Backward bifurcation analysis on Tuberculosis disease transmission with saturated treatment

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Abstract. In this research article, the authors intend to introduced an SEI (Susceptible-Exposed-Infectious) Tuberculosis model to consider the limitation of medical resources using a saturated treatment function. This is important to analyze the effect of hospital capacity in the success of Tuberculosis prevention strategy. Mathematical analysis was conducted to determine and analyze the existence and local stability criteria for equilibrium points, and how they related to the basic reproduction number of the model. The stability criteria of the endemic equilibrium point were analyzed using the center manifold theory. Our analysis showed that the saturated treatment rate might lead our proposed model to exhibit backward bifurcation at a basic reproduction number equal to one, and this phenomena appears related to the size of the treatment saturated parameter. Local sensitivity analysis was given to give a suggestion about how to avoid the occurrence of backward bifurcation phenomena. To support our analytical results, some simulations were presented at the end of the work.

1. Introduction
Tuberculosis (TB) is an infectious disease caused by *Mycobacterium tuberculosis*, and spread through cough or direct contact with an infected individual. TB frequently attacks lungs. The latency period of TB needs several years, which make TB categorized as a slow infection disease. Infected individual who in latency period usually do not show any symptoms [1]. Because of the long term of latency period, individuals may conduct reinfection [2]. Recently, many interventions introduced to avoid new infection or to cure infected individuals, such as by vaccine, medical treatment, etc [3, 9].

Mathematical models can be useful tools to analyze, predict, and anticipate the possible outcome when some interventions planned to be applied to the community. Huo et al. [4] present how self-treatment at home can be used to eradicate TB, Chasanah et al. [5] and Aldila et al. [6] consider vaccination strategies on an age-structured population for TB, Rahmawati et al. [7] analyze the effect of chemotherapy for TB treatment as some example from many introduced TB model, and Simorangkir et al. [8] consider the effect of hospitalization on TB control program. Recently, Baba et al. [10] introduced a simple SEI (Susceptible-Exposed-Infected) model with a saturated infection rate to understand how TB transmission can be suppressed using chemoprophylaxis as an optimal control problem. Mathematical analysis of the equilibrium points, basic reproduction number, and optimal control simulation were conducted in this paper. They have shown that a unique endemic equilibrium point appear from their model when the basic reproduction number greater than one and no endemic equilibrium point when the basic reproduction number is less than one.
Due to the emergence of TB treatment, which need to isolate the infected individual from general social contact with susceptible population make a medical resource become so limited, such as for isolation room, number of beds in the hospital, etc. Motivated by this, here we modify the SEI model introduced by Baba et al. [10] by considering the limitation of medical resources. Modification of the model lies in the modification of the treatment rate. Instead of treating it as a constant linear rate, we prefer to use it as a saturated function depending on the number of infected individuals. Using this approach, we may model the fact that it is more difficult to treat TB infected individuals when the number of infected individuals is in a high number, probably because of the limitation of room in the hospital, medical treatment resources, number of doctors, etc. Comparing with authors in [10], we treat our population as a constant in time, which allows us to reduce the dimension from three into the only two-dimensional system. The other difference is instead of a unique endemic equilibrium, our modified model may give a multiple endemic equilibrium when the basic reproduction number is less than one. Sensitivity analysis is conducted to find the most crucial parameter, which determines the magnitude of the basic reproduction number, and the existence of backward bifurcation.

This paper is organized as follows. In Section 2, a modification of proposed model by authors [10] introduced. Non-dimensionalization for parameters and variables is conducted in this section, followed by an analysis of the well-posed properties of our model. Mathematical analysis of the existence and local stability of equilibrium points is conducted in Section 3. In Section 4, backward bifurcation analysis is conducted using the center manifold theory. We give some numerical simulations in Section 5 and close with some conclusions in the last section.

