Lifting systems of Galois representations associated to abelian varieties

Rutger Noot

2nd October 2006

Abstract

This paper treats what we call ‘weak geometric liftings’ of Galois representations associated to abelian varieties. This notion can be seen as a generalization of the idea of lifting a Galois representation along an isogeny of algebraic groups. The weaker notion only takes into account an isogeny of the derived groups and disregards the centres of the groups in question. The weakly lifted representations are required to be geometric in the sense of a conjecture of Fontaine and Mazur. The conjecture in question states that any irreducible geometric representation is a twist of a subquotient of an étale cohomology group of an algebraic variety over a number field.

It is shown that a Galois representation associated to an abelian variety admits a weak geometric lift to a group with simply connected derived group. In certain cases, such a weak geometric lift is itself associated to an abelian variety. This means that the conjecture of Fontaine and Mazur is confirmed for these representations. In other cases, one may find a lift which can not be found back in the étale cohomology of any abelian variety. The Fontaine–Mazur conjecture remains open for these representations. Nevertheless, certain consequences of the conjecture can be established.

2000 Mathematics Subject Classification 11G10, 11F80, 14K15

Introduction

This paper is motivated by a conjecture of Fontaine and Mazur, conjecture 1 of [FM95] and generalizes the observations made in [Noo01]. The conjecture

*Partially supported by the European Marie Curie Research Training Network contract MRTN-CT2003-504917
in question aims to characterize the Tate twists of the irreducible subquotients of the Galois representations arising from the étale cohomology of algebraic varieties over number fields.

To explain the conjecture, fix a number field $F$ and let $\mathcal{G}_F$ be the absolute Galois group of $F$. In [FM95, §1], a representation of $\mathcal{G}_F$ on a finite dimensional $\mathbb{Q}_p$-vector space $V_p$ is called geometric if it is unramified outside a finite set of non-archimedean places of $F$ and if, for each non-archimedean valuation $\mathfrak{o}$ of $F$ of residue characteristic $p$, the restriction to the inertia group at $\mathfrak{o}$ is potentially semi-stable in the sense of Fontaine, see also [4.1]. An irreducible representation of $\mathcal{G}_F$ on $V_p$ is said to come from algebraic geometry if it is isomorphic to a subquotient of the Galois representation on a cohomology group $H^1_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)(m)$ for some smooth and proper $F$-scheme $X$ and integers $i$ and $m$. The Tate twist $(m)$ has the effect of multiplying the action of $\mathcal{G}_F$ on $H^1_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ by the $m$th power of the cyclotomic character. Fontaine and Mazur have expressed the conjecture that an irreducible $\mathbb{Q}_p$-linear representation $V_p$ of $\mathcal{G}_F$ is geometric if and only if it comes from algebraic geometry.

The ‘if’-part of the conjecture being resolved (see [Isu99]), this paper is concerned with some reflexions on the implication ‘only if’ in the conjecture. We will actually investigate a particular type of geometric representations.

The construction of these representations starts with the choice of an abelian variety $A$ over $F$. For such an abelian variety, the representation of $\mathcal{G}_F$ on $H^1_{\text{ét}}(A_{\overline{F}}, \mathbb{Q}_p)$ factors through a map

$$\rho_{A,p}: \mathcal{G}_F \to G_A(\mathbb{Q}_p),$$

where $G_A$ is the Mumford–Tate group of $A$. After reading [Win95], one is tempted to look for an isogeny $\pi: \tilde{G} \to G_A$ and a map $\tilde{\rho}_p: \mathcal{G}_F \to \tilde{G}(\mathbb{Q}_p)$ such that $\rho_{A,p} = \pi \circ \tilde{\rho}_p$ and such that $\tilde{\rho}_p$ defines a geometric representation of $\mathcal{G}_F$ on $\tilde{V}_p$ for any $\mathbb{Q}_p$-linear representation $\tilde{V}_p$ of $\tilde{G}/\mathbb{Q}_p$.

This is not exactly the point of view adopted in this paper. As the conjecture of Fontaine–Mazur for representations with abelian image is quite well understood, cf. [FM95, §6], the centres of the groups occurring above can be considered to be less interesting than the derived groups. For this reason, given the representation $\rho_{A,p}: \mathcal{G}_F \to G_A(\mathbb{Q}_p)$, we will search for a linear algebraic group $\tilde{G}$, an isogeny $\pi^\text{der}: \tilde{G}^\text{der} \to G_A^\text{der}$ and a map $\tilde{\rho}_p: \mathcal{G}_F \to \tilde{G}(\mathbb{Q}_p)$ such that $\pi^\text{ad} \circ \rho_{A,p} = \tilde{\pi}^\text{ad} \circ \tilde{\rho}_p$. Here $\pi^\text{ad}: G_A \to G_A^\text{ad}$ and $\tilde{\pi}^\text{ad}: \tilde{G} \to \tilde{G}^\text{ad}$ are the natural
projections and the groups $G_{A}^{\text{ad}}$ and $\tilde{G}_{A}^{\text{ad}}$ have been identified using the map induced by $\pi_{\text{der}}$. If such a representation $\tilde{\rho}_{p}$ defines a geometric representation of $\mathcal{G}$ on any $\mathbb{Q}_{p}$-linear representation $\tilde{V}_{p}$ of $\tilde{G}/\mathbb{Q}_{p}$, then we will say that it is a weak geometric lift of $\rho_{A,p}$. An advantage of working with weak geometric liftings is that, by ‘correcting’ liftings using characters, it is easier to produce geometric representations than if one considers liftings along isogenies.

It turns out that for any abelian variety $A/F$, there exist a finite extension $F'/F$, a group $\tilde{G}$ such that $\tilde{G}_{\text{der}}$ is the universal cover of $G_{A}$ and, for every prime number $p$, a weak geometric lift $\tilde{\rho}_{p}: \mathcal{G}_{F'} \to \tilde{G}(\mathbb{Q}_{p})$ of the restriction of $\rho_{A,p}$ to $\mathcal{G}_{F}$. We refer to corollary 5.11 for a precise statement. Moreover, it follows from corollary 5.12 that every weak geometric lift of a $\rho_{A,p}$ is dominated by a weak geometric lift to a group $\hat{G}$ with $\hat{G}_{\text{der}}$ simply connected. We can even conveniently ‘normalize’ the group $\tilde{G}$.

It is natural to ask if the conjecture of Fontaine and Mazur is true for any weak geometric lift of a representation associated to an abelian variety. This question is not answered in this paper, but a number of partial results are obtained.

In section 4 we prove the following results. By combining proposition 3.4 and remark 4.10, it follows that for every abelian variety $A/F$, there exists a group $\tilde{G}$, a map $\tilde{G}_{\text{der}} \to G_{A}^{\text{der}}$ and, after replacing $F$ by a finite extension, a system $(\tilde{\rho}_{p})$ of weak geometric liftings of the $\rho_{A,p}$ such that

- the group $\tilde{G}_{\text{der}}$ is ‘not far’ from the universal cover of $G_{A}^{\text{der}}$ and
- for any representation $\tilde{V}$ of $\tilde{G}$, the system of representations of $\mathcal{G}$ on the $\tilde{V} \otimes \mathbb{Q}_{p}$ is isomorphic to the system of $p$-adic representation associated to an abelian variety.

To be more precise where the first property is concerned, it means in the first place that the group $\tilde{G}_{\text{der}}/\mathcal{C}$ is the product of its simple factors $\tilde{G}_{i}$. Secondly, it follows from well-known facts on the Mumford–Tate groups of abelian varieties that these factors are all of classical type ($A$, $B$, $C$ or $D$). For the $\tilde{G}_{i}$ which are of type $A_{k}$, $B_{k}$ or $C_{k}$, being ‘not far’ from the universal cover means that they are simply connected. Where the factors $\tilde{G}_{i}$ of type $D_{k}$ are concerned, the condition is more difficult to state. We have to distinguish two subtypes, $D_{k}^{R}$ and $D_{k}^{H}$, see 2.5 for the definitions. The $\tilde{G}_{i}$ which are of type $D_{k}^{R}$ in this classification are also simply connected. The factors $\tilde{G}_{i}$ of type $D_{k}^{H}$ are $h$-maximal in the sense
of \( \widetilde{G} \), which means that every such \( \widetilde{G}_i \) is a quotient of its universal cover by a subgroup of order 2. Over \( \mathbb{R} \), these \( h \)-maximal groups are orthogonal groups.

By theorem 3.9, the above system \((\tilde{\rho}_p)\) is maximal in the following sense. For any group \( G' \), any isogeny \( G'^{\text{der}} \to G_A^{\text{der}} \) such that \( G'^{\text{der}} \) is a quotient of \( \widetilde{G}^{\text{der}} \) and any weak geometric lift \( \rho'_p: \mathfrak{H}_F \to G'(\mathbb{Q}_p) \) of \( \rho_{A,p} \) (allowing a finite extension of \( F \)), the representation \( \rho'_p \) belongs to the tannakian category generated by the \( p \)-adic Galois representations associated to abelian varieties and the representations with finite image.

The above statements about Galois representations follow from analog properties of Hodge structures associated to abelian varieties. If \( A_\mathbb{C} \) is an abelian variety over \( \mathbb{C} \), then the Hodge structure on the first Betti cohomology group \( H^1_{\text{B}}(A_\mathbb{C}(\mathbb{C}), \mathbb{Q}) \) is determined by a morphism \( h: S \to G_{A/R} \), where \( S = \mathbb{C}^\times \) as algebraic groups over \( \mathbb{R} \). It is shown in section 2 that the group \( \widetilde{G} \) above is actually the Mumford–Tate group of an abelian variety \( B_\mathbb{C} \) and that the Hodge structure of \( B \) corresponds to a morphism \( \tilde{h}: S \to \widetilde{G}_{/\mathbb{R}} \) such that \( h \) and \( \tilde{h} \) have the same projection to \( G_{A/R}^{\text{ad}} = \widetilde{G}_{/\mathbb{R}}^{\text{ad}} \). To underscore the analogy with the construction of weak geometric liftings of Galois representations, such an abelian variety \( B \) will be called a \emph{weak Mumford–Tate lift} of \( A \), cf. 2.1. This notion is very close to the notion of \( A_\mathbb{C} \) and \( B_\mathbb{C} \) being ‘adjoint-isogenous’ in the sense of definition 6.1 of Vasiu’s e-print [Vas03] and our variety \( B_\mathbb{C} \) should be equal to the variety obtained by Vasiu’s ‘shifting process’, loc. cit. 6.4.

In section 3 we prove some properties comparing the fields of definition of an abelian variety and a weak Mumford–Tate lift. Using the theory of absolute Hodge motives, the above statements concerning Galois representations are then derived from the corresponding statements about the Hodge structures associated to weak Mumford–Tate liftings. The arguments are based on those used in [Noo01] and [Pau04].

In the above statements, the abelian varieties with Mumford–Tate group of type \( D^H_k \) stand out as an exception to the general situation. These varieties deserve a separate treatment, and this is the subject of section 5. Let \( F \) be a number field as above, \( A \) an abelian variety over \( F \) and assume that, for the Mumford–Tate group \( G_A \), the derived group \( G_A^{\text{der}} \) is isomorphic to a product \( \prod G_{A,i} \) of \( h \)-maximal groups of type \( D^H_k \). It can be shown that such abelian varieties do indeed exist. It turns out that the associated system \((\rho_{A,p})\) of \( p \)-adic Galois representations admits a system of non-trivial weak geometric liftings.
The corollary 5.9 states that there exists a group \( \tilde{G} \) such that \( \tilde{G}^{\text{der}} \to G_{A}^{\text{der}} \) is the universal cover and, as always after replacing \( F \) by a finite extension, a system \((\tilde{\rho}_{p})\) of weak geometric liftings of the \( \rho_{A,p} \). These representations are called the \( p \)-adic Galois representations of \textit{lifted abelian} \( D_{k}^{H} \)-type. The construction makes use, again, of a lifting \( \tilde{h}: S \to \tilde{G}_{/R} \) of \( h: S \to G_{A/R} \). This time, the existence of the system \((\tilde{\rho}_{p})\) follows from [Win95].

When considering weak geometric liftings of Galois representations associated to abelian varieties, these representations of lifted abelian \( D_{k}^{H} \)-type are the only ‘new’ representations that one may encounter. To be precise, let \( A/F \) be an abelian variety with Mumford–Tate group \( G_{A} \). For any group \( G' \) and isogeny \( G'^{\text{der}} \to G_{A}^{\text{der}} \) and any weak geometric lift \( \rho'_{p}: \mathcal{G}_{F} \to G'(\mathbb{Q}_{p}) \) of \( \rho_{A,p} \), the representation \( \rho'_{p} \) belongs to the tannakian category generated by the \( p \)-adic Galois representations associated to abelian varieties, the representations of lifted abelian \( D_{k}^{H} \)-type and those with finite image, see corollary 5.12.

In the final section 6, we study the representations \( \tilde{\rho}_{p} \) of lifted abelian \( D_{k}^{H} \)-type. It is proved in 6.2 that such a representation \( \tilde{\rho}_{p} \) generally does not belong to the tannakian category generated by the Galois representations associated to abelian varieties. The question if \( \tilde{\rho}_{p} \) belongs to the tannakian category generated by the Galois representations associated to algebraic varieties, as predicted by the Fontaine–Mazur conjecture, remains open.

Combined with ‘standard’ conjectures, the conjecture of Fontaine and Mazur implies that the \( \tilde{\rho}_{p} \) have certain ‘motivic’ properties. For example, any Frobenius element in \( \mathcal{G}_{F} \) should act semi-simply on any representation \( \tilde{V}_{p} \) of \( \tilde{G}_{/\mathbb{Q}_{p}} \) and the eigenvalues of the image should be Weil numbers, i. e. algebraic integers of which all complex absolute values coincide. Several of these motivic properties are proved in the final part of section 6.

In addition to their relevance to the Fontaine–Mazur conjecture, the construction of weak geometric liftings has applications to the study of Galois representations associated to abelian varieties. We have seen that for an abelian variety \( A/F \), and for \( F \) large enough, the representations \( \rho_{A,p}: \mathcal{G}_{F} \to G_{A}(\mathbb{Q}_{p}) \) have weak geometric liftings \( \rho_{B,p}: \mathcal{G}_{F} \to G_{B}(\mathbb{Q}_{p}) \), where \( B/F \) is an abelian variety and \( G_{B}^{\text{der}} \to G_{A}^{\text{der}} \) is not far from the universal cover. The Galois representation on \( H^{1}_{\text{et}}(A_{F}, \mathbb{Q}_{p}) \) belongs to the tannakian category generated by \( H^{1}_{\text{et}}(B_{F}, \mathbb{Q}_{p}) \) and the representations associated to abelian varieties of CM-type. This can be useful because the representation \( H^{1}_{\text{et}}(B_{F}, \mathbb{Q}_{p}) \) is in general easier to study than
H^1_{et}(A_F, Q_p). A first example of such an application can be found in the paper [Pau04] of F. Paugam. In [Vas03], A. Vasiu uses his technique of adjoint isogenous abelian varieties to study new instances of the Mumford–Tate conjecture. Other applications are to follow in forthcoming publications.

Acknowledgements. This paper has benefited from discussions and correspondence with a number of people and I thank them heartily for their contribution. In particular, I would like to thank D. Blasius and P. Deligne for (independently) pointing out, after my previous work on the conjecture of Fontaine and Mazur, that the case of abelian varieties with Mumford–Tate group of type D^H_k deserved special attention.

1 Preliminaries

1.1 Basic notations. For any field $F$, we denote by $\bar{F}$ an algebraic closure of $F$. The absolute Galois group of $F$ is the group $\mathcal{G}_F = \text{Aut}_F(\bar{F})$.

For any prime number $p$, the field $Q_p$ is the $p$-adic completion of $Q$ and $C_p$ is the completion of an algebraic closure $\bar{Q}_p$.

If $G$ is a group and $K$ a field, then $\text{Rep}_K(G)$ is the category of finite dimensional $K$-linear representations of $G$. If $G$ is a topological group (a Galois group for example) and $K$ a topological field, then the representations in $\text{Rep}_K(G)$ are assumed to be continuous. If $G$ is a linear algebraic group, then $\text{Rep}_K(G)$ is the category of algebraic representations of $G$.

If $G$ is an algebraic group, a quasi-cocharacter of $G$ is an element of the direct limit

$$\lim_{\overset{k\to}\longrightarrow} \text{Hom}(G_m(k), G),$$

where the transition map $G_m(k_\ell) \to G_m(k)$ is the morphism $z \mapsto z^\ell$. Giving a quasi-cocharacter of $G$ is equivalent to giving an integer $k$ and a cocharacter $\mu: G_m \to G$. Intuitively the quasi-cocharacter given by $(k, \mu)$ is the $k$th root of $\mu$.

1.2 Absolute Hodge motives. We will freely use the language of tannakian categories. Everything we need here can be found in [DM82], which is to be considered the authoritative reference for all notions used but not explained in this paper. In particular, if $\mathcal{C}$ is a tannakian category, then the subcategory
⊗-generated by a collection $X$ of objects of $C$ is the smallest subcategory of $C$ containing all objects which are isomorphic to a subquotient of a polynomial expression with coefficients in $\mathbb{N}$ in the objects contained in $X$. In such a polynomial expression, $+$ and $\cdot$ are to be interpreted as $\oplus$ and $\otimes$ respectively.

For any field $F$ of characteristic 0, let $\text{Mot}_{AH}(F)$ be the category of motives for absolute Hodge cycles as described in [DM82], especially section 6 of that paper. It is constructed as Grothendieck’s category of Chow motives except where it concerns the morphisms which are given by absolute Hodge classes, not by cycle classes as for Grothendieck motives. It must be pointed out that by definition, the morphisms between two motives $M_1$ and $M_2$ defined over $F$ are the appropriate absolute Hodge classes on the product which are defined over $F$, i.e. Hodge classes on $(M_1 \times M_2)_F$ invariant under the action of $\mathcal{G}_F$.

Motives are graded objects, each motive $M$ is a finite direct sum $\bigoplus M_i$, where each $M_i$ is a pure motive of weight $i$. The Tate motive $\mathbb{Q}(1)$ is the dual of $h^2(P_1)$ in the category $\text{Mot}_{AH}(F)$.

1.3 Mumford–Tate groups. In what follows, we will assume that $F$ is contained in $\mathbb{C}$ and we write $\bar{F}$ for the algebraic closure of $F$ in $\mathbb{C}$. Assume that $M_1, \ldots, M_r$ are absolute Hodge motives over $F$, and let $\mathcal{C} = \langle M_1, \ldots, M_r, \mathbb{Q}(1) \rangle$ be the tannakian subcategory of $\text{Mot}_{AH}(F)$ which is $\otimes$-generated by $M_1, \ldots, M_r$ and the Tate motive $\mathbb{Q}(1)$. Let $H_B(\cdot, \mathbb{Q})$ be the fibre functor of $\mathcal{C}$ over $\mathbb{Q}$ defined by the Betti cohomology $H^*_B(X(\mathbb{C}), \mathbb{Q})$ of complex algebraic varieties. Then the Mumford–Tate group $G$ of $\mathcal{C}$ is by definition the automorphism group of $H_B(\cdot, \mathbb{Q})$. The connected component of $G$ is reductive and $G$ acts on $H_B(M, \mathbb{Q})$ for every object $M$ of $\mathcal{C}$. The fibre functor $H_B(\cdot, \mathbb{Q})$ induces an equivalence between $\mathcal{C}$ and the category $\text{Rep}_\mathbb{Q}(G)$. In particular, for $M, M' \in \mathcal{C}$, one has

$$\text{Hom}_\mathcal{C}(M, M') = \text{Hom}_G(H_B(M, \mathbb{Q}), H_B(M', \mathbb{Q})),$$

i.e. the action of $G$ fixes all absolute Hodge classes defined over $F$ on all objects of $\mathcal{C}$. Moreover, $G$ is the smallest $\mathbb{Q}$-algebraic group with this property. Note that the connected component of $G$ is the Mumford–Tate group of the subcategory of $\text{Mot}_{AH}(\bar{F})$ which is $\otimes$-generated by $M_{1/F}, \ldots, M_{r/F}$ and $\mathbb{Q}(1)$. When considering only one motive $M$, the Mumford–Tate group $G_M$ of $M$ is the Mumford–Tate group of the subcategory $\langle M, \mathbb{Q}(1) \rangle$ of $\text{Mot}_{AH}(F)$. 
For a general subcategory $\mathcal{C}$ of $\text{Mot}_{\text{AH}}(F)$, we define the Mumford–Tate group in a similar way, obtaining a pro-algebraic group.

1.4 Subcategories of $\text{Mot}_{\text{AH}}$. An Artin motive is an object of $\text{Mot}_{\text{AH}}(F)$ with finite Mumford–Tate group. The category of Artin motives is the tannakian subcategory of $\text{Mot}_{\text{AH}}(F)$ generated by the finite $F$-schemes. An abelian motive over $F$ is an $\text{AH}$-motive which belongs to the tannakian subcategory of $\text{Mot}_{\text{AH}}(F)$ generated by the motives of abelian varieties, the Tate motive and the Artin motives. We will write $(\text{Artin})_F$ for the category of Artin motives and $(\text{AV})_F$ for the category of abelian motives over $F$. An abelian variety is potentially of CM-type if the connected component of its Mumford–Tate group is commutative and $(\text{CM})_F$ is the tannakian subcategory of $\text{Mot}_{\text{AH}}(F)$ generated by the Artin motives, $\mathbb{Q}(1)$ and the motives of the abelian varieties which are potentially of CM-type.

