TWISTED TOROIDAL LIE ALGEBRAS AND
MOODY-RAO-YOKONUMA PRESENTATION

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Abstract. Let \( \mathfrak{g} \) be an affine Kac-Moody algebra, and \( \mu \) a diagram automorphism of \( \mathfrak{g} \). In this paper, we give an explicit realization for the universal central extension \( \hat{\mathfrak{g}}[\mu] \) of the twisted loop algebra of \( \mathfrak{g} \) related to \( \mu \), which provides a Moody-Rao-Yokonuma presentation for the algebra \( \hat{\mathfrak{g}}[\mu] \) when \( \mu \) is non-transitive, and the presentation is indeed related to the quantization of toroidal Lie algebras.

1. Introduction

Let \( \mathfrak{g} \) be a (twisted or untwisted) affine Kac-Moody algebra (without derivation), and \( \bar{\mathfrak{g}} \) the quotient algebra of \( \mathfrak{g} \) modulo its center. When \( \mathfrak{g} \) is of untwisted type, the universal central extension \( \hat{\mathfrak{g}} \) of the loop algebra \( \mathbb{C}[t_1, t_1^{-1}] \otimes \bar{\mathfrak{g}} \) is called a toroidal Lie algebra. This algebra was first introduced by Moody-Rao-Yokonuma in [MRY], where the authors introduced the famous Moody-Rao-Yokonuma (MRY) presentation. The presentation makes it more effective to study representations of toroidal Lie algebras in a manner similar to that of untwisted affine Lie algebras [MRY, T1, T2, FJW, JM, JMX, C, JMT]. Moreover, it turns out that the classical limit of the quantum toroidal algebra is just the MRY presentation of the toroidal Lie algebra ([GKV, J]).

Let \( \mu \) be a diagram automorphism of \( \mathfrak{g} \) of order \( N \), and \( \bar{\mu} \) the automorphism on \( \bar{\mathfrak{g}} \) induced from \( \mu \). The twisted loop algebra \( \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) \) of \( \bar{\mathfrak{g}} \) is defined as follows

\[
\mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}t_1^n \otimes \bar{\mathfrak{g}}(n),
\]

where \( \bar{\mathfrak{g}}(n) = \{ x \in \bar{\mathfrak{g}} \mid \bar{\mu}(x) = \xi^n x \} \) and \( \xi = e^{2\pi \sqrt{-1}/N} \). In this paper, we study the universal central extension \( \hat{\mathfrak{g}}[\mu] \) of \( \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) \), and give Moody-Rao-Yokonuma presentation for \( \hat{\mathfrak{g}}[\mu] \) when \( \mu \) is non-transitive. One may expect that the MRY

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presentation could be used to study the representation and quantization for the twisted toroidal extended affine Lie algebras ([MRY, GKV, J, BL]).

An extended affine Lie algebra (EALA) is a complex Lie algebra, together with a non-zero finite dimensional Cartan subalgebra and a non-degenerate invariant symmetric bilinear form, that satisfies a list of natural axioms ([H-KT, AABGP, N1]). The root system of an EALA is a disjoint union of isotropic and non-isotropic root systems, and the rank of the free abelian group generated by the isotropic root system is defined to be the nullity of the EALA ([AABGP]). Indeed, the nullity 0 EALAs are finite dimensional simple Lie algebras over the complex number field, and the nullity 1 EALAs are precisely the affine Kac-Moody algebras ([ABGP]). We remark that the nullity 2 EALAs are the most important class of EALAs other than the finite dimensional simple Lie algebras and affine Kac-Moody algebras, which are closely related to the singularity theory studied by Saito and Slodowy ([Sa, Sl]). And the nullity 2 EALAs are classified in [ABP2] (also see [GP]).

For a given EALA $\mathfrak{L}$, the subalgebra of $\mathfrak{L}$ generated by the set of non-isotropic root vectors is called the core of $\mathfrak{L}$ ([AABGP]). We denote by $E_2$ the class of all Lie algebras that are isomorphic to the centerless cores (cores modulo their centers) of EALAs with nullity 2. Let $\mathfrak{sl}_n(\mathbb{C}_q)$ ($n \geq 2$) be the special linear Lie algebra over the quantum torus $\mathbb{C}_q$ in two variables ([BGK]). It was proved in [ABP2] that any Lie algebra in $E_2$ is either isomorphic to $\mathfrak{sl}_n(\mathbb{C}_q)$ with $q \in \mathbb{C}^\times$ not a root of unity, or isomorphic to a Lie algebra of the form $\mathfrak{L}(\bar{g}, \mu)$ with $\mu$ non-transitive. The universal central extension $\hat{\mathfrak{sl}}_n(\mathbb{C}_q)$ of $\mathfrak{sl}_n(\mathbb{C}_q)$ is given in [BGK], and its MRY presentation is obtained in [VV]. The purpose of this paper is to study the universal central extension $\hat{\mathfrak{g}}[\mu]$ of $\mathfrak{L}(\bar{g}, \mu)$, and the MRY presentation for $\hat{\mathfrak{g}}[\mu]$ with non-transitive diagram automorphism $\mu$.

This paper is organized as follows. In Section 2, we recall some facts for the affine Kac-Moody algebras which will be used later on. In Section 3, we show that any diagram automorphism $\mu$ of an affine Kac-Moody algebra $\mathfrak{g}$ can be lifted to an automorphism $\hat{\mu}$ for the universal central extension $\hat{\mathfrak{g}}$ of $\mathfrak{L}(\bar{g}, \text{id})$. The Lie subalgebra of $\hat{\mathfrak{g}}$ fixed by $\hat{\mu}$ is denoted by $\hat{\mathfrak{g}}[\mu]$. We claim that $\hat{\mathfrak{g}}[\mu]$ is the universal central extension of $\mathfrak{L}(\bar{g}, \mu)$ (Theorem 3.3), and state the MRY presentation for $\hat{\mathfrak{g}}[\mu]$ with $\mu$ non-transitive (Theorem 3.6). Section 4 and Section 5 are devoted to the proofs of Theorems 3.3 and 3.6.

We denote the sets of non-zero complex numbers, non-zero integers, and positive integers respectively by $\mathbb{C}^\times$, $\mathbb{Z}^\times$ and $\mathbb{Z}_+$. For $M \in \mathbb{Z}_+$, we set $\xi_M = e^{2\pi \sqrt{-1}/M}$ and $\mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$.

2. Diagram automorphisms of affine Kac-Moody algebras

2.1. Affine Kac-Moody algebras. In this subsection, we collect some basics about affine Kac-Moody algebras that will be used later on.

Let $A = (a_{ij})_{i,j=0}^\infty$ be a generalized Cartan matrix (GCM) of affine type, and $\mathfrak{g}$ the affine Kac-Moody algebra (without derivation) associated to the GCM $A$. We
denote the set \( \{0, 1, \ldots, \ell\} \) by \( I \). By definition, the Lie algebra \( \mathfrak{g} \) is generated by the Chevalley generators \( \alpha_i^\vee, e_i^\pm, i \in I \) with the defining relations \((i, j \in I)\)
\[
[a_i^\vee, a_j^\vee] = 0, \quad [a_i^\vee, e_j^\pm] = \pm a_{ij} e_j^\pm, \quad [e_i^+, e_j^-] = \delta_{ij} a_i^\vee, \quad \text{ad}(e_i^\pm)^{1-a_{ij}}(e_j^\pm) = 0 \quad (i \neq j).
\]

Let \( \Delta \) be the root system (including 0) of \( \mathfrak{g} \), \( \Delta^\times \) the set of real roots in \( \Delta \), and \( \Delta^0 = \Delta \setminus \Delta^\times = \mathbb{Z}\delta_2 \) the set of imaginary roots in \( \Delta \). Then \( \mathfrak{g} \) has a root space decomposition \( \mathfrak{g} = \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \). Let \( P = \{\alpha_i, i \in I\} \) be the simple root system of \( \mathfrak{g} \) such that \( e_i^\pm \in \mathfrak{g}_{\pm \alpha_i} \) for \( i \in I \), and \( \mathcal{Q} = \oplus_{i \in I} \mathbb{Z} \alpha_i \) the lattice of \( \mathfrak{g} \). Then the root space decomposition naturally induces a \( \mathcal{Q} \)-grading on \( \mathfrak{g} \). In addition, let \( \mathfrak{g} \) be the quotient algebra of \( \mathfrak{g} \) modulo its center. Then the \( \mathcal{Q} \)-grading on \( \mathfrak{g} \) naturally induces a \( \mathcal{Q} \)-grading \( \mathfrak{g} = \oplus_{\alpha \in \mathcal{Q}} \mathfrak{g}_\alpha \) on \( \mathfrak{g} \).

Now we recall the twisted loop realization of the affine Kac-Moody algebra \( \mathfrak{g} \) (see [K, Chapters 7 and 8]). Using the notations given in [K, Chapter 4, Table Aff 1-3], we assume that the GCM \( A \) is of type \( X_n^{(r)} \).

We start with a finite dimensional simple Lie algebra \( \hat{\mathfrak{g}} \) of type \( X_n \). Let \( \hat{\alpha}_i^\vee, \hat{E}_i^\pm \) \((i = 1, 2, \ldots, n)\) be the Chevalley generators of \( \hat{\mathfrak{g}} \), and \( \hat{\mathfrak{h}} = \oplus_{i=1}^n \mathbb{C} \hat{\alpha}_i^\vee \) a Cartan subalgebra of \( \hat{\mathfrak{g}} \). We denote by \( \hat{\Delta} \) the root system (containing 0) of \( \hat{\mathfrak{g}} \) with respect to \( \hat{\mathfrak{h}} \). Then \( \hat{\mathfrak{g}} \) has a root space decomposition \( \hat{\mathfrak{g}} = \oplus_{\alpha \in \hat{\Delta}} \hat{\mathfrak{g}}_\alpha \) such that \( \hat{\mathfrak{g}}_0 = \hat{\mathfrak{h}} \). Let \( \hat{\Pi} \) be a fixed simple root system of \( \hat{\Delta} \), and \( \hat{\Delta}_+ \) the set of positive roots with respect to \( \hat{\Pi} \). In addition, for each \( \hat{\alpha} \in \hat{\Delta}_+ \), there exist \( \hat{E}_\alpha^\pm \in \hat{\mathfrak{g}}_{\pm \alpha} \) and \( \hat{\alpha}^\vee \in \hat{\mathfrak{h}} \), such that \( \{\hat{E}_\alpha^+, \hat{\alpha}^\vee, \hat{E}_\alpha^-\} \) form a \( \mathfrak{sl}_2 \) triple. Moreover, for a simple root \( \hat{\alpha}_i \in \hat{\Pi} \), we assume that \( \hat{E}_{\hat{\alpha}_i}^\pm = \hat{E}_i^\pm \).

