Rigidity of Einstein Metrics as Critical Points of Some Quadratic Curvature Functionals on Complete Manifolds

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Abstract
In this paper, we consider some rigidity results for the Einstein metrics as the critical points of some known quadratic curvature functionals on complete manifolds, characterized by some point-wise inequalities. Moreover, we also provide rigidity results by the integral inequalities involving the Weyl curvature, the traceless Ricci curvature and the Sobolev constant, accordingly.

Keywords Critical metric · Sobolev constant · Einstein · rigidity

Mathematics Subject Classification Primary 53C24 · Secondary 53C21

1 Introduction
In this paper, we always assume that \( M^n \) is a complete manifold of dimension \( n \geq 3 \) and \( g \) is a Riemannian metric on \( M^n \) with the Riemannian curvature tensor \( R_{ijkl}, \) the Ricci tensor \( R_{ij} \) and the scalar curvature \( R. \) It is well known that Einstein metrics are critical points for the Einstein–Hilbert functional

\[
\mathcal{H} = \int_M R
\]
on the space of unit volume metrics $\mathcal{M}_1(M^n)$. In [4], Catino considered the following family of quadratic curvature functionals

$$\mathcal{F}_t = \int_M |R_{ij}|^2 + t \int_M R^2, \quad t \in \mathbb{R} \tag{1.1}$$

which are also defined on $\mathcal{M}_1(M^n)$, and proved some related rigidity results. Furthermore, it has been observed in [2] that every Einstein metric is a critical point of $\mathcal{F}_t$ for all $t \in \mathbb{R}$ (see (2.17) in Sect. 2). But the converse of this conclusion is not true in general.

Therefore, it is natural to study canonical metrics which arise as solutions of Euler–Lagrange equations for more general curvature functionals. For instance, Anderson [1] proved that every complete three-dimensional critical metric for the Ricci functional $\mathcal{F}_0$ with non-negative scalar curvature is flat. In [3], Catino gave a characterization of complete critical metrics for the functional

$$\mathcal{H} = \int_M R^2$$

with non-negative scalar curvature in every dimension. For the $\sigma_2$-curvature functional on $M^3$, Catino [7] proved that flat metrics are the only complete metrics with non-negative scalar curvature.

By imposing some inequalities conditions involving the Weyl tensor $W$ and the traceless Ricci tensor $\tilde{\text{Ric}}$, with components $W_{ijkl}$ and $\tilde{R}_{ij}$, respectively, we are able to prove a few rigidity results for the Einstein metrics considered as the critical points of the functional $\mathcal{F}_t$ given by (1.1).

Firstly, we define two constants $D_n$ and $E_n$ by

$$D_n = \begin{cases} \frac{4}{n^2}, & \text{if } n = 3, 4, 5, 6; \\ \left(\frac{4}{n}-1\right)^2, & \text{if } n \geq 7; \end{cases} \tag{1.2}$$

$$E_n = \begin{cases} \sqrt{6}, & \text{if } n = 4; \\ \frac{4(n-1)}{n(n-2)} \left(\frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}\right)^{-1}, & \text{if } n \geq 6. \end{cases} \tag{1.3}$$

Then our main results in this paper can be stated as follows:

**Theorem 1.1** Let $(M^n, g)$ be a complete Riemannian manifold of dimension $n \geq 3$ with positive scalar curvature and

$$\int_M |\tilde{\text{Ric}}|^2 < \infty. \tag{1.4}$$

Suppose that $g$ is a critical metric for the functional $\mathcal{F}_t$ over $\mathcal{M}_1(M^n)$ with

$$t \leq \frac{5}{12}, \quad \text{when } n = 3;$$

$$t \leq \frac{\Gamma_{n-1}}{50(n-1)}, \quad \text{when } n \geq 4. \tag{1.5}$$
If
\[ W + \frac{\sqrt{nD_n}}{\sqrt{2(n-2)}} \text{Ric} \otimes g \leq -\frac{2\sqrt{2}[(n-2) + n(n-1)t]}{n\sqrt{n-1}(n-2)} R, \] (1.6)
then \( M^n \) is Einstein. In particular, when \( n = 3 \), \( M^3 \) must be of constant positive sectional curvature; when \( n \geq 4 \), if the parameter \( t \) also satisfies that
\[
\begin{align*}
&\left\{ \begin{array}{c}
-\frac{\sqrt{2}+1}{6\sqrt{2}} < t \leq -\frac{51}{150}, \\
-\frac{3}{20} \left( 1 + \frac{\sqrt{15}}{8} \right) < t \leq -\frac{8}{25}, \\
-(\frac{n-2}{n(n-1)} + \frac{1}{2\sqrt{\sqrt{\pi n} C_n}} \sqrt{\frac{n-2}{n-1}}) < t \leq -\frac{13n-1}{50(n-1)},
\end{array} \right. \\
\text{when } n = 4, 5, 6, \quad (1.7)
\end{align*}
\]
then \( M^n \) must be of constant positive sectional curvature.

We recall that the Sobolev constant \( Q_g(M) \) is defined by
\[
Q_g(M) = \inf_{0 \neq u \in C_0^\infty(M)} \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right)}{\left( \int_M |u|^\frac{2n}{n-2} \right)^\frac{n-2}{n}}. \tag{1.8}
\]

For example, any simply connected complete manifold with \( W \equiv 0 \) has positive Sobolev constant (see [20]). Moreover, it is easy to see from (1.8) that
\[
Q_g(M) \left( \int_M |u|^\frac{2n}{n-2} \right)^\frac{n-2}{n} \leq \int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) \tag{1.9}
\]
for any smooth \( L^{\frac{2n}{n-2}} \)-function \( u \). With the help of (1.9), we can prove the following rigidity result:

**Theorem 1.2** Let \((M^n, g)\) be a complete Riemannian manifold of dimension \( n \geq 3 \) with positive scalar curvature and \( g \) be a critical metric for the functional \( F_t, t < -\frac{1}{2} \), over \( \mathcal{M}_1(M^n) \) satisfying
\[
\int_M |\text{Ric}|^2 < \infty. \tag{1.10}
\]

If \( Q_g(M) > 0 \) and
\[
\left( \int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \text{Ric} \otimes g \right| \right)^\frac{n}{2} < \sqrt{\frac{n-1}{2(n-2)} Q_g(M)}, \tag{1.11}
\]
then \( M^n \) is Einstein. In particular,

(1) when \( n = 3, 4, 5 \), \( M^n \) must be of constant positive sectional curvature;
(2) when \( n \geq 6 \), if we replace (1.11) with

\[
\left( \int_M |W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \hat{\text{Ric}} \otimes g |^2 \right)^{\frac{n}{2}} < E_n Q_g(M),
\]

then \( M^n \) must be of constant positive sectional curvature.

