Sufficiently Small $\bar{\theta}$ in $SU(3)^3 \times S_3$ Unification Model

K. Chalut, H. Cheng, P.H. Frampton, K. Stowe and T. Yoshikawa

Department of Physics and Astronomy, University of North Carolina, Chapel Hill, NC 27599-3255.

Abstract

Since CP violation in weak decays is successfully described by the KM mechanism, the strong CP problem cannot easily be accommodated. This leads us to reconsider the issue. If the axion and massless up quark are abandoned, we must extend the standard model. Extension to $SU(3)^3 \times S_3$ unification leads to the following situation: if CP is a high-energy symmetry and the appropriate symmetry-breaking hierarchy of scales is in place, then the $\bar{\theta}$ parameter of the QCD sub-theory is guaranteed to be sufficiently small. We find $\bar{\theta} < 10^{-11}$ while the empirical limit from the neutron electric dipole moment requires only that $\bar{\theta} < 1.3 \times 10^{-10}$.

Emails: hawcheng@hotmail.com, chalut@phy.duke.edu, frampton@physics.unc.edu, stowek@physics.unc.edu, tadashi@physics.unc.edu
1 Introduction

Within the last year, we have learned a great deal about the nature of CP violation. Results from the B Factories \cite{1,2} have shown unambiguously that there is a large CP asymmetry in the decay mode ($B^0, \bar{B}^0 \to \Psi K_S$). The resultant value for the parameter $\sin 2\beta$ is $\sin 2\beta = 0.79 \pm 0.10$ which is eight standard deviations from zero and fully consistent with the prediction of the standard model \cite{3}.

From the viewpoint of fundamental theory, the data suggest that explicit CP violation is at work and disfavor models based on soft CP violation \cite{4} which generically, although not universally, predicted a value of $\sin 2\beta$ too small to be observable in the present B Factories.

One advantage of soft CP violation was that it offered a natural solution of the strong CP problem: the value of $\theta$ vanished at tree-level by virtue of the assumption that CP was an exact symmetry of the lagrangian and radiative corrections then contributed a sufficiently small value of $\theta$ to be acceptable.

It now appears that this approach is disfavored and therefore one must reconsider the strong CP problem. Two other well-known approaches are, in our opinion, equally as disfavored as soft CP violation:

(i) The invisible axion suffers from unacceptable fine-tuning when gravitational effects are considered\cite{5}, actually a greater degree of fine-tuning than required to solve the original issue.

(ii) The massless up quark is strongly disfavored by the most recent analysis of lattice gauge theories\cite{6}.

Thus we are led to reconsider the strong CP problem. In particular, it requires some extension of the standard model. The Left-Right\cite{7} model has been much studied from this point of view\cite{8} and the conclusion is that additional discrete symmetries must be added to do the job.

In the present work we present a unified extension of the standard model in which, provided CP is an exact high-energy symmetry of the lagrangian $\mathcal{L}$ and the hierarchy of symmetry-breaking scales is taken care of properly, the strong CP problem is resolved.
2 The Model

We consider unification of the standard model in the semi-simple gauge group $SU(3)^3$ as suggested in [9]. In particular we follow the notation of [10] for the fields and subscripts.

The unification group is $G = SU(3)_C \times SU(3)_L \times SU(3)_R \times S_3$ where $SU(3)_C$ is color, $SU(3)_L$ contains the $SU(2)_L$ of electroweak interactions as in [11], and the remaining $U(1)_Y$ is distributed between $SU(3)_L$ and $SU(3)_R$. In addition, there is a permutation $S_3$ symmetry relating the three $SU(3)$ gauge groups. This $S_3$ symmetry includes a cyclic $Z_3$ subgroup and three pair switching permutations. One of them has a role of parity symmetry.

