Smearing of chaos in sandwich $pp$-waves

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Abstract

Recent results demonstrating the chaotic behavior of geodesics in non-homogeneous vacuum $pp$-wave solutions are generalized. Here we concentrate on motion in non-homogeneous sandwich $pp$-waves and show that chaos smears as the duration of these gravitational waves is reduced. As the number of radial bounces of any geodesic decreases, the outcome channels to infinity become fuzzy, and thus the fractal structure of the initial conditions characterizing chaos is cut at lower and lower levels. In the limit of impulsive waves, the motion is fully non-chaotic. This is proved by presenting the geodesics in a simple explicit form which permits a physical interpretation, and demonstrates the focusing effect. It is shown that a circle of test particles is deformed by the impulse into a family of closed hypotrochoidal curves in the transversal plane. These are deformed in the longitudinal direction in such a way that a specific closed caustic surface is formed.

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1 Introduction

The widely known class of plane-fronted gravitational waves with parallel rays (pp-waves) has become a paradigm of exact radiative spacetimes in general relativity. Found by Brinkmann in 1923 \[1\] and later rediscovered independently by several authors, it has been studied for decades. The metric of vacuum pp-waves can be written in the standard form \[2\]

\[\begin{align*}
    ds^2 &= 2 \, d\zeta \, d\bar{\zeta} - 2 \, du \, dv - (f + \bar{f}) \, du^2 ,
\end{align*}\]  

where \( f(u, \zeta) \) is an arbitrary function of the retarded time \( u \) and the complex coordinate \( \zeta \) spanning the plane wave surfaces \( u = u_0 = \text{const} \). When \( f \) is linear in \( \zeta \), the metric (1) represents Minkowski universe since the only non-trivial components of the curvature tensor are proportional to \( f, \zeta \zeta \). Thus the simplest case for which (1) describes gravitational waves arises for \( f = h(u) \zeta^2 \), where an arbitrary function \( h(u) \) characterizes the ‘profile’ of the wave. Solutions of this type are called plane waves (or homogeneous pp-waves) and have been investigated extensively in the literature (see \[2\],\[3\] for Refs). This important class of exact radiative spacetimes has also been used for the construction of sandwich waves (see e.g. \[4\]-\[7\]) for which \( h(u) \) is non-vanishing on a finite interval of \( u \) only. Also, the very interesting problem of the collision of two plane waves has been thoroughly studied (see \[8\] for review of the topic and Refs.)

Surprisingly, more general non-homogeneous pp-waves with generic or sandwich wave-profiles have not been investigated as much (a physical interpretation of some solutions of this type was proposed e.g. in \[9\],\[10\]). On the other hand, much attention has been payed to impulsive waves described by the metric (1) with the function \( f \) of the form \( f = \delta(u) F(\zeta) \), where \( \delta(u) \) is the Dirac distribution. These solutions have attracted attention recently not only for the investigation of gravitational radiation emitted during the collision of black holes \[11\],\[12\], but also in the context of field and string theories in a study of the scattering processes at extremely high Planckian energies, see \[13\] and elsewhere. Such impulsive waves can naturally be understood as distributional limits of suitable sequences of sandwich waves regularizing the Dirac \( \delta \). Spacetimes of this type can also be obtained by boosting the Schwarzschild, other Kerr-Newman, and dilaton black holes \[14\]-\[19\], or axially symmetric solutions of the Weyl class \[20\] to the speed of light. Some of these solutions can be interpreted as impulsive pp-waves generated by null particles of an arbitrary multipole structure \[21\]. Another method for constructing general impulsive pp-wave spacetimes has been proposed in \[22\]: the ‘scissors-and-paste’ approach is based on the removal of a null hyperplane from Minkowski spacetime and re-attaching the parts by making a formal identification with a ‘warp’. This corresponds to a specific coordinate shift \[23\].

Impulsive pp-waves can alternatively be introduced using a different coordinate system
in which the metric is explicitly continuous [11], [22]-[25]. It has only recently been shown [26] that these approaches are equivalent in a rigorous sense. They lead to identical (unique) particle motion, i.e. the corresponding geodesics in (1), see [27]-[29], agree with those obtained from the continuous form of the metric.

In our recent works [30], [31] we demonstrated that geodesic motion in the class of non-homogeneous vacuum pp-wave spacetimes is chaotic. This is the first example of chaos in exact radiative spacetimes (chaotic behavior of geodesics in various black-hole spacetimes is well known, see e.g. [32]-[36], and references therein). It is the purpose of the presented paper to generalize these results which remained restricted to spacetimes (1) with \( f = \frac{4}{n}C\zeta^n \), \( n = 3, 4, \ldots \), i.e. those having a constant profile, \( h(u) = C = \text{const} \). In particular, we wish to investigate the behavior of geodesics in non-homogeneous sandwich pp-waves and in the corresponding impulsive limits.

In section 2 we briefly summarize previous results concerning chaotic motion in pp-waves with a constant profile. Also, we introduce the concepts and quantities necessary for our subsequent investigation. Section 3 is devoted to geodesics in the simplest sandwich pp-waves having a ‘square’ profile. A rigorous characterization of the level of the fractal structure of basin boundaries (which separate initial conditions leading to different outcomes) by the ‘number of radial bounces’ of geodesics is introduced. This enables us to describe quantitatively how chaos smears as the sequence of sandwich waves approaches the impulsive limit. The same behavior of geodesics is investigated within the class of smooth sandwich waves in section 4, where a relation between the smearing of chaotic behavior and the gradual vanishing of the outcome channels is also discussed. In section 5 explicit (i.e. non-chaotic) geodesics in the resulting non-homogeneous impulsive pp-waves are presented and interpreted physically. Finally, in section 6 we discuss the effect of the focusing of geodesics and describe the deformation of a disc of test particles.

