Towards the understanding of fully-heavy tetraquark states from various models

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We use a color-magnetic interaction model, a traditional constituent quark model and a multiquark color flux-tube model to systematically investigate the properties of the fully-heavy tetraquark states $[Q_1Q_2][Q_3Q_4]$ ($Q = c, b$) with the help of the Gaussian expansion method. Numerical results indicate that the color-magnetic interaction model can not completely absorb QCD dynamic effects through effective constituent quark mass in the states $[Q_1Q_2][Q_3Q_4]$. In addition, the model may overestimate the color-magnetic interaction in the extension from heavy mesons to the states $[Q_1Q_2][Q_3Q_4]$ under the assumption of same spatial configurations. The Coulomb interaction plays a critical role in the dynamical model calculations on the heavy hadrons, which is the direct reason why none of bound states $[Q_1Q_2][Q_3Q_4]$ can be found in the dynamical models. The color configuration $[Q_1Q_2][Q_3Q_4][a_1]$ should be taken seriously in the ground states due to the strong Coulomb attraction. The color configuration $[Q_1Q_2][Q_3Q_4][a_1]$ is absolutely dominant in the excited states mainly because of the lower kinetic induced by the Coulomb interaction.

I. INTRODUCTION

The dynamics in fully-heavy tetraquark states is very simple, which only includes perturbative QCD one gluon exchange (OGE) interaction and quark confinement potential. They can provide a unique environment to examine the non-relativistic quark model with QCD effective potentials if they do exist in the form of compact bound states than the loosely bound hadron molecular states because of the lack of the light mesons exchange between two $QQ$-mesons. The question of whether there exist such bound states has been debated for more than forty years \cite{1}, which has received much attention from the different theoretical frameworks, such as the non-relativistic quark models \cite{2}–\cite{5}, the color-magnetic interaction model \cite{6}–\cite{9}, the QCD sum rules \cite{10}, the Bethe-Salpeter equation \cite{11}, MIT bag model \cite{12}, the lattice QCD \cite{13} et al. The conclusions were model dependent. Taking the state $b\bar{b}b\bar{b}$ as an example, it can exist as a stable state against strong interaction in the spin-spin interaction model \cite{14} while it is not stable in the string model \cite{15}. On the experimental side, the ATLAS, CMS and LHCb collaborations have measured the cross section for double charmonium production \cite{16}. Recently, the LHCb collaboration investigated the $\Upsilon_{\mu^+\mu^-}$ invariant-mass distribution to search for a possible fully-heavy tetraquark state $b\bar{b}b\bar{b}$, and observed no significant excess \cite{17}. The existence of the fully-heavy tetraquark states has been still controversial so far.

It is necessary to carry out a dynamical investigation on the natures of the fully-heavy tetraquark states from various theoretical frameworks although the states are still missing in experiments, which is propitious to widen our understanding on the structures of the states. In this work, we prepare to make a systematic research on the states $[Q_1Q_2][Q_3Q_4]$ from the perspective of the phenomenological models, in which the color-magnetic interaction model, traditional constituent quark model and multiquark color flux-tube model are involved. The color-magnetic interaction models have various versions \cite{18}, the model with reference mass scale is employed here. The traditional constituent quark model includes the OGE interaction and two-body confinement potential proportional to color charge. The multiquark color flux-tube model based on the lattice QCD color flux-tube picture and the traditional quark model has been developed, which contains a multibody confinement potential instead of two-body one. The model was recently applied to systematically investigate on the states $[c\bar{s}][\bar{c}s]$ \cite{19}. Furthermore, the conclusions of other phenomenological models are involved to make a comprehensive understanding on the fully-heavy tetraquark states.

This paper is organized as follows. After the introduction section, the descriptions of three models are given in Sec. II. The wavefunction of the states $[Q_1Q_2][Q_3Q_4]$ is shown in Sec. III. The numerical results and discussions of the states are presented in Sec. IV. A brief summary is listed in the last section.

II. THREE MODELS

A. Color-magnetic interaction model (CMIM)

The color-magnetic interaction was deduced from the spin-dependent part of the OGE interaction \cite{21}. In addition, the effective quark masses are involved in the CMIM Hamiltonian, which is assumed to be able to absorb various QCD dynamic effects in principle. The model can give a convincing explanation of the mass splitting of ordinary hadrons. This mechanism has been applied to investigate the properties of the $H$-particle and the heavy pentaquark $Qqqqq$ \cite{21, 22}. Recently, the mechanism was
also widely utilized to study the natures of multiquark states to explain some hadrons \[18\].

The CMIM Hamiltonian of $n$-body ground states acting on the color and spin degrees of freedom reads

$$ H_{cm}^n = - \sum_{i<j}^n C_{ij} \lambda_i^c \cdot \lambda_j^c \sigma_i \cdot \sigma_j, $$

(1)

where $\lambda^c$ and $\sigma$ represent the Gell-Mann matrices and the Pauli matrices, respectively. In the conventional mesons and baryons, the color factor is frozen as a result of constant color factor, and the calculation of color-magnetic interaction reduces to the simple algebra of the spin-spin operator $\sigma_i \cdot \sigma_j$. In multiquark states, the calculation is complicated because of various color configurations. The coefficient $C_{ij}$ describes the effective coupling constant between the $q_i$ or $\bar{q}_i$ and $q_j$ or $\bar{q}_j$, which incorporates the effects from the spatial configuration and the quark mass. In general, it is difficult to exactly obtain the effect from the spatial configuration because of no knowing the spatial wave function. Therefore, it is assumed that the interactional systems with the same flavors share the same size.

For $n$-body ground states, the color-magnetic interaction Hamiltonian leads to the mass formula

$$ M = \sum_{i=1}^n m_i + \langle H_{cm}^n \rangle $$

(2)

where $m_i$ is the effective mass of the $q_i$ or $\bar{q}_i$, which includes the constituent quark mass and various dynamic effects in the system. In principle, the values of $m_i$ and $C_{ij}$ should be different in the various hadron environment. For the simplicity and model universality, they are usually extracted from the masses of conventional hadrons and then are extended to multiquark systems, which would be considered to cause the uncertainty on mass estimations \[18\]. Therefore, various CMIMs were proposed by modifying the mass formula or choosing refined parameters in calculations to obtain reasonable description for hadron spectra \[18\].

