Lipschitz spaces generated by the Sobolev-Poincaré inequality and extensions of Sobolev functions. I.

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Abstract

Let \( d \) be a metric on \( \mathbb{R}^n \) and let \( C^m,(d)(\mathbb{R}^n) \) be the space of \( C^m \)-function on \( \mathbb{R}^n \) whose partial derivatives of order \( m \) belong to the space \( \text{Lip}\,(\mathbb{R}^n;d) \). We show that the homogeneous Sobolev space \( L^{m+1}_p(\mathbb{R}^n), p > n \), can be represented as a union of \( C^{m,.(d)}(\mathbb{R}^n) \)-spaces where \( d \) belongs to a family of metrics on \( \mathbb{R}^n \) with certain “nice” properties. This enables us in several important cases to give intrinsic characterizations of the restrictions of Sobolev spaces to arbitrary closed subsets of \( \mathbb{R}^n \). In particular, we generalize the classical Whitney extension theorem for the space \( C^m(\mathbb{R}^n) \) to the case of the Sobolev space \( L^m_p(\mathbb{R}^n) \) whenever \( m \geq 1 \) and \( p > n \).

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1. Introduction.

Let \( p \in [1, \infty] \) and let \( L^1_p(\mathbb{R}^n) \) be the homogeneous Sobolev space consisting of all (equivalence classes of) real valued functions \( F \in L^1_{p,\text{loc}}(\mathbb{R}^n) \) whose distributional partial derivatives of the first order belong to the space \( L_p(\mathbb{R}^n) \). We equip \( L^1_p(\mathbb{R}^n) \) with the seminorm

\[
\|F\|_{L^1_p(\mathbb{R}^n)} := \|\nabla F\|_{L^p(\mathbb{R}^n)}.
\]

In the case where \( p = \infty \), the Sobolev space \( L^1_\infty(\mathbb{R}^n) \) can be identified with the space \( \operatorname{Lip}(\mathbb{R}^n) \) of Lipschitz functions on \( \mathbb{R}^n \). It is known that the restriction \( \operatorname{Lip}(\mathbb{R}^n)|_E \) of the Lipschitz space \( \operatorname{Lip}(\mathbb{R}^n) \) to an arbitrary closed set \( E \subset \mathbb{R}^n \) coincides with the space \( \operatorname{Lip}(E) \) of Lipschitz functions on \( E \). See e.g., [19]. Furthermore, the classical Whitney extension operator linearly and continuously maps the space \( \operatorname{Lip}(E) \) into the space \( \operatorname{Lip}(\mathbb{R}^n) \) (see e.g., [29], Chapter 6).

In [26] we have shown that an analogous result also holds for all \( p \) in the range \( n < p < \infty \), namely, that the same classical linear Whitney extension operator provides an extension to \( L^1_p(\mathbb{R}^n) \), \( p > n \), even an almost optimal extension, of each function on \( E \) which is the restriction to \( E \) of a function in \( L^1_p(\mathbb{R}^n) \). Using this property of the Whitney extension operator, in [26] we give several intrinsic characterizations of the restriction \( L^1_p(\mathbb{R}^n)|_E \) whenever \( p > n \).

In the paper under consideration we present alternative and simpler proofs of some of these results and give new characterizations of the trace space \( L^1_p(\mathbb{R}^n)|_E \). We also explain the above-mentioned phenomenon of the “universality” of the Whitney extension operator for the scale \( L^1_p(\mathbb{R}^n) \), \( p > n \).

Let \( d \) be a metric on \( \mathbb{R}^n \) and let \( \operatorname{Lip}(\mathbb{R}^n; d) \) be the space of functions on \( \mathbb{R}^n \) satisfying the Lipschitz condition with respect to the metric \( d \). \( \operatorname{Lip}(\mathbb{R}^n; d) \) is equipped with the standard seminorm

\[
\|F\|_{\operatorname{Lip}(\mathbb{R}^n; d)} := \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|F(x) - F(y)|}{d(x, y)}.
\]

We show that for each \( p \in (n, \infty) \) the Sobolev spaces \( L^1_p(\mathbb{R}^n) \) can be represented as a union of Lipschitz spaces \( \operatorname{Lip}(\mathbb{R}^n; d) \) where \( d \) belongs to a certain family \( \mathcal{D} \) of metrics on \( \mathbb{R}^n \):

\[
L^1_p(\mathbb{R}^n) = \bigcup_{d \in \mathcal{D}} \operatorname{Lip}(\mathbb{R}^n; d). \tag{1.1}
\]

Before we describe main ideas of our approach we need to define several notions and fix some notation: Throughout this paper, the word “cube” will mean a closed cube in \( \mathbb{R}^n \) whose sides are parallel to the coordinate axes. We let \( Q(c, r) \) denote the cube in \( \mathbb{R}^n \) centered at \( c \) with side length \( 2r \). Given \( \lambda > 0 \) and a cube \( Q \) we let \( \lambda Q \) denote the dilation of \( Q \) with respect to its center by a factor of \( \lambda \). (Thus \( \lambda Q(c, r) = Q(c, \lambda r) \).) By \( |A| \) we denote the Lebesgue measure of a measurable set \( A \subset \mathbb{R}^n \).

Representation (1.1) is based on the following idea: We slightly modify the classical Sobolev-Poincaré inequality for \( L^1_p(\mathbb{R}^n) \)-functions, \( p > n \), in such a way that the modified inequality can be interpreted as a Lipschitz condition with respect to a certain metric on \( \mathbb{R}^n \).
Let us recall a variant of the Sobolev-Poincaré inequality for $p > n$. Let $q \in (n, p]$ and let $F \in L^1_p(\mathbb{R}^n)$ be a continuous function. Then for every cube $Q \subset \mathbb{R}^n$ and every $x, y \in Q$ the following inequality
\[
|F(x) - F(y)| \leq C(n, q) \text{diam} Q \left( \frac{1}{|Q|} \int_Q \|\nabla F(u)\|^q \, du \right)^{\frac{1}{q}}
\] (1.2)
holds. See, e.g. [17], p. 61, or [18], p. 55.

In particular, by (1.2), for every $x, y \in \mathbb{R}^n$
\[
|F(x) - F(y)| \leq \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^q(u) \, du \right)^{\frac{1}{q}}
\] (1.3)
where
\[
Q_{xy} := Q(x, \|x - y\|)
\]
and $h = C(n, q)\|\nabla F\|$.

Let $h \in L^1_{p,\text{loc}}(\mathbb{R}^n)$ be an arbitrary non-negative function. Inequality (1.3) motivates us to introduce a function
\[
\delta_q(x, y : h) = \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^q \, du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n.
\] (1.4)

By this inequality, for each $p \in (n, \infty)$ and every $F \in L^1_p(\mathbb{R}^n)$ there exists a non-negative function $h \in L^1_p(\mathbb{R}^n)$ such that $\|h\|_{L^\infty(\mathbb{R}^n)} \leq C(n, p, q)\|F\|_{L^1_p(\mathbb{R}^n)}$ and
\[
|F(x) - F(y)| \leq \delta_q(x, y : h) \quad \text{for every} \quad x, y \in \mathbb{R}^n.
\] (1.5)

Conversely, let $n < p < \infty$ and let $F$ be a continuous function on $\mathbb{R}^n$. Suppose that there exists a non-negative function $h \in L^1_p(\mathbb{R}^n)$ such that (1.5) holds. Then $F \in L^1_p(\mathbb{R}^n)$ and $\|F\|_{L^1_p(\mathbb{R}^n)} \leq C(n, p, q)\|h\|_{L^p(\mathbb{R}^n)}$. See [13] and the sufficiency part of Theorem 1.4, Section 3.

Thus the Sobolev space $L^1_p(\mathbb{R}^n)$ is completely determined by the “Lipschitz-like” conditions (1.5) with respect to the functions $\delta_q(h)$ generated by functions $h \in L^1_p(\mathbb{R}^n)$. Of course, these observations lead us to the desired representation (1.1) provided $\delta_q(h)$ is a metric on $\mathbb{R}^n$.

However, in general, $\delta_q(h)$ is not a metric. Moreover, there are examples of non-negative functions $h \in L^1_p(\mathbb{R}^n)$ such that $\delta_q(h)$ is not equivalent to any metric on $\mathbb{R}^n$. See, e.g., Semmes [23], p. 586.

Nevertheless, we prove that under a certain additional restriction on $h$, there does exist a metric $d$ on $\mathbb{R}^n$ such that $d \sim \delta_q(h)$. More specifically, we show that $\delta_q(h)$ is equivalent to a metric on $\mathbb{R}^n$ whenever $h^q$ belongs to the Muckenhoupt class of $A_1$-weights. See Section 2 for definition and main properties of $A_1$-weights.
We express our result in terms of the geodesic distance \( d_q(h) \) associated with the function \( \delta_q(h) \). Given \( x, y \in \mathbb{R}^n \) this distance is defined by the formula

\[
d_q(x, y : h) := \inf \sum_{i=0}^{m-1} \delta_q(x_i, x_{i+1} : h)
\]

(1.6)

where the infimum is taken over all finite sequences of points \( \{x_0 = x, x_1, ..., x_m = y\} \) in \( \mathbb{R}^n \). Notice that \( d_q(h) \leq \delta_q(h) \). Furthermore, \( \delta_q(h) \) is equivalent to a metric on \( \mathbb{R}^n \) if and only if \( \delta_q(h) \) is equivalent to \( d_q(h) \). In Section 2 we prove the following

**Theorem 1.1** Let \( n \leq q < \infty \) and let \( h \in L_{q,\text{loc}}(\mathbb{R}^n) \) be a non-negative function such that \( h^q \in A_1(\mathbb{R}^n) \). Then for every \( x, y \in \mathbb{R}^n \) the following inequality holds.

\[
d_q(x, y : h) \leq \delta_q(x, y : h) \leq C d_q(x, y : h)
\]

(1.7)

holds. Here \( C \) is a constant depending only on \( n, q \), and the “norm” \( \|h^q\|_{A_1} \).

**Remark 1.2** David and Semmes [7] noted that (1.7) holds for \( q = n \). Semmes [23] proved Theorem 1.1 for a certain family of metric spaces (containing \( \mathbb{R}^n \)) and parameters \( q \) big enough. We refer the reader to these papers and also to the works [11, 15, 22, 24, 25] and references therein for various properties of weights \( h \) satisfying inequalities (1.7). 

**Remark 1.3** If \( h \in L_{q,\text{loc}}(\mathbb{R}^n) \) is a weight such that \( h^q \in A_1(\mathbb{R}^n) \), then for every \( s \in (0, q] \) and every \( x, y \in \mathbb{R}^n \) the following inequality

\[
\delta_s(x, y : h) \leq \delta_q(x, y : h) \leq \|h^q\|_{A_1}^{\frac{1}{q}} \delta_s(x, y : h)
\]

(1.8)

holds. See, e.g., [23], p. 579, or [10]; see also Remark 2.2.

Combining these inequalities with Theorem 1.1 we conclude that for every \( q \in [n, \infty) \) and every \( s \in (0, q] \)

\[
\delta_s(x, y : h) \sim d_s(x, y : h) \sim d_q(x, y : h), \quad x, y \in \mathbb{R}^n,
\]

with constants in the equivalences depending only on \( n, q, s, \) and \( \|h^q\|_{A_1} \). 

Let us formulate the first main result of the paper.

**Theorem 1.4** Let \( n \leq q < p < \infty \). There exists a positive constant \( \eta \) depending only on \( n, p, \) and \( q \) such that the following is true:

A continuous function \( F \in L_1^p(\mathbb{R}^n) \) if and only if there exists a non-negative function \( h \in L_q(\mathbb{R}^n) \) such that \( h^q \in A_1(\mathbb{R}^n) \), \( \|h^q\|_{A_1} \leq \eta \), and for every \( x, y \in \mathbb{R}^n \) the following inequality

\[
|F(x) - F(y)| \leq d_q(x, y : h)
\]

(1.9)

holds. Furthermore,

\[
\|F\|_{L_1^p(\mathbb{R}^n)} \sim \inf \|h\|_{L_p(\mathbb{R}^n)}.
\]

The constants of this equivalence depend only on \( n, p, \) and \( q \).
This result shows that for each $q \in [n, p)$ the Sobolev space $L^1_p(\mathbb{R}^n)$ can be represented in the form

$$L^1_p(\mathbb{R}^n) = \bigcup_{d \in D_{p,q}(\mathbb{R}^n)} \text{Lip}(\mathbb{R}^n; d)$$

(1.10)

where

$$D_{p,q}(\mathbb{R}^n) := \{ d_q(h) : h \in L_p(\mathbb{R}^n), h \geq 0, h^q \in A_1(\mathbb{R}^n), \|h^q\|_{A_1} \leq \eta(n, p, q) \}. \quad (1.11)$$

We apply Theorem 1.4 to the study of the extension and restriction properties of Sobolev functions. In Section 4 we characterize the restrictions of functions from the space $L^1_p(\mathbb{R}^n)$ to an arbitrary closed subset $E$ of $\mathbb{R}^n$. Recall that, when $p > n$, it follows from the Sobolev embedding theorem that every function $F \in L^1_p(\mathbb{R}^n)$ coincides almost everywhere with a continuous function on $\mathbb{R}^n$. This fact enables us to identify each element $F \in L^1_p(\mathbb{R}^n)$, $p > n$, with its unique continuous representative. This identification, in particular, implies that $F$ has a well-defined restriction to any given subset of $\mathbb{R}^n$.

As usual, we define the trace space $L^1_p(\mathbb{R}^n)|_E$ of all restrictions of $L^1_p(\mathbb{R}^n)$-functions to $E$ by

$$L^1_p(\mathbb{R}^n)|_E := \{ f : E \to \mathbb{R} : \text{there exists } F \in L^1_p(\mathbb{R}^n) \cap C(\mathbb{R}^n) \text{ such that } F|_E = f \}. \quad (1.12)$$

We equip this space with the standard quotient space seminorm

$$\|f\|_{L^1_p(\mathbb{R}^n)|_E} := \inf \{ \|F\|_{L^1_p(\mathbb{R}^n)} : F \in L^1_p(\mathbb{R}^n) \cap C(\mathbb{R}^n), F|_E = f \}.$$

As is customary, we refer to $\|F\|_{L^1_p(\mathbb{R}^n)|_E}$ as the trace norm of the function $F$ (in $L^1_p(\mathbb{R}^n)$).

The main result of Section 4, Theorem 1.5, provides an explicit formula for the order of magnitude of the trace norm of a function $f : E \to \mathbb{R}$ in the space $L^1_p(\mathbb{R}^n)|_E$ whenever $p \in (n, \infty)$.

**Theorem 1.5** Let $n < p < \infty$ and let $f : \to \mathbb{R}$ be a continuous function defined on a closed set $E \subset \mathbb{R}^n$. Then

$$\|f\|_{L^1_p(\mathbb{R}^n)|_E} \sim \left\{ \int_{\mathbb{R}^n} \left( \sup_{y,z \in E} \frac{|f(y) - f(z)|}{\|x - y\| + \|x - z\|} \right)^p \, dx \right\}^{1/p}. \quad (1.12)$$

The constants of this equivalence depend only on $n$ and $p$.

**Remark 1.6** As we have mentioned above, in [26] we show that the classical extension operator constructed by Whitney in [30] for the space $C^1(\mathbb{R}^n)$ provides an almost optimal extension operator for the trace space $L^1_p(\mathbb{R}^n)|_E$ whenever $n < p < \infty$. In this paper we present a proof of the trace criterion (1.12) which does not use the Whitney extension method. More specifically, this proof uses only Theorem 1.4 and McShane’s trace theorem [19]. See Remark 4.1.
Remark 1.7 Theorem [14] and certain special properties of metrics from $\mathcal{D}_{p,q}(\mathbb{R}^n)$ also enable us to explain the above-mentioned phenomenon of the universality of the Whitney extension operator for the scale of the Sobolev spaces $L^1_p(\mathbb{R}^n)$, $p > n$.

Let us introduce a special family of metrics on $\mathbb{R}^n$. Let $\omega$ be a concave non-decreasing continuous function on $[0, \infty)$ such that $\omega(0) = 0$. We refer to $\omega$ as a “modulus of continuity”. By $\mathcal{MC}$ we denote the family of all “moduli of continuity”. For every $\omega \in \mathcal{MC}$ the function

$$d_\omega(x, y) = \omega(\|x - y\|), \quad x, y \in \mathbb{R}^n,$$

is a metric on $\mathbb{R}^n$. The metric space $(\mathbb{R}^n, d_\omega)$ is known in the literature as the metric transform of $\mathbb{R}^n$ by $\omega$ or the $\omega$-metric transform of $\mathbb{R}^n$.

We show that for each metric $d \in \mathcal{D}_{p,q}(\mathbb{R}^n)$ there exists a mapping $\mathbb{R}^n \ni x \mapsto \omega_x \in \mathcal{MC}$ such that

$$d(x, y) \sim \omega_x(\|x - y\|), \quad x, y \in \mathbb{R}^n,$$

with constants depending only on $n, p$, and $q$. This equivalence motivates us to refer to the metric space $(\mathbb{R}^n, d)$ as a variable metric transform of $\mathbb{R}^n$.

It is well-known that for each modulus of continuity $\omega \in \mathcal{MC}$ the Whitney extension operator provides an almost optimal extension of every function $f \in \text{Lip}(\mathbb{R}^n, d_\omega)|_E$ to a function from Lip$(\mathbb{R}^n, d_\omega)$. (See, e.g., [29], 2.2.3.) In other words, the Whitney operator is a universal extension operator for the scale Lip$(\mathbb{R}^n; d)$ of the Lipschitz spaces with respect to variable metric transforms $d \in \mathcal{D}_{p,q}(\mathbb{R}^n)$. This and representation (1.10) imply the universality of the Whitney extension operator for the scale $L^1_p(\mathbb{R}^n)$, $p > n$. 

