TRIPLE CORRELATION SUMS OF COEFFICIENTS OF CUSP FORMS

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ABSTRACT. We produce nontrivial asymptotic estimates for shifted sums of the form \( \sum a(h)b(m)c(2m - h) \), in which \( a(n), b(n), c(n) \) are un-normalized Fourier coefficients of holomorphic cusp forms. These results are unconditional, but we demonstrate how to strengthen them under the Riemann Hypothesis. As an application, we show that there are infinitely many three term arithmetic progressions \( n - h, n, n + h \) such that \( a(n - h)a(n)a(n + h) \neq 0 \).

1. Introduction

Convolution sums formed from coefficients of modular forms have frequent applications throughout number theory. Let \( d(n) \) denote the divisor function and let \( r_d(n) \) denote the number of representations of \( n \) as a sum of \( d \) squares. Correlation sums of the approximate forms

\[
\sum_{n \leq X} d(n)d(n + h) \quad \text{or} \quad \sum_{\substack{n \leq X \\ h \leq Y}} r_d(n)r_d(n + h)
\]

appear in off-diagonal terms for fourth moment estimates of the Riemann zeta function (as in [HB79]), for second moment estimates of the Gauss circle problem (as in [Ivi03]), and for second moment estimates of the \( d \)-dimensional Gauss circle problem (as in [HKLDW18a]). These correlation sums are well-studied and many techniques have been developed to understand their asymptotic behavior.

Triple correlation sums of the form

\[
\sum a(n - h)b(n)c(n + h)
\]
can also exhibit distinguished behavior, though they are far less understood. Blomer [Blo17] used the spectral theory of automorphic forms to produce asymptotics for partially smoothed triple correlation sums of the form

$$\sum_h W(h) \sum_{N \leq n \leq 2N} d(n - h)a(n)d(n + h),$$

where $a(n)$ is any sequence of complex numbers and $W$ is a smooth bump function. Lin [Lin18] built on Blomer’s analysis to establish similar bounds for triple correlation sums of Fourier coefficients of cusp forms. In particular, letting $A(n)$ denote the normalized Fourier coefficients of a Hecke eigenform, Lin proves that

$$\sum_h W(h) \sum_{N \leq n \leq 2N} A(n - h)A(n)A(n + h) \ll N^\epsilon \min \left(NH, \frac{N^2}{H^{1/2}} \right).$$

This estimate is nontrivial when $H \geq N^{2/3+\epsilon}$ and otherwise matches the trivial bound from bounding by the length of the sum. Singh [Sin18] used the circle method of Heath-Brown to prove that

$$\frac{1}{H} \sum_h W_1(h) \sum_{n \leq N} A(n)B(n + h)C(n + 2h)W_2 \left(\frac{n}{N} \right) \ll N^{1-\delta}$$

when $H \gg N^{1/2+\epsilon}$ for some $\delta > 0$, where $A(n)$, $B(n)$, and $C(n)$ are normalized coefficients of holomorphic cusp forms or Maass eigenforms on $SL(2, \mathbb{Z})$. Singh’s result allows a more concentrated sum in $H$ at the cost of a slightly different form of triple correlation.

In this paper, we consider yet another form of triple correlation between coefficients of cusp forms. Namely, we consider triple correlation sums of the form

$$\sum_{m,h} a(h)b(m)c(2m - h)e^{-m/X}e^{-h/Y}$$

in which $a(n)$, $b(n)$, and $c(n)$ denote non-normalized coefficients of a holomorphic cuspidal Hecke eigenforms $f_1$, $f_2$, and $f_3$, respectively, each of even weight $k$, level $N$, and trivial nebentypus.

To attain heuristic estimates for sums of this form, note that when $Y = O(X)$,

$$\sum_{m \leq X \atop h \leq Y} a(h)b(m)c(2m - h) \ll \sum_{m \leq X \atop h \leq Y} h^{k/2+1+\epsilon}m^{k-1+\epsilon} \ll \begin{cases} X^{k-1+1+\epsilon}Y^{k+1+\epsilon} & \text{naively} \\ X^{k-1+1+\epsilon}Y^{k+1+\epsilon} & \text{double square-root cancellation} \end{cases}$$

The naive estimate follows from termwise application of Deligne’s bound for each coefficient and a bound by absolute values. The second estimate follows from assuming that there is square-root type cancellation in both the
m and h sums. This would occur if the m and h sums experience independent random sign changes, but whether or not that occurs is unknown.

Our first theorem for these correlation sums is that square-root cancellation occurs in both the m and h sums at all scales.

**Theorem 1.1.** Let \(a(\cdot), b(\cdot),\) and \(c(\cdot)\) denote the coefficients of holomorphic cuspidal eigenforms of weight \(k,\) level \(N,\) and trivial nebentypus. Then for any \(\epsilon > 0,\)

\[
\sum_{m,h \geq 1} a(h)b(m)c(2m - h) e^{-m/X} e^{-h/Y} \ll X^{k-1+\Theta + 1/2+\epsilon} Y^{k-1/2} - \Theta + 1/2 + \epsilon.
\]

Here \(\Theta < 7/64\) refers to the best bound towards Selberg’s Eigenvalue Conjecture.

Under the assumption of the Riemann Hypothesis, we can prove that there is square-root cancellation in \(X\) and \(3/4\)-type cancellation in \(Y.\)

**Theorem 1.2.** Assume the Riemann Hypothesis. Then with the same notation as above and for any \(\epsilon > 0,\) we have

\[
\sum_{m,h \geq 1} a(h)b(m)c(2m - h) e^{-m/X} e^{-h/Y} \ll X^{k-1+\Theta + 1/2+\epsilon} Y^{k-1/2} - \Theta + 1/2 + \epsilon.
\]

There are famous sums with conjectured \(3/4\)-type cancellation, including the Gauss Circle problem and the Dirichlet Divisor problem. The work of Chandrasekharan and Narasimhan [CN62] implies that

\[
\frac{1}{X} \int_1^X \sum_{n \leq t} a(n) dt \ll X^{k-1/2+1/4+\epsilon},
\]

indicating that the sums of coefficients of cusp forms experience \(3/4\)-type cancellation on average. It appears that this large degree of cancellation carries into the \(h\)-sum. We will see below that we do not expect more-than-square-root cancellation in the \(m\)-sum.

