Abstract

We propose a linear algorithm for determining two function parameters by their linear combination. These functions must satisfy the first order differential equations with polynomial coefficients and our parameters are the coefficients of these polynomials. The algorithm consists of sequential solution by least squares method of two linear problems - first, differential equation polynomial coefficients determining for linear combination of two given functions and second - determining functions parameters by these polynomial coefficients. Numerical modeling carried by this scheme gives an good accordance under weak normal noise (with dispersion \(< 5\%\)).

Introduction

In practice we often run against a problem of determining two functions parameters from certain classes by their linear combination. In physics problems of that kind usually called the problems of signals, modes or rays separation. The examples of these certain classes are exponents with polynomial arguments, harmonics functions with polynomial phase and rational functions \(^5\). Such problems can arise from experiment interpretation when function class in linear combination is derived from some theoretical consideration, but parameters of these functions are unknown.
One of the most common methods for functions parameters determining is the least squares method, which in general case boils down to absolute minimization of the function from given class which depends on many unknown parameters. In linear case these problem has a simple solution - essentially, the linear system equation solution for given matrix. In our case, when parameters dependence is nonlinear, this problem is sufficiently hard and does not have an analytical solution and so (when direct parameters search is not acceptable by resources conditions) the problem usually attacks with some step-by-step function quasilinearization (Newton’s methods)\textsuperscript{1}. However, these methods has some serious faults - strong dependence on initial approximation and need for big computational resources. These conditions cannot be met if each function computing needs too much time or if we do not have good initial approximation and we are not sure in unique global minimum existence.

We propose an algorithm for transformation of the above-mentioned problem to problem of finding minimum for functionals with linear dependence on parameters and so we may apply the well-known methods based on linear systems equations solution. We restrict our attention to the case of functions satisfying the first order differential equation with polynomial coefficients. This problem becomes linear if we shifted from function parameters determining to the problem of differential equations parameters determining (due to the fact that differential equations are linear with respect to these polynomial coefficients)

The method similar to ours was developed by Kulikov \textsuperscript{2} for differential equations with constant coefficients, but in our case this method cannot be applied because our function does not satisfy differential equation with constant coefficients.

1. The problem.

Let \( f_1(t), f_2(t) \) - unknown vector-valued functions satisfying the differential equation with polynomial coefficients:

\[
[D + \hat{P}_t(t)] f_i(t) = 0 \quad (1)
\]

where

\[
\hat{P}_t(t) = \hat{M}_i \sum_{j=0}^{N_i} p_{ij} t^j \quad (2)
\]

- polynomial coefficients of degree \( N_1 \geq N_2 \) with unknown coefficients \( p_{ij} \),
matrixes $\hat{M}_1, \hat{M}_2$ commute ($\hat{M}_1\hat{M}_2 = \hat{M}_2\hat{M}_1$) and are given, but operators themselves does not equal, $\hat{P}_1(t) \neq \hat{P}_2(t)$. Suppose that we know the linear combination of these functions on interval $t \in [0, 1]$:

$$F(t) = a_1 f_1(t) + a_2 f_2(t),$$

and that linear combination is bounded on this interval ($\sup(F(t)) < \infty$), but coefficients $a_1, a_2 \neq 0$ are unknown.

Our goal is to transform the problem of determining nonlinear parameters $p_{ij}$ in model function

$$F_{mod}(t; p_{ij}) = a_1 f_1(t; p_{1j}) + a_2 f_2(t; p_{2j})$$

(4)

to the sequential series of linear problems.

2. The algorithm description.

Main idea. Consider the problem of determining $N_1 + N_2 + 2$ unknown parameters $p_{ij}$ for two vector-valued functions by their linear combination (3). From (1)-(2) we can see that unknown parameters $p_{ij}$ are nonlinear one for functions $f_i(t; p_{ij})$. So we have a problem of determining of $N_1 + N_2 + 2$ unknown parameters $p_{ij}, a_1, a_2$ by given function $F(t)$, and only two of these parameters are linear for this function.

