Model Selection with Gini Indices under Auto-Calibration

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Abstract

The Gini index does not give a strictly consistent scoring rule in general. Therefore, maximizing the Gini index may lead to wrong decisions. The main issue is that the Gini index is a rank-based score that is not calibration-sensitive. We show that the Gini index allows for strictly consistent scoring if we restrict to the class of auto-calibrated regression models.

Keywords. Regression model, binary classification, Gini index, Gini score, consistency, consistent scoring, auto-calibration, Lorenz curve, concentration curve, cumulative accuracy profile, CAP, receiver operating characteristics, ROC, area under the curve, AUC, accuracy ratio, Somers’ D, forecast-dominance.

1 Introduction

The Gini index (Gini score, accuracy ratio) is a popular tool for model selection in machine learning, and there are versions of the Gini index that are used to evaluate actuarial pricing models and financial credit risk models; see Frees et al. [7, 8], Denuit et al. [3], Engelmann et al. [6] and Tasche [17]. However, in general, the Gini index does not give a (strictly) consistent scoring rule; Example 3 of Byrne [1] gives a counterexample. (Strict) consistency is an important property in model selection because it ensures that maximizing the Gini index does not lead to a wrong model choice; see Gneiting [10] and Gneiting–Raftery [11]. The Gini index can be obtained from Somers’ D [16], which essentially considers Kendall’s τ; see Newson [14]. Intuitively, this tells us that the Gini index is a rank-based score that is not calibration-sensitive. The missing piece to make the Gini index a strictly consistent scoring rule is to restrict it to the class of auto-calibrated regression models, this is proved in Theorem 4.5 below; for auto-calibration we refer to Krüger–Ziegel [12], Denuit et al. [2] and Section 7.4.2 of Wüthrich–Merz [18].

Organization. In the next section, we introduce the notion of strictly consistent scoring rules. In Section 3, we discuss the Gini index as it is usually used in the machine learning community. In Section 4, we introduce and discuss the property of having an auto-calibrated regression model (forecasts), and we prove that the Gini index gives a strictly consistent scoring rule if we restrict to the class of auto-calibrated regression models. This makes the maximization of the Gini index a sensible model selection tool on the class of auto-calibrated regression models. Finally, in Section 5 we conclude.

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2 Consistent scoring rules

Let \((Y, X)\) be a random tuple on a sufficiently rich probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with real-valued non-negative response \(Y\) having finite mean and with covariates \(X\). Denote by \(\mathcal{F}\) the family of potential distributions of \((Y, X)\) being supported on \(\mathcal{Y} \times \mathcal{X}\). Let \(F_{Y|X}\) be the conditional distribution of \(Y\), given \(X\). For any model \((Y, X) \sim F \in \mathcal{F}\), we consider the conditional mean functional \(T\)

\[
F_{Y|X} \rightarrow T(F_{Y|X}) = \mu^\dagger(X) = \mathbb{E}[Y|X],
\]

where \(X \mapsto \mu^\dagger(X) = \mathbb{E}[Y|X]\) denotes the true regression function of the chosen model. The main task in regression modeling is to find this unknown true regression function \(\mu^\dagger(\cdot)\) from i.i.d. data \((Y_i, X_i), 1 \leq i \leq n\), having the same distribution as \((Y, X)\).

Choose a scoring function \(S : \mathcal{Y} \times \mathbb{R} \rightarrow \mathbb{R}\) giving us the score \(\mathbb{E}[S(Y, \hat{\mu}(X))]\) for regression function \(X \mapsto \hat{\mu}(X)\) and \((Y, X) \sim F \in \mathcal{F}\). A scoring rule is obtained by selecting the argument(s) \(\hat{\mu}^*(\cdot)\) that maximize the score over the regression functions \(\hat{\mu}(\cdot)\), subject to existence,

\[
\hat{\mu}^*(\cdot) \in \arg\max_{\hat{\mu}(\cdot)} \mathbb{E}[S(Y, \hat{\mu}(X))],
\]

under the given model choice \((Y, X) \sim F \in \mathcal{F}\).

A scoring rule is called consistent on \(\mathcal{F}\) for the conditional mean functional \(T\), if for any model \((Y, X) \sim F \in \mathcal{F}\) with conditional distributions \(F_{Y|X}\) of \(Y\), given \(X\), we have \(S(Y, T(F_{Y|X})) \in L^1(\mathbb{P})\), and for any regression function \(X \mapsto \hat{\mu}(X)\) with \(S(Y, \hat{\mu}(X)) \in L^1(\mathbb{P})\) we have

\[
\mathbb{E}[S(Y, T(F_{Y|X}))] \geq \mathbb{E}[S(Y, \hat{\mu}(X))].
\]

A scoring rule is called strictly consistent on \(\mathcal{F}\) for the conditional mean functional \(T\), if it is consistent on \(\mathcal{F}\), and if an identity in (2.2) holds if and only if \(\hat{\mu}(X) = T(F_{Y|X}) = \mu^\dagger(X)\), a.s.