2 Chaotic motion in non-homogeneous pp-waves with a constant profile

As shown in [30], [31] the geodesic equations for (1) reduce to

\[
\ddot{\zeta} + \frac{1}{2} \ddot{\zeta} \dot{U}^2 = 0 ,
\]

\[
u(\tau) = U\tau + \tilde{U} ,
\]

\[
v(\tau) = \frac{1}{2}U^{-1} \int \left[ 2\dot{\zeta}^{2} - (f + \bar{f})U^{2} - \epsilon \right] d\tau + \tilde{V} ,
\]

where \( \tau \) is an affine parameter, dot denotes \( d/d\tau \), \( U, \tilde{U}, \tilde{V} \) are constants, and \( \epsilon = -1, 0, +1 \) for timelike, null or spacelike geodesics, respectively. It suffices to find solutions of Eq. (2).
Introducing real coordinates $x$ and $y$ by $\zeta = x + iy$, this system follows from the Hamiltonian

$$H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + V(x, y, u),$$

where the potential is $V(x, y, u) = \frac{1}{2} U^2 \text{Re} f$. For non-homogeneous $pp$-waves given by

$$f = \frac{2}{n} h(u) \zeta^n, \quad n = 3, 4, \cdots,$$

the corresponding polynomial potential

$$V(x, y, u) = \frac{1}{n} U^2 h(u) \zeta^n$$

at any $u = u_0$ is called ‘$n$-saddle’. It can by visualized in polar coordinates $\rho, \phi$ where $\zeta = \rho \exp(i\phi)$, in which it takes the form $V(\rho, \phi, u_0) = \frac{1}{n} U^2 h(u_0) \rho^n \cos(n\phi)$.

It was shown in a series of mathematical papers [37]-[40] that motion in the Hamiltonian system (5) with a polynomial potential (7) where $h(u) = C = \text{const.}$ is chaotic. Note that, interestingly, in the simplest case $n = 3$ the corresponding ‘monkey saddle’ potential (after removing the factor $CU^2$) is $V(x, y) = \frac{1}{3} x^3 - xy^2$, so that we get the particular case of the famous Hénon-Heiles Hamiltonian [41] (with missing quadratic terms) which is a ‘textbook’ example of a chaotic system. This was investigated by Rod [37] who concentrated on the topology of bounded orbits in the energy manifolds $H = E > 0$. The sets of orbits asymptotic to the basic unstable periodic orbits (denoted by $\Pi_j$) as $\tau \to \pm \infty$ intersect transversely. This proves the existence of homoclinic and heteroclinic orbits and indicates the complicated structure of the flow. These results were later refined in [38] by showing that $\Pi_j$ are hyperbolic, so that they admit stable and unstable asymptotic manifolds. Finally, in [39] the above Hamiltonian was presented as an example of a system for which the Smale horseshoe map can explicitly be embedded as a subsystem along the homoclinic and heteroclinic orbits. It was shown in [37]-[40] that similar results hold for a general potential (7) with $h = C$. Therefore, geodesic motion in all non-homogeneous $pp$-wave spacetimes with the corresponding function (6) is chaotic.

In order to support these arguments for the chaotic behavior, we investigated [30]-[31] the structure of motion by a fractal method. Complementary to the analysis described above, we concentrated on unbounded geodesics. The fractal method (advanced in relativity in [33]-[34] and elsewhere) starts with a definition of several distinct asymptotic outcomes (given here by ‘types of ends’ of all trajectories). Subsequently, a set of initial conditions is evolved until one of the outcome states is reached. Chaos is established if the basin boundaries which separate initial conditions leading to different outcomes are fractal. Such fractal partitions are the result of chaotic dynamics and measure an extremely sensitive dependence of the evolution on the choice of initial conditions. We have observed exactly these structures in the system studied. We integrated numerically the equations of motion given by (2), (6) for $h = C$. The
initial conditions (without loss of generality) were chosen such that the geodesics started (from rest) at \( \tau = 0 \) from a unit circle in the \((x, y)\)-plane. The initial positions were parametrized by an angle \( \phi \in [-\pi, \pi] \) such that \( x(0) = \cos \phi, \ y(0) = \sin \phi \). In Fig. 1 we present typical trajectories for \( n = 3, 4, 5 \). Each unbounded geodesic escapes to infinity where the curvature singularity is located only along one of the \( n \) distinct outcome channels in the potential with the radial axis \( \phi_j = (2j - 1)\pi/n, \ j = 1, \cdots, n \) (in fact, it oscillates around the axis with frequency growing to infinity and amplitude approaching zero \[30\]). These channels represent possible outcomes of our system and we label them by the corresponding values of \( j \). In certain regions the function \( j(\phi) \) representing a portrait of the basin structure depends sensitively on initial position given by \( \phi \) — see Fig. 1 where the (finite resolution) results for \( n = 3, 4, 5 \) are shown. In the same diagrams we plot the function \( \tau_s(\phi) \) which takes the value of \( \tau \) when the singularity is reached by a given geodesic. The boundaries between the outcomes are fractal which we confirmed in \[31\] on the enlarged detail, on the detail of the detail etc. up to the sixth level (where numerical errors became significant). At each level the structure has the property that between two connected sets of geodesics with channels \( j_1 \) and \( j_2 \neq j_1 \) there is always a smaller set of geodesics with channel \( j_3 \) such that \( j_3 \neq j_1 \) and \( j_3 \neq j_2 \). This has a counterpart in \( \tau_s(\phi) \), see Fig. 1. The value of \( \tau_s \) diverges on each discontinuity of \( j(\phi) \), i.e., on any fractal basin boundary. There is an infinite number of peaks, each corresponding to an unstable trapped orbit which never 'decides' on a particular outcome to infinity. Also, \( \tau_s \) increases as one zooms into the higher levels of the fractal. This is natural since higher levels are generated by geodesics which undergo ‘more bounces’ in the inner region before escaping through one of the outcome channels.

