New merit functions for multiobjective optimization and their properties

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Abstract

A merit (gap) function is a map that returns zero at the solutions of problems and strictly positive values otherwise. Its minimization is equivalent to the original problem by definition, and it can estimate the distance between a given point and the solution set. Ideally, this function should have some properties, including the ease of computation, continuity, differentiability, boundedness of the level set, and error boundedness. In this work, we propose new merit functions for multiobjective optimization with lower semicontinuous objectives, convex objectives, and composite objectives, and we show that they have such desirable properties under reasonable assumptions.

1 Introduction

Multiobjective optimization is an important field of research with many practical applications. It minimizes several objective functions at once, but usually, there does not exist a single point that minimizes all objective functions simultaneously. Therefore, we use the concept of Pareto optimality. We call a point Pareto optimal if there does not exist another point with the same or smaller objective function values and with at least one objective function value being strictly smaller. However, for non-convex problems, it is difficult to get Pareto optimal solutions. Thus, we use the concept of Pareto stationarity. A point is called Pareto stationary if there does not exist a descent direction from it.

Many algorithms for getting Pareto optimal or Pareto stationary solutions have been developed, including the scalarization approaches \textsuperscript{19,20,47}, the metaheuristics \textsuperscript{18}, and the descent methods \textsuperscript{11,12,14,41}. However, from a practical point of view (e.g., estimating convergence rates and describing the duality gap), it is essential to know how far a feasible point is from the Pareto set. Within the context of optimization and equilibrium problems, one of the most commonly used tools to meet such needs is the merit (gap) functions \textsuperscript{2,23}, which return
zero at the problem’s solutions and strictly positive values otherwise. Merit functions were first proposed by Auslender [2] in 1976 for variational inequality problems, and became widely known after Hearn [23] re-proposed the same functions and named them gap functions when studying the dual gap for convex programming problems in 1982. If we can minimize the merit function globally, we can obtain the solution of the original problem. For this reason, the merit functions should have the following properties:

- Their values at given points can be quickly evaluated;
- Continuity;
- (Directional) differentiability;
- Their stationary points solve the original problem;
- Level-boundedness, i.e., their level sets are bounded;
- They provide error bounds, i.e., they are lower bounded by some multiple of the distance between a given point and the solution set of the original problem.

It is worth commenting that merit functions have been extensively studied for more general problems such as quasi-variational inequalities [13,16,21,36]. For a comprehensive survey of merit functions for variational inequality and complementarity problems, we refer the reader to the work of Fukushima [15].

This paper is not the first attempt to develop merit functions for multiobjective or vector problems. However, such studies are relatively new compared to the history of research on merit functions for single-objective or scalar problems [2,15,22]. First, in 1998, Chen, Goh, and Yang [8] developed the merit function for polyhedral-constrained convex multiobjective optimization. Afterward, various merit functions were considered for vector variational inequalities [17,0,28,29,31,15,46] and vector equilibrium problems [25,29,30,32,35]. In 2010, Li and Mastroeni [31] introduced gap functions with error bounds, and Charitha and Dutta [7] studied regularized gap functions and D-gap functions with continuity, directional differentiability, and error bounds, both for finite-dimensional convex-constrained vector variational inequalities. On the other hand, the merit functions with error bounds for convex-constrained multiobjective or vector optimization were considered in 2009 for linear objectives [33] and in 2017 for convex objectives [10]. Also, Soleimani-damaneh [40] defines the merit function for multiobjective optimization problems in Banach space, and discusses differentiability by Clarke’s generalized gradients. However, most of those papers do not discuss all of the desired properties mentioned above. In particular, few discussions exist, related to the computing methods of the merit functions.

Let us now consider the following multiobjective optimization problem:

$$\min_{x \in S} F(x),$$  \hspace{1cm} (1)
where \( F : S \to \mathbb{R}^m \) is a vector-valued function with \( F := (F_1, \ldots, F_m)\top \), and \( S \subseteq \mathbb{R}^n \) is nonempty, closed, and convex. Here, we propose the following three merit functions for (1): a simple one for lower semicontinuous problems, a regularized one for convex problems, and a regularized and partially linearized one for composite problems, i.e., problems with each objective being the sum of a differentiable but not necessarily convex function and a convex but not necessarily differentiable one. In Table 1, we summarize the properties of those merit functions, which will be shown in the subsequent sections. There, ‘Sol.’ represents the types of Pareto solutions for (1) corresponding to the minima (zero points) of the merit functions. Moreover, ‘SP,’ ‘LB,’ and ‘EB’ indicate each \( F_i \)’s sufficient conditions so that stationary points of the merit functions can solve (1), the merit functions are level-bounded, and the merit functions provide error bounds, respectively. The simple one connects its minima and the weak Pareto solutions of (1) but does not have good properties in other aspects. The regularized one has better properties but requires the convexity of \( F_i \). The convexity assumption is relaxed in the regularized and partially linearized one, which is also easy to compute for particular problems.

The outline of this paper is as follows. In Section 2 we introduce some notations and concepts used in the subsequent discussion. Section 3 proposes different merit functions for multiobjective optimization with lower semicontinuous objectives, convex objectives, and composite objectives, along with methods for evaluating the function values, the differentiability, and the stationary point properties. Furthermore, sufficient conditions for them to be level-bounded and to provide error bounds are given in Sections 5 and 6, respectively.

2 Preliminaries

2.1 Notations and definitions

Let us present some notions and definitions used in this paper. We use the symbol \( \| \cdot \| \) for the Euclidean norm in \( \mathbb{R}^n \). For \( u, v \in \mathbb{R}^n \), the notation \( u \leq v \) \((u < v)\) means that \( u_i \leq v_i \) \((u_i < v_i)\) for all \( i = 1, \ldots, m \). The zero vector is denoted by 0 without mentioning the dimension. We also define the standard simplex \( \Delta^m \subseteq \mathbb{R}^m \) by

\[
\Delta^m := \left\{ \lambda \in \mathbb{R}^m \mid \lambda \geq 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}.
\]

(2)

Let \( S \subseteq \mathbb{R}^n \) be a nonempty closed convex set, and let \( x \in S \). The normal cone of \( S \) at \( x \), denoted by \( N_S(x) \), is defined as

\[
N_S(x) := \left\{ z \in \mathbb{R}^n \mid z^\top (y - x) \leq 0, \forall y \in S \right\}.
\]

(3)

Furthermore, the convex hull of a set \( A \subseteq \mathbb{R}^n \), denoted by \( \text{conv}(A) \), is the smallest convex set containing \( A \).

Now, we introduce some definitions of functions. A function \( h : S \to \mathbb{R} \cup \{\infty\} \) is lower semicontinuous at \( x \in S \) if \( h(x) \leq \liminf_{k \to \infty} h(x_k) \) for any sequence
Table 1: Properties of our proposed merit functions

(a) Proposed merit functions and their properties

|                | Obj. | Sol. | Cont. | Diff. | SP   | LB   | EB   |
|----------------|------|------|-------|-------|------|------|------|
| Simple         | Cont. |      | LSC   | ×     | ×    | LB   |      |
| Regularized    | Conv. |      | Cont. | DD    | SC   | Conv., LB | PL   |
| Regularized and partially linearized | Comp. | PS   | Cont. | DD    | SC, $C^2$ | Conv., LB, etc. |      |

(b) Table of abbreviations

| Abbreviation | Description                                   |
|--------------|-----------------------------------------------|
| Obj.         | Objective functions                           |
| Sol.         | Solutions                                     |
| Cont.        | Continuity                                    |
| Diff.        | Differentiability                              |
| SP           | Stationary points                             |
| LB           | Level-boundedness                             |
| EB           | Error bounds                                  |
| Cont.        | Continuity                                    |
| Comp.        | Compositeness                                 |
| WPO          | Weak Pareto optimality                        |
| PS           | Pareto stationarity                           |
| LSC          | Lower semicontinuity                          |
| DD           | Directional differentiability                  |
| SC           | Strict convexity                              |
| $C^2$        | Twice continuous differentiability            |
| PL           | Multiobjective proximal-PL inequality         |
\{x^k\} \subseteq S convergent to x. In particular, if h is lower semicontinuous at every point on S, we say that h is lower semicontinuous or closed on S. On the other hand, h is proper if its effective domain, defined by \(\text{dom}(h) := \{x \in S \mid h(x) < \infty\}\), is not empty. Moreover, we call

\[
h'(x; d) := \lim_{t \to 0} \frac{h(x + td) - h(x)}{t}
\]

the directional derivative of \(h: S \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}\) at \(x \in S\) with \(h(x) < \infty\) in the direction \(d \in \mathbb{R}^n\). Note that \(h'(x; d) = \nabla h(x)^\top d\) when h is differentiable at x, and \(\nabla\) denotes transpose. In addition, given \(\sigma > 0\), we say that h is \(\sigma\)-convex if

\[
h(\alpha x + (1 - \alpha)y) \leq \alpha h(x) + (1 - \alpha)h(y) - \frac{\alpha(1 - \alpha)\sigma}{2} \|x - y\|^2
\]

for all \(x, y \in S\) and \(\alpha \in [0, 1]\). In particular, 0-convexity is equivalent to the usual convexity, and when \(\sigma > 0\), h is called strongly convex. For a convex function \(h: S \to \mathbb{R} \cup \{\infty\}\), we define the subdifferential of h at \(x \in S\) as

\[
\partial h(x) = \{h' \in \mathbb{R}^n \mid h(y) \geq h(x) + (h')^\top(y - x) \text{ for all } y \in S\}.
\] (4)

We now suppose that \(h: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}\) is a closed, proper, and convex function. Then, the Moreau envelope or Moreau-Yosida regularization \(\mathcal{M}_h: \mathbb{R}^n \to \mathbb{R}\) is given by

\[
\mathcal{M}_h(x) := \min_{y \in \mathbb{R}^n} \left\{ h(y) + \frac{1}{2} \|x - y\|^2 \right\}.
\] (5)

The minimization problem in (5) has a unique solution because of the strong convexity of its objective functions. By this solution, the proximal operator is defined as

\[
\text{prox}_h(x) := \arg\min_{y \in \mathbb{R}^n} \left\{ h(y) + \frac{1}{2} \|x - y\|^2 \right\}.
\] (6)

The proximal operator is non-expansive, i.e., \(\|\text{prox}_h(x) - \text{prox}_h(y)\| \leq \|x - y\|\). This also means that \(\text{prox}_h\) is 1-Lipschitz continuous. Moreover, when h is the indicator function of \(C \subseteq \mathbb{R}^n\), i.e,

\[
\iota_C(x) := \begin{cases} 0, & x \in C, \\ \infty, & x \notin C, \end{cases}
\] (7)

where \(C \subseteq \mathbb{R}^n\) is a nonempty closed convex set, the proximal operator of h reduces to the projection onto C, i.e.,

\[
\text{prox}_{\iota_C}(x) = \text{proj}_C(x) := \arg\min_{y \in C} \|x - y\|.
\] (8)

Recall that h is closed, proper, and convex. Even if h is non-differentiable, its Moreau envelope \(\mathcal{M}_h\) is known to be differentiable.
Theorem 2.1. [3, Theorem 6.60] Let $h: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a closed, proper, and convex function. Then, $M_h$ has an 1-Lipschitz continuous gradient given by

$$\nabla M_h(x) = x - \text{prox}_h(x).$$

We also refer to the so-called second prox theorem as well as a corollary quickly derived from it.

