Adjunctions between Eilenberg-Moore categories and a PBW-type theorem

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Abstract

Recently, Dotsenko and Tamaroff have shown that a morphism of $T \to S$ of monads over a category $\mathcal{C}$ satisfies the PBW-property if and only if it makes $S$ into a free right $T$-module. We consider an adjunction $\Psi = (G, F)$ between categories $\mathcal{C}, \mathcal{D}$, a monad $S$ on $\mathcal{C}$ and a monad $T$ on $\mathcal{D}$. We show that a morphism $\phi : (\mathcal{C}, S) \to (\mathcal{D}, T)$ that is well behaved with respect to the adjunction $\Psi$ has a PBW-property if and only if it makes $S$ satisfy a certain freeness condition with respect to $T$-modules with values in $\mathcal{C}$.

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1 Introduction

Let $\mathcal{C}$ be a category and let $\phi : T \to S$ be a morphism of monads on $\mathcal{C}$. Let $\mathcal{C}^S$ and $\mathcal{C}^T$ denote respectively the Eilenberg-Moore categories with respect to $S$ and $T$. Then, if $x \in \mathcal{C}$ carries the structure of an $S$-algebra, given by a structure map $\lambda_x : Sx \to x$, the composition $Tx \xrightarrow{\phi} Sx \xrightarrow{\lambda_x} x$ gives $x$ the structure of a $T$-algebra. This determines a functor $\phi_* : \mathcal{C}^S \to \mathcal{C}^T$. Moreover, this functor $\phi_*$ has a left adjoint

$$\phi^! : \mathcal{C}^T \to \mathcal{C}^S$$

which may be viewed as a “universal enveloping algebra.” In $[3]$, Dotsenko and Tamaroff prove a striking result; they say that a morphism of monads on $\mathcal{C}$ has the PBW property if the underlying object of the universal enveloping algebra of an object in $\mathcal{C}^T$ does not depend on its $T$-algebra structure. Then, Dotsenko and Tamaroff $[3]$ show that, if the Eilenberg-Moore categories contain reflexive coequalizers, $\phi : T \to S$ satisfies the PBW-property if and only if $\phi$ makes $S$ into a free right $T$-module.

In our situation, we start with an adjunction $\Psi = (G, F)$ between categories $\mathcal{C}$ and $\mathcal{D}$. We consider a monad $S$ on $\mathcal{C}$ and a monad $T$ on $\mathcal{D}$. By a $\Psi$-morphism $\phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \to (\mathcal{D}, T)$, we mean a pair of natural transformations $\phi_G : GT \to TS$ and $\phi_F : TF \to FS$ satisfying certain condition that make the “morphism of monads” compatible with the adjunction $\Psi = (G, F)$. We then define a PBW-property similar to that of $[3]$. Our aim is to show that a $\Psi$-morphism has the PBW-property if and only if it satisfies a certain freeness condition with respect to $T$-modules with values in $\mathcal{C}$.

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2 Adjunctions between Eilenberg-Moore categories

We begin by fixing some notation. Suppose that \((G, F)\) is a pair of adjoint functors between categories \(\mathcal{A}\) and \(\mathcal{B}\). Accordingly, for objects \(a \in \mathcal{A}\) and \(b \in \mathcal{B}\), we have an isomorphism

\[
\mathcal{B}(Ga, b) \cong \mathcal{A}(a, Fb)
\]  

(2.1)

For a morphism \(g \in \mathcal{B}(Fa, b)\), we denote by \(g^R\) the corresponding element in \(\mathcal{A}(a, Gb)\). Similarly, for any \(f \in \mathcal{A}(a, Fb)\), we will denote by \(f^L\) the corresponding element in \(\mathcal{B}(Ga, b)\).

A monad on a category \(\mathcal{C}\) consists of an endofunctor \(S : \mathcal{C} \rightarrow \mathcal{C}\) along with natural transformations \(\mu(S) : S^2 \rightarrow S\) and \(\eta(S) : 1 \rightarrow S\) satisfying the usual associativity and unit conditions. We will denote this monad as a pair \((S, \mu, \eta)\). Given a monad \(\rho : \mathcal{C} \rightarrow \mathcal{D}\), we denote by \(S\rho\) the corresponding element in \(\mathcal{D}\).

