Non-Abelian statistics of Majorana zero modes in the presence of an Andreev bound state

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The low-energy Andreev bound states (ABSs) mixing with the Majorana zero modes (MZMs) may destroy the non-Abelian braiding statistics of the MZMs. We numerically studied the braiding properties of MZMs when an ABS is present. Numerical simulation results support the argument that the ABS can be regarded as a pair of weakly coupled MZMs. The non-Abelian braiding properties of MZMs exhibit oscillation behaviour with respect to the braiding time if the ABS-related dynamic phase is present. Remarkably, such dynamic phase can be eliminated by tuning the magnetic field or gate voltage. In this way, the non-Abelian braiding statistics independent of the braiding time retrieves so that the topological quantum computation could still be robust even when the ABS is engaged.

PACS numbers:

Introduction — Majorana zero mode (MZM) is deemed as the most promising candidate for topological quantum computation (TQC) [1–2] for its non-Abelian statistics. The exploration for MZMs in topological superconductors (TSCs) has been drawing extensive attention in the last decade [3–12]. To date, TSC has been realized in various experimental platforms and the signals for MZMs have also been reported [13–29]. The semiconductor-conventional superconductor heterostructure, first experimentally realized one among these systems, is regarded as one of the most promising platforms for the realization of TQC. Experimentally, however, a roadblock in such system is that other low-energy modes, e.g., Andreev bound states (ABSs), may blend with the MZMs [27–36]. Such ABSs are widely viewed as a pair of weakly coupled MZMs with finite separation [30–32, 33]. It would conceal the information of one MZM due to the weak hybridization and masquerade as a single MZM. For example, in the tunneling experiments, the ABS will also induce a zero-bias conductance peak, which is very similar to the case of MZM [30]. Thus, ABSs and MZMs are hard to be distinguished through normal measurement such as the local electric transport.

Recently, various experimental schemes have been proposed to distinguish these two types of states. These proposals can be classified into two categories: measuring the response to the local perturbations [30–33] or detecting the non-local conductance correlations [37–43]. Nevertheless, all these methods strongly depend on the detailed properties of the materials, therefore other possibilities besides MZMs and ABSs are still hard to be ruled out. As a result, there is still no smoking-gun evidence for the identification of the MZMs yet. One convincing way to distinguish the MZM from the ABS is based on its non-Abelian statistics [44–46] which is a global property instead of a local one. However, only few studies have paid attention to this topic yet [47]. Therefore, as the first step toward distinguishing the MZM from the ABS, it is essential to theoretically investigate the difference between the braiding properties of MZM and ABS. What’s more, since MZM and ABS are usually mixed
with each other, it is necessary to study the braiding statistics in the presence of both MZMs and ABS. Such investigation will also shed light on how to realize TQC when ABS is also engaged.

Based on the cross-shaped junction shown in Fig. 1(d) [38, 39], we numerically studied the non-Abelian braiding of the MZMs in a semiconductor-superconductor nanowire. The braiding properties of the MZMs are robust against perturbations as expected. On the contrary, the braiding results exhibit a dynamic-phase-induced periodic oscillation if one pair of MZMs is replaced by an ABS. Further investigation implies that an ABS can be decomposed into two weakly coupled MZMs. In this way, the braiding results in the presence of an ABS can be explained by combining the non-Abelian geometric phase with the ABS-related dynamic phase. Such dynamic phase can be eliminated by reversing the eigenenergy at the middle of the symmetric braiding protocol, which is widely adopted in the geometric quantum computation (GQC). In this way, the non-Abelian braiding properties will come back to its original form, which implies that the TQC could still be robust even when the ABS is engaged.

**MZM and ABS in the semiconductor-superconductor nanowire** — The tight-binding model for a one-dimensional s-wave superconductor with Rashba spin-orbit coupling can be described as:

\[
H_{1D} = \sum_{\mathbf{R},d,\alpha} -t_0 (\psi^\dagger_{\mathbf{R}+d,\alpha} \psi_{\mathbf{R},\alpha} + h.c.) - \mu \psi^\dagger_{\mathbf{R},\alpha} \psi_{\mathbf{R},\alpha} + U_{\mathbf{R}} \psi^\dagger_{\mathbf{R},\alpha} \psi_{\mathbf{R},\alpha} + \Delta \psi^\dagger_{\mathbf{R},\alpha} \psi^\dagger_{\mathbf{R},-\alpha} + h.c. + \sum_{\mathbf{R},\alpha,\beta} \Delta e^{i\phi} \psi^\dagger_{\mathbf{R},\alpha} \psi^\dagger_{\mathbf{R},-\alpha} + \Delta e^{i\phi} (\sigma \cdot \mathbf{d})_{\alpha\beta} \psi_{\mathbf{R},\alpha} \psi_{\mathbf{R},\beta}
\]

