Noncommutativity, Cosmology and $\Lambda$

M. Sabido$^1$, S. Pérez-Payán$^2$
Departamento de Física de la Universidad de Guanajuato,
A.P. E-143, C.P. 37150, León, Guanajuato, México
E-mail: $^1$msabid@fisica.ugto.mx
E-mail: $^2$sinuhe@fisica.ugto.mx

E. Mena
Centro Universitario de la Ciénega, Universidad de Guadalajara
Ave. Universidad 1115 Edif. B, C.P. 47820 Ocotlán, Jalisco, México
E-mail: emena@cuci.udg.mx

C. Yee-Romero
Departamento de Matemáticas, Facultad de Ciencias
Universidad Autónoma de Baja California, Ensenada, Baja California, México
E-mail: carlos.yee@uabc.edu.mx

Abstract. The effects of phase space deformations in standard scalar field cosmology are studied. The deformation is introduced by modifying the symplectic structure of the minisuperspace variables to have a deformed Poisson algebra among the coordinates and the canonical momenta. The resulting theory is knowns as noncommutative cosmology. We show the effects of this deformation in scalar field cosmology and establish a relationship between the noncommutative parameters and $\Lambda$. This work is based on the papers, arXiv:1111.6137 and arXiv:1111.6136.

1. Introduction
The initial interest in noncommutative field theory [1] slowly but steadily permeated in the realm of gravity, from which several approaches to noncommutative gravity [2] where proposed. All of these formulations showed that the end result of a noncommutative theory of gravity, is a highly nonlinear theory. In order to study the effects of noncommutativity on different aspects of the universe, noncommutative cosmology was presented in [3]. The authors noticed that the noncommutative deformations modify the noncommutative fields, and conjectured that the effects of the full noncummutative theory of gravity should be reflected in the minisuperspace variables. This was achieved by introducing the Moyal product of functions in the Wheeler-DeWitt equation, in the same manner as is done in noncommutative quantum mechanics [4, 5].

Although the noncommutative deformations of the minisuperspace where originally analyzed at the quantum level, by an effective noncommutativity on the minisuperspace, classical noncommutative formulations have been proposed. The main idea of this classical
noncommutativity is based on the assumption that modifying the Poisson brackets of the classical theory gives the noncommutative equations of motion [3, 5, 4, 6].

We will work with an FRW universe and a scalar field. This model has been used to explain several aspects of our universe, like inflation, dark energy and dark matter. Our approach to noncommutativity is through its introduction in a phase space constructed in the minisuperspace variables, and is achieved by modifying the symplectic structure, in the same manner as in [4, 5, 6]. It will be showed that in the absence of a cosmological constant, the behavior of the scale factor can account for a late time acceleration, furthermore it can be seen from the solution that there is no initial singularity. We follow this line of reasoning and give a proposal for the origin of $\Lambda$, by considering noncommutative deformation of a 5 dimensional theory of gravity with a compact extra dimension.

We will start in Section 2, by introducing the commutative model. In Section 3, the noncommutative model is presented, as well as the dynamics of the cosmological model. We will show that with our approach late time acceleration can be accounted, contrary to the commutative case. In section 4, we present a model where the cosmological constant arises from the phase space deformation. Finally, the last section is devoted for conclusions and final remarks.

2. The Commutative Model

As already suggested, cosmology presents an attractive arena for noncommutative models, both in the quantum as well as classical level. One of the features of noncommutative field theories is UV/IR mixing, this effectively mixes short scales with long scales, from this fact one may expect that even if noncommutativity is present at a really small scale, by this UV/IR mixing, the effects might be present at an older time of the universe.

We start with a homogeneous and isotropic universe endowed with a flat Friedmann-Robertson-Walker metric
\[ ds^2 = -N^2(t)dt^2 + a^2(t)\left[dr^2 + r^2d\Omega^2\right], \]

where as usual $a(t)$ is the scale factor of the universe and $N(t)$ is the lapse function. The action we will be working with is the Einstein-Hilbert action and a scalar field $\phi$ as the matter content for the model. In units $8\pi G = 1$, the action takes the form
\[ S = \int dt \left\{ -\frac{3a\dot{a}^2}{N} + a^3\left(\frac{\dot{\phi}^2}{2N} + NV(\phi)\right)\right\}, \]

here $V(\phi)$ is the scalar potential. From now on we will restrict to the case of constant potential ($V(\phi) = -\Lambda$).

