TOPOLOGY OF SYMPLECTIC CALABI-YAU 4-MANIFOLDS VIA ORBIHOLD COVERING

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Abstract. We propose to study symplectic Calabi-Yau 4-manifolds through orbifold coverings. The program will revolve around a class of symplectic 4-orbifolds with vanishing canonical class. We show that for any such symplectic 4-orbifold X, there is a canonically constructed symplectic 4-orbifold Y, together with a cyclic orbifold covering \( \pi: Y \to X \), such that Y has at most isolated Du Val singularities and a trivial orbifold canonical line bundle. The (minimal) resolution of Y is a symplectic Calabi-Yau 4-manifold endowed with a natural symplectic finite cyclic action, extending the deck transformations of the orbifold covering \( \pi: Y \to X \).

The second main result is concerned with configurations of symplectic surfaces in a rational 4-manifold. We introduce a successive symplectic blowing-down procedure, which, under suitable assumptions, converts a given symplectic configuration into a “symplectic arrangement” of pseudoholomorphic curves in \( \mathbb{CP}^2 \) with controlled singularities and intersection properties. The procedure is reversible: by successively blowing up the symplectic arrangement in \( \mathbb{CP}^2 \), one recovers the original symplectic configuration up to a smooth isotopy. This procedure will be a fundamental tool for studying the topology of the symplectic 4-orbifold X in our program.

1. Introduction

1.1. Symplectic Calabi-Yau 4-manifolds. The work of Taubes [25] has taught us the importance of the canonical class in understanding the topology of symplectic 4-manifolds. In particular, when the 4-manifold is minimal, the sign of the pairing of the canonical class and the symplectic form, i.e., \( c_1(K_M) \cdot [\omega] \), plays a critical role. In this paper, we are interested in the case where \( c_1(K_M) \cdot [\omega] = 0 \), which is equivalent to the condition that \( c_1(K_M) \) is a torsion class. In this case, the work of Taubes [25] implies some very strong constraints on the Seiberg-Witten invariant of the 4-manifold. In particular, it follows that either \( c_1(K_M) = 0 \), or it is torsion of order 2. Furthermore, in the latter case the 4-manifold must have the integral homology and intersection form of an Enriques surface (cf. [17]). In the former case where \( c_1(K_M) = 0 \), the 4-manifold is usually referred to as symplectic Calabi-Yau 4-manifold.

Empirical evidence seems to suggest that symplectic Calabi-Yau 4-manifolds form a very restricted class of smooth 4-manifolds. The only known examples are orientable \( T^2 \)-bundles over \( T^2 \) and the complex K3 surfaces (more generally, the only

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examples of symplectic 4-manifolds with canonical class of torsion of order 2 are the Enriques surfaces). Moreover, it is known that the basic constructions in symplectic 4-manifolds, i.e., knot surgery and normal connected sum, cannot produce any new examples from existing ones. On the other hand, applications of gauge theory have established very strong homological constraints on these 4-manifolds: a symplectic Calabi-Yau 4-manifold must have the rational homology and intersection form of a $T^2$-bundle over $T^2$ or the integral homology and intersection form of a $K3$ surface (cf. [21, 17, 18, 3]). Furthermore, by a covering trick, interesting constraints on the fundamental group were also obtained [12]. Finally, we point out that due to Taubes work [25], there is currently no smooth invariant of 4-manifolds which can distinguish homeomorphic symplectic Calabi-Yau 4-manifolds. For a comprehensive discussion on this subject, we refer the readers to [19].

In order to gain further insight on the smooth classification or the classification of the fundamental groups, we recently initiated a study of symplectic Calabi-Yau 4-manifolds which, in addition, possess a certain finite symmetry property (cf. [8]). The most general setup goes as follows: we consider symplectic 4-manifold $(M, \omega)$ with torsion canonical line bundle $K_M$, which is equipped with a symplectic $G$-action by a finite group $G$. Moreover, we assume that the $G$-action satisfies the following condition:

\[ (*) \text{ There exists a } g \in G, \text{ such that either } g \text{ fixes a 2-dimensional surface in } M, \text{ or } g \text{ fixes an isolated point } p \in M, \text{ where the complex linear representation of } g \text{ on } T_p M \text{ has determinant } \neq 1. \]

(We remark that if the symplectic 4-manifold $M$ admits a hyperelliptic Lefschetz pencil, then there is a natural symplectic $\mathbb{Z}_2$-action on $M$, lifting the hyperelliptic involution on the fibers of the Lefschetz pencil. The $\mathbb{Z}_2$-action on $M$ obeys the condition \((*)\). For more information concerning the monodromy representation of a symplectic Lefschetz pencil on $M$, see Baykur-Hayano [4].)

The basic idea is as follows. The $G$-action defines $M$ as a regular orbifold covering, $\pi : M \to X =: M/G$, of the symplectic quotient orbifold. Thus in principle, we can recover the topology of $M$ from the topology of the orbifold $X$. With this understood, recall that in [7], we constructed a canonical symplectic resolution of the symplectic 4-orbifold $X = M/G$, which is a symplectic 4-manifold denoted by $M_G$. Furthermore, we consider the pre-image $D$ of the singular set of $X$ in $M_G$, which in general is a disjoint union of configurations of symplectic surfaces. In this way, we can analyze the topology of the orbifold $X$ via the pair $(M_G, D)$. Now here is the key consequence of the assumption \((*)\) above: the resolution $M_G$ is a rational or ruled symplectic 4-manifold if and only if the $G$-action obeys \((*)\), cf. [7], Theorem 1.9.

