Hermitian modular forms congruent to 1 modulo $p$.

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Abstract

For any natural number $\ell$ and any prime $p \equiv 1 \pmod{4}$ not dividing $\ell$ there is a Hermitian modular form of arbitrary genus $n$ over $L := \mathbb{Q}[\sqrt{-\ell}]$ that is congruent to 1 modulo $p$ which is a Hermitian theta series of an $O_L$-lattice of rank $p-1$ admitting a fixed point free automorphism of order $p$. It is shown that also for non-free lattices such theta series are modular forms.

1 Introduction.

The purpose of the present note is to generalize the construction of Siegel modular forms that are congruent to 1 modulo a suitable prime $p$ given in [3] to the case of Hermitian modular forms over $L := \mathbb{Q}[\sqrt{-\ell}]$. For $\ell = 1$ and $\ell = 3$ this was done in [11], in fact we use the same strategy by constructing an even unimodular lattice $\Lambda$ as an ideal lattice in $K := \mathbb{Q}[\sqrt{-\ell}, \zeta_p]$ for any prime $p \equiv 1 \pmod{4}$ not dividing $\ell$. The existence of $\Lambda$ essentially follows from class field theory and is predicted by [2, Théorème 2.3, Proposition 3.1 (1)] (see also [1, Corollary 2]). Since the ring of integers $O_K$ is in general not a principal ideal domain the lattice $\Lambda$ is not necessarily a free $O_K$-module. We are not aware of an explicit statement in the literature that the genus $n$ Hermitian theta series $\theta^{(n)}(\Lambda)$ of such a lattice $\Lambda$ is a modular form for the full modular group. Therefore the first section sketches a proof. In fact the proofs in the literature never seriously use the fact that the lattice is a free $O_K$-module. The next section applies the results of [2] and [1] to the special case of the field $K = \mathbb{Q}[\sqrt{-\ell}, \zeta_p]$ and proves the existence of a Hermitian $O_K$-lattice $\Lambda_h$ that is an even unimodular $\mathbb{Z}$-lattice (with respect to the trace of the Hermitian form). The invariance under $O_K$ yields both, a Hermitian $O_L$-module structure on $\Lambda_h$ and an $O_L$-linear automorphism (the multiplication by the primitive $p$-th root of unity $\zeta_p \in O_K$) of order $p$ acting fixed point freely on $\Lambda_h \setminus \{0\}$. Therefore all but the first coefficient in $\theta^{(n)}(\Lambda_h)$ are multiples of $p$ yielding the desired Hermitian modular form.

2 Hermitian theta-series are Hermitian modular forms.

Let $\ell \in \mathbb{N}$ such that $-\ell$ is a fundamental discriminant (which means that either $\ell \equiv -1 \pmod{4}$ is square-free or $\ell = 4m$, where $m \equiv 2$ or 1 (mod 4) is square-free). Let $L := \mathbb{Q}[\sqrt{-\ell}]$. For any natural number $\ell$ and any prime $p \equiv 1 \pmod{4}$ not dividing $\ell$ there is a Hermitian modular form of arbitrary genus $n$ over $L := \mathbb{Q}[\sqrt{-\ell}]$ that is congruent to 1 modulo $p$ which is a Hermitian theta series of an $O_L$-lattice of rank $p-1$ admitting a fixed point free automorphism of order $p$. It is shown that also for non-free lattices such theta series are modular forms.
\( \mathbb{Q}[\sqrt{-\ell}] \) be the imaginary quadratic number field of discriminant \(-\ell\), with ring of integers \( O_L \) and inverse different

\[
O_L^* := \{ a \in L \mid \text{Tr}_{L/\mathbb{Q}}(aO_L) \subset \mathbb{Z} \} = \sqrt{-\ell}^{-1}O_L.
\]

Let \((V, h)\) be a finite dimensional positive definite Hermitian vector space over \( L \). This section extends the results in [4] to not necessarily free Hermitian \( O_L \)-lattices in \((V, h)\). Note that we use a different scaling for the Hermitian form resulting in the additional factor of 2 in the definition of the Hermitian Siegel theta series below. It is already stated in [4] that the authors restrict to free lattices “for convenience” and that the same results hold in the more general context. The full modular group

\[
\Gamma_n := \langle \left( \begin{array}{cc} I_n & B \\ 0 & I_n \end{array} \right), \left( \begin{array}{cc} U & 0 \\ 0 & U^{-1} \end{array} \right), \left( \begin{array}{cc} 0 & -I_n \\ I_n & 0 \end{array} \right) \mid B \in O_L^{n \times n} \text{ Hermitian }, U \in \text{GL}_n(O_L) \rangle
\]

(see [7], [5, Anhang V], [8]) for the proof that these matrices really generate) acts on the Hermitian half space by

\[
Z \mapsto Z + B, \ Z \mapsto U^t Z U, \ Z \mapsto -Z^{-1}
\]

for the respective generators.

