Statistics of time delay in quantum chaotic transport

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Abstract

We study the statistical properties of the time delay matrix $Q$ in the context of quantum transport through a chaotic cavity, in the absence of time-reversal invariance. First, we approach the problem from the point of view of random matrix theory, and obtain exact results that provide the average value of any polynomial function of $Q$. We then consider the problem from the point of view of the semiclassical approximation, obtaining the entire perturbation series for some energy-dependent correlation functions. Using these correlation functions, we show agreement between the random matrix and the semiclassical approaches for several statistical properties.

1 Introduction

Quantum scattering processes at energy $E$ can be described by the scattering matrix $S(E)$, which transforms incoming wavefunctions into outgoing wavefunctions. This matrix is necessarily unitary, in order to enforce conservation of probability and, consequently, conservation of charge. Another important operator is the Wigner-Smith time delay matrix $Q$ \[1, 2\], a hermitian matrix related to the energy derivative of $S$. Its eigenvalues are the delay times of the system, and its normalized trace is the Wigner time delay, $\tau_W = \frac{1}{M} \text{Tr} Q$. These quantities contain information about the time a particle spends inside a scattering region. A thorough discussion can be found in the review \[3\].

We consider a scattering region (‘cavity’) inside of which the classical dynamics is strongly chaotic, connected to the outside world by small, perfectly transparent, openings. This can be realized in experiments with microwave cavities \[4, 5, 6, 7\], quantum dots \[8, 9, 10, 11\] and compound nuclei \[12\]. In this case, there is a well defined classical decay rate $\Gamma$, such that the total probability of a particle to be found inside the cavity decays exponentially in time, $\sim e^{-\Gamma t}$. The quantity $\tau_D = 1/\Gamma$ is called the classical ‘dwell time’.

In the semiclassical regime (when $\hbar \rightarrow 0$ and the electron wavelength is much smaller than the cavity size), the $S$ and $Q$ matrices are strongly oscillating functions of the energy and a statistical approach is advantageous. One such approach is based on random matrix theory (RMT). Its main hypothesis is that $S$ behaves like a random unitary matrix, distributed in the unitary group according to some probability measure (in the presence of time-reversal invariance, $S$ must also be symmetric; we do not consider that situation in this work). If the openings are perfectly transparent, this distribution is the normalized Haar measure of the group. A typical ergodicity hypothesis is that the energy average of an observable for a fixed system is equal to an average over many different, yet similar, systems (ensemble average). We denote these two averages by the same symbol, $\langle \cdot \rangle$. In particular, the average of the Wigner time delay is equal to the classical dwell time \[13, 14\], $\langle \tau_W \rangle = \tau_D$.

The RMT approach has had much success in describing so-called transport statistics \[15, 16, 17, 18\], such as conductance, shot-noise, their variances, etc. RMT can be applied to time delay, but usually this is not done starting from the $S$ matrix, but rather
from the Hamiltonian of the system. This allows better control of the energy dependence and calculation of correlation functions, but requires mapping the problem to a nonlinear supersymmetric $\sigma$-model \[19, 20, 21, 22\].

On the other hand, Brouwer, Frahm and Beenakker \[23\] succeeded in finding the joint probability distribution for the eigenvalues of the time delay matrix $Q$, let us denote them by $\tau_1, ..., \tau_M$, where $M$ is the total number of scattering channels. This allowed the calculation of marginal distributions \[24\], distribution of Wigner time delay (for $M = 2$ \[25\] and in the limit $M \gg 1$ \[26\]), and the ensemble average of linear moments \[27, 28\],

$$\mathcal{M}_n = \frac{1}{M} \text{Tr}[Q^n] = \sum_{i=1}^{M} \tau_i^n. \quad (1)$$

A few more general, non-linear, moments have also been computed \[29, 30\]. A recent review, also considering extension to non-ideal openings and other symmetry classes, can be found in \[31\].

In the first part of this work, we advance the RMT approach to statistics of time delay, obtaining an explicit formula for arbitrary moments of $Q$, i.e. quantities of the kind

$$\mathcal{M}_{n_1, n_2, ...} = \frac{1}{M} \text{Tr}[Q^{n_1}] \frac{1}{M} \text{Tr}[Q^{n_2}] ..., \quad (2)$$

for any finite set of positive integers $n_1, n_2, ...$. This allows the calculation of the average value of any observable which is polynomial in $Q$. Our method starts from the result of \[23\] and is based on Schur function expansions and determinant evaluations. Importantly, our results are not perturbative in the number of channels, being valid at finite values of $M$.

A different way of treating the problem of quantum chaotic transport is the semiclassical approximation, in which elements of the $S$ matrix are written as sums over classical scattering trajectories \[32\]. Calculation of energy-averaged transport statistics then require so-called action correlations, sets of trajectories having the same total action, leading to constructive interference. Using only identical trajectories and ergodicity arguments \[33, 34, 35\] one can recover some semiclassical large-$M$ asymptotics. Quantum corrections, important at finite $M$, can be related to non-identical trajectories having close encounters \[36\], and may be obtained systematically \[37, 38, 39\].

The semiclassical approach has also been used to understand time delay. Interestingly, in this case one can use the periodic orbits \[40\] that live in the fractal chaotic saddle of the system \[41\] (sometimes called ‘the repeller’). This approach was used to compute correlation functions \[42, 43, 44\] up to first few orders in perturbation theory in $1/M$. It is actually equivalent \[45\] to the one based on scattering trajectories \[40\]. Berkolaiko and Kuipers treated the linear moments $\mathcal{M}_n$ semiclassically, initially in the large-$M$ limit \[47\] and later up to the first finite-$M$ corrections \[48\], showing agreement with the corresponding RMT predictions. These works actually consider the more general problem of an energy-dependent correlation function

$$C_n(\epsilon) = \frac{1}{M} \text{Tr} \left[ S^\dagger \left( E - \frac{\epsilon \hbar}{2T_D} \right) S \left( E + \frac{\epsilon \hbar}{2T_D} \right) \right]^n, \quad (3)$$

from which the moments $\mathcal{M}_n$ can be recovered by differentiation (yet another semiclassical approach to time delay, that avoids correlation functions, has recently been introduced by Kuipers, Savin and Sieber \[49\]).

In the second part of this work, we advance the semiclassical approach to the statistics of time delay, deriving from it a formula for correlation functions $C_n(\epsilon)$. This formula is a Taylor series in $\epsilon$, the coefficients of which are rational functions of $M$ expressed as finite sums involving characters of the symmetric group and Stirling numbers. Our method is an
extension of a recently introduced semiclassical matrix model for transport statistics [50]. The structure of our formula for $C_n(\epsilon)$ suggests that the agreement between semiclassics and RMT holds exactly in $M$ for all non-linear moments $\mathcal{M}_{n_1,n_2,...}$ (which we computed in the first part). However, even though this can be checked in many cases using the computer, we come short of explicitly showing it in full generality.

This paper is organized as follows. In the next Section we present and discuss our results, before entering into details of calculations. Section 3 contains an exposition of some preliminary material. Section 4 has the derivation of our random matrix theory results for the general moments (2), while Section 5 contains our semiclassical approach to the correlation function (3).

