The weak rate of convergence for the Euler-Maruyama approximation of one-dimensional stochastic differential equations involving the local times of the unknown process

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Abstract

In this paper, we consider the weak convergence of the Euler-Maruyama approximation for one dimensional stochastic differential equations involving the local times of the unknown process. We use a transformation in order to remove the local time $L^a_t$ from the stochastic differential equations of type:

$$X_t = X_0 + \int_0^t \varphi(X_s)dB_s + \int_\mathbb{R} \nu(da)L^a_t$$

where $B$ is a one-dimensional Brownian motion, $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, and $\nu$ is a bounded measure on $\mathbb{R}$ and we provide the transformation for Euler-maruyama for the stochastic differential equations without local time. After that, we conclude the approximation of Euler-maruyama $X^n_t$ of the above mentioned equation, and we provide the rate of weak convergence $\text{Error} = E|G(X_T) - G(X^n_T)|$, for any function $G$ in a certain class.

Keywords: Euler-Maruyama approximation, weak convergence, stochastic differential equation, local time, bounded variation.

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1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a process stochastic involving the local time defined by the stochastic differential equations:

$$X_t = X_0 + \int_0^t \varphi(X_s)dB_s + \int_\mathbb{R} \nu(da)L^a_t$$

where $B$ is a one-dimensional Brownian motion, $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, and $\nu$ is a bounded measure on $\mathbb{R}$, $L^a_t$ denotes the local time at $a$ for the time $t$ of the semimartingale $X$.

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In [14], if \( \nu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) (i.e. \( \nu(da) = g(a)da \)) then (1) becomes the usual Itô equation:

\[
X_t = X_0 + \int_0^t \varphi(X_s) dB_s + \int_0^t (g\varphi^2)ds
\]  

(2)

In general, we consider the following stochastic differential equation (SDE) with coefficients \( b \) and \( \sigma \), driven by a Brownian motion \( B \) in \( \mathbb{R} \):

\[
X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s)ds
\]  

(3)

where the drift coefficient \( b \) and the diffusion coefficient \( \sigma \) are Borel-measurable functions from \( \mathbb{R} \) into \( \mathbb{R} \) and \( X_0 \) is an \( \mathbb{R} \)-valued random variable, which is independent of \( B \) and \( b \).

The continuous Euler scheme \( \{X^n_t, 0 \leq t \leq T\} \) for the SDE (3) on the time interval \([0, T]\) is defined as follows: \( X^n_0 = X_0 \), and

\[
X^n_t = X^n_0 + \int_0^t \sigma(X^n_{s\wedge k}) dB_s + \int_0^t b(X^n_{s\wedge k})ds,
\]  

(4)

For \( \eta_k \leq t \leq \eta_{k+1}, k = 0, 1, 2, ..., n - 1 \), where \( 0 = \eta_0 \leq \eta_1 \leq \cdots \leq \eta_n = T \) is a sequence of random partitions of \([0, T]\). The weak convergence of the stochastic differential equations has been studied by Avikainen [1], Bally and Talay [2], Mikulevičius and Platen [16], D.Talay and L.Tubaro [21], and the weak convergence of SDEs with discontinuous coefficients has been studied by Chan and Stramer [8], Yan [23], Arturo Kohatsu-Higa [12].

We use the same criteria that mentioned in [11], that the definition of weak convergence with order \( \gamma > 0 \) is that for all functions \( f \) in a certain class, there exists a positive constant \( C \), which does not depend on \( \Delta t \), such that:

\[
|E[f(X_T)] - E[f(X^n_T)]| \leq C(\Delta t)^\gamma
\]  

(5)

The stochastic differential equations of the type (1) have been studied by Stroock and Yor [20], Portenko [19], Le Gall [13, 14], Blei and Engelbert [6], Bass and Chen [4] and the approximation of Euler-Maruyama for SDE of type (1) has been studied by, e.g., Benabdallah, Elkettani and Hiderah [5].

In [14], when \( \nu = \beta \delta_0 \) (\( \delta(0) \) denotes the Dirac measure at 0) and \( \varphi = 1 \), we get

\[
X_t = X_0 + B_t + \beta L^0_t(X), |\beta| \leq 1
\]  

(6)

The solution of equation (6) is the well-known process called the skew Brownian motion which has been studied by Harrison and Shepp [10], Ouknine [17, 18], Bouhadou and Ouknine [7], Lejay [15], Barlow [3], Étore and Martinez [9], Walsh [22].