A CMIM with a reference mass scale has been developed by modifying the mass formula to avoid generally overestimated masses and presented as the following form \[18\],

$$ M = M_{\text{ref}} - \langle H_{cm}^n \rangle_{\text{ref}} + \langle H_{cm}^n \rangle $$

(3)

$M_{\text{ref}}$ and $\langle H_{cm}^n \rangle_{\text{ref}}$ are respectively a reference threshold and its color-magnetic interaction energy. The parameters $C_{ij}$ related to ground heavy-meson states are taken from Ref. \[18\], which are used in the present work and listed in Table I. More details about the model can be found in Ref. \[18\].

According to the mass formula, one can define a binding energy as

$$ \Delta E = M - M_{\text{ref}} = \langle H_{cm}^n \rangle - \langle H_{cm}^n \rangle_{\text{ref}} $$

(4)

to identify whether a state is stable against strong interaction. If $\Delta E > 0$, the state can fall apart into the two mesons through quark rearrangement. If $\Delta E < 0$, the strong decay into the two mesons is forbidden and therefore the decay must be weak or electromagnetic interaction.

**B. Constituent quark model (CQM)**

Constituent quark model is formulated under the assumption that the hadrons are color singlet nonrelativistic bound states of constituent quarks with effective masses and interactions. One expects the dynamics of the model to be governed by QCD. The perturbative effect is well the OGE interaction, which takes its standard form and is listed in the following \[23\],

$$ V^{\text{OGE}}_{ij} = \frac{\alpha_s}{4} \lambda_i^c \cdot \lambda_j^c \left( \frac{1}{r_{ij}} - \frac{2\pi\delta(r_{ij})\sigma_i \cdot \sigma_j}{3m_im_j} \right), $$

The color-magnetic interaction proportional to the spin-color factor $\lambda_i^c \cdot \lambda_j^c \sigma_i \cdot \sigma_j$ in the OGE interaction leads to mass splitting among different color-spin configurations. $\alpha_s$ is a running strong coupling constant in the perturbative QCD \[24\],

$$ \alpha_s(\mu^2) = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}}. $$

(5)

In this work, we take the following form,

$$ \alpha_s(\mu_{ij}^2) = \frac{\alpha_0}{\ln \frac{\mu_{ij}^2}{\mu_0^2}}, $$

(6)

where $\mu_{ij}$ is the reduced mass of two interacting particles. The function $\delta(r_{ij})$ should be regularized \[25\],

$$ \delta(r_{ij}) = \frac{1}{4\pi r_{ij}^2(\mu_{ij})} e^{-r_{ij}/r_0(\mu_{ij})}, $$

(7)

where $r_0(\mu_{ij}) = r_0/\mu_{ij}$. $\Lambda_0$, $\alpha_0$, $\mu_0$ and $\tilde{r}_0$ are adjustable model parameters determined by fitting experimental data of heavy mesons.

Color confinement is one of the most prominent features of QCD and should play an essential role in the low energy hadron physics. At present it is still impossible for us to derive color confinement analytically from the QCD Lagrangian. In the CQM, it can be phenomenologically described as the sum of two-body interactions proportional to the color charges and $r_{ij}^2$ \[26\],

$$ V^{\text{con}} = -\alpha_s \sum_{i>j}^n \lambda_i^c \cdot \lambda_j^c r_{ij}^2 $$

(8)

| $C_{ij}$ | $C_{eq}$ | $C_{ez}$ | $C_{cc}$ | $C_{hq}$ | $C_{hs}$ | $C_{hb}$ | $C_{hc}$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| Value   | 6.6     | 6.7     | 5.3     | 2.1     | 2.3     | 2.9     | 3.3     |
where \( r_{ij} \) is the distance between the \( q_i \) or \( \bar{q}_i \) and the \( q_j \) or \( \bar{q}_j \). The model can automatically prevent overall color singlet multiquark states disintegrating into several color subsystems by means of color confinement with an appropriate \( SU(3) \) Casimir constant \([22]\). The model also allows a multiquark system disintegrating into color-singlet clusters, and it leads to interacting potentials within mesonlike \( qq \) and baryonlike \( qqq \) subsystems in accord with the empirically known potentials \([22]\). However, the model is known to be flawed phenomenologically because it leads to power law van der Waals forces between color-singlet hadrons. In addition, it also leads to anticonfinement for symmetrical color structure in the multiquark system \([28]\).

The completely Hamiltonian for the heavy mesons and fully-heavy tetraquark states \([Q_1Q_2][Q_3Q_4]\) can be presented as

\[
H_n = \sum_{i=1}^{n} \left( m_i + \frac{p_i^2}{2m_i} \right) - T_C + \sum_{i>j}^{n} V_{ij}^{\text{con}} + V^{\text{con}}.
\]

\( T_c \) is the center-of-mass kinetic energy of the states and should be deducted; \( p_i \) and \( m_i \) are the momentum and mass of the \( q_i \) or \( \bar{q}_i \), respectively.

In order to avoid the misjudgement of the behavior of model dynamics due to inaccurate numerical results, a high precision computational method is therefore indispensable. The Gaussian expansion method (GEM) \([29]\), which has been proven to be rather powerful to solve few-body problem in nuclear physics, is therefore widely used to study multibody systems. According to the GEM, the two-body relative motion wave function of heavy mesons can be written as,

\[
\phi_{lm}^G(r) = \sum_{n=1}^{n_{\text{max}}} c_n N_{nt} \alpha^{n-1} e^{-\nu_n r^2} Y_{lm}(\hat{r}) \tag{9}
\]

Gaussian size parameters are taken as geometric progression

\[
\nu_n = \frac{1}{r_n^2}, \quad r_n = r_1 a^{n-1}, \quad a = \left( \frac{r_{n_{\text{max}}}}{r_1} \right)^{\frac{1}{n_{\text{max}}-1}} \tag{10}
\]

The coefficient \( c_n \) is determined by the dynamics of systems. More details can be found in Ref. \([29]\). With \( r_1 = 0.2 \text{ fm} \), \( r_{n_{\text{max}}} = 2.0 \text{ fm} \) and \( n_{\text{max}} = 7 \), the converged numerical results can be arrived at.