If $A$ is an abelian variety over $F$, then $h(A) = \bigoplus_i H^i(A)$, so in most questions concerning abelian motives, it suffices to consider the motives $h^1(A)$ instead of the $h(A)$.

1.5 Betti realization. For any absolute Hodge motive $M$, the Betti realization $H_B(M, \mathbb{Q})$ carries a Hodge structure. Giving this Hodge structure is equivalent to giving an action of the group $S = \mathbb{C}^\times$, viewed as an algebraic group over $\mathbb{R}$, on the $\mathbb{R}$-vector space $H_B(M, \mathbb{Q}) \otimes \mathbb{R}$. As the the Mumford–Tate group $G_M$ of $M$ fixes all (absolute) Hodge classes, this action is given by a morphism of algebraic groups $h: S \to G_M/\mathbb{R}$. The couple $(G_M, h)$ is the Mumford–Tate datum associated to $M$. More generally, if $\mathcal{C}$ is a $\otimes$-subcategory of $\text{Mot}_{\text{AH}}(F)$, with Mumford–Tate group $G$, then there is a morphism $h: S \to G/\mathbb{R}$ which functorially defines the Hodge structures on $H_B(M, \mathbb{Q})$ for all objects $M$ of $\mathcal{C}$.

We restrict our attention to the category of abelian motives $(\text{AV})_F$. As explained in [Del82, 6.25], it follows from [Del82, Theorem 2.11] that the Betti realization induces an equivalence of $(\text{AV})_F$ with its essential image in the category Hodge of Hodge structures.

Let $\mathcal{C}$ be a $\otimes$-subcategory of $\text{Mot}_{\text{AH}}(F)$ and let $(G, h)$ be the associated Mumford–Tate datum. For any object $M$ of $\mathcal{C}$, the Betti realization $H_B(M, \mathbb{Q})$ is a representation of $G$ and the Hodge structure on $H_B(M, \mathbb{Q})$ is given by the action of $S$ on $H_B(M, \mathbb{Q}) \otimes \mathbb{R}$ induced by $h: S \to G/\mathbb{R}$. Since every $\mathbb{Q}$-linear
representation of $G$ is the Betti realization of an object of $\mathcal{C}$, this construction defines a $\otimes$-equivalence of $\text{Rep}_\mathbb{Q}(G)$ with the essential image of $\mathcal{C}$ in $\text{Hodge}$.

If $A$ is an abelian variety, its Mumford–Tate group $G_A$ is equal to the Mumford–Tate group of $h^1(A)$. The connected component of $G_A$ coincides in turn with the Mumford–Tate group of the Hodge structure $H^1_b(A(\mathbb{C}), \mathbb{Q})$.

Let $(G, h)$ be a Mumford–Tate datum. Composition of $h$ with the cocharacter $G_{m/\mathbb{C}} \to S/\mathbb{C}$ dual to $z: S/\mathbb{C} \to G_{m/\mathbb{C}}$ gives rise to Hodge cocharacter $\mu: G_{m/\mathbb{C}} \to G_{M/\mathbb{C}}$. Alternatively, $\mu$ is determined by the condition that $G_{m/\mathbb{C}}$ acts on the factor $V^{p,q}$ of the Hodge decomposition as multiplication by $z^p$. Conversely, a pure Hodge structure is determined by giving its weight and the Hodge cocharacter.

1.6 $p$-adic Galois representations. Let $p$ be a prime number. The étale cohomology with coefficients in $\mathbb{Q}_p$ of algebraic varieties over $\overline{F}$ defines the $p$-adic étale realization on the category of absolute Hodge motives. The $p$-adic étale realization of a motive $M$ over $F$ is a $\mathbb{Q}_p$-vector space $H_{\text{et}}(M, \mathbb{Q}_p)$ endowed with a continuous action of the group $G_F$. It follows from standard tannakian theory that the image of $G_F$ in $\text{GL}(H_{\text{et}}(M, \mathbb{Q}_p))$ lies in $G_M(\mathbb{Q}_p)$, where $G_M$ is the Mumford–Tate group of $M$. The representation thus gives rise to a morphism $\rho_{M,p}: G_F \to G_M(\mathbb{Q}_p)$.

For any prime number $p$, let $(\text{Artin})-\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$ be the tannakian subcategory of $\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$ generated by the $p$-adic étale realizations of the objects of $(\text{Artin})_F$. This category coincides with the category of $\mathbb{Q}_p$-linear representations of $\mathcal{G}_F$ with finite image. Similarly, we let $(\text{AV})-\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$ be the tannakian subcategory of $\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$ generated by the $p$-adic étale realizations of the objects of $(\text{AV})_F$. As before, in the case of an abelian variety $A$, it is usually sufficient to consider just the representation on the first étale cohomology group $H^1_{\text{et}}(A, \mathbb{Q}_p)$.

Restricting to an inertia group at a $p$-adic place of $F$, these representations give rise to representations of Hodge–Tate type, cf. [Fon94, §3] or [Ser78]. If $\mathcal{I} \subset \mathcal{G}_F$ is such an inertia subgroup and $V_p$ is a $\mathbb{Q}_p$-linear representation of $\mathcal{I}$ of Hodge–Tate type, there is a canonical decomposition

$$V_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p = \bigoplus_{p,q} V^{p,q},$$
the Hodge–Tate decomposition of $V_p \otimes \mathbb{C}_p$. The Hodge–Tate cocharacter is the cocharacter $\mu: G_{m}/\mathbb{C}_p \to \text{GL}(V_p \otimes \mathbb{C}_p)$ such that action of $G_{m}/\mathbb{C}_p$ on $V_p^\tau$ is the multiplication by $x^p$. It can be shown (see [Ser78, §1]) that the connected component $H$ of the Zariski closure of the image of $\mathcal{F}$ in $\text{GL}(V_p)$ coincides with the smallest subgroup $H \subset \text{GL}(V_p)$ such that $\mu$ factors through $H/\mathbb{C}_p$.

## 2 Mumford–Tate liftings of abelian varieties over $\mathbb{C}$

### 2.1 Mumford–Tate liftings

Let $M/\mathbb{C}$ be an abelian motive, $G_M$ its Mumford–Tate group and $(G_M, h_M)$ the associated Mumford–Tate datum. This means that the Hodge structure on $V_M = H_B(M, \mathbb{Q})$ is defined by $h_M: S \to G_M/\mathbb{R}$.

We say that an abelian motive $N$ with Mumford–Tate datum $(G_N, h_N)$ provides a Mumford–Tate lift of $M$ if there exists a central isogeny $\pi: G_N \to G_M$ such that $\pi_R \circ h_N = h_M$. We say that $M$ is Mumford–Tate liftable if there exists an abelian motive $N/\mathbb{C}$ giving a Mumford–Tate lift of $M$ and such the morphism $\pi: G_N \to G_M$ is not an isomorphism. We say that $M$ is Mumford–Tate unliftable if it is not Mumford–Tate liftable.

If there exists a central isogeny $\pi^\text{der}: G^\text{der}_N \to G^\text{der}_M$ such that

$$\pi^\text{ad}_N \circ h_N = \pi^\text{ad}_M \circ h_M$$

then $N$ provides a weak Mumford–Tate lift of $M$. Here the maps $\pi^\text{ad}_M$ and $\pi^\text{ad}_N$ are the projections $G_M \to G^\text{ad}_M$ and $G_N \to G^\text{ad}_N$. As it is a central isogeny, it follows that $\pi^\text{der}$ induces an isomorphism $G^\text{ad}_M \cong G^\text{ad}_N$, giving a sense to the equality $\pi^\text{ad}_N \circ h_N = \pi^\text{ad}_M \circ h_M$. Finally, $M$ is essentially Mumford–Tate unliftable if there does not exists any abelian motive $N/\mathbb{C}$ giving a weak Mumford–Tate lift of $M$ for which $\pi^\text{der}$ is not an isomorphism.

We will often write M-T (un)liftable instead of Mumford–Tate (un)liftable.

### 2.2 Remark

If $N$ is a weak Mumford–Tate lift of $M$, then $M$ and $N$ are adjoint-isogenous in the sense of [Vas03, 6.1] and conversely, if $M$ and $N$ are adjoint-isogenous and if there is an isogeny $G^\text{der}_N \to G^\text{der}_M$, then $N$ is a weak Mumford–Tate lift of $M$.

### 2.3 The Mumford–Tate datum of an abelian variety

Let $A/\mathbb{C}$ be an abelian variety, $(G_A, h)$ the associated Mumford–Tate datum and $\mu: G_{m/\mathbb{C}} \to G_{A/\mathbb{C}}$ be
the Hodge cocharacter. It follows from [Del79], in particular from 1.3 and 2.3, that the simple factors of $G_{A/R}^{ad}$ are absolutely simple and of classical type ($A$, $B$, $C$ or $D$). The group $G_{Q}$ acts on the set of these simple factors and each factor is conjugate to a non-compact one. The compact factors are exactly the factors to which $h$ projects trivially.

Let $H$ be a $Q$-simple factor $G_{A}^{ad}$ and decompose $H/C = \prod_{i \in I_H} H/C_i$ for some finite set $I_H$ with $G_{Q}$-action. Here we have identified $Q$ with the algebraic closure of $Q$ in $C$. Note that the complex conjugation in $G_{Q}$ acts trivially on $I$ because the simple factors of $G_{A/R}^{ad}$ are absolutely simple. Let $I_{H,c}$ be the set of indices such that the Hodge cocharacter $\mu$ projects trivially to $H/C_i$ and let $I_{H,nc}$ be the complement of $I_{H,c}$ in $I_H$. The $i \in I_{H,c}$ are exactly the indices for which the corresponding real factor of $H/R$ is compact. For each $i \in I_{H,nc}$, the Hodge cocharacter lifts to a quasi-cocharacter $\tilde{\mu}_i$ of the universal cover $\tilde{H}/C_i$ of $H/C_i$. If the simple factors of $H/C$ are of type $A$, $B$ or $C$ then it follows from [Del79, 1.3] that $\tilde{H}/C_i$ admits a faithful representation $W/C_i$ such that $\tilde{\mu}_i$ acts on $W/C_i$ with exactly two weights. Contemplating the tables [Del79, 1.3.9] or [Pin98, Table 4.2] one sees that the highest weight of $W/C_i$ is

- either $\varpi_1$ or $\varpi_k$ if the simple factors of $H/C$ are of type $A_k$, 
- $\varpi_k$ if these simple factors are of type $B_k$ and 
- $\varpi_1$ if the factors are of type $C_k$. 

For each $i \in I_H$, we define a representation $V/C_i$ of $\tilde{H}/C_i$ as follows. Let $V/C_i$ be the direct sum of the representations with highest weights $\varpi_1$ and $\varpi_k$ if $H$ is of type $A_k$ with $k \geq 2$ and define $V/C_i$ to be the representation with highest weight $\varpi_1$ (resp. $\varpi_k$) if $H$ is of type $A_1$ or $C_k$ (resp. $B_k$). For $i \in I_{H,nc}$, the cocharacter $\tilde{\mu}_i$ still acts on each irreducible factor of $V/C_i$ with exactly two weights. The product over $i \in I_H$ of the $\tilde{H}/C_i$ descends to an algebraic group $\tilde{H}$ over $Q$ and a multiple of the direct sum of the $V/C_i$ descends to a faithful $Q$-linear representation $V$ of $\tilde{H}$.

2.4 Remarks. If the simple factors of $H/C$ are of type $B_k$ or $C_k$ then the condition that $\tilde{\mu}_i$ acts on $W/C_i$ with exactly two weights uniquely determines the highest weight. Similarly, if the simple factors of $H/C$ are of type $A_k$ and if $\tilde{\mu}_i$ is not
dual to $\alpha_1$, only the representations with highest weight $\varpi_1$ or $\varpi_k$ fulfill this condition. Obviously, all these representations are faithful.

In the case where the simple factors of $H/C$ are of type $A_k$ and where $\tilde{\mu}_i$ is dual to $\alpha_1$, the cocharacter $\tilde{\mu}_i$ acts with exactly two weights on the representation with highest weight $\varpi_s$ for any $1 \leq s \leq k$, see [Pin98, Table 4.2]. Only for $s = 1$ and for $s = k$ does one obtain a faithful representation.

Also note that $V/C, \iota$ is self dual in each case, see the aforementioned table in Pink’s paper.

2.5 Factors of type $D_k$. In what follows, we will focus on the factors of $G^{ad}$ of type $D_k$, first considering the case $k \geq 5$. Fix a $\mathbb{Q}$-simple factor $H$ of $G^{ad}$ such that the simple factors of $H/C$ are of type $D_k$ with $k \geq 5$. Let $H/C = \prod_{i \in I_H} H/C_i$, and let $I_{H,C}$ and $I_{H,nc}$ be as before. The Dynkin diagram of $H/C$ is the disjoint union, indexed by $I_H$, of diagrams of type $D_k$ with $k \geq 5$, see [Del79, 1.3] that, for each $\iota \in I_{H,nc}$, the conjugacy class of the cocharacter $\mu_{\iota}$ is dual to an endpoint $\alpha^{(1)}(\iota)$ of the $\iota$-component of the Dynkin diagram.

2.6 Lemma. Assume that $k \geq 5$ and that $\alpha^{(1)}(\iota) = \alpha_1$ for some $\iota \in I_{H,nc}$. Then $\alpha^{(\kappa)} = \alpha_1$ for all $\kappa \in I_{H,nc}$.

Proof. Assume that $\alpha^{(1)}(\iota) = \alpha_1$ for $\iota \in I_{H,nc}$ and let $G^{der}_{A/C, \iota}$ be the simple factor of $G^{der}_{A/C}$ projecting onto $H/C, \iota$. Via the representation of $G_A$ on $V = H^1_B(A(C), \mathbb{Q})$, the Hodge cocharacter $\mu$ acts on $V/C$ with two weights. By the table in [Del79, 1.3], this implies that every non-trivial irreducible direct factor of the representation of $G^{der}_{A/C, \iota}$ on $V/C$ has highest weight $\varpi_{k-1}$ or $\varpi_k$. As $H$ is $\mathbb{Q}$-simple, the same thing is true for the representation on $V/C$ of the simple factors of $G^{der}_{A/C}$ mapping to the factors $H/C, \kappa$ for the other $\kappa \in I_H$. Using the tables [Del79, 1.3.9], the fact that $\mu$ acts with two weights implies that all non-trivial $\mu_{\kappa}$ are dual to $\alpha_1$. 

If any factor of $H/C$ satisfies the conditions of the lemma, then we say that $H$ and the factors of $H/R$ and $H/C$ are of type $D^R_k$. As $H$ was assumed to be $\mathbb{Q}$-simple, the vertices $\alpha^{(1)}(\iota)$ form a single orbit for the action of $G_C$ in this case. In the opposite case we say that they are of type $D^H_k$. In the latter case, the conjugacy class of the projection of $\mu$ to any factor of $H/C$ is either trivial or dual to $\alpha_{k-1}$ or $\alpha_k$. 
Let $H/\mathcal{C}_\iota$ be a factor of $G^\text{ad}_{\mathcal{A}/\mathcal{C}}$ of type $D^R_k$ and let $\tilde{H}/\mathcal{C}_\iota$ be its universal cover. For each $\iota \in I_H$, let $V/\mathcal{C}_\iota$ be the direct sum of the representations of $H/\mathcal{C}_\iota$ with highest weights $\omega_{k-1}$ and $\omega_k$. As above, for $\iota \in I_{H,nc}$, the projection of the Hodge cocharacter $\mu$ to $H/\mathcal{C}_\iota$ lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of $\tilde{H}/\mathcal{C}_\iota$ and $\tilde{\mu}_\iota$ acts on $V/\mathcal{C}_\iota$ with two weights $\pm 1/2$. The product over $\iota \in I$ of the $\tilde{H}/\mathcal{C}_\iota$ and a multiple of the direct sum of the $V/\mathcal{C}_\iota$ descend to an algebraic group $\tilde{H}$ over $\mathbb{Q}$ and a faithful $\mathbb{Q}$-linear representation $V$ of $\tilde{H}$.

Next assume that $H/\mathcal{C}_\iota$ is a factor of $G^\text{ad}_{\mathcal{A}/\mathcal{C}}$ of type $D^H_k$. As before, for $\iota \in I_{H,nc}$, the projection of the Hodge cocharacter lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of the universal cover $\tilde{H}/\mathcal{C}_{\iota'}$ but this time the group $\tilde{H}/\mathcal{C}_\iota$ does not have a faithful representation on which $\tilde{\mu}_\iota$ acts with two weights. This property can only be achieved for a quotient of $\tilde{H}/\mathcal{C}_\iota$ which can be constructed as follows. For each $\iota \in I_H$, let $V/\mathcal{C}_\iota$ be the representation of $\tilde{H}/\mathcal{C}_\iota$ with highest weight $\omega_\iota$. We will refer to the quotient of $\tilde{H}/\mathcal{C}_\iota$ which acts faithfully on $V/\mathcal{C}_\iota$ as the $h$-maximal cover of $H/\mathcal{C}_\iota$. This $h$-maximal cover descends over $\mathbb{R}$ to an inner form of the orthogonal group $SO(2k)$. For each $\iota \in I_{H,nc}$, the quasi-cocharacter $\tilde{\mu}_\iota$ acts on $V/\mathcal{C}_\iota$ with weights $\pm 1/2$. The product over $\iota \in I$ of these $h$-maximal covers descends to an algebraic group over $\mathbb{Q}$ and a multiple of the direct sum of the $V/\mathcal{C}_\iota$ descends to a faithful $\mathbb{Q}$-linear representation of this group.

### 2.7 Factors of type $D_4$

We finally consider the case where $k = 4$, so let $H$ be a $\mathbb{Q}$-simple factor of $G^\text{ad}_{\mathcal{A}/\mathcal{C}}$ such that the simple factors of $H/\mathcal{C}$ are of type $D_4$. In this case, the automorphism group of the Dynkin diagram permutes the set of endpoints, so the types $D^R_4$ and $D^H_4$ can not be distinguished in the same manner as before. The fundamental difference between the two types is the existence of a $\mathbb{Q}$-linear representation $V$ as above. We formalize this as follows.

Let $H/\mathcal{C} = \prod_{\iota \in I_H} H/\mathcal{C}_\iota$ and $I_{H,nc}$ and $I_{H,\iota}$ as in the general case. For each $\iota \in I_{H,nc}$ let $\mu_\iota : G_{m/\mathcal{C}} \to H/\mathcal{C}_\iota$ be the projection of the Hodge cocharacter. The conjugacy class of each $\mu_\iota$ is dual to an endpoint of the corresponding component of the Dynkin diagram. Let $\Delta$ be the set of these vertices, it contains exactly one endpoint of the $\iota$-component of the Dynkin diagram if $\iota \in I_{H,nc}$ and no vertices in the other components. Choose $\iota \in I_{H,nc}$, let $G^\text{der}_{\mathcal{A}/\mathcal{C}_\iota}$ be the almost simple factor of $G^\text{der}_{\mathcal{A}/\mathcal{C}}$ projecting onto $H/\mathcal{C}_\iota$ and let $\omega$ be the highest weight of some non trivial factor of the representation of $G^\text{der}_{\mathcal{A}/\mathcal{C}_\iota}$ on $H^1_B(A(\mathcal{C}), \mathbb{Q})$. As the Hodge cocharacter acts on $H^1_B(A(\mathcal{C}), \mathbb{C})$ with two weights, the arguments of [Del79],
1.3, 2.3] imply that \( \varpi \) corresponds to an endpoint \( \alpha \) of the Dynkin diagram of \( H \) and that the \( \mathcal{O}_Q \)-orbit \( \Delta' \) of \( \alpha \) does not meet \( \Delta \). In particular, there exists a \( \mathcal{O}_Q \)-stable set \( \Delta' \) of endpoints such that \( \Delta \cap \Delta' = \emptyset \). The set \( \Delta' \) contains at least one endpoint in each connected component of the Dynkin diagram.

If there exists a \( \mathcal{O}_Q \)-stable set \( \Delta'_{\text{max}} \) of endpoints of the Dynkin diagram with \( \Delta \cap \Delta'_{\text{max}} = \emptyset \) containing two endpoints in each connected component, then we say that \( H \) and the factors of \( H/R \) and \( H/C \) are of type \( D^R \). If there does not exist such a \( \Delta'_{\text{max}} \), they are of type \( D^H \).

Assume that \( H \) is a factor of \( G^{\text{ad}}_{A/C} \) of type \( D^R_4 \), and let \( \iota \in I_H \). Then the \( \iota \)-component of the Dynkin diagram contains two endpoints \( \alpha_\iota \) and \( \beta_\iota \in \Delta'_{\text{max}} \). Let \( V/C_{\iota} \) be the direct sum of the representations of the universal cover \( \tilde{H}/C_{\iota} \) with highest weights corresponding to \( \alpha_\iota \) and \( \beta_\iota \) respectively. It is a faithful representation of \( \tilde{H}/C_{\iota} \) and as \( \alpha_\iota, \beta_\iota \notin \Delta \), it follows from [Del79, 1.3.9] that, for \( \iota \in I_{H,nc} \), the lifting \( \tilde{\mu}_\iota \) of \( \mu_\iota \) to \( \tilde{H}/C_{\iota} \) acts on \( V/C_{\iota} \) with weights \( \pm 1/2 \). The product of the \( \tilde{H}/C_{\iota} \) and a multiple of the direct sum of the \( V/C_{\iota} \) descend to a \( Q \)-algebraic group \( \tilde{H} \) with a faithful \( Q \)-linear representation. This cover of a \( Q \)-simple factor of \( G^{\text{ad}}_A \) of type \( D^H_4 \) is the \( h \)-maximal cover of \( G^{\text{ad}}_A \).

