Konishi Anomalies and Curves without Adjoint

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Abstract

Generalized Konishi anomaly relations in the chiral ring of $\mathcal{N}=1$ supersymmetric gauge theories with unitary gauge group and chiral matter field in two-index tensor representations are derived. Contrary to previous investigations of related models we do not include matter multiplets in the adjoint representation. The corresponding curves turn out to be hyperelliptic. We also point out equivalences to models with orthogonal or symplectic gauge groups.
1 Introduction

Supersymmetric field theories are among the most interesting theoretical laboratories to investigate gauge theories in the strongly coupled regime. It is often possible to obtain exact results on the non perturbative dynamics. A recent prominent example is the usage of a generalized form of the well-known Konishi anomaly [1] to compute the exact effective superpotential [2], see also [3]. This usage of generalized Konishi anomalies was motivated by the the Dijkgraaf-Vafa conjecture [4]. It says that the exact effective superpotential of a confining $\mathcal{N}=1$ supersymmetric gauge theory can be computed using a matrix model whose action is given by the tree level superpotential of the field theory.

The prototypical example in this line of research is $\mathcal{N}=2$ supersymmetric gauge theory softly broken to $\mathcal{N}=1$ by a superpotential for the chiral multiplet in the adjoint representation. Let us briefly recall the construction of [2]. With the help of the generalized Konishi anomalies one finds algebraic relations for the gauge invariant operators

$$
T(z) = \left\langle \text{tr} \left( \frac{1}{z - \phi} \right) \right\rangle, \quad R(z) = -\frac{1}{32\pi^2} \left\langle \text{tr} \left( \frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{z - \phi} \right) \right\rangle, \quad (1)
$$

where $\phi$ is the scalar component of the chiral multiplet in the adjoint and $\mathcal{W}_\alpha$ the gaugino field. It turns out that $R(z)$ takes values on a hyperelliptic Riemann surface and that $T(z)$ defines a meromorphic differential on it. The superpotential can be evaluated in terms of period integrals on this Riemann surface.

Much work has been devoted to generalize the approach of [2] to theories with different gauge groups and matter content, see the review [5] for a list of references. However, most of the work so far has been based on the presence of a chiral multiplet in the adjoint representation of the gauge group. One exception to this is [6], [7] where orthogonal gauge groups with symmetric tensors and symplectic gauge groups with antisymmetric tensor have been considered. These theories also presented a puzzle in how to compute correctly the effective superpotential that took some time to resolve and understand completely [8], [9], [10], [11]. Konishi anomalies for theories with chiral spectrum and no additional adjoints have also been studied in [12], [13], [14].

Motivated by these developments we will investigate generalized Konishi anomaly relations for theories with unitary gauge groups and chiral matter
multiplets in the symmetric and antisymmetric representation. Contrary to the SO/Sp case the two-index representations of unitary groups are complex, and therefore we need two chiral multiplets in conjugate representations. A different possibility is to combine a chiral multiplet in the antisymmetric representation with a chiral multiplet in the conjugate symmetric representation and eight fundamentals to cancel the chiral anomaly. Not including any further matter fields there are then three models we can consider.

The first two only differ in the choice of symmetry of the chiral multiplets

\[ X^T = \epsilon X, \quad Y^T = \epsilon Y, \]

with \( \epsilon = \pm 1 \). The gauge transformations are

\[ X \rightarrow UXU^T, \quad Y \rightarrow U^*YU^\dagger, \]

where the star denotes complex conjugation and \( U \) is a unitary \( N \times N \) matrix.

The third model has a chiral fermion spectrum, it consists of an antisymmetric field \( A \) a symmetric field \( S \) and eight fundamentals \( Q_f \), where \( f = 1 \ldots 8 \) denotes the flavor index.

\[ S^T = S, \quad A^T = -A, \]

with gauge transformations

\[ S \rightarrow U^*SU^\dagger, \quad A \rightarrow UA^T, \quad Q_f \rightarrow UQ_f. \]

For all models we will consider a polynomial tree level superpotential in \( XY \) or \( AS \) respectively. In the chiral case we will also include a coupling between the symmetric field and the eight fundamentals. We will discuss the classical vacua first and compute then the generalized Konishi anomalies. We also will show that one can define holomorphic matrix models \[15\] whose loop equations can be mapped in an large \( N \) expansion to the Konishi anomalies of the gauge theories. Models with this spectrum and an additional chiral multiplet in the adjoint representation have been investigated in \[16\], \[17\], \[18\].

After completion of this work \[23\] appeared where the non-chiral theory with antisymmetric representation is also studied. The curve defined by the Konishi anomaly there is a double cover of the one discussed here.
2 Non-chiral Models

In this section we will investigate the models with chiral multiplets in mutually conjugate (anti)symmetric tensor representations and tree level superpotential

\[ W = \sum_{k=0}^{d} \frac{g_{k}}{k+1} \text{tr} [(XY)^{k+1}] . \]  

(6)

2.1 Classical Moduli Space

As is well-known the moduli space of a supersymmetric gauge theory can be obtained as the critical points of the superpotential modulo complexified gauge transformations. In the case at hand it is useful to consider the variation

\[ X \frac{\partial W}{\partial X} = XY \sum_{k=0}^{d} g_{k} (XY)^{k} . \]  

(7)

Note that \( XY \) transforms in the adjoint representation

\[ (XY) \rightarrow U(XY)U^{-1} . \]  

(8)

This can be used to diagonalize \( XY = \text{diag}(\xi_{1}1_{N_{1}}, \ldots, \xi_{d+1}1_{N_{d+1}}) \) where the eigenvalues have to fulfill

\[ \xi W(\xi) = 0 . \]  

(9)

Note that independent of the particular choice of the couplings \( g_{k} \xi = 0 \) is always a possible eigenvalue in a vacuum! We find therefore \( XY = \text{diag}(0_{N_{0}}, \xi_{1}1_{N_{1}}, \ldots, \xi_{d}1_{N_{d}}) \). Since the non-vanishing eigenvalues come from fields in the symmetric or antisymmetric representation the gauge symmetry breaking pattern is

\[ U(N) \longrightarrow \left\{ \begin{array}{ll} U(N_{0}) \bigotimes_{i=1}^{d} SO(N_{i}) , & \epsilon = +1 \\ U(N_{0}) \bigotimes_{i=1}^{d} Sp(N_{i}) , & \epsilon = -1 \end{array} \right. , \]  

(10)

where\(^{1} \) \( N = \sum_{i=0}^{d} N_{i} \).