When $M_G$ is rational or ruled, the diffeomorphism class of $M_G$ can be easily determined from the homological data and fixed-point set data of the $G$-action on $M$. With this understood, the analysis of the orbifold $X$, in particular the recovering of $M$, is largely reduced to the problem of understanding the embedding of the symplectic configurations of $D$ into the rational or ruled 4-manifold $M_G$. We record below the relevant result from [8], which indicates some initial success of our program.
**Theorem:** (Theorem 1.1 in [8]) Suppose $M$ is a symplectic Calabi-Yau 4-manifold with $b_1 > 0$ which is endowed with a finite symplectic $G$-action. If the resolution $M_G$ is irrational ruled, or $M_G$ is rational and $G = \mathbb{Z}_2$, then $M$ must be diffeomorphic to a $T^2$-bundle over $T^2$ with homologically essential fibers.

First, we remark that by the above theorem, the case where $M_G$ is irrational ruled is completely settled. This is because if $M$ has $b_1 = 0$ (and a torsion canonical class), $M_G$ cannot be irrational ruled (cf. [7]). Hence we only need to focus on the case where $M_G$ is rational. Secondly, we note that for the purpose of simplification, one may impose additional conditions on the $G$-action without compromising our overall goal. More concretely, we may assume the group $G$ is generated by the element $g$ in (*), in particular, $G$ is cyclic, and furthermore, when $g$ fixes a 2-dimensional surface in $M$, we may even assume $G$ is of prime order without compromising the condition (*). In particular, the singular set of the quotient orbifold $X$ consists of a disjoint union of 2-dimensional components $\{\Sigma_i\}$ and a set of isolated points $\{q_j\}$.

With this understood, the purpose of this paper is to introduce further techniques for the analysis of the orbifold $X$ (which are complementary to those developed in the earlier paper [8]). There are two, somewhat, separate threads.

For the first one, we single out a class of symplectic 4-orbifolds which have vanishing canonical class (the quotient orbifolds $X = M/G$ included). Our study of symplectic Calabi-Yau 4-manifolds will revolve around these 4-orbifolds. We show that for any such 4-orbifold $X$, there is a symplectic Calabi-Yau 4-manifold endowed with a symplectic finite cyclic action, which is an orbifold covering of $X$ in a suitable sense (see Theorems 1.1 and 1.2 for more details). This fundamental existence result provided a basis for the future development of our project.

For the second thread, we introduce a successive symplectic blowing-down procedure to analyze configurations of symplectic surfaces in a rational 4-manifold. The blowing-down procedure is reversible, which translates the study of configurations in the rational 4-manifold to the study of certain “symplectic arrangements” of pseudoholomorphic curves in $\mathbb{CP}^2$. Thus the relevant issues concerning the topology of the symplectic 4-orbifold $X$, which can be formulated in terms of embeddings of symplectic configurations in a rational 4-manifold, are turned into more amenable questions concerning symplectic arrangements of pseudoholomorphic curves in $\mathbb{CP}^2$.

1.2. **Symplectic 4-orbifolds with vanishing canonical class.** We shall consider a class of symplectic 4-orbifolds, to be denoted by $X$ throughout, which are specified by conditions (i)-(iii) listed below. We denote the underlying space of $X$ by $|X|$.

(i) Note that the canonical line bundle $K_X$ is an orbifold complex line bundle, and has a well-defined first Chern class $c_1(K_X) \in H^2(|X|, \mathbb{Q})$. We assume $c_1(K_X) = 0$.

(ii) We assume the singular set of $X$ consists of a disjoint union of embedded surfaces $\{\Sigma_i\}$ and a set of isolated points $\{q_j\}$, where we denote by $m_i > 1$ the order of isotropy along $\Sigma_i$, and by $G_j$ the isotropy group at $q_j$. Note that the symplectic $G_j$-action on the uniformizing system (i.e., orbifold chart) centered at $q_j$ naturally defines $G_j$ as a subgroup of $U(2)$, i.e., $G_j \subset U(2)$. With this understood, we let $H_j$ be the normal subgroup of $G_j$ which consists of elements
with determinant 1 (i.e., $H_j = G_j \cap SU(2)$), and let $m_j$ be the order of the quotient group $\Gamma_j := G_j/\overline{H}_j$, which is easily seen cyclic. (The singular point $q_j$ is called a Du Val singularity if and only if $m_j = 1$.)

(iii) We set $n := \operatorname{lcd} \{m_i, m_j\}$ to be the least common multiple of $m_i, m_j$, and we assume $n > 1$, which means that either there is a 2-dimensional component $\Sigma_i$ in the singular set, or there is an isolated singular point $q_j$ with $m_j > 1$.

Note that the orbifold $X = M/G$ in the preceding subsection satisfies (1)-(iii) above. However, in this paper we do not assume $X$ necessarily arises in this way.

Theorem 1.1. There exists a symplectic 4-orbifold $\tilde{Y}$, together with a cyclic, symplectic orbifold covering $\pi : \tilde{Y} \to X$, which has the following properties.

(1) The orbifold $\tilde{Y}$ has at most Du Val singularities, which are given by the set $\pi^{-1}(\{q_j | H_j \neq \{1\}\})$.

(2) The canonical line bundle $K_\tilde{Y}$ is trivial as an orbifold complex line bundle. Moreover, there exists a nowhere vanishing section $s$ of $K_\tilde{Y}$ such that the induced $\mathbb{Z}_n$-action on $K_\tilde{Y}$ by the deck transformations is given by the multiplication of $\exp(2\pi i/n)$, i.e., $s \mapsto \exp(2\pi i/n) \cdot s$, for some generator of $\mathbb{Z}_n$.

