Foliated and compactly supported isotopies of regular neighborhoods

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Abstract. Let \( F \) be a foliation with a “singular” submanifold \( B \) on a smooth manifold \( M \) and \( p : E \to B \) be a regular neighborhood of \( B \) in \( M \). Under certain “homogeneity” assumptions on \( F \) near \( B \) we prove that every leaf preserving diffeomorphism \( h \) of \( M \) is isotopic via a leaf preserving isotopy to a diffeomorphism which coincides with some vector bundle morphism of \( E \) near \( B \). This result is mutually a foliated and compactly supported variant of a well known statement that every diffeomorphism \( h \) of \( \mathbb{R}^n \) fixing the origin is isotopic to the linear isomorphism induced by its Jacobi matrix of \( h \) at \( 0 \). We also present applications to the computations of the homotopy type of the group of leaf preserving diffeomorphisms of \( F \).

1. Introduction

Let \( M \) be a smooth \( n \)-manifold and \( B \) be a submanifold whose connected components may have distinct dimensions. Then a singular foliation on \( M \) of dimension \( k \) and a singular set \( B \) is a partition \( F \) of \( M \) such that every connected component of \( B \) is an element of \( F \), and the induced partition of \( M \setminus B \) is a \( k \)-dimension foliation in a usual sense, e.g. [7].

Denote by \( D(F) \) the group of diffeomorphisms of \( M \) leaving invariant each leaf of \( F \), and by \( D_{fix}(F, B) \), resp. \( D_{nb}(F, B) \), its subgroup consisting of diffeomorphisms fixed on \( B \), resp. on some neighborhood of \( B \). The paper is motivated by study of the homotopy type of \( D(F) \).

Notice that we have a natural group homomorphism \( \rho : D(F) \to D(B) \) associating to each \( h \in D(F) \) its restriction to \( B \), that is \( \rho(h) = h|_B \). Evidently, \( D_{fix}(F, B) \) is the kernel of \( \rho \).

The present authors shown in [18] that if \( F \) is of codimension 1 and its singularities are of Morse-Bott type, then \( \rho \) is a locally trivial fibration over its image. For instance, this holds when there exists a Morse-Bott function \( f : U \to \mathbb{R} \) on some neighborhood \( U \) of \( B \) such that \( F \) in \( U \) consists of connected components of level sets of \( f \) and \( B \) is a union of all critical submanifolds of \( f \).

That statement can be regarded as a foliated analogue of well known results by J. Cerf [8], R. Palais [30] and E. Lima [22] on local triviality of the restriction maps for embeddings.

In particular, one gets a long exact sequence of homotopy groups of the fibration \( \rho \):

\[
\cdots \to \pi_{i+1}(D(B)) \to \pi_i(D_{fix}(F, B)) \to \pi_i(D(F)) \to \pi_i(D(B)) \to \cdots ,
\]

(1.1)
and thus can reduce the computation of the homotopy type of $D(F)$ to two more simple problems of computing the homotopy types of $D(B)$ and $D_{fix}(F, B)$. This is especially useful if the dimension of $B$ is 1 or 2 since in those cases the homotopy type of $D(B)$ is explicitly computed, [33, 9, 11, 10, 15].

In the present paper we make further step in studying the homotopy type of $D(F)$ and obtain certain information about $D_{fix}(F, B)$. Let $p : E \to B$ be a structure of a vector bundle on some regular neighborhood $E$ of $B$ in $M$, and $GL(E, B)$ the group of $C^\infty$ vector bundle self-isomorphisms $h : E \to E$ over $id_B$, i.e. satisfying $p = h \circ p$. We will prove that, under certain homogeneity assumptions on $F$ near $B$, the group $D_{fix}(F, B)$ can be deformed into a subgroup $D_{fix}(F, B, p)$ consisting of diffeomorphisms which coincide near $B$ with vector bundle isomorphisms of $E$, see Theorem 4.1.1. This is mutually a parametrized, foliated, and compactly supported variant of the well known result that each self diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ with $h(0) = 0$ is isotopic to the linear isomorphism given by the Jacobi matrix of $h$ at 0. Moreover, the result also include not only foliations but even “continuous” and possibly “fractal” partitions having certain “homogeneity” properties, see Example 5.2.3.

Evidently, we have another restriction homomorphism $\rho : D_{fix}(F, B, p) \to GL(E, B)$ associating to each $h \in D_{fix}(F, B, p)$ a unique vector bundle isomorphism $\hat{h} \in GL(E, B)$ such that $h = \hat{h}$ near $B$, and the kernel of $\rho$ is $D_{nb}(F, B)$. This allows to reduce the study of the homotopy type of $D_{fix}(F, B)$ to two more simple groups: the image of $D_{fix}(F, B, p)$ in $GL(E, B)$ and the group $D_{nb}(F, B)$.

Further notice that for connected $B$ the group $GL(E, B)$ can be identified with the space of $C^\infty$ sections of some principal $GL(\mathbb{R}^n)$-bundle, and thus can be studied by purely homotopic methods, see Lemma 4.3.1. On the other hand, the group $D_{nb}(F, B)$ consists of diﬀeomorphisms supported out of the set $B$ of singular leaves, so in this group we “get rid of the singularity $B$”. In particular, if $M$ is compact, then $D_{nb}(F, B)$ is the group of compactly supported diﬀeomorphisms of the non-singular (usual) foliation on $M \setminus B$, which is widely studied, [35, 21, 36, 31, 12, 13, 32]. The applications of obtained results and concrete computations will be presented elsewhere.

1.1. Main technical result. Recall that every diﬀeomorphism $h : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is isotopic to the linear map $J_h : \mathbb{R}^m \to \mathbb{R}^m$ defined by the Jacobi matrix $J_h$ of $h$ at 0 via the following isotopy $H : [0; 1] \times \mathbb{R}^m \to \mathbb{R}^m$, $H_\tau(v) = \frac{h(\tau v)}{\tau}$ for $\tau > 0$, and $H_0(v) = J_h v$. The smoothness of $H$ is guaranteed by the Hadamard lemma, see Lemma 6.1.1. This kind of isotopies is used for the proof of existence of isotopies between (open) regular neighborhoods of a submanifold in an ambient manifold, e.g. [16, Chapter 4, Theorem 5.3]. On the other hand, such isotopies are not compactly supported and therefore it is not always possible to extend them globally. The following Theorem 1.1.1 is a compactly supported variant of the above observation, where we replace $\tau$ with some function $\phi$.

Let us fix once and for all a $C^\infty$ function $\mu : \mathbb{R} \to [0; 1]$ such that $\mu = 0$ on $[0; 0.2]$ and $\mu = 1$ on $[0.8; +\infty)$. Let also $p : E \to B$ be a $C^\infty$ vector bundle over a smooth manifold $B$ equipped with some orthogonal structure (see §3.1), $\|\cdot\| : E \to [0; +\infty)$ be the corresponding norm,

$$R_\varepsilon = \{x \in E \mid \|x\| \leq \varepsilon\}$$

for $\varepsilon > 0$ be the closed tubular neighborhood of $B$ in $E$, $N \subset E$ be a smooth submanifold being also a neighborhood of $B$, and

$$C_0^\infty(N, E) = \{h \in C^\infty(N, E) \mid h(B) \subset B\},$$
$$\mathcal{E}_0(N, E) = \{ h \in \mathcal{C}^\infty_0(N, E) \mid h \text{ is an embedding}\}.$$  

Notice that each $h \in \mathcal{C}^\infty_0(N, E)$ induces a certain vector bundle morphism $T_{ph} h : E \to E$ which can be regarded as a “partial derivative of $h$ along $B$ in the direction of fibers”, see below §3.3 and (3.5).

Now, given a function $\delta : \mathcal{E}_0(N, E) \to (0; +\infty)$ continuous with respect to the weak Whitney topology $\mathcal{W}_\infty$, see §2.1, define the following maps

$$\phi : \mathcal{E}_0(N, E) \times [0; 1] \times E \to [0; 1], \quad \phi(h, t, x) = t + (1 - t)\mu(\|x\|), \quad (1.2)$$

$$H : \mathcal{E}_0(N, E) \times (0; 1] \to \mathcal{C}^\infty_0(N, E), \quad H(h, t) = \frac{h(\phi(h, t, x) \cdot x)}{\phi(h, t, x)}. \quad (1.3)$$

Evidently, for all $h \in \mathcal{E}_0(N, E)$ we have that

(A) $H$ is $\mathcal{W}_\infty$-continuous, i.e. continuous with respect to the corresponding topologies $\mathcal{W}_\infty$;

(B) $\phi(h, t, x) > 0$ for $t > 0$, so $H$ is well-defined;

(C) $\phi(h, 1, x) \equiv 1$, whence $H(h, 1) = h$, i.e. $H_1 = id_{\mathcal{E}_0(N, E)}$;

(D) $\phi(h, t, x) \equiv 1$ for $\|x\| \geq 0.8\delta(h)$, whence $H(h, 1) = h$ on $E \setminus R_{0.8\delta(h)}$.

Notice that, depending on $\delta$, the map $H(h, t)$ is not necessarily an embedding, even if $h$ is so. One of the difficulties which is not presented in the case when $B$ is a point, is that an embedding $h : N \to E$ does not necessarily preserves the transversal directions of fibers of $E$ at $B$. Moreover, the deformation (1.3) always preserves projection of the image of the tangent spaces of fibers under the tangent map $Th$ onto $TB$, and therefore $H(h, t)$ can not be made arbitrary close to $h$ in Whitney topology $\mathcal{W}_r$ for $r \geq 1$, see Lemma 6.3.1(b).

Nevertheless, the following theorem claims that if $\delta$ is sufficiently small, one can guarantee that $H(h, t)$ for all $t \in (0; 1]$ is an embedding, and has a limit at $t \to 0$ which coincides with a vector bundle morphism.

**Theorem 1.1.1 (Linearization theorem).** Let $B$ be a compact manifold, and $p : E \to B$ a $\mathcal{C}^\infty$ vector bundle equipped with some orthogonal structure. Then there exists a strictly positive $\mathcal{W}_\infty$-continuous function $\delta : \mathcal{E}_0(N, E) \to (0; +\infty)$ such that the image of the map (1.3) is contained in $\mathcal{E}_0(N, E)$ and $H$ extends to a $\mathcal{W}_\infty$-continuous map $H : \mathcal{E}_0(N, E) \times [0; 1] \to \mathcal{E}_0(N, E)$ such that for all $h \in \mathcal{E}_0(N, E)$,

1. $H(h, t)(R_{\delta(h)}) \subset N$ for $t \in [0; 1]$;
2. $H(h, 0)$ coincide with the vector bundle morphism $T_{ph} h : E \to E$ on $R_{0.2\delta(h)}$.

In fact we will prove a more general statement about “linearization” of smooth maps between maps of pairs of manifolds $(M, B) \to (N, C)$, Theorem 6.4.1.

**1.2. Structure of the paper.** In Section 2 we recall necessary definitions and introduce notation which will be used throughout the paper. In particular, we discuss Whitney topologies and principal fibrations. In Section 3 we consider several constructions related with vector bundles. Section 4 we will deduce from Theorem 1.1.1 the first application to homotopy type of certain diffeomorphism groups in a non-foliated case. Let $\mathcal{D}_{\text{inv}}(M, B)$ be the group of diffeomorphisms of a manifold $M$ which leave invariant a compact submanifold $B$, $p : E \to B$ a regular neighborhood of $B$, and $\mathcal{D}_{\text{inv}}(M, B, p)$ the subgroup of $\mathcal{D}_{\text{inv}}(M, B)$ consisting of diffeomorphisms which coincide near $B$ with some vector bundle morphisms of
$E$. We will prove in Theorem 4.1.1 that the inclusion $\mathcal{D}_{ins}(M, B, p) \subset \mathcal{D}_{ins}(M, B)$ is a homotopy equivalence, and also establish Lemmas 4.2.1 and 4.3.1 allowing to reduce the study of the homotopy type of $\mathcal{D}_{ins}(M, B)$ to several simpler groups, see §4.4.

In Section 5, for a special class of singular foliations having certain homogeneity property, we deduce from linearization Theorem 4.1.1 a foliated version (Theorem 5.1.5). We also consider particular cases corresponding to foliations on vector bundles (Theorem 5.1.8, and especially Corollary 5.1.9 for trivial vector bundles), and for foliations by level sets of functions having only isolated “homogeneous” critical points (Theorem 5.3.2).

In Section 6 we establish several preliminary variants of linearization theorem. They are based of the Hadamard lemma and its modifications. First we consider maps between total spaces of vector bundles $p : E \to B$ and $q : F \to C$ sending zero sections to zero sections.

Every such map $h : E \to F$ induces a certain vector bundle morphism $T_{fib}h : E \to F$ which can be regarded as a “derivative of $h$ along $B$ in the direction of fibres”, see §3. We prove that $h$ is homotopic to $T_{fib}h$ (Lemma 6.2.2). Next, we modify the that lemma to prove that $h$ is homotopic to a map which coincides with $T_{fib}h$ near $B$, and the corresponding homotopy $\{h_t\}_{t \in [0;1]}$ is supported in an arbitrary small neighborhood of $B$ (Corollary 6.3.2). In order to apply the latter corollary to diffeomorphisms we will also give estimations on ranks along $B$ of the tangent maps of $h_t$, see §6.5 and Theorem 6.4.1. Finally in §7 we prove Theorem 6.4.1 and in §8 deduce from it Theorem 4.1.1.

2. Preliminaries

Throughout the paper by a manifold we will mean a $C^\infty$ manifold which may be non-compact and have a boundary.

2.1. Whitney topologies. Let $A, B$ be two manifolds. Then for each $l \in \{0, 1, \ldots, \infty\}$ and $r \geq l$ the space $\mathcal{C}^l(A, B)$ admits two topologies, weak $W^r$ and strong $S^r$, satisfying the following relations: $W^r \subset W^s$ and $S^r \subset S^s$ for $r < s$, $W^\infty = \bigcup_{0 \leq r < \infty} W^r$, $S^\infty = \bigcup_{0 \leq r < \infty} S^r$, $W^r \subset S^r$, and they coincide if $A$ is compact, e.g. [16, 29].

Let $C, D, E, F$ be some other manifolds, $\mathcal{X} \subset \mathcal{C}^l(A, B) \times C$ and $\mathcal{Y} \subset \mathcal{C}^l(D, E) \times F$ two subsets, and $r, s \in \{0, 1, \ldots, \infty\}$. Endow $\mathcal{C}^l(A, B)$ with the topology $W^r$ and $\mathcal{C}^l(D, E)$ with the topology $W^s$. Then a map $\mathcal{X} \to \mathcal{Y}$ (resp. $\mathcal{X} \to F$, $E \to \mathcal{Y}$) is continuous with respect to the induced topologies on $\mathcal{X}$ and $\mathcal{Y}$ will be called $W^{r,s}$-continuous (resp. $W^r$, $W^s$-continuous).

Similarly, a map $f : \mathcal{X} \to \mathcal{Y}$ is a $W^{r,s}$-homotopy equivalence, if it is a homotopy equivalence when $\mathcal{X}$ and $\mathcal{Y}$ are endowed with the topologies $W^r$ and $W^s$ respectively. The following lemma is a direct consequence of the relation $W^\infty = \bigcup_{0 \leq r < \infty} W^r$:

**Lemma 2.1.1.** Suppose that for each $s < \infty$ there exists $r < \infty$ such that $f : \mathcal{X} \to \mathcal{Y}$ is $W^{r,s}$-continuous. Then $f$ is $W^{\infty, \infty}$-continuous. \hfill $\Box$

One can define $S^{r,s}$- and $S^r$-continuity corresponding to strong topologies and establish analogue of Lemma 2.1.1 in a similar way. The following lemma is also trivial:

**Lemma 2.1.2.** The evaluation map $ev : \mathcal{C}^l(A, B) \times A \to B$, $ev(h, a) = h(a)$, is $W^l$-continuous, therefore $W^r$- and $S^r$-continuous for all $r \geq 0$. \hfill $\Box$

Let $B$ be a manifold, and $U \subset B \times \mathbb{R}^m$ an open subset. For every compact subset $K \subset U$, $n \geq 1$, and finite $r \geq 0$ define respectively an $(r, K)$-seminorm along $\mathbb{R}^m$ and an $(r, K)$-norm along $\mathbb{R}^m$

$$
| \cdot |_{r,m,K}, \quad \| \cdot \|_{r,m,K} : C^\infty(U, \mathbb{R}^n) \to [0; +\infty)
$$
by the following formulas: if \( h = (h_1, \ldots, h_n) : U \to \mathbb{R}^n \) is a \( C^\infty \) map, then
\[
|h|_{r,m,K} = \sum_{\alpha=(\alpha_1, \ldots, \alpha_m)} \sum_{|\alpha|=l} \sup_{v \in K} \left| \frac{\partial^{(\alpha)} h_i}{\partial v_1^{\alpha_1} \cdots \partial v_m^{\alpha_m}}(x, v) \right|,
\]
where \( v = (v_1, \ldots, v_m) \) coordinates in \( \mathbb{R}^m \), \( x \in B \), and \( |\alpha| = \alpha_1 + \cdots + \alpha_m \).

Thus an \((r, K)\)-seminorm along \( \mathbb{R}^m \) (resp. an \((r, K)\)-norm along \( \mathbb{R}^m \)) give bounds on partial derivatives in \( \mathbb{R}^m \) of coordinate functions of \( h \) of order \( r \) (resp. of all orders up to \( r \)) on \( K \).

In particular, if \( B \) is a point, so \( U \) is actually a subset of \( \mathbb{R}^m \), then \( |h|_{r,m,K} \) and \( \|h\|_{r,m,K} \) will be called the \((r, K)\)-seminorm (resp. the \((r, K)\)-norm) of \( h \). In this case, if \( K \) is itself a smooth submanifold of \( U \), then \( \|\cdot\|_{r,K} \) is a metric on \( C^l(K, \mathbb{R}^n) \) which generates \( W^r \) topology.

In particular, for \( h \in C^l(U, \mathbb{R}^m) \), \( r \in \{0, 1, \ldots, \infty\} \), \( \varepsilon > 0 \), and a compact subset \( K \subset U \), the following set
\[
N_{K,\varepsilon}^r(h) = \{ \tilde{h} \in C^l(U, \mathbb{R}^m) \mid \|\tilde{h} - h\|_{r,K} < \varepsilon \}.
\]
is an \( W^r \)-open neighborhood of \( h \) in \( C^l(U, \mathbb{R}^m) \).

### 2.2. Homomorphisms as principal fibrations

We recall here a simple lemma which will be used several times. For a topological group \( G \) denote by \( e_G \) its unit element, by \( G_0 \) the path component of \( e_G \) in \( G \), and by \( \pi_0 G \) the set of path components of \( G \). Then \( G_0 \) is a normal subgroup of \( G \) and we have a natural bijection \( \pi_0 G = G/G_0 \) which allows to endow \( \pi_0 G \) with a groups structure.

Let \( \phi : G \to H \) be a continuous homomorphism of topological groups. Say that \( \phi \) admits a local section if there exists an open neighborhood \( U \subset H \) of \( e_H \) and a continuous map \( s : U \to G \) being a section of \( \phi \), i.e. \( \phi \circ s = \text{id}_U \).

**Lemma 2.2.1.** Let \( \phi : G \to H \) be a continuous homomorphism of topological groups with kernel \( K \). Denote \( K' = K \cap G_0 \) and assume that \( H \) is paracompact and Hausdorff. If \( \phi \) admits a local section \( s : U \to G \), then

1. \( \phi : G \to q(G) \) is a locally trivial principal \( K \)-fibration over the image \( q(G) \);
2. \( q(G) \) is a union of path components of \( H \);
3. the restriction \( \phi : G_0 \to H_0 \) is surjective and is a principal \( K' \)-fibration;
4. we have a long exact sequence of homotopy groups
\[
\cdots \to \pi_k(K_0, e_G) \to \pi_k(G_0, e_G) \to \pi_k(H_0, e_H) \to \pi_{k-1}(K_0, e_G) \to \cdots
\]
\[
\cdots \to \pi_1(H_0, e_H) \to K/K_0 \to G/G_0 \to G/K;
\]
5. if \( \phi \) admits a global section \( s : \phi(G) \to G \), then \( G \) is homeomorphic with \( K \times \phi(G) \);
6. if \( H \) is a subgroup of \( G \) and \( \phi \) is a retraction onto \( H \), i.e. \( \phi(h) = h \) for all \( h \in H \), then the inclusion \( H \subset G \) is a global section of \( \phi \), so \( G \) is homeomorphic with \( K \times H \).

