Free Probability and Combinatorics

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Abstract

A combinatorial approach to free probability theory has been developed by Roland Speicher, based on the notion of noncrossing cumulants, a free analogue of the classical theory of cumulants in probability theory. We review this theory, and explain the connections between free probability theory and random matrices. We relate noncrossing cumulants to classical cumulants and also to characters of large symmetric groups. Finally we give applications to the asymptotics of representations of symmetric groups, specifically to the Littlewood-Richardson rule.

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1. Introduction

Free probability has been introduced by D. Voiculescu [21] as a means of studying the group von Neumann algebras of free groups, using probabilistic techniques. His theory has become very successful when he discovered a deep relation with the theory of random matrices, and solved some old questions in operator algebra, see [3], [7], [24] for an overview. A purely combinatorial approach to Voiculescu’s definition of freeness has been given by R. Speicher [19], [20], building on G. C. Rota’s [16] approach to classical probability. It is based on the notion of noncrossing partitions, also known as “planar diagrams” in quantum field theory, and provides unifying concepts for many computations in free probability. Noncrossing partitions turn out to be connected with the geometry of the symmetric group, and this leads to some new understanding of the asymptotic behaviour of the characters and representations of large symmetric groups. Our aim is to survey these results, we shall start with the basic definition of freeness, then explain its connection to random matrix theory. In the third section we review Speicher’s theory. In the fourth section we show how noncrossing cumulants arise naturally in connection

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with classical cumulants associated with random matrices, and with characters of symmetric groups. Finally in section 5 we explain the asymptotic behaviour of representations of symmetric groups in terms of free probability concepts.

2. Freeness and random matrices

The usual framework for free probability is a von Neumann algebra $A$, equipped with a faithful, tracial, normal state $\tau$. To any self-adjoint element $X \in A$ one can associate its distribution, the probability measure on the real line, uniquely determined by the identity $\tau(X^n) = \int_{\mathbb{R}} x^n \mu(dx)$ for all $n \geq 1$. This makes it natural to think of the elements of $A$ as noncommutative random variables, and of $\tau$ as an expectation map, and one usually calls noncommutative probability space such a pair $(A, \tau)$. Although a great deal of the theory, especially the combinatorial side, can be developed in a purely algebraic way, assuming only that $A$ is a complex algebra with unit, and $\tau$ a complex linear functional, we shall stick to the von Neumann framework in the present exposition.

Given $(A, \tau)$, one considers a family $\{A_i; i \in I\}$ of von Neumann subalgebras. This family is called a free family if the following holds: for any $k \geq 1$ and $k$-tuple $a_1, \ldots, a_k \in A$ such that

- each $a_j$ belongs to some algebra $A_i$, with $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$,
- $\tau(a_j) = 0$ for all $j$,

one has $\tau(a_1 \ldots a_k) = 0$.

Moreover, a family of elements of $A$ is called free if the von Neumann algebras each of them generates form a free family. Freeness is a noncommutative notion analogous to the independence of $\sigma$-fields in probability theory, but which incorporates also the notion of algebraic independence.

Observe that if $a_1$ and $a_2$ are free elements in $(A, \tau)$, and one defines the centered elements $\hat{a}_i = a_i - \tau(a_i)1$ then one can compute

$$\tau(a_1 a_2) = \tau(\hat{a}_1 \hat{a}_2) + \tau(a_1)\tau(a_2) = \tau(a_1)\tau(a_2)$$

where the freeness condition has been used to get $\tau(\hat{a}_1 \hat{a}_2) = 0$. Actually, if $\{A_i; i \in I\}$ is a free family, it is not difficult to see that one can compute the value of $\tau$ on any product of the form $a_1 \ldots a_k$, where each $a_j$ belongs to some of the $A_i$'s, in terms of the quantities $\tau(a_{j_1} \ldots a_{j_l})$ where all the elements $a_{j_1}, \ldots, a_{j_l}$ belong to the same subalgebra. This implies that the value of $\tau$ on the algebra generated by the family $\{A_i; i \in I\}$ is completely determined by the restrictions of $\tau$ to each of these subalgebras. However the problem of finding an explicit formula is nontrivial, and this is where combinatorics comes in. We shall describe Speicher’s theory of noncrossing cumulants, which solves this problem, in the next section, but before that we explain how free probability is relevant to understand large random matrices.

