't Hooft expansion of $\text{SO}(N)$ and $\text{Sp}(N)$ $\mathcal{N} = 4$ SYM revisited

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ABSTRACT: We study the 't Hooft expansion of $d = 4$ $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with the gauge group $\text{SO}(N)$ or $\text{Sp}(N)$. We consider the $1/N_5$ expansion with fixed $g_s N_5$, where $g_s$ denotes the string coupling of bulk type IIB string theory on $\text{AdS}_5 \times \mathbb{R}P^5$ and $N_5$ refers to the RR 5-form flux through $\mathbb{R}P^5$. $N_5$ differs from $N$ due to a shift coming from the RR charge of O3-plane. As an example, we consider the $1/N_5$ expansion of the free energy of $\mathcal{N} = 4$ SYM on $S^4$ and the 1/2 BPS circular Wilson loops in the fundamental representation of $\text{SO}(N)$ or $\text{Sp}(N)$. We find that the $1/N_5$ expansion is more “closed string like” than the ordinary $1/N$ expansion.

KEYWORDS: $1/N$ Expansion, AdS-CFT Correspondence, Matrix Models

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1 Introduction

d = 4 \mathcal{N} = 4 \text{ supersymmetric Yang-Mills (SYM) theory with the gauge group } \text{SO}(N) \text{ or } \text{Sp}(N) \text{ is realized as a worldvolume theory on D3-branes in the presence of orientifold 3-plane, and it is holographically dual to the type IIB string theory on } \text{AdS}_5 \times \mathbb{R}P^5 [1]. \text{ In [2, 3], the } 1/2 \text{ BPS circular Wilson loops in } \text{SO}(N) \text{ and } \text{Sp}(N) \mathcal{N} = 4 \text{ SYM are studied in the } 1/N \text{ expansion with fixed } g_s N, \text{ where } g_s \text{ denotes the string coupling of bulk type IIB string theory. Note that } g_s \text{ is proportional to the square of the Yang-Mills coupling constant } g_{YM}^2. \text{ In [4, 5], it is suggested that in the context of AdS/CFT correspondence, it is more natural to consider the } 1/N_5 \text{ expansion with fixed } g_s N_5, \text{ where } N_5 \text{ refers to the RR 5-form flux through } \mathbb{R}P^5 \text{ in the bulk type IIB string theory on } \text{AdS}_5 \times \mathbb{R}P^5

\begin{align}
N_5 &= \int_{\mathbb{R}P^5} G_5 \frac{2}{2\pi} \tag{1.1} \\
\end{align}

As shown in [6–8], \( N_5 \) is shifted from \( N \) due to the RR charge of orientifold 3-plane

\begin{equation}
N_5 = \begin{cases} 
\frac{N}{2} - \frac{1}{4}, & \text{for } \text{SO}(N), \\
\frac{N}{2} + \frac{1}{4}, & \text{for } \text{Sp}(N). 
\end{cases} \tag{1.2}
\end{equation}

Rather surprisingly, the \( 1/N_5 \) expansion with fixed \( g_s N_5 \) has not been fully explored in the literature before, as far as we know. In this paper, we will study the \( 1/N_5 \) expansion for the partition function of \( \mathcal{N} = 4 \) SYM on \( S^4 \) as well as the \( 1/2 \) BPS circular Wilson loops in the fundamental representation of \( \text{SO}(N) \) or \( \text{Sp}(N). \)\(^1\) In the original \( 1/N \) expansion, both

\(^1\)Note that \( \text{SO}(2n) \) gauge theories and \( \text{Sp}(2n) \) gauge theories are formally related by the replacement \( n \rightarrow -n \) [9, 10].
even and odd powers of $1/N$ appear in the expansion of the 1/2 BPS Wilson loops [2, 3]. It turns out that the $1/N_5$ expansion is more “closed string like”: in the $1/N_5$ expansion of the partition function, only the even powers of $1/N_5$ appear and in the $1/N_5$ expansion of 1/2 BPS Wilson loops only the odd powers of $1/N_5$ appear, except for a constant term. This is consistent with the general property of holography where D-branes/O-planes are replaced by a closed string background in the bulk gravitational picture.\(^2\)

