Numerical constructing of two-dimensional surface contain the piecewise smooth subdomains

F S Khayrullin and D D Mingaliev
Department of Theoretical Mechanics and Strength of Materials
Kazan national technological university
Kazan, Russia

E-mail: x_farid@mail.ru

Abstract. The paper proposes a numerical method for constructing a two-dimensional smoothing function defined by a system of points, the "bending" condition based on the theory of shells of the Tymoshenko type. As approximating functions we use polynomials of different orders to define the unknown functions inside the domain and on the boundaries. This makes it possible to numerically construct the middle surfaces of shells of complex shape, while the conditions for the continuity of the first derivatives of the unknown functions are not required, and a relatively small amount of initial information is used.

1. Fundamental relations, method of solution
When the shell have not canonical form, then its median surface is usually given by a grid of discrete points, along which an interpolation or smoothing function is constructed [1, 2, 3]. Spline functions, mostly cubic spline functions, are often used as approximating functions. However, in this case, it is required to set the values of the first derivatives on the boundaries of the domain, but is not always possible. Polynomials can be used as approximating functions [4] and it is not required to set the values of the first derivatives on the boundaries of the domain.

When using the variational method for solving the problem, there are terms in the defining functional that include second-order partial derivatives of the desired function, which, in a mechanical sense, ensure the minimum "bending" of the surface given by this function A numerical method for constructing a smoothing function was proposed in [5], the theory of shells of the Tymoshenko type is based on the "bending" condition. In this paper, in the development of the proposed method, it is proposed to use polynomials of different orders as approximating functions to define the unknown functions inside the domain and on the boundaries. It is provide to construct numerical middle surface of shells of complex shape and using a few initial data and obtaining a more optimal solution.

It is required to construct a two-dimensional surface \( \Omega \) composed of curvilinear quadrangular surface elements \( \Omega_k \) of class \( C^1 \). It is assumed that for the desired subdomain \( \Omega_k \), we can introduce a certain surface \( P_k \) (Fig. 1) with a curvilinear orthogonal coordinate system \( \alpha_1, \alpha_2 \) is given in the lines of principal curvatures. In the future, the index \( k \), which determines the belonging to the subdomain, will be neglected.
The position of the subdomain \( \Omega_k \) is determined by a system of points \( (\alpha_{1i}^k, \alpha_{2i}^k, w_i^k) i = 1, I_k \), and \( w_i^k \) - is the distance along the normal to the domain \( P_k \) from the point \((\alpha_{1i}^k, \alpha_{2i}^k)\). Some of these points will be located on the boundary lines of the subdomain and the rest points inside the subdomain.

![Diagram](image)

**Figure 1.**

First, determine the projections \( \lambda_j^* \) to the surface \( P_k \) of the boundary lines \( \lambda_j \) by the coordinates of the points \( \alpha_{1j}^k, \alpha_{2j}^k \) on the boundary lines of the subdomain. For the construction of these lines using the example of the line \( \lambda_j^* \). For the desired line, an initial approximation \( \lambda_j^0 \) is introduced, which must be sufficiently close to this line and its equation can be written in an analytical form \( \alpha_2 = f_1^0(\alpha_1) \). To determine the line \( \lambda_j^* \) equation, we use the functional

\[
\phi_j^0(\bar{f}_1, \bar{\varphi}_1) = \frac{1}{2} \int_{\lambda_j^*} \left\{ p \left[ \frac{1}{A_{1j}^0 A_{2j}^0} \frac{d A_{1j}^0}{d \alpha_1} \frac{d A_{2j}^0}{d \alpha_1} \right]^2 + \left[ \frac{1}{A_{1j}^0 A_{2j}^0} \frac{d (A_{1j}^0 \bar{f}_1)}{d \alpha_1} - \bar{\varphi}_1 \right]^2 \right\} ds + \sum_{m=1}^{N} p_m \left[ \bar{f}_1(\alpha_{1m}) - \bar{f}_1 \right]^2 \tag{1}
\]

where \( \bar{f}_1(\alpha_1), \bar{\varphi}_1(\alpha_1) \) - is equation of line \( \lambda_j^* \) and first derivative of \( f_1(\alpha_1) \);