We use the spectral theory of modular forms to prove these theorems. In the analysis, several lines of spectral poles appear. Applying a theorem from [LD19], it follows that any one of these poles guarantees non-vanishing of infinitely many triples \(a(h)b(n)c(2n - h).\)

**Theorem 1.3.** Maintaining the same notation as the two previous theorems, fix \(0 < \alpha < 1.\) Suppose there exists a non-constant Maass form \(\mu_j\) on \(\Gamma_0(2N)\) with Laplacian eigenvalue \(\lambda > \frac{1}{4}\) such that \(\langle f_2(2z)f_3(z)y^k, \mu_j \rangle \neq 0.\) Then

\[
\bigg| \sum_{m \leq X} \sum_{h \leq 2X} \frac{a(h)b(m)c(2m - h)}{h^{1/2} + \epsilon} \bigg| = \Omega(X^{k-1/2}).
\]

Therefore infinitely many terms of the dual-sequence

\[
\{a(h)b(m)c(2m - h)\}_{m,h \in \mathbb{N}}
\]

are nonzero.
One interpretation of the preceding theorem is that the $X$ sum often has no more than square-root cancellation. The power of $h$ appearing in the theorem is mostly technical, and does not affect this interpretation.

Applied to the case when $f_1 = f_2 = f_3 = f$, we get the following corollary.

**Corollary 1.4.** Suppose that there exists a non-constant Maass form $\mu_j$ on $\Gamma_0(2N)$ with Laplacian eigenvalue $\lambda > \frac{1}{4}$ such that $\langle f(2z)f(y^k)\mu_j, \mu_j \rangle \neq 0$.

Then there are infinitely many three-term arithmetic progressions $n - h, n, n + h$ such that

$$a(n - h)a(n)a(n + h) \neq 0.$$ 

**Remark 1.5.** The results above may be further generalized to concern triples of modular forms $f_i$ which do not necessarily have the same level, weight, or nebentypus. The restrictions we impose are used to simplify the exposition of the proof; loosening them would not significantly alter the overall argument.

**Motivation from the Congruent Number Problem**

Our initial motivation to understand sums of this form came from the congruent number problem. Recall that a congruent number is an integer which appears as the area of a right triangle with rational-length sides. The congruent number problem is the classification problem of determining which integers are congruent. It is a well-studied classical problem; see [Con08] for a nice survey.

There is a well-known correspondence between three-term arithmetic progressions of squares and congruent numbers, in which the common difference in the progression is a congruent number. Let $\theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z) = \sum_{n \geq 0} r_1(n)e(nz)$ denote the classical theta function, where $e(z) = e^{2\pi iz}$. Here, $r_1(n)$ is essentially twice the square-indicator function, except that $r_1(0) = 1$. Then if $r_1(h)r_1(m)r_1(2m - h) \neq 0$, the triple $(h, m, 2m - h)$ is a three-term arithmetic progression of squares and $m - h$ is congruent.

Understanding sums of the shape

$$\sum r_1(h)r_1(m)r_1(2m - h)$$

would open up new approaches to understanding the distribution of congruent numbers. Sums of the above shape can be attained from the primary sum

$$\sum_{m,h=1}^{\infty} r_1(m - h)r_1(m)r_1(m + h)r_1(th)$$

where $t$ is a square-free integer, studied by the authors in [HKLDW18b], after summing over all such $t$, although the implicit dependence of the error term on $t$ prevents detailed analysis.

Heuristically, by replacing holomorphic cusp forms with the classical theta function $\theta(z) = \sum e(n^2 z)$, where $e(z) = e^{2\pi iz}$, the main results of this paper
would describe sums of the shape (1.1). Furthermore, one would attain meromorphic continuation for the series

$$\tilde{D}(s, w) = \sum_{m,h \geq 1} \frac{r_1(h)r_1(m)r_1(2m-h)}{m^{s-\frac{1}{2}}h^w}.$$  

This series would provide additional tools to investigate the asymptotic behavior of congruent numbers.

There are significant challenges to carrying out this heuristic: it’s necessary to work in higher level with additional cusps; $\theta(z)$ is half-integral weight; and perhaps most significantly, $\theta(z)$ is not cuspidal. The authors hope to continue this investigation in later work.

2. Methodology and Notation

Let $f_1(z) = \sum a(n)e(nz)$ be a holomorphic cuspidal Hecke eigenform of weight $k$ on $\Gamma_0(N)$ with trivial nebentypus and real coefficients; similarly, define $f_2(z)$ and $f_3(z)$ with respective coefficients $b(n)$ and $c(n)$. We will investigate the meromorphic continuation of the shifted multiple Dirichlet series

$$D(s, w) := \sum_{m,h \geq 1} \frac{a(h)b(m)c(2m-h)}{m^{s+k-1}h^w},$$

defined initially for Re $s$, Re $w$ sufficiently large. Ultimately, we will show that this double Dirichlet series has meromorphic continuation to $\mathbb{C}^2$ and has polynomial growth in vertical strips (away from poles).

Let $V(z)$ denote the product $V(z) = y^k f_2(2z)f_3(z)$. Note that $f_2(2z)$ is a holomorphic cusp form of weight $k$ and level $2N$. Define the level $2N$ Poincaré series as

$$P_h(z, s; 2N) := \sum_{\gamma \in \Gamma_0 \backslash \Gamma_0(2N)} \text{Im} (\gamma z)^s e(h\gamma z)$$

and note that this converges locally uniformly on the upper-half plane $\mathbb{H}$ and belongs to $L^2(\Gamma_0(2N) \backslash \mathbb{H})$.