In Section 5 we generalize representation (1.10) to the case of the homogeneous Sobolev space $L^m_p(\mathbb{R}^n)$ of order $m \geq 1$. We recall that this space consists of all (equivalence classes of) real valued functions $F \in L^p,\text{loc}(\mathbb{R}^n)$, $p \in [1, \infty]$, whose distributional partial derivatives of order $m$ belong to $L^p(\mathbb{R}^n)$. The space $L^m_p(\mathbb{R}^n)$ is equipped with the seminorm

$$\|F\|_{L^m_p(\mathbb{R}^n)} := \|\nabla^m F\|_{L^p(\mathbb{R}^n)}$$

where

$$\nabla^m F(x) := \left( \sum_{|\alpha|=m} (D^\alpha F(x))^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

(1.14)

When $p > n$, by the Sobolev embedding theorem, every function $F \in L^m_p(\mathbb{R}^n)$ coincides almost everywhere with a $C^{m-1}$-function, so that we can identify each element $F \in L^m_p(\mathbb{R}^n)$ with its unique $C^{m-1}$-representative. As usual, given a closed set $E \subset \mathbb{R}^n$ we let $L^m_p(\mathbb{R}^n)|_E$ denote the trace of the space $L^m_p(\mathbb{R}^n)$ to $E$. This space is normed by

$$\|f\|_{L^m_p(\mathbb{R}^n)|_E} := \inf \{\|F\|_{L^m_p(\mathbb{R}^n)} : F \in L^m_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n), F|_E = f\}.$$

Let $d$ be a metric on $\mathbb{R}^n$ and let $C^{m,(d)}(\mathbb{R}^n)$ be the space of functions $F \in C^m(\mathbb{R}^n)$ whose partial derivatives of order $m$ are Lipschitz continuous on $\mathbb{R}^n$ with respect to $d$. 

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This space is equipped with the seminorm
\[
\|F\|_{C^m,d}(\mathbb{R}^n) := \sum_{|\alpha| = m} \|D^\alpha F\|_{\text{Lip}(\mathbb{R}^n; d)}.
\]

Using the same approach as for the case \(m = 1\) we prove that
\[
L^m_p(\mathbb{R}^n) = \bigcup_{d \in \mathcal{D}_{p,q}(\mathbb{R}^n)} C^{m-1,d}(\mathbb{R}^n)
\]
provided \(n \leq q < p\). See Theorem 5.3 and Remark 5.4.

This representation enables us to prove two extension theorems. The first of them is

\textbf{Theorem 1.8} Let \(p \in (n, \infty)\), \(m \in \mathbb{N}\), and let \(J = \{P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\}\) be a polynomial field defined on a closed set \(E \subset \mathbb{R}^n\).

There exists a function \(F \in L^m_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n)\) such that \(T^{m-1}_x[F] = P_x\) for every \(x \in E\) if and only if the following quantity
\[
\mathcal{N}_{m,p}(J) := \sum_{|\alpha| \leq m-1} \sup_{y,z \in E, y \neq z} \frac{|D^\alpha P_y(y) - D^\alpha P_z(y)|}{\|y - z\|^{m-|\alpha|}}
\]
is finite. See also Glaeser [11].

\textbf{Theorem 1.8} Let \(p \in (n, \infty)\), \(m \in \mathbb{N}\), and let \(J = \{P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\}\) be a polynomial field defined on a closed set \(E \subset \mathbb{R}^n\).

There exists a function \(F \in L^m_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n)\) such that \(T^{m-1}_x[F] = P_x\) for every \(x \in E\) if and only if the following inequality
\[
\mathcal{N}_{m,p}(J) := \sum_{|\alpha| \leq m-1} \left( \int_{\mathbb{R}^n} \left( \sup_{y,z \in E} \frac{|D^\alpha P_y(y) - D^\alpha P_z(y)|}{\|x - y\|^{m-|\alpha|} + \|x - z\|^{m-|\alpha|}} \right)^p \, dx \right)^{\frac{1}{p}} < \infty \quad (1.16)
\]
holds. Furthermore,
\[
\mathcal{N}_{m,p}(J) \sim \inf \left\{ \|F\|_{L^m_p(\mathbb{R}^n)} : F \in L^m_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n), T^{m-1}_x[F] = P_x \text{ for every } x \in E \right\}.
\]
The constants of this equivalence depend only on \(n, m\) and \(p\).

Clearly, for \(m = 1\) the above equivalence coincides with (1.12). Similar to the case \(m = 1\) (see Remark 1.6), we show that the extension operator constructed by Whitney [30] for the space \(C^m(\mathbb{R}^n)\) provides an almost optimal extension for every polynomial field \(\{P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\}\) on \(E\) whenever \(n < p < \infty\) and \(E\) is an arbitrary closed subset in \(\mathbb{R}^n\).
Our second extension result provides a constructive characterization of the traces of Sobolev $L^p_m$-functions in one dimensional case.

Let $E$ be a closed subset of $\mathbb{R}$. We recall that Whitney \[31\] constructed a continuous linear extension operator $T_E$ for the trace space $L^m_\infty(\mathbb{R})|_E$. The operator norm of $T_E$ is bounded by a constant depending only on $m$. In the forthcoming paper \[28\] we show that very same Whitney extension operator $T_E$ also provides an almost optimal extension of each function $f \in L^m_p(\mathbb{R})|_E$ to a function from $L^m_p(\mathbb{R})$ whenever $1 < p < \infty$.

This leads us to an analogue of the Whitney trace criterion \[31\] for the space $L^m_\infty(\mathbb{R})$. Recall that, by this criterion, for arbitrary positive integer $m$ and every function $f \in L^m_p(\mathbb{R})|_E$ of order $m$:

$$\|f\|_{L^m_\infty(\mathbb{R})|_E} \sim \sup_{S \subset E, \#S = m+1} |\Delta^m f[S]|.$$  \hfill (1.17)

In this formula given a finite set $S = \{x_0, x_1, \ldots, x_m\} \subset \mathbb{R}$ the quantity $\Delta^m f[S]$ denotes the divided difference of $f$ on $S$ of order $m$:

$$\Delta^m f[S] = \Delta^m f[x_0, x_1, \ldots, x_m] = \sum_{i=0}^{m} \frac{f(x_i)}{\omega^m_S(x_i)},$$

where $\omega^m_S(x) = (x - x_0) \cdot (x - x_1) \cdots (x - x_m)$. In (1.17) the symbol $\#$ denotes the number of points of a set. (See also J. Merrien \[20\] and Glaeser \[11\].)

In \[28\] we present a counterpart of formula (1.17). We prove that for every $1 < p < \infty$, every closed set $E \subset \mathbb{R}$ and every function $f$ on $E$

$$\|f\|_{L^m_p(\mathbb{R})|_E} \sim \left\{ \int_{\mathbb{R}} \sup_{S \subset E, \#S = m+1} \left( \frac{|\Delta^m f[S]| \cdot \text{diam } S}{\text{diam}(\{x\} \cup S)} \right)^p \right\}^{\frac{1}{p}}$$

$$\sim \left\{ \int_{\mathbb{R}} \sup_{x_0 < x_1 < \ldots < x_m, \ x_i \in E} \frac{|\Delta^{m-1} f[x_0, \ldots, x_{m-1}] - \Delta^{m-1} f[x_1, \ldots, x_m]|^p}{|x - x_0|^p + |x - x_m|^p} \right\}^{\frac{1}{p}}.$$  

The constants in these equivalences depend only on $p$ and $m$.

As we have noted above, the extension operator $T_E$ constructed by Whitney \[31\] for the space $C^m(\mathbb{R})|_E$ provides an almost optimal extension for every trace space $L^m_p(\mathbb{R})|_E$ and every closed set $E \subset \mathbb{R}$ whenever $p > 1$. Since $T_E$ is linear, we obtain the following result.

**Theorem 1.9** For every closed set $E \subset \mathbb{R}$ and every $p > 1$ there exists a linear extension operator which maps the trace space $L^m_p(\mathbb{R})|_E$ continuously into $L^m_p(\mathbb{R})$. Its operator norm is bounded by a constant depending only on $p$.

Note that G. K. Luli \[21\] proved the existence of a continuous linear extension operator for the space $L^m_p(\mathbb{R})|_E$, $p > 1$, for every finite set $E \subset \mathbb{R}$. We also remark that an analog of Theorem 1.9 for the space $L^2_p(\mathbb{R}^2)$, $p > 2$, has been proven in recent works of A. Israel \[14\] and the author \[27\]. Quite recently C. Fefferman, A. Israel and G. K. Luli \[29\] proved the existence of a continuous linear extension operator for the space $L^m_p(\mathbb{R}^n)|_E$ whenever $n < p < \infty$ and $E \subset \mathbb{R}^n$ is an arbitrary closed set.
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2. Metrics on $\mathbb{R}^n$ generated by the Sobolev-Poincaré inequality.

Let us fix additional notation. Throughout the paper $\gamma, \gamma_1, \gamma_2, \ldots$, and $C, C_1, C_2, \ldots$ will be generic positive constants which depend only on parameters determining function spaces $(p, q, n, m, \text{etc})$. These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(n, p, q)$. We write $A \sim B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$.

Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by $\|x\| := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ we denote the uniform norm in $\mathbb{R}^n$. Let $A, B \subset \mathbb{R}^n$. We put $\text{diam } A := \sup\{\|a - a'\| : a, a' \in A\}$ and $\text{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}$. For each pair of points $z_1$ and $z_2$ in $\mathbb{R}^n$ we let $(z_1, z_2)$ denote the open line segment joining them. Finally given a cube $Q$ in $\mathbb{R}^n$ by $c_Q$ we denote its center, and by $r_Q$ a half of its side length. (Thus $Q = Q(c_Q, r_Q)$.)

We let $P_m(\mathbb{R}^n)$ denote the space of polynomials of degree at most $m$ defined on $\mathbb{R}^n$. Finally given a function $g \in L_{1, \infty}(\mathbb{R}^n)$ we let $M[g]$ denote its Hardy-Littlewood maximal function:

$$M[g](x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |g| dx, \quad x \in \mathbb{R}^n.$$  

Here the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ containing $x$.

2.1 Proof of Theorem 1.1. Let $w$ be a weight on $\mathbb{R}^n$, i.e., a non-negative locally integrable function. Let $q > 0$ and let $\varphi_q(w) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a function defined by the formula

$$\varphi_q(x, y : w) := \|x - y\| \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q w(u) du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n. \quad (2.1)$$

**Proposition 2.1** Let $n \leq q < \infty$. Then for every $x, y \in \mathbb{R}^n$ and every finite family of points $x_0 = x, x_1, \ldots, x_{m-1}, x_m = y$ in $\mathbb{R}^n$ the following inequality

$$\varphi_q(x, y : w) \leq 16 \sum_{i=0}^{m-1} \varphi_q(x_i, x_{i+1} : w) \quad (2.2)$$

holds.
Proof. Let $K$ be an arbitrary cube in $\mathbb{R}^n$ and let $x, y \in K$. Prove that

$$
\|x - y\| \left(\frac{1}{|K|} \int_K w(u) \, du\right) \leq 16 \sum_{i=0}^{m-1} \varphi_q(x_i, x_{i+1} : w). \quad (2.3)
$$

Let

$$
\tilde{Q} = Q(x, 2\|x - y\|) = 2Q_{xy}.
$$

Consider two cases.

**The first case.** Suppose that there exists $j \in \{0, \ldots, m - 1\}$ such that $x_j \in 2Q_{xy}$, but $x_j + 1 \notin 4Q_{xy}$.

Hence $\|x - x_j\| \leq 2\|x - y\|$ and $\|x - x_{j+1}\| \geq 4\|x - y\|$ so that

$$
\|x_i - x_{j+1}\| \geq 2\|x - y\|. \quad (2.4)
$$

Since $x, y \in K$, we have $\text{diam } K \geq \|x - y\|$ which easily implies the inclusion

$$
2Q_{xy} \subset 5K. \quad (2.5)
$$

Hence $x_j \in 2Q_{xy} \subset 5K$.

Consider the following two subcases. First let us assume that

$$
x_{j+1} \in 8K.
$$

Since $x_j \in 2Q_{xy} \subset 5K$, we have $x_j, x_{j+1} \in 8K$. Hence, by (2.4) and definition (2.1) of $\varphi_q$, we obtain

$$
\|x - y\| \left(\frac{1}{|K|} \int_K w(u) \, du\right) \leq \|x_j - x_{j+1}\| \left(\frac{1}{|K|} \int_K w(u) \, du\right)
\leq 8^n \|x_j - x_{j+1}\| \left(\frac{1}{|8K|} \int_{8K} w(u) \, du\right)
\leq 8^n \varphi_q(x_j, x_{j+1} : w).
$$

Since $n \leq q$, we have

$$
\|x - y\| \left(\frac{1}{|K|} \int_K w(u) \, du\right) \leq 8\varphi_q(x_j, x_{j+1} : w)
$$

proving (2.3) in the case under consideration.

Now consider the second subcase where

$$
x_{j+1} \notin 8K.
$$
Since $x_j \in 5K$, we conclude that $\|x_j - x_{j+1}\| \geq 3r_K$.

Let $\overline{Q} := Q(x_j, 2\|x_j - x_{j+1}\|)$. Since $x_j \in 5K$, for every $z \in K$ we have

$$\|x_j - z\| \leq \|x_j - c_K\| + \|c_K - z\| \leq 5r_K + r_K = 6r_K \leq 2\|x_j - x_{j+1}\| = r_{\overline{Q}}$$

proving that $\overline{Q} \supsetneq K$. Clearly, $r_K \leq r_{\overline{Q}}$ and $\overline{Q} \ni x_j, x_{j+1}$.

Hence

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} = 2^\frac{n}{q} \|x - y\| r_K^{\frac{1}{q}} \left( \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^\frac{n}{q} \|x - y\| r_{\overline{Q}}^{\frac{2}{q}} \left( \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}}$$

Since $x, y \in K$, we have $\|x - y\| \leq \text{diam } K = 2r_K$. Combining this with inequality $r_K \leq r_{\overline{Q}}$, we obtain

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^\frac{n}{q} + 1 \cdot r_K^{\frac{1}{q}} \left( \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^\frac{n}{q} + 1 \cdot r_{\overline{Q}}^{\frac{1}{q}} \left( \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}}$$

$$= 2^\frac{n}{q} + 1 \cdot r_{\overline{Q}} \left( \frac{1}{|\overline{Q}|} \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}}.$$ 

Since $r_{\overline{Q}} = 2 \|x_j - x_{j+1}\|$ and $\frac{n}{q} \leq 1$, we have

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^4 \|x_j - x_{j+1}\| \left( \frac{1}{|\overline{Q}|} \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}}.$$ 

But $x_j, x_{j+1} \in \overline{Q}$ so that, by definition $\varphi_\omega$,

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^4 \varphi_\omega(x_j, x_{j+1} : w)$$

proving $\varphi_\omega$.

The second case. Suppose that the assumption of the first case is not satisfied, i.e., for each $i \in \{0, \ldots, m - 1\}$ such that $x_i \in 2Q_{xy}$ we have $x_{i+1} \in 4Q_{xy}$.

Let us define a number $j \in \{0, 1, \ldots, m - 1\}$ as follows. If $\{x_0, x_1, \ldots, x_m\} \subset 2Q_{xy}$, we put $j = m - 1$. If $\{x_0, x_1, \ldots, x_m\} \not\subset 2Q_{xy}$, then there exists $j \in \{0, 1, \ldots, m - 1\}$, such that

$$\{x_0, x_1, \ldots, x_j\} \subset 2Q_{xy}$$

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but
\[ x_{j+1} \notin 2Q_{xy}. \]

Note that, by the assumption, \( x_{j+1} \in 4Q_{xy} \).

Prove that
\[
\| x - y \| \left( \frac{1}{|K|} \int_{K} w(u) \, du \right)^{\frac{1}{q}} \leq 16 \sum_{i=0}^{j} \varphi_{q}(x_{i}, x_{i+1} : w).
\]

In fact, since \( x_{j+1} \notin 2Q_{xy} = Q(x, 2\|x - y\|) \) we have
\[
\| x_{0} - x_{j+1} \| = \| x - x_{j+1} \| \geq 2 \| x - y \|
\]
so that
\[
\sum_{i=0}^{j} \| x_{i} - x_{i+1} \| \geq \| x_{0} - x_{j+1} \| \geq 2 \| x - y \|. \tag{2.6}
\]

Recall that, by (2.5), \( 2Q_{xy} \subset 5K \) so that \( 10K \supset 4Q_{xy} \). Hence
\[
x_{0}, x_{1}, ..., x_{j}, x_{j+1} \in 10K. \tag{2.7}
\]

By (2.6),
\[
\| x - y \| \left( \frac{1}{|K|} \int_{K} w(u) \, du \right)^{\frac{1}{q}} \leq \frac{1}{2} \left( \sum_{i=0}^{j} \| x_{i} - x_{i+1} \| \right) \left( \frac{1}{|K|} \int_{K} w(u) \, du \right)^{\frac{1}{q}} \leq \frac{(10)^{\frac{q}{r}}}{2} \left( \sum_{i=0}^{j} \| x_{i} - x_{i+1} \| \right) \left( \frac{1}{|10K|} \int_{10K} w(u) \, du \right)^{\frac{1}{q}}.
\]

Since \( 10^{\frac{q}{r}} \leq 10 \), we obtain
\[
\| x - y \| \left( \frac{1}{|K|} \int_{K} w(u) \, du \right)^{\frac{1}{q}} \leq 5 \left( \sum_{i=0}^{j} \| x_{i} - x_{i+1} \| \right) \left( \frac{1}{|10K|} \int_{10K} w(u) \, du \right)^{\frac{1}{q}}.
\]

But, by (2.7), \( x_{i}, x_{i+1} \in 10K \) for every \( 0 \leq i \leq j \), so that
\[
\| x_{i} - x_{i+1} \| \left( \frac{1}{|10K|} \int_{10K} w(u) \, du \right)^{\frac{1}{q}} \leq \varphi_{q}(x_{i}, x_{i+1} : w).
\]

Hence
\[
\| x - y \| \left( \frac{1}{|K|} \int_{K} w(u) \, du \right)^{\frac{1}{q}} \leq 5 \sum_{i=0}^{j} \varphi_{q}(x_{i}, x_{i+1} : w) \leq 5 \sum_{i=0}^{m-1} \varphi_{q}(x_{i}, x_{i+1} : w)
\]

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proving inequality (2.3).

