Lagrangian characterization of sub-Alfvénic turbulence energetics

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The energetics of strongly magnetized turbulence has so far resisted all attempts to understand them. Numerical simulations of compressible turbulence reveal that kinetic energy can be orders of magnitude larger than fluctuating magnetic energy. We solve this lack-of-balance puzzle by calculating the energetics of compressible and sub-Alfvénic turbulence based on the dynamics of coherent cylindrical fluid parcels. Using a Lagrangian formulation, we prove analytically that the bulk of the magnetic energy transferred to kinetic is the energy stored in the coupling between the initial and fluctuating magnetic field. The analytical relations are in striking agreement with numerical data, up to second order terms.

I. Introduction

Magnetohydronamic (MHD) turbulence is involved in a plethora of physical phenomena [1–5]. The interplay between kinetic and magnetic energy is important for understanding such processes [6–14]. The energetics of MHD turbulence depend on the initial magnetization of the fluid [15–19], which can be quantified in terms of the Alfvén Mach number ($M_A$, the ratio between turbulent velocities over the characteristic propagation speed $V_A$ of magnetized fluctuations). Here we focus on sub-Alfvénic turbulence ($M_A \lesssim 1$), which is encountered in systems such as tokamaks [20–22], in the interstellar medium [23–27], and the Sun [28–31].

In sub-Alfvénic, and compressible turbulence, the amplitude of magnetic perturbations ($\delta B$) is smaller than $B_0$, hence magnetic fluctuations propagate in the form of small-amplitude waves [32]. Direct numerical simulations suggest that a fluid, which is constantly perturbed (forced), can maintain large amounts of kinetic energy, such that $\rho \langle u^2 \rangle / 2 \gg \langle \delta B^2 \rangle / 8\pi$ [19, 33–36]. Furthermore, numerical results show that the ratio between kinetic ($E_{\text{kinetic}}$) and fluctuating magnetic ($E_{\text{b,harmonic}}$) energy depends on the strength of $B_0$ [15–19].

The discrepancy between $E_{\text{kinetic}}$ and $E_{\text{b,harmonic}}$ has been phenomenologically attributed to the coupling between $B_0$ and $\delta B$ (i.e., $B_0 \cdot \delta B$), which should store most of the magnetic energy [35–38]; the coupling potential is realizable

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only in compressible fluids [39–42]. Thus, the magnetic coupling seems to hold the key for exploring turbulence energetics in sub-Alfvénic turbulence. However, there is yet lack of first-principles understanding of the role of \( \mathbf{B}_0 \cdot \delta \mathbf{B} \) in turbulence dynamics and energetics.

We present an analytical theory of the role of the coupling potential in the energy exchange of sub-Alfvénic, and compressible turbulence. We use a Lagrangian formulation of coherent [43] flux tube segments. The motion of coherent tubes is the net effect of magnetic and velocity perturbations propagating through the surface of the tube. We calculate the energy exchange between kinetic and magnetic forms as a function of \( \mathcal{M}_A \), and find remarkable agreement with MHD numerical simulations.

II. Setup

Statistical properties of strongly magnetized turbulence are axially symmetric, with \( \mathbf{B}_0 \) being the axis of symmetry [44, 45]. For this reason, we consider a fluid consisting of coherent flux tube segments [e.g., 46] (or fluid parcels) with coordinates \((r(t), \phi(t), z(t))\), Fig. 1. We assume the following initial conditions: 1) uniform temperature; 2) uniform density \( \rho \); 3) no bulk velocity; 4) uniform static magnetic field \( \mathbf{B}_0 = B_0 \mathbf{\hat{z}} \). We ignore gravity.

