Optimal Reliability in Design for Fatigue Life

Part I – Existence of Optimal Shapes

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Abstract: The failure of a component often is the result of a degradation process that originates with the formation of a crack. Fatigue describes the crack formation in the material under cyclic loading. Activation and deactivation operations of technical units are important examples in engineering where fatigue and especially low-cycle fatigue (LCF) play an essential role. A significant scatter in fatigue life for many materials results in the necessity of advanced probabilistic models for fatigue. Moreover, optimization of reliability is of vital interest in engineering, where with respect to fatigue the cost functionals are motivated by the predicted probability for the integrity of the component after a certain number of load cycles. The natural mathematical language to model failure, here understood as crack initiation, is the language of spatio-temporal point processes and their first failure times. The local crack formation intensities thereby need to be modeled as a function of local stress states and thus as a function of the derivatives of the displacement field $u$ obtained as the solution to the PDE of linear elasticity. This translates the problem of optimal reliability in the framework of shape optimization. The cost functionals derived in this way for realistic optimal reliability problems are too singular to be $H^1$-lower semi-continuous as many damage mechanisms, like LCF, lead to crack initiation as a function of the stress at the component’s surface. Realistic crack formation models therefore impose a new challenge to the theory of shape optimization. In this work, we have to modify the existence proof of optimal shapes, for the case of sufficiently smooth shapes using elliptic regularity, uniform Schauder estimates and compactness of strong solutions via the Arzela-Ascoli theorem. This result applies to a variety of crack initiation models and in particular applies to a recent probabilistic model for LCF.

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1 Introduction

A design being made from material will fail if the material degradation due to loading exceeds certain limits. Reliability, i.e. the absence of failure, is thus the ultimate goal of structural design. Whether a design will operate safely under certain load conditions depends on the failure mechanisms which are very diverse for different material classes and operation conditions. Degradation can occur as a function of operating time, which is e.g. the case for creep damage. Or it can occur when small plastic deformations under cyclic loading pile up and result in a crack. This is called fatigue which can be differentiated into high-cycle fatigue (HCF) and low-cycle fatigue (LCF) \cite{9, 37}.

A common feature of crack initiation over diverse classes of material and damage mechanisms is its probabilistic nature, see e.g. \cite{37}. A natural mathematical language to capture the

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random times and locations of crack initiation is the language of point processes [30]. Associated to a spatio-temporal point process is a failure time, which is well known from the theory of renewable models in reliability statistics [19]. The probability that the time to failure – here understood as the time that passes until the formation of the first crack – will be larger than a certain warranty time or service interval length $t^*$ is an important quantity in engineering. The maximization of this quantity sets the problem of optimal reliability.

A central part of engineering is about the choice of the form $\Omega \subseteq \mathbb{R}^3$ of the component. Given material, surface and volume loads that act on the component, it is desirable to optimize failure and survival probabilities for the component at $t^*$ as a function of $\Omega$. This embeds the quest of optimal reliability into the established field of shape optimization [3, 5, 10, 12, 14, 26, 17, 25, 43] or synonymously PDE constrained optimization with shape control. The natural choice for the PDE is the PDE of (linear and elliptic) elasticity [11, 13, 20, 16].

In [3, 5, 6, 14], the authors show generalized optimal shapes with elasticity state equation by the homogenization method, which translates shape optimization to a sizing problem. In particular this allows compliance optimization, which otherwise leads to an ill-posed problem. The usage of generalized shapes in design however poses some additional problems. The existence results in [25] and references therein are based on shapes that fulfill some compactness properties and cost functionals with lower $H^1$-semicontinuity. Fujii [22] gives an interesting method to prove lower semicontinuity for convex cost functionals given by volume integrals of derivatives of the solution $u$ with respect to weak $H^1$-topologies for scalar elliptic PDE.

Although we essentially follow the book by Mäkinen and Haslinger, we are able to deal with cost functionals that live on the boundary and depend on stress or even higher derivatives on the bulk or boundary, which is not covered in [25]. Also, we do not require any convexity assumption. Further results on optimal shapes are e.g. [18] dealing with the case of quasilinear cost functionals and [32] for quadratic cost functionals not depending on derivatives of the solution. For results of eigenvalue optimization see [14] and references given there.

In particular, cost functionals motivated by reasonable point process models of the material failure mechanisms however fall into this singular class. This is due to the fact that many failure mechanisms are stress driven and generate surface cracks, as it is the case for LCF. The associated crack intensities thus depend locally on $\nabla u$ restricted to $\partial \Omega$, which is an operation that is not well defined for the weak $H^1$-solutions $u$ to the elasticity PDE. As a consequence, the theory of weak solutions as in [25] is insufficient for our purposes and the shape optimization formalism needs to be extended to strong solutions with the help of elliptic regularity theory [1, 2, 20, 23]. This makes it necessary to revise existing proofs for the existence of optimal shapes with a PDE constraint given by linear elasticity. In particular, stronger $C^4$-boundary regularity is considered that is not needed in the articles cited above.

Note that in the first sections, we treat the stochastic subject of optimal reliability which leads to cost functionals that are given by volume and surface integrals. In Section 4 to 6 we focus on a class of shape optimization problems which include the objective of optimal reliability under linear elastic PDE constraints. Readers only interested in the general theory of shape optimization can also read these sections independently. The main result regarding the existence of optimal shapes is Theorem 6.4. The paper is organized as follows:

In Section 2 we review the theory of point processes essentially following [30] with applications to reliability statistics [19] and formulate the optimal reliability problem. Here, the notions of crack initiation models and their associated failure times and probabilities are crucial. In particular, if interaction between cracks is neglected, the Poisson point process (PPP)
plays a central role. Here, we connect the topic of optimal reliability to shape optimization with the elasticity PDE as state equation and classify PPP models according to their singularity. In Section 3 we introduce a recently proposed and experimentally validated crack initiation model for LCF in polycrystalline metal. This model has been applied to gas turbine compressor design and extended to themomechanical design of cooled gas turbine blades. Here, we give a detailed account of its mathematical properties and show that it fits to the framework of Section 2.

Section 4 sets the stage for shape optimization essentially following with an emphasis on uniform $C^k$-regularity of shapes in the local epigraph parametrization of shapes with $k = 4$. The following Section 5 treats existence and regularity of strong solutions to the elasticity PDE based on regularity theory for elliptic systems. In order to check uniform regularity we revisit several results in this field and check uniformity of estimates as a function of our parametrization of shapes. In conclusion we are able to show that the solutions to the elasticity PDE on admissible domains $\Omega$ fulfill uniform $[C^{3,\phi}(\Omega)]^3$-bounds where this space stands for 3-times differentiable functions with $\phi$-Hölder continuous third derivative, $\phi \in (0,1)$. We also show that such functions can be extended to $[C^{3,\phi}(\Omega^{\text{ext}})]^3$ with $\Omega^{\text{ext}} \subseteq \mathbb{R}^3$ being some bounded Lipschitz domain that contains all admissible shapes $\Omega$. This section contains the technical core of this paper.

In the final Section 6 we derive our main result regarding the existence of optimal shapes under linear elastic PDE constraints (Theorem 6.1). Here, we apply the results of Section 5 to the framework of shape optimization developed in Section 3. By the Arzela-Ascoli theorem, balls in $[C^{3,\phi}(\Omega)]^3$ are compact. It is therefore easy to show that the graph $\mathcal{G}$ that consists of pairs $(\Omega, u(\Omega))$ of admissible shapes – endowed with a suitable $C^4$-topology – and the associated solution to the elasticity state equation with $C^{3,\phi}$-topology possesses the compactness of $\mathcal{G}$ required in the general shape optimization formalism. For a minimizing sequence $(\Omega_n)_{n \in \mathbb{N}}$ of admissible shapes with respect to a cost functional $J(\Omega, u(\Omega))$, the implied $[C_0^{3,\phi}(\Omega^{\text{ext}})]^3$-convergence of a subsequence of extended solutions $(u^{\text{ext}}(\Omega_{nk}))_{nk \in \mathbb{N}}$ makes it easy to check the (lower semi-) continuity properties for a large class of rather singular cost functionals $J(\Omega, u)$. This concludes the proof of existence of optimal admissible shapes. In particular, our proof covers a large class of cost functionals that include optimal reliability problems introduced in Sections 2 and the probabilistic LCF model from Section 3 in particular.

2 A Mathematical Setting for Optimal Reliability

Let us consider a bounded, open domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial \Omega$. Let $\Omega$ be interpreted as the portion of the space filled with matter, i.e. the shape of the component. Let $\mathcal{T} = \mathbb{N}_0$ or $= \mathbb{R}_+$ be the time axis, where time is either measured in discrete units (load cycles) of in natural time. Let $dt$ denote either the continuous or discrete Lebesgue measure on $\mathcal{T}$.

We define $\mathcal{C} = \mathcal{T} \times \Omega$ as the configuration space of crack initiations at time $t \in \mathcal{T}$ and at location $x \in \Omega$ and endow $\mathcal{C}$ with the standard metric topology.

Let $\mathcal{B} = \mathcal{B}(\mathcal{C})$ be the space of Radon measures on $\mathcal{C}$, i.e. the set of measures $\gamma$ on the measurable space $(\mathcal{C}, \mathcal{B}(\mathcal{C}))$ such that $\gamma(A) < \infty$ for $A \in \mathcal{B}(\mathcal{C})$ bounded. $\mathcal{B}(\mathcal{C})$ denotes the Borel sigma algebra of the topological space $\mathcal{C}$. By $\mathcal{B}_c$ we denote the counting measures, i.e. the Radon measures $\gamma \in \mathcal{B}$ such that $\gamma(B) \in \mathbb{N}_0$ for bounded, measurable $B \subseteq \mathcal{C}$. In the given context $\gamma \in \mathcal{B}_c$ encodes one particular history of (multiple) crack initiations on the component $\Omega$. Given such a history and some $B \in \mathcal{B}(\mathcal{C})$, $\gamma(B)$ gives the number of cracks that initiated.
with time-location instances $c = (t, x) \in B$. Note that $\partial \Omega \subset \bar{\Omega}$, thus surface crack formation as in the case of LCF can be modeled by measures $\gamma$ with support in $\mathcal{I} \times \partial \Omega \subseteq \mathcal{C}$.

The occurrence of the first crack on $\bar{\Omega}$ will be interpreted as a failure event. For the crack initiation history $\gamma$, we define the failure time $\tau : \mathcal{R}_c \rightarrow \mathcal{T} = \mathcal{T} \cup \{\infty\}$ as

$$\tau(\gamma) = \inf\{t > 0 : \gamma(c_t) > 0\},$$

where $c_t = \{(\tau, x) : \tau \leq t\}$. For $B \in \mathcal{B}(\mathcal{C})$ with bounded diameter, the restriction of $\gamma \in \mathcal{R}_c$ to $B$ can be written as (see [30, Chapter 2])

$$\gamma |_B = \sum_{j=1}^n b_j \delta_{c_j}, \quad c_j \in \mathcal{C}, c_i \neq c_j \text{ for } i \neq j, \quad b_j \in \mathbb{N}. \tag{2}$$

This decomposition is unique up to order. $\delta_c$ stands for the Dirac measure in $c$. The Radon counting measure $\gamma$ is called simple, if $\forall B \in \mathcal{B}(\mathcal{C})$ in (2) we have $b_j = 1$ for all $j = 1, \ldots, n$.

The simplicity of crack initiation histories $\gamma$ is a natural condition, namely two cracks that originate at the same time at the same place are considered as the same crack.

Next we have to take into account that crack initiation is a random process. By $\mathcal{N}(\mathcal{R}_c)$ we denote the standard sigma algebra on the space of Radon counting measures generated by the mappings $\gamma \mapsto \int f \, d\gamma$ with $f \in \mathcal{C}^0(\mathcal{C})$, the space of compactly supported continuous functions on $\mathcal{C}$. The failure time $\tau : (\mathcal{R}_c, \mathcal{N}(\mathcal{R}_c)) \rightarrow (\mathcal{T}, \mathcal{B}(\mathcal{T}))$ is easily seen to be measurable. The following definition can be found in [30]:

**Definition 2.1 (Point Process)**

Let $(\mathcal{X}, \mathcal{A}, P)$ be a probability space.

(i) A point process on $\mathcal{C}$ is a measurable mapping $\gamma : (\mathcal{X}, \mathcal{A}, P) \rightarrow (\mathcal{R}_c, \mathcal{N}(\mathcal{R}_c))$.

(ii) The point process $\gamma$ is simple, if $\gamma(., \omega)$ is simple for $P$-almost all $\omega \in \mathcal{X}$.

(iii) A point process $\gamma$ is non-atomic, if $P(\gamma(\{c\}) > 0) = 0 \forall c \in \mathcal{C}$.

(iv) A point process $\gamma$ has independent increments, if for $B_1, \ldots, B_n \in \mathcal{B}(\mathcal{C})$ mutually disjoint, the random variables $\gamma(B_1), \ldots, \gamma(B_n)$ are independent.

Random crack initiation histories are naturally modeled as simple point processes. The interpretation of the additional assumption that $\gamma$ does not possess 'atoms' is that there is no location $x \in \bar{\Omega}$ such that a crack will originate exactly in $x$ with a probability larger than zero.

**Definition 2.2 (Crack Initiation Process)**

(i) A crack initiation process $\gamma$ is a simple, non-atomic point process on $\mathcal{C}$.

(ii) The time to crack initiation $T : \mathcal{X} \rightarrow \mathcal{T}$ associated with $\gamma$ is the random variable $T = \tau(\gamma)$.