We can arrive at this conclusion also in a slightly different way. For definiteness let us study the case of symmetric representations. Since the gauge group is complexified we can bring any vev of the field \( Y \) to the form

\(^{1}\text{Our conventions are } SP(2) \equiv SU(2).\)
\[ Y = \text{diag}(0_{N_0}, 1_{\tilde{N}}). \] This breaks the gauge group in a first step to \( U(N_0) \otimes SO(\tilde{N}) \). Now we write
\[
X = \begin{pmatrix}
 a & b \\
 b^T & c
\end{pmatrix}
\]  
(11)
where \( a \) is a \( N_0 \times N_0 \) matrix, \( b \) a \( N_0 \times \tilde{N} \) matrix and \( c \) a \( \tilde{N} \times \tilde{N} \) matrix. Notice now that \( a = b = 0 \) lies on the extended gauge orbit of \( U(N_0, \mathbb{C}) \) which leaves us with the matrix \( c \) that transforms as a symmetric representation of \( SO(\tilde{N}, \mathbb{C}) \) and can be diagonalized to \( c = \text{diag}(\xi_1 1_{N_1}, \ldots, \xi_d 1_{N_d}) \). In this way we arrive at the gauge breaking pattern (10) for \( \epsilon = +1 \). The case with antisymmetric representations works analogous. We can now choose \( Y = \text{diag}(0_{N_0}, J_{\tilde{N}}) \) where
\[
J_{\tilde{N}} = 1_{N/2} \otimes i\sigma_2 \quad \text{where} \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .
\]  
(12)
This breaks the gauge group to \( U(N_0) \otimes SP(\tilde{N}) \). The same arguments as before tell us that from \( X \) we obtain only a field \( c \) transforming under the antisymmetric representation of \( SP(\tilde{N}) \) and \( cJ_{\tilde{N}} \) can be brought into the form \( c = -\text{diag}(\xi_1 1_{N_1/2}, \ldots, \xi_d 1_{N_d/2}) \otimes i\sigma_2 \) that breaks the group further to the second line in (10).

We thus find that the moduli space consists of isolated points given by the eigenvalues \( \xi_i \) of the adjoint valued matrix \( XY \) fulfilling \( \xi_i W(\xi_i) = 0 \) and that the gauge breaking pattern is described by (10).

### 2.2 Konishi Anomalies

Following [2] we will now discuss the chiral ring relations that follow from generalized forms of the Konishi Anomalies. It is useful to remember the general chiral ring relation
\[
W^{(r)}_\alpha \Phi^{(r)} \equiv 0 .
\]  
(13)
Here \( \Phi^{(r)} \) is a chiral operator transforming in the representation \( r \) of the gauge group and \( W_\alpha \) is the chiral operator corresponding to the gaugino field. Equation (13) holds as an equivalence relation inside the chiral ring. In our case this can be written more explicitly with the help of matrix representations as
\[
W_\alpha X \equiv W_\alpha X + XW^T_\alpha \equiv 0 ,
\]  
(14)
\[
W_\alpha Y \equiv -Y W_\alpha - W^T_\alpha Y \equiv 0 ,
\]  
(15)

where $\mathcal{W}_\alpha$ is in the fundamental $N \times N$ matrix representation and juxtaposition stands for matrix multiplication. The chiral operator $X Y$ commutes with $\mathcal{W}_\alpha$ since it transforms according to the adjoint representation. Using these relations we see that the chiral ring of gauge and Lorentz invariant operators is spanned by $\text{tr} [(X Y)^k], \text{tr} [\mathcal{W}_\alpha \mathcal{W}_\alpha (X Y)^k]$. If we do not restrict to Lorentz singlets we also have $\text{tr} [\mathcal{W}_\alpha (X Y)^k]$ these have however vanishing expectation value in a supersymmetric vacuum. The vacuum amplitudes of the gauge and Lorentz invariant single trace operators can be formally summed up in the generating functions

$$T(z) := \left\langle \text{tr} \left( \frac{1}{z - X Y} \right) \right\rangle, \quad R(z) := \left\langle -\frac{1}{32\pi^2} \text{tr} \left( \frac{\mathcal{W}^2}{z - X Y} \right) \right\rangle \quad (16)$$

The generalized Konishi anomaly is the anomalous Ward identity for a holomorphic field transformation:

$$\mathcal{O}^{(r)} \longrightarrow \mathcal{O}^{(r)} + \delta \mathcal{O}^{(r)} \quad (17)$$

As a relation in the chiral ring it can be written as

$$\left\langle \delta \mathcal{O}_I \frac{\partial \mathcal{W}_I}{\partial \mathcal{O}_I} + \frac{1}{32\pi^2} \mathcal{W}_I^J \mathcal{W}_{\alpha, J}^K \frac{\partial (\delta \mathcal{O}_K)}{\partial \mathcal{O}_I} \right\rangle = 0 \quad (18)$$

where the capital indices enumerate a basis of the representation $r$.

We will investigate the generalized Konishi relations corresponding to the field transformations:

$$\delta_1 X = \frac{1}{z - X Y} X, \quad \delta_2 X = -\frac{1}{32\pi^2} \frac{\mathcal{W}^2}{z - X Y} X \quad (19)$$

Let us first show that these variations are indeed symmetric or antisymmetric respectively

$$(\delta_1 X)^T = X^T \left( \frac{1}{z - X Y} \right)^T = X^T \frac{1}{z} \sum\limits_{n=0}^{\infty} \frac{(Y^T X^T)^n}{z^n} = \frac{1}{z} \sum\limits_{n=0}^{\infty} \frac{(X^T Y^T)^n}{z^n} X^T = \frac{\epsilon}{z} \sum\limits_{n=0}^{\infty} \frac{(XY)^n}{z^n} X = \epsilon \delta_1 X \quad (20)$$

$$= \frac{1}{z} \sum\limits_{n=0}^{\infty} \frac{(X^T Y^T)^n}{z^n} X^T = \frac{\epsilon}{z} \sum\limits_{n=0}^{\infty} \frac{(XY)^n}{z^n} X = \epsilon \delta_1 X \quad (21)$$

and similarly for the variation with the $\mathcal{W}^2$ insertion where one also has to use the chiral ring relations (14).
We will evaluate the tree level term and the anomalous term of the Konishi relation \(^{(18)}\) for the variation \(\delta_1 X\) now separately. The tree level term is

\[
\left\langle \text{tr} \left( \delta_1 X \frac{\partial W(XY)}{\partial X} \right) \right\rangle = \left\langle \text{tr} \left( \frac{XYW'(XY)}{z - XY} \right) \right\rangle = c(z) + T(z) z W'(z), \tag{22}
\]

where

\[
- \text{tr} \left( \frac{z W'(z) - XY W'(XY)}{z - XY} \right) = c(z) \tag{23}
\]

is a polynomial of degree \(d\)!

