Poynting flux in the neighbourhood of a point charge in arbitrary motion and radiative power losses

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Received 26 June 2015, revised 4 April 2016
Accepted for publication 5 May 2016
Published 27 May 2016

Abstract
We examine the electromagnetic fields in the neighbourhood of a ‘point charge’ in arbitrary motion and thereby determine the Poynting flux across a spherical surface of vanishingly small radius surrounding the charge. We show that the radiative power losses from a point charge turn out to be proportional to the scalar product of the instantaneous velocity and the first time-derivative of the acceleration of the charge. This may seem to be discordant with the familiar Larmor formula where the instantaneous power radiated from a charge is proportional to the square of acceleration. However, it seems that the root cause of the discrepancy actually lies in Larmor’s formula, which is derived using the acceleration fields but without due consideration for the Poynting flux associated with the velocity-dependent self-fields ‘co-moving’ with the charge. Further, while deriving Larmor’s formula, one equates the Poynting flux through a surface at some later time to the radiation loss by the enclosed charge at the retarded time. Poynting’s theorem, on the other hand, relates the outgoing radiation flux from a closed surface to the rate of energy decrease within the enclosed volume, all calculated for the same given instant only. Here we explicitly show the absence of any Poynting flux in the neighbourhood of an instantly stationary point charge, implying no radiative losses from such a charge, which is in complete conformity with energy conservation. We further show how Larmor’s formula is still able to serve our purpose in the vast majority of cases. It is further shown that Larmor’s formula in general violates momentum conservation and, in the case of synchrotron radiation, leads to a potentially incorrect conclusion about the pitch angle changes of the radiating charges, and that only the radiation reaction formula yields a correct result, consistent with special relativity.
1. Introduction

In classical electrodynamics there is an old unsolved enigma: what exactly constitutes the instantaneous rate of radiative power loss from an arbitrarily moving point charge? Is it proportional to the square of the acceleration (Larmor’s formula) or is it proportional to the instant velocity multiplied by the rate of change of the acceleration, as inferred from the radiation reaction formula? This puzzle, a paradox, has now existed without a proper, universally acceptable, solution for more than a century. The conventional wisdom is that something may be lacking in the radiation losses derived from the radiation–reaction formula, which is thought to be perhaps not as rigorously derived as Larmor’s formula, the latter derived by calculating the Poynting flux across a spherical surface of a large enough radius centred on the charge \[1–4\]. The Poynting flux appears to be independent of the radius of the sphere only if the acceleration-dependent term is used, and one usually assigns this to be the power lost by the charge into radiation. We demonstrate here that a mathematical subtlety has been missed in the application of Poynting’s theorem to derive Larmor’s formula, and that a proper mathematical procedure shows that the radiation losses can be considered to take place only when there is a change in the acceleration of the charge. For this we derive the power loss formula from the Poynting flux in the neighbourhood of the charge. We show that the radiative power is proportional to the scalar product of the instantaneous velocity and the first time-derivative of the acceleration of the charge. This expression for power loss was hitherto obtained in the literature only from a detailed derivation of the self-force of an accelerated charge sphere of a vanishingly small radius \[5–8\]. But here we have derived the power loss directly from the Poynting flow for a ‘point charge’.

That Larmor’s formula could lead to incorrect conclusions can be seen by examining the case in the instantaneous rest-frame of an accelerated charge. Such a charge has no velocity at that instant and hence no kinetic energy that could be lost into the radiation. Even if some external agency, assumedly imparting acceleration to the charge, were considered to be supplying the energy going into the radiation, it could not have provided the power necessary for radiation in this case since work done by this external agency will also be zero as the system has a zero velocity at that instant. However, according to Larmor’s formula, the radiated power is directly proportional to \(\text{square of the instantaneous value of the acceleration}\) of the charged particle, even for an instantly stationary charge. We explicitly calculate the Poynting flux passing through a spherical surface of vanishingly small dimensions surrounding the charge, in its instantaneous rest-frame, and from that we show that the Poynting flux in the rest frame is zero, indicating thereby the absence of radiative losses, in conformity with energy conservation. This in turn removes the need for the acceleration-dependent internal-energy term, introduced in the literature on an ad hoc basis \[6, 9, 10\], to comply with law of energy conservation. We shall further demonstrate that in the instantaneous rest-frame of a \(\text{uniformly accelerated charge}\), the Poynting flux is zero not only in its immediate neighbourhood, but at all radial distances from the charge. This in turn implies an absence of radiation for a charge supported in a gravitational field, which conforms with the strong principle of equivalence. Finally we would show how Larmor’s formula could lead to an incorrect inference about the dynamics of a charge radiating by a synchrotron process, in particular about its pitch angle changes as the charge loses energy by radiation, and where only the radiation reaction formula leads to a correct result.
2. Larmor’s formula for radiation from an accelerated charge