**Theorem 1.** Let \( \Lambda_h \leq (V, h) \) be an \( O_L \)-lattice such that the \( O_L \)-dual lattice

\[
\Lambda_h^* := \{ v \in V \mid h(v, \Lambda_h) \subset O_L \} = \sqrt{-\ell}^{-1} \Lambda_h = (O_L^*)^{-1} \Lambda_h.
\]

Then its Hermitian theta series

\[
\theta^{(n)}(\Lambda_h)(Z) := \sum_{(x_1, \ldots, x_n) \in \Lambda_h^*} \exp(2\pi i \text{trace}(h(x_i, x_j)Z))
\]

is a Hermitian modular form for the full modular group \( \Gamma_n \). Here \( \text{trace} : L^{n \times n} \to \mathbb{Q} \) denotes the composition of the matrix trace with the trace of \( L \) over \( \mathbb{Q} \).

**Proof.** For \( x := (x_1, \ldots, x_n) \in \Lambda_h^* \) the Hermitian matrix \( H := H_x := (h(x_i, x_j)) \in (O_L^*)^{n \times n} \), so for any Hermitian matrix \( B \in O_L^{n \times n} \) the trace \( \text{trace}(HB) \) is in \( \mathbb{Z} \). This shows the invariance of \( \theta^{(n)}(\Lambda_h) \) under \( Z \mapsto Z + B \). Similarly

\[
\text{trace}(H_x U^t Z U) = \text{trace}(U H_x U^t Z) = \text{trace}(H_{xU} Z)
\]

so the transformation \( Z \mapsto U^t Z U \) for \( U \in \text{GL}_n(O_L) \) just changes the order of summation in \( \theta^{(n)}(\Lambda_h) \). It remains to prove the theta-transformation formula

\[
(\star) \quad \theta^{(n)}(\Lambda_h)(-Z^{-1}) = \text{det}(Z/i)^d \theta^{(n)}(\Lambda_h)(Z)
\]

also for non-free \( O_L \)-lattices \( \Lambda_h \) of dimension \( d \) that satisfy \( \Lambda_h^* = (O_L^*)^{-1} \Lambda_h \). But Poisson summation only depends on the abelian group structure, not on the underlying module, so the usual proof (see for instance [9, p. 111]) can be adopted to the situation here (for details we refer to [6]): Using the Identity Theorem, it suffices to prove (\( \star \)) for \( Z = iY \), \( Y \) Hermitian.
positive definite. Let $\varphi : \mathbb{R}^{2dn} \to \mathbb{C}^{d \times n}$ be the obvious isomorphism and consider $\Lambda^*_h$ as a lattice $\tilde{\Lambda}$ in $\mathbb{C}^{d \times n}$ choosing coordinates with respect to an orthonormal basis of $(\mathbb{C}^d, h)$. Then there is some $F \in \mathbb{R}^{2dn \times 2dn}$ such that $\tilde{\Lambda} = \varphi(F\mathbb{Z}^{2dn})$ and $H_{\varphi(x)} = \overline{\varphi(Fx)}^t \varphi(Fx)$. Then

$$\theta^\alpha(\Lambda_h)(iY) = \sum_{g \in \mathbb{Z}^{2dn}} \psi(g)$$

where

$$\psi : \mathbb{R}^{2dn} \to \mathbb{C}, x \mapsto \exp(-2\pi \text{trace}(\overline{\varphi(Fx)}^t \varphi(Fx)Y)).$$

The condition $\Lambda^*_h = (O_L^*)^{-1} \Lambda_h$ implies that $|\det(F)| = 1$ and we can apply the usual Poisson summation to get the result as in [9, pp. 110-112].

3 Congruences of Hermitian theta-series.

Let $p$ be a prime $p \equiv 1 \pmod{4}$ such that $\ell$ is not a multiple of $p$. This section constructs a Hermitian $O_L$-lattice $(\Lambda, h)$ of rank $p - 1$ admitting an automorphism of order $p$ such that the $\mathbb{Z}$-lattice $(\Lambda, \text{Tr}_{L/Q}(h))$ is a positive definite even unimodular lattice. The existence of such a lattice follows from the much more general result [2, Théorème 2.3] together with [2, Proposition 3.1] which are based on Artin’s reciprocity law in global class field theory (see [10, Theorem (V.3.5)]). For our special case it is however more convenient to use [1, Corollary 2], which is essentially a consequence of [2, Théorème 2.3].

To this aim we consider the number field $K = \mathbb{Q}[\sqrt{-\ell}, \zeta_p] = LM$ with $M = \mathbb{Q}[\zeta_p]$, where $\zeta_p = \exp(2\pi i/p)$ is a primitive $p$-th root of unity. Then $K$ is an abelian number field of degree $2(p - 1)$ over $\mathbb{Q}$ which is a multiple of 8. The field $K$ is totally complex and admits an involution $\overline{\cdot}$, the complex conjugation, with fixed field $F$ the totally real subfield of $K$.

