Ultrafilter extensions of linear orders

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Abstract

Ultrafilter extensions of arbitrary first-order models were defined in [1]. Here we consider the case when the models are linearly ordered sets. We explicitly calculate the extensions of a given linear order and the corresponding operations of minimum and maximum on a set. We show that the extended relation is not more an order but is close to the natural linear ordering of nonempty half-cuts of the set and that the two extended operations define a skew lattice structure on the set of ultrafilters.

1. Preliminaries

Ultrafilter extensions of arbitrary first-order models were defined in [1]. If \((X, F, \ldots, P, \ldots)\) is a model with the universe \(X\), operations \(F, \ldots\), and relations \(P, \ldots\), it canonically extends to the model \((\beta X, \tilde{F}, \ldots, \tilde{P}, \ldots)\) (of the same language), where \(\beta X\) is the set of ultrafilters over \(X\), the operations \(\tilde{F}, \ldots\) extend the operations \(F, \ldots\), and the relations \(\tilde{P}, \ldots\) extend the relations \(P, \ldots\). Here \(X\) is considered as a subset of \(\beta X\) by identifying each element \(x\) in \(X\) with the principal ultrafilter \(\tilde{x}\) given by \(x\). The main result of [1] shows that, roughly speaking, the construction smoothly generalizes the Stone–Čech compactification of a discrete space to the situation when the space carries a first-order structure.

The principal precursor of this construction was ultrafilter extensions of semigroups, the technique invented in 60s and then used to obtain significant results in number theory, algebra, and topological dynamics; the book [2] is a comprehensive treatise of this field. For the general definition of the extension, a description of topological properties of the extended models, and the precise formulation of the aforementioned result, we refer the reader to [1].

In this note we consider a rather special case of models, namely, linearly ordered sets. We shall deal only with binary relations and operations. If \(R\) is a binary relation on a set \(X\), it extends to the binary relation \(\tilde{R}\) on the set \(\beta X\) defined by

\[
u \tilde{R} v \iff \{x \in X : \{y \in X : x R y\} \in v\} \in u
\]

for all ultrafilters \(u, v \in \beta X\), and if \(F\) is a binary operation on \(X\), it extends to the binary operation \(\tilde{F}\) on \(\beta X\) defined by

\[
S \in \tilde{F}(u, v) \iff \{x \in X : \{y \in X : F(x, y) \in S\} \in v\} \in u
\]

for all \(u, v \in \beta X\) and all \(S \subseteq X\). The relations and operations considered here are definable from a given linear order \(<\), namely, the orders \(<\) and \(\leq\), the converse orders \(>\) and \(\geq\), and the operations of minimum and maximum.
As usually, a transitive binary relation is a pre-order (order, strict order) iff it is reflexive (reflexive and antisymmetric, irreflexive); it is called linear, or total, iff it is connected, i.e. any two distinct elements are comparable. We use the standard notation ≤ for (pre-)orders and < for strict orders, and its variants. By a linearly ordered set we mean as \((X, \leq)\) as well as \((X, <)\), and usually write simply \(X\). A subset \(I\) of a linearly ordered \(X\) is its initial segment iff it is downward closed, i.e. iff \(y \in I\) implies \(x \in I\) for all \(x < y\); final segments are upward closed subsets. A pair \((I, J)\) is a cut of a linearly ordered set \(X\) iff \(I\) and \(J\) are an initial and a final segments of \(X\) forming its partition (so \(x < y\) for all \(x \in I\) and \(y \in J\)); we shall call \(I\) and \(J\) the (left and right) half-cuts. A cut \((I, J)\) is proper iff both \(I\) and \(J\) are nonempty, a jump iff \(I\) has the greatest element and \(J\) the smallest one, a gap iff neither of these two happens, and a Dedekind cut iff only one happens, i.e. either \(I\) has the greatest element but \(J\) does not have the smallest one, or conversely. A linearly ordered set is dense iff it has no jumps, complete iff it has no proper gaps, and continuous iff it is dense and complete (so has only Dedekind cuts). The Dedekind completion of \(X\) is the smallest complete set containing \(X\), which is obtained by adding to \(X\) all its proper gaps; if one adds also improper gaps, the resulting set is the smallest ordered compactification of \(X\) (w.r.t. the interval topology). Arbitrary ordered compactification of \(X\) has either one or two elements filling each proper gap of \(X\), so the family of all ordered compactifications of \(X\) is isomorphic to the powerset of the set of its proper gaps (see [4]–[6] and recent review [7]). For more on linearly ordered sets we refer the reader to [8].

N. L. Poliakov asked me about the ultrafilter extensions of a linear order and the corresponding operations of minimum and maximum. He hypothesized that the ultrafilter extension of a linear order on a set is a linear pre-order whose quotient is isomorphic to the natural ordering of cuts of the set. Here it is proved that his attempt to describe the extension works for well-orders and in general is, though not correct, rather close to be correct: the extension itself is not a pre-order but a certain its combination with the extension of the converse order gives a pre-order whose quotient is isomorphic to the natural ordering of half-cuts.