The scalar bosons and each family of fermions is assigned to a 27 dimensional representation of $G$. They transform under $G$ as the representation $\Psi = \psi_L(3,\bar{3},1) + \psi_R(\bar{3},1,3) + \psi_\ell(1,3,\bar{3})$, where

$$\psi_L(3,\bar{3},1) : U_C \begin{pmatrix} u_1 & d_1 & B_1 \\ u_2 & d_2 & B_2 \\ u_3 & d_3 & B_3 \end{pmatrix} \nabla_L,$$

$$\psi_R(\bar{3},1,3) : W_R \begin{pmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \\ d_1 & d_2 & d_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \U_C,$$

$$\psi_\ell(1,3,\bar{3}) : V_L \begin{pmatrix} E^0 & E^- & e^- \\ E^+ & \bar{E}^0 & \nu \\ e^+ & N^0 & \bar{N}^0 \end{pmatrix} \W_R.$$

$U, W, and V$ are group elements of the three $SU(3)$ groups. The left-handed quarks and anti-right-handed quarks will be found in $\psi_L$ and $\psi_R$ respectively, and the leptons are found inside $\psi_\ell$. There are a heavy weak singlet quark $B$ with charge $-1/3$ and a heavy lepton doublet $E^0$ and $E^+$ with charge 0 and 1. In addition, there is a neutral chiral state $N^0$ which may be called neutretto.

The scalars will acquire vacuum expectation values (VEV’s) which are arranged as

$$\langle \phi_\ell(1,3,\bar{3}) \rangle = \begin{pmatrix} u & 0 & 0 \\ 0 & u & u \\ 0 & w & v \end{pmatrix}$$

The ‘constants’ $u, v, and w$ here merely represent orders of magnitude, rather than specific values. For the scales we assume, as in [10], that the hierarchy $v \gg w \gg u$ is incorporated in the $SU(3)^3 \times S_3$ model with whatever fine-tuning is necessary to accomplish it. Our aim here is to show only that the strong CP parameter $\bar{\theta}$ is under control without further fine tuning.

The unification scale $v$ breaks $SU(3)^3 \times S_3$ down to $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \times P$. At the scale $w$, the symmetry breaks down to the standard model $SU(3)_C \times SU(2)_L \times SU(1)_Y$. Finally, the scale $u$ accomplishes the electroweak breaking. If there
is only one scalar field, the VEV can always be diagonalized, so that it is impossible to
have the standard model at an intermediate scale. Hence it is necessary to assume at least
two 27's, and, in the interest of economy, we will assume exactly two 27's. Hence, we will
assume two 27's of scalars for Higgs fields, three 27's of fermions for the generations and
one 24 of gauge boson in this model.

We must treat one of the pair switching permutations in the $S_3$ symmetry as parity
symmetry\[1]\]. Let us write down the action of $P$ on all the fields:

$$
\begin{align*}
\psi^A_L(\vec{x}, t) &\rightarrow \psi^A_R(-\vec{x}, t), & \phi^i_L(\vec{x}, t) &\rightarrow \phi^i_R(-\vec{x}, t), & C^\mu_a(\vec{x}, t) &\rightarrow C^\mu_a(-\vec{x}, t) \\
\psi^A_R(\vec{x}, t) &\rightarrow \psi^A_L(-\vec{x}, t), & \phi^i_R(\vec{x}, t) &\rightarrow \phi^i_L(-\vec{x}, t), & L^\mu_a(\vec{x}, t) &\rightarrow R^\mu_a(-\vec{x}, t) \\
\psi^A_L(\vec{x}, t) &\rightarrow \psi^A_R(-\vec{x}, t), & \phi^i_R(\vec{x}, t) &\rightarrow \phi^i_L(-\vec{x}, t), & R^\mu_a(\vec{x}, t) &\rightarrow L^\mu_a(-\vec{x}, t)
\end{align*}
$$

where $C$, $R$, and $L$ are the gauge fields, the lowering of the index $\mu$ to $\mu$ indicates reversal
of the spatial components, $A = 1, 2, 3$ is a family index and $i = 1, 2$ is a gauge boson index.
The daggers ($\dagger$) represent the fact that not only are each component of these matrices
complex conjugated, but the $SU(3)_L$ and $SU(3)_R$ indices are exchanged as well.