Thus (2.3) is proven for an arbitrary family of points \( x_0 = x, x_1, ..., x_m = y \) in \( \mathbb{R}^n \). Taking the supremum in this inequality over all cubes \( K \ni x, y \), we finally obtain the statement of the proposition. \( \square \)

We recall that a non-negative function \( w \in L_{1, \text{loc}}(\mathbb{R}^n) \) is said to be \( A_1 \)-weight if there exists a constant \( \lambda > 0 \) such that for every cube \( Q \subset \mathbb{R}^n \) the following inequality

\[
\frac{1}{|Q|} \int_Q w(u) \, du \leq \lambda \text{ess inf}_Q w
\]

holds. See, e.g. [10]. We put

\[
\|w\|_{A_1} = \inf \lambda.
\]

Clearly, a weight \( w \in A_1 \) if and only if there exists \( \lambda > 0 \) such that

\[
M[w] \leq \lambda w(x) \quad \text{a.e. on } \mathbb{R}^n.
\]

Furthermore,

\[
\|w\|_{A_1} \sim \sup_{\mathbb{R}^n} \frac{M[w](x)}{w(x)}.
\]

Let us also notice the following important property of \( A_1 \)-weights: for every \( w \in A_1(\mathbb{R}^n) \) and every two cubes \( K, Q \) in \( \mathbb{R}^n \) such that \( Q \subset K \) the following inequality

\[
\frac{1}{|K|} \int_K w(u) \, du \leq \|w\|_{A_1} \frac{1}{|Q|} \int_Q w(u) \, du
\]

holds.

These properties of \( A_1 \)-weights and Proposition 2.1 enable us to finish the proof of Theorem 1.1 as follows.

The first inequality in (1.7) is trivial so let us prove that for every \( x, y \in \mathbb{R}^n \)

\[
\delta_q(x, y : h) \leq C \, d_q(x, y : h)
\]

provided \( q \in [n, \infty) \) and \( w := h^q \in A_1(\mathbb{R}^n) \). Here \( C \) is a constant depending only on \( n, q \), and \( \|w\|_{A_1} \). In fact, by (1.4), for every \( x, y \in \mathbb{R}^n \)

\[
\delta_q(x, y : h) := \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^q \, du \right)^\frac{1}{q} = \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} w(u) \, du \right)^\frac{1}{q}
\]

so that, by (2.1),

\[
\delta_q(x, y : h) \leq \|x - y\| \sup_{Q_{xy}} \left( \frac{1}{|Q|} \int_Q w(u) \, du \right)^\frac{1}{q} = \varphi_q(x, y : w).
\]

On the other hand, the cube \( Q_{xy} \ni x, y \) and \( \text{diam} \, Q_{xy} = 2\|x - y\| \) so that for every cube \( K \ni x, y \) we have \( Q_{xy} \subset 3K \). Hence, by (2.9),

\[
\frac{1}{|K|} \int_K w \, du \leq \frac{3^n}{|3K|} \int_{3K} w(u) \, du \leq 3^n \|w\|_{A_1} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} w(u) \, du \right).
\]

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Combining this inequality with (2.1), we obtain

\[ \varphi_q(x, y : w) := \|x - y\| \sup_{K \ni x, y} \left( \frac{1}{|K|} \int_K w \, du \right)^{\frac{1}{q}} \leq 3^\frac{n}{q} \|w\|_{A_1}^{\frac{1}{q}} \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} w \, du \right)^{\frac{1}{q}} \]

proving that

\[ \varphi_q(x, y : w) \leq C\delta_q(x, y : h). \]  

(2.12)

Let \(\{x_0 = x, x_1, ..., x_m = y\}\) be an arbitrary family of points in \(\mathbb{R}^n\). Then, by (2.11) and Proposition 2.1,

\[ \delta_q(x, y : h) \leq \varphi_q(x, y : w) \leq 16 \sum_{i=0}^{m-1} \varphi_q(x_i, x_{i+1} : w) \]

so that, by (2.12),

\[ \delta_q(x, y : h) \leq C \sum_{i=0}^{m-1} \delta_q(x_i, x_{i+1} : h). \]

Taking the infimum in this inequality over all families of points \(\{x_0 = x, x_1, ..., x_m = y\}\), we obtain the required inequality (2.10). See (1.6).

Theorem 1.1 is completely proved. \(\square\)

**Remark 2.2** Prove inequality (1.8). Since \(0 < s \leq q\), the first inequality in (1.8) is elementary. Prove the second inequality. Let \(w := h^q\). Then, by (2.8), for every \(x, y \in \mathbb{R}^n\)

\[ \frac{1}{|Q_{xy}|} \int_{Q_{xy}} w(u) \, du \leq \|w\|_{A_1} \text{ess inf}_{Q_{xy}} w \]

so that

\[ \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} w(u) \, du \right)^{\frac{1}{q}} \leq \|w\|_{A_1}^{\frac{1}{q}} \left( \text{ess inf}_{Q_{xy}} w^q \right)^{\frac{1}{q}} \leq \|w\|_{A_1}^{\frac{1}{q}} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} w^q(u) \, du \right)^{\frac{1}{q}}. \]

Hence

\[ \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^q(u) \, du \right)^{\frac{1}{q}} \leq \|h^q\|_{A_1}^{\frac{1}{q}} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^q(u) \, du \right)^{\frac{1}{q}} \]

proving (1.8). \(\triangleleft\)

### 2.2 Variable metric transforms.

Let \(q \in [n, \infty)\) and let \(\eta > 0\). Define a family of metrics on \(\mathbb{R}^n\)

\[ M_{q, \eta}(\mathbb{R}^n) = \{d_q(h) : h^q \in A_1(\mathbb{R}^n), \|h^q\|_{A_1} \leq \eta\}. \]
In this subsection we present several simple but important properties of metrics from the family $M_{q,\eta}(\mathbb{R}^n)$. Note that, by Theorem 1.1 and Remark 1.3, for every $s \in (0, q]$ and every metric $d = d_q(h) \in M_{q,\eta}(\mathbb{R}^n)$ such that $h^q \in A_1(\mathbb{R}^n)$, $\|h^q\|_{A_1} \leq \eta$ we have

$$d(x, y) \sim \delta_s(x, y : h) = \|x - y\| \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^s du\right)^{\frac{1}{s}}, \quad x, y \in \mathbb{R}^n. \quad (2.13)$$

with constants depending only on $n, q,$ and $\eta$.

**Claim 2.3** Let $n \leq s \leq q$ and let $d = d_q(h) \in M_{q,\eta}(\mathbb{R}^n)$ where $h \in A_1(\mathbb{R}^n)$, $\|h^q\|_{A_1} \leq \eta$. Then for every $x, y \in \mathbb{R}^n$ the following inequality holds with constants depending only on $n, q,$ and $\eta$. Here the infimum is taken over all cubes $Q \ni x, y$.

**Proof.** Since $x, y \in Q_{xy} = Q(x, \|x - y\|)$ and diam $Q_{xy} = 2\|x - y\|$, 

$$I(x, y) := \inf_{Q \ni x, y} \text{diam } Q \left(\frac{1}{|Q|} \int_Q h^s du\right)^{\frac{1}{s}} \leq \text{diam } Q_{xy} \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^s du\right)^{\frac{1}{s}} \leq 2\|x - y\| \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^s du\right)^{\frac{1}{s}}$$

so that, by (2.13),

$$I(x, y) \leq C(n, q)d(x, y).$$

On the other hand, for every cube $Q \subset \mathbb{R}^n$ such that $Q \ni x, y$ we have $3Q \ni Q_{xy}$. Hence, by (2.13),

$$d(x, y) \sim \|x - y\| \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^s du\right)^{\frac{1}{s}} \sim \|x - y\|^{\frac{1}{s}} \left(\int_{Q_{xy}} h(u)^s du\right)^{\frac{1}{s}} \sim (\text{diam } Q)^{1 - \frac{n}{s}} \left(\int_{3Q} h(u)^s du\right)^{\frac{1}{s}} \leq C \text{diam } Q \left(\frac{1}{|3Q|} \int_{3Q} h(u)^s du\right)^{\frac{1}{s}}.$$
Since $h \in A_1(\mathbb{R}^n)$ and $\|h^q\|_{A_1} \leq \eta$, by (2.9),
\[
\left(\frac{1}{|3Q|} \int_{3Q} h(u)^q du \right)^{\frac{1}{q}} \leq \left(\frac{1}{|3Q|} \int h(u)^q du \right)^{\frac{1}{q}} \leq \eta \left(\frac{1}{|Q|} \int_{Q} h(u)^q du \right)^{\frac{1}{q}}.
\]
Hence
\[
d(x, y) \leq C \text{diam} Q \left(\frac{1}{|Q|} \int_{Q} h^q du \right)^{\frac{1}{q}},
\]
proving the claim. \(\square\)

**Claim 2.4** Every metric $d \in M_{q, \eta}(\mathbb{R}^n)$ has the following properties:

(a). There exists a mapping $\mathbb{R}^n \ni x \mapsto \omega_x \in \mathcal{MC}$ such that
\[
d(x, y) \sim \omega_x(\|x - y\|), \quad x, y \in \mathbb{R}^n,
\]
with constants depending only on $n, q$, and $\eta$;

(b). Let $x, y, z \in \mathbb{R}^n$ and let $\lambda \geq 1$ be a constant such that $\|y - z\| \leq \lambda\|x - z\|$. Then
\[
d(y, z) \leq C d(x, z) \quad (2.15)
\]
and
\[
\frac{d(x, z)}{\|x - z\|} \leq C \frac{d(y, z)}{\|y - z\|} \quad (2.16)
\]
where $C$ is a constant depending only on $n, q, \eta$, and $\lambda$.

(c). There exists a constant $C = C(n, q, \eta) > 0$ such that for every $x, y \in \mathbb{R}^n$ and $z \in (x, y)$ the following inequality
\[
d(x, z) + d(z, y) \leq C d(x, y)
\]
holds.

**Proof.** (a). Let $d = d_q(h)$ where $h \in A_1(\mathbb{R}^n)$ and $\|h^q\|_{A_1} \leq \eta$. Fix $x \in \mathbb{R}^n$. By $v_x$ we denote a function on $\mathbb{R}_+$ such that $v_x(0) = 0$ and
\[
v_x(t) := t \left(\frac{1}{|Q(x, t)|} \int_{Q(x,t)} h^q du \right)^{\frac{1}{q}}, \quad t > 0.
\]
Then, by (2.13),
\[
d(x, y) \sim v_x(\|x - y\|), \quad y \in \mathbb{R}^n. \quad (2.17)
\]
Clearly, since \( q \geq n \), the function

\[
v_x(t) = 2^{-\frac{n}{q}} t^{1 - \frac{n}{q}} \left( \int_{Q(x,t)} h^q \, du \right)^{\frac{1}{q}}
\]

is non-decreasing. On the other hand, by (2.17), for every \( 0 < t_1 < t_2 \)

\[
v_x(t_2)/t_2 = \left( \frac{1}{|Q(x,t_2)|} \int_{Q(x,t_2)} h^q \, du \right)^\frac{1}{q} \leq \|h^q\|_{A_1} \left( \frac{1}{|Q(x,t_1)|} \int_{Q(x,t_1)} h^q \, du \right)^\frac{1}{q}
\]

so that

\[
v_x(t_2)/t_2 \leq \eta v_x(t_1)/t_1, \quad 0 < t_1 < t_2.
\] (2.18)

It is well known that every non-negative non-decreasing function \( v_x \) on \( \mathbb{R}_+ \) satisfying inequality (2.18) is equivalent to a concave non-decreasing function. (In particular, \( v_x \) is equivalent to its least concave majorant.) Thus there exists a modulus of continuity \( \omega_x \in \mathcal{MC} \) such that \( \omega_x(t) \sim v_x(t) \) on \( \mathbb{R}_+ \) with constants of the equivalence depending only \( \eta \). Combining this equivalence with equivalence (2.17) we obtain the statement (a) of the claim.

(b). By part (a) of the claim there exists a concave majorant \( \omega_z \in \mathcal{MC} \) such that \( d(a,z) \sim \omega_z(\|a - z\|) \), \( a \in \mathbb{R}^n \). Since \( \omega_z \) is non-negative concave and non-decreasing, the function \( \omega_z(t)/t \) is non-increasing. Hence for every \( t_1, t_2 > 0 \) such that \( t_1 \leq \lambda t_2 \) we have

\[
\omega_z(t_1) \leq \lambda \omega_z(t_2) \quad \text{and} \quad \omega_z(t_2)/t_2 \leq \lambda \omega_z(t_1)/t_1.
\]

It remains to put \( t_1 := \|y - z\| \) and \( t_2 := \|x - z\| \) and the part (b) of the claim follows.

(c). Since \( z \in (x,y) \), we have \( \|y - z\|, \|x - z\| \leq \|x - y\| \) so that by part (b) of the claim

\[
d(x,z) \leq C d(x,y) \quad \text{and} \quad d(y,z) \leq C d(x,y)
\]

proving the statement (c) and the claim. \( \square \)

**Remark 2.5** Equivalence (2.14) motivates us to refer to the metric space \((\mathbb{R}^n, d)\) where \( d \in M_{q,\eta} (\mathbb{R}^n) \) as a *variable metric transform* of \( \mathbb{R}^n \); see Remark 1.7. This equivalence shows that given \( x \in \mathbb{R}^n \) the local behavior of the metric \( d \) is similar to that of a certain regular metric transform \( d_{\omega_x} := \omega_x(\|x - \cdot\|) \) where \( \omega_x \in \mathcal{MC} \) is a “modulus of continuity”. The function \( \omega_x \) varies from point to point, and this is the main difference between a regular metric transform (where \( \omega_x \) is constant, i.e., the same “modulus of continuity” \( \omega \) for all \( x \in \mathbb{R}^n \)) and a variable metric transform.

Nevertheless, in spite of \( \omega_x \) changes, the metric \( d \in M_{q,\eta} (\mathbb{R}^n) \) preserves several important properties of regular metric transforms.

For instance, let \( E \subset \mathbb{R}^n \) be a closed set and let \( x \in \mathbb{R}^n \setminus E \). Let \( \hat{x} \in E \) be an almost nearest point to \( x \) on \( E \) with respect to the Euclidean distance. Then \( \hat{x} \) is an almost nearest to \( x \) point with respect to the variable majorant \( d \) as well.
Another example is the standard Whitney covering of $\mathbb{R}^n \setminus E$ by a family of Whitney’s cubes. See, e.g. [29]. It is well known that this covering is universal with respect to the family

$$\mathcal{MT} = \{(\mathbb{R}^n, d_\omega), \ \omega \in \mathcal{MC}\}$$

of all metric transforms, i.e., it provides an almost optimal Whitney type extension construction for the family of Lipschitz spaces with respect to metric transforms. As we shall see below the same property holds for variable metric transforms as well.

Thus there exists a more or less complete analogy between extension methods for regular and variable metric transforms. In the next sections we present several applications of this approach to extensions of Sobolev functions.

3. Sobolev $L^1_\omega$-space as a union of Lipschitz spaces.

Proof of Theorem \[\text{1.4} \ (\text{Sufficiency})\]. Let $n \leq q < p$. Let $F \in C(\mathbb{R}^n)$ and let $h \in L^p_\omega(\mathbb{R}^n)$ be a non-negative function such that

$$|F(x) - F(y)| \leq d_q(x, y : h), \quad x, y \in \mathbb{R}^n.$$ 

Prove that $F \in L^1_\omega(\mathbb{R}^n)$ and

$$\|F\|_{L^1_\omega(\mathbb{R}^n)} \leq C(n, q, p)\|h\|_{L^p(\mathbb{R}^n)}. \quad (3.1)$$

Since $d_q \leq \delta_q$, for every $x, y \in \mathbb{R}^n$ we have

$$|F(x) - F(y)| \leq \delta_q(x, y : h) = \|x - y\| \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^q du\right)^{\frac{1}{q}}.$$

Let $Q$ be a cube in $\mathbb{R}^n$ and let

$$F_Q := \frac{1}{|Q|} \int_Q F(u) \, du.$$

Then

$$\frac{1}{|Q|} \int_Q |F(u) - F_Q| \, du \leq \sup_{x, y \in Q} |F(x) - F(y)| \leq C \sup_{x, y \in Q} \|x - y\| \left(\frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^q du\right)^{\frac{1}{q}} \leq C(diam Q)^{1 - \frac{n}{q}} \sup_{x, y \in Q} \left(\int_{Q_{xy}} h(u)^q du\right)^{\frac{1}{q}}.$$

Clearly, $Q_{xy} \subset 3Q$ for every $x, y \in Q$. Hence

$$\frac{1}{|Q|} \int_Q |F(u) - F_Q| \, du \leq C(diam Q)^{1 - \frac{n}{q}} \left(\int_{3Q} h(u)^q du\right)^{\frac{1}{q}}.$$ 

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so that
\[
\frac{1}{|Q|} \int_Q |F(u) - F_Q| \, du \leq C \text{diam } Q \left( \frac{1}{|3Q|} \int_{3Q} h(u)^q \, du \right)^{\frac{1}{q}}. \tag{3.2}
\]

Since \(1 \leq q < p\), for an arbitrary cube \(Q \subset \mathbb{R}^n\) we have
\[
\frac{1}{|Q|} \int_Q |F(u) - F_Q| \, du \leq C \text{diam } Q \left( \frac{1}{|3Q|} \int_{3Q} h(u)^p \, du \right)^{\frac{1}{p}}.
\]

In [16] it is proven that each function \(F\) satisfying this condition belongs to \(L^1_p(\mathbb{R}^n)\) and its gradient \(\|\nabla F(x)\| \leq C(n, p) h(x)\) a.e. on \(\mathbb{R}^n\). (See also [13], p. 266.) This of course implies the required inequality (3.1).