\( \bar{f}_1(\alpha_1) = f_1(\alpha_1) - f_1^0(\alpha_1), \bar{\varphi}_1(\alpha_1) = \varphi_1(\alpha_1) - \varphi_1^0(\alpha_1) \) - deviations of the unknown functions from certain initial approximations \( f_1^0(\alpha_1), \varphi_1^0(\alpha_1) \); \( \bar{f}_1, \bar{\varphi}_1 \) - function values \( \bar{f}_1(\alpha_1) \) in the points specified on the line \( \lambda_j^* \); \( A_{1j}^0, A_{2j}^0 \) - the Lyam coefficients on the line \( \lambda_j^* \); \( p, p_j \) - weighting factors of Lagrange; \( I_1 \) - the number of approximation points on the boundary line \( \lambda_j^* \).

For the required functions, in addition to equation (1), boundary conditions can be given. For example, if it is necessary to ensure the immobility of points at the ends \( \lambda_j^* \), then we equate to zero the values of the function \( \bar{f}_1(\alpha_1) \) at these points. If it is required to ensure the impossibility of turning at the ends, then at these points we assume that \( \bar{\varphi}_1(\alpha_1) = 0 \).

Mechanical sense of functional (1) is the total energy of the beam bending considering the shear strains. Here, the interpolation conditions for the passage of the desired line near the given points and the condition for the minimal bending of the boundary line are combined.

The solutions of equations (1) have the form:
\[ \left\{ \vec{f}_1, \vec{\varphi}_1 \right\} = \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ B_{mn}^1, B_{mn}^2 \right\} \cdot t_m(\beta_1), \quad (2) \]

\[ \{ B_{mn}^1, B_{mn}^2 \}^T - \text{vector of unknown constants}; \quad \beta_1 - \text{dimensionless arc coordinate on the line } \lambda_1^0 \quad [5]; \]

\[ t_m(\beta_1) - \text{form functions that have the form} \quad [5]: \]

\[ t_1(\beta_1) = 1 - \beta_1, \quad t_2(\beta_1) = \beta_1, \quad t_m(\beta_1) = t_1(\beta_1) [t_2(\beta_1)]^{m-2} \quad (m = 3, M). \quad (3) \]

To construct a surface \( \Omega_k \), we use the functional

\[ \Phi(w, \psi_1, \psi_2) = \int_{\Omega_0} \left\{ q \cdot \left[ \frac{1}{A_1^0 A_2^0} \frac{\partial (A_1^0 \psi_1)}{\partial \alpha_1} \right]^2 + q \cdot \left[ \frac{1}{A_1^0 A_2^0} \frac{\partial (A_1^0 \psi_2)}{\partial \alpha_2} \right]^2 + \left( \psi_1 - \frac{1}{A_2^0} \frac{\partial w}{\partial \alpha_2} \right)^2 + \left( \psi_2 - \frac{1}{A_1^0} \frac{\partial w}{\partial \alpha_1} \right)^2 \right\} d\Omega_0 + \sum_{j=1}^{J} q_j \left[ w(\alpha_{1j}, \alpha_{2j}) - w_j \right]^2, \quad (4) \]

where \( w, \psi_1, \psi_2 \) - the equation of the surface function \( \Omega_k \) which approximating the first derivatives of the function \( w \); \( q, q_j \) - weighting factors of Lagrange;

Mechanical sense of functional \( (4) \) is the total energy of the shell bending of the Tymoshenko type, where the components of displacements of the shell in the middle surface are not taken into account and Poisson's ratio is assumed to be zero. The interpolation conditions for the passage of the desired line near the given points and the condition for the minimum bending of the surface \( \Omega_k \) are combined.

