Marginal Reduced High-degree CKF and its Application in Nonlinear Rapid Transfer Alignment

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Abstract: The error model is not perfect when the nonlinear transfer alignment is applied to the inertial navigation system of shipborne aircraft. In this paper, a new model incorporating acceleration error of lever-arm is proposed. Considering the influence of flexure deformation and dynamic lever-arm, it is suitable for the case of large azimuth misalignment. The state filtering estimation selects the High-degree Cubature Kalman filter (HCKF). A reduced HCKF algorithm with marginalized-sampling-based cubature transformation is designed according to the state equation and measurement equation structure of the transfer alignment model, and it is proved. This algorithm simplifies the measurement update process, and applies the marginal sampling algorithm to the time update process, achieving the purpose of reducing the calculation amount of HCKF. In the simulation, the velocity plus attitude matching method is used. The experimental results show that compared with the transfer alignment model without considering the dynamic lever-arm, this model has a higher accuracy, and the alignment precision and time also meet the requirements.

1. Introduction

Transfer alignment is a common method of moving base alignment [1-2]. It can improve alignment accuracy and shorten alignment period when master inertial navigation system (MINS) with high accuracy of shipborne is used to share information with slave inertial navigation system (SINS) of the low accuracy of carrier aircraft [3-5]. Transfer alignment contains two processes, coarse alignment and fine alignment [1]. In the coarse alignment process, MINS provides navigation information such as attitude, velocity and position for SINS to accomplish one binding, and then start attitude calculation and navigation calculation. The fine alignment process is in fact an estimation process using Kalman filter based on the transfer alignment error model. Lever-arm effect error and flexural deformation error of shipborne are two main error sources in practical MINS and SINS alignment process. Compensation is made in measurement of filters on velocity error caused by lever-arm effect during fine alignment process, which can weaken influence brought by lever-arm effect. Structure of ship will change under the effect of external factors such as airflow and wave in real sailing, which brings dynamic lever-arm caused by flexural deformation that effects the length of lever-arm [7-15]. [6-9] sets incorporate error modeling with flexure of carrier and lever-arm effect of transfer alignment on the condition that three misalignments are small enough. In fact, the actual position of carrier aircraft is flexible on the board. Since MINS is mount in navigation room below the board, the azimuth misalignment angle between MINS and SINS is so big that the incorporate error modeling of transfer alignment based on small misalignment assumption isn’t applicable any more. It’s necessary to make
further research on incorporate error modeling of nonlinear transfer alignment. Q.W. Gao proposes nonlinear transfer alignment model that is suitable for large misalignment, however, it doesn’t set modeling for actual misalignment. In addition, the measurement equation is nonlinear, which reduces accuracy of filter and numerical stability. K. Chen proposes an attitude matching method that is applicable under the condition that installation misalignment is large on the basis of compensating installation misalignment by attitude measurement, but the installation misalignment needs to be known in advance, which limits its application. As can be seen, all proposed transfer alignment models in [7-11] are established in computer navigation frame. Kain proposes transfer alignment error model based on measurement misalignment in 1989, introduces computer body frame and establishes a new “velocity plus attitude” matching rapid transfer alignment error model [12]. [13-16] establishes nonlinear rapid transfer alignment model but without further considering comprehensive influences brought by flexural deformation and lever-arm effect.

Aiming at nonlinear transfer alignment problem of inertial navigation system with large azimuth misalignment in carrier aircraft, and considering two main errors (due to carrier based flexural deformation and lever-arm effect), a nonlinear incorporated error model is thus proposed and established. The paper is organized as follows. Incorporate model of nonlinear rapid transfer alignment is in Section 2. The traditional High-degree Cubature Kalman filter (CKF), the linear structure of proposed model, and the marginal reduced High-degree CKF (M-RHCKF) is Section 3. Simulation results are used to verify the proposed model and to demonstrate the effectiveness of proposed filter algorithm, which is in Section 4. The conclusion is given in Section 5 eventually.

2. Incorporate Model of Nonlinear Rapid Transfer Alignment

2.1 Definition of Coordinate

The derivations of proposed model mainly involves the following coordinates, relation between them is shown in Figure.1.

Besides, a standard frame so is introduced here. The s_o-frame is coincident with the s-frame when flexural deformation angles are zero. Navigation frame n chooses local geography coordinate; the origin of ship borne frame m is located in the center of ship borne, and the misalignment between m-frame and the standard frame s_o is defined as actual misalignment \( \phi_a \); flexural deformation angle \( \theta \) is the angle between s_o-frame and actual aircraft frame s; measurement misalignment \( \phi_m \) is defined as the misalignment between s-frame and computer body frame s'.

2.2 Flexure Deformation and Dynamic Lever-arm Analysis of Transfer Alignment

In general, Markov process can better describe flexure deformation [6-10, 17], the flexure deformation angle at this moment is assumed as the second-order Markov process stimulated by white noise, which can be described as follows

\[
\begin{align*}
\dot{\theta} &= \omega_{\dot{\theta}} \\
\dot{\omega}_{\dot{\theta}} &= -\beta^2 \theta - 2\beta \omega_{\dot{\theta}} + w
\end{align*}
\]

(1)

Where, \( \theta = [\theta_x \ \theta_y \ \theta_z]^T \) denotes flexure deformation angle vector, \( \omega \) denotes flexure deformation angular velocity, \( \beta = 2.146 / \tau \) is a constant depending on correlation time,
\[ \text{Var}(w) = Q = 4\beta^3\sigma^3. \]

where, \( \sigma = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z \end{bmatrix}^T \) denotes covariance of three flexure deformation angles, \( \tau = \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^T \) denotes three correlation time, \( w \) denote inspired noise whose covariance is \( Q \).

