The tangle-free hypothesis
on random hyperbolic surfaces

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Abstract

This article introduces the notion of $L$-tangle-free compact hyperbolic surfaces, inspired by the identically named property for regular graphs. Random surfaces of genus $g$, picked with the Weil-Petersson probability measure, are $(a \log g)$-tangle-free for any $a < 1$. This is almost optimal, for any surface is $(4 \log g + O(1))$-tangled. We establish various geometric consequences of the tangle-free hypothesis at a scale $L$, amongst which the fact that closed geodesics of length $\leq \frac{L}{4}$ are simple, disjoint and embedded in disjoint hyperbolic cylinders of width $\geq \frac{L}{4}$.

Introduction

In this article, we introduce the tangle-free hypothesis on compact (connected, oriented) hyperbolic surfaces (without boundary), and explore some of its geometric implications, with a special emphasis on random surfaces, which we show are almost optimally tangle-free.

This work follows several recent articles aimed at adapting results on random regular graphs in both geometry and spectral theory to the setting of random hyperbolic surfaces – see [24, 25, 15, 26, 32, 21] for instance. Though the initial motivation was to provide some useful tools for spectral theory, the results and techniques developed here are purely geometric. Several of our results are significant improvements of useful properties of geodesics on compact hyperbolic surfaces, allowed by the random setting: the length scale at which they apply goes from constant to logarithmic in the genus.

A key innovation of this article is finding an elementary geometric condition which is simultaneously easy to prove for random surfaces, and has far-reaching consequences on their geometry (notably their geodesics) at a large scale. Similar
geometric assumptions have been made recently by Mirzakhani and Petri [25, Proposition 4.5] and Gilmore, Le Masson, Sahlsten and Thomas [15]. The use of the tangle-free hypothesis simplifies and improves the result in [15], and generalises one consequence of [25, Proposition 4.5] to a larger scale.

The tangle-free hypothesis for hyperbolic surfaces

Let us first define what we mean by tangle-free and contrast it with existing concepts in the graph theoretic and hyperbolic surface literature. Heuristically speaking, we shall say that a surface is tangle-free if it does not contain embedded pairs of pants or one-holed tori with ‘short’ boundaries. More precisely, we make the following definition.

**Definition 1.** Let $X$ be a compact hyperbolic surface and $L > 0$. Then, $X$ is said to be $L$-tangle-free if all embedded pairs of pants and one-holed tori in $X$ have total boundary length larger than $2L$. Otherwise, $X$ is $L$-tangled.

To be precise, we emphasise that a pair of pants and a one-holed torus are respectively surfaces of signature $(0,3)$ and $(1,1)$, and the embedded surfaces we consider have totally geodesic boundary. The total boundary length is defined as the sum of the length of all the boundary geodesics. One should note that we could also have defined the notion of tangle-free using the maximum boundary length (the length of the longest boundary geodesic) and the results of this paper would follow through (up to changes of constants).

It may not be so clear to the reader why we call such a property tangle-free. In order to clarify this, we prove that, when a surface is tangled, it contains a non-simple geodesic; that is, a tangled geodesic in the literal sense of the word.

**Proposition A (Proposition 2).** Any $L$-tangled surface contains a self-intersecting geodesic of length smaller than $2L + 2\pi$.

Tangle-free graphs

One can motivate the study of this geometric property of surfaces through the medium of regular graphs. Indeed, the naming of this property is inspired by a similar notion Bordenave introduced in [8] in order to prove Friedman’s theorem [14] regarding the spectral gap of the Laplacian on large regular graphs (we shall come back to this result in more detail at the end of the introduction). A graph $G = (V,E)$ is said to be $L$-tangle-free if, for any vertex $v$, the ball for the graph distance $\dist_G$

$$B_L(v) = \{ w \in V : \dist_G(v, w) \leq L \},$$

contains at most one cycle. This definition might seem quite different to the surface definition given above, but we shall prove that balls on tangle-free surfaces contain at most one ‘cycle’ in the following sense.
Proposition B (Proposition 11). If a surface $X$ is $L$-tangle-free, then for any point $z \in X$, the ball
\[ B_L(z) = \left\{ w \in X : \text{dist}_X(z,w) < \frac{L}{8} \right\} \]
is isometric to a ball in the hyperbolic plane or a hyperbolic cylinder.

It is worth noting that in the original proof by Friedman [14], there is also a notion of 'supercritical tangle' in a graph, which are small subgraphs with many cycles. In a sense, pairs of pants or one-holed tori with small total boundary lengths can be seen as analogues of these bad tangles for surfaces.

Admissible values of $L$

Let us now discuss typical values that $L$ can take in Definition 1 both for being tangle-free and tangled. Throughout, we shall use the notation $A = O(B)$ to indicate that there is a constant $C > 0$ such that $|A| \leq C|B|$ with $C$ independent of all other variables such as the genus.

It is clear that a surface of injectivity radius $r$ is $r$-tangle-free, for it has no closed geodesic of length smaller than $2r$. In a deterministic setting, it is hard to say much more than this.

On the other hand, we know that a hyperbolic surface of genus $g$ admits a pants decomposition with all boundary components smaller than the Bers constant $B_g$ – see [12, Chapter 5]. We know that $B_g \geq \sqrt{6g} - 2$ [12, Theorem 5.1.3], and the best known upper bounds on $B_g$ are linear in $g$ [13, 30]. All surfaces of genus $g$ are $\frac{3}{2}B_g$-tangled. This bound however is rather loose, since it follows from cutting all of the surface into pairs of pants rather than isolating a single short pair of pants. In light of this, we in fact prove the following, using a method based on Parlier’s work [30].

Proposition C (Proposition 13). Any hyperbolic surface of genus $g$ is $L$-tangled for $L = 4 \log g + O(1)$.

Random graphs and surfaces

How tangle-free can a typical surface be? Can $L$ be much larger than the injectivity radius for a large class of surfaces? An instructive method to answer these questions is to consider the setting of random surfaces, and to find an $L$ for which most surfaces are $L$-tangle-free.

For $d$-regular graphs with $n$ vertices, sampled with the uniform probability measure $\mathbb{P}_n^{(d)}$, Bordenave proved [8] that for any real number $0 < a < \frac{1}{4}$,
\[ \mathbb{P}_n^{(d)}(G \text{ is } (a \log_{d-1}(n))\text{-tangle-free}) \xrightarrow{n \to \infty} 1. \]

This is a key ingredient in Bordenave’s proof of Friedman’s theorem [8].
In this article, we will consider the Weil-Petersson probability $P_{gWP}$ on the set of closed hyperbolic surfaces of genus $g$. However, one should note that there exist other non-equivalent random surface models such as that of Brooks and Makover [9] or Magee, Naud and Puder [21]. We introduce the model in detail in Section 2, and then prove that, in this setting, random surfaces are tangle-free at a scale $\log g$ with high probability.

**Theorem D** (Theorem 5). For any real number $0 < a < 1$,

$$P^{WP}_{g}(X \text{ is } (a \log g)\text{-tangle-free}) = 1 - O\left(\frac{(\log g)^2}{g^{1-a}}\right).$$

Since any surface of genus $g$ is $(4 \log g + O(1))$-tangled, random surfaces are almost as tangle-free as possible. The scale $\log g$ is a very large scale on a random hyperbolic surface of high genus: by work of Mirzakhani [24] the diameter of such a surface is $\leq 40 \log g$ with high probability. Mirzakhani and Petri [25] also proved that the mean value of its injectivity radius goes to a constant value $\approx 0.807$ as $g$ approaches infinity, hence proving that random surfaces of high genus have short closed geodesics. These closed geodesics do not bound any pair of pants.

**Geometric implications of the tangle-free hypothesis**

The $L$-tangle-free hypothesis has various consequences on the local geometry of the surface at a scale (roughly) $L$, which we explore in Section 3. This will be particularly interesting when $L$ is large; in the case of random surfaces notably, where $L = a \log g$ for $a < 1$. All the results are stated for any $L$-tangle-free surface, with a general $L$ and no other assumption, so that they can be directly applied to another setting in which a tangle-free hypothesis is established.

First and foremost, we analyse the embedded cylinders around simple closed geodesics. In a hyperbolic surface with no further geometric assumptions to it, the standard collar theorem [12, Theorem 4.1.1] proves that the collar of width $\operatorname{arcsinh} \left(\sinh \left(\frac{\ell}{2}\right)^{-1}\right)$ around a simple closed geodesic of length $\ell$ is an embedded cylinder; moreover, at this width, disjoint simple closed geodesics have disjoint collars. The width of this deterministic collar is optimal and very satisfying for small $\ell$. For larger values of $\ell$ however, it becomes very poor. Under the tangle-free hypothesis, we are able to obtain significant improvements to the collar theorem that remedy this issue at larger scales.

