DYNAMICS OF A NONLOCAL SIS EPIDEMIC MODEL WITH FREE BOUNDARY

JIA-FENG CAO, WAN-TONG LI AND FEI-YING YANG
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

(Communicated by Yuan Lou)

Abstract. This paper is concerned with the spreading or vanishing of an epidemic disease which is characterized by a diffusion SIS model with nonlocal incidence rate and double free boundaries. We get the full information about the sufficient conditions that ensure the disease spreading or vanishing, which exhibits a detailed description of the communicable mechanism of the disease. Our results imply that the nonlocal interaction may enhance the spread of the disease.

1. Introduction. As seen in its long history, epidemic disease has caused orders of magnitude more deaths throughout the world than wars or famines. In order to get a better understanding of the transmission mechanism, many researchers devoted to the mathematical modelling and dynamical behaviors of epidemic diseases [21]. We find that aforementioned models are strongly based on the homogeneity of the space. Concerning the fact that many physical aspects of the environment such as climate, chemical composition or physical structure can vary from place to place, as early as in 1957, Kendall [23] proposed the spatially dependent integro-differential SIR epidemic system

\[
\begin{align*}
S_t &= -\beta S(x,t) \int_{-\infty}^{+\infty} K(x - y)I(y,t)dy, \\
I_t &= \beta S(x,t) \int_{-\infty}^{+\infty} K(x - y)I(y,t)dy - \gamma I(x,t), \\
R_t &= \gamma I(x, t)
\end{align*}
\]

(1)

to describe the spread of an epidemic disease in a one-dimensional habitat, where \( S, I \) and \( R \) obey the susceptible-infective-removed (SIR) scheme just as that showed in the Kermack-McKendrick epidemic equations; the constant \( \beta \) represents the contact rate and \( \gamma \) is the recovery rate. In addition, \( K(x - y) \geq 0 \) denotes the probability density that weights the contributions of infectious at location \( y \) to the infection of susceptible individuals at location \( x \). Moreover, Kendall [24] established that if \( \frac{\beta S_0}{\gamma} > 1 \) and \( c \geq c^* \), system (1) admits a traveling wavefront solution of the form

2010 Mathematics Subject Classification. 35K57, 35R20, 92D25.
Key words and phrases. SIS model, reaction-diffusion, free boundary, spreading-vanishing dichotomy, nonlocal incidence rate.

247
(S(x + ct), I(x + ct), R(x + ct)) and has no such solution for c < c∗, for which c denotes the wave speed, S0 is the population density at the beginning of the epidemic with everyone susceptible and c∗ > 0 is called the minimal wave speed. We refer readers to Aronson [3] and Mollison [32] for some relevant progress on epidemic waves of (1). In particular, we refer to Ai et al. [1] about the existence and uniqueness (for sufficient large wave speed) of traveling wavefront solutions for spatial SIRS epidemic models by assuming that the kernel function K(·) is nonnegative, symmetric and satisfies ∫−∞+∞ K(x)dx = 1 and K(x) = 0 for |x| ≥ ϵ, where ϵ > 0 is small.

Concerning the fact that population individuals are distributed in space randomly, and typically interact with the physical environment and other organisms in their spatial neighborhood randomly, too, Mottoni et al. [33] added diffusion term into the Kendall model as

\[
\begin{align*}
S_t &= \Delta S + \sigma - \mu S - S \int_{\Omega} K(x, y)I(y, t)dy, \\
I_t &= d\Delta I + S \int_{\Omega} K(x, y)I(y, t)dy - \gamma I
\end{align*}
\]

with Neumann boundary condition, where Ω ⊂ R^N (N ≥ 1) is an open bounded domain with piecewise smooth boundary, K(·, ·) : Ω × Ω → [0, ∞) is smooth, nonnegative (typically, with compact support), symmetric (i.e. K(x, y) = K(y, x) for any x, y ∈ Ω), not identically equals to zero, and satisfies ∫Ω K(x, x)dx ≤ 1 for any y ∈ Ω. For (2), Mottoni et al. established the stability and attractivity of the stationary solutions, especially in the case that the initial data are spatially inhomogeneous.

It is worth pointing out that excepting the nonlocal interaction as in (2), Wang et al. [42] incorporated the time delay as a latent period into the spread of the diffusive Kermack-McKendrick epidemic disease, that is, the incidence rate is characterized by βS(x, t) ∫−∞+∞ K(x−y, t−s)I(y, t)dyds, and obtained the threshold dynamics for the spread of the disease (see [42] for the detailed description about the kernel function K). For more relevant work on the existence of traveling waves of reaction-diffusion equations with nonlocal interaction and time delay, we refer readers to Ducrot et al. [15], Faria et al. [16] and references cited therein. We also refer to Lou [31] for some challenging mathematical problems in evolution of dispersal and population dynamics.

Recently, free boundary problems have been studied intensively in many fields. In particular, the well-known Stefan condition has been used to describe the spreading front in many applied problems. For example, it was used to describe the melting of ice in contact with water [35], the wound healing [6], the tumor growth [7] and so on. In order to get a more precise prediction of the location of the spreading front of an invading species, Du et al. [10] firstly studied the spreading-vanishing dichotomy of some invasion species which is described by a diffusive logistic model in the homogenous environment of one dimensional space. Since then, more results for more general free boundary problems have been obtained, for example, see [8, 9, 14, 12, 13, 34, 41] for single species model and [20, 37, 11, 18, 19, 36, 39] for Lotka-Vottera systems. In particular, Kim et al. [20] investigated the following free boundary diffusion SIR epidemic model...
for more relevant theoretical advances. For other epidemic models with free boundary and related results, one refer to [22, 30, 17] for more relevant theoretical advances.

It is noted that for the nonlocal diffusive SI epidemic model (2), if we add the recovery class term, that is, making (2) becomes following

\[
\begin{cases}
S_t = \Delta S + \sigma - \mu S - \beta SI, & r > 0, t > 0, \\
I_t = \Delta I + \beta SI - (\tau_2 + \gamma)I, & 0 < r < h(t), t > 0, \\
R_t = \Delta R + \gamma I - \tau_3 R, & 0 < r < h(t), t > 0, \\
S_r(t,0) = I_r(t,0) = R_r(t,0), & t > 0, \\
I(t,x) = R(t,x) = 0, & r \geq h(t), t > 0, \\
\frac{d}{dt}h(t) = -\mu_1 I_r(t,h(t)), h(0) = h_0, & t > 0, \\
S(0,x) = S_0(x), I(0,x) = I_0(x), R(0,x) = R_0(x), & 0 \leq r \leq h_0,
\end{cases}
\]

where \( r = |x| \) and \( x \in \mathbb{R}^N \), \( h(t) \) is the moving boundary to be determined, \( \tau_1, \tau_2, \tau_3 \) are positive constants that denote the death rates of each class respectively, and \( \mu_1 \) has the same meaning as that in (4). They proved some sufficient conditions that ensure the disease vanishing or spreading. For other related results, one refer to [22, 30, 17] for other epidemic models with free boundary and [28, 38, 40, 43, 44] for more relevant theoretical advances.

Motivated by above, we will discuss the dynamics of the following nonlocal epidemic model with free boundary as

\[
\begin{cases}
S_t = \Delta S + \sigma - \mu S - \beta SI, & x \in \Omega, t > 0, \\
I_t = \Delta I + \beta S \int_{\Omega} K(x,y)I(y,t)dy + \gamma I, & x \in \Omega, t > 0, \\
N_t = \Delta N + \sigma - \mu N, & x \in \Omega, t > 0, \\
I_t = \Delta I + \beta (N-I) \int_{\Omega} K(x,y)I(y,t)dy - (\mu + \gamma)I, & x \in \Omega, t > 0.
\end{cases}
\]

Motivated by above, we will discuss the dynamics of the following nonlocal epidemic model with free boundary as

\[
\begin{cases}
N_t = \Delta N + \sigma - \mu N, & x \in \mathbb{R}, t > 0, \\
I_t = \Delta I + \beta (N-I) \int_{\mathbb{R}} K(x,y)I(y,t)dy - (\mu + \gamma)I, & x \in (g(t), h(t)), t > 0, \\
N(x,0) = N_0(x), I(x,0) = I_0(x), & x \in \mathbb{R}, \\
I(x,t) = 0, & x \in \mathbb{R} \setminus (g(t), h(t)), t > 0, \\
g'(t) = -\mu_1 I_x(g(t),t), g(0) = -h_0, & t > 0, \\
h'(t) = -\mu_1 I_x(h(t),t), h(0) = h_0, & t > 0
\end{cases}
\]
satisfying
\[
\begin{align*}
N_0(x) &\in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ and } N_0(x) > 0 \text{ for } x \in \mathbb{R}, \\
I_0(x) &\in C^2([-h_0, h_0]), \quad I_0(x) > 0 \text{ for } x \in (-h_0, h_0), \\
I_0(x) &= 0 \text{ for } x \in (-\infty, -h_0] \cup [h_0, \infty).
\end{align*}
\]

We see that problem (6)-(7) indicates that the whole class exhibit themselves in the whole area \( \mathbb{R} \), while the individuals which are infected occupy initial region \([-h_0, h_0]\) at the beginning of the stage and spread further into the environment from two ends of the initial region. The spreading frontiers expand at a speed that is proportional to the infectious gradient at the front, which gives rise to the Stefan conditions \( g'(t) = -\mu_1 I_x(g(t), t) \) and \( h'(t) = -\mu_1 I_x(h(t), t) \). Furthermore, \( S(x, t) \int_\mathbb{R} K(x, y) I(y, t) dy \) implies that the infectious at location \( y \) can contact with the susceptible individuals at location \( x \) and then make them become infectious, which gives rise to the nonlocal effect. This work can be regarded as a continuation of our previous paper [5] in which we investigated a diffusion SIRS model with double free boundaries and bilinear incidence, and got the sufficient conditions that ensure disease spreading or vanishing and an estimate for the spreading speed if the disease spreading happens.