Figure 2: Diagram of dynamic lever arm caused by \( \theta_z \)

As illustrated in Figure 2, the length of initial lever-arm between MINS and SINS is \( r_0 = [x_0, y_0, z_0]^T \), flexure deformation angle is \( \theta = [\theta_x, \theta_y, \theta_z]^T \) which is small. Firstly take the x-axis flexure deformation into consideration, the other two axes are the same as x-axis. Mark the x-axis flexure deformation angle \( \angle A'OB \) as \( \theta_x \), initial lever-arm of y-axis is \( OA = y_0 \). When flexure deformation exists on carrier base, \( OA \) turns into the arc \( OA' \), and \( E \) denotes the center of the circle corresponding to arc \( OA' \), \( EO \) and \( EA' \) are corresponding radius of arc \( OA' \), the length of \( OA' \) and \( OA \) is the same. Mark \( C \) as the point of vertical intersection of point \( D \) and \( OA' \). It can be known from geometric principle that

\[
\begin{align*}
OD &= A'D = \frac{OA'}{2\angle ODC} = \frac{y_0}{2\theta_x} \quad (2) \\
OA &= 2OC = \frac{\sin \theta_x y_0}{\theta_x} \quad (3) \\
OB &= \cos \theta_x OA = \frac{\sin \theta_x \cos \theta_x y_0}{\theta_x} \quad (4) \\
AB &= \sin \theta_x OA' = \frac{\sin^2 \theta_x y_0}{\theta_x} \quad (5) \\
\delta r_y &= AB = y_0 \left(1 - \frac{\sin \theta_x \cos \theta_x}{\theta_x} \right) \quad (6) \\
\delta r_z &= A'B = \frac{\sin^2 \theta_x y_0}{\theta_x} \quad (7)
\end{align*}
\]

As mentioned, flexure deformation angle of y-axis and z-axis are the same. Table 1 denotes the lever-arm error caused by \( \theta \) on the other two axes.

Table 1: Lever-arm error on y-axis and z-axis caused by flexural deformation angle.

| Lever-arm error of y-axis | Lever-arm error of z-axis |
|---------------------------|--------------------------|
| \( \theta_x \) | \( \frac{x_0}{2\theta_x} \) | \( \frac{\sin \theta_x x_0}{\theta_x} \) |
| \( \theta_y \) | / | \( \frac{\sin \theta_x x_0}{\theta_x} \) |
| \( \theta_z \) | \( \frac{-x_0}{2\theta_x} \) | \( \frac{\sin \theta_x x_0}{\theta_x} \) |
The length of lever-arm influenced by dynamic lever-arm $\delta r$ can be obtained

$$ r = r_0 + \delta r = \begin{bmatrix} \sin^2 \theta_2 z_0 + x_0 \\ \sin \theta_2 \cos \theta_2 \theta_2 \\ \sin^2 \theta_3 z_0 + y_0 \\ \sin \theta_3 \cos \theta_3 \theta_3 \\ \sin^2 \theta_4 z_0 + z_0 \\ \sin \theta_4 \cos \theta_4 \theta_4 \end{bmatrix} + \begin{bmatrix} \theta_2 z_0 + x_0 \\ \theta_3 x_0 + y_0 \\ \theta_4 y_0 + z_0 \end{bmatrix}$$

(8)

When flexure deformation angle satisfies the small assumption, $\sin \theta \approx \theta$, $\cos \theta \approx 1$, then Equation(8) can be further reduced as

$$ r = \begin{bmatrix} \theta_2 z_0 + x_0 \\ \theta_3 x_0 + y_0 \\ \theta_4 y_0 + z_0 \end{bmatrix} = r_0 + L_0 \theta$$

(9)

Dynamic lever-arm $\delta r$ can be described as

$$ \delta r = L_0 \theta$$

(10)

Where, $L_0 = \begin{bmatrix} z_0 \\ x_0 \\ y_0 \end{bmatrix}$.

Equation (10) can be differentiated on both sides. The first-order and second-order derivatives can be obtained

$$ \dot{r} = L_0 \dot{\theta}$$

(11)

$$ \ddot{r} = L_0 \ddot{\theta}$$

(12)

Combing with Equation (1), Equation (12) can be further written as

$$ \ddot{r} = L_0 \left(-\beta^2 \theta - \beta \dot{\theta} + w\right) = -L_0 \dot{\theta} - L_2 \theta + L_0 w$$

(13)

Where, $L$ and $L_0$ are

$$ L = \begin{bmatrix} 2z_0 \beta_z \\ 2x_0 \beta_x \\ 2y_0 \beta_y \end{bmatrix}$$

(14)

$$ L_0 = \begin{bmatrix} z_0 \beta_z \\ x_0 \beta_x \\ y_0 \beta_y \end{bmatrix}$$

(15)

As is illustrated in Figure 3, ship borne is swaying while sailing, so it’s a common that the ship borne swaying center is the mounting position of master inertial components. However, the mounting position of slave inertial components isn’t coincident with ship borne swaying center, thus making disturbance acceleration included in accelerometer of SINS. Therefore, the lever-arm effect error is described as the phenomenon that velocity information calculated by MINS and SINS is different, and the lever-arm acceleration error is the difference between specific force information sensitized by master inertial components and slave inertial components.
\( \alpha_x, y, z \) denotes the inertial frame, \( \alpha_{x_m}, y_m, z_m \) denotes the master inertial navigation carrier frame, \( \alpha_x, y, z_s \) the slave inertial navigation carrier frame, and according to vector relation shown in Figure.3, we can get

\[
R'_i = R'_m + r'
\] (16)