(3) The symplectic $\mathbb{Z}_n$-action on $\tilde{Y}$ by the deck transformations has the following fixed-point set structure: for each $i$, every component in $\pi^{-1}(\Sigma_i)$ is fixed by an element of order $m_i$ in $\mathbb{Z}_n$, and for each $j$ with $m_j > 1$, every point in $\pi^{-1}(q_j)$ is fixed by an element of order $m_j$ in $\mathbb{Z}_n$. The number of components in $\pi^{-1}(\Sigma_i)$ is $n/m_i$ and the number of points in $\pi^{-1}(q_j)$ is $n/m_j$, for each $i, j$.

Let $\tilde{Y}$ be the symplectic (minimal) resolution of $Y$. Then $\tilde{Y}$ is a symplectic Calabi-Yau 4-manifold, and furthermore, the symplectic $\mathbb{Z}_n$-action on $Y$ naturally extends to a symplectic $\mathbb{Z}_n$-action on $\tilde{Y}$ (cf. [7]). We note that $\tilde{Y}$ only depends on the partial resolution $\tilde{X}^0$ of $X$, i.e., the symplectic 4-orbifold obtained by only resolving the Du Val singularities of $X$. Moreover, it is easy to see that the quotient orbifold $\tilde{Y}/\mathbb{Z}_n = \tilde{X}^0$ if and only if for each $j$ with $m_j > 1$, the subgroup $H_j = G_j \cap SU(2)$ is trivial. Finally, note that $\tilde{Y} = Y$ and $Y/\mathbb{Z}_n = X$ if and only if for each $j$, $m_j > 1$ and $H_j$ is trivial.

We shall call $Y$ or $\tilde{Y}$ the Calabi-Yau cover of $X$.

Remarks: (1) The construction of $Y$ is a standard affair in the algebraic geometry setting (see e.g. [2]). Our construction may be regarded as a topological version of it. Note that $Y$ is not necessarily the same as $M$ even if $X$ arises as $M/G$.

(2) When $X$ admits a complex structure under which $X$ becomes a complex orbifold with vanishing canonical class, it follows easily from the proof of Theorem 1.1 that $\tilde{Y}$ is a complex surface whose canonical line bundle is topologically trivial. (If $X = M/G$, then $M$ is also a complex surface, but with only torsion canonical line bundle.) In this case, the diffeomorphism class of the 4-manifold (either $\tilde{Y}$ or $M$) is completely understood (which belongs to one of the existing examples).

(3) With Theorem 1.1 at hand, one may attempt to construct new examples of symplectic Calabi-Yau 4-manifolds via the Calabi-Yau cover of a suitable $X$.

One of the basic questions concerning the orbifold $X$ is whether it admits a complex structure. For such an orbifold $X$, the 4-manifold (either $\tilde{Y}$ or $M$) always has the
standard smooth structure. On the other hand, in order to construct new symplectic Calabi-Yau 4-manifolds via the Calabi-Yau cover, the 4-orbifold has to be one without any complex structures. Currently, we do not have any potential examples; in particular, for all the finite group actions on symplectic Calabi-Yau 4-manifolds considered in [8] where $M_G$ is rational, the corresponding fixed-point set structure can always be realized by a holomorphic action.

Another basic problem is to determine the fundamental group of the complement of the singular set of $X$. It is clear that this information would lead to useful information about the fundamental group of the symplectic Calabi-Yau 4-manifold which is an orbifold covering of $X$. We remark that the fundamental group is the most useful and the only known invariant that can be used to distinguish a symplectic Calabi-Yau 4-manifold from the existing examples.

Finally, besides the case where $X$ admits a complex structure, there is another scenario which is favorable in determining the diffeomorphism class of the symplectic Calabi-Yau 4-manifold. To describe it, we recall that in [7], it was shown that the underlying space $|X|$ has a natural symplectic 4-orbifold structure with only isolated singular points $\{q_j\}$, and furthermore, each 2-dimensional singular component $\Sigma_i$ descends to an embedded symplectic surface $B_i$ in $|X|$. With this understood, we consider the scenario where $|X|$ contains an embedded symplectic sphere $F$ with the following significance: $F \cdot F = 0$, and $F$ lies in the complement of $\{q_j\}$ and intersects transversely and positively with each $B_i$. Given such a sphere $F$, we may slightly perturb it if necessary (cf. [13]), so that there is a compatible almost complex structure $J$ on $|X|$, making $F$ and each $B_i$ $J$-holomorphic. Then by standard result in Gromov theory (see [15]), the deformation of the $J$-holomorphic sphere $F$ in $|X|$ is unobstructed. This gives rise to a (singular) $J$-holomorphic $S^2$-fibration structure on $|X|$, which, under favorable conditions, can be lifted to an elliptic fibration structure on the symplectic Calabi-Yau 4-manifold (cf. [8], proof of Theorem 1.1).

With the preceding understood, we shall next explain how to approach these issues concerning the orbifold $X$ through the symplectic resolution $\tilde{X}$ of $X$ constructed in [7]. To this end, we recall that $\tilde{X}$ is simply the symplectic (minimal) resolution of the isolated singularities $\{q_j\}$ of the symplectic 4-orbifold $|X|$. Furthermore, it was shown that $\tilde{X}$ is either rational or ruled. With this understood, for each $j$ we let $\{F_{j,k} | k \in I_j\}$ be the exceptional set in the minimal resolution of $q_j$, and let $D_j := \cup_{k \in I_j} F_{j,k}$ be the configuration of symplectic spheres in $\tilde{X}$. Note that the pre-image $D$ of the singular set of $X$ in the resolution $\tilde{X}$ is given by the union $\cup_i B_i \cup \cup_j D_j$.

