Numerical analysis of two Galerkin discretizations with graded temporal grids for fractional evolution equations

Binjie Li*, Tao Wang†, and Xiaoping Xie‡

1School of Mathematics, Sichuan University
2South China Research Center for Applied Mathematics and Interdisciplinary Studies, South China Normal University

March 9, 2020

Abstract

Two numerical methods with graded temporal grids are analyzed for fractional evolution equations. One is a low-order discontinuous Galerkin (DG) discretization in the case of fractional order $0 < \alpha < 1$, and the other one is a low-order Petrov Galerkin (PG) discretization in the case of fractional order $1 < \alpha < 2$. By a new duality technique, pointwise-in-time error estimates of first-order and $(3 - \alpha)$-order temporal accuracies are respectively derived for DG and PG, under reasonable regularity assumptions on the initial value. Numerical experiments are performed to verify the theoretical results.

Keywords: fractional diffusion-wave equation, graded temporal grid, convergence

1 Introduction

Let $X$ be a separable Hilbert space with inner product $(\cdot, \cdot)_X$. Assume that the linear operator $A : D(A) \subset X \to X$ is densely defined and admits a bounded inverse $A^{-1} : X \to X$, which is compact, symmetric and positive. Consider the following time fractional evolution equation:

$$
(D_0^\alpha (u - u_0))(t) + Au(t) = 0, \quad 0 < t \leq T,
$$

where $\alpha \in (0, 2) \setminus \{1\}$, $0 < T < \infty$, $u_0 \in X$ and $D_0^\alpha$ is a Riemann-Liouville fractional derivative operator of order $\alpha$. Note that (1) is usually called as a time fractional diffusion or wave equation when $A$ is a second order elliptic operator.
There are quite a few research works on the numerical treatment of time fractional evolution equations. Let us briefly introduce four types of numerical methods for the discretization of time fractional evolution equations. The first-type methods use convolution quadrature to approximate the fractional integral (derivative). These methods is very effective, but they require the temporal grid to be uniform (cf. [15, 16, 2, 35, 5]). The second-type methods use L1 scheme to approximate the fractional derivative (cf. [30, 4, 31, 11, 14]). Such methods are popular and easy to implement. The third-type methods are spectral methods (cf. [8, 32, 13, 19, 33]), which use nonlocal basis functions to approximate the solution. The accuracy of spectral methods is high, provided that the solution or data is smooth enough. The fourth-type methods are finite element methods (cf. [22, 23, 20, 18, 9, 12]), which use local basis functions to approximate the solution. These methods are time-stepping, and easy to design high order schemes. It should be mentioned that the finite element method is identical to the L1 scheme in some cases (cf. [6, 11]).

Most of the convergence analyses for the numerical methods mentioned above are based on the assumption that the exact solution is smooth enough. However, the solution of a fractional equation generally has singularity near the origin despite how smooth the data is (cf. [5, 7]). In fact, the main difficulty is to derive the error estimates without any regularity restriction on the solution, especially for the case with nonsmooth data. When using uniform temporal grids, the Laplace transform technique is a powerful tool for error estimation in case of nonsmooth data (cf. [16, 2, 20, 4, 31, 11]). We note that the non-uniform temporal grids is also useful to handle the singularity of fractional equations (cf. [21, 29, 14, 24]), but the corresponding numerical analysis seems rather complicated.

McLean and Mustapha [21] analyzed DG methods with graded temporal grids for a variant form of (1):

\[
\partial_t u + D^{1-\alpha}_{0+} Au(t) = 0, \quad 0 < t \leq T,
\]

\[
u(0) = u_0,
\]

which is obtained by applying \(D^{1-\alpha}_{0+}\) to the both sides of (1). For (2) with \(0 < \alpha < 1\), they derived first-order temporal accuracy for a piecewise-constant DG under the condition that \(u_0 \in D(A^\nu)\) for \(\nu > 0\). For the case \(1 < \alpha < 2\), they proved optimal error bounds for the piecewise-constant DG and a piecewise-linear DG under the condition that

\[
t\|A\partial_t u(t)\|_X + t^2\|A\partial_{tt} u(t)\|_X \leq C t^{\sigma - 1}, \quad 0 < t \leq T,
\]

\[
\|\partial_t u(t)\|_X + t\|\partial_{tt} u(t)\|_X \leq C t^{\sigma - 1}, \quad 0 < t \leq T,
\]

where \(\sigma > 0\) is a constant. For a fractional reaction-subdiffusion equation, Mustapha [24] derived second-order temporal accuracy for the L1 scheme with graded temporal grids under the condition that

\[
\|u(t)\|_{H^2} \leq C, \quad \|\partial_t u(t)\|_{H^2} + t^{1-\alpha/2}\|\partial_{tt} u(t)\|_{H^1} + t^{2-\alpha/2}\|\partial_{ttt} u(t)\|_{H^1} \leq C t^{\sigma - 1},
\]

for all \(0 < t \leq T\).

Though being equivalent to (2) in some sense, equation (1) leads to different kinds of numerical methods. For the fractional diffusion equation with nonsmooth data, Li et al. [10] obtained optimal error estimates for a low order DG. It
should be noticed that their analysis is optimal in the sense of some space-time Sobolev norms, which is not very sharp compared with the pointwise-in-time error estimates. For a fractional diffusion equation, Stynes et al. [29] analyzed the L1 scheme with graded temporal grids and derived temporal accuracy $O(N^{-α-2})$ ($N$ is the number of nodes in the temporal grid) under the condition that
\[
\|\partial_x^4 u(t)\|_{L^\infty} \leq C, \quad \|\partial_t u(t)\|_{L^\infty} \leq Ct^{-α-2}, \quad 0 < t \leq T.
\]
Liao et al. [14] obtained temporal accuracy $O(N^{-α-2})$ for a reaction-subdiffusion equation by assuming that
\[
\|\partial_x^4 u(t)\|_{L^2} \leq C, \quad \|\partial_{tt} u(t)\|_{L^2} \leq Ct^{-2}, \quad 0 < t \leq T,
\]
where $σ \in (0, 2) \setminus \{1\}$. Although the regularity assumptions above are reasonable in some situations, it is worthwhile to carry out error estimation for some numerical methods with lesser regularity assumptions on the data. Moreover, as far as we know, there is no rigorous numerical analysis for (1) with $1 < α < 2$ and graded temporal grids.

In this paper, we consider the DG and PG approximations for time fractional evolution equation (1) with $0 < α < 1$ and $1 < α < 2$ respectively. These methods are identical to the L1 scheme when the temporal grid is uniform. We develop a new duality technique for the pointwise-in-time error estimation, which is inspired by the local error estimation for the standard linear finite element method [28, 1]. The key point of the analysis is the weighted estimate of a “regularized Green function” (cf. Lemmas 3.3 and 4.2). For $0 < α < 1$ and $u_0 \in D(A^\nu)$ with $0 < \nu \leq 1$, we obtain the first-order temporal accuracy for the DG approximation with graded grids (cf. Theorem 3.1). For $1 < α < 2$ and $u_0 \in D(A^\nu)$ with $1/2 < \nu \leq 1$, we obtain the $(3-α)$-order temporal accuracy for the PG approximation with graded grids (cf. Theorem 4.1).

The rest of this paper is organized as follows. Section 2 gives some notations and basic results, including Sobolev spaces, fractional calculus operators, spectral decomposition of $A$, solution theory and discretization spaces. Section 3 and Section 4 establish the error estimates for problem (1) with $0 < α < 1$ and $1 < α < 2$ respectively. Section 5 performs two numerical experiments to verify the theoretical results. The last section is a conclusion.

2 Preliminaries

Throughout this paper, we will use the following conventions: if $ω \subset \mathbb{R}$ is an interval, then $\langle p, q \rangle_ω$ denotes the Lebesgue or Bochner integral $\int_ω pq$ for scalar or vector valued functions $p$ and $q$ whenever the integral makes sense; for a Banach space $W$, we use $\langle \cdot, \cdot \rangle_W$ to denote a duality paring between $W^*$ (the dual space of $W$) and $W$; the notation $C_\nu$ denotes a positive constant depending only on its subscript(s), and its value may differ at each occurrence; for any function $v$ defined on $(0, T)$, by $v(t-)$, $0 < t \leq T$ we mean $\lim_{s \to t-} v(s)$ whenever this limit exists; given $0 < a \leq T$, the notation $(a-t)_+$ denotes a function of variable $t$ defined by
\[
(a-t)_+ := \begin{cases} 
    a-t & \text{if } 0 \leq t < a, \\
    0 & \text{if } a \leq t \leq T.
\end{cases}
\]
Sobolev spaces. Assume that $-\infty < a < b < \infty$. For any $m \in \mathbb{N}$, define

$$\mathcal{D}^m(a,b) := \left\{ v \in H^m(a,b) : \left. v^{(k)}(a) = 0 \right\} \quad \forall 0 \leq k < m \right\}$$

and endow this space with the norm

$$\| v \|_{\mathcal{D}^m(a,b)} := \| v^{(m)} \|_{L^2(a,b)} \quad \forall v \in \mathcal{D}^m(a,b),$$

where $H^m(a,b)$ is an usual Sobolev space and $v^{(k)}$, $1 \leq k \leq m$, is the $k$-th order weak derivative of $v$. For any $m \in \mathbb{N}_{>0}$ and $0 < \theta < 1$, define

$$\mathcal{D}^{m-1+\theta}(a,b) := \left( \mathcal{D}^{m-1}(a,b), \mathcal{D}^m(a,b) \right)_{\theta,2},$$

where $(\cdot, \cdot)_{\theta,2}$ means the interpolation space defined by the $K$-method [17]. The space $0H^\gamma(a,b)$, $0 \leq \gamma < \infty$, is defined analogously. For each $-\infty < \gamma < 0$, we use $0H^\gamma(a,b)$ and $0H^{-\gamma}(a,b)$ to denote the dual spaces of $0H^{-\gamma}(a,b)$ and $0H^{-\gamma}(a,b)$, respectively. The embedding $L^2(a,b) \hookrightarrow 0H^{-\gamma}(a,b)$, $\gamma > 0$, is understood in the conventional sense that

$$\langle v, w \rangle_{0H^\gamma(a,b)} := \langle v, w \rangle_{(a,b)} \quad \forall w \in 0H^{-\gamma}(a,b), \quad \forall v \in L^2(a,b).$$

We will also use the following space:

$$H^{2\theta}(a,b) := \left( L^2(a,b), H^2(a,b) \right)_{\theta,2}, \quad \theta \in (0,1).$$

Note that if $0 < \gamma < 1/2$ then

$$0H^\gamma(a,b) = 0H^{-\gamma}(a,b) = H^{\gamma}(a,b) \quad \text{with equivalent norms.}$$

