Noether Symmetry in $f(T)$ Theory

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ABSTRACT

As is well known, symmetry plays an important role in the theoretical physics. In particular, the well-known Noether symmetry is a useful tool to select models motivated at a fundamental level, and find the exact solution to the given Lagrangian. In the present work, we try to consider Noether symmetry in $f(T)$ theory. At first, we briefly discuss the Lagrangian formalism of $f(T)$ theory. In particular, the point-like Lagrangian is explicitly constructed. Based on this Lagrangian, the explicit form of $f(T)$ theory and the corresponding exact solution are found by requiring Noether symmetry. In the resulting $f(T) = \mu T^n$ theory, the universe experiences a power-law expansion $a(t) \sim t^{2n/3}$. Furthermore, we consider the physical quantities corresponding to the exact solution, and find that if $n > 3/2$ the expansion of our universe can be accelerated without invoking dark energy. Also, we test the exact solution of this $f(T)$ theory with the latest Union2 Type Ia Supernovae (SNIa) dataset which consists of 557 SNIa, and find that it can be well consistent with the observational data in fact.

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I. INTRODUCTION

The current accelerated expansion of our universe has been one of the most active fields in modern cosmology. As is well known, it could be due to an unknown energy component (dark energy) or a modification to general relativity (modified gravity). Today, modified gravity theory has been a competitive alternative to the familiar dark energy scenario.

In analogy to the well-known \( f(R) \) theory, recently a new modified gravity theory, namely the so-called \( f(T) \) theory, has been proposed to drive the current accelerated expansion without invoking dark energy. It is a generalized version of the so-called teleparallel gravity originally proposed by Einstein. In teleparallel gravity, the Weitzenböck connection is used, rather than the Levi-Civita connection which is used in general relativity. Following, here we briefly review the key points of teleparallel gravity and \( f(T) \) theory. In this work, we consider a spatially flat Friedmann-Robertson-Walker (FRW) universe whose spacetime is described by

\[
ds^2 = -dt^2 + a^2(t)dx^2,
\]

where \( a \) is the scale factor. The orthonormal tetrad components \( e_i(x^\mu) \) relate to the metric through

\[
g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu,
\]

where Latin \( i, j \) are indices running over 0, 1, 2, 3 for the tangent space of the manifold, and Greek \( \mu, \nu \) are the coordinate indices on the manifold, also running over 0, 1, 2, 3. In teleparallel gravity, the gravitational action is

\[
S_T = \int d^4x|e|T,
\]

where \( |e| = \det (e^i_\mu) = \sqrt{-g} \), and for convenience we use the units \( 16\pi G = h = c = 1 \) throughout. The torsion scalar \( T \) is defined by

\[
T \equiv S^\rho_{\mu\nu}T^\rho_{\mu\nu},
\]

where

\[
T^\rho_{\mu\nu} \equiv -e_i^\rho \left( \partial_\mu e^i_\nu - \partial_\nu e^i_\mu \right),
\]

\[
K^\mu\nu_{\rho} \equiv -\frac{1}{2} \left( T^\mu\nu_{\rho} - T^\nu\mu_{\rho} - T^\rho\mu_{\nu} \right),
\]

\[
S^\rho_{\mu\nu} \equiv \frac{1}{2} \left( K^\mu\nu_{\rho} + \delta^\rho_\mu T^\theta_\nu - \delta^\rho_\nu T^\theta_\mu \right).
\]

For a spatially flat FRW universe, from Eqs. (4) and (11), one has

\[
T = -6H^2,
\]

where \( H \equiv \dot{a}/a \) is the Hubble parameter, and a dot denotes a derivative with respect to cosmic time \( t \). In analogy to the well-known \( f(R) \) theory, one can replace \( T \) in the gravitational action by any function \( f(T) \), and then obtain the so-called \( f(T) \) theory. In \( f(T) \) theory, the modified Friedmann equation and Raychaudhuri equation are given by

\[
12H^2f_T + f = \rho,
\]

\[
48H^2f_{TT}\dot{H} - f_T \left( 12H^2 + 4\dot{H} \right) = f = p,
\]