**Theorem E** (Theorem 6). On a $L$-tangle-free hyperbolic surface, the collar of width $\frac{L-\ell}{2}$ around a closed geodesic of length $\ell < L$ is isometric to a cylinder.

This implies that we can find wide collars around geodesics of size $a \log g$, $a < 1$, on random surfaces; as a comparison, the width of the deterministic collar around such a geodesic decreases like $g^{-\frac{1}{2}}$. By a volume argument, Theorem 6 is optimal up to multiplication of the width by a factor two.
The methodology to prove this result is to examine the topology of an expanding neighbourhood of the geodesic. Since the two simplest hyperbolic subsurfaces (namely the pair of pants and one-holed torus) cannot be encountered up to a scale $\sim L$ due to the tangle-free hypothesis, the neighbourhood remains a cylinder.

An immediate consequence of this improved collar theorem is a bound on the number of intersections of a closed geodesic of length $\ell < L$ and any other geodesic of length $\ell'$. We prove in Corollary 7 that two such geodesics intersect at most $1 + \frac{\ell'}{\ell}$ times (and we can remove the 1 if the two geodesics are closed). Therefore, two closed geodesics of length $< \frac{L}{2}$ do not intersect; Proposition 8 furthermore states that the collars of width $\frac{L}{2} - \ell$ around two such geodesics are disjoint.

As well as the neighbourhood of geodesics, one can look at the geometric consequences that the tangle-free hypothesis has on the neighbourhood of points. To this end, we explore the set of geodesic loops based at a point on the surface on length scales up to $L$. As has already been mentioned above in Proposition 11, which establishes a link between our tangle-free definition and that of graphs, on an $L$-tangle-free surface, balls of radius $\frac{L}{8}$ are isometric to balls in either the hyperbolic plane or a hyperbolic cylinder. There are several ways to prove this property, some of which are similar to the proof of the improved collar theorem. In order to present different methods, we rather deduce it from the following slightly more general result.

**Theorem F (Theorem 9).** If $z$ is a point on a $L$-tangle-free surface, and $\delta_z$ is the shortest geodesic loop based at $z$, then any other loop $\beta$ based at $z$ such that $\ell(\delta_z) + \ell(\beta) < L$ is homotopic to a power of $\delta_z$.

Another consequence of Theorem 9 is Corollary 12, which states that any closed geodesic of length $< L$ on a $L$-tangle-free surface is simple. Put together, these observations imply the following corollary.

**Corollary G.** On a $L$-tangle-free hyperbolic surface,

1. all closed geodesics of length $< L$ are simple;
2. all closed geodesics of length $< \frac{L}{2}$ are pairwise disjoint;
3. all closed geodesics of length $< \frac{L}{4}$ are embedded in pairwise disjoint cylinders of width $\geq \frac{L}{4}$.

In the random case, this result is an improvement of the very useful collar theorem II [12, Theorem 4.1.6], which states that all closed geodesics of length $< 2 \text{arcsinh} 1$ on a hyperbolic surface are simple and do not intersect.

Short closed geodesics in random hyperbolic surfaces have been studied by Mirzakhani and Petri [24, 25]. One can deduce from [25, Proposition 4.5] and Markov’s inequality that, for any fixed $M$,

$$1 - \mathbb{P}^{WP}_g (\text{all closed geodesics of length } < M \text{ are simple}) \leq \frac{C_M}{g}$$
for a constant $C_M > 0$, when we prove that, for any real number $0 < a < 1$,

$$1 - P_g^\text{WP} \text{ (all closed geodesics of length } < a \log g \text{ are simple)} \leq C \frac{(\log g)^2}{g^{1-a}}$$

for a constant $C > 0$. In order to push the proof in [25] to a scale $\log g$, one would need to use strong properties of the Weil-Petersson volumes and deal with technical estimates, while our approach is quite elementary in both the geometric and probabilistic sense.

As illustrated in Section 2, the tools used to study random surfaces in the Weil-Petersson setting require to reduce problems to the study of multicurves. Knowing that all closed geodesics of length $< \frac{2}{3} \log g$ form a multicurve can be useful to the understanding of other properties of random surfaces.

Furthermore, McShane and Parlier proved in [22] that for any $g \geq 2$,

$$P_g^\text{WP} \text{ (the simple length spectrum has no multiplicity)} = 1,$$

where the simple length spectrum of a surface is the list of all the lengths of its simple closed geodesics. Corollary 12 then implies the following.

**Corollary 1.** For any $a \in (0, 1)$, if $\mathcal{L}(X)$ denotes the length spectrum of $X$, then

$$P_g^\text{WP} \left( \mathcal{L}(X) \cap [0, a \log g] \text{ has no multiplicity} \right) = 1 - O \left( \frac{(\log g)^2}{g^{1-a}} \right).$$

This could be surprising since, by the work of Horowitz and Randol, for any compact hyperbolic surface, the length spectrum has unbounded multiplicity [12, Theorem 3.7.1]. However, these high multiplicities are constructed in embedded pairs of pants, and therefore it is natural that their lengths are large for tangle-free surfaces.

**Motivations in spectral theory**

To conclude this introduction we will outline the connection between the geometry of hyperbolic surfaces and their spectral theory and in particular discuss how the tangle-free hypothesis and its implications on the geometry of surfaces on $\log g$ scales, which is a crucial scale in spectral theory, could be used to tackle some open problems in this area. As promised, let us first return to the relation of the tangle-free hypothesis with spectral graph theory and contrast this with that of surfaces.

**Friedman’s theorem**

Let $G$ be a $d$-regular graph, and $A$ be its adjacency matrix. We will call eigenvalues of $G$ the eigenvalues of the matrix $A$. They are linked to the eigenvalues of the Laplacialian $\Delta$ through the relation $-A + d \text{Id} = \Delta$. The value $d$ is always an
eigenvalue of $G$ corresponding to constant functions, and $-d$ is an eigenvalue if and only if $G$ is bipartite; both $d$ and $-d$ are referred to as trivial eigenvalues. Friedman’s theorem [14], first conjectured by Alon [3], states that for any $\varepsilon > 0$,

$$\mathbb{P}^{(d)}_n \left( \forall \lambda \text{ non-trivial eigenvalue of } G, |\lambda| < 2\sqrt{d-1} + \varepsilon \right) \xrightarrow{n \to +\infty} 1.$$  

This means that large random regular graphs have an optimal spectral gap, by a result of Alon [28].

Let us compare this to what one may expect of surfaces. We will refer to the spectrum of a compact hyperbolic surface $X$ as meaning the spectrum of the (positive) Laplace-Beltrami operator $\Delta$ on $X$. It is a non-decreasing sequence of eigenvalues $(\lambda_n)_{n \geq 0}$, $\lambda_n \geq 0$. The value $\lambda_0 = 0$ is known as the trivial eigenvalue; it is simple and the corresponding eigenfunctions are the constant functions. The equivalent surface conjecture of the Friedman theorem was formulated by Wright [35], and states that for any small enough $\varepsilon > 0$,

$$\mathbb{P}_{\text{WP}}^g \left( \lambda_1 \geq \frac{1}{4} - \varepsilon \right) \xrightarrow{g \to +\infty} 1.$$  

Note that this conjecture could concern any reasonable probabilistic setting, and the remarkable properties of the Weil-Petersson model (like Wolpert’s magic formula [33] and Mirzakhani’s integration formula [23]) make it an excellent candidate. Recently, Magee, Naud and Puder [21] have proved that if $X$ is a surface such that $\lambda_1(X) \geq \frac{1}{4}$ (such a surface exists [18]), then for any $\varepsilon > 0$,

$$\mathbb{P}_{\text{RC}}^n \left( \lambda_1(Y) \geq \frac{3}{16} - \varepsilon \right) \xrightarrow{n \to +\infty} 1$$  

where $Y$ is sampled uniformly amongst the finite number of degree $n$ covers of $X$. The conjecture with $\frac{1}{4}$ instead of $\frac{3}{16}$ is still open in this setting too.

**Short cycles on graphs and surfaces**

In spectral theory, when studying large-scale limits ($n \to +\infty$ for a graph, $g \to +\infty$ for a surface), it is important to know that the small-scale geometry of the object will not affect the spectrum. Often, a simple assumption to avoid this is to assume the injectivity radius to be large.

Unfortunately, random regular graphs have an asymptotically non-zero probability of having a small injectivity radius (see [34]). The same occurs with surfaces taken with the Weil-Petersson probability, by work of Mirzakhani [24]. As a consequence, in both cases, if we want to prove results true with probability approaching 1 in the large-scale limit, one needs to impose weaker and more typical geometric conditions.