For each $j$, there is a set of rational numbers $\{a_{j,k}\}$, which is uniquely determined by the following set of equations: set $c_1(D_j) := \sum_{k \in I_j} a_{j,k} F_{j,k}$, then

$$c_1(D_j) \cdot F_{j,l} + F_{j,l}^2 + 2 = 0, \quad \forall l \in I_j.$$  

Furthermore, it is well-known that each $a_{j,k} \leq 0$, and $m_j = 1$ if and only if $a_{j,k} = 0$ for all $k \in I_j$. With this understood, we observe that (cf. [7], Proposition 3.2) the assumption $c_1(K_X) = 0$ implies that

$$c_1(K_{\tilde{X}}) = \sum_i \frac{1 - m_i}{m_i} B_i + \sum_j c_1(D_j).$$
Note that if we let \( g_i \) denote the genus of \( B_i \), then \( B_i^2 = 2m_i(g_i - 1) \) for each \( i \) by the adjunction formula.

Now assume \( \tilde{X} \) is rational. Then in most of the cases the diffeomorphism class of \( \tilde{X} \) is determined by the number

\[
c_1(K_{\tilde{X}})^2 = \sum_i \frac{2(1-m_i)}{m_i}(g_i - 1) + \sum_j c_1(D_j)^2,
\]

which depends only on the topological type of the singular set of \( X \); in the case when \( c_1(K_{\tilde{X}})^2 = 8 \), further information from \( D \) can be used to determine whether \( \tilde{X} \) is even or odd. It follows easily that the orbifold \( X \) is completely determined by the data \( \{(g_i, m_i)\}, \{D_j\} \), and the embedding of \( D = \cup_i B_i \cup \cup_j D_j \) in \( \tilde{X} \). With this understood, note that in particular, the fundamental group of the complement of the singular set of \( X \) is the same as \( \pi_1(\tilde{X} \setminus D) \), and if the embedding of \( D \) in \( \tilde{X} \) is smoothly equivalent to a holomorphic embedding, then the orbifold \( X \) admits a complex structure. Similarly, the existence of the symplectic sphere \( \tilde{F} \) in \( |X| \) can be formulated easily in terms of the resolution \( \tilde{X} \), relative to the embedding \( D \subset \tilde{X} \).

On the other hand, by exploiting the symplectic \( \mathbb{Z}_n \)-action on \( \tilde{T} \) and relying on the fixed-point set analysis in [8], the following further information on the symplectic 4-orbifold \( X \) as well as the Calabi-Yau covers \( Y \) and \( \tilde{Y} \) is obtained.

**Theorem 1.2.** The Calabi-Yau covers \( Y \) and \( \tilde{Y} \) are classified according to the topology of \( X \) as follows:

1. Suppose \( b_1(X) > 0 \). Then the singular set of \( X \) consists of only tori with self-intersection zero. In this case, \( Y = \tilde{Y} \), which is a \( T^2 \)-bundle over \( T^2 \).
2. Suppose \( b_1(X) = 0 \). Then the Calabi-Yau cover \( \tilde{Y} \) is an integral homology \( K3 \) surface, unless \( X \) falls into one of the following two cases: (i) the singular set of \( X \) consists of 9 non-Du Val isolated points of isotropy of order 3, or (ii) the singular set of \( X \) consists of 5 isolated points of isotropy of order 5 which are all of type (1, 2). In both cases (i) and (ii), \( Y = \tilde{Y} \), which is a symplectic Calabi-Yau 4-manifold with \( b_1 = 4 \).

In Example 2.2 of Section 2, we give various examples of orbifold \( X \) which belong to neither (i) or (ii), so the Calabi-Yau cover \( \tilde{Y} \) is an integral homology \( K3 \) surface (in fact in these examples, \( X \) is naturally a complex orbifold, so \( \tilde{Y} \) is a \( K3 \) surface).

Note that \( \tilde{X} \) is rational if and only if \( b_1(X) = 0 \). As a consequence of Theorem 1.2, it suffices to only consider the case where \( \tilde{X} \) is rational.

### 1.3. Successive symplectic blowing-down.

Now we describe the successive symplectic blowing-down procedure. First, we shall adopt the following notations: we set \( X_N := \mathbb{CP}^2 \# N \mathbb{CP}^2 \), which is equipped with a symplectic structure denoted by \( \omega_N \). In order to emphasize the dependence of the canonical class on the symplectic structure, we shall denote by \( K_{\omega_N} \) the canonical line bundle of \( (X_N, \omega_N) \).

The successive symplectic blowing-down procedure, to be applied to \( (X_N, \omega_N) \) for \( N \geq 2 \), depends on a choice of the so-called reduced basis of \( (X_N, \omega_N) \). To explain this notion, we let \( \mathcal{E}_{X_N} \) be the set of classes in \( H^2(X_N) \) which can be represented by a smooth \((-1\)-sphere, and let \( \mathcal{E}_{\omega_N} := \{ E \in \mathcal{E}_{X_N} | c_1(K_{\omega_N}) \cdot E = -1 \} \). Then each class in \( \mathcal{E}_{\omega_N} \) can be represented by a symplectic \((-1\)-sphere; in particular, \( \omega_N(E) > 0 \) for any \( E \in \mathcal{E}_{\omega_N} \). With this understood, a basis \( \{ H, E_1, E_2, \cdots, E_N \} \) of \( H^2(X_N) \) is called a reduced basis of \( (X_N, \omega_N) \) if the following are true:
it has a standard intersection form, i.e., $H^2 = 1$, $E_i^2 = -1$ and $H \cdot E_i = 0$ for any $i$, and $E_i \cdot E_j = 0$ for any $i \neq j$;

- $E_i \in \mathcal{E}_{\omega_N}$ for each $i$, and moreover, the following area conditions are satisfied for $N \geq 3$: $\omega_N(E_N) = \min_{E \in \mathcal{E}_{\omega_N}} \omega_N(E)$, and for any $2 < i < N$, $\omega_N(E_i) = \min_{E \in \mathcal{E}_{\omega_N}} \omega_N(E)$, where $\mathcal{E}_i := \{ E \in \mathcal{E}_{\omega_N} | E \cdot E_j = 0 \ \forall j > i \}$ for any $i < N$;

- $c_1(K_{\omega_N}) = -3H + E_1 \cdots + E_N$.