We perturb the magnetic field of a coherent fluid parcel by \( \delta \mathbf{B} \) such that \( j \mathbf{B}_0 \cdot j \delta \mathbf{B} \), which applies to sub-Alfvénic turbulence. Magnetic perturbations tend to redistribute the magnetic flux within a fluid. For ideal-MHD (flux-freezing) conditions, the magnetic flux is preserved. Thus, the perturbed volume’s surface \( \mathcal{S} \) follows the magnetic field lines. The motion of the field lines, and hence of \( \mathcal{S} \), can be either parallel or perpendicular to \( \mathbf{B}_0 \): 1) squeezing and stretching of \( \mathcal{S} \) along \( \mathbf{B}_0 \) leads to parallel motions, \( \dot{z} \neq 0 \); 2) fluctuations of \( r \) lead to perpendicular motions, \( \dot{r} \neq 0 \); 3) twisting of the perturbed volume leads to rotational motions, \( \dot{\phi} \neq 0 \). The motion of \( \mathcal{S} \) is coherent, which means that the ensemble of sub-volumes embedded within the perturbation volume moves on average as \( \mathcal{S} \). This naturally defines \( z \), and \( r \) as the coherence lengths of the perturbed volume parallel and perpendicular to \( \mathbf{B}_0 \) respectively. We invoke as a boundary condition the presence of a local environment beyond the coherence length of the fluid parcel (“pressure wall”).

The flux freezing theorem can be expressed as,

\[
\frac{d \mathbf{\vec{B}}}{dt} \cdot \mathcal{S} = -\mathbf{\vec{B}} \cdot \frac{d \mathcal{S}}{dt}.
\]

The cross sections of the perturbed volume perpendicular and parallel to the initial field \( \mathbf{\vec{B}}_0 \) are \( \mathcal{S}_\perp = 2\pi rz\mathbf{\hat{r}} \), and \( \mathcal{S}_\parallel = \pi r^2 \mathbf{\hat{z}} \) respectively. The cross section related to the rotational motion is \( \mathcal{S}_\phi = zr\mathbf{\hat{\phi}} \). The total magnetic field in cylindrical coordinates can be expressed as \( \mathbf{\vec{B}} = \delta B_r \mathbf{\hat{r}} + \delta B_\phi \mathbf{\hat{\phi}} + (B_0 + \delta B_\parallel) \mathbf{\hat{z}} \). From Eq. 1 we obtain that when \( |\mathbf{\vec{B}}_0| \gg |\delta \mathbf{\vec{B}}| \), magnetic perturbations along \( \mathcal{S}_\parallel \) are associated with a movement such that,

\[
\dot{r}(t) = -\frac{\delta \mathbf{\vec{B}}_\parallel(t)}{2B_0 + \delta B_\parallel} \approx -\frac{\delta \mathbf{\vec{B}}_\parallel(t)}{2B_0} r_0,
\]
Fig. 1 A magnetized fluid consisting of multiple coherent cylindrical fluid parcels. Red arrows show the initial magnetic field morphology. Untwisted fluid parcels are elongated, $L_{\parallel} \gg L_{\perp}$, and their motion is longitudinal along or perpendicular to $\vec{B}_0$. In sub-Alfvénic turbulence, the motion of these fluid parcels can be decomposed into two independent velocity components, parallel (black arrows) and perpendicular (orange arrows) to $\vec{B}_0$.

where we have considered that the initial dimension of the perturbed volume $r_0$ is much larger than its perturbations.

Along $\vec{S}_{\perp}$ we find that,

$$\dot{z}(t) = -\left(\frac{\delta \dot{B}_r(t)}{\delta B_r(t)} - \frac{\delta \dot{B}_\parallel(t)}{2B_0}\right) \frac{\delta B_r(t)}{\delta B_r(t)} z(t) \approx -\frac{\delta \dot{B}_r(t)}{\delta B_r(t)} z(t), \quad (3)$$

while the azimuthal velocity along $\vec{S}_\phi$ is,

$$u_\phi \equiv \dot{\phi}(t) r(t) \approx -\left(\frac{\delta \dot{B}_r(t)}{\delta B_r(t)} - \frac{\delta \dot{B}_\phi(t)}{\delta B_\phi(t)}\right) r(t). \quad (4)$$