2 Results and Discussion

We start by extending the RMT approach and computing all nonlinear statistics of the time delay matrix. For example, the average value of the moments $\mathcal{M}_n$ were found in [27] for general number of channels $M$, but expressed as a sum with $M$ terms. Our results imply the following simple general formula, which contains a sum with only $n$ terms:

$$\langle \mathcal{M}_n \rangle = \tau^n D^n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{[M-k]^n}{[M+k]^n},$$

where

$$[x]^n = x(x+1)\cdots(x+n-1), \quad [x]_n = x(x-1)\cdots(x-n+1),$$

are the raising and falling factorials.

As another example, the first four cumulants of the Wigner time delay were computed in [29] using some nonlinear differential equation for their generating function. This amounts to finding the value of $\langle \tau^j_W \rangle$ for $j$ up to 4. Our results imply the explicit general formula

$$\langle \tau^j_W \rangle = \tau^n D^n \sum_{\lambda \vdash n} d^2_{\lambda} \frac{[M]_{\lambda}^j}{[M]_{\lambda}},$$

where the sum is over all partitions of $n$, the length of a partition $\lambda$ is denoted $\ell(\lambda)$ (these concepts are discussed in Section 3) and

$$[M]_{\lambda} = \prod_{i=1}^{\ell(\lambda)} [M-i+1]^{\lambda_i}, \quad [M]_{\lambda} = \prod_{i=1}^{\ell(\lambda)} [M+i-1]_{\lambda_i}$$

are generalizations of the rising and falling factorials. The quantity $d_{\lambda}$ is the dimension of the irreducible representation of the permutation group labeled by $\lambda$, and it is given by

$$d_{\lambda} = n! \prod_{i=1}^{\ell(\lambda)} \frac{1}{(\lambda_i-i+\ell(\lambda))!} \prod_{j=i+1}^{\ell(\lambda)} (\lambda_i-\lambda_j-i+j).$$

The above examples are derived from particular cases of our most general result, which is the following

**Theorem:** Let $Q$ be the $M$-dimensional time delay matrix of a chaotic cavity with no time-reversal symmetry. Let $\lambda \vdash n$ and let $s_{\lambda}(Q)$ be a Schur function of matrix argument. Then, in terms of the quantities defined above, we have

$$\langle s_{\lambda}(Q) \rangle = (M\tau_D)^n \frac{d_{\lambda} [M]_{\lambda}^\lambda}{n! [M]_{\lambda}}.$$
The functions \( s_\lambda(Q) \) are actually homogeneous symmetric polynomials in the eigenvalues of \( Q \). Since any symmetric polynomial in these variables can be expressed as a linear combination of Schur functions, this can be seen as a complete solution to the problem of computing the average value of polynomial (or analytic functions, if we allow infinite series) of \( Q \), such as the quantities \( M_{n_1,n_2,...} \) defined in (3). For instance, the first of these which are neither of the form (4) nor of the form (6) are

\[
\langle M_{2,1} \rangle = \frac{2M^2(M^2 + 2)}{(M^2 - 1)(M^2 - 4)},
\]

and

\[
\langle M_{2,2} \rangle = \frac{4M^2(M^4 + 8M^2 - 3)}{(M^2 - 1)(M^2 - 4)(M^2 - 9)}, \quad \langle M_{3,1} \rangle = \frac{6M^2(M^2 + 1)^2}{(M^2 - 1)(M^2 - 4)(M^2 - 9)}.
\]

In the second part of this work, we develop a new formulation for the semiclassical approach to time delay. Following our previous work on transport statistics [50], this is based on a matrix integral which is designed to have the correct diagrammatic expansion. In this way, we find for example that

\[
C_1 = \frac{1}{1 - i\epsilon} - \frac{\epsilon^2}{M^2(1 - i\epsilon)^6} - \frac{\epsilon^2(1 + 12i\epsilon - 8\epsilon^2)}{M^4(1 - i\epsilon)^6} + O(1/M^6),
\]

and

\[
C_2 = \frac{(1 - 2i\epsilon - 2\epsilon^2)}{(1 - i\epsilon)^4} - \frac{\epsilon^2(4 + 8i\epsilon - 7\epsilon^2 - 2i\epsilon)}{M^2(1 - i\epsilon)^8} + O(1/M^4).
\]

The leading order part of these functions appear in the Appendix of [47].

Solving exactly our matrix integral, we arrive at our most general result, which is a formula for the correlation functions in the form of a Taylor series:

\[
C_n(\epsilon) = \frac{1}{Mn!} \sum_{m=0}^{\infty} \frac{M^{im}}{m!} \sum_{\lambda,\mu} \sum_{\lambda,\mu} d_\lambda d_\mu \chi_\lambda(n) \frac{[M]_\lambda}{[M]_\mu} F_{\lambda,\mu},
\]

where \( \chi \) are the characters of the permutation group (see Section 3) and \( F_{\lambda,\mu} \) is some nontrivial function for which we have an explicit form (see Section 5.4.1).

Following [47], the average value of moments \( M_m \) can be obtained as

\[
\langle M_m \rangle = \frac{\tau_m^D}{i^m m!} \left[ \frac{d^m}{d\epsilon^m} \sum_{n=1}^{m} (-1)^{m-n} \binom{m}{n} C_n(\epsilon) \right]_{\epsilon=0}.
\]

These quantities have been computed semiclassically up to the first few orders in perturbation theory in 1/M in [48]. Using the above expression for \( C_n(\epsilon) \) we could compute them in closed form as rational functions of \( M \) up to \( m = 8 \) and check that the results agree with the RMT prediction (4). Unfortunately, we could not establish this agreement in general, because of the complicated nature of the function \( F_{\lambda,\mu} \).

As any reader who compares Sections 4 and 5 will notice, the semiclassical calculation is much more complicated than the RMT one, so much so that it may seem hardly worth it. We can raise two points in its defence. First, it provides the energy-dependent correlation functions, which have more information than the energy-independent RMT statistics [9]. For instance, correlation functions are required in order to develop a semiclassical treatment of Andreev systems, in line with [51 52]. Second, the semiclassical approximation is in principle able to go beyond RMT by including Ehrenfest time effects (see e.g. [53 54 55].
These possible developments are outside the scope of the present work, but we hope they will attract attention in the future.

A last remark about our semiclassical calculation. It is based on an integral over $N$-dimensional complex matrices, and requires that we take the limit $N \to 0$. This limit is needed to enforce that our semiclassical expansions do not contain periodic orbits. It is easily taken in the perturbative framework (see Section 5.2), i.e. order by order in $1/M$. However, we cannot rigorously justify it for the exact calculation. This is why we do not claim our semiclassical results as theorems. We believe the nature of this limit is an interesting open problem that deserves further study.

3 Preliminaries

3.1 Partitions and permutations

A weakly decreasing sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called a partition of $n$, denoted by $\lambda \vdash n$ or by $|\lambda| = n$, if $\sum \lambda_i = n$. Each of the integers is a part, and the total number of parts is the length $\ell(\lambda)$.

Partitions of $n$ label the conjugacy classes of the permutation group $S_n$: the cycle type of a permutation $\pi$ is a partition whose parts are the lengths of the cycles of $\pi$, and two permutations $\pi, \sigma$ have the same cycle type if and only if they are conjugated, i.e. if there exists $\tau$ such that $\pi = \tau \sigma \tau^{-1}$. Let $C_\lambda$ denote the set of permutations with cycle type $\lambda$, and $|C_\lambda|$ the number of elements in $C_\lambda$.

