Regularization of Ill-Posed Problems with Unbounded Operators. *†

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Abstract
Variational regularization and the quasisolutions method are justified for unbounded closed operators.

1 Introduction. Variational regularization.

There is a large literature on methods for solving ill-posed problems: variational regularization, method of quasisolutions, iterative and projection regularization [2]-[8]. The case of ill-posed problems with a closed linear operator was discussed in [5], and the case of nonlinear, possibly unbounded, operators possibly unbounded, does not seem to be discussed. In the theory of ill-posed problems the following well-known result ([1, Lemma 1.5.8]) is often used: if $A$ is an injective and continuous mapping from a compact set $M$ of a Banach space into a set $N := AM$, then the inverse mapping $A^{-1} : N \rightarrow M$ is continuous. In [7, p.112] the usual assumption about continuity of $A$ in the above result is replaced by the assumption about closedness of $A$.

In this short note ill-posed problems are studied in the case of the mapping $A$ not necessarily continuous, but closed, possibly nonlinear. Our argument is very simple and the result is fairly general.

Let $A$ be an injective, possibly nonlinear, closed operator on a Banach space $X$, and the equation

$$Ay = f$$  \hspace{1cm} (1.1)

has a solution $y$. Our arguments hold in metric spaces as well without changes.

Assume that $A^{-1}$ is not continuous. This implies that problem (1.1) is ill-posed. Let

$$\|f_\delta - f\| \leq \delta.$$  \hspace{1cm} (1.2)

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Given $f_\delta$ and $A$, one wants to construct $u_\delta = R_\delta(f_\delta)$, such that $\|u_\delta - y\| \to 0$ as $\delta \to 0$, where $y$ solves (1.1). If $u_\delta$ is constructed, then the operator $R_\delta$ yields a stable approximation of the solution $y$ to (1.1).

Let us first describe the method of variational regularization in our case.

Define the functional

$$F(u) := \|A(u) - f_\delta\| + \delta \phi(u), \quad (1.3)$$

and assume that $\phi(u) \geq 0$ is a functional, such that for any constant $c > 0$ the set

$$\{u : \phi(u) \leq c\} \text{ is precompact in } X. \quad (1.4)$$

The functional $F$ depends on $\delta$ and $f_\delta$, but for simplicity of writing we do not show this dependence explicitly. Assume that $D(A) \subset D(\phi)$, the domain of definition of $\phi$, contains $D(A)$. This assumption implies that $y \in D(\phi)$, so that $\phi(y) < \infty$. Define $D(F) = D(A)$. If $A$ were bounded, defined on all of $X$, then one would assume $y \in D(\phi)$ and $D(F) = D(\phi)$. If $A$ were unbounded and $D(\phi) \subset D(A)$, then one would assume that $y \in D(\phi)$ and $D(F) = D(\phi)$.

Denote

$$0 \leq m := \inf_{u \in D(A)} F(u). \quad (1.5)$$

The number $m = m(\delta) \geq 0$. Let $u_j$ be a minimizing sequence $u_j \in D(F)$ for the functional $F$, such that:

$$F(u_j) \leq m + \frac{1}{j} = m + \delta, \quad \frac{1}{j} \leq \delta. \quad (1.6)$$

Denote by $u_\delta := u_j(\delta)$ a member $u_j(\delta)$ of this minimizing sequence, where $j(\delta)$ is chosen so that $\frac{1}{j(\delta)} \leq \delta$. There are many such $j(\delta)$ and we fix one of them, for example, the minimal one. Since

$$F(y) \leq \delta + \delta \phi(y) := c_1 \delta, \quad c_1 := 1 + \phi(y), \quad (1.7)$$

one has:

$$m \leq c_1 \delta, \quad (1.8)$$

and

$$F(u_\delta) \leq m + \delta \leq c \delta, \quad c := c_1 + 1. \quad (1.9)$$

Thus $\delta \phi(u_\delta) \leq c \delta$, and

$$\phi(u_\delta) \leq c. \quad (1.10)$$

Let us now take $\delta \to 0$. By (1.4) and (1.10) one can select a convergent in $X$ subsequence of the set $u_\delta$, which we denote also $u_\delta$, such that

$$\|u_\delta - u\| \to 0 \text{ as } \delta \to 0, \quad (1.11)$$

where $u$ is the limit of $u_\delta$. 

