On quantum interactive proofs with short messages

Attila Pereszlényi*

Centre for Quantum Technologies, National University of Singapore

September 5, 2011

Abstract

This paper proves one of the open problems posed by Beigi et al. in [BSW11]. We consider quantum interactive proof systems where in the beginning the verifier and prover send messages to each other with the combined length of all messages being at most logarithmic (in the input length); and at the end the prover sends a polynomial-length message to the verifier. We show that this class has the same expressive power as \( \text{QMA} \).

1 Introduction

Quantum interactive proof systems (QIP) were introduced by [Wat99, Wat03] as a natural extension of interactive proofs (IP) to the quantum computational setting. They have been extensively studied and now it’s known that the power of quantum interactive proof systems is the same as the classical one, i.e., \( \text{QIP} = \text{IP} = \text{PSPACE} \) [JJUW10]. Furthermore, quantum interactive proof systems still have the same expressive power if we restrict the number of messages to three and have exponentially small one-sided error [KW00]. If the interaction is only one message from the prover to the verifier then the class is called QMA, which is the quantum analogue of \( \text{NP} \) and \( \text{MA} \). QMA can also be made to have exponentially small error, and has natural complete problems [AN02].

Several variants of QIP and QMA have also been studied. We now focus on the case where some or all of the messages are small, meaning at most logarithmic in the input length. These cases are usually not interesting in the classical setting since a logarithmic-length message can be eliminated by the verifier by enumerating all possibilities. This is not true in the quantum case, indeed the variant of QMA where we have two unentangled logarithmic-length proofs contains \( \text{NP} \) [BT09]; hence not believed to be equal to \( \text{BQP} \). On the other hand, if QMA has one logarithmic-length proof then it has the same expressive power as \( \text{BQP} \) [MW05].

Beigi et al. [BSW11] proved that in other variants of quantum interactive proof systems the short message can also be eliminated without changing the power of the proof system. Among others they showed that in the setting when the verifier sends a short message to the prover and the prover responds with an ordinary, polynomial-length message, the short message can be discarded, and hence the class has the same power as QMA. They have raised the question if this is also true if we replace the short question of the verifier with a ‘short interaction’. I.e., consider quantum interactive proof systems where in the beginning the verifier and prover send messages to each other with the combined length of all messages being at most logarithmic, and at the end the prover sends a polynomial-length message to the verifier. We show that this class has the power of QMA, or in other words, the short interaction can be discarded. This is formalized by the following theorem.

*E-mail: attila.pereszlenyi@gmail.com.
Theorem 1.1. Let \( c, s : \mathbb{N} \to (0,1) \) be polynomial-time computable functions such that \( c(n) - s(n) \in 1/poly(n) \). Then \( \text{QIP}_{\text{short}}(\log(n), c, s) = \text{QMA} \).

Here \( \text{QIP}_{\text{short}}(\log(n), c, s) \) is the class described above, with completeness-soundness gap being separated by some inverse polynomial function of the input length. For a rigorous description of the class see Definition \( 2.1 \) and for the notation see the discussion in Section \( 2 \).

The remainder of the paper is organized as follows. Section 2 discusses the background theorems and definitions needed for the rest of the paper. The proof of the main theorem is split into two. Section 3 describes how to deal with the short interaction, and using that Section 4 finishes the proof by discussing how to handle the last round. This part closely follows the corresponding proof in \( \text{BSW11} \).

# 2 Preliminaries

We assume familiarity with quantum information \( \text{[Wat08b]} \), computation \( \text{[NC00]} \) and computational complexity \( \text{[Wat08a]} \); such as mixed states, unitary operations, quantum channels, representations of quantum channels, quantum de Finetti theorems, state tomography and complexity classes like \( \text{QMA} \) and \( \text{QIP} \). The purpose of this section is to present the notations and background information (definitions, theorems) required to understand the following two sections.

