FINITENESS AND VANISHING THEOREMS FOR COMPLETE OPEN RIEMANNIAN MANIFOLDS

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Let $M^n$ denote an $n$-dimensional complete open Riemannian manifold. In [AG] Abresch and Gromoll introduced a new concept of “diameter growth.” Roughly speaking, one would like to measure the essential diameter of ends at distance $r$ from a fixed point $p \in M^n$. They showed that $M^n$ is homotopy equivalent to the interior of a compact manifold with boundary if $M^n$ has nonnegative Ricci curvature and diameter growth of order $o(r^{1/n})$, provided the sectional curvature is bounded from below. It is well known that any complete open manifold with nonnegative sectional curvature has finite topological type. This is a weak version of the Soul Theorem of Cheeger-Gromoll [CG]. Examples of Sha and Yang show that this kind of finiteness result does not hold for complete open manifolds with nonnegative Ricci curvature in general (see [SY1, SY2]), and additional assumptions are therefore required.

We will use a concept of the essential diameter of ends slightly stronger than that of [AG]: For any $r > 0$, let $B(p, r)$ denote the geodesic ball of radius $r$ around $p$. Let $C(p, r)$ denote the union of all unbounded connected components of $M^n \setminus B(p, r)$. For $r_2 > r_1 > 0$, set $C(p; r_1, r_2) = C(p; r_1) \cap B(p, r_2)$. Let $1 > \alpha > \beta > 0$ be fixed numbers. For any connected component $\Sigma$ of $C(p; \alpha r, \beta r)$, and any two points $x, y \in \Sigma \cap \partial B(p, r)$, consider the distance $d_r(x, y) = \inf \text{Length}(\phi)$ between $x$ and $y$ in $C(p, \beta r)$, where the infimum is taken over all smooth curves $\phi \subset C(p, \beta r)$ from $x$ to $y$. Set $\text{diam}(\Sigma \cap \partial B(p, r), C(p, \beta r)) = \sup d_r(x, y)$, where $x, y \in \Sigma \cap \partial B(p, r)$. Then the diameter of ends at distance $r$ from $p$ is defined by

$$\text{diam}(p, r) = \sup \text{diam}(\Sigma \cap \partial B(p, r), C(p, \beta r)),$$

where the supremum is taken over all connected components $\Sigma$ of $C(p; \alpha r, \beta r)$. The diameter defined here is not smaller than that defined by Abresch and Gromoll. Our definition will be essential in Lemma 3 and its applications.

The purpose of this note is to announce the following results.

**Theorem A.** Let $M$ be a complete open Riemannian manifold with sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some base point $p \in M$,

$$\lim_{r \to +\infty} \sup \text{diam}(p, r) < \frac{\ln 2}{K}.$$
Then $M$ is homotopy equivalent to the interior of a compact manifold with boundary.

**Theorem B.** Let $M^n$ be an $n$-dimensional complete open Riemannian manifold. Suppose that the sectional curvature $K_M \geq -K^2$ for some constant $K > 0$. Assume that for some $2 \leq k \leq n - 1$, $M^n$ has nonnegative $k$th-Ricci curvature and that for some $p \in M^n$,

$$\limsup_{r \to +\infty} \frac{\text{diam}(p, r)}{r^{1/k}} < \left[ \frac{2(k + 1)}{k} \left( \frac{(k - 1) \ln 2}{2kK} \right) \right]^{1/(k+1)}.$$ 

Then $M^n$ is homotopy equivalent to the interior of a compact manifold with boundary.

**Theorem C.** Let $M^n$ be an $n$-dimensional complete open Riemannian manifold. Assume that for some $1 \leq k \leq n - 1$, $M^n$ has positive $k$th-Ricci curvature everywhere and that for some $p \in M^n$, $M^n$ has diameter growth of order $o(r)$, i.e.

$$\limsup_{r \to +\infty} \frac{\text{diam}(p, r)}{r} = 0.$$ 

Then $M^n$ has the homotopy type of a CW-complex with cells of dimensions $\leq k - 1$.

The precise condition that $M^n$ have nonnegative (positive) $k$th-Ricci curvature at some point $x \in M^n$ is that for all $v$ in the span of any orthonormal set $\{e_1, \ldots, e_{k+1}\}$ in $T_x M^n$,

$$\sum_{i=1}^{k+1} \langle R(e_i, v) v, e_i \rangle \geq 0 \ (> 0),$$

where $R(x, y)z$ denotes the curvature tensor of $M^n$ (cf. also [H] for the definition of $k$th-Ricci curvature).

**Remark 1.** (1) In Theorem A the upper bound $\ln 2/K$ must depend on $K$. Otherwise, the connected sum of infinitely many copies of $S^2 \times S^2$ (see [AG]) provides an easy counterexample.

(2) Theorem B generalizes the Abresch-Gromoll Theorem [AG].

(3) The condition in Theorem C can be weakened to that $M^n$ has nonnegative $k$th-Ricci curvature everywhere and positive $k$th-Ricci curvature outside a compact subset of $M^n$ (see Lemma 5).

(4) The same argument as in [AG] shows that any complete open Riemannian manifold with nonnegative Ricci curvature must have diameter growth of order $o(r)$. We do not know whether the condition in Theorem C on diameter growth is necessary. Examples in [SY1, SY2, We and GM] have diameter growth of order at most $o(r)$.

