Combined variable field theory in FLRW cosmology

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Abstract

This work aims to study consequences of combined variable field theory developed in [1] by analyzing FLRW models of gravity. It shows $\Phi_{q \rightarrow 0} \rightarrow \infty$ is the Big bang in combined variable field theoretic settings. Nature of the Universe depends on the new parameter $k$ appearing in the theory. Flat Universe has two disconnected branches namely expanding and collapsing Universe. The combined variable field for closed Universe is complex in general. Whereas, the theory of open Universe is necessarily self-coupled combined variable field theory. But in the large $r$ limit, it reduces to the theory of flat Universe. Resolution of the classical Big bang is a result of quantization program. Quantum theory predicts that the size of Universe was non-zero in the beginning. Combined variable field quantum in absence of scalar field coupling is interpreted as a quantum of space. The energy spectrum of this combined variable field quantum varies with time.

1 Introduction

I will be working with massless scalar field throughout this paper. Theory formulated in [1] is recast into FLRW model of the Universe. So that consequences of the theory can be studied and results can be compared. The theory differs from other canonical theories such as loop quantum gravity and Wheeler-DeWitt theory. Unlike other canonical theories, $(\phi, q_{\alpha})$ are kinematical variables in combined variable field theory. The Universe is seen as a field. Similar to loop quantum cosmology, quantization of gravity is a result of quantization program and not an assumption. Theory suggests that the Universe began with finite, non-zero size. As Universe expanded, the energy of a quantum of space decreased sharply. In large $q$ limit, approaches to 3 units of Planck energy.

When classical theory shows singularity, quantum theory expected to take over which would be free from those singularities. Therefore the Big bang singularity expected to get resolved in the quantum theory of gravity. There have been several attempts to quantize the gravity. Loop quantum gravity which is also built under canonical formulation of general relativity is one of the promising theory among other theories. Loop quantum cosmology which is a branch of loop quantum gravity resolves the Big bang singularity. One can refer [2] in order to understand quantization program and resolution of the
Big bang. An important lesson taken from loop quantum gravity is that the fields evolve with respect to one another. As mentioned in above paragraph, reason to choose massless scalar field because it acts as a clock. There are various papers suggesting this such as [2]. Whereas [3] discusses in detail about possibilities and limits under which scalar field can be used as a clock. These two points are key ingredients for this work.

History taught us that quantizing non-relativistic theory and quantizing special relativistic theory is not equivalent. Quantization of non-relativistic theory happens to be straightforward whereas quantization of relativistic theory results in quantum field theory. When one classically evolves initial matter field configuration (which curves the space-time), underline space-time also evolves. This evolution is unique. But in case of quantum evolution, for a particular matter field configuration virtual multi-curvature states are possible. This very fact leads to combined variable field $\Phi$ distributed over metric and evolves with matter field. This is achieved by re-interpreting ($\hat{H}\Phi = 0$) $\Phi$ as a classical combined variable field (refer [1] for details).

Section 2 is a brief overview of combined variable field theory formulated in [1]. Equations have been re-derived in the FLRW cosmological context. Hubble parameter can be obtained from Hamiltonian dynamics but it does not have natural combined variable field theoretic extension. Section 3 splits further into three parts $\kappa = 0$ (flat Universe), $\kappa = +1$ (closed Universe) and $\kappa = -1$ (open Universe). Section 4 contains quantum analog of section 3. In the end, section 5 concludes earlier three sections. Throughout this paper I am going to work in units of $16\pi G = 1$, $\hbar = 1$ and $c = 1$.