We denote the set of functions of \( n \) that are upper-bounded by some polynomial in \( n \) by \( \text{poly}(n) \). If the argument is clear, we omit it and just write \( \text{poly} \). Similarly, we write \( \log(n) \) or \( \log \) for the set of functions that are in \( O(\log(n)) \). We try to follow the notations used in \( \text{Wat08b, BSW11} \). When we talk about a quantum register (\( R \)) of size \( k \), we mean the object made up of \( k \) qubits. It has associated Hilbert space \( R = \mathbb{C}^{2^k} \). \( L(R) \) denotes the space of all linear mappings from \( R \) to itself, and the set of all density operators on \( R \) is denoted by \( D(R) \).

The adjoint of \( X \in L(R) \) is denoted by \( X^\dagger \), and the trace norm of \( X \) by \( \|X\|_{\text{Tr}} \). The trace distance between \( X \) and \( Y \) is defined as

\[
\frac{1}{2} \|X - Y\|_{\text{Tr}}.
\]

A quantum channel (\( \Phi \)) is a completely positive and trace-preserving linear map of the form \( \Phi : L(Q) \to L(R) \). The set of all such channels is denoted by \( C(Q, R) \). For any \( \Phi \in C(\mathbb{C}^{2^k}, \mathbb{C}^{2^\ell}) \) the normalized Choi–Jamiołkowski representation of \( \Phi \) is defined to be

\[
\rho_\Phi \in D(\mathbb{C}^{2^k} \otimes \mathbb{C}^{2^\ell}), \quad \rho_\Phi = \frac{1}{2^k} \sum_{x,y \in \{0,1\}^k} \Phi(|x\rangle\langle y|) \otimes |x\langle y|.
\]

It can be generated by applying \( \Phi \) on one half of \( k \) pairs of qubits in the state \( \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \).

If we are given \( \rho_\Phi \) and an arbitrary \( \sigma \in D(\mathbb{C}^{2^k}) \) then there exist a simple procedure which produces \( \Phi(\sigma) \) with probability \( 1/4^k \). We will refer to it as ‘post-selection’. For details see \( \text{BSW11} \) Section 2.1.

When we talk about a polynomial-time quantum algorithm, we mean a quantum circuit containing Hadamard (\( H \)), \( \pi/8 \) (\( T \)) and controlled-not (\( \text{CNOT} \)) gates, and which can be generated by a classical algorithm in polynomial-time. The classes \( \text{QMA} \) and \( \text{QIP} \) have been defined in \( \text{[AN02]} \) and \( \text{[Wat03]} \) respectively, and we will use those definitions. Now we want to define the quantum interactive proof systems where in the beginning there is a \( \log(n) \)-long interaction which is followed by a \( \text{poly}(n) \)-length message from the prover. Note that in this
setting we can assume, without loss of generality, that all the messages except the last one are one qubits, and the total number of rounds is at most \(O(\log n)\). This is because we can add dummy qubits that are interspersed with the qubits sent by the other party. We define the class according to this observation.

**Definition 2.1.** Let the class \(QIP_{\text{short}}(m, c, s)\) be the set of languages for which there exist a quantum interactive proof system with the following properties. The completeness parameter is \(c\) and the soundness is \(s\). The proof system consists of \(m\) rounds, each round is a question-answer pair. All questions and answers are one qubits except for the last answer which is \(\text{poly}(n)\) qubits, where \(n\) is the length of the input. See Figure 1 for an example with \(m = 3\).

![Figure 1: The interaction in the proof system of Definition 2.1 in case \(m = 3\).](image)

A similar class, \(QIP(\log, \text{poly}, c, s)\) was defined in [BSW11] to be the class of problems for which there exist a one round quantum interactive proof system, with completeness and soundness parameters \(c\) and \(s\). Additionally the verifier’s question has length \(\log(n)\), and the prover’s answer is \(\text{poly}(n)\) qubits.

**Remark 2.2.** The following inclusion is trivially true between the above classes.

\[
QIP(\log, \text{poly}, c, s) \subseteq QIP_{\text{short}}(\log(n), c, s),
\]

for all values of \(c\) and \(s\).

In [BSW11] it was proven that in their setting the question from the verifier is unnecessary. Or more precisely:

**Theorem 2.3 (BSW11).** Let \(c, s : \mathbb{N} \to (0, 1)\) be polynomial-time computable functions such that \(c(n) - s(n) \in 1/\text{poly}(n)\). Then \(QIP(\log, \text{poly}, c, s) = \text{QMA}\).

