Difference Sets are Not Multiplicatively Closed

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Abstract: We prove that for any finite set $A \subset \mathbb{R}$ its difference set $D := A - A$ has large product set and quotient set: that is, $|DD|, |D/D| \gg |D|^{1+c}$, where $c > 0$ is an absolute constant. A similar result takes place in the prime field $\mathbb{F}_p$ for sufficiently small $D$. It gives, in particular, that multiplicative subgroups of size less than $p^{4/5 - \epsilon}$ cannot be represented in the form $A - A$ for any $A \subset \mathbb{F}_p$.

Key words and phrases: sumsets, additive energy

1 Introduction

Let $A, B \subset \mathbb{R}$ be finite sets. Define the sum set, the difference set, the product set and the quotient set of $A$ and $B$ to be

$$A + B := \{a + b : a \in A, b \in B\}, \quad A - B := \{a - b : a \in A, b \in B\}$$

$$AB := \{ab : a \in A, b \in B\}, \quad A/B := \{a/b : a \in A, b \in B, b \neq 0\},$$

respectively. The Erdős–Szemerédi sum–product conjecture [5] says that for any $\epsilon > 0$ one has

$$\max\{|A + A|, |AA|\} \gg |A|^{2-\epsilon}.$$

Thus, it asserts that for an arbitrary subset of real numbers (or integers) either the sumset or the product set of this set is large. Modern bounds concerning the conjecture can be found in [24], [10], [11].

We consider the following sum–product type question.
**Problem.** Let $A, P \subseteq \mathbb{R}$ be two finite sets, $P \subseteq A - A$. Suppose that $|PP| \leq |P|^{1+\varepsilon}$, where $\varepsilon > 0$ is a small parameter. In other words, $P$ has small product set. Is it true that there exists $\delta = \delta(\varepsilon) > 0$ with
\[
\sum_{x \in P} |\{a_1 - a_2 = x : a_1, a_2 \in A\}| \ll |A|^{2-\delta}?
\] (1.1)

Thus, we consider a set $P$ with small product set and we want to say something nontrivial about the additive structure of $A$, that is, the sum of additive convolutions over $P$. The question plays an important role in recent papers [15], [22] and [28], [29]. Even the famous unit distance problem of Erdős (see [4] and also the survey [20]) can be considered as a question of that type. Indeed, the unit distance problem is to find a good upper bound for
\[
\int_{S^1} |\{a_1 - a_2 = x : a_1, a_2 \in A\}| \, dx,
\]
where $A$ is a finite subset of the Euclidean plane (which we consider as the complex plane) and $S^1$ is the unit circle. Since the unit circle $S^1$ is a subgroup, it trivially has small product set: $S^1 \cdot S^1 = S^1$.

Let us return to (1.1). In [15] Roche–Newton and Zhelezov studied some sum–product type questions and obtained the following result (in principle, the method of their paper allows one to obtain subexponential bounds for the sum from (1.1) but not of the required form). The multiplicative energy of $A$, denoted by $E^\times(A)$, is the number of solutions to the equation $ab = cd$, where $a, b, c, d \in A$.

**Theorem 1** For any $\varepsilon > 0$ there are constants $C'(\varepsilon), C''(\varepsilon) > 0$ such that for any set $A \subseteq \mathbb{C}$ one has
\[
E^\times(A - A) \leq \max\{C''(\varepsilon)|A|^{3+\varepsilon}, |A - A|^3 \exp(-C'(\varepsilon) \log^{1/3-o(1)} |A|)\}. \tag{1.2}
\]

Thus, bound (1.2) says us that the difference set $D = A - A$ enjoys a non–trivial upper bound for its multiplicative energy. The proof used a deep result of Sanders [16] and the Subspace Theorem of Schmidt (see, e.g., [20]). Roughly speaking, thanks to the Subspace Theorem, Roche–Newton and Zhelezov obtained an upper bound for the sums from (1.1) for $P$ equal to a multiplicative subgroup of $\mathbb{C}$ of small rank (so $P$ automatically has small product set for trivial reasons) and using Sanders’ structural result they extended it to general sets with small multiplicative doubling – see details in [15].

Theorem 1 has the following consequence.

**Corollary 2** Let $A \subseteq \mathbb{R}$ be a finite set, let $D = A - A$, and let $\varepsilon > 0$ be a real number. Then for some constant $C'(\varepsilon) > 0$ one has
\[
|DD|, |D/D| \gg_{\varepsilon} |D| \cdot \min\{|D|^3|A|^{-(3+\varepsilon)}, \exp(C'(\varepsilon) \log^{1/3-o(1)} |A|)\}. \tag{1.3}
\]

Avoiding using either of the strong results of Sanders and Schmidt, we prove the following result (see Theorem 20 of section 4).
Theorem 3 Let $A \subset \mathbb{R}$ be a finite set and let $D = A - A$. Then

$$|DD|, |D/D| \gg |D|^{1 + \frac{1}{2} \log^{-\frac{1}{2}} |D|}. \quad (1.4)$$

The bound (1.4) can be considered as a new necessary condition for a set to be a difference set of the form $A - A$. Namely, any such set must have a large product set and a large quotient set.

One might think that the optimal version of (1.4) should state that $|DD|, |D/D| \gg |D|^{2 - \varepsilon}$ for arbitrary $\varepsilon > 0$, but this is not true: see Proposition 22 which gives examples of sets $A$ with $|DD|, |D/D| \ll |D|^{3/2}$.

Also, it was conjectured in [15] that if $|(A + A)/(A + A)| \ll |A|^2$ or $|(A - A)/(A - A)| \ll |A|^2$, then $|A \pm A| \ll |A|$. The authors obtained some first results in this direction. A refined version of Theorem 3, Theorem 20 below, implies the following result.

Corollary 4 Let $A \subset \mathbb{R}$ be a finite set. Suppose that $|(A - A)/(A - A)| \ll |A|^2$ or $|(A - A)(A - A)| \ll |A|^2$. Then

$$|A - A| \ll |A|^{2 - \frac{1}{2} \frac{3}{\log 3} |A|}. \quad \text{(1.5)}$$

A simple consequence of the conjectured bound (1.1) is that $A - A \neq P$ for sets $P$ with small product/quotient set. Our weaker estimate (1.4) gives the same, so, in particular, geometric progressions are not (symmetric) difference sets of the form $A - A$. An analog of geometric progressions in prime fields $\mathbb{F}_p$ are multiplicative subgroups. Obtaining an appropriate version of Theorem 3 in the finite-fields setting and using further tools, we obtain the following theorem.