## 2 Overview

ADM formulation recasts gravity as a gauge field theory and thus brings closer to other special relativistic field theories. Zero Hamiltonian is a hallmark of background independent theories.

$$\int_M d^3x \left( N^a H_a + |N| \left( H_{\text{scalar}} + H_\phi \right) \right) = 0 \tag{1}$$

As mentioned in section 2, of paper [1], the full Hamiltonian constraints includes scalar Hamiltonian constraints, vector (or Diffeomorphism) constraints $H_a$ and matter field part. The scalar field taken here is a massless scalar field. The shift vector $N^a$ and lapse function $N$ are Lagrange multipliers. Shift vector together with the lapse function tells how neighbouring slices of hypersurfaces are deformed.

$$ds^2 = \left( N^2 - N_a N^a \right) dt^2 - 2N_a dt dx^a - q_{ab} dx^a dx^b \tag{2}$$

$q_{ab}$ is metric defined over 3 dimensional spatial manifold. By separating space and time dependent parts of gravitational field as well as scalar field ([1], section 2, (19)) and re-defining (rather a canonical transformation in the [1], section 2, (32)), we get

$$H_{\text{total}} = \frac{1}{2} p_\phi^2 - \frac{1}{2} |\dot{q}|^2 p_\rho^2 + 2\alpha \dot{q} p_k^k - V(q) \tag{3}$$
scalar field potential is 0 in this case. \( \eta_i := q_i \dot{q}_j \). Combined variable field theory requires \( \alpha = i \) and lapse function chosen as (refer [1], section 2)

\[
N^a = \frac{-\alpha}{\int_M d^3x \left( q_{ac}(x) \dot{D}_b(x) \dot{P}^{bc}(x) \right)} \quad N = \frac{\sqrt{\det q_{a}(t)}}{\int_M d^3x \left( \int_{abca}(x) \dot{D}_b(x) \dot{P}^{bc}(x) \right)} \tag{4}
\]

Where \( D_b \) is unique torsion-free covariant differential compatible with \( q_{ab} \) and \( \dot{f}_{abcd} := \frac{1}{\sqrt{\det \dot{q}_{ab}(x)}} \left( \dot{q}_{ab}(x) \dot{q}_{cd}(x) - \dot{q}_{ac}(x) \dot{q}_{bd}(x) - \dot{q}_{bc}(x) \dot{q}_{ad}(x) \right) \tag{5} \)

\( \dot{P}^{ab}(t, \bar{x}) := \left( \sqrt{\det \dot{q}_{ab}(\bar{x})} \left( \dot{K}^{ab}(\bar{x}) - \dot{K}(\bar{x}) q^{ab}(\bar{x}) \right) \right) \tag{6} \)

\( \dot{\bar{x}} \) indicates spatial component. \( \dot{K}_{ab} \) and \( \dot{K} \) are extrinsic curvature tensor and extrinsic curvature scalar respectively. In absence of scalar field coupling combined field potential is purely gravitational in nature. Borrowing combined variable field potential from (27), section 2, [1]

\[
V(\bar{x}) = \det(q_{a}(t)) \frac{\int_M d^3x \sqrt{\det \dot{q}_{ab}(x)} R^{(3)}(t, \bar{x})}{\int_M d^3x \left( \int_{abca}(x) \dot{D}_b(x) \dot{P}^{bc}(x) \right)} \tag{7}
\]

\( R^{(3)}(t, \bar{x}) \) is intrinsic curvature scalar in 3 dimensions. Spatial part of potential can be seen as a ratio of intrinsic spatial action to extrinsic spatial action. This form of combined potential is a result of separation of temporal and spatial dependence of field variables.

• FLRW model

3-Metric for FLRW models is given as

\[
q_{ab} := q_a(t) \text{ diag} \left( \frac{1}{1 - \kappa r^2}, r^2, r^2 \sin^2 \theta \right) \tag{8}
\]

\( \kappa \) is 0 for flat Universe, \( +1 \) for closed Universe and \( -1 \) for open Universe. Total Hamiltonian is obtained by using (39), section 2 of [1] for this FLRW metric

\[
H_{\text{total}} = \frac{1}{2} \dot{P}^2 - \frac{1}{2} (3qP)^2 + 6\alpha qP - V(\bar{q}) \tag{9}
\]

The third term is diffeomorphism term. In the paper [1], shift vector was chosen in such a way as to make combined variable field theory consistent. Here, only to make calculation easier I have chosen the right side of (25), section 2, [1] to be \( \alpha \). Notice that the dynamics depends neither on lapse function nor on the shift vector. Therefore, I choose \( \alpha = 0 \) to make calculation easier. But for combined variable field theory, the choice is given by (25), section 2, [1].

