Private Online Prefix Sums via Optimal Matrix Factorizations

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Abstract

Motivated by differentially-private (DP) training of machine learning models and other applications, we investigate the problem of computing prefix sums in the online (streaming) setting with DP. This problem has previously been addressed by special-purpose tree aggregation schemes with hand-crafted estimators. We show that these previous schemes can all be viewed as specific instances of a broad class of matrix-factorization-based DP mechanisms, and that in fact much better mechanisms exist in this class.

In particular, we characterize optimal factorizations of linear queries under online constraints, deriving existence, uniqueness, and explicit expressions that allow us to efficiently compute optimal mechanisms, including for online prefix sums. These solutions improve over the existing state-of-the-art by a significant constant factor, and avoid some of the artifacts introduced by the use of the tree data structure.

1. Introduction

We consider the task of privately estimating the prefix sums of a column vector \( \mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n \) with each \( x_i \in [-\zeta, \zeta] \). That is, we wish to output a noisy estimate \((\mathbf{v}^{\text{priv}})\) of the vector \( S\mathbf{x} = (x_1, x_1 + x_2, \ldots, \sum_{i=1}^{n} x_i)^\top \) satisfying \((\epsilon, \delta)\) differential privacy (DP). We focus on the online version of this problem (Dwork et al., 2010; Chan et al., 2011), where the data samples \( x_i \) arrive in a stream, and one must estimate and release a private estimate of \( S_t = \sum_{i=1}^{t} x_i \) when the \( t \in [n] \)-th data sample arrives. In the sequel we refer to this requirement as the streaming constraint.

In a variety of private learning tasks this streaming-cumulative-sum problem appears naturally, e.g., for online PCA (Dwork et al., 2014), online marginal estimation (Dwork et al., 2010; Chan et al., 2011; Cormode et al., 2019), online top-k selection (Cardoso and Rogers, 2021), and for training ML models (Smith et al., 2017; Kairouz et al., 2021).

We are motivated especially by the problem of training large ML models on private data. In this context, Kairouz et al. (2021) makes the fundamental point that for learning algorithms based on stochastic gradient descent (SGD), the key DP primitive is not independently estimating each gradient privately (as in, e.g., Abadi et al. (2016)), but rather accurately estimating cumulative sums of gradients. An intuitive way to see this is that for SGD with constant learning rate \( \eta \) on parameters \( \theta \) starting from 0, the \( t \)th iterate is simply \( \theta_t = -\eta \sum_{i=1}^{t} g_i \) where \( g_i \) is the gradient computed on

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step $i$. That is, the learned model parameters $\theta_t$ are exactly a scaled version of the cumulative sum of gradients so far; hence, it is the total squared error in these cumulative sums that matters most, not the error in the private estimates of each individual $g_t$. This observation can be made rigorous in the analysis of SGD with noisy gradients, as in Theorem 5.1 of Kairouz et al. (2021), and serves as a primary motivation for our work. The role of the streaming constraint is clear in this setting: since SGD is an iterative algorithm, $\theta_t$ must necessarily depend only on information available up to time $t$.

**Tree-aggregation mechanisms** A natural approach to computing DP prefix sums in the streaming setting was presented concurrently by Dwork et al. (2010) and Chan et al. (2011), and is in fact used in all of the references above: We construct a complete binary tree $T$ from the elements of $x$, with each $x_i$ a leaf of $T$, and the internal nodes representing the sum of all their children leaves. We consider $T$ to be the private object subject to release. Therefore we add appropriately calibrated noise to each node of this tree, and express our estimate of $Sx$ as a linear combination of the values at particular nodes of $T$. The streaming constraint can be reflected in constraining the linear combinations used to compute the estimate for any index to contain only nodes which depend only on smaller indices of the input.

The tree-aggregation method was substantially improved by Honaker (2015), who proposed using different linear combinations of the nodes of $T$ to estimate the vector $Sx$, effectively reducing the variance by leveraging redundant (but independently noised) information in the tree.¹ Honaker (2015) introduced both estimation-from-above and estimation-from-below, with estimation from below respecting our notion of streaming constraints, but expected error improving when estimates incorporate information ‘from above’ in the binary tree.

These tree aggregation approaches all suffer from non-uniform errors. For example, for binary trees the error in $S_t$ can vary by $\Omega(\log n)$ between $t = 2^i$ (when error is relatively low) and $t = 2^i - 1$, as illustrated in Fig. 1 for the Honaker estimator. As a result, any algorithm relying on the tree data structure also must carefully choose $n$. For example, Kairouz et al. (2021) observed that for ML applications, it is advantageous (in terms of prediction accuracy) to train for exactly a power of two steps.

**Our goal** In this work, we are primarily motivated by the following question: Can we achieve practical improvements in privacy/utility tradeoffs for the streaming cumulative sum problem? We answer this question in the affirmative, as demonstrated in Fig. 1: the mechanisms we design improve average errors by a factor of almost 2, while also removing the artifact of highly variable per-step errors observed for tree aggregation. Along the way, we resolve a number of other interesting open questions (see contributions below).

**A matrix factorization view** Our key tool in addressing this question is based on a generalization of the class of mechanisms we consider from tree-based algorithms to arbitrary matrix factorizations. We begin by observing that the transformation $x \mapsto Sx$ can be viewed as a matrix-vector

¹. For example, there is a node representing the sum $x_1 + x_2 + x_3 + x_4$, but we can improve the variance of this estimate by also considering the two nodes representing $x_1 + x_2$ and $x_3 + x_4$ which taken together provide an independent estimate of the same prefix sum.
Figure 1: Variance of each cumulative sum $S_i$ for $t$ from 1 to 4096 for our optimal estimator and the binary tree streaming Honaker estimator-from-below with 4096 leaf nodes at equivalent levels of privacy.

multiplication with

$$S := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{1}$$

Let $S = W_{n \times d} \cdot H_{d \times n}$ be any factorization of the matrix $S$. Then, for suitable noise $z$, we can define a corresponding DP mechanism by

$$v^{\text{priv}} = W(Hx + z) \quad \text{ where } \quad z \sim N(0, \sigma^2 I). \tag{2}$$

If the column $\ell_2$ norms of $H$ are bounded by $\Gamma$, then the overall $\ell_2$-sensitivity of $x \rightarrow Hx$ is upper bounded by $2\zeta \Gamma$. Thus, Eq. (2) satisfies $(\varepsilon, \delta)$-DP if we take $\sigma^2 = \frac{4\Gamma^2 \zeta^2 \log(1/\delta)}{\varepsilon^2}$, where we apply the Gaussian mechanism to $Hx$ and view the final multiplication by $W$ as post-processing.

For ease of presentation we consider $x_i \in \mathbb{R}$, but our approach extends naturally to the case where $x_i \in \mathbb{R}^m$ and we view $x$ as a vector of vectors. This requires applying the high-dimensional Gaussian mechanism to noise the vector-valued quantities $[Hx]_i$ and appropriately computing sensitivity based on e.g. $\|x_i\|_2 \leq \zeta$.

While there are standard algorithms (Li et al., 2015) that can solve for a factorization of $S$ that provides the optimal $\ell_2$-error for this mechanism class via convex relaxations of the problem, they do not provide any method for enforcing streaming constraints and hence do not immediately apply to our setting. Several other aspects of these solutions remain unclear, in particular: whether

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2. we define the $\ell_2$-sensitivity of a query $f$ on the vector (database) by: $\text{sens}(f) = \sup_{(x, \tilde{x}), d_H(x, \tilde{x})=2} \|f(x) - f(\tilde{x})\|_2$, where $d_H(\cdot, \cdot)$ is the Hamming distance between two vectors. This is also called the replace one notion of sensitivity Dwork and Roth (2014).
the optima defined by Li et al. (2015) are unique; how efficiently these optima can be computed; and whether these optima yield privatizing algorithms for which efficient implementations exist in high-dimensional settings.

