INTEGRATION BY PARTS FOR NONSYMMETRIC FRACTIONAL-ORDER OPERATORS ON A HALFSPACE

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ABSTRACT. For a strongly elliptic pseudodifferential operator $L$ of order $2a$ ($0 < a < 1$) with real kernel, we show an integration-by-parts formula for solutions of the homogeneous Dirichlet problem, in the model case where the operator is $x$-independent with homogeneous symbol, considered on the halfspace $\mathbb{R}_+^n$. The new aspect compared to $(-\Delta)^a$ is that $L$ is nonsymmetric, having both an even and an odd part. Hence it satisfies a $\mu$-transmission condition where generally $\mu \neq a$.

We present a complex method, relying on a factorization in factors holomorphic in $\xi$ in the lower or upper complex halfplane, using order-reducing operators combined with a decomposition principle originating from Wiener and Hopf. This is in contrast to a real, computational method presented very recently by Dipierro, Ros-Oton, Serra and Valdinoci. Our method allows $\mu$ in a larger range than they consider.

Another new contribution is the (model) study of “large” solutions of nonhomogeneous Dirichlet problems when $\mu > 0$. Here we deduce a “halfways Green’s formula” for $L$:

$$\int_{\mathbb{R}_+^n} Lu \bar{v} \, dx - \int_{\mathbb{R}_+^n} u \bar{L}^* v \, dx = c \int_{\mathbb{R}_-^{n-1}} \gamma_0(u/x_0^{n-1}) \gamma_0(\bar{v}/x_0^{n-1}) \, dx',$$

when $u$ solves a nonhomogeneous Dirichlet problem for $L$, and $v$ solves a homogeneous Dirichlet problem for $L^*$; $\mu^* = 2a - \mu$. Finally, we show a full Green’s formula, when both $u$ and $v$ solve nonhomogeneous Dirichlet problems; here both Dirichlet and Neumann traces of $u$ and $v$ enter, as well as a first-order pseudodifferential operator over the boundary.

1. Preface.

The fractional Laplacian $(-\Delta)^a$ ($0 < a < 1$) and its generalizations have received much attention lately, because of applications in probability, finance, mathematical physics and differential geometry. (References to applications are listed e.g. in our earlier works [G14-G19].) It is interesting to observe that the operator has been attacked from several angles: Roughly speaking, 1) there is a probabilistic approach, based on the fact that it is the infinitesimal generator of a Lévy process, 2) there is an approach by potential theoretic methods searching for similarities with the usual Laplacian $\Delta$, and 3) there is an approach

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by Fourier transformation using that the fractional Laplacian is a pseudodifferential operator. The first two methods have filled the major part of contributions in recent years; they deal mainly with real functions and real integral operators (singular integral operators with real kernels). The third method gave some early results in the 1960s and '70s, but has only been reintroduced fairly recently for this particular type of operators; it uses complex functions and distribution theory, and complex symbols and kernels of operators, in an essential way.

In the study of boundary value problems for these operators on open subsets $\Omega$ of $\mathbb{R}^n$, there are often two distinguished steps in the procedure: One step is to analyze a model case, where the domain $\Omega$ has the simple form of a ball or a halfspace, and the operator is translation-invariant; the other step is to piece the resulting information together to treat cases where $\Omega$ is more general, or the operator is $x$-dependent, or both things happen.

The purpose of the present paper is to show how the complex method can be used to advantage to give satisfactory results in the model case for operators $L$ that generalize the fractional Laplacian by being nonsymmetric, but still have real homogeneous kernels. (An example is $L = (-\Delta)^{\frac{1}{2}} + b \cdot \nabla$, $b \in \mathbb{R}^n$; the fractional Laplacian with critical drift.)

A direct inspiration for this work was a correspondence with the authors X. Ros-Oton, S. Dipierro, J. Serra and E. Valdinoci of the recent paper [DRSV20], treating generators $L$ of $\alpha$-stable processes and in particular showing an integration-by-parts formula. For the proof of the formula in the model case of the operator on a halfspace, they use an entirely real method passing through cumbersome integral operator formulas and calculations with special functions, taking up many pages. As we suggested to them, there is a complex method relying on the knowledge around pseudodifferential operators, leading to the result in an instructive way.

The complex method is presented here. The essence of the difference between the strategy of [DRSV20] and this work lies in the decomposition of $L$, where [DRSV20] uses a factorization into two real factors $L = L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}$, and we decompose $L$ in terms with symbol holomorphic in the lower, resp. upper complex plane (by principles originating from Wiener and Hopf).

An additional gain is that we do not need certain restrictions on the numerical range of the symbol of $L$ imposed in [DRSV20], hence can allow negative transmission numbers $\mu$ or $\mu^*$, in the considerations of homogeneous Dirichlet problems.

As a new contribution, we moreover consider “large” solutions (with the singularity $x_n^{\mu-1}$ at the boundary), solving nonhomogeneous Dirichlet problems. Here we show when $\mu, \mu^* > 0$ how a “halfways Green’s formula” can be deduced from the integration-by-parts formula. Furthermore we work out a full Green’s formula with nontrivial Dirichlet and Neumann traces.

The present note is only concerned with model cases on the halfspace $\mathbb{R}^n_+$. For the transition to cases with more general domains $\Omega$, the localization method of [DRSV20] would most likely be a useful tool.

**Outline.** In Section 2, the operator $L = \text{OP}(A(\xi) + iB(\xi))$ is introduced, and a value $\mu$ for which it satisfies the $\mu$-transmission condition is determined. Section 3 recalls mapping properties and regularity results from [G15] for $L$ on $\mathbb{R}^n_+$ with homogeneous Dirichlet condition. In Section 4, the integration-by-parts formula is shown in the model case on $\mathbb{R}^n_+$ by an extension of methods from [G16] and [G18]. Section 5 introduces large solutions and the nonhomogeneous Dirichlet problem, and shows how the result of Section 4 implies
a halfways Green’s formula. Section 6 gives the proof of the full Green’s formula, drawing on methods from [G18]. In Section 7, the formulas are extended to general functions $u, u', v$ in Sobolev-type and Hölder-type function spaces.

2. Introduction.

In the following we consider pseudodifferential operators $P = \text{OP}(p(\xi))$ on $\mathbb{R}^n$, where the symbols are independent of $x$:

\begin{equation}
Pu = \mathcal{F}^{-1}(p(\xi)(\mathcal{F}u)(\xi)), \quad \text{where } \mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) \, dx.
\end{equation}

$P$ can also be written as a singular integral operator

\begin{equation}
(Pu)(x) = \text{PV} \int_{\mathbb{R}^n} (u(x) - u(x + y)) K(y) \, dy, \quad K(y) = \mathcal{F}^{-1}p,
\end{equation}

where $\text{PV}$ indicates principal value.

In a recent paper [DRSV20], Dipierro, Ros-Oton, Serra and Valdinoci have considered such problems without the customary assumption that $P$ is symmetric; we shall study such cases more closely.

Assume that $p(\xi)$ is $C^\infty$ for $\xi \neq 0$ and homogeneous of degree $m > 0$; then $K(y)$ is a distribution homogeneous of degree $-n - m$. [DSRV20] moreover assumes that $K$ is real (in order to work with real integral operators), but without the earlier imposed assumption that $K$ should be even, i.e. invariant under the mapping $y \mapsto -y$. In the present case, $K$ splits in an even and an odd part,

\begin{equation}
K(y) = K_e(y) + K_o(y), \quad K_e(y) = \frac{i}{2}(K(y) + K(-y)), \quad K_o(y) = \frac{i}{2}(K(y) - K(-y)).
\end{equation}

It is checked from the properties of the Fourier transform that here $p(\xi) = \mathcal{A}(\xi) + i\mathcal{B}(\xi)$, where $\mathcal{A}(\xi) = \text{Re} \, p(\xi) = \mathcal{F}K_e$ and $\mathcal{B}(\xi) = \text{Im} \, p(\xi) = \frac{1}{i} \mathcal{F}K_o$ are real, and $\mathcal{A}(\xi)$ is even, $\mathcal{B}(\xi)$ is odd. The assumptions in [DSRV20] imply that $p(\xi)$ is strongly elliptic, i.e., $\mathcal{A}(\xi) > 0$ for $\xi \neq 0$. Let us give $p$ the name $\mathcal{L}(\xi) = \mathcal{A}(\xi) + i\mathcal{B}(\xi)$ and denote the operator $L = \text{OP}(\mathcal{L}(\xi))$.

It is moreover assumed in [DSRV20] that $L$ is the infinitesimal generator of an $\alpha$-stable $n$-dimensional Lévy process; this puts certain limitations on the numerical range of $\mathcal{L}$ (cf. (2.9) below) and the form of the integral operator, which we need not impose in our treatment. We shall simply assume, in addition to the homogeneity, that $\mathcal{L}(\xi)$ is strongly elliptic and that $K(y) = \mathcal{F}^{-1}\mathcal{L}$ is real.

For a homogeneous symbol $p(\xi)$ of degree $m$, the $\mu$-transmission condition with respect to the boundary of $\mathbb{R}^n_+$ was defined in [G15, Sect. 3] (with earlier input from Hörmander [H66,H85] and Eskin [E81]) as a measurement of how much $p$ deviates from being symmetric on the normal $(\xi_1, \ldots, \xi_{n-1}, \xi_n) = (0,1)$ to $\partial \mathbb{R}^n_+$:

\begin{equation}
p(0, -1) = e^{i\pi(m-2\mu)}p(0, 1)
\end{equation}

(note that this only determines $\mu$ mod 1). When $p$ is even in $\xi_n$, (2.4) holds with $\mu = m/2$. The case $m = 2a > 0$, $\mu = a$ has been amply treated in works of Grubb, Ros-Oton and Serra with coauthors, and many others. In particular, when $p$ is even in $\xi$, (2.4) holds with
\[ \mu = m/2 \] also for rotations of \( \mathbb{R}^n \); hence on a general domains \( \Omega \subset \mathbb{R}^n \) it holds regardless of the direction of the normal to \( \partial \Omega \).

What is the index \( \mu \) in the case \( \mathcal{L}(\xi) = A(\xi) + iB(\xi) \)? The order \( m \) will be denoted \( 2a \), \( 0 < a < 1 \). We have:

\begin{align*}
(2.5) \quad & \mathcal{L}(0, 1) = A(0, 1) + iB(0, 1), \quad \mathcal{L}(0, -1) = A(0, -1) + iB(0, -1) = A(0, 1) - iB(0, 1), \\
& \text{in short,} \\
(2.6) \quad & A(0, 1)^{-1}\mathcal{L}(0, \xi_n) = \begin{cases} 
1 + ib & \text{for } \xi_n = 1 \\
1 - ib & \text{for } \xi_n = -1
\end{cases}, \quad \text{with } b = \frac{B(0, 1)}{A(0, 1)}.
\end{align*}

In particular, \( \mathcal{L}(0, 1) \) and \( \mathcal{L}(0, -1) \) have the same length. The angle \( \theta \) between the positive real axis and \( \mathcal{L}(0, 1) \) is

\[ \theta = \arctan b, \quad \text{and we set } \pi^{-1}\theta = \delta, \quad \text{whereby } \mathcal{L}(0, 1) = e^{i\pi\delta} |\mathcal{L}(0, 1)|. \]

Moreover,

\[ \mathcal{L}(0, -1) = e^{-i2\theta} \mathcal{L}(0, 1) = e^{-i\pi2\delta} \mathcal{L}(0, 1) = e^{i\pi(2a - 2(a + \delta))} \mathcal{L}(0, 1), \]

so (2.4) holds with

\[ \mu = a + \delta; \quad \delta = \pi^{-1} \arctan \frac{B(0, 1)}{A(0, 1)} \in ] - \frac{1}{2}, \frac{1}{2} [. \]

(all other possible \( \mu \) equal this one plus an integer). As far as we understand the analysis in [E81, Ex. 6.1], the value \( \mu \) in (2.8) equals the factorization index, called \( \kappa \) in [E81] and \( \mu_0 \) in [G15].

There is the difficulty in comparisons with the Russian works Eskin [E81] and the subsequent developments by Shargorodsky [S95] and Chkadua and Duduchava [CD01], that they use a Fourier transformation having the exponential factor \( e^{+ix\cdot\xi} \) with plus instead of minus. This does not change the overall objective, but makes it hard to check exact formulas.

In the notation of [DRSV20], \( m/2 \) is called \( s \) (or \( \alpha \)), and our formula (2.8) for \( \mu \) corresponds to their formula (1.9) for \( \gamma \), consistent also with a related formula in Fernandez-Real and Ros-Oton [FR18, Prop. 2.2] concerning the special case \( \mathcal{L} = (-\Delta)^{1/2} + b \cdot \nabla, b \in \mathbb{R}^n \), where \( a = \frac{1}{2} \). The extra restrictions imposed in [DRSV20] require that (with \( 0 < a < 1 \))

\[ \mu \in ]0, 2a[ \cap ]2a - 1, 1[, \]

for their operators. We only assume \( a \in ]0, 1[ \) and \( |\delta| < \frac{1}{2} \) (whereby \( \mu \) can take any value in \( ] - \frac{1}{2}, \frac{1}{2} [. \))

For completely general strongly elliptic symbols \( p(\xi) \) (where one drops the requirement of having a real kernel), \( p(0, 1) \) and \( p(0, -1) \) can have different lengths, so that \( \mu \) satisfying (2.4) may be complex.
3. Regularity in a model case.

For a fractional-order \( \psi \) do \( P \) on \( \mathbb{R}^n \), one can define a homogeneous Dirichlet problem over an open subset \( \Omega \subset \mathbb{R}^n \) as follows: For \( f \) given on \( \Omega \), find \( u \) satisfying

\[
r^+ Pu = f \text{ in } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \Omega;
\]

this is sometimes called the \textit{restricted fractional Dirichlet problem}. Here \( r^+ \) indicates restriction from \( \mathbb{R}^n \) to \( \Omega \); in the cases we study here, \( \Omega = \mathbb{R}^n_+ \). When \( P \) is strongly elliptic, a solvable (perhaps just Fredholm solvable) problem in low-order \( L^2 \)-Sobolev spaces can be formulated using variational theory, when \( \Omega \) is bounded or \( P \) has a suitable lower boundedly.

