THE DESCRIPTIVE LOOK AT THE SIZE OF SUBSETS OF GROUPS

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ABSTRACT. We explore the Borel complexity of some basic families of subsets of a countable group (large, small, thin, sparse and other) defined by the size of their elements. Applying the obtained results to the Stone-Čech compactification $\beta G$ of $G$, we prove, in particular, that the closure of the minimal ideal of $\beta G$ is of type $F_{\sigma\delta}$.

Given a group $G$, we denote by $P_G$ and $F_G$ the Boolean algebra of all subsets of $G$ and its ideal of all finite subsets. We endow $P_G$ with the topology arising from identification (via characteristic functions) of $P_G$ with $\{0,1\}^G$. For $K \in F_G$ the sets
\[
\{X \in P_G : K \subseteq X\}, \quad \{X \in P_G : X \cap K = \emptyset\}
\]
form the sub-base of this topology.

After the topologization, each family $F$ of subsets of a group $G$ can be considered as a subspace of $P_G$, so one can ask about Borel complexity of $F$, the question typical in the Descriptive Set Theory (see [1]). We ask these questions for the most intensively studied families in Combinatorics of Groups. For the origins of the families defined in section 1, see the survey [2]. The main results are in section 2 and its applications to $\beta G$, the Stone-Čech compactification of a discrete group $G$, in section 3. We conclude the paper with some comments.

1. DIVERSITY OF SUBSETS OF A GROUP

A subset $A$ of a group $G$ is called
- **large** if $G = F \cdot A$ for some $F \in F_G$;
- **extralarge** if $A \cap L$ is large for each large subset $L$;
- **small** if $L \setminus A$ is large for each large subset $L$;
- **thick** if, for any $F \in F_G$ there exist $g \in G$ such that $F \cdot g \subseteq A$;
- **prethick** if $F \cdot A$ is thick for some $F \in F_G$.

Some evident or easily verified (see [3]) relationships: $A$ is large if and only if $G \setminus A$ is not thick, $A$ is small if and only if $A$ is not prethick if and only if $G \setminus A$ is extralarge. The family of all small subsets of $G$ is an ideal in $P_G$.

A subset $A$ of a group $G$ is called
- **$P$-small** if there exists an injective sequence $(g_n)_{n \in \omega}$ in $G$ such that the subsets $\{g_n \cdot A : n \in \omega\}$ are pairwise disjoint;
- **weakly $P$-small** if, for any $n \in \omega$, there exists $g_0, \ldots, g_n$ such that the subsets $g_0 \cdot A, \ldots, g_n \cdot A$ are pairwise disjoint;
- **almost $P$-small** if there exists an injective sequence $(g_n)_{n \in \omega}$ in $G$ such that $g_n \cdot A \cap g_m \cdot A$ is finite for all distinct $n, m$;
- **near $P$-small** if, for every $n \in \omega$, there exists $g_0, \ldots, g_n$ such that $g_i \cdot A \cap g_j \cdot A$ is finite for all distinct $i, j \in \{0, \ldots, n\}$.

Every infinite group $G$ contains a weakly $P$-small set, which is not $P$-small, see [4]. Each almost $P$-small subset can be partitioned into two $P$-small subsets [5]. Every countable Abelian group contains a near $P$-small subset which is neither weakly nor almost $P$-small [6].

A subset $A$ of a group $G$ with the identity $e$ is called
- **thin** if $g \cdot A \cap e$ is finite for each $g \in G \setminus \{e\}$;
- **sparse** if, for every infinite subset $Y$ of $G$, there exists a non-empty finite subset $F \subseteq Y$ such that $\bigcap_{g \in F} g \cdot A$ is finite.

Key words and phrases. Borel complexity, Stone-Čech compactification.
The union of two thin subsets need not to be thin, but the family of all sparse subsets is an ideal in $P_G$ [7]. For plenty of modifications and generalizations of thin and sparse subsets see [5], [8], [9], [10], [11].