2.8 Remark. For all factors \( H/C_{\iota} \) of type \( D \), the representation \( V/C_{\iota} \) is self dual. As in the case of the factors of types \( A, B \) and \( C \) this can be read off from [Pin98, Table 4.2].

2.9 Theorem. Let \( A/C \) be an abelian variety and let \( (G_A, h) \) be the associated Mumford–Tate datum. Then the following conditions are equivalent.

1. \( A \) is essentially Mumford–Tate unliftable.

2. The group \( G^{\text{der}}_{A/C} \) is the product of its simple factors. The simple factors of types \( A_k, B_k, C_k \) and \( D_k^R \) are simply connected and the factors of type \( D_k^H \) are \( h \)-maximal in the sense defined above.
2.10 Definition. Let $A/\mathbb{C}$ be an abelian variety, $G_A$ its Mumford–Tate group and $V_A = H^1_B(A(\mathbb{C}), \mathbb{Q})$. If $G_A^{\text{ad}}$ is $\mathbb{Q}$-simple, then we say that $A$ is Mumford–Tate decomposed if the following conditions hold.

- There are a totally real field $K_0$ and an absolutely simple algebraic group $G^s$ over $K_0$ such that $G_A^{\text{der}} \cong \text{Res}_{K_0/\mathbb{Q}} G^s$.

- There is a faithful representation $V^s$ of $G^s$ such that the representation of $G_A^{\text{der}}$ on $V$ is isomorphic to $\text{Res}_{K_0/\mathbb{Q}} V^s$.

- There is no proper non-trivial abelian subvariety $B$ of $A$ with Mumford–Tate group $G_B$ verifying $G_B^{\text{der}} = G_A^{\text{der}}$.

We say that an abelian variety $A/\mathbb{C}$ is Mumford–Tate decomposed if it is isogenous to a product $\prod A_i$ such that each $A_i$ is Mumford–Tate decomposed with $G_A^{\text{ad}}_i$ simple and if $G_A^{\text{ad}}$ is the product of the $G_A^{\text{ad}}_i$. An abelian variety over a number field $F \subset \mathbb{C}$ is Mumford–Tate decomposed if $A/F$ is Mumford–Tate decomposed.

2.11 Remark. The upshot of this definition is that the abelian varieties arising from the construction of [Del79, 2.3] are Mumford–Tate decomposed. The representation of $G_A^{\text{der}}$ on $H^1_B(A(\mathbb{C}), \mathbb{Q})$ is a direct sum of the representations of the $G_A^{\text{der}}_i$ constructed in 2.3 and 2.5. See also the proofs given below.

The notion of being (essentially) Mumford–Tate unliftable is a condition on the Mumford–Tate datum of an abelian variety whereas the notion of being Mumford–Tate decomposed pertains to the action of the Mumford–Tate group on the first Betti cohomology group.

2.12 Theorem. For every abelian variety $A/\mathbb{C}$ there exists a weak Mumford–Tate lift $B/\mathbb{C}$ of $A$ such that $B$ is essentially Mumford–Tate unliftable and Mumford–Tate decomposed.

Proofs of 2.9 and 2.12. These results can be derived from the work of Satake, see for example [Del79]. The same argument can be found in [Vas03], see §4 and paragraphs 6.3 and 6.4 in particular. The strategy of the proof is as follows. It is first shown that the condition 2.9.2 implies the condition 2.9.1. We then prove that any abelian variety admits a weak Mumford–Tate lift satisfying 2.9.2 and which is Mumford–Tate decomposed. Thanks to the fact that 2.9.2 implies 2.9.1.
this M-T lift is also M-T unliftable. Finally, the proof that the condition 2.9.1 implies 2.9.2 is a formality.

First assume that $A$ verifies the conditions of 2.9.2. We will show that it is essentially M-T unliftable, so let $B$ be a weak Mumford–Tate lift of $A$ and let $G_B$ be its Mumford–Tate group. It has to be proved that the map $G_B^{\text{der}} \rightarrow G_A^{\text{der}}$ is an isomorphism. It suffices to prove this after extension of scalars to $\mathbb{C}$ and as the only non-simply connected factors of $G_A^{\text{der}}/\mathbb{C}$ are the factors of type $D^H_k$, we only need to consider these factors.

Consider a factor $H/\mathbb{C}$ of $G_A^{\text{der}}$ of type $D^H_k$ to which the Hodge cocharacter projects non-trivially, assuming at first that $k \geq 5$. As we saw, the conjugacy class of the projection of the Hodge cocharacter to $H/\mathbb{C}$ is dual to one of the vertices $\alpha_{k-1}$ or $\alpha_k$ of the Dynkin diagram. Let $\tilde{H}/\mathbb{C}$ be the factor of $G_B^{\text{der}}$ mapping onto $H/\mathbb{C}$. An appropriate direct factor $W$ of the representation of $\tilde{H}/\mathbb{C}$ on $H^1_B(B(\mathbb{C}), Q) \otimes \mathbb{C}$ provides a faithful representation on which $\tilde{\mu}$ acts with two weights $\pm 1/2$. As explained in 2.5, it follows from the tables in [Del79, 1.3.9], that the highest weight of every irreducible direct factor of $W$ is $\varpi_1$ and hence that $\tilde{H}/\mathbb{C}$ is isomorphic to $H/\mathbb{C}$. This proves that 2.9.2 implies 2.9.1 if $k \geq 5$.

In the case where $k = 4$, first note that $G_A^{\text{der}}$ is the product of its $\mathbb{Q}$-simple factors. Let $H$ be a simple factor of type $D^H_4$ and let $\tilde{H}$ be the factor of $G_B^{\text{der}}$ mapping onto $H$. Consider $H^1_B(B(\mathbb{C}), Q)$ as a representation of $\tilde{H}$ and let $W$ be a direct factor which is a faithful representation of $\tilde{H}$. Let $\tilde{H}_{/\mathbb{C},t}$ be any factor of $\tilde{H}/\mathbb{C}$ and let $W_{/\mathbb{C},t}$ be any irreducible direct factor of the restriction of $W \otimes \mathbb{C}$ to $\tilde{H}_{/\mathbb{C},t}$. Then the lifting to $\tilde{H}_{/\mathbb{C},t}$ of the Hodge cocharacter either acts trivially on $W_{/\mathbb{C},t}$ or with exactly two weights. It follows from 2.7 that $\tilde{H}$ is the $h$-maximal cover of $H$ and as $H$ was $h$-maximal by hypothesis it follows that $\tilde{H} \cong H$. This proves that 2.9.2 implies 2.9.1 in case $k = 4$.

We next show that if $A$ is any abelian variety, then there exists a weak Mumford–Tate lift $B$ of $A$ with Mumford–Tate group $G_B$ satisfying the conditions of 2.9.2 and which is Mumford–Tate decomposed. This fact readily follows from [Del79, 2.3] and the discussions in 2.3, 2.5 and 2.7, we recall the argument.

Fix a $\mathbb{Q}$-simple factor $H^{\text{ad}}$ of $G_A^{\text{ad}}$. It is of the form $\text{Res}_{K_0/\mathbb{Q}} H^{s,\text{ad}}$ for some totally real number field $K_0$ and some absolutely simple adjoint group $H^{s,\text{ad}}$ over $K_0$. As usual, decompose $H_{/\mathbb{C},t}^{\text{ad}} = \prod_{I_H} H_{/\mathbb{C},I_H}$ where $I_H$, $I_{H,\text{nc}}$ and $I_{H,c}$ are as before.

Unless $H_{/\mathbb{C},t}^{\text{ad}}$ is of type $D^H_k$, we let $\tilde{H}^{s,\text{der}}$ be the simply connected cover of
Lifting Galois representations of abelian varieties

17

$H^{s,\text{ad}}$. In the remaining case we let $\tilde{H}^{s,\text{der}}$ be the $h$-maximal cover of $H^{s,\text{ad}}$, in the sense of [2,3] resp. [2,7]. In all cases, put

$$\tilde{H}^{\text{der}} = \text{Res}_{K_0/Q} \tilde{H}^{s,\text{der}}.$$  

For each $\iota \in I_{H,nc}$, the projection of the Hodge cocharacter to $H^{\text{ad}}/C_{\iota}$ lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of $\tilde{H}^{\text{der}}/C_{\iota}$. The faithful representation $V$ of $\tilde{H}^{\text{der}}$ constructed in [2,3] and [2,5] resp. [2,7] is of the form $V = \text{Res}_{K_0/Q} V^s$ for some faithful $\mathbb{Q}$-linear representation $V^s$ of $H^{s,\text{der}}$. These data have the following properties.

- There is a map $\tilde{H}^{\text{der}} \to G^{\text{der}}_A$ such that $\tilde{\mu} = \prod_{\iota \in I_{H,nc}} \tilde{\mu}_\iota$ lifts the Hodge cocharacter.

- If $\iota \in I_{H,nc}$, and if $W$ is an irreducible factor of $V_{C,\iota}$, then the cocharacter $\tilde{\mu}$ acts either trivially on $W$ or with two rational weights $r$ and $r + 1$.

There exist a torus $T$ over $\mathbb{Q}$ acting $\tilde{H}^{\text{der}}$-linearly on $V$ and a quasi-cocharacter $\mu_T$ of $T/C$ such that the product $\tilde{\mu}_T$ acts trivially on the $V_{C,\iota}$ for $\iota \in I_{H,nc}$ and with weights $\pm 1/2$ for $\iota \in I_{H,nc}$. In fact, $T$ is characterized by the condition that $T/C$ is the group of the automorphisms of $V \otimes C$ acting by scalar multiplication on each isobaric component of $V \otimes C$ as $\tilde{H}^{\text{der}}$-representation. The existence of $\mu_T$ follows from the fact that if $\tilde{\mu}$ acts non-trivially on an isobaric component then it acts with two weights $r$ and $r + 1$. Let $H'$ be the image of $\tilde{H}^{\text{der}} \times T$ in $GL(V)$ and let $\mu' = \tilde{\mu}_T$.

We choose a quadratic and totally imaginary extension $L$ of $K_0$ and consider the $\mathbb{Q}$-algebraic group $L^\times$. The natural action of $L^\times$ on $L$ gives rise to a $\mathbb{Q}$-linear representation $W$ of $L^\times$. One has

$$(L^\times)/C \cong \bigoplus_{\iota \in I} G^2_{m/C}$$

and one defines a quasi-cocharacter $\nu$ of $(L^\times)/C$ by $\nu_\iota(z) = (z^{1/2}, z^{1/2})$ for $\iota \in I_{nc}$ and $\nu_\iota(z) = (z, 1)$ for $\iota \in I_c$. The action of $H' \times L^\times$ on $V \otimes K_0 W$ defines a faithful representation of a quotient $\tilde{H}$ of $H' \times L^\times$ with derived group $\tilde{H}^{\text{der}}$ in which $(\tilde{\mu}, \nu)$ acts with weights 0 and 1. Let $\tilde{h}: S \to \tilde{H}_R$ be defined by

$$\tilde{h}(z, \bar{z}) = (\mu' \nu)(z)(\mu' \nu)(\bar{z}).$$

Shrinking the centre of $\tilde{H}$, we may assume that the image of $\tilde{h}$ is Zariski dense in $\tilde{H}$. We claim that this defines the Shimura datum $(\tilde{H}, \tilde{h})$ associated to an abelian
variety \( B_1 \). To see why this is the case, note that \( \tilde{h} \) defines a \( \mathbb{Q} \)-Hodge structure of type \((1, \, 0), (0, \, 1)\) on \( V \otimes_{\mathbb{K}_0} W \). In order to establish that this Hodge structure comes from an abelian variety it is sufficient to show that it is polarizable, cf. [Del72, 2.3]. First use that, by loc. cit. 2.11, the element \( \mathrm{ad}(h_A(i)) \) defines a Cartan involution of \( G^\mathrm{der}_A \) and hence that \( \mathrm{ad}(\tilde{h}(i)) \) is a Cartan involution of \( \tilde{H}^\mathrm{der} / \mathbb{R} \). Next, one checks that the weight \( w_h : \mathbb{G}_m \to \tilde{H} \) is central, defined over \( \mathbb{Q} \) and that the quotient of the center of \( \tilde{H} \) by \( w_h(\mathbb{G}_m) \) is compact. The last statement is deduced from the fact that this center is contained in a product of CM tori. It now follows that \( \mathrm{ad}(\tilde{h}(i)) \) is a Cartan involution of \( \tilde{H}^\mathrm{der} / w(\mathbb{G}_m) \) and by [Del79, 1.1.18(b)] this implies that the Hodge structure on \( V \otimes_{\mathbb{K}_0} W \) is polarizable.

The group \( \tilde{H}^\mathrm{ad} \) is \( \mathbb{Q} \)-simple, \( \tilde{H}^\mathrm{der} \) is its \( h \)-maximal cover and \( B_1 \) is Mumford–Tate decomposed by construction. Since \( G_A \) is the Mumford–Tate group of an abelian variety, \( G^\mathrm{der}_A \) is a quotient of the \( h \)-maximal cover of \( G^\mathrm{ad}_A \), so there is an isogeny \( \tilde{H}^\mathrm{der} \to G^\mathrm{der}_A \) lifting \( \tilde{H}^\mathrm{ad} \to G^\mathrm{ad}_A \).

Applying this to all \( \mathbb{Q} \)-simple factors of \( G^\mathrm{ad}_A \) we obtain Mumford–Tate decomposed abelian varieties \( B_i \) such that \( B = \prod B_i \) is a weak M-T lift of \( A \) verifying the conditions of 2.9.2. By construction, if \( G_B \) (resp. \( G_{B_i} \)) denotes the Mumford–Tate group of \( B \) (resp. \( B_i \)), then \( G^\mathrm{der}_B = \prod G^\mathrm{der}_{B_i} \). The first part of this proof implies that \( B \) is essentially M-T unifiable. This terminates the proof of theorem 2.12.

Finally, let \( A \) be essentially M-T unifiable, i.e. the condition 2.9.1 is satisfied. The theorem 2.12 implies that \( A \) has a weak M-T lift \( B \) with Mumford–Tate group \( G_B \) satisfying the condition of 2.9.2. As \( A \) is essentially M-T unifiable, we must have \( G^\mathrm{der}_A \cong G^\mathrm{der}_B \), which implies that \( G_A \) also verifies 2.9.2. \( \square \)

2.13 Remarks.

2.13.1 In the above proof, fix a \( \mathbb{Q} \)-simple factor \( H^\mathrm{ad} \) of \( G^\mathrm{ad}_{A} \), let \( \iota \in I_H \) and consider an irreducible factor \( W \) of the representation \( V_{C, \iota} \) of \( H^\mathrm{der}_{C, \iota} \). It follows from [Del79, 1.3] that the highest weight of \( W \) is a fundamental weight of \( H^\mathrm{der}_{C, \iota} \). More precisely, according to the type of \( H^\mathrm{der}_{C, \iota} \) and the quasi-cocharacter \( \tilde{\mu}_\iota \), it is the weight given by the tables 1.3.9 of Deligne’s paper or [Pin98, Table 4.2].

2.13.2 With the same notations, assume that \( H^\mathrm{ad} \) is not of type \( A_k \) with \( k \geq 2 \). In the above construction of the essentially M-T unifiable variety \( B_i \) corresponding to this factor, we then have \( r = 1/2 \). This means that the construction of the
intermediate group $H'$ can be shunted in this case. It follows that the Mumford–Tate group of $B$ is contained in the image of $H^{\text{der}} \times L^\times$ in $\text{GL}(V \otimes_{K_0} W)$.

This argument is also valid for factors of type $A_k$ for which all the numbers $r$ are equal to $1/2$.

2.13.3 Instead of using [Del72] to prove the fact $\tilde{h}$ defines a polarizable Hodge structure on $V \otimes_{K_0} W$, one may also explicitly construct a polarization. This is not difficult, using the autoduality of $V$ as representation of $H^{\text{der}}$.

2.14 Examples.

2.14.1 Let $A/\mathbb{C}$ be an abelian variety arising from Mumford’s construction, see [Mum69]. In this case one has $G_{A/\mathbb{Q}}^{\text{der}} \cong \SL_2^3/\tilde{N}$, where

$$\tilde{N} = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i = \pm 1 \text{ for } i = 1, 2, 3 \text{ and } \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1\}.$$  

The Mumford–Tate group of the M-T unliftable and M-T decomposed weak M-T lift $B$ of $A$ satisfies $G_{B/\mathbb{Q}}^{\text{der}} \cong \SL_2^3$. This is the example studied in detail in [Noo01].

2.14.2 There exist simple abelian varieties $A/\mathbb{C}$ for which $G_A^{\text{ad}}$ is not simple. One can construct such an example where $G_A^{\text{der}} \cong G_1 \times G_1/N$ with

$$G_1/\mathbb{Q} \cong G_2/\mathbb{Q} \cong \SL_2^2$$

and $N/\mathbb{Q} = \{(+1, +1)\} \subset \SL_2^2 \cong G_i$ embedded diagonally into $G_1 \times G_2$. In this case, the M-T unliftable and M-T decomposed weak M-T lift $B$ of $A$ is a product $B = B_1 \times B_2$ and $G_{B_i}^{\text{der}} = G_i$.

2.14.3 There exist simple abelian varieties $A/\mathbb{C}$ for which $G_A^{\text{ad}}$ is absolutely simple of type $D_R^k$, with $k$ even, $\mathcal{O}_Q$ acting trivially on the Dynkin diagram and where $V = H^1_B(A(\mathbb{C}), \mathbb{Q})$ decomposes over $\mathcal{O}$ as a multiple of the representation of $G_A^{\text{der}}$ with highest weight $\varpi_k$. Since $G_A^{\text{der}}$ acts faithfully on $V$, it is not simply connected. Let $B$ be the M-T unliftable and M-T decomposed weak M-T lift of $A$. Then $G_B^{\text{der}}$ is the universal cover of $G_A^{\text{der}}$ and $B$ is a product $B \cong A \times B'$, where $W = H^1_B(B'(\mathbb{C}), \mathbb{Q})$ decomposes over $\mathcal{O}$ as a multiple of the representation of $G_B^{\text{der}}$ with highest weight $\varpi_{k-1}$. 


2.15 Corollary. For every abelian motive $M/\mathbb{C}$ there exists an essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety $B/\mathbb{C}$ which provides a weak Mumford–Tate lift for $M$.

Proof. There exist an abelian variety $A/\mathbb{C}$ and a surjection of the corresponding Mumford–Tate groups $G_A \to G_M$ commuting with the maps $h_A$ and $h_M$. Let $B'/\mathbb{C}$ be the essentially Mumford–Tate unliftable and Mumford–Tate decomposed weak Mumford–Tate lift for $A$ provided by theorem 2.13. This gives a morphism $\pi_{B'} : G^\text{der}_{B'} \to G^\text{der}_M$. The fact that $B'$ is Mumford–Tate decomposed implies that there is an isogeny $B' \sim \prod_{i \in I} B_i$, where the $B_i$ are M-T decomposed abelian varieties such that the groups $G^\text{der}_{B_i}$ are $\mathbb{Q}$-simple, and such that $G^\text{der}_{B'} \cong \prod_{i \in I} G^\text{der}_{B_i}$. Let $J \subset I$ be the subset of indices $i$ such that $G^\text{der}_{B_i}$ is not in the kernel of $\pi_{B'}$ and let $B = \prod_{i \in J} B_i$. Then $G_B = \prod_{i \in J} G_{B_i}$ and $(\pi_{B'})_{|G^\text{der}_B}$ is an isogeny from $G^\text{der}_B$ onto $G^\text{der}_M$, so $B$ verifies the condition of the corollary. \qed

3 Mumford–Tate liftings and motives

3.1 Proposition. Suppose that $A/\mathbb{C}$ and $B/\mathbb{C}$ are abelian varieties over $\mathbb{C}$, let $(G_A, h_A)$ and $(G_B, h_B)$ be the associated Mumford–Tate data and assume that there exists an isomorphism $G^\text{ad}_A \cong G^\text{ad}_B$ such that $\pi^\text{ad}_A \circ h_A = \pi^\text{ad}_B \circ h_B$.

Let $F \subset \mathbb{C}$ be an algebraically closed field. Then there exists an abelian variety $A/F$ such $A \otimes_F \mathbb{C} = A/\mathbb{C}$ if and only if there exists an abelian variety $B/F$ such that $B \otimes_F \mathbb{C} = B/\mathbb{C}$.