Then comes the question of \textit{regularity}, namely what smoothness can be obtained for \( u \) when \( f \) has a given smoothness. Ros-Oton and Serra gave interesting results for the fractional Laplacian in Hölder spaces in [RS14], followed up with generalizations in subsequent papers, and the present author treated the question in [G15] in \( L^2 \)-Sobolev-type spaces for general \( x \)-dependent \( m \)-order pseudodifferential operators \( P \) and smooth domains, satisfying the \( \mu \)-transmission condition with an arbitrary complex \( \mu \) (followed up in other function spaces in [G14]).

Recall the notation: \( \mathcal{E}'(\mathbb{R}^n) \) is the space of distributions with compact support, \( S(\mathbb{R}^n) \) is the Schwartz space of \( C^\infty \)-functions \( f \) on \( \mathbb{R}^n \) such that \( x^\beta D^\alpha f \) is bounded for all \( \alpha, \beta \), \( S'(\mathbb{R}^n) \) is its dual space of temperate distributions. Denote \( r^+ S(\mathbb{R}^n) = S(\mathbb{R}^n_+) \).

The basic strategy of [G15], when \( \Omega = \mathbb{R}^n_+ \), is to compose \( P \) to the left and right with the order-reducing operators \( \Xi_{\mu-m}^- = \text{OP}(((\xi') - i\xi_n)^{\mu-m}) \) and \( \Xi_+^\mu = \text{OP}(((\xi') + i\xi_n)^{-\mu}) \), arriving at a zero-order operator \( Q = \Xi_{\mu-m}^- P \Xi_+^\mu \) satisfying the 0-transmission condition, as studied in the Boutet de Monvel calculus (cf. e.g. [B71,G96,S01,G09]), where results are obtained in standard Sobolev spaces. Strictly speaking, one needs truly pseudodifferential versions \( \Lambda_{\mu-m}^\mu, \Lambda_+^\mu \) of \( \Xi_{\mu-m}^\mu, \Xi_+^\mu \) to apply the pseudodifferential calculus properly, but the use of \( \Xi_{\mu-m}^\mu, \Xi_+^\mu \) suffices for some purposes.

Since \( \Xi_+^\mu \) is closely linked with multiplication by \( x_\mu^\mu \) on \( \mathbb{R}^n_+ \), one can show that \( P \) operates nicely on functions \( u = e^+ x_\mu^\mu v \) with \( v \in C^\infty(\mathbb{R}^n_+) \); here \( e^+ \) denotes extension by zero. Moreover, when \( u \) solves (3.1), then the boundary value of \( u/x_\mu^\mu \) for \( x_\mu \to 0^+ \) exists. To keep notation simple, we now just recall the smoothest solution space (more general spaces are recalled in Section 7 below), defined by:

\[
\mathcal{E}_\mu(\mathbb{R}^n_+) = e^+ x_\mu^\mu C^\infty(\mathbb{R}^n_+) \text{ when } \text{Re} \mu > -1;
\]

and \( \mathcal{E}_\mu(\mathbb{R}^n_+) \) is defined successively as the linear hull of first-order derivatives of elements of \( \mathcal{E}_{\mu+1}(\mathbb{R}^n_+) \) when \( \text{Re} \mu \leq -1 \) (then distributions supported in the boundary can occur).

It is shown in [G15, Th. 4.2, 4.4] that when \( P \) is of order \( m \) and satisfies the \( \mu \)-transmission condition, then

\[
u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n) \implies r^+ Pu \in C^\infty(\mathbb{R}^n_+) \cap \bigcap_s \mathcal{H}^s(\mathbb{R}^n_+),
\]

and when \( P \) moreover is elliptic with factorization index \( \mu_0 \) (\( \equiv \mu \) mod 1), one can conclude the other way: When \( u \in \mathcal{H}^\sigma(\mathbb{R}^n_+) \) for some \( \sigma > \mu_0 - \frac{1}{2} \),

\[
r^+ Pu \in C^\infty(\mathbb{R}^n_+) \cap \bigcap_s \mathcal{H}^s(\mathbb{R}^n_+) \implies u \in \mathcal{E}_{\mu_0}(\mathbb{R}^n_+).\]
Our notation for Sobolev spaces is recalled below in the start of Section 4.

For the operator $L = \text{OP}(\mathcal{A} + i\mathcal{B})$ considered in Section 2, with a smooth symbol $\mathcal{L}$ homogeneous of degree $m = 2a > 0$ for $\xi \neq 0$, $L$ is covered by the considerations of [G15], with $\mu = \mu_0 = a + \delta$. Namely, we can modify $\mathcal{L}(\xi)$ for $|\xi| \leq 1$ to a symbol $\mathcal{L}_1(\xi)$ that is $C^\infty$ also for $|\xi| \leq 1$; then the remainder $\mathcal{R} = \text{OP}(\mathcal{L}(\xi) - \mathcal{L}_1(\xi))$ is smoothing (sends general distributions into $C^\infty(\mathbb{R}^n)$), and has a bounded symbol. Then $r^+L$ sends $\mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n_+) \cap \bigcap_s \mathcal{H}^s(\mathbb{R}^n_+)$. The solutions of the Dirichlet problem with right-hand side in $C^\infty(\mathbb{R}^n_+) \cap \bigcap_s \mathcal{H}^s(\mathbb{R}^n_+)$ are in $\mathcal{E}_\mu(\mathbb{R}^n_+)$. Thus we have as a special case of [G15]:

**Theorem 3.1.** Let $0 < a < 1$, and let $L = \text{OP}(\mathcal{L}(\xi))$, where $\mathcal{L}(\xi)$ is homogeneous of order $2a$, with even real part $\mathcal{A}(\xi)$ and odd imaginary part $i\mathcal{B}(\xi)$, such that $\mathcal{L}(\xi) = \mathcal{A}(\xi) + i\mathcal{B}(\xi)$ is $C^\infty$ for $\xi \neq 0$, and $\mathcal{A}(\xi) > 0$ for $\xi \neq 0$. Let $\mu$ and $\mu^*$ be defined by

$$\mu = a + \delta; \quad \delta = \pi^{-1}\text{Arctan}\frac{\mathcal{B}(0, 1)}{\mathcal{A}(0, 1)} \in ] - \frac{1}{2}, \frac{1}{2}], \quad \mu^* = a - \delta.$$  

Then with $\sigma > \mu - \frac{1}{2}$,

$$u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n) \implies r^+Lu \in C^\infty(\mathbb{R}^n_+) \cap \bigcap_s \mathcal{H}^s(\mathbb{R}^n_+),$$

$$r^+Lu \in C^\infty(\mathbb{R}^n_+) \cap \bigcap_s \mathcal{H}^s(\mathbb{R}^n_+), u \in \dot{H}^\sigma(\mathbb{R}^n_+) \implies u \in \mathcal{E}_\mu(\mathbb{R}^n_+).$$

The analogous result holds for $L^* = \text{OP}(\mathcal{A}(\xi) - i\mathcal{B}(\xi))$ with $\mu$ replaced by $\mu^*$.

There are similar results in spaces based on $L_p$-Sobolev spaces and other function spaces, see [G15,G14].

4. Integration by parts in the halfspace case.

To keep notation simple, we restrict the attention to real $\mu$; this covers the case of strongly elliptic homogeneous symbols with a real kernel function. Recall the notation for $L_2$-Sobolev spaces:

$$H^s(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_2(\mathbb{R}^n) \},$$

$$\mathcal{H}^s(\mathbb{R}^n_+) = r^+H^s(\mathbb{R}^n),$$

the restricted space,

$$\dot{H}^s(\mathbb{R}^n_+) = \{ u \in H^s(\mathbb{R}^n) \mid \text{supp } u \subset \mathbb{R}^n_+ \},$$

the supported space.

Here $\mathcal{H}^s(\mathbb{R}^n_+)$ identifies with the dual space of $\dot{H}^{-s}(\mathbb{R}^n_+)$ for all $s \in \mathbb{R}$ (the duality extending the $L_2(\mathbb{R}^n_+)$ scalar product). When $|s| < \frac{1}{2}$, there is an identification of $\dot{H}^s(\mathbb{R}^n_+)$ with $\mathcal{H}^s(\mathbb{R}^n_+)$ (more precisely $e^s\mathcal{H}^s(\mathbb{R}^n_+)$). The trace operator $\gamma_0: u \mapsto \lim_{x_n \to 0^+} u(x', x_n)$ extends to a continuous mapping $\gamma_0: \mathcal{H}^s(\mathbb{R}^n_+) \to H^{s - \frac{1}{2}}(\mathbb{R}^{n-1})$ for $s > \frac{1}{2}$.

Recall that $\Xi^0_+ = \text{OP}((\langle \xi \rangle' \pm i\xi_n)\hat{u})$; here if homogeneity is important, $\langle \xi \rangle'$ can be replaced by $|\xi'|$, a $C^\infty$-function that equals $|\xi'|$ for $|\xi'| \geq 1$ and ranges in $[\frac{1}{2}, 1]$ when $|\xi'| \leq 1$. These operators have the mapping properties:

$$\Xi^t_+ : \dot{H}^s(\mathbb{R}^n_+) \ni \dot{H}^{s-t}(\mathbb{R}^n_+), \quad r^+\Xi^t_+ e^s: \mathcal{H}^s(\mathbb{R}^n_+) \ni \mathcal{H}^{s-t}(\mathbb{R}^n_+),$$

all $s, t \in \mathbb{R}$.
and for each $t \in \mathbb{R}$, the operators $\Xi^t_+ \text{ and } r^+\Xi^t_+ e^+$, also denoted $\Xi^t_{-,+}$, identify with each other's adjoints over $\overline{\mathbb{R}}_+^n$. Recall also the simple composition rules (as noted e.g. in [GK93, Th. 1.2]):

\begin{equation}
\Xi^s_+ \Xi^t_+ = \Xi^{s+t}_+ \quad \Xi^s_{-,+} \Xi^t_{-,+} = \Xi^{s+t}_{-,+} \text{ for } s, t \in \mathbb{R}.
\end{equation}

Recall from [G15] that for all $\mu$,

\begin{equation}
\mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \subset \Xi^\mu_+ e^+(C^\infty(\overline{\mathbb{R}}_+^n) \cap \int_s \mathcal{H}^\mu(\mathbb{R}^n_+)).
\end{equation}

A sharpening: $e^+ x_\mu^s S(\overline{\mathbb{R}}_+^n) = \Xi^\mu_+ e^+ S(\overline{\mathbb{R}}_+^n)$, is shown below in Lemma 6.1.

To show integration-by-parts formulas, we shall use methods from [G16] in a simplified version. First we have from [G16, Th. 3.1]:

**Theorem 4.1.** Let $0 < a < 1$, $|\delta| < \frac{1}{2}$, and $\mu = a + \delta$. Let $w \in C^\infty(\overline{\mathbb{R}}_+^n) \cap \int_s \mathcal{H}^\mu(\mathbb{R}^n_+)$, and let $u' \in \mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$. Denote $w' = r^+\Xi^a_+ u'$; correspondingly $u' = \Xi^\mu_+ e^+ w'$ in view of [G15, Prop. 1.7]. Then

\begin{equation}
\int_{\mathbb{R}^n_+} \Xi^\mu_+ e^+ w \partial_n u' \, dx = (\gamma_0 w, \gamma_0 u')_{L^2(\mathbb{R}^{n-1})} + (w, \partial_n u')_{L^2(\mathbb{R}^n_+)}.
\end{equation}

Here if $\mu \leq 0$, the left-hand side is interpreted as the duality

\begin{equation}
\langle \Xi^\mu_+ w, \partial_n u' \rangle_{\mathcal{H}^{\frac{1}{2}+\epsilon}_-(\mathbb{R}^n_+), \mathcal{H}^{-\frac{1}{2}-\epsilon}_-(\mathbb{R}^n_+)} \text{, also equal to } \int_{\mathbb{R}^n_+} \Xi^\mu_+ e^+ w \overline{\partial_n u'} \, dx.
\end{equation}

**Proof.** In view of (4.1), $w \in \mathcal{H}^s(\mathbb{R}^n_+)$ for all $s$ implies that $r^+\Xi^\mu_+ e^+ w \in \mathcal{H}^s(\mathbb{R}^n_+)$ for all $s$. By (3.3), $w' = r^+\Xi^a_+ u' \in C^\infty(\overline{\mathbb{R}}_+^n) \cap \int_s \mathcal{H}^\mu(\mathbb{R}^n_+)$. Consider first the case where $\mu > 0$. Since $u' \in \mathcal{E}_\mu(\overline{\mathbb{R}}_+^n)$ with compact support and is continuous on $\mathbb{R}^n$, $\partial_n u' \in \mathcal{E}_{\mu-1}(\overline{\mathbb{R}}_+^n)$ with compact support. Here $x_n^\mu u'$ is integrable over compact sets. Altogether, the integrand $r^+\Xi^a_+ e^+ w \partial_n u'$ is the product of $x_n^\mu u'$ with a compactly supported smooth function (on $\overline{\mathbb{R}}_+^n$), so the integral is well-defined.