Since there is no complete non-compact Einstein manifold with positive constant scalar curvature, the following results follows easily from the above two theorems:

**Corollary 1.3** Suppose that \((M^n, g)\) is a complete non-compact Riemannian manifold of dimension \( n \geq 3 \) with positive scalar curvature and that \( g \) is a critical metric for the functional \( \mathcal{F}_t \) over \( \mathcal{H}_1(M^n) \) with \( t \) satisfying (1.5). If (1.6) holds, then it must hold that

\[
\int_M |\hat{\text{Ric}}|^2 = \infty.
\]

**Corollary 1.4** Suppose that \((M^n, g)\) is a complete non-compact Riemannian manifold of dimension \( n \geq 3 \) with positive scalar curvature and that \( g \) is a critical metric for the functional \( \mathcal{F}_t \) over \( \mathcal{H}_1(M^n) \) with \( t < -\frac{1}{2} \). If \( Q_g(M) > 0 \) and (1.11) holds, then it must hold that

\[
\int_M |\hat{\text{Ric}}|^2 = \infty.
\]

**Remark 1.1** Noticing that if \( M^n \) has constant sectional curvature, then both the Weyl tensor \( W \) and the traceless Ricci tensor \( \hat{\text{Ric}} \) are identically equal to zero, implying that the function

\[
|W + \frac{\sqrt{n D_n}}{\sqrt{2(n-2)}} \hat{\text{Ric}} \otimes g |
\]

is identically zero. But in general the right-hand side of (1.6) is strictly positive by (1.5) and the assumption that \( R > 0 \). Therefore, Theorem 1.1 actually reveals a pinching phenomenon for the function (1.15). Similarly, when \( n = 3, 4, 5 \), Theorem 1.2 also discloses a different pinching phenomenon for the quantity

\[
\int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \hat{\text{Ric}} \otimes g \right|^\frac{n}{2}.
\]

**Remark 1.2** The first two authors of the present paper have previously proved in [19] some rigidity results on Einstein metrics as the critical points of the same quadratic curvature functionals for the case of compact manifolds. Our theorems are then the generalizations of those conclusions to the case of complete manifolds.
2 Some Necessary Lemmas

Recall that the Weyl curvature $W_{ijkl}$ of a Riemannian manifold $(M^n, g)$ with $n \geq 3$ is related to the Riemannian curvature $R_{ijkl}$ by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} \left( R_{ikgjl} - R_{ilgjk} + R_{jlgik} - R_{jkgil} \right) + \frac{R}{(n-1)(n-2)} (g_{ikgjl} - g_{ilgjk}). \quad (2.1)$$

Since the traceless Ricci curvature $\hat{R}_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, $(2.1)$ can be written as

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} \left( \hat{R}_{ikgjl} - \hat{R}_{ilgjk} + \hat{R}_{jlgik} - \hat{R}_{jkgil} \right) - \frac{R}{n(n-1)} (g_{ikgjl} - g_{ilgjk}). \quad (2.2)$$

Furthermore, the Cotton tensor $C$ is defined to be with the components

$$C_{ijk} := R_{k,i,j} - R_{k,i,j} - \frac{1}{2(n-1)} (R_{i,jgk} - R_{j,igk}) = \hat{R}_{k,i,j} - \hat{R}_{k,i,j} + \frac{n-2}{2n(n-1)} (R_{i,jgk} - R_{j,igk}), \quad (2.3)$$

where the indices after a comma denote the covariant derivatives.

We define the $(a, b)$-Cotton tensor $C^{(a,b)}_{ijk}$ to be with the components as follows:

$$C^{(a,b)}_{ijk} = \hat{R}_{k,i,j} - a \hat{R}_{k,i,j} + \frac{n-2}{2n(n-1)} (R_{i,jgk} - b R_{j,igk}), \quad (2.4)$$

where $a, b$ are two real constants. Clearly, the $(1, 1)$-Cotton tensor $C^{(1,1)}_{ijk}$ is exactly the Cotton tensor $C_{ijk}$ defined by (2.3). In order to state our results, we first introduce the following cut-off function. We denote by $p \in M^n$ and $B_r$ a fixed point and the geodesic sphere of $M^n$ of radius $r$ centered at $p$, respectively. Let $\phi_r$ be the non-negative cut-off function defined on $M^n$ satisfying

$$\phi_r = \begin{cases} 1, & \text{on } B_r \\ 0, & \text{on } M^n \setminus B_{r+1} \end{cases} \quad (2.5)$$

with $|\nabla \phi_r| \leq 2$ on $B_{r+1} \setminus B_r$.

First, we give an integral estimate on complete manifolds as follows:

**Lemma 2.1** Let $M^n$ be a complete manifold. Then for any two positive constants $\epsilon_1, \epsilon_2$, we have

$$\int_M |\nabla \hat{Ric}|^2 \phi_r^2$$

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Using the Ricci identity, we have
\[
\frac{1}{a^2 + 1 + \epsilon_1} \int_M \left[ 2a \left( W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} - \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} - \frac{1}{n-1} R |\hat{\text{Ric}}|^2 \right) + |C(a,b)|^2 + \left( -\frac{(n-2)^2(b-1)^2}{4n(n-1)^2} + \frac{(n-2)^2 a}{2n(n-1) - \epsilon_2} \right) |\nabla R|^2 \right] \phi_r^2
\]
\[-\left( \frac{4}{\epsilon_1 + \frac{n-2}{n\epsilon_2}} \right) a^2 \int_M |\text{Ric}|^2 |\nabla \phi_r|^2. \tag{2.6}
\]

**Proof** From (2.4), we have
\[
|C(a,b)|^2 = \sum_{i,j,k} |\hat{R}_{ij,i} - a \hat{R}_{ki,j}|^2 + \frac{(n-2)^2}{4n^2(n-1)^2} \sum_{i,j,k} |R_{i g j k} - b R_{j g i k}|^2
\]
\[+ \frac{n-2}{n(n-1)} (\hat{R}_{ij,i} - a \hat{R}_{ki,j})(R_{i g j k} - b R_{j g i k})
\]
\[= (1 + a^2) \sum_{i,j,k} |\hat{R}_{ij,i}|^2 - 2a \hat{R}_{ij,i} \hat{R}_{ki,j} + \frac{(n-2)^2}{4n^2(n-1)^2} (n(1 + b^2)
\]
\[- 2b |\nabla R|^2 - \frac{(n-2)^2}{2n^2(n-1)} (a + b) |\nabla R|^2
\]
\[= (1 + a^2) \sum_{i,j,k} |\hat{R}_{ij,i}|^2 - 2a \hat{R}_{ij,i} \hat{R}_{ki,j}
\]
\[+ \frac{(n-2)^2}{4(n-1)^2} \left( \frac{(b-1)^2}{n} - \frac{2(n-1)a}{n^2} \right) |\nabla R|^2, \tag{2.7}
\]

where we used the second Bianchi identity $\hat{R}_{ij,j} = \frac{n-2}{2n} R_{i j}$. Multiplying both sides of (2.7) by $\phi_r^2$ and integrating it yield
\[
\int_M |C(a,b)|^2 \phi_r^2 = (1 + a^2) \int_M |\nabla \text{Ric}|^2 \phi_r^2 - 2a \int_M \hat{R}_{ij,i} \hat{R}_{ki,j} \phi_r^2
\]
\[+ \frac{(n-2)^2}{4(n-1)^2} \left( \frac{(b-1)^2}{n} - \frac{2(n-1)a}{n^2} \right) \int_M |\nabla R|^2 \phi_r^2. \tag{2.8}
\]