where a subscript \( T \) denotes a derivative with respect to \( T \), and \( \rho, p \) are the total energy density and pressure, respectively. In an universe with only pressureless matter, obviously we have \( p = \rho_m = 0 \) and \( \rho = \rho_m = \rho_{m0}a^{-3} \), where the subscript “0” indicates the present value of the corresponding quantity, and we have set \( a_0 = 1 \). It is well known that when \( f(T) = T \) the familiar general relativity can be completely recovered.
In fact, $f(T)$ theory was firstly used to drive inflation by Ferraro and Fiorini [9]. Later, Bengochea and Ferraro [3], as well as Linder [4], proposed to use $f(T)$ theory to drive the current accelerated expansion of our universe without invoking dark energy. Very soon, $f(T)$ theory attracted much attention in the community. We refer to e.g. [7, 8, 10–15, 41] for relevant works.

So far, the specified forms of function $f(T)$ in the literature are written by hand. There is no natural guidance from fundamental physics on the form of $f(T)$ in fact. In the present work, we try to address this issue. As is well known, symmetry plays an important role in the theoretical physics. In particular, the well-known Noether symmetry is an useful tool to select models motivated at a fundamental level. In the literature, Noether symmetry has been extensively used in scalar field cosmology [16, 17], non-minimally coupled cosmology [18], $f(R)$ theory [19–21], scalar-tensor theory [22], higher order gravity theory [22], multiple scalar fields [24], vector field [25], fermion field [26], tachyon field [27], non-flat cosmology [28], quantum cosmology [29], Bianchi universe [30], Brans-Dicke theory [31], dilaton [32], induced gravity theory [33], gravity with variable $G$ and $\Lambda$ [34], Gauss-Bonnet gravity [35], and so on. Now, we try to consider Noether symmetry in $f(T)$ theory in the present work.

This paper is organized as follows. In Sec. II, we briefly discuss the Lagrangian formalism of $f(T)$ theory. In particular, the point-like Lagrangian is explicitly constructed. In Sec. III, we consider Noether symmetry in $f(T)$ theory. The explicit form of $f(T)$ theory and the corresponding exact solution are found by requiring Noether symmetry. In Sec. IV, we further discuss the physical quantities corresponding to the exact solution. In Sec. V, we test the exact solution of $f(T)$ theory found in Sec. III with the latest Union2 Type Ia Supernovae (SNIa) dataset which consists of 557 SNIa. Finally, some brief concluding remarks are given in Sec. VI.

II. LAGRANGIAN FORMALISM OF F(T) THEORY

In the study of Noether symmetry, the point-like Lagrangian plays an important role. In this section, we discuss the Lagrangian formalism of $f(T)$ theory. As mentioned above, the relevant action of $f(T)$ theory is given by

$$S = \int d^4x |\epsilon| f(T) + S_m,$$

(11)

where $S_m$ is the action of pressureless matter minimally coupled with gravity, and we assume that the radiation can be ignored. Following e.g. [20, 21], to derive the cosmological equations in the FRW metric, one can define a canonical Lagrangian $L = L(a, \dot{a}, T, \dot{T})$, whereas $Q = \{a, T\}$ is the configuration space, and $TQ = \{a, \dot{a}, T, \dot{T}\}$ is the related tangent bundle on which $L$ is defined. The scale factor $a(t)$ and the torsion scalar $T(t)$ are taken as independent dynamical variables. One can use the method of Lagrange multipliers to set $T$ as a constraint of the dynamics (nb. Eq. (8)). Selecting the suitable Lagrange multiplier and integrating by parts, the Lagrangian $L$ becomes canonical [20, 21]. In our case, we have

$$S = 2\pi^2 \int dt a^3 \left[ f(T) - \lambda \left( T + 6 \frac{\dot{a}^2}{a^2} \right) - \frac{\rho_{m0}}{a^3} \right],$$

(12)

where $\lambda$ is a Lagrange multiplier. The variation with respect to $T$ of this action gives

$$\lambda = f_T.$$

(13)

Therefore, the action (12) can be rewritten as

$$S = 2\pi^2 \int dt a^3 \left[ f(T) - f_T \left( T + 6 \frac{\dot{a}^2}{a^2} \right) - \frac{\rho_{m0}}{a^3} \right],$$

(14)

and then the point-like Lagrangian reads (up to a constant factor $2\pi^2$)

$$L(a, \dot{a}, T, \dot{T}) = a^3 (f - f_T T - 6f_T a\dot{a}^2 - \rho_{m0}).$$

(15)
As is well known, for a dynamical system, the Euler-Lagrange equation is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \tag{16} \]

where \( q_i \) are the generalized coordinates of the configuration space \( Q \), and in our case \( q_i = a \) and \( T \).

