Constructing the ultimate theory of grand unification

V.V.Kiselev\textsuperscript{1,2} V.V.Shakirov\textsuperscript{2}

\textsuperscript{1}Russian State Research Center “Institute for High Energy Physics”, Pobeda 1, Protvino, Moscow Region, 142281, Russia
Fax: +7-4967-744937

\textsuperscript{2}Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudny, Moscow Region, 141701, Russia

In accordance with known phenomenological facts on leptons and quarks in the Standard Model as well as on the scale of neutrino masses and introducing the supersymmetry, we logically substantiate the unique composition of fundamental representation for the fermionic multiplet of gauge group $E_8$.

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I. INTRODUCTION

In the framework of local quantum field theory the idea of grand unification of gauge interactions (GUT) is logically based on three facts \textsuperscript{1,2}:

1. the known fermions are chiral fields,

2. the sum of electric charges of those fermions (quarks and leptons) in each generation is equal to zero,

3. the renormalization group evolution of coupling constants for the electroweak and strong interactions (the gauge group of Standard Model (SM) is $G_{SM} = U_Y(1) \times SU(2)_L \times SU_c(3)$) leads to the almost precise equality of those three “running” charges to each other at the scale of “unification” $\Lambda_{GUT}$ about $10^{16}$ GeV, above which the propagation of heavy fields becomes essential and, hence, the underlying gauge symmetry is restored with the single coupling constant.

According to the structure of gauge group $G_{SM}$ the actual fields have got 4 independent quantum numbers: the hypercharge $Y$, the projection of electroweak isospin for the left doublets and right singlets $T_3$, and the colored analogues of “isospin” $\lambda_3^c$ and “strangeness” $\lambda_8^c$, that sets the rang of group equal to 4. Under the grand unification the amount of quantum numbers cannot decrease, so that the rang of GUT group cannot be less than 4. However, the introduction of common coupling constant of interactions suggests that the GUT group is simple, therefore, the sum of quantum numbers in the multiplet has to be equal to zero (the trace of generator in the simple nonabelian group equals zero), that exactly makes the property 2 pointed out above, while in accordance with item 1 in this scheme, the group itself should allow for non-real (non-selfconjugative) representations corresponding to chiral fields.

In this way the ratios of charges in the multiplet are completely determined by the structure of group, hence, there is the charge quantization, for instance, the quantization of electric charge of fermions. Emphasize that there should exist an irreducible representation, which includes the known charged fermions only, because the addition of new charged fermions to such the representation would make the arbitrariness in the quantization of electric charges, if only the sum of charges of additional fermions would not be equal to zero. In the later case, we could conclude that the additional fermions compose an irreducible representation of the same group, which produces the quantization of charges for the known fermions. In this respect, a simple group of minimal admissible rang should regard to the quantization of charges for the chiral fields of fermions in the SM, and the problem to find such the group was actual. This problem has been successfully solved in \textsuperscript{1}: the group of charge quantization is $SU(5)$, and it has got the minimally admissible rang equal to 4, while the known fermions of SM compose two fundamental representations, the antiquintet and decuplet of left chiral fields\textsuperscript{1} (the formulation with two conjugated fundamental representations is equivalent to the given one).

In Section\textsuperscript{11} we recall the properties of primary set for the chiral fields of SM and expand the action of charge quantization-group to the Higgs sector of SM, which gives the natural introduction of Dirac masses for quarks and charged leptons under the spontaneous breaking the symmetry by the vacuum state of scalar field. Further, we involve the supersymmetry, that makes the addition of chiral superpartners of Higgs fields into the multiplet of representation for the GUT group to be necessary. The charges of these superpartners are uniquely quantized.

Next, we argue for the necessity of introducing a special mechanism, which should generate the required scale for

\textsuperscript{1} In this case the hypercharge $Y$ is proportional to one of Cartan generator in the group $SU(5)$, while a coefficient can be easily calculated from the common normalization of group generators, i.e. by the same sum of charge squares in the multiplet, that is essential in the derivation of statement on the equality of “running” coupling constants at the scale of unification (the main content of item 3 in the list above).
the neutrino masses as follows from the phenomenology: the only way to relate the scale of electroweak symmetry breaking to the neutrino masses is the mixing of electroweak singlet neutrino with a neutral heavy Majorana particle, that completes the composition of multiplet of gauge group for the single generation, i.e. the fundamental representation of group $E_8$ with the unique determination of quantum numbers for all of components.

In Section III we consider $n_g$ generations for the chiral fermions of group $E_8$ as the components of multiplet unified with the Majorana superpartners of gauge vector fields in $E_6$, i.e. with the gaugino. In this way, the supersymmetry requires to include $n_g$ conjugated fundamental representations of $E_6$ for the chiral fields into such the common multiplet in order to make the unification with the Majorana gaugino from the adjoint representation of $E_6$. Then, those fields can be unified in the same extended selfconjugated representation. Moreover, the equivalence of generations for the chiral fields means, in fact, the introduction of unitary transformations of generations themselves. Therefore, we have to include the “horizontal” symmetry of generations into the ultimate group of GUT. The Majorana superpartners of gauge vector bosons in $SU_H(n_g)$ as well as the gauge bosons of horizontal symmetry are singlets with respect to the group $E_6$, and they should be added into the complete multiplet of fermions, too. In this step, we have to take into account for that the gauginos of $E_6$ are singlets with respect to the group of horizontal transformations for the fermion generations. Fortunately, the fermionic multiplet with the required properties does exist, and it is unique, whereas $n_g = 3$, while the group of symmetry is the exceptional group $E_8$. Finally, we discuss some consequences following from the formulation of ultimate theory of grand unification as well as its properties.

