Campedelli surfaces with fundamental group of order 8

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Abstract

Let \( S \) be a Campedelli surface (a minimal surface of general type with \( p_g = 0, K^2 = 2 \)), and \( \pi: Y \rightarrow S \) an etale cover of degree 8. We prove that the canonical model \( \overline{Y} \) of \( Y \) is a complete intersection of four quadrics \( \overline{Y} = Q_1 \cap Q_2 \cap Q_3 \cap Q_4 \subset \mathbb{P}^6 \). As a consequence, \( Y \) is the universal cover of \( S \), the covering group \( G = \text{Gal}(Y/S) \) is the topological fundamental group \( \pi_1 S \) and \( G \) cannot be the dihedral group \( D_4 \) of order 8.

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1 Introduction

Let \( Y \) be a minimal surface of general type with \( K_Y^2 = 16 \) and \( p_g = 7, q = 0 \), having a free action by a group \( G \) of order 8. Write \( \varphi: Y \rightarrow \overline{Y} \subset \mathbb{P}^6 \) for the 1-canonical map, with image \( \overline{Y} \). We prove the following:

Theorem 1.1 The surface \( \overline{Y} \subset \mathbb{P}^6 \) is the complete intersection of 4 quadrics. It is isomorphic to the canonical model of \( Y \).

Theorem 1.1 is known if \( G = \mathbb{Z}_2^3 \) by Miyaoka [Mi], Theorem B; in this case there are four linearly independent diagonal quadrics through \( \overline{Y} \), which necessarily form a regular sequence defining \( \overline{Y} \). We thus assume throughout that \( G \) is a group of order 8 and contains an element of order 4.

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Corollary 1.2 Let $S$ be a Campedelli surface and $\pi: Y \to S$ an etale cover of degree 8. Then $Y$ is the universal cover of $S$ and the covering group $G = \text{Gal}(Y/S)$ is the topological fundamental group $\pi_1 S$.

Corollary 1.3 The dihedral group $D_4$ of order 8 is not the fundamental group of a Campedelli surface.

The proof of Theorem 1.1 consists of two parts, the first of which is now quite standard (compare Reid [Re2], Naie [Na], Konno [Ko]):

Proposition 1.4 (i) The canonical linear system $|K_Y|$ on $Y$ is free and defines a morphism $\varphi: Y \to \overline{Y} \subset \mathbb{P}^6$ that is birational to its image.

(ii) If $\overline{Y}$ is not a complete intersection of four quadrics, its quadric hull (the intersection of all quadrics containing $\overline{Y}$) is a 3-fold $X$ of degree 4, 5 or 6.

(iii) Moreover, in these three cases, $\overline{Y}$ is contained in a hypersurface $F_d$ of $\mathbb{P}^6$ not containing $X$, of degree $d = 6, 4, 3$ respectively.

The second part analyses the possible cases $\overline{Y} \subset X$, with ad hoc arguments involving the $G$-action to rule out each case; see Section 3.

1.1 The background

A Campedelli surface is a surface $S$ of general type with $p_g = 0$, $K^2 = 2$. The algebraic fundamental group $\pi_1^{\text{alg}}(S)$ classifies finite etale covers $Y \to S$, and is the profinite completion of the topological fundamental group $\pi_1 S$. Results of Beauville [Be] and Reid [Re1, Re2] (see also Mendes Lopes and Pardini [MP]) guarantee that $S$ has no irregular covers, and that an etale cover $Y \to S$ has degree $\leq 9$. The reasons underlying [Re1], Theorem 1.1 and all related results are as follows:

Principle 1.5 (1) The automorphism group $G$ acts on any intrinsically defined feature of $Y$: for example, the base points or base $-2$-cycles of $|K_Y|$ occur in multiples of 8.

(2) If a subgroup $H \subset G$ normalises a subscheme $Z \subset Y$, its order $|H|$ divides the Euler characteristic $\chi(O_Z)$; for example, if $Y$ has an intrinsically defined genus $g$ pencil $\psi: Y \to \mathbb{P}^1$ and $H \subset G$ fixes $P \in \mathbb{P}^1$ then $|H|$ divides $\chi(O_F) = g - 1$, where $F = \psi^* P$. 