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**Outline of Proofs.** Throughout this part we assume that $M^n$ denotes a complete open Riemannian manifold of dimension $n$ and $p$ is a point of
$M^n$ fixed during the discussion. For arbitrary $t \geq 0$, let $R_t(p) = \{y(t); y$ is a ray emanating from $p\}$, which is a closed subset of the distance sphere $S(p,t)$. Set $B^t_p(x) = t - d(x,R_t(p))$ for any $x \in M^n$. It is easy to see that $B^t_p(x)$ is increasing in $t$ and $|B^t_p(x)| \leq d(p,x)$ for any $x \in M^n$. The generalized Busemann function $B_p$ is defined as $B_p(x) = \lim_{t \to +\infty} B^t_p(x)$, which is a Lipschitz function with Lipschitz constant 1. The excess function $E_p$ is defined as $E_p(x) = d(p,x) - B_p(x)$. We will introduce a new function $L_p$ which plays an essential role in the study of the generalized Busemann function $B_p$. Set $L_p(x) = d(x,R_t(p))$, where $t = d(p,x)$. Since $B^t_p(x)$ is increasing in $t$, it is easy to see that $E_p(x) < L_p(x)$ and $d(p,x) - L_p(x) \geq B_p(x)$ for all $x \in M^n$. A more detailed discussion for generalized Busemann functions has been given by H. Wu [W1]. For the purpose of this note, we need the following

**Lemma 1.** For any $q \in M^n$, there exists a ray $\sigma_q(t)$ emanating from $q$ such that for all $t \geq 0$, the function $B^{q,t}_p(x)$ defined by $B^q_p(q) + t - d(x,\sigma_q(t))$ supports $B_p(x)$ at $q$, namely $B^{q,t}_p(x) \leq B_p(x)$ for all $x \in M^n$ and $B^{q,t}_p(q) = B_p(q)$.

**Lemma 2.** Suppose that $M^n$ has sectional curvature $K_M \geq -K^2$ for some $K > 0$, then for any critical point $q$ with respect to $p$, $E_p(q) \geq \frac{1}{K} \left( e^{Kd(p,q)} \right)$.

Notice that $E_p(x) \leq L_p(x)$ for all $x \in M^n$. Thus if $\limsup_{d(p,x) \to +\infty} L_p(x) < \frac{\ln 2}{K}$, Lemma 2 shows that outside a compact subset there is no critical point with respect to $p$, Theorem A follows from this argument and the following

**Lemma 3.** Suppose that $M^n$ has diameter growth of order $o(r)$. Then there exists an $R > 0$ such that for any $x \in M^n \setminus B(p,R)$,

\[(1) \quad L_p(x) \leq \text{diam}(p,d(p,x)),\]

and the Busemann function $B_p$ is proper.

Notice that $d(p,x) - L_p(x) \geq B_p(x)$ for all $x \in M^n$. It is clear that (1) implies that $g(x) \equiv d(p,x) - L_p(x)$ is proper, and so is $B_p(x)$.

One can obtain a better estimate for $E_p \leq L_p$ in terms of $L_p$ if $M^n$ has nonnegative $k$th-Ricci curvature.

**Lemma 4.** Suppose that $M^n$ has nonnegative $k$th-Ricci curvature for some $2 \leq k \leq n-1$, then for all $x \in M^n$ with $L_p(x) < d(p,x)$,

\[(2) \quad E_p(x) \leq \frac{2k}{k-1} \left[ \frac{k}{2(k+1)} \times \frac{L_p(x)^{k+1}}{d(p,x) - L_p(x)} \right]^{1/k} .\]

The proof of Lemma 4 depends on Lemma 1 and the maximum principle. Theorem B therefore follows from Lemmas 2, 3, and 4. For the proof of Theorem C, we need Lemma 3 and the following
Lemma 5. Suppose that for some $1 \leq k \leq n - 1$, $M^n$ has nonnegative $k$th-Ricci curvature everywhere and positive $k$th-Ricci curvature outside a compact subset. If the Busemann function $B_p$ is proper, then there exists a $C^2$ function $\chi(t)$ such that $\chi \circ B_p$ is proper and strictly $k$-convex. Therefore $M^n$ has the homotopy type of a CW-complex with cells of dimensions $\leq k - 1$.

Compare [W2] for a definition of $k$-convexity. It seems to be crucial that the Busemann function $B_p$ is proper. The first assertion in Lemma 5 follows from Lemma 1. If we assume that $\chi \circ B_p$ is proper and strictly $k$-convex, then the last assertion in Lemma 5 follows from Wu’s Smoothing Theorem [W2] and the standard Morse Theory [M]. This proves Theorem C.

Remark 2. An observation of Cheeger-Gromoll ([CG], sharpened in [GW]) is that if $M^n$ has nonnegative sectional curvature outside a compact subset, then $M^n$ has finite topological type and $B_p$ is a proper function. If an addition, $M^n$ has nonnegative $k$th-Ricci curvature everywhere and positive $k$th-Ricci curvature outside a compact subset, then $M^n$ has the homotopy type of a CW-complex with finitely many cells of dimensions $\leq k - 1$ (cf. [W2]).

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