On convolved generalized Fibonacci and Lucas polynomials

José L. Ramírez
Instituto de Matemáticas y sus Aplicaciones, Universidad Sergio Arboleda, Bogotá 110221, Colombia

Abstract

We define the convolved \( h(x) \)-Fibonacci polynomials as an extension of the classical convolved Fibonacci numbers. Then we give some combinatorial formulas involving the \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials. Moreover we obtain the convolved \( h(x) \)-Fibonacci polynomials from a family of Hessenberg matrices.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art (e.g., see [1]). The Fibonacci numbers \( F_n \) are the terms of the sequence \( 0, 1, 2, 3, 5, \ldots \), wherein each term is the sum of the two previous terms, beginning with the values \( F_0 = 0 \) and \( F_1 = 1 \).

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature. In particular, a generalization is the \( k \)-Fibonacci Numbers. For any positive real number \( k \), the \( k \)-Fibonacci sequence, say \( \{F_{k,n}\}_{n \in \mathbb{N}} \), is defined recurrently by

\[
F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1
\]

In [2], \( k \)-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [2–7].

The convolved Fibonacci numbers \( F^{(r)}_n \) are defined by

\[
(1 - x - x^2)^{-r} = \sum_{j=0}^{\infty} F^{(r)}_j x^j, \quad r \in \mathbb{Z}^+.
\]

If \( r = 1 \) we have classical Fibonacci numbers. These numbers have been studied in several papers; see [8–10]. Convolved \( k \)-Fibonacci numbers have been studied in [11].

Large classes of polynomials can be defined by Fibonacci-like recurrence relation and yield Fibonacci numbers [1]. Such polynomials, called Fibonacci polynomials, were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials \( F_n(x) \) studied by Catalan are defined by the recurrence relation

\[
F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \geq 1.
\]

The Fibonacci polynomials studied by Jacobsthal are defined by

\[
J_0(x) = 1, \quad J_1(x) = 1, \quad J_{n+1}(x) = J_n(x) + xJ_{n-1}(x), \quad n \geq 1.
\]

The Lucas polynomials \( L_n(x) \), originally studied in 1970 by Bicknell, are defined by

\[
L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad n \geq 1.
\]
In [12], the authors introduced the \( h(x) \)-Fibonacci polynomials. That generalize Catalan’s Fibonacci polynomials \( F_n(x) \) and the \( k \)-Fibonacci numbers \( F_{k,n} \). In this paper, we introduce the convolved \( h(x) \)-Fibonacci polynomials and we obtain new identities.

2. Some properties of \( h(x) \)-Fibonacci polynomials and \( h(x) \)-Lucas polynomials

**Definition 1.** Let \( h(x) \) be a polynomial with real coefficients. The \( h(x) \)-Fibonacci polynomials \( \{F_{h,n}(x)\}_{n \in \mathbb{N}} \) are defined by the recurrence relation

\[
F_{h,0}(x) = 0, \quad F_{h,1}(x) = 1, \quad F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geq 1.
\]

For \( h(x) = x \) we obtain Catalan’s Fibonacci polynomials, and for \( h(x) = k \) we obtain \( k \)-Fibonacci numbers. For \( k = 1 \) and \( k = 2 \) we obtain the usual Fibonacci numbers and the Pell numbers.

The characteristic equation associated with the recurrence relation (5) is \( \nu^2 = h(x) \nu + 1 \). The roots of this equation are

\[
r_1(x) = \frac{-h(x) + \sqrt{h(x)^2 - 4}}{2}, \quad r_2(x) = \frac{-h(x) - \sqrt{h(x)^2 - 4}}{2}.
\]

Then we have the following basic identities:

\[
r_1(x) + r_2(x) = h(x), \quad r_1(x) - r_2(x) = \sqrt{h(x)^2 - 4}, \quad r_1(x)r_2(x) = -1.
\]

Some of the properties that the \( h(x) \)-Fibonacci polynomials verify are summarized below (see [12] for the proofs).

- Binet formula: \( F_{h,n}(x) = \frac{r_1^n(x) - r_2^n(x)}{r_1(x) - r_2(x)} \).
- Combinatorial formula: \( F_{h,n}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \left( \binom{n-1}{i} h_{n-i} - i \right) x^n \).
- Generating function: \( g_h(t) = \frac{1}{1 - h(x)t} \).