The Yukawa couplings are given by

$$
\mathcal{L}_Y = -Z_3 \{ f_{iAB} \text{Tr}(\phi^i_L \psi^A_R \psi^B_L) + \text{h.c.} \}
$$

where $Z_3$ simply implies that we must include cyclic permutations to assure the $Z_3$ symmetry
is respected. Under the symmetry $P$, we can relate the terms to their hermitian conjugates,
so that $f_{iAB}^T = f_{iBA}$, or, thinking of these as matrices, $f_i^T = f_i$. If the scalar VEV's
are real (proved below), this will result in Hermitian quark mass matrices, or $M^a_q = M_q$.
Since the determinant of a Hermitian matrix is real, this in turn would result in vanishing
$\theta$ at tree level. It remains only to prove the $<\phi^i_L>$ are real.

3 Reality of $<\phi^i_L>$ at tree level

The scalar potential responsible for the symmetry breaking involves only the $\phi_L$ portions
which acquire VEV’s. The portion of the scalar potential that is relevant is given by

$$
\mathcal{L}_i = -m^2_{ij} \text{Tr}(\phi^i_L \phi^j_L) + \{ \gamma_{ijk} \epsilon_{\alpha \beta \gamma} \epsilon^{\delta \rho \sigma} \phi^i_{\ell \beta} \phi^j_{\ell \gamma} \phi^k_{\ell \rho} \phi^l_{\ell \sigma} + \text{h.c.} \} \\
+ \lambda_{ijkm} \text{Tr}(\phi^i_L \phi^j_L \phi^k_L \phi^m_L) + \eta_{ijkm} \text{Tr}(\phi^i_L \phi^j_L \phi^k_L \phi^m_L).
$$

All the coefficients in this potential are real because of parity symmetry and the hermiticity.
Hermiticity and parity imply

$$
m^2_{ij} = m^2_{ji}, \quad \gamma_{ijk} = \gamma^*_{ijk}, \quad \lambda_{ijkm} = \lambda_{jimk} = \lambda^*_{jimk},
$$

$$
\eta_{ijkm} = \eta_{mijk} = \eta_{kmij} = \eta_{jkm} = \eta^*_{mkji}.
$$
From these conditions we can find \( m_{ij}, \gamma_{ijk} \) and \( \lambda_{ijkm} \) must be real values. \( \eta_{ijkm} \) is also real because the indices take on only the values 1 or 2. Thus the whole potential \( \mathcal{L}_t \) is real.

Under the condition that all the constants in the potential are real, using the degree of freedom of gauge symmetry (rotation), we can take the VEV as follows;

\[
\begin{pmatrix}
\langle \phi^1_t \rangle \\
\langle \phi^2_t \rangle
\end{pmatrix} = \begin{pmatrix}
u & 0 & 0 \\
0 & u_2 & 0 \\
0 & 0 & v
\end{pmatrix}
\]

where \( u_i \) are real values.

As a minimum case to realize the SM, we can assume that \( u_1, u_4, u_5, x = 0 \). Then,

\[
\begin{pmatrix}
\langle \phi^1_t \rangle \\
\langle \phi^2_t \rangle
\end{pmatrix} = \begin{pmatrix}
u & 0 & 0 \\
0 & u_2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The remaining phase of the VEV is only \( \alpha_3 \). The stationary condition for the remaining phase \( \alpha_3 \) appear in only the cubic interactions because the other terms are completely real.

\[
\frac{dV}{d\alpha_3} = -12\gamma_{112}u_2u_3v \sin \alpha_3 = 0.
\]

So this condition shows \( \alpha_3 = 0 \) (or \( \pi \)) and all VEVs are real in this minimum case. This confirms the earlier assertion that \( \bar{\theta} = 0 \) at tree level, and it remains to confirm whether the loop corrections to \( \bar{\theta} \) are sufficiently small to satisfy the empirical constraint \( \bar{\theta} < 1.3 \times 10^{-10} \) [12].