Let \( \nu \) be a diagram automorphism of \( \hat{\mathfrak{g}} \) of order \( r \). By definition, there exists a permutation \( \nu \) on the set \( \{1, 2, \ldots, n\} \), such that
\[
\nu(\hat{E}_i^\pm) = \hat{E}_{\nu(i)}^\pm \quad \text{and} \quad \nu(\hat{\alpha}_i^\vee) = \hat{\alpha}_{\nu(i)}^\vee \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

For each \( x \in \hat{\mathfrak{g}} \) and \( m \in \mathbb{Z} \), we set
\[
x_{[m]} = r^{-1} \sum_{p \in \mathbb{Z}_r} \xi_r^{-mp}\nu^p(x) \quad \text{and} \quad \hat{\mathfrak{g}}_{[m]} = \{x_{[m]} \mid x \in \hat{\mathfrak{g}}\}.
\]

And define the Lie algebra
\[
\text{Aff}(\hat{\mathfrak{g}}, \nu) = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} \mathfrak{l}_2^m \otimes \hat{\mathfrak{g}}_{[m]} \oplus \mathbb{C} \mathfrak{k}_2
\]

with the Lie bracket given by
\[
[\mathfrak{l}_2^m \otimes x + a_1 \mathfrak{k}_2, \mathfrak{l}_2^m \otimes y + a_2 \mathfrak{k}_2] = \mathfrak{l}_2^{m_1+m_2} \otimes [x, y] + (x, y)\delta_{m_1+m_2,0}m_1 \mathfrak{k}_2,
\]

where \( m_1, m_2 \in \mathbb{Z} \), \( x \in \hat{\mathfrak{g}}_{[m_1]}, y \in \hat{\mathfrak{g}}_{[m_2]} \), \( a_1, a_2 \in \mathbb{C} \) and \( \langle \cdot, \cdot \rangle \) is the normalized symmetric invariant bilinear form on \( \hat{\mathfrak{g}} \).
We denote by
\[
\hat{\theta} = \begin{cases} 
\text{the highest root of } \hat{\mathfrak{g}}, & \text{if } r = 1 \text{ or } X_n = A_{2\ell}, r = 2; \\
\hat{\alpha}_1 + \cdots + \hat{\alpha}_\ell, & \text{if } X_n = D_{\ell+1}, r = 2, 3; \\
\hat{\alpha}_1 + \cdots + \hat{\alpha}_{2\ell-2}, & \text{if } X_n = A_{2\ell-1}, r = 2; \\
\hat{\alpha}_1 + 2\hat{\alpha}_2 + 2\hat{\alpha}_3 + \hat{\alpha}_4 + \hat{\alpha}_5 + \hat{\alpha}_6, & \text{if } X_n = E_6, r = 2.
\end{cases}
\]
And for each \(i = 1, 2, \ldots, n\), we let \(r_i\) be the cardinality of the set \(\{\hat{\nu}^k(i) \mid k \in \mathbb{Z}_r\}\).

If the GCM \(A\) is of type \(A_{2\ell}^{(2)}\), we set
\[
E_i^{\pm} = r_i\hat{E}_{i[0]}^{\pm}, \quad E_\ell^{\pm} = \hat{E}_{\ell[1]}^{\pm}, \quad E_0^{\pm} = 2\sqrt{2}\hat{E}_{0[0]}^{\pm}, \quad H_i = r_i\hat{\alpha}_i^{\vee}, \quad H_\ell = -\hat{\theta}^{\vee}, \quad H_0 = 4\hat{\alpha}_0^{\vee},
\]
where \(i = 1, \ldots, \ell - 1\). Otherwise, we set
\[
E_i^{\pm} = r_i\hat{E}_{i[0]}^{\pm}, \quad H_i = r_i\hat{\alpha}_i^{\vee}, \quad E_0^{\pm} = r\hat{E}_{0[0]}^{\pm}, \quad H_0 = -r\hat{\theta}^{\vee}, \quad i = 1, \ldots, \ell.
\]
It was proved in [K, Theorem 8.3] that we could identify \(\mathfrak{g}\) with \(\text{Aff}(\hat{\mathfrak{g}}, \hat{\nu})\) by
\[
\alpha_i^{\vee} = r\alpha_i^{-1}k_2 + 1 \otimes H_0, \quad \epsilon^{\pm}_i = t^{\pm 1} \otimes E_i^{\pm}, \quad \alpha_i^{\vee} = 1 \otimes H_i, \quad \epsilon^{\pm}_i = 1 \otimes E_i^{\pm}, \quad i \neq \epsilon,
\]
where \(\epsilon = 0, a_0 = 1\) expect that the GCM \(A\) is of type \(A_{2\ell}^{(2)}\), in which case \(\epsilon = \ell, a_0 = 2\). From now on, we will often use the following identifications
\[
\mathfrak{g} = \text{Aff}(\hat{\mathfrak{g}}, \hat{\nu}) \quad \text{and} \quad \tilde{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} C\ell_{2m} \otimes \hat{\mathfrak{g}}_{[m]}
\]
without further explanation.

Let \(\hat{Q} = \bigoplus_{i=1}^{n} \hat{\alpha}_i\) be the root lattice of \(\hat{\mathfrak{g}}\). Note that \(\hat{\nu}\) induces an automorphism of \(\hat{Q}\) such that \(\hat{\nu}(\hat{\alpha}_i) = \hat{\alpha}_{\nu(i)}\) for \(i = 1, 2, \ldots, n\). Set
\[
\hat{Q}_{[0]} = \{\alpha_{[0]} = r^{-1} \sum_{p \in \mathbb{Z}_r} \hat{\nu}^p(\hat{\alpha}) \mid \hat{\alpha} \in \hat{Q}\} \subset \hat{\mathfrak{h}}^*.
\]
Then the root lattice \(Q\) of \(\mathfrak{g}\) is equivalent to \(\hat{Q}_{[0]} \oplus \mathbb{Z}\delta_2\) and the simple root system \(\Pi\) of \(\mathfrak{g}\) is equivalent to
\[
\{\alpha_\ell = -\hat{\theta}_{[0]} + \delta_2, \quad \alpha_{\ell-\epsilon} = \hat{\alpha}_{\ell[0]}, \quad \alpha_i = \hat{\alpha}_{i[0]}, \quad i \neq \epsilon, \ell - \epsilon\}.
\]
We extend the normalized bilinear form \(\langle \cdot, \cdot \rangle\) on \(\hat{\mathfrak{g}}\) to a symmetric invariant bilinear form on \(\mathfrak{g}\) by letting
\[
\langle t_{m_1}^{m_1} \otimes x + a_1k_2, t_{m_2}^{m_2} \otimes y + a_2k_2 \rangle = \delta_{m_1+m_2,0} \langle x, y \rangle,
\]
where \(m_1, m_2 \in \mathbb{Z}, \ x \in \hat{\mathfrak{g}}_{[m_1]}, \ y \in \hat{\mathfrak{g}}_{[m_2]}\) and \(a_1, a_2 \in \mathbb{C}\). Since the restriction of \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{h}\) is non-degenerate, we get a non-degenerate bilinear form \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{h}^*\) by duality. In addition, the bilinear form \(\langle \cdot, \cdot \rangle\) can be extend to a symmetric bilinear form on \(Q\) by letting
\[
\langle \alpha + m\delta_2, \beta + n\delta_2 \rangle = \langle \alpha, \beta \rangle,
\]
where \(\alpha, \beta \in \hat{Q}_{[0]}\) and \(m, n \in \mathbb{Z}\).
2.2. **Diagram automorphisms.** Throughout this paper, we let $\mu$ be a permutation of $I$ with order $N$ such that $a_{ij} = a_{\mu(i)\mu(j)}$ for $i, j \in I$. It is known that $\mu$ induces a diagram automorphism $\mu$ of $\mathfrak{g}$ such that

$$
\mu(\alpha^i_j) = \alpha^i_{\mu(j)} , \quad \mu(e^i_{\mu(i)}) = e^i_{\mu(i)} , \quad i \in I.
$$

(2.4)

This subsection is devoted to an explicit description of the action of $\mu$ on $\mathfrak{g}$.

It is immediate to see that the permutation $\mu$ induces an automorphism of $Q$ such that $\mu(\delta_2) = \delta_2$. Recall from [K, Proposition 8.3] that the finite dimensional simple Lie algebra $\mathfrak{g}$ can be generated by the elements $E^i_+$, $i \in I$ defined in (2.1). Then we have that

**Lemma 2.1.** (a) The action

$$
E^i_+ \mapsto E^i_{\mu(i)} , \quad i \in I,
$$

(2.5)

defines (uniquely) an automorphism $\hat{\mu}$ of $\hat{\mathfrak{g}}$.

(b) The Cartan subalgebra $\mathfrak{h}$ of $\hat{\mathfrak{g}}$ is stable under $\hat{\mu}$, and

$$
\hat{\mu}(\hat{\nu}(h)) = \hat{\nu}(\hat{\mu}(h)), \quad \forall h \in \mathfrak{h}.
$$

(2.6)

(c) There is a homomorphism $\rho_\mu : \hat{Q} \to \mathbb{Z}$ of abelian groups such that

$$
\rho_\mu(\hat{\nu}(\hat{\alpha})) = \rho_\mu(\hat{\alpha}), \quad \mu(\hat{\alpha}|0) = \hat{\mu}(\hat{\alpha}|0) + \rho_\mu(\hat{\alpha})\delta_2, \quad \forall \alpha \in \hat{Q}.
$$

(2.7)

(d) For $\alpha \in \Delta$, $x \in \hat{\mathfrak{g}}_\alpha$ and $m \in \mathbb{Z}$, we have that

$$
\hat{\mu}(x|_m) = \hat{\mu}(x|_{m + \rho_\mu(\hat{\alpha})}).
$$

(2.8)

Proof. We first consider the case $\hat{\nu} = \text{id}$. For each $\hat{\alpha} \in \hat{Q}$, write

$$
\mu(\hat{\alpha}) = \hat{\mu}(\hat{\alpha}) + \rho_\mu(\hat{\alpha})\delta_2 \quad \text{with} \quad \hat{\mu}(\hat{\alpha}) \in \hat{Q}.
$$

Then the map

$$
\hat{\mu} : \hat{Q} \to \hat{Q}, \quad \hat{\alpha} \mapsto \hat{\mu}(\hat{\alpha})
$$

is an automorphism of $\hat{Q}$ (with order $N$) and the map

$$
\rho_\mu : \hat{Q} \to \mathbb{Z}, \quad \hat{\alpha} \mapsto \rho_\mu(\hat{\alpha})
$$

is a homomorphism of abelian groups. We define a linear map $\hat{\mu}$ on $\hat{\mathfrak{g}}$ as follows

$$
\hat{\mu} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}, \quad \hat{E}^+_\hat{\alpha} \mapsto \hat{\mu}(\hat{E}^+_\hat{\alpha}), \quad \hat{\alpha}^+ \mapsto \hat{\mu}(\hat{\alpha}^+), \quad \forall \alpha \in \hat{\Delta}^+,
$$

where $\hat{\mu}(\hat{E}^+_\hat{\alpha})$ are the elements in $\hat{\mathfrak{g}}_{\pm \hat{\mu}(\hat{\alpha})}$ determined by the following equation

$$
\mu(1 \otimes \hat{E}^+_\hat{\alpha}) = \iota^\rho_\mu(\pm \hat{\alpha}) \otimes \hat{\mu}(\hat{E}^+_\hat{\alpha}).
$$

It is easy to see that $\hat{\mu}$ is an automorphism of $\hat{\mathfrak{g}}$ (with order $N$). Moreover, one can check that the automorphism $\hat{\mu}$ and the homomorphism $\rho_\mu$ defined above satisfy all the assertions in the lemma.

Next we consider the case that $\hat{\nu} \neq \text{id}$. If $\mu = \text{id}$, then we only need to take $\hat{\mu} = \text{id}$ and $\rho_\mu = 0$. So we assume further that $\mu$ is nontrivial. Then either
that, if \( X \) and if \( X \) functional on \( \dot{\omega} \) satisfies the properties (2.5) and (2.6). This proves the assertion (a) and (b).

It is obvious that the property (2.7) holds true for all \( \dot{\omega} \). Finally, it can be checked case by case that, the property (2.8) holds true for every \( m \).

Let \( \dot{\omega} \) be as in Lemma 2.1. Since the bilinear form \( \langle \cdot, \cdot \rangle \) is non-degenerated on \( \mathfrak{h} \), we may and do identify \( \mathfrak{h} \) with its dual space \( \mathfrak{h}^* \), and extend \( \rho_\mu \) to a linear functional on \( \mathfrak{h} \) by \( \mathbb{C} \)-linearity. The following result is an explicit description of the action of the diagram automorphism \( \mu \).
Moreover, the commutator relations among these elements are as follows:

\[
(3.1)
\]

Proof. Using Lemma 2.1 and the identification (2.2), one can check that the action given in (2.9) defines an automorphism of \( \hat{g} \) such that the equation (2.4) holds, as desired. \( \square \)

3. The Lie algebra \( \hat{g}[\mu] \) and its MRY presentation

In this section, we define the twisted toroidal Lie algebra \( \hat{g}[\mu] \) and state its Moody-Rao-Yokonuma presentation.

3.1. The Lie algebra \( \hat{g}[\mu] \). In this subsection, we introduce the definition of the Lie algebra \( \hat{g}[\mu] \).