Whether the assumption of independent increments is realistic for random crack initiation can be disputed. This approach however should be valid if we are only interested in the component’s history until the formation of the first crack initiation. For a model of interacting crack networks, see e.g. [33].

We now apply some standard results from the theory of point processes to the present context:
Proposition 2.3 (Crack Initiation Processes and Poisson Point Processes)

(i) Any crack initiation process \( \gamma \) on \( \mathcal{C} \) with independent increments is a Poisson point process, i.e. there exists a unique Radon measure \( \rho \in \mathcal{R} \) such that

\[
P(\gamma(B) = n) = e^{-\rho(B)} \frac{\rho(B)^n}{n!} \quad \forall B \in \mathcal{B}(\mathcal{C}) \text{ bounded.}
\]

\( \rho \) is called the intensity measure of \( \gamma \).

(ii) The distribution function \( F_T \) of the time to crack initiation \( T \) is given by \( F_T(t) = 1 - e^{-H(t)} \) with cumulative hazard function \( H(t) = \rho(\mathcal{C}) \).

(iii) If \( \rho(\mathcal{C}) = \infty \), then \( P(T = \infty) = 0 \) and \( T \) can be modified to \( T : (\mathcal{X}, \mathcal{A}) \to (\mathcal{S}, \mathcal{B}(\mathcal{F})) \).

Proof: Assertion (i) is proven in [30, Corollary 6.7] and (ii) then follows from \( F_T(t) = 1 - P(T > t) = 1 - P(\gamma(\mathcal{C}) = 0) = 1 - e^{-\rho(\mathcal{C})} = 1 - e^{-H(t)} \).

Finally, consider (iii): If \( \rho(\mathcal{C}) = \infty \), we have by lower continuity of radon measures that \( H(t) \to \infty \) and thus \( S(t) = e^{-H(t)} = P(T > t) \to 0 \) as \( t \to \infty \). This implies by lower continuity of \( P \) that \( T < \infty \) holds \( P \) almost sure and we can redefine \( T = 0 \) on the null set \( \{T = \infty\} \) without changing the probability law of \( T \).

The reliability of the component \( \Omega \) at some warranty time \( t^* \) or after the passage of a service interval of duration \( t^* \) depends on the loads that act on \( \Omega \), the material and shape \( \Omega \) itself. As in many design applications the loads and the material are given, the choice of the shape \( \Omega \) is the crucial design task. A natural question arising is: Is there a shape with optimal reliability? The answer crucially depends on an assignment of failure probabilities – i.e. probability of crack initiation on \( \Omega \) until \( t^* \) – to the shape \( \Omega \). The following definition collects some basic requirements:

Definition 2.4 (Crack Initiation Model)

Let \( \mathcal{O} \) be some collection of admissible domains contained in \( \Omega^\text{ext} \subseteq \mathbb{R}^3 \) and let \( f, g : \mathcal{C}^\text{ext} \times \mathcal{O}^\text{ext} \to \mathbb{R}^3 \) be vector fields of some spaces \( \mathcal{Y}_\text{vol} \) and \( \mathcal{Y}_\text{sur} \), respectively. For \( \Omega \in \mathcal{O} \), \( g \mid _{\partial \Omega} \) is interpreted as the history – or load collective – of surface force densities on \( \partial \Omega \) and \( f \mid _{\Omega} \) as the history of volume force densities on \( \Omega \).

A crack initiation model is a mapping \( \gamma \) from \( \mathcal{O} \times \mathcal{Y}_\text{vol} \times \mathcal{Y}_\text{sur} \) to the space of all crack initiation processes on \( \mathcal{S} \times \Omega^\text{ext} \) mapping \( (\Omega, f, g) \) to \( \gamma_{\Omega, f,g} \) such that

(i) \( \gamma_{\Omega, f,g}(\mathcal{S} \times \Omega^\text{ext} \setminus \bar{\Omega}) = 0 \) \( P \)-almost surely;

(ii) \( \gamma_{\Omega, f,g} \) \( P \)-almost surely depends only on \( f \mid _{\mathcal{S} \times \Omega} \) and \( g \mid _{\mathcal{S} \times \partial \Omega} \).

Obviously, for any crack initiation model, we obtain an induced mapping of \( (\Omega, f, g) \) to the associated crack initiation time random variables \( T_{\Omega, f,g} \) associated with \( \gamma_{\Omega, f,g} \). We can now define the optimal reliability problem, given the fixed load histories \( f \) and \( g \) and a fixed time \( t^* \in \mathcal{S} \). For notational simplicity, we suppress \( f \) and \( g \) dependency in the following.

Definition 2.5 (Optimal Reliability Problem)

Given \( t^* \in \mathcal{S} \), \( f \in \mathcal{Y}_\text{vol} \), \( g \in \mathcal{Y}_\text{sur} \) and a crack initiation model \( \gamma \), find \( \Omega^* \in \mathcal{O} \) such that

\[
P(T_{\Omega^*} \leq t^*) \leq P(T_{\Omega} \leq t^*) \quad \forall \Omega \in \mathcal{O}.
\]
Now, we want to construct crack initiation models with independent increments based on the PDE of linear isotropic elasticity. This establishes the link between the optimal reliability problem of Definition 2.5 and PDE constrained shape optimization. For simplicity, we restrict ourselves to the case that the load vector fields \( f \) and \( g \) are independent of \( t \) such that our model is based on one well defined load cycle, see Figure 1. The time \( t \) then counts the number of such load cycles.

Let \( \nu \) be the outward normal of the boundary \( \partial \Omega \) and let \( \partial \Omega = \partial \Omega_D \cup \partial \Omega_N \) be a partition where \( \partial \Omega_D \) is clamped and on \( \partial \Omega_N \) a force surface density \( g \mid_{\partial \Omega_N} \) is imposed. Then, according to [16] the mixed problem of linear isotropic elasticity is described by:

\[
\begin{align*}
\nabla \cdot \sigma(u) + f &= 0 & \text{in } \Omega, \\
\sigma(u) &= \lambda (\nabla \cdot u) I + \mu (\nabla u + \nabla u^T) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega_D, \\
\sigma(u) \cdot \nu &= g & \text{on } \partial \Omega_N. \\
\end{align*}
\]

Here, \( \lambda > 0 \) and \( \mu > 0 \) are the Lamé coefficients and \( u : \Omega \to \mathbb{R}^3 \) is the displacement field on \( \Omega \). \( I \) is the identity on \( \mathbb{R}^3 \). The linearized strain rate tensor \( \varepsilon(u) : \Omega \to \mathbb{R}^{3 \times 3} \) is defined as \( \varepsilon(u) = \frac{1}{2} (\nabla u + \nabla u^T) \). Approximate numerical solutions can be computed by a finite element approach, confer [28] and [16].

**Definition 2.6 (Local Crack Initiation Model)**

Let \( \mathcal{O} \times \mathcal{Y}_{\text{vol}} \times \mathcal{Y}_{\text{sur}} \) be such that for all \( f \in \mathcal{Y}_{\text{vol}}, g \in \mathcal{Y}_{\text{sur}} \) and \( \Omega \in \mathcal{O} \) there exists a unique (weak) solution \( u(\Omega) \) to (3). Let furthermore \( \varrho_{\text{sur}} : \mathcal{T} \times \mathbb{R}^d \to \mathbb{R}_+ \) and \( \varrho_{\text{vol}} : \mathcal{T} \times \mathbb{R}^d \to \mathbb{R}_+ \) with \( d = 3 + \sum_{j=0}^{r} 3^{j+1} = 3 + \frac{3^r + 1}{2} \) be measurable, nonnegative functions with \( r \in \mathbb{N} \) the order of the model. Suppose that the \( l \)-th weak derivative \( \nabla^l u \) of \( u \) are measurable functions for \( l = 0, \ldots, r \) and that the trace \( \nabla^l u \mid_{\partial \Omega} \) is well defined in the sense of measurable functions. Here, \( (\nabla^l u)_{j_1 \ldots j_l} \) stands for \( \frac{\partial^l u}{\partial x_{j_1} \ldots \partial x_{j_l}} \). Then, we define:

(i) A \( r \)-th order local crack initiation model with independent increments and linear elasticity state equation is defined by this data by setting \( \gamma_{\Omega} \) to be the Poisson point process on \( \mathcal{E}^{\text{ext}} \) associated to the intensity measure

\[
\rho_{\Omega}(B) = \int_{B \cap (\mathcal{T} \times \Omega)} \varrho_{\text{vol}}(t, x, u, \nabla u, \ldots, \nabla^r u) \, dt \, dx + \int_{B \cap (\mathcal{T} \times \partial \Omega)} \varrho_{\text{sur}}(t, x, u, \nabla u, \ldots, \nabla^r u) \, dt \, dA \quad \forall B \in \mathcal{B}(\mathcal{E}^{\text{ext}}),
\]

provided that the resulting measures are Radon measures on \( \mathcal{E}^{\text{ext}} \).

(ii) \( \gamma \) is said to be strain driven, if \( \varrho_{\text{sur}} \) and \( \varrho_{\text{vol}} \) depend only on the elastic strain tensor field \( \varepsilon(u) \). As the elastic stress tensor field \( \sigma(u) \) can be obtained from \( \varepsilon(u) \) and vice versa, strain and stress driven crack initiation models are synonymous.

(iii) If \( \varrho_{\text{sur}} = 0 \), then \( \gamma \) is volume driven and if \( \varrho_{\text{vol}} = 0 \) it is surface driven.

(iv) We say that the \( r \)-th order crack initiation model has \( s \)-regular intensity functions, \( s \in \mathcal{T} \), if \( \varrho_{\text{sur}}, \varrho_{\text{vol}} \) are in \( C^0(\mathcal{T}) \otimes C^s(\mathbb{R}^d) \).
From the above definition it is now clear that the optimal reliability problem – confer Definition 2.5 – in the case described $r$-th order local crack initiation model is just a PDE constrained shape optimization problem:

**Lemma 2.7 (Optimal Reliability and PDE Constrained Optimization)**

For a $r$-th order local crack initiation model with elasticity state equation as defined in Definition 2.6, the optimal reliability problem given in Definition 2.5 is equivalent to the shape optimization problem given in Definition 4.1 below with

$$J(\Omega, y) = \int_{\Omega} F_{\text{vol}}(x, y, \nabla y, \ldots, \nabla^r y) \, dx + \int_{\partial \Omega} F_{\text{sur}}(x, y, \nabla y, \ldots, \nabla^r y) \, dA$$

with $F_{\text{vol}}(\cdot) = \int_{0}^{t^*} \varrho_{\text{vol}}(t, \cdot) \, dt$ and $F_{\text{sur}}(\cdot) = \int_{0}^{t^*} \varrho_{\text{sur}}(t, \cdot) \, dt$ and $y : \Omega^{\text{ext}} \to \mathbb{R}^3$ sufficiently regular.

In particular, for the case of $s$-regular intensity functions, $F_{\text{vol}}, F_{\text{sur}} \in C^s(\mathbb{R}^d)$.

**Proof:** By Proposition 2.3 (ii) the optimal reliability problem is equivalent to $\rho_{\Omega^{\text{ext}}}((\epsilon_{t^*})^+) \leq \rho_{\Omega^{\text{ext}}}(\epsilon_{t^*})$ for all $\Omega \in \mathcal{O}$. Now apply Definition 2.6 (i), Fubini’s lemma for positive functions and the properties of Bochner integrals.

3 A Probabilistic Crack Initiation Model for LCF

A somewhat general approach should contain at least one relevant example. In the following we present probabilistic LCF crack initiation as a surface and strain driven crack initiation model with elasticity state equation. In [39] this model has been numerically implemented and applied to gas turbine design. An extension to thermomechanical equations can be found in [40]. Experimental validation is presented in [41].

One of the paradoxes that comes with the use of the equations of isotropic elasticity in the context of fatigue comes from the fact that any perfectly elastic deformation of a material is completely reversible and therefore does not lead to degradation. The time-honored elastic-plastic stress conversion resolves this issue and is widely used in engineering. We therefore choose it as the basis of our model.

Defining the von Mises stress as $\sigma_v = \sqrt{\frac{2}{3} \text{tr}(\sigma'^2)}$, with $\sigma' = \sigma - \frac{1}{3} \text{tr}(\sigma) I$ the trace-free part of $\sigma$ capturing non-hydrostatic stresses, only, we present the Ramberg-Osgood equation which is used to locally derive strain levels from scalar comparison stresses $\sigma_v$, confer [38]. This equation describes stress-strain curves of metals near their yield points.

**Notation (Ramberg-Osgood Equation)**

Let $K > 0$ denote the strain hardening coefficient and $n' > 0$ the strain hardening exponent. Then, the Ramberg-Osgood relation between an elastic-plastic comparison strain $\varepsilon_{\text{el-pl}}^{\text{el-pl}} \in \mathbb{R}_+$ and an elastic-plastic comparison stress $\sigma_v^{\text{el-pl}} \in \mathbb{R}_+$ is given by

$$\varepsilon_{\text{el-pl}}^{\text{el-pl}} = \frac{\sigma_v^{\text{el-pl}}}{E} + \left( \frac{\sigma_v^{\text{el-pl}}}{K} \right)^{1/n'}$$  (5)
with Young’s modulus $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$. The equation defines the comparison strain $\varepsilon^{\text{el}-\text{pl}}$. We also write $\varepsilon^{\text{el}-\text{pl}} = \text{RO}(\sigma^{\text{el}-\text{pl}})$. The obvious problem is that the elastic-plastic comparison stress $\sigma^{\text{el}-\text{pl}}$ needs to be defined on the basis of the elastic von Mises stress $\sigma_v$ which can be approximated by a finite element calculation. This is accomplished by the method of stress shakedown by Neuber$^3$, confer [35] and [9].