Using the Ricci identity, we have
\[
\hat{R}_{ij,i} \hat{R}_{ki} = (\hat{R}_{kj,ji} + \hat{R}_{pj} \hat{R}_{pkij} + \hat{R}_{kp} \hat{R}_{pji}) \hat{R}_{ki}
\]
\[= \frac{n-2}{2n} R_{ki} \hat{R}_{ki} + \hat{R}_{pj} \hat{R}_{ki} \hat{R}_{pkij} + \hat{R}_{kp} \hat{R}_{ki} \hat{R}_{pi} + \frac{R}{n} |\hat{\text{Ric}}|^2
\]
\[= \frac{n-2}{2n} R_{ki} \hat{R}_{ki} - W_{ijkl} \hat{R}_{ikj} \hat{R}_{jl} + \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki}
\]
\[+ \frac{1}{n-1} R |\hat{\text{Ric}}|^2, \tag{2.9}
\]
which shows

\[-2a \int_M \hat{R}_{k,j,i} \hat{R}_{k,i,j} \phi_r^2 = 2a \int_M \hat{R}_{k,j,i} \hat{R}_{k,i} \phi_r^2 + 2a \int_M \hat{R}_{k,j,i} \hat{R}_{k} (\phi_r^2)_{j} = -2a \int_M \left( W_{ijkl} \hat{R}_{ikl} - \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{j} \hat{R}_{k} \right) \phi_r^2 + \frac{1}{n-1} R |\hat{R}||\phi_r| \phi_r^2 + 2a \int_M \hat{R}_{k,j,i} \hat{R}_{k} (\phi_r^2)_{j}. \tag{2.10} \]

Using the Cauchy inequality, we have

\[2a \int_M \hat{R}_{k,j,i} \hat{R}_{k} (\phi_r^2)_{j} \leq \epsilon_1 \int_M |\nabla \hat{R}||\phi_r| \phi_r^2 + \frac{4a^2}{\epsilon_1} \int_M |\hat{R}||\nabla \phi_r| \phi_r^2 \tag{2.11} \]

and

\[\frac{(n-2)a}{n} \int_M R_{k} \hat{R}_{k} \phi_r^2 \]

\[= - \frac{(n-2)^2a}{2n^2} \int_M |\nabla R|^2 \phi_r^2 - \frac{(n-2)a}{n} \int_M R_{k} \hat{R}_{k} (\phi_r^2)_{i} \leq \left( \epsilon_2 - \frac{(n-2)^2a}{2n^2} \right) \int_M |\nabla R|^2 \phi_r^2 + \frac{(n-2)^2a^2}{n \epsilon_2} \int_M |\hat{R}||\nabla \phi_r| \phi_r^2. \tag{2.12} \]

where we used the inequality

\[- \frac{(n-2)a}{n} \int_M R_{k} \hat{R}_{k} (\phi_r^2)_{i} \]

\[= -2 \sum_k \int_M (R_{k} \phi_r) \left( \frac{(n-2)a}{n} \sum_i \hat{R}_{k} (\phi_r)_{i} \right) \leq \epsilon_2 \sum_k \int_M R_{k}^2 \phi_r^2 + \frac{(n-2)^2a^2}{n \epsilon_2} \sum_{i,k} \int_M (\hat{R}_{k} (\phi_r)_{i})^2 \leq \epsilon_2 \int_M |\nabla R|^2 \phi_r^2 + \frac{(n-2)^2a^2}{n \epsilon_2} \int_M |\hat{R}||\nabla \phi_r| \phi_r^2. \tag{2.13} \]

Substituting (2.11) and (2.12) into (2.10) yields

\[-2a \int_M \hat{R}_{k,j,i} \hat{R}_{k,i,j} \phi_r^2 \leq -2a \int_M \left( W_{ijkl} \hat{R}_{ikl} - \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{j} \hat{R}_{k} \right) \phi_r^2 + \frac{1}{n-1} R |\hat{R}||\phi_r| \phi_r^2 + \epsilon_1 \int_M |\nabla \hat{R}||\phi_r| \phi_r^2 \tag{2.14} \]
\[ + \left( \epsilon_2 - \frac{(n-2)^2a}{2n^2} \right) \int_M |\nabla R|^2 \phi_r^2 + \left( \frac{4}{\epsilon_1} \right) \]
\[ + \frac{(n-2)^2}{n\epsilon_2} a^2 \int_M |\text{Ric}|^2 |\nabla \phi_r|^2 , \tag{2.15} \]

which together with (2.8) gives the desired estimate (2.6). \hfill \Box

For complete manifold \((M^n, g)\) with the metric \(g\) critical for \(\mathcal{F}_t\) over \(\mathcal{M}_1(M^n)\), we have the following result.

**Lemma 2.2** Let \(M^n\) be a complete manifold and \(g\) be a critical for \(\mathcal{F}_t\) over \(\mathcal{M}_1(M^n)\). Then for any two positive constants \(\epsilon_3 \in (0, 1)\) and \(\epsilon_4\), we have

\[ \int_M |\nabla \hat{\text{Ric}}|^2 \phi_r^2 \leq \frac{1}{1 - \epsilon_3} \int_M \left[ 2W_{ijkl} \hat{\hat{R}}_{jl} \hat{\hat{R}}_{ik} - \frac{4}{n-2} \hat{\hat{R}}_{ij} \hat{\hat{R}}_{jk} \hat{\hat{R}}_{ki} \right. \]
\[ + 2(n-2) + 2n(n-1)t \frac{\text{R}|\hat{\text{Ric}}|^2}{n(n-1)} + \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) |\nabla R|^2 \phi_r^2 \]
\[ + \frac{1}{1 - \epsilon_3} \left( \frac{1+2t^2}{\epsilon_4} + \frac{1}{\epsilon_3} \right) \int_M |\hat{\hat{\text{Ric}}}|^2 |\nabla \phi_r|^2 . \tag{2.16} \]

**Proof** It has been shown by Catino in [4, Proposition 2.1] that a metric \(g\) is critical for \(\mathcal{F}_t\) over \(\mathcal{M}_1(M^n)\) if and only if it satisfies the following equations

\[ \Delta \hat{\hat{R}}_{ij} = (1 + 2t)R_{ij} - \frac{1+2t}{n}(\Delta R)g_{ij} - 2R_{ikjl} \hat{\hat{R}}_{kl} \]
\[ - \frac{2 + 2nt}{n} \text{R} \hat{\hat{R}}_{ij} + \frac{2}{n} |\hat{\text{Ric}}|^2 g_{ij}, \tag{2.17} \]
\[ [n + 4(n-1)t] \Delta R = (n-4)[|R_{ij}|^2 + t R^2 - \lambda], \tag{2.18} \]

where \(\lambda = \mathcal{F}_t(g)\).