The structure of this note is as follows. In Section 2 we define, for every ultrafilter \(u\) over a linearly ordered set, its support, which is either the element generating \(u\) if \(u\) is principal, or a half-cut otherwise. We note that the natural linear ordering of supports connects to the largest linearly ordered compactification of the set. In Section 3 we describe the ultrafilter extensions of given order relations \(<, \leq, >, \geq\) in terms of supports of ultrafilters. Then we show that \(\leq, \geq\) do not share many features of orders, however, can be “amalgamed” into a linear pre-order inducing the natural ordering of supports. In Section 4 we describe ultrafilter extensions of the operations min and max in terms of supports, show that, except for commutativity, \(\text{min}\) and \(\text{max}\) have the usual features of min and max on a linearly ordered set and, actually, turn out the set of ultrafilters into a distributive skew lattice of a special form. Finally, we show that the equivalence \(D\) on the skew lattice coincides with the equality of supports and the quotient lattice \(\beta X/D\) is isomorphic to the set of supports with its operations min and max. We conclude by asking about properties of ultrafilter extensions of partially ordered sets and related algebras. The note is quite easy and self-contained.

\(^1\)Personal communication. July, 2013.
2. Supports of ultrafilters over linearly ordered sets

Let $X$ be a linearly ordered set. For any ultrafilter $u$ over $X$ define the initial segment $I_u$ and the final segment $J_u$ of $X$ as follows:

$$I_u = \bigcap \{ I \in u : I \text{ is an initial segment of } X \},$$

$$J_u = \bigcap \{ J \in u : J \text{ is a final segment of } X \}.$$

**Lemma 1.** Let $X$ be a linearly ordered set and $u$ an ultrafilter over $X$.

1. If $u$ is principal, then $I_u \cap J_u = \{ x \}$ where $u = \overline{x}$.
2. If $u$ is non-principal, then $(I_u, J_u)$ is a cut, and either $I_u$ or $J_u$, but not both, is in $u$.
3. If $I_u$ is in $u$, then so are all final segments of $I_u$, $S \cap I_u$ is cofinal in $I_u$ for all $S \in u$, and $I_u$ does not have the greatest element whenever $u$ is non-principal.
4. If $J_u$ is in $u$, then so are all initial segments of $J_u$, $S \cap J_u$ is coinitial in $J_u$ for all $S \in u$, and $J_u$ does not have the least element whenever $u$ is non-principal.

**Proof.** Easy. \(\square\)

Define the **support** $\text{supp}(u)$ of an ultrafilter $u \in \beta X$ by

$$\text{supp}(u) = \begin{cases} 
\{ x \} & \text{if } u = \overline{x} \text{ and } x \in X, \\
I_u & \text{if } u \in \beta X \setminus X \text{ and } I_u \in u, \\
J_u & \text{if } u \in \beta X \setminus X \text{ and } J_u \in u.
\end{cases}$$

Thus supports of ultrafilters over $X$ are subsets of $X$ which are either singletons, or initial segments without the last point, or else final segments without the first point, and it is clear that any subset of one of the three forms is the support of some ultrafilter.

**Example.** If $X$ is well-ordered and $u \in \beta X \setminus X$, then $\text{supp}(u) = I_u$. If $\alpha$ is an ordinal and $u \in \beta \alpha \setminus \alpha$, then $\text{supp}(u)$ is a limit ordinal $\beta \leq \alpha$.

This notion of supports, however, should be slightly refined. Let $X$ have no end-points (e.g. $X$ is the set $\mathbb{Z}$ of integers with their natural ordering), and let $u \in \beta X$ have all initial segments of $X$ and $v \in \beta X$ all final segments of $X$. Then $\text{supp}(u) = J_u = X$ and $\text{supp}(v) = I_v = X$, which shows that our notion cannot distinguish ultrafilters “concentrated” at the beginning and at the end of the set. There are several ways to correct this. E.g. in such cases we could define the supports as $\{-\infty\}$ and $\{+\infty\}$ (in fact, adding end-points to the set); or we could define the support of an $u$ as a pair — either $(I_u, J_u)$ or $(J_u, I_u)$ depending on what of $I_u$ and $J_u$ is in $u$. We prefer, however, to keep the definition above but understand henceforth the expressions “$\text{supp}(u) = I_u$” by “$u$ is non-principal and all final segments of $I_u$ are in $u$” and “$\text{supp}(u) = J_u$” by “$u$ is non-principal and all initial segments of $J_u$ are in $u$”.