Consider $n$ random $N \times N$ matrices $X_1^{(N)}, \ldots, X_n^{(N)}$, of the form

$$X_j^{(N)} = U_j D_j^{(N)} U_j^*$$

(2.1)
3. Noncrossing partitions and cumulants

A partition of the set \{1, \ldots, n\} is said to have a crossing if there exists a quadruple \((i, j, k, l)\), with \(1 \leq i < j < k < l \leq n\), such that \(i\) and \(k\) belong to some class of the partition and \(j\) and \(l\) belong to another class. If a partition has no crossing, it is called noncrossing. The set of all noncrossing partitions of \(\{1, \ldots, n\}\) is denoted by \(NC(n)\). It is a lattice for the refinement order, which seems to have been first systematically investigated in [10].

Let \((A, \tau)\) be a noncommutative probability space, then we shall define a family \(R^{(n)}\) of \(n\)-multilinear forms on \(A\), for \(n \geq 1\), by the following formula

\[
\tau(a_1 \ldots a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, \ldots, a_n). \tag{3.1}
\]

Here, for \(\pi \in NC(n)\), one has defined

\[
R[\pi](a_1, \ldots, a_n) = \prod_{V \in \pi} R^{(|V|)}(a_V)
\]

where \(a_V = (a_{j_1}, \ldots, a_{j_k})\) if \(V = \{j_1, \ldots, j_k\}\) is a class of the partition \(\pi\), with \(j_1 < j_2 < \ldots < j_k\) and \(|V| = k\) is the number of elements of \(V\). In particular \(R[1_n] = R^{(n)}\) if \(1_n\) is the partition with only one class. Thus one has, for \(n = 3\),

\[
\tau(a_1a_2a_3) = R^{(3)}(a_1, a_2, a_3) + R^{(2)}(a_1, a_2)R^{(1)}(a_3) + R^{(2)}(a_1, a_3)R^{(1)}(a_2) + R^{(2)}(a_2, a_3)R^{(1)}(a_1) + R^{(1)}(a_1)R^{(1)}(a_2)R^{(1)}(a_3).
\]
Observe that
\[
\tau(a_1 \ldots a_n) = R^{(n)}(a_1, \ldots, a_n) + \text{terms involving } R^{(k)} \text{ for } k < n
\]
so that the \( R^{(n)} \) are well defined by (3.1) and can be computed by induction on \( n \).

They are called the noncrossing (or sometimes free) cumulant functionals on \( A \).

The formula (3.1) can be inverted to yield
\[
R^{(n)}(a_1, \ldots, a_n) = \sum_{\pi \in \text{NC}(n)} \text{Moeb}([\pi, 1_n])\tau[\pi](a_1, \ldots, a_n).
\]

Here \( \tau[\pi](a_1, \ldots, a_n) = \prod_{\nu \in \pi} \tau(a_{j_1} \ldots a_{j_k}) \) where \( V = \{j_1, \ldots, j_k\} \) are the classes of \( \pi \), and \( \text{Moeb} \) is the M"obius function of the lattice \( \text{NC}(n) \), see [20].

For example, one has
\[
\begin{align*}
R^{(1)}(a_1) &= \tau(a_1); \\
R^{(2)}(a_1, a_2) &= \tau(a_1a_2) - \tau(a_1)\tau(a_2); \\
R^{(3)}(a_1, a_2, a_3) &= \tau(a_1a_2a_3) - \tau(a_1)\tau(a_2)a_3 - \tau(a_2)\tau(a_1)a_3 \\
&\quad - \tau(a_3)\tau(a_1a_2) + 2\tau(a_1)\tau(a_2)\tau(a_3).
\end{align*}
\]

Note that when the lattice of all partitions is used instead of noncrossing partitions, then one gets the usual family of cumulants (see Rota [16]), with another M"obius function.

The connection between noncrossing cumulants and freeness is the following result from section 4 of [19].

**Theorem 2.** Let \( \{A_i; i \in I\} \) be a free family of subalgebras of \( (A, \tau) \), and \( a_1, \ldots, a_n \in A \) be such that \( a_j \) belongs to some \( A_{i_j} \) for each \( j \in \{1, 2, \ldots, n\} \). Then one has \( R^{(n)}(a_1, \ldots, a_n) = 0 \) if there exists some \( j \) and \( k \) with \( i_j \neq i_k \).