On the other hand, in the framework of superstring theory the phenomenology of Yukawa sector in the SM points to the emphasis of specific way for the compactification of extra dimensions, namely, the F-theory [2,3] with two hierarchical scales of energy at $\Lambda \ll M$, where $M$ is the stringy Planckian scale, while $\Lambda$ is the GUT scale. In this scenario, the unique role of group $E_8$ is also noticed [4].

Thus, to our point of view, we successfully construct the only true solution for the problem of finding the group of grand unification as well as its fermionic multiplet by starting from the given data of phenomenology and by involving the supersymmetry. This solution gets the common with the mentioned scheme of superstring model, but, as we will see, the proposed way essentially differs from the superstring models, since the states with Planckian scale of masses are included in the fundamental multiplet of gauge group in contrast to multiplets inspired by the superstring theory, when the multiplet is composed of light sub-Planckian states. In this study we put the mechanism of breaking both the gauge symmetry and the supersymmetry beyond the current consideration.

II. THE SINGLE GENERATION

Following the principles listen in Introduction, we are considering the logical construction of multiplet for the single generation of chiral fermionic fields in the SM.

A. The Standard Model: chiral fermions and Higgs scalar

The observed leptons and quarks of SM compose the set of left-handed chiral spinors of weak doublets and singlets over the group $SU(2)_L$:}

$\begin{pmatrix}
\nu_L, \\
e^L, \\
u^c_L, \\
e^c_L
\end{pmatrix}$, 

(1)

where the superscript “$c$” refers to the charged conjugated particles, i.e. antiparticles. The scalar Higgs field is the weak doublet:

$\begin{pmatrix}
H^0 \\
H^-
\end{pmatrix}$, 

(2)

which has got zero lepton and baryonic numbers

$B[H] = L[H] = 0$.

The masses of Dirac kind for the charged leptons and quarks are given by matrices for the chiral spinors, so that, for instance, in the case of electrons

$(e, e^c)_L \left( \begin{array}{cc} 0 & m_e \\ m_e & 0 \end{array} \right) \left( \begin{array}{c} e \\ e^c \end{array} \right)_L$, 

(3)

where the two-component indices of chiral spinors are conjuated by means of spinor metric being the completely antisymmetric tensor of second rang, i.e. the Levi-Civita tensor. In this way, the masses themselves are generated by the spontaneous breaking of initial gauge group $G_{SM}$ due to both the vacuum expectation value of neutral component of Higgs field $\langle H^0 \rangle = v/\sqrt{2}$ and the Yukawa-contact, gauge-invariant interaction of Higgs scalar with the fermions, so that $m_e = \lambda_e \langle H^0 \rangle$.

It is important to note that the observed hierarchy of masses for three known generations of fermions (two generations are much lighter than the third generation) can be naturally explained by the single scale of energy, i.e by the vacuum expectation value of Higgs scalar. Indeed, the introduction of symmetric, so called “democratic” matrix for the masses of Dirac spinors $\psi_{\alpha}$ distinguished only by the marker of generation $\alpha = \{1, 2, 3\}$ into the corresponding term of lagrangian $\mathcal{L}_M$ with the unified coupling constant $\lambda$ in the form of

$\mathcal{M}_D = \lambda \langle H^0 \rangle \left( \begin{array}{ccc} 1 & \lambda & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \rightarrow \mathcal{L}_M = \bar{\psi} \mathcal{M}_D \psi$, 

(4)
leads to two massless generations and the single heavy generation, while small corrections to the democracy of generations cause the opportunity to describe the mass spectrum of charged leptons and quarks by making use of “seesaw” mechanism \([7]\), i.e. by a small mixing of light states with the heavy state, but, importantly, without any introduction of new physical scales relevant to the differences of masses between the electron, muon and \(\tau\)-lepton, for instance.

In the case of quarks, those corrections to the mass matrix of “democratic” kind cause the formation of matrix for the mixing of charged weak currents, i.e. the Cabibbo–Kobayashi–Maskawa matrix \([8]\), too. Hence, this matrix of current mixing has also got the hierarchy, since it is formed by matrices of transitions from the initial basis of quark fields to the basis of physical massive states for the generations with different values of weak isospin. The transition matrices carry all of mentioned properties of symmetry for the generation in the Yukawa sector of theory. Therefore, we do not require the introduction of additional scales except the vacuum expectation value of Higgs field in the case of masses for charged fermions.