The number of permutations in $S_n$ which have exactly $k$ cycles is the (unsigned) Stirling number of the first kind, $[n]_k$. These numbers also appear when we expand the rising factorial,

$$[x]^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad (16)$$

For any finite group, there are as many irreducible representations as there are conjugacy classes. Therefore, partitions of $n$ also label the irreducible representations of $S_n$. The trace of permutation $\pi$, in the representation labeled by $\lambda$, is denoted as $\chi_\lambda(\pi)$ and called its character. The character of the identity, $\chi_\lambda(1) = d_\lambda$, is the dimension of the representation, for which there is the explicit formula (8). Characters satisfy orthogonality relations,

$$\sum_{\tau \in S_n} \chi_\mu(\tau) \chi_\lambda(\tau \sigma) = \frac{n!}{d_\lambda} \chi_\lambda(\sigma) \delta_{\mu,\lambda}. \quad (17)$$

3.2 Symmetric functions

Let $X$ be a matrix of dimension $N$, with eigenvalues $x_i$, $1 \leq i \leq N$. Power sum symmetric functions of matrix argument are defined as

$$p_\lambda(X) = \prod_{i=1}^\ell(\lambda) p_{\lambda_i}(X), \quad p_n(X) = \text{Tr}[X^n] = \sum_{i=1}^N x_i^n. \quad (18)$$

They are clearly symmetric functions of the eigenvalues.

Another important family of symmetric functions are Schur functions, related to power sums by

$$s_\lambda(X) = \frac{1}{n!} \sum_{\mu \vdash n} |C_\mu| \chi_\lambda(\mu) p_{\mu}(X), \quad p_\lambda(X) = \sum_{\mu \vdash n} \chi_\lambda(\mu) s_\mu(X). \quad (19)$$
These functions can also be written as a ratio of determinants,

\[ s_\lambda(X) = \frac{\det(x_i^{\lambda_j - j + N})}{\Delta(X)}, \]

where

\[ \Delta(X) = \det(x_i^{j-1}) = \prod_{i=1}^{N} \prod_{j=i+1}^{N} (x_j - x_i), \]

is the Vandermonde determinant. The value of the Schur function when all arguments are equal to 1 is

\[ s_\lambda(1^N) = \frac{d_\lambda}{n!} [N]^\lambda, \]

where \([N]^\lambda\) is the generalization of the rising factorial defined in (7). Noticing that in the formula for \(d_\lambda\) there appears the Vandermonde for \(x_i = \lambda_i - i\), it is also possible to show that

\[ \Delta(\{\lambda_i - i\}) = s_\lambda(1^N) \prod_{j=1}^{N-1} j!. \]

Let \(d\vec{x} = dx_1 \cdots dx_N\). In view of the identity

\[ \int d\vec{x} \det(f_i(x_j)) \det(g_i(x_j)) = N! \det \left( \int dx f_i(x) g_j(x) \right), \]

easily proved using the Leibniz formula for the determinant, the representation (20) of Schur functions shall be useful for performing multidimensional integrals involving these functions.

### 3.3 Weingarten functions

Given \(j = (j_1, j_2, \ldots, j_n)\), \(m = (m_1, m_2, \ldots, m_n)\) and \(\tau \in S_n\), define the function

\[ \delta_\tau[j, m] = \prod_{k=1}^{n} \delta_{j_k m_{\tau(k)}}. \]

Let \(U(N)\) be the group of \(N \times N\) unitary complex matrices \(U\) and let \(dU\) denote its normalized Haar measure. Then the so-called Weingarten function of this group is defined by

\[ \int dU \prod_{k=1}^{n} U_{a_k b_k} U_{c_k d_k}^\dagger = \sum_{\sigma, \tau \in S_n} Wg_N(\tau \sigma^{-1}) \prod_{k=1}^{n} \delta_{\rho[a]} \delta_\tau[bc], \]

where \(U^\dagger\) denotes the transpose conjugate of \(U\). The character expansion of this function is known

\[ Wg_N(g) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{d_\lambda}{[N]^\lambda} \chi_\lambda(g), \]

and the orthogonality of characters implies the following identity:

\[ \sum_{\tau \in S_n} \chi_\lambda(\tau \theta) Wg_N^U(\tau \sigma^{-1}) = \frac{\chi_\lambda(\theta \sigma^{-1})}{[N]^\lambda}. \]
4 Random Matrix Theory approach

We wish to compute the average value of a Schur function of the time delay matrix, \( s_{\lambda} (Q) \). This will be done using the following result obtained in [23]: if \( \gamma = Q^{-1} \), then the probability distribution of this matrix is

\[
P(\gamma) = \frac{1}{Z} |\Delta(\gamma)|^2 \det(\gamma)^M e^{-M \tau D \text{Tr} \gamma}.
\] (29)

where

\[
Z = \int_0^\infty |\Delta(\gamma)|^2 \det(\gamma)^M e^{-M \tau D \text{Tr} \gamma} d\gamma
\] (30)

is a normalization constant.

Let \( \tau_i, 1 \leq i \leq M \) be the eigenvalues of \( Q \) and \( \gamma_i = 1/\tau_i \) be the eigenvalues of \( \gamma \). The normalization constant is computed using Eq. [24]:

\[
Z = \int_0^\infty d\tilde{\gamma} \det(\gamma_j^{M+i-1} e^{-M \tau D \gamma_j}) \det(\gamma_i^{j-1}) = \frac{M!}{(M \tau_D)^{2M^2}} \det((M+j+i-2)!).
\] (31)

Standard determinant manipulations yield

\[
Z = \frac{1}{(M \tau_D)^{2M^2}} \prod_{j=1}^{M} j!(M+j-1)!. (32)
\]

The quantity we are after is

\[
\langle s_{\lambda} (Q) \rangle = \frac{1}{Z} \int_0^\infty d\tilde{\gamma} |\Delta(\gamma)|^2 \det(\gamma)^M e^{-M \tau D \text{Tr} \gamma} \langle s_{\lambda}(\gamma^{-1}) \rangle.
\] (33)

Writing the Schur function as a determinant, as in Eq. [20], and using the following identity for the Vandermonde,

\[
\Delta(\gamma^{-1}) = \frac{(-1)^{M(M-1)/2} \Delta(\gamma)}{\det \gamma^{M-1}},
\] (34)

we arrive at

\[
\langle s_{\lambda} (Q) \rangle = \frac{(-1)^{M(M-1)/2}}{Z} \int_0^\infty d\tilde{\gamma} \det(\gamma_j^{2M+i-2} e^{-M \tau D \gamma_j}) \det(\gamma_i^{-\lambda_j+j-M}).
\] (35)

Using Eq. [24] again we have

\[
\langle s_{\lambda} (Q) \rangle = \frac{(-1)^{M(M-1)/2}}{Z(M \tau_D)^{2M^2-n} M! \det((M-\lambda_j+j+i-2)!)}.
\] (36)

