MONGE-AMPÈRE MEASURES FOR CONVEX BODIES AND BERNSTEIN-MARKOV TYPE INEQUALITIES

D. BURNS, N. LEVENBERG, S. MA’U AND SZ. RÉVÉSZ

Abstract. We use geometric methods to calculate a formula for the complex Monge-Ampère measure \((dd^cV_K)^n\), for \(K \subset \mathbb{R}^n \subset \mathbb{C}^n\) a convex body and \(V_K\) its Siciak-Zaharjuta extremal function. Bedford and Taylor had computed this for symmetric convex bodies \(K\). We apply this to show that two methods for deriving Bernstein-Markov-type inequalities, i.e., pointwise estimates of gradients of polynomials, yield the same results for all convex bodies. A key role is played by the geometric result that the extremal inscribed ellipses appearing in approximation theory are the maximal area ellipses determining the complex Monge-Ampère solution \(V_K\).

1. Introduction.

For a function \(u\) of class \(C^2\) on a domain \(\Omega \subset \mathbb{C}^n\), the complex Monge-Ampère operator applied to \(u\) is

\[(dd^c u)^n := i\partial\partial u \wedge \cdots \wedge i\partial\partial u.\]

For a plurisubharmonic (psh) function \(u\) which is only locally bounded, \((dd^c u)^n\) is well-defined as a positive measure. Given a bounded set \(E \subset \mathbb{C}^n\), we define the Siciak-Zaharjuta extremal function

\[V_E(z) := \sup\{u(z) : u \in L(\mathbb{C}^n), \ u \leq 0 \text{ on } E\}\]

where \(L(\mathbb{C}^n)\) denotes the class of psh functions \(u\) on \(\mathbb{C}^n\) with \(u(z) \leq \log^+ |z| + c(u)\). If \(E\) is non-pluripolar, the upper-regularized function

\[V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)\]

Date: February 1, 2008.

Supported in part by NSF grants DMS-0514070 (DB).

Supported in part by the Hungarian National Foundation for Scientific Research, Project #s T-049301 T-049693 and K-61908 (SzR).

This work was accomplished during the fourth author’s stay in Paris under his Marie Curie fellowship, contract # MEIF-CT-2005-022927.
is a locally bounded psh function which satisfies \((dd^c V^*_E)^n = 0\) outside of \(E\) and the total mass of \((dd^c V^*_E)^n\) is \((2\pi)^n\).

In this paper we consider \(E = K \subset \mathbb{R}^n\) a convex body, that is, a compact, convex set with non-empty interior. In this situation the function \(V_K = V_K^*\) is continuous but it is not necessarily smooth, even if \(K\) is smoothly bounded and strictly convex. Indeed, for \(K = \mathbb{B}_R^n\), the unit ball in \(\mathbb{R}^n \subset \mathbb{C}^n\), Lundin found \([9], [1]\) that
\[
V_K(z) = \frac{1}{2} \log h(|z|^2 + |z \cdot z - 1|),
\]
where \(|z|^2 = \sum |z_j|^2, z \cdot z = \sum z_j^2\), and \(h(\frac{1}{2}(t + \frac{1}{t})) = t\), for \(1 \leq t \in \mathbb{R}\).

In this example, the Monge-Ampère measure \((dd^c V_K)^n\) has the explicit form
\[
(dd^c V_K)^n = n! \text{vol}(K) \frac{dx}{(1 - |x|^2)^{\frac{1}{m}}} := n! \text{vol}(K) \frac{dx_1 \wedge \cdots \wedge dx_n}{(1 - |x|^2)^{\frac{1}{m}}}.
\]

The main result of this paper is a general formula for this measure (see Theorem 4.1 and Corollary 4.5).

**Theorem 1.1.** Let \(K\) be a convex body and \(V_K\) its Siciak-Zaharjuta extremal function. The limit
\[
\delta^K_B(x, y) = \delta_B(x, y) := \lim_{t \to 0^+} \frac{V_K(x + ity)}{t}
\]
exists for each \(x \in K^o\) and \(y \in \mathbb{R}^n\) and for \(x \in K^o\)
\[
(dd^c V_K)^n = \lambda(x) dx \text{ where } \lambda(x) = n! \text{vol}(\{y : \delta_B(x, y) \leq 1\}^*).
\]
Moreover, \((dd^c V_K)^n\) puts no mass on the boundary \(\partial K\) (relative to \(\mathbb{R}^n\)).

Here, for a symmetric convex body \(E\) in \(\mathbb{R}^n\),
\[
E^* := \{y \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } x \in E\}
\]
is also a symmetric convex body in \(\mathbb{R}^n\), called the polar of \(E\). The quantity \(\delta_B(x, y)\) is continuous on \(K^o \times \mathbb{R}^n\) and for each fixed \(x \in K^o, y \to \delta_B(x, y)\) is a norm on \(\mathbb{R}^n\); i.e., \(\delta_B(x, y) \geq 0\); \(\delta_B(x, \lambda y) = \lambda \delta_B(x, y)\) for \(\lambda \geq 0\); and \(\delta_B(x, y_1 + y_2) \leq \delta_B(x, y_1) + \delta_B(x, y_2)\) (see \([3]\)).

For a symmetric convex body, i.e., \(K = -K\), Bedford and Taylor \([3]\) showed the existence of the limit \((1.2)\) and proved the formula \((1.3)\) using the description of the Monge-Ampère solution given by Lundin \([10]\). The present paper relies on the description of \(V_K\) given in \([6], [7]\) for general convex bodies \(K\). \([6]\) showed the existence, through each point \(z \in \mathbb{C}^n \setminus K\), of a holomorphic curve on which \(V_K\) is harmonic,
while [7] showed that for many \( K \) (all \( K \) in \( \mathbb{R}^2 \)) these curves give a continuous foliation of \( \mathbb{C}^n \setminus K \) by holomorphic curves. It also showed that these curves are algebraic curves of degree 2, and interpreted them in terms of a (finite dimensional) variational problem among real ellipses contained in \( K \). The real points of such a quadratic curve describe an ellipse within \( K \) of maximal area in its class of competitors. These competitor classes are specified by the points \( c \) on the hyperplane \( H \) at infinity in \( \mathbb{P}^n \) through which the quadratic curves pass. The geometry of these foliations is our main tool.

The norm \( \delta_B(x, y) \) is also related to Bernstein-Markov inequalities for real, multivariate polynomials on \( K \). This will be explained in section 3, specifically, in equation (3.2) and the remarks after it. Conversely, a key role in the proof of the main result Theorem 1.1 is played by the observation (Proposition 3.2) that the extremal inscribed ellipses appearing in a geometric approach – the “inscribed ellipse method” of Sarantopoulos [14], [13] – to Bernstein-Markov inequalities are the maximal area ellipses appearing in the determination of the Monge-Ampère solution as described above. A corollary of our main result is that the inscribed ellipse method and the pluripotential-theoretic method, due to Baran [11, 12] for obtaining Bernstein-Markov-type estimates are equivalent for all convex bodies. This was straightforward for symmetric convex bodies. It was proved for simplices and conjectured for the general case as “Hypothesis A” in [12].

The remainder of the paper is organized as follows. In section 2 we recall in more detail some of the features of the leaf structure for the Monge-Ampère foliation. In section 3 we review the maximal inscribed ellipse problem from [14], [13], its relation to Bernstein-Markov inequalities from approximation theory and to the Monge-Ampère maximal ellipses in section 2. We also sketch the relation to the extremal function \( V_K \) for symmetric convex bodies [3], [2]. Finally, in section 4 using details of the Monge-Ampère foliation and its continuity, we prove the main results.