A reduced basis $\{H, E_1, E_2, \cdots, E_N\}$ obeys the following constraints in symplectic areas, which are crucial in the applications of such bases:

- $\omega_N(H) > 0$, and for any $j > i$, $\omega_N(E_i) \geq \omega_N(E_j)$ (note that, without loss of generality, we may assume $\omega_N(E_1) \geq \omega_N(E_2)$);

- for any $i \neq j$, $H - E_i - E_j \in \mathcal{E}_{\omega_N}$, so that $\omega_N(H - E_i - E_j) > 0$;

- $\omega_N(H - E_i - E_j - E_k) \geq 0$ for any distinct $i, j, k$.

Reduced bases always exist, see [20] for more details. We remark that a reduced basis is not necessarily unique, however, the symplectic areas of its classes

$$(\omega_N(H), \omega_N(E_1), \omega_N(E_2), \cdots, \omega_N(E_N))$$

uniquely determine the symplectic structure $\omega_N$ up to symplectomorphisms, cf. [16]. Finally, we mention the following crucial result from [16]:

Suppose $N \geq 2$. Then for any $\omega_N$-compatible almost complex structure $J$, any class $E \in \mathcal{E}_{\omega_N}$ which has the minimal symplectic area can be represented by an embedded $J$-holomorphic sphere. In particular, for $N \geq 3$, the class $E_N$ in a reduced basis $H, E_1, \cdots, E_N$ can be represented by a $J$-holomorphic $(-1)$-sphere for any given $J$.

This technical result allows us to associate to any reduced basis $\{H, E_1, E_2, \cdots, E_N\}$ a symplectic blowing-down procedure by blowing down successively $(-1)$-classes of minimal areas. When $N \geq 3$, it successively blows down the classes $E_N, E_{N-1}, \cdots, E_3$ (cf. Lemma 3.1), reducing $(X_N, \omega_N)$ to $(X_2, \omega_2)$. To further blow down $(X_2, \omega_2)$, we first introduce the following terminology: the symplectic structure $\omega_N$ is called odd if $\omega_N(H - E_1 - 2E_2) \geq 0$, and is called even if otherwise. (Note that this definition does not depend on the choice of the reduced basis.) With this understood, it is easily seen that $\omega_N$ is odd if and only if $E_2$ has the minimal area among the classes in $\mathcal{E}_2$, i.e., $\omega_N(E_2) = \min_{E \in \mathcal{E}_2} \omega_N(E)$. Thus when $\omega_N$ is odd, we can further blow down $E_2$ to reach $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. If $\omega_N$ is even, then $\omega_N(H - E_1 - E_2) = \min_{E \in \mathcal{E}_2} \omega_N(E)$. In this case, by blowing down the $(-1)$-class $H - E_1 - E_2 \in \mathcal{E}_2$, we reach the final stage $S^2 \times S^2$.

Since at each stage of the blowing-down procedure the $(-1)$-class has minimal area, it can be represented by a $J$-holomorphic $(-1)$-sphere for any given $J$. This property allows us to construct, in a canonical way, the descendant of a given set of symplectic surfaces $D = \cup_k F_k$ in $(X_N, \omega_N)$ under the successive blowing-down procedure. Without loss of much generality, we shall assume $D$ satisfies the following condition:

(†) Any two symplectic surfaces $F_k, F_l$ in $D$ are either disjoint, or intersect transversely and positively at one point, and no three distinct components of $D$ meet in one point.

Further assumptions on $D$ are required so that the procedure can be successively reversed. In order to explain this, observe that the class of each $F_k$ in $D$ can be written
as \( F_k = aH - \sum_{i=1}^{N} b_i E_i \) with respect to the reduced basis \( \{ H, E_1, E_2, \cdots, E_N \} \), where \( a, b_i \in \mathbb{Z} \). We shall call the numbers \( a \) and \( b_i \) the \( a \)-coefficient and \( b_i \)-coefficients of \( F_k \).

(See Section 3 of [8] for some general properties of the \( a \)-coefficient and \( b_i \)-coefficients of a symplectic surface.) The expression \( F_k = aH - \sum_{i=1}^{N} b_i E_i \) is called the homological expression of \( F_k \) (with respect to the reduced basis).

The assumptions are concerned with the homological expressions of the components \( F_k \) whose \( a \)-coefficients are zero. It was shown in [8] that such a component must be a symplectic sphere, and the \( b_i \)-coefficients are equal to 1 except for one of them, which equals \(-1\). Let \( S \subset D \) be any such symplectic sphere, and we write the homological expression of \( S \) as

\[
S = E_n - E_{l_1} - E_{l_2} - \cdots - E_{l_s}, \quad \text{where } n < l_s \text{ for all } s.
\]

Then one can show that there are at most two components \( F_k \) in \( D \), such that the expression of \( F_k \) contains the class \( E_n \) and \( F_k \) has a zero \( a \)-coefficient (cf. Lemma 3.3).

With this understood, we impose the following two assumptions for any such \( S \subset D \):

(a) Suppose there are two symplectic spheres \( S_1, S_2 \subset D \) whose \( a \)-coefficients equal zero and whose homological expressions contain the class \( E_n \). Then for any class \( E_i \) which appears in \( S \), but does not appear in \( S_1 \) nor \( S_2 \), there is at most one component \( F_k \) of \( D \) other than \( S \), whose homological expression contains \( E_i \) with \( F_k \cdot E_i = 1 \).