2. The construction of the model

As previously mentioned in Sec. 1, we modify the Tuberculosis model introduced by authors in [10] by changing the treatment rate into a saturated treatment. This change is important to describe the limitation of the hospital or any other medical resources to cure or to treat the infected individual. The model based on an SEI (Susceptible-Exposed-Infected) model. The model is constructed following the transmission diagram given in Fig. 1.

![Transmission diagram of TB spread considering saturated incidence and treatment rate. The red arrows indicate infection term, while black arrows indicate transition, death rate or newborn terms.](image)

Figure 1. Transmission diagram of TB spread considering saturated incidence and treatment rate. The red arrows indicate infection term, while black arrows indicate transition, death rate or newborn terms.
Susceptible individual ($S$) increases due to newborn $\Lambda$, and decreases due to natural death rate $\mu$, and infection caused by direct contact with $I$ with saturated incidence rate $\frac{\beta}{1+b^*I}$, where $\beta$ is the infection rate and $b^*$ is the saturated incidence parameter. Exposed compartment ($E$) increases due to new infection with proportion $p$ caused by slow infection from $S$, and recovery from $I$ with saturated recovery rate $\frac{\gamma}{1+a^*I}$, where $\gamma$ is the recovery rate, and $a^*$ is the saturated recovery rate. Parameter $a^*$ describes the limitation of medical resources. If $b^* = 0$, then the community has an unlimited number of beds in the hospital, doctor, room for isolation, etc. On the other hand, if $b^* \to \infty$, then we have that the community has very limited resources to accelerate the recovery for the infected individual. Furthermore, $E$ decreases due to the natural death rate $\mu^*$ and progression to be active TB with the rate of $\alpha^*$. Lastly, the infected individual $I$ increases due to new fast infection from $S$, the progression of active TB from $E$, and decreases caused by recovery and natural death rate. Therefore, the model that governs our described above illustration is given by the following system of ordinary differential equations.

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \frac{\beta^*}{(1+b^*I)} S \frac{I}{N} - \mu^* S, \\
\frac{dE}{dt} &= (1-p) \frac{\beta^*}{(1+b^*I)} S \frac{I}{N} - \alpha^* E - \mu^* E + \frac{\gamma^*}{(1+a^*I)} I, \\
\frac{dI}{dt} &= p \frac{\beta^*}{(1+b^*I)} S \frac{I}{N} + \alpha^* E - \mu^* I - \frac{\gamma^*}{(1+a^*I)} I.
\end{align*}
\]  

(1)

where $N = S + E + I$. This model has a non-negative initial condition as

\[ S(0) \geq 0, E(0) \geq 0, I(0) \geq 0. \]  

(2)

All parameters in system (1) are assumed non-negative. We also assumed that number of newborn are equal to the number of death, we have that $\frac{dN}{dt} = \Lambda - \mu N = 0$ which gave us $N$ is always constant.

**Non-dimensionalization of the TB model (1)**

Before we proceed to analyze the well-posed criteria of our model, and further analyzing the bifurcation phenomena, we will do a non-dimensionalization to system (1). Let

\[
\begin{align*}
x_1 &= \frac{S}{N}, \quad x_2 = \frac{E}{N}, \quad x_3 = \frac{I}{N}, \quad \tau = \mu^* t, \\
\beta = \frac{\beta^*}{\mu}, \quad \gamma = \frac{\gamma^*}{\mu}, \quad \alpha = \frac{\alpha^*}{\mu}, \quad a = a^* N, \quad b = b^* N.
\end{align*}
\]  

(3)

Since the total of human population is constant, we have that $x_1 = 1 - x_2 - x_3$. Substituting equations (3) into system (1) yields

\[
\begin{align*}
\frac{dx_2}{dt} &= (1-p) \frac{(1-x_2-x_3)\beta x_3}{(1+bx_3)} - \alpha x_2 - x_2 + \frac{\gamma x_3}{1+ax_3}, \\
\frac{dx_3}{dt} &= p \frac{(1-x_2-x_3)\beta x_3}{(1+bx_3)} + \alpha x_2 - x_3 - \frac{\gamma x_3}{1+ax_3}.
\end{align*}
\]  

(4)

Instead of analyzing the TB model in system (1), we will analyze model (4), which is a non-dimensional version of the TB model (1).
Positivity and boundedness of solutions
To make the non-dimensional version of TB model (4) to be meaningful, we will show that the model has a non-negative solution for all time \( t > 0 \).