Remarks 2.1

• Strict consistency implies that the true regression function \(\mu^\dagger(\cdot)\) is the unique maximizer in (2.1), and it can be estimated by score maximization (assuming it is contained in the set over which we optimize, which we generally do). Empirically, we then consider for i.i.d. data \((Y_i, X_i), 1 \leq i \leq n\),

\[
\arg\max_{\hat{\mu}(\cdot)} \frac{1}{n} \sum_{i=1}^n S(Y_i, \hat{\mu}(X_i)),
\]

where we still need to ensure that we can exchange the limit \(n \rightarrow \infty\) and the arg-max operator to asymptotically select the true regression function \(\mu^\dagger(\cdot)\) under strict consistency.

• Formula (2.2) states unconditional consistency as we average over the distribution of \(X\), For conditional consistency in \(X\) and its relation to the unconditional version we refer to Section 2.2 in Dimitriadis et al. [5]. A point prediction version of consistency is given in Definition 1 in Gneiting [10].

• For scoring rule (2.1) we consider a maximization. By a sign switch we can turn this into a minimization problem, and in that case we rather speak about expected loss minimization.

• Typically, we restrict (2.1) to smaller classes of regression functions \(X \mapsto \hat{\mu}(X)\).

In the sequel, we will require continuity for these smaller classes, and, further below, we require the auto-calibration property. This requires that the true regression function \(\mu^\dagger(\cdot)\) has this continuity, auto-calibration it will satisfy automatically, see Lemma 4.1 below.
3 The Gini index in machine learning

In the sequel we assume \( \hat{\mu}(X) \) to have a continuous distribution \( F_{\hat{\mu}(X)} \) for all \( (Y, X) \sim F \) and for any considered regression function \( X \mapsto \hat{\mu}(X) \). This implies \( F_{\hat{\mu}(X)}(F_{\hat{\mu}(X)}^{-1}(\alpha)) = \alpha \) for all \( \alpha \in (0, 1) \), and with \( F_{\hat{\mu}(X)}^{-1} \) denoting the left-continuous generalized inverse of \( F_{\hat{\mu}(X)} \).

In machine learning (ML) one considers the cumulative accuracy profile (CAP) defined by

\[
\alpha \in (0, 1) \mapsto \text{CAP}_{Y, \hat{\mu}(X)}(\alpha) = \frac{1}{E[Y]} \mathbb{E}\left[ Y \mathbb{1}_{\{ \hat{\mu}(X) > F_{\hat{\mu}(X)}^{-1}(1-\alpha) \}} \right] \in [0, 1].
\]

In actuarial science, the CAP is also called concentration curve (up to sign switches), see Denuit–Trufin [4]. The CAP measures a rank-based correlation between the prediction \( \hat{\mu}(X) \) and the response \( Y \).

The Gini index (Gini score, Gini ratio, Gini coefficient, accuracy ratio) in ML is defined by

\[
G_{Y, \hat{\mu}(X)}^\text{ML} = \int_0^1 \text{CAP}_{Y, \hat{\mu}(X)}(\alpha) \, d\alpha - 1/2
\]

where we additionally assume that \( Y \) has an (unconditional) continuous distribution \( F_Y \). For a geometric interpretation see Figure [1] (lhs) and formula (3.1), below.

Remarks 3.1

- The denominator in (3.1) does not use the regression function \( \hat{\mu}(\cdot) \), i.e., it has no impact on model selection by maximizing the Gini index \( G_{Y, \hat{\mu}(X)}^\text{ML} \) over \( \hat{\mu}(\cdot) \). Hence, for scoring we can focus on the term in the enumerator

\[
\int_0^1 \text{CAP}_{Y, \hat{\mu}(X)}(\alpha) \, d\alpha = \frac{1}{E[Y]} \mathbb{E}\left[ Y \int_0^1 \mathbb{1}_{\{ \hat{\mu}(X) > F_{\hat{\mu}(X)}^{-1}(1-\alpha) \}} \, d\alpha \right]
= \frac{1}{E[Y]} \mathbb{E}\left[ Y \mathbb{P}\left( F_{\hat{\mu}(X)}^{-1}(U) < \hat{\mu}(X) \bigg| \hat{\mu}(X) \right) \right]
= \frac{1}{E[Y]} \mathbb{E}\left[ Y F_{\hat{\mu}(X)}(\hat{\mu}(X)) \right],
\]

for an independent \((0, 1)\)-uniform random variable \( U \) and where we use continuity of \( F_{\hat{\mu}(X)} \).

This shows that the Gini index in ML is not calibration-sensitive because \( F_{\hat{\mu}(X)}(\hat{\mu}(X)) \) has a \((0, 1)\)-uniform distribution, i.e., the specific distribution of \( \hat{\mu}(X) \) does not matter, but only its correlation with \( Y \) matters.