It is easy to see from (2.17) that

\[ \hat{\hat{R}}_{ij} \Delta \hat{\hat{R}}_{ij} = (1 + 2t)\hat{\hat{R}}_{ij} R_{ij} - 2R_{ikjl} \hat{\hat{R}}_{kl} \hat{\hat{R}}_{ij} - \frac{2 + 2nt}{n} \text{R} |\hat{\text{Ric}}|^2 \]
\[ = (1 + 2t)\hat{\hat{R}}_{ij} R_{ij} - \frac{2(n-2) + 2n(n-1)t}{n(n-1)} \frac{\text{R}|\hat{\text{Ric}}|^2}{n(n-1)} \]
\[ + \frac{4}{n-2} \hat{\hat{R}}_{ij} \hat{\hat{R}}_{jk} \hat{\hat{R}}_{ki} - 2W_{ikjl} \hat{\hat{R}}_{kl} \hat{\hat{R}}_{ij} . \tag{2.19} \]

Thus,

\[ \int_M |\nabla \hat{\text{Ric}}|^2 \phi_r^2 = - \int_M \hat{\hat{R}}_{ij} \Delta \hat{\hat{R}}_{ij} \phi_r^2 - \int_M \hat{\hat{R}}_{ij} \hat{\hat{R}}_{i,j,k} (\phi_r^2)_k \]

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\[
= \int_M \left( 2W_{ijkl} \hat{R}_{ji} \hat{R}_{ik} - \frac{4}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \\
+ \frac{2(n-2)+2n(n-1)t}{n(n-1)} R|\hat{Ric}|^2 \right) \phi_r^2 \\
- (1 + 2t) \int_M \hat{R}_{ij} R_{ij} \phi_r^2 - \int_M \hat{R}_{ij} \hat{R}_{ij,k} (\phi_r^2)_k.
\]

(2.20)

Applying the inequality

\[
- \int_M \hat{R}_{ij} \hat{R}_{ij,k} (\phi_r^2)_k \leq \epsilon_3 \int_M |\nabla \hat{Ric}|^2 \phi_r^2 + \frac{1}{\epsilon_3} \int_M |\hat{Ric}|^2 |\nabla \phi_r|^2
\]

and

\[
-(1 + 2t) \int_M \hat{R}_{ij} R_{ij} \phi_r^2 \leq \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) \int_M |\nabla R|^2 \phi_r^2 \\
+ \frac{(1 + 2t)^2}{\epsilon_4} \int_M |\hat{Ric}|^2 |\nabla \phi_r|^2
\]

into (2.20) gives the desired estimate (2.16).

Lemma 2.3

On every Einstein manifold \((M^n, g)\), we have

\[
\frac{1}{2} \Delta |W|^2 \geq \frac{n+1}{n-1} |\nabla |W||^2 + \frac{2}{n} R |W|^2 - 2C_n |W|^3,
\]

(2.21)

where \(C_n\) is defined by

\[
C_n = \begin{cases} 
\frac{\sqrt{6}}{4}, & \text{if } n = 4; \\
\frac{\sqrt{10}}{15}, & \text{if } n = 5; \\
\frac{n-2}{n(n-1)} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}, & \text{if } n \geq 6.
\end{cases}
\]

(2.22)

In particular, if the scalar curvature of Einstein metric \(g\) is positive, then it is of constant positive sectional curvature, provided either

\[
C_n |W| < \frac{1}{n} R,
\]

(2.23)

or

(1) for \(n \neq 5\),

\[
\left( \int_M |W|^\frac{n}{2} \right)^{\frac{2}{n}} \leq E_n Q_g(M),
\]

(2.24)

where \(E_n\) is given by (1.3);

(2) for \(n = 5\),

\[
\left( \int_M |W|^\frac{5}{2} \right)^{\frac{2}{5}} \leq \frac{2\sqrt{15} - 4}{\sqrt{10}} Q_g(M).
\]

(2.25)
Proof In [14] (or see [10]), it has been proved for an Einstein manifold that
\[
\frac{1}{2} \Delta |W|^2 = |\nabla W|^2 + \frac{2}{n} R |W|^2 - 2(2W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij})
\geq \frac{n+1}{n-1} |\nabla |W||^2 + \frac{2}{n} R |W|^2 - 2(2W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij}), \tag{2.26}
\]
where the inequality in (2.26), we used the refined Kato inequality (see [6]) of an Einstein manifold.

When \( n = 4, 5 \), we have (see [5,10])
\[
|2W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij}| \leq \frac{\sqrt{6}}{4} |W|^3 \tag{2.27}
\]
and
\[
|2W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij}| \leq \frac{4\sqrt{10}}{15} |W|^3, \tag{2.28}
\]
respectively.

When \( n = 6 \), making use of the inequality proved by Li and Zhao [18] (see the formulas (13) and (14) in [18], or see [8]):
\[
2|W_{ijkl} W_{ipkq} W_{pjql}| = |W_{ijkl}(W_{ipkq} W_{pjql} - W_{jpqk} W_{piql})| 
\leq \frac{n-2}{\sqrt{n(n-1)}} |W|^3
\]
and Huisken [17]:
\[
|W_{ijkl} W_{klpq} W_{pqij}| \leq \frac{n^2 - n - 4}{\sqrt{(n-2)(n-1)n(n+1)}} |W|^3
\]
gives
\[
|2W_{ijkl} W_{ipkq} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqij}|
\leq 2|W_{ijkl} W_{ipkq} W_{pjql}| + \frac{1}{2} |W_{ijkl} W_{klpq} W_{pqij}|
\leq \left[ \frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2 - n - 4}{2\sqrt{(n-2)(n-1)n(n+1)}} \right] |W|^3. \tag{2.29}
\]
Hence, the desired estimate (2.21) follows by inserting (2.27)-(2.29) into (2.26).
Since the scalar curvature of Einstein metric $g$ is positive, $M^n$ must be of compact from the Myer’s Theorem. The estimate (2.23) comes from integrating both sides of (2.21).

Let $v = |W|$. Then (2.21) becomes

$$v \Delta v \geq \frac{2}{n-1} |\nabla v|^2 + \frac{2}{n} R v^2 - 2C_n v^3,$$  

(2.30)

and hence

$$v^\alpha \Delta v^\alpha = v^\alpha [\alpha(\alpha - 1)v^{\alpha-2}|\nabla v|^2 + \alpha v^{\alpha-1} \Delta v]$$

$$= (1 - \frac{1}{\alpha}) |\nabla v^\alpha|^2 + \alpha v^{2\alpha-2} v \Delta v$$

$$\geq (1 - \frac{n-3}{(n-1)\alpha}) |\nabla v^\alpha|^2 - 2\alpha C_n v^{2\alpha+1} + \frac{2\alpha}{n} R v^{2\alpha}$$  

(2.31)

which shows

$$0 \geq \left(2 - \frac{n-3}{(n-1)\alpha}\right) \int_M |\nabla v^\alpha|^2 - 2\alpha C_n \int_M v^{2\alpha+1} + \frac{2\alpha}{n} R \int_M v^{2\alpha}.$$  

(2.32)

Therefore, for $2 - \frac{n-3}{(n-1)\alpha} > 0$, by virtue of (1.9) with $u = v^\alpha$, we have

$$0 \geq \left(2 - \frac{n-3}{(n-1)\alpha}\right) Q_g(M) \left(\int_M v^{\frac{2n\alpha}{n-2}}\right)^{\frac{n-2}{n}} - 2\alpha C_n \int_M v^{2\alpha+1} + \frac{2\alpha}{n} R \int_M v^{2\alpha}$$

$$\geq \left[2 - \frac{n-3}{(n-1)\alpha}\right] Q_g(M) - 2\alpha C_n \left(\int_M v^{\frac{n}{2}}\right)^{\frac{3}{2}} \left(\int_M v^{\frac{2n\alpha}{n-2}}\right)^{\frac{n-2}{n}}$$

$$+ \left[2\alpha - \frac{n-2}{4(n-1)} \left(2 - \frac{n-3}{(n-1)\alpha}\right)\right] R \int_M v^{2\alpha}.$$  