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It seems most likely that all groups of order $\leq 9$ except the dihedral groups of order 8 and 6 occur as $\pi_1 S$. The case $|\pi_1 S| = 9$ was treated in detail in Mendes Lopes and Pardini [MP2]. Here we treat $|\pi_1 S| = 8$, patching up the incomplete manuscript [Re2]. Naie [Na] obtained similar results for $|\pi_1 S| = 6$ using similar methods. Campedelli surfaces with $\pi_1 = \mathbb{Z}/8$ and $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ are contained in passing in Barlow [Ba]. Beauville [Be2] constructs a family of Calabi–Yau 3-folds with $\pi_1$ the quaternion group $H_8$, and Campedelli surfaces with the same $\pi_1$ are obtained by taking the unique invariant section $X_1 = 0$ of this.

1.2 Representations of $G$ and proof of Corollary 1.3

Let $Y \to S$ be the universal cover of a Campedelli surface with group $G$. Then $G$ acts naturally on $H^0(K_Y)$ and $H^0(2K_Y)$. Since the $G$-action is free, $H^0(K_Y)$ is the regular representation of $G$ minus the trivial rank 1 representation, and $H^0(2K_Y)$ is three times the regular representation (for example, by [Re4, Corollary 8.6]). Finally, the $G$-equivariant multiplication map

$$S^2 H^0(K_Y) \to H^0(2K_Y)$$

is surjective by Theorem 1.1.

These remarks allow one to show that the group $G$ is not the dihedral group, and to describe explicitly $Y$ and the $G$-action for all the remaining groups of order 8.

Let $G = D_4$ be the dihedral group of order 8. Write 1 for the trivial rank 1 representation, and $\rho$ for the sole irreducible rank 2 representation; let $\chi_1 := \bigwedge^2 \rho$, $\chi_2$ and $\chi_3$ be the remaining rank 1 representations. Then

$$H^0(K_Y) = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \rho^{\oplus 2},$$

$$H^0(2K_Y) = 1^{\oplus 3} \oplus \chi_1^{\oplus 3} \oplus \chi_2^{\oplus 3} \oplus \chi_3^{\oplus 3} \oplus \rho^{\oplus 6}. \tag{1.2}$$

Using the decomposition of $H^0(K_Y)$, one computes:

$$S^2 H^0(K_Y) = 1^{\oplus 6} \oplus \chi_1^{\oplus 2} \oplus \chi_2^{\oplus 4} \oplus \chi_3^{\oplus 4} \oplus \rho^{\oplus 6}. \tag{1.3}$$

Clearly the equivariant map (1.1) cannot be surjective. This contradicts Theorem 1.1 and proves Corollary 1.3.
2 Proof of Proposition 1.4

The canonical map \( \varphi : Y \to \mathbb{P}^6 \) is a morphism by Ciliberto, Mendes Lopes and Pardini [CMP Proposition 5.2] and is birational to its image \( \overline{Y} \) by [CMP, Proposition 5.3]. Thus \( \overline{Y} \) is an irreducible surface of degree 16. Since

\[
\dim S^2 H^0(Y, K_Y) = \binom{8}{2} = 28 \quad \text{and} \quad h^0(Y, 2K_Y) = \chi(O_Y) + K_Y^2 = 24,
\]

the multiplication map \( S^2 H^0(Y, K_Y) \to H^0(Y, 2K_Y) \) has kernel of dimension \( \geq 4 \); that is, \( \overline{Y} \) is contained in at least 4 linearly independent quadrics.

Let \( Q_1, Q_2, Q_3, Q_4 \) be four linearly independent quadrics through \( Y \). We are home if \( Y \) is an irreducible component of \( Q_1 \cap \cdots \cap Q_4 \). For in turn, if any of \( Q_1 \) or \( Q_1 \cap Q_2 \) or \( Q_1 \cap Q_2 \cap Q_3 \) or \( Q_1 \cap \cdots \cap Q_4 \) is reducible, then \( \deg \overline{Y} < 16 \). This is impossible, so \( \overline{Y} = Q_1 \cap \cdots \cap Q_4 \) is a complete intersection of 4 quadrics. Then \( \overline{Y} \) is Gorenstein with \( K_{\overline{Y}} = O_{\overline{Y}}(1) \) and \( K_Y = \varphi^* K_{\overline{Y}} \). Therefore it has canonical singularities and is the canonical model of \( Y \).