The electromagnetic field \((\mathbf{E}, \mathbf{B})\) of an arbitrarily moving charge \(e\) is given by \([1, 11]\)

\[
\mathbf{E} = \left[ \frac{e(\mathbf{n} - \mathbf{v}/c)}{\gamma^2 r^2 (1 - \mathbf{n} \cdot \mathbf{v}/c)^2} + \frac{e \mathbf{n} \times \{ (\mathbf{n} - \mathbf{v}/c) \times \dot{\mathbf{v}} \}}{re^2 (1 - \mathbf{n} \cdot \mathbf{v}/c)^3} \right]_{\text{ret}},
\]

\[
\mathbf{B} = \mathbf{n} \times \mathbf{E},
\]

where the quantities in square brackets are to be evaluated at the retarded time. More specifically, \(\mathbf{v}, \dot{\mathbf{v}},\) and \(\gamma = 1/\sqrt{1 - (\mathbf{v}/c)^2}\) represent respectively the velocity, acceleration and the Lorentz factor of the charge at the retarded time, while \(r = \mathbf{r}\) is the radial vector from the retarded position of the charge to the field point where electromagnetic fields are being evaluated.

To calculate the electromagnetic power (radiation) passing through a spherical surface \(\Sigma\) around the charge, we make use of the radial component of the Poynting vector \([1]\)

\[
\mathbf{n} \cdot \mathbf{S} = \frac{c}{4\pi} (\mathbf{n} \times (\mathbf{E} \times \mathbf{B}) = \frac{c}{4\pi} (\mathbf{n} \times \mathbf{E}) \cdot \mathbf{B} = \frac{c}{4\pi} (\mathbf{n} \times \mathbf{E})^2.
\]

Using only the acceleration fields (second term on the right-hand side in equation (1)), which are transverse in nature (perpendicular to \(\mathbf{n}\)) and assuming a non-relativistic motion, one obtains for the Poynting flux

\[
P = \oint_{\Sigma} \mathbf{n} \cdot \mathbf{S} = \frac{e^2 \mathbf{[v^2]}_{\text{ret}}}{2e^3} \int_0^\pi d\theta \sin^3 \theta = \frac{2e^2}{3c^2} [\mathbf{v^2}]_{\text{ret}}.
\]

This Poynting flux through the spherical surface is independent of its radius \(r\), and one infers that this radiated power crossing the surface at time say, \(t_0\), must equal the mechanical energy loss rate \((\propto v^2)\) of the charge at the retarded time \(t_0 - r/c\), when it had an acceleration \(\mathbf{v}\). This is Larmor’s famous result for an accelerated charge that the radiative power loss at any time is proportional to the square of its acceleration at that instant; the power loss occurring presumably out of the kinetic energy of the charge.

It is usually assumed that the acceleration fields, which fall with distance as \(1/r\), solely represent the radiation from a charge, since the contribution of the velocity fields \((\propto 1/r^2)\) appears to be negligible for a large enough value of \(r\). Of course that would not be the case if the velocity were also changing monotonically with time, as for example in the case of a uniform acceleration where the retarded value of the velocity will be having a term \(\mathbf{v} \propto -\mathbf{v} r/c\) and in order to calculate the field at any event (space and time location), for a larger \(r\), we will be going proportionally further back in retarded time \((t_0 - r/c)\) and consequently the term \(\mathbf{v}/c^2 r\) in the velocity fields might become comparable to the acceleration field term \(\mathbf{v}/c^2 r\) in equation (1). In fact this makes it imperative in Larmor’s derivation that the motion considered is in a sense repetitive (though not necessarily in regular cycles), where the velocity will not be increasing monotonically with acceleration and thence the acceleration too would not be a constant and will have a finite time-derivative.

3. A mathematical subtlety missed in the derivation of Larmor’s formula

In the textbook derivation of Larmor’s formula, a mathematical subtlety seems to have hitherto been overlooked while applying Poynting’s theorem which, strictly speaking, relates the outgoing radiation flux from a closed surface to the rate of energy decrease within the enclosed volume, all calculated only for the same instant of time. While deriving Larmor’s
formula (equation (4)), one equated the Poynting flux through a surface at a time \( t_0 \) to the radiation loss of the charge at the retarded time \( t_0 - r/c \). It is true that the electromagnetic fields at \( r \) at time \( t_0 \) are determined by the charge position and motion at the retarded time \( t_0 - r/c \), and one might expect a similar causal relation for the radiated power. However, the relation between the Poynting flux through a sphere to the rate of the energy loss in the enclosed volume is given by Poynting’s theorem, which is valid strictly when all quantities are evaluated for the same instant of time. Poynting’s theorem does not state any relation between the radiation flux at \( t_0 \) and the energy loss rate of the charge at \( t_0 - r/c \). We may make \( r \) as small as we want but \( t_0 > t_0 - r/c \) always.