We can also observe that by the identification of $e^+ \mathcal{H}^t(\mathbb{R}^n_+)$ and $\mathcal{H}^t(\mathbb{R}^n_+)$ for $|t| < \frac{1}{2}$, $e^+ w' \in \mathcal{H}^{\frac{1}{2}-\epsilon}(\mathbb{R}^n_+)$ for any $\epsilon \in ]0,1[$, so

\[
\partial_n u' = \partial_n \Xi^\mu_+ e^+ w' \in \partial_n \mathcal{H}^{\frac{1}{2}-\epsilon}(\mathbb{R}^n_+) \subset \mathcal{H}^{\frac{1}{2}-\epsilon}(\mathbb{R}^n_+).
\]

Thus the integral may be written as the duality

\[
I = \langle r^+\Xi^\mu_+ e^+ w, \partial_n u' \rangle_{\mathcal{H}^{\frac{1}{2}+\epsilon}_-(\mathbb{R}^n_+), \mathcal{H}^{-\frac{1}{2}-\epsilon}_-(\mathbb{R}^n_+)}.\]

Note that $r^+\Xi^\mu_+ e^+; \mathcal{H}^{\frac{1}{2}+\epsilon}(\mathbb{R}^n_+) \rightarrow \mathcal{H}^{\frac{1}{2}-\mu+\epsilon}(\mathbb{R}^n_+)$ has the adjoint $\Xi^\mu_+; \mathcal{H}^{-\frac{1}{2}-\epsilon}(\mathbb{R}^n_+) \rightarrow \mathcal{H}^{-\frac{1}{2}+\epsilon}(\mathbb{R}^n_+)$. This allows to continue the calculation of $I$ as follows:

\[
I = \langle w, \Xi^\mu_+ \partial_n u' \rangle_{\mathcal{H}^{\frac{1}{2}+\epsilon}_-(\mathbb{R}^n_+), \mathcal{H}^{-\frac{1}{2}-\epsilon}_-(\mathbb{R}^n_+)} = \langle w, \partial_n \Xi^\mu_+ u' \rangle = \langle w, \partial_n e^+ w' \rangle.
\]
Here \( w' \) itself is a nice function on \( \mathbb{R}_+^n \), but the extension \( e^+ w' \) to \( \mathbb{R}^n \) has the jump \( \gamma_0 w' \) at \( x_n = 0 \), and there holds the formula

\[
\partial_n e^+ w' = (\gamma_0 w')(x') \otimes \delta(x_n) + e^+ \partial_n w'.
\]

where \( \otimes \) indicates a product of distributions with respect to different variables \((x' \text{ resp. } x_n)\). (4.6) is a distribution version of Green’s formula (cf. e.g. [G96] (2.2.38)-(2.2.39)); it has been much used in the literature on boundary value problems (early in the theory in e.g. Seeley [S66], Boutet de Monvel [B66], Hörmander [H66]). Recall moreover from distribution theory (cf. e.g. [G09] p. 307) that the “two-sided” trace operator \( \tilde{\gamma}_0 : v(x) \mapsto \tilde{\gamma}_0 v = v(x',0) \) has the mapping \( \tilde{\gamma}_0^* : \varphi(x') \mapsto \varphi(x') \otimes \delta(x_n) \) as adjoint, with continuity properties

\[
\tilde{\gamma}_0 : H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^n) \to H^{\varepsilon}(\mathbb{R}^{n-1}), \quad \tilde{\gamma}_0^* : H^{-\varepsilon}(\mathbb{R}^{n-1}) \to H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n), \text{ for } \varepsilon > 0.
\]

Here \( \tilde{\gamma}_0^* \varphi \) is supported in \( \{x_n = 0\} \), hence lies in \( H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n) \). We can then write

\[
\partial_n e^+ w' = \tilde{\gamma}_0^* (\gamma_0 w') + e^+ \partial_n w' \text{ on } \mathbb{R}^n.
\]

Since \( w \in \overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n) \), it has an extension \( W \in H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^n) \) with \( w = r^+ W \), and \( \gamma_0 w = \tilde{\gamma}_0 W \). Then

\[
\langle w, \tilde{\gamma}_0^* (\gamma_0 w') \rangle_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)} = \langle W, \tilde{\gamma}_0^* (\gamma_0 w') \rangle_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)};
\]

this is verified e.g. by approximating \( \tilde{\gamma}_0^* (\gamma_0 w') \) in \( H^{-\frac{1}{2}-\varepsilon} \)-norm by a sequence of functions in \( C_0^\infty(\mathbb{R}_+^n) \). Here we can use (4.7) to write

\[
\langle W, \tilde{\gamma}_0^* (\gamma_0 w') \rangle_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)} = \langle \gamma_0 W, \gamma_0 w' \rangle_{H^{\varepsilon}(\mathbb{R}^{n-1}), H^{-\varepsilon}(\mathbb{R}^{n-1})} = \langle \gamma_0 w, \gamma_0 w' \rangle_{L_2(\mathbb{R}^{n-1})}.
\]

In the last step we used that since both \( \gamma_0 w \) and \( \gamma_0 w' \) are in \( H^{\varepsilon}(\mathbb{R}^{n-1}) \subset L_2(\mathbb{R}^{n-1}) \), the duality over the boundary is in fact an \( L_2(\mathbb{R}^{n-1}) \)-scalar product.

Then finally

\[
I = \langle w, \partial_n e^+ w' \rangle_{H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n), H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)} = \langle \gamma_0 w, \gamma_0 w' \rangle_{H^{\varepsilon}(\mathbb{R}^{n-1}), H^{-\varepsilon}(\mathbb{R}^{n-1})} = \langle \gamma_0 w, \gamma_0 w' \rangle_{L_2(\mathbb{R}^{n-1})}.
\]

where we used that \( w' \in \bigcap_\mu \overline{H}^{\varepsilon}(\mathbb{R}_+^n) \). This shows (4.2) in the case where \( \mu > 0 \).

**Case** \( \mu \leq 0 \). The case \( \mu \leq 0 \) can occur only when \( a < \frac{1}{2} \) and \( \delta < 0 \). As already noted, \( r^+ \Xi^+ e^+ w = \Xi^+ e^+ \) is in any \( \overline{H}^{\varepsilon}(\mathbb{R}_+^n) \)-space, so we can choose \( t \) to fit with a \( \overline{H}^{-t}(\mathbb{R}_+^n) \)-space for the right-hand factor \( \partial_n w' \). This factor belongs (cf. (4.3)) to

\[
\mathcal{E}_{\mu-1}(\mathbb{R}_+^n) \subset \Xi^{1-\mu} e^+ (\mathcal{C}_0(\mathbb{R}_+^n)) \cap \bigcap_\varepsilon \Xi^{-\varepsilon}(\mathbb{R}_+^n) = \overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n) \subset H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n),
\]
so we can take \( t = \frac{1}{2} + \varepsilon - \mu \), showing the first interpretation in (4.5).

For the second interpretation, note that \( \Xi_{-+}^\mu = \Xi_{-+}^\delta \Xi_{-+}^a \) (cf. (4.2)), where \( \Xi_{-+}^\delta \) can be transposed to the right-hand side as the adjoint \( \Xi_{++}^\delta \). Then the integrand is a product of functions \( \Xi_{-+}^a w \in C^\infty(\mathbb{R}_+^n) \) and \( \Xi_{++}^\delta \partial_n u' \in \mathcal{E}_{a-1}(\mathbb{R}_+^n) \), integrable up to the boundary and belonging to \( L_1(\mathbb{R}_+^n) \), and \( I \) may be written

\[
I = \langle r^+ \Xi_{-+}^a e^+ w, \Xi_{++}^\delta \partial_n u' \rangle_{\mathcal{H}^{1/2-\varepsilon_+}(\mathbb{R}_+^n),H^{1/2-\varepsilon_+}(\mathbb{R}_+^n)} = \int_{\mathbb{R}_+^n} \Xi_{-+}^a e^+ w \Xi_{++}^\delta \partial_n u' \, dx.
\]

We proceed by carrying \( \Xi_{-+}^a \) over to the right factor, using again (4.2) and the adjoint properties, which gives

\[
I = \langle w, \Xi_{++}^\mu \partial_n u' \rangle_{\mathcal{H}^{1/2-\varepsilon}(\mathbb{R}_+^n),H^{1/2-\varepsilon}(\mathbb{R}_+^n)}.
\]

From here on, the proof is completed as in the case \( \mu > 0 \). \( \square \)

An immediate consequence of Theorem 4.1 is the following integration-by-parts result for a very special operator:

**Theorem 4.2.** Let \( \mu = a + \delta \), \( \mu^* = a - \delta \) with \( a, \delta \) as in Theorem 4.1, and consider \( P = \Xi_{-+}^\mu \Xi_{++}^a \). Let \( u \in \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \) and \( u' \in \mathcal{E}_{\mu^*}(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \). Then

\[
\int_{\mathbb{R}_+^n} Pu \partial_n u' \, dx + \int_{\mathbb{R}_+^n} \partial_n u P^* u' \, dx = \Gamma(\mu + 1) \Gamma(\mu^* + 1) \int_{\mathbb{R}^{n-1}} \gamma_0(u/x_\mu) \gamma_0(u'/x_{\mu^*}) \, dx'.
\]

**Proof.** We apply Theorem 4.1 to the integrals in the left-hand side of (4.9). When \( u \in \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \), then \( w = r^+ \Xi_{-+}^\mu u \in \bigcap_s \mathcal{H}^s(\mathbb{R}_+^n) \). Hence \( r^+ Pu = r^+ \Xi_{-+}^\mu e^+ w \in \bigcap_s \mathcal{H}^s(\mathbb{R}_+^n) \), and an application of Theorem 4.1 with \( \mu \) replaced by \( \mu^* \) gives:

\[
\int_{\mathbb{R}_+^n} Pu \partial_n u' \, dx = (\gamma_0 w, \gamma_0 w')_{L^2(\mathbb{R}^{n-1})} + (w, \partial_n w')_{L^2(\mathbb{R}_+^n)},
\]

where \( w' = r^+ \Xi_{++}^\mu u' \).

Using that \( P^* = \Xi_{++}^\mu \Xi_{-+}^a \), we can apply the analogous argument to show that the conjugate of \( \int_{\mathbb{R}_+^n} \partial_n u P^* u' \, dx \) satisfies

\[
\int_{\mathbb{R}_+^n} P^* u' \partial_n u \, dx = (\gamma_0 w', \gamma_0 w)_{L^2(\mathbb{R}^{n-1})} + (w', \partial_n w)_{L^2(\mathbb{R}_+^n)},
\]

with the same definitions of \( w' \) and \( w \). It follows by addition that

\[
\int_{\mathbb{R}_+^n} Pu \partial_n u' \, dx + \int_{\mathbb{R}_+^n} \partial_n u P^* u' \, dx = 2(\gamma_0 w, \gamma_0 w')_{L^2(\mathbb{R}^{n-1})} + (w, \partial_n w')_{L^2(\mathbb{R}_+^n)} + (\partial_n w, w')_{L^2(\mathbb{R}_+^n)} = (\gamma_0 w, \gamma_0 w')_{L^2(\mathbb{R}^{n-1})};
\]

where \( \gamma_0 \) is the Dirac mass at the origin.
in the last step we used that \( \int_{\mathbb{R}^n_+} (w \partial_n \bar{w}' + \partial_n w \bar{w}') \, dx = -\int_{\mathbb{R}^{n-1}} \gamma_0 w \gamma_0 \bar{w}' \, dx' \). Insertion of 
\( \gamma_0 w = \gamma_0 \Xi_+ \mu \) as in Theorem 4.3 and reproved below in (6.9) for \( \mu - 1 \) and \( \gamma_0 w' = \gamma_0 \Xi_+ u' = (1 + \mu^*) \gamma_0 (u'/x^*_n) \) gives (4.9). □

We now turn to the operator \( L = \text{OP}(A(\xi) + iB(\xi)) \) considered in Theorem 3.1. Define
\[
Q = \Xi_-^\mu L \Xi_+^\mu = \text{OP}(q(\xi)), \quad q(\xi) = (\langle \xi' \rangle - i \xi_n)^{-\mu} \mathcal{L}(\xi)(\langle \xi' \rangle + i \xi_n)^{-\mu},
\]
so that \( L = \Xi_-^\mu Q \Xi_+^\mu \).

Since \( \mathcal{L} \) is \( C^\infty \) only for \( \xi \neq 0 \), and just Hölder continuous of order \( 2a \) at 0, we shall write it as the sum of a smooth symbol and a symbol with small support:
\[
\mathcal{L}(\xi) = \mathcal{L}_\varphi(\xi) + \varphi(\xi) \mathcal{L}(\xi),
\]
where \( \varphi(\xi) \) is a function in \( C^\infty_0(\mathbb{R}^n, [0, 1]) \) supported for \( |\xi| \leq 1 \) and equal to 1 for \( |\xi| \leq \frac{1}{2} \); so that \( \mathcal{L}_\varphi(\xi) = (1 - \varphi(\xi)) \mathcal{L}(\xi) \) is in \( S^{2a}(\mathbb{R}^n \times \mathbb{R}^n) \). Then similarly,
\[
q(\xi) = q_\varphi(\xi) + \varphi(\xi) q(\xi),
\]
where \( q_\varphi(\xi) \) is a smooth symbol of order 0. Its principal part \( q^0 = (\langle \xi' \rangle - i \xi_n)^{-\mu} \mathcal{L}(\xi)(\langle \xi' \rangle + i \xi_n)^{-\mu} \) is homogeneous of degree 0 in \( \xi \) and satisfies the 0-transmission condition:
\[
q^0(0, 1) = (-i)^{-a+\delta} \mathcal{L}(0, 1) i^{-a-\delta} = i^{-2\delta} \mathcal{L}(0, 1) = e^{-i\pi\delta} \mathcal{L}(0, 1),
\]
\[
q^0(0, -1) = (+i)^{-a+\delta} \mathcal{L}(0, -1) (-i)^{-a-\delta} = i^{2\delta} e^{i\pi(2a-2(a+\delta))} \mathcal{L}(0, 1) = e^{-i\pi\delta} \mathcal{L}(0, 1) = q^0(0, 1).
\]

Note that
\[
s_0 \equiv q^0(0, 1) = e^{-i\pi\delta} \mathcal{L}(0, 1) = |\mathcal{L}(0, 1)|.
\]

It is checked by use of a Taylor expansion of \( \langle \xi' \rangle = |\xi'|(1 + |\xi'|^{-2})^{\frac{1}{2}} \) that also the lower-order terms in \( q_\varphi \) fulfill the rules for the 0-transmission condition (cf. [G15]).

**Theorem 4.3.** Consider \( L = \text{OP}(\mathcal{L}(\xi)), a, \delta, \mu, \mu^* \) as described in Theorem 3.1. For \( u \in \mathcal{E}_\mu(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n) \), \( u' \in \mathcal{E}_{\mu^*}(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n) \), there holds
\[
\int_{\mathbb{R}^n_+} Lu \partial_n u' \, dx + \int_{\mathbb{R}^n_+} \partial_n u L \bar{u}' \, dx = \Gamma(\mu + 1) \Gamma(\mu^* + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0 (u'/x^*_n) \gamma_0 (\bar{u}'/x^*_n) \, dx',
\]
where \( s_0 = |\mathcal{L}(0, 1)| = (\mathcal{A}(0, 1)^2 + \mathcal{B}(0, 1)^2)^{\frac{1}{2}} \).