The set of supports carries a natural linear order: $\text{supp}(u) < \text{supp}(v)$ iff either the cut given by $\text{supp}(u)$ is less than the cut given by $\text{supp}(v)$, or $\text{supp}(u)$ is the initial segment and $\text{supp}(v)$ is the final segment of the same cut. All possible cases are listed in the following...
supp(\(u\)) < supp(\(v\)) | supp(\(v\)) = \(\{y\}\) | supp(\(v\)) = \(I_v\) | supp(\(v\)) = \(J_v\) \\
--- | --- | --- | --- \\
\(\text{supp}(u) = \{x\}\) | \(x < y\) | \(x < \sup I_v\) | \(x \leq \inf J_v\) \\
\(\text{supp}(u) = I_u\) | \(\sup I_u \leq y\) | \(\sup I_u < \sup I_v\) | \(\sup I_u \leq \inf J_v\) \\
\(\text{supp}(u) = J_u\) | \(\inf J_u < y\) | \(\inf J_u < \sup I_v\) | \(\inf J_u < \inf J_v\)

(\text{which should be read as follows: “if supp}(\(u\)) = \{x\}\text{ and supp}(\(v\)) = \{y\}\text{, then supp}(u) < supp(v)\text{ is equivalent to } x < y\text{”, etc.}\text{) providing that sup and inf are in the Dedekind completion of } X.\)

Given a linearly ordered set \(X\), let \(s(X)\) denote the set of the supports of ultrafilters over \(X\) with their natural ordering. The transition from a linearly ordered set \(X\) to the linearly ordered set \(s(X)\) is a procedure similar to the Dedekind completion of \(X\); or, rather, the ordered compactification of \(X\); however, while the latter two add to the set only its gaps, the former one adds all its unbounded \textit{half-cuts} (rather than cuts), i.e. initial segments without the greatest element and final segments without the least element. Note also that both completion and compactification procedures are idempotent (i.e. their iterations do not change sets) while our construction is not.

If \(X\) is of the order-type \(\tau\), let \(s(\tau)\) denote the order-type of \(s(X)\). As it is customarily in linear order theory, the letters \(\zeta, \eta, \lambda\) are used to denote the order-types of the sets \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}\) of integers, rationals, reals, respectively; the multiplication of order-types is antilexicographic (e.g. \(2\omega = \omega, \omega 2 = \omega + \omega\)); for more details see [3].

\textbf{Examples.} 1. \(s(\omega) = \omega + 1\). Moreover, for all ordinals \(\alpha, s(\omega + \alpha) = \omega + \alpha + 1\).
2. \(s(\zeta) = 1 + \zeta + 1\). Moreover, for all ordinals \(\alpha, s(\alpha \zeta) = 1 + \alpha \zeta + 1\).
3. \(s(\lambda) = 1 + 3\lambda + 1\). Moreover, for all continuous order-types \(\tau, s(\tau) = 1 + 3\tau + 1\).
4. \(s(\eta) = 1 + \sum_{x \in \mathbb{R}} \tau_x + 1\) where \(\tau_x = 3\) if \(x \in \mathbb{Q}\), and \(\tau_x = 2\) otherwise. Moreover, for all dense order-type \(\tau, s(\tau) = 1 + \sum_{x \in \mathbb{R}} \tau_x + 1\) where \(\tau_x = 3\) if \(x \in \mathbb{X}\), and \(\tau_x = 2\) otherwise, whenever \(X\) is any set of the order-type \(\tau\) and \(Y\) the Dedekind completion of \(X\).

\textbf{3. Ultrafilter extensions of linear orders}

The following theorem describes the ultrafilter extensions of linear orders in terms of supports.

\textbf{Theorem 1.} For all ultrafilters \(u, v\) over a linearly ordered set \(X\),

\(u \prec v \iff \text{supp}(u) < \text{supp}(v) \lor \text{supp}(u) = \text{supp}(v) = I_u = I_v,\)
\(u \preceq v \iff \text{supp}(u) < \text{supp}(v) \lor \text{supp}(u) = \text{supp}(v) = I_u = I_v \lor \exists x (u = v = x),\)
\(u \succ v \iff \text{supp}(u) > \text{supp}(v) \lor \text{supp}(u) = \text{supp}(v) = J_u = J_v,\)
\(u \succeq v \iff \text{supp}(u) > \text{supp}(v) \lor \text{supp}(u) = \text{supp}(v) = J_u = J_v \lor \exists x (u = v = x).\)

Consequently, on non-principal ultrafilters, \(\prec\) coincides with \(\preceq\) and \(\succ\) coincides with \(\succeq\).
Proof. Let \( X_{<x} \) denote the initial segment \( \{ y \in X : y < x \} \), and \( X_{\leq x}, X_{>x}, X_{\geq x} \) have the expected meaning. By definition, \( u \prec v \) means \( \{ x : X_{>x} \in v \} \in u \). First, we observe that

\[
\begin{array}{cccc}
\text{supp}(v) = \{ y \} & \text{supp}(v) = I_v & \text{supp}(v) = J_v \\
X_{<x} \in v & y < x & \sup I_v \leq x & \inf J_v < x \\
X_{\leq x} \in v & y \leq x & \sup I_v \leq x & \inf J_v < x \\
X_{>x} \in v & x < y & x < \sup I_v & x \leq \inf J_v \\
X_{\geq x} \in v & x \leq y & x < \sup I_v & x \leq \inf J_v \\
\end{array}
\]

