Composite observer-based integral sliding mode dynamical tracking control for nonlinear systems subject to actuator faults and mismatched disturbances

Bei Liu, Yang Yi, Hong Shen and Chengbo Niu

Abstract
This brief proposes a novel composite observer-based integral sliding mode tracking control algorithm for a class of nonlinear systems affected by both actuator faults and mismatched disturbances. First, different types of observers, including the extended state observer, the fault diagnosis observer, and the disturbance observer, are integrated to estimate the unknown system state, actuator faults, and mismatched disturbances timely. Then, in accordance with the estimation information, the integral sliding surface and the integral sliding mode controller are proposed, which can tolerate the actuator faults and reject the mismatched disturbances. Meanwhile, the state trajectories can be driven into the specified sliding surface in a finite time. Furthermore, not only the stability, but the favorable dynamical tracking and the output constraints of closed-loop augmented systems can be guaranteed. Finally, the validities of the proposed algorithm are embodied by the simulation results of typical A4D systems.

Keywords
Anti-disturbance control, dynamical tracking control, fault-tolerant control, composite observer design, integral sliding mode

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Introduction
In industrial systems, actuators are usually vulnerable to sudden faults which might result in closed-loop instability and even cause catastrophic accidents. Therefore, developing effective fault diagnosis (FD) and reliable fault-tolerant control (FTC) algorithms has aroused extensive interest among researchers such as Ding and Yang et al. Under such a background, a mass of different theories and sources were documented during decades. Liu et al. proposed a semi-supervision FD method based on attitude information for a satellite on-orbit operation. Based on switching strategy, a new adaptive FTC method was investigated by Ouyang and Lin in the strict-feedback systems subjected to actuator failures and uncertain parameters.

In addition, unknown disturbances or uncertainties widely exist in various kinds of practical control systems, for example, helicopters with vibration suppression and sampled-data control systems with periodic disturbances. Hence, how to reject or attenuate unknown disturbances and achieve the preferable performance of systems becomes a crucial problem. After decades of development, many outstanding results have also been achieved, that is, $H_\infty$ or $L_2/L_\infty$ robust control and output regulation theory. Recently, a disturbance observer–based control (DOBC) scheme utilizes feedforward–feedback method to effectively estimate the dynamics of uncertainties or external disturbances. Some relatively complete theoretical proofs including stability and robustness of systems have been obtained when using DOBC method. Because of the simplicity of the implementation, DOBC approaches have been successfully applied in many different practical systems, such as those by Yi et al., Li et al., Guo and Cao, and Yi et al.

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Sliding mode control (SMC) has gained significant attention because of good suppression of external disturbances. However, it might be sensitive against uncertainties or disturbances when reaching the sliding surface. So an integral sliding mode control (ISMC) scheme is proposed and well developed, which can eliminate the reaching phase needed in a conventional SMC strategy. Wang et al. proposed a novel event-triggered ISMC strategy for higher-order multi-agent systems subject to external disturbances. In Yang et al.’s study, based on the ISMC and a designed disturbance observer (DO), a novel control scheme is presented to reduce chattering and exhibit the properties of nominal performance recovery. In order to precisely estimate the unknown external disturbances and improve the disturbance-handling capability, Pandey et al. proposed an ISMC combined with a DO. The DO-based ISMC approach has also been developed by Zhang et al. for systems with mismatched disturbances.

Here, our motivations of this study can be composed of the following points. First, most of the existing results only focus on single anti-disturbance or FTC problem. Theoretically, it is urgent to discuss an effective control algorithm when both external disturbances and sudden faults simultaneously affect the controlled systems. Second, the matched disturbances and the measurable state are two widely used for assumption conditions, which can cause a lot of constraints in engineering practices. It is necessary to discuss the mismatched disturbances and unmeasurable state. Finally, both the favorable dynamical tracking and the output constraint are two important indicators to measure the performance of a controlled system. So it is requisite to design algorithms for optimizing them under the framework of anti-disturbance or FTC.

Based on the analysis above and compared with aforementioned literatures, this paper focuses on a class of complex systems, which are subjected to mismatched disturbances, actuator faults and unmeasurable states. By integrating the designed extended state observer (ESO), fault diagnosis observer (FDO) with DO, a novel composite observer is built for realizing the estimation of unknown extended state, actuator faults as well as mismatched disturbances. Compared with those previous results of single FTC or anti-disturbance control, it is meaningful to discuss an effective control algorithm for the systems when suffering with both external disturbances and sudden faults simultaneously. Furthermore, the ISMC approach can effectively reject the actuator faults and the mismatched disturbances. The favorable reaching ability of integral sliding surface (ISS) is also guaranteed in a finite time. In addition to the stability and robustness discussed by Guo and Cao and Zhang et al., a convex optimization-based method is presented not only to guarantee the convergence of the tracking error to zero but to compress the system output within a bounded range.