**Definition 2.** Let \( h(x) \) be a polynomial with real coefficients. The \( h(x) \)-Lucas polynomials \( \{L_{h,n}(x)\}_{n \in \mathbb{N}} \) are defined by the recurrence relation

\[
L_{h,0}(x) = 2, \quad L_{h,1}(x) = h(x), \quad L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), \quad n \geq 1.
\]

For \( h(x) = x \) we obtain the Lucas polynomials, and for \( h(x) = k \) we have the \( k \)-Lucas numbers. For \( k = 1 \) we obtain the usual Lucas numbers.

Some properties that the \( h(x) \)-Lucas numbers verify are summarized below (see [12] for the proofs).

- Binet formula: \( L_{h,n}(x) = r_1(x)^n + r_2(x)^n \).
- Relation with \( h(x) \)-Fibonacci polynomials: \( L_{h,n}(x) = F_{h,n-1}(x) + F_{h,n+1}(x) \), \( n \geq 1 \).

3. Convolved \( h(x) \)-Fibonacci polynomials

**Definition 3.** The convolved \( h(x) \)-Fibonacci polynomials \( F_{h,j}^{(r)}(x) \) are defined by

\[
F_{h,j}^{(r)}(t) = (1 - h(x)t - t^2)^{-r} = \sum_{j=0}^{\infty} F_{h,j+1}(x)t^j, \quad r \in \mathbb{Z}^+.
\]

Note that

\[
F_{h,j+1}^{(r)}(x) = \sum_{j_1, j_2, \ldots, j_m} F_{h,j_1+1}(x)F_{h,j_2+1}(x) \cdots F_{h,j_m+1}(x).
\]

Moreover, using a result of Gould [13, p. 699] on Humbert polynomials (with \( n = j \), \( m = 2 \), \( x = h(x)/2 \), \( y = -1 \), \( p = -r \) and \( C = 1 \)), we have

\[
F_{h,j+1}^{(r)}(x) = \sum_{l=0}^{\lfloor i/2 \rfloor} \binom{j + r - l - 1}{j - l} \binom{j - l}{l} h(x)^{j - 2l}.
\]

If \( r = 1 \) we obtain the combinatorial formula of \( h(x) \)-Fibonacci polynomials. In Table 1 some polynomials of convolved \( h(x) \)-Fibonacci polynomials are provided. The purpose of this paper is to investigate the properties of these polynomials.
Theorem 5. The following identities hold:

1. \( F_{h_{1,n}}^{(r)}(x) = rh(x) \).
2. \( F_{h_{n,n}}^{(r)}(x) = F_{h_{n,n-1}}^{(r-1)}(x) + h(x)F_{h_{n,n-2}}^{(r)}(x) \), \( n \geq 2 \).
3. \( nF_{h_{n,n-1}}^{(r)}(x) = r(h(x)F_{h_{n,n}}^{(r-1)}(x) + 2F_{h_{n,n-2}}^{(r-1)}(x)) \), \( n \geq 1 \).

Proof.

1. Taking \( j = 1 \) in (9), we obtain
\[
F_{h_{1,n}}^{(r)}(x) = \left( \begin{array}{c} r \\ 1 \end{array} \right) h(x) = rh(x).
\]
2. This identity is obtained from observing that
\[
\sum_{j=0}^{\infty} F_{h_{j+1}}^{(r)}(x) t^j = (h(x) t + t^2) \sum_{j=0}^{\infty} F_{h_{j+1}}^{(r)}(x) t^j + \sum_{j=0}^{\infty} F_{h_{j+1}}^{(r-1)}(x) t^j.
\]
3. Taking the first derivative of \( g_h^{(r)}(t) = (1 - h(x) t - t^2)^{-r} \), we obtain
\[
(g_h^{(r)}(t))' = \sum_{j=1}^{\infty} F_{h_{j+1}}^{(r)}(x) t^{j-1} = r \left( \frac{1}{1 - h(x) t - t^2} \right) ^{r-1} \left( \frac{h(x) + 2t}{1 - h(x) t - t^2} \right) = r(h(x) + 2t) g_h^{(r-1)}(t)
\]
Therefore the identity is clear. \( \square \)

In the next theorem we show that the convolved \( h(x) \)-Fibonacci polynomials can be expressed in terms of \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials. This theorem generalizes Theorem 4 of [10] and Theorem 4 of [11].