One of most common methods for determining the unknown parameters by given function is the least squares method, which in our case essentially the minimization of deviation functional by all set of parameters

$$\Omega(p_{ij}, a_1, a_2) = \int_0^1 (F(t) - F_{mod}(t; p_{ij}, a_1, a_2))^2 dt$$

(5)

where $F_{mod}(t; p_{ij}, a_1, a_2)$ - model function (4) with given parameters dependence and $F(t)$ - experimental function.

Least squares method gives simple analytical solution which stable small errors in $F(t)$ when all the parameters for functional (3) are linear for model function (4). In our case that is not so and thus direct least squares approach is not acceptable because of complex functional behavior and need for big computational resources. So we come to idea to develop a linear algorithm due to specific properties of our problem.

Although our parameters $p_{ij}, a_1, a_2$ are nonlinear for model function (4), differential equations (1) are linear with respect to $p_{ij}$ and does not depend on...
scalar factor for \( f_i(t) \). Thus it makes sense to analyze differential equations themselves and this approach may be more constructive than direct approach.

Formally, the problem of two functions parameters determining boils down to determining of the unknown coefficients \( p_{ij} \) in differential equations (1) by given linear combination of unknown functions (3). If the function \( 3 \) satisfies the differential equation \( \hat{L}(t, p_{ij}) F_{\text{mod}}(t; p_{ij}) = 0 \), where all the unknown parameters are linear, then we may obtain these parameters by minimization of the functional:

\[
\Omega(p_{ij}) = \int_0^1 (\hat{L}(t, p_{ij}) F(t))^2 dt
\]  

(6)

In order to diminish the influence of numerical differentials we may integrate our differential equation thus transform it to integral equation. By solving the linear problem for functional (4) we obtain the unknown parameters for functions \( f_i(t) \). If these parameters are obtained, then our initial problem (3) is linear for remaining parameters \( a_1, a_2 \) and so it has an analytical solution.

Differential equation for sum of unknown functions. Our next goal is to construct a differential equation for function \( F(t) \). Our initial equation are homogeneous, and so differential equation for sum (3) does not depend on parameters \( a_1, a_2 \).

Suppose that we have an equation:

\[
[\hat{A}D^2 + \hat{B}D + \hat{C}] F = 0
\]

Then we have:

\[
[\hat{A}D^2 + \hat{B}D + \hat{C}] f_i = 0
\]

(8)

From system (8) we obtain a linear system for unknown functions \( A(t), B(t), C(t) \). If the operators \( P_i \) commute, then this system has a simple solution:

\[
\begin{align*}
A &= P_2 - P_1 \\
B &= (P_2^2 - P_1^2) - \frac{d}{dt}(P_2 - P_1) \\
C &= P_2 P_1 (P_2 - P_1) - (P_1 \frac{dP_2}{dt} - P_2 \frac{dP_1}{dt})
\end{align*}
\]

(9)

As far as our functions \( P_1(t), P_2(t) \) are polynomials of given degrees, then the functions (9) also are polynomials; for example:
\[ B_k(t) = \sum_{k=0}^{N_1+N_2} \beta_k t^k \hat{Y}_k \]  

(10)

where \( \beta_k \) - new coefficients, which has explicit dependence on parameters \( p_{ij} \), and \( \hat{Y}_k \) - matrixes which are explicitly obtained from matrixes \( \hat{M}_1, \hat{M}_2 \).

By using differential equation (7), (9) for linear combination of initial functions (3) we may construct two-step algorithm for determining parameters, with each step linear. In order to do this, transform our problem to two linear problems.