In the next sections we prove that the seemingly stronger class of Definition 2.1 also has the power of QMA if \(m = O(\log n)\). For this we will need the following theorems.

**Theorem 2.4 (NC00, Chapter 4.5.2).** An arbitrary unitary operator on \(\ell\) qubits can be implemented using a circuit containing \(O(\ell^2 4^\ell)\) single qubit and CNOT gates.

The next theorem follows from the Solovay–Kitaev theorem [Kit97], and also appears in [NC00].

**Theorem 2.5.** For any unitary operator \(U\) on one qubit and \(\varepsilon > 0\), there exist a circuit \(C_{U, \varepsilon}\) such that \(C_{U, \varepsilon}\) is made up of \(O(\log^3(1/\varepsilon))\) gates from the set \(\{H, T\}\), and for all \(|\varphi\rangle \in \mathbb{C}^2\) it holds that

\[
\frac{1}{2} \| U |\varphi\rangle \langle \varphi| U^* - |\xi\rangle \langle \xi| \|_{\text{Tr}} \leq \varepsilon,
\]

where \(|\xi\rangle \defeq C_{U, \varepsilon}(|\varphi\rangle)\).
The following is corollary to Theorem 2.4 and 2.5.

**Corollary 2.6.** For any unitary operator $U$ on $\ell$ qubits and $\varepsilon > 0$, there exist a circuit $C_{U,\varepsilon}$ such that $C_{U,\varepsilon}$ is made up of $O\left(5^\ell \cdot \log^3(5\ell/\varepsilon)\right)$ gates from the set \{H, T, CNOT\}, and for all $|\varphi\rangle \in \mathbb{C}^2$, it holds that

$$\frac{1}{2} \left\| U |\varphi\rangle \langle \varphi | U^* - |\xi\rangle \langle \xi | \right\|_\text{Tr} \leq \varepsilon,$$

where $|\xi\rangle \equiv C_{U,\varepsilon}(|\varphi\rangle)$.

**Lemma 2.7** (Lemma 1 of [BSWIT]). Let $\rho \in D(\mathbb{C}^{2^q})$ be a state on $q = O(\log n)$ qubits. For any $\varepsilon \in 1/\text{poly}(n)$, choose $N$ such that $N \geq 2^{10q}/\varepsilon^3$, and $N \in \text{poly}(n)$. If $\rho^{\otimes N}$ is given to a poly($n$)-time quantum machine, then it can perform quantum state tomography, and get a classical description $\xi \in L(\mathbb{C}^{2^n})$ of $\rho$, which with probability at least $1 - \varepsilon$ satisfies

$$\|\rho - \xi\|_\text{Tr} < \varepsilon.$$

**Theorem 2.8** (quantum de Finetti theorem [CKMR07]: this form is from [Wat08b]). Let $X_1, \ldots, X_n$ be identical quantum registers, each having associated space $\mathbb{C}^d$, and let $\rho \in D(\mathbb{C}^{dn})$ be the state of these registers. Suppose that for all permutation $\pi \in S_n$ it holds that $\rho = W_\pi \rho W_\pi^*$, where $W_\pi$ permutes the contents of $X_1, \ldots, X_n$ according to $\pi$. Then for any choice of $k \in \{2, 3, \ldots, n-1\}$ there exists a number $N \in \mathbb{N}$, a probability vector $p \in \mathbb{R}^N$, and a collection of density operators $\{\sigma_i : i \in \{1, 2, \ldots, N\}\} \subset D(\mathbb{C}^d)$ such that

$$\left\| \rho^{X_1 \cdots X_k} - \sum_{i=1}^N p_i \sigma_i^\otimes k \right\|_\text{Tr} < \frac{4d^2k}{n}.$$

### 3 Compressing the prover’s private space

This section proves that during the initial short interaction the prover doesn’t need to keep too many qubits in its private memory. This will make it easy to give the description of the actions of the prover during these rounds as a classical proof. This idea of upper-bounding the prover’s private space has also appeared in [KM03].