Now we evaluate the anomalous term.

\[
\left\langle \text{Tr} \left( W^\alpha \cdot W_\alpha \cdot \frac{\partial (\delta X)}{\partial X} \right) \right\rangle = \text{tr} \left\langle \left( W^\alpha \cdot W_\alpha \frac{\partial (\delta X)}{\partial X} + W^\alpha \frac{\partial (\delta X)}{\partial X} W_\alpha - W_\alpha \frac{\partial (\delta X)}{\partial X} W^\alpha T - \frac{\partial (\delta X)}{\partial X} W^\alpha W_\alpha T \right) \right\rangle, \tag{24}
\]

\[
= \left\langle W^\alpha W_\alpha \frac{\partial (\delta X)}{\partial X} + W^\alpha \frac{\partial (\delta X)}{\partial X} W_\alpha - W_\alpha \frac{\partial (\delta X)}{\partial X} W^\alpha T - \frac{\partial (\delta X)}{\partial X} W^\alpha W_\alpha T \right\rangle \tag{25}
\]

where \(\text{Tr}\) denotes the trace in the (anti)symmetric representation and \(\text{tr}\) the trace in the fundamental. We evaluate further

\[
\left\langle (W^\alpha)^i k \frac{\partial (\delta X_{km})}{\partial X_{im}} \right\rangle = \left\langle W^\alpha W_\alpha \frac{\partial (\delta X)}{\partial X} + W^\alpha \frac{\partial (\delta X)}{\partial X} W_\alpha - W_\alpha \frac{\partial (\delta X)}{\partial X} W^\alpha T - \frac{\partial (\delta X)}{\partial X} W^\alpha W_\alpha T \right\rangle \tag{26}
\]

and

\[
\left\langle (W^\alpha)^i k \frac{\partial (\delta X_{km})}{\partial X_{il}} W_{\alpha, l} \right\rangle = \left\langle W^\alpha W_\alpha \frac{\partial (\delta X)}{\partial X} + W^\alpha \frac{\partial (\delta X)}{\partial X} W_\alpha - W_\alpha \frac{\partial (\delta X)}{\partial X} W^\alpha T - \frac{\partial (\delta X)}{\partial X} W^\alpha W_\alpha T \right\rangle \tag{27}
\]

where in the last term we took the vacuum expectation value of the spinor values single trace operators to vanish. The third term in (24) gives the same result as (26) and the last term gives the same result as (26).
Let us now compute the Konishi anomaly relation for the variation $\delta_2 X$. Here it is useful to notice that

$$(\delta_2 X)_{km} = (W^2)_{l}^{i} (\delta_1 X)_{lm} \quad (28)$$

Using this and the previously derived relation for $\delta_1 X$ we find for the tree level term

$$\left\langle \text{tr} \left( \delta_2 X \frac{\partial W_{\text{tree}}}{\partial X} \right) \right\rangle = f(z) + zW'(z)R(z), \quad (29)$$

where

$$f(z) = \left\langle \text{tr} \left( \frac{W^2(zW'(z) - XYW'(XY))}{32\pi^2(z - XY)} \right) \right\rangle. \quad (30)$$

For the anomalous term we find now

$$\left\langle \text{Tr} \left( W^\alpha W^\beta \frac{\partial (\delta X)}{\partial X} \right) \right\rangle = \frac{z}{2} R^2(z) \quad (31)$$

Taking these results together we obtain

$$\frac{z}{2} R(z)^2 - zW'(z)R(z) - f(z) = 0, \quad (32)$$

$$zR(z)T(z) - zW'(z)T(z) - 2\epsilon zR'(z) - c(z) = 0. \quad (33)$$

As is well-known in the case of adjoint representations, the equation for $R(z)$ defines a hyperelliptic Riemann surface. We can set $R = y + W'$ and find thus

$$y^2 = (W')^2 + \frac{2f(z)}{z}. \quad (34)$$

This equation has the somewhat unusual feature that $y$ tends to $\infty$ as $z$ goes to 0. This curve is a double cover of the $z$-plane with branchpoints at $y = 0$ and a distinguished branchpoint at $y = \infty$, $z = 0$. This branchpoint is always present and at the fixed locus $z = 0$ unless the coefficient $f_0$ in the polynomial $f(z) = \sum_{i=0}^{d} f_i z^i$ vanishes. Notice further that if in addition the coefficient $c_0$ in $c(z) = \sum_{i=0}^{d} c_i z^i$ vanishes the Konishi relations (32), (33) take the same form as the ones for $SO$ gauge group with symmetric matter ($\epsilon = +1$) or $SP$ gauge group with antisymmetric matter ($\epsilon = -1$). This could have been expected of course from our analysis of the classical moduli space which showed that in a generic vacuum with $\xi \neq 0$ the gauge group is either $SO$ or $SP$. The special point $\xi = 0$ in the classical moduli space gives
rise to the special fixed point at $z = 0$ in the quantum theory. As usual the
gaugino condensates in the factor groups and their ranks are given by period
integrals on

$$S_i = \oint_{A_i} y dz , \quad N_i = \oint_{A_i} T(z) dz . \tag{35}$$

where $A_i$ are compact cycles surrounding the cuts of $y$.

### 2.3 Matrix Model

Based on string theory considerations Dijkgraaf and Vafa conjectured
that the exact effective superpotential of a confining $\mathcal{N}=1$ supersymmetric
gauge theory can be computed with the help of a simple matrix model. This
conjecture has been proved for many different models. The perturbative part
of the conjecture can be proved using superfield techniques in perturbation
theory. A different approach is based on comparing the $1/\hat{N}$ expansion
of the loop equations of the matrix model with the Konishi anomaly relations
of the field theory. This approach is non-perturbative in nature although it
has to be emphasized that the Konishi anomaly relation as stated in (18) is
proven to be exact only in perturbation theory. Of course, once a one-to-one
map to the loop equations of the matrix model (and therefore string theory)
is found, one can take this as evidence for the non-perturbative exactness of
the Konishi anomaly relations.

