Scaling Relationships between van Genuchten Model Parameters and Hydraulic Conductivity

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Abstract. The soil-water characteristic curve (SWCC) is indispensable for predicting the behaviour of groundwater flow in vadose zones. It remains difficult to determine SWCC, particularly for fractured rocks, under field conditions when little or no test data is available. It is generally believed that the van Genuchten model (VG) parameters $\alpha$ and $n$ are related to pore size distribution (PSD) of soil and can be estimated from PSD data. In this study, correlations are proposed for predicting the VG parameters $\alpha$ and $n$ from hydraulic conductivity $K$. The VG parameters were obtained by curve-fitting the experimental data of 993 soil samples collected from UNSODA database and published papers. The hydraulic conductivity $K$ of the soil samples varies in six orders of magnitude from $1.30 \times 10^{-9}$ m/s to $1.22 \times 10^{-3}$ m/s. The statistical distribution of the VG parameters $\alpha$ and $n$ is, respectively, examined using hypothesis test and Anderson-Darling test, showing that $\alpha$ follows a normal distribution after a Box-Cox transformation, and $n$ follows a three-parameter lognormal distribution. The mean and standard deviation of $\alpha$ can be best estimated from $K$ by $\bar{\alpha} = 4.9K^{0.23}$ and $\sigma_\alpha = 1.4K^{0.13}$, and the parameter $n$ can be similarly estimated by $\bar{n} = 1.5 + 50K^{0.5}$ and $\sigma_n = 6.6K^{0.18}$. These correlations are also validated against site-scale test data at Yucca Mountain and published data on fractured rocks and applied to estimate the VG parameters of fractured rocks in the left bank slope at the site of Changbo Hydropower Project. The compiled data and the proposed equations are useful for estimating a narrower range of VG parameters from the more easily-obtained $K$ values at site scale or in case of unavailable test data, and these estimates can be used as input for inverse modelling and simulation of unsaturated flow in porous or fractured media.

1. Introduction

Unsaturated flow is an important subsurface process, and the modelling of unsaturated flow in porous/fractured media becomes increasingly vital for applications such as slope stability analysis [1], nuclear waste disposal [2], waste landfill [3], and contaminant transport [4]. This modelling is enabled on the premise that the soil-water characteristic curve (SWCC) of the media could be properly determined.
Quite a variety of techniques have been developed to measure matric/total suction of soil, such as tensiometer, pressure plate, Tempe cell, TDR technique, centrifuge permeameter method, relative humidity method, electrical/thermal conductivity sensor method, and filter paper method. The testing is mostly time-consuming, and full of uncertainties when the experimental conditions are not well controlled, resulting in high variability of measurements even for the same type of soil. More importantly, most of these techniques are hard to apply for site-scale testing, with the measurements less representative of site conditions. As an alternative, predictive models are proposed to estimate the SWCC of soils, which can be grouped into three broad categories [5]: (i) to estimate water contents at various soil suctions [6], (ii) to estimate the fitting parameters of SWCC models from soil properties [7], and (iii) to estimate SWCC from the particle size distribution (PSD) of soil with physico-empirical models [8]. These models are of practical use, but may still be hard to apply when the related representative soil properties are unavailable.

One controversy over SWCC is its applicability in fractured rocks. Numerical simulations have shown that the VG model is also suitable for fractured rocks once the model parameters could be properly determined [9-11]. Given that the hydraulic conductivity and SWCC of geological materials are both governed by their structures, correlations may exist between the SWCC model parameters and hydraulic conductivity. In this study, scaling relationships are proposed to predict the van Genuchten model (VG) parameters $\alpha$ and $n$ from hydraulic conductivity $K$ through a broad synthesis of published data. The flexibility of the proposed relationships in fractured rock formations are validated against test data at Yucca Mountain and published data on fractured rocks, and then applied to estimate the VG parameters of fractured rocks at the dam site of Changbo Hydropower Station. The results show that the proposed equations are useful for estimating a narrower range of VG parameters from the more easily-obtained $K$ values at site scale or in case of unavailable test data, and these estimates can be used as input for inverse modelling and simulation of unsaturated flow in porous or fractured media.

