Gravitational and electromagnetic fields near a de Sitter-like infinity

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We present a characterization of general gravitational and electromagnetic fields near de Sitter-like conformal infinity which supplements the standard peeling behavior. This is based on an explicit evaluation of the dependence of the radiative component of the fields on the null direction from which infinity is approached. It is shown that the directional pattern of radiation has a universal character that is determined by the algebraic (Petrov) type of the spacetime. Specifically, the radiation field vanishes along directions opposite to principal null directions.

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A direct observation of gravitational waves will be properly understood only when it can be compared with reliable predictions supplied by numerical relativity. To make such predictions is difficult since, among other things, no rigorous statements are available which relate the properties of sufficiently general strong sources to the radiation fields produced. Only a few explicit radiative solutions of Einstein’s equations are known which can be used as test beds for numerical codes (see e.g. [1, 2]), in particular, spacetimes representing “uniformly simple vacuum solutions which differ on an arbitrary given Cauchy surface by a finite but sufficiently small amount from de Sitter data [8], while an analogous result for data close to Minkowski ($\Lambda = 0$) is still under investigation [4]. Thus, many vacuum asymptotically simple spacetimes with de Sitter-like $I^+$ do exist. Assuming their existence, Penrose proved already in 1965 [3, 4] that both the gravitational and the electromagnetic fields satisfy the peeling-off property with respect to null geodesics reaching any point of $I^+$. This means that along a null geodesic parametrized by an affine parameter $\eta$, the part of any spin-$s$ zero rest-mass field proportional to $\eta^{-(k+1)}$, $k = 0, 1, \ldots, 2s$, has, in general, $2s - k$ coincident principal null directions. In particular, the part of the field that falls off as $\eta^{-1}$ is a radiation (“null”) field. The peeling-off property is easier to prove with $\Lambda > 0$ than in asymptotically Minkowskian spacetimes when $I^+$ is a null hypersurface [5]. With a spacelike $I^+$, however, one can approach any point on $I^+$ from infinitely many different null directions and, consequently, the radiation field becomes mixed up with other components of the field when the null geodesic is changed. This fact of the “origin dependence” of the radiation field in case of a spacelike $I^+$ has been repeatedly emphasized by Penrose [3, 5]. Exactly this directional radiation pattern, i.e., the dependence of fields (with respect to appropriate tetrad) on the direction along which the null geodesic reaches a point on a spacelike $I^+$, is analyzed in the present work.

SPACETIME INFINITY, FIELDS AND TETRADS

Following general formalism [3, 8], a spacetime $\mathcal{M}$ with physical metric $g$ can be embedded into a larger conformal manifold $\tilde{\mathcal{M}}$ with conformal metric $\tilde{g}$ related to $g$ by $\tilde{g} = \omega^2 g$. Here, the conformal factor $\omega$, negative in $\mathcal{M}$, vanishes on the boundary of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ called conformal infinity $\mathcal{I}$. It is spacelike if the gradient $d\omega$ on $\mathcal{I}$ has timelike character. This may be either future infinity $I^+$ or past infinity $\mathcal{I}^-$. Near $\mathcal{I}^+$ we decompose $\tilde{g}$ into a spatial 3-metric $\tilde{\mathcal{I}}^+ \tilde{g}$ tangent to $\mathcal{I}^+$ and a part orthogonal to $\mathcal{I}^+$:

$$g = \omega^{-2}(-\tilde{\mathcal{N}}^2 d\omega^2 + \mathcal{I}^+ \tilde{g}).$$

(1)

The conformal lapse function $\tilde{\mathcal{N}}$ can be chosen to be con-
stant on \( \mathcal{I}^\pm \), e.g., equal to \( \ell = \sqrt{3/\Lambda} \). The form (1) allows us to define a timelike unit vector \( \mathbf{n} \) normal to \( \mathcal{I}^\pm \),
\[
\mathbf{n}^\mu = \omega^{-1} \tilde{N} \, g^{\mu\nu} \, \mathbf{d}_\nu \omega.
\] (2)