For \( M_1, M_2 \in \mathbb{Z}_+ \), let \( \mathcal{K}_{M_1, M_2} \) be the \( \mathbb{C} \)-vector space spanned by the symbols

\[
t_1^{m_1}t_2^{m_2}k_1, \quad n_1^{m_1}n_2^{m_2}k_2, \quad m_1 \in M_1\mathbb{Z}, \quad m_2 \in M_2\mathbb{Z},
\]

subject to the relation

\[
m_1^{n_1}n_2^{m_2}k_1 + m_2^{n_1}n_2^{m_2}k_2 = 0.
\]

We define

\[
\hat{g} = \bigoplus_{m,n \in \mathbb{Z}} \mathbb{C}t_1^{m}t_2^{n} \otimes \hat{g}[n] \oplus \mathcal{K}_{1, r} \subset (\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \otimes \hat{g}) \oplus \mathcal{K}_{1, r}
\]

to be a Lie algebra with Lie bracket given by

\[
\left[ t_1^{m_1}t_2^{n_2} \otimes x, t_1^{m_2}t_2^{n_2} \otimes y \right] = t_1^{m_1+n_1}t_2^{m_2+n_2} \otimes [x, y] + \langle x, y \rangle \left( \sum_{i=1}^{2} m_i t_1^{m_1+n_1}t_2^{m_2+n_2}k_i \right),
\]

where \( x \in \hat{g}[m_1], \ y \in \hat{g}[n_2], \ m_1, m_2, n_1, n_2 \in \mathbb{Z} \) and \( \mathcal{K}_{1, r} \) is the center space. It follows from MRY SM that the projective map

\[
\psi : \hat{g} \to \bigoplus_{m,n \in \mathbb{Z}} \mathbb{C}t_1^{m}t_2^{n} \otimes \hat{g}[n] = \mathbb{C}[t_1, t_1^{-1}] \otimes \hat{g}
\]

is the universal central extension of the loop algebra \( \mathcal{L}(\hat{g}, \text{id}) \) of \( \hat{g} \).

For convenience, we view \( \mathbb{C}[t_1, t_1^{-1}] \otimes g \) as a subspace of \( \hat{g} \) in the following way

\[
t_1^{m_1} \otimes x = t_1^{m_1}t_2^{m_2} \otimes \hat{x} + at_1^{m_1}k_2,
\]

for \( x = t_2^{m_2} \otimes \hat{x} + ak_2 \in g, \ m_1 \in \mathbb{Z} \). Then it is easy to see that the Lie algebra \( \hat{g} \) is spanned by the elements

\[
t_1^{m_1} \otimes x, \quad k_1, \quad n_1^{m_1}n_2^{m_2}k_1, \quad x \in g, \ m_1, n_1 \in \mathbb{Z}, \ n_2 \in r\mathbb{Z}^\times.
\]

Moreover, the commutator relations among these elements are as follows:
Thus, by taking the restriction of (3.6)

It follows from (3.5) that the automorphism of presented in Section 4.

The following theorem is the first main result of this paper, whose proof will be presented in Section 4.

\[ [t_1^{m_1} \otimes x, t_1^{n_1} \otimes y] = t_1^{m_1+n_1} \otimes [x, y] + m_1 \delta_{m_1,n_1} \langle x, y \rangle k_1. \]

If \( x = t_2^{m_2} \otimes \dot{x}, y = t_2^{n_2} \otimes \dot{y} \) and \( \alpha + \beta \in \Delta^0 \setminus \{0\} \), then

\[ [t_1^{m_1} \otimes x, t_1^{n_1} \otimes y] = t_1^{m_1+n_1} \otimes [x, y] + \langle \dot{x}, \dot{y} \rangle \frac{m_1 n_2 - m_2 n_1}{m_2 + n_2} t_1^{m_1+n_1} t_2^{m_2+n_2} k_1. \]

Observe that the Lie algebra \( \hat{g} \) is generated by the elements

\[ t_1^m \otimes e_i^\pm, \quad t_1^m \otimes \alpha_i^\pm, \quad k_i, \quad i \in I, \ m \in \mathbb{Z}. \]

Similar to (2.4), the permutation \( \mu \) induces an automorphism of \( \hat{g} \) as follows.

Lemma 3.2. The action

\[ t_1^m \otimes e_i^\pm \mapsto \xi^{-m} t_1^m \otimes e_{\mu(i)}^\pm, \quad t_1^m \otimes \alpha_i^\pm \mapsto \xi^{-m} t_1^m \otimes \alpha_{\mu(i)}^\pm, \quad k_i \mapsto k_i \]

for \( i \in I, \ m \in \mathbb{Z} \), defines an automorphism \( \hat{\mu} \) of \( \hat{g} \).

Proof. We define a linear transformation \( \hat{\mu} \) on \( \hat{g} \) by letting

\[
\begin{align*}
t_1^m \otimes x &\mapsto \xi^{-m} t_1^m \otimes \mu(x), \\
t_1^m \otimes h &\mapsto \xi^{-m} (t_1^m \otimes \mu(h)) - \frac{m_1}{m_2} \rho_\mu(h) t_1^m t_2^m k_1, \\
k_1 &\mapsto k_1, \quad t_1^m t_2^m k_1 \mapsto \xi^{-m} t_1^m t_2^m k_1,
\end{align*}
\]

where \( m_1, n_1 \in \mathbb{Z}, \ x \in \mathfrak{g}_\alpha, \alpha \in \Delta^\times \cup \{0\}, \ h = t_2^{m_2} \otimes \hat{h}, \ m_2 \in \mathbb{Z}^\times, \ n_2 \in r\mathbb{Z}^\times \) and \( \hat{h} \in \mathfrak{h}_{|m_2|}. \) Note that \( \rho_\mu(h) \neq 0 \) only if \( m_2 \in r\mathbb{Z} \), and so \( \hat{\mu} \) is well-defined.

By using the explicit action of \( \mu \) given in Proposition 2.2 and the commutate relations of \( \hat{g} \) given in Lemma 3.1 one can easily verify that the map \( \hat{\mu} \) is an automorphism of \( \hat{g} \). Moreover, it is obvious that the actions of \( \hat{\mu} \) on those generators in (3.4) coincide with that in (3.5). This completes the proof.

We define \( \hat{g}[\mu] \) to be the subalgebra of \( \hat{g} \) fixed by \( \hat{\mu} \). Recall from Introduction that \( \hat{\mu} \) is the automorphism of \( \hat{g} \) induced from \( \mu \), and that \( \mathcal{L}(\hat{g}, \hat{\mu}) \) is the twisted loop algebra of \( \hat{g} \) related to \( \hat{\mu} \). Note that \( \mathcal{L}(\hat{g}, \hat{\mu}) \) is the subalgebra of \( \mathcal{L}(\hat{g}, \text{id}) \) fixed by the automorphism

\[ \xi^{-d_1} \otimes \hat{\mu} : \mathcal{L}(\hat{g}, \text{id}) \rightarrow \mathcal{L}(\hat{g}, \text{id}), \quad t_1^m \otimes x \mapsto \xi^{-m} t_1^m \otimes \hat{\mu}(x), \ m \in \mathbb{Z}, \ x \in \hat{g}. \]

It follows from (3.5) that

\[ \psi \circ \hat{\mu} = (\xi^{-d_1} \otimes \hat{\mu}) \circ \psi. \]

Thus, by taking the restriction of \( \psi \) on \( \hat{g}[\mu] \), one gets a Lie algebra homomorphism

\[ \psi_\mu = \psi|_{\hat{g}[\mu]} : \hat{g}[\mu] \rightarrow \mathcal{L}(\hat{g}, \hat{\mu}). \]
Theorem 3.3. The Lie algebra homomorphism \( \psi_\mu : \hat{\mathfrak{g}}[\mu] \to L(\hat{\mathfrak{g}}, \bar{\mu}) \) is the universal central extension of the twisted loop algebra \( L(\hat{\mathfrak{g}}, \bar{\mu}) \).

3.2. The MRY presentation. Here we state an MRY presentation for \( \hat{\mathfrak{g}}[\mu] \).
Throughout this subsection, we may always assume that \( \mu \) is non-transitive. Observe that a diagram automorphism on \( \mathfrak{g} \) is transitive if and only if \( \mathfrak{g} \) is of type \( A_\ell^{(1)} \) \( (\ell \geq 1) \), and the diagram automorphism is an order \( \ell + 1 \) rotation of the Dynkin diagram.

We first introduce some notations. Set \( V = \mathbb{R} \otimes_{\mathbb{Z}} Q \) and extend \( (\cdot, \cdot) \) (see (2.3)) to a bilinear form on \( V \) by \( \mathbb{R} \)-linearity. For \( i, j \in I \), we set
\[
\check{\alpha}_i = \frac{1}{N} \sum_{k \in \mathbb{Z}_N} \alpha_{\mu^k(i)} \quad \text{and} \quad \check{a}_{ij} = 2 \frac{(\check{\alpha}_i, \check{\alpha}_j)}{(\check{\alpha}_i, \check{\alpha}_i)}.
\]
We fix a representative subset of \( I \) as follows
\[
\check{I} = \{ i \in I | \mu^k(i) \geq i \text{ for } k \in \mathbb{Z}_N \}.
\]
It was proved in [ABP2, Proposition 12.1.10] (see also [FSS]) that the folded matrix \( \check{A} = (\check{a}_{ij})_{i,j \in \check{I}} \) of the GCM \( A \) associated to \( \mu \) is also a GCM of affine type.

For \( i \in I \), we denote by \( \mathcal{O}(i) \subset I \) the orbit containing \( i \) under the action of the group \( \langle \mu \rangle \). The following result was proved in [ABP2, Lemma 12.1.5].

Lemma 3.4. For each \( i \in I \), exactly one of the following holds
(a) The elements \( \alpha_p, p \in \mathcal{O}(i) \) are pairwise orthogonal;
(b) \( \mathcal{O}(i) = \{ i, \mu(i) \} \) and \( a_{i\mu(i)} = -1 = a_{\mu(i)i} \).

As in [ABP2], for \( i \in I \), we set
\[
s_i = \begin{cases} 
1, & \text{if (a) holds in Lemma 3.4;} \\
2, & \text{if (b) holds in Lemma 3.4.}
\end{cases}
\]
Now we introduce the following definition.

Definition 3.5. Define \( M(\mathfrak{g}, \mu) \) to be the Lie algebra generated by the elements
\[
\begin{align*}
&h_{i,m}, \quad x_{i,m}^\pm, \quad c, \quad i \in I, \ m \in \mathbb{Z},
\end{align*}
\]
and subject to the relations
\[
\begin{align*}
&T0 \quad h_{\mu(i),m} = \xi^m h_{i,m}, \quad x_{\mu(i),m}^\pm = \xi^m x_{i,m}^\pm, \\
&T1 \quad [c, h_{i,n}] = 0 = [c, x_{i,n}^\pm], \\
&T2 \quad [h_{i,m}, h_{j,n}] = \sum_{k \in \mathbb{Z}_N} mN(\alpha_{\mu^k(j)}^\vee, \alpha_{\mu^k(i)}^\vee) \delta_{m+n,0} \xi^k c, \\
&T3 \quad [h_{i,m}, x_{j,n}^\pm] = \pm \sum_{k \in \mathbb{Z}_N} a_{i\mu^k(j)} x_{j,m+n}^\pm \xi^k c.
\end{align*}
\]
(T4) \([x_{i,m}^+, x_{j,n}^-] = \sum_{k \in \mathbb{Z}_N} \delta_{i,m}^{\mu(k)} \left( h_{j,m+n} + \frac{m N(\alpha_i^\vee, \alpha_l^\vee)}{2} \delta_{m,n,0} \right) \xi^{km}, \)

(T5) \((\text{ad} x_{i,0}^+)^{1-\delta_{ij}} (x_{j,m}^+) = 0, \) if \(\delta_{ij} \leq 0, \)

(T6) \([x_{i,m_1}^+, \ldots, [x_{i,m_s}^+, x_{i,m_{s+1}}]] = 0. \)

In view of (3.4) and (3.5), we know that the Lie algebra \(\hat{g}[\mu] \) is generated by the following elements

\[(3.8) \quad t_i^m \otimes e_i^\pm, \quad t_i^m \otimes \alpha^\vee_i \otimes, \quad k_1, \quad i \in I, \quad m \in \mathbb{Z}, \]

where \(x(m) = \sum_{p \in \mathbb{Z}_N} \xi^{-pm} \mu^p(x) \) for \(x \in g. \) The following theorem is the second main result of this paper, whose proof will be presented in Section 5.