Where, \( R_m \) is the distance from the center of MINS to inertial frame projected into \( i \)-frame, \( R'_i \) the distance from the center of SINS to inertial frame projected into \( i \)-frame, and \( r \) the length of lever-arm between MINS and SINS projected to \( i \)-frame. Differentiating on both sides of Equation (16)

\[
\dot{R}'_i = \dot{R}'_m + \dot{r}' = \dot{R}'_m + C_m \omega_m \times r
\]

Further differentiating on both sides of Equation (17), assuming matrix differential equation

\[
\dot{C}_m = C'_m \left[ \omega_m \times \right]
\]

we can get

\[
\dot{R}'_i = \dot{R}'_m + C'_m \left( \left[ \omega_m \times \right] \left[ \omega_m \times \right] r_m \right) + \left[ \omega_m \times \right] r_m + 2 \left[ \omega_m \times \right] \dot{r}_m + \ddot{r}_m
\]

(18)

\[
f'^i = f'_m \left[ \right] \left[ \omega_m \times \right] r_m + \left[ \omega_m \times \right] r_m + 2 \left[ \omega_m \times \right] \dot{r}_m + \ddot{r}_m
\]

(19)

\[
f'^i = f'^m + \left[ \omega_m \times \right] \left[ \omega_m \times \right] r_m + \left[ \omega_m \times \right] r_m + 2 \left[ \omega_m \times \right] \dot{r}_m + \ddot{r}_m
\]

(20)

Define the lever-arm acceleration error being \( \dot{f}'_m = f'^m - f'_m \). In actual sailing, it’s easy that ship body has flexure deformation and the length of lever-arm is usually large, so the derivative needs to be considered, which is shown as follows

\[
\dot{f}'_m = \left[ \omega_m \times \right] \left[ \omega_m \times \right] r_m + \left[ \omega_m \times \right] r_m + 2 \left[ \omega_m \times \right] \dot{r}_m + \ddot{r}_m
\]

(21)

2.3 Attitude Error Model of MINS and SINS

When flexure deformation exists between MINS and SINS, gyro of slave inertial components not only sensitizes the angular rate caused by carrier movement but also sensitizes the flexure angular rate \( \omega_f \) caused by flexure deformation. MINS isn’t sensitive to flexure angular rate because it’s mounted in
swaying center of ship body. Therefore the relation of angular rate between MINS and SINS can be expressed as

$$\omega_{s} = C_{s}^{m} \omega_{s}^{m} + C_{s}^{m} \omega_{s}^{s} + \omega_{s}^j$$

(22)

Here, \( C_{s}^{m} \) is the coordinate transformation matrix from the m-frame to the s-frame, including actual misalignment \( \phi \) and flexure deformation \( \theta \). It’s commonly known that there is no error in MINS during the transfer alignment process, so the bias \( \epsilon_s \) of SINS gyro is the only parameter needed to be considered. Considering that there’s no relative rotation between standard aircraft frame \( s_0 \) and carrier coordinate \( m \), then substitute \( \dot{\phi} = 0 \) and \( \omega_{s}^j = \omega_{s}^j \) into Equation (22), we can get

$$\omega_{s} = C_{s}^{m} \omega_{s}^{m} + \omega_{s}^j$$

(23)

Assuming that the flexure deformation angle is small, the above equation can be written as

$$\omega_{s} = (I - \theta \times) C_{s}^{m} \omega_{s}^{m} + \omega_{s}^{j}$$

(24)

It can be easily derived from attitude matrix differential equation as follows [12-16]

$$\dot{C}_{s}^{n} = C_{s}^{n}(\omega_{s}^{m} \times)$$

(25)

$$\dot{C}_{s}^{s} = C_{s}^{s}(\omega_{s}^{s} + \epsilon_{s}^{s}) \times)$$

(26)

Direction cosine matrix \( C_{s}^{s} \) from s'-frame to n-frame is expressed as

$$C_{s}^{s} = C_{s}^{n} C_{s}^{m}$$

(27)

Differentiating on both sides of Equation (27), we can get

$$\dot{C}_{s}^{s} = \dot{C}_{s}^{n} C_{s}^{m} + C_{s}^{n} \dot{C}_{s}^{s}$$

$$\dot{C}_{s}^{s} = \dot{C}_{s}^{n} C_{s}^{m} + C_{s}^{n} \dot{C}_{s}^{s}$$

(28)

Then substituting Equation (25) and (26) into Equation (27), multiplying \( C_{s}^{n} \) on both sides, we can obtain

$$\dot{C}_{s}^{n} = C_{s}^{n}(\omega_{s}^{s} + \epsilon_{s}^{s}) \times - (\omega_{s}^{m} \times) C_{s}^{n}$$

(29)

Rate \( \omega_{s}^{s} \) and rate \( C_{s}^{n} \) satisfy the differential equation shown as follows

$$\dot{C}_{s}^{n} = C_{s}^{n}(\omega_{s}^{s} \times)$$

(30)

Inserting Equation (30) into Equation (29), the following Equation (31) can be derived as

$$\dot{C}_{s}^{n} = (\omega_{s}^{s} + \epsilon_{s}^{s}) \times - (\omega_{s}^{m} \times) C_{s}^{n}$$

(31)

Modifying Equation (31) into vector format

$$\dot{\omega}_{s} = -C_{s}^{n} \omega_{s}^{m} - \omega_{s}^{s} + \epsilon_{s}^{s}$$