An \(S\)-algebra, or more precisely, an algebra over the monad \((S, \mu, \eta)\), is a pair \((x, \lambda_x) \in \mathcal{C}\) along with a morphism \(\lambda_x : Sx \rightarrow x\) which satisfies associativity and unit conditions with respect to the action of \(S\), i.e., \(\lambda_x \circ \mu(S)x = \lambda_x \circ (S\lambda_x)\) and \(1_x = \lambda_x \circ \eta(S)x\). The category of algebras over the monad \((S, \mu, \eta)\) is the Eilenberg-Moore category which we denote by \(\mathcal{C}^S\). A morphism \(f : (x, \lambda_x) \rightarrow (x', \lambda_{x'})\) in \(\mathcal{C}^S\) is given by \(f : x \rightarrow x'\) such that \(\lambda_{x'} \circ Sf = f \circ \lambda_x\).

Given a monad \(\mathcal{C}^S\), we also denote by \(\rho(S) : \mathcal{C} \rightarrow \mathcal{C}^S\) the free functor which is left adjoint to the forgetful functor \(\pi(S) : \mathcal{C}^S \rightarrow \mathcal{C}\). The following result is well known.

**Lemma 2.1.** Let \((F, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)\) be a morphism of monads. Then, there is a functor \(\hat{\phi}_F : \mathcal{C}^S \rightarrow \mathcal{D}^T\) that fits into the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}^S & \xrightarrow{\hat{\phi}_F} & \mathcal{D}^T \\
\pi(S) \downarrow & & \downarrow \pi(T) \\
\mathcal{C} & \xrightarrow{\phi} & \mathcal{D}
\end{array}
\]  

(2.3)

**Proof.** We consider \((y, \lambda_y) \in \mathcal{C}^S\). It may be verified that the composition

\[
\lambda_{Fy} : T[Fy] \xrightarrow{\phi_F(y)} FSy \xrightarrow{F\lambda_y} Fy
\]  

(2.4)

gives \(Fx\) the structure of an algebra over \((\mathcal{D}, T)\). This defines a functor \(\hat{\phi}_F : \mathcal{C}^S \rightarrow \mathcal{D}^T\) that makes the diagram (2.3) commute.

\(\square\)

We now fix a pair \(\Psi = (G : \mathcal{D} \rightarrow \mathcal{C}, F : \mathcal{C} \rightarrow \mathcal{D})\) of adjoint functors between \(\mathcal{D}\) and \(\mathcal{C}\). Our aim is to construct a commutative diagram of left adjoints corresponding to the commutative square of right adjoints in (2.3).

**Definition 2.2.** Let \((\mathcal{C}, S)\) and \((\mathcal{D}, T)\) be monads and \(\Psi = (G, F)\) be an adjunction between \(\mathcal{D}\) and \(\mathcal{C}\). A \(\Psi\)-morphism of monads is a pair \(\phi = (\phi_G, \phi_F)\) such that:

(1) \(\phi_G : GT \rightarrow SG\) is a natural transformation which satisfies

\[
\begin{array}{ccc}
GT^2 & \xrightarrow{\phi_G^T} & SG \\
\mu(GT) \downarrow & & \downarrow \mu(SG) \\
GT & \xrightarrow{\phi_G} & SG
\end{array}
\]  

(2.5)
(2) \( \phi_F : TF \rightarrow FS \) is a natural transformation which satisfies

\[
T^2F \xrightarrow{T\phi_F} TFS \xrightarrow{\phi_F S} FS^2
\]

\[
\mu(T)F \xrightarrow{\phi_F} TF \xrightarrow{\phi_F} FS
\]

(3) For any morphism \( f : Gx \rightarrow y \) in \( \mathcal{C} \), the following two equivalent conditions are satisfied

\[
\frac{GTx \xrightarrow{GT(f^R)} GTFy}{\phi_G(x)} \Leftrightarrow \frac{Tx \xrightarrow{T(f^R)} TFy}{\phi_F(y)}
\]

Lemma 2.3. Let \( \phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T) \) be a \( \Psi \)-morphism of monads. Let \( (x, \lambda_x) \in \mathcal{D}^T \). Then, the following two compositions are identity

\[
SGx \xrightarrow{SG\eta(T)x} SGTx \xrightarrow{(\mu(S)Gx) \circ (SG\phi_G(x))} SGx
\]

Proof. From the condition (2.5), we observe that

\[
Gx \xrightarrow{G\eta(T)(x)} GTx \xrightarrow{\phi_G(x)} SGx \Rightarrow \frac{SGx \xrightarrow{SG\eta(T)x} SGTx}{SGx \xrightarrow{SG\eta(T)x} SGTx}
\]

Since \( (\mu(S)Gx) \circ (SG\eta(T)x) = id \), it follows from (2.9) that \( (\mu(S)Gx) \circ (SG\phi_G(x)) \circ (SG\eta(T)x) = id \). Additionally, since \( (x, \lambda_x) \) is an algebra over \( (\mathcal{D}, T) \), we have \( (SG\lambda_x) \circ (SG\eta(T)x) = SG(\lambda_x \circ \eta(T)x) = id \).