(2.33)

When $n \neq 5$, taking $\frac{1}{\alpha} = \frac{n-1}{n-3} \left(1 + \sqrt{1 - \frac{8(n-3)}{n(n-2)}}\right)$ which satisfies

$$\frac{2\alpha}{n} - \frac{n-2}{4(n-1)} \left(2 - \frac{n-3}{(n-1)\alpha}\right) = 0.$$  

Thus, if

$$\left(\int_M |W|^\frac{2}{n}\right)^{\frac{n}{2}} \leq \frac{1}{2C_n\alpha} \left(2 - \frac{n-3}{(n-1)\alpha}\right) Q_g(M)$$

$$= \frac{4(n-1)}{n-2} \frac{Q_g(M)}{nC_n},$$  

(2.34)
(2.33) shows that $W = 0$ and it is of constant positive sectional curvature. When $n = 5$, (2.33) becomes

$$0 \geq \left(2 - \frac{1}{2\alpha}\right)Q_g(M) - 2\alpha C_5 \left(\int_M v^2 \right)^{\frac{3}{5}} \left(\int_M v^{\frac{10\alpha}{3}} \right)^{\frac{3}{5}}$$

$$+ \left[\frac{2\alpha}{5} - \frac{3}{16} \left(2 - \frac{1}{2\alpha}\right)\right] R \int_M v^{2\alpha}.$$  

(2.35)

Taking $\alpha = \frac{\sqrt{15}}{8}$ in (2.35) gives

$$0 \geq \left[\frac{2\sqrt{15} - 4}{\sqrt{15}} Q_g(M) - \frac{\sqrt{15}}{4} C_5 \left(\int_M v^2 \right)^{\frac{3}{5}} \left(\int_M v^{\frac{10\alpha}{3}} \right)^{\frac{3}{5}}$$

$$+ \left(\sqrt{\frac{3}{20}} - \frac{3}{8}\right) R \int_M v^{2\alpha},$$  

(2.36)

which shows that $W = 0$ and it is of constant positive sectional curvature provided (2.25) holds.

We complete the proof of Lemma 2.3. $\square$

The following lemma comes from [9,16] (for the case of $\lambda = \frac{2}{n-2}$, see [5,11,15]):

**Lemma 2.4** On every Riemannian manifold $(M^n, g)$, for any $\lambda \in \mathbb{R}$, the following estimate holds

$$- W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} + \lambda \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki}$$

$$\leq \sqrt{\frac{n-2}{2(n-1)}} \left(\left|W\right|^2 + \frac{2(n-2)\lambda^2}{n} |\hat{\text{Ric}}|^2 \right)^{\frac{1}{2}} |\hat{\text{Ric}}|^2$$

$$= \sqrt{\frac{n-2}{2(n-1)}} \left|W + \frac{\lambda}{\sqrt{2n}} |\hat{\text{Ric}}| \otimes g\right| |\hat{\text{Ric}}|^2.$$  

(2.37)

### 3 Proof of Results

#### 3.1 Proof of Theorem 1.1

By combining (2.6) with (2.16), we derive

$$\frac{1}{1 - \epsilon_3} \int_M \left[2 W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} - \frac{4}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki}$$

$$+ \frac{2(n-2) + 2n(n-1)t}{n(n-1)} R |\hat{\text{Ric}}|^2 + \left(\epsilon_4 + \frac{(n-2)(1+2t)}{2n}\right) |\nabla R|^2 \right] \phi_r^2$$

$$+ \frac{1}{1 - \epsilon_3} \left(\frac{1+2t}{\epsilon_4} + \frac{1}{\epsilon_3}\right) \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2$$
\[
\begin{align*}
\geq & \frac{1}{a^2 + 1 + \epsilon_1} \int_M \left[ 2a \left( W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} - \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} - \frac{1}{n-1} R |\hat{\text{Ric}}|^2 \right) \\
& + |C(a, b)|^2 + \left( - \frac{(n-2)^2(b-1)^2}{4n(n-1)^2} + \frac{(n-2)^2 a}{2n(n-1)} - \epsilon_2 \right) |\nabla R|^2 \right] \phi_r^2 \\
& - \left( \frac{4}{\epsilon_1} + \frac{(n-2)^2}{n \epsilon_2} \right) \frac{a^2}{a^2 + 1 + \epsilon_1} \int_M |\hat{\text{Ric}}| |\nabla \phi_r|^2,
\end{align*}
\]

which gives

\[
\begin{align*}
\left[ \frac{1}{1 - \epsilon_3} \left( \frac{(1 + 2t)^2}{\epsilon_4} + \frac{1}{\epsilon_3} \right) + \left( \frac{4}{\epsilon_1} + \frac{(n-2)^2}{n \epsilon_2} \right) \frac{a^2}{a^2 + 1 + \epsilon_1} \right] \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2 \\
\geq & -2 \frac{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)}{(1 - \epsilon_3)(a^2 + 1 + \epsilon_1)} \int_M W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} \phi_r^2 \\
& + \frac{\frac{4}{\epsilon_1} a^2 - 2(1 - \epsilon_3)a + \frac{\frac{4}{\epsilon_1} (1 + \epsilon_1)}{n}}{(1 - \epsilon_3)(a^2 + 1 + \epsilon_1)} \frac{n}{n-2} \int_M \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \phi_r^2 \\
& - \frac{2}{n(n-1)} \frac{a(1 - \epsilon_3) + (a^2 + 1 + \epsilon_1)((n-2) + n(n-1)t)}{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)} \int_M R |\hat{\text{Ric}}|^2 \phi_r^2 \\
& - \frac{1}{1 - \epsilon_3} \left( \epsilon_4 + \frac{(n-2)(1 + 2t)}{2n} \right) \int_M |\nabla R|^2 \phi_r^2 \\
& + \frac{1}{a^2 + 1 + \epsilon_1} \int_M |C(a, b)|^2 \phi_r^2. \tag{3.2}
\end{align*}
\]