**Notation (Elastic-Plastic Stress Conversion and Shakedown)**

Given $\sigma_v \in \mathbb{R}_+$, the associated elastic-plastic comparison stress $\sigma^{\text{el}-\text{pl}}_v$ is defined as the positive solution to the following equation

$$\frac{(\sigma_v)^2}{E} = \sigma^{\text{el}-\text{pl}}_v \varepsilon^{\text{el}-\text{pl}} = \frac{(\sigma^{\text{el}-\text{pl}}_v)^2}{E} + \sigma^{\text{el}-\text{pl}}_v \left(\frac{\sigma^{\text{el}-\text{pl}}_v}{K}\right)^{1/n'}.$$

From the the elastic von Mises comparison stress $\sigma_v$, we can thus calculate the elastic-plastic von Mises stress $\sigma^{\text{el}-\text{pl}}_v$ by solving (6) and thus we are able to obtain $\varepsilon^{\text{el}-\text{pl}}$ from [35]. We also write $\sigma^{\text{el}-\text{pl}}_v = \text{SD}(\sigma_v)$.

In structural analysis, fatigue describes the damage or failure of material under cyclic loading, confer [9] and [37]. Compared to the static case material is damaged by much lower load amplitudes of cyclic loading. Figure 1 shows a triangle shaped uniaxial load-time-curve as an example, where $\sigma_a = \frac{1}{2}(\sigma_{\text{max}} - \sigma_{\text{min}})$, is the elastic von Mises comparison amplitude. For more details on surface driven LCF failure mechanism with respect to polycrystalline metal, we refer to [9, 37, 21, 44]. In fatigue the number of cycles until failure is determined and if the tests are strain controlled so-called $E - N$ diagrams can be created, see the test points in Figure 2.

Figure 2 also shows the relationship of strain amplitude $\varepsilon^{\text{el}-\text{pl}}_a = \text{RO} \circ \text{SD}(\sigma_a)$ and the life time $N_i$ to crack initiation measured in cycles. The Coffin-Manson-Basquin (CMB) or Wöhler equation$^4$ connects the number of cycles to crack initiation $N_i$ with the plastic strain amplitude $\varepsilon^{\text{pl}}_a$.

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$^3$The stress shakedown is based on an energy-conservation ansatz, confer [29] for details on Neuber shakedown in conjunction with equivalent stresses. As an alternative to Neuber’s rule we could have also used Glinka’s method, see e.g. [31].

$^4$A discussion of the physical origin of this equation can be found in [44].
Notation (Coffin-Manson-Basquin Equation)
The CMB-equation connects the (deterministic) time to crack initiation \( N_i \) with the elastic-plastic strain amplitude \( \varepsilon_a^{el-pl} \) via
\[
\varepsilon_a^{el-pl} = \text{CMB}(N_i) = \frac{\sigma_f'}{E}(2N_i)^b + \varepsilon_f'(2N_i)^c. \tag{7}
\]
Here, \( \sigma_f' \) and \( \varepsilon_f' \) are positive and \( b \) and \( c \) negative material parameters.

In the following, we will assume for simplicity that the lower edge of the load cycle is stress-free, corresponding to \( f_{\text{min}} = 0 \) and \( g_{\text{min}} = 0 \) in (3) and thus \( \sigma_a = \sigma_v/2 \), where we set \( \sigma = \sigma_{\text{max}}, f = f_{\text{max}} \) and \( g = g_{\text{max}} \).

In deterministic design, the lifetime of a component under cyclic loading corresponds to the loading conditions at the part’s surface position of highest stress. Safety factors are additionally imposed to account for the stochastic nature of LCF and size effects\(^5\). We refer to this method as the safe-life approach in fatigue design which is widely used in engineering, confer [9, 37, 41, 42]. The following lemma collects the mathematical properties of this procedure that will be of importance:

**Lemma 3.1** \( \varphi = \text{CMB}^{-1} \circ \text{RO} \circ \text{SD} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), that maps elastic von Mises comparison stress to a predicted life time, satisfies:
(i) \( \varphi \) is bijective and strictly monotonically decreasing;
(ii) \( \lim_{\sigma_v \rightarrow 0} \varphi(\sigma_v) = \infty \);  
(iii) \( \varphi \) is in \( C^\infty(\mathbb{R}^+) \).

**Proof:** It is elementary to check that \( \text{RO} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is bijective, an element of \( C^\infty(\mathbb{R}^+) \) and strictly monotonically increasing. As inverse of strictly monotonically increasing (decreasing) \( C^\infty(\mathbb{R}^+) \)-function, the same applies to \( \text{SD} \) (\( \text{CMB}^{-1} \), which is strictly monotonically decreasing, however). The three assertions of the lemma now follow immediately. \[\square\]

Notation (Deterministic LCF-Life at a Surface Point)
Let \( \mathbb{R}^{3 \times 3} \) be the space of real \( 3 \times 3 \) matrices. Define \( N_{\text{det}} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^+ \cup \{ \infty \} \) via
\[
N_{\text{det}}(M) = \varphi\left( [\lambda \text{tr}(M)I + \mu(M + M^T)]_+ \right), \tag{8}
\]
with \( \varphi \) as in Lemma 3.1 extended by \( \varphi(0) = \infty \). Given a solution \( u \in [H^1(\Omega)]^3 \) to (3) such that the trace \( \nabla u \mid_{\partial\Omega} \) can be reasonably defined and can be represented as a continuous function, \( N_{\text{det}}(\nabla u(x)) \) is the predicted deterministic LCF-life at the point \( x \in \partial\Omega \).

Although somewhat out of the main line of this article, we state the following definition that represents (up to safety factors) the traditional engineering lifing approach:

**Definition 3.2** (Deterministic Optimal LCF-Lifing Problem)
Let the setting of the previous Definition of LCF-life at a surface point be given.
(i) The deterministic life time to crack initiation for LCF is defined as
\[
T_{\text{det},\Omega} = \inf \{ N_{\text{det}}(\nabla u(x)) \mid x \in \partial\Omega \}.
\]

\(^5\)Note that different geometries of test specimens lead to different Wöhler/CMB curves, confer [37].
(ii) Given a set of admissible shapes $\mathcal{O}$ and loads $f \in \mathcal{V}_{\text{vol}}$, $g \in \mathcal{V}_{\text{sur}}$ such that the above setting holds for all $\Omega \in \mathcal{O}$, we define the deterministic optimal LCF lifting problem as follows

Find $\Omega^* \in \mathcal{O}$ such that $T_{\text{det},\Omega^*} \geq T_{\text{det},\Omega} \forall \Omega \in \mathcal{O}$.

It is a usual approach in reliability statistics [19], to choose the deterministic life prediction as a scale variable of a failure time distribution. Moreover, Weibull distributions are widely used in technical reliability analysis. Recall that the Weibull distribution with scale parameter $\eta$ and shape parameter $m$ is defined by the cumulative distribution function $F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^m}$ for $t > 0$ and zero otherwise.

**Definition 3.3 (Local Weibull Model for LCF)**

For $m \geq 1$, the strain and surface driven crack initiation model (confer Definition 2.6) with independent increments of 1-st order that is defined by

$$\varrho_{\text{vol}} = 0, \quad \varrho_{\text{sur}}(t, \nabla u) = \frac{m}{N_{\text{det}}(\nabla u)} \left(\frac{t}{N_{\text{det}}(\nabla u)}\right)^{m-1},$$

is called the local (probabilistic) Weibull model for LCF. The associated optimal reliability problem as given in Definition 2.5 and Lemma 2.7 is called the optimal reliability problem for LCF. Here, we employ the convention $\frac{1}{\infty} = 0$.

The model could be defined for both, $\mathcal{T} = \mathbb{N}_0$ and $\mathcal{T} = \mathbb{R}_+$, but the second option is used more often. In this case, $\mathcal{F}_{\text{sur}}(\nabla u) = \left(\frac{t}{N_{\text{det}}(\nabla u)}\right)^m$.

**Proposition 3.4 (Properties of the local Weibull Model for LCF)**

Let the conditions of Definition 2.6 be fulfilled such that $\frac{1}{N_{\text{det}}(\nabla u)} \in L^m(\partial \Omega)$, and for all $f \in \mathcal{V}_{\text{vol}}$, $g \in \mathcal{V}_{\text{sur}}$ and $\Omega \in \mathcal{O}$. Then,

(i) The local probabilistic Weibull model for LCF actually induces a 1st order local crack initiation model, i.e. the associated measures $\rho_{\Omega}$ are Radon measures.

(ii) The intensities of this model are 0-regular, i.e. are continuous functions of $\nabla u$.

(iii) The crack initiation time $T_{\Omega}$ is Weibull distributed with shape parameter $m$ and scale parameter $\eta = \left\| \frac{1}{N_{\text{det}}(\nabla u)} \right\|^{-1}_{L^m(\partial \Omega)}$.

**Proof:** (i) To see the finiteness of the intensity measure on sets of finite diameter, note that for any such set $B \subseteq \mathcal{B}(\mathcal{E}^{\text{ext}})$ we find $t \in \mathcal{T}$ such that $B \subseteq \mathcal{E}^{\text{ext}}_t$ and thus, with $\mathcal{F}_{\text{sur}}(\nabla u) = \left(\frac{t}{N_{\text{det}}(\nabla u)}\right)^m$, confer Definition 3.3

$$\rho_{\Omega}(B) \leq \rho_{\Omega}(\mathcal{E}^{\text{ext}}_t) = \int_{\mathcal{E}^{\text{ext}}_t \cap (\mathcal{T} \times \partial \Omega)} \varrho_{\text{sur}}(t, \nabla u) dAdt = t^m \left\| \frac{1}{N_{\text{det}}(\nabla u)} \right\|^{m}_{L^m(\partial \Omega)} < \infty. \quad (10)$$

(ii) Obviously, the elastic von Mises stress depends continuously on the components of $\nabla u$. Taking into account the convention $\frac{1}{\infty} = 0$ and Lemma 3.1 we see that $\frac{1}{N_i(\nabla u)}$ depends
continuously on the components of $\nabla u$. This implies $0$-regularity of the intensity function $\nu$ in the sense of Definition 2.6 (iv).

(iii) The right hand side of (10) equals the cumulative hazard function $H(t)$ of $T_\Omega$. Comparing this to the Weibull cumulative hazard function $H_{\text{Wei}}(t) = (t^\eta)^m$, the assertion follows.

Higher order regularity of the intensity functions crucially depends on the material parameters. We postpone a detailed account to [24].

4 Shape Optimization and $C^k$-Admissible Domains

In this section, we introduce into an abstract setting of shape optimization and discuss so-called $C^k$-admissible domains. This leads to a theoretical frame for an existence proof of optimal designs in Section 6 with respect to a general class of cost functionals which include some functionals given in Lemma 2.7, namely those coming from 3-rd order local crack initiation model with 0-regular intensity functions, see Definition 2.6 (i–iv). In particular this includes the local, probabilistic Weibull model for LCF, confer Definition 3.3 and Proposition 3.4, but also the deterministic optimal LCF lifting problem, confer Definition 3.2.

First, we introduce basic notations and closely follow Section 2.4 in [25]. These notations have to be precisely concretized in each class of shape optimization problems, as we will do so at the end of this section and in Section 6 after we have established the $C^k$-admissible domains and the state problem, respectively. For further introductions into shape optimization confer [43, 12, 14], for example.

**Notation (Family of Admissible Domains, State Space)**

Let $\bar{\Omega}$ denote a family of admissible domains and let $V(\Omega)$ for every $\Omega \in \bar{\Omega}$ denote a state space of real functions defined in $\Omega$.

**Notation (Convergence of Sets and of Functions with Variable Domains)**

Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence in $\bar{\Omega}$ and let $\Omega \in \bar{\Omega}$. Then $\Omega_n \to \Omega$ as $n \to \infty$ denotes the convergence of $(\Omega_n)_{n \in \mathbb{N}}$ against $\Omega$. If $(y_n)_{n \in \mathbb{N}}$ is a sequence of functions with $y_n \in V(\Omega_n)$ for every $n \in \mathbb{N}$ and if $y \in V(\Omega)$ then $y_n \to y$ as $n \to \infty$ denotes the convergence of $(y_n)_{n \in \mathbb{N}}$ against $y$. Moreover, it is assumed that any subsequence of a convergent sequence converges against the limit of the original one.

In every $\Omega \in \bar{\Omega}$ one solves a state problem which can be a PDE or a variational inequality, for example. Assuming that every state problem has a unique solution and associating with any $\Omega \in \bar{\Omega}$ the corresponding unique solution $u(\Omega) \in V(\Omega)$ one obtains the map $u : \Omega \mapsto u(\Omega) \in V(\Omega)$. Let $\mathcal{O}$ be a subfamily of $\bar{\Omega}$, then $\mathcal{G} = \{(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O}\}$ is called the graph of the mapping $(u(\cdot))$ restricted to $\mathcal{O}$.

**Definition 4.1 (Cost Functional, Optimal Shape Design Problem)**

A cost functional $J$ on $\bar{\Omega}$ is given by a map $J : (\Omega, y) \mapsto J(\Omega, y) \in \mathbb{R}$, where $\Omega \in \bar{\Omega}$ and $y \in V(\Omega)$. Let $\mathcal{O}$ be a subfamily of $\bar{\Omega}$ and for every $\Omega \in \mathcal{O}$ let $u(\Omega)$ be the unique solution of a state problem given in $\Omega$. An optimal shape design problem can then be defined by

$$\left\{ \begin{array}{l}
\text{Find } \Omega^* \in \mathcal{O} \text{ such that } \\
J(\Omega^*, u(\Omega^*)) \leq J(\Omega, u(\Omega)) \quad \forall \Omega \in \mathcal{O}.
\end{array} \right. \quad (11)$$
Now, we present a statement regarding the existence of optimal shapes. Note that this theorem will structure the following section where we prove of our main existence results.