Our goal of this paper is that under the assumption that the SDE (1) has a weak solution and that it is unique, we study the conditions under which the Euler scheme \( \{X^n_t : 0 \leq t \leq T\} \) converges weakly to the exact solution \( \{X_t : 0 \leq t \leq T\} \) of the SDE (1).

We face two major problems: the presence of local time in equation (1) and the inability to provide a simple discretization scheme, and the presence of a discontinuity for the diffusion coefficient \( \varphi \).

Our paper is divided as follows: In section (2), we present the important propositions, assumptions and stochastic differential equations involving local time. Our main results on the weak convergence of Euler scheme of one-dimensional stochastic differential equations involving the local time are given in section (3). Some numerical examples are given in section (4). All proofs of the theorems are given in section (5).
2 Preliminaries and approximation

In this section, we provide the definitions, important propositions, assumptions and stochastic differential equations involving local time.

Definition 2.1. Let

\[ T_f(x) := \sup_{N} \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})|, \]

where the supremum is taken over \( N \) and all partitions \(-\infty < x_0 < x_1 < ... < x_N = x\) be the total variation function of \( f \). Then we say that \( f \) is a function of bounded variation, if \( V(f) := \lim_{x \to \infty} T_f(x) \) is finite, and call \( V(f) \) the (total) variation of \( f \).

Definition 2.2. We say that a function \( f \) has at most polynomial growth in \( \mathbb{R} \) is there exist an integer \( k \) and a constant \( C > 0 \) such that

\[ |f(x)| \leq C(1 + |x|^k) \]

for any \( x \in \mathbb{R} \).

Remark 2.3. Let \( C^k_p(\mathbb{R}) \) denote the space of all \( C^k \)-functions of polynomial growth (together with their derivatives).

Note 2.4. \( BV(\mathbb{R}) \) will denote the space of all functions \( \varphi : \mathbb{R} \to \mathbb{R} \) of bounded variation on \( \mathbb{R} \) such that:

1. \( \varphi \) is right continuous.
2. There exists an \( \varepsilon > 0 \) such that \( \varphi(x) \geq \varepsilon \) for all \( x \).

If \( \varphi \) is in \( BV(\mathbb{R}) \), \( \varphi(x^-) \) will denote the left-limit of \( \varphi \) at point \( x \) and \( \varphi'(dx) \) will be the bounded measure associated with \( \varphi \).

Note 2.5. \( M(\mathbb{R}) \) will denote the space of all bounded measures \( \nu \) on \( \mathbb{R} \) such that:

\[ |\nu(\{x\})| < 1, \forall x \in \mathbb{R} \]

We have the stochastic differential equation (1), if \( \varphi \) is in \( BV(\mathbb{R}) \) and \( \nu \) is in \( M(\mathbb{R}) \), then the stochastic differential equation (1) has a unique strong solution as soon as \( |\nu(da)| < 1 \).

Theorem 2.6. Let \( \varphi \) be in \( BV(\mathbb{R}) \) and \( \nu \) be in \( M(\mathbb{R}) \). Then existence and pathwise uniqueness of solution hold for (1).

Proposition 2.7. Let \( \nu \) be in \( M(\mathbb{R}) \). There exists a function \( f \) in \( BV(\mathbb{R}) \), unique up to a multiplicative constant, such that:

\[ f'(dx) + (f(x) + f(x^-))\nu(dx) = 0 \tag{7} \]

If we require that \( f(x) \xrightarrow{x \to -\infty} 1 \), then \( f \) is unique and is given by:

\[ f(x) = f_\nu(x) = \exp(-2\nu([-\infty; x])) \prod_{y \leq x} \left( \frac{1 - \nu(\{y\})}{1 + \nu(\{y\})} \right) \tag{8} \]

Where \( \nu^c \) denotes the continuous part of \( \nu \).