From now on, let $L \in \text{QIP}_{\text{short}}(m+1, c, s)$, let $V$ be the verifier and $P$ be the (honest or dishonest) prover. We now describe what happens in each but the last round, so we can give a name to all quantum registers in the process.

In the beginning of round $i$ for $i \in \{1, 2, \ldots, m\}$ the prover is holding some private register from the previous round which we call $P_{i-1}$. The verifier is holding the answer from the previous round ($A_{i-1}$) and his private register ($V_{i-1}$). Note that $A_{i-1}$ is made up of one qubit and $V_{i-1}$ is poly($n$) qubits. The verifier applies a unitary transformation $W_i$ on $A_{i-1}V_{i-1}$, and gets the registers $Q_i$ and $V_i$ as the output. The one qubit $Q_i$ holds the $i$th question, while the poly-qubit $V_i$ is the verifier’s new private register. The verifier then sends $Q_i$ to the prover, who applies the unitary $U_i$ on $P_{i-1}Q_i$ and gets $P_i$ and $A_i$ as the output. At the end of the round the prover sends back $A_i$ to the verifier. Figure 2 shows a schematic of this procedure. Note that it is without loss of generality that for all $i, j \in \{1, 2, \ldots, m\}$, $P_i$ has the same number of qubits as $P_j$, and the same is true for all $V_i$s. Moreover all $Q_i$s and $A_i$s are one qubits. Since we introduce new registers whenever an operation takes place, we can talk about ‘the state of a
register' without confusion. Without loss of generality we set the state of $P_0 A_0 V_0$ to be $|00 \ldots 0\rangle$. We didn’t put any upper bound on the size of register $P_i$. However, the prover doesn’t need arbitrary big private space, neither in the honest nor in the dishonest case. This is formalized by the following lemma.

**Figure 2:** Schematic of round $i$ in a QIP$_{short}$ proof system.

**Lemma 3.1.** Fix any verifier strategy $V$ and prover strategy $P$ for an input to the problem $L$, as described above. Then we can construct another prover $P'$ which makes the verifier accept with exactly the same probability as $P$, and if $V$ interacts with $P'$ then for all $i \in \{1, 2, \ldots, m\}$, at most $2i$ qubits of $P_i$ will have state different than $|0\rangle$.

**Proof.** We prove the lemma by induction on $i$. I.e., we modify $P$ round-by-round to satisfy the statement of the lemma. If $i = 0$ then $P_0$ already has all qubits in state $|0\rangle$, so we are done.

Now suppose that for some $i$, we have that at most $2i$ qubits of $P_i$ has state different than $|0\rangle$. Then we will show that at most $2i + 2$ qubits of $P_{i+1}$ has state different than $|0\rangle$, by modifying $P$, such that the acceptance probability will stay the same. So let us consider the situation in round $(i + 1)$, right after the verifier sent $Q_{i+1}$ to the prover. See Figure 2. Let’s denote the part of $P_i$ that contains something other than $|0\rangle$ by $P'_{i}$. By the induction hypothesis, the size of $P'_{i}$ is $\leq 2i$. Since the joint state of $P'_{i} Q_{i+1} V_{i+1}$ is pure, there exist a unitary $X_{i+1}$ acting on $V_{i+1}$ and transforming it to $W_{i+1}$ of size $\leq 2i + 1$ and a register containing only $|00 \ldots 0\rangle$. Suppose $V$ performs this operation and let us not worry now about whether it is doable in polynomial time.

Now $P'$ performs $U_{i+1}$ and gets $P_{i+1}$ and $A_{i+1}$. Now the joint state of $P_{i+1} A_{i+1} W_{i+1}$ is pure, so by the same argument as before, we have that there exist a unitary $Y_{i+1}$ that transforms $P_{i+1}$ into $P'_{i+1}$ of size $\leq 2i + 2$ and some register containing only $|00 \ldots 0\rangle$. Our $P'$ performs this $Y_{i+1}$ as well. Now $V$ performs $X_{i+1}^{-1}$ and gets back $V_{i+1}$. Additionally $P'$ in the next round will perform $Y_{i+1}^{-1}$ just before it is about to perform $U_{i+2}$, so at that point the state of the whole system will be the same as in the original proof system, so the acceptance probability won’t change. Note that $V$ doesn’t actually need to perform $X_{i+1}^{-1}$ since it is followed by $X_{i+1}$. So we don’t need to modify $V$ at all. This finishes the proof of the lemma.