- Since typically the true data model \((Y, X) \sim F\) is not known, the Gini index in ML (3.1) is replaced by an empirical version

\[
\hat{G}_{Y, \hat{\mu}(X)}^\text{ML} = \int_0^1 \hat{\text{CAP}}_{Y, \hat{\mu}(X)}(\alpha) \, d\alpha - 1/2 \leq 1,
\]

where we set

\[
\hat{\text{CAP}}_{Y, \hat{\mu}(X)}(\alpha) = \frac{1}{n} \sum_{i=1}^n \frac{1}{\sum_{i=1}^n Y_i} \sum_{i=1}^n Y_i \mathbb{1}_{\{ \hat{\mu}(X_i) > \hat{\mu}(X_{(i(1-\alpha)n)}) \}}
\]

for i.i.d. data \((Y_i, X_i)\), \(1 \leq i \leq n\), having the same distribution as \((Y, X)\), and for order statistics \( \hat{\mu}(X_{(1)}) < \hat{\mu}(X_{(2)}) < \ldots < \hat{\mu}(X_{(n)}) \): note that by assumption the distribution of \( \hat{\mu}(X) \) is continuous which implies that all observations \( \hat{\mu}(X_i) \) are mutually different for \( 1 \leq i \leq n \), and we have a strict ordering in the order statistics.
Let us further comment on (3.4). First, if we mirror the CAP at the diagonal we have
\[
\text{CAP}^-_{Y,\hat{\mu}(\mathbf{x})}(\alpha) = \frac{1}{\mathbb{E}[Y]} \mathbb{E} \left[ Y I\{\hat{\mu}(\mathbf{x}) \leq \hat{F}^{-1}_{\hat{\mu}(\mathbf{x})}(\alpha)\} \right] \tag{3.5}
\]
\[
= 1 - \frac{1}{\mathbb{E}[Y]} \mathbb{E} \left[ Y I\{\hat{\mu}(\mathbf{x}) > \hat{F}^{-1}_{\hat{\mu}(\mathbf{x})}(\alpha)\} \right] = 1 - \text{CAP}_{Y,\hat{\mu}(\mathbf{x})}(1-\alpha).
\]
For an empirical version of the mirrored CAP we replace the above expression by
\[
\hat{\text{CAP}}^-_{Y,\hat{\mu}(\mathbf{x})}(\alpha) = \frac{1}{\hat{F}_{\hat{\mu}(\mathbf{x})}(m)} \sum_{i=1}^{n} Y_i I\{\hat{\mu}(\mathbf{x}_i) \leq m\} \quad \text{and} \quad \hat{F}^{-1}_{\hat{\mu}(\mathbf{x})}(\alpha) = \hat{\mu}(\mathbf{X}_{(\lceil \alpha n \rceil)}).
\]
This justifies the choice in (3.4). Similarly, we have for the denominator in (3.3)
\[
\hat{\text{CAP}}^-_{Y,Y}(\alpha) = \frac{1}{\hat{F}_{Y}(m)} \sum_{i=1}^{n} Y_i I\{Y_i \leq \hat{F}^{-1}_{Y}(\alpha)\} \quad \text{(s)} \quad \frac{1}{n} \sum_{i=1}^{\lceil \alpha n \rceil} Y_i \frac{1}{n} \sum_{i=1}^{n} Y_i, \tag{3.6}
\]
for the identity \((s)\) to hold for any \(\alpha \in (0,1)\), we need to assume that we have a strict ordering \(Y_{(1)} < Y_{(2)} < \ldots < Y_{(n)}\), i.e., that there are no ties in the observations \((Y_i)_{1 \leq i \leq n}\), which is the case because \(Y\) was assumed to have a continuous distribution \(F_Y\). This then motivates to set
\[
\hat{\text{CAP}}_{Y,Y}(\alpha) = 1 - \hat{\text{CAP}}^-_{Y,Y}(1-\alpha) = \frac{1}{\hat{F}_{Y}(m)} \sum_{i=1}^{n} Y_i I\{Y_i > \hat{F}^{-1}_{Y}(1-\alpha)\} \quad \text{(s)} \quad \frac{1}{n} \sum_{i=1}^{\lceil \alpha n \rceil} Y_i \frac{1}{n} \sum_{i=1}^{n} Y_i^{(i)} \tag{3.7}
\]
If we have a perfect joint ordering between \((Y_i)_{1 \leq i \leq n}\) and \((\hat{\mu}(\mathbf{X}_i))_{1 \leq i \leq n}\), the upper bound in (3.3) is attained, see (3.4) and (3.7). This is the motivation for the scaling in (3.1).

In the definition of the Gini index in ML (3.1) we have assumed that \(Y\) has a continuous distribution \(F_Y\). This is not the case for discrete responses \(Y\). Therefore, in the discrete case we need to replace the denominator in (3.1) by a different object. For illustrative purposes we show the binary classification case in the next example.