The organization of this paper is as follows. In Section 2, we prove the general existence and uniqueness result, which implies in particular that problem (6)-(7) has a unique positive solution defined for all \( t > 0 \), the method is inspired by [10, 8, 12, 26]. In Section 3, we firstly analyze an eigenvalue problem and discuss the property of its principal eigenvalue \( \lambda_1 \). Then we define \( \mathcal{R}_0^F(t) \) as a critical function and run our discussions by comparing \( \mathcal{R}_0^F(0) \) with 1. Meanwhile, we propose a comparison principle for our free boundary problem which will be frequently used in this paper. Section 4 is concerned with some sufficient conditions that ensure the disease vanishing and Section 5 exhibits some conditions that make the disease spread. Finally, we give a brief discussion.

2. Solutions of (6) and (7). It is quite well understood now that the global existence and uniqueness of solutions of (6) are deduced from the local existence, uniqueness and a priori estimates, see [10, 29, 37, 38]; and the local existence and uniqueness are obtained by contraction mapping theorem. Firstly, we introduce a standard hypotheses on kernel \( K(\cdot, \cdot) \) as follows:

\( (K) \): \( K \in C(\mathbb{R} \times \mathbb{R}) \) is nonnegative, symmetric (i.e. \( K(x, y) = K(y, x) \) for any \( x, y \in \mathbb{R} \) with \( x \neq y \)), and satisfies \( \int_{\mathbb{R}} K(x, y) dx = 1 \) for any \( y \in \mathbb{R} \).

We prove the following local existence and uniqueness result by the contraction mapping theorem and then use a priori estimates to show that the solution is defined for all \( t > 0 \). The proof can be done by modifying the arguments of [10] (see also [37, 38]).

Theorem 2.1. Assume that \( (K) \) holds. Then for any given \( (N_0, I_0) \) satisfying (7), there is a \( T > 0 \) such that problem (6) admits a unique solution

\[
(N, I; g, h) \in C^{1+\alpha, \frac{1+\alpha}{2}}(\mathbb{R} \times [0, T]) \times C^{1+\alpha, \frac{1+\alpha}{2}}(D_T) \times [C^{1+\frac{\alpha}{2}}([0, T])]^2
\]

satisfying

\[
\|N\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\mathbb{R} \times [0, T])} + \|I\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(D_T)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0, T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0, T])} \leq C,
\]

for all \( t \leq T \).
where $\alpha \in (0, 1)$ and $D_T = \{(x, t) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in [0, T]\}$, $C$ and $T$ depend on $h_0, \alpha, \|N_0\|_{C^2(\mathbb{R})}$ and $\|I_0\|_{C^2([-h_0, h_0])}$.

**Proof.** Inspired by the method in [8, 17], we first straighten the free boundaries. Let $\zeta(y)$ be a function in $C^3((0, +\infty))$ satisfying

$$
\zeta(y) = 1 \text{ if } |y - h_0| < \frac{h_0}{8}, \quad \zeta(y) = 0 \text{ if } |y - h_0| > \frac{h_0}{2}, \quad |\zeta'(y)| < \frac{5}{h_0} \text{ for all } y
$$

and take $\xi(y) = -\zeta(-y)$. Consider the transformation (see Du et al. [10])

$$(r, t) \rightarrow (x, t), \text{ where } x = r + \zeta(r)(h(t) - h_0) + \xi(r)(g(t) + h_0), r \in \mathbb{R},$$

which changes $x = h(t)$ and $x = g(t)$ to the fixed lines $r = h_0$ and $r = -h_0$, respectively.

The above transformation $x \rightarrow r$ is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ as long as $|h(t) - h_0| \leq \frac{h_0}{8}$ and $|g(t) + h_0| \leq \frac{h_0}{8}$. Now, direct calculations show that

$$
\frac{\partial r}{\partial x} = \frac{1}{1 + \zeta'(r)(h(t) - h_0) + \zeta'(r)(g(t) + h_0)} := \sqrt{A(h(t), g(t), r)},
$$

$$
\frac{\partial r}{\partial t} = -\frac{\zeta'(r)(h(t) - h_0) + \zeta'(r)(g(t) + h_0)}{1 + \zeta'(r)(h(t) - h_0) + \zeta'(r)(g(t) + h_0)} := C(h(t), g(t), r)
$$

and

$$
\frac{\partial^2 r}{\partial x^2} = -\frac{\zeta''(r)(h(t) - h_0) + \zeta''(r)(g(t) + h_0)}{[1 + \zeta'(r)(h(t) - h_0) + \zeta'(r)(g(t) + h_0)]^2} := B(h(t), g(t), r).
$$

If we set

$$
N(x, t) = N(r + \zeta(r)(h(t) - h_0) + \xi(r)(g(t) + h_0), t) = u(r, t),
$$

$$
I(x, t) = I(r + \zeta(r)(h(t) - h_0) + \xi(r)(g(t) + h_0), t) = v(r, t)
$$

and $K(x, y) = K(r + \zeta(r)(h(t) - h_0) + \xi(r)(g(t) + h_0), y) = \tilde{K}(r, y)$ with $\tilde{K}$ satisfying the condition $(K)$, then

$$
N_t = u_t + u_r \frac{\partial r}{\partial t} = u_t + C(h(t), g(t), r)u_r, \quad I_t = v_t + v_r \frac{\partial r}{\partial t} = v_t + C(h(t), g(t), r)v_r,
$$

and

$$
N_x = u_r \frac{\partial r}{\partial x} = \sqrt{A(h(t), g(t), r)}u_r, \quad N_{xx} = A(h(t), g(t), r)u_{rr} + B(h(t), g(t), r)u_r,
$$

$$
I_x = v_r \frac{\partial r}{\partial x} = \sqrt{A(h(t), g(t), r)}v_r, \quad I_{xx} = A(h(t), g(t), r)v_{rr} + B(h(t), g(t), r)v_r.
$$

Hence, the free boundary problem (6) becomes following

$$
\begin{align*}
u_t - A\nu_{rr} - (B - C)\nu_r &= \sigma - \mu u, & r \in \mathbb{R}, \ t > 0, \\
v_t - A\nu_{rr} - (B - C)\nu_r &= \beta(u - v)(\tilde{K} \ast v) - (\mu + \gamma)v, & r \in (-h_0, h_0), \ t > 0, \\
v(r, t) &= 0, & r \in \mathbb{R} \setminus (-h_0, h_0), \ t > 0, \\
u(r, 0) &= u_0(r), & v(r, 0) = v_0(r) = I_0(r), & r \in \mathbb{R}, \\
h'(t) &= -\mu_1\nu_r(h_0, t), & h(0) = h_0, & t > 0, \\
g'(t) &= -\mu_1\nu_r(-h_0, t), & g(0) = h_0, & t > 0,
\end{align*}
$$

where $A = A(h(t), g(t), r), B = B(h(t), g(t), r)$ and $C = C(h(t), g(t), r)$.

(8)
Denote \( h^* = -\mu_1 v_0'(h_0), g^* = -\mu_1 v_0'(-h_0) \), and for \( 0 < T < \frac{h_0}{\mu_1 \|v_0\|_{L^\infty(\Delta_T)}} \), define 
\( \Delta_T = [-h_0, h_0] \times [0, T] \), \( \Delta_T = \mathbb{R} \times [0, T] \).

\[
\begin{align*}
U_T &= \left\{ u \in C(\Delta_T) \left| (u(y,0) = u_0(y), \|u - u_0\|_{L^\infty(\Delta_T)} \leq 1 \right. \right\}, \\
V_T &= \left\{ v \in C(\Delta_T) \left| (v(y,0) = v_0(y), \|v - v_0\|_{L^\infty(\Delta_T)} \leq 1 \right. \right\}, \\
H_T &= \left\{ h \in C^1([0,T]), h(0) = h_0, h'(0) = h^*, \|h' - h^*\|_{C([0,T])} \leq 1 \right\}
\end{align*}
\]

and
\[
G_T = \left\{ g \in C^1([0,T]), g(0) = -h_0, g'(0) = g^*, \|g' - g^*\|_{C([0,T])} \leq 1 \right\}.
\]

It is easily seen that \( D := U_T \times V_T \times H_T \times G_T \) is a complete metric space with the metric

\[
d((u_1, v_1, g_1, h_1), (u_2, v_2, g_2, h_2)) = \|u_1 - u_2\|_{L^\infty(\Delta_T)} + \|v_1 - v_2\|_{C(\Delta_T)} + \|g'_1 - g'_2\|_{C([0,T])} + \|h'_1 - h'_2\|_{C([0,T])}.
\]

Noticing that for \( h_1, h_2 \in H_T \) and \( g_1, g_2 \in G_T \), we have \( h_1(0) = h_2(0) = h_0 \) and \( g_1(0) = g_2(0) = -h_0 \), then

\[
\|h_1 - h_2\|_{C([0,T])} \leq T \|h'_1 - h'_2\|_{C([0,T])} \quad \text{and} \quad \|g_1 - g_2\|_{C([0,T])} \leq T \|g'_1 - g'_2\|_{C([0,T])}.
\]