1.1. Contributions

We present explicit and efficiently computable optimal streaming factorizations of any $S$ via a fixed point algorithm. We emphasize the direct nature of our solutions to this streaming factorization problem by stating a consequence of our main theoretical result, Theorem 3.1:

**Theorem 1.1** Finding a streaming factorization for any lower-triangular $S$ to ensure $(\varepsilon, \delta)$ differential privacy while minimizing expected reconstruction error can be translated to a strictly convex program over compact domain with solution which is unique in a natural sense (see Corollary 3.3). Further, this unique minimum can be expressed in explicit terms as the fixed point of a well-behaved vector-valued mapping, defined in Eq. (12).

We make several important and novel observations in the process of analyzing the problem and deriving the Theorem above.

- The tree-based aggregation mechanisms in the literature can be viewed uniformly and profitably as instantiations of this matrix-factorization framework. Indeed, formulating an appropriate optimization problem for these factorizations allows us to immediately derive the results of Honaker (2015). This mapping effectively takes the problem from the discrete setting to the continuous setting, allowing for use of a new set of tools.

- Tree-based mechanisms require the DP release of $n \log n$ noised values; expressed in the matrix framework this corresponds to $W \in \mathbb{R}^{n \times n \log_2 n}$ and $H \in \mathbb{R}^{n \log_2 n \times n}$. However, consideration of square (and real) factorizations $W \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{R}^{n \times n}$ is actually sufficient to achieve optimal reconstruction error for real matrices $S$. (Theorem 2.1).

- The presence of streaming constraints does not affect the optimal reconstruction error achievable by the matrix mechanism, assuming the matrix $S$ itself can be computed in a streaming manner.

- The optimal square factorization of any lower-triangular $S$ under streaming constraints is unique in a natural sense, and can be computed by solving an explicit fixed-point problem with unique solution. The equivalence between these two problems in fact yields nontrivial statements in linear algebra and real analysis which may be of independent interest; see Corollary 3.1 and Corollary 3.2.

- The solutions for prefix-sum $S$ bypass the artifacts of the binary tree data structure mentioned above.

- These solutions yield an improvement of approximately a factor of two over existing methods in terms of expected $\ell_2$-norm reconstruction error (see Table 1); this can be taken as an equivalent reduction of $\varepsilon$ by a factor of two for a fixed value of expected reconstruction error. In the small-$\varepsilon$ regime, often required for training a machine learning model with strong privacy guarantees, this factor of two can have a significant impact. While the optimal factorizations
we compute may not be directly amenable to efficiency at runtime, we show (Section 4.2) that structured approximations yield mechanisms with efficiency equal to that of tree aggregation while preserving essentially all of the utility improvement.

- We provide a lower bound of $\Omega \left( \frac{\log n \sqrt{\log(1/\delta)}}{\varepsilon} \right)$ on the $\ell_2$-error achievable by any matrix-factorization based DP algorithm for prefix-sum computation; though the argument is not ours, we include it in Appendix C, in particular in Proposition C.1 and Proposition C.2.

Overall, we contribute significant progress on a core primitive of applied differential privacy. We highlight that though this problem occupies a prominent position in modern applications of differential privacy, the state of the art in its core operation has seen very little improvement for nearly a decade.

1.2. Preliminaries: Privacy Notion and Notations

(\varepsilon,\delta)-differential privacy We use (\varepsilon,\delta)-differential privacy in the replacement model. Differential Privacy (DP) (Dwork et al., 2006b,a) is a formal method for quantifying the privacy leakage from the output of a data analysis procedure. A randomized algorithm $M : \mathcal{D}^* \to \mathcal{Y}$ is $(\varepsilon,\delta)$-DP if, for all neighbouring dataset pairs $D, D' \in \mathcal{D}^*$ and all measurable sets of outputs $S \subseteq \mathcal{Y}$, we have

$$\Pr[M(D) \in S] \leq e^\varepsilon \cdot \Pr[M(D') \in S] + \delta.$$  

We define two data sets to be neighbouring one can be obtained from the other by replacing a single row in the data set. In our context, vectors $x$ and $y$ are neighboring if they differ in at most one element, by magnitude at most one.

It is worth mentioning that our algorithms are primarily based on Gaussian mechanism Dwork et al. (2006a), hence our privacy guarantees seamlessly extend to other notions of privacy like zCDP Bun and Steinke (2016), and RDP Mironov (2017).

Notation We will consider $S$ in the sequel to represent a square and full-rank matrix, and let $n$ represent its number of rows and columns, which we consider fixed for the purposes of the paper. We will at times specialize to the prefix-sum $S$ defined in Eq. (1), but we will always do so explicitly.

We will always interpret ‘positive definite’ and ‘positive semi-definite’ to also imply symmetry of the associated matrix. We will use ‘PSD’ to refer to the space of positive semidefinite symmetric matrices, and ‘PD’ to refer to the space of positive definite symmetric matrices. Bold lowercase letters (e.g. $x, y$) will represent vectors, while bold uppercase letters ($A, B$) will be matrices. We use $X^\dagger$ to denote the Moore-Penrose pseudoinverse of a matrix $X$, and $X^*$ denotes the conjugate transpose. The $t$th entry in a vector $x$ is denoted $[x]_t$, and we index sub-matrices of a matrix $M$ using Python numpy-like syntax, e.g., $M[i,:]$ denotes the $i$th row of $M$.

Organization We begin by diving deeper into the matrix mechanism for streaming operators in Section 2. We close this section by proving that consideration of square, streaming factorizations is sufficient to achieve optimal loss. In Section 3, we abstract away from the particular instantiation of prefix-sum $S$, and prove an equivalence between optimal factorizations and solutions of an explicit fixed-point problem. From this equivalence, we derive interesting corollaries, both for the optimization problem (e.g. uniqueness in a natural sense, solutions for real $S$ may be taken to be real), and for the fixed point problem and associated representations. We investigate the numerical
consequences of this optimal factorization in Section 4, providing approximation strategies which can compute factorizations with efficient implementations from our optima while retaining much of the improvement in expected reconstruction error. Finally, in Section 5 we sketch the questions raised by this work, and the directions we plan to pursue.

2. The matrix mechanism for streaming linear operators

We focus on providing differential privacy by the matrix mechanism of Eq. (2), as described in Li et al. (2015). We can explicitly compute the expected mean-square error on the privatized vector given a noise distribution $\mathcal{D}$ as:

$$L(W, H; \mathcal{D}) = \mathbb{E}_{z \sim \mathcal{D}} \left[ \| W (Hx + z) - Sx \|_2^2 \right] = \mathbb{E}_{z \sim \mathcal{D}} \left[ \| Wz \|_2^2 \right].$$

In particular, to achieve a particular DP guarantee, $\mathcal{D}$ needs to be appropriately scaled to the sensitivity of $Hx$. Reasoning similarly to Section 5.1 of Li et al. (2015), we may compute the expected squared reconstruction error of our privatized factorization Eq. (3) for a given level of privacy when $\mathcal{D}$ is chosen to be Gaussian. For completeness, we include details of this calculation in Appendix A.1.

$$L(W, H) = \left( \max_{i \in [1, \ldots, n]} \| H_{[,i]} \|_2 \right) \| W \|_F^2$$

Since the Moore-Penrose pseudoinverse yields the minimal $\ell^2$-norm solution to a set of underdetermined linear equations (see Theorem 2.1.1 of Campbell and Meyer (1979)), we have

$$W_H^* = SH^\dagger$$

where $W_H^*$ is the optimal $W$ for a fixed $H$.

2.1. Tree aggregation and decoding as matrix factorization

The tree data structure $T$ described in Section 1 is linear in the vector $x$; all of its internal nodes are linear combinations of the entries of $x$. Therefore the mapping $x \rightarrow T$ can be represented as multiplication by a matrix.