This paper is organized as follows. In section 2, we study the $1/N_5$ expansion of the partition function of $\mathcal{N} = 4$ SYM, which is inversely proportional to the volume of the gauge group $SO(N)$ or $Sp(N)$. We find that the volume of the gauge group is characterized by a universal function $V(N_5)$ for both $SO(N)$ and $Sp(N)$, up to an overall factor $2^{\pm N_5}$. It turns out that the $1/N_5$ expansion of $\log V(N_5)$ contains only even powers of $1/N_5$. In section 3, we study the $1/N_5$ expansion of the $1/2$ BPS circular Wilson loops of $\mathcal{N} = 4$ SYM in the fundamental representation of $SO(N)$ or $Sp(N)$. Finally, we conclude in section 4 with some discussions. In appendix A, we present a proof of the relation (3.4).

2 $1/N_5$ expansion of the volume of SO(N) and Sp(N)

In this section, we consider the $1/N_5$ expansion of the free energy of $\mathcal{N} = 4$ SYM on $S^4$ with the gauge group $G = SO(N)$ or $G = Sp(N)$. As shown by Pestun [12], the partition function of $\mathcal{N} = 4$ SYM on $S^4$ reduces to a Gaussian matrix model owing to the supersymmetric localization

$$Z_G = \frac{1}{\text{vol}(G)} \int_{\text{Lie}(G)} dM e^{-\frac{1}{2g_s} \text{Tr} M^2},$$

where the integral of $M$ is over the Lie algebra of gauge group $G$. Since the integral of $M$ is Gaussian, the $g_s$-dependence of $Z_G$ is rather simple

$$Z_G = \frac{(2\pi g_s)^{\frac{1}{2} \dim G}}{\text{vol}(G)},$$

and $Z_G$ is essentially determined by the volume of the gauge group $G$. Thus, in what follows we will consider the $1/N_5$ expansion of $\text{vol}(G)$.

The volume of $SO(N)$ is given by [13–15]

$$\text{vol}[SO(N)] = \frac{2^N\pi^{\frac{1}{2} N(N+1)}}{\prod_{k=1}^N \Gamma(k/2)} = \begin{cases} \frac{2^{\frac{1}{2}}(2\pi)^{n^2}}{(n-1)! \prod_{i=1}^{n-1} (2i-1)!}, & (N = 2n), \\
\frac{2^{n+\frac{1}{2}}(2\pi)^{n^2+n}}{\prod_{i=1}^n (2i-1)!}, & (N = 2n + 1). \end{cases} \quad (2.3)$$

Our definition of the volume of $SO(2n)$ is the same as that in [15], but the volume of $SO(2n + 1)$ differs from [15] by a factor of $(\pi/2)^{\frac{1}{2}}$. The volume of $Sp(N)$ is given by [15]

$$\text{vol}[Sp(2n)] = \frac{2^{-n}(2\pi)^{n^2+n}}{\prod_{i=1}^n (2i-1)!}. \quad (2.4)$$

\(^2\)The question of open versus closed string expansions for the expectation values of Wilson loops of $\mathcal{N} = 4$ SYM was addressed for $G = SU(N)$ in [11].
From the definition of $N_5$ in (1.2), one can rewrite the above volumes in terms of $N_5$ as
\[
\text{vol}(G) = 2^\pm N_5 V(N_5),
\]
\[
V(N_5) = \frac{2^N_5 \pi^{(N_5+1/4)(N_5+3/4)} G_2(1/2)}{G_2 \left( N_5 + \frac{3}{2} \right) G_2 \left( N_5 + \frac{5}{2} \right)},
\]
where the $\pm$ sign corresponds to $G = \text{SO}(N)$ and $G = \text{Sp}(N)$, respectively, and $G_2(z)$ denotes the Barnes $G$-function.