Consider the determinant \( \det((x_j + i)!). \) Suppose we factor out a term \( (x_j + i)! \) from each row. The remaining determinant has the following structure: its \( ij \) element is a monic polynomial in \( x_j \) of degree \( i - 1 \). It is well known that it therefore must be equal to the Vandermonde \( \Delta(x) \). Applying this argument to (35) we get

\[
\langle s_{\lambda} (Q) \rangle = \frac{1}{Z(M \tau_D)^{2M^2-n}} s_{\lambda}(1^M) \prod_{j=1}^{M} j!(M-\lambda_j+j-1)!
\] (37)

where we used \( \Delta(\{M-\lambda_i+i-2\}) = (-1)^{M(M-1)/2} \Delta(\{\lambda_i\}) \) and the special value of the Vandermonde, Eq. [23]. Plugging in the values of \( s_{\lambda}(1^M) \) and \( Z \), we get

\[
\langle s_{\lambda} (Q) \rangle = (M \tau_D)^n \frac{d\lambda}{n!} [M]^{\lambda} \prod_{j=1}^{M} \frac{(M-\lambda_j+j-1)!}{(M+j-1)!},
\] (38)
or, in terms of the generalized falling factorial defined in (7), our claimed result,

\[
\langle s_\lambda(Q) \rangle = (M\tau_D)^n \frac{d_\lambda [M]^\lambda}{n! [M]^\lambda}.
\] (39)

The relation between power sums and Schur functions, Eq. (19), allows the calculation of more familiar quantities, such as

\[
\langle M_n \rangle = \frac{1}{M} \langle p_n(Q) \rangle = \frac{1}{M} \sum_{\lambda \vdash n} \chi_\lambda(n) \langle s_\lambda(Q) \rangle.
\] (40)

The character \( \chi_\lambda(n) \) is different from zero only if \( \lambda = (n-k, 1^k) \) (so-called hook partitions), and is equal to \((-1)^k\) in this case. On the other hand, the dimension \( d_\lambda \) becomes \( \binom{n-1}{k} \) for hooks, and with this we arrive at our example (4). The other example we mentioned in Section 2 was

\[
\langle \tau^n_W \rangle = \frac{1}{M^n} \langle p_{(1,1,\ldots,1)}(Q) \rangle = \frac{1}{M^n} \sum_{\lambda \vdash n} d_\lambda \langle s_\lambda(Q) \rangle.
\] (41)

Finally, consider the general moments \( M_{n_1,n_2,\ldots} \). Without any loss of generality, we may assume that \( \mu = (n_1,n_2,\ldots) \) is a partition of some integer, \(|\mu|\). Then, we have

\[
\langle M_{n_1,n_2,\ldots} \rangle = \frac{1}{M^{|\mu|}} \sum_{\lambda \vdash |\mu|} \chi_\lambda(\mu) \langle s_\lambda(Q) \rangle.
\] (42)

Using this expression, we recover our examples (10) and (11).

5 Semiclassical approach

In the semiclassical limit \( \hbar \to 0, M \to \infty \), the element \( S_{oi} \) of the \( S \) matrix may be approximated by a sum over trajectories \( \gamma \) starting at channel \( i \) and ending at channel \( o \)

\[
S_{oi} = \frac{1}{\sqrt{T_H}} \sum_{\gamma_{i\to o}} A_\gamma e^{iS_\gamma/\hbar}.
\] (43)

The phase \( S_\gamma \) is the action of \( \gamma \), while \( A_\gamma \) is related to its stability. The prefactor contains the so-called Heisenberg time, \( T_H = M\tau_D \).

Consider the correlation function

\[
C_n(\epsilon) = \frac{1}{M} \text{Tr} \left[ S^{n} \left( E - \frac{\epsilon \hbar}{2\tau_D} \right)^n S \left( E + \frac{\epsilon \hbar}{2\tau_D} \right)^n \right].
\]

Expanding the trace, we find a multiple sum over trajectories,

\[
C_n = \frac{1}{MT^n_H} \prod_{k=1}^n \sum_{i_k,o_k,\gamma_k,\sigma_k} A_{\gamma_k} A_{\sigma_k}^* e^{i(S_{\gamma_k} - S_{\sigma_k})/\hbar} e^{\frac{i\hbar}{2\tau_D}(T_{\gamma_k} + T_{\sigma_k})},
\] (44)

such that \( \gamma_k \) goes from \( i_k \) to \( o_k \), while \( \sigma_k \) goes from \( i_k \) to \( o_{k+1} \), i.e. \( \sigma \) trajectories implement a cyclic permutation on the labels of the channels. The channels labels are all being summed from 1 to \( M \).

In (44) we have used

\[
S_\gamma(E + \frac{\epsilon \hbar}{2\tau_D}) = S_\gamma(E) + \frac{\epsilon \hbar}{2\tau_D} T_\gamma,
\] (45)

where \( T_\gamma \) is the total duration of \( \gamma \). The quantity \( A_\gamma = \Pi_k A_{\gamma_k} \) is a collective stability, while \( S_\gamma = \sum_k S_{\gamma_k} \) and \( T_\gamma = \sum_k T_{\gamma_k} \) are the collective action and duration of the \( \gamma \) trajectories, and analogously for \( \sigma \).

The result of the sum (44) is, for a chaotic system, a strongly fluctuating function of the energy. A local energy average is thus introduced which, under the stationary phase
approximation, requires $\gamma$ and $\sigma$ to have almost the same collective action. In the past years [36], it has been established that these action correlations arise when each $\sigma$ follows closely a certain $\gamma$ for a period of time, and some of them exchange partners at so-called encounters. A $q$-encounter is a region where $q$ pieces of trajectories run nearly parallel and $q$ partners are exchanged. This theory has been presented in detail in [37, 60]. We consider only systems not invariant under time-reversal, so $\sigma$ trajectories never run in the opposite sense with respect to $\gamma$ trajectories.

For example, we show in Figure 1a a situation contributing to the second correlation function, $C_2(\epsilon)$. Trajectory $\gamma_1$ starts in channel $i_1$ and ends in channel $o_1$, while $\gamma_2$ starts in channel $i_2$ and ends in channel $o_2$. On the other hand, $\sigma_1$ and $\sigma_2$ are initially almost identical to $\gamma_1$ and $\gamma_2$, respectively, but they exchange partners in a 2-encounter. Later, $\gamma_2$ has a 3-encounter with itself, inside of which the pieces of $\sigma_1$ are connected differently. We also show in Figure 1b a situation contributing to $C_3(\epsilon)$ which has no encounters, but has coinciding channels. There are two major simplifications done here for visual clarity: 1) The encounters are greatly magnified, to show their internal structure; 2) The actual trajectories are extremely convoluted and chaotic. Many other examples of correlated trajectories can be found in previous work such as [36, 37, 38, 39, 47, 48, 50].

Correlated sets of trajectories contributing to the semiclassical calculation of correlation functions can be depicted in the form of ribbon graphs, as suggested in [61, 62]. The $q$-encounters become vertices of valence $2q$. Channels also become vertices, but their valence depends on whether there are coinciding channels or not. The pieces of trajectories connecting vertices become fat edges, or ribbons. Each ribbon is bordered by one $\gamma$ and one $\sigma$, and these trajectories traverse the encounter vertices in a well defined rotation sense: a trajectory arriving from one ribbon departs via the adjacent ribbon (graphs endowed with a cyclic order around vertices are also called maps). We show in Figure 2 the ribbon graphs corresponding to the trajectories shown in Figure 1.