Next, we prove the existence and uniqueness result by using the contraction mapping theorem. First, for any given \((u, v, g, h) \in D\), we have

\[
|h(t) - h_0| \leq T(1 + h^*) < \frac{h_0}{8} \quad \text{and} \quad |g(t) + h_0| \leq T(1 + g^*) < \frac{h_0}{8},
\]

which lead to \( \frac{7}{8}h_0 \leq b(t), -g(t) \leq \frac{7}{8}h_0 \). Therefore, the transformation \((r, t) \to (x, t)\) as well as \( A, B \) and \( C \) are well defined. Applying standard \( L^p \) theory and then the Sobolev embedding theorem \([27]\), we can find that for any \((u, v, g, h) \in D\), the following initial boundary value problem

\[
\begin{align*}
\begin{cases}
\tilde{u}_t - A\tilde{u}_{rr} - (B - C)\tilde{u}_r &= \sigma - \mu u, & r \in \mathbb{R}, t > 0, \\
\tilde{v}_t - A\tilde{v}_{rr} - (B - C)\tilde{v}_r &= \beta(u - v)(\tilde{K} * v) - (\mu + \gamma)\tilde{v}, & r \in (-h_0, h_0), t > 0, \\
\tilde{v}(x, t) &= 0, & r \in \mathbb{R} \setminus (-h_0, h_0), t > 0, \\
\tilde{u}(r, 0) &= N_0(r), & \tilde{v}(r, 0) = I_0(r), & r \in \mathbb{R}
\end{cases}
\end{align*}
\]

admits a unique solution \((\tilde{u}, \tilde{v}) \in W^{2,1}_p(\Delta_T) \times W^{2,1}_p(\Delta_T) \hookrightarrow C^{1+\alpha, \frac{1+\alpha}{p}}(\Delta_T) \times C^{1+\alpha, \frac{1+\alpha}{p}}(\Delta_T) \) with \( p > \frac{3}{3 - \alpha} \), and

\[
\|\tilde{u}\|_{C^{1+\alpha, \frac{1+\alpha}{p}}(\Delta_T)}, \|\tilde{v}\|_{C^{1+\alpha, \frac{1+\alpha}{p}}(\Delta_T)} \leq C_1 = C_1(h_0, \alpha, \|N_0\|_{C^2(\mathbb{R})}, \|I_0\|_{C^2((-h_0, h_0))}).
\]

Define

\[
\tilde{h}(t) = h_0 - \mu_1 \int_0^t \tilde{v}_r(h_0, \tau) d\tau \quad \text{and} \quad \tilde{g}(t) = -h_0 - \mu_1 \int_0^t \tilde{v}_r(-h_0, \tau) d\tau.
\]

Then we have

\[
\begin{align*}
\tilde{h}'(t) &= -\mu_1 \tilde{v}_r(h_0, t), \quad \tilde{h}'(0) = -\mu_1 \tilde{v}'_0(h_0) = h^*, \quad \tilde{h}(0) = h_0, \\
\tilde{g}'(t) &= -\mu_1 \tilde{v}_r(-h_0, t), \quad \tilde{g}'(0) = -\mu_1 \tilde{v}'_0(-h_0) = g^*, \quad \tilde{g}(0) = -h_0
\end{align*}
\]
and hence
\[
\hat{h}', \  \hat{g}' \in C^{\frac{\alpha}{2}}([0,T]) \quad \text{with} \quad \|\hat{h}'\|_{C^{\frac{\alpha}{2}}([0,T])}, \  \|\hat{g}'\|_{C^{\frac{\alpha}{2}}([0,T])} \leq \mu_1 C_1 := C_2. \tag{12}
\]

In what follows, we define a map \( F : D \rightarrow C(\Delta_T) \times C(\Delta_T) \times [C^1([0,T])]^2 \) by
\[
F(u,v;g,h) = (\hat{u}; \hat{v}; \hat{g}; \hat{h}).
\]
It is clear that \((u,v;g,h) \in D\) is a fixed point of \( F \) if and only if it solves (5). By (11) and (12), we see that
\[
\|\hat{h}' - h^*\|_{C([0,T])} \leq \|\hat{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^\frac{\alpha}{2} \leq C_2 T^\frac{\alpha}{2},
\]
\[
\|\hat{g}' - g^*\|_{C([0,T])} \leq \|\hat{g}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^\frac{\alpha}{2} \leq C_2 T^\frac{\alpha}{2},
\]
\[
\|\hat{u} - u_0\|_{C(\Delta_T)} \leq \|\hat{u} - u_0\|_{C^{0,\frac{1+\alpha}{4}}(\Delta_T)} T^\frac{1+\alpha}{4} \leq C_1 T^\frac{1+\alpha}{4}
\]
and
\[
\|\hat{v} - v_0\|_{C(\Delta_T)} \leq \|\hat{v} - v_0\|_{C^{0,\frac{1+\alpha}{4}}(\Delta_T)} T^\frac{1+\alpha}{4} \leq C_1 T^\frac{1+\alpha}{4}.
\]

Therefore if we take \( T < \min \left\{ C_2^{-\frac{2}{\alpha}}, \ C_1^{-\frac{4}{1+\alpha}} \right\} \), then \( F \) maps \( D \) into itself.

Now, we are in the position to prove that \( F \) is a contraction mapping on \( D \) for \( T > 0 \) small. Just as in the proof of Theorem 2.1 in Du et al. [10], the left part relies on the \( L^p \) estimates for parabolic equations and Sobolev embedding theorem, we will not repeat them again. Therefore, we obtain a unique fixed point \((u,v;g,h)\) of operator \( F \) in \( D \). Moreover, by the Schauder’s estimate, we get additional regularity for \((u,v;g,h)\) as a solution of (5). Namely, \( h(t), \ g(t) \in C^{1+\frac{\alpha}{2}}([0,T]), \ u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Delta_T) \) and \( v \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Delta_T) \). In other words, \((u,v;g,h)\) is the unique classical solution of problem (5). This completes the proof.

To show that the local solution obtained in Theorem 2.1 can be extended to all \( t > 0 \), we need the following estimate.

**Theorem 2.2.** Assume that (K) holds and let \((N,I;g,h)\) be the solution of problem (6) for \( t \in (0,T_0) \) with some \( T_0 \in (0,\infty) \). Then, we have
\[
0 < N(x,t) \leq M_1 \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad t \in (0,T_0),
\]
\[
0 < I(x,t) \leq M_2 \quad \text{for} \quad x \in (g(t),h(t)) \quad \text{and} \quad t \in (0,T_0)
\]
and \( 0 < h'(t), -g'(t) \leq M_3 \) for \( t \in (0,T_0) \), in which \( M_1, \ M_2 \) and \( M_3 \) are positive constants independent of \( T_0 \).

**Proof.** For any \((x,t) \in \mathbb{R} \times (0,\infty)\), it follows from the comparison principle that
\[
N(x,t) \leq \hat{N}(t) := \frac{\sigma}{\mu} + \left( \hat{N}(0) - \frac{\sigma}{\mu} \right) e^{-\mu t},
\]
which is a solution of
\[
\hat{N}'(t) = \sigma - \mu \hat{N}(t), \quad t > 0, \quad \hat{N}(0) = \|N_0\|_{L^\infty}.
\]
Therefore, we obtain that
\[
N(x,t) \leq M_1 := \sup_{t \geq 0} \hat{N}(t) \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad t \in (0,T_0).
\]
Note that the bound of \( N(x,t) \) immediately deduces that \( I(x,t) \) satisfies
\[
I(x,t) \leq \hat{I}(t) = M_1 e^{B M_1 t} \left( e^{B M_1 t} - 1 + \frac{M_1}{\|I_0\|_{L^\infty}} \right)^{-1},
\]
where \( \bar{I}(t) \) is the solution of
\[
\bar{I}'(t) = \beta \bar{I}(t) \left[ M_1 - \bar{I}(t) \right], \quad \bar{I}(0) = \|I_0\|_{L^\infty}.
\]
It is clear that \( \bar{I}(t) \) is continuous on \( t \geq 0 \). In addition, we see that \( \bar{I}(t) \) is nondecreasing in \( t \) if \( \|I_0\|_{L^\infty} \leq M_1 \) and decreasing if \( \|I_0\|_{L^\infty} > M_1 \). Therefore, for any \( x \in (g(t), h(t)) \) and \( t \in (0, T_0) \), there is
\[
I(x, t) \leq \sup_{t \geq 0} \bar{I}(t) = \max \left\{ M_1, \|I_0\|_{L^\infty} \right\} := M_2.
\]

Applying the strong maximum principle and the Hopf lemma to the equations of \( N \) and \( I \) resulting that
\[
N(x, t), \quad I(x, t) > 0, \quad I_x(h(t), t), -I_x(g(t), t) < 0 \quad \text{for} \quad x \in (g(t), h(t)) \quad \text{and} \quad t \in (0, T_0).
\]
Thus, we have \( h'(t), -g'(t) > 0 \) for \( t \in (0, T_0) \) by the Stefan conditions.