### 4 The contribution from loop diagrams

Some possible CP violating effects remain in interactions among \( \phi^i_L(3, \bar{3}, 1), \phi^i_R(\bar{3}, 1, 3) \) and \( \phi^i(R, \bar{3}, 3) \). Here we consider only such terms which can contribute to the imaginary part of quark mass terms at either one or two loop level:

\[
V_{CPviolating} = A^{L}_{ijkm} T^i_L T^k_L \phi^m_L + A^{R}_{ijkm} T^i_R T^k_R \phi^m_R + B^{L}_{ijkm} T^i_L T^j_L \phi^k_L \phi^m_L + B^{R}_{ijkm} T^i_R T^j_R \phi^k_R \phi^m_R + C_{ijkm} \phi^i_L \phi^j_L \phi^k_L \phi^m_L + D_{ijkm} \phi^i_R \phi^j_R \phi^k_R \phi^m_R + h.c.
\]

where \( i, j, k, m \) are Higgs scalar indices. Under P invariance and hermiticity, we get the following conditions for each coupling constant,

\[
\begin{align*}
A^{L}_{ijkm} &= A^{L}_{jimk} = A^{R}_{ijkm} = A^{R}_{jimk}, \\
B^{L}_{ijkm} &= B^{L}_{jimk} = B^{R}_{ijkm} = B^{R}_{jimk}, \\
C_{ijkm} &= C^{*}_{ijkm}, \\
D_{ijkm} &= D^{*}_{imk}.
\end{align*}
\]
\(A^{L,R}\) and \(B^{L,R}\) are real if \(i = j\) and \(k = m\). \(C\) and \(D\) are also real if \(k = m\). However there is no constraint for the other cases \(i \neq j\) or \(k \neq m\). So these constants are in general complex. This model has \(\mathbf{P}\) symmetry between the scales \(v\) and \(w\) so that \(\hat{\phi}_L\) and \(\hat{\phi}_R\) are degenerate. Above the scale \(w\) any imaginary contributions are cancelled and \(\bar{\theta}\) vanishes in all orders. Below the scale \(w\), \(\bar{\theta}\) may have non zero value through loop diagrams. The imaginary parts of these interaction contribute to the quark mass through Yukawa couplings

\[-\mathcal{L}_Y = Z_3 \left[ f_{iAB} Tr \left( \phi^i_L \psi^A_R \psi^B_L \right) + g_{iAB} Tr \left\{ \left( \phi^i_L \right)_{\beta \alpha} (\psi^A_R)_{\sigma} (\psi^B_L)_{\rho} \bar{\gamma} \varepsilon_{\alpha \beta \gamma \varepsilon} \right\} + h.c. \right].\]

(17)

The up-type quark masses arise from the VEV of \(\phi^2_L\) and down-type quark masses come from \(\phi^1_L\) if we choose the sets of VEV as Eq.(11). These Yukawa couplings are proportional to the mass of quarks and leptons for each family.

\[f_{2AB} = \frac{m_u A}{u} f_{2AB}\]

(18)

\[f_{1AB} = \frac{m_d A}{u} f_{1AB}\]

(19)

\[g_{iAB} = \frac{m_l u}{u} g_{iAB}\]

(20)

In the case of the coupling to charged Higgs, \(f_{iAB}\) includes the Kobayashi-Maskawa matrix. We investigate the loop effects to \(\bar{\theta}\) through these interactions in the follow subsections.