**Example 3.2 (binary classification)** We consider a binary classification example with true regression function
\[
\mathbf{X} \mapsto p^*(\mathbf{X}) = \mathbb{E}[Y|\mathbf{X}] = \mathbb{P}[Y=1|\mathbf{X}] \in (0,1).
\]
Gini index in ML

function in the discrete case by we need to modify (3.1). Starting from the right-hand side of (3.6), we define the empirical $F$ only applies for a continuous distribution $F_Y$ of $Y$. For the Gini index in ML we need to calculate the denominator of (3.1). However, this formula up to $\alpha$ for i.i.d. data ($X, Y_i$), $1 \leq i \leq n$. In the Bernoulli case, this function is identically equal to zero up to $\alpha \leq 1 - \sum_{i=1}^n Y_i/n$, these describes the number of zeros among the observations $(Y_i)_{1 \leq i \leq n}$, and afterwards it increases to 1. Since this increase is only described on the discrete grid with span $1/n$, we linearly interpolate between these points. This provides a straight line between $1 - \sum_{i=1}^n Y_i/n$ and 1 with slope $n/\sum_{i=1}^n Y_i$. Under this linear interpolation, we get the area (integral)

$$\int_{0}^{1} \text{CAP}_{Y,Y}^-(\alpha) \, d\alpha = \frac{1}{2n} \sum_{i=1}^{n} Y_i.$$  

By the law of large numbers, the latter converges to $p^\dagger/2 = \mathbb{E}[p^\dagger(X)]/2 = \mathbb{E}[Y]/2$, a.s., as $n \to \infty$. This motivates in the (discrete) binary classification case the following definition of the Gini index in ML

$$G_{Y, \hat{p}(X)}^{\text{ML}} = \frac{1/2 - \int_{0}^{1} F_{\hat{p}(X)|Y=1}(\hat{p}(X)(1 - \alpha)) \, d\alpha}{(1 - p^\dagger)/2}. \quad (3.8)$$

In the binary classification case, the CAP can be related to the receiver operating characteristics (ROC) curve. The area under the curve (AUC) of the ROC curve has a one-to-one relationship to the Gini index in ML (3.8) in the Bernoulli case, we refer to Section 5 in Tasche [17]. We mention this because the ML community more frequently uses the AUC than the Gini index for model selection.

In general, in the discrete case we replace the integral in the denominator in (3.1) by the term

$$\frac{1}{4\mathbb{E}[Y]} \mathbb{E} \left[ \left( Y - \check{Y} \right) \right], \quad (3.9)$$

where $\check{Y}$ is an independent copy of $Y$. This latter quantity (3.9) can be calculated for any distribution $F_Y$ of $Y$, and in the continuous case we precisely receive the denominator in (3.1). The binary classification case (3.9) provides us with $(1 - p^\dagger)/2$ which gives (3.8).
4 Auto-calibration and consistency of the Gini index

Let \((Y, X) \sim F\). A regression function \(X \mapsto \hat{\mu}(X)\) is auto-calibrated for \(Y\) if, a.s.,

\[
\hat{\mu}(X) = \mathbb{E}[Y \mid \hat{\mu}(X)].
\]

Auto-calibration is an important property in insurance pricing, as it implies that every cohort of insurance policies paying the same price \(\hat{\mu}(X)\) is in average self-financing, because the price \(\hat{\mu}(X)\) exactly covers the expected claim \(Y\) of that cohort. I.e., we do not have any systematic cross-financing between the price cohorts. This is the core of risk classification in insurance. It also implies unbiasedness on the portfolio level

\[
\mathbb{E} [\hat{\mu}(X)] = \mathbb{E}[Y], \tag{4.1}
\]

which is a minimal requirement in insurance pricing. Typically, there are many auto-calibrated regression functions \(\hat{\mu}(X)\) for \(Y\), i.e., there are many systems of self-financing pricing cohorts.

**Lemma 4.1** The true regression function \(X \mapsto \mu^\dagger(X) = \mathbb{E}[Y \mid X]\) is auto-calibrated for \(Y\), and it strictly dominates in convex order any other auto-calibrated regression function \(X \mapsto \hat{\mu}(X)\) for \(Y\).

**Proof.** To prove auto-calibration of \(\mu^\dagger\) we apply the tower property to the \(\sigma\)-algebras \(\sigma(\mu^\dagger(X)) \subset \sigma(X)\) which gives, a.s.,

\[
\mathbb{E}[Y \mu^\dagger(X)] = \mathbb{E} [\mathbb{E}[Y \mid X] \mu^\dagger(X)] = \mathbb{E} [\mu^\dagger(X) \mu^\dagger(X)] = \mu^\dagger(X).
\]

For any convex function \(\psi\), auto-calibration, the tower property for \(\sigma(\hat{\mu}(X)) \subset \sigma(X)\) and Jensen’s inequality give

\[
\mathbb{E}\left[\psi(\hat{\mu}(X))\right] = \mathbb{E}\left[\psi(\mathbb{E}[Y \mid \hat{\mu}(X)])\right] = \mathbb{E}[\psi(\mathbb{E}[Y \mid X] \hat{\mu}(X))] = \mathbb{E}\left[\psi \left( \mathbb{E} \left( \mu^\dagger(X) \hat{\mu}(X) \right) \right) \right] \\
\leq \mathbb{E}\left[\psi \left( \mu^\dagger(X) \hat{\mu}(X) \right) \right] = \mathbb{E}\left[\psi \left( \mu^\dagger(X) \right) \right],
\]