Fractional calculus operators. Assume that $-\infty < a < b < \infty$. For $-\infty < \gamma < 0$, define

$$\left( \mathcal{D}_{\alpha+}^\gamma v \right)(t) := \frac{1}{\Gamma(-\gamma)} \int^t_a \left( t - s \right)^{-\gamma-1} v(s) \, ds, \quad a < t < b,$$

$$\left( \mathcal{D}_{\alpha-}^\gamma v \right)(t) := \frac{1}{\Gamma(-\gamma)} \int^b_t \left( s - t \right)^{-\gamma-1} v(s) \, ds, \quad a < t < b,$$

for all $v \in L^1(a,b)$, where $\Gamma(\cdot)$ is the gamma function. In addition, let $\mathcal{D}_{\alpha+}^0$ and $\mathcal{D}_{\alpha-}^0$ be the identity operator on $L^1(a,b)$. For $j - 1 < \gamma < j$ with $j \in \mathbb{N}_{>0}$, define

$$\mathcal{D}_{\alpha+}^\gamma v := \mathcal{D}^j \mathcal{D}_{\alpha+}^{\gamma-j} v,$$

$$\mathcal{D}_{\alpha-}^\gamma v := (- \mathcal{D})^j \mathcal{D}_{\alpha-}^{\gamma-j} v,$$

for all $v \in L^1(a,b)$, where $\mathcal{D}$ is the first-order differential operator in the distribution sense. The vector-valued version fractional calculus operators are defined analogously. Assume that $0 < \beta \leq \gamma < \beta + 1/2$. For any $v \in 0H^\beta(a,b)$, define $\mathcal{D}_{\alpha+}^\gamma v \in 0H^{\beta-\gamma}(a,b)$ by that

$$\langle \mathcal{D}_{\alpha+}^\gamma v, w \rangle_{0H^{\beta-\gamma}(a,b)} := \langle \mathcal{D}_{\alpha+}^\gamma v, \mathcal{D}_{\alpha-}^{-\beta} w \rangle_{(a,b)}$$

\[ \text{4} \]
for all \( w \in 0^H \gamma^{-\beta}(a, b) \). For any \( v \in 0^H \beta(a, b) \), define \( D^\gamma_{b-} v \in 0^H \gamma^{-\beta}(a, b) \) by that
\[
\langle D^\gamma_{b-} v, w \rangle_{\gamma^{-\beta}(a, b)} := \langle D^\gamma_{b-} v, D^\gamma_{a+} w \rangle_{(a, b)}
\]
for all \( w \in 0^H \gamma^{-\beta}(a, b) \). By Lemma A.2 and a standard density argument, it is easy to verify that the above definitions are well-defined and that if
\[
\langle D^\gamma_{b-} v, w \rangle_{\gamma^{-\beta}(a, b)}
\]
both make sense by the definition, then they are identical.

**Spectral decomposition of** \( A \). Assume that the separable Hilbert space \( X \) is infinite dimensional. It is well known that (cf. [34]) there exists an orthonormal basis, \( \{ \phi_n : n \in \mathbb{N} \} \subset D(A) \), of \( X \) such that
\[
A\phi_n = \lambda_n \phi_n,
\]
where \( \{ \lambda_n : n \in \mathbb{N} \} \) is a positive non-decreasing sequence and \( \lambda_n \to \infty \) as \( n \to \infty \). For any \( -\infty < \beta < \infty \), define
\[
D(A^{\beta/2}) := \left\{ \sum_{n=0}^{\infty} c_n \phi_n : \sum_{n=0}^{\infty} \frac{\lambda_n^{\beta/2} c_n^2}{n} < \infty \right\}
\]
and equip this space with the norm
\[
\| \sum_{n=0}^{\infty} c_n \phi_n \|_{D(A^{\beta/2})} := \left( \sum_{n=0}^{\infty} \frac{\lambda_n^{\beta/2} c_n^2}{n} \right)^{1/2}.
\]

**Solution theory.** For any \( \beta > 0 \), define the Mittag-Leffler function \( E_{\alpha,\beta}(z) \) by
\[
E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k \alpha + \beta)} \quad \forall z \in \mathbb{C},
\]
which admits the following growth estimate (cf. [26]):
\[
|E_{\alpha,\beta}(-t)| \leq \frac{C_{\alpha,\beta}}{1 + t} \quad \forall t > 0.
\] (3)
For any \( \lambda > 0 \), a straightforward calculation yields
\[
D^\alpha_{0+} (E_{\alpha,1}(-\lambda t^\alpha) - 1) + \lambda E_{\alpha,1}(-\lambda t^\alpha) = 0 \quad \forall t \geq 0.
\] (4)
Therefore, the solution to problem (1) is of the form (cf. [27])
\[
u(t) = \sum_{n=0}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha)(u_0, \phi_n) X \phi_n, \quad 0 \leq t \leq T.
\] (5)
For any \( 0 < t \leq T \), a straightforward calculation gives
\[
u'(t) = -\sum_{n=0}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)(u_0, \phi_n) X \phi_n,
\]
\[
u''(t) = -\sum_{n=0}^{\infty} \lambda_n t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)(u_0, \phi_n) X \phi_n.
\]
Hence, for $1 < \alpha < 2$, by (3) we obtain that
\[
\ell^{-1} \|u'(t)\|_X + \|u''(t)\|_X \leq C_\alpha \ell^{\alpha/2-2} \|u_0\|_{D(A)'} ,
\]
(6)
\[
\ell^{-1} \|u'(t)\|_{D(A)^{1/2}} + \|u''(t)\|_{D(A)^{1/2}} \leq C_\alpha \ell^{\alpha(\nu-1/2)-2} \|u_0\|_{D(A)} ,
\]
(7)
where $0 \leq \nu \leq 1$.

**Discretization spaces.** Let $t_j := (j/J)^\sigma T$ for each $0 \leq j \leq J$, where $J \in \mathbb{N}_{>0}$ and $\sigma \geq 1$. Define
\[
W_\sigma := \{ v \in L^\infty(0,T; D(A)^{1/2}) : v \text{ is constant on } (t_{j-1}, t_j) \text{ for each } 1 \leq j \leq J \},
\]
(8)
\[
W^e_\sigma := \{ v \in C([0,T]; D(A)^{1/2}) : v \text{ is linear on } (t_{j-1}, t_j) \text{ for each } 1 \leq j \leq J \}.
\]
For the particular case $D(A) = \mathbb{R}$, we use $W_\sigma$ and $W^e_\sigma$ to denote $W_\sigma$ and $W^e_\sigma$, respectively. Assume that $Y = X$ or $\mathbb{R}$. For any $v \in L^1(0,T;Y)$ and $w \in C([0,T];Y)$, define $Q_\sigma v \in L^\infty(0,T;Y)$ and $I_\sigma w \in C([0,T];Y)$ respectively by
\[
(Q_\sigma v)(t) := \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v \quad \text{and} \quad (I_\sigma w)(t) := \frac{t_j - t}{t_j - t_{j-1}} w(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}} w(t_j)
\]
for all $t_{j-1} < t < t_j$ and $1 \leq j \leq J$. In the sequel, we will always assume that $\sigma \geq 1$.

**3 Fractional diffusion equation** $\ (0 < \alpha < 1)$

This section considers the following discretization: seek $U \in W_\sigma$ such that
\[
\int_0^T ((D_0^\alpha + A)U, V) \, dt = \int_0^T (D_0^\alpha u_0, V) \, dt \quad \forall V \in W_\sigma .
\]
(9)

**Remark 3.1.** By (5), a straightforward calculation yields that
\[
\int_0^T ((D_0^\alpha u + A)u, V) \, dt = \int_0^T (D_0^\alpha u_0, V) \, dt \quad \forall V \in W_\sigma .
\]
(10)

**Remark 3.2.** We note that when using uniform temporal grids, the discretization (8) is equivalent to the $L^1$ scheme [6].

**Theorem 3.1.** Assume that $u_0 \in D(A')$ with $0 < \nu \leq 1$. Then
\[
\|u - U\|_{L^\infty(0,T;X)} \leq C_{\alpha,\sigma,\nu,T} \ell^{-\min\{\alpha,\nu,1\}} \|u_0\|_{D(A')} .
\]
(11)

The main task of the rest of this section is to prove Theorem 3.1. To this end, we proceed as follows. Assume that $\lambda > 0$. For any $y \in \alpha H^{\alpha/2}(0,T)$, define $\Pi_\lambda^\alpha y \in W_\sigma$ by that
\[
\langle (D_0^\alpha + \lambda)(y - \Pi_\lambda^\alpha y), w \rangle_{\alpha H^{\alpha/2}(0,T)} = 0 \quad \forall w \in W_\sigma .
\]
(12)
For each $1 \leq m \leq J$, define $G_\lambda^m \in W_\sigma$ by that $G_\lambda^m |_{(t_m, T)} = 0$ and
\[
\langle w, (D_{t_m}^\alpha + \lambda)G_\lambda^m \rangle_{(0,t_m)} = \frac{1}{t_m - t_{m-1}} \int_{t_{m-1}}^{t_m} w
\]
for all $w \in W_\sigma$. In addition, let $G_\lambda^{m+1} := 0$ and, for each $1 \leq j \leq m$, let
\[
G_\lambda^j := \lim_{t \to t_j-} G_\lambda^m (t).
\]
**Remark 3.3.** The $G^m_{\lambda,k}$ can be viewed as a regularized Green function with respect to the operator $D_{m,-}^\alpha + \lambda$.

**Lemma 3.1.** For each $1 \leq m \leq J$,

\begin{align}
G^m_{\lambda,m} > G^m_{\lambda,m-1} > \ldots > G^m_{\lambda,1} > 0, \\
G^m_{\lambda,m} &= \frac{1}{(t_m - t_{m-1})^{1-\alpha}/\Gamma(2-\alpha) + \lambda(t_m - t_{m-1})}, \\
G^m_{\lambda,m} &= \sum_{j=1}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) (t_j^{1-\alpha} \lambda = \sum_{j=1}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) (t_j^{1-\alpha} - (t_j - t_k)^{1-\alpha} + \lambda(t_j - t_k)) \lambda,j \Gamma(2-\alpha)t_1), \tag{15}
\end{align}

Proof. Let us first prove that

\[ G^m_{\lambda,j+1} > G^m_{\lambda,j} \quad \text{for all} \quad 1 \leq j < m. \tag{16} \]

For any $1 \leq k < m$, by (12) we obtain

\[ \sum_{j=k}^{m} (G^m_{\lambda,j} - G^m_{\lambda,j+1}) ((t_j - t_{k-1})^{1-\alpha} = \mu(t_k - t_{k-1}) G^m_{\lambda,k} = 0, \]

where $\mu := \lambda \Gamma(2-\alpha)$, so that a simple algebraic computation yields

\[ \sum_{j=k}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) ((t_j - t_{k-1})^{1-\alpha} = \lambda(t_k - t_{k-1}) G^m_{\lambda,k} = \mu(t_k - t_{k-1}) \lambda,j \Gamma(2-\alpha)t_1). \tag{17} \]

Inserting $k = m - 1$ into the above equation and noting the fact $G^m_{\lambda,m} > 0$ indicate $G^m_{\lambda,m} > G^m_{\lambda,m-1}$. Assume that $G^m_{\lambda,j+1} > G^m_{\lambda,j}$ for all $1 \leq j < m$, where $2 \leq k < m$. Multiplying both sides of (17) by

\[ \frac{(t_m - t_{k-1})^{1-\alpha} - (t_m - t_{k-1})^{1-\alpha} + \mu(t_k - t_{k-1})}{(t_m - t_{k-1})^{1-\alpha} - (t_m - t_{k-1})^{1-\alpha} + \mu(t_k - t_{k-1})}, \]

from Lemma B.2 we obtain

\[ \sum_{j=k}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) ((t_j - t_{k-2})^{1-\alpha} = \lambda(t_k - t_{k-1})^\alpha + \mu(t_k - t_{k-2}) \lambda,j \Gamma(2-\alpha)t_1). \]

Similarly to (17), we have

\[ \sum_{j=k-1}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) ((t_j - t_{k-2})^{1-\alpha} = \lambda(t_k - t_{k-1})^\alpha + \mu(t_k - t_{k-2}) \lambda,j \Gamma(2-\alpha)t_1). \]

Combining the above two equations yields $G^m_{\lambda,k} > G^m_{\lambda,k-1}$. Therefore, (16) is proved by induction.