Write \( \text{Quad}(\overline{Y}) \subset \mathbb{P}^6 \) for the quadric hull of \( \overline{Y} \), the intersection of all the quadrics through \( \overline{Y} \), following [Re3] and Konno [Ko]. The alternative to \( \overline{Y} \) a complete intersection of four quadrics is that \( \text{Quad}(\overline{Y}) \) has a component \( X \) strictly containing \( \overline{Y} \). Then \( X \) is a 3-fold of degree 4, 5 or 6 and is the unique component of \( \text{Quad}(\overline{Y}) \) containing \( \overline{Y} \).

Indeed, by elementary inequalities due to Castelnuovo, an irreducible \( m \)-fold \( X \) spanning \( \mathbb{P}^N \) is contained in at most

\[
\left( \binom{N-m+2}{2} - \min\{\deg X, 2(N-m) + 1\} \right)
\]

linearly independent quadrics. See for example the discussion in [Re3] or [Ko, Corollary 1.5]. The equality \( X = \text{Quad}(\overline{Y}) \) follows by [Ko, Corollary 2.6]. The estimate on \( d \) follows from (2.1) or by [Ko, Proposition 1.3].

Finally, in the three cases for \( d \), crude estimates give that the restriction map

\[
H^0(\mathbb{P}^6, O(k)) \to H^0(O_X(k))
\]

has rank

\[
= 252 \quad \text{for} \quad d = 4, k = 6, \quad \text{whereas} \quad h^0(6K_Y) = 248; \\
\geq 105 \quad \text{for} \quad d = 5, k = 4, \quad \text{whereas} \quad h^0(4K_Y) = 104; \\
\geq 58 \quad \text{for} \quad d = 6, k = 3, \quad \text{whereas} \quad h^0(3K_Y) = 56
\]

(compare [Re3] and [Ko, Lemma 1.8]). This proves Proposition 1.4.
3 Proof of Theorem 1.1

We exclude the cases of Proposition 1.4, (ii) by studying the $G$-action on $Y \subset X$, treating separately the cases $\deg X = 4, 5$ or $6$. In any case, $X$ is linearly normal, since $Y \to Y \subset \mathbb{P}^6$ is given by the complete canonical system, and is regular, since $Y$ is.

3.1 $G$-invariant linear systems on $Y$

The following lemmas group together a number of restrictions on $G$-invariant linear systems on $Y$, that we use several times in what follows. Their proofs are applications of Principle 1.5.

Lemma 3.1 A $G$-invariant linear system $|D|$ on $Y$ with $D^2 = 2$ has a fixed part.

**Proof** Assume by contradiction that $|D|$ has no fixed part. Since $G$ acts on the base locus of $|D|$, $D^2 = 2$ implies $|D|$ is free. Hence $|D|$ defines a $G$-equivariant 2-to-1 morphism $Y \to \mathbb{P}^2$. Since we assume that $G$ has an element of order 4, this contradicts Beauville [Be, Corollary 5.8].

Lemma 3.2 Let $|F|$ be a $G$-invariant pencil on $Y$ with $K_Y F \leq 8$. Then $|F|$ is free and $K_Y F = 8$.

**Proof** Since $K_Y F \leq 8$, the index theorem gives $F^2 \leq (K_Y F)^2 / K_Y^2 \leq 4$. Now $F^2$, equal to the degree of the base locus of $|F|$, is divisible by 8 by Principle 1.5, so $F^2 = 0$ and $|F|$ is free. If $K_Y F < 8$, the general $F \in |F|$ is nonsingular of genus $g \leq 4$, contradicting [CMP, Lemma 2.2], so $K_Y F = 8$.

Proposition 3.3 Let $\mathcal{D} \subset |K_Y|$ be a $G$-invariant subsystem of projective dimension $\geq 3$. Then one of the following holds:

(1) $\mathcal{D}$ is free; or

(2) $\mathcal{D}$ has base locus consisting of 8 transversal base points.