Theorem 2.2. [3, Theorem 6.39] Let $h: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a closed, proper, and convex function. Then, for any $x \in \mathbb{R}^n$, we have

$$(x - \text{prox}_h(x))^\top (y - \text{prox}_h(x)) \leq h(y) - h(\text{prox}_h(x)) \quad \text{for all} \ y \in \mathbb{R}^n.$$

Corollary 2.3. Let $h: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a closed, proper, and convex function. Then, it follows that

$$\|x - \text{prox}_h(x)\|^2 \leq h(x) - h(\text{prox}_h(x)) \quad \text{for all} \ x \in \mathbb{R}^n.$$

We move on to Hölder and Lipschitz continuities. We call $h: \mathbb{R}^n \rightarrow \mathbb{R}$ to be locally Hölder continuous with exponent $\beta > 0$ if for every bounded set $\Omega \subseteq \mathbb{R}^n$ there exists $L > 0$ such that

$$|h(x) - h(y)| \leq L \|x - y\|^\beta \quad \text{for all} \ x, y \in \Omega.$$

In particular, when $L$ does not depend on $\Omega$, we say that $h$ is Hölder continuous with exponent $\beta > 0$. Moreover, we refer to the (local) Hölder continuity with exponent 1 as the (local) Lipschitz continuity. When $h$ is Lipschitz continuous, we call $L$ the Lipschitz constant, and we also say that $h$ is $L$-Lipschitz continuous. As the following lemma shows, many functions with good properties are locally Lipschitz continuous.

Lemma 2.4. Continuously differentiable functions and finite-valued convex functions are locally Lipschitz continuous.

Proof. The former is due to the mean value theorem, and the latter is from [44].

Finally, we recall a fact on sensitivity analysis for the following parameterized optimization problem:

$$\min_{x \in X} h(x, \xi),$$

(9)

depending on the parameter vector $\xi \in \Xi$. Here, $h: X \times \Xi \subseteq \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R} \cup \{\infty\}$ is the objective function. We assume that $X \subseteq \mathbb{R}^p$ and $\Xi \subseteq \mathbb{R}^q$ are nonempty and closed. Let us write the optimal value function of (9) as

$$\phi(\xi) := \inf_{x \in X} h(x, \xi)$$

(10)

and the associated set as

$$\Phi(\xi) := \{x \in X \mid \phi(\xi) = h(x, \xi)\}.$$  

(11)

The following proposition describes the directional differentiability of the optimal value function $\phi$. 

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Proposition 2.5. [6, Proposition 4.12] Let $\xi^0 \in \Xi$. Suppose that

(i) the function $h(x, \xi)$ is continuous on $X \times \Xi$;

(ii) there exist $\alpha \in \mathbb{R}$ and a compact set $C \subseteq X$ such that for every $\hat{\xi}$ near $\xi^0$, the level set $\text{lev}_\alpha h(\cdot, \hat{\xi})$ is nonempty and contained in $C$;

(iii) for any $x \in X$ the function $h_x(\cdot) := h(x, \cdot)$ is directionally differentiable at $\xi^0$;

(iv) if $\xi \in \Xi$, $t_k \searrow 0$, and $\{x^k\}$ is a sequence in $C$ given by (ii), then $\{x^k\}$ has a limit point $\bar{x}$ such that

$$\limsup_{k \to \infty} \frac{h(x^k, \xi^0 + t_k(\xi - \xi^0)) - h(x^k, \xi^0)}{t_k} \geq h'_x(\xi^0; \xi - \xi^0).$$

Then, the optimal value function $\phi$ given by (10) is directionally differentiable at $\xi^0$ and

$$\phi'(\xi^0; \xi - \xi^0) = \inf_{x \in \Phi(\xi^0)} h'_x(\xi^0; \xi - \xi^0).$$

2.2 Optimality, stationarity, and level-boundedness for multiobjective optimization

We first introduce the concept of optimality and stationarity for (1). Note that vector-to-vector inequalities are componentwise, as defined at the beginning of the previous section. Recall that $x^* \in S$ is Pareto optimal if there is no $x \in S$ such that $F(x) \leq F(x^*)$ and $F(x) \neq F(x^*)$. Likewise, $x^* \in S$ is weakly Pareto optimal if there does not exist $x \in S$ such that $F(x) < F(x^*)$. It is known that Pareto optimal points are always weakly Pareto optimal, and the converse is not true necessarily. When $F$ is directionally differentiable, we also say that $\bar{x} \in S$ is Pareto stationary [41] if

$$\max_{i=1,\ldots,m} F'_i(\bar{x}; z - \bar{x}) \geq 0 \quad \text{for all } z \in S.$$ 

We state below the relation between the three concepts of Pareto optimality.

Lemma 2.6. [4, Lemma 2.2] When $F$ is directionally differentiable, the following three statements hold.

(i) If $x \in S$ is weakly Pareto optimal for (1), then $x$ is Pareto stationary.

(ii) Let every component $F_i$ of $F$ be convex. If $x \in S$ is Pareto stationary for (1), then $x$ is weakly Pareto optimal.

(iii) Let every component $F_i$ of $F$ be strictly convex. If $x \in S$ is Pareto stationary for (1), then $x$ is Pareto optimal.
Now, let us extend the level-boundedness \cite[Definition 1.8]{[38]} for a scalar-valued function \( f : S \to \mathbb{R} \), i.e., \( \text{lev}_\alpha f := \{ x \in S \mid f(x) \leq \alpha \} \) is bounded for any \( \alpha \in \mathbb{R} \), to a vector-valued function as follows.

**Definition 2.7.** A vector-valued function \( F : S \to \mathbb{R}^m \) is level-bounded if the level set \( \text{lev}_\alpha F := \{ x \in S \mid F(x) \leq \alpha \} \) is bounded for all \( \alpha \in \mathbb{R}^m \).

If \( F_i \) is level-bounded for all \( i = 1, \ldots, m \), then \( F = (F_1, \ldots, F_m)^\top \) is also level-bounded. Note that even if \( F \) is level-bounded, every \( F_i \) is not necessarily level-bounded (e.g. \( F(x) = (x - 1, -2x + 1)^\top \)). Now, we show the existence of weakly Pareto optimal points under the level-boundedness assumption. Recall that throughout the paper, the feasible set \( S \) is nonempty, closed, and convex.

**Theorem 2.8.** If \( F \) is closed and level-bounded, then (1) has a weakly Pareto optimal solution.

**Proof.** Let \( F \) be closed and level-bounded. Then the level set \( \text{lev}_\alpha F := \{ x \in S \mid F_i(x) \leq \alpha \text{ for all } i = 1, \ldots, m \} \) is bounded for all \( \alpha \in \mathbb{R} \). Now, we have

\[
\text{lev}_\alpha F = \{ x \in S \mid \max_{i=1, \ldots, m} F_i(x) \leq \alpha \} = \text{lev}_\alpha \left( \max_{i=1, \ldots, m} F_i \right),
\]

so \( \max_i F_i \) is also level-bounded. Moreover, since \( F_i \) is closed for each \( i = 1, \ldots, m \), \( \max_i F_i \) is also closed. Thus, the problem

\[
\begin{align*}
\min_{i=1, \ldots, m} & \quad \max_i F_i(x) \\
\text{s.t.} & \quad x \in S
\end{align*}
\]

has a global optimal solution \( x^* \). This gives

\[
\max_{i=1, \ldots, m} F_i(x^*) \leq \max_{i=1, \ldots, m} F_i(x) \quad \text{for all } x \in S.
\]

Since \( \max_{i=1, \ldots, m} a_i - \max_{i=1, \ldots, m} (a_i - b_i) \leq \max b_i \), we have \( \max_{i=1, \ldots, m} F_i(x) - \max_{i=1, \ldots, m} (F_i(x) - F_i(x^*)) \leq \max_{i=1, \ldots, m} F_i(x^*) \) for all \( x \in S \), which together with the above inequality gives \( \max_{i=1, \ldots, m} (F_i(x^*) - F_i(x)) \leq 0 \) for all \( x \in S \). As this means \( F_i(x^*) \leq F_i(x) \) for all \( i = 1, \ldots, m \) and all \( x \in S \), we obtain the result. \( \square \)

### 3 New merit functions for multiobjective optimization

A merit function associated with an optimization problem is a function that returns zero at their solutions and strictly positive values otherwise, which implies that it is nonnegative \cite{[22][23]}. This section proposes different types of merit functions for the multiobjective optimization problem (1), considering three cases, respectively, when the objective function \( F \) is lower semicontinuous, when it is convex, and when it has a composite structure.
3.1 A simple merit function for lower semicontinuous multiobjective optimization

First, we assume only lower semicontinuity on $F$ and propose a simple merit function $u_0: S \to \mathbb{R} \cup \{\infty\}$ as follows:

$$u_0(x) := \sup_{y \in S} \min_{i=1,\ldots,m} \{F_i(x) - F_i(y)\}.$$  \hfill (12)

When $F$ is linear, this merit function has already been discussed in \cite{33}, but here we consider the more general nonlinear cases. We now show that $u_0$ is a merit function in the sense of weak Pareto optimality.

**Theorem 3.1.** Let $u_0$ be defined by \hfill (12). Then, we have $u_0(x) \geq 0$ for all $x \in S$. Moreover, $x \in S$ is weakly Pareto optimal for \hfill (1) if and only if $u_0(x) = 0$.

**Proof.** Let $x \in S$. By the definition \hfill (12) of $u_0$, we get

$$u_0(x) = \sup_{y \in S} \min_{i=1,\ldots,m} \{F_i(x) - F_i(y)\} \geq \min_{i=1,\ldots,m} \{F_i(x) - F_i(x)\} = 0.$$  

On the other hand, again considering the definition \hfill (12) of $u_0$, we obtain

$$u_0(x) = 0 \iff \min_{i=1,\ldots,m} \{F_i(x) - F_i(y)\} \leq 0 \text{ for all } y \in S.$$  

So, there does not exist $y \in S$ such that

$$F_i(x) - F_i(y) > 0 \text{ for all } i = 1,\ldots,m,$$

which means that $x$ is weakly Pareto optimal for \hfill (1) by definition. \hfill \Box

The following theorem is clear from the lower semicontinuity of $F_i$ and \cite[Theorem 10.3]{43}.

**Theorem 3.2.** The function $u_0$ defined by \hfill (12) is lower semicontinuous on $S$.

Theorems 3.1 and 3.2 imply that if $u_0(x^k) \to 0$ holds for some bounded sequence $\{x^k\}$, its accumulation points are weakly Pareto optimal. Thus, we can use $u_0$ to measure the convergence rate of multiobjective optimization methods (i.e., \cite{42}).

Moreover, Theorem 3.1 implies that we can get weakly Pareto optimal solutions via the following single-objective optimization problem:

$$\min_{x \in S} u_0(x).$$

However, in some cases, such as when $F_i$ is not bounded from below on $S$ for all $i = 1,\ldots,m$, we cannot guarantee that $u_0$ is finite-valued. Moreover, even if $u_0$ is finite-valued, $u_0$ does not preserve the differentiability of the original objective function $F$. 

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3.2 A regularized merit function for convex multiobjective optimization

Here, we suppose that each component $F_i$ of the objective function $F$ of (1) is convex. Then, we define a regularized merit function $u_\ell: S \to \mathbb{R}$ with a given constant $\ell > 0$, which overcomes the shortcomings mentioned at the end of the previous subsection, as follows:

$$u_\ell(x) := \max_{y \in S} \min_{i=1,\ldots,m} \left\{ F_i(x) - F_i(y) - \frac{\ell}{2} \| x - y \|^2 \right\}. \tag{13}$$

Note that the strong concavity of the function inside $\max_{y \in S}$ and \[8\] imply that there exists a unique solution $U_\ell(x) \in S$ given by

$$U_\ell(x) := \arg\max_{y \in S} \min_{i=1,\ldots,m} \left\{ F_i(x) - F_i(y) - \frac{\ell}{2} \| x - y \|^2 \right\} \tag{14}$$

and $u_\ell$ is finite-valued. Like $u_0$, we can show that $u_\ell$ is also a merit function in the sense of weak Pareto optimality.

**Theorem 3.3.** Let $u_\ell$ be defined by (13) for some $\ell > 0$. Then, we have $u_\ell(x) \geq 0$ for all $x \in S$. Moreover, $x \in S$ is weakly Pareto optimal for (1) if and only if $u_\ell(x) = 0$.