Then the classical unfolding computation shows that the Petersson inner product $\langle V, P_h(\cdot, \overline{\gamma}; 2N) \rangle$ gives the double correlation sum

$$\langle V, P_h(\cdot, \overline{\gamma}; 2N) \rangle = \frac{\Gamma(s+k-1)}{(8\pi)^{s+k-1}} \sum_{m=1}^{\infty} \frac{b(m)c(2m-h)}{m^{s+k-1}}.$$  

We recognize $D(s, w)$ as the sum

$$D(s, w) = \frac{(8\pi)^{s+k-1}}{\Gamma(s+k-1)} \sum_{h \geq 1} \frac{a(h)\langle V, P_h(\cdot, \overline{\gamma}; 2N) \rangle}{h^w},$$

which converges absolutely for Re $s$ and Re $w$ sufficiently large. In Section 3, we replace $P_h$ with its spectral expansion to obtain an alternate description of $D(s, w)$. 
In Section 4, we use this spectral expansion to study the meromorphic properties of $D(s, w)$. The broad methodology of this section is similar to classical ideas of Selberg, more recently refined in the appendix to Sar- nak [Sar01], the work of Blomer and Harcos [BH08], and the work of Hoff- stein, the first author, and Reznikov [HH16]. The observation that it is possible to multiply by $a(h)h^{-w}$ and still make sense of the resulting spectral decomposition has been observed by Hoffstein, and is used in the first author’s thesis.

The main result of Section 4 is to show that $D(s, w)$ has meromorphic continuation to $\mathbb{C}$ and to describe the nature of the leading poles. With this description, the remainder of the paper is very straightforward. The final section then shows that $D(s, w)$ has polynomial growth in vertical strips and proves the main theorems.

We note that it is possible to obtain various weighted averages of triple correlation sums from the meromorphic properties of $D(s, w)$ by adapting the methods that yield the main results in this paper; these might be applied to yield interesting results in the future.

3. Spectral Expansion

In this section, we use the spectral expansion of $P_h(z, s; 2N)$ to rewrite $D(s, w)$ in a way that exposes its meromorphic properties.

By Selberg’s Spectral Theorem (as in [IK04, Theorem 15.5]), the Poincaré series $P_h(z, s; 2N)$ has a spectral expansion of the form

$$P_h(z, s; 2N) = \sum_j \langle P_h(\cdot, s; 2N), \mu_j(z) \rangle \mu_j(z) + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot, s; 2N), E_a(\cdot, \frac{1}{2} + it; 2N) \rangle E_a(z, \frac{1}{2} + it; 2N) dt,$$

where $a$ ranges over the cusps of $\Gamma_0(2N) \setminus \mathbb{H}$; $E_a$ denotes Eisenstein series associated to $a$; and $\{\mu_j\}$ denotes an orthonormal basis of the residual and cuspidal spaces, consisting of the constant form $\mu_0$ and of Hecke-Maass forms $\mu_j$ for $L^2(\Gamma_0(2N) \setminus \mathbb{H})$ with associated types $1/2 + it_j$. The inner product of the Poincaré series against the constant form $\mu_0$ vanishes, so we omit further mention of it.

Here, $E_a(z, s; 2N)$ is the Eisenstein series of level $2N$ defined by

$$E_a(z, s; 2N) = \sum_{\gamma \in \Gamma_0(2N)} \text{Im} (\sigma_a^{-1} \gamma z)^s,$$

where $\Gamma_a \subset \Gamma_0(2N)$ is the stabilizer of the cusp $a$ and $\sigma_a \in \text{PSL}_2(\mathbb{R})$ satisfies $\sigma_a \infty = a$, induces an isomorphism $\Gamma_a \cong \Gamma_\infty$ via conjugation, and is unique up to right translation. We refer to the sum over $j$ as the “discrete part of the spectrum” and the sum of integrals of Eisenstein series as the “continuous part of the spectrum.”
By replacing $P_h$ with its spectral expansion in $\langle V, P_h \rangle$, we obtain the expansion

$$\langle V(z), P_h(z, \sqrt{s}; 2N) \rangle = \sum_j \overline{\langle P_h(\cdot, \sqrt{s}; 2N), \mu_j \rangle} \langle V, \mu_j \rangle$$

(3.1)

$$+ \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot, \sqrt{s}; 2N), E_a(\cdot, \frac{1}{4} + it; 2N) \rangle \langle V, E_a(\cdot, \frac{1}{4} + it; 2N) \rangle dt.$$  

The Fourier expansions of the Maass forms and Eisenstein series are known and can be used to understand the inner products against the Poincaré series. The Maass forms have Fourier expansions of the form

$$\mu_j(z) = \sqrt{y} \sum_{|m| \neq 0} \rho_j(m) K_{it_j}(2\pi|m|y)e(mx),$$

where $K_{it_j}$ is a $K$-Bessel function. For each Maass form $\mu_j$, there is a constant $\rho_j(1)$ such that for each prime $p$ with $\gcd(p, 2N) = 1$, the coefficient $\rho_j(p)$ can be written $\rho_j(p) = \rho_j(1)\lambda_j(p)$, where $\lambda_j(p)$ is the eigenvalue of the $p$-th Hecke operator. In level 1, this common constant is the first coefficient $\rho_j(1)$. By a minor abuse of notation, we continue to use the notation $\rho_j(1)$ even though the $m = 1$ Fourier coefficient might be zero. Thus we write $\rho_j(h) = \rho_j(1)\lambda_j(h)$.

The Eisenstein series have Fourier expansions

$$E_a(z, s; 2N) = \delta_{a, \infty} y^s + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2}) \rho_a(s, 0) y^{1-s}}{\Gamma(s)}$$

$$+ \frac{2\pi^s \sqrt{y}}{\Gamma(s)} \sum_{m \neq 0} |m|^{-\frac{1}{2}} \rho_a(s, m) K_{\frac{1}{4} - \frac{s}{2}}(2\pi|m|y)e(mx)$$

with computable coefficients $\rho_a(s, m)$.