The precise definition of the matrix model has been worked out in where the need of a holomorphic definition has been emphasized. We will
follow this viewpoint here. The partition function of the holomorphic matrix
model is given by

$$Z = \frac{1}{|G|} \int_{\Gamma} d\hat{X} d\hat{Y} e^{-\frac{1}{\kappa} \text{tr}[W(\hat{X}\hat{Y})]} , \tag{36}$$

where $\hat{X}$ and $\hat{Y}$ are the matrices corresponding to the chiral multiplets
$X, Y$ of the gauge theory. They are thus complex $\hat{N} \times \hat{N}$ matrices obeying $(\hat{X}^T, \hat{Y}^T) = \epsilon(\hat{X}, \hat{Y})$. $|G|$ is a normalization factor including the volume
of the gauge group and $\Gamma$ is a suitably chosen path in the configuration space
$\mathcal{M}$ of the matrices $\hat{X}, \hat{Y}$ with $\dim_\mathbb{R}(\Gamma) = \dim_\mathbb{C}(\mathcal{M})$.

The matrix model action $W(\hat{X}\hat{Y})$ has the gauge symmetry

$$\hat{X} \rightarrow g\hat{X}g^T , \quad \hat{Y} \rightarrow (g^{-1})^T \hat{Y} g^{-1} , \tag{37}$$
where \( g \in GL(\hat{N}, \mathbb{C}) \). Before attempting to evaluate the path integral we therefore have to fix the gauge. Doing so we have to treat the two cases in slightly different ways.

Let us start with \( \epsilon = +1 \), i.e. \( \hat{X}, \hat{Y} \) being symmetric. We chose the gauge \( \hat{Y} = \mathbf{1}_{\hat{N}} \). We could implement this via the BRST formalism. However, with this gauge choice the ghost sector decouples from the \( d\hat{X} \) integrations. Furthermore, it is clear that only the symmetric part of the \( gl(\hat{N}, \mathbb{C}) \) valued ghost fields contribute. The gauge fixing is not complete, gauge transformations leaving \( \hat{Y} = \mathbf{1}_{\hat{N}} \) invariant, i.e. obeying \( gg^T = 1 \) survive. Therefore after this partial gauge fixing we are left with a matrix model based on an \( SO(\hat{N}, \mathbb{C}) \) gauge group and a symmetric field \( \hat{X} \)! At this point we can do a further gauge fixing \( \hat{X} = \text{diag}(\lambda_1, \ldots, \lambda_{\hat{N}}) \). This time of course the ghost sector contributes non-trivially and leads to the insertion of the Vandermonde determinant in the integral, which of course is the same one as for the \( SO \)-theory with symmetric matter. We find therefore [18]

\[
Z_{\epsilon=+1} = \frac{1}{|G'|} \int \prod_{k=1}^{\hat{N}} d\lambda_k \prod_{m<n} |\lambda_m - \lambda_n| e^{-\frac{\hat{N}}{\kappa} \sum_i W(\lambda_i)} .
\]

In the intermediate steps leading to this formula we have to chose an appropriate path \( \Gamma \) in the matrix configuration space including the ghosts such that all the integrals converge and \( \lambda_i \in \mathbb{R} \). This is by now well-studied in many examples and we therefore do not give any more details. The interested reader is instead referred to [15] and chapter 6.1 in [16].

At this point one might conclude that the loop equation for our model is the same as the one for the \( SO \) model with symmetric matter and that therefore a relation to the Konishi anomalies of the field theory can not be established or could at most be established for the vacua with \( \xi \neq 0 \). This would be wrong for the following reason. The loop equations in the eigenvalue representation for the \( SO \) model follow from the identity

\[
0 = \int \prod_{k=1}^{\hat{N}} d\lambda_k \left[ \sum_r \frac{\partial}{\partial \lambda_r} \left( \frac{1}{z - \lambda_r} \prod_{m<n} |\lambda_m - \lambda_n| e^{-\frac{\hat{N}}{\kappa} \sum_i W(\lambda_i)} \right) \right] .
\]

It is the insertion of \( \frac{1}{z - \lambda_r} \) that is problematic. It corresponds to the variation \( \delta \hat{X} = \frac{1}{z - \hat{X}} \). This variation is of course valid in the model based on the \( SO \) gauge group. In our model we should however not forget that the underlying
gauge symmetry is $GL(\hat{N}, \mathbb{C})$ and that $\hat{X}$ is a symmetric two-tensor under this symmetry. This reasoning shows that such an insertion is not gauge invariant in our model. Rather than (39) the underlying gauge symmetry instructs us to use the identity

$$0 = \int \prod_{k=1}^{\hat{N}} d\lambda_k \left[ \sum_r \frac{\partial}{\partial \lambda_r} \left( \frac{\lambda_r}{z - \lambda_r} \prod_{m<n} |\lambda_m - \lambda_n|^{-\frac{1}{\kappa}} e^{-\frac{1}{\kappa} \sum \lambda_i W(\lambda_i)} \right) \right], \quad (40)$$

corresponding to the variation $\delta \hat{X} = \frac{1}{z - \lambda_i} \hat{X}$ respecting the $GL(\hat{N}, \mathbb{C})$ gauge symmetry. More explicitly the identity (40) is

$$\left\langle \sum_{i=r}^{\hat{N}} \left( \frac{1}{z - \lambda_r} + \frac{\lambda_r}{(z - \lambda_r)^2} - \frac{1}{\kappa} \frac{\lambda_r W'(\lambda_r)}{z - \lambda_r} + \sum_{s \neq r, \lambda_s} \frac{1}{z - \lambda_s} \right) \right\rangle = 0 \quad (41)$$

Here we defined

$$\left\langle \sum_r f(\lambda_r) \right\rangle = \int \prod_{k=1}^{\hat{N}} d\lambda_k \sum_r f(\lambda_r) \prod_{m<n} |\lambda_m - \lambda_n|^{-\frac{1}{\kappa}} e^{-\frac{1}{\kappa} \sum \lambda_i W(\lambda_i)} \cdot (42)$$

We further define the matrix model resolvent as

$$\omega(z) = \kappa \sum_{i=1}^{\hat{N}} \frac{1}{z - \lambda_i}, \quad (43)$$

and a polynomial

$$\hat{f}(z) = -\kappa \sum_{i=1}^{\hat{N}} \frac{z W'(z) - \lambda_i W'(\lambda_i)}{z - \lambda_i}. \quad (44)$$

Using

$$\kappa \sum_i \left( \frac{1}{z - \lambda_i} + \frac{\lambda_i}{(z - \lambda_i)^2} \right) = -z \omega'(z) \quad (45)$$

and

$$\sum_{i \neq j} \frac{\lambda_i}{z - \lambda_i} \frac{1}{\lambda_i - \lambda_j} = \frac{1}{2} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \left( \frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_j}{z - \lambda_j} \right) = \frac{z}{2} \left( \sum_{i,j} \frac{1}{(z - \lambda_i)(z - \lambda_j)} - \sum_i \frac{1}{(z - \lambda_i)^2} \right) \quad (46)$$

$^2$The sums in the following formula are double sums over both indices $i$ and $j$. 