Theorem 5 Let $p$ be a prime number, let $\Gamma \subset \mathbb{F}_p$ be a multiplicative subgroup, let $|\Gamma| < p^{3/4}$, and let $\xi \neq 0$ be an arbitrary residue. Suppose that for some $A \subset \mathbb{F}_p$ one has

$$A - A \subseteq \xi \Gamma \bigcup \{0\}. \quad (1.5)$$

Then $|A| \ll |\Gamma|^{4/9}$. If $|\Gamma| \geq p^{3/4}$, then $|A| \ll |\Gamma|^{4/3} p^{-2/3}$. In particular, for any $\varepsilon > 0$ and sufficiently large $\Gamma, |\Gamma| \leq p^{4/5-\varepsilon}$ we have that $A - A \neq \xi \Gamma \bigcup \{0\}$.

Results on representations of sets (and multiplicative subgroups, in particular) as sumsets or product sets can be found in [2], [3], [17], [22], [25]. For example, it was proved in [25] that any set satisfying (1.5) has size $|A| \leq |\Gamma|^{1 + o(1)}$. In [22] the author refined this result in the special case when $A$ is a multiplicative subgroup and obtained the estimate $|A| \leq |\Gamma|^{1/3 + o(1)}$. In our new Theorem 5 we have to deal with the general case of arbitrary set $A$ and it is the first result of such type.

The results of the article allow us to make a first tiny tiny step towards answering a beautiful question of P. Hegarty [6].
Problem. Let $P \subseteq A + A$ be a strictly convex (concave) set. Is it necessarily true that $|P| = o(|A|^2)$?

Recall that a sequence of real numbers $A = \{a_1 < a_2 < \cdots < a_n\}$ is called strictly convex (concave) if the consecutive differences $a_i - a_{i-1}$ are strictly increasing (decreasing). It is known that from a combinatorial point of view sets $P$ with small product set have behaviour similar to convex (concave) sets (see [19] or discussion before Corollary 36), but in the opinion of the author they have a simpler structure. The following proposition is a consequence of Theorem 3.

**Corollary 6** Let $A \subset \mathbb{R}$ and let $D = A - A$. Suppose that $|DD| \leq M|D|$ or $|D/D| \leq M|D|$, where $M \geq 1$. Then

$$|A| \ll_M 1.$$  

Thus, Corollary 6 proves the conjecture of Hegarty in the case of pure difference sets $P$ instead of sumsets and where we assume that $P$ has small product/quotient set instead of assuming convexity.

In the proof we develop some ideas from [22], combining them with the Szemerédi–Trotter Theorem (see section 3), as well as with a new simple combinatorial observation, see formula (4.3). The last formula tells us that if one forms a set $D/D$, $D := A - A$ (which is known as $Q[A]$ in the literature, see e.g. [27]), then the set $D/D$ contains a large subset $R \subseteq D/D$ which is additively rich. Namely, we consider

$$R = R[A] = \left\{ \frac{a_1 - a}{a_2 - a} : a, a_1, a_2 \in A, a_2 \neq a \right\} \subseteq D/D$$

and note that $R = 1 - R$. By the Szemerédi–Trotter Theorem the existence of such additive structure in $R$ means that the product of $R$ is large and hence the product of $D$ is large as well.

The paper is organized as follows. In section 2 we give a list of the results, which will be further used in the text. In the next section we discuss some consequences of the Szemerédi–Trotter Theorem in its uniform and modern form. In Section 4 we prove our main Theorem 20 which implies Theorem 3 and Corollary 4. In the next section we deal with the prime fields case and obtain Theorem 5 above. Finally, the constants in Theorem 3 and Corollary 4 can be improved in the case of the quotient set $D/D$ but it requires much more work – see section 6. In the appendix we discuss some generalizations of the quantities from section 3.

Let us conclude with a few comments regarding the notation used in this paper. All logarithms are to base 2. The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. When the constants in the signs depend on some parameter $M$, we write $\ll_M$ and $\gg_M$.

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## 2 Preliminaries

Let $G = (G, +)$ be an abelian group with the group operation $+$. We begin with the famous Plünnecke–Ruzsa inequality (see e.g. [27]).
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**Lemma 7** Let \( A, B \subseteq G \) be two finite sets with \( |A + B| \leq K|A| \). Then for all positive integers \( n, m \) we have the inequality
\[
|nB - mB| \leq K^{n+m}|A|.
\] (2.1)

Furthermore, for any \( 0 < \delta < 1 \) there exists \( X \subseteq A \) such that \( |X| \geq (1 - \delta)|A| \) and such that for any integer \( k \) one has
\[
|X + kB| \leq (K/\delta)^k|X|.
\] (2.2)

We also need Ruzsa’s triangle inequality (see, e.g., [27]).

**Lemma 8** Let \( A, B, C \subseteq G \) be three finite sets. Then
\[
|C||A - B| \leq |A - C||B - C|.
\] (2.3)

In this paper we have to deal with the quantity \( T(A, B, C, D) \), see [14], [22] (\( T \) standing for collinear triples)
\[
T(A, B, C, D) := \sum_{c \in C, d \in D} E^+(A - c, B - d) = \left| \{(a - c)(b - d) = (a' - c)(b' - d) : a, a' \in A, b, b' \in B, c \in C, d \in D\} \right|.
\] (2.4)

If \( A = B \) and \( C = D \), then we write \( T(A, C) \) for \( T(A, A, C, C) \) and we write \( T(A) \) for \( T(A, A, A, A) \).

In [7] Jones proved a good upper estimate for the quantity \( T(A) \), \( A \subseteq \mathbb{R} \) (another proof was obtained by Roche–Newton in [14]). Upper bounds for \( T(A) \) when \( A \) belongs to a prime field can be found in [1] and [22].

**Theorem 9** Let \( A \subseteq \mathbb{R} \) be finite. Then
\[
T(A) \ll |A|^4 \log |A|.
\] (2.5)

**Theorem 10** Let \( p \) be a prime number and let \( A \subseteq \mathbb{F}_p \) be a subset with \( |A| < p^{2/3} \). Then
\[
T(A) \ll |A|^{9/2}.
\]
Theorem 11 Let \( \mathcal{P} \) be a set of points and let \( \mathcal{L} \) be a pseudo-line system. Then
\[
I(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{2/3} |\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|.
\]

Remark 12 If we redefine a pseudo-line system as a family of continuous plane curves with \( O(1) \) points in common and if any two points are simultaneously incident to at most \( O(1) \) curves, then Theorem 11 remains true: see e.g. [27, Theorem 8.10] for precise bounds in this direction.

Now let us recall the main result of [23].

Theorem 13 Let \( \Gamma \subseteq \mathbb{F}_p \) be a multiplicative subgroup, let \( k \geq 1 \) be a positive integer, and let \( x_1, \ldots, x_k \) be different nonzero elements of \( \mathbb{F}_p \). Also, let
\[
32k2^{20k\log(k+1)} \leq |\Gamma|, \quad p \geq 4k|\Gamma|(|\Gamma|^{1/\tau} + 1).
\]
Then
\[
|\Gamma \cap (\Gamma + x_1) \cap \ldots \cap (\Gamma + x_k)| \leq 4(k+1)(|\Gamma|^{1/\tau} + 1)^{k+1}.
\]
(2.6)
The same holds if one replaces \( \Gamma \) in (2.6) by any cosets of \( \Gamma \).