(should be read: “if \( \text{supp}(v) = \{ y \} \), then \( X_{<x} \in v \) is equivalent to \( y < x \)”, etc.), and so

\[
\begin{array}{cccc}
\text{supp}(v) = \{ y \} & \text{supp}(v) = I_v & \text{supp}(v) = J_v \\
\{ x : X_{<x} \in v \} & X_{>y} & X_{\sup I_v} & X_{\inf J_v} \\
\{ x : X_{\leq x} \in v \} & X_{\geq y} & X_{\sup I_v} & X_{\inf J_v} \\
\{ x : X_{>x} \in v \} & X_{<y} & X_{\inf I_v} & X_{\sup J_v} \\
\{ x : X_{\geq x} \in v \} & X_{\leq y} & X_{\inf I_v} & X_{\sup J_v} \\
\end{array}
\]

(should be read: “if \( \text{supp}(v) = \{ y \} \), then \( \{ x : X_{<x} \in v \} \) equals \( X_{>y} \)” etc.). Repeating this observation once more, we characterize \( \{ x : X_{>x} \in v \} \in u \) as follows:

\[
\begin{array}{cccc}
\text{supp}(v) = \{ y \} & \text{supp}(v) = I_v & \text{supp}(v) = J_v \\
\{ x : X_{>x} \in v \} \in u & \text{supp}(v) = \{ y \} & \text{supp}(v) = I_v & \text{supp}(v) = J_v \\
\text{supp}(u) = \{ x \} & x < y & x < \sup I_v & x \leq \inf J_v \\
\text{supp}(u) = I_u & \sup I_u \leq y & \sup I_u \leq \sup I_v & \sup I_u \leq \inf J_v \\
\text{supp}(u) = J_u & \inf J_u < y & \inf J_u < \sup I_v & \inf J_u < \inf J_v \\
\end{array}
\]

(should be read: “if \( \text{supp}(u) = \{ x \} \) and \( \text{supp}(v) = \{ y \} \), then \( \{ x : X_{>x} \in v \} \in u \) is equivalent to \( x < y \)” etc.). And comparing this with \([1]\), we see that \( \{ x : X_{>x} \in v \} \in u \) holds iff either \( \text{supp}(u) \prec \text{supp}(v) \) or \( \text{supp}(u) = I_u = \text{supp}(v) = I_v \), as required.

Next, we have

\[
\tilde{u} \preceq v \iff \{ x : \{ y : x \leq y \} \in v \} \in u \iff \{ x : \{ y : x < y \lor x = y \} \in v \} \in u
\]

\[
\iff \{ x : \{ y : x < y \} \in v \} \in u \lor \{ x : \{ y : x = y \} \in v \} \in u
\iff \tilde{u} \preceq v \lor \tilde{u} \equiv v.
\]

And as easy to see, \( \tilde{u} \equiv v \) means \( u = v = \bar{x} \) for some \( x \).

The relations \( \tilde{x} \) and \( \tilde{y} \) are handled dually: by definition, \( u \succ v \) means \( \{ x : X_{<x} \in v \} \in u \); by \([2]\), we get

\[
\begin{array}{cccc}
\{ x : X_{<x} \in v \} \in u & \text{supp}(v) = \{ y \} & \text{supp}(v) = I_v & \text{supp}(v) = J_v \\
\text{supp}(u) = \{ x \} & y < x & \sup I_v \leq x & \inf J_v < x \\
\text{supp}(u) = I_u & y < \sup I_u & \sup I_v < \sup I_u & \inf J_v < \sup I_u \\
\text{supp}(u) = J_u & y \leq \inf J_u & \sup I_v \leq \inf J_u & \inf J_v \leq \inf J_u \\
\end{array}
\]
and comparing this with (1), we see that \( \{ x : X_{<x} \in v \} \in u \) holds iff either \( \text{supp}(v) < \text{supp}(u) \) or \( \text{supp}(u) = J_u = \text{supp}(v) = J_v \), as required. And \( u \gtrless v \) is equivalent to \( u \gtrless v \lor u \gtrless v \).

As easy to see from the established theorem, the relations extending linear orders generally have only a few features of linear orders.