whenever these exist. This proves that \(\mu^\dagger\) dominates in convex order any other auto-calibrated regression function \(\hat{\mu}\) for \(Y\). Assume that there exists an auto-calibrated regression function \(X \mapsto \hat{\mu}(X)\) for \(Y\) such that for any convex function \(\psi\) we have an equality in the previous calculation, whenever these exist. This implies that \(\mu^\dagger(X)\) is \(\sigma(\hat{\mu}(X))\)-measurable. Auto-calibration and the tower property for \(\sigma(\hat{\mu}(X)) \subset \sigma(X)\) then provide, a.s.,

\[
\hat{\mu}(X) = \mathbb{E}[Y \mid \hat{\mu}(X)] = \mathbb{E}[Y \mid X] \hat{\mu}(X)] = \mathbb{E}\left[\mu^\dagger(X) \mid \hat{\mu}(X)\right] = \mu^\dagger(X).
\]

This proves the statement of strict convex order. \(\square\)

The next proposition is a consequence of Lemma 4.1 and of Theorem 3.1 in Krüger–Ziegel [12].

**Proposition 4.2** The true regression function \(X \mapsto \mu^\dagger(X)\) forecast-dominates any auto-calibrated regression function \(X \mapsto \hat{\mu}(X)\) for \(Y\) meaning that

\[
\mathbb{E}\left[-D_\psi(Y, \mu^\dagger(X))\right] \geq \mathbb{E}\left[-D_\psi(Y, \hat{\mu}(X))\right],
\]

for any convex function \(\psi\) where the above exists, and with Bregman divergence given by

\[
D_\psi(y, m) = \psi(y) - \psi(m) - \psi'(m)(y - m) \geq 0,
\]

for \(y, m \in \mathbb{R}\) and \(\psi'\) is a (sub-)gradient of the convex function \(\psi\).
Proposition 4.2 says that every negative Bregman divergence provides a consistent scoring rule for the conditional mean regression functional $T$ under auto-calibration for $Y$. This statement motivates the common practice in model selection of minimizing (out-of-sample) deviance losses, as deviance losses are special cases of Bregman divergences; see Chapters 2 and 4 in Wüthrich–Merz [18]. For more information on this topic we refer to Krüger–Ziegel [12], Theorem 7 in Gneiting [10] and Savage [15], the latter two references state that Bregman divergences provide the only strictly consistent scoring functions for mean estimation.

The definition of the Gini index [9] in economics slightly differs from the ML version (3.1). Assume $\hat{F}_{\hat{\mu}}(X)$ is a continuous distribution. It is then based on the Lorenz curve [13] given by

$$\alpha \in (0, 1) \mapsto L_{\hat{\mu}}(X)\left(F_{\hat{\mu}}^{-1}(\alpha)\right) = \frac{1}{E[\hat{\mu}(X)]} \mathbb{E}\left[\hat{\mu}(X) \mathbb{I}\{\hat{\mu}(X) \leq F_{\hat{\mu}}^{-1}(\alpha)\}\right] \in [0, 1].$$

Note that we have the property $\text{CAP}_{\hat{\mu}(X), \hat{\mu}(X)} = L_{\hat{\mu}}(X)(F_{\hat{\mu}}^{-1}(\alpha))$, see (3.5).

The Gini index in economics has many (equivalent) definitions, we use the following two

$$G_{\hat{\mu}}(X) = 1 - 2 \int_0^1 L_{\hat{\mu}}(X)\left(F_{\hat{\mu}}^{-1}(\alpha)\right) d\alpha = \frac{1}{2E[\hat{\mu}(X)]} \mathbb{E}\left[\left|\hat{\mu}(X) - \hat{\mu}(Z)\right|\right], \quad (4.2)$$

where $\hat{\mu}(Z)$ is an independent copy of $\hat{\mu}(X)$. The first definition in (4.2) is based on a continuous distribution $F_{\hat{\mu}}(X)$, whereas the second one can be used for any distribution $F_{\hat{\mu}}(X)$, we also refer to (3.9).

![Gini index in ML (CAP)](image)

![Gini index in economics (Lorenz curve)](image)

Figure 1: (lhs) cumulative accuracy profile (CAP) and (rhs) Lorenz curve.

There are three differences between the Gini index in ML and the one in economics, see Figure 1: (i) $G_{\hat{\mu}}^{\text{ec}}(X)$ considers a mirrored version of the curves compared to $G_{Y,\hat{\mu}}^{\text{ML}}(X)$; (ii) $G_{Y,\hat{\mu}}^{\text{ML}}(X)$ depends on $Y$ and $\hat{\mu}(X)$, $G_{\hat{\mu}}^{\text{ec}}(X)$ only depends on $\hat{\mu}(X)$; (iii) scalings are different leading to areas B and C, respectively, in Figure 1. The two Gini indices are geometrically obtained by, see Figure 1.

$$G_{Y,\hat{\mu}}^{\text{ML}}(X) = \frac{\text{area}(A)}{\text{area}(A + B)} \quad \text{and} \quad G_{\hat{\mu}}^{\text{ec}}(X) = \frac{\text{area}(A)}{\text{area}(A + C)} = 2\text{area}(A) = 1 - 2\text{area}(C). \quad (4.3)$$

For an equivalence in (4.2) we need that $F_{\hat{\mu}}(X)$ is continuous, otherwise one should choose the term on the right-hand side as the definition of the Gini index in economics.
Property 3.1 of Denuit–Trufin [4] gives the following nice result.