2. Scaling between VG parameters and hydraulic conductivity
The VG-Mualem model [12-13] is the most widely-applied equations for description of water retention and flow behaviors in unsaturated soils, which reads:

$$S_e = \left[1 + (\alpha \psi)^n \right]^{-\frac{1}{n}}$$ (1)

$$k_r = S_e^{\frac{1}{2}} \left[1 - (1 - S_e^{\frac{1}{n}})^n \right]^2$$ (2)

where $S_e = (\theta_s - \theta_r)/(\theta_s - \theta_l)$ is the effective saturation, $\theta$ is the volumetric water content, $\theta_s$ and $\theta_l$ are the saturated and residual water content, $k_r$ is the relative permeability, $\psi$ is the suction, and $\alpha$, $m$, and $n$ are model parameters, with $m = 1 - 1/n$.

The model contains 4 independent parameters ($\theta_s$, $\theta_l$, $\alpha$ and $n$), among which $\theta_l$ and $\theta_s$ can be relatively easily estimated. The parameters $\alpha$ and $n$, however, are of great significance to the shape of SWCC [14], and need to be determined by curve-fitting the test data.

2.1. Statistics of VG parameters
A set of SWCC data on 993 soil samples was collected from UNSODA database and published papers. For each soil sample, there are at least 5 ($\theta$, $\psi$) pairs of test data, and the VG parameters were determined by curve-fitting the test data with the lscurvefit function in MATLAB. The hydraulic conductivity $K$ of the soil samples at saturated state varies in 6 orders of magnitude from $1.30 \times 10^{-9}$ to $1.22 \times 10^{-3}$ m/s. The selected soil samples can be categorized into four textural groups: sands (sand, loamy sand, sandy loam), loams (loam, clay loam), silts (silt, silty loam), and clays (clay, sandy clay, silty clay, silty clay loam). The geometric mean $\bar{\psi}$, standard deviation $\sigma$ and coefficient of variation CV of the VG parameters for each textural group are listed in Table 1. It can be
observed that the geometric means of $\alpha$ and $n$ decrease as the soil particles become finer, which can be explained by higher adsorptive force and capillary force in smaller pores [15]. Further observed are high CV values for $\alpha$ and $n$.

Table 1. Statistics of the VG parameters for each textural group.

| Texture | $\alpha$ (kPa$^{-1}$) | $n$ | No. of data |
|---------|----------------------|-----|-------------|
|         | $\bar{x}$  | $\sigma$ | CV | $\bar{x}$  | $\sigma$ | CV |             |
| Sands   | 0.275   | 0.382   | 0.919 | 1.856 | 1.279 | 0.689 | 420        |
| Loams   | 0.178   | 0.415   | 1.122 | 1.381 | 0.569 | 0.392 | 122        |
| Silts   | 0.120   | 0.391   | 1.329 | 1.429 | 0.559 | 0.373 | 230        |
| Clays   | 0.098   | 0.454   | 1.426 | 1.315 | 0.580 | 0.423 | 221        |
| All     | 0.171   | 0.408   | 1.134 | 1.561 | 0.994 | 0.578 | 993        |

The statistical distributions of $\alpha$ and $n$ are examined using hypothesis test and Anderson-Darling test. The histogram and probability density function of the Box-Cox transformed $\alpha$ values and $n$ values of the soil samples are plotted in Figures 1 and 2, respectively. The plots show that the parameter $\alpha$ roughly follows a normal distribution after a Box-Cox transformation, and the parameter $n$ roughly follows a three-parameter lognormal distribution.

2.2. Scaling between VG parameters and $K$

It has been well understood that there is a strong correlation between particle size distribution, pore size distribution and the SWCC [8]. The parameter $\alpha$ of the VG model is approximately equal to the inverse of soil’s air-entry value, which is inversely proportional to the pore radius according to the Young-Laplace equation. The $n$ value is related to the pore size distribution index, with the more uniform pore sizes corresponding to larger values of $n$. According to Poiseuille’s law and Darcy’s law, the hydraulic conductivity of porous media is related to its porosity, PSD and interconnectivity of pores. Therefore, it is reasonable to suppose that there exist some correlations between the parameters $\alpha$, $n$ and hydraulic conductivity $K$.