Let a charge moving with a constant acceleration \( \mathbf{v} \) be instantly stationary (i.e., \( \mathbf{v} = 0 \)) at a time \( t = 0 \), then we can write for the electromagnetic fields

\[
E = \left[ \frac{e \mathbf{n}}{r^2} + \frac{e \mathbf{n} \times (\mathbf{n} \times \mathbf{v})}{mc^2} \right]_{t=0}.
\]

Equation (5) gives for any time \( t = \tau \) the electromagnetic field on a spherical surface \( \Sigma \) of radius \( r = c\tau \) centred on the charge position at \( t = 0 \). Then Larmor’s formula says that the Poynting flux through the spherical surface at time \( t = \tau \)

\[
P = \frac{2e^2}{3c}v^2
\]

is the energy loss rate of the charge at the retarded time \( t = 0 \). However, that does not seem consistent with the law of energy conservation since the charge was instantly stationary and had a nil kinetic energy at \( t = 0 \) (we do not entertain the possibility that the radiation losses may be out of the proper mass of the charge, though such a thing has been considered in the literature [12]).

Poynting’s theorem relates the mechanical work done on the charge by the electromagnetic fields (which in this case comprises its self-fields as there are no other fields or charges within the system being considered here) to the Poynting flux through a surface enclosing the charge, both quantities evaluated for the same time, say \( t_0 \). Now the fields at \( t_0 \) are determined by the motion of the charge at the retarded time \( t_0 - r/c \). Thus, in order to calculate the power loss by an instantly stationary charge at \( t = 0 \) one must consider the Poynting flux through a surface surrounding the charge at \( t = 0 \) itself, even though the fields at the surface are determined from the time retarded positions of the charge when presumably was not yet stationary, say, at \( t = -r/c \). As we shall show later, the velocity and acceleration fields at the surface at time \( t = 0 \), arising from the retarded position of the charge at \( t = -r/c \), neatly cancel each other, resulting in a nil Poynting flux through the surface, consistent with the charge possessing a nil kinetic energy at the time \( t = 0 \).

Further, in the derivation of Larmor’s formula, one calculated the radiated power from the Poynting flux due to acceleration fields under the assumption that these contribute exclusively to the radiation and ignored the velocity fields which may also have a transverse component and thus a finite Poynting flux. Actually, as the charge undergoes an acceleration, the energy in the self-fields in its neighbourhood must be ‘co-moving’ with the changing velocity of the charge (after all the self-fields cannot be lagging behind the charge) and there would be a Poynting flux due to that. Therefore, not all of the Poynting flux may constitute radiation; some of it is just field energy dragged along the particle as it moves. The radiated energy is the part that propagates off to infinity and is detached from the charge [4], i.e., it no longer remains a part of the self-fields of the charge.

It is well known that the self-field energy of a charge moving with a uniform velocity is different for different values of the velocity (see e.g. [13]). After all, when a charge is
accelerated, depending upon the change in velocity, its self-field energy must change too. But that change in self-field energy cannot come from the velocity fields alone (the first term on the right-hand side of equation (1)), which contains no information about the change that might take place in the velocity of the charge. Therefore the acceleration fields, to some extent at least, must represent the changes taking place in the energy in self-fields which are attached to the charge. On the other hand, in the standard picture of ‘radiation’, acceleration fields are considered, exclusively and wholly, to represent power that is ‘lost’ by the charge irreversibly as radiation and that of course is the genesis of Larmor’s formula of radiative losses. Thus we see that there is something amiss in the standard picture which does not take into account at all the contribution of the acceleration fields towards the changing self-field energy of the accelerating charge. Actually, radiation will be that part of the Poynting flux which is over and above the flux value arising from the ‘present’ velocity of the charge.

We can find out the Poynting flux from the self-fields being ‘dragged’ along the charge due to its ‘present’ velocity $v_0$, by a comparison with the Poynting flux of a charge moving with a constant (non-relativistic) velocity, $v_0$ (i.e., $\dot{v} = 0$), for which we get the transverse component of the electric field from equation (1) as

$$E = -\frac{c n \times v_0}{r^2 c}.$$  

Accordingly we obtain the Poynting flux arising from the velocity fields of a charge moving with a velocity $v_0$ as

$$\frac{e^2 c}{2} \int_0^\pi d\theta \sin^3 \theta r^2 \left[ \frac{v_0}{r^2 c} \right]^2 = \frac{2e^2 v_0^2}{3r^2 c},$$

which is true for all values of $r$.