The integrals over \( \mathbb{R}^n_+ \) are interpreted as in Theorem 4.4 if \( \mu \) or \( \mu^* \leq 0 \).

**Proof.** We first consider the contribution from \( \varphi \mathcal{L} \). This can be handled in the same way as in the treatment of the smoothing term \( \mathcal{S} \) in the proof of [G16, Cor. 3.5], showing that \( \varphi \mathcal{L} \) contributes with zero.
Now consider the contribution from $\mathcal{L}_\varphi$. To simplify the notation we omit the subscript \( \varphi \) from now on. It is accounted for in [G16] that \( q_\varphi \) (from here on simplified to \( q \)) lies in \( S^0(\mathbb{R}^{n-1}, \mathcal{H}_0) \), hence has a decomposition
\[
q(\xi', \xi_n) = q(0, 1) + f_+(\xi) + f_-(\xi), \quad q(0, 1) = s_0
\]
(cf. (4.12)), where
\[
f_+(\xi', \xi_n) = h^+ q \in S^0(\mathbb{R}^{n-1}, \mathcal{H}^+), \quad f_-(\xi', \xi_n) = h^- q \in S^0(\mathbb{R}^{n-1}, \mathcal{H}^-).
\]
The terminology with \( h^\pm \) and \( \mathcal{H}_d, \mathcal{H}^\pm \), etc., originating from a decomposition principle ascribed to Wiener and Hopf, and set up in this way in Boutet de Monvel [B71], is summarized in [G16] and in [G18], and described at length e.g. in [G96], [G09] and Schrohe [S01]. Let us here just recall that the functions \( f(\xi_n) \in \mathcal{H}_0 \) are the Fourier transforms of \( \tilde{f}(x_n) \in e^+ S(\mathbb{R}_+^n) \oplus e^- S(\mathbb{R}_-^n) \oplus \mathbb{C} \delta = \mathcal{F}^{-1}(\mathcal{H}^+ \oplus \mathcal{H}^- \oplus \mathbb{C}) \), where \( h^+ \) corresponds to the projection onto \( e^+ S(\mathbb{R}_+^n) \), and \( h^- \) corresponds to the projection onto \( e^- S(\mathbb{R}_-^n) \). Here \( S(\mathbb{R}_\pm^m) = r^{\pm} \mathbb{S}(\mathbb{R}) \), the Schwartz space. (More details in [G18, (A.2ff.).]

In [G16] it is moreover shown that \( q \) has a factorization into two such factors, by making use of the strong ellipticity that allows passing to \( \log q \) and back, but we do not need the factorization nor the ellipticity here, only the sum decomposition (4.14). This simplifies the proof and is inspired from [G18].

Denote
\[
(4.15) \quad \text{OP}(f_+) = F_+, \quad \text{OP}(f_-) = F_-, \quad \text{so} \quad F_-^* = \text{OP}(\tilde{f}_-),
\]
where \( \tilde{f}_- \in S^0(\mathbb{R}^{n-1}, \mathcal{H}^+) \). Then
\[
(4.16) \quad L = \Xi_- \Xi_+^* Q \Xi_+^\mu, \quad Q = \text{OP}(q) = s_0 + F_+ + F_-.
\]
The contribution from the first term in \( Q \) is \( s_0 \Xi_- \Xi_+^\mu \Xi_+^\mu \), which we have already treated in Theorem 4.2; it gives the right-hand side in (4.13).

It remains to treat the terms with \( F_+ \) and \( F_- \) and to show that they together contribute with 0. Here we note that as in [G15, Th. 4.2], using that \( u \) is supported in \( \mathbb{R}_+^n \),
\[
(4.17) \quad r^\pm \Xi_+^\mu F_+ \Xi_+^\mu u = r^\pm \Xi_+^\mu e^+(F_+ \Xi_+^\mu) u, \quad r^\pm \Xi_- \Xi_+^\mu F_- \Xi_+^\mu u = r^+(\Xi_- \Xi_+^\mu F_-) e^+ \Xi_+^\mu u.
\]
(In the various calculations, the extension by 0 on \( \mathbb{R}_\pm^n \) is sometimes tacitly understood.)

For the given \( u, u' \), denote
\[
(4.18) \quad w = r^\pm \Xi_+^\mu u, \quad w_1 = r^\pm F_+ w = r^\pm F_+ \Xi_+^\mu u, \quad w' = r^\pm \Xi_+^\mu u', \quad w_2 = r^\pm F_+ w' = r^\pm F_+ \Xi_+^\mu u';
\]
they all lie in \( C^\infty(\mathbb{R}_+^n) \cap \bigcap_s \mathcal{S}^s(\mathbb{R}_+^n) \).

For the first term in (4.13), we proceed as follows: By Theorem 4.1 (applied with \( \mu \) replaced by \( \mu^* \)), we have for the contribution from \( F_+ \), in view of (4.17),
\[
(4.19) \quad \int_{\mathbb{R}_+^n} \Xi_- \Xi_+^\mu F_+ \Xi_+^\mu u \partial_n \bar{u}' \, dx = \int_{\mathbb{R}_+^n} \Xi_- \Xi_+^\mu e^+ w_1 \partial_n \bar{u}' \, dx
\]
\[
= (\gamma_0 w_1, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w_1, \partial_n u')_{L_2(\mathbb{R}_+^n)}.
\]
We can likewise use the argumentation of Theorem 4.1 to show for the contribution from $F_-$, after a transposition:

\[
\int_{\mathbb{R}_+^n} \Xi^{\mu^*} F_- \Xi^{\mu^*} u \partial_n \bar{u}' \, dx = \int_{\mathbb{R}_+^n} \Xi^{\mu^*} u (\Xi^{\mu^*} F_-)^* \partial_n u' \, dx \\
= \int_{\mathbb{R}_+^n} \Xi^{\mu^*} u F_- \Xi^{\mu^*} \partial_n u' \, dx = (\gamma_0 w, \gamma_0 w'_2)_{L^2(\mathbb{R}^{n-1})} + (w, \partial_n w'_2)_{L^2(\mathbb{R}_+^n)}.
\]

Now there is a special observation that allows removing the boundary terms appearing here, similarly as at [G16,(3.30)]: With $w_1$ defined in (4.18) we have, using that the boundary value of a function supported in $\mathbb{R}^n_+$ can be described by the convention for trace operators in the Boutet de Monvel calculus ([G18, (A.15), (A.1)]):

\[
\gamma_0 w_1 = \gamma_0 (F_+ w) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \xi} \int_{\mathbb{R}} f_+ (\xi', \xi_n) \mathcal{F}(e^+ w) \, d\xi' d\xi_n.
\]

This equals 0 for the following reason: Both $f_+$ and $\mathcal{F}(e^+ w)$ are in $\mathcal{H}^+$ as functions of $\xi_n$, in particular $O((\xi_n)^{-1})$, whereby the integrand is $O((\xi_n)^{-2})$ and extends holomorphically into the lower imaginary halfplane $\mathbb{C}_-$; then the integral over $\mathbb{R}$ can be transformed to a closed contour in $\mathbb{C}_-$ and therefore vanishes.

Similarly, $\gamma_0 w_2' = 0$. Thus only the integrals over $\mathbb{R}_+^n$ remain, and we find altogether:

\[
\int_{\mathbb{R}_+^n} \Xi^{\mu^*} (F_+ + F_-) \Xi^{\mu^*} u \partial_n \bar{u}' \, dx = (w_1, \partial_n w')_{\mathbb{R}_+^n} + (w, \partial_n w'_2)_{\mathbb{R}_+^n}.
\]

The analogous arguments apply to the contribution from $F_+ + F_-$ in the second term in (4.13) (after conjugation), with $\mu$ and $\mu^*$ interchanged, and lead to:

\[
\int_{\mathbb{R}_+^n} \partial_n u (\Xi^{\mu^*} (F_+ + F_-) \Xi^{\mu^*})^* u' \, dx = (\partial_n w, w'_2)_{\mathbb{R}_+^n} + (\partial_n w_1, w')_{\mathbb{R}_+^n}.
\]

Addition of the right-hand sides of (4.20) and (4.21) gives

\[
(w_1, \partial_n w')_{\mathbb{R}_+^n} + (w, \partial_n w'_2)_{\mathbb{R}_+^n} + (\partial_n w, w'_2)_{\mathbb{R}_+^n} + (\partial_n w_1, w')_{\mathbb{R}_+^n} = \int_{\mathbb{R}_+^n} (\partial_n (w \bar{w}'_2) + \partial_n (w_1 \bar{w}')) \, dx = -\int_{\mathbb{R}^{n-1}} (\gamma_0 w \gamma_0 \bar{w}'_2 + \gamma_0 w_1 \gamma_0 \bar{w}') \, dx' = 0,
\]

where we have again used the vanishing of $\gamma_0 w_1$ and $\gamma_0 w'_2$.

Adding all the contributions from $F_+, F_-$ and $s_0$, we arrive at (4.13). \(\square\)

Theorem 4.3 is proved by real methods in [DRSV20, Prop. 5.1], for operators $L$ that in addition satify (2.9), and are infinitesimal generators of $\alpha$-stable $n$-dimensional Lévy processes. It can be remarked that in the above method, we only work with the symbol $\mathcal{L}(\xi)$ of $L$ and not with the corresponding distribution kernel, so that we avoid the issue of singularities of the kernel.

The results can undoubtedly be extended to $x$-dependent operators by efforts as in [G16], as long as $\mu$ is constant in $x'$. For variable $\mu(x')$, other tools are needed. There is a method in [DRSV20] showing how one extends the integration-by-parts formula from the halfspace case to general domains for the $x$-independent operators considered there.
5. The nonhomogeneous Dirichlet problem and a halfways Green’s formula.

Along with the homogeneous Dirichlet problem (3.1), one can consider a nonhomogeneous Dirichlet problem if the scope expanded to allow so-called ”large solutions”, behaving like \( d^{\mu - 1} \) near the boundary (with \( d(x) = \text{dist}(x, \partial \Omega) \)); such solutions blow up at the boundary when \( \mu < 1 \).

For elliptic operators \( P \) satisfying the \( \mu \)-transmission condition, we can then pose the nonhomogeneous problem

\[
(5.1) \quad r^+ Pu = f \quad \text{in} \quad \Omega, \quad \gamma_0(u/d^{\mu - 1}) = \psi \quad \text{on} \quad \partial \Omega, \quad u = 0 \quad \text{on} \quad \mathbb{R}^n \setminus \Omega;
\]

assuming \( \mu > 0 \). It has good solvability properties when \( u \) is sought in \( E^{\mu - 1}_-(\Omega) \) and related Sobolev-type spaces, cf. [G15, Sect. 5 and Th. 6.1]. (We now take \( \mu > 0 \), since \( E^{\mu - 1}_-(\Omega) \) is defined via (3.2) then; for lower \( \mu \), the interpretation of the boundary condition may be more complicated.) The homogeneous Dirichlet problem fits in here as the problem with \( \psi = 0 \) in (5.1); for \( f \in C^\infty(\Omega) \) the solutions with vanishing \( \gamma_0(u/d^{\mu - 1}) \) are in \( E^{\mu}_-(\Omega) \).

The interest of problem (5.1) for the fractional Laplacian \( (-\Delta)^a \) (where \( \mu = a \)) was also pointed out in Abatangelo [A15] (independently of [G15]); the boundary condition there is given in a less explicit way, except when \( \Omega \) is a ball.

One can now set up a “halfways Green’s formula” describing the difference of integrals of \( Pu \) times \( v \) and \( u \) times \( P^* v \) as a boundary integral, when \( u \) is a large solution for \( P \) (say in \( E^{\mu - 1}_-(\Omega) \)) and \( v \) is an ordinary solution for \( P^* \). Such a formula was shown in [G18] when \( P \) is even (so that \( \mu = a \)), and a related formula was shown in [A15, formula 1.2(9)] for \( (-\Delta)^a \).

It was observed in [G18] for even operators that the halfways Green’s formula is essentially equivalent with the integration-by-parts formula (in Cor. 4.5 there, the conclusion in one direction is explained; in fact the same ingredients allow the other conclusion, as also done below).

We shall generalize this to the present operators, and can thereby rather easily deduce a halfways Green’s formula in the case of \( L \) on a halfspace, by manipulations with the result of Theorem 4.3. The word “halfways” refers to taking one function \( u \) as a solution of the nonhomogeneous Dirichlet problem (for \( L \)) and the other function \( v \) as a solution of the homogeneous Dirichlet problem (for \( L^* \)). (When both functions are solutions of nonhomogeneous Dirichlet problems, one should get a full Green’s formula with more boundary terms; this is carried out in Section 6 below.)

**Theorem 5.1.** Let \( L = \text{OP}(\mathcal{L}(\xi)) \), \( a, \delta, \mu, \mu^* \) and \( s_0 \) be as described in Theorems 3.1 and 4.3, and assume moreover that \( \mu, \mu^* > 0 \).

For \( u \in E^{\mu - 1}_-(\mathbb{R}^n_+) \cap E'(\mathbb{R}^n) \) and \( v \in E^{\mu^*}_+(\mathbb{R}^n) \cap E'((\mathbb{R}^n_+)^c) \), there holds

\[
(5.2) \quad \int_{\mathbb{R}^n_+} Lu \, \overline{v} \, dx - \int_{\mathbb{R}^n_+} u \, L^* v \, dx = -\Gamma(\mu)\Gamma(\mu^* + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_n^{\mu - 1}) \gamma_0(\overline{v}/x_n^{\mu^*}) \, dx'.
\]

As shown for the fractional Laplacian in [A15], this type of formula can be used to derive a representation of a solution of (5.1) in terms of the data.

Note that since \( \mu = a + \delta \) and \( \mu^* = a - \delta \) with \( |\delta| < \frac{1}{2} \), the extra requirement on positivity of \( \mu, \mu^* \) is always satisfied when \( a \geq \frac{1}{2} \).

