CLOSURES OF HESSENBERG SCHUBERT CELLS IN REGULAR SEMISIMPLE HESSENBERG VARIETIES

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ABSTRACT. It is well known that the Schubert cells in the flag variety $GL_n(\mathbb{C})/B$ satisfies closure relations which can be characterized purely combinatorially in terms of Bruhat order on the symmetric group $\mathfrak{S}_n$. This closure result is fundamental in the study of the geometry and combinatorics of $GL_n(\mathbb{C})/B$. The main result of this manuscript is a closure relation among the Hessenberg analogues of the Schubert cells (which we call Hessenberg Schubert cells) in a regular semisimple Hessenberg variety, which can also be stated entirely in terms of Bruhat order. Our analysis is inspired by and depends upon work of Cho, Hong, and Lee, but our proof also depends on the combinatorics of subsets of Weyl type, first introduced by Sommers and Tymoczko. Our result opens the door to a plethora of open questions, a select few of which we sketch in the last section. The Appendix, written by Michael Zeng, proves a lemma concerning subsets of Weyl type which is required in our arguments.

1. INTRODUCTION

The main result of this paper is a Hessenberg generalization of the classical result in the geometry of Schubert varieties which characterizes the closure of a Schubert cell in the flag variety $GL(n, \mathbb{C})/B$ in terms of Bruhat order on the symmetric group $\mathfrak{S}_n$. This closure relation is fundamental in Schubert calculus and in the study of the geometry of flag varieties (and related spaces). Moreover, the study of Hessenberg varieties has recently garnered great interest in different research communities due to their connections (which have come to light in the past decade) to many areas, not the least of which is the famously unsolved Stanley–Stembridge conjecture in algebraic combinatorics [24, 6, 13]. More generally, research on Hessenberg varieties lies in the rich intersection of Lie theory, algebraic geometry, symplectic and Poisson geometry, representation theory, and combinatorics (see [3] for a recent overview of the subject). We expect that a Hessenberg version of the well-known closure relations among Schubert varieties will be of wide interest and lead to significant advancements in this area. Indeed, we sketch – in our last section – a select few among the many research avenues suggested by our main result.

We briefly recount more context; for details see Section 2. Hessenberg varieties are subvarieties of the full flag variety $GL(n, \mathbb{C})/B$, where $B$ denotes the Borel subgroup of upper-triangular matrices. In this manuscript we focus on the special case of regular semisimple Hessenberg varieties, which (for a fixed positive integer $n$) are parametrized by a choice of diagonalizable matrix $S \in \mathfrak{gl}_n(\mathbb{C})$ with distinct eigenvalues and a nondecreasing Hessenberg function $h : [n] \to [n]$ such that $h(i) \geq i$. We denote the corresponding regular semisimple Hessenberg variety by $\mathcal{Hess}(S, h)$. First introduced by DeMari, Procesi, and Shayman in [12], these varieties $\mathcal{Hess}(S, h)$ are intimately related to the open Stanley–Stembridge conjecture, through an action of the symmetric group $\mathfrak{S}_n$ on $H^*_\mathbb{Q}(\mathcal{Hess}(S, h))$ which was defined by Tymoczko in [25]. The action on equivariant cohomology descends to an action on ordinary cohomology, and the Stanley–Stembridge conjecture roughly states that this $\mathfrak{S}_n$-representation on $H^*(\mathcal{Hess}(S, h))$ has a basis permuted by the dot action.
It should be emphasized that much of the topology and geometry of general Hessenberg varieties remains mysterious despite the recent surge of interest in the subject. For any choice $X \in \mathfrak{gl}_n(\mathbb{C})$ (not necessarily regular or semisimple) and Hessenberg function $h$, there is an associated Hessenberg variety $\mathcal{Hess}(X, h)$ (see (2.3) below). Special cases include varieties well-studied in other research areas, such as Springer fibers, the permutohedral variety, the Peterson variety, and the flag variety itself (this occurs when $h(i) = n$ for all $i$). The geometry of $\mathcal{Hess}(X, h)$ varies significantly as $X$ and $h$ change. For example, if $X$ is a regular matrix, $\mathcal{Hess}(X, h)$ is irreducible of dimension $N_h = \sum_{i=1}^n (h(i) - i)$ [21] - this generalizes known geometric properties of the flag variety. In contrast, if $X$ is nilpotent, but not regular, $\mathcal{Hess}(X, h)$ is frequently reducible and it is, in general, unknown what its irreducible components and dimension are. One exception is the case of the Springer fibers (occurring when $X$ is nilpotent and $h(i) = i$), where the dimension and irreducible components have a concrete description using the combinatorics of standard tableaux [22].

Schubert varieties, on the other hand, have been studied for decades and are much better understood. Recall that the flag variety $GL_n(\mathbb{C})/B$ decomposes into a disjoint union of $B$-orbits, known as Schubert cells, each indexed by an element of $\mathfrak{S}_n$, and isomorphic to affine space. The closure of a Schubert cell is called a Schubert variety; the theory of Schubert varieties is a key example of the interplay between combinatorics and geometry in the flag variety. In particular, a famous and fundamental result is that each Schubert variety is a union of a certain subset of Schubert cells, characterized combinatorially in terms of the Bruhat order on $\mathfrak{S}_n$. More precisely, for $u, v \in \mathfrak{S}_n$, $u \leq v$ in Bruhat order if and only if the Schubert cell indexed by $u$ is contained in the Schubert variety for $v$. Our main result generalizes this key result to the context of Hessenberg varieties.

Let $h : [n] \to [n]$ be a Hessenberg function and consider the intersection $\Omega_{w, h}$ of $\mathcal{Hess}(\mathfrak{S}, h)$ with the (opposite) Schubert cell indexed by $w \in \mathfrak{S}_n$. We call these (opposite) Hessenberg Schubert cells. In this paper we study their closures $\Omega_{w, h} := \overline{\Omega_{w, h}}$ in $\mathcal{Hess}(\mathfrak{S}, h)$. Our analysis is inspired by recent work of Cho, Hong, and Lee in [10], where they view $\Omega_{w, h}$ as Białynicki-Birula strata, but we introduce an additional point of view into this analysis: specifically, we begin with a partition of $\mathfrak{S}_n$ into sets $\mathcal{W}(\mathfrak{S}, h)$ defined by particular subsets $\mathcal{S}$ in the type A root system, called subsets of Weyl type (with respect to $h$) (cf. Definition 2.8). This partition was introduced by Sommers–Tymoczko in [23] and subsequently used by the second author in [21] to prove that the Betti numbers of regular Hessenberg varieties are palindromic in all Lie types. Appendix A below by Michael Zeng proves that each set $\mathcal{W}(\mathfrak{S}, h)$ is a weak Bruhat interval. Let $w_s \in \mathcal{W}(\mathfrak{S}, h)$ denote the maximal element in this interval. Our main result (Theorem 4.8) proves that

\begin{equation}
\Omega_{w_s, h} = \bigcup_{w \leq u} \Omega_{w, h}.
\end{equation}

Thus, in direct analogy with the Schubert variety case, our result shows that the Bruhat order determines the closures of these Hessenberg Schubert cells in $\mathcal{Hess}(\mathfrak{S}, h)$. Moreover, (1.1) demonstrates that the partition $\mathfrak{S}_n = \bigsqcup_{\mathcal{S}} \mathcal{W}(\mathfrak{S}, h)$ merits more attention, as it captures both geometric and topological properties of $\mathcal{Hess}(\mathfrak{S}, h)$. We intend to pursue this in future work.