where \(\mathbf{R}\) denotes the lattice site, \(\mathbf{d}\) is the unit vector that \(d_x\) and \(d_y\) connect the nearest neighbor sites along the \(x\)- and \(y\)-direction, respectively. Besides, \(\alpha\) and \(\beta\) are spin indices, \(t_0\) denotes the hopping amplitude, \(\mu\) is the chemical potential, \(U_{\mathbf{R}}\) is the Rashba coupling strength, and \(V\) is the Zeeman energy induced by the magnetic field along the \(x\)- or the \(y\)-direction. The superconducting pairing amplitude is denoted as \(\Delta\), and \(\phi\) is the pairing phase. The practical parameters can be obtained from a recent experiment [33]. Here, we adopt \(\Delta = 250\mu eV\), \(t_0 = 10\Delta\), and \(U_{\mathbf{R}} = 2\Delta\) in the following calculations.

As the localized state bound with the impurity or spatial defect, the ABS is absent in a clean system. Experimentally, the ABS is usually presented with the quantum dot (QD) confinement at the end of the nanowire [33, 39]. Thus, for obtaining an ABS, a QD confinement potential should be included. As depicted in Fig. 1(b), a sinusoidal local chemical potential in the form of \(V_d(R) = -V_D \cos(2\pi \frac{R - L_D/2}{L_D})\) is presented at the right end of the nanowire, in which \(L_D = 10a\) is the half length of the QD and \(V_D = 0.1t_0\) is the depth of the potential well in the QD. When the external magnetic field is turned on, as shown in Fig. 1(c), a low-energy ABS will be trapped in the QD before the system entering into the topologically non-trivial phase. In the condition that the magnetic field is larger than \(\Delta\), the topological phase transition happens and therefore two MZMs will distribute non-locally at both ends of the nanowire [Fig. 1(a)]. On the contrary, the ABS is a localized state distributed at one end of the nanowire [Fig. 1(b)]. Remarkably, the ABS’s energy is close to zero at \(V = 0.6\Delta\). In such condition, the MZM and the ABS is hard to be distinguished by the local conductance measurement.

**Non-Abelian braiding of MZMs** — The non-Abelian braiding of MZMs can be simulated based on the cross-shaped junction [Fig. 1(d)]. Each of the four arms in the junction is a topologically non-trivial semiconductor-superconductor nanowire [\(V_z = 2\Delta\) as indicated by the vertical black line in Fig. 1(c)]. Four gates (G1, G2, G3, and G4) are situated near the cross point, each arm can be connected to (separated from) the others by turning off (on) the gate voltage in the corresponding gate. Initially, gate voltages in G1 and G3 are turned on while in G2 and G4 are turned off, hence three pairs of MZMs (\(\gamma_{2j-1}\) and \(\gamma_{2j}\)) with \(j = 1, 2, 3\) are localized at the ends of the three divided parts. The aim of our braiding operation is swapping the positions of \(\gamma_2\) and \(\gamma_3\). The nanowire along the \(y\)-direction is an auxiliary one in such operation. The braiding protocol takes three steps (the time cost for each step is \(T\)) to swap \(\gamma_2\) and \(\gamma_3\) spatially. In the first step, G1 is turned off and then G2 is turned on, hence \(\gamma_2\) is transmitted to the top of G2. In the second step, G3 is turned off and then G1 is turned on, so that \(\gamma_3\) is moved to the original position of \(\gamma_2\). In the third step, G2 is turned off and then G3 is turned on, as a result, the spatial positions of \(\gamma_2\) and \(\gamma_3\) are swapped.