It remains to show that \( h'(t), -g'(t) \leq M_3 \) for all \( t \in (0, T_0) \) with some \( M_3 \) independent of \( T_0 \). We just give the proof of \( h(t) \), the proof for \( g(t) \) is similar. As in Du et al. [10], define
\[
\Omega_M := \{(x, t) : h(t) - M^{-1} < x < h(t), \quad 0 < t < T_0\}
\]
and
\[
\bar{I}(x, t) = M_2 \left[ 2M(h(t) - x) - M^2(h(t) - x)^2 \right],
\]
in which \( M > 0 \) is a constant that will be chosen later. For any \((x, t) \in \Omega_M\), we have
\[
\bar{I}_t = 2MM_2 \left[ 1 - M(h(t) - x) \right] h'(t) \geq 0, \quad -\Delta \bar{I} = 2M^2M_2,
\]
\[
\beta(N - I) \int_R K(x, y)I(y, t)dy - (\mu + \gamma)I \leq \beta M_1M_2
\]
and
\[
\bar{I}_t - \Delta \bar{I} \geq 2M^2M_2 \geq \beta M_1M_2 \quad \text{if} \quad 2M^2 \geq \beta M_1.
\]
Furthermore, we have
\[
I(h(t) - M^{-1}, t) = M_2 \geq I(h(t) - M^{-1}, t) \quad \text{and} \quad I(h(t), t) = I(h(t), t) = 0.
\]
On the other hand, for \( x \in [h_0 - M^{-1}, h_0] \), there are
\[
\bar{I}(x, 0) = M_2 \left[ 2M(h_0 - x) - M^2(h_0 - x)^2 \right] \geq MM_2(h_0 - x),
\]
\[
I_0(x) = -\int_{h_0}^{h_0} \bar{I}'(s)ds \leq (h_0 - x)\|I_0\|_{C([-h_0, h_0])}
\]
and
\[
\bar{I}(x, 0) \geq I_0(x) \quad \text{in} \quad [h_0 - M^{-1}, h_0] \quad \text{if} \quad MM_2 \geq \|I_0\|_{C([-h_0, h_0])}.
\]
Hence, if we choose
\[
M = \max \left\{ \frac{\|I_0\|_{C([-h_0, h_0])}}{M_2}, \sqrt{\frac{\beta M_1}{2}} \right\},
\]
then the maximum principle deduces that \( \bar{I}(x, t) \geq I(x, t) \) for \( (x, t) \in \Omega_M \). Therefore, we acquire that \( I_x(h(t), t) \geq \bar{I}_x(h(t), t) = -2MM_2 \) and then
\[
h'(t) = -\mu_1I_x(h(t), t) \leq 2\mu_1MM_2 := M_3.
\]
This completes the proof. \( \square \)

**Theorem 2.3.** Assume that (K) holds, then the solution of problem [\( \square \)] exists and is unique for all \( t \in (0, \infty) \).
Theorem 3.1. Thus, the following property holds.

Proof. It follows from the uniqueness of the solutions of \( \theta \) that there is some \( T_{\text{max}} > 0 \) such that \( [0, T_{\text{max}}) \) is the maximal time interval that the solution exists. It remains to show \( T_{\text{max}} = \infty \). We will get our conclusion by deriving a contradiction. Suppose that \( T_{\text{max}} < \infty \), then there exist positive constants \( M_1, M_2 \) and \( M_3 \) independent of \( T_{\text{max}} \) such that

\[
0 < N(x, t) \leq M_1 \quad \text{for } x \in \mathbb{R} \text{ and } t \in [0, T_{\text{max}}),
\]

\[
0 < I(x, t) \leq M_2 \quad \text{for } x \in [g(t), h(t)] \text{ and } t \in [0, T_{\text{max}}),
\]

\[
0 < h'(t), -g'(t) \leq M_3 \quad \text{for } t \in [0, T_{\text{max}})
\]

with the help of Theorem 2.2. According to the standard \( L^p \) estimates, the Sobolev embedding theorem and the H"older estimates for parabolic equations, for some fixed \( \gamma^* \in (0, T_{\text{max}}) \), we can find a positive constant \( M_4 \) depending on \( \gamma^*, M_1, M_2 \) and \( M_3 \) such that

\[
\|N(\cdot, t)\|_{C^2(\mathbb{R})}, \|I(\cdot, t)\|_{C^2([g(t), h(t)])} \leq M_4
\]

for all \( t \in [\gamma^*, T_{\text{max}}) \). Then, according to the proof of Theorem 2.1, there is a \( \tau > 0 \) depending on \( M_i (i = 1, 2, 3, 4) \) such that the solution of problem \( \theta \) with initial time \( T_{\text{max}} - \frac{\tau}{\gamma} \) can be extended uniquely to \( T_{\text{max}} - \frac{\tau}{\gamma} + \tau \), which contradicts to the definition of \( T_{\text{max}} \). Thus, our results follow. \( \square \)

We note that Theorem 2.2 implies that free boundaries \( h(t) \) and \( -g(t) \) are increasing in time and bounded, then \( h_* := \lim_{t \to \infty} h(t) \in (h_0, \infty) \) and \( g_* := \lim_{t \to \infty} g(t) \in (-\infty, -h_0) \) are well-defined.

3. Critical function and comparison principle. In this section, we mainly analyze an eigenvalue problem and discuss the property of its principal eigenvalue, and then propose a function \( R_0(t) \) as a critical term to determine the spreading or vanishing of the disease. These results will play an important role in discussing the main results and have their own interests.

Firstly, let us consider the following eigenvalue problem

\[
\begin{aligned}
\phi_{xx} + \beta \sigma \mu \int_\Omega K(x, y) \phi(y) dy - (\mu + \gamma) \phi + \lambda \phi &= 0 \quad \text{in } \Omega,

\phi &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(13)

where \( \Omega \subset \mathbb{R} \) is a bounded domain with \( \partial \Omega \) of class \( C^{2,\alpha} \). It is well-known that \( \theta \) admits a unique principal eigenvalue (Krein-Rutman theorem), denoted by \( \lambda_1(\Omega) \), and there is a positive eigenfunction \( \phi_1 \) with \( \|\phi_1\|_{L^\infty} = 1 \) corresponding to \( \lambda_1(\Omega) \). By the variational method, \( \lambda_1(\Omega) \) can be characterized by

\[
\lambda_1(\Omega) = \inf_{\phi \in W_0^{1,2}(\Omega), \phi \neq 0} \left\{ \int_\Omega |\nabla \phi|^2 dx - \beta \int_\Omega \int_\Omega K(x, y) \phi(x) \phi(y) dy dx + \mu + \gamma \right\}.
\]

(14)

Thus, the following property holds.

Theorem 3.1. (\cite{4} Corollary 2.3) For \( \Omega_1, \Omega_2 \subset \mathbb{R} \) with \( \Omega_1 \subset \Omega_2 \), there is \( \lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \), with strict inequality if \( \Omega_1 \setminus \Omega_2 \) is an open set.

Inspired by the work in \cite{1}, let us introduce the basic reproduction number by

\[
R_0(\Omega) = \sup_{\phi \in W_0^{1,2}(\Omega), \phi \neq 0} \left\{ \frac{\beta \sigma \mu \int_\Omega \int_\Omega K(x, y) \phi(x) \phi(y) dy dx}{\int_\Omega |\nabla \phi|^2 dx + (\mu + \gamma)} \right\}.
\]

(15)
Note that $\lambda_1(\Omega)$ has the same sign as $1 - R_0(\Omega)$, see the proof in [2]. Meanwhile, for $\Omega_1 \subset \Omega_2 \subset \mathbb{R}$, we have $R_0(\Omega_1) \leq R_0(\Omega_2)$, with strict inequality if $\Omega_1 \setminus \Omega_2$ is an open set. Moreover, for any $\phi \in W^{1,2}_0(\Omega)$ and $\|\phi\|_{L^2(\Omega)} = 1$, there holds $R_0(\Omega) \leq \frac{\beta^\sigma}{\mu(\mu+\gamma)}$ due to

$$\int_\Omega \int_\Omega K(x,y) \phi(x) \phi(y) dy dx \leq \int_\Omega \int_\Omega K(x,y) \frac{\phi^2(x)}{2} + \frac{\phi^2(y)}{2} dy dx \leq 1. \quad (16)$$

Here, $\frac{\beta^\sigma}{\mu(\mu+\gamma)}$ is the basic reproduction number of problem (4) with $\beta SI$ in the place of $\beta S \int_\Omega K(x,y)I(y,t)dy$ and fixed boundary, and in which whether the disease dies out or not mainly depends on $\frac{\beta^\sigma}{\mu(\mu+\gamma)} > 1$ or $\frac{\beta^\sigma}{\mu(\mu+\gamma)} < 1$.

**Remark 1.** For any $\phi \in W^{1,2}_0(\Omega)$ and $\|\phi\|_{L^2(\Omega)} = 1$, define

$$\varphi(x) = \begin{cases} \phi(x), & x \in \Omega, \\ 0, & x \in \mathbb{R} \setminus \Omega. \end{cases} \quad (17)$$

It is easily seen that $\int_\mathbb{R} \int_\mathbb{R} K(x,y) \varphi(x) \varphi(y) dy dx = 1$ if and only if $\varphi$ is a constant value function. In this case, $\varphi = 0$, which is impossible. Thus, we get $R_0(\Omega) < \frac{\beta^\sigma}{\mu(\mu+\gamma)}$.

Clearly, if we choose $\Omega = (-h_0, h_0)$, then (15) becomes

$$R_0(\Omega) = \sup_{\phi \in W^{1,2}_0(\Omega(t)), \phi \neq 0} \left\{ \frac{\beta^\sigma}{\mu} \int_\Omega \int_\Omega K(x,y) \phi(x) \phi(y) dy dx \right\}. \quad (18)$$

Define

$$\Pi = \left\{ \phi \mid \phi \in W^{1,2}_0(\Omega(t)), \phi(x) > 0 \text{ in } \Omega(t) \text{ and } \phi = 0 \text{ on } \mathbb{R} \setminus \Omega(t) \right\}.$$ 

Now, for our free boundary problem (6), we introduce a function $R_0^F(t)$ as follows

$$R_0^F(t) = R_0^F(g(t), h(t)) = \sup_{\phi \in \Pi} \left\{ \frac{\beta^\sigma}{\mu} \int_\mathbb{R} \int_\mathbb{R} K(x,y) \phi(x) \phi(y) dy dx \right\}. \quad (19)$$

Then, it follows from (14) that $R_0^F(0) < 1$ if $\lambda_1(0) = \lambda_1(\Omega(0)) > 0$ and $R_0^F(0) > 1$ if $\lambda_1(0) < 0$, where $\Omega(0) = (-h_0, h_0)$ (resp. $\Omega(t) = (g(t), h(t))$).