In particular, $\mathcal{D}$ is not composed with a pencil.
Proof If \( D \) has a nonzero fixed part \( Z \), write \( D = M + Z \) with mobile \(|M|\). The \( G \)-action takes \( Z \) to itself, so \( Z \) is the pullback from \( S = Y/G \) of a divisor \( Z_0 \) that satisfies \( K_S Z_0 \equiv Z_0^2 \) mod 2; therefore \( MZ = (K_Y - Z)Z \) is divisible by 16. Connectedness of canonical divisors gives \( MZ > 0 \), and thus \( 16 \leq MZ \leq K_Y M \leq K_Y^2 = 16 \). We get:

\[
M^2 = K_Y Z = 0 \quad \text{and} \quad K_Y M = 16, \quad Z^2 = -16.
\]

Since \(|M|\) is mobile and \( M^2 = 0 \), it is contained in a multiple of a \( G \)-invariant free pencil, say \(|M| \subset |nF|\) with \( K_Y F = 16/n \); Lemma 3.2 implies \( n \leq 2 \). But \( n = \dim |nF| \geq \dim D \geq 3 \), a contradiction.

Therefore \( D \) has no fixed part. Since \( G \) acts on the base scheme of \( D \), the number \( \nu \) of base points is divisible by 8. If \( \nu > 8 \) or \( \nu = 8 \) and the base points are not transversal, two curves of \( D \) have no free intersections, hence \( D \) is composed with a pencil. Write \( D = nF \), with \(|F|\) a \( G \)-invariant pencil and \( n \geq 3 = \dim D \). Then \( F^2 = 0 \) by Lemma 3.2 contradicting \( 16 = D^2 = n^2 F^2 \).

3.2 The case \( \deg X = 4 \)

In this case, by Fujita [Fu1], \( X \) is either a quartic scroll \( \mathbb{F}(a,b,c) \) with \( a + b + c = 4 \), or the cone over the Veronese surface \( V_4 \subset \mathbb{P}^5 \). By Proposition 1.4 (iii), there is a sextic hypersurface containing \( Y \) and not containing \( X \).

If \( X \) is a scroll, the birational transform of its unique ruling by planes is a \( G \)-invariant pencil \(|F|\) on \( Y \) with \( K_Y F \leq 6 \), contradicting Lemma 3.2.

If \( X \) is the cone over \( V_4 \), the linear subsystem \( D \subset |K_Y| \) formed by hyperplanes through its vertex define a \( G \)-equivariant map \( \psi: Y \to V_4 \). By Proposition 3.3 \( D \) is either free or has 8 simple base points. In the latter case, \( \deg \psi = 2 \) contradicts [Be Corollary 5.8], as in Lemma 3.1. So \( D \) is free, and \( \psi: Y \to V_4 \cong \mathbb{P}^2 \) is a morphism of degree 4. The \( G \)-action on \( \mathbb{P}^2 \) fixes some point \( P \in \mathbb{P}^2 \) by Lemma 3.4 below, whereas \( \psi^{-1} P \) consists of \( \leq 4 \) points or trees of \(-2\)-curves, on which \( G \) cannot act freely. This is a contradiction.

Lemma 3.4 Let \( G \) be a group of order \( 2^r \) acting on \( \mathbb{P}^2 \). Then there is a point \( P \in \mathbb{P}^2 \) fixed by the whole of \( G \).
Indeed, $G$ has nontrivial centre, so a central element $g$ of order 2. The action of $g$ on $\mathbb{P}^2$ must fix an isolated point $P$ and a line $L$. For any $h \in G$, by the conjugacy principle, the element $hgh^{-1}$ is an involution with isolated fixed point $h(P)$. But $hgh^{-1} = g$, so that $h(P) = P$.

### 3.3 The case $\deg X = 5$

This is the hard case of the proof, and we break it into several steps.

**Step 1.** $X$ is a normal del Pezzo variety with $K_X = \mathcal{O}_X(-2)$. Recall from the start of the proof that we assume that $X$ is linearly normal and regular. By [Fu3, Theorem 2.1] (or [Fu2] in the nonsingular case) $X$ is either a normal del Pezzo variety of index 2 or a cone from a point vertex over a (weak) del Pezzo surface $V_5 \subset \mathbb{P}^5$. If $X$ is a cone, the subsystem $D \subset |K_Y|$ given by hyperplanes through its vertex defines a $G$-equivariant map $\psi: Y \to V_5$. By Proposition 3.3, $\psi$ is onto the surface $V_5$, and

$$\deg V_5 \cdot \deg \psi = 5 \deg \psi = 8 \text{ or } 16$$

provides a contradiction.