This result leads to an explicit expression for \( \tau(a_1 \ldots a_n) \), where \( a_1, \ldots, a_n \) is an arbitrary sequence in \( A \), such that each \( a_j \) belongs to one of the algebras \( A_i; i \in I \). By Theorem 2, in the right hand side of (3.1), the terms corresponding to partitions \( \pi \) having a class containing two elements \( j, k \) such that \( a_j \) and \( a_k \) belong to distinct algebras give a zero contribution. Thus we have to sum over partitions in which all \( j \)'s belonging to a certain block of the partition are such that \( a_j \) belongs to the same algebra. Since we can express noncrossing cumulants in terms of moments we get the formula for \( \tau(a_1 \ldots a_n) \) in terms of the restrictions of \( \tau \) to each of the subalgebras \( A_i \). Noncrossing cumulants are a powerful tool for making computations in free probability, see [11], [12], [13], [14], [15], for some applications. We give a simple illustration below.

Let \( X_1 \) and \( X_2 \) be two self-adjoint elements which are free, then the distribution of \( X_1 + X_2 \), depends only on the distributions of \( X_1 \) and \( X_2 \) and can be computed as follows. Let \( R^{(n)}(X_1, X_1, \ldots, X_1) \) and \( R^{(n)}(X_2, \ldots, X_2) \), for \( n \geq 1 \), be the noncrossing cumulants of \( X_1 \) and \( X_2 \); then one can expand \( R^{(n)}(X_1 + X_2, X_1 + X_2, \ldots, X_1 + X_2) \) by multilinearity as \( \sum_{i_1, \ldots, i_n} R^{(n)}(X_{i_1}, \ldots, X_{i_n}) \) where the sum is over all sequences of 1 and 2. By Theorem 2, all terms vanish except \( R^{(n)}(X_1, \ldots, X_1) \) and \( R^{(n)}(X_2, \ldots, X_2) \). It follows that
\[
R^{(n)}(X_1 + X_2, \ldots, X_1 + X_2) = R^{(n)}(X_1, \ldots, X_1) + R^{(n)}(X_2, \ldots, X_2)
\]
allowing the computation of the moments of \( X_1 + X_2 \), hence its distribution, in terms of the distributions of \( X_1 \) and \( X_2 \). It remains to give a compact form to the relation between moments and noncrossing cumulants. For any self-adjoint element \( X \) with distribution \( \mu \), let

\[
G_X(z) = \frac{1}{z} + \sum_{k=1}^{\infty} z^{-k-1} \tau(X^k) = \int_{\mathbb{R}} \frac{1}{z-x} \mu(dx)
\]

be its Cauchy transform, and let

\[
K(z) = \frac{1}{z} + \sum_{k=0}^{\infty} R_k z^k
\]

be the inverse series for composition.

**Theorem 3.** [19]

One has \( R_k = R^{(k)}(X, \ldots, X) \) for all \( k \).

The operation which associates to the two distributions of \( X_1 \) and \( X_2 \) the distribution of their sum is called the free convolution of measures on the real line, and was introduced by D. Voiculescu, who first considered the coefficients \( R_k \) and proved the formula for the free convolution of two measures, using very different methods [22].

Combining theorems 1 and 2, given two large random matrices of known spectra one can predict the spectral distribution of their sum, with a good accuracy and probability close to 1. It is illuminating to look at the following example. The histogram below is made of the 800 eigenvalues of a random matrix of the form \( \Pi_1 + \Pi_2 \) where \( \Pi_1 \) and \( \Pi_2 \) are two orthogonal projections onto some random subspaces of dimension 400 in \( \mathbb{C}^{800} \), chosen independently. The curve \( y = \frac{40}{\pi \sqrt{2-x(2-x)}} \) which corresponds to the large \( N \) limit predicted by free probability has been drawn.
4. Noncrossing cumulants, random matrices and characters of symmetric groups