From now onwards, we assume that the Eilenberg-Moore categories \( \mathcal{E}^S \) and \( \mathcal{D}^T \) both contain reflexive coequalizers. For more on this condition, we refer the reader to [1, 2, § 9.3] and [3].

Lemma 2.4. Let \( \phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T) \) be a \( \Psi \)-morphism of monads. Let \( (x, \lambda_x) \in \mathcal{D}^T \). Then, the coequalizer

\[
Coeq\left(SGTx \xrightarrow{(\mu(S)Gx) \circ (SG\phi_G(x))} SGx\right)
\]

in \( \mathcal{C} \) is equipped with the structure of an \( S \)-algebra. This determines a functor \( \hat{\phi}_G : \mathcal{D}^T \rightarrow \mathcal{E}^S \).

Proof. We observe that each of the objects in (2.5) is canonically equipped with the structure of an \( S \)-algebra and each arrow in (2.5) is a morphism of \( S \)-algebras. Applying Lemma 2.5, it follows that the coequalizer in (2.10) is a reflexive coequalizer. By assumption, \( \mathcal{E}^S \) contains reflexive coequalizers and the result follows.

Proposition 2.5. Let \( \Psi = (G : \mathcal{D} \rightarrow \mathcal{E}, F : \mathcal{C} \rightarrow \mathcal{D}) \) be a pair of adjoint functors between \( \mathcal{D} \) and \( \mathcal{C} \). Let \( \phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T) \) be a \( \Psi \)-morphism of monads. Then, \( \phi_G : \mathcal{D}^T \rightarrow \mathcal{E}^S \) is left adjoint to the functor \( \hat{\phi}_F \) and fits into the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{E}^S & \xrightarrow{\hat{\phi}_G} & \mathcal{D}^T \\
\rho(S) & \uparrow & \rho(T) \\
\mathcal{E} & \xleftarrow{G} & \mathcal{D}
\end{array}
\]
Proof. We consider \((x, \lambda_x) \in \mathcal{D}^T\), \((y, \lambda_y) \in \mathcal{C}^S\) and a morphism \(\beta : x \to F y\) which is a map of \(T\)-algebras. In particular, \(\beta \in \mathcal{D}(x,Fy)\). Accordingly, we have a morphism \(\beta^L : G x \to y\) in \(\mathcal{C}\). From the coequalizer in (2.10) we see that in order to construct a morphism \(\alpha : \hat{\phi}_G(x) \to y\) we must show that

\[
\lambda_y \circ S \beta^L \circ (\mu(S)G x) \circ (S \hat{\phi}_G(x)) = \lambda_y \circ S \beta^L \circ (SG \lambda_x)
\]  

(2.12)

Since \(\beta : x \to F y\) is a morphism of \(T\)-algebras, the following diagrams commute

\[
\begin{array}{ccc}
T x & \xrightarrow{T \beta} & TF y \\
\downarrow{\lambda_x} & & \downarrow{F \lambda_y} \\
x & \xrightarrow{\beta} & F y
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccc}
G T x & \xrightarrow{G T \beta} & GTF y \\
\downarrow{G \lambda_x} & & \downarrow{\lambda_y} \\
G x & \xrightarrow{\beta^L} & y
\end{array}
\]  

(2.13)

Here the right hand side diagram follows from the left hand side diagram using the adjointness of \((G, F)\). Applying the functor \(S\) to the right hand side diagram in (2.13) and composing with \(\lambda_y\), we obtain

\[
\begin{array}{ccc}
SG T x & \xrightarrow{SG T \beta} & SGTF y \\
\downarrow{SG \lambda_x} & & \downarrow{S \lambda_y} \\
SG x & \xrightarrow{s \beta^L} & Sy \\
\downarrow{S \phi_G(x)} & & \downarrow{\lambda_y} \\
SG x & \xrightarrow{S \beta^L} & Sy \\
\downarrow{S \phi_G(x)} & & \downarrow{\lambda_y} \\
SG x & \xrightarrow{S \beta^L} & Sy
\end{array}
\]  

(2.14)