Thus, the theorem above asserts that \( |\Gamma \cap (\Gamma + x_1) \cap \ldots \cap (\Gamma + x_k)| \ll_k |\Gamma|^{1/\tau + \alpha_k} \), provided that \( 1 \ll_k |\Gamma| \ll_k p^{1-\beta_k} \), where \( \alpha_k, \beta_k \) are some sequences of positive numbers, and \( \alpha_k, \beta_k \to 0 \) as \( k \to \infty \).

We finish this section with a result from [22], see Lemma 19.

Lemma 14 Let \( \Gamma \subseteq \mathbb{F}_p \) be a multiplicative subgroup, and \( k \) be a positive integer. Then for any nonzero distinct elements \( x_1, \ldots, x_k \) of \( \mathbb{F}_p \) one has
\[
|\Gamma \cap (\Gamma + x_1) \cap \ldots \cap (\Gamma + x_k)| = \frac{|\Gamma|^{k+1}}{(p-1)^k} + \theta k2^{k+3} \sqrt{p},
\]
(2.7)
where \( |\theta| \leq 1 \).

3 Some consequences of the Szemerédi–Trotter Theorem

In this section we discuss some implications of Szemerédi–Trotter Theorem 11, which are given in a modern form (see e.g. [13]). We start with a definition from [21].

Definition 15 A finite set \( A \subseteq \mathbb{R} \) is said to be of Szemerédi–Trotter type (abbreviated as SzT–type) if there exists a parameter \( D(A) > 0 \) such that inequality
\[
|\{ s \in A - B \mid |A \cap (B + s)| \geq \tau \}| \leq \frac{D(A)|A||B|^2}{\tau^3},
\]
(3.1)
holds for every finite set \( B \subseteq \mathbb{R} \) and every real number \( \tau \geq 1 \).
So, $D(A)$ can be considered as the infimum of numbers such that (3.1) takes place for any $B$ and $\tau \geq 1$ but, of course, the definition is applicable just for sets $A$ with small quantity $D(A)$.

Now we can introduce a new characteristic of a set $A \subset \mathbb{R}$, which can be considered as a generalization of $D(A)$.

**Definition 16** Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be any function. A finite set $A \subset \mathbb{R}$ is said to be of Szemerédi–Trotter type relative to $\Phi$ (abbreviated as SzT$_{\Phi}$–type) if there exists a parameter $D_\Phi(A) > 0$ such that inequality

$$\left|\{x : |\{(a,b) \in A \times B, a = \Phi(b,x)\}| \geq \tau\}\right| \leq \frac{D_\Phi(A)|A||B|^2}{\tau^3},$$

holds for every finite set $B \subset \mathbb{R}$ and every real number $\tau \geq 1$.

The usual quantity $D(A)$ corresponds to the case $\Phi(x,y) = x + y$.

Any SzT–type set has small number of solutions of a wide class of linear equations, see, e.g., [10, Corollary 8] (where nevertheless another quantity $D(A)$ was used) and [21, Lemmas 7, 8], say. This is another illustration of that fact in a more general context.

**Corollary 17** Let $A \subset \mathbb{R}$ be a finite set of SzT$_{\Phi}$–type. For any finite $B \subset \mathbb{R}$ let

$$\alpha_{A,B}(x) = \alpha_{A,B}^\Phi(x) := \left|\{(a,b) \in A \times B : a = \Phi(b,x)\}\right|.$$

Then

$$\left|\{x : \alpha_{A,B}(x) > 0\}\right| \gg |A||B|^{1/2}D_\Phi^{-1/2}(A),$$

and

$$E_2^\Phi(A,B) := \sum_{x} \alpha_{A,B}^2(x) \ll D_\Phi^{1/2}(A) \cdot |A||B|^{3/2}.$$  

**Proof.** Let $D = D_\Phi(A)$. Put $S = \{x : \alpha_{A,B}(x) > 0\}$ and $\tau = |A||B|/(2|S|)$. Then

$$|A||B| = \sum_{x \in S} \alpha_{A,B}(x) \leq 2 \sum_{x \in S'} \alpha_{A,B}(x),$$

where $S' = \{x \in S : \alpha_{A,B}(x) \geq \tau\}$. By our assumption $A$ is a set of SzT$_{\Phi}$–type. Thus, arranging $\alpha_{A,B}(x_1) \geq \alpha_{A,B}(x_2) \geq \ldots$, we get $\alpha_{A,B}(x_j) \leq (D|A||B|^2)^{1/3}j^{-1/3}$. Whence

$$|A||B| \ll \sum_{j=1}^{T} (D|A||B|^2)^{1/3}j^{-1/3} \ll (D|A||B|^2)^{1/3}T^{-2/3},$$

where $T^{1/3} = (D|A||B|^2)^{1/3}\tau^{-1}$. It follows that

$$|A||B| \ll D|A||B|^2\tau^{-2} \ll D|A|^{-1}|S|^2$$
as required. Estimate (3.4) can be obtained similarly, or see the proof of Lemma 7 in [21]. Also notice that this bound, combined with the Cauchy–Schwarz inequality, implies (3.3). This concludes the proof. □

In [12] authors considered the quantity

$$\tilde{d}_+ (A) = \inf \min_{f, c} \frac{|f(A) + C|^2}{|A||C|},$$

(3.5)

where the infimum is taken over convex/concave functions $f$ and proved that $D_+ (A) \ll \tilde{d}(A)$. In a similar way, one can consider the quantity $D_\times (A)$, which corresponds to $\Phi(x, y) = xy$, and obtain

$$D_\times (A) \ll \inf \min_{f, c} \frac{|f(A) + C|^2}{|A||C|},$$

(3.6)

where the infimum is taken over all functions $f$ such that for any $b \neq b'$ the function $f(xb) - f(xb')$ is monotone. The last bound is a generalization of [8, Lemma 15]. For the rigorous proof of formula (3.6) and the proof of a similar upper bound for the quantity $D_\Phi(A)$, see the appendix.

4 The proof of the main result

First of all let us derive a simple consequence of Theorem 11.