Besides free probability theory, noncrossing partitions appear in several areas of mathematics. We indicate some relevant connections. The first is with the theory of map enumeration initiated by investigations of theoretical physicists in two-dimensional quantum field theory. The noncrossing partitions appear there under the guise of planar diagrams, the Feynman diagrams which dominate the matrix integrals in the large $N$ limit. This is of course related to the fact that large matrices model free probability. We shall not discuss this further here, but refer to [26] for an accessible introduction. Another place where noncrossing partitions play a role, which is closely related to the preceding, is the geometry of the symmetric group, more precisely of its Cayley graph. Consider the (unoriented) graph whose vertex set is the symmetric group $\Sigma_n$, and such that $\{\sigma_1, \sigma_2\}$ is an edge if and only if $\sigma_1^{-1}\sigma_2$ is a transposition, i.e. this is the Cayley graph of $\Sigma_n$ with respect to the generating set of all transpositions. The distance on the graph is given by
\[ d(\sigma_1, \sigma_2) = n - \text{number of orbits of } \sigma_1^{-1}\sigma_2 := |\sigma_1^{-1}\sigma_2|. \]

The lattice of noncrossing partitions can be imbedded in $\Sigma_n$ in the following way [10], given a noncrossing partition of $\{1, \ldots, n\}$, its image is the permutation $\sigma$ such that $\sigma(i)$ is the element in the same class as $i$, which follows $i$ in the cyclic order $12\ldots n$. One can check [1] that the image of $NC(n)$ is the set of all permutations satisfying $|\sigma| + |\sigma^{-1}c| = |c|$ where $c$ is the cyclic permutation $c(i) = i + 1 \mod(n)$, in other words, this set consists of all permutations which lie on a geodesic from the identity to $c$ in the Cayley graph. These facts are at the heart of the connections between free probability, random matrices and symmetric groups. As an illustration we shall see how free cumulants arise from asymptotics of both random matrix theory and symmetric group representation theory.

Recall that cumulants (also called semi-invariants, see e.g. [17]) of a random variable $X$ with moments of all orders, are the coefficients in the Taylor expansion of the logarithm of its characteristic function, i.e.
\[ \log E[e^{itX}] = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} C_n(X). \]

We shall consider random variables of the following form $Y^{(N)} = NX^{(N)}$ where $X^{(N)} = UD^{(N)}U^*$ is a random matrix chosen as in [24] and $X_{1,1}^{(N)}$ is its upper left coefficient. Assume now that the moments of $X^{(N)}$ converge
\[ \frac{1}{N} Tr((X^{(N)})^k) \to_{N\to\infty} \int_{\mathbf{R}} x^k \mu(dx) \]
for some probability measure $\mu$ on $\mathbf{R}$, with noncrossing cumulants $R_n(\mu)$, then one has
\[ \lim_{N\to\infty} \frac{1}{N^2} C_n(Y^{(N)}) = \frac{1}{n} R_n(\mu). \]
This was first observed by P. Zinn-Justin [25], a proof using representation theory has been found by B. Collins [6].

We have related noncrossing cumulants to usual cumulants via random matrix theory, we shall see that that noncrossing cumulants are also useful in evaluating characters of symmetric groups. The precise relation however is not obvious at first sight.

Let us recall a few facts about irreducible representations of symmetric groups. It is well known that they can be parametrized by Young diagrams. In the following it will be convenient to represent a Young diagram by a function $\omega : \mathbb{R} \to \mathbb{R}$ such that $\omega(x) = |x|$ for $|x|$ large enough, and $\omega$ is a piecewise affine function, with slopes $\pm 1$, see the following picture which shows the Young diagram corresponding to the partition $8 = 3 + 2 + 2 + 1$.

![Young Diagram](image)

Alternatively we can encode the Young diagram using the local minima and local maxima of the function $\omega$, denoted by $x_1, \ldots, x_k$ and $y_1, \ldots, y_{k-1}$ respectively, which form two interlacing sequences of integers. These are (-3,-1,2,4) and (-2,1,3) respectively in the above picture. Associated with the Young diagram there is a unique probability measure $m_\omega$ on the real line, such that

$$\int_\mathbb{R} \frac{1}{z - x} m_\omega(dx) = \frac{\prod_{i=1}^{k-1} (z - y_i)}{\prod_{i=1}^{k} (z - x_i)}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$.