The Poynting flux in equation (8) at time $t$ due to the ‘present’ velocity of the accelerating charge can be related to the formerly named radiated power (equation (6)) by substitution of

$$v_0 = \dot{v} r / c$$

(9)

for a constant acceleration as we assumed earlier to get

$$\frac{2e^2 v_0^2}{3r^2 c} = \frac{2e^2 \dot{v}^2 r^2}{3r^2 c^3} = \frac{2e^2 \dot{v}^2}{3c^3}.$$

Thus we see that in the case of a constant acceleration, the hitherto termed radiation losses (à la Larmor’s formula) are nothing but the Poynting flow due to the movement of self-fields along with the charge, due to its ‘present’ velocity $v_0$.

4. Poynting flux in the neighbourhood of a moving point charge

Using the vector identity $v = n (v \cdot n) - n \times (n \times v)$, we can decompose the electric field in equation (1) in terms of the radial (along $n$) and transverse components as [14, 15]

$$E = \left[ \frac{en}{\gamma^2 r^2 (1 - n \cdot v / c)^2} + \frac{en \times \{(n - v/c) \times (v + \gamma^2 v r/c)\}}{\gamma^2 r^2 c (1 - n \cdot v / c)^3} \right]_{\text{ret}},$$

the quantities on the right-hand side are evaluated at the retarded time.

This is a general expression for the electric field of a charge, with the radial and transverse terms fully separated. The second term on the right-hand side includes transverse terms both from the velocity and acceleration fields together, that should be responsible for the net Poynting flux through a surface surrounding the charge.

We assume that the motion of the charged particle is non-relativistic and it varies slowly so that during the light-travel time across the region, any change in its velocity, acceleration
or other higher time derivatives is relatively small. This is equivalent to the conditions that 
\(|v|/c \ll 1, |v| r/c \ll c, |v| r/c \ll |v|, etc. Therefore we keep only linear terms \(v, v\) etc., in our formulation. Then from equation (10) we can write

\[
\mathbf{n} \times \mathbf{E} = -e \left[ \frac{\mathbf{n} \times (v + \dot{v} \frac{r}{c})}{r^2 c} \right]_{\text{ret}}.
\]

Then using equation (11) for the transverse electric field, one gets the electromagnetic power passing through the surface \(\Sigma\) as

\[
P = \int_{\Sigma} \mathbf{n} \cdot \mathbf{S} = \frac{e^2 c}{2} \int_0^\pi d\theta \sin^3 \theta \ r^2 \left[ \frac{(v + \dot{v} \frac{r}{c})^2}{r^2 c} \right]_{\text{ret}}.
\]

or

\[
P = \frac{2e^2}{3c} \left[ \frac{(v + \dot{v} \frac{r}{c})^2}{r^2} \right]_{\text{ret}}.
\]

Here the Poynting flux through a surface at a time \(t_o\) is written in terms of the charge motion at the retarded time \(t_o - r/c\), i.e., \(v\) and \(\dot{v}\) in equation (13) are values at the retarded time. Also, if we ignore velocity \(v\) in equation (13), we get the usual Larmor formula (cf equation (4)).

As we discussed above, to correctly apply the Poynting theorem, we must express equation (13) in terms of the charge motion at \(t_o\). For this we can make a Taylor series expansion of the velocity and acceleration at the retarded time \(t_o - r/c\) in terms of the charge motion at present time \(t_o\) as

\[
v = v_o - \frac{\dot{v}_o r}{c} + \frac{\ddot{v}_o r^2}{2c^2} + \cdots,
\]

(14)

\[
\ddot{v} = \ddot{v}_o - \frac{\dddot{v}_o r}{c} + \cdots,
\]

(15)

all quantities on the right-hand side are evaluated at the present time \(t_o\).

Substituting for \(v\) and \(\dot{v}\) from equations (14) and (15) into equation (13), we get

\[
P = \frac{2e^2 (v_o - \dot{v}_o r^2/2c^2)^2}{3r^2 c}.
\]

(16)

where we notice that the acceleration-dependent term does not appear in the power formula as it gets cancelled. We should point out here that the spherical surface \(\Sigma\) of radius \(r\) considered here is centred around the retarded position of the charge, which is somewhat off from the present position of the charge due to its motion. However, the charge and its neighbourhood are still enclosed well within the surface \(\Sigma\) (for \(|v|/c \ll 1\) and Poynting’s theorem is as much applicable to the radiation flux through this surface as to a surface centred on the charge.

Now dropping terms of order \(r^2\) or its higher powers, that will become zero as \(r \to 0\), we obtain

\[
P = \frac{2e^2 \dot{v}_o^2}{3r^2 c} - \frac{2e^2 \dot{v}_o \cdot \ddot{v}_o}{3c^3}.
\]

(17)

This is the Poynting flux in the neighbourhood of a point charge, instead of Larmor’s formula as given by equation (4). The first term on the right-hand side is the self-Coulomb field energy of the charge moving with a ‘present’ velocity \(v_o\) (as per equation (8)), while the second term
is the same as the power loss derived in the literature earlier because of drag against the self-force of an accelerated charged sphere [5–8].