**Proof.** Let $x \in S$. The definition (13) of $u_\ell$ yields

$$u_\ell(x) = \max_{y \in S} \min_{i=1,\ldots,m} \left\{ F_i(x) - F_i(y) - \frac{\ell}{2} \| x - y \|^2 \right\} \geq \min_{i=1,\ldots,m} \left\{ F_i(x) - F_i(x) - \frac{\ell}{2} \| x - x \|^2 \right\} = 0,$$

which proves the first statement.

We now prove the second statement. First, assume that $u_\ell(x) = 0$. Then, (13) again gives

$$\min_{i=1,\ldots,m} \left\{ F_i(x) - F_i(y) - \frac{\ell}{2} \| x - y \|^2 \right\} \leq 0 \quad \text{for all } y \in S.$$

Let $z \in S$ and $\alpha \in (0, 1)$. Since the convexity of $S$ implies that $x + \alpha(z - x) \in S$, by substituting $y = x + \alpha(z - x)$ into the above inequality, we get

$$\min_{i=1,\ldots,m} \left\{ F_i(x) - F_i(x + \alpha(z - x)) - \frac{\ell}{2} \| \alpha(z - x) \|^2 \right\} \leq 0.$$

The convexity of $F_i$ leads to

$$\min_{i=1,\ldots,m} \left\{ \alpha(F_i(x) - F_i(z)) - \frac{\ell}{2} \| \alpha(z - x) \|^2 \right\} \leq 0.$$
Dividing both sides by $\alpha$ and letting $\alpha \downarrow 0$, we have
$$\min_{i=1,\ldots,m} \{F_i(x) - F_i(z)\} \leq 0.$$ 
Since $z$ can take an arbitrary point in $S$, it follows from (12) that $u_0(x) = 0$. Therefore, from Theorem 3.1 $x$ is weakly Pareto optimal.

Now, suppose that $x$ is weakly Pareto optimal. Then, it follows again from Theorem 3.1 that $u_0(x) = 0$. It is clear that $u_\ell(x) \leq u_0(x)$ from the definitions (12) and (13) of $u_0$ and $u_\ell$. So, we get $u_\ell(x) = 0$. \hfill \Box

Then, we can also show the continuity of $u_\ell$ and $U_\ell$ without any particular assumption.

**Theorem 3.4.** For each $\ell > 0$, $u_\ell$ and $U_\ell$ defined by (13) and (14) are locally Lipschitz continuous and locally Hölder continuous with exponent $1/2$ on $S$, respectively.

**Proof.** The optimality condition of the maximization problem associated with (13) and (14) and [4, Proposition A.22] give
$$\ell(x - U_\ell(x)) \in \text{conv} \left( \frac{\partial F_i(U_\ell(x)) + N_S(U_\ell(x))}{\partial x} \right) \quad \text{for all } x \in S,$$

where $N_S$ denotes the normal cone to the convex set $S$ defined by (3) and
$$\mathcal{I}(x) = \arg\min_{i=1,\ldots,m} \{F_i(x) - F_i(U_\ell(x))\}.$$

Considering the definitions (3) and (4) of the normal cone and subdifferential, for each $x \in S$ there exists $\lambda(x) \in \Delta^m$, where $\Delta^m$ is the standard simplex given by (2), such that $\lambda_j(x) = 0$ for all $j \notin \mathcal{I}(x)$ and
$$\ell(x - U_\ell(x))^T (z - U_\ell(x)) \leq \sum_{i=1}^m \lambda_i(x) \{F_i(z) - F_i(U_\ell(x))\} \quad \text{for all } z \in S.$$

For any bounded set $\Omega \subseteq S$, let $x^1, x^2 \in \Omega$. Adding the two inequalities obtained by substituting $(x, z) = (x^1, U_\ell(x^2))$ and $(x, z) = (x^2, U_\ell(x^1))$ into the above
Hence, the above two inequalities show
\[ \ell (U_\ell(x^1) - U_\ell(x^2) - (x^1 - x^2)) = (U_\ell(x^1) - U_\ell(x^2)) \]
\[ \leq \sum_{i=1}^{m} (\lambda_i(x^2) - \lambda_i(x^1)) \{ F_i(U_\ell(x^1)) - F_i(U_\ell(x^2)) \} \]
\[ = \sum_{i=1}^{m} \lambda_i(x^1) \{ F_i(x^1) - F_i(U_\ell(x^1)) \} + \sum_{i=1}^{m} \lambda_i(x^2) \{ F_i(x^2) - F_i(U_\ell(x^2)) \} \]
\[ + \sum_{i=1}^{m} \lambda_i(x^1) \{ F_i(U_\ell(x^2)) - F_i(x^1) \} + \sum_{i=1}^{m} \lambda_i(x^2) \{ F_i(U_\ell(x^1)) - F_i(x^2) \} \]
\[ \leq \sum_{i=1}^{m} \lambda_i(x^1) \{ F_i(x^1) - F_i(U_\ell(x^1)) \} + \sum_{i=1}^{m} \lambda_i(x^2) \{ F_i(x^2) - F_i(U_\ell(x^2)) \} \]
\[ + \sum_{i=1}^{m} \lambda_i(x^1) \{ F_i(U_\ell(x^2)) - F_i(x^1) \} + \sum_{i=1}^{m} \lambda_i(x^2) \{ F_i(U_\ell(x^1)) - F_i(x^2) \} \]
\[ = \sum_{i=1}^{m} (\lambda_i(x^2) - \lambda_i(x^1)) \{ F_i(x^1) - F_i(x^2) \} \leq \frac{1}{2} \max_{\ell \in I, i \in \mathcal{I}(x)} |F_i(x^1) - F_i(x^2)|, \]
where the second equality holds from the definition of \( \mathcal{I}(x) \) and since \( \lambda(x) \in \Delta^m \) and \( \lambda_j(x) \neq 0 \) for all \( j \in \mathcal{I}(x) \). Dividing by \( \ell \) and adding \( (1/4)\|x^1 - x^2\|^2 \) in both sides of the inequality, we have
\[ \|U_\ell(x^1) - U_\ell(x^2) - \frac{1}{2} (x^1 - x^2)\|^2 \leq \frac{1}{4} \|x^1 - x^2\|^2 + \frac{2}{\ell} \max_{\ell \in I, i \in \mathcal{I}(x)} |F_i(x^1) - F_i(x^2)|. \]
Taking the square root of both sides, we obtain
\[ \|U_\ell(x^1) - U_\ell(x^2) - \frac{1}{2} (x^1 - x^2)\| \leq \sqrt{\frac{1}{4} \|x^1 - x^2\|^2 + \frac{2}{\ell} \max_{\ell \in I, i \in \mathcal{I}(x)} |F_i(x^1) - F_i(x^2)|}. \]
Then, it follows from the triangle inequality that
\[ \|U_\ell(x^1) - U_\ell(x^2)\| \leq \frac{1}{2} \|x^1 - x^2\| + \sqrt{\frac{1}{4} \|x^1 - x^2\|^2 + \frac{2}{\ell} \max_{\ell \in I, i \in \mathcal{I}(x)} |F_i(x^1) - F_i(x^2)|}. \]
Since Theorem 2.4 implies that \( F_i \) is locally Lipschitz continuous on \( S \), there exists \( L_i(\Omega) > 0 \) such that
\[ |F_i(x^1) - F_i(x^2)| \leq L_i(\Omega) \|x^1 - x^2\| \quad (15) \]
Hence, the above two inequalities show \( U_\ell \)'s local Hölder continuity with exponent 1/2.
On the other hand, the definition \[13\] of \( u_\ell \) gives
\[
u_\ell(x^1) = \max_{y \in C} \min_{i=1, \ldots, m} \left[ F_i(x^1) - F_i(y) - \frac{\ell}{2} \| x^1 - y \|^2 \right]
\]
\[
\geq \min_{i=1, \ldots, m} \left[ F_i(x^1) - F_i(U_\ell(x^2)) - \frac{\ell}{2} \| x^1 - U_\ell(x^2) \|^2 \right].
\]
Reducing \( u_\ell(x^2) \) from both sides yields
\[
u_\ell(x^1) - u_\ell(x^2) \geq \min_{i=1, \ldots, m} \left[ F_i(x^1) - F_i(U_\ell(x^2)) - \frac{\ell}{2} \| x^1 - U_\ell(x^2) \|^2 \right] - u_\ell(x^2).
\]
\[13\] and \[14\] lead to
\[
u_\ell(x^1) - u_\ell(x^2) \geq \min_{i=1, \ldots, m} \left[ F_i(x^1) - F_i(U_\ell(x^2)) - \frac{\ell}{2} \| x^1 - U_\ell(x^2) \|^2 \right] - \min_{i=1, \ldots, m} \left[ F_i(x^1) - F_i(U_\ell(x^2)) - \frac{\ell}{2} \| x^2 - U_\ell(x^2) \|^2 \right].
\]
As it follows that that \( \min_{i=1, \ldots, m} v_i^1 - \min_{i=1, \ldots, m} v_i^2 \geq \min_{i=1, \ldots, m} (v_i^1 - v_i^2) \) for any \( v^1, v^2 \in \mathbb{R}^m \), we obtain
\[
u_\ell(x^1) - u_\ell(x^2) \geq \min_{i=1, \ldots, m} \left[ F_i(x^1) - F_i(x^2) - \frac{\ell}{2} \left( x^1 + x^2 - 2U_\ell(x^2) \right)^\top (x^1 - x^2) \right].
\]
Cauchy-Schwarz inequality and \[15\] implies
\[
u_\ell(x^1) - u_\ell(x^2) \geq - \left[ \max_{i=1, \ldots, m} L_i(\Omega) + \frac{\ell}{2} \| x^1 + x^2 - 2U_\ell(x^2) \| \right] \| x^1 - x^2 \|.
\]
Since \( U_\ell(x) \) is bounded for \( x \in \Omega \) due to the continuity, and the above inequality holds even if we interchange \( x^1 \) and \( x^2 \), we can show the local Lipschitz continuity of \( u_\ell \).

On the other hand, since \( \min_{i=1, \ldots, m} q_i = \min_{\lambda \in \Delta^m} \sum_{i=1}^m q_i \) for any \( q \in \mathbb{R}^m \) with the standard simplex \( \Delta^m \) defined by \[2\], \( u_\ell \) given by \[13\] can also be expressed as
\[
u_\ell(x) = \max_{y \in S} \min_{\lambda \in \Delta^m} \sum_{i=1}^m \lambda_i \left( F_i(x) - F_i(y) - \frac{\ell}{2} \| x - y \|^2 \right).
\]
We can see that \( S \) is convex, \( \Delta^m \) is compact and convex, and the function inside \( \min_{\lambda \in \Delta^m} \) is convex for \( \lambda \) and concave for \( y \). Therefore, Sion’s minimax theorem \[39\] leads to
\[
u_\ell(x) = \min_{\lambda \in \Delta^m} \max_{y \in S} \sum_{i=1}^m \lambda_i \left( F_i(x) - F_i(y) - \frac{\ell}{2} \| x - y \|^2 \right)
\]
\[
= \min_{\lambda \in \Delta^m} \left\{ \sum_{i=1}^m \lambda_i F_i(x) - \ell M \sum_{i=1}^m \lambda_i F_i + 1_{S_\ell}(x) \right\}.
\]
(16)
where $\mathcal{M}$ and $\iota$ denote the Moreau envelope and the indicator function defined by (5) and (7), respectively. Thus, for each $\ell > 0$, we can evaluate $u_\ell$ through the following $m$-dimensional, simplex-constrained, convex optimization problem:

$$
\begin{align*}
\min_{\lambda \in \mathbb{R}^m} & \sum_{i=1}^m \lambda_i F_i(x) - \ell \mathcal{M}_{\iota} \sum_{i=1}^m \lambda_i F_i(x) \\
\text{s.t.} & \quad \lambda \geq 0 \quad \text{and} \quad \sum_{i=1}^m \lambda_i = 1.
\end{align*}
$$

As the following theorem shows, the objective function of (17) is continuously differentiable.