3 Motion in shock waves and smearing of chaos in the impulsive limit of simplest sandwich waves

It can immediately be observed that the above results can easily be applied to a description of geodesics in shock \( pp \)-waves given by the metric \([\mathbb{R}], (\mathbb{R})\) with \( h(u) = C\Theta(u) \), where \( \Theta(u) \) is the Heaviside step function. It is natural to consider free test particles which are at rest \((\dot{x} = 0 = \dot{y})\) in the flat Minkowski half-space \( u < 0 \). (The Minkowski coordinates are given by \( x_M = \sqrt{2}x, \ y_M = \sqrt{2}y, \ z_M = (v - u)/\sqrt{2}, \ t_M = (v + u)/\sqrt{2} \).) At \( u = 0 \) these particles are hit by the shock and subsequently for \( u > 0 \) they move in the wave with a constant profile \( h = C \). The behavior of the corresponding geodesics has been summarized in section 2. In particular, the trajectories and the fractal structure of basin boundaries have again the form indicated in Fig. 1. Therefore, the geodesic motion in non-homogeneous shock \( pp \)-waves is chaotic in a rigorous sense.

However, it could be argued that these above results concern very specific and rather
‘unrealistic’ classes of pp-wave solutions for which the profile function $h(u)$ in (5) is constant on an infinite interval of the retarded time $u$. Such waves have an ‘infinite duration’ and a constant ‘strength’. It is the purpose of the presented work to investigate more realistic non-homogeneous pp-waves, namely sandwich waves of this type described by functions $h(u)$ having only finite support.

For an investigation of sandwich waves it is convenient to parametrize the geodesics by the coordinate $u$ instead of the parameter $\tau$; for $\zeta(u)$ we get from Eqs. (2), (3), (6)

$$\zeta'' + h(u)\zeta^{n-1} = 0,$$

where prime denotes $d/du$. For timelike geodesics we can simply substitute $u = U\tau + \bar{U}$ in the result in order to obtain the dependence on the proper time.

It is natural to start with the simplest sandwich waves having a ‘square’ profile

$$h(u) = \frac{1}{\bar{a}} \left[ \Theta(u) - \Theta(u-a) \right],$$

where $a$ and $\bar{a}$ are positive constants. There are flat Minkowski regions in front of the wave ($u < 0$) and behind the wave ($u > a$). Within the wavezone ($0 < u < a$) the amplitude is constant, $h = 1/\bar{a}$, and we can use the results described above. In particular, we can study a deformation of a ring of particles in the $(x,y)$-plane which are at rest in front of the wave. The particles start moving at $u = 0$ and then follow exactly the same trajectories $\zeta(u)$ as shown in Fig. 1 for $u < a$. At $u = a$ the influence of the sandwich wave ends, the potential (7) defining the outcome channels vanishes, and for $u > a$, the particles move uniformly in different directions along straight lines in the flat space behind the wave. Consequently, the fractal structure of basin boundaries indicating chaos is ‘cut’ at some level given by the value of $a$: there is not enough time for particles to bounce arbitrarily many times between the potential walls. As $a \to 0$, the number of bounces tends to zero so that the fractal structure is completely erased. In other words, for smaller $a$, the dependence on initial conditions is less sensitive since the prediction of the outcome can be done with only a lower resolution. The narrower the sandwich wave, the ‘less chaotic’ the corresponding geodesic motion. Finally, in the impulsive case given by the limit $a = \bar{a} \to 0$ of (4), i.e. $f = \frac{2}{n} \delta(u)\zeta^n$, $n \geq 3$, the motion is non-chaotic. This effect can be called a ‘smearing of chaos’ in the impulsive limit of non-homogeneous sandwich pp-waves.

This behavior can be described formally. Let us define the **number of bounces** $N$ of the geodesic as the number of (local) maxima of the function $\rho(u) = |\zeta(u)| = \sqrt{x^2(u) + y^2(u)}$ measuring the radial distance from the origin of the $(x,y)$-plane. $N$ represents the number of times the geodesic crosses the phase-space surface of section $\rho' = 0$ (with $\rho'' < 0$) before escaping to infinity. This is a more appropriate measure here than the twist number used in
As in the previous section, we consider a family of geodesics starting from rest from a unit circle $\rho(0) = 1$ with the angle $\phi$ parametrizing their initial positions. In Fig. 2 we plot the function $N(\phi)$ for the geodesics given by Eq. (8) with $h = 1$ and $n = 3$. The sequence of graphs shows the zooming in of the fractal interval around the value $\phi \approx 0$. Also, we plot the function $u_s(\phi)$ which takes the value of $u$ when the singularity at $\rho = \infty$ is reached by a given geodesic. It is obvious that the fractal structure described by the function $N(\phi)$ corresponds to the structure given by $u_s(\phi)$. Also, $N$ is a precise definition of the ‘level’ of the fractal since we have demonstrated that the $N$-th level of the fractal structure is given by those geodesics which bounce exactly $N$ times before they choose one of the outcome channels. Now we can describe how the fractal structure indicating chaos arises.

Let us denote by $I_k$ the set of initial conditions $\phi$ generating geodesics with $N(\phi) \leq k$, where $k = 0, 1, 2, \cdots$. Clearly, $I_0 \subset I_1 \subset I_2 \subset \cdots \subset (0, 2\pi)$, and the complement of $\lim_{k \to \infty} I_k$ describes the fractal boundary intimately related to the fractal structure of the outcome basin boundaries (although the discontinuities in $N(\phi)$ and $j(\phi)$ do not coincide). Now, we define a sequence of real numbers $a_k = \max\{u_s(\phi) \mid \phi \in I_k\}$. Their physical meaning is the following: at $u > a_k$ all geodesics generated by $I_k$ have already fallen into the singularity at $\rho = \infty$ along the three channels given by $\phi_j$ (after performing at most $k$ bounces). From numerical simulations visualized in Fig. 2 we obtained the values $a_0 = 6.3, a_1 = 9.3, a_2 = 12.2, a_3 = 15.1, a_4 = 18.0, a_5 = 21.0$.