Equations of motion for gravitational part are given by

\[
\dot{q} := \{ q, H_{\text{total}} \} = -9q^2P \quad \dot{P} := \{ P, H_{\text{total}} \} = -9qP^2 \tag{10}
\]

These are coupled equations in \( q \) and \( P \). First we obtained \( P \) as a function of \( q \) by taking ratio and integrating

\[
P = \sqrt{q^2 + C} \tag{11}
\]
Finally, we get the solution to an equation of motion for \( q \) by using above equation

\[
q = \sqrt{C} \tan \left[ \sin^{-1} \left( \frac{1}{9Ct} \right) \right]
\]  

(12)

The domain of time is \( t \in (\frac{1}{9C}, \infty) \). As time increases \( q \) decreases and tends to zero in the limit \( t \to \infty \). One can relate this result to standard cosmology using \( d\tau = N(t)dt \). Invariant length element in standard cosmologies is \( ds^2 = d\tau^2 - a(\tau)^2(f(r)dr^2 + r^2d\Omega^2) \). Here, \( \tau \) represents time used in the standard cosmologies. Also note that the scalar field of this theory is related to the scalar field used in the standard cosmologies by a canonical transformation given in [1], section 2, (32). This is the reason why the form of \( q(t) \) and Hubble parameter \( \frac{\dot{a}}{a} \) is different from that of standard cosmologies. One can in principle work with the same scalar field (i.e. without canonically transforming it). In that case, combined variable theory developed could be different. Further more, whether that theory has same dynamics or not, remains a question. In fact, without that transformation, whether it has simpler combined variable field theoretic extension itself is a question. This goes beyond scope of this paper and explorations in that direction will be done in future work.

Equations of motion for scalar field part are given by

\[
\dot{\phi} := \{\phi, H_{\text{total}}\} = P_{\phi}
\]

\[
P_{\phi} := \{P_{\phi}, H_{\text{total}}\} = 0
\]  

(13)

\( P_{\phi} \) is a constant of motion and therefore \( \phi \) is a linear function of time. Notice that time ‘t’ used in \( q(t) \) is not a real physical time. Hamiltonian vanishes on the constraint surface. Therefore, Hamiltonian evolution is a gauge transformation and not a physical time evolution. Thus, evolution is better understood in terms of variation of \( q \) with respect to the scalar field (i.e. \( q(\phi) \)). Above result shows that \( q_{t=\infty} \rightarrow 0 \) and \( q \) increases non-linearly with a decrease of scalar field \( \phi \). In absence of a scalar field, variation of \( q \) can not be realized (in isotropic space).

On re-interpreting \((\dot{H}\Phi = 0)\) \( \Phi \) as classical combined variable field (refer section 3, [1]), we get

\[
\left( \partial_{\phi}^2 - \partial_i \eta_{ij} \partial^j + V(q, \phi) \right) \Phi(\phi, q) = 0
\]  

(14)

Where \( \eta_{ij} := q_{ij} \) serves as a spatial metric for combined variable field theory. Note that the \( H \) we used is the one with a correct choice of shift vector used in (25), section 2, [1] and not \( \alpha = 0 \). For FLRW case, by using metric defined in (8) above combined variable field equation is re-calculated as

\[
\left( \frac{\partial^2}{\partial q^2} - 9q^2 \frac{\partial^2}{\partial q^2} - 12 \frac{\partial}{\partial q} + V(q) \right) \Phi(r, q, \phi) = 0
\]  

(15)

Stress-energy tensor (refer section 3, [2]) can be recalculated for FLRW theory. Energy density \( (\rho) \) and pressure \( (p) \) for combined variable field

\[
\rho = T^0_0 = \frac{1}{2} \left( (\partial_{\phi} \Phi)^2 + \eta_{ij} \partial^i \Phi \partial^j \Phi + V(q) \Phi^2 \right)
\]  

(16)

\[
p = -T^1_1 = -T^2_2 = -T^3_3 = \eta_{ij} \partial^i \Phi \partial^j \Phi + \frac{1}{2} \left( (\partial_{\phi} \Phi)^2 - \eta_{ij} \partial^i \Phi \partial^j \Phi - V(q) \Phi^2 \right)
\]  

(17)
In the limit \( q \frac{\partial \Phi}{\partial q} \to 0, \)

\[
\rho = \frac{1}{2} \left( (\partial_\phi \Phi)^2 + V(q) \Phi^2 \right)
\]

(18)

\[
p = \frac{1}{2} \left( (\partial_\phi \Phi)^2 - V(q) \Phi^2 \right)
\]

(19)

In the limit \( \partial_\phi \Phi \to 0 \) pressure becomes negative.