**Proof.** (1) Note that the natural action of \( K \) on \( G \) by left shifts turns the canonical projection \( p : G \to G/K \) to a principal \( K \)-fibration. We also have a continuous bijection \( \hat{\phi} : G/K \to q(G) \) such that \( \phi = \hat{\phi} \circ p \). Then the existence of a section \( s : U \to G \) implies that

- \( \hat{\phi} \) is a homeomorphism, so the map \( \phi : G \to q(G) \) is a principal \( K \)-fibration;
- that fibration \( \phi : G \to q(G) \) is locally trivial.

(2) & (3) Recall that every locally trivial fibration over a paracompact Hausdorff space satisfies path lifting property, e.g. [34, Chapter 2, §7, Corollary 1.2]. This implies that
\[ \phi(G_0) = H_0 \] and the image of \( \phi \) must consist of path components of \( H \). Hence \( \phi : G_0 \to H_0 \) is a principal \((K \cap G_0)\)-fibration;

(4) That sequence is the long exact sequence of homotopy groups of the locally trivial fibration \( \phi : G \to \phi(G) \), where we take to account that \( \pi_k(G,e_G) \cong \pi_k(G_0,e_G) \) for \( k \geq 1 \), \( \pi_0G = G/G_0 \), and \( G/K = \phi(G)/(G/G_0)/(K/K_0) = \pi_0G/\pi_0K \).

(5) & (6) If \( \phi \) admits a global section, then it is a trivial principal \( K \)-fibration. \( \square \)

3. Vector bundles

Let \( B \) be a connected manifold of dimension \( b \) and \( p : E \to B \) a smooth vector bundle over \( B \) of rank \( m \), so \( \dim(E) = b + m \). We will identify \( B \) with the image of the zero section \( \zeta : B \to E \), and thus \( p \) will be a smooth retraction of \( E \) onto \( B \).

For each \( x \in B \) let \( E_x := p^{-1}(x) \) be the corresponding fiber over \( x \). By definition \( E_x \) is endowed with a structure of an \( m \)-dimensional vector space. In particular, let \( \delta : \mathbb{R} \times E \to E \) be the “multiplication by scalars” map. To simplify notations put \( tx := \delta(t,x) \). Then \( t(sx) = (ts)x \), \( p(tx) = p(x) \), \( 0x = p(x) \), \( 1x = x \) for all \( s,t \in \mathbb{R} \) and \( x \in E \). In particular, if \( x \in B \), so \( x = p(x) \), then \( tx = tp(x) = t \cdot (0x) = (t \cdot 0)x = 0x = x \) for all \( t \in \mathbb{R} \).

A subset \( N \subset E \) will be called star-like, whenever \( tx \in N \) for each \( t \in [0;1] \) and \( x \in N \).

3.1. Trivialized atlas. A trivialized local chart of \( p \) is an open embedding \( \Phi : V \times \mathbb{R}^m \to E \) making commutative the following diagram:

\[
\begin{array}{ccc}
V \times \mathbb{R}^m & \xrightarrow{\Phi} & E \\
\downarrow & & \downarrow p \\
V & \xleftarrow{\Phi|_V} & B
\end{array}
\]

and being linear on fibres (in particular \( \Phi \) is a vector bundle morphism from a trivial vector bundle), where \( V \subset \mathbb{R}^b \) is an open subset.

A trivialized atlas of \( p \) is a collection of trivialized local charts

\[ \xi = \{ \Phi_i : V_i \times \mathbb{R}^m \to E \}_{i \in \Lambda} \] (3.1)

such that \( E = \bigcup_{i \in \Lambda} \Phi_i(V_i \times \mathbb{R}^m) \). Notice that if \( V_i \cap V_j \neq \emptyset \), then the map

\[ \Phi_j^{-1} \circ \Phi_i : (V_i \cap V_j) \times \mathbb{R}^m \to (V_i \cap V_j) \times \mathbb{R}^m \]

is given by \( \Phi_j^{-1} \circ \Phi_i(x,v) = (x,A_{ij}(x)v) \), where \( x \in V \cap V, v \in \mathbb{R}^m \), and \( A_{ij} : V_i \cap V_j \to \text{GL}(\mathbb{R}^m) \) are \( C^\infty \) maps called transition functions and satisfying the standard cocycle relations.

Such an atlas \( \xi \) will be called an orthogonal structure on \( E \) whenever each \( A_{ij} \) takes values in the orthogonal group \( O(m) \). It is well known that every vector bundle admits an orthogonal structure, e.g. [17, Chapter 3, §9].

If \( \xi \) is an orthogonal structure on \( E \), then one can define a norm \( \| \cdot \| : E \to [0;+\infty) \) as follows. Let \( w \in E \), and \( \Phi_i : V_i \times \mathbb{R}^m \to E \) be a trivialized local chart of \( p \) from \( \xi \) such that \( w \in \Phi_i(V_i \times \mathbb{R}^m) \), so \( w = \Phi_i(x,v) \), where \( x \in V_i \), and \( v = (v_1, \ldots, v_m) \in \mathbb{R}^m \). Put \( \|w\| := \sqrt{v_1^2 + \cdots + v_m^2} \). Since all transition functions take values in the orthogonal group, \( \|w\| \) does not depend on a particular choice of a trivialized local chart \( \Phi_i \) whose image contains \( x \).
3.2. Vertical subbundle. Denote by \( \pi : TE \to E \) the tangent bundle of \( E \). Let \( U \) be a subset of \( E \), and \( i : U \subset E \) be the corresponding inclusion map. Then we will use the following notations:

\[
T_U E := \bigcup_{x \in U} T_x E \equiv i^*(TE), \\
\text{Vert}(U) := \bigcup_{x \in U} T_x E_{p(x)} \equiv \ker(Tp : T_U E \to TB).
\]

Thus \( T_U E \) is union of all tangent spaces of \( E \) at points of \( U \), which can also be regarded as the pull back of \( \pi \) by the inclusion \( i : U \subset E \). Also \( \text{Vert}(U) \) is the subset of \( T_U E \) consisting of tangent vectors to fibres of \( p \) at points of \( U \), which can also be described as the kernel of the restriction to \( T_U E \) of the tangent map to the projection \( E \supset U \xrightarrow{p} B \).

Consider the case when \( U = B \). Then we have an inclusion \( T\zeta : TB \subset T_B E \) induced by the zero section \( \zeta : B \to E \), and also the projection \( Tp : T_B E \to TB \), which gives a canonical direct sum splitting:

\[
T_B E \cong TB \oplus \text{Vert}(B). \tag{3.2}
\]

The following statement is well known and straightforward:

**Lemma 3.2.1.** There is a canonical vector bundle isomorphism

\[
\psi : E \to \text{Vert}(B) \tag{3.3}
\]

defined as follows. Let \( x \in E, b = p(x) \in B \), and \( \gamma_x : \mathbb{R} \to E_b \subset \text{Vert}(B) \) be the curve defined by \( \gamma_x(t) = tx \), so \( \frac{d}{dt} \gamma_x(t) = x \) for all \( t \in \mathbb{R} \). Then \( \psi(x) := \frac{d}{dt} \gamma_x(1) \in T_x E_x \) is the tangent vector to \( \gamma \) at \( x \).

Hence we get from (3.2) another canonical direct sum splitting:

\[
T_B E \cong TB \oplus E, \tag{3.4}
\]

which will play an important role in this paper.

3.3. Tangent map along fibers. Now let \( q : F \to C \) be another vector bundle, \( N \subset E \) be a neighborhood of \( B \), and \( h : N \to F \) be a \( C^1 \) mapping such that \( h(B) \subset C \). Then \( Th(T_B E) \subset T_C F \). Hence \( h \) induces the following vector bundle morphism:

\[
T_{\text{fib}} h : E \cong \text{Vert}(B) \subset T_B E \xrightarrow{T \text{h}} T_C F \cong TC \oplus \text{Vert}(C) \xrightarrow{\text{pr}_2} \text{Vert}(C) \cong F. \tag{3.5}
\]

We will call \( T_{\text{fib}} h : E \to F \) the tangent map of \( h \) along \( B \) in the direction of fibers. Evidently, for each \( v \in E \) we have that

\[
T_{\text{fib}} h(v) = \lim_{t \to 0} \frac{1}{t} h(tv). \tag{3.6}
\]

**Example 3.3.1.** Let \( h : \mathbb{R}^m \to \mathbb{R}^n \) be a \( C^1 \) map such that \( h(0) = 0 \). One can regard it as a map between total spaces of trivial vector bundles over a point. Then \( T_{\text{fib}} h : \mathbb{R}^m \to \mathbb{R}^n \) is just the tangent map of \( h \) at 0.

**Example 3.3.2.** More generally, let \( p : \mathbb{R}^b \times \mathbb{R}^m \to \mathbb{R}^b \) and \( q : \mathbb{R}^c \times \mathbb{R}^n \to \mathbb{R}^c \) be trivial vector bundles, and \( h = (f, g) : \mathbb{R}^b \times \mathbb{R}^m \to \mathbb{R}^c \times \mathbb{R}^n \) a \( C^1 \) map such that \( h(\mathbb{R}^b \times 0) \subset \mathbb{R}^c \times 0 \). Let also \( v = (v_1, \ldots, v_m) \) be coordinates in \( \mathbb{R}^m \), \( g = (g_1, \ldots, g_n) : \mathbb{R}^b \times \mathbb{R}^m \to \mathbb{R}^n \) the coordinate functions of \( g \), and \( S(x, v) = \left( \frac{\partial g_j}{\partial v_i}(x, v) \right)_{i=1,\ldots,n, j=1,\ldots,m} \) the matrix of partial derivatives of \( g \) in \( v \). Then \( T_{\text{fib}} h(x, v) = (f(x, 0), S(x, v)v) \).
Moreover, it also follows from \(W\). It follows from (D) that usually one is able to compute only the weak homotopy type of infinite dimensional spaces. Let also \(h \in \mathcal{D}(M, B)\) be a regular neighborhood of \(B\) in \(M\), i.e. a \(C^\infty\) retraction admitting a vector bundle structure, where \(E\) is some open neighborhood of \(B\) in \(M\). Define the following groups:

- \(\mathcal{D}_{\text{inv}}(M, B)\) is the group of \(C^\infty\) diffeomorphisms \(h\) of \(M\) such that \(h(B) = B\);
- \(\mathcal{D}_{\text{inv}}(M, B, p)\) is the subgroup of \(\mathcal{D}_{\text{inv}}(M, B)\) consisting of diffeomorphisms \(h\) of \(M\) for which there exists a vector bundle isomorphism \(\hat{h} : E \to E\) and an open neighborhood \(U_h\) of \(B\) in \(E\) such that \(h|_{U_h} = \hat{h}|_{U_h}\);
- \(\mathcal{D}_{\text{fix}}(M, B)\) is a subgroup of \(\mathcal{D}_{\text{inv}}(M, B)\) consisting of diffeomorphisms fixed on \(B\);
- \(\mathcal{D}_{\text{fix}}(M, B, p) := \mathcal{D}_{\text{fix}}(M, B) \cap \mathcal{D}_{\text{inv}}(M, B, p)\);
- \(\mathcal{D}_{\text{nb}}(M, B)\) is the group of diffeomorphisms of \(M\) fixed near \(B\), i.e. having support in \(M \setminus B\);
- \(\text{GL}(E)\) is the group of vector bundle automorphisms \(g\) of \(E\) such that \(g(E) = E\);
- \(\text{GL}(E, B)\) is the subgroup of \(\text{GL}(E)\) consisting of vector bundle morphisms fixed on \(B\) or, equivalently, leaving invariant the intersection of each fiber of \(p\) with \(E\).

The following theorem is a direct consequence of the linearization Theorem 1.1.1. Notice that usually one is able to compute only the weak homotopy type of infinite dimensional spaces especially with respect to strong \(S^\infty\) Whitney topologies, e.g. [1, 5, 2, 3, 19].

**Theorem 4.1.1.** Let \(B\) be a compact proper submanifold of a manifold \(M\) and \(p : E \to B\) be a regular neighborhood of \(B\) in \(M\). Then the inclusion of pairs

\[
(D_{\text{fix}}(M, B, p), D_{\text{inv}}(M, B, p)) \subset (D_{\text{fix}}(M, B), D_{\text{inv}}(M, B))
\]

is mutually \(W^{\infty, \infty}\)- and \(S^{\infty, \infty}\)-homotopy equivalence.

**Proof.** Let \(H : \mathcal{E}_0(N, E) \times [0; 1] \to \mathcal{E}_0(N, E)\) be the same as in Theorem 1.1.1. Define another map \(G : D_{\text{inv}}(M, B) \times [0; 1] \to D_{\text{inv}}(M, B)\) by

\[
G(h, t)(x) = \begin{cases} H(h, t)(x), & x \in E, \\ h(x), & x \in M \setminus E. \end{cases}
\]

It follows from (D) that \(G(h, t)\) is a \(C^\infty\) diffeomorphism of \(M\), and thus \(G\) is well-defined. Moreover, it also follows from \(W^{\infty, \infty}\)-continuity of \(H\), see (B), that \(G\) is also \(W^{\infty, \infty}\)-continuous.

We claim that the map \(G\) is a deformation of the pair \((D_{\text{fix}}(M, B), D_{\text{inv}}(M, B))\) into the pair \((D_{\text{fix}}(M, B, p), D_{\text{inv}}(M, B, p))\), i.e. it is a homotopy between \(\text{id}_{D_{\text{inv}}(M, B)}\) and the map whose image is contained in \(D_{\text{inv}}(M, B, p)\) and such that \(D_{\text{fix}}(M, B)\) and \(D_{\text{inv}}(M, B, p)\), and therefore \(D_{\text{fix}}(M, B, p) = D_{\text{fix}}(M, B) \cap D_{\text{inv}}(M, B, p)\), are invariant under \(G\). In particular, the inclusion (4.1) is a homotopy equivalence. This follows from (i)-(iv) below.

---

1 Recall that \(B\) is a proper submanifold of \(M\), if \(\partial B = \partial M \cap B\) and this intersection is transversal.
(i) Indeed, by (C), $G_1(h)(x) = h(x)$ for all $h \in \mathcal{D}_{inv}(M, B)$ and $x \in M$. In other words, $G_1 = \text{id}_{\mathcal{D}_{inv}(M, B)}$.

(ii) Also, by Theorem 1.1.1, for each $h \in \mathcal{D}_{inv}(M, B)$ the map $G_0(h)$ coincides near $B$ with the vector bundle morphism $T_h b h$. In particular, $G_0(\mathcal{D}_{inv}(M, B), p) \subset \mathcal{D}_{inv}(M, B, p)$.

(iii) If $h \in \mathcal{D}_{inv}(M, B, p)$, i.e. it coincides with some vector bundle morphism $\hat{h} : E \to E$ on some neighborhood $W$ of $B$, then by formula (1.3) for $G$ we have that

$$G_t(h)(x) = \frac{h(\phi(h,t,x),x)}{\phi(h,t,x)} = \frac{\hat{h}(\phi(h,t,x))h(x)}{\phi(h,t,x)} = \hat{h}(x) = h(x)$$

for $x \in W$ and $t > 0$. In other words, $G(\mathcal{D}_{inv}(M, B, p) \times (0;1]) \subset \mathcal{D}_{inv}(M, B, p)$. Also, as just shown in (iv), $G(\mathcal{D}_{inv}(M, B, p) \times 0) \subset \mathcal{D}_{inv}(M, B, p)$ as well.

(iv) Finally, by (D), $G_t(h)(x) = h(x)$ for all $(h, t, x) \in \mathcal{D}_{inv}(M, B) \times [0;1] \times B$. In particular, if $h \in \mathcal{D}_{fix}(M, B)$, i.e. $h$ is fixed on $B$, then so is each $G_t(h)$. In other words, $G(\mathcal{D}_{fix}(M, B) \times (0;1]) \subset \mathcal{D}_{fix}(M, B)$.

2) Let us deduce from (D) that $G$ is also $S^{\infty,\infty}$ continuous. Let $(h, t) \in \mathcal{D}_{inv}(M, B) \times [0;1]$ and $U$ be an $S^{\infty}$-neighborhood of $G_t(h)$ in $\mathcal{D}_{inv}(M, B)$. We need to find an $S^{\infty}$-neighborhood of $(h, t)$ in $\mathcal{D}_{inv}(M, B) \times [0;1]$ such that $G(U) \subset U$.

By definition, $S^{\infty} = \bigcup_{r=0}^{\infty} S^r$, so one can assume that $U$ is $S^r$-open for some finite $r$. In turn, the topology $S^r$ on $C^\infty(M, M)$ is induced from the topology $S^0$ on the space maps $C^\infty(M, J^r(M, M))$ with respect to the natural inclusion $j^r : C^\infty(M, M) \hookrightarrow C^\infty(M, J^r(M, M))$ called $r$-jet prolongation. Thus one can assume that there exists a locally finite cover $\alpha = \{K_i\}_{i \in \Lambda}$ of $M$ by compact subsets and a family of open subsets $\{\bar{U}_{i, r}\}_{i \in \Lambda}$ of $J^r(M, M)$ such that $U = \bigcap_{i \in \Lambda} K_i \cap \bigcup_{i \in \Lambda} \bar{U}_{i, r}$, where as usual $[K_i, U_{i, r}] = \{f \in C^\infty(M, M) \mid j^r(K_i) \subset U_i\}$. Since $V$ is compact, there are only finitely many $K_{i_1}, \ldots, K_{i_n} \in \alpha$ intersecting $V$. Denote $K = \bigcup_{j=1}^{n} K_{i_j}$. Then $V \subset \bigcup_{j=1}^{n} K_{i_j}$. As $G$ is $W^{\infty,\infty}$-continuous, there exists finite $r' \geq 0$, a $W^{r'}$-neighborhood $V$ of $h$ in $\mathcal{D}_{inv}(M, B)$, and a closed neighborhood $J \subset [0;1]$ of $t$ such that $G(V \times J) \subset \bigcap_{j=1}^{n} K_{i_j} \cap \bigcup_{i \in \Lambda} U_{i, r}$.

Now let $r'' = \max\{r, r'\}$. Then $W := (V \times J) \cap \bigcap_{i \in \Lambda \setminus \{i_1, \ldots, i_n\}} [K_i, U_i] \times [0;1]$ is $S^{r''}$ open in $C^\infty(M, M)$.

Note that $(h, t) \in W$. Indeed, by assumption, $(h, t) \in V \times J$. Also, as $K_i \subset M \setminus V$ for $i \in \Lambda \setminus \{i_1, \ldots, i_n\}$, we have by (D) that $h|_{K_i} = G(h, t)|_{K_i}$, whence $h(K_i) = G(h, t)(K_i) \subset U_i$.

Finally, we claim that $G(W) \subset U$. Indeed, let $(g, s) \in W$. Then $(g, s) \in V \times J$, whence $G(g, s)(K_{i_j}) \subset U_{i_j}$ for $j = 1, \ldots, l$. On the other hand, again due to (D), $g|_{K_{i_j}} = G(g, s)|_{K_{i_j}}$ for $i \in \Lambda \setminus \{i_1, \ldots, i_l\}$, whence $G(g, s)(K_{i_j}) = g(K_{i_j}) \subset U_{i_j}$. This proves that $G$ is also $S^{\infty,\infty}$ continuous. \hfill \Box

4.2. Applications to the homotopy type of $\mathcal{D}_{inv}(M, B)$. Notice that all the above groups can be collected into the following commutative diagram:

$$
\begin{array}{ccccccccc}
\mathcal{D}_{fix}(M, B) & \rightarrow & \mathcal{D}_{inv}(M, B) & \rightarrow & \mathcal{D}(B) \\
\sim & & \sim & & \sim & & \sim & & \sim \\
\mathcal{D}_{nh}(M, B) & \rightarrow & \mathcal{D}_{inv}(M, B) & \rightarrow & \mathcal{D}(B) \\
\rho \downarrow & & \rho \downarrow & & \rho \downarrow & & \rho \downarrow & & \rho \downarrow \\
\mathcal{D}_{fix}(M, B, p) & \rightarrow & \mathcal{D}_{inv}(M, B, p) & \rightarrow & \mathcal{D}(B) \\
\GL(E, B) & \rightarrow & \GL(E) & \rightarrow & \mathcal{D}(B) \\
\end{array}
$$

(4.3)
where the vertical inclusions $\hookrightarrow\tilde{\to}$ are homotopy equivalences due to Theorem 4.1.1, and

$$\kappa : D_{\text{inv}}(M, B) \to D(B), \quad \kappa(h) = h|_B,$$

$$\tilde{\rho} : \text{GL}(E) \to D(B), \quad \tilde{\rho}(h) = h|_B,$$

$$\rho : D_{\text{inv}}(M, B, p) \to \text{GL}(E), \quad \rho(h) = T_{\rho_B} h.$$

are natural restriction homomorphisms being $W^{-r}$-continuous for all $r \in \{0, \ldots, \infty\}$. The following statement allows to reduce the study of the homotopy type of $D_{\text{inv}}(M, B)$ to simpler groups, see §4.4.