Now, we quantitatively characterize the behavior of geodesics in the sandwich wave given by (8). The amplitude within the wavezone is $h = 1/\tilde{a}$ but it follows from Eq. (8) that we can obtain the corresponding values of $u_s(\phi)$ from those for $h = 1$ (discussed above) by a simple rescaling $u_s(\phi) = \sqrt{\tilde{a}} u^{h=1}_s(\phi)$, where $u^{h=1}_s(\phi)$ is drawn in Fig. 2. Notice that $u_s \to 0$ as $h \to \infty$. It is obvious that all geodesics for which $u_s(\phi) \leq a$, i.e. $u^{h=1}_s(\phi) \leq a/\sqrt{\tilde{a}}$, have time enough to fall into the singularity (in the impulsive limit, $a = \tilde{a} \to 0$, this condition cannot be satisfied so that none of the geodesics considered ends in the singularity). Since $N(\phi)$ is independent of the rescaling of $u$, the fractal structure of initial conditions given by $I_k$ and the values of $a_k$ do not depend on $\tilde{a}$. Consequently, the condition $a/\sqrt{\tilde{a}} \geq a_k$ guarantees that all geodesics starting from the initial set $I_k$ will manage to fall into the singularity at $\rho = \infty$ along the three channels (bouncing at most $k$-times before that). For a sequence of sandwich waves with $\tilde{a} = a$ leading to the impulsive limit as $a \to 0$, this condition reduces to a simple relation $a \geq a_k^2$ which can be interpreted as follows. In order to emerge the $k$-th level of the fractal structure of motion the sandwich wave must have at least the duration $a$ such that $a \geq a_k^2$.

For example, if $a > 441$ then all geodesics with $N \leq 5$ bounces have time enough to choose the

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¹Note that actually the particle does not stop at this section — only its radial velocity vanishes — so that the event could better be called a ‘turn’ or a ‘radial deflection’ rather then a ‘bounce’ on the potential wall, but we use this last word for its natural intuitive meaning.
corresponding channel and fall along this into the singularity, i.e. \( I_5 \) is fully developed. For smaller \( a \) the fractal structure is ‘cut’ at lower levels \( I_k \) so that the geodesic motion becomes ‘less chaotic’. For \( a < a_0^2 \approx 40 \) the fractal structure vanishes since there is no time for any geodesic (performing at least one bounce) to choose the outcome channel and fall into the singularity. Thus, for \( a \to 0 \) the motion is regular. This effect occurs in all non-homogeneous sandwich \( pp \)-waves with arbitrary exponent \( n = 3, 4, \ldots \), only the specific values of \( a_k \) for a given \( n \) are different: for higher \( n \) the values of \( a_k \) are smaller.

4 Motion in smooth asymptotic sandwich waves

We have observed that, in non-homogeneous sandwich \( pp \)-waves with a short duration, many particles still remain in the inner region when the potential (7) defining the outcome channels vanishes. These particles move through various points in different directions having different velocities when the sandwich wave ends. Subsequently, they move uniformly in the Minkowski space behind the wave and it is obvious that they will not follow former channels to the singularity at \( \rho = \infty \). In fact, any point is accessible by some trajectory. The narrower the sandwich wave, the greater the number of particles moving ‘outside’ the channels.

In order to illustrate such behavior of geodesics (and to emphasize this important aspect of the impulsive limit) we consider another type of sandwich waves (8) with the profile function

\[
h(u) = \frac{\tilde{b}}{\cosh^2(2bu)},
\]

where \( b, \tilde{b} \) are positive constants. Note that the corresponding radiative spacetimes are curved everywhere, becoming flat (exponentially fast) only asymptotically as \( u \to \pm \infty \). On the other hand, these waves are smooth since all derivatives of \( h(u) \) are continuous. Moreover, (10) is more appropriate for numerical integration of geodesics (contrary to the case of simplest waves given by Eq. (9) we need not apply the junction conditions which guarantee the continuity of motion at \( u = 0 \) and \( u = a \) where \( h(u) \) is discontinuous). Again, we consider a ring of free test particles which are at rest at \( u = 0 \) (other initial conditions, e.g. a ring which is at rest at \( u = -\infty \) lead to analogous results). In Fig. 3 we show their geodesic motion in the wave with \( n = 3 \) and \( h(u) \) of the form (10), for different values of the parameter \( b = \tilde{b} \). For very small values of \( b \) the profile remains almost constant on a large interval of \( u \) and the trajectories ‘coincide’ with those shown in Fig. 1. The outcome channels are very well defined and the motion is chaotic. However, with a growing value of \( b \) these channels become fuzzy. It is harder and harder to distinguish between them, so that the basin boundaries lose their fractal structure. For \( b \to \infty \), the profile function (10) approaches the Dirac delta (in a distributional sense) and again, in the impulsive limit the geodesic motion is not chaotic but regular.
This effect is quantitatively characterized in Fig. 4 where we plot the function \( j(\phi) \) for different values of \( b \). For this purpose we define three outcome windows as small intervals \( \Delta \phi \) of angles around the radial axes \( \phi_j \) of the three outcome channels (localized sufficiently far away from the origin); here we consider \( |\phi - \phi_j| < \Delta \phi = 0.1 \). When a geodesic starting at \( \phi \) passes through some outcome window we assign the corresponding value of \( j(\phi) \in (1, 2, 3) \) to it. If the trajectory does not pass through any of the three windows then we define \( j(\phi) = 0 \) which means that the geodesic does not approach infinity along the outcome channels \( \phi_j \). From a sequence presented in Fig. 4 it is obvious that the number of geodesics localized outside the channels increases with a growing value of \( b \), i.e. as the sandwich profiles approach the impulsive limit. In the same diagrams we also plot the function \( u_s(\phi) \). One can observe that the number of geodesics reaching the singularity in finite values of \( u_s \) rapidly decreases with a growing \( b \). In particular, all geodesics outside the channel windows (with \( j(\phi) = 0 \)) have \( u_s(\phi) = \infty \). There is a critical value \( b_c \approx 0.09 \) such that for \( b > b_c \) all geodesics considered have \( u_s = \infty \), see Fig. 4. These geodesics reaching \( \rho = \infty \) at \( u_s = \infty \) are, in fact, given by specific solutions of Eq. (8) for which the condition \( |h(\rho)| \ll \rho^{1-n} \) is satisfied for large values of \( u \). Consequently, \( |x''| \leq |\zeta''| \ll 1, |y''| \leq |\zeta''| \ll 1 \), i.e. the acceleration is negligible and asymptotically the particles move uniformly along straight lines. The coordinates \( x \) and \( y \) depend linearly on \( u \), so that \( \rho = \infty \) is reached at \( u = \infty \). Note that for the smooth profile (10) of the wave with \( n = 3 \) this condition reduces to \( \cosh(2bu) \gg \sqrt{b/\rho(u)} \). Once this condition is satisfied for some \( u \), it is valid for all greater values of \( u \) since the left hand side grows exponentially while the right hand side only linearly as \( u \to \infty \).