**Proposition 4.3** Under auto-calibration of the regression function \(X \mapsto \hat{\mu}(X)\) for \(Y\) we have the identity \(\text{CAP}_{Y,\hat{\mu}(X)}(\alpha) = 1 - L_{\hat{\mu}(X)}(F^{-1}_{\hat{\mu}(X)}(1 - \alpha))\) for all \(\alpha \in (0, 1)\).

**Proof.** Using the tower property, auto-calibration of \(\hat{\mu}\) for \(Y\) and unbiasedness [4] give us

\[
\text{CAP}_{Y,\hat{\mu}(X)}(\alpha) = \frac{1}{E[Y]} E\left[Y \mathbb{1}_{\{\hat{\mu}(X) > F^{-1}_{\hat{\mu}(X)}(1 - \alpha)\}}\right] = \frac{1}{E[\hat{\mu}(X)]} E\left[E[Y | \hat{\mu}(X)] \mathbb{1}_{\{\hat{\mu}(X) > F^{-1}_{\hat{\mu}(X)}(1 - \alpha)\}}\right] = 1 - \frac{1}{E[\hat{\mu}(X)]} E\left[\mathbb{1}_{\{\hat{\mu}(X) \leq F^{-1}_{\hat{\mu}(X)}(1 - \alpha)\}}\right].
\]

This proves the claim. \(\square\)

Thus, under auto-calibration for \(Y\), the CAP and the Lorenz curve coincide (up to mirroring/sign switching). This gives us the following corollary.

**Corollary 4.4** Under auto-calibration of the regression function \(X \mapsto \hat{\mu}(X)\) for \(Y\) we have for the Gini indices

\[
G^\text{ML}_{Y,\hat{\mu}(X)} = \frac{G^\text{eco}_\hat{\mu}(X)}{2 \int_0^1 \text{CAP}_{Y,Y}(\alpha) \, d\alpha - 1}.
\]

**Proof.** Proposition [4] gives us for the Gini index in ML

\[
G^\text{ML}_{Y,\hat{\mu}(X)} = \frac{\int_0^1 \text{CAP}_{Y,\hat{\mu}(X)}(\alpha) \, d\alpha - 1/2}{\int_0^1 \text{CAP}_{Y,Y}(\alpha) \, d\alpha - 1/2} = \frac{1/2 - \int_0^1 L_{\hat{\mu}(X)}(F^{-1}_{\hat{\mu}(X)}(1 - \alpha)) \, d\alpha}{\int_0^1 \text{CAP}_{Y,Y}(\alpha) \, d\alpha - 1/2} = \frac{1 - 2 \int_0^1 L_{\hat{\mu}(X)}(F^{-1}_{\hat{\mu}(X)}(\beta)) \, d\beta}{\int_0^1 \text{CAP}_{Y,Y}(\alpha) \, d\alpha - 1},
\]

where the last step uses the change of variable \(\alpha \mapsto \beta = 1 - \alpha\). This proves the claim. \(\square\)

This says that under auto-calibration for the response both Gini indices (the ML score and the version in economics) provide the same scoring rule because the (positive) denominator in (4.4) does not depend on the specific choice of the regression function \(\hat{\mu}(\cdot)\). Moreover, the same arguments apply to the Gini indices in non-continuous cases, e.g., in the binary classification (Bernoulli) case (3.8).

**Theorem 4.5** The true regression function \(X \mapsto \mu^\dagger(X)\) maximizes the Gini index (in ML) among all auto-calibrated regression functions \(X \mapsto \hat{\mu}(X)\) for \(Y\), i.e., \(G^\text{ML}_{Y,\mu^\dagger(X)} > G^\text{ML}_{Y,\hat{\mu}(X)}\) unless \(\hat{\mu}(X) = \mu^\dagger(X)\), a.s.

**Proof.** Conditionally, given \(\hat{\mu}(Z)\), \(m \mapsto |m - \hat{\mu}(Z)|\) is a convex function in \(m \in \mathbb{R}\). Using formula (4.2), independence between \(\hat{\mu}(X)\) and \(\hat{\mu}(Z)\) in (4.1) and Lemma [4] we obtain inequality, a.s.,

\[
E \left[|\hat{\mu}(X) - \hat{\mu}(Z)| \right] \leq E \left[|\mu^\dagger(X) - \hat{\mu}(Z)| \right],
\]

where \(\mu^\dagger(X)\) is independent of \(\hat{\mu}(Z)\). Using the tower property, applying the same argument to the exchanged role of \(\mu^\dagger(X)\) and \(\hat{\mu}(Z)\), using unbiasedness [4] and using Corollary [4] provides \(G^\text{ML}_{Y,\mu^\dagger(X)} \geq G^\text{ML}_{Y,\hat{\mu}(X)}\).