5. Radiation losses from a charge in arbitrary motion

As we mentioned earlier, in order to relate the Poynting flux to the power loss by the charge, we should distinguish between the Poynting flux due to the self-fields carried along the charge due to its ‘present’ velocity and the remainder not represented in the motion of the charge and is thus detached from it and represents irretrievable radiative losses from the charge. The first term on the right-hand side in equation (17) is the Poynting flow due to the movement of self-fields along with the charge, because to its ‘present’ velocity $v$, as the Poynting flux is the same as in equation (8) for all values of $r$. Of course both expressions diverge as $r \to 0$, but that is unavoidable as it is due to the ‘point’ nature of the considered charge distribution, and the same divergence to infinity for $r \to 0$ is present even in the Coulomb field energy of a stationary charge. Thus in equation (17) it is only the remainder, distance-independent second term, not contained in the velocity fields of the charge, that could therefore be considered to be detached from the charge and hence radiated away.

In equation (9) the charge had a zero initial velocity. For a non-zero initial velocity $v$ the Poynting flux in equation (13) is again due to the ‘present’ velocity, $v_0 = v + \dot{v}r/c$, provided the acceleration $\dot{v}$ is a constant. Here there is no excess flux (than what is required for its present velocity) that could be called radiative loss. In fact, in such a case even if we let $r$ become very large, the Poynting flux will still represent the large velocity of the charge unless there is a change in acceleration, and it is only in the latter case that the Poynting flux at large $r$ will not match the value required for the actual motion (with a changing acceleration) of the charge at that time. This remains true even if the charge achieves a relativistic motion. For example, in the case of a uniform acceleration, where the charge may acquire relativistic velocity for a large enough time (we assume a one-dimensional motion with $v \parallel \dot{v}$), then instead of equation (13) we obtain for the Poynting flux [15]

$$P = \frac{2e^2(\gamma v + \gamma^3 \dot{v} r/c)^2}{3r^2c} = \frac{2e^2(\gamma v)^2}{3r^2c},$$

(18)

which is nothing but a relativistic generalization of equation (8), since $\gamma v + \gamma^3 \dot{v} r/c = (\gamma v)_0$ for a uniform acceleration.

From detailed analytical calculations it has been shown [14, 15] that for a uniformly accelerated charge, the total energy in the fields (i.e., including both the velocity and acceleration field terms from equation (1)) at any time is just equal to the self-energy of a charge moving uniformly with a velocity equal to the instantaneous ‘present’ velocity of the accelerated charge (even though the detailed field configurations may differ in the two cases). In fact as we showed above, for a uniformly accelerated charge (with $\dot{v} = 0$), all the Poynting flux in the acceleration fields goes towards making the change in the self-field energy of the charge as its velocity changes due to the acceleration.

Thus only when a rate of change of acceleration is present that we get an excess Poynting flux than that needed for the present velocity of the charge, and this excess electromagnetic power constitutes the radiative loss by the charge. The standard Larmor expression, in general, comprises the Poynting flux even of the self-field corresponding to the ‘present’ motion of the accelerating charge, along with the radiative losses, if any.

According to Poynting’s theorem, the rate of change of the mechanical energy ($E_{me}$) of the charges plus the electromagnetic field energy ($E_{em}$) enclosed within a volume could be
equated to the negative of the net Poynting flow through a surface surrounding the volume, with all quantities evaluated at the same time \( t_0 \)

\[
\frac{d\mathcal{E}_{\text{me}}}{dt} + \frac{d\mathcal{E}_{\text{em}}}{dt} = -\int_{\Sigma} (\mathbf{n} \cdot \mathbf{S}). \quad (19)
\]

Now in case of a ‘point’ charge, the volume integral of electromagnetic energy, \( \mathcal{E}_{\text{em}} \), diverges. However, we can instead use a comparison with a uniformly moving charge with velocity equal to the ‘present’ velocity of the accelerated charge. It has been explicitly shown \([14, 15]\) that for a uniformly accelerated charge, the field energy as well as the Poynting flux are equal to that of a uniformly moving charge with velocity equal to the ‘present’ velocity of the uniformly accelerated charge, and there is no other flux that could be called radiation loss. Thus it is only the excess Poynting flux due to the rate of change of acceleration that represents radiative power loss. Therefore, from a comparison with equation (8), we infer that it is only the second term in equation (17), viz.

\[
P = -\frac{2e^2V_0 \cdot V_0}{3c^3}, \quad (20)
\]

which is the excess outgoing power from the charge and therefore represents instantaneous radiative losses. Larmor’s formula does not separate out, from the radiated power, the contribution of acceleration fields (the second term on the right-hand side of equation (1)) to the Poynting flux that is due to the changing energy in velocity fields. Of course in the vast majority of cases where the energy in velocity fields does not keep on increasing indefinitely, Larmor’s formula yields a time-averaged value of power loss undergone by the radiating charge.