### 4.1 One loop

There are contributions from one loop diagrams to the imaginary part of the quark mass from the Feynman graphs of Figs. 1, 2, 3, and 4. These one loop diagrams are proportional to

\[\frac{1}{(4\pi)^2} \frac{1}{M^2_{\phi^i_L} - M^2_{\phi^i_R}} \log \frac{M^2_{\phi^i_L}}{M^2_{\phi^i_R}},\]

(21)

where \(M_{\phi}\) is the mass of the colored heavy higgs mass. Above the scale \(w\) these masses became degenerate and the difference between \(M_{\phi^i_L}\) and \(M_{\phi^i_R}\) therefore arises from the difference of contribution at scale \(w\) after breaking the \(\mathbf{P}\) symmetry. In general, we can write the masses as follows:

\[M^2_{\phi^i_L} = \alpha^i_L v^2 + \beta^i_L w^2\]

(22)

\[M^2_{\phi^i_R} = \alpha^i_R v^2 + \beta^i_R w^2\]

(23)

By the relations of Eq.(16) and the hermiticity of the Yukawa couplings, the imaginary part of the sum of the two diagrams in each of Figs. 1-4 is suppressed by a factor
Thus, all one-loop diagrams have a suppression factor \( \frac{w^2}{v^2} \) in \( \bar{\theta} \).

The largest contribution to \( \bar{\theta} \) comes from the loop correction to the third family quark mass because the Yukawa coupling and the mass are much larger than the other family. When the scalar has the largest VEV \( v \), the estimates of these one loop diagrams are

\[
\bar{\theta} = \frac{1}{m} \text{Im}[\text{Fig.1}] \sim v \text{ Im}[D_{112}] \hat{f}_{133} \hat{f}_{233} \frac{1}{(4\pi)^2} \left( \frac{w}{v} \right)^2 \frac{m_t m_b}{u^2} \frac{m_t}{m_t + m_b} \quad (25)
\]

\[
\bar{\theta} = \frac{1}{m} \text{Im}[\text{Fig.2}] \sim u v \hat{f}_{133} \hat{f}_{233} \frac{1}{(4\pi)^2} \left( \frac{w}{v} \right)^2 \frac{m_E}{u^2} \frac{m_t m_b}{m_b} \left( \frac{\text{Im}[C_{1212}]}{m_t} + \frac{\text{Im}[C_{1112}]}{m_t} \right) \quad (26)
\]

where \( D_{112} \) has a mass dimension and it will be at largest order of scale \( v \). \( m_E \) is the mass of heavy lepton which is proportional to the lepton Yukawa coupling. In the third family case,

\[
m_E = \hat{g}_{133} \frac{m_e}{u} v. \quad (27)
\]
In Fig. 1, the contribution to bottom quark mass is negligible because it proportional to the neutrino mass. So largest contribution comes from the loop correction to bottom mass of Fig. 2. If the size of the suppression factor \((\frac{w}{v})^2\) is \(O(10^{-7})\) as estimated in ref. 10, the largest contribution will be \(10^{-11}\) except for further unknown parameters expected to be less than one.

Figs. 3 and 4 give contributions to \(\bar{\theta}\) smaller than the previous two because these diagrams contain leptonic Yukawa coupling \(g_{133}\), necessarily smaller than its quark counterpart.

Hence, in summary, the contributions from all one-loop diagrams to \(\bar{\theta}\) add to a value smaller than \(10^{-11}\).
4.2 Two loops

The effective complex quartic interactions of $\phi_i^\ell$ will appear through loops of colored scalars as illustrated by the four Feynman graphs of Fig. 5 which involve the $A_{ijkl}^{L,R}$ (recall that $A^R$ is related to $A^L$) quartic couplings of Eq. (13). There are similar graphs for the $B_{ijkl}^{L,R}$ and $C_{ijkl}$ quartic couplings, as well as a similar box diagram involving the cubic term $D_{ijkl}$ in Eq. (13). The loop diagram in Fig. 5 is proportional to $A_{rijxy}^L A_{klxyz}^L$ (or $A_{rijxy}^R A_{klxyz}^R$), where $x, y$ show the indices of the colored scalars. If $i = l$ and $j = k$, this combination will be real because of the relations of the coefficients, $A_{ijkl}^{L,R} = A_{ijlx}^{L,R}$ in Eq. (14). The case of $i = j, k = l$ is also real by adding the exchanged diagrams between the indices of the colored scalars $x$ and $y$.