**Corollary 3.2.** Without loss of generality, for all $i \in \{1, 2, \ldots, m\}$ we can assume that $P_i$ is
made up of at most 2m qubits, both in the honest and dishonest case. Furthermore, the action of the prover in round i is still a unitary, transforming $P_{i-1}Q_{i}$ to $P_{i}A_{i}$.

4 Proof of the main theorem

This section finishes the proof of the main theorem by using the result from the previous section. The idea is that the prover’s unitaries in the first $m$ rounds can be given as classical descriptions of quantum circuits. Using this, the QMA verifier can approximately produce the state of the whole system appearing before the last round in the QIP protocol. This means the prover’s private space, the answer to the verifier and the verifier’s private space. To simulate the last round, we don’t need to care about the prover’s private space, so we treat its operation as a quantum channel, acting on the the private space of the prover and the question from the verifier. Since the input is on log-many qubits, to perform the action of this channel, we can use the same method as in [BSW11, Section 3]. The detailed proof is as follows.

Proof of Theorem 1.1. The inclusion $\text{QMA} \subseteq \text{QIP}_{\text{short}}(\log(n), c, s)$ is trivial, so we only need to prove $\text{QIP}_{\text{short}}(\log(n), c, s) \subseteq \text{QMA}$. Just as above, let $L \in \text{QIP}_{\text{short}}(m + 1, c, s)$, where $m = O(\log n)$, and let $V$ be the corresponding verifier. We will construct a verifier $W$ for the QMA proof system. Because of Corollary 3.2, we can assume that any prover strategy in the first $m$ rounds are unitary operators on $2^m$ qubits, say $U_1, \ldots, U_m$. The constructed $W$ expects to get as part of the proof, the classical descriptions of circuits $C_{U_1,1/3^n}, \ldots, C_{U_m,1/3^n}$, i.e., the circuits that approximate the prover’s operators with precision $1/3^n$. According to Corollary 2.6 the length of this proof is $O\left(m \cdot 5^{2m} \cdot \log^3\left(5^{2m} \cdot 3^n\right)\right) \in \text{poly}(n)$. $W$ uses this classical proof to simulate the first $m$ rounds of the proof system, and produce the state of registers $P_mA_mV_m$, which we denote by $|\psi\rangle$. Note that since each circuit approximates the corresponding unitary with precision $1/3^n$, after applying log-many of them, it is true that

$$\frac{1}{2} \| |\psi\rangle\langle \psi| - |\phi\rangle\langle \phi| \|_{\text{Tr}} \leq \frac{m}{2^n} \leq \frac{1}{2^n},$$

for sufficiently large $n$; where $|\phi\rangle$ is the state of $P_mA_mV_m$ in the case where the unitaries $U_1, \ldots, U_m$ were applied instead of the circuits.
We are left with specifying how $W$ simulates the prover in the last $(m+1)$th round. We use exactly the same method as in the proof of Theorem 2.3 appeared in [BSW11], and our proof closely follows that proof as well. Since we are in the last round, we don’t have to keep track of the prover’s private space, so we can just describe it’s strategy as a quantum channel that transforms registers $P_m Q_{m+1}$ to $A_{m+1}$. Let’s call this channel $\Phi \in C(S, R)$ from now on; where $S$ is the joint space associated to registers $P_m Q_{m+1}$, and $R$ is the space associated to $A_{m+1}$. The input space $S$ is on $q \equiv 2m + 1 = O(\log n)$ qubits and the output space $R$ is on $\text{poly}(n)$ qubits. $W$ expects to get $\rho_\Phi^{((N+k)}}$ as the quantum part of it’s proof, where $\rho_\Phi \in D(R \otimes S)$ is the normalized Choi–Jamiołkowski representation of $\Phi$, for $N$ and $k$ to be specified later. Let’s divide up the quantum certificate given to $W$ into registers $R_1, S_1, R_2, S_2, \ldots, R_{N+k}, S_{N+k}$, where the space of each $R_i$ is $R$, and the space of each $S_i$ is $S$. $W$ expects each $R_i, S_i$ to contain a copy of $\rho_\Phi$. To simulate the last round of the interactive proof system, $W$ does the following.