One representation of this matrix is particularly simple, and can be recursively described. This construction begins with the $1 \times 1$ matrix $[1]$, which we will denote by $M_1$; we will define $M_k$ to be the matrix constructed by duplicating $M_{k-1}$ on the diagonal, and adding one more row of constant 1s. That is,

$$M_1 := (1), M_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_4 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and so on. Each entry of the vector $M_k x$ can be seen readily to correspond to a node of the binary tree $T$ constructed from $x$, assuming $x$ has $2^{k-1}$ entries.
With this construction, it is straightforward to represent both vanilla differentially-private binary tree aggregation and the Honaker variant as instantiations of our matrix factorization framework. For a vector \( x \) with \( n = 2^k - 1 \) entries, vanilla binary-tree aggregation can be represented as \( H = M_k \), \( W \) an appropriate \( \{0, 1\} \)-valued matrix satisfying \( WH = S \) for prefix-sum \( S \). The Honaker estimators can both be computed as (real-valued) matrices also satisfying \( WH = S \), and are in fact optimal, as we show (proof deferred to Appendix A):

**Proposition 2.1** For a vector \( x \) with \( n = 2^k - 1 \) entries, the (non-streaming) Honaker fully efficient estimator represents the minimal-loss factorization for prefix sum \( S = WH \), for \( H = M_k \). Remarkably, this estimator is precisely \( SH^\dagger \). The streaming Honaker estimator-from-below represents the minimal loss under streaming constraints, and can be expressed similarly row-by-row with a constrained pseudoinverse of \( H \).

### 2.2. Incorporating streaming constraints

As discussed in Section 1, leveraging differentially-private aggregation in an iterative algorithm like SGD critically requires the ability to compute the aggregation in an online fashion. This requirement introduces structural constraints on the matrices \( W \) and \( H \) which represent valid factorizations of \( S \) for this streaming setting. We will see that the streaming constraints can be captured quite simply by restriction to a class of matrices; in particular, it is sufficient if \( WH = S \) and \( W \) and \( H \) are both square, lower-triangular matrices. To see this, we can imagine that we release and noise \( [Hx]_t \) on step \( t \) (as the lower-triangular property implies this only depends on \( x_1, \ldots, x_t \)), and then reconstruct an estimate of \( S_t \) based on \( W_{[t:]} \) (as the lower-triangular property implies this only depends on the values released up through step \( t \)).

We may remove the explicit dependence of Eq. (4) on its \( H \)-term by considering the optimization problem to be constrained. That is, since \( S = HW \) implies \( S = \alpha H^\dagger \alpha W \), for any linear space of matrices \( V \),

\[
\min_{H \in V} \mathcal{L} \left( SH^\dagger, H \right) = \min_{H \in V} \left( \max_{i \in [1, \ldots, n]} \left\| H_{[i,:]} \right\|_2^2 \right) \left\| SH^\dagger \right\|_F^2 = \min_{H \in V, \max_{i \in [1, \ldots, n]} \left\| H_{[i,:]} \right\|_2^{-1}} \left\| SH^\dagger \right\|_F^2 .
\] (7)

Define the matrix-valued mapping

\[
\psi(X) := P \text{CholeskyFactor}(PX^*P^*)^*P^*,
\] (8)

where \( \text{CholeskyFactor} \) represents the action of taking a Cholesky factorization (returning a lower-triangular matrix \( L \) such that \( LL^* = A \)), and \( P \) is a matrix with ones on the antidiagonal and zeros elsewhere. Note that \( P \) is an involution, \( P^2 = I \).

**Theorem 2.1** Let \( S \) be full-rank and square, and let \( S = WH \) be any factorization. Define

\[
H_{lt} := \psi \left( H^*H \right).
\]

Then:
• $H_l$ is the unique square, positive semi-definite, lower-triangular matrix such that $H_l^*H_l = H^*H$.

• The column norms of $H_l$ are exactly those of $H$.

• The factorization pair $(SH_l^*, H_l)$ achieves at most the same loss as $(W, H)$, for loss defined as in Eq. (4).

The proof of this theorem consists of a relatively straightforward verification of the claimed properties of $\psi$, and can be found in Appendix A.3.

Remark. We emphasize that Theorem 2.1 yields a somewhat surprising result: while using the matrix mechanism to make differentially private a linear operator $S$, we lose nothing by considering square factorizations. If $S$ is additionally lower-triangular, we lose nothing by considering each term $W$ and $H$ of this factorization to be lower triangular as well; therefore we lose nothing by restricting ourselves to streaming factorizations. Indeed, early on in the research presented in this work, we conjectured both of these facts to be false.

Remark. Theorem 2.1 makes clear the role played by the matrix $P$. Our desired factorization of $H^*H$ must have an upper-triangular matrix on the left of the product, whereas a Cholesky factorization yields a lower-triangular matrix in this position.

3. Computing optimal $W, H$ pairs

In this section, we abstract away from our motivating prefix-sum $S$, deriving expressions, algorithms and consequences for optimal matrix factorizations of any square, full-rank matrix $S$.

Let $V$ be the space of $n \times n$ lower-triangular matrices with column norms at most 1. Following Eq. (7), we are interested in solving

$$H^* = \arg \min_{H \in V} \left\| SH \right\|_F^2 = \arg \min_{H \in V} \text{tr} \left( (SH^{-1})^* SH^{-1} \right) = \arg \min_{H \in V} \text{tr} \left( S^* S (H^* H)^{-1} \right).$$ (9)

Let $X := H^* H$. The constraint that $H$ has column norms at most 1 translates to the constraint that $X$ has diagonal entries at most 1. By definition $X$ must be symmetric and positive definite. Theorem 2.1 shows that the lower-triangular restriction on $H$ imposes no further constraints on $X$. Therefore, the problem Eq. (9) is equivalent to:

$$X^* = \arg \min_{X \text{ is PD}, X_{[i,i]} \leq 1, 1 \leq i \leq n} \text{tr}(S^* SX^{-1})$$ (10)

and given a solution of this problem, Theorem 2.1 immediately tells us how to recover our streaming solution; it is $\psi(X^*)$.

The problem Eq. (10) has several structural properties which make it advantageous to study in comparison to Eq. (9). We collect these properties in the following Lemmas, whose proofs we delay to Appendix B.

Lemma 3.1 There exists $\varepsilon > 0$ depending on $S$ such that any solution $X^*$ of Eq. (10) has all eigenvalues at least $\varepsilon$; moreover, we may consider the optimization problem of Eq. (10) to be constrained to a compact, convex set without losing any optima.
Remark. Lemma 3.1 shows, since we are considering a loss function which is continuous over this compact set, that at least one true minimizer of our loss exists, and that this minimizer must be an element of our compact, convex set.

**Lemma 3.2** For any full-rank \( S \), \( X \mapsto \text{tr}(S^*SX^{-1}) \) is strictly convex over symmetric, positive-definite matrices.

Remark. Lemma 3.2 shows that our minimizer of Eq. (10) is in fact unique.

We may leverage the relatively straightforward computational nature of the problem Eq. (10) to derive explicit formulae for its optimum. The following two Lemmas lay the groundwork for these formulae.

**Lemma 3.3** The problem Eq. (10) is equivalent to the problem:

\[
X^\star = \arg\min_{X \text{ is PD}, X_{[i,i]} = 1, 1 \leq i \leq n} \text{tr}(S^*SX^{-1})
\]

That is, any solution of Eq. (10) satisfies \( X_{[i,i]} = 1 \) for \( 1 \leq i \leq n \).

**Lemma 3.4** Minimizers \( X^\star \) of Eq. (10) are in one-to-one correspondence with fixed points of the function

\[
\phi : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+
\]

\[
\phi(v) = \text{diagpart} \left( \sqrt{\text{diag}(v)^{1/2} S^*S \text{diag}(v)^{1/2}} \right).
\]

Moreover, for such a fixed point \( \lambda \), the associated minimizer \( X^\star \) may be written as

\[
X^\star = \text{diag}(\lambda)^{-1/2} \left( \text{diag}(\lambda)^{1/2} S^*S \text{diag}(\lambda)^{1/2} \right)^{1/2} \text{diag}(\lambda)^{-1/2},
\]

and therefore this pair \( (X^\star, \lambda) \) satisfies the following equation:

\[
S^*S = X^\star \text{diag}(\lambda) X^\star.
\]

Collectively, Lemma 3.2 - Lemma 3.4 yield our main result.