Figure 3 plots the variations of $\alpha$ and $n$ with $K$. Although the data points seem to be quite scattered, there is a clear trend that the values of $\alpha$ and $n$ increase and have higher variability as $K$ increases. As shown in Figure 3(a), the $\alpha$ value averaged at each 0.2-interval of $\lg K$ can be well correlated to $K$ by a power-law function with $R^2 = 0.95$:

$$\bar{\alpha} = 4.9K^{-0.23}$$

(3)
where $K$ is in unit of m/s, and $\alpha$ is in unit of kPa$^{-1}$. This scaling shows that as $K$ increases from $1.0 \times 10^{-9}$ to $1.0 \times 10^{-3}$ m/s, the mean of $\alpha$ increases from 0.042 kPa$^{-1}$ to 1.0 kPa$^{-1}$.

Similarly, as shown in Figure 3(b), the $n$ value averaged at each 0.2-interval of $\lg K$ can be correlated to $K$ by the following function with $R^2=0.88$:

$$\bar{n} = 1.5 + 50K^{0.5}$$

(4)

This scaling indicates that the mean $n$ value increases from 1.50 to 3.08 as $K$ increases from $1.0 \times 10^{-9}$ to $1.0 \times 10^{-3}$ m/s.

As shown in Figure 4, the standard deviations of the $\alpha$ and $n$ values also increase with $K$, and can be best described with the following power-law correlations, respectively:

$$\sigma_\alpha = 1.4K^{0.13}$$

(5)

$$\sigma_n = 6.6K^{0.18}$$

(6)

Equations (3)-(6) are of practical significance to predict the range of $\alpha$ or $n$ at a given $K$, because the value of $K$ could be much more easily and economically measured under field conditions. The lower bounds of $\alpha$ and $n$ are known as $\alpha=0$ and $n=1$. The upper bounds of $\alpha$ and $n$ can be estimated as $\alpha_{UB} = \bar{\alpha} + 3\sigma_\alpha$ and $n_{UB} = \bar{n} + 3\sigma_n$, respectively, as plotted in Figure 3. As a result of higher heterogeneity of pore structures for higher permeability media, the range between the lower and upper bounds of both $\alpha$ and $n$ increases with increasing $K$ value.

![Figure 3. Variation of (a) parameter $\alpha$, and (b) parameter $n$ with hydraulic conductivity $K$.](image)

As shown in Figure 4, the standard deviations of the $\alpha$ and $n$ values also increase with $K$, and can be best described with the following power-law correlations, respectively:

![Figure 4. Variation of standard deviation for (a) $\alpha$ data, and (b) $n$ data with hydraulic conductivity $K$.](image)
Fractured rocks are of much higher heterogeneity and larger REV size than soils. Therefore, there is a controversy about if the SWCC similarly exists in fractured rocks. This hypothesis has been verified by numerical simulations [9-11], but it remains difficult to determine the model parameters representative of the site conditions. Figure 3 also plots the available (a, n, K) data for intact rocks, fractures and fractured rocks at Yucca Mountain and in published data, showing that most of the measurements are within the predicted upper bound. It is not surprising that a small number of data points fall outside the upper bound, due to higher heterogeneity and more complex flow geometries of fractured rocks as compared to soils. The plot evidences that the VG model parameters of fractured rocks can be estimated, in relatively narrow range and high confidence, from the easily-obtained K values using the proposed correlations, which is exactly the motivation of this study.

3. Application of the scaling relationships

3.1. Site description

The proposed scaling relationships are applied to estimate the VG parameters of fractured rocks in the left bank slope at the site of Changbo Hydropower Project. This project is located in the upper Jinsha River in Southwest China. The project mainly consists of a gate dam of 58 m high and an underground tunnel and cavern system in the left bank for water diversion and power generation, as shown in Figure 5. The underground caverns are situated in the wedge-shaped zone formed by the Jinsha River and the Maiqu stream. The bedrocks outcropping at the site are slate and schist belonging to the upper Triassic system (T1-2zh2) and the lower Permian system (Pgij). The permeability of the fractured rocks at the dam site was measured by a total of 209 conventional packer tests in 16 boreholes. The mean monthly rainfall and evaporation at the dam site are plotted in Figure 6.