1. Randomly permute the pairs $(R_1, S_1), \ldots, (R_{N+k}, S_{N+k})$, according to a uniformly chosen permutation, and discard all but the first $(N+1)$ pairs.

2. Perform quantum state tomography on the registers $(S_2, \ldots, S_{N+1})$, and reject if the resulting approximation is not within trace-distance $\delta/2$ of the completely mixed state $(1/2^n) \mathbb{1}$, for $\delta$ to be specified below.

3. Simulate the channel specified by $(R_1, S_1)$ by post-selection. Reject if post-selection fails, otherwise simulate the last operation of $V$ and accept if and only if $V$ accepts.

Let $g(n) \in \text{poly}(n)$ be such that $c(n) - s(n) \geq 1/g(n)$. We now set the parameters.

$$
\varepsilon \equiv \frac{1}{4^{g+1}n}, \quad \delta \equiv \frac{\varepsilon^2}{4}, \quad N \equiv \left\lceil \frac{210q}{(\delta/2)^5} \right\rceil, \quad k \equiv \left\lceil \frac{4q+1(N+1)}{\varepsilon} \right\rceil
$$

Note that $1/\varepsilon, 1/\delta, N, k \in \text{poly}(n)$.

**Completeness.** Suppose there exist a $P$ that causes $V$ to accept with probability $\geq c$. Let the certificate to $W$ be the classical descriptions of circuits $C_{U_1,1/3^n}, \ldots, C_{U_m,1/3^n}$, together with the state $\rho_\Phi^{(N+k)}$, where each $R_i, S_i$ contains a copy of $\rho_\Phi$, for $i \in \{1, 2, \ldots, N+k\}$. After simulating the first $m$ rounds, $W$ produces $|\psi\rangle$ which is $\leq 1/2^n$ far from the correct $|\phi\rangle$ in the trace distance, just as described above. Note that in the simulation of the last round, step 1 doesn’t change the state of registers $(R_1, S_1), \ldots, (R_{N+1}, S_{N+1})$. According to Lemma 2.7, $W$ rejects in step 2 with probability $\leq \delta/2$. In step 3, post-selection succeeds with probability $1/4^q$. If $W$ was using $|\phi\rangle$ instead of $|\psi\rangle$ the probability of acceptance would be at least

$$
\left(1 - \frac{\delta}{2}\right) \frac{c}{4^q}
$$

So using $|\psi\rangle$, the probability that $W$ accepts is at least

$$
\left(1 - \frac{\delta}{2}\right) \frac{c}{4^q} - \frac{1}{2^n} \geq \frac{c}{4^q} - \varepsilon\frac{1}{2^n}.
$$

**Soundness.** Suppose that all $P$ causes $V$ to accept with probability $\leq s$. Note that, without loss of generality any classical proof specifies some set of unitaries that correspond to a valid prover strategy. Hence it is still true, that after $W$ simulates the first $m$ rounds using the given circuits, it ends up with a state $|\psi\rangle$ that is at most $1/2^n$ far from a state $|\phi\rangle$, where $|\phi\rangle$ can be produced by some $P$ interacting with $V$. 

7
Now consider the situation that the state of \((S_1, \ldots, S_{N+1})\) before step 3 has the form
\[
\sigma \otimes (N+1), \tag{1}
\]
for some \(\sigma \in D(S)\). (The classical part of the proof has been used up and discarded before step 1.) We consider two cases:

- Suppose that \(\|\sigma - (1/2^q) 1\|_{Tr} < \delta\). Let the state of \((R_1, S_1)\) before step 3 be \(\xi \in D(R \otimes S)\), so we have \(Tr_R(\xi) = \sigma\). Because of the same argument as in \cite{BSW11}, there exists a state \(\tau \in D(R \otimes S)\) such that \(Tr_R(\tau) = (1/2^q) 1\) and \(1/2 \|\tau - \xi\|_{Tr} \leq \varepsilon\). Given this \(\tau\), the post-selection in step 3 succeeds with probability \(1/4^q\), so the acceptance in step 3 occurs with probability at most
\[
\frac{s}{4^q} + \frac{1}{2^n} + \varepsilon.
\]

- If \(\|\sigma - (1/2^q) 1\|_{Tr} \geq \delta\), then in step 2 \(W\) will accept with probability \(\leq \delta/2\).