**Theorem 4.2 (Existence of An Optimum in Shape Design Problems)**

Let $\hat{\Omega}$ be a family of admissible domains and $\mathcal{O}$ a subfamily. Moreover, let $J$ be a cost functional on $\hat{\Omega}$ and assume that every $\Omega \in \hat{\Omega}$ has a state problem with state space $V(\Omega)$ where each such state problem has a unique solution $u(\Omega) \in V(\Omega)$. Finally, conjecture

(i) Compactness of $\mathcal{G} = \{(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O}\}$:

Every sequence $(\Omega_n, u(\Omega_n))_{n \in \mathbb{N}} \subset \mathcal{G}$ has a subsequence $(\Omega_{n_k}, u(\Omega_{n_k}))_{k \in \mathbb{N}}$ which satisfies

$$\Omega_{n_k} \overset{\hat{\Omega}}{\rightarrow} \Omega, \quad k \to \infty,$$

$$u(\Omega_{n_k}) \rightharpoonup u(\Omega), \quad k \to \infty$$

for some $(\Omega, u(\Omega)) \in \mathcal{G}$.

(ii) Lower semi-continuity of $J$:

Let $(\Omega_n)_{n \in \mathbb{N}}$ with $\Omega_n \in \hat{\Omega}$, $n \in \mathbb{N}$, and $(y_n)_{n \in \mathbb{N}}$ with $y_n \in V(\Omega_n)$, $n \in \mathbb{N}$, be sequences and let $\Omega$ and $y$ be some elements in $\hat{\Omega}$ and in $V(\Omega)$, respectively. Then

$$\Omega_n \overset{\hat{\Omega}}{\rightarrow} \Omega, \quad y_n \rightharpoonup y, \quad n \to \infty$$

$$\implies \liminf_{n \to \infty} J(\Omega_n, y_n) \geq J(\Omega, y).$$

Under these assumptions, the optimal shape design problem (11) possesses at least one solution.

According to the previous theorem and abstract setting we have to define the family of admissible domains $\hat{\Omega}$. In this work we consider so-called $C^k$-admissible domains which have smooth boundaries. For the sake of simplicity, we only optimize a part of the boundary of these shapes. This method is described in Section 2.8 of [43] and in [25], too. Importantly, $C^k$-admissible domains satisfy compactness properties as required in Theorem 4.2 according to the Arzela-Ascoli theorem.

At first, admissible domains are defined which are determined by uniformly bounded functions. Later on, these functions are assumed to be sufficiently smooth to obtain $C^k$-admissible domains. On these domains we will later impose boundary value problems (BVPs) of linear elasticity which are characterized by disjoint Dirichlet and Neumann boundaries. In the following we employ so-called uniform cone properties which are defined in Appendix A of [25].

**Definition 4.3 (Basic Design, Design Variables, Admissible Domains)**

Let $\hat{\Omega} \subset \mathbb{R}^3$ be a simply connected and bounded domain which is called basic design under the following assumptions:

(i) $\hat{\Omega}$ has a uniform cone property and a $C^k$-boundary for some $k \in \mathbb{N}, k \geq 1$.

(ii) There is some $\alpha_{\min} \in \mathbb{R}$ so that the cross section $\Omega_{2d} = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1, x_2, \alpha_{\min}) \in \hat{\Omega}\} \subset \mathbb{R}^2$ is a nonempty domain in $\mathbb{R}^2$.  

---

6This leads to a local description of $\partial \hat{\Omega}$ by a finite number of hemisphere transformations of class $C^k$, confer Definitions A.1 and A.2 and Remark A.3.
(iii) There are some $z \in \hat{\Omega}$ and $r > 0$ such that $B(z, r) \subset \hat{\Omega}$ and $z_3 + r < \alpha_{\min}$.

For $\alpha_{\max} > \alpha_{\min}$ and positive constants $L_1, L_2, L_3$ the elements of

$$\tilde{U}^{ad} = \left\{ \alpha \in C^k(\overline{\Omega}_{2d}) \mid \alpha_{\min} \leq \alpha \leq \alpha_{\max} \text{ in } \overline{\Omega}_{2d}, \alpha|_{\partial \Omega_{2d}} = \alpha_{\min}, \int_{\Omega_{2d}} \alpha(x)dx = L_1, \|\alpha\|_{C^k} \leq L_2, \left|\alpha^{(k)}(x) - \alpha^{(k)}(y)\right| \leq L_3\|x - y\|_2 \forall x, y \in \overline{\Omega}_{2d} \right\}$$

are called design variables. Let $\alpha \in \tilde{U}^{ad}$ define the set

$$\Omega(\alpha) = \{x \in \hat{\Omega} \mid x_3 \leq \alpha_{\min}\} \setminus B(z, r) \cup \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \Omega_{2d}, \alpha_{\min} < x_3 < \alpha(x_1, x_2)\},$$

see Figure 3. Varying only functions $\alpha \in \tilde{U}^{ad}$ the domains $\tilde{\Omega} = \{\Omega(\alpha) \mid \alpha \in \tilde{U}^{ad}\}$ with Lipschitz continuous boundaries are called admissible domains. Finally, choose an open superset for all admissible domains such as $\Omega^{ext} = B(z, r^{ext})$ with $r^{ext} > 0$ sufficiently large.

**Lemma 4.4** $\tilde{U}^{ad}$ is compact in $(C^k(\overline{\Omega}_{2d}), \|\cdot\|_{C^k})$.

**Proof:** Applying the Arzela-Ascoli theorem several times to

$$\left\{ \alpha \in C^k(\overline{\Omega}_{2d}) \mid \|\alpha\|_{C^k} \leq L_2, \left|\alpha^{(k)}(x) - \alpha^{(k)}(y)\right| \leq L_3\|x - y\|_2 \forall x, y \in \overline{\Omega}_{2d} \right\}$$

(12)

shows the compactness in $(C^k(\overline{\Omega}_{2d}), \|\cdot\|_{C^k})$. $\tilde{U}^{ad}$ being a closed subset of (12) proves the statement of this lemma.

**Definition 4.5** ($C^k$-Convergence of Sets)

$\Omega(\alpha_n) \xrightarrow{\text{C}} \Omega(\alpha)$ as $n \to \infty$ is defined by $\alpha_n \to \alpha$ in $C^k(\overline{\Omega}_{2d})$ as $n \to \infty$, where $\alpha, \alpha_n$ and $\Omega(\alpha), \Omega(\alpha_n)$ for $n \in \mathbb{N}$ are defined as in Definition 4.3.
In order to apply regularity results of linear elasticity we have to require a sufficiently smooth boundary of the admissible domains. Therefore, additional boundary conditions on the design variables $\alpha$ are introduced which enable the construction of such domains.

**Definition 4.6 (C^k-Admissible Domains)**

Let $\tilde{U}^{ad}$ be the set of design variables of Definition 4.3 and let $S_\beta : \partial \Omega_2 \rightarrow \mathbb{R}$ be functions for multi-indices $\beta$ with $1 \leq |\beta| \leq k$. Define $U^{ad} = \{ \alpha \in \tilde{U}^{ad} | |\nabla^\beta \alpha|_{\partial \Omega_2} = S_\beta \forall |\beta| \in \{1,\ldots,k\} \}$. Choosing $S_\beta$ so that $\Omega(\alpha)$ has a $C^k$-boundary for every $\alpha \in U^{ad}$ the set $O = \{ \Omega(\alpha) | \alpha \in U^{ad} \}$ denotes the family of so-called $C^k$-admissible domains.

**Lemma 4.7** $U^{ad}$ is compact in $\left(C^k(\overline{\Omega}_2), \| \cdot \|_{C^k} \right)$.

**Proof:** Note that $U^{ad}$ is a closed subset of $\tilde{U}^{ad}$, where $\tilde{U}^{ad}$ is already compact according to Lemma 4.4.

## 5 Uniform Schauder Estimates

In this section, we review regularity results in linear elasticity and show that certain estimates that lead to the mentioned regularity are uniform in $O$. From the shape optimization perspective, these results will influence our choice of state space and of the definition of convergence of functions with variable domains. But they also provide the necessary input in order to define what is meant by sufficiently regularity in Definition 2.6 and provide candidates for the vector spaces $V_{vol/surf}$ in that definition for local crack initiation models of order up to three.

Recall the mixed problem (3). Regularity results for the mixed problem depend crucially on the properties of the domain’s boundary in which the elasticity equations are posed. Theorem 6.3-5 of [13] ensures the existence of a weak solution of the mixed problem:

**Theorem 5.1 (Existence of a Weak Solution)**

Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipschitz continuous boundary and let $\partial \Omega_D \subset \partial \Omega$ be measurable where $\partial \Omega_D$ has a positive area. Let the Lamé coefficients $\lambda, \mu$ be positive constants and let $f \in [L^{6/5}(\Omega)]^3$, $g \in [L^{4/3}(\partial \Omega_N)]^3$ where $\partial \Omega_N = \partial \Omega \setminus \partial \Omega_D$. Moreover, define on $V_{DN} = \{ v \in [H^1(\Omega)]^3 | v = 0$ a.e. on $\partial \Omega_D \}$

$$B(u,v) = \int_\Omega \lambda (\nabla \cdot u)(\nabla \cdot v) \, dx + \int_\Omega 2\mu \text{tr}(\varepsilon(u)\varepsilon(v)) \, dx,$$

$$L(v) = \int_\Omega f \cdot v \, dx + \int_{\partial \Omega_N} g \cdot v \, dA.$$  

Then, there exists a unique $u \in V_{DN}$ that satisfies

$$B(u,v) = L(v) \quad \forall v \in V_{DN}$$

and additionally $J(u) = \inf \{ J(v) | v \in V_{DN} \}$, where $J(v) = \frac{1}{2} B(v,v) - L(v)$.  

The unique solution $u \in V_{DN}$ of [13] is called the weak solution of [3]. The following inequality can be found in [11]: If $\Omega \subset \mathbb{R}^3$ is a domain the so-called Korn’s second inequality

$$c\|v\|_{[H_0^1(\Omega)]^3} \leq \left( \int_\Omega \text{tr}(|\varepsilon(v)|^2) \, dx \right)^{1/2} = \|\varepsilon(v)\|_{H^1(\Omega)}^{1/2}$$

(14)
holds for all \( v \in V_{DN} \). We now consider the co-called disjoint displacement-traction problem of linear elasticity \(^3\), where \( \partial \Omega_D \cap \partial \Omega_D = \emptyset \). This BVP will have additional regularity properties if \( \partial \Omega \) and the forces \( f \) and \( g \) are sufficiently regular, confer Theorem 6.3-6 of \(^3\) and remarks thereafter:

**Theorem 5.2 (Regularity for the Disjoint Displacement-Traction Problem)**
Let \( \Omega \subset \mathbb{R}^3 \) be a domain with a \( C^4 \)-boundary, let \( f \in [W^{2,p}(\Omega)]^3 \) and let \( g \in [W^{1-1/p, p}(\partial \Omega)]^3 \) for some \( p \geq 6/5 \). Consider on \( \Omega \) a disjoint displacement-traction problem. Then, there exists a unique solution \( u \in V_{DN} \backslash B(u, v) = L(v) \) for all \( v \in V_{DN} \), where \( V_{DN}, B \) and \( L \) are defined as in Theorem 5.4. Moreover, \( u \) is an element of \([W^{4,p}(\Omega)]^3\).

The key to the proof is to employ the fact that the previous problems (pure displacement, pure traction, disjoint displacement-traction problem) are uniformly elliptic and satisfy the so-called supplementary and complementing conditions. These conditions are introduced in \(^2\), where Schauder estimates are also described in detail which can be applied to solutions of mixed problems. The Schauder estimates are an important ingredient in our proof of existence for optimal shapes in this section.

From now on, we consider on \( C^4 \)-admissible shape \( \Omega(\alpha) \in O \) the disjoint displacement-traction problem of linear elasticity as follows:

**Definition 5.3 (State Problem \( P(\alpha) \))**
The state problem \( P(\alpha) \) for a \( C^k \)-admissible shape \( \Omega(\alpha) \in O \) is defined to be given by the elasticity equation \(^6\) with \( \Omega = \Omega(\alpha) \) where \( \partial \Omega_D(\alpha) = \partial B(z, r) \) is the complete interior boundary, \( \partial \Omega_N(\alpha) = \partial B(z, r) \backslash \partial \Omega_D(\alpha) \) the exterior boundary, and \( \nu \) is the normal of \( \partial \Omega_N(\alpha) \). See Figure 8 too. We choose \( V(\Omega(\alpha)) = [C^3(\Omega(\alpha))]^3 \) as state space \(^6\) for \( P(\alpha) \).

As the crucial step in this section, we use the Schauder estimates of Theorem 9.3 in \(^2\) and validate if the corresponding assumptions are satisfied. At first, we present two lemmas which show the existence of sufficiently regular hemisphere transformations — confer Definition A.1 — and the validity of a certain inequality, respectively. These two lemmas will be used in the proof for the next theorem. In the following statements Banach spaces \( C^{q, \phi} \) of Hölder continuous \( C^q \)-functions for \( \phi \in (0, 1) \) occur, whose definition can e.g. be found in Section 1 of \(^7\) and in Section 7 of \(^2\).