### 2.1 Flat Universe

Solution to (15) is obtained using separation of variables. Let \( \Phi := T(\phi)Q(q) \),

\[
Q(q) \frac{\partial^2 T}{\partial \phi^2} = 9T(\phi)q^2 \frac{\partial^2 Q(q)}{\partial q^2} + 12T(\phi)q \frac{\partial Q(q)}{\partial q}
\]

Dividing this equation by \( T(\phi)Q(q) \) we get

\[
\frac{1}{T} \frac{\partial^2 T}{\partial \phi^2} = k^2 = \frac{9q^2 \frac{\partial^2 Q}{\partial q^2}}{Q} + \frac{12q \frac{\partial Q}{\partial q}}{Q}
\]

(20)

Left side of equation is purely scalar field dependent and right side is purely gravitational field dependent. Parameter \( k \) therefore must be a constant. Solution to the gravitational part as well as scalar field part is given respectively as

\[
Q(q) = q^{-\frac{1}{2}} \left( C_1 q^{\frac{1}{2} \sqrt{1+4k^2}} + C_2 q^{-\frac{1}{2} \sqrt{1+4k^2}} \right) \quad \text{and} \quad T(\phi) = e^{k\phi} + B_2 e^{-k\phi}
\]

(21)

\[
\Phi(\phi, q) = q^{-\frac{1}{2}} \left( C_1 q^{\frac{1}{2} \sqrt{1+4k^2}} + C_2 q^{-\frac{1}{2} \sqrt{1+4k^2}} \right) \left( B_1 e^{k\phi} + B_2 e^{-k\phi} \right)
\]

Constants \( k, C_1, C_2, B_1 \) and \( B_2 \) can be settled by applying appropriate boundary conditions. Combined variable field has singularity at \( q = 0 \) indicating the Big bang.

For \( \sqrt{1+4k^2} > 1 \): Combined variable field has two branches. The first, indicating collapsing Universe and the second indicating expanding Universe.

For \( \sqrt{1+4k^2} = 1 \): The first branch is steady state Universe whereas second is expanding Universe.

For \( 0 \leq \sqrt{1+4k^2} < 1 \): There exists only expanding Universe.

For \( \sqrt{1+4k^2} \in \mathbb{I} \): Combined variable field is complex. This can be seen as linear combination of two real scalar combined variable fields.

The left side of (20) is

\[
\frac{1}{T} \frac{\partial^2 T}{\partial \phi^2} = k^2
\]

(22)

It is already shown in section 2 that scalar field is a linear function of time. Therefore \( T(\phi) \) of combined variable field is equivalent to \( \phi(t) \) of a scalar field.

• Remarks:
For $k^2 < -\frac{1}{4}$, complex nature of combined variable field possibly indicating the existence of two independent real combined variable fields or in other words existence of two independent Universes. For $k^2 > -\frac{1}{4}$, there exists just one combined variable field. If one demands the existence of just our Universe then this condition puts a restriction on parameter $k$. $\lim_{q \to 0} \Phi = \infty$ is the Big Bang singularity. This parameter $k$ decides nature of the Universe. Expansion or collapse of the Universe is a result of $\Phi \to 0$. Using (18) and (19) we can also see that the combined variable field satisfies $\rho = p$ in the limit $q \frac{\partial \Phi}{\partial q} \to 0$.