**Theorem 3.5.** Let $x \in S$ be given. The objective function of (17) is continuously differentiable at every $\lambda \in \mathbb{R}^m$ and

$$
\nabla_{\lambda} \left[ \sum_{i=1}^m \lambda_i F_i(x) - \ell \mathcal{M}_{\iota} \sum_{i=1}^m \lambda_i F_i(x) \right] = F(x) - F \left( \text{prox}_{\iota} \sum_{i=1}^m \lambda_i F_i(x) \right),
$$

where $\text{prox}$ denotes the proximal operator (6).

**Proof.** Define

$$h(y, \lambda) := \sum_{i=1}^m \lambda_i F_i(y) + \frac{1}{2} \|x - y\|^2.$$

We now check that $\mathcal{M}_{\iota} \sum_{i=1}^m \lambda_i F_i(x) = \min_{y \in S} h(y, \lambda)$ satisfies the assumptions (i)–(iv) of Theorem 2.5. First, since $F$ is finite-valued and convex, $h$ is clearly continuous on $S \times \Delta^m$ (Assumption (i)). Moreover, $h(y, \cdot) := h(y, \cdot)$ is continuously differentiable and

$$\nabla_{\lambda} h(y, \lambda) = F(y)$$

for any $y \in S$ (Assumption (iii)). Furthermore, $\text{prox}_{\iota} \sum_{i=1}^m \lambda_i F_i(x) = \arg \min_{y \in S} h(y, \lambda)$ is also continuous at every $\lambda \in \mathbb{R}^m$ from [33, Exercise 7.38] (Assumptions (ii) and (iv) with $C = \{\text{prox}_{\iota} \sum_{i=1}^m \lambda_i F_i(x)\}$). Therefore, all assumptions of Theorem 2.5 are satisfied. Since $\text{prox}_{\iota} \sum_{i=1}^m \lambda_i F_i(x)$ is unique, we obtain the desired result.

Therefore, when $\text{prox}_{\iota} \sum_{i=1}^m \lambda_i F_i(x)$ is easy to compute, we can solve (17) using well-known convex optimization techniques such as the interior point method [5]. If $n \gg m$, this is usually faster than solving the $n$-dimensional problem directly to compute (13).

Let us now write the optimal solution set of (17) by

$$\Lambda(x) := \arg \min_{\lambda \in \Delta^m} \left\{ \sum_{i=1}^m \lambda_i F_i(x) - \ell \mathcal{M}_{\iota} \sum_{i=1}^m \lambda_i F_i(x) \right\}.
$$

Then, we can show the directional differentiability of $u_\ell$, as in the following theorem.
Theorem 3.6. Let $x \in S$. For each $\ell > 0$, the merit function $u_\ell$ defined by (13) has a directional derivative

$$u_\ell'(x; z - x) = \inf_{\lambda \in \Lambda(x)} \left\{ \sum_{i=1}^{m} \lambda_i F_i'(x; z - x) - \ell \left( x - \text{prox}_{\sum_{i=1}^{m} \lambda_i F_i + \iota S}(x) \right)^\top (z - x) \right\}$$

for all $z \in S$, where $\Lambda(x)$ is given by (18), and $\text{prox}$ denotes the proximal operator (6). In particular, if $\Lambda(x)$ is a singleton, i.e., $\Lambda(x) = \{\lambda(x)\}$, and $F_i$ is continuously differentiable at $x$, then $u_\ell$ is continuously differentiable at $x$, and we have

$$\nabla u_\ell(x) = \sum_{i=1}^{m} \lambda_i(x) \nabla F_i(x) - \ell \left( x - \text{prox}_{\sum_{i=1}^{m} \lambda_i(x) F_i + \iota S}(x) \right)^\top (z - x).$$

Proof. Let

$$h(x, \lambda) := \sum_{i=1}^{m} \lambda_i F_i(x) - \ell \text{M}^\top \sum_{i=1}^{m} \lambda_i F_i + \iota S(x).$$

Since $\text{M}^\top \sum_{i=1}^{m} \lambda_i F_i + \iota S(x)$ is continuous at every $(x, \lambda) \in S \times \Delta^m$ from Theorem 7.37, $h$ is also continuous on $S \times \Delta^m$. Moreover, Theorem 2.1 implies that for all $x, z \in S$ the function $h_\lambda(\cdot) := h(\cdot, \lambda)$ has a directional derivative:

$$h_\lambda'(x; z - x) = \sum_{i=1}^{m} \lambda_i F_i'(x; z - x) - \ell \left( x - \text{prox}_{\sum_{i=1}^{m} \lambda_i(x) F_i + \iota S}(x) \right)^\top (z - x).$$

As $\text{prox}_{\sum_{i=1}^{m} \lambda_i F_i + \iota S}(x)$ is continuous at every $(x, \lambda) \in S \times \Delta^m$ (cf. Exercise 7.38), $h_\lambda'(x; z - x)$ is also continuous at every $(x, z, \lambda) \in S \times S \times \Delta^m$. The discussion above and the compactness of $\Delta^m$ show that all assumptions of Theorem 2.5 are satisfied. So, we get the desired result.

From Theorems 3.3 and 3.6, the weakly Pareto optimal solutions for (1) are the global optimal solutions of the following directionally differentiable single-objective optimization problem:

$$\min_{x \in S} u_\ell(x).$$

(19)

Since $u_\ell$ is generally non-convex, (19) may have local optimal solutions or stationary points that are not globally optimal. As the following example shows, such stationary points are not necessarily Pareto stationary for (1).

Example 3.7. Let $m = 1, \ell = 1, S = \mathbb{R}$ and $F_1(x) = |x|$. Then, we have

$$\text{M}F_1(x) = \begin{cases} x^2/2, & \text{if } |x| < 1, \\ |x| - 1/2, & \text{otherwise.} \end{cases}$$
Hence, we can evaluate $u_1$ as follows:

$$u_1(x) = \begin{cases} |x| - x^2/2, & \text{if } |x| < 1, \\ 1/2, & \text{otherwise.} \end{cases}$$

It is stationary for (19) at $|x| \geq 1$ and $x = 0$ but minimal only at $x = 0$. Furthermore, the stationary point of $F_1$ is only $x = 0$.

However, if we assume the strict convexity of each $F_i$, then the stationary point of (19) is Pareto optimal for (1) and hence global optimal for (19). Note that this assumption does not assert the convexity of $u_\ell$.

**Theorem 3.8.** Suppose that $F_i$ is strictly convex for all $i \in \{1, \ldots, m\}$. If $x \in S$ is a stationary point of (19), i.e.,

$$u_\ell'(x; z - x) \geq 0 \quad \text{for all } z \in S,$$

then $x$ is Pareto optimal for (1).

**Proof.** Let $\lambda \in \Lambda(x)$, where $\Lambda(x)$ is given by (18). Then, Theorem 3.6 gives

$$\sum_{i=1}^{m} \lambda_i F_i'(x; z - x) - \ell \left( x - \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x) \right) \geq 0 \quad \text{for all } z \in S.$$

Substituting $z = \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x)$ into the above inequality, we get

$$\sum_{i=1}^{m} \lambda_i F_i'(x; \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x) - x) + \ell \left\| x - \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x) \right\|^2 \geq 0.$$

On the other hand, Theorem 2.3 yields

$$\left\| x - \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x) \right\|^2 \leq \frac{1}{\ell} \sum_{i=1}^{m} \lambda_i \left\{ F_i(x) - F_i(\text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x)) \right\}.$$

Combining the above two inequalities, we have

$$\sum_{i=1}^{m} \lambda_i F_i'(x; \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x) - x) \geq \sum_{i=1}^{m} \lambda_i \left\{ F_i(\text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x)) - F_i(x) \right\}. $$

Since $F_i$ is strictly convex for all $i \in \{1, \ldots, m\}$, the above inequality implies that $x = \text{prox}_{P} \sum_{i=1}^{m} \lambda_i F_i + \ell S(x)$, and hence $u_\ell(x) = 0$. This means that $x$ is Pareto optimal for (1) from the strict convexity of $F_i$, Theorem 2.6 (i), (iii) and Theorem 3.3.\qed
3.3 A regularized and partially linearized merit function for composite multiobjective optimization

Now, let us consider the composite case, i.e., each component $F_i$ of the objective function $F$ of (11) has the following structure:

$$F_i(x) := f_i(x) + g_i(x), \quad i = 1, \ldots, m,$$

(20)

where $f_i: S \rightarrow \mathbb{R}$ is continuously differentiable but not necessarily convex, and $g_i: S \rightarrow \mathbb{R}$ is convex but not necessarily differentiable. Note that since $g_i$ is finite and convex, there exists a directional derivative $g'_i(x; z-x)$ for any $x, z \in S$. Such composite objective functions have many applications, particularly in machine learning. Since they are generally non-convex, we can regard them as a relaxation of the assumptions of the previous subsection. For (11) with objective function (20), we propose a regularized and partially linearized merit function $w_\ell: S \rightarrow \mathbb{R}$ with a given $\ell > 0$ as follows:

$$w_\ell(x) := \max_{y \in S} \min_{i = 1, \ldots, m} \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2 \right\}. \quad (21)$$

Like $u_\ell$, the convexity of $g_i$ leads to the finiteness of $w_\ell$ and the existence of a unique solution that attains $\max_{y \in S}$. As the following remark shows, $w_\ell$ generalizes other kinds of merit functions.

**Remark 1.**
(i) When $g_i = 0$, $w_\ell$ corresponds to the regularized gap function [7] for vector variational inequality.
(ii) When $f_i = 0$, $w_\ell$ matches $u_\ell$ defined by (13).

As shown in the following theorem, $w_\ell$ is a merit function in the sense of Pareto stationarity.

**Theorem 3.9.** Let $w_\ell$ be given by (21) for some $\ell > 0$. Then, we have $w_\ell(x) \geq 0$ for all $x \in S$. Furthermore, $x \in S$ is Pareto stationary for (11) if and only if $w_\ell(x) = 0$.

**Proof.** We first show the nonnegativity of $w_\ell$. Let $x \in S$. The definition of $w_\ell$ gives

$$w_\ell(x) = \sup_{y \in S} \min_{i = 1, \ldots, m} \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2 \right\}$$

$$\geq \min_{i = 1, \ldots, m} \left\{ \nabla f_i(x)^\top (x - x) + g_i(x) - g_i(x) - \frac{\ell}{2} \|x - x\|^2 \right\} = 0.$$ 

Let us prove the second statement. Assume that $w_\ell(x) = 0$. Then, again using the definition of $w_\ell$, we get

$$\min_{i = 1, \ldots, m} \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2 \right\} \leq 0 \quad \text{for all } y \in S.$$
Let \( z \in S \) and \( \alpha \in (0, 1) \). Since \( S \subseteq \mathbb{R}^n \) is convex, \( x, z \in S \) implies \( x + \alpha(z - x) \in S \). Therefore, by substituting \( y = x + \alpha(z - x) \) into the above inequality, we obtain

\[
\min_{i=1,\ldots,m} \left\{ -\nabla f_i(x)^\top (\alpha(z - x)) + g_i(x) - g_i(x + \alpha(z - x)) - \frac{\ell}{2}\|\alpha(z - x)\|^2 \right\} \leq 0.
\]

Dividing both sides by \( \alpha \) yields

\[
\min_{i=1,\ldots,m} \left\{ -\nabla f_i(x)^\top (z - x) - \frac{g_i(x + \alpha(z - x)) - g_i(x)}{\alpha} - \frac{\ell\alpha}{2}\|z - x\|^2 \right\} \leq 0.
\]

By taking \( \alpha \searrow 0 \) and multiplying both sides by \(-1\), we get

\[
\max_{i=1,\ldots,m} F_i'(x; z - x) \geq 0,
\]

which means that \( x \) is Pareto stationary for (1).