It is also noted in [13] that for the case \(1 \leq q < p\) inequality (3.1) can be directly deduced from (3.2) and a theorem of Calderón [4]; see also [5]. Recall that Calderón’s theorem states that a function \(F \in L^1_p(\mathbb{R}^n), 1 < p < \infty\), provided \(F^\sharp \in L^p_p(\mathbb{R}^n)\). Here \(F^\sharp\) is the sharp maximal function of \(F\)
\[
F^\sharp(x) := \sup_{r>0} \frac{1}{r} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |F(u) - F_{Q(x,r)}| \, du.
\]
Furthermore, \(\|F\|_{L^1_h(\mathbb{R}^n)} \leq C(n, p) \|F^\sharp\|_{L^p_h(\mathbb{R}^n)}\).

In fact, by (3.2), for every \(x \in \mathbb{R}^n\)
\[
F^\sharp(x) \leq C \sup_{r>0} \frac{1}{r} \text{diam } Q(x,r) \left( \frac{1}{|3Q(x,r)|} \int_{3Q(x,r)} h(u)^q \, du \right)^{\frac{1}{q}}
\leq C \sup_{r>0} \left( \frac{1}{|3Q(x,r)|} \int_{3Q(x,r)} h(u)^q \, du \right)^{\frac{1}{q}} \leq C \left( \mathcal{M}[h^q](x) \right)^{\frac{1}{q}}.
\]

Hence
\[
\|F^\sharp\|_{L^p_h(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^n} \mathcal{M}[h^q]^\frac{p}{q}(x) \, dx \right)^{\frac{1}{p}}.
\]

Since \(p/q > 1\), by the Hardy-Littlewood maximal theorem,
\[
\|F^\sharp\|_{L^p_h(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^n} (h^q)^\frac{p}{q}(x) \, dx \right)^{\frac{1}{p}} = C \|h\|_{L^p_h(\mathbb{R}^n)}
\]
proving that \(F^\sharp \in L^p_h(\mathbb{R}^n)\). Hence, by Calderón’s theorem [4], \(F \in L^1_p(\mathbb{R}^n)\) and
\[
\|F\|_{L^1_h(\mathbb{R}^n)} \leq C \|F^\sharp\|_{L^p_h(\mathbb{R}^n)} \leq C \|h\|_{L^p_h(\mathbb{R}^n)}
\]

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proving inequality (3.1) and the sufficiency part of Theorem 1.4.

\textbf{(Necessity).} Suppose that \( F \in L^p_1(\mathbb{R}^n) \). Let \( n < q < p \). Fix a constant \( \sigma \in (q, p) \), say \( \sigma = (p + q)/2 \). We put \( h_1(x) := \| \nabla F(x) \| \) and \( h_2(x) := \mathcal{M}[h_1^\sigma]^\frac{1}{\sigma}(x), \quad x \in \mathbb{R}^n \). Notice that

\[
h_1 = (h_1^\sigma)^{\frac{1}{\sigma}} \leq (\mathcal{M}[h_1^\sigma])^{\frac{1}{\sigma}} = h_2 \quad \text{a.e. on } \mathbb{R}^n
\]

so that, by (1.2),

\[
|F(x) - F(y)| \leq C(n, q)\|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h_2^\sigma(u) \, du \right)^{\frac{1}{\sigma}} = \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^\sigma(u) \, du \right)^{\frac{1}{\sigma}}
\]

where \( h = C(n, q)h_2 \).

Coifman and Rochberg [6] proved that if \( g \in L_{1,loc}(\mathbb{R}^n) \) and \( \mathcal{M}[g](x) < \infty \) a.e., then \( \mathcal{M}[g]^\sigma \in A_1(\mathbb{R}^n) \) for every \( 0 < \theta < 1 \) and \( \| \mathcal{M}[g]^\sigma \|_{A_1} \leq \gamma(n, \theta) \).

Let us apply this result to the function \( g = h^\sigma = C\mathcal{M}[h_1^\sigma]^\frac{1}{\sigma} \). Since \( 0 < \theta = q/\sigma < 1 \), the function \( h^\sigma \in A_1(\mathbb{R}^n) \) and

\[
\|h^\sigma\|_{A_1} = \|C\mathcal{M}[h_1^\sigma]^\theta\|_{A_1} = \|\mathcal{M}[h_1^\sigma]^\theta\|_{A_1} \leq \gamma(n, \theta) = \gamma(n, \frac{p+q}{2}) = \eta(n, p, q).
\]

Since \( p/\sigma > 1 \), by the Hardy-Littlewood maximal theorem,

\[
\|h\|_{L^p_1(\mathbb{R}^n)} = C \left( \int_{\mathbb{R}^n} \mathcal{M}[h_1^\sigma]^\frac{1}{\sigma}(x) \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n} (h_1^\sigma)^\frac{1}{\sigma}(x) \, dx \right)^{\frac{1}{p}} = C \left( \int_{\mathbb{R}^n} \|\nabla F\|^p(x) \, dx \right)^{\frac{1}{p}} = C\|F\|_{L^p_1(\mathbb{R}^n)}.
\]

Thus we have proved that for every \( q \in (n, p) \) there exists a non-negative function \( h \in L^p_1(\mathbb{R}^n) \) such that

\[
h^\sigma \in A_1(\mathbb{R}^n), \quad \|h^\sigma\|_{A_1} \leq \eta(n, p, q), \quad \|F\|_{L^p_1(\mathbb{R}^n)} \leq C\|h\|_{L^p_1(\mathbb{R}^n)} \tag{3.3}
\]

and

\[
|F(x) - F(y)| \leq \delta_q(x, y : h), \quad \text{for every } x, y \in \mathbb{R}^n. \tag{3.4}
\]

Thus it remains to show the existence of such a function \( h \) for \( q = n \). Let \( h \) be a function satisfying inequalities (3.3) and (3.4) for \( q := (p + n)/2 \). Prove that for some positive constant \( C = C(n, p) \) the function \( Ch \) satisfies (3.3) and (3.4) whenever \( q = n \).

In fact, for every cube \( Q \subset \mathbb{R}^n \)

\[
\frac{1}{|Q|} \int_Q h_n(u) \, du \leq \left( \frac{1}{|Q|} \int_Q h^\sigma(u) \, du \right)^{\frac{1}{\sigma}} \leq \|h^\sigma\|_{A_1}^{\frac{1}{\sigma}} (\text{ess inf}_Q h^\sigma)^{\frac{1}{\sigma}} = \|h^\sigma\|_{A_1}^{\frac{1}{\sigma}} \text{ess inf}_Q h^n
\]

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so that \( h^n \in A_1(\mathbb{R}^n) \) and

\[
\| h^n \|_{A_1} \leq \| h^n \|_{A_1}^{\frac{q}{A}} \leq \eta(n, p, q) \frac{n}{q} \leq \eta(n, p, q).
\]

Furthermore, by (1.8),

\[
\delta_q(x, y : h) \leq \| h_q \|_{A_q} \delta_n(x, y : h) \leq C \delta_n(x, y : Ch), \quad x, y \in \mathbb{R}^n.
\]

Since \( \| (Ch)^n \|_{A_1} = \| h^n \|_{A_1} \), the function \( Ch \) satisfies conditions (3.3) and (3.4) if \( q = n \).

Now combining inequality (3.4) and definition (1.6) we finally obtain the required estimate (1.9).

The proof of Theorem 1.4 is complete. \( \square \)

4. Sobolev \( L_p^1 \)-functions on closed subsets of \( \mathbb{R}^n \): a proof of Theorem 1.5.

Let \( f \in L_p^1(\mathbb{R}^n)|_E \) and let \( F \in L_p^1(\mathbb{R}^n) \) be an arbitrary continuous function such that \( F|_E = f \). Given \( x \in \mathbb{R}^n \) let

\[
f_{\infty, E}(x) := \sup_{y, z \in E} \frac{|f(y) - f(z)|}{\|x - y\| + \|x - z\|}.
\]

Prove that \( f_{\infty, E} \in L_p(\mathbb{R}^n) \) and

\[
\| f_{\infty, E} \|_{L_p(\mathbb{R}^n)} \leq C(n, p) \| F \|_{L_p^1(\mathbb{R}^n)}.
\]

In fact, let \( q := \frac{n + p}{2} \). Given \( x \in \mathbb{R}^n \) and \( y, z \in E \) let \( K := Q(x, \|x - y\| + \|x - z\|) \). Then, by the Sobolev-Poincaré inequality (1.2),

\[
|f(y) - f(z)| = |F(y) - F(z)| \leq C \text{diam } K \left( \frac{1}{|K|} \int_K \| \nabla F(u) \|^{q} du \right)^{\frac{1}{q}} \leq C(\|x - y\| + \|x - z\|) \left( \frac{1}{|K|} \int_K \| \nabla F(u) \|^{q} du \right)^{\frac{1}{q}}
\]

where \( C = C(n, p) \). Hence

\[
\frac{|f(y) - f(z)|}{\|x - y\| + \|x - z\|} \leq C \left( \frac{1}{|K|} \int_K \| \nabla F(u) \|^{q} du \right)^{\frac{1}{q}} \leq C M \| \nabla F \|^{\frac{q}{q}}(x).
\]

Taking the supremum in this inequality over all \( y, z \in E \) we obtain

\[
f_{\infty, E}(x) \leq C M \| \nabla F \|^{\frac{q}{q}}(x)^{\frac{1}{q}}, \quad x \in \mathbb{R}^n.
\]
Hence, by the Hardy-Littlewood maximal theorem,

$$\|f^2_{\infty,E}\|_{L^p(\mathbb{R}^n)} \leq C\|\mathcal{M}[f^2]\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla F\|_{L^p(\mathbb{R}^n)} = C\|F\|_{L^p(\mathbb{R}^n)}$$

proving (4.2).

Let

$$I_p(f; E) := \left\{ \int_{\mathbb{R}^n} \left( \sup_{y,z \in E} \frac{|f(y) - f(z)|}{\|x - y\| + \|x - z\|} \right)^p \right\}^{\frac{1}{p}}.$$

Then, by inequality (4.2),

$$I_p(f; E) = \|f^2_{\infty,E}\|_{L^p(\mathbb{R}^n)} \leq C\|F\|_{L^p(\mathbb{R}^n)}.$$

Taking the infimum in the right hand side of this inequality over all $F \in L^1_p(\mathbb{R}^n)$ such that $F|_E = f$ we obtain

$$I_p(f; E) \leq C(n, p)\|f\|_{L^p(\mathbb{R}^n)}|_E.$$

Prove that $f \in L^1_p(\mathbb{R}^n)|_E$ and $\|f\|_{L^p(\mathbb{R}^n)}|_E \leq CI_p(f; E)$ provided $f$ is a continuous function on $E$ and $I_p(f; E) < \infty$.

Let $\theta := (q + p)/2$; thus $n < q = (p + n)/2 < \theta < p$. Let $h_1 := \mathcal{M}[(f^2_{\infty,E})^\theta]$. Clearly, $f^2_{\infty,E} \leq h_1$ a.e. on $\mathbb{R}^n$.

Let $x, y \in E$ and let $u \in Q_{xy} = Q(x, \|x - y\|)$. Then, by definition (4.1),

$$|f(x) - f(y)| \leq f^2_{\infty,E}(u)(\|u - x\| + \|u - y\|)$$

so that

$$|f(x) - f(y)| \leq 3\|x - y\|f^2_{\infty,E}(u) \quad \text{for every} \quad u \in Q_{xy}.$$

Integrating this inequality over the cube $Q_{xy}$ we obtain

$$|f(x) - f(y)| \leq 3\|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} f^2_{\infty,E}(u) \, du \right) \leq 3\|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h_1(u) \, du \right)^\frac{1}{\theta}.$$

Hence

$$|f(x) - f(y)| \leq 3\|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h_1^q(u) \, du \right)^\frac{1}{q}.$$

so that

$$|f(x) - f(y)| \leq \delta_q(x, y : h_2), \quad x, y \in E,$$

where $h_2 := 3h_1$.

By a result of Coifman and Rochberg [6] which we have mentioned in Section 3, the weight

$$h_2^q = 3^q h_1^q = 3^q \mathcal{M}[(f^2_{\infty,E})^\theta] \in A_1(\mathbb{R}^n)$$

(4.4)
and

\[ \| h_2^2 \|_{A_1} \leq \eta(n, p, q). \]  

(4.5)

Furthermore, since \( 1 < \theta < p \), by the Hardy-Littlewood maximal theorem,

\[ \| h_2 \|_{L^p(\mathbb{R}^n)} \leq C \| \mathcal{M}(f_{\infty, E}^2) \|_{L^p(\mathbb{R}^n)} \leq C \| f_{\infty, E}^2 \|_{L^p(\mathbb{R}^n)} = CI_p(f; E). \]  

(4.6)

Thus the geodesic distance

\[ d = d_q(x, y : h_2), \quad x, y \in \mathbb{R}^n, \]

associated with the function \( \delta_q(h_2) \), see (1.6), belongs to the family \( D_{p,q}(\mathbb{R}^n) \) of metrics defined by (1.1).

By Theorem 1.1,

\[ \delta_q(x, y : h_2) \leq C(n, q) d_q(x, y : h_2) \quad \text{for all} \quad x, y \in \mathbb{R}^n, \]

so that, by (1.3),

\[ |f(x) - f(y)| \leq C d_q(x, y : h_2), \quad x, y \in E. \]

Hence \( f \in \text{Lip}(E; d_q(h_2)) \) and \( \| f \|_{\text{Lip}(E; d_q(h_2))} \leq C. \)

Let us extend the function \( f \) from the set \( E \) to all of \( \mathbb{R}^n \) using the McShane extension formula

\[ F(x) = \inf_{y \in E} \{ f(y) + C d_q(x, y : h_2) \}, \quad x \in E. \]

Then, by McShane’s theorem [10], the extension \( F \in \text{Lip}(\mathbb{R}^n; d_q(h_2)) \) and

\[ \| F \|_{\text{Lip}(\mathbb{R}^n; d_q(h_2))} \leq \| f \|_{\text{Lip}(E; d_q(h_2))} \leq C. \]

Let \( h := C h_2 \). We have proved that \( F \) satisfies the Lipschitz condition

\[ |F(x) - F(y)| \leq C d_q(x, y : h_2) = d_q(x, y : h), \quad x, y \in \mathbb{R}^n, \]

with respect to the metric \( d_q(h) \). Furthermore, \( h^q = C^q h_2^q \in A_1(\mathbb{R}^n) \) and \( \| h^q \|_{A_1} = \| h_2^q \|_{A_1} \leq \eta(n, p, q) \). Hence, by Theorem 1.4

\[ F \in L^1_p(\mathbb{R}^n) \quad \text{and} \quad \| F \|_{L^1_p(\mathbb{R}^n)} \leq C(n, p) \| h \|_{L^p(\mathbb{R}^n)} \leq C(n, p) \| h_2 \|_{L^p(\mathbb{R}^n)} \]

so that, by (1.6), \( \| F \|_{L^1_p(\mathbb{R}^n)} \leq CI_p(f; E). \)

Since \( F \in L^1_p(\mathbb{R}^n) \), the function \( f = F|_E \in L^1_p(\mathbb{R}^n)|_E \) and

\[ \| f \|_{L^1_p(\mathbb{R}^n)|_E} \leq \| F \|_{L^1_p(\mathbb{R}^n)} \leq CI_p(f; E) \]

proving the theorem. \( \square \)

**Remark 4.1** Let us present main steps of an extension algorithm which we used in the proof of Theorem 1.5.

Let \( q = (n + p)/2 \) and \( \theta = (q + p)/2 \). Let \( f \) be a continuous function defined on \( E \).
Step 1. We construct the sharp maximal function
\[ f^\sharp_{\infty,E}(x) = \sup_{y,z \in E} \frac{|f(x) - f(y)|}{\|x - y\| + \|y - z\|}, \quad x \in \mathbb{R}^n. \]

Step 2. We define a weight \( h : \mathbb{R}^n \to \mathbb{R}_+ \) by the formula
\[ h = \mathcal{M}^\theta\left[ (f^\sharp_{\infty,E})^q \right]^{\frac{1}{q}}. \]

Step 3. Using formula (1.6) we construct the geodesic distance \( d = d_q(x, y : h) \) associated with the “pre-metric”
\[ \delta_q(x, y : h) = \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h^q(u) du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n. \]

Step 4. We extend the function \( f : E \to \mathbb{R} \) to a function \( F \) defined on all of \( \mathbb{R}^n \) using the McShane’s extension formula
\[ F(x) = \inf_{y \in E} \{ f(y) + C d_q(x, y : h) \}. \]

where \( C \) is the constant from inequality (1.7).

By Theorem 1.5, the function \( F \) provides an almost optimal extension of the function \( f \) to a function from the Sobolev space \( L^1_p(\mathbb{R}^n) \) whenever \( f^\sharp_{\infty,E} \in L^p(\mathbb{R}^n) \).

Of course, the extension operator \( f \mapsto F \) is non-linear. Notice that this algorithm is new even for families of “nice” sets \( E \subset \mathbb{R}^n \) (like, say, closures of Lipschitz domains etc.).

As we have mentioned above, the classical Whitney extension method also provides an almost optimal extensions of functions from the trace space \( L^1_p(\mathbb{R}^n)|_E \) to functions from \( L^1_p(\mathbb{R}^n) \). See [26]. We give an alternative proof of this property of the Whitney extension operator in the next section. \(<\)

5. Sobolev-Poincaré inequality and extensions of \( L^m_p \)-functions.

In this section we generalize the approach presented in the previous sections to the case of the Sobolev spaces \( L^m_p(\mathbb{R}^n) \) whenever \( p > n \). In particular, we shall prove representation (1.15).