The solutions of equation \( (4) \) have the form \[ [6]: \]

\[ U = \sum_{m=1}^{M_1} D_m t_m(\beta_1) t_1(\beta_2) + \sum_{m=1}^{M_2} D_m t_m(\beta_1) t_2(\beta_2) + \sum_{m=1}^{N_1} D_m t_n(\beta_2) t_1(\beta_1) + \]

\[ + \sum_{m=3}^{N} D_m t_n(\beta_2) t_2(\beta_1) + \sum_{m=3}^{M} \sum_{n=3}^{N} D_{mn} t_m(\beta_1) t_n(\beta_2), \quad (5) \]

where \( U = \{ w, \psi_1, \psi_2 \}^T \) - vector of unknown functions; \( D_{nm} = \{ D_{mn}^1, D_{mn}^2, D_{mn}^3 \}^T \) - vector of unknown constants; \( M_1, M_2, N_1, N_2 \) - order of approximating functions on boundary lines \( \lambda_j \); \( M, N \) - order of approximating functions inside the subdomain \( \Omega_k \).

The curvilinear coordinate system \( \beta_1, \beta_2 \) and the functions of the form \( (3) \) are chosen in this way so that on the boundary lines the coordinate grid is uniform and the desired functions \( (5) \) are approximated by one-dimensional polynomials of one coordinate, that makes it possible to form one surface from these subdomains \( \Omega_k \) in the future. The order of the approximating functions on each of the boundaries will be its own.
Substituting the approximating functions (5) in equation (4), if necessary, satisfying the boundary conditions, with respect to unknown parameters \( D_{nm} = \{D^1_{nm}, D^2_{nm}, D^3_{nm}\} \), we obtain a system of equations.

2. Determining of geometrical parameters of the surface

The position vector of a point \( M \) on the subdomain \( \Omega_k \) we can find by the formula

\[
\rho = r(\alpha_1, \alpha_2) + w(\alpha_1, \alpha_2) \cdot \overline{n}_0(\alpha_1, \alpha_2)
\]

where \( r(\alpha_1, \alpha_2) \) is the position vector of a point \( M_0 \) on the subdomain \( \Omega^0_k \) (Fig. 1), \( \overline{n}_0(\alpha_1, \alpha_2) \) is the normal vector to the subdomain \( \Omega^0_k \).

Due to the fact that the coordinate grid in the subdomain \( \Omega^0_k \) is orthogonal, the tangent vectors to the coordinate lines in the subdomain \( \Omega^0_k \) have the form

\[
\rho_i = (1 + w \cdot k_{0i}) r_i + w_j \cdot \overline{n}_0, \quad i = 1,2
\]

where \( k_{0i} \) is the main curvatures of the coordinate lines in the subdomain \( \Omega^0_k \), \( ( )_i \) is the partial derivatives with respect to \( \alpha_1, \alpha_2 \).

Taking into account formulas (6), the coefficients of the first quadratic form of the surface \( \Omega_k \) are written as follows:

\[
a_{11} = A^2_{01} \left(1 + w \cdot k_{01}\right)^2 + \psi_2^2,
\]

\[
a_{22} = A^2_{02} \left(1 + w \cdot k_{02}\right)^2 + \psi_1^2,
\]

\[
a_{12} = A_{01} A_{02} \psi_1 \psi_2,
\]

where \( A_{01}, A_{02} \) is the coefficients of the first quadratic form of the surface \( \Omega^0_k \).

The normal vector to surface \( \Omega^0_k \) is given by:

\[
\overline{n} = \frac{A_{01} A_{02}}{\sqrt{a}} \left[(1 + w \cdot k_{01}) \left(1 + w \cdot k_{02}\right) \overline{n}_0 - (1 + w \cdot k_{02}) \psi_2 \overline{e}_01 - (1 + w \cdot k_{01}) \psi_1 \overline{e}_02\right]
\]

where \( \overline{e}_01, \overline{e}_02 \) is a unit vectors the coordinate lines in the subdomain \( \Omega^0_k \) and \( a \) is a discriminant given by formula:

\[
a = A^2_{01} A^2_{02} \left[\left(1 + w \cdot k_{01}\right)^2 + \psi_2^2\right] \left[\left(1 + w \cdot k_{02}\right)^2 + \psi_1^2\right] - \psi_1^2 \psi_2^2
\]