Our Theorem 4.8, which proves (1.1), is a substantial step forward in our understanding of the geometric structure of regular semisimple Hessenberg varieties. The explicit formula of (1.1) connects this structure to the Bruhat order, and is surprising since $\mathcal{Hess}(\mathfrak{S}, h)$ is not $B$-invariant. Since the geometry and topology of Schubert varieties and combinatorial properties of the Bruhat order are fundamental to the study of flag varieties, we expect (1.1) to play an analogous key role in the future study of Hessenberg varieties, both in $GL_n(\mathbb{C})/B$ and in other Lie types. We record several specific open questions and problems for future research in Section 5 below.
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2. Background

2.1. Hessenberg Varieties and Hessenberg Schubert cells. Hessenberg varieties in Lie type A are subvarieties of the (full) flag variety $GL_n(\mathbb{C})/B$ where $B$ is the Borel subgroup of upper triangular matrices in $GL_n(\mathbb{C})$. Let $G = GL_n(\mathbb{C})$ and let $B_-$ denote the Borel subgroup of lower triangular matrices. The following two cell decompositions of $G/B$ (both called a Bruhat decomposition of $G/B$) are well-studied:

\begin{equation}
G/B = \bigsqcup_{w \in \mathcal{S}_n} X^\circ_w = \bigsqcup_{w \in \mathcal{S}_n} \Omega^\circ_w
\end{equation}

where $X^\circ_w := BwB/B$ is the Schubert cell and $\Omega^\circ_w = B_-wB/B$ is the opposite Schubert cell. The closure $X_w := \overline{X^\circ_w}$ (respectively $\Omega_w := \overline{\Omega^\circ_w}$) is called the Schubert variety (respectively opposite Schubert variety) for $w \in \mathcal{S}_n$. It is an important and well-known fact that

\begin{equation}
X_w = \bigsqcup_{u \leq w} X^\circ_u \quad \text{and} \quad \Omega_w = \bigsqcup_{u \geq w} \Omega^\circ_u
\end{equation}

where $\leq$ denotes the Bruhat order on $\mathcal{S}_n$. The main theorem of this paper generalizes (2.2) to the setting of Hessenberg Schubert varieties.

We denote the root system of $\text{gl}_n(\mathbb{C})$ by $\Phi = \{t_i - t_j \mid 1 \leq i \neq j \leq n\}$ with positive roots $\Phi^+ = \{t_i - t_j \in \Phi \mid i < j\}$, negative roots $\Phi^- = \{t_i - t_j \in \Phi \mid i > j\}$, and simple positive roots $\Delta = \{t_i - t_{i+1} \mid 1 \leq i \leq n-1\}$. Given $w \in \mathcal{S}_n$ the inversion set of $w$ is

$$N(w) := \{t_i - t_j \in \Phi^+ \mid w(t_i - t_j) \in \Phi^-\} = \Phi^+ \cap w^{-1}(\Phi^-).$$

We set $\ell(w) := |N(w)| = |\{i > j \mid w(i) < w(j)\}|$. It is known that $X^\circ_w \simeq \mathbb{C}^{\ell(w)}$ and $\Omega^\circ_w \simeq \mathbb{C}^{N-\ell(w)}$ where $N = \sum_{i=1}^n (n - i) = \dim_{\mathbb{C}} GL_n(\mathbb{C})/B$.

A Hessenberg variety in $G/B$ is specified by two pieces of data: a Hessenberg function, that is, a nondecreasing function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $h(i) \geq i$ for all $i$, and a choice of an element $X$ in $\text{gl}(n, \mathbb{C})$. We frequently write a Hessenberg function by listing its values in sequence, i.e., $h = (h(1), h(2), \ldots, h(n))$. The Hessenberg variety associated to the linear operator $X$ and Hessenberg function $h$ and is defined as

\begin{equation}
\text{Hess}(X, h) = \{gB \mid Xg_1 \in \text{span}_{\mathbb{C}} \{g_1, \ldots, g_{h(i)}\}\}
\end{equation}

where $g_1, \ldots, g_n$ denote the columns of $g \in GL_n(\mathbb{C})$. In this paper, we focus on the case when $X$ is a regular semisimple operator $S$ (i.e., diagonalizable with distinct eigenvalues); more specifically, we fix $S$ to be a diagonal matrix with distinct eigenvalues. We refer to the corresponding Hessenberg variety $\text{Hess}(S, h)$ as a regular semisimple Hessenberg variety. It was established in [12] that $\text{Hess}(S, h)$ is a smooth, irreducible variety.

For each $w \in \mathcal{S}_n$ we consider the Hessenberg Schubert cell, defined as

$$X^\circ_{w,h} = X^\circ_w \cap \text{Hess}(S, h) = BwB/B \cap \text{Hess}(S, h)$$
and also the **opposite Hessenberg Schubert cell**, defined as

\[(2.4) \quad \Omega_{w,h}^o = \Omega_w^o \cap \text{Hess}(S, h) = B_wB/B \cap \text{Hess}(S, h).\]

From this, we obtain decompositions of the regular semisimple Hessenberg variety

\[(2.5) \quad \text{Hess}(S, h) = \bigsqcup_{w \in \mathcal{G}_n} X_{w,h}^o = \bigsqcup_{w \in \mathcal{G}_n} \Omega_{w,h}^o,\]

where \(X_{w,h}^o \simeq \mathcal{C}_h(w)\) and \(\Omega_{w,h}^o \simeq \mathcal{C}_{N,h} - \ell_h(w)\) for

\[(2.6) \quad \ell_h(w) := \{|i > j \mid w(i) < w(j) \text{ and } i \leq h(j)\}\].

When \(h = (n, n, \ldots, n)\) then \(\text{Hess}(S, h) = G/B\) and we recover the Bruhat decomposition of \(GL_n(\mathbb{C})/B\) from (2.1). We now define the **Hessenberg Schubert variety** for \(w \in \mathcal{G}_n\) to be \(X_{w,h} := X_{w,h}^o\) and the **opposite Hessenberg Schubert variety** to be \(\Omega_{w,h} := \Omega_{w,h}^o\). Our main result gives a combinatorial characterization of \(\Omega_{w,h}\) for certain permutations \(w\).

**2.7. Remark.** One may define the Hessenberg Schubert cells and opposite Hessenberg Schubert cells in the language of Białynicki-Birula strata, as in [10]. However, our definition is equivalent, and the Morse-theoretic point of view is not necessary for our purposes.

**2.2. Subsets of Weyl type and acyclic orientations.** As mentioned in the Introduction, one of the contributions of this article is to introduce the theory of subsets of Weyl type into the study of Hessenberg Schubert closure relations and the related geometry and combinatorics. Indeed, it is through the language of subsets of Weyl type that we are able to prove our main theorem. In this section we briefly introduce the relevant terminology necessary for our results.

Let \(h : [n] \rightarrow [n]\) be a Hessenberg function. Then \(h\) determines a subset of \(\Phi^+\) defined by

\[\Phi_h^+ = \{t_i - t_j \in \Phi \mid i < j \text{ and } j \leq h(i)\}\]

and similarly, we let \(\Phi_h^- = \{t_j - t_i \mid i < j \text{ and } j \leq h(i)\} \subseteq \Phi^-\). Note that \(\ell_h(w) = |N(w) \cap \Phi_h^+|\) where \(\ell_h\) is the Hessenberg length function defined in (2.6) above.

**2.8. Definition.** Given a subset \(S \subseteq \Phi_h^+\) we say that \(S\) is **\(\Phi_h^+\)-closed** if for all \(\alpha, \beta \in S\) such that \(\alpha + \beta \in \Phi_h^+\), then \(\alpha + \beta \in S\) as well. Given such a subset \(S \subseteq \Phi_h^+\), we say that \(S\) is a **subset of Weyl type (with respect to \(h\))** if both \(S\) and its complement \(\Phi_h^+ \setminus S\) are \(\Phi_h^+\)-closed. Denote the set of all subsets \(S \subseteq \Phi_h^+\) of Weyl type (with respect to \(h\)) by \(\mathcal{W}_h\).

It is a well known theorem of Kostant [17, Prop. 5.10] that \(S \subseteq \Phi^+\) is a subset of Weyl type if and only if \(S = N(w)\) for some \(w \in \mathcal{G}_n\). Sommers and Tymoczko generalized that result to the setting of subsets of Weyl type in \(\Phi_h^+\). The following summarizes their results from [23] in the form most useful for our purposes.

**2.9. Theorem (Sommers–Tymoczko [23]).** Let \(h : [n] \rightarrow [n]\) be a Hessenberg function and \(S \in \mathcal{W}_h\).

1. There exists \(w \in \mathcal{G}_n\) such that \(S = N(w) \cap \Phi_h^+\), and \(S\) is a subset of Weyl type with respect to \(h\) if and only if it is of this form.

2. There exists a unique element \(z_S \in \mathcal{G}_n\) satisfying both \(S = N(z_S) \cap \Phi_h^+\) and \(z_S^{-1}(-\Delta) \cap \Phi^+ \subseteq \Phi_h^+\).

3. \(N(z_S) \subseteq N(y)\) for any \(y \in \mathcal{G}_n\) with \(S \subseteq N(y)\).