Using the compatibility condition in (2.10) in Definition 2.2 we also have the following commutative diagram

\[
\begin{array}{ccc}
SG T x & \xrightarrow{SG T \beta} & SGTF y \\
\downarrow{S \phi_G(x)} & & \downarrow{S \phi_G(x) \mu(S)y} \\
SSG x & \xrightarrow{S S \beta^L} & SSy \\
\downarrow{\mu(S)G x} & & \downarrow{\mu(S)y} \\
SG x & \xrightarrow{S \beta^L} & Sy \\
\downarrow{\mu(S)G x} & & \downarrow{\lambda_y} \\
SG x & \xrightarrow{S \beta^L} & Sy
\end{array}
\]  

(2.15)

Combining (2.14) and (2.15), the equality in (2.12) follows. It may be verified directly that \(\alpha : \hat{\phi}_G(x) \to y\) is a morphism of \(S\)-algebras. Conversely, if we have an \(S\)-algebra morphism \(\hat{\phi}_G(x) \to y\), we have an induced map \(G x \xrightarrow{\mu(S)G x} SG x \to y\), which corresponds to a morphism \(x \to F y\). Again, it may be verified directly that this is a \(T\)-algebra morphism and these two associations are inverses of each other.

Accordingly, we now have \(\hat{\phi}_G : \mathcal{D}^T \to \mathcal{C}^S\) which is left adjoint to \(\hat{\phi}_F : \mathcal{C}^S \to \mathcal{D}^T\). Then, \(\hat{\phi}_G \circ \rho(T)\) is left adjoint to \(\pi(T) \circ \hat{\phi}_F\). By Lemma 2.1, we have \(\pi(T) \circ \hat{\phi}_F = F \circ \pi(S)\) and hence the diagram (2.11) of left adjoints must commute. 

\[
\square
\]

3 The Main theorem

We begin with the following definition.

**Definition 3.1.** Let \(\mathcal{C}, \mathcal{D}\) be categories and let \((\mathcal{D}, T)\) be a monad. By a \(T\)-module taking values in \(\mathcal{C}\), we will mean a pair \((P, \nu(P))\) such that

(a) \(P : \mathcal{D} \to \mathcal{C}\) is a functor
Lemma 3.3. Let the following diagram commute.\[\begin{array}{ccc}
PT & \xrightarrow{\nu(P)T} & PT \\
\nu(P) & \downarrow & \nu(P) \\
PT & \xrightarrow{\nu(P)} & P
\end{array}\] A morphism \(\alpha : (P, \nu(P)) \rightarrow (P', \nu(P'))\) of such \(T\)-modules consists of a natural transformation \(\alpha : P \rightarrow P'\) of functors satisfying \(\nu(P') \circ (\alpha T) = \alpha \circ \nu(P)\). We will denote by \(\text{Mod}_C^T\) the category of \(T\)-modules taking values in \(\mathcal{C}\).

**Proposition 3.2.** Let \(\mathcal{E}, \mathcal{D}\) be categories and let \((\mathcal{D}, T)\) be a monad. There is a canonical functor
\[
\Sigma_T^C : \text{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Mod}_C^T \quad P \mapsto (PT, P\mu(T))
\]
which is left adjoint to the forgetful functor \(\text{Mod}_C^T \rightarrow \text{Fun}(\mathcal{D}, \mathcal{E})\).

**Proof.** Since \((\mathcal{D}, T)\) is a monad, it is easy to verify that for any \(P \in \text{Fun}(\mathcal{D}, \mathcal{E})\), the pair \((PT, P\mu(T))\) satisfies the conditions in \((3.1)\) for being a \(T\)-module with values in \(\mathcal{E}\). It remains to show that for any \((P', \nu(P')) \in \text{Mod}_C^T\), we have isomorphisms
\[
\text{Mod}_C^T((PT, P\mu(T)), (P', \nu(P'))) \cong \text{Fun}(\mathcal{D}, \mathcal{E})(P, P')
\]
Indeed, given \(\alpha : P \rightarrow P'\) in \(\text{Fun}(\mathcal{D}, \mathcal{E})\), we have \(\alpha^L : (PT, P\mu(T)) \rightarrow (P', \nu(P'))\) defined by setting
\[
\alpha^L : PT \xrightarrow{\alpha T} P'T \xrightarrow{\nu(P')} P'
\]
Conversely, for \(\beta : (PT, P\mu(T)) \rightarrow (P', \nu(P'))\) in \(\text{Mod}_C^T\), we have the transformation \(\beta^R : P \rightarrow P'\) given by
\[
\beta^R : P \xrightarrow{P\nu(T)} PT \xrightarrow{\beta} P'
\]
Because \((\mathcal{D}, T)\) is a monad, it is clear that these two associations are inverse to each other. \(\square\)

Following Proposition 3.2, we will say that \((PT, P\mu(T))\) is the free \(T\)-module in \(\mathcal{E}\) associated to the functor \(P : \mathcal{D} \rightarrow \mathcal{E}\).