Now, we prove the converse by indirect proof. Suppose that \( w_\ell(x) > 0 \). Then, from the definition of \( w_\ell \), there exists some \( y \in S \) such that

\[
\min_{i=1,\ldots,m} \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2}\|x - y\|^2 \right\} > 0.
\]

Since \( g_i \) is convex, we obtain

\[
\min_{i=1,\ldots,m} \left\{ \nabla f_i(x)^\top (x - y) - g_i'(x; y - x) - \frac{\ell}{2}\|x - y\|^2 \right\} > 0.
\]

Thus, we have

\[
\max_{i=1,\ldots,m} F_i'(x; y - x) \leq -\frac{\ell}{2}\|x - y\|^2 < 0,
\]

which shows that \( x \) is not Pareto stationary for (1).

While \( u_0 \) and \( u_\ell \) given by (12) and (13) are merit functions in the sense of weak Pareto optimality, \( w_\ell \) defined by (21) is a merit function only in the sense of Pareto stationarity. As indicated by the following example, even if \( w_\ell(x) = 0 \), \( x \) is not necessarily weakly Pareto optimal for (1).

**Example 3.10.** Consider the single-objective function \( F: \mathbb{R} \to \mathbb{R} \) defined by \( F(x) := f(x) + g(x) \), where

\[
f(x) := -x^2 \quad \text{and} \quad g(x) := 0,
\]

and set \( S = \mathbb{R} \). Then, we have

\[
w_\ell(0) = \max_{y \in \mathbb{R}} \left\{ f'(0)(0 - y) + g(0) - g(y) - \frac{\ell}{2}(y - 0)^2 \right\} = \max_{y \in \mathbb{R}} \left\{ -\frac{\ell}{2}y^2 \right\} = 0,
\]

but \( x = 0 \) is not global minimal (i.e., weakly Pareto optimal) for \( F \).
We now define the optimal solution mapping $W_\ell: S \to S$ associated with (21) by

$$W_\ell(x) := \arg\max_{y \in S} \min_{i=1,\ldots,m} \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2 \right\}.$$  

From the optimality condition of the maximization problem associated with (21) and (22) and [4, Proposition A.22], we obtain

$$\ell(x - W_\ell(x)) \in \text{conv} \left\{ \nabla f_i(x) + \partial g_i(W_\ell(x)) \right\} + N_S(W_\ell(x)) \quad \text{for all } x \in S,$$

where $N_S$ is the normal cone (3) to the convex set $S$ and

$$\mathcal{I}(x) := \arg\min_{i=1,\ldots,m} \left[ \nabla f_i(x)^\top (x - W_\ell(x)) + g_i(x) - g_i(W_\ell(x)) \right].$$

Therefore, from (3) and (4), for any $x \in S$ there exists $\lambda(x)$ belonging to the unit $m$-simplex $\Delta^m$ defined by (2) such that $\lambda_j(x) = 0$ for all $j \notin \mathcal{I}(x)$ and

$$\ell(x - W_\ell(x)) \leq \sum_{i=1}^m \lambda_i(x) \left[ \nabla f_i(x)^\top (z - W_\ell(x)) + g_i(z) - g_i(W_\ell(x)) \right]$$

for all $z \in S$. Particularly, if we substitute $z = x$, we get

$$\ell\|x - W_\ell(x)\|^2 \leq w_\alpha(x) + \frac{\ell}{2} \|x - W_\ell(x)\|^2,$$

which reduces to

$$w_\ell(x) \geq \frac{\ell}{2} \|x - W_\ell(x)\|^2. \quad (24)$$

We can also show the continuity of $w_\ell$ and $W_\ell$.

**Theorem 3.11.** For all $\ell > 0$, $w_\ell$ and $W_\ell$ defined by (21) and (22) are continuous on $S$. Moreover, if every $\nabla f_i$ is locally Lipschitz continuous for $i = 1, \ldots, m$, $w_\ell$ and $W_\ell$ are locally Lipschitz continuous and locally Hölder continuous with exponent $1/2$, respectively, on $S$.

**Proof.** Let $\Omega$ be a bounded subset of $S$ and let $x^1, x^2 \in \Omega$. Adding the two inequalities gotten by substituting $(x, z) = (x^1, W_\ell(x^2))$ and $(x, z) = (x^2, W_\ell(x^1))$
into (23), we obtain
\[
\frac{1}{\ell} \left( W_\ell(x^1) - W_\ell(x^2) - (x^1 - x^2) \right)^\top \left( W_\ell(x^1) - W_\ell(x^2) \right)
\]
\[
\leq \sum_{i=1}^{m} \lambda_i(x^1) \left[ \nabla f_i(x^1)^\top (x^1 - W_\ell(x^1)) + g_i(x^1) - g_i(W_\ell(x^1)) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^2) \left[ \nabla f_i(x^2)^\top (x^2 - W_\ell(x^2)) + g_i(x^2) - g_i(W_\ell(x^2)) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^1) \left[ \nabla f_i(x^1)^\top (W_\ell(x^2) - x^1) + g_i(W_\ell(x^2)) - g_i(x^1) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^2) \left[ \nabla f_i(x^2)^\top (W_\ell(x^1) - x^2) + g_i(W_\ell(x^1)) - g_i(x^2) \right].
\]

Since \( \lambda_j(x) \neq 0 \) for \( j \in \mathcal{I}(x) \), we have
\[
\frac{1}{\ell} \left( W_\ell(x^1) - W_\ell(x^2) - (x^1 - x^2) \right)^\top \left( W_\ell(x^1) - W_\ell(x^2) \right)
\]
\[
\leq \min_{i=1, \ldots, m} \left[ \nabla f_i(x^1)^\top (x^1 - W_\ell(x^1)) + g_i(x^1) - g_i(W_\ell(x^1)) \right]
\]
\[
+ \min_{i=1, \ldots, m} \left[ \nabla f_i(x^2)^\top (x^2 - W_\ell(x^2)) + g_i(x^2) - g_i(W_\ell(x^2)) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^1) \left[ \nabla f_i(x^1)^\top (W_\ell(x^2) - x^1) + g_i(W_\ell(x^2)) - g_i(x^1) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^2) \left[ \nabla f_i(x^2)^\top (W_\ell(x^1) - x^2) + g_i(W_\ell(x^1)) - g_i(x^2) \right]
\]
\[
\leq \sum_{i=1}^{m} \lambda_i(x^2) \left[ \nabla f_i(x^1)^\top (x^1 - W_\ell(x^1)) + g_i(x^1) - g_i(W_\ell(x^1)) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^1) \left[ \nabla f_i(x^2)^\top (x^2 - W_\ell(x^2)) + g_i(x^2) - g_i(W_\ell(x^2)) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^1) \left[ \nabla f_i(x^1)^\top (W_\ell(x^2) - x^1) + g_i(W_\ell(x^2)) - g_i(x^1) \right]
\]
\[
+ \sum_{i=1}^{m} \lambda_i(x^2) \left[ \nabla f_i(x^2)^\top (W_\ell(x^1) - x^2) + g_i(W_\ell(x^1)) - g_i(x^2) \right].
\]
Therefore, simple calculations give
\[
\frac{1}{\ell} \| W_\ell(x^1) - W_\ell(x^2) \|^2 \leq \frac{1}{\ell} (W_\ell(x^1) - W_\ell(x^2))^\top (x^1 - x^2) \\
+ \sum_{i=1}^m [\lambda(x^2) - \lambda(x^1)] [g_i(x^1) - g_i(x^2) + \nabla f_i(x^1)^\top (x^1 - x^2) \\
- (\nabla f_i(x^1) - \nabla f_i(x^2))^\top x^2] \\
+ \sum_{i=1}^m \lambda_i(x^1)(\nabla f_i(x^1) - \nabla f_i(x^2))^\top W_\ell(x^2) + \sum_{i=1}^m \lambda_i(x^2)(\nabla f_i(x^2) - \nabla f_i(x^1))^\top W_\ell(x^1).
\] (25)

When \( x^1 \to x^2 \), the right-hand side tends to zero, which means the continuity of \( W_\ell \) on \( C \). Therefore, from the definition, we can also say that \( w_\ell \) is continuous on \( C \) immediately.

Assume that each \( \nabla f_i, i = 1, \ldots, m \) is locally Lipschitz continuous. Since \( g_i \) is also locally Lipschitz continuous from Theorem 2.4, we can prove the local Hölder continuity of \( W_\ell \) from (25). On the other hand, the definitions (21) and (22) of \( w_\ell \) and \( W_\ell \) give
\[
w_\ell(x^1) - w_\ell(x^2) \\
= \min_{i=1,\ldots,m} \left[ \nabla f_i(x^1)^\top (x^1 - W_\ell(x^1)) + g_i(x^1) - g_i(W_\ell(x^1)) \right] - \frac{\ell}{2} \| x^1 - W_\ell(x^1) \|^2 \\
- \max_{y \in S} \min_{i=1,\ldots,m} \left[ \nabla f_i(x^2)^\top (x^2 - y) + g_i(x^2) - g_i(y) - \frac{\ell}{2} \| x^2 - y \|^2 \right] \\
\leq \min_{i=1,\ldots,m} \left[ \nabla f_i(x^1)^\top (x^1 - W_\ell(x^1)) + g_i(x^1) - g_i(W_\ell(x^1)) \right] - \frac{\ell}{2} \| x^1 - W_\ell(x^1) \|^2 \\
- \min_{i=1,\ldots,m} \left[ \nabla f_i(x^2)^\top (x^2 - W_\ell(x^1)) + g_i(x^2) - g_i(W_\ell(x^1)) \right] + \frac{\ell}{2} \| x^2 - W_\ell(x^1) \|^2 \\
\leq \max_{i=1,\ldots,m} \left[ (\nabla f_i(x^1) - \nabla f_i(x^2))^\top (x^1 - W_\ell(x^1)) + \nabla f_i(x^2)^\top (x^1 - x^2) + g_i(x^1) - g_i(x^2) \right] \\
- \frac{\ell}{2} (x^1 - x^2)^\top (x^1 + x^2 - 2W_\ell(x^1)) \\
\leq \| x^1 - W_\ell(x^1) \| \max_{i=1,\ldots,m} \| \nabla f_i(x^1) - \nabla f_i(x^2) \| \\
+ \max_{i=1,\ldots,m} \| \nabla f_i(x^2) \| \| x^1 - x^2 \| + \max_{i=1,\ldots,m} | g_i(x^1) - g_i(x^2) | \\
+ \frac{\ell}{2} \| x^1 + x^2 - 2W_\ell(x^1) \| \| x^1 - x^2 \|,
\]
where the first inequality comes from the inequality \( \min_{i=1,\ldots,m} v_i^1 - \min_{i=1,\ldots,m} v_i^2 \geq \min_{i=1,\ldots,m} (v_i^1 - v_i^2) \) for any \( v^1, v^2 \in \mathbb{R}^m \), and the third inequality follows from the Cauchy-Schwarz inequality. The above inequality holds even if we interchange \( x^1 \) and \( x^2 \). Furthermore, \( W_\ell(x) \) and \( \nabla f_i(x) \) are bounded for any \( x \in \Omega \) due to their continuity. Therefore, local Lipschitz continuity of \( \nabla f_i \) and \( g_i \) implies the local Lipschitz continuity of \( w_\ell \). □
On the other hand, in the same way as the derivation of (16), Sion’s minimax theorem [39] gives another representation of \( w_\ell \) for \( \ell > 0 \) as follows:

\[
 w_\ell(x) = \min_{\gamma \in \Delta^m} \max_{y \in S} \sum_{i=1}^{m} \gamma_i \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \| x - y \|^2 \right\}, 
\]

where \( \Delta^m \) denotes the standard simplex [2]. Moreover, simple calculations show that

\[
 w_\ell(x) = \min_{\gamma \in \Delta^m} \left[ \sum_{i=1}^{m} \gamma_i g_i(x) + \frac{1}{2\ell} \left\| \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right\|^2 - \min_{y \in S} \left\{ \sum_{i=1}^{m} \gamma_i g_i(y) + \frac{\ell}{2} \left\| x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) - y \right\|^2 \right\} \right] 
\]

\[
 = \min_{\gamma \in \Delta^m} \left[ \sum_{i=1}^{m} \gamma_i g_i(x) + \frac{1}{2\ell} \left\| \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right\|^2 - \ell M + \sum_{i=1}^{m} \gamma_i g_i + \iota S \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right], 
\]

where \( M \) and \( \iota \) is given by (5) and (7), respectively. In other words, we can compute \( w_\ell \) via the following \( m \)-dimensional, simplex-constrained, and convex optimization problem:

\[
 \min_{\gamma \in \mathbb{R}^m} \sum_{i=1}^{m} \gamma_i g_i(x) + \frac{1}{2\ell} \left\| \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right\|^2 - \ell M + \sum_{i=1}^{m} \gamma_i g_i + \iota S \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) 
\]

s.t. \( \gamma \geq 0 \) and \( \sum_{i=1}^{m} \gamma_i = 1 \).