(32)

Substituting Equation (24) into Equation (32), and the modified equation is shown as

$$\dot{\omega}_{s} = (I - \theta \times) C_{s}^{m} - \omega_{s}^{m} \omega_{s}^{j} + \epsilon_{s}^{j}$$

(33)

Let \( \omega_{s}^{j} \) denotes the angular rate of s’-frame with respect to m-frame, expressed in s’-frame, the relation between \( \omega_{s}^{j} \) and measurement misalignment \( \phi \) can be obtained

$$\dot{\phi} = C_{s}^{j} \omega_{s}^{j}$$

(34)

Where
\[ C_w^{-1} = \begin{bmatrix}
\cos \phi_w & \cos \phi_w & 0 & \sin \phi_w & \cos \phi_w \\
\sin \phi_w & \sin \phi_w & \cos \phi_w & -\sin \phi_w & \cos \phi_w \\
-\sin \phi_w & 0 & \cos \phi_w & 0 & \cos \phi_w
\end{bmatrix} \]  
(35)

After inserting Equation (34) into Equation (33), the attitude error equation of nonlinear rapid transfer alignment is
\[ \phi_t = C_w^{-1} \left[ (I - \theta \times) C_w - C_w^\prime \right] \omega_m + \omega'_0 + \epsilon'_t \]  
(36)

### 2.4 Velocity Error Model of MINS and SINS

Velocity differential equations of MINS and SINS are respectively shown as
\[ v'_m = C_w f_m^\prime - (2\omega'_m + \omega'_s) \times v'_m + g'_m \]  
(37)
\[ v'_s = C_s f'_s - (2\omega'_s + \omega'_s) \times v'_s + g'_s \]  
(38)

Velocity error is defined as \( \delta v = v'_m - v'_s \), the equation \( f_m^\prime = f'_s + f'_s + \nabla'_s \) can be satisfied due to the effect brought by acceleration error of dynamic lever-arm. Equation (38) minus Equation (37) is Equation (39).
\[ \delta v = C'_w \left( I - C'_w C'_w \right) f'_s - (2\omega'_s + \omega'_s) \delta v + C'_s \left( f'_s + \nabla'_s \right) \]  
(39)

Commonly, lever-arm velocity error caused by fixed lever-arm can be compensated in the measurement of velocity error. However, the velocity error caused by dynamic lever-arm acceleration can’t be compensated. Inserting Equation (21) into Equation (39), the nonlinear incorporate velocity error equation can be written as
\[ \delta v = C'_w \left( I - C'_w C'_w \right) \left( I + \theta \times \right) f'_s - (2\omega'_s + \omega'_s) \delta v + C'_s \left( f'_s + \nabla'_s \right) \]  
(40)

Where, \( M_1 = 2 [\omega'_s \times] L_0 - L_1, M_2 = [\omega'_s \times] [\omega'_s \times] + [\omega'_s \times] \).

### 3. Marginal Reduced High-degree Cubature Kalman Filter

#### 3.1 State Equation and Measurement Equation of Transfer Alignment

After differentiating on both sides of Equation (10), the differential equation of dynamic lever-arm \( \delta r \) can be get
\[ \delta \dot{r} = L_i \omega'_{ih} \]  
(41)

\( \epsilon'_s \) and \( \nabla'_s \) are described by random constant usually, the differential equation of inertial unit is derived as
\[ \epsilon'_s = 0 \]  
(42)
\[ \nabla'_s = 0 \]  
(43)

The nonlinear incorporate transfer alignment state equation is composed of Equation (1), (36), (40), (41), (42) and (43).

Applying “velocity plus attitude” pattern, after solving the velocity difference between MINS and SINS and making lever-arm velocity compensation for standard lever-arm \( r_0 \), velocity measurement can be solved. Attitude measurement is get by multiplying the attitude matrix of MINS and SINS, whose expression is shown as
\[ C'_n = C'_a C'_m \]  
(44)
It's obvious that measurement misalignment $\phi$ can denote state variable and measurement at the same time, so measurement equation turns to simple line equation, measurement matrix $H$ is written as

$$H = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times12} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times12} \end{bmatrix} \quad (45)$$

### 3.2 Reduced High-degree Cubature Kalman Filter (RHCKF)

Filter accuracy of HCKF can reach fifth-order \[18-19\], thus making HCKF have clear superiority than traditional CKF and UKF whose filter accuracy can only reach the third-order. However, the number of cubature points applied in HCKF during time update process and measure update process reaches up to $(2n^2 + 1)$, which brings huge calculating amount and limits its application as well. In order to maintain the high accuracy of filter algorithm, also known from Equation (45) that the measurement equation of proposed rapid transfer alignment model is linear, traditional measurement update of HCKF is replaced by reduced measurement update so that only needs cubature transformation for once in time update can the whole flow of algorithm finished (see specific proof procedure in appendix).