Since \(\delta/2 \leq s/4^q + 1/2^n + \varepsilon\) then in both cases acceptance occurs with probability \(\leq s/4^q + 1/2^n + \varepsilon\).

Now suppose that the state of \((S_1, \ldots, S_{N+1})\) before step 3 has the form
\[
\sum_i p_i \sigma_i \otimes (N+1), \tag{2}
\]
for some probability vector \(p\) and some set \(\{\sigma_i\} \subset D(S)\). Since (2) is a convex combination of states of the form (1), acceptance will occur with probability \(\leq s/4^q + 1/2^n + \varepsilon\). In the real scenario, by Theorem 2.8, it is true that the state of \((S_1, \ldots, S_{N+1})\) after step 1 will be \(\varepsilon\) close to a state of the form (2), in the trace distance. So the probability of acceptance of \(W\) will be \(\leq s/4^q + 2\varepsilon + 1/2^n\). Since
\[
\frac{\varepsilon}{4^q} - \varepsilon - \frac{1}{2^n} - \left(\frac{s}{4^q} + 2\varepsilon + \frac{1}{2^n}\right) \geq \frac{1}{h(n)},
\]
for some \(h(n) \in \text{poly}(n)\), it holds that \(L \in \text{QMA}\). \(\square\)

References

- [AN02] Dorit Aharonov and Tomer Naveh. Quantum NP - a survey. 2002, quant-ph/0210077.
- [BSW11] Salman Beigi, Peter Shor, and John Watrous. Quantum interactive proofs with short messages. *Theory of Computing*, 7(1):101–117, 2011.
- [BT09] Hugue Blier and Alain Tapp. All languages in NP have very short quantum proofs. In *Third International Conference on Quantum, Nano and Micro Technologies*, pages 34–37, 2009.
- [CKMR07] Matthias Christandl, Robert König, Graeme Mitchison, and Renato Renner. One-and-a-half quantum de Finetti theorems. *Communications in Mathematical Physics*, 273:473–498, 2007.
- [JJUW10] Rahul Jain, Zhengfeng Ji, Sarvagya Upadhyay, and John Watrous. QIP = PSPACE. In *Proceedings of the 42nd annual ACM Symposium on Theory of Computing*, STOC ’10, pages 573–582, 2010.
[Kit97] A. Yu Kitaev. Quantum computations: algorithms and error correction. *Russian Mathematical Surveys*, 52(6):1191, 1997.

[KM03] Hirotada Kobayashi and Keiji Matsumoto. Quantum multi-prover interactive proof systems with limited prior entanglement. *Journal of Computer and System Sciences*, 66(3):429–450, 2003.

[KW00] Alexei Kitaev and John Watrous. Parallelization, amplification, and exponential time simulation of quantum interactive proof systems. In *Proceedings of the 32nd annual ACM Symposium on Theory of Computing*, STOC ’00, pages 608–617, 2000.

[MW05] Chris Marriott and John Watrous. Quantum Arthur–Merlin games. *Computational Complexity*, 14:122–152, 2005.

[NC00] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[Wat99] John Watrous. PSPACE has constant-round quantum interactive proof systems. In *40th Annual Symposium on Foundations of Computer Science*, pages 112–119, 1999.

[Wat03] John Watrous. PSPACE has constant-round quantum interactive proof systems. *Theoretical Computer Science*, 292(3):575–588, 2003.

[Wat08a] John Watrous. Quantum computational complexity. April 2008, arXiv:0804.3401.

[Wat08b] John Watrous. Theory of quantum information. Lecture notes from Fall 2008, http://www.cs.uwaterloo.ca/~watrous/quant-info/, 2008.