LETTER TO THE EDITOR

Quasinormal spectrum and quantization of charged black holes

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Abstract

Black-hole quasinormal modes have been the subject of much recent attention, with the hope that these oscillation frequencies may shed some light on the elusive theory of quantum gravity. We study analytically the asymptotic quasinormal spectrum of a charged scalar field in the (charged) Reissner–Nordström spacetime. We find an analytic expression for these black-hole resonances in terms of the black-hole physical parameters: its Bekenstein–Hawking temperature $T_{BH}$, and its electric potential $\Phi$. We discuss the applicability of the results in the context of black-hole quantization. In particular, we show that according to Bohr’s correspondence principle the asymptotic resonance corresponds to a fundamental area unit $\Delta A = 4\hbar \ln 2$.

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Everything in our past experience in physics tells us that general relativity and quantum theory are approximations, special limits of a single, universal theory. However, despite the flurry of research in this field we still lack a complete theory of quantum gravity. In many respects, the black hole plays the same role in gravitation that the atom played in the nascent of quantum mechanics [1]. It is therefore believed that black holes may play a major role in our attempts to shed light on the nature of a quantum theory of gravity.

The quantization of black holes was proposed long ago by Bekenstein [2, 3], based on the remarkable observation that the horizon area of a non-extremal black hole behaves as a classical adiabatic invariant. In the spirit of the Ehrenfest principle [4]—any classical adiabatic invariant corresponds to a quantum entity with a discrete spectrum—and based on the idea of a minimal increase in black-hole surface area [2], Bekenstein conjectured that the horizon area of a quantum black hole should have a discrete spectrum of the form

$$A_n = \gamma \ell_P^2 \cdot n, \quad n = 1, 2, 3, \ldots,$$

where $\gamma$ is a dimensionless constant, and $\ell_P = (G\hbar/c^3)^{1/2}$ is the Planck length (we use units in which $G = c = \hbar = 1$ henceforth). This type of area quantization has since been reproduced based on various other considerations (see, e.g., [5] for a detailed list of references).
In order to determine the value of the coefficient $\gamma$, Mukhanov and Bekenstein [6–8] have suggested, in the spirit of the Boltzmann–Einstein formula in statistical physics, to relate $g_n = \exp[S_{\text{BH}}(n)]$ to the number of the black-hole microstates that correspond to a particular external macrostate, where $S_{\text{BH}}$ is the black-hole entropy. In other words, $g_n$ is the degeneracy of the $n$th area eigenvalue. Now, the thermodynamic relation between black-hole surface area and entropy, $S_{\text{BH}} = A/4\hbar$, can be met with the requirement that $g_n$ has to be an integer for every $n$ only when

$$\gamma = 4 \ln k,$$

where $k$ is some natural number.

Identifying the specific value of $k$ requires further input. This information may emerge by applying Bohr’s correspondence principle to the (discrete) quasinormal mode (QNM) spectrum of black holes [9]. Gravitational waves emitted by a perturbed black hole are dominated by this ‘quasinormal ringing’, damped oscillations with a discrete spectrum (see, e.g., [10] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [11–14]. These black-hole resonances are the characteristic ‘sound’ of the black hole itself, depending on its parameters: mass, charge and angular momentum.

It turns out that for a Schwarzschild black hole, for a given angular harmonic index $l$ there exist an infinite number of (complex) quasinormal frequencies, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [15, 16]. On the other hand, it was found numerically [15, 17, 18] that the real part of the Schwarzschild gravitational resonances approaches an asymptotic constant value. Based on Bohr’s correspondence principle, it was argued [9] that the asymptotic resonances are given by\(^1\) (we assume a time dependence of the form $e^{-i\omega t}$)

$$\omega = \pm T_{\text{BH}}^l \ln 3 - i2\pi T_{\text{BH}}^l (n + \frac{1}{2}),$$

where $T_{\text{BH}}^l = 1/8\pi M$ is the Bekenstein–Hawking temperature of the Schwarzschild black hole. An analytical proof of this equality was later given in [19].

The emission of a quantum of frequency $\omega$ results in a change $\Delta M = h\omega R$ in the black-hole mass. Assuming that $\omega$ corresponds to the asymptotically damped limit (3) (see footnote 1), and using the first law of black-hole thermodynamics $\Delta M = \frac{1}{2} T_{\text{BH}}^l \Delta A$, this implies a change $\Delta A = 4\hbar \ln 3$ in the black-hole surface area. Thus, the correspondence principle, as applied to the black-hole resonances, provides the missing link, and gives evidence in favour of the value $k = 3$.