Next, we denote the vectors of an orthonormal tetrad as \( \mathbf{t}, \mathbf{q}, \mathbf{r}, \mathbf{s} \), where \( \mathbf{t} \) is a unit timelike vector and the remaining three are unit spacelike vectors. With this tetrad we associate a null tetrad \( \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{m} \), such that \( \mathbf{k} \cdot \mathbf{l} = -1, \mathbf{m} \cdot \mathbf{m} = 1 \),
\[
\mathbf{k} = \frac{1}{\sqrt{2}} (\mathbf{t} + \mathbf{q}), \quad \mathbf{l} = \frac{1}{\sqrt{2}} (\mathbf{t} - \mathbf{q}), \\
\mathbf{m} = \frac{1}{\sqrt{2}} (\mathbf{r} - i \mathbf{s}), \quad \mathbf{m} = \frac{1}{\sqrt{2}} (\mathbf{r} + i \mathbf{s}).
\] (3)

Various specific tetrads introduced below will be distinguished by an additional label in subscript.

As usual, we parametrize the Weyl tensor \( C_{\alpha\beta\gamma\delta} \) (representing the gravitational field) by 5 complex coefficients
\[
\Psi_0 = C_{\alpha\beta\gamma\delta} k^\alpha m^\beta k^\gamma m^\delta, \quad \Psi_1 = C_{\alpha\beta\gamma\delta} k^\alpha m^\beta l^\gamma m^\delta, \\
\Psi_2 = -C_{\alpha\beta\gamma\delta} k^\alpha m^\beta l^\gamma l^\delta, \quad \Psi_3 = C_{\alpha\beta\gamma\delta} k^\alpha m^\beta \bar{l}^\gamma m^\delta, \\
\Psi_4 = C_{\alpha\beta\gamma\delta} k^\alpha m^\beta \bar{l}^\gamma l^\delta, \quad \Psi_5 = C_{\alpha\beta\gamma\delta} \bar{k}^\alpha \bar{m}^\beta \bar{l}^\gamma m^\delta.
\] (4)

The null vector \( \Psi_0 \) is “comparable”, independent of their parametrization, \( \tilde{N} = \tilde{N} \big|_{\mathcal{I}^\pm} \) is the same for all geodesics ending at point \( \mathcal{I}^\pm \). We require that the approach of geodesics to \( \mathcal{I}^\pm \) is “comparable”, independent of their direction, we assume also \( \omega = \text{constant} \).

The conformal factor \( \omega \) and lapse \( \tilde{N} \) can be expanded along the geodesic in powers of \( 1/\eta \):
\[
\omega \approx \omega_0 \eta^{-1} + \ldots, \quad \tilde{N} \approx \tilde{N}_0 + \ldots.
\] (5)

The value \( \tilde{N}_0 = \tilde{N} \big|_{\mathcal{I}^\pm} \) is the same for all geodesics ending at point \( \mathcal{I}^\pm \). We require that the approach of geodesics to \( \mathcal{I}^\pm \) is “comparable”, independent of their direction, we assume also \( \omega \) to be constant. This is equivalent to fixing the energy \( E_o = -\mathbf{p} \cdot \mathbf{n} \) (\( \mathbf{p} = \frac{D\mathbf{z}}{d\eta} \) being 4-momentum) at a given small value of \( \omega \), i.e., at given proximity from the reference tetrad adapted to the conformal infinity \( \mathcal{I}^\pm \).