Theorem 5. Let \( j \geq 0 \) and \( r \geq 1 \). We have
\[
F_{h_{j+1}}^{(r)}(x) = \sum_{l=0}^{r-1} \left( \begin{array}{c} r \\ l \end{array} \right) \frac{1}{(h(x)^2 + 4)^{l(r)}} L_{h_{j+1-l}}^{(r)}(x)
\]
\[
+ \sum_{l=0}^{r-1} \left( \begin{array}{c} r \\ l \end{array} \right) \frac{1}{(h(x)^2 + 4)^{l(r-1)}} F_{h_{j+1-l}}^{(r-1)}(x)
\]

Proof. Given \( \alpha, \beta \in \mathbb{C} \), such that \( \alpha \beta \neq 0 \) and \( \alpha \neq \beta \). Then we have the following partial fraction decomposition:
\[
(1 - \alpha z)^{-r} (1 - \beta z)^{-r} = \sum_{l=0}^{r-1} \left( \begin{array}{c} r \\ l \end{array} \right) \frac{(\alpha^l \beta^{l-1})}{(\alpha - \beta)^{l-1}} (1 - \alpha z)^{-l-r} + \sum_{l=0}^{r-1} \left( \begin{array}{c} r \\ l \end{array} \right) \frac{\alpha^l \beta^{l-1}}{(\beta - \alpha)^{l-1}} (1 - \beta z)^{-l-r},
\]
where \( \left( \begin{array}{c} t \\ 0 \end{array} \right) = \frac{(t - 1) e^{-t}}{2} \) with \( t \in \mathbb{R} \). Using the Taylor expansion
\[
(1 - z)^l = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^j
\]
Then \((1 - xz)^{-1} - \beta z)^{-1} = \sum_{j=0}^{\infty} (j)z^j\), where
\[
\gamma(j) = \sum_{l=0}^{r-1} \frac{(-1)^j \alpha^l \beta^l}{l!} \left( \frac{1 - r}{l} \right) x^l + \sum_{l=0}^{r-1} \frac{(-1)^j \beta^l}{l!} \left( \frac{1 - r}{l} \right) y^l = \sum_{l=0}^{r-1} \frac{(-1)^j \beta^l}{l!} \left( \frac{1 - r}{l} \right) y^l
\]
Note that \(1 - h(x)z - z^2 = (1 - r_1(x)z)(1 - r_2(x)z)\). On substituting these values of \(x = r_1(x)\) and \(\beta = r_2(x)\) and using the identities (6), we obtain
\[
F_{h(x)}^{(r)}(x) = \sum_{l=0}^{r-1} \frac{(-1)^j \alpha^l \beta^l}{l!} \left( \frac{1 - r}{l} \right) x^l + \sum_{l=0}^{r-1} \frac{(-1)^j \beta^l}{l!} \left( \frac{1 - r}{l} \right) y^l = \sum_{l=0}^{r-1} \frac{(-1)^j \beta^l}{l!} \left( \frac{1 - r}{l} \right) y^l
\]
Since that \((-1)^j \frac{(-1)^j}{l!} = \frac{r + l - 1}{l!} \frac{r + l - j}{j!} \frac{1}{l!} \frac{1}{l!} \frac{1}{l!} (h(x)^2 + 1)^{-1}
\]
From the above equality and Binet formula, we obtain the Eq. (10).

4. Hessenberg matrices and convolved \(h(x)\)-Fibonacci polynomials

An upper Hessenberg matrix, \(A_n\), is an \(n \times n\) matrix, where \(a_{i,j} = 0\) whenever \(i > j + 1\) and \(a_{i,1} \neq 0\) for some \(j\). That is, all entries below the superdiagonal are 0 but the matrix is not upper triangular:
\[
A_n = \begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\
  a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\
  0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
  0 & 0 & 0 & \cdots & 0 & a_{n,n}
\end{pmatrix}
\]

We consider a type of upper Hessenberg matrix whose determinants are \(h(x)\)-Fibonacci numbers. Some results about Fibonacci numbers and Hessenberg can be found in [14]. The following known result about upper Hessenberg matrices will be used.