Proof. This is proved as in [Noo01, 4.5]. Let $h_A : S \to G_A/\mathbb{R}$ and $h_B : S \to G_B/\mathbb{R}$ be the maps defining the Hodge structures on $H^1(B(C), \mathbb{Q})$ and $H^1(B(C), \mathbb{Q})$ respectively. Let $X_A$ and $X_B$ be the $G_A(\mathbb{R})$- and $G_B(\mathbb{R})$-conjugacy classes of $h_A$ and $h_B$. The main theorem of [Del79] implies that for all compact open subgroups $K_A \subset G_A(\mathbb{A}_f)$ and $K_B \subset G_B(\mathbb{A}_f)$, one can construct quasi-canonical models $K_A M/Q(G_A, X_A)^0$ and $K_B M/Q(G_B, X_B)^0$ over $\overline{\mathbb{Q}}$ of the corresponding connected Shimura varieties. Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

For $K_A$ and $K_B$ sufficiently small, there exist “universal” abelian schemes $\mathcal{A} \to K_A M/Q(G_A, X_A)^0$ and $\mathcal{B} \to K_B M/Q(G_B, X_B)^0$ and points

$$a \in K_A M/Q(G_A, X_A)^0(\mathbb{C}), \quad b \in K_B M/Q(G_B, X_B)^0$$

such that $A = \mathcal{A}_a$ and $B = \mathcal{B}_b$. 
Let $G^\text{ad} = G^\text{ad}_A \cong G^\text{ad}_B$. One can choose $K_A$, $K_B$ and $K^\text{ad} \subset G^\text{ad}(A_f)$ such that, in addition to the above conditions, there is a diagram

$$K_A M/\mathbb{Q}(G_A, X_A)^0 \rightarrow K^\text{ad} M/\mathbb{Q}(G^\text{ad}, X^\text{ad})^0 \leftarrow K_B M/\mathbb{Q}(G_B, X_B)^0,$$

in which each arrow is a quotient map for the action of a finite group and hence is a finite morphism (cf. [Del79, 2.7.11(b)]) and such that $a$ and $b$ have the same image in $K^\text{ad} M/\mathbb{Q}(G^\text{ad}, X^\text{ad})(\mathbb{C})^0$. The proposition follows. \hfill $\square$

3.2 Corollary. The statement of the proposition is true in particular if $B/\mathbb{C}$ provides a (weak) Mumford–Tate lift of $A/\mathbb{C}$.

3.3 Proposition. Let $F \subset \mathbb{C}$ be an algebraically closed field and let $A$ and $B$ abelian varieties over $F$ such that $B/\mathbb{C}$ provides a Mumford–Tate lift of $A/\mathbb{C}$. Then the motive $h^1(A)$ belongs to the category $\langle h^1(B), \mathbb{Q}(1) \rangle$.

The map $G_B \rightarrow G_A$ induced by the Betti realization of this inclusion is the map given by the structure of $B$ as a Mumford–Tate lift of $A$.

**Proof.** This generalises [Noo01, 4.8–4.11].

As explained in [1.5], it follows from [DM82, 6.25] that the Betti realization induces an equivalence of $\langle h^1(B), \mathbb{Q}(1) \rangle$ with $\langle H^1_B(B(\mathbb{C}), \mathbb{Q}), \mathbb{Q}(1) \rangle$, the tannakian subcategory of the category of Hodge structures generated by $H^1_B(B(\mathbb{C}), \mathbb{Q})$ and the Tate Hodge structure $\mathbb{Q}(1)$. To prove the proposition, it therefore suffices to show that the Hodge structure $H^1_B(A(\mathbb{C}), \mathbb{Q})$ belongs to $\langle H^1_B(B(\mathbb{C}), \mathbb{Q}), \mathbb{Q}(1) \rangle$. For the rest of the proof, we write $V_A = H^1_B(A(\mathbb{C}), \mathbb{Q})$ and $V_B = H^1_B(B(\mathbb{C}), \mathbb{Q})$.

By definition of the Mumford–Tate group, the underlying vector space of any object of the category $\langle V_B, \mathbb{Q}(1) \rangle$ of Hodge structures naturally carries the structure of a representation of $G_B$. This gives a $\otimes$-equivalence of $\langle V_B, \mathbb{Q}(1) \rangle$ with the subcategory $\langle V_B, \mathbb{Q}(1) \rangle_{\text{Rep}}$ of $\text{Rep}_{\mathbb{Q}}(G_B)$. As we saw in [1.5], for any object of $W$ the latter category, the Hodge structure is given by composing the morphism $h_B: S \rightarrow G_B/\mathbb{R}$ with the action of $G_B$ on $W$.

Let $(G_A, h_A)$ be the Mumford–Tate datum associated to $A$. By hypothesis, there exists a central morphism $\pi: G_B \rightarrow G_A$ such that $h_A = \pi_\mathbb{R} \circ h_B$. This makes every $\mathbb{Q}$-linear representation $W$ of $G_A$ into a representation of $G_B$ and,
for any such \( W \), it carries the Hodge structure on \( W \) defined by \( h_A \) into the Hodge structure defined by \( h_B \). To prove the proposition it is therefore sufficient to show that \( V_A \), considered as representation of \( G_B \), belongs to \( \langle V_B, \mathbb{Q}(1) \rangle_{\text{Rep}} \).

This follows immediately, because \( V_B \) is a faithful representation of \( G_B \) and \( V_B \otimes \mathbb{Q}(1) \) is its dual, so one has

\[
\langle V_B, \mathbb{Q}(1) \rangle_{\text{Rep}} = \text{Rep}_\mathbb{Q}(G_B),
\]

cf. [DM82, 2.20] and its proof.

\[\text{3.4 Proposition.}\] Let \( A \) and \( B \) be abelian varieties over an algebraically closed field \( F \subset \mathbb{C} \) such that \( B/\mathbb{C} \) provides a weak Mumford–Tate lift of \( A/\mathbb{C} \). Then \( h^1(A) \) belongs to \( \langle h^1(B), (\mathbb{CM})_F \rangle \).

Taking the Betti realization, this inclusion induces a map between the corresponding Mumford–Tate groups. On the derived group, this map is the map \( \pi_{\text{der}}: G^\text{der}_B \to G^\text{der}_A \) given by the structure of \( B \) as weak Mumford–Tate lift of \( A \).

\[\text{Proof.}\] Let \( T_A \) and \( T_B \) be the connected components of the centres of \( G_A \) and of \( G_B \) respectively. We write \( V_A = H^1_B(A(\mathbb{C}), \mathbb{Q}) \) and \( V_B = H^1_B(B(\mathbb{C}), \mathbb{Q}) \) as in the proof of 3.3 and consider all spaces we encounter as representations of \( H = T_A \times T_B \times G^\text{der}_B \). In the case of \( V_A \), the group \( H \) acts via \( H \rightarrow T_A \times G^\text{der}_B \rightarrow G_A \) and in the case of \( V_B \) it acts via \( H \rightarrow T_B \times G^\text{der}_B \rightarrow G_B \).

The groups \( T_A \) and \( T_B \) act on \( V_A \) and \( V_B \). Composing these representations with the projections \( H \rightarrow T_A \) and \( H \rightarrow T_B \) respectively we obtain representations \( V^c_A \) and \( V^c_B \) of \( H \). As \( W = V_B \oplus V^c_A \oplus V^c_B \) is a faithful representation of \( H \), it follows that \( V_A \) belongs to the subcategory of \( \text{Rep}_\mathbb{Q}(H) \otimes \)-generated by \( W \) and its dual. As there is an isomorphism of Hodge structures \( V^\vee_B \cong V_B \otimes \mathbb{Q}(1) \), this in turn implies that \( V_A \) belongs to the tannakian subcategory of \( \text{Rep}_\mathbb{Q}(H) \) generated by \( V_B \) and the abelian representations of \( H \), i.e. the representations where \( H \) acts through a commutative quotient.

We fix an irreducible direct factor \( W_2 \) of \( V_A \) in \( \text{Rep}_\mathbb{Q}(H) \). There is an irreducible abelian \( \mathbb{Q} \)-linear representation \( W_3 \) of \( H \) such that \( W_2 \) is isomorphic to a subobject of \( W_3 \otimes V^\vee_B \). Replace \( V^\vee_B \) by an irreducible direct factor \( W_1 \) such that \( W_2 \) still is a subobject of \( W_3 \otimes W_1 \). The projection of \( W_1 \otimes W^\vee_1 \) onto the trivial...
representation induces a surjection $W_3 \otimes W_1 \otimes W_1^\vee \twoheadrightarrow W_3$. The composite of this surjection with the map

$$W_2 \otimes W_1^\vee \hookrightarrow W_3 \otimes W_1 \otimes W_1^\vee$$

deduced from the inclusion $W_2 \hookrightarrow W_3 \otimes W_1$ is a map $W_2 \otimes W_1^\vee \rightarrow W_3$. It is not difficult to check that this map is non-trivial and as $W_3$ was assumed to be irreducible, this implies that the map is surjective. This proves that $W_3$ belongs to the tannakian subcategory of $\text{Rep}_{\mathbb{Q}}(H)$ which is $\otimes$-generated by $W_1^\vee$ and $W_2$.

It follows that the action of $H$ on $W_3$ factors through $G_A \times G_B$ and as $W_3$ is an abelian representation of $H$, it is also an abelian representation of $G_A \times G_B$. This means that $W_3$ carries a Hodge structure of CM-type and hence that $V_A$ belongs to $\langle V_B, (\text{CM})-\text{Hodge} \rangle$ as required.

3.5 Corollary. Let $A$ and $B$ be abelian varieties over $F$ and let $G_A$ and $G_B$ be their respective Mumford–Tate groups. Assume that $B$ provides a weak Mumford–Tate lift of $A$ and that $G_A^{\text{der}} \cong G_B^{\text{der}}$. Then the categories $\langle h^1(A), (\text{CM})_F \rangle$ and $\langle h^1(B), (\text{CM})_F \rangle$ coincide.

3.6 Corollary. Let $F \subset \mathbb{C}$ be an algebraically closed field and let $M$ be an object of $(\text{AV})_F$. Then there exist an essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety $A$ over $F$ such that $M$ belongs to $\langle h^1(A), (\text{CM})_F \rangle$.

4 Motivic liftings of Galois representations

4.1 Assume that $F \subset \mathbb{C}$ is a number field, $M$ an abelian motive over $F$ and $(G_M, h_M)$ the Mumford–Tate datum associated to $M$. As we recalled in [1.6], the $p$-adic Galois representation associated to the étale realization of $M$ factors through a morphism

$$\rho_{M,p} : G_F \longrightarrow G_M(\mathbb{Q}_p).$$

There exists a finite extension $F' \supset F$ such that the Mumford–Tate group of $M_{F'}$ is connected. The Mumford–Tate group of $M_{F'}$ is then equal to the connected component of $G_M$. In what follows, we will assume that $G_M$ is already connected (replacing $F$ by a finite extension if necessary).
It follows from theorems of Tsuji, [Tsu99] and De Jong, [dJ96] that, for every $p$, the representation $\rho_{M,p}$ of $\mathcal{G}_F$ on $H_{\text{ét}}(M,\mathbb{Q}_p)$ is geometric in the sense of Fontaine and Mazur, [FM95, §1]. Here a representation of $\mathcal{G}_F$ on a finite dimensional $\mathbb{Q}_p$-vector space is called geometric if

- it is unramified outside a finite set of non-archimedean places of $F$ and
- for each valuation $\mathfrak{v}$ of $F$ of residue characteristic $p$, the restriction to the inertia group $\mathcal{I}_{F,\mathfrak{v}}$ is potentially semi-stable (cf. [Font94]).

More generally, for a linear algebraic group $G$ over $\mathbb{Q}_p$ and a continuous morphism $\rho_p : \mathcal{G}_F \rightarrow G(\mathbb{Q}_p)$, we will say that $\tilde{\rho}_p$ is geometric if there exists a faithful representation $V_p$ of $G$ such that the resulting representation of $\mathcal{G}_F$ on $V_p$ is geometric. This is the case if and only if the representation of $\mathcal{G}_F$ on $W_p$ is geometric for any representation $W_p$ of $G$.

4.2 Definition. Assume that $G$ and $\tilde{G}$ are linear algebraic groups over $\mathbb{Q}$ or over $\mathbb{Q}_p$ and that $\rho_p : \mathcal{G}_F \rightarrow G(\mathbb{Q}_p)$ and $\tilde{\rho}_p : \mathcal{G}_F \rightarrow G(\mathbb{Q}_p)$ are geometric Galois representations. We will say that $\tilde{\rho}_p$ is a geometric lift of $\rho_p$ if there exists a central isogeny $\pi : \tilde{G} \rightarrow G$ such that $\rho_p = \pi \circ \tilde{\rho}_p$. If $\rho_p$ does not admit any geometric lift with ker $\pi$ non-trivial, it will be called geometrically unliftable.

If there exists a central isogeny $\pi^{\text{der}} : \tilde{G}^{\text{der}} \rightarrow G^{\text{der}}$ such that there is an equality $\pi^{\text{ad}} \circ \tilde{\rho}_p \circ \pi^{\text{ad}} = \pi^{\text{ad}} \circ \rho_p$ then $\tilde{\rho}_p$ is said to be a weak geometric lift of $\rho_p$. As in [2.1], the maps $\pi^{\text{ad}}$ and $\tilde{\pi}^{\text{ad}}$ are the projections $G \rightarrow G^{\text{ad}}$ and $\tilde{G} \rightarrow \tilde{G}^{\text{ad}}$. The central isogeny $\pi^{\text{der}}$ induces an isomorphism $G^{\text{ad}} \cong \tilde{G}^{\text{ad}}$, giving a sense to the equality $\pi^{\text{ad}} \circ \tilde{\rho}_p = \pi^{\text{ad}} \circ \rho_p$. We will say that $\rho_p$ is essentially geometrically unliftable if it does not admit a weak geometric lift with ker $\pi^{\text{der}}$ non-trivial.

4.3 Proposition. Suppose that $F \subset \mathbb{C}$ is a number field, $A/F$ an abelian variety with connected Mumford–Tate group $G_A$ and that $B$ is an abelian variety over a finite extension $F' \supset F$ such that $B/\mathbb{C}$ provides a Mumford–Tate lift of $A/\mathbb{C}$.

The number field $F'$ can be chosen such that, for every prime number $p$, the Galois representation $\rho_{A,p}$ of $\mathcal{G}_{F'}$ on $H^1_{\text{ét}}(A_{F'},\mathbb{Q}_p)$ belongs to the subcategory of $\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{F'}) \otimes -$-generated by $H^1_{\text{ét}}(B_{F'},\mathbb{Q}_p)$ and $\mathbb{Q}_p(1)$. For every $p$, the morphism $G_B \rightarrow G_A$ then realises the Galois representation $\rho_{B,p}$ as a geometric lift of $\rho_{A,p}$.

Proof. By proposition 5.3, the motive $h^1(A_{\hat{F}})$ belongs to $\langle h^1(B_{\hat{F}},\mathbb{Q}(1)) \rangle$ and, for $F'$ large enough, this is already the case over $F'$. Taking the $p$-adic étale re-
alizations, this implies the corresponding statement for the Galois representations. The inclusion of $h^1(A_{F'})$ in $\langle h^1(B_{F'}), \mathbb{Q}(1) \rangle$ corresponds to a morphism $G_B \to G_A$ and taking the $p$-adic étale realizations this gives rise to a commutative diagram

$$
\begin{array}{ccc}
G_B(\mathbb{Q}_p) & \xrightarrow{\rho_{B,p}} & G_{A}(\mathbb{Q}_p) \\
\downarrow & & \downarrow \\
\mathcal{G}_{F'} & \xrightarrow{\rho_{A,p}} & \mathcal{G}_F
\end{array}
$$

proving the proposition.

\[4.4\] As above, assume that $F$ is a number field contained in $\mathbb{C}$. Following the notations of [FM95, §6], let $(\text{CM})\text{-Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$, or $(\text{CM})\text{-Rep}$ if no confusion is likely, denote the tannakian subcategory of $\text{Rep}_{Q_p}(\mathcal{G}_F)$ consisting of the potentially abelian geometric representations, in other words, the geometric representations such that the restriction to a subgroup of $\mathcal{G}_F$ of finite index has abelian image. It follows from [FM95, §6] that $(\text{CM})\text{-Rep}$ is the tannakian subcategory of $\text{Rep}_{Q_p}(\mathcal{G}_F)$ generated by the representations factoring through finite groups and the representations of the form $H^1_{et}(A, \mathbb{Q}_p)$ for $A/F$ an abelian variety which is potentially of CM-type. Thus, $(\text{CM})\text{-Rep}$ is the tannakian category of the $p$-adic étale realizations of the objects of $(\text{CM})_F$.

\[4.5\] Proposition. Suppose that $F \subset \mathbb{C}$ is a number field, $A/F$ an abelian variety with connected Mumford–Tate group $G_A$ and that $B$ is an abelian variety over a finite extension $F' \supset F$ such that the Mumford–Tate group $G_B$ is connected and $B/\mathbb{C}$ provides a weak Mumford–Tate lift of $A/\mathbb{C}$.

For every prime number $p$, the representation $\rho_{A,p}$ of $\mathcal{G}_F$ on $H^1_{et}(A_{\overline{F}}, \mathbb{Q}_p)$ is an object of the subcategory $\langle H^1_{et}(B_{\overline{F}}, \mathbb{Q}_p), (\text{CM})\text{-Rep}_{\mathbb{Q}_p}(\mathcal{G}_F) \rangle$ of $\text{Rep}_{Q_p}(\mathcal{G}_F)$. Via the map $G^\text{der}_B \to G^\text{der}_A$, the Galois representation $\rho_{B,p}$ provides a weak geometric lift of $(\rho_{A,p})_{|\mathcal{G}_{F'}}$.

\[Proof.\] It follows from proposition [3.4] that there is a finite extension $F''$ of $F'$ such that the motive $h^1(A_{F''})$ belongs to $\langle h^1(B_{F''}), (\text{CM})_{F''} \rangle$. As the Mumford–Tate group of $\langle h^1(B_{F''}), (\text{CM})_{F''} \rangle$ is a pro-algebraic group with derived group $G^\text{der}_B$, this gives rise to an isogeny $\pi: G^\text{der}_B \to G^\text{der}_A$ and $\pi$ induces an isomorphism $\pi^\text{ad}: G^\text{ad}_B \to G^\text{ad}_A$. Taking $p$-adic étale realizations, there is a commutative dia-
This proves all statements of the proposition with $F''$ instead of $F'$.

Faltings’ theorem, [Fal83, Satz 4] implies that if $C$ is the commuting algebra of $G_A$ in $\text{End}(H^1_B(A(C), \Q))$, then $C \otimes \Q$ is the commuting algebra of $\rho_{A,p}(\Ad_{F''})$ in $\text{End}(H^1_{\text{et}}(A(C), \Q_p))$. This implies that that the centralizer of the image of $\rho_{A,p}$ in $G^\text{ad}_{A}(\Q_p)$ is trivial. It follows from lemma [4.5] that the diagram [4.5.*] also commutes with $\Ad_{F''}$ replaced by $\Ad_{F'}$. Lemma [4.7] implies that the representation of $\Ad_{F'}$ on $H^1_{\text{et}}(A_F, \Q_p)$ is an object of $\langle H^1_{\text{et}}(B_F, \Q_p), (\CM)\text{-Rep}_{\Q_p}(\Ad_{F'}) \rangle$. \[\square\]

**4.6 Lemma.** Assume that $p$ is a prime number and that $G$ is a connected linear algebraic group over $\Q_p$. Let $F$ be a number field and let $\rho_1, \rho_2 : \Ad_F \to G(\Q_p)$ be Galois representations coinciding on $\Ad_{F'}$ for some finite extension $F'$ of $F$. Also assume that the centralizer of $\rho_1(\Ad_{F'}) = \rho_2(\Ad_{F'})$ in $G$ is trivial. Then $\rho_1 = \rho_2$ on $\Ad_F$.

Proof. It is sufficient to treat the case where $F'$ is a Galois extension of $F$.

Let $\delta : \Ad_F \to G(\Q_p)$ be defined by $\delta(\sigma) = \rho_1(\sigma)\rho_2(\sigma)^{-1}$. This map satisfies the cocycle condition $\delta(\sigma\tau) = \delta(\sigma)(\rho_2(\sigma)\delta(\tau)\rho_2(\sigma)^{-1})$. This implies that $\delta$ is constant on the classes $\sigma \Ad_{F'}$ for $\sigma \in \Ad_F$. As $\Ad_{F'}$ is normal, it follows that $\delta$ is also constant on the classes $\Ad_{F'}\sigma$ and it follows that for all $\sigma \in \Ad_F$ and $\tau \in \Ad_{F'}$ one has $\delta(\sigma) = \rho_2(\tau)\delta(\sigma)\rho_2(\tau)^{-1}$. Therefore $\delta(\sigma)$ lies in the centralizer of $\rho_2(\Ad_{F'})$ and we conclude that $\delta$ is trivial. \[\square\]

**4.7 Lemma.** Let $p$ be a prime number, $G_1$ and $G_2$ connected linear algebraic groups over $\Q_p$ and $\pi^\text{der} : G^\text{der}_1 \to G^\text{der}_2$ a central isogeny. Let $V_1$ be a faithful $\Q_p$-linear representation of $G_1$ and let $V_2$ be any $\Q_p$-linear representation of $G_2$.

Let $F$ be a number field and let $\rho_i : \Ad_F \to G_i(\Q_p)$ (for $i = 1, 2$) be geometric Galois representations. Assume that $V^\text{i}_1$ lies in the subcategory of $\text{Rep}_{\Q_p}(\Ad_F) \otimes$-generated by $V_1$ and $(\CM)\text{-Rep}_{\Q_p}(\Ad_F)$ and that $\pi^\text{ad}_1 \circ \rho_1 = \pi^\text{ad}_2 \circ \rho_2$, where the $\pi^\text{ad}_i : G_i \to G^\text{ad}_i$ are the canonical projections and $G^\text{ad}_2 = G^\text{ad}_1$ is the identification induced by $\pi^\text{der}$. Then $V_2$ is an object of the subcategory of $\text{Rep}_{\Q_p}(\Ad_F) \otimes$-generated by $V_1$ and $(\CM)\text{-Rep}_{\Q_p}(\Ad_F)$.
Proof. Fix an object $V_3$ of $(\text{CM})\text{-Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$ such that $V_3'$ lies in the subcategory of $\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$ $\otimes$-generated by $V_1$ and $V_3$. Let $T_3$ be the Zariski closure of the image of $\mathcal{G}_F$ in $\text{GL}(V_3)$ and let $\rho_3 : \mathcal{G}_F \to T_3(\mathbb{Q}_p)$ be the morphism giving the action of the image of $\mathcal{G}_F$ on $V_3$.