Following previous work on transport and on closed systems, Kuipers and Sieber obtained some diagrammatic rules [45], that determine how much a given graph contributes to the correlation function. The contribution of a graph factorizes into the contributions of individual vertices and edges: an encounter vertex of valence $2q$ gives rise to $-M(1-iq\epsilon)$; channels of any valence give rise to $M$; each ribbon gives rise to $[M(1-i\epsilon)]^{-1}$. These rules were then used in several works dealing with time delay statistics [47, 48, 51, 52].

Notice that there are no periodic orbits in a ribbon graph that arises from the semiclassical expansion of time delay. This means that we may start from $i_1$ and follow $\sigma_1$ up to $o_1$, then follow $\gamma_1$ in reverse back to $i_1$, then $\sigma_2$ to $o_2$, then $\gamma_2$ in reverse back to $i_2$, and so on, and traverse every border of every ribbon exactly once. This means that the graph has

---

**Figure 1**: a) Correlated trajectories contributing to $C_2$. Solid lines are $\gamma_1$ (going from $i_1$ to $o_1$) and $\gamma_2$ (going from $i_2$ to $o_2$), dashed lines are $\sigma_1$ (going from $i_1$ to $o_2$) and $\sigma_2$ (going from $i_2$ to $o_1$). In this situation we have one 2-encounter and one 3-encounter (the encounters are greatly magnified). b) Correlated trajectories contributing to $C_3$, in a case with coinciding channels. In both figures the chaotic nature of the trajectories is not shown.
The contribution of a graph will be proportional to $M^{V-E-1}$, where $V$ is the total number of vertices (including channels) and $E$ is the total number of edges. The Euler characteristic of a ribbon graph is $V - E + F$, where $F$ is the number of faces ($F = 1$ in our case). The Euler characteristic is also equal to $2 - 2g$, where $g$ is called the genus. Therefore, the $1/M^2g$ expansion coming from semiclassical diagrammatics is actually what is called a genus expansion: the contribution of a graph is proportional to $1/M^2g$. Graphs with $g = 0$ are called planar (they can be drawn on the plane so that the ribbons never cross each other), and they give the leading order contribution.

The graph in Figure 2a, for example, contributes

$$\frac{(1 - 2i\epsilon)(1 - 3i\epsilon)}{M^2(1 - i\epsilon)^3}$$

(46)

to $C_2$. Notice that it is not a planar graph, since there is a crossing between two of the ribbons. This particular graph actually has $g = 1$ (this means it may be drawn on a torus without any crossings). The graph in Figure 2b, on the other hand, is planar and contributes $(1 - i\epsilon)^{-3}$ to $C_3$.

**5.1 Gaussian integrals and Wick diagrammatics**

We shall introduce a certain Gaussian matrix integral and formulate it diagrammatically, using Wick’s rule. This procedure has been discussed in detail for hermitian matrices in [63] and in [64]. The only difference compared to the present work is that we integrate over non-hermitian matrices. Our diagrams are then interpreted as providing the semiclassical formulation of the time delay problem. The same approach was used to treat transport statistics in [50].

Let $Z$ denote a general complex matrix of dimension $N$, and define

$$\langle\langle f(Z, Z^\dagger)\rangle\rangle = \frac{1}{Z} \int dZe^{-i\Omega[V(Z, Z^\dagger)]} f(Z, Z^\dagger),$$

(47)

where the normalization constant (not to be confused with the normalization constant of Section 4) is

$$Z = \int dZe^{-i\Omega[V(Z, Z^\dagger)]}.$$

(48)
We see (47) as an average value, but we use the symbol $\langle \cdot \rangle$ to differentiate it from the true physical average we considered in previous sections. For example, since the elements are actually independent, it is clear that
\[
\langle Z_{m_3} Z_{q_1}^\dagger \rangle = \frac{\delta_{m_3} \delta_{j_1}}{\Omega}.
\] (49)

Integrals over a product of matrix elements can be computed using the so-called Wick’s rule, which states that we must sum, over all possible pairings between $Z$’s and $Z^\dagger$’s, the product of the average values of the pairs. Namely,
\[
\left\langle \prod_{k=1}^n Z_{m_k j_k} Z_{q_k}^\dagger \right\rangle = \sum_{\sigma \in S_n} \prod_{k=1}^n \langle Z_{m_k j_k} Z_{q_{\sigma(k)}}^\dagger \rangle. \tag{50}
\]
If we the quantity we wish to average involves traces of $ZZ^\dagger$, all we need to do is expand these traces in terms of matrix elements and apply Wick’s rule. Most importantly, we can then employ a diagrammatic technique.

For example, suppose we wish to compute
\[
\left\langle \text{Tr}[(ZZ^\dagger)^2] \text{Tr}[(ZZ^\dagger)^3] Z_{i_1 o_1} Z_{o_2 i_1} Z_{i_2 o_2} Z_{o_1 i_2} \right\rangle. \tag{51}
\]
We start by writing it as
\[
\sum_{m_1, \ldots, m_5} \sum_{j_1, \ldots, j_5} \left\langle \prod_{k=1}^2 Z_{m_k j_k} Z_{j_k}^\dagger \prod_{s=3}^5 Z_{m_s j_s} Z_{j_s}^\dagger \right\rangle Z_{i_1 o_1} Z_{o_2 i_1} Z_{i_2 o_2} Z_{o_1 i_2}, \tag{52}
\]
where all sums run from 1 to $N$ (in the first product we mean $m_3 \equiv m_1$, while in the second product we mean $m_6 \equiv m_3$). The diagrammatics consists in picturing the matrix elements as pairs of arrows. Arrows that represent elements from $Z$ have a marked end at the head, while arrows that represent elements from $Z^\dagger$ have a marked end at the tail. Arrows representing matrix elements coming from traces are arranged in clockwise order around vertices, so that all marked ends are on the outside. Finally, the elements that do not come from traces are arranged surrounding the other ones, also in clockwise order. Since this is most easily explained by means of an image, we show it in Figure 3(a).

Once we have arranged the arrows, Wick’s rule consists in making all possible connections between them, using the marked ends. Clearly, this produces a ribbon graph. According to Eq. (49), when computing the value of a graph, each ribbon gives rise to a factor $\Omega^{-1}$. For the example in Figure 3(a), there are 7! possible connections. We show two of them in Figures 3(b,c). The coupling in Figure 3(b) leads to the identifications
\[
i_1 = m_1, \quad i_2 = m_2 = m_3 = m_4 = m_5 \quad o_1 = j_2 = j_3 = j_4 = j_5, \quad o_2 = j_1,
\] (53)
and gives a contribution of $\Omega^{-7}$ to the average (51). Notice how this coupling is similar to Figure 2. On the other hand, the coupling in Figure 3(c) leads to the identifications
\[
i_1 = m_1, \quad i_2 = m_2 = m_4 = m_5 \quad o_1 = j_2 = j_3 = j_5, \quad o_2 = j_1.
\] (54)
In this case the indices $m_3$ and $j_4$ remain free to be summed over. Therefore, this coupling gives a contribution of $N^2 \Omega^{-7}$ to the average (51).