Now, let us consider the free energy coming from the volume of the gauge group $G$
\[
- \log[\text{vol}(G)] = \mp N_5 \log 2 - \log V(N_5).
\]

The $1/N_5$ expansion of the Barnes $G$-function in (2.5) can be computed by integrating the asymptotic expansion of the $\Gamma$-function\(^3\)
\[
\log \Gamma(z + a) = (z + a - 1/2) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{k=2}^{\infty} \frac{(-1)^k B_k(a)}{k(k-1)z^{k-1}},
\]
where $B_k(a)$ denotes the Bernoulli polynomial. After some algebra, we find
\[
- \log V(N_5) = c_0 + N_5^2 \left( \frac{3}{2} + \log \frac{N_5}{\pi} \right) - \frac{5}{48} \log N_5
\]
\[
+ \sum_{g=2}^{\infty} N_5^{2-2g} \left[ \frac{B_{2g}(1/4)}{2g(g-1)} - \frac{B_{2g-1}(1/4)}{4(g-1)(2g-1)} \right],
\]
where $c_0$ is some constant. As we mentioned in the Introduction, the $1/N_5$ expansion of $\log V(N_5)$ has only even powers of $1/N_5$.

This $1/N_5$ expansion (2.8) should be compared with the $1/N$ expansion appearing in the topological string. In the case of topological string, the natural expansion parameter is $1/N_{\text{top}}$, where $N_{\text{top}}$ is given by \cite{15}
\[
N_{\text{top}} = N \mp 1.
\]

Here the upper and lower sign correspond to $G = \text{SO}(N)$ and $G = \text{Sp}(N)$, respectively. The $1/N_{\text{top}}$ expansion of the volume of $G$ is computed in \cite{15}
\[
- \log[\text{vol}(G)] = \frac{1}{2} \sum_{g} \left( \chi(\mathcal{M}_g) N_{\text{top}}^{2-2g} \pm \chi(\mathcal{M}_{1g}) N_{\text{top}}^{1-2g} \right),
\]
where $\chi(\mathcal{M}_g)$ and $\chi(\mathcal{M}_{1g})$ denote the Euler characteristic of the moduli space of Riemann surfaces of genus $g$ with zero and one cross-cap, respectively \cite{16}
\[
\chi(\mathcal{M}_g) = \frac{B_{2g}}{2g(2g-2)}, \quad \chi(\mathcal{M}_{1g}) = \frac{2^{2g-1} B_{2g}(1/2)}{2g(2g-1)}.
\]

One can see that the $1/N_{\text{top}}$ expansion of $\text{vol}(G)$ contains both even and odd powers of $1/N_{\text{top}}$, while the $1/N_5$ expansion of $\text{vol}(G)$ contains only even powers of $1/N_5$ except for

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\(^3\)See e.g. http://dlmf.nist.gov/5.11.E8.
the first term of (2.6). This difference comes from the different definition of $N_5$ in (1.2) and $N_{\text{top}}$ in (2.9). As discussed in [15, 17], the shift of $N$ in $N_{\text{top}}$ (2.9) comes from the RR charge of topological O-plane, which differs from the RR charge of O3-plane in type IIB string theory. Thus the $1/N_{\text{top}}$ expansion (2.10) cannot be applied to our case of $N = 4$ SYM. In the holographic duality between $\mathcal{N} = 4$ SYM with the gauge group $SO(N)$ or $Sp(N)$ and the type IIB string theory on $AdS_5 \times \mathbb{R}P^5$, we should use the $1/N_5$ expansion (2.8), instead of the $1/N_{\text{top}}$ expansion (2.10).

3 1/2 BPS Wilson loops in the fundamental representation of $SO(N)$ and $Sp(N)$

In this section, we consider the $1/N_5$ expansion of the 1/2 BPS circular Wilson loops in the fundamental representation of $G = SO(N)$ or $G = Sp(N)$. As shown in [12, 18, 19], the expectation value of the 1/2 BPS circular Wilson loop is given by the Gaussian matrix model

$$W_G = \langle Tr e^M \rangle, \quad (3.1)$$

where the expectation value is defined by the Gaussian measure (2.1). Note that in our definition of $W_G$ we do not divide it by the dimension $N$ of the fundamental representation. The Gaussian integral (3.1) can be evaluated by the method of orthogonal polynomials and the result is written in terms of the Laguerre polynomials [2]

$$W_{SO(2n)} = 2e^{\frac{1}{2}g_s} \sum_{i=0}^{n-1} L_{2i}(-g_s),$$

$$W_{SO(2n+1)} = 1 + 2e^{\frac{1}{2}g_s} \sum_{i=0}^{n-1} L_{2i+1}(-g_s), \quad (3.2)$$