(b) Suppose there is only one symplectic sphere \( S_1 \subset D \) whose \( a \)-coefficient equals zero and whose homological expression contains the class \( E_n \). Then there is at most one class \( E_i \) in \( S \), which does not appear in \( S_1 \), but either appears in the expression of more than one components \( F_k \neq S \), or appears in the expression of only one component \( F_k \neq S \) but with \( F_k \cdot E_i > 1 \).

(We remark that when \( S \) is a \((-2)\)-sphere or \((-3)\)-sphere, and \( S_1, S_2 \) are disjoint from \( S \), the assumptions (a) and (b) are automatically satisfied.)

With the preceding understood, we now state the theorem concerning the descendant of \( D \) under the successive symplectic blowing-down procedure. For simplicity, we shall only discuss the case where the symplectic structure \( \omega_N \) is odd, which is the most relevant case for us. The case where \( \omega_N \) is even can be similarly dealt with.

**Theorem 1.3.** Let \( D = D_N = \cup_k F_k \) be a union of symplectic surfaces in \( (X_N, \omega_N) \), where \( N \geq 2 \) and \( \omega_N \) is odd, such that \( D \) satisfies the condition \((\dagger)\). For any reduced basis \( \{ H, E_1, E_2, \cdots, E_N \} \) of \( (X_N, \omega_N) \) such that the assumptions (a) and (b) are satisfied for the homological expressions of the components \( F_k \) in \( D_N \), there is a well-defined successive symplectic blowing-down procedure associated to the reduced basis, blowing down the classes \( E_N, E_{N-1}, \cdots, E_2 \) successively, such that \( (X_N, \omega_N) \) is reduced to \( (X_1, \omega_1) \) (note that \( X_1 = \mathbb{C}P^2 \# \mathbb{C}P^2 \)), and \( D_N \) is transformed to its descendant \( D_1 \) in \( (X_1, \omega_1) \), which is a union of \( J_1 \)-holomorphic curves with respect to some \( \omega_1 \)-compatible almost complex structure \( J_1 \) on \( (X_1, \omega_1) \), where the singularities and the intersection pattern of the components of \( D_1 \) are canonically determined by the homological expressions of the components \( F_k \) of \( D_N \). Moreover, under any of the conditions (c),(d),(e) listed below, one can further blow down the class \( E_1 \) to reach
in the final stage of the successive blowing-down, with the descendant $D_0$ of $D_1$ in $\mathbb{CP}^2$ having the same properties of $D_1$:

(c) The classes $E_1, E_2$ have the same area, i.e., $\omega_N(E_1) = \omega_N(E_2)$.

(d) The class $E_1$ appears in the homological expression of a component $F_k \subset D_N$ with ($+1$)-coefficient (i.e., the corresponding $b_1$-coefficient for $E_1$ equals $-1$). (Note that $F_k$ necessarily has zero $a$-coefficient, and is a symplectic sphere.)

(e) There is a component $F_k$ of $D_N$ with the following property: let $a, b$ be the $a$-coefficient and the $b_1$-coefficient for $E_1$ respectively, then $2b < a$ holds true.

More specifically, let $E$ be the subset of $\{E_2, E_3, \cdots, E_N\}$ which is the complement of those classes that appear in a zero $a$-coefficient component of $D_N$ with a ($-1$)-coefficient, plus the class $E_1$ if $\mathbb{CP}^2$ is the final stage of the successive blowing-down. Then the new intersection points in $D_1$ or $D_0$ are labelled by the elements of $E$. For each new intersection point $E_i$ labelled by $E_i \in E$, there is a small 4-ball $B(E_i)$ centered at $E_i$, with standard symplectic structure and complex structure, such that $D_1 \cap B(E_i)$ or $D_0 \cap B(E_i)$ consists of a union of holomorphic discs intersecting at $E_i$, which are either embedded or singular at $E_i$ with a singularity modeled by equations of the form $z_1^n = a z_2^m$ in some compatible complex coordinates $(z_1, z_2)$ (i.e., the link of the singularity is always a torus knot). The orders of tangency of the intersections at $E_i$ as well as the singularity types in $B(E_i)$ are completely and canonically determined by the pattern of appearance of the class $E_i$ and the classes not contained in $E$ in the homological expressions of the components $F_k$ in $D_N$. Finally, a component of $D_N$ descends to a component in $D_1$ or $D_0$ if and only if it has nonzero $a$-coefficient (a component with zero $a$-coefficient disappears).

Remarks: (1) We shall call $D_0$ or $D_1$ a symplectic arrangement of pseudoholomorphic curves.

(2) Two situations of the new intersection points are worth mentioning, as they occur more generically: let $E_n \in E$ be any element.

- If $E_n$ does not appear in the expression of a component of $D_N$ with ($+1$)-coefficient, then the descendants of the components of $D_N$ containing $E_n$ will intersect the 4-ball $B(\hat{E}_n)$ in a union of holomorphic discs, which are all embedded and intersecting at $\hat{E}_n$ transversely.
- If $E_n$ appears in the expression of a component $S$ of $D_N$ with ($+1$)-coefficient, where

$$S = E_n - E_{l_1} - E_{l_2} - \cdots - E_{l_s}$$

such that the classes $E_{l_i}$ in $S$ do not appear in the expression of any component of $D_N$ with ($+1$)-coefficient, then the holomorphic discs in $B(\hat{E}_n)$ are all embedded, and moreover, each $E_{l_s}$ determines a complex line (through the origin $E_n$) in $B(\hat{E}_n)$, such that the descendants of the components of $D_N$ containing $E_{l_s}$ will intersect the 4-ball $B(\hat{E}_n)$ in a union of holomorphic discs which are all tangent to the complex line determined by $E_{l_s}$, with a tangency of order 2.