The reminder is organized as follows. Section “System description” introduces the system description and the composite observer design can be found in section “Composite observer design.” The ISMC approach and the multi-objective performance analysis are discussed in sections “ISS and ISM controller design” and “Dynamical performance analysis.” Simulation illustrations are given in section “Numerical illustrations,” and section “Conclusion” draws the conclusions.

System description
In this brief, we research the following nonlinear system with actuator faults and exogenous disturbances given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + f(t) + Eh(t) \\
y(t) &= Cx(t) + Dh(t)
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector which is assumed to be unavailable. \(u(t) \in \mathbb{R}^m\) and \(y(t) \in \mathbb{R}\) denote the control input and system output. \(f(t)\) and \(h(t)\), respectively, represent unknown actuator faults and external disturbances. \(A, B, C, D,\) and \(E\) are the given constant matrices.

It is obvious that the existence of unknown faults and disturbances seriously affects the dynamical performance of the system. From equation (1), we can see the actuator faults and the systematic disturbances occur in different channels. Different observers will be respectively designed to estimate them. In this paper, the actuator faults are assumed to happen suddenly and the exogenous disturbances are regarded as existing all the time. In order to design effective control input, the following two assumptions are needed in this brief.

Assumption 1. \(\text{rank}(CB) = \text{rank}(B) = m\)

Assumption 2. The actuator faults \(f(t)\) and mismatched disturbances \(h(t)\) are assumed to satisfy \(\|f(t)\| \leq \varepsilon,\|h(t)\| \leq \rho,\) and \(\|h(t)\| \leq \rho_1\), where \(\varepsilon, \rho,\) and \(\rho_1\) are three positive scalars.

Similarly with Guo and Cao and Yang et al., the disturbances \(h(t)\) are supposed to be formulated by an exogenous system described by

\[
\begin{align*}
\dot{\xi}(t) &= M\xi(t) \\
h(t) &= N\xi(t)
\end{align*}
\]

where \(M\) and \(N\) are designed coefficient matrices.

For the sake of realizing good dynamical tracking performance, the state system is extended as

\[
\tilde{x}(t) = \begin{bmatrix} x^T(t), \int_0^t e^T(\theta)d\theta \end{bmatrix}^T
\]
where \( e(t) = y_d - y(t) \) is the tracking error, and \( y_d \) is the desired output. Based on equations (1) and (3), the augmented system can be deduced as

\[
\begin{align*}
\dot{x}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}(u(t) + \hat{f}(t)) + \tilde{E}\hat{h}(t) + \tilde{F}y_d \\
y(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}\hat{h}(t)
\end{align*}
\]

where

\[
\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C^T \\ 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D^T \\ 0 \end{bmatrix},
\]

\[\tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0 \\ -I \end{bmatrix}\]

**Composite observer design**

In order to design the effective control input, the extended state \( \tilde{x}(t) \), the actuator faults \( \hat{f}(t) \), and the mismatched disturbances \( \hat{h}(t) \) need to be described or estimated. In this section, by integrating the ESO, FDO with DO, a composite observer is constructed to estimate unknown state, faults, and mismatched disturbances simultaneously.

Specifically, three different kinds of observers are expressed as follows

\[
\begin{align*}
\dot{x}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}(u(t) + \hat{f}(t)) + \tilde{E}\hat{h}(t) + \tilde{F}y_d \\
\dot{y}(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}\hat{h}(t) \\
\dot{f}(t) &= -L_1y(t) + \rho(t) \\
\rho(t) &= L_1\tilde{C}\tilde{B}(u(t) - L_1y(t)) + L_1\tilde{C}(\tilde{A}\tilde{x}(t) + \tilde{B}u(t) + \tilde{E}\hat{h}(t) + \tilde{F}y_d) \\
\dot{h}(t) &= N_1\hat{h}(t) \\
\dot{z}(t) &= -L_2y(t) + \phi(t) \\
\phi(t) &= (M + L_2\tilde{C}\tilde{EN})(\hat{v}(t) - L_2y(t)) + L_2\tilde{C}(\tilde{A}\tilde{x}(t) + \tilde{B}u(t) + \tilde{F}y_d) \\
\end{align*}
\]

where \( \tilde{x}(t), \hat{h}(t), \xi(t), \hat{f}(t), \) and \( y(t) \) stand for the estimation of \( \tilde{x}(t), \hat{h}(t), \xi(t), \hat{f}(t), \) and \( y(t) \) respectively. \( \rho(t) \) and \( \phi(t) \) are two designed auxiliary variables. \( L_1, L_2, \) and \( L_2 \) represent the gains of ESO, FDO, and DO, which will be solved later.