2. Review of the variational problem.

Let \( K \subset \mathbb{R}^n \subset \mathbb{C}^n \) be a convex body, and consider \( \mathbb{C}^n \subset \mathbb{P}^n \), the complex projective space with \( H := \mathbb{P}^n \setminus \mathbb{C}^n \) the hyperplane at infinity. Let \( \sigma : \mathbb{P}^n \rightarrow \mathbb{P}^n \) be the anti-holomorphic map of complex conjugation, which preserves \( \mathbb{C}^n \) and \( H \), and is the identity on \( \mathbb{R}^n \). Let \( H_{\mathbb{R}} \) denote the
real points of $H$ (fixed points of $\sigma$ in $H$). For any non-zero vector $c \in C^n$, let $\sigma(c) = \overline{c}$, and $[c] \in H$ the point in $H$ given by the direction of $c$. If $[c] \neq [\overline{c}]$, then $c, \overline{c}$ span a complex subspace $V \subset C^n$ of dimension two, which is real, that is, invariant under $\sigma$; hence $V$ is the complexification of a two-dimensional real subspace $V_0 \subset R^n$. If we translate $V$ by a vector $A \in \mathbb{R}^n$, we get a complex affine plane $V + A$ invariant by $\sigma$ and containing the real form $V_0 + A$, the fixed points of $\sigma$ in $V + A$. Associated to the point $[c] \in H$, we consider holomorphic maps $f : \triangle \to \mathbb{P}^n$, $\triangle$ the unit disk in $C$, such that $f(0) = [c]$, and $f(\partial \triangle) \subset K$. Such maps can be extended by Schwarz reflection to maps (still denoted) $f : \mathbb{P}^1 \to \mathbb{P}^n$ by the formula

$$f(\tau(z)) = \sigma(f(z)) \in \mathbb{P}^n$$

where $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is the inversion $\tau(z) = 1/\overline{z}$. In particular, such maps have the form

$$f(z) = \rho \frac{C}{\zeta} + A + \rho \overline{C}\overline{\zeta},$$

where $[c] = [C]$, i.e., $C = \lambda c$, for some $\lambda \in C$, $A \in \mathbb{R}^n$, and $\rho > 0$. Then $f(\mathbb{P}^1) \subset \mathbb{P}^n$ is a quadratic curve, and restricted to $\partial \triangle$, the unit circle in $C$, $f$ gives a parametrization of a real ellipse inside the planar convex set $K \cap \{V_0 + A\}$, with center at $A$. According to [7], the extremal function $V_K$ is harmonic on the holomorphic curve $f(\triangle \setminus \{0\}) \subset C^n \setminus K$ if and only if the area of the ellipse bounded by $f(\partial \triangle)$ is maximal among all those of the form (2.2). For a fixed, normalized $C$, this is equivalent to varying $A \in \mathbb{R}^n$ and $\rho > 0$ among the maps in (2.2) with $E = f(\partial \triangle) \subset K$ in order to maximize $\rho$. Fixing $C$ amounts to prescribing the orientation (major and minor axis) and eccentricity of a family of inscribed ellipses in $K$. We will call an extremal ellipse $E$ a maximal area ellipse, or simply $a$–maximal. In the case where $\partial K$ contains no parallel faces, for each $[c] \in H$ there is a unique $a$–maximal ellipse (Theorem 7.1, [7]); we denote the corresponding map by $f_c$. In this situation, the collection of complex ellipses $\{f_c(\triangle \setminus \{0\}) : [c] \in H\}$ form a continuous foliation of $C^n \setminus K$. In simple terms, this means that if $z, z'$ are distinct points in $C^n \setminus K$, with $|z - z'|$ small, lying on leaves $L(z) := f_c(\triangle \setminus \{0\})$ and $L'(z) := f_c'(\triangle \setminus \{0\})$, then the corresponding leaf parameters in (2.2) are close; i.e., $C \sim C'$, $\rho \sim \rho'$ and $A \sim A'$ (and of course $|\zeta| \sim |\zeta'|$ where $f_c(\zeta) = z$ and $f_c'(\zeta') = z'$). Any convex body in $\mathbb{R}^2$ admits a
continuous foliation; this follows from Proposition 9.2 or Theorem 10.2 in [7]. Moreover, if we let \( C \) denote the set of all convex bodies \( K \subset \mathbb{R}^n \) admitting a continuous foliation, then \( C \) is dense in the Hausdorff metric in the set \( \mathcal{K} \) of all convex bodies \( K \subset \mathbb{R}^n \). This follows, for example, from the fact that strictly convex bodies \( K \) belong to \( C \) (cf., Theorem 7.1 of [7]). In addition, all symmetric convex bodies admit a continuous foliation.

For convenience, instead of using the holomorphic curves \( f(\triangle \setminus \{0\}) \) we will work with the holomorphic curve \( f(\mathbb{C} \setminus \triangle) \); thus \( V_K \) being harmonic on this curve means that

\[
(2.3) \quad V_K(f(\zeta)) = \log |\zeta| \text{ for } |\zeta| \geq 1.
\]

3. INSCRIBED ELLIPSE PROBLEM.

Let \( K \subset \mathbb{R}^n \) be a convex body. Consider the following geometric problem: fix \( x \in K^o \) and a non-zero vector \( y \in \mathbb{R}^n \) and consider all ellipses \( \mathcal{E} \) lying in \( K \) which contain \( x \) and have a tangent at \( x \) in the direction \( y \). We write \( y \in T_x\mathcal{E} \). That is, \( \mathcal{E} = \mathcal{E}_b = \mathcal{E}_b(x, y) \) is given by a parameterization

\[
(3.1) \quad \theta \rightarrow r(\theta) := a \cos \theta + by \sin \theta + (x - a)
\]

where \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^+ \) are such that \( r(\theta) \in K \) for all \( \theta \). The problem is to maximize \( b \) among all such ellipses. We will call such an ellipse a maximal inscribed ellipse (for \( x, y \)) or simply \( b \)-maximal. Note that \( r(0) = x \) and \( r'(0) = by \); thus one is allowed to vary \( a \) and \( b \) in (3.1). Often we will normalize and assume that \( y \) is a unit vector.

An observation which will be used later is that if we fix \( a \), then \( \mathcal{E}_b \) lies “inside” \( \mathcal{E}_y \) if \( b < b' \) with two common points \( x \) and \( x - 2a \).

We give some motivation for studying this problem; this goes back to Sarantapoulos (cf., [14] or [13]). For any such ellipse \( \mathcal{E} \), if \( p \) is a polynomial of \( n \) real variables of degree \( d \), say, with \( ||p||_K \leq 1 \), then \( t(\theta) := p(r(\theta)) \) is a trigonometric polynomial of degree at most \( d \) with \( ||t(\theta)||_{[0,2\pi]} \leq ||p||_K \leq 1 \) (since \( \mathcal{E} \subset K \)). By the Bernstein-Szegö inequality for trigonometric polynomials,

\[
\frac{|t'(\theta)|}{\sqrt{||t||_{[0,2\pi]}^2 - t(\theta)^2}} \leq d.
\]
From the chain rule,
\[ |t'(0)| = |\nabla p(x) \cdot r'(0)| = |D_{y_0}p(x)| = b|D_y p(x)|.\]
Thus
\[ b|D_y p(x)| = |t'(0)| \leq d \sqrt{||t||^2_{[0,2\pi]} - t(0)^2} \leq d \sqrt{1 - p(x)^2};\]
i.e.,
\[ \frac{1}{\deg p} \frac{|D_y p(x)|}{\sqrt{1 - p(x)^2}} \leq \frac{1}{b}. \tag{3.2} \]

The left-hand-side is related to what we shall refer to as a Bernstein-Markov inequality\footnote{For any univariate algebraic polynomial of degree not exceeding \( n \), the sharp uniform estimate for the derivative \( \|p'\|_{\infty,[-1,1]} \leq n\|p\|_{\infty,[-1,1]} \) is due to Markov, while the pointwise estimate \( |p'(x)|\sqrt{1 - x^2} \leq n\|p\|_{\infty,[-1,1]} \) is known as Bernstein's Inequality, see e.g. \cite{4}, pages 232-233. In approximation theory, these types of derivative estimates – or, in the multivariate case, gradient and directional derivative estimates – are usually termed Bernstein and/or Markov type inequalities.}: it relates the directional derivative of \( p \) at \( x \) in the direction \( y \) with the sup-norm of \( p \) on \( K \) (the “1” on the right-hand-side of (3.2)) and the degree of \( p \). This motivates the definition of the Bernstein-Markov pseudometric (cf., \cite{5}): given \( x \in K, y \in \mathbb{R}^n \), let
\[ \delta^K_M(x; y) = \delta_M(x; y) := \sup_{\|p\|_K \leq 1, \deg p \geq 1} \frac{1}{\deg p} \frac{|D_y p(x)|}{\sqrt{1 - p(x)^2}}. \]
(This definition makes sense for general compacta in \( \mathbb{R}^n \)). Inequality (3.2) says that whenever you have an inscribed ellipse \( E_b = E_b(x, y) \) through \( x \) with tangent at \( x \) in the direction of \( y \), the number \( 1/b \) gives an upper bound on the Bernstein-Markov pseudometric:
\[ \delta_M(x; y) \leq 1/b. \]
The bigger you can make \( b \), the better estimate you have.