**Theorem 3.1** The optimization problem

\[
X^\star = \arg\min_{X \text{ is PD}, X_{[i,i]} \leq 1, 1 \leq i \leq n} \text{tr}(S^*SX^{-1})
\]

can be solved by solving a strictly convex problem over a compact, convex set, for which a unique minimizer exists. Further, this solution can be expressed as Eq. (13) for a unique setting of \( \lambda \in \mathbb{R}^n \), the unique fixed point of \( \phi \) defined by Eq. (12).

**Proof**

By Lemma 3.3, we may pass to the equality-constrained version of Eq. (15) immediately, without losing any optima. Lemma 3.1 allows us to pass to a compact, convex set similarly. Finally,
Lemma 3.2 tells us that our objective function is strictly convex (in fact strongly convex) on this compact set. Therefore the solution to Eq. (15) is unique.

The claimed expressions for this solution, and the uniqueness of the fixed point of φ, then follow from Lemma 3.4.

Remark. This Theorem bears some relationship to Theorem 4 of Li et al. (2015): both translate the problem of finding factorizations with minimal reconstruction error into convex problems. Theorem 3.1 differs from the Theorem of Li et al. (2015) in two respects. The first is uniqueness; Theorem 4 of Li et al. (2015) formulates our minimization problem as a semidefinite program, which does not in general imply uniqueness of solution (Zhu, 2010). The second is the explicit nature of our solutions, e.g. the formula Eq. (13). Finally, the two lines of argumentation are of quite different flavors; ours proceeds through a direct application of matrix calculus, and this is the source of our explicit expressions.

Remark. Existence of a fixed point for φ can be derived explicitly from the definition of φ via Brouwer’s fixed point theorem by examining the action of φ on the trace of the intermediate matrices, via an argument first due to Sergey Denisov (personal communication). Uniqueness of this fixed point does not seem to follow as easily. Further, we conjecture, analogously to the mapping $X \to \sqrt{X}$, that iterations of φ converge exponentially quickly to this unique fixed point; indeed, numerically this is true. We note that given such convergence, we obtain a parameter-free algorithm for computing the optimal factorizations of any S, and it is this algorithm we use to compute the numerical results in Section 4.

Several interesting results follow immediately by combining this Theorem with preceding observations.

Corollary 3.1 There is a unique setting of $\lambda$ which satisfies Eq. (14) for $X$ with all 1s on the diagonal, symmetric and positive definite. This $\lambda$ uniquely determines $X$.

Proof This corollary is the direct result of combining the uniqueness of Theorem 3.1 with the mapping between optimizers and fixed points of φ described by Lemma 3.4.

Remark. We state this Corollary separately to emphasize that the uniqueness of this factorization is a nontrivial linear-algebraic fact; naïvely, Eq. (21) defines $n$ nonlinear equations in $n$ unknowns. Indeed, it is easy to see that removing the requirement of positive-definiteness on $X$ yields multiple solutions; $-X$ represents a solution to Eq. (21) if $X$ is a solution.

Corollary 3.2 $\phi$ as defined in Eq. (12) has a unique fixed point.

Proof This corollary follows by precisely the same reasoning as Corollary 3.1.

Corollary 3.3 Constraining $H$ to be streaming achieves the same loss value as unconstrained $H$. The square, positive semi-definite, lower-triangular minimum to the problem Eq. (9) is unique.

Proof The first part of this Corollary was already noted as a consequence of Theorem 2.1, in a remark following that Theorem’s proof. The uniqueness of the square, PSD, lower-triangular minimum to Eq. (9) follows directly from the uniqueness stated in Theorem 3.1 and Theorem 2.1.
Table 1: Values of $\sqrt{\mathcal{L}}$ for the expected squared reconstruction error $\mathcal{L}$ defined in Eq. (4) (which implies equivalent levels of privacy). The “Efficient” column gives $\sqrt{\mathcal{L}}$ for the structured approximation $\hat{W}$ of $W^*$ with parameters $(d, r)$ described in Section 4.2. When $n = 2^i$ we choose $d + r = i$, so that the mechanism based on $\hat{W}$ has memory and computation efficiency comparable to the Honaker approach.

| $n$   | Honaker | $(W^*, H^*)$ | Efficient | $(d, r)$ |
|-------|---------|-------------|-----------|----------|
| $2^8 = 256$ | 74.4 | 40.4 | 40.4 | (4, 4) |
| $2^9 = 512$ | 116.5 | 62.0 | 62.2 | (5, 4) |
| $2^{10} = 1024$ | 180.8 | 94.6 | 95.5 | (5, 5) |
| $2^{11} = 2048$ | 278.3 | 143.6 | 145.8 | (6, 5) |
| $2^{12} = 4096$ | 425.6 | 217.3 | 224.0 | (6, 6) |

4. Optimal factorizations for prefix sums

In this section we investigate the consequences of these results for the streaming private prefix-sum problem which has served as our motivation, specializing to prefix sum $S$.

4.1. Comparison to tree aggregation

Recall that our Proposition 2.1 shows that the Honaker estimator-from-below represents the optimal streaming $W$ matrix for binary-tree $H$. Our factorizations improve further upon the expected reconstruction error for this optimum, and remove the tree-based artifacts discussed in Section 1.

We may immediately compare values of the loss defined in Eq. (4) for differentially private prefix sum’s previous state of the art and our optima. These loss values can be found in Table 1, with the $(W^*, H^*)$ column representing the optimum found by computing an approximate fixed-point to $\phi$ (Eq. (12)) by simply iterating the mapping from a random initial vector.  

The relationship between the binary expansion of an index and the variance around its estimate is gone for these optimal solutions, which distribute the variance load much more evenly across observations, as shown in Fig. 1. We expect this property to yield significantly better performance for training a machine learning model with differential privacy.

4.2. Computational efficiency

Our primary goal has been to develop mechanisms with best-possible privacy vs utility tradeoffs in the streaming setting. When $x_i$ is scalar (as in our presentation), a $(W^*, H^*)$ factorization immedi-

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3. In particular, we iterated $\phi$ until a relative norm tolerance of $1e-5$ was reached, that is, until $\frac{\|\phi(x) - x\|}{\|x\|} < 10^{-5}$. 

---
\textbf{Eq. (2)). For these purposes an optimal } \textbf{H} \textbf{defined in Eq. (3), as well as to appropriately calibrate the noise to achieve a DP guarantee (see Eq. (3)).} Finally, \textbf{can be updated in time } \beta \textbf{accumulators} \textbf{2013). Given such a representation, the cost of computing \textbf{applications we would like to think of } x_i \textbf{as a high-dimensional vector, i.e., } x_i \in \mathbb{R}^m \textbf{for some large } m. \textbf{In particular, in the context of machine learning with the DP-FTRL algorithm (Kairouz et al., 2021), } x_i \textbf{might represent the } i\text{th gradient update to a deep network, with } m \textbf{potentially } 10^6 \textbf{or more. As observed in Section 2, our mechanism extends to this scenario. However, the } (\textbf{W}, \textbf{H}) \textbf{we compute via the results of Section 3 are dense, and do not obviously admit a computationally-efficient implementation of the associated DP mechanism. In contrast, tree aggregation (including, with a careful implementation, the streaming Honaker estimator) allows implementations with only } \log(n) \textbf{overhead; that is, each DP estimate of the partial sum } S_i \textbf{can be computed in time and space } O(m \log(n)). \textbf{In this section, we demonstrate empirically that the optimal } (\textbf{W}^*, \textbf{H}^*) \textbf{can be approximated by structured matrices in such a way as to be competitive with the tree-aggregation approach in terms of computation and memory, but retain the advantage of substantially improved utility. Recalling } \textbf{Eq. (2), we wish to compute } \nu_t^{\text{priv}} \textbf{efficiently for each step } t. \textbf{We observe immediately}

\[
\nu_t^{\text{priv}} = [\textbf{W}(\textbf{H}x + \textbf{z})]_t = S_t + [\textbf{W}z]_t.
\]