In the approximate expressions we have employed that $|\vec{B}_0| \gg |\delta \vec{B}|$, which implies that parallel and perpendicular motions are independent. The coupling of parallel and perpendicular motions becomes inevitable when $|\vec{B}_0| \sim |\delta \vec{B}|$.
We henceforth use the following notation: \( z = L_\parallel \), and \( r = L_\perp \). From Eqs. 2, and 3 we derive,

\[
\delta B_\parallel(t) \propto -B_0 \log L_\perp(t),
\]

\[
\delta B_\perp(t) \propto L_\parallel^{-1}(t).
\]

The difference in the scaling is due to the Lorenz force by \( \vec{B}_0 \), which affects perpendicular motions, while it has no effect on parallel motions.

### III. Lagrangian formulation

We employ the Lagrangian of the perturbed volume. We place the reference frame at the center of mass of the target volume, hence there is no bulk velocity term in the Lagrangian. Therefore, all the velocity components are due to internal motions induced by magnetic perturbations. We focus on low plasma-beta fluids where thermal pressure is subdominant; these fluids are highly compressible, \( M_s > 1 \).

When \( M_A \ll 1 \), magnetic tension dominates over magnetic pressure [32]. The large tension rapidly suppresses transverse oscillations and induces large restoring torques. Thus, twisting would have minimum contribution to the dynamics [e.g.,47] and motions would be mostly longitudinal (\( \phi, \delta B_\phi \approx 0 \)). Since \( u_\phi \to 0 \) then, due to Eqs. 3, and 4, \( L_\parallel \gg L_\perp \). This implies that untwisted coherent structures are stretched towards the \( \vec{B}_0 \) axis, which is consistent with the anisotropic properties of sub-Alfvénic turbulence [9, 15, 44, 48–55].

The local perturbed Lagrangian [56–58] of the fluid parcel can be split into parallel (\( || \)) and perpendicular (\( \perp \)) terms to \( \vec{B}_0 \), since parallel and perpendicular motions are independent:

\[
\delta \mathcal{L} = \left( \frac{1}{2} \rho u_\parallel^2 - \frac{\delta B_\perp^2}{8\pi} \right) + \left( \frac{1}{2} \rho u_\perp^2 - \frac{B_0 \delta B_\parallel}{4\pi} - \frac{\delta B_\parallel^2}{8\pi} \right).
\]

For untwisted fluid parcels, \( \delta \mathcal{L} \) is independent of \( \delta B_\phi \), hence \( \delta B_\perp = \delta B_\parallel \). Quantities \( u_\parallel \) and \( u_\perp \) are the parallel and perpendicular velocity components, which correspond to the motion of \( L_\parallel \) and \( L_\perp \) respectively, \( u_\parallel = \dot{L}_\perp, u_\perp = \dot{L}_\parallel \). Then, due to Eqs. 2, and 3, \( \delta B_\parallel \), and \( \delta B_\perp \) are generalized coordinates of \( \delta \mathcal{L} \). From Eq. 6, we obtain that \( L_\parallel(t) = C/\delta B_\perp(t) \), where \( C \) is a constant determined from the initial conditions. With this expression we eliminate \( L_\parallel \) from the Lagrangian,
which up to second order terms, is:

\[
\delta L_\perp (\delta B_\perp, \delta \hat{B}_\perp) \approx \frac{1}{2} \rho C_s^2 \delta \frac{\hat{B}_\perp^2}{\hat{B}_\perp^2} - \frac{\delta B_\perp^2}{8\pi},
\]

\[
\delta L_\parallel (\delta B_\parallel, \delta \hat{B}_\parallel) \approx \frac{1}{8} \rho \frac{\delta B_\parallel^2}{B_0^2} \frac{L_\perp^2}{L_\perp^2} - \frac{B_0 \delta B_\parallel}{4\pi} - \frac{\delta B_\parallel^2}{8\pi},
\]

where \( L_\perp (t = 0) = L_{\perp,0} \). \( L_\parallel \) and \( L_\perp \) are unconstrained, hence the dynamics of the untwisted perturbed volume are self-similar, and \( L_\parallel \gg L_\perp \). Below we solve the Euler-Lagrange equations for \( \delta L_\parallel \) and \( \delta L_\perp \).