Let $T_1$ and $T_2$ be the connected components of the centres of $G_1$ and of $G_2$ respectively. Put $H = T_1 \times T_2 \times T_3 \times G_1^\text{der}$ and consider the natural maps $H \to G_1$ and $H \to G_2$. Via these maps, we consider $V_1$ and $V_2$ as representations of $H$. It can be shown exactly as in the proof of proposition 3.4 that $V_2$ belongs to the subcategory of $\text{Rep}_{\mathbb{Q}_p}(T_3 \times G_A \times G_B)$ generated by $V_1$ and the abelian representations. For any abelian representation $W_3$ of $T_3 \times G_A \times G_B$, the induced Galois representation belongs to $(\text{CM})\text{-Rep}(\mathcal{G}_F)$ so this implies that $V_2$ belongs to the subcategory $\langle V_1, (\text{CM})\text{-Rep}(\mathcal{G}_F) \rangle$ of $\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)$.

4.8 We keep the above notations, i.e. $F \subset \mathbb{C}$ is a number field, $A/F$ an abelian variety and $G_A$ its Mumford–Tate group, which is assumed to be connected. For each prime number $p$, we denote by $\rho_{A,p} : \mathcal{G}_F \to G_A(\mathbb{Q}_p)$ the $p$-adic Galois representation associated to $A$.

Fix an algebraic group $\mathcal{G}$ over $\mathbb{Q}$ and a central isogeny $\tilde{G}_A^\text{der} \to G_A^\text{der}$ and suppose that for every $p$ in a set $P$ of prime numbers $\tilde{\rho}_p : \mathcal{G}_F \to \tilde{G}(\mathbb{Q}_p)$ is a weak geometric lift of $\rho_{A,p}$.

4.9 Theorem. Assume that any simple factor of $\tilde{G}_A^\text{der}$ lying over a factor of $G_A^\text{der}$ of type $D_k^H$ is a quotient of the $h$-maximal cover, cf. 2.5. Let $F'$ be a finite extension of $F$ and $B$ an essentially Mumford–Tate unliftable abelian variety over $F'$ with connected Mumford–Tate group $G_B$ such that $B_{/\mathbb{C}}$ provides a weak Mumford–Tate lift of $A_{/\mathbb{C}}$. For each prime number $p$, let $V_{B,p} = H^1_{\text{et}}(B_{\mathbb{F}_p}, \mathbb{Q}_p)$ be the $p$-adic representation of $\mathcal{G}_{F'}$ associated to $B$.

Then the map $G_B^\text{der} \to G_A^\text{der}$ lifts to $G_B^\text{der} \to \tilde{G}^\text{der}$. For every $p \in P$

- the representation $\rho_{B,p}$ is a weak geometric lift of the restriction $(\tilde{\rho}_p)|_{\mathcal{G}_{F'}}$ and

- for every representation $\tilde{V}_p$ of $\tilde{G}/\mathbb{Q}_p$, the Galois representation on $\tilde{V}_p$ is an object of $\langle V_{B,p}, (\text{CM})\text{-Rep} \rangle$.

Proof. We fix a faithful self-dual representation $\tilde{V}$ of $\tilde{G}$ and for each prime number $p$ we write $\tilde{V}_p = \tilde{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. As $\tilde{V}$ generates the tannakian category of representations of $\tilde{G}$, it is sufficient to prove the corollary for the representations $\tilde{V}_p$. 
Since $B$ is essentially Mumford–Tate unliftable, it follows from theorem 2.9 that $G^\text{der}_B$ is $h$-maximal. It follows that the map $G^\text{der}_B \to G^\text{der}_A$ lifts to a map $G^\text{der}_B \to \tilde{G}^\text{der}$. Let $\rho_{B,p} : \mathcal{G}_{F'} \to G_B(Q_p)$ be the map giving the Galois representation on $V_{B,p}$. Write $\pi^\text{ad}_A : G_A \to G^\text{ad}_A$, $\pi^\text{ad}_B : G_B \to G^\text{ad}_B$ and $\tilde{\pi}^\text{ad} : \tilde{G} \to \tilde{G}^\text{ad}$ for the projections. For any $p \in P$, the proposition 4.5 and the fact that $\tilde{\rho}_p$ is a weak geometric lift of $\rho_{A,p}$ imply that $\tilde{\pi}^\text{ad} \circ \tilde{\rho}_{p|\mathcal{G}_{F'}} = \pi^\text{ad}_A \circ \rho_{A,p|\mathcal{G}_{F'}} = \pi^\text{ad}_B \circ \rho_{B,p}$. The remaining statement of the theorem follows from lemma 4.7. \hfill \Box

4.10 Important remark. Concerning the abelian variety $B/F'$ which appears in propositions 4.3 and 4.5 and in theorem 4.9, it follows from theorem 2.12 and proposition 3.1 that there exist a number field $F'$ and an essentially M-T decomposed weak Mumford–Tate lift $B/F'$ of $A$ as in the propositions and in the theorem. The condition that $G_B$ is connected can be forced by replacing $F'$ by a finite extension.

4.11 Corollary. Let notations be as in 4.8 with $\tilde{G}^\text{der} \to G^\text{der}_A$ satisfying the hypotheses of the theorem. Then, for every $p \in P$ and every representation $\tilde{V}_p$ of $\tilde{G}/Q_p$, the induced representation of $G_{F'}$ on $\tilde{V}_p$ occurs in the $p$-adic étale realization of an abelian motive.

Proof. The remark and the theorem imply that there is a finite extension $F'$ of $F$ such that the representation of $\mathcal{G}_{F'}$ on $\tilde{V}_p$ occurs in the $p$-adic étale realization of an object $M'$ of $(AV)_{F'}$. The representation of $\mathcal{G}_{F'}$ on $\tilde{V}_p$ then occurs in the $p$-adic étale realization of the Weil restriction $\text{Res}_{F'/F}M'$, which is also in $(AV)_{F'}$. \hfill \Box

5 Abelian varieties with Mumford–Tate group of type $D^H_k$

5.1 The following notation and hypotheses will be in force until definition 5.10. We let $A/C$ be a simple abelian variety and $(G_A,h_A)$ the associated Mumford–Tate datum. This implies that $G_A$ is connected. We assume throughout that $A$ is essentially Mumford–Tate unliftable, Mumford–Tate decomposed and that $G_A$ is of type $D^H_k$ with $k \geq 4$.

It follows that there exist a totally real number field $K_0$ and an absolutely simple algebraic group $G^s/K_0$ such that $G^\text{der}_A = \text{Res}_{K_0/Q}G^s$. By assumption, the group $G^s/K_0$ is $h$-maximal in the sense of 2.5. The representation of $G^\text{der}_A$ on $H^1_B(A(C),Q)$ decomposes over $\overline{Q}$ as a multiple of the direct sum of the standard (orthogonal) representations of the different factors of $G^\text{der}_A/Q$. As in [Del79]
and in §2 of this paper, denote by \( I = \{ \iota: K_0 \hookrightarrow \mathbb{C} \} \) the set of complex embeddings of \( K_0 \). As \( K_0 \) is totally real, \( I \) is also the set of real embeddings of \( K_0 \). The Dynkin diagram of \( G^\text{der}_{A/\mathbb{C}} \) is a disjoint union, indexed by \( I \), of diagrams of type \( D_k \). The Hodge cocharacter \( \mu_A: G_{m/\mathbb{C}} \rightarrow G_{/\mathbb{C}} \) associated to \( A \) projects trivially on some factors of \( G^\text{ad}_{/\mathbb{C}} \). On the other factors, the conjugacy class of the projection is dual to one of the vertices \( \alpha_{k-1} \) or \( \alpha_k \) (or possibly \( \alpha_1 \) if \( k = 4 \)) of the corresponding component of the Dynkin diagram. Without loss of generality, we will henceforth assume that, on these factors, it corresponds to \( \alpha_k \). As in section 2, let \( I_c \subset I \) be the set of embeddings corresponding to the factors onto which \( \mu_A \) projects trivially and let \( I_{nc} = I - I_c \). Recall that \( I_c \) is the set of embeddings \( \iota: K_0 \hookrightarrow \mathbb{R} \) such that the factor of \( G^\text{der}_{A/\mathbb{R}} \) corresponding to \( \iota \) is compact and \( I_{nc} \) is the set of real embeddings of \( K_0 \) for which the corresponding factor of \( G^\text{der}_{A/\mathbb{R}} \) is non-compact.

As the Hodge filtration and its complex conjugate are opposite filtrations, the complex conjugate of \( \mu \) is conjugate to \( \mu^{-1} \), up to a central cocharacter. This implies that complex conjugation acts on the Dynkin diagram by the main involution and hence that it acts trivially if \( k \) is even and exchanges \( \alpha_{k-1} \) and \( \alpha_k \) on every factor if \( k \) is odd. It follows that \( G_{/\mathbb{Q}} \) (or \( \text{Aut}(\mathbb{C}) \)) acts on the Dynkin diagram through \( \text{Gal}(K/\mathbb{Q}) \) for a number field \( K \supset K_0 \) with \( [K:K_0] = 1 \) or \( 2 \) and which is totally real if \( k \) is even, a CM field if \( k \) is odd. In particular \( [K:K_0] = 2 \) if \( k \) is odd. Note that the statement is also true for \( k = 4 \), because it follows from the definition of the case \( D^H_4 \) (see 2.5) that the vertices \( \alpha_1 \) of the connected components of the Dynkin diagram form a \( G_{/\mathbb{Q}} \)-orbit, so the stabilizer of a connected component is of order at most 2.

5.2 Construction of \( \tilde{G} \) and \( \tilde{\mu} \). We aim to construct an algebraic group \( \tilde{G}/\mathbb{Q} \) such that \( \tilde{G}^\text{der} \) is simply connected and that \( \tilde{G}^\text{ad} = G^\text{ad}_A \), together with a cocharacter \( \tilde{\mu} \) of \( \tilde{G}_{/\mathbb{C}} \) such that \( \tilde{\pi}^\text{ad} \circ \tilde{\mu} = \pi^\text{ad} \circ \mu \). The argument is strongly inspired by [Del79], see also the proof of theorems 2.9 and 2.12. Here, as before, \( \pi^\text{ad} \) and \( \tilde{\pi}^\text{ad} \) are the projections \( G_A \rightarrow G^\text{ad}_A \) and \( \tilde{G} \rightarrow G^\text{ad}_A \) respectively.

Let \( \tilde{G}^s/K_0 \) be the simply connected cover of \( G^s \). Consider the direct sum of the representations of \( \tilde{G}^s_{/\mathbb{Q}} \) with highest weights \( \omega_{k-1} \) and \( \omega_k \). A multiple of that representation can be defined over \( K_0 \), let \( W^s \) be the resulting representation of \( \tilde{G}^s \). By construction, there is a decomposition \( W^s \otimes_{K_0} K = W^s_1 \oplus W^s_2 \), where \( W^s_1 \) (resp. \( W^s_2 \)) is a multiple of the representation with highest weight.
A non-trivial element of $\text{Gal}(K/K_0)$ exchanges the factors $W_1^s$ and $W_2^s$ of this decomposition. If $[K : K_0] = 2$, then the composite map $W^s \subset W^s \otimes_{K_0} K \to W_1^s$ is an isomorphism of $K_0$-vector spaces and endows $W^s$ with a structure of $K$-vector space.

Put $\tilde{G}^{\text{der}} = \text{Res}_{K_0/Q} \tilde{G}^s$ and let $W$ be the rational representation of $\tilde{G}^{\text{der}}$ deduced from $W^s$. Since $W$ is the underlying $Q$-vector space of $W^s$, it carries a structure of $K$-vector space. The cocharacter $\pi^{\text{ad}} \circ \mu$ of $G^{\text{ad}}$ lifts to a quasi-cocharacter $\nu$ of $\tilde{G}^{\text{der}}_{/C}$, cf. [1.1].

5.3 Lemma. There is a decomposition

$$W \otimes Q C \cong \bigoplus_{i \in I} \left( W_1^{(i)} \oplus W_2^{(i)} \right) \quad (5.3.\ast)$$

such that, for each $i \in I$, the representation $W_1^{(i)}$ (resp. $W_2^{(i)}$) is a multiple of the irreducible representation with highest weight $\omega_{k-1}$ (resp. $\omega_k$) of the factor of $G^{\text{der}}_{A/C}$ corresponding to $i$.

The weights of $\nu$ on the $W_j^{(i)}$ are trivial for $j = 1, 2$ if $i \in I_c$. For $i \in I_{nc}$, the weights of $\nu$ on $W_1^{(i)}$ are of the form $\frac{k-2}{4} - m_1$ and those on $W_2^{(i)}$ are of the form $\frac{k}{4} - m_2$ with $m_1, m_2 \in \mathbb{Z}$. If $k$ is even $m_1$ runs from 0 to $\frac{k-2}{2}$ and $m_2$ from 0 to $\frac{k}{2}$. For $k$ odd, both $m_1$ and $m_2$ run from 0 to $\frac{k-1}{2}$.

Proof. The decomposition of $W^s$ induces the decomposition $(5.3.\ast)$. The quasi-cocharacter $\nu$ projects trivially to the factors of $\tilde{G}^{\text{der}}_{/C}$ corresponding to the $i \in I_c$ and non-trivially to the other factors. The highest weights of $\nu$ on $W_1^{(i)}$ and $W_2^{(i)}$ can easily be deduced from the information collected in [Del79, Table 1.3.9], it is the rational number corresponding to $\omega_{k-1}$ resp. $\omega_k$ in that table. The lowest weight of $\nu$ on $W_1^{(i)}$ is the opposite of the number corresponding to $\omega_{k-1}$ if $k$ is even and the opposite of the number corresponding to $\omega_k$ if $k$ is odd. For $W_2^{(i)}$, the converse is the case.

We continue the construction of $\tilde{G}$ and $\tilde{\mu}$. The case where $k$ is even and the case where $k$ is odd will be treated separately.

5.4 The case where $k$ is even. In this case, $K$ is totally real and either $K = K_0$ or $[K : K_0] = 2$. Let $L'$ be a totally imaginary quadratic extension of $K_0$, put $L = KL'$ and define the algebraic torus $T_L$ over $Q$ by $T_L = \ker(N_{L/K}) \subset L^\times$. 

where $N_{L/K} : L^\times \to K^\times$ is the field norm. The group $T_L$ naturally acts on $L$ and this gives rise to a $\mathbb{Q}$-linear representation $V_L$ of $T_L$. We have a natural structure of $K$-vector space on $V_L$. For each embedding $\iota : K \hookrightarrow \mathbb{C}$ we choose a complex embedding of $L$ extending $\iota$ and this gives an identification

$$ T_{L/\mathbb{C}} = \bigoplus_{\iota \in \tilde{I}} G_{m/\mathbb{C}}, $$

where $\tilde{I}$ is the set of complex embeddings of $K$.

From now on, we will further distinguish the cases where $[K : K_0] = 1$ and where $[K : K_0] = 2$.

**The subcase where $K = K_0$.** Consider the decomposition (5.3.4) of $W \otimes \mathbb{Q} \mathbb{C}$. Since $W^s = W_1^s \oplus W_2^s$ is a decomposition of $W^s$ as a direct sum of $K_0$-vector spaces, we obtain a $\mathbb{Q}$-linear decomposition $W = W_1 \oplus W_2$. For $\iota \in I_{nc}$, the weights of $\nu$ on $W_1^{(\iota)}$ are in $\frac{1}{2} + \mathbb{Z}$ if $k \equiv 0 \pmod{4}$ and in $\mathbb{Z}$ if $k \equiv 2 \pmod{4}$. For the weights on $W_2^{(\iota)}$, the converse is the case.

Let $T = T_L \times T_L$ and define $W_3$ (resp. $W_4$) be the representation of $T$ given by the action of the first (resp. the second) factor $T_L$ on $V_L$. We define a quasi-cocharacter

$$ \nu_L : G_{m/\mathbb{C}} \to T/\mathbb{C} = \bigoplus_{\iota \in \tilde{I}} G_{m/\mathbb{C}}, $$

by

$$ \nu_L(z)_t = \begin{cases} 
(1, 1) & \text{if } t \in I_c \\
(\sqrt{z}, 1) & \text{if } t \in I_{nc} \text{ and } k \equiv 0 \pmod{4} \\
(1, \sqrt{z}) & \text{if } t \in I_{nc} \text{ and } k \equiv 2 \pmod{4}, 
\end{cases} $$

where $\nu_L(z)_t$ is the component of $\nu_L(z)$ in the factor of $T/\mathbb{C}$ corresponding to $t \in I$. Finally, let $\tilde{V}$ be the representation of $\tilde{G}^{\text{der}} \times T$ defined by

$$ \tilde{V} = W_1 \otimes_{K_0} W_3 \oplus W_2 \otimes_{K_0} W_4 $$

and let $\tilde{G}$ be the image of $\tilde{G}^{\text{der}} \times T$ in $\text{GL}(\tilde{V})$. This will not cause any confusion, since $\tilde{G}^{\text{der}}$ acts faithfully on $\tilde{V}$, so the derived group of $\tilde{G}$ is $\tilde{G}^{\text{der}}$. The weights of $(\nu, \nu_L)$ on $\tilde{V}$ are in $\mathbb{Z}$, so the projection of the quasi-cocharacter $(\nu, \nu_L)$ from $(\tilde{G}^{\text{der}} \times T)/\mathbb{C}$ to $\tilde{G}/\mathbb{C}$ is a true cocharacter $\tilde{\mu}$ of $\tilde{G}/\mathbb{C}$. By construction,

$$ \tilde{\pi}^{\text{ad}} \circ \tilde{\mu} = \tilde{\pi}^{\text{ad}} \circ \nu = \pi^{\text{ad}} \circ \mu, $$

so we have constructed the couple $(\tilde{G}, \tilde{\mu})$ in this particular subcase.
The subcase where \([K : K_0] = 2\). The structure of \(K\)-vector space on \(W\) gives rise to a decomposition
\[
W \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{i : \tilde{I} \to \mathbb{C}} W^{(i)}.
\]

Recall that \(\tilde{I}\) is the set of complex embeddings of \(K\). Put \(\tilde{I}_c = \{ i \in \tilde{I} \mid i_{|K_0} \in I_c \}\) and \(\tilde{I}_{nc} = \{ i \in \tilde{I} \mid i_{|K_0} \in I_{nc} \}\). For \(i \in \tilde{I}_c\), the weights of \(\nu\) on \(W^{(i)}\) are all \(0\), for \(i \in \tilde{I}_{nc}\), these weights are either in \(\mathbb{Z}\) or in \(\frac{1}{2} + \mathbb{Z}\). Let \(\tilde{I}_{nc,1}\) be the set of \(i \in \tilde{I}_{nc}\) where the former possibility occurs and \(\tilde{I}_{nc,2}\) its complement in \(\tilde{I}_{nc}\). Note that for any \(i \in I_{nc}\) one of the embeddings \(K \hookrightarrow \mathbb{C}\) restricting to \(i\) lies in \(\tilde{I}_{nc,1}\) and the other one lies in \(\tilde{I}_{nc,2}\).

This time we define a quasi-cocharacter
\[
\nu_L : G_{m/\mathbb{C}} \to T_{L/\mathbb{C}} = \bigoplus_{\tilde{i} \in \tilde{I}} G_{m/\mathbb{C}},
\]
by
\[
\nu_L(z)_i = \begin{cases} 
1 & \text{if } \tilde{i} \in \tilde{I}_c \cup \tilde{I}_{nc,1} \\
\sqrt{z} & \text{if } \tilde{i} \in \tilde{I}_{nc,2}.
\end{cases}
\]
As above, \(\nu_L(z)_i\) is the component of \(\nu_L(z)\) in the factor of \(T_{L/\mathbb{C}}\) corresponding to \(\tilde{i} \in \tilde{I}\). Finally, let \(\tilde{V} = W \otimes_K V_L\) as representation of \(\tilde{G}^{\text{der}} \times T\) and let \(\tilde{G}\) be the image of \(\tilde{G}^{\text{der}} \times T\) in \(\text{GL}(\tilde{V})\). Once again, there is no risk of confusion, because the derived group of \(\tilde{G}\) is \(\tilde{G}^{\text{der}}\). Since the weights of \((\nu, \nu_L)\) on \(\tilde{V}\) are in \(\mathbb{Z}\), the projection of \((\nu, \nu_L)\) from \((\tilde{G}^{\text{der}} \times T)/\mathbb{C}\) to \(\tilde{G}/\mathbb{C}\) is a true cocharacter \(\tilde{\mu}\) of \(\tilde{G}/\mathbb{C}\). By construction, \(\tilde{\pi}^{\text{ad}} \circ \tilde{\mu} = \tilde{\pi}^{\text{ad}} \circ \nu = \pi^{\text{ad}} \circ \mu\), so this achieves the construction of the couple \((\tilde{G}, \tilde{\mu})\) in the case where \(k\) is even.

5.5 The case where \(k\) is odd. In this case, \(K\) is a totally imaginary quadratic extension of \(K_0\). In order to unify this case with the previous one as much as possible, we put \(L = K\). Let \(T_L\) be the \(\mathbb{Q}\)-algebraic torus \(\ker(N_{L/K_0})\), where \(N_{L/K_0} : L^\times \to K_0^\times\) is the field norm. As in the beginning of 5.4, the action of \(T_L\) on \(L\) by multiplication on the left gives rise to a representation \(V_L\) of \(T_L\).