Specific flow of proposed algorithm is shown as follows

- **Initialization of filter**
  
  $$\begin{align*}
  \hat{X}_0 &= \text{E}[X_0] \\
  P_0 &= \text{E}\left((X_0 - \hat{X}_0)(X_0 - \hat{X}_0)^\top\right) \quad (46)
  \end{align*}$$

- **Time update**

  a) The calculation of cubature points

  SVD decomposition can better solve the problem of the ill-condition of covariance matrix compared with traditional Cholesky decomposition, which doesn’t require that the covariance matrix must be symmetric and positive definite, thus making the whole algorithm have higher numerical stability and filter accuracy. Applying SVD decomposition to covariance matrix $P_{k-1/k-1}$, the decomposition result is shown as

  $$P_{k-1/k-1} = U_{k-1/k-1}S_{k-1/k-1}V_{k-1/k-1}^\top \quad (47)$$

  $$X_{\epsilon, k-1/k-1} = U_{k-1/k-1}S_{k-1/k-1}\xi_i + \hat{X}_{k-1/k-1} \quad (48)$$

  Where, $S = \text{diag}(s_1, s_2, s_3, \ldots, s_r)$ denotes singular value of $P_{k-1/k-1}$, $s_1 \geq s_2 \geq s_3 \geq \ldots \geq s_r \geq 0$, $U \in \mathbb{R}^{nxn}$, $V \in \mathbb{R}^{nxn}$. $\xi_i$ denotes integrate point set, when the fifth-order cubature principle is used, the number of cubature point is $2n^2 + 1$, specific expression of $\xi_i$ is written as \[8,9\]

  $$\xi_i = \begin{cases} 
  0, \ldots, 0^\top \quad i = 0 \\
  \sqrt{(n+2)s_i^+} & i = 1, \ldots, n(n-1)/2 \\
  -\sqrt{(n+2)s_i^-} & i = n(n-1)/2 + 1, \ldots, n(n-1) \\
  \sqrt{(n+2)s_i^+} & i = n(n-1)+1, \ldots, 3n(n-1)/2 \\
  -\sqrt{(n+2)s_i^-} & i = 3n(n-1)/2 + 1, \ldots, 2n(n-1) \\
  \sqrt{(n+2)s_i^+} & i = 2n(n-1)+1, \ldots, n(2n-1) \\
  -\sqrt{(n+2)s_i^-} & i = n(2n-1)+1, \ldots, 2n^2 
  \end{cases} \quad (49)$$

  Where denotes the state dimension of system, $e_i$ denotes $n$ dimensional unit vector whose $i$ th element equals to 1, specific expression of $s_i^+$ and $s_i^-$ are
\[ s_j^+ = \left\{ \frac{1}{\sqrt{2}} (e_j + e_{j'}) : l < k, l, k = 1, 2, \cdots, n \right\} \]
\[ s_j^- = \left\{ \frac{1}{\sqrt{2}} (e_j - e_{j'}) : l < k, l, k = 1, 2, \cdots, n \right\} \]  
(50)

b) The calculation of cubature point transmitted by state equation
\[ X_{i,k|k-1}^* = f(X_{i,k-1|k-1}) \]  
(51)

c) The calculation of one step prediction of state
\[ \hat{x}_{k|k-1} = \sum_{i=0}^{2n^2} \omega_i \left( X_{i,k|k-1}^* \right) \]  
(52)

Where, \( \omega_i \) denotes the weight of cubature points, which is shown as follows
\[
\begin{align*}
\omega_0 &= \frac{2}{n + 2} & i &= 0 \\
\omega_i &= \frac{1}{(n + 2)^2} & i &= 1, 2, \cdots, 2n(n - 1) \\
\omega_{2n+i} &= \frac{4 - n}{2(n + 2)^2} & i &= 2n(n - 1) + 1, \cdots, 2n^2 
\end{align*}
\]  
(53)

d) The calculation of predicted error covariance matrix
\[ P_{k,k-1} = \sum_{i=0}^{2n^2} \omega_i \left( X_{i,k|k-1}^* - \hat{x}_{i,k|k-1} \right) \left( X_{i,k|k-1}^* - \hat{x}_{i,k|k-1} \right)^T + Q_{k-1} \]  
(54)

 Reduced measurement update
e) The calculation of updated state cubature point
\[ X_{i,k|k} = S_{k|k-1} \hat{z}_{i|k-1} + \hat{x}_{i,k|k-1} \]  
(55)

Where, \( P_{k,k-1} = S_{k|k-1} (S_{k|k-1})^T \).
f) The calculation of cubature point transmitted by measurement equation
\[ Z_{i,k|k-1} = H_k X_{i,k|k-1} \]  
(56)
g) The calculation of measurement prediction
\[ \hat{z}_{k|k-1} = \sum_{i=0}^{2n^2} \omega_i Z_{i,k|k-1} = H_k \hat{x}_{k|k-1} \]  
(57)
h) The calculation of measurement error covariance matrix and predicted cross-correlation covariance matrix
\[ P_{zz,k|k-1} = H_k P_{k|k-1} H_k^T + R_k \]  
(58)
\[ P_{xz,k|k-1} = P_{k|k-1} H_k^T \]  
(59)

 Filter update
i) The calculation of filter gain matrix, filter state and covariance matrix
\[ K_k = P_{zz,k|k-1} (P_{zz,k|k-1})^{-1} \]  
(60)
\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (z_k - \hat{z}_{k|k-1}) \]  
(61)
\[ P_{k|k} = P_{k|k-1} - K_k P_{zz,k|k-1} (K_k)^T \]  
(62)

It’s obvious that RHCKF only needs one time cubature transformation during time update but not measurement update. So RHCKF based on SVD decomposition demands less calculation volume and shows stronger numerical stability compared with traditional HCKF.
3.3 Partially Linear Structure of Nonlinear Rapid Transfer Alignment Model

For a nonlinear rapid transfer alignment model, every state variable of the proposed model needs cubature point samples in time update of HCKF, which brings huge calculating amount and decreases numerical stability as well. Through careful analysis of the nonlinear rapid transfer alignment model, it’s found that the nonlinearity of the model is caused by the measurement misalignment $\phi_m$, the velocity error $\delta v$ and the actual misalignment $\phi$. Flexure deformation angle $\theta$ in the attitude and velocity error equation can be written as “plus” expression similar with gyro bias $\varepsilon$ and accelerometer bias $\nabla$ because it’s small. Then partially linear structure of state equation included in proposed model is simplified.