In particular, for any positive \( \epsilon_1, \epsilon_3 \), we have \( a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1) > 0 \). Hence, (3.2) is equivalent to

\[
\begin{align*}
\frac{(1 - \epsilon_3)(a^2 + 1 + \epsilon_1)}{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)} & \left[ \frac{1}{1 - \epsilon_3} \left( \frac{(1 + 2t)^2}{\epsilon_4} + \frac{1}{\epsilon_3} \right) \\
& + \left( \frac{4}{\epsilon_1} + \frac{(n-2)^2}{n \epsilon_2} \right) \frac{a^2}{a^2 + 1 + \epsilon_1} \right] \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2 \\
\geq & -2 \int_M W_{ijkl} \hat{R}_{jl} \hat{R}_{ik} \phi_r^2 \\
& + \frac{\frac{4}{\epsilon_1} a^2 - 2(1 - \epsilon_3)a + \frac{\frac{4}{\epsilon_1} (1 + \epsilon_1)}{n}}{(1 - \epsilon_3)(a^2 + 1 + \epsilon_1)} \frac{n}{n-2} \int_M \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \phi_r^2 \\
& - \frac{2}{n(n-1)} \frac{a(1 - \epsilon_3) + (a^2 + 1 + \epsilon_1)((n-2) + n(n-1)t)}{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)} \int_M R |\hat{\text{Ric}}|^2 \phi_r^2 \\
& + \frac{(1 - \epsilon_3)(a^2 + 1 + \epsilon_1)}{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)} \left[ - \frac{1}{1 - \epsilon_3} \left( \epsilon_4 + \frac{(n-2)(1 + 2t)}{2n} \right) \right].
\end{align*}
\]
\[
\frac{1}{a^2 + 1 + \epsilon_1} \left( - \frac{(n-2)^2(b-1)^2}{4n(n-1)^2} + \frac{(n-2)^2a}{2n(n-1)} - \epsilon_2 \right) \int_M |\nabla R|^2 \phi_r^2 \\
+ \frac{1 - \epsilon_3}{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)} \int_M |C^{(a,b)}|^2 \phi_r^2.
\] (3.3)

Now, we fix \(\epsilon_1, \epsilon_3\) and minimize the function

\[
f(a) = \frac{\frac{4}{n}a^2 - 2(1 - \epsilon_3)a + \frac{4}{n}(1 + \epsilon_1)}{a^2 - (1 - \epsilon_3)a + (1 + \epsilon_1)}
\]

by taking

\[
a = \sqrt{1 + \epsilon_1},
\]

and then (3.3) becomes

\[
\frac{\sqrt{1 + \epsilon_1}(1 - \epsilon_3)}{2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)} \left[ \frac{1}{1 - \epsilon_3} \left( \frac{(1 + 2\lambda)^2}{\epsilon_4} + \frac{1}{\epsilon_3} \right) \\
+ \frac{1}{2} \left( \frac{4}{\epsilon_1} + \frac{(n-2)^2}{n \epsilon_2} \right) \right] \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2 \\
\geq - \int_M W_{ijkl} \hat{R}_{ji} \hat{R}_{ki} \phi_r^2 + \frac{\frac{4}{n} \sqrt{1 + \epsilon_1} - (1 - \epsilon_3)}{n} \frac{n}{2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)} \frac{n}{n-2} \int_M \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \phi_r^2 \\
- \frac{1}{n(n-1)} \frac{n(1 - \epsilon_3)}{2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)} \frac{n}{n-2} \int_M R |\hat{\text{Ric}}|^2 \phi_r^2 \\
+ \frac{\sqrt{1 + \epsilon_1}(1 - \epsilon_3)}{2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)} \left[ \frac{1}{2(1 + \epsilon_1)} \left( - \frac{(n-2)^2(b-1)^2}{4n(n-1)^2} \\
+ \frac{(n-2)^2}{2n(n-1)} \sqrt{1 + \epsilon_1} - \epsilon_2 \right) - \frac{1}{1 - \epsilon_3} \left( \epsilon_4 + \frac{(n-2)(1 + 2\lambda)}{2n} \right) \right] \int_M |\nabla R|^2 \phi_r^2 \\
+ \frac{1 - \epsilon_3}{2\sqrt{1 + \epsilon_1}[2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)]} \int_M |C^{(a,b)}|^2 \phi_r^2.
\] (3.4)

Applying the inequality (2.37) with

\[
\lambda = \frac{\frac{4}{n} \sqrt{1 + \epsilon_1} - (1 - \epsilon_3)}{n} \frac{n}{2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)} \frac{n}{n-2}
\]

into (3.4) and taking \(b = 1\), we have

\[
\frac{\sqrt{1 + \epsilon_1}(1 - \epsilon_3)}{2\sqrt{1 + \epsilon_1} - (1 - \epsilon_3)} \left[ \frac{1}{1 - \epsilon_3} \left( \frac{(1 + 2\lambda)^2}{\epsilon_4} + \frac{1}{\epsilon_3} \right) \\
+ \frac{1}{2} \left( \frac{4}{\epsilon_1} + \frac{(n-2)^2}{n \epsilon_2} \right) \right] \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2
\]
\begin{align*}
\text{On the other hand, if } D_n \leq 0, \text{ we have}
\int_M \left[ -\sqrt{\frac{n-2}{2(n-1)}} W + \frac{\sqrt{n}}{2(n-2)} \frac{4}{n} \sqrt{1+\epsilon_1} - (1-\epsilon_3) \right] \text{Ric} \otimes g \\
- \frac{1}{n(n-1)} n(1-\epsilon_3) + 2\sqrt{1+\epsilon_1}(n-2) + n(n-1)t \int_M |\text{Ric}|^2 \phi_r^2 \\
+ \frac{\sqrt{1+\epsilon_1}(1-\epsilon_3)}{2\sqrt{1+\epsilon_1} - (1-\epsilon_3)} \left[ \frac{1}{2(1+\epsilon_1)} \left( \frac{(n-2)^2}{2n(n-1)} \sqrt{1+\epsilon_1} - \epsilon_2 \right) \\
- \frac{1}{1-\epsilon_3} \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) \right] \int_M |\nabla R|^2 \phi_r^2 \\
+ \frac{1-\epsilon_3}{2\sqrt{1+\epsilon_1}[2\sqrt{1+\epsilon_1} - (1-\epsilon_3)]} \int_M |C(a,b)|^2 \phi_r^2. \quad (3.5)
\end{align*}

For all \( \epsilon_1, \epsilon_3 \), we have
\[
\left( \frac{4}{n} \sqrt{1+\epsilon_1} - (1-\epsilon_3) \right)^2 = \left( 1 - \frac{2 - \frac{4}{n}}{2 - \frac{1-\epsilon_3}{\sqrt{1+\epsilon_1}}} \right)^2 < D_n, \quad (3.6)
\]

where \( D_n \) is given by (1.2). Using the fact that \( W \) is perpendicular to \( \text{Ric} \otimes g \), we have
\[
\left| W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \frac{4}{n} \sqrt{1+\epsilon_1} - (1-\epsilon_3) \right| \text{Ric} \otimes g \\
= |W|^2 + \frac{n}{2(n-2)^2} \left( \frac{4}{n} \sqrt{1+\epsilon_1} - (1-\epsilon_3) \right)^2 |\text{Ric} \otimes g|^2 \\
< |W|^2 + \frac{nD_n}{2(n-2)^2} |\text{Ric} \otimes g|^2 \\
= \left| W + \frac{\sqrt{nD_n}}{\sqrt{2(n-2)}} \text{Ric} \otimes g \right|^2, \quad (3.7)
\]

which shows
\[
\left| W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \frac{4}{n} \sqrt{1+\epsilon_1} - (1-\epsilon_3) \right| \text{Ric} \otimes g \\
< \left| W + \frac{\sqrt{nD_n}}{\sqrt{2(n-2)}} \text{Ric} \otimes g \right|. \quad (3.8)
\]