**IV. Solutions of \( \delta L_\parallel \)**

From the Euler-Lagrange equation of \( \delta L_\parallel \) we obtain:

\[
\delta \frac{\hat{B}_\parallel}{B_0} = \delta B_\parallel (t) + B_0 \frac{4V_A^2}{L_\perp^2},
\]

where \( V_A = B_0/\sqrt{4\pi\rho} \) is the Alfvénic speed.

Initially we compress the perturbed volume perpendicularly to \( \overline{B}_0 \), then release it and let the compression propagate (initial conditions: \( u_\perp (t = 0) = 0, \delta B_\parallel (t = 0) = \delta B_\parallel_{\text{max}} \); we could initiate the fluid parcel at \( \delta B_\parallel (t = 0) = -\delta B_\parallel_{\text{max}} \), but in that case \( u_\perp (t = 0) \neq 0 \).) Solutions of Eq. 10 are harmonic, but are valid only in the linear regime. From the jump conditions, we obtain analytically that when the sonic Mach number (\( M_s \)) is \( M_s \approx 1 \), an isothermal shock, perpendicular to \( \overline{B}_0 \) forms when:

\[
\delta B_\parallel \lesssim \frac{B_0}{2} \left( M_s^2 - 1 \right).
\]

Thus, in sub-Alfvénic turbulence, \( M_s < 1 \), magnetized shocks form when \( \delta B_\parallel < 0 \). This means that \( \delta B_\parallel \) will never perform a full harmonic cycle, hence solutions of Eq. 10 are valid only at early times when perturbations are quasi-linear.

Keeping the dominant term of the expansion of the harmonic solutions we derive that,

\[
\delta B_\parallel (t) \approx \delta B_\parallel_{\text{max}} - 2B_0 \frac{V_A^2}{L_\perp^2} t^2.
\]

The above solution, through Eq. 2 yields,

\[
u_\perp (t) \approx \frac{2V_A^2}{L_\perp} t.
\]

From Eqs. 12, and 13 we obtain that as the magnetic field of the perturbed volume decompresses, \( u_\perp \) increases. When the shock is formed, the perturbed volume instantaneously bounces off its environment, which acts as a pressure wall [59]. At the post-shock phase the motion is reversed and the perturbed volume will start contracting until \( +\delta B_\parallel_{\text{max},p} \).

The post-shock solutions are obtained from Eq. 10 with initial conditions: \( u_p (t = 0) > 0 \), and \( \delta B_\parallel_{p} (t = 0) < 0 \), where
the subscript $p$ denotes post-shock quantities; acceleration in the post-shock phase is negative. At the post-shock phase, the solution of $\delta B_\parallel$ is:

$$\delta B_\parallel(t) \approx -\delta B_\parallel,p + u_p(t = 0) t - \frac{2B_0 V_A^2}{L_{\perp,0}^2} t^2. \quad (14)$$

At the post-shock phase, the magnetic field increases until $+\delta B_{\parallel,\max,p}$, which is smaller than the initial magnetic field increase ($+\delta B_{\parallel,\max}$) of the pre-shock phase, because energy has been dissipated by the shock [60, 61]. When the perturbed volume reaches $+\delta B_{\parallel,\max,p}$, velocity is zero, and the motion is reversed. Then, the volume starts expanding until it forms a shock again. Overall, the perturbed volume would perform damped oscillations until all the energy is dissipated [59, 62].

Fluids in nature are commonly assumed to be constantly perturbed until turbulence reaches a steady state [63–65]. Various driving mechanisms could maintain turbulent energy in nature [66–76]. In our model, turbulent driving is equivalent to adding externally kinetic energy to the perturbed volume, such that the initial velocity at the post-shock phase, $u_p(t = 0)$, is sufficient to compress the perturbed volume until it reaches the maximum compression it had in the pre-shock phase, $\delta B_{\parallel,\max,p} = +\delta B_{\max,\parallel}$.