In respect of masses, the situation is crucially changed if we consider the phenomenology of neutrino, since the maximal mass of neutrinos is restricted by the constraint \(m_\nu < 0.2 \text{ eV} \)[2]. Therefore, the analogous mechanism for the generation of neutrino masses suggests that the initial Yukawa coupling constant in the matrix of Dirac masses of democratic kind should be twelve orders of magnitude less than the constant for the charged leptons, that evidently points to the essential difference in the mechanisms of mass generation for the charged and neutral particles\(^2\). Thus, the neutrinos involve the necessity of modifying the theory of mass generation by introducing an additional scale of energy\(^3\), whereas the naive introduction of low energy scale is in contradiction with the phenomenology, because, except the neutrino masses, we do not observe any unusual phenomena beyond the SM at energies much less than the scale of electroweak symmetry breaking, that witnesses on the fact that the low scale is not related to the dynamics at low energies, but it is reducible from the dynamics at high energy scales, which is not yet observed at the energies about the vacuum expectation value of Higgs field. Nevertheless, the “seesaw” mechanism allows us to make the step to the derivation of GUT structure in this sector, too (see Section \[11]\). However, in this way, the relation of masses with the vacuum expectation values suggests the introduction of additional neutral field with a new vacuum expectation value, that will be explicitly done below.

B. The charge quantization

As was mentioned in Introduction, the solution of electric charge quantization for the fermionic fields of SM is the unification group \(SU(5)\), whereas the fields are decomposed into fundamental representations, so that the antiquintet \(\bar{5}\) is given by

\[
\begin{pmatrix}
d^c \\
u \\
e^c
\end{pmatrix}_L,
\]

while the decuplet \(10\) is composed of \(u, d, u^c, e^c\) in the form of antisymmetric matrix \(5 \times 5\) (see the review on the group theory in \[9]\).

From the form of antiquintet we would see that the vector leptoquarks transforming the leptons into the \(d^c\)-antiquarks, are the part of gauge bosons of \(SU(5)\). If we assume that the same group is responsible for the charge quantization of scalar Higgs bosons, then we arrive to the conclusion that the Higgs scalars necessary compose the following antiquintet \(\bar{5}\):

\[
\begin{pmatrix}
K^c \\
H^0 \\
H^-
\end{pmatrix},
\]

with the color antitriplet-scalar leptoquark \(K^c\), possessing the charge \(Q = \frac{1}{3}\), the lepton number \(L = -1\) and the baryon number \(B = -\frac{1}{3}\), because this scalar antiquintet interacts with the same gauge vector bosons as the chiral antiquintet does\(^4\).

C. The supersymmetry

We can quite naturally suggest that the ultimate theory of grand unification is supersymmetric. Then the fermions and Higgs bosons are components of chiral superfields, including corresponding superpartners for the particles of SM (see the textbook \[11\]). A new quantum number is the \(R\)-parity, it marks the particles and their superpartners: the particles are assigned to positive parity \(R = +1\), while the superpartners have got the negative \(R\)-parity.

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\(^2\) The difference of Yukawa coupling constants in the democratic symmetry for generations of quarks and charged leptons is restricted by two orders of magnitude, that can be certainly explained by the renormalization group evolution of those constants from the scale of true “democracy”, when the constants have got the same order of magnitude, to the scale of breaking down the electroweak symmetry, since the renormalization group equations depend on the quantum numbers, which are different for the quark and charged leptons because of different values of weak isospin projection.

\(^3\) The single or a few.

\(^4\) If one does not use the argumentation on the transitions between the components of fermionic and scalar antiquintet due to the emission of gauge bosons with identical quantum numbers, then the triplet field \(K\) can get other values of lepton and baryon numbers (see review \[10\]).
In terms of chiral superfields the supersymmetric lagrangian of interactions is constructed as the polynomials of third power, whereas the mass term for the fields with the positive and negative projections of weak isospin appears only after the introduction of two Higgs scalars with conjugated quantum numbers,

\[
\begin{pmatrix}
  K^+ \\
  H^0_d \\
  H^+ \\
  H^0_u
\end{pmatrix},
\begin{pmatrix}
  K^- \\
  H^0_d \\
  H^- \\
  H^0_u
\end{pmatrix}.
\]

This fact is the consequence of supersymmetry of lagrangian, that leads to doubling the number of Higgs scalars. The doubling is also necessary for the cancelation of triangle quantum anomaly for the chiral currents: the introduction of fermionic superpartners for the single Higgs boson of SM would produce the additional contribution of this superpartner into the triangle diagram, and, hence, to the quantum anomaly, while the conjugated superpartner cancels the anomaly.

Thus, the chiral sector should be extended by addition of superpartners for the Higgs particles, i.e. due to the antiquintet 5 and quintet 5 with respect to SU(5):

\[
\begin{pmatrix}
  \chi^0_L \\
  \chi^+_L \\
  \chi^0_d_L
\end{pmatrix},
\begin{pmatrix}
  \chi^-_L \\
  \chi^+_L \\
  \chi^0_d_L
\end{pmatrix},
\]

with the negative R-parity.

### D. Neutrino masses

As was mentioned in Introduction, the observed scale of neutrino masses is not truly fundamental, this scale is reducible from factors given dynamically, hence, the neutrino masses can be calculated in terms of high energy-scales and their ratios. In this way, the primary scales of energy originate from the dynamics of gauge theory.

So, the mass of Dirac kind \( m_D \) is related to the breaking of electroweak symmetry, and \( m_D \sim v \sim 10^2 \) GeV, so that this kind of mass breaks the weak isospin, but it conserves both the lepton and baryon numbers, as well as the \( R \)-parity. Therefore, neutrinos could get the masses of Dirac kind under the introduction of electroweak left-handed singlet antineutrino \( \nu^e_L \) (the leptonic number is \( L = -1 \), the \( R \)-parity is equal to \( R = +1 \) as for the ordinary matter, and the baryon number is equal to zero \( B = 0 \), of course). However, the single Dirac contribution to the neutrino mass is not enough in order to get the necessary scale of neutrino masses.