Free indices arise from closed loops in the ribbon graph. Each such loop increases by one the number of faces of the graph (every graph has at least one face). Therefore, the power of $N$ in the contribution of a given coupling is always one less than the number of faces in the graph.

It should be clear that this theory is very close to the semiclassical approach to time delay, provided we choose $\Omega = M(1 - i\epsilon)$. However, the ribbon graphs in the semiclassical theory always have a single face. As we have just mentioned, this corresponds to keeping only those Wick couplings whose contribution does not depend on $N$. Since all contributions are proportional to a positive power of $N$, we could simply let $N \to 0$. 

5.2 Matrix integrals for correlation functions

Let $\xi = (12\cdots n)$ be the cyclic permutation of the first $n$ positive integers, and let $\vec{i} = (i_1,\ldots,i_n)$ and $\vec{o} = (o_1,\ldots,o_n)$. Introduce the integral

$$G_n(M,\epsilon,N,\vec{i},\vec{o}) = \frac{1}{MZ} \int dZe^{-M\sum_{q\geq1} \frac{(1-\epsilon q)}{q}\text{Tr}(ZZ^\dagger)^q} \prod_{k=1}^n Z_{i_k o_k} Z^\dagger_{\xi(k)k}. \quad (55)$$

This can be seen as a Gaussian average as the ones considered previously, if we understand the first term in the exponent, $e^{-M(1-i\epsilon)\text{Tr}(ZZ^\dagger)}$, to be part of the measure. Accordingly, we set

$$Z = \int dZe^{-M(1-i\epsilon)\text{Tr}(ZZ^\dagger)}.$$ \quad (56)

The rest of the exponential can be Taylor expanded as

$$e^{-M\sum_{q\geq2} \frac{(1-\epsilon q)}{q}\text{Tr}(ZZ^\dagger)^q} = \sum_{t=0}^\infty \frac{(-M)^t}{t!} \left( \sum_{q\geq2} \frac{(1-i\epsilon q)}{q}\text{Tr}(ZZ^\dagger)^q \right)^t. \quad (57)$$

For now, we consider this as a formal power series and integrate term by term, employing Wick’s rule and its diagrammatical representation previously discussed. By construction, encounter vertices of valence $2q$ will be accompanied by the factor $-M(1-i\epsilon q)$, giving the correct semiclassical diagrammatic rules.

The integral (55) is therefore designed to automatically produce all the required ribbon graphs for the semiclassical evaluation of the correlation function $C_n$. The exponential produces all possible encounters, while the matrix elements in the last product play the role
of the channels. In line with Eq. (44), we must sum over all channels from 1 to \( M \), i.e. we must consider the quantity

\[
G_n(M, N, \epsilon) = \sum_{i, \tilde{\epsilon}} G_n(M, N, \epsilon, \tilde{\epsilon}, \tilde{\sigma}) = \sum_{i_1, \ldots, i_n=1}^M \sum_{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n=1}^M G_n(M, N, \epsilon, \tilde{\epsilon}, \tilde{\sigma}).
\]  

(58)

The matrix integral produces more graphs than needed, but we have provided for this overcounting. For example, the Taylor series of the exponential naturally has a \( t! \) in the denominator, which is responsible for eliminating the symmetry associated with shuffling the vertices, when there are \( t \) of them. Also, graphs are produced that differ from each other only by the rotation of a vertex. This is why we have divided \( \text{Tr}[(ZZ^\dagger)^q] \) by \( q \); it remedies the overcounting that would be caused by the possible \( q \) rotations of the vertex.

As we have discussed, in order to select only those ribbon graphs with a single face it is necessary to take the limit \( N \to 0 \) at the end of the calculation. Therefore, the correlation function will be given by

\[
C_n(M, \epsilon) = \lim_{N \to 0} G_n(M, N, \epsilon).
\]  

(59)

It is not very difficult to implement Eq. (55) in a computer and obtain the first few orders in \( 1/M \) for the first few correlation functions (the integral is not to be performed numerically, of course, but using Wick’s rule together with the covariance (49)). This leads to the results in (12)-(13). Notice that letting \( N \to 0 \) in this context presents no difficulty.

The remainder of this paper is dedicated to the exact solution of the matrix integral (55), and the calculation of its limit as \( N \to 0 \).

5.3 Exact Solution

5.3.1 Angular integration

Introduce the singular value decomposition \( Z = UDV \), where \( D \) is real, positive and diagonal while \( U \) and \( V \) are unitary. Let \( X = D^2 \) be a matrix with the same eigenvalues as \( ZZ^\dagger \), and denote these eigenvalues by \( x_i, 1 \leq i \leq N \). It is known \([65]\) that the measure \( dZ \) is expressed in these new variables as

\[
dZ = c_N |\Delta(X)|^2 d\hat{x} dU dV,
\]  

(60)

where \( c_N \) depends only on the dimension, \( dU \) is the normalized Haar measure on the unitary group \( \mathcal{U}(N) \), and the Vandermonde squared is the Jacobian of the transformation. This is a generalization of the transformation from cartesian to polar coordinates in the complex plane. We shall first perform the angular integration over \( U \) and \( V \).

A minor point to be mentioned is that \( dV \) is not the same as the normalized Haar measure. This is related to the fact that in the singular value decomposition there is a certain ambiguity, as we may freely conjugate \( D \) by a diagonal unitary matrix. The matrix \( V \) is thus uniquely determined only as an element of the coset \( \mathcal{U}(N)/[\mathcal{U}(1)]^N \). However, the functions we shall integrate, polynomials in matrix elements as those in Section 3.3, are all invariant under multiplication by a diagonal unitary matrix, and in this context \( dV \) behaves just like the Haar measure, up to normalization.

The only part of the integral in (55) that depends on the angular variables \( U \) and \( V \) is the last product. Thus, the angular integral is

\[
\mathcal{A} = \int dU dV \prod_{k=1}^n U_{ik} j_k D_{jk} V_{jk o_k} V_{o_k}^{\dagger}(\epsilon) m_k D_{m_k} U_{m_k j_k}^{\dagger},
\]  

(61)

which can be expressed terms of Weingarten functions as

\[
\mathcal{A} = \sum_{\sigma \tau \rho \theta \in S_n} W_{\sigma N}^V (\rho \theta^{-1}) W_{\tau \sigma^{-1}}^V p_{\tau^{-1} \sigma} (X) \delta_{\rho} [i, i] \delta_{\rho} [\sigma, \xi (\sigma)],
\]  

(62)
where we have used that
\[
\prod_{k=1}^{n} \sum_{j,k,m_k} D_{jk} D_{mk} \delta_r[j,m] \delta_\theta[j,m] = \prod_{k=1}^{n} \sum_{j_k} \delta_{r^{-1}\theta}[j,j] = p_{r^{-1}\theta}(X). \tag{63}
\]

The quantity we are after, Eq. (58), requires summation over the indices \( i \) and \( o \). It is easy to see that
\[
\sum_{i_1,\ldots,i_M=1}^{M} \delta_\sigma[i,i] = p_\sigma(1^M), \quad \sum_{o_1,\ldots,o_M=1}^{M} \delta_\rho[o,\xi(o)] = p_\rho(1^M). \tag{64}
\]

Notice that the channel labels in the original matrix integral (55) are all constrained to be between 1 and \( N \). Nevertheless, we are summing them from 1 to \( M \). We are thus assuming \( N \geq M \). However, this will not deter us from letting \( N \to 0 \) later.