2.2 Closed Universe

For closed Universe (i.e. $\kappa = +1$), spatial part of the metric is $q_{ab} = \text{diag}\left(\frac{1}{1-r^2}, r^2, r^2 \sin^2 \theta\right)$. Therefore, intrinsic curvature is positive i.e. $R^{(3)} = \frac{6}{r^2}$. Using unit vector $n_a$ normal to the surface $r = \text{constant}$ spatial part of extrinsic curvature tensor and extrinsic curvature scalar can be calculated.

$$K_{ab} := D_a n_b = \text{diag} \left(-r \sqrt{\frac{1}{1-r^2}}, \frac{r - r^3}{\sqrt{1-r^2}}, \frac{r (1 + r^2) \sin^2 \theta}{\sqrt{1-r^2}}\right)$$

$$K = \frac{2 - 3r^2}{r \sqrt{1-r^2}}$$

These curvatures are defined on 3-manifold which can be pulled back to 4-dimensional spacetime manifold using projections, lapse function and shift vector. Spatial part of combined potential is a ratio of intrinsic action to extrinsic action as shown in (7).

$$V(r,q) = 2 \left(\frac{2880 \left(-r + r^3 + \sqrt{1-r^2} \sin^{-1} r\right)}{\sqrt{1-r^2} (r \sqrt{1-r^2} (3105 - 6890r^2 + 8312r^4 - 4944r^6 + 1152r^8) + 735 \sin^{-1} r)}\right) \frac{\partial}{\partial q}$$

Combined potential varies quadratically with $q$ having positive $r$ dependent coupling. This coupling increases with $r$. On implementing energy conservation (which $\int d^3q \rho$ where $\rho$ is energy density of the combined variable field)
one can infer that the combined variable field behaves as an oscillator. Solution to the equation of motion

\[ \frac{1}{T} \frac{\partial^2 T}{\partial \phi^2} = k^2 = \frac{9q^2}{Q} \frac{\partial^2 Q}{\partial q^2} + \frac{12q}{Q} \frac{\partial Q}{\partial q} - 2V(r)q^2 \]  

is given as

\[ \Phi(\phi, q) = C_1 J \left( \frac{\sqrt{1 + 4k^2}}{6}, -i q \frac{\sqrt{2}V}{3} \right) \left( B_1 e^{k\phi} + B_2 e^{-k\phi} \right) \]

\[ + C_2 Y \left( \frac{\sqrt{1 + 4k^2}}{6}, -i q \frac{\sqrt{2}V}{3} \right) \left( B_1 e^{k\phi} + B_2 e^{-k\phi} \right) \]  

**Remarks:**

The combined variable field for closed Universe is complex in general. This can also be seen as a linear combination of two independent real combined variable fields. By analyzing in further detail, one can find out the nature of closed Universe depending on the initial conditions and the parameter \( k \). This will be done in the future work. Equations (18) and (19) suggest that near \( r = 0 \), \( \rho = p \) and near \( r = 1 \), \( \rho = -p \).

### 2.3 Open Universe

For open Universe (i.e. \( \kappa = -1 \)), spatial part of the metric is \( q_{ab} = \text{diag} \left( \frac{1}{1+r^2}, r^2, r^2 \sin^2 \theta \right) \). Therefore, an intrinsic curvature is positive i.e. \( R^{(3)} = -\frac{2}{q} \). The spatial part of extrinsic curvature tensor as well as extrinsic curvature scalar are given as

\[ K_{ab} = \text{diag} \left( r \sqrt{\frac{1}{1+r^2}}, r \frac{1}{(1+r^2)^{3/2}}, \frac{r \sin^2 \theta}{(1 + r^2)^{3/2}} \right) \]

\[ K = \frac{2 + 3r^2}{r \sqrt{\frac{1}{1+r^2}}} \]

Then combined variable field potential is obtained using these curvatures.

\[ V(r, q) = -2 \left( \frac{2880 (r + r^3 - \sqrt{1 + r^2} \sinh^{-1} r)}{3105r + 9995r^3 + 15202r^5 + 13256r^7 + 6096r^9 + 1152r^{11} + 735 \sqrt{1 + r^2} \sinh^{-1} r} \right)^2 \]  

(26)
Remarks:

This combined potential is also quadratic in $q$ but comes with a negative signature. As mentioned in [1] for negative potential $\Phi = 0$ is not a minima. Instead $\Phi_{\text{vac}} = \pm \sqrt{-V(r, q)}$ is a minima. where $\alpha$ is self-coupling constant. Therefore for negative spatial curvature, theory is a self-coupled combined variable field theory. But in the large $r$ limit, theory becomes a free combined variable field theory or the theory of flat Universe.