**Lemma 1.** Let the initial condition of system (4) be \( x_2(0) = x_{20} \geq 0, x_3(0) = x_{30} \geq 0 \), then the solution of system (4) are non-negative for all \( t > 0 \).

**Proof.** Using the concept of vector field, since the initial condition is non-negative, we have
\[
\begin{align*}
\frac{dx_2}{dt}(x_2 = 0, x_3 \geq 0) &= (1 - p) \frac{(1 - x_2)\beta x_3}{1 + bx_3} + \frac{\gamma x_3}{1 + ax_3} \geq 0, \\
\frac{dx_3}{dt}(x_2 \geq 0, x_3 = 0) &= \alpha x_2 \geq 0.
\end{align*}
\]
Therefore, we have that the vector field of system (4) are always pointed onward whenever it approaching 0. Therefore, we have that \( x_2(t) \) and \( x_3(t) \) are always positive for all \( t > 0 \).

Since we have that all solution of \( x_2 \) and \( x_3 \) from system (4) are always positive, and \( x_1 + x_2 + x_3 = 1 \), we have that \( x_2 \) and \( x_3 \) are bounded above by 1. Therefore, we consider the feasible region of system (4) on \( \Omega \), where
\[
\Omega = \{ (x_2(t), x_3(t)) \in \mathbb{R}_+^2 : x_2(t) + x_3(t) \leq 1 \}.
\]

**Lemma 2.** The region \( \Omega \subset \mathbb{R}_+^2 \) is positively invariant for the model (4) with non-negative initial condition \( x_2(0) = x_{20} \geq 0, x_3(0) = x_{30} \geq 0 \) in \( \mathbb{R}_+^2 \).

3. Basic reproduction number and the equilibrium points
System (4) has two types of equilibrium point. The first equilibrium point is the disease-free equilibrium point which is given by
\[
E_0 = (x_2, x_3) = (0, 0). \tag{5}
\]
Using the next-generation matrix approach, the basic reproduction number of system (4) is given by:
\[
R_0 = \frac{\beta(p + \alpha)}{\alpha + \gamma + 1}. \tag{6}
\]
In our model, the basic reproduction number describes the number of new tuberculosis infected cases caused by one primer infection in a completely susceptible population, during it infection period. In many mathematical models, such as \([12, 13, 14, 15, 16, 17, 18]\), the disease-free equilibrium point is locally asymptotically stable when \( R_0 < 1 \). In our case, a similar result occurs, which is stated in the following lemma.

**Lemma 3.** The disease-free equilibrium \( E_0 \) of TB-model (4) is locally asymptotically stable (LAS) if \( R_0 < 1 \), and unstable otherwise.