We consider the presence of an external driver, which ensures that $\delta B_\parallel$ fluctuations, and hence energy, are maintained in a quasi-static state. In addition, we consider that the fluid is ergodic [77, 78]. For ergodic fluids, $\delta B_\parallel$ would oscillate ballistically, $\delta B_\parallel \propto t^2$, between $+\delta B_{\parallel,\max}$ and $-\delta B_{\parallel,\max}$, as we argue below, with period $T_b = 4L_{\perp,0}V_A^{-1} \sqrt{\delta B_{\parallel,\max}/2B_0}$.

When we initially compress the magnetic field of the perturbed volume along $\vec{B}_0$, then due to Eq. 5, $L_{\perp}$ decreases. This forces the surface of the environment of the perturbed volume to increase by equal amounts. Thus, the $+\delta B_{\parallel,\max}$ initial increase of the magnetic field of the perturbed volume, causes the magnetic field of the environment to decrease by $-\delta B_{\parallel,\max}$, due to flux freezing. If the fluid is ergodic, then different fluid parcels correspond to different oscillation phases of the target fluid parcel [77, 78]. Therefore, the $-\delta B_{\parallel,\max}$ of the environment, corresponds to the maximum decrease of the magnetic field strength of the target volume. Non-linear effects can break the symmetry between $+\delta B_{\parallel,\max}$ and $-\delta B_{\parallel,\max}$, but ergodicity is only weakly broken when $\vec{B}_0 \neq 0$ [79].

The perturbed volume would spend most of its time in the compressed state, since there the velocity is minimum. On the other hand, the velocity of the fluid parcel is maximum when $\delta B_\parallel < 0$, and hence the fluid parcel would spend minimum time there. As a result, due to ergodicity, the majority of fluid parcels at a given time would be compressed ($\delta B_\parallel > 0$), which is verified by numerical simulations [36].

V. Solutions of $\delta L_\perp$

From the Euler-Lagrange equation of $\delta L_\perp$ we obtain,

$$\delta B_\perp(t)\delta B_\perp(t) - 2\delta B_\perp^2(t) + \frac{\delta B_{\parallel}^2(t)}{4\pi \rho C^2} = 0. \quad (15)$$
would oscillate periodically, since there would be no energy losses as the target fluid parcel expands, its environment along the \( L \) axis contracts, provided that the fluid has fixed boundaries. Due to the expansion of the target volume, the initial velocity of the environment would be \( -u_{\|,\max} \), which results in negative sign in the denominator of Eq. 16, and hence \( \delta B_{\perp} \) increases in the environment. On the other hand, \( \delta B_{\perp} \) in the target volume stops increasing when \( t_c = L_{\|,0}/(fV_A) \), because \( \delta B_{\perp} \) in the environment becomes infinite.

In sub-Alfvénic flows \( |\vec{B}_0| \gg |\delta \vec{B}| \), so this infinity should be treated as an asymptotic behaviour of \( \delta B_{\perp} \): there is a physical limit above which \( \delta B_{\perp} \) cannot grow. After \( t_c \), the motion is reversed and the environment starts expanding along \( \vec{B}_0 \), causing the target volume to contract with \( \delta B_{\perp} \) growing as \( \delta B_{\perp}(t) = fB_0/(2 - fV_AL_{\|,0}^{-1}) \) until it reaches \( \delta B_{\perp,\max} \). If the interaction between the target fluid parcel and its environment were elastic, then the target volume would oscillate periodically, since there would be no energy losses, between \( \delta B_{\perp,\max} \) and \( \delta B_{\perp,\max}/2 \) with period \( T_{\parallel} = 2L_{\|,0}/(fV_A) \).