3 Quantum Theory

3.1 Overview

Quantization program of [1], section 3.2 shows that the Hamiltonian operator for free combined variable field can be written in terms of collection of an infinite harmonic oscillators and $|\phi, \vec{q}\rangle$ are eigen states of the Hamiltonian.

$$\hat{H} = \int d^Dq \left|\omega(\phi, \vec{q}) - D\right| \hat{\eta} + \int d^Dq \frac{1}{2} \omega(\phi, \vec{q}) \delta(0)$$

(27)

$D$ is dimensional parameter depends on how many components of the metric are time dependent. $\omega(\phi, \vec{q})$ is a solution to Riccati equation appearing at the end of this subsection. Unlike standard Hamiltonians, this Hamiltonian gives $\phi$ evolution. At first glance, it may sound weird but gravity being a dynamical theory of space-time it does not evolve with respect to any external time. Instead, it evolves with respect to a scalar field. This Hamiltonian is a function over the phase space $(\Phi, \Pi)$. The only non-trivial commutation relation (refer (57) section 3.2, [1]) between these fields is given by

$$\left[\hat{\Phi}(\phi, \vec{q}), \hat{\Pi}(\phi, \vec{q})\right] = i\delta(\vec{q}, \vec{q}')$$

(28)

and only non-trivial commutation relation (refer (64) section 3.2, [1]) between creation and annihilation operators satisfy

$$\left[\hat{a}(\phi, \vec{q}), \hat{a}^\dagger(\phi, \vec{q}')\right] = \left(\omega(\phi, \vec{q}) - D\right) \delta(\vec{q}, \vec{q}')$$

(29)
$D$ is a number of spatial dimensions. A number operator is defined as $\hat{n} := \hat{a}^{\dagger}\hat{a}$ for $\omega - D > 0$ whereas for $\omega - D < 0$ it is $\hat{n} := \hat{a}\hat{a}^{\dagger}$. This is because of role of creation and annihilation operator gets reversed in these two domains. $\omega$ is a solution to following Riccati equation (refer (62), section 3.2, [1])

$$V(\phi, \vec{q}) = \omega^2 - \frac{\partial}{\partial \vec{q}}(\hat{q}\omega) = \omega^2 - D\omega - \frac{\partial}{\partial \vec{q}}(30)$$

This equation arises in the process of quantization. Refer to quantum theory section 3.2 of [1] for more details.

### 3.2 Flat Universe

For flat (3 dimensional) Universe with a massless scalar field, the solution to Riccati equation

$$\omega^2 - 3\omega - 3q\frac{\partial}{\partial q} = 0 \quad (31)$$

$$\omega = \frac{3}{1 + Aq} \quad (32)$$

Where $A$ is constant which can be fixed by applying proper boundary conditions. Hamiltonian operator for flat Universe with massless scalar field is

$$\hat{H} = \int d^3q \left[ \frac{3}{1 + Aq} - 3 \right] \hat{h} + \int d^3q \frac{1}{2} \frac{3}{1 + Aq} \delta(\vec{0}) \quad (33)$$

For $A = 0$: $\epsilon = 0$ the theory is classical and does not have quantum analog.

For $A > 0$: $\epsilon$ increases with $q$ and approach to 3. Classical nature of the Universe at the big bang (at $q = 0$) which later becomes more and more quantum makes this condition unphysical.

For $A < 0$: $\lim_{q \to \infty} \epsilon \to 3$. Diverging nature of $\epsilon$ for $q \neq 0$ tells us that the Universe did not begin at $q = 0$ instead the Universe began at $q \neq 0$ ($= q_0$). Constant $A$ therefore can be identified as $A = -\frac{1}{q_0^2}$.