(3) The successive blowing-down procedure is purely a symplectic operation; there are no holomorphic analogs. Note that the descendant $D_0$ or $D_1$ depends on the choice of the reduced basis, which in general is not necessarily uniquely determined by
the symplectic structure $\omega_N$. On the other hand, there is also flexibility in choosing the symplectic structure $\omega_N$ (cf. [8], Lemma 4.1). Hence it is not clear if there is a descendant $D_0$ or $D_1$ that is determined by $D_N$ itself.

(4) The successive blowing-down procedure is reversible; by reversing the procedure (with either symplectic blowing-up or holomorphic blowing-up), one can recover $D_N \subset X_N$ up to a smooth isotopy.

With the preceding understood, we now explain how Theorem 1.3 can be used to study the relevant topology of the symplectic 4-orbifold $X$. Consider the resolution $\tilde{X}$ and the symplectic configuration $D = \cup_i B_i \cup_j D_j$ in $\tilde{X}$, and we assume $\tilde{X} = X_N$ for some $N \geq 2$. (Note that $D$ satisfies the condition (†).) Then as we explained in [8], one can fix certain symplectic structures $\omega_N$ on $X_N$, so that for any reduced basis $\{H, E_1, E_2, \ldots, E_N\}$, there are only finitely many possible sets of homological expressions for the components in $D$ (this depends only on the topological type of the singular set of $X$). Assume one can also arrange such that $\omega_N$ is odd.

With this understood, we fix any possible homological expressions of the components of $D$. Assuming the assumptions (a) and (b) are satisfied, and furthermore, one of the conditions (c), (d), (e) is satisfied, then by the successive symplectic blowing-down procedure, we obtain a possible symplectic arrangement of pseudoholomorphic curves in $\mathbb{CP}^2$, to be denoted by $\hat{D}$. The embedding of $D$ in $\tilde{X} = X_N$ with the fixed homological expressions for its components is then converted to the symplectic arrangement $\tilde{D}$ in $\mathbb{CP}^2$. With this understood, we observe:

- If there is a pseudoholomorphic, equisingular deformation of $\hat{D}$ to a complex (i.e., holomorphic) arrangement in $\mathbb{CP}^2$, then one can show that the embedding of $D$ in $\tilde{X} = X_N$ is smoothly equivalent to a holomorphic embedding. This allows us to show that the orbifold $X$ admits a complex structure. The deformation problem of $\hat{D}$ can be attacked using tools from the Gromov theory. However, in general it is a rather delicate problem, even in the case of symplectic line arrangements (cf. [22]).
- Determining the fundamental group of the complement of $\hat{D}$ in $\mathbb{CP}^2$ will help us understanding the fundamental group of $\tilde{X} \setminus D$. We remark that when $\hat{D}$ is an arrangement of plane curves (i.e., a complex arrangement), there is a large body of work in the literature concerning the topology of the complement of $\hat{D}$, cf. e.g. the survey article [1].
- If the symplectic arrangement $\hat{D}$ can be realized, then an embedding of $D$ in $\tilde{X}$ is realized. This gives rise to a construction of the orbifold $X$. For the realization of symplectic line arrangements, see [22].

For an illustration, we shall examine some concrete examples.

**Example 1.4.** (1) Consider the case where $X$ has a singular set described in (i) of Theorem 1.2(2), i.e., the singular set consists of 9 isolated non-Du Val singularities of isotropy of order 3. In this case, the symplectic configuration $D$ is a disjoint union of 9 symplectic $(-3)$-spheres, to be denoted by $F_1, F_2, \ldots, F_9$. Note that the canonical
class of the resolution $\tilde{X}$ is given by
\[ c_1(K_{\tilde{X}}) = -\frac{1}{3}(F_1 + F_2 + \cdots + F_9). \]

It follows immediately that $\tilde{X} = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2}$.

The following is a set of possible homological expressions for $F_1, F_2, \ldots, F_9$:
- $H - E_i - E_r - E_s - E_t$, $H - E_i - E_u - E_v - E_w$, $H - E_i - E_x - E_y - E_z$,
- $H - E_j - E_r - E_u - E_x$, $H - E_j - E_s - E_v - E_y$, $H - E_j - E_t - E_w - E_z$,
- $H - E_k - E_r - E_v - E_z$, $H - E_k - E_s - E_w - E_x$, $H - E_k - E_t - E_u - E_y$.

Each class can be represented by a symplectic $(-3)$-sphere, each pair of distinct classes has zero intersection number, and the sum of the 9 classes equals $-3c_1(K_{\tilde{X}})$. Furthermore, one can arrange so that the symplectic structure on $\tilde{X}$ is odd, e.g., when $F_1, F_2, \ldots, F_9$ have the same area (cf. [8], Lemma 4.1).

It is easy to see that the assumptions (a) and (b) are satisfied, and also, the condition (e) is satisfied. Thus by the successive blowing-down procedure, we obtain a symplectic arrangement $\hat{D}$ in $\mathbb{CP}^2$, which is a union of 9 symplectic lines (i.e., a symplectic $(-3)$-sphere of degree 1) intersecting at 12 points. Note that each line contains 4 intersection points, each intersection point is contained in 3 lines, so $\hat{D}$ has an incidence relation which is the same as that of the dual configuration of the famous Hesse configuration (cf. [14]). In particular, $\hat{D}$ can be realized by an arrangement of complex lines.