**Lemma 4.2.1.** Suppose $B$ is compact. Then in the diagram (4.3) for every consecutive pair of horizontal, vertical or diagonal arrows of the form $P \hookrightarrow Q \tilde{\to} R$, the second arrow $\alpha : Q \to R$ admits a local section $s : U \to Q$ defined on some neighborhood $U$ of the unit of $R$, and therefore it is a locally trivial principal $P$-fibration over its image $\alpha(P)$. In particular, the image of $\alpha$ is an union of path components of $R$.

**Proof.** Fix an orthogonal structure on $E$ and let $\|\cdot\| : E \to [0; +\infty)$ be the corresponding norm. Let also $\mu : [0; 1] \to [0; 1]$ be any $C^\infty$ function such that $\mu([0; 0.2]) = 0$ and $\mu([0.8; 1]) = 1$.

1) **Proof for $\kappa : D_{\text{inv}}(M, B) \to D(B)$.** As mentioned above, due to [8, 30, 22], the restriction map $\tilde{\kappa} : D(M) \to \text{Emb}(B, M)$ is a locally trivial fibration over its image. In particular, there exists a $W^\infty$-neighborhood $\tilde{U}$ of the identity inclusion $i : B \subset M$ and a section $\tilde{\lambda} : \tilde{U} \to D(M)$ of $\tilde{\kappa}$. Notice that $D(B)$ can be identified with the subspace of $\text{Emb}(B, M)$ consisting of embeddings $h : B \to M$ such that $h(B) = B$. Then $U := \tilde{U} \cap D(B)$ is a neighborhood of $\text{id}_B$ in $D(B)$, and it is evident that $\tilde{\lambda}(U) \subset D_{\text{inv}}(M, B)$. Hence $\tilde{\lambda}|_U : U \to D_{\text{inv}}(M, B)$ is the desired section of $\kappa$.

2) **Proof for $\tilde{\rho} : \text{GL}(E) \to D(B)$.** Recall that for compact $B$ the group $D(B)$ is locally contractible with respect to the topology $W^\infty$. Moreover, (see e.g. the proof main theorem in [22]), there exists a $W^\infty$ open neighborhood $U$ of $\text{id}_B$ in $D(B)$, and a $W^{\infty, \infty}$ continuous map $H : U \times [0; 1] \times B \to B$ such that

(a) $H(\text{id}_B, t, x) = x$ for all $t \in [0; 1]$;

(b) $H(h, 0, x) = x$ and $H(h, 1, x) = h(x)$ for all $h \in U$ and $x \in B$;

(c) for each $h \in U$ the map $H_h : [0; 1] \times B \to B$, $H_h(t, x) = H(h, t, x)$, is a $C^\infty$ isotopy (between $\text{id}_B$ and $h$).

Fix any affine connection $\nabla$ on $E$. Then for every $C^\infty$ curve $\gamma : [0; 1] \to B$ there is a “parallel transport with respect to $\nabla$” being a family of linear isomorphisms $\Gamma_t : E_{\gamma(0)} \to E_{\gamma(t)}$, $t \in [0; 1]$, between fibers over the points of $\gamma$ such that $\Gamma_0$ is the identity, e.g. [20, Theorem 9.8].

Let us mention that if $\gamma$ is a constant path, then $\Gamma_t$ is the identity for all $t \in [0; 1]$.

Moreover, $\Gamma$ smoothly depends on $\gamma$ in the following sense: if $G : [0; 1] \times B \to B$ is a $C^\infty$ isotopy such that $G_0 = \text{id}_B$, then it induces a unique $C^\infty$ isotopy $\hat{G} : [0; 1] \times E \to E$ such that $\hat{G}_0 = \text{id}_E$, each $\hat{G}_t$ is a vector bundle morphism over $G_t$.

In particular, there exists a $W^\infty$-continuous map $\hat{H} : U \times [0; 1] \times E \to E$ such that for every $h \in U$, the map $\hat{H}(h, \cdot, \cdot) : [0; 1] \times E \to E$ is an isotopy being a lifting of the isotopy $H_h$ and consisting of vector bundle isomorphisms of $E$. Then the required section $\hat{\sigma} : U \to \text{GL}(E)$ of $\hat{\rho}$ can be given by $\hat{\sigma}(h) = \hat{H}(h, 1, \cdot) : E \to E$. 


3) Proof for $\kappa : D_{inv}(M, B, p) \to D(B)$. Let $\tilde{H} : U \times [0; 1] \times E \to E$ be the map constructed in 2). It has the following two properties:

(i) $\tilde{H}(id_B, t, x) = x$ for all $x \in E$ and $t \in [0; 1]$;
(ii) $\tilde{H}(h, 0, x) = x$, for all $h \in U$ and $x \in E$.

Define the following map $\lambda : U \to \mathcal{C}^\infty(M, M)$ by

$$\lambda(h)(x) = \begin{cases} \tilde{H}(h, 1 - \mu(\|x\|), x), & x \in E, \\ x, & x \in M \setminus E. \end{cases}$$

Then $\lambda(h)$ is indeed $\mathcal{C}^\infty$ due to (ii), and $\lambda(h)|_B = h$ for all $h \in U$. It is also easy to see that $\lambda$ is $\mathcal{W}^{\infty, \infty}$-continuous.

Moreover, it follows from (i) that $\lambda(id_B)(x) = x$ for all $x \in M$. In other words, $\lambda(id_B) = id_M$, whence $V := U \cap \lambda^{-1}(D(M))$ is an open neighborhood of $id_B$ in $D(B)$ such that $\lambda(V) \subset D(M)$.

Also notice that $\lambda(h)(x) = \tilde{H}(h, 1, x) = \tilde{\sigma}(h)(x)$ for $\|x\| \leq 0.2$, so $\lambda(h)$ coincides with vector bundle morphism $\tilde{\sigma}(h)$ of $E$ near $B$, so $\tilde{\sigma}|_V = \rho \circ \lambda$. In other words, $\lambda(V) \subset D_{inv}(M, B, p)$, and thus $\lambda|_V : V \to D_{inv}(M, B, p)$ is the desired section of $\kappa$.

4) Proof for $\rho : D_{inv}(M, B, p) \to GL(E, B)$. Let $\sigma : GL(E, B) \to \mathcal{C}^\infty(M, M)$ be a map defined by

$$\sigma(h)(x) = \begin{cases} (1 - \mu(\|x\|))h(x) + \mu(\|x\|)x, & x \in E, \\ x, & x \in M \setminus E. \end{cases}$$

Evidently,

(a) $\sigma$ is $\mathcal{W}^{r, r}$-continuous for all $r \geq 0$;
(b) $\sigma(h)(x) = h(x)$ for all $x \in R_{0.2}$;
(c) $\sigma(h)(x) = x$ on $M \setminus R_{0.8}$;
(d) $\sigma(id_E)(x) = x$ for all $x \in M$.

Since the group $D(M)$ of diffeomorphisms of $M$ is $\mathcal{W}^\infty$-open in $\mathcal{C}^\infty(M, M)$ and $\sigma$ is $\mathcal{W}^{\infty, \infty}$ continuous, the following set $W := \sigma^{-1}(D(M))$ is open in $GL(E, B)$. Moreover, $id_M \in W$ due to (d). Also, due to (b), $\rho \circ \sigma(h) = h$ and $\sigma(h) \in D_{fix}(M, B)$ for all $h \in GL(E, B)$.

Thus, the restriction $\sigma|_W$ is the desired local section of $\rho$ on a neighborhood of $id_E$.

5) Proof for $\rho : D_{inv}(M, B, p) \to GL(E)$. By 2), 3), 4), there exist

- a neighborhood $U$ of $id_B$ in $D(B)$ and a section $\tilde{\sigma} : V \to GL(E)$ of $\tilde{\rho}$;
- a neighborhood $V \subset U$ of $id_B$ in $D(B)$ and a section $\lambda : V \to D_{inv}(M, B, p)$ of $\kappa$ such that $\tilde{\sigma}|_V = \rho \circ \lambda$;
- a neighborhood $W$ of $id_E$ in $GL(E, B)$ and a section $\sigma : W \to D_{fix}(M, B, p)$ of $\rho$.

Since, by 2), the map $\tilde{\rho} : GL(E) \to D(B)$ is a locally trivial principal $GL(E, B)$-fibration, the following subset of $GL(E)$:

$$\mathcal{O} := \{g \circ \tilde{\sigma}(h) \mid h \in V, g \in W\}$$

is an open neighborhood of $id_E$ in $GL(E)$. Moreover, for each $k \in \mathcal{O}$ the representation $k = g \circ \tilde{\sigma}(h)$ with $h \in V$ and $g \in W$ is unique, and $g$ and $h$ continuously depend on $k$.

Indeed,

$$\tilde{\rho}(k) = \tilde{\rho}(g \circ \tilde{\sigma}(h)) = \tilde{\rho}(g) \circ \tilde{\rho}(\tilde{\sigma}(h)) = id_B \circ h = h,$$
\[ k \circ (\hat{\sigma}(\hat{\rho}(k)))^{-1} = k \circ h^{-1} = g. \]

Now the section \( \sigma' : \mathcal{O} \to \mathcal{D}_{\text{inv}}(M,B,p) \) of \( \rho : \mathcal{D}_{\text{inv}}(M,B,p) \to \text{GL}(E) \) can be given by the following formula: if \( k = g \circ \hat{\sigma}(h) \), then \( \sigma'(k) := \sigma(g) \circ \lambda(h) \).

\[ \square \]

4.3. Fibration of automorphisms of the vector bundle. Denote by \( \text{GL}(E,B) \) the group of \( \mathcal{C}^\infty \) vector bundle isomorphisms \( h : E \to E \) over \( \text{id}_B \), that is \( \rho \circ h = \rho \). We recall here that \( \text{GL}(E,B) \) can be identified with the space of \( \mathcal{C}^\infty \) sections of a certain principal \( \text{GL}(\mathbb{R}^m) \)-fibration \( E_{\text{aut}} \to B \), see Lemma 4.3.1 below.

Let \( p_m = \bigoplus_m p : \bigoplus_m E \to B \) be the Whitney sum of \( m \) copies of \( E \), so each element of \( \bigoplus_m E \) is an ordered \( m \)-tuple of vectors \( (v_1, \ldots, v_m) \) belonging to the same fiber of \( p \). Denote by \( Q \subset \bigoplus_m E \) the subset consisting of linearly independent \( m \)-tuples, called frames. Then the group \( \text{GL}(\mathbb{R}^m) \) freely acts on \( Q \), so that the restriction \( p_m|_Q : Q \to B \) is a principal \( \text{GL}(\mathbb{R}^m) \)-bundle over \( B \), and \( p : E \to B \) is the associated \( \mathbb{R}^m \)-bundle corresponding to the natural action of \( \text{GL}(\mathbb{R}^m) \) on \( \mathbb{R}^m \).

Let \( X = \{(a,b) \in Q \times Q \mid p_m(a) = p_m(b)\} \) be the fiberwise product of the total space \( Q \), so it consists of pairs of frames over the same point, and the correspondence \( (a,b) \mapsto p_m(a) \), is the pull back of the fibration \( p_m : Q \to B \) corresponding to the same map \( p : E \to B \).

Notice that for every such pair \( (a,b) \in X \) there exists a unique matrix \( A \in \text{GL}(\mathbb{R}^m) \) such that \( b = aA \). Thus \( (a,b) \) can be regarded as an automorphism of the fiber \( E_{p_m(a)} \) written in the basis \( a \).

Let us get rid of dependence on the basis. To do that notice that the group \( \text{GL}(\mathbb{R}^m) \) naturally acts from the right on \( X \) by the rule, if \( A \in \text{GL}(\mathbb{R}^m) \), and \( (a,b) \in X \), then \( (a,b)A = (aA,bA) \). Let \( E_{\text{aut}} = X/\text{GL}(\mathbb{R}^m) \) be the quotient space. Then we have a natural projection \( q : E_{\text{aut}} \to B \), and the fibers \( q^{-1}(x) \) can be regarded as automorphisms of \( E_x \).

Let \( S(E_{\text{aut}}) \) be the space of \( \mathcal{C}^\infty \)-sections of \( q \). Then \( S(E_{\text{aut}}) \) is a group with respect to the pointwise composition of automorphisms. Indeed, the multiplication in the fibers of \( S(E_{\text{aut}}) \) is defined as follows. For \( (a,b), (c,d) \in X \) denote by \( [a,b] \) its equivalence class in \( E_{\text{aut}} \). Let \( (a,b),(c,d) \in X \) be two pairs of frames in the same fiber of \( S(E_{\text{aut}}) \), i.e. \( p_m(a) = p_m(c) \). Then there is a unique matrix \( U \in \text{GL}(\mathbb{R}^m) \) such that \( bU = c \), and we put \( [a,b] \cdot [c,d] := [aU,d] \). Notice that this definition does not depend on a choice of representatives: if \( A, B \in \text{GL}(\mathbb{R}^m) \) are any matrices, so \( (aA,bA),(cB,dB) \) are some other representatives of \( (a,b) \) and \( (c,d) \), then \( bA(A^{-1}UB) = cB \), whence

\[ [aA,bA] \cdot [cB,dB] := [aA(A^{-1}UB),dB] = [aUB,dB] = [aU,d]. \]

One easily checks that this operation is associative, the unit is the class \([a,a]\) for any frame \( a \), while \([a,b]^{-1} = [b,a]\).

We also have a free action of \( \text{GL}(\mathbb{R}^m) \) on \( E_{\text{aut}} \) by \([a,b]A = [aA,b] \), which is transitive on each fiber of \( q \), so \( q : E_{\text{aut}} \to B \) is a principal \( \text{GL}(\mathbb{R}^m) \)-fibration.

**Lemma 4.3.1.** There is a bijection \( \theta : \text{GL}(E,B) \to S(E_{\text{aut}}) \) being a \( \mathcal{W}^{r,r} \)- as well as \( S^{r,r} \)-homeomorphism for all \( r \in \{0,1,\ldots,\infty\} \).

**Proof.** 1) First suppose that \( p \) is a trivial vector bundle, so \( E = B \times \mathbb{R}^m \). Then \( q \) is also a trivial fibration, whence \( E_{\text{aut}} = B \times \text{GL}(\mathbb{R}^m) \) and \( \mathcal{C}^\infty \) sections are just \( \mathcal{C}^\infty \) maps \( A : B \to \text{GL}(\mathbb{R}^m) \). Furthermore, each \( h \in \text{GL}(E,B) \) is a \( \mathcal{C}^\infty \) map given by \( h : B \times \mathbb{R}^m \to B \times \mathbb{R}^m \), \( h(x,v) = (x,A_h(x)v) \), for some map \( A_h : B \to \text{GL}(\mathbb{R}^m) \). It follows from the formula for \( h \) that \( A_h \) is \( \mathcal{C}^\infty \) iff \( h \) is so. Then \( \theta \) can be defined by \( \theta(h) = A_h \).
2) Consider the general case. Let \( h \in \text{GL}(E, B) \). Then \( h(E_x) = E_x \), and the restriction \( h|_{E_x} : E_x \to E_x \) is a linear isomorphism for every \( x \in B \). Hence we can define the following section \( \theta(h) : B \to E_{\text{aut}} \) of \( q \) by \( \theta(h)(x) = h|_{E_x}, \ x \in B \). Passing to local coordinates, see 1), we obtain that \( \theta(h) \) is \( C^\infty \), and thus we get a well-defined map \( \theta : \text{GL}(E, B) \to \mathcal{S}(E_{\text{aut}}) \). Moreover, again it follows from 1) that \( \theta \) is a continuous bijection and its inverse is also continuous with respect to all \( W^r \) and \( S^r \) topologies. \( \square \)

4.4. Conclusion. Using short exact sequences of the corresponding fibrations, the above observations allow to reduce (at least partially) the computation of the homotopy types of \( \mathcal{D}_{\text{inv}}(M, B) \) to study

- the identity path components of simpler groups \( \mathcal{D}_{\text{nb}}(M, B), \text{GL}(E, B) \), and \( \mathcal{D}(B) \),

and the images of all arrows \( \kappa \) and \( \rho \) in the corresponding groups \( \pi_0\mathcal{D}(B), \pi_0\text{GL}(E, B) \), and \( \pi_0\text{GL}(E) \).

The group \( \mathcal{D}(B) \) might be simpler, since \( \dim(B) < \dim(M) \), and the homotopy types of \( \mathcal{D}_{\text{nb}}(B) \) for manifolds of dimensions 0, 1, 2 are completely known, and they also mostly computed for \( \dim(B) = 3 \).

Also the group \( \text{GL}(E, B) \) reduces to the space of sections of certain \( \text{GL} \)-fibration over \( B \), see Lemma 4.3.1, and can be studied by purely homomopical methods of obstruction theory.

Finally, the group \( \mathcal{D}_{\text{nb}}(M, B) \), on the one hand, might be regarded simpler in a “conceptual” sense: if we regard \( B \) as a “singularity”, then \( \mathcal{D}_{\text{nb}}(M, B) \) consists of diffeomorphisms supported out of that singularity. On the other hand, if \( M \) is compact, then \( \mathcal{D}_{\text{nb}}(M, B) \) can be identified with the group of compactly supported diffeomorphism of \( M \setminus B \). Such groups are widely studied, e.g. [35, 21, 36, 31, 12, 13, 32].

5. Linearization theorem for leaf preserving diffeomorphisms

In this section we will apply linearization theorem to leaf preserving diffeomorphisms for singular foliations, see Theorems 5.1.5, 5.1.8 below.

5.1. Homogeneous partitions. Let \( \mathcal{F} \) be a partition of a manifold \( M \). The elements of \( \mathcal{F} \) will also be called leaves. A subset \( B \subset M \) is \( \mathcal{F} \)-saturated, if \( B \) is a union of leaves of \( \mathcal{F} \). For an open subset \( U \subset M \) denote by \( \mathcal{F}|_U \) the partition of \( U \) into path components of the non-empty intersections \( U \cap \omega \) for all \( \omega \in \mathcal{F} \). We will call \( \mathcal{F}|_U \) the restriction of \( \mathcal{F} \) onto \( U \).

Let \( U \subset M \) be a subset. Then a map \( h : U \to M \) will be called

- \( \mathcal{F} \)-foliated if for each \( \omega \in \mathcal{F} \) the image \( h(\omega \cap U) \) is contained in some (possibly distinct from \( \omega \)) leaf of \( \mathcal{F} \);

- \( \mathcal{F} \)-leaf preserving if \( h(\omega \cap U) \subset \omega \) for all \( \omega \in \mathcal{F} \).

Denote by \( \mathcal{D}(\mathcal{F}) \), the group of \( \mathcal{F} \)-leaf preserving diffeomorphisms of \( M \), and for each subset \( B \subset M \) put:

\[
\mathcal{D}_{\text{inv}}(\mathcal{F}, B) := \mathcal{D}(\mathcal{F}) \cap \mathcal{D}_{\text{inv}}(M, B), \quad \mathcal{D}_{\text{fix}}(\mathcal{F}, B) := \mathcal{D}(\mathcal{F}) \cap \mathcal{D}_{\text{fix}}(M, B),
\]

\[
\mathcal{D}_{\text{nb}}(\mathcal{F}, B) := \mathcal{D}(\mathcal{F}) \cap \mathcal{D}_{\text{nb}}(M, B).
\]

If \( B \) is a submanifold with a regular neighborhood \( p : E \to B \), then we also denote:

\[
\mathcal{D}_{\text{inv}}(\mathcal{F}, B, p) := \mathcal{D}(\mathcal{F}) \cap \mathcal{D}_{\text{inv}}(M, B, p), \quad \mathcal{D}_{\text{fix}}(\mathcal{F}, B, p) := \mathcal{D}(\mathcal{F}) \cap \mathcal{D}_{\text{fix}}(M, B, p).
\]
Definition 5.1.1. Let $B$ be an $\mathcal{F}$-saturated submanifold of $M$ with a regular neighborhood $p : E \to B$. Say that a neighborhood $U$ of $B$ in $E$ is $\mathcal{F}$-homogeneous (with respect to $p$), whenever it has the following property:

- if $x, y \in U$ and $\tau > 0$ are such that $\tau x, \tau y \in U$ and $x, y$ belong to the same element of $\mathcal{F}$, then $\tau x, \tau y$ also belong to the same element $\mathcal{F}$.