Given a fixed \(S \in \mathcal{W}_h\), we now consider

\[\mathcal{W}(S, h) := \{w \in \mathcal{G}_n \mid N(w) \cap \Phi_h^+ = S\} \subseteq \mathcal{G}_n\]
i.e., $\mathcal{W}(S, h)$ is the set of permutations whose associated subset of Weyl type is exactly $S$. Note that $\mathcal{W}(S, h)$ is always non-empty for any $S \in \mathcal{W}_h$ by Theorem 2.9, and we obtain a partition $\mathfrak{S}_n = \bigsqcup_{S \in \mathcal{W}_h} \mathcal{W}(S, h)$. Recall that (left) weak Bruhat order is the partial order on $\mathfrak{S}_n$ defined by
\[ u \leq_L v \text{ if } v = s_{i_1} \cdots s_{i_k} u \text{ for simple reflections } s_{i_1}, \ldots, s_{i_k} \text{ such that } \ell(v) = \ell(u) + k. \]

The weak Bruhat order is stronger than Bruhat order in the sense that $u \leq_L v$ implies $u \leq v$ for all $u, v \in \mathfrak{S}_n$. Note that $u \leq_L v$ if and only if $N(u) \subseteq N(v)$ [5, Prop. 3.1.3]. The following lemma tells us that $\mathcal{W}(S, h)$ is a weak Bruhat interval. A proof can be found in Appendix A.

2.10. **Lemma.** Let $h : [n] \rightarrow [n]$ be a Hessenberg function and $S \in \mathcal{W}_h$. There exist elements $z_S, w_S \in \mathcal{W}(S, h)$ such that $\mathcal{W}(S, h)$ is precisely the weak (left) Bruhat interval
\[ \{ z_S, w_S \} = \{ v \in \mathfrak{S}_n \mid z_S \leq_L v \leq_L w_S \}. \]

In order to apply the results of [10] needed below, we now introduce a graph $\Gamma_h$ uniquely determined by a Hessenberg function $h$.

2.11. **Definition.** Let $h : [n] \rightarrow [n]$ be a Hessenberg function. The incomparability graph $\Gamma_h = (V(\Gamma_h), E(\Gamma_h))$ is the graph on vertex set $V(\Gamma_h) = [n]$ with edges $E(\Gamma_h) := \{ \{i, j\} \mid j < i \text{ and } i \leq h(j) \}$.

2.12. **Example.** The incomparability graph $\Gamma_h$ for $h = (2, 4, 4, 4)$ and $h = (3, 4, 5, 5, 5)$ are given below.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\hline
2 & 1 & 3 & 4 \\
3 & 2 & 4 & 5 \\
4 & 3 & 5 & 4 \\
5 & 4 & 4 & 3 \\
\end{array}
\]

The incomparability graph for $h$ plays a key role in the results of [15] and also appears in [10, 11] (with the notation $G_{e,h}$). An acyclic orientation $o$ of $\Gamma_h$ is an assignment of a direction (i.e. orientation) to each edge $e \in E(\Gamma_h)$ such that the resulting oriented graph contains no directed cycles. Given $S \in \mathcal{W}_h$, we obtain an orientation $o_h(S)$ of $\Gamma_h$ defined by the rule
\[ j \leftarrow i \text{ for } \{i, j\} \in E(\Gamma_h) \text{ with } j < i \text{ if and only if } t_j - t_i \in S. \]

In other words, $o_h(S)$ is obtained by orienting edges corresponding to the roots in $S$ to the left, and oriented all other edges to the right. The following observation connects acyclic orientations and subsets of Weyl type.

2.14. **Lemma.** The set of all acyclic orientations of $\Gamma_h$ is precisely $\{ o_h(S) \mid S \in \mathcal{W}_h \}$.

**Sketch of Proof.** Consider first the case in which $h = (n, n, \ldots, n)$ so $\Gamma_h = K_n$ is the complete graph on $n$ vertices. It is a simple exercise to show that there are precisely $n!$ acyclic orientations of $K_n$, each uniquely determined by a permutation $w$ according to the rule that $j \leftarrow i$ if and only if $j < i$ and $w(j) > w(i)$, or equivalently, if $t_j - t_i \in N(w)$. Since every subset of Weyl type of the form $N(w)$ for a unique $w \in \mathfrak{S}_n$, the claim now follows.

Returning to the case of an arbitrary Hessenberg function, note that $\Gamma_h$ is a subgraph of $K_n$. Let $S \in \mathcal{W}_h$. By Theorem 2.9 there exists $w \in \mathfrak{S}_n$ such that $S = N(w) \cap \Phi^n_+$. The orientation $o_h(S)$ defined in (2.13) is the orientation induced by the that of $w$ on $K_n$ as in the previous paragraph. This shows that $o_h(S)$ is an acyclic orientation. Since every acyclic orientation of $\Gamma_h$ is the restriction of an acyclic orientation on $K_n$ it follows that every acyclic orientation is of this form. \[ \square \]

2.15. **Remark.** In [11], the authors define and study an equivalence class of permutations. In their notation, the equivalence class $[w]_h$ from [11, Definition 3.4] is precisely the set $\mathcal{W}(S, h)$ above. We also note that in [10, 11] the authors use the language of subgraphs $G_{w,h}$ of $\Gamma_h$ for different permutations $w$, but this data can be equivalently characterized by acyclic orientations, which is what we choose to do in this manuscript.
The relation between \( \mathcal{W}(S, h) \) (and specifically the maximal element \( w_S \) of \( \mathcal{W}(S, h) \)) to the acyclic orientation \( o_h(S) \) is developed further in Section 3 below.

3. Reachability

As mentioned above, a key point of this manuscript is to connect the work of Cho, Hong, and Lee in [10] – in which they use the combinatorial notion of reachability to study Hessenberg Schubert cells – to the theory of subsets of Weyl type. In this section, we make this connection precise. The essential result is Proposition 3.12, which is the technical engine driving the proof of our main theorem (Theorem 4.8).

We begin with the definition of reachability, taken from [10], relating two vertices \( i \) and \( j \) on the incompatibility graph \( \Gamma_h \).

3.1. **Definition.** Let \( h : [n] \to [n] \) be a Hessenberg function and \( \Gamma_h \) its associated incompatibility graph, equipped with an acyclic orientation \( o_h(S) \). Suppose \( i > j \). We say that \( i \) is reachable from \( j \) with respect to \( S \) (or \( o_h(S) \)) if \( i = v_m \) and \( j = v_0 \) and there exists a sequence of vertices \( j = v_0 < v_1 < \cdots < i = v_m \) of \( \Gamma_h \) such that there is an oriented edge from each \( v_k \) to \( v_{k+1} \), i.e., there is a sequence of oriented edges of the form \( j = v_0 \to v_1 \to \cdots \to v_{m-1} \to v_m = i \) in \( \Gamma_h \) (equipped with the orientation \( o_h(S) \)). We allow \( m \) to be 0, that is, \( j \) is always reachable from \( j \).

We say that a vertex \( k \) in an oriented graph is a **source** (with respect to the given orientation) if all edges adjacent to \( k \) point “out” of \( k \), i.e., is of the form \( k \to i \) for all \( i \) adjacent to \( k \). Since our incompatibility graphs have vertices labelled by sets of positive integers \( \{1, 2, \ldots, n\} \), we say that a vertex \( k \) is the **largest source** if \( k \) is a source and moreover, if \( j \) is another source, then \( k > j \). The next lemma shows that any vertex larger than the largest source \( k \) is reachable from \( k \).

3.2. **Lemma.** Let \( h : [n] \to [n] \) be a Hessenberg function, \( \Gamma_h \) its associated incompatibility graph, and let \( o_h(S) \) be an acyclic orientation of \( \Gamma_h \) for \( S \in \mathcal{W}_h \). Suppose that \( k \) is the largest source of \( \Gamma_h \) with respect to \( o_h(S) \) and suppose that \( i \) is a vertex with \( i > k \). Then \( i \) is reachable from \( k \).

**Proof.** To prove the claim of the lemma, it would suffice to show that the set

\[
\{ i > k \mid i \text{ is not reachable from } k \}
\]

is empty. For the sake of obtaining a contradiction, suppose not, and consider the oriented subgraph \( \Gamma' \) induced by the vertices in set \( \{ i > k \mid i \text{ is not reachable from } k \} \). Since the original oriented graph \( \Gamma_h \) is acyclic, so is the subgraph. Any acyclic orientation must have a source, so there exists a vertex \( i' \) which is a source of \( \Gamma' \). We claim that the the vertex \( i' \) must also be a source of the original oriented graph \( \Gamma_h \). To see this, we must show that for any vertex \( j \) in \( \Gamma_h \) with and edge between \( j \) and \( i' \), the edge must be oriented from \( i' \) to \( j \), i.e., of the form \( i' \to j \). To argue this, we first observe that \( i' > h(k) \), because if \( i' \leq h(k) \) then by definition of \( \Gamma_h \) there would be an edge from \( k \) to \( i' \), and since \( k \) is a source by assumption, the edge between \( k \) and \( i' \) would be oriented as \( k \to i' \), making \( i' \) reachable from \( k \). This contradicts the assumption that \( i' \) is in the set \( \{ i > k \mid i \text{ is not reachable from } k \} \). This implies that there is no edge in \( \Gamma_h \) between \( i' \) and any vertex \( j \) with \( j \leq k \). Thus we may now assume without loss of generality that \( j > k \). We take cases.