For instance, Brooks and Lindenstrauss [11] and Brooks and Le Masson [10] studied eigenfunctions on regular graphs of size $n \to +\infty$, under assumptions on
the number of cycles up to a certain length $L$. This parameter $L$ can always be taken to be the injectivity radius, but in the case of random graphs, it can be increased to be of order $\log n$. In a recent article of Gilmore, Le Masson, Sahlsten and Thomas [15], a similar geometric hypothesis on the number of geodesic loops shorter than a scale $L$ based at each point is made, in order to control the $L^p$-norms of eigenfunctions of the Laplacian on hyperbolic surfaces. The authors prove it holds for random surfaces of high genus $g$ at a scale $L = c \log g$, but the proof provides no explicit value of the constant $c > 0$.

This limitation could be seen as originating from the methodology used to study the geometry of the surfaces. In essence, the authors prove that the loop condition is implied by a geometric condition, which is typical. This condition however is quite complex, and both the proof of its sufficiency and typicality are rather technical, leaving the local geometry of the random surfaces that are selected to remain quite opaque.

It follows from Corollary 10 that the constant $c$ in [15] can be taken to be any value $< \frac{1}{4}$. In turn, this improves (and makes precise) the rate of convergence of the probability for which the $L^p$-norm estimates in [15] hold. This is rather demonstrative of the capabilities of the tangle-free geometric condition allowing for a firm grasp over $\log(g)$ scale geometries for spectral theoretic purposes.

**Benjamini-Schramm convergence**

The notion of Benjamini-Schramm convergence is another way to study spectral properties of graphs and surfaces in the large-scale limit despite the existence of short cycles. In both cases, there is a general definition of Benjamini-Schramm convergence of a sequence to a limiting object [7, 1, 2], but when the limit is the infinite $d$-regular tree (for graphs) or the hyperbolic plane (for surfaces), the definition is equivalent to a simpler characterisation. A sequence of hyperbolic surfaces $(X_g)_{g \to \infty}$ converges to the hyperbolic plane if and only if

$$\forall R > 0, \quad \frac{\text{Vol}\{z \in X_g : \text{injrad}_{X_g}(z) < R\}}{\text{Vol}(X_g)} \xrightarrow{g \to \infty} 0,$$

and the definition for graphs is the same, replacing volumes by cardinalities.

Random graphs and surfaces satisfy this property for an $R$ proportional to $\log n$ and $\log g$ respectively, and this has consequences on their eigenvalues and eigenfunctions (see Anantharaman, Le Masson [5] and Anantharaman [4] for graphs, Le Masson, Sahlsten [20] and Monk [26] for surfaces). The notion of Benjamini-Schramm convergence and the tangle-free hypothesis correspond to assuming the objects have few cycles, but in different ways. The former means that the points which are the base of at least one short loop are scarce on the surface, while the latter implies that no point has more than one loop. Both approaches can be useful in different settings.
Outline of the paper

The paper is organised as follows:

- Section 1: tangled surfaces have tangled geodesics.
- Section 2: random surfaces are \((a \log g)\)-tangle-free for any \(a < 1\).
- Section 3: geometric consequences of the tangle-free hypothesis.
- Section 4: any surface of genus \(g\) is \((4 \log g + O(1))\)-tangled.

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1 Tangled surfaces have tangled geodesics

The aim in this section is to prove that being tangled implies having a tangled geodesic - that is to say a non-simple closed geodesic of length \(\leq 2L + O(1)\).

**Proposition 2.** Let \(X\) be a compact hyperbolic surface and \(L > 0\). Assume that \(X\) is \(L\)-tangled. Then, there exists a closed geodesic \(\gamma\) in \(X\) of length smaller than \(2L + 2\pi\) with one self-intersection.

The geodesic we construct is what is called a figure eight. Any non-simple geodesic on a hyperbolic surface has length greater than \(4 \text{arcsinh } 1 \approx 3.52\ldots\), and this result is sharp (see [12, Theorem 4.2.2]).

**Proof.** It suffices to prove that there is such a geodesic in any pair of pants or one-holed torus of total boundary length smaller than \(2L\).

![Construction of a short self-intersecting geodesic](image)

Figure 1: Construction of a short self-intersecting geodesic.
Let us first consider a hyperbolic pair of pants boundary lengths $\ell_1$, $\ell_2$, $\ell_3$, such that $\ell_1 + \ell_2 + \ell_3 < 2L$. We construct a closed curve with one self-intersection as represented in Fig. 1a. By [12, Formula 4.2.3],
\[
\cosh\left(\frac{\ell(\gamma)}{2}\right) = 2\cosh\left(\frac{\ell_1}{2}\right)\cosh\left(\frac{\ell_3}{2}\right) + \cosh\left(\frac{\ell_2}{2}\right) \leq 3e^L.
\]
Since $\cosh x \geq \frac{e^x}{2}$, we deduce that the length of $\gamma$ is smaller than $2L + 2\log 6$.

We use a different proof in the one-holed torus case, because we do not have access to several small geodesics straight away. Let us study a one-holed torus boundary geodesic of boundary length $\alpha$ as represented in Fig. 1a. By [12, Formula 4.2.3],
\[
\cosh\left(\frac{\ell(\gamma)}{2}\right) = 2\cosh\left(\frac{\ell_1}{2}\right)\cosh\left(\frac{\ell_3}{2}\right) + \cosh\left(\frac{\ell_2}{2}\right) \leq 3e^L.
\]

By the collar theorem [12, Theorem 4.1.1], when $w$ is small enough, $C_w$ is a half-cylinder with Fermi coordinates $(\rho, t)$, in which the hyperbolic metric is $ds^2 = d\rho^2 + \cosh^2 \rho \, dt^2$. This isometry has to break down at some point, because the area of the one-holed torus is $2\pi$, and as long as the isometry holds
\[
\text{Vol}(C_w) = \int_0^\ell \int_0^w \cosh \rho \, d\rho \, dt = \ell \sinh w \leq 2\pi.
\]

We pick $w_+$ to be the supremum of the widths for which the isometry holds. By continuity, $w_+$ satisfies inequality (1).

The fact that the isometry ceases implies that there is (at least) one self-intersection point $z$ at the boundary of $C_{w_+}$. By definition, there are two distinct geodesic segments $c_1$, $c_2$ of length $w_+$ from $\partial T$ to $z$. Furthermore, these segments are orthogonal to the inner boundary $\beta_{w_+} := \partial C_{w_+} \setminus \partial T$ of the $w_+$-neighbourhood $C_{w_+}$. By minimality of $w_+$, the two tangents of $\beta_{w_+}$ at $z$ are aligned, and therefore the two segments $c_1$, $c_2$ connect to form a geodesic segment $c$ from $\partial T$ to itself.

The regular neighbourhood of $\partial T$ and $c$ is a topological pair of pants, with three boundary components, $\gamma_1$, $\partial T$, $\gamma_2$. Neither $\gamma_1$ nor $\gamma_2$ is contractible because they are freely homotopic to geodesic bigons ($c$ and a portion of $\partial T$). Then, replacing $\gamma_1$ and $\gamma_2$ by the closed geodesics $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ in their respective free homotopy classes yields a decomposition of the handle into a pair of pants of boundary components $(\tilde{\gamma}_1, \partial T, \tilde{\gamma}_2)$. Let $\gamma$ denote the figure-eight geodesic constructed in the pair-of-pants case, which is freely homotopic to $a_1ca_2^{-1}c_2$, where $a_1$ and $a_2$ are the portions of $\partial T$ delimited by $c$ as represented in Fig. 1b. We shall estimate the length of $\gamma$.

Let $\varepsilon > 0$. We observe that the portion $c_\varepsilon$ of the geodesic segment $c$ outside of $C_{w_+ - \varepsilon}$ is a geodesic segment of length $2\varepsilon$, connecting two points of $\beta_{w_+ - \varepsilon}$. Let $a_1^\varepsilon$, $a_2^\varepsilon$ be the two portions of $\beta_{w_+ - \varepsilon}$ delimited by $c_\varepsilon$. Then, the loop $a_1^\varepsilon c_\varepsilon (a_2^\varepsilon)^{-1} c_\varepsilon$ is freely homotopic to $a_1ca_2^{-1}c$ and hence $\gamma$. Its length is equal to
\[
\ell(\beta_{w_+ - \varepsilon}) + 4\varepsilon = \ell \cosh(w_+ - \varepsilon) + 4\varepsilon \xrightarrow{\varepsilon \to 0} \ell \cosh(w_+).
\]
By minimality of the geodesic representative in a free homotopy class, 
\[ \ell(\gamma) \leq \ell \cosh(w_+) = \ell \sqrt{1 + \sinh^2(w_+)} \leq \ell \sqrt{1 + \frac{4\pi^2}{\ell^2}} \leq 2L + 2\pi \]
by equation (1), which allows us to conclude.