Evidently, if $U$ is $\mathcal{F}$-homogeneous, then so is any other neighborhood $V \subset U$ of $B$ in $E$.

Before stating our main result Theorem 5.1.5 we will present a class of partitions admitting $\mathcal{F}$-homogeneous neighborhoods of saturated submanifolds which will be useful to keep in mind.

Definition 5.1.2. Let $f : M \to R$ be a map into some set $R$, and $B \subset M$ a subset. Then by an $(f, B)$-partition we will call a partition $\mathcal{F}_{f, B}$ of $M$ into path components of $B$ and path components of the sets $f^{-1}(c) \setminus B$ for all $c \in R$.

Lemma 5.1.3. Let $p : E \to B$ be a vector bundle and $f : E \to R$ a continuous function such that

1. $f$ is homogeneous of some degree $k > 0$ (possibly fractional) on fibers, i.e. $f(\tau v) = \tau^k f(v)$ for all $\tau > 0$ and $v \in E$;
2. for some $a < 0$ and $b > 0$ the path components of $f^{-1}(a)$ and $f^{-1}(b)$ are closed in $E$.

Such a function will be called admissible homogeneous. Let also $\mathcal{F}$ be the $(f, B)$-partition of $E$. Then

(a) each neighborhood $U$ of $B$ in $E$ is $\mathcal{F}$-homogeneous;
(b) $\varnothing \setminus \omega \subset B$ for all $\omega \in \mathcal{F}$.

Proof. First we show that for any $c \neq 0$ the path components of the set $f^{-1}(c)$ are closed in $E$. For definiteness assume that $c > 0$. Put $\tau = (c/b)^{1/k}$. Then we have a well-defined homeomorphism $h : E \to E$, $h(v) = \tau v$.

Notice that $h(f^{-1}(b)) = f^{-1}(c)$. Indeed, if $f(v) = b$, then

$$f(h(v)) = f(\tau v) = \tau^k f(v) = (c/b) \cdot b = c.$$ 

Hence $h$ maps the path components of $f^{-1}(b)$ onto path components of $f^{-1}(c)$. Since path components of $f^{-1}(b)$ are closed in $E$, so must be path components of $f^{-1}(c)$. The proof for $c < 0$ is similar and uses $a$ instead of $b$.

Further let us mention that $f(B) = 0$. Indeed, let $v \in B$, and $\tau > 1$. Then $v = \tau v$ and $f(v) = f(\tau v) = \tau^k f(v)$, which is possible only when $f(v) = 0$.

We will show that $E$ is $\mathcal{F}$-homogeneous. Then, as mentioned above, any other neighborhood of $B$ in $E$ is $\mathcal{F}$-homogeneous as well. Let $\omega$ be a leaf of $\mathcal{F}$, $x, y \in \omega$ any two points, and $\tau > 0$. It is necessary to show that $\tau x, \tau y$ also belong to the same leaf and $\varnothing \setminus \omega \subset B$. Consider the following cases.

(i) If $\omega$ is a path component of $B$, then $\tau x = x, \tau y = y$, so they belong to the same leaf $\omega$ of $\mathcal{F}$. Moreover, since $B$ is closed in $E$ and contains $\omega$, we have that $\varnothing \subset B$, whence $\varnothing \setminus \omega \subset B$ as well.

(ii) Suppose $\omega$ is a path component of the set $f^{-1}(c) \setminus B$ for some $c \in R$, so there exists a path $\gamma : [0; 1] \to \omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In particular, $f \circ \gamma : [0; 1] \to \mathbb{R}$ is a constant map into $c \in \mathbb{R}$. Then $\tau \gamma : [0; 1] \to E$ is a path connecting $\tau x$ and $\tau y$. Since $\gamma([0; 1]) \subset E \setminus B$, we also have that $\tau \gamma([0; 1]) \subset E \setminus B$. Moreover, $f(\tau \gamma(t)) = \tau^k f(\gamma(t)) = \tau^k c$, $t \in [0; 1]$. Thus $\tau \gamma$ is a path between $\tau x$ and $\tau y$ in some path component of $f^{-1}(\tau^k c) \setminus B$, being by definition a leaf of $\mathcal{F}$. This proves (a).
If $c \neq 0$, then $\omega$ is a path component of $f^{-1}(c)$ and is closed as shown above. Hence $\overline{\omega} \setminus \omega = \emptyset \subset B$.

On the other hand, suppose $c = 0$. Since $\omega \subset f^{-1}(0) \setminus B$, and $f^{-1}(0)$ is closed, we have that $\overline{\omega} \subset f^{-1}(0)$, whence $\overline{\omega} \setminus \omega \subset f^{-1}(0) \setminus (f^{-1}(0) \setminus B) = B$. \hfill \Box

Fiberwise homogeneous functions on tangent and cotangent bundles are widely studies, see e.g. [4, 6] and references therein.

Let us also mention that the function $f$ in Lemma 5.1.3 is assumed to be only continuous, whence the corresponding homogeneous partition $\mathcal{F}$ is not necessarily smooth, and might have even a «fractal» structure.

Example 5.1.4. Let $p : E = B \times \mathbb{R}^m \to B$ be a trivial vector bundle over connected manifold $B$, $k > 0$ a positive real number, $g : S^{m-1} \to \mathbb{R}$ a continuous (possibly even nowhere differentiable) function, and $f : E \to \mathbb{R}$ a function given by

$$f(x, v) = \begin{cases} 0, & v = 0, \\ \|v\|^k \cdot g(v/\|v\|), & v \neq 0, \end{cases}$$

where $\|v\| = \sqrt{v_1^2 + \cdots + v_m^2}$ is the usual Euclidean norm of a vector $v = (v_1, \ldots, v_m)$. Notice that if $g(v) = 0$ for some $v \in S^{m-1}$, then $f$ is zero on the line $\mathbb{R}v$ passing through the origin and $v$.

Evidently, if $k = 1$ and $g \equiv 1$ is constant, then $f(x, v) = \|v\|$. Also if $k = 2$, $m = 2$, and $g : S^1 \to \mathbb{R}$ is given by $g(\phi) = \cos(\phi) \sin(\phi) = \frac{\sin(2\phi)}{2}$, then $f(x, u, v) = uv$ for $(u, v) \in \mathbb{R}^2$.

We claim that $f$ satisfies assumptions of Lemma 5.1.3.

(1) Indeed, let $\tau > 0$ and $(x, v) \in B \times \mathbb{R}^m$. Then $\tau(x, v) = (x, \tau v)$. Hence if $v = 0$, then $f(\tau(x, 0)) = f(x, 0)) = 0 = \tau^k f(x, v)$. On the other hand, if $v \neq 0$, then

$$f(x, \tau v) = \|\tau v\|^k \cdot g(\tau v/\|\tau v\|) = \tau^k \|v\|^k \cdot g(v/\|v\|) = \tau^k f(x, v),$$

so $f$ is homogeneous of degree $k$.

(2) Let $G_- = B \times g^{-1}((-\infty, 0))$. Then for each $a < 0$ we have a homeomorphism $\alpha : f^{-1}(a) \to G_-$, $\alpha(x, v) = (x, v/\|v\|)$, whose inverse is given by $\alpha^{-1}(x, v) = (x, -|a/g(v)|^{1/k}v)$. Indeed, if $(x, v) \in G_-$, so $\|v\| = 1$, then $f \circ \alpha^{-1}(x, v) = -|a/g(v)|g(v) = -|a| = a$.

Since $G_-$ is an open subset of the manifold $B \times S^{m-1}$, its path components are open closed in $G_-$. Hence each path component $\omega$ of $f^{-1}(a)$ is open closed in $f^{-1}(a)$. But $f^{-1}(a)$ is closed in $B \times \mathbb{R}^m$, whence $\omega$ is closed in $B \times \mathbb{R}^m$ as well.

Theorem 5.1.5 (Foliated linearization theorem). Let $\mathcal{F}$ be a partition of $M$, $B$ an $\mathcal{F}$-saturated submanifold with a regular neighborhood $p : E \to B$. Suppose that there exists a compact neighborhood $U \subset E$ of $B$ such that

(a) $U$ is $\mathcal{F}$-homogeneous;
(b) $U \cap (\overline{\omega} \setminus \omega) \subset B$ for each $\omega \in \mathcal{F}$.

Then the inclusion of pairs of groups of $\mathcal{F}$-leaf preserving diffeomorphisms:

$$\left(\mathcal{D}_{\text{fix}}(\mathcal{F}, B, p), \mathcal{D}_{\text{inv}}(\mathcal{F}, B, p)\right) \subset \left(\mathcal{D}_{\text{fix}}(\mathcal{F}, B), \mathcal{D}_{\text{inv}}(\mathcal{F}, B)\right),$$

is mutually a $W^{\infty,\infty}$- and $S^{\infty,\infty}$-homotopy equivalence.

Proof. Let $G : \mathcal{D}_{\text{inv}}(M, B) \times [0; 1] \to \mathcal{D}_{\text{inv}}(M, B)$ be the deformation of the group $\mathcal{D}_{\text{inv}}(M, B)$ into $\mathcal{D}_{\text{inv}}(M, B, p)$ constructed in Theorem 4.1.1 for $N = U$. It suffices to check
that \(G(\mathcal{D}_{\text{inv}}(F, B) \times [0; 1]) \subseteq \mathcal{D}_{\text{inv}}(F, B)\). Then the restriction of \(G\) to \(\mathcal{D}_{\text{inv}}(F, B) \times [0; 1]\) will be a deformation of the right pair in (5.1) into the corresponding left pair.

Let \(h \in \mathcal{D}_{\text{inv}}(F, B)\) and \(V = R_{0,88(h)}\). Let also \(x \in M\), and \(\omega\) be the leaf of \(F\) containing \(x\). We need to show that \(G(x, t) \in \omega\) for all \(t \in [0; 1]\). This will imply that \(G_t(h)\) is \(F\)-leaf preserving.

1) If \((x, t) \in (B \cup \overline{M \setminus V}) \times [0; 1]\), then by (D), formula (4.2) for \(G\), and (iv) from the proof of Theorem 4.1.1, \(G(h, t)(x) = h(x) \in \omega\), since \(h\) is \(F\)-leaf preserving.

2) Suppose \((x, t) \in V \times (0; 1]\) and let \(\tau = \phi(x, t)\). Then by (1.2), \(\tau > 0\). We claim that the four points \(x, \tau x, h(\tau x), G(h, t)(x) := \frac{1}{\tau}(h(\tau x))\) belong to \(U\).

Indeed, \(x \in V \subset U\) by definition. Since \(V\) is star-like and \(\tau = \phi(x, t) \in [0; 1]\), we have that \(\tau x \in V \subset U\). Moreover, by Theorem 1.1.1, \(H(h, t)(V) \subset N = U\) for all \(t \in [0; 1]\), and thus \(H(h, t)(\tau x) \in U\) as well. For \(t = 1\), this implies that \(h(\tau x) = H(h, 1)(\tau x) = G(h, 1)(\tau x) \in U\). Also for other \(t\), we get that \(\frac{1}{\tau}(h(\tau x)) = H(h, t)(\tau x) = G(h, t)(x) \in U\) as well.

Since \(h\) is \(F\)-leaf preserving, the points \(\tau x\) and \(h(\tau x)\) belong to the same leaf of \(F\). Moreover, as \(U\) is \(F\)-homogeneous, their respective images under multiplication by \(\frac{1}{\tau}\), that is \(x := \frac{1}{\tau}x\) and \(G(h, t)(x) = \frac{1}{\tau}h(\tau x)\), must also belong to the same leaf of \(F\). In other words, \(G(h, t)(x) \in \omega\).

3) It remains to show that \(G(x, 0) \in \omega\) if \(x \in V\). Due to 1) and 2), \(G(h, t)(x) \in U \cap \omega\) for \(t \in (0; 1]\). Suppose \(G(x, 0) \notin \omega\). Then by continuity of \(G\), \(G(x, 0) \in U \cap (\mathcal{W} \setminus \omega) \subset B\). But \(G_0\) is a diffeomorphism of \(M\) which coincide with \(h\) on \(B\), and therefore it yields a self-bijection of \(B\). Hence \(x \in B\), and thus by 1), \(G(x, 0) = h(x) \in \omega\). \(\square\)

**Remark 5.1.6.** Theorem 5.1.5 implies that for each \(h \in \mathcal{D}_{\text{inv}}(F, B)\) the corresponding vector bundle morphism \(T_{hB}h : E \to E\) also preserves leaves of \(F\) near \(B\). This is close to [24, Lemma 36] claiming that if \(h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\) is a \(C^1\) diffeomorphism, and \(f : \mathbb{R}^n \to \mathbb{R}\) a continuous homogeneous function such that \(f \circ h = f\), i.e. \(h\) preserves level sets of \(f\), then \(f \circ T_{hB}h = f\) as well.

**Remark 5.1.7.** As in the non-foliated case we have a homomorphism \(\rho : \mathcal{D}_{\text{inv}}(F, B, p) \to \text{GL}(E)\) associating to each \(h \in \mathcal{D}_{\text{inv}}(F, B, p)\) the unique vector bundle morphism \(\rho(h) : E \to E\) such that \(h = \rho(h)\) near \(B\). In particular, \(\rho(h)\) preserves leaves of \(F\) “near \(B\)” in the sense that \(h\) does so. However, the description of the image of that map and the question whether \(\rho : \mathcal{D}_{\text{inv}}(F, B, p) \to \rho(\mathcal{D}_{\text{inv}}(F, B, p))\) is a locally trivial fibration, are essentially harder than in the non-foliated case and essentially depend on the structure of \(F\).

We will now consider the case when the map \(\rho\) is indeed a locally trivial fibration over its image.

Let \(p : E \to B\) be a vector bundle over a compact manifold \(B\), so \(\text{GL}(E)\) can be regarded as a subgroup of \(\mathcal{D}_{\text{inv}}(E, B)\), and \(\text{GL}(E, B)\) as a subgroup of \(\mathcal{D}_{\text{fix}}(E, B)\).

**Theorem 5.1.8.** Let \(f : E \to \mathbb{R}\) be an admissible homogeneous function, and \(F\) be the \((f, B)\)-partition of \(E\). Then the homomorphism \(\rho : \mathcal{D}_{\text{inv}}(F, B, p) \to \text{GL}(E)\) is a retraction onto its image, \(\ker(\rho) = \mathcal{D}_{\text{fix}}(F, B)\),

\[
\rho(\mathcal{D}_{\text{fix}}(F, B, p)) = \mathcal{D}_{\text{fix}}(F, B, p) \cap \text{GL}(E, B) = \mathcal{D}_{\text{fix}}(F, B) \cap \text{GL}(E, B), \tag{5.2}
\]

\[
\rho(\mathcal{D}_{\text{inv}}(F, B, p)) = \mathcal{D}_{\text{inv}}(F, B, p) \cap \text{GL}(E) = \mathcal{D}_{\text{inv}}(F, B) \cap \text{GL}(E). \tag{5.3}
\]
Whence we have the following homotopy equivalences ($\simeq$) and homeomorphisms ($\cong$):
\[
\begin{align*}
\mathcal{D}_{\text{fix}}(\mathcal{F}, B) &\simeq \mathcal{D}_{\text{fix}}(\mathcal{F}, B, p) \cong \mathcal{D}_{\text{nb}}(\mathcal{F}, B) \times \rho(\mathcal{D}_{\text{fix}}(\mathcal{F}, B, p)), \\
\mathcal{D}_{\text{inv}}(\mathcal{F}, B) &\simeq \mathcal{D}_{\text{inv}}(\mathcal{F}, B, p) \cong \mathcal{D}_{\text{nb}}(\mathcal{F}, B) \times \rho(\mathcal{D}_{\text{inv}}(\mathcal{F}, B, p))
\end{align*}
\] (5.4) (5.5)
with respect to $\mathcal{W}^\infty$ and $\mathcal{S}^\infty$ topologies.

**Proof.** (1) The following statements are evident:
\[
\ker(\rho) = \mathcal{D}_{\text{nb}}(\mathcal{F}, B),
\]
\[
\mathcal{D}_{\text{inv}}(\mathcal{F}, B, p) \cap \text{GL}(E) = \mathcal{D}_{\text{inv}}(\mathcal{F}, B) \cap \text{GL}(E),
\]
\[
\mathcal{D}_{\text{fix}}(\mathcal{F}, B, p) \cap \text{GL}(E) = \mathcal{D}_{\text{fix}}(\mathcal{F}, B) \cap \text{GL}(E, B).
\]
Moreover, (5.2) follows from (5.3) and the observation that $h = \rho(h)$ on $B$, so $h$ is fixed on $B$ iff $\rho(h)$ is fixed on $B$.

Also, by Lemma 5.1.3, $\mathcal{F}$ satisfies assumptions of Theorem 5.1.5, whence we get the homotopy equivalences in (5.4) and (5.5).

(2) Let us show that $\rho(\mathcal{D}_{\text{inv}}(\mathcal{F}, B, p)) \subset \mathcal{D}_{\text{inv}}(\mathcal{F}, B) \cap \text{GL}(E)$. Let $h \in \mathcal{D}_{\text{inv}}(\mathcal{F}, B, p)$ and $h' = \rho(h) \in \text{GL}(E)$. We should show the vector bundle morphism $h'$ is also $\mathcal{F}$-leaf preserving, i.e. for each $v \in E$, its image $h'(v)$ belong to the same leaf as $v$.

Indeed, since $h = h'$ near $B$, there exists small $\tau > 0$ such that $h(\tau v) = h'(\tau v) = \tau h'(v)$, where the latter equality holds since $h'$ is a vector bundle morphism (linear of fibres). As $h$ is $\mathcal{F}$-leaf preserving, $\tau v$ and $\tau h'(v) = h(\tau v)$ belong to the same leaf of $\mathcal{F}$. Hence by $\mathcal{F}$-homogeneity of $U = E$, $v$ and $h'(v)$ also belong to the same leaf of $\mathcal{F}$.

(3) Now let $h \in \mathcal{D}_{\text{inv}}(\mathcal{F}, B) \cap \text{GL}(E)$. Then $\rho(h)$ and $h$ are two vector bundle isomorphisms which coincide near $B$. Then they should coincide on all of $E$. In other words, $\rho(h) = h$, and thus $g$ is a retraction of $\mathcal{D}_{\text{inv}}(\mathcal{F}, B, p)$ onto its subgroup $\mathcal{D}_{\text{inv}}(\mathcal{F}, B) \cap \text{GL}(E)$, which proves (5.3). Together with Lemma 2.2.1 this also implies the homeomorphism in (5.5). \[\square\]

Let $g : \mathbb{R}^m \to \mathbb{R}$ be an admissible homogeneous function with respect to the vector bundle $\mathbb{R}^m \to 0$ over a point, so $g$ is homogeneous of some (possibly fractional) degree $k > 0$, and the path components of its level sets $g^{-1}(c), c \neq 0$, are closed. Let $\mathcal{G}$ be the $(g, 0)$-partition of $\mathbb{R}^m$. Then Theorem 5.1.8 holds for $\mathcal{G}$. It will be convenient to denote the image of the retraction $\rho$ by
\[
\mathcal{L}^*(g) := \rho(\mathcal{D}_{\text{inv}}(\mathcal{G}, 0, p)) = \mathcal{D}_{\text{inv}}(\mathcal{G}, 0) \cap \text{GL}(\mathbb{R}^m).
\]
Then $\mathcal{L}^*(f)$ is closed in $\mathcal{D}_{\text{inv}}(\mathcal{G}, 0, p)$, as a retract of the Hausdorff space $\mathcal{D}_{\text{inv}}(\mathcal{G}, 0, p)$, whence it is also closed in $\text{GL}(\mathbb{R}^m)$. In particular, $\mathcal{L}^*(f)$ is a Lie subgroup of $\text{GL}(\mathbb{R}^m)$. It consists of linear automorphisms of $\mathbb{R}^m$ leaving invariant the leaves of $\mathcal{G}$.

Let also
\[
\mathcal{L}(g) = \{ A \in \text{GL}(\mathbb{R}^m) \mid g(Av) = g(v), v \in \mathbb{R}^m \}
\]
be the group of all linear automorphisms preserving $g$. Then $\mathcal{L}^*(g) \subset \mathcal{L}(g)$.