On the other hand, if
\[
n + 2[(n-2) + n(n-1)t] \leq 0, \quad (3.9)
\]

then we have
\[
2[(n-2) + n(n-1)t] < \frac{n(1-\epsilon_3) + 2\sqrt{1+\epsilon_1}(n-2) + n(n-1)t}{2\sqrt{1+\epsilon_1} - (1-\epsilon_3)}
\]
\begin{align}
&\frac{n^{1-\epsilon_3}}{\sqrt{1+\epsilon_1}} + 2[(n-2) + n(n-1)t] \\
&= \frac{2 - \frac{1-\epsilon_3}{\sqrt{1+\epsilon_1}}}{2 - \frac{1-\epsilon_3}{\sqrt{1+\epsilon_1}}} \\
&< \frac{n + 2[(n-2) + n(n-1)t]}{2 - \frac{1-\epsilon_3}{\sqrt{1+\epsilon_1}}} \\
&< \frac{n + 2[(n-2) + n(n-1)t]}{2} \tag{3.10}
\end{align}

and hence

\begin{equation}
\frac{n(1-\epsilon_3) + 2\sqrt{1+\epsilon_1}[(n-2) + n(n-1)t]}{2\sqrt{1+\epsilon_1} - (1-\epsilon_3)} < 0. \tag{3.11}
\end{equation}

Moreover, taking \(\epsilon_2 = \epsilon_1\sqrt{1+\epsilon_1}\) and \(\epsilon_4 = \frac{\epsilon_1(1-\epsilon_3)}{2\sqrt{1+\epsilon_1}}\), we have

\begin{align}
(n-2)^2 \frac{1-\epsilon_3}{4n(n-1)\sqrt{1+\epsilon_1}} - \epsilon_2 \frac{1-\epsilon_3}{2(1+\epsilon_1)} - \epsilon_4 - \frac{(n-2)(1+2t)}{2n} \\
= \left(\frac{(n-2)^2}{4n(n-1)} - \epsilon_1\right) \frac{1-\epsilon_3}{\sqrt{1+\epsilon_1}} - \frac{(n-2)(1+2t)}{2n} \tag{3.12}
\end{align}

Since,

\begin{equation}
\left(\frac{(n-2)^2}{4n(n-1)} - \epsilon_1\right) \frac{1-\epsilon_3}{\sqrt{1+\epsilon_1}} \to \frac{(n-2)^2}{4n(n-1)} \tag{3.13}
\end{equation}

as \(\epsilon_1 \to 0\) and \(\epsilon_3 \to 0\), there exist two positive constants \(\overline{\epsilon}_1, \overline{\epsilon}_3\) such that

\begin{align}
\frac{(n-2)^2}{4n(n-1)\sqrt{1+\overline{\epsilon}_1}} - \overline{\epsilon}_2 \frac{1-\overline{\epsilon}_3}{2(1+\overline{\epsilon}_1)} - \overline{\epsilon}_4 - \frac{(n-2)(1+2t)}{2n} \\
> \frac{6(n-2)^2}{25n(n-1)} - \frac{(n-2)(1+2t)}{2n} \tag{3.14}
\end{align}

Hence, if

\begin{equation}
\frac{6(n-2)^2}{25n(n-1)} - \frac{(n-2)(1+2t)}{2n} \geq 0 \tag{3.15}
\end{equation}

then (3.14) shows

\begin{equation}
\frac{(n-2)^2}{4n(n-1)\sqrt{1+\overline{\epsilon}_1}} - \overline{\epsilon}_2 \frac{1-\overline{\epsilon}_3}{2(1+\overline{\epsilon}_1)} - \overline{\epsilon}_4 - \frac{(n-2)(1+2t)}{2n} > 0 \tag{3.16}
\end{equation}
which is equivalent to

\[
\frac{1}{2(1+\epsilon_1)} \left( \frac{(n-2)^2}{2n(n-1)} \sqrt{1+\epsilon_1 - \epsilon_2} \right) - \frac{1}{1-\epsilon_3} \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) > 0.
\]  

(3.17)

It is easy to check that the scope of \( t \) satisfying both (3.9) and (3.15) is equivalent to (1.5). Therefore, if (1.5) and (1.6) both hold, then we have

\[
\frac{\sqrt{1+\epsilon_1}(1-\epsilon_3)}{2\sqrt{1+\epsilon_1} - (1-\epsilon_3)} \left[ \frac{1}{1-\epsilon_3} \left( (1+2t)^2 \frac{2\sqrt{1+\epsilon_1}}{\epsilon_1(1-\epsilon_3)} + \frac{1}{\epsilon_3} \right) \right]
\]

\[
+ \frac{1}{2} \left( \frac{4}{\epsilon_1} + \frac{(n-2)^2}{n\epsilon_1\sqrt{1+\epsilon_1}} \right) \int_M [\text{Ric}^2 |\nabla \phi_r|^2]
\]

\[
\geq \int_M \left[ - \sqrt{\frac{n-2}{2(n-1)}} W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \frac{4}{n} \sqrt{1+\epsilon_1} - (1-\epsilon_3) \text{Ric} \otimes g \right] |\text{Ric}|^2 |\phi_r|^2
\]

\[
- \frac{1}{n(n-1)} \frac{n(1-\epsilon_3) + 2\sqrt{1+\epsilon_1}((n-2) + n(n-1)t)}{2\sqrt{1+\epsilon_1} - (1-\epsilon_3)} R |\text{Ric}|^2 |\phi_r|^2
\]

\[
\geq 0
\]  

(3.18)

from (3.5).

Since

\[
\int_M |\text{Ric}|^2 < \infty,
\]  

(3.19)

then we have

\[
\int_M |\text{Ric}|^2 |\nabla \phi_r|^2 \to 0,
\]  

(3.20)

as \( r \to \infty \), which together with (3.18) shows that \( M^n \) is Einstein. In this case, (1.6) becomes

\[
|W| \leq - \frac{2\sqrt{2}((n-2) + n(n-1)t)}{n \sqrt{(n-1)(n-2)}} R,
\]  

(3.21)

which gives

\[
C_n |W| \leq - \frac{2\sqrt{2}((n-2) + n(n-1)t)}{n \sqrt{(n-1)(n-2)}} C_n R.
\]  

(3.22)

When \( n = 3 \), we have \( W = 0 \) automatically. When \( n \geq 4 \) and \( t \) satisfies (1.7), we have

\[
- \frac{2\sqrt{2}((n-2) + n(n-1)t)}{n \sqrt{(n-1)(n-2)}} C_n R < \frac{1}{n} R
\]  

(3.23)
which, combining with (2.23), shows that $W = 0$ and hence $M^n$ is of constant positive sectional curvature.