Assume there exists an auto-calibrated regression function \(\hat{\mu}\) for \(Y\) such that \(G^\text{ML}_{Y,\mu^\dagger(X)} = G^\text{ML}_{Y,\hat{\mu}(X)}\). Using auto-calibration of \(\hat{\mu}\) for \(Y\) and the tower property, we receive for P-a.e. \(\omega \in \Omega\)

\[
\hat{\mu}(X)(\omega) = E[Y | \hat{\mu}(X)](\omega) = E[E[Y | \hat{\mu}(X)](\omega) = E[\mu^\dagger(X)](\hat{\mu}(X))(\omega).
\]

\footnote{Note that the denominator in (4.4) is positive for every non-deterministic \(Y\). This follows from the fact that the denominator is equal to twice (3.9) which is positive unless \(Y\) is deterministic.}
Denote by \( \Omega_1 \subset \Omega \) a set of full measure 1 on which (4.5) holds. On \( \Omega_1 \), the predictor \( \hat{\mu}(X) \) is between the conditional essential infimum and supremum of \( \mu^!(X) \), given \( \hat{\mu}(X) \), because it corresponds to the conditional expectation of \( \mu^!(X) \), given \( \hat{\mu}(X) \). Consider the case of sample points \( \omega \in \Omega_1 \) where the conditional essential infimum and supremum of \( \mu^!(X) \), given \( \hat{\mu}(X) \), do not coincide, and denote the corresponding set of sample points by \( \Omega_2 \subset \Omega_1 \). On \( \Omega_2 \), the predictor \( \hat{\mu}(X) \) is strictly between the conditional essential infimum and supremum of \( \mu^!(X) \), given \( \hat{\mu}(X) \), due to the conditional expectation property (4.5). We have using (4.5) and independence between \( \hat{\mu}(X) \) and \( \hat{\mu}(Z) \)

\[
\begin{align*}
E[||\hat{\mu}(X) - \hat{\mu}(Z)||] &= E\left[ E \left[ \mu^!(X) | \hat{\mu}(X) \right] - \hat{\mu}(Z) \right] = E \left[ E \left[ \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \right] \\
&= E \left[ \left( \mathbb{1}_{\Omega_1} + \mathbb{1}_{\Omega_2} \right) \left| \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \right].
\end{align*}
\]

We calculate the first term on the right-hand side of (4.6)

\[
E \left[ \mathbb{1}_{\Omega_2} \left| \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \right] = \int_{\Omega_2} \left( \int_{\Omega} E \left[ \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \left| \omega, \tilde{\omega} \right. \right. dP(\tilde{\omega}) \right. dP(\omega).
\]

We study the inner integral for fixed sample point \( \omega \in \Omega_2 \). Jensen’s inequality gives us

\[
\int_{\Omega} \left| E \left[ \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \left| \omega, \tilde{\omega} \right. \right. dP(\tilde{\omega}) < \int_{\Omega} \left| E \left[ \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \left| \omega, \tilde{\omega} \right. \right. dP(\tilde{\omega}).
\]

where we receive a strict inequality for \( \omega \in \Omega_2 \) because of the following items: (1) on \( \Omega_2 \), \( \mu^!(X) \) is non-deterministic, conditionally given \( \hat{\mu}(X) \), (2) \( m \mapsto [m - \hat{\mu}(Z)] \) is a convex function, (3) \( \mu^!(X) \) has the same distribution (and support) as \( \hat{\mu}(X) \), and (4) \( \mu^!(X) \) and \( \hat{\mu}(Z) \) are independent. Items (1)-(4) imply that on a set of positive \( P(\tilde{\omega}) \)-measure we receive a strict Jensen’s inequality, because on this set, \( \hat{\mu}(Z) \) is strictly within the conditional essential infimum and supremum of (the non-deterministic) \( \mu^!(X) \), given \( \hat{\mu}(X) \).

Assume \( P[\Omega_2] > 0 \), i.e., strict inequality (4.7) occurs on a set of positive measure. Applying Jensen’s inequality also to the other term in (4.6) we receive strict inequality

\[
E[||\hat{\mu}(X) - \hat{\mu}(Z)||] < E \left[ E \left[ \mu^!(X) - \hat{\mu}(Z) \left| \hat{\mu}(X), \hat{\mu}(Z) \right. \right] \right].
\]

This strict inequality contradicts our assumption \( G^\text{ML}_{Y,\mu^!(X)} = G^\text{ML}_{Y,\hat{\mu}(X)} \). Therefore, \( P[\Omega_2] = 0 \), which implies

\[
P[\Omega_2 \cap \Omega_1] = P[\Omega_2] = 1.
\]

On the set \( \Omega_2 \cap \Omega_1 \), we have \( \mu^!(X) = \hat{\mu}(X) \), which proves the claim.