**Corollary 5.1.9.** Let $g : \mathbb{R}^m \to \mathbb{R}$ be an admissible homogeneous function with respect to the vector bundle $\mathbb{R}^m \to 0$ over a point. Let also $p : E = B \times \mathbb{R}^m \to B$ be a trivial vector bundle over a compact manifold, and $f : B \times \mathbb{R}^m \to \mathbb{R}$ the function given by $f(x, v) = g(v)$. Then $f$ is admissible homogeneous, so Theorem 5.1.8 holds for the $(f, B)$-partition $\mathcal{F}$. Moreover, in this case there is a **homeomorphism**:
\[
\sigma : \rho(\mathcal{D}_{\text{inv}}(\mathcal{F}, B, p)) \cong \mathcal{C}^\infty(B, \mathcal{L}^*(g)) \times \mathcal{D}(B),
\] (5.6)
such that \( \sigma(\rho(D_{fix}(\mathcal{F}, B, p))) = C^\infty(B, \mathcal{L}^*(g)) \times \text{id}_B \). Hence we get the following homotopy equivalences \((\simeq)\) and homeomorphisms \((\cong)\):

\[
\begin{align*}
D_{fix}(\mathcal{F}, B) \simeq D_{fix}(\mathcal{F}, B, p) & \cong D_{nb}(\mathcal{F}, B) \times C^\infty(B, \mathcal{L}^*(g)), \\
D_{inv}(\mathcal{F}, B) \simeq D_{inv}(\mathcal{F}, B, p) & \cong D_{nb}(\mathcal{F}, B) \times C^\infty(B, \mathcal{L}^*(g)) \times D(B) \cong D_{fix}(\mathcal{F}, B, p) \times D(B)
\end{align*}
\]

with respect to \(W^\infty\) and \(S^\infty\) topologies.

**Proof.** 1) Let us show that \( f \) is admissible homogeneous. Notice that for each \( \tau > 0 \) we have that

\[
f(\tau(x, v)) = f(x, \tau v) = g(\tau v) = \tau^k g(v) = \tau^k f(x, v),
\]

so \( f \) is homogeneous of the same degree as \( g \). Moreover, for each \( c \neq 0 \) each path component \( \beta \) of \( f^{-1}(c) \) is a product of some path component \( A \) of \( B \) with some path component \( \alpha \) of \( g^{-1}(c) \). But \( \alpha \) is closed since \( g \) is admissible homogeneous, while \( A \) is closed since \( B \) is locally path connected. Hence \( \beta = A \times \alpha \) is closed in \( E \). Therefore \( f \) is admissible homogeneous.

2) Let us construct a homeomorphism \( \sigma \). Recall that we have a natural homomorphism \( \rho : GL(E) \to D(B), \rho(h) = h|_B \), whose kernel is \( GL(E, B) \). Since \( p \) is trivial, \( \rho \) admits a global section \( s : D(B) \to GL(E), s(\phi)(x, v) = (h(x), v) \), being \( W^{r-} \) and \( S^{r-} \)-continuous for all \( r \in \{0, 1, \ldots, \infty\} \). Hence by Lemma 2.2.1 we have a homeomorphism \( GL(E) \cong GL(E, B) \times D(B) \).

On the other hand, since \( p \) is a trivial vector bundle, there is another homeomorphism \( \theta : GL(E, B) \to C^\infty(B, GL(\mathbb{R}^m)) \) given in the part 1) of the proof of Lemma 4.3.1. This finally gives a homeomorphism

\[
\sigma : GL(E) \cong C^\infty(B, GL(\mathbb{R}^m)) \times D(B).
\]

Now let \( h \in GL(E) \) and \( \sigma(h) = (\alpha, \beta) \). Then \( h \in \rho(D_{inv}(\mathcal{F}, B, p)) \) iff \( \alpha : B \to GL(\mathbb{R}^m) \) is contained in \( \mathcal{L}^*(g) \). Moreover, in this case \( h \in \rho(D_{fix}(\mathcal{F}, B, p)) \) iff in addition \( \beta = \text{id}_B \). \( \Box \)

**5.2. Examples.** Consider several computation of the group \( \mathcal{L}^*(g) \) in Corollary 5.1.9 when \( g : \mathbb{R}^m \to \mathbb{R} \) is a non-zero homogeneous polynomial of some degree \( k \) for which \( 0 \) is a unique critical point. Assume also that \( B \) is a compact connected manifold.

**Example 5.2.1** (Case \( m = 1 \)). Then \( g : \mathbb{R} \to \mathbb{R} \) is given by \( g(v) = C v^k \) for some integer \( k > 0 \) and \( C \neq 0 \). Hence \( (f, B) \)-foliation on \( B \times \mathbb{R} \) is \( \mathcal{F} = \{ B \times v \mid v \in \mathbb{R} \} \) and it does not depend on \( k \) and \( C \). Notice that in this case \( \mathcal{L}^*(g) = \{ 1 \} \), whence \( C^\infty(B, \mathcal{L}^*(g)) \) consists of a unique map, and therefore

\[
D_{fix}(\mathcal{F}, B) \simeq D_{nb}(\mathcal{F}, B), \quad D_{inv}(\mathcal{F}, B) \simeq D_{nb}(\mathcal{F}, B) \times D(B).
\]

**Example 5.2.2** (Case \( m = 2 \)). Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a homogeneous polynomial of some degree \( k \geq 2 \). Then

\[
g = L_1 \cdots L_p Q_1 \cdots Q_q
\]

is a product of linear \( \{ L_i(u, v) = a_i u + b_i v \}_{i=1,\ldots,p} \) and irreducible over \( \mathbb{R} \) quadratic factors \( \{ Q_j(u, v) = c_j u^2 + d_j uv + e_j v^2 \}_{j=1,\ldots,q} \). Moreover, \( 0 \in \mathbb{R}^2 \) is a unique critical point of \( g \) iff \( g \) has no multiple linear factors. It is shown in [26, Lemma 6.2] that if in addition \( \deg g \geq 3 \), then \( \mathcal{L}(g) \) is a finite dihedral group, and hence by linear change of coordinates one can assume that \( \mathcal{L}(f) \subset O(2) \).

Again consider several cases.
(2.1) Suppose \( g \) is an irreducible over \( \mathbb{R} \) quadratic form. Then one can assume that \( g(x, y) = x^2 + y^2 \); see Figure 5.1(a). Hence \( \mathcal{L}^*(g) = O(2) \), so it is a disjoint union of two circles. Since \( B \) is connected,

\[
\mathcal{C}^\infty(B, \mathcal{L}^*(g)) = \mathcal{C}^\infty(B, S^1 \sqcup S^1) = \mathcal{C}^\infty(B, S^1) \sqcup \mathcal{C}^\infty(B, S^1) \cong \mathcal{C}^\infty(B, S^1) \times \mathbb{Z}_2
\]

It is easy to see that the path components of the space \( \mathcal{C}^\infty(B, S^1) \) (in the topology \( W^\infty \)) are homotopy equivalent to the circle. More precisely, let \( x \in B \) be any point, \( \hat{K} := \mathcal{C}^\infty((B, x), (\mathbb{R}, 0)) \) be the space of \( \mathcal{C}^\infty \)-functions on \( B \) taking zero value at \( x \). Evidently, \( \hat{K} \) is contractible and we claim that each path component of \( \mathcal{C}^\infty(B, S^1) \) is homeomorphic with \( \hat{K} \times S^1 \).

Indeed, since \( \mathcal{C}^\infty(B, S^1) \) is a topological group with respect to point-wise multiplication, all its path components are homeomorphic each other. Let \( C_0 \) be the path component of \( \mathcal{C}^\infty(B, S^1) \) consisting of null-homotopic maps. Then the “evaluation at \( x \) map” \( \delta : C_0 \to S^1 \), \( \delta(h) = h(x) \), is a continuous surjective homomorphism with kernel \( K \) consisting of maps taking value \( 1 \in S^1 \) at \( x \). Moreover, \( \delta \) has a section \( s : S^1 \to C_0 \), associating to each \( u \in S^1 \) the constant map \( s(u) : B \to S^1 \) into the point \( u \). Hence by Lemma 2.2.1, we have a homeomorphism \( C_0 \cong K \times S^1 \). Let \( p : \mathbb{R} \to S^1 \), \( p(t) = e^{2\pi it} \), be the universal cover of \( S^1 \). Then each \( h \in K \) (being null-homotopic) lifts to a unique continuous function \( \hat{h} : B \to \mathbb{R} \) such that \( \hat{h}(x) = 0 \), i.e. \( \hat{h} \in \hat{K} \) and one easily check that the correspondence \( h \mapsto \hat{h} \) is a homeomorphism of \( K \) onto \( \hat{K} \). Thus \( C_0 \cong \hat{K} \times S^1 \), and therefore is homotopy equivalent to \( S^1 \).

Further, since \( S^1 \) is a \( K(\mathbb{Z}, 1) \)-space, \( \pi_0 \mathcal{C}^\infty(B, S^1) \) is isomorphic with the first integral cohomology group \( H^1(B, \mathbb{Z}) \). Hence we have the following homotopy equivalences:

\[
\begin{align*}
\mathcal{D}_{fix}(\mathcal{F}, B) &\simeq \mathcal{D}_{nb}(\mathcal{F}, B) \times S^1 \times \mathbb{Z}_2 \times H^1(B, \mathbb{Z}) \cong \mathcal{D}_{nb}(\mathcal{F}, B) \times O(2) \times H^1(B, \mathbb{Z}), \\
\mathcal{D}_{inv}(\mathcal{F}, B) &\simeq \mathcal{D}_{fix}(\mathcal{F}, B) \times \mathcal{D}(B).
\end{align*}
\]

(2.2) Suppose \( g \) is a product of at least 2 irreducible over \( \mathbb{R} \) quadratic forms, see Figure 5.1(b) and (c). Then the origin \( 0 \in \mathbb{R}^2 \) is a global extreme of \( g \), and the level sets of \( g \) are concentric closed curves wrapping around 0. In particular, they are connected, whence \( \mathcal{F} = \{ B \times g^{-1}(c) \mid c \in \mathbb{R} \} \), and \( \mathcal{L}^*(g) = \mathcal{L}(g) \) is a finite dihedral group. As \( B \) is connected, \( \mathcal{C}^\infty(B, \mathcal{L}^*(g)) \) consists of constant maps only, whence we get the following homotopy

\[
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.1}
\caption{Foliation by level sets of homogeneous polynomials}
\end{figure}
\]
equivalences:
\[ \mathcal{D}_{fix}(\mathcal{F}, B) \simeq \mathcal{D}_{nb}(\mathcal{F}, B) \times \mathcal{L}(g), \quad \mathcal{D}_{inv}(\mathcal{F}, B) \simeq \mathcal{D}_{fix}(\mathcal{F}, B) \times \mathcal{L}(g) \times \mathcal{D}(B). \]

(2.3) Suppose \( g = L_1 Q_1 \cdots Q_q \) has a unique linear factor, see Figure 5.1(e). Then one can assume that \( L_1(u, v) = v \), so the level sets of \( g \) are connected and consist of \( u \)-axis and a curves “parallel” to it. Therefore \( \mathcal{L}^*(g) = \mathcal{L}(g) \). Moreover, every \( A \in \mathcal{L}(g) \subset O(2) \) must preserve \( u \)-axis. Hence we have the following two cases

- either \( \mathcal{L}(g) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \) is trivial, and we have homotopy equivalences as in (5.9);
- or \( \mathcal{L}(g) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z}_2 \), so \( g \) must satisfy the identity \( g(-u, v) = g(u, v) \), and then
  \[ \mathcal{D}_{fix}(\mathcal{F}, B) \simeq \mathcal{D}_{nb}(\mathcal{F}, B) \times \mathbb{Z}_2, \quad \mathcal{D}_{inv}(\mathcal{F}, B) \simeq \mathcal{D}_{nb}(\mathcal{F}, B) \times \mathcal{D}(B). \]

(2.4) Suppose \( g \) is a product of exactly two linear factors. Then by linear change of coordinates, one can assume that \( g(u, v) = uv \), see Figure 5.1(d). Hence \( \mathcal{L}^*(g) = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \mid t > 0 \right\} \). This group is isomorphic with \( \mathbb{R} \), and therefore contractible. Hence so is \( \mathcal{C}_\infty(B, \mathcal{L}^*(g)) \), and therefore we have the homotopy equivalences as in (5.9).

(2.5) Finally, assume that \( g \) has at least two linear factors and \( \deg g \geq 3 \), see Figure 5.1(e). Then \( \mathcal{L}(g) \subset O(2) \), however every non-unit element \( A \in \mathcal{L}(g) \) interchanges path components of level sets of \( g \). Hence \( \mathcal{L}^*(g) = \{ E \} \) is a trivial group, whence we have homotopy equivalences as in (5.9).

EXAMPLE 5.2.3. Consider an example of a non-smooth foliation. Let \( m = 2 \), and \( g : \mathbb{R}^2 \to \mathbb{R} \), \( g(u, v) = |u| + |v| \), see Figure 5.2(a). Then \( g \) is homogeneous of degree 1, and its level sets are concentric squares centered at the origin. One easily sees that \( \mathcal{L}(g) \cong \mathbb{D}_4 \) is a dihedral subgroup of \( O(2) \) generated by rotation by \( \frac{\pi}{4} \) and reflection with respect to \( u \)-axis. Hence
\[ \mathcal{D}_{fix}(\mathcal{F}, B) \simeq \mathcal{D}_{nb}(\mathcal{F}, B) \times \mathbb{D}_4, \quad \mathcal{D}_{inv}(\mathcal{F}, B) \simeq \mathcal{D}_{nb}(\mathcal{F}, B) \times \mathbb{D}_4 \times \mathcal{D}(B). \]
See also Figure 5.2(b) and (c) for other examples of non-smooth foliations.

\[
\begin{array}{ccc}
\text{(a)} \ |u| + |v| & \text{(b)} \ |u|^{0.6} + |v|^{0.6} & \text{(c)} \ |u|^{0.7} + |v|^{0.7} + |0.5u + v|^{0.7} \\
\end{array}
\]

**Figure 5.2.** Non smooth foliations level sets of homogeneous functions

5.3. **Functions with isolated homogeneous singularities.** Let \( M \) be a smooth \( m \)-
manifold, \( P \) be either the real line or the circle, and \( \mathcal{F}(M, P) \) the subset of \( \mathcal{C}_\infty(M, P) \) consisting of maps \( f : M \to P \) having the following properties:

- (i) \( f \) takes constant values on boundary components of \( M \);
- (ii) all critical points of \( f \) are isolated and belong to \( \text{Int}M \);
- (iii) for every critical point \( x \) of \( f \) there exist local charts \( (\mathbb{R}^m, 0) \xrightarrow{\phi_x} (M, x) \) and \( (\mathbb{R}, 0) \xrightarrow{\psi_x} (P, f(x)) \), and a homogeneous polynomial \( g_x : \mathbb{R}^m \to \mathbb{R} \) such that \( g_x = \psi_x^{-1} \circ f \circ \phi_x \).

on some open neighborhood \( U_x \) of 0 in \( \mathbb{R}^m \).
Condition (iii) can be expressed so that $f$ is a homogeneous polynomial (with respect to some local charts) near each of its critical points. Due to Morse Lemma, every non-degenerate critical point satisfies (iii), so $\mathcal{F}(M, P)$ contains all Morse functions satisfying (i) and (ii).

**Remark 5.3.1.** Suppose $M$ is a compact surface. Let also $f \in \mathcal{F}(M, P)$, and $\mathcal{F}$ be the $(f, B)$-partition of $M$. Then it is shown in [25, 27, 28] that the path components of the group $\mathcal{D}(\mathcal{F})$ of $\mathcal{F}$-leaf preserving diffeomorphisms are either contractible or homotopy equivalent to the circle.

Let $f \in \mathcal{F}(M, P)$, $B$ be the set of critical points of $f$, and $\mathcal{F}$ be the $(f, B)$-partition of $M$, so its leaves are critical points of $f$ and connected components of the sets $f^{-1}(c) \setminus B$ for all $c \in P$. Since $B$ is finite, each $x \in B$ is a leaf of $\mathcal{F}$. In particular, each $h \in \mathcal{D}(\mathcal{F})$ is fixed on $B$.

Denote by $\mathcal{D}_{\text{lin}}^*(\mathcal{F})$ the subgroup of $\mathcal{D}(\mathcal{F})$ consisting of diffeomorphisms $h : M \to M$ having the following property: for each $x \in B$ there exist (depending on $h$)

- an open neighborhood $U'_x \subset U_x$ of $0$ in $\mathbb{R}^m$,
- a linear isomorphism $\tilde{h}$ of $\mathbb{R}^m$ belonging to $\mathcal{L}^*(g_x)$, i.e. preserving leaves of $(g_x, 0)$-partition, such that

$$h(U'_x) \subset U_x, \quad \phi^{-1}_x \circ h \circ \phi_x = \tilde{h} \text{ on } U'_x.$$

**Theorem 5.3.2.** Let $f \in \mathcal{F}(M, P)$, $B$ be the set of critical points of $f$, and $\mathcal{F}$ be the $(f, B)$-partition of $M$. Then the inclusion $\mathcal{D}_{\text{lin}}^*(\mathcal{F}) \subset \mathcal{D}(\mathcal{F})$ is a $\mathcal{W}^{\infty, \infty}$-homotopy equivalence.

**Proof.** Notice that, by definition, for each $x \in B$ the set $E_x := \phi_x(\mathbb{R}^m)$ is an open neighborhood of $x$, while the constant map $p_x : E_x \to x$ is regular neighborhood of $x$. One can decrease $U_x$ and assume that $\phi_x(U_x) \cap \phi_{x'}(U_{x'}) = \emptyset$ for $x \neq x' \in B$. Hence, changing $\phi_x$ out of $U_x$ we can also assume that $E_x \cap E_{x'} = \emptyset$.

Denote $E = \bigcup_{x \in B} E_x$. Then the map $p : E \to B$, $p(E_x) = x$ is a trivial vector bundle over $B$. It also follows from definition, that $\mathcal{D}_{\text{lin}}^*(\mathcal{F})$ coincides with the group $\mathcal{D}_{\text{fix}}(\mathcal{F}, B, p)$.

Let $U \subset E$ be any compact neighborhood of $B$. We claim that $U$ is $\mathcal{F}$-homogeneous and $U \cap (\overline{\omega} \setminus \omega) \subset B$ for each $\omega \in \mathcal{F}$. Then it will follow from Theorem 5.1.5, that the inclusion $\mathcal{D}_{\text{lin}}^*(\mathcal{F}) \equiv \mathcal{D}_{\text{fix}}(\mathcal{F}, B, p) \subset \mathcal{D}(\mathcal{F})$ is a homotopy equivalence.

Consider the following function $g : E \to \mathbb{R}$ given by $g(v) = g_x \circ \phi_x^{-1}(v)$ for $v \in E_x$. Then $g$ satisfies assumptions (1) and (2) of Lemma 5.1.3. Indeed, by definition $g$ is homogeneous on fibres, so (1) holds. Moreover, each level set $g_x^{-1}(c), c \in \mathbb{R}$, admits a triangulation, e.g. [23, Theorem 1]. Hence $g_x^{-1}(c)$ is locally path connected (even locally contractible), whence its path components are closed $g_x^{-1}(c)$. But the latter set is closed in $E_x$ and hence in $E$. Therefore, the path components of level sets of $g$ must be closed in $E$.

Now by Lemma 5.1.3, any neighborhood $U \subset E$ of $B$ is $\mathcal{F}$-homogeneous and $U \cap (\overline{\omega} \setminus \omega) \subset B$ for each $\omega \in \mathcal{F}$. □

### 6. Linearization of smooth maps between vector bundles

In Theorem 1.1.1 we considered smooth self maps of a total space of some vector bundle $p : E \to B$ which leaves invariant the zero section $B$. In this section we will consider smooth maps between total spaces of vector bundles which send zero section to zero section, and formulate a more general statement (Theorems 6.4.1) about deformations of such maps to their “derivative along fibers”.

6.1. Maps into Euclidean spaces. Let \( B \subset \mathbb{R}^b \) be a \( C^1 \) submanifold (possibly with corners) and \( f = (f_1, \ldots, f_c) : B \to \mathbb{R}^c \) be a \( C^1 \) map. For each \( j = 1, \ldots, b \) define the map
\[
\frac{\partial f_j}{\partial v_j} : B \to \mathbb{R}^c, \quad \frac{\partial f_j}{\partial v_j} = (\frac{\partial f_1}{\partial v_j}, \ldots, \frac{\partial f_c}{\partial v_j}),
\]
whose coordinate functions are partial derivatives of coordinate functions of \( f \) in \( v_j \). Such a map will often be denoted by \( f_{v_j}' \).

For a continuous path \( \gamma = (\gamma_1, \ldots, \gamma_c) : [a; b] \to \mathbb{R}^c \) we put
\[
\int_a^b \gamma(s)ds := \left( \int_a^b \gamma_1(s)ds, \ldots, \int_a^b \gamma_c(s)ds \right).
\]

The following variant of Hadamard lemma is easy and we leave it for the reader.