We again consider the decomposition (5.4.4) and the subsets \(\tilde{I}_c\) and \(\tilde{I}_{nc}\) of the set \(\tilde{I}\) of complex embeddings of \(K = L\). By lemma 5.3, the weights of \(\nu\) on \(W^{(i)}\) are either in \(\frac{1}{2} + \mathbb{Z}\) or in \(-\frac{1}{2} + \mathbb{Z}\) for \(i \in \tilde{I}_{nc}\) and trivial for \(i \in \tilde{I}_c\). Let \(\tilde{I}_{nc,1}\) the set of \(i \in \tilde{I}_{nc}\) for which the weights of \(\nu\) on \(W^{(i)}\) are in \(\frac{1}{2} + \mathbb{Z}\) and let \(\tilde{I}_{nc,2}\) be the complement of \(\tilde{I}_{nc,1}\) in \(\tilde{I}_{nc}\). For each \(i \in I_{nc}\), one of the embeddings
K \hookrightarrow \mathbb{C} restricting to \iota lies in \tilde{I}_{nc,1} and the other one lies in \tilde{I}_{nc,2}. Thus, the map \( r: \tilde{I} \rightarrow I \) given by \( \iota \mapsto \iota|_{K_0} \) induces a bijection of \( \tilde{I}_{nc,1} \) with \( I_{nc} \). For \( I_c \), we arbitrarily fix a subset \( \tilde{I}_{c,1} \subset \tilde{I}_{c} \) such that \( r \) induces a bijection of \( \tilde{I}_{c,1} \) with \( I_c \).

Putting \( \tilde{I}_1 = \tilde{I}_{nc,1} \cup \tilde{I}_{c,1} \), we get an identification \( T_L/C = \bigoplus_{\tilde{\iota} \in \tilde{I}_1} G_{m/C} \).

Under this identification the factor \( G_{m/C} \) corresponding to \( \tilde{\iota} \) acts on the factor of \( V_L/C \) corresponding to \( \tilde{\iota} \) by multiplication in case \( \tilde{\iota} \in \tilde{I}_{c,1} \) and by multiplication with the inverse if \( \tilde{\iota} \not\in \tilde{I}_{nc,1} \).

The quasi-cocharacter \( \nu_L: G_{m/C} \rightarrow T_L/C \) is defined by
\[
\nu_L(z)_{\tilde{\iota}} = \begin{cases} 1 & \text{if } \tilde{\iota} \in \tilde{I}_{c,1} \\ z^{-1/4} & \text{if } \tilde{\iota} \in \tilde{I}_{nc,1} \end{cases}
\]

We put \( \tilde{\nu} = W \otimes_L V_L \) as representation of \( \tilde{G}^{\text{der}} \times T \) and again let \( \tilde{G} \) be the image of \( \tilde{G}^{\text{der}} \times T \) in \( \text{GL}(\tilde{\nu}) \). As before, the derived group of \( \tilde{G} \) is \( \tilde{G}^{\text{der}} \) and the projection of \((\nu, \nu_L)\) from \((\tilde{G}^{\text{der}} \times T)/C\) to \( \tilde{G}/C \) is a true cocharacter \( \tilde{\mu} \) of \( \tilde{G}/C \). We obviously have equalities \( \tilde{\pi}^{\text{ad}} \circ \tilde{\mu} = \tilde{\pi}^{\text{ad}} \circ \nu = \pi^{\text{ad}} \circ \mu \), so this settles the construction of the couple \((\tilde{G}, \tilde{\mu})\) in this case.

The preceding discussion establishes the following theorem.

5.6 Theorem. Let \( A/C \) be a simple abelian variety, \((G_A, h_A)\) its Mumford–Tate datum and \( \mu_A \) the Hodge cocharacter associated to \( A \). Assume that \( A \) is essentially Mumford–Tate unliftable and that \( G_A \) is of type \( D^H_k \) with \( k \geq 4 \). Then there exist an algebraic group \( \tilde{G} \) with \( \tilde{G}^{\text{der}} \) simply connected, an identification \( \tilde{G}^{\text{ad}} = G_A^{\text{ad}} \) and a cocharacter \( \tilde{\mu}: G_{m/C} \rightarrow \tilde{G}/C \) such that \( \tilde{\pi}^{\text{ad}} \circ \tilde{\mu} = \pi^{\text{ad}} \circ \mu_A \).

5.7 Remark. In all cases above, the group \( T_{L/R} \) occurring in the construction of \( \tilde{G} \) is compact. It thus follows that \( \tilde{G}_{ab}^{R} \) is compact. It is left to the reader to construct an isomorphism \( \tilde{G}^{ab} \cong T_L \) and to compute the composite \( T_L \subset \tilde{G} \rightarrow \tilde{G}^{ab} \cong T_L \).

5.8 Suppose that \( F \subset \mathbb{C} \) is a number field and that \( A \) is an abelian variety over \( F \) with connected Mumford–Tate group such that the conditions of the theorem
are verified for $A_{/\mathbb{C}}$ and its Mumford–Tate datum $(G_A, h_A)$. Let $\tilde G$ and $\mu$ be as in the conclusion of the theorem.

Let $\tilde G^{ab} = \tilde G / \tilde G^{d_{er}}$ and define $\mu^{ab}$ to be the composite of the cocharacter $\bar{\mu}: \mathbb{G}_{m_{/\mathbb{C}}} \to \tilde G^{ab}$ with the natural projection $\tilde G^{ab} / \mathbb{C} \to \tilde G^{ab}_{/\mathbb{C}}$. Define

$$h^{ab}: S \to \tilde G^{ab}_{/\mathbb{R}} \quad z \mapsto \mu^{ab}(z)\mu^{ab}(z).$$

The remark 5.7 implies that $\tilde G^{ab}_{/\mathbb{R}}$ is compact. As the weight $w_h: \mathbb{G}_{m_{/\mathbb{R}}} \to \tilde G^{ab}_{/\mathbb{R}}$ is trivial, [[Del82b] 1.1.18(b)] implies that $h^{ab}$ defines a polarizable Hodge structure on any $\mathbb{Q}$-linear representation $V^{ab}$ of $\tilde G^{ab}$.

Fix a faithful rational representation $V^{ab}$ of $\tilde G^{ab}$. It follows from proposition A.1 of [[Del82b] and the remark preceding it that the Hodge structure on $V^{ab}$ is the Betti realization of an absolute Hodge motive $M^{ab}_{/\mathbb{Q}}$ belonging to $(\text{CM})_{/\mathbb{Q}}$. Note that using the isomorphism $\tilde G^{ab} \cong \mathbb{T}_L$ from remark 5.7, this motive can also be constructed explicitly. Replacing $F$ by a finite extension and fixing an embedding $F \subset \mathbb{Q}$ we can assume that $M^{ab}_{/\mathbb{Q}}$ descends to a motive $M^{ab}$ over $F$. At the cost of a further finite extension of $F$, we can also assume that the Mumford–Tate group of $M^{ab}$ is connected and hence contained in $\tilde G^{ab}$. This implies that for every $p$, the $p$-adic realization of $M^{ab}$ factors through a map $\rho^{ab}_p: \mathcal{G}_F \to \tilde G^{ab}(\mathbb{Q}_p)$. Note that the $\rho^{ab}_p$ form a compatible system of representations (see 5.7) and that the $\rho^{ab}_p$ do not depend on the choice of the representation $V^{ab}$ of $\tilde G^{ab}$.

For each prime number $p$, let $\Sigma_p$ be the set of $p$-adic places of $F$. There is a finite set $\Sigma$ of places $v$ of $F$ such that $\mathcal{A}$ and $M^{ab}$ have good reduction outside $\Sigma$. For every prime number $p$, the representations $\rho_{A_p}$ and $\bar{\rho}^{ab}_p$ are unramified at all $v \notin \Sigma \cup \Sigma_p$. For any finite extension $F'$ of $F$, let $\Sigma'_p$ be the set of $p$-adic places of $F'$ and let $\Sigma'$ be the set of places lying over the $v \in \Sigma$.

Let $G' = G^{ad}_{A} \times \tilde G^{ab}$ and let $\pi': \tilde G \to G'$ be defined by the projections of $\tilde G$ onto $\tilde G^{ab}$ and $G^{ad}_{A} = \tilde G^{ad}$. For each $p$, define $\rho^{ad}_p = \pi^{ad} \circ \rho_{A_p}: \mathcal{G}_F \to G^{ad}_{A}(\mathbb{Q}_p)$ and put

$$\rho'_p = (\rho^{ad}_p, \rho^{ab}_p): \mathcal{G}_F \to G'(\mathbb{Q}_p).$$

5.9 Corollary. There exist a finite extension $F'$ of $F$ and a system of geometric Galois representations $\bar{\rho}_p: \mathcal{G}_{F'} \to \tilde G(\mathbb{Q}_p)$ lifting the restrictions $\rho'_p: \mathcal{G}_{F'} \to G'(\mathbb{Q}_p)$. The system $(\bar{\rho}_p)$ can be chosen in such a way that each $\bar{\rho}_p$ is unramified at $v$ for each $v \notin \Sigma' \cup \Sigma'_p$. 
For each prime number $p$, the representation $\overline{\rho}_p$ is a weak geometric lift of $\rho_{A,p}$ and it is essentially geometrically unliftable.

Proof. Since $G_A$ is the Mumford–Tate group of $A$, the adjoint representation of $G_A$ gives rise to a motive $M_{\text{ad}}^\text{d}$ belonging to the subcategory of $\text{Mot}_{\text{AH}}(F)$ which is $\otimes$-generated by $h^1(A)$ and the Tate motive. The Mumford–Tate group of $M_{\text{ad}}^\text{d}$ is the adjoint group $G_A^{\text{ad}} = \tilde{G}^{\text{ad}}$.

Consider the object $M' = M_{\text{ad}}^\text{d} \times M_{\text{ab}}^\text{d}$ of $\text{Mot}_{\text{AH}}(F)$. Its Mumford–Tate group is contained in $G'$ and the Hodge cocharacter $\mu' : G_{m,C} \to G'_C$ associated to its Betti realization is the product $(\mu_{\text{ad}}, \mu_{\text{ab}})$. It follows that $\mu' = \pi'_C \circ \mu$. On the other hand, for each $p$, the Galois representation on the $p$-adic realization of the motive $M'$ is the above map $\rho'_p$. The motive $M'$ is an abelian motive, so it follows from [Bla94, Theorem 0.3] that the $p$-adic comparison maps linking the $p$-adic and the DeRham realizations of $M'$ are compatible with absolute Hodge classes. The main theorem 2.1.7 of [Win95] implies that there is a finite extension $F' \supset F$ such that every $\rho'_p |_{G_{F'}}$ lifts to a representation $\overline{\rho}_p : G_{F'} \to \tilde{G}(Q_p)$ satisfying the conditions of the corollary.

It is clear from the above that, for each $p$, the map $\overline{\rho}_p$ provides a weak geometric lift of $\rho_{A,p}$. As $\tilde{G}^{\text{der}}$ is simply connected, it is also clear that the $\overline{\rho}_p$ are essentially geometrically unliftable.

5.10 Definition. The representations $\overline{\rho}_p$ constructed as in the corollary from Mumford–Tate decomposed abelian varieties will be called representations of lifted abelian $D^H_k$-type.

5.11 Corollary. Let $F \subset C$ be a number field and $A/F$ an abelian variety with connected Mumford–Tate group $G_A$ and associated system of Galois representations $(\rho_{A,p})$. Then there exist an linear algebraic group $\tilde{G}$ over $Q$ such that $\tilde{G}^{\text{der}}$ is the universal cover of $G_A^{\text{der}}$, a finite extension $F' \supset F$ and a system of weak geometric lifts

$$\overline{\rho}_p : G_{F'} \to \tilde{G}(Q_p)$$

of the restrictions to $G_{F'}$ of the $\rho_{A,p}$.

Proof. By theorem 2.12 and proposition 5.1 there exist a number field $F'$ and an essentially M-T unliftable and M-T decomposed weak Mumford–Tate lift $B/F'$ of $A$ with connected Mumford–Tate group $G_B$. It follows from proposition 4.5.
that the system of Galois representations \((\rho_{B,p})\) associated to \(B\) is a system of weak geometric lifts of the system \((\rho_{A,p})\).

Let \(B_i/C \sim \prod B_i/C\) be the isogeny decomposition as in definition 2.10. After enlarging \(F'\), we can assume that this decomposition exists over \(F'\). In that case, each \(B_i\) is Mumford–Tate decomposed with \(G_{B_i}^{ad}\), simple and \(G_{B_i}^{ad}\) is the product of the \(G_{B_i}^{ad}\). For each \(i\), let \(\rho_{B_i,p} : \mathcal{G}_F \to G_{B_i}(\mathbb{Q}_p)\) be the \(p\)-adic Galois representation associated to \(B_i\). For every \(i\) such that \(G_{B_i}\) is of type \(D_k\), put \(\tilde{G}_i = G_{B_i}\) and \(\tilde{\rho}_{i,p} = \rho_{B_i,p}\). For each \(i\) such that \(G_{B_i}\) is of type \(D_k\), let \(\tilde{\rho}_{i,p} : \mathcal{G}_{F'} \to \tilde{G}_i(\mathbb{Q}_p)\) be the representation of lifted abelian \(D_k\)-type resulting from corollary 5.9, replacing \(F'\) by a finite extension again if necessary. Let \(\tilde{G} = \prod \tilde{G}_i\) and let \(\tilde{\rho}_p : \mathcal{G}_{F'} \to \tilde{G}(\mathbb{Q}_p)\) be the product of the maps \(\tilde{\rho}_{i,p}\).

For each prime number \(p\), the representation \(\tilde{\rho}_p\) is a weak geometric lift of \(\rho_{B,p}\) and therefore also of \(\rho_{A,p}\). As \(\tilde{G}^{\text{der}}\) is simply connected, the corollary follows. \(\square\)

5.12 Corollary. Let \(F\) be a number field, \(M\) an abelian motive and \(\rho' : \mathcal{G}_F \to G'(\mathbb{Q}_p)\) a weak geometric lift of the associated \(p\)-adic Galois representation. Then, after replacing \(F\) by a finite extension and for any linear representation \(V'_p\) of \(G'/\mathbb{Q}_p\), the representation of \(\mathcal{G}_F\) on \(V'_p\) deduced from \(\rho'_p\) lies in the tannakian subcategory of \(\text{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F)\) generated by the \(p\)-adic representations associated to abelian motives and the representations of lifted abelian \(D_k\)-type.

Proof. It is sufficient to prove the corollary for one fixed faithful self-dual representation \(V'_p\) of \(G'\).

Let \(A\) be the essentially Mumford–Tate unifiable and Mumford–Tate decomposed abelian variety supplied by applying corollary 5.10 to \(M\), so \(M\) is an object of \((h^1(A), \mathbb{Q}(1))\). This inclusion corresponds to a map \(G_A \to G_M\). Supposing isogeny factors of \(A\), we can assume that the induced map \(G_A^{\text{der}} \to G_M^{\text{der}}\) is an isogeny. After replacing \(F\) by a finite extension, define the group \(\tilde{G}\) and the representation \(\tilde{\rho}_p : \mathcal{G}_F \to \tilde{G}(\mathbb{Q}_p)\) as in the proof of 5.11.

As \(\tilde{G}^{\text{der}}\) is the simply connected cover of \(G_A^{\text{der}}\), the isogeny

\[
\tilde{G}^{\text{der}} \to G_A^{\text{der}} \to G_M^{\text{der}}
\]

lifts to an isogeny \(\tilde{G}^{\text{der}} \to G'^{\text{der}}\). This results in an identification of the adjoint groups \(G^{\text{ad}} = G_M^{\text{ad}} = G_A^{\text{ad}} = \tilde{G}^{\text{ad}}\) and, by construction, the projections to \(G_M^{\text{ad}}(\mathbb{Q}_p)\) of the representations \(\rho'_p, \rho_{M,p}, \rho_{A,p}\) and \(\tilde{\rho}_p\) all coincide. The corollary
Lifting Galois representations of abelian varieties

now follows from lemma 4.7 with $G_1 = \tilde{G}, V_1$ any faithful self-dual representation of $G_1, G_2 = G'$ and $V_2 = V'_p$.

6 Galois representations of lifted abelian $D_k^H$-type

6.1 We revert to the notations of 5.8, in particular $A/F$ is essentially Mumford–Tate unliftable, $G_A$ is of type $D_k^H$ and $\tilde{G}$ is the group constructed in theorem 5.6. We will also assume $A$ to be Mumford–Tate decomposed. For the first part of this section, we fix a prime number $p$ and assume that there exists a weak geometric lift $\tilde{\rho}_p: G_F \to \tilde{G}(\mathbb{Q}_p)$ of $\rho_{A,p}$ as described in 6.1, with $k \geq 5$. Assume moreover that $\rho_{A,p}(G_F) \subset G_A(\mathbb{Q}_p)$ is Zariski dense.

Let $\tilde{V}_p$ be a faithful $\mathbb{Q}_p$-linear representation of $\tilde{G}_{/\mathbb{Q}_p}$. Then there is no finite extension $F'$ of $F$ such that the representation of $G_{F'}$ on $\tilde{V}_p$ induced by $\tilde{\rho}_p$ belongs to the category $(\text{AV})-\text{Rep}_{\mathbb{Q}_p}(G_{F'}).$

Proof. Replace $F$ by a finite extension and assume that the representation of $G_F$ on $\tilde{V}_p$ belongs to $(\text{AV})-\text{Rep}_{\mathbb{Q}_p}(G_F)$. After further enlarging $F$ and applying proposition 5.3 and remark 5.10, it follows that there is an essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety $B/F$ such that $\tilde{V}_p$ belongs to $\langle H^1_{\text{et}}(B_{/\mathbb{Q}_p}), (\text{CM})-\text{Rep} \rangle$.

Let $B \sim \prod_{j=1}^{m-1} B_j$ be the decomposition from the definition 2.10, so that $G_B^\text{der} = \prod G_{B_j}^\text{der},$ each $G_{B_j}^\text{ad}$ is simple and $G_B^\text{ad} = \prod G_{B_j}^\text{ad}.$ For $j = 1, \ldots, m-1,$ let $\rho_j = \rho_{B_j,p},$ put $V_j = H^1_{\text{et}}(B_j/F, \mathbb{Q}_p)$ and let $H_j$ be the Zariski closure of $\rho_j(G_F)$ in $G_{B_j}.$ Let $V_m$ be an object of $(\text{CM})-\text{Rep}$ such that $\tilde{V}_p$ belongs to $\langle V_1, \ldots, V_m \rangle$, let $H_m$ be the Zariski closure of the image of the Galois representation on $V_m$ and let $\rho_m: G_F \to H_m(\mathbb{Q}_p)$ be the corresponding morphism. Finally, let $H \subset \prod_{j=1}^m H_j$ be the Zariski closure of the image of

$$(\rho_1, \ldots, \rho_m): G_F \to \prod_{j=1}^m H_j(\mathbb{Q}_p).$$
and \( \sigma: \mathcal{G}_F \to H(\mathbb{Q}_p) \) the induced representation. At the cost of a further finite extension of \( F \), we can assume \( H \) to be connected. The fact that \( \tilde{V}_p \) belongs to the tannakian category generated by the \( V_j \) implies that there is a surjection \( \pi_p: H \to \tilde{G}/\mathbb{Q}_p \) such that \( \tilde{\rho}_p = \pi_p \circ \sigma \).

For any \( p \)-adic place \( \mathfrak{o} \) of \( F \), the cocharacter \( \mu_{\tilde{\rho}_p,\mathfrak{o}}: \mathbb{G}_m/\mathbb{C}_p \to \tilde{G}/\mathbb{C}_p \) associated to the Hodge–Tate decomposition corresponding to \( (\tilde{\rho}_p)|_{\mathfrak{o}} \) lifts to the cocharacter \( \mu_{\sigma,\mathfrak{o}}: \mathbb{G}_m/\mathbb{C}_p \to H/\mathbb{C}_p \) associated to the Hodge–Tate decomposition corresponding to \( \sigma|_{\mathfrak{o}} \). There is at least one simple factor \( G'/\mathbb{C}_p \) of \( \tilde{G}^{\text{ad}} \) to which \( \mu_{\tilde{\rho}_p,\mathfrak{o}} \) projects non-trivially. It follows from [Win88, Proposition 7] and the fact that [Bla94, Theorem 0.3] implies [Win88, Conjecture 1] that this projection is dual to the root \( \alpha_k \) of this factor.

Let \( H'/\mathbb{C}_p \) be the simple isogeny factor of \( H^{\text{der}} \) which surjects onto \( G'/\mathbb{C}_p \), such a factor exists by hypothesis. As \( \tilde{G}^{\text{der}} \) is simply connected, \( H'/\mathbb{C}_p \) is simply connected as well. The Hodge–Tate cocharacter \( \mu_{\sigma,\mathfrak{o}} \) lifts to a quasi-cocharacter \( \mu'_{\mathfrak{o}} \) of \( H'/\mathbb{C}_p \). This quasi-cocharacter is still dual to the vertex \( \alpha_k \) of the Dynkin diagram. Over \( \mathbb{C}_p \), any faithful representation \( W' \) of \( H'/\mathbb{C}_p \) whose highest weights are fundamental weights contains a direct factor with highest weight \( \omega_k \) or \( \omega_{k-1} \) and it follows from lemma 5.3 applied to \( H'/\mathbb{C}_p \), with \( \mu'_{\mathfrak{o}} \) playing the role of \( \nu \), that \( \mu'_{\mathfrak{o}} \) has at least three weights on this factor. We will show that this leads to a contradiction.