The expression for power loss (equation (20)) is the same as hitherto obtained in the literature from a detailed derivation of the self-force of an accelerated charge sphere of a vanishingly small radius \([5, 6, 8]\). But here we have derived the power loss due to radiation reaction from the Poynting flow for a ‘point source’ and which we showed to be different from the familiar Larmor formula. This should now also obviate the need for the internal-energy term, introduced on an ad hoc basis \([6, 9, 10]\), with a desire to make the power loss due to radiation reaction comply with the instantaneous radiative losses, hitherto thought to be given by Larmor’s formula.

We can generalize the formula for radiative losses to a relativistic case, using the condition of relativistic covariance (see e.g., \([2, 3]\))

\[
P = -\frac{2e^2\gamma_0^2}{3c^3} \left[ V_0 \cdot V_0 + 3\gamma_0^2 \frac{(V_0 \cdot V_0)^2}{c^2} \right]. \quad (21)
\]

Equation (21) should be contrasted with Liénard’s formula for radiative power losses from a charge moving with an arbitrary velocity, obtained by a relativistic transformation of Larmor’s formula \([1, 4, 17]\)

\[
P = \frac{2e^2\gamma}{3c^3} \left[ V^2 - \frac{(V \times V)^2}{c^2} \right] = \frac{2e^2\gamma^4}{3c^3} \left[ V^2 + \gamma^2 \frac{(V \cdot V)^2}{c^2} \right]. \quad (22)
\]

Equations (21) and (22) apparently look very different. Later we shall explore the difference that the two formulas make in their applicability.
6. The absence of radiation from an instantly stationary point charge

From equations (20) or (21), \( P = 0 \) if \( \mathbf{v}_0 = 0 \), which means there is no radiation from an instantaneously stationary point charge. Actually, it is evident from equation (17) itself that there is no Poynting flux anywhere in the entire vicinity of the instantly stationary charge. From this one readily infers an answer to the question whether for such a charge there are any energetic or radiative losses at that moment: the answer is an emphatic NO. The expected radiation flux term from Larmor’s formula, proportional to the square of the acceleration and independent of the radius of the sphere \( \left(-\frac{2e^2\mathbf{v}^2}{3c^3}\right) \), is certainly not present in the case of an instantly stationary charge. In all neighbourhoods of the charge in its instantaneous rest-frame, the transverse terms of the time-retarded velocity fields cancel the acceleration fields which were responsible in Larmor’s formula for a distance-independent radiation power through a surface surrounding the charge. From the absence of any Poynting flux in its neighbourhood it is clear that no net energy loss is taking place by the charge at that instant.

It is interesting to note that in the case of a uniformly accelerated charge, the acceleration fields get cancelled completely by the transverse term of the velocity fields not merely in the neighbourhood but at all distances, in its instantaneous rest-frame. For a uniformly accelerated charge, even when the motion might have been relativistic at the retarded time, for all values of \( r \) the time-retarded value of the expression \( \gamma \mathbf{v} + \gamma^3 \mathbf{v}r/c \) represents the ‘present’ velocity, \((\gamma \mathbf{v})_0\), which becomes zero in the instantaneous rest-frame of the charge. Actually for a larger \( r \), we need to go further back in time to get the time-retarded value of velocity which is directly proportional to \( t-r/c \) for a uniform acceleration. This results in

\[
\gamma \mathbf{v} + \gamma^3 \mathbf{v} \frac{r}{c} = (\gamma \mathbf{v})_0 = 0,
\]

with all transverse field in equation (10) getting cancelled for all \( r \), implying zero Poynting flux and hence nil radiation in the instantaneous rest-frame. This argument was used to show the absence of radiation for a charge supported in a gravitational field [14, 15], in conformity with the strong principle of equivalence. Incidentally Pauli [16] first brought it to notice that the magnetic field is zero throughout the instantaneous rest-frame of a uniformly accelerated charge, indicating the absence of radiation in this case.

Thus one arrives at the conclusion that there is a nil radiative loss rate from a charge that is instantaneously stationary, in conformity with the requirement that in the instantaneous rest-frame of the charge, because of a nil kinetic energy and a nil rate of work being done on the charge by the external agency responsible for the acceleration of the charge, nothing could have provided the necessary power that can go into the radiation.

Here, however, a possible objection could be raised. After all, for any motion of the charge, at any time one can find an inertial frame in which the charge is momentarily at rest, and thus not radiating according to the above arguments. Does it mean that the charge does not radiate at all?