Substituting Eq. (15) into the Euler-Lagrange equation (16), we obtain

\[ a^3 f_{TT} \left( T + 6 \frac{\ddot{a}^2}{a^2} \right) = 0, \tag{17} \]

\[ f - f_T T + 2 f_T H^2 + 4 \left( f_T \frac{\ddot{a}}{a} + H f_{TT} T \right) = 0. \tag{18} \]

If \( f_{TT} \neq 0 \), from Eq. (17), it is easy to find that

\[ T = -6 \frac{\ddot{a}^2}{a^2} = -6H^2, \tag{19} \]

i.e., the relation (8) is recovered. Generally, this is the Euler constraint of the dynamics. Substituting Eq. (19) into Eq. (18) and using \( \ddot{a}/a = H^2 + \dot{H} \), we get

\[ 48H^2 f_{TT} \dot{H} - 4 f_T \left( 3H^2 + \dot{H} \right) - f = 0, \tag{20} \]

i.e., the modified Raychaudhuri equation (10) is recovered (note that \( p = p_m = 0 \)). On the other hand, it is also well known that the total energy (Hamiltonian) corresponding to Lagrangian \( L \) is given by

\[ H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L. \tag{21} \]

Substituting Eq. (15) into Eq. (21), we have

\[ H(a, \dot{a}, T, \dot{T}) = a^3 \left( -6 f_T \frac{\ddot{a}^2}{a^2} - f + f_T T + \frac{p_m a}{a^3} \right). \tag{22} \]

Considering the total energy \( H = 0 \) (Hamiltonian constraint) \[16, 20, 21\] and using Eq. (19), we get

\[ 12H^2 f_T + f = \frac{p_m a}{a^3}, \tag{23} \]

i.e., the modified Friedmann equation (9) is also recovered (note that \( \rho = \rho_m = \rho_m a^3/a^3 \)). So far, we have shown that the point-like Lagrangian given in Eq. (15) can yield all the correct equations of motion, and hence it is the desired one.

III. NOETHER SYMMETRY IN \( f(T) \) THEORY

As is well known (see e.g. \[16, 32\]), Noether symmetry is an useful tool to select models motivated at a fundamental level, and find the exact solution to the given Lagrangian. In this section, we try to consider Noether symmetry in \( f(T) \) theory.

Following e.g. \[20, 21\], the generator of Noether symmetry is a vector

\[ X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial T} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{T}}, \tag{24} \]

where \( \alpha = \alpha(a, T) \) and \( \beta = \beta(a, T) \) are both functions of the generalized coordinates \( a \) and \( T \). Noether symmetry exists if the equation

\[ L X \mathcal{L} = X \mathcal{L} = \alpha \frac{\partial \mathcal{L}}{\partial a} + \beta \frac{\partial \mathcal{L}}{\partial T} + \dot{\alpha} \frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\beta} \frac{\partial \mathcal{L}}{\partial \dot{T}} = 0 \tag{25} \]
has solution, where \( L_X L \) is the Lie derivative of the Lagrangian \( L \) with respect to the vector \( X \). Of course, according to the well-known Noether theorem, there will be a constant of motion (Noether charge) \([20, 21]\), namely
\[
Q_0 = \sum \alpha_i \frac{\partial L}{\partial \dot{q}_i} = \alpha \frac{\partial L}{\partial \dot{a}} + \beta \frac{\partial L}{\partial \dot{T}} = \text{const.} \quad (26)
\]

Note that the meaning of \( L_X L = 0 \) is that \( L \) is constant along the flow (possibly a local flow) generated by \( X \), namely Eq. (25) is identically verified all over \( T \) \([16]\). Its explicit evaluation gives an expression of second degree in \( \dot{a} \) and \( \dot{T} \), whose coefficients are functions of \( a \) and \( T \) only. Therefore, they should be zero separately \([16, 20, 21]\).

