A NOTE ON UNFREE GAUGE SYMMETRY

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Abstract. We study the general structure of field theories with the unfree gauge symmetry where the gauge parameters are restricted by differential equations. The examples of unfree gauge symmetries include volume preserving diffeomorphisms in the unimodular gravity and various higher spin field theories with transverse gauge symmetries. All the known examples of the models with unfree gauge symmetry share one common feature. They admit local quantities which vanish on shell, though they are not linear combinations of Lagrangian equations and their derivatives. We term these quantities as mass shell completion functions.

In the case of usual gauge symmetry with unconstrained gauge parameters, the irreducible gauge algebra involves the two basic constituents: the action functional and gauge symmetry generators. For the case of unfree gauge symmetry, we identify two more basic constituents: operators of gauge parameter constraints and completion functions. These two extra constituents are involved in the algebra of unfree gauge symmetry on equal footing with action and gauge symmetry generators. Proceeding from the algebra, we adjust the Faddeev-Popov (FP) path integral quantization scheme to the case of unfree gauge symmetry. The modified FP action involves the operators of the constraints imposed on the gauge parameters, while the corresponding BRST transformation involves the completion functions. The BRST symmetry ensures gauge independence of the path integral. We provide two examples which admit the alternative unconstrained parametrization of gauge symmetry and demonstrate that they lead to the equivalent FP path integral.

1. Introduction

In this article we study the gauge symmetry with unfree transformation parameters. The usual definition of gauge symmetry implies that the transformation parameters are arbitrary functions of space-time coordinates. The set of the gauge symmetry generators can be over-complete that results in gauge symmetries of symmetries. This is known as a reducible gauge symmetry. At the level of corresponding Batalin-Vilkovisky (BV) formalism and even when the Faddeev-Popov (FP) method works well, the symmetry for symmetry requires to introduce ghosts for ghosts. The unfree gauge symmetry is a different phenomenon. It is the symmetry with the transformation
parameters subject to the differential equations. Some articles term these equations as constraints imposed on the gauge transformation parameters \[3\], \[4\].

One of the best known examples of the field theory with unfree gauge symmetry is the unimodular gravity \[5\], where the gauge symmetry reduces to the volume-preserving diffeomorphisms, so called \(T\)-diff's. The \(T\)-diff is the usual diffeomorphism with the unfree transformation parameter which is supposed to be a transverse vector field. The gauge parameter is constrained by the differential equation: the divergence should vanish of the vector. Also notice that the \(T\)-diff’s form a subalgebra in the full algebra of diffeomorphisms. In some cases, this subalgebra can be treated differently from the entire diffeomorphism in the Hamiltonian BFV-BRST\(^1\) quantization of gravity \[6\] even if the unimodularity condition is not imposed. While the unimodular gravity is equivalent to General Relativity (GR) at classical level, at least in most of physical problems, some subtleties are known about possible distinctions at quantum level, for discussion see \[7\] and references therein. Even though the unimodular gravity can be considered as an equivalent reformulation of the GR, these two theories differ by the gauge symmetry, so the general structure of unfree gauge symmetry remains the problem of interest for this model. Various advantages and disadvantages are known of different forms of gauge symmetry in locally equivalent formulations of gravity, see \[8\] for discussion and further references.

Besides the \(T\)-diff’s in gravity, various models with unfree gauge symmetry are known among higher spin field theories, see \[9\], \[10\] and the references therein. The model of irreducible spin 2 traceless field is known \[11\], \[12\] with the vector gauge parameter restricted by transversality equation. This model corresponds to the linearized unimodular gravity. The work \[9\] can be viewed as an extension of the model \[11\] to the higher spin fields. The unfree gauge symmetry of the model of \[9\] can be viewed as a higher spin extension of the linearized \(T\)-diff's. The Maxwell-like models of ref. \[10\] describe higher spin fields in terms of a tracefull tensors. These models also have the gauge symmetry with transverse parameters.

In the article \[3\], the issue of the unfree gauge theory is considered for general linear local field theory. It is found that any linear field theory always admits unconstrained parametrization of gauge symmetry, with the gauge parameters being arbitrary functions unrestricted by any equation. The unconstrained form of the gauge symmetry is reducible, in general, and the reducibility order is always finite, being bounded by the space-time dimension. This means, the

\(^1\)Batalin-Fradkin-Vilkovisky (BFV); Becchi-Ruet-Stora-Tyutin (BRST)
generators of unfree gauge symmetry in linear theory can be always replaced by equivalent (maybe over-complete) generating set with unconstrained parameters. In the article [3], the reducible unconstrained gauge symmetry generating sets are explicitly identified for various higher spin theories where the gauge symmetry has been previously known only in the constrained form.

As far as every theory admits unconstrained (possibly reducible) parametrization of gauge symmetry at least at linearized level, it may seem unnecessary to study the unfree gauge symmetry at all. We can mention at least three reasons why it can be the issue of interest.

First, the unfree gauge symmetry parametrization is equivalent, in principle, to the unconstrained but reducible parametrization. From technical viewpoint, and geometrically, these two parameterizations can be quite different, so each one may have its own advantages and disadvantages. For example, in the model of irreducible higher spin field described by traceless tensors proposed in ref. [9], the unfree gauge transformations involve only first order derivatives of the gauge parameters, while the parameters are symmetric traceless tensors. These parameters are constrained by the first order differential equations. The unconstrained reducible transformations are found for this model in the work [3]. The gauge parameters of unconstrained symmetry are the tensors with a two-row Young tableaux, and the transformations involve the higher order derivatives of the parameters. The sequence of symmetry for symmetry transformations of the free gauge parameters involve the tensor parameters with various Young tableaux, depending on spin. Obviously, these two algebraic structures are essentially different while they describe the gauge symmetry of the same field theory. Each of them can provide different insight into the dynamics once the issues are considered of including consistent interactions or quantization.

Second, the unconstrained gauge symmetry parametrization, with possible reducibility, is proven to always exist [3] only in the linear theory. So, if the theory has the gauge symmetry with constrained gauge parameters at interacting level, the unconstrained equivalent can be non-existent at all for the nonlinear model, and vice versa. The examples are known of the field theories which admit different unconstrained parametrization of gauge symmetry, including reducible and irreducible generating sets at linear level [3], [13]. Also the example is known of the model where the reducible generating set of the linear theory admits consistent inclusion of interaction, while the irreducible generating set obstructs the same interaction [13]. So, the reducible and irreducible parametrization of gauge symmetry can be inequivalent with respect to deformations, while both generating sets can correspond to the same model at linear level.
The third reason is that the clarification of general unfree gauge symmetry structure can be considered as a matter of principle for the general theory of gauge systems. The general structure of gauge algebra is well known in the theories with unconstrained gauge parameters both with irreducible and reducible generating sets of gauge symmetry transformations. In particular, all the structure relations are known for the gauge generators, the generators of symmetry for symmetry, and the structure functions involved in the off-shell disclosure of the symmetry. At the level of the BRST field-anti-field formalism, all the structure relations are generated by the BV master equation. As a standard reference to the gauge algebra structure, we mention the textbook [13]. The general gauge algebra also has well developed formalism for not necessarily Lagrangian gauge field theories [15], [16]. One of the important distinctions of the gauge algebra in the non-Lagrangian case is that the gauge symmetries are not necessarily paired with gauge identities. In the Lagrangian case the same gauge generators are involved in the gauge symmetry transformations and in the gauge identities between the equations of motion. In non-Lagrangian case, the generators of gauge symmetries and gauge identities can be different in general, so the gauge algebra involves more generating elements. The non-Lagrangian extension of the master equation [16], [17] generates a more reach gauge algebra that involves more structures comparing to the Lagrangian case. Any deformation of the theory, be it inclusion of interaction or quantization, should consistently deform all the structure relations of the gauge algebra. Somewhat similar phenomenon we shall observe in the theory of the systems with unfree gauge symmetry. There are extra generating elements in the gauge algebra besides the action and the generators of gauge symmetry transformations. These extra elements contribute to the gauge identities. In this sense, the unfree gauge symmetry is similar to the gauge algebra of non-Lagrangian systems. Any deformation of the theory with unfree gauge symmetry, either by inclusion of interaction or by quantization, should be consistent with deformation of corresponding gauge algebra structures. That is why it seems important to clarify the structure of the gauge algebra with unfree parameters.

In this work we consider three aspects of the problem of unfree gauge symmetry. At first we identify the basic generating elements of the unfree gauge symmetry algebra. Besides the action and the generators of gauge symmetry, there are two extra constituents having no analogues in the usual unconstrained gauge algebra. The first extra element is the operator of constraint imposed onto the gauge parameter. The second one is the structure we term the completion function.
It is explained in the next section. These two elements define both the Noether identities and unfree gauge symmetry of the action. The gauge symmetry transformations and gauge identities have to satisfy the compatibility conditions that can be viewed as the higher structure relations of the gauge algebra in the case of unfree gauge parameters. These relations are distinct from the analogues in the case of the gauge symmetry with unconstrained gauge parameters. The consistent deformation of any model with unfree gauge parameters should be compatible with these structure relations.