Finally, we wish to find the critical value of \( b_c \). From Fig. 4 it is obvious that for \( b = b_c \) even the geodesic starting at \( u = 0 \) from rest from the unit circle at \( \phi = \pi \) (and similarly from \( \phi = \pm \pi/3 \)) will reach \( \rho = \infty \) with an infinite value of \( u_s \). In real coordinates, this geodesic is given by \( x'' + h(u)x^2 = 0, y(u) = 0 \), with \( x(0) = -1 \) and \( x'(0) = 0 \). Introducing \( \psi = \ln |x| \) we get the equation \( \psi'' = h(u) \exp(\psi) - \psi'^2, \psi(0) = \psi'(0) = 0 \); the corresponding solutions for various values of \( b = \tilde{b} \) are plotted in Fig. 5. For small \( b \) the function \( \psi(u) \) is convex and diverges at finite values of \( u \). For large values of the parameter \( b \), it becomes concave so that the geodesics approaches infinity only as \( u \to \infty \). The boundary between these two types of behavior defines \( b_c \). From numerical simulations we obtained an approximate value \( b_c = 0.0872374 \), see Fig. 5.

5 Motion in impulsive waves

We have demonstrated above that chaos disappears when the sequence of non-homogeneous sandwich \( pp \)-waves approaches the impulsive limit. In this section we concentrate on a description of motion in these impulsive spacetimes. In particular, we present and discuss an analytic
solution to the geodesic equations, thus proving explicitly that the motion is fully non-chaotic.

For this purpose it is convenient to use a coordinate system for impulsive \( pp \)-wave space-times which is continuous for all values of \( u \) [24], [25]. Following [25], the metric can be written in the form

\[
\text{ds}^2 = 2 |d\eta - \frac{1}{2} u \Theta(u) \left[ d^2 F(\eta)/d\eta^2 \right] d\eta|^2 - 2 dudr. \tag{11}
\]

The transformation relating (11) and (1) with \( f = \delta(u)F(\zeta) \) is

\[
\zeta = \eta - \frac{1}{2} u \Theta(u) (dF/d\eta), \tag{12}
\]

\[
v = r - \frac{1}{2} \Theta(u)(F + \vec{F}) + \frac{1}{4} u \Theta(u) |dF/d\eta|^2. \tag{13}
\]

Obviously, there are privileged geodesics \( \eta = \eta_0 = \text{const.} \) in (11) corresponding to free particles which remain at rest in flat Minkowski half-space \( u < 0 \) in front of the impulsive wave. After the passage of the impulse these geodesics are still given by \( \eta = \eta_0 \) but the flat half-space \( u > 0 \) behind the wave is naturally described by coordinates \( \zeta \) and \( v \) (see (11)) in which the motion is uniform. From Eq. (12) it follows that, for impulsive waves given by \( F(\zeta) = \frac{2}{n} C \zeta^n \) (so that \( h(u) = C \delta(u) \)), the motion in the transversal plane is described simply by \( \zeta(u) = \eta_0 - C\Theta(u) u \tilde{\eta}_0^{n-1} \). If we parametrize the initial position of each particle by \( \zeta(u < 0) = \eta_0 = \rho_0 \exp(\imath \phi_0) \), the motion behind the impulse \( (u > 0) \) is explicitly given by

\[
x(u) = \rho_0 \cos \phi_0 - C \rho_0^{n-1} \cos[(n-1)\phi_0] u, \\
y(u) = \rho_0 \sin \phi_0 + C \rho_0^{n-1} \sin[(n-1)\phi_0] u, \tag{14}
\]

so that

\[
\rho^2(u) = \rho_0^2 + C^2 \rho_0^{2n-2} u^2 - 2C \rho_0^n \cos(n\phi_0) u. \tag{15}
\]

The motion is uniform, i.e. the trajectories are straight lines with the velocity of each particle \( (x', y') = C \rho_0^{n-1}(- \cos[(n-1)\phi_0], \sin[(n-1)\phi_0]) \) being constant. Thus, the inclination of the straight trajectory in the transversal \( (x, y) \)-plane is \( \alpha = \tan(y'/x') = \pi + (1-n)\phi_0 \), and the speed is \( \sqrt{x'^2 + y'^2} = |C| \rho_0^{n-1} \). Notice that the speed depends on \( \rho_0 \) while the direction of motion on \( \phi_0 \) only. The above geodesics can easily be visualized. We do not present their trajectories which would be very similar to those in sufficiently narrow sandwich waves, such as the \( b = \bar{b} = 0.5 \), \( n = 3 \) case shown in Fig. 3. Instead, we draw in Fig. 6 ‘complementary’ pictures showing a deformation of a ring of free test particles (for \( \rho_0 = 1 = C, n = 3 \)). The initial circle at \( u = 0 \) is continuously deformed into smooth curves with \( n \) growing loops which are visualized here as sequences of 4, 8, 13 and 22 consecutive steps \( \Delta u = 0.1 \), i.e. the largest connected curve describes the deformation of the ring at \( u = 2.2 \). Similarly, in Fig. 7 the deformation is shown for \( n = 3, 4, 5, 6 \) and \( u = 0.25, 0.5, 0.75, 1 \). The most distant particles are those which started at \( \phi_0 = \phi_j = (2j-1)\pi/n, j = 1, 2, \cdots, n \) with \( \cos(n\phi_0) = -1 \) so that
\[ \rho(u) = |\rho_0 + C\rho_0^{n-1}u|; \]  
they exactly follow the radial axes of the outcome channels which would be present for corresponding pp-waves with a constant profile. On the other hand, for \( \phi_0 \) given by \( \cos(n\phi_0) = 1 \), we get \( \rho(u) = |\rho_0 - C\rho_0^{n-1}u| \).