$$W_{Sp(2n)} = 2e^{\frac{1}{2}g_s} \sum_{i=0}^{n-1} L_{2i+1}(-g_s).$$

One can check that they are correctly normalized as

$$W_{SO(N)} \big|_{g_s=0} = N, \quad W_{Sp(N)} \big|_{g_s=0} = N. \quad (3.3)$$

As explained in appendix A, the derivative of $W_G$ with respect to $g_s$ has a simple form

$$\partial_{g_s} W_{SO(N)} = e^{\frac{1}{2}g_s} L_{N-2}^{(2)}(-g_s), \quad \partial_{g_s} W_{Sp(N)} = e^{\frac{1}{2}g_s} L_{N-1}^{(2)}(-g_s), \quad (3.4)$$

which are both written in terms of $N_5$ as

$$\partial_{g_s} W_G = e^{\frac{1}{2}g_s} L_{2N_5-\frac{3}{2}}^{(2)}(-g_s). \quad (3.5)$$

3.1 1/N_5 expansion of $W_G$

In this subsection, we consider the $1/N_5$ expansion of $W_G$ with fixed ’t Hooft parameter $\lambda$

$$\lambda = 8g_s N_5. \quad (3.6)$$
To do this, it is useful to express $W_G$ as a contour integral [20]. Let us first consider $W_{SO(N)}$ for definiteness. Using the series expansion of the Laguerre polynomial

$$L_n^{(\alpha)}(-g_s) = \sum_{i=0}^{n} \frac{(n+\alpha)\cdots(n-i)}{i!} g_s^i,$$  \hspace{1cm} (3.7)

$\partial_g W_{SO(N)}$ in (3.4) is written as

$$\partial_g W_{SO(N)} = e^{\frac{i}{2}g_s} \sum_{i=0}^{N-2} \frac{N}{N-2-i}! g_s^i,$$

$$= e^{\frac{i}{2}g_s} \sum_{i=0}^{N-2} \frac{N}{i}! (N-2-i)!\! g_s^i,$$

$$= e^{\frac{i}{2}g_s} \int\! \frac{dw}{2\pi i} \sum_{i=0}^{N-2} \frac{N}{i}! (N-2-i)!\! w^{i-1} g_s^i,$$

$$= e^{\frac{i}{2}g_s} \int\! \frac{dw}{2\pi i} \frac{(1+w)^N}{w^3} e^{\frac{g_s}{2}w},$$  \hspace{1cm} (3.8)

where the contour of $w$-integral is a circle surrounding $w = 0$ counterclockwise. By the change of variable $w = e^{2z} - 1$ we find

$$\partial_g W_{SO(N)} = \int\! \frac{dz}{2\pi i} e^{(2N-1)z} e^{\frac{g_s}{2}z} \coth z,$$

$$= \int\! \frac{dz}{2\pi i} e^{4N_5z+\frac{g_s}{2}z} \coth z,$$  \hspace{1cm} (3.9)

where the contour of $z$-integral is around $z = 0$. For the $Sp(N)$ case, one can show that $\partial_g W_{Sp(N)}$ is also given by the same formula (3.9). Thus we find

$$\partial_g W_G = \int\! \frac{dz}{2\pi i} e^{4N_5z+\frac{g_s}{2}z} \coth z.$$  \hspace{1cm} (3.10)

Finally, integrating this expression with respect to $g_s$, we arrive at

$$W_G = \pm \frac{1}{2} + \int\! \frac{dz}{2\pi i} \frac{1}{\sinh z \sinh 2z} e^{4N_5z+\frac{g_s}{2}z} \coth z.$$  \hspace{1cm} (3.11)

Here we have determined the integration constant by the normalization condition (3.3).