With these expansions, one can explicitly compute the inner products as

$$\overline{\langle P_h(\cdot, \sqrt{s}; 2N), E_a(\cdot, \frac{1}{4} + it; 2N) \rangle} = \rho_a(1) \Gamma(s - \frac{1}{4} + it) \Gamma(s - \frac{1}{4} - it)$$

$$\frac{4^{s-1} \pi^{s-\frac{1}{2} - it} h^{s-\frac{1}{2} - it} \Gamma(s) \Gamma(\frac{1}{2} + it)}{\Gamma(s - \frac{1}{4} + it) \Gamma(s - \frac{1}{4} - it)};$$

$$\overline{\langle P_h(\cdot, \sqrt{s}; 2N), \mu_j \rangle} = \frac{\rho_j(h) \sqrt{\pi} \Gamma(s - \frac{1}{4} + it_j) \Gamma(s - \frac{1}{4} - it_j)}{(4\pi h)^{s-\frac{1}{2}} \Gamma(s)}.$$

This computation is another application of unfolding, in which one uses the integral identity found in [GR15, 6.621(3)] to understand the integrals involving $K$-Bessel functions.

Substituting these expressions into (3.1) gives the following spectral expansion.
Lemma 3.1 (Spectral expansion). The inner product $\langle V, P_h(\cdot, \overline{s}; 2N) \rangle$ has the spectral expansion

$$\langle V(z), P_h(z, \overline{s}; 2N) \rangle = \sum_j \rho_j(h) \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(s)} \langle V, \mu_j \rangle$$

$$+ \sum_a \int_{-\infty}^{\infty} \rho_a(\frac{1}{2} + it, h) \frac{\Gamma(s - \frac{1}{2} + it) \Gamma(s - \frac{1}{2} - it)}{\Gamma(s) \Gamma(\frac{1}{2} + it)} \langle V, E_a(\cdot, \frac{1}{2} + it; 2N) \rangle dt$$

for $\Re s$ sufficiently large. We refer to the sum indexed by $j$ as the “discrete part” and the sum indexed by the cusps $a$ as the “continuous part” of the spectral expansion.

4. The double sum $D(s, w)$

To study the meromorphic continuation for $D(s, w)$, the double Dirichlet series given by

$$\sum_{m, h \geq 1} \frac{a(h)b(m)c(2m - h)}{m^{s+k-1}h^w} = \frac{(8\pi)^{s+k-1}}{\Gamma(s+k-1)} \sum_{h \geq 1} \frac{a(h)\langle V, P_h(\cdot, \overline{s}; 2N) \rangle}{h^w},$$

we substitute the inner products with the spectral expansion obtained in Lemma 3.1. We split our analysis into two parts based on the natural subdivision of $\langle V, P_h \rangle$ into discrete and continuous spectral terms. We also discuss the convergence of each part in turn.

4.1. Discrete Spectrum. The discrete component of $D(s, w)$ is obtained from the discrete part of the spectral expansion in Lemma 3.1 upon multiplying by $a(h)(8\pi)^{s+k-1}/(h^w\Gamma(s+k-1))$ and summing over $h$. After simplification, the discrete component is

$$2^{s-2}(8\pi)^k \sum_j \frac{\Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j)}{\Gamma(s) \Gamma(s+k-1)} \langle V, \mu_j \rangle \rho_j(1) \sum_{h \geq 1} \frac{a(h)\lambda_j(h)}{h^{s+w-\frac{1}{2}}}. \tag{4.1}$$

In simplifying this expression, we have exchanged the order of summation; this needs justification.

It is clear that the $h$-sum converges absolutely for $\Re(s + w)$ sufficiently large. The behavior of $\langle V, \mu_j \rangle \rho_j(1)$ is of polynomial growth in $|t_j|$ on average. In particular, Reznikov’s appendix to [HH16] prove the following lemma.

Lemma 4.1. (Reznikov’s appendix to [HH16]) Suppose $f$ and $g$ are two weight $k$ cuspidal modular forms on the congruence subgroup $\Gamma_0(N)$. Then for any $\epsilon > 0$,

$$\sum_{|t_j| \sim T} \rho_j(1) \langle f, \text{Im}(\cdot)^k, \mu_j \rangle \ll T^{k+1+\epsilon}. $$
For any $s$ away from poles, Stirling’s approximation shows that the gamma functions give exponential decay in $|t_j|$, and thus the sum over $j$ converges locally normally. Thus the double sum converges absolutely, and the sums can be reordered.

4.2. The $h$ sum. We can recognize the $h$-sum,

$$\rho_j(1) \sum_{h \geq 1} \frac{a(h) \lambda_j(h)}{h^s},$$

as a Rankin–Selberg convolution $L$-function which is obtained by unfolding the inner product of the form $\langle \mu_j \Im(\cdot^{k/2} f_1), E \rangle$, where $E$ is an appropriately chosen Eisenstein series.

For this application, we use the weight $k$ Eisenstein series

$$E_\infty^k(z, w; 2N) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(2N)} \Im(\gamma z)^w J(\gamma, z)^{-k},$$

where $J(\gamma, z) = j(\gamma, z)/|j(\gamma, z)|$ and $j(\gamma, z) = (cz + d)$ is our automorphic multiplier. Then another unfolding computation shows that

$$\langle \mu_j \Im(\cdot^{k/2} f_1), E_\infty^k(\cdot, \pi; 2N) \rangle = \sqrt{\pi} \Gamma(s + \frac{k}{2} - \frac{1}{2} - it_j) \Gamma(s + \frac{k}{2} - \frac{1}{2} + it_j) \sum_{h \geq 1} \frac{a(h) \rho_j(h)}{h^{s + \frac{k}{2} - 1}}.$$

Let $\zeta^{(2N)}(s)$ denote the completed zeta function with the Euler factors corresponding to divisors of $2N$ omitted. Then the completed Eisenstein series

$$\zeta^{(2N)}(2s) E_\infty^k(z, s; 2N)$$

has a functional equation of the shape $s \mapsto 1 - s$ and poles at most at $s = 0$ and $s = 1$. We therefore define the Rankin-Selberg convolution $L$-function $L(s, \mu_j \otimes f_1)$ as

$$L(s, \mu_j \otimes f_1) = \zeta^{(2N)}(2s) \sum_{h \geq 1} \frac{a(h) \rho_j(h)}{h^{s + \frac{k}{2} - 1}}.$$

This $L$-function can be completed and satisfies the functional equation

$$\Lambda(s, \mu_j \otimes f_1) := L(s, \mu_j \otimes f_1) \Gamma(s + \frac{k}{2} - \frac{1}{2} - it_j) \Gamma(s + \frac{k}{2} - \frac{1}{2} + it_j)$$

$$= \Lambda(1 - s, \mu_j \otimes f_1).$$

Further, the meromorphic behavior of the Eisenstein series guarantees that the completed $L$-function has poles at most at $s = 0$ and $s = 1$.