\textbf{Hence, the trusted curator need not actually use the matrix } \textbf{H} \textbf{in the implementation of the mechanism at all; instead, on each round } t \textbf{it is only necessary to compute the total noise vector added to } S_t \textbf{, namely } y := [\textbf{W}z]_t = \textbf{W}_{[t,:]} \cdot \textbf{z}. \textbf{If } s_t \in \mathbb{R}^m \textbf{and } \textbf{W} \textbf{is dense, naively this takes } O(mn) \textbf{operations, which is likely prohibitive.} \textbf{However, having a structured matrix } \textbf{W} \textbf{that allows efficient multiplication with } \textbf{z} \textbf{mitigates this problem. We propose the following construction, which empirically provides a good approximation while also allowing computational efficiency. Let } \textbf{D}^{(d)} \textbf{denote the lower-triangular banded matrix formed by taking the first } d \textbf{ diagonals of } \textbf{W}, \textbf{so } \textbf{D}^{(0)} \textbf{is the all-zero matrix, } \textbf{D}^{(1)} \textbf{is main diagonal of } \textbf{W}, \textbf{and } \textbf{D}^{(2)} \textbf{contains the main diagonal and one below it, etc. Let } \textbf{M}^{(d)} \in \{0, 1\}^{m \times n} \textbf{ contain a 1 in the place of each non-zero element of } \textbf{W} \textbf{ not captured in } \textbf{D}^{(d)} \textbf{ and zero elsewhere, so in particular } \textbf{W} = \textbf{W} \odot \textbf{M}^{(d)} + \textbf{D}^{(d)} \textbf{ where } \odot \textbf{ is elementwise multiplication. Then, we propose the representation}

\[
\hat{\textbf{W}} = (\textbf{AB}^\top) \odot \textbf{M}^{(d)} + \textbf{D}^{(d)},
\]

\textbf{where } \textbf{A}, \textbf{B} \in \mathbb{R}^{n \times r}. \textbf{Finding a low-rank factorization } \textbf{AB}^\top \textbf{ which minimizes } \|\hat{\textbf{W}} - \textbf{W}\|_F^2 \textbf{ can be cast as a matrix completion problem, as we only care about approximating with } \textbf{AB}^\top \textbf{ the entries of } \textbf{W} \textbf{ selected by } \textbf{M}^{(d)}. \textbf{For these experiments we used an alternating least squares solver with a regularization penalty of } 10^{-6} \textbf{ on } \|\textbf{A}\|_F^2 + \|\textbf{B}\|_F^2. (Srebro et al., 2005; Koren et al., 2009; Jain et al., 2013). \textbf{Given such a representation, the cost of computing } [\textbf{W}z]_t \textbf{ is } O((d + k)n): \textbf{we maintain accumulators } \beta \textbf{ which on round } t \textbf{ contain } \beta = \textbf{B}^\top \textbf{z}^t_1 \textbf{ where } \textbf{z}^t_1 = (z_1, \ldots, z_t, 0, \ldots, 0)^\top \in \mathbb{R}^n; \textbf{β can be updated in time } kmn \textbf{ on each step. Then, } [\textbf{A}\beta]_t \textbf{ can similarly be computed in time } mk. \textbf{Finally, } \textbf{D}^{(d)}\textbf{z} \textbf{can be computed in time } dm. \textbf{Details are given via pseudo-code in Appendix D.}}

\textbf{Columns 3 and 4 in Table 1 shows empirically that this approximation recovers almost all of the accuracy improvement of } (\textbf{W}^*, \textbf{H}^*) \textbf{ at comparable computational efficiency to tree aggregation with the Honaker estimator (that is, we choose } d + k = \log_2(n)). \textbf{While a paired } \textbf{H} \textbf{is not used directly in computing the private estimates of } S_t \textbf{, it is necessary in order to compute the loss } \mathcal{L} \textbf{ defined in Eq. (3), as well as to appropriately calibrate the noise to achieve a DP guarantee (see Eq. (2)). For these purposes an optimal } \textbf{H}_\textbf{W} \textbf{ can be found analogous to Eq. (5) as } \textbf{H}_\textbf{W} = \textbf{W}^\top \textbf{S}. \textbf{12}
5. Future work and open problems.

The present work presents an effectively complete solution to the problem of computing optimal factorizations of linear operators for the purpose of $(\varepsilon, \delta)$ differential privacy in the streaming setting. Still, important questions remain open.

One relatively inflexible aspect of the development presented here is the necessity of knowing in advance the number of steps (in our formalism $n$, the dimensionality of the vector $x$). This knowledge does not appear to be superfluous. Designing an adaptive algorithm for which the number of observations need not be known in advance, and which is optimal or near-optimal, has clear advantages. One might begin concretely by asking: do our optimal matrices $(W^*, H^*)$ approach a limit in any natural sense as $n \to \infty$?

Finally, from the purely mathematical perspective, though Corollary 3.1 implies that there is a unique fixed point to the mapping $\phi$ defined above (already a nontrivial fact) and empirically iterates of $\phi$ converge exponentially quickly to this fixed point, we have no proof of this statement or direct expression for the fixed point. An explicit expression for the fixed point of $\phi$ in terms of $S$ would be ideal; however, it is not clear that such an explicit expression must exist in general. An analytic expression for the fixed point of $\phi$ specialized to prefix-sum $S$ would already be an interesting result in our context. This seems to be a nontrivial problem in analysis, whose solution would in some sense completely close the space of problems pursued in the present work.

Acknowledgments

We thank Zachary Charles and Thomas Steinke for their valuable feedback and insights. In particular Thomas pointed us to the Speyer’s argument of the lower bound, and Zach discussed several of the technical arguments with the authors.

References

Martin Abadi, Andy Chu, Ian Goodfellow, H. Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security, Oct 2016. doi: 10.1145/2976749.2978318. URL http://dx.doi.org/10.1145/2976749.2978318.

Mark Bun and Thomas Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds. In Theory of Cryptography Conference, pages 635–658. Springer, 2016.

S. L. Campbell and C. D. Meyer. Generalized inverses of linear transformations / S. L. Campbell, C. D. Meyer. Pitman London ; San Francisco, 1979. ISBN 0273084224.

Adrian Rivera Cardoso and Ryan Rogers. Differentially private histograms under continual observation: Streaming selection into the unknown. CoRR, abs/2103.16787, 2021. URL https://arxiv.org/abs/2103.16787.

T.-H. Hubert Chan, Elaine Shi, and Dawn Song. Private and continual release of statistics. ACM Trans. on Information Systems Security, 14(3):26:1–26:24, November 2011.

Graham Cormode, Tejas Kulkarni, and Divesh Srivastava. Answering range queries under local differential privacy. Proceedings of the VLDB Endowment, 12(10):1126–1138, 2019.
Cynthia Dwork and Aaron Roth. The algorithmic foundations of differential privacy. *Foundations and Trends in Theoretical Computer Science*, 9(3–4):211–407, 2014.

Cynthia Dwork, Krishnaram Kenthapadi, Frank McSherry, Ilya Mironov, and Moni Naor. Our data, ourselves: Privacy via distributed noise generation. In *Advances in Cryptology—EUROCRYPT*, pages 486–503, 2006a.

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In *Proc. of the Third Conf. on Theory of Cryptography (TCC)*, pages 265–284, 2006b. URL http://dx.doi.org/10.1007/11681878_14.

Cynthia Dwork, Moni Naor, Toniann Pitassi, and Guy N. Rothblum. Differential privacy under continual observation. In *Proc. of the Forty-Second ACM Symp. on Theory of Computing (STOC’10)*, pages 715–724, 2010.

Cynthia Dwork, Kunal Talwar, Abhradeep Thakurta, and Li Zhang. Analyze gauss: optimal bounds for privacy-preserving principal component analysis. In *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*, pages 11–20, 2014.