Once we expand
\[
p_{r^{-1}\theta}(X) = \sum_{\lambda \vdash n} \chi_\lambda(r^{-1}\theta)s_\lambda(X), \tag{65}
\]
we get
\[
\sum_{i,\delta} A = \sum_{\lambda \vdash n} \sum_{\sigma,\rho \in S_n} W_{g_N}(\rho^{-1}) W_{g_N}(\tau^{-1}) \chi_\lambda(\tau^{-1}\theta)s_\lambda(X) p_\sigma(1^M)p_\rho(1^M). \tag{66}
\]

Using identity Eq.(28) twice leads to
\[
\sum_{\rho \in S_n} \sum_{\lambda \vdash n} \frac{1}{([N]^\lambda)^2} \chi_\lambda(\rho^{-1}) p_\sigma(1^M)p_\rho(1^M). \tag{67}
\]
Using another identity,
\[
\sum_{\sigma \in S_n} \chi_\lambda(\rho^{-1}) p_\sigma(1^M) = \chi_\lambda(\rho)[M]^\lambda, \tag{68}
\]
twice finally leads to
\[
\sum_{i,\delta} A = \sum_{\lambda \vdash n} \chi_\lambda(\delta) \left( \frac{[M]^\lambda}{[N]^\lambda} \right)^2 s_\lambda(X). \tag{69}
\]

### 5.3.2 Eigenvalue integration

So far, the quantity we are after is given by
\[
\mathcal{G}_n = \sum_{\bar{i},\bar{o}} G_n = \sum_{\lambda \vdash n} \chi_\lambda(\delta) \left( \frac{[M]^\lambda}{[N]^\lambda} \right)^2 \mathcal{R}, \tag{70}
\]
where \( \mathcal{R} \) is the radial integral over the eigenvalues of \( ZZ^\dagger \). It is equal to
\[
\mathcal{R} = \frac{c_N}{MZ} \int_0^1 d\bar{x} \det (1 - X)^M e^{M \text{Tr}[\frac{X}{1-X}] \Delta(x)]^2 s_\lambda(X), \tag{71}
\]
where we have used that
\[
e^{-M \sum_{q=1}^{(1-q\delta)(1)\text{Tr}X^q} = \det [(1 - X)^M] e^{M \text{Tr}(\frac{X}{1-X})}. \tag{72}
\]
From the well known Schur function expansion,
\[
e^{M \text{Tr}(\frac{X}{1-X})} = \sum_{m=0}^{\infty} \frac{(M i)^n}{m!} \sum_{\mu = m} d_\mu s_\mu \left( \frac{X}{1-X} \right), \tag{73}
\]
we get
\[ R = \sum_{m=0}^{\infty} \frac{(Mi \epsilon)^m}{m!} \sum_{\mu-m} d_{\mu} \mathcal{I}_{\lambda, \mu}, \]  \tag{74}

where
\[ \mathcal{I}_{\lambda, \mu} = \frac{c_N}{Z} \int_0^1 dx \det (1 - X)^M |\Delta(x)|^2 s_\mu \left( \frac{X}{1 - X} \right) s_\lambda(X). \]  \tag{75}

Using the determinantal form of the Schur functions, the identity
\[ \Delta \left( \frac{X}{1 - X} \right) = \frac{\Delta(X)}{\det(1 - X)^{N-1}} \]  \tag{76}
and the integral identity Eq. (24), one can show that
\[ \mathcal{I}_{\lambda, \mu} = \frac{c_N N!}{Z} \det \left( \frac{(M - \mu_j + j - 1)! (\lambda_i - i + \mu_j - j + 2N)!}{(M + 2N + \lambda_i - i)!} \right). \]  \tag{77}

Two factorials can be taken out of the determinant, and we can write
\[ \mathcal{I}_{\lambda, \mu} = \frac{c_N N!}{Z} \prod_{j=1}^{N} \frac{(M - \mu_j + j - 1)!}{(M + 2N + \lambda_j - j)!} \det ((\lambda_i - i + \mu_j - j + 2N)!). \]  \tag{78}

Introducing \((M + j - 1)!\) in the product, we get
\[ \mathcal{I}_{\lambda, \mu} = \frac{c_N N!}{Z} \prod_{j=1}^{N} \frac{(M + N - j)!}{(M + 2N + \lambda_j - j)!} \det ((\lambda_i - i + \mu_j - j + 2N)!). \]  \tag{79}

\subsection{5.4 The \(N \to 0\) limit}
We must now take the \(N \to 0\) limit. This is a delicate procedure. We can only do it for quantities that are analytic functions of \(N\). For example, using the singular value decomposition, the normalization constant (56) becomes
\[ Z = c_N \int_0^{\infty} dx e^{-M(1-i\epsilon)X} |\Delta(x)|^2 = \frac{c_N}{[M(1-i\epsilon)]^{N^2}} \prod_{j=1}^{N} j!(N-j)! \]  \tag{80}

It is perfectly fine to take the limit in the denominator. In the rest of the expression, we must leave \(N\) intact for now. In this sense, we write
\[ Z \to c_N \prod_{j=1}^{N} j!(N-j)! \]  \tag{81}

The quantity \(\mathcal{I}_{\lambda, \mu}\) contains the factor
\[ \prod_{j=1}^{N} \frac{(M + N - j)!}{(M + 2N + \lambda_j - j)!}. \]  \tag{82}

First, we let \(N \to 0\) inside the product, to get
\[ \prod_{j=1}^{N} \frac{(M - j)!}{(M + \lambda_j - j)!}. \]  \tag{83}
This still depends on $N$ via the limit of the product. However, $\lambda_j = 0$ for $j > \ell(\lambda)$. Hence, if we assume $N \geq \ell(\lambda)$, we can write this as

$$
\prod_{j=1}^{\ell(\lambda)} \frac{(M-j)!}{(M+\lambda_j-j)!} = \frac{1}{[M]^\lambda}, \tag{84}
$$

which is independent of $N$. Now, in all rigor we are not allowed to take $N \to 0$ after assuming $N \geq \ell(\lambda)$. We do it anyway, and write

$$
\mathcal{I}_{\lambda,\mu} \to \frac{N!}{\mathcal{Z}[M]^{[M]^\lambda}} \det \left( \left( \lambda_i - i + \mu_j - j + 2N! \right) \right). \tag{85}
$$

Further, we factor out the smallest factor from each row of the determinant, producing $\prod_{j=1}^{N} (N + \lambda_j - j + \mu_N)!$. If we assume that $N \geq \ell(\mu)$, then $\mu_N = 0$. Hence, using (81),

$$
\mathcal{I}_{\lambda,\mu} \to \frac{N!}{[M]^{\mu}[M]^\lambda \prod_{j=1}^{N} (N + \lambda_j - j)!} \det \left( \frac{(\lambda_i - i + \mu_j - j + 2N)!}{(\lambda_i - i + N)!} \right). \tag{86}
$$

We again consider $N \geq \ell(\lambda)$ first and $N \to 0$ later, to arrive at

$$
\mathcal{I}_{\lambda,\mu} \to \frac{[N]^\lambda}{[M]^{\mu}[M]^\lambda \prod_{j=1}^{N} (N + \lambda_j - j)!} \det \left( \frac{(\lambda_i - i + \mu_j - j + 2N)!}{(\lambda_i - i + N)!} \right). \tag{87}
$$