In this application, we can rewrite the $h$ sum as

$$\sum_{h \geq 1} \frac{a(h) \rho_j(h)}{h^{s + w - \frac{1}{2}}} = \frac{L(s + w - \frac{k}{2}, \mu_j \otimes f_1)}{\zeta^{(2N)}(2s + 2w - k)}.$$

As a result, we have the following lemma.
Lemma 4.2. The discrete component of $D(s,w)$ in (4.1) can be rewritten as

\[
\frac{(8\pi)^{s+k-1}}{\Gamma(s+k-1)} \sum_{j} \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2} + it_j)\Gamma(s - \frac{1}{2} - it_j)}{\Gamma(s)} \times \langle V, \mu_j \rangle \frac{L(s + w - \frac{k}{2}, \mu_j \otimes f_1)}{\zeta(2N)(2s + 2w - k)}.
\]

Furthermore, the discrete component has meromorphic continuation to $\mathbb{C}^2$.

We note that for $s$ and $w$ in any compact set away from poles, the gamma functions give exponential decay in $|t_j|$, giving locally normal convergence. There are potential poles at $2s + 2w - k = \rho$ where $\rho$ is a zero of the zeta function. There are also potential poles when $s - \frac{1}{2} \pm it_j = -n$ for $n \in \mathbb{Z}_{\geq 0}$ from the gamma functions. In light of the Selberg Eigenvalue Conjecture, we expect that the leading poles of the latter type in $s$ form an infinite family with the same real part; therefore later analysis will not give a consistent asymptotic leading term coming from the discrete spectrum.

Remark 4.3. When $w$ is a nonpositive integer, the poles from $s - \frac{1}{2} \pm it_j = -n$ do not occur. This can be observed by rewriting the discrete spectrum in terms of $\Lambda(\cdot, \mu_j \otimes f_1)$. This reveals pairs of gamma factors of the shape $\Gamma(s - \frac{1}{2} \pm it_j)/\Gamma(s - w - \frac{1}{2} \pm it_j)$, indicating the cancellation.

4.3. Continuous Spectrum. The continuous component of $D(s,w)$ is obtained from the continuous part of the spectral expansion in Lemma 3.1 after multiplying by $a(h)(8\pi)^{s+k-1}/(h \Gamma(s+k-1))$ and summing over $h$. After simplification, the continuous component is

\[
\frac{2^{s+3k-3} \pi ^{k-\frac{1}{2}}}{\Gamma(s+k-1)} \sum_{a} \int_{-\infty}^{\infty} \pi^{-it} \frac{\Gamma(s - \frac{1}{2} + it)\Gamma(s - \frac{1}{2} - it)}{\Gamma(s)\Gamma(\frac{1}{2} + it)} \langle V, E_a(\cdot, \frac{1}{2} + it; 2N) \rangle \times \sum_{h \geq 1} a(h) \rho_a(\frac{1}{2} + it, h) \frac{h^{w+s-\frac{1}{2}-it}}{h^{w+s-\frac{1}{2}-it}} dt.
\]

Classical bounds on $a(h)$ and Lemma 3.4 of [Blo04] imply that $a(h)\rho_a(1/2 + it, h)$ has at most mild polynomial growth in $t$ and $h$. The growth in $t$ is analogous to the growth in $t_j$ in the discrete spectrum. The following lemma follows from Stirling’s formula and the same result in Reznikov’s appendix to [HH16] that Lemma 4.1 is derived from.

Lemma 4.4. (Reznikov’s Appendix to [HH16]) With the notation above, we have the bound

\[
\int_{-T}^{T} \frac{|\langle V, E_a(\cdot, \frac{1}{2} + it; 2N) \rangle|}{|\Gamma(\frac{1}{2} + it)|} dt \ll T^{1+k+\epsilon}.
\]

It follows from Stirling’s formula and Proposition 4.1 of [HH16] that the integral has exponential decay in $t$. Thus for $\text{Re}(s+w)$ sufficiently large, this converges absolutely.
As with the discrete spectrum, we will recognize the sum over $h$ as a Rankin-Selberg convolution and use this convolution to produce a meromorphic continuation of the continuous component of $D(s, w)$.

Explicit computation shows that

$$
\langle E_a(\cdot, \frac{1}{2} + it; 2N) \operatorname{Im}(\cdot)^{\frac{s}{2}} f_1(\cdot), E_{\infty}^k(\cdot, \overline{\tau}) \rangle
= 2\pi^{\frac{s}{2}+it} \frac{\Gamma(\frac{s}{2}+it)}{\Gamma(\frac{s}{2}+it)} \sum_{h \geq 1} a(h) \rho_a(\frac{s}{2}+it, h) \int_0^\infty y^{s+\frac{k}{2}+\frac{1}{2}} K_{it}(2\pi h y) e^{-2\pi h y} \frac{dy}{y}
= 2\pi^{1+it} (4\pi)^{s+\frac{k}{2}+\frac{1}{2}} \frac{\Gamma(s+k-\frac{1}{2}+it)\Gamma(s+k-\frac{1}{2}-it)}{\Gamma(\frac{1}{2}+it)\Gamma(s+k)} \sum_{h \geq 1} a(h) \rho_a(\frac{s}{2}+it, h) \frac{h^{s-it+\frac{k}{2}-\frac{1}{2}}}{h^{s-it+\frac{k}{2}-\frac{1}{2}}}
.(4.5)
$$