Suppose that \( k < j < i' \). If there is no edge between \( j \) and \( i' \) in \( \Gamma_h \) then there is nothing to prove, so suppose there is an edge. If \( j \) is not reachable by \( k \), then \( j \) is in the set \( \{ i > k \mid i \text{ is not reachable from } k \} \) and we have assumed that \( i' \) is a source of the subgraph, so we must have \( i' \to j \). On the other hand, if \( j \) is reachable by \( k \), then by reasoning similar to the above, we must also have \( j \leftarrow i' \) since otherwise \( i' \) would be reachable by \( k \).

Next suppose that \( j' > i' \). Again, if there is no edge between \( j \) and \( i' \) then there is nothing to prove, so we may suppose \( \{i', j'\} \) is an edge of \( \Gamma_h \). If \( j' \) is not reachable by \( k \) then by the same reasoning as above we already know the edge is oriented as \( i' \to j \), so suppose \( j' \) is reachable by
k. We wish to show \( i' \to j \) in \( o_h(S) \). Suppose for the sake of contradiction that \( i' \leftarrow j \) instead. Since \( j' \) is reachable from \( k \), there exists a sequence \( v_0 < v_1 < \cdots < v_m \) of vertices such that \( k = v_0 \to j_1 \to \cdots \to v_{m-1} \to v_m = j' \) in \( o_h(S) \). Let \( v_t \) be the largest vertex in that list such that \( v_t < i' \). Since \( \{v_t, v_{t+1}\} \) is an edge in \( \Gamma_h \) and \( i < v_{t+1} \), we can conclude that \( \{v_t, i\} \) is also an edge in \( \Gamma_h \). Since \( v_t \) is reachable from \( k \), by the same reasoning as above we know that \( v_t \leftarrow i' \) and we may visualize the graph as

\[
v_t \leftarrow i' \leftarrow v_{t+1} \cdots \to v_m = j'
\]

in \( o_h(S) \), which shows that there is a cycle starting and ending at \( i' \), contradicting the fact that \( o_h(S) \) is an acyclic orientation. Thus we must have \( i' \to j \). We have shown that \( i' \) is a source in \( \Gamma_h \), contradicting that \( k \) is the largest source. Therefore, (3.3) is indeed empty as was to be shown. \( \square \)

The next lemma shows that reachability implies an inequality among entries in the one-line notation of \( w \) for \( w \in \mathcal{W}(S, h) \).

3.4. Lemma. Let \( j \leq i \) be two vertices in \( \Gamma_h \) and let \( S \in \mathcal{W}_h \). If \( i \) is reachable from \( j \) with respect to \( o_h(S) \), then \( w(j) \leq w(i) \) for all \( w \in \mathcal{W}(S, h) \).

Proof. Let \( w \in \mathcal{W}(S, h) \). By assumption on the \( i \) and \( j \), we have a sequence of vertices \( v_0 < v_1 < \cdots < v_m \) such that \( j = v_0 \to v_1 \to \cdots \to v_{m-1} \to v_m = i \) in \( o_h(S) \). Thus for each \( \ell \) such that \( 1 \leq \ell \leq m \), we have \( t_{v_{\ell-1}} - t_{v_\ell} \in \Phi_h^+ \setminus S \) and since \( S = N(w) \cap \Phi_h^+ \), we know \( \{v_{\ell-1}, v_{\ell}\} \) is an edge of \( \Gamma_h \), we get \( t_{v_{\ell-1}} - t_{v_{\ell}} \notin N(w) \). This means \( w(v_{\ell-1}) = w(v_{\ell}) \) for all \( 1 \leq \ell \leq m \), and by putting the inequalities together we obtain \( w(j) \leq w(i) \), as desired. \( \square \)

It also turns out that the location of the 1 in the one-line notation of \( w \in \mathcal{W}(S, h) \) is significant.

3.5. Lemma. Let \( h : [n] \to [n] \) be a Hessenberg function and \( S \in \mathcal{W}_h \). If \( w \in \mathcal{W}(S, h) \), then \( w^{-1}(1) \) is a source of \( \Gamma_h \) equipped with the orientation \( o_h(S) \).

Proof. Suppose \( j = w^{-1}(1) \). It follows directly from the definition of the inversion set that

\[
\{t_1 - t_j, t_2 - t_j, \ldots, t_{j-1} - t_j\} \subseteq N(w)
\]

since \( w(j) = 1 \) is strictly smaller than any \( w(1), w(2), \ldots, w(j-1) \). By similar reasoning, since \( w(j) = 1 \) is also smaller than \( w(j+1), \ldots, w(n) \), we have

\[
\{t_j - t_{j+1}, t_j - t_{j+2}, \ldots, t_j - t_n\} \subseteq \Phi_h^+ \setminus N(w).
\]

Since \( w \in \mathcal{W}(S, h) \) by hypothesis, we have that \( S = N(w) \cap \Phi_h^+ \). From the definition of \( o_h(S) \) in (2.13) we see that the edges \( \{k, j\} \) with \( k < j \) must be oriented to the left and the edges for \( k > j \) are oriented toward the right. Thus \( j \) is a source, as desired. \( \square \)

We now give an inductive construction using the Hessenberg function \( h \) and graph \( \Gamma_h \) that will be useful in what follows. Let \( k \in [n] \). Consider the smaller graph on \( n-1 \) vertices which is obtained from \( \Gamma_h \) by deleting vertex \( k \) and all adjacent edges to \( k \), and for convenience in our arguments below, relabeling the vertex set to be \( \{2, 3, \ldots, n\} \) (so \( \{1, 2, \ldots, k-1\} \) gets relabelled as \( \{2, 3, \ldots, k\} \) respectively, and the labels of \( \{k+1, \ldots, n\} \) are unchanged). Alternatively, if the Hessenberg function \( h \) is visualized as a collection of boxes in an \( n \times n \) array where the \( (i, j) \)-th box (in row \( i \) and column \( j \)) is said to be in the collection if \( i \leq j \), or, if \( j < i \leq h(j) \). Then the collection of boxes in the \( (n-1) \times (n-1) \) array corresponding to \( h^{(k)} \) is obtained from that of \( h \) by deleting the \( k \)-column and the \( k \)-th row and interpreting the result as an \( (n-1) \times (n-1) \) array. We let \( h^{(k)} : \{2, 3, \ldots, n\} \to \{2, 3, \ldots, n\} \) denote the Hessenberg function whose associated graph \( \Gamma_{h^{(k)}} \) is precisely the graph just described.
Set $\Psi := \{t_i - t_j \mid i \neq j \text{ and } i,j \in \{2, \ldots, n\}\}$. Then $\Psi$ corresponds to a root system of type $A_{n-2}$, and the only difference between $\Psi$ and the standard root system is that we have shifted the index set to be $\{2, 3, \ldots, n\}$ instead of $\{1, 2, \ldots, n-1\}$. Fix $k \in [n]$. In analogy with how we defined $\Phi^+_h$ above, let us define $\Psi^+_h := \{t_i - t_j \in \Psi \mid i < j \text{ and } j \leq h(k(i))\}$. From the description of the graph $\Gamma_{h(k)}$ it is not hard to check that

\begin{equation}
\Psi^+_h = u_k(\Phi^+_h) \cap \Psi
\end{equation}

where $u_k := s_1s_2 \cdots s_{k-1} \in \mathcal{S}_n$ (in one-line notation we have $u_k = (2, 3, \ldots, k, 1, k + 1, k + 2, \ldots, n)$). Here we take $u_1 := e \in \mathcal{S}_n$. The set $\Psi^+_h$ is the analogue of the set $\Phi^+_h$ with the exception that the indices start at 2 instead of 1. Given $S \in W_h$, we can consider the induced orientation of $\Gamma_{h(k)}$, which must necessarily be acyclic. Let $S' \subseteq \Psi^+_h$ denote the corresponding subset of Weyl type. By (3.6) we have

\[ S' = u_k(S) \cap \Psi. \]

The next lemma relates certain elements in $\mathcal{W}(S, h)$ with those in $\mathcal{W}(S', h(k))$.