Let
\[ \delta^* (x, y) := \sup \{ b : E_b(x, y) \subset K \}. \tag{3.3} \]
Note that \( b^*(x, ty) = b^*(x, y)/t \) for \( t > 0 \). In the symmetric case, this is intimately related to \( V_K \):

**Proposition 3.1.** If \( K = -K \), then \( \delta_M(x; y) = \frac{1}{b^*(x, y)} \). Moreover,
\[ \delta_M(x; y) = \lim_{t \to 0^+} \frac{V_K(x + ity)}{t}. \]
As in the introduction, define
\begin{equation}
\delta^K_B(x; y) = \delta_B(x, y) := \lim_{t \to 0^+} \frac{V_K(x + ity)}{t},
\end{equation}
provided this limit exists. For symmetric convex bodies $K$, the proposition says that the limit does exist and we have
\begin{equation}
\delta_B(x, y) = \delta_M(x, y) = \frac{1}{b^*(x, y)}.
\end{equation}
Moreover, for each fixed $x \in K^o$, the function $y \to \frac{1}{b^*(x, y)}$ is a norm (cf., Proposition 3.3). A proof of the existence of the limit was given by Bedford and Taylor [3]. We sketch an alternate proof due to Baran [2].

**Step 1:** $V_K(z) = \sup\{\log|h(z \cdot Z)| : Z \in K^*\}$ where $h(w) = w + \sqrt{w^2 - 1}$ is the standard Joukowski map and
\begin{equation}
K^* := \{Z : x \cdot Z \leq 1 \text{ for all } x \in K\}
\end{equation}
is the polar of $K$ (cf., [10], or [2], Proposition 1.15).

**Step 2:** We have the following explicit estimates on $h$: if $|\alpha| < 1$, $|\beta| \leq \sqrt{1 - |\alpha|}$, and $0 < \epsilon \leq 1/2$, then
\begin{equation}
(1 - \epsilon) \frac{|\beta|}{\sqrt{1 - \alpha^2}} \leq \frac{1}{\epsilon} \log|h(\alpha + i\epsilon\beta)| \leq \frac{|\beta|}{\sqrt{1 - \alpha^2}}
\end{equation}
(the inequality on the right-hand-side is valid without the restriction $|\beta| \leq \sqrt{1 - |\alpha|}$; cf., [2], Proposition 1.13). This states precisely that $\log|h|$ is Lipschitz as you approach $(-1, 1)$ vertically and the Lipschitz constant grows like one-over-the-distance to the boundary points.

Now fix $x \in K^o$ and $y \in \mathbb{R}^n$; then for any $Z \in K^*$ and for $t > 0$ small, since $(x + ity) \cdot Z = x \cdot Z + ity \cdot Z$,
\begin{equation}
(1 - \epsilon) \frac{t|y \cdot Z|}{\sqrt{1 - (x \cdot Z)^2}} \leq \frac{1}{t} \log|h((x + ity) \cdot Z)| \leq \frac{t|y \cdot Z|}{\sqrt{1 - (x \cdot Z)^2}}.
\end{equation}
This gives
\begin{equation}
\lim_{t \to 0^+} \frac{V_K(x + ity)}{t} = \sup\{\frac{|y \cdot Z|}{\sqrt{1 - (x \cdot Z)^2}} : Z \in K^*\}.
\end{equation}
To relate this with \( b^*(x; y) \), in the symmetric case, the \( b-\)maximal ellipse is easily seen to be an \( a-\)maximal ellipse (see the next proposition for a generalization of this), and the linear polynomial \( p \) that maps the support “strip” of this ellipse to \([-1, 1]\) (i.e., it maps one parallel support hyperplane to \(-1\) and the other to \(+1\)) is easily seen to give

\[
\frac{1}{\deg p} \frac{|D_y p(x)|}{\sqrt{1 - p(x)^2}} = \frac{1}{b^*(x, y)}
\]

so that we have the equality \( \delta_M(x, y) = \frac{1}{b^*(x, y)} \). Thus, the first part of the proposition is proved. Moreover, we have the following formula for \( b^*(x, y) \):

\[
b^*(x; y) = \inf \{ \frac{\sqrt{1 - (x \cdot w)^2}}{|y \cdot w|} : w \in K^* \}.
\]

To see this, in the symmetric case one considers symmetric ellipses in \((3.1)\), i.e., \( a := x \), and, from the definition of \( b^*(x; y) \) and \( K^* \) we can write

\[
b^*(x; y) = \sup \{ b : \sup_{w \in K^*} |x \cos t \cdot w + y b \sin t \cdot w| = 1 \}
\]

\[
= \sup \{ b : \sup_{w \in K^*} [(w \cdot x)^2 + b^2(w \cdot y)^2] = 1 \}.
\]

Basically unwinding things shows that this is the reciprocal of \((3.7)\). For details we refer the reader to [11] or [5].

A key geometric observation which will be used in the next section is the following.

**Proposition 3.2.** For any convex body \( K \), a \( b-\)maximal ellipse \( \mathcal{E} \) is also an \( a-\)maximal ellipse.

**Proof.** First observe that an \( a-\)maximal ellipse \( \mathcal{E} \) is characterized by the property that no translate of \( \mathcal{E} \) lies entirely in the interior \( K^0 \) of \( K \). For if \( \mathcal{E} + v \subseteq K^0 \) for some \( v \neq 0 \), one can dilate \( \mathcal{E} + v \) to get an ellipse with the same orientation and eccentricity as \( \mathcal{E} \) which lies in \( K \) but has larger area. Conversely, if \( \mathcal{E} \) is not an \( a-\)maximal ellipse, then one can find an ellipse \( \mathcal{E}' \) with the same orientation and eccentricity as \( \mathcal{E} \) which lies in \( K \) but has larger area. The convex hull \( H \) of \( \mathcal{E} \cup \mathcal{E}' \) lies in \( K \) and we can translate \( \mathcal{E} \) within \( H \) to an ellipse \( \mathcal{E}'' \) lying in the two-dimensional surface \( S(\mathcal{E}') \) determined by \( \mathcal{E}' \); if \( \mathcal{E}'' \) does not lie in the “interior” of \( S(\mathcal{E}') \), we simply translate it within this surface (since the area of \( \mathcal{E}' \) is greater than that of \( \mathcal{E}'' \)) until it does.
Indeed, we need a slightly more precise statement: \( \mathcal{E} \) is not an \( a \)-maximal ellipse if and only if there is a unit vector \( v \) and \( \delta > 0 \) such that \( \mathcal{E} + sv \subset K^o \) for \( 0 < s < \delta \), i.e., all translates by a small amount in some direction stay in \( K^o \). This follows from the previous paragraph if we observe the following fact: if \( K \) is a convex body, \( u \in K \) and \( u + v \in \partial K^o \), then the entire half-open segment \((u, u + v]\) lies in \( K^o \).

