LOCAL UNCERTAINTY PRINCIPLES FOR THE TWO-SIDED GABOR QUATERNION FOURIER TRANSFORM

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Abstract.

Regarding the important applications of Gabor transform in time-frequency analysis and signal analysis, Actually, in this paper, we consider the Gabor quaternion Fourier transform (GQFT), and we prove a version of Benedicks-type uncertainty principle for GQFT and some local concentration uncertainty principles.

keywords: Quaternion algebra, Quaternion Fourier transform, Gabor Fourier transform, concentration theorem, Benedicks theorem.

1 Introduction

The two sided quaternion Fourier transform has become increasingly important tool in image processing. Thus, the two-sided quaternion Fourier transform gives us a simple representation of signals with several components that can be controlled simultaneously. The quaternion Fourier transform QFT was first introduced by Ell [17]. In [19], Hitzer has proved important properties of QFT. In [24], the authors used the QFT in color image analysis.

In 1940 [13], Dennis Gabor has given a powerful tool in signal analysis. So, he introduced a windowed function for studying a time-frequency signals. This idea becomes a useful tool to obtain information about a signal in limited region. Motivated by the applications of QFT and the Gabor Fourier transform (GFT) cited above, in [15] the others have given an extension for GFT to the quaternion case. They have defined the two-sided Gabor quaternion Fourier transform (GQFT). Some useful results of GQFT are derived, like Plancherel and reconstruction formulas. Also, in [15] the authors have demonstrated a version of the Heisenberg uncertainty principle and Logarithmic inequality for the GQFT.

The uncertainty principle state that, we cannot give simultaneously the position and momentum of a particle. Therefore, if try to limit the region of one we lose control of the other. The uncertainty principles have many applications in quantum physics and signal analysis. There are many versions of uncertainty principles, for example, Donoho-stark [12], Benedicks theorem [7]. The aim of this paper is to demonstrate some uncertainty for the GQFT. The paper is organized as follows. In the second
section, we remind some harmonic analysis properties for the two-sided quaternion Gabor Fourier transform proved in [15]. In the third section, we give a version of the Benedicks-type theorem and some local uncertainty principles.

1.1 Definition and properties of quaternion \( \mathbb{H} \):

Considering the classical notations, the quaternion algebra \( \mathbb{H} \) is the set of all elements \( q \) such that

\[
q = q_1 + iq_2 + jq_3 + kq_4 \in \mathbb{H}, \quad q_1, q_2, q_3, q_4 \in \mathbb{R},
\]

with \( i, j \) and \( k \) are three imaginary units obey the Hamilton’s multiplication rules,

\[
ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \tag{1.1}
\]

Due to (1.1) \( \mathbb{H} \) is non-commutative algebra.

The conjugate of quaternion \( q \) is obtained by changing the sign of the pure part, i.e.

\[
\overline{q} = q_1 - iq_2 - jq_3 - kq_4
\]

The quaternion conjugation is a linear anti-involution

\[
\overline{p + q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\overline{p}, \quad \forall p, q \in \mathbb{H}
\]

The modulus \( |q| \) of a quaternion \( q \) is giving by

\[
|q| = \sqrt{q\overline{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}, \quad |pq| = |p||q|.
\]

the modulus of \( \mathbb{H} \) has the following properties,

\[
|pq| = |p||q|, \quad |q| = |\overline{q}|, \quad p, q \in \mathbb{H}
\]

In particular, when \( q = q_1 \) is a real number, the module \( |q| \) reduces to the ordinary Euclidean modulus, i.e. \( |q| = \sqrt{q_1^2} \). A quaternion valued function \( f : \mathbb{R}^2 \to \mathbb{H} \) can also be written as

\[
f(x, y) := f_1(x, y) + if_2(x, y) +jf_3(x, y) + kf_4(x, y),
\]

where \( (x, y) \in \mathbb{R} \times \mathbb{R} \).