In order to study the 't Hooft expansion of $W_G$, it is convenient to further rewrite the second term of (3.11) as

$$\int\! \frac{dz}{2\pi i} \frac{1}{\sinh z \sinh 2z} e^{4N_5z+\frac{g_s}{2}z} \coth z$$

$$= \int\! \frac{dz}{2\pi i} \left( \frac{1}{\sinh^2 z} - \frac{2\sinh^2 z}{\sinh z \sinh 2z} \right) e^{4N_5z+\frac{g_s}{2}z} \coth z$$

$$= \int\! \frac{dz}{2\pi i} \left( -\frac{1}{g_s} e^{4N_5z+\frac{g_s}{2}z} \coth z \right) - \int\! \frac{dz}{2\pi i} \frac{2\sinh^2 z}{\sinh z \sinh 2z} e^{4N_5z+\frac{g_s}{2}z} \coth z$$

$$= \frac{4N_5}{g_s} \int\! \frac{dz}{2\pi i} e^{4N_5z+\frac{g_s}{2}z} \coth z - \int\! \frac{dz}{2\pi i} \frac{2\sinh^2 z}{\sinh z \sinh 2z} e^{4N_5z+\frac{g_s}{2}z} \coth z.$$  \hspace{1cm} (3.12)
One can show that the first term of (3.12) is equal to the 1/2 BPS Wilson loop of U(2N) \( \mathcal{N} = 4 \) SYM [19]

\[
\frac{4N_5}{g_s} \oint \frac{dz}{2\pi i} e^{4N_5z + \frac{2\pi}{e} \coth z} = e^{\frac{1}{2}g_s} L_{2N_5-1}^{(1)}(-g_s) = W_{U(2N_5)}. \tag{3.13}
\]

Let us consider the ’t Hooft expansion of \( W_{U(2N_5)} \) in (3.13) following the approach of [20]. By rescaling \( z \to g_s z \), \( W_{U(2N_5)} \) is written as

\[
W_{U(2N_5)} = 4N_5 \oint \frac{dz}{2\pi i} e^{\frac{1}{2}((\lambda z + z^{-1}) + \frac{2\pi}{e} g_s z - \frac{1}{2} z^{-1}).} \tag{3.14}
\]

The first part of the exponential \( e^{\frac{1}{2}((\lambda z + z^{-1})} \) is essentially the generating function of the modified Bessel function of the first kind \( I_n(x) \)

\[
e^{\frac{1}{2}((\lambda z + z^{-1})} = \sum_{n \in \mathbb{Z}} \hat{I}_n z^n, \tag{3.15}
\]

where \( \hat{I}_n \) is given by

\[
\hat{I}_n = \frac{I_n(\sqrt{\lambda})}{(\sqrt{\lambda})^n}. \tag{3.16}
\]

The second part of the exponential in (3.14) can be expanded in \( g_s \) as

\[
e^{\frac{2\pi}{e} g_s z - \frac{1}{2} z^{-1}} = 1 + \frac{z}{6} g_s^2 + \left( \frac{z^2}{72} - \frac{z^3}{90} \right) g_s^4 + O(g_s^6). \tag{3.17}
\]

Then, taking the residue at \( z = 0 \) we find the small \( g_s \) expansion of \( W_{U(2N_5)} \)

\[
W_{U(2N_5)} = \frac{\lambda}{2g_s} \left[ \hat{I}_1 + \hat{I}_2 \frac{g_s^2}{6} + \left( \frac{\hat{I}_3}{72} - \frac{\hat{I}_4}{90} \right) g_s^4 + O(g_s^6) \right]. \tag{3.18}
\]

Note that the small \( g_s \) expansion with fixed \( \lambda = 8g_s N_5 \) is basically the same as the \( 1/N_5 \) expansion since \( g_s \) and \( 1/N_5 \) are related by

\[
g_s = \frac{\lambda}{8N_5}. \tag{3.19}
\]

Next consider the second term of (3.12), which we will denote by \( W_T \)

\[
W_T = -\frac{g_s}{4} \oint \frac{dz}{2\pi i \sinh g_s z \sinh 2g_s z} e^{\frac{1}{2}((\lambda z + z^{-1}) + \frac{2\pi}{e} g_s z - \frac{1}{2} z^{-1}).} \tag{3.20}
\]