**Theorem 3.6.** The assignment

\[c \mapsto k_1, \quad h_{i,m} \mapsto t_i^m \otimes \alpha^\vee_i \otimes, \quad x_{i,m} \mapsto t_i^m \otimes e_i^\pm, \quad i \in I, \quad m \in \mathbb{Z} \]

determines a Lie algebra isomorphism from \(\mathcal{M}(g, \mu) \) to \(\hat{g}[\mu]. \)

When \(g \) is of untwisted type and \(\mu = 1, \) Theorem 3.6 was proved in [MRY].

4. Proof of Theorem 3.6

4.1. Multiloop algebras. We start by recalling the definition of multiloop algebras (see [ABFP]). Let \(k \) be an arbitrary Lie algebra, and let \(\sigma_1, \sigma_2, \ldots, \sigma_s \) be pairwise commuting automorphisms on \(k. \) From now on, we denote by

\[k^{\sigma_1, \sigma_2, \ldots, \sigma_s} \]

the fixed point subalgebra of \(k \) under the automorphisms \(\sigma_1, \sigma_2, \ldots, \sigma_s. \) Suppose further that each automorphism \(\sigma_i \) has a finite period \(M_i, \) i.e. \(\sigma_i^{M_i} = 1, \) for \(i = 1, \ldots, s. \) The **multiloop algebra** associated to \(k, \) \(\sigma_1, \sigma_2, \ldots, \sigma_s \) is by definition the following subalgebra of \(\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes k:

\[\mathcal{L}_{M_1, M_2, \ldots, M_s}(k, \sigma_1, \sigma_2, \ldots, \sigma_s) = \bigoplus_{m_1, m_2, \ldots, m_s} k^{m_1, m_2, \ldots, m_s} \otimes (k^{m_1, m_2, \ldots, m_s}), \]

where

\[k^{m_1, m_2, \ldots, m_s} = \{ x \in k \mid \sigma_i(x) = \xi_{M_i}^{m_i} x, \quad i = 1, 2, \ldots, s \}, \]

and when each \(M_i \) is the order of \(\sigma_i \) we often write

\[\mathcal{L}(k, \sigma_1, \sigma_2, \ldots, \sigma_s) = \mathcal{L}_{M_1, M_2, \ldots, M_s}(k, \sigma_1, \sigma_2, \ldots, \sigma_s). \]

Let \(\sigma \) be an automorphism of \(k, \) and \((c_1, c_2, \ldots, c_s) \) an \(s\)-tuple in \((\mathbb{C}^*)^s. \) Let

\[c_1^{-d_1} \otimes c_2^{-d_2} \otimes \cdots \otimes c_s^{-d_s} \otimes \sigma \]

be the automorphism of \(\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes k \) defined by:

\[t_i^{m_1} t_2^{m_2} \ldots t_s^{m_s} \otimes x \mapsto c_1^{-m_1} c_2^{-m_2} \cdots c_s^{-m_s} t_i^{m_1} t_2^{m_2} \cdots t_s^{m_s} \otimes \sigma(x), \]
where \( x \in \mathfrak{k} \) and \( m_i \in \mathbb{Z} \). It is obvious that the multiloop algebra \( L_{M_1, \ldots, M_s}(\mathfrak{k}, \sigma_1, \ldots, \sigma_s) \) is the subalgebra of \( \mathbb{C}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}] \otimes \mathfrak{k} \) fixed by the following commuting automorphisms
\[
\xi_{M_1}^{-d_1} \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_1, \ 1 \otimes \xi_{M_2}^{-d_2} \otimes \cdots \otimes 1 \otimes \sigma_2, \ldots, \ 1 \otimes \cdots \otimes 1 \otimes \xi_{M_s}^{-d_s} \otimes \sigma_s.
\]

4.2. The functor \( u_{ce} \).

In this subsection, we recall the endofunctor \( u_{ce} \) on the category of Lie algebras introduced in [N2]. Let \( \mathfrak{k} \) be an arbitrary Lie algebra, and \( B \) the subspace of \( \mathfrak{k} \otimes \mathfrak{k} \) spanned by all elements of the form
\[
x \otimes y + y \otimes x \quad \text{and} \quad x \otimes [y, z] + y \otimes [z, x] + z \otimes [x, y], \quad x, y, z \in \mathfrak{k}.
\]
We define
\[
uce(\mathfrak{k}) = \mathfrak{k} \otimes \mathfrak{k} / B
\]
to be a Lie algebra with Lie bracket
\[
[x \otimes x', y \otimes y']_{\nuce(\mathfrak{k})} = [x, x'] \otimes [y, y'] + B.
\]
Then we have the following well-defined Lie algebra homomorphism
\[
uce : \nuce(\mathfrak{k}) \rightarrow [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad x \otimes y \mapsto [x, y],
\]
which is in fact a central extension of \([\mathfrak{k}, \mathfrak{k}]\).

Let \( f : \mathfrak{k} \rightarrow \mathfrak{k}_0 \) be a homomorphism of Lie algebras. Then the map
\[
uce(f) : \nuce(\mathfrak{k}) \rightarrow \nuce(\mathfrak{k}_0)
\]
\[
x \otimes y \mapsto f(x) \otimes f(y),
\]
is also a Lie algebra homomorphism. Note that \( \nuce \) is a covariant functor. Therefore, if \( f \) is an isomorphism, then so is \( \nuce(f) \).

We say that a homomorphism \( \hat{f} : \nuce(\mathfrak{k}) \rightarrow \nuce(\mathfrak{k}_0) \) covers \( f : \mathfrak{k} \rightarrow \mathfrak{k}_0 \) if
\[
uce_0 \circ \hat{f} = f \circ \nuce.
\]
The following results were proved in [N2].

**Proposition 4.1.** Let \( \mathfrak{k} \) be a perfect Lie algebra. Then

(a) the map \( \nuce : \nuce(\mathfrak{k}) \rightarrow \mathfrak{k} \) is the universal central extension of \( \mathfrak{k} \), and \( \ker(\nuce) \) is the center of \( \nuce(\mathfrak{k}) \) when \( \mathfrak{k} \) is centerless;

(b) for any homomorphism \( f : \mathfrak{k} \rightarrow \mathfrak{k}_0 \) of Lie algebras, the map \( \nuce(f) \) is the unique homomorphism from \( \nuce(\mathfrak{k}) \) to \( \nuce(\mathfrak{k}_0) \) that covers \( f \).

We also record the following trivial result as a lemma that will be used later on.

**Lemma 4.2.** Let \( \sigma_1, \ldots, \sigma_s \) and \( \tau_1, \ldots, \tau_s \) be pairwise commuting automorphisms of Lie algebras \( \mathfrak{k} \) and \( \mathfrak{k}_0 \), respectively. Assume that there is a homomorphism \( \gamma : \mathfrak{k} \rightarrow \mathfrak{k}_0 \) such that \( \gamma \circ \sigma_i = \tau_i \circ \gamma \) for each \( i = 1, \ldots, s \). Then one has that

(a) if the map \( \nuce(\gamma) \) is injective, then
\[
\nuce(\gamma)(\nuce(\mathfrak{k})_{\nuce(\sigma_1), \ldots, \nuce(\sigma_s)}) = \nuce(\mathfrak{k}_0)^{\nuce(\tau_1), \ldots, \nuce(\tau_s)} \cap \mathrm{im}(\nuce(\gamma));
\]
(b) if the map $\gamma$ is an isomorphism, then

$$\text{ucc}(\gamma) : \text{ucc}(\theta)^{\text{ucc}(\sigma_1), \ldots, \text{ucc}(\sigma_s)} \cong \text{ucc}(\theta_0)^{\text{ucc}(\tau_1), \ldots, \text{ucc}(\tau_s)}.$$ 

Suppose now that $\sigma_1$ and $\sigma_2$ are two commuting automorphisms of $\hat{g}$ with periods $M_1$ and $M_2$, respectively. We define

$$\hat{L}_{M_1,M_2}(\hat{g}, \sigma_1, \sigma_2) = L_{M_1,M_2}(\hat{g}, \sigma_1, \sigma_2) \oplus K_{M_1,M_2}$$

to be the Lie algebra with Lie bracket as in (4.1). In particular, we have that $\hat{g} = \hat{L}_{1,1}(\hat{g}, \text{id}, \nu)$. It was proved in [Su] that $\hat{L}_{M_1,M_2}(\hat{g}, \sigma_1, \sigma_2)$ is the universal central extension of $L_{M_1,M_2}(\hat{g}, \sigma_1, \sigma_2)$. For convenience, when $M_i$ is the order of $\sigma_i$ for $i = 1, 2$, we also write $\hat{L}(\hat{g}, \sigma_1, \sigma_2) = \hat{L}_{M_1,M_2}(\hat{g}, \sigma_1, \sigma_2)$.

For $\sigma \in \text{Aut}(\hat{g})$ and $c_1, c_2 \in \mathbb{C}^\times$, one can easily verify that the assignment

$$t_m^1 t_m^2 \otimes x \mapsto c_1^{-m_1} c_2^{-m_2} t_m^1 t_2^m \otimes \sigma(x), \quad x \in \hat{g}, \quad m_1, m_2 \in \mathbb{Z},$$

$$t_m^1 t_m^2 k_i \mapsto c_1^{-m_1} c_2^{-m_2} t_m^1 t_m^2 k_i, \quad i = 1, 2,$$

determines an automorphism on $\hat{L}(\hat{g}, \text{id}, \text{id}) = \text{ucc}(L(\hat{g}, \text{id}, \text{id}))$. Note that this automorphism covers $c_1^{-d_1} \otimes c_2^{-d_2} \otimes \sigma$, and hence coincides with $\text{ucc}(c_1^{-d_1} \otimes c_2^{-d_2} \otimes \sigma)$ (Proposition 4.1 (b)). Using this, it is easy to see that

$$\hat{L}_{M_1,M_2}(\hat{g}, \sigma_1, \sigma_2) = (\hat{L}(\hat{g}, \text{id}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes 1^{-d_2} \otimes \sigma_1), \text{ucc}(1^{-d_1} \otimes \xi_{M_2}^{-d_2} \otimes \sigma_2)}.$$ 

In other words, we have the following isomorphism

$$\text{ucc} \left( L(\hat{g}, \text{id}, \text{id})^{\xi_{M_1}^{-d_1} \otimes 1^{-d_2} \otimes \sigma_1, 1^{-d_1} \otimes \xi_{M_2}^{-d_2} \otimes \sigma_2} \right)$$

$$\cong \text{ucc} \left( L(\hat{g}, \text{id}, \text{id})^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes 1^{-d_2} \otimes \sigma_1), \text{ucc}(1^{-d_1} \otimes \xi_{M_2}^{-d_2} \otimes \sigma_2)} \right).$$

4.3. Automorphism groups. In this subsection we collect some basics on the automorphism group of $\hat{g}$, one may consult [ABP2, Section 6] for details. Let $\text{Aut}(A)$ be the group of diagram automorphisms of $\hat{g}$. Define the outer automorphism group of $\hat{g}$ to be

$$\text{Out}(A) = \langle \omega \rangle \times \text{Aut}(A),$$

where $\omega$ is the Chevalley involution of $\hat{g}$. Let $\text{Hom}(Q, \mathbb{C}^\times)$ denote the set of group homomorphisms from $Q$ to $\mathbb{C}^\times$, which is viewed as a group under pointwise multiplication. The group $\text{Hom}(Q, \mathbb{C}^\times)$ can be identified as a subgroup of $\text{Aut}(\hat{g})$ in the following way:

$$(4.2) \quad \text{Hom}(Q, \mathbb{C}^\times) \hookrightarrow \text{Aut}(\hat{g}), \quad \rho \mapsto (x \mapsto \rho(\alpha) x), \quad x \in \hat{g}_\alpha, \ \alpha \in \Delta.$$ 

Define the inner automorphism group of $\hat{g}$ to be

$$\text{Aut}^0(\hat{g}) = \langle \exp(\text{ad} x_\alpha) \mid \alpha \in \Delta^\times \rangle \cdot \text{Hom}(Q, \mathbb{C}^\times).$$

Consider now the group homomorphism

$$\bar{\chi} : \text{Aut}(\hat{g}) \to \text{Aut}(\hat{g}),$$
where \( \bar{\chi}(\tau) = \bar{\tau} \) is the automorphism of \( \bar{g} \) induced from \( \tau \). Note that the restriction of \( \bar{\chi} \) on \( \text{Out}(A) \) and \( \text{Hom}(Q, \mathbb{C}^\times) \) are both injective. Thus we may view them as subgroups of \( \text{Aut}(\bar{g}) \). The following statements were proved in [ABP2 Proposition 6.1.5 and Proposition 6.1.8].