In our case, substituting Eq. (15) into Eq. (25) and using the relations \( \dot{\alpha} = (\partial \alpha / \partial a) \dot{a} + (\partial \alpha / \partial T) \dot{T} \), \( \dot{\beta} = (\partial \beta / \partial a) \dot{a} + (\partial \beta / \partial T) \dot{T} \), we obtain
\[
3\alpha a^2 (f - f_{TT}) - 3\alpha^3 a f_{TT} - 6\alpha^2 \left( \alpha f_T + \beta a f_{TT} + 2 a f_T \frac{\partial \alpha}{\partial a} \right) - 12\alpha \dot{a} \dot{a} \frac{\partial \alpha}{\partial T} = 0. \quad (27)
\]

As mentioned above, requiring the coefficients of \( \dot{a}^2 \), \( \dot{T}^2 \) and \( \dot{a} \dot{T} \) in Eq. (27) to be zero, we find that
\[
\frac{\partial \alpha}{\partial T} = 0, \quad (28)
\]
\[
\alpha f_T + \beta a f_{TT} + 2 a f_T \frac{\partial \alpha}{\partial a} = 0, \quad (29)
\]
\[
3\alpha a^2 (f - f_{TT}) - 3\alpha^3 a f_{TT} = 0. \quad (30)
\]

In particular, the constraint \([31]\) is sometimes called Noether condition \([20]\). The corresponding constant of motion (Noether charge) given in Eq. (26) reads
\[
Q_0 = -12\alpha f_T \dot{a} = \text{const.} \quad (31)
\]

A solution of Eqs. (28), (29) and (30) exists if explicit forms of \( \alpha \) and \( \beta \) are found; and if at least one of them is different from zero, Noether symmetry exists \([20]\). Obviously, from Eq. (28), it is easy to see that \( \alpha \) is independent of \( T \), and hence it is a function of \( a \) only, i.e., \( \alpha = \alpha(a) \). On the other hand, from Eq. (30), we have
\[
\beta a f_{TT} = 3\alpha (f - f_{TT}). \quad (32)
\]

Multiplying \( T \) in both sides of Eq. (29), and then substituting Eq. (32) into it, we obtain
\[
f_T T \left( 2a \frac{\partial \alpha}{\partial a} - 2\alpha \right) + 3\alpha f = 0. \quad (33)
\]

Fortunately, one can perform a separation of variables, and recast Eq. (33) as
\[
1 - \frac{a \frac{d\alpha}{da}}{\alpha \frac{d\alpha}{da}} = \frac{3f}{2f_T T}. \quad (34)
\]

Since its left-hand side is a function of \( a \) only and its right-hand side is a function of \( T \) only, they must be equal to a same constant in order to ensure that Eq. (34) always holds. For convenience, we let this constant be \( 3/(2n) \), and then Eq. (34) can be separated into two ordinary differential equations, i.e.,
\[
nf = f_T T, \quad (35)
\]
\[
1 - \frac{a \frac{d\alpha}{da}}{\alpha \frac{d\alpha}{da}} = \frac{3}{2n}. \quad (36)
\]

It is easy to find the solutions of these two ordinary differential equations, namely
\[
f(T) = \mu T^n, \quad (37)
\]
\[
\alpha(a) = \alpha_0 a^{1-3/(2n)}, \quad (38)
\]
where $\mu$ and $\alpha_0$ are integral constants. Obviously, $f(T)$ and $\alpha(a)$ are both power-law forms. Substituting Eqs. (37) and (38) into Eq. (32), we find that

$$\beta(a, T) = -\frac{3\alpha_0}{n} a^{-3/(2n)} T. \quad (39)$$

So far, we have found the explicit non-zero solutions of $f(T)$ and $\alpha$. Therefore, Noether symmetry exists. Finally, we try to find out the exact solution of $a(t)$ for this type of $f(T)$. Substituting Eqs. (37), (38) and (19) into Eq. (31), we can obtain an ordinary differential equation of $a(t)$, namely

$$a c_1 \dot{a} = c_2, \quad (40)$$

where

$$c_1 = \frac{3}{2n} - 1, \quad c_2 = \left[ \frac{Q_0}{-12\alpha_0\mu_0(-6)^{n-1}} \right]^{1/(2n-1)} \quad (41)$$

It is easy to find that the solution of Eq. (40) is given by

$$a(t) = -(1 + c_1)(c_3 - c_2 t)^{1/(1+c_1)} = (-1)^{1+2n/3} \frac{3}{2n} (c_2 t - c_3)^{2n/3}, \quad (42)$$

where $c_3$ is an integral constant. Obviously, in the late time $|c_2 t| \gg |c_3|$ and the universe experiences a power-law expansion. In fact, we can make it clearer. Requiring $a(t = 0) = 0$, it is easy to see that the integral constant $c_3$ is zero in fact. So, we have

$$a(t) \sim t^{2n/3}, \quad (43)$$

where its prefactor $(-1)^{1+2n/3} \frac{3}{2n} c_2^{2n/3}$ is not important. Note that $n > 0$ is required to ensure that the universe is expanding.