Theorem (4.5) proves that the Gini index gives a strictly consistent scoring rule on the class of auto-calibrated regression functions that are \( X \)-measurable, because the true regression function \( X \mapsto \mu^!(X) \) maximizes this Gini index. A bigger Gini index can only be achieved by a larger information set than the \( \sigma \)-algebra generated by \( X \).

The following proposition generalizes Property 5.1 of Denuit et al. [2], which gives a method of restoring auto-calibration for a general regression function \( X \mapsto \hat{\mu}(X) \).

**Proposition 4.6** Consider a regression function \( X \mapsto \hat{\mu}(X) \). The following regression function is auto-calibrated for \( Y \)

\[
X \mapsto \hat{\mu}^{\text{(auto)}}(X) = E \left[ Y \left| \hat{\mu}(X) \right. \right] .
\]

**Proof.** Note that \( \hat{\mu}^{\text{(auto)}}(X) \) is \( \sigma(\hat{\mu}(X)) \)-measurable. This implies \( \sigma(\hat{\mu}^{\text{(auto)}}(X)) \subset \sigma(\hat{\mu}(X)) \). Henceforth, using the tower property, a.s.,

\[
E \left[ Y \left| \hat{\mu}^{\text{(auto)}}(X) \right. \right] = E \left[ E [Y \left| \hat{\mu}(X) \right. \right] \hat{\mu}^{\text{(auto)}}(X) \right] = E \left[ \hat{\mu}^{\text{(auto)}}(X) \right] \hat{\mu}^{\text{(auto)}}(X) = \hat{\mu}^{\text{(auto)}}(X).
\]

This completes the proof.
5 Conclusions

In general, one should not use the Gini index for model selection because it does not give a strictly consistent scoring rule and, thus, may lead to wrong decisions. We have shown in Theorem 4.5 that if we restrict Gini index scoring to the class of auto-calibrated regression functions for the given response, the Gini index allows for strictly consistent scoring. This also translates to the binary classification case where the (machine learning version of the) Gini index has an equivalent formulation in terms of the area under the curve (AUC) of the receiver operating characteristics (ROC) curve, we refer to Tasche [17]. We only need to ensure that the binary classification model is auto-calibrated for the Bernoulli response to receive a strictly consistent scoring rule from the AUC.

References

[1] Byrne, S. (2016). A note on the use of empirical AUC for evaluating probabilistic forecasts. Electronic Journal of Statistics 10, 380-393.
[2] Denuit, M., Charpentier, A., Trufin, J. (2021). Autocalibration and Tweedie-dominance for insurance pricing in machine learning. Insurance: Mathematics & Economics 101/B, 485-497.
[3] Denuit, M., Sznajder, D., Trufin, J. (2019). Model selection based on Lorenz and concentration curves, Gini indices and convex order. Insurance: Mathematics & Economics 89, 128-139.
[4] Denuit, M., Trufin, J. (2021). Lorenz curve, Gini coefficient, and Tweedie dominance for autocalibrated predictors. LIDAM Discussion Paper ISBA 2021/36.
[5] Dimitriadis, T., Fissler, T., Ziegel, J.F. (2020). The efficiency gap. arXiv, 2010.14146.
[6] Engelmann, B., Hayden, E., Tasche, D. (2003). Testing rating accuracy. Risk 16/1, 82-86.
[7] Frees, E.W., Meyers, G., Cummings, A.D. (2011). Summarizing insurance scores using a Gini index. Journal of the American Statistical Association 106, 1085-1098.
[8] Frees, E.W., Meyers, G., Cummings, A.D. (2013). Insurance ratemaking and a Gini index. Journal of Risk and Insurance 81, 335-366.
[9] Gini, C. (1912). Variabilità e Mutuabilità. Contributo allo Studio delle Distribuzioni e delle Relazioni Statistiche. C. Cuppini, Bologna.
[10] Gneiting, T. (2011). Making and evaluating point forecasts. Journal of the American Statistical Association 106/494, 746-762.
[11] Gneiting, T., Raftery, A.E. (2007). Strictly proper scoring rules, prediction, and estimation. Journal of the American Statistical Association 102/477, 359-378.
[12] Krüger, F., Ziegel, J.F. (2021). Generic conditions for forecast dominance. Journal of Business & Economics Statistics 39/4, 972-983.
[13] Lorenz, M.O. (1905). Methods of measuring the concentration of wealth. Publications of the American Statistical Association 9/70, 209-219.
[14] Newson, R. (2002). Parameters behind “nonparametric” statistics: Kendall’s tau, Somers’ D and median differences. Stata Journal 2/1, 45-64.
[15] Savage, L.J. (1971). Elicitable of personal probabilities and expectations. Journal of the American Statistical Association 66/336, 783-810.
[16] Somers, R.H. (1962). A new asymmetric measure of association for ordinal variables. *American Sociological Review* **27/6**, 799-811.

[17] Tasche, D. (2006). Validation of internal rating systems and PD estimates. *arXiv:0606071*.

[18] Wüthrich, M.V., Merz, M. (2022). *Statistical Foundations of Actuarial Learning and its Applications*. Springer Actuarial, in press.