The mass of \( ud \)-quark is taken to be one third of that of nucleon, other adjustable model parameters in Table III can be determined by approximately strict solving two-body Schrödinger equation to fit the masses of the ground states of heavy mesons in Table III. At the same time, we also give the values of various parts of the model Hamiltonian. \( E_k \), \( V^{\text{con}} \), \( V^{\text{em}} \) and \( V^{\text{clb}} \) represent kinetic, confinement potential, color-magnetic interaction and Coulomb interaction, respectively. It can be found from Table III that the Coulomb interaction provides an extremely strong short-range attraction, which is the main reason why a quark and an antiquark can form a bound state.

In the color-magnetic interaction model, the matrix elements \( \langle \sigma_i \cdot \sigma_j \rangle = -3 \) and \( 1 \) for spin \( S = 0 \) and spin \( S = 1 \), respectively. Assuming the same spatial configuration, the model, which does not explicitly involve any dynamic effect, gives a ratio \( 3 : 1 \) for spin \( S = 0 \) and \( S = 1 \) mesons with the same flavors, such as \( D \) and \( D^* \). However, it is because of the color-magnetic interaction that the sizes of the two mesons have a litter difference, see Table III. The ration of the color-magnetic interaction in the dynamical calculation is not strict \( 3 : 1 \) because of the same reason, which is between \( 3 : 1 \) and \( 4 : 1 \). It is therefore approximately reasonable to describe the mass splitting of mesons by the color magnetic interaction in the color-magnetic interaction model.

### Table II: Model parameters, quark mass and \( \Lambda_0 \) unit in MeV, \( a_{\alpha} \) unit in MeV fm \( \alpha^2 \), \( r_0 \) unit in MeV fm and \( \alpha_0 \) is dimensionless.

| Para. | \( m_{u,d} \) | \( m_s \) | \( m_c \) | \( m_b \) | \( a_c \) | \( a_0 \) | \( \Lambda_0 \) | \( r_0 \) |
|-------|---------------|----------|----------|---------|--------|--------|-------------|--------|
| Valu. | 313           | 494      | 1664     | 5006    | -150   | 4.25   | 40.85       | 119.3  |

### Table III: Ground state heavy-meson spectra and the values of various parts of the Hamiltonian in MeV and the average distance in fm.

| States | PDG | \( E_z \) | \( E_k \) | \( V^{\text{con}} \) | \( V^{\text{em}} \) | \( V^{\text{clb}} \) | \( \langle r^2 \rangle \) |
|--------|-----|----------|----------|--------------------|----------------|----------------|----------------|
| \( D^0 \) | 1869 | 1886 | 737 | 200 | -92 | -937 | 0.50 |
| \( D^+ \) | 2007 | 2000 | 633 | 226 | 27 | -862 | 0.53 |
| \( D_s^+ \) | 1969 | 1982 | 693 | 151 | -105 | -914 | 0.43 |
| \( D_s^* \) | 2112 | 2109 | 560 | 170 | 29 | -816 | 0.47 |
| \( \eta_c \) | 2980 | 2965 | 679 | 75 | -123 | -995 | 0.31 |
| \( J/\Psi_c \) | 3097 | 3103 | 488 | 97 | 29 | -838 | 0.35 |
| \( B^0 \) | 5280 | 5261 | 664 | 197 | -34 | -885 | 0.50 |
| \( B^+ \) | 5325 | 5305 | 623 | 207 | 11 | -855 | 0.51 |
| \( B_s^0 \) | 5366 | 5346 | 612 | 143 | -42 | -868 | 0.42 |
| \( B_s^+ \) | 5416 | 5399 | 555 | 155 | 13 | -824 | 0.44 |
| \( B_s^* \) | 6277 | 6244 | 644 | 54 | -79 | -1044 | 0.26 |
| \( \Upsilon \) | 9301 | 9376 | 740 | 24 | -96 | -1305 | 0.17 |

### C. Multiquark color flux-tube model (MCFTM)

Details of the multiquark color flux-tube model based on traditional constituent quark models and the lattice QCD color flux-tube picture can be found in our previous paper \([10]\). Only prominent characteristics of the model are presented here. Within the framework of color flux-tube picture, the quark and antiquark in a meson are linked with a three-dimensional color flux tube. A two-body confinement potential can be written as

\[
V^{\text{con}}(r) = K r^2, \tag{11}
\]

where \( r \) is distance between the quark and antiquark and the parameter \( K \) is the stiffnesses of a three-dimension color flux-tube and is determined by fitting the heavy-meson spectra, where \( K = -a_c \lambda_1 \cdot \lambda_2 = 800 \text{ MeV fm}^{-2} \).
The states \([Q_1Q_2][Q_3Q_4]\) favor a compact tetraquark configuration than a loosely bound hadron molecular configuration. According to a double Y-shaped color flux-tube structure, a four-body quadratic confinement potential can be written as,

\[
V^{\text{con}}(4) = K \left[ (r_1 - y_{12})^2 + (r_2 - y_{12})^2 + (r_3 - y_{34})^2 + (r_4 - y_{34})^2 + \kappa_d(y_{12} - y_{34})^2 \right],
\]

in which \(r_1, r_2, r_3, \) and \(r_4\) respectively represent the position of the \(Q_1, Q_2, Q_3, \) and \(Q_4\) two Y-shaped junctions \(y_{12}\) and \(y_{34}\) are variational parameters determined by taking the minimum of the confinement potential. The relative stiffness parameter \(\kappa_d\) is equal to \(\frac{A}{C_d}\), where \(C_d\) is the eigenvalue of the Casimir operator associated with the \(SU(3)\) color representation at either end of the color flux-tube, such as \(C_3 = \frac{4}{3}, C_6 = \frac{10}{3},\) and \(C_8 = 3\).