By defining the errors as \( e_x(t) = \tilde{x}(t) - \hat{x}(t), \)
\( e_f(t) = \hat{f}(t) - \tilde{f}(t), \)
\( e_{\xi}(t) = \hat{\xi}(t) - \tilde{\xi}(t), \)
the dynamics of estimation errors can be, respectively, expressed by

\[
\begin{align*}
\dot{e}_x(t) &= (\tilde{A} - L_1\tilde{C})e_x(t) + \tilde{B}e_f(t) + (\tilde{E} - L_2\tilde{D})e_{\xi}(t) \\
\dot{e}_f(t) &= \tilde{f}(t) + L_1\tilde{C}\tilde{B}e_f(t) + L_1\tilde{C}\tilde{A}e_x(t) + L_1\tilde{C}\tilde{EN}e_{\xi}(t) + L_1\tilde{D}\hat{h}(t) \\
\dot{e}_{\xi}(t) &= (M + L_2\tilde{C}\tilde{EN})e_{\xi}(t) + L_2\tilde{C}\tilde{A}e_x(t) + L_2\tilde{C}\tilde{B}e_f(t) + L_2\tilde{C}\tilde{D}\hat{h}(t)
\end{align*}
\]

Then, by using equations (8)-(10), the composite estimation error vector is illustrated as

\[
\begin{align*}
\dot{e}_x(t) &= \begin{bmatrix} \tilde{A} - L_1\tilde{C} & \tilde{B} & \tilde{E} - L_2\tilde{D} \end{bmatrix} e(t) \\
&= \begin{bmatrix} L_1\tilde{C}\tilde{A} & L_1\tilde{C}\tilde{B} & L_1\tilde{C}\tilde{EN} \\ L_2\tilde{C}\tilde{A} & L_2\tilde{C}\tilde{B} & M + L_2\tilde{C}\tilde{EN} \end{bmatrix} e(t) \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_1\tilde{D} \end{bmatrix} \hat{h}(t) + \begin{bmatrix} 1 \end{bmatrix} \tilde{f}(t) + \begin{bmatrix} 0 \\ L_2\tilde{D} \end{bmatrix} \hat{h}(t)
\end{align*}
\]

where \( e_x(t) = [e_x(t), e_f(t), e_{\xi}(t)]^T \).

**Theorem 1.** For the known parameters \( \mu_i > 0, i = 1, 2, \) if there exist matrices \( P > 0, P_i > 0, i = 1, 2, R \) and \( R_i > 0, i = 1, 2, \) satisfying the following inequality

\[
[\begin{array}{ccc}
P_{i1} & P_{i2} & 0 \\
P_{i2} & P_{i3} & 0 \\
0 & 0 & R_i \\
\end{array}] < 0
\]

with

\[
[\begin{array}{ccc}
P_{11} & P_{12} & 0 \\
P_{12} & P_{13} & 0 \\
0 & 0 & R_1 \\
\end{array}] < 0
\]

Then the estimation errors \( e_x(t), e_f(t), \) and \( e_{\xi}(t) \) can be proved to be ultimately uniformly bounded (UUB). The observer gains can be calculated by \( L = P^{-1}R, L_1 = P_i^{-1}R_1, \) and \( L_2 = P_2^{-1}R_2 \).

**Proof.** Select the following Lyapunov function as

\[
\Phi = e_x^T(t)P_xe_x(t) + e_f^T(t)P_f e_f(t) + e_{\xi}^T(t)P_{\xi}e_{\xi}(t)
\]

Differentiating \( \Phi \) with respect to time \( t \), we have

\[
\Phi = e_x^T(t)P_x(\tilde{A} - L_1\tilde{C})e_x(t) + 2e_x^T(t)P_x(\tilde{B})e_f(t) + 2e_f^T(t)P_f(\tilde{E} - L_2\tilde{D})e_{\xi}(t) + 2e_f^T(t)P_fL_1\tilde{C}\tilde{B}e_f(t) + 2e_x^T(t)P_L_1\tilde{C}\tilde{EN}e_{\xi}(t) + 2e_x^T(t)P_L_1\tilde{D}\hat{h}(t) + 2e_f^T(t)P_L_1\tilde{D}\hat{h}(t) + e_{\xi}^T(t)P_{\xi}L_2\tilde{C}\tilde{EN}e_{\xi}(t)
\]

\[
\leq e_x^T(t)\Psi e_x(t) + \mu_1^2\|e_f(t)\|^2 + \mu_2^2\|\hat{h}(t)\|^2
\]
where

\[
\Psi = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
* & \Omega_{22} & \Omega_{23} \\
* & * & \Omega_{33}
\end{bmatrix}
\]