**Corollary 1.** Let \( X \) be a linearly ordered set.

1. For all non-principal ultrafilters \( u, v \) over \( X \),

\[
(u \gtrless v \lor u \gtrless v) \land \neg (u \gtrless v \land u \gtrless v).
\]

More precisely, if \( u, v \) have distinct supports, then

\[
u \gtrless v \iff \neg (v \gtrless u) \iff \neg (u \gtrless v) \iff \text{supp}(u) < \text{supp}(v),
\]

and if \( u, v \) have the same support, then

\[
u \gtrless v \iff \neg (u \gtrless v) \iff \text{supp}(u) = \text{supp}(v) = I_u = I_v,
\]

\[
u \gtrless v \iff \neg (u \gtrless v) \iff \text{supp}(u) = \text{supp}(v) = J_u = J_v.
\]

2. The relations \( \gtrless, \lessgtr, \gtrless \) are transitive, but non-antisymmetric, non-connected, and neither reflexive nor irreflexive.

**Proof.** 1. It immediately follows from Theorem 1. Alternatively, we can see this without Theorem 1, from a general argument: start from the corresponding formula about \( < \) and \( > \) and observe that connectives commute with ultrafilter quantifiers.

2. Transitivity is also immediate by Theorem 1. Moreover, by clause 1, we have the following description of points of reflexivity and irreflexivity:

\[
u \gtrless u \iff \neg (u \gtrless u) \iff \text{supp}(u) = I_u,
\]

\[
u \gtrless u \iff \neg (u \gtrless u) \iff \text{supp}(u) = J_u,
\]

and if we pick \( u \neq v \), the following equivalences describe non-antisymmetry:

\[
u \gtrless u \land v \gtrless u \iff \text{supp}(u) = \text{supp}(v) = I_u = I_v,
\]

\[
u \gtrless v \land \nu \gtrless u \iff \text{supp}(u) = \text{supp}(v) = J_u = J_v,
\]

and non-connectedness:

\[
\neg (u \gtrless v) \land \neg (v \gtrless u) \iff \text{supp}(u) = \text{supp}(v) = J_u = J_v,
\]

\[
\neg (u \gtrless v) \land \neg (v \gtrless u) \iff \text{supp}(u) = \text{supp}(v) = I_u = I_v.
\]

Of course, the existence of two distinct ultrafilters \( u, v \) with any of the required properties assumes a dose of AC, as even the existence of one such ultrafilter does. In some cases (e.g. if \( X \) is well-orderable), the existence of one such ultrafilter implies the existence of two ultrafilters. \( \square \)
Let us emphasize that, although for \( u \neq v \) the formula \( u \lesssim v \lor u \gtrsim v \) looks like connectedness and the formula \( \neg(u \lesssim v \land u \gtrsim v) \) looks like antisymmetry, they actually are not these properties since \( u \leq v \) and \( v \geq u \) are not the same. Instructively, this shows that the ultrafilter extension of a relation does not commute with taking of the inverse.

Combining \( u \lesssim v \) and \( v \gtrsim u \), however, we can get a kind of their “commutator”, which behaves closer to a linear order. Define a relation \( \preceq \) on ultrafilters by

\[
\preceq \leftrightarrow u \lesssim v \lor v \gtrsim u.
\]

It is clear from the previous that \( u \preceq v \) is equivalent to \( u \lesssim v \lor v \gtrsim u \lor \exists x (u = v = \tilde{x}) \).

We put also

\[
u \equiv v \leftrightarrow u \preceq v \land v \preceq u.
\]

**Corollary 2.** For all ultrafilters \( u, v \) over a linearly ordered set \( X \),

\[
\begin{align*}
u \preceq v &\leftrightarrow \text{supp}(u) \leq \text{supp}(v), \\
u \equiv v &\leftrightarrow \text{supp}(u) = \text{supp}(v).
\end{align*}
\]

Thus \( \preceq \) is a linear pre-order, \( \equiv \) is an equivalence, and the quotient set \( \beta X/\equiv \) with the induced linear order is isomorphic to the set \( s(X) \) of supports with their natural ordering.

**Proof.** By Theorem 1, we have

\[
\begin{align*}
u \preceq v &\leftrightarrow u \lesssim v \lor v \gtrsim u \\
&\leftrightarrow \text{supp}(u) < \text{supp}(v) \lor \text{supp}(u) = \text{supp}(v) = I_u = I_v \\
&\quad \lor \text{supp}(u) = \text{supp}(v) = J_u = J_v \lor \exists x (u = v = \tilde{x}) \\
&\leftrightarrow \text{supp}(u) \leq \text{supp}(v),
\end{align*}
\]

as required. The equivalence class \( \{v : v \equiv u\} \) of \( u \) is hence \( \{v : \text{supp}(v) = \text{supp}(u)\} \), and the claim follows.

**Corollary 3.** If \( \leq \) is a well-order, then \( \lesssim \) coincides with \( \leq \) and is a pre-well-order.

**Proof.** As noted above, for all non-principal ultrafilters \( u \) over a well-ordered set \( X \), \( \text{supp}(u) = I_u \). Hence, \( u \lesssim v \) is equivalent to \( \text{supp}(u) \leq \text{supp}(v) \) by Theorem 1 and thus to \( u \preceq v \) by Corollary 2.