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we find the loop equation
\[
\left\langle \frac{1}{2}z\omega(z)^2 - \frac{\kappa}{2}z\omega'(z) - zW'(z)\omega(z) - \hat{f}(z) \right\rangle = 0.
\] (47)

In the case when \( \epsilon = -1 \) the discussion is similar. We can chose in a first step the gauge \( \hat{Y} = \mathbf{J}_N \). This leaves us with the residual gauge group \( SP(\hat{N}, \mathbb{C}) \) and a matter field \( \hat{X} \) in the symmetric representation. The second step in the gauge fixing procedure is now to set \( \hat{X}\mathbf{J}_N = \text{diag}(\lambda_1, \ldots, \lambda_{\hat{N}/2}) \otimes \mathbf{1}_2 \). The eigenvalue representation of the partition function coincides therefore with the one of the model base on \( SP \) gauge groups and antisymmetric matter fields [11]

\[
Z_{\epsilon = -1} = \int \prod_{k=1}^{\hat{N}/2} d\lambda_k \prod_{m<n} (\lambda_m - \lambda_n)^4 e^{-\frac{2}{\kappa} \sum_i W(\lambda_i)}.
\] (48)

The same reasoning as before leads us to consider

\[
0 = \int \prod_{k=1}^{\hat{N}/2} d\lambda_k \left[ \sum_r \frac{\partial}{\partial \lambda_r} \left( \frac{\lambda_r}{z - \lambda_r} \prod_{m<n} (\lambda_m - \lambda_n)^4 e^{-\frac{2}{\kappa} \sum_i W(\lambda_i)} \right) \right],
\] (49)

The resolvent is now defined as

\[
\omega(z) = 2\kappa \sum_{i=1}^{\hat{N}/2} \frac{1}{z - \lambda_i},
\] (50)

and the matrix model polynomial is defined by

\[
\hat{f}(z) = -2\kappa \sum_{i=1}^{\hat{N}} \frac{zW'(z) - \lambda_i W'(\lambda_i)}{z - \lambda_i}.
\] (51)

The analogous calculations as before lead now to the loop equation.

Concluding our results for the loop equations of the matrix models we find that

\[
\left\langle \frac{z}{2}\omega(z)^2 - \frac{z}{2}\omega'(z) - zW'(z)\omega(z) - \hat{f}(z) \right\rangle = 0,
\] (52)

where the resolvent is defined by (43) or (50) respectively.

To make contact with the Konishi anomaly relations in the field theories we expand in orders of \( \kappa \) which is equivalent to an expansion in \( 1/\hat{N} \)

\[
\langle \omega(z) \rangle = \sum_{k=0}^{\infty} \kappa^k \omega_k.
\] (53)
The expansion of the loop equations is
\[
\frac{z}{2} \omega_0(z)^2 - z W'(z) \omega(z) - \hat{f}(z) = 0, \mathcal{O}(0) \tag{54}
\]
\[
z \omega_0(z) \omega_1(z) - \frac{z}{2} \epsilon \omega'_0(z) - z W'(z) \omega_1(z) = 0, \mathcal{O}(\kappa), \tag{55}
\]
We also consider the differential operator \( \delta = \sum_i N_i \frac{\partial}{\partial S_i} \) and apply it to the loop equation at \( \mathcal{O}(0) \) to find
\[
z \omega_0(z) \delta \omega_0(z) - z W'(z) \delta \omega_1(z) - \delta \hat{f}(z) = 0. \tag{56}
\]
Now it is easy to see that the Konishi anomalies are formally reproduced by the first to terms in the \( 1/N \) expansion of the matrix model loop equations if we set
\[
\omega_0(z) = R(z), \quad \delta \omega_0(z) + 4 \omega_1(z) = T(z) \tag{57}
\]
\[
\hat{f}(z) = f(z), \quad \delta \hat{f}(z) = c(z). \tag{58}
\]
We also identify the filling fractions of the matrix model with the gaugino bilinears in the gauge theory \( \kappa N_i = S_i \) The identification of \( T(z) \) in terms of matrix model quantities implies also the relation between the free energy \( F = -\kappa^2 \log(Z) \) and the field theory superpotential
\[
W_{\text{eff}} = \delta F_0 + 4 F_1 \tag{59}
\]
where we expanded \( F = \sum_{k=0}^{\infty} \kappa^k F_k \). This relation has also been found for the theories in \([6], [17], [18]\).

2.4 Non-Perturbative Superpotential and Normalization

In principle the formula \((59)\) proves the relation between the effective superpotential and the matrix model partition function only up to an integration constant that is independent of the tree-level couplings \( g_k \). This integration constant is commonly taken as the Veneziano-Yankielowicz part of the gaugino superpotential. However, as emphasized already in \([4], [2]\) the matrix model does contain indeed all the information to compute also this non-perturbative contribution to the superpotential. We will illustrate this in the simplest example of a tree level superpotential consisting solely of a mass term, \( W = \text{mtr}(XY) \).
The matrix model partition function is

\[ Z = \frac{K}{\text{vol}(U(\hat{N}))} \int_{\Gamma} d\hat{X} d\hat{Y} \ e^{-mtr(\hat{X}\hat{Y})}. \] (60)

We have made the dependence on the volume of the gauge group explicit and also consider an additional normalization constant \( K \). We have also set \( \kappa = 1 \), \( F_0 \) and \( F_1 \) are therefore identified by their scaling in \( \hat{N} \).

The Gaussian integration leads to\(^3\)

\[ F = \log(K) - \log(\text{vol}(U(\hat{N}))) + \frac{1}{2} \hat{N}(\hat{N} + \epsilon) \log(\frac{\pi}{m}) . \] (61)

Using

\[ \log(\text{vol}(U\hat{N}))) = -\frac{\hat{N}^2}{2} \log\left(\frac{\hat{N}}{2\pi e^{3/2}}\right) + O(\hat{N}^0) , \] (62)

and writing \( K = \alpha^{\hat{N}^2} \beta^{\bar{\hat{N}}} \) we find

\[ F = -\log(Z) = -\hat{N}^2 \log(\alpha) - \hat{N} \log(\beta) - \frac{\hat{N}^2}{2} \log\left(\frac{\hat{N}}{2e^{3/2}m}\right) - \frac{\epsilon}{2} \hat{N} \log\left(\frac{\pi}{m}\right) . \] (63)

Setting \( \hat{N} = S \) we therefore have

\[ F_0 = -\frac{S^2}{2} \log\left(\frac{\alpha^2 S}{2e^{3/2}m}\right) , \quad F_1 = -\frac{\epsilon}{2} S \log\left(\frac{m}{\pi^{3/2}}\right) \] (64)

Setting now \( \alpha = \frac{\sqrt{2}}{\Lambda} \) and \( \beta = \left(\frac{\Lambda}{\pi}\right)^{\epsilon/2} \) where \( \Lambda \) is interpreted as the scale of the underlying gauge theory and using (59) we compute the superpotential

\[ W_{\text{eff}} = S \log\left(\frac{\Lambda^{2N-2\epsilon} m^{N+2\epsilon}}{S^N}\right) + NS , \] (65)

which is the expected Veneziano-Yankielowicz superpotential with the scale matching \( \Lambda_{\text{low}}^{2N} = \Lambda^{2N-2\epsilon} m^{N+2\epsilon} \).