2 Random surfaces are \((a \log g)\)-tangle-free

In this section, we will show that, for any \(0 < a < 1\), typical surfaces of genus \(g\) are \((a \log g)\)-tangle-free. By typical we mean in the probabilistic sense for the Weil-Petersson model of random surfaces. To be precise we shall introduce this model briefly here, a more thorough overview can be found in [17] or [35].

2.1 Teichmüller and moduli spaces

For integers \(g, n\) such that \(2g - 2 + n > 0\), fix a connected and oriented smooth surface \(S_{g,n}\) of genus \(g\) and with \(n\) numbered boundary components. Let us also fix a length vector \(\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}^n_{>0}\). Define the Teichmüller space \(\mathcal{T}_{g,n}(\ell)\) by

\[ \mathcal{T}_{g,n}(\ell) = \left\{ (X, f) : \begin{array}{l} f : S_{g,n} \to X \text{ diffeomorphism} \\ X \text{ hyperbolic surface} \\ i\text{-th boundary component of length } \ell_i \text{ for } 1 \leq i \leq n \end{array} \right\} / \sim, \]

where \(\sim\) is the equivalence relation \((X_1, f_1) \sim (X_2, f_2)\) if and only if there exists an isometry \(h : X_1 \to X_2\) such that \(f_2 \circ h \circ f_1^{-1} : S_{g,n} \to S_{g,n}\) is isotopic to the identity.

The elements of \(\mathcal{T}_{g,n}(\ell)\) are surfaces with a marking. Many surfaces are isometric, but have a different marking. If one wants to pick a random surface, it is more natural to take it in the moduli space

\[ \mathcal{M}_{g,n}(\ell) = \left\{ \begin{array}{l} \text{hyperbolic surfaces of genus } g \\ \text{with } n \text{ boundary components} \\ \text{i-th component of length } \ell_i \end{array} \text{ for } 1 \leq i \leq n \right\} / \{\text{isometry}\} \]

where the quotient is over the set of isometries that preserve the \(i\)-th component setwise, for all \(i \in \{1, \ldots, n\}\). The moduli space can be obtained as a quotient of the Teichmüller space by the action of the mapping class group

\[ \mathcal{M}_{g,n}(\ell) = \mathcal{T}_{g,n}(\ell) / \text{MCG}(S_{g,n}). \]

We recall that \(\text{MCG}(S_{g,n})\) is the group of orientation preserving diffeomorphisms of \(S_{g,n}\) that setwise preserve the boundary components of the surface, up to isotopy, and it acts on the Teichmüller space by precomposition of the marking.

In the case when \(n = 0\) (and the surface is compact, with no boundary), we will suppress the mention of \(n\) (and the empty vector \(\ell\)), and write \(S_g, \mathcal{M}_g, \mathcal{T}_g\).
2.2 The Weil-Petersson probability

The Teichmüller space $T_{g,n}(\ell)$ possesses a natural symplectic structure, the Weil-Petersson form $\omega_{WP}^{g,n,\ell}$, which is invariant under the action of the mapping class group and therefore descends to the moduli space.

The symplectic form induces a volume form $d\text{Vol}_{WP}^{g,n,\ell} = \frac{1}{N!}(\omega_{WP}^{g,n,\ell})^\wedge N$ for $N = 3g - 3 + n$, called the Weil-Petersson volume form. The volume of the moduli space is a finite quantity $V_{g,n}(\ell) := \text{Vol}_{WP}^{g,n,\ell}(\mathcal{M}_{g,n}(\ell))$.

When $n = 0$ (and the surface is compact, with no boundary), we write $\text{Vol}_{WP}^{g}$ and $V_{g}$ to simplify notations. We will see in the next section why we need to introduce these volumes for surfaces with boundary components, even when we only want to study boundary-free compact surfaces.

We can normalise $\text{Vol}_{WP}^{g}$ and obtain the Weil-Petersson probability measure $\mathbb{P}_{g}^{WP} = \frac{1}{V_{g}}\text{Vol}_{WP}^{g}$ on the moduli space $\mathcal{M}_{g}$. The Weil-Petersson form can be expressed in Fenchel-Nielsen coordinates thanks to Wolpert’s theorem [33]. This geometric expression has deep consequences, and is what ultimately allows for explicit computations in this model.

2.3 Mirzakhani’s integration formula

In this subsection, we explain how Mirzakhani’s integration formula [23] can be used to compute expectations of a certain class of functions known as geometric functions. Knowing how to compute expectations then allows one to estimate the probability of certain events by, for instance, using Markov’s inequality $P(|X| > a) \leq \frac{1}{a} E(|X|)$.

**Definition 2.** A geometric function is a function $\mathcal{M}_{g} \to \mathbb{R}$ that can be written as:

$$F_{\Gamma}(X) = \sum_{(\gamma_{1},\ldots,\gamma_{k}) \in O(\Gamma)} F(\ell_{X}(\gamma_{1}),\ldots,\ell_{X}(\gamma_{k})),$$

where:

- $F : \mathbb{R}_{\geq 0}^{k} \to \mathbb{R}$ is a positive measurable function
- $\Gamma$ is a multi-curve on $S_{g}$, and $O(\Gamma)$ is the orbit of $\Gamma$ under the action by the mapping class group $\text{MCG}(S_{g})$
- for a closed curve $\gamma$ on $S_{g}$ and $(X,f) \in T_{g}$, $\ell_{X}(\gamma)$ is the length of the unique closed geodesic freely homotopic to the image of $\gamma$ on $X$ under the marking map $f$. 

Though a fixed term of the sum in the previous definition only really makes sense for an element of the Teichmüller space, the summation over the orbit makes it invariant under the action of the mapping class group, and hence a well-defined function on the moduli space $\mathcal{M}_g$.

The following result is an expression of the integral of any geometric function as an integral over $\mathbb{R}^k_{\geq 0}$. In order to write the formula, we must understand the surface resulting in cutting $S_g$ by the curves in $\Gamma$. For this, we observe that the cut surface $S_g \setminus \Gamma$ can be written as the disjoint union $\bigsqcup_{i=1}^q S_{g_i,n_i}$ of its connected pieces.

The $k$ curves of $\Gamma$ form $2k$ boundary components of the cut surface. If the multi-curve $\Gamma$ had lengths $\ell \in \mathbb{R}^k_{\geq 0}$ on $X$, then these lengths become the boundary lengths of the surface $X$ cut along $\Gamma$. Each component $S_{g_i,n_i}$ therefore has a length vector $\ell^{(i)} \in \mathbb{R}^{n_i}_{\geq 0}$. We then define

$$V_g(\Gamma, \ell) := \prod_{i=1}^q V_{g_i,n_i}(\ell^{(i)}).$$

Mirzakhani’s integration formula can then be formulated as follows.

**Theorem 3** ([23]). Given a multi-curve $\Gamma$ and a function $F : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ there exists a constant $0 < C_{\Gamma} \leq 1$ dependent only on $\Gamma$ for which

$$\int_{\mathcal{M}_g} F^\Gamma(X) \text{dVol}^\text{WP}_g(X) = C_{\Gamma} \int_{\mathbb{R}^k_{\geq 0}} F(x) V_g(\Gamma, \ell) \ell_1 \cdots \ell_k \text{d}\ell_1 \cdots \text{d}\ell_k.$$

### 2.4 Volume estimates

The previous formula indicates that in order to estimate expectations, we need to understand the asymptotic behaviour of Weil-Petersson volumes. In our proof, we will only use a handful of them, grouped in the following Lemma.

**Lemma 4** (Lemmas 3.2 and 3.3 [24]). Given $g, n \geq 0$ such that $2g - 2 + n > 0$,

1. $\ell_1 \cdots \ell_n V_{g,n}(\ell_1, \ldots, \ell_n) \leq 2^n \prod_{i=1}^n \sinh(\frac{\ell_i}{2}) V_{g,n},$

2. $V_{g,n+2} \leq V_{g+1,n},$

3. there exists a constant $C$ independent of $g$ and $n$ such that

$$V_{g,n} \leq C \frac{V_{g,n+1}}{2g - 2 + n},$$

4. there exists a constant $C_n$ independent of $g$ such that for any integers $n_1, n_2$ satisfying $n_1 + n_2 = n$,

$$\sum_{g_1 + g_2 = g} V_{g_1,n_1+1} V_{g_2,n_2+1} \leq C_n \frac{V_{g,n}}{g}.$$
2.5 Probabilistic result

We can now state and prove our probabilistic result.