James Honaker. Efficient use of differentially private binary trees. *Theory and Practice of Differential Privacy (TPDP 2015)*, London, UK, 2015.

Prateek Jain, Praneeth Netrapalli, and Sujay Sanghavi. Low-rank matrix completion using alternating minimization. In *Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing (STOC)*. Association for Computing Machinery, 2013. ISBN 9781450320290.

Peter Kairouz, Brendan McMahan, Shuang Song, Om Thakkar, Abhradeep Thakurta, and Zheng Xu. Practical and private (deep) learning without sampling or shuffling. In *ICML*, 2021.

Yehuda Koren, Robert Bell, and Chris Volinsky. Matrix factorization techniques for recommender systems. *Computer*, 42(8), 2009. ISSN 0018-9162. doi: 10.1109/MC.2009.263. URL https://doi.org/10.1109/MC.2009.263.

H. W. Kuhn and A. W. Tucker. Nonlinear programming. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, pages 481–492, Berkeley and Los Angeles, 1951. University of California Press.
Appendix A. Supplementary material for Section 2

A.1. Derivation of Eq. (4)

Let the random vector $v$ in $\mathbb{R}^3$ be distributed according to $\mathcal{N}(0, \Gamma)$, where

$$
\Gamma := \text{sens}(x \mapsto Hx) = \sup_{(x, \tilde{x}) \in \mathcal{N}(\mathbb{R}^n)} \|Hx - H\tilde{x}\|_2 = \sup_{(x, \tilde{x}) \in \mathcal{N}(\mathbb{R}^n)} \|H(x - \tilde{x})\|_2
$$

for $\mathcal{N}(\mathbb{R}^n)$ denoting the neighboring relation defined in Section 1.2. Any pair of vectors in the supremum above differ in at most one entry, and this entry differs by magnitude at most 1; therefore for any $x, \tilde{x}$ in the supremum above,

$$
H(x - \tilde{x}) = \alpha \hat{H}_{[i, \cdot]}
$$

for some $i$, where $|\alpha| \leq 1$ and $\hat{H}_{[i, \cdot]}$ represents the $i^{th}$ column of the matrix representation of $H$. Therefore

$$
\Gamma \leq \max_{i \in [1, \ldots, n]} \| \hat{H}_{[i, \cdot]} \|_2.
$$
That this maximum is achieved by the supremum defining \( \Gamma \) is seen quite simply by noting that any \((x, \tilde{x})\) such that \(x - \tilde{x} = e_i\) (the \(i^{th}\) standard basis vector for \(\mathbb{R}^n\)) is an element of \(N(\mathbb{R}^n)\). So

\[
\Gamma = \max_{i \in [1,\ldots,n]} \|H(:,i)\|_2,
\]

the max of the \(\ell^2\)-column norms of \(H\).

We proceed to directly compute \(\mathbb{E} \left[ \|Wv\|_2^2 \right] \). Recall that for \(v \sim \mathcal{N}(0, \Gamma I)\), \(b \sim (v_1, \ldots, v_n)^T\), where each \(v_i\) is an independent zero-mean real-valued normal random variable of standard deviation \(\Gamma\), so

\[
Wv = (\langle W_{[1,:]}, v \rangle, \ldots, \langle W_{[n,:]}, v \rangle)^T
\]

where similarly to the above, \(W_{[i,:]}\) represents the \(i^{th}\) row of \(W\).

Therefore

\[
\mathbb{E} \left[ \|Wv\|_2^2 \right] = \mathbb{E} \left[ \sum_{i=1}^n \left| \langle W_{[i,:]}, v \rangle \right|^2 \right] = \mathbb{E} \left[ \sum_{i=1}^n \sum_{j=1}^n W_{ij} v_j \right] = \Gamma^2 \sum_{i=1}^n \sum_{j=1}^n |W_{ij}|^2
\]

by noting that \(\mathbb{E} \left[ v_i v_j \right] = c^2 \delta_{ij}\).

We recognize the double-sum on the right-hand side as the squared Frobenius norm of the matrix \(W\). Therefore

\[
\mathcal{L}(W, H) = \left( \max_{i \in [1,\ldots,n]} \|H(:,i)\|_2^2 \right) \|W\|_F^2.
\]

### A.2. Proof of Proposition 2.1

**Proof**

We begin by recalling a geometric property of the Moore-Penrose pseudoinverse. Theorem 2.1.1 of Campbell and Meyer (1979) states that for any matrix \(A \in \mathbb{C}^{m \times n}\), vector \(b \in \mathbb{C}^m\), the vector \(A^\dagger b\) is the minimal least-squares solution to \(Ax = b\). Notice that this statement is implicitly a statement of uniqueness; \(A^\dagger b\) is the unique minimal-norm solution to \(Ax = b\), assuming feasibility of this equation.

Since the square of the Frobenius norm of the matrix \(W\) is the sum of the squared norms of its rows, we may apply this Theorem row-by-row to \(W\) to demonstrate that the minimal Frobenius norm solution \(W\) to \(S = WH\) is \(SH^\dagger\).

Given the equality between Eq. (3), Eq. (4), we may translate this property of the Moore-Penrose pseudoinverse to a statistical perspective. That is, for a fixed matrix \(H\), \(SH^\dagger\) represents the minimal-variance unbiased linear estimator for \(Sx\) given the noisy estimates \(Hx + b\). This is precisely the definition of Honaker’s fully efficient estimator in Section 3.4 of Honaker (2015), and so shows the first statement of this proposition.

The second follows similarly, but leveraging instead the geometric properties of the constrained pseudoinverse. These properties are collected in Theorem 3.6.3 of Campbell and Meyer (1979), and allow us to compute directly the optimal \(W\) under constraints that certain entries in each row must be 0, corresponding to capacity to compute our factorization in a streaming manner.
By construction of the matrices $M_k$, the streaming constraints correspond to restricting the linear estimator computed from a binary tree to depend only on the information below the nodes corresponding to the 1s in a binary expansion of the index of the partial sum under consideration. This is precisely the definition of the estimator from below in Section 3.2 of Honaker (2015).

A.3. Proof of Theorem 2.1

Proof

Let $\tilde{H}$ be an element of some factorization pair $\left(\widetilde{W}, \tilde{H}\right)$. Let $X := \tilde{H}^*\tilde{H}$, and note that $X$ must be invertible since $S = \tilde{W}\tilde{H}$ and $S$ is full-rank. Further, $X$ is clearly positive semi-definite by definition, and by invertibility then is positive definite.

Let $L = \text{CholeskyFactor}(PXP^*)$, so that $H_r = PL^*P$ and $LL^* = PXP^*$. Recalling the involution property of $P$, we see

$$\tilde{H}^*\tilde{H} = (PL^*P^*)(PL^*P^*) = PLP^*PL^*P^* = P(PXP^*)P^* = P^2XP^2* = X.\
$$

That $H_r$ is lower triangular can be seen directly. Multiplying a matrix by $P$ on the right inverts the order of the columns; on the left inverts the order of the rows. Since $\text{CholeskyFactor}(PXP^*)^*$ is upper-triangular, the result of these two inversions is lower-triangular. Uniqueness of this $H_r$ follows from uniqueness of the Cholesky decomposition for positive definite matrices (see e.g. Corollary 7.2.9 of Horn and Johnson (1990)), since a lower-triangular matrix factorizing $H^*H$ is mapped one-to-one to a Cholesky factorization by Eq. (8). This shows the first claim of the Theorem.

Since the column norms of both $H_r$ and $H$ are precisely the elements on the diagonal of $X$, the column norms of $H_r$ are exactly those of $H$. Therefore we have the second claim of the Theorem.