The inner product of two quaternion valued functions \( f, g \) defined on \( \mathbb{R}^2 \) is defined as follows

\[
<f, g>_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} f(x) \overline{g(x)} \, dx
\]

When \( f = g \), we obtain the associated norm giving by

\[
\|f\|_2^2 = < f, f >_{2} = \int_{\mathbb{R}^2} |f(x)|^2 \, dx
\]
we define the space $L^2(\mathbb{R}^2, \mathbb{H})$ of all squared integrable functions by

$$L^2(\mathbb{R}^2, \mathbb{H}) = \{ f : \mathbb{R}^2 \to \mathbb{H} \| f \|_2 < \infty \}$$

For quaternion measurable function $f$ and $p$ nonzero integer, we define

$$\| f \|_{L^p(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} |f(x)|^p dx$$

$$L^p(\mathbb{R}^2, \mathbb{H}) = \{ f : \mathbb{R}^2 \to \mathbb{H} \| f \|_{L^p(\mathbb{R}^2, \mathbb{H})} < \infty \}$$

Convolution of two functions two measurable functions $f$ and $g$ over $\mathbb{R}^2$ is given by,

$$f \ast g(y) = \int_{\mathbb{R}^2} f(x)g(x-y)dx$$

2 The two-sided Gabor Quaternionic Fourier Transform (GQFT)

In this section, we start by defining the two-sided Gabor quaternion Fourier transform GQFT, and we reminder some properties, which will be used to prove the principle results.

**Definition 2.1** (Quaternion Fourier transform). The two-sided quaternion Fourier transform (QFT) of a quaternion function $f \in L^1(\mathbb{R}^2, \mathbb{H})$ is the function $F_q(f) : \mathbb{R}^2 \to \mathbb{H}$ defined by:

$$F_q(f)(w) = \int_{\mathbb{R}^2} e^{-2\pi i x_1 \omega_1} f(x)e^{-2\pi i x_2 \omega_2} dx$$

where $dx = dx_1 dx_2$ (2.1)

According to [15] the GQFT is given by

**Definition 2.2.** We define the GQFT of $f \in L^2(\mathbb{R}^2, \mathbb{H})$ with respect to non-zero quaternion window function $\varphi \in L^2(\mathbb{R}^2, \mathbb{H})$ as,

$$G_\varphi f(\omega, b) = \int_{\mathbb{R}^2} e^{-2\pi i x_1 \omega_1} f(x)\varphi(x-b)e^{-2\pi i x_2 \omega_2} dx$$

(2.2)

We note by

$$\| G_\varphi \{ f \} \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})}^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\varphi f(\omega, b) d\omega db$$

Some important properties of GQFT have been derived [13], which we will use to prove some uncertainty principle and some inequalities.

**Theorem 2.3** (Inversion formula). Let $\varphi$ be a quaternion window function. Then for every function $f \in L^2(\mathbb{R}^2, \mathbb{H})$ can be reconstructed by:

$$f(x) = \frac{1}{\| \varphi \|_2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{2i\pi x_1 \omega_1} G_\varphi f(\omega, b)e^{2i\pi x_2 \omega_2} \varphi(x-b)d\omega db$$
Theorem 2.4 (Plancherel theorem). Let $\varphi$ be quaternion window function and $f \in L^2(\mathbb{R}^2, \mathbb{H})$, then we have
\[
\|G_{\varphi}\{f\}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} = \|f\|_2 \|\varphi\|_2^2
\] 

3 Uncertainty Principles

Theorem 3.1 (Hausdorff-Young inequality). If $1 \leq p \leq 2$ and letting $p'$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$, then, for all $f \in L^p$ we have
\[
\|F_{q}\{f\}\|_{q, p'} \leq \|f\|_p
\] 
where,
\[
\|F_{q}\{f\}\|_{q, p'} = \left( \int_{\mathbb{R}^2} |F_{q}\{f\}(\omega)|^{p'} d\omega \right)^{\frac{1}{p'}}
\]
with
\[
|F_{q}\{f\}(\omega)|_{q} = |F_{q}\{f_0\}(\omega)| + |F_{q}\{f_1\}(\omega)| + |F_{q}\{f_2\}(\omega)| + |F_{q}\{f_3\}(\omega)|
\]

4 Local Uncertainty Principle

Now, we give some versions of local uncertainty principles like Benedicks uncertainty principle. First, we start by giving the following Hausdorff-Young’s lemma for the GQFT.