Next, inserting $k = 1$ into (17) yields

\[ \sum_{j=1}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) ((t_j - t_1)^{1-\alpha} = \lambda(t_1)^\alpha + \mu(t_1) \lambda,j \Gamma(2-\alpha)t_1). \tag{18} \]
Since
\[ t^1_{m-\alpha} - (t_j - t_1)^{1-\alpha} + \mu t_1 > t^1_{m-\alpha} - (t_m - t_1)^{1-\alpha} + \mu t_1 \quad \forall 1 \leq j \leq m - 1, \]
from (16) and (18) it follows that
\[ \sum_{j=1}^{m-1} (G^m_{\lambda,j+1} - G^m_{\lambda,j}) < G^m_{\lambda,m}. \]
This implies \( G^m_{\lambda,1} > 0 \) and hence proves (13) by (16).

Finally, (14) is evident by (12), and dividing both sides of (18) by \( t^1_{m-\alpha} - (t_m - t_1)^{1-\alpha} + \mu t_1 \) proves (15). This completes the proof.

**Lemma 3.2.** For each \( 1 \leq k \leq J \),
\[ \sum_{j=1}^{k} j^{(\sigma-1)(\alpha-1)} \| (I - Q_T)(t_k - t)^{-\alpha} \|_{L^1(t_{j-1}, t_j)} \leq C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)}, \tag{19} \]
\[ \sum_{j=1}^{k} j^{-\sigma+\alpha+1} \| (I - Q_T)(t_k - t)^{-\alpha} \|_{L^1(t_{j-1}, t_j)} \leq C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)k^{-\alpha}}. \tag{20} \]

**Proof.** A straightforward calculation gives
\[ k^{(\sigma-1)(\alpha-1)} \| (I - Q_T)(t_k - t)^{-\alpha} \|_{L^1(t_{k-1}, t_k)} \leq C_{\alpha} k^{(\sigma-1)(\alpha-1)} (t_k - t_{k-1})^{1-\alpha} \leq C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)} \]
and
\[ \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} \| (I - Q_T)(t_k - t)^{-\alpha} \|_{L^1(t_{j-1}, t_j)} \leq C_{\alpha} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} (t_k - j^{-\alpha}) - (t_k - (j-1)^{-\alpha}) \leq C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} (j^{\sigma} - (j-1)^{\sigma}) \cdot (k^{\sigma} - j^{\sigma})^{-\alpha} \cdot (k^{\sigma} - (j-1)^{\sigma})^{-\alpha} \leq C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} j^{2(\sigma-1)} (k^{\sigma} - j^{\sigma})^{-\alpha-1} = C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha+1)} (k^{\sigma} - j^{\sigma})^{-\alpha-1} \leq C_{\alpha,\sigma,T} J^{\sigma(1-\alpha)} \quad \text{(by Lemma B.4)}. \]

Combining the above two estimates proves (19). Similarly, a simple calculation gives
\[ k^{-\sigma+\alpha+1} \| (I - Q_T)(t_k - t)^{-\alpha} \|_{L^1(t_{k-1}, t_k)} \leq C_{\alpha} k^{-\sigma+\alpha+1} (t_k - t_{k-1})^{1-\alpha} \leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)k^{-\alpha}}. \]
We have

Lemma 3.3. For each \(1 \leq m \leq J\),

\[
\sum_{j=1}^{m} \left( \frac{m}{j} \right)^{(\sigma-1)(1-\alpha)} \| (I - Q_{\tau}) D_{t_{m}}^{\alpha} - G_{\lambda}^{\alpha} \|_{L^{1}(t_{j-1}, t_{j})} \leq C_{\alpha, \sigma, T}.
\]

(21)

Proof. For each \(1 \leq j \leq m\), let

\[
\eta_{j}^{m} := \frac{(J/j)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} j^{(\sigma-1)(\alpha-1)} j^{\sigma(1-\alpha)}.
\]

(22)

Since

\[
(D_{t_{m}}^{\alpha} - G_{\lambda}^{\alpha})(t) = \sum_{j=1}^{m} (G_{\lambda,j}^{\alpha} - G_{\lambda,j+1}^{\alpha}) \frac{(t_{j} - t)^{-\alpha}}{\Gamma(1 - \alpha)},
\]

we have

\[
\sum_{j=1}^{m} \eta_{j}^{m} \| (I - Q_{\tau}) D_{t_{m}}^{\alpha} - G_{\lambda}^{\alpha} \|_{L^{1}(t_{j-1}, t_{j})}
\]

\[
\leq \frac{1}{\Gamma(1 - \alpha)} \sum_{j=1}^{m} \eta_{j}^{m} \sum_{k=j}^{m} (G_{\lambda,k}^{\alpha} - G_{\lambda,k+1}^{\alpha}) \| (I - Q_{\tau})(t_{k} - t)^{-\alpha} \|_{L^{1}(t_{j-1}, t_{j})}
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \sum_{k=1}^{m} (G_{\lambda,k}^{\alpha} - G_{\lambda,k+1}^{\alpha}) \sum_{j=1}^{k} \eta_{j}^{m} \| (I - Q_{\tau})(t_{k} - t)^{-\alpha} \|_{L^{1}(t_{j-1}, t_{j})}
\]

\[
\leq C_{\alpha, \sigma, T} \sum_{k=1}^{m} (G_{\lambda,k}^{\alpha} - G_{\lambda,k+1}^{\alpha}) \frac{J^{\sigma(1-\alpha)}}{(J/m)^{\sigma\alpha} + \lambda}
\]

\[
\times \sum_{j=1}^{k} \left( (J/j)^{\sigma\alpha} + \lambda \right) j^{(\sigma-1)(\alpha-1)} \| (I - Q_{\tau})(t_{k} - t)^{-\alpha} \|_{L^{1}(t_{j-1}, t_{j})}
\]

\[
\leq C_{\alpha, \sigma, T} \sum_{k=1}^{m} \frac{(J/k)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} |G_{\lambda,k}^{\alpha} - G_{\lambda,k+1}^{\alpha}| \quad \text{(by (19) and (20)).}
\]
Therefore, from Lemma 3.1 and the inequality
\[ t_k^{1-\alpha} - (t_k - t_1)^{1-\alpha} + \lambda \Gamma(2-\alpha)t_1 \geq C_{\alpha,\frac{\tau}{m}} (J/k)^{\sigma\alpha} + \lambda \frac{(J/m)^{\sigma\alpha} + \lambda}{(J/m)^{\alpha}} \]
it follows that
\[ \sum_{j=1}^{m} \eta_j^m \|(I - Q_{\tau}) D_{t_m - \vec{G}_{\lambda, m}^m} \|_{L^1(t_j-1, t_j)} \leq C_{\alpha,\sigma,T} G_{\lambda,m}^m. \]
In addition, by (14) and (22), it holds
\[ \frac{m^m}{G_{\lambda,m}^m} \geq C_{\alpha,\sigma,T} \frac{(J/j)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} j^{(\sigma-1)(\alpha-1)} j^{\sigma(1-\alpha)} (m^{(\sigma-1)(1-\alpha)} j^{\sigma(1-\alpha)} + \lambda m^{\sigma-1} J^{-\sigma}) \]
\[ \geq C_{\alpha,\sigma,T} (J/j)^{\sigma\alpha} + \lambda j^{(\sigma-1)(\alpha-1)} m^{(\sigma-1)(1-\alpha)} \]
\[ \geq C_{\alpha,\sigma,T} (m/j)^{(\sigma-1)(1-\alpha)}. \]
Consequently, combining the above two estimates proves (21) and thus concludes the proof. □

**Remark 3.4.** $D_{t_m - \vec{G}_{\lambda}^m}$ is a non-smooth function in $L^1(0,T)$, but it is smoother away from $t_m$. This is the starting point of Lemma 3.3.

**Lemma 3.4.** If $y \in \alpha H^{\alpha/2}(0,T) \cap C(0,T)$, then
\[ (\Pi^\lambda y - Q_{\tau} y)(t_m - \cdot) = \langle (I - Q_{\tau}) y, (I - Q_{\tau}) D_{t_m - \vec{G}_{\lambda}^m} \rangle_{(0,t_m)} \] (23)
for each $1 \leq m \leq J$.

**Proof.** A straightforward calculation gives
\[ \langle (\Pi^\lambda y - Q_{\tau} y)(t_m - \cdot) \rangle = \langle (\Pi^\lambda y - Q_{\tau} y, (D_{t_m - \vec{G}_{\lambda}^m} + \lambda) G_{\lambda}^m) \rangle_{(0,t_m)} \] (by 12)
\[ = \langle (\Pi^\lambda y - Q_{\tau} y, (D_{t_m - \vec{G}_{\lambda}^m} + \lambda) G_{\lambda}^m) \rangle_{(0,T)} \] (by the fact $G_{\lambda}^m|_{(t_m,T)} = 0$)
\[ = \langle (D_{t_m - \vec{G}_{\lambda}^m} + \lambda)(\Pi^\lambda y - Q_{\tau} y), G_{\lambda}^m \rangle_{{\|H^{\alpha/2}}(0,T)} \] (by Lemma A.3)
\[ = \langle (I - Q_{\tau}) y, (D_{t_m - \vec{G}_{\lambda}^m} + \lambda) G_{\lambda}^m \rangle_{(0,T)} \] (by Lemma A.3)
\[ = \langle (I - Q_{\tau}) y, (D_{t_m - \vec{G}_{\lambda}^m} + \lambda) G_{\lambda}^m \rangle_{(0,t_m)} \] (by the fact $G_{\lambda}^m|_{(t_m,T)} = 0$).

Hence, (23) follows from the equality
\[ \langle (I - Q_{\tau}) y, (D_{t_m - \vec{G}_{\lambda}^m} + \lambda) G_{\lambda}^m \rangle_{(0,t_m)} = \langle (I - Q_{\tau}) y, (I - Q_{\tau}) D_{t_m - \vec{G}_{\lambda}^m} \rangle_{(0,t_m)}, \]
which is easily derived by the definition of $Q_{\tau}$. This completes the proof. □