**Lemma 6.1.1** (e.g. [14, Lemma II.6.10]). Let \( f : B \times \mathbb{R}^m \to \mathbb{R}^n \) be a \( C^r \) map, \( 1 \leq r \leq \infty \). Then for all \( (x, v) \in B \times \mathbb{R}^m \) and \( a, b \in \mathbb{R} \) we have the following identity:
\[
| f(x, bv) - f(x, av) | = \int_a^b \frac{\partial}{\partial s} \left( f(x, sv) \right)ds = \sum_{j=1}^m v_j \int_a^b f_{v_j}'(x, sv)ds,
\]
where we assume that the middle and right parts of this identity are zero whenever \( v = 0 \).

In particular, if \( f(x, 0) \equiv 0 \) for all \( x \in B \), then \( f(x, v) = \sum_{i=1}^m v_i \int_0^1 f_{v_i}'(x, sv)ds \). Moreover, define the following \( C^{r-1} \) map \( \alpha : [0; 1] \times B \times \mathbb{R}^m \to \mathbb{R}^n \) by
\[
\alpha(\tau, x, v) = \sum_{i=1}^m v_i \int_0^1 f_{v_i}'(x, s\tau v)ds.
\]
Then
(a) \( f(x, \tau v) = \tau \alpha(\tau, x, v) \), so \( f(x, v) = \alpha(1, x, v) \) and \( \alpha(\tau, x, v) = \frac{f(x, \tau v)}{\tau} \) for \( \tau > 0 \);
(b) \( \alpha(0, x, v) = \sum_{j=1}^m v_j f_{v_j}'(x, 0) \), so for each \( x \in B \) we have a well-defined linear map \( \mathbb{R}^m \to \mathbb{R}^n, v \mapsto \alpha(0, x, v) \);
(c) for all \( i = 1, \ldots, b, j = 1, \ldots, m, x \in B, \) and \( \tau \in [0; 1] \)
\[
\alpha'_{x_i}(\tau, x, 0) = 0, \quad \alpha'_{v_j}(\tau, x, 0) = f_{v_j}'(x, 0).
\]

6.2. Maps between total spaces of vector bundles. Let \( p : E \to B \) and \( q : F \to C \) be vector bundles of ranks \( m \) and \( n \) respectively. As above, we identify \( B \) (resp. \( C \)) with the submanifolds of \( E \) (resp. \( F \)) via the corresponding zero section.

Given a star-like neighborhood \( N \subset E \) of \( B \) define the following subsets of \( C^\infty(N, F) \).

Let
- \( C^\infty_0(N, F) \) be the set of smooth maps \( h : N \to F \) such that \( h(B) \subset C \), whence \( h \) induces a vector bundle morphism \( T_{fib}h : E \to F \) being a tangent map to \( h \) in the direction of fibers, see (3.5);  
- \( C^\infty_{vert}(N, F) \) be the subset of \( C^\infty_0(N, F) \) consisting of maps \( h : N \to F \) such that its tangent map \( Th : TN \to TF \) sends vertical vectors at \( B \) to vertical vectors at \( C \), that is \( Th(Vert(B)) \subset Vert(C) \);  
- \( C^\infty_{sub}(N, F) \) be the subset of \( C^\infty_0(N, F) \) consisting of maps \( h \) which coincide with some vector bundle morphism \( g : E \to F \) on some neighborhood of \( B \); in fact \( g = T_{fib}h \).
\( \mathcal{L}(N,F) \) be the space of restrictions to \( N \) of vector bundle morphisms \( E \to F \), so \( h \equiv T_{\text{fb}}h|_N : N \to F \) for all \( h \in \mathcal{L}(N,F) \).

Since each vector bundle morphism \( g : E \to F \) is uniquely determined by its restriction to arbitrary neighborhood of \( B \), the restriction map \( \mathcal{L}(E,F) \to \mathcal{L}(N,F) \), \( g \to g|_N \), is a bijection. It is also evident that

\[
\mathcal{L}(E,F) \equiv \mathcal{L}(N,F) \subset C^\infty_{c,\text{tr}}(N,F) \subset C^\infty_{\text{vert}}(N,F) \subset C^\infty_0(N,F). \quad (6.3)
\]

Our aim is to prove that if \( B \) and \( N \) are compact, then the inclusions (6.3) are homotopy equivalences with respect to topologies \( \mathcal{W}^\infty \), see Lemma 6.2.2(i). First we introduce several notations and recall some definitions.

Let \( h : N \to F \) be a \( C^\infty \) map, \( y \in N \) a point, and \( \Psi : W \times \mathbb{R}^n \to F \) a trivialized local chart of \( q \) whose image contains \( h(y) \). Then there exists an open neighborhood \( U'' \subset N \) of \( y \) contained in the image of some trivialized local chart \( \Phi : V \times \mathbb{R}^m \to E \) and such that \( h(U'') \subset \Psi(W \times \mathbb{R}^n) \). In other words, denoting \( U = \Phi^{-1}(U'') \), we get the following commutative diagram:

\[
\begin{array}{ccc}
V \times \mathbb{R}^m & \xrightarrow{\Phi} & U \xleftarrow{(f,g)} W \times \mathbb{R}^n \\
\downarrow & & \downarrow \\
E & \leftarrow N & \xrightarrow{h} \rightarrow F
\end{array}
\]

We will call the map

\[
V \times \mathbb{R}^m \supset U \xrightarrow{(f,g)} W \times \mathbb{R}^n,
\]

a local representation of \( h \) at \( y \) (with respect to the local trivializations \( \Phi \) and \( \Psi \)).

Assuming that \( W \) is an open subset of \( \mathbb{R}^c \), let also \( f = (f_1, \ldots, f_c) : U \to W \) and \( g = (g_1, \ldots, g_n) : U \to \mathbb{R}^n \) be the coordinate functions of the maps \( f \) and \( g \) being in turn the coordinate functions of \( h \) in the above trivialized local representation.

Let \( J_h = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} : U \to M(c+n,b+m) \) be the map associating to each \( w = (x,v) \in U \) the Jacobi matrix of \( h \) at \( w \), where

\[
P(w) = \left( \frac{\partial f_i}{\partial x_j}(w) \right)_{i=1,\ldots,c, j=1,\ldots,b}, \quad Q(w) = \left( \frac{\partial f_i}{\partial v_j}(w) \right)_{i=1,\ldots,c, j=1,\ldots,m},
\]

\[
R(w) = \left( \frac{\partial g_i}{\partial x_j}(w) \right)_{i=1,\ldots,n, j=1,\ldots,b}, \quad S(w) = \left( \frac{\partial g_i}{\partial v_j}(w) \right)_{i=1,\ldots,n, j=1,\ldots,m},
\]

are the matrices consisting of the corresponding derivatives of coordinate functions of \( f \) and \( g \) in \( x \in V \) and \( v \in \mathbb{R}^m \). The following lemma directly follows from definitions and we leave it for the reader.

**Lemma 6.2.1.** Let \( h : N \to F \) be a \( C^\infty \) map.

(a) The following conditions are equivalent:

(i) \( h \in C^\infty_0(N,F) \), i.e. \( h(B) \subset C \);

(ii) for any local trivialization (6.4), \( g(x,0) = 0 \in \mathbb{R}^n \) for all \( (x,0) \in U \).

In the latter case \( R(x,0) = 0 \), i.e. \( J_h(x,0) = \begin{pmatrix} P(x,0) & Q(x,0) \\ R(x,0) & S(x,0) \end{pmatrix} \).

(b) For each \( (x,0) \in B \) the ranks of submatrices \( P(x,0) \) and \( S(x,0) \) do not depend on a particular local representation of \( h \) at \( (x,0) \).

(c) Suppose \( h \in C^\infty_0(N,F) \). Then the following conditions are equivalent:

(i) \( h \in C^\infty_{\text{vert}}(N,F) \), that is \( T_{h(\text{Vert}(B))} \subset \text{Vert}(C) \);
Hence for some non-degenerate matrices which are preserved by local trivializations. Hence, passing to another local presentation will equal bundle morphism $h$. 

**6.2.2 (Hadamard Lemma for Vector Bundles)**

□

(c) Notice that $w$ needs to show that $h(x, v) = (f(x), S(x, 0) v)$ for all $(x, v) \in U$. Since multiplication by scalars preserve zero sections, we have that $h(x, 0) = (f(x), 0)$, $h(x) = (f(x), S(x, 0) v)$, and in particular, $h$ does not depend in $w$. Hence $\text{rank}(LPX) = \text{rank}(P)$ and $\text{rank}(MSC) = \text{rank}(S)$. 

All other statements are easy and we leave their proof for the reader. 

**Lemma 6.2.2 (Hadamard lemma for vector bundles).** For each $h \in \mathcal{C}_i^\infty(N, F)$ there exists a unique $\mathcal{C}^\infty$ homotopy $\alpha : N \times [0; 1] \to F$ such that for every $\tau \in [0; 1]$ 

(a) $h(\tau w) = \tau h(w)$ for all $w \in N$, and, in particular, $\alpha_1 = h$; 

(b) $\alpha_\tau|_B = h|_B : B \to C$, and in particular, $\alpha_\tau \in \mathcal{C}_i^\infty(N, F)$; 

(c) $\alpha_0 = \text{T}_w h|_N : N \to F$; 

(d) $J_{x,v}(\tau w) = \left( \begin{array}{cc} P(x,0) & \tau Q(x,0) \\ \tau x,0 & \tau \end{array} \right)$ for $(x, v) \in U$; 

(e) if $h \in \mathcal{C}_i^\infty(N, F)$, then $\alpha_\tau \in \mathcal{C}_i^\infty(N, F)$; 

(f) if $h \in \mathcal{C}_i^\infty(N, F)$, then $h = \alpha_\tau$ near $B$, and in particular, $\alpha_\tau \in \mathcal{C}_i^\infty(N, F)$; 

(g) if $h \in \mathcal{L}(N, F)$, then $\alpha_\tau = h$ on all of $N$; 

(h) the induced map $A : \mathcal{C}_i^\infty(N, F) \to \mathcal{C}^\infty(N \times [0; 1], F)$, $A(h) = \alpha$, is $W^{r+1}|_r$-continuous for every $r \geq 0$; hence it is $W^{\infty, \infty}$-continuous by Lemma 2.1.1; 

(i) if $N$ is compact, then the inclusions (6.3) are $W^{\infty, \infty}$-homotopy equivalences.

**Proof.** Since $N$ and $F$ are star-like neighborhoods of $B$ and $C$ respectively, the map $\alpha$ is uniquely determined by the formula: $\alpha(\tau, w) = \frac{1}{\tau} h(\tau w)$ for $(w, \tau) \in [0; 1] \times N$.

Hence we need to show that $\alpha$ extends to a $C^\infty$ map $[0; 1] \times N \to F$. Consider a local representation (6.4) of $h$. Then $\alpha(\tau, x, v) = \left( \sum_{j=1}^m v_j \int_0^1 \frac{\partial g}{\partial v_j}(x, s \tau v) ds \right)$. 

(b) Let $w \in B$. Since multiplication by scalars preserve zero sections, we have that $\tau w = w$. Hence $\alpha(\tau, w) = \frac{1}{\tau} h(\tau w) = \frac{1}{\tau} h(w) = h(w)$ for $\tau > 0$. Then by continuity of $\alpha$ and $h$ we see that $\alpha(0, w) = h(w)$ as well.

(c) Notice that $\alpha(0, x, v) = \left( \sum_{j=1}^m v_j \int_0^1 \frac{\partial g}{\partial v_j}(x, s \tau v) ds \right)$, so $\alpha_0$ is linear on fibres, and thus a vector bundle morphism.
Statement (d) follows from (6.5) for direct computation. In turn, statement (e) follows from (d) and Lemma 6.2.1(c).

(f) If \( h \in C^\infty_{t,nb}(N,F) \), so there exists a neighborhood \( W \) of \( U \cap (W \times 0) \) such that \( h(x,v) = (f(x),S(x,0)v) \) for all \( (x,v) \in W \). Then for \( (x,v) \in W \) and \( \tau > 0 \) we have that

\[
\alpha(\tau, x, v) = (f(x), \frac{1}{\tau}S(x,0)(\tau v)) = (f(x), S(x,0)v) = h(x,v).
\]

By continuity of \( \alpha \) this also holds for \( \tau = 0 \).

(g) If \( h \in \mathcal{L}(N,F) \), then \( h \) commutes with multiplication by scalars, whence

\[
\alpha(\tau, w) = \frac{1}{\tau}h(\tau w) = \frac{1}{\tau}h(w) = h(w)
\]

for all \( w \in N \). Hence this also holds for \( \tau = 0 \).

(h) Formula (6.5) shows that \( \alpha \) continuously depends on 1-jet of \( h \), i.e. on the partial derivatives of \( h \) up to order 1, that is \( A \) is \( W^{1,0} \)-continuous. More generally, differentiating right hand side of (6.5) in \( x \) and \( v \) we see that \( r \)-jet of \( \alpha, r \geq 1 \), also continuously depends on \((r+1)\)-jet of \( h \), i.e. on partial derivatives of \( h \) up to order \( r+1 \), so \( A \) is in fact \( W^{r+1,r} \)-continuous. We leave the details for the reader.

(i) It follows from (b) that we have a well-defined map

\[
G : C^\infty_0(N,F) \times [0;1] \to C^\infty_0(N,F), \quad G(h, \tau)(y) = A(h)(y, \tau).
\]

Suppose \( N \) is compact. Then similarly to (h) one can show that \( G \) is \( W^{r+1,r} \)-continuous for all \( r \geq 0 \), and therefore it is \( W^{\infty,\infty} \)-continuous.

Moreover, by (e), (f), (g) the spaces \( C^\infty_{vert}(N,F) \) and \( C^\infty_{t,nb}(N,F) \) invariant under \( G \), \( \mathcal{L}(N,F) \) is fixed, and, due to (c), \( G_1(C^\infty_0(N,F)) \subset \mathcal{L}(N,F) \). In other words, \( G \) is a strong deformation retraction of \( C^\infty_0(N,F), C^\infty_{vert}(N,F), C^\infty_{t,nb}(N,F) \) onto \( \mathcal{L}(N,F) \), and thus the inclusions (6.3) are \( W^{\infty,\infty} \)-homotopy equivalences.

**Remark 6.2.3.** If \( N \) or \( B \) are non-compact, then \( G \) is not compactly supported, and therefore one can not guarantee that \( A \) and \( G \) are continuous between the corresponding strong topologies.

### 6.3. Compactly supported linearizations

Notice that the support of the homotopy \( \alpha \) constructed in Lemma 6.2.2 may coincide with all of the neighborhood \( N \) of \( B \). Therefore it makes difficult to extend such homotopies to all of \( E \). Nevertheless, we will shows that it is possible to make \( G(h, \tau) \) to coincide with \( h \) out of some smaller neighborhood \( U \subset N \) of \( B \). The idea is to replace \( \tau \) with a certain function \( \phi \) depending on \( h, v \) and \( t \in [0;1] \). This will give another proof that the last three spaces (6.3) are homotopy equivalent. Further, we will extend the proof to embeddings and diffeomorphisms, see Theorems 6.4.1.

Denote

\[
\mathcal{Y} := C^\infty_0(N,F) \times [0;+\infty) \times [0;1], \quad \mathcal{Y}_0 := C^\infty_0(N,F) \times (0;+\infty) \times [0;1].
\]

Now, similarly to §1.1, let us fix:

- a \( C^\infty \) function \( \mu : \mathbb{R} \to [0;1] \) such that \( \mu = 0 \) on \([0;0.2]\) and \( \mu = 1 \) on \([0.8;+\infty)\);
- an orthogonal structure on \( E \), and let \( \| \cdot \| : E \to [0;+\infty) \) be the corresponding norm;
- the function \( \phi : \mathcal{Y}_0 \times E \to [0;1] \) given by

\[
\phi(h, \sigma, t, w) = t + (1-t)\mu\left(\frac{\|w\|}{\sigma}\right);
\]
the map $\mathcal{H} : \mathcal{Y} \to C_0^\infty(N, F)$ defined by

$$
\mathcal{H}(h, \sigma, t)(w) = \begin{cases} 
\alpha_h(\phi(h, \sigma, t, w), & \sigma > 0, \\
h(w), & \sigma = 0,
\end{cases}
$$

(6.7)

where $\alpha_h : N \times [0; 1] \to F$ is the $C^\infty$ homotopy constructed in Lemma 6.2.2; note that for $\sigma > 0$ and $\phi(h, \sigma, t, w) \neq 0$ we have that

$$
\mathcal{H}(h, \sigma, t)(w) := \alpha_h(\phi(h, \sigma, t, w), w) = \frac{h(\phi(h, \sigma, t, w)w)}{\phi(h, \sigma, t, w)}.
$$

(6.8)

and one more map $\tilde{\mathcal{H}} = \text{ev} \circ (\mathcal{H} \times \text{id}_N) : \mathcal{Y} \times N \to F$ given by

$$
\tilde{\mathcal{H}}(h, \sigma, t, w) = \mathcal{H}(h, \sigma, t)(w),
$$

(6.9)

which we will call the $\mathcal{H}$-evaluation map, see Lemma 2.1.2.

Let us mention that

(i) if $\|w\| \leq 0.2\sigma$, then $\phi(h, \sigma, t, w) = t$, whence $\mathcal{H}(h, \sigma, t, w) = \alpha_h(t, w)$;

(ii) if $\|w\| \geq 0.8\sigma$, then $\phi(h, \sigma, t, w) = 1$, whence $\mathcal{H}(h, \sigma, t, w) = h(w)$.

**Lemma 6.3.1.** Suppose $N$ is compact. Then the following statements hold.

(a) $\mathcal{H}$ is $W^{0,0}$-continuous, whence due to Lemma 2.1.2, $\tilde{\mathcal{H}}$ is $W^r$-continuous for all $r \geq 0$.

(b) However, if $B$ is not finite (i.e. not a compact 0-manifold), then for any topology $\tau$ on $C_0^\infty(N, F)$ and $r \geq 1$, it is not continuous as a map

$$
\mathcal{H} : (C_0^\infty(N, F), \tau) \times [0; +\infty) \times [0; 1] \to (C_0^\infty(N, F), W^r)
$$

into $W^r$-topology of $C_0^\infty(N, F)$.

(c) On the other hand, the restriction of $\mathcal{H}$ to $\mathcal{Y}_0$, i.e. when $\sigma > 0$, is $W^{r+1,r}$-continuous for all $r \geq 0$, and therefore it is $W^{\infty,\infty}$-continuous.

**Proof.** Statement (a) directly follows from (ii) and Lemma 6.2.2(b), while statement (c) follows from formulas for $\mu$, $\phi$, and Lemma 6.2.2(h).

(b) Notice that for each $h \in C_0^\infty(N, F)$ and $\sigma > 0$ we have that $\mathcal{H}(h, \sigma, 1) = h$ and $\mathcal{H}(h, \sigma, 0) = \alpha_h$ near $B$. Now let $w \in B$ and $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be the Jacobi matrix of $h$ at $w$. Then, by Lemma 6.2.2(d), $\begin{pmatrix} P & 0 \\ R & S \end{pmatrix}$ is the Jacobi matrix of $\alpha_h$ at $v$. If $\text{dim}(B) \geq 1$, then there always exists $h \in C_0^\infty(N, F)$ with $Q \neq 0$. Then, for such $h$, decreasing $\sigma$ to 0 one can not make the 1-jet of $\alpha_h$ to be arbitrary close to the 1-jet of $h$ near $v$. In other words, the restriction of $\mathcal{H}$ to $\{h\} \times [0; 1] \to C_0^\infty(N, F)$ is not continuous into $W^1$-topology on $C_0^\infty(N, F)$, and therefore into any other $W^r$-topology with $r \geq 1$. Hence $\mathcal{H} : (C_0^\infty(N, F), \tau) \times [0; +\infty) \times [0; 1] \to (C_0^\infty(N, F), W^r)$ is not continuous for any topology $\tau$ on $C_0^\infty(N, F)$ and $r \geq 1$.

**Corollary 6.3.2.** Suppose $N$ is compact and for some $\varepsilon > 0$ the tubular neighborhood $R_\varepsilon$ of $B$ is contained in $\text{Int} N$. Let also $\delta : C_0^\infty(N, F) \to (0; \varepsilon)$ be any $W^r$-continuous function for some $r \geq 1$, and $H : C_0^\infty(N, F) \times [0; 1] \to C_0^\infty(N, F)$ the map given by

$$
H(h, t)(w) = \mathcal{H}(h, \delta(h), t, w).
$$

(6.10)

Then the following statements hold.