For the purposes of introducing the deformation to the minisuperspace variables an appropriate redefinition needs to be made, we make the following change of variables
\[ x = m^{-1}a^{3/2}\sinh(m\phi), \quad y = m^{-1}a^{3/2}\cosh(m\phi). \]

where $m^{-1} = 2\sqrt{2/3}$. In these new variables we calculate the Hamiltonian and is given by
\[ H_c = N\left(\frac{1}{2}P_x^2 + \frac{\omega^2}{2}x^2\right) - N\left(\frac{1}{2}P_y^2 + \frac{\omega^2}{2}y^2\right), \]

where $\omega^2 = -\frac{3}{4}\Lambda$. This is the canonical Hamiltonian which is a first-class constraint as is usual in general relativity. Since we do not have second class constraints in the model we will continue
to work with the usual Poisson brackets and the relations of commutation between the phase space variables
\[ \{ x_i, x_j \} = 0, \quad \{ P_{x_i}, P_{y_j} \} = 0, \quad \{ x_i, P_{x_j} \} = \delta_{ij}. \] (5)
At this point, we have a minisuperspace spawned by the new variables \((x, y)\). To find the dynamics of this model, we need to solve the equations of motions, these are derived as usual by using Hamilton’s equations. For the particular model the equations are
\[ \dot{x} = -P_{x}, \quad \dot{y} = P_{y}, \]
\[ \dot{P}_{x} = \omega^{2} x, \quad \dot{P}_{y} = -\omega^{2} y, \] (6)
these equations can easily be integrated
\[ x(t) = X_{0} \cos(|\omega|t + \delta_{x}), \quad y(t) = Y_{0} \cos(|\omega|t + \delta_{y}). \] (7)
In order to satisfy the Hamiltonian constraint we introduce the solutions to the Hamiltonian, this gives a relationship between the integration constants, it easy to verify that \(X_{0} = \pm Y_{0}\). Finally we write the solution in the original variables
\[ a^{3}(t) = V_{0} [\cos (\delta_{x} - \delta_{y}) + \cos (2|\omega|t + \delta_{x} + \delta_{y})], \] (8)
where \(\delta_{x} - \delta_{y} = 2n\pi\) in order to have a positive volume for the universe, due to the periodic nature of the functions we can choose \(n = 0\). From the equation for the volume we can see, that you get a periodic universe, furthermore we get zero volume when \(t = \frac{(2n+1)}{4|\omega|}\pi\).

3. Deformed Space Model
The original ideas of deformed phase space, or more precisely deformed minisuperspace, where done in connection with noncommutative cosmology [3]. As already mentioned, in order to avoid the complications of a noncommutative theory of gravity, they introduce a deformation to the minisuperspace in order to incorporate noncommutativity. Usually the deformation is introduced by the Moyal brackets, which is based in the Moyal product. To study the behavior of the model in a deformed phase space framework such that the minisuperspace variables do not commute with each other; noncommutativity between the phase space variables can be understood by replacing the usual product with the star product, also know as the Moyal product law between two arbitrary functions of position and momentum, as
\[ (f \star g)(x) = \exp \left[ \frac{1}{2} \alpha^{ab} \partial^{(1)}_{a} \partial^{(2)}_{b} \right] f(x_{1})g(x_{2})|_{x_{1}=x_{2}=x}, \] (9)
such that
\[ \alpha = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{pmatrix}, \] (10)
where the \(2 \times 2\) matrices \(\theta\) and \(\beta\) are assumed to be constant, antisymmetric and represent the noncommutativity in the coordinates and momenta respectively and \(\sigma\) is a product of \(\theta\) and \(\beta\). With this product law, the \(\alpha\)-deformed Poisson brackets can be written as
\[ \{ f, g \}_{\alpha} = f \star_{\alpha} g - g \star_{\alpha} f. \] (11)
An alternative an far more useful construction, is based on symplectic manifolds [8]. Once the deformation has been done one arrives to a modified Poisson algebra
\[ \{ x_{i}, x_{j} \}_{\alpha} = \theta_{ij}, \quad \{ x_{i}, p_{j} \}_{\alpha} = \delta_{ij} + \sigma_{ij}, \quad \{ p_{i}, p_{j} \}_{\alpha} = \beta_{ij}. \] (12)
Making the following transformation on the classical phase space variables \( \{x, y, p_x, p_y\} \)

\[
\begin{align*}
\hat{x} &= x + \frac{\theta}{2} p_y, \\
\hat{y} &= y - \frac{\theta}{2} p_x,
\end{align*}
\]

\[
\begin{align*}
\hat{p}_x &= p_x - \frac{\beta}{2} y, \\
\hat{p}_y &= p_y + \frac{\beta}{2} x,
\end{align*}
\]