It follows from Theorems 2.2 and 3.1 that $R_0^F(t)$ enjoys the following property.

**Theorem 3.2.** $R_0^F(t)$ is strictly increasing in $t$, that is, if $t_1 < t_2$, then $R_0^F(t_1) < R_0^F(t_2)$. Moreover, we have $R_0^F(\infty) := \lim_{t \to +\infty} R_0^F(t) \leq \frac{\beta^\sigma}{\mu(\mu+\gamma)}$.

**Remark 2.** Note that in [25][28], they could give the exact limit of $R_0^F(t)$ which is defined in as $t \to +\infty$ and $h_\infty = \infty$. For (10), we have

$$R_0^F(\infty) = \sup_{\phi \in \Pi} \left( \frac{\beta^\sigma}{\mu} \int_\mathbb{R} \int_\mathbb{R} K(x,y) \phi(x) \phi(y) dy dx \right) (\mu + \gamma)^{-1}$$

if $h_\infty, -g_\infty = \infty$. Hence, we obtain that $R_0^F(\infty) < \frac{\beta^\sigma}{\mu(\mu+\gamma)}$ due to Remark 1. In other words, we have $R_0^F(t) < \frac{\beta^\sigma}{\mu(\mu+\gamma)}$ for all $t > 0$, which is different from the one that the infective class is just in contact with the adjacent individuals, see for example [25].
In what follows, we will regard $R^F_0(t)$ as a critical term to run our discussions. Many conclusions below mainly depend on the constructions of some suitable upper and lower solution and then the comparison principle for free boundary problem is essential here. The following comparison principle is a similar result just as in [10].

**Lemma 3.3.** Suppose that $T \in (0, \infty)$, $\tilde{h}, \tilde{g} \in C^4([0, T])$, $\tilde{N} \in C^{2,1}(\mathbb{R} \times (0, T])$ and $\tilde{I} \in C(D^*_{2T}) \cap C^2(D^*_{2T})$ with $D^*_T = \{(x, t) \in \mathbb{R}^2 : \tilde{g}(t) < x < \tilde{h}(t), \quad 0 < t \leq T\}$; $\bar{h}, g \in C^1([0, T])$, $\bar{N} \in C^{2,1}(\mathbb{R} \times (0, T])$ and $\bar{I} \in C(D^*_{2T}) \cap C^2(D^*_{2T})$ with $D^*_T = \{(x, t) \in \mathbb{R}^2 : g(t) < x < \bar{h}(t), \quad 0 < t \leq T\}$ satisfying

$$
\begin{aligned}
\tilde{N}_t - \Delta \tilde{N} &\geq \sigma - \mu \tilde{N}, \\
\tilde{I}_t - \Delta \tilde{I} &\geq \beta(\tilde{N} - \tilde{I})(K \ast \tilde{I}) - (\mu + \gamma) \tilde{I}, \\
\bar{N}_t - \Delta \bar{N} &\leq \sigma - \mu \bar{N}, \\
\bar{I}_t - \Delta \bar{I} &\leq \beta(\bar{N} - \bar{I})(K \ast \bar{I}) - (\mu + \gamma) \bar{I}, \\
\tilde{I}(x, t) = 0, \\
\bar{I}(x, t) = 0,
\end{aligned}
$$

with $\tilde{N}(x, 0) \geq N_0(x)$, $\tilde{I}(x, 0) \geq I_0(x)$, $\bar{N}(x, 0) \leq \bar{N}_0(x)$, $\bar{I}(x, 0) \leq I_0(x)$ for $x \in \mathbb{R}$ and $\bar{g}(0) < -\bar{h}_0$, $\tilde{h}(0) > \bar{h}_0$, $g(0) > -\bar{h}_0$, $\tilde{h}(0) < \bar{h}_0$. Then the unique solution $(N, I; g, \bar{h})$ of free boundary problem [10] satisfies

$$
N \leq \tilde{N}, \quad N \geq \bar{N}, \quad I \leq \tilde{I}, \quad I \geq \bar{I}, \quad g \geq \bar{g}, \quad g \leq \tilde{g}, \quad h \leq \tilde{h} \quad \text{and} \quad h \geq \bar{h}
$$

for $x \in \mathbb{R}$ and $0 < t \leq T$.

**Proof.** The proof can be obtained by the minor modification of Lemma 3.5 in [10].

4. **Conditions for vanishing.** In this section, we give some sufficient conditions that ensure the disease vanishing. If $h_\infty, -g_\infty < \infty$ and $\lim_{t \to +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0$, we say that the disease vanishing happens; while if $h_\infty, -g_\infty = \infty$, then the domain $(g(t), h(t))$ becomes the whole region $\mathbb{R}$, in this case, we say that the disease spreading happens.

**Theorem 4.1.** If $h_\infty, -g_\infty < \infty$, then $\lim_{t \to +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0$.

**Proof.** First of all, we show that for any integer $n \geq 0$ and $0 < \alpha < 1$, there exists a constant $\tilde{C} = C(\mu_1, g_\infty, h_\infty, M_3)$ with $M_3$ is defined in Theorem 2.2 such that

$$
||g||_{C^{\alpha}[n+1, n+3]} + ||h||_{C^{\alpha}[n+1, n+3]} \leq \tilde{C}.
$$

Our proof here is motivated by Theorem 2.1 in Wang [31] (see also Theorem 4.1 in [40]). We straighten the free boundaries by the following transformations. That is, let

$$
s = \frac{2x}{h(t) - g(t)} - \frac{h(t) + g(t)}{h(t) - g(t)}, \quad N(x, t) = u(s, t), \quad I(x, t) = v(s, t)
$$

and $K(x, y) = K \left( \frac{h(t) - g(t)}{h(t) - g(t)} s + \frac{h(t) + g(t)}{h(t) - g(t)} y \right) = \tilde{K}(s, y)$ with $\tilde{K}$ satisfying the condition $(K)$. Thus, we arrive at

$$
\frac{\partial s}{\partial x} = \frac{2}{h(t) - g(t)} := \sqrt{F(g(t), h(t))}
$$
and
\[\frac{\partial s}{\partial t} = -\frac{h'(t) - g'(t)}{h(t) - g(t)} \frac{h'(t) + g'(t)}{h(t) - g(t)} := E(g(t), h(t), s).\]

Then direct calculations show that \(v(s, t)\) satisfies
\[
\begin{cases}
  v_t + Ev_s - Fv_{ss} = \beta(u - v)(\bar{K} * v) - (\mu + \gamma)v, & s \in (-1, 1), \ t > 0, \\
  v(s, t) = 0, & s \in \mathbb{R} \setminus (-1, 1), \ t > 0, (22) \\
  v(s, 0) = I_0(s) \geq 0, & s \in [-1, 1],
\end{cases}
\]

where \(E = E(g(t), h(t), s)\) and \(F = F(g(t), h(t))\). It is clear that (22) is an initial-boundary value problem with fixed boundary. For any integer \(n \geq 0\), define
\[
  u^n(s, t) = u(s, t + n) \quad \text{and} \quad v^n(s, t) = v(s, t + n).
\]

Then problem (22) becomes following
\[
\begin{cases}
  v^n_t + E^n v^n_s - F^n v^n_{ss} = \beta(u^n - v^n)(\bar{K} * v^n) - (\mu + \gamma)v^n, & s \in (-1, 1), \ t \in [0, 3], \\
  v^n(s, t) = 0, & s \in \mathbb{R} \setminus (-1, 1), \ t \in [0, 3], \\
  v^n(s, 0) = I^n(s) \geq 0, & s \in [-1, 1],
\end{cases}
\]

where \(E^n = E(t + n)\) and \(F^n = F(t + n)\). With the help of Theorem 2.2, we note that \(u^n, v^n, E^n\) and \(F^n\) are bounded uniformly on \(n\), and
\[
\max_{0 \leq t_1, t_2 \leq 3, \ |t_1 - t_2| \leq \tau} \left| F^n(t_1) - F^n(t_2) \right| \leq \frac{8(h^n(t) - g^n(t))' (h^n(t) - g^n(t))^3}{(h^n(t) - g^n(t))^3} \leq \frac{2M_3}{h_0^3} \tau \to 0 \quad \text{as} \quad \tau \to 0
\]
with \(h^n(t) = h(t + n)\) and \(g^n(t) = g(t + n)\). In addition, we have \(F^n \geq \frac{4}{(h_\infty - g_\infty)^2}\) for all \(n \geq 0\) and \(0 \leq t \leq 3\) as if \(h(t) \leq h_\infty < \infty\) and \(g(t) \geq g_\infty > -\infty\).