In the second instance, we extend the FP path integral quantization scheme to the theories with unfree gauge transformation parameters. The key idea of the extension is that the ghosts for the unfree gauge symmetry should be unfree themselves. To put it different, the ghost should be constrained by the same equations as imposed on the gauge parameters. Of course, the FP recipe remains applicable as far as there are no off-shell disclosure of the gauge algebra. For the general case of open unfree algebra, the extension of the BV master equation has to be worked out. This problem will be addressed elsewhere, though in the concluding section we mention some clues to the issue. We also consider the BRST symmetry of the FP action. In the case of unfree symmetry, the BRST transformation squares to zero in general not identically, but modulo constraints imposed on the ghosts.

Third, in Section 4, we consider the specific models with unfree gauge symmetry to exemplify the general formalism and to verify the general conclusions by alternative methods admitted by specific models. In this section, we also exemplify the way in which the BRST symmetry can distinguish the physical vertices from the nonphysical ones in the models with unfree gauge parameters.

2. COMPLETION OF LAGRANGIAN EQUATIONS AND UNFREE GAUGE SYMMETRY

We begin this section by noticing the phenomenon which is common for all the known examples of the field theories with unfree gauge transformation parameters. In all these models, one can find the on-shell vanishing local quantities such that do not reduce to differential consequences of Lagrangian equations. These quantities vanish by virtue of Lagrangian equations and the boundary conditions imposed on the fields.
For example, for the spin 2 field theory with the traceless tensor $h_{\mu\nu}$ proposed in the article [11], the Lagrangian equations have the differential consequence,

$$\partial_\mu \tau \approx 0, \quad \tau = \partial_\mu \partial_\nu h^{\mu\nu}, \quad h^{\mu}_{\mu} \equiv 0.$$  \hspace{0.5cm} (1)

Throughout the paper, the on-shell equality is denoted as $\approx$. Maxwell-like field theories of higher spins proposed in [10] have the same consequence while the tensor $h_{\mu\nu}$ is tracefull. This means, the quantity $\tau$ should be constant on-shell. Given zero boundary conditions for the fields at the space infinity, the constant should be zero\(^2\). With the usual boundary conditions, we have the on-shell vanishing local quantity

$$\tau \approx 0,$$ \hspace{0.5cm} (2)

while $\tau$ does not reduce to linear combination of Lagrangian equations and their derivatives. Also notice that the equation (2) does not restrict solutions of the Lagrangian equations, given the boundary conditions. For the Maxwell-like equations of higher spin fields, this fact is emphasized in the article [4].

Given these observations made in the specific models with unfree gauge symmetry, below we consider the general field theory where the system of Lagrangian equations is incomplete in certain sense. We mean that the local on-shell vanishing quantities $\tau_a$ exist such that

$$\tau_a(\phi) \approx 0, \quad \tau_a(\phi) \neq K^i_a \partial_i S(\phi).$$ \hspace{0.5cm} (3)

Hereinafter, we use the DeWitt condensed notation\(^3\), in particular the indices $a$ and $i$ are condensed. By $K^i_a(\phi)$ we mean the (rectangular) matrix of the local differential operator. To put it slightly different, the Lagrange equations $\partial_i S(\phi) \approx 0$ are incomplete in the sense that the on shell vanishing local quantity does not necessarily reduce to the linear combination with local coefficients of the left hand sides of Lagrangian equations.

\(^2\)Notice that the Cauchy data are not included into the boundary conditions. The boundary conditions define the class of admissible fields, while the Cauchy data concern the initial state of the fields. This can be rephrased in slightly different wording: the boundary conditions define the configuration space of the fields, while Cauchy data define the initial field configuration and velocity.

\(^3\)All the indices are condensed, in the sense that they include the space-time coordinates. In particular, the fields $\phi^i$ are labeled by the index $i$ which includes all the discrete indices, and the space-time point $x$. For example, for the vector field $A_\mu(x)$, the condensed index would include $\mu$ and $x$. Summation in the condensed index includes integration over space-time. The derivatives in $\phi^i$ are understood as variational, so $\partial_i S(\phi)$ is the left hand side of Lagrange equations.
We term the local quantities $\tau_a(\phi)$ as the generating set of completion functions if any on-shell vanishing local quantity can be spanned in the left hand sides of Lagrange equations and $\tau$’s:

$$O(\phi) \approx 0 \iff O(\phi) = V^i(\phi) \partial_i S(\phi) + V^a(\phi) \tau_a(\phi) .$$

(4)

The coefficients $V^i, V^a$ stand for the local differential operators.

The usual definitions of the gauge field theory, see [14], assume that the Lagrange equations are complete, in the sense that any on-shell vanishing local quantity is a linear combination of the lhs of the equations and their derivatives. As we learn from the examples, this assumption is not true for the models with unfree gauge parameters. The key observation is that the generating set for the on-shell vanishing local quantities includes both Lagrangian equations and the completion functions (3).

Notice that the generating set of completion functions is defined modulo linear combinations. The generating sets of completion functions are considered equivalent if they differ by the lhs of Lagrangian equations,

$$\tau'_a(\phi) \sim \tau_a(\phi) , \quad \tau'_a(\phi) = \tau_a(\phi) + \theta^i_a(\phi) \partial_i S(\phi) .$$

(5)

The set of completion functions (3) can be over-complete, and/or it can be dependent with the Lagrangian equations. This means, the identities are possible among Lagrangian equations and the completion functions:

$$\Gamma^i_a(\phi) \partial_i S(\phi) + \Gamma^a_\alpha(\phi) \tau_\alpha(\phi) \equiv 0 .$$

(6)

The coefficients $\Gamma^i_a(\phi), \Gamma^a_\alpha(\phi)$ are the matrices of the differential operators. We interpret the above relations as the most general form of Noether identities in the system where the Lagrangian equations are incomplete in the above mentioned sense. In Section 4 we provide explicit examples of the field theories where the gauge identities involve the completion functions.

It may seem that the equations (3) could be considered on equal footing with the Lagrangian equations

$$\partial_i S(\phi) \approx 0 ,$$

(7)

so we have just general non-Lagrangian theory with equations of motion (3) and (7) with gauge identities (6). General gauge algebra for not necessarily Lagrangian theory defined just by equations of motion is well known [16]. The issues of locality of gauge algebra of not necessarily
Lagrangian theories are described in details in reference [18]. In fact, there is a subtlety which makes a difference between the general non-Lagrangian equations and incomplete Lagrangian system. In the latter case, the mass shell is defined by Lagrangian equations (7), while the completion equations (3) do not restrict the solutions of Lagrangian equations, given the boundary conditions. Once the mass shell is a zero locus of critical points of the action \( S(\phi) \), the gauge symmetry should leave the action invariant, while the symmetry of the general equations is defined irrespectively to the existence of action at all. It is the difference which leads to the constraints on gauge parameters as we shall see below.

Notice that the gauge identity generators \( \Gamma \) are defined by the relations (6) modulo natural ambiguity. The generators \( \Gamma \) and \( \Gamma' \) are considered equivalent once they differ by certain on-shell vanishing terms:

\[
\Gamma'_{\alpha}(\phi) - \Gamma_{\alpha}(\phi) = E^{ij}_{\alpha}(\phi)\partial_i S(\phi) + E^{ia}_{\alpha}(\phi)\tau_a(\phi), \quad E^{ij}_{\alpha} = -E^{ji}_{\alpha};
\]

\[
\Gamma'_{a}(\phi) - \Gamma_{a}(\phi) = E^{ab}_{\alpha}(\phi)\tau_b(\phi) - E^{ia}_{\alpha}(\phi)\partial_i S(\phi), \quad E^{ab}_{\alpha} = -E^{ba}_{\alpha}.
\]

If the identity generators \( \Gamma \) are replaced by \( \Gamma' \) in relations (6), all the coefficients \( E \) will drop out from the identities. To put it different, the right hand sides of the relations (8), (9) are understood as trivial generators of gauge identities. If the Lagrange equations were complete in the above mentioned sense, no completion functions \( \tau_a \) would be admitted by the theory. If \( \tau_a \) were not involved, the relations (5) would correspond to the usual definition of trivial gauge generators in Lagrangian theory. In the incomplete case, the gauge identities can involve completion functions, that is why the definition is modified of the trivial generators.

Also notice that the change of the generating set of completion functions (5) results in the corresponding change of the gauge generators \( \Gamma_{\alpha} \)

\[
\tau_{a}(\phi) \mapsto \tau'_{a}(\phi) = \tau_{a}(\phi) + \theta_{a}(\phi)\partial_{i}S(\phi), \quad \Gamma_{\alpha} \mapsto \Gamma'_{\alpha} = \Gamma_{\alpha} + \theta_{a}(\phi)\Gamma_{a},
\]

(10)

The left hand sides of the Lagrangian equations \( \partial_{i}S(\phi) \) and completion functions \( \tau_{a}(\phi) \) constitute the generating set of the on-shell vanishing quantities, as stated by relations (11). Much like that, the operators \( \Gamma_{\alpha}, \Gamma_{a} \) are assumed to form the generating set for the gauge identities. Any generator of gauge identity is assumed to be a linear combination of \( \Gamma_{\alpha} \) modulo trivial generators:

\[
L^{i}(\phi)\partial_{i}S(\phi) + L^{a}(\phi)\tau_{a}(\phi) \equiv 0 \Rightarrow L^{i}(\phi) \approx k^{i}(\phi)\Gamma^{i}_{\alpha}(\phi), \quad L^{a}(\phi) \approx k^{a}(\phi)\Gamma^{a}_{\alpha}(\phi).
\]

(11)
In this article we assume that the generators $\Gamma_\alpha$ are independent in the sense that they can not be linearly combined with on-shell non vanishing coefficients into a trivial generator. In the other words, the gauge identities (6) are assumed irreducible. In principle, this assumption restricts generality, though no example is known at the moment of field theory with reducible unfree gauge symmetry.