Moreover, the following theorem proves that the objective function of (28) is continuously differentiable.

**Theorem 3.12.** Let \( x \in S \) be given. The objective function of (28) is continu-
ously differentiable at every $\gamma \in \mathbb{R}^m$ and

$$\nabla_{\gamma} \left[ \sum_{i=1}^{m} \gamma_i g_i(x) + \frac{1}{2\ell} \left\| \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right\|^2 - \ell \mathcal{M}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right]$$

$$= g(x) - g \left( \text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right)$$

$$- J_f(x) \left( \text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) - x \right),$$

where $\text{prox}$ is the proximal operator \cite{combettes2011}, and $J_f(x)$ is the Jacobian matrix at $x$ given by

$$J_f(x) := (\nabla f_1(x), \ldots, \nabla f_m(x))^\top.$$

**Proof.** Let

$$\theta(y, \lambda) := \sum_{i=1}^{m} \lambda_i g_i(y) + \frac{\ell}{2} \left\| x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) - y \right\|^2.$$

Then, $\theta$ is continuous, $\theta_y(\cdot) := \theta(y, \cdot)$ is continuously differentiable, and

$$\nabla_{\gamma} \theta_y(\gamma) = g(y) + J_f(x) \left( y - x + \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right).$$

Moreover, $\text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} (x) = \arg\min_{y \in S} \theta(y, \lambda)$ is also continuous at every $\gamma \in \mathbb{R}^m$ (cf. \cite[Exercise 7.38]{combettes2011}). The above discussion implies that every assumption in Theorem 2.5 is satisfied, as well as the proof of Theorem 3.5. Combined with the uniqueness of $\text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} (x)$, we get

$$\nabla_{\gamma} \left[ \ell \mathcal{M}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right]$$

$$= g \left( \text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right)$$

$$+ J_f(x) \left( \text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + \iota \gamma} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) - x + \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right).$$

On the other hand, we have

$$\nabla_{\gamma} \left[ \sum_{i=1}^{m} \gamma_i g_i(x) + \frac{1}{2\ell} \left\| \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right\|^2 \right] = g(x) + \frac{1}{\ell} J_f(x) \sum_{i=1}^{m} \gamma_i \nabla f_i(x).$$

Adding the above two equalities, we obtain the desired result. \qed
Thus, like \( \text{[17]} \), \( \text{[28]} \) is solvable with convex optimization techniques such as the interior point method \( \text{[2]} \) when we can quickly evaluate \( \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \lambda_i g_i + \epsilon S} (\cdot) \). When \( n \gg m \), this usually gives a faster way to compute \( w_\ell \). Note, for example, that if \( g_i(x) = 0 \) for all \( i = 1, \ldots, m \), then \( \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \lambda_i g_i + \epsilon S} \) reduces to the projection onto \( S \) from \( \text{[8]} \). Moreover, for example, if \( g_i(x) = g_i(x) \) for any \( i = 1, \ldots, m \), or if \( g_i(x) = g_i(x)_{\ell_i} \) and the index sets \( I_i \subseteq \{1, \ldots, n\} \) do not overlap each other, then \( \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \lambda_i g_i} \) is computable with each \( \text{prox}_g \), when \( S = \mathbb{R}^n \).

Now, define the optimal solution set of \( \text{(28)} \) by

\[
\Gamma(x) = \arg\min_{\gamma \in \Delta^m} \left\{ \sum_{i=1}^m \gamma_i g_i(x) + \frac{1}{2\ell} \left\| \sum_{i=1}^m \gamma_i \nabla f_i(x) \right\|^2 - \ell \mathcal{M}_{\frac{1}{\ell} \sum_{i=1}^m \gamma_i g_i + \epsilon S} \left( x - \frac{1}{\ell} \sum_{i=1}^m \gamma_i \nabla f_i(x) \right) \right\}.
\]

Then, in the same manner as Theorem \( \text{[3.0]} \) we obtain the following theorem.

**Theorem 3.13.** Let \( x \in S \). Assume that \( f_i \) is twice continuously differentiable at \( x \). Then, for all \( \ell > 0 \), the merit function \( w_\ell \) defined by \( \text{[21]} \) has a directional derivative

\[
w_\ell'(x; z - x) = \inf_{\gamma \in \Gamma(x)} \left\{ \sum_{i=1}^m \gamma_i g_i'(x; z - x) - \ell \left[ I - \frac{1}{\ell} \sum_{i=1}^m \gamma_i \nabla^2 f_i(x) \right] \left[ x - \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \gamma_i g_i + \epsilon S} \left( x - \frac{1}{\ell} \sum_{i=1}^m \gamma_i \nabla f_i(x) \right) \right] - \frac{1}{\ell} \sum_{i=1}^m \gamma_i \nabla f_i(x) \right\}^T (z - x) \]

for all \( z \in S \), where \( \text{prox} \) and \( \Gamma \) is given by \( \text{[6]} \) and \( \text{[29]} \), respectively, and \( I \) is the \( n \)-dimensional identity matrix. In particular, if \( \Gamma(x) \) is a singleton, i.e., \( \Gamma(x) = \{\gamma(x)\} \), and \( g_i \) is continuously differentiable at \( x \), then \( w_\ell \) is continuously differentiable at \( x \), and we have

\[
\nabla w_\ell(x) = \sum_{i=1}^m \gamma_i(x) \nabla F_i(x) - \ell \left[ I - \frac{1}{\ell} \sum_{i=1}^m \gamma_i(x) \nabla^2 f_i(x) \right] \left[ x - \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \gamma_i(x) g_i + \epsilon S} \left( x - \frac{1}{\ell} \sum_{i=1}^m \gamma_i(x) \nabla f_i(x) \right) \right].
\]

If the convex part \( g_i \) is the same regardless of \( i \), we get the following corollary without assuming the differentiability of \( g_i \).

**Corollary 3.14.** Let \( x \in S \) and \( \ell > 0 \). Assume that \( f_i \) is twice continuously differentiable at \( x \) and \( g_i = g_i^* \) for all \( i = 1, \ldots, m \), and recall that \( w_\ell \) and \( \text{prox} \)
be defined by \( (6) \) and \( (21) \), respectively. If \( \Gamma(x) \) given by \( (29) \) is a singleton, i.e., \( \Gamma(x) = \{ \gamma(x) \} \), then the function \( w_\ell - g_1 \) is continuously differentiable at \( x \), and we have

\[
\nabla_x (w_\ell(x) - g_1(x)) = -\ell \left[ I - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i(x) \nabla^2 f_i(x) \right] \left[ x - \text{prox}_{\ell g_1 + z} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i(x) \nabla f_i(x) \right) \right] + \sum_{i=1}^{m} \gamma_i(x) \nabla f_i(x).
\]

Theorem 3.14 implies that, under certain conditions, the merit function \( w_\ell = (w_\ell - g_1) + g_1 \) is composite, i.e., the sum of a continuously differentiable function and a convex one.

Theorems 3.9 and 3.13 show that the Pareto stationary points for \( (1) \) are global optimal for the following directionally differentiable single-objective optimization problem:

\[
\min_{x \in S} w_\ell(x).
\]

Moreover, when the assumptions of Theorem 3.14 hold, we can apply first-order methods such as the proximal gradient method \[17\] to \( (30) \). On the other hand, if we consider Theorem 3.7 with \( f_i = 0 \), we can see that the stationary point for \( (30) \) is not necessarily Pareto stationary for \( (1) \). However, if \( f_i \) is convex and twice continuously differentiable, and \( F_i \) is strictly convex, then we can prove that every stationary point of \( (30) \) is Pareto optimal for \( (1) \), i.e., global optimal for \( (19) \).

Note that this assumption does not assert the convexity of \( w_\ell \).

**Theorem 3.15.** Let \( x \in S \) and \( \ell > 0 \). Suppose that \( f_i \) is convex and twice continuously differentiable at \( x \), and \( F_i \) is strictly convex for any \( i = 1, \ldots, m \). If \( x \) is stationary for \( (30) \), i.e.,

\[
w_\ell'(x; z - x) \geq 0 \quad \text{for all } z \in S,
\]

then \( x \) is Pareto optimal for \( (1) \).

**Proof.** Let \( z \in S \) and \( \gamma \in \Gamma(x) \), where \( \Gamma(x) \) is defined by \( (29) \). Then, it follows from Theorem 3.13 that

\[
\sum_{i=1}^{m} \gamma_i g_i'(x; z - x) - \ell \left( I - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla^2 f_i(x) \right) \left[ x - \text{prox}_{\sum_{i=1}^{m} \gamma_i g_i + z} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right] - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) (z - x) \geq 0.
\]
Substituting \( z = \text{prox}_{\frac{1}{L} \sum_{i=1}^{m} \lambda_i G_i} (x) \), we have

\[
\sum_{i=1}^{m} \gamma_i F_i' \left( \text{prox}_{\frac{1}{L} \sum_{i=1}^{m} \lambda_i G_i} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) - x \right) + \ell \left| \text{prox}_{\frac{1}{L} \sum_{i=1}^{m} \lambda_i G_i} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right|^2 \geq 0.
\]

Since the convexity of \( f_i \) implies that \( \nabla^2 f_i(x) \) is positive semidefinite, we get

\[
\sum_{i=1}^{m} \gamma_i F_i' \left( \text{prox}_{\frac{1}{L} \sum_{i=1}^{m} \lambda_i G_i} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) - x \right) + \ell \left| \text{prox}_{\frac{1}{L} \sum_{i=1}^{m} \lambda_i G_i} \left( x - \frac{1}{\ell} \sum_{i=1}^{m} \gamma_i \nabla f_i(x) \right) \right|^2 \geq 0.
\]

Therefore, with similar arguments used in the proof of Theorem 3.8, we obtain \( x = \text{prox}_{\frac{1}{L} \sum_{i=1}^{m} \lambda_i G_i} (x - (1/\ell) \sum_{i=1}^{m} \lambda_i \nabla f_i(x)) \), and thus \( w_\ell(x) = 0 \).

Since \( F_i \) is strictly convex, \( x \) is Pareto optimal for (1) from Theorem 2.6 (iii) and Theorem 3.9.

\[ \Box \]

### 4 Relation between different merit functions

This section assumes that the problem has a composite structure \[20\] and discusses the connection between the merit functions proposed in Sections \[3.1, 3.2]\.

First, we show some inequalities between different types of merit functions.

**Theorem 4.1.** Let \( u_0, u_\ell, \) and \( w_\ell \) be defined by \[12\], \[13\] and \[21\], respectively, for all \( \ell > 0 \). Then, the following statements hold.