It follows that the continuous spectrum can be written as

$$
\frac{2^{3s+2w+3k-5\pi s+w+k-2}}{i\Gamma(s+k-1)} \sum_a \int_{-i\infty}^{i\infty} \frac{\Gamma(s+w)\Gamma(s-\frac{1}{2}+z)\Gamma(s-\frac{1}{2}-z)}{\Gamma(s)\Gamma(s+w-\frac{1}{2}+z)\Gamma(s+w-\frac{1}{2}-z)}
\times \langle E_a(\cdot, \frac{1}{2} + z; 2N) \operatorname{Im}(\cdot)^{\frac{s}{2}} f_1(\cdot), E_{\infty}^k(\cdot, s+w-\frac{k}{2}) \rangle \langle V, E_a(\cdot, \frac{1}{2} - \overline{\tau}; 2N) \rangle dz.
.(4.6)
$$

Each part in this expression for the continuous spectrum has a clear meromorphic continuation, but the integral entangles poles in $s$ with those of $z$. Considering the poles of the gamma functions and Eisenstein series, it is immediately clear that the continuous component has meromorphic continuation to the region defined by $\operatorname{Re}(s+w)-\frac{k}{2}>\frac{1}{2}$ and $\operatorname{Re} s > \frac{1}{2}$. This is sufficient to prove the primary theorems in the next section. But we also explore how to delicately and iteratively extend the meromorphic continuation of the continuous spectrum by carefully shifting the line of integration and collecting residual terms.

Initially take $\operatorname{Re} w$ large. For small $\epsilon > 0$, let $\theta > 0$ be in the interval $(\frac{1}{2}, \frac{1}{2} + \epsilon)$. Shift the $z$-contour of integration to the right along a contour $C$ which bends to remain in the zero-free region of $\zeta(1-2z)$, avoiding the potential poles from these zeroes in $E_a(\cdot, \frac{1}{2} - \overline{\tau}; 2N)$. Taking $\epsilon$ sufficiently small, this shift of contour passes a pole at $z = s - \frac{1}{2}$ with residual term

$$
\mathcal{R}^- = 2^{3s+2w+3k-4\pi s+w+k-1} \frac{\Gamma(s+w)\Gamma(2s-1)}{\Gamma(s+k-1)\cdot 2\pi i} \sum_a \frac{\Gamma(s+w)\Gamma(2s-1)}{\Gamma(s)\Gamma(2s+w-1)\Gamma(w)} \times \langle E_a(\cdot, s; 2N) \operatorname{Im}(\cdot)^{\frac{s}{2}} f_1(\cdot), E_{\infty}^k(\cdot, s+w-\frac{k}{2}) \rangle \langle V, E_a(\cdot, 1-\overline{\tau}; 2N) \rangle.
$$

Note that the residual term $\mathcal{R}^-$ has clear meromorphic continuation to $\mathbb{C}^2$ and is analytic in the region defined by $\operatorname{Re}(s+w)-\frac{k}{2}>\frac{1}{2}$ and $\operatorname{Re} s > 0$, except for a potential pole at $s = \frac{1}{2}$ from $\Gamma(2s-1)$.

The deformation of the contour integral in (4.6) along the contour $C$ is analytic for $\operatorname{Re}(s)$ to the right of the contour $\frac{1}{2}-C$ and to the left of the line $\frac{1}{2}+\epsilon$. Examining a value of $s$ with real part left of the line $\frac{1}{2}$ but still within
the region to the right of the contour \( C \), we can deform the contour back to the line \( \text{Re} z = 0 \). This passes the pole at \( z = \frac{1}{2} - s \) from the other gamma function, giving the residual term

\[
\mathcal{R}^+ = \frac{2^{3s+2w+3k-4} \pi^{s+w+k-1}}{\Gamma(s+k-1) \cdot 2\pi i} \sum_a \frac{\Gamma(s+w)\Gamma(2s-1)}{\Gamma(s)\Gamma(w)\Gamma(2s+w-1)} \times \langle E_a(\cdot,1-s;2N) \text{Im} (\cdot) \frac{1}{2} f_1(\cdot), E^k_{\infty}(\cdot,s+w-\frac{k}{2}) \rangle \langle V, E_a(\cdot,\overline{s};2N) \rangle.
\]

For \( \text{Re} w \) sufficiently large, the now un-deformed contour integral in (4.6) is analytic for \( s \) with \( -\frac{1}{2} < \text{Re} s < \frac{1}{2} \). Thus the continuous spectrum has meromorphic continuation to the region \( \text{Re}(s+w) - \frac{k}{2} > \frac{1}{2} \) and \( \text{Re} s > -\frac{1}{2} \), and the only poles in this region occur in the two residual terms \( \mathcal{R}^- \) and \( \mathcal{R}^+ \).

As in [HH16, §4, p. 481–483] or [HKLDW17, §4], it is possible to iterate this argument: for each pair of conflated poles in \( s \) and \( z \), one can shift and unshift the contour of integration to extend the region of meromorphy at the cost of introducing additional residual terms. Each residual term has clear meromorphic continuation to \( \mathbb{C}^2 \), and thus so does the continuous component.

**Remark 4.5.** The authors have employed this iterative technique of disambiguating poles in appearing in the continuous spectrum several times in the past after specializing to the case where \( w = 0 \). In those cases, the pair of residual terms \( \mathcal{R}^- \) and \( \mathcal{R}^+ \) are anti-symmetric.

### 4.4. Polar behavior of \( D(s,w) \)

Having described the meromorphic continuation of the discrete and continuous parts of \( D(s,w) \), we now summarize the polar behavior of \( D(s,w) \) necessary for the proof of the theorems in the next section.