**Theorem 5.** For any real number $0 < a < 1$,

$$
\mathbb{P}_g^{WP}(X \text{ is } (a \log g)\text{-tangle-free}) = 1 - O\left(\frac{(\log g)^2}{g^{1-a}}\right).
$$

**Proof.** Let us list all the topological types of embedded one-holed tori or pair of pants in a genus $g$ surface (see 2):

(i) a curve separating a one-holed torus;

(ii) three curves cutting $S_g$ into a pair of pants and a component $S_{g-2,3}$;

(iii) three curves cutting $S_g$ into a pair of pants and two components $S_{g_1,1}$ and $S_{g_2,2}$ such that $g_1 + g_2 = g - 1$;

(iv) three curves cutting $S_g$ into a pair of pants and three connected components $S_{g_1,1}$, $S_{g_2,1}$ and $S_{g_3,1}$ with $1 \leq g_1 \leq g_2 \leq g_3$ and $g_1 + g_2 + g_3 = g$.

![Figure 2: The different topological ways to embed a one-holed torus or pair of pants in a surface of genus $g$.](image)

For any topological situation, we will consider a multicurve $\alpha$ on the base surface $S_g$ realising the topological configuration and study the counting function

$$
N^\alpha_L(X) = \#\{\beta \in O(\alpha) : \ell_X(\beta) \leq 2L\},
$$

where the length of a multi-curve is defined as the sum of its components. Then, the probability of finding a component in the topological situation $\alpha$ of total boundary length $\leq 2L$ can be bounded by Markov’s inequality:

$$
\mathbb{P}_g^{WP}(N^\alpha_L(X) \geq 1) \leq \mathbb{E}_g^{WP}[N^\alpha_L(X)].
$$

We observe that $N^\alpha_L(X)$ is a geometric function, and its expectation can therefore be computed using Mirzakhani’s integration formula (3). This reduces the problem to estimating integrals with Weil-Petersson volumes, which we will now detail.
In case (i), the integral that appears is
\[ \int_0^{2L} V_{1,1}(\ell)V_{g-1,1}(\ell) \, d\ell. \]
From \([27]\), it is known that \(V_{1,1}(\ell) = \frac{\ell^2}{24} + \frac{\pi^2}{6}\). Moreover, by Lemma 4,
\[ \ell V_{g-1,1}(\ell) \leq 2e^2 V_{g-1,1}. \]
It follows that the probability is smaller than
\[ \frac{V_{g-1,1}}{V_g} \int_0^{2L} 2 \left( \frac{\ell^2}{24} + \frac{\pi^2}{6} \right) e^2 \, d\ell = O \left( \frac{V_{g-1,1}}{V_g} L^2 e^L \right) = O \left( \frac{(\log g)^2}{g^{1-a}} \right) \]
where the last bound is deduced from Lemma 4 parts (2) and (3) and taking \(L = a \log g\).

In case (ii), the integral that appears is
\[ \frac{1}{V_g} \iiint_{0 \leq \ell_1 + \ell_2 + \ell_3 \leq 2L} V_{0,3}(\ell_1, \ell_2, \ell_3)V_{g-2,3}(\ell_1, \ell_2, \ell_3) \ell_1 \ell_2 \ell_3 \, d\ell_1 \, d\ell_2 \, d\ell_3. \]
Due to the fact that \(V_{0,3}(\ell_1, \ell_2, \ell_3) = 1\) and by Lemma 4(1), we need to estimate
\[ \frac{V_{g-2,3}}{V_g} \iiint_{0 \leq \ell_1 + \ell_2 + \ell_3 \leq 2L} \exp \left( \frac{\ell_1 + \ell_2 + \ell_3}{2} \right) \, d\ell_1 \, d\ell_2 \, d\ell_3 = O \left( \frac{(\log g)^2}{g^{1-a}} \right) \]
by Lemma 4 (2-3).

Let us now bound the sum of all the topological situations of case (iii). By the same manipulations, we obtain that the probability is
\[ O \left( \frac{L^2 e^L}{V_g} \sum_{g_1 + g_2 + g_3 = g} V_{g_1,1}V_{g_2,2} \right) = O \left( \frac{(\log g)^2}{g^{1-a}} \right) \]
by Lemma 4(4) and then Lemma 4(2-3).

Finally, in the last case we have to estimate
\[ \sum_{g_1 + g_2 + g_3 = g} \sum_{\text{all } g_1, g_2, g_3} V_{g_1,1}V_{g_2,1}V_{g_3,1} \leq C_0 \sum_{g_1 = 1}^{\frac{g-2}{2}} \sum_{g_2 + g_3 = g-g_1} V_{g_1,1}V_{g-g_1,0} \frac{1}{g-g_1} \]
where \(C_0\) is the constant from Lemma 4(4). We observe that \(g - g_1 \geq \frac{2}{3} g\) and use Lemma 4(3) to conclude that the probability is
\[ O \left( \frac{(\log g)^2}{V_g g^{2-a}} \sum_{g_1 = 1}^{\frac{g-2}{2}} V_{g_1,1}V_{g-g_1,1} \right) = O \left( \frac{(\log g)^2}{g^{3-a}} \right) \]
by Lemma 4(4).
Remark. In the cases (i), (iii) and (iv), there is a separating geodesic of length \( \leq 2a \log g \). Therefore, we could have bounded these probabilities by the probability of having a separating geodesic of length \( \leq 2a \log g \), which has been estimated by Mirzakhani in [24, Theorem 4.4]. This approach yields the same end result, but the authors decided to detail the four cases for the sake of self-containment. Furthermore, this more detailed study allows us to see that the most likely cases are cases (i.) and (ii.), and therefore we expect the first length at which the surface is tangled to be obtained by one of these two topological situations.

3 Geometry of tangle-free surfaces

The aim of this section is to provide information about geodesics and neighbourhoods of points on tangle-free surfaces. The results will be expressed in terms of an arbitrary \( L \)-tangle-free surface \( X \), but can also been seen as result that are true with high probability for \( L = a \log g \), \( a < 1 \) due to Theorem 5.

3.1 An improved collar theorem

Theorem 6. Let \( L > 0 \), and \( X \) be a \( L \)-tangle-free hyperbolic surface. Let \( \gamma \) be a simple closed geodesic of length \( \ell < L \). Then, for \( w := \frac{L-\ell}{2} \), the neighbourhood 

\[ C_w(\gamma) = \{ z \in X : \text{dist}(z,\gamma) < w \} \]

is isometric to a cylinder.

The collar theorem [12] is a similar result, with the width \( \text{arcsinh}\left(\sinh(\ell/2)^{-1}\right) \).

We recall that, in the random case, for \( a < 1 \), with high probability, we can take \( L = a \log g \). This result therefore is a significant improvement for geodesics of length \( b \log g \), \( 0 < b < a \). We obtain a collar of width \( w = \frac{a-b}{2} \log g \), which is expanding with the genus, as opposed to the deterministic collar, of width \( \asymp g^{-\frac{1}{2}} \).

For very short geodesics, the width of this new collar is \( \asymp \frac{a}{2} \log g \). It might seem less good than the deterministic collar, which is of width \( \asymp -\log(\ell) \). However, by Theorem 4.2 in [24], the injectivity radius of a random surface is greater than \( g^{-\frac{1}{2}} \) with probability \( 1 - O(g^{-a}) \). Under this additional probabilistic assumption, the two collars are of similar sizes.

Proof. For small enough \( w \), the neighbourhood \( C_w(\gamma) \) is a cylinder, with two boundary components \( \gamma_w^+ \) and \( \gamma_w^- \). Let us assume that, for a certain \( w \), the topology of the neighbourhood changes. There are two ways for this to happen (and both can happen simultaneously) – see Fig. 3.

(A) One boundary component, \( \gamma_w^+ \) or \( \gamma_w^- \), self-intersects.

(B) The two boundary components \( \gamma_w^+ \) and \( \gamma_w^- \) intersect one another.
In both cases, let $z \in X$ denote one intersection point. Since the distance between $z$ and $\gamma$ is $w$, there are two distinct geodesic arcs $c_1$, $c_2$ of length $w$, going from $z$ to points of $\gamma$, and intersecting $\gamma$ perpendicularly. Both $c_1$ and $c_2$ are orthogonal to the boundaries of the cylinder and the two boundaries are tangent to one another by minimality of the width $w$. As a consequence, the curve $c = c_1^{-1}c_2$ is a geodesic arc.

The regular neighbourhood of the curves $\gamma$ and $c$ has Euler characteristic $-1$. There are two possible topologies for this neighbourhood.