Now, let us see that the gauge identities (6) define the gauge symmetry of the action, while the gauge transformation parameters cannot be free, once the completion functions are involved. Consider the gauge transformation of the fields

$$\delta_\epsilon \phi^i = \Gamma^i_\alpha (\phi) \epsilon^\alpha,$$

(12)

where $\epsilon^\alpha$ are the gauge transformation parameters. With the account of the identity (6), the gauge variation of the action reads:

$$\delta_\epsilon S(\phi) \equiv \epsilon^\alpha \Gamma^i_\alpha (\phi) \partial_i S(\phi) \equiv -\epsilon^\alpha \Gamma^a_\alpha (\phi) \tau_a (\phi).$$

(13)

If the completion functions $\tau_a (\phi)$ were not not involved into the gauge identities (6), i.e. if $\Gamma^a_\alpha = 0$, the action would remain intact under the gauge variation (12) with free parameters $\epsilon^\alpha$. Once $\Gamma^a_\alpha \neq 0$, the action is invariant under the transformation (12) if the gauge parameters are constrained by the equations:

$$\epsilon^\alpha \Gamma^a_\alpha (\phi) = 0.$$  

(14)

As we see, the gauge identities (6) involving completion functions (3) result in the gauge symmetry of the action

$$\delta_\epsilon S(\phi) \equiv 0,$$

(15)

though the gauge parameters have to be constrained by equations (14). The quantity $\Gamma^a_\alpha$, being involved in the gauge identity (6) as a coefficient at completion function, defines the restriction imposed onto gauge parameter. With this regard, we term $\Gamma^a_\alpha$ as operator of gauge parameter constraint.

As we have mentioned above, the generating set of completion functions is defined modulo lhs of Lagrangian equations (5). This leads to the ambiguity in the definition of the gauge symmetry generators (10). This ambiguity does not contribute to the gauge transformations because the gauge parameters are unfree (14).
As we have already said, the completion equations (3) do not impose restrictions on the solutions of the Lagrange equations (7) with given boundary conditions. This means that the mass shell remains invariant under the transformations which leave the action intact. In particular, any on shell vanishing local quantity should remain vanishing on shell after the gauge transformation (12), (14),

\[ O(\phi) \approx 0 \quad \Rightarrow \quad \delta_c O(\phi) \approx 0. \]  

Once the unfree gauge variation (12), (14) vanishes on shell of the local quantity \( O \), off shell this means

\[ \delta_c O(\phi) \approx 0 \quad \Leftrightarrow \quad \Gamma^i_A \partial_i S(\phi) + V^a_A(\phi) \tau_a(\phi) + W_a(\phi) \Gamma^a(\phi). \]  

For the gauge symmetry with constrained gauge parameters, the last two terms can arise once the gauge symmetry is unfree. This is a distinction from the theory with unconstrained gauge parameters. Also notice that the last one of these two terms does not vanish on shell.

For the requirement of the gauge invariance of the mass shell (16) to be satisfied, it is sufficient that it is satisfied for the generating set of local quantities vanishing on the shell. The set includes Lagrangian equations \( \partial_i S(\phi) \) and the completion functions (3), so these quantities have to be on shell gauge invariant,

\[ \delta_c \partial_i S(\phi) \approx 0, \quad \delta_c \tau_a(\phi) \approx 0. \]  

Making use of relations (17) we get the off shell action of gauge generators on Lagrangian equations and completion functions

\[ \Gamma^i_A(\phi) \partial_i \tau_a(\phi) = U^{i}_{\alpha a}(\phi) \partial_i S(\phi) + U^b_{\alpha a}(\phi) \tau_b(\phi) + W_{ab}(\phi) \Gamma^a(\phi). \]  

The specifics of the unfree gauge symmetry is seen at first in the terms which involve the operator of gauge parameter constraint \( \Gamma^a_A \). These terms do not necessarily vanish on shell. They originate from the fact that the mass shell is invariant under the transformations with the parameters restricted by the equations (14). We also mention that the structure functions \( W_{ab}(\phi) \) involved in the off shell non-vanishing terms in relation (19) are on shell symmetric

\[ W_{ab}(\phi) - W_{ba}(\phi) \approx 0. \]
This property can be deduced by making use of consequences of the identity (6) and the assumption of completeness of the generating set (11).

Let us also notice, that the structure functions $U$ in the right hand sides of the relations (19), (20) vanish in the linear theory. Unlike that, the structure functions $W$ (that do not have any analogue in the theory with free gauge parameters) do not necessarily vanish even in the linear theory. In Section 4 we provide an explicit example of linear field theory with the structure relations (19) involving non-trivial structure functions $W$.

Notice that any gauge symmetry transformation of the mass shell should be spanned by the gauge generators $\Gamma^i_\alpha$ once the gauge parameters are restricted by the equations (14). This fact is a consequence of the completeness assumption (11). For the gauge symmetry transformations (12), this means the commutator of the generators should be spanned by the generators modulo trivial ones defined by relations (8), and up to the terms that do not contribute to the transformations once the parameters are unfree (14). Explicitly, the commutators read

$$
\Gamma^i_\alpha(\phi)\partial_i \Gamma^j_\beta(\phi) - \Gamma^j_\beta(\phi)\partial_i \Gamma^i_\alpha(\phi) = U^{\gamma}_{\alpha\beta}(\phi) \Gamma^j_\gamma(\phi) \\
+ E^{ij}_{\alpha\beta}(\phi) \tau_a(\phi) + E^{ij}_{\alpha\beta}(\phi) \partial_i S(\phi) = R^i_{\alpha a}(\phi) \Gamma^a_\beta(\phi) - R^i_{\beta a}(\phi) \Gamma^a_\alpha(\phi).$$

The following relations should be hold between the unfree gauge symmetry generators $\Gamma^i_\alpha$ and the gauge parameter constraints operators $\Gamma^a_\alpha$:

$$
\Gamma^i_\alpha(\phi)\partial_i \Gamma^a_\beta(\phi) - \Gamma^i_\beta(\phi)\partial_i \Gamma^a_\alpha(\phi) = U^{\gamma}_{\alpha\beta}(\phi) \Gamma^a_\gamma(\phi) + \\
R^a_{\alpha b}(\phi) \Gamma^b_\beta(\phi) - R^a_{\beta b}(\phi) \Gamma^b_\alpha(\phi) + E^{ab}_{\alpha\beta}(\phi) \tau_b(\phi) - E^{ai}_{\alpha\beta}(\phi) \partial_i S(\phi).$$

To get this relation, we contract the gauge identity (6) with a test function with the property (14), and then compute the gauge variation of obtained expression. After that the roles of the test function and gauge parameter are interchanged. The difference of the gauge variations is an identity between the Lagrangian equations and completion functions (3). Under the assumption (11) of completeness this means that the coefficients at completion functions are proportional to the gauge generators $\Gamma^a_\alpha$ modulo trivial generators (8) and gauge parameter constraints (14). The formula (23) express no more than this fact.

In the first relations of the unfree gauge algebra (6), two extra constituents are involved – the completion functions (3), and the operators of gauge parameter constraints (14). These quantities
have no direct analogue either in Lagrangian gauge theory with unconstrained gauge parameter or in the general non-Lagrangian gauge system. Also notice that non-Lagrangian field equations could be also incomplete in the same sense, in principle. Hence the unfree gauge symmetry could occur for non-variational field equations, though no explicit examples are known yet of this phenomenon. Further compatibility conditions are possible for the unfree gauge algebra relations involving higher structure functions, much like the case with free gauge transformations parameters.

With unconstrained gauge parameters, all the structure relations of gauge algebra are generated by the BV master equation. In non-Lagrangian case, instead of the master action, the operator $Q$ is constructed of the BRST transformation [16] with the initial data defined by the classical field equations, their gauge symmetries and gauge identities. The relation $Q^2 = 0$ replaces the master equation in this case, and all the gauge algebra structure relations are generated by this equation. The set of ghosts is more general in non-Lagrangian theory than in the Lagrangian case. For the recipe of BRST embedding for not necessarily Lagrangian field equations we refer to the article [16]. In particular, anti-fields are assigned to the field equations, not to the fields, while there is no pairing between fields and equations once the equations are not supposed to be the variational derivatives of any action. As the gauge identities are not necessarily paired with gauge symmetries in non-Lagrangian theory, the ghosts are not necessarily dual to the ghost anti-fields in this case [16]. Much like the non-Lagrangian field theory, the case of unfree gauge symmetry would require a more general set of ghosts than the case without constraints on the gauge parameters. In this article, we do not work out a procedure for the field-anti-field BV-BRST embedding of a system with general unfree gauge symmetry. We restrict the consideration by the case where no higher structure functions appear, and the FP recipe is sufficient. This is briefly considered in the next section.