The minimum of the confinement potential \(V^{\text{con}}_{\text{min}}(4)\) can be obtained by taking the variation of \(V^{\text{con}}(4)\) with respect to \(y_{12}\) and \(y_{34}\), and it can be expressed as

\[
V^{\text{con}}_{\text{min}}(4) = K \left( R_1^2 + R_2^2 + \frac{\kappa_d}{1 + \kappa_d} R_3^2 \right),
\]

The canonical coordinates \(R_i\) have the following forms,

\[
\begin{align*}
R_1 &= \frac{1}{\sqrt{2}}(r_1 - r_2), & R_2 &= \frac{1}{\sqrt{2}}(r_3 - r_4),
R_3 &= \frac{1}{\sqrt{4}}(r_1 + r_2 - r_3 - r_4),
R_4 &= \frac{1}{\sqrt{4}}(r_1 + r_2 + r_3 + r_4).
\end{align*}
\]

The use of \(V^{\text{con}}_{\text{min}}(4)\) can be understood here as that the gluon field readjusts immediately to its minimal configuration.

The OGE interaction is also involved in the multiquark color flux-tube model, which is the same as that of the CQM. It’s worth mentioning that the multiquark color flux-tube model is not a completely new model but the updated version of the traditional CQM based on the color flux-tube picture of hadrons in the lattice QCD. In fact, it merely modifies the two-body confinement potential into the multibody one to describe multiquark states with multibody interaction. The MCFSTM reduces to the CQM in mesons while the MCFSTM can obtain different results from the CQM in multiquark states.

### III. WAVEFUNCTION

Numerical results of the states \([Q_1Q_2][Q_3Q_4]\) can be obtained by solving a fourbody Schrödinger equation with its complete wavefunctions of the states including all possible flavor-spin-color-spatial channels that contribute to a given well defined parity, isospin, and total angular momentum. Within the framework of the diquark-antidiquark configuration, the wavefunction of the state \([Q_1Q_2][Q_3Q_4]\) can be constructed as a sum of the following direct products of color \(\chi_c\), isospin \(\eta_i\), spin \(\chi_s\) and spatial \(\phi^G_{im}\) terms

\[
\psi_{[Q_1Q_2][Q_3Q_4]} = \sum_{\alpha} \xi_\alpha \left[ \phi^G_{i_{m_a}}(r) \chi_{s_a} \right]_{m_a} \left[ \phi^G_{i_{m_b}}(R) \times \chi_{s_b} \right]_{m_b}\]

The subscripts \(a\) and \(b\) in the intermediate quantum numbers represent the diquark \([Q_1Q_2]\) and antidiquark \([Q_3Q_4]\), respectively. The summing index \(\alpha\) stands for all possible flavor-spin-color-spatial intermediate quantum numbers.

The relative spatial coordinates \(r, R\) and \(X\) and center of mass \(R_c\) in the states \([Q_1Q_2][Q_3Q_4]\) can be defined as,

\[
\begin{align*}
\mathbf{r} &= r_1 - r_2, & \mathbf{R} &= r_3 - r_4,
\mathbf{X} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 - m_3 \mathbf{r}_3 + m_4 \mathbf{r}_4}{m_1 + m_2 - m_3 + m_4},
\mathbf{R}_c &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 + m_4 \mathbf{r}_4}{m_1 + m_2 + m_3 + m_4}.
\end{align*}
\]

In the dynamical calculation, the relative motion wave functions \(\phi^G_{i_{m_a}}(r)\), \(\phi^G_{i_{m_b}}(R)\) and \(\phi^G_{i_{m_c}}(X)\) can be expressed as the superposition of many different size Gaussian functions with well-defined quantum numbers, which share the exactly same form with that of heavy mesons, to obtain accurate numerical results. For the sake of saving space, their explicit expressions are not presented here. The heavy quarks have isospin zero so they do not contribute to the total isospin. The flavor wavefunction is therefore symmetrical if \(Q_1\) and \(Q_2\) \((Q_3\) and \(Q_4)\) are identical particles.

The color representation of the antidiquark \([Q_3Q_4]\) (diquark \([Q_1Q_2]\)) maybe antisymmetrical \(3\) \((\bar{3})\) or symmetrical \(6\) \((\bar{6})\). Coupling the diquark and the antidiquark into an overall color singlet according to color coupling rule only have two ways: \([\bar{3}]_{\bar{3}} \otimes [3]_3\) and \([\bar{3}]_{\bar{3}} \otimes [3]_3\). The spin of the diquark \([Q_1Q_2]\) is coupled to \(s_a\) and that of the antidiquarks \([Q_3Q_4]\) to \(s_b\). The total spin wavefunction of the tetraquark state \([Q_1Q_2][Q_3Q_4]\) can be written as \(S = s_a \oplus s_b \oplus s_c\). Then we have the following basis vectors as a function of the total spin \(S, 0 = 1 \oplus 0 \oplus 0, 1 = 1 \oplus 1, 1 \oplus 0 \oplus 0, 1 \oplus 0 \oplus 1, 2 = 1 \oplus 1\).

Taking all degrees of freedom of identical particles in the diquark (antidiquark) into account, the Pauli principle must be satisfied by imposing restrictions on their quantum numbers to satisfy antisymmetry. The S-wave diquark (antidiquark) with two identical quarks (antiquarks) has two possible configurations, \([Q_1Q_2]_{3_{c}}^{\bar{3}_{c}}\) and \([Q_1Q_2]_{\bar{6}_{c}}^{\bar{6}_{c}}\), where the superscript and subscript denote the spin and color representation, respectively. The possible color-flavor-spin functions of
the states $[cc][\bar{c}c]$, $[bb][\bar{c}c]$ and $[bb][\bar{b}\bar{b}]$ states can be written as,

\begin{align*}
0^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
1^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
2^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
\end{align*}

those of the states $[cc][\bar{b}\bar{d}]$ and $[bb][\bar{c}\bar{c}]$ reads,

\begin{align*}
0^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
1^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
2^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
\end{align*}

those of the state $[cb][\bar{c}d]$ reads,

\begin{align*}
0^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
1^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
2^+ : & \quad \left[ [Q_1Q_2]_{3} \right]_{3} \left[ [\bar{Q}_3\bar{Q}_4]_{3} \right]_{1} \\quad \left[ [Q_1Q_2]_{6} \right]_{6} \left[ [\bar{Q}_3\bar{Q}_4]_{6} \right]_{1}, \\
\end{align*}

In the following, we will extend the three models to study the properties of the fully-heavy tetraquark states with the well-defined wavefunction.