Lemma 18 Let $A, B, C, D \subseteq \mathbb{R}$ be four finite sets. Then for any nonzero $\alpha$ one has

$$|A \cap (B + \alpha)| \ll (|C||D|)^{-1/3}(|AC||BD|)^{2/3} + |D|^{-1}|BD| + |C|^{-1}|AC|. \quad (4.1)$$

In a dual way

$$|A \cap (\alpha/B)| \ll (|C||D|)^{-1/3}(|A + C||B + D|)^{2/3} + |D|^{-1}|B + D| + |C|^{-1}|A + C|. \quad (4.2)$$

Proof. We can remove the origin from $A, B, C, D$ if we want. First, note that the number of solutions of the equation $a - b = \alpha$ with $\alpha \in A$ and $b \in B$ is equal to $|A \cap (B + \alpha)|$. Hence,

$$|A \cap (B + \alpha)| \ll (|C||D|)^{-1} \{pc^{-1} - p_\alpha d^{-1} = \alpha : p \in AC, p_s \in BD, c \in C, d \in D\}.$$

Let $\mathcal{P}$ be the set of points $\{(c^{-1}, p_s) : c \in C, p_s \in BD\}$ and let $\mathcal{L}$ be the set of lines $\{l_s\}$ where $s \in AC, t \in D^{-1}$ and $l_s = \{(x, y) : sx - ty = \alpha\}$. Clearly, $|\mathcal{P}| = |C||BD|$ and $|\mathcal{L}| = |D||AC|$. Using Theorem 11 with the set of points equal to $\mathcal{P}$ and the set of lines equal to $\mathcal{L}$, we get

$$|A \cap (B + \alpha)| \ll (|C||D|)^{-1} (|\mathcal{P}||\mathcal{L}|)^{2/3} + |\mathcal{P}| + |\mathcal{L}| \ll$$

$$\ll (|C||D|)^{-1} (|C||D||AC||BD|)^{2/3} + |C||BD| + |D||AC| \ll$$
\[ \ll \frac{1}{3} |AC||BD|^{1/3} + |C|-1|BD| + |C|-1|AC|. \]

Similarly, in order to prove (4.2), we consider the equation \( ab = \alpha, a \in A, b \in B \), and also the equation 
\[(p - c)(p_s - d) = \alpha, p \in A + C, p_s \in B + D, \] which correspond to the curves 
\( l_{p,d} = \{(x,y) : (p - x)(y - d) = \alpha\} \) and the points 
\( \mathcal{P} = C \times (B + D) \). It is easy to check that any two curves 
\( l_{p,d}, l'_{p',d'} \) have at most two points in common. After that one applies the Szemerédi–Trotter Theorem one more time and 
performs the calculations above. This completes the proof. \(\square\)

Take arbitrary finite sets \( A, B \subset \mathbb{R}, |B| > 1 \) and consider the quantity 
\[ R[A,B] = \left\{ \frac{a-b}{b_1-b} : a \in A, b, b_1 \in B, b_1 \neq b \right\}. \]

Note that for any \( x \) one has \( R[A+x,B+x] = R[A,B] \) and that for all nonzero \( \lambda \) one has \( R[\lambda A, \lambda B] = R[A,B] \).

Now let us make a crucial observation about the set \( R[A,B] \), which says that the set is somehow 
additively structured. Because \( \frac{a-b}{b_1-b} - 1 = -\frac{a-b_1}{b-b_1} \), we have that 
\[ R[A,B] = 1 - R[A,B]. \] (4.3)

Write \( R[A] \) for \( R[A,A] \) and note that \( R^{-1}[A] = R[A] \). Geometrically, the set 
\[ R[A] = \left\{ \frac{a_2-a}{a_1-a} : a, a_1, a_2 \in A, a_1 \neq a \right\} \]
is the set of all fractions of (oriented) lengths of segments \([a_1,a], [a,a_2]\). With this point of view the formulas 
\( R^{-1}[A] = R[A] \) and \( R[A] = 1 - R[A] \) become almost trivial. Also note that \( 0, 1 \in R[A] \) and that 
putting \( D = A - A \), we have \( R[A] \subseteq D/D \subseteq R[A] \cdot R[A] \). Thus, the products \( (D/D)^n, n \in \mathbb{N} \) are controlled 
by \( R^m[A], m \in \mathbb{N} \) and vice versa.

**Question.** Let \( A \subset \mathbb{R} \) be a finite set. Is it true that \( |R[A]| \gg |A - A| \)? Similarly, is it true that \( |R[A]| \gg |A/A| ? \)

A simple consequence of Theorem 9 is the following lower bound for the size of the set \( R[A] \), see [7], [14].

**Theorem 19** Let \( A \subset \mathbb{R} \) be a finite set. Then 
\[ |R[A]| \gg \frac{|A|^2}{\log |A|}. \] (4.4)

Now we can prove the main result of this section. The author thanks Misha Rudnev, who pointed out to the author how to improve Theorems 20, 24 below.
**Theorem 20** Let $A \subset \mathbb{R}$ be a finite set and let $D = A - A$. Then
\[ |DD|, |D/D| \gg |D|^{3/2} |R[A]|^{1/2} \gg |D|^{3/2} |A|^{1/2} \log^{-1/4} |A|. \] (4.5)

In particular,\[ |DD|, |D/D| \gg |D|^{1+12/\log^{-1/4} |A|}. \] (4.6)

**Proof.** Put $R = R[A]$. Using identity (4.3) as well as Lemma 18 with $A = -R, B = R, C = D = A - A$ and $\alpha = -1$, we get
\[ |R| = |-R \cap (R - 1)| \ll |D|^{-2/3} |RD|^{4/3} \leq |D|^{-2/3} |DD/D|^{4/3}. \] (4.7)

Applying Lemma 7 in its multiplicative form with $A = D, B = D$ or $B = D^{-1}$, we have
\[ |DD/D| \leq |D|^{-2} \min\{|DD|, |D/D|\}^{3}. \]

Combining the last two bounds, we obtain the first inequality from (4.5). The second inequality follows from Theorem 19. The trivial estimates $|A| \leq |D| \leq |A|^2$ imply formula (4.6). This completes the proof. \( \Box \)

From identity (4.3) and the results of section 3, it follows that the set $R$ has some interesting properties, which we give in the next proposition, one consequence of which is that $R$ has a large product set $|RR| \gg |R|^{5/4}$.

**Proposition 21** Let $A \subset \mathbb{R}$ be a finite set, and $R = R[A]$. Then $D_{\times}(R) \ll |RR|^2/|R|^2$. In particular, for any $B \subset \mathbb{R}$ one has
\[ |RB| \gg \frac{|R|^2 |B|^{1/2}}{|RR|}. \] (4.8)

**Proof.** Put $f(x) = \ln(x - 1)$. Then one can check that for any $b \neq b'$ the function $f(xb) - f(xb')$ is monotone. Formula (3.6) gives us that
\[ D_{\times}(R) \leq \min_{C} \frac{|f(R) + C|^2}{|R||C|}. \]

Now putting $C = \ln R$ and using identity (4.3), we deduce that $D_{\times}(R) \leq |RR|^2/|R|^2$. The last bound combined with formula (3.3) of Corollary 17 implies (4.8). This completes the proof. \( \Box \)

One can obtain the inequality $|RR| \gg |R|^{5/4}$ by taking $B = R$ in the general formula (4.8). Actually, since Proposition 21 gives the stronger result $D_{\times}(R) \ll |RR|^2/|R|^2$, one can apply methods of [21] to derive the inequality $|RR| \gtrsim |R|^{14/9} \cdot D_{\times}(R)^{-5/9}$ and hence the inequality $|RR| \gtrsim |R|^{24/19}$.