(i) If \( f_i \) is \( \mu_i \)-convex for some \( \mu_i \in \mathbb{R} \) and \( \mu = \min_{i=1,...,m} \mu_i \), then we have

\[
\begin{cases}
  u_0(x) \leq w_\mu(x) & \text{and} & u_\ell(x) \leq w_{\mu+\ell}(x), & \text{if } \mu \geq 0, \\
  u_{\mu+\ell}(x) \leq w_\ell(x), & \text{otherwise}.
\end{cases}
\]

for all \( \ell > 0 \) and \( x \in S \).

(ii) If \( \nabla f_i \) is \( L_i \)-Lipschitz continuous for some \( L_i > 0 \) and \( L = \max_{i=1,...,m} L_i \), then we get

\[
 u_{L+\ell}(x) \leq w_\ell(x), \quad u_0(x) \geq w_L(x), \quad \text{and} \quad u_\ell(x) \geq w_{L+\ell}(x)
\]

for all \( \ell > 0 \) and \( x \in S \).
Proof. (i) Let $i \in \{1, \ldots, m\}$. The $\mu_i$-convexity of $f_i$ gives
\[ f_i(x) - f_i(y) \leq \nabla f_i(x)^\top (x - y) - \frac{\mu_i}{2} \|x - y\|^2. \]
By the definition of $\mu$, we get
\[ f_i(x) - f_i(y) \leq \nabla f_i(x)^\top (x - y) - \frac{\mu}{2} \|x - y\|^2. \]
Thus, recalling (20), we have
\[
\begin{align*}
F_i(x) - F_i(y) & \leq \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\mu}{2} \|x - y\|^2, \\
F_i(x) - F_i(y) - \frac{\ell}{2} \|x - y\|^2 & \leq \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\mu + \ell}{2} \|x - y\|^2, \\
F_i(x) - F_i(y) - \frac{-\mu + \ell}{2} \|x - y\|^2 & \leq \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2,
\end{align*}
\]
so the desired inequalities are clear from (12), (13) and (21).

(ii) Let $i \in \{1, \ldots, m\}$. Suppose that $\nabla f_i$ is $L_i$-Lipschitz continuous. Then, the descent lemma [3, Proposition A.24] yields
\[ |f_i(y) - f_i(x) - \nabla f_i(x)^\top (y - x)| \leq L_i \frac{\ell}{2} \|x - y\|^2. \]
By the definition of $L$, we have
\[ |f_i(y) - f_i(x) - \nabla f_i(x)^\top (y - x)| \leq \frac{L}{2} \|x - y\|^2. \]
This gives
\[
\begin{align*}
F_i(x) - F_i(y) - \frac{L + \ell}{2} \|x - y\|^2 & \leq \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2, \\
F_i(x) - F_i(y) & \geq \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{L}{2} \|x - y\|^2, \\
F_i(x) - F_i(y) - \frac{\ell}{2} \|x - y\|^2 & \geq \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{L + \ell}{2} \|x - y\|^2.
\end{align*}
\]
Therefore, we immediately get $u_{L+\ell}(x) \leq w_\ell(x)$, $u_0(x) \geq w_L(x)$, and $u_\ell(x) \geq w_{L+\ell}(x)$ for all $x \in S$ by (12), (13) and (21).

Second, we present the relation between coefficients and the proposed merit functions’ values.

Theorem 4.2. Recall that $w_\ell$ is defined by (21) for all $\ell > 0$. Let $r$ be an arbitrary scalar such that $r \geq \ell$. Then, we get
\[ w_r(x) \leq w_\ell(x) \leq \frac{r}{\ell} w_r(x) \quad \text{for all } x \in S. \]
Proof. Let \( x \in S \). Since \( r \geq \ell > 0 \), the definition (21) of \( w_r \) and \( w_\ell \) clearly gives the first inequality. Thus, we prove the second one. From the definition (21) of \( w_\ell \), we have

\[
\begin{align*}
    w_\ell(x) &= \sup_{y \in S} \min_{i=1, \ldots, m} \left\{ \nabla f_i(x)^\top (x - y) + g_i(x) - g_i(y) - \frac{\ell}{2} \|x - y\|^2 \right\} \\
    &= \frac{r}{\ell} \sup_{y \in S} \min_{i=1, \ldots, m} \left\{ \nabla f_i(x)^\top \left( \frac{\ell}{r} (x - y) \right) + \frac{\ell}{r} (g_i(x) - g_i(y)) - \frac{r}{2} \left\| \frac{\ell}{r} (x - y) \right\|^2 \right\} \\
    &\leq \frac{r}{\ell} \sup_{y \in S} \min_{i=1, \ldots, m} \left\{ \nabla f_i(x)^\top \left( \frac{\ell}{r} (x - y) \right) + g_i(x) - g_i \left( x - \frac{\ell}{r} (x - y) \right) \right. \\
    &\quad \left. - \frac{r}{2} \left\| \frac{\ell}{r} (x - y) \right\|^2 \right\}
\end{align*}
\]

where the first inequality follows from the convexity of \( g_i \). Since \( S \) is convex, \( x, y \in S \) implies \( x - (\ell/r)(x - y) \in S \). Therefore, from the definition (21) of \( w_r \), we get

\[
w_\ell(x) \leq \frac{r}{\ell} w_r(x).
\]

Considering Remark 1 (ii) we get the following corollary.

**Corollary 4.3.** Assume that each component \( F_i \) of the objective function \( F \) of (1) is convex. Recall that \( u_\ell \) is defined by (13) for all \( \ell > 0 \). Let \( r \) be an arbitrary scalar such that \( r \geq \ell \). Then, we get

\[
u_r(x) \leq u_\ell(x) \leq \frac{r}{\ell} u_r(x) \quad \text{for all } x \in S.
\]

Remark 2. For unconstrained problems, we can consider the following inequality.

\[
w_L(x) \geq \tau u_0(x) \quad \text{for all } x \in \mathbb{R}^n \text{ for some } \tau > 0,
\]

which is an extension of the proximal-PL inequality for scalar optimization [27]. Under this condition, we can prove that proximal gradient methods for multiobjective optimization [41] have linear convergence rate [42]. Note that this inequality holds particularly if each \( f_i \) is strongly convex from Theorem 4.1 (i) and Theorem 4.2.

5 Level-boundedness of the proposed merit functions

Recall that we call a function *level-bounded* if every level set is bounded. This is an important property because it ensures that the sequences generated by descent methods have accumulation points. We state below sufficient conditions for the level-boundedness of the merit functions proposed in Section 4.
Theorem 5.1. Consider $u_0$, $u_\ell$ and $w_\ell$ defined in (12), (13) and (21), respectively, for all $\ell > 0$. Then, the following statements hold.

(i) If $F_i$ is level-bounded for all $i = 1, \ldots, m$, then $u_0$ is level-bounded.

(ii) If $F_i$ is convex and level-bounded for all $i = 1, \ldots, m$, then $u_\ell$ is level-bounded for all $\ell > 0$.

(iii) Suppose that $F$ has the composite structure (20). If $f_i$ is $\mu_i$-convex for some $\mu_i \in \mathbb{R}$ or $\nabla f_i$ is $L_i$-Lipschitz continuous for some $L_i > 0$, and $F_i$ is convex and level-bounded for all $i = 1, \ldots, m$, then $w_\ell$ is level-bounded for all $\ell > 0$.

Proof. (i) Suppose, contrary to our claim, that $u_0$ is not level-bounded. Then, there exists $\alpha \in \mathbb{R}$ such that 
\[
\{ x \in S \mid u_0(x) \leq \alpha \}
\] is unbounded. By the definition (13) of $u_0$, the inequality $u_0(x) \leq \alpha$ can be written as
\[
\sup_{y \in S} \min_{i=1,\ldots,m} \{ F_i(x) - F_i(y) \} \leq \alpha.
\] This implies that for some fixed $z \in S$, there exists $j \in \{1, \ldots, m\}$ such that
\[
F_j(x) \leq F_j(z) + \alpha.
\] Therefore, it follows that
\[
\{ x \in S \mid u_0(x) \leq \alpha \} \subseteq \bigcup_{j=1}^m \{ x \in S \mid F_j(x) \leq F_j(z) + \alpha \}.
\] Since $F_i$ is level-bounded for all $i = 1, \ldots, m$, the right-hand side must be bounded, which contradicts the unboundedness of the left-hand side.

(ii) Recall the definitions (2), (5) and (6) of $\Delta^m$, $M$, and $\text{prox}$. Eq. (16) gives
\[
u_\ell(x) = \min_{\lambda \in \Delta^m} \left\{ \sum_{i=1}^m \lambda_i F_i(x) - \ell M \frac{1}{\sum_{i=1}^m \lambda_i F_i + \iota}(x) \right\}
\]
\[
= \min_{\lambda \in \Delta^m} \sum_{i=1}^m \lambda_i \left\{ F_i(x) - F_i \left( \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \lambda_i F_i + \iota}(x) \right) \right\}
\]
\[
\geq \frac{1}{2} \min_{i=1}^m \sum_{i=1}^m \lambda_i \left\{ F_i(x) - F_i \left( \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \lambda_i F_i + \iota}(x) \right) \right\}
\]
\[
= \frac{1}{2} \min_{i=1,\ldots,m} \left\{ F_i(x) - F_i \left( \text{prox}_{\frac{1}{\ell} \sum_{i=1}^m \lambda_i F_i + \iota}(x) \right) \right\},
\]
where the inequality follows from Theorem 2.3. Therefore, with similar arguments given in the proof of statement (i), we can show the level-boundedness of $u_\ell$ by contradiction.
From Theorems 4.1 and 4.2 there exist some \( a > 0 \) and \( r > 0 \) such that \( u_r(x) \leq aw_L(x) \) for all \( x \in S \). Since statement (ii) implies that \( u_r \) is level-bounded, \( w_L \) is also level-bounded.

As indicated by the following example, our proposed merit functions are not necessarily level-bounded even if \( F \) is level-bounded.

**Example 5.2.** Consider the bi-objective function \( F: \mathbb{R} \to \mathbb{R}^2 \) with each component given by

\[
F_1(x) := x^2, \quad F_2(x) := 0.
\]

Then, the merit function \( u_0 \) defined by (13) is written as

\[
u_0(x) = \sup_{y \in \mathbb{R}} \min \{F_1(x) - F_1(y), F_2(x) - F_2(y)\} = \sup_{y \in \mathbb{R}} \min \{(x^2 - y^2), 0\} = 0.
\]

On the other hand, \( F \) is level-bounded because \( \lim_{\|x\| \to \infty} F_1(x) = \infty \).

### 6 The multiobjective proximal-PL inequality and error bounds

In this section, we extend the proximal-PL inequality introduced in [27] and shows that it induces the proposed merit function’s error bound. This section assumes that (1) has the composite structure (20); note that if \( f = 0 \) or \( g = 0 \), the assumption also holds for Sections 3.1 and 3.2.

We first define the multiobjective proximal-PL inequality.

**Definition 6.1.** Assume that \( f_i \) is \( L_i \)-Lipschitz continuous with \( L_i > 0 \) for all \( i = 1, \ldots, m \) and let \( L := \max_{i=1, \ldots, m} L_i \). We say that (1) satisfies the multiobjective proximal-PL inequality if there exists \( \tau > 0 \) such that

\[
w_L(x) \geq \tau u_0(x) \quad \text{for all } x \in S
\]

with \( u_0 \) and \( w_L \) given by (12) and (21).

If \( m = 1 \), (31) reduces to the proximal-PL inequality [27].

We state below some sufficient conditions for (31).

**Proposition 6.2.**

(i) When \( f_i \) is \( \mu_i \)-convex with \( \mu_i > 0 \), (31) holds with \( \tau := \min(\mu/L, 1) \), where \( \mu := \min_{i=1, \ldots, m} \mu_i \).