First let us recall the Sobolev-Poincaré inequality for \( L^m_p(\mathbb{R}^n) \)-functions whenever \( p > n \). Let \( q \in (n, p) \) and let \( F \in L^m_p(\mathbb{R}^n) \). Given \( x \in \mathbb{R}^n \) let \( P_x := T^{m-1}_x[F] \) be the Taylor polynomial of \( F \) at \( x \) of order \( m - 1 \). Then for every \( x, y \in \mathbb{R}^n \) and every multiindex \( \beta, |\beta| \leq m - 1 \),
\[ |D^\beta P_x(x) - D^\beta P_y(x)| \leq C \|x - y\|^{|\beta|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} (\nabla^m F(u))^q du \right)^{\frac{1}{q}}. \] (5.1)

In particular, for every \( \alpha, |\alpha| = m - 1 \), the following inequality
\[ |D^\alpha F(x) - D^\alpha F(y)| \leq C \|x - y\|^{|\alpha|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} (\nabla^m F(u))^q du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n, \]

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Let \( d \) be a metric on \( \mathbb{R}^n \). In Section 1 we have introduced the space \( C^{m,(d)}(\mathbb{R}^n) \) which consists of all \( C^m \)-functions \( F \) on \( \mathbb{R}^n \) satisfying the following condition: there exists a constant \( \lambda = \lambda(F) > 0 \) such that for every \( \alpha, |\alpha| = m \), and every \( x, y \in \mathbb{R}^n \)

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq \lambda d(x, y).
\]

In other words, \( F \in C^{m,(d)}(\mathbb{R}^n) \) if its partial derivatives \( D^\alpha F \) of order \( |\alpha| = m \) belong to \( \text{Lip}(\mathbb{R}^n, d) \). The space \( C^{m,(d)}(\mathbb{R}^n) \) is normed by

\[
\|F\|_{C^{m,(d)}(\mathbb{R}^n)} = \sum_{|\alpha|=m} \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|D^\alpha F(x) - D^\alpha F(y)|}{d(x, y)}. \tag{5.2}
\]

We say that a metric \( d \) on \( \mathbb{R}^n \) is \textit{pseudoconvex} if there exists a constant \( \lambda_d \geq 1 \) such that for every \( x, y, z \in \mathbb{R}^n, z \in (x, y) \), the following inequality

\[
d(x, z) + d(x, y) \leq \lambda_d d(x, y) \tag{5.3}
\]

holds.

Let \( F \in C^m(\mathbb{R}^n), y \in \mathbb{R}^n \), and let

\[
T_y^m[F](x) = \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha F(y)(x - y)^\alpha, \quad x \in \mathbb{R}^n,
\]

be the Taylor polynomial of \( F \) of degree \( m \) at \( y \). The following claim is a variant of the Taylor formula for the space \( C^{m,(d)}(\mathbb{R}^n) \) whenever \( d \) is a pseudoconvex metric.

**Claim 5.1** Let \( d \) be a pseudoconvex metric on \( \mathbb{R}^n \). Then for every \( F \in C^{m,(d)}(\mathbb{R}^n) \) and every multi-index \( \beta \) with \( |eta| \leq m \) the following inequality

\[
|D^\beta F(x) - D^\beta (T_y^m[F])(x)| \leq C\|F\|_{C^{m,(d)}(\mathbb{R}^n)} \|x - y\|^{m-|\beta|} d(x, y), \quad x, y \in \mathbb{R}^n,
\]

holds. Here \( C = C(n, m, \lambda_d) \).

**Proof.** For \( |eta| = m \) the claim follows from definition (5.2) of the norm in the space \( C^{m,(d)}(\mathbb{R}^n) \), so we can assume that \( |eta| < m \).

Let us make use of the following well known identity:

\[
F(x) = T_y^m[F](x) + m \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - y)^\alpha \int_0^1 (1 - t)^{m-1}(D^\alpha F(x + t(x - y)) - D^\alpha F(y))dt
\]

provided \( m > 0 \), \( x, y \in \mathbb{R}^n \), and \( F \in C^m(\mathbb{R}^n) \).

Let us apply this identity to \( D^\beta F \). We obtain

\[
D^\beta F(x) = T_y^{m-|\beta|}[D^\beta F](x) + (m - |\beta|) \sum_{|\alpha|=m-|\beta|} \frac{1}{\alpha!} (x - y)^\alpha \\
\cdot \int_0^1 (1 - t)^{m-|\beta|-1}\left\{D^\alpha(D^\beta F)(x + t(x - y)) - D^\alpha(D^\beta F)(y)\right\}dt.
\]
Since
\[ T^m - |\beta| [D^\beta F](x) = D^\beta (T^m_y [F])(x) \quad \text{for every} \quad x, y \in \mathbb{R}^n, \]
we have
\[ D^\beta F(x) - D^\beta (T^m_y [F])(x) = (m - |\beta|) \sum_{|\alpha| = m - |\beta|} \frac{1}{\alpha!} (x - y)^\alpha \]
\[ \cdot \int_0^1 (1 - t)^{m - |\beta| - 1} \left\{ D^{\alpha + \beta} F(x + t(y - x)) - D^{\alpha + \beta} F(y) \right\} dt. \]
Hence
\[ |D^\beta F(x) - D^\beta (T^m_y [F])(x)| \leq m \sum_{|\alpha| + |\beta| = m} \| x - y \|^{|\alpha|} \sup_{z \in [x, y]} |D^{\alpha + \beta} F(z) - D^{\alpha + \beta} F(y)|. \]
Combining this inequality with definition (5.2) we obtain
\[ |D^\beta F(x) - D^\beta (T^m_y [F])(x)| \leq m \| x - y \|^{m - |\beta|} \| F \|_{C^{m,0}(\mathbb{R}^n)} \sup_{z \in [x, y]} d(z, y). \]
Since \( d \) is pseudoconvex, by (5.3),
\[ |D^\beta F(x) - D^\beta (T^m_y [F])(x)| \leq \lambda_d m \| x - y \|^{m - |\beta|} \| F \|_{C^{m,0}(\mathbb{R}^n)} d(x, y) \]
proving the claim. \( \square \)

Let us apply this result to a metric which belongs to the family \( D_{p,q}(\mathbb{R}^n) \) whenever \( q \in [n, p) \), see (1.11). Notice that, by part (c) of Claim 2.1 every metric \( d \in D_{p,q}(\mathbb{R}^n) \) is quasiconvex. This property of \( d \) and Claim 2.1 imply the following

**Proposition 5.2** Let \( d \in D_{p,q}(\mathbb{R}^n) \) where \( q \in [n, p) \). Then for every \( F \in C^{m,0}(\mathbb{R}^n) \), every \( x, y \in \mathbb{R}^n \) and every \( \beta, |\beta| \leq m \), the following inequality
\[ |D^\beta F(x) - D^\beta (T^m_y [F])(x)| \leq C \| F \|_{C^{m,0}(\mathbb{R}^n)} \| x - y \|^{m - |\beta|} \]
holds. Here \( C = C(n, q, p) \).

Our next result is a generalization of Theorem 1.4 to the case of the Sobolev space \( L^m_p(\mathbb{R}^n) \).

**Theorem 5.3** Let \( m \) be a positive integer and let \( n \leq q < p < \infty \). There exists a positive constant \( \eta = \eta(n, p, q) \) depending only on \( n, q, \) and \( p \) such that the following statement is true:

A \( C^{m-1} \)-function \( F \) belongs to \( L^m_p(\mathbb{R}^n) \) if and only if there exists a non-negative function \( h \in L^q_p(\mathbb{R}^n) \) such that \( h^q \in A_1(\mathbb{R}^n) \), \( \| h^q \|_{A_1} \leq \eta \), and for every multiindex \( \beta, |\beta| = m - 1 \), and every \( x, y \in \mathbb{R}^n \) the following inequality
\[ |D^\beta F(x) - D^\beta F(y)| \leq d_q(x, y : h) \] (5.4)
holds. Furthermore,
\[ \| F \|_{L^m_p(\mathbb{R}^n)} \sim \inf \| h \|_{L^q_p(\mathbb{R}^n)}. \]
The constants of this equivalence depend only on \( m, n, q, \) and \( p \).
Proof. The case $m = 1$ is proven in Theorem 1.3. The case $m > 1$ easily follows from this result.

(Necessity.) Suppose that $F \in L_p^m(\mathbb{R}^n)$. Then $D^\beta F \in L_p^1(\mathbb{R}^n)$ for every $\beta$ with $|\beta| = m - 1$ and

$$\|D^\beta F\|_{L_p^1(\mathbb{R}^n)} \leq C \|F\|_{L_p^m(\mathbb{R}^n)}. \quad (5.5)$$

By Theorem 1.3 there exists a function $h_\beta \in L_p(\mathbb{R}^n)$ satisfying the following conditions: $h_\beta^q \in A_1(\mathbb{R}^n)$, $\|h_\beta^q\|_{A_1} \leq \eta = \eta(n, p, q)$,

$$\|h_\beta\|_{L_p(\mathbb{R}^n)} \leq C \|D^\beta F\|_{L_p^1(\mathbb{R}^n)}, \quad (5.6)$$

and

$$|D^\beta F(x) - D^\beta F(y)| \leq d_q(x, y : h_\beta), \quad x, y \in \mathbb{R}^n. \quad (5.7)$$

Let

$$h := \left( \sum_{|\beta| = 1} h_\beta^q \right)^{\frac{1}{q}}.$$

Since $h_\beta \leq h$, by definitions (1.4) and (1.6), $d_q(h_\beta) \leq d_q(h)$ for every $\beta, |\beta| = m - 1$. This inequality and (5.7) imply (5.4). On the other hand, by (5.5) and (5.6), $\|h\|_{L_p(\mathbb{R}^n)} \leq C\|F\|_{L_p^m(\mathbb{R}^n)}$.

It remains to prove that $h^q \in A_1(\mathbb{R}^n)$. In fact, since $\|h_\beta^q\|_{A_1} \leq \eta, |\beta| = m - 1$, by definition (2.8), for every cube $Q \subset \mathbb{R}^n$ we have

$$\frac{1}{|Q|} \int_Q h^q(u) \, du = \sum_{|\beta| = m-1} \frac{1}{|Q|} \int_Q h_\beta^q(u) \, du \leq \eta \sum_{|\beta| = m-1} \text{ess inf}_Q h_\beta^q \leq \eta \text{ess inf}_Q \left( \sum_{|\beta| = m-1} h_\beta^q \right) = \eta \text{ess inf}_Q h^q.$$

Hence $\|h\|_{A_1} \leq \eta$ proving the necessity.

(Sufficiency.) Suppose that $F \in C^{m-1}(\mathbb{R}^n)$ and there exists a function $h \in L_p(\mathbb{R}^n)$ such that $h^q \in A_1(\mathbb{R}^n)$, $\|h^q\|_{A_1} \leq \eta$, and for every $\beta, |\beta| = m - 1$, inequality (5.4) is satisfied. Then, by Theorem 1.3 $D^\beta F \in L_p^1(\mathbb{R}^n)$ and

$$\|D^\beta F\|_{L_p^1(\mathbb{R}^n)} \leq C\|h\|_{L_p(\mathbb{R}^n)}$$

so that $F \in L_p^m(\mathbb{R}^n)$ and $\|F\|_{L_p^m(\mathbb{R}^n)} \leq C\|h\|_{L_p(\mathbb{R}^n)}$.

The proof of Theorem 5.3 is complete. □

Remark 5.4 It is obvious that Theorem 5.3 implies representation (1.15). Furthermore, by this theorem, for each $F \in L_p^{m+1}(\mathbb{R}^n)$

$$\|F\|_{L_p^{m+1}(\mathbb{R}^n)} \sim \inf \{ \|h\|_{L_p(\mathbb{R}^n)} : d = d_q(h) \in D_{p,q}(\mathbb{R}^n), \|F\|_{C^{m,d}(\mathbb{R}^n)} \leq 1 \}$$

with constants in this equivalence depending only on $n, q,$ and $p$. □
5.1 A Whitney-type extension theorem for the space $C^{m,(d)}(\mathbb{R}^n)$.

We turn to the proof of Theorem [13.8]. This proof is based on representation [13.13] and the following Whitney-type extension theorem for spaces $C^{m,(d)}(\mathbb{R}^n)$ where $d \in \mathcal{D}_{p,q}(\mathbb{R}^n)$ is a metric transform.

**Theorem 5.5** Let $p \in (n, \infty)$, $q \in [n,p)$, $m \in \mathbb{N}$, and let $d \in \mathcal{D}_{p,q}(\mathbb{R}^n)$. Let

$$J = \{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\}$$

be a polynomial field defined on a closed set $E \subset \mathbb{R}^n$. There exists a function $F \in C^{m,(d)}(\mathbb{R}^n)$ such that $T^m_x[F] = P_x$ for every $x \in E$ if and only if the following quantity

$$\mathcal{L}_{m,d}(J) := \sum_{|\alpha| \leq m} \sup_{x,y \in E, x \neq y} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|}{\|x - y\|^{m-|\alpha|} d(x,y)}$$

is finite. Furthermore,

$$\mathcal{L}_{m,d}(J) \sim \inf \{\|F\|_{C^{m,(d)}(\mathbb{R}^n)} : F \in C^{m,(d)}(\mathbb{R}^n), T^m_x[F] = P_x \text{ for all } x \in E\}$$

with constants depending only on $n, p, q$, and $m$.

**Proof.** Suppose that the quantity

$$\mathcal{T}(J) := \inf \{\|F\|_{C^{m,(d)}(\mathbb{R}^n)} : F \in C^{m,(d)}(\mathbb{R}^n), T^m_x[F] = P_x \text{ for every } x \in E\}$$

is finite. Then, by Proposition [5.2], for every function $F \in C^{m,(d)}(\mathbb{R}^n)$ such that $T^m_x[F] = P_x$, $x \in E$, the following inequality

$$\mathcal{L}_{m,d}(J) \leq C \|F\|_{C^{m,(d)}(\mathbb{R}^n)}$$

holds. Taking the infimum in this inequality over all such function $F$ we obtain that $\mathcal{L}_{m,d}(J) \leq C \mathcal{T}(J)$ with $C = C(n, p, q, m)$.

Prove that $\mathcal{T}(J) \leq C \mathcal{L}_{m,d}(J)$ where $C$ is a constant depending only on $n, p, q$, and $m$.

Let $\lambda > 0$ and let $J = \{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\}$ be a polynomial field on $E$ such that for every multiindex $\alpha$, $|\alpha| \leq m$, and every $x, y \in E$ we have

$$|D^\alpha P_x(x) - D^\alpha P_y(x)| \leq \lambda \|x - y\|^{m-|\alpha|} d(x,y). \quad (5.8)$$

Let $F : \mathbb{R}^n \to \mathbb{R}$ be an extension of the field $J$ obtained by the Whitney extension method. Prove that $F$ belongs to the space $C^{m,(d)}(\mathbb{R}^n)$ and the norm of $F$ in this space does not exceed $C(n, q, p, m)\lambda$.

But first let us recall the Whitney extension construction. Since $E$ is a closed set, the set $\mathbb{R}^n \setminus E$ is open so that it admits a Whitney covering $W_E$ by a family $W_E$ of non-overlapping cubes. See, e.g., [29], or [12]. These cubes have the following properties:

(i). $\mathbb{R}^n \setminus E = \cup \{Q : Q \in W_E\}$;

(ii). For every cube $Q \in W_E$ we have

$$\text{diam } Q \leq \text{dist}(Q, E) \leq 4 \text{ diam } Q. \quad (5.9)$$

We are also needed certain additional properties of Whitney’s cubes which we present in the next lemma. These properties easily follow from constructions of the Whitney covering given in [29] and [12].

Given a cube $Q \subset \mathbb{R}^n$ let $Q^* := \frac{9}{8}Q$.
Lemma 5.6

(1) If \( Q, K \in W_E \) and \( Q^* \cap K^* \neq \emptyset \), then
\[
\frac{1}{4} \text{diam } Q \leq \text{diam } K \leq 4 \text{diam } Q;
\]

(2) For every cube \( K \in W_E \) there are at most \( N = N(n) \) cubes from the family \( W_E^* := \{Q^* : Q \in W_E\} \) which intersect \( K^* \);

(3) If \( Q, K \in W_E \), then \( Q^* \cap K^* \neq \emptyset \) if and only if \( Q \cap K \neq \emptyset \).

Let \( \Phi_E := \{\varphi_Q : Q \in W_E\} \) be a smooth partition of unity subordinated to the Whitney decomposition \( W_E \). Recall the main properties of this partition.

Lemma 5.7

The family of functions \( \Phi_E \) has the following properties:

(a) \( \varphi_Q \in C^\infty(\mathbb{R}^n) \) and \( 0 \leq \varphi_Q \leq 1 \) for every \( Q \in W_E \);

(b) \( \text{supp } \varphi_Q \subset Q^*(:= \frac{9}{8}Q), Q \in W_E \);

(c) \( \sum_{Q \in W_E} \varphi_Q(x) = 1 \) for every \( x \in \mathbb{R}^n \setminus S \);

(d) For every cube \( Q \in W_E \), every \( x \in \mathbb{R}^n \) and every multiindex \( \beta, |\beta| \leq m \), the following inequality
\[
|D^\beta \varphi_Q(x)| \leq C(n,m) (\text{diam } Q)^{-|\beta|}
\]
holds.