(13)

now the algebra reads

\[
\{\hat{y}, \hat{x}\} = \theta, \quad \{\hat{u}, \hat{p}_x\} = \{\hat{y}, \hat{p}_y\} = 1 + \sigma, \quad \{\hat{p}_y, \hat{p}_x\} = \beta,
\]

(14)

where \( \sigma = \theta \beta / 4 \). Now that we have constructed the modified phase space, we apply the transformation to the Hamiltonian, where after defining

\[
\omega_1^2 = \frac{4(\beta - \omega^2 \theta)}{4 - \omega^2 \theta^2}, \quad \omega_2^2 = \frac{4(\omega^2 - \beta^2 / 4)}{4 - \omega^2 \theta^2},
\]

(15)

we get

\[
\hat{H} = \frac{1}{2} \left\{ \hat{p}_x^2 - \hat{p}_y^2 + \omega_1^2 (\hat{x} \hat{p}_y - \hat{y} \hat{p}_x) + \omega_2^2 (\hat{x}^2 - \hat{y}^2) \right\}.
\]

(16)

We can construct a bidimensional vector potential \( \hat{A}_x = -\frac{\omega_1^2}{2} \hat{y}, \hat{A}_y = \frac{\omega_2^2}{2} \hat{x} \) from were a magnetic field \( B = \omega_1^2 \) is calculated, this result allow us to write the effects of the noncommutative deformation as minimal coupling on the Hamiltonian, \( \hat{H} = [(p_x - \hat{A}_x)^2 + \omega_2^2 \hat{x}^2] - [(p_y - \hat{A}_y)^2 + \omega_2^2 \hat{y}^2] \), this expression can be rewritten in terms of the magnetic B-field as

\[
\hat{H} = [(\hat{p}_x^2 + (\omega_2^2 - B^2 / 4) \hat{x}^2] - [(\hat{p}_y^2 + (\omega_2^2 - B^2 / 4) \hat{y}^2]
\]

\[
+ B(\hat{y} \hat{p}_x - \hat{x} \hat{p}_y),
\]

(17)

where \( \omega_3^2 = \omega_1^2 + \omega_2^2 \). This is a typical result in 2-dimensional noncommutativity, where the effects of the noncommutative deformation can be encoded in a perpendicular constant magnetic field.

To obtain the dynamics for our model, we can derive the equations of motion from the Hamiltonian (29)

\[
\dot{\eta} = \{x, \hat{H}\} = \frac{1}{2} [2 \hat{p}_x + \omega_1^2 y],
\]

\[
\dot{\zeta} = \{y, \hat{H}\} = \frac{1}{2} [-2 \hat{p}_y + \omega_1^2 x],
\]

\[
\dot{\hat{p}}_x = \{ \hat{p}_x, \hat{H}\} = \frac{1}{2} [-\omega_1^2 \hat{p}_y - 2 \omega_2^2 \hat{x}],
\]

\[
\dot{\hat{p}}_y = \{ \hat{p}_y, \hat{H}\} = \frac{1}{2} [-\omega_1^2 \hat{p}_x + 2 \omega_2^2 \hat{y}],
\]

(18)

defining new variables \( \eta \) and \( \zeta \) as \( \eta = \hat{x} + \hat{y}, \zeta = \hat{y} - \hat{x} \), this set of equations reduce to

\[
\dot{\eta} - \omega_1^2 \eta + \frac{1}{4} (4 \omega_2^2 + \omega_1^4) \eta = 0,
\]

\[
\dot{\zeta} + \omega_1^2 \zeta + \frac{1}{4} (4 \omega_2^2 + \omega_1^4) \zeta = 0,
\]

(19)

solving this equations we can get the solutions in terms of the noncommutative variables \( \hat{x}(t) \) and \( \hat{y}(t) \)

\[
\hat{x}(t) = \frac{1}{2} \left[ e^{\frac{\omega_1^4}{4} t} \cosh (\omega_2^2 t) - e^{-\frac{\omega_1^4}{4} t} \cosh (\omega_2^2 t) \right],
\]