As a preparation for the proof, we recall an elementary observation:
Lemma 5.2. Let \( \mu > 0 \). Then any function \( u \in \mathcal{E}_\mu^{-1}(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \) can be written as

\[
(5.3) \quad u = \partial_n U + u_1, \text{ where } U, u_1 \in \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n).
\]

There is a detailed proof of this on page 494 in [G15]. We also note an easy identity:

Theorem 5.3. Consider \( L = \text{OP}(\mathcal{L}(\xi)) \), \( a, \delta, \mu, \mu^* \) as described in Theorem 3.1. For \( w, w' \in \dot{H}^a(\mathbb{R}_+^n) \), there holds

\[
(5.4) \quad \langle r^+ Lw, w' \rangle_{\dot{H}^{-a}(\mathbb{R}_+^n), \dot{H}^a(\mathbb{R}_+^n)} - \langle w, r^+ L^* w' \rangle_{\dot{H}^a(\mathbb{R}_+^n), \dot{H}^{-a}(\mathbb{R}_+^n)} = 0.
\]

In particular,

\[
(5.5) \quad \int_{\mathbb{R}_+^n} Lw \, w' \, dx - \int_{\mathbb{R}_+^n} w \, L^* w' \, dx = 0
\]

holds when \( w \in \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \), \( w' \in \mathcal{E}_{\mu^*}(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \).

Proof. For \( w, w' \in H^a(\mathbb{R}^n) \) we have

\[
\langle Lw, w' \rangle_{H^{-a}(\mathbb{R}^n), H^a(\mathbb{R}^n)} - \langle w, L^* w' \rangle_{H^a(\mathbb{R}^n), H^{-a}(\mathbb{R}^n)} = 0,
\]

since \( L \) and its adjoint \( L^* \) are bounded from \( H^a(\mathbb{R}^n) \) to \( H^{-a}(\mathbb{R}^n) \) (having \( x \)-independent symbols that are \( O(\|\xi\|^{2a}) \)). When moreover \( w, w' \in \dot{H}^a(\mathbb{R}_+^n) \),

\[
\langle Lw, w' \rangle_{H^{-a}(\mathbb{R}^n), H^a(\mathbb{R}^n)} = \langle r^+ Lw, w' \rangle_{\dot{H}^{-a}(\mathbb{R}_+^n), \dot{H}^a(\mathbb{R}_+^n)},
\]

\[
\langle w, L^* w' \rangle_{H^a(\mathbb{R}^n), H^{-a}(\mathbb{R}^n)} = \langle w, r^+ L^* w' \rangle_{\dot{H}^a(\mathbb{R}_+^n), \dot{H}^{-a}(\mathbb{R}_+^n)};
\]

e.g. for the first expression, we can approximate \( w' \) in \( \dot{H}^a(\mathbb{R}_+^n) \) by functions \( \varphi \in C_0^\infty(\mathbb{R}_+^n) \), where it clearly holds. Thus (5.4) holds for \( w, w' \in \dot{H}^a(\mathbb{R}_+^n) \).

According to [G15, Prop. 4.1], \( \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \subset H^{\mu(s)}(\mathbb{R}_+^n) \) for any \( s > \mu - \frac{1}{2} \); here \( H^{\mu(s)}(\mathbb{R}_+^n) = \dot{H}^s(\mathbb{R}_+^n) \) if \( s \in [\mu - \frac{1}{2}, \mu + \frac{1}{2}] \) (cf. [G15, Def. 1.4 ff. or Th. 5.4]). Since \( \mu = a + \delta \) with \( |\delta| < \frac{1}{2} \), the value \( s = a \) satisfies

\[
\mu - \frac{1}{2} = a + \delta - \frac{1}{2} < a < a + \delta + \frac{1}{2} = \mu + \frac{1}{2},
\]

so \( H^{\mu(a)}(\mathbb{R}_+^n) = \dot{H}^a(\mathbb{R}_+^n) \). Thus \( \mathcal{E}_{\mu(a)}(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \subset \dot{H}^a(\mathbb{R}_+^n) \). Similarly, since \( \mu^* = a - \delta \) with \( |\delta| < \frac{1}{2} \), \( \mathcal{E}_{\mu^*}(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \subset \dot{H}^a(\mathbb{R}_+^n) \). Then formula (5.4) is valid for such functions, where it may be written in terms of integrals (5.5).

\[\square\]

Proof of Theorem 5.1. Use Lemma 5.2 to write the given \( u \) as in (5.3), \( u = \partial_n U + u_1 \).

The contribution from \( u_1 \) is dealt with by Theorem 5.3, showing that

\[
(5.6) \quad \int_{\mathbb{R}_+^n} Lu_1 \, v \, dx - \int_{\mathbb{R}_+^n} u_1 \, L^* v \, dx = 0.
\]
For the contribution from $\partial_n U$, we note that, writing $U = x_n^\mu w$ for $x_n > 0$, $w \in C^\infty(\mathbb{R}_+^n)$,

$$\partial_n U = \partial_n (x_n^\mu w) = \mu x_n^{\mu-1} w + x_n^\mu \partial_n w$$

for $x_n > 0$,

so the weighted boundary value for $x_n \to 0^+$ satisfies

$$\gamma_0(\partial_n U/x_n^{\mu-1}) = \mu \gamma_0 w = \mu \gamma_0 (U/x_n^\mu).$$

Now by Theorem 4.3 applied to $U$ and $v$,

$$\Gamma(\mu + 1)\Gamma(\mu^* + 1) \int_{\mathbb{R}_+^{n-1}} s_0 \gamma_0(U/x_n^\mu) \gamma_0(v/x_n^{\mu^*}) \, dx' = \int_{\mathbb{R}_+^n} LU \partial_n \bar{v} \, dx + \int_{\mathbb{R}_+^n} \partial_n U \overline{L*v} \, dx$$

$$= -\int_{\mathbb{R}_+^n} \partial_n (LU) \bar{v} \, dx + \int_{\mathbb{R}_+^n} \partial_n U \overline{L*v} \, dx = -\int_{\mathbb{R}_+^n} L(\partial_n U) \bar{v} \, dx + \int_{\mathbb{R}_+^n} \partial_n U \overline{L*v} \, dx;$$

giving

$$\int_{\mathbb{R}_+^n} L(\partial_n U) \bar{v} \, dx - \int_{\mathbb{R}_+^n} \partial_n U \overline{L*v} \, dx = \Gamma(\mu)\Gamma(\mu^* + 1) \int_{\mathbb{R}_+^{n-1}} s_0 \gamma_0(\partial_n U/x_n^{\mu-1}) \gamma_0(v/x_n^{\mu^*}) \, dx'.$$

Adding (5.6) to this, we find the desired formula (5.2).  \(\square\)

6. The full Green’s formula.

The formulas in Theorem 4.1 and 5.3 were shown by very basic tools in Fourier theory and distribution theory, and are surprising by containing only local terms in the integrals over the boundary, both in the sense that the trace operators $\gamma_0^{\mu-1}$ and $\gamma_0^\mu$ are local, and that they enter with the local coefficient $s_0$. For completeness, we shall now show a full Green’s formula (of which they are corollaries). Here we need to draw on some further tools from the Boutet de Monvel calculus, and the resulting integrals over the boundary contain local trace operators but also a nonlocal coefficient in the form of a first-order pseudodifferential operator over the boundary. The proof follows the scheme worked out in [G18] for the case $\mu = a$, of course with simplifications due to considering only a model problem.

Recall the notation from [G15] and [G18, Sect. 3]¹: When $u \in e^x x_n^{\mu-1} S(\mathbb{R}_+^n)$ with $\mu > 0$, we have an expansion at the boundary defined by Taylor expanding $w = u/x_n^{\mu-1}$ for $x_n \geq 0$ and renormalizing the coefficients:

$$(6.1) \quad u(x) = u_0(x')I^{\mu-1}(x_n) + u_1(x')I^\mu(x_n) + \cdots + u_k(x')I^{\mu-1+k}(x_n) + O(x_n^{\mu+k}),$$

where $I^\mu(x_n)$ is defined as in [G15]:

$$(6.2) \quad I^\mu(x_n) = H(x_n)x_n^\mu/\Gamma(\mu + 1) \quad \text{when Re} \mu > -1, \text{here } \partial_{x_n} I^{\mu+1} = I^\mu;$$

¹In [G18, Sect. 3], the indication $e^x S(\mathbb{R}_+^n)$ should be replaced by $e^x x_n^{\alpha-1} S(\mathbb{R}_+^n)$ in the occurrences on page 756, 757 and 758.
$H$ being the Heaviside function $1_{\{x_n \geq 0\}}$. The Gamma factor serves to normalize $I^\mu$ so that $I^\mu = \partial_{x_n} I^{\mu+1}$; this formula is also used to define the distribution $I^\mu$ for lower Re $\mu$.

The coefficients $u_k(x') \in \mathcal{S}(\mathbb{R}^{n-1})$ in (6.1) are denoted $\gamma_k^{\mu-1} u$; here

$$
\gamma_k^{\mu-1} u = \Gamma(\mu + k) \gamma_k(u/x_n^{\mu-1}), \quad \text{in particular}
$$

$$
\gamma_0^{\mu-1} u = u_0 = \Gamma(\mu) \gamma_0(u/x_n^{\mu-1}),
\gamma_1^{\mu-1} u = u_1 = \Gamma(\mu + 1) \gamma_1(u/x_n^{\mu-1}),
$$

where $\gamma_k w = \lim_{x_n \to 0^+} \partial^n_k w$. The cases $k = 0$ and 1 are viewed as the Dirichlet resp. Neumann traces of $u \in \mathcal{E}_{\mu-1}(\mathbb{R}_+^n)$. It is useful to involve also another expansion obtained by Taylor expanding $x_n^{-\mu} e^{(\xi^0) x_n} \mathcal{F}_{x' \to \xi^0} u$; here the partially Fourier transformed terms have a factor $e^{-\sigma x_n}$, $\sigma = \langle \xi^0 \rangle$:

$$
\mathcal{F}_{x' \to \xi^0} u \equiv \hat{u}(\xi^0, x_n) = \hat{\phi}_0(\xi^0) I^{\mu-1}(x_n) e^{-\sigma x_n} + \hat{u}'(\xi^0, x_n)
$$

$$
= \hat{\phi}_0(\xi^0) I^{\mu-1}(x_n) e^{-\sigma x_n} + \hat{\phi}_1(\xi^0) I^{\mu}(x_n) e^{-\sigma x_n} + \hat{u}''(\xi^0, x_n)
$$

$$
\sim \sum_{k \geq 0} \hat{\phi}_k(\xi^0) I^{\mu+1+k}(x_n) e^{-\sigma x_n},
$$

where the $\varphi_k$ are in $\mathcal{S}(\mathbb{R}^{n-1})$. Furthermore,

$$
\mathcal{F} u \sim \sum_{k \geq 0} \hat{\varphi}_k(\xi^0)(\sigma + i \xi_n)^{-\mu-k},
$$

in view of the formula

$$
\mathcal{F}_{x_n \to \xi_n} [I^\mu(x_n) e^{-\sigma x_n}] = (\sigma + i \xi_n)^{-\mu-1}.
$$

(The two expansions (6.1) and (6.5) are examined in [G15, Sect. 5] as a tool for the study of nonhomogeneous boundary problems.) By the expansion (6.4), we can write

$$
u = U_0 + u' = U_0 + U_1 + u'' \sim \sum_{k \geq 0} U_k, \text{ with}
$$

$$
U_k = \mathcal{F}_{\xi^0 \to x'}^{\mu-1} [\hat{\varphi}_k(\xi^0) I^{\mu+1+k}(x_n) e^{-\sigma x_n}] = \mathcal{F}^{-1} [\hat{\varphi}_k(\xi^0)(\sigma + i \xi_n)^{-\mu-k}].
$$

Since $\mathcal{F}_{\xi^0 \to x'}^{\mu-1} [\hat{\varphi}_k(\xi^0) H(x_n) e^{-\langle \xi^0 \rangle x_n}] \in e^+ \mathcal{S}(\mathbb{R}_+^n)$, one has that $U_0 \in e^+ x_n^{\mu-1} \mathcal{S}(\mathbb{R}_+^n)$, $U_1 \in e^+ x_n^{\mu} \mathcal{S}(\mathbb{R}_+^n)$, and $U_k \in e^+ x_n^{\mu+k-1} \mathcal{S}(\mathbb{R}_+^n)$ in general.

There is a one-to-one correspondence between the coefficient sets $\{u_0, \ldots, u_k\}$ and $\{\varphi_0, \ldots, \varphi_k\}$ for any $k$, that follows by comparing the Taylor expansions of $\hat{u} = \mathcal{F}_{x' \to \xi^0}(u/x_n^{\mu-1})$ and $\hat{w}_e = \mathcal{F}_{x' \to \xi^0}(e^{\sigma x_n} u/x_n^{\mu-1})$. We just need the transition formula for the first two coefficients, namely, when the $I^\mu$-factors are taken into account:

$$
\varphi_0 = \Gamma(\mu) \gamma_0 \hat{w}_e = \Gamma(\mu) \gamma_0 \hat{u} = \hat{u}_0,
$$

$$
\varphi_1 = \Gamma(\mu + 1) \gamma_0 (\partial_n \hat{w}_e) = \Gamma(\mu + 1) \gamma_0 (\sigma \hat{w}_e + e^{\sigma x_n} \partial_n \hat{w})
$$

$$
= \Gamma(\mu + 1) \Gamma(\mu)^{-1} \sigma \hat{u}_0 + \Gamma(\mu + 1) \gamma_1 \hat{u} = \mu \sigma \hat{u}_0 + \hat{u}_1;
hence with \( \langle D' \rangle = \text{OP}(\langle \xi' \rangle) \),

\[
(6.8) \quad \varphi_0 = u_0, \quad \varphi_1 = u_1 + \mu \langle D' \rangle u_0.
\]

Note that since \( \gamma_0^{\mu-1}U_0 = \varphi_0 = u_0 = \gamma_0^{\mu-1}u \), the first trace of \( u' = u - U_0 \) is zero, so \( u' \in e^+x_n^\mu \mathcal{S}(\mathbb{R}_+^n) \). Similarly, having the two first traces equal to zero, \( u'' = u - U_0 - U_1 \) is in \( e^+x_n^{\mu+1} \mathcal{S}(\mathbb{R}_+^n) \). In general, \( u^{(k)} = u - (U_0 + \cdots + U_{k-1}) \) is in \( e^+x_n^{\mu+k-1} \mathcal{S}(\mathbb{R}_+^n) \).