To define an interpretation null tetrad \( \mathbf{k}_i, \mathbf{l}_i, \mathbf{m}_i, \mathbf{m}_i \), we have to specify it in a comparable way for all geodesics along different directions. The geodesics reach the same point \( P_+ \) and we prescribe its form there. We require the null vector \( \mathbf{k}_i \) to be proportional to the tangent vector of the geodesic,
\[
\mathbf{k}_i = \frac{1}{\sqrt{2N_+}} \frac{D\mathbf{z}}{d\eta};
\] (6)

so that, as \( \eta \to \infty \), the interpretation tetrad is “infinitely boosted” with respect to an observer with 4-velocity \( \mathbf{n} \). To see this explicitly, we introduce an auxiliary tetrad \( \mathbf{t}, \mathbf{q}, \mathbf{r}, \mathbf{s} \), adapted to the conformal infinity, \( \mathbf{t}_i = \mathbf{n} \), with \( \mathbf{q}_i \) oriented along the spatial direction of the geodesic,
\[
\mathbf{q}_i \propto \mathbf{k}_i = (\mathbf{k}_i \cdot \mathbf{n}) \mathbf{n} + \mathbf{k}_i,
\] (7)

and we choose the remaining spatial vectors \( \mathbf{r}_i, \mathbf{s}_i \) to coincide with those of the interpretation tetrad. Using Eqs. (9), (10) and the definition of \( \mathbf{l}_i \), we get
\[
\mathbf{k}_i = B_i \mathbf{k}_i = \eta^{-1} \frac{1}{\sqrt{2}} (\mathbf{n} + \mathbf{q}_i), \quad \mathbf{m}_i = \mathbf{m}_i, \\
\mathbf{l}_i = B_i^{-1} \mathbf{l}_i = \eta \frac{1}{\sqrt{2}} (\mathbf{n} - \mathbf{q}_i), \quad \mathbf{m}_i = \mathbf{m}_i, \\
B_i = 1/\eta \text{ being the boost parameter. Under such a boost the field components \( \Psi_0, \Psi_2 \) transform as \( \frac{1}{\eta} \),}
\]

Together with behavior (22) of the field components in a tetrad adapted to \( \mathcal{I}^\pm \) we obtain the peeling-off property.

**DIRECTIONAL PATTERN OF RADIATION**

We shall now derive the directional dependence of radiation near a point \( P_+ \) at \( \mathcal{I}^\pm \). It is necessary to parametrize the direction of the null geodesic reaching \( P_+ \). This can be done with respect to a suitable reference tetrad \( \mathbf{t}_0, \mathbf{q}_0, \mathbf{r}_0, \mathbf{s}_0 \), with the time vector \( \mathbf{t}_0 \) adapted to the conformal infinity, \( \mathbf{t}_0 = \mathbf{n} \), and spatial directions chosen arbitrarily. It is convenient to choose them in accordance with the spacetime geometry. Privileged choices will be discussed later (cf. Fig. 1).

The unit spatial direction \( \mathbf{q}_0 \) of a general null geodesic near \( \mathcal{I}^\pm \) can be expressed in terms of spherical angles \( \theta, \phi \), with respect to the reference tetrad,
\[
\mathbf{q}_0 = \cos \theta \, \mathbf{q}_0 + \sin \theta \cos \phi \, \mathbf{r}_0 + \sin \theta \sin \phi \, \mathbf{s}_0.
\] (8)
It is useful to introduce the **stereographic representation** of the angles $\theta, \phi$,

$$R = \tan(\theta/2) \exp(-i\phi). \quad (15)$$

Then the null rotation (16) with $K = R$ transforms $k_0$ into $k$ with its spatial direction $k^\perp \propto q$, specified by $\theta, \phi$.

Now, the interpretation tetrad is related by boost (12) to the tetrad $t_r, q_r, r_t, s_r$, which is a spatial rotation of the reference tetrad. If we choose

$$r_t = -\sin \theta \cos \phi \, (q_o + \tan \frac{\theta}{2} (\cos \phi \, r_o + \sin \phi \, s_o)) + r_o, \quad s_r = -\sin \theta \sin \phi \, (q_o + \tan \frac{\theta}{2} (\cos \phi \, r_o + \sin \phi \, s_o)) + s_o, \quad (16)$$

the spatial rotation is a composition of the null rotations with $l$ fixed, $k$ fixed and the boost, given by parameters $K = R$, $L = -R/(1 + |R|^2)$, $B = (1 + |R|^2)^{-1}$. This decomposition into elementary Lorentz transformations enables us to calculate the field components in the interpretation tetrad. We start with $\Psi^o_j$, or $\Psi^o_j$ in the reference tetrad and we characterize these in terms of algebraically privileged **principal null directions** (PNDs).