### VI. Energetics

Turbulence nonlinearities lead to shock formation, hence to energy diffusion. The driver maintains the energy constant before and after the shock. This enables us to approximate the fully-developed, steady-state turbulence energetics with the proposed analytical formulation, which preserves the energy.

For an ergodic fluid [77, 78], the volume-averaged energetics \( \langle f \rangle_V \) at a given time are equivalent to the time-averaged energetics \( \langle f \rangle_t \) of a fluid parcel. We next compute analytically the \( \langle f \rangle_t \) energy contribution of each Lagrangian term (Eq. 7) and their relative ratios. We compare the energy ratios against the \( \langle f \rangle_V \) numerical values. The

\[
\delta B_{\perp}(t) \approx \frac{fB_0}{1 \pm fV_AL_{\|,0}^{-1}}, \quad u_{\|}(t) \approx \pm fV_A,
\]

where \( L_{\|}(t = 0) = L_{\|,0}, \) and \( f = \delta B_{\perp,\max}/B_0 \ll 1 \). In the above equations signs depend on the initial conditions. Initially we consider that \( \delta B_{\perp}(t = 0) = \delta B_{\perp,\max}, \) and \( u_{\|}(t = 0) = u_{\perp,\max} \), which leads to positive signs.

If the initial velocity along \( \vec{B}_0 \) is zero, then both \( u_{\|} \) and \( \delta B_{\perp} \) would remain static. The coupling of parallel and perpendicular motions (Eq. 3) would induce parallel motions when \( \delta B_{\perp} \neq 0 \), even if \( u_{\|}(0) = 0 \). However, since we have neglected the coupling of motions, we initiate \( u_{\|} \) from the initial conditions.

From Eq. 6 we obtain that the free streaming of the perturbed volume causes \( L_{\|} \) to expand as:

\[
L_{\|}(t) \approx L_{\|,0} \left( 1 + \frac{fV_A}{L_{\|,0}} t \right)
\]

As the target fluid parcel expands, its environment along the \( \vec{B}_0 \) axis contracts, provided that the fluid has fixed boundaries. Due to the expansion of the target volume, the initial velocity of the environment would be \( -u_{\|,\max} \), which results in negative sign in the denominator of Eq. 16, and hence \( \delta B_{\perp} \) increases in the environment. On the other hand, \( \delta B_{\perp} \) in the target volume stops increasing when \( t_c = L_{\|,0}/(fV_A) \), because \( \delta B_{\perp} \) in the environment becomes infinite. In sub-Alfvénic flows \( |\vec{B}_0| \gg |\delta \vec{B}| \), so this infinity should be treated as an asymptotic behaviour of \( \delta B_{\perp} \): there is a physical limit above which \( \delta B_{\perp} \) cannot grow. After \( t_c \), the motion is reversed and the environment starts expanding along \( \vec{B}_0 \), causing the target volume to contract with \( \delta B_{\perp} \) growing as \( \delta B_{\perp}(t) = fB_0/(2 - fV_AL_{\|,0}^{-1}) \) until it reaches \( \delta B_{\perp,\max} \). If the interaction between the target fluid parcel and its environment were elastic, then the target volume would oscillate periodically, since there would be no energy losses, between \( \delta B_{\perp,\max} \) and \( \delta B_{\perp,\max}/2 \) with period \( T_{\parallel} = 2L_{\|,0}/(fV_A) \).

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\*This solution is obtained by considering that the initial conditions in the reversed motion of the fluid parcel are: \( \delta B_{\perp}(0) = \delta B_{\perp,\max}/2, u_{\|}(0) = -u_{\|,\max}, \) and \( L_{\|}(0) = 2L_{\|,0} \). These values correspond to the solutions of Eqs. 16, and 17 for \( t = t_c \).

\*Shocks can form in parallel motions and diffuse energy. External driving can maintain the maximum amplitude of fluctuations constant.
numerical data correspond to simulations of ideal, isothermal MHD turbulence without self-gravity; turbulence is forced and maintained in a quasi-static state. The simulations we have used are forced with a mixture of compressible and incompressible modes, but the driving does not affect the energetics of sub-Alfvénic, and compressible turbulence [35].