This probability measure is supported by the set \{x_1, \ldots, x_k\} and is called the transition measure of the diagram, see [8]. Let $\sigma$ denote the conjugacy class in $\Sigma_n$ of a permutation with $k_2$ cycles of length 2, $k_3$ of length 3, etc.. Here $k_2, k_3, \ldots$ are fixed while we let $n \to \infty$. Denote by $\chi_\omega$ the normalized character of $\Sigma_n$ associated with the Young diagram $\omega$, then the following asymptotic evaluation holds uniformly on the set of $A$-balanced Young diagrams, i.e. those whose longest row and longest column are less that $A\sqrt{n}$ (where $A$ is some constant $> 0$),

$$\chi_\omega(\sigma) = \prod_{j=2}^{\infty} n^{-jk_j} R_{j+1}^{k_j}(\omega) + O(n^{-|\sigma|/2}). \quad (4.1)$$

Note that $R_k$ is scaled by $\lambda^k$ if we scale the diagram $\omega$ by a factor $\lambda$, therefore the first term in the right hand side is of order $O(n^{\sum_j (j+1)k_j/2 - \sum_j jk_j}) = O(n^{-|\sigma|/2})$,\[\]
this gives the order of magnitude of the character of a fixed conjugacy group for an
$A$-balanced diagram.

In [2] a proof of (4.1) has been given, using in an essential way the Jucys-Murphy operators. Another proof, leading to an exact formula for characters of
cycles due to S. Kerov [9], was shown to me later by A. Okounkov [15], see [5].

5. Representations of large symmetric groups

The asymptotic formula (4.1) shows in particular that irreducible characters of
symmetric groups become asymptotically multiplicative i.e. for permutations with
disjoint supports $\sigma_1$ and $\sigma_2$, one has

$$\chi_\omega(\sigma_1\sigma_2) = \chi_\omega(\sigma_1)\chi_\omega(\sigma_2) + O(n^{-1-|\sigma_1\sigma_2|/2}) \quad (5.1)$$

uniformly on $A$-balanced diagrams. Conversely, given a central, normalized, positive
definite function on $\Sigma_n$, a factorization property such as (5.1) implies that the
positive function is essentially an irreducible character [5]. More precisely, recall
that a central normalized positive definite function $\psi$ on $\Sigma_n$ is a convex combination
of normalized characters, and as such it defines a probability measure on the set
of Young diagrams. For any $\varepsilon, \delta > 0$, for all $n$ large enough, if an approximate
factorization such as (5.1) holds for $\psi$, then there exists a curve $\omega$, such that the
measure on Young diagrams associated with $\psi$ puts a mass larger than $1 - \delta$ on
Young diagrams which lie in a neighbourhood of this curve, of width $\varepsilon \sqrt{n}$. Therefore
one can say that condition (5.1) on a positive definite function implies that the
representation associated with this function is approximately isotypical, i.e. almost
all Young diagrams occurring in the decomposition have a shape close to a certain
definite curve.

Using this fact it is possible to understand the asymptotic behaviour of several
operations in representation theory. Consider for example the operation of
induction. One starts with two irreducible representations of symmetric groups $\Sigma_{n_1}, \Sigma_{n_2}$,
corresponding to two Young diagrams $\omega_1$ and $\omega_2$. One can then induce the product
representation $\omega_1 \otimes \omega_2$ of $\Sigma_{n_1} \times \Sigma_{n_2}$ to $\Sigma_{n_1+n_2}$. This new representation is reducible
and the multiplicities of irreducible representations can be computed using a com-
binatorial device, the Littlewood-Richardson rule. This rule however gives little
light on the asymptotic behaviour of the multiplicities. Using the factorization-
concentration result, one can prove that when $n_1$ and $n_2$ are very large, but of
the same order of magnitude, then there exists a curve, which depends on $\omega_1$ and
$\omega_2$, and such that the typical Young diagram occurring in the decomposition of the
induced representation, is close to this curve. As we saw in section 4, one can asso-
ciate a probability measure on the real line to any Young diagram. The description
of the typical shape of Young diagram which occurs in the decomposition of the
induced representation is easier if we use this correspondance between probability
measures and Young diagrams, indeed the probability measure associated with the
shape of the typical Young diagram corresponds to the free convolution of the two
probability measures [2].
There are analogous results for the restriction of representations from large symmetric groups to smaller ones. There the corresponding operation on probability measure is called the free compression, it corresponds at the level of the large matrix approximation, to taking a random matrix with prescribed eigenvalue distribution, as in section 2, and extracting a square submatrix. Finally there are also results for Kronecker tensor products of representations. Here a central role is played by the well known Kerov-Vershik limit shape, whose associated probability measure is the semi-circle distribution with density \( \frac{1}{\pi \sqrt{4-x^2}} \) on the interval \([-2,2]\), see [2].

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