Given a cube \( Q \in W_E \) by \( a_Q \) we denote a point nearest to \( Q \) on the set \( E \). Notice the following property of \( a_Q \) which follows from inequality (5.9):
\[
a_Q \in 9Q \quad \text{for every cube} \quad Q \in W_E. \tag{5.10}
\]

By \( P^{(Q)} \) we denote the polynomial \( P_{a_Q} \). Finally, we define the extension \( F \) by the Whitney extension formula:
\[
F(x) := \begin{cases} P_x(x), & x \in E, \\ \sum_{Q \in W_E} \varphi_Q(x)P^{(Q)}(x), & x \in \mathbb{R}^n \setminus E. \end{cases} \tag{5.11}
\]

Let us note that the metric \( d \) is continuous with respect to the Euclidean distance, i.e., for every \( x \in \mathbb{R}^n \)
\[
d(x, y) \to 0 \quad \text{as} \quad \|x - y\| \to 0. \tag{5.12}
\]

In fact, since \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \), by definition \( \mathcal{D}_{p,q}(\mathbb{R}^n) \), \( d = d_q(h) \) where \( h \in L_{q,\text{loc}}(\mathbb{R}^n) \) is a non-negative function such that \( h^q \in A_1(\mathbb{R}^n) \) and \( \|h^q\|_{A_1} \leq \eta(n,p,q) \). Then, by Theorem \( 1.11 \)
\[
d(x, y) \sim \delta_q(x, y : h) = \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h(u)^q du \right)^{\frac{1}{q}} = 2^n \|x - y\|^{1 - \frac{n}{q}} \left( \int_{Q_{xy}} h(u)^q du \right)^{\frac{1}{q}}.
\]

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Since $n \leq q$, $h \in L_{q,\text{loc}}(\mathbb{R}^n)$ and $\text{diam} Q_{xy} = 2 \|x - y\| \to 0$ as $\|x - y\| \to 0$, we have $d(x, y) \to 0$ proving (5.12).

Hence, by (5.8), for every multiindex $\beta$, $|\beta| \leq m$, we have

$$D^\beta P_x(x) - D^\beta P_y(y) = o(\|x - y\|^{m - |\beta|}), \quad x, y \in E,$$

so that the $m$-jet $\{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\}$ satisfies the hypothesis of the Whitney extension theorem [30]. By this theorem the extension $F : \mathbb{R}^n \to \mathbb{R}$ defined by the formula (5.11) is a $C^m$-function such that

$$D^\beta F(x) = D^\beta P_x(x) \text{ for every } x \in E \text{ and every } \beta, |\beta| \leq m. \quad (5.13)$$

Prove that $\|F\|_{C^m(\mathbb{R}^n)} \leq C\lambda$, i.e., for every multiindex $\alpha$, $|\alpha| = m$, and every $x, y \in \mathbb{R}^n$ the following inequality

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C\lambda d(x, y). \quad (5.14)$$

holds.

Consider four cases.

The first case: $x, y \in E$. Since $P_y \in \mathcal{P}_m(\mathbb{R}^n)$, for every multiindex $\alpha$ of order $m$ the function $D^\alpha P_y$ is a constant function. In particular, $D^\alpha P_y(x) = D^\alpha P_y(y)$. Hence, by (5.13),

$$|D^\alpha F(x) - D^\alpha F(y)| = |D^\alpha P_x(x) - D^\alpha P_y(y)| = |D^\alpha P_x(x) - D^\alpha P_y(x)|$$

so that, by (5.8),

$$|D^\alpha F(x) - D^\alpha F(y)| \leq \lambda d(x, y)$$

proving (5.14) in the case under consideration.

The second case: $x \in E$, $y \in \mathbb{R}^n \setminus E$. Given a Whitney cube $K \in W_E$ let

$$T(K) := \{Q \in W_E : Q \cap K \neq \emptyset\} \quad (5.15)$$

be a family of all Whitney cubes touching $K$.

**Lemma 5.8** Let $K \in W_E$ be a Whitney cube and let $y \in K^* = \frac{9}{8}K$. Then for every multiindex $\alpha$ the following inequality

$$|D^\alpha F(y) - D^\alpha P_{aK}(y)| \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam} K)^{|\xi| - |\alpha|} |D^\xi P_{aq}(a_K) - D^\xi P_{aK}(a_K)|$$

holds. Here $C = C(n, m, \alpha)$.

**Proof.** We notice that, by part (2) of Lemma [5.6], $\#T(K) \leq N(n)$, and, by part (3) of this lemma,

$$T(K) = \{Q \in W_E : Q^* \cap K^* \neq \emptyset\}.$$

Recall $P^{(Q)} = P_{aq}$ and $a_Q \in 9Q$ for every $Q \in W_E$. Let us estimate the quantity

$$I := |D^\alpha F(y) - D^\alpha P_{aK}(y)|.$$
By formula (5.11) and by part (c) of Lemma 5.6

\[ \mathbf{F}(y) - \mathbf{P}_{a\mathbf{K}}(y) = \sum_{Q \in \mathbf{W}_{E}} \varphi_{Q}(y)(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y)) \]

so that, by part (b) of Lemma 5.6 and by definition (5.15),

\[ \mathbf{F}(y) - \mathbf{P}_{a\mathbf{K}}(y) = \sum_{Q \in \mathbf{T}(\mathbf{K})} \varphi_{Q}(y)(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y)). \]

Hence

\[ I := |D^{\alpha}\mathbf{F}(y) - D^{\alpha}\mathbf{P}_{a\mathbf{K}}(y)| \leq \sum_{Q \in \mathbf{T}(\mathbf{K})} |D^{\alpha}(\varphi_{Q}(y)(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y)))| \]

so that

\[ I \leq \sum_{Q \in \mathbf{T}(\mathbf{K})} A_{Q}(y; \alpha) \quad (5.16) \]

where

\[ A_{Q}(y; \alpha) := |D^{\alpha}(\varphi_{Q}(y)(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y)))|. \]

Let \( Q \in \mathbf{T}(\mathbf{K}) \). Then

\[ A_{Q}(y; \alpha) \leq C \sum_{|\beta| + |\gamma| = |\alpha|} |D^{\beta}\varphi_{Q}(y)||D^{\gamma}(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y))| \]

so that, by part (d) of Lemma 5.7

\[ A_{Q}(y; \alpha) \leq C \sum_{|\beta| + |\gamma| = |\alpha|} (\text{diam } Q)^{-|\beta|} |D^{\gamma}(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y))|. \]

Since \( Q \cap \mathbf{K} \neq \emptyset \), by part (1) of Lemma 5.6\(^{5.6}\) diam \( Q \sim \text{diam } \mathbf{K} \) so that

\[ A_{Q}(y; \alpha) \leq C \sum_{|\beta| + |\gamma| = |\alpha|} (\text{diam } \mathbf{K})^{-|\beta|} |D^{\gamma}(P^{(Q)}(y) - \mathbf{P}_{a\mathbf{K}}(y))|. \quad (5.17) \]

Let us estimate the distance between \( a_{Q} \) and \( y \). By (5.9),

\[ \|a_{Q} - y\| \leq \text{diam } \mathbf{K}^{*} + \text{diam } Q + \text{dist}(Q, \mathbf{E}) \leq 2 \text{diam } \mathbf{K} + \text{diam } Q + 4 \text{diam } Q. \]

Since \( Q \cap \mathbf{K} \neq \emptyset \), by part (1) of Lemma 5.6\(^{5.6}\)

\[ \|a_{Q} - y\| \leq 2 \text{diam } \mathbf{K} + 4 \text{diam } \mathbf{K} + 16 \text{ diam } \mathbf{K} = 22 \text{ diam } \mathbf{K}. \quad (5.18) \]

Let

\[ \mathbf{P}_{Q} := P^{(Q)} - \mathbf{P}_{a\mathbf{K}} = \mathbf{P}_{aQ} - \mathbf{P}_{a\mathbf{K}}. \]
Let us estimate the quantity $|D^\gamma \tilde{P}_Q(y)|$. Since $\tilde{P}_Q \in \mathcal{P}_m(\mathbb{R}^n)$, we can represent this polynomial as

$$\tilde{P}_Q(z) = \sum_{|\xi| \leq m} \frac{1}{\xi!} D^\xi \tilde{P}_Q(a_K) (z - a_K)^\xi.$$

Hence

$$D^\gamma \tilde{P}_Q(z) = \sum_{|\gamma| \leq |\xi| \leq m} \frac{1}{(\xi - \gamma)!} D^\xi \tilde{P}_Q(a_K) (z - a_K)^{\xi - \gamma}$$

so that

$$|D^\gamma \tilde{P}_Q(y)| \leq C \sum_{|\gamma| \leq |\xi| \leq m} |D^\xi \tilde{P}_Q(a_K)| \|y - a_K\|^{\xi - |\gamma|} \leq C \sum_{|\gamma| \leq |\xi| \leq m} (\text{diam } K)^{\xi - |\gamma|} |D^\xi \tilde{P}_Q(a_K)|.$$

Combining this inequality with (5.17) we obtain

$$A_Q(y; \alpha) \leq C \sum_{|\beta| + |\gamma| = |\alpha|} (\text{diam } K)^{-|\beta|} |D^\gamma \tilde{P}_Q(y)|$$

$$\leq C \sum_{|\beta| + |\gamma| = |\alpha|} (\text{diam } K)^{-|\beta|} \sum_{|\gamma| \leq |\xi| \leq m} (\text{diam } K)^{\xi - |\gamma|} |D^\xi \tilde{P}_Q(a_K)|$$

$$\leq C \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - |\alpha|} |D^\xi \tilde{P}_Q(a_K)|.$$

Hence, by (5.16),

$$I \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - |\alpha|} |D^\xi \tilde{P}_Q(a_K)|$$

proving the lemma.

\[ \square \]

**Lemma 5.9** Let $x \in E$ and let $K \in W_E$ be a Whitney cube. Then for every $y \in K$ and every $\alpha, |\alpha| = m$, the following inequality

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C(n, m) \left\{ |D^\alpha P_x (x) - D^\alpha P_{a_K}(x)| \right. + \left. \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m} |D^\xi P_{aq}(a_K) - D^\xi P_{a_K}(a_K)| \right\}$$

holds.

**Proof.** We have

$$|D^\alpha F(x) - D^\alpha F(y)| \leq |D^\alpha F(x) - D^\alpha P_{a_K}(y)| + |D^\alpha P_{a_K}(y) - D^\alpha F(y)| = I_1 + I_2.$$

Since $P_{a_K} \in \mathcal{P}_m(\mathbb{R}^n)$ and $|\alpha| = m$, the function $D^\alpha P_y$ is a constant function so that

$$D^\alpha P_{a_K}(y) = D^\alpha P_{a_K}(x).$$

Since $x \in E$, by (5.13), $D^\alpha F(x) = D^\alpha P_x(x)$ so that, by (5.8),

$$I_1 := |D^\alpha F(x) - D^\alpha P_{a_K}(y)| = |D^\alpha P_x(x) - D^\alpha P_{a_K}(x)|.$$
It remains to apply Lemma 5.8 to the quantity $I_2 := |D^\alpha F(y) - D^\alpha P_{a_K}(y)|$, and the lemma follows. □

Let us prove inequality (5.14) for arbitrary $x \in E$, $y \in \mathbb{R}^n \setminus E$ and $\alpha$, $|\alpha| = m$. Let $y \in K$ for some $K \in W_E$. By Lemma 5.9,

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C\{J_1 + J_2\}$$

where

$$J_1 := |D^\alpha P_x(x) - D^\alpha P_{a_K}(x)|$$

and

$$J_2 := \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m} |D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)|$$

First let us estimate $J_2$. By (5.8), for every $\xi$, $|\xi| \leq m$, and every $Q \in T(K)$, we have

$$|D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)| \leq \lambda \|a_Q - a_K\|^{|\xi|} d(a_Q, a_K).$$

Hence

$$J_2 \leq \lambda \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m} \|a_Q - a_K\|^{|\xi|} d(a_Q, a_K).$$

But, by (5.18),

$$\|a_Q - a_K\| \leq \|a_Q - y\| + \|y - a_K\| \leq 23 \text{ diam } K \quad (5.19)$$

proving that

$$J_2 \leq C(n, m) \lambda \sum_{Q \in T(K)} d(a_Q, a_K). \quad (5.20)$$

Now prove that for some constant $C = C(n, p, q)$

$$d(a_Q, y) \leq C d(x, y) \quad \text{for every } Q \in T(K). \quad (5.21)$$

In fact, by (5.18), $\|a_Q - y\| \leq 22 \text{ diam } K$. Since $x \in E$ and $y \in K$, by (5.9),

$$\text{diam } K \leq 4 \text{ dist}(K, E) \leq 4 \|x - y\|$$

so that $\|a_Q - y\| \leq 88 \|x - y\|$. Hence, by part (b) of Claim 2.4 see inequality (2.15),

$$d(a_Q, y) \leq C(n, p, q) d(x, y)$$

proving (5.21).

Now we have

$$d(a_Q, a_K) \leq d(a_Q, y) + d(y, a_K) \leq C d(x, y)$$

so that, by (5.20),

$$J_2 \leq C \lambda \#T(K) d(x, y) \leq C \lambda d(x, y).$$

See part (2) of Lemma 5.6.
On the other hand, by (5.8) and (5.21),
\[
J_1 := |D^\alpha P_x(x) - D^\alpha P_{a_K}(x)| \leq \lambda d(x, a_K) \leq \lambda (d(x, y) + d(y, a_K)) \leq C \lambda d(x, y).
\]
Finally,
\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C \{J_1 + J_2\} \leq C \lambda d(x, y).
\]

The third case: \(y \in K, K \in W_E\) and \(x \in \mathbb{R}^n \setminus K^*\).
Since \(K^* = \frac{9}{8}K\) and \(x \notin K^*\), we have
\[
\|x - y\| \geq \frac{1}{16} \text{diam } K.
\]
Let \(a \in E\) be a point nearest to \(x\) on \(E\). Then
\[
\|a - x\| = \text{dist}(x, E) \leq \text{dist}(y, E) + \|x - y\| \leq \text{dist}(K, E) + \text{diam } K + \|x - y\|
\]
so that, by (5.9),
\[
\|a - x\| \leq 4 \text{ diam } K + \text{diam } K + \|x - y\| \leq 81\|x - y\|.
\]
Hence, by part (b) of Claim 2.4, see (2.15), \(d(a, x) \leq C d(x, y)\).
We have
\[
\|y - a\| \leq \|x - y\| + \|x - a\| \leq 82\|x - y\|
\]
so that again, by (2.15), \(d(y, a) \leq C d(x, y)\). We obtain
\[
|D^\alpha F(x) - D^\alpha F(y)| \leq |D^\alpha F(x) - D^\alpha F(a)| + |D^\alpha F(a) - D^\alpha F(y)|
\]
so that, by the result proven in the second case,
\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C \lambda (d(x, a) + d(y, a)) \leq C \lambda d(x, y).
\]

The fourth case: \(y \in K, x \in K^*\) where \(K \in W_E\). The proof of inequality (5.14) in this case is based on the next

**Lemma 5.10** Let \(K \in W_E\) be a Whitney cube and let \(x, y \in K^*\). Then for every multi-index \(\alpha, |\alpha| = m\), the following inequality
\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C \frac{\|x - y\|}{\text{diam } K} \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi|-m} |D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)|
\]
holds. Here \(C\) is a constant depending only on \(n\) and \(m\).

**Proof.** Notice that the function \(F \in C^\infty(\mathbb{R}^n \setminus E)\), see formula (5.11), so that, by the Lagrange theorem, for every \(\alpha, |\alpha| = m\), there exists \(z \in [x, y]\) such that
\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C \|x - y\| \sum_{|\beta|=m+1} |D^\beta F(z)|.
\]

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Since \( x, y \in K^* \), the point \( z \in K^* \) as well.

Since the polynomial \( P_{a_K} \in P_m(\mathbb{R}^n) \), for every multiindex \( \beta \) of order \(|\beta| = m + 1\) we have \( D^\beta P_{a_K} = 0 \) so that, by Lemma 5.8,

\[
|D^\beta F(z)| = |D^\beta F(z) - D^\beta P_{a_K}(z)| \\
\leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - |\beta|} |D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)| \\
= C (\text{diam } K)^{-1} \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m} |D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)|.
\]

Combining this inequality with (5.22) we obtain the statement of the lemma. \( \square \)

We are in a position to prove inequality (5.14) for arbitrary \( y \in K \) and \( x \in K^* \). By inequality (5.8), for every cube \( Q \in T(K) \) and every \( \xi, |\xi| \leq m \),

\[
|D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)| \leq \lambda d(a_Q, a_K)(\text{diam } K)^{m-|\xi|}
\]

so that, by Lemma 5.22,

\[
I := |D^\alpha F(x) - D^\alpha F(y)| \\
\leq C \frac{\|x - y\|}{\text{diam } K} \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m} |D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)| \\
\leq C \frac{\|x - y\|}{\text{diam } K} \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m} (\lambda d(a_Q, a_K)(\text{diam } K)^{m-|\xi|}) \\
\leq C\lambda \frac{\|x - y\|}{\text{diam } K} \sum_{Q \in T(K)} d(a_Q, a_K).
\]

Notice that, by (5.19) and (5.2),

\[
\|a_Q - y\| \leq \|a_Q - a_K\| + \|a_K - y\| \leq 23 \text{ diam } K + \text{ diam } K + \text{ dist}(K, E) \\
\leq 23 \cdot 4 \text{ dist}(K, E) + 4 \text{ dist}(K, E) + \text{ dist}(K, E) \\
= 97 \text{ dist}(K, E) \leq 97 \|y - a_K\|.
\]

Hence, by part (b) of Claim 2.4 see (2.15), \( d(a_Q, y) \leq C d(y, a_K) \) so that

\[
d(a_Q, a_K) \leq d(a_Q, y) + d(y, a_K) \leq C d(y, a_K).
\]

This implies the following inequality

\[
I \leq C\lambda (\# T(K)) \frac{\|x - y\|}{\text{diam } K} d(y, a_K).
\]

But, by part (2) of Lemma 5.6 \( \# T(K) \leq N(n) \) so that

\[
I \leq C\lambda \frac{\|x - y\|}{\text{diam } K} d(y, a_K).
\]
Since \( x, y \in K^* \), the distance \( \| x - y \| \leq \text{diam } K^* = \frac{9}{8} \text{diam } K \). But
\[
\text{diam } K \leq 4 \text{dist}(K, E) \leq 4\| y - a_K \|
\]
so that \( \| x - y \| \leq 5\| y - a_K \| \). Therefore, by part (b) of Claim 2.4 see (2.13),
\[
\frac{\| x - y \|}{\| y - a_K \|} \text{d}(y, a_K) \leq C \text{d}(x, y).
\]

On the other hand,
\[
\| y - a_K \| \leq \text{diam } K + \text{dist}(a_K, K) = \text{diam } K + \text{dist}(K, E)
\]
\[
\leq \text{diam } K + 4 \text{diam } K = 5 \text{diam } K.
\]
Finally,
\[
I \leq C\lambda \frac{\| x - y \|}{\text{diam } K} \text{d}(y, a_K) \leq 5 C\lambda \| x - y \| \| y - a_K \| \text{d}(y, a_K) \leq C\lambda \text{d}(x, y).
\]
The proof of Theorem 5.5 is complete. \( \square \)

### 5.2 Extensions of \( L^m_p(\mathbb{R}^n) \)-jets: a proof of Theorem 1.8

Let \( m \geq 1 \) and let \( J : E \to P_{m-1}(\mathbb{R}^n) \) be a mapping on \( E \) which every point \( x \in E \) assigns a polynomial \( P_x = J(x) \in P_{m-1}(\mathbb{R}^n) \). We consider \( J \) as a polynomial field
\[
J = \{ P_x \in P_{m-1}(\mathbb{R}^n) : x \in E \}
\]
on \( E \) and refer to \( J \) as a polynomial jet on \( E \) of order \( m - 1 \) or \((m - 1)\)-jet.