The coefficients of the second quadratic form of the surface \( \Omega_k \):

\[
b_{11} = \frac{A_{01} A_{02}}{\sqrt{a}} \left[(1 + w \cdot k_{01}) \left(1 + w \cdot k_{02}\right) \left(b_{011} + w_{11}\right) - (1 + w \cdot k_{02}) \psi_2 \left(\Gamma^1_{011} + 2 A_{01} k_{01} \psi_2\right) A_{01} - (1 + w \cdot k_{01}) \psi_1 \Gamma^2_{011} A_{02}\right],
\]

\[
b_{22} = \frac{A_{01} A_{02}}{\sqrt{a}} \left[(1 + w \cdot k_{01}) \left(1 + w \cdot k_{02}\right) \left(b_{022} + w_{22}\right) - (1 + w \cdot k_{01}) \psi_1 \left(\Gamma^1_{022} + 2 A_{02} k_{02} \psi_1\right) A_{02} - (1 + w \cdot k_{02}) \psi_2 \Gamma^1_{022} A_{01}\right],
\]
\[ b_{12} = \frac{A_{01}A_{02}}{\sqrt{a}} \left[ (1 + w \cdot k_{01}) (1 + w \cdot k_{02}) w_{11} - (1 + w \cdot k_{02}) \psi_2 \left( \Gamma^i_{012} + A_2 k_{01} \psi_1 \right) A_{01} - (1 + w \cdot k_{01}) \psi_1 \left( \Gamma^i_{012} + A_1 k_{02} \psi_2 \right) A_{02} \right], \]

where \( b_{011}, b_{022} \) is the coefficients of the second quadratic form, \( \Gamma^i_{0j} \) is Christoffel symbols of the surface \( \Omega_k^0 \).

Thus, all geometric parameters of the subdomain \( \Omega_k \) are determined by the geometric parameters of the subdomain \( \Omega_k^0 \) and by the numerically constructed functions \( w, \psi_1, \psi_2 \).

3. Calculation results

On the basis of the presented method the calculation algorithm was developed, Fortran program was compiled, and numerical results were obtained.

For example, we consider the construction in the Cartesian coordinate system of a surface by points located on a part of a spherical surface

\[ z = \sqrt{50^2 - x^2 - y^2} - 40^2, \quad x^2 + y^2 \leq 30^2 \tag{7} \]

\( P_k \) is a reduced surface where \( z = 0 \). In connection with symmetry, a subdomain is considered where \( x \geq 0, y \geq 0 \).

To approximation the surface it was taken next points:

- \( x = 0, y = 0; \quad x = 30, y = 0; \quad x = 0, y = 30; \quad x = 21.21, y = 21.21 \) - are the corner points of the subdomain;
- \( x = 7.5, y = 0; \quad x = 15, y = 0; \quad x = 22.5, y = 0 \) - is the points on the boundary line \( \lambda_1^1 \);
- \( x = 5.85, y = 29.42; \quad x = 11.48, y = 27.72; \quad x = 16.67, y = 24.94 \) - is the points on the boundary line \( \lambda_2^1 \);
- \( x = 0, y = 7.5; \quad x = 0, y = 15; \quad x = 0, y = 22.5 \) - is the points on the boundary line \( \lambda_3^1 \);
- \( x = 29.42, y = 5.85; \quad x = 27.72, y = 11.48; \quad x = 24.94, y = 16.67 \) - is the points on the boundary line \( \lambda_4^1 \);
- \( x = 6.93, y = 2.87; \quad x = 13.86, y = 5.74; \quad x = 20.79, y = 8.61; \quad x = 5.3, y = 5.3; \quad x = 10.61, y = 10.61 \);
- \( x = 15.91, y = 15.91; \quad x = 2.87, y = 6.93; \quad x = 5.74, y = 13.86; \quad x = 8.61, y = 20.79 \) - is the points inside the subdomain.

Z coordinates of the approximation points were found coordinates were found from equation (7).

Thus, on each of the boundary lines three points are given without consideration of the corner points, nine points are defined inside the domain. In total, there were 25 points.

First, boundary lines \( \lambda_2^1, \lambda_4^1 \) are constructed, since their equations are used to construct the desired surface. The lines \( \lambda_1^1, \lambda_3^1 \) are straight lines and their equations are known.