**Lemma 5.4** Each \( \Omega \in O \) satisfies a hemisphere property where the corresponding hemisphere transformations are of class \( C^{3, \phi} \) for \( \phi \in (0, 1) \) and have a uniform bound \( \kappa \) with respect to \( O \).

**Proof:** Regarding Definition A.1, we have to show that every \( x \in \Omega \) within a certain distance \( d > 0 \) of \( \partial \Omega \) has a neighborhood \( U_x \) with \( B(x, d/2) \subset U_x \) and

\[
\overline{U_x} \cap \overline{\Omega} = T_x(\Sigma_{R(x)}), \quad 0 < R(x) \leq 1, \quad \overline{U_x} \cap \partial \Omega = T_x(F_{R(x)}),
\]

for some hemisphere \(^9\) \( \Sigma_{R(x)} \) and transformations \( T_x, T_x^{-1} \) of class \( C^{3, \phi} \). At first, we consider \( x \in \Omega \) within a sufficiently small distance \( d > 0 \) of \( \Gamma(\alpha) \), where \( \Gamma(\alpha) \) denotes the portion of \( \partial \Omega = \partial B(\alpha) \) which is determined by the design variable \( \alpha \), see Figure 8.

\(^3\)\(C^4\) is needed for Theorem 5.2. Therefore, set \( k = 4 \) in the definition of \( O \).

\(^6\)Note that this decomposition of the boundary depends continuously on \( \alpha \in U^{ad} \) and the two-dimensional Lebesgue measure of \( \partial \Omega_D(\alpha) \) is greater than a positive constant for all \( \alpha \in U^{ad} \).

\(^7\)This allows to analyze third order local crack initiation models.

\(^8\)\(F_{R(x)}\) denotes the flat boundary of the hemisphere \( \Sigma_{R(x)} \).
Because of the definition of the basic design and of the admissible shapes $\alpha \in U^{ad}$ there is a $C^4$-extension $\alpha^{ext} : \Omega^{ext}_{2d} \to \mathbb{R}$ of $\alpha$ with $\Omega_{2d} \subset \Omega^{ext}_{2d}$ which describes a portion $\Gamma^{ext}(\alpha)$ of the boundary $\partial \Omega(\alpha)$ beyond $\Gamma(\alpha)$ and where $\Omega^{ext}_{2d}$ is the image of a $C^4$-diffeomorphism $\tilde{T}_{2d} : B(0, R) \subset \mathbb{R}^2 \to \Omega^{ext}_{2d}$ for some $R > 0$. This extension is needed in order to consider all $x \in \Omega(\alpha)$ within a sufficiently small distance $d > 0$ of $\Gamma(\alpha)$. Now, we are able to define a hemisphere transformation with the required properties:

$$T_x : \Sigma_R \to \mathbb{R}^2, \quad T_x(x_1, x_2, x_3) = \begin{pmatrix} (\tilde{T}_{2d}(x_1, x_2))_1 \\ (\tilde{T}_{2d}(x_1, x_2))_2 \\ \alpha^{ext}((\tilde{T}_{2d}(x_1, x_2))_1, (\tilde{T}_{2d}(x_1, x_2))_2) - x_3 \end{pmatrix},$$

(15)

see Figure 4 too. The neighborhood $U_x$ can be chosen so that $U_x \cap \Omega = T_x(\Sigma_R)$ and $U_x \cap \partial \Omega = T_x(F_R)$. This can be achieved by sufficiently expanding $T_x(\Sigma_R)$ beyond $\alpha^{ext}$. Because of the definition of the basic design and of the design variables $\alpha$ one can find a bound for the norms of the hemisphere transformations which is valid for all $\alpha \in U^{ad}$. Analogously the remaining hemisphere transformations can be constructed which are of a finite number and all have a uniform bound denoted by $\kappa$, confer the last part of the proof of Lemma 6.2 too.

The following lemma contains an inequality which can be found in Section 7 of [1]. Using only a few additional technical arguments, a statement about the inequality’s constant $C$ can be added regarding its dependency on cone properties of the underlying domain $\Omega$.

**Lemma 5.5** Let $\mathcal{M}$ be a set of bounded domains in $\mathbb{R}^n$ with a uniform cone property and let $\Omega \in \mathcal{M}$. Then, for every $\varepsilon > 0$ there is a $C(\varepsilon) > 0$ uniform with respect to $\mathcal{M}$ such that $\|v\|_{C^0(\Omega)} \leq \varepsilon \|v\|_{C^1(\Omega)} + C \int_{\Omega} |v| dx$ holds for all $v \in C^1(\Omega)$.

**Proof:** Without loss of generality let $v$ be non-negative. Furthermore, assume that there is a $x_0$ with $v(x_0) = \|v\|_{C^0(\Omega)}$. The case that there is no such $x_0$ can be treated similar. Consider
the cone with height $\|v\|_{C^0(\Omega)}$ and ground area $B(x_0, R)$ with

$$R = \min \left( \frac{\|v\|_{C^0(\Omega)}}{n \cdot \|Dv\|_{C^0(\Omega)}}, C_1(\theta, h, r, n) \right)$$

for some constant $C_1(\theta, h, r, n)$. Figure 5 shows for the one-dimensional case that the ratio $\|v\|_{C^0(\Omega)} / \|Dv\|_{C^0(\Omega)}$ can be used as radius for a cone which possesses the height $\|v\|_{C^0(\Omega)}$ and a segment that is included under the graph of $v$ if $\Omega = [a, b]$ is sufficiently large. Constant $C_1(\theta, h, r, n)$ considers the case when $\Omega$ is not sufficiently large. Thus, it follows from the cone property that there is a constant $C_2(\theta, h, r, n)$ such that a segment of the cone fits under the graph of $v$ and has the volume

$$\min \left( \frac{\|v\|_{C^0(\Omega)}^n}{\|Dv\|_{C^0(\Omega)}^n}, 1 \right) C_2(\theta, h, r, n) \cdot \|v\|_{C^0(\Omega)}$$

which is a lower bound of the integral $\int_{\Omega} v dx$.

![Figure 5: Cone segment under the graph of $v$: The corresponding ground area is determined by the greatest increase of $v$ and the cone property of $\Omega$. If $\Omega$ is not sufficiently large a truncated cone segment has to be considered.](image)

If the minimum equals the first argument we show that for every $\varepsilon > 0$ there exists a $C(\varepsilon, \theta, h, r, n) > 0$ such that

$$\|v\|_{C^0(\Omega)} \leq \varepsilon (\|v\|_{C^0(\Omega)} + \|Dv\|_{C^0(\Omega)}) + C \frac{\|v\|_{C^0(\Omega)}^{n+1}}{\|Dv\|_{C^0(\Omega)}^{n}}$$

which is equivalent to $1 \leq \varepsilon (1 + t) + Ct^{-n}$ for $t = \|Dv\|_{C^0(\Omega)} / \|v\|_{C^0(\Omega)}$. The function on the right side has its global minimum at $t_0 = \frac{n+1}{\sqrt{nC/\varepsilon}}$ for positive values of $t$. Because the minimum is

$$\varepsilon \left( 1 + \frac{n+1}{\sqrt{nC/\varepsilon}} \right) + \left( \frac{n}{C^{1/n} \varepsilon} \right)^{-n/(n+1)}$$

it is greater or equal than 1 if $C \geq (n/\varepsilon)^n$.

If the minimum of the volume of the cone segment equals $C_2(\theta, h, r, n)\|v\|_{C^0(\Omega)}$ we show that for every $\varepsilon > 0$ there exists a $C(\varepsilon, \theta, h, r, n) > 0$ such that

$$\|v\|_{C^0(\Omega)} \leq \varepsilon (\|v\|_{C^0(\Omega)} + \|Dv\|_{C^0(\Omega)}) + C \|v\|_{C^0(\Omega)}$$
where $\gamma$ and $\beta$ define $\delta_{\beta\gamma}$ to be one if $\gamma = \beta$ and to be zero in any other case. A vector and a matrix of multi-indices is given by $\varrho(i)$ and $\beta(ij)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, respectively, where each $\varrho(i)$ and $\beta(ij)$ represents a multi-index. The $k$-th component is given by $\varrho(i)_{k}$ and $\beta(ij)_{k}$, respectively.

**Theorem 5.6** Let the state problem $\mathcal{P}(\alpha)$ be given on a domain $\Omega(\alpha) \in \mathcal{O}$. Suppose that the Lamé coefficients are constants. Moreover, let $\varrho \in [C^{1,\varphi}(\Omega)^{n}]^{3}$ and $g \in [C^{2,\varphi}(\Omega)^{3}]^{n}$ and $\varphi \in (0,1)$. Then, there are hemisphere transformations of class $C^{5,\varphi}$ with uniform bound $\kappa$ and a unique solution $u \in V(\Omega(\alpha))$ of $\mathcal{P}(\alpha)$ which also belongs to $[C^{3,\varphi}(\Omega(\alpha))]^{3}$ and satisfies

$$
\|u\|_{C^{3,\varphi}(\Omega)^{3}} \leq C \left( \|f\|_{C^{1,\varphi}(\Omega)^{3}} + \|g\|_{C^{2,\varphi}(\Omega)^{3}} + \|u\|_{C^{0}(\Omega)^{3}} \right)
$$

(16)

for any $\varphi \in (0,\varphi)$ and some constant $C$. The term $\|u\|_{C^{0}(\Omega)^{3}}$ can be replaced by $\int_{\Omega} |u| dx$ and $C$ can be chosen uniformly with respect to $\mathcal{O}$.

**Proof:** Let first $\varrho \in [C^{2,\varphi}(\Omega^{ext})]^{3}$ such that its restriction to any $\Omega(\alpha)$ is in $W^{2,p}(\Omega(\alpha))$ for $p > 1$ arbitrary. This additional condition will be eliminated in Lemma 5.8 below.

The existence of a unique solution $u \in V(\Omega(\alpha))$ is a consequence of Theorem 5.6 because $\Omega(\alpha)$ has $C^{1}$-boundary we have a unique solution $u \in [W^{4,p}(\Omega(\alpha))]^{3}$ for arbitrary $p \geq 6/5$. Then, the general Sobolev inequalities – confer Section 5.6 in [20] – will lead to $u \in [C^{3,\varphi}(\Omega(\alpha))]^{3}$ if $p$ is sufficiently large [12].

Now, we show that the assumptions for the Schauder estimates of Theorem 9.3 in [2] are satisfied. Main assumptions are complementing and supplementary boundary conditions, uniform ellipticity with the corresponding constant $\kappa$, and the existence of hemisphere transformations of class $C^{5,\varphi}$ with corresponding norms uniformly bounded by a constant $\kappa$, confer Sections 1, 2, 7 and 9 of [2] for a detailed description. We start by rewriting the equations of $\mathcal{P}(\alpha)$ in the form of $\sum_{j=1}^{3} l_{ij}(x, \nabla) u_{j}(x) = f_{i}(x)$ for $x \in \Omega$, $i = 1, 2, 3$ and of $\sum_{j=1}^{3} B_{hj}(x, \nabla) u_{j}(x) = g_{h}(x)$ for $x \in \partial \Omega_{N}$ and $h = 1, 2, 3$, see (1.1) and (2.1) in [2]. Therefore, we write

$$
l_{ij}(x, \Xi) = \sum_{|\varrho|=0}^{2} a_{ij,\varrho}(x) \Xi^{\varrho} \quad 
= \sum_{|\varrho|=2} \left[ \mu (\delta_{ij} (\delta_{\varrho(2,0,0)} + \delta_{\varrho(0,2,0)} + \delta_{\varrho(0,0,2)}) + \delta_{\varrho(ij)}) + \lambda \sum_{\beta(i), \gamma > 0} \delta_{\varrho\beta\gamma} \right] \Xi^{\varrho},
$$

(17)

where $\gamma(ij) = (\delta_{i1} + \delta_{1j}, \delta_{2i} + \delta_{j2}, \delta_{3i} + \delta_{3j})$ and where $\sum_{\beta(i), \gamma > 0}$ is the sum over all multi-indices with $\beta(i), \gamma > 0$. Corresponding to the Neumann boundary conditions on $\partial \Omega_{N}$ we write

$$
B_{hj}(x, \Xi) = \sum_{|\varrho|=0}^{1} b_{hj,\varrho} \Xi^{\varrho} = \sum_{|\varrho|=1} (\lambda \mu_{h}(x) \delta_{\varrho\beta(j)} + \delta_{hj} \mu_{\varrho\beta} \Xi^{\varrho} + \mu_{\varrho\beta} \Xi^{\varrho} \delta_{\varrho\beta(h)}),
$$

(18)

Note that $f \mid_{\Omega(\alpha)}$ and $g \mid_{\Omega(\alpha)}$ are continuously differentiable.

Recall Definition [A.1] where also constant $\varphi$ is defined.
where \( \beta(1) = (1,0,0), \beta(2) = (0,1,0), \beta(3) = (0,0,1) \), \( \nu_{(1,0,0)} = \nu_1, \nu_{(0,1,0)} = \nu_2, \nu_{(0,0,1)} = \nu_3 \).

Regarding Theorem 9.3 in [2] and equations (17) and (15), we have \( s_1 = 0, t_j = 2 \) and \( r_h = -1 \) for the Neumann condition and \( r_h = -2 \) for the the Dirichlet condition\(^{13}\) which implies \( l_0 = \max\{0, r_h\} = 0 \) and allows to choose \( l = l_0 \) for \( i,j,h \in \{1, 2, 3\} \).