3. **Faddeev-Popov path integral for systems with unfree gauge symmetry**

We begin with a geometric remark that the ghosts are the coordinates on the fibers of the same bundle as the gauge transformation parameters [16]. The difference is that the Grassmann parity of the ghosts is shifted by 1 with respect to the parity of the gauge parameters, and the ghosts are assigned with the ghost number grading 1. Because of this geometric reason, the ghosts have to be constrained by the same equations as the gauge transformation parameters [14]. So, once
the ghosts $C^\alpha$ are assigned to the unfree gauge transformations $[12]$, they should be subjected to the equations
\[
\Gamma^\alpha_\alpha(\phi)C^\alpha = 0, \quad gh(C^\alpha) = 1, \quad \varepsilon(C^\alpha) = 1, \quad (24)
\]
where $\Gamma^\alpha_\alpha(\phi)$ is the operator of gauge parameter constraint $[14]$. The path integration should be done over the surface of the ghost constraints, not by free ghosts $C^\alpha$.

Now, let us discuss the gauge fixing in the theory with unfree gauge symmetry. Let us denote the number of unfree gauge parameters by $m$, and the number of gauge parameter constraints will be $n$. If one could locally solve the equations $[14]$ and find the unconstrained gauge parameters, without symmetries for symmetries, their number would be $\bar{m} = m - n$. This means, $\bar{m}$ independent conditions are required to fix the gauge. Denote the independent gauges $\chi^I(\phi)$. The index $I$ is condensed, so it includes the space coordinates $x^\mu$. The dimension of digital part of the index should be $\bar{m}$. If we use the independent gauge fixing conditions, the number of unfree gauge parameters will exceed the number number of gauges, so FP matrix will be rectangular,
\[
\frac{\delta \chi^I}{\delta C^\alpha} = \Gamma^i_\alpha(\phi)\partial_i \chi^I(\phi). \quad (25)
\]

The rectangular matrix cannot be invertible, nor can it have the determinant, while it can be considered non-degenerate in certain sense. Consider the equation for the null-vectors $u_I(\phi)$ of the matrix,
\[
\Gamma^i_\alpha(\phi)\partial_i \chi^I u_I(\phi) \approx 0. \quad (26)
\]
If the general solution for $u_I$ does not involve arbitrary functions of all the space-time coordinates $x^\mu$, the FP matrix is considered non-degenerate. To put it different, the general gauge orbit is transverse to zero locus of $\chi^I(\phi)$. Some gauge variations $[12]$ can be tangential to the surface $\chi^I(\phi) = 0$, though the corresponding parameters are constrained much stronger that just by the condition $[14]$, so they can involve arbitrary functions of less than $d$ coordinates in $d$-dimensional space-time. This definition of admissible gauge fixing condition is applicable also in the case with free gauge parameters. For example, the Lorentz gauge in Maxwell electrodynamics has the d’Alembert operator as the FP matrix. Relation $[26]$ in this case is just d’Alembert equation, so the general solution involves arbitrary functions of $d - 1$ coordinates (e.g., Cauchy data).

Given the admissible gauge fixing conditions, the anti-ghosts
\[
\bar{C}_I, \quad gh(\bar{C}_I) = -1, \quad \varepsilon(\bar{C}_I) = 1 \quad (27)
\]
are assigned to $\chi^I(\phi)$. The number of the anti-ghosts is $\bar{m}$, and it is less than the number of ghosts, $\bar{m} = m - n$, where $m$ is the number of ghosts and $n$ is the number of ghost constraints \cite{24}. Unlike the case of unconstrained gauge symmetry, there is no pairing between ghosts and anti-ghosts if the gauges are chosen independent.

Given the gauges, and FP matrix, we can consider the adjustment of the FP path integral to the case of unfree gauge symmetry. Once the ghosts are subject to the equations (24) in the case of unfree gauge symmetry, the FP path integral for the transition amplitude has to be restricted to the ghost constraint surface, so it reads

$$Z_{FP} = \int \prod_{i,\alpha,a,I} \left[ d\phi^i dC^\alpha d\bar{C}_I \right] \delta(\chi^I(\phi)) \delta(\Gamma^\alpha_\beta(\phi)C^\beta) \exp \frac{i}{\hbar} \{ S(\phi) + \bar{C}_I \Gamma_i^I \partial_i \chi^I(\phi)C^\alpha \}.$$  (28)

This amplitude could be viewed as an implicit expression of the standard FP integral over the original fields and independent ghosts introduced for the gauge transformations with free parameters if one could find the local unconstrained parametrization of the gauge symmetry. If, for example, one could explicitly rearrange the gauge generators into unconstrained ones and zero operators, the delta functions of ghost constraints would just remove the ghosts for the vanishing parameters, so the expression (28) would reproduce the usual FP amplitude.

Consider Fourier representation for the delta-functions of the gauges and ghost constraints

$$\prod_I \delta(\chi^I(\phi)) = \int \prod_I [d\pi_I] \exp \frac{i}{\hbar} \pi_I \chi^I(\phi), \quad \varepsilon(\pi_I) = gh(\pi_I) = 0; \quad (29)$$

$$\prod_a \delta(\Gamma^\alpha_\beta(\phi)C^\beta) = \int \prod_a [d\bar{C}_a] \exp \frac{i}{\hbar} \bar{C}_a \Gamma^b_\beta(\phi)C^\beta, \quad \varepsilon(\bar{C}_a) = 1, \quad gh(\bar{C}_a) = -1. \quad (30)$$

Substituting (29), (31) into (28), we bring the FP integral to Feynman’s form

$$Z = \int [d\varphi] \exp \frac{i}{\hbar} S_{FP}(\varphi), \quad \varphi = (\phi^i, \pi_\alpha, C^\alpha, \bar{C}_I, \bar{C}_a), \quad (31)$$

where the FP action reads

$$S_{FP} = S(\phi) + \pi_I \chi^I(\phi) + \bar{C}_I \Gamma_i^I(\phi) \partial_i \chi^I(\phi)C^\alpha + \bar{C}_a \Gamma^a_\alpha(\phi)C^\alpha.$$  (32)

Once the Fourier multipliers $\bar{C}_a$ to the ghost constraints $\Gamma^a_\alpha(\phi)C^\alpha$ have the ghost number $-1$, these can be considered as anti-ghosts, on equal footing with the anti-ghosts $\bar{C}_I$ assigned to the gauge fixing conditions $\chi^I(\phi)$. With this regard, the total number of anti-ghosts in the FP action (32)
becomes equal to the total number of ghosts, while the ghosts are not constrained anymore. The matrix of ghost-anti-ghost bilinear form in the FP action (32) is squared, and it is non-degenerate. So, the integral (31) is regular both in ghosts and zero ghost number variables, including original fields and Lagrange multipliers to the gauges.

Let us discuss the independence of the FP path integral of the choice of gauge fixing conditions \( \chi^I(\phi) \). The BRST symmetry is an appropriate tool for the control of gauge independence of the path integral, so let us seek for the BRST transformation.

Given the gauge identities (6), the natural candidate for the BRST symmetry generator read

\[
Q = C^\alpha \Gamma^i_\alpha \partial_i + \pi_I \frac{\partial}{\partial C_I} + \tau_a \frac{\partial}{\partial \bar{C}_a} + o(C^2). \tag{33}
\]

Let us consider for simplicity the abelian case, when the generators \( \Gamma^i_\alpha \) commute, and do not act on \( \Gamma^a_\alpha \), and therefore no \( C^2 \) terms can appear in the BRST transformation. In this case the FP action is obviously \( Q \)-invariant, because of the gauge identity (6),

\[
\begin{align*}
Q S_{FP} = C^\alpha \left( \Gamma^i_\alpha \partial_i S + \Gamma^a_\alpha \tau_a \right) + \bar{C}_I C^\alpha C^\beta \left( \Gamma^i_\alpha \Gamma^j_\beta \partial_i \partial_j \chi^I \right) &\equiv 0. \tag{34}
\end{align*}
\]

\( Q \) does not square to zero identically unless the structure functions \( W_{ab} \) vanish in relation (19):

\[
Q^2 = C^\alpha \Gamma^i_\alpha \partial_i \tau_a \frac{\partial}{\partial C_a} \equiv W_{ab} C^\alpha \Gamma^a_\alpha \frac{\partial}{\partial C_a}. \tag{35}
\]

Notice that the path integral (28) is localized at the ghost constraint surface (24), where the BRST transformation (33) is truly nilpotent. If we considered the constraints on the ghosts (24) as a part of mass shell, the BRST-generator \( Q \) would square to zero on shell. In the theories with open gauge algebra, the BRST symmetry of the gauge fixed theory typically holds only on shell.