**Proof.** Linearizing system (4) in \( E_0 \) yields
\[
J|_{DFE} = \begin{bmatrix}
-\alpha - 1 & (1 - p)\beta + \gamma \\
\alpha & \beta p - \gamma - 1
\end{bmatrix}. \tag{7}
\]
The characteristic equation is given by
\[
\lambda^2 + (-\beta p + \alpha + \gamma + 2) \lambda + (\alpha + \gamma + 1)(1 - R_0) = 0. \tag{8}
\]
Therefore, the roots of equation (8) have negative real part if \( R_0 < 1 \) and \( -\beta p + \alpha + \gamma + 2 < 0 \iff \frac{\beta p}{\alpha + \gamma} < 1 \). Since \( R_0 > \frac{\beta p}{\alpha + \gamma} \), then we have that \( E_0 \) is LAS if \( R_0 < 1 \).
The next equilibrium is the endemic equilibrium point, which is given by
\[ E^\dagger = (x_2, x_3) = (x_2^*, x_3^*), \] (9)
where
\[ x_2^* = \frac{x_3^*(a\beta px_3^{*2} + abx_3^{*2} - a\beta px_3^{*2} + b\gamma x_3^{*2} + p\beta x_3^{*2} + ax_3^{*2} + bx_3^* - \beta p + \gamma + 1)}{a\alpha bx_3^{*2} - a\beta px_3^{*2} + a\alpha x_3^{*2} + \alpha bx_3^* - p\beta x_3^* + \alpha}, \]
and \( x_3^* \) is taken from the positive root of quadratic equation given by:
\[ B_2 (x_3^*)^2 + B_1 (x_3^*) + B_0 = 0 \] (10)
where
\[ B_2 = -a (\alpha + 1) (b + \beta), \]
\[ B_1 = ((a - 1) \alpha + ap - \gamma - 1) \beta + (-a - b) \alpha - a + (-\gamma - 1) b, \]
\[ B_0 = \alpha \beta + \beta p - \alpha - \gamma - 1 = \left( \frac{p\beta}{\alpha + \gamma + 1} - 1 \right) \alpha + \gamma + 1 = (R_0 - 1)(\alpha + \gamma + 1). \]

Since \( B_2 < 0 \) and \( B_0 > 0 \iff R_0 > 1 \), we have the following lemma.

**Lemma 4.** System (4) always has a unique endemic equilibrium point if \( R_0 > 1 \).

Since equation (10) is quadratic, we have that system (4) has:

(i) One endemic equilibrium point if \( R_0 > 1 \) (Lemma 4), or \( R_0 < 1 \), \( B_1 < 0 \) and \( B_1^2 - 4B_2B_0 = 0 \).

(ii) Two endemic equilibrium points if \( R_0 < 1 \), \( B_1 < 0 \), and \( B_1^2 - 4B_2B_0 > 0 \).

(iii) No endemic equilibrium point otherwise.

### 4. Backward bifurcation analysis

#### 4.1. Existence of backward bifurcation

In this section, we explore the phenomena of backward bifurcation of system (4) using the center manifold theorem, given by Theorem in[19]. For more example about the implementation of this method, please see [16, 17, 18]. Let \( \beta \) be the bifurcation parameter. Solving \( R_0 = 1 \) respect to \( \beta \) yields
\[ \beta^* = \frac{(\alpha + \gamma + 1)}{(p + \alpha)}. \] (11)

Substituting \( \beta^* \) and \( E_0 \) into the Jacobian of system (4) gives us:
\[ J^* = \begin{bmatrix} -\alpha - 1 & \frac{(1-p)(\alpha+\gamma+1)}{p+\alpha} + \gamma \\ \alpha & -1 + \frac{p(\alpha+\gamma+1)}{p+\alpha} - \gamma \end{bmatrix} \] (12)

Matrix \( J^* \) has two eigenvalues, i.e \( \lambda_1 = 0 \) and \( \lambda_2 = \frac{(\alpha^2+\alpha\gamma+2\alpha+p)}{\alpha+p} < 0 \). Since we have simple zero eigenvalue of and the other eigenvalue is negative, we can use the center-manifold theorem to analyze the dynamic of system (4) near \( \beta = \beta^* \).
Next, we calculate the left and the right eigenvector of $J^*$ (associated with the zero eigenvalue). The right eigenvector is given by $w = (w_1, w_2)^T$ where

$$w_1 = \frac{(\gamma + 1 - p) w_2}{p + \alpha},$$

$$w_2 = w_2 > 0.$$

Similarly, we have the left eigenvector of system (4) associated with the zero eigenvalue is given by $v = (v_1, v_2)^T$ where