with

\[
\begin{align*}
\Omega_{11} &= \text{sym}(P \hat{A} - PL \hat{C}) \\
\Omega_{12} &= PB + (P_1 L_1 \hat{C} A)^	op \\
\Omega_{13} &= P(\hat{E} - LD)N + (P_2 L_2 \hat{C} \hat{A})^	op \\
\Omega_{22} &= \text{sym}(P_1 L_1 \hat{C} B) + \mu_1^2 P_1 P_1 \\
\Omega_{23} &= P_1 L_1 CEN + (P_2 L_2 \hat{C} \hat{B})^T \\
\Omega_{33} &= \text{sym}(P_2 M + P_2 L_2 CEN) + \mu_2^2 P_2 L_2 \hat{D} \hat{D}^T L_2^T P_2
\end{align*}
\]

Using Schur complement lemma, and defining \( R = PL, R_1 = P_1 L_1, \) and \( R_2 = P_2 L_2, \) we can conclude that \((11) \Leftrightarrow \Psi < 0.\) So equation \((11)\) can be deduced as

\[
\Phi \leq -\kappa_1 e_t^2(t) e_t(t) + \mu_1^2 \varepsilon_t^2(t) + \mu_2^2 \beta^2
\]

where \(\kappa_1 > 0\) is a designed constant. Furthermore, we can get that \(\| e_t(t) \|^2 \leq \kappa_1^{-1}(\mu_1^2 \varepsilon_t^2(t) + \mu_2^2 \beta^2),\) which can reflect the composite estimation error \(e_t(t)\) is UUB.

Following the dynamical estimation of the extended state, actuator faults and mismatched disturbances, ISS, and the corresponding integral sliding mode (ISM) controller will be designed for achieving satisfactory control performances in the next section.

**ISS and ISM controller design**

In this part, the ISS is organized as

\[
s(t) = \hat{G} \left(y(t) - y(0) - \hat{C} \int_0^t \hat{A} \hat{x}(\tau) + \hat{B} u_h(\tau) d\tau + \hat{B} f(\tau) + \hat{E} \hat{h}(\tau) + \hat{F} y_i d\tau d\tau \right) = 0
\]

where \(\hat{G}\) is a designed matrix, and \(\hat{G} \hat{C} \hat{B}\) is assumed as non-singular.

In equation \((16)\), \(u_h(t)\) is the nominal control law and is designed by

\[
u_h(t) = -K_h \hat{x} - K_e \hat{x}(t) - K_h \hat{h}(t) - \hat{f}(t)
\]

where \(K\) is the controller gain. In order to realize the dynamical compensation for unknown extended state and mismatched disturbances, we respectively define the gain \(K_x\) and \(K_h\) as \(K_x = (\hat{C} \hat{B})^T \hat{C} \hat{A}\) and \(K_h = (\hat{C} \hat{B})^T \hat{C} \hat{E},\) where \((\hat{C} \hat{B})^T = ((\hat{C} \hat{B})^T (\hat{C} \hat{B})^{-1} (\hat{C} \hat{B})^T).

Furthermore, the ISM controller can be constructed as

\[
u(t) = \nu_h(t) + \nu_N(t)
\]

where \(\nu_N(t)\) is a discontinuous control law.

It is noted that Theorem 1 has proved the boundedness of estimation errors. For obtaining the appropriate \(u_N(t)\), the following assumption needs further to be satisfied.

**Assumption 3.** The estimation errors \(e_x(t), e_f(t),\) and \(e_h(t)\) are respectively supposed to be bounded by \(\| e_x(t) \| \leq \theta_1, \| e_f(t) \| \leq \theta_2,\) and \(\| e_h(t) \| \leq \theta_3,\) where \(e_x(t) = \hat{h}(t) - \hat{h}(t), \theta_1, \theta_2,\) and \(\theta_3\) are known positive scalars.

So we select

\[
u_N(t) = - (\kappa_2 + \theta_1) \hat{B}^T \hat{C} \hat{G} T \hat{s}(t) - (\hat{G} \hat{C} \hat{B})^{-1} \hat{G} \hat{C} \hat{E} \hat{E}^T \hat{G} T \hat{s}(t) \theta_1 - (\hat{G} \hat{C} \hat{B})^{-1} \hat{G} \hat{C} \hat{E} \hat{E}^T \hat{G} T \hat{s}(t) \theta_2 - (\hat{G} \hat{C} \hat{B})^{-1} \hat{G} \hat{D} \hat{D}^T \hat{G} T \hat{s}(t) \theta_3
\]

where \(\kappa_2\) is a positive scalar.