$$\text{Im}[A_{ixy}^L A_{kxy}^L + A_{iyx}^L A_{kxy}^L] = \text{Im}[A_{ixy}^L A_{kxy}^L + A_{ixy}^R A_{kxy}^R] = 0$$  \hspace{1cm} (28)

Other combinations with at least one index different are also cancelled by adding the diagrams of $\phi_R$ as in Fig. 5. The difference between the diagrams of $\phi_L$ and $\phi_R$ comes only from mass splittings. Hence the remaining imaginary part of the effective quartic interactions is suppressed by a factor $\left(\frac{w}{v}\right)^2$. The effective couplings arising from $B_{ijkl}^{L,R}, C_{ijkl}$ and $D_{ijkl}$ interactions are also suppressed by the same factor.

Two-loop diagrams which illustrate such contributions to $\bar{\theta}$ from effective quartic interactions are shown in Figs. 6-7. In Fig. 6, there are possible $A_{ijkl}^{L,R}$ and, separately, $B_{ijkl}^{L,R}$ contributions but the largest contribution comes from $B_{ijkl}^{L,R}$ interactions with one large VEV $v$. We estimate:

$$\bar{\theta}_{\text{Fig.6}} \sim u v \text{ Im}[B_{22xy}^L B_{12yx}^L] \frac{1}{(4\pi)^4} \left(\frac{w}{v}\right)^2 \frac{m_B}{v^2} \left(\hat{f}_{233} \frac{m_t^2}{u^2} \frac{1}{m_t} + \hat{f}_{133} \hat{f}_{233} \frac{m_t m_b}{u^2} \frac{1}{m_b}\right),$$  \hspace{1cm} (29)

where $m_B$ is the heavy quark mass $m_B = \frac{m_b}{u} v$ coming from the Yukawa coupling to $\phi_1^\ell$. So if the factor $\left(\frac{w}{v}\right)^2$ is $O(10^{-7})$, the size will become $O(10^{-13})$ except for unknown parameters expected to be smaller than one.

The contribution from Fig. 7 is estimated as:

$$\bar{\theta}_{\text{Fig.7}} \sim u v \frac{1}{(4\pi)^4} \left(\frac{w}{v}\right)^2 \frac{m_B m_t^2}{v^2} \hat{f}_{233}^2 \left(\text{Im}[C_{12xy} C_{22yx}] \frac{1}{m_t} + \text{Im}[C_{11xy} C_{22yx}] \frac{1}{m_b}\right).$$  \hspace{1cm} (30)

The contribution is largest for the bottom quark mass but even for it the contribution to $\bar{\theta}$ will be smaller than $10^{-11}$.

Further examples of two-loop contributions are depicted in Figs. 8, 9, 10. All such other two loop contributions to $\bar{\theta}$ are smaller than $10^{-13}$. 


Figure 5: CP-violating quartic interaction induced by $A_{ijk\ell}^{L,R}$ couplings.

Figure 6: Two-loop contribution to $\bar{\theta}$ involving $A_{ijkl}^{L,R}$ or $B_{ijkl}^{L,R}$ couplings.

Figure 7: Two-loop contribution to $\bar{\theta}$ involving $C_{ijkl}$ couplings.
Figure 8: A further two-loop contribution to $\bar{\theta}$

Figure 9: An even further two-loop contribution to $\bar{\theta}$

Figure 10: A final example of a two-loop contribution to $\bar{\theta}$
5 Discussion

Because of the observation of large CP asymmetries in decay of \((B^0, \bar{B}^0)\) at B Factories\cite{1, 2} the option of soft CP violation\cite{3, 4, 5} as solution of strong CP is disfavored though not excluded. Two popular alternative solutions, a massless up quark and an (invisible) axion have theoretical and empirical difficulties\cite{5, 7}.