We finish this section with an example of a set $A$ with a difference set that has small product set and small quotient set.
Proposition 22. For any integer \( n \geq 1 \) there is a set \( A \subset \mathbb{R} \), \(|A| = n\) such that \(|DD|, |D/D| \leq 25|D|^{3/2}\), where \( D = A - A \).

Proof. Let \( A = \{2, 4, \ldots, 2^n\} \). Then it is easy to see that \( n^2/2 \leq |D| = n^2 - n + 1 \leq n^2 \). Furthermore,
\[
DD = \{(2^i - 2^j)(2^k - 2^l) : i, j, k, l \in [n]\} = \{2^{i+j}(2^{i-j} - 1)(2^{k-l} - 1) : i, j, k, l \in [n]\}
\]
and hence \(|DD| \leq (2n)^3 \leq (2\sqrt{2|D|})^3 \leq 25|D|^{3/2}\). Similarly,
\[
D/D = \{(2^i - 2^j)/(2^k - 2^l) : i, j, k, l \in [n]\} = \{2^{i-l}(2^{i-j} - 1)/(2^{k-l} - 1) : i, j, k, l \in [n]\}
\]
and again \(|D/D| \leq 25|D|^{3/2}\). This completes the proof. \(\square\)

5. The finite fields case

In this section \( p \) is a prime number. An analog of Lemma 18 in the prime fields setting is the following, see [1].

Lemma 23. Let \( A, B \subset \mathbb{F}_p \) be two subsets with \(|A| = |B| < p^{2/3}\). Then for any nonzero \( \alpha \in \mathbb{F}_p \) one has
\[
|A \cap (B + \alpha)| \ll |A|^{-1/2}|AB|^{4/3}. \tag{5.1}
\]
Moreover, for any \( C, D \subset \mathbb{F}_p \) with \(|C|, |D| < p^{2/3}\) we have the inequality
\[
|A \cap (B + \alpha)| \ll (|C||D|)^{-1/4}(|AC||BD|)^{2/3}. \tag{5.2}
\]

Sketch of the proof. Actually, bound (5.1) was proved in [1] in a particular case \( A = B \) but since the proof uses a variant of the Szemerédi–Trotter Theorem in \( \mathbb{F}_p \), namely,
\[
\mathcal{J}(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{3/4} |\mathcal{L}|^{2/3} + |\mathcal{L}| + |\mathcal{P}| \tag{5.3}
\]
for any set \( \mathcal{P} \) of the form \( \mathcal{P} = A \times \mathbb{B} \) with \(|\mathcal{B}| \leq |\mathcal{A}| < p^{2/3}\) (see [1]), the arguments of the proof of Lemma 18 are preserved. In order to obtain (5.2) we use the same method with \( \mathcal{P} = C^{-1} \times D^{-1} \) and \( \mathcal{L} = \{l_{s,t}\} \), where \( s \in AC, t \in BD \) and \( l_{s,t} := \{ (x, y) : sx - ty = \alpha \} \), so \(|\mathcal{L}| = |AC||BD|\). Then one has \( \mathcal{J}(\mathcal{P}, \mathcal{L}) \geq |C||D||A \cap (B + \alpha)| \) — see the proof of Lemma 18. Let \( m = \min\{|A|, |B|\} \). Suppose that
\[
|A \cap (B + \alpha)| \gg (|C||D|)^{-1/4}(|AC||BD|)^{2/3} \tag{5.4}
\]
because otherwise there is nothing to prove. It follows that
\[
(|AC||BD|)^8 \ll m^{12} (|C||D|)^3.
\]
If the second term in (5.3) dominates, then in view of the last inequality, we obtain
\[
m^6 (|C||D|)^{3/2} \gg (|AC||BD|)^4 \gg (|C||D|)^9.
\]
Thus, \(|C||D| \ll m^{4/5}\), and from (5.4) we have
\[m^{6/5} \gg |A \cap (B + \alpha)||(|C||D|)^{1/4} \gg (|AC||BD|)^{2/3} \geq (|A||B|)^{2/3} \geq m^{4/3},\]
which is a contradiction for large \(m\). Similarly, if the third term in (5.3) is the largest one, then
\[(|C||D|)^3 \gg (|AC||BD|)^8 \geq (|C||D|)^8,\]
which is a contradiction for large \(C, D\). In the remaining case, we obtain
\[|C||D||A \cap (B + \alpha)| \ll \mathcal{J}(\mathcal{P}, \mathcal{L}) \ll |\mathcal{P}|^{3/4}|\mathcal{L}|^{2/3} \leq (|C||D|)^{3/4}(|AC||BD|)^{2/3}\]
as required.

Now let us prove an analog of Theorem 20.

**Theorem 24** Let \(A \subset \mathbb{F}_p\) be a finite set and let \(D = A - A\). Suppose that \(|R[A]| \leq cp^{5/9}\), where \(c > 0\) is an absolute constant. Then
\[|DD|, |D/D| \gg |D|^{19/3} |R[A]|^{1/3} \gg |D|^{10/3} |A|^{1/3}.\]  
(5.5)

**Proof.** The arguments are the same as in the proof of Theorem 20. Put \(R = R[A]\). Suppose that \(|D| < p^{2/3}\). Then, using identity (4.3) and Lemma 23 (with \(A = -R, B = R, \alpha = -1, C = D = A - A\)) instead of Lemma 18, we get
\[|R| = |R \cap (R - 1)| \ll |D|^{-1/2} |RD|^{4/3} \leq |D|^{-1/2} |DD/D|^{4/3}.\]  
(5.6)

By Lemma 7, we have
\[|DD/D| \leq |D|^{-2} \min\{ |DD|, |D/D|\}^{3}.\]
Combining the last two bounds, we obtain the first inequality from (5.5). Because \(|A| \leq |D| < p^{2/3}\), we see that Theorem 10 implies that \(|R[A]| \gg |A|^{3/2}\) and this gives us the second inequality.

Now assume that \(|D| \geq p^{2/3}\). We shall show that
\[|DD|, |D/D| \gg |D|^{10/3} |R|^{1/3}.\]  
(5.7)
If inequality (5.7) does not hold for \(|DD|\), then
\[|D| \leq |DD|, |D/D| \ll |D|^{10/3} |R|^{1/3}.\]
Hence,
\[|R| \gg |D|^{5/6} \geq p^{5/9},\]
which is a contradiction. Again, the second inequality in (5.5) follows from Theorem 10 and the fact that \(|A| \leq |R| \ll p^{5/9} < p^{2/3}\). This completes the proof.  