**3.7. Lemma.** Let $w \in \mathcal{S}_n$, $y \in \langle s_2, s_3, \ldots, s_{n-1} \rangle$, and let $k$ be a source of $\Gamma_h$ with respect to $o_h(S)$ for $S \in W_h$. Let $u_k$ as above and assume $w = yu_k$. Then $y \in \mathcal{W}(S', h(k))$ if and only if $w \in \mathcal{W}(S, h)$.

**Proof.** First suppose $w \in \mathcal{W}(S, h)$, so $N(w) \cap \Phi^+_h = S$. On the other hand, since the decomposition $w = yu_k$ satisfies $\ell(w) = \ell(y) + \ell(u_k)$ (note $u_k$ is a minimal coset representative of $\mathcal{S}_{n-1} \setminus \mathcal{S}_n$) we have $N(w) = N(u_k) \cup u_k^{-1}N(y)$ (cf. for instance [16, Section 1.7]). Combining these facts we obtain

\begin{equation}
S = (N(u_k) \cup u_k^{-1}N(y)) \cap \Phi^+_h \iff u_k(S) = (u_k N(u_k) \cup N(y)) \cap u_k \Phi^+_h
\end{equation}

and thus

\[ S' = u_k(S) \cap \Psi = N(y) \cap \Psi \cap u_k \Phi^+_h = N(y) \cap \Psi^+_h \]

where the first and third equalities follows from (3.6) and the second equality follows from an explicit computation of $u_k N(u_k)$ which shows that $u_k N(u_k) \cap \Psi = \emptyset$. This proves $y \in \mathcal{W}(S', h(k))$.

To see the other direction, suppose $y \in \mathcal{W}(S', h(k))$. We want to show $w \in \mathcal{W}(S, h)$, for which it suffices to prove (3.8). Since $k$ is a source we have

\[ S = \{t_j - t_i \in S \mid i, j \in [n] \setminus \{k\}\} \cup \{t_j - t_k \mid j < k, k \leq h(j)\}. \]

Since $u_k([n] \setminus \{k\}) = \{2, \ldots, n\}$ we have

\begin{equation}
\{t_j - t_i \in S \mid i, j \in [n] \setminus \{k\}\} = u_k^{-1}(S')
\end{equation}

and now the claim follows from the observations that (3.6) implies

\[ u_k^{-1}(S') = \{t_j - t_i \in S \mid i, j \in [n] \setminus \{k\}\} = u_k^{-1}(N(y) \cap \Psi^+_h) = u_k^{-1}N(y) \cap u_k^{-1}(\Psi) \cap \Phi^+_h \]

and that $N(u_k) \cap \Phi^+_h = \{t_j - t_k \mid j, k \leq h(j)\}$. This completes the proof. \hfill \Box

The next lemma is a kind of converse to Lemma 3.5.

**3.10. Lemma.** Let $S \in W_h$. If $k$ is a source of $\Gamma_h$ with respect to the orientation $o_h(S)$, then there exists $w \in W_h(S)$ such that $w^{-1}(1) = k$.

**Proof.** Since $W(S', h(k))$ is non-empty, there exists $y \in W(S', h(k)) \subseteq \langle s_2, \ldots, s_{n-1} \rangle$. Consider $w = yu_k$. By Lemma 3.7, $w \in \mathcal{W}(S, h)$, and by construction $w(k) = 1$. \hfill \Box

The relationship is even tighter between the largest source and the maximal element $w_S$.

**3.11. Lemma.** Let $h : [n] \to [n]$ be a Hessenberg function and $S \in W_h$. Then the vertex $k$ is the largest source of $\Gamma_h$ with respect to $o_h(S)$ if and only if $w_S^{-1}(1) = k$. 

Proof. Suppose $k$ is the largest source of $o_h(S)$. By Lemma 3.10 there exists $w \in \mathcal{W}(S, h)$ such that $w(1) = k$ so we may write $w = yu_k$ for some $y \in \langle s_2, \ldots, s_{n-1} \rangle$. Let $j := w_S^{-1}(1)$ or equivalently, $w_S(j) = 1$. Then by Lemma 3.5 we know $j$ must be a source, and since $k$ is the largest source, we conclude $j \leq k$. To complete the argument that $j = w_S^{-1}(1) = k$ we need to show $k \leq j$. To see this, write $w_S = ysu_j$ for some $ys \in \langle s_2, \ldots, s_n \rangle$, which is possible since $j = w_S^{-1}(1)$. We know that $w_S$ is the maximal element of $\mathcal{W}_h(S)$, so

$$w \leq w_S \Rightarrow yu_k \leq ysu_j \Rightarrow u_k \leq u_j \Rightarrow k \leq j$$

where the second implication follows from the fact that the map from $\mathcal{S}_n$ to the shortest coset representatives $\{u_1 := e, u_2, \ldots, u_n\}$ of the subgroup $\langle s_2, s_3, \ldots, s_n \rangle$ is order-preserving [5, Prop. 2.5.1]. Thus $j = w_S^{-1}(1) = k$ as desired. The converse follows by similar reasoning. \hfill \Box

We can now prove one of our important technical results, which characterizes reachability in terms of the maximal elements $w_S$ for $S \in \mathcal{W}_h$. This reformulation is the tool which allows us to prove the closure relations in the next section.

3.12. Proposition. Let $h : [n] \rightarrow [n]$ be a Hessenberg function and $S \in \mathcal{W}_h$. Suppose $j \leq i$. Then $i$ is reachable from $j$ with respect to $S$ if and only if $w_S(j) \leq w_S(i)$.

Proof. First suppose that $j \leq i$ and $i$ is reachable from $j$. We wish to show that $w_S(j) \leq w_S(i)$. This follows immediately from Lemma 3.4.

So now suppose $j \leq i$ and that $w_S(j) \leq w_S(i)$. We need to show that $i$ is reachable from $j$. We proceed to prove the contrapositive statement by an induction on $n$. When $n = 1$, we have $j = i = 1$ and $w_S = e$ and the claim is trivial. Now suppose $n \geq 2$ and that the claim is true for $n - 1$. Note that we may assume $j \neq i$ since if $j = i$ then $i$ is reachable from $j$ by convention and the claim is immediate. So suppose $j < i$ and additionally suppose that $i$ is not reachable from $j$. Let $k := w_S^{-1}(1)$ and write $w_S = ysu_k$ for some $ys \in \langle s_2, \ldots, s_{n-1} \rangle$. If $i = k$, then our claim follows immediately since $w_S(k) = w_S(i) = 1 < w_S(j)$. Next, notice that by Corollary 3.11, $k$ is the largest source of $\Gamma_h$ with respect to $o_h(S)$. If $j = k$, then by Lemma 3.2 we would have that $i$ is reachable from $j$, but we have assumed that $i$ is not reachable from $j$, so we conclude $j \neq k$. So we now have that $i \neq k, j \neq k$ and $j < i$. Now we consider the graph $\Gamma_h(k)$ obtained from $\Gamma_h(S)$ with acyclic orientation $o_h(S')$ induced by $o_h(S)$. By Lemma 3.7, since $w_S \in \mathcal{W}(S, h)$ we know $ys \in \mathcal{W}(S', h(k))$. We wish to show $ys$ is the maximal element in $\mathcal{W}(S', h(k))$. To see this, suppose $zs$ is the maximal element. First observe $ys \leq zs$ since $zs$ is maximal. On the other hand, $zsu_k \in \mathcal{W}(S, h)$ by Lemma 3.7 and since $w_S$ is maximal, we conclude $zsuk \leq w_S = ysu_k$. Now the properties of Bruhat order imply $zs \leq ys$. Hence $ys = zs$ and $ys$ is maximal in $\mathcal{W}(S', h(k))$. Our assumptions imply that $i' = u_k(i)$ is not reachable from $j' = u_k(j)$. As $\Gamma_h(k)$ is a Hessenberg graph on $n - 1$ vertices and $ys$ is maximal, we know $ys(i') < ys(j')$ by induction. Since $w_S = ysu_k$, we conclude $w_S(i) < w_S(j)$ as desired. \hfill \Box

As a first geometric application of the results in this section, we prove the following. Recall that if $v \in \mathcal{W}(S, h)$ then $v \leq_L w_S$ so there exists (by definition of weak Bruhat order) an element $u \in \mathcal{S}_n$ such that $w_S = u^{-1}v$ with $\ell(w_S) = \ell(v) + \ell(u)$.