Suppose that \( \mathcal{E} \) given by
\[
\theta \to a \cos \theta + by \sin \theta + (x - a)
\]
is a \( b \)-maximal ellipse for \( x, y \). For the sake of obtaining a contradiction, we assume that \( \mathcal{E} \) is not an \( a \)-maximal ellipse. By the previous paragraph, we can find a nonzero vector \( v \) and \( \delta > 0 \) so that \( \mathcal{E} + sv \subset K^o \) for \( 0 < s < \delta \). For \( 0 < \epsilon < \delta / 2 \), consider the ellipse \( \tilde{\mathcal{E}}(\epsilon) \) given by
\[
(3.8) \quad r_\epsilon(\theta) = (a - \epsilon v) \cos \theta + by \sin \theta + x - (a - \epsilon v).
\]
We claim that \( \tilde{\mathcal{E}}(\epsilon) \subset K^o \). Assuming this is the case, note that \( r_\epsilon(0) = x \in \tilde{\mathcal{E}}(\epsilon) \) and \( r'_\epsilon(0) = by \); in particular, the “\( b \)” for \( \tilde{\mathcal{E}}(\epsilon) \) is the same as the “\( b \)” for \( \mathcal{E} \). Since \( \tilde{\mathcal{E}}(\epsilon) \subset K^o \), we can modify \( \tilde{\mathcal{E}}(\epsilon) \) to an ellipse \( \tilde{\mathcal{E}}(\epsilon)' \) containing \( x \) and lying in \( K \) by replacing \( b \) in \( (3.8) \) by \( b' > b \) contradicting the assumption that \( \mathcal{E} \) is a \( b \)-maximal ellipse for \( x, y \).

To verify that \( \tilde{\mathcal{E}}(\epsilon) \subset K^o \), observe that for each fixed \( \theta \), the point
\[
(a - \epsilon v) \cos \theta + by \sin \theta + x - (a - \epsilon v)
\]
\[
= a \cos \theta + by \sin \theta + (x - a) + \epsilon v(1 - \cos \theta)
\]
on \( \tilde{\mathcal{E}}(\epsilon) \) lies on the ellipse \( \mathcal{E}_{s_\epsilon} := \mathcal{E} + \epsilon(1 - \cos \theta)v \) where \( s_\epsilon = \epsilon(1 - \cos \theta) \leq 2\epsilon < \delta \). Thus \( \tilde{\mathcal{E}}(\epsilon) \subset K^o \).

For use in the next section, we prove some results about the function \( b^*(x, y) \).

**Proposition 3.3.** For a convex body \( K \subset \mathbb{R}^n \), \( b^*(x, y) \) defined in \( (3.3) \) is a continuous function of \( x \in K^o \) and \( y \in \mathbb{R}^n \). Moreover, for each fixed \( x \in K^o \), \( y \to 1/b^*(x, y) \) is a norm in \( \mathbb{R}^n \).

**Proof.** For the continuity of \( b^*(x, y) \), we first verify uppersemicontinuity of this function. Fix a convex body \( K \) and fix \( x \in K^o \) and \( y \in \mathbb{R}^n \). Let
\{x_j\} \subset \mathbb{K}^o \text{ with } x_j \to x \text{ and } \{y_j\} \subset \mathbb{R}^n \text{ with } y_j \to y. \text{ Let } r_j(\theta) = a_j \cos \theta + b^*(x_j, y_j)y_j \sin \theta + (x_j - a_j)

parameterize a \(b\)--maximal ellipse \(E_j\) for \(K\) through \(x_j\) in the direction \(y_j\). Take a subsequence \(\{j_k\}\) of positive integers so that the numbers \(\{b^*(x_{j_k}, y_{j_k})\}\) converge to a number \(\bar{b}\); and take a further subsequence (which we still call \(\{j_k\}\)) so that the vectors \(\{a_{j_k}\} \subset \mathbb{R}^n\) converge to \(a \in \mathbb{R}^n\). Consider the ellipse \(E\) where

\[
r(\theta) = a \cos \theta + \bar{b} y \sin \theta + (x - a).
\]

Since \(x_{j_k} \to x\), \(y_{j_k} \to y\), \(b^*(x_{j_k}, y_{j_k}) \to \bar{b}\) and \(a_{j_k} \to a\), the functions \(r_{j_k}\) converge uniformly to \(r\) (equivalently, the ellipses \(E_{j_k}\) converge in the Hausdorff metric to \(E\)). Thus \(E\) is an inscribed ellipse for \(K\) through \(x\) in the direction of \(y\); hence \(\bar{b} \leq b^*(x, y)\); i.e.,

\[
\limsup_{x' \to x, \; y' \to y} b^*(x', y') \leq b^*(x, y).
\]

To verify lowersemicontinuity of \(b^*(x, y)\), we fix \(x \in \mathbb{K}^o\), \(y \in \mathbb{R}^n\) and \(b' < b^*(x, y)\), and we show there is a \(\delta > 0\) such that for all \(|x' - x| < \delta\), \(|y' - y| < \delta\) there is an inscribed ellipse \(E'\) through \(x'\) with tangent direction \(y'\) of the form

\[
\theta \to a' \cos \theta + b' y' \sin \theta + (x' - a').
\]

Let \(E\) be a \(b\)--maximal ellipse through \(x\) in the direction \(y\) given by

\[
r(\theta) = a \cos \theta + b* (x, y) y \sin \theta + (x - a).
\]

If \(x - 2a \in \mathbb{K}^o\), then for \(b' < b\) the ellipse \(E_{b'}\)

\[
r_{b'}(\theta) = a \cos \theta + b' y \sin \theta + (x - a)
\]

lies fully in \(\mathbb{K}^o\) (for \(E_{b'}\) lies entirely “inside” of \(E\) except for the common points \(x, x - 2a\), and \(x \in \mathbb{K}^o\)). Then any sufficiently small translation \(E'\)

\[
\theta \to a \cos \theta + b' y \sin \theta + (x' - a)
\]

of \(E_{b'}\) by \(x' - x\) keeps \(E'\) in \(\mathbb{K}^o\); hence replacing \(y\) by \(y'\) sufficiently close to \(y\) yields \(E''\)

\[
\theta \to a \cos \theta + b' y' \sin \theta + (x' - a)
\]

in \(\mathbb{K}\).

If \(x - 2a \notin \mathbb{K}^o\), we first modify \(E_{b'}\) to \(E_{b', a'}:\)

\[
r_{b', a'}(\theta) = a' \cos \theta + b' y \sin \theta + (x - a')
\]
with \(a' - a = \delta(a - x)\) with \(\delta > 0\) sufficiently small so that \(E_{\theta, a'} \subset K^o\). This is possible since the vectors

\[
r_{\theta, \lambda y}(\theta) - r_{\lambda y}(\theta) = (a - \lambda y)(1 - \cos \theta)\delta(1 - \cos \theta)(a - a')
\]

point in the same direction for all \(\theta\). Note that

\[
r_{\theta, \lambda y}(0) = x\quad \text{and} \quad r'_{\theta, \lambda y}(0) = b'y
\]

so that once again any sufficiently small translation \(E'\)

\[
\theta \to a' \cos \theta + b'y \sin \theta + (x' - a')
\]

of \(E_{\theta, a'}\) by \(x' - x\) keeps \(E'\) in \(K^o\). Again, replacing \(y\) by \(y'\) sufficiently close to \(y\) yields \(E''\)

\[
\theta \to a' \cos \theta + b'y' \sin \theta + (x' - a')
\]

in \(K\). This completes the proof that \((x, y) \to b^{*}(x, y)\) is continuous.