Choosing \(p \gg 1\), we can use the interior \(L^p\) estimate to derive that there exists a positive constant \(C^*\) independent of \(n\) such that \(\|v^n\|_{W^{2, p}([-1, 1] \times [1, 3])} \leq C^*\) for all \(n \geq 0\). In view of the embedding theorem, it follows that \(\|v^n\|_{C^{1, \alpha}}((-1, 1] \times [1, 3]) \leq C^*\) for all \(n \geq 0\), which implies that \(\|v\|_{C^{1, \alpha}}((-1, 1] \times [1, 3]) \leq C^*\) for all \(n \geq 0\), where \(I_n = [-1, 1] \times [n + 1, n + 3]\). All of those combined with
\[
h'(t) = -\mu I_x(h(t), t), \quad I_x(h(t), t) = \frac{2}{h(t) - g(t)} v_x(1, t),
\]
and \(0 < h'(t), -g'(t) \leq M_3\) allows us to derive (21). Since these rectangles \(I_n\) overlap and constant \(C^*\) is independent of \(n\), it follows that
\[
\|g\|_{C^{1, \alpha/2}([-1, 1])} + \|h\|_{C^{1, \alpha/2}([-1, 1])} \leq \tilde{C} \quad \text{and} \quad \|v\|_{C^{1, \alpha}}([-1, 1] \times [1, 3]) \leq C^* \quad (24)
\]
Moreover, we arrive at \(g'(t) \to 0, h'(t) \to 0\) as \(t \to \infty\) since \(g(t)\) and \(h(t)\) are bounded.

Now we return to the proof of \(L^1(\cdot, t) = C((g(t), h(t))) = 0\). Suppose to the contrary that \(\limsup_{t \to +\infty} \|I(\cdot, t)\|_{C((g(t), h(t)))} = \epsilon > 0\). Then there exists a sequence \((x_k, t_k)\) in \((g(t), h(t)) \times (0, +\infty)\) such that \(I(x_k, t_k) \geq \frac{\epsilon}{2}\) for all \(k \in \mathbb{N}\) and \(t_k \to \infty\) as \(k \to \infty\). Since \(-\infty < g_\infty < g(t) < x_k < h(t) < h_\infty < \infty\), we then have that a subsequence of \(\{x_k\}\) converges to \(x_0 \in (g_\infty, h_\infty)\). Without loss of generality, we assume \(x_k \to x_0\) as \(k \to \infty\). Define
\[
N_k(x, t) = N(x, t_k + t) \quad \text{and} \quad I_k(x, t) = I(x, t_k + t)
\]
for \((x, t) \in (g(t_k + t), h(t_k + t)) \times (-t_k, \infty)\). It follows from the parabolic regularity theory that \(\{(N_k, I_k)\}\) has a subsequence \(\{(N_{k_i}, I_{k_i})\}\) such that \((N_{k_i}, I_{k_i}) \to (\hat{N}, \hat{I})\) as \(i \to \infty\), where \((\hat{N}, \hat{I})\) satisfies

\[
\begin{align*}
&\begin{cases}
\dot{\hat{N}}_i - \Delta \hat{N} = \sigma - \mu \hat{N}, & g_\infty < x < h_\infty, \ t \in \mathbb{R}, \\
\dot{\hat{I}}_i - \Delta \hat{I} = \beta(\hat{N} - \hat{I}) \int_{\mathbb{R}} K(x, y) \hat{I}(y, t) \, dy - (\mu + \gamma) \hat{I}, & g_\infty < x < h_\infty, \ t \in \mathbb{R}.
\end{cases}
\end{align*}
\]

Note that \(\hat{I}(x_0, 0) \geq \frac{\gamma}{2}\), therefore \(\hat{I} > 0\) in \((g_\infty, h_\infty) \times (-\infty, +\infty)\). Recalling that

\[
\beta(\hat{N} - \hat{I}) \int_{\mathbb{R}} K(x, y) \hat{I}(y, t) \, dy - (\mu + \gamma) \hat{I} \geq -\beta \|\hat{I}(\cdot, t)\|_{L^\infty([g(t), h(t)])} \hat{I} - (\mu + \gamma) \hat{I}.
\]

Putting \(\hat{M} = \|\hat{I}(\cdot, t)\|_{L^\infty([g(t), h(t)])} + (\mu + \gamma)\), then using Hopf lemma to the equation

\[
\dot{\hat{K}}_i - \Delta \hat{K} \geq -\hat{M} \hat{K} \text{ at } (h_\infty, 0) \text{ and } (g_\infty, 0) \text{ yields that } \hat{K}_i(h_\infty, 0) \leq -\sigma_0 < 0 \text{ and } \hat{K}_i(g_\infty, 0) \geq \sigma_0 > 0 \text{ respectively for some } \sigma_0 > 0.
\]

Since \(h'\) \to 0, \(g'\) \to 0 as \(t \to \infty\), then by Stefan conditions we have \(I_x(h(t_k), t_k) \to 0\) and \(I_x(g(t_k), t_k) \to 0\) as \(t_k \to \infty\). However, it is clear that \(\|\|C^{1+\alpha, 1+\alpha}_{\mathbb{R}^+ \times [1, \infty)}(I_x(g(t_k), h(t_k))) \leq C^*\) exhibits \(I_x(h(t_k), t_k) = (I_k)_x(h(t_k), 0) \to I_x(h_\infty, 0)\) and \(I_x(g(t_k), t_k) = (I_k)_x(g(t_k), 0) \to I_x(g_\infty, 0)\) as \(k \to \infty\), which contradicts to \(I_x(h_\infty, 0) \leq -\sigma_0 < 0\) and \(I_x(g_\infty, 0) \geq \sigma_0 > 0\). Hence, there must holds

\[
limit_{t \to +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0.
\]

This completes the proof. \(\square\)

**Theorem 4.2.** \(h_\infty, -g_\infty < \infty\) provided that if \(\mu_1 \leq \frac{1}{4h_\infty M_1}\) and \(h_0 \leq \sqrt{\frac{1}{4h_\infty M_1}}\), where \(M_1\) is the super bounds of \(N(x, t)\) obtained in Theorem 2.2 and \(M^* = \frac{4}{3} \|I_0\|_{L^\infty}\).

**Proof.** We are going to construct a suitable upper solution to (8) and then apply Lemma 3.3 to prove this theorem. As in [10 20], we define

\[
\delta(t) = 2h_0(2 - e^{-\eta t}), \quad \vartheta(t) = -\delta(t), \quad t \geq 0,
\]

\[
\hat{N}(x, t) = M_1, \quad V(y) = 1 - y^2, \quad -1 \leq y \leq 1,
\]

\[
\hat{I}(x, t) = \begin{cases}
M^* e^{-\eta t} V \left( \frac{x}{\delta(t)} \right), & \vartheta(t) \leq x \leq \delta(t), \\
0, & x > \delta(t) \text{ or } x < \vartheta(t),
\end{cases}
\]

where \(\eta\) and \(M^*\) are positive constants which will be chosen later. We notice that

\[
V'' = -2 < 0 \text{ and } \delta'(t) = 2h_0 \eta e^{-\eta t}, \quad \delta^2(t) = 4h_0^2(2 - e^{-\eta t})^2.
\]

A series of detailed calculations yield \(\hat{N}_i - \Delta \hat{N} = 0 \geq \sigma - \mu \hat{N}\) and

\[
\begin{align*}
&\hat{I}_i - \Delta \hat{I} - \beta(\hat{N} - \hat{I}) \int_{\mathbb{R}} K(x, y) \hat{I}(y, t) \, dy + (\mu + \gamma) \hat{I} \\
&\geq \hat{I}_i - \Delta \hat{I} - \beta \hat{N} \int_{\mathbb{R}} K(x, y) \hat{I}(y, t) \, dy + (\mu + \gamma) \hat{I} \\
&= M^* e^{-\eta t} \left[ -\eta V - x \frac{\delta'(t)}{\delta^2(t)} V' - \frac{V''}{\delta^2(t)} - \beta M_1 \int_{\mathbb{R}} K(x, y) V \left( \frac{y}{\delta(t)} \right) \, dy + (\mu + \gamma) V \right] \\
&\geq M^* e^{-\eta t} \left[ -\eta - \beta M_1 + \frac{1}{8h_0^2} \right] \geq 0
\end{align*}
\]
provided that if \( h_0 \leq \sqrt{\frac{1}{16M_1}} \) and \( \eta = \frac{1}{10h_0^2} \). On the other hand, we have

\[
-\mu_1 \vec{I}_x(\delta(t), t) = 2\mu_1 M^* e^{-\eta t} \frac{\delta(t)}{\delta(t)} \quad \text{and} \quad -\mu_1 \vec{I}_x(\vartheta(t), t) = 2\mu_1 M^* e^{-\eta t} \frac{\vartheta(t)}{\vartheta(t)}.
\]

Then \( \delta'(t) \geq -\mu_1 \vec{I}_x(\delta(t), t) \) and \( \vartheta'(t) \leq -\mu_1 \vec{I}_x(\vartheta(t), t) \) if \( \mu_1 < \frac{1}{8M^*} \). Moreover, following \( \delta(t) = 2h_0(2 - e^{-\eta t}) \leq 4h_0 \), there hold

\[
\bar{N}(x, 0) \geq N_0(x) \quad \text{and} \quad \bar{I}(x, 0) = M^* \left( 1 - \frac{x^2}{4h_0^2} \right) \geq \frac{3}{4} M^*
\]

for \( x \in [-h_0, h_0] \). Now, we choose \( M^* = \frac{4}{3} \| I_0 \|_{L^\infty} \), then for \( h_0 \leq \sqrt{\frac{1}{16M_1}} \), there is

\[
\begin{align*}
\bar{N}_t - \Delta \bar{N} &\geq \sigma - \mu \bar{N}, & x \in \mathbb{R}, \ t > 0, \\
\bar{I}_t - \Delta \bar{I} &\geq \beta(\bar{N} - \bar{I})(K \ast \bar{I}) - (\mu + \gamma)\bar{I}, & x \in (\vartheta(t), \delta(t)), \ t > 0, \\
\bar{I}(x, t) &\geq 0, & x \geq \delta(t) \text{ or } x \leq \vartheta(t), \ t > 0, \\
\delta'(t) &\geq -\mu_1 \bar{I}_x(\delta(t), t), \ \delta(0) = 2h_0 > h_0, & t > 0, \\
\vartheta'(t) &\leq -\mu_1 \bar{I}_x(\vartheta(t), t), \ \vartheta(0) = -2h_0 < -h_0, & t > 0, \\
\bar{N}(x, 0) &\geq N_0(x), \ \bar{I}(x, 0) \geq I_0(x), & x \in \mathbb{R}.
\end{align*}
\]

Thus, following from Lemma 3.3, we have \( g(t) \geq \vartheta(t) \) and \( h(t) \leq \delta(t) \) for \( t > 0 \). An immediate result is \( h_\infty, -g_\infty < \lim_{t \to \infty} \delta(t) = 4h_0 < \infty \). This completes the proof. \( \square \)

**Remark 3.** Compared Theorem 4.2 with Lemma 4.1 in [26], due to the effect of the nonlocal term, we find that our result shows that the disease will die out eventually with a smaller \( h_0 \) but with no restriction on the relation between \( \frac{\beta \sigma}{\mu (\rho + \gamma)} \) and 1.