Initially, the effective low-energy Hamiltonian describing each separated arm is in the form of \(H_{eff} = i\epsilon_j \gamma_{2j-1}\gamma_{2j}\) (\(j = 1, 2, 3\)). Here, \(\epsilon_j\) is the coupling energy between MZMs induced by the finite-size effect, which is exponentially small and can be neglected in the most cases. Thus, the eigenstates are in the wavefunctions of \(\psi^\dagger_j(0) = (\gamma_{2j-1} \pm i\gamma_{2j})/\sqrt{2}\). During the braiding process, the wavefunction evolves as \(|\psi^\dagger_j(t)\rangle = U(t)|\psi^\dagger_j(0)\rangle\), where \(U(t) = \hat{T}\exp[i\int_0^t dt H(\tau)]\) is the time-evolution operator (\(\hat{T}\) is the time-ordering operator). Since \(\epsilon_j\) is exponentially small, the dynamic phase accumulated can be neglected. However, a topological geometric phase \(\pi\) is picked up during the braiding hence we have \(\gamma_2 \rightarrow \gamma_3\) and \(\gamma_3 \rightarrow -\gamma_2\) [44]. Thus, if \(\gamma_2\) and \(\gamma_3\) are swapped once, then the wavefunction will evolve into \(\psi^\dagger_1(3T) = (\gamma_1 \pm i\gamma_3)/\sqrt{2}\) and \(\psi^\dagger_3(3T) = (-\gamma_2 \pm i\gamma_4)/\sqrt{2}\). After \(\gamma_2\) and \(\gamma_3\) are swapped twice in succession, the wavefunctions are in the forms of \(\psi^\dagger_1(6T) = (\gamma_1 \mp i\gamma_3)/\sqrt{2} = \psi^\dagger_1(0)\) and \(\psi^\dagger_2(6T) = (-\gamma_3 \pm i\gamma_4)/\sqrt{2} = -\psi^\dagger_2(0)\). Our numerical simulation results [Fig. 1(e)] confirm that \(\psi^\dagger_j\) evolves...
Non-Abelian braiding in the presence of ABS — Naively, the presence of the ABSs are supposed to ruin the non-Abelian statistics of the MZMs. However, based on the simple assumption that an ABS can be decomposed into two MZMs [30, 32, 33], we found that the braiding operation between MZMs can still be performed in the cross-shaped junction if only one pair of MZMs is replaced by an ABS. This suggests that the possible non-Abelian braiding properties of MZMs could still be exhibited even in the presence of an ABS.

As shown in Fig. 2(a), the left arm of the cross-shaped junction is tuned into the ABS region, while the other three arms are still in the topologically non-trivial region supporting MZMs as before. At the beginning, the low-energy ABS can be regarded as a pair of coupled MZMs $\gamma_1$ and $\gamma_2$ which are spatially separated with a limited distance. In this point of view, the braiding in the presence of ABS is equivalent to the exchange between one “free” MZM $\gamma_3$ and another MZM $\gamma_2$ bounded in the ABS. A complete braiding operation, which swaps $\gamma_2$ and $\gamma_3$ twice in succession, can be decomposed into five operation steps: step 1, 3, and 5 are the coupling (fusion) process with nonvanishing dynamic phase that forms an ABS; step 2 and 4 are the exchange operation swapping one “free” MZM and another MZM bounded in the ABS.

FIG. 2: (a) The braiding process with an ABS engaged. The low-energy ABS can be regarded as a pair of coupled MZMs $\gamma_1$ and $\gamma_2$ which are spatially separated with a limited distance. In this point of view, the braiding in the presence of ABS is equivalent to the exchange between one “free” MZM $\gamma_3$ and another MZM $\gamma_2$ bounded in the ABS. (b) An illustration for the braiding operation in the presence of the ABS. A complete braiding operation, which swaps $\gamma_2$ and $\gamma_3$ twice in succession, can be decomposed into five operation steps: step 1, 3, and 5 are the coupling (fusion) process with nonvanishing dynamic phase that forms an ABS; step 2 and 4 are the exchange operation swapping one “free” MZM and another MZM bounded in the ABS.

FIG. 3: (a) Energy spectrum of the bulk states and the ABS in the cross-shaped junction during the braiding process. Here we set the Zeeman field in the left arm of the cross-junction as $V_4 = 0.6\Delta$, while the Zeeman field in the other three arms is still $V_4 = 2\Delta$. In such case, an ABS is presented at the left arm while two pairs of MZMs are presented in the other three arms as before. (b) Numerical simulation of the braiding results as functions of the braiding time-cost $T$.