The BRST symmetry of the FP action (32) means that the path integral is independent from the choice of gauge fixing condition \( \chi(\phi) \). This can be seen in the same way as for the case with unconstrained gauge transformation parameters. Consider the infinitesimal change of the gauge

\[
\chi(\phi) \mapsto \chi(\phi) + \delta \chi(\phi). \tag{36}
\]

Let us make the BRST transformation of all the fields, including ghosts, anti-ghosts and Lagrange multipliers \( \varphi = (\phi', \pi_I, C^\alpha, \bar{C}_I, \bar{C}_a) \) with the transformation parameter \( \delta \Psi \) induced by the change
of gauge (36):

$$\varphi \mapsto \varphi_{\Psi} = \varphi + \delta \varphi, \quad \delta \varphi = (Q \varphi) \delta \Psi, \quad \delta \Psi = \frac{i}{\hbar} \bar{C}_I \delta \chi^I(\phi).$$

(37)

Once the FP action (32) is BRST-invariant (33), the infinitesimal change of fields can contribute to the path integral only through the transformation Jacobian. Up to the first order in $\delta \chi(\phi)$, the Jacobian reads

$$\det \left( \frac{\partial \varphi_{\Psi}}{\partial \varphi} \right) = \exp \left( \frac{i}{\hbar} \left( \pi_I \delta \chi^I(\phi) + \bar{C}_I \Gamma^i_{\alpha}(\phi) \frac{\partial \delta \chi^I}{\partial \phi^i} + (\text{div} Q) \bar{C}_I \delta \chi^I \right) \right),$$

(38)

where $\text{div} Q$ is a divergence of the BRST transformation vector $Q$ (33). The divergence of $Q$ is a simplest characteristic class of any gauge system [19]. Complete classification of characteristic classes of gauge systems can be found in [20]. The divergence of BRST transformation is usually termed as a modular class. The one-loop anomaly is known to be proportional to the modular class. If the modular class vanishes (hence, the theory is free from anomaly), the Jacobian (38) reproduces the change of the gauge fixing condition (36) in the FP path integral (28). In this way, one can see that the FP path integral (28) with the gauge $\chi(\phi)$ is connected by the change of fields (37) with the integral involving the gauge $\chi(\phi) + \delta \chi(\phi)$.

As we have seen, the FP path integral (28) does not depend on the choice of gauge fixing condition in the sense that the infinitesimal change of the gauge can be compensated by the change of the integration variables. The path integral quantization recipe (28) applies to the theories with unfree gauge symmetry when no higher structure functions are involved in the gauge algebra. Once the higher structure functions are involved, e.g., when the gauge transformations do not commute off shell, the extension of the BV field-anti-field formalism has to be worked out for the case of unfree gauge algebra. This will be done elsewhere, while some clues to the extension are discussed in the conclusion of the article.

4. Examples

In this section we exemplify the general structures of theories with unfree gauge symmetry by two linear models: traceless spin two free field with the action proposed in [11], and the Maxwell-like Lagrangian [10] for the tracefull second rank tensor field. Even in the linear models, the distinctive structures of the unfree gauge symmetry turn out non-trivial. At first, we demonstrate the non-trivial completion functions (3) in these models. Then, we observe that the gauge identities (6) involve, besides the Lagrangian equations and gauge symmetry generators, also completion
functions and the operators of gauge symmetry constraints \((14)\). We also see that the structure functions \(W_{ab}\) \((19), (21)\) can be non-trivial already at linear level in the models with unfree gauge symmetry.

The considered models admit at least three different ways of quantization that allows one to verify the results by the cross check. First, besides the unfree irreducible parametrization of gauge symmetry with the parameters constrained by the differential equations, these models admit an alternative parametrization with free gauge parameters, though with the symmetry of symmetry. The FP quantization rules involving ghosts-for-ghosts are well known for the theories with reducible gauge symmetry. Second, by inclusion appropriate auxiliary fields into the action, both models can be equivalently reformulated in the way with unconstrained irreducible gauge symmetry. In this form, the usual FP quantization rules apply, while the auxiliary fields can be eliminated from the path integral by imposing the gauge fixing conditions such that the extra fields are forced to vanish. And third, the modification of FP recipe \((31), (32)\) can be directly applied to both models in the original form with unfree gauge symmetry, without any reformulation. As we shall see, all three ways lead to the same result in these models, so the examples confirm proposed ansatz \((31), (32)\). Even though the models are linear, and the ghost contributions to the path integral reduce to the determinants of the field independent operators, it can be considered as a reasonable test for the correctness of the path integral \((31), (32)\) for the theories with unfree gauge symmetry. The reason is that the perturbative inclusion of interactions would deform the ghost terms of free theory, not replace them by the structures with a different constant part.

For the second example – the Maxwell-like Lagrangian – besides the free theory we consider the specific cubic vertex found in the article \([4]\). A complete classification of consistent cubic interactions of higher spin fields is established in the article \([21]\) making use of light-cone formalism. The new vertex does not correspond to any cubic interaction in this classification, though it seems admissible from the viewpoint of the usual Noether procedure of inclusion interactions. While the vertex is local, it can be removed by a nonlocal change of fields noticed in the article \([4]\). Given the general BRST differential \((33)\) of the theory with the unfree gauge symmetry, we shall demonstrate that the above mentioned vertex is BRST exact, i.e. it is the BRST variation of local quantity. This means, the vertex is trivial from the viewpoint of the local gauge field theory that explains why it does not have the place on the list of admissible interactions of the article \([21]\).
4.1. Linearized unimodular gravity. Consider symmetric traceless second rank tensor field $h_{\mu\nu}(x), h'^{\nu}_{\nu}(x) \equiv 0$ in $d = 4$ Minkowski space with the action

$$S[h(x)] = \int Ld^4x, \quad L = \frac{1}{2} \left( \partial_{\mu} h_{\nu\rho} \partial^{\rho} h'^{\nu}_{\nu} - 2 \partial_{\mu} h_{\nu\rho} \partial^{\rho} h'^{\nu}_{\nu} \right).$$  \hspace{1cm} (39)

The signature of the metric is mostly negative. The Lagrange equations read

$$\frac{\delta S}{\delta h^{\mu\nu}} = -\Box h_{\mu\nu} + \partial_{\mu} \partial^{\rho} h_{\rho\nu} + \partial_{\nu} \partial^{\rho} h_{\rho\mu} - \frac{1}{2} \eta_{\mu\nu} \partial^{\rho} \partial_{\rho} h_{\lambda\lambda} \approx 0, \quad \Box = \partial_{\rho} \partial^{\rho}.$$  \hspace{1cm} (40)

Taking the divergence of the equations, we arrive at the differential consequence (Cf. (1)):

$$\partial_{\mu} \tau \approx 0, \quad \tau = \partial^{\nu} \partial^{\lambda} h_{\nu\lambda}.$$  \hspace{1cm} (41)

Provided for zero boundary conditions for the fields at the space infinity, we obtain a single completion function

$$\tau(x) \approx 0,$$  \hspace{1cm} (42)

which has the form (3) where the condensed index $a$ is just the Minkowski space point $x$. Any on shell vanishing local quantity is spanned by the Lagrangian equations (40) and completion function (42):

$$O(h, \partial h, \partial^2 h, \partial^3 h, \ldots) \approx 0 \quad \Leftrightarrow \quad O = \hat{V}^{\mu\nu}, \frac{\delta S}{\delta h^{\mu\nu}} + \hat{V} \tau,$$  \hspace{1cm} (43)

where $\hat{V}^{\mu\nu}, \hat{V}$ are the local differential operators. This means that completion functions and left hand sides of Lagrangian equations constitute the generating set for the on-shell vanishing local quantities.

The generating (40), (42) of this set are dependent. The gauge identities (6) between the Lagrangian equations (40) and completion function (42) read

$$\partial^{\nu} \frac{\delta S}{\delta h^{\mu\nu}} - \frac{1}{2} \partial_{\mu} \tau \equiv 0.$$  \hspace{1cm} (44)

From this relation one can find the identity generators (6) in this model

$$\Gamma^i_{\rho} \equiv \Gamma^{\mu\nu}_{\rho} = \delta^\mu_\rho \partial^\nu + \delta^\nu_\rho \partial^\mu, \quad \Gamma^n_{\rho} \equiv \Gamma_{\rho} = \partial_{\rho}.$$  \hspace{1cm} (45)

In accordance with the general structure of unfree gauge algebra described in Section 2, the coefficient at the completion function in the gauge identity (6) should define the operator of the gauge parameter constraint (14), while the coefficient at the Lagrangian equation defines the
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generator of unfree gauge symmetry (12). Given the explicit form of the identity generators (45),
the unfree gauge transformation (12) and the gauge parameter constraint (14) in this model should
read
\[ \delta_\epsilon h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu, \quad \partial_\mu \epsilon^\mu = 0. \]  (46)

In this way we see that the involvement of the completion function in the gauge identity (44)
defines the linearized T-diff as a gauge symmetry of the model. The unfree gauge variation (46)
obviously leaves the action (39) intact
\[ \delta_\epsilon S[h(x)] = \int \tau \partial_\mu \epsilon^\mu = 0. \]  (47)

Let us also verify the gauge-invariance of the mass shell and identify the structure function \( W \) (19) of this model
\[ \delta_\epsilon \left( \frac{\delta S}{\delta h^{\mu\nu}} \right) = \frac{1}{2} \partial_\mu \partial_\nu (\partial_\rho \epsilon^\rho) = 0, \quad \delta_\epsilon \tau = 2 \Box \partial_\nu \epsilon^\nu = 0. \]  (48)

As is seen, the structure function (19) does not vanish,
\[ W = 2\Box. \]  (49)

Once the d’Alembertian is self-adjoint, the structure function \( W \) is symmetric indeed, cf. (21).