To show that \(y \to 1/b^{*}(x, y)\) is a norm, observe that \(b^{*}(x, \lambda y) = \frac{1}{\lambda}b^{*}(x, y)\) for \(\lambda > 0\) and \(0 < b^{*}(x, y) < +\infty\) if \(y \neq 0 \in \mathbb{R}^n\). Thus \(b^{*}(x, 0) = +\infty\) so that \(1/b^{*}(x, y) \geq 0\) with equality precisely when \(y = 0\); and \(1/b^{*}(x, \lambda y) = \lambda/b^{*}(x, y)\) for \(\lambda \geq 0\). To verify subadditivity in \(y\), fix \(x \in K^o\) and \(y_1, y_2 \in \mathbb{R}^n\). Let \(E_1\) and \(E_2\) be \(b\)-maximal ellipses through \(x\) in the directions \(y_1\) and \(y_2\) as in (3.1):

\[
\theta \to r_j(\theta) := a_j \cos \theta + b_j y_j \sin \theta + (x - a_j), \quad j = 1, 2;
\]

here, \(b_j := b^{*}(x, y_j)\). Consider the convex combination

\[
\frac{b_2}{b_1 + b_2} r_1(\theta) + \frac{b_1}{b_1 + b_2} r_2(\theta)
\]

\[
= \frac{a_1 b_2}{b_1 + b_2} \cos \theta + \frac{b_1 b_2}{b_1 + b_2} y_1 \sin \theta + \frac{b_2}{b_1 + b_2} (x - a_1)
\]

\[
+ \frac{a_2 b_1}{b_1 + b_2} \cos \theta + \frac{b_1 b_2}{b_1 + b_2} y_2 \sin \theta + \frac{b_1}{b_1 + b_2} (x - a_2)
\]

\[
= \frac{a_1 b_2 + a_2 b_1}{b_1 + b_2} \cos \theta + \frac{b_1 b_2}{b_1 + b_2} (y_1 + y_2) \sin \theta + x - \frac{a_1 b_2 + a_2 b_1}{b_1 + b_2}.
\]

By convexity, this ellipse, through \(x\) in the direction \(y_1 + y_2\), lies in \(K\) so that

\[
b^{*}(x, y_1 + y_2) \geq \frac{b_1 b_2}{b_1 + b_2}.
\]

Unwinding, this says that

\[
\frac{1}{b^{*}(x, y_1 + y_2)} \leq \frac{1}{b_1} + \frac{1}{b_2}.
\]
as desired.

For future use, we mention that in $\mathbb{R}^2$, if $\mathcal{E}$ is an $a$–maximal ellipse for $K$, then either

1. $\mathcal{E} \cap \partial K$ contains exactly two points $a_1, a_2$, in which case the tangent lines to $\mathcal{E}$ at $a_1, a_2$ are parallel and determine a strip $S$ containing $K$ and $\mathcal{E}$ is an $a$–maximal ellipse for any rectangular truncation $T$ of $S$ with $\mathcal{E} \subset K \subset T$; or

2. $\mathcal{E} \cap \partial K$ contains $m \geq 3$ points $a_1, ..., a_m$, in which case either a subset of three points from $\{a_1, ..., a_m\}$ can be found so that the tangent lines to $\mathcal{E}$ at these three points bound a triangle $T$ containing $K$ and $\mathcal{E}$ is an $a$–maximal ellipse for $T$, or a rectangular truncation $T$ of a strip $S$ with $\mathcal{E} \subset K \subset T$ can be found so that $\mathcal{E}$ is an $a$–maximal ellipse for $T$.

4. Main result.

For any compact set $K \subset \mathbb{R}^n$ with non-empty interior, take $x \in K^o$ and $y \in \mathbb{R}^n \setminus \{0\}$. Then we always have the pointwise inequality

\begin{equation}
\delta_{M}(x, y) \leq \delta_{B}^{(i)}(x, y) := \liminf_{t \to 0^+} \frac{V_{K}(x + ity)}{t}.
\end{equation}

This follows from Proposition 2.1 in [5]. In particular, this inequality holds for any convex body $K$, with equality in case $K$ is symmetric (as we saw in the previous section). In this section, we prove that the limit $\lim_{t \to 0^+} \frac{V_{K}(x + ity)}{t}$ exists and equals $1/b^*(x, y)$. This verifies “Hypothesis A” in [12] for convex bodies $K \subset \mathbb{R}^n$, i.e., the inscribed ellipse method and the pluripotential-theoretic method for obtaining Bernstein-Markov-type estimates for convex bodies are equivalent.

Let $K$ be an arbitrary convex body in $\mathbb{R}^n$. Fix $x \in K^o$ and $y \in \mathbb{R}^n \setminus \{0\}$. Take a $b$–maximal ellipse $\mathcal{E}$ through $x$ with tangent direction $y$ at $x$. We will normalize and assume that $y$ is a unit vector; moreover, it will be convenient to have the center at $a$ instead of $x - a$. Thus we write

\begin{equation}
\theta \to r(\theta) = (x - a) \cos \theta + b^*(x, y)y \sin \theta + a, \; \theta \in [0, 2\pi]
\end{equation}

This is an $a$–maximal ellipse $\mathcal{E}$ by Proposition 3.2, i.e., $\mathcal{E}$ forms the real points of a leaf $L$

\begin{equation}
f(\zeta) = (x - a)[\frac{1}{2}(\zeta + 1/\zeta)] + b^*(x, y)y[\frac{1}{2}(\zeta - 1/\zeta)] + a, \; |\zeta| \geq 1
\end{equation}
of our foliation for the extremal function $V_K$. We can compare this “$b-$maximal” form of the leaf with its $a-$maximal form \(2.2\):

\[
f(\zeta) = A + c\zeta + \frac{c}{\zeta}, \quad |\zeta| \geq 1,
\]

where, for simplicity, we write $c := \rho C$ in \(2.2\). Thus, from \(2.3\), $V_K(f(\zeta)) = \log |\zeta|$ for $|\zeta| \geq 1$.

In these coordinates $V_K(f(\zeta)) = \log |\zeta|$. We first show that

\[
\lim_{r \to 1^+} \frac{f(r) - f(1)}{r - 1} = ib^*(x, y).
\]

This follows from the calculation

\[
f(r) - f(1) = (x - a)\left(\frac{(r - 1)^2}{2r}\right) + ib^*(x, y)\left(\frac{(r - 1)(r + 1)}{2r}\right).
\]

Thus the real tangent vector to the real curve $r \to f(r)$, $r \geq 1$ as $r \to 1^+$ is in the direction $ib^*(x, y)$. Now $f(1) = x$ and $x \in K$ so $V_K(f(1)) = V_K(x) = 0$; and, since $f$ is a leaf of our foliation, $V_K(f(r)) = \log r$. Hence

\[
\frac{V_K(f(r)) - V_K(f(1))}{r - 1} = \frac{\log r}{r - 1}
\]

so that

\[
\lim_{r \to 1^+} \frac{V_K(f(r)) - V_K(f(1))}{r - 1} = \lim_{r \to 1^+} \frac{\log r}{r - 1} = 1.
\]

This elementary calculation shows that for any convex body $K \subset \mathbb{R}^n$,

\[
\lim_{r \to 1^+} \frac{V_K(f(r)) - V_K(f(1))}{b^*(x, y)(r - 1)} = \frac{1}{b^*(x, y)};
\]

i.e., the curvilinear limit along the curve $f(r)$ in the direction of $iy$ at $x$ exists and equals $\frac{1}{b^*(x, y)}$.

Note that

\[
f(r) - f(1) = f(r) - x = ib^*(x, y)y(r - 1) + 0((r - 1)^2),
\]

so that the point $x + ib^*(x, y)y(r - 1)$ is $O((r - 1)^2)$ close to the point $f(r)$. We use the explicit form \(4.3\) of the leaf to verify the existence of the limit in the directional derivative $\delta_B(x, y)$.