Again, the first part of the exponential has the expansion (3.15) and the rest of the integrand can be expanded in \( g_s \) as

\[
\frac{8 \sinh^2 \frac{g_s z}{2}}{\sinh g_s z \sinh 2g_s z} e^{\frac{2\pi}{e} g_s z - \frac{1}{2} z^{-1}} = 1 + \left( \frac{z}{6} - \frac{3z^2}{4} \right) g_s^2 + \left( \frac{z^2}{72} - \frac{49z^3}{360} + \frac{3z^4}{8} \right) g_s^4 + O(g_s^6). \tag{3.21}
\]
Taking the residue at \( z = 0 \), we find the small \( g_s \) expansion of \( W_T \) with fixed \( \lambda \)

\[
W_T = -\frac{g_s}{4} \left[ \tilde{I}_1 + \left( \frac{\tilde{I}_2}{6} - \frac{3\tilde{I}_3}{4} \right) g_s^2 + \left( \frac{\tilde{I}_3}{72} - \frac{49\tilde{I}_4}{360} + \frac{3\tilde{I}_5}{8} \right) g_s^4 + O(g_s^6) \right]. \tag{3.22}
\]

To summarize, we find that \( W_G \) is decomposed as

\[
W_G = \pm \frac{1}{2} + W_{U(2N_5)} + W_T, \tag{3.23}
\]

and the last two terms are expanded as

\[
W_{U(2N_5)} = \sum_{g=0}^{\infty} a_g(\lambda) g_s^{2g-1} = \sum_{g=0}^{\infty} a_g(\lambda) \left( \frac{\lambda}{8N_5} \right)^{2g-1},
\]

\[
W_T = \sum_{g=0}^{\infty} b_g(\lambda) g_s^{2g+1} = \sum_{g=0}^{\infty} b_g(\lambda) \left( \frac{\lambda}{8N_5} \right)^{2g+1}, \tag{3.24}
\]

where \( a_g(\lambda) \) and \( b_g(\lambda) \) are some functions of \( \lambda \) whose explicit forms can be found in (3.18) and (3.22). One can see that \( W_{U(2N_5)} \) and \( W_T \) are both expanded in \( 1/N_5 \) with only odd powers of \( 1/N_5 \).

### 3.2 Relation to the ordinary \( 1/N \) 't Hooft expansion

Let us compare our \( 1/N_5 \) expansion of \( W_G \) with the ordinary \( 1/N \) expansion of \( W_G \). For definiteness, we consider the \( G = SO(N) \) case. The \( 1/N \) expansion of \( W_{SO(N)} \) is studied in [2, 3] where the 't Hooft coupling \( \lambda' \) is defined as

\[
\lambda' = 8g_s N. \tag{3.25}
\]

To this end, it is convenient to start with the expression of \( W_{SO(N)} \) found in [2]\(^4\)

\[
W_{SO(N)} = e^{\frac{1}{2}g_s L^{(1)}_{N-1}(-g_s)} - \frac{1}{2} \int_0^{g_s} dx e^{\frac{1}{2}x L^{(1)}_{N-1}(-x)} = W_{U(N)}(g_s) - \frac{1}{2} \int_0^{g_s} dx W_{U(N)}(x). \tag{3.26}
\]

From the known \( 1/N \) expansion of the \( 1/2 \) BPS Wilson loop in \( U(N) \) \( \mathcal{N} = 4 \) SYM [19], one can easily compute the \( 1/N \) expansion of \( W_{SO(N)} \)

\[
W_{SO(N)} = \frac{1}{2} + \frac{2\sqrt{2N}}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'/2}) - \frac{1}{2} I_0(\sqrt{\lambda'/2}) + \frac{\lambda' I_2(\sqrt{\lambda'/2})}{96N} - \frac{\lambda'^{3/2} I_3(\sqrt{\lambda'/2})}{384\sqrt{2N^2}} + O(N^{-3}). \tag{3.27}
\]

One can check that the \( 1/N \) expansion in (3.27) and our \( 1/N_5 \) expansion are related by the change of parameters \( (\lambda', N) \rightarrow (\lambda, g_s) \)

\[
\lambda' = 2\lambda + 4g_s, \quad N = \frac{\lambda}{4g_s} + \frac{1}{2}. \tag{3.28}
\]

\(^4\)See also appendix A for a derivation of this expression.
Plugging this relation into (3.27) and expanding in $g_s$, we find
\[
W_{\text{SO}(N)} = \frac{1}{2} + \frac{\lambda}{2g_s} \left[ \hat{I}_1 + \frac{\hat{I}_2}{6} g_s^2 \right] - \frac{g_s}{4} \hat{I}_1 + \mathcal{O}(g_s^3).
\]  
(3.29)
This agrees with our result of $1/N_5$ expansion (3.18) and (3.22) up to this order $\mathcal{O}(g_s^3)$, as expected.