2. Results

For a group $G$, we denote by $L_G$, $EL_G$, $S_G$, $T_G$, $PT_G$ the sets of all large, extralarge, small, thick and prethick subsets of $G$, respectively.

Theorem 1. For a countable group $G$, we have: $L_G$ is $F_\sigma$, $T_G$ is $G_\delta$, $PT_G$ is $G_\delta\sigma$, $S_G$ and $EL_G$ are $F_\sigma\delta$.

Proof. We take $F, H \in F_G$ and prove the following auxiliary claim: the set $T(F, H) = \{ A \in P_G : H \subseteq FA \}$ is open.

Indeed, let $F = \{ g_1, \ldots, g_n \}$ and $(H_1, \ldots, H_n)$ is a partition of $H$. Then the set $\{ A \in P_G : H_1 \subseteq g_1 A, \ldots, H_n \subseteq g_n A \}$ is open. It follows that $T(F, H)$ is open.

Now, the set $T_H(F) = \bigcup \{ T(F, Hg) : g \in G \}$ is open and the set $T(F) = \bigcap \{ T_H(F) : H \in F_G \}$ is $G_\delta$. We note that $T_G = T(\{ e \})$ and $PT_G = \bigcup \{ T(F) : F \in F_G \}$ so $T_G$ is $G_\delta$ and $PT_G$ is $G_\delta\sigma$.

Since $S_G = P_G \setminus T_G$, $S_G$ is $F_\delta\sigma$. The mapping defined by $A \mapsto G \setminus A$ is a homeomorphism of $P_G$, so $L_G$ is homeomorphic to $P_G \setminus T_G$ and $EL_G$ is homeomorphic to $S_G$. Hence, $L_G$ is $F_\sigma$ and $EL_G$ is $F_\delta\sigma$.

Theorem 2. For a countable group $G$, the sets of thin, weakly $P$-small and near $P$-small subsets of $G$ are $F_\delta\sigma$.

Proof. Given $F \in F_G$ and $g \in G \setminus \{ e \}$, the set $X(F, g) = \{ A \in P_G : ga \not\subseteq A \text{ for each } a \in A \setminus F \}$ is closed. The set $X(g) = \bigcup \{ X(F, g) : F \in F_G \}$ is $F_\sigma$, and $\bigcap \{ X(g) : g \in G \setminus \{ e \} \}$ is the set of all thin subsets.

For $n \in \omega$, $[G]^n$ denotes the family of all $n$-subsets of $G$. Given $F \in [G]^n$, the set $Y(F) = \{ A \in P_G : gA \cap hA = \emptyset \text{ for all distinct } g, h \in F \}$ is closed, and the set of all weakly $P$-small subsets of $G$ coincides with

$$\bigcap_{n \in \omega} \bigcup \{ Y(F) : F \in [G]^n \}.$$

Given $F \in [G]^n$ and $H \in F_G$, the set $Y(F, H) = \{ A \in P_G : g(A \setminus H) \cap h(A \setminus H) = \emptyset \text{ for all distinct } g, h \in F \}$ is closed, and the set of all near $P$-small subsets of $G$ coincides with

$$\bigcap_{n \in \omega} \bigcup \{ Y(F, H) : F \in [G]^n, H \in F_G \}.$$ 

We recall that a topological space $X$ is Polish if $X$ is homeomorphic to a separable complete metric space. A subset $A$ of a Polish space $X$ is analytic if $A$ is a continuous image of some Polish space, and $A$ is coanalytic if $X \setminus A$ is analytic.

Using the classical tree technique [11] adopted to groups in [10], we get.

Theorem 3. For a countable group $G$, the ideal of sparse subsets is coanalytic and the set of $P$-small subsets is analytic in $P_G$.

3. Applications to $\beta G$

Given a discrete group $G$, we identify the Stone-Čech compactification $\beta G$ with the set of all ultrafilters on $G$ and consider $\beta G$ as a right-topological semigroup (see [12]). Each non-empty closed subspace $X$ of $\beta G$ is determined by some filter $\varphi_X$ on $G$:

$$X = \bigcap \{ \overline{\Phi} : \Phi \in \varphi \}, \quad \overline{\Phi} = \{ p \in \beta G : \Phi \in p \}.$$ 

On the other hand, each filter $\Phi$ on $G$ is a subspace of $P_G$, so we can ask about complexity of $X$ as the complexity of $\varphi_X$ in $P_G$.