**First step.** Our first goal is to determine the coefficients of the polynomials \( A(t), B(t), C(t) \) under assumption that all these coefficients are independent. This problem is linear since all the coefficients are linear in differential equation (7):

\[
\begin{align*}
N_1 \sum_{k=0} \alpha_k t^k \hat{X}_k D^2 + N_1+N_2 \sum_{k=0} \beta_k t^k \hat{Y}_k D + 2N_1+N_2 \sum_{k=0} \gamma_k t^k \hat{Z}_k F &= 0 
\end{align*}
\]

(11)

So the problem of obtaining the coefficients \( \alpha_k, \beta_k, \gamma_k \) is the linear least squares problem for \( 4N_1 + 2N_2 + 3 \) linear parameters, and we easily solve this problem [3].

In order to diminish the numerical errors let us transform our differential equation (7) to integral one by integrating it two times:

\[
\begin{align*}
\hat{A}(t) F(t) + \int_0^t \left( \hat{B}(x) - 2 \frac{d\hat{A}(x)}{dx} \right) F(x) dx + \\
+ \int_0^t dx \int_0^x dy \left( \hat{C}(y) - \frac{d\hat{B}(y)}{dy} + \frac{d^2\hat{A}(y)}{d^2y} \right) F(y) dy + \hat{L}_1 t = \hat{L}_0 
\end{align*}
\]

(12)

This integral equation also linear with respect to parameters \( \alpha_k, \beta_k, \gamma_k \), and all its coefficients are known functions - certain combinations of the function \( F(t) \) and its integrals. Since all the coefficients are known and additional integration constants (which are also must be determined) are linear, then linear least squares method gives us an analytical solution. To exclude the trivial solution we must set one of the parameters (for example \( L_0 \)) equal to 1 and after that we may find parameters \( \alpha_k, \beta_k, \gamma_k, L_1 \) by minimization of the functional:
\[ \Omega(L_1, \alpha_k, \beta_k, \gamma_k) = \int_0^1 dt \left( \tilde{A}(t)F(t) + \int_0^t \left( \tilde{B}(x) - \frac{2}{dx} \frac{d\tilde{A}(x)}{dx} \right) F(x)dx + \int_0^t dx \int_0^x \frac{d\tilde{C}(y)}{dy} + \frac{d^2\tilde{A}(y)}{dx^2} \right) F(y)dy + \hat{L}_1 t - 1 \right)^2 = \min \] (13)

Since this problem is linear with respect to all parameters \( \alpha_k, \beta_k, \gamma_k, L_1 \), we may obtain the solution by linear least squares method.

**Second step.** Our second goal is to determine the unknown coefficients of the initial polynomials \( \hat{P}_1(t), \hat{P}_2(t) \) by given functions \( A(t), B(t), C(t) \). This problem also can transform to the linear problem. Really, after solving the problem (13) for functions \( A(t), B(t), C(t) \), we have three functions:

\[
\begin{align*}
K_1(t) &= A(t) = P_2 - P_1 \\
K_2(t) &= B(t) + \frac{dA(t)}{dt} = P_2 - P_1^2 \\
K_3(t) &= C(t) = P_2P_1(P_2 - P_1) - (P_1 \frac{dp_2}{dt} - P_2 \frac{dp_1}{dt})
\end{align*}
\] (14)

Under our assumptions we obtain functions \( K_1(t), K_2(t), K_3(t) \) only up to constant factor (due to excluding of the trivial solution). So we derive from system (14) two equations which is invariant under multiplication of the functions \( K_1(t), K_2(t), K_3(t) \) by the constant factor:

\[
\begin{align*}
P_1(t) + P_2(t) &= K_2/K_1 \\
P_1(t)P_2(t) &= [K_3 - (K_1 \frac{d}{dt} (K_2/K_1) - (K_2/K_1) \frac{d}{dt} K_1)]/K_1
\end{align*}
\] (15)