Now, consider the path integral quantization of the model (39). At first, we shall apply the
recipe (31), (32) to get the transition amplitude for the theory proceeding from the original action
(39) and unfree gauge symmetry (46). Then, we shall consider the two equivalent reformulations of
the model. One of these reformulations makes use the same action (39) while the gauge symmetry
is parameterized in a different way. The gauge parameters are unconstrained but the symmetry
is reducible: it admits a sequence of gauge symmetry for symmetry. This allows one to quantize
the model along the usual lines, by introducing ghosts for ghosts. One more reformulation makes
use of the fact that the model (39) can be viewed as a partially gauge-fixed version of linearized
Einstein’s gravity with the partial gauge \( h^\mu_\mu = 0 \). Choosing the complete gauge fixing conditions
involving this partial gauge, one can explicitly integrate out the trace of \( h_{\mu\nu} \) and get the path
integral for the model (39).

To simplify the comparison of the results of three methods, it is convenient to impose the same
gauge fixing conditions in the sector of the original fields in all the schemes. We choose the
independent gauge-fixing conditions,
\[
\chi^i = \partial_j h^{ji} - \frac{1}{2} \partial^i h^j_j = 0.
\] (50)

Here, and below in this section, the Latin indices \(i, j = 1, 2, 3\) label the space components of the Minkowski tensors or coordinates.

Let us begin with applying the quantization receipt (31), (32) to the model (39). The unfree gauge symmetry generators (12) and gauge parameter constraint operators (14) are defined for this model by relations (46). Substituting (39), (46), and the gauge-fixing conditions (50) into the general prescription (31), (32) we arrive at the path integral
\[
Z = \int [dhdb^i dC^i d\bar{C}^i] \exp \left( \frac{i}{\hbar} S_{FP} \right),
\]
\[
S_{FP} = \int \left( L + \bar{C}^i \Delta C_i + \bar{C} \partial_\mu C^\mu + (\partial^i h_{ij} - 1/2 \partial^j h^j_j) b^i \right) d^4 x,
\]
\[
\Delta = \sum_{i=1}^{3} \partial_i^2.
\] (51)

The FP action (51) is invariant with respect to the action of the BRST symmetry operator (33),
\[
Q = (\partial_\mu C^\nu + \partial_\nu C^\mu) \frac{\delta}{\delta h_\mu^\nu} + b^i \frac{\delta}{\delta \bar{C}^i} + \tau \frac{\delta}{\delta \bar{C}}.
\] (52)

Now, let us consider an alternative parametrization of gauge symmetry. The action (39) admits an unconstrained reducible parametrization of gauge symmetry
\[
\delta \xi_{\mu\nu} = \partial_\mu \epsilon^\nu_{\rho\sigma} + \partial_\nu \epsilon^\mu_{\rho\sigma} \xi_{\rho\sigma}, \quad \bar{\xi}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \xi^{\rho\sigma}, \quad \delta \xi S(h) \equiv 0, \quad \forall \xi.
\] (53)

The transformation parameter is an antisymmetric tensor \(\xi_{\mu\nu} = -\xi_{\nu\mu}\) whose components are arbitrary functions of space-time coordinates. The gauge symmetry (53) is reducible. The gauge transformations for the gauge parameters read
\[
\delta_{(1)} \xi_{\mu\nu} = \partial_\mu \xi_{(1)\nu} - \partial_\nu \xi_{(1)\mu}, \quad \delta_{(2)} \xi_{\mu\nu} = \partial_\mu \xi_{\nu}.
\] (54)

Once the gauge transformations are reducible, the minimal set of the BV fields and anti-fields includes ghosts for ghosts and conjugate anti-fields. The gradings read of the fields and anti-fields
\[
gh h_{\mu\nu} = 0, \quad gh C_{\mu\nu} = 1, \quad gh C^{(1)}_{\mu} = 2, \quad gh C^{(2)} = 3.
\]
\[
gh h^*_{\mu\nu} = -1, \quad gh C^*_{\mu\nu} = -2, \quad gh C^{(1)*}_{\mu} = -3, \quad gh C^{(2)*} = -4.
\] (55) (56)
The minimal BV-action for the theory reads

$$ S_{\text{min}} = \int \left( L + h^*_{\mu\nu}(\partial^\mu \partial_\nu \widetilde{C}^{\rho\nu} + \partial' \partial_\rho \widetilde{C}^{\rho\mu}) + C^*_{\mu\nu}(\partial^\mu C^{(1)\nu} - \partial' C^{(1)\mu}) + C^{(1)*}_{\mu} \partial^\mu C^{(2)} \right) d^4x. \quad (57) $$

To introduce the fields and anti-fields in non-minimal sector, we assume that all the gauges for original fields and ghosts are independent. This is a slight deviation from the usual scheme of BV quantization of the theories with reducible gauge symmetry where the reducibility of gauge fixing conditions are assumed to follow the reducibility pattern of gauge symmetry generators. In fact, this assumption can be bypassed, as we see in this example. The independent gauges can exist both for the fields and for the ghosts, even if the gauge symmetries, and symmetries for symmetries are reducible. In the case at hands, it is convenient to use the independent gauges because this simplifies comparison with the result obtained in terms of irreducible generators of gauge symmetry.

Let us choose the same independent gauge fixing conditions (50) for the original fields $h_{\mu\nu}$ as we applied above. For the ghosts $C_{\mu\nu}$ and the next level ghosts for ghosts $C_{\mu}^{(1)}$, we can also choose the irreducible gauge fixing conditions:

$$ \chi^{(1)}(C) \equiv C_{0i} = 0, \quad \chi^{(2)}(C^{(1)}) \equiv C_{0}^{(1)} = 0. \quad (58) $$

Now let us introduce anti-ghosts $\bar{C}$ and Lagrange multipliers $b$ for every gauge fixing condition. The ghost number of the multiplier should be opposite to the number of the gauge fixing condition, while the ghost number of the anti-ghost is shifted by $-1$ with respect to the number of the multiplier. The anti-field has the opposite number to that of the field shifted by minus one. All that means, we introduce the following variables of the non-minimal sector:

$$ \text{gh} C_i = -1, \quad \text{gh} C^{(1)}_i = -2, \quad \text{gh} C^{(2)} = -3, \quad \text{gh} b_i = 0, \quad \text{gh} b^{(1)} = -1, \quad \text{gh} b^{(2)} = -2. \quad (59) $$

$$ \text{gh} \bar{C}^*_i = 0, \quad \text{gh} \bar{C}^{(1)*}_i = 1, \quad \text{gh} \bar{C}^{(2)*} = 2, \quad \text{gh} b^*_i = -1, \quad \text{gh} b^{(1)*} = 0, \quad \text{gh} b^{(2)*} = 1. \quad (60) $$

As all the gauges are independent, there is no gauge symmetry for the anti-ghosts, and the Lagrange multipliers for the gauges. That is why no ghosts for ghosts are introduced in the non-minimal sector.
The non-minimal BV-action is introduced in the form
\[
S_{\text{nonmin}} = S_{\text{min}} + \int \left( \overline{C}^* b_i + \overline{C}^{(1)*} b^{(1)}_i + \overline{C}^{(2)*} b^{(2)} \right) d^4x. \tag{61}
\]

Given the gauge fixing conditions (58), the gauge-fixing fermion reads
\[
\psi = \int \left( \overline{C}^i (\partial^i h_{ij} - 1/2 \partial_j h^i_j) \right) + \overline{C}^{(1)i} C_{0i} + \overline{C}^{(2)} C^{(1)}_0 \right) d^4x. \tag{62}
\]

The gauge-fixing for anti-fields, being defined as \( \varphi^* = \partial \psi / \partial \varphi \), reads:

\[
h^*_0i = 0 \quad h^*_ij = -1/2(\partial_i \overline{C}_j + \partial_j \overline{C}_i - \delta_{ij} \partial_k \overline{C}^k) ;
\]

\[
C^*_{ij} = 0, \quad C^*_{0i} = C^{(1)}_i, \quad C^{(1)*}_i = 0, \quad C^{(1)*}_0 = \overline{C}^{(2)} ;
\]

\[
\overline{C}^*_i = \partial^i h_{ij} - 1/2 \partial_j h^i_j, \quad \overline{C}^{(1)*}_i = C_{0i}, \quad \overline{C}^{(2)*} = C^{(1)}_0 . \tag{63}
\]

Gauge fixed action reads
\[
S_\psi = \int \left( L - (\partial_i \overline{C}_j + \partial_j \overline{C}_i - \delta_{ij} \partial_k \overline{C}^k) \partial^i \partial^j \overline{C}^{*ij} + \overline{C}_{0i}(\partial \partial^{0} C^{(1)i} - \partial^{i} C^{(1)0}) + \overline{C}^{(2)} \partial^0 C^{(2)} + \right.
\]

\[
\left. + (\partial^i h_{ij} - 1/2 \partial_j h^i_j) b^i + C_{0i} b^{(1)i} + C^{(1)*}_0 b^{(2)} \right) d^4x. \tag{64}
\]