(ii) Assume that \( f_i(x) := h(A_i x) \) with some strongly convex function \( h_i \) and linear transformation \( A_i \), \( g_i := 0 \), and \( S = \mathcal{X} \) is a polyhedral set. If each \( \min_{x \in S} F_i(x) \) has a nonempty set \( X_i^* \) for \( i = 1, \ldots, m \), then (31) holds with some constant \( \tau \).
Proof. (i) Since $f_i$ is strongly convex, Theorem 4.1(i) gives
\[ u_0(x) \leq w_\mu(x) \text{ for all } x \in S. \] (32)

Applying Theorem 4.2 to the above inequality implies
\[ u_0(x) \leq \max \left( \frac{L}{\mu}, 1 \right) w_L(x) \text{ for all } x \in S, \] (33)
which means
\[ w_L(x) \geq \min \left( \frac{\mu}{L}, 1 \right) u_0(x) \text{ for all } x \in S. \] (34)

(ii) Since $S$ is polyhedral, we can write it as $\{ x \in \mathbb{R}^n \mid Bx \leq c \}$ for some matrix $B$ and vector $c$. We now show that for all $i = 1, \ldots, m$ there exists some $z_i$ such that
\[ X_i^* = \{ x \in \mathbb{R}^n \mid Bx \leq c \text{ and } A_i x = z_i \}. \]

To obtain a contradiction, suppose that there exists $x^1 \in X_i^*$ and $x^2 \in X_i^*$ such that $A_i x^1 \neq A_i x^2$. Clearly, we have $f_i(x^1) = f_i(x^2)$. Since $h_i$ is strongly convex, we get
\[ f_i(x^1) = \frac{1}{2} f_i(x^1) + \frac{1}{2} f_i(x^2) = \frac{1}{2} h_i(A_i x^1) + \frac{1}{2} h_i(A_i x^2) \]
\[ > h_i \left( A_i \left( \frac{1}{2} x^1 + \frac{1}{2} x^2 \right) \right) = f_i \left( \frac{1}{2} x^1 + \frac{1}{2} x^2 \right), \]
which contradicts the fact that $x^1 \in X_i^*$. Therefore, we can use Hoffman’s error bound [24], and so there exists some $\rho_i > 0$ such that for any $x \in S$, there exists $x_i^* \in X_i^*$ with
\[ \|x - x_i^*\| \leq \rho_i \max \left[ \begin{pmatrix} B \\ A_i \\ -A_i \end{pmatrix} x - \begin{pmatrix} c \\ z_i \\ -z_i \end{pmatrix}, 0 \right]. \]

Note that we take the max operator componentwise on the right-hand side. Since $Bx - c \leq 0$ for all $x \in S$, we have
\[ \|x - x_i^*\| \leq \rho_i \max \left[ \begin{pmatrix} A_i \\ -A_i \end{pmatrix} x - \begin{pmatrix} z_i \\ -z_i \end{pmatrix}, 0 \right] \text{ for all } x \in S, \]
which yields
\[ \|x - x_i^*\|^2 \leq \rho_i^2 \|A_i x - z_i\|^2 \text{ for all } x \in S. \]

Since $\text{proj}_{X_i^*}(x) \in X_i^*$, it follows that
\[ \|x - \text{proj}_{X_i^*}(x)\|^2 \leq \|x - x_i^*\|^2 \leq \rho_i^2 \|A_i (x - \text{proj}_{X_i^*}(x))\|^2 \text{ for all } x \in S. \] (35)
Now, suppose that \( x \in S \). From the definition (12) of \( u_0 \), we get

\[
u_0(x) = \sup_{z \in S} \min_{i=1,\ldots,m} [F_i(x) - F_i(z)] \\
\leq \min_{i=1,\ldots,m} \sup_{z \in S} [F_i(x) - F_i(z)] = \min_{i=1,\ldots,m} \left[ F_i(x) - F_i \left( \text{proj}_{X_i}^* (x) \right) \right],
\]

where the second equality holds because \( \text{proj}_{X_i}^* (x) = \arg \min_{z \in S} F_i(z) \). Assuming that \( h_i \) is \( \sigma_i \)-convex with \( \sigma_i > 0 \), it follows that

\[
u_0(x) \leq \min_{i=1,\ldots,m} \left[ \nabla h_i(A_i x) \top A_i \left( x - \text{proj}_{X_i}^* (x) \right) - \frac{\sigma_i}{2} \| A_i \left( x - \text{proj}_{X_i}^* (x) \right) \|^2 \right]
= \min_{i=1,\ldots,m} \left[ \nabla f_i(x) \top \left( x - \text{proj}_{X_i}^* (x) \right) - \frac{\sigma_i}{2} \| A_i \left( x - \text{proj}_{X_i}^* (x) \right) \|^2 \right].
\]

Applying (35) to the above inequality leads to

\[
u_0(x) \leq \min_{i=1,\ldots,m} \left[ \nabla f_i(x) \top \left( x - \text{proj}_{X_i}^* (x) \right) - \frac{\sigma_i}{2} \| x - \text{proj}_{X_i}^* (x) \|^2 \right].
\]

Let \( e \in \Delta^m \) with \( \Delta^m \) given by (2). Since \( \min_{i=1,\ldots,m} \inf_{v_i} = \min_{e \in \Delta^m} \sum_{i=1}^m e_i v_i \) for any \( v \in \mathbb{R}^m \), we get

\[
u_0(x) \leq \min_{e \in \Delta^m} \sum_{i=1}^m e_i \left[ \nabla f_i(x) \top \left( x - \text{proj}_{X_i}^* (x) \right) - \frac{\sigma_i}{2} \| x - \text{proj}_{X_i}^* (x) \|^2 \right]
\leq \min_{e \in \Delta^m} \sup_{z \in \mathbb{R}^m} \sum_{i=1}^m e_i \left[ \nabla f_i(x) \top (x - z) - \frac{\sigma_i}{2} \| x - z \|^2 \right]
= \sup_{z \in S} \min_{e \in \Delta^m} \sum_{i=1}^m e_i \left[ \nabla f_i(x) \top (x - z) - \frac{\sigma_i}{2} \| x - z \|^2 \right]
= \sup_{z \in S} \min_{i=1,\ldots,m} \left[ \nabla f_i(x) \top (x - z) - \frac{\sigma_i}{2} \| x - z \|^2 \right]
\leq \min_{i=1,\ldots,m} \frac{\sigma_i}{2} \rho_i^2 (x),
\]

where the first equality follows from the Sion’s minimax theorem (33), and the third equality comes from the definition (21) of \( w_{\rho_i^2} / \min_{i=1,\ldots,m} \sigma_i \). Thus, Theorem 4.2 gives

\[
u_0(x) \leq \max \left( \frac{L \rho_i^2}{\min_{i=1,\ldots,m} \sigma_i} \right) \min_{i=1,\ldots,m} \sigma_i \rho_i^2 (x),
\]

which completes the proof.

We now show that the multiobjective proximal-PL inequality (31) leads to the error-bound property.
Theorem 6.3. Let \( x \in S \). Suppose that \( f_i \) is \( L_i \)-smooth with \( L_i > 0 \) for each \( i = 1, \ldots, m \), \( L := \max_{i=1,\ldots,m} L_i \), and the multiobjective proximal-PL inequality (31) holds with \( \tau > 0 \). Then, the trajectory \( \{ W_L^k(x) := W_L \circ \cdots \circ W_L(x) \} \) converges linearly to a weakly Pareto optimal point \( x^* \) and

\[
W_L(x) := \max_{i=1,\ldots,m} L_i .
\]

converges linearly to a weakly Pareto optimal point \( x^* \) and

\[
u_0(x) \geq \frac{\tau L}{8} \| x - x^* \|^2 \geq \frac{\tau L}{8} \min_{z \in X^*} \| x - z \|^2,
\]

where \( u_0 \) and \( W_L \) are given by (12) and (22), respectively, and \( X^* \) denotes the set of weakly Pareto optimal solutions.

Proof. Recall that \( u_0 \) is non-negative due to Theorem 3.1. We have

\[
\sqrt{u_0(x)} - \sqrt{u_0(W_L(x))} = \frac{u_0(x) - u_0(W_L(x))}{\sqrt{u_0(x)} + \sqrt{u_0(W_L(x))}}.
\]

The definition (12) of \( u_0 \) gives

\[
u_0(x) - u_0(W_L(x)) \geq \min_{i=1,\ldots,m} [F_i(x) - F_i(W_L(x))] \geq w_L(x),
\]

where the second inequality follows from the descent lemma [5, Proposition A.24], (21), and (22). Note that this inequality, together with (31), proves that \( \{ W_L^k(x) \} \) converges linearly to zero. On the other hand, since \( u_0(x) \geq u_0(W_L(x)) \) because of Theorem 3.9 and the above inequality, we get

\[
\sqrt{u_0(x)} + \sqrt{u_0(W_L(x))} \leq 2 \sqrt{u_0(x)} \leq 2 \sqrt{w_L(x)/\tau},
\]

where the second inequality comes from (31). Then, the above three inequalities show

\[
\sqrt{u_0(x)} - \sqrt{u_0(W_L(x))} \geq \frac{w_L(x)}{2 \sqrt{w_L(x)/\tau}} = \frac{1}{2} \sqrt{\tau w_L(x)}.
\]

Therefore, it follows from (24) that

\[
\sqrt{u_0(x)} - \sqrt{u_0(W_L(x))} \geq \frac{\sqrt{\tau L}}{2 \sqrt{2}} \| x - W_L(x) \|.
\]

More generally, we arrive at

\[
\sqrt{u_0(W_L^k(x))} - \sqrt{u_0(W_L^{k+1}(x))} \geq \frac{\sqrt{\tau L}}{2 \sqrt{2}} \| W_L^k(x) - W_L^{k+1}(x) \|
\]

for all \( k = 0, 1, \ldots \). Adding up the above inequality from \( k = k_1 \) to \( k = k_2 - 1 \) yields

\[
\sqrt{u_0(W_L^{k_1}(x))} - \sqrt{u_0(W_L^{k_2}(x))} \geq \frac{\sqrt{\tau L}}{2 \sqrt{2}} \sum_{k=k_1}^{k_2-1} \| W_L^k(x) - W_L^{k+1}(x) \|.
\]
Thus, the triangle inequality implies

\[ \sqrt{u_0(W_{k_1}^L(x))} - \sqrt{u_0(W_{k_2}^L(x))} \geq \frac{\sqrt{\tau L}}{2\sqrt{2}} \|W_{k_1}^L(x) - W_{k_2}^L(x)\|. \]  \hspace{1cm} (37)

As \( k_1, k_2 \to \infty \), the left-hand side tends to zero. Therefore, the right-hand side also tends to zero because of the non-negativity of the norm. This means that \( \{W_{k}^L(x)\} \) is the Cauchy sequence, which is convergent to some weakly Pareto optimal point \( x^* \). Substituting \( k_1 = 0 \) and \( k_2 = \infty \) into (37) leads to

\[ \sqrt{u_0(x)} \geq \frac{\sqrt{\tau L}}{2\sqrt{2}} \|x - x^*\|. \]

This theorem also presents the error-bound property of \( w_\ell \) and \( u_\ell \) for any \( \ell > 0 \) because of \( (31) \), Theorem 4.2, and Theorem 4.1 (ii).

7 Conclusion

We first proposed a simple merit function for (1) in the sense of weak Pareto optimality and showed its lower semicontinuity. We also defined a regularized merit function when \( F \) is convex and discussed its continuity, the way of evaluating it, its differentiability, and the properties of its stationary points. Furthermore, when each \( F_i \) is composite, we introduced a regularized and partially linearized merit function in the sense of Pareto stationarity and showed similar properties. In addition, we gave sufficient conditions for the proposed merit functions to be level-bounded and to provide error bounds.

We can consider a natural extension of our proposed merit functions for vector problems with an infinite number of objective functions. We can also regard the generalization of other merit functions for scalar problems, such as the implicit Lagrangian \( [34] \) and the squared Fischer-Burmeister function \( [26] \), to multiobjective and vector problems. These will be some subjects for future works.

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