We derive here the expression for the accretion luminosity, $L(\infty)$, as seen by a distant inertial observer $S_\infty$, for the case of spherical accretion onto a static compact object having a surface gravitational red-shift $z_x$. It is found that the “efficiency” for conversion of mass energy into accretion energy is given by $\epsilon = z_x / (1 + z_x)$. And since the maximum value of $z_x$ permitted by General Theory of Relativity (GTR) is 2, the maximum theoretical value of the accretion efficiency is 66.66%.

I. INTRODUCTION

One of the key concepts in astrophysics is that of accretion of the ambient gas by a compact object by virtue of its gravitational field. In the test particle assumption, as the accreting material traverses down the gravitational potential well of the compact object having a mass $M_x$ and a hard surface of radius $R_x$, the infalling kinetic energy is released when the gas “hits” the “hard surface”. The rate of energy release by this process is given by the depth of the potential well: and, for an accretion rate $\dot{M}$, the accretion luminosity is given by

$$L = \frac{GM_x \dot{M}}{R_x} \quad (1)$$

where $G$ is the gravitation constant. When the gas possesses viscosity, it gets heated during the infall unlike an idealized test particle or a dust, and radiates even before it heats the “hard surface”. In this case, the kinetic energy released upon impact is less than the corresponding free fall value. Accordingly, the amount of accretion energy released after the impact is less than the corresponding value for a dust. However, the net energy released over the entire process is same in either case. Thus, viscosity or no viscosity, the eventual expression for $L$ remains unchanged.

A related question is the idea of the maximum luminosity or the Eddington luminosity at which the repulsive effect of the emitted radiation cancels the gravitational attraction on the infalling particles [1]:

$$L_{ed} = \frac{4\pi GM_x c}{\kappa} \quad (2)$$

where $c$ is the speed of light and $\kappa$ is the appropriate opacity of the accreting plasma. If we are considering emission of electromagnetic radiation only, for a fully ionized H-plasma, the expression for opacity is given by $\kappa = \sigma_T / m_p$, where $\sigma_T = 6.65 \times 10^{-25} \text{cm}^2$ is the Thompson cross section and $m_p$ is the proton mass, so that $\kappa \approx 0.4 \text{cm}^2/\text{g}$. The corresponding Eddington luminosity is given by:

$$L_{ed} = 1.26 \times 10^{38} \left( \frac{M_x}{M_\odot} \right) \text{ergs/s} \quad (3)$$

However, at very high accretion rates, astrophysical plasma may cool by emitting neutrinos, and in such cases, the value of $L_{ed}$ could be much higher because of the tiny neutrino-matter interaction cross-sections [1]:

$$L_{ed} \approx 2 \times 10^{34} \left( \frac{T_\nu}{8 \text{MeV}} \right)^{-2} \text{ergs/s} \left( \frac{M_x}{M_\odot} \right) \quad (4)$$

where $T_\nu$ is the temperature of the emitted neutrinos.

It is obvious that all the above expressions are correct only for the Newtonian gravity and the aim of this paper is to arrive at the correct General Relativistic (GTR) expressions for the above quantities.

II. ENERGY AND LUMINOSITY IN GTR

In GTR, the global notions like the total mass-energy of an isolated body $M$, its self-gravitational energy, $E_g$, and the binding energy, $E_B$, are meaningfully definable only with respect to an observer $S_\infty$ situated at infinite spatial...
distance. This is so because, the spacetime seen by him is flat (Galilean) and where the notions of global energy-momentum conservation can be readily invoked. At the same time it is known that the energy of a particle/photon emitted from the surface of the compact object is reckoned to be lower by a factor of \((1 + z)\) by \(S_\infty\). Consequently, if the temperature/energy of the particles/photons near the surface of the compact object is \(T\), its value as measured by \(S_\infty\) is \([1,2,3,4]\):

\[
T_\infty = T(1 + z)^{-1}; \quad T = T_\infty(1 + z)
\]

(5)

Also, since the clocks move slower near the compact object by the same factor, the local value of luminosity is higher by a factor of \((1 + z)^2\):

\[
L = L(\infty)(1 + z)^2; \quad L(\infty) = L(1 + z)^{-2}
\]

(6)

This fact is well known and yet we discussed it here for the sake of completion. Accordingly, the fact that, the expression for Eddington luminosity gets modified in GTR is well discussed in literature \([1]\):

\[
L_{ed} = \frac{4\pi GM_x c}{\kappa}(1 + z)
\]

(7)

and

\[
L_{ed}(\infty) = \frac{4\pi GM_x c}{\kappa(1 + z)}
\]

(8)

Here, note that, the definition of \(E_{ed}\) does not depend on the actual definition of \(L\), and the GTR correction arises from the fact that in an external Schwarzschild spacetime, the definition of gravitational acceleration gets modified. It is also pointed out that the notion of Eddington luminosity is relevant only locally, i.e, at the source of the radiation. Thus, it is actually, the Eq. (7) which is relevant here. However, surprisingly, what has not been discussed in literature is how the basic expression for accretion luminosity should change in GTR (although, we will point out later that the expression for the binding energy of a rotating pressureless infinitesimally thin disk is known). We shall obtain this GTR expression for accretion luminosity in a remarkably simple manner.