Consider that the complementing conditions and the supplementary boundary conditions are satisfied, confer Section 6.3 of [13]. The existence of appropriate hemisphere transformations of class \( C^{3,\phi} \) is shown in Lemma 5.4, where the constants \( d, \kappa \) can be chosen uniformly with respect to \( O \). According to Section 1.3 of [13] the minor constant \( \Delta_{\partial \Omega} \) is positive and determined by the Lamé coefficients which are constant in our case.

Now, we analyze the effect of the hemisphere transformation \( T_x \) on the ellipticity constant and follow Section 9 of [2]. Let \( c(\nabla) = \sum_{\phi} c_\phi \nabla^\phi = \sum_{i_1,i_2,...} c_{i_1,i_2,...} \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \) be an arbitrary linear combination of differentiation operators. It is transformed into \( \hat{c}(\hat{\nabla}) = \sum_{i_1,i_2,...} \hat{c}_{i_1,i_2,...} \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \), where \( \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \) and each \( \hat{c}_{i_1,i_2,...} \) is a linear combination of the \( c_{i_1,i_2,...} \) with coefficients that are products of the \( \frac{\partial x_{i_1}}{\partial x_{i_1}}, \frac{\partial x_{i_2}}{\partial x_{i_2}} \cdots \). Correspondingly, we obtain \( c(\xi) = \hat{c}(\hat{\xi}) \) with \( \hat{\xi} = \frac{\partial x}{\partial x_{i_1}} \xi_i \). According to Section 1 in [2] uniform ellipticity is described by the inequality \( A^{-1} |\xi|^{2m} \leq |L(x, \Xi)| \leq A |\xi|^{2m} \) for all \( \Xi \in \mathbb{R}^{n+1} \) and all \( x \in \Omega \), where \( L(x, \Xi) \) is the characteristic determinant of the PDE-system. The determinant is invariant under the hemisphere transformation in the sense that \( \hat{L}(\hat{x}, \hat{\xi}) = L(x, \xi) \). As the first derivatives of \( T_x \) and \( T_x^{-1} \) exist and as these maps are in \( x \) uniformly bounded with respect to the norm \( \| \cdot \|_{C^{3,\phi}} \) there is a constant \( \omega \) such that \( \omega^{-1} \| \xi \|_2 \leq \| \hat{\xi} \|_2 \leq \omega \| \xi \|_2 \) for every \( x \in \mathcal{A} \), \( \xi \in \Sigma_{R(x)} \) and \( \hat{\xi} = T_x(\xi) \). Confer Chapter I.6.2 of [23], too. Finally, this results in the uniform ellipticity of the transformed system with the new ellipticity constant \( A\omega^{2m} \).

Applying the Schauder estimates to both cases of Neumann boundary and homogeneous Dirichlet conditions yields the inequality statement, which even holds for \( 0 < \varphi < \phi \). A uniform choice of the constant \( C \) in (16) with respect to \( O \) is justified by the previous analysis of the constants \( d, \kappa, \Delta_{\partial \Omega} \) and \( A \). The replacement of \( \| u \|_{[C^{0}(\Omega)]^3} \) by \( \int_{\Omega} |u| \, dx \) is a consequence of a uniform cone property of \( O \) and of Lemma 5.5\(^{14}\).

By means of the following theorem and the properties of \( C^k \)-admissible domains we can show compactness of \( G = \{ (\Omega, u(\Omega)) \mid \Omega \in \mathcal{O} \} \) where \( u(\Omega) \) uniquely solves \( \mathcal{P}(\Omega) \).

**Theorem 5.7** There is a positive constant \( C \) such that the solution \( u \in V(\Omega(\alpha)) \) of the previous theorem satisfies \( \| u \|_{[C^{3,\varphi}(\Omega)]^3} \leq C \) for any \( \varphi \in (0, \phi) \), where \( C \) can be chosen uniformly with respect to \( O \) and forces \( f \in [C^{1,\varphi}(\Omega^{\text{ext}})]^3, g \in [C^{2,\phi}(\Omega^{\text{ext}})]^3 \).

**Proof:** First note that the norms \( \| f \|_{[C^{1,\varphi}(\Omega)]^3} \) and \( \| g \|_{[C^{2,\phi}(\Omega^{\text{ext}})]^3} \) in (16) are uniformly (in \( \Omega \in \mathcal{O} \)) bounded by \( \| f \|_{[C^{1,\varphi}(\Omega^{\text{ext}})]^3} \) and \( \| g \|_{[C^{2,\phi}(\Omega^{\text{ext}})]^3} \), respectively.

The main part of the proof thus consists of showing that there is a constant \( C \) independent of \( \Omega \in \mathcal{O} \) such that \( \| u \|_{H^{1}(\Omega(\alpha))} \leq C \). We follow ideas of the proof of Lemma 2.24 in [25].

The weak formulation (13) of \( \mathcal{P}(\alpha) \) can be rewritten in the form
\[
\int_{\Omega} \text{tr}(\sigma(u) \varepsilon(v)) \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\partial \Omega} g \cdot v \, dA \quad \forall v \in V_{DN},
\]
\(^{13}\)Note that homogeneous Dirichlet conditions on \( \partial \Omega_D \) lead to \( B_{h_j}(x, \Xi) = \delta_{h_j}, g_h(x) = 0 \) for all \( x \in \partial \Omega_D \) and to \( r_h = -2 \).

\(^{14}\)Recall that we will treat the case \( f \in [C^{1,\varphi}(\Omega^{\text{ext}})]^3 \) in Lemma 5.8 where then \( 0 < \varphi < \phi \) is required.
where \( \sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl} \xi_{kl} \). The ellipticity constant \( C_{ijkl} = \delta_{ij} \delta_{kl} \lambda + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \) is given by the Lamé coefficients \( \lambda, \mu \) and so a constant element of \( C^{\alpha}(\bar{\Omega}_{\text{ext}}) \). Moreover, the constant satisfies the symmetries \( C_{ijkl} = C_{jikl} = C_{klij} \) and there exists a constant \( q > 0 \) such that \( C_{ijkl}(x) \xi_{ij} \xi_{kl} \geq q \xi_{ij} \xi_{kl} \) for all \( x \in \bar{\Omega}_{\text{ext}} \). This results in

\[
B_\alpha(v, v) = \int_{\Omega(\alpha)} \text{tr}(\sigma(v) \varepsilon(v)) \, dx \geq q \int_{\Omega(\alpha)} \text{tr}(\varepsilon(v)^2) \, dx = q \| \varepsilon(v) \|_{L^2(\Omega(\alpha))}^3
\]

for all \( v \in V_{DN} \). Because of the assumptions for \( f \) and \( g \)

\[
|L_\alpha(v)| = \left| \int_{\Omega(\alpha)} f \cdot v \, dx + \int_{\partial \Omega_\alpha} g \cdot v \, dA \right| \leq C \| v \|_{H^1(\Omega(\alpha))}^3
\]

holds and the constant is uniform in \( \alpha \in U^{rad} \) as a result of the uniform bounds on \( \| f \|_{C^{1,\phi}(\Omega)}^3 \) and \( \| g \|_{C^{2,\phi}(\Omega)}^3 \). This independence is a consequence of \( \| v \|_{H^2(\Omega)} \leq C \| v \|_{H^1(\Omega)} \) with \( C \) depending only on the uniform Lipschitz constant of the boundary, confer \([34]\). By the same reason, the volume of \( \partial \Omega \) is uniformly bounded. The previous inequalities and the weak equation \([13]\) lead to \( q\|\varepsilon(u)\|_{H^0(\Omega(\alpha))}^3} \leq C \| u \|_{H^1(\Omega(\alpha))}^3 \). This and Korn’s second inequality \([13]\) imply

\[
q\|\varepsilon(u)\|_{H^0(\Omega(\alpha))}^3} \leq C \| u \|_{H^1(\Omega(\alpha))}^3 \leq C \| \varepsilon(u) \|_{H^0(\Omega(\alpha))}^3},
\]

where \( C \) also depends on the constant of Korn’s second inequality. As \([30]\) shows for the local epigraph parametrization, this constant is uniform with respect to a class of domains possessing a uniform cone property. Applying once more Korn’s second inequality one obtains \( \| u \|_{H^1(\Omega(\alpha))}^3 \leq C \) with \( C \) independent of \( \alpha \).

This result, the inequality \( \| v \|_{L^3(\Omega)} \leq \sqrt{\text{vol}(\Omega)} \| v \|_{L^2(\Omega)} \) for \( v \in L^2(\Omega) \) and the results of the previous theorem show the statement of this theorem.

The following lemma closes the gap left open in the proof of Theorem 5.6 as so far we have only proven these statements for \( f \in [C^{2,\phi}(\bar{\Omega}_{\text{ext}})]^3 \) in order to be able to apply Theorem 5.2—see the beginning of the proof of Theorem 5.6. Since the right hand side of the Schauder estimate \([16]\) only depends on the \( C^{1,\phi} \)-norm of \( f \), it is possible to overcome this restriction:

**Lemma 5.8** Suppose that the statements of Theorem 5.6 and Theorem 5.7 hold for \( f \in [C^{2,\phi}(\bar{\Omega}_{\text{ext}})]^3 \). Then, they extend to \( f \in [C^{1,\phi}(\bar{\Omega}_{\text{ext}})]^3 \) with the same uniform constant \( C \).

**Proof:** Finally, we have to consider the case \( f \in [C^{1,\phi}(\bar{\Omega}_{\text{ext}})]^3 \). Then, there exists a sequence \( (f_n)_{n \in \mathbb{N}} \subset [C^{2,\phi}(\bar{\Omega}_{\text{ext}})]^3 \) such that \( f_n \to f \) in \([C^{1,\phi}(\bar{\Omega}_{\text{ext}})]^3 \) as \( n \to \infty \). Given \( \Omega(\alpha) \), denote the sequence of solutions to \( \mathcal{P}(\alpha)_n \) with volume force \( f_n \) by \( u_n, n \in \mathbb{N} \). We can now apply Theorem 5.7 to \( u_n \) with \( \varphi = \phi \) due to \( f_n \in [C^{2,\phi}(\bar{\Omega}_{\text{ext}})]^3 \), confer the last paragraph in the proof of Theorem 5.6. Thus, \( (u_n)_{n \in \mathbb{N}} \) is uniformly bounded in \([C^{3,\phi}(\bar{\Omega}_{\text{ext}})]^3 \). According to compact embeddings of Hölder spaces (a consequence of Arzelà Ascoli), there is a subsequence \( (u_{nk})_{n \in \mathbb{N}} \) such that \( u_{nk} \to u \) in \([C^{3,\phi}(\bar{\Omega}_{\text{ext}})]^3 \) for any \( \varphi \in (0, \phi) \). We deduce that \( u \) fulfills \( \mathcal{P}(\alpha) \) due to pointwise convergence of first and second derivatives. As the Schauder estimate \([16]\) holds for all \( u_{nk} \), it carries over to \( u \) by \( C^{3,\phi} \)-continuity in \( u \) of both sides and the \( C^{1,\phi} \)-continuity in \( f \) of the right hand side of that inequality.
6 Existence of Optimal Shapes

In this final section, we exploit the results of the previous section in order to prove existence of solutions to shape optimization problems which are given by a very general class of cost functionals. These cost functionals are not constrained by convexity assumptions and the state problems are described by mixed problems of linear elasticity, see Definition 5.3. We optimize shapes within the family of \(C^k\)-admissible domains. The class of shape optimization problems is large enough to include those originating from optimal reliability of local crack initiation models up to third order, confer Lemma 2.7 and Definition 2.6.

Since the cost functionals include surface integrals which lead to a loss of regularity according to the trace theorem and (higher) derivatives of \(u\), we have to resort to strong regularity of solutions. As already announced in Section 4 only a part of the boundary is subject of optimization. The abstract setting of Section 4 and Theorem 4.2 determine the structure of this section which leads to our existence results.

As our state space \(V(\Omega(\alpha))\) is \([C^3(\Omega(\alpha))]^3\), we employ the following definition with \(q = 3\) for the convergence of functions with variable domains in \(\mathcal{O}\), also confer Section 2.5.2 in [20]. Following the previous section, we consider \(C^4\)-admissible shapes \(\mathcal{O}\).

**Definition 6.1 (\(C^q\)-Convergence of Functions with Variable Domains)**
Recalling the sets \(\mathcal{O}\) and \(\Omega^{\text{ext}}\) of Definition 4.3 let \(p_Q : [C^q(\Omega)]^3 \to [C^q(\Omega^{\text{ext}})]^3\) be the extension operator which can be derived from Lemma 4.4 for \(q \in \mathbb{N} \setminus \{0\}\). For \(u \in [C^q(\Omega)]^3\) set \(u^{\text{ext}} = p_Q u\). For \((\Omega_l)_{l \in \mathbb{N}} \subset \mathcal{O}\), \(\Omega \in \mathcal{O}\) and \((u_l)_{l \in \mathbb{N}}\) with \(u_l \in [C^q(\Omega)]^3\), \(l \in \mathbb{N}\), and \(u \in [C^q(\Omega)]^3\) the expression \(u_l \to u\) as \(l \to \infty\) is defined by \(u_l^{\text{ext}} \to u^{\text{ext}}\) in \([C^q(\Omega^{\text{ext}})]^3\).