**Theorem 4.6.** The multiple Dirichlet series \( D(s,w) \) has meromorphic continuation to \( \mathbb{C}^2 \). For \( \text{Re} s + \text{Re} w > \frac{k}{2} \) and \( \text{Re} s > 0 \), \( D(s,w) \) has potential poles at

1. \( s = \frac{1}{2} \pm it_j - r \), where \( r \) is a nonnegative integer, arising from \( \Gamma(s - \frac{1}{2} \pm it_j) \) in (4.3)
2. \( 2s + 2w - k = \rho \), where \( \rho \) is a zero of \( \zeta^{(2N)}(s) \), arising from \( \zeta^{(2N)}(2s+2w-k) \) in (4.3) or \( E^k_{\infty}(\cdot,s+w-\frac{k}{2}) \) in (4.6)
3. \( s = \frac{1}{2} \) arising from \( \Gamma(2s-1) \) in the residual terms \( \mathcal{R}^- \) and \( \mathcal{R}^+ \)

### 5. Bounds on Triple Correlation Sums

In this section we consider a double integral transform of the form

\[
\int_{\sigma_w - i\infty}^{\sigma_w + i\infty} \int_{\sigma_s - i\infty}^{\sigma_s + i\infty} D(s,w)X^{s+k-1}Y^w\Gamma(s+k-1)\Gamma(w)ds dw = \sum_{h,m \geq 1} a(h)b(m)c(2m - h)e^{-m/X}e^{-h/Y}
\]
in order to prove our theorems concerning the sizes of the triple correlation sums. Initially, we take the lines of integration to be $\text{Re } s > 1$ and $\text{Re } w > \frac{k+1}{2}$, within the domain of absolute convergence of $D(s, w)$.

Our main theorem follows quickly from the meromorphic description of $D(s, w)$ and from recognizing that $D(s, w)$ grows at most polynomially in vertical strips.

5.1. **Polynomial growth.** We now examine the discrete component (4.3). Let $\sigma_s := \text{Re } s$ and $\sigma_w := \text{Re } w$. Stirling’s approximation demonstrates that the exponential contribution from the gamma factors is

$$\exp \left( \pi |\text{Im } s| - \pi \max(|t_j|, |\text{Im } s|) \right),$$

wherein we’ve used that $|a+b|+|a-b| = 2 \max(a, b)$ to simplify. There is no other source of exponential growth. Thus for $|t_j| > |\text{Im } s|^{1+\epsilon}$, the sum over $t_j$ has exponential decay and rapidly converges; this is quickly seen to not be the dominant contribution. For $|t_j| \leq |\text{Im } s|^{1+\epsilon}$, there is no exponential contribution, and more care must be given.

It follows from Stirling’s approximation that the polynomial contribution from the gamma factors is

$$\frac{(1 + |\text{Im } s + t_j|)^{\sigma_s-1}(1 + |\text{Im } s - t_j|)^{\sigma_s-1}}{(1 + |\text{Im } s|)^{2\sigma_s+k-2}}.$$  

In the region $|t_j| \leq |\text{Im } s|^{1+\epsilon}$, this is clearly of polynomial growth in $|\text{Im } s|$. To understand the rest of the $j$-sum, it is necessary to decouple the growth in $t_j$ from the $L$-function. Writing each $L$-function as $L(s, \mu_j \otimes \overline{f_1}) = \rho_j(1)\overline{L}(s, \mu_j \otimes \overline{f_1})$ effectively focuses the $j$-dependence into the $\rho_j(1)$ term, while classical Hecke and convexity bounds can handle $\overline{L}(s, \mu_j \otimes \overline{f_1})$.

In particular, the convexity bound, the functional equation (4.2), and the classical bound $1/\zeta(1+z) \ll |\log z|$ [THB86, 3.11.10] show that for $\text{Re}(s+w) \geq \frac{k}{2}$, we have the bound

$$\frac{\overline{L}(s+w - \frac{k}{2}, \mu_j \otimes \overline{f_1})}{\zeta(2s+2w-k)} \ll (1 + |\text{Im } s + \text{Im } w|)^{2+\epsilon}$$

for any $\epsilon > 0$. More generally, away from poles, this is of polynomial growth in $|\text{Im } s|$ and $|\text{Im } w|$ in vertical strips.

Bounding by absolute values and applying Lemma 4.1 to the sum over those $t_j$ with $|t_j| \leq |\text{Im } s|^{1+\epsilon}$, it follows that, away from poles, the discrete component is of polynomial growth in $|\text{Im } s|$ and $|\text{Im } w|$ in vertical strips.

The continuous component (4.4) is very similar and gives almost the same bound. The four gamma factors containing an $s$ are identical to the four appearing in the discrete spectrum, except with $t$ in place of $t_j$; correspondingly the analysis with Stirling’s formula carries over. There is rapid exponential decay when $|t| > |\text{Im } s|^{1+\epsilon}$, and polynomial growth otherwise.
The convexity bound shows that, for $\text{Re}(s + w) > \frac{k}{2}$,

$$\left| \sum_h a(h) \rho_a \left( \frac{1}{2} + it, h \right) \frac{1}{h^{s+w-\frac{1}{2}-it}} \right| \ll (1 + |\text{Im } s + \text{Im } w| + |t|)^{2+\epsilon}$$

for any $\epsilon > 0$. We note that to get this convexity bound, we regard this Dirichlet series as being of Rankin-Selberg type. One can use the functional equation of the Eisenstein series in (4.5) to understand the functional equation of this Dirichlet series. More generally, away from poles it is clear that this Rankin-Selberg type Dirichlet series has polynomial growth in vertical strips.

Finally, combining these bounds together with Lemma 4.4, we see that the integral has exponential decay and doesn’t contribute meaningfully for $|t| > |\text{Im } s|^{1+\epsilon}$, and otherwise is of polynomial growth in $|\text{Im } s|$ and $|\text{Im } w|$.

Thus the continuous component is of polynomial growth in vertical strips, away from poles.

5.2. Proofs of Main Results. We first prove a result analogous to simultaneous square-root cancellation in each variable.

**Theorem 5.1.** Let $a(\cdot)$ denote the coefficients of a holomorphic cuspidal eigenform of weight $k$, level $N$, and trivial nebentypus. Then for any $\epsilon > 0$,

$$\sum_{m, h \geq 1} a(h)b(m)c(2m - h)e^{-m/X}e^{-h/Y} \ll X^{k-1+\Theta + \frac{1}{2} + \epsilon}Y^{k-1+\frac{1}{2} - \Theta + \epsilon},$$

where $\Theta < 7/64$ is the best bound towards the non-holomorphic Ramanujan-Petersson conjecture.