**Theorem 4.3.** Assume that \( R_*^\nu(0) < 1 \). Then \( h_\infty, -g_\infty < \infty \) provided that \( \| I_0 \|_{L^\infty} \) and \( h_0 \) are sufficiently small.

**Proof.** As in the proof of Theorem 4.2, we will construct a suitable upper solution to (8). Since \( R_*^\nu(0) < 1 \), then we have \( \lambda_1(0) > 0 \) and \( 0 < \phi(x) \leq 1 \) in \((-h_0, h_0)\) such that

\[
\begin{align*}
\phi_{xx} + \frac{\beta \sigma}{\mu} \int_{\mathbb{R}} K(x, y)\phi(y)dy - (\mu + \gamma)\phi + \lambda_1 \phi &= 0, \quad -h_0 < x < h_0, \\
\phi(-h_0) &= \phi(h_0) = 0.
\end{align*}
\]

Hence, there exists a small \( g > 0 \) such that

\[
\lambda_1(0) \phi \geq g(1 + g^2) + \frac{\beta \sigma}{\mu} g (2 + g).
\]

Recalling the fact that there exists \( T_1 > 0 \) such that \( 0 < N(x, t) \leq \frac{g}{\mu} \) in \([g(t), h(t)] \times [T_1, +\infty)\), then as in [10, 26], we define

\[
\delta(t) = h_0(1 + g - \frac{g}{2} e^{-\nu t}), \quad N(x, t) = \frac{\sigma}{\mu}, \quad t \geq T_1,
\]

\[
I(x, t) = \begin{cases} xe^{-\nu t} \phi \left( \frac{xh_0}{\delta(t)} \right), & -\delta(t) \leq x \leq \delta(t), \ t \geq T_1, \\ 0, & x > \delta(t) \text{ or } x < -\delta(t), \ t \geq T_1. \end{cases}
\]
A series of detailed calculations yield $\dot{N} - \Delta \bar{N} = 0 = \sigma - \mu \bar{N}$ and
\[
\dot{I} - \Delta \bar{I} - \beta(N - \bar{I}) \int_{\mathbb{R}} K(x, y) \bar{I}(y, t) dy + (\mu + \gamma) \bar{I}
\]
\[
\geq \dot{I} - \Delta \bar{I} - \beta N \int_{\mathbb{R}} K(x, y) \bar{I}(y, t) dy + (\mu + \gamma) \bar{I}
\]
\[
= \varepsilon e^{-\mu t} \left[ -\phi - \frac{xh_0\phi'(t)}{\delta^2(t)} \phi' - \frac{h_0^2}{\delta^2(t)} \phi'' - \beta \frac{\sigma}{\mu} \int_{\mathbb{R}} K(x, y) \bar{I}(y, t) dy + (\mu + \gamma) \phi \right]
\]
\[
\geq \varepsilon e^{-\mu t} \left\{ -\phi + \frac{\lambda_1(0) \phi}{(1 + \varrho)^2} + \beta \frac{\sigma}{\mu} \left[ \frac{1}{(1 + \varrho)^2} - 1 \right] \int_{\mathbb{R}} K(x, y) \bar{I}(y, t) dy \right\}
\]
\[
\geq 0
\]
provided that
\[
-\phi + \frac{\lambda_1(0) \phi}{(1 + \varrho)^2} + \beta \frac{\sigma}{\mu} \left[ \frac{1}{(1 + \varrho)^2} - 1 \right] \int_{\mathbb{R}} K(x, y) \bar{I}(y, t) dy \geq 0.
\]
That is
\[
\lambda_1(0) \phi \geq \varrho(1 + \varrho)^2 + \beta \frac{\sigma}{\mu} \left[ (1 + \varrho)^2 - 1 \right],
\]
which is correct since we already have $\boxed{28}$. On the other hand, there hold
\[
-\mu_1 \dot{I}_x(\delta(t), t) = -\mu_1 \varepsilon e^{-\mu t} \phi'(h_0) \frac{h_0}{\delta(t)} \quad \text{and} \quad -\mu_1 \dot{I}_x(-\delta(t), t) = -\mu_1 \varepsilon e^{-\mu t} \phi'(-h_0) \frac{h_0}{\delta(t)}.
\]
Noticing that $\phi'(-h_0) = -\phi'(h_0) > 0$, then for $0 < \varepsilon \leq \frac{\varrho^2 h_0}{2 \mu \phi'(h_0)}$, we have
\[
\delta'(t) \geq -\mu_1 \dot{I}_x(\delta(t), t) \quad \text{and} \quad -\delta'(t) \leq -\mu_1 \dot{I}_x.
\]
Moreover, if $\bar{N}(x, 0) \geq \|N_0\|_{L^\infty}$ and $\|I_0\|_{L^\infty} \leq \bar{I}(x, 0) = \varepsilon \phi \left( \frac{x}{1 + \varrho^2} \right) \leq \varepsilon$, then
\[
(\bar{N}, \bar{I}, -\delta(t), \delta(t)) \text{ is an upper solution of } (0).
\]
Hence, there holds $h(t), -g(t) \leq \delta(t)$, and then $h_\infty, -g_\infty \leq \lim_{t \to \infty} \delta(t) = h_0(1 + \varrho) < \infty$. This completes the proof. $\square$

By the monotonicity and continuity of $R^F_0(t)$, we find that if $R^F_0(0) < 1$, there exists some $t_0 > 0$ such that $R^F_0(t_0) = 1$. Thus, for any fixed $t^* \in (0, t_0)$, we have $R^F_0(t^*) < 1$ and $\lambda_1(t^*) > 0$. By repeating the proof of Theorem 4.3 in which $h_0$ and $\lambda_1(0)$ are replaced by $h(t^*)$ and $\lambda_1(t^*)$ respectively, we can obtain that $h_\infty, -g_\infty < \infty$ if $R^F_0(t^*) < 1$ as well as $\|I_0\|_{L^\infty}$ and $h_0$ are sufficiently small. In addition, theorems 4.2 and 4.3 show that the disease vanishes in the long run with an exponential decay as long as the initial values and initial region are sufficiently small.

5. Conditions for spreading. This section is devoted to the sufficient conditions that make the disease spread. For the case of $R^F_0(0) \geq 1$, we have the following spreading result.

**Theorem 5.1.** Assume that $R^F_0(0) \geq 1$. Then $\lim_{t \to +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} > 0$ and $h_\infty, -g_\infty = \infty$.

**Proof.** For the case that $R^F_0(0) > 1$, we have $\lambda_1(0) < 0$. Recalling the fact that $\lim_{t \to +\infty} N(x, t) = \frac{\mu}{\varrho}$ uniformly in any bounded subset of $\mathbb{R}$, therefore, for some $\epsilon_0 > 0$ small enough, there exists $T^* > 0$ such that $N(x, t) \geq \frac{\mu}{\varrho} - \epsilon_0$ for all $x \in [g(t), h(t)]$ and $t \geq T^*$. Let
\[
N(x, t) = \frac{\sigma}{\mu} - \epsilon_0 \quad \text{and} \quad I(x, t) = e^t \phi(x) \quad \text{for} \quad (x, t) \in [g(t), h(t)] \times [T^*, +\infty),
\]
where \(0 < \phi(x) \leq 1\) is the eigenfunction corresponding to \(\lambda_1(0)\) and \(\epsilon^* > 0\) is small. We note that for sufficiently small \(\epsilon_0\) and \(\epsilon^*\), there is \(\lambda_1(0) \leq -\beta (\epsilon_0 + \epsilon^*)\). Therefore, we arrive at \(N_\epsilon - \Delta N = 0 \leq \sigma - \mu N\) and

\[
I_t - \Delta I - \beta (N - I) \int_R K(x, y)I(y, t)dy + (\mu + \gamma)I =
\]

\[
= -\epsilon^*\phi'' - \beta \left(\frac{\sigma}{\mu} - \epsilon_0 - \epsilon^*\phi\right) \int_R K(x, y)\epsilon^*\phi(y)dy + (\mu + \gamma)\epsilon^*\phi
\]

\[
= \epsilon^* \left[-\phi'' - \beta \left(\frac{\sigma}{\mu} - \epsilon_0 - \epsilon^*\phi\right) \int_R K(x, y)\phi(y)dy + (\mu + \gamma)\phi\right]
\]

\[
= \epsilon^* \left[\lambda_1(0)\phi + \beta (\epsilon_0 + \epsilon^*) \int_R K(x, y)\phi(y)dy\right]
\]

\[
\leq \epsilon^* [\lambda_1(0) + \beta (\epsilon_0 + \epsilon^*)]
\]

\[
\leq 0.
\]

It follows from the comparison principle that

\[
\liminf_{t \to +\infty} \|I(\cdot, t)||_{C([g(t), h(t)])} \geq \liminf_{t \to +\infty} \|I(\cdot, t)||_{C([g(t), h(t)])} = \|\epsilon^*\phi(x)||_{C([g(t), h(t)])} > 0,
\]

then our conclusion follows and \(h_\infty, -g_\infty = \infty\) by Theorem 4.1.