**Proposition 4.3.** The homomorphism \( \bar{\chi} \) is an isomorphism. Furthermore

\[
\text{Aut}(g) = \text{Aut}^0(g) \rtimes \text{Out}(A), \quad \text{Aut}(\bar{g}) = \text{Aut}^0(\bar{g}) \rtimes \text{Out}(A),
\]

where \( \text{Aut}^0(\bar{g}) = \bar{\chi}(\text{Aut}^0(g)) \).

By Proposition 4.3, we have the following projections

\[
p : \text{Aut}(g) \to \text{Out}(A) \quad \text{and} \quad \bar{p} : \text{Aut}(\bar{g}) \to \text{Out}(A)
\]

such that \( \bar{p} \circ \bar{\chi} = p \). An automorphism \( \sigma \) of \( g \) (resp. \( \bar{g} \)) is said to be of the first kind if \( p(\sigma) \) (resp. \( \bar{p}(\sigma) \)) lies in \( \text{Aut}(A) \). Otherwise, we say that \( \sigma \) is of the second kind.

### 4.4. Universal central extensions

This subsection is devoted to a proof of the following theorem.

**Theorem 4.4.** Let \( \bar{\eta} \) be an automorphism of \( \bar{g} \) of the first kind with period \( M \). Then the Lie algebra \( \text{uce}(\mathcal{L}(\bar{g}, \text{id}))^{\text{uce}(\xi_M^{-1} \otimes \bar{\eta})} \) is the universal central extension of the loop algebra \( \mathcal{L}_M(\bar{g}, \bar{\eta}) = \mathcal{L}(\bar{g}, \text{id}) \).

Recall that the automorphism \( \hat{\mu} \) of \( \hat{g} = \text{uce}(\mathcal{L}(\bar{g}, \text{id})) \) covers the automorphism \( \xi^{-d_1} \otimes \hat{\mu} \) of \( \mathcal{L}(\bar{g}, \text{id}) \) (see [3.6]), and so coincides with \( \text{uce}(\xi^{-d_1} \otimes \hat{\mu}) \) (Proposition 4.1(b)). Thus, Theorem 3.3 is just a special case of Theorem 4.4.

We first prove some technique lemmas. Let \( \bar{\sigma} \) be an automorphism of \( \bar{g} \) with period \( M \). It is known that the twisted loop algebra of \( \bar{g} \) related to \( \bar{\sigma} \) is independent from the choice of its periods ([ABP1, Lemma 2.3]). In the following, we extend this result to their universal central extensions.

**Lemma 4.5.** Let \( \bar{\sigma} \) be an automorphism of \( \bar{g} \) of finite period, and \( M, M' \) two periods of \( \bar{\sigma} \). Then

\[
(4.3) \quad \text{uce}(\mathcal{L}(\bar{g}, \text{id}))^{\text{uce}(\xi_{M'}^{-1} \otimes \bar{\sigma})} \cong \text{uce}(\mathcal{L}(\bar{g}, \text{id}))^{\text{uce}(\xi_{M}^{-1} \otimes \bar{\sigma})}.
\]

**Proof.** We may (and do) assume that \( M' = bM \) for some \( b \in \mathbb{Z}_+ \). Consider the natural imbedding

\[
i_b : \mathcal{L}(\bar{g}, \text{id}) \to \mathcal{L}(\bar{g}, \text{id}), \quad t_1^m \otimes x \mapsto t_1^{bm} \otimes x,
\]

where \( m \in \mathbb{Z} \) and \( x \in \bar{g} \). It is clear that the image of \( i_b \) is the Lie algebra \( \mathcal{L}_b(\bar{g}, \text{id}) = \mathcal{L}_{b, r}(\bar{g}, \text{id}, \nu) \) and that

\[
(4.4) \quad (\xi_{M'}^{-d_1} \otimes \bar{\sigma}) \circ i_b = i_b \circ (\xi_{M}^{-d_1} \otimes \bar{\sigma}).
\]
Using Proposition 4.1 (b), it is easy to see that the action of $\text{uce}(i_b)$ on the center of $\text{uce}(\mathcal{L}(\hat{g}, \text{id})) = \text{uce}(\mathcal{L}_{1, r}(\hat{g}, \text{id}, \tilde{\nu}))$ is given by

$$t_1^{m_1} t_2^{m_2} k_i \mapsto t_1^{b m_1} t_2^{b m_2} k_i, \quad i = 1, 2, \ m_1 \in \mathbb{Z}, \ m_2 \in r\mathbb{Z}.$$ 

This implies that

$$\text{im}(\text{uce}(i_b)) = \text{uce}(\mathcal{L}_{b, r}(\hat{g}, \text{id}, \tilde{\nu}))$$

and that

$$\text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes 1 - d_2 \otimes 1)} = \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes 1 - d_2 \otimes 1)}$$

$$(\text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id})))^{\text{uce}(\xi_b^{-d_1} \otimes 1 - d_2 \otimes 1)} = \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes 1)}.$$ 

Note that we also have

$$\text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)} \subset \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)_{\mathcal{M}}}$$

$$\text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)_{\mathcal{M}}} = \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes 1)}.$$ 

This together with \[\text{(4.6)}\] gives that

$$\text{im}(\text{uce}(i_b)) \cap \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)} = \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)}.$$ 

Now the assertion is implied by \[\text{(4.3)}, \ (4.5), \ (4.7)\] and Lemma 4.2 (a). \(\square\)

Let $\sigma$ be an automorphism of $\hat{g}$ with period $M$. Now $\hat{g} = \mathcal{L}(\hat{g}, \tilde{\nu})$ itself is a twisted loop algebra and so is independent from the choice of the period of $\tilde{\nu}$. Namely, if $M'$ is another period of $\tilde{\nu}$, then one has the natural isomorphism $\hat{g} \cong \mathcal{L}(\hat{g}, \tilde{\nu})$. Via this isomorphism, $\sigma$ induces an automorphism, say $\sigma'$, of $\mathcal{L}_{M'}(\hat{g}, \tilde{\nu})$ with period $M$. Similar to Lemma 4.5, we have that

**Lemma 4.6.** Let $\sigma, M, M'$ and $\sigma'$ be as above. Then one has that

$$\text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)} \cong \text{uce}(\mathcal{L}(\mathcal{L}_{M'}(\hat{g}, \tilde{\nu}), \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma')}.$$ 

**Proof.** Set $b = M' / r$ and define the embedding

$$j_b : \hat{g} = \mathcal{L}(\hat{g}, \tilde{\nu}) \to \mathcal{L}(\hat{g}, \text{id}), \quad t_2^{m_2} \otimes x \mapsto t_2^{b m_2} \otimes x, \ m_2 \in \mathbb{Z}, \ x \in \hat{g}.$$ 

Then the image of $j_b$ is the Lie algebra $\mathcal{L}_{M'}(\hat{g}, \tilde{\nu})$ and

$$j_b \circ \sigma = \sigma' \circ j_b.$$ 

Moreover, the action of $\text{uce}(1^{-d_1} \otimes j_b)$ on the center of $\text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))$ is given by

$$t_1^{m_1} t_2^{m_2} k_i \mapsto t_1^{b m_1} t_2^{b m_2} k_i, \quad i = 1, 2, \ m_1 \in \mathbb{Z}, \ m_2 \in r\mathbb{Z}.$$ 

This implies that

$$\text{im}(\text{uce}(1^{-d_1} \otimes j_b)) \cap \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)} = \text{uce}(\mathcal{L}(\hat{g}, \text{id}, \text{id}))^{\text{uce}(\xi_b^{-d_1} \otimes \sigma)}.$$ 

The map $\text{uce}(1^{-d_1} \otimes j_b)$ is injective.
and that
\[(4.11) \quad \text{im}(\text{ucc}(1^{-d_1} \otimes j_b)) = \text{ucc}(L_{1,M'}(\hat{g}, \text{id}, \nu)) = \text{ucc}(L(M'(\hat{g}, \nu), \text{id})).\]

Then the lemma follows from (4.9), (4.10), (4.11) and Lemma 4.2 (a). \qed

Using Lemma 4.3, we have the following result.

**Lemma 4.7.** Let $\bar{\sigma}$ be an automorphism of $\mathfrak{g}$ with period $M$. Then
\[(4.12) \quad \text{ucc}(L(\bar{g}, \text{id}))^{\text{ucc}(\xi_{M}^{-d_1} \otimes \bar{\sigma})} \cong \text{ucc}(L(\bar{g}, \text{id}))^{\text{ucc}(\xi_{M}^{-d_1} \otimes \bar{p}(\bar{\sigma}))}.\]

**Proof.** Recall the isomorphism $\bar{\chi} : \text{Aut}(\mathfrak{g}) \to \text{Aut}(\hat{\mathfrak{g}})$ given in Proposition 4.3. Then we may choose an automorphism $\sigma$ of $\mathfrak{g}$ such that $\sigma^M = \text{id}$ and $\chi(\sigma) = \bar{\sigma}$. This together with [KW, Lemma 4.31] gives that there exists a $\rho \in \text{Hom}(\hat{Q}, \mathbb{C}^\times)$ such that
\[\rho \bar{p}(\bar{\sigma}) = \bar{p}(\bar{\sigma}) \rho, \quad \rho^M = \text{id} \quad \text{and} \quad \bar{\sigma} \text{ is conjugate to } \bar{p}(\bar{\sigma}) \rho.\]

Note that the automorphisms $\rho$ and $\bar{p}(\bar{\sigma})$ of $\hat{\mathfrak{g}}$ satisfy all the assumptions stated in [ABP1, Theorem 5.1]. Then it follows from [ABP1, (5.3)] that the automorphism $\xi_{M}^{-d_1} \otimes \rho \bar{p}(\bar{\sigma})$ is conjugate to $\xi_{M}^{-d_1} \otimes \bar{p}(\bar{\sigma})$. This together with Lemma 4.2 (b) and Lemma 4.3 gives that
\[\text{ucc}(L(\bar{g}, \text{id}))^{\text{ucc}(\xi_{M}^{-d_1} \otimes \bar{\sigma})} \cong \text{ucc}(L(\bar{g}, \text{id}))^{\text{ucc}(\xi_{M}^{-d_1} \otimes \bar{p}(\bar{\sigma}))}\]
\[\cong \text{ucc}(L(\bar{g}, \text{id}))^{\text{ucc}(\xi_{M}^{-d_1} \otimes \bar{p}(\bar{\sigma}))} \cong \text{ucc}(L(\bar{g}, \text{id}))^{\text{ucc}(\xi_{M}^{-d_1} \otimes \bar{p}(\bar{\sigma}))}.\]

Therefore, we complete the proof. \qed

Let $\text{Hom}(\hat{Q}, \mathbb{C}^\times)$ be the set of group homomorphisms from $\hat{Q}$ to $\mathbb{C}^\times$. Similar to [H.2], we may (and do) view $\text{Hom}(\hat{Q}, \mathbb{C}^\times)$ as a subgroup of $\text{Aut}(\hat{\mathfrak{g}})$. From now on, let $\bar{\eta}$ be as in Theorem 4.4. The following characterization of $L_M(\hat{g}, \bar{\eta})$ plays a key role in the proof of Theorem 4.4.