### III. ACCRETION LUMINOSITY

Since, in GTR, the concept of the mass of an isolated body (for instance \(M_x\)) is defined with respect to \(S_\infty\), it is necessary that the accretion rate is measured by the same distant observer:

\[
\dot{M}_\infty \sim \frac{\Delta M}{\Delta t} |_\infty
\]

(9)

Accretion energy arises from the conversion of infall kinetic energy into random energy. If a hard surface is present, the entire available kinetic energy is converted into random energy, and, on the other hand, if at a given region \(R > R_x\), where no hard surface is present, the gas can still convert part of its infall kinetic energy into random energy and even radiate it. In the limit of infinite viscosity, the energy emitted at \(R\) could be equal to the infall kinetic energy, and, thus, the maximum accretion energy at any region, \(R \geq R_x\), is determined by the free fall kinetic energy. If \(v\) is the speed of the test particle measured in a Local Inertial Frame (LIF), in, Newtonian physics

\[
\text{Kinetic Energy Per Unit Mass} = \frac{1}{2}v^2
\]

(10)

Also, for free-fall, one has

\[
\frac{1}{2}v^2 = \frac{GM_x}{R}
\]

(11)

And, as is well known, the two foregoing equations trivially explain the origin of the Newtonian formula for accretion luminosity (Eq. 1).

For a transition to GTR, first we recall that, by the Principle of Equivalence, GTR reduces to the Special Theory of Relativity in the LIF. So, in the LIF, we have \([1,2,3,4]\):
\[ \text{Kinetic Energy Per Unit Mass} = \left[ 1 - \frac{v^2}{c^2} \right]^{-1/2} c^2 = (\gamma - 1) c^2 \quad (12) \]

where \( \gamma \) is the Lorentz factor of the fluid. However, in an external Schwarzschild geometry, in the absence of angular momentum, the equation (11) remains exactly valid \([1,2,3,4]\), so that

\[ \text{Kinetic Energy Per Unit Mass} = \left[ 1 - \frac{2GM_x}{Rc^2} \right]^{-1/2} - 1 \] \quad (13)

The right hand side of the above equation can be identified with the gravitational redshift for photons (or any particles) emitted at \( R = R \):

\[ \text{Kinetic Energy Per Unit Mass} = (\gamma - 1)c^2 = zc^2 \quad (14) \]

It is also clear that \( zc^2 \) is the escape kinetic energy or gravitational binding energy of a test particle at \( R = R \). As discussed before, because of gravitational redshift, energy released at \( R = R \) will appear lower to \( S_\infty \) by a factor \((1 + z)\). Therefore, the maximum accretion energy per unit mass or, the accretion efficiency seen by \( S_\infty \) will be

\[ \epsilon = \frac{z_x}{1 + z_x} \quad (15) \]

And hence the accretion luminosity measured by \( S_\infty \) is

\[ L(\infty) = \frac{z_x}{1 + z_x} \dot{M}_\infty c^2 \quad (16) \]

Eq.(16) obviously reduces to the Newtonian Eq.(1) for \( GM_x/R_x c^2 \ll 1 \). Also, the accretion luminosity, measured locally will be higher by a factor \((1 + z)^2\):

\[ L = z_x (1 + z_x) \dot{M}_\infty c^2 \quad (17) \]

Now, we would like to point out an early work which handled the more difficult problem of finding the gravitational binding energy of an idealized rotating infinitesimally thin disk of a dust (pressure =0) \([5]\):

\[ E_B = M_0 c^2 \left[ \frac{z_c}{1 + z_c} - \frac{2\omega J}{M_0 c^2} \right] \quad (18) \]

where \( M_0 \) is the “rest mass” (measured by \( S_\infty \) of course), \( \omega \) is the angular speed, and \( J \) is the angular momentum of the disk. \( z_c \) is the redshift at the “center” of the infinite disk. Here the binding energy of the disk (as measured by \( S_\infty \)) is given by:

\[ E_B = (M_0 - M) c^2 \quad (19) \]

In the absence of rotation, the Eq.(18) would give \( E_B = \frac{z_c}{1 + z_c} M_0 c^2 \). And although, the problem of spherical accretion discussed by us is technically different from the above problem, clearly, there is an inner physical link and agreement.