Lemma 4.1. Let $\varphi \in L^p(\mathbb{R}^2, \mathbb{H})$, $f \in L^q(\mathbb{R}^2, \mathbb{H})$ and $p, q \in [1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have
\[
\|G_{\varphi}\{f\}(\omega, b)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} \leq \|f\|_{L^q(\mathbb{R}^2, \mathbb{H})} \|\varphi\|_{L^p(\mathbb{R}^2, \mathbb{H})}
\]

Proof. We have
\[
|G_{\varphi}\{f\}(\omega, b)| = \left| \int_{\mathbb{R}^2} e^{-2\pi i x_1 \omega_1} f(x) \overline{\varphi(x-b)} e^{-2\pi i x_2 \omega_2} dx \right| 
\]
\[
\leq \int_{\mathbb{R}^2} |f(x)||\overline{\varphi(x-y)}| dx
\]
Using Hölder inequality we get our result
\[
\|G_{\varphi}\{f\}(\omega, b)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} \leq \|f\|_{L^q(\mathbb{R}^2, \mathbb{H})} \|\varphi\|_{L^p(\mathbb{R}^2, \mathbb{H})}
\]

\[\square\]

Theorem 4.2. Let $\varphi$ a quaternion windowed function, with $\|\varphi\|_2 = 1$. For $f \in L^2(\mathbb{R}^2, \mathbb{H})$ that $\|f\|_2 = 1$, then for $\Sigma \subset \mathbb{R}^2 \times \mathbb{R}^2$ and $0 \geq \varepsilon < 1$ such that,
\[
\int_{\Sigma} \int_{\Sigma} |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db \geq 1 - \varepsilon.
\]
we have $m(\Sigma) \geq 1 - \varepsilon$.

Where $m(\Sigma)$ is the Lebesgue measure of $\Sigma$. 
Proof. For \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \) we have by the lemma 4.1

\[
\|G_\varphi \{ f \}(\omega, b)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} \leq \|f\|_2 \|\varphi\|_2
\]

From the relation above, we get

\[
1 - \varepsilon \leq \int_{\Sigma} \int_{\mathbb{R}^2} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db \leq \|G_\varphi \{ f \}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} m(\Sigma)
\]

\[
\leq m(\Sigma) \|f\|_2^2 \|\varphi\|_2^2 = m(\Sigma)
\]

then,

\[
1 - \varepsilon \leq m(\Sigma)
\]

\( \square \)

Theorem 4.3. Let \( \Sigma \subset \mathbb{R}^2 \times \mathbb{R}^2 \) such that \( 0 < m(\Sigma) < 1 \). Then for all \( f, \varphi \in L^2(\mathbb{R}^2, \mathbb{H}) \), we have,

\[
\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|\varphi\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq \frac{1}{\sqrt{1 - m(\Sigma)}} \left( \int \int_{\Sigma} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db \right)^{\frac{1}{2}}
\]

(4.2)

Proof. For every function \( f \in L^2(\mathbb{R}^2, \mathbb{H}) \),

\[
\|G_\varphi \{ f \}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})}^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db
\]

\[
= \int_{\Sigma} \int_{\mathbb{R}^2} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db + \int_{\Sigma^c} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db
\]

\[
\leq m(\Sigma) \|\varphi\|_2^2 \|f\|_2^2 + \int_{\Sigma^c} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db
\]

Then,

\[
\int_{\Sigma^c} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db \geq \|G_\varphi \{ f \}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})}^2 - m(\Sigma) \|\varphi\|_2^2 \|f\|_2^2
\]

Using the Plancherel formula (2.3), we get

\[
\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|\varphi\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq \frac{1}{\sqrt{1 - m(\Sigma)}} \left( \int \int_{\Sigma^c} |G_\varphi \{ f \}(\omega, b)|^2 d\omega db \right)^{\frac{1}{2}}
\]

\( \square \)

Remark 4.4. This shows that for a non-zero function \( f \), if its Gabor transform \( G_\varphi \{ f \} \) is concentrated on a set \( \Sigma \) of volume such that, \( 0 < m(\Sigma) < 1 \), then \( f \equiv 0 \) or \( \varphi \equiv 0 \).