We also observe:

\[
(6.9) \quad \gamma_0^{\mu-1}u = \gamma_0 \Xi^{\mu-1}u.
\]

This follows since

\[
\gamma_0 \Xi_+^{\mu-1}u = \gamma_0 \Xi_+^{\mu-1}(U_0 + \cdots + U_k + u^{(k)}),
\]

where

\[
\gamma_0 \Xi_+^{\mu-1}U_j = \gamma_0 \mathcal{F}^{-1}((\sigma + i\xi_n)^{\mu-1}\hat{\varphi}_j(\sigma + i\xi_n)^{-\mu-j}) = \gamma_0 \mathcal{F}^{-1}(\hat{\varphi}_j(\sigma + i\xi_n)^{-1-j})
\]

if \( j = 0 \), \( 0 \) if \( j > 0 \),

\[
\Xi_+^{\mu-1}u^{(k)} \in \Xi_+^{\mu-1}H^{k-2}(\mathbb{R}_+^n) = H^{k-1-\mu}(\mathbb{R}_+^n)
\]

for large \( k \); then \( \gamma_0 \Xi_+^{\mu-1}u^{(k)} = 0 \).

Note also:

\[
\Xi_+^\mu U_k = \mathcal{F}^{-1}[(\sigma + i\xi_n)^{\mu}\hat{\varphi}_k(\xi') \sigma(\sigma+i\xi_n)^{-\mu-k}]
\]

\[
= \mathcal{F}^{-1}_{\xi' \to x'}[\hat{\varphi}_k(\xi')I^{k-1}e^{-\sigma x_n}] \text{ for } k \in \mathbb{N}_0, \text{ in particular,}
\]

\[
\Xi_+^\mu U_0 = \mathcal{F}^{-1}[(\sigma + i\xi_n)^{\mu}\hat{\varphi}_0(\xi')(\sigma + i\xi_n)^{-\mu} = \varphi_0(x') \otimes \delta(x_n),
\]

\[
\Xi_+^\mu U_1 = \mathcal{F}^{-1}[(\sigma + i\xi_n)^{\mu}\hat{\varphi}_1(\xi')(\sigma + i\xi_n)^{-\mu-1} = \mathcal{F}^{-1}_{\xi' \to x'}[\hat{\varphi}_1(\xi')He^{-\sigma x_n}].
\]

One more conclusion will be drawn; we here refer to the space \( H^{(\mu-1)(s)}(\mathbb{R}_+^n) \) recalled in Section 7, defined as when \( s > \mu - \frac{1}{2} \).

**Lemma 6.1.** For \( \mu > 0 \), \( e^+x_n^{\mu-1} \mathcal{S}(\mathbb{R}_+^n) = \Xi_+^{\mu+1}e^+\mathcal{S}(\mathbb{R}_+^n) \).

**Proof.** We have seen above that when \( u \in e^+x_n^{\mu-1} \mathcal{S}(\mathbb{R}_+^n) \), then for any \( k \in \mathbb{N}_0, u = U_0 + \cdots + U_k + u^{(k)} \), where the \( U_j \) are defined in (6.7), and \( u^{(k)} \in e^+x_n^{\mu-1+k} \mathcal{S}(\mathbb{R}_+^n) \).

Here

\[
U_j = \mathcal{F}^{-1}[\hat{\varphi}_j(\sigma + i\xi_n)^{-\mu-j}] = \Xi_+^{\mu+1} \mathcal{F}^{-1}(\hat{\varphi}_j(\sigma + i\xi_n)^{-j-1}) \in \Xi_+^{\mu+1}e^+ \mathcal{S}(\mathbb{R}_+^n)
\]

for all \( j \). The remainder \( u^{(k)} \) is in \( e^+x_n^{\mu-1+k} \mathcal{S}(\mathbb{R}_+^n) \), which for \( k \to \infty \) converges to \( \mathcal{S}(\mathbb{R}_+^n) = \{ v \in \mathcal{S}(\mathbb{R}^n) \mid \text{supp } v \subset \mathbb{R}_+^n \} \), where all boundary values at \( x_n = 0 \) vanish. Here \( \Xi_+^{\mu+1} \mathcal{S}(\mathbb{R}_+^n) = \mathcal{S}(\mathbb{R}_+^n) \), since \( \Xi_+^{\mu+1} \) is a homeomorphism on \( \mathcal{S}(\mathbb{R}^n) \), and preserves support in \( \mathbb{R}_+^n \), and so does its inverse \( \Xi_+^{\mu-1} \). Thus letting \( k \to \infty \) in the decomposition, we find the inclusion \( e^+x_n^{\mu-1} \mathcal{S}(\mathbb{R}_+^n) \subset \Xi_+^{\mu+1}e^+ \mathcal{S}(\mathbb{R}_+^n) \).

The opposite inclusion follows in a similar way: For \( u \) given in \( \Xi_+^{\mu+1}e^+ \mathcal{S}(\mathbb{R}_+^n) \), let \( v \in e^+ \mathcal{S}(\mathbb{R}_+^n) \) be such that \( u = \Xi_+^{\mu+1}v \). By a Taylor expansion in \( x_n \) of \( \mathcal{F}_{x' \to x'}(e^{\sigma x_n}v(x', x_n)) \), we find an expansion of \( \hat{\varphi} = \mathcal{F}_{x' \to x'}v \) in terms \( I^j(x_n)\hat{\varphi}_j(\xi')e^{-\sigma x_n} \), so that an application
of $\Xi_{-}^{-\mu+1}$ to the corresponding term in $v$ gives $U_j = \mathcal{F}^{-1}[\hat{\varphi}_j(\sigma + i\xi_n)^{-\mu-j}]$. As we know from the preceding analysis, these $U_j$ are also in $e^{+x_n^{\mu+1}}S(\mathbb{R}^n_+)$. The remainder in $v$ after $k$ terms is in $e^{+x_n^{\mu+1}}S(\mathbb{R}^n_+)$, which for $k \to \infty$ converges to $\hat{\mathcal{S}}(\mathbb{R}^n_+)$. $\Xi_{-}^{-\mu+1}$ can be applied along the way, and in the limit we use that $\Xi_{-}^{-\mu+1}\hat{\mathcal{S}}(\mathbb{R}^n_+) = \hat{\mathcal{S}}(\mathbb{R}^n_+)$, to reach the conclusion.  

All the above is a straightforward use of the Fourier transform and distribution theory. Further below, we shall moreover need to apply some basic constant-coefficient rules from the Boutet de Monvel calculus. They are covered e.g. by the Appendix of [G18], which we shall not repeat here.

Including that terminology, we observe that there are several important descriptions of $U_0$ (recall that $\varphi_0 = u_0$):

$$U_0 = \mathcal{F}^{-1}_{\xi \to x} [\hat{u}_0(\xi')I^{\mu-1}(x_n)e^{-\sigma x_n}] = \mathcal{F}^{-1}_{\xi \to x} [\hat{u}_0(\xi')(\sigma + i\xi_n)^{-\mu}]$$

(6.11)

$$U_0 = \Xi_{+}^{\mu-1} \mathcal{F}^{-1}_{\xi \to x} [\hat{u}_0(\xi')(\sigma + i\xi_n)^{-1}] = \Xi_{+}^{1-\mu} e^\mu K_0 u_0,$$

$$U_0 = I^{\mu-1}(x_n)\mathcal{F}^{-1}_{\xi \to x} [\hat{u}_0(\xi')e^{-\sigma x_n}] = I^{\mu-1}(x_n)e^\mu K_0 u_0 = \frac{1}{\Gamma(\mu)} x_n^{\mu-1} e^\mu K_0 u_0,$$

where $K_0$ is the well-known Poisson operator $K_0 \varphi = \mathcal{F}^{-1}_{\xi \to x} [\hat{\varphi}(\xi')r^+ e^{-\langle \xi', x \rangle}]$, defining a right-inverse of $\gamma_0$ satisfying $(1 - \Delta)^\mu K_0 = 0$ on $\mathbb{R}^n_+$; its symbol is $(\langle \xi' \rangle + i\xi_n)^{-1}$. There is also defined a related right-inverse of $\gamma_0^{-1}$, namely the operator

$$K_0^{\mu-1} = \Xi_{+}^{1-\mu} e^\mu K_0;$$

it is a Poisson-like operator, solving the nonhomogeneous Dirichlet problem for $(1 - \Delta)^\mu$ with zero interior data. Moreover, in view of (6.11),

$$K_0^{\mu-1} = I^{\mu-1}(x_n)e^\mu K_0 = \frac{1}{\Gamma(\mu)} x_n^{\mu-1} e^\mu K_0,$$

and $U_0 = K_0^{\mu-1} u_0$. There is a detailed study of the role of $K_0^{\mu-1}$ and higher-order Poisson-like operators in [G19].

Now consider our operator $L$, written as $\Xi_{+}^\mu Q \Xi_{+}^\mu$ as in Section 4 (disregarding a smoothing term that does not contribute to the formula). We shall show a Green’s formula that allows writing $\langle r^+ L u, v \rangle - \langle u, r^+ L^* v \rangle$ as a boundary integral when both $u$ and $v$ have non-trivial Dirichlet and Neumann traces. The boundary terms are local, except for a term involving the Dirichlet traces of $u$ and $v$ with a pseudodifferential coefficient $(\mu - \mu^*) \langle D' \rangle + B$.

**Theorem 6.2.** Let $L$, $a$, $\delta$, $\mu$, $\mu^*$ and $s_0$ be as described in Theorems 3.1 and 4.3, and assume that $\mu, \mu^* > 0$. For $u \in \mathcal{E}_{\mu-1}(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n)$, $v \in \mathcal{E}_{\mu^*-1}(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n)$, there holds:

$$\langle r^+ L u, v \rangle = \langle \frac{u}{\pi^{\mu-\frac{1}{2}+\epsilon}} H_{\mu-\frac{1}{2}-\epsilon}, \frac{v}{\pi^{\mu-\frac{1}{2}+\epsilon}} \rangle - \langle u, r^+ L^* v \rangle$$

(6.14)

$$= \langle s_0\gamma_0^{\mu^{-1}} u, \gamma_0^{\mu^{-1}} v \rangle - \langle s_0\gamma_0^{\mu^{-1}} u, \gamma_1^{\mu^{-1}} v \rangle + \langle [s_0(\mu - \mu^*) \langle D' \rangle + B] \gamma_0^{\mu^{-1}} u, \gamma_0^{\mu^{-1}} v \rangle,$$

with $L_2(\mathbb{R}^{n-1})$-scalar products in the right-hand side; it may also be written

$$\int_{\mathbb{R}^n_+} L u \bar{v} \, dx - \int_{\mathbb{R}^n_+} u \overline{L^* v} \, dx$$

(6.15)

$$= \int_{\mathbb{R}^{n-1}} \left( s_0\Gamma(\mu + 1) \Gamma(\mu^*) \gamma_1 \left( \frac{u}{x^{\mu^*_0}} \right) \gamma_0 \left( \frac{\bar{v}}{x^{\mu^*_0}} \right) - s_0\Gamma(\mu) \Gamma(\mu^* + 1) \gamma_0 \left( \frac{u}{x^{\mu^*_0}} \right) \gamma_1 \left( \frac{\bar{v}}{x^{\mu^*_0}} \right) + [s_0(\mu - \mu^*) \langle D' \rangle + B] \Gamma(\mu) \Gamma(\mu^*) \gamma_0 \left( \frac{u}{x^{\mu^*_0}} \right) \gamma_0 \left( \frac{\bar{v}}{x^{\mu^*_0}} \right) \right) \, dx'.$$
Here, with

\[ q(\xi) = (\langle \xi' \rangle - i\xi_n)^{-\mu} \mathcal{L}(\xi)(\langle \xi' \rangle + i\xi_n)^{-\mu}, \]

the first-order pseudodifferential operator \( B \) on \( \mathbb{R}^{n-1} \) equals \( \text{OP}(b(\xi')) \), where \( b(\xi') \) is the jump at \( x_n = 0 \) of \( \mathcal{F}^{-1}_{x_n \to x_n}(q - s_0) \).

**Remark 6.3.** It is also shown in the proof, using terminology from the Boutet de Monvel calculus, that when \( q \) is decomposed as \( q(\xi) = s_0 + f_+(\xi) + f_-(\xi) \) as in (4.14), \( f_+ = h^+q \) and \( f_- = h^-q \), then

\[ b(\xi') = b(\xi') - \overline{b'}(\xi'), \quad \text{where } b = \frac{1}{2\pi} \int^+ f_+ d\xi_n, \quad b' = \frac{1}{2\pi} \int^+ f_- d\xi_n. \]

**Proof of Theorem 6.2.** Take \( u \in \mathcal{E}_{\mu-1}(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n) \), \( v \in \mathcal{E}_{\mu^* - 1}(\mathbb{R}^n_+) \cap \mathcal{E}'(\mathbb{R}^n) \), and write them as

\[ u = u' + U_0 \quad \text{with} \quad U_0 = K_0^{\mu-1}u_0, \quad v = v' + V_0 \quad \text{with} \quad V_0 = K_0^{\mu^* - 1}v_0, \]

according to the above notation with \( \mu \) for \( u, \mu^* \) for \( v \). Here \( U_0 = K_0^{\mu-1}u_0 \in \Xi_{\mu} \in \mathcal{E}(\mathbb{R}^n_+) \)

\[ = e^+ x_n^{\mu-1}S(\mathbb{R}^n_+) \quad (\text{cf. Lemma 6.1}), \]

whereas \( u' \) lies in the smaller space \( e^+ x_n^{\mu^*}S(\mathbb{R}^n_+) = \Xi_{\mu^*} \in \mathcal{E}(\mathbb{R}^n_+) \), as accounted for before the theorem.