If \(R_t^F(0) = 1\), then for any \(t_0 > 0\), we have \(h(t_0) > h_0\). It follows from Theorem 3.2 that \(R_t^F(t_0) > R_0^F(0)\). Replacing the initial time 0 by \(t_0\), then we arrive at \(h_\infty, -g_\infty = +\infty\).

**Theorem 5.2.** Assume that \(R_0^F(0) < 1 < \frac{\beta\sigma}{\mu(\mu + \gamma)}\), then the disease will spread as time elapse if \(\|N_0\|_{L^\infty}\) and \(\|I_0\|_{L^\infty}\) are sufficiently large.

**Proof.** Here, we will construct a lower solution to (6) to illustrate our conclusion. As in Theorem 5.1 we have \(N(x, t) \geq \frac{\sigma}{\mu} - \epsilon_0\) for all \(x \in [g(t), h(t)]\), \(t \geq T^*\) and \(\epsilon_0 > 0\) small enough. Define

\[
\delta(t) = \sqrt{1 + \sigma}, \quad N(x, t) = \frac{\sigma}{\mu} - \epsilon_0 \quad \text{for} \quad t > T^*,
\]

\[
I(x, t) = \begin{cases} 
M^{**} \left\{ \begin{array}{ll}
\frac{M^{**}}{\sigma}\epsilon^* \phi \left(\frac{x}{\sqrt{1 + \sigma}}\right), & \delta(t) < x < \delta(t), t > T^*, \\
0, & x \geq \delta(t) \quad \text{or} \quad x \leq -\delta(t), t > T^*,
\end{array} \right.
\end{cases}
\]

where

\[
M^{**} > \frac{(T + 1)^k}{2\mu\phi'(h_0)} > 0 \quad \text{for} \quad 0 < \phi(x) \leq 1
\]

and 0 < \(\phi(x) \leq 1\) denotes the associated eigenfunction of eigenvalue problem

\[
\phi_{xx} + \frac{\text{sgn}(x)}{2}\phi_x + \beta \frac{\sigma}{\mu} \int_{-1}^{1} K(x, y)\phi(y)dy - (\mu + \gamma)\phi + \lambda_1(0)\phi = 0, \quad -1 < x < 1,
\]

\[
\phi(-1) = \phi(1) = 0.
\]

Note that \(\lambda_1(0) > 0\) since \(R_0^F(0) < 1\). Direct calculations show that \(N_\epsilon - \Delta N \leq \sigma - \mu N\) and

\[
I_t - \Delta I - \beta (N - I) \int_R K(x, y)I(y, t)dy + (\mu + \gamma)I
\]
we have lim for all such $R$ the initial values are sufficiently large even though $R$ Remark 4.

provided that $\beta/\mu$ and $\lambda(0)$ Lemma 3.3 that $h$ we arrive at

and

provided that $M** > -(T+1)^k/2\mu \phi'(-1) > 0$

and

and

provided that $\parallel N_0 \parallel_{L^\infty}$ and $\parallel I_0 \parallel_{L^\infty}$ are sufficiently large. Then, it follows from Lemma 3.3 that $h(t), -g(t) \geq \delta(t)$ for all $t \in [0, T]$, an immediate result is

In addition, Theorem 3.2 implies that there exists some $T^{**} > 0$ such that $R_0^F(t) > 1$ for all $t > T^{**}$. Thus, if we choose $T > \max\{T^*, T^{**}\}$, then $R_0^F(T) > 1$. Hence, we have limit as $t \to \infty$ $\parallel I(t) \parallel_{L^\infty} > 0$ and $\parallel g(t) \parallel_{L^\infty} > 0$ by Theorem 5.1.

Remark 4. Theorem 5.2 indicates that the disease will spread in the long run if the initial values are sufficiently large even though $R_0^F(0) < 1$. Furthermore, by $R_0^F(0) < 1 < \frac{\beta}{\mu(\mu+\gamma)}$, we have $R_0^F(t) > 1$ for all $t > t_0$, and $t_0 > 0$ is the one such that $R_0^F(t_0) = 1$. Hence, for any fixed $t_0 > 0$, we have $R_0^F(t_0) > 1$. Therefore, for such a $t_0$, the disease will spread in the long run by Theorem 5.1.

6. Discussion. As far as we concerned that the threshold for many epidemiology models is the so-called basic reproduction number $R_0$, which is defined as the average number of secondary infections produced when one infected individual is introduced into a host population where everyone is susceptible. It is shown in [2] (the basic reproduction number is defined for the involved SIS PDE model there) that regardless of the initial values and initial habit region, if $R_0 < 1$, the unique disease-free equilibrium is globally asymptotic stable and if $R_0 > 1$, the disease-free equilibrium is unstable and there is a unique endemic equilibrium, see also [20] and the references therein.
Our results here indicate that the spreading of the disease is not only determined by the reproduction number, but also the initial size $[-h_0, h_0]$ as well as the initial functions $S_0$ (resp. $N_0$) and $I_0$, which seems more reasonable and acceptable. Meanwhile, we note that $R^F_0(t)$ defined as a critical function and $R^F_0(0)$ defined as a threshold are not bigger than $\frac{\beta\sigma}{\mu(\mu+\gamma)}$, which gives rise to a phenomenon that the disease spreads more easily due to the free boundary in the senses that: if $R^F_0(0) \geq 1$, then $\lim_{t \to +\infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} > 0$ and $h_\infty, -g_\infty = \infty$; if $R^F_0(0) < 1$, then the disease will spread as time elapses if $\|N_0\|_{L^\infty}$ and $\|I_0\|_{L^\infty}$ are sufficiently large. Precisely, due to the appearance of the nonlocal effect, we obtain that the disease becomes more easy to spread, see Remark 1. That is, the nonlocal interaction may enhance the spread of the disease.

Moreover, it is noticed that if $K(x, y) = K(x - y)$ and $K(x) = \delta(x)$, where $\delta(\cdot)$ is the Dirac delta function, then $R^F_0(0)$ in (19) becomes

$$R^F_0(t) = \beta\sigma \frac{\pi^2}{\mu(h(t) - g(t))^2 + \mu + \gamma}^{-1}$$

and hence there holds $R^F_0(\infty) = \frac{\beta\sigma}{\mu(\mu+\gamma)}$ as if $h_\infty, -g_\infty = \infty$ in these conditions.

Acknowledgments. The authors are grateful to anonymous referee for his/her very valuable comments and suggestions helping to the improvement of the original manuscript. The second author was supported in part by NSF of China (11271172).

REFERENCES

[1] S. Ai and R. Albashaireh, Traveling waves in spatial SIRS models, J. Dynam. Differential Equations, 26 (2014), 143–164.

[2] L. J. S. Allen, B. M. Bolker, Y. Lou and A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, Discrete Contin. Dyn. Syst., 21 (2008), 1–20.

[3] D. Aronson, The asymptotic speed of propagation of a simple epidemic, Research Notes in Math., 14 (1977), 1–23, Pitman, London.

[4] R. Cantrell and C. Cosner, Spatial Ecology via Reaction-Diffusion Equations Wiley Series in Mathematical and Computational Biology, 2003.

[5] J. Cao, W. T. Li, J. Wang and F. Yang, A free boundary problem of a diffusive SIRS model with nonlinear incidence, submitted, 2015.

[6] X. Chen and A. Friedman, A free boundary problem arising in a model of wound healing, SIAM J. Math. Anal., 32 (2000), 778–800.

[7] X. Chen and A. Friedman, A free boundary problem for an elliptic-hyperbolic system: an application to tumor growth, SIAM J. Math. Anal., 35 (2003), 974–986.

[8] Y. Du and Z. Guo, Spreading-Vanishing dichotomy in a diffusive logistic model with a free boundary II, J. Differential Equations, 250 (2011), 4336–4366.

[9] Y. Du, Z. Guo and R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, J. Funct. Anal., 265 (2013), 2089–2112.

[10] Y. Du and Z. Lin, Spreading-Vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., 42 (2010), 377–405.

[11] Y. Du and Z. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), 3105–3132.

[12] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, J. Eur. Math. Soc., 17 (2015), 2673–2724.

[13] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, SIAM J. Math. Anal., 46 (2014), 375–396.

[14] Y. Du and X. Liang, Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model, Ann. Inst. H. Poincaré Anal. Non Linéaire, 32 (2015), 279–305.

[15] A. Ducrot and P. Magal, Travelling wave solutions for an infection-age structured model with diffusion, Proc. Roy. Soc. Edinburgh Sect. A, 139 (2009), 459–482.
P. Zhou and D. Xiao, The diffusive logistic model with a free boundary in heterogeneous environment, *J. Differential Equations*, 256 (2014), 1927–1954.

Received February 2016; revised March 2016.

E-mail address: caojf07@lzu.edu.cn
E-mail address: wtli@lzu.edu.cn (Corresponding author)
E-mail address: yangfy@lzu.edu.cn