Note that, in the original $1/N$ expansion (3.27) both even and odd powers of $N^{-1}$ appear. On the other hand, in our case (3.23) only the odd powers of $g_s$ arise, except for the constant term $\pm 1/2$ in (3.23). Although our decomposition (3.23) is similar to (3.26), we stress that they are different. In particular, our $W_T$ is not equal to the second term of (3.26).

4 Conclusions and outlook

In this paper, we have studied the $1/N_5$ expansion of the volume of the gauge group $G$ and the $1/2$ BPS Wilson loops in the fundamental representation of $G$ in $\mathcal{N} = 4$ SYM with $G = \text{SO}(N)$ or $G = \text{Sp}(N)$.

Due to the shift of $N$ coming from the RR charge of O3-plane (1.2), the $1/N_5$ expansion with fixed ’t Hooft parameter $\lambda = 8g_sN_5$ is different from the ordinary $1/N$ expansion. We found that the $1/N_5$ expansion looks more “closed string like” than the ordinary $1/N$ expansion. For instance, we found that the $1/N_5$ expansion of the volume of $G$ contains only the even powers of $1/N_5$, except for the first term $\mp N_5 \log 2$ in (2.6). This is different from the $1/N_{\text{top}}$ expansion of $\text{vol}(G)$ in topological string [15].

It would be interesting to find a mathematical meaning, if any, of the coefficient of $N_5^{2-2g}$ in (2.8) as a certain quantity on the moduli space of Riemann surfaces of genus $g$.

We have also studied the $1/N_5$ expansion of the $1/2$ BPS Wilson loop $W_G$ in the fundamental representation of $G = \text{SO}(N)$ or $G = \text{Sp}(N)$. We found that $W_G$ is decomposed as (3.23). Except for the constant term $\pm 1/2$ in (3.23), $W_{U(2N_5)}$ and $W_T$ are both expanded in $1/N_5$ with only odd powers of $1/N_5$. It is tempting to speculate that $W_{U(2N_5)}$ and $W_T$ correspond to the untwisted and the twisted sector of bulk type IIB string theory on $AdS_5 \times \mathbb{RP}^5$. It would be interesting to understand the bulk gravitational interpretation of the decomposition (3.23) more clearly.

It would be interesting to extend our analysis to more general observables in $\mathcal{N} = 4$ SYM with the gauge group $\text{SO}(N)$ or $\text{Sp}(N)$, such as an integrated four-point correlator [5] and the $1/2$ BPS Wilson loop in the spinor representation of $\text{SO}(N)$ [2, 3], to name a few. We leave this as an interesting future problem.

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A Proof of (3.4)

In this appendix, we present a proof of the relation (3.4). For definiteness we consider $W_{\text{SO}(2n)}$. To this end, we can use the fact that the Laguerre polynomial is written as a
matrix element of the harmonic oscillator (see e.g. [21])

\[ \langle i | e^{\sqrt{g_s}(a+a^\dagger)} | j \rangle = \langle j | e^{\sqrt{g_s}(a+a^\dagger)} | i \rangle = \sqrt{\frac{i^j}{j^i g_s L}} L_i^{(j-i)} (-g_s), \]  

(A.1)

where

\[ [a, a^\dagger] = 1, \quad a|0\rangle = 0, \quad |k\rangle = \left(a^\dagger\right)^k \sqrt{k!}|0\rangle. \]  

(A.2)

Then \( W_{SO(2n)} \) in (3.2) is written as

\[ W_{SO(2n)} = 2 \sum_{i=0}^{n-1} \langle 2i | e^{\sqrt{g_s}(a+a^\dagger)} | 2i \rangle \]

\[ = \sum_{k=0}^{2n-1} \left[ 1 + (-1)^k \right] \langle k | e^{\sqrt{g_s}(a+a^\dagger)} | k \rangle. \]  