**Lemma 3.5.** Assume that $y \in \alpha H^{\alpha/2}(0,T) \cap C^1(0,T)$ satisfies
\[ |y'(t)| \leq t^{-r}, \quad 0 < t \leq T, \] (24)
where $0 < r < 1$. Then
\[ \| (I - \Pi^\lambda y \|_{L^\infty(0,T)} \leq C_{\alpha,\sigma,T} J^{-\min\{\sigma(1-r),1\}}. \] (25)
Proof. For any $1 \leq m \leq J$,
\[
|\langle \Pi^m y - Q_r y \rangle(t_m^-)|
= |\langle (I - Q_r)y, (I - Q_r)D_{t_m}^\infty - G^\infty_{(t, t_m)} \rangle| \quad \text{(by Lemma 3.4)}
\leq \sum_{j=1}^m \|\langle (I - Q_r)y \rangle_{L^\infty(t_{j-1}, t_j)}\| \|\langle (I - Q_r)D_{t_m}^\infty - G^\infty_{(t_{j-1}, t_j)} \rangle\|
\leq \max_{1 \leq j \leq m} (m/j)^{(\sigma-1)(\alpha-1)} \|\langle (I - Q_r)y \rangle_{L^\infty(t_{j-1}, t_j)}\|
\times \sum_{j=1}^m (m/j)^{(\sigma-1)(\alpha-1)} \|\langle (I - Q_r)D_{t_m}^\infty - G^\infty_{(t_{j-1}, t_j)} \rangle\|
\leq C_{\alpha, r, T} \max_{1 \leq j \leq m} (m/j)^{(\sigma-1)(\alpha-1)} \|\langle (I - Q_r)y \rangle_{L^\infty(t_{j-1}, t_j)}\| \quad \text{(by (21))}
\leq C_{\alpha, r, T} \|\langle (I - Q_r)y \rangle_{L^\infty(0, t_m)}\|.
\]
It follows that
\[
\|\langle \Pi^m y - Q_r y \rangle \rangle_{L^\infty(0, T)} = \max_{1 \leq m \leq J} \|\langle \Pi^m y - Q_r y \rangle(t_m^-)\| \leq C_{\alpha, r, T} \|\langle (I - Q_r)y \rangle_{L^\infty(0, T)}\|,
\]
and hence
\[
\|\langle (I - \Pi^m y) \rangle_{L^\infty(0, T)} \leq \|\langle (I - Q_r)y \rangle_{L^\infty(0, T)}\| + \|\langle (\Pi^m y - Q_r y) \rangle_{L^\infty(0, T)}\|
\leq C_{\alpha, r, T} \|\langle (I - Q_r)y \rangle_{L^\infty(0, T)}\|.
\]
In addition, by (24) we obtain
\[
\|\langle (I - Q_r)y \rangle_{L^\infty(0, T)} \leq \max_{1 \leq j \leq J} \|\langle (I - Q_r)y \rangle_{L^\infty(t_{j-1}, t_j)}\|
\leq \max_{1 \leq j \leq J} (t_{j}^{1-r} - t_{j-1}^{1-r})/(1-r)
\leq C_{\alpha, r, T} \max_{1 \leq j \leq J} j^{\sigma(1-r) - 1}J^{-\min(\sigma(1-r), 1)}
\leq C_{\alpha, r, T} J^{-\min(\sigma(1-r), 1)}.
\]
Finally, combining the above two estimates proves (25) and hence this lemma. \qed

Finally, we are in a position to prove Theorem 3.1 as follows.

**Proof of Theorem 3.1.** For each $n \in \mathbb{N}$, let
\[
u^n(t) := (u(t), \phi_n)_{X}, \quad 0 \leq t \leq T.
\]
By (5) we have
\[
u^n(t) = E_{\alpha, 1}(-\lambda_n t^\alpha)(u_0, \phi_n)_{X}, \quad 0 \leq t \leq T.
\]
A straightforward calculation gives
\[
u^n(t) = -\lambda_n t^{\alpha-1}E_{\alpha, \alpha}(-\lambda_n t^\alpha)(u_0, \phi_n)_{X}\phi_n, \quad 0 \leq t \leq T,
\]
and hence (3) implies
\[
|\nu^n(t)| \leq C \lambda_n^\alpha t^{\alpha-1}|u_0, \phi_n)_{X}|, \quad 0 \leq t \leq T.
\]
By (8), (9) and (11) we have \( U = \sum_{n=0}^{\infty} (\Pi \lambda_n^* u^n) \phi_n \), so that
\[
\| u - U \|_{L^\infty(0,T;X)} = \sup_{0 < t < T} \left( \sum_{n=0}^{\infty} \| (u^n - \Pi \lambda_n^* u^n)(t) \|^2 \right)^{1/2} \\
\leq \left( \sum_{n=0}^{\infty} \| (I - \Pi \lambda_n^*) u^n \|_{L^\infty(0,T)}^2 \right)^{1/2} \\
\leq C_{\alpha,\sigma,T} J^{-\min\{\sigma \alpha,1\}} \left( \sum_{n=0}^{\infty} \lambda_n^{2\sigma} (u_0, \phi_n)^2 \right)^{1/2} \tag{by Lemma 3.5 and (26)} \\
= C_{\alpha,\sigma,T} J^{-\min\{\sigma \alpha,1\}} \| u_0 \|_{D(A^{\nu})}.
\]
This proves (10) and thus concludes the proof. \( \blacksquare \)

4 Fractional wave equation \((1 < \alpha < 2)\)

This section considers the following discretization: seek \( U \in W^c_\tau \) such that
\[
U(0) = u_0 \quad \text{and} \\n\int_0^T (D^{\alpha-1} u + AU, V)_X \, dt = 0 \quad \forall V \in W_\tau.
\] (27)

Remark 4.1. By (5), a straightforward calculation gives that
\[
\int_0^T (D^{\alpha-1} u + AU, V)_X \, dt = 0 \quad \forall V \in W_\tau.
\] (28)

Remark 4.2. We note that when using uniform temporal grids, the discretization (27) is equivalent to the L1 scheme (cf. [11]).

Theorem 4.1. Assume that \( u_0 \in D(A^\nu) \) with \( 1/2 < \nu \leq 1 \). If
\[
\sigma > \frac{3 - \alpha}{\alpha(\nu - 1/2)},
\] (29)
then
\[
\max_{1 \leq m \leq J} \| (u - U)(t_m) \|_X \leq C_{\alpha,\sigma,T} J^{\alpha-3} \| u_0 \|_{D(A^\nu)}.
\] (30)

The main task of the rest of this section is to prove the theorem above. For each \( 1 \leq m \leq J \), define \( G^m \in W_\tau \) by that \( G^m|_{(t_m,T)} = 0 \) and that
\[
\langle w, D^{\alpha-1}_m G^m \rangle_{(0,t_m)} = \langle 1, w \rangle_{(0,t_m)} \quad \forall w \in W_\tau.
\] (31)

Let \( G^m_{m+1} = 0 \) and, for each \( 1 \leq j \leq m \), let
\[
G^m_j := \lim_{t \to t_j^-} G^m(t).
\]

Since
\[
D^{\alpha-1}_m G^m = \sum_{j=1}^{m} (G^m_j - G^m_{j+1}) \frac{(t_j - t)^{1-\alpha}}{\Gamma(2-\alpha)},
\]
a straightforward calculation yields, from (31), that
\[ \sum_{j=k}^{m} (G_j^m - G_{j+1}^m) ((t_j - t_{k-1})^{2-\alpha} - (t_j - t_k)^{2-\alpha}) = \Gamma(3 - \alpha)(t_k - t_{k-1}) \] (32)
for each 1 \leq k \leq m.

**Remark 4.3.** Although $G^m$ is not a regularized Green function, it has similar properties.

**Lemma 4.1.** For any $1/2 < \beta < 1$ and $1 \leq k \leq J$,
\[ \sum_{j=1}^{k} (j/J)^{(1-\alpha)} ((t_k - t_{j-1})^{1-\beta} - (t_k - t_j)^{1-\beta}) \leq C_{\alpha,\sigma,T}(k/J)^{(2-\alpha-\beta)}. \] (33)

**Proof.** An elementary calculation gives
\[ k^{(1-\alpha)}(k^\sigma - (k-1)^\sigma)^{1-\beta} \leq C_\sigma k^{(1-\alpha)} k^{(\sigma-1)(1-\beta)} = C_\sigma k^{(2-\alpha-\beta)+\beta-1} \]
and
\[ \sum_{j=1}^{k-1} j^{(1-\alpha)} (k^\sigma - (j-1)^\sigma)^{1-\beta} \leq C_\sigma (1 - \beta) \sum_{j=1}^{k-1} j^{(1-\alpha)} (k^\sigma - j^\sigma)^{1-\beta} \]
\[ \leq C_\sigma (1 - \beta) \sum_{j=1}^{k-1} j^{2\alpha-\sigma-1} (k^\sigma - j^\sigma)^{1-\beta} \]
\[ \leq C_{\alpha,\sigma} k^{(2-\alpha-\beta)} \] (by Lemma B.5).

It follows that
\[ \sum_{j=1}^{k} (j/J)^{(1-\alpha)} ((t_k - t_{j-1})^{1-\beta} - (t_k - t_j)^{1-\beta}) \]
\[ = J^{-\sigma(2-\alpha-\beta)\beta} \sum_{j=1}^{k} j^{(1-\alpha)} (k^\sigma - (j-1)^\sigma)^{1-\beta} \]
\[ \leq C_{\alpha,\sigma,T} J^{-\sigma(2-\alpha-\beta)} (k^{\sigma(2-\alpha-\beta)+\beta-1} + k^{\sigma(2-\alpha-\beta)}) \]
\[ \leq C_{\alpha,\sigma,T}(k/J)^{(2-\alpha-\beta)}. \]
This proves (33) and hence this lemma. \(\square\)

**Lemma 4.2.** For any $1/2 < \beta < 1$ and $1 \leq m \leq J$,
\[ \sum_{j=1}^{m} (j/J)^{(1-\alpha)} \|D_{t_m}^3 - G^m\|_{L^1(t_j-1,t_j)} \leq C_{\alpha,\sigma,T}. \] (34)

**Proof.** By (32) and Lemma B.3, an inductive argument yields that
\[ G_1 > G_2 > \ldots > G_m = \Gamma(3 - \alpha)(t_m - t_{m-1})^{\alpha-1}. \] (35)
Plugging $k = 1$ into (32) shows
\[
\sum_{j=1}^{m} (g_j^m - g_{j+1}^m) (t_j^2 - (t_j - t_1)^2) = \Gamma(3 - \alpha) t_1,
\]
and hence
\[
\sum_{j=1}^{m} \frac{t_j^2 - (t_j - t_1)^2}{\Gamma(3 - \alpha) t_1} (g_j^m - g_{j+1}^m) = 1.
\]
From (35) and the inequality
\[
\frac{t_j^2 - (t_j - t_1)^2}{\Gamma(3 - \alpha) t_1} \geq C_{\alpha,\sigma,T} (j/J)^{\sigma(1 - \alpha)},
\]
it follows that
\[
\sum_{j=1}^{m} (j/J)^{\sigma(1 - \alpha)} (g_j^m - g_{j+1}^m) \leq C_{\alpha,\sigma,T}.
\] (36)
Since
\[
D_t^{\beta} - g^m = \sum_{j=1}^{m} (g_j^m - g_{j+1}^m) \frac{(t_j - t)^{-\beta}}{\Gamma(1 - \beta)},
\]
we obtain
\[
\sum_{j=1}^{m} (j/J)^{\sigma(1 - \alpha)} \|D_t^{\beta} - g^m\|_{L^1(t_j-1,t_j)} \leq \sum_{j=1}^{m} (j/J)^{\sigma(1 - \alpha)} \sum_{k=1}^{m} (g_k^m - g_{k+1}^m) \frac{\|t_j - t_{j-1}\|^{1-\beta} - \|t_k - t_{k-1}\|^{1-\beta}}{\Gamma(2 - \beta)} \quad (by \ (35)) \]
\[
= \sum_{k=1}^{m} (g_k^m - g_{k+1}^m) \sum_{j=1}^{m} (j/J)^{\sigma(1 - \alpha)} \frac{\|t_j - t_{j-1}\|^{1-\beta} - \|t_k - t_{k-1}\|^{1-\beta}}{\Gamma(2 - \beta)} \quad (by \ Lemma \ 4.1 \ and \ (35)) \]
\[
\leq C_{\alpha,\sigma,T} \sum_{k=1}^{m} (k/J)^{\sigma(2 - \alpha - \beta)} (g_k^m - g_{k+1}^m) \quad (by \ Lemma \ 4.1 \ and \ (35)).
\]
This proves (34) and thus completes the proof.

\begin{remark}
For more details about proving (35), we refer the reader to the proof of (13).
\end{remark}

\begin{lemma}
Assume that $y \in C^2([0,T]; X)$ satisfies
\[
t^{-1} \|y'(t)\|_{X} + \|y''(t)\|_{X} \leq t^{-r}, \quad 0 < t \leq T,
\] (37)
where $0 < r < 2$. For each $1 \leq j \leq J$, the following three estimates hold:
if $\sigma < 2/(3 - r)$, then
\[
\|D_0^{\alpha-2}(I-Q_t)y\|_{L^\infty(t_{j-1},t_j; X)} \leq C_{\alpha,\sigma,r,T} J^{-\sigma(3-\alpha-r)} j^{-\sigma\alpha} (j^{(3-r)+\alpha-3} + 1); \quad (38)
\]
if $\sigma = 2/(3 - r)$, then
\[
\|D_0^{\alpha-2}(I-Q_t)y\|_{L^\infty(t_{j-1},t_j; X)} \leq C_{\alpha,\sigma,r,T} J^{-\sigma(3-\alpha-r)} j^{-\sigma\alpha} (j^{(3-r)+\alpha-3} + \ln j); \quad (39)
\]
if $\sigma > 2/(3 - r)$, then
\[
\|D_0^{\alpha-2}(I-Q_t)y\|_{L^\infty(t_{j-1},t_j; X)} \leq C_{\alpha,\sigma,r,T} J^{-\sigma(3-\alpha-r)} j^{(3-\alpha-r)+\alpha-3}. \quad (40)
\]
\end{lemma}
Proof. We only present a proof of (40), the proofs of (38) and (39) being similar. Since the case $r = 1$ can be proved analogously, we assume that $r \neq 1$.