In the discretization on state equation, the discrete interval is $dt$, the nonlinear rapid transfer alignment model of discrete format can be written as:

$$
\begin{align*}
\phi(k+1) &= \phi(k) + dt \cdot C_W^{-1}(k)C_{\phi}^{0}(k)\omega_{mm}^m - dt \cdot C_W^{-1}(k)C_{\phi}'^{0}(k)\omega_{mm}^m \\
\delta v(k+1) &= \delta v(k) + dt \cdot C_W^{-1}(k)\omega_{mm}^m \\
\phi(k+1) &= \phi(k) + dt \cdot (B(k) + R_x)\Theta(k) + dt \cdot C_\omega\nabla_s(k) + dt \cdot M_1\delta r(k) + dt \cdot M_2\delta \phi(k)
\end{align*}
$$

Where, $\phi(k+1) = \phi(k) + dt \cdot (B(k) + R_x)\Theta(k) + dt \cdot C_\omega\nabla_s(k) + dt \cdot M_1\delta r(k) + dt \cdot M_2\delta \phi(k)$

$$
\begin{align*}
\phi(k+1) &= \phi(k) + dt \cdot C_W^{-1}(k)C_{\phi}^{0}(k)\omega_{mm}^m - dt \cdot C_W^{-1}(k)C_{\phi}'^{0}(k)\omega_{mm}^m \\
\delta v(k+1) &= \delta v(k) + dt \cdot C_W^{-1}(k)\omega_{mm}^m \\
\phi(k+1) &= \phi(k) + dt \cdot (B(k) + R_x)\Theta(k) + dt \cdot C_\omega\nabla_s(k) + dt \cdot M_1\delta r(k) + dt \cdot M_2\delta \phi(k)
\end{align*}
$$

Where the coefficient matrix $A$ and $B$ are:

$$
A = \begin{bmatrix}
C_W^{-1}(2.1)C_{\omega}^{0}(3) - C_{\omega}^{0}(3)C_{\omega}(2) & C_W^{-1}(2.2)C_{\omega}^{0}(3) - C_{\omega}^{0}(3)C_{\omega}(2) & C_W^{-1}(3.1)C_{\omega}^{0}(3) - C_{\omega}^{0}(3)C_{\omega}(2) & C_W^{-1}(3.2)C_{\omega}^{0}(3) - C_{\omega}^{0}(3)C_{\omega}(2) & C_W^{-1}(3.3)C_{\omega}^{0}(3) - C_{\omega}^{0}(3)C_{\omega}(2)
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
C_W^{-1}(2.1)f_{\epsilon}^{0}(3) - C_{\omega}^{0}(3)f_{\epsilon}^{0}(2) & C_W^{-1}(2.2)f_{\epsilon}^{0}(3) - C_{\omega}^{0}(3)f_{\epsilon}^{0}(2) & C_W^{-1}(3.1)f_{\epsilon}^{0}(3) - C_{\omega}^{0}(3)f_{\epsilon}^{0}(2) & C_W^{-1}(3.2)f_{\epsilon}^{0}(3) - C_{\omega}^{0}(3)f_{\epsilon}^{0}(2) & C_W^{-1}(3.3)f_{\epsilon}^{0}(3) - C_{\omega}^{0}(3)f_{\epsilon}^{0}(2)
\end{bmatrix}
$$

Where, $C_{\omega} = C_{\omega}^{0}C_{\omega}^{m}$, $C_{\omega} = C_{\omega}^{0}C_{\omega}^{m}$. Mark that $a = [\phi_m \delta v \phi]^T$, $b = [\epsilon \nabla \theta \omega_{\phi} \delta r]^T$, Equation (63) can be written as Equation (64).

$$
x(k+1) = \begin{bmatrix}
[a(k+1) \ b(k+1)]^T
\end{bmatrix}
= \psi(a(k)) + \gamma(a(k)) \cdot b(k) \tag{64}
$$

Where, $\psi(a(k))$ and $\gamma(a(k))$ are respectively expressed as:

$$
\psi(a(k)) = \begin{bmatrix}
\phi(k) + dt \cdot C_W^{-1}(k)C_{\phi}^{0}(k)\omega_{mm}^m - dt \cdot C_W^{-1}(k)C_{\phi}'^{0}(k)\omega_{mm}^m \\
\delta v(k) + dt \cdot C_W^{-1}(k)\omega_{mm}^m \\
\phi(k)
\end{bmatrix}
$$

$$
\gamma(a(k)) = \begin{bmatrix}
0_{15 \times 1}
\end{bmatrix}
$$
\[
\gamma(a(k)) = \begin{bmatrix}
    dt \cdot C^{-1}_w(k) & 0_{3 \times 3} & -dt \cdot A(k) & dt \cdot C^{-1}_w(k) & 0_{3 \times 3} \\
    0_{3 \times 3} & dt \cdot C^\eta_s & -dt \cdot (B(k) + L_2) & dt \cdot M_1 & dt \cdot M_2 \\
    0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
    I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
    0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
    0_{3 \times 3} & 0_{3 \times 3} & dt \cdot I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
    0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & dt \cdot L_0 & I_{3 \times 3} \\
\end{bmatrix}
\]

(66)

### 3.4 M-HCKF

\[
y = f(x) = f\left[\begin{bmatrix} a \\ b \end{bmatrix}\right] = \psi(a) + \gamma(a)b
\]