(20)
Figure 1. Dynamics of the phase space deformed model for the values $X_0 = Y_0 = 1, \delta_2 = \delta_1 = 0, \omega = 0$ and $\beta = 1$. The solid line corresponds to the volume of the universe, calculated with the noncommutative model. The dotted line corresponds to the volume of the de Sitter spacetime. For large values of $t$ the behavior is the same.

\[
\dot{y}(t) = \frac{1}{2} \left[ e^{\frac{\omega}{4} t} \cosh \left( \omega^2 t \right) + e^{-\frac{\omega}{4} t} \cosh \left( \omega^2 t \right) \right],
\]

where $\omega^2 = (3\omega_1^4 + 16\omega_2^2)/4$.

Up to this point we have obtained the equations of motion using the deformed Poisson algebra (14). In order to find a solution we define the new variables $\eta$ and $\zeta$ which decouple equations (19) into two differential equations. In the original variables the volume of the universe is given by

\[
a^3(t) = V_0 \cosh^2 \left( \frac{1}{4} t \beta \right),
\]

where we have taken the lim $\omega' \to 0$. From the definition of $\omega$, this limit means that $\Lambda = 0$, there is no cosmological constant. From Figure 1 we can notice two things: first we can see that with this construction the volume of the universe is not zero, there is no initial singularity for this cosmological model. Secondly, for large values of $t$ our cosmological model behaves like a de Sitter cosmology. Comparing the models in the late time limit enables us to show that for late times, the noncommutative parameter $\beta$ can play the role of the cosmological constant.

4. A model for $\Lambda$

Following the results of the previous section we will present a model for $\Lambda$. We will work with an empty (4+1) theory of gravity with cosmological constant $\Lambda$

\[
I = \int \sqrt{-g} \left( R - \Lambda \right) dt d^3x d\rho,
\]

where $\{t, x^i\}$ are the coordinates of the 4-dimensional space time and $\rho$ represents the coordinate of the fifth dimension. We are interested in Kaluza-Klein cosmology, so an FRW type metric is assumed, which is of the form

\[
d \mathcal{s} = -dt^2 + \frac{a^2(t)dr^i dr^i}{\left( 1 + \frac{\rho^2}{4} \right)^2} + \phi^2(t) d\rho^2,
\]
where $\kappa = 0, \pm 1$ and $a(t), \phi(t)$ are the scale factors of the universe and the compact dimension. Substituting this metric into the action (23) and integrating over the spatial dimensions, we obtain an 4-dimensional effective lagrangian

$$L = \frac{1}{2} \left( a\phi \dot{a}^2 + a^2 \dot{a} \phi - \kappa a \phi + \frac{1}{3} \Lambda a^3 \phi \right). \tag{25}$$

For the purposes of simplicity and calculations, we can rewrite this lagrangian in a more convenient way

$$L = \frac{1}{2} \left[ (\dot{x}^2 - \omega^2 x^2) - (\dot{y}^2 - \omega^2 y^2) \right], \tag{26}$$

where the new variables where defined as

$$x = \frac{1}{\sqrt{8}} \left( a^2 + a \phi - \frac{3\kappa}{\Lambda} \right), \quad y = \frac{1}{\sqrt{8}} \left( a^2 - a \phi - \frac{3\kappa}{\Lambda} \right), \tag{27}$$

and $\omega = -\frac{2\Lambda}{3}$. The equations of motion are those derived from varying the action action (26). The Hamiltonian constraint for the model is calculated from the action and is given by

$$H = \frac{1}{2} \left[ (\dot{x}^2 + \omega^2 x^2) - (\dot{y}^2 + \omega^2 y^2) \right] = 0, \tag{28}$$

which describes an isotropic oscillator-ghost-oscillator system. Now that we have defined the deformed phase space, we can see the effects on the proposed cosmological model. From the action (26) we can obtain the hamiltonian constraint (28), inserting relations (14) into the constraint we get

$$H \Psi(\hat{u}, \hat{v}) = \left\{ \left( \hat{p}_u - \frac{2(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \hat{v} \right)^2 - \left( \hat{p}_v + \frac{2(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \hat{u} \right)^2 \right\}, \tag{29}$$

making the following identifications

$$\omega^2 \equiv \frac{4(\beta - \theta \omega^2)^2}{(4 - \omega^2 \theta^2)^2} + \frac{4(\omega^2 - \beta^2/4)}{4 - \omega^2 \theta^2}, \tag{30}$$