**Theorem 4.1.** Let $K$ be a convex body in $\mathbb{R}^n$. Then the limit in the definition of the directional derivative exists and equals $\frac{1}{b^*(x, y)}$:

\[
\delta_B(x, y) := \lim_{t \to 0^+} \frac{V_K(x + ity)}{t} = \frac{1}{b^*(x, y)}.
\]
Proof. If we can show
\[
\lim_{r \to 1^+} \frac{V_K(f(r)) - V_K(x + ib^*(x, y)y(r - 1))}{b^*(x, y)(r - 1)} = 0,
\]
then using (4.5) and the preceding discussion, we will have
\[
\lim_{t \to 0^+} \frac{V_K(x + ity)}{t} = \frac{1}{b^*(x, y)}.
\]
We first consider the case when \(K\) admits a continuous foliation; i.e., \(K \in C\). Consider a fixed point \(w := x + ib^*(x, y)y(r - 1) \in C^n\). This belongs to some foliation leaf \(M\) which we write in the form (4.4):
\[
g(\zeta) = \alpha + \gamma\zeta + \bar{y}/\zeta : C \setminus \Delta \to M \subset C^n.
\]
We need to use the facts that when \(r \to 1^+\), then \(w \to x \in L\), and, by continuity of the foliation, the leaf parameters for \((g, M)\) should converge to those of \((f, L)\); i.e., \(\alpha \to A\) and \(\gamma \to c\). We remark that if we compare (4.3) and (4.4), writing \(b := b^*(x, y)\) we have the relations
\[
A = a \quad \text{and} \quad c = \frac{1}{2}(x - a + iby).
\]
Here we suppress a rotational invariance: the substitution \(\zeta' := \zeta e^{i\varphi}\) for any fixed constant \(\varphi\) describes the same leaf with a different parametrization; thus we fix its value so that \(\xi := g(1) = 2\Re\gamma + \alpha\) is closest to \(x := f(1) = 2\Re c + A\), i.e., \(|g(1) - f(1)| \leq |g(e^{i\theta}) - f(1)|\) for all \(\theta\). To emphasize, writing the leaf \((f, L)\) in \(b\)-maximal form (4.3),
\[
f(\zeta) = (x - a)\frac{1}{2}(\zeta + \frac{1}{\zeta}) + by\frac{i}{2}(\zeta - \frac{1}{\zeta}) + a
\]
where, from (4.8) and the fact that \(y\) is a unit vector, \(b := 2|\Im c| > 0\) and \(y := \frac{2}{b}\Im c \in \mathbb{R}^n\). Now, apriori, we do not know if \((g, M)\) is \(b\)-maximal (aposteriori, it is: see Corollary 4.3). However, we may still write this leaf in the form
\[
g(\zeta) = (\xi - \alpha)\frac{1}{2}(\zeta + \frac{1}{\zeta}) + \beta\eta\frac{i}{2}(\zeta - \frac{1}{\zeta}) + \alpha
\]
with \(\beta := 2|\Im \gamma| > 0\) and \(\eta := \frac{2}{\beta}\Im \gamma \in \mathbb{R}^n\). Note that continuity of the foliation implies \(\beta > 0\) since \(b > 0\); indeed, \(\beta \sim b, \xi \sim x, \eta \sim y, \alpha \sim a, \text{and } \gamma \sim c\).
Since \(w \in M\), there is a point \(\omega \in \mathbb{C} \setminus \Delta\) with \(g(\omega) = w\). Our task is to calculate \(V_K(w) = V_K(g(\omega))\). On a leaf of the foliation we have the formula \(V_K(g(\omega)) = \log |\omega|\), so it suffices to compute \(\log |\omega|\). The representation of \(w\) as \(g(\omega)\) means that for \(j = 1, \ldots, n\),

\[
 x_j + iby_j(r - 1) = w_j = g_j(\omega) = (\xi_j - \alpha_j) \frac{1}{2}(\omega + \frac{1}{\omega}) + \beta \eta_j \frac{i}{2}(\omega - \frac{1}{\omega}) + \alpha_j.
\]

Since \(y\) and \(\eta\) are unit vectors which are close to each other, we can choose a coordinate \(j\) with \(y_j \neq 0, \eta_j \neq 0\). For this coordinate \(j\), the previous displayed equation gives

\[
 \frac{1}{2}(\xi_j - \alpha_j + i\beta \eta_j)\omega^2 + (\alpha_j - x_j - iby_j(r - 1))\omega + \frac{1}{2}(\xi_j - \alpha_j - i\beta \eta_j) = 0,
\]

a quadratic equation in \(\omega\). Corresponding to the double mapping properties of the Joukowski map \(\frac{1}{2}(\zeta + 1/\zeta)\), there are two roots, one in \(|\zeta| < 1\) and one in \(|\zeta| > 1\), the latter being our \(\omega\) as we considered the mapping of the exterior of the unit disc \(\Delta\). For convenience, put \(\rho := b(r - 1)y_j\). Since \(by_j \neq 0, \rho \asymp r - 1\). By the quadratic formula,

\[
 \omega_{1,2} = \frac{x_j - \alpha_j + i\rho \pm \sqrt{(\alpha_j - x_j - i\rho)^2 - (\xi_j - \alpha_j)^2 - \beta^2 \eta_j^2}}{\xi_j - \alpha_j + i\beta \eta_j}.
\]

Set \(Q := \beta^2 \eta_j^2 + (\xi_j - \alpha_j)^2 - (x_j - \alpha_j)^2 \sim b^2 y_j^2 > 0\) by continuity of the leaf parameters and choice of \(j\). Using this and the simple formula

\[
 \sqrt{A + 2B} = \sqrt{A} + B/\sqrt{A} + O(B^2/A^{3/2}),
\]

valid uniformly for \(|B| < A/3\), say, we can rewrite the square root as

\[
 \pm \sqrt{(x_j - \alpha_j)^2 - i2(\alpha_j - x_j)\rho - \rho^2 - (\xi_j - \alpha_j)^2 - \beta^2 \eta_j^2}
 = \pm i \sqrt{Q + 2i(\alpha_j - x_j)\rho + \rho^2}
 = \pm i \left\{ \sqrt{Q} + \frac{(\alpha_j - x_j)\rho}{\sqrt{Q}} + O(\rho^2) \right\}
 = \pm \left\{ \frac{(x_j - \alpha_j)\rho}{\sqrt{Q}} + i\sqrt{Q} + O(\rho^2) \right\}.
\]
Put $P := \xi_j - \alpha_j + i\beta \eta_j$. Then
\[|\omega_{1,2}P|^2 = |(x_j - \alpha_j)(1 \pm \frac{\rho}{\sqrt{Q}}) + i[\pm \sqrt{Q} + \rho] + O(\rho^2)|^2\]
\[= [(\alpha_j - x_j) \pm \frac{\rho(\alpha_j - x_j)}{\sqrt{Q}}]^2 + [\pm \sqrt{Q} + \rho]^2 + O(\rho^2)\]
\[= (\alpha_j - x_j)^2 \pm \frac{2\rho(\alpha_j - x_j)^2}{\sqrt{Q}} + Q \pm 2\rho\sqrt{Q} + O(\rho^2)\]
\[= (\alpha_j - x_j)^2 + Q \pm \frac{2\rho}{\sqrt{Q}}(Q + (\alpha_j - x_j)^2) + O(\rho^2).\]

We have the identity $|P|^2 = (\alpha_j - x_j)^2 + Q$; dividing by this quantity on both sides yields
\[|\omega_{1,2}|^2 = 1 \pm \frac{2\rho}{\sqrt{Q}} + O(\rho^2).\]

Fixing the branch of the square-root with $\sqrt{Q} > 0$, it is clear that among the choice of signs in $\pm$ the one equal to the sign of $y_j$ leads to the larger absolute value (and the one with $|\omega|$ exceeding 1); hence for such $\omega$ with $g(\omega) = w$,
\[|\omega|^2 = 1 + \frac{2\rho}{\sqrt{Q}} + O(\rho^2) = 1 + \frac{2b|y_j|(r - 1)}{\sqrt{Q}} + O((r - 1)^2),\]
so that
\[\log |\omega|^2 = log|1 + \frac{2b|y_j|(r - 1)}{\sqrt{Q}} + O((r - 1)^2)| = \frac{2b|y_j|(r - 1)}{\sqrt{Q}} + O((r - 1)^2).\]

Hence
\[\frac{V_K(x + iby(r - 1))}{r - 1} = \frac{V_K(g(\omega))}{r - 1} = \frac{\log |\omega|}{r - 1} = \frac{1}{2} \frac{\log |\omega|^2}{r - 1} = \frac{b|y_j|}{\sqrt{Q}} + O(r - 1).\]

Recall once again that the continuity of the foliation, as $r \to 1$ we have $\xi_j \to x_j$, $\alpha_j \to a_j$, $\beta \to b$, and thus $\sqrt{Q} \to b|y_j|$. Hence
\[\lim_{r \to 1^+} \frac{V_K(x + iby(r - 1))}{r - 1} = 1 = \lim_{r \to 1^+} \frac{V_K(f(r))}{r - 1}.
\]

This verifies (4.6) and hence (4.7) in the case when $K$ admits a continuous foliation.
For a general convex body $K$, we use the previous case and an appropriate approximation argument to verify (4.7). To emphasize the set(s) under discussion, we write $b^*(K; x, y) := b^*(x, y)$ for the set $K$.