(67)

Where, \(\psi(\cdot)\) denotes nonlinear function, \(\gamma(\cdot)\) denotes nonlinear or linear function. Assuming that state vector \(x\) obeys Gaussian distribution, its mean and variance can be expressed as

\[
E[x] = E[a b]^T = \begin{bmatrix} E(a) \\ E(b) \end{bmatrix}
\]

(68)

\[
P_x = \begin{bmatrix}
    \text{cov}(a,a) & \text{cov}(a,b) \\
    \text{cov}(b,a) & \text{cov}(b,b) \\
\end{bmatrix} = \begin{bmatrix} P_{aa} & P_{ab} \\
    P_{ba} & P_{bb} \end{bmatrix}
\]

(69)

The mean of Gaussian random variable \(b\) condition on Gaussian random variable \(a\) and the covariance of \(b\) condition on \(a\) are written as

\[
E(b | a) = E(b) + P_{ba} (P_{aa})^{-1} (a - E(a))
\]

(70)

\[
P_{ba} = P_{bb} - P_{ba} (P_{aa})^{-1} P_{ab}
\]

(71)

Then the mean and variance of random variable \(y\) are shown as follows

\[
E(y) = \int \left(\psi(a) + \gamma(a)b\right) p(a,b) da
\]

\[
= \int \phi(a) \cdot p(a) da
\]

(72)

\[
P_y = \int \left[(y - E(y))(y - E(y))^T \cdot p(a,b) da
\]

\[
= \int \left(\psi(a) + \gamma(a)b - E(y)\right) \cdot \left(\psi(a) + \gamma(a)b - E(y)\right)^T \cdot p(a,b) da
\]

(73)

\[
= \int \left(\phi(a) - E(y)\right) \left(\phi(a) - E(y)\right)^T \cdot p(a) da + \gamma(a) P_{ba} \gamma(a)^T \cdot p(a) da
\]

Where, \(\phi(a) = \psi(a) + \gamma(a) \cdot E(b | a)\). To obtain the analytical solution of Equation (72) and Equation (73) is difficult and time consuming, instead so the Gaussian approximation sum filter is
used to get their numerical integration solution. Combining with HCKF proposed in session 2.2, the mean and the variance are calculated by high-degree cubature principle, $E(y)$ and $P_y$ are expressed as

$$E(y) = \sum_{i=0}^{2n^2} \phi_v(\xi_i)$$  \hspace{1cm} (74)

$$P_y = \sum_{i=0}^{2n^2} \phi_v \left( (\phi_v(\xi_i) - E(y))(\phi_v(\xi_i) - E(y))^T \right)$$  \hspace{1cm} (75)

Replacing Equations (52) and (54) with Equations (74) and (75), applying reduced measurement update at the same time, M-RHCKF is derived. Comparing the proposed M-RHCKF and traditional HCKF, from the aspect of time update, it’s found that traditional HCKF needs to sample on all states, but M-RHCKF only needs to sample on the nonlinear part of state equation, so M-RHCKF reduces the amount of sampling cubature points of five dimensions namely $\epsilon$, $\nabla$, $\theta$, $\omega_{\mu}$ and $\delta_r$, thus the calculating amount of M-RHCKF obvious reduces up to 62.5% that of HCKF. Considering other aspects of measurement update, M-RHCKF has following characteristics: the measurement equation of rapid transfer alignment equation being linear, adopting reduced measurement update, and simplifying one time cubature points transmitted process. As results, it has higher numerical stability and less calculating amount.

### 4. Simulation

The flow chart for the simulation of the proposed method for transfer alignment is illustrated in Figure 4. Initial position of carrier is assumed to be 30 degrees north latitude and 127 degrees east longitude, the height 0m, the initial velocity 10m/s, and the ship borne is in uniform linear motion. The time of simulation lasts 120s, with calculating cycle of SINS being 0.01s. Table 2 shows the specifications of gyro and accelerometer of SINS, assuming that MINS of ship borne has no error.
Table 2 Specifications of inertial measurement module

| Specifications                  | Value  |
|--------------------------------|--------|
| Gyro constant drift (°/h)      | 0.1    |
| Gyro random drift (°/h)        | 0.01   |
| Accelerometer constant drift/μg| 100    |
| Accelerometer random drift/μg  | 10     |

Assuming that the ship borne is moving under the effect of sea wave in the following form

\[
\begin{align*}
\theta &= \Theta_m \sin(\omega_m t + \alpha_m) \\
\gamma &= \gamma_m \sin(\omega_i t + \alpha_i) \\
\varphi &= \varphi_m \sin(\omega_i t + \alpha_i)
\end{align*}
\]

Where, \(\Theta_m, \gamma_m\) and \(\varphi_m\) are swaying amplitude of ship borne (\(\Theta_m =12^\circ\), \(\gamma_m =15^\circ\), \(\varphi_m =10^\circ\)), \(\omega_i = 2\pi / T_i (i = \theta, \gamma, \varphi)\) denotes the swaying frequency (\(T_{\theta} = 8\ s, T_{\gamma} = 10\ s, T_{\varphi} = 6\ s\)), \(\alpha_i (i = \theta, \gamma, \varphi)\) denotes initial phase (\(\alpha_{\theta} = 0^\circ, \alpha_{\gamma} = 0^\circ, \alpha_{\varphi} = 30^\circ\)), the length of standard lever-arm is \(r_0 = [10\text{m} \ 10\text{m} \ 20\text{m} ]^T\), actual misalignment is \(\phi_i = [0.5^\circ \ 0.6^\circ \ 10^\circ ]^T\), the correlation time of second-order Markov process is \(\tau_i = 60\ s (i = x, y, z)\).