3.13. Proposition. Suppose $S \in \mathcal{W}_h$ and $v \in \mathcal{W}(S, h)$ and let $w_S = u^{-1}v$ for $u \in \mathcal{S}_n$ where $\ell(w_S) = \ell(v) + \ell(u)$. Then

$$\Omega_{v, h} = u \left( \Omega_{w_S} \cap \mathcal{Hess}(u^{-1}Sw, h) \right).$$

Proof. Let $b$ denote a lower triangular matrix with 1’s on the diagonal. Recall that $wbB \in \Omega_w$ if and only if $b_{ij} = 0$ for all $i > j$ such that $w(i) < w(j)$, or equivalently, $b_{ij} = 0$ for all $t_j - t_i \in N(w)$. Our assumptions imply that $v \leq_L w_S$ so $N(v) \subseteq N(w_S)$. 

\hfill 9
Suppose \( wSB \in \Omega^o_{wS} \), so \( b_{ij} = 0 \) for all \( t_j - t_i \in N(w_S) \) implying that \( b_{ij} = 0 \) for all \( t_j - t_i \in N(v) \) also. Thus \( uwSB = vB \in \Omega^o_v \). We now have
\[
 u\Omega^o_{wS} \subseteq \Omega^o_v \quad \Rightarrow \quad u\Omega^o_{wS} \cap \text{Hess}(S, h) \subseteq \Omega^o_v \cap \text{Hess}(S, h)
\]
\[
 \Rightarrow \quad u \left( \Omega^o_{wS} \cap \text{Hess}(u^{-1}Su, h) \right) \subseteq \Omega^o_{v,h}. 
\]
Here the second implication uses the fact that \( \text{Hess}(S, h) = u\text{Hess}(u^{-1}Su, h) \).

To prove the opposite inclusion, suppose \( vB \in \Omega^o_{v,h} \). By [10, Corollary 3.9], \( b_{ij} = 0 \) for all \( i > j \) such that \( i \) is not reachable from \( j \). By Proposition 3.12, this implies \( b_{ij} = 0 \) for all \( i > j \) such that \( w_S(i) < w_S(j) \) so \( wSB \in \Omega^o_{wS} \). Furthermore, since \( vB \in \text{Hess}(S, h) \) it follows immediately that
\[
 u^{-1}vB = wSB \in \text{Hess}(u^{-1}Su, h). 
\]
Thus \( u^{-1}vB \in \Omega^o_{wS} \cap \text{Hess}(u^{-1}Su, h) \), as desired. 

\[
\Box 
\]

4. Closures of Hessenberg Schubert cells

In this section we state and prove the main result of this paper, Theorem 4.8. To do so, we need some terminology and notation from the work of Cho, Hong, and Lee [10]. First let \( n \) be a positive integer and let \( 1 \leq j \leq n \). Denote by \( I_{j,n} \) the set of strictly increasing ordered \( j \)-tuples of positive integers \( \overline{i} = (i_1, i_2, \ldots, i_j) \in \mathbb{Z}^j \) satisfying \( 1 \leq i_1 < i_2 < \cdots < i_j \leq n \). Next, suppose given a permutation \( w \in S_n \) and \( \overline{i} \in I_{j,n} \). We can consider first the \( j \)-tuple of not-necessarily-strictly-increasing integers \((w(i_1), w(i_2), \ldots, w(i_j)) \in \mathbb{Z}^j \), and then we can re-order the entries in such a way that they are strictly increasing; we denote the result as \( w \cdot \overline{i} \) and by construction, \( w \cdot \overline{i} \in I_{j,n} \).

Second, we use the notation \( p_{\overline{i}} \) to denote the usual \( \overline{i} \)-th Plücker coordinate \( p_{\overline{i}} \) corresponding to the \( k \times k \) minor corresponding to the \( \overline{i} = (i_1, \ldots, i_k) \) rows and leftmost \( k \) columns of \( g \in GL(n, \mathbb{C}) \). Assembling all Plücker coordinates for varying sizes of minors yields the well-known Plücker embedding of the flag variety \( GL(n, \mathbb{C})/B \) into \( \prod_{k=1}^{n-1} \mathbb{P}(\mathbb{C}^{\binom{k}{2}}) \).

Finally, we need to define a certain subset \( J_{w,h,k} \) associated to a permutation \( w \in S_n \), a Hessenberg function \( h : [n] \to [n] \), and an integer \( k, 1 \leq k \leq n \). To define it, we need the following terminology from [10], which extends the notion of reachability (described in Section 3) to two subsets of \([n]\).

4.1. Definition. Let \( A = \{ 1 \leq a_1 < a_2 < \cdots < a_r \leq n \} \) and \( B = \{ 1 \leq b_1 < b_2 < \cdots < b_r \leq n \} \) be two subsets of \([n]\) of the same cardinality \( r \). We say \( A \) is reachable from \( B \) if there exists a permutation \( \sigma \in S_n \) such that \( a_{\sigma(i)} \) is reachable from \( b_i \) for all \( 1 \leq i \leq r \).

We can now define a certain subset of \( I_{j,n} \) [10, Equation (3.2)] as follows:
\[
 J_{w,h,j} := \{ \overline{i} = (i_1, i_2, \cdots, i_j) \mid \{i_1, \cdots, i_j\} \text{ is reachable from } \{1, 2, \cdots, j\}\}.
\]

With this notation in place, we can state the following result of Cho, Hong, and Lee [10, Corollary 3.17].

4.3. Proposition (Cho–Hong–Lee). Let \( h : [n] \to [n] \) be a Hessenberg function and \( w \in S_n \). Then
\[
 \Omega_{w,h} = \bigcap_{k=1}^{n-1} \{ gB \in \text{Hess}(S, h) \mid p_{\overline{i}}(gB) = 0 \text{ for all } \overline{i} \in I_{k,n} \setminus J_{w,h,k}\}. 
\]

Here and below we define and use a partial order \( \leq \) on \( I_{k,n} \) as follows:
\[
(i_1, i_2, \cdots, i_k) \geq (j_1, j_2, \cdots, j_k) \quad \text{if and only if} \quad i_\ell \geq j_\ell \quad \text{for all } \ell, 1 \leq \ell \leq k.
\]

In order to use Proposition 4.3 for our purposes, we need the following lemma, which characterizes the set \( J_{w,h,j} \) defined in (4.2) in terms of maximal elements of subsets of Weyl type. This connects the discussion to that of Section 3 and allows us to use the results therein.
4.5. **Lemma.** For all $S \in \mathcal{W}_h$ and all $k$ such that $1 \leq k \leq n - 1$, we have

$$J_{w,h,k} = \{(i_1, \ldots, i_k) \in I_{k,n} \mid w_S \cdot (i_1, \ldots, i_k) \geq w_S \cdot (1, \ldots, k)\}$$

where $J_{w,h,k}$ is the set defined by (4.2).

**Proof.** By its definition in (4.2), and from the definition of reachability, $J_{w,h,k}$ consists of sets $(i_1, \ldots, i_k) \in I_{k,h}$ such that there exists a permutation $\sigma \in \mathfrak{S}_n$ with the property that $i_{\sigma(\ell)}$ is reachable from $\ell$ for all $1 \leq \ell \leq k$. On the other hand, from Proposition 3.12 it immediately follows that this is equivalent to $w_S(i_{\sigma(\ell)}) \geq w_S(\ell)$ for all $1 \leq \ell \leq k$. The claim now follows from our definition of the notation $w_S \cdot (i_1, \ldots, i_k) \geq w_S \cdot (1, \ldots, k)$.

We will need the following description of the Bruhat order in $\mathfrak{S}_n$, which we now briefly recall (cf. [5, Propositions 2.4.8, 2.5.1, and Theorem 2.6.3]).

4.6. **Lemma.** Let $w, v \in \mathfrak{S}_n$. Then $w \leq v$ in Bruhat order if and only if

$$w \cdot (1, \ldots, k) \leq v \cdot (1, \ldots, k)$$

for all $1 \leq k \leq n - 1$.

We are now ready to state and prove our main theorem.