(1) $H(h, t) = h$ on $N \setminus R_\varepsilon$ for all $h \in C_0^\infty(N, F)$ and $t \in [0; 1]$;

(2) $H$ is a $W^{\infty,\infty}$-continuous.
(3) \( H \) is a deformation of \( C_0^\infty(N, F) \) into \( C_{\text{t,lb}}^\infty(N, F) \) which leaves \( C_{\text{vert}}^\infty(N, F) \) invariant, that is
\[
\begin{align*}
(a) & \quad H_1 = \text{id}_{C_0^\infty(N, F)}, \\
(b) & \quad H_0(C_0^\infty(N, F)) \subset C_{\text{t,lb}}^\infty(N, F), \\
(c) & \quad H(C_{\text{vert}}^\infty(N, F) \times [0; 1]) \subset C_{\text{vert}}^\infty(N, F), \\
(d) & \quad H(C_{\text{t,lb}}^\infty(N, F) \times [0; 1]) \subset C_{\text{t,lb}}^\infty(N, F),
\end{align*}
\]
In particular, the inclusions (6.3) are \( W^{\infty, \infty} \)-homotopy equivalences.

**Proof.** Statement (1) directly follows from (ii).

(2) Since the topology \( W^r \) is finer than the \( W^r \) one, it follows that \( \delta \) is \( W^{\infty, \infty} \)-continuous. Together with \( W^{\infty, \infty} \)-continuity of \( H \) for \( \sigma > 0 \) this implies \( W^{\infty, \infty} \)-continuity of \( H \).

(3) Statement (a) is evident, and the inclusions (b), (c), (d) follow respectively from statements (c), (e) and (f) of Lemma 6.2.2.

**Definition 6.3.3.** Let \( \mathcal{X} \subset C_0^\infty(N, F) \) be a subset, and \( \delta : \mathcal{X} \to (0; +\infty) \) a \( W^r \)-continuous function, for some \( r \geq 1 \). Then the map \( H : \mathcal{X} \times [0; 1] \to C_0^\infty(N, F) \) given by (6.10) will be called the \( \delta \)-linearizing homotopy. We will sometimes denote \( H(h, \sigma, t) : \mathcal{X} \to C_0^\infty(N, F) \) by \( H_{h, \sigma, t} \).

The following lemma allows to detect subsets \( \mathcal{X} \subset C_0^\infty(N, F) \) invariant under \( \delta \)-linearizing homotopies.

**Lemma 6.3.4.** Let \( \mathcal{X} \subset C_0^\infty(N, F) \) be a subset and \( r \geq 1 \). Suppose \( N \) is compact and for every \( h \in \mathcal{X} \) there exists a \( W^r \)-neighborhood \( U_h \) of \( h \) in \( \mathcal{X} \) and \( \varepsilon_h > 0 \) such that
\[
\mathcal{H}(U_h \times [0; \varepsilon_h] \times [0; 1]) \subset \mathcal{X}.
\]
Then there exists a continuous function \( \delta : \mathcal{X} \to (0; +\infty) \) such that the image of the corresponding \( \delta \)-linearizing homotopy \( H : \mathcal{X} \times [0; 1] \to C_0^\infty(N, F) \) is contained in \( \mathcal{X} \). Hence, the inclusion \( \mathcal{X} \cap C_{\text{t,lb}}^\infty(N, F) \subset \mathcal{X} \) is a \( W^{\infty, \infty} \)-homotopy equivalence.

**Proof.** Notice that \( C_0^\infty(N, F) \) and thus \( \mathcal{X} \) are metrizable with respect to the topology \( W^r \). Therefore, they are paracompact, so there exists a locally finite cover \( \{U_{\lambda}\}_{\lambda \in \Lambda} \) of \( \mathcal{X} \) being a refinement of the cover \( \{U_{\lambda}\}_{h \in \mathcal{X}} \), i.e. for every \( \lambda \in \Lambda \) there exists \( h_{\lambda} \in \mathcal{X} \) such that \( U_{\lambda} \subset U_{h_{\lambda}} \). In particular, \( \mathcal{H}(U_{\lambda} \times [0; \varepsilon_{h_{\lambda}}] \times [0; 1]) \subset \mathcal{X} \). Using partition of unity one can further construct a continuous function \( \delta : C_0^\infty(N, F) \to (0; \infty) \) such that \( \delta < \varepsilon_{h_{\lambda}} \) on \( U_{\lambda} \). Hence if \( h \in U_{\lambda} \), then \( \delta(h) < \varepsilon_{h_{\lambda}} \) and therefore \( H(h, t) = H(h, \delta(h), t) \in \mathcal{X} \) for all \( t \in [0; 1] \).

**Remark 6.3.5.** Notice that a priori an \( \delta \)-linearizing homotopy does not preserve useful open subsets like immersions, embeddings, submersions, and diffeomorphisms open in \( W^r \) topologies for \( r \geq 1 \). Moreover, disregarding for the moment Lemma 6.3.1(b), suppose that \( \mathcal{H} \) is \( W^{s+r} \)-continuous for some \( s \geq 0 \) and \( r \geq 1 \); then for every \( W^1 \)-open subset \( U \subset C_0^\infty(N, F) \) there exists a \( W^1 \)-continuous function \( \delta : U \to (0; +\infty) \) such that the image of the \( \delta \)-linearizing homotopy \( H : U \times [0; 1] \to C_0^\infty(N, F) \) is contained in \( U \). In particular, this would give a deformation of \( U \) onto \( U \cap C_{\text{t,lb}}^\infty(N, F) \).

Indeed, notice that \( \mathcal{H}(h, 0, t) = h \) for all \( h \in U \) and \( t \in [0; 1] \). In other words, \( \mathcal{H}^{-1}(U) \) is a \( W^s \)-open neighborhood of \( U \times \{0\} \times [0; 1] \). Then by paracompactness of \( U \times \{0\} \times [0; 1] \) one can find a \( W^s \)-continuous function \( \delta : U \to (0; +\infty) \) such that \( \mathcal{H}(h, \sigma, t) \in U \) for all \( h \in U \), \( \sigma \in [0; \delta(h)] \), \( t \in [0; 1] \), i.e. the corresponding \( \delta \)-linearizing homotopy preserves \( U \).

In particular, this would give a proof that for sufficiently small function \( \delta \), the \( \delta \)-linearizing homotopy preserves open \( C^1 \)-embeddings, e.g. diffeomorphisms. We will prove that this is
nevertheless true, though, due to discontinuity of $\mathcal{H}$, one needs more delicate arguments and estimates.

Namely, Lemma 6.3.1 shows that non-zero submatrices matrices $Q$ are the obstruction of continuity of $\mathcal{H}$ into $\mathcal{W}^1$-topology on $C^\infty_0(N, F)$. We will show that this is a unique obstruction: at least “the remaining part of 1-jets of $\mathcal{H}$ corresponding to submatrices $P$, $R$, and $S$ continuously depend on 3-jet of $h$, see Corollary 6.5.2 below.

6.4. Horizontal and vertical ranks of maps belonging to $C^\infty_0(N, F)$. Let $h \in C^\infty_0(N, F)$, and $\hat{h} = (f, g) : U \to W \times \mathbb{R}^n$ be a local representation of $h$ at some point $w \in B$ with respect to some trivialized local charts $\Phi$ and $\Psi$. In particular, $\Phi^{-1}(w) = (x, 0)$, and the Jacobi matrix of $\hat{h}$ at $(x, 0)$ is of the form $J_h = \begin{pmatrix} P & Q \\ 0 & S \end{pmatrix}$ and the ranks of $P$ and $S$ do not depend on a particular local representation of $h$, see Lemma 6.2.1. We will call $\text{rank}(P(x, 0))$ and $\text{rank}(S(x, 0))$ respectively the horizontal and the vertical ranks of $h$ at $w$.

For $a, b \geq 0$ denote by $C^\infty_{a,b}(N, F)$, resp. $C^\infty_{0}(N, F)$, the subsets of $C^\infty_0(N, F)$ consisting of maps $h$ for which the horizontal rank $\geq a$, resp. vertical rank $\geq b$, at each $w \in B$. Also we put

$$C^\infty_{+a,b}(N, F) := C^\infty_{-0,a}(N, F) \cap C^\infty_{+0,b}(N, F)$$

Evidently, these spaces are $\mathcal{W}^1$-open in $C^\infty_0(N, F)$. Moreover, some of them can be described in another terms. Recall that

$$\dim(B) = b, \quad \dim(E) = b + m, \quad \dim(C) = c, \quad \dim(F) = c + n.$$  

Then, for instance,

- $C^\infty_{-a,0}(N, F) = C^\infty_{+0,0}(N, F) = C^\infty_{+0,0}(N, F) = C^\infty_0(N, F)$;
- $C^\infty_{-b,0}(N, F) = \{ h \in C^\infty_0(N, F) | h|_B : B \to C \text{ is an immersion} \}$;
- $C^\infty_{+a,0}(N, F) = \{ h \in C^\infty_0(N, F) | h \text{ is transversal to } C \text{ along } B \}$;
- if $b = c$ and $m = n$, then

$$C^\infty_{+b,m}(N, F) = \{ h \in C^\infty_0(N, F) | h \text{ is a local diffeomorphism near } B \}.$$  

THEOREM 6.4.1. Suppose $N$ and $B$ are compact, and let $\varepsilon > 0$ be such that $R_\varepsilon \subset N$. Let $\mathcal{X}$ be one of the following spaces:

1) $C^\infty_{-a}(N, F)$ for some $a \geq 0$;
2) $C^\infty_{+b}(N, F)$ for some $b \geq 0$;
3) $C^\infty_{+a,b}(N, F)$ for some $a, b \geq 0$;
4) $\mathcal{E}_{+b,m}(N, F) = \{ h \in C^\infty_{+b,m}(N, F) | h \text{ is an embedding} \}$;
5) $\mathcal{E}_{0}(N, F) = \{ h \in C^\infty_0(N, F) | h \text{ is an embedding} \}$ for the case when $\dim(B) = \dim(C)$ and $\dim(E) = \dim(F)$.

Then there exists a $\mathcal{W}^3$-continuous function $\delta : \mathcal{X} \to (0; \varepsilon)$ such that the image of the corresponding $\delta$-linearizing homotopy $H : \mathcal{X} \times [0; 1] \to C^\infty_0(N, F)$ is contained in $\mathcal{X}$. Therefore, the inclusion $\mathcal{X} \cap C^\infty_{i,m}(N, F) \subset \mathcal{X}$ is a $\mathcal{W}^{\infty,\infty}$-homotopy equivalence.

The proof will be given in §7. First we will establish certain inequalities for matrices $P$ and $S$, see Lemma 6.5.1.

6.5. Certain estimations for linearizing homotopies. We first consider the case local case of trivial vector bundles, $p : \mathbb{R}^b \times \mathbb{R}^m \to \mathbb{R}^b$ and $q : \mathbb{R}^c \times \mathbb{R}^n \to \mathbb{R}^c$. 
Let $U \subset \mathbb{R}^b \times \mathbb{R}^m$ be an open set. Then $C^\infty_0(U, \mathbb{R}^c \times \mathbb{R}^n)$ is the space of all $C^\infty$ maps $h = (f, g) : U \to \mathbb{R}^c \times \mathbb{R}^n$ such that $h(U \cap (\mathbb{R}^b \times 0)) \subset \mathbb{R}^c \times 0$. In other words, $g(x, 0) = 0$ for all $(x, 0) \in U$, whence by (6.1)
\begin{equation}
    g(x, v) = \sum_{j=1}^m v_j \int_0^1 g'_{v_j}(x, sv)ds.
\end{equation}

Let $\left(\begin{array}{ll} P_h & Q_h \\ R_h & S_h \end{array}\right) : U \to M(\mathbb{C} + \mathbb{n}, \mathbb{b} + \mathbb{m})$ be the Jacobi matrix map of $h$.

Fix $\sigma > 0$ and let $\phi : [0; 1] \times \mathbb{R}^m \to [0; 1]$, $\phi(t, v) = t + (1 - t)\mu(\|v\|/\sigma)$, be the function given by (6.6) but for simplicity we omit $h$ and the constant $\sigma$ from notation. Then the $\sigma$-linearizing homotopy $H : [0; 1] \times U \to \mathbb{R}^c \times \mathbb{R}^n$ for $h$ is given by
\begin{equation}
    H(t, x, v) := H_{h,\sigma,t}(x, v) = \left(f(x, \phi(t, v)v), \sum_{j=1}^m v_j \int_0^1 g'_{v_j}(x, s\phi(t, v)v)ds\right).
\end{equation}

Denote by $\left(\begin{array}{ll} P_h & Q_h,\sigma,t \\ R_h & S_h,\sigma,t \end{array}\right) : U \to M(\mathbb{C} + \mathbb{n}, \mathbb{b} + \mathbb{m})$ the Jacobi matrix (regarded as a matrix valued map from $U$) of the mapping $H_{h,\sigma,t} : U \to \mathbb{R}^c \times \mathbb{R}^n$. Since $H_{h,\sigma,1} = h$ for all $\sigma > 0$, we also have that $\left(\begin{array}{ll} P_h & Q_h \\ R_h & S_h \end{array}\right)$ is the Jacobi matrix map of $h$ and it does not depend on $\sigma$ as well.

Define also the following $C^\infty$ maps $\beta, \gamma : [0; 1] \times U \to \mathbb{R}^c \times \mathbb{R}^n$
\begin{equation}
    \beta(t, x, v) = f(x, v) - f(x, \phi(t, v)x, v) \overset{(6.1)}{=} \sum_{j=1}^m v_j \int_0^1 f'_{v_j}(x, sv)ds.
\end{equation}
\begin{equation}
    \gamma(t, x, v) = g(x, v) - \sum_{j=1}^m v_j \int_0^1 g'_{v_j}(x, s\phi(t, v)v)ds \overset{(6.11)}{=} \sum_{j=1}^m v_j \int_0^1 (g'_{v_j}(x, sv) - g'_{v_j}(x, s\phi(t, v)v))ds
\end{equation}
\begin{equation}
    = \sum_{j=1}^m v_j \int_0^1 \left(1 - \int_0^1 g''_{v_j}(x, s\tau v)d\tau\right)s\,ds,
\end{equation}
being coordinate functions on the difference $h(x, v) - H(t, x, v)$. It will also be convenient to denote $\beta_t(x, v) := \beta(t, x, v)$ and $\gamma_t(x, v) := \gamma(t, x, v)$.

**Lemma 6.5.1.** Let $K \subset U$ be any compact subset. Then for each $t \in [0; 1]$ we have the following inequalities:
\begin{equation}
    |\beta_t|_{0,K} \leq \sigma|f|_{1,m,K},
\end{equation}
\begin{equation}
    |\gamma_t|_{0,K} \leq \sigma^2|g|_{2,m,K},
\end{equation}
\begin{equation}
    |P_h,\sigma,t - P_h|_{0,K} \leq \sigma|f|_{2,m,K},
\end{equation}
\begin{equation}
    |R_h,\sigma,t - R_h|_{0,K} \leq \sigma^2|g|_{3,m,K},
\end{equation}
\begin{equation}
    |S_h,\sigma,t - S_h|_{0,K} \leq \sigma \cdot (2|g|_{2,m,K} + \sigma|g|_{3,m,K} + m\mu_1 \cdot |g|_{2,m,K}).
\end{equation}

**Proof.** (6.15) Let
\begin{equation}
    \beta_t = (\beta_{1,t}, \ldots, \beta_{c,t}) : [0; 1] \times U \to \mathbb{R}^c,
\end{equation}
\begin{equation}
    f = (f_1, \ldots, f_c) : U \to \mathbb{R}^c
\end{equation}
be the coordinate functions of $\beta$ and $f$ respectively. Then, due to (6.13),

$$\beta_{i,t}(x,v) := \sum_{j=1}^{m} v_j \int_{\phi(t,v)}^{1} \frac{\partial f_i}{\partial v_j}(x,sv)ds, \quad i = 1, \ldots, c.$$ 

Let $(t, x, v) \in [0; 1] \times K$. If $||v|| \geq 0.8\sigma$, then $\beta_{i,t}(x,v) = 0$. On the other hand, if $||v|| < \sigma$ then $|\beta_{i,t}(x,v)| \leq \delta \sum_{j=1}^{m} |\frac{\partial f_i}{\partial v_j}(x,v)|$. Hence

$$|\beta_{i,t}|_{0,K} := \sum_{i=1}^{c} \sup_{(x,v) \in K} |\beta_{i,t}(x,v)| =: \delta |f|_{1,m,K}.$$ 

The inequality (6.16) for $|\gamma_{i,t}|_{0,K}$ follows from (6.14) in a similar way.

(6.17) Since $\phi$ does not depend on $x$, formula (6.13) also implies that

$$\beta'_{x_p}(t, x, v) = \sum_{i=1}^{m} v_i \int_{\phi(t,v)}^{1} f''_{x_p v_i}(x,sv)ds, \quad p = 1, \ldots, b,$$

which in turn gives the inequality (6.17) for the matrices $P$.

Further notice that $\phi'_{v_q}(t, v) = (1 - t)\mu'\left(\frac{||v||}{\sigma}\right)\frac{v_q}{||v||\sigma}$, which implies that

$$\sigma \sum_{q=1}^{m} \sup_{||v|| \leq \sigma} |\phi'_{v_q}(t,v)| \leq m|\mu|_{1}.$$ 

We also have from (6.14) that

$$\gamma'_{x_p}(t, x, v) = \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j \int_{0}^{\phi(t,v)} f''_{x_p v_i v_j}(x, s\tau v) d\tau ds,$$

$$\gamma'_{v_q}(t, x, v) = 2 \sum_{i=1}^{m} v_i \int_{0}^{1} f''_{v_i v_j}(x, s\tau v) d\tau s +$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j \int_{0}^{\phi(t,v)} f''_{v_i v_j v_j}(x, s\tau v) d\tau s^2 ds -$$

$$- \phi'_{v_q}(t, v) \sum_{i=1}^{m} \sum_{j=1}^{m} v_i v_j \int_{0}^{\phi(t,v)} f''_{v_i v_j}(x, s\phi(t,v) v) d\tau s ds,$$

which together with (6.20) imply inequalities (6.18) and (6.19) for the matrices $R$ and $S$ respectively. We leave the details for the reader. □

**Corollary 6.5.2.** The following map

$$\theta : C^\infty_0(U, \mathbb{R}^c \times \mathbb{R}^b) \times [0; +\infty) \times [0; 1] \to C^0(U, M(c,b) \times M(n,b) \times M(n,m)),$$

defined by

$$\theta(h, \sigma,t) = \begin{cases} (P_{h,\sigma,t}, R_{h,\sigma,t}, S_{h,\sigma,t}), \quad \sigma > 0, \\
(P_h, R_h, S_h), \quad \sigma = 0, \end{cases}$$

is $W^{3,0}$-continuous. Hence the following “$\theta$-evaluation” map

$$\tilde{\theta} : C^\infty_0(U, \mathbb{R}^c \times \mathbb{R}^n) \times [0; +\infty) \times [0; 1] \times U \times \mathbb{R}^b \times \mathbb{R}^b \times \mathbb{R}^m \to \mathbb{R}^c \times \mathbb{R}^c \times \mathbb{R}^n,$$
\[ \tilde{\theta}(h, \sigma, t, w, \tilde{u}, \tilde{v}, \tilde{w}) = \begin{cases} (P_{h, \sigma, t} \tilde{u}, R_{h, \sigma, t} \tilde{v}, S_{h, \sigma, t} \tilde{w}), & \sigma > 0, \\ (P_h \tilde{u}, R_h \tilde{v}, S_h \tilde{w}), & \sigma = 0, \end{cases} \]

is $W^r$-continuous for all $r \geq 3$.

**Proof.** It is a direct consequence of inequalities (6.17)-(6.19) whose right hand sides are of the form $c(h, \sigma, t)$, where $c(h, \sigma, t)$ continuously depends on $\sigma$, $t$ and partial derivatives of $h$ up to order 3. We leave the details for the reader. \hfill \Box

Say that an $(a \times b)$-matrix $A$ has maximal rank, whenever $\text{rank}(A) = \min\{a, b\}$.

**Corollary 6.5.3.** Let $a \geq 0$, $K \subset U$ be a compact subset such that $\text{rank}(P(x, 0)) \geq a$ for all $w \in K$. Then there exist $\eta, \varepsilon > 0$ such that $\text{rank}(P_{h, \sigma, t}(w)) \geq a$ for all $(h, \sigma, t, w) \in N^2_{K, \eta}(h) \times [0; \varepsilon] \times [0; 1] \times K$.