As \( W = \bigoplus_{j=1}^{m-1} V_j \) is the Galois representation on \( H^1_{\text{ét}}(B_F, \mathbb{Q}_p) \), the Hodge–Tate cocharacter \( \mu_{\sigma,\mathfrak{o}} \) acts on \( W \otimes \mathbb{C}_p \) with two weights. By construction, \( W \) is a faithful representation of \( H^{\text{der}} \) so \( W \otimes \mathbb{Q}_p \mathbb{C}_p \) is a faithful representation of \( H'/\mathbb{C}_p \). Let \( W' \) be the direct sum of the direct factors of the representation of \( H'/\mathbb{C}_p \) on \( W \otimes \mathbb{C}_p \) on which \( H'/\mathbb{C}_p \) acts non-trivially. The quasi-cocharacter \( \mu'_{\mathfrak{o}} \) then acts with exactly two weights on \( W' \). It follows from [Del79, 1.3.7] that for every irreducible direct factor \( W'' \) of \( W' \), the highest weight is a fundamental weight of \( H'/\mathbb{C}_p \). We have shown above that this implies that \( \mu'_{\mathfrak{o}} \) has at least three weights on \( W'' \), so we arrive at the contradiction we were looking for. \( \square \)

6.3 Remark. It follows from theorems 2.9 and 2.12 and their proof, that for every \( k \geq 4 \) there exists an essentially M-T unliftable and M-T decomposed abelian variety \( A/\mathbb{C} \) such that \( G_A \) is of type \( D_k^{11} \). It follows from [Noo95, Theorem 1.7] that there also exists such an abelian variety which can be defined over a number field \( F \) and for which \( \rho_{A,p}(\mathcal{G}_F) \subset G_A(\mathbb{Q}_p) \) is Zariski dense. Thus, for
every \( k \geq 5 \), there is an abelian variety for which the hypothesis of the theorem is fulfilled.

6.4 The case \( k = 4 \). We consider the case of Galois representations of lifted abelian \( D^H_4 \)-type, so we keep all the notations of 6.1 but fix \( k = 4 \). In this case, the method of the proof of the above theorem gives a more limited statement. The reason for this lies in the facts that the representation of \( G_F \) on \( \tilde{V}_p \) may be reducible and that the group \( \tilde{G}/\mathbb{C} \) does have faithful representations in which the Hodge cocharacter acts with only two weights.

It is left to the reader to verify the following statement, whose proof is completely analogous to the proof of theorem 6.2.

6.5 Proposition. Let notations and hypotheses be as in theorem 6.2, but with \( k = 4 \). Let \( \tilde{V} \) be a faithful \( \mathbb{Q} \)-linear representation of \( \tilde{G} \). Then there is no finite extension \( F' \) of \( F \) such that the representation of \( G_{F'} \) on \( \tilde{V} \otimes \mathbb{Q}_p \) induced by \( \tilde{\rho}_p \) belongs to the category \((\text{AV})\text{-Rep}_{\mathbb{Q}_p}(G_{F'})\).

6.6 Remark. In the situation of the proposition, assume that \( p \) is a prime number such that the image of \( G_{\mathbb{Q}_p} \) in the automorphism group of the Dynkin diagram is equal to the image of \( G_{\mathbb{Q}} \) in this automorphism group. Then the statement of the proposition (and of theorem 6.2) holds for every \( \mathbb{Q}_p \)-linear representation \( \tilde{V}_p \) of \( \tilde{G}/\mathbb{Q}_p \). If \( G_{\mathbb{Q}} \) acts on the Dynkin diagram of \( G_{\mathbb{Q}} \) through a cyclic group, then the images of \( G_{\mathbb{Q}} \) and \( G_{\mathbb{Q}_p} \) in the automorphism group of the Dynkin diagram coincide for infinitely many \( p \).

On the other hand, there also exist examples of Galois representations of lifted abelian \( D^H_4 \)-type where the proof of the theorem fails for every \( p \). One can find an example of this situation by taking an abelian variety for which \( G_{\mathbb{Q}} \) acts on the Dynkin diagram of the Mumford–Tate group through \((\mathbb{Z}/2\mathbb{Z})^2\). For every prime number \( p \), the image of \( G_{\mathbb{Q}_p} \) in the automorphism group of the Dynkin diagram is then of order at most 2.

6.7 Frobenius elements. Let \( v \) be a valuation of \( F \). By \( \text{Fr}_v \in G_F \), we will denote a geometric Frobenius element at \( v \), i.e. an element of the decomposition group of a place \( \bar{v} \) of \( \bar{F} \) lying over \( v \) such that, on the residue field \( \bar{k}_v \) of \( \bar{F} \) at \( \bar{v} \), \( \text{Fr}_v \) induces the inverse of the map \( x \mapsto x^{q_v} \), where \( q_v \) is the order of the residue
field $k_v$ of $F$ at $v$. Note that $\mathrm{Fr}_v$ is defined only up to conjugation and up to multiplication by an element of $\mathcal{O}_{F,v}$. This implies that the image of $\mathrm{Fr}_v$ in a representation which is unramified at $v$ is defined up to conjugation and that the eigenvalues and the characteristic polynomial of the image of $\mathrm{Fr}_v$ in such a representation are well defined.

Let $V_p$ be a $\mathbb{Q}_p$-linear representation of $\mathcal{G}_F$ on an étale cohomology group of a proper and smooth $F$-variety and let $v$ be a place where this variety has good reduction. It follows from the Weil conjectures, proved by Deligne, that the eigenvalues of a Frobenius element $\mathrm{Fr}_v$ at $v$ are algebraic integers and that all complex absolute values of all these eigenvalues coincide. If the Fontaine–Mazur conjecture is true, then every geometric representation of $\mathcal{G}_F$ should have this property. We will verify this for the representations of lifted abelian $D^H_k$-type. It is also shown that $\mathrm{Fr}_v$ acts semi-simply in any representation of lifted abelian $D^H_k$-type, a property which is conjectured for all representations coming from the cohomology of algebraic varieties.

The Weil conjectures also imply that, for varying $p$, the $p$-adic étale cohomology groups of a proper and smooth variety form a compatible system $(V_p)$ of Galois representations. This means that there is a finite set $\Sigma$ of valuations of $F$ such that $V_p$ is unramified at $v$ for all $v \not\in \Sigma$ with $v(p) = 0$ and that for all $v \not\in \Sigma$ there is a polynomial $P^v \in \mathbb{Q}[X]$ which is equal to the characteristic polynomial of $\mathrm{Fr}_v$ acting on $V_p$ for all $p$ with $v(p) = 0$. We will prove a result in this direction for the representations of lifted abelian $D^H_k$-type.

It should be pointed out that this property does not follow from the conjecture of Fontaine and Mazur. However, combined with the Mumford–Tate conjecture, the Fontaine–Mazur conjecture implies that for any geometric $p$-adic representation $W_p$ of $\mathcal{G}_F$, there should exist a number field $E$ and a compatible system (indexed by the primes $p$ of $E$) of $E_p$-linear representations of $\mathcal{G}_F$ such that $W_p$ occurs in this system. The characteristic polynomials of $\mathrm{Fr}_v$ acting on the $V_p$ would then lie in $E[X]$ and be independent of $p$, for all $p$ with $v(p) = 0$.

**6.8 Proposition.** Let $F \subset \mathbb{C}$ be a number field and let $\tilde{\rho}_p : \mathcal{G}_F \to \tilde{\mathcal{G}}(\mathbb{Q}_p)$ be a representation of lifted abelian $D^H_k$-type, with $k \geq 4$. Let $\Sigma$ be the set of places of bad reduction defined in 5.8 and let $\tilde{V}$ be an irreducible $\mathbb{Q}$-linear representation of $\tilde{\mathcal{G}}$.

Then, for each $v \not\in \Sigma \cup \Sigma_p$, the Frobenius element $\tilde{\rho}_p(\mathrm{Fr}_v)$ acts semi-simply on $\tilde{V}_p = \tilde{V} \otimes \mathbb{Q}_p$ and its eigenvalues are algebraic integers with all complex absolute values equal to 1.
Proof. It suffices to prove the proposition for the faithful representation $\tilde{V}$ constructed in 5.4 and 5.5 (according to the parity of $k$) and after replacing $F$ by a finite extension. Let $\bar{\rho}_p$ be lifted from the representation associated to the M-T decomposed and M-T unliftable abelian variety $A/F$, with Mumford–Tate group $G_A$. The derived group $G_A^{\text{der}}$ is of the form $\text{Res}_{K_0/Q}G^{s,\text{der}}$ and we have $\bar{G}^{\text{der}} = \text{Res}_{K_0/Q}\bar{G}^{s,\text{der}}$ where $\bar{G}^{s,\text{der}}$ is the universal cover of $G^{s,\text{der}}$. By construction, the representation $\tilde{V}$ carries a structure of $K_0$-vector space compatible with the structure of a Weil restriction on $\bar{G}^{\text{der}}$, so the action of $\bar{G}$ on $\tilde{V}$ commutes with the action of $K_0$.

It is enough to prove the proposition for the representation of $G^{\text{der}}_F$ on the tensor product $\bar{W}_p = \bar{V}_p \otimes_{K_0} \bar{V}_p$, where $K_0,p = K_0 \otimes Q Q_p$. The representation of $\bar{G}_{Q_p}^{\text{der}}$ on $\bar{W}_p$ factors through a representation of $G^{\text{der}}_A/Q_p$. In fact, $\bar{W}_p$ is a faithful representation of $G^{\text{der}}_A/Q_p$, but this plays no role in what follows. By theorem 4.9, $\bar{W}_p$ occurs in the $p$-adic realization of an abelian motive $M$. The proof of 4.9 even shows that we can find such a motive $M$ in the subcategory of $(A \mathcal{V})_F$ generated by $h^{1}(A)$ and $(\mathbf{CM})_F$ so it follows that $M$ has potentially good reduction at $v$ for each $v \not\in \Sigma$. After replacing $F$ by a finite extension, we can assume that $M$ has good reduction at all $v \not\in \Sigma$. This implies that for any valuation $v \not\in \Sigma \cup \Sigma_p$, the Frobenius element $\bar{\rho}_p(F_{v})$ acts semi-simply on $\bar{W}_p$ and that each eigenvalue $\lambda$ is an algebraic integer with all complex absolute values of the form $N_v^w(\lambda)/2$, where $N_v$ is the cardinal of the residue field of $F$ at $v$ and $w(\lambda)$ is an integer.

To show that all $w(\lambda)$ are equal to 0, it suffices to show that the Betti realization of $M$ is pure of weight 0. This realization can be determined as follows. The cocharacter $\bar{\mu}$ constructed in 5.6 defines a map $\bar{h}_C: S/C \rightarrow \bar{G}/C$ given by $\bar{h}_C(z, \bar{z}) = \bar{\mu}(z)\bar{\mu}(\bar{z})$. This map descends to a map $\hat{h}: S \rightarrow \bar{G}/R$ lifting $h^{\text{ad}}_A: S \rightarrow G^{\text{ad}}_A$ and these data define a Hodge structure on $\tilde{V}$ on which $K_0$ acts by endomorphisms. The tensor product $\bar{W} = \bar{V} \otimes_{K_0} \bar{V}$ is a representation of $G_A^{\text{der}}$, and the Hodge structure on $\bar{W}$ derived from the Hodge structure on $\tilde{V}$ is the Hodge structure on the Betti realization of the abelian motive $M$. Since $\bar{\mu}$ and its complex conjugate are inverse to each other, it follows that the Hodge structure on $\tilde{V}$ is of weight 0 and thus the same thing holds for $\bar{W}$. \hfill \Box

6.9 We keep the notations of proposition 6.8. In particular, $\Sigma$ is the set of places of bad reduction defined in 5.8. For the representation $\tilde{V}$ however, we fix the $Q$-linear representation of $\bar{G}$ constructed in 5.4 or 5.5 as in the proof of propo-
It follows from corollary 5.9 that, after replacing $F$ by a finite extension, there is a system of weak geometric lifts $\tilde{\rho}_p^v : \mathcal{G}_F \to \tilde{G}(\mathbb{Q}_p)$ of the $\rho_{A,p}$, for varying $p$. From now on we will assume that we dispose of such a system.

The rest of the paper concerns the variation of the characteristic polynomials of the $\tilde{\rho}_p^v(\text{Fr}_v)$ on the $\tilde{V}_p = \tilde{V} \otimes \mathbb{Q}_p$, for a fixed valuation $v \not\in \Sigma$ of $F$ and varying $p$ with $v(p) = 0$. We will deduce our result on the characteristic polynomials from the statement 6.12 which is independent of the choice of a representation of $\tilde{G}$.

As in the proof of proposition 6.8, assume that $\tilde{\rho}_p$ is lifted from the representation associated to the Mumford–Tate decomposed and essentially M-T unliftable abelian variety $A/F$, with Mumford–Tate group $G_A$. Without loss of generality, we can assume that $G_A$ is of the type constructed in remark 2.13.2. We also keep the notations $G_{s,\text{der}}^\text{der} = \text{Res}_{K_0/\mathbb{Q}}G^s,\text{der}$ and $\tilde{G}^\text{der} = \text{Res}_{K_0/\mathbb{Q}}\tilde{G}^s,\text{der}$, for a totally real number field $K_0$ and semi-simple groups $G^s,\text{der}$ and $\tilde{G}^s,\text{der}$ over $K_0$. As explained in 2.13.2, there is a totally imaginary quadratic extension $L$ of $K_0$ such that $G_A$ is isogenous to a subgroup of $G^\text{der}_A \times L^\times$. It follows from loc. cit. that $L$ is contained in the centre of $\text{End}^0(A/\mathbb{C}) = \text{End}(A/\mathbb{C}) \otimes \mathbb{Z} \mathbb{Q}$. In this case, there exists a linear algebraic group $\tilde{G}^s$ over $K_0$ such that $G = \text{Res}_{K_0/\mathbb{Q}}\tilde{G}^s$.

Let $\tilde{N} \subset \tilde{G}^\text{der} \subset G$ be the centre of $\tilde{G}^\text{der}$. It is the kernel of the natural map $\tilde{G} \to G' = \tilde{G}^\text{ad} \times \tilde{G}^\text{ab}$. There is an isomorphism $\tilde{N} \cong \text{Res}_{K_0/\mathbb{Q}}\tilde{N}^s$, where $\tilde{N}^s$ is the centre of $\tilde{G}^s,\text{der}$, a finite group scheme over $K_0$ of order 4, geometrically isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ if $k$ is even and to $\mathbb{Z}/4\mathbb{Z}$ if $k$ is odd. Note that, by construction, each $\tilde{\rho}_p$ is determined up to a finite character $\mathcal{G}_F \to \tilde{N}(\mathbb{Q}_p)$ which is unramified outside $\Sigma \cup \Sigma_p$. This ambiguity explains the fact that in the next proposition the characteristic polynomials may vary with $p$.

For each valuation $v$ of $F$, each prime number $p$ and each $\varepsilon \in \tilde{N}(\mathbb{Q}_p)$, let $\tilde{P}_{t,\varepsilon,p}(X) \in \mathbb{Q}_p[X]$ be the characteristic polynomial

$$\tilde{P}_{t,\varepsilon,p}(X) = \det_{\tilde{\rho}_p^v}(\varepsilon \tilde{\rho}_p^v(\text{Fr}_v) - X \cdot \text{id}) .$$

We write $\tilde{P}_{\varepsilon,p}(X) = \tilde{P}_{t,1,p}(X)$ for the characteristic polynomial of $\tilde{\rho}_p^v(\text{Fr}_v)$.

**6.10 Proposition.** Assume that for some, hence any, prime number $\ell$, the rank of the Zariski closure of the image of $\rho_{A,\ell}$ is equal to the rank of $G_A$. Then there exist a set $\Sigma_{Fr} \supset \Sigma$ of valuations of $F$ of Dirichlet density 0 and, for each $v \not\in \Sigma_{Fr}$, a polynomial $\tilde{P}_v \in \mathbb{Q}_p[X]$ such that for every prime number $p$ with $v(p) = 0$, one has $\tilde{P}_{t,\varepsilon,p} = \tilde{P}_v$ for some $\varepsilon = \varepsilon_p \in \tilde{N}(\mathbb{Q}_p)$. 

Rutger Noot
6.11 Corollary. Let notations and hypotheses be as in the proposition. Let $M_{\text{nor}}$ be a normal closure of $K_0/Q$. For each prime number $p$, let $M_{\text{nor}}^p$ be the image of $M_{\text{nor}}$ in $\overline{Q}_p$ and put $M_p = M_{\text{nor}}^p \cap Q_p$. Then, for every $v \not\in \Sigma_{Fr}$, the characteristic polynomial $P_p^v$ of $\rho_p(F_{\ell,v})$ lies in $M_p[X]$.

Proofs. The $\ell$-independence of the rank of the Zariski closure of the image of $\rho_{A,\ell}$ follows from [Ser85, 2.2.4]. The corollary easily follows from the proposition. To prove the proposition we will reformulate the result in terms of the the quotient variety of $\tilde{G}$ by a subgroup $\text{Aut}' \subset \text{Aut}(\tilde{G}^{\text{der}})$.

If $k \geq 5$, we put $\text{Aut}'(\tilde{G}_s^{\text{der}}) = \text{Aut}(\tilde{G}_s^{\text{der}})$ and $\text{Out}'(\tilde{G}_s^{\text{der}}) = \text{Out}(\tilde{G}_s^{\text{der}})$. If $k = 4$ then $\text{Out}(\tilde{G}_s^{\text{der}}) \cong S_3$, acting naturally on the vertices $\alpha_1, \alpha_3, \alpha_4$ of the Dynkin diagram. Let $\text{Out}' \subset \text{Out}(\tilde{G}_s^{\text{der}})$ be the stabiliser of $\alpha_1$ and $\text{Aut}'(\tilde{G}_s^{\text{der}})$ the inverse image of $\text{Out}'$ in $\text{Aut}(\tilde{G}_s^{\text{der}})$.

In either case, $\text{Aut}'(\tilde{G}_s^{\text{der}})$ is an extension of $\text{Out}'(\tilde{G}_s^{\text{der}}) \cong \mathbb{Z}/2\mathbb{Z}$ by $\tilde{G}_s^{\text{ad}}$. It acts on the centre of $\tilde{G}_s^{\text{der}}$ through its quotient $\text{Out}'(\tilde{G}_s^{\text{der}})$ and by going through the constructions in 5.2 to 5.5 (sub)case by (sub)case, it is not very difficult to check that this action extends to an action of $\text{Out}'(\tilde{G}_s^{\text{der}})$ on the centre of $\tilde{G}_s$. We thus obtain an action of $\text{Aut}'(\tilde{G}_s^{\text{der}})$ on $\tilde{G}_s$ and, taking Weil restrictions, an action of $\text{Aut}' = \text{Res}_{K_0/Q}\text{Aut}'(\tilde{G}_s^{\text{der}})$ on $\tilde{G}_s$.

Let $\text{Cl}(\tilde{G})$ be the categorical quotient of $\tilde{G}$ by this action, see [MF82, Chapter 1]. This means that if $R = \Gamma(\tilde{G}, \mathcal{O}_{\tilde{G}})$ is the affine coordinate ring of $\tilde{G}$ then the quotient is given by $\text{Cl}(\tilde{G}) = \text{Spec}(R^{\text{Aut}'})$. For each prime number $p$, we denote by $\text{Cl}(\tilde{\rho}_p) : \mathcal{G}_F \to \text{Cl}(\tilde{G})(Q_p)$ the map deduced from $\tilde{\rho}_p$. The proposition 6.10 then follows from the following, more precise, statement.

6.12 Proposition. Assume that for some, hence any, prime number $\ell$, the rank of the Zariski closure of the image of $\rho_{A,\ell}$ is equal to the rank of $G_A$. Then there are a set $\Sigma_{Fr} \supset \Sigma$ of valuations of $F$ of Dirichlet density 0 and elements $\text{Cl}(F_{\ell,v}) \subset \text{Cl}(\tilde{G})(Q)$, for $v \not\in \Sigma_{Fr}$, such that for every prime number $p$ there exists $\epsilon_p \in \tilde{N}(Q_p)$ such that the conjugacy class of $\epsilon_p \tilde{\rho}_p(F_{\ell,v})$ is $\text{Cl}(\tilde{F}_{\ell,v})$.

The proof requires several lemmas and the following notation. The group $\text{Aut}'$ defined above also acts on $G_A^{\text{der}}$. As $\text{Out}'(\tilde{G}_s^{\text{der}}) \subset \text{Out}(G_A^{\text{der}})$ acts trivially on the centre of $G_A^{\text{der}}$, this action extends to an action of $\text{Aut}'$ on $G_A$ with trivial action on the centre. As above, we write $\text{Cl}(G_A)$ for the categorical quotient of $G_A$ by this action and let $\text{Cl}(\rho_{A,p}) : \mathcal{G}_F \to \text{Cl}(G_A)(Q_p)$ be the map induced by $\rho_{A,p} : \mathcal{G}_F \to G_A(Q_p)$, for each prime number $p$. 