Taking the time-derivative of the ISS equation \((16),\) we can obtain

\[
\dot{s}(t) = \hat{G} (\hat{C} \hat{x}(t) + \hat{D} \hat{h}(t) - \hat{C} \hat{A} \hat{x}(t) + \hat{B} u_h(t) + \hat{B} f(t) + \hat{E} \hat{h}(t) + \hat{F} y_i d\tau d\tau)
\]

Substituting the ISM controller equation \((18)\) into equation \((20),\) we have

\[
\dot{s}(t) = \hat{G} (\hat{C} \hat{B} u_h(t) + \hat{C} \hat{E} \hat{e}_h(t) + \hat{C} \hat{A} \hat{e}_x(t) + \hat{C} \hat{E} e_h(t) + \hat{D} \hat{h}(t))
\]

**Remark 1.** In the process of controller design, some appropriate adjustments will be considered due to the discontinuity of equation \((18).\) For example, \((\hat{G} \hat{C} \hat{B})^{-1} (\hat{G} \hat{C} \hat{E} \hat{E}^T \hat{G} T \hat{s}(t)) \theta_1\) should be rewritten as \((\hat{G} \hat{C} \hat{B})^{-1} (\hat{G} \hat{C} \hat{E} \hat{E}^T \hat{G} T \hat{s}(t)) \theta_1 + \delta \theta_2,\) where \(\delta\) is a small positive scalar.

Now, we put forward to the following theorem to prove a reachability condition of the augmented system equation \((4).\)

**Theorem 2.** Considering the designed ISS equation \((16),\) if the ISM controller is chosen as equations \((17)-(19),\) then the reachability condition of the augmented system equation \((4)\) can be satisfied, in other words, the state trajectories of the augmented system equation \((4)\) can
be proved to be ultimately driven into the ISS equation (16) within a finite time $T_r$, where $T_r$ can be calculated by

$$T_r \leq \frac{\sqrt{2}}{\kappa_2 \| B^T C^T G \|} \sqrt{\Phi_1(0)} \quad (22)$$

**Proof.** A Lyapunov function candidate is chosen as

$$\Phi_1(s(t), t) = \frac{1}{2} s^T(t) s(t) \quad (23)$$

From equations (19) and (21), the derivative of $\Phi_1(s(t), t)$ with respect to time $t$ is

$$\dot{\Phi_1} = s^T(t) \dot{G}(\tilde{C} \tilde{B} u_N(t)) + \tilde{C} \dot{B} \dot{e}_y(t) + \tilde{C} \dot{A} \dot{e}_x(t)$$

$$+ \tilde{C} \dot{E} \dot{e}_y(t) + \tilde{D} \dot{h}(t)$$

$$= -(\kappa_2 + \theta_1) \| B^T C^T G \| \theta_0 + s^T(t) \tilde{G} \tilde{C} \dot{B} \dot{e}_y(t)$$

$$- \| \tilde{E} \| \| \tilde{C} \| \| s(t) \| \theta_0 + s^T(t) \tilde{G} \tilde{C} \dot{A} \dot{e}_x(t)$$

$$- \| \tilde{D} \| \| \tilde{G} \| \| s(t) \| \rho + s^T(t) \tilde{G} \tilde{D} \dot{h}(t)$$

$$\leq -\kappa_2 \| B^T C^T G \| \| s(t) \| \leq 0 \quad (24)$$

which implies that the state trajectories of the augmented system equation (4) must converge into the ISS equation (16).

From equations (23) and (24), one has

$$\frac{d\Phi_1(t)}{dt} \leq -\sqrt{2\kappa_2} \| B^T C^T G \| \sqrt{\Phi_1(t)} \quad (25)$$

The inequality equation (25) can be further transformed into

$$dt \leq \frac{-d\Phi_1(t)}{\sqrt{2\kappa_2} \| B^T C^T G \|} \sqrt{\Phi_1(t)} \quad (26)$$

By adopting $T_r$ as the time to reach the ISS equation (16), we can know $\Phi_1(T_r) = 0$. Integrating inequality equation (26) from 0 to $T_r$ yields

$$\int_0^{T_r} dt \leq \int_0^{\Phi_1(0)} \frac{d(\sqrt{\Phi_1(t)})^2}{\sqrt{2\kappa_2} \| B^T C^T G \|} \sqrt{\Phi_1(t)} \quad (27)$$

then we can get

$$T_r \leq \frac{\sqrt{2}}{\kappa_2 \| B^T C^T G \|} \sqrt{\Phi_1(0)} \quad (28)$$

Thus, the state trajectories can be guaranteed to drive into the ISS equation (16) within a finite time $T_r$. This proof is completed.

Next, we will propose a convex optimization method to calculate the controller gain $K$ of the ISMC law equation (18) and further analyze the stability, dynamical tracking performance, and output constraints of closed-loop augmented system.