Note that an analog of Proposition 21 also holds in \(\mathbb{F}_p\) because of an appropriate version of Theorem 11 (see [1]). Also note that one can relax the condition \(|R[A]| \leq cp^{5/9}\) at the cost of a weaker bound (5.5).

Now we can obtain Theorem 5 from the introduction.
\textbf{Theorem 25} Let \( \Gamma \subset \mathbb{F}_p \) be a multiplicative subgroup with \( |\Gamma| < p^{3/4} \) and let \( \xi \neq 0 \) be an arbitrary residue. Suppose that for some \( A \subset \mathbb{F}_p \) one has
\[
A - A \subseteq \xi\Gamma \bigcup \{0\}.
\] (5.8)
Then \( |A| \ll |\Gamma|^{4/9} \). If \( |\Gamma| \geq p^{3/4} \), then \( |A| \ll |\Gamma|^{4/3}p^{-2/3} \). In particular, for any \( \epsilon > 0 \) and sufficiently large \( \Gamma, |\Gamma| \leq p^{4/5-\epsilon} \) we have that
\[
A - A \neq \xi\Gamma \bigcup \{0\}.
\]

\textbf{Proof.} We can assume that \( |A| > 1 \). Let \( \Gamma_* = \Gamma \cup \{0\} \) and let \( R = R[A] \). Then in the light of our condition (5.8), we have
\[
R \subseteq (\xi\Gamma \cup \{0\})/\xi\Gamma = \Gamma_*.
\]
In particular, \( |R| \leq |\Gamma| + 1 < p^{3/4} + 1 \). We know by (5.8) that \( |A| \leq |\Gamma|^{1/2+o(1)} \ll p^{2/3} \) (see [25], [23], [22] or just Theorem 13). Hence, using Theorem 10, we obtain that \( |R| \gg |A|^{3/2} \). Applying (5.6) and the last inequality, we get
\[
|A|^{3/2} \ll |R| = |-R \cap (R - 1)| \leq |\Gamma_* \cap (1 - \Gamma_*)|.
\]
Using Stepanov’s method from [9] or just the case \( k = 1 \) of Theorem 13, we obtain
\[
|A|^{3/2} \ll |\Gamma_* \cap (1 - \Gamma_*)| \leq |\Gamma \cap (1 - \Gamma)| + 2 \ll |\Gamma|^{2/3}
\]
as required.

Now let us suppose that \( |\Gamma| \geq p^{3/4} \). Using formula (2.7) of Lemma 14 with parameter \( k = 1 \) and the previous calculations, we get
\[
|A|^{3/2} \ll |\Gamma_* \cap (1 - \Gamma_*)| \leq |\Gamma \cap (1 - \Gamma)| + 2 \ll |\Gamma|^2/p
\]
as required. Finally, it is known that if \( A + B = \xi\Gamma \) or \( A + B = \xi\Gamma \bigcup \{0\} \), then \( |A| \sim |B| \sim |\Gamma|^{1/2+o(1)} \), see [25], [23], [22]. Thus, we have \( \xi\Gamma \bigcup \{0\} \neq A - A \) for sufficiently large \( \Gamma \). This completes the proof. \( \square \)

\section{The case \( D/D \)}

The bound (4.6) as well as a similar estimate (5.5) can be improved in the case of the quotient set \( D/D \). We thank Misha Rudnev and Oliver Roche–Newton who pointed out an idea for how to do this.

First of all, let us generalize our main identity \( R[A] = 1 - R[A] \) as well as Theorem 19. Let \( A \subset \mathbb{R} \) be a subset with \( |A| > 1 \) and let \( D := A - A \). Also, let \( X \subseteq D \setminus \{0\} \) be an arbitrary subset such that \( X = -X \). Let
\[
R_X[A] = \left\{ \frac{a_2 - a}{a_1 - a} : a, a_1, a_2 \in A, a_1 - a \in X \right\}.
\]
Then
\[
R_X[A] = 1 - R_X[A].
\] (6.1)
Note that \( R_X[A] \subseteq D / X \). Finally, let

\[
\sigma_X(A) := \sum_{x \in X} |A \cap (A + x)|.
\]

Let us derive a consequence of Theorem 9.

**Theorem 26** Let \( A \subset \mathbb{R} \) be finite and let \( X \subseteq (A - A) \setminus \{0\} \) be a set with \( X = -X \). Then

\[
|R_X[A]| \gg \frac{\sigma_X^2(A)}{|A|^2 \log |A|}.
\]  

(6.2)

**Proof.** We have

\[
|A| \sigma_X(A) = \sum_{x \in X} \sum_{y} |A \cap (A + x) \cap (A + y)| = \sum_{\lambda \in R_X[A]} \sum_{x \in X} |A \cap (A + x) \cap (A + \lambda x)|.
\]

Using the Cauchy–Schwarz inequality, we get

\[
|A|^2 \sigma_X^2(A) \leq |R_X[A]| \sum_{\lambda} \left( \sum_{x} |A \cap (A + x) \cap (A + \lambda x)| \right)^2 = |R_X[A]| T(A).
\]  

(6.3)

Applying Theorem 9, we obtain the result. This completes the proof. \( \square \)

Note that we do not use the fact that \( A \) is a subset of the reals to get formula (6.3), but just that \( A \) belongs to some field.

Now we obtain the main result of this section.

**Theorem 27** Let \( A \subset \mathbb{R} \) be a finite set and let \( D = A - A \). Then

\[
|D/D| \gg |D| \frac{1}{2} |A| \frac{1}{2} \log^{-\frac{1}{2}} |A| \gg |D|^{1+\frac{1}{2}} \log^{-\frac{1}{2}} |D|.
\]  

(6.4)

In particular, if \( |(A - A)/(A - A)| \ll |A|^2 \), then \( |A - A| \ll |A|^2 \frac{1}{2} \log \frac{1}{2} |A| \).

**Proof.** Put \( L = \log |A| \). Let

\[
D' = \{ x \in D : |A \cap (A + x)| > |A|^2 / (2|D|) \}.
\]

We have \( \sigma_{D'}(A) \geq |A|^2 / 2 \). Now let

\[
D'_j = \{ x \in D' : |A|^2 / (2|D|) \cdot 2^{j-1} < |A \cap (A + x)| \leq |A|^2 / (2|D|) \cdot 2^j \},
\]

where \( j \geq 1 \). By the pigeonhole principle there exists \( j \ll L \) such that

\[
\sigma_{D'_j}(A) \gg |A|^2 / L.
\]  