Actually one could look at this intriguing problem from two different aspects. The first one is the ‘power loss’ by the accelerated ‘point’ charge, \(|d(1/2mv^2)/dt| = |mv dv/dt|\), which will be zero when \( \mathbf{v} = 0 \), even if only for an instant in the instantaneous rest frame. But it will be finite in a frame where \( \mathbf{v} \) is finite. The second one is from an observer’s point of view. After all one may rightly object that the energy in electromagnetic fields may be transformed but cannot be made totally zero by changing to any frame if it is finite as measured in at least one other frame, of say, a distant observer. To understand this let us say that one observer (in the instantaneous rest frame) instantly measures a zero Poynting flux through a spherical surface.
Now another observer (in a moving frame) will see these not as simultaneous measurements, but made at different times at different parts of the spherical surface, and in his own set of simultaneous measurements (that is in the second observer’s time) could still see a finite flux passing through the surface. Thus field energy–momentum, which is volume integrals over extended regions of space, even if it turns out instantly to be zero in one frame, need not necessarily be zero in other frames too.

7. Applicability of the old versus new formulas

Recently it has been shown [25] that Larmor’s formula is compatible with the power loss from the radiation reaction, and that the apparent discrepancy in the two formulations arises only because the two are inadvertently expressed in two different time systems. If the motion of the charge is expressed in terms of the values of its parameters (velocity, acceleration and higher time-derivatives, if any) at the present time, then we arrive at the formula for radiation losses due to the radiation reaction. On the other hand, expressing the charge motion in terms of the parameter values at the retarded time gives us the familiar Larmor radiation formula. In particular, it was explicitly shown [25] that for a charge moving in a circle (e.g., in a cyclotron or a synchrotron case), the two formulas yield the same power loss rate at every instant. The instantaneous power rate is identical in the case of a circular motion can be directly seen from equations (21) and (22) as well, because $\mathbf{v} \cdot \mathbf{v} = 0$ also implies $\mathbf{v} \cdot \mathbf{v} + \mathbf{v}^2 = 0$, and then the two expressions for power losses (equations (21) and (22)) yield equal values.

Of course this compatibility does not mean that the two formulas, in general, give identical results. In order to understand the difference in the two formulas when applied to a charge with an arbitrary motion, let us first consider a harmonically oscillating charge (like in a radio antenna)

$$x = x_0 \cos(\omega t + \phi).$$

Then

$$v = \dot{x} = -\omega x_0 \sin(\omega t + \phi),$$

$$\ddot{v} = \ddot{x} = -\omega^2 x_0 \cos(\omega t + \phi) = -\omega^2 x,$$

$$\mathbf{v} \cdot \mathbf{v} = \omega^2 x_0 \sin(\omega t + \phi) = -\omega^2 v.$$  

Then Larmor’s formula yields the radiative power $\propto \mathbf{v}^2 + \omega^2 x_0^2 \cos^2(\omega t + \phi)$, while the power loss from the radiation reaction turns out to be $\propto -\mathbf{v} \cdot \mathbf{v} = \omega^2 x_0^2 \sin^2(\omega t + \phi)$. Though the two expressions yield equal radiated energy when integrated or averaged over a complete cycle, the instantaneous rates are quite different; in fact, the two rates of power loss will be out of phase with each other. When the instantaneous power loss as calculated from the radiation reaction will be maximum, the radiation loss estimated from Larmor’s formula will be minimum and vice versa. For instance, when $\omega t + \phi = 0$ and $v = 0$ from equation (25), the radiation reaction equation gives zero power loss, Larmor’s formula yields a maximum power loss rate. Thus the two formulas may give the same result for the power loss in a time averaged sense; however, the strictly instantaneous rates could be very different. As any actual motion of the charge could be Fourier analysed, the above statement would be true for individual Fourier components though the detailed time behaviour of the combined effect could be quite different. However, in spite of this difference, the time averaged power losses would be similar. It also implies that the power spectrum, which gives average power in the cycle for each frequency component, would be the same. Of course there will be cases where
a Fourier analysis is not possible, for instance, in the case of a uniformly accelerated charge. In such cases the two formulas could yield conflicting answers.

8. Radiation reaction in a synchrotron source

It might seem that the difference between radiative losses given by Larmor’s formula or those inferred from the radiation reaction formula might be more of an academic interest or at most making a difference only in some very special cases like that of a uniformly accelerated charge. First thing, uniformly accelerated charges are not a rare breed. All charges stationary in a gravitational field of a star or other celestial bodies, including that of Earth, belong to this category. Secondly there could be other cases where a reasoning based on Larmor’s formula (or its relativistic generalization Liénard’s result) could lead to potentially incorrect conclusions. An example is the calculation of dynamics of relativistic charges radiating in a synchrotron source (be it an astrophysical or a laboratory phenomenon).