**Lemma 4.8.** There exist finite order automorphisms $\hat{\rho}$ and $\hat{\tau}$ of $\hat{\mathfrak{g}}$ such that
\[\hat{\rho} \in \text{Hom}(\hat{Q}, \mathbb{C}^\times), \quad \hat{\rho} \hat{\nu} = \hat{\nu} \hat{\rho}, \quad (\hat{\nu} \hat{\rho}) \hat{\tau} = \hat{\tau}(\hat{\nu} \hat{\rho}) \quad \text{and} \quad L_{M_1, M_2}(\hat{g}, \hat{\tau}, \hat{\nu} \hat{\rho}) \cong L_M(\hat{g}, \bar{\eta}),\]

where $M_1$ and $M_2$ are some periods of $\hat{\tau}$ and $\hat{\nu} \hat{\rho}$, respectively.

**Proof.** By [ABP2, Theorem 10.1.1], there exist finite order automorphisms $\hat{\tau}$ and $\hat{\sigma}$ such that $L(\hat{g}, \bar{\eta}) \cong L(\hat{g}, \hat{\tau}, \hat{\sigma})$. Up to conjugation, we may assume that $\hat{\sigma}$ is of the form $\hat{\rho} \hat{\vartheta}$, where $\hat{\rho} \in \text{Hom}(\hat{Q}, \mathbb{C}^\times)$ and $\hat{\vartheta}$ is a diagram automorphism of $\hat{g}$ such that $\hat{\rho} \hat{\vartheta} = \hat{\vartheta} \hat{\rho}$. If $\mathfrak{g}$ is of untwisted type, then it follows from the proof of [ABP2, Theorem 10.1.1] that one may take $\hat{\vartheta} = \text{id}$. If $\mathfrak{g}$ is of twisted type, then by comparing the classification results (the relative and absolute types) given in [ABP2, Tables 3] and [GP, Table 9.2.4], we find out that the diagram automorphism $\hat{\vartheta}$ can also be taken to be $\hat{\nu}$. \qed
Denote by (4.13) \( \varphi \circ (\xi_{M_2}^{-d_2} \otimes \breve{\nu}) \circ \varphi^{-1} = \xi_{M_2}^{-d_2} \otimes \breve{\nu} \).

Notice that the automorphisms \( \breve{\rho} \) and \( \breve{\nu} \) satisfy the assumptions given in [ABP1] Theorem 5.1. Thus, there is an automorphism \( \varphi \) of \( \mathcal{L}(\hat{g}, \text{id}) \) such that

\[
\varphi \circ (\xi_{M_2}^{-d_2} \otimes \breve{\nu}) \circ \varphi^{-1} \equiv \xi_{M_2}^{-d_2} \otimes \breve{\nu}.
\]

Denote by \( \tau' \) the automorphism

\[
\varphi \circ (1^{-d_2} \otimes \breve{\tau}) \circ \varphi^{-1}
\]

of \( \mathcal{L}(\hat{g}, \text{id}) \). Then \( \tau' \) commutes with the automorphism \( \xi_{M_2}^{-d_2} \otimes \breve{\nu} \), and hence preserves the Lie algebra \( \mathcal{L}_{M_2}^2(\hat{g}, \breve{\nu}) \). Write \( \tau'' \) for the restriction of \( \tau' \) on \( \mathcal{L}_{M_2}^2(\hat{g}, \breve{\nu}) \), and \( \breve{\tau} \) for the automorphism of \( \hat{g} \) induced from \( \tau'' \) via the isomorphism \( \hat{g} \cong \mathcal{L}_{M_2}^2(\hat{g}, \breve{\nu}) \).

So by definition we have that

\[
\mathcal{L}(\hat{g}, \text{id}) \xi_{M_1}^{-d_1} \otimes 1^{-d_2} \otimes 1^{-d_1} \xi_{M_2}^{-d_2} \otimes \nu' \equiv \mathcal{L}(\mathcal{L}(\hat{g}, \text{id}), \text{id}) \xi_{M_1}^{-d_1} \otimes 1^{-d_1} \otimes (\xi_{M_2}^{-d_2} \otimes \nu')
\]

\[
\cong \mathcal{L}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id}) \xi_{M_1}^{-d_1} \otimes \tau'' \cong \mathcal{L}(\hat{g}, \breve{\tau}).
\]

**Lemma 4.9.** One has that

\[
\text{ucc}(\mathcal{L}(\hat{g}, \text{id}), \text{id}))^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau')} \cong \text{ucc}(\mathcal{L}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id})),
\]

\[
\cong \text{ucc}(\mathcal{L}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id})),
\]

\[
\cong \text{ucc}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id})).
\]

**Proof.** Due to the isomorphisms

\[
\text{ucc}(\mathcal{L}(\hat{g}, \text{id}), \text{id}))^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau')} \cong \text{ucc}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id}),
\]

it suffices to show that the restriction of \( \text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau') \) on \( \text{ucc}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id}) \) coincides with \( \text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau'') \). Set \( t = \mathcal{L}(\hat{g}, \text{id}, \text{id}) \) and \( t_0 = \mathcal{L}(\hat{g}, \breve{\nu}, \text{id}) \).

Then by definition one has that

\[
\text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau') = (\xi_{M_1}^{-d_1} \otimes \tau') \circ \text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau'') = (\xi_{M_1}^{-d_1} \otimes \tau'') \circ \text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau') \circ \text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau'').
\]

This implies that the restriction of \( \text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau') \) on \( \text{ucc}(t_0) \) covers \( \xi_{M_1}^{-d_1} \otimes \tau'' \). Combining with Proposition 4.11 (b), we complete the proof. \( \square \)

Now, by using Lemma 4.9, Lemma 4.2 (b) and Lemma 4.6 we can extend the isomorphisms given in (4.14) to their universal central extensions as follows:

\[
\text{ucc}(\mathcal{L}(\hat{g}, \text{id}), \text{id}))^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes 1^{-d_2} \otimes \breve{\tau})} \equiv \text{ucc}(\mathcal{L}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id})),
\]

\[
\cong \text{ucc}(\mathcal{L}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id})),
\]

\[
\cong \text{ucc}(\mathcal{L}(\hat{g}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau')},
\]

\[
\cong \text{ucc}(\mathcal{L}(\mathcal{L}(\hat{g}, \breve{\nu}), \text{id})),
\]

\[
\cong \text{ucc}(\mathcal{L}(\hat{g}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-d_1} \otimes \tau')}.
\]
Combining Lemma 4.8 with (4.14), we get the isomorphism
\[ \mathcal{L}_{M_1}(\hat{\mathfrak{g}}, \hat{\tau}) \cong \mathcal{L}_{M}(\hat{\mathfrak{g}}, \tilde{\eta}). \]

By using [ABP2, Theorem 10.1.1 and Corollary 10.1.5], we get that \( \tilde{\tau} \) is of the first kind. Moreover, it follows from [ABP2, Theorem 13.2.3] that the diagram automorphism \( \tilde{\rho}(\tilde{\tau}) \) is conjugate to \( \tilde{\rho}(\tilde{\eta}) \). Thus, one can conclude from Lemma 4.7 and Lemma 4.4 that
\[
\text{ucc}(\mathcal{L}(\hat{\mathfrak{g}}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-1} \otimes \tilde{\tau})} \cong \text{ucc}(\mathcal{L}(\hat{\mathfrak{g}}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-1} \otimes \tilde{\rho}(\tilde{\tau}))}
\cong \text{ucc}(\mathcal{L}(\hat{\mathfrak{g}}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-1} \otimes \tilde{\rho}(\tilde{\eta}))}.
\]

Combining with (4.15), we get that
\[
\text{ucc}(\mathcal{L}(\hat{\mathfrak{g}}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-1} \otimes \tilde{\eta})} \cong \text{ucc}(\mathcal{L}(\hat{\mathfrak{g}}, \text{id}))^{\text{ucc}(\xi_{M_1}^{-1} \otimes \tilde{\rho}(\tilde{\eta}))}^{\text{ucc}(\xi_{M_1}^{-1} \otimes \tilde{\rho}(\tilde{\eta}))}.
\]
is central closed. This completes the proof of Theorem 4.4.

5. Proof of Theorem 3.6

Throughout this section, we assume that the diagram automorphism \( \mu \) is non-transitive.

5.1. Root system of \( \hat{\mathfrak{g}}[\mu] \). In this subsection, we determine the non-isotropic roots in \( \hat{\mathfrak{g}}[\mu] \). As indicated in [ABP2, Section 14], this affords an explicit realization of all nullity 2 reduced extended affine root systems given by Saito ([Sa]).

Recall that \( V = \mathbb{R} \otimes \mathbb{Z} \), and we extend \( \mu \) to a linear automorphism on \( V \) by \( \mathbb{R} \)-linearity. We denote by \( V_\mu \) the fixed point subspace of \( V \) under the isometry \( \mu, \pi_\mu : V \to V_\mu \), and \( \hat{Q}_\mu \) the abelian group \( \pi_\mu(Q) \times \mathbb{Z} \).

Define a \( Q \times \mathbb{Z} \)-grading on \( \hat{\mathfrak{g}} = \oplus_{(\alpha, n) \in Q \times \mathbb{Z}} \hat{\mathfrak{g}}_{\alpha, n} \) by letting
\[
t_1^{n_1} \otimes x \in \hat{\mathfrak{g}}_{\alpha, n_1}, \quad k_1 \in \hat{\mathfrak{g}}_{0,0}, \quad t_1^{n_1} \otimes k_1 \in \hat{\mathfrak{g}}_{n_2,0, n_1},
\]
where \( x \in \hat{\mathfrak{g}}_\alpha, \alpha \in \Delta, n_1 \in \mathbb{Z} \) and \( n_2 \in r\mathbb{Z} \). The above grading induces a \( \hat{Q}_\mu \)-grading \( \hat{\mathfrak{g}}[\mu] = \oplus_{(\alpha, n) \in \hat{Q}_\mu} \hat{\mathfrak{g}}[\mu]_{\alpha, n} \) on \( \hat{\mathfrak{g}}[\mu] \) such that for any \( (\alpha, n) \in \hat{Q}_\mu \),
\[
\hat{\mathfrak{g}}[\mu]_{\alpha, n} = \{ x \in \hat{\mathfrak{g}}[\mu] : x \otimes k_1 \in \hat{\mathfrak{g}}[\mu], k_1 \in \hat{\mathfrak{g}}[\mu]_{0,0}, \beta \in \mathbb{Q}, \pi_\mu(\beta) = \alpha \}.
\]
Notice that this is the unique \( \hat{Q}_\mu \)-grading on \( \hat{\mathfrak{g}}[\mu] \) such that
\[
t_1^{n_1} \otimes \hat{e}_{i(n)}^+ \in \hat{\mathfrak{g}}[\mu]_{\pm \alpha_i, n_1}, \quad t_1^{n_1} \otimes \hat{\alpha}_i^+ \in \hat{\mathfrak{g}}[\mu]_{\alpha_i, n_1}, \quad k_1 \in \hat{\mathfrak{g}}[\mu]_{0,0}, \quad i \in I \text{ and } n \in \mathbb{Z}.
\]

Consider now the following subsets of \( \hat{Q}_\mu \):
\[
\Phi_\mu = \{ (\alpha, n) \in \hat{Q}_\mu | \hat{\mathfrak{g}}[\mu]_{\alpha, n} \neq 0 \},
\]
\[
\hat{Q}_\mu^\times = \{ (\alpha, n) \in \hat{Q}_\mu | (\alpha, \alpha) \neq 0 \},
\]
\[
\Phi_\mu^\times = \Phi_\mu \cap \hat{Q}_\mu^\times = \{ (\alpha, n) \in \hat{Q}_\mu | (\alpha, \alpha) \neq 0 \}.
\]
It obvious that $\Phi_\mu \subset \pi_\mu(\Delta) \times \mathbb{Z}$ and so we have
\begin{equation}
\Phi_\mu^x \subset \pi_\mu(\Delta)^x \times \mathbb{Z},
\end{equation}
where $\pi_\mu(\Delta)^x = \{ \alpha \in \pi_\mu(\Delta) \mid (\alpha, \alpha) \neq 0 \}$. By definition, for each $i \in I$ we have that $\tilde{\alpha}_i = \pi_\mu(\alpha_i)$. In addition, for $i \in I$ with $s_i = 2$, we have that $2\tilde{\alpha}_i = \pi_\mu(\alpha_i + \alpha_{\mu(i)})$. This shows that
\begin{equation}
k_i \tilde{\alpha}_i \in \pi_\mu(\Delta)^x, \quad 1 \leq k_i \leq s_i, \ i \in \tilde{I}.
\end{equation}

For $i \in I$, we let $N_i$ be the cardinality of the orbit $O(i)$ in $I$ and set $d_i = \frac{N}{N_i}$. Denote by $\tilde{W}$ the Weyl group of the folded GCM $\tilde{A}$. Then we have the following description of the set $\Phi_\mu^x$.