**IV. PHYSICAL QUANTITIES CORRESPONDING TO THE EXACT SOLUTION**

Here, we further consider the physical quantities corresponding to the exact solution found in the previous section. Firstly, from Eq. (43), we find that the Hubble parameter is

$$H \equiv \frac{\dot{a}}{a} = \frac{2n}{3} t^{-1}, \quad (44)$$

and the deceleration parameter is given by

$$q \equiv -\frac{\ddot{a}}{aH^2} = \frac{3}{2n} - 1. \quad (45)$$

When $n > 3/2$, the expansion of our universe can be accelerated (note that $n < 0$ is not acceptable since the universe contracts in this case). Secondly, from Eqs. (19) and (37), one can find that the effective dark energy density and pressure from torsion are given by

$$\rho_{de} = 6H^2 - f - 12H^2 f_T, \quad (46)$$

$$p_{de} = -\rho_{de} - 4 \left( 12H^2 f_T - f_T + 1 \right) \dot{H}. \quad (47)$$

Substituting Eqs. (44), (19) and (37) into Eqs. (46) and (47), we can find that the equation-of-state parameter (EoS) of the effective dark energy from torsion is given by

$$w_{de} \equiv \frac{p_{de}}{\rho_{de}} = -\frac{8n(n-1)3^n}{8n^2 \cdot 3^n - 3\mu(8^n - n \cdot 21^{n/3})(-n^2)^n \mu(1-n)}. \quad (48)$$
Using the well-known relation between the total EoS $w_{tot}$ and the deceleration parameter $q$ (see e.g. [36]), it is easy to see that

$$w_{tot} \equiv \frac{\rho_{tot}}{\rho_{tot}} = \frac{1}{3} (2q - 1) = \frac{1}{n} - 1 .$$

(49)

Again, when $n > 3/2$ we have $w_{tot} < -1/3$ and then the expansion of our universe can be accelerated (note that $n < 0$ is not acceptable since the universe contracts in this case). It is worth noting that if $n > 1$, from Eq. (45) we have $w_{de} \to 1/n - 1 = w_{tot}$ in the late time $t \to \infty$. This is not surprising. Note that the well-known relation (see e.g. [36])

$$w_{tot} = \Omega_{de} w_{de} + \Omega_m w_m ,$$

where the EoS of pressureless matter $w_m = 0$, and $\Omega_i$ are the fractional energy densities of the effective dark energy and pressureless matter. Although $w_{tot}$ is constant, $w_{de}$ and $\Omega_{de}$ are dependent on time $t$. If $w_{de} < w_m = 0$, it is inevitable that $\Omega_{de} \to 1$ and then $w_{de} \to w_{tot}$ in the late time $t \to \infty$. Finally, we turn to the fractional energy densities. Using Eqs. (48), (49), (50) and $w_m = 0$, we find that the fractional density of the effective dark energy from torsion is given by

$$\Omega_{de} = \frac{w_{tot}}{w_{de}} = 1 - \frac{3\mu (8^n - n \cdot 2^{1+3n}) (-n^2)^n t^{2(1-n)}}{8n^2 \cdot 3^n} ,$$

(51)

and then the fractional density of pressureless matter $\Omega_m = 1 - \Omega_{de}$ is ready, namely

$$\Omega_m = \frac{3\mu (8^n - n \cdot 2^{1+3n}) (-n^2)^n t^{2(1-n)}}{8n^2 \cdot 3^n} .$$

(52)

It is worth noting that $\Omega_m$ coming from Eq. (52) contains only two parameters $n$ and $\mu$. On the other hand, by definition $\Omega_m \propto \rho_m / H^2$ and $\rho_m = \rho_{m0} / a^3$, while the prefactor in $a(t)$ contains $c_2$ which is given in Eq. (11), we see that $\Omega_m$ contains five parameters $\rho_{m0}$, $n$, $\mu$, $\alpha_0$ and $Q_0$ in this way. Requiring the equality between these two $\Omega_m$, it is easy to see that at least one of these five parameters $\rho_{m0}$, $n$, $\mu$, $\alpha_0$ and $Q_0$ is not independent.