It is therefore of interest to find simple extensions which can solve strong CP. In the present article we have shown how unification in a group \(SU(3)^3 \times S_3\) leads to a natural \(P\) parity operation and to sufficient suppression of \(\bar{\theta}\) provided CP is a high-energy symmetry and the hierarchy of symmetry-breaking scales is in place.

Acknowledgments

This work was supported in part by the US Department of Energy under Grant No. DE-FG02-97ER-41036.
Appendices

Appendix A. The Stationary conditions.

The stationary conditions for each VEVs are

\[ v^2 [m_{11}^2 + 2(\lambda_{1111} + \eta_{1111})v^2 + (\lambda_{1122} + \lambda_{2211} + \eta_{1221} + \eta_{2112})w^2 + 2\lambda_{1111}u_2^2 + (\lambda_{1122} + \lambda_{2211})u_3^2] + 6\gamma_{1112} u_2 u_3 = 0 \] (31)

\[ m_{22}^2 + 2(\lambda_{2222} + \eta_{2222})w^2 + (\lambda_{1122} + \lambda_{2211} + \eta_{1221} + \eta_{2112})v^2 + (\lambda_{1122} + \lambda_{2211})u_2^2 + 2\lambda_{2222}u_3^2 = 0 \] (32)

\[ u_2^2 [m_{11}^2 + 2(\lambda_{1111} + \eta_{1111})v^2 + (\lambda_{1122} + \lambda_{2211})w^2 + 2(\lambda_{1111} + \eta_{1111})u_2^2 + (\lambda_{1122} + \lambda_{2211})u_3^2] + 6\gamma_{1112} u_2 u_3 = 0 \] (33)

\[ u_3^2 [m_{22}^2 + (\lambda_{1122} + \lambda_{2211})v^2 + 2\eta_{2222}w^2 + (\lambda_{1122} + \lambda_{2211})u_2^2 + 2(\lambda_{2222} + \eta_{2222})u_3^2] + 6\gamma_{1112} u_2 u_3 = 0 \] (34)

At \( m_{11}, v \gg w \gg u_2, u_3 \), by neglecting the terms for \( w \) and \( u_2, u_3 \) in Eq.(31), the size of \( v \) is

\[ v^2 = \frac{-m_{11}^2}{2(\lambda_{1111} + \eta_{1111})} \] (35)

Then, Eq.(32) is

\[ m_{22}^2 + 2(\lambda_{2222} + \eta_{2222})w^2 + (\lambda_{1122} + \lambda_{2211} + \eta_{1221} + \eta_{2112})v^2 = 0 \] (36)

Hence, to realize the hierarchy of the scales \( v \gg w \), there are two possibilities. One is the terms of scale \( w^2 \) remain after cancelling between the terms of \( v^2 \) and \( m_{22}^2 \). The other one is the \( m_{22} \) is the quantity of order \( w \) and the size of the coefficients satisfy a condition as following,

\[ \lambda_{1122} + \lambda_{2211} + \eta_{1221} + \eta_{2112} \leq \left(\frac{w}{v}\right)^2, \] (37)

\[ \lambda_{iiii} + \eta_{iiii} \sim O(1), \] (38)

where \( i = 1, 2 \). These are the coefficients in the potential of \( \phi_\ell \), the magnitude will correspond to the coefficients \( A^{L,R}_{ijkl}, B^{L,R}_{ijkl} \) and \( C_{ijkl} \) of Eq.(13) by \( S_3 \) symmetry.