Let us first prove that

$$\sup_{t_{j-1} \leq a < t_j} \left\| \left( (a-t)^{1-\alpha}, (I-Q_r)y^r \right)_{(0,t_{j-1})} \right\|_X \leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} J^{-\sigma(3-\alpha-r) + \alpha - 3}$$

(41)

for each $2 \leq j \leq J$. Since the case $j = 2$ can be easily verified, we assume that $3 \leq j \leq J$. Let $t_{j-1} \leq a < t_j$. By the definition of $Q_r$, we have

$$\left\| \left( (a-t)^{1-\alpha}, (I-Q_r)y^r \right)_{(0,t_{j-1})} \right\|_X \leq I_1 + I_2 + I_3,$$

(42)

where

$$I_1 := \left\| \langle (I-Q_r)(a-t)^{1-\alpha}, (I-Q_r)y^r \rangle_{(0,t_1)} \right\|_X,$$

$$I_2 := \sum_{k=2}^{j-2} \left\| \langle (I-Q_r)(a-t)^{1-\alpha}, (I-Q_r)y^r \rangle_{(t_{k-1}, t_k)} \right\|_X,$$

$$I_3 := \left\| \langle (I-Q_r)(a-t)^{1-\alpha}, (I-Q_r)y^r \rangle_{(t_{j-2}, t_{j-1})} \right\|_X.$$

By (37) and the facts $\sigma > 2/(3-r)$ and $t_{j-1} \leq a$, a routine calculation yields the following three estimates:

$$I_1 \leq \left\| (I-Q_r)(a-t)^{1-\alpha} \right\|_{L^\infty(0,t_1)} \left\| (I-Q_r)y^r \right\|_{L^1(0,t_1,X)} \leq C_{\alpha, r} (a-t_1)^{1-\alpha} - a^{1-\alpha}) t_1^{-r},$$

$$\leq C_{\alpha, r} \left( (t_{j-1} - t_1)^{1-\alpha} - t_{j-1}^{1-\alpha} \right) t_1^{2-r},$$

$$\leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} \left( (j-1)^{-\sigma} - 1 \right)^{1-\alpha} - (j-1)^{\sigma(1-\alpha)}$$

$$\leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} j^{-\sigma},$$

$$\leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} j^{\sigma(3-\alpha-r) + \alpha - 3},$$

$$I_2 \leq C_{\alpha} \sum_{k=2}^{j-2} \left\| (I-Q_r)y^r \right\|_{L^\infty(t_{k-1}, t_k; X)} (t_k - t_{k-1}) \left( (a-t_k)^{1-\alpha} - (a-t_{k-1})^{1-\alpha} \right)$$

$$\leq C_{\alpha, r} \sum_{k=2}^{j-2} \left| t_k^{-r} - t_{k-1}^{-r} \right| (t_k - t_{k-1}) \left( (t_{j-1} - t_k)^{1-\alpha} - (t_{j-1} - t_{k-1})^{1-\alpha} \right)$$

$$\leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} \sum_{k=2}^{j-2} k^{\sigma(1-\alpha)-1} k^{2(\sigma-1)} (j^\sigma - k^\sigma)^{-\alpha}$$

$$= C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} \sum_{k=2}^{j-2} k^{3\sigma - \sigma - 3} (j^\sigma - k^\sigma)^{-\alpha}$$

$$\leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} j^{\sigma(3-\alpha-r) + \alpha - 3} \quad \text{(by Lemma B.4)}$$

and

$$I_3 \leq C_{\alpha} \left\| (I-Q_r)y^r \right\|_{L^\infty(t_{j-2}, t_{j-1}; X)} \left( (a-t_{j-2})^{2-\alpha} - (a-t_{j-1})^{2-\alpha} \right)$$

$$\leq C_{\alpha, r} \left| t_{j-1}^{-r} - t_{j-2}^{-r} \right| (t_{j-1} - t_{j-2})^{2-\alpha},$$

$$\leq C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} j^{\sigma(1-\alpha) - 1} j^{\sigma(2-\alpha)}$$

$$= C_{\alpha, \sigma, r, T} J^{-\sigma(3-\alpha-r)} j^{\sigma(3-\alpha-r) + \alpha - 3}.$$
Since \(a, t_{j-1} \leq a < t_j\), is arbitrary, combining (42) and the above three estimates proves (41) for \(3 \leq j \leq J\).

Next, let us prove that (40) holds for all \(2 \leq j \leq J\). For any \(t_{j-1} \leq a < t_j\),

\[
(D_{0^+}^{-2}(y' - Q_T y'))(a) = (\mathbb{I}_4 + \mathbb{I}_5)/\Gamma(2 - \alpha),
\]

where

\[
\mathbb{I}_4 := \langle (a - t)^{1-\alpha}, (I - Q_T)y' \rangle_{(t_{j-1},a)},
\]

\[
\mathbb{I}_5 := \langle (a - t)^{1-\alpha}, (I - Q_T)y' \rangle_{(0,t_{j-1})}.
\]

We have

\[
\|\mathbb{I}_4\|_X \leq C_{\alpha}(a - t_{j-1})^{2-\alpha}\|\Lambda Q_T y'\|_{L^\infty(t_{j-1},t_j;X)}
\]

\[
\leq C_{a,r}(t_j - t_{j-1})^{2-\alpha}\big|t_j^{1-r} - t_{j-1}^{1-r}\big| \quad \text{by (37)}
\]

\[
\leq C_{a,\sigma,r,T}\beta(3-\alpha-r)\gamma(3-\alpha-r+1-\alpha)
\]

and, by (41),

\[
\|\mathbb{I}_5\|_X \leq C_{a,\sigma,r,T}\beta(3-\alpha-r)\gamma(3-\alpha-r+1-\alpha).
\]

Combining the above two estimates and (43) gives

\[
\|\big(D_{0^+}^{-2}(y' - Q_T y')\big)(a)\|_X \leq C_{a,\sigma,r,T}\gamma(3-\alpha-r)\gamma(3-\alpha-r+1-\alpha).
\]

Hence, the arbitrariness of \(t_{j-1} \leq a < t_j\) proves (40) for \(2 \leq j \leq J\).

Finally, for any \(0 < a \leq t_1\),

\[
\|\big(D_{0^+}^{-2}(I - Q_T)y'\big)(a)\|_X
\]

\[
\leq C_{a,r}\int_0^a (a - t)^{1-\alpha}(t^{1-r} + t_1^{1-r}) \, dt \quad \text{by (37)}
\]

\[
\leq C_{a,r}a^{3-\alpha-r}\int_0^1 (1 - s)^{1-\alpha}s^{1-r} \, ds + a^2a^{1-\alpha_1-1}
\]

\[
\leq C_{a,r}a^{3-\alpha-r} \leq C_{a,\sigma,r,T}\beta(3-\alpha-r).
\]

This proves (40) for \(j = 1\) and thus concludes the proof. \(\square\)

For any \(y \in H^{(\alpha+1)/2}(0, T)\), define \(\mathcal{P}_\tau y \in \mathcal{W}_\tau^r\) by

\[
\left\{ \begin{array}{l}
(y - \mathcal{P}_\tau y)(0) = 0, \\
\langle D_{0^+}^{\alpha-1} (y - \mathcal{P}_\tau y)', w \rangle_{H^{(\alpha-1)/2}(0, T)} = 0 \quad \forall w \in \mathcal{W}_\tau,
\end{array} \right.
\]

and define \(\Xi_\tau y \in \mathcal{W}_\tau^r\) by

\[
\left\{ \begin{array}{l}
(y - \Xi_\tau y)(0) = 0, \\
\langle D_{0^+}^{\alpha-1} (y - \Xi_\tau y)', \lambda (y - \Xi_\tau y), w \rangle_{H^{(\alpha-1)/2}(0, T)} = 0 \quad \forall w \in \mathcal{W}_\tau.
\end{array} \right.
\]
Lemma 4.4. If $\alpha - 1 < \beta < 1$ and $y \in H^{(\alpha+1)/2}(0, T)$, then
\[
(y - P_\tau y)(t_m) = \left\langle D_{0+}^{\alpha-1-\beta, \eta} (Q_\tau - I) y', D_{t_m}^\beta G^m \right\rangle_{(0, t_m)} \tag{46}
\]
for each $1 \leq m \leq J$.

Proof. A straightforward calculation gives
\[
(y - P_\tau y)(t_m) = (I_x y - P_\tau y)(t_m) = \left\langle (I_x y - P_\tau y)', 1 \right\rangle_{(0, t_m)}
\]
by (41)
\[
= \left\langle (I_x y - P_\tau y)', D_{t_m}^{\alpha-1} G^m \right\rangle_{(0, t_m)} \tag{46}
\]
for each $1 \leq m \leq J$.

For any $\alpha - 1 < \beta < 1$,
\[
\left\langle D_{0+}^{\alpha-1} (I_x y - y)', G^m \right\rangle_{H^{(\alpha+1)/2}(0, T)} = \left\langle D_{0+}^\beta, D_{0+}^{\alpha-1-\beta} (I_x y - y)', G^m \right\rangle_{H^{(\alpha+1)/2}(0, T)}
\]
by Lemma A.3
\[
= \left\langle D_{0+}^{\alpha-1-\beta} (I_x y - y)', D_{t_m}^\beta G^m \right\rangle_{(0, t_m)} \tag{46}
\]
for each $1 \leq m \leq J$.

Combining the above two equations proves (46) and hence this lemma. \qed

For any
\[
y \in H^{(\alpha+1)/2}(0, T; X) := \left\{ \sum_{n=0}^\infty c_n \phi_n : \sum_{n=0}^\infty \|c_n\|^2_{H^{(\alpha+1)/2}(0, T)} < \infty \right\},
\]
define
\[
P_\tau^X y := \sum_{n=0}^\infty \left( P_\tau (y, \phi_n)_X \right) \phi_n. \tag{47}
\]

Remark 4.5. By (44), (47), Lemma A.1 and Lemma A.2, we obtain
\[
\|P_\tau^X y\|_{H^{(\alpha+1)/2}(0, T; X)} \leq C_{\alpha, T} \|y\|_{H^{(\alpha+1)/2}(0, T; X)} \tag{48}
\]
for all $y \in H^{(\alpha+1)/2}(0, T; X)$.