$H(t) = i \sum_{\gamma} \epsilon_{\gamma}(t) \gamma_i \gamma_j$, in which the corresponding time-evolution operator $U(t) = \hat{T} \exp[i \int_0^T d\tau H(\tau)]$ is equivalent to a unitary transformation on $\gamma_i$ and $\gamma_j$ [30]:

$$\gamma_i = \cos(\theta_k)\gamma_i + \sin(\theta_k)\gamma_j,$$

$$\gamma_j = -\sin(\theta_k)\gamma_i + \cos(\theta_k)\gamma_j.$$  (2)

where $\theta_k/2 = \int \epsilon_{\gamma}(t) dt$ is the dynamic phase induced by the coupling energy between the two MZMs which form the ABS.

With such a dynamic phase being taken into account, the braiding in the presence of ABS can be clearly described as below. In the first step of the braiding operation, one MZM in the ABS is moved to the top arm. The coupling strength between $\gamma_1$ and $\gamma_2$ varies during such moving process. This will induce a unitary evolution as $\gamma_1 \rightarrow \tilde{\gamma}_1 = \cos(\theta_1)\gamma_1 + \sin(\theta_1)\gamma_2$ and $\gamma_2 \rightarrow \tilde{\gamma}_2 = -\sin(\theta_1)\gamma_1 + \cos(\theta_1)\gamma_2$, in which $\theta_{1/2} = \int_0^T \epsilon_{12}(t) dt$ is the dynamic phase accumulated. Such dynamic phase will significantly alter the braiding properties below. In the second step of the braiding operation, $\gamma_3$ is moved to the original position of $\gamma_2$. In the third step of the braiding operation, $\gamma_2$ instead of $\gamma_2$ is moved to the original position of $\gamma_3$, which gives rise to $\gamma_2 \rightarrow \tilde{\gamma}_3$ and $\gamma_3 \rightarrow -\tilde{\gamma}_2$. Similarly, the dynamic phase induced by the coupling between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ will also alter the braiding results in the second half of the complete braiding process that $\gamma_2$ and $\gamma_3$ are swapped twice in succession. Such complete braiding process can be effectively decomposed into five
operation steps as shown in Fig. 2(b). Operation steps 1, 3 and 5 are the fusion (coupling) process with nonvanishing dynamic phase that forms an ABS, while operation steps 2 and 4 are the exchange operation swapping one “free” MZM and another MZM bounded in the ABS. Therefore, the wavefunctions evolve as \( \gamma_3 \rightarrow -\gamma_3 \) and \( \gamma_2 \rightarrow -\gamma_2 \) after the complete braiding process, in which \( \gamma_3 = -\sin(\theta_2)\gamma_1 + \cos(\theta_2)\gamma_3 \) and \( \theta_2/2 = \int_{t_1}^{t_f} \epsilon_3(t)dt \) is the dynamic phase accumulated due to the fusion (coupling) energy between \( \gamma_3 \) and \( \gamma_1 \).

The numerical simulation results support the analysis above pretty well. Fig. 3(a) shows that the ABS’s eigenenergy becomes relatively larger and cannot be neglected during the braiding. Such larger energy will significantly alter the braiding results. For example, \( \psi_2^\dag(0) = (\gamma_3 + i\gamma_4)/\sqrt{2} \) will finally evolve into \( \psi_2^\dag(6T) = (-\gamma_3 + i\gamma_4)/\sqrt{2} \) (the other states will also show the similar behavior in the wavefunction evolution with some additional trivial dynamic phase accumulated), see (31), hence the weight of \( \psi_2^\dag(6T) \) on \( \psi_2^\dag(0) \) is \((1 - \cos(\theta_2))/2\), on \( \psi_2^\dag(0) \) is \((1 + \cos(\theta_2))/2\), and on \( \psi_2^\dag(0) = \sin(\theta_2)/2 \), in which \( \theta_2 \) is the dynamic phase accumulated during \( t \in [T, 4T] \) due to the fusion (coupling) energy [red curves in Fig. 3(a)]. The mean value of such fusion energy is about \( \bar{\epsilon}_2 \approx 10^{-2}\Delta \), which gives rise to the oscillation period of \( \Delta T = \frac{\pi}{\bar{\epsilon}_2} \approx 300 \Delta \). The numerical results shown in Fig. 3(b) is fully consistent with these analytical predictions.