**Lemma 3.3.** Let \(\Psi = (G, F)\) be an adjunction and let \(\phi = (\phi_G, \phi_F) : (\mathcal{E}, S) \rightarrow (\mathcal{D}, T)\) be a \(\Psi\)-morphism of monads. Then, \(SG : \mathcal{D} \rightarrow \mathcal{E}\) is canonically equipped with the structure of a \(T\)-module taking values in \(\mathcal{E}\).

**Proof.** We set \(\nu' : SGT \xrightarrow{S\phi_G} S^2G \xrightarrow{\mu(S)G} SG\). In order to show that this determines a \(T\)-module, we note that the following diagram commutes.

\[
\begin{array}{ccc}
SGT^2 & \xrightarrow{S\phi_G T} & SSGT \\
| & \downarrow{S\phi_G} & | \\
SSGT & \xrightarrow{\mu(S)GT} & SGT \\
| & \downarrow{S\phi_G} & | \\
SGT & \xrightarrow{S\phi_G} & S^2G \\
| & \downarrow{S\mu(S)G} & | \\
SSG & \xrightarrow{\mu(S)G} & SG
\end{array}
\]

In \((3.6)\), the left hand square is obtained by applying \(S\) to the commutative square in \((2.6)\). Using the triangle in \((2.5)\), we also have the commutative diagram

\[
\begin{array}{ccc}
SGT & \xrightarrow{S\phi_G} & SSG \\
| & \downarrow{S\phi_G} & | \\
SG & \xrightarrow{\mu(S)G} & SG
\end{array}
\]

This proves the result. \(\square\)

\[\text{(b) } \nu(P) : PT \rightarrow P\text{ is a natural transformation satisfying}
\]
\[
\begin{array}{ccc}
PT^2 & \xrightarrow{P\mu(T)} & PT \\
\nu(P) & \downarrow & \nu(P) \\
PT & \xrightarrow{P\nu(T)} & PT
\end{array}\]

\[\begin{aligned}
(3.1)
\end{aligned}\]
Definition 3.4. Let $\Psi = (G, F)$ be an adjunction and let $\phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)$ be a $\Psi$-morphism of monads. We will say that $\phi$ satisfies the PBW-property if there exists a functor $Q : \mathcal{D} \rightarrow \mathcal{C}$ which fits into the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}^T & \xrightarrow{\hat{\phi}_G} & \mathcal{C}^S \\
\phi(T) & \downarrow & \phi(S) \\
\mathcal{D} & \xrightarrow{Q} & \mathcal{C}
\end{array}
$$

(3.8)

Proposition 3.5. Let $\Psi = (G, F)$ be an adjunction and let $\phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)$ be a $\Psi$-morphism of monads. Then, if $\phi = (\phi_G, \phi_F)$ has the PBW property, $\nu' : SG T \rightarrow S^2 G \rightarrow S^3 G \rightarrow SG$ gives $SG$ the structure of a free $T$-module in $\mathcal{C}$.

Proof. Since $\phi$ has the PBW property, we have $Q : \mathcal{D} \rightarrow \mathcal{C}$ which fits into the commutative diagram (3.8). We first claim that $SG = QT$. Using (2.11) and (3.8), we have the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\rho(T)} & \mathcal{D}^T \\
\downarrow{\phi(G)} & & \downarrow{\phi(T)} \\
\mathcal{C} & \xrightarrow{\rho(S)} & \mathcal{C}^S
\end{array}
$$

(3.9)

From (3.9), it is clear that for any $x \in \mathcal{D}$, we have $QTx = SGx \in \mathcal{C}$. It remains to show that $QT = SG$ as objects of $\mathcal{C}$. For this, we note the following commutative diagram

$$
\begin{array}{ccc}
SGT y & \xrightarrow{\mu(G)GT y} & SGGT y \\
\downarrow{\phi_G y} & & \downarrow{\phi_G y} \\
SG y & \xrightarrow{\mu(G)y} & S y
\end{array}
$$