By Eq. (4) and recalling that $\mathcal{L}(W, H) \geq \mathcal{L}(SH^*, H)$, we see the third claim will follow from $\mathcal{L}(SH^*, H) = \mathcal{L}(SH_r^*, H_r)$. But this follows immediately from permuting the trace, in combination with the Theorem’s previous two claims:

$$\|SH^*\|_F^2 = \text{tr} \left( (SH^*)^*SH^* \right) = \text{tr} \left( \tilde{H}^*S^*SH^* \right) = \text{tr} \left( S^*SH^*H^* \right) = \text{tr} \left( S^*S \left( \tilde{H}^*H \right)^* \right).$$

(16)

Appendix B. Proofs of Lemmas in Section 3.

We begin with a standalone proposition of matrix calculus, which will be used in the sequel.

Proposition B.1 Let $B$, $X$, $Y$ be symmetric positive definite matrices, and define

$$f(t) = \text{tr} \left( B \left( tX + (1 - t)Y \right)^{-1} \right).$$

17
Then:

\[ f'(t) = \text{tr}(YZBZ) - \text{tr}(XZBZ) \]

and

\[ f''(t) = \text{tr}(YZBZYZ) + \text{tr}(YZYZBZ) + \text{tr}(XZBZXZ) + \text{tr}(XZXZBZ) . \]

**Proof**

We calculate for \( Z = (tX + (1-t)Y)^{-1} \):

\[ f'(t) = \text{tr}(YZBZ) - \text{tr}(XZBZ) = g_1(t) - g_2(t). \] (17)

This gives the claimed expression for \( f' \). Further,

\[ g'_1(t) = \text{tr}(YZBZYZ) + \text{tr}(YZYZBZ) - \text{tr}(XZBZYZ) - \text{tr}(XZYBZ) , \]

\[ g'_2(t) = \text{tr}(YZBZXZ) + \text{tr}(YZXZBZ) - \text{tr}(XZBZXZ) - \text{tr}(XZXZBZ). \] (18)

By permuting the trace, we see that

\[ \text{tr}(YZBZXZ) = \text{tr}(XZYZBZ) \]

\[ \text{tr}(YZXZBZ) = \text{tr}(XZBZYZ) \] (19)

Putting together Eq. (17) - Eq. (19) shows that the mixed terms cancel, and so:

\[ f''(t) = \text{tr}(YZBZYZ) + \text{tr}(YZYZBZ) + \text{tr}(XZBZXZ) + \text{tr}(XZXZBZ) \]

**Proof (Lemma 3.1)**

For any positive definite \( X \) and \( Y \), we have by Theorem 9 of Merikoski and Kumar (2004):

\[ \sigma_{\text{min}}(X)\sigma_{\text{max}}(Y) \leq \sigma_{\text{max}}(XY). \] (20)

We may set \( X = I \) in Eq. (10) to see that the minimum value of \( \text{tr} (S^*X^{-1}) \) over PSD matrices with 1s on the diagonal must be at most \( \text{tr} (S^*S) \).

For any \( Y \) which solves Eq. (10), therefore,

\[ \text{tr}(S^*S) \geq \sigma_{\text{min}}(S^*S) \sigma_{\text{min}}(Y)^{-1} \]

and so

\[ \sigma_{\text{min}}(Y) \geq \frac{\sigma_{\text{min}}(S^*S)}{\text{tr}(S^*S)} = \beta > 0. \]

So restricting Eq. (10) to the space of positive definite matrices with minimum eigenvalue at least \( \beta \) does not lose any optima. Call this restricted set \( K \).

The constraint that the diagonal elements of \( X \) are identical to 1 implies in particular that the trace of \( X \) is at most \( n \); therefore \( K \) is bounded as a subset of \( \mathbb{R}^{n \times n} \). It is closed because it contains
all its limit points by continuity of eigenvalues, and it is convex since convex combinations preserve
the property that the diagonal is 1 and the lower bound on the minimum eigenvalue (as implied by
Weyl’s inequality, Theorem 4.3.1 of Horn and Johnson (1990)).

**Proof (Lemma 3.2)**
Note that this Lemma is implied by showing that \( f'' > 0 \) for \( f \) as defined in Proposition B.1.
But this is immediate since by assumption \( X \) and \( Y \) are positive definite, and therefore every term
in the expression for \( f'' \) is positive.

**Proof (Lemma 3.3)**
Let \( X \) be symmetric, positive definite such that there is some \( i \) for which \( X_{i,i} = \beta < 1 \).
Then we claim that for the matrix \( Y \) defined by \( Y_{jk} = \delta_{ij}\delta_{ik} \), the derivative of our loss of
interest evaluated at \( X \) in the direction of \( Y \) is negative.

Let \( B = S^*S \), and let
\[
h(t) = \text{tr} \left( B (X + tY)^{-1} \right).
\]
As in Proposition B.1, we calculate:
\[
h'(t) = -\text{tr} \left( Y (X + tY)^{-1} B (X + tY)^{-1} \right),
\]
so
\[
h'(0) = -\text{tr} \left( YX^{-1}BX^{-1} \right) = - \left( X^{-1}BX^{-1} \right)_{i,i} < 0
\]
by the positive-definiteness of \( X^{-1}BX^{-1} \).
Moreover, by continuity of eigenvalues and symmetry of \( Y \), for sufficiently small \( \alpha > 0 \),
\( X + \alpha Y \) is symmetric, positive definite, and has diagonal elements at most 1.
Therefore, since the directional derivative evaluated above is negative, for sufficiently small \( \alpha \),
\( X + \alpha Y \) is an element of the constraint set with strictly smaller loss than \( X \), and \( X \) cannot be a
solution of Eq. (10).

**Proof (Lemma 3.4)** By Lemma 3.3, we may consider the equality-constrained problem Eq. (11);
we will take advantage of this for simplicity. We introduce a Lagrange multiplier, defining
\[
f(X) = \text{tr}((S^*SX)^{-1}) + \sum_{i=1}^{n} \lambda_i X_{i,i}. \tag{21}
\]
Differentiating Eq. (21) with respect to \( X \), we find
\[
\frac{\partial f}{\partial X} = -(X^{-1}S^*SX^{-1}) + (\text{diag} (\lambda) X) \odot I \tag{22}
\]
where \( \odot \) represents the Kronecker (elementwise) product.
Since we have exercised our right to consider the equality-constrained problem, we note that
this constraint yields
\[
(\text{diag} (\lambda) X) \odot I = \text{diag} (\lambda)
\]
since \( X \) is constant 1 on the diagonal in the equality-constrained problem Eq. (11). Therefore, our necessary condition for optimality becomes

\[
S^* S = X \text{diag} (\lambda) X.
\]  

(23)

which is precisely Eq. (14). We note that Eq. (23) implies that \( \text{diag}(\lambda) \) and \( X \) are in particular invertible.

We recognize that solving for \( X \) corresponds to solving for a generalized matrix square root. Defining

\[
X = \text{diag}(\lambda)^{-1/2} \left( \text{diag}(\lambda)^{1/2} S^* S \text{diag}(\lambda)^{1/2} \right)^{1/2} \text{diag}(\lambda)^{-1/2},
\]

we see that Eq. (23) is satisfied, and \( X \) is positive definite by e.g. Theorem 2 of Wigner (1963). Moreover, this solution is unique, as can be derived by uniqueness of the symmetric matrix square root. Therefore Eq. (13) must be satisfied at an optimizer \( X^* \).

Since \( X \) must have constant 1s on the diagonal, the expression Eq. (13) implies that

\[
\text{diagpart} \left( \sqrt{\text{diag}(\lambda)^{1/2} S^* S \text{diag}(\lambda)^{1/2}} \right) = \lambda,
\]

and that therefore \( \lambda \) is a fixed point of the mapping \( \phi \) defined above.

We have shown that an optimizer corresponds to a fixed point of \( \phi \); the other direction is immediate, since Lemma 3.2 and the affine structure of the equality constraints in Eq. (11) imply by the Karush-Kuhn-Tucker conditions Kuhn and Tucker (1951) that representation of \( X^* \) as Eq. (13) for \( \lambda \) a fixed point of \( \phi \) is not simply necessary, but also sufficient for optimality.

Appendix C. A lower bound for squared error of DP prefix sum.