4.8. **Theorem.** Let $h : [n] \to [n]$ be a Hessenberg function and fix $S \in \mathcal{W}_h$. Then

$$\Omega_{w_S,h} = \bigcup_{w \leq u} \Omega_{u,h} = \Omega_{w_S} \cap \mathcal{Hess}(S, h).$$

**Proof.** We begin by showing the second equality in (4.9). From the definition (2.4) of the opposite Hessenberg Schubert cells and the well-known fact that the closure of an opposite Schubert cell $\Omega_w$ is given by $\bigcup_{w \leq u} \Omega_u$, it follows that the middle expression in (4.9) is equal to $\Omega_{w_S} \cap \mathcal{Hess}(S, h)$ as desired. Since the opposite Schubert variety and $\mathcal{Hess}(S, h)$ are both closed, it follows that the closure of $\Omega_{w_S,h} = \Omega_{w_S} \cap \mathcal{Hess}(S, h)$ is also contained in $\Omega_{w_S} \cap \mathcal{Hess}(S, h)$. This shows the inclusion

$$\Omega_{w_S,h} \subseteq \bigcup_{w \leq u} \Omega_{u,h} = \Omega_{w_S} \cap \mathcal{Hess}(S, h).$$

It now suffices to show the other inclusion, namely, that $\bigcup_{w \leq u} \Omega_{u,h} \subseteq \Omega_{w_S,h}$. To see this it would suffice to show that $\Omega_{u,h} \subseteq \Omega_{w_S,h}$ for all $u \geq w_S$. Moreover, by Proposition 4.3, this is equivalent to showing that

$$p_{w_S}(buB) = 0 \text{ for all } \mathfrak{i} \in I_{k,n} \setminus J_{w,h,k} \text{ for all } k, 1 \leq k \leq n - 1$$

for all $b \in B_-$.

We begin by showing the case $b = e$, i.e. the trivial group element, which means we wish to show that

$$p_{w_S}(u) = 0 \text{ for all } \mathfrak{i} \in I_{k,n} \setminus J_{w,h,k}, 1 \leq k \leq n - 1.$$  

Fix $\mathfrak{i} \in I_{k,n} \setminus J_{w,h,k}$. To prove (4.10), it would be enough to show that the $k \times k$ minor of $u$ corresponding to columns $1, \ldots, k$ and rows $w_S(i_1), \ldots, w_S(i_k)$ contains a row equal to 0. Denote this minor by $\bar{u}$. Suppose for the sake of contradiction that $\bar{u}$ has no zero rows. Since $u$ is a permutation matrix, this implies that each row of $\bar{u}$ contains exactly one nonzero entry equal to 1. Since these rows correspond to rows $w_S(i_1), \ldots, w_S(i_k)$ of $u$, it follows that these rows of $u$ must each contain a 1 within the first $k$ columns. Since the rows of $u$ containing a 1 in the first $k$ columns are precisely rows $u(1), \ldots, u(k)$ we have,

$$\{u(1), u(2), \ldots, u(k)\} = \{w_S(i_1), w_S(i_2), \ldots, w_S(i_k)\} \Rightarrow u(1, 2, \ldots, k) = u_S \cdot (i_1, i_2, \ldots, i_k).$$

As $u \geq w_S$ we now conclude by Lemma 4.6 that $w_S \cdot (i_1, i_2, \ldots, i_k) \geq w_S(1, 2, \ldots, k)$. This contradicts our assumption that $\mathfrak{i} \notin J_{w,h,k}$ by Lemma 4.5, proving (4.10).
Now suppose \( b \in B_\cdot \) and \( buB \in \text{Hess}(S, h) \). We wish to show \( buB \in \Omega_{w, h} \), for which it suffices, as in the case above, to show that \( p_{w, S}(bu) = 0 \) for all \( \mathfrak{i} \in I_{k,n} \setminus J_{w, h,k} \) where \( 1 \leq k \leq n-1 \).

Let us denote the \((i, j)\)-th matrix entry of \( b \in B_\cdot \) by \( x_{ij} \), so \( b = (x_{ij}) \). Since \( b \) is lower-triangular, we know that \( x_{ij} = 0 \) if \( i < j \). Now let \( y_{ij} \) denote the \((i, j)\)-th matrix entry of the product \( bu \).

In computing the Plücker coordinate \( p_{w, S}(bu) \), we need to know the entries of \( bu \) in the rows labelled by \( w_S(i_1), w_S(i_2), \ldots, w_S(i_k) \) and the \( k \) leftmost \( k \) rows. Focusing attention on these entries, and using that \( u \) is a permutation matrix, it is straightforward to compute that

\[
y_{w_S(i),a} = \begin{cases} x_{w_S(i),u(a)} & \text{if } u(a) < w_S(i) \\ 0 & \text{else} \end{cases}
\]

From the definition of determinants it is immediate that

\[
p_{w_S, S}(bu) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} y_{w_S(i_1), \sigma(1)} y_{w_S(i_2), \sigma(2)} \cdots y_{w_S(i_k), k}
\]

so in order to prove that the LHS is equal to 0 it would suffice to show that every summand appearing in the sum on the RHS is equal to 0. This is what we prove next.

For the sake of a contradiction, suppose there exists \( \sigma \in \mathfrak{S}_k \) such that

\[
y_{w_S(i_1), \sigma(1)} y_{w_S(i_2), \sigma(2)} \cdots y_{w_S(i_k), k} \neq 0.
\]

This means that each \( y_{w_S(i_\ell), \sigma(\ell)} = x_{w_S(i_\ell), u(\sigma(\ell))} \) is non-zero, for all \( 1 \leq \ell \leq k \). It follows that \( w_S(\ell) \geq u(\sigma(\ell)) \) for all \( 1 \leq \ell \leq k \), which in turn implies that \( u \cdot (1, 2, \ldots, k) \leq w_S(i_1, i_2, \ldots, i_k) \).

On the other hand, we have assumed that \( u \geq w_S \) in Bruhat order, so the tableau criterion in Lemma 4.6 implies \( u \cdot (1, 2, \ldots, k) \geq w_S(1, 2, \ldots, k) \). By transitivity we obtain \( w_S(i_1, i_2, \ldots, i_k) \geq w_S(1, 2, \ldots, k) \), but from Lemma 4.5 we see that this contradicts the assumption that \( \mathfrak{i} \in I_{k,n} \setminus J_{w, h,k} \). Thus such a \( \sigma \in \mathfrak{S}_k \) cannot exist, and we conclude that each summand in the RHS of (4.11) is 0, as was to be shown. Thus \( p_{w, S}(bu) = 0 \) for \( \mathfrak{i} \in I_{k,n} \setminus J_{w, h,k} \), concluding the proof.

By translating by \( w_0 \), we easily obtain the analogous result for Hessenberg Schubert cells.

**4.12. Corollary.** Let \( h : [n] \rightarrow [n] \) be a Hessenberg function and \( S \in \mathcal{W}_h \). Then

\[
X_{z_S, h} = \bigcup_{u \leq z_S} X_{u, h}.
\]

**Proof.** Let \( w_0 = [n, n-1, 1, \ldots, 2, 1] \) denote the longest element in \( \mathfrak{S}_n \) and \( \bar{S} = \Phi^+_h \setminus S \in \mathcal{W}_h \).

Since \( \Omega_{w, h} \) is the Hessenberg Schubert variety \( \Omega_{w, h} = \mathcal{W}_h \setminus \mathcal{H} \), we have \( w_0 \Omega_{w, h} = X_{w_0 w}^\circ \). Using the fact that \( w_0 w_S = z_S \) as in the proof of Corollary A.3 we obtain

\[
\begin{align*}
w_0 \left( \Omega_{w, S}^\circ \cap \mathcal{H}(w_0 S w_0, h) \right) &= X_{z_S, h}^\circ \cap \mathcal{H}(S, h) = X_{z_S, h}^\circ
\end{align*}
\]

Since left multiplication respects closure relations in the flag variety, the corollary now follows by application of Theorem 4.8 to the Hessenberg Schubert variety \( \Omega_{w, S}^\circ \cap \mathcal{H}(w_0 S w_0, h) \) and the fact that \( w_S \leq u \iff z_S \geq w_0 u \) since left multiplication by \( w_0 \) is an order reversing involution of \( \mathfrak{S}_n \). \( \square \)

We also obtain the following description of the \( T \)-fixed point \( \Omega_{v, h}^T \) in the opposite Hessenberg Schubert variety \( \Omega_{v, h} \). This gives a reformulation of [10, Theorem 3.5] in the language of Bruhat order.