The mass term of Majorana kind suggests the conservation of electroweak symmetry due to the zero values of electroweak charges of Majorana particle, i.e. the Majorana component of \( \nu_S \) should be the singlet with respect to the group of electroweak symmetry. Moreover, the term quadratic in \( \nu_S \) has to conserve both the lepton and baryon numbers. Therefore, the Majorana component is the sterile particle with zero values of lepton and baryon numbers (sterino), while, under the condition of \( R \)-parity conservation, its coupling to the neutrinos \( \{ \nu_L, \nu^e_L \} \) leads to \( R[\nu_S] = +1 \) and gives the zero constants of mixing with the neutral components of Higgs superpartners \( \chi^0_u \) and \( \chi^0_d \), because their \( R \)-parity is negative.

Therefore, the mass matrix of neutrinos is the matrix of \( 3 \times 3 \) for the components \( \{ \nu_L, \nu^e_L, \nu_S \} \), and it take the form of

\[
\mathcal{M} = \begin{pmatrix}
  0 & m_D & 0 \\
  m_D & 0 & \Lambda \\
  0 & \Lambda & M
\end{pmatrix},
\]

where \( m_D \) is the mass of Dirac kind, the Majorana mass is denoted by symbol \( M \), while the scale \( \Lambda \) determines the mixing between the electroweak singlet antineutrino \( \nu^e_L \) and sterino \( \nu_S \).

This mechanism suggests the explicit breaking of lepton number\(^5\) at the scale of \( \Lambda \), therefore, we could naturally put \( \Lambda \sim \Lambda_{\text{GUT}} \), because the grand unification of gauge interactions puts leptons and quarks into a common multiplet and makes them indistinguishable under the gauge transformations, hence, it is the most probable that breaking the lepton number is related to the dynamical scale of breaking the symmetry of grand unification.

As for the term of Majorana mass with the scale of \( M \), it does not generate any breaking of quantum numbers entering the charges of SM and marking the leptons and baryons. This fact can point to the relation with more universal dynamical characteristics, such as the breaking the local supersymmetry, i.e. the supergravity. Therefore, in the way of working hypothesis we could set that \( M \) gets the scale characteristic for the gravity, i.e. it is of the order of Planckian mass \( m_{\text{Pl}} \). This suggestion means that the mechanism generating the low scale of neutrino masses leads to the theory of grand unification, which essentially differs from models inspired by the superstring theory, when the fermionic multiplet of GUT is composed of particles with masses of the GUT scale or scales much less than it, because one considers the particles in the low energy spectrum, while the fields of Planckian scale are referred to the spectrum of excitations of superstring. In our construction the sterino as well as some other fields, which would be introduced below, explicitly belong to the sector of particles with the Planckian scale of mass.

The eigen values of matrix \( \mathcal{M} \) as well as the corresponding quantum states can be easily calculated, if we suggest the hierarchy of scales \( m_D, \Lambda \) and \( M \). Indeed, setting the parameters of matrix to be real, we perform, first, the rotation for two heavy states by the angle \( \theta_{23} \),

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\(^5\) If we set the nonzero value for the lepton number of sterino as for the left-handed electroweak doublet with neutrino, then the mixing of sterino with the singlet antineutrino would conserve the lepton number, but the lepton number would be broken by the introduction of Majorana mass term of sterino, which would be not the true Majorana particle, in this case.
into account for the mass hierarchy and after the definition of the matrix (12) allows for the consideration in the framework of the weak isospin and the lepton number. It seems that the weak isospin and the lepton number are conserved, if we assume the conservation of the parity. Therefore, we have to introduce the conjugated multiplet is called the mirror matter.

The introduction of nonzero vacuum expectation value 

By the spontaneous breaking of gauge symmetry the masses of chiral fermions are generated due to the vacuum expectation values of scalar fields, which appear for the electrically neutral fields, only, in the case of the broken electroweak symmetry, that does not exhibit the breaking of 

where we especially show the leading corrections to the masses of heavy neutrinos in order to demonstrate the valid relation of matrix trace to its initial value $\text{tr} M = M$. There is the hierarchy of neutrino masses in the following form:

$\tan 2\theta_{23} = \frac{2\Lambda}{M}, \quad (10)$

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix},$$

where $c_{23} = \cos \theta_{23}, s_{23} = \sin \theta_{23}$, so that the mass matrix gets the form of

$$\mathcal{M}_{23} = U_{23}^\dagger M U_{23} \approx \begin{pmatrix} 0 & m_D & m_D \Delta \Lambda \\ m_D & -\Lambda \Delta \frac{X}{M} & 0 \\ m_D \Delta \Lambda \frac{X}{M} & 0 & M + \Lambda \Delta \frac{X}{M} \end{pmatrix},$$

where we have used the smallness of ratio $\Lambda/M \ll 1$, hence,

$$\sin \theta_{23} \approx \frac{\Lambda}{M}, \quad \cos \theta_{23} \approx 1 - \frac{\Lambda^2}{2M^2}.$$