4. Ultrafilter extensions of operations min and max

Here we describe the ultrafilter extensions of the minimum and maximum operations on a given linearly ordered set. Firstly we do this in terms of the extensions of the order and the converse order.

**Theorem 2.** If \( X \) is a linearly ordered set and \( u, v \) are ultrafilters over \( X \), then

\[
\begin{align*}
\inf(u, v) = u &\leftrightarrow \sup(u, v) = v \leftrightarrow u \leq v \lor u = v, \\
\inf(u, v) = v &\leftrightarrow \sup(u, v) = u \leftrightarrow u \geq v \lor u = v.
\end{align*}
\]
Proof. We have, for all $S \subseteq X$,

$S \in \tilde{\text{min}}(u, v) \iff \{ x : \{ y : \min(x, y) \in S \} \in v \} \in u$

$\iff \{ x : \{ y : (x \leq y \land x \in S) \lor (x \geq y \land y \in S) \} \in v \} \in u$

$\iff \{ \{ x : \{ y : x \leq y \} \in v \} \in u \land S \in u \} \lor \{ \{ x : \{ y : x \geq y \} \in v \} \in u \land S \in v \}$

$\iff (u \leq v \land S \in u) \lor (u \geq v \land S \in v)$.

Therefore,

$$\tilde{\text{min}}(u, v) = \begin{cases} u & \text{if } u \leq v, \\ v & \text{if } u \geq v. \end{cases}$$

The dual argument gives

$$\tilde{\text{max}}(u, v) = \begin{cases} u & \text{if } u \geq v, \\ v & \text{if } u \leq v. \end{cases}$$

Recalling now that for any $u \in \beta X$ either $u \leq u$ or $u \geq u$ (if $u$ is non-principal, this depends on what of $I_u$ or $J_u$ is the support of $u$), we complete the proof.

Now we are able to describe $\tilde{\text{min}}$ and $\tilde{\text{max}}$ in terms of supports.

**Corollary 4.** If $X$ is a linearly ordered set and $u, v$ are ultrafilters over $X$, then

$$\tilde{\text{min}}(u, v) = u \iff \tilde{\text{max}}(u, v) = v$$

$$\iff \text{supp}(u) < \text{supp}(v) \lor \text{supp}(u) = \text{supp}(v) = I_u = I_v \lor u = v,$$

$$\tilde{\text{min}}(u, v) = v \iff \tilde{\text{max}}(u, v) = u$$

$$\iff \text{supp}(v) < \text{supp}(u) \lor \text{supp}(u) = \text{supp}(v) = J_u = J_v \lor u = v.$$

**Proof.** Theorems 1 and 2.

**Example.** If $X$ is $\omega$ with the natural ordering, we get $\tilde{\text{max}}(u, v) = v$ if $v$ is non-principal, and $\tilde{\text{max}}(u, v) = u$ if $v$ is principal and $u$ non-principal. This was noted in [2], Exercise 4.1.11.

Turning to algebraic properties of $\tilde{\text{min}}$ and $\tilde{\text{max}}$, we recall some facts about skew algebras. $(X, \cdot)$ is a skew semilattice, or shorter, a band, iff $\cdot$ is associative and idempotent, and a semilattice iff it is moreover commutative. A band is rectangular, or nowhere commutative, iff it satisfies $xyx = x$, or equivalently, $xy \neq yx \lor x = y$. Bands satisfying the stronger condition $xy = x \lor xy = y$ are sometimes called quasi-trivial, see e.g. [8]; they are easily characterized as groupoids (i.e. algebras with one binary operation) in which each non-empty subset forms a subgroupoid. A complete description of all varieties of bands can be found in any of [9]–[11]; for more on various special classes of semigroups see e.g. [12]. The congruence $D$ on a band $X$ is defined by letting, for all $x, y \in X$,

$$x D y \iff x y x = x \land y x y = y.$$

The quotient $X/D$ of a band $X$ is a semilattice and $D$-equivalence classes are rectangular subbands of $X$; moreover, $X/D$ is the largest semilattice quotient of $X$ (i.e. any homomorphism of $X$ into any semilattice $Y$ is decomposed into the canonical homomorphism
of $X$ onto $X/D$ and a homomorphism of $X/D$ into $Y$) and the $D$-equivalence class of each
$x \in X$ is the largest rectangular subband containing $x$.