(6.5)
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Set $\Delta = |A|^2/(2|D|) \cdot 2^j$ and redefine $D'$ to be the set $D'$. Further, instead of using the Plünnecke–Ruzsa inequality (2.1), choose $\delta = 1/2$ and apply estimate (2.2) in its multiplicative form with $A = D'$ and $B = D^{-1}$. It gives us a set $X \subseteq A$ with $|X| \geq |D'|/2$ such that

$$|XBB| = |X/(DD)| = |DD/X| \ll |D/D|^2|X|/|D|^2. \quad (6.6)$$

Considering the set $X \cup (-X)$, we see that the last bound holds for this new set with possibly bigger constants. Redefining $X$ to be $X \cup (-X)$ we conclude that the bound holds for a symmetric set. After that, using the arguments of the proof of Theorem 20, identity (6.1), and Lemma 18 with $A = -R_X[A], B = R_X[A], C = D = A - A$ and $\alpha = -1$, we get

$$|R_X[A]| = |R_X[A] \cap (1 - R_X[A])| \ll |D|^{-2/3}|R_X[A] \cdot D|^{4/3} \leq |D|^{-2/3}|DD/X|^{4/3}.$$  

Substituting the bound (6.6) as well as estimate (6.2) of Theorem 26, we obtain

$$\frac{\sigma_X^{3/2}(A)|D|^{1/2}}{|A|^{7/2}L^{3/4}} \ll |R_X[A]|^{3/4}|D|^{1/2} \ll |DD/X| \ll \frac{|D/D|^2|X|}{|D|^2}. \quad (6.7)$$

We have $\sigma_X(A) \geq \Delta|X|2^{-1}$ and hence

$$|D/D| \gg |D|^{5/2}|X|^{1/2}\Delta^{3/2}/|A|^{7/2}L^{3/4}. \quad (6.8)$$

In view of inequality (6.5) and the bound $|X| \geq |D'|/2$, we know that

$$\Delta|X| \geq \sigma_X(A) \gg \sigma_{D'}(A) \gg |A|^2/L.$$  

Substituting the last estimate into (6.8) and recalling that $\Delta \gg |A|^2/|D|$, we obtain, finally,

$$|D/D| \gg \frac{|D|^{5/2}\Delta}{|A|^{7/2}L^{3/4}} \gg \frac{|D|^{3/2}|A|^{3/2}}{L^{5/4}}.$$  

Thus, we have

$$|D/D| \gg \frac{|A|^{3/4}|D|^{3/4}}{L^{5/8}} \gg \frac{|D|^{9/8}}{L^{5/8}}$$

as required. This completes the proof. \hfill $\square$

The same arguments, combined with the method of proof of Theorem 24, give us the following theorem.

**Theorem 28** Let $A \subseteq \mathbb{F}_p$ be a finite set, $D = A - A$. Suppose that $|A| \leq cp^{10/27}\log^{4/9}|A|$, where $c > 0$ is an absolute constant. Then

$$|D/D| \gg \frac{|D|^{11/2} |A|^{9/10} \log^{-4/5} |A|}. \quad (6.9)$$
We finish our paper by studying the quantity \( D \).

The last inequality contradicts the assumption that \( 1 \).

If there is no such a function \( F \), then the quantity \( d \)
form a pseudo-line system, \( l_{b,s} = \{(x,y) : s = F(\Phi(b,x),y)\} \).

\begin{align*}
|X|^3/2 \Delta^{3/2} |D|^{3/8} \leq |A|^{15/8} \leq |A|^3/4 |D|^{3/8} \leq |DD/X| \leq |D/D|^2 |X|/|D|^2.
\end{align*}

Of course one needs to apply Theorem 10 instead of Theorem 9 in the proof. Using the last formula we obtain after some calculations the result in the case \( |D| < p^{2/3} \). For larger sets we suppose that (6.9) does not hold and obtain that
\[ |D| \leq |DD|, |D/D| \leq |D|^{11/12} |A|^{9/2} \log^{-1/2} |A| \]
and hence that
\[ p^{10/27} \leq |D|^{5/9} \ll |A| \log^{-4} |A|. \]
The last inequality contradicts the assumption that \( |A| \ll p^{10/27} \log^{4/9} |A| \). This completes the proof. \( \square \)

7 Appendix

We finish our paper by studying the quantity \( D_\Phi(A) \) and obtaining the bound (3.6).

**Definition 29** For any function \( \Phi : \mathbb{R}^2 \to \mathbb{R} \) and any finite set \( A \subset \mathbb{R} \), let
\[ d_\Phi(A) = \inf_{F} \min_{C} \frac{|F(A,C)|^2}{|A||C|}, \tag{7.1} \]
where the infimum is taken over all functions \( F \) such that
1) for any given \( b \) and \( s \) the curves
\[ l_{b,s} := \{(x,y) : s = F(\Phi(b,x),y)\} \tag{7.2} \]
form a pseudo-line system,
2) the number of solutions to the equation \( s = F(\Phi(b,x),y) \) is at least two,
3) the set \( C \) in (7.1) is chosen over all nonempty subsets of \( \mathbb{R} \) such that for every \( a \in A \) we have that \( |F(\{a\},C)| = |C| \) and that \( |F(\tilde{A},C)| \geq |\tilde{A}| \) for every \( \tilde{A} \subseteq A \).

If there is no such a function \( F \), then the quantity \( d_\Phi(A) \) is not defined.

Let us give some examples.

**Example 30** Suppose that for any \( b \neq b' \) the function \( F(\Phi(b,x),y) - F(\Phi(b',x),y) \) does not depend on \( y \) and is a monotone function of \( x \). Then the curves defined in (7.2) form a pseudo-line system. For example, if \( F(x,y) = x + y \), then it is enough to ensure that \( \Phi(b,x) - \Phi(b',x) \) is monotone. If \( F(x,y) = xy \), then the monotonicity of \( \Phi(b,x)/\Phi(b',x) \) ensures that the curves defined in (7.2) form a pseudo-line system.

**Example 31** It is easy to see that
\[ D_+(A) \leq \tilde{d}_+(A). \]
Indeed, just put \( F(x,y) = f(x) + y \) and check that the curves \( l_{b,s} = \{(x,y) : s = f(b+x)+y\} \) satisfy all the required properties of Definition 29 (e.g., the difference \( F(\Phi(b,x),y) - F(\Phi(b',x),y) = f(b+x) - f(b'+x) \) does not depend on \( y \) and is monotone).
Let us show some simple properties of the quantity $d_\Phi(A)$.

**Lemma 32** Let $A \subset \mathbb{R}$ be a finite set and let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the quantity $d_\Phi(A)$ is defined. Then

- $1 \leq d_\Phi(A) \leq |A|$.
- For any $A' \subseteq A$ one has $d_\Phi(A') \leq d_\Phi(A) \cdot |A|/|A'|$.

**Proof.** By the assumption, for any $C$ in the infimum from (7.1) we have that $|F(A, C)| \geq \max\{|A|, |C|\}$. Thus, $\frac{|F(A, C)|^2}{|A||C|} \geq 1$, and hence $d_\Phi(A) \geq 1$. Taking $C$ equal to a one–element set and applying a trivial bound $|F(A, C)| \leq |A|$, we obtain that $d_\Phi(A) \leq |A|$.