Theorem 4.5. Let \( s > 0 \). There exists a constant \( C_s > 0 \) such that, for \( f, \varphi \in L^2(\mathbb{R}^2, \mathbb{H}) \)

\[
\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|\varphi\|_{L^2(\mathbb{R}^2, \mathbb{H})} \leq C_s \left( \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} |(\omega, y)|^{2s} |G_\varphi \{ f \}(\omega, y)|^2 d\omega dy \right)^{\frac{1}{2}}
\]

(4.3)
Proof. Let $0 < r \leq 1$ be a real number and $B_r = \{(\omega, b) \in \mathbb{R}^2 \times \mathbb{R}^2 : |(\omega, b)| < r\}$ the ball of center 0 and radius $r$ in $\mathbb{R}^2 \times \mathbb{R}^2$. Fix $0 < t \leq 1$ small enough such that $m(B_t) < 1$. Therefore by the inequality (2) we obtain

$$
\|f\|^2_{L^2(\mathbb{R}^2; H)} \|\varphi\|^2_{L^2(\mathbb{R}^2; H)} \leq \frac{1}{t^2(1 - m(B_t))} \int_{|(\omega, b)| > t} |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db
$$

$$
\leq \frac{1}{t^2(1 - m(B_t))} \int_{|(\omega, b)| > t} |(\omega, b)|^2 |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db
$$

$$
\leq \frac{1}{t^2(1 - m(B_t))} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |(\omega, b)|^2 |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db
$$

we take the square of the two sided of the inequality

$$
\|f\|^2_{L^2(\mathbb{R}^2; H)} \|\varphi\|^2_{L^2(\mathbb{R}^2; H)} \leq \frac{1}{t^2(1 - m(B_t))} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |(\omega, b)|^2 |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db
$$

we get

$$
\|f\|_{L^2(\mathbb{R}^2; H)} \|\varphi\|_{L^2(\mathbb{R}^2; H)} \leq \frac{1}{t^2 \sqrt{1 - m(B_t)}} \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} |(\omega, b)|^2 |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db \right)^{\frac{1}{2}}
$$

We obtain the desired result by taking $C_s = t^4 \sqrt{1 - m(B_t)} \square$

4.1 Benedicks-type uncertainty principle

First, we remind the definitions of some notions.

**Definition 4.6.** Let $\Sigma$ a measurable subset of $\mathbb{R}^2 \times \mathbb{R}^2$ and $\varphi \in L^2(\mathbb{R}^2; H)$ a nonzero window function. Then,

1. We say that $\Sigma$ is weakly annihilating, if any function $f \in L^2(\mathbb{R}^2; H)$ vanishes when its GQFT $G_{\varphi}\{f\}$ with respect to window $\varphi$ is supported in $\Sigma$.
2. We say that $\Sigma$ is strongly annihilating, if there exists a constant $C(\Sigma) > 0$, such that for every function $f \in G_{\varphi}\{f\}$

$$
\|f\|^2_{L^2(\mathbb{R}^2; H)} \|\varphi\|^2_{L^2(\mathbb{R}^2; H)} \leq C(\Sigma) \int_{\mathbb{R}^2 \times \mathbb{R}^2} |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db
$$

The constant $C(\Sigma)$ will be called the annihilation constant of $C(\Sigma)$

**Lemma 4.7.** Let $\varphi$ be a nonzero window function. Then,

1. If $\|P_{\Sigma} P_{\varphi}\| < 1$, then for all $f \in L^2(\mathbb{R}^2; H)$,

$$
\|f\|^2_{L^2(\mathbb{R}^2; H)} \|\varphi\|^2_{L^2(\mathbb{R}^2; H)} \leq \frac{1}{\sqrt{1 - \|P_{\Sigma} P_{\varphi}\|^2}} \int_{\Sigma^c} |G_{\varphi}\{f\}(\omega, b)|^2 d\omega db \tag{4.4}
$$