The semigroup $\beta G$ has the minimal ideal $K_G$ which play one of the key parts in combinatorial applications of $\beta G$. By [3, Theorem 1.5], the closure $cl(K_G)$ is determined by the filter of all extralarge subsets of $G$. If $G$ is countable, applying Theorem 1, we conclude that $cl(K_G)$ has the Borel complexity $F_\sigma\delta$.

An ultrafilter $p$ on $G$ is called strongly prime if $p \notin cl(G^*G^*)$, where $G^*$ is a semigroup of all free ultrafilters on $G$. We put $X = cl(G^*G^*)$ and choose the filter $\varphi_X$ which determine $X$. By [7], $A \in \varphi_X$ if and only if $G \setminus A$ is sparse. If $G$ is countable, applying Theorem 3, we conclude that $\varphi_X$ is coanalytic in $P_G$. 

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Let \((g_n)_{n \in \omega}\) be an injective sequence in \(G\). The set
\[
\{g_{i_1}g_{i_2} \cdots g_{i_n} : 0 \leq i_1 < i_2 < \ldots < i_n < \omega\}
\]
is called an \(FP\)-set. By the Hindman Theorem 5.8 [12], for every finite partition of \(G\), at least one cell of the partition contains an \(FP\)-set. We denote by \(FP_G\) the family of all subsets of \(G\) containing some \(FP\)-set. A subset \(A\) of \(G\) belongs to \(FP_G\) if and only if \(A\) is an element of some idempotent of \(\beta G\). By analogy with Theorem 3, we can prove that \(FP_G\) is analytic in \(P_G\).

4. Comments and open questions

1. Answering a question from [13], Zakrzewski proved [14] that, for a countable amenable group \(G\), the ideal of absolute null subsets has the Borel complexity \(F_{\sigma \delta}\). Each absolute null subset is small but, for every \(\epsilon > 0\), there exists a small subset \(A\) of \(G\) such that \(\mu(A) > 1 - \epsilon\) for some Banach measure \(\mu\) on \(G\) (see [15]).

2. The classification of subsets of a group by their size can be considered in much more general context of Asymptology (see [15]). In this context, large, thick and small subsets play the parts of dense, open and nowhere dense subsets of a uniform topological space. For dynamical look at the subsets of a group see [16].

3. The following type of subsets of a group arised in Asymptology [11]. A subset \(A\) of a group \(G\) is called scattered if \(A\) has no subsets coarsely equivalent to the Cantor macrocube. Each sparse subset is scattered, each scattered subset is small and the set of all scattered subsets of a countable group is an ideal in \(P_G\).

By Theorem 1 [11], a subset \(A\) of a group \(G\) is scattered if and only if \(A\) contains no piecewise shifted \(FP\)-sets.

Let \((g_n)_{n \in \omega}\) be an injective sequence in \(G\) and let \((b_n)_{n \in \omega}\) is a sequence in \(G\).

The set
\[
\{g_{i_1}g_{i_2} \cdots g_{i_n}b_{i_n} : 0 \leq i_1 < i_2 < \ldots < i_n < \omega\}
\]
is called a piecewise shifted \(FP\)-set.

Using this combinatorial characterization and the tree technique from [10], we can prove that the ideal of scattered subsets of a countable group \(G\) is coanalytic in \(P_G\).

4. By [17], every meager topological group \(G\) can be represented as the product \(G = CN\) of some countable subset \(C\) and nowhere dense subset \(N\). Every infinite group \(G\) can be represented as a union of some countable family of small subsets [3].

*Can every infinite group \(G\) be represented as the product \(G = CS\) of some countable subset \(C\) and small subset \(S\)*?

The answer is positive if either \(G\) is amenable or \(G\) has a subgroup of countable index.

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