**Notation (Local Cost Functional)**
Find an optimal shape \(\Omega(\alpha) \in \mathcal{O}\) which minimizes a local functional of the form \(J(\Omega, u) = J_{\text{vol}}(\Omega, u) + J_{\text{sur}}(\Omega, u)\) with

\[
J_{\text{vol}}(\Omega, u) = \int_{\Omega} F_{\text{vol}}(x, u, \nabla u, \nabla^2 u, \nabla^3 u) \, dx,
\]
\[
J_{\text{sur}}(\Omega, u) = \int_{\partial \Omega} F_{\text{sur}}(x, u, \nabla u, \nabla^2 u, \nabla^3 u) \, dA
\]

(19)

and \(u\) uniquely given by \(\Omega(\alpha)\) as the solution of the state problem \(\mathcal{P}(\alpha)\).

**Lemma 6.2** Let the setting of Theorem 6.1 be given on an arbitrary sequence of domains \((\Omega(\alpha_n))_{n \in \mathbb{N}} \subset \mathcal{O}\). For \(\phi \in (0, 1)\) let \(f \in [C^{1,\phi}(\Omega^{\text{ext}})]^3\) and \(g \in [C^{2,\phi}(\Omega^{\text{ext}})]^3\). Let \((\alpha_n, u_n)_{n \in \mathbb{N}}\) be a sequence of admissible shapes \(\alpha_n \in \mathcal{U}_{\text{ad}}\) and of the corresponding solutions \(u_n \in V(\Omega(\alpha_n))\) of \(\mathcal{P}(\alpha_n)\). Then, there is a subsequence \((\alpha_{n_k}, u_{n_k})_{k \in \mathbb{N}}\) such that \(\Omega(\alpha_{n_k}) \stackrel{\cdot}{\to} \Omega(\alpha)\) and \(u_{n_k} \to u\) as \(n_k \to \infty\) for some \(\alpha \in \mathcal{U}_{\text{ad}}\) and for the corresponding solution \(u \in V(\Omega(\alpha))\) of \(\mathcal{P}(\alpha)\).

**Proof:** Due to Lemma 6.1 there is a subsequence \((\alpha_{n_i})_{i \in \mathbb{N}}\) with \(\Omega(\alpha_{n_i}) \stackrel{\cdot}{\to} \Omega(\alpha)\) as \(i \to \infty\) for some \(\alpha \in \mathcal{U}_{\text{ad}}\). According to Theorem 6.1 \(u_{n_i} \in [C^{3,\varphi}(\Omega(\alpha_{n_i}))]^3\) holds for some \(\varphi \in (0, \phi)\) and every \(n_i \in \mathbb{N}\). Moreover, according to Theorem 5.2 there is a constant \(C > 0\) independent

15 Confer Appendix B.3.5 in [20] for more details.
16 The uniqueness of \(u\) is realized by Theorem 4.6.
of every $\Omega \in \mathcal{O}$ such that $\|u_{n_k}\|_{C^3,\varphi}(\Omega) \leq C$. Because of Lemma \[A.4\] there is a constant $C$ such that $u_{n_k}^\text{ext} = p(\alpha_{n_k})u_{n_k}$ satisfies $\|u_{n_k}^\text{ext}\|_{C^3,\varphi}(\Omega) \leq C\|u_{n_k}\|_{C^3,\varphi}(\Omega)$. The constant can be chosen to be independent of the design variables $\alpha$ which we will show at the end of the proof. The previous inequalities result in a uniform bound for all $\|u_{n_k}^\text{ext}\|_{C^3,\varphi}(\Omega)$. As the unit ball in $C^3,\varphi(\Omega)$ is compact in $C^3(\Omega)$ due to the Arzela-Ascoli theorem, we find a further subsequence $(\alpha_{n_k}, u_{n_k})_{n_k \in \mathbb{N}}$ such that

$$\Omega(\alpha_{n_k}) \overset{C}{\longrightarrow} \Omega(\alpha), \quad u_{n_k}^\text{ext} \overset{C^3}{\longrightarrow} v \quad \text{as} \quad n_k \to \infty \quad \text{for some} \quad v \in [C^3(\Omega)]^3.$$  

Because of $u_{n_k}^\text{ext} \overset{C^3}{\longrightarrow} v$ as $n_k \to \infty$ the function $v_{\Omega(\alpha)}$ solves $P(\alpha)$. According to Theorem \[5.2\] $P(\alpha)$ has a unique solution $u \in V(\Omega)$ and $v$ is an extension $u^\text{ext}$ of $u$.

Finally, we show the statement that the constant $C$ of Lemma \[A.4\] can be chosen uniformly with respect to the set of admissible shapes $\mathcal{O}$. The proof of this extension lemma, confer the proof of Lemma 6.37 in \[23\], is based on $C^{k,\varphi}$-diffeomorphisms $\psi$ that locally straighten the boundary $\partial\Omega$. For $u \in C^{k,\varphi}(\Omega)$ one considers $\tilde{u}(y) = u \circ \psi^{-1}(y)$, where $\psi = (y_1, \ldots, y_{n-1}, y_n) = (y', y_n)$ and $y_n > 0$. An extension into $y_n < 0$ can then be defined by

$$\tilde{u}(y', y_n) = \sum_{i=1}^{k+1} c_i \tilde{u}(y', -y_n/i), \quad y_n < 0,$$

where $c_1, \ldots, c_{k+1}$ are determined by the equations $\sum_{i=1}^{k+1} c_i (-1/i)^m = 1$ with $m = 0, \ldots, k$. The proof in \[23\] then shows that $w = \tilde{u} \circ \psi$ provides a $C^{k,\varphi}$ extension of $u$ into $\Omega \cup B$ for some balls $B$ and corresponding $C^{k,\varphi}$-diffeomorphisms $\psi$. Considering a finite-covering argument of $\partial\Omega$ and a partition of unity (subordinate to this covering) prove the existence of an extension $w \in C^{k,\varphi}(\Omega^\text{ext})$. The inequality

$$\|w\|_{C^{k,\varphi}(\Omega^\text{ext})} \leq C\|u\|_{C^{k,\varphi}(\Omega)} \quad (20)$$

for a constant $C$ depending only on $k$, $\Omega$ and $\Omega^\text{ext}$ is a consequence of

$$K^{-1}\|x - y\| \leq \|x' - y'\| \leq K\|x - y\|, \quad x' = \psi(x), \quad y' = \psi(y), \quad (21)$$

confer proof of Lemma 6.37 and equations (6.29) and (6.30) in \[23\]. As we are employing a hypograph representation for the admissible shapes we can subdivide the problem by first considering the part of the boundary that is fixed and then the part that varies with the admissible design variables $\alpha \in U^{\text{ad}}$. The fixed part can be treated uniformly for all admissible shapes. Thus, there are always the same $C^{k,\varphi}$-diffeomorphisms $\psi$ with respect to the admissible shapes for that part which can be used for the construction of the extension as described above. For the varying part of the boundary we explicitly give the form of the $C^{k,\varphi}$-diffeomorphisms depending on the admissible shape. Following the proof of Lemma 6.4 we obtain

$$\psi(x) = \begin{pmatrix} x_1 \\ x_2 \\ \alpha(x_1, x_2) - x_3 \end{pmatrix}, \quad \psi^{-1}(x') = \begin{pmatrix} x'_1 \\ x'_2 \\ \alpha(x'_1, x'_2) - x'_3 \end{pmatrix}.$$  

Because the norms of the admissible design variables $\alpha \in U^{\text{ad}}$ are uniformly bounded so are the norms of $\psi$ and $\psi^{-1}$ and thereby constant $K$ of \[24\] can be chosen uniformly with respect to the admissible shapes. This finally shows that $C$ of \[20\] can also be chosen uniformly.

\[\Box\]
As already mentioned we consider functionals which are volume or surface integrals with continuous integrands. In order to show continuity of the functionals we apply Lebesgue’s dominated convergence theorem.

**Lemma 6.3 (Continuity of Local Cost Functionals)**

Let \( \mathcal{F}_{\text{vol}}, \mathcal{F}_{\text{sur}} \in C^0(\mathbb{R}^d) \) (with \( d \) as in Definition 2.7 with \( r = 3 \)) and let the set \( \mathcal{O} \) only consist of \( C^0 \)-admissible shapes. For \( \Omega \in \mathcal{O} \) and \( u \in [C^3(\Omega)]^3 \) consider the volume integral \( J_{\text{vol}}(\Omega, u) \) and the surface integral \( J_{\text{sur}}(\Omega, u) \) of (19), respectively.

Let \( (\Omega_n)_{n \in \mathbb{N}} \subset \mathcal{O} \) with \( \Omega_n \xrightarrow[]{} \Omega \) as \( n \to \infty \) and let \( (u_n)_{n \in \mathbb{N}} \subset [C^3(\Omega_n)]^3 \) be a sequence with \( u_n \rightharpoonup u \) as \( n \to \infty \) for some \( u \in [C^3(\Omega)]^3 \). Then,

(i) \( J_{\text{vol}}(\Omega_n, u_n) \to J_{\text{vol}}(\Omega, u) \) as \( n \to \infty \).

(ii) If the set \( \mathcal{O} \) only consists of \( C^1 \)-admissible shapes one obtains \( J_{\text{sur}}(\Omega_n, u_n) \to J_{\text{sur}}(\Omega, u) \) as \( n \to \infty \) as well.

**Proof:** (i) At first, we consider the volume integral. Using the characteristic function one obtains

\[
J_{\text{vol}}(\Omega_n, u_n) = \int_{\Omega_n} \chi_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, u_n^\text{ext}, \nabla u_n^\text{ext}, \nabla^2 u_n^\text{ext}, \nabla^3 u_n^\text{ext}) \, dx.
\]

Because of \( \mathcal{F}_{\text{vol}} \in C^0(\mathbb{R}^d) \) and \( u_n \rightharpoonup u \) as \( n \to \infty \) there is a constant \( C > 0 \) such that the inequality \( |\chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^\text{ext}(x), \nabla u_n^\text{ext}(x), \nabla^2 u_n^\text{ext}(x))| \leq C \) holds for all \( n \in \mathbb{N} \) almost everywhere in \( \Omega_n^\text{ext} \). Moreover, \( \Omega_n \xrightarrow[]{} \Omega \) and \( u_n^\text{ext} \to u^\text{ext} \) in \([C^3(\Omega^\text{ext})]^3 \) and \( \mathcal{F}_{\text{vol}} \in C^0(\mathbb{R}^d) \) ensure the existence of

\[
\lim_{n \to \infty} \left( \chi_{\Omega_n}(x) \cdot \mathcal{F}_{\text{vol}}(x, u_n^\text{ext}(x), \nabla u_n^\text{ext}(x), \nabla^2 u_n^\text{ext}(x)) \right) = \chi(x) \cdot \mathcal{F}_{\text{vol}}(x, u^\text{ext}(x), \nabla u^\text{ext}(x), \nabla^2 u^\text{ext}(x), \nabla^3 u^\text{ext}(x))
\]

for all \( x \in \Omega^\text{ext} \). As the integrands are pointwise and uniformly in \( \Omega^\text{ext} \) bounded, Lebesgue’s dominated convergence theorem can now be applied to permute integral and limit:

\[
\lim_{n \to \infty} J_{\text{vol}}(\Omega_n, u_n) = \lim_{n \to \infty} \int_{\Omega_n} \chi_{\Omega_n} \mathcal{F}_{\text{vol}}(x, u_n^\text{ext}, \nabla u_n^\text{ext}, \nabla^2 u_n^\text{ext}, \nabla^3 u_n^\text{ext}) \, dx
= \int_{\Omega} \lim_{n \to \infty} \left( \chi_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, u_n^\text{ext}, \nabla u_n^\text{ext}, \nabla^2 u_n^\text{ext}, \nabla^3 u_n^\text{ext}) \right) \, dx
= \int_{\Omega} \mathcal{F}_{\text{vol}}(x, u, \nabla u, \nabla^2 u, \nabla^3 u) \, dx = J_{\text{vol}}(\Omega, u).
\]

(ii) With respect to the surface integral similar arguments can be used and we only address technical steps which are special to integrating over a surface. Because of their definition the boundary of every \( \Omega \in \mathcal{O} \) is a differentiable submanifold and the surface integral

\[
J_{\text{sur}}(\Omega_n, u_n) = \int_{\partial \Omega_n} \mathcal{F}_{\text{sur}}(x, u_n, \nabla u_n, \nabla^2 u_n, \nabla^3 u_n) \, dA
\]
Because all prerequisites of Lemma 6.3 are satisfied we obtain as \( n \to \infty \) shape \( \Omega^* \) optimal shape via Lemma 6.2 there exists a subsequence \((\alpha_n)_{n \in \mathbb{N}} \). Let \((\alpha_n)_{n \in \mathbb{N}} \) be a minimizing sequence of \( \inf \{ J(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O} \} \), where \( u_n = u(\Omega_n) = u(\Omega(\alpha_n)) \) is the unique solution of the state problem \( \mathcal{P}(\alpha) \). Then, because of Lemma 6.2 there exists a subsequence \((\alpha_{n_k}, u_{n_k})_{k \in \mathbb{N}} \) such that \( \Omega(\alpha_{n_k}) \xrightarrow{d} \Omega(\alpha) \) and \( u_{n_k} \rightharpoonup u \) as \( n_k \to \infty \) for some \( \alpha \in U^{ad} \) and for the corresponding solution \( u \in V(\Omega(\alpha)) \) of \( \mathcal{P}(\alpha) \). Since all prerequisites of Lemma 6.3 are satisfied we obtain \( J(\Omega_{n_k}, u_{n_k}) \to J(\Omega, u) \) as \( n_k \to \infty \). Because \((\alpha_{n_k}, u_{n_k})_{k \in \mathbb{N}} \) is also a minimizing sequence of \( \inf \{ J(\Omega, u(\Omega)) \mid \Omega \in \mathcal{O} \} \) the admissible shape \( \Omega^* = \Omega(\alpha) \) is an optimal shape.