It is important to note that for higher order tree-level superpotentials and \( \epsilon = -1 \) we have to take into account the effects described in [8], [10], [11]. For vacua in which the gauge group is broken to \( SP(N_i) \) with an antisymmetric matter field we should keep the corresponding \( S_i \) gaugino condensates different from zero even in the case \( N_i = 0 \) ("\( SP(0) \)" - gauge group factors).

\(^3\)The Path \( \Gamma \) in the integral can be chosen such that \( X^\dagger = Y ! \)
3 Chiral Theory

In this section we want to study briefly the theory with chiral fermion spectrum. It has a chiral multiplet $A$ in the antisymmetric representation, a chiral multiplet $S$ in the conjugate symmetric representation and eight chiral multiplets in the fundamental representations which are needed to cancel the chiral anomaly. The superpotential we choose is given by

$$W = \text{tr} [V(AS)] + \sum_{f=1}^{8} Q_f S Q_f,$$

where $V(z)$ is an even polynomial of order $d+1$. Because of the symmetry properties of $A$ and $S$ odd powers in $V$ vanish

$$\text{tr} [(AS)^{2n+1}] = 0.$$  \hspace{1cm} (67)

Let us first study the classical moduli space. The equations of motion are

$$SAV' + SQ_f = 0,$$
$$V'(AS)AS = 0,$$
$$SQ_f = 0.$$  \hspace{1cm} (68-70)

We can proceed now as in the non-chiral theory. By complexified gauge transformations we can bring $S$ into the form $S = \text{diag}(0_{N_0}, 1_{\tilde{N}})$. This breaks the gauge group in a first step to $U(N_0) \otimes SO(\tilde{N})$. The fundamentals have to lie in the kernel of the matrix $S$ and $A$ we can divide into $N_0 \times \tilde{N}$ blocks

$$Q_f = \begin{pmatrix} \tilde{q}_f \\ 0_{\tilde{N}} \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ -b^T & c \end{pmatrix},$$

where $\tilde{q}_f$ are $N_0$ dimensional row vectors, $a$ is an antisymmetric $N_0 \times N_0$ matrix, $b$ is a $N_0 \times \tilde{N}$ matrix and $c$ is an antisymmetric $\tilde{N} \times \tilde{N}$ matrix; $a = 0$, $b = 0$ and $q_f = 0$ all lie on an extended gauge orbit of $U(N_0, \mathbb{C})$. This leaves $c$ which transforms as an adjoint under the $SO(\tilde{N})$ residual gauge symmetry. By an $SO(N_0, \mathbb{C})$ gauge transformation we can bring it to the form $c = \text{diag}(\lambda_1 1_{N_1} \otimes J_2, \ldots, \lambda_{N_d} 1_{N_d} \otimes J_2)$ which breaks the $SO(\tilde{N})$ to a product of unitary gauge groups. Of course the non-zero $\lambda_i$ have to fulfill $V'(\lambda_i) = 0$. We find therefore the classical moduli space to consist of isolated
points parametrized by $\lambda_i$ fulfilling $\lambda_i V'(\lambda_i) = 0$. The gauge breaking pattern in a general vacuum is

\[ U(N) \to U(N_0) \bigotimes_{i=1}^{d} U(N_i) \ , \ \sum_{i=1}^{d} 2N_i + N_0 = N. \quad (72) \]

The generalized Konishi Anomalies we are interested in follow from the field transformations

\[ \delta_1 A = \frac{1}{2} \left( \frac{1}{z - AS} A + A \frac{1}{z + SA} \right) = \frac{z}{z^2 - (AS)^2} A \quad (73) \]

\[ \delta_2 A = \frac{1}{2} \left( \frac{\mathcal{W}^2}{32\pi^2} \frac{1}{z - AS} A + \frac{\mathcal{W}^2}{32\pi^2} A \frac{1}{z + SA} \right) = \frac{\mathcal{W}^2}{32\pi^2 z^2 - (AS)^2} A \quad (74) \]

\[ \delta Q_f = \rho_{fg} \frac{1}{z - AS} Q_g, \quad (75) \]

where $\rho_{fg}$ is an arbitrary matrix in flavor space. The resolvents we are looking for are

\[ R(z) = -\frac{1}{32\pi^2} \left\langle \text{tr} \left( \frac{\mathcal{W}^2}{z - (AS)} \right) \right\rangle , \ T(z) = \left\langle \text{tr} \left( \frac{\mathcal{W}^2}{z - (AS)} \right) \right\rangle , \quad (76) \]

(67) implies that the resolvents can also be written as

\[ R(z) = -\frac{1}{32\pi^2} \left\langle \text{tr} \left( \frac{\mathcal{W}^2 z}{z^2 - (AS)^2} \right) \right\rangle , \ T(z) = \left\langle \text{tr} \left( \frac{z}{z^2 - (AS)^2} \right) \right\rangle \quad (77) \]

The computation of the Konishi anomalies is now straightforward and rather similar to the non-chiral case. For this reason we will not give any details here. We define the order $d - 1$ polynomials even in $z$

\[ c(z) = -\text{tr} \left( \frac{z V'(z) - AS V'(AS)}{z^2 - (AS)^2} \right). \quad (78) \]

and

\[ f(z) = \text{tr} \left( \frac{\mathcal{W}^2 (z V'(z) - AS V'(AS))}{32\pi^2 (z^2 - (AS)^2)} \right). \quad (79) \]

The generalized Konishi anomalies can be written then as

\[ \frac{1}{2} R(z)^2 - V'(z) R(z) - f(z) = 0, \quad (80) \]

\[ T(z) R(z) - \frac{2}{z} R(z) - V'(z) T(z) - c(z) = 0, \quad (81) \]

\[ \left\langle Q_f \frac{1}{z - AS} S Q_g \right\rangle = \frac{1}{2} R(z) \delta_{fg}. \quad (82) \]

\[ 15 \]
The relations (80) and (82) are precisely the same as the ones for the model with $SO$ gauge group and a chiral multiplet in the adjoint representation [20]. This should be contrasted to the chiral model with additional adjoint in (??) whose generalized Konishi anomaly relations for $R(z)$ and $T(z)$ have been found to be the same as the ones for the $SO$ model with chiral multiplet in the symmetric representation. This implies an exact non-perturbative equivalence between these theories. An exact non-perturbative equivalence for the non-chiral model with antisymmetric representations in the large $N$ limit has been discussed recently in [21]. Similar equivalences have been found in [22] for $U(N)$ theories with matter in adjoint and fundamental representations.