(A.3)

Thus \( W_{SO(2n)} \) is naturally decomposed as

\[ W_{SO(2n)} = W^+_{SO(2n)} + W^-_{SO(2n)}, \]  

(A.4)

where

\[ W^+_{SO(2n)} = \sum_{k=0}^{2n-1} \langle k | e^{\sqrt{g_s}(a+a^\dagger)} | k \rangle, \]

\[ W^-_{SO(2n)} = \sum_{k=0}^{2n-1} (-1)^k \langle k | e^{\sqrt{g_s}(a+a^\dagger)} | k \rangle. \]  

(A.5)

Note that \( W^+_{SO(2n)} \) is equal to the Wilson loop of \( U(2n) \) \( \mathcal{N} = 4 \) SYM [19]. The sum over \( k \) in \( W^+_{SO(2n)} \) can be simplified as

\[ \sqrt{g_s} W^+_{SO(2n)} = \sum_{k=0}^{2n-1} \langle k | [a, e^{\sqrt{g_s}(a+a^\dagger)}] | k \rangle \]

\[ = \sum_{k=0}^{2n-1} \left[ \sqrt{k+1} \langle k+1 | e^{\sqrt{g_s}(a+a^\dagger)} | k \rangle - \sqrt{k} \langle k | e^{\sqrt{g_s}(a+a^\dagger)} | k-1 \rangle \right] \]  

(A.6)

\[ = \sqrt{2n} \langle 2n | e^{\sqrt{g_s}(a+a^\dagger)} | 2n-1 \rangle \]

\[ = \sqrt{g_s} e^{1/2g_s L^{(1)}_{2n-1}}. \]

In the last step we used (A.1). Thus we find

\[ W^+_{SO(2n)} = W_{U(2n)} = e^{1/2g_s L^{(1)}_{2n-1}} (-g_s), \]  

(A.7)

which agrees with the known result of \( W_{U(2n)} \) in [19].
Next, let us consider the $g_s$-derivative of $W^-_{SO(2n)}$

$$
\partial_{g_s} W^-_{SO(2n)} = \frac{1}{2\sqrt{g_s}} \sum_{k=0}^{2n-1} (-1)^k \langle k | e^{\sqrt{g_s}(a - a^\dagger)} (a + a^\dagger) | k \rangle
$$

$$
= \frac{1}{2\sqrt{g}} \sum_{k=0}^{2n-1} (-1)^k \left[ \sqrt{k} \langle k | e^{\sqrt{g}(a - a^\dagger)} | k - 1 \rangle + \sqrt{k + 1} \langle k | e^{\sqrt{g}(a - a^\dagger)} | k + 1 \rangle \right]
$$

$$
= \frac{1}{2\sqrt{g_s}} \sum_{k=0}^{2n-1} \left[ (-1)^k \sqrt{k} \langle k | e^{\sqrt{g_s}(a - a^\dagger)} | k - 1 \rangle - (-1)^{k+1} \sqrt{k + 1} \langle k | e^{\sqrt{g_s}(a - a^\dagger)} | k \rangle \right]
$$

$$
= -\frac{1}{2\sqrt{g_s}} \sqrt{2n} \langle 2n | e^{\sqrt{g_s}(a - a^\dagger)} | 2n - 1 \rangle
$$

$$
= -\frac{1}{2} e^{\frac{1}{2} g_s} L_{2n-1}^{(1)} (-g_s).
$$

(A.8)

Finally, we find

$$
\partial_{g_s} W_{SO(2n)} = \partial_{g_s} W^+_{SO(2n)} + \partial_{g_s} W^-_{SO(2n)}
$$

$$
= \partial_{g_s} \left[ e^{\frac{1}{2} g_s} L_{2n-1}^{(1)} (-g_s) \right] - \frac{1}{2} e^{\frac{1}{2} g_s} L_{2n-1}^{(1)} (-g_s)
$$

$$
= e^{\frac{1}{2} g_s} L_{2n-2}^{(2)} (-g_s).
$$

(A.9)

This proves (3.4) for the $SO(2n)$ case. $SO(2n + 1)$ and $Sp(N)$ cases can be proved in a similar manner.

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