Matrix (12) allows for the consideration in the framework of stationary perturbation theory, so that under taking into account for the mass hierarchy and after the definition of composite scale $\Lambda_X = \Lambda^2/M \gg m_D$ we get

$$m_1 \approx \frac{m_D^2}{\Lambda_X}, \quad m_2 \approx \Lambda_X - m_1, \quad m_3 \approx M - \Lambda_X, \quad (13)$$

where we especially show the leading corrections to the masses of heavy neutrinos in order to demonstrate the valid relation of matrix trace to its initial value $\text{tr} M = M$. There is the hierarchy of neutrino masses in the following form:

$$m_1 \ll m_2 \ll m_3.$$

E. The vacuum

The listing of admissible vacuum expectation values allows us to make a question on the masses of Higgs particle superpartners: it is clear that the triplet higgsinos $\{\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3\}$ can be supermassive due to the scale of $M$, only, because their mass term does not break down both the weak isospin and the lepton number. It seems that the introduction of masses for the doublet higgsinos would follow the same arguments, but then it would be problematic to introduce the scale for the breaking of electroweak symmetry, that is inherently related to the scale of mixing the neutral fields $\{H^0_u, H^0_d\}$ and their superpartners due to the $\mu$-term in the lagrangian of superfields in the case of supersymmetry. This is the well known problem of doublet-triplet splitting for the Higgs fields. The problem is reduced to the hierarchy of scales, and it is essential for the proton lifetime, since the light neutrinos have non-zero masses.
triplet leptoquarks of Higgs sector are excluded by experimental constraints on the proton decay rate. However, the mechanism of generating the hierarchy of scales for the breaking of electroweak symmetry and gauge group of GUT, $v \ll \Lambda$, can be naturally invented in the case of several generations of fermions as would be discussed in Section III.

F. The multiplet of $E_6$

Thus, summing up the components necessary for the construction of chiral multiplet of single generation as described above, we get the 27-plet with the quantized charges:

1. leptons $e_L, e'_L, \nu_L, \nu'_L$ (4 components),
2. sterino $\nu_S$ (1 component),
3. color-triplet leptoquark higgsino $\chi$, weak isospin-doublets $\chi^0, \chi^-$ and the corresponding conjugated higgsinos (10 components),
4. colored quarks $u_L, u'_L, d_L, d'_L$ (12 components).

Such the set exactly and uniquely corresponds to the irreducible fundamental representation of group $E_6$. Significantly, the group has no chiral quantum anomalies.

III. GENERATIONS

The further unification of fermions into the multiplet of ultimate theory of grand unification has to account for the existence of several generations of fermions in SM, as well as for the superpartners of gauge bosons of group $E_6$, i.e. gauginos.

A. Gauginos of $E_6$

In the supersymmetric theory, the gauginos are transformed under the adjoint representation of symmetry group and they are Majorana particles. Therefore, for the group of $E_6$ there is the 78-plet of gauginos [9]. Then, the unification of gauginos with the chiral fermions of fundamental $E_6$-representation, i.e. with the 27-plet, into the common multiplet requires the introduction of conjugated chiral fundamental representation of $E_6$, i.e. 27-plet. Certainly, such the extension permits the self-conjugation of unified multiplet. The conjugated fundamental multiplet of $E_6$ is called the “mirror” matter. Each the generation of SM is associated with the generation of mirror matter in the ultimate GUT.

B. The horizontal symmetry of generations

If we temporary miss the Yukawa sector responsible for the introduction of fermion masses and current mixing, then the observed fermion generations become equivalent to each other with respect to the transformations of gauge group. Therefore, we can introduce the gauge symmetry of generations, that unitary mixes the generations, i.e. the “horizontal” group of symmetry, the group of unitary rotations of generations $SU_H(n_g)$, where $n_g$ is the number of generations. It transforms the fundamental representations of $E_6$. The actual number of generations is not less than 3.

The gauge bosons of horizontal symmetry are singlets with respect to the group $E_6$. The Majorana superpartners of those gauge bosons should be included into the fermionic multiplet of ultimate GUT. The number of gauge bosons of horizontal symmetry is equal to $n_g^2 - 1$.

Finally, the gauginos of $E_6$ do not carry any quantum numbers of generations, i.e. they are singlets with respect to the group of horizontal symmetry.

Thus, the self-conjugated multiplet of ultimate GUT includes

- $n_g$ fundamental representations of $E_6$ and $n_g$ conjugated representations of mirror matter,
- $n_g^2 - 1$ self-conjugated gauginos of horizontal symmetry as singlets with respect to $E_6$,
- gauginos of $E_6$ being singlets with respect to the horizontal symmetry.