Similarly, $\text{Aut}'$ acts naturally on $\tilde{G}^{\text{ad}}$ and on $\tilde{G}^{\text{ab}}$ so we deduce an action on $G' = \tilde{G}^{\text{ad}} \times \tilde{G}^{\text{ab}}$. Let $\text{Cl}(\tilde{G}^{\text{ad}})$ (resp. $\text{Cl}(G')$) be the quotient of $\tilde{G}^{\text{ad}}$ (resp. $G'$) by Aut'. The maps $G_A \to \tilde{G}^{\text{ad}} \to G'$ and $\tilde{G} \to G'$ are Aut’-equivariant and therefore induce maps $\text{Cl}(G_A) \to \text{Cl}(G')$ and $\text{Cl}(\tilde{G}) \to \text{Cl}(G')$.

Recall from 5.8 and corollary 5.9 that the $\tilde{\rho}_p$ lift the representations

$$\rho'_p = (\rho^{\text{ad}}_p, \rho^{\text{ab}}_p) : \mathcal{S}_F \to G'(\mathbb{Q}_p).$$

Let the $\text{Cl}(\rho'_p) : \mathcal{S}_F \to \text{Cl}(G')(\mathbb{Q}_p)$ be the induced maps.

6.13 Lemma. For each $v \notin \Sigma$, there is an element

$$\text{Cl}(\text{Fr}_{A,v}) \in \text{Cl}(G_A)(\mathbb{Q})$$

such that for every prime number $p$ with $v(p) = 0$ we have $\text{Cl}(\rho'_p)(\text{Fr}_v) = \text{Cl}(\text{Fr}_{A,v})$.

Proof. We use the notation of 5.9. The hypotheses that $G_A$ arises from the construction of 2.13.2 imply that $G^{\text{der}}_A \times L^\times$ acts on $H^1_b(A(C), \mathbb{Q})$ and that this representation is a Weil restriction of $W \otimes_{K_0} V^s$, where $W$ is the representation of $L^\times$ on $L$ by left multiplication and $V^s$ is a multiple of the representation of $G^{s,\text{der}}$ of highest weight $\omega_1$. It follows in particular that $L$ lies in the centre of $\text{End}^0(A_{/C}) = \text{End}(A_{/C}) \otimes \mathbb{Z} \mathbb{Q}$.

Since $v \notin \Sigma$, the abelian variety $A$ has good reduction $A_v$ at $v$ so we can identify $\text{End}^0(A_{/C})$ with a subalgebra of $\text{End}^0(A_{\theta})$. Here $A_{\theta}$ is the base extension of $A_v$ to the algebraic closure of the residue field of $F$ at $v$. We obtain an embedding $L \subset \text{End}^0(A_{\theta})$. Let $\pi_v : A_v \to A_{\theta}$ be the Frobenius endomorphism. For each prime number $p$ different from the residue characteristic at $v$, there is a canonical, hence $L$-equivariant, identification $H^1_{\text{et}}(A_{/F}, \mathbb{Q}_p) = H^1_{\text{et}}(A_{\theta}, \mathbb{Q}_p)$. Under this identification, the action of $\rho_p(\text{Fr}_v)$ on the left hand side corresponds to the action of $\pi_v$ on the right hand side.

Since $\pi_v$ is semi-simple and lies in the centre of $\text{End}^0(A_{/C})$, the subalgebra $M = L[\pi_v] \subset \text{End}^0(A_{/C})$ is a product of number fields $M_i$. Each $M_i$ is of the form $\mathbb{Q}(\alpha_i)$ for some $\alpha_i \in \text{End}^0(A_{/C})$ and it is a standard fact (cf. [Mum70, §19, theorem 4]) that the characteristic polynomial of $\alpha_i$ acting on $H^1_{\text{et}}(A_{/C}, \mathbb{Q}_p)$ has coefficients in $\mathbb{Q}$ and is independent of $p$. This implies that there exists an $M$-module $U_{\theta}$ such that for every prime number $p$ with $v(p) = 0$ there is an isomorphism $H^1_{\text{et}}(A_{/\theta}, \mathbb{Q}_p) \cong U_{\theta} \otimes \mathbb{Q} \mathbb{Q}_p$ of $M \otimes \mathbb{Q} \mathbb{Q}_p$-modules. Let $\mathbb{Q}[X] \in L[X]$ be
the characteristic polynomial of $\pi_v \in M$ acting on $U_v$ as an $L$-linear endomorphism. For every $p$ with $\nu(p) = 0$, the characteristic polynomial of $\rho_{A,p}(\text{Fr}_v)$, acting $L \otimes \mathbb{Q} Q_p$-linearly on $H^1_{\text{et}}(A_F, Q_p)$, is equal to $Q$. This result is due to Shimura, see [Shi67, 11.10.1].

Consider the map $G^{s,\text{der}} / \mathfrak{G} \to A^n / \mathfrak{G}$ corresponding to the characteristic polynomial in the representation with highest weight $\omega_1$. It factors through the quotient of $G^{s,\text{der}}$ for the action of $\text{Aut}'(G^{s,\text{der}})$, giving a map $\text{Cl}(G_A) \to \text{Res}_{L/\mathbb{Q}} A^n_L$. This map is easily seen to be injective on geometric points. The fact that $Q$ is the characteristic polynomial of $\rho_{A,p}(\text{Fr}_v)$ for every $p$ with $\nu(p) = 0$ implies that $Q$ is the image in $\text{Res}_{L/\mathbb{Q}} A^n_L$ of an element $\text{Cl}(\text{Fr}_{A,v}) \in \text{Cl}(G_A)(\mathbb{Q})$ and that this element $\text{Cl}(\text{Fr}_{A,v})$ verifies the condition of the lemma.

**6.14 Corollary.** For each $v \notin \Sigma$, there is an element

$$\text{Cl}(\text{Fr}_v') \in \text{Cl}(G')(\mathbb{Q})$$

such that for every prime number $p$ with $\nu(p) = 0$ we have $\text{Cl}(\rho_p')(\text{Fr}_v) = \text{Cl}(\text{Fr}_v')$.

**Proof.** By 5.8, the $\rho^a_p : \mathfrak{G} \to \tilde{G}^a(\mathbb{Q}_p)$ form a compatible system, so all $\rho^a_p(\text{Fr}_v)$ are defined over $\mathbb{Q}$ and coincide. This gives rise to an element $\text{Fr}_v^a \in G^a(\mathbb{Q})$. On the adjoint side, let $\text{Cl}(\text{Fr}_v') \in \text{Cl}(\tilde{G}^a)(\mathbb{Q})$ be the image of $\text{Cl}(\text{Fr}_{A,v})$ and let $\text{Cl}(\text{Fr}_v')$ be the image of $(\text{Cl}(\text{Fr}_v^a), \text{Fr}_v^a)$ in $\text{Cl}(G')(\mathbb{Q})$.

For each $p$, the representation $\tilde{\rho}_p$ is a weak geometric lift of $\rho_{A,p}$ so the projection $\tilde{\rho}_p^a : \mathfrak{G} \to \tilde{G}^a(\mathbb{Q}_p)$ coincides with the composite of $\rho_{A,p}$ with the projection $G_A(\mathbb{Q}_p) \to \tilde{G}^a(\mathbb{Q}_p)$. It follows that $\text{Cl}(\text{Fr}_v')$ is the element promised by the corollary.

**6.15 Lemma.** Let $M$ be either the group scheme $\tilde{N} = \text{Res}_{K_0/\mathbb{Q}} \tilde{N}^s$ (the centre of $\tilde{G}^\text{der}$, cf. 5.9) or

$$\text{Out}'(\tilde{G}) = \text{Aut}'(\tilde{G}) / \tilde{G}^\text{ad} \cong \text{Res}_{K_0/\mathbb{Q}} H_2$$

and let $P$ be a co-finite set of prime numbers. Then the natural map

$$H^1_{\text{et}}(\text{Spec}(\mathbb{Q}), M) \to \prod_{p \in P} H^1_{\text{et}}(\text{Spec}(\mathbb{Q}_p), M)$$

is injective.

**Proof.** In the lemma, $M$ denotes either $\tilde{N}$ or $\text{Out}'(\tilde{G})$. We put $M' = \tilde{N}^s$ in the former and $M' = \mu_2$ in the latter case. Let $f : \text{Spec}(K_0) \to \text{Spec}(\mathbb{Q})$ be the
natural morphism. This implies that $\mathcal{M} = f_* \mathcal{M}'$ as étale sheaves and since $f_*$ is exact by [Mil80, II, Corollary 3.6], this gives an isomorphism

$$H^1_{\text{ét}}(\text{Spec}(\mathbb{Q}), \mathcal{M}) \cong H^1_{\text{ét}}(\text{Spec}(\mathbb{K}_0), \mathcal{M}')$$

cf. [Ser94, I. 2.5] for the interpretation in terms of Galois cohomology. Similarly, for every prime number $p$, there is an isomorphism

$$H^1_{\text{ét}}(\text{Spec}(\mathbb{Q}_p), \mathcal{M}) \cong \prod_{p \mid p} H^1_{\text{ét}}(\text{Spec}(\mathbb{K}_0, p), \mathcal{M}')$$

obtained from the above identification by base change to $\text{Spec}(\mathbb{Q}_p)$. To prove the lemma, it is sufficient to prove the injectivity of the map

$$H^1_{\text{ét}}(\text{Spec}(\mathbb{K}_0), \mathcal{M}') \to \prod_{p \in P'} H^1_{\text{ét}}(\text{Spec}(\mathbb{K}_0, p), \mathcal{M}')$$

where $P'$ is the set of primes of $\mathbb{K}_0$ lying over the rational primes in $P$. In the case where $\mathcal{M} = \text{Out}'(\tilde{G})$, the group scheme $\mathcal{M}' = \mu_2$ is constant and as $H^1_{\text{ét}}(\text{Spec}(\mathcal{K}), \mu_2) \cong \text{Hom}(\mathcal{g}_K, \mu_2(\mathcal{K}))$ for any field $\mathcal{K}$, the lemma follows.

In the case where $\mathcal{M}' \cong \tilde{N}_s$, there are two possibilities, $\mathcal{M}'$ is geometrically isomorphic to either $(\mathbb{Z}/2\mathbb{Z})^2$ or $\mathbb{Z}/4\mathbb{Z}$. In both cases the group scheme becomes trivial over an extension of degree (at most) 2. If $\mathcal{M}'$ is already trivial over $\mathbb{K}_0$, then the above argument applies. If $\mathcal{M}'$ is geometrically isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and trivial over the quadratic extension $\mathbb{K} \supset \mathbb{K}_0$, then $\mathcal{M}'$ is a Weil restriction and the lemma follows by the same argument as before.

We are left with the case where $\mathcal{M}'_K \cong (\mathbb{Z}/4\mathbb{Z})_\mathcal{K}$ for a quadratic extension $\mathbb{K} \supset \mathbb{K}_0$. The proof in this remaining case, which also applies in the other cases where $\mathcal{M}' \cong N^s$, makes use of the long exact cohomology sequences associated to the short exact sequence $1 \to \mu_2 \to \mathcal{M}' \to \mu_2 \to 1$ of étale sheaves on $\text{Spec}(\mathbb{K}_0)$. This gives a commutative diagram with exact rows,

$$
\begin{array}{c}
\mu_2(\mathbb{K}_0) \rightarrow H^1(\mathbb{K}_0, \mu_2) \rightarrow H^1(\mathbb{K}_0, \mathcal{M}') \rightarrow H^1(\mathbb{K}_0, \mu_2) \\
\downarrow \quad \downarrow \text{loc}_{\mu_2} \quad \downarrow \text{loc}_{\mathcal{M}'} \quad \downarrow \text{loc}_{\mu_2} \\
\prod \mu_2(\mathbb{K}_0, p) \rightarrow \prod H^1(\mathbb{K}_0, p, \mu_2) \rightarrow \prod H^1(\mathbb{K}_0, p, \mathcal{M}') \rightarrow \prod H^1(\mathbb{K}_0, p, \mu_2).
\end{array}
$$

Here all $H^1$ are étale cohomology groups over the spectrum of the specified field. These groups can be identified with the corresponding Galois cohomology groups. The products in the second row are over all $p \in P'$. 
The image of $i$ identifies with $K_0^x / \langle K_0^{x^2}, \alpha \rangle$ for some $\alpha \in K_0^x$, where $\langle K_0^{x^2}, \alpha \rangle$ is the subgroup of $K_0^x$ generated by $\alpha$ and the squares in $K_0^x$. Similarly, the image of each $i_p$ is identified with $K_0^x / \langle K_0^{x^2}, \alpha \rangle$. If $x \in \ker \text{loc}_{M'}$, then the injectivity of $\text{loc}_{\mu_2}$ implies that $x$ lies in the image of $i$. Let $y \in K_0^x$ represent the preimage of $x$ in $K_0^x / \langle K_0^{x^2}, \alpha \rangle$. Since $x \in \ker \text{loc}_{M'}$, the element $y$ lies in $\langle K_0^{x^2}, \alpha \rangle$ for each $p \in P'$, so $y$ is a square in $K_0(\sqrt{\alpha})$ locally at each place above a $p \in P'$. It follows that $y$ is a square in $K_0(\sqrt{\alpha})$, hence it maps to 1 in $K_0^x / \langle K_0^{x^2}, \alpha \rangle$.

**Proof of proposition 6.12.** The map $\tilde{G} \to G'$ induces a map $\text{pr}: \text{Cl}(\tilde{G}) \to \text{Cl}(G')$. Obviously,

$$\text{Cl}(\rho'_p) = \text{pr}_{Q_p} \circ \text{Cl}(\tilde{\rho}_p): \mathcal{G}_F \to \text{Cl}(G')(Q_p)$$

for each $p$.

For the groups $\tilde{G}$ and $G'$, let $\text{Conj}(\tilde{G})$ and $\text{Conj}(G')$ be the varieties of geometric conjugacy classes, i.e. the quotients for the actions of $\tilde{G}^{ad}$. Note that $\text{Cl}(\tilde{G})$ (resp. $\text{Cl}(G')$) is the quotient of $\text{Conj}(\tilde{G})$ (resp. $\text{Conj}(G')$) by $\text{Out}'(\tilde{G})$ and that there is a commutative diagram

$$\begin{array}{ccc}
\text{Conj}(\tilde{G}) & \longrightarrow & \text{Cl}(\tilde{G}) \\
\downarrow & & \downarrow \text{pr} \\
\text{Conj}(G') & \longrightarrow & \text{Cl}(G').
\end{array}$$

There is a Zariski closed subset $B \subset \text{Cl}(G')$ such that $\text{Conj}(G') \to \text{Cl}(G')$ and $\text{pr}$ are both unramified outside $B$. Let the $\text{Cl}(F_{\tilde{G}})$ be as in corollary 6.14. The hypothesis on the rank of the Zariski closure of the image of $\rho_{A,\ell}$ and the Chebotarev density theorem imply that there is a set $\Sigma_{F_{\ell}}$ of places of $F$ of Dirichlet density 0 such that $\text{Cl}(F_{\tilde{G}}) \in B(Q)$ if and only if $v \notin \Sigma_{F_{\ell}}$. To prove the proposition, it suffices to show that for every $v \notin \Sigma_{F_{\ell}}$, the fibre of $\text{pr}$ over $\text{Cl}(F_{\tilde{G}})$ contains an element $\text{Cl}(\tilde{F}_{v}) \in \text{Cl}(\tilde{G})(Q)$.

Fix $v \notin \Sigma_{F_{\ell}}$. The fibre of $\text{Conj}(G') \to \text{Cl}(G')$ over $\text{Cl}(F_{\tilde{G}})$ is a $\text{Out}'$-torsor which admits a $Q_p$-valued point $\text{Conj}(\tilde{\rho}_p(F_{\tilde{G}}))$ for almost every $p$. It follows from lemma 6.13 that there exists $\text{Conj}(\tilde{F}_{v}) \in \text{Conj}(G')(Q)$ above $\text{Cl}(F_{\tilde{G}})$. The fibre of $\text{Conj}(\tilde{G}) \to \text{Conj}(G')$ over this element is a $\tilde{N}$-torsor. As this fibre identifies with $\text{pr}^{-1}(\text{Cl}(F_{\tilde{G}}))$, this last fibre is also a $\tilde{N}$-torsor. It has a $Q_p$-valued point $\text{Cl}(\tilde{\rho}_p(F_{\tilde{G}}))$ for almost every $p$, so the promised existence of $\text{Cl}(\tilde{F}_{v}) \in \text{Cl}(\tilde{G})(Q)$ follows from another application of lemma 6.13. \qed
References

[Bla94] D. Blasius. A $p$-adic property of Hodge classes on abelian varieties. In U. Jansen, S. Kleiman, and J.-P. Serre, editors, *Motives*, Proc. Sympos. Pure Math. 55, Part 2, pages 293–308. Amer. Math. Soc., 1994.

[Del72] P. Deligne. La conjecture de Weil pour les surfaces $K3$. *Invent. Math.* 15 (1972), 206–226.

[Del79] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In A. Borel and W. Casselman, editors, *Automorphic forms, representations, and L-functions*, Proc. Sympos. Pure Math. XXXIII, Part 2, pages 247–289. Amer. Math. Soc., 1979.

[Del82a] P. Deligne. Hodge cycles on abelian varieties. In P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, editors, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math. 900, chapter I, pages 9–100. Springer-Verlag, 1982.

[Del82b] P. Deligne. Motifs et groupes de Taniyama. In P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, editors, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math. 900, chapter IV, pages 261–279. Springer-Verlag, 1982.

[dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Publ. Math. de l’I. H. E. S.* 83 (1996), 51–93.

[DM82] P. Deligne and J. S. Milne. Tannakian categories. In P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, editors, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math. 900, chapter II, pages 101–228. Springer-Verlag, 1982.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.* 73, 3 (1983), 349–366. Erratum in [Fal84]. English translation published as [Fal86].

[Fal84] G. Faltings. Erratum: “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern”. *Invent. Math.* 75, 2 (1984), 381.
[Fal86] G. Faltings. Finiteness theorems for abelian varieties over number fields. In G. Cornell and J. H. Silverman, editors, *Arithmetic geometry*, chapter II, pages 9–27. Springer-Verlag, 1986. English translation of [Fal83] and [Fal84].

[FM95] J.-M. Fontaine and B. Mazur. Geometric Galois representations. In J. Coates and S. T. Yau, editors, *Elliptic curves, modular forms, & Fermat’s last theorem*, Ser. Number Theory I, pages 41–78. International Press Inc., 1995.

[Fon94] J.-M. Fontaine. Représentations $p$-adiques semi-stables. In *Périodes $p$-adiques, séminaire de Bures 1988*, Astérisque 223, pages 113–184. Soc. Math. France, 1994.

[MF82] D. Mumford and J. Fogarty. *Geometric invariant theory*, Ergeb. Math. Grenzgeb. 34. Springer-Verlag, second enlarged edition edition, 1982.

[Mil80] J. S. Milne. *Étale cohomology*. Princeton Univ. Press, 1980.

[Mum69] D. Mumford. A note of Shimura’s paper “Discontinuous groups and abelian varieties”. *Math. Ann.* 181 (1969), 345–351.

[Mum70] D. Mumford. *Abelian varieties*. Oxford university press, 1970.

[Noo95] R. Noot. Abelian varieties—Galois representations and properties of ordinary reduction. *Compos. Math.* 97, 1–2 (1995), 161–171.

[Noo01] R. Noot. Lifting Galois representations, and a conjecture of Fontaine and Mazur. *Doc. Math.* 6 (2001), 419–445.

[Pau04] F. Paugam. Galois representations, Mumford-Tate groups and good reduction of abelian varieties. *Math. Ann.* 329, 1 (2004), 119–160.

[Pin98] R. Pink. $\ell$-adic algebraic monodromy groups, cocharacters, and the Mumford–Tate conjecture. *J. Reine Angew. Math.* 495 (1998), 187–237.

[Ser78] J-P. Serre. Groupes algébriques associés aux modules de Hodge–Tate. In *Journées de géométrie algébrique de Rennes (III)*, Astérisque 65, pages 155–188. Soc. Math. France, 1978. Also published in [Ser86, 119].
[Ser85] J-P. Serre. Résumé des cours et travaux. In *Annuaire du Collège de France 1984–1985*, pages 84–91, 1985. Also published in [Ser00], 135.

[Ser86] J-P. Serre. *Œuvres — Collected papers*. Springer-Verlag, 1986.

[Ser94] J-P. Serre. *Cohomologie galoisienne*, Lecture Notes in Math. 5. Springer-Verlag, fifth edition, 1994.

[Ser00] J-P. Serre. *Œuvres — Collected papers, Volume IV*. Springer-Verlag, 2000.

[Shi67] G. Shimura. Algebraic number fields and symplectic discontinuous groups. *Ann. of Math. (2)* **86** (1967), 503–592.

[Tsu99] T. Tsuji. p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. *Invent. Math.* **137**, 2 (1999), 233–411.

[Vas03] A. Vasiu. The Mumford–Tate conjecture and Shimura varieties, Part I. e-print, arXiv:math.NT/0212066, 2003. Version 3, 16 Sep. 2003, http://www.arxiv.org/abs/math/0212066v3.

[Win88] J.-P. Wintenberger. Motifs et ramification pour les points d’ordre fini des variétés abéliennes. In *Séminaire de théorie des nombres de Paris 1986-87*, Progr. Math. 75, pages 453–471. Birkhäuser, 1988.

[Win95] J.-P. Wintenberger. Relèvement selon une isogénie de systèmes ℓ-adiques de représentations galoisiennes associés aux motifs. *Invent. Math.* **120**, 2 (1995), 215–240.

Rutger Noot
I. R. M. A.
Université Louis Pasteur and CNRS
7 rue René Descartes
67084 Strasbourg, France
rutger.noot@math.u-strasbg.fr
http://www-irma.u-strasbg.fr/~noot