It can be observed that for small values of \( u \) the deformation of the ring agrees with that shown in Fig 8. of Ref. [31] describing chaotic motion in the non-homogeneous pp-waves with a constant profile. The principal difference occurs for large \( u \). In the impulsive case there are no subsequent loops arising from more and more bounces in the inner region so that the circle can not be deformed in a fractal way with different segments moving to different outcome channels. Instead, there are no channels to the singularity, the trajectory of each particle is explicitly given by (14), and the motion is non-chaotic.

These results can easily be generalized to the case when the test particles are not at rest initially. From the metric (1) it is obvious that, using the coordinate \( \zeta \), the motion must be uniform in both flat half-spaces, i.e. \( \zeta(u < 0) = \psi_0u + \eta_0, \zeta(u > 0) = \chi_0u + \zeta_0 \), where \( \psi_0, \eta_0, \chi_0, \zeta_0 \) are constants. The relation between these parameters can again be found using the continuous form of the impulsive metric (11). By solving the corresponding geodesic equations we could obtain \( \eta(u) \). Although this function is very complicated for \( u > 0 \), it has the property that \( \eta(u) \) and \( \eta'(u) \) are continuous (even at \( u = 0 \)). Using this fact and Eq. (12), which for \( F(\zeta) = \frac{2}{n}C\zeta^n \) reduces to \( \zeta(u > 0) = \eta(u) - Cu\bar{\eta}^{n-1}(u) \), we obtain

\[ \lim_{u \to 0} \zeta(u > 0) = \zeta_0 = \eta(0) = \eta_0 \text{ and } \zeta'(u > 0) = \chi_0 = \eta'(0) - C\bar{\eta}^{n-1}(0) = \psi_0 - C\bar{\eta}^{n-1}_0, \text{ i.e.} \]

\[ \zeta(u) = [\psi_0 - C\Theta(u)\bar{\eta}^{n-1}_0]u + \eta_0 . \]

Therefore, the trajectory of each particle in the transversal plane is a continuous but refracted straight line with a discontinuity in the velocity at \( u = 0 \) given by \( \Delta\zeta' \equiv \chi_0 - \psi_0 = -C\bar{\eta}^{n-1}_0 \).

Notice that the value of the jump depends on the position \( \eta_0 \) of the particle at \( u = 0 \) only, not on its actual velocity. By parametrizing \( \eta_0 = \rho_0 \exp(i\phi_0) = \zeta_0 \) we immediately obtain the discontinuity in velocity, \( \Delta x', \Delta y' = C\rho_0^{n-1}(\cos((n-1)\phi_0), \sin((n-1)\phi_0)) \). Thus, the result is trivial: if the particle is not at rest initially, behind the wave its constant velocity merely superimposes to the effect of a characteristic jump in velocity given by the impulse.

By denoting \( \cot\alpha_x \equiv x'(u < 0) = \Re e \psi_0, \cot\beta_x \equiv -x'(u > 0) = -\Re e \chi_0 \), and similarly \( \cot\alpha_y \equiv y'(u < 0) = \Im m \psi_0, \cot\beta_y \equiv -y'(u > 0) = -\Im m \chi_0 \), we can rewrite the expression for \( \Delta\zeta' \) as \( \cot\alpha_x + \cot\beta_x = C\rho_0^{n-1}\cos((n-1)\phi_0) \) and \( \cot\alpha_y + \cot\beta_y = -C\rho_0^{n-1}\sin((n-1)\phi_0) \), which generalize to non-axisymmetric cases the ‘refraction formula’ for deflection on (null) geodesics in the axisymmetric Aichelburg & Sexl spacetime [14] (see e.g. [23], [12], [13]).

So far, we have concentrated on a description of motion in the transversal plane. We should also comment on behavior in the longitudinal direction. From the form of the metric (11) it follows that, for geodesics \( \eta = \eta_0 \), the coordinate \( r(u) \) is given by \( r(u) = s_0u + r_0, s_0, r_0 \) being
constants. Moreover, \( r(u) \) is continuous for all \( u \). Due to the Eq. (13), there is a jump in

\[
v(u) = s_0 u + r_0 + C^2 \rho_0^{2n-2} u \Theta(u) - \frac{2}{n} C \rho_0^n \cos(n \phi_0) \Theta(u)
\]

at \( u = 0 \) given by \( \Delta v = -\frac{2}{n} C \rho_0^n \cos(n \phi_0) \) depending both on \( \rho_0 \) and \( \phi_0 \). Since the Minkowski coordinates are \( z_M(u) = \frac{1}{\sqrt{2}} [v(u) - u] \), \( t_M(u) = \frac{1}{\sqrt{2}} [v(u) + u] \), there is a discontinuity \( \Delta z_M = \Delta t_M = \frac{1}{\sqrt{2}} \Delta v \) at \( u = 0 \). This ‘shift’ effect of impulsive wave on geodesics is well-known for the axisymmetric Aichelburg & Sexl spacetime [14], see [11], [23], [42], [43] and elsewhere.