### Dynamical performance analysis

The equivalent controller can be obtained by solving the equation $\dot{\tilde{x}}(t) = 0$ for $u(t)$

$$u(t) = -e(t)(-\tilde{C} \tilde{B})^+ (\tilde{C} \tilde{A} \dot{e}_x(t) + \tilde{C} \dot{E} \dot{e}_y(t) + \tilde{D} \dot{h}(t)) \quad (29)$$

Then, we have

$$u(t) = u(t) + u_{eq}(t) \quad (30)$$

By substituting equations (17), (29), and (30) into the augmented system equation (4), the sliding dynamics of $\tilde{x}(t)$ can be described as

$$\dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{B}(u(t) + f(t)) + \tilde{E} \dot{h}(t) + \tilde{F} \dot{y}_d$$

$$= \tilde{A} \tilde{x}(t) - \tilde{B} K \tilde{x}(t) - \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{C} \tilde{A} \tilde{x}(t)$$

$$+ \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{C} \dot{E} \dot{e}_y(t) + \tilde{D} \dot{h}(t) + \tilde{F} \dot{y}_d$$

$$= \left( \tilde{A} - \tilde{B} K - \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{C} \tilde{A} \right) \tilde{x}(t) + \tilde{B} K \tilde{e}_x(t)$$

$$+ \left( I - \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{C} \right) \dot{E} \dot{e}_y(t) - \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{D} \dot{h}(t)$$

$$+ \tilde{F} \dot{y}_d \quad (31)$$

The following theorem discusses the moving trajectories of the state vector $\tilde{x}(t)$ in the ISS equation (16). Meanwhile, both the favorable dynamical tracking performance and the output constraint can be verified.

**Theorem 3.** For the designed parameters $\lambda_i > 0$, $i \in \{1, 2, 3, 4\}$, if we can find matrices $Q = P^{-1}_3 > 0$ and $R_3$ such that the following inequalities

$$\begin{bmatrix}
\Theta_{11} & \tilde{B} & \Theta_{13} & \Theta_{14} & \tilde{F} \\
* & -\lambda_1^{-2} I & 0 & 0 & 0 \\
* & * & -\lambda_2^{-2} I & 0 & 0 \\
* & * & * & -\lambda_3^{-2} I & 0 \\
* & * & * & * & -\lambda_4^{-2} I
\end{bmatrix} < 0 \quad (32)$$

with

$$\Theta_{11} = \text{sym}(\tilde{A} Q - \tilde{B} R_3 - \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{C} A Q) + Q$$

$$\Theta_{13} = (I - \tilde{B} (\tilde{C} \tilde{B})^+ \tilde{C}) E$$

$$\Theta_{14} = \tilde{B} (\tilde{C} \tilde{B})^+ D$$

(33)
are solvable, then the sliding blocks of the state of closed-loop augmented system equation (31) are guaranteed to converge into a compact set $\Omega_{\Omega(t)}$, where $\Omega_{\Omega(t)} = \{\tilde{x}(t) \mid \|\tilde{x}(t)\| \leq \lambda_{\text{max}}(P_3)\sigma\}$, $\sigma$ will be defined in the proof. Furthermore, the tracking error and the system output are proved to satisfy the conditions $\lim_{t \to \infty} y(t) = y_d$ and $|y(t)| \leq y_d \forall t \geq 0$, respectively. Meanwhile, the gain matrix is given by $K = R_3 Q^{-1}$.

**Proof.** Design a suited Lyapunov function as

$$\Phi_2(\tilde{x}(t), t) = \tilde{x}^T(t) P_3 \tilde{x}(t)$$

(35)

The derivative of $\Phi_2(\tilde{x}(t), t)$ is deduced as

$$\Phi_2 = 2 \tilde{x}^T(t) P_3 \left( \tilde{A} - \tilde{B} \tilde{K} - \tilde{B} (\tilde{C} \tilde{B}^T + \tilde{C} \tilde{A}) \right) \tilde{x}(t) + 2 \tilde{x}^T(t) P_3 \tilde{B} K e^{-\gamma t} + 2 \tilde{x}^T(t) P_3 \left( I - \tilde{B} (\tilde{C} \tilde{B} + \tilde{C}) \right) \tilde{E} h(t) - 2 \tilde{x}^T(t) P_3 \tilde{B} (\tilde{C} \tilde{B} + \tilde{C}) \tilde{E} \tilde{E}^T \tilde{x}(t) + \lambda_1^2 \left[ \tilde{x}(t) \right] + \lambda_2^2 \left[ \tilde{x}(t) \right] + \lambda_3^2 \left[ \tilde{x}(t) \right] + \lambda_4^2 \left[ \tilde{x}(t) \right] + \lambda_5^2 \left[ \tilde{x}(t) \right]$$