- If it is a pair of pants, then it has three boundary components. Neither of them is contractible on the surface $X$. Indeed, one component is freely homotopic to $\gamma$, and the two others to $c$ and a portion of $\gamma$, which are geodesic bigons. Therefore, when we replace the boundary components of the regular neighbourhood by the closed geodesic in their free homotopy classes, we obtain a pair of pants or a one-holed torus (if two of the boundary components are freely homotopic to one another), of total boundary length smaller than $2\ell + 4w$.

- Otherwise, it is a one-holed torus. Its boundary component is not contractible, because there is no hyperbolic surface of signature $(1,0)$. Therefore, the closed geodesic in its free homotopy class separates a one-holed torus with boundary length smaller than $2\ell + 4w$ from $X$.

In both cases, by the tangle-free hypothesis, $2L < 2\ell + 4w$, which allows us to conclude. \hfill $\square$

**Remark.** Let $\mathcal{A}_g \subset \mathcal{M}_g$ be the event “the surface has a simple closed geodesic of length between 1 and 2”. By work of Mirzakhani and Petri [25],

$$\mathbb{P}^{WP}_{g} (\mathcal{A}_g) \xrightarrow{g \to +\infty} 1 - \exp \left( - \int_{1}^{2} e^{t} + e^{-t} - 2 \frac{2t}{2t} dt \right) > 0,$$
so this event has asymptotically non-zero probability.

Let \( X \) be an element of \( A_g \) which is also \((a \log g)\)-tangle-free, and let \( \gamma \) be a closed geodesic on \( X \) of length \( \ell \in [1, 2] \). Then, the collar \( C_w(\gamma) \) given by Theorem 6 has volume

\[
\text{Vol}(C_w(\gamma)) = 2\ell \sinh w \geq 2 \sinh \left( \frac{a}{2} \log g - 1 \right) \sim g^{\frac{a}{2}} \quad \text{as } g \to +\infty.
\]

However, \( \text{Vol}(C_w(\gamma)) \leq \text{Vol} X = 2\pi(2g - 2) \). This leads to a contradiction for \( g \) approaching \(+\infty\) as soon as \( a > 2 \). Hence, for large \( g \), the elements of \( A_g \) are \((a \log g)\)-tangle-free for \( a > 2 \):

\[
\limsup_{g \to +\infty} \mathbb{P}_g^\text{WP}(X \text{ is } (a \log g)\text{-tangled}) \geq \lim_{g \to +\infty} \mathbb{P}_g^\text{WP}(A_g) > 0.
\]

Therefore, for all \( a > 2 \), random surfaces do not have high probability of being \((a \log g)\)-tangle-free.

By taking \( a \) close to but larger than 1, this same line of reasoning and the fact that we know surfaces to be \((a \log g)\)-tangle-free with high probability implies that the improved collar cannot be much larger than \( L - \ell \). As a consequence, our result is optimal up to multiplication by 2.

### 3.2 Number of intersections of geodesics

A consequence of this improved collar theorem is a bound on the number of intersections of a short closed geodesic with any other geodesic.

**Corollary 7.** Let \( L > 0 \), and \( X \) be a \( L\)-tangle-free hyperbolic surface.

Let \( \gamma \) be a simple closed geodesic of length \( \ell(\gamma) \) on \( X \). Then, for any geodesic \( \gamma' \) transverse to \( \gamma \), the number of intersections \( i(\gamma, \gamma') \) between \( \gamma \) and \( \gamma' \) satisfies

\[
i(\gamma, \gamma') \leq \frac{\ell(\gamma')}{L - \ell(\gamma)} + 1.
\]

In the case where \( \gamma' \) is also closed, then

\[
i(\gamma, \gamma') \leq \frac{\ell(\gamma')}{L - \ell(\gamma)}.
\]

In particular, if \( \ell(\gamma) + \ell(\gamma') < L \), then \( \gamma \) and \( \gamma' \) do not intersect.

**Proof.** By Theorem 6, \( \gamma \) is embedded in an open cylinder \( C \) of width \( w = \frac{L - \ell(\gamma)}{2} \).

Let us parametrize the geodesic \( \gamma' : [0, 1] \to X \). The set of times when \( \gamma' \) visits the cylinder can be decomposed as

\[
\bigsqcup_{i=1}^{k} (t_i^-, t_i^+), \quad 0 \leq t_1^- < t_1^+ \leq \ldots \leq t_k^- < t_k^+ \leq 1,
\]
Figure 4: Illustration of the proof of Corollary 7.

as represented in Fig. 4. The restriction $c_i$ of $\gamma'$ between $t^-_i$ and $t^+_i$ is a geodesic in the cylinder $C$, transverse to the central geodesic $\gamma$. Therefore, if $c_i$ intersects $\gamma$, then it does at most once. Let $I \subset \{1, \ldots, k\}$ be the set of $i$ such that $c_i$ intersect $\gamma$. We have that $i(\gamma, \gamma') = \#I \leq k$.

We assume that $\#I \geq 2$ (otherwise there is nothing to prove). Any geodesic intersecting the central geodesic transversally travels through the entire cylinder, and is therefore of length greater than $2w$. As a consequence, for any $i \in I$ different from 1 and $k$, $\ell(c_i) \geq 2w$. Also, if $i = 1$ or $k$ belongs in $I$, then $\ell(c_i) \geq w$. This leads to our claim, because

$$(i(\gamma, \gamma') - 1)(L - \ell(\gamma)) = (\#I - 1) \cdot 2w \leq \sum_{i \in I} \ell(c_i) \leq \ell(\gamma').$$

The case when the curve $\gamma'$ is closed can be obtained observing that, in this case, $\ell(c_1)$ and $\ell(c_k)$ also are greater than $2w$ (when 1 or $k$ belongs in $I$).

Proposition 8. Let $L > 0$, and $X$ be a $L$-tangle-free hyperbolic surface. Let $\gamma$, $\gamma'$ be two distinct simple closed geodesics such that $\ell(\gamma) + \ell(\gamma') < L$. Then, the distance between $\gamma$ and $\gamma'$ is greater than $L - \ell(\gamma) - \ell(\gamma')$.

In particular, if $\ell(\gamma), \ell(\gamma') < \frac{L}{2}$, then the collars of width $\frac{L}{2} - \ell(\gamma)$ around $\gamma$ and $\frac{L}{2} - \ell(\gamma')$ around $\gamma'$ are two disjoint embedded cylinders.

Proof. We already know, owed to Corollary 7, that $\gamma$ and $\gamma'$ do not intersect. Let $c$ be a length-minimising curve with one endpoint on $\gamma$ and the other on $\gamma'$ (see Fig. 5). Then, by minimality, $c$ is simple and only intersects $\gamma$ and $\gamma'$ at endpoints. The regular neighbourhood $R$ of $\gamma$, $\gamma'$ and $c$ is a topological pair of pants of total boundary length less than $2(\ell(\gamma) + \ell(\gamma') + \ell(c))$. Since $\gamma$ and $\gamma'$ are non-contractible and not freely homotopic to one another, the third boundary component is not contractible and $R$ corresponds to an embedded pair of pants.
or one-holed torus on $X$. By the tangle-free hypothesis, $\ell(\gamma) + \ell(\gamma') + \ell(c) \geq L$, and therefore the distance between $\gamma$ and $\gamma'$ is greater than $L - \ell(\gamma) - \ell(\gamma')$. This implies our claim.

\[ \Box \]

### 3.3 Short loops based at a point

Let us now study short loops based at a point on a tangle-free surface.

**Theorem 9.** Let $L > 0$, and $X$ be a $L$-tangle-free hyperbolic surface. Let $z \in X$, and let $\delta_z$ be the shortest geodesic loop based at $z$.

Let $\beta$ be a (non necessarily geodesic) loop based at $z$, such that $\ell(\beta) + \ell(\delta_z) < L$. Then $\beta$ is homotopic with fixed endpoints to a power of $\delta_z$.

The result is empty if the injectivity radius of the point $z$ is greater than $\frac{L}{4}$. The “shortest geodesic loop” $\delta_z$ is not necessarily unique. It will be as soon as the injectivity radius at $z$ is smaller than $\frac{L}{4}$. More precisely, we directly deduce from Theorem 9 the following corollary, which was used in [15] for random surfaces (with a length $L = a \log g$, but the value of $a$ was not explicit). Note the similarity of this result to the classical Margulis lemma [31]. In particular, we obtain an explicit constant for the Margulis lemma in the case of tangle-free surfaces in the same way that the classical collar theorem provides.