**Proof.** Consider the inverse Mellin transform (5.1). By Theorem 4.6, $D(s, w)$ is holomorphic when $\text{Re } s > \frac{k+1}{2} - \text{Re } w$ and $\text{Re } s > \frac{1}{2} + \max(\text{Re } t_j)$, where $t_j$ ranges over the types of the Maass forms in the discrete spectrum. Shifting lines to $\text{Re } s = \frac{1}{2} + \Theta + \epsilon$ and $\text{Re } w = \frac{k-1}{2} + \frac{1}{2} - \Theta + \epsilon$ avoids all poles. The shifted integral clearly converges since $D(s, w)$ is of polynomial growth in $|\text{Im } s|$ and $|\text{Im } w|$ while $\Gamma(s + k - 1)\Gamma(w)$ have exponential decay in $|\text{Im } s|$ and $|\text{Im } w|$.

Assuming the Riemann Hypothesis, it is possible to further shift the $w$ variable by an additional $1/4$ before encountering the poles from the zeta functions in the denominator. Thus we also have the following theorem.

**Theorem 5.2.** Assume the Riemann Hypothesis. Using the same notation as in Theorem 5.1, for any $\epsilon > 0$ we have

$$\sum_{m, h \geq 1} a(h)b(m)c(2m - h)e^{-m/X}e^{-h/Y} \ll X^{k-1+\frac{1}{2} + \Theta + \epsilon}Y^{k-1+\frac{1}{2} - \Theta + \epsilon}.$$
6. Nonvanishing result

We now prove that, under mild hypotheses, infinitely many products $a(h)b(m)c(2m-h)$ do not vanish. To do this, we specialize $w$ and examine under what conditions the residues of poles coming from the discrete spectrum do not vanish. We note that it would also be possible to consider poles corresponding to zeros of $\zeta(2N)$, occurring in the continuous spectrum.

We fix $w = k^2 + \frac{3}{2}$ and examine potential poles with $\text{Re } s > 0$. This guarantees that $\text{Re } s + \text{Re } w > \frac{k}{2}$. Theorem 4.6 indicates that there are potential poles at $s = \frac{1}{2} \pm it_j$ occurring from the discrete spectrum.

From the description of the discrete spectrum in Lemma 4.2, we compute that the residue at $s = \frac{1}{2} \pm it_j$ is

$$\frac{(8\pi)^{k-\frac{1}{2} \pm it_j}}{\Gamma(k - \frac{1}{2} \pm it_j)} \frac{\sqrt{\pi}}{\Gamma(\pm 2it_j)} \Gamma(1/2 \pm it_j) \langle V, \mu_j \rangle L(2 \pm it_j, \mu_j \otimes \overline{f_1}) \zeta(2N)(4 \pm 2it_j).$$

The ratios of Gamma functions and powers of 2 and $\pi$ are some nonzero constant $C_j$. It remains only to consider the inner product $\langle V, \mu_j \rangle$, the $L$-function, and the $\zeta$-function.

The zeta function is considered far within the region of absolute convergence, and thus is evaluated far from poles and zeros. Similarly, the Rankin–Selberg convolution $L(2 \pm it_j, \mu_j \otimes \overline{f_1})$ is evaluated within its domain of absolute convergence. As both $L(s, \mu_j)$ and $L(s, f_1)$ have Euler products, the general theory of Rankin–Selberg convolutions guarantees that $L(s, \mu_j \otimes \overline{f_1})$ has an Euler product (see for instance Chapter 12 of [Gol06]). As no factor of the Euler product is zero within the domain of absolute convergence, we see that $L(2 \pm it_j, \mu_j \otimes \overline{f_1}) \neq 0$. Thus every factor in the residue is nonzero with the possible exception of $\langle V, \mu_j \rangle$.

Let us suppose that there is a Maass form $\mu_j$ such that $\langle V, \mu_j \rangle \neq 0$ and such that $it_j \neq 0$ is real. Let us fix that form $\mu_j$.

**Remark 6.1.** Note that the Laplacian eigenvalue of $\mu_j$ is $\lambda = (\frac{1}{2} + it_j)(\frac{1}{2} - it_j)$ and that $\lambda$ is real and nonnegative. Thus $it_j$ is either purely real or purely imaginary. If $it_j$ is purely real, then the Maass form is exceptional.

The condition that $t_j \neq 0$ is real is thus equivalent to $\lambda > \frac{1}{4}$.

The Dirichlet series

$$D(s, k^2 + \frac{3}{2}) = \sum_{m \geq 1} \left( \sum_{h \geq 1} \frac{a(h)c(2m-h)}{h^{\frac{k}{2} + \frac{3}{2}}} \right) b(m) m^s$$

can be regarded as a single Dirichlet series in $s$ with meromorphic continuation to $\mathbb{C}$. Note that summing over $h \geq 1$ is equivalent to summing over $h \leq 2X$, as for $h > 2X$ we have $c(2m-h) = 0$. As $\langle V, \mu_j \rangle \neq 0$, we see that this Dirichlet series has a pair of poles at $s = k - \frac{1}{2} \pm it_j$.

Then Theorem 1 of [LD19] applies and yields the following result.
Theorem 6.2. Let \( \mu_j \) be a non-constant Maass form such that \( \langle V, \mu_j \rangle \neq 0 \), and let \( MT(X) \) denote the sum of the residues of \( D(s, \frac{k}{2} + \frac{3}{2})X^s/s \) at all real poles \( s = \sigma \) with \( \sigma \geq k - \frac{1}{2} \). Then
\[
\sum_{m \leq X} \sum_{h \leq 2X} \frac{a(h)b(m)c(2m - h)}{h^{\frac{k}{2} + \frac{1}{2}}} - MT(X) = \Omega_{\pm}(X^{k-\frac{1}{2}})
\]

This is a more precise statement of Theorem 1.3. As an immediate corollary, it follows that infinitely many triples \( a(h)b(m)c(2m - h) \) are non-vanishing.

Remark 6.3. The main term \( MT(X) \) consists of residues at exceptional eigenvalues and the residue at \( s = k - \frac{1}{2} \). In cases where we expect Selberg’s Eigenvalue Conjecture to hold, the main term will arise entirely out of the pole at \( s = k - \frac{1}{2} \).

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