A charge in a constant uniform magnetic field moves in a helical path with a velocity component parallel to the magnetic field $v_\parallel = v \cos \theta$ that remains unaffected by the magnetic field. Here $\theta$ is the pitch angle, i.e., angle of the velocity vector of the charge with respect to the magnetic field vector. Now all the radiated power from a highly relativistic charge in a synchrotron case, as calculated from Larmor’s formula (or rather from Liénard’s formula), is within a narrow cone of angle $1/\gamma$ around the instantaneous direction of motion [1, 18, 19]. Therefore, any radiation reaction on the charge would be in a direction opposite to its instantaneous velocity vector [20]. This means that the direction of motion of the charge will not be affected, implying no change in the pitch angle of the charge. The subsequent formulation depends on these arguments, and the dynamics, as well as the lifetimes of the synchrotron electrons, are accordingly calculated. The formulas have appeared in various review articles and textbooks [18, 21, 22], and have been widely used for the last 50 years.

However, there is something amiss in the above arguments and it turns out that this picture is not consistent with the special theory of relativity. In a synchrotron case, component $v_\parallel$ of a charge remains constant, while magnitude of $v_\perp$ decreases continuously and as a consequence the pitch angle of the radiating charge in general changes [23, 24]. This can be seen in the gyro-centre (GC) frame, which moves with a velocity $v_\parallel$ with respect to the lab-frame $S$ and in which the charge therefore has only a circular motion in a plane perpendicular to the magnetic field (with a pitch angle $\theta = 90^\circ$). In the GC frame, due to radiative losses, there will be a decrease in the velocity, which is solely in a plane perpendicular to the magnetic field.

Now $v_\parallel$ is the constant relative velocity between two reference frames, therefore even in the lab frame, the parallel component of velocity of the charge should remain unchanged. However, the magnitude of the perpendicular component of velocity is continuously decreasing because of radiative losses, therefore the pitch angle of the charge given by $\tan \theta = v_\perp/v_\parallel$ should decrease continuously with time and the velocity vector of the charge should increasingly align with the magnetic field vector. While from Larmor’s formulation it has been inferred that the pitch angle of the radiating charge does not change, from special relativistic arguments we conclude that the pitch angle of the radiating charge would continuously decrease, with the velocity vector of the charge gradually becoming aligned with the magnetic field direction [24]. A careful examination of the effect of the radiation reaction in the synchrotron case on the dynamics of the charged particle, viz. the pitch angle of the radiating charge decreases, which appears contrary to conventional wisdom, but is in complete agreement with conclusions based on special relativistic arguments [23, 24].
It is quite intriguing that in spite of the instantaneous power loss rate being the same in the two cases, the effect on the charge motion is quite different in each case. In particular, Larmor’s formula does not yield results compatible with special relativity. There is actually an inherent inconsistency in estimating radiation reaction on a charge from Larmor’s formula. The radiation pattern of an accelerated charge has a \( \sin^2 \phi \) dependence about the direction of acceleration [1, 2, 4]. Due to this azimuthal symmetry the net momentum carried by the radiation is nil. Therefore the charge cannot also be losing momentum, even though it is undergoing radiative losses. Thus we have a paradox of a radiating charge losing its kinetic energy but without a corresponding change in its linear momentum. Such a paradox does not appear in the radiation reaction formulation, which alone seems to beget results consistent with special relativity.

9. Conclusions

From the electromagnetic fields in the neighbourhood of a ‘point charge’ in arbitrary motion, we determined the Poynting flux across a spherical surface of vanishingly small radius surrounding the charge. From that we showed that the radiative power loss turns out to be proportional to the scalar product of the instantaneous velocity and the first time-derivative of the acceleration of the charge. The discordance of these results with the familiar Larmor formula was traced to the fact that firstly, in the textbook derivation of Larmor’s formula, one did not properly consider the contribution of the velocity-dependent self-fields to the Poynting flux. Secondly, one equated the Poynting flux through a surface at a later time to the radiation loss by the charge at the retarded time, which is not what Poynting’s theorem states. A consistent picture was shown to emerge only when a mathematically correct procedure is followed. We showed that Larmor’s formula, in general, gives the radiative power loss only in a time-averaged sense and does not always yield an instantaneous value of radiative loss from a point charge. In particular, even for an instantly stationary accelerated charge, Larmor’s formula predicts a finite rate of radiative losses proportional to the square of acceleration of the charge, in violation of energy conservation as the charge has no kinetic energy that could be lost into radiated power. However, a proper examination of the electromagnetic fields in the neighbourhood of the point charge and the Poynting flux across a surface surrounding the point charge shows the absence of any such radiation flux in the instantaneous rest frame, where the contribution of acceleration fields is cancelled by the velocity fields. Further, we showed that in the case of synchrotron radiation, Larmor’s formula leads to a potentially incorrect conclusion about the constancy of the pitch angle of a radiating charge, a result inconsistent with special relativity, and that only the radiation reaction formula yields a correct result about the decrease in pitch angle due to radiation losses, with the velocity vector becoming gradually aligned with the magnetic field direction.

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