**Proposition 5.1.** One has that
\begin{equation}
\Phi_\mu^x = \{(w(k_i \tilde{\alpha}_i), p) \mid w \in \tilde{W}, i \in \tilde{I}, 1 \leq k_i \leq s_i, p \in (k_i - 1)d_i + k_id_i \mathbb{Z} \},
\end{equation}
and that
\begin{equation}
dim \tilde{\mathfrak{h}}[\mu]_{\alpha,p} = 1, \ \forall (\alpha, p) \in \Phi_\mu^x.
\end{equation}

Before proving Proposition 5.1, we first give a characterization of the set $\pi_\mu(\Delta)^x$. This result is a slight generalization of [ABP2, Proposition 12.1.16].

**Lemma 5.2.** One has that
\begin{equation}
\pi_\mu(\Delta)^x = \{ w(k_i \tilde{\alpha}_i) \mid w \in \tilde{W}, i \in \tilde{I}, 1 \leq k_i \leq s_i \}.
\end{equation}

**Proof.** For convenience, we set
\begin{equation}
\tilde{\Delta}^w = \{ w(k_i \tilde{\alpha}_i) \mid w \in \tilde{W}, i \in \tilde{I}, 1 \leq k_i \leq s_i \}.
\end{equation}
We first show that
\begin{equation}
\tilde{W}(\pi_\mu(\Delta)) \subset \pi_\mu(\Delta).
\end{equation}
Let $r_{\tilde{\alpha}_i}, i \in \tilde{I}$ denote the reflections associated to $\tilde{\alpha}_i$. Note that the Weyl group $\tilde{W}$ is generated by these reflections. Thus we only need to show that
\begin{equation}
r_{\tilde{\alpha}_i}(\pi_\mu(\Delta)) \subset \pi_\mu(\Delta), \ i \in \tilde{I}.
\end{equation}
If $s_i = 1$, it is shown in the proof of [ABP2, Proposition 12.1.16] that for each $\alpha \in \Delta$, the following relation holds true
\begin{equation}
r_{\tilde{\alpha}_i}(\pi_\mu(\alpha)) = \pi_\mu \left( \prod_{p \in O(i)} r_{\alpha_p} \right)(\alpha) \in \pi_\mu(\Delta).
\end{equation}
If $s_i = 2$, then $2\tilde{\alpha}_i = \alpha_i + \alpha_{\mu(i)} \in \Delta$ as $a_{i,\mu(i)} = -1$. Note that $\pi_\mu(\tilde{\alpha}_i) = \tilde{\alpha}_i$ and hence $(\tilde{\alpha}_i, \pi_\mu(\alpha)) = (\tilde{\alpha}_i, \alpha)$ for all $\alpha \in \Delta$. This implies that
\begin{equation}
r_{\tilde{\alpha}_i}(\pi_\mu(\alpha)) = \pi_\mu(\alpha) - 2(\tilde{\alpha}_i, \pi_\mu(\alpha)) \tilde{\alpha}_i = \pi_\mu \left( \alpha - 2(\tilde{\alpha}_i, \alpha) \tilde{\alpha}_i \right).
\end{equation}
\[ = \pi_\mu \left( \alpha - 2 \frac{(\hat{\alpha}_i, \alpha)}{2\hat{\alpha}_i, 2\hat{\alpha}_i} 2\hat{\alpha}_i \right) = \pi_\mu (r_{2\hat{\alpha}_i}(\alpha)) \in \pi_\mu(\Delta). \]

Thus we complete the proof of the assertion (5.8) and hence the assertion (5.7). Now, as the reflections preserve the bilinear form \((\ , \ )\), we have that

\[ W(\pi_\mu(\Delta)^\times) \subset \pi_\mu(\Delta)^\times. \]

This together with (5.3) gives that

\[ \hat{\Delta}^{en} \subset W(\pi_\mu(\Delta)^\times) \subset \pi_\mu(\Delta)^\times. \]

For the reverse inclusion, observe first that any non-zero element \(\beta \in \pi_\mu(\Delta)\) can be written uniquely in the form \(\beta = \sum_{i \in \hat{I}} n_i \hat{\alpha}_i\), where the \(n_i\) are either all non-negative integers or all non-positive integers. Set \(n = \sum_{i \in \hat{I}} n_i\). Assume that \(\beta \in \pi_\mu(\Delta)^\times\). We then show that \(\beta \in \hat{\Delta}^{en}\) by using induction on \(n\). Without loss of generality, we may assume that \(n > 0\). Since \((\beta, \beta) > 0\), there are some \(i \in \hat{I}\) such that \((\beta, \hat{\alpha}_i) > 0\) and that \(n_i > 0\). If \(r_{\hat{\alpha}_i}(\beta)\) is positive, then we are done by the induction hypothesis. If \(r_{\hat{\alpha}_i}(\beta)\) is negative, then \(\beta = q\hat{\alpha}_i\) for some positive integer \(q\). This implies that \(\beta = \pi_\mu(\alpha)\) for some

\[ \alpha = \sum_{p \in \mathcal{O}(i)} m_p \alpha_p \in \Delta \text{ with } \sum_{p \in \mathcal{O}(i)} m_p = q. \]

If \(s_i = 1\), then \(q\) must equal to 1 as all \(\alpha_p, p \in \mathcal{O}(i)\) are pairwise orthogonal. If \(s_i = 2\), then \(q\) can be 1 or 2, as \(|\mathcal{O}(i)| = 2\) and \(a_{ij} = -1\). This completes the proof. \(\square\)

As a by-product of Lemma 5.2 we have that

**Corollary 5.3.** Let \(i, j \in I\) with \(\hat{\alpha}_{ij} \leq 0\). Then for every \(p \in \mathbb{Z}\), the elements

\[ ((1 - \hat{\alpha}_{ij})\hat{\alpha}_i + \hat{\alpha}_j, p), ((s_i + 1)\hat{\alpha}_i, p) \]

are contained in \(\hat{Q}_\mu^\times\) but not contained in \(\hat{\Phi}_\mu^\times\).

**Proof.** By Lemma 5.2 it suffices to show that if \(\hat{\alpha}_{ij} \leq 0\), then \((1 - \hat{\alpha}_{ij})\hat{\alpha}_i + \hat{\alpha}_j\) is non-isotropic. Otherwise,

\[ 0 = 2 \frac{(\hat{\alpha}_i, (1 - \hat{\alpha}_{ij})\hat{\alpha}_i + \hat{\alpha}_j)}{(\hat{\alpha}_i, \hat{\alpha}_i)} = 2(1 - \hat{\alpha}_{ij}) + \hat{\alpha}_{ij} = 2 - \hat{\alpha}_{ij}, \]

a contradiction. \(\square\)

Let \(\mathfrak{g}\) be the subalgebra of \(\hat{\mathfrak{g}}[\mu]\) generated by the elements \(\alpha_i^\vee, e_i^\pm; i \in \hat{I}\). Then by applying Corollary 5.3 we have that

**Corollary 5.4.** The Lie algebra \(\mathfrak{g}\) is isomorphic to the derived subalgebra of the Kac-Moody algebra associated to \(\hat{A}\).
Lemma 5.5. Proof of Proposition 5.1. Using (5.2) and Lemma 5.2, we know that any element in $\Phi^I_\mu$ has the form
\[
(\tilde{\omega}(k_i\check{\alpha}_i), p), \quad \tilde{\omega} \in \tilde{W}, \quad 1 \leq k_i \leq s_i, \quad i \in \check{I}, \quad p \in \mathbb{Z}.
\]

Regard $\hat{g}[\mu]$ as a module of the affine Kac-Moody algebra $\hat{g}$ (Corollary 5.4) via the adjoint action. Then it is integrable, and for each $p \in \mathbb{Z}$, the graded subspace $\hat{g}[\mu]_p$ of $\hat{g}[\mu]$ is a $\hat{g}$-submodule, where
\[
\hat{g}[\mu]_p = \bigoplus_{\check{\alpha} \in \pi_p(\Delta)} \hat{g}[\mu]_{\check{\alpha}, p}.
\]

Using this and the standard $\mathfrak{sl}_2$-theory, we obtain that $(\tilde{\omega}(k_i\check{\alpha}_i), p) \in \Phi^I_\mu$ if and only if $(k_i\check{\alpha}_i, p) \in \Phi^I_\mu$. Moreover, we have that $\dim \hat{g}[\mu]_{\tilde{\omega}(k_i\check{\alpha}_i), p} = \dim \hat{g}[\mu]_{k_i\check{\alpha}_i, p}$. So we only need to treat the case that $\tilde{\omega} = 1$.

We first consider the case that $k_i = 1$. Note that for each $i \in \check{I}$,
\[
\hat{g}[\mu]_{\check{\alpha}_i, p} = \mathbb{C} \ell^{p}_{1} \otimes e_{i(p)}^{+} = \mathbb{C} \sum_{s \in \mathbb{Z}_{N_i}} \left( \sum_{k \in \mathbb{Z}_{d_i}} \xi_{-d_i}^{-kp} \right) \xi_{-s}^{-kp} \otimes e_{\mu(i)}^{+}.
\]

This together with the fact
\[
\sum_{k \in \mathbb{Z}_{d_i}} \xi_{-d_i}^{-kp} \neq 0 \iff p \in d_i \mathbb{Z}
\]
gives that $(\check{\alpha}_i, p) \in \Phi^I_\mu$ if and only if $p \in d_i \mathbb{Z}$. Next, for the case $k_i = 2$ (and hence $s_i = 2$), we have that
\[
\hat{g}[\mu]_{2\check{\alpha}_i, p} = \mathbb{C} \ell^{p}_{1} \otimes [e_{\mu(i)}^{+}, e_{\mu(i)}^{+}]_{(p)}.
\]

This together with the fact
\[
\mu \left( [e_{i(p)}^{+}, e_{\mu(i)}^{+}] \right) = [e_{\mu(i)}^{+}, e_{i(p)}^{+}] = -[e_{i(p)}^{+}, e_{\mu(i)}^{+}].
\]
gives that $(2\check{\alpha}_i, p) \in \Phi^I_\mu$ if and only if $p \in d_i + N \mathbb{Z}$. Therefore, we complete the proof of Proposition 5.1.

5.2. Proof of Theorem 3.6. We start with the following lemma.

Lemma 5.5. The action
\[
c \mapsto k_1, \quad h_{i,m} \mapsto t^{m}_{1} \otimes \alpha_{(m)}^{i}, \quad x^{\pm}_{i,m} \mapsto t^{m}_{1} \otimes e_{i(m)}^{\pm}, \quad i \in I, \quad m \in \mathbb{Z},
\]
determines (uniquely) a surjective Lie homomorphism from $\mathcal{M}(\mathfrak{g}, \mu)$ to $\hat{g}[\mu]$. 