Here we set down an argument due to Speyer, showing a lower bound for the norm of vectors with specified inner products; these inner products correspond to our prefix sum \( S \) matrix.

The structure of this argument yields a lower bound for all applications of the matrix mechanism; we will therefore split the presentation of this argument into two components, the first of which shows a generic lower bound, and the second of which instantiates this lower bound for the prefix sum \( S \) under consideration.

**Proposition C.1** Let \( S \) be any \( n \times n \) matrix, and let \( \{\sigma_i\} \) denote the singular values of \( S \). Applying the matrix mechanism to this factorization for \((\varepsilon, \delta)\) differential privacy results in expected square error at least

\[
\frac{1}{n} \left( \frac{(n+1)/2}{\sum_{j=1}^{|(n+1)/2|} \sigma_{2j-1}} \right)^2.
\]

(24)

**Proof**

Note that we are essentially replicating the argument of Speyer. Let \( S = AB \) be some matrix factorization. Let \( \{\alpha_i\} \) and \( \{\beta_i\} \) be the singular values of \( A \) and \( B \), respectively. Then:
\[ \|A\|_F^2 = \text{tr}(A^*A) = \sum \alpha_i^2, \]

and
\[ \max_{i \in [1, \ldots, n]} \|B_{[i,i]}\|_2^2 \geq \frac{1}{n} \text{tr}(B^*B) = \frac{1}{n} \sum \beta_i^2. \]

Therefore, for \( L \) as defined in Eq. (4),
\[ L(A, B) \geq \frac{1}{n} \left[ \sum \alpha_i^2 \right]^{1/2} \left[ \sum \beta_i^2 \right]^{1/2} \geq \frac{1}{n} \left( \sum \alpha_i \beta_i \right)^2 \]

where the second inequality is Cauchy-Schwarz.

Weyl’s inequality for singular values shows that \( \alpha_i \beta_i \geq \sigma_{2i-1}^2 \) (see Theorem 9 of Merikoski and Kumar (2004)). Therefore:
\[ L(A, B) \geq \frac{1}{n} \left[ \frac{(n+1)/2}{\sum_{j=1}^{(n+1)/2} \sigma_{2j-1}} \right]^2, \]

which is precisely the conclusion of this Proposition.

\( \blacksquare \)

**Proposition C.2** Let \( S \) be the \( n \times n \) prefix-sum matrix defined in Eq. (1). The singular values \( \sigma_i \) of this \( S \) satisfy:
\[ \sum_{j=1}^{(n+1)/2} \sigma_{2j-1} \geq \frac{n}{2\pi} \log(n). \quad (25) \]

**Proof** Again following Speyer, note that for \( S \) defined in Eq. (1),
\[ S^{-1} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \quad (26) \]

and
\[ (S^*S)^{-1} := \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 & \end{pmatrix}. \quad (27) \]

\( (S^*S)^{-1} \) has eigenvalues \( \sigma_k^{-2} \); we will show the eigenvalues of \( (S^*S)^{-1} \) are \( 4 \sin^2 \left( \frac{(2k-1)\pi}{4n+2} \right) \).

Denoting by \( p_n \) the characteristic polynomial of \( (S^*S)^{-1} \), we note:
\[ p_n(x) = (x - 2) p_{n-1}(x) - p_{n-2}(x). \]  

(28)

We recall that the Chebyshev polynomials \( T_n \) (see e.g. DLMF Chapter 18 for a reference) satisfy the recursion:

\[ T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x) \]

for appropriate initialization \( T_0, T_1 \). Clearly, then, defining \( \psi_n(x) = T_n(ax + b) \), the polynomials \( \psi_n \) satisfy

\[ \psi_n(x) = 2(a x + b) \psi_{n-1}(x) - \psi_{n-2}(x) \]

and this property uniquely determines the polynomials \( \psi_n \). Setting \( a = 1/2, b = -1 \), we recover Eq. (28), and therefore our characteristic polynomial \( p_n \) is a shifted version of the \( n^{th} \) Chebyshev polynomial.

Mapping the roots of the Chebyshev polynomial to our \( p_n \), we see that the eigenvalues of \( (S^*S)^{-1} \) are

\[ 4 \sin^2 \left( \frac{(2k - 1)\pi}{4n + 2} \right) \]

and therefore the singular values of \( S \) are

\[ \sigma_k = \frac{1}{2 \sin \left( \frac{(2k - 1)\pi}{4n + 2} \right)}. \]

(29)

Therefore our claim comes down to estimating the sum

\[ \sum_{j=1}^{[(n+1)/2]} \frac{1}{2 \sin \left( \frac{(4j-3)\pi}{4n+2} \right)}. \]

(30)

We bound \( \sin x < x \), so that

\[ \sum_{j=1}^{[(n+1)/2]} \frac{1}{2 \sin \left( \frac{(4j-3)\pi}{4n+2} \right)} \geq \frac{4n + 2}{2\pi n} \sum_{j=1}^{[(n+1)/2]} \frac{1}{4j - 3} \geq \frac{1}{2\pi} \sum_{j=1}^{[(n+1)/2]} \frac{2}{j - 3/2}. \]

(31)

We will compare this sum term-by-term with the integral \( \int_{1}^{n} \frac{1}{x} \, dx \). That is, we claim:

\[ \frac{2}{2j - 3/2} \geq \int_{2j-1}^{2j+1} \frac{1}{x} \, dx. \]

(32)

We may compute

\[ \int_{2j-1}^{2j+1} \frac{1}{x} \, dx = \log(2j + 1) - \log(2j - 1) = \log \left( 1 + \frac{2}{2j - 1} \right) \leq \frac{2}{2j - 1}. \]

Since \( 2j - 3/2 < 2j - 1 \), our claimed inequality Eq. (32) follows.

Therefore
\[
\sum_{j=1}^{\lfloor (n+1)/2 \rfloor} \frac{2}{j - 3/2} \geq \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} \int_{2j-1}^{2j+1} \frac{1}{x} \, dx \geq \int_1^n \frac{1}{x} \, dx = \log(n).
\]

Combining this estimate with Eq. (31) shows our proposition.

\[\text{Remark.} \quad \text{These two propositions together immediately imply that expected squared error for any matrix factorization-based approach to differentially private prefix sums must have at least } \Omega(\sqrt{n \log(n)}) \text{ expected error in } \ell^2.\]

## Appendix D. Pseudo-code for efficient implementation via structured matrices

The following pseudo-code makes explicit the algorithm proposed in Section 4.2:

\textbf{Algorithm 1} An efficient implementation (executed by the trusted curator)

1: Parameters:
2: Matrix \( D^{(d)} \) containing \( d \in \{0, \ldots, n\} \) diagonals from \( W \)
3: Matrices \( A, B \in \mathbb{R}^{n \times r} \)
4: Noise vector \( z \in \mathbb{R}^n \) with entries \( z_i \)
5: Observations \( x \in \mathbb{R}^n \) with entries \( x_i \)
6: \( \beta := 0 \in \mathbb{R}^r \)
7: \( S := 0 \)
8: for \( i \) in \( 1, \ldots, n \) do
9: \( S +:= x_i \) \hfill \( \triangleright \) Maintain the un-noised cumulative sum
10: \( y := 0 \in \mathbb{R} \)
11: for \( k \) in \( 0, \ldots, \min(i - 1, d - 1) \) do
12: \( y +:= D^{(d)}_{[i,i-k]} z_{i-k} \) \hfill \( \triangleright \) Read or recompute \( z_i, \ldots, z_{i-d-1}; \) \( d \) multiplies
13: end for
14: if \( i > d \) then
15: \( i' \leftarrow i - d \)
16: \( \beta +:= z_{i'} B_{[r,:]} \) \hfill \( \triangleright \) \( r \) multiplies
17: \( y +:= A_{[i,:]} \cdot \beta \) \hfill \( \triangleright \) \( r \) multiplies
18: end if
19: Release \( S + y \) \hfill \( \triangleright \) A DP estimate of \( \sum_{t=1}^{i} x_i \)
20: end for