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From (1.2), (1.3), (1.9) and (1.10), one gets
\[ 0 = \lim_{\delta \to 0} F(u_\delta) = \lim_{\delta \to 0} \| A(u_\delta) - f_\delta \| = \lim_{\delta \to 0} \| A(u_\delta) - f \|. \] (1.12)

Since \( A \) is closed, (1.11) and (1.12) imply
\[ \lim_{\delta \to 0} A(u_\delta) = A(u), \quad 0 = \| A(u) - f \|. \] (1.13)

Since \( A \) is injective, (1.13) and (1.1) imply \( u = y \), so
\[ \lim_{\delta \to 0} \| u_\delta - y \| = 0. \] (1.14)

Since the limit \( y \) of any subsequence \( u_\delta \) is unique, the whole sequence \( u_\delta \) converges to \( y \).

We have proved the following result:

**Theorem 1.1.** Assume that (1.4) holds, \( \phi \geq 0 \), \( A : D(A) \to X \) is a closed, injective, possibly nonlinear unbounded operator, \( A(y) = f \), and \( A^{-1} \) is not continuous. Let \( u_\delta \) be constructed as above so that (1.9) holds. Then (1.14) holds.

In section 2 the method of quasisolutions is discussed in the case of possibly unbounded and nonlinear operators.

## 2 Quasisolutions for unbounded operators.

In this section the assumptions about equation (1.1) and the operator \( A \) are the same as in section 1, in particular, \( A^{-1} \) is not continuous, so that solving equation (1.1) is an ill-posed problem.

Choose a compactum \( K \subset X \) such that the solution of (1.1) \( y \in K \). Consider the problem
\[ \| A(u) - f_\delta \| = \inf := \mu, \quad u \in K \subset D(A). \] (2.1)

The infimum \( \mu = \mu(\delta) \geq 0 \) depends on \( f_\delta \) also, but we do not show this dependence explicitly. Let \( u_j \) be a minimizing sequence:
\[ \| A(u_j) - f_\delta \| \leq \mu + \frac{1}{j}. \] (2.2)

Choose \( j = j(\delta) \) such that \( \frac{1}{j} \leq \delta \) and denote \( u_j := u_\delta. \)

Then
\[ \| A(u_\delta) - f_\delta \| \leq \mu + \delta. \] (2.3)

Since \( \| A(y) - f_\delta \| \leq \delta \), it follows that \( \mu \leq \delta \), so
\[ \| A(u_\delta) - f_\delta \| \leq 2\delta. \] (2.4)
Since \( \{u_\delta\} \subset K \) one can select a convergent (to some \( u \)) subsequence, denoted also \( \{u_\delta\} \):

\[
\|u_\delta - u\| \to \text{ as } \delta \to 0. \tag{2.5}
\]

From (2.4) it follows that \( A(u_\delta) \) converges to \( f \):

\[
\|A(u_\delta) - f\| \leq \|A(u_\delta) - f_\delta\| + \|f_\delta - f\| \leq 3\delta \to 0 \text{ as } \delta \to 0. \tag{2.6}
\]

Since \( A \) is closed, it follows from (2.5) and (2.6) that

\[
A(u) = f. \tag{2.7}
\]

Injectivity of \( A \), equation (2.7), and the equation \( A(y) = f \) imply \( u = y \). We have proved:

**Theorem 2.1.** Assume that \( A : D(A) \to X \) is a closed, injective, possibly nonlinear and unbounded, operator, \( A(y) = f \), (1.2) holds, \( K \) is a compact set in \( X \), and \( y \in K \). If \( u_\delta \in K \) satisfies (2.4), then (1.14) holds.

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