Principal null directions of the gravitational (or electromagnetic) field are null directions $k$ such that $\Psi_0 = 0$ (or $\Phi_0 = 0$) in a null tetrad $k, l, m, n$ (a choice of $l, m, n$ is irrelevant). In the tetrad related to the reference tetrad by null rotation (16), such a condition for $\Psi_0$ takes the form of a quartic (or quadratic for $\Phi_0$) equation for $K$, cf. Eqs. (7). The roots $K = R_n, n = 1, 2, 3, 4$ (or $K = R^o_n, n = 1, 2$) of this equation thus parametrize PNDs $k_n$ (or $k^o_n$). As follows from the note after Eq. (15), the angles $\theta_n$, $\phi_n$ of these PNDs are related to $R_n$ exactly by Eq. (16).

In a generic situation we have $\Psi^o_i \neq 0$ (or $\Phi^o_i \neq 0$), and we can express the remaining components of the Weyl (or electromagnetic) tensor in terms of roots $R_n$ (or $R^o_n$)

$$\Psi^o_3 = -\frac{1}{4} \Psi^o_4 (R_1 + R_2 + R_3 + R_4), \quad \Psi^o_2 = \frac{1}{6} \Psi^o_4 (R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_3 + R_2 R_4 + R_3 R_4), \quad \Psi^o_1 = -\frac{1}{4} \Psi^o_4 (R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4), \quad \Psi^o_0 = \Psi^o_2 R_1 R_2 R_3 R_4, \quad \Phi^o_1 = -\frac{1}{2} \Phi^o_2 (R^n_1 + R^n_2), \quad \Phi^o_0 = \Phi^o_2 R^n_1 R^n_2. \quad (17)$$

Transforming these to tetrad $k_r, l_r, m_r, n_r$ we obtain

$$\Psi^i_4 = \Psi^o_4 (1 + |R|^2)^{-2} \times (1 - \frac{R^i_1}{R^o_1})(1 - \frac{R^i_2}{R^o_2})(1 - \frac{R^i_3}{R^o_3})(1 - \frac{R^i_4}{R^o_4}), \quad \Phi^i_4 = \Phi^o_4 (1 + |R|^2)^{-1} \times (1 - \frac{R^i_1}{R^o_1})(1 - \frac{R^i_2}{R^o_2})(1 - \frac{R^i_3}{R^o_3}). \quad (19)$$

Here, the complex number $R_a$,

$$R_a = -\bar{R}^{-1} = -\cot(\theta/2) \exp(-i\phi), \quad (20)$$

characterizes a spatial direction opposite to the direction given by $R$, i.e., the *antipodal* direction with $\theta_0 = \pi - \theta$ and $\phi_0 = \phi + \pi$. Finally, we express the leading term of the field components in the interpretation tetrad. The freedom in the choice of the vectors $m_l, \bar{m}_n$ changes just a phase of the field components, so only their modulus has a physical meaning. It is also known that as a consequence of field components, field components in a reference tetrad near $\mathcal{I}^+$ behave as

$$\Psi^o_j \approx \Psi^o_{j+} \eta^{-3}, \quad \Phi^o_j \approx \Phi^o_{j+} \eta^{-2}. \quad (21)$$

Combining Eqs. (13) and (19), (20), we thus obtain

$$|\Psi^i_4| \approx |\Psi^o_{i+} \eta^{-1} \cos^4(\theta/2)| 	imes |1 - \frac{R^i_1}{R^o_1}|\! |1 - \frac{R^i_2}{R^o_2}|\! |1 - \frac{R^i_3}{R^o_3}|\! |1 - \frac{R^i_4}{R^o_4}|, \quad (22)$$

$$|\Phi^i_4| \approx |\Phi^o_{i+} \eta^{-1} \cos^2(\theta/2)| \left|1 - \frac{R^i_1}{R^o_1}\right|\! \left|1 - \frac{R^i_2}{R^o_2}\right|. \quad (23)$$