Lemma 4.5. Assume that $y \in H^{(\alpha+1)/2}(0, T; X) \cap C^2([0, T]; X)$ satisfies
\[
t^{-1} \|y'(t)\|_X + \|y''(t)\|_X \leq t^{-r}, \quad 0 < t \leq T,
\]
where $0 < r < 2$. Then
\[
\| (y - P_\tau^X y)(t_m) \|_X \leq C_{\alpha, \sigma, T} J^{-\sigma(2-r)} \max_{1 \leq j \leq m} j^{2\sigma - \sigma + \alpha - 3} \tag{49}
\]
for each $1 \leq m \leq J$. 

17
Proof. For each $n \in \mathbb{N}$, let

$$y^n(t) := (y(t), \phi_n)_X, \quad 0 \leq t \leq T.$$ 

A straightforward calculation gives

$$\left\| (y - \mathcal{P}^X_n y)(t_m) \right\|_X = \left( \sum_{n=0}^{\infty} \left[ \left\| (y^n - \mathcal{P}_T y^n) \right\|_X \right]^2 \right)^{1/2} \quad \text{(by (47))}$$

$$= \left( \sum_{n=0}^{\infty} \left\| D_{0+}^{\alpha - 1-\beta} (I - Q_r) (y^n)' \right\|_2 \right)^{1/2} \quad \text{(by Lemma 4.4)}$$

$$\leq \int_0^{t_m} \left( \sum_{n=0}^{\infty} \left\| D_{0+}^{\alpha - 1-\beta} (I - Q_r) (y^n)' \right\|^2 \right)^{1/2} dt \quad \text{(by the Minkowski inequality)}$$

$$= \int_0^{t_m} \left\| D_{0+}^{\alpha - 1-\beta} (I - Q_r) y' \right\|_X \left\| D_{t_m}^\alpha - G^m \right\| dt,$$

for any $\alpha - 1 < \beta < 1$. From Lemma 4.2 it follows that

$$\left\| (y - \mathcal{P}^X y)(t_m) \right\|_X \leq \sum_{j=1}^m (J/j)^{\sigma (\alpha - 1)} \left\| D_{t_m}^\alpha - G^m \right\|_{L^1(t_j - 1, t_j)} \times \max_{1 \leq j \leq m} (J/j)^{\sigma (1-\alpha)} \left\| D_{0+}^{\alpha - 1-\beta} (I - Q_r) y' \right\|_{L^\infty(t_j - 1, t_j; X)}$$

$$\leq C_{\alpha,\sigma, r} \max_{1 \leq j \leq m} (J/j)^{\sigma (1-\alpha)} \left\| D_{0+}^{\alpha - 2} (I - Q_r) y' \right\|_{L^\infty(t_j - 1, t_j; X)}.$$

Passing to the limit $\beta \to 1$ then yields

$$\left\| (y - \mathcal{P}^X y)(t_m) \right\|_X \leq C_{\alpha,\sigma, r} \max_{1 \leq j \leq m} (J/j)^{\sigma (1-\alpha)} \left\| D_{0+}^{\alpha - 2} (I - Q_r) y' \right\|_{L^\infty(t_j - 1, t_j; X)},$$

so that a straightforward calculation proves (49) by Lemma 4.3. This completes the proof.

Lemma 4.6. Assume that $y \in H^{(\alpha+1)/2}(0, T; X) \cap C^2((0, T]; D(A^{1/2}))$ satisfies

$$t^{-r} \| y'(t) \|_{D(A^{1/2})} + \| y''(t) \|_{D(A^{1/2})} \leq t^{-r}, \quad 0 < t \leq T,$$

where $0 < r < 2$. If $\sigma > (3 - \alpha)/(2 - r)$, then

$$\| (I - \mathcal{P}_r) y \|_{L^{2/\alpha}(0, T; D(A^{1/2}))} \leq C_{\alpha,\sigma, r, T} J^{\alpha - 3}. \quad (50)$$

Proof. A simple modification of the proof of (49) yields

$$\max_{1 \leq m \leq J} \| (y - \mathcal{P}_r y)(t_m) \|_{D(A^{1/2})} \leq C_{\alpha,\sigma, r, T} J^{\alpha - 3}, \quad (51)$$

which implies

$$\| (\mathcal{I}_r - \mathcal{P}_r) y \|_{L^\infty(0, T; D(A^{1/2}))} \leq C_{\alpha,\sigma, r, T} J^{\alpha - 3}.$$

It follows that

$$\| (\mathcal{I}_r - \mathcal{P}_r) y \|_{L^{2/\alpha}(0, T; D(A^{1/2}))} \leq C_{\alpha,\sigma, r, T} J^{\alpha - 3}.$$ 

In addition, a routine calculation gives

$$\| (I - \mathcal{I}_r) y \|_{L^{2/\alpha}(0, T; D(A^{1/2}))} \leq C_{\alpha,\sigma, r, T} J^{-2}.$$ 

Combining the above two estimates proves (50) and hence this lemma.
Lemma 4.7. If $y \in H^{(\alpha+1)/2}(0,T)$, then
\[
\left| (y - \Xi^\lambda_n y)(t_m) \right| \leq C_{\alpha,T} \left( \| (y - \mathcal{P}_\tau y)(t_m) \| + \lambda^{1/2} \| (I - \mathcal{P}_\tau)y \|_{L^{2/\alpha}(0,t_m)} \right)
\]
for each $1 \leq m \leq J$.

Proof. Letting $\theta := (\Xi^\lambda - \mathcal{P}_\tau)y$, by (44), (45) and Lemma A.3 we obtain
\[
\langle D_{0+}^{\alpha+1} \theta', \theta' \rangle_{(0,t_m)} + \lambda \langle \theta, \theta' \rangle_{(0,t_m)} = \lambda \langle y - \mathcal{P}_\tau y, \theta' \rangle_{(0,t_m)},
\]
so that using Lemmas A.1 and A.2 and integration by parts yields
\[
\| \theta' \|_{H^{(\alpha+1)/2}(0,t_m)} + \lambda \| \theta(t_m) \|^2 \leq C_{\alpha}\lambda \| (I - \mathcal{P}_\tau)y \|_{L^{2/\alpha}(0,t_m)} \| \theta' \|_{L^{2/\alpha}(0,t_m)}.
\]
Since
\[
\| \theta' \|_{L^{2/\alpha}(0,t_m)} \leq C_{\alpha,T} \| \theta' \|_{H^{(\alpha+1)/2}(0,t_m)},
\]
it follows that
\[
\| \theta(t_m) \| \leq C_{\alpha,T} \lambda^{1/2} \| (I - \mathcal{P}_\tau)y \|_{L^{2/\alpha}(0,t_m)}.
\]
Hence, (52) follows from the triangle inequality
\[
\left| (y - \Xi^\lambda_n y)(t_m) \right| \leq |\theta(t_m)| + |(y - \mathcal{P}_\tau y)(t_m)|.
\]
This completes the proof. 

Proof of Theorem 4.1. For each $n \in \mathbb{N}$, let
\[
u^n(t) := (u(t), \phi_n)_X, \quad 0 < t \leq T.
\]
By (27), (28), (45) and Lemma A.3, we have
\[
U = \sum_{n=0}^\infty (\Xi^\lambda u^n) \phi_n,
\]
so that
\[
\| (u - U)(t_m) \|_X = \left( \sum_{n=0}^\infty \| (u^n - \Xi^\lambda u^n)(t_m) \|^2 \right)^{1/2} \leq C_{\alpha,T} \left( \| (u - \mathcal{P}_\tau^X u)(t_m) \|_X + \sum_{n=0}^\infty \lambda_n \| (I - \mathcal{P}_\tau^X u)^n \|^2_{L^{2/\alpha}(0,t_m)} \right)^{1/2} \quad \text{(by (52)).}
\]
Applying the Minkowski inequality gives
\[
\left( \sum_{n=0}^\infty \lambda_n \| (I - \mathcal{P}_\tau^X u)^n \|^2_{L^{2/\alpha}(0,t_m)} \right)^{1/2} \leq \| (I - \mathcal{P}_\tau^X u) \|_{L^{2/\alpha}(0,t_m;D(A^{1/2}))}.
\]
The above two estimates yield
\[
\| (u - U)(t_m) \|_X \leq C_{\alpha,T} \left( \| (u - \mathcal{P}_\tau^X u)(t_m) \|_X + \| (I - \mathcal{P}_\tau^X u) \|_{L^{2/\alpha}(0,t_m;D(A^{1/2}))} \right).
\]
In addition, using (6), (29) and Lemma 4.5 gives
\[
\| (u - \mathcal{P}_\tau^X u)(t_m) \|_X \leq C_{\alpha,\sigma,\nu,T} J^\alpha T^{-3} \| u_0 \|_{D(A^\nu)},
\]
and using (7), (29) and Lemma 4.6 shows
\[
\| (I - \mathcal{P}_\tau^X u) \|_{L^{2/\alpha}(0,T;D(A^{1/2}))} \leq C_{\alpha,\sigma,\nu,T} J^\alpha T^{-3} \| u_0 \|_{D(A^\nu)}.
\]
Finally, combining the above three estimates proves (30) and thus concludes the proof. 

\[\text{\hfill \square}\]
5 Numerical experiments

This section performs two numerical experiments to verify Theorems 3.1 and 4.1, respectively, in the following settings:

\[
\begin{aligned}
T &= 1; \\
X &:= \{ w \in H^1_0(0,1) : w \text{ is linear on } ((m-1)/2^{11}, m/2^{11}) \text{ for all } 1 \leq m \leq 2^{11} \}; \\
A : X \to X &\text{ is defined by that, for any } v \in X, \int_0^1 (Av)w = -\int_0^1 v'w' \quad \forall w \in X.
\end{aligned}
\]

**Experiment 1.** The purpose of this experiment is to verify Theorem 3.1. Let \(u_0\) be the \(L^2\)-orthogonal projection of \(x^0.51(1-x), 0 < x < 1\), onto \(X\). Define

\[ E_1 := \| U^* - U \|_{L^\infty(0,T;L^2(0,1))}, \]

where \(U^*\) is the numerical solution of discretization (8) with \(J = 2^{15}\) and \(\sigma = 2/\alpha\). Clearly, regarding \(\nu\) as 0.5 is reasonable. The numerical results in Tables 1, 2 and 3 illustrate that \(E_1\) is close to \(O(J^{-\min\{\sigma\alpha/2,1\}})\), which agrees well with the estimate (10) in Theorem 3.1.

| \(J\) | \(\sigma = 1\) Order | \(\sigma = 5\) Order | \(\sigma = 10\) Order |
|------|-----------------|-----------------|-----------------|
| 2^7  | 2.12e-1 - |  1.60e-2 -  |  6.58e-4 -  |
| 2^8  | 2.05e-1 0.05  |  1.99e-2 0.55  |  3.23e-4 1.03 |
| 2^10 | 1.97e-1 0.06  |  7.54e-3 0.54  |  1.58e-4 1.03 |
| 2^12 | 1.89e-1 0.06  |  5.23e-3 0.53  |  7.64e-5 1.05 |