\section*{IV. NUMERICAL RESULTS AND DISCUSSIONS}

Recently, various versions of color-magnetic interaction models were widely utilized to investigate the properties of the states $[Q_1Q_2][\bar{Q}_3\bar{Q}_4]$. Berezhnoy et al applied a color-magnetic model, in which tetraquark mass can be determined by solving a two-particle Schrödinger equation with pointlike diquark (antidiquark) in color $3$, $(\bar{3})$, to research the states $[cc][\bar{c}c]$, $[bb][\bar{b}\bar{b}]$ and $[bc][\bar{b}\bar{c}]$. With the exception of the tensor states $[cc][\bar{c}c]$ and $[bc][\bar{b}\bar{c}]$, the lowest states with other quantum numbers are all below relevant two meson thresholds. Karliner et al. studied the $0^+$ states $[cc][\bar{c}c]$ and $[bb][\bar{b}\bar{b}]$ with the color-magnetic interaction model motivated by the QCD-string junction picture. Their masses are, respectively, $6192 \pm 25$ MeV and $18826 \pm 25$ MeV. It was noted that an experimental search for these states in the predicted mass range is highly desirable. Wu et al systematically investigated the mass spectra of the states $[Q_1Q_2][\bar{Q}_3\bar{Q}_4]$ with a color-magnetic interaction model with a reference mass scale. It was found that the states $[bb][\bar{b}\bar{c}]$ and $[bc][\bar{b}\bar{c}]$ are possible stable or narrow resonance states.

One should note that all color-magnetic interaction models ignore the spatial degree of freedom so that everything in the models depends only on the color-spin algebra. The generalization of the color-magnetic interaction models from conventional hadrons to multiquark states is implemented under the assumption that the spatial configurations of each $qq$, $q\bar{q}$ and $q\bar{q}$ pairs are the same in multiquark states as in ordinary hadrons. The well-known H-particle predicted in the color-magnetic interaction model was below the $\Lambda\Lambda$ threshold about $80$ MeV. However, the state was above the threshold in the nonrelativistic quark model involving the color-magnetic interaction with spatial degree of freedom and other various dynamics once $SU(3)$ flavor symmetry is broken. The state was once very fashionable and was searched for in many experiments. The high-sensitivity search at Brookhaven gave no evidence for the production of the state. Recently, the theoretical case for the state continues to be strong and has been strengthened by the NPLQCD and HALQCD collaborations that both observed the state. The high-statistics search for the state production shown that no indication of the state with a mass near the $\Lambda\Lambda$ threshold was seen.

In view of the inherent defects of the color-magnetic models and the experience of the H-particle, it is therefore necessary to make a systematically dynamical investigation on the properties of the states $[Q_1Q_2][\bar{Q}_3\bar{Q}_4]$ with quark models containing various QCD dynamic effects. The MCFTM and CQM are therefore involved, in which the masses of all possible states $[Q_1Q_2][\bar{Q}_3\bar{Q}_4]$ can be obtained by solving a four-body Schrödinger equation with the well-defined trial wavefunctions and presented in Table IV. The notations $3$, $(3)$, and $6$, $(6)$ stand for the color configurations $[[Q_1Q_2]_{3}][[\bar{Q}_3\bar{Q}_4]_{3}]_{1}$ and $[[Q_1Q_2]_{6}][[\bar{Q}_3\bar{Q}_4]_{6}]_{1}$, respectively. C.C. represents the coupling of the two color configurations. The masses of the states $[Q_1Q_2][\bar{Q}_3\bar{Q}_4]$ with two color configurations and their individual proportion in the eigen states can be achieved by the eigen wavefunction of the states and listed in Table IV. In order to facilitate the comparison, we also reproduce the masses of the states $[Q_1Q_2][\bar{Q}_3\bar{Q}_4]$ in the CMIM with the approximation $C_{QQ} = C_{Q\bar{Q}}$, which are represented in Table IV.

It can be found from Table IV that the masses predicted by the CMIM are lower $300\text{-}500$ MeV than those predicted by other two models involving QCD dynamic effects. The masses predicted by the MCFTM are lower $30\text{-}120$ MeV than those by the CQM. Comparing the masses with the lowest two meson thresholds $T_{M_1,M_2}$, the binding energy $\Delta E$ in the MCFTM can be calculated and are presented in Table IX. One can find that none of these states can exist as a bound state because all states are hundreds of MeV above the corresponding threshold in the MCFTM while the masses of the states predicted by the CMIM are close to the corresponding threshold. In order to unveil the underlying cause, we give the values of various parts in the Hamiltonian and the average distances $\langle r_{ij}^2 \rangle \hat{z}$ and $\langle \mathbf{X}^2 \rangle \hat{z}$ by using the eigen wavefunction, which are shown in Table IV and IX, respectively.

The investigation on the spectrum of heavy-mesons in Sec. II indicates that the Coulomb interaction is significant in the formation of heavy-mesons. It can be found from Table IX that the interaction also plays a decisive
### TABLE IV: The mass spectra of the ground states $[Q_1 Q_2] [Q_3 Q_4]$ in the three models, unit in MeV.

| Model | CMIM | MCFTM | CQM |
|-------|------|-------|-----|
| Flavor | $J^P$ | $E_0, E_{cc}$ | $3 \otimes 3_c$ | $6_c \otimes 6_c$ | $6_e \otimes 6_e$ | $3 \otimes 3_c$ | $6_e \otimes 6_e$ | $E_0, E_{cc}$ |
| $[cc][cc]$ | $1^+$ | 0.00, 100% | 0.00, 6139 | 6463, 100% | 6407 | 6573, 36% | 6537, 64% | 6491 |
| | $2^+$ | 56.53, 100% | 56.53, 6194 | 6486, 100% | 6486 | 6607, 100% | 6607 |
| $[cc][bb]$ | $0^+$ | 0.00, 100% | 0.00, 6139 | 6463, 100% | 6407 | 6573, 36% | 6537, 64% | 6491 |
| | $1^+$ | 4.27, 100% | 4.27, 12660 | 62945, 100% | 62945 | 13024, 100% | 13024 |
| $[bb][bb]$ | $0^+$ | 0.00, 100% | 0.00, 6139 | 6463, 100% | 6407 | 6573, 36% | 6537, 64% | 6491 |
| | $1^+$ | 39.47, 100% | 39.47, 12695 | 62960, 100% | 62960 | 13041, 100% | 13041 |
| $[cc][cb]$ | $1^+$ | 0.00, 100% | 0.00, 6139 | 6463, 100% | 6407 | 6573, 36% | 6537, 64% | 6491 |
| | $2^+$ | 30.93, 100% | 30.93, 18921 | 61387, 100% | 61387 | 19429, 100% | 19429 |
| $[bb][cb]$ | $0^+$ | 22.93, 100% | 22.93, 18921 | 61387, 100% | 61387 | 19429, 100% | 19429 |
| | $1^+$ | 15.85, 100% | 15.85, 18921 | 61387, 100% | 61387 | 19429, 100% | 19429 |
| $[cb][cb]$ | $1^+$ | 0.00, 100% | 0.00, 6139 | 6463, 100% | 6407 | 6573, 36% | 6537, 64% | 6491 |
| | $2^+$ | 33.07, 100% | 33.07, 15806 | 61812, 100% | 61812 | 16274, 100% | 16274 |