Elimination of the dynamic phase — As a dynamical effect, such oscillation behaviour is expected to be removed by eliminating the dynamic phase. A direct way is to decrease the ABS’s fusion energy during the braiding process. In Fig. 3 we set the chemical potential in the cross point of the junction as \( \mu_c = \mu = -2t_0 \). This means that the arms are perfectly connected to each other when the gate voltage is turned off. It is worth noting that the fusion energy significantly decreases if these arms are not perfectly connected to each other [52]. As shown in Fig. 4(a), in the case of \( \mu_c = -2t_0 - 5.6\Delta \), the eigenenergy of the ABS during the braiding process is reduced to \( \sim 10^{-3}\Delta \). Therefore, the braiding results oscillate very slowly with respect to the braiding time \( T \) [Fig. 4(c)]. In addition, for relatively small braiding time \( T \sim 200/\Delta \), the dynamic phase is much smaller compared with the geometric phase so that the braiding results retrieve its original form.

In the traditional GQC, the dynamic phase can be eliminated through the spin-echo technique [53][55] which reverses the sign of the eigenenergy at the middle of the symmetric braiding protocol. Similar technique is also a powerful method to cancel out the ABS-related dynamic phase. Noticing that the spectrum of the ABS [Fig. 4(c)] is nearly symmetric about the zero-energy in the vicinity of \( V_z = 0.6\Delta \). Hence, it is possible to reverse the ABS’s eigenenergy through modulating the Zeeman energy. For example, if the magnetic field is \( V_z = 0.62\Delta \) during \( t \in [0, T/2] \), then the sign of the ABS’s eigenenergy could be reversed by setting the magnetic filed as \( V_z = 0.58\Delta \) during \( t \in [T/2, T] \). Such dynamic phase elimination works better if the magnetic field is modulated in a more smooth way. As shown in Fig. 4(b), we choose a sinusoidal magnetic field as \( V_z = [0.603 + 0.02 \cos(t/T - \pi)]\Delta \) (since the spectrum is not perfectly symmetric, the mean value of \( V_z \) will slightly deviate from 0.6\Delta), hence the eigenenergy of the ABS changes its sign after \( t = T/2 \). In this way, the dynamic phase can be canceled out so that the MZMs’ non-Abelian statistics retrieves as shown in Fig. 4(d). Finally, as shown in Fig. 4(c) and 4(d), the braiding time should satisfy the adiabatic condition as \( T \gtrsim 200/\Delta \) to avoid the mixing between the subgap states and the bulk states. Since the \( \Delta \) is typically in the order of \( \sim 1\text{meV} \), the braiding time \( T \) should be in the order \( \sim 0.1\text{ns} \), which could be realized based on the state of art of the terahertz technology.

Discussion — We have shown that the Non-Abelian statistics of MZMs can still be preserved in the presence
of ABS. It would suggest that ABS will provide more advantages in the following TQC. In the TQC, the information stored in the qubit could be read out by measuring the parity of two combined MZMs \cite{1806.02801, 1807.06632}. Since MZMs are usually non-locally distributed, it means that the reading out technique would be very difficult. While in the case of ABS, however, the reading out technique could be rather easy since a ABS consists of two little separated MZMs. Moreover, since the ABS is deemed as two weakly coupled MZMs with finite distance, the dynamic phase elimination method discussed above can also be performed for the finite-size-induced partially overlapped MZMs. In one of our previous work \cite{2017YFA0303301, 2019YFA0308403}, we have revealed that the spectrum of such partially overlapped MZMs will cross at zero-energy with definite parity by modulating the Zeeman field or gate voltage. It implies that the dynamic phase can be canceled out by modulating either the Zeeman field or the gate voltage. Therefore, the TQC can be realized even in shorter TSC nanowire.

Acknowledgement. — Wenqin Chen and Jiachen Wang contributed equally to this work. This work is financially supported by NSFC (Grants No. 11974271) and NBRPC (Grants No. 2017YFA0303301, and No. 2019YFA0308403).
See Supplementary materials.

Actually the bulk gap also becomes larger with the mismatch of the chemical potentials. The bulk gap is about $0.1\Delta$ in Fig. 3 while increases to about $0.2\Delta$ with the mismatch of the chemical potentials. Thus the corresponding adiabatic condition changes from $T \gtrsim 400/\Delta$ to $T \gtrsim 200/\Delta$.