Since \( v \in \Xi_{\mu^*} e^+S(\mathbb{R}^n_+) \subset \Xi_{\mu^*} \quad \text{and} \quad r^+Lu \in C^\infty(\mathbb{R}^n_+) \cap \bigcap_{\eta} \mathcal{H}^\infty(\mathbb{R}^n_+) \), the scalar product of \( r^+Lu \) and \( v \) can be interpreted as the duality

\[ I = \langle r^+ Lu, v \rangle_{\Pi^{1/2}+\mu^*+\varepsilon(\mathbb{R}^n_+), H^{1/2+\varepsilon}(\mathbb{R}^n_+)}, \]

as in the proof of Theorem 4.1 above. It splits into four pieces:

\[ I = I_1 + I_2 + I_3 + I_4, \]

\[ I_1 = \langle r^+ Lu', v' \rangle, \quad I_2 = \langle r^+LK_0^{\mu-1}u_0, v' \rangle, \]

\[ I_3 = \langle r^+ Lu', K_0^{\mu^* - 1}v_0 \rangle, \quad I_4 = \langle r^+LK_0^{\mu^* - 1}u_0, K_0^{\mu^* - 1}v_0 \rangle. \]

\( I_1 \) will be kept unchanged, to match a similar term with \( L^* \) later.

For \( I_2 \) we observe, using (6.10), (6.11) and the representation \( Q = \text{OP}(q), q = s_0 + f_+ + f_- \) from (4.14):

\[ I_2 = \langle r^+LK_0^{\mu-1}u_0, v' \rangle_{\Pi^{1/2+\varepsilon, H^{1/2-\varepsilon}}, H^{1/2-\varepsilon}} \]

\[ = \langle \Xi_{\mu^*} r^+Q \Xi_{\mu^*} e^+K_0^{\mu-1}u_0, v' \rangle_{\Pi^{1/2+\varepsilon, H^{1/2-\varepsilon}}} \]

\[ = \langle r^+Q(u_0(x' \otimes \delta(x_n))), \Xi_{\mu^*} v' \rangle_{\Pi^{1/2+\varepsilon, H^{1/2-\varepsilon}}} \]

\[ = \langle r^+ \text{OPK}(f_+u_0, \Xi_{\mu^*} v')_{\Pi^{1/2+\varepsilon, H^{1/2-\varepsilon}}}. \]
The last equality came from the rule from the Boutet de Monvel calculus that the mapping $u_0 \mapsto r^+ Q(u_0(x') \otimes \delta(x_n))$ is the Poisson operator with symbol $h^+ q(\xi) = f_+(\xi)$; the contributions from $s_0$ and $f_-(\xi)$ vanish since they map into distributions supported in $\mathbb{R}_-^n$.

Next, consider $I_3$. Here, by (6.12) and (4.2),

$$I_3 = \langle r^+ Lu', K_0^{\mu^*-1} v_0 \rangle = \langle \Xi_{\mu^*}^{-1} r^+ Q \Xi_{\mu^*}^{1} u', \Xi_{\mu^*} e^+ K_0 v_0 \rangle_{\mathcal{F}_{\mu^*-\frac{1}{2} + \epsilon}, \mathcal{H}_{\mu^*-\frac{1}{2} - \epsilon}}$$

(6.21)

$$= \langle \Xi_{\mu^*}^{-1} \Xi_{\mu^*} r^+ Q \Xi_{\mu^*}^{1} u', K_0 v_0 \rangle_{\mathcal{F}_{\frac{1}{2} + \epsilon}, \mathcal{H}_{\frac{1}{2} - \epsilon}}$$

$$= \langle K_0^{\mu^*} Q \Xi_{\mu^*}^{1} u', v_0 \rangle_{H^e(\mathbb{R}^n), H^{-e}(\mathbb{R}^n)} = (K_0^{\mu^*} Q \Xi_{\mu^*}^{1} u', v_0)_{L_2(\mathbb{R}^n)},$$

since $v_0$ is in $L_2(\mathbb{R}^n)$. It is used that $\mathcal{F}$ identifies with $\mathcal{F}$ for $|t| < \frac{1}{2}$ (there the indication $e^+$ is understood).

Denote $\Xi_{\mu^*} u' = w \in e^+ \mathcal{F}_{\frac{1}{2} + \epsilon}(\mathbb{R}_+^n)$. Since $K_0^{\mu^*}$ is the trace operator with symbol $(\langle \xi' \rangle - i\xi_n)^{-1}$, the rules of calculus give that

$$K_0^{\mu^*} r^+ \Xi_{\mu^*}^{1} Q w = \text{OPT}(h^-(\langle \xi' \rangle - i\xi_n)^{-1}(\langle \xi' \rangle - i\xi_n) q(\xi)) w$$

$$= \text{OPT}(h^- q(\xi)) w = \text{OPT}(s_0 + f_-(\xi)) w,$$

and hence, in view of (6.9), (6.10),

$$I_3 = (\text{OPT}(s_0 + f_-) \Xi_{\mu^*}^{1} u', v_0)_{L_2(\mathbb{R}^n)} = ((s_0 \gamma_0 + \text{OPT}(f_-)) \Xi_{\mu^*}^{1} u', v_0)_{L_2(\mathbb{R}^n)}$$

(6.22)

$$= (s_0 \gamma_0 u', v_0)_{L_2(\mathbb{R}^n)} + (\text{OPT}(f_-) \Xi_{\mu^*}^{1} u', v_0)_{L_2(\mathbb{R}^n)}$$

$$= (s_0 (\gamma_1 u + \mu \langle D' \rangle \gamma_0 \mu^{-1} u, v_0)_{L_2(\mathbb{R}^n)} + (\text{OPT}(f_-) \Xi_{\mu^*}^{1} u', v_0)_{L_2(\mathbb{R}^n)}.$$

Finally, consider $I_4$. Beginning as in the treatment of $I_3$, we find:

$$I_4 = \langle r^+ LK_0^{\mu^*-1} u_0, K_0^{\mu^*-1} v_0 \rangle_{H^e(\mathbb{R}^n), H^{-e}(\mathbb{R}^n)}$$

(6.23)

$$= \langle \Xi_{\mu^*}^{-1} r^+ Q \Xi_{\mu^*}^{1} e^+ K_0 u_0, \Xi_{\mu^*} e^+ K_0 v_0 \rangle_{H^e(\mathbb{R}^n), H^{-e}(\mathbb{R}^n)}$$

$$= (K_0^{\mu^*} \Xi_{\mu^*}^{1} e^+ K_0 u_0, v_0)_{L_2(\mathbb{R}^n)},$$

where $B = K_0^{\mu^*} \Xi_{\mu^*}^{1} e^+ K_0$ is a certain $\psi$do on $\mathbb{R}^{n-1}$ of order 1. We can reduce this expression by rules of calculus involving the so-called plus-integral, cf. [G18,(A.14)ff., (A.15)]. Since $K_0$ has symbol $(\langle \xi' \rangle + i\xi_n)^{-1}$, the symbol of the Poisson operator $r^+ Q \Xi_{\mu^*}^{1} e^+ K_0$ is $h^+ q = f_+$, which by composition with $\Xi_{\mu^*}^{1}$ to the left gives a Poisson operator with symbol $h^+((\langle \xi' \rangle - i\xi_n) f_+)$. The symbol $b(\xi')$ of $B$ is calculated as in [G18,(4.15)]:

$$b(\xi') = \frac{1}{2\pi} \int_{1}^{+} ((\langle \xi' \rangle - i\xi_n)^{-1} h^+((\langle \xi' \rangle - i\xi_n) f_+)) d\xi_n$$

(6.24)

$$= \frac{1}{2\pi} \int_{1}^{+} f_+ d\xi_n = \lim_{z_n \to 0^+} \bar{q}(\xi', z_n),$$
where \( \bar{q}(\xi', z_n) \) stands for \( \mathcal{F}_{\xi_n \to z_n}^{-1} q(\xi) \).

The same arguments apply to \( I' = \langle u, r^+ L^* v \rangle \), after a conjugation and an exchange of \( \mu, \mu^* \) by \( \mu^*, \mu \). This gives:

\[
I' = I'_1 + I'_2 + I'_3 + I'_4,
\]

where \( I'_1 = \langle u', r^+ P^* v' \rangle_{H^{\mu - \frac{1}{2}, -}, H^\mu_{\frac{1}{2} + \varepsilon}} \), \( I'_2 = \langle \Xi^+_{\mu'} u', \text{OPK}(\bar{f})v_0 \rangle_{H^{\mu - \frac{1}{2}, -}, H^\mu_{\frac{1}{2} + \varepsilon}} \), \( I'_3 = \langle s_0 u_0, \gamma_1^{\mu^* - 1} v + \mu^* \langle D' \gamma_0^{\mu^* - 1} v \rangle \rangle \), \( I'_4 = \langle u_0, B' v_0 \rangle_{L_2(\mathbb{R}^{n-1})} \),

where \( B' = \text{OP}(b'(\xi)) \), with

\[
b'(\xi') = \frac{1}{2\pi} \int_0^\pi h^+ \bar{q}(\xi) \, d\xi_n = \frac{1}{2\pi} \int_0^\pi \bar{f} \, d\xi_n, \quad \bar{f} = \lim_{z_n \to 0^-} \bar{q}(\xi', z_n).
\]

Now to calculate \( I - I' \), we first find

\[
I_1 - I'_1 = 0
\]

by Theorem 5.3, which clearly covers \( u' \in e^+ x_n^\mu S(\mathbb{R}^n_+) \), \( v' \in e^+ x_n^{\mu^*} S(\mathbb{R}^n_+) \). Next,

\[
I_2 - I'_3 = \langle \text{OPK}(f_+) u_0, \Xi^+_{\mu'} v' \rangle - \langle s_0 u_0, \gamma_1^{\mu^* - 1} v + \mu^* \langle D' \gamma_0^{\mu^* - 1} v \rangle \rangle = \langle \text{OPK}(f_+) u_0, \Xi^+_{\mu'} v' \rangle - \langle \text{OPK}(f_+) u_0, \Xi^+_{\mu'} v' \rangle - \langle s_0 u_0, \gamma_1^{\mu^* - 1} v + \mu^* \langle D' \gamma_0^{\mu^* - 1} v \rangle \rangle = -\langle s_0 u_0, \gamma_1^{\mu^* - 1} v + \mu^* \langle D' \gamma_0^{\mu^* - 1} v \rangle \rangle,
\]

using that the adjoint of the trace operator \( \text{OPT}(\bar{f}_+) \) is the Poisson operator \( \text{OPK}(f_+) \).

There is a similar calculation of \( I_3 - I'_2 \), so we find

\[
I_2 + I_3 - I'_2 - I'_3 = \langle s_0 \gamma_1^{\mu - 1} u, \gamma_0^{\mu^* - 1} v \rangle - \langle s_0 \gamma_1^{\mu - 1} u, \gamma_1^{\mu^* - 1} v \rangle + \langle s_0 (\mu - \mu^*) \langle D' \gamma_0^{\mu^* - 1} u, \gamma_0^{\mu^* - 1} v \rangle \rangle.
\]

(Since \( u' \) and \( v' \) are solutions of homogeneous Dirichlet problems, whereas \( K_0^{\mu - 1} u_0 \) and \( K_0^{\mu^* - 1} v_0 \) solve nonhomogeneous Dirichlet problems, this could also have been derived using the halfways Green’s formula.) Finally,

\[
I_4 - I'_4 = \langle (B - B'^*) u_0, v_0 \rangle_{L_2(\mathbb{R}^{n-1})} = \langle (B - B'^*) \gamma_0^{\mu - 1} u, \gamma_0^{\mu^* - 1} v \rangle_{L_2(\mathbb{R}^{n-1})},
\]

where \( B = B - B'^* \) satisfies, with \( b(\xi') = b(\xi') - \bar{f}(\xi') \),

\[
b(\xi') = \lim_{z_n \to 0^+} \bar{q}(\xi', z_n) - \lim_{z_n \to 0^-} \bar{q}(\xi', z_n),
\]

the jump at \( z_n = 0 \) in the bounded part of \( \bar{q}(\xi', z_n) \). Adding the terms, we find the assertion in the theorem. \( \square \)
7. Results in other function spaces.

For the interested reader, we shall go rapidly through some results in function spaces of finite regularity, that hold along with the above statements shown for $C^\infty$-functions.

First we recall the definitions of the spaces; more details are given in [G15,G14].

The $\mu$-transmission spaces $H^{\mu(s)}(\mathbb{R}^n_+)$ are defined as $\Xi^{-\mu} e^+ \mathcal{H}^{s-\mu}(\mathbb{R}^n_+)$ for $s > \mu - \frac{1}{2}$, and satisfy when $\mu > -1$:

\[
H^{\mu(s)}(\mathbb{R}^n_+) = \begin{cases} \hat{H}^s(\mathbb{R}^n_+) & \text{if } s \in ]\mu - \frac{1}{2}, \mu + \frac{1}{2}[; \nonumber \\
\cap e^+ x_\mu' \mathcal{H}^{s-\mu}(\mathbb{R}^n_+) + \hat{H}^{s-\varepsilon}(\mathbb{R}^n_+) & \text{if } s > \mu + \frac{1}{2}; \nonumber 
\end{cases}
\]

where $-\varepsilon$ is active if $s - \mu - \frac{1}{2}$ is integer. Here the trace operator $\gamma_k^n$ maps $H^{\mu(s)}(\mathbb{R}^n_+)$ continuously into $H^{s-\mu-\frac{1}{2}}(\mathbb{R}^{n-1})$ when $s > \mu + k + \frac{1}{2}$; note that the range is in $L_2(\mathbb{R}^{n-1})$.

**Lemma 7.1.** $\mathcal{E}_\mu(\mathbb{R}^n_+) \cap\mathcal{E}'(\mathbb{R}^n)$ is dense in $H^{\mu(s)}(\mathbb{R}^n_+)$, for all $s > \mu - \frac{1}{2}$.