An important conclusion can be drawn if we also consider the field transformation

$$\delta S = \frac{1}{2} \left( S \frac{1}{z - AS} + \frac{1}{z + SA} S \right) = S \frac{z}{z^2 - (AS)^2}. \quad (83)$$

The corresponding Konishi anomaly is

$$c(z) + V'(z)T(z) + \left< Q_f \frac{1}{z^2 - (AS)^2} SQ_f \right> = R(z)T(z) + 2 \frac{z}{z} R(z). \quad (84)$$

Let us keep formally the number of flavors arbitrary. From (82) we find

$$\left< Q_f \frac{z}{z^2 - (AS)^2} SQ_f \right> = \frac{N_f}{2}R(z). \quad (85)$$

Here we are using $\left< Q_f \frac{z}{z^2 - (AS)^2} SQ_f \right> = \left< Q_f \frac{1}{z - (AS)} SQ_f \right>$ which holds because of the symmetry of $(Q_f \otimes Q_f)_{ij}$. The important point is that (84) coincides with (81) only if $N_f = 8!$! So consistency of the Konishi anomaly equations allows us to obtain the same constraint on the number of fundamental flavors as the cancellation of the chiral anomaly. This has already been observed in the theory with adjoint in [18].

Let us now study the corresponding matrix model. It is given by

$$Z = \frac{1}{G} \int_{\Gamma} d\hat{A} d\hat{S} d\hat{Q} e^{-\frac{1}{G} \left[ V(\hat{A}\hat{S}) + Q_f \hat{S} \hat{Q}_f \right]} \quad (86)$$

Of course this matrix model has to be understood again in the holomorphic context [15].
To go to an eigenvalue representation we have to possibilities. First we can choose the gauge \( S = 1^N \). In this case the integration over the \( \hat{Q}_f \) degrees of freedom are trivial and contribute only an overall factor to the normalization. The same is true for the integration over the ghosts that are needed in the gauge fixing. This leaves the residual gauge group \( SO(\hat{N}) \) and \( \hat{A} \) that transforms as an adjoint under this gauge group. So we end up with the eigenvalue model for an \( SO(\hat{N}) \) matrix model and an adjoint field \( \hat{A} \) [20].

\[
Z = \frac{1}{G'} \int \prod_{i=1}^{\hat{N}} d\lambda_i \prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 e^{-\frac{2}{\kappa} \sum_n V(\lambda_n)}
\]  

(87)

In the non-chiral models we had to take care now to keep track of the original gauge symmetry which enforced a different derivation of the loop equations. What is then the correct Ward identity leading to the loop equations in the chiral model? It can of course be read off from the transformation that lead to the Konishi anomalies. In particular for the field \( \hat{A} \) this is

\[
\delta \hat{A} = \frac{z}{z^2 - (\hat{A}S)^2} \hat{A}.
\]

(88)

The corresponding Ward identity in the eigenvalue representation is

\[
0 = \frac{1}{G'} \int \prod_{i=1}^{\hat{N}} d\lambda_i \sum_{r+1}^{\hat{N}} \frac{\partial}{\partial \lambda_r} \left( \frac{z \lambda_r}{z^2 - \lambda_r^2} \prod_{k<l} (\lambda_k^2 - \lambda_l^2)^2 e^{-\frac{2}{\kappa} \sum_n V(\lambda_n)} \right).
\]

(89)

This is however up to the overall factor of \( z \) precisely the Ward identity that leads to the loop equation for the \( SO \) model with adjoint! Indeed we also find that the definition of the matrix model resolvent

\[
\omega(z) = \left\langle \kappa \text{tr} \left( \frac{z}{z^2 - (\hat{A}S)^2} \right) \right\rangle = \left\langle \kappa \sum_{i=1}^{\hat{N}} \frac{z}{z^2 - \lambda_i^2} \right\rangle,
\]

(90)

matches also the the one in the \( SO \) model [20]. So we infer that indeed we find the well-known loop equation

\[
\left\langle \frac{1}{2} \omega(z)^2 - V'(z)\omega(z) - \frac{\kappa}{2z} \omega(z) - \bar{f}(z) \right\rangle = 0.
\]

(91)

Finally let us investigate the variations

\[
\delta \hat{S} = \hat{S} \frac{z}{z^2 - (\hat{A}S)^2}, \quad \delta \hat{Q}_f = \frac{\rho f_g}{z^2 - (\hat{A}S)^2} \hat{Q}_g.
\]

(92)
and also keep the number of flavors explicit. We find then

\begin{equation}
\left\langle \frac{1}{2} \omega(z)^2 + \frac{\kappa}{2z} \omega(z) - \kappa \hat{Q}_f \hat{S} \frac{1}{z^2 - (\hat{A} \hat{S})^2} \hat{Q}_f - V'(z) \omega(z) - \hat{f}(z) \right\rangle = 0,
\end{equation}

\begin{equation}
\left\langle \hat{Q}_f \hat{S} \frac{z}{z^2 - (\hat{A} \hat{S})^2} \hat{Q}_f \right\rangle = \frac{\hat{N}_f}{2} \omega(z).
\end{equation}

We therefore that the two ways of deriving the loop equations lead to the same result only if \( \hat{N}_f = 2 \! \). This mismatch in the number of flavors between the field theory and the matrix model has already appeared before in \cite{18}. It therefore of course not surprising at all that we find the same mismatch here.

The polynomial \( \tilde{f}(z) \) in the above equations is

\begin{equation}
\tilde{f}(z) = \text{tr} \left( \frac{z V'(z) - \hat{A} \hat{S} V'(\hat{A} \hat{S})}{z^2 - (\hat{A} \hat{S})^2} \right).
\end{equation}

The matrix model loop equations match the Konishi anomaly relations in precisely the same way as in the non-chiral case. We expand the resolvent as in (53) and find also for the chiral theory the relations (57), (58) and (59).