Let us prove the second part of the lemma. Take a set $C$ and a function $F$ such that $d_\Phi(A) \geq \frac{|F(A, C)|^2}{|A||C|} - \varepsilon$, where $\varepsilon > 0$ is arbitrary, such that for any element $a \in A$ we have that $|F(\{a\}, C)| \geq |C|$ and that for all $\tilde{A} \subseteq A$ the inequality $|F(\tilde{A}, C)| \geq |\tilde{A}|$ holds. Clearly, for an arbitrary $a \in A'$ we have again $|F(\{a\}, C)| \geq |C|$ and for all $\tilde{A} \subseteq A'$ we have the inequality $|F(\tilde{A}, C)| \geq |\tilde{A}|$. Using the trivial inequality $|F(A', C)| \leq |F(A, C)|$, we obtain

$$d_\Phi(A') \leq \frac{|F(A', C)|^2}{|A'||C|} \leq \frac{|F(A, C)|^2}{|A'||C|} \leq (d_\Phi(A) + \varepsilon) \cdot |A|/|A'|$$

as required. \qed

If $A = \{0\}$, $F(x, y) = xy$, and $\Phi(x, y) = xy$, say, then it is easy to see that $d_\Phi(A) = 0$. Thus, we need the property $|F(\{a\}, C)| \geq |C|$ for every $a \in A$ to obtain $d_\Phi(A) \geq 1$.

Now we can prove the main technical statement of this section.

**Proposition 33** Let $A, B \subset \mathbb{R}$ be finite sets and let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be any function such that for any fixed $z$ and $a \in A$ we have the inequality $|\{b \in B : \Phi(b, z) = a\}| \leq M$, where $M > 0$ is an absolute constant. Then for every real number $\tau \geq 1$ one has

$$|\{x : |\{(a, b) \in A \times B, a = \Phi(b, x)\}| \geq \tau\}| \ll_M \frac{d_\Phi(A)|A||B|^2}{\tau^3}. \quad (7.3)$$

In other words, $A$ has SzT$\Phi$–type with $O_M(d_\Phi(A))$.

**Proof.** First of all note that we can assume that $\tau \leq |B|$, since otherwise the left–hand side of inequality (7.3) is equal to 0. Further, by hypothesis, for any fixed $z$ and any $a \in A$ one has $|\{b \in B : \Phi(b, z) = a\}| \leq M$ and hence

$$\tau \leq \min\{|B|, M|A|\}. \quad (7.4)$$

Now take an arbitrary nonempty set $C \subset \mathbb{R}$ and any $x$ from the set in the left–hand side of (7.3). Then for every $c \in C$ one has

$$F(\Phi(b, x), c) = F(a, c) := s \in F(A, C), \quad (7.5)$$
where $F$ is any function from the infimum in the definition of the quantity $d_0(A)$. Consider the curves $l_{b,s} = \{(x, y) : s = F(\Phi(b, x), y)\}$, $b \in B$, $s \in F(A, C)$. By assumption, the set $\mathcal{L} = \{l_{b,s}\}$ forms a pseudo-line family. In particular, $|\mathcal{L}| = |B||F(A, C)|$. Also, let $\mathcal{P}$ be the set of all intersection points defined by the curves. Identity (7.5) tells us that the point $(x, c)$ belongs to at least $\tau$ curves $l_{b,s}$. Hence, $(x, c)$ belongs to a narrow family of points $\mathcal{P}_\tau$ of intersections of at least $\tau$ curves $l_{b,s}$. Using the Szemerédi–Trotter Theorem 11, we obtain

$$|C| \cdot \{|(a, b) \in A \times B, a = \Phi(b, x) \geq \tau\} \leq |\mathcal{P}_\tau| \ll \frac{|F(A, C)|^2|B|^2}{\tau^3} + \frac{|F(A, C)||B|}{\tau} \ll_M \frac{|F(A, C)|^2|B|^2}{\tau^3}.$$ 

To derive the last inequality we have used estimate (7.4), the bound $|F(A, C)| \geq |A|$ and the trivial inequality

$$\tau^2 \leq (\min\{|B|, M|A|\})^2 \leq |B| \cdot M|F(A, C)|.$$

This concludes the proof.

**Example 34** Let $F(x, y) = x + y$ and let $\Phi(x, y) = y \sin x$. Then the lines $l_{b,s} = \{(x, y) : s = y + x \sin b\}$ form a pseudo-line system. Nevertheless, for any fixed $z$ one has $|\Phi^{-1}(z, z)| = +\infty$ or zero. So, the pseudo-line condition does not imply any bounds for the size of preimages of the map $\Phi$.

**Remark 35** It is easy to see that it is enough to check the condition $|\{b : \Phi(b, z) = a\}| \leq M$, $a \in A$ just for $z$ belonging to the set from the left-hand side of (7.3).

Now let us derive some consequences of Proposition 33. Inequality (7.6) in the corollary below was obtained in [18]; see Lemma 2.6. Here we give a more systematic proof. The corollary asserts that $D_+(A) < 4$ for any convex/concave set $A$. On the other hand, from Proposition 33 it follows that $D_+(A) \leq M^2$ for any $A$ with $|AA| \leq M|A|$ or $|A/A| \leq M|A|$. This demonstrates some combinatorial similarity between convex sets and sets with small product/quotient set.

**Corollary 36** Let $A \subset \mathbb{R}$ be a finite convex set. Then $D_+(A) < 4$. Furthermore, if $A' \subseteq A$ is a subset of $A$, then for an arbitrary set $B \subset \mathbb{R}$ one has

$$|A' + B| \gg |A'|^{3/2}|B|^{1/2}|A|^{-1/2}. \quad (7.6)$$

**Proof.** By assumption, $A = g(I)$, where $I = \{1, 2, \ldots, |A|\}$ and $g$ is a convex function. In view of Example 31, we have $D_+(A) \leq \tilde{d}_+(A)$. Moreover, substituting $f = g^{-1}$ in formula (3.5) and taking $C = I$, we obtain

$$D_+(A) \leq \tilde{d}_+(A) \leq \frac{|I + I|^2}{|A||I|} = \frac{(2|I| - 1)^2}{|I|^2} < 4.$$ 

Inequality (7.6) follows from Corollary 17 and the second part of Lemma 32. This completes the proof. □

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To derive the bound (3.6) from Proposition 33, just put $F(x, y) = f(x) + y$ and consider the curves

$$l_{b,s} = \{(x, y) : s = f(xb) + y\}.$$

These curves have infinite cardinality and form a pseudo-line system. Clearly, for any nonempty $A, C$ such that $0 \notin A, C$ one has $|F(\{a\}, C)| = |C|$ for every $a \in A$, and $|F(\tilde{A}, C)| \geq |\tilde{A}|$ holds for an arbitrary $\tilde{A} \subseteq A$.

Thus, formula (3.6) follows.

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