The Table 1 gives the values of the geometric parameters of the surface obtained by using different orders of the approximating function in formula (5). Here \( r_1 \) is the radius of curvature along the \( x \) coordinate line. The symmetry conditions were performed along the boundary lines \( \lambda_1^1, \lambda_3^1 \): \( \psi_1 = 0 \ for \ y = 0, \ \psi_2 = 0 \ for \ x = 0 \), along the lines \( \lambda_2^1, \lambda_4^1 \) it was assumed that \( w = 0 \).

The results are presented along the line \( y = 0 \) for different values of the \( x \) coordinate. Points with coordinates \( x = 3.75, x = 11.25 \) are located between the points of approximation. In the first column, the first value shows the order of the approximating function at the boundaries, the second - inside the domain. It can be seen from the table that even with the order of 3x3 approximation, the Lame coefficients are determined exactly, the radius of curvatures with an order of approximation higher than the fourth are determined to within five percent. The precision at the points of approximation and between them is the same.
### Table 1. Comparison results.

| Order | $x$  | $w$  | $A_1$ | $r_1$ | $\Delta r_1$, % |
|-------|------|------|-------|-------|------------------|
| 3x3   | 0    | 10   | 1     | 53.1  | 6.2              |
|       | 3.75 | 9.85 | 1     | 50.9  | 1.8              |
|       | 7.5  | 9.43 | 1.01  | 48.9  | 2.2              |
|       | 11.25| 8.73 | 1.03  | 47.2  | 5.6              |
|       | 15   | 7.71 | 1.05  | 45.8  | 8.4              |
| 3x4   | 0    | 10   | 1     | 51    | 2                |
|       | 3.75 | 9.84 | 1     | 50.5  | 1                |
|       | 7.5  | 9.43 | 1.01  | 49.6  | 0.8              |
|       | 11.25| 8.72 | 1.03  | 48.3  | 3.4              |
|       | 15   | 7.71 | 1.05  | 46.9  | 6.2              |
| 4x4   | 0    | 10   | 1     | 48.9  | 2.2              |
|       | 3.75 | 9.86 | 1     | 49.9  | 0.2              |
|       | 7.5  | 9.43 | 1.01  | 49.9  | 0.2              |
|       | 11.25| 8.72 | 1.03  | 49.1  | 1.8              |
|       | 15   | 7.70 | 1.05  | 47.7  | 4.6              |
| 4x5   | 0    | 10   | 1     | 52.2  | 4.4              |
|       | 3.75 | 9.85 | 1     | 50.4  | 0.8              |
|       | 7.5  | 9.43 | 1.01  | 48.6  | 2.8              |
|       | 11.25| 8.73 | 1.03  | 47.2  | 5.6              |
|       | 15   | 7.71 | 1.05  | 46.2  | 7.6              |
| 5x5   | 0    | 10   | 1     | 49.9  | 0.2              |
|       | 3.75 | 9.86 | 1     | 49.3  | 1.4              |
|       | 7.5  | 9.44 | 1.01  | 48.9  | 2.2              |
|       | 11.25| 8.72 | 1.03  | 48.3  | 3.4              |
|       | 15   | 7.70 | 1.05  | 47.6  | 4.8              |

### References

[1] Grigolyuk E I and Chulkov P P 1973 *Stability and Vibrations of Three-Layer Shells* (Moscow: Nauka) 180 (77)

[2] Kornishin M S, Paimushin V N and Snigirev V F 1989 *Vychislitel'nya geometriya v zadachakh mekhaniki obolochek* (Moscow: Nauka) 208 (120)

[3] Marchuk G I 1982 *Methods of Numerical Mathematics* (Springer-Verlag New York) 510

[4] Sezrautdinov M N and Nedorezov O A 1990 About the approximation of the middle shell surface *Investigations in the theory of shells. Proceedings of the seminar* 97–102 (230)

[5] Khairullin F S and Serzautdinov M N 2006 Method of parametrization of the middle surface of a thin-walled structural element *Proceedings of universities. Aviation equipment* 414–16

[6] Khayrullin F S and Mingaliev D D 2017 Stress and strain definition for thin shells, which based on approximation functions of different degrees *Vestnik of KSTU* 20 (14) 102–104