A similar statement holds for matrices $S$, but one should replace $N^2_{K, \eta}(h)$ with $N^3_{K, \eta}(h)$.

**Proof.** 1) Suppose that $\text{rank}(P(w)) \geq a$ for all $w \in K$. Since $K$ is compact and $h$ is $C^1$-differentiable (so $P$ is continuous), there exists $c_0 > 0$ such that $\text{rank}(P') \geq a$ for every matrix $P' \in M(c, b)$ satisfying $\|P' - P(w)\| < c_0$ for some $w \in K$.

Let $\tilde{h} = (\hat{f}, \hat{g}) \in C^\infty_0(U, \mathbb{R}^c \times \mathbb{R}^m)$ and $(\hat{P}, \hat{Q})$ be its Jacobi matrix map. Then, due to (6.17), for every $w \in K$ we have that

\[
|P_{h, \sigma, t}(w) - P(w)| \leq |P_{h, \sigma, t}(w) - \hat{P}(w)| + |\hat{P}(w) - P(w)| \\
\leq \sigma \|\hat{f}\|_{2,K} + \|\tilde{h} - h\|_{1,K} \\
\leq \sigma \|\tilde{h}\|_{2,K} + \sigma \|\tilde{h} - h\|_{2,K} + \|\tilde{h} - h\|_{1,K} \\
\leq \sigma \|\tilde{h}\|_{2,K} + (\sigma + 1) \|\tilde{h} - h\|_{2,K}. \tag{6.21}
\]

Choose small $\eta, \varepsilon > 0$ so that

\[
\varepsilon \|\tilde{h}\|_{2,K} + (\varepsilon + 1) \eta < c_0.
\]

Then for all $(\tilde{h}, \sigma, t, w) \in N^2_{K, \eta}(h) \times [0; \varepsilon] \times [0; 1] \times K$ we have that $|P_{h, \sigma, t}(w) - P(w)| < c_0$, so $\text{rank}(P_{h, \sigma, t}(w)) \geq a$.

2) Similarly, suppose that $\text{rank}(S(w)) \geq a$ for all $w \in K$. Then there exists $c_1 > 0$ such that $\text{rank}(S') \geq a$ for every matrix $S' \in M(n, m)$, satisfying $\|S' - S(w)\| < c_1$ for some $w \in K$. Assume that $\sigma < 1$ and let $c_2 := 3 + m|\mu|_1$. Then due to (6.19) for each $\tilde{h} = (\hat{f}, \hat{g}) \in C^\infty_0(U, \mathbb{R}^c \times \mathbb{R}^m)$ we have that

\[
|S_{h, \sigma, t}(w) - S(w)| \leq |S_{\tilde{h}, \sigma, t}(w) - \tilde{S}(w)| + |\tilde{S}(w) - S(w)| \\
\leq \sigma (2 |\tilde{g}|_{2,m,K} + |\tilde{g}|_{3,m,K} + m|\mu|_1 \cdot |\tilde{g}|_{2,m,K}) + \|\tilde{h} - h\|_{1,K} \\
\leq \sigma c_2 \|\tilde{h}\|_{3,K} + \|\tilde{h} - h\|_{1,K} \\
\leq \sigma c_2 \|\tilde{h}\|_{3,K} + (\sigma c_2 + 1) \|\tilde{h} - h\|_{3,K}. \tag{6.22}
\]

Choose $\eta, \varepsilon > 0$ so that $\varepsilon < 1$ and

\[
\varepsilon c_2 \|\tilde{h}\|_{3,K} + (\varepsilon c_2 + 1) \eta < c_1.
\]

Then, for all $(\tilde{h}, \sigma, t, w) \in N^3_{K, \eta}(h) \times [0; \varepsilon] \times [0; 1] \times K$ we have that $|S_{\tilde{h}, \sigma, t}(w) - S(w)| < c_1$, so $\text{rank}(S_{\tilde{h}, \sigma, t}(w)) \geq a$. \hfill \Box
Corollary 6.5.4. Assume that $m \leq n$, $b \leq c$. Suppose a map $h \in C_0^\infty(U, \mathbb{R}^c \times \mathbb{R}^n)$ is injective on some compact subset $K \subset U$, and

$$\text{rank}(P_h(x,0)) = b, \quad \text{rank}(S_h(x,0)) = m,$$

for every $(x,0) \in K \cap (\mathbb{R}^b \times 0)$. In other words, those matrices induce injective linear maps, which is the same here as having maximal ranks. Then there exist $\varepsilon, \eta > 0$ such that for all $(\tilde{h}, \sigma, t) \in N^3_{K,\eta}(h) \times [0; \varepsilon] \times [0; 1]$ the map $H_{\tilde{h},\sigma,t}$ is injective on $K$.

Proof. Suppose our statement fails. Then there exist a sequence

$$(h_i, \sigma_i, t_i) \in C_0^\infty(U, \mathbb{R}^c \times \mathbb{R}^n) \times (0; +\infty) \times [0; 1], \quad i \in \mathbb{N},$$

and two sequences $\{(x_i, v_i)\}_{i \in \mathbb{N}}, \{(y_i, w_i)\}_{i \in \mathbb{N}} \subset K$ of mutually distinct points such that

$$\lim_{i \to \infty} \|h - h_i\|_{3,K} = 0, \quad \lim_{i \to \infty} \sigma_i = 0,$$

$$H_{h_i,\delta_i,t_i}(x_i, v_i) = H_{h_i,\delta_i,t_i}(y_i, w_i), \quad i \in \mathbb{N},$$

One can assume, in addition, that

1. each $h_i$ is an embedding near $K$, since $h$ is so and embeddings near a compact subset are open in any topology $W^r$ for $r \geq 1$;
2. $\lim_{i \to \infty} \langle x_i, v_i \rangle = \langle x, v \rangle$, $\lim_{i \to \infty} \langle y_i, w_i \rangle = \langle y, w \rangle$, $\lim_{i \to \infty} t_i = t$ for some $(x, v), (y, w) \in K$ and $t \in [0; 1]$, due to compactness of $K \times [0; 1]$.

Since the $H$-evaluation map is $W^3$-continuous, see Lemma 6.3.1, we obtain that

$$h(x,v) = \lim_{i \to \infty} H_{h_i,\delta_i,t_i}(x_i, v_i) = \lim_{i \to \infty} H_{h_i,\delta_i,t_i}(y_i, w_i) = h(y, w),$$

whence $\langle x, v \rangle = \langle y, w \rangle$ since $h$ is injective on $K$.

We claim that then $v = w$. Indeed, suppose $\|v\| > a$ for some $a > 0$. Then for sufficiently large $i$ we have that $\delta_i < a < \min\{\|v_i\|, \|w_i\|\}$. Hence by (ii) before Lemma 6.3.1

$$h_i(x_i, v_i) = H_{h_i,\delta_i,t_i}(x_i, v_i) = H_{h_i,\delta_i,t_i}(y_i, w_i) = h_i(y_i, w_i),$$

and therefore $\langle x_i, v_i \rangle = \langle y_i, w_i \rangle$ since each $h_i$ is injective. This contradicts to the assumption that those points are distinct.

Let $H_{h_i,\sigma_i,t_i} = (f_{h_i,\sigma_i,t_i}, g_{h_i,\sigma_i,t_i}) : U \to \mathbb{R}^c \times \mathbb{R}^n$ be the coordinate functions of $H_{h_i,\sigma_i,t_i}$. Consider two cases.

1) Suppose that $v_i \neq w_i$ for all $i \in \mathbb{N}$. Then we can assume that the following sequence of unit vectors $u_i := \frac{v_i - w_i}{\|v_i - w_i\|} \in \mathbb{R}^m, i \in \mathbb{N}$, converges to some unit vector $u$. Then, by $W^3$-continuity of $\theta$-evaluation map $\tilde{\theta}$, see Corollary 6.5.2,

$$S_h(x,0)u = \lim_{i \to \infty} \frac{g_{h_i,\delta_i,t_i}(x_i, v_i) - g_{h_i,\delta_i,t_i}(y_i, w_i)}{\|v_i - w_i\|} \equiv 0,$$

which contradict to the assumption that $S_h(x,0)$ induces an injective linear map.

2) Otherwise, we can assume that $x_i \neq y_i$ for all $i \in \mathbb{N}$. Then, as in the previous case, we can assume that the following sequence of unit vectors $\tau_i := \frac{x_i - y_i}{\|x_i - y_i\|} \in \mathbb{R}^b, i \in \mathbb{N}$, converges to some unit vector $\tau$. Then, by $W^3$-continuity of $\theta$-evaluation map,

$$P_h(x,0)\tau = \lim_{i \to \infty} \frac{f_{h_i,\delta_i,t_i}(x_i, v_i) - f_{h_i,\delta_i,t_i}(y_i, w_i)}{\|x_i - y_i\|} \equiv 0,$$

which contradict to the assumption that $P_h(x,0)$ induces an injective linear map. \hfill \square
7. Proof of Theorem 6.4.1

Due to Lemma 6.3.4 it suffices to show that for each \( h \in X \) there exists a \( \mathcal{W}^3 \)-neighborhood \( \mathcal{U} \) in \( X \) and \( \varepsilon > 0 \) such that

\[
\mathcal{H}(U \times [0; \varepsilon] \times [0; 1]) \subset X.
\]

It will be convenient to say that \( (\mathcal{U}, \varepsilon) \) is admissible for \( h \) with respect to \( X \).

1) Suppose \( X = C_{a}(N, F) \). As \( B \) is compact, there exist finitely many local trivializations of \( h \)

\[
\hat{h}_i = (f_i, g_i) : \mathbb{R}^b \times \mathbb{R}^m \supset U_i \rightarrow W_i \times \mathbb{R}^n, \quad i = 1, \ldots, s,
\]

with respect to some trivialized local charts \( \Phi_i \) and \( \Psi_i \), and for each \( i \) a compact subset \( K_i \subset U_i \) such that \( B = \bigcup_i \Phi_i(K_i) \). Then, by Corollary 6.5.3, there exist \( \eta_i, \varepsilon_i > 0 \) such that

\[
\text{rank}(P_{h', \sigma, t}(w)) \geq a \quad \text{for all} \quad (h', \sigma, t, w) \in \mathcal{N}_{K_i, \eta_i}(\hat{h}_i) \times [0; \varepsilon_i] \times [0; 1] \times K_i.
\]

Define the following \( \mathcal{W}^2 \)-open neighborhood of \( h \) in \( C_{\alpha}(N, F) \):

\[
\mathcal{U}_i = \{ \hat{h} \in C_{\infty}^0(N, F) | \hat{h}(\Phi_i(K_i)) \subset \Psi_i(W_i \times \mathbb{R}^n) \text{ and } \|\Psi_i^{-1} \circ \hat{h} \circ \Phi_i - \hat{h}\|_{2, K_i} < \eta_i \}.
\]

Then the pair \( (\bigcap_{i=1}^s \mathcal{U}_i, \min \varepsilon_i) \) is admissible for \( h \) with respect to \( C_{\alpha}(N, F) \).

2) The proof for \( X = C_{\infty}^1(N, F) \) is literally the same and based on the part of Corollary 6.5.3 for matrices \( S \), but in that case \( \mathcal{U} \) will be only \( \mathcal{W}^3 \)-open.

3) Suppose \( X = C_{+a, b}(N, F) = C_{\alpha}(N, F) \cap C_{+a, b}(N, F) \). Let \( (\mathcal{U}, \varepsilon) \) and \( (\mathcal{U}_-, \varepsilon_-) \) be admissible for \( h \) pairs with respect to \( C_{\alpha}(N, F) \) and \( C_{+a, b}(N, F) \) correspondingly. Then \( (\mathcal{U} \cap \mathcal{U}_-, \min\{\varepsilon, \varepsilon_-\}) \) is admissible for \( h \) with respect to \( C_{+a, b}(N, F) \).

4) Let \( X = E_{+b, m}(N, F) \) be the subset of \( C_{+b, m}(N, F) \) consisting of embeddings and \( h \in X \). In this case we should have that \( b \leq c \) and \( m \leq n \).

Let \( R_a \) be a tubular neighborhood of \( B \) in \( E \) contained in \( N \). Then similarly to the previous cases 1) and 2) and also by Corollary 6.5.4 there exist finitely many local trivializations \( \hat{h}_i = (f_i, g_i) : \mathbb{R}^b \times \mathbb{R}^m \supset U_i \rightarrow W_i \times \mathbb{R}^n \) of \( h \), \( i = 1, \ldots, s \), with respect to some trivialized local charts \( \Phi_i \) and \( \Psi_i \), and for each \( i \) a pair of compact subset \( K_i, L_i \subset U_i \) with non-empty interiors, \( \eta_i, \varepsilon_i > 0 \) such that

(a) \( K_i \subset \text{Int}L_i \) and \( B \subset \bigcup_{i=1}^s \Phi_i(\text{Int}K_i) \);

(b) \( \text{rank}(P_{h', \sigma, t}(w)) = b, \text{rank}(S_{h', \sigma, t}(w)) = m \), and \( H_{h', \sigma, t}|_{L_i} : L_i \rightarrow W_i \times \mathbb{R}^n \) is injective for all \( (h', \sigma, t, w) \in \mathcal{N}_{K_i, \eta_i}(\hat{h}_i) \times [0; \varepsilon_i] \times [0; 1] \times L_i \);

(c) in particular, the rank of the Jacobi matrix of \( H_{h', \sigma, t} \) at each \( w \in L_i \) is equal to \( b + m = \dim(N) \), whence \( H_{h', \sigma, t} \) is an embedding near \( L_i \).

For \( i = 1, \ldots, s \) define the following \( \mathcal{W}^3 \)-open neighborhood of \( h \) in \( E_{+b, m}(N, F) \):

\[
\mathcal{U}_i = \{ \hat{h} \in E_{+b, m}(N, F) | \hat{h}(\Phi_i(L_i)) \subset \Psi_i(W_i \times \mathbb{R}^n) \text{ and } \|\Psi_i^{-1} \circ \hat{h} \circ \Phi_i\|_{3, L_i} < \eta_i \}.
\]

Put \( \mathcal{U}' = \bigcap_{i=1}^s \mathcal{U}_i \) and \( \varepsilon' = \min \varepsilon_i \). Then for all \( (\hat{h}, \sigma, t) \in \mathcal{U}' \times [0; \varepsilon'] \times [0; 1] \)

(d) \( H_{h, \sigma, t} = \hat{h} \) on \( N \setminus R_{\varepsilon'} \), and in particular, \( H_{h, \sigma, t} \) is an embedding on \( N \setminus R_{\varepsilon'} \);

(e) \( H_{h, \sigma, t} \) is an embedding near each compact set \( \Phi_i(L_i), i = 1, \ldots, s \), due to (c).

However, this only implies that \( H_{h, \sigma, t} \) is an immersion. To make \( H_{h, \sigma, t} \) embedding on all of \( N \) we will decrease \( \mathcal{U}' \) and \( \varepsilon' \) as follows.
By (a), there exists $\varepsilon'' \in (0; \varepsilon')$ such that $R_{\varepsilon''} \subset \bigcup_{i=1}^{s} \Phi_i(\text{Int}K_i)$. Since $h$ is injective, one can choose open neighborhoods $V, W \subset F$ of disjoint compact sets $h(B)$ and $h(N \setminus R_{\varepsilon''})$ respectively such that $V \cap W = \emptyset$. Define the following $W^0$-open neighborhood $V$ of $h$ in $E_{+,b,m}(N,F)$:

$$V = \{ \tilde{h} \in E_{+,b,m}(N,F) \mid \tilde{h}(\Phi_i(K_i)) \subset V, \ \tilde{h}(N \setminus \Phi_i(L_i)) \subset W, \ i = 1, \ldots, s \}$$

Note that by Lemma 6.3.1 the map $\mathcal{H} : C^0_\delta(N,F) \times [0; +\infty) \times [0; 1] \to C^0_\delta(N,F)$ is $W^{0,0}$-continuous, and $\mathcal{H}(h,0,t) = h$. Therefore, $\mathcal{H}^{-1}(V)$ is a $W^0$-open neighborhood of $h \times 0 \times [0; 1]$, whence there exists another neighborhood $U \subset U' \cap \mathcal{H}^{-1}(V)$ and $\varepsilon < b$ such that $\mathcal{H}(U \times [0; \varepsilon] \times [0; 1]) \subset U' \cap V$.

We claim that $\mathcal{H}(U \times [0; \varepsilon] \times [0; 1])$ consists of injective maps, which will finally imply that $\mathcal{H}(U \times [0; \varepsilon] \times [0; 1]) \subset E_{+,b,m}^\infty(N,F)$, i.e. that $(U, \varepsilon)$ is admissible for $h$ with respect to $E_{+,b,m}^\infty(N,F)$.

Indeed, suppose there exist $(\tilde{h}, \sigma, t) \in U \times [0; \varepsilon] \times [0; 1]$ and two distinct points $w_1 \neq w_2 \in N$ such that $H_{\tilde{h},\sigma,t}(w_1) = H_{\tilde{h},\sigma,t}(w_2)$. If $w_1 \in \Phi_i(K_i)$ for some $i \in \{1, \ldots, s\}$, then, due to injectivity of $H_{\tilde{h},\sigma,t}$ on $\Phi_i(L_i)$, we must have that $w_2 \in N \setminus \Phi_i(L_i)$. But then by (d),

$$H_{\tilde{h},\sigma,t}(w_1) = H_{\tilde{h},\sigma,t}(w_2) \in H_{\tilde{h},\sigma,t}(\Phi_i(K_i)) \cap H_{\tilde{h},\sigma,t}(N \setminus \Phi_i(L_i)) \subset V \cap W = \emptyset,$$

which gives a contradiction.

Therefore $w_1, w_2 \in N \setminus \bigcup_{i=1}^{s} \Phi_i(\text{Int}K_i) \subset N \setminus R_{\varepsilon''} \subset N \setminus R_\varepsilon$. Since $\sigma < \varepsilon$, we have that $\tilde{h}(w_1) = H_{\tilde{h},\sigma,t}(w_1) = H_{\tilde{h},\sigma,t}(w_2) = \tilde{h}(w_2)$ which is also impossible, since $\tilde{h}$ is injective.

5) If $b = \dim(B) = \dim(C) = c$ and $b + m = \dim(E) = \dim(F) = c + n$, then $E_0(N,F) \equiv E_{+,b,m}^\infty(N,F)$. Therefore 5) is a particular case of 4).

8. Proof of Theorem 1.1.1

Let $B$ be a compact manifold, $p : E \to B$ be a vector bundle equipped with some orthogonal structure, and $N \subset E$ be a smooth submanifold being also a neighborhood of $B$.

Evidently that for every $W^\infty$-continuous function $\delta : C^0_\delta(N,E) \to (0; +\infty)$ the map (1.3) is a $\delta$-linearizing homotopy, whence by Corollary 6.3.2 it always extends to a $W^\infty$-continuous map $H : E_0(N,E) \times [0; 1] \to C^0_\delta(N,E)$. Moreover, $H(h,1) = T_{h_0}h = \alpha(0,\cdot)$ on $R_{0,2\delta(h)}$, see (6.7), so statement (2) of Theorem 1.1.1 also always hold.

We will now choose $\delta$ so that the image of $H$ will be contained in $E_0(N,E)$. Let $\tilde{B}$ be a connected component of $B$ contained in a connected component $\tilde{E}$ of $E$, $\tilde{N} = N \cap \tilde{E}$, $b = \dim(B)$, and $m = \dim(M) - b$. Then by 5) of Theorem 6.4.1 there exists a $W^2$-continuous function $\delta^{b,m} : C^0_\delta(\tilde{N},E) \to (0; +\infty)$ such that the corresponding $\delta^{b,m}$-linearizing homotopy $H^{b,m} : C^0_\delta(\tilde{N},E) \times [0; 1] \to C^0_\delta(\tilde{N},E)$ leaves invariant the set $E_{+,b,m}^\infty(\tilde{N},E) = E(\tilde{N},E)$ of embeddings $\tilde{N} \subset \subset E$.

Let $\delta : C^0_\delta(N,E) \to (0; +\infty)$ be the minimum of all functions $\delta^{b,m}$, where $b$ runs over dimensions of connected components of $B$, and $b + m$ runs over dimensions of connected components of $E$. Then for every $h \in E_0(N,E)$ and $t \in [0; 1]$ the map $H(h,t) : N \to E$ is an embedding.

Finally, due to the estimate (6.16), one can additionally decrease $\delta$ so that $H(h,t)(R_{\delta(h)}) \subset N$ for all $h \in E_0(N,E)$ and $t \in [0; 1]$, which proves statement (1) of Theorem 1.1.1 as well. □
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