**Step 2.** $\overline{Y} \cap \text{Sing } X$ is a finite set. If $\text{Sing } X$ is positive dimensional, it contains a single line $L$ ([Fu3, Theorem 2.7]). Apply Proposition 3.3 to the subsystem $D \subset |K_Y|$ given by hyperplanes of $\mathbb{P}^6$ through $L$; then $D$ has no fixed part, so $L$ is not contained in $\overline{Y}$.

**Step 3.** The general section $C$ of $\overline{Y}$ is nonsingular. Let $\Sigma$ be a general hyperplane section of $X$ and set $C = \Sigma \cap \overline{Y}$. The surface $\Sigma$ is a (possibly singular) del Pezzo surface of degree 5, nonsingular along $C$ by Step 2, so that $C$ is a Cartier divisor on $\Sigma$. Write $A = -K_\Sigma = \mathcal{O}_\Sigma(1)$ for the restriction of a hyperplane to $\Sigma$. Since $AC = -K_\Sigma C = 16$, the index theorem gives $C^2 \leq (AC)^2/A^2 = 256/5$, so $C^2 \leq 51$. The curve $C$ is the birational image of a general canonical curve of $Y$, so has geometric genus 17. On the other hand, the arithmetic genus of $C \subset \Sigma$ is given by $2p_a C - 2 = C^2 + K_\Sigma C = C^2 - 16$.

There are thus two possibilities:

(a) $C^2 = 48$ and $C$ is nonsingular, or

(b) $C^2 = 50$ and $C$ has a single node or cusp.
If case (b) holds for the general hyperplane section of \( \overline{Y} \), the codimension 1 part of the singular locus of \( \overline{Y} \) is a line \( L \), necessarily invariant under the action of \( G \). The system of hyperplanes through \( L \) then give the same contradiction to Proposition 3.3 as in Step 2.

**Step 4. Conclusion of the proof.** We continue to use the notation of Step 3. The canonical class of \( C \) calculated on \( Y \) is \( K_C = (K_Y + C)|_C = O_C(2A) \). Calculated on \( \Sigma \), it is \( (K_\Sigma + C)|_C = O_C(-A + C) \). Therefore the Cartier divisor \( D = C - 3A \) on \( \Sigma \) restricted to \( C \) is linearly equivalent to zero. Consider the exact sequence of sheaves on \( \Sigma \):

\[
0 \to O_\Sigma(-3A) \to O_\Sigma(D) \to O_C \to 0.
\]

Since \( H^1(\Sigma, -3A) = 0 \) by Kodaira vanishing, or by well known results on del Pezzo surfaces, it follows that \( h^0(O_\Sigma(D)) = 1 \), so \( D \) is a Cartier divisor linearly equivalent to an effective divisor.

Now \( -K_\Sigma D = AD = 1 \), and \( D^2 = 48 - 96 + 45 = -3 \). This is a contradiction. Indeed, \( AD = 1 \) and \( A \) very ample implies that \( D \) is a line on \( \Sigma \). But then \( D \) is nonsingular, and because it is a Cartier divisor, \( \Sigma \) is nonsingular near \( D \), so \( D^2 = -1 \).

### 3.4 The case \( \deg X = 6 \)

Assume \( \deg X = 6 \). By Proposition 1.4 and its proof, the linear system of cubics of \( \mathbb{P}^6 \) containing \( \overline{Y} \) restricts on \( X \) to a positive dimensional linear system \( |N| \) of surfaces of degree 2. Now \( X \) is not ruled by planes (because it is linearly normal of degree 6 and regular), so that the moving part of \( |N| \) must be a pencil of quadrics.

The birational transform of \( |N| \) on \( Y \) is then a \( G \) invariant pencil \( |F| \) with \( K_Y F \leq 6 \) and contradicts Lemma 3.2.

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