(36)

where

$$\Omega = \text{sym} \left( P_3 \tilde{A} - P_3 \tilde{B} \tilde{K} - P_3 \tilde{B} (\tilde{C} \tilde{B} + \tilde{C}) \right) + \lambda_1^2 P_3 \tilde{B} \tilde{B}^T P_3 + \lambda_2^2 P_3 (I - \tilde{B} (\tilde{C} \tilde{B} + \tilde{C}) \left[ \tilde{x}(t) \right] + \lambda_3^2 \left[ \tilde{x}(t) \right] + \lambda_4^2 \left[ \tilde{x}(t) \right] + \lambda_5^2 \left[ \tilde{x}(t) \right]$$

(37)

Similarly with the previous theorem proof, equation (32) $\Rightarrow \Omega < -Q$ can be deduced using Schur complement lemma and pre-multiplying and post-multiplying matrix diag$(Q, I, I, I, I)$ to two sides of equation (32). As a result, equation (36) is reformulated as

$$\Phi_2 \leq -\tilde{x}^T(t) P_3 \tilde{x}(t) + \lambda_1 \left[ K \right] + \lambda_2^2 \left[ \tilde{x}(t) \right] + \lambda_3^2 \left[ \tilde{x}(t) \right] + \lambda_4^2 \left[ \tilde{x}(t) \right] + \lambda_5^2 \left[ \tilde{x}(t) \right]$$

(38)

It is not hard to see if $\tilde{x}^T(t) Q \tilde{x}(t) > \lambda_1 \left[ K \right] + \lambda_2^2 \left[ \tilde{x}(t) \right] + \lambda_3^2 \left[ \tilde{x}(t) \right] + \lambda_4^2 \left[ \tilde{x}(t) \right] + \Phi_2 < 0$ can be arrived. Therefore for any $\tilde{x}(t)$, we deduce the following inequality

$$\tilde{x}^T(t) P_3 \tilde{x}(t) \leq \left\{ \tilde{x}^T(0) P_3 \tilde{x}(0) \right\} + \lambda_1 \left[ K \right] + \lambda_2^2 \left[ \tilde{x}(t) \right] + \lambda_3^2 \left[ \tilde{x}(t) \right] + \lambda_4^2 \left[ \tilde{x}(t) \right]$$

(39)

is true for any $t \geq 0$, which implies the closed-loop system equation (31) is assured of stability and the sliding blocks of the state $\tilde{x}(t)$ can be proved to converge into the compact set $\Omega_{\Omega(t)}$.

Next, we will discuss two issues of dynamical tracking and output constraint. By squaring the system output $y(t)$, one has

$$|y(t)|^2 = \left( \tilde{C} \tilde{x}(t) + \tilde{D} h(t) \right) \left( \tilde{C} \tilde{x}(t) + \tilde{D} h(t) \right)$$

(40)

Furthermore, by using Schur complement to inequality equation (34), we can get

$$\begin{bmatrix} P_3 & 0 & 0 \\ 0 & (y_d - \sigma) I & -1 \\ 0 & \tilde{C} & \tilde{D} \end{bmatrix} \left[ \begin{bmatrix} \tilde{C} \\ \tilde{D} \end{bmatrix} \right] \tilde{x}(t) + \tilde{h}(t) > 0$$

(41)

By combining equations (39) and (40) with (41), we have

$$y_d^{-1} \tilde{x}^T(t) \leq \tilde{x}(t) P_3 \tilde{x}(t) + (y_d - \sigma) \leq y_d$$

(42)

Without loss of generality, the desired output $y_d$ is chosen as positive. Hence, the constraint condition of output can be achieved by $|y(t)| \leq y_d$. On one hand, since the term $\int e(t) dt$ is one of the variables of $\tilde{x}(t)$, we can assert that the variable $\int e(t) dt$ must be bounded when $t \to + \infty$. On the other hand, due to the constraint condition of system output, the sign of $e(t) = y_d - y(t)$ will not change for any $t \geq 0$. To sum up, we can deduce that the dynamics of tracking error will eventually be satisfied as $\lim_{t \to \infty} y(t) = y_d$. This proof is completed.