**Theorem 6.** Let \(a_1, p_{ij}(i < j)\) be arbitrary elements of a commutative ring \(R\), and let the sequence \(a_1, a_2, \ldots\) be defined by:
\[
a_{n+1} = \sum_{i=1}^{n} p_{i,n} a_i, \quad (n = 1, 2, \ldots).
\]
If
\[
A_n = \begin{pmatrix}
p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\
-1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\
0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
0 & 0 & 0 & \cdots & -1 & a_{n,n}
\end{pmatrix}
\]
then
\[
a_{n+1} = a_1 \det A_n.
\]
In particular, if
\[
F_{n}^{(h)} = \begin{pmatrix}
h(x) & 1 & 0 & \cdots & 0 & 0 \\
-1 & h(x) & 1 & \cdots & 0 & 0 \\
0 & -1 & h(x) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & h(x) & 1 \\
0 & 0 & 0 & \cdots & -1 & h(x)
\end{pmatrix}
\]
then from Theorem 6 we have that
\[
\det F_n^{(b)} = F_{h,n+1}(x), \quad (n = 1, 2, \ldots).
\]
(14)

It is clear that the principal minor \(M^{(b)}(i)\) of \(F_n^{(b)}\) is equal to \(F_{h,i}(x)F_{h,n-i+1}(x)\). It follows that the principal minor \(M^{(b)}(i_1, i_2, \ldots, i_l)\) of the matrix \(F_n^{(b)}\) is obtained by deleting rows and columns with indices \(1 \leq i_1 < i_2 < \cdots < i_l \leq n\):
\[
M^{(b)}(i_1, i_2, \ldots, i_l) = F_{h,i_1}(x)F_{h,i_2-i_1}(x) \cdots F_{h,n-i_l+1}(x)F_{h,n-i_l+1}(x).
\]
(15)

Then we have the following theorem.

**Theorem 7.** Let \(S^{(b)}_{n-l}(l = 0, 1, 2, \ldots, n - 1)\) be the sum of all principal minors of \(F_n^{(b)}\) or order \(n - l\). Then
\[
S^{(b)}_{n-l} = \sum_{j_1+j_2+\cdots+j_{l+1}=n-l} F_{h,j_1+1}(x)F_{h,j_2+1}(x) \cdots F_{h,j_{l+1}+1}(x) = F^{(l+1)}_{h,n-l+1}(x).
\]
(16)

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

**Corollary 8.** The convolved \(h(x)\)-Fibonacci polynomials \(F_{h,n-l+1}^{(l+1)}(x)\) is equal, up to the sign, to the coefficient of \(t^l\) in the characteristic polynomial \(p_n(t)\) of \(F_n^{(b)}\).

**Corollary 9.** The following identity holds:
\[
F_{h,n-l+1}^{(l+1)}(x) = \sum_{i=0}^{\lfloor n-l/2 \rfloor} \binom{n-i}{i} \binom{n-2i}{l} h(x)^{n-2i-l}.
\]

**Proof.** The characteristic matrix of \(F_n^{(b)}\) has the form
\[
\begin{pmatrix}
t - h(x) & 1 & 0 & \cdots & 0 & 0 \\
-1 & t - h(x) & 1 & \cdots & 0 & 0 \\
0 & -1 & t - h(x) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & t - h(x) & 1 \\
0 & 0 & 0 & \cdots & -1 & t - h(x)
\end{pmatrix}
\]
(17)

Then \(p_n(t) = F_{n+1}(t - h(x))\), where \(F_{n+1}(t)\) is a Fibonacci polynomial. Then from Corollary 8 and the following identity for Fibonacci polynomial [5]:
\[
F_{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^{n-2i},
\]
we obtain that
\[
F_{n+1}(t - h(x)) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n-2i}{l} (-1)^{n-1-h(x)^{n-2i-l}} t^l.
\]

Therefore the corollary is obtained. □

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