(3.10)

which gives for any $y \in \mathcal{D}$ the morphism from the coequalizer $QT = Q\hat{\phi}(T y, \lambda_T y) = \pi(S)\hat{\phi}(G y) \rightarrow S y$. The condition (2.11) gives us the commutative diagram

$$
\begin{array}{ccc}
SGT y & \xrightarrow{\mu(G)GT y} & SGGT y \\
\downarrow{\phi_G y} & & \downarrow{\phi_G y} \\
SG y & \xrightarrow{\mu(G)y} & S y
\end{array}
$$

(3.11)

and also

$$
\begin{array}{ccc}
SGT y & \xrightarrow{\mu(G)GT y} & SGGT y \\
\downarrow{\phi_G y} & & \downarrow{\phi_G y} \\
SG y & \xrightarrow{\mu(G)y} & S y
\end{array}
$$

(3.12)
Putting together \[3.11\] and \[3.12\] and considering the coequalizers defining $\hat{\phi}_G(Ty, \lambda_{Ty})$ and $\hat{\phi}_G(TTy, \lambda_{TTy})$, we see that $QT = SG$ is compatible with the right action of $T$ on both sides. This proves the result.

**Proposition 3.6.** Let $\Psi = (G, F)$ be an adjunction and let $\phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)$ be a $\Psi$-morphism of monads. Then, $SG \in \text{Mod}_T^C$ is isomorphic to a free $T$-module taking values in $\mathcal{C}$.

**Proof.** We take $Q : \mathcal{D} \rightarrow \mathcal{C}$ so that $SG \cong QT$ in $\text{Mod}_T^C$. We claim that $Q \pi(T) = \pi(S) \hat{\phi}_G$. Using the definition of $\hat{\phi}_G$ in \[2.10\], this means that we must show that the coequalizer

\[
\text{Coeq} \left( SGT x \xrightarrow{(\mu(S)Gx)\circ(S\phi_G(x))} SGx \right) \cong Q(x)
\]

in $\mathcal{C}$ for any $T$-algebra $(x, \lambda_x) \in \mathcal{T}$. Since $SG \cong QT$ as $T$-modules, it follows that the composition $SGT \xrightarrow{\mu(S)G} SSG \xrightarrow{\mu(S)G} SG$ giving the $T$-module structure on $SG$ corresponds to the composition $QTT \xrightarrow{Q\mu(T)} QT$ giving the $T$-module structure on $QT$. Accordingly, we can prove \[3.13\] by showing that

\[
\text{Coeq} \left( QTT x \xrightarrow{Q\mu(T)x} QT x \xrightarrow{Q\lambda x} QT x \right) \cong Q(x)
\]

We will prove this by showing that the following is a split coequalizer diagram in $\mathcal{C}$:

\[
\begin{array}{ccc}
QT x & \xrightarrow{\epsilon = Q\pi(T)x} & QTT x \\
\downarrow{\gamma = Q\mu(T)x} & & \downarrow{\beta = Q\eta(T)x} \\
QT x & \xrightarrow{\delta = QT\lambda x} & Qx \\
\end{array}
\]

(3.15)

Since $(x, \lambda_x)$ is a $T$-algebra, we see that $\alpha\gamma = \alpha\delta$. Since $T$ is a monad, we get $\alpha\beta = id$ and $\gamma\epsilon = id$. We also observe directly that $\delta\epsilon = \beta\alpha$. We conclude that $Qx = \text{Coeq}(\gamma, \delta)$, which proves the result.

**Theorem 3.7.** Let $\Psi = (G, F)$ be an adjunction and let $\phi = (\phi_G, \phi_F) : (\mathcal{C}, S) \rightarrow (\mathcal{D}, T)$ be a $\Psi$-morphism of monads. Then, the following are equivalent.

1. There exists a functor $Q : \mathcal{D} \rightarrow \mathcal{C}$ such that there are isomorphisms in $\mathcal{C}$

\[
\hat{\phi}_G(x, \lambda_x) = Q(x) \quad \forall (x, \lambda_x) \in \mathcal{T}
\]

natural with respect to morphisms in $\mathcal{T}$.

2. The functor $SG$, along with the map $\nu' : SGT \xrightarrow{S\phi_G} S^2G \xrightarrow{\mu(S)G} SG$, is isomorphic to a free $T$-module with values in $\mathcal{C}$.

**Proof.** (1) $\Rightarrow$ (2) follows from Proposition \[3.5\] and (2) $\Rightarrow$ (1) follows from Proposition \[3.6\].

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