V. COSMOLOGICAL TEST

As is well known, the current accelerated expansion of our universe was firstly found from the observation of distant Type Ia Supernovae (SNIa) [1]. Here, we would like to test the exact solution of $f(T)$ theory found in Sec. III with the latest Union2 SNIa dataset [32] which consists of 557 SNIa.

The data points of the 557 Union2 SNIa compiled in [37] are given in terms of the distance modulus $\mu_{obs}(z_i)$. On the other hand, the theoretical distance modulus is defined as

$$\mu_{th}(z_i) \equiv 5 \log_{10} D_L(z_i) + \mu_0 ,$$

(53)

where $z = 1/a - 1$ is redshift, $\mu_0 \equiv 42.38 - 5 \log_{10} h$ and $h$ is the Hubble constant $H_0$ in units of 100 km/s/Mpc, whereas

$$D_L(z) = (1 + z) \int_0^z \frac{dz}{E(z; \mathbf{p})} ,$$

(54)

in which $E \equiv H / H_0$, and $\mathbf{p}$ denotes the model parameters. Correspondingly, the $\chi^2$ from the 557 Union2 SNIa is given by

$$\chi^2_{\mathbf{p}}(\mathbf{p}) = \sum_i \frac{[\mu_{obs}(z_i) - \mu_{th}(z_i)]^2}{\sigma^2(z_i)} ,$$

(55)

where $\sigma$ is the corresponding 1σ error. The parameter $\mu_0$ is a nuisance parameter but it is independent of the data points. One can perform an uniform marginalization over $\mu_0$. However, there is an alternative
way. Following [38, 39], the minimization with respect to \( \mu_0 \) can be made by expanding the \( \chi^2_\mu \) of Eq. (55) with respect to \( \mu_0 \) as

\[
\chi^2_\mu(p) = \tilde{A} - 2\mu_0 \tilde{B} + \mu_0^2 \tilde{C},
\]

where

\[
\tilde{A}(p) = \sum_i \frac{[\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0, p)]^2}{\sigma^2_{\mu_{\text{obs}}}(z_i)},
\]

\[
\tilde{B}(p) = \sum_i \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i; \mu_0 = 0, p)}{\sigma^2_{\mu_{\text{obs}}}(z_i)}, \quad \tilde{C} = \sum_i \frac{1}{\sigma^2_{\mu_{\text{obs}}}(z_i)}.
\]

Eq. (56) has a minimum for \( \mu_0 = \tilde{B}/\tilde{C} \) at

\[
\chi^2_\mu(p) = \tilde{A}(p) - \frac{\tilde{B}(p)^2}{\tilde{C}}.
\]

Since \( \chi^2_{\mu, \text{min}} = \tilde{\chi}^2_{\mu, \text{min}} \) obviously (up to a constant), we can instead minimize \( \tilde{\chi}^2_\mu \) which is independent of \( \mu_0 \). As is well known, the best-fit model parameters are determined by minimizing \( \chi^2 = \tilde{\chi}^2_\mu \). As in [38, 40], the 68.3% confidence level is determined by \( \Delta \chi^2 \equiv \chi^2 - \chi^2_{\text{min}} \leq 1.0, 2.3 \) and 3.53 for \( n_p = 1, 2 \) and 3, respectively, where \( n_p \) is the number of free model parameters. Similarly, the 95.4% confidence level is determined by \( \Delta \chi^2 \equiv \chi^2 - \chi^2_{\text{min}} \leq 4.0, 6.17 \) and 8.02 for \( n_p = 1, 2 \) and 3, respectively. Note that the corresponding \( \Delta \chi^2 \) can be determined by \( \mu_0 = \tilde{B}/\tilde{C} \) for the best-fit parameters.