Simplifying this expression, we get
\[
S_\psi = \int \left( L + 1/2 \epsilon^{ijkl} \overline{C}_i \partial \partial^0 \overline{C}_{jk} + \overline{C}_{0i} \partial \partial^{0} C^{(1)i} + \overline{C}^{(2)} \partial^0 C^{(2)} + \right.
\]

\[
\left. + (\partial^i h_{ij} - 1/2 \partial_j h^i_j) b^i + C_{0i} b^{(1)i} + C^{(1)*}_0 b^{(2)} \right) d^4x. \tag{65}
\]

Let us show the path integral with the action above,
\[
Z = \int [d\varphi] \exp i \frac{\hbar}{\hbar} S_\psi(\varphi), \quad \varphi = (h_{ij}, C_\mu, \overline{C}_i, b^i, C^{(1)\mu}_i, \overline{C}^{(1)i}, b^{(1)i}, C^{(2)}, \overline{C}^{(2)}, b^{(2)}), \tag{66}
\]

can brought to the form (51). The main steps are as follows. First, the time derivatives at ghosts \( C_{ij} \) and \( C^{(1)i} \) are absorbed by the field redefinition,

\[
C_{ij} = \partial_0 C_{ij} , \quad C^{(1)} = \partial_0 C^{(1)i} . \tag{67}
\]

The differential change of variables should change the integration measure by the Jacobian, being the determinant of corresponding differential operator. As we use the same operator for changing the variables of the opposite Grassmann parity, the Jacobians cancel each other. After that,
the reducibility ghosts of minimal and non-minimal sector (except $C^{(2)}$ and $\bar{C}^{(2)}$) are integrated out making use of corresponding gauge fixing conditions. This does not add any factor to the integration measure at this step. The intermediate result for the path integral reads

$$Z = \int [d\varphi] \exp \left\{ \frac{i}{\hbar} \int \left( L + \frac{1}{2} \varepsilon^{ikl} \bar{C}_i \Delta C_{kl} + \bar{C}^{(2)} \partial_0 C^{(2)} + (\partial^j h_{ij} - 1/2 \partial_i h^j_{\ j}) b^i \right) d^4 x \right\},$$  

$$\varphi = (h_{\mu\nu}, b^i, C_{ij}, \bar{C}^i, C^{(2)}, \bar{C}^{(2)}).$$  

(68)

At the third step, the path integral by the ghost number 3, $-3$ variables $C^{(2)}, \bar{C}^{(2)}$ is replaced by the equivalent expression involving the new ghost 1, $-1$ variables $C^0, \bar{C}$,

$$\int [d\bar{C}^{(2)} dC^{(2)}] \exp \left\{ \frac{i}{\hbar} \int \bar{C}^{(2)} \partial_0 C^{(2)} d^4 x \right\} = \int [d\bar{C} dC^0] \exp \left\{ \frac{i}{\hbar} \int \bar{C} \partial_0 C^0 d^4 x \right\},$$

$$gh C^0 = 1, \quad gh \bar{C} = -1.$$  

(69)

At the fourth step, we perform the off-diagonal shift of the ghost field $C^0$,

$$C^0 \rightarrow C^0 + 1/2 \partial_0^{-1} \varepsilon^{ijk} \partial_i C_{jk}.$$  

(70)

The Jacobian of this variable change is unit, so the integration measure is preserved. After the transformations (69), (70) are done, we rewrite the path integral (68) in the form

$$Z = \int [d\varphi] \exp \left\{ \frac{i}{\hbar} \int \left( L + \frac{1}{2} \varepsilon^{ikl} \bar{C}_i \Delta C_{kl} + \bar{C} (\partial_0 C^0 + 1/2 \varepsilon^{ijk} \partial_i C_{jk}) + (\partial^j h_{ij} - 1/2 \partial_i h^j_{\ j}) b^i \right) d^4 x \right\}, \quad \varphi = (h_{\mu\nu}, b^i, C_{ij}, \bar{C}^i, C^0, \bar{C}).$$  

(71)

The modified FP action (51) follows from this formula if the ghosts $C^0, 1/2 \varepsilon^{ijk} C_{jk}$ are considered as the time and space components of 4-vector $C^\mu = (C^0, 1/2 \varepsilon^{ijk} C_{jk})$.

Consider one more way to deduce the path integral for the theory (39). Introduce the action functional for the linearized Einstein’s gravity,

$$S_{gr}[h(x)] = \int L_{gr} d^4 x, \quad L_{gr} = \frac{1}{2} (\partial_{\mu} h_{\nu\rho} \partial^\mu h^{\nu\rho} - 2 \partial_{\mu} h_{\nu\rho} \partial^\nu h^{\mu\rho} + 2 \partial_{\rho} h^{\mu\nu} \partial^\nu h - \partial_{\mu} h \partial^\mu h).$$  

(72)

The dynamical field in the model is the tracefull second-rank symmetric tensor $h_{\mu\nu}(x)$, and the notation is used $h \equiv h^{\mu\mu}$. The action functional (72) is invariant under linearized diffeomorphism gauge transformation,

$$\delta_{\xi} h_{\mu\nu} = \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}.$$  

(73)
The gauge parameter is the vector $\xi$, which is unconstrained. The convenient gauge fixing for the linearized Einstein’s theory includes relations (50) and zero trace condition,

$$\chi_i \equiv \partial^j h_{ij} - \frac{1}{2} \partial_i h^i_j = 0, \quad h = 0.$$  \hfill (74)

With this gauge imposed, the trace of the metric is excluded, and the classical theory coincides with the model (39) in the gauge (50). Consider now the conventional FP for the model (72) in the gauge (74):

$$S_{FP} = \int \left( L_{gr} + \bar{C}^i \Delta C_i + \bar{C} \partial_\mu C^\mu + (\partial^j h_{ij} - 1/2 \partial_i h^i_j) b^i + h b \right) d^4x.$$  \hfill (75)

The multiplier $b$ and the trace of the metric $h$ can be integrated out. After that, the path integral (75) takes the form (51). In this way, one can see that the conventional FP path integral for linearized gravity in the gauge $h^\mu_\mu = 0$ and (50) reproduces the answer (51) constructed by the general recipe (31), (32) for the linearized unimodular gravity. This example confirms once again the general prescription (31), (32) for path integral in the theory with unfree gauge symmetry.

4.2. Maxwell-like theory of symmetric tensor field. Consider the symmetric tracefull second rank tensor field $h_{\mu\nu}(x)$, $h^\mu_\mu = h$ in $d = 4$ Minkowski space. The action reads as in the previous case (39), where the tensor $h_{\mu\nu}$ is tracefull. Many of the relations of the previous subsection hold true for this model if the tensor is understood as tracefull. So, we provide the relations which cannot be obtained in this way, otherwise we refer to the previous section.

The Lagrangian equations for the Maxweel-like model of second rank tensor field read

$$\frac{\delta S}{\delta h_{\mu\nu}} \equiv -\Box h_{\mu\nu} + \partial_\mu \partial^\rho h_{\rho\nu} + \partial_\nu \partial^\rho h_{\rho\mu} \approx 0.$$  \hfill (76)

These equations have differential consequence (41). The completion function is the same as in the previous example, $\tau = \partial_\mu \partial_\nu h^{\mu\nu}$ (42). The gauge identities (6) have slightly different form,

$$\partial^\nu \frac{\delta S}{\delta h^{\mu\nu}} - \partial_\mu \tau \equiv 0.$$  \hfill (77)

The identity generators read

$$\Gamma^\mu_\rho = \delta^\mu_\rho \partial^\nu + \delta^\nu_\rho \partial^\mu, \quad \Gamma_\rho = 2 \partial_\rho.$$  \hfill (78)
There is additional factor 2 in $\Gamma_\rho$ comparing to (44), (45). The constrained gauge transformation is the linearized T-diff (46). The set of on-shell vanishing quantities includes the Lagrangian equations (76) and completion functions (42), which are gauge-invariant.

The quantization of the Maxwell-like theory proceeds along the same lines as the linearized unimodular gravity. All the quantization schemes turn out equivalent in this case much like the previous one, so we write down the FP action (32) in the gauge (50) omitting the details of derivation

$$S_{FP} = \int \left( L + \bar{C}^i \Delta C_i + \bar{C} \partial_\mu C^\mu + (\partial^j h_{ij} - 1/2 \partial_i h^i_j) b^j \right) d^4 x.$$ (79)

The BRST symmetry generator (33) reads in this case

$$Q = (\partial^\mu C^\nu + \partial^\nu C^\mu) \frac{\delta}{\delta h_{\mu\nu}} + b^i \frac{\delta}{\delta C^i} + 2 \tau \frac{\delta}{\delta C}.$$ (80)

The action (79) is BRST invariant with respect to this transformation.