We need to prove two inequalities:

\begin{equation}
\frac{1}{b^*(K; x, y)} \leq \liminf_{t \to 0^+} \frac{V_K(x + ity)}{t},
\end{equation}

and

\begin{equation}
\frac{1}{b^*(K; x, y)} \geq \limsup_{t \to 0^+} \frac{V_K(x + ity)}{t}.
\end{equation}

Note that if $K$ and $\kappa$ are two convex bodies, we have for any $y \in \mathbb{R}^n$ the inequalities

\begin{equation}
b^*(K; x, y) \begin{cases} 
\leq b^*(\kappa; x, y) & \text{if } K \subset \kappa \\
\geq b^*(\kappa; x, y) & \text{if } \kappa \subset K,
\end{cases}
\end{equation}

and for any $y \in \mathbb{R}^n$ and any $t \in \mathbb{R}$ the inequalities

\begin{equation}
V_K(x + ity) \begin{cases} 
\geq V_\kappa(x + ity) & \text{if } K \subset \kappa \\
\leq V_\kappa(x + ity) & \text{if } \kappa \subset K.
\end{cases}
\end{equation}

Fix $\alpha < 1$ arbitrarily close to 1 and choose $\delta > 0$ small (to be determined later in terms of $\alpha$). From the discussion at the end of section 2, $C$ is dense in $K$; thus we can find $\kappa \in C$ such that the Hausdorff distance between $\kappa$ and $K$ is at most $\delta$. Take an $\alpha$-dilated (at $x$) copy $K_1$ of $K$ and an $\alpha$-dilated copy $\kappa_1$ of $\kappa$. Then $x \in K_1^0$, and if $\delta = \delta(\alpha)$ is sufficiently small, we have $\kappa_1 \subset K$. We also take the $1/\alpha$-dilated copies $K_2$ and $\kappa_2$ of $K$ and $\kappa$. Again for small enough $\delta$, we will have $K \subset \kappa_2$. Note that $\kappa_2$ is the $\alpha^{-2}$-dilated copy of $\kappa_1$; hence $b^*(\kappa_2; x, y) = \alpha^{-2}b^*(\kappa_1; x, y)$. Therefore, using (4.7) for $\kappa_1$ and $\kappa_2$, we obtain

\begin{equation}
\lim_{t \to 0^+} \frac{V_{\kappa_2}(x + ity)}{t} = \frac{1}{b^*(\kappa_2; x, y)} = \alpha^2 \frac{1}{b^*(\kappa_1; x, y)} = \alpha^2 \lim_{t \to 0^+} \frac{V_{\kappa_1}(x + ity)}{t}.
\end{equation}
Using (4.13), (4.11) and (4.12), we obtain

$$\frac{1}{b^*(K; x, y)} \leq \frac{1}{b^*(\kappa_1; x, y)} = \lim_{t \to 0^+} \frac{V_{\kappa_1}(x + ity)}{t} = \frac{1}{\alpha^2} \liminf_{t \to 0^+} \frac{V_{\kappa_2}(x + ity)}{t};$$

and, in a similar fashion we get

$$\frac{1}{b^*(K; x, y)} \geq \frac{1}{b^*(\kappa_2; x, y)} = \lim_{t \to 0^+} \frac{V_{\kappa_2}(x + ity)}{t} = \frac{1}{\alpha^2} \limsup_{t \to 0^+} \frac{V_{\kappa_1}(x + ity)}{t} \geq \frac{1}{\alpha^2} \limsup_{t \to 0^+} \frac{V_K(x + ity)}{t}.$$

(4.14)

Since $\alpha$ can be taken arbitrarily close to 1, (4.9) and (4.10) follow from (4.14) and (4.15).

□

Remark 4.2. Observe that the essential property used to verify (4.6) for $K \in \mathcal{C}$ is (2.3); i.e., that $V_K(f(\zeta)) = \log |\zeta|$ on $L$ (i.e., for $|\zeta| \geq 1$), which is equivalent to the $a-$maximality of the real ellipse $E \subset L$ (see section 2).

Corollary 4.3. For any convex body $K$, an ellipse $E \subset K$ is $a-$maximal if and only if it is $b-$maximal for all $x \in K^o$ and $y \in T_xE$.

Proof. That $b-$maximality implies $a-$maximality was proved in Proposition 3.2. For the converse, we first suppose that $K \in \mathcal{C}$. Let $E$ be an $a-$maximal ellipse. Fix $x \in K^o$ and $y \in T_xE$ a unit vector. Then $E = f(\partial \Delta)$ where

$$f(\zeta) = (x - \alpha)\frac{1}{2} (\zeta + \overline{\zeta}) + \beta y \frac{i}{2} (\zeta - \overline{\zeta}) + \alpha$$

and $V_K(f(\zeta)) = \log |\zeta|$ for $|\zeta| \geq 1$, so that, from the remark,

$$1/\beta = \lim_{r \to 1^+} \frac{V_K(f(r))}{\beta(r - 1)} = \lim_{r \to 1^+} \frac{V_K(x + i\beta y(r - 1))}{\beta(r - 1)}.$$
Writing $t = \beta(r - 1)$ in the limit on the right and using Theorem 4.1,

$$1/\beta = \lim_{t \to 0^+} \frac{V_K(x + ity)}{t} = \frac{1}{b^*(x, y)}$$

so that $E$ is $b-$maximal for $x, y$.

Now let $K \in \mathcal{K}$ be an arbitrary convex body. We consider first the case where $E$ is the unique $a-$maximal ellipse for $[c] \in H$; i.e., for its orientation and eccentricity, and we again write $E = f(\partial \Delta)$ as in (4.16). Take a sequence $\{K_j\} \subset \mathcal{C}$ with $K_j \searrow K$. For each $j$, let $E_j$ be the unique $a-$maximal ellipse for $K_j, [c]$, and let $f_j$ denote the corresponding leaf. Then (cf., [7]) $f_j \to f$ uniformly so that $E_j \to E$.

As in the proof of Theorem 4.1, we may write

$$f_j(\zeta) = (x_j - \alpha_j) \frac{1}{2} (\zeta + \frac{1}{\zeta}) + \beta_j y_j \frac{i}{2} (\zeta - \frac{1}{\zeta}) + \alpha_j.$$

By the first part of the proof, $E_j$ is $b$-maximal in $K_j$, so that $\beta_j = b^*(x_j, y_j, K_j)$. The uniform convergence of $f_j$ to $f$ implies that $\alpha_j \to \alpha$, $x_j \to x$ and $y_j \to y$. Moreover, since $x \in K^\circ$, for $j$ sufficiently large, $x_j \in K^\circ$. From the continuity of $b^*$ (Proposition 3.3) and the fact that $K_j \subset K$,

$$b^*(x, y, K) = \lim_{j \to \infty} b^*(x_j, y_j, K) \leq \lim_{j \to \infty} b^*(x_j, y_j, K_j) = \lim_{j \to \infty} \beta_j = \beta.$$

Hence $\beta = b^*(x, y, K)$ and $E$ is $b-$maximal.