### 3.2 Proof of Theorem 1.2

From (2.37), it is easy to see

$$2W_{ijkl} \hat{R}_{jl} \hat{R}_{ki} - \frac{4}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \leq \sqrt{\frac{2(n-2)}{n-1}} |W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \hat{\text{Ric}} \otimes g| |\hat{\text{Ric}}|^2.$$  

(3.24)

Applying (3.24) into (2.16) and using the Kato inequality, we obtain

$$\int_M |\nabla \hat{\text{Ric}}|^2 \phi_r^2 \leq \int_M |\nabla \text{Ric}|^2 \phi_r^2 \leq \frac{1}{1 - \epsilon_3} \int_M \left[ \sqrt{\frac{2(n-2)}{n-1}} |W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \hat{\text{Ric}} \otimes g| |\hat{\text{Ric}}|^2 \right]$$

$$+ \frac{2(n-2) + 2n(n-1)t}{n(n-1)} R \hat{\text{Ric}} |\hat{\text{Ric}}|^2 \phi_r^2$$

$$+ \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) \int_M |\nabla R|^2 \phi_r^2$$

$$+ \frac{1}{1 - \epsilon_3} \left( \frac{(1+2t)^2}{\epsilon_4} + 1 \right) \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2.$$  

(3.25)

Taking $u = |\hat{\text{Ric}}| \phi_r$ in (1.9) and applying (3.25) yield

$$Q_g(M) \left( \int_M (|\hat{\text{Ric}}| \phi_r)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\leq \int_M \left[ |\nabla (|\hat{\text{Ric}}| \phi_r)|^2 + \frac{n-2}{4(n-1)} R |\hat{\text{Ric}}|^2 \phi_r^2 \right]$$

$$\leq (1 + \epsilon_5) \int_M |\nabla \hat{\text{Ric}}|^2 \phi_r^2 + \left( 1 + \frac{1}{\epsilon_5} \right) \int_M |\hat{\text{Ric}}|^2 |\nabla \phi_r|^2$$

$$+ \frac{n-2}{4(n-1)} \int_M R |\hat{\text{Ric}}|^2 \phi_r^2$$

$$\leq \frac{1 + \epsilon_5}{1 - \epsilon_3} \sqrt{\frac{2(n-2)}{n-1}} \int_M \left[ W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \hat{\text{Ric}} \otimes g |\hat{\text{Ric}}|^2 \phi_r^2 \right]$$

$$+ \frac{1}{n-1} \left[ \frac{n-2}{4} + \frac{2(n-2) + 2n(n-1)t}{n} \frac{1 + \epsilon_5}{1 - \epsilon_3} \right] \int_M R |\hat{\text{Ric}}|^2 \phi_r^2$$

$$+ \left( 1 + \epsilon_5 \right) \left( \frac{1}{\epsilon_5} + \frac{1}{1 - \epsilon_3} \left( \frac{(1+2t)^2}{\epsilon_4} + 1 \right) \right) \int_M |\nabla R|^2 \phi_r^2$$

$$+ \left( 1 + \epsilon_5 \right) \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) \int_M \left| \nabla \phi_r \right|^2.$$  

(3.26)
Inserting the following Hölder inequality

\[
\int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \text{Ric} \otimes g \right| \left| \text{Ric} \right|^2 \phi_r^2 \\
\leq \left( \int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \text{Ric} \otimes g \right|^2 \right)^{\frac{2}{n}} \left( \int_M \left( \left| \text{Ric} \right| \phi_r \right) \frac{2n}{n-2} \right)^{\frac{n-2}{n}}
\]

into (3.26) deduces

\[
\left[ Q_g(M) - \frac{1}{1 - \epsilon_3} \frac{2(n-2)}{n-1} \left( \int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \text{Ric} \otimes g \right|^2 \phi_r^2 \right) \right] \times \left( \int_M \left( \left| \text{Ric} \right| \phi_r \right) \frac{2n}{n-2} \right)^{\frac{n-2}{n}} \\
\leq \frac{1}{n-1} \left[ \frac{n-2}{4} + \frac{2(n-2) + 2n(n-1)t}{n-1} \right] \int_M \left| \text{Ric} \right| \phi_r^2 \\
+ (1 + \epsilon_5) \left[ \frac{1}{\epsilon_3} \left( \frac{1 + 2t}{\epsilon_3} + \frac{1}{\epsilon_4} \right) \right] \int_M \left| \text{Ric} \right| \phi_r^2 \\
+ (1 + \epsilon_5) \left( \epsilon_4 + \frac{(n-2)(1+2t)}{2n} \right) \int_M \left| \text{Ric} \right| \phi_r^2.
\]

We check that if \( t < -\frac{1}{2} \), then \( (n-2) + n(n-1)t < 0 \) and

\[
\frac{-8[(n-2) + n(n-1)t]}{n(n-2)} > 1.
\]

Hence for all \( \epsilon_3, \epsilon_5 \), it holds that

\[
\frac{1 - \epsilon_3}{1 + \epsilon_5} < \frac{-8[(n-2) + n(n-1)t]}{n(n-2)},
\]

which is equivalent to

\[
\frac{n-2}{4} + \frac{2(n-2) + 2n(n-1)t}{n-1} \frac{1 + \epsilon_5}{1 - \epsilon_3} < 0.
\]

If (1.11) occurs, then

\[
\left( \int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \text{Ric} \otimes g \right|^2 \phi_r^2 \right)^{\frac{2}{n}} < \sqrt{\frac{n-1}{2(n-2)} Q_g(M)} < 1.
\]
In this case, there exist $\tilde{\varepsilon}_3, \tilde{\varepsilon}_5$ such that
\[
\frac{1 - \tilde{\varepsilon}_3}{1 + \tilde{\varepsilon}_5} = \left( \int_M \left| W + \frac{\sqrt{n}}{\sqrt{n(n-2)}} \text{Ric} \otimes g \right|^{\frac{n}{2}} \right)^{\frac{2}{n}},
\] (3.31)
which, from (3.27), shows that
\[
0 \leq \frac{1}{n-1} \left[ \frac{n-2}{4} + \frac{2(n-2) + 2n(n-1)t}{n} \frac{1 + \tilde{\varepsilon}_5}{1 - \tilde{\varepsilon}_3} \right] \int_M R |\text{Ric}|^2 \phi_r^2
+ (1 + \tilde{\varepsilon}_5) \left[ \frac{1}{\tilde{\varepsilon}_5} + \frac{1}{1 - \tilde{\varepsilon}_3} \left( - \frac{2n(1+2t)}{n-2} + \frac{1}{\tilde{\varepsilon}_3} \right) \right] \times
\int_M |\text{Ric}|^2 |\nabla \phi_r|^2
\] (3.32)
by taking $\varepsilon_4 = -\frac{(n-2)(1+2t)}{2n}$. It follows from (1.10) that
\[
\int_M |\text{Ric}|^2 |\nabla \phi_r|^2 \rightarrow 0
\] (3.33)
as $r \rightarrow \infty$, which together with (3.32) gives that $M^n$ is Einstein.

Since $M^n$ is Einstein, then (1.11) becomes
\[
\left( \int_M |W|^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \sqrt{\frac{n-1}{2(n-2)}} Q_g(M).
\] (3.34)

When $n = 4, 5$, we can check that
\[
\sqrt{\frac{n-1}{2(n-2)}} < E_n,
\] (3.35)
which implies from Lemma 2.3 that $W = 0$ and hence $M^n$ is of constant positive sectional curvature. When $n \geq 6$, it is easy to check
\[
\frac{4}{n} \sqrt{\frac{2(n-1)}{n-2}} < \frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2 - n - 4}{2\sqrt{(n-2)(n-1)n(n+1)}},
\] (3.36)
which is equivalent to
\[
E_n < \sqrt{\frac{n-1}{2(n-2)}}.
\] (3.37)

Therefore, if (1.12) holds, then $M^n$ is also of constant positive sectional curvature.

We complete the proof of Theorem 1.2.
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