Next, we apply the previous theorem to the cost functional from Lemma 2.7 and we get solutions to the optimal reliability problem as an easy corollary to Theorem 6.4.

**Theorem 6.5 (Optimal Reliability)**

Let \( \mathcal{O} \) be \( C^4 \)-admissible domains as described in Definition 4.6 and let \( \mathcal{F}_{vol} = C^1(\mathcal{O}^{ext})^3 \) and \( \mathcal{F}_{sur} = C^2(\mathcal{O}^{ext})^3 \). Let furthermore \( \gamma \) be a local crack initiation model as in Definition 2.6 with order \( r \leq 3 \) and 0-regular intensity functions \( q_{vol} \) and \( q_{sur} \). Let \( T \) be failure time defined as the formation of the first crack associated to \( \gamma \). Let furthermore \( t^* \in \mathcal{T} \) be given.

(i) Then, there exists at least one shape \( \Omega^* \in \mathcal{O} \) that minimizes the probability of failure up to time \( t^* \), confer Definition 2.7.

---

\( \mathcal{O} \) consists of \( C^4 \)-admissible shapes.
(ii) In particular, this applies to the local Weibull model for LCF, confer Definition 3.3.

Proof: (i) As all prerequisites of Theorem 6.4 are satisfied by Lemma 2.7 we directly obtain the existence of an optimal shape $\Omega^* \in \mathcal{O}$ from Theorem 6.4.

(ii) Confer Proposition 3.4 and note that for $\Omega \in \mathcal{O}$ the condition $\frac{1}{N_{\text{det}}}(\nabla u(\Omega)) \in L^m(\partial \Omega)$ is fulfilled since $N_{\text{det}}(\nabla u(\Omega))$ is continuous and hence bounded on $\partial \Omega$. Now apply (i).

The strong convergence properties underlying Theorem 6.4 can also be applied to cost functionals that do not belong to the class (19). As an example, one can also prove existence of solutions to the deterministic optimal LCF lifting problem.

**Theorem 6.6 (Deterministic LCF and Shape Optimization)**

Under the setting of Theorem 6.5 there exists a solution $\Omega^* \in \mathcal{O}$ to the deterministic optimal LCF lifting problem of Definition 3.2.

Proof: Define the cost functional $J_{\text{sur}}(\Omega, u) = -\inf_{x \in \partial \Omega} N_{\text{det}}(\nabla u(x)) = -T_{\text{det}, \Omega}$ on $\mathcal{O} \times \left[ C^{3,\phi}(\Omega) \right]$ that needs to be minimized. The proof of Theorem 6.4 yields all arguments needed except for the continuity property of $J_{\text{sur}}$. But this follows from uniform convergence required in Definition 6.1.

We have proven the existence of optimal designs for a general class of cost functionals without convexity constraints and including higher order derivatives of the state solutions. The technically relevant case of optimal reliability for probabilistic models of LCF and its deterministic counterpart is included. In [24], we address sensitivity analysis and the existence of shape gradients in the probabilistic framework.

### Appendix

**Definition A.1 (Hemisphere Property, Hemisphere Transformation)**

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\Gamma$ be a regular portion of $\partial \Omega$ and let $A \subset \Omega$ be a subdomain such that $\partial A \cap \partial \Omega$ is in the interior of $\Gamma$ in the $n$-dimensional sense and let $d > 0$ be a positive constant. If every $x \in A$ within a distance $d$ of $\partial \Omega$ has a neighborhood $U_x$ with $U_x \cap \partial \Omega \subset \Gamma$, $B(x, d/2) \subset U_x$ and

$$U_x \cap \Omega = T_x(\Sigma_R(x)), \quad U_x \cap \partial \Omega = T_x(F_R(x)), \quad 0 < R(x) \leq 1$$

for some hemisphere $\Sigma_R(x)$ and functions $T_x, T^{-1}_x$ of some class $C^{k,\phi}$, $A$ is said to satisfy a hemisphere property. Moreover, the functions $T_x$ are called hemisphere transformations.

**Definition A.2 (C^{k,\phi}-Boundary, Lipschitz Boundary)**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $k \in \mathbb{N}$ and $0 \leq \phi \leq 1$. If every $x_0 \in \partial \Omega$ there exists $B = B(x_0, r)$ for some $r > 0$ and an injective $C^{k,\phi}$-map $\psi : B \rightarrow D \subset \mathbb{R}^n$ such that $\psi(B \cap \partial \Omega) \subset \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \}$, $\psi(B \cap \partial \Omega) \subset \partial \mathbb{R}^n_+$ and $\psi^{-1} \in C^{k,\phi}(D)$. If the maps $\psi$ are only Lipschitz continuous $\Omega$ has a Lipschitz boundary.

**Remark A.3**

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and let $k \in \mathbb{N}$ and $0 \leq \phi \leq 1$. If every $x_0 \in \partial \Omega$ has a neighbourhood in which the boundary is locally described by a graph of a $C^{k,\phi}$-function of $n - 1$ of the coordinates $x_1, \ldots, x_n$ the domain $\Omega$ has a $C^{k,\phi}$-boundary. Note that the converse is true if $k \geq 1$. 


Lemma A.4 (Extension Lemma) \cite{23}, Part I, Section 6

Let $\Omega^{ext} \subset \mathbb{R}^n$ be open and let $\Omega$ be a $C^{k,\phi}$-domain with $\overline{\Omega} \subset \Omega^{ext}$, $k \geq 1$ and $0 \leq \phi < 1$. For $\phi = 0$ it is $C^{k,0} = C^k$. If $u \in C^{k,\phi}(\Omega)$ there is a function $w \in C^{k,\phi}_0(\Omega^{ext})$ such that $w = u$ in $\overline{\Omega}$ and $\|w\|_{C^{k,\phi}(\Omega^{ext})} \leq C\|u\|_{C^{k,\phi}(\Omega)}$ for a constant $C$ depending only on $k, \Omega$ and $\Omega^{ext}$.

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References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions I, Communications On Pure And Applied Mathematics, Vol. XII, pp. 623-727, 1959.

[2] S. Agmon, A. Douglis and L. Nirenberg, Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions II, Communications On Pure And Applied Mathematics, Vol. XVII, pp. 35-92, 1964.

[3] G. Allaire, E. Bonneter, G. Francfort and F. Jouve, Shape Optimization by the Homogenization Method, Numer. Math., 76, 1997, 27-68.

[4] G. Allaire and R. V. Kohn, Optimal design for minimum weight and compliance in plane stress using extremal microstructures, Eur. J. Mech. A Solids 12 (1993), 839-878.

[5] G. Allaire and F. Jouve, Existence of Minimizers for Non-Quasiconvex Functionals Arising in Optimal Design, Ann. I.H.P., Anal. Nonlin., 15, 3, 1998, 301-339.

[6] G. Allaire and R. V. Kohn, Optimal design for minimum weight and compliance in plane stress using extremal microstructures, Eur. J. Mech. A Solids 12 (1993), 839-878.

[7] H. Alt, Lineare Funktionalanalysis, fifth edition, Springer, Berlin, 2006.

[8] J.S. Arora, Introduction to Optimum Design, McGraw-Hill Book Company, New York, 1989.

[9] M. Bäker, H. Harders and J. Rösler, Mechanical Behaviour of Engineering Materials: Metals, Ceramics, Polymers, and Composites, German edition published by Teubner Verlag (Wiesbaden, 2006), Springer, Berlin Heidelberg 2007.

[10] A. Borzi and V. Schulz, Computational Optimization of Systems governed by Partial Differential Equations, SIAM series on computational engineering, SIAM 2012.

[11] D. Braess, Finite Elemente - Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie, fourth edition, Springer, Berlin, 2007.

[12] D. Bukur and G. Buttazzo, Variational Methods in Shape Optimization Problems, first edition, Birkhäuser, Boston, 2005.
REFERENCES

[13] P. Ciarlet, *Mathematical Elasticity - Volume I: Three-Dimensional Elasticity*, Studies in Mathematics and its Applications, Vol. 20, North-Holland, Amsterdam, 1988.

[14] M. C. Delfour and J.-P. Zolesio, *Shapes and geometries*, (2nd Ed), Advances in Design and Control, SIAM 2011.

[15] A. Douglis and L. Nirenberg, *Interior Estimates for Elliptic Systems of Partial Differential Equations*, Communications On Pure And Applied Mathematics, Vol. VIII, pp. 503-538, 1955.

[16] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements*, Springer, New York, 2004.

[17] K. Eppler, *Efficient Shape Optimization Algorithms for Elliptic Boundary Value Problems*, Habilitationsschrift, Univ. Chemnitz (2007).

[18] K. Eppler and A. Unger, *Boundary control of semilinear elliptic equations – existence of optimal solutions*, Control and Cybernetics vol. 26 (1997) No. 2, 249–259.

[19] L. A. Escobar and W. Q. Meeker, *Statistical Methods for Reliability Data*, Wiley-Interscience Publication, New York, 1998.

[20] L. C. Evans, *Partial Differential Equations*, second edition, AMS, Providence, 2010.

[21] B. Fedelich, *A stochastic theory for the problem of multiple surface crack coalescence*, International Journal of Fracture 91 (1998) 2345.

[22] N. Fujii, *Lower Semicontinuity in Domain Optimization Problems*, Journal of Optimization Theory and Applications, Vol. 59, No. 3, 1988.

[23] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.

[24] H. Gottschalk, R. Krause and S. Schmitz, *Optimal Reliability in Design for Fatigue Life, Part II - Shape Optimization and Sensitivity Analysis*, in preparation.

[25] J. Haslinger and R. A. E. Mäkinen, *Introduction to Shape Optimization - Theory, Approximation and Computation*, SIAM - Advances in Design and Control, 2003.

[26] E.J. Haug and J.S. Arora, *Applied Optimal Design*, John Wiley & Sons, New York, 1979.

[27] E.J. Haug, K.K. Choi and V. Komkov, *Design Sensitivity Analysis of Structural Systems*, Academic Press, Orlando, 1986.

[28] R. B. Hetnarski and M. Reza Eslami, *Thermal Stresses - Advanced Theory and Applications*, Solid Mechanics and Its Applications, Vol. 158, Springer, Berlin, 2009.

[29] M. Hoffmann and T. Seeger, *A Generalized Method for Estimating Elastic-Plastic Notch Stresses and Strains, Part 1: Theory*, Journal of Engineering Materials and Technology, 107, pp. 250-254, 1985.
REFERENCES

[30] O. Kallenberg, *Random Measures*, Akademie-Verlag, Berlin 1983.

[31] M. Knop, R. Jones, L. Molent and L. Wang, *On Glinka and Neuber methods for calculating notch tip strains under cyclic load spectra*, International Journal of Fatigue, Vol. 22, (2000) 743–755.

[32] W. B. Liu, P. Neittaanmäki and D. Tiba, *Existence for shape optimization problems in arbitrary dimension*, SIAM J. Contr. Optimization 41 (2003) 1440-1454.

[33] N. Malésys, L. Vincent and F. Hild, *A probabilistic model to predict the formation and propagation of crack networks in thermal fatigue*, International Journal of Fatigue 31, 3 (2009) 565-574.

[34] J. Necas, *Les methodes directes en theorie des equations elliptiques*, first edition, Academia, Prag, 1967.

[35] H. Neuber, *Theory of Stress Concentration for Shear-Strained Prismatical Bodies with Arbitrary Nonlinear Stress-Strain Law*, J. Appl. Mech. 26, 544, 1961.

[36] J. A. Nitsche, *On Korn’s second inequality*, RAIRO Anal. Numer. 15, pp. 237-248, 1981.

[37] D. Radaj and M. Vormwald, *Ermüdungsfestigkeit*, third edition, Springer, Berlin Heidelberg, 2007.

[38] W. Ramberg and W. R. Osgood, *Description of Stress-Strain Curves by Three Parameters*, Technical Notes - National Advisory Committee For Aeronautics, No. 902, Washington DC., 1943

[39] S. Schmitz, G. Rollmann, H. Gottschalk and R. Krause, *Risk estimation for LCF crack initiation*, Proc. ASME Turbo Expo 2013, GT2013-94899, arXiv:1302.2909v1.

[40] S. Schmitz, G. Rollmann, H. Gottschalk and R. Krause, *Probabilistic analysis of the LCF crack initiation life for a turbine blade under thermo-mechanical loading*, to appear Proc. Int. Conf. LCF 7 (September 13).

[41] S. Schmitz, T. Seibel, T. Beck, G. Rollmann, R. Krause and H. Gottschalk, *A probabilistic Model for LCF*, Computational Materials Science 79 (2013), 584-590.

[42] G. Schott, *Werkstoffermüdung - Ermüdungsfestigkeit*, Deutscher Verlag für Grundstoffindustrie, forth edition, Stuttgart, 1997.

[43] J. Sokolowski and J.-P. Zolesio, *Introduction to Shape Optimization - Shape Sensitivity Analysis*, first edition, Springer, Berlin Heidelberg, 1992.

[44] D. Sornette, T. Magnin and Y. Brechet, *The Physical Origin of the Coffin-Manson Law in Low-Cycle Fatigue*, Europhys. Lett., 20 (5), pp. 433-438, 1992.

[45] J. L. Thompson, *Some Existence Theorems for the Traction Boundary Value Problem of Linearized Elastostatics*, Arch. Rational Mech. Anal., 32, pp. 369-399, 1969.