In the case where $E$ is not the unique $a-$maximal ellipse for the corresponding $[c] \in H$, it is an $a-$maximal ellipse for this $[c]$ and a “strip” $S$, i.e., a closed body $S$ bounded by two parallel hyperplanes $P_1, P_2$ with $K \subset S$ (see section 7 of [7]). If $E$ is given by

$$r(\theta) = f(e^{i\theta}) = (x - \alpha) \cos \theta + \beta y \sin \theta + \alpha$$

then there is $\theta_0$ such that $r(\theta_0) \in P_1$ and $r(\theta_0 + \pi) \in P_2$. It is therefore sufficient to show that any ellipse $\mathcal{E} \subset K$ that intersects $P_1, P_2$ is $b$-maximal for any $x \in \mathcal{E} \cap K^\circ, y \in T_x \mathcal{E}$. Take a sufficiently large convex set $T$ that is symmetric about the center of the ellipse (e.g., a large box) so that $K \subset T \subset S$. Clearly $\mathcal{E}$ is $a$-maximal for $T$. Since $T$ is symmetric, $T \in \mathcal{C}$, so by the first part of the proof, $\mathcal{E}$ is $b$-maximal for $T, x, y$. Hence it is also $b$-maximal for $K, x, y$. □

We turn to the Monge-Ampere measure. We know that $(dd^c V_K)^n$ is supported in $K$ and is absolutely continuous with respect to Lebesgue
measure $dx$ on $K^o$, i.e., $(dd^cV_K)^n = \tilde{\lambda}(x)dx$ for a locally integrable non-negative function $\tilde{c}$ on $K^o$ [3]. Baran has proved the following (see [2], Propositions 1.10, 1.11 and Lemma 1.12).

**Proposition 4.4.** Let $D \subset \mathbb{C}^n$ and let $\Omega := D \cap \mathbb{R}^n$. Let $u$ be a nonnegative psh function on $D$ which satisfies:

i. $\Omega = \{ u = 0 \}$

ii. $(dd^c u)^n = 0$ on $D \setminus \Omega$

iii. $(dd^c u)^n = \lambda(x) dx$ on $\Omega$ where $c \in L^1_{\text{loc}}(\Omega)$

iv. for all $x \in \Omega$, $y \in \mathbb{R}^n$, the limit

$h(x, y) := \lim_{t \to 0^+} \frac{u(x + ity)}{t}$ exists and is continuous on $\Omega \times i\mathbb{R}^n$

v. $x \in \Omega, y \to h(x, y)$ is a norm.

Then

$$\lambda(x) = n! \text{vol} \{ y : h(x, y) \leq 1 \}^*$$

and $\lambda(x)$ is a continuous function on $\Omega$.

We now obtain the generalization of (1.3).

**Corollary 4.5.** Let $K$ be a convex body in $\mathbb{R}^n$. Then

$$(dd^cV_K)^n = \lambda(x) dx \text{ for } x \in K^o$$

where $\lambda(x) = n! \text{vol} \{ y : \delta_B(x, y) = \frac{1}{|x-y|} \leq 1 \}^*$ is continuous. Moreover, $(dd^cV_K)^n$ puts no mass on the boundary $\partial K$ (relative to $\mathbb{R}^n$).

**Proof.** The formula for $(dd^cV_K)^n$ on $K^o$ is immediate from Theorem 4.1, Proposition 4.4 and the paragraph preceding it, and Proposition 3.3. To show that $(dd^cV_K)^n$ puts no mass on the boundary $\partial K$, we proceed as in [3]. Let $\{K_j\}$ be a sequence of convex bodies in $\mathbb{R}^n$ with real-analytic boundaries $\{\partial K_j\}$ such that $K_j$ increases to $K$. Then $\partial K_j$ is pluripolar so that $(dd^cV_{K_j})^n$ puts no mass on $\partial K_j$ (cf., Proposition 4.6.4 of [3]). Writing $(dd^cV_{K_j})^n := \lambda_j(x) dx$ for $x \in K_j^o$, we have

$$(2\pi)^n = \int_{K_j^o} \lambda_j(x) dx = \int_{K^o} \lambda_j(x) dx$$

where we extend $\lambda_j(x)$ to be zero outside of $K_j^o$. Since $\int_{K^o} \lambda_j(x) dx \leq (2\pi)^n < \infty$, and the density functions $\lambda_j(x)$ increase almost everywhere on $K^o$ to $\lambda(x)$ (cf., (4.11)), by dominated convergence we have

$$(2\pi)^n = \lim_{j \to \infty} \int_{K^o} \lambda_j(x) dx = \int_{K^o} \lambda(x) dx.$$
Thus \((dd^cV_K)^n\) puts no mass on the boundary \(\partial K\).

We end this note with a final remark on Bernstein-Markov inequalities. Baran [2] conjectured that we have equality \(\delta_M(x, y) = \delta_B(x, y)\) in (4.1) for general convex bodies. With respect to this conjecture, we make the following observation: if we can prove \(\delta_M(x, y) = \delta_B(x, y)\) for a triangle \(T\) in \(\mathbb{R}^2\), then equality holds for all convex bodies in \(\mathbb{R}^2\). For let \(K\) be a convex body in \(\mathbb{R}^2\). Fix \(x \in K^o\) and \(|y| = 1\). Take a \(b\)-maximal ellipse \(E = E(x, y)\) for \(K\) with parameter \(b = b^*(x, y)\) and, as in (1) or (2) at the end of section 3, take a rectangle or triangle \(T\) containing \(K\) in which \(E\) is an \(a\)-maximal ellipse. Then since \(\delta_B^T(x, y) = \frac{1}{b^*(x, y)} = \delta_B^K(x, y)\), we have

\[
\delta_M^K(x, y) \geq \delta_M^T(x, y) = \delta_B^T(x, y) = \delta_B^K(x, y).
\]

From (4.1), \(\delta_M^K(x, y) \leq \delta_B^K(x, y)\) and equality holds.

References

[1] M. Baran, Plurisubharmonic extremal functions and complex foliations for the complement of convex sets in \(\mathbb{R}^n\), Michigan Math. J. 39 (1992), 395-404.

[2] M. Baran, Complex equilibrium measure and Bernstein type theorems for compact sets in \(\mathbb{R}^n\), Proc. AMS 123 (1995), no. 2, 485-494.

[3] E. Bedford, B. A. Taylor, The complex equilibrium measure of a symmetric convex set in \(\mathbb{R}^n\), Trans. AMS 294 (1986), 705-717.

[4] P. Borwein, T. Erdélyi, Polynomials and polynomial inequalities, Graduate Texts in Mathematics, Springer Verlag, 1995.

[5] L. Bos, N. Levenberg and S. Waldron, Pseudometrics, distances, and multivariate polynomial inequalities, submitted for publication.

[6] D. Burns, N. Levenberg, S. Ma’u, Pluripotential theory for convex bodies in \(\mathbb{R}^N\), Math. Zeitschrift 250 (2005), no. 1, 91-111.

[7] ________, Exterior Monge-Ampère Solutions, submitted for publication (and on Math arXiv, math.CV/0607643).

[8] M. Klimek, Pluripotential Theory, Clarendon Press, Oxford, 1991.

[9] M. Lundin, The extremal plurisubharmonic function for the complement of the disk in \(\mathbb{R}^2\), unpublished preprint, 1984.

[10] ________, The extremal plurisubharmonic function for the complement of convex, symmetric subsets of \(\mathbb{R}^n\), Michigan Math. J. 32 (1985), 197-201.

[11] L. Milev and Sz. Révész, Bernstein’s inequality for multivariate polynomials on the standard simplex, J. Inequal. Appl. (2005), no. 2, 145-163.
[12] Sz. Révész, A comparative analysis of Bernstein type estimates for the derivative of multivariate polynomials, *Ann. Polon. Math.* 88 (2006), no. 3, 229-245.

[13] Sz. Révész and Y. Sarantopoulos, A generalized Minkowski functional with applications in approximation theory, *J. Convex Analysis* 11 (2004), no. 2, 303-334.

[14] Y. Sarantopoulos, Bounds on the derivatives of polynomials on Banach spaces *Math. Proc. Camb. Phil. Soc.* 110 (1991), 307-312.

**Univ. of Michigan, Ann Arbor, MI 48109-1043 USA**
*E-mail address*: dburns@umich.edu

**Indiana University, Bloomington, IN 47405 USA**
*E-mail address*: nlevenbe@indiana.edu

**Indiana University, Bloomington, IN 47405 USA**
*E-mail address*: sinmau@indiana.edu

**A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, 1364 Hungary.**
*E-mail address*: revesz@renyi.hu

**AND**

**Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75005 Paris, France**
*E-mail address*: revesz@ihp.jussieu.fr