$$v_1 = v_1 > 0,$$

$$v_2 = \frac{(\alpha + 1) v_1}{\alpha}.$$

To show the existence of backward bifurcation, we calculate the following second order partial derivative of $f_2 := \frac{dx_2}{dt}$ and $f_3 := \frac{dx_3}{dt}$ at $E_0$, and obtain:

$$\frac{\partial^2 f_2}{\partial x_2^2} = 0$$

$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_2} = \frac{(p - 1) \beta}{(bx_3 + 1)^2}$$

$$\frac{\partial^2 f_2}{\partial x_3^2} = -2 \frac{(p - 1) (ax_3 + 1)^3 (-1 + (x_2 - 1) b) \beta - 2 a \gamma (bx_3 + 1)^3}{(bx_3 + 1)^3 (ax_3 + 1)^3}$$

$$\frac{\partial^2 f_3}{\partial x_2^2} = 0$$

$$\frac{\partial^2 f_3}{\partial x_2 \partial x_3} = \frac{\partial^2 f_3}{\partial x_3 \partial x_2} = \frac{(p - 1) \beta}{(bx_3 + 1)^2}$$

$$\frac{\partial^2 f_3}{\partial x_3^2} = 2 a \left( \beta p (-1 + (x_2 - 1) b) a^2 + b^3 \gamma \right) x_3^3 + 6 \left( \beta p (-1 + (x_2 - 1) b) a + b^2 \gamma \right) a x_3^2$$

$$+ 6 a \left( \beta p (-1 + (x_2 - 1) b) + b \gamma \right) x_3 + 2 a \gamma + 2 \beta p (-1 + (x_2 - 1) b)$$

$$\frac{(bx_3 + 1)^3 (ax_3 + 1)^3}{(bx_3 + 1)^3 (ax_3 + 1)^3} +$$

Now, we calculate the coefficient $A$ and $B$ defined in Theorem 4.1 [19] of Castillo-Chaves and Song as follows

$$A = \sum_{k,i,j=2}^{3} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} (0, 0)$$

$$= -2 \frac{-a \alpha \gamma - a \gamma p + a^2 b + a b \gamma + \alpha b p + b p \gamma + \alpha^2 + \alpha b + 2 \gamma \alpha + b p + \gamma^2 + 2 \alpha + 2 \gamma + 1}{\alpha (p + \alpha)}$$

$$B = \sum_{k,i=2}^{3} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta} (0, 0)$$

$$= \frac{p + \alpha}{\alpha} > 0$$

Since the coefficient of $B$ is always positive, system (4) exhibits a backward bifurcation at $R_0 = 1$ if $A > 0$, equivalent if

$$a > \frac{(\alpha + \gamma + 1) (\alpha b + b p + \alpha + \gamma + 1)}{\gamma (p + \alpha)}.$$  \hspace{1cm} (13)$$

We have establish the following conclusion.
**Theorem 1.** System (4) undergoes a backward bifurcation at $R_0 = 1$ whenever the inequality (13) holds. On the other hand, system (4) undergoes forward bifurcation at $R_0 = 1$ otherwise.

4.2. Numerical experiment on the existence of backward bifurcation

In this section, we illustrate how backward bifurcation of system (4) appears when Theorem 4 holds. To perform the numerical experiment, we use the following parameter values:

$$\alpha = 0.17, \quad \gamma = 0.67, \quad p = 0.5, \quad b = 0.4.$$  

Using above parameters value, we have that $a$ which hold inequality (13) is $8.640498997$. Therefore, whenever $a > 8.640498997$, backward bifurcation occurs at $R_0 = 1$. In this case, we choose $a = 20$ to conduct the backward bifurcation. Next, we calculate the fold point respect to above parameters. In our case, the fold point is taken $R_0$ which make the discriminant of equation (10) equal to zero. Substituting above parameter values, $\beta^*$ and $a = 20$ into equation (10) yields

$$(−9.360 − 64.26268657 R_0)x_3^2 + (31.74686566 R_0 − 24.136)x_3 + 1.840000000 R_0 − 1.84 = 0,$$

which gave us $R_0 = R_c = 0.9379273945$ as the fold point. Plotting above polynomials, we have Fig. 2 shows a backward bifurcation driven by saturated parameter for treatment.