![Figure 1](image_url)  

**FIG. 1:** The \( \chi^2 \) and likelihood \( L_{\chi^2} \propto e^{-\chi^2/2} \) as functions of parameter \( n \).

In our case, from Eqs. (43) and (44), we have \( H \propto a^{-3/(2n)} \), and then

\[
E = \frac{H}{H_0} = a^{-3/(2n)} = (1 + z)^{3/(2n)}.
\]

We present the corresponding \( \chi^2 \) and likelihood \( L_{\chi^2} \propto e^{-\chi^2/2} \) as functions of parameter \( n \) in Fig. 1. The best fit has \( \chi^2_{\text{min}} = 549.222 \), and the best-fit parameter is

\[
n = 2.417^{+0.147}_{-0.131} \text{ (with 1σ uncertainty)} +_{0.312}^{0.250} \text{ (with 2σ uncertainty).}
\]
The corresponding $h = 0.692$ for the best fit. Obviously, $n > 3/2$ as required by the accelerated expansion of our universe. In Fig. 2 we show the Hubble diagram for the best fit, comparing with the 557 Union2 SN Ia data points. Obviously, our $f(T)$ theory with solution (43) can be well consistent with the latest 557 Union2 SN Ia dataset.

![Hubble diagram](image)

**FIG. 2**: The Hubble diagram for the best fit (red solid line), comparing with the 557 Union2 SN Ia data points (black diamonds). See the text for details.

### VI. CONCLUDING REMARKS

As is well known, symmetry plays an important role in the theoretical physics. In particular, the well-known Noether symmetry is an useful tool to select models motivated at a fundamental level, and find the exact solution to the given Lagrangian. In the present work, we try to consider Noether symmetry in $f(T)$ theory. At first, we briefly discuss the Lagrangian formalism of $f(T)$ theory. In particular, the point-like Lagrangian is explicitly constructed. Based on this Lagrangian, the explicit form of $f(T)$ theory and the corresponding exact solution are found by requiring Noether symmetry. In the resulting $f(T) = \mu T^n$ theory, the universe experiences a power-law expansion $a(t) \sim t^{2n/3}$. Furthermore, we consider the physical quantities corresponding to the exact solution, and find that if $n > 3/2$ the expansion of our universe can be accelerated without invoking dark energy. Also, we test the exact solution of this $f(T)$ theory with the latest Union2 SN Ia dataset which consists of 557 SN Ia, and find that it can be well consistent with the observational data in fact.

Some remarks are in order. Firstly, although there are many works concerning Noether symmetry in the literature, we still find something new by considering Noether symmetry in $f(T)$ theory. For instance, we take $f(R)$ theory for comparison. It is well known that the equations of motion in $f(R)$ theory are 4th order while they are 2nd order in $f(T)$ theory; $R$ contains $\ddot{a}$ and $\dot{a}$, but $T$ contains only $\dot{a}$. Therefore, the corresponding point-like Lagrangian of $f(R)$ theory contains $\dot{R}$, but the one of $f(T)$ theory does not contain $\dot{T}$. Note that in the $L_XE = 0$ equation, the Euler-Lagrange equation and the total energy (Hamiltonian) corresponding to Lagrangian, $\partial L/\partial \dot{q}_i$ plays an important role. The absence of $\dot{T}$ in the point-like Lagrangian of $f(T)$ theory makes difference and brings something new. Secondly, unlike other theories (e.g., $f(R)$ theory, which has been studied for more than ten years), $f(T)$ theory became an active field just from 2010. Due to its relatively short history, many aspects of $f(T)$ theory are unexplored in fact. For instance, to our knowledge, the reconstruction of $f(T)$ from cosmological observations is absent in the literature so far, unlike $f(R)$ theory. We do not know what forms of $f(T)$ are the suitable ones, or
what forms of $f(T)$ have a good motivation from fundamental theories and principles. So far, all $f(T)$ forms considered in the literature are written by hand. In such a situation, it is important to find a $f(T)$ form from some principles. Of course, one might argue that Noether symmetry is not really a physical guiding principle. But, why we deny to use Noether symmetry to shed some light on the unknown side of $f(T)$ theory? So, we consider that it is better to keep an open mind. Finally, we have shown in Sec. V that the resulting $f(T)$ theory from Noether symmetry can be well consistent with the observational data. We are not playing mathematical games with Noether symmetry. Instead, we have shown that the $f(T)$ motivated by Noether symmetry can be tested by realistic cosmological data, and then it can be considered as a serious theory of modified gravity and an alternative to dark energy.

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