### TABLE V: The values of various parts of the Hamiltonian in the MCFTM, unit in MeV.

| Flavor | $J^P$ | $E_4$ | $E_k$ | $V_{min}$ | $V_{cm}$ | $V_{C.C.}$ | $T_{M_1 M_2}$ | $\Delta E$ | $\Delta E_k$ | $\Delta V_{min}$ | $\Delta V_{cm}$ | $\Delta V_{C.C.}$ |
|--------|--------|--------|--------|-----------|-----------|-----------|-------------|------------|-------------|----------------|----------------|---------------|
| $[cc][cc]$ | $1^+$ | 6463 | 887 | 192 | -51 | -1279 | $\eta_c \eta_c$ | 477 | -471 | 42 | 195 | 711 |
| | $2^+$ | 6463 | 887 | 192 | -51 | -1279 | $\eta_c \eta_c$ | 477 | -471 | 42 | 195 | 711 |
| $[cc][bb]$ | $0^+$ | 12906 | 853 | 131 | -27 | -1392 | $B_c B_c$ | 418 | -435 | 24 | 132 | 696 |
| | $1^+$ | 12945 | 877 | 135 | 6 | -1324 | $B_c^* B_c$ | 365 | -359 | 17 | 65 | 452 |
| $[bb][cb]$ | $0^+$ | 19329 | 865 | 69 | -26 | -1605 | $\eta_c \eta_b$ | 577 | -615 | 21 | 166 | 1005 |
| | $1^+$ | 19373 | 826 | 68 | 3 | -1550 | $\eta_b \chi(1S)$ | 511 | -474 | 14 | 75 | 895 |
| $[cc][cb]$ | $0^+$ | 9670 | 858 | 161 | -38 | -1295 | $\eta_c B_c$ | 461 | -465 | 32 | 164 | 730 |
| | $1^+$ | 9683 | 838 | 165 | -25 | -1295 | $\eta_c B_c$ | 461 | -465 | 32 | 164 | 730 |
| $[bb][cb]$ | $0^+$ | 16126 | 856 | 97 | -27 | -1483 | $B_c \eta_b$ | 506 | -528 | 19 | 148 | 866 |
| | $1^+$ | 16107 | 905 | 94 | -19 | -1483 | $B_c \eta_b$ | 418 | -337 | 6 | 58 | 696 |
| $[cb][cb]$ | $0^+$ | 12829 | 932 | 123 | -84 | -1483 | $\eta_c \eta_c$ | 394 | -87 | 24 | 135 | 817 |
| | $1^+$ | 12881 | 816 | 134 | -31 | -1379 | $\eta_c \chi(1S)$ | 430 | -423 | 29 | 68 | 756 |

### TABLE VI: The average distances $(r_{ij}^2)^{1/2}$ and $(X^2)^{1/2}$ of the ground states $[Q_1 Q_2] [Q_3 Q_4]$ in the MCFTM, unit in fm.

| State | $J^P$ | $(r_{12}^2)^{1/2}$ | $(r_{34}^2)^{1/2}$ | $(r_{13}^2)^{1/2}$ | $(r_{24}^2)^{1/2}$ | $(r_{14}^2)^{1/2}$ |
|-------|-------|------------------|------------------|------------------|------------------|------------------|
| $(r_{12}^2)^{1/2}$ | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| $(r_{34}^2)^{1/2}$ | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| $(r_{13}^2)^{1/2}$ | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| $(r_{24}^2)^{1/2}$ | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| $(r_{14}^2)^{1/2}$ | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| $(X^2)^{1/2}$ | 0.31 | 0.36 | 0.37 | 0.25 | 0.29 | 0.30 | 0.17 | 0.20 | 0.21 | 0.28 | 0.29 | 0.33 | 0.20 | 0.20 | 0.24 | 0.20 | 0.22 | 0.22 |
role in the states \([Q_1Q_2][Q_3Q_4]\). The Coulomb interaction provides very strong attraction in the heavy-mesons and the states \([Q_1Q_2][Q_3Q_4]\). The interaction depends on \(\frac{1}{r}\) and the color factor \(\langle \lambda_i^c \cdot \lambda_j^c \rangle\), the strength of which is related to the color factor \(\langle \lambda_i^c \cdot \lambda_j^c \rangle\). In the heavy quark sector, the large quark mass allows two particles to approach each other as a result of small kinetic, which helps to strengthen the Coulomb interaction. In the mesons, \(\langle \lambda_i^c \cdot \lambda_j^c \rangle = -\frac{3}{5}\), which is stronger than those of the states \([Q_1Q_2][Q_3Q_4]\) in Table \(\text{VIII}\). The states \([Q_1Q_2][Q_3Q_4]\) are therefore looser than heavy mesons \(\eta_c, \Psi, B_c, \eta_b\) and \(Y\), see the distances in Table \(\text{III}\) and \(\text{VI}\). In addition, the value of the Coulomb interaction in the states \([Q_1Q_2][Q_3Q_4]\) are higher 500-1000 MeV than those of their corresponding two-meson thresholds, see the value \(\Delta V_{\text{CQM}}\) in Table \(\text{V}\) which is mainly reason resulting in none of bound states in the quark models with QCD dynamic effects. It is therefore difficult for the CMIM to completely absorb the strong Coulomb interaction effects in the states \([Q_1Q_2][Q_3Q_4]\) by the effective constituent quark mass.