\section{Conclusions}

We have investigated Konishi anomaly relations and loop equation in the corresponding matrix models for theories with unitary gauge groups but without basic fields in the adjoint representation. Of course a composite adjoint field is easily constructed as a bilinear of the basic fields in two-index tensor representations. It turns out that this composite adjoint field can be used to define generalized Konishi anomaly relations or loop equations that are rather similar as the ones for the models with basic adjoint field\footnote{Composite adjoints and Konishi anomalies have also been discussed very recently in the context of chiral quiver theories in \cite{14}}. In contrast to the models with additional adjoint field the curves we found are hyperelliptic! In the non-chiral case we found a somewhat new and interesting feature in the hyperelliptic curve defined by the Konishi anomalies. There was a branch point fixed at the origin and related to the special vacuum at \( \xi = 0 \).
With view of all the interesting results that have been obtained in the last years for the models with basic adjoint fields it is clear that there is still much work to be done in these new models. A detailed study of the phase structure should lead to a deeper understanding of the physics of the special vacuum with fixed branchpoint at the origin. A different question is how these models are realized in string theory either using D5-branes wrapped on orientifolds of resolutions of generalized conifolds in type IIB string theory or as intersecting brane configurations a la Hanany-Witten in the type IIA/M-theory approach. We hope to come back to these questions in a future publication.

References

[1] K. i. Konishi, “Anomalous Wt Identities And Chiral Properties Of Supersymmetric Gauge Theories,” IFUP-TH-84/4 Talk given at 19th Rencontre de Moriond: Electroweak Interactions and Unified Theories, Feb 26-Mar 4, 1984

[2] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, “Chiral rings and anomalies in supersymmetric gauge theory,” JHEP 0212, 071 (2002) [arXiv:hep-th/0211170].

[3] A. Gorsky, “Konishi anomaly and N = 1 effective superpotentials from matrix models,” Phys. Lett. B 554, 185 (2003) [arXiv:hep-th/0210281].

[4] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” arXiv:hep-th/0208048
R. Dijkgraaf and C. Vafa, “On geometry and matrix models,” Nucl. Phys. B 644, 21 (2002) arXiv:hep-th/0207106,
R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” Nucl. Phys. B 644, 3 (2002) arXiv:hep-th/0206255.

[5] R. Argurio, G. Ferretti and R. Heise, “An introduction to supersymmetric gauge theories and matrix models,” arXiv:hep-th/0311066

[6] P. Kraus and M. Shigemori, “On the matter of the Dijkgraaf-Vafa conjecture,” JHEP 0304, 052 (2003) arXiv:hep-th/0303104.
P. Kraus, A. V. Ryzhov and M. Shigemori, “Loop equations, matrix
models, and $N = 1$ supersymmetric gauge theories,” JHEP 0305, 059 (2003) [arXiv:hep-th/0304138].

[7] L. F. Alday and M. Cirafici, “Effective superpotentials via Konishi anomaly,” JHEP 0305 (2003) 041 [arXiv:hep-th/0304119].

[8] F. Cachazo, “Notes on supersymmetric Sp(N) theories with an antisymmetric tensor,” arXiv:hep-th/0307063.

[9] M. Matone, “The affine connection of supersymmetric SO(N)/Sp(N) theories,” JHEP 0310 (2003) 068 [arXiv:hep-th/0307285].

[10] K. Landsteiner and C. I. Lazaroiu, “On Sp(0) factors and orientifolds,” Phys. Lett. B 588, 210 (2004) [arXiv:hep-th/0310111].

[11] K. Intriligator, P. Kraus, A. V. Ryzhov, M. Shigemori and C. Vafa, “On low rank classical groups in string theory, gauge theory and matrix models,” Nucl. Phys. B 682, 45 (2004) [arXiv:hep-th/0311181].

[12] A. Brandhuber, H. Ita, H. Nieder, Y. Oz and C. Romelsberger, “Chiral rings, superpotentials and the vacuum structure of $N = 1$ supersymmetric gauge theories,” Adv. Theor. Math. Phys. 7, 269 (2003) [arXiv:hep-th/0303001].

[13] R. Argurio, G. Ferretti and R. Heise, “Chiral SU(N) gauge theories and the Konishi anomaly,” JHEP 0307, 044 (2003) [arXiv:hep-th/0306125].

[14] E. Di Napoli, V. S. Kaplunovsky and J. Sonnenschein, “Chiral Rings of Deconstructive $[SU(n_c)]^N$ Quivers,” arXiv:hep-th/0406122.

[15] C. I. Lazaroiu, “Holomorphic matrix models,” JHEP 0305 (2003) 044 [arXiv:hep-th/0303008].

[16] A. Klemm, K. Landsteiner, C. I. Lazaroiu and I. Runkel, “Constructing gauge theory geometries from matrix models,” JHEP 0305, 066 (2003) [arXiv:hep-th/0303032].

[17] S. G. Naculich, H. J. Schnitzer and N. Wyllard, “Cubic curves from matrix models and generalized Konishi anomalies,” JHEP 0308, 021 (2003) [arXiv:hep-th/0303268].
S. G. Naculich, H. J. Schnitzer and N. Wyllard, “Matrix-model description of \( N = 2 \) gauge theories with non-hyperelliptic Seiberg-Witten curves,” Nucl. Phys. B 674 (2003) 37 [arXiv:hep-th/0305263].

[18] K. Landsteiner, C. I. Lazaroiu and R. Tatar, “Chiral field theories, Konishi anomalies and matrix models,” JHEP 0402, 044 (2004) [arXiv:hep-th/0307182]; K. Landsteiner, C. I. Lazaroiu and R. Tatar, “Puzzles for matrix models of chiral field theories,” Fortsch. Phys. 52 (2004) 590 [arXiv:hep-th/0311103].

[19] R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa and D. Zanon, “Perturbative computation of glueball superpotentials,” Phys. Lett. B 573 (2003) 138 [arXiv:hep-th/0211017].

[20] S. K. Ashok, R. Corrado, N. Halmagyi, K. D. Kennaway and C. Romelsberger, “Unoriented strings, loop equations, and \( N = 1 \) superpotentials from matrix models,” Phys. Rev. D 67, 086004 (2003) [arXiv:hep-th/0211291].

[21] A. Armoni, A. Gorsky and M. Shifman, “An exact relation for \( N = 1 \) orientifold field theories with arbitrary superpotential,” arXiv:hep-th/0404247.

[22] R. Argurio, “Equivalence of effective superpotentials,” arXiv:hep-th/0405250.

[23] R. Argurio, “Effective superpotential for U(\( N \)) with antisymmetric matter,” arXiv:hep-th/0406253.