Furthermore, it was later suggested to use the black-hole QNM frequencies to fix the value of the Immirzi parameter in loop quantum gravity, a viable approach to the quantization of general relativity [20–22] (see, however, other contradicting results in [24]). It should be

\(^1\) To understand the original argument, it is useful to recall that in the early development of quantum mechanics, Bohr suggested a correspondence between classical and quantum properties of the hydrogen atom, namely, ‘transition frequencies at large quantum numbers should equal classical oscillation frequencies’. The black hole is in many senses the ‘hydrogen atom’ of general relativity. It was therefore suggested [9] to use the discrete set of black-hole resonances in order to shed some light on the quantum properties of a black hole. However, there is one important difference between the hydrogen atom and a black hole: while a (classical) atom emits radiation spontaneously according to the (classical) laws of electrodynamics, a classical black hole does not emit radiation. This crucial difference hints that one should look for the highly damped black-hole oscillations (let $\omega = \omega_g - i\omega_I$, then $\tau \equiv \omega_I^{-1}$ is the effective relaxation time for the black hole to return to a quiescent state after emitting radiation. Hence, the relaxation time $\tau \to 0$ as $\omega_I \to \infty$, implying no radiation emission, as should be the case for a classical black hole). Note also that the asymptotic real value of the Schwarzschild black-hole resonances is universal in the sense that it is independent of the harmonic index $l$, as expected, if it is to describe a fundamental characteristic of the black hole itself.
mentioned that the generality of the idea to link the black-hole quasinormal resonances with the quantization of surface area is still an open question [25–28].

The intriguing proposals outlined above [9, 20] have triggered a flurry of research attempting to calculate the asymptotic ringing frequencies of various types of black holes (for a detailed list of references see, e.g., [23]).

It should be emphasized, however, that former analytical studies of the asymptotic QNM spectrum did not include chemical potentials, such as rotation or electric charge. These potentials enter into the first law of black-hole thermodynamics for rotating black holes, or for a charged scalar field in the (charged) Reissner–Nordström (RN) spacetime (see equation (14) below). In contrast to the Schwarzschild black hole, in these cases there is no simple one-to-one correspondence between the energy of the emitted quanta and the resulting change in black-hole surface area. Thus, the inclusion of chemical potentials may allow a deeper test of the applicability of Bohr’s correspondence principle to the quantization of black holes. In this work, we study the asymptotic resonances of a charged scalar field in the RN spacetime, and provide analytical formulae for the corresponding QNM spectrum. This is done by using a similarity between the QNMs of the charged scalar field and the known asymptotic spectrum of the natural field.

The dynamics of a charged scalar field in the RN spacetime is governed by the Klein–Gordon equation [29]

\[ \Delta \frac{d^2 R_i}{dr^2} + (2r - 2M) \frac{dR_i}{dr} - l(l + 1)R_i + \frac{r^4}{\Delta} \left( \omega - \frac{eQ}{r} \right)^2 R_i = 0, \quad (4) \]

where \( \Delta \equiv (r - r_+)(r - r_-) \), and \( r_\pm = M \pm (M^2 - Q^2)^{1/2} \) are the black hole (event and inner) horizons. Here \( e \) is the charge coupling constant (\( e \) stands for \( e/\hbar \), and has dimensions of (length)^{-1}). The black-hole QNMs correspond to solutions of the wave equation with the physical boundary conditions of outgoing waves at spatial infinity and ingoing waves crossing the event horizon [30]. Such boundary conditions single out a discrete set of resonances \( \{\omega_n\} \).

The solution to the wave equation may be expressed as

\[ R_i = e^{i\omega r} (r - r_-)^{-1/2} e^{i\sigma \sqrt{r(r - r_+)(r - r_-)}} \sum_{n=0}^{\infty} d_n \left( \frac{r - r_+}{r - r_-} \right)^n, \quad (5) \]

where \( \sigma \equiv r_+^2 \left( \omega - \frac{eQ}{r_+} \right) / (r_+ - r_-) \).

The sequence of expansion coefficients \( \{d_n : n = 1, 2, \ldots \} \) is determined by a recurrence relation of the form [15]

\[ \alpha_n d_{n+1} + \beta_n d_n + \gamma_n d_{n-1} = 0, \quad (6) \]

with initial conditions \( d_0 = 1 \) and \( \alpha_0 d_1 + \beta_0 d_0 = 0 \). The quasinormal frequencies are determined by the requirement that the series in equation (6) be convergent; that is, \( \Sigma d_n \) exists and is finite [15].

We find that the physical content of the recursion coefficients \( \alpha_n, \beta_n \) and \( \gamma_n \) becomes clear when they are expressed in terms of the black-hole physical parameters: the Bekenstein–Hawking temperature \( T_{BH} = (r_+ - r_-)/4\pi \) and the black-hole electric potential \( \Phi = Q/r_+ \), where \( \Lambda = 4\pi r_+^2 \) is the black-hole surface area. The recursion coefficients obtain a surprisingly simple form in terms of these physical quantities,

\[ \alpha_n = (n + 1)(n + 1 - 2i\beta_+ \hat{\omega}), \quad (7) \]

\[ \beta_n = -2 \left( n + \frac{1}{2} - 2i\beta_+ \hat{\omega} \right) \left( n + \frac{1}{2} - 2i\omega r_+ + ieQ \right) - \frac{1}{2} - l(l + 1), \quad (8) \]

and

\[ \gamma_n = [n - 2i(2M\omega - eQ)](n - 2i\beta_+ \hat{\omega}), \quad (9) \]

where \( \beta_+ = (4\pi T_{BH})^{-1} \) is the black-hole inverse temperature and \( \hat{\omega} \equiv \omega - e\Phi \).
We shall show that the quasinormal spectrum of a charged scalar field in the RN spacetime is closely related to the corresponding spectrum of a natural scalar field. The asymptotic spectrum of a natural scalar field in the RN spacetime is determined by the equation \cite{31}:

\[ 2e^{\pm 2\pi \beta \omega} + 3e^{\pm 2\pi M \omega} = -1. \] (10)