A. Kinetic energy

The averaged total kinetic energy \( E_{\text{kinetic}} \) of the fluid parcel is,

\[
\frac{1}{2} \rho \left( \langle u_\perp^2 \rangle + \langle u_\parallel^2 \rangle \right) \approx \frac{B_0 \delta B_{\parallel,\text{max}}}{6\pi} + \frac{\delta B_{\perp,\text{max}}^2}{8\pi},
\]

where brackets denote averaging over a single period. The kinetic energy is dominated, to first order, by \( u_\perp \). Thus, the average Alfvénic Mach number is, to first order,

\[
M_A \equiv \frac{\langle v^2 \rangle^{1/2}}{v_A} \approx \sqrt{\frac{4\delta B_{\parallel,\text{max}}}{3B_0}}.
\]

B. Harmonic potential

From Eqs. 12, and 16 we find that \( \langle \delta B_{\parallel}^2 \rangle = \frac{7}{15} \delta B_{\parallel,\text{max}}^2 \), and \( \langle \delta B_{\perp}^2 \rangle = \frac{\delta B_{\perp,\text{max}}^2}{2} \). The total average harmonic potential energy \( E_{\text{harmonic}} \) density is equal to,

\[
\frac{\langle \delta B_{\perp}^2 \rangle}{8\pi} \approx \frac{\delta B_{\parallel,\text{max}}^2}{8\pi} \left( \frac{7}{15} + \frac{\zeta^2(M_A)}{2} \right),
\]

where \( \zeta = \delta B_{\perp,\text{max}} / \delta B_{\parallel,\text{max}} \). Sub-Alfvénic turbulence is anisotropic [15, 44, 48, 49], with the anisotropy between \( \delta B_{\perp} \) and \( \delta B_{\parallel} \) depending on \( M_A \) [80]. To account for this property we assume that \( \zeta \) is a function of \( M_A \). When \( M_A \to 0 \), \( B_0 \) suppresses any bending of the magnetic field lines with the amplitude of \( \delta B_{\parallel} \) being larger than that of \( \delta B_{\perp} \) [80], hence \( \zeta \to 0 \); this is also a consequence of \( \nabla \cdot \vec{B} = 0 \) for anisotropic fluid parcels with \( L_\parallel \gg L_\perp \). For \( M_A \to 1 \), fluctuations tend to become more isotropic, and hence \( \zeta \to \sqrt{2} \). These limiting behaviors are consistent with numerical simulations [36, 80].

C. Coupling potential

According to Eq. 18, \( \vec{B}_0 \cdot \delta \vec{B} \) contributes to \( E_{\text{kinetic}} \) as,

\[
\frac{B_0 \delta B_{\parallel,\text{max}}}{6\pi} = \frac{\sqrt{15}}{7} \frac{B_0 \langle \delta B_{\parallel}^2 \rangle^{1/2}}{6\pi} \approx \frac{B_0 \langle \delta B_{\parallel}^2 \rangle^{1/2}}{4\pi}.
\]

This equation demonstrates that the energy stored in the coupling potential \( E_{\text{coupling}} \) is in equipartition with the average kinetic energy, when turbulence is sub-Alfvénic.
Fig. 2 Comparison between analytical and numerical results. Solid and dashed thick black lines correspond to the $E_{\text{harmonic}}/E_{\text{coupling}}$ ratio obtained analytically for $M_A \to 0$ ($\zeta = 0$) and $M_A \to 1$ ($\zeta = \sqrt{2}$) respectively. Numerical data are shown with colored dots. The blue line corresponds to the analytically-obtained $E_{\text{kinetic}}/E_{\text{coupling}}$ ratio, while colored triangles show the same quantities calculated from numerical data. Red boxes correspond to $E_{\text{kinetic}}/E_{m,\text{total}}$. The thin green line shows energy terms in equipartition. The colorbar shows the sonic Mach number ($M_s$) of the simulations.
D. Energetics ratios