From this system we can easily obtain an analytical expressions for functions \( \hat{P}_1(t), \hat{P}_2(t) \):

\[
P_{1,2}(t) = S_{1,2}(t)
\] (16)

where \( S_{1,2}(t) \) has explicit expression as functions of \( K_1(t), K_2(t), K_3(t) \) due to Viet theorem (13):

\[
\begin{align*}
S_1(t) + S_2(t) &= K_2/K_1 \\
S_1(t)S_2(t) &= [K_3 - (K_1 \frac{d}{dt} (K_2/K_1) - (K_2/K_1) \frac{d}{dt} K_1)]/K_1
\end{align*}
\] (17)

The coefficients of these polynomials can be found from functional minimum condition:

\[
\Omega_i(p_{ij}) = \int_0^1 (P_i(t; p_{ij}) - S_i(t))^2 dt, \quad \text{(18)}
\]
where analytical expression for $P_i(t; \tilde{p}_{ij})$ are given by (3), and $S_i(t)$ are determined by the first step (17).

Since our model polynomials $P_i(t; \tilde{p}_{ij})$ are linear with respect to $\tilde{p}_{ij}$ (3), and $S_i(t)$ are determined by the first step (17), the problem of determining the polynomial coefficient is linear and so can be solved by linear least squares method [3]. Thus our algorithm transforms initial nonlinear problem to sequential two-step linear problem for differential equation (7) and each step is linear least squares problem for certain set of parameters.

### 3. Modeling results.

Our algorithm have been tested for two problems - separation of two overlapping Gaussian functions $f_i(t) = R_i \exp(-\alpha_i t^2 + \beta_i t)$, satisfying the differential equation

$$[D - (2\alpha_i t + \beta_i)]f_i = 0,$$

and separation of two vector-valued signals with linear frequency modulation

$$f_i(t) = R_i \begin{pmatrix} \cos(\alpha_i t + \beta_i t^2) \\ \sin(\alpha_i t + \beta_i t^2) \end{pmatrix},$$

satisfying the differential equation

$$\begin{bmatrix} D + (\alpha_i + 2\beta_i t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{bmatrix} f_i = 0$$

In the first case algorithm is valid under $10 - 15\%$ normal noise (For function maximal amplitude), in the second case - under $3 - 5\%$ normal noise.

### 4. Discussion.

In practice we often have the case when operators in equation (1) are not polynomials but a rational functions:

$$\hat{P}_i(t) = \hat{M}_i \frac{\sum_{j=0}^{N_{pi}} \tilde{p}_{ij} t^j}{\sum_{j=0}^{N_{qi}} q_{ij} t^j},$$

under assumption that $P_i(t)$ has no singularities.
Such class includes rational functions, exponents and harmonic functions with rational argument. Our method also can be applied to that extended class. The reason for this is that functions $K_1(t), K_2(t), K_3(t)$ will be also rational so after multiplying our differential equation (7) by some polynomial we can obtain the differential equation with polynomial coefficients. Since determining $P_i(t)$ by $K_1(t), K_2(t), K_3(t)$ does not depend on constant factor (even if this factor is nonconstant polynomial), then the expressions (16-17) are also valid in this For determining all the coefficients we must multiply all the expressions by $P_i(t)$ denominator, and set one of the coefficients $q_{ij}$ (for example, under highest degree) to 1 to exclude the trivial solution:

$$
\dot{M}_i \sum_{j=0}^{N_{pi}} p_{ij} t^j - \sum_{j=0}^{N_{pi}-1} q_{ij} t^j \hat{S}_i(t) = t^{N_{qi}} \hat{S}_i(t)
$$

(23)

This problem also linear with respect to parameters $p_{ij}, q_{ij}$ and thus can be solved by linear least squares method.

**Conclusion.**

We propose an algorithm for determining parameters of two functions satisfying differential equations of the first order with polynomial coefficients by their linear combination. This algorithm transform initial nonlinear problem to sequential solving of the two linear problems - determining of polynomial coefficients for linear differential equation of the second order with polynomial coefficients, and determining initial polynomial coefficients from results or the first step. Numerical testing shows us that the algorithm is valid under weak normal noise (with dispersion < 5%).

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