Figure 5 and Figure 6 show the curves of attitude misalignment \(\phi\) and actual misalignment \(\phi_i\) filtered by M-RHCKF and RHCKF respectively. As can be seen, the attitude misalignment and actual misalignment convergence to steady state after approximately 10s and 20s respectively, showing the effectiveness of the marginal sampling algorithm. For rapid transfer alignment model which adopts “velocity plus attitude”, M-RHCKF and RHCKF can both effectively estimate the attitude misalignment \(\phi\) and \(\phi_i\) even though the ship borne doesn’t make accelerating maneuvering.

In this paper, 50 Monte Carlo runs are carried out for the two algorithms. The root-mean-square error (RMSE) statistics of the attitude misalignment, velocity error and actual misalignment are shown in Table 3. It can be seen from Table 3 and Table 4 that the errors of attitude misalignment, velocity and actual misalignment of proposed M-RHCKF and HCKF are almost the same which is shown in Table 3 but M-RHCKF costs shorter filtering time than HCKF that is shown in Table 4, so the proposed M-RHCKF is more suitable for transfer alignment than HCKF. Figure 7 shows the curve of flexure deformation angle \(\theta\), and Figure 8 shows the curve of dynamic lever-arm \(\delta r\). The filter successfully tracks the change of flexure deformation angle and dynamic lever-arm after 10s approximately. Knowing from the length of standard lever-arm \(r_0\) and combing with (10), it’s learned that error of dynamic lever-arm along x-axis is the biggest because the component of \(r_0\) along z-axis is the biggest. So the error of dynamic lever-arm along y-axis is the smallest, which agrees with the true value curve of \(\delta r\) in Figure 8.
Figure 5 M-RHCKF and RHCKF filter estimation of the misalignment angle

Figure 6 M-RHCKF and RHCKF filter estimation of the misalignment angle

Table 3. The RMSE of attitude misalignment, velocity error and actual misalignment of M-RHCKF and HCKF

| RMSE(°) | Segment(80s-100s) | Segment(100s-120s) |
|---------|------------------|------------------|
| $\phi_x$ (°) | M-RHCKF | HCKF | M-RHCKF | HCKF |
| 1.3972 | 1.4017 | 1.5135 | 1.5051 |
| $\phi_y$ (°) | HCKF | M-RHCKF | HCKF | M-RHCKF |
| 0.9949 | 0.9950 | 1.0709 | 1.0739 |
\[
\begin{array}{cccc}
\phi_z (') & 3.3656 & 3.3543 & 3.7350 & 3.7826 \\
\delta v_x (m/s) & 0.0260 & 0.0238 & 0.0356 & 0.0373 \\
\delta v_y (m/s) & 0.0275 & 0.0234 & 0.0355 & 0.0382 \\
\phi_{ax} (') & 3.6633 & 3.1703 & 3.1635 & 3.3411 \\
\phi_{ay} (') & 4.3537 & 4.6430 & 4.4387 & 4.1305 \\
\phi_{az} (') & 3.3102 & 3.5955 & 3.8336 & 3.7913 \\
\end{array}
\]

Table 4. The filtering time of two algorithms

| Algorithm   | Total running time | Single running time |
|-------------|-------------------|--------------------|
| M-RHCKF    | 2044.6554s        | 40.8931s           |
| HCKF        | 4798.7623s        | 95.9752ss          |

Figure 7  M-RHCKF filter estimation of the flexural deformation angle

Figure 8  M-RHCKF filter estimation of the deformation dynamic lever arm
Figure.9 The RMSE of attitude misalignment of M-RHCKF under 3 different conditions

Figure.10 The RMSE of velocity error of M-RHCKF under 3 different conditions

Figure.11 The RMSE of actual misalignment of M-RHCKF under 3 different conditions
Figure 9–Figure 11 respectively illustrate the RMSE statistics of attitude misalignment, velocity error and actual misalignment of proposed M-RHCKF under 3 different conditions, namely considering dynamic lever-arm (Condition A), without considering dynamic lever-arm (Condition B) and without considering dynamic lever-arm but with 4 times the length of initial lever-arm (Condition C). As illustrated in Figure 9–Figure 11, conclusions can be obtained as follows:

a. The proposed incorporate transfer alignment model of Condition A corresponds to actual situation and has better filter performance compared to Condition B because dynamic lever-arm is established and the acceleration error of dynamic lever-arm are included.

b. The filtering performance under Condition C is the worst among above 3 conditions because the velocity error model mismatches with actual situation caused by that the dynamic lever-arm is not included in velocity error model. Furthermore, the effect dynamic lever-arm has on filtering performance is worse due to the reason that the length of initial lever-arm increases to 3 times longer than that under Condition B.

5. Conclusion

1) The nonlinear incorporated rapid transfer alignment model is proposed and validated by simulation results. It predicts the flexure deformation angle and dynamic lever-arm accurately. The model doesn’t require the small value assumption of either the actual misalignment or the actual attitude, thus have wider range of application.

2) The M-RHCKF is proposed based on the characteristic of transfer alignment model. The model is partially linear for large azimuth misalignment. M-RHCKF has smaller calculation amount compared with HCKF and satisfies the accuracy and time requirement of the transfer alignment process.

3) The research on restraining the effect of flexure deformation and dynamic lever-arm through improving nonlinear filter algorithm needs to be further studied, because accurate parameters of flexure deformation and dynamic lever-arm can’t be obtained in the real world environment.

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