Table 1: \(\alpha = 0.2\)

| \(J\) | \(\sigma = 1\) Order | \(\sigma = 2\) Order | \(\sigma = 4\) Order |
|------|-----------------|-----------------|-----------------|
| 2^7  | 1.36e-1 - |  3.42e-2 -  |  3.02e-3 -  |
| 2^8  | 1.14e-1 0.25  |  2.29e-2 0.58  |  1.45e-3 1.06 |
| 2^9  | 9.47e-2 0.27  |  1.55e-2 0.56  |  7.04e-4 1.04 |
| 2^10 | 7.76e-2 0.29  |  1.06e-2 0.55  |  3.43e-4 1.04 |

Table 2: \(\alpha = 0.5\)

| \(J\) | \(\sigma = 1\) Order | \(\sigma = 2\) Order | \(\sigma = 2.5\) Order |
|------|-----------------|-----------------|-----------------|
| 2^7  | 6.20e-2 - |  6.95e-3 -  |  3.77e-3 -  |
| 2^8  | 4.46e-2 0.48  |  3.89e-3 0.84  |  1.82e-3 1.05 |
| 2^9  | 3.22e-2 0.47  |  2.19e-3 0.83  |  8.81e-4 1.05 |
| 2^10 | 2.34e-2 0.46  |  1.24e-3 0.82  |  4.26e-4 1.05 |

Table 3: \(\alpha = 0.8\)

**Experiment 2.** The purpose of this experiment is to verify Theorem 4.1. Let \(u_0\) be the \(L^2\)-orthogonal projection of \(x^{1.51}(1-x)^2, 0 < x < 1\), onto \(X\). Let

\[ E_2 := \max_{1 \leq j \leq J} \| (U^* - U)(t_j) \|_{L^2(\Omega)}, \]

where \(U^*\) is the numerical solution of discretization (8) with \(J = 2^{15}\) and \(\sigma = 2/\alpha\). Clearly, regarding \(\nu\) as 0.5 is reasonable. The numerical results in Tables 1, 2 and 3 illustrate that \(E_2\) is close to \(O(J^{-\min\{\sigma\alpha/2,1\}})\), which agrees well with the estimate (10) in Theorem 4.1.
where \( U^* \) is the numerical solution of discretization (27) with \( J = 2^{15} \) and 
\[ \sigma = 2(3 - \alpha)/\alpha. \] Evidently, regarding \( u_0 \in D(A) \) is reasonable. The numerical results in Table 4 clearly demonstrate that \( E_2 \) is close to \( O(J^{\alpha-3}) \), which agrees well with Theorem 4.1.

\[
\begin{array}{cccccc}
\hline
\alpha = 1.2 & \alpha = 1.5 & \alpha = 1.8 \\
\hline
J & E_2 & Order & E_2 & Order & E_2 & Order \\
2^5 & 2.44e-5 & 1.27e-4 & 7.97e-4 & - & - & - \\
2^7 & 6.81e-6 & 1.84 & 4.58e-5 & 1.47 & 3.57e-4 & 1.16 \\
2^8 & 1.90e-6 & 1.84 & 1.65e-5 & 1.47 & 1.57e-4 & 1.18 \\
2^9 & 5.35e-7 & 1.83 & 5.97e-6 & 1.47 & 6.87e-5 & 1.20 \\
\hline
\end{array}
\]

Table 4: \( \sigma = 2(3 - \alpha)/\alpha \)

6 Conclusions

For the fractional evolution equation, we have analyzed a low-order discontinuous Galerkin (DG) discretization with fractional order \( 0 < \alpha < 1 \) and a low-order Petrov Galerkin (PG) discretization with fractional order \( 1 < \alpha < 2 \). When using uniform temporal grids, the two discretizations are equivalent to the L1 scheme with \( 0 < \alpha < 1 \) and the L1 scheme with \( 1 < \alpha < 2 \), respectively. For the DG discretization with graded temporal grids, sharp error estimates are rigorously established for smooth and nonsmooth initial data. For the PG discretization, the optimal \( (3 - \alpha) \)-order temporal accuracy is derived on appropriately graded temporal grids. The theoretical results have been verified by numerical results.

However, our analysis of the PG discretization requires \( u_0 \in D(A^\nu) \) with \( 1/2 < \nu \leq 1 \). Hence, how to analyze the case \( 0 < \nu \leq 1/2 \) remains an open problem. It appears that the results and techniques developed in this paper can be used to analyze the semilinear fractional diffusion-wave equations with graded temporal grids, and this is our ongoing work.

Acknowledgements

Binjie Li was supported in part by the National Natural Science Foundation of China (NSFC) Grant No. 11901410, Xiaoping Xie was supported in part by the National Natural Science Foundation of China (NSFC) Grant No. 11771312, and Tao Wang was supported in part by the China Postdoctoral Science Foundation (CPSF) Grant No. 2019M66294.

References

[1] S. C. Brenner and R. Scott. The mathematical theory of finite element methods. Springer-Verlag New York, 3 edition, 2008.

[2] E. Cuesta, C. Lubich, and C. Palencia. Convolution quadrature time discretization of fractional diffusion-wave equations. Math. Comput., 75(254):673–696, 2006.
[3] V. Ervin and J. Roop. Variational formulation for the stationary fractional advection dispersion equation. *Numer. Meth. Part. D. E.*, 22(3):558–576, 2006.

[4] B. Jin, R. Lazarov, and Z. Zhou. An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data. *IMA J. Numer. Anal.*, 36:197–221, 2016.

[5] B. Jin, R. Lazarov, and Z. Zhou. Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. *SIAM J. Sci Comput.*, 38(1):A146–A170, 2016.

[6] B. Jin, B. Li, and Z. Zhou. Discrete maximal regularity of time-stepping schemes for fractional evolution equations. *Numer. Math.*, 138(1):101-131, 2018.

[7] B. Li and X. Xie Regularity of solutions to time fractional diffusion equations. *Discrete Contin. Dyn. Syst. B*, 24(7):3195–3210; arXiv:1704.00147, 2019.

[8] B. Li, H. Luo, and X. Xie A time-spectral algorithm for fractional wave problems. *J. Sci. Comput.*, 7(2):1164–1184, 2018.

[9] B. Li, H. Luo, and X. Xie A space-time finite element method for fractional wave problems. *Numer. Algor.*, doi:10.1007/s11075-019-00857-w; arXiv:1803.03437, 2018.

[10] B. Li, H. Luo, and X. Xie Analysis of a time-stepping scheme for time fractional diffusion problems with nonsmooth data. *SIAM J. Numer. Anal.*, 57(2):779–798, 2019.

[11] B. Li, T. Wang, and X. Xie Analysis of the L1 scheme for fractional wave equations with nonsmooth data. submitted, arXiv:1908.09145v2.

[12] B. Li, T. Wang, and X. Xie Analysis of a time-stepping discontinuous Galerkin method for fractional diffusion-wave equations with nonsmooth data. *J. Sci. Comput.*, Doi:10.1007/s10915-019-01118-7, 2020.

[13] X. Li and C. Xu A space-time spectral method for the time fractional diffusion equation. *SIAM J. Numer. Anal.*, 47(3):2108–2131, 2009.

[14] H. Liao, D. Li, and J. Zhang Sharp error estimate of the nonuniform L1 formula for linear reaction-subdiffusion equations. *SIAM J. Numer. Anal.*, 56(2):1112–1133, 2018.

[15] C. Lubich. Convolution quadrature and discretized operational calculus. *Numer. Math.*, 52(2):129–145, 1988.

[16] C. Lubich, I. Sloan, and V. Thomée. Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comput.*, 65(213):1–17, 1996.

[17] A. Lunardi. *Interpolation theory*. Edizioni della Normale, Pisa, 2018.
[18] H. Luo, B. Li, and X. Xie. Convergence analysis of a petrov–galerkin method for fractional wave problems with nonsmooth data. *J. Sci. Comput.*, 80(2):957–992, 2019.

[19] Z. Mao and S. Jie. Efficient spectral-Galerkin methods for fractional partial differential equations with variable coefficients. 307(1):243–261, 2016.

[20] W. McLean and K. Mustapha. Time-stepping error bounds for fractional diffusion problems with non-smooth initial data. *J. Comput. Phys.*, 293(C):201–217, 2015.

[21] W. McLean and K. Mustapha. Convergence analysis of a discontinuous Galerkin method for a sub-diffusion equation. *Numer. Algor.*, 52(1):69–88, 2009.

[22] K. Mustapha and W. McLean. Uniform convergence for a discontinuous galerkin, time-stepping method applied to a fractional diffusion equation. *IMA J. Numer. Anal.*, 32(3):906–925(20), 2012.

[23] K. Mustapha and W. McLean. Superconvergence of a discontinuous galerkin method for fractional diffusion and wave equations. *SIAM J. Numer. Anal.*, 51(1):491–515, 2013.

[24] K. Mustapha. An L1 approximation for a fractional reaction-diffusion equation, a second-order error analysis over time-graded meshes, arXiv:1909.06739v1, 2019.

[25] K. Mustapha, B. Abdallah, and K. Furati. A discontinuous Petrov-Galerkin method for time-fractional diffusion equations. *Fuel*, 58(12):896–897, 2014.

[26] I. Podlubny. *Fractional differential equations*. Academic Press, 1998.

[27] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.*, 382(1):426–447, 2011.

[28] A. H. Schatz. Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids: part i. global estimates. *Math. Comput.*, 67(223):877–899, 1998.

[29] M. Stynes, E. O'Riordan, and J. L. Gracia. Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation. *SIAM J. Numer. Anal.*, 55(2):1057–1079, 2017.

[30] Z. Sun and X. Wu. A fully discrete difference scheme for a diffusion-wave system. *Appl. Numer. Math.*, 56(2):193 – 209, 2006.

[31] Y. Yan, M. Khan, and N. J. Ford. An analysis of the modified L1 scheme for time-fractional partial differential equations with nonsmooth data. *SIAM J. Numer. Anal.*, 56(1):210–227, 2018.

[32] Y. Yang, Y. Chen, Y. Huang, and H. Wei. Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis. *Comput. Math. Appl.*, 73(6):1218–1232, 2017.
[33] M. Zayernouri, M. Ainsworth, and G. E. Karniadakis. A unified Petrov-Galerkin spectral method for fractional pdes. *Comput. Methods Appl. Mech. Eng.*, 283:1545–1569.

[34] Eberhard Zeidler. *Applied Functional Analysis: Applications to Mathematical Physics*. 2009.