Then, the expansion of multiplet under the direct product of groups for the horizontal symmetry and the symmetry of single generation $SU_H(n_g) \times E_6$ should get the form

$$(n_g, 27) + (\bar{n}_g, \overline{27}) + (1, 78) + (n_g^2 - 1, 1).$$

It is interesting that there is the unique simple group with the required expansion of irreducible multiplet at $n_g \geq 1$. This is the exceptional group $E_8$ [9], whereas $n_g = 3$, so that $E_8 \supset SU(3) \times E_6$, while the fundamental representation of minimal dimension is the self-conjugated 248-plet 9:

$$248 = (3, 27) + (\overline{3}, \overline{27}) + (1, 78) + (8, 1).$$

Therefore, the offered logics for the construction of ultimate GUT leads to the unique result, the group $E_8$. The inherent attribute of such the construction is the introduction of mirror generations of matter and Higgs particles. In this way, the condition of $R$-parity conservation can result in that the mass terms in the sector of matter are generated due to the vacuum expectation values of scalar particles in the mirror world, that does not change the justification of our approach to the construction of fermionic multiplet in GUT. In addition, we get the problem of decoupling for the mirror matter, probably being superheavy, that is shortly considered in the end of next subsection.
C. The mechanism of hierarchy

The existence of several generations for the Higgs superfields allows us naturally to introduce the hierarchy of scales for the breaking of electroweak and GUT symmetries in the same manner as was done for the hierarchy of masses for the fermion generations: the mass matrix of doublet components of Higgs particles can take the form corresponding to the “democracy” of Higgs particle generations. Therefore, the only generation would be supermassive, i.e. it would get the mass of the order of Planck scale $M$. For instance, the mixing given by the single scale $M$ for three generations of neutral higgsinos could be written in the form

$$L_{ud} = \lambda_{ud}^{-1}(\chi_{d1}, \chi_{d2}, \chi_{d3}) \left( \begin{array}{ccc} M & M & M \\ M & M & M \\ M & M & M \end{array} \right) \left( \begin{array}{c} \chi_{d1}^0 \\ \chi_{d2}^0 \\ \chi_{d3}^0 \end{array} \right),$$

that is supersymmetrically reflected in the mass matrix for the scalar Higgs particles.

Two lightest generations of Higgs doublets would get non-zero $\mu$-terms due to a small breaking of democratic symmetry in the Higgs sector, whereas the hierarchy is also admissible as for the quarks and charged leptons. Therefore, two lightest generations of Higgs doublets would split, so that the only generation of Higgs particles would determine the physics at the electroweak scale, while two other generations remain heavy with the Planckian and sub-Planckian masses. In this way, the evolution of bare masses to lower scales for the heavy generations of Higgs doublets is frozen, at least, under a sub-Planckian scale, where the heavy states have to decouple, hence, their propagation can be neglected. Then, these heavy generations do not acquire any non-zero vacuum expectations, i.e. they do not contribute to the masses of gauge bosons in the SM. In contrast, the primary mass-term of the lightest generation of Higgs particles will evolve up to the electroweak scale, whereas the minimum of potential is shifted from instable zero field to its vacuum expectation value of actual magnitude.

It is interesting to note that we have to introduce another kind of symmetry for the color-triplet components of Higgs particles: the nondegenerate matrix of generation masses of diagonal kind with the elements of the same order of $M$, but without any mixing. This method lifts the problem of splitting between the triplet and doublet components of Higgs field, because we can formulate quite the natural mechanism for the splitting with the only scale of $M$. Indeed, for the triplet higgsinos we get

$$L_{\chi} = \lambda_{\chi}^{-1}(\chi_1, \chi_2, \chi_3) \left( \begin{array}{ccc} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{array} \right) \left( \begin{array}{c} \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \right),$$

that corresponds to the analogous term for the scalar Higgs triplets.

Thus, the problem of scale hierarchy becomes closely related to the problem of triplet-doublet splitting for the Higgs fields. Moreover, these problems get the solution with the single dynamical scale $M$, and they are reduced to the mechanism of small breaking of “democratic” symmetry of generations in the Higgs sector, that is quite analogous to the problem on the mechanism of breaking of the “democratic” symmetry of generations in the Yukawa sector of theory.

In addition, we have to emphasize that two kinds of mass-matrices: non-generate and “democratic” ones, suggest the explicit breaking of SU(5)-symmetry in the Yukawa sector of theory. In this way, the scale of such breaking is Planckian, i.e. it is higher than $\Lambda$.

The same idea could be useful for the decoupling of mirror matter: in the mirror Higgs sector the mixing is determined by a nondegenerate diagonal matrix with the sterile scale $M \sim m_{\nu_{1}}$, hence, all of Higgs particles and fermions with the Dirac mass in the mirror world are supermassive. Consequently, the breaking of electroweak symmetry is decoupled from the mirror world with the scale of $M$. The masses of mirror neutrinos are Planckian, too (see Appendix D).

D. The supersymmetry

The unification of fermionic components for chiral superfields of matter and conjugated components of mirror matter into the common multiplet with the Majorana components of real gauge superfields points to the fact that the massless particles with the spin of $1, \frac{1}{2}$ and 0 enter the same superfield of $E_8$, that assumes two steps with increment of $\frac{1}{2}$ within the representation of supermultiplet, i.e. the supersymmetry of $N = 2$. However, this extended supersymmetry is not important for our consideration of fermionic multiplet, hence, it can be broken at a Planckian scale.