$(X, +, \cdot)$ is a skew lattice iff both $(X, +)$ and $(X, \cdot)$ are bands and the following
absorption laws hold:

\[ x(x + y) = x + xy = x, \]
\[ (x + y)y = xy + y = y. \]

A commutative skew lattice is a lattice. A skew lattice is rectangular iff both its bands are
rectangular and dualize each other: $x + y = yx$. In a skew lattice $X$, the congruences $D$ for
$+$ and $\cdot$ coincide, $X/D$ is the largest lattice quotient of $X$, and $D$-equivalence classes are
maximal rectangular skew lattices. A skew lattice is distributive iff each of its operations
is left and right distributive w.r.t. another one:

\[ x(y + z) = xy + xz, \quad x + yz = (x + y)(x + z), \]
\[ (x + y)z = xz + yz, \quad xy + z = (x + z)(y + z). \]

(Note that distributivity implies “a half” of the absorption identitys above.) If a skew
lattice $X$ is distributive, so is the lattice $X/D$. We point out that skew lattices, introduced
(with slightly different absorption lawss) in [13], were intensively studied in past decades,
see e.g. [14, 15].

Our following result shows that the ultrafilter extensions of linearly ordered sets with
its minimum and maximum operations provide natural instances of skew lattices.

**Corollary 5.** The operations $\widetilde{\min}$ and $\widetilde{\max}$ are associative, idempotent and even quasi-
trivial, non-commutative, and distributive w.r.t. each other. Therefore, $(\beta X, \widetilde{\min}, \widetilde{\max})$ is
a distributive skew lattice.

**Proof.** As well-known, associativity is stable under ultrafilter extensions (see [2]), so the
operations $\widetilde{\min}$ and $\widetilde{\max}$ are associative. On the other hand, it can be shown that neither
commutativity, nor idempotency, nor distributivity is not stable (see [16]).

It is clear from Theorem 2 that $\widetilde{\min}$ and $\widetilde{\max}$ are indeed non-commutative on distinct
non-principal ultrafilters with the same support. Indeed, if $\text{supp}(u) = \text{supp}(v) = I_u = I_v$,
then $\widetilde{\min}(u, v) = \widetilde{\max}(v, u) = u$ and $\widetilde{\min}(v, u) = \widetilde{\max}(v, u) = v$, and similarly for the dual
case. So we get the following description of points of non-commutativity:

\[ \widetilde{\min}(u, v) \neq \widetilde{\min}(v, u) \leftrightarrow \widetilde{\max}(u, v) \neq \widetilde{\max}(v, u) \leftrightarrow u \neq v \land \text{supp}(u) = \text{supp}(v). \quad (3) \]

It is evident from Theorem 2 also that $\widetilde{\min}$ and $\widetilde{\max}$ are idempotent. But this can be
seen without Theorem 2 from a general fact: each of $\min$ and $\max$ satisfies quasi-triviality,
which is obviously stronger than idempotency and is stable under the ultrafilter extension
(see [16]).

Next, distributivity is verified by a direct calculation. E.g. check the identity

\[ \widetilde{\max}(u, \widetilde{\min}(v, w)) = \widetilde{\min}(\widetilde{\max}(u, v), \widetilde{\max}(u, w)). \quad (4) \]
Recall that we have either \( u \leq v \) or \( u \geq v \), but if one of \( u, v \) is non-principal, not both (Corollary 1). Hence, for any \( u, v, w \) we have exactly 8 conjunctions of possible relationships between each pair of them:

\[
\begin{align*}
(i) & \quad u \leq v \land u \leq w \land v \leq w, \\
(ii) & \quad u \leq v \land u \leq w \land v \geq w, \\
(iii) & \quad u \leq v \land u \geq w \land v \leq w, \\
(iv) & \quad u \leq v \land u \geq w \land v \geq w, \\
(v) & \quad u \geq v \land u \leq w \land v \leq w, \\
(vi) & \quad u \geq v \land u \leq w \land v \geq w, \\
(vii) & \quad u \geq v \land u \geq w \land v \leq w, \\
(viii) & \quad u \geq v \land u \geq w \land v \geq w.
\end{align*}
\]

It is immediate from Theorem 2 that the identity (4) holds in all of these cases except for cases (iii) and (vi), where it may appear that it fails. However, these two cases are in fact degenerate because of transitivity of \( \leq \) and \( \geq \) (Corollary 1). E.g. in case (iii), \( u \leq v \leq w \) gives \( u \leq w \), which together with \( u \geq w \) gives \( u = w = \bar{x} \) for some \( x \), whence it follows \( u = v = w \), which of course gives the required identity.

Finally, to handle absorption let check e.g. that

\[\widetilde{\min}(u, \widetilde{\max}(u, v)) = u.\]

But this easily follows from Theorem 2: if \( \widetilde{\max}(u, v) = u \) then the left term \( \widetilde{\min}(u, u) \) equals \( u \) by idempotency, while if \( \widetilde{\max}(u, v) = v \) then the left term \( \widetilde{\min}(u, v) \) equals \( u \) because \( \min(u, v) = u \) is equivalent to \( \max(u, v) = v. \]

Thus, among the obvious features of the operations \( \min \) and \( \max \), only commutativity fails under ultrafilter extensions. Modulo the equivalence \( \equiv \), however, the operations \( \widetilde{\min} \) and \( \widetilde{\max} \) become commutative and actually the corresponding minimum and maximum.