Equation (10) suggests that the natural spectrum (for which \( \hat{\omega} = \omega \)) depends on the combinations \( \beta \omega + 2M \omega \) appearing in equations (7)–(9), but does not depend explicitly on \( \omega r_+ \). From equations (7)–(9) one learns that the analogy between the asymptotic spectrum of a natural scalar field and the corresponding spectrum of a charged field is obtained by applying the transformations \( \beta \omega + \hat{\omega} \rightarrow \beta \omega + \hat{\omega} \) and \( 2M \omega \rightarrow 2M \omega - eQ \) in equation (10). Using these transformations, one finds that the asymptotic quasinormal mode spectrum of a charged scalar field is given by

\[ 2e^{\pm 2\pi \beta \omega (\omega - \Phi)} + 3e^{\pm 2\pi (2M \omega - eQ)} = -1. \] (11)

For charged black holes that satisfy the condition \( eQ \gg r_+ / r_- \) (this condition also reads \( Q / M \gtrsim (\hbar / \alpha A)^{1/6} \), where \( \alpha \) is the fine structure constant), one of the exponents (depending on the sign of \( eQ \)) in equation (11) is negligible as compared to the other, thus yielding two families of QNM resonances

\[ \omega = \pm T_{BH} \ln 2 + \frac{eQ}{r_+} - i2\pi T_{BH} \left( n + \frac{1}{2} \right), \] (12)

and

\[ \omega = \mp T_{BH}^s \ln 3 + \frac{eQ}{r^s_+} - i2\pi T_{BH}^s \left( n + \frac{1}{2} \right), \] (13)

where \( r^s_+ = 2M \) is the Schwarzschild radius, and the upper/lower signs correspond to positive/negative values of \( eQ \), respectively.

The emission of a quantum of frequency \( \omega \) and an electric charge \( e \) results in a change \( \Delta M = \hbar \omega R \) in the black-hole mass, and a change \( \Delta Q = e \) in its charge. Substituting the fundamental resonance, equation (12), into the first law of black-hole thermodynamics

\[ \Delta M = \frac{1}{4} T_{BH} \Delta A + \Phi \Delta Q, \] (14)

one obtains the corresponding change in black-hole surface area

\[ \Delta A = 4\hbar \ln 2. \] (15)

Remarkably, this fundamental change in black-hole surface area is in accord with the Bekenstein–Mukhanov general prediction, equation (2). In particular, it should be stressed that this area spacing turns out to be independent of the black-hole parameters, \( M \) and \( Q \), and also independent of the charged-field parameters, \( e \) and \( l \).

Note that the fundamental area spacing equation (15) agrees with that found for natural fields in the RN spacetime \cite{31}. (In \cite{31} it was suggested that one treats \( M \) slightly imaginary, thus yielding \( \omega R \rightarrow T_{BH} \ln 2 \) for natural fields.)

Note also that the second quasinormal branch, equation (13), yields a non-universal area spacing, one which depends on the black-hole parameters. This could indicate that the area quantization of charged black holes may be more complicated than in the simple Schwarzschild case. This can also be inferred from the results of \cite{31}. It seems that further research is needed in order to resolve this issue.

\footnote{The final result of \cite{31} for the asymptotic QNM spectrum of natural fields in the RN spacetime can be written as equation (10) by simply multiplying both sides of equation (66) in \cite{31} by \( e^{-\beta \omega} \).}

\footnote{Note that these solutions have the desired symmetry, namely that if \( \omega(eQ) \) is a QNM frequency, then \( -\omega^*(-eQ) \) is also a solution.
In summary, motivated by novel results in the theory of black-hole quantization, we have studied analytically the QNM spectrum of a charged scalar field in the charged RN spacetime. It was shown that the asymptotic resonances can be expressed in terms of the black-hole physical parameters: its temperature $T_{BH}$ and electric potential $\Phi$.

The case of a charged field is interesting from a physical point of view, since it introduces a chemical potential into the system (in the form of the black-hole electric potential $\Phi$). This enabled us to test the applicability of Bohr’s correspondence principle to the quantization of black holes in generalized situations, in which there is no one-to-one correspondence between the energy of the emitted quantum and the resulting change in black-hole surface area. We have shown that according to the Bohr correspondence principle, the emission of a charged quantum from a charged RN black hole induces a fundamental change in black-hole surface area, $\Delta A = 4\hbar \ln 2$. Remarkably, this area unit is universal in the sense that it is independent of the black-hole parameters, and does not depend on the charged-field parameters.

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