Only $A < 0$ is a physical choice. Because for $A = 0$, the Universe is completely classical and in case of $A > 0$, the Universe is classical at $q = 0$ (i.e. at the Big Bang). $A = -1/q_0$ has two branches $q < q_0$ and $q > q_0$. If in the
beginning our Universe had $q < q_0$, the Universe would have been collapsing Universe. But for $q > q_0$, the Universe is expanding. As $q$ increases, the energy of the combined variable field quantum decreases and approaches to 3 units of Planck energy. The Hamiltonian operator is a collection of an infinite combined variable field quantum. Creation operator acting on the vacuum state $|0\rangle$ produces $|1, q\rangle$. Quantum vacuum (which is infinite!) may thought to be a sea of constantly creating and annihilating combined variable field quantum.

### 3.3 Closed Universe

$\epsilon(r, q)$ is obtained using solution to Riccati equation (refer (62), section 3.2, [1]) as shown below

$$\omega^2 - 3\omega - 3q \frac{\partial \omega}{\partial q} = V(r)q^2 \quad (34)$$

$$\omega(\phi, q) = \frac{(3A + q \sqrt{-3V}) \sin(q \sqrt{-\frac{V}{3}}) + (3 - Aq \sqrt{-3V}) \cos(q \sqrt{-\frac{V}{3}})}{A \sin(q \sqrt{-\frac{V}{3}}) + \cos(q \sqrt{-\frac{V}{3}})} \quad (35)$$

Above plot of $\epsilon(r, q)$ verses $r$ shows that the combined variable field quantum have different energy spectrum at different $r$ and at different times as well (through time-dependent fields). $\epsilon$ hits singularity at $r = 0$. Above result is true for any value of $A$.

### 3.4 Open Universe

As mentioned in the classing part of the theory, this theory is self-coupled combined variable field theory. Combined potential in this case is $\frac{1}{2}V(r, q)\Phi^2 + \frac{1}{4}\alpha \Phi^4$. But I have taken a zeroth order approximation of the theory in order to understand zeroth order quantum effects. This approximation is sensible because combined potential is negligible in large $r$ limit and therefore theory (at least classically) becomes equivalent to that of the theory with flat Universe.
The form of $\omega$ required is same as that of given in (35) but combined potential is different. In the limit $r \to \infty$, the theory becomes a classical theory (as $\epsilon_{r=\infty} \to 0$). The role of creation and annihilation operator is exactly opposite compared to that of the theory with $\kappa = +1$. Detail analysis of this theory will be carried out in the next work.

4 Conclusion

Classical theory shows nature of the Universe can be expanding, static or oscillatory depends upon the parameter $k$. Further investigation is required in order understand the physics of parameter $k$. lim$_{\epsilon \to 0} \Phi \to \infty$ is the combined variable field theoretic understanding of the Big Bang. This theory does not have direct analog of Hubble parameter because metric field and scalar field are not observables. Expansion of the Universe is a result of the combined variable field $\Phi \to 0$ or energy density $\rho$ of combined variable field approaching 0. In case of flat Universe, similar to other standard cosmologies this theory also has two disconnected branches, expanding and collapsing branch. The combined variable field for closed Universe is complex and can be thought of a linear combination of two independent real combined variable fields. Whereas, the theory of open Universe is self-coupled combined variable field theory. Further detail analysis of these two theories will be done in the future work. The equation of state for flat Universe in the large $q$ limit shows combined variable field behaves as a normal scalar field. The equation of state for closed Universe shows both solutions normal and dark energy. Near $r = 0$ it behaves as a normal field but near $r = 1$ it behaves as a dark energy field. The equation of state for open Universe, in general, behaves as a normal field.

Quantum theory of flat Universe resolves the Big Bang singularity through quantum dynamics. It suggests that the Universe began with finite and non-zero size with $q_0$ being metric at the beginning. The spectrum of combined variable field quantum which is interpreted as a quantum of space had very high energy. It then decreased and in the limit $q \to \infty$ it approaches to $3E_{Planck}$. This interpretation of combined variable field quantum being a quantum of space is not possible in presence of scalar field couplings. In general quantum of combined variable field is neither a quantum of space nor a quantum of scalar field. It is a quantum of both combined together. Quantum theory of
closed and open Universe shows that the energy of a quantum of space varies also with $r$.

5 References

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