IV. CONCLUSION

Thus, we have shown how one can logically arrive to the unique variant of ultimate supersymmetric theory of grand unification with the group $E_6$ for the fundamental 27-plet of single generation of fermionic matter and higgsinos, as well as with the group $E_8$ for the fundamental 248-plet of three generations of fermions and gauginos.
Of course, the consideration of those groups themselves is not new in GUT: see original articles [12–14] and the comprehensive review on \( E_6 \) in [10], as well as the papers on \( E_8 \) [15, 16]. We especially note article [17], where the supersymmetric \( E_6 \) GUT is built in 2 steps by unifying two coupling constants at one scale \( \Lambda \) and further reaching the final unification of third and combined coupling constant at another scale \( M \gg \Lambda \), that is similar to our approach. This mechanism produces a new insight on the doublet-triplet splitting problem \( \Lambda \). However, our construction allows us not only uniquely to substantiate the set of irreducible representation in explicit nickname-components, but to introduce the natural mechanism for generating both the hierarchy of scales and the decoupling of superheavy states. This mechanism is certainly differs from the approaches on the market: the mechanism by Dimopoulos and Wilczek [19], the pseudosymmetry with the Nambu-Goldstone boson [20, 21] etc. [22, 23].

Let us emphasize that the offered schemes for the mass-matrices are realistic, but for their consistent substantiation certainly requires to write down both a Yukawa sector of interactions and a spectrum of vacuum expectations values for the scalar fields in the explicit form as well as to specify the potentials of self-action for these fields, that result in the spontaneous breaking of symmetry. These items are predominant directions for the further development of our model.

It is interesting to note that the algebras of isometries in the projective spaces of real \( \mathbb{R} \) and complex \( \mathbb{C} \) numbers, as well as quaternions \( \mathbb{H} \) are isomorphic to the algebras of generators for the infinite series of classical simple compact groups of special orthogonal, unitary and symplectic transformations [24]:

\[
\text{isom}(\mathbb{R}P^n) \cong \mathfrak{so}(n+1), \\
\text{isom}(\mathbb{C}P^n) \cong \mathfrak{su}(n+1), \\
\text{isom}(\mathbb{H}P^n) \cong \mathfrak{sp}(n+1),
\]

while the exceptional groups \( E_6 \), \( E_7 \), and \( E_8 \) are related to octonions \( \mathbb{O} \), so that there is the isomorphism for the isometries of projective planes [24]:

\[
E_6 \cong \text{Isom}(\mathbb{C}P^2), \\
E_7 \cong \text{Isom}(\mathbb{H}P^2), \\
E_8 \cong \text{Isom}(\mathbb{O}P^2),
\]

and the number of such the groups are finite because of the non-associativity of octonion algebra. Thus, the ultimate group of GUT is mathematically based on the maximal generalization of numbers, i.e. octonions.

The octonions are the non-associative generalization of quaternions, which are widely used in the representation of hermitian matrices of \( 2 \times 2 \) for the description of Lorentz symmetry of Minkowskian space-time. In addition, there are the following isomorphisms for the algebras of special linear transformations of two-component vectors:

\[
\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, 1), \\
\mathfrak{sl}(2, \mathbb{H}) \cong \mathfrak{so}(5, 1), \\
\mathfrak{sl}(2, \mathbb{O}) \cong \mathfrak{so}(9, 1),
\]

that probably points to a relation of ultimate group of GUT to the 10-dimensional space-time, which naturally appears in the theory of superstrings. In this respect, the relation of two ways for constructing the GUT becomes more predictable: the first way is given by the phenomenology of local quantum field theory as we have done in this paper, while the second way originates from the F-theory in superstrings [5]. These ways lead to the common result, i.e. the group \( E_8 \). However, remember that the multiplet constructed in this paper contains the fields with the masses of Planckian scale, that essentially differs from the multiplets in the models inspired by superstrings.

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### Appendix A: The mixing of mirror neutrinos

In the case of mirror neutrinos: the doublet, singlet and sterile particles, the mixing matrix takes the following form up to the overall Yukawa factor:

\[
\mathcal{M}_{\text{mir}} = \begin{pmatrix} 0 & M' & 0 \\ M' & 0 & \Lambda \\ 0 & \Lambda & M \end{pmatrix},
\]

where \( M' \sim M \sim m_{\text{PP}} \). Since \( \det \mathcal{M}_{\text{mir}} = -MM' \), this matrix is not degenerate, and we expect that all of massive states have got the same scale of mass of the order of \( M \). Indeed, let us start with the rotation of two states under the matrix

\[
U_{12}^{\text{mir}} = \begin{pmatrix} c_{12}^{\text{mir}} & s_{12}^{\text{mir}} & 0 \\ -s_{12}^{\text{mir}} & c_{12}^{\text{mir}} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where \( c_{12}^{\text{mir}} = s_{12}^{\text{mir}} = 1/\sqrt{2} \), so that

\[
U_{12}^{\text{mir}} \cdot \mathcal{M}_{\text{mir}} \cdot U_{12}^{\text{mir}} = \begin{pmatrix} -M' & 0 & -\Lambda \sqrt{2} \\ 0 & M' & \Lambda \sqrt{2} \\ -\Lambda \sqrt{2} & \Lambda \sqrt{2} & M \end{pmatrix},
\]

and further, we make the rotation by the angle \( \theta_{23}^{\text{mir}} \), given by the condition

\[
\tan 2\theta_{23}^{\text{mir}} = \frac{\Lambda \sqrt{2}}{M - M'},
\]
after which the mass matrix of mirror neutrinos $\tilde{M}_{\text{mir}} = U_{23}^\dagger U_{12}^\dagger \cdot M \cdot U_{12}^\dagger U_{23}^\dagger$ gets the form