![Figure 2](image)

**Figure 2.** Backward bifurcation diagram when the backward bifurcation generated using the treatment saturated parameter ($a$). Red and blue curves present endemic and disease-free equilibrium, respectively. Dot, and solid curves present unstable and stable equilibrium, respectively.

From Fig. 2, there exist two types of flow. For all initial conditions in the first flow, when $R_0 < R_c$, then the disease-free equilibrium is stable. When $R_0 = R_c$, hysteresis occurs, which followed with a new "large" endemic equilibrium. On the other hand, when the initial
condition follows the second flow (started with stable endemic equilibrium solution), the endemic equilibrium remains stable, until it reaches $R_0 = 1$. When $R_0 = 1$, the stable equilibrium jumped from stable endemic equilibrium into new stable disease-free equilibrium. Therefore, it is very crucial to understand the effect of different initial condition of our system when backward bifurcation occurs, since a small change of $R_0$ could lead to a "shocking" new large endemic size.

4.3. The effect of saturated treatment on the backward bifurcation

The impact of the saturated treatment parameter $a$ on the backward bifurcation will be assessed by analyzing the bifurcation coefficient $A$ as follows. Differentiating $A$ with respect to $a$ gives

$$\frac{\partial A}{\partial a} = \frac{2\gamma}{\alpha}.$$ 

Hence, the bifurcation parameter $A$ is an increasing function respect to $a$. Therefore, saturated treatment increases the possibility of backward bifurcation of system (4). Furthermore, when $a = 0$, then we have that

$$A = -2 \frac{(1 + \alpha + \gamma)(ba + bp + \alpha + \gamma + 1)}{\alpha(\alpha + p)} < 0,$$

which indicate our system exhibit forward bifurcation at $R_0 = 1$ if medical resources limitation do not exist.

4.4. Local sensitivity analysis of bifurcation parameter $A$ and $R_0$ respect to model parameters

From above analysis, we can see that the existence of backward bifurcation is highly depend on $A$ and $R_0$. For this reason, we carry out the local sensitivity analysis.

**Definition 1.** (See [20]) The normalized local sensitivity index of $\Gamma$ respect to parameter $\zeta$ is given by

$$\Delta_\zeta = \frac{\partial \Gamma}{\partial \zeta} \frac{\zeta}{\Gamma}.$$  \hspace{1cm} (14)

This indices allow us to find the most critical parameter that strongly affect the size of $R_0$ or $A$. Using Eq. (14), and parameter values

$$\alpha = 0.17, \quad \gamma = 0.67, \quad p = 0.5, \quad b = 0.4, \quad a = 20, \quad \beta = 2.4,$$

the sensitivity indices of $R_0$ is given by

$$\Delta_\alpha = 0.161, \quad \Delta_\gamma = -0.364, \quad \Delta_p = 0.746, \quad \Delta_b = 0, \quad \Delta_a = 0, \quad \Delta_\beta = 1.$$ 

It can be seen from the sensitivity indices of $R_0$ respect to all parameter, the saturation parameters $a$ and $b$ do not effect the size of $R_0$. The increase of $\alpha, p$, and $\beta$ will increase $R_0$. For example, increases value of $\alpha$ by 10% will increase value of $R_0$ 1.61%. On the other hand, we can see that the increase of $\gamma$ will decrease value of $R_0$. Therefore, from above analysis, we can conclude that the most crucial controllable parameter in $R_0$ is $\beta$. Therefore, it is very reasonable to reduce contact with infected people to eradicate TB from community.