We compute the $E_{\text{kinetic}}/E_{\text{coupling}}$ ratio:

$$
\frac{E_{\text{kinetic}}}{E_{\text{coupling}}} = \frac{2\pi \rho \langle u^2 \rangle}{B_0 \langle \delta B^2 \rangle^{1/2}} \approx 1 + \frac{9}{16} M_A^2 \zeta^2(M_A). \quad (22)
$$

For $M_A \to 0$, $E_{\text{coupling}} \approx E_{\text{kinetic}}$, while for $M_A \to 1$, $E_{\text{kinetic}} \geq E_{\text{coupling}}$, $E_{\text{kinetic}}$ becomes larger than $E_{\text{coupling}}$, since $u_\parallel$ contributes more in $E_{\text{kinetic}}$ as $M_A$ increases. When $M_A \to 1$, $\zeta \approx \sqrt{2}$ so the $E_{\text{kinetic}}/E_{\text{coupling}}$ ratio in trans-Alfvénic turbulence scales as:

$$
\frac{E_{\text{kinetic}}}{E_{\text{coupling}}} \approx 1 + \frac{9}{8} M_A^2. \quad (23)
$$

For $M_A \to 0$, $E_{\text{kinetic}} \approx E_{\text{coupling}}$.

Regarding the $E_{\text{harmonic}}/E_{\text{coupling}}$ ratio we find that,

$$
\frac{\langle \delta B_{\text{tot}}^2 \rangle}{2B_0 \langle \delta B_{\|}^2 \rangle^{1/2}} = \frac{3}{8} \sqrt{\frac{15}{7}} M_A^2 \left( \frac{7}{15} \zeta^2(M_A) \right), \quad (24)
$$

which, for the two limiting cases of $\zeta$, becomes,

$$
\frac{E_{\text{harmonic}}}{E_{\text{coupling}}} \approx \begin{cases} 0.25 M_A^2, & M_A \to 0 \\ 0.80 M_A^2, & M_A \to 1 \end{cases}. \quad (25)
$$

E. Numerical simulations

In the figure, we compare the analytically-calculated energy ratios against numerical results from the literature [36]. Lines correspond to the analytical relations for $E_{\text{harmonic}}/E_{\text{coupling}}$ (Eq. 25), and $E_{\text{kinetic}}/E_{\text{harmonic}}$ (Eq. 23), while colored markers correspond to the numerical values. The numerical data behave as predicted by the analytical relations. Accounting for the contribution from both $\vec{B}_0 \cdot \delta \vec{B}$ and $\delta B^2$, the total energy stored ($E_{\text{m,total}} = E_{\text{coupling}} + E_{\text{harmonic}}$) in magnetic fluctuations is very close to equipartition with kinetic energy, as shown by the red boxes.

VII. Summary

This work presents a Lagrangian description of the energy transfer between kinetic and magnetic fluctuations of compressible and sub-Alfvénic fluids. From the flux-freezing theorem, we showed that $\delta B_{\|}$ and $\delta B_\perp$ are generalized coordinates of the local perturbed Lagrangian. We derived analytically the relations which connect kinetic and magnetic energy of sub-Alfvénic and compressible fluids, as a function of $M_A$. We conclude that when $M_A \leq 1$, the total
magnetic energy density transferred to kinetic is equal to \( \left( 2B_0 \sqrt{\langle \delta B^2 \rangle} + \langle \delta B^2 \rangle \right) / 8\pi \). The consistency between our analytical relations and numerical data is remarkable and for this reason we believe that the formalism presented here could offer new insights into MHD turbulence. The consistency between our analytical relations and numerical data is remarkable. It is not the first time that simple analytical arguments agree quantitatively with numerical simulations of nonlinear problems [e.g. 81]. However, an analytical theory is always advantageous, since it allows us to get a deeper understanding of complex problems. For this reason, we believe that the formalism presented here could offer new insights into strongly magnetized turbulence.

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