Let
\[
JL^m_p(\mathbb{R}^n)|_E = \{ J : E \to P_{m-1}(\mathbb{R}^n) : \exists F \in L^m_p(\mathbb{R}^n) \text{ such that } T^{m-1}_x[F] = P_x \forall x \in E \}
\]
be the space of traces to \( E \) of all \((m - 1)\)-jets generated by \( L^m_p(\mathbb{R}^n) \)-functions. We norm this space by the standard trace norm
\[
\| J \|_{JL^m_p(\mathbb{R}^n)|_E} := \inf \{ \| F \|_{L^m_p(\mathbb{R}^n)} : F \in L^m_p(\mathbb{R}^n), T^{m-1}_x[F] = P_x \text{ for all } x \in E \}. \tag{5.23}
\]

In these settings the result of Theorem 1.8 can be reformulated as follows: for every \((m - 1)\)-jet
\[
J = \{ P_x \in P_{m-1}(\mathbb{R}^n) : x \in E \} \in JL^m_p(\mathbb{R}^n)|_E
\]
the following equivalence
\[
\| J \|_{JL^m_p(\mathbb{R}^n)|_E} \sim N_{m,p}(J)
\]
holds with constants depending only on \( n, m, \) and \( p \). Recall that the quantity \( N_{m,p}(J) \) is defined in the statement of Theorem 1.8.

Prove that
\[
N_{m,p}(J) \leq C(n, p, m) \| J \|_{JL^m_p(\mathbb{R}^n)|_E}. \tag{5.24}
\]
Let \( F \in L^p_m(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n) \) be an arbitrary function such that \( P_x = T_x^{m-1}[F] \) for all \( x \in E \). Let \( x \in \mathbb{R}^n \), \( y \), \( z \in E \), and let

\[
K := Q(x, 2\|x - y\| + 2\|x - z\|).
\]

By the Sobolev-Poincaré inequality (5.1), for every multiindex \( \beta, |\beta| \leq m - 1 \), we have

\[
|D^\beta P_y(y) - D^\beta P_z(y)| \leq C \|y - z\|^{m-|\beta|} \left( \frac{1}{|Q_{yz}|} \int_Q |\nabla^m F(u)|^q du \right)^{\frac{1}{q}}
\]

\[
\leq C \|y - z\|^{m-|\beta|} \left( \frac{1}{|Q_{yz}|} \int_Q |\nabla^m F(u)|^q du \right)^{\frac{1}{q}}.
\]

Since \( Q_{yz} \subset K \) and \( m - |\beta| - \frac{m}{q} > 0 \) provided \( |\beta| \leq m - 1 \), we obtain

\[
|D^\beta P_y(y) - D^\beta P_z(y)| \leq C (\|x - y\| + \|x - z\|)^{m-|\beta|} \left( \frac{1}{|K|} \int_K |\nabla^m F(u)|^q du \right)^{\frac{1}{q}}
\]

\[
\leq C (\|x - y\|^{m-|\beta|} + \|x - z\|^{m-|\beta|}) |K|^{-\frac{n}{q}} \left( \frac{1}{|K|} \int_K |\nabla^m F(u)|^q du \right)^{\frac{1}{q}}.
\]

Hence

\[
\frac{|D^\beta P_y(y) - D^\beta P_z(y)|}{\|x - y\|^{m-|\beta|} + \|x - z\|^{m-|\beta|}} \leq C \left( \frac{1}{|K|} \int_K |\nabla^m F(u)|^q du \right)^{\frac{1}{q}}.
\]

(5.25)

Let us introduce a sharp maximal function \( J^\#_E : \mathbb{R}^n \to \mathbb{R}_+ \) for the jet \( J = \{P_x : x \in E\} \) by letting

\[
J^\#_E(x) := \sum_{|\beta| \leq m-1} \sup_{y,z \in E} \frac{|D^\beta P_y(y) - D^\beta P_z(y)|}{\|x - y\|^{m-|\beta|} + \|x - z\|^{m-|\beta|}}, \quad x \in \mathbb{R}^n.
\]

(5.26)

Clearly, by (1.16),

\[
\|J^\#_E\|_{L^p(\mathbb{R}^n)} \sim \mathcal{N}_{m,p}(J).
\]

(5.27)

Prove that \( \|J^\#_E\|_{L^p(\mathbb{R}^n)} \leq C \|F\|_{L^p_m(\mathbb{R}^n)} \). By inequality (5.25),

\[
\sup_{y,z \in E} \frac{|D^\beta P_y(y) - D^\beta P_z(y)|}{\|x - y\|^{m-|\beta|} + \|x - z\|^{m-|\beta|}} \leq C \left\{ \mathcal{M}(|\nabla^m F(u)|^q)(x) \right\}^{\frac{1}{q}}.
\]

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so that
\[ J_E^\# (x) \leq C \{ \mathcal{M}[|\nabla^{m} F|^q](x) \}^{\frac{1}{q}}, \quad x \in \mathbb{R}^n. \]

Since \( q < p \), by the Hardy-Littlewood maximal theorem,
\[
\| J_E^\# \|_{L_p(\mathbb{R}^n)} \leq C \{ \mathcal{M}[|\nabla^{m} F|^q] \}^{\frac{1}{q}} \| F \|_{L_p(\mathbb{R}^n)} \leq C \| \nabla^{m} F \|_{L_p(\mathbb{R}^n)} = C \| F \|_{L_p(\mathbb{R}^n)}
\]
so that, by (5.27), \( N_{m,p}(J) \leq C \| F \|_{L_p(\mathbb{R}^n)} \). Taking the infimum in this inequality over all functions \( F \in L_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n) \) such that \( T_x^{m-1}[F] = P_x \) for every \( x \in E \), we obtain the required inequality (5.24).

Prove that
\[
\| J \|_{L_p^m(\mathbb{R}^n)|_E} \leq C(n, p, m) N_{m,p}(J).
\] (5.28)

Let \( q := (n + p)/2 \) and \( \theta := (q + p)/2 \). Let \( h_1 := \mathcal{M}[|J_E^\#|^\theta]^\frac{1}{\theta} \). Clearly, \( J_E^\# \leq h_1 \) a.e. on \( \mathbb{R}^n \).

Let \( x, y \in E \) and let \( u \in Q_{xy} = Q(x, ||x - y||) \). Then
\[
\| u - x \|, \| u - y \| \leq 2 ||x - y||
\]
so that, by (5.26),
\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq (||u - x||^{m-|\beta|} + ||u - y||^{m-|\beta|}) J_E^\#(u) \leq C ||x - y||^{m-|\beta|} J_E^\#(u)
\]
for every \( \beta, |\beta| \leq m - 1 \), and every \( u \in Q_{xy} \). Integrating this inequality over the cube \( Q_{xy} \) with respect to \( u \), we obtain
\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq C ||x - y||^{m-|\beta|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} J_E^\#(u) \, du \right)^{\frac{1}{\theta}}
\]
so that
\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq C ||x - y||^{m-|\beta|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h_1(u) \, du \right)^{\frac{1}{\theta}}
\]

Hence
\[
\frac{|D^\beta P_x(x) - D^\beta P_y(x)|}{||x - y||^{m-|\beta|-1}} \leq C ||x - y|| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h_1^q(u) \, du \right)^{\frac{1}{q}}
\]
proving that
\[
\frac{|D^\beta P_x(x) - D^\beta P_y(x)|}{||x - y||^{m-|\beta|-1}} \leq \delta_q(x, y : h_2), \quad x, y \in E,
\] (5.29)
where \( h_2 := C h_1 \). See \((3.1)\). 

Similar to \((3.4)\), \((3.5)\) and \((4.6)\) we show that
\[
A_1(\mathbb{R}^n), \quad \|h_2\|_{A_1} \leq \eta(n, p, q),
\]
and
\[
\|h_2\|_{L_p(\mathbb{R}^n)} \leq C \|J^h_\delta \|_{L_p(\mathbb{R}^n)}. \tag{5.31}
\]

Let \( d = d_q(x : y : h_2), x, y \in \mathbb{R}^n \), be the geodesic metric associated with \( \delta_q(h_2) \), see \((1.6)\).

By \((5.30)\) and \((5.31)\), the metric \( d \in D_{p,q}(\mathbb{R}^n) \), see \((1.11)\). Hence, by Theorem \(1.1\)
\[
\delta_q(x : y : h_2) \leq C d_q(x : y : h_2), \quad x, y \in \mathbb{R}^n,
\]
where \( C = C(n, p) \).

This inequality and \((5.29)\) imply that for every \( x, y \in E \) and every \( \beta, |\beta| \leq m - 1, \)
\[
\frac{|D^{\beta}P_x(x) - D^{\beta}P_y(x)|}{\|x - y\|^{m-|\beta|-1}} \leq C d_q(x : y : h_2) = C d(x, y).
\]

Hence
\[
S[J] := \sum_{|\beta| \leq m-1} \sup_{x, y \in E, x \neq y} \frac{|D^{\beta}P_y(y) - D^{\beta}P_x(y)|}{\|x - y\|^{m-|\beta|-1} d(x, y)} \leq C(n, m, p), \quad x, y \in \mathbb{R}^n,
\]
so that, by Theorem \(5.5\), the jet \( J \in JC^{m-1,(d)}(\mathbb{R}^n)|_E \).

Thus there exists a function \( F \in C^{m-1,(d)}(\mathbb{R}^n) \) such that \( T^{m-1}_x[F] = P_x \) for every \( x \in E \). Furthermore
\[
\|F\|_{C^{m-1,(d)}(\mathbb{R}^n)} \leq C S[J] \leq C(n, m, p),
\]
so that for every \( \alpha, |\alpha| = m - 1, \) and every \( x, y \in \mathbb{R}^n \)
\[
|D^{\alpha}F(x) - D^{\alpha}F(y)| \leq C d_q(x : y : h_2) = d_q(x : y : h_3)
\]
where \( h_3 = C h_2 \). Since \( h_2 \in A_1(\mathbb{R}^n) \) and \( \|h_2\|_{A_1} \leq \eta(n, p, q) \), the same is true for the function \( h_3 \) as well.

Hence, by Theorem \(5.3\), the function \( F \in L^m_p(\mathbb{R}^n) \) and
\[
\|F\|_{L^m_p(\mathbb{R}^n)} \leq C \|h_3\|_{L^p(\mathbb{R}^n)} \leq C \|h_2\|_{L^m_p(\mathbb{R}^n)},
\]
so that, by \((5.31)\) and \((5.27)\),
\[
\|F\|_{L^p(\mathbb{R}^n)} \leq C \|J^h_\delta\|_{L^p(\mathbb{R}^n)} \leq C N_{m,p}(J).
\]

Finally,
\[
\|J\|_{L^p(\mathbb{R}^n)|_E} \leq \|F\|_{L^m_p(\mathbb{R}^n)} \leq C(n, m, p) N_{m,p}(J)
\]
proving inequality \((5.28)\).

Theorem \(1.8\) is completely proved. \( \square \)

### 5.3 A variational criterion for the traces of \( L^m_p(\mathbb{R}^n) \)-jets.

In [20] we have proved a criterion which provides a characterization of the trace space \( L^1_p(\mathbb{R}^n)|_E \) in terms of certain local oscillations of functions on subsets of the set \( E \). The next theorem generalizes this result to the case of jet-spaces generated by \( L^m_p(\mathbb{R}^n) \)-functions.
Theorem 5.11 Let \( m \in \mathbb{N} \) and let \( p \in (n, \infty) \). Let \( J = \{ P_x \in \mathcal{P}_{m-1} : x \in E \} \) be a polynomial field on a closed set \( E \subset \mathbb{R}^n \).

There exists a \( C^m \)-function \( F \in L^m_p(\mathbb{R}^n) \) such that \( T^{m-1}_x(F) = P_x \) for every \( x \in E \) if and only if there exists a constant \( \lambda > 0 \) such that for every finite family \( \{ Q_i : i = 1, \ldots, k \} \) of disjoint cubes in \( \mathbb{R}^n \), every \( x_i, y_i \in (\gamma Q_i) \cap E \), and every multiindex \( \beta, |\beta| \leq m - 1 \), the following inequality

\[
\sum_{i=1}^k \left| \frac{D^\beta P_x(x_i) - D^\beta P_y(x_i)}{(\text{diam } Q_i)^{m-|\beta|p-n}} \right|^p \leq \lambda
\]

holds. Here \( \gamma > 1 \) is an absolute constant.

Furthermore,

\[
\|J\|_{L^m_p(\mathbb{R}^n) | E} \sim \inf \lambda^{\frac{1}{p}}.
\]

See (5.23). The constants in this equivalence depend only on \( n, m \) and \( p \).

Proof. (Necessity.) Let

\[
J = \{ P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E \}
\]

be a polynomial field defined on \( E \). Let \( \gamma > 1 \) be a constant and let \( F \in L^m_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n) \) be an arbitrary function such that \( P_x = T^{m-1}_x[F] \) for all \( x \in E \). Let \( Q \) be a cube in \( \mathbb{R}^n \) and let \( x, y \in K := \gamma Q \). Then, by the Sobolev-Poincaré inequality (5.1), for every multiindex \( \beta, |\beta| \leq m - 1 \), we have

\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq C \|x - y\|^{m-|\beta|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} (\nabla^m F(u))^q du \right)^{\frac{1}{q}}
\]

\[
\leq C \|x - y\|^{m-|\beta|} \frac{n}{p} \left( \int_{Q_{xy}} (\nabla^m F(u))^q du \right)^{\frac{1}{q}}.
\]

Clearly, \( Q_{xy} \subset 2K \) and \( m - |\beta| - \frac{n}{p} > 0 \) whenever \( n < p \) and \( |\beta| \leq m - 1 \) so that

\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq C (\text{diam } Q)^{m-|\beta|} \left( \frac{1}{2K} \int_{2K} (\nabla^m F(u))^q du \right)^{\frac{1}{q}}.
\]

Therefore for every \( z \in Q \) we have

\[
|D^\beta P_x(x) - D^\beta P_y(x)|^p \leq C (\text{diam } Q)^{p(m-|\beta|)} (\mathcal{M}[\nabla^m F^q](z))^{\frac{p}{q}}.
\]

Integrating this inequality over cube \( Q \) (with respect to \( z \)) we obtain

\[
\frac{|D^\beta P_x(x) - D^\beta P_y(x)|^p}{(\text{diam } Q)^{(m-|\beta|)p-n}} \leq C \int_Q (\mathcal{M}[\nabla^m F^q](z))^{\frac{p}{q}} \, dz.
\]
Hence,
\[ I_\beta := \sum_{i=1}^{m} \left| \frac{D^\beta P_{x_i}(x_i) - D^\beta P_{y_i}(x_i)}{(\text{diam } Q_i)^{(m-|\beta|)p-n}} \right|^p \leq C \sum_{i=1}^{m} \int_{Q_i} \left( \mathcal{M} [\nabla^m F]^q(z) \right)^\frac{p}{q} dz \]
\[ \leq C \int_{\mathbb{R}^n} \left( \mathcal{M} [\nabla^m F]^q(z) \right)^\frac{p}{q} dz \]
so that, by the Hardy-Littlewood maximal theorem,
\[ I_\beta \leq C \int_{\mathbb{R}^n} (\nabla^m F)^p(z) dz \leq C \|F\|^p_{L^p_\ell (\mathbb{R}^n)}. \quad (5.33) \]
This proves (5.32) with \( \lambda = C \|F\|^p_{L^p_\ell (\mathbb{R}^n)} \). Furthermore, taking the infimum in (5.33) over all functions \( F \in L^m_p(\mathbb{R}^n) \) such that \( T^m_{x^{-1}}[F] = P_{x} \) on \( E \) we obtain that
\[ I_\beta \leq C \|J\|_{L^m_p(\mathbb{R}^n)|_E} \]
proving the necessity.

(Sufficiency). Let \( \gamma := 10^4 \). Let \( J = \{P_{x} \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\} \) be a polynomial field on \( E \). Suppose that there exists a constant \( \lambda > 0 \) such that for every family \( \{Q_i : i = 1, ..., k\} \) of pairwise disjoint cubes in \( \mathbb{R}^n \), every \( x_i, y_i \in (\gamma Q_i) \cap E \), and every multiindex \( \beta, |\beta| \leq m-1 \), inequality (5.32) is satisfied. Here \( \gamma \) is a certain absolute constant which we determine below.