A solution of the geodesic (and geodesic deviation) equations in general impulsive pp-wave spacetime has been presented in [27]-[29]. Starting from the distributional form of the metric (1), one has to deal with ill-defined products of distributions. In order to obtain correct results in a mathematically rigorous fashion, careful regularization procedures and delicate manipulations during calculation of the distributional limit are required. In fact, the Colombeau theory of generalized functions providing a suitable consistent framework has to be applied. Using this rigorous solution concept, it was shown in [28] that in the impulsive limit the geodesics are totally independent of the regularization, i.e. on the particular shape of the sandwich wave (the impulsive wave ‘totally forgets its seed’). Here we obtained an identical explicit form of geodesics starting from the continuous form of the impulsive metric (11). It was demonstrated in [26] that these two approaches are equivalent in a mathematical sense (even if the transformation (12), (13) is discontinuous). However, our main goal here was not to re-derive the known geodesics but to concentrate on their physical description and visualization.

6 Focusing of geodesics and caustic properties

Obviously, geodesics with parallel trajectories in flat half-space in front of a sandwich or impulse wave are refracted. A natural question arises whether a specific character of the refraction leads to some form of focusing of the corresponding geodesics. Such an astigmatic focusing effect is well-known for plane waves, and results in interesting caustic properties thoroughly investigated in [6]. For principal reasons (non-existence of a diagonal Rosen form of the metric, see Eq. (15) in [26], or the presence of chaos in geodesic motion) it would be a very difficult task to reproduce these results for non-homogeneous sandwich waves. Therefore, we restrict ourselves to impulsive gravitational waves only.

It immediately follows from Eqs. (14), (17) that particles staying at fixed \( x_0, y_0, z_0 \) in front of the impulse, move at \( u > 0 \) according to

\[
x_M(u) = \sqrt{2} x(u) , \quad y_M(u) = \sqrt{2} y(u) , \quad \sqrt{2} [z_M(u) - z_0] = C^2 \rho_0^{2n-2} u - \frac{2}{n} C \rho_0^n \cos(n \phi_0) ,
\]
where $x_M, y_M, z_M$ are Minkowski coordinates behind the impulse, $x(u)$ and $y(u)$ are given by (13). In particular, for impulsive plane waves ($n = 2$) we get $x_M(u) = (1 - Cu)x_0$, $y_M(u) = (1 + Cu)y_0$ and $2\sqrt{2}[z_M(u) - z_0] = C(1 + Cu)y_0^2 - C(1 - Cu)x_0^2$, where $x_0 = \sqrt{2}p_0\cos\phi_0$, $y_0 = \sqrt{2}p_0\sin\phi_0$. Consequently, at $u = u_f \equiv 1/C > 0$ one gets $x_M(u_f) = 0$, $y_M(u_f) = 2y_0$ and $\sqrt{2}[z_M(u_f) - z_0] = Cy_0^2$, so that all particles meet at $x_M = 0$, independently of their initial position $x_0$ in the $z_0$-plane. The generically quadratic surface (hyperbolic paraboloid) $u = \text{const.}$ degenerates at $u = u_f$ to a parabolic caustic line $x_M = 0$, $z_M = C\sqrt{2}y_M^2 + z_0$. This is shown in Fig. 8 which visualizes the coordinate singularity formation. It plays a crucial role in colliding plane-wave solutions where the corresponding global ‘fold’ singularities arise, as described in (14). Consequently, these spacetimes are globally hyperbolic in contrast to single plane-wave solutions where the focusing effect on null cones forbids an existence of a spacelike hypersurface which would be adequate for the global specification of Cauchy data (14).

Here we wish, however, to investigate key features of the focusing effect in more general non-homogeneous impulsive $pp$-wave spacetimes with $n = 3, 4, 5, \ldots$. From the deformation of a ring of free particles indicated in Fig. 7 it follows that only $n$ privileged particles can collide at the origin (in contrast to the plane-wave case where particles with arbitrary $x_0$ are focused along the $x_M = 0$ line). It can easily be derived from Eq. (13) by setting $\rho(u) = 0$ that only those $n$ particles with $\cos(n\phi_0) = 1$ reach the origin of the transversal $(x, y)$-plane, simultaneously at $u = u_f \equiv \rho_0^{2-n}/C$. Notice that particles starting from a larger circle focus sooner. This is understandable: they have to travel a bigger distance $\rho_0$, however, their speed is also bigger, $C\rho_0^{n-1}$, so that the travel time is indeed $\rho_0^{2-n}/C$. Note that the occurrence of such focusing is associated with the coordinate singularity in the Rosen form of the metric (14).

At $u = u_c$ the circle is deformed into a curve with $n$ cusps, see Fig. 7. The value of $u_c$ can be found from the condition $(\partial\phi/\partial\phi_0)|_{\cos(n\phi_0)=1} = 0$, where $\phi \equiv \arctan[y(u)/x(u)]$, with $x(u), y(u)$ being given by (12). This leads to equation $(n - 1)C^2\rho_0^{2n-2}u_c^n + (n - 2)C\rho_0^n u_c - \rho_0^2 = 0$ admitting a unique positive solution $u_c = \rho_0^{2-n}/[C(n - 1)] = u_f/(n - 1)$. For $\rho_0 = 1 = C$ we get a simple expression $u_c = 1/(n - 1)$ which is in agreement with Fig. 7.

In fact, the shape of deformation of the initial circle $\rho = \rho_0$ of test particles in the transversal plane has a beautiful geometrical interpretation: at any time $u$, the circle is deformed into a curve called a hypotrochoid. Such a curve is generated by a point $p$ attached to a small circle of radius $B$ rolling around the inside of a large fixed circle of radius $A$, with $H$ being the distance from $p$ to the centre of the rolling circle (14). The Eqs. (14) for particle motion behind the impulsive wave are just the parametric equations for a hypotrochoid with the identification

$$A = \frac{n}{n - 1}\rho_0, \quad B = \frac{1}{n - 1}\rho_0, \quad H = Cu\rho_0^{n-1},$$

where $\phi_0$ is the parameter. We observe that $A = nB$ so that the hypotrochoids are closed curves with $n$ loops in our case. Also, the parameter $H$ grows linearly with $u$. At $u = 0$ we
have $H = 0$ and the curve is just the initial circle of radius $A - B = \rho_0$. At $u = u_c$ the parameter has the value $H = \rho_0/(n-1) = B$ and the hypotrochoid reduces to the curve called a \textit{hypocycloid} which has $n$ cusps. Note that it degenerates for $n = 2$ to a line segment, a 3-cusped hypocycloid is called a deltoid, a 4-cusped hypocycloid is called an astroid (see Fig. 7 for $n = 3$ and $n = 5$ case). Finally, at the focusing time $u = u_f > u_c$ we get $H = \rho_0 = A - B$ so that the hypotrochoid $n$-times intersects the origin. The corresponding curve is called a \textit{rose} since it resembles a flower with $n$ petals (for $n = 2$ the rose degenerates to a line since $u_f = u_c$).