**4.13. Corollary.** Let \( h : [n] \rightarrow [n] \) be a Hessenberg function and \( S \in \mathcal{W}_h \). Suppose \( v \in \mathcal{W}(S, h) \) and let \( w_S = u^{-1} v \) for \( u \in \mathfrak{S}_n \) such that \( \ell(w_S) = \ell(v) + \ell(u) \). Then

\[
\Omega_{v, h}^T = u[w_S, w_0]
\]

where \([w_S, w_0]\) denotes the Bruhat interval of \( w \in \mathfrak{S}_n \) such that \( w_S \leq w \).
Proof. Since left multiplication respects closure relations in the flag variety, by Proposition 3.13 we have $\Omega_{w,h} = u \left( \Omega^0_{w_S} \cap \text{Hess}(u^{-1}S u, h) \right)$. Theorem 4.8 now yields

$$
\left( \Omega^0_{w_S} \cap \text{Hess}(u^{-1}S u, h) \right)^T = [w_S, w_0]
$$

and the corollary follows. \hfill \Box

5. Future Directions

In this section we sketch a select few among the many avenues of future research that our results inspire. We emphasize that, for an arbitrary subvariety of the flag variety which is paved by affines, it is in general difficult to compute the closures of the paving’s affine cells. For example, a well-known result of Spaltenstein shows that type A Springer fibers are paved by affines [22], but there is still no general characterization for the closures of the affine cells – except in very restricted cases. On the other hand, the closures of Schubert cells are fully governed by Bruhat order on $S_n$, and much of the geometry of Schubert variety is controlled by the combinatorics of this ordering (see e.g. [9, 18, 8]).

Our main result proves that, surprisingly, closures of certain Hessenberg Schubert cells in $\text{Hess}(S, h)$ are also dictated by the Bruhat order. In particular, for the maximal element $w_S \in \mathcal{W}(S, h)$, the equality (4.9) shows that the opposite Hessenberg Schubert variety $\Omega_{w_S, h}$ is precisely the intersection of the opposite Schubert variety $\Omega_{w_S}$ with the Hessenberg variety $\text{Hess}(S, h)$. Thus, not only do Hessenberg Schubert varieties satisfy closure relations governed by Bruhat order, they are in a tight relation with the Schubert varieties themselves. This leads to many open questions on Hessenberg Schubert varieties, some of which we record below.

Firstly, recall that singular Schubert varieties can be characterized in terms of pattern avoidance [19]. We may therefore ask:

5.1. Question. For which $S \in \mathcal{W}(h)$ is $\Omega_{w_S, h}$ singular? Can the set of all such $w_S$ in $S_n$ be characterized in terms of pattern avoidance?

Similarly, one can ask what features of the determinantal varieties known as matrix Schubert varieties (see [20, Ch.15-16]) generalize to the Hessenberg setting. This requires the development of commutative algebra techniques to study determinantal ideals in the Hessenberg context. As a first step, we may pose – for example – the following problem.

5.2. Problem. Define, and construct Gröbner bases for, the ideals defining the Hessenberg analogues of matrix Schubert varieties.

Thirdly, an opposite Hessenberg Schubert variety defines a class $\sigma^T_{w,h}$ in the equivariant cohomology $H^*_T(\text{Hess}(S, h))$, see [10]. In the special case of $GL_n(\mathbb{C})/B$, these are the famous equivariant Schubert classes, and the value $\sigma^T_{w,h}(v)$ of the localization of $\sigma^T_{w,h}$ at $v \in S_n$, computed by Andersen–Jantzen–Soergel [4] and independently by Billey in [7], is a fundamental formula in Schubert calculus and the study of the geometry of Schubert and flag varieties [26]. In the more general Hessenberg context, the classes $\sigma^T_{w,h}$ were studied by Cho, Hong, and Lee [10], and we may hence ask for an analogous formula, starting with the permutations $w_S$ studied above.

5.3. Problem. Let $\{\sigma^T_{w,h}\} \subset H^*_T(\text{Hess}(S, h))$ denote the basis of $H^*_T(\text{Hess}(S, h))$ corresponding to the Hessenberg Schubert varieties $\Omega_{w,h}$ as in [10]. For each $S \in \mathcal{W}(S, h)$, give an explicit combinatorial formula for the localization $\sigma^T_{w_S,h}(v)$ in the spirit of Billey–Anderson–Jantzen–Soergel.

Fourthly, recall that the regular semisimple Hessenberg varieties $\text{Hess}(S, h)$ form a flat family over $A^1$, whose singular fibers are regular nilpotent Hessenberg varieties [1]. In general, given
a regular matrix $X \in \mathfrak{gl}_n(\mathbb{C})$ of any Jordan type we call $\text{Hess}(X, h)$ a regular Hessenberg variety.

As the Jordan form of $X$ varies, the regular Hessenberg variety interpolates between the regular semisimple Hessenberg varieties studied in this manuscript and the regular nilpotent Hessenberg varieties, whose cohomology captures the $\mathfrak{S}_n$-invariants of the dot action [6, 2]. This motivates the following.

5.4. **Question.** Consider the regular Hessenberg variety $\text{Hess}(X, h)$ where $X$ is a regular matrix of any Jordan type. Is there a statement analogous to that of Theorem 4.8 which describes the Hessenberg Schubert varieties $\Omega_w \cap \text{Hess}(X, h)$?

Finally, we note that Hessenberg varieties can be defined more generally than the $GL(n, \mathbb{C})$ case studied here. Indeed, one can consider the semisimple Hessenberg varieties defined using the adjoint action of an arbitrary algebraic group $G$, as in [12]. Thus we close this list of open problems with the following.

5.5. **Question.** Does the analogous statement of Theorem 4.8 hold for other Lie types?

**APPENDIX A. PROOF OF LEMMA 2.10 BY MICHAEL ZENG**

We now present a proof of Lemma 2.10, written by Michael Zeng as part of an undergraduate research project with the second author. Let $w_0 = [n, n - 1, \ldots, 2, 1]$ denote the longest element of $\mathfrak{S}_n$. Consider the map

$$\varphi : \mathfrak{S}_n \to \mathfrak{S}_n, \quad \varphi(w) = w_0w.$$  

Since

$$N(w_0w) = \Phi^+ \setminus N(w)$$  

for all $w \in \mathfrak{S}_n$, it follows that $\varphi$ defines an order-reversing involution with respect to the weak order, that is, $u \leq_L v \iff \varphi(u) \geq_L \varphi(v)$ [5, Prop. 3.1.5].

**A.2. Lemma.** Let $S \in W_h$. The restriction of $\varphi$ to $W(S, h)$ is a bijection between $W(S, h)$ and $W(\Phi^+_h \setminus S, h)$.

**Proof.** Since $\varphi$ is an involution, it suffices to show that $\varphi(w) \in W(\Phi^+_h \setminus S, h)$ for all $w \in W(S, h)$. Intersecting both sides of equation (A.1) with $\Phi^+_h$ we obtain $N(w_0w) \cap \Phi^+_h = \Phi^+_h \setminus N(w) \cap \Phi^+_h$, which is equivalent to $N(w_0w) \cap \Phi^+_h = \Phi^+_h \setminus S$. Thus $\varphi(w) = w_0w \in W(\Phi^+_h \setminus S, h)$, as desired. $\square$

Using the previous lemma and Theorem 2.9, we obtain the following corollary.

**A.3. Corollary.** For each $S \in W_h$, $W(S, h)$ has a unique maximal element with respect to weak Bruhat order.

**Proof.** Since $S$ is a subset of Weyl type with respect to $h$, the complement $\bar{S} = \Phi^+_h \setminus S$ is also a subset of Weyl type with respect to $h$. By Theorem 2.9(3), the set $W(\Phi^+_h \setminus S, h)$ has a unique minimal element with respect to the weak Bruhat order, denoted $z_S$. By Lemma A.2, $\varphi_S(z_S) \in W(S, h)$.

Since $\varphi$ is an order reversing involution and $z_S$ is the unique minimal element in $W(\Phi^+_h \setminus S, h)$, we conclude that $\varphi(z_S)$ is the unique maximal element in $W(S, h)$. $\square$

Finally, we prove Lemma 2.10: $W(S, h)$ is an interval in the weak Bruhat order.

**Proof of Lemma 2.10.** Let $z_S$ and $w_S$ be the unique minimum and maximum elements, respectively, of $W(S, h)$ with respect to the weak (left) Bruhat order. Such elements must exist by Theorem 2.9(3) and Corollary A.3. The inclusion $W(S, h) \subseteq [z_S, w_S]_L$ is immediate. To show the converse, let $w \in [z_S, w_S]_L$. This means $z_S \leq_L w \leq_L w_S$, or equivalently, $N(z_S) \subseteq N(w) \subseteq N(w_S)$. Taking the intersection with $\Phi^+_h$ at each term in this chain, we obtain $N(z_S) \cap \Phi^+_h \subseteq N(w) \cap \Phi^+_h \subseteq N(w_S) \cap \Phi^+_h$. Since $z_S, w_S \in W(S, h)$ we have $N(z_S) \cap \Phi^+_h = N(w_S) \cap \Phi^+_h = S$, implying $N(w) \cap \Phi^+_h = S$. This proves $w \in W(S, h)$, as desired. $\square$
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