Proposition 2.8. Let \( \varphi \) be in \( BV(\mathbb{R}) \) and \( \nu \) be in \( M(\mathbb{R}) \) and \( f_\nu \) be define by(8) and set:

\[ F_\nu(x) = \int_0^x f_\nu(y)dy \tag{9} \]

Then \( X \) is a solution of equation (1), if and only if \( Y := F_\nu(X) \) is a solution of:

\[ dY_t = (\varphi f_\nu) \circ F^{-1}(Y_t)dB_t \tag{10} \]
The Euler scheme \( Y_t^n : 0 \leq t \leq T \) for the SDE (10) on the time interval \([0, T]\) is defined as follows: \( Y_0^n = Y_0 \), and
\[
Y_t^n = Y_{\eta_k}^n + (\varphi f_\nu) \circ F^{-1}(Y_{\eta_k}^n)(B_t \circ B_{\eta_k})
\]
for \( \eta_k < t \leq \eta_{k+1}, k = 0, 1, 2, ..., n \) where \( 0 = \eta_0 \leq \eta_1 \leq ... \leq \eta_n = T \). This Euler scheme can be written as
\[
Y_t^n = Y_0^n + \int_0^t (\varphi f_\nu) \circ F^{-1}(Y_u^n)dB_s
\]
for \( t \geq 0 \). In [4], the SDE (1) has existence of strong solutions and pathwise uniqueness when \( \varphi \) is a function on \( \mathbb{R} \) that is bounded above and below by positive constants and such that there is a strictly increasing function \( f \) on \( \mathbb{R} \) such that
\[
|\varphi(x) - \varphi(y)|^2 \leq |f(x) - f(y)|, x, y \in \mathbb{R}
\]
and \( \nu \) is a finite measure with \( |\nu(\{a\})| \leq 1 \) for every \( a \in \mathbb{R} \).

**Theorem 2.9.** Suppose \( \varphi \) is a measurable function on \( \mathbb{R} \) that is bounded above and below by positive constants and suppose that there is a strictly increasing function \( f \) on \( \mathbb{R} \) such that
\[
|\varphi(x) - \varphi(y)|^2 \leq |f(x) - f(y)|, x, y \in \mathbb{R}
\]
For any finite signed measure \( \nu \) on \( \mathbb{R} \) such that \( \nu(\{x\}) < \frac{1}{2} \) for each \( x \in \mathbb{R} \) and every \( x_0 \in \mathbb{R} \), the SDE
\[
X_t = X_0 + \int_0^t \varphi(X_s)dB_s + \int_{\mathbb{R}} \nu(da)L^a_t, t \geq 0
\]
has a continuous strong solution and the continuous solution is pathwise unique.

Define
\[
\pi(x) := \begin{cases} 
-\frac{\log(1-2x)}{2x}, & \text{if } x \in (-\infty, 0) \cup (0, \frac{1}{2}) \\
1, & \text{if } x = 0.
\end{cases}
\]
Let \( \mu(dx) := \pi(\nu(\{x\}))\nu(dx) \) which is a finite signed measure. Define
\[
S(x) := \int_0^x e^{-2\pi(-\infty,y)}dy
\]
Since \( \mu \) is a finite measure, \( S' \) is right continuous and strictly positive. Hence \( S \) is increasing and one-to-one. Let \( S^{-1} \) denote the inverse of \( S \), let \( S'_l \) denote the left continuous version of \( S' \), i.e., the left hand derivative of \( S' \), and let \( \sigma'_l \) denote the left hand derivative of \( S^{-1} \). Since \( \mu \) is a finite signed measure, \( S' \) is of bounded variation. Let \( \{Y_t, t \leq 0\} \) solve
\[
dY_t = (S'_l(\varphi)) \circ S^{-1}(Y_t)dB_t, Y_0 = S(X_0)
\]
Let \( X = S^{-1}(Y) \), and \( Y_t \) define by (15), in [4], they must show that \( X \) is a solution to (14).

3 The main results

In this section, we provide the main theorems. we study two cases: the continuous of function \((\varphi f_\nu) \circ F^{-1}(\cdot)\) and the discontinuous of function \((\varphi f_\nu) \circ F^{-1}(\cdot)\).
3.1 Main Theorems

In this section, we present the following results on the rates of the Euler-Maruyama approximation. Let $X_t$ be defined as in equation (1), $X^n_t$ be the Euler scheme for equation (1), let $F$ be defined by equation (9), and let $f_\nu$ be from the proposition (2.7). Here, we suppose that the assumptions of proposition (2.8) satisfied, and the constant $C$ may change from line to line and from theorem to theorem.

**Theorem 3.1.** For any function $g : \mathbb{R} \to \mathbb{R}$, if for each $G := g \circ F^{-1} \in C^2_p(\gamma + 1)(\mathbb{R})$, and $\psi(.) := (\varphi.f_\nu) \circ F^{-1}(.)$ is continuous, then, there exists a constant $C > 0$, which does not depend on $n$, such that

$$
E |g(X^n_T) - g(X_T)| \leq C/n^\gamma
$$

holds.