**Remark 2.** In order to obtain the gains of ESO, FDO, and the controller gain, we need to solve a series of linear matrix inequalities (LMIs). In Theorem 1, the number of variables in LMI equation (12) is six and they are $P$, $R$, $P_1$, $P_2$, $R_1$, and $R_1$. In Theorem 3, the variables in LMI equation (32) are $Q$ and $R_3$. It can be seen that the number of variables of LMIs is not very much. Meanwhile, the structure of each matrices are conventional, and their dimensions are also not very high. So we can obtain a relatively low computational complexity in the process of looking for a feasible solution by using LMI toolbox.
Numerical illustrations

Similarly with Guo and Cao9 and Yi et al.,12 when staying in a flight condition with 0.9 Mach and 15,000 ft altitude, the longitudinal model of A4D aircraft is depicted as

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + f(t) + Eh(t) \\
y(t) &= Cx(t) + Dh(t)
\end{align*}
\]

(43)

where \(x(t) \in \mathbb{R}^4\) is the state of the aircraft. \(x_1(t)\) is the forward velocity (ft \(\cdot\) s\(^{-1}\)), \(x_2(t)\) is the attack angle (rad), \(x_3(t)\) and \(x_4(t)\) are the velocity of pitch (rad \(\cdot\) s\(^{-1}\)), and the angle of pitch (rad), respectively. \(u(t)\) is the elevator deflection (°). Specifically, the system matrices are obtained by

\[
A = \begin{bmatrix}
-0.06050 & 32.37 & 0 & 32.2 \\
-0.00014 & -1.475 & 1 & 0 \\
-0.0111 & -34.72 & -2.793 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
-0.1064 \\
-33.8 \\
0
\end{bmatrix}
\quad E = \begin{bmatrix}
0.1 \\
0 \\
-3 \\
0.1
\end{bmatrix}
\]

\[
C = [1 \ 1 \ 1 \ 1], \quad D = 0
\]

In order to show the advancement of designed fault-tolerant and anti-disturbance algorithm, the following actuator faults and mismatched disturbances model are considered to occur in the A4D system.

The time-varying faults are assumed to occur in 10 s as

\[
f(t) = \begin{cases}
0, & \text{if } t < 10 \\n1 + 0.2 \sin(t), & \text{if } t \geq 10
\end{cases}
\]

(44)

The harmonic disturbances are assumed to be described by equation (2) with

\[
M = \begin{bmatrix}
0 & 3 \\
-3 & 0
\end{bmatrix}, \quad N = \begin{bmatrix}
0.5 \\
0
\end{bmatrix}
\]

Meanwhile, selecting \(G = 0.0259\), defining the parameters \(\mu_i = 1, \ i = 1, 2, \lambda_i = 1, \ i = 1, 2, 3, 4, \ k_2 = 0.15, \theta_x = 0.5, \theta_r = 0.3, \theta_h = 2\) and solving inequalities equations (12), (32), and (34), the observer gains \(L, L_1,\) and \(L_2,\) the control gains \(K, K_h,\) and \(K_e\) can be computed as

\[
L = [13.8566 \ -0.2264 \ -2.8468 \ 3.3263 \ 1.0000] \\
L_1 = 6.4762, \quad L_2 = [1.3843 \ 0.3617]^T, \quad K_h = 0.0826 \\
K = \begin{bmatrix}
0.2751 \\
82.1745 \\
-2.1624 \\
-99.5676 \\
-5.0043
\end{bmatrix}^T, \quad K_x = \begin{bmatrix}
0.0021 \\
0.1128 \\
0.0234 \\
-0.9497 \\
0
\end{bmatrix}^T
\]

Supposed that the initial values are, respectively, denoted as \(y(0) = -0.6\) and \(\xi(0) = [-2, 1]^T\). Then, we can figure out the reaching time satisfies \(T_r \leq 13.6235\).

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The expected tracking signal is chosen as \(y_d = 10\).

Figures 1–4 express the satisfactory estimation and stability performance for the state of the A4D system. It can be observed that the designed ISM controller has a
favorable anti-disturbance and fault-tolerant capacity. The dynamical trajectories of the time-varying faults and the harmonic disturbances with their estimation values are illustrated in Figures 5 and 6, respectively. Figures 7 and 8 show the dynamics of the ISS equation (16) with reaching time and the tracks of the ISM controller equation (18). The controlled output and its estimation is shown in Figure 9, and the dynamical tracking performance can be embodied by Figure 10. The convergence of the tracking error is guaranteed to
zero, and the system output compressed within a bounded range ([-10,10]) effectively.

**Conclusion**

For a class of nonlinear systems subject to actuator faults and mismatched disturbances, a novel optimal ISM tracking control algorithm has been proposed in this brief. By constructing a composite observer constituted with ESO, FDO, and DO, the unknown system state, the faults, and the disturbances are accurately estimated. On the basis of estimation information, the ISM controller is offered to drive the state of the augmented system into the proposed ISS in a finite time. A convex optimization-based method is put forward such that the augmented system is stable. Meanwhile, the system output is proved to keep within a given boundary and eventually tracks to a desired reference signal.

It is noted when facing those different types of faults or exogenous disturbances, the multi-source fault-tolerant and anti-disturbance problem will be discussed in the future. On the contrary, how to reduce the computational complexity and the chattering problem will be further considered.

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