**Corollary 10.** Let $L > 0$, and $X = \mathcal{H}/\Gamma$ be an $L$-tangle-free hyperbolic surface. Then, for any $z \in \mathcal{H}$, the set $\{T \in \Gamma : \text{dist}_\mathcal{H}(z, T \cdot z) < \frac{L}{4}\}$ is:

- reduced to the identity element (when the injectivity radius at $z$ is $\geq \frac{L}{4}$),
- or included in the subgroup $\langle T_0 \rangle$ generated by the element $T_0 \in \Gamma$ corresponding to the shortest geodesic loop through $z$.

We recall that any compact hyperbolic surface is isometric to a quotient of the hyperbolic plane $\mathcal{H}$ by a Fuchsian co-compact group $\Gamma \subset \text{PSL}_2(\mathbb{R})$ – see [19] for more details.

We could prove Theorem 9 using the same method as we used for Theorem 6 and Corollary 7, expanding a cylinder around $\delta_z$. However, our initial proof used
a different method, which we decided to present here, in order to expose different ways to use the tangle-free hypothesis.

![Illustrations of the proof of Theorem 9.](image)

(a) Case \( k = 0 \).

(b) Case \( k > 0 \).

Figure 6: Illustrations of the proof of Theorem 9.

Proof of Theorem 9. By replacing \( \beta \) by a new curve in its homotopy class, we can assume that \( \beta \) has a finite number of self-intersections, and of intersections with \( \delta_z \), while still satisfying the length condition.

We now prove this result by induction on the number of self-intersections \( k \geq 0 \) of \( \beta \). We start with the base case of \( k = 0 \) so that \( \beta \) is simple. We parametrise \( \beta : [0,1] \to X \). Let \( 0 = t_0 < t_1 < \ldots < t_I = 1 \) be the times when \( \beta \) meets \( \delta_z \).

Let \( 0 \leq i < I \), and \( \beta_i \) be the restriction of \( \beta \) to \([t_i, t_{i+1}]\) – see Fig. 6a. Then, the regular neighbourhood \( R \) of \( \delta_z \) and \( \beta_i \) has Euler characteristic \(-1\), and total boundary length \( \leq 2(\ell(\delta_z) + \ell(\beta_i)) < 2L \). If \( R \) is a topological one-holed torus, then by the tangle-free hypothesis, its boundary component is contractible, which is impossible for there is no hyperbolic surface of signature \((1,0)\).

Therefore, \( R \) is a topological pair of pants. By the tangle-free hypothesis, one of its boundary components is contractible. It can not be the component corresponding to \( \delta_z \), so it is another one. Hence, \( \beta_i \) is homotopic with fixed endpoints to a portion \( \delta_z^{(i)} \) of \( \delta_z \).

As a consequence, \( \beta = \beta_0 \ldots \beta_{I-1} \) is homotopic with fixed endpoints to the product

\[
c = \delta_z^{(0)} \delta_z^{(1)} \ldots \delta_z^{(I-1)}.
\]

\( c \) goes from \( z \) to \( z \) following only portions of \( \delta_z \). Therefore, \( c \) is homotopic with fixed endpoints to a power \( \delta_z^i \) of \( \delta_z \).

We now move forward to the case \( k > 0 \). We assume the result to hold for any smaller \( k \). The idea is to find a way to cut \( \beta \) into smaller loops on which to apply the induction hypothesis; the construction is represented in Fig. 6b.

Let \( \ell = \ell(\beta) \). We pick a length parametrisation of \( \beta : \mathbb{R}/\ell \mathbb{Z} \to X \) such that \( \beta(0) = z \). We look for the first intersection point of \( \beta \), starting a 0, but looking in
both directions:
\[
\ell_+ = \min \{ t \geq 0 : \exists s \in (t, \ell) \text{ such that } \beta(s) = \beta(t) \}
\]
\[
\ell_- = \min \{ t \geq 0 : \exists s \in (t, \ell) \text{ such that } \beta(-s) = \beta(-t) \}.
\]
Up to a change of orientation of \( \beta \), we can assume that \( \ell_+ \leq \ell_- \). Then, we set
\[
t = \max \{ s \in (\ell_+, \ell) : \beta(s) = \beta(\ell_+) \}
\]
to be the last time at which \( \beta \) visits \( \beta(\ell_+) \), so that the restriction of \( \beta \) to \([\ell_+, t]\) is a loop \( \beta_+ \). The curve has no self-intersection between \( \ell - \ell_- \) and \( \ell \), so \( t \leq \ell - \ell_- \).

Let us apply the induction hypothesis to the two loops \( c_+ \beta_+ c_-^{-1} \) and \( c_+ c_- \). It will follow that they, and hence \( \beta \), are homotopic with fixed endpoints to a power of \( \delta_z \).

\( \beta_+ \) is a sub-loop of \( \beta \). As a consequence, \( c_+ c_- \) has less self-intersections than \( \beta \), and hence strictly less than \( k \). Furthermore, it is shorter, so it satisfies the length hypothesis \( \ell(c_+ c_-) + \ell(\delta_z) < L \). So we can apply the induction hypothesis.

\( c_+ \) is simple and does not intersect \( \beta_+ \) (except at its endpoint). As a consequence, we can find a curve \( b \) homotopic to \( c_+ \beta_+ c_-^{-1} \) with as many self-intersections as \( \beta_+ \). \( \beta_+ \) is a strict sub-loop of \( \beta \), so this intersection number is strictly smaller than \( k \). The length of \( b \) can be taken as close as desired to that of \( c_+ \beta_+ c_-^{-1} \). Moreover,
\[
\ell(c_+ \beta_+ c_-^{-1}) = 2\ell_+ + \ell(\beta_+) \leq \ell_+ + \ell_- + \ell(\beta_+) \leq \ell(\beta)
\]
so \( b \) can be chosen to satisfy the length hypothesis \( \ell(\delta_z) + \ell(b) < L \), and we can apply the induction hypothesis to it.

**3.4 Neighbourhood of a point and graph definition**

Now that we know about short loops based at a point, we can understand the geometry (and topology) of balls on a tangle-free surface.

**Proposition 11.** Let \( L > 0 \), and \( X \) be a \( L \)-tangle-free hyperbolic surface. For a point \( z \) in \( X \), let \( B_{\delta}(z) := \{ w \in X : \text{dist}_X(z, w) < \frac{L}{\delta} \} \). Then, \( B_{\delta}(z) \) is isometric to a ball in either the hyperbolic plane (whenever the injectivity radius at \( z \) is \( \geq \frac{L}{\delta} \)) or a hyperbolic cylinder.

In the second case, since the injectivity radius at \( z \) is greater than \( \frac{L}{\delta} \), the ball \( B_{\delta}(z) \) is not contractible on \( X \); it is therefore homeomorphic to a cylinder (see Fig. 7). In a sense, this corollary proves that our notion of tangle-free implies the natural translation of the notion of tangle-free for graphs. Indeed, the ball \( B_{\delta}(z) \) has
either no non-contractible geodesic loop, or only one (and its iterates). We could have picked Proposition 11 to be a definition for tangle-free, but we consider the pair of pants definition to be both convenient to use and natural in the context of hyperbolic geometry and the Weil-Petersson model.

Proof. In order to prove this result, we will work in the universal cover $\mathcal{H}$ of $X$. Let us write $X = H/\Gamma$, for a co-compact Fuchsian group $\Gamma$.

Let $z$ be a point on $X$ of injectivity radius smaller than $\frac{L}{8}$ (otherwise, the conclusion is immediate). Then, the shortest geodesic loop $\beta$ based at $z$ satisfies $\ell(\beta) < \frac{L}{4}$.

Let $\tilde{z} \in \mathcal{H}$ be a lift of $z$, $\tilde{\beta}$ be a lift of $\beta$ starting at $\tilde{z}$, and $\tilde{B}$ be the ball of radius $\frac{L}{8}$ around $\tilde{z}$ in $\mathcal{H}$. Let $T_\beta \in \Gamma$ be the covering transformation corresponding to $\beta$. The quotient $C = \mathcal{H}/\langle T_\beta \rangle$ is a hyperbolic cylinder. The ball $\tilde{B}$ is projected on a ball $B_0$ on $C$. Let us prove that the projection from $B_0$ on $C$ to $B$ on $X$ is an isometry.