Next we give a sufficient condition under which the Euler scheme converges weakly to the weak solution of SDE (2) case: discontinuous of function $(\varphi.f_\nu) \circ F^{-1}(.)$. Let $\psi_1(z) := \lim_{y \to z} \psi_2(y) > 0$

**Theorem 3.2.** Suppose that $\psi := (\varphi.f) \circ F^{-1}(.)$ has at most linear growth with $D_\psi$ of Lebesgue measure zero and $E(Y_0)^4 < \infty$. If $\psi_1(z) > 0$ for $z \in D_\psi$, and $G := g \circ F^{-1} \in C^2_p(\gamma + 1)(\mathbb{R})$, then the Euler scheme of SDE(1) converges weakly to the unique weak solution of SDE(1), where $D_\psi$ is the set of discontinuous points of $\psi(.)$.

If we suppose that the conditions of theorem (2.9) satisfied, we show the following note.

**Note 3.3.** The same results in the theorems (3.1,3.2) stay valid, if the conditions of theorem (2.9) are hold.

4 Some examples

**Example 4.1.** In equation (1), let $\nu(dx) = \alpha \delta_0(dx), |\alpha| < 1$ where $\delta_0$ is the Dirac measure at 0 ) and

$$
\varphi = \begin{cases} 
\frac{1+\alpha}{1-\alpha} \exp \left( -\frac{\alpha}{1+\alpha} x \right) & \text{if } x \geq 0 \\
\exp(x) & \text{if } x < 0
\end{cases}
$$

Using proposition (2.7), we have

$$
f_\nu(x) = \begin{cases} 
\frac{1-\alpha}{1+\alpha} & \text{if } x \geq 0 \\
1 & \text{if } x < 0
\end{cases}
$$

Using proposition (2.8), we have

$$
F(x) = \begin{cases} 
\frac{1-\alpha}{1+\alpha} x & \text{if } x \geq 0 \\
x & \text{if } x < 0
\end{cases}
$$

Then

$$
\psi(y) := (\varphi.f) \circ F^{-1}(y) = \begin{cases} 
\exp(-y) & \text{if } y \geq 0 \\
\exp(y) & \text{if } y < 0
\end{cases}
$$

We note $\psi$ is continuous, then for any function $g : \mathbb{R} \to \mathbb{R}$, such that $g \circ F^{-1} \in C^2_p(\gamma + 1)(\mathbb{R})$, for any $\gamma > 0$, for example

$$
g(x) = \frac{1}{1 + \left( \frac{1-\alpha}{1+\alpha} x \right)^2} 1_{x \geq 0} + \frac{1}{1 + \frac{1}{x^2} 1_{x < 0}}
$$

there exist a constant $C > 0$, which does not depend on $n$, such that
\[ |E[g(X^n_T)] - E[g(X_T)]| \leq \frac{C}{n} \]

**Example 4.2.** In equation (1), let \( \nu(x) = \alpha \delta_0(x) \), \( |\alpha| < 1 \), where \( \delta_0 \) is the Dirac measure at 0, and \( E(X_0)^4 = E(Y_0)^4 < \infty \) we define \( \varphi = 1 \), such that

Using proposition (2.7), we have

\[ f(x) = f_\nu(x) = \begin{cases} 1 - \alpha & \text{if } x \geq 0 \\ \frac{1}{1+\alpha} & \text{if } x < 0 \end{cases} \]

And by using proposition (2.8), we have

\[ F_\nu(x) = \begin{cases} \frac{1-\alpha}{1+\alpha} x & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}, \quad F_\nu^{-1}(y) = \begin{cases} \frac{1+\alpha}{\alpha} y & \text{if } y \geq 0 \\ y & \text{if } y < 0 \end{cases} \]

Then, we obtain

\[ \psi(y) = (\varphi \circ f_\nu \circ F_\nu^{-1})(y) = \begin{cases} \frac{1-\alpha}{1+\alpha} & \text{if } y \geq 0 \\ \frac{1}{1+\alpha} & \text{if } y < 0 \end{cases} \] (16)

And

\[ Y_t = Y_0 + \int_0^t \psi(Y_s) dB_s \] (17)

The Euler scheme \( \{ Y^n_t : 0 \leq t \leq T \} \) for the SDE (17), on the time interval \([0,T]\) is defined as follows: \( Y^n_0 = Y_0 \), and

\[ Y^n_t = Y^n_{\eta_k} + \psi(Y^n_{\eta_k})(B_t - B_{\eta_k}) \] (18)