**Corollary 6.** Let \( X \) be a linearly ordered set.

1. For all \( u, v \in \beta X \),

\[
\begin{align*}
    u \leq v & \iff \widetilde{\min}(u, v) = u \lor \widetilde{\min}(v, u) = u & \iff \widetilde{\min}(u, v) = u \lor \widetilde{\min}(u, v) \neq \widetilde{\min}(v, u) \\
    & \iff \widetilde{\max}(u, v) = v \lor \widetilde{\max}(v, u) = v & \iff \widetilde{\max}(u, v) = v \lor \widetilde{\max}(u, v) \neq \widetilde{\max}(v, u),
\end{align*}
\]

and

\[
    u \equiv v \iff \widetilde{\min}(u, v) \neq \widetilde{\min}(v, u) \lor u = v \iff \widetilde{\max}(u, v) \neq \widetilde{\max}(v, u) \lor u = v.
\]

2. The equivalence \( \equiv \) is a congruence of the skew lattice \( (\beta X, \widetilde{\min}, \widetilde{\max}) \) and actually coincides with \( D \).

3. The quotient \( \beta X/\equiv \) with the operations induced by \( \widetilde{\min} \) and \( \widetilde{\max} \) is isomorphic to the lattice \( s(X) \) with its minimum and maximum operations.

4. The \( \equiv \)-equivalence class of each \( u \in \beta X \) is either a left-zero band for \( \widetilde{\min} \) and a right-zero band for \( \widetilde{\max} \), or conversely, a right-zero band for \( \widetilde{\min} \) and a left-zero band for \( \widetilde{\max} \).

**Proof.** 1. By reflexivity of \( \leq \) and Theorem 2, we have

\[
    u \leq v \iff u \leq v \lor v \geq u \\
    \iff u \leq v \lor v \geq u \lor u = v \iff \widetilde{\min}(u, v) = u \lor \widetilde{\min}(v, u) = u.
\]
The equivalences
\[ u \equiv v \iff \tilde{\min}(u,v) \neq \tilde{\min}(v,u) \lor u = v \]
and
\[ u \preceq v \iff \tilde{\min}(u,v) \neq \tilde{\min}(v,u) \lor \tilde{\min}(u,v) = u \]
can be deduced either directly from Theorem 2 or by using (3). The characterizations using \( \tilde{\max} \) are obtained similarly.

2, 3. By Corollary 3, \( u \equiv v \) is equivalent to \( \text{supp}(u) = \text{supp}(v) \), which holds for non-principal \( u, v \) either if the support is \( I_u = I_v \) or if it is \( J_u = J_v \). Now it is easily follows from Corollary 4 that \( \equiv \) is a congruence of \( (\beta X, \tilde{\min}, \tilde{\max}) \) and its quotient is isomorphic the lattice \( (s(X), \text{min}, \text{max}) \). As the congruence \( D \) has the largest lattice quotient, we conclude that \( D \subseteq \equiv \). To verify the converse inclusion \( \equiv \subseteq D \), it suffices to show the following implication:
\[ u \equiv v \rightarrow \tilde{\min}(\tilde{\min}(u,v), u) = u. \]
This is immediate in the case \( \tilde{\min}(u,v) = u \) as well as in the case \( \tilde{\min}(u,v) = v \land \min(v,u) = u \). In the remaining case \( \min(u,v) = \min(v,u) = v \) we use Theorem 2 to conclude that \( u = v \) and so the implication holds too.

4. By Corollary 4, if \( \text{supp}(u) = I_u \) then the \( \equiv \)-equivalence class of \( u \) is a left-zero band for \( \text{min} \) and a right-zero band for \( \text{max} \), and dually if \( \text{supp}(u) = J_u \). If \( \text{supp}(u) = \{x\} \) then of course the class is the singleton \( \{u\} \).

We see that the ultrafilter extension of a given linear order, as well as of other relations and operations definable via it, allows a clear and easy description. This is so, roughly speaking, because the theory of linear orders is easy. Ultrafilters having the same supports behave in the same way, so each \( \equiv \)-equivalence class can be identified with the filter that is its intersection (i.e. with a filter that is generated either (i) by one point, or (ii) by all final segments of an initial segment without the last element, or else (iii) by all initial segments of a final segment without the first element). These filters, in turn, can be identified with ultrafilters on the Boolean algebra of definable subsets, which has a rather simple structure since the theory is easy.

Task. Study ultrafilter extensions of partially ordered sets and related algebras (semilattices, lattices, Boolean algebras, etc.), and also their skew generalizations.

It may be hypothesized that the extensions are again some skew algebras, however, a proof requires new arguments since now ultrafilters can be concentrated on antichains.

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