The long-range confinement interaction contributes a little to the masses and binding energy of the states \([Q_1Q_2][Q_3Q_4]\) because of the small distances, see the values \(V_{\text{CQM}}\) and \(\Delta V_{\text{CQM}}\) in Table \(\text{V}\). The mass difference, about 30-120 MeV, between the CQM and MCFTM in Table \(\text{IV}\) originates from different types of confinement potential. The multibody confinement potential based on the lattice color flux-tube picture is thought to be closer to real physical images than two-body one related to color charges, which plays significant roles in many interesting places of hadron physics, such the formation and decay via strong interaction, quark pair creation and hadron structure. The multibody confinement potential can reduce the mass of multiquark states. Similar quark models with multibody confinement potential have been extensively applied to study the properties of multiquark states.

### Table VII: Energy of various parts of the Hamiltonian in MeV and the average distances in fm in the MCFTM.

| \(LS\) | \(J^P\) | States | Mass. prop. | \(E_k\) | \(V_{\text{CQM}}\) | \(V_{\text{CQM}}\) | \(\langle r_{12}^2 \rangle\) | \(\langle r_{13}^2 \rangle\) | \(\langle r_{14}^2 \rangle\) | \(\langle r_{24}^2 \rangle\) | \(\langle r_{23}^2 \rangle\) | \(\langle X^2 \rangle\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(3c \otimes 3c\) | 6454, 56% | 878 | 188 | -11 | -1258 | 0.42 | 0.42 | 0.45 | 0.45 | 0.45 | 0.33 |
| \(0^+\) | 667, 44% | 899 | 199 | 17 | -1306 | 0.46 | 0.46 | 0.43 | 0.43 | 0.43 | 0.28 |
| C.C. | 6407 | 887 | 192 | -51 | -1279 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.31 |
| \(3c \otimes 3c\) | 6730, 98% | 783 | 283 | 4 | -997 | 0.47 | 0.47 | 0.61 | 0.61 | 0.61 | 0.52 |
| \(0^+\) | 6888, 2% | 910 | 274 | 12 | -966 | 0.51 | 0.51 | 0.54 | 0.54 | 0.54 | 0.40 |
| C.C. | 6277 | 785 | 283 | -2 | -997 | 0.47 | 0.47 | 0.61 | 0.61 | 0.61 | 0.51 |
| \([cc][cc]\) | 10 | 1^- | 66, 99% | 688, 2% | 7213 | 739 | 10 | -772 | 0.55 | 0.55 | 0.63 | 0.63 | 0.63 | 0.50 |
| \([cc][cc]\) | 12939, 41% | 847 | 127 | -3 | -1372 | 0.27 | 0.41 | 0.37 | 0.37 | 0.37 | 0.29 |
| \(0^+\) | 66, 99% | 859 | 135 | 13 | -1411 | 0.33 | 0.42 | 0.35 | 0.35 | 0.35 | 0.23 |
| C.C. | 12906 | 853 | 131 | -27 | -1392 | 0.30 | 0.42 | 0.36 | 0.36 | 0.36 | 0.25 |
| \([bb][bb]\) | 10 | 1^- | 66, 99% | 66, 99% | 13370, 1% | 884 | 186 | 9 | -1051 | 0.36 | 0.48 | 0.44 | 0.44 | 0.44 | 0.32 |
| \(0^+\) | 13204 | 727 | 201 | 6 | -1071 | 0.30 | 0.46 | 0.52 | 0.52 | 0.52 | 0.45 |
| C.C. | 13204 | 728 | 201 | 4 | -1071 | 0.30 | 0.46 | 0.52 | 0.52 | 0.52 | 0.45 |
| \([bb][bb]\) | 10 | 1^- | 66, 99% | 19367, 38% | 899 | 63 | -6 | -1615 | 0.24 | 0.24 | 0.26 | 0.26 | 0.26 | 0.19 |
| \(0^+\) | 19352, 62% | 884 | 72 | 9 | -1638 | 0.28 | 0.28 | 0.25 | 0.25 | 0.25 | 0.16 |
| C.C. | 19329 | 865 | 69 | -26 | -1605 | 0.27 | 0.27 | 0.26 | 0.26 | 0.26 | 0.17 |
| \([3c \otimes 3c]\) | 19636, 99% | 700 | 110 | 4 | -1204 | 0.29 | 0.29 | 0.39 | 0.39 | 0.39 | 0.33 |
| \(0^+\) | 19792, 99% | 854 | 104 | 6 | -1198 | 0.32 | 0.32 | 0.32 | 0.32 | 0.32 | 0.23 |
| C.C. | 19635 | 701 | 110 | 2 | -1204 | 0.29 | 0.29 | 0.39 | 0.39 | 0.39 | 0.33 |
| \([3c \otimes 3c]\) | 19812, 99% | 659 | 157 | 6 | -1035 | 0.31 | 0.31 | 0.50 | 0.50 | 0.50 | 0.45 |
| \(0^+\) | 20105, 99% | 898 | 136 | 4 | -960 | 0.36 | 0.36 | 0.39 | 0.39 | 0.39 | 0.29 |
| C.C. | 19812 | 659 | 157 | 6 | -1035 | 0.31 | 0.31 | 0.50 | 0.50 | 0.50 | 0.45 |

### Table VIII: Color matrix elements, \(\hat{O}_{ij} = \lambda_i^c \cdot \lambda_j^c\).

| \(\langle \hat{O}_{ij} \rangle\) | \(\langle \hat{O}_{12} \rangle\) | \(\langle \hat{O}_{13} \rangle\) | \(\langle \hat{O}_{24} \rangle\) | \(\langle \hat{O}_{14} \rangle\) | \(\langle \hat{O}_{23} \rangle\) |
|---|---|---|---|---|---|
| \(3c \otimes 3c, \hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) |
| \(6c \otimes 6c, \hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) | \(\hat{O}_{ij}\) |
| \(\hat{O}_{ij}\) | 0 | -2\sqrt{2} | -2\sqrt{2} | 2\sqrt{2} | 2\sqrt{2} |
the same spatial configuration in despite of hadron environments, such as the $Qar{Q}$ in the conventional mesons and multiquark states. The dynamical calculations on the heavy-mesons and $[Q_1Q_2][Q_3Q_4]$ states indicate that the difference of their distances is apparent, see Tables IV and V which is contradict with the CMIM assumption of the same spatial configuration. Furthermore, it can be found from Table IV and V that the color-magnetic interactions of the states $[Q_1Q_2][Q_3Q_4]$ in the CMIM are overestimated relative to that in the dynamical models due to the spatial assumption, which results in the appearance of bound states in the CMIM. In addition, the difference of confinement potential based on string and junction $\Delta V_{\text{conf}}^\text{min}$ is not a constant, which depends on the specific state. However, the added constant term $S$ may be thought of as representing the contribution of two additional QCD strings and one junction $S$. In this way, the predictive power of the color-magnetic mechanism needs to be checked on a large scale and in full detail by more sophisticated models with various QCD dynamic effects.