In order to do so, we shall establish that for any $\tilde{w} \in \tilde{B}$, the set of transformations $T \in \Gamma$ such that $T \cdot \tilde{w} \in \tilde{B}$ is included in $\langle T_\beta \rangle$. Since any two points in $\tilde{B}$ are at a distance at most $\frac{L}{4}$, this will follow from proving

$$\Gamma_L(\tilde{w}) := \left\{ T \in \Gamma : \text{dist}_{\mathcal{H}}(\tilde{w}, T \cdot \tilde{w}) < \frac{L}{2} \right\} \subset \langle T_\beta \rangle.$$  

Let $c$ be the shortest path from $\tilde{w}$ to $\tilde{z}$. The path $c \tilde{\beta} (T_\beta \circ c^{-1})$ is a path from $\tilde{w}$ to $T_\beta \cdot \tilde{w}$. Its length is $2\ell(c) + \ell(\beta) < 2 \times \frac{L}{8} + \frac{L}{4} = \frac{L}{2}$. As a consequence, $T_\beta$ belongs in $\Gamma_L(\tilde{w})$. Then, $\Gamma_L(\tilde{w})$ is not reduced to $\{\text{id}\}$. By Corollary 10, it is included in a cyclic subgroup $\langle T_0 \rangle$. $T_\beta$ hence is a power of $T_0$, but $T_\beta$ is primitive. Therefore, $T_\beta = T_0^{\pm 1}$, and the conclusion follows. \qed
3.5 Short geodesics are simple

**Corollary 12.** Let $L > 0$, and $X$ be a $L$-tangle-free hyperbolic surface. Any primitive closed geodesic on $X$ of length $< L$ is simple.

This consequence of Theorem 9 can also be deduced from the fact that the shortest non-simple primitive closed geodesic on a compact hyperbolic surface is a figure eight geodesic [12, Theorem 4.2.4], which is embedded in a pair of pants or one-holed torus.

**Proof.** Let us assume by contradiction that $\gamma$ is not simple; we can then pick an intersection point $z$. This allows us to write $\gamma$ as the product of two geodesic loops $\gamma_1, \gamma_2$ based at $z$. Since $\ell(\gamma_1) + \ell(\gamma_2) < L$, one of them is $< L/2$. Up to a change of notation, we take it to be $\gamma_1$.

Let $\delta_z$ be the shortest geodesic loop based at $z$. By definition, $\ell(\delta_z) \leq \ell(\gamma_1)$. So $\gamma_1$ and $\gamma_2$ both satisfy the length hypothesis of Theorem 9:

\[
\ell(\gamma_1) + \ell(\delta_z) \leq 2\ell(\gamma_1) < L
\]
\[
\ell(\gamma_2) + \ell(\delta_z) \leq \ell(\gamma) < L.
\]

Therefore, they are both homotopic with fixed endpoints to powers of $\delta_z$, which implies $\gamma$ is too. So $\gamma$ is freely homotopic to a power $j$ of the simple closed geodesic $\gamma_0$ in the free homotopy class of $\delta_z$. By uniqueness, $\gamma = \gamma_0^j$. $\gamma$ is primitive, so $j = 0$ or 1. But $\gamma$ is not contractible (so $j \neq 0$) and not simple (so $j \neq 1$): we reach a contradiction, which allows us to conclude.

**Remark.** Put together, Corollary 12 and 7 imply that all primitive closed geodesics of length $< L/2$ are simple and disjoint. Any such family of curves has cardinality at most $2g - 2$. But we know that the number of primitive closed geodesics of length $< L/2$ on a fixed surface is equivalent to $\frac{2}{L} e^{\frac{L}{2}}$ as $L \to +\infty$ [16, 12]. This can be seen as another indicator that, if $X$ is $L$-tangle-free of large genus, then we expect $L$ to be at most logarithmic in $g$.

4 Any surface of genus $g$ is $(4 \log g + O(1))$-tangled

We recall that any surface is $L$-tangled for $L = \frac{3}{2} B_g$, the Bers constant, because it can be entirely decomposed in pairs of pants of maximal boundary length smaller than $B_g$. The best known estimates on the Bers constant $B_g$ are linear in the genus $g$ [13, 30], which is pretty far off the $c \log g$ we obtained for random surfaces. This is not a surprise, because in order to prove that a surface is tangled, we only need to find one embedded pair of pants or one-holed torus. In Buser and Parlier’s estimates on $B_g$ [12, 30], the pair of pants decomposition is constructed by successively exhibiting short curves on the surface; the first ones are of length $\simeq \log g$, but as the construction goes on, and we find $2g - 2$ curves to entirely cut the surface, a linear factor appears.
In our case, we only need to stop the construction as soon as we manage to separate a pair of pants. Following Parlier’s approach in [30] to bound the Bers constant, we prove the following.

**Proposition 13.** There exists a constant $C > 0$ such that, for any $g \geq 2$, any compact hyperbolic surface of genus $g$ is $L$-tangled for $L = 4 \log g + C$.

This goes to prove that random hyperbolic surfaces are almost optimally tangle-free, despite the possibility of having a small injectivity radius.

The proof relies on the following two Lemmas, which are all used by Parlier [30]. Lemma 14, due to Bavard [6], allows us to find a small geodesic loop on our surface.

**Lemma 14.** Let $X$ be a hyperbolic surface of genus $g$. For any $z \in X$, the length of the shortest geodesic loop through $z$ is smaller than

$$2 \arccosh \left( \frac{1}{2 \sin \frac{\pi}{12g-6}} \right) = 2 \log g + O(1).$$

Some problems will arise in the proof if the geodesic loop we obtain using this result is too small. These difficulties can be solved by assuming a lower bound on the injectivity radius of the surface; for instance, for random surfaces, with high probability, one can assume that the injectivity radius is bounded below by $g^{-c}$ for a $c > 0$ [24]. However, such an assumption makes the final inequality weaker.

Another way to fix this issue, used in [30], is to expand all the small geodesics, and by this process obtain a new surface, with an injectivity radius bounded below, and in which the lengths of all the curves are longer. For our purposes, we only need to expand one curve. This is achieved by the following Lemma.

**Lemma 15** (Theorem 3.2 in [29]). Let $S_{g,n}$ be a base surface with $n > 0$ boundary components. Let $(X,f) \in T_{g,n}(\ell_1,\ldots,\ell_n)$ and $\varepsilon_1,\ldots,\varepsilon_n \geq 0$. Then, there exists a marked hyperbolic surface $(\tilde{X},\tilde{f})$ in $T_{g,n}(\ell_1 + \varepsilon_1,\ldots,\ell_n + \varepsilon_n)$ such that, for any closed curve $c$ on the base surface $S_{g,n}$, $\ell_X(c) \leq \ell_{\tilde{X}}(c)$.

We are now able to prove the result.

**Proof.** Let $\gamma$ be the systole of $X$ which is necessarily simple. We cut the surface $X$ along this curve, and obtain a (possibly disconnected) hyperbolic surface $X_{\text{cut}}$ with two boundary components. By the extension Lemma (applied to both components separately if need be), there exists a surface $X_{\text{cut}}^+$ such that:

- the boundary components $\beta_1, \beta_2$ in $X_{\text{cut}}^+$ are of length $1 \leq \ell \leq 2 \log g + O(1)$.
- for any closed curve $c$ not intersecting $\gamma$, $\ell_{X_{\text{cut}}}(c) \leq \ell_{X_{\text{cut}}^+}(c)$.

We shall find a pair of pants in $X_{\text{cut}}^+$, and use the relationship between lengths in $X$ and $X_{\text{cut}}^+$ to conclude.
For $w > 0$, let us consider the $w$-neighbourhood of one component $\beta_1$ of the boundary of $X^+_{\text{cut}}$

$$C_w(\beta_1) = \{ z \in X^+_{\text{cut}} : \text{dist}(z, \beta_1) < w \}.$$ 

For small enough $w$, $C_w(\beta_1)$ is a half-cylinder. However, there is a $w$ at which this isometry stops. This $w$ can be bounded by a volume argument: as long as $C_w$ is a half-cylinder,

$$\text{Vol}(C_w(\beta_1)) = \ell \sinh w \leq \text{Vol} X = 2\pi(2g - 2).$$

However, $\ell \geq 1$. It follows that $w \leq \log g + O(1)$.

There are two reasons for this isometry to stop.

- The half-cylinder self-intersects inside the surface (see Fig. 3a). Then, one can construct an embedded pair of pant on $X^+_{\text{cut}}$, of total boundary length $\leq 2\ell + 4w$. This pair of pant will also be one on $X$, with shorter boundary components.

- The half-cylinder reaches the boundary of $X^+_{\text{cut}}$. It can only do so by intersecting the component $\beta_2$. Then, one can construct an embedded pair of pant on $X^+_{\text{cut}}$ of boundaries shorter than $\ell$, $\ell$, and $2\ell + 2w$, which corresponds to a one-holed torus on $X$, of boundary shorter than $2\ell + 2w$ (see Fig. 3b, but expanding only a half-cylinder).

We can conclude that the surface $X$ is $L$-tangled, for $L = \ell + 2w \leq 4\log g + O(1)$.

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