If we can assume that these coefficients of the mixing terms between \( \phi_1 \) and \( \phi_2 \) are suppressed by the factor, \( \bar{\theta} \) from loop diagrams is also suppressed by the smallness of the coefficients, because the imaginary parts come only from such mixing interactions by the feature of the coefficients of Eq.(13).
Appendix B. Another option to reduce $\bar{\theta}$.

In the text, we assumed VEVs of $\phi_i$ as in Eq.(11) and then found the size of $\bar{\theta}$ will be smaller than $10^{-11}$. We mention another option to reduce $\bar{\theta}$. We may choose alternatively the set of the VEVs as follows:

$$
\left\langle \phi_i^1 \right\rangle = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & v \end{pmatrix} \quad \text{and} \quad \left\langle \phi_i^2 \right\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix}
$$

(39)

In this case, we have only one constraint to the Yukawa couplings because the quarks and leptons get mass only from the VEV of $\phi_1$. So the Yukawa couplings are

$$
f_{1AB} = \frac{m_A^u}{u_1} \quad (40)$$

$$
g_{1AB} = \frac{m_A^l}{u_2} \quad (41)
$$

If we set all other Yukawa couplings to zero, the contributions of all loop diagrams discussed in the text to $\bar{\theta}$ disappear because, by the relations among the coefficients in Eq.(13), at least one coupling to the quark line in these graphs must be one of $\phi_2$, $f_{2AB}$, $g_{2AB}$. Hence, if there are no such couplings, an imaginary part of quark mass appears only in one-loop diagrams with additional propagators, leading to a suppression of at least $(\frac{w}{v})^4 \sim 10^{-14}$. 


References

[1] B. Aubert et al. [BABAR Collaboration], Phys.Rev.Lett. 87, 091801 (2001).

[2] K. Abe et al. [Belle Collaboration], Phys.Rev.Lett. 87, 091802 (2002).

[3] A.B. Carter and A.I. Sanda, Phys. Rev. D23, 1567 (1981).
    I.I. Bigi and A.I. Sanda, Nucl. Phys. B193, 85 (1981).
    H. Quinn and A.I. Sanda, Eur. Phys. Journ. C15, 626 (2000).

[4] P.H. Frampton, S.L. Glashow and T. Yoshikawa, Phys.Rev.Lett.87, 011801 (2001).

[5] R. Holman, S.D.H. Hsu, T.W. Kephart, E.W. Kolb, R. Watkins and L.M. Widrow, Phys. Lett. B282, 132 (1992).

[6] D.R. Nelson, G.T. Fleming and G.W. Kilcup. hep-lat/0112029.

[7] R.N. Mohapatra and G. Senjanovic, Phys.Lett. B79, 283 (1978).
    R.N. Mohapatra, A. Rasin and G. Senjanovic, Phys.Rev.Lett. 79, 4744 (1997).

[8] See e.g. K.S. Babu and R.N. Mohapatra, Phys. Rev. D41, 1286 (1990).

[9] A. De Rujula, H. Georgi, and S.L. Glashow, in *Fifth Workshop on Grand Unification* edited by K. Kang, H. Fried, and P. Frampton ( World Scientific, Singapore, 1984) p.88.

[10] E.D. Carlson and M.Y. Wang, hep-ph/9211279.

[11] P.H. Frampton, Phys. Rev. Lett. 69, 2889 (1992).

[12] M.V. Romalis, W.C. Griffith and E.N. Fortson, Phys.Rev.Lett.86, 2505 (2001).

[13] P.H. Frampton and T.W. Kephart, Phys. Rev. Lett. 66, 1666 (1991).
    P.H. Frampton and D. Ng, Phys. Rev. D43, 3034 (1991).

[14] A.W. Ackley, P.H. Frampton, B. Kayser and C.N. Leung, Phys. Rev. D50, 3860 (1994).

[15] P.H. Frampton and S.L. Glashow, Phys. Rev. D55, 1691 (1997).
    P.H. Frampton and M. Harada, Phys. Rev. D59, 036004 (1999).