Let us prove that under these conditions the Whitney extension \( F : \mathbb{R}^n \to \mathbb{R} \) of the jet \( J \) defined by the formula (5.11) has the following properties:

(i). \( F \in C^{m-1}(\mathbb{R}^n) \) and \( T^m_{x^{-1}}[F] = P_{x} \) for every \( x \in E \);

(ii). \( F \in L^m_p(\mathbb{R}^n) \) and
\[ \|F\|_{L^m_p(\mathbb{R}^n)} \leq C \lambda^{\frac{1}{p}}. \quad (5.34) \]

Prove (i). For every multiindex \( \beta, |\beta| \leq m-1 \), and every \( x, y \in E \), by (5.32),
\[ \left| \frac{D^\beta P_{x}(x) - D^\beta P_{y}(x)}{(\text{diam } Q_{xy})^{(m-|\beta|)p-n}} \right|^p \leq \lambda. \]
Recall that \( Q_{xy} = Q(x, \|x - y\|) \) so that \( \text{diam } Q_{xy} = 2\|x - y\| \). Hence
\[ |D^\beta P_{x}(x) - D^\beta P_{y}(x)| \leq C \lambda^{\frac{1}{p}} \|x - y\|^{|m-|\beta||-1} \cdot \|x - y\|^{1-\frac{n}{p}}. \]

But \( n < p \) so that
\[ D^\beta P_{x}(x) - D^\beta P_{y}(x) = o(\|x - y\|^{m-|\beta|}) \quad \text{as} \quad y \to x, y \in E. \]
Thus the jet \( J = \{P_{x} \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\} \) satisfies the hypothesis of the Whitney extension theorem [30]. This theorem implies the statement (i).
Prove (ii). Let us fix a multiindex $\beta$ of order $|\beta| = m - 1$ and prove that $D^\beta F \in L^1_p(\mathbb{R}^n)$ and 
\[ \|D^\beta F\|_{L^1_p(\mathbb{R}^n)} \leq C \lambda^{\frac{1}{p}}. \]

We will make use of a characterization of Sobolev spaces which follows from results proven in [2]: Let $p > n$ and let $G \in C(\mathbb{R}^n)$. Suppose that there exists a constant $\tau > 0$ such that the following inequality
\[ \sum_{i=1}^k \frac{|G(x_i) - G(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \tau \] holds for every finite family $\{Q_i : i = 1, ..., k\}$ of pairwise disjoint equal cubes and all $x_i, y_i \in Q_i$. Then $G \in L^1_p(\mathbb{R}^n)$ and $\|G\|_{L^1_p(\mathbb{R}^n)} \leq C(n, p) \tau^{\frac{1}{p}}$.

We are also needed the following combinatorial

**Theorem 5.12** ([3, 8]) Let $A = \{Q\}$ be a collection of cubes in $\mathbb{R}^n$ with covering multiplicity $\text{M}(A) < \infty$. Then $A$ can be partitioned into at most $N = 2^{n-1}(\text{M}(A) - 1) + 1$ families of disjoint cubes.

Recall that covering multiplicity $\text{M}(A)$ of a family of cubes $A$ is the minimal positive integer $M$ such that every point $x \in \mathbb{R}^n$ is covered by at most $M$ cubes from $A$.

**Lemma 5.13** Let $Q = \{Q_1, ..., Q_k\}$ be a family of pairwise disjoint equal cubes in $\mathbb{R}^n$ such that
\[ \text{dist}(c_{Q_i}, E) \leq 40 \text{ diam } Q_i, \quad i = 1, ..., k. \]
Then for every $x_i, y_i \in Q_i$ the following inequality
\[ \sum_{i=1}^k \frac{|D^\beta F(x_i) - D^\beta F(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq C \lambda \]
holds.

**Proof.** Fix $i \in \{1, ..., k\}$. Let $Q = Q_i \in Q$ so that
\[ \text{dist}(c_{Q_i}, E) \leq 40 \text{ diam } Q. \] Let $p_Q \in E$ be a point nearest to $Q$ on $E$. Then
\[ \|c_Q - p_Q\| \leq \text{diam } Q + 40 \text{ diam } Q = 41 \text{ diam } Q = 82r_Q. \]
Hence $p_Q \in 82Q$.

Let $x_Q = x_i$ and $y_Q = y_i$. (Recall that $Q = Q_i$ for some $i \in \{1, ..., k\}$.) Then
\[ I_Q := \frac{|D^\beta F(x_Q) - D^\beta F(y_Q)|^p}{(\text{diam } Q)^{p-n}} \leq 2^p \left\{ \frac{|D^\beta F(x_Q) - D^\beta F(p_Q)|^p}{(\text{diam } Q)^{p-n}} + \frac{|D^\beta F(p_Q) - D^\beta F(y_Q)|^p}{(\text{diam } Q)^{p-n}} \right\} \]
so that

\[
I := \sum_{Q \in \mathcal{Q}} I_Q \leq C \left\{ \sum_{Q \in \mathcal{Q}} \frac{|D^\beta F(x_Q) - D^\beta F(p_Q)|^p}{(\text{diam } Q)^{p-n}} \right. \\
+ \left. \sum_{Q \in \mathcal{Q}} \frac{|D^\beta F(p_Q) - D^\beta F(y_Q)|^p}{(\text{diam } Q)^{p-n}} \right\} = C \{I_1 + I_2\}.
\]

Recall that \(p_Q \in E\) so that \(D^\beta F(p_Q) = D^\beta P_{p_Q}(p_Q)\).

Now suppose that \(x_Q \in E\). Then \(D^\beta F(x_Q) = D^\beta P_{x_Q}(x_Q)\). Furthermore, since \(P_{x_Q} \in \mathcal{P}_{m-1}(\mathbf{R}^n)\) and \(|\beta| = m-1\), the function \(D^\beta P_{x_Q}\) is constant so that \(D^\beta F(x_Q) = D^\beta P_{x_Q}(p_Q)\). Hence

\[
\frac{|D^\beta F(x_Q) - D^\beta F(p_Q)|^p}{(\text{diam } Q)^{p-n}} = \frac{|D^\beta P_{x_Q}(p_Q) - D^\beta P_{p_Q}(p_Q)|^p}{(\text{diam } Q)^{p-n}}
\]

so that, by assumption \((5.32)\) (with \(|\beta| = m-1\)), we have

\[
I_1 := \sum_{Q \in \mathcal{Q}} \frac{|D^\beta F(x_Q) - D^\beta F(p_Q)|^p}{(\text{diam } Q)^{p-n}} \leq \lambda
\]

provided \(x_Q \in E\) for every \(Q \in \mathcal{Q}\).

Thus later on we can assume that \(x_Q \in \mathbf{R}^n \setminus E\) for all \(Q \in \mathcal{Q}\).

Let \(K_Q \in W_E\) be a Whitney cube which contains \(x_Q\). Recall that given \(H \in W_E\) by \(a_H\) we denote a point nearest to \(H\) on \(E\). Also by \(T(K_Q)\) we denote the family of Whitney’s cubes intersecting \(K_Q\). See \((5.13)\).

Let

\[
S(Q) := |D^\beta F(x_Q) - D^\beta F(p_Q)|
\]

and let

\[
V(Q) := |D^\beta P_{p_Q}(p_Q) - D^\beta P_{a_{K_Q}}(p_Q)|.
\]

Given \(H \in T(K_Q)\) and a multiindex \(\xi\) with \(|\xi| \leq m-1\) let

\[
L(\xi : H, Q) := |D^\xi P_{a_H}(a_{K_Q}) - D^\xi P_{a_{K_Q}}(a_{K_Q})|.
\]

Then, by Lemma \((5.9)\)

\[
S(Q) \leq C \left\{ V(Q) + \sum_{H \in T(K_Q)} \sum_{|\xi| \leq m-1} \frac{L(\xi : H, Q)}{(\text{diam } K_Q)^{m-1-|\xi|}} \right\}.
\]

Since \(#T(K_Q) \leq N(n)\), see Lemma \((5.6)\) we have

\[
\frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \leq \frac{CV(Q)^p}{(\text{diam } Q)^{p-n}} + C \sum_{H \in T(K_Q)} \sum_{|\xi| \leq m-1} \left( \frac{\text{diam } K_Q}{\text{diam } Q} \right)^{p-n} \frac{L(\xi : H, Q)^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}}.
\]

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Prove that \( a_{KQ} \in \gamma Q \). In fact, since \( x_Q \in KQ \cap Q \), we have
\[
\text{diam } KQ \leq 4 \text{ dist}(KQ, E) \leq 4 \text{ dist}(x_Q, E) \leq 4(\|x_Q - c_Q\| + \text{dist}(c_Q, E)) \leq 4 \text{ diam } Q + 4 \text{ dist}(c_Q, E)
\]
so that, by (5.36),
\[
\text{diam } KQ \leq 4 \text{ diam } Q + 4 \cdot 40 \text{ diam } Q = 164 \text{ diam } Q.
\]
In particular, by this inequality,
\[
\frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \leq C \left\{ \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} + \sum_{H \in T(KQ)} \sum_{|\xi| \leq m-1} \frac{L(\xi : H, Q)^p}{(\text{diam } KQ)^{(m-|\xi|)p-n}} \right\}.
\]
Since \( KQ \cap Q \neq \emptyset \), we have
\[
\|c_Q - a_{KQ}\| \leq \text{dist}(a_{KQ}, KQ) + \text{diam } KQ + \text{diam } Q = \text{dist}(KQ, E) + \text{diam } KQ + \text{diam } Q \leq 4 \text{ diam } KQ + \text{diam } KQ + \text{diam } Q \leq 5 \cdot 164 \text{ diam } Q + \text{diam } Q \leq (\gamma/2) \text{ diam } Q.
\]
(Recall that \( \gamma := 10^4 \).) Hence \( a_{KQ} \in \gamma Q \).

Prove that \( a_H \in \gamma KQ \) whenever \( H \in T(KQ) \). Since \( H \cap KQ \neq \emptyset \) and \( \text{diam } H \leq 4 \text{ diam } KQ \), see Lemma [5.6], we have
\[
\|c_{KQ} - a_H\| \leq \text{dist}(a_H, E) + \text{diam } H + \text{diam } KQ = \text{dist}(H, E) + \text{diam } H + \text{diam } KQ \leq 4 \text{ diam } H + \text{diam } H + \text{diam } KQ \leq 5 \cdot 4 \text{ diam } KQ + \text{diam } KQ = 21 \text{ diam } KQ.
\]
Hence
\[
a_H \in 42KQ \quad \text{for every } \quad H \in T(KQ) \tag{5.37}
\]
proving that \( a_H \subset \gamma KQ \).

By \( H_Q \) we denote a cube \( H \in T(KQ) \) for which the quantity
\[
\sum_{|\xi| \leq m-1} \frac{L(\xi : H, Q)^p}{(\text{diam } KQ)^{(m-|\xi|)p-n}}
\]
is maximal on \( T(KQ) \). Since \( \# T(KQ) \leq N(n) \), we obtain the following inequality
\[
\frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \leq C \left\{ \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} + \sum_{|\xi| \leq m-1} \frac{L(\xi : H_Q, Q)^p}{(\text{diam } KQ)^{(m-|\xi|)p-n}} \right\}.
\]
Hence

\[ I_2 := \sum_{Q \in Q} \frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \]

\[ \leq C \left\{ \sum_{Q \in Q} \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} + \sum_{|\xi| \leq m-1} \sum_{Q \in Q} \frac{L(\xi : H_Q, Q)^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}} \right\} \]

\[ = C \{ I_2^{(1)} + I_2^{(2)} \}. \]

Since \( p_Q \in 82Q \subset \gamma \) and \( a_{K_Q} \in \gamma \) we can apply assumption (5.32) to the family of cubes \( Q \). By this assumption \( I_2^{(1)} \leq \lambda \).

Let \( K := \{ K_Q : Q \in Q \} \). We know that the cubes of this family are non-overlapping, but, in general, they are not disjoint so that we can not apply the assumption of the theorem to \( K \). Nevertheless, the family \( K \) as a subfamily of \( W_E \) has covering multiplicity bounded by a constant \( N(n) \). By Theorem 5.12 \textit{it can be partitioned into at most \( M(n) \) families of pairwise disjoint cubes} so that, without loss of generality, we may assume \( K \) itself consists of pairwise disjoint cubes.

Since \( a_{H_Q}, a_{K_Q} \in \gamma K_Q \) for every \( Q \in Q \), by assumption (5.32),

\[ I_2^{(2)} := \sum_{|\xi| \leq m-1} \sum_{Q \in Q} \frac{|D^\xi P_{ah_Q}(a_{K_Q}) - D^\xi P_{aK_Q}(a_{K_Q})|^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}} \leq \sum_{|\xi| \leq m-1} \lambda \leq C \lambda \]

proving the lemma. \( \square \)

**Lemma 5.14** Let \( Q = \{Q_1, ..., Q_k\} \) be a family of pairwise disjoint equal cubes in \( \mathbb{R}^n \) such that

\[ \text{diam } Q_i < \frac{1}{40} \text{ dist}(c_Q, E), \quad i = 1, ..., k. \]

Then

\[ \sum_{i=1}^{k} \frac{|D^\beta F(x_i) - D^\beta F(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq C \lambda \]

for every choice of points \( x_i, y_i \in Q_i \).

**Proof.** For every cube \( Q \in Q \), by (5.38), \( \text{dist}(c_Q, E) > 40 \text{ diam } Q > 0 \) so that \( Q \subset \mathbb{R}^n \setminus E \).

Let \( K_Q \) be a Whitney cube which contains \( c_Q \). Prove that \( Q \subset K_Q = \frac{9}{8} K_Q \).

In fact,

\[ \text{diam } Q < \frac{1}{40} \text{ dist}(c_Q, E) \leq \frac{1}{40} \{ \text{diam } K_Q + \text{ dist}(K_Q, E) \} \]

\[ \leq \frac{1}{40} \{ \text{diam } K_Q + 4 \text{ diam } K_Q \} = \frac{1}{8} \text{ diam } K_Q. \]

Hence for every \( z \in Q \) we have

\[ \| z - c_{K_Q} \| \leq \| z - c_Q \| + \| c_Q - c_{K_Q} \| \leq \frac{1}{2} \text{ diam } Q + \frac{1}{2} \text{ diam } K_Q \]

\[ \leq \frac{1}{2} \cdot \frac{1}{8} \text{ diam } K_Q + \frac{1}{2} \text{ diam } K_Q = (\frac{1}{8} + 1) r_{K_Q} \]

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so that \( Q \subset \frac{2}{5} K_Q = K_Q^* \). Hence the points

\[
x_Q = x_i, \ y_Q = y_i \in K_Q^*
\]

provided \( Q = Q_i \).

Let \( K \in W_E \) and let

\[
Q(K) := \{ Q \in Q : K = K_Q \} = \{ Q \in Q : c_Q \in K \}.
\]

By \( K \) we denote a family of Whitney’s cubes \( K \) for which \( Q(K) \neq \emptyset \).

Then for each \( K \in \mathcal{K} \), by Lemma \ref{Lemma5.10}

\[
|D^\beta F(x_Q) - D^\beta F(y_Q)| \leq C \frac{\|x_Q - y_Q\|}{\text{diam } K} \sum_{H \in T(K)} \sum_{|\xi| \leq m} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|}{(\text{diam } K)^{m-|\xi|}}.
\]

By (5.39),

\[
a_K, a_H \in 42K \quad \text{for every} \quad H \in T(K).
\]

Now we have

\[
I_K := \sum_{Q \in Q(K)} \frac{|D^\beta F(x_Q) - D^\beta F(y_Q)|^p}{(\text{diam } Q)^{p-n}}
\]

\[
\leq C \left\{ \sum_{Q \in Q(K)} \left( \frac{\|x_Q - y_Q\|}{\text{diam } Q} \right)^p |Q| \right\} \left\{ \sum_{H \in T(K)} \sum_{|\xi| \leq m} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|}{(\text{diam } K)^{m-|\xi|}} \right\}^p
\]

\[
\leq C |K| \left\{ \sum_{H \in T(K)} \sum_{|\xi| \leq m} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|}{(\text{diam } K)^{m-|\xi|}} \right\}^p.
\]

Since \( \# T(K) \leq N(n) \), see Lemma \ref{Lemma5.6} we obtain

\[
I_K \leq C \sum_{H \in T(K)} \sum_{|\xi| \leq m} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{(m-|\xi|)p-n}}.
\]

Let \( \tilde{H} \in T(K) \) be a cube such that the quantity

\[
\sum_{|\xi| \leq m} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{(m-|\xi|)p-n}}
\]
takes the maximal value on $T(K)$. Then

$$I_K \leq C \sum_{|\xi| \leq m} \frac{|D^\xi P_{a_K}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{(m-|\xi|)p-n}}.$$  

Hence

$$I := \sum_{i=1}^k \frac{|D^\beta F(x_i) - D^\beta F(y_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \sum_{K \in \mathcal{K}(Q)} I_K \leq C \sum_{|\xi| \leq m} \sum_{K \in \mathcal{K}(Q)} \frac{|D^\xi P_{a_K}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{(m-|\xi|)p-n}}.$$  

As in the proof of the previous lemma, using Theorem 5.12 we can assume that the cubes of the family $K(Q)$ are pairwise disjoint. This and inclusions (5.39) enable us to apply assumption (5.32) to the last sum of the above inequality. By this assumption

$$I \leq C \sum_{|\xi| \leq m} \lambda \leq C \lambda$$

proving the lemma. □

Combining Lemma 5.13 and Lemma 5.14 with the criterion (5.35) we conclude that $F$ is a $C^{m-1}$-smooth function such that for every multiindex $\beta$ of order $m - 1$ the function $D^\beta F \in L_1^1(\mathbb{R}^n)$ and $\|D^\beta F\|_{L_1^1(\mathbb{R}^n)} \leq C \lambda^{\frac{1}{\beta}}$. Hence $F \in L_p^m(\mathbb{R}^n)$ and $\|F\|_{L_p^m(\mathbb{R}^n)} \leq C \lambda^{\frac{1}{\beta}}$.

The proof of Theorem 5.11 is complete. □

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