**Proof.** This is a corollary to Lemma 6.1, where the identity $e^+ x_\mu' S(\mathbb{R}^n_+) = \Xi^{-\mu} e^+ S(\mathbb{R}^n_+)$ for $\mu > -1$ was proved. Note that since $S(\mathbb{R}^n_+)$ is densely embedded in $\mathcal{H}^{s-\mu}(\mathbb{R}^n_+)$, $\Xi^{-\mu} e^+ S(\mathbb{R}^n_+) = e^+ x_\mu' S(\mathbb{R}^n_+)$ is densely embedded in $\Xi^{-\mu} e^+ H^{s-\mu}(\mathbb{R}^n_+) = H^{\mu(s)}(\mathbb{R}^n_+)$. When $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ for $|x| \leq 1$, then if $v \in e^+ x_\mu' S(\mathbb{R}^n_+)$, $\varphi(\delta x)v \to v$ in $e^+ x_\mu' S(\mathbb{R}^n_+)$ for $\delta \to 0$, and a fortiori in $H^{\mu(s)}(\mathbb{R}^n_+)$. For a $u \in H^{\mu(s)}(\mathbb{R}^n_+)$, one can for any $k \in \mathbb{N}$ find $u_k \in e^+ x_\mu' S(\mathbb{R}^n_+)$ such that $\|u - u_k\|_{H^{\mu(s)}} \leq 1/k$; then $\|u - \varphi(\delta x)u_k\|_{H^{\mu(s)}} \leq 2/k$ for $\delta$ sufficiently small, and $\varphi(\delta x)u_k \in \mathcal{E}_{\mu-1}(\mathbb{R}^n_+) \cap\mathcal{E}'(\mathbb{R}^n)$. \hfill \Box

**Lemma 7.2.** The mapping $\partial_n$ sends $H^{\mu(s)}(\mathbb{R}^n_+)$ continuously into $H^{(\mu-1)(s-1)}(\mathbb{R}^n_+)$ for all $\mu$, all $s > \mu - \frac{1}{2}$.

**Proof.** This follows from

\[
\partial_n H^{\mu(s)}(\mathbb{R}^n_+) = \partial_n \Xi^{-\mu} e^+ \mathcal{H}^{s-\mu}(\mathbb{R}^n_+) = (\partial_n + \langle D' \rangle - \langle D' \rangle) \Xi^{-\mu} e^+ \mathcal{H}^{s-\mu}(\mathbb{R}^n_+) \nonumber \\
\cap \Xi^{\mu+1} e^+ \mathcal{H}^{s-\mu}(\mathbb{R}^n_+) + \Xi^{\mu} e^+ \mathcal{H}^{s-1-\mu}(\mathbb{R}^n_+) \nonumber \\
= H^{(\mu-1)(s-1)}(\mathbb{R}^n_+) + H^{(\mu)(s-1)}(\mathbb{R}^n_+) = H^{(\mu-1)(s-1)}(\mathbb{R}^n_+), \nonumber 
\]

using that $\partial_n + \langle D' \rangle = \text{OP}(i\xi_n + \langle \xi' \rangle) = \Xi^1$. \hfill \Box

**Theorem 7.3.** For $s > \mu + \frac{1}{2}$, $s' > \mu^* + \frac{1}{2}$, Theorem 4.3 extends to $u \in H^{\mu(s)}(\mathbb{R}^n_+)$, $u' \in H^{\mu^*(s')}(\mathbb{R}^n_+)$, in the form

\[
\langle r^+ Lu, \partial_n u' \rangle_{\mathcal{H}^{-\mu^*+\frac{1}{2}+\varepsilon}(\mathbb{R}^{n-1} \cap\mathbb{R}^n_+), \mathcal{H}^{-\mu^*-\frac{1}{2}+\varepsilon}(\mathbb{R}^n_+) + \langle \partial_n u, L^* u' \rangle_{\mathcal{H}^{-\mu^*+\frac{1}{2}-\varepsilon}(\mathbb{R}^{n-1} \cap\mathbb{R}^n_+), \mathcal{H}^{-\mu^*-\frac{1}{2}+\varepsilon}(\mathbb{R}^n_+)} \nonumber \\
= \Gamma(\mu + 1) \Gamma(\mu^* + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_\mu')(\mu^*+1) dx'. \nonumber 
\]

**Proof.** When $u' \in H^{\mu^*(s')}(\mathbb{R}^n_+)$ with $s' > \mu^* + \frac{1}{2}$, then since (a fortiori) $s' - \mu^* > \frac{1}{2} - \varepsilon$, we have using Lemma 7.2:

\[
\partial_n u' \in H^{(\mu^*-1)(s'-1)}(\mathbb{R}^n_+) = \Xi^{-\mu^*+1} e^+ \mathcal{H}^{-\mu^*-\varepsilon}(\mathbb{R}^n_+) \subset \Xi^{-\mu^*+1} e^+ \mathcal{H}^{-\mu^*-\frac{1}{2}-\varepsilon}(\mathbb{R}^n_+) \nonumber \\
= \Xi^{-\mu^*+1} H^{-\mu^*+\frac{1}{2}-\varepsilon}(\mathbb{R}^n_+) = H^{\mu^*-\frac{1}{2}-\varepsilon}(\mathbb{R}^n_+). \nonumber 
\]
for small $\varepsilon > 0$. At the same time, when $u \in H^\mu(s)(\mathbb{R}_+^n)$ with $s > \mu + \frac{1}{2}$, then (by [G15, Th. 4.2])

$$r^+ Lu \in \mathcal{H}^{s-2a}_+(\mathbb{R}_+^n) \subset \mathcal{H}^{\mu+\frac{1}{2}-2a+\varepsilon}_+(\mathbb{R}_+^n) = \mathcal{H}^{-\mu-\frac{1}{2}+\varepsilon}_+(\mathbb{R}_+^n)$$

for small $\varepsilon > 0$. Thus the first duality in (7.2) has a sense for such $u, u'$.

Similarly, the second duality in (7.2) has a sense when $s > \mu + \frac{1}{2}$, $s' > \mu^* + \frac{1}{2}$.

In view of Lemma 7.1, we can choose sequences of approximating functions $(u_k)_{k \in \mathbb{N}} \subset \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}_+^n)$ converging to $u$, $(u'_k)_{k \in \mathbb{N}} \subset \mathcal{E}_{\mu^*}(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}_+^n)$ converging to $u'$, in the abovementioned norms. The formula (7.2) holds for the pair $\{u_k, u'_k\}$ by Theorem 4.3, each $k$, and converges to the formula for the given $u, u'$ when $k \to \infty$. \(\square\)

The halfways Green’s formula similarly extends:

**Theorem 7.4.** Let $\mu, \mu^* > 0$. For $s > \mu + \frac{1}{2}$, $s' > \mu^* + \frac{1}{2}$, Theorem 5.1 extends to $u \in H^{(\mu-1)(s)}(\mathbb{R}_+^n)$, $v \in H^{\mu^*(s')}(\mathbb{R}_+^n)$, in the form

$$
\langle r^+ Lu, v \rangle_{\mathcal{H}^{-\mu^*+\frac{1}{2}-2a+\varepsilon}_+(\mathbb{R}_+^n)} - \langle u, L^* v \rangle_{\mathcal{H}^{\mu+\frac{1}{2}-2a-\varepsilon}_+(\mathbb{R}_+^n)} = -\Gamma(\mu)\Gamma(\mu^* + 1) \int_{\mathbb{R}_+^{n-1}} s_0 \gamma_0(u/x_n^{\mu-1}) \gamma_0(\bar{v}/x_n^{\mu^*}) \, dx'.
$$

(7.3)

**Proof.** When $u \in H^{(\mu-1)(s)}(\mathbb{R}_+^n)$ with $s > \mu + \frac{1}{2}$, then since (a fortiori) $s - \mu + 1 > \frac{1}{2} - \varepsilon$, we have:

$$
u \in H^{(\mu-1)(s)}(\mathbb{R}_+^n) = \Xi_+^{\mu-1} e^{\mathcal{H}^s-\mu+1}_+(\mathbb{R}_+^n) \subset \Xi_+^{\mu+1} e^{\mathcal{H}^s-\mu+1}_+(\mathbb{R}_+^n)
$$

$$= \Xi_+^{\mu+1} \mathcal{H}^{\frac{1}{2}-\varepsilon}_+(\mathbb{R}_+^n) = \mathcal{H}^{\mu-\frac{1}{2}-\varepsilon}_+(\mathbb{R}_+^n),$$

for small $\varepsilon > 0$. At the same time, when $v \in H^{\mu^*(s')}(\mathbb{R}_+^n)$ with $s' > \mu^* + \frac{1}{2}$, then

$$v^+ L^* u \in \mathcal{H}^{s'-2a}(\mathbb{R}_+^n) \subset \mathcal{H}^{\mu^*+\frac{1}{2}-2a+\varepsilon}(\mathbb{R}_+^n) = \mathcal{H}^{-\mu-\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n)$$

for small $\varepsilon > 0$. Thus the second duality in (7.3) has a sense for such $u, v$.

We also have, since $v \in H^{\mu^*(s')}(\mathbb{R}_+^n)$ with $s' > \mu^* + \frac{1}{2}$, that

$$v \in H^{\mu^*(s')}(\mathbb{R}_+^n) = \Xi_+^{\mu^*} e^{\mathcal{H}^s-\mu^*}_+(\mathbb{R}_+^n) \subset \Xi_+^{\mu^*} e^{\mathcal{H}^s-\mu^*}_+(\mathbb{R}_+^n)
$$

$$= \Xi_+^{\mu^*} \mathcal{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n) = \mathcal{H}^{\mu^*+\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n),$$

and since $u \in H^{(\mu-1)(s)}(\mathbb{R}_+^n)$ with $s > \mu - 1 + \frac{1}{2} = \mu - \frac{1}{2}$, then (by [G15, Th. 4.2])

$$r^+ L u \in \mathcal{H}^{s-2a}(\mathbb{R}_+^n) \subset \mathcal{H}^{\mu-\frac{1}{2}-2a+\varepsilon}(\mathbb{R}_+^n) = \mathcal{H}^{-\mu-\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n)$$

for small $\varepsilon > 0$. Thus the first duality in (7.3) is well-defined.

The formula (7.3) is then deduced from (5.1) by approximation, as in the proof of Theorem 7.3. \(\square\)

By a similar proof, the full Green’s formula extends:
Theorem 7.5. Let $\mu, \mu^* > 0$. For $s > \mu + \frac{1}{2}$, $s' > \mu^* + \frac{1}{2}$, Theorem 6.2 extends to $u \in H^{(\mu-1)(s)}(\mathbb{R}^n_+)$, $v \in H^{(\mu^*-1)(s')}(\mathbb{R}^n_+)$, in the form

$$(7.4) \quad (r^+Lu, v)_{\mathcal{P}^{(\mu^*+\frac{1}{2}+\varepsilon),\mu^*+\frac{1}{2}+\varepsilon}} - (u, r^+L^*v)_{\mathcal{P}^{(\mu-\frac{1}{2}-\varepsilon),\mu-\frac{1}{2}-\varepsilon}} = \langle s_0\gamma_0^{\mu-1}u, \gamma_0^{\mu^*-1}v \rangle - \langle s_0\gamma_0^\mu u, \gamma_0^{s-1}v \rangle + \langle [s_0(\mu - \mu^*)(D') + B]\gamma_0^{\mu-1}u, \gamma_0^{s-1}v \rangle,$$

with $L_2(\mathbb{R}^{n-1})$-scalar products in the right-hand side.

As a corollary, it is found that the formulas hold for $u, u', v$ in certain Hölder-type spaces with parameter $s$, contained in the Sobolev-type spaces with parameter $s - \varepsilon$. The transmission spaces are defined in [G14] by the usual formula $C^{\mu(s)}_*(\mathbb{R}^n_+) = \Xi^{\mu}_e^{-\mu}C^{s-\mu}_*(\mathbb{R}^n_+)$ for $s > \mu - 1$, where $C^s_*$ stands for the standard Hölder space when $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and the Hölder-Zygmund space otherwise, and there is a description like (7.1):

$$(7.5) \quad C^{\mu(s)}_*(\mathbb{R}^n_+) = \begin{cases} \dot{C}^s_*(\mathbb{R}^n_+) & \text{if } s \in ]\mu - 1, \mu[, \\ \{ e^{+x_n^\mu C^{s-k}_*}f \} & \text{if } s > \mu, \end{cases}$$

where $-\varepsilon$ is active if $s - \mu$ is integer. For $s > \mu + k$, $k \in \mathbb{N}_0$, $\gamma_0^\mu$ maps $C^{\mu(s)}_*(\mathbb{R}^n_+)$ into $C^{s-k}_*(\mathbb{R}^{n-1}_+)$. The operator $r^+L$ maps $C^{\mu(s)}_*(\mathbb{R}^n_+)$ into $C^{s-2a}_*(\mathbb{R}^n_+)$ when $s > \mu - 1$. When $s > \mu > 0$ and $s - \mu \notin \mathbb{N}$, $u \in C^{\mu(s)}_*(\mathbb{R}^n_+) \implies u/x_n^\mu \in C^{s-k}_*(\mathbb{R}^n_+)$ in view of (7.5).

Since $C^s_*(\mathbb{R}^n) \subset H^{s-\varepsilon}(\mathbb{R}^n)$ (any $\varepsilon > 0$) with consequential embeddings for all the derived spaces, we can formulate, as special cases of Theorems 7.3–7.5:

Corollary 7.6. Let $s > \mu + \frac{1}{2}$ and $s' > \mu^* + \frac{1}{2}$, and $s' > 2a$.

The formula in Theorem 7.3 holds when $u \in C^{\mu(s)}_*(\mathbb{R}^n_+)$, $u' \in C^{\mu^*(s')}_*(\mathbb{R}^n_+)$. When $\mu, \mu^* > 0$, the formula in Theorem 7.4 holds when $u \in C^{\mu(s)}_*(\mathbb{R}^n_+)$, $v \in C^{\mu^*(s')}_*(\mathbb{R}^n_+)$, and the formula in Theorem 7.5 holds when $u \in C^{\mu^*(s')}_*(\mathbb{R}^n_+)$.

Since $s, s' > 2a$ here, $L$ and $L^*$ map into the continuous functions on $\mathbb{R}^n_+$, so the formulas can be written with integrals over $\mathbb{R}^n_+$ in the place of dualities, as in Theorem 4.3 (when $\mu, \mu^* > 0$), Theorem 5.1 and Theorem 6.2.

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