### 5.4.1 The determinant

We need to consider the determinant

$$
\mathcal{D} = \det \left( \frac{(\lambda_i - i + \mu_j - j + 2N)!}{(\lambda_i - i + N)!} \right) = \det \left( \frac{(a_i + b_j)!}{a_i!} \right), \tag{88}
$$

where

$$
a_i = \lambda_i - i + N, \quad b_j = \mu_j - j + N. \tag{89}
$$

Each column consists of raising factorials, i.e. we have

$$
(a_i + 1)(a_i + 2)\cdots(a_i + b_j) = \frac{[a_i]^{b_j+1}}{a_i}. \tag{90}
$$

We therefore expand each column using identity Eq.(16), in terms of unsigned Stirling numbers of the first kind,

$$
\frac{[a_i]^{b_j+1}}{a_i} = \sum_{k_j=1}^{b_j+1} \binom{b_j+1}{k_j} a_i^{k_j-1} = \sum_{k_j=0}^{b_j} \binom{b_j+1}{k_j+1} a_i^{k_j}. \tag{91}
$$

The determinant is then given by

$$
\mathcal{D} = \prod_{j=1}^{N} \sum_{k_j=0}^{b_j} \binom{b_j+1}{k_j+1} \det \left( a_i^{k_j} \right). \tag{92}
$$

Introducing $k_j = \omega_j - j + N$ we have

$$
\mathcal{D} = \prod_{j=1}^{N} \sum_{\omega_j-\omega_j-N+1}^{\mu_j-j+1} \binom{\mu_j-j+N+1}{\omega_j-j+N+1} \det \left( a_i^{\omega_j-j+N} \right). \tag{93}
$$
Notice that $\omega$ is not a partition, since its elements are not necessarily ordered, and they can be negative. Still, the last determinant, if it does not vanish, can be turned into a Schur function by simply re-ordering the columns. Let $\bar{\omega}$ be the partition that is created in this way, and $|\bar{\omega}|$ the number it partitions. For instance, if $\omega = (1,1,-1,1)$ we have

$$\det \left( a^N_i \ a^{N-1}_i \ a^{N-4}_i \ a^{N-3}_i \ a^{N-5}_i \ldots \right) = -\det \left( a^N_i \ a^{N-1}_i \ a^{N-3}_i \ldots \right),$$

so the corresponding partition is $\bar{\omega} = (1,1)$ and $|\bar{\omega}| = 2$. As we can see, the reordering of the columns may lead to a change in sign. Let $\eta(\omega)$ denote this sign, so that

$$\det \left( a^{\omega_j-j+N}_i \right) = \eta(\omega) \Delta(a) s_{\bar{\omega}}(a) = \eta(\omega) \frac{d_\lambda}{n!} [N]^\lambda s_{\bar{\omega}}(a) \prod_{j=1}^{N-1} j!.$$

We must consider the $N \to 0$ limit of

$$s_{\bar{\omega}}(a) = \frac{1}{|\bar{\omega}|!} \sum_{\rho \in \bar{\omega}} |C_\rho| \chi_{\bar{\omega}}(\rho) p_\rho(a).$$

The limit of $p_\rho(a)$ can be obtained simply removing from this quantity everything that scales with $N$:

$$\lim_{N \to 0} p_\rho(\{\lambda_i - i + N\}) = \left( \prod_{q=1}^{\ell(\rho)} \left( \sum_{i=1}^{\ell(\lambda)} (\lambda_i - i)^q - (-i)^q \right) \right) \equiv f_\rho(\lambda).$$

We can finally write

$$\frac{\mathcal{D}}{n! \prod_{j=1}^{N-1} j!} \to \frac{d_\lambda}{n!} [N]^\lambda F_{\lambda,\mu},$$

where the function $F_{\lambda,\mu}$ is given by

$$F_{\lambda,\mu} = \frac{\ell(\mu)}{\ell(\rho)} \sum_{j=1}^{\omega_j = j - \ell(\mu)} \left[ \frac{\mu_j - j + 1}{\omega_j - j + 1} \right] \frac{\eta(\omega)}{|\omega|!} \sum_{\rho \in \omega} |C_\rho| \chi_{\omega}(\rho) f_\rho(\lambda).$$

### 5.4.2 Final Result

It is time to put the pieces back together. We have to plug the limiting value of $\mathcal{D}$ into the expression for $I_{\lambda,\mu}$, Eq. (87), put this into the expression for the radial integral, Eq. (74), and finally arrive at the quantity we want, which is $G_n$, Eq. (70). After some cancelations, we get that the limit as $N \to 0$ of $G_n$, which is nothing but the semiclassical expression for the correlation function $C_n(\epsilon)$, is given by

$$\lim_{N \to 0} G_n = C_n(\epsilon) = \frac{1}{M n!} \sum_{m=0}^{\infty} \frac{(Me\epsilon)^m}{m!} \sum_{\mu = m} \sum_{\lambda = n} d_\lambda d_\mu \chi_\lambda(\xi) \frac{[M]^\lambda}{[M]^\mu} F_{\lambda,\mu}.$$ 

This expression is perhaps not as simple we one might hope for, specially the $F_{\lambda,\mu}$ part. This complication is probably due to the fact that we are using a Taylor series in $\epsilon$. We know that, at each order in $1/M$, the correlation functions are rational functions of $\epsilon$, with the denominator being a power of $(1-\epsilon i)$. Maybe if this fact could be explicitly incorporated into the calculation somehow, the resulting expression would be more manageable.

For the simplest correlation function, explicit calculations suggest that the following expression holds:

$$C_1(\epsilon) = \sum_{n=1}^{\infty} \frac{(Mi\epsilon)^n}{n} \sum_{k=0}^{n-1} \frac{1}{[M+k]^n},$$

(101)
which is indeed in agreement with the first 3 orders in $1/M$ as computed from (12).

The average value of linear moments $M_m$ is given by

$$\langle M_m \rangle = \frac{\tau_m}{i^m m!} \left[ \frac{d^m}{de^m} \sum_{n=1}^{m} (-1)^{m-n} \binom{m}{n} C_n(e) \right]_{e=0}. \tag{102}$$

Therefore, if the identity

$$\frac{1}{n!} \sum_{n=1}^{m} (-1)^{m-n} \binom{m}{n} \sum_{\lambda \vdash n} d_{\lambda} \chi_{\lambda}(n) [M]^{\lambda} F_{\lambda,\mu} = [M]^{\mu} \chi_{\mu}(m), \tag{103}$$

is true, then the semiclassical formula for $\langle M_m \rangle$ becomes exactly equal to the RMT prediction (10). We have checked that (103) indeed holds for all $\mu \vdash m$ up to $m = 8$ (in doing so one needs only deal with hook partitions, for otherwise the character $\chi_{\lambda}(n)$ vanishes). This guarantees agreement between the semiclassical and RMT calculations up to the first 8 moments. Incidentally, since both expressions for $M_m$ are written as a sum over $\langle s_\lambda \rangle$, this suggests that the agreement between these approaches extends to all Schur functions, and hence to all statistics, as would be expected.

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