2. If $\Sigma$ is strongly annihilating, $\|P_{\Sigma} P_{\varphi}\| < 1$

**Proof.** For every $f \in L^2(\mathbb{R}^2; H)$; we have

$$
\|G_{\varphi}(f)\|^2_{L^2(\mathbb{R}^2; H)} = \|G_{\varphi}\{f\}\chi_{\Sigma}\|^2_{L^2(\mathbb{R}^2; H)} + \|G_{\varphi}\{f\}\chi_{\Sigma^c}\|^2_{L^2(\mathbb{R}^2; H)} \tag{4.5}
$$
with $\mathcal{G}_\varphi \{ f \} \chi_\Sigma = P_\Sigma P_\varphi (\mathcal{G}_\varphi \{ f \})$
and from the Plancherel formula \ref{eq:plancherel} \[ \| \mathcal{G}_\varphi \{ f \} \chi_\Sigma \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} \leq \| P_\Sigma P_\varphi \|_2 \| \mathcal{G}_\varphi \{ f \} \|_{L^2(\mathbb{R}^2, \mathbb{H})} \]
\[ = \| P_\Sigma P_\varphi \|_2 \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]
Thus, by the equation \ref{eq:inequality}
\[ \| \mathcal{G}_\varphi \{ f \} \chi_\Sigma \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{H})} \geq (1 - \| P_\Sigma P_\varphi \|_2^2) \| \varphi \|_{L^2(\mathbb{R}^2, \mathbb{H})} \| f \|_{L^2(\mathbb{R}^2, \mathbb{H})}. \]

Now, we give an analogue of Benedicks-type theorem, which state that, for a subset $\Sigma$ of the form $\Sigma = S \times B_R \subset \mathbb{R}^2 \times \mathbb{R}^2$, such that $0 < m(S) < \infty$ and $B_R$ is the ball of centre 0 and the radius $R$. Then $\Sigma$ is weakly annihilating.

Now, we give the Benedicks theorem,

**Theorem 4.8** (Benedicks-type theorem for $\mathcal{G}_\varphi \{ f \}$). Let $r, R > 0$. Let $\varphi \in L^2(\mathbb{R}^2, \mathbb{H}) \cap L^\infty(\mathbb{R}^2, \mathbb{H})$ be nonzero window function such that $\text{supp}\varphi \subseteq B_r$ and let $\Sigma = S \times B_R \subset \mathbb{R}^2 \times \mathbb{R}^2$, be a subset of finite measure $0 < m(\Sigma) < \infty$. Then, $\text{Im} P_\varphi \cap \text{Im} P_\Sigma = \{ 0 \}$
i.e. $\Sigma$ is weakly annihilating.

**Proof.** Let $F \in \text{Im} P_\varphi \cap \text{Im} P_\Sigma$, then, there exists a function $f \in L^2(\mathbb{R}^2, \mathbb{H})$, such that, $F = \mathcal{G}_\varphi \{ f \}$ and $\text{supp}\{ F \} \subset \Sigma$.
Then,
\[ F(\omega, b) = \mathcal{F}_q (f(\cdot)\overline{\varphi(\cdot - b)})(\omega) \]
thus,
\[ \text{supp}\{ f(\cdot)\overline{\varphi(\cdot - b)} \} \subset S. \]
On other hand $\text{supp}\{ \varphi \} \subset B_r$,
we have
\[ \text{supp}\{ f(\cdot)\overline{\varphi(\cdot - b)} \} \subset B_{r+R}. \]
Hence, by the Benedicks theorem for two sided quaternion Fourier transform \cite{Benedicks}, we deduce that
\[ f(\cdot)\overline{\varphi(\cdot - b)} \equiv 0, \text{ then } F = 0. \]
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