The following lemma is well known.

**Lemma 2.** $K/F$ is unramified at all finite primes.

**Proof.** The discriminant $d_{K/F}$ of $K/F$ divides the discriminant of any $F$-basis of $K$ that consists of integral elements. For $B_1 = (1, \sqrt{-\ell})$ one finds $d_{B_1} = \det(\text{Tr}_{K/F}(b_i b_j)) = -4\ell$ and for $B_2 = (1, \zeta_p)$ one get $d_{B_2} = \zeta_p^{-2}(\zeta_p^2 - 1)^2$ which generates an ideal of norm $p^2$ in $F$. Since $p$ is an odd prime not dividing $\ell$, the gcd of these two discriminants is 1 and hence $d_{K/F} = 1$ which implies the lemma.

Since all real embeddings of $F$ extend to complex embeddings of $K$ and $[K : \mathbb{Q}] = 2(p - 1) \equiv 0 \pmod{8}$ [1, Corollary 2] yields the existence of a fractional $O_K$-ideal $\mathcal{A}$ in $K$ and a totally positive element $d \in F$ such that the $O_K$-module $\mathcal{A}$ together with the symmetric integral bilinear form

$$b_d : \mathcal{A} \times \mathcal{A} \to \mathbb{Z}, \ (x, y) \mapsto \text{trace}_{K/Q}(dxy)$$

is an even unimodular $\mathbb{Z}$-lattice $\Lambda := (\mathcal{A}, b_d)$. This means that $b_d(x, x) \in 2\mathbb{Z}$ for all $x \in \mathcal{A}$ and

$$\Lambda^\# := \{ x \in K \mid b_d(x, y) \in \mathbb{Z} \text{ for all } y \in \mathcal{A} \} = \Lambda.$$
Corollary 3. The $O_L$-lattice $\Lambda_h := (A, h(x, y) := \text{Tr}_{K/L}(dx \overline{y}))$ is a Hermitian $O_L$-lattice with automorphism $x \mapsto \zeta_p x$ of order $p$ such that $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$.

Proof. Since $A$ is an ideal of $K$, the multiplication by $\zeta_p \in O_K$ preserves the lattice $A$. It also respects the Hermitian form $h$, because

$$h(\zeta_p x, \zeta_p y) = \text{Tr}_{K/L}(d\zeta_p x \overline{\zeta_p y}) = \text{Tr}_{K/L}(d\zeta_p \zeta_p^{-1} x y) = h(x, y).$$

The fact that $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$ follows from the unimodularity of the integral lattice $\Lambda$: For $y \in K$ we obtain

$$b_d(x, y) = \text{trace}_{L/Q}(h(x, y)) \in \mathbb{Z} \text{ for all } x \in A \Leftrightarrow h(x, y) \in O_L^* \text{ for all } x \in A$$

using the fact that $A$ is an $O_L$-module and $h$ is Hermitian over $O_L$. Hence $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$. 

Together this implies the existence of a Hermitian modular form of weight $p - 1$ that is congruent to 1 modulo $p$ for more general imaginary quadratic number fields than those treated in [11]:

Theorem 4. Let $L = \mathbb{Q}[\sqrt{-\ell}]$ be an imaginary quadratic number field ($-\ell$ a fundamental discriminant) and let $p$ be a prime $p \equiv 1 \pmod{4}$ not dividing $\ell$. Then for arbitrary genus $n \geq 1$ there is a Hermitian modular form

$$F_p^{(n)} \in M_{p-1}(SU_n(O_L))$$

for the full modular group over the ring of integers $O_L$ of $L$ such that

$$F_p^{(n)} \equiv 1 \pmod{p}.$$

Proof. Corollary 3 constructs a Hermitian $O_L$-lattice $\Lambda_h$ of rank $p - 1$ admitting an automorphism of order $p$ (which necessarily acts fixed point freely) such that $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$. By Theorem 1 its Siegel theta series is a Hermitian modular form for the full modular group. Since $\Lambda_h$ admits a fixed point free automorphism of order $p$, all the representation numbers

$$R_A := |\{(x_1, \ldots, x_n) \in \Lambda^n \mid (h(x_i, x_j)) = A\}|$$

for any non-zero Hermitian matrix $A \in L^{n \times n}$ are multiples of $p$ and hence

$$F_p^{(n)} := \theta_{\Lambda_h}^{(n)} \equiv 1 \pmod{p}$$

provides the desired Hermitian modular form. 

Since the root lattice $E_8$ is the unique even unimodular lattice of dimension 8, we obtain the following corollary.

Corollary 5. Let $\ell \in \mathbb{N}$ be not a multiple of 5. Then the root lattice $E_8$ has a Hermitian structure as a lattice $\Lambda_h$ over the ring of integers of $\mathbb{Q}[\sqrt{-\ell}]$ such that $\text{Aut}(\Lambda_h)$ contains an element of order 5.
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