$$
\tilde{M}_{\text{mir}} = \begin{pmatrix}
-M' & \Lambda \frac{\sin(\theta_{23})}{\sqrt{2}} & -\Lambda \frac{\cos(\theta_{23})}{\sqrt{2}} \\
\Lambda \frac{\sin(\theta_{23})}{\sqrt{2}} & M_2 - \tilde{\Lambda} & 0 \\
-\Lambda \frac{\cos(\theta_{23})}{\sqrt{2}} & 0 & M_3 + \tilde{\Lambda}
\end{pmatrix},
$$

where we denote

$$
M_2 = M'(c_{23}^{\text{mir}})^2 + M(s_{23}^{\text{mir}})^2; \\
M_3 = M'(s_{23}^{\text{mir}})^2 + M(c_{23}^{\text{mir}})^2; \\
\tilde{\Lambda} = \Lambda \frac{\sin(2\theta_{23})}{\sqrt{2}}.
$$

The perturbation theory is applicable to matrix (A2), so that the mirror neutrinos get the masses

$$
m_1^{\text{mir}} \approx -M' - \frac{\Lambda^2(s_{23}^{\text{mir}})^2}{2(M' + M_2)}, \\
m_2^{\text{mir}} \approx M_2 - \tilde{\Lambda} + \frac{\Lambda^2(s_{23}^{\text{mir}})^2}{2(M' + M_3)}, \\
m_3^{\text{mir}} \approx M_2 + \tilde{\Lambda} + \frac{\Lambda^2(c_{23}^{\text{mir}})^2}{2(M' + M_3)},
$$

As we have expected these values have got the same order of magnitude of Planckian scale. At this scale the breaking of lepton number for the mirror neutrinos becomes essential due to the mixing of initial states as given by both the rotations described above and the corrections in the perturbation theory. But we do not consider the mixing here.

[1] H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).
[2] K. Nakamura et al. [Particle Data Group], J. Phys. G 37, 075021 (2010).
[3] C. Beasley, J. J. Heckman and C. Vafa, JHEP 0901, 058 (2009) [arXiv:0802.3391 [hep-th]].
[4] C. Beasley, J. J. Heckman and C. Vafa, JHEP 0901, 059 (2009) [arXiv:0806.0102 [hep-th]].
[5] J. J. Heckman, [arXiv:1001.0577 [hep-th]].
[6] J. J. Heckman, A. Tavanfar and C. Vafa, JHEP 1008, 040 (2010) [arXiv:0906.0581 [hep-th]].
[7] H. Fritzsch, Phys. Lett. B 70, 436 (1977); H. Harari, H. Haut and J. Weyers, Phys. Lett. B 78, 459 (1978); H. Fritzsch, Nucl. Phys. B 155, 189 (1979); Y. Koide, Phys. Rev. D 28, 252 (1983); P. Kaus and S. Meshkov, Mod. Phys. Lett. A 3, 1251 (1988) [Erratum-ibid. A 4, 603 (1989)]; Y. Koide, Phys. Rev. D 39, 1391 (1989); H. Fritzsch and J. Plankl, Phys. Lett. B 237, 451 (1990).
[8] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).
[9] R. Slansky, Phys. Rept. 79, 1 (1981).
[10] J. L. Hewett and T. G. Rizzo, Phys. Rept. 183, 193 (1989).
[11] S. Weinberg, “The quantum theory of fields”, Volume III “Supersymmetry”, Cambridge University Press, 2000.
[12] F. Gursey, P. Ramond and P. Sikivie, Phys. Lett. B 60, 177 (1976).
[13] S. F. King, S. Moretti and R. Nevzorov, Phys. Rev. D 73, 035009 (2006) [arXiv:hep-ph/0510419].
[14] S. F. King, S. Moretti and R. Nevzorov, Phys. Lett. B 634, 278 (2006) [arXiv:hep-ph/0511256].
[15] I. Bars and M. Gunaydin, Phys. Rev. Lett. 45, 859 (1980).
[16] V. Braun, Y. H. He, B. A. Ovrut and T. Pantev, JHEP 0506, 039 (2005) [arXiv:hep-th/0502155].
[17] W. Kilian and J. Reuter, Phys. Lett. B 642, 81 (2006) [arXiv:hep-ph/0606277].
[18] F. Braam, A. Knochel and J. Reuter, JHEP 1006, 013 (2010) [arXiv:1001.4074 [hep-ph]].
[19] S. Dimopoulos and F. Wilczek, In the Proceedings of 19th International School of Subnuclear Physics: The Unity of the Fundamental Interactions, Erice, Italy, 31 Jul - 11 Aug 1981, pp 237-249.
[20] R. Barbieri, G. R. Dvali and A. Strumia, Nucl. Phys. B 391, 487 (1993).
[21] L. Randall and C. Csaki, [arXiv:hep-ph/9508208].
[22] G. F. Giudice and A. Masiero, Phys. Lett. B 266, 480 (1988).
[23] J. E. Kim and H. P. Nilles, Mod. Phys. Lett. A 9, 3575 (1994) [arXiv:hep-ph/9406296].
[24] J. C. Baez, [arXiv:math/0105155].