**DISCUSSION**

These expressions characterize the asymptotic behavior of fields near de Sitter-like infinity. In a general spacetime there are four spatial directions along which the radiative component of the gravitational field $\Psi^o_4$ vanishes, namely directions $R_a = R_n$, $n = 1, 2, 3, 4$ (or two such directions for electromagnetic field $\Phi^o_4$). In fact, their spatial parts $-k^a_n$ are exactly opposite to the projections of the principal null directions $k^\perp_n$ onto $\mathcal{I}^+$.

In algebraically special spacetimes, some PNDs coincide and Eq. (23) simplifies. Moreover, it is always possible to choose the “canonical” reference tetrad: (i) the vector $q_o$ oriented along the spatial projection of the degenerate (multiply) PND, say $k^\perp_1$ (i.e. $k_o \propto k_1$); (ii) the $q_o-r_o$ plane oriented so that it contains the spatial projection of one of the remaining PNDs (for type N spacetimes this choice is arbitrary). Using such a reference tetrad, the degenerate PND $k^\perp_4$ is given by $\theta_4 = 0$, i.e., $R_4 = 0$ (cf. Eq. (17)), whereas one of the remaining PNDs, say $k^\perp_3$, has $\phi_1 = 0$, i.e., $R_3 = \tan^{\frac{\pi}{2}}$ is real.

Thus, for Petrov type N spacetimes (with quadruple PND) $R_1 = R_2 = R_3 = R_4 = 0$, so the asymptotic behavior of gravitational field (23) becomes

$$|\Psi^i_4| = |\Psi^o_{i+} \eta^{-1} \cos^4(\theta/2)|. \quad (24)$$

The corresponding directional pattern of radiation is illustrated in Fig. 11(N). It is axisymmetric, with maximum value at $\theta = 0$ along the spatial projection of the quadruple PND onto $\mathcal{I}^+$. Along the opposite direction, $\theta = \pi$, the field vanishes.

In the Petrov type III spacetimes, $R_1 = \tan^{\frac{\theta}{2}}$, $R_2 = R_3 = R_4 = 0$, so (23) implies

$$|\Psi^i_4| = |\Psi^o_{i+} |\eta^{-1} \cos^4(\theta) | \left|1 + \frac{\theta}{\theta_0} \tan^{\frac{\theta}{2}} e^{i\phi}\right|. \quad (25)$$

The directional pattern of radiation is shown in Fig. 11(III). The field vanishes along $\theta = \pi$ and along $\theta = \pi - \theta_1$, $\phi = \pi$ which are spatial directions opposite to the PNDs.
To summarize: it is well-known that for null $I^+$ the null direction $l$, which is complementary to the tangent vector $k_i$ of the null geodesic $z(\eta)$ reaching $I^+$, is tangent to $I^+$ and does not depend on the choice of $z(\eta)$. This is not the case when $I^+$ is spacelike. The radiation field ($\eta^{-1}$ term) is thus “less invariant” [6]. We have shown that the dependence of this field on the choice of $z(\eta)$ has a universal character that is determined by the algebraic (Petrov) type of the fields. In particular, we have proved that the radiation vanishes along directions opposite to PNDs. In a generic direction the radiative component of the fields generated by any source is nonvanishing. Thus, unlike in asymptotically flat spacetimes, the absence of $\eta^{-1}$ component cannot be used to distinguish nonradiative sources: for a de Sitter-like infinity the radiative component reflects not only properties of the sources but also their “kinematic” relation to an observer at infinity. Intuitively, near spacelike $I^+$ our observer is, in general, moving “nonradially” from sources and thus measures “infinitely boosted” fields. Some important questions, such as how energy is radiated away in asymptotically de-Sitter spacetimes, still remain open. Since vacuum energy seems to be dominant in our universe, these appear to be of considerable interest.

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