Proof. One needs to check that the generators \( \alpha_{i(m)}^+, \alpha_{i(m)}^-, k_1, i \in I, m \in \mathbb{Z} \) of \( \hat{g}[\mu] \) satisfy the defining relations (T0-T6) of \( \mathcal{M}(\mathfrak{g}, \mu) \). The relations (T0-T4) follow from a direct verification by using formula (3.1), and the relations (T5-T6) are immediate from Proposition 5.4 \( \square \)

Denote by \( \phi_\mu : \mathcal{M}(\mathfrak{g}, \mu) \rightarrow \hat{g} \) the Lie homomorphism given in Lemma 5.5 and \( \bar{\phi}_\mu = \psi_\mu \circ \phi_\mu : \mathcal{M}(\mathfrak{g}, \mu) \rightarrow \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) \) the composition of the map \( \phi_\mu \) and the universal central extension \( \psi_\mu : \hat{g}[\mu] \rightarrow \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) \). By the universal property of \( \psi_\mu \), we know that Theorem 3.6 is implied by the following result.

**Proposition 5.6.** The Lie homomorphism \( \bar{\phi}_\mu : \mathcal{M}(\mathfrak{g}, \mu) \rightarrow \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) \) is a central extension.

The rest part of this subsection is devoted to a proof of Proposition 5.6. Notice that there is a (unique) \( \hat{Q}_\mu \)-grading \( \mathcal{M}(\mathfrak{g}, \mu) = \bigoplus_{(\alpha, n) \in \hat{Q}_\mu} \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, n} \) on \( \mathcal{M}(\mathfrak{g}, \mu) \) such that
\[
\deg c = (0, 0), \quad \deg h_{i, m} = (0, m) \quad \text{and} \quad \deg x_{i, m}^{\pm} = (\pm \alpha_i, m), \quad i \in I, m \in \mathbb{Z}.
\]

We also introduce a \( \hat{Q}_\mu \)-grading structure \( \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) = \bigoplus_{(\alpha, n) \in \hat{Q}_\mu} \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu})_{\alpha, n} \) so that the quotient map \( \psi_\mu : \hat{g}[\mu] \rightarrow \mathcal{L}(\bar{\mathfrak{g}}, \bar{\mu}) \) is graded. It is obvious that the homomorphism \( \phi_\mu \) is \( \hat{Q}_\mu \)-graded (see (5.1)) and so is the homomorphism \( \bar{\phi}_\mu \).

Let \( \mathcal{M}(\mathfrak{g}, \mu)^\pm \) be the subalgebra of \( \mathcal{M}(\mathfrak{g}, \mu) \) generated by \( \{ x_{i, m}^\pm \mid i \in I, m \in \mathbb{Z} \} \), and \( \mathcal{M}(\mathfrak{g}, \mu)^0 \) the subalgebra of \( \mathcal{M}(\mathfrak{g}, \mu) \) generated by \( \{ h_{i, m} \mid i \in I, m \in \mathbb{Z} \} \). Then we have the following triangular decomposition of \( \mathcal{M}(\mathfrak{g}, \mu) \), whose proof is straightforward and omitted.

**Lemma 5.7.** One has that \( \mathcal{M}(\mathfrak{g}, \mu) = \mathcal{M}(\mathfrak{g}, \mu)^+ \oplus \mathcal{M}(\mathfrak{g}, \mu)^0 \oplus \mathcal{M}(\mathfrak{g}, \mu)^- \).

Recall from Lemma 5.2 that \( \pi_\mu(\Delta)^\times = \{ \bar{w}(k_i \tilde{\alpha}_i) \mid \bar{w} \in \bar{W}, i \in \bar{I}, 1 \leq k_i \leq s_i \} \).

**Lemma 5.8.** Let \( (\alpha, p) \in \hat{Q}_\mu^\times \). Then the following results hold true

1) if \( \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, p} \neq 0 \), then \( \alpha \in \pi_\mu(\Delta)^\times \);

2) if \( \alpha = \bar{w}(\tilde{\alpha}_i) \) for some \( i \in \bar{I} \) and \( \bar{w} \in \bar{W} \), then the dimension of the graded subspace \( \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, p} \) is 1 if \( p \in d_i \mathbb{Z} \), and is 0 otherwise;

3) if \( \alpha = \bar{w}(2\tilde{\alpha}_i) \) for some \( i \in \bar{I} \) with \( s_i = 2 \) and \( \bar{w} \in \bar{W} \), then the dimension of the graded subspace \( \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, p} \) is 1 if \( p \in d_i + N \mathbb{Z} \), and is 0 otherwise.

Proof. Denote by \( \mathcal{M}_0(\mathfrak{g}, \mu) \) the subalgebra of \( \mathcal{M}(\mathfrak{g}, \mu) \) generated by the elements \( h_{i, 0}, x_{i, 0}^\pm, i \in \bar{I} \).

Then one concludes from the relations (T2)-(T5) that \( \mathcal{M}_0(\mathfrak{g}, \mu) \) is the derived subalgebra of the Kac-Moody algebra associated to \( \bar{A} \). Viewing \( \mathcal{M}(\mathfrak{g}, \mu) \) as an
\( \mathcal{M}_0(\mathfrak{g}, \mu) \)-module by the adjoint action, we see from (T3)-(T6) that the \( \mathcal{M}_0(\mathfrak{g}, \mu) \)-module \( \mathcal{M}(\mathfrak{g}, \mu) \) is integrable. Moreover, for each \( p \in \mathbb{Z} \), the subspace
\[
\mathcal{M}(\mathfrak{g}, \mu)_p = \bigoplus_{(\alpha, p) \in \hat{Q}_\mu} \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, p}
\]
of \( \mathcal{M}(\mathfrak{g}, \mu) \) is an \( \mathcal{M}_0(\mathfrak{g}, \mu) \)-submodule. A standard \( \mathfrak{sl}_2 \)-theory argument gives that
\[
\dim \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, p} = \dim \mathcal{M}(\mathfrak{g}, \mu)_{\bar{\alpha}, p}, \quad \bar{w} \in \hat{W}.
\]
Assume now that \( \mathcal{M}(\mathfrak{g}, \mu)_{\alpha, p} \neq 0 \) for some \((\alpha, p) \in \hat{Q}_\mu\). We now prove that \( \alpha \in \pi_\mu(\Delta)^\times \) by using induction on \( \text{ht} \alpha \). Here and as before, \( \text{ht} \alpha = \sum_{i \in \hat{I}} n_i \) if \( \alpha = \sum_{i \in \hat{I}} n_i \alpha_i \). By Lemma 5.7 the integers \( n_i, i \in \hat{I} \) are either all non-negative or all non-positive. We assume that \( \text{ht} \alpha > 0 \), so that all \( n_i \) are non-negative. Then there exist some \( i \in \hat{I} \) such that \( (\hat{\alpha}_i, \alpha) > 0 \) and \( n_i > 0 \). If \( \text{ht} r_{\hat{\alpha}_i}(\alpha) > 0 \), then we are done by the induction hypothesis. Otherwise \( \text{ht} r_{\hat{\alpha}_i}(\alpha) < 0 \) and so \( \alpha = k\hat{\alpha}_i \) for some positive integer \( k \). But the relation (T6) forces that \( 1 \leq k \leq s_i \). This proves the assertion (1).

The assertion (2) is implied by (T0) as \( \mathcal{M}(\mathfrak{g}, \mu)_{\bar{\alpha}, p} = \mathbb{C}x_{i,p}^+ \). As for the assertion (3), we have that \( N_i = 2 \) and \( \alpha_{\mu(i)} = -1 \) in this case. Then by the assertion (2) and Lemma 5.7 we get that
\[
(5.10) \quad \mathcal{M}(\mathfrak{g}, \mu)_{2\hat{\alpha}_i, p} = \sum_{m+n=p, m,n \in (N/2)\mathbb{Z}} [\mathcal{M}(\mathfrak{g}, \mu)_{\hat{\alpha}_i, m}, \mathcal{M}(\mathfrak{g}, \mu)_{\hat{\alpha}_i, n}].
\]
So the proof of the assertion (3) can be reduced to the proof of the following facts: \( \mathcal{M}(\mathfrak{g}, \mu)_{2\hat{\alpha}_i, p} = 0 \) if \( p \in N\mathbb{Z} \), and \( \dim \mathcal{M}(\mathfrak{g}, \mu)_{2\hat{\alpha}_i, p} = 1 \) if \( p \in N/2 + N\mathbb{Z} \). We first show that \( \mathcal{M}(\mathfrak{g}, \mu)_{2\hat{\alpha}_i, p} = 0 \) if \( p \in N\mathbb{Z} \). By (5.10), this is implied by
\[
(5.11) \quad [x_{i,M/2}, x_{i,N/2}^+] = 0, \quad \text{if } m \equiv n \pmod{2}.
\]
Using (T4), we have that
\[
[x_{i,M/2}, x_{i,N/2}^-] = \frac{N}{2} h_{i,(m+n)N/2} + a c,
\]
for some \( a \in \mathbb{C} \). And by (T3), we have that
\[
[h_{i,M/2}, x_{i,N/2}^+] = \frac{2 - (-1)^m}{2} x_{i,(m+n)N/2}^+.
\]
Thus, if \( m \equiv n \pmod{2} \), then
\[
[[x_{i,0}, x_{i,0}^-], [x_{i,M/2}, x_{i,N/2}^+]] = \frac{N^2}{2} [x_{i,M/2}, x_{i,N/2}^+],
\]
\[
[[x_{i,M/2}, x_{i,N/2}^+], x_{i,0}^-] = 0.
\]
Combining with (T6), we get that
\[
\frac{N^2}{2} [x_{i,M/2}, x_{i,N/2}^+] = [[x_{i,0}, x_{i,0}^-], [x_{i,M/2}, x_{i,N/2}^+]]
\]
This completes the verification of (5.11).

We now turn to prove the fact that \( \dim M(g, \mu)_{2\alpha_i, p} = 1 \) if \( p \in N/2 + N\mathbb{Z} \). It follows from (T3) and (T4) that

\[
[x_{i,0}^-, [x_{i,0}, M(g, \mu)_{2\alpha_i, p}]] \subset \mathcal{C}h_{i,p}.
\]

It is immediate from the (T2), (T3) and (T4) that \( \mathbb{C}x_{i,0}^+ + \mathbb{C}x_{i,0}^- + \mathbb{C}h_{i,0} \cong \mathfrak{s}\mathfrak{l}_2 \).

Then by (5.10), (5.12) and the assertion (1), we find that the space spanned by

\[
M(g, \mu)_{k\alpha_i, p, h_{i,p}, k = \pm 1, \pm 2}
\]

is an irreducible \( \mathfrak{s}\mathfrak{l}_2 \)-module. This gives that \( \dim M(g, \mu)_{2\alpha_i, p} \leq 1 \). But one can conclude from Proposition 5.1 that

\[
\dim M(g, \mu)_{2\alpha_i, p} \geq \dim \hat{\mathfrak{g}}[\mu]_{2\alpha_i, p} = 1,
\]
as \( \phi_\mu \) is a graded surjective homomorphism. Thus we complete the proof of the assertion (3).

Now we are in a position to complete the proof of Proposition 5.6. It follows from Proposition 5.1 and Lemma 5.8 that

\[
ker \hat{\phi}_\mu \subset M(g, \mu)^{iso} = \bigoplus_{(\alpha, p) \in Q_0^\mu} M(g, \mu)_{\alpha, p},
\]

where

\[
Q_0^\mu = \{ (\alpha, p) \in \hat{Q}_\mu | (\alpha, \alpha) = 0 \}.
\]

Note that \( \hat{Q}_0^\mu + \hat{Q}_\mu^\times \subset \hat{Q}_\mu^\times \), which in particular shows that

\[
[x_{i,m}^+, M(g, \mu)^{iso}] \cap M(g, \mu)^{iso} = \{0\}, \quad \text{for } i \in I, m \in \mathbb{Z}.
\]

Finally, Proposition 5.6 is implied by (5.13) and (5.14), as the Lie algebra \( M(g, \mu) \) is generated by the elements \( x_{i,m}^\pm, i \in I, m \in \mathbb{Z} \).

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