For \( \eta_k < t \leq \eta_{k+1}, k = 0, 1, 2, ..., n \), where \( 0 = \eta_0 \leq \eta_1 \leq ... \leq \eta_n = T \), the coefficient of equation (17) is discontinuous at point 0, and, we have:

- \( D_\psi = \{0\} \)
- \( \psi(z) := \lim \sup_{z \to y} \psi(y) > 0 \)
- \( \psi(\cdot) \) is at most linear growth with \( D_\psi \)

We have the Euler scheme of SDE(18) converges weakly to the unique weak solution of SDE((17)), then the Euler scheme \( X^n_t \) converges weakly to the unique weak solution \( X_t \) of SDE((1)).

## 5 Proofs of theorems

Before proving the theorems below, we introduce some notations.

**Note 5.1.** The constant \( C \) may change from line to line and from theorem to theorem.

Let \( B \) be a one-dimensional brownian motion \( (B_t)_{t \in [0, T]} \), and \( \varphi \) is in \( BV(\mathbb{R}) \) and \( \nu \) is in \( M(\mathbb{R}) \) and

\[ X_t = X_0 + \int_0^t \varphi(X_s) dB_s + \int_0^t \nu(da) L^\alpha_t. \] (19)

and \( Y_t \) is solution of the equation

\[ dY_t = (\varphi \circ f_\nu \circ F_\nu^{-1})(Y_t) dB_t. \] (20)

for which, as we get by the proposition(2.8), where \( F \) be defined by the equation (9), \( f_\nu \) get by the proposition(2.7), and let \( Y^n_t \) is a solution by Euler scheme for equation(10) and defined by equation(12).
5.1 Proof of theorem (3.1)

**Definition 5.2.** In [11], a time discrete approximation $X^n$ converges weakly with order $\gamma > 0$ to $X$ at time $T$ as $n \to \infty$, if for each $G \in C^{2(\gamma+1)}_p(\mathbb{R})$, there exists a constant $C > 0$, which does not depend on $n$, such that

$$|E(G(X^n_T)) - E(G(X_T))| \leq \frac{C}{n^{\gamma}}$$

Using the above definition, we can prove theorem (3.1).

1. The proposition (2.8) is satisfied, then $X_t = F^{-1}(Y_t)$ is solution uniqueness of the SDE (1).
2. $G := g \circ F^{-1} \in C^{2(\gamma+1)}_p$
3. By using definition (5.2), we have

$$|E[g(X_T)] - E[g(X^n_T)]| = |E[g \circ F^{-1}(Y_T)] - E[g \circ F^{-1}(Y^n_T)]|$$

$$= |E[G(Y_T)] - E[G(Y^n_T)]| \leq \frac{C}{n^{\gamma}}$$

5.2 Proof of theorem (3.2)

If we have SDE of type

$$Y_t = Y_0 + \int_0^t \sigma(Y_s)dB_s$$

and the Euler scheme of equation (21) give by

$$Y_t = Y_0 + \int_0^t \sigma(Y_{\eta_n(s)})dB_s$$

Here, we present the following theorem (see [23]):

**Theorem 5.3.** Suppose that $\sigma(y)$ has at most linear growth with $D_\sigma$ of Lebesgue measure zero and $E(Y_0)^4 < \infty$. If $\sigma_1(z) := \lim_{z \to y} \frac{\sigma_2(y)}{y}$ for $z \in D_\phi$, then the Euler scheme of SDE(21) converges weakly to the unique weak solution of SDE(21), where $D_\sigma$ is the set of discontinuous points of $\sigma(y)$.

Using the above theorem, we can prove theorem (3.2).

1. The proposition (2.8) is satisfied, then $X_t = F^{-1}(Y_t)$ is solution uniqueness of the SDE (1).
2. $G := g \circ F^{-1} \in C^{2(\gamma+1)}_p$
3. $(\varphi,f_\nu) \circ F^{-1}(\cdot)$ has at most linear growth with $D_\sigma$ of Lebesgue measure zero.
4. $E(Y_0)^4 < \infty$
5. By using definition (5.2) and theorem (5.3), we have

$$|E[g(X_T)] - E[g(X^n_T)]| = |E[g \circ F^{-1}(Y_T)] - E[g \circ F^{-1}(Y^n_T)]|$$

$$= |E[G(Y_T)] - E[G(Y^n_T)]| \leq \frac{C}{n^{\gamma}}$$
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