$$A_{\hat{u}} \equiv \frac{-2(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \hat{v}, \quad A_{\hat{v}} \equiv \frac{2(\beta - \theta \omega^2)}{4 - \omega^2 \theta^2} \hat{u},$$

we can rewrite (29) in a much simpler and suggestive form

$$H = \left\{ \left( \hat{p}_u - A_{\hat{u}} \right)^2 + \omega^2 \hat{u}^2 \right\} - \left( \left( \hat{p}_v - A_{\hat{v}} \right)^2 + \omega^2 \hat{v}^2 \right), \tag{31}$$

which is a two dimensional anisotropic ghost-oscillator [7]. From (31) we can see that the terms $(p_i - A_i)$ can be associated to a minimal coupling term. We found that $\omega$ is defined in terms of the cosmological constant, then modifications to the oscillator frequency will imply modifications to the effective cosmological constant. In our case we get

$$\omega^2 = \frac{4(\omega^2 - \beta^2/4)}{4 - \omega^2 \theta^2}. \tag{32}$$
This \( \tilde{\omega} \) was obtained by a definition of the effective cosmological constant \( \tilde{\Lambda}_{\text{eff}} = -\frac{3}{2} \tilde{\omega}^2 \) and finally get a redefinition of the effective cosmological constant due to noncommutative parameters

\[
\tilde{\Lambda}_{\text{eff}} = \frac{4(\Lambda_{\text{eff}} + \frac{3}{8} \beta^2)}{4 - \frac{4}{3} \theta^2 | \Lambda_{\text{eff}} |}.
\]

(33)

In the case where there is no deformation on the coordinates, the noncommutative parameter \( \theta = 0 \) and we have that the frequency and the effective cosmological constant are given by

\[
\tilde{\omega}^2 = \omega^2 - \frac{\beta^2}{4}, \quad \text{and} \quad \tilde{\Lambda}_{\text{eff}} = \Lambda_{\text{eff}} + \frac{3\beta^2}{8}.
\]

(34)

We can see that noncommutative parameter \( \beta \) and \( \Lambda_{\text{eff}} \) compete to give the effective cosmological constant \( \tilde{\Lambda}_{\text{eff}} \). If we consider the case of a flat universe with a vanishing \( \Lambda_{\text{eff}} \) we see, that \( \tilde{\Lambda}_{\text{eff}} = \frac{3\beta^2}{8} \). This gives the origin of the cosmological constant in connection with the deformation parameter. As already argued the presence of the deformation parameters \( \theta \) and \( \beta \) gives an alternative approach to the origin of the cosmological constant, in this approach the cosmological constant is given by the parameter \( \beta \) and is considered a new fundamental constant. Then in this case, one can trace the origin of \( \Lambda \) to the noncommutative deformation between the canonical momenta associated to the 4 dimensional scale factor and the canonical momenta associated to the compact extra dimension.

Discussion and Outlook

In this letter we have constructed a deformed phase space model of scalar field cosmology. The deformation is introduced by modifying the symplectic structure of the minisuperspace variables, in order to have a deformed Poisson algebra among the coordinates and momenta. This construction is consistent with the assumption taken in noncommutative quantum cosmology [3, 4, 5, 6], and enable us to study the effects of phase space deformations in scalar field cosmology.

The deformed phase space model is obtained making the transformation (13) on the canonical Hamiltonian (4) which allow us to work out a Hamiltonian that depends on the deformed variables \( \hat{x}_i \) and \( \hat{p}_i \). To obtain the noncommutative equations of motion for the model, an in order to find solutions, a convenient change of variables was made. Finally, with the solutions, we where able to return to the original variables, and find that the volume of the universe is given by equation (22).

We found that with our model the volume of the universe behaves like a de Sitter cosmology for large values of \( t \) even when \( \Lambda = 0 \). This allows us to get a relation between the cosmological constant and the deformed parameter for the momenta by comparing the late time evolution of the volume of the noncommutative model with the volume of a de Sitter universe. Evidence of a possible relationship between the late time acceleration of the universe and the noncommutative parameters has been accumulating [7, 9, 10], our results also point in this direction, based on this observation in a model that gives some insight of the origin of \( \Lambda \).

Finally, we constructed 5 dimensional model in where it is conjectured that the cosmological constant has its origin in the noncommutative deformation between the canonical momenta associated to the 4 dimensional scale factor and the canonical momenta associated to the compact extra dimension.

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