![Figure 5. Landscape at the Changbo dam site.](image1)

![Figure 6. Mean monthly precipitation and evaporation at the dam site.](image2)

3.2. Saturated-unsaturated flow modelling

A 2D FE mesh along section I-I (Figure 5) was created for saturated-unsaturated flow modelling, in which the topographical and geological conditions were well represented (Figure 7). The saturated-unsaturated flow was simulated by the discretized parabolic variational inequality (PVI) method [16], using the FE code THYME [17]. The boundary conditions of the model were set as follows: The riverbed surface was prescribed as a water head boundary. The lateral boundaries on the mountain side and on the river side were assumed as groundwater divides and set as no-flow boundary. The bottom of the model was also set as a no-flow boundary. The slope surface and the surfaces of the exploratory adits were set as the third-type boundary. The initial head distribution in the saturated zone was determined by estimating a water table that best fits the groundwater level observed in boreholes with a steady-state flow model. The initial head distribution in unsaturated zone was roughly estimated by assuming a vertically linear reduction of saturation to 60% at the ground surface. The saturated and
residual water contents ($\theta_s$ and $\theta_r$) were estimated by the porosity and void structure of the fractured rocks [18]. The upper bounds of the VG parameters $\alpha$ and $n$ were estimated from the representative value of $K$ according to Equations (3)-(6), and the representative values of $\alpha$ and $n$ were then determined by calibration with field observations. Table 2 lists the estimated unsaturated hydraulic parameters for the rocks at the site. The specific storage coefficients of fractured rocks and faults are listed in Table 3.

![Figure 7. 2D finite element mesh for the study area: (a) overall view, and (b) enlarged view for the near-bank region.](image)

**Table 2.** Hydraulic properties of rocks at the dam site.

| Rock            | $K$ (m/s) | $\theta_s$ | $\theta_r$ | $\alpha$ (kPa$^{-1}$) | $n$       |
|-----------------|-----------|------------|------------|------------------------|-----------|
| Loose sediments | $2.00 \times 10^{-5}$ | 0.26       | 0.05       | 0-1.256               | 0.055     |
| High-moderate PZ| $1.57 \times 10^{-5}$ | 0.18       | 0.03       | 0-1.205               | 0.042     |
| Moderate PZ     | $1.82 \times 10^{-6}$ | 0.18       | 0.02       | 0-0.851               | 0.020     |
| Moderate-low PZ | $5.95 \times 10^{-7}$ | 0.10       | 0.02       | 0-0.718               | 0.005     |
| Low PZ          | $4.36 \times 10^{-7}$ | 0.08       | 0.01       | 0-0.685               | 0.003     |
| Faults $^a$     | $2.00 \times 10^{-6}$ | 0.20       | 0.04       | 0-0.864               | 0.303     |

$^a$ The listed $K$ value of faults represents the in-plane hydraulic conductivity ($K_\parallel$), and the component of $K$ perpendicular to the plane ($K_\perp$) is about one or two orders of magnitude smaller than the in-plane component.
Table 3. Specific storage coefficients of fractured rocks and faults at the site*.

| Rock                  | $E_0$ (GPa) | $\mu$ | $K_b$ (GPa) | $\alpha_c$ (GPa$^{-1}$) | $\alpha_w$ (GPa$^{-1}$) | $\phi$ | $S_s$ (m$^{-1}$) |
|-----------------------|-------------|-------|-------------|--------------------------|--------------------------|--------|------------------|
| Loose sediments       | 3.5         | 0.30  | 2.92        | 0.342                    | 0.47                     | 0.26   | 4.55 × 10$^{-6}$ |
| High-moderate PZ      | 5.5         | 0.25  | 3.67        | 0.272                    | 0.47                     | 0.18   | 3.50 × 10$^{-6}$ |
| Moderate PZ           | 10.0        | 0.23  | 6.17        | 0.162                    | 0.47                     | 0.10   | 2.05 × 10$^{-6}$ |
| Moderate-low PZ       | 12.0        | 0.22  | 7.14        | 0.140                    | 0.47                     | 0.10   | 1.83 × 10$^{-6}$ |
| Low PZ                | 18.0        | 0.22  | 10.71       | 0.093                    | 0.47                     | 0.08   | 1.28 × 10$^{-6}$ |
| Faults                | 2.0         | 0.35  | 2.22        | 0.450                    | 0.47                     | 0.20   | 5.34 × 10$^{-6}$ |