It should be emphasized that the above behavior of geodesics in the $(x, y)$-plane describes only part of the overall deformation. We have to consider not only the transversal deformation but also the motion in the $z_M$-direction described by (19), as we have done for the plane-wave case in Fig. 8. At $u = 0$ the longitudinal deformation $\sqrt{2}[z_M(u) - z_0]$ caused by the impulse is given by the second term on the right hand side of Eq. (19), which is exactly the shift $\Delta v = -\frac{1}{n}C \rho_0^2 \cos(n\phi_0)$. For $u > 0$ the first term increases with $u$; at $u_w = \frac{2}{n} \rho_0^2 - n/C = \frac{2}{n} u_f$ the deformation of the whole disc $\rho \leq \rho_0$, $z_0 = \text{const.}$, becomes non-negative and subsequently grows to positive values, uniformly with $u$. In combination with the deformation in the transversal plane we get an interesting new effect of ‘wrapping up’ of the inner part of the disc into a closed caustic surface as shown in Fig. 9. Note that this behavior is specific for non-homogeneous \textit{pp}-waves and does not occur in impulsive plane waves since $u_w = u_f$ when $n = 2$.

7 Conclusions

We have shown for non-homogeneous \textit{pp}-wave spacetimes that the motion of free test particles is chaotic. However, for non-homogeneous sandwich gravitational waves the chaotic behavior smears as the duration of the wave tends to zero. In the limit of impulsive waves the motion is regular. The focusing effect of non-homogeneous impulsive waves can also be described explicitly and leads to a formation of a specific closed caustic surface. It would be an interesting task to investigate from this point of view geodesics in other classes of exact radiative spacetimes and elucidate an open question whether such a type of behavior is common in general relativity.

Also, it is obvious that sandwich \textit{pp}-wave solutions may serve as local models of gravitational waves far away from a radiating source. In realistic situations these waves should contain non-homogeneous components whose influence on test particles has been described above. In particular, chaotic-type effects on null geodesics could in principle be observable after the passage of the sandwich wave as characteristic “chaotic deformation patterns” of an observer’s view of the sky, i.e. as peculiar changes of positions of stars and galaxies. These specific astronomical consequences will be investigated elsewhere.
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Figure Captions:

**Fig.1** Geodesics starting from a unit circle escape to infinity only along one of the $n$ channels (left). The functions $j(\phi)$ and $\tau_s(\phi)$ indicate that basin boundaries separating different outcomes are fractal (right).

**Fig.2** Plots of the functions $N(\phi)$ and $u_s(\phi)$ in the sequence of ever narrower intervals exhibit their fractal structure. Obviously, the number $N$ of bounces of each geodesic is an appropriate measure of the fractal level.

**Fig.3** Geodesics in the smooth sandwich gravitational waves given by Eq. (10), $n = 3$. For small values of the parameter $b = \tilde{b}$, most of them escape to infinity only along the well-defined three channels. For larger values of $b$ these channels become fuzzy and the basin boundaries separating different outcomes are losing their fractal structure. In the impulsive limit ($b \rightarrow \infty$) the motion is regular.

**Fig.4** Plots of the functions $j(\phi)$ and $u_s(\phi)$ describing behavior of geodesics in smooth sandwich gravitational waves. When the parameter $b = \tilde{b}$ is small, most geodesics reach the singularity in finite value of $u_s$. For $b > b_c \approx 0.0872374$ all geodesics have $u_s = \infty$. Also, for higher $b$ the number of geodesics outside the three outcome windows ($j = 0$) is greater.

**Fig.5** The function $\psi(u) = \ln |x(u)|$ describing specific geodesics changes its character at the critical value $b_c \approx 0.0872374$. For $b > b_c$ the geodesics approaches $\rho = \infty$ only as $u \rightarrow \infty$.

**Fig.6** The deformation of a ring of free test particles which are at rest initially in the transversal $(x, y)$-plane by the influence of an impulsive gravitational wave with $f = \frac{2}{3}\delta(u)\zeta^3$ at $u = 0.4, 0.8, 1.3$ and $2.2$ with steps $\Delta u = 0.1$. Bellow is the 3D visualization of the deformation $(x(u), y(u))$.

**Fig.7** Deformation of a unit ring of particles by impulsive waves with $f = \frac{2}{n}\delta(u)\zeta^n$ at $u = 0.25, 0.5, 0.75, 1$ for $n = 3, 4, 5, 6$. At $u = u_f \equiv \rho_0^{2-n}/C = 1$ all the privileged $n$ particles focus at the origin of the transversal plane. The curves are hypotrochoids, as explained further in the text.

**Fig.8** Deformation of a disc consisting of free particles $\rho_0 \leq 1$, $z_0 = 0$. By an impulsive plane wave ($n = 2, C = 1$) the dics is deformed into a hyperbolic paraboloidal surface at $u =$const.$\geq 0$ in the Minkowski space behind the wave. At $u = u_f = 1$ the surface degenerates to a caustic parabolic line.

**Fig.9** Deformation of a disc $\rho_0 \leq 1$, $z_0 = 0$ of particles by an impulsive non-homogeneous wave $f = \frac{2}{3}\delta(u)\zeta^3$ at $u = 0, 0.25, 0.5, 0.75, 1$ and $1.25$ in the Minkowski space behind the wave. At $u = u_f = 1$ all the privileged three particles collide and with the deformation in the longitudinal $z_M$-direction a closed caustic surface is created.
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