Using similar approaches and parameter values as previous experiment, we calculate sensitivity indices of $A$ respect to all parameters. The sensitivity indices of $A$ is given by

$$\Delta_\alpha = -0.963, \quad \Delta_\gamma = 1.242, \quad \Delta_p = 0.168, \quad \Delta_b = -0.041, \quad \Delta_a = 0.014, \quad \Delta_\beta = 0.$$ 

It can be seen that $\Delta_\beta = 0$ since we already choose $\beta$ as the bifurcation parameter. Therefore, the most crucial parameter in determining $A$ is the recovery rate $\gamma$. Larger value of $\gamma$, will increase value of $A$. Similar interpretation for the value applied for this simulation.
5. Simulation of the TB model

5.1. Effect of saturated treatment parameter

This simulation is conducted to understand the effect of limitation of medical resources. In this section, we use the following parameters:

\[ \alpha = 0.17, \quad \gamma = 0.67, \quad p = 0.5, \quad b = 0.4, \quad \beta = 2.7, \]

which gave us \( R_0 = 0.983 \), and use various value of \( a \). Using this set of parameters, we have the critical parameter \( a \) which become the threshold of backward bifurcation parameter as in Eq. (13) is \( 8.640498997 \). The result of the simulation is given in Fig. 3. It can be seen that when \( a < 8.640498997 \), backward bifurcation do not occur, and since \( R_0 < 1 \), we have the blue curve tends to the disease free equilibrium \( E_0 \). On the other hand, when \( a > 8.640498997 \), backward bifurcation occur which lead to a stable endemic equilibrium (red curve) even though \( R_0 < 1 \). Please note that in this scenario, a stable disease free equilibrium always exist. This confirm our result in Lemma 3 and Theorem 1.

![Figure 3. Trajectories of \( x_2 \) and \( x_3 \) for different values of \( a \). Blue curve is when \( a < 8.640498997 \), while the red curve for \( a > 8.640498997 \).](image)

5.2. Autonomous simulation

In this section, we simulate effect of the combination between medical treatment \( \gamma \) and saturated treatment parameter \( a \). In this scenario, we use

\[ \alpha = 0.17, \quad a = 20, \quad p = 0.5, \quad b = 0.4, \quad \beta = 2.7, \]

while \( \gamma \) will varying. It can be seen from Fig. 4 that greater treatment rate needed to make the system tend to the disease-free equilibrium point. The cyan curve tends to \( E_0 \), while red curve tends to \( E^\dagger \). The arrow downward arrow indicates the effect of greater value of \( \gamma \). This confirms our sensitivity analysis results, that increasing treatment rate will reduce \( R_0 \).
Figure 4. Trajectories of $x_2$ and $x_3$ for different values of $a$. Blue curve show the dynamic of population tends to the disease-free equilibrium, while the red curve tends to the endemic equilibrium.

6. Conclusion
In this paper, we modify the TB model introduced by Baba et al. [10] by considering the treatment rate as a saturated treatment function. The non-dimensionalization process allows us to reduce the complexity of our model and make all parameters and variables become nondimensional. By means of the next-generation matrix approach, we obtained the basic reproduction number ($R_0$) of our proposed model. We find that the tuberculosis-free equilibrium point is locally asymptotically stable if $R_0 < 1$, and unstable otherwise. Furthermore, using the Castillo-Song theorem [19], we proved that our model might exhibit a backward bifurcation at $R_0 = 1$ when the saturated treatment parameter larger than its minimum threshold. Backward bifurcation which driven from saturated treatment parameter made TB eradication program become more difficult, since (i) the endemic equilibrium may still exist even $R_0 < 1$, (ii) small changes of infection parameter ($\beta$), may lead the system from stable TB-free equilibrium point into "jump" new large endemic equilibrium point.

It is very interesting to compare our model with the model proposed by the author in [10] with changing the treatment parameter into a saturated treatment function. A completely different qualitative behavior of the model occurs caused by the saturated treatment. As proposed by the author in [10], to understand the limitation of the budget for the TB eradication program, the model re-constructed as an optimal control problem. Therefore, it is also interesting to consider our model in this article as an optimal control problem. We leave this task for future research.

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