* The specific storage coefficients $S_s$ of rocks were determined by $S_s = \rho g (\alpha_c + \phi \alpha_w)$, where $\alpha_c = 1/K_b$ is the compressibility of rock, $\alpha_w$ is the compressibility of water, $\phi$ is the porosity, $\rho$ is the density of water, $g$ is the gravitational acceleration, and $K_b$ is the bulk modulus of rock estimated by $K_b = E_0/3(1-2\mu)$, where $E_0$ is the deformation modulus of rock, and $\mu$ is the Poisson ratio of rock. The values of $E_0$, $\mu$ and $\phi$ for various zones of rocks listed in the table were determined by site characterization.

To eliminate the influence of initial head distribution on the groundwater flow, a natural precipitation and evaporation process was first simulated, which lasted for 30 years until the groundwater flow in each season became stable. The site exploration process of 9 years was then modelled. The initial time step and maximum time step in each simulation were taken as 7 hours and 7 days, respectively. Figure 8 shows the change of calculated groundwater level between wet and dry seasons in the slope, showing that the numerical results agree rather well with borehole groundwater level observations. The seasonal fluctuation of groundwater level is both influenced by rainfall and river water level, mostly within a range of 10-20 m. The numerical simulation shows that the mean rainfall infiltration rate at the dam site is about 17.21%, which is reasonable in the metamorphic rocks.

Figure 9 plots the water head contours in December 2020, three years after the exploratory adits PD-s3, PD-s4, PD-c1, PD-c2 and PD-c4 were excavated. The excavation of the adits led to a significant discharge of groundwater and drawdown of water table. The total discharge from the adit system varied drastically during the excavation of the adits, and decreased to an amount about 0.6 L/s in December 2020. The discharges from the adits PD-c1, PD-c2 and PD-c4 were negligible, which is consistent with field observations.

4. Discussion and conclusions
In this study, power-law scaling relationships are proposed to estimate the VG parameters ($\alpha$, $n$) of porous/fractured media from hydraulic conductivity $K$ by compiling test data on 993 soil samples. It is found that the mean, standard deviation and the upper-bound estimates of $\alpha$ and $n$ all increase with $K$, due mainly to higher heterogeneity of the media and more complexity of flow geometries as $K$ increases. These correlations are practically useful for estimating a narrower range of $\alpha$ and $n$ values from the representative value of $K$, because the latter can be much more easily and economically determined with higher confidence. The estimated narrower ranges of $\alpha$ and $n$ can then be used for
inverse modelling or calibration of field-scale saturated-unsaturated flow in porous media or fractured rocks, as illustrated by an example at the Changbo dam site.

It should be noted, however, that there are quite a few simplifications and assumptions in the proposed methodology. First, there are many factors affecting the retention behavior of unsaturated soils, such as initial water content, pore ratio, soil texture, bulk density, suction measurement methodologies, soil stress history, and stress state [19-20]. The soil-water characteristic curve data on 993 soil samples used in this study is collected from independent tests where the SWCCs may be affected by one or more external influencing factors. Consequently, the correlation of the VG parameters to hydraulic conductivity may not be well represented in some of the tests. This limitation could be hopefully overcome if a larger size of test data is collected. Second, the equations proposed in this study is somewhat empirical. Different from Mishra and Parker’s [21] conclusion based on capillary model that the $a$ value is exactly proportional to $K^{0.5}$, the exponent in this study is 0.23, indicating a discrepancy of the ideal model from experimental results. Yang et al. [22] evidenced that the linear relationship between the local capillary pressure $\psi$ and $K^{-0.5}$ may not well describe the data with smaller sand proportions. Deriving a correlation with solid theoretical background is meaningful, and will be our future work. Finally, the case study is oversimplified, and is not sufficiently illustrative. A 3D numerical simulation with dual porosity model is preferable in our future study.

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