[35] F. Zeng, C. Li, F. Liu, and I. Turner. The use of finite difference/element approaches for solving the time-fractional subdiffusion equation. *SIAM J. Sci. Comput.*, 35(6):2976–3000, 2013.
A  Properties of fractional calculus operators

Lemma A.1. For any \( v \in 0^{H}(a, b) \) with \( 0 < \gamma < 1/2, \)
\[
\cos(\gamma \pi) \|D_{a+}^{\gamma} v\|_{L^2(a,b)}^{2} \leq \langle D_{a+}^{\gamma} v, D_{a+}^{\gamma} v \rangle_{(a,b)} \leq \sin(\gamma \pi) \|D_{a+}^{\gamma} v\|_{L^2(a,b)}^{2}.
\]
\[
\cos(\gamma \pi) \|D_{b-}^{\gamma} v\|_{L^2(a,b)}^{2} \leq \langle D_{a+}^{\gamma} v, D_{b-}^{\gamma} v \rangle_{(a,b)} \leq \sin(\gamma \pi) \|D_{b-}^{\gamma} v\|_{L^2(a,b)}^{2}.
\]

Lemma A.2. For any \( v \in 0^{H}(a, b) \) and \( w \in 0^{H}(a, b) \) with \( 0 < \gamma < \infty, \)
\[
C_1 \|v\|_{a^{H}\gamma(a,b)} \leq \|D_{a+}^{\gamma} v\|_{L^2(a,b)} \leq C_2 \|v\|_{a^{H}\gamma(a,b)},
\]
\[
C_1 \|w\|_{a^{H}\gamma(a,b)} \leq \|D_{b-}^{\gamma} w\|_{L^2(a,b)} \leq C_2 \|w\|_{a^{H}\gamma(a,b)},
\]

where \( C_1 \) and \( C_2 \) are two positive constants depending only on \( \gamma. \)

Lemma A.3. Assume that \( v \in 0^{H/2}(a, b) \) and \( w \in 0^{H/2}(a, b) \) with \( 0 < \gamma < 1. \)
Then
\[
\langle D_{a+}^{\gamma} v, w \rangle_{a^{H/2}(a,b)} = \langle D_{b-}^{\gamma} w, v \rangle_{a^{H/2}(a,b)}.
\]

If \( D_{a+}^{\gamma} v \in L^{2/(1+\gamma)}(a, b), \) then
\[
\langle D_{a+}^{\gamma} v, w \rangle_{a^{H/2}(a,b)} = \langle D_{b-}^{\gamma} w, v \rangle_{(a,b)}.
\]

If \( D_{b-}^{\gamma} w \in L^{2/(1+\gamma)}(a, b), \) then
\[
\langle D_{b-}^{\gamma} w, v \rangle_{a^{H/2}(a,b)} = \langle D_{a+}^{\gamma} v, w \rangle_{(a,b)}.
\]

For the proof of Lemma A.1, we refer the reader to [3]. For the proof of
Lemma A.2, we refer the reader to [18]. Since the proof of Lemma A.3 is a
standard density argument by Lemmas A.1 and A.2, it is omitted here.

B  Some inequalities

Lemma B.1. For any \( 0 < \beta < 1 \) and \( 0 \leq t < a < b < c < d, \)
\[
\frac{(d-t)^{1-\beta} - (d-a)^{1-\beta}}{(d-a)^{1-\beta} - (d-b)^{1-\beta}} \geq \frac{(c-t)^{1-\beta} - (c-a)^{1-\beta}}{(c-a)^{1-\beta} - (c-b)^{1-\beta}}.
\]

Proof. Let
\[
w(y) := \begin{cases} \beta/(1-\beta) & \text{if } y = 1, \\ \frac{1-y^{-\beta}}{y^{-\beta}-1} & \text{if } y \in [0, \infty) \setminus \{1\}. \end{cases}
\]

A routine argument proves that \( w \) is strictly decreasing on \([0, \infty), \) so that
\[
w((d-t-x)/(d-a-x)) < w((d-b-x)/(d-a-x)) \quad \forall 0 \leq x \leq d-c.
\]

It follows that, for any \( 0 \leq x \leq d-c, \)
\[
\frac{(d-a-x)^{1-\beta} - (d-t-x)^{1-\beta}}{(d-t-x)^{1-\beta} - (d-a-x)^{1-\beta}} \leq \frac{(d-b-x)^{1-\beta} - (d-a-x)^{1-\beta}}{(d-a-x)^{1-\beta} - (d-b-x)^{1-\beta}}
\]
which implies
\[
\frac{((d-a-x)^{-\beta} - (d-t-x)^{-\beta})((d-a-x)^{1-\beta} - (d-b-x)^{1-\beta})}{((d-b-x)^{-\beta} - (d-a-x)^{-\beta})((d-t-x)^{1-\beta} - (d-a-x)^{1-\beta})} < 0.
\]

25
for all $0 \leq x \leq d - c$. A simple calculation then yields $g'(x) < 0$ for all $0 \leq x \leq d - c$, where

$$g(x) := \frac{(d - t - x)^{1-\beta} - (d - a - x)^{1-\beta}}{(d - a - x)^{1-\beta} - (d - b - x)^{1-\beta}}, \quad 0 \leq x \leq d - c.$$  

This proves $g(d - c) < g(0)$, namely (57), and thus concludes the proof. \qed

**Lemma B.2.** For any $0 < \beta < 1$, $\mu \geq 0$ and $0 \leq t < a < b < c < d$,

$$\frac{(d - t)^{1-\beta} - (d - a)^{1-\beta} + \mu(a - t)}{(d - a)^{1-\beta} - (d - b)^{1-\beta} + \mu(b - a)} > \frac{(c - t)^{1-\beta} - (c - a)^{1-\beta} + \mu(a - t)}{(c - a)^{1-\beta} - (c - b)^{1-\beta} + \mu(b - a)}.$$  

(58)

**Proof.** Define

$$g(s) := (d - b + s(b - a))^{1-\beta} - (c - b + s(b - a))^{1-\beta} \quad \forall 0 \leq s \leq a.$$  

By the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$g(1) - g(0) = g'(\theta) = (1 - \beta) \left( (d - b + \theta(b - a))^{1-\beta} - (c - b + \theta(b - a))^{1-\beta} \right) (b - a).$$

Since

$$(d - b + \theta(b - a))^{1-\beta} - (c - b + \theta(b - a))^{1-\beta} > (d - a)^{1-\beta} - (c - a)^{1-\beta},$$

it follows that

$$g(1) - g(0) > (1 - \beta)(d - a)^{1-\beta} - (c - a)^{1-\beta}(b - a),$$

which implies

$$\frac{1}{b - a} > \frac{(1 - \beta)(d - a)^{1-\beta} - (c - a)^{1-\beta}}{(d - a)^{1-\beta} - (d - b)^{1-\beta} - (c - a)^{1-\beta} + (c - b)^{1-\beta}}.$$  

(59)

Hence, by the estimate

$$(d - s)^{1-\beta} - (c - s)^{1-\beta} < (d - a)^{1-\beta} - (c - a)^{1-\beta} \quad \forall 0 \leq s < a,$$

we obtain

$$\frac{1}{b - a} > \frac{(1 - \beta)(d - s)^{1-\beta} - (c - s)^{1-\beta}}{(d - a)^{1-\beta} - (d - b)^{1-\beta} - (c - a)^{1-\beta} + (c - b)^{1-\beta}} \quad \forall 0 \leq s < a.$$  

(60)

Integrating both sides of the above equation with respect to $s$ from $t$ to $a$ yields

$$\frac{a - t}{b - a} > \frac{(c - t)^{1-\beta} - (c - a)^{1-\beta} - (d - t)^{1-\beta} + (d - a)^{1-\beta}}{(c - a)^{1-\beta} - (c - b)^{1-\beta} - (d - a)^{1-\beta} + (d - b)^{1-\beta}}.$$  

(61)

Let

$$\mathcal{A} := (d - t)^{1-\beta} - (d - a)^{1-\beta}, \quad \mathcal{B} := (d - a)^{1-\beta} - (d - b)^{1-\beta},$$

$$\mathcal{C} := (c - t)^{1-\beta} - (c - a)^{1-\beta}, \quad \mathcal{D} := (c - a)^{1-\beta} - (c - b)^{1-\beta},$$

$$\mathcal{M} := \mu(a - t), \quad \mathcal{N} := \mu(b - a).$$

Since Lemma B.1 implies $\mathcal{A}\mathcal{D} > \mathcal{B}\mathcal{C}$ and (61) implies $\mathcal{M}(\mathcal{D} - \mathcal{B}) > \mathcal{N}(\mathcal{C} - \mathcal{A})$, we obtain

$$(\mathcal{A} + \mathcal{M})(\mathcal{D} + \mathcal{N}) > (\mathcal{B} + \mathcal{N})(\mathcal{C} + \mathcal{M}),$$

which proves (58). This completes the proof. \qed
Lemma B.3. For any \(1 < \beta < 2\) and \(0 \leq t < a < b \leq c\),

\[
\frac{(c-t)^{2-\beta} - (c-a)^{2-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} \leq \frac{a-t}{b-a}.
\]  
(62)

Proof. By the mean value theorem, there exists \(0 < \theta < 1\) such that

\[
(c-a)^{2-\beta} - (c-b)^{2-\beta} = (2-\beta)(c-b+\theta(b-a))^{1-\beta}(b-a),
\]
and so

\[
\frac{(2-\beta)(c-a)^{1-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} = \left(\frac{c-a}{c-b+\theta(b-a)}\right)^{1-\beta} \frac{1}{b-a} < \frac{1}{b-a}.
\]

Since

\[
(c-a)^{1-\beta} \geq (c-s)^{1-\beta}
\]
for all \(0 \leq s \leq a\),

it follows that

\[
\frac{(2-\beta)(c-s)^{1-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} < \frac{1}{b-a}
\]
for all \(0 \leq s \leq a\).

Hence, for any \(0 \leq t < a\),

\[
\int_t^a \frac{(2-\beta)(c-s)^{1-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} \, ds < \int_t^a \frac{1}{b-a} \, ds,
\]
which implies (62). This completes the proof.

Lemma B.4. If \(\beta > -1\) and \(\gamma > 1\), then

\[
\sum_{j=1}^{k-1} j^\beta (k^\sigma - j^\sigma)^{-\gamma} \leq C_{\beta,\gamma,\sigma} k^\beta (k^\sigma - 1)^{-\gamma \sigma}
\]
(63)

for all \(k \geq 2\).

Proof. A routine calculation gives

\[
C_0 \leq \frac{j^\beta (k^\sigma - j^\sigma)^{-\gamma}}{(j-x)^\beta (k^\sigma - (j-x)^\sigma)^{-\gamma}} \leq C_1,
\]
for all \(2 \leq j \leq k-1\) and \(0 < x \leq 1\), where \(C_0\) and \(C_1\) are two positive constants depending only on \(\beta, \gamma\) and \(\sigma\). Hence,

\[
\sum_{j=1}^{k-1} j^\beta (k^\sigma - j^\sigma)^{-\gamma}
\]

\[
\leq C_{\beta,\gamma,\sigma} \int_1^{k-1} x^\beta (k^\sigma - x^\sigma)^{-\gamma} \, dx
\]

\[
\leq C_{\beta,\gamma,\sigma} k^{-\sigma \gamma + \beta + 1} \int_{k^{-\sigma}}^{(k-1)/k^\beta} s^{(1+\beta)/\sigma - 1} (1-s)^{-\gamma} \, ds
\]

\[
\leq C_{\beta,\gamma,\sigma} k^{-\sigma \gamma + \beta + 1} \int_0^{((k-1)/k)^\beta} s^{(1+\beta)/\sigma - 1} (1-s)^{-\gamma} \, ds
\]

\[
\leq C_{\beta,\gamma,\sigma} k^{-\sigma \gamma + \beta + 1} \left(1 - ((k-1)/k)^\beta\right)^{1-\gamma}
\]

\[
\leq C_{\beta,\gamma,\sigma} k^{-\sigma \gamma + \beta + 1 + \gamma - 1}
\]

\[
= C_{\beta,\gamma,\sigma} k^\beta (k^\sigma - 1)^{-\gamma \sigma}.
\]

This proves the lemma.
A trivial modification of the proof of Lemma B.4 yields the following estimate.

**Lemma B.5.** If \( \beta > -1 \) and \( 1/2 \leq \gamma < 1 \), then

\[
\sum_{j=1}^{k-1} j^\beta (k^\sigma - j^\sigma)^{-\gamma} \leq C_{\beta,\sigma} (1 - \gamma)^{-1} k^{\beta - \sigma \gamma + 1}
\]

for all \( k \geq 2 \).