The investigations on the ground states $[Q_1Q_2][Q_3Q_4]$ preferred the color configuration $3_c \otimes 3_c$, in the color-magnetic mechanism $[4]$. However, the interactions between the $[Q_1Q_2]6_c$ and $[Q_3Q_4]6_c$ in the color configuration $6_c \otimes 6_c$ are attractive although the interactions in the $[Q_1Q_2]6_c$ and $[Q_3Q_4]6_c$ are repulsive, which are much stronger than those of the color configuration $3_c \otimes 3_c$, because the strength of the interaction depends on the color factors listed in Table VIII. Therefore, the final result, which is mainly dominated by the Coulomb interaction, of the color configuration $6_c \otimes 6_c$, relies on the distance $\langle X^2 \rangle^{\frac{1}{2}}$ between the $[Q_1Q_2]6_c$ and $[Q_3Q_4]6_c$. The heavier the heavy quark mass, the smaller the distance $\langle X^2 \rangle^{\frac{1}{2}}$, the stronger the Coulomb interaction, the bigger the proportion of the color configuration $6_c \otimes 6_c$, which can be found from the group $[cc][cc]-[cc][bb]-[bb][bb]$ with $0^+$ and the group $[cc][bb]-[bb][cb]$ with $0^+$ and $1^+$. The two color configurations can couple each other through mainly the color-magnetic interaction, the strength of which is inversely proportional to the interacting quark masses. The proportion of the color configuration $6_c \otimes 6_c$ in the CQM is bigger than that in the MCFTM because the confinement potential involving the color factor can strengthen the coupling due to the different distances in the two color configurations, see Table VIII. In the CMM, the proportion in the group does not change because it only determined by spin-color structure due to the absence of the spatial degree of freedom. In a word, the color configuration $6_c \otimes 6_c$ can not be ignored but should be taken seriously in the investigation on the ground fully-heavy tetraquark states, which is supported by other conclusion in other two models with QCD dynamic effects [4].

Taking the states $[cc][cc], [cc][bb]$ and $[bb][bb]$ as an example, it can be found from Table VII that the distance $\langle X^2 \rangle^{\frac{1}{2}}$ rapidly increases with the increase of $L$ in the excited states $[Q_1Q_2][Q_3Q_4]$. As a result, the Coulomb interaction between the $[Q_1Q_2]$ and $[Q_3Q_4]$ rapidly decreases, the color configuration $6_c \otimes 6_c$ is faster than the color configuration $3_c \otimes 3_c$, because the interaction strength of the former is stronger than that of the latter. The Coulomb interaction in the color configuration $3_c \otimes 3_c$ is stronger than that in the color configuration $6_c \otimes 6_c$, because the Coulomb interaction in the $[Q_1Q_2]3_c$ and $[Q_3Q_4]3_c$ is strong attractive while those in the $[Q_1Q_2]6_c$ and $[Q_3Q_4]6_c$ are repulsive. In addition, the kinetic $E_k$ of the color configuration $3_c \otimes 3_c$ is obvious lower, more than 100 MeV, than that of the color configuration $6_c \otimes 6_c$, because of the big distance induced by the relative weak Coulomb interaction between the $[Q_1Q_2]3_c$ and $[Q_3Q_4]3_c$, see Table VII which is the main reason resulting in the mass difference between two color configurations. The coupling between two color configurations is very weak because of the weak color-magnetic interaction. In this way, the proportion of the color configuration $6_c \otimes 6_c$ is small in the excited states while the color configuration $3_c \otimes 3_c$ is absolutely dominant.

Other different versions of non-relativistic quark models involving the OGE interaction and color confinement potential were also employed to investigate the fully-heavy tetraquark states $[3]$, which presented similar mass spectra to our models. The masses of the states in those quark models are much higher, about 300-500 MeV, than the corresponding thresholds, which indicates that there does not exist a bound state in the scheme of those quark models. However, the non-relativistic model with a Cornell-inspired potential, in which a four-body problem is simplified into three two-body problems, predicted that the lowest S-wave $[cc][cc]$ might be below their thresholds of spontaneous dissociation into low-lying charmonium pairs $[37]$. So far, none of fully-heavy tetraquark state has been found in experiments. Whether the states exist or not awaits more experimental judgement in the future, which facilitates to construct effective phenomenological models.

V. SUMMARY

In this work, we use the CMIM and two quark models with the OGE interaction and color confinement potential, CQM with two-body confinement and MCFTM with multibody one based on the lattice color flux-tube picture, to systematically investigate the properties of the states $[Q_1Q_2][Q_3Q_4]$ with the help of Gaussian expansion method, which is a high-precision numerical method. The difference between the two confinement potentials in the states is 30-120 MeV. The multibody confinement potential is usually thought to be closer to real physical images than two-body one related to color charges. The masses of the ground states predicted by the CMIM are close to the corresponding two heavy-meson threshold while those predicted by the quark models with QCD dynamic effects are higher about hundreds of MeV mainly because of the strong Coulomb interaction, which indicates that the CMIM can not completely absorb QCD
The interaction in the states [Q1Q2][Q3Q4] is weaker than that in the ground states [Q1Q2][Q3Q4] in the quark models involving QCD dynamic effects. The color configuration |[Q1Q2][Q3Q4]| can not be ignored in the ground states owing to the strong Coulomb interaction. The color configuration |[Q1Q2][Q3Q4]| is absolutely dominant in the excited states, which is determined by the small kinetic induced by the Coulomb interaction.

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