### IV. DISCUSSION

GTR gives an absolute upper limit on the compactness of cold bodies in hydrostatic equilibrium. This limit is independent of the details of the equation of state of the stellar matter, and is given by \([2,3,4]\)

\[ \frac{2GM_x}{R_x c^2} \leq \frac{8}{9}; \quad z_x \leq 2; \quad \gamma_x \leq 3 \quad (20) \]

Thus, the theoretical maximum accretion efficiency is \( \approx 66.66\% \). We recall that, the accretion efficiency for a canonical neutron star is \( \sim 10 - 15\% \). In contrast, for strict spherical accretion onto a Schwarzschild black hole, no energy is emitted when the accreted material hits the “event horizon” because event horizon is not a “hard surface” and is only a fictitious membrane with \( z = \infty \). In this case accretion luminosity, would be only due to viscous and radiative properties of the infalling gas, and could be very low. This specific feature is invoked for the observational search
for BHs. However, in astrophysics, accretion matter usually possesses angular momentum, and accretion is usually mediated by an “accretion disk”. For thin and sufficiently viscous accretion disks around Schwarzschild BHs, the “inner edge” is located at three Schwarzschild radius: \( R_i = 6GM/c^2 \). It is found that, the maximum accretion efficiency in such a case is \( \epsilon \approx 5.72\% \) [1,2], and the accretion energy output comes in the form of photons or neutrinos because there is no emission of gravitational radiation in a stationary gravitational field.

On the other hand, the net maximum efficiency for BH accretion could be much higher when the BH is rotating, or, in other words, if we have a Kerr BH. The maximum value of \( \epsilon \) is realized for a maximally rotating BH, i.e., for which one has \( J = GM^2/c \), and is given by \( \epsilon \approx 42.3\% \) [1,2]. Further, the accreting matter in the rotating Kerr spacetime may emit radiation by means of gravitational waves, and, it can not be ascertained, what fraction of this efficiency corresponds to emission of photons and neutrinos. In any case, the value of \( \epsilon \approx 66.66\% \) obtainable for a compact object with \( z_x = 2 \) is indeed the corresponds to the highest theoretical accretion efficiency.

Now, the natural question would be whether, this high accretion efficiency can be realized in Nature. The canonical NS is a hard -surface compact object with a modest value of \( 0.1 < z_x < 0.2 \). And GTR actually allows existence of static compact objects upto \( z_x = 2 \). If in future, there is much improved understanding of nuclear equation of state (EOS) at extremely high temperatures and densities, and further, if it is possible to solve the GTR collapse equations with full generality and without making tacit simplification, it might be possible to theoretically infer the existence of Ultra Compact Objects even in the context of normal QCD. Note that, a total accumulated theoretical and numerical uncertainty of \( \sim 10\% \) can push the \( 2GM/Re^2 = 8/9 \) state (\( z_x = 2 \)) to an apparently event horizon state with \( 2GM/Re^2 = 1 \) (\( z = \infty \)). Thus, in the absence of an exact EOS and radiation transport ansatz at arbitrary high density and temperature, the reliability of any numerical (approximate) study of the GTR collapse problem is rather poor once we proceed beyond the NS stage.

It may be pointed out that some effective field theories of strong interaction allow confinement of not only quarks but also of nucleons at densities well below the nuclear (rest mass) density, \( \rho_{nu} \approx 2.8 \times 10^{14} \text{ g cm}^{-3} \) [6,7,8]. Compact objects proposed on this idea are called Q-stars (Q stands for a conserved quantum number and nor for “quark”). Q-stars could be very massive \( \geq 100M_\odot \). For some choice of the model parameters, the maximum value of \( 2GM/Re^2 < 1/3 \) [9] corresponding to \( z_x \leq 0.73 \). Clearly, since Q-stars may have densities below the nuclear densities, depending on the model parameters, it may also be possible to have such stars with \( z < 0.1 \), stars less compact than a canonical NS. Presumably, a different EOS may lead to higher values of \( z_x \). In other words, it is possible that, beyond the White dwarf stage, Nature actually allows existence of compact objects having a fairly wide range of \( z_x \) with an upper bound at \( z_x = 2.0 \). In fact, at present, there is no proper explanation for the cosmic gamma ray bursts which may involve release of \( \sim 10^{55} \) ergs of neutrinos [10]. It is presumable that the gamma ray bursts are birth pangs of relativistic ultra compact objects.

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