Let us consider the Maxwell-like Lagrangian with specific cubic vertex found in the article [4]:

$$L(g) = \frac{1}{2} (\partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho} - 2 \partial_\mu h_{\nu\rho} \partial^\nu h^{\mu\rho}) - g(\partial_\mu \partial_\nu h^{\mu\nu}) h_{\rho\lambda} h^{\rho\lambda},$$ (81)

where $g$ the coupling constant. Up to the first order in $g$, the unfree gauge transformation for the action read,

$$\delta_i h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu, \quad \partial_\mu \epsilon^\mu + g(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) h^{\mu\nu} = 0.$$ (82)

Notice that the symmetry transformation remains unchanged at this level, while the gauge parameter constraint operator is deformed. The vertex (81) is gauge invariant with respect to the gauge transformation above, with account for the deformation of the constraint. This cubic interaction seems admissible and non-trivial from the viewpoint of the Noether procedure for inclusion of gauge invariant interaction applied in the work [4] along the usual lines of theory with the free gauge parameters, and accompanied by deformation of the gauge parameter constraint. On the other hand, this vertex does not fit into the known classification of cubic interactions of higher spin fields [21]. In the article [4] this discrepancy is explained in the following way. The authors observe that this interaction vertex can be interpreted as due to non-local redefinitions of the fields,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + g \frac{\partial_\mu \partial_\nu}{\Box} h_{\rho\lambda} h^{\rho\lambda}.$$ (83)
Proceeding from this nonlocal substitution, it is concluded that inclusion of the vertex does not alter the physical properties of the model. Below, we shall demonstrate, without any recourse to non-local manipulations, that the local vertex \( (81) \) is trivial indeed, given the BRST symmetry \( (80) \) of the FP action \( (79) \).

At first, let us construct the FP action \( (32) \) for the model with the interaction vertex \( (81) \). To do that, we have to upload into the general formula \( (32) \) all the specific ingredients of the model: the Lagrangian \( (81) \); the gauge fixing conditions \( (50) \); the gauge generators and gauge parameter constraint operators \( (82) \). The result reads

\[
S_{FP}(g) \equiv S_{FP}^{(0)} + gS_{FP}^{(1)} = \int \left( L(g) + \bar{C}^i \Delta C_i + \bar{C}(\partial_\mu C^\mu + g(\partial_\mu C_\nu + \partial_\nu C_\mu)h^{\mu\nu} + (\partial^ijh_{ij} - 1/2\partial_i h^i_j)b^i \right) d^4x. 
\]

The cubic terms include both the original vertex and the ghost contributions:

\[
S_{FP}^{(1)} = \int \left( -(\partial_\mu \partial_\nu)h^{\mu\nu}h_{\rho\lambda}h^{\rho\lambda} + \bar{C}(\partial_\mu C_\nu + \partial_\nu C_\mu)h^{\mu\nu} \right) d^4x. 
\]

Even though the symmetry is abelian, and the gauge generators are constants, the action involves the cubic term with ghosts \( \bar{C}(\partial_\mu C_\nu + \partial_\nu C_\mu)h^{\mu\nu} \). This term originates from the contribution of the gauge parameter constraint operator to the FP action \( (32) \). As the unfree gauge parameter is constrained at interacting level by the equation \( (82) \) involving \( h \), the field contributes to the ghost term. It is easy to see that the full cubic part of the FP action \( (85) \) is BRST-exact with respect to the BRST differential of the free theory \( (80) \):

\[
S_{FP}^{(1)} = Q\Psi, \quad \Psi = -\frac{1}{2} \int \bar{C} h_{\rho\lambda} h^{\rho\lambda} d^4x. 
\]

The potential \( \Psi \) is local, so the vertex is trivial indeed from the viewpoint of local BRST cohomology.

### 5. Concluding remarks

Let us first summarize the results of the article, and then discuss the remaining problems.

In this article we study the general phenomenon of the gauge symmetry with unfree gauge parameters. We proceed from the conjecture that the system of Lagrangian equations is incomplete in certain sense: given the boundary conditions, the local on shell vanishing quantities exist such
that they do not reduce to combination of the equations and their derivatives. We choose the generating set for on-shell vanishing local quantities (4) which includes Lagrangian equations and completion functions (3). The latter quantities vanish on shell, while they are not differential consequences of Lagrangian equations. In general, the gauge identities of the theory involve both Lagrangian equations and completion functions (6). It is the structure of gauge identities which leads to the constraints (14) on the gauge symmetry parameters of the action (13). In the usual case, when the Lagrangian field theory does not admit the completion functions, the gauge algebra involves the two primary constituents: the action functional and the generating set of the gauge symmetry transformations. Given these two constituents, all the higher structures of gauge algebra are defined by the compatibility conditions of gauge identities. Once the Lagrangian equations admit completion functions, the gauge identities (6) involve two more ingredients: the completion functions (3), (4) and the gauge parameter constraint operators (14). With this regard, the higher structure relations of the unfree generated gauge algebra involve more structure functions comparing to the algebra with unconstrained gauge parameters. We deduce the structure relations of the unfree gauge algebra up to the level which corresponds to the Lie algebra in the case of unconstrained gauge symmetry. We observe that the unfree gauge algebra can involve non-trivial structure constants which do not necessarily vanish even in the linear theory. This does not have a direct analogue even at the linear level of the theories with unconstrained gauge parameters. As an example, we can mention the structure function $W_{ab}$ involved in the relation (19) which follows from the fact that mass shell is invariant under the transformations with unfree gauge parameters. If all the structure functions are constants, we suggest the extension of the FP path integral construction to the case of unfree gauge symmetry. The path integral (31), (32) explicitly involves the operators of gauge parameter constraint. The FP action is BRST invariant, while the BRST differential (33) has a distinction from the case of unconstrained gauge symmetry, as the completion functions are explicitly involved. The BRST invariance ensures the gauge independence of the path integral.

In Section 4, we consider two examples of the field theories with unfree gauge transformation parameters. Both models admit alternative parametrization of gauge symmetry with unconstrained parameters, though with the gauge symmetry of symmetry. Also, by inclusion of auxiliary fields, they can be equivalently reformulated as theories with irreducible gauge symmetry and unconstrained gauge parameters. In this way, one can verify the path integral quantization recipe (31),
by comparing the transition amplitude with the ones deduced by usual FP rules based on
the alternative gauge symmetry parameterizations. All the answers coincide for the amplitude.
Notice that the second example also demonstrates how the BRST symmetry of the theory with
unfree gauge parameters can be helpful for separation of nontrivial interaction vertices from the
trivial ones. The vertex has been previously known in this model which looks eligible from the
viewpoint of Noether procedure, while it should not appear from the viewpoint of known classi-

cification of admissible cubic interactions. The BRST complex, which makes a due account for the
gauge parameter constraints and completion functions, identifies this vertex as BRST exact, and
thereby trivial.

Let us mention some remaining problems concerning general structure of the field theories with
unfree gauge parameters, and possible solutions.

At first, notice that the quantization recipe (31), (32) involves independent gauge fixing condi-
tions. In many cases the independent gauge fixing conditions cannot be consistent with Poincaré
or AdS symmetry of the theory with unfree gauge symmetry. The reason is obvious: for example,
given the vector gauge parameter in \( d \) dimensions restricted by the transversality equation, the
number of independent gauge fixing conditions should be \( d - 1 \), so they cannot be tensors. It is
unlikely to find \( d - 1 \) appropriate scalars to fix the gauge. Explicitly covariant gauges are admissi-

ble, for example \( \partial \cdot h \) in Maxwell-like theory [10] or in the model of traceless higher spin fields [9].
These gauge fixing conditions are obviously over-complete and therefore they should be on-shell
reducible. The reducibility is obvious indeed, \( \partial \cdot \partial \cdot h \approx 0 \). Reducibility of the gauge condition
would require the ghosts for ghosts with the higher negative ghost numbers (anti-ghosts), while no
ghosts for ghosts are introduced with positive ghost numbers. This asymmetry between ghost and
anti-ghost sectors does not have direct counterpart in the case of reducible gauge symmetry where
the over-complete set of gauge generators is mirrored by the reducible set of gauge fixing condi-
tions. That is why the ghosts-for-ghosts are accompanied by anti-ghosts for anti-ghosts. It should
be examined, however, that the asymmetry between ghost and anti-ghost sector is consistent with
the usual physical interpretation of BRST cohomology groups.

The second open problem is the extension of the BV field-anti-field formalism to the class of
field theories with unfree gauge symmetry. To begin with the problem, the field-anti-field content
of the theory has to be modified comparing to the case of gauge symmetry with unconstrained
parameters. If the field-anti-field space remained the same, the BV master equation would generate
the usual gauge algebra relations where no place is left for the operators of gauge parameter constraints \((14)\). The general idea of finding an appropriate field-anti-field space extension is that the ghost constraints \((24)\) and completion equations \((3)\) have to be considered as equations of motion, on equal footing with the original Lagrangian equations. This brings the theory, at least for a while, to the realm of not necessarily Lagrangian systems. In not necessarily Lagrangian case, the anti-fields are assigned to the equations \([16]\), not to the fields, while in Lagrangian case this would the same. The specifics of the ghost constraint equation \((24)\) is that it has the ghost number one. This means, the corresponding anti-field should have zero ghost number, while the anti-field to the completion equation would have the ghost number \(-1\). These two extra-anti-fields are the dual variables, and they have the opposite parity, so it is natural to expect that they should be taken as conjugate with respect to the anti-bracket. Proceeding from this general setup, we expect to develop the BV formalism for the systems with general unfree gauge symmetry in the future work.

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