(2) Consider the case where $X$ has a singular set as in (ii) of Theorem 1.2(2). In this case, $D$ is a disjoint union of 5 pairs of a symplectic $(-3)$-sphere and a symplectic $(-2)$-sphere, denoted by $F_{1,k}, F_{2,k}$ for $k = 1, 2, \cdots, 5$, where each pair of symplectic spheres $F_{1,k}, F_{2,k}$ intersect transversely and positively in one point. Moreover,
\[ c_1(K_X) = -\frac{1}{5} \sum_{k=1}^{5} (2F_{1,k} + F_{2,k}). \]

It follows easily that $\tilde{X} = \mathbb{CP}^2 \# 11\overline{\mathbb{CP}^2}$.

The following is a set of possible homological expressions for $F_{1,k}, F_{2,k}$, $1 \leq k \leq 5$:
- $F_{1,1} = H - E_i - E_v - E_s - E_t$, $F_{2,1} = H - E_r - E_{i_5} - E_{i_{10}}$,
- $F_{1,2} = H - E_i - E_{i_6} - E_{i_7} - E_{i_8}$, $F_{2,2} = H - E_r - E_{i_3} - E_{i_9}$,
- $F_{1,3} = H - E_{i_2} - E_{i_5} - E_{i_6} - E_{i_7}$, $F_{2,3} = H - E_r - E_{i_4} - E_{i_6}$,
- $F_{1,4} = H - E_{i_3} - E_{i_6} - E_{i_8} - E_{i_{10}}$, $F_{2,4} = H - E_r - E_{i_2} - E_{i_7}$,
- $F_{1,5} = H - E_{i_4} - E_{i_7} - E_{i_9} - E_{i_{10}}$, $F_{2,5} = H - E_r - E_{i_1} - E_{i_8}$

where the symplectic structure on $\tilde{X}$ is odd. Again, the assumptions (a), (b) and the condition (e) are satisfied, so we can blow down $\tilde{X}$ and transform $D$ to a symplectic arrangement $\hat{D} \subset \mathbb{CP}^2$. In this case, $\hat{D}$ is also a symplectic line arrangement, consisting of 10 lines which intersect at 16 points. There are 5 original intersection points, i.e., those inherited from $D$, and 11 new intersection points corresponding to the 11 $E_i$-classes. The original intersection points are double points, and among the 11 new intersection points, 10 are triple points and one point is contained in 5 lines. We note that this incidence relation is realized by the real line arrangement $A_1(2m)$ for $m = 5$. (Recall that $A_1(2m)$, for $m \geq 3$, is the arrangement of $2m$ lines in $\mathbb{RP}^2$, of which $m$
are the lines determined by the edges of a regular \( m \)-gon in \( \mathbb{R}^2 \), while the other \( m \) are the lines of symmetry of that \( m \)-gon, cf. [14].) In particular, \( \mathcal{D} \) can be realized by the complexification of a real line arrangement.

**Example 1.5.** Here we consider the orbifold \( X \) in Example 1.4(1) again, but with the following possible set of homological expressions for \( F_1, F_2, \ldots, F_9 \):

- \( F_1 = E_u - E_j - E_w, \)
- \( F_2 = E_y - E_k - E_z, \)
- \( F_3 = H - E_i - E_r - E_s - E_t, \)
- \( F_4 = H - E_i - E_u - E_v - E_w, \)
- \( F_5 = H - E_i - E_x - E_y - E_z, \)
- \( F_6 = H - E_j - E_r - E_u - E_x, \)
- \( F_7 = H - E_k - E_r - E_v - E_y, \)
- \( F_8 = 2H - E_x - E_t - E_u - E_y - E_j - E_v - E_z, \)
- \( F_9 = 2H - E_s - E_t - E_u - E_y - E_k - E_x - E_w. \)

Again, the assumptions (a), (b) and the condition (e) are satisfied. In this case, the symplectic arrangement \( \hat{D} \) is a union of 5 symplectic lines and 2 symplectic spheres of degree 2, consisting of the descendants of \( F_k \) for \( 3 \leq k \leq 9 \). As for the intersection points, note that \( \mathcal{E} = \{E_s, E_t, E_x, E_u, E_r, E_i, E_w, E_y\} \), so there are totally 8 intersection points labelled by these classes. Moreover, each of the 6 intersection points \( \hat{E}_s, \hat{E}_t, \hat{E}_x, \hat{E}_u, \hat{E}_r, \hat{E}_i \) is a triple point; it is contained in 3 components in \( \hat{D} \) intersecting at it transversely. As for \( \hat{E}_u \) and \( \hat{E}_y \), let’s denote by \( \hat{F}_k \) the descendant of \( F_k \) in \( \hat{D} \), for \( 3 \leq k \leq 9 \). Then the class \( E_j \) (resp. \( E_w \)) determines a complex line in the 4-ball \( B(\hat{E}_a) \), such that \( \hat{F}_6 \) and \( \hat{F}_8 \) (resp. \( \hat{F}_4 \) and \( \hat{F}_9 \)) are tangent to it at \( \hat{E}_u \), with the intersection of \( \hat{F}_6 \) and \( \hat{F}_8 \) (resp. \( \hat{F}_4 \) and \( \hat{F}_9 \)) at \( \hat{E}_u \) being of tangency of order 2.

Similar discussions apply to \( \hat{E}_y \), and the classes \( E_k, E_z \). Finally, we remark that we do not know whether \( \hat{D} \) is realized by a complex arrangement, or even by a symplectic arrangement.

The organization of this paper is as follows. In Section 2, we begin with proofs of Theorems 1.1 and 1.2, followed by Example 2.2. Here we list a few examples of the orbifold \( X \) which arise as the quotient orbifold of a holomorphic cyclic action on a hyperelliptic surface or a complex torus. The singular sets of these orbifolds are different from the ones in (i) or (ii) of Theorem 1.2, so the corresponding Calabi-Yau cover \( \hat{Y} \) is a K3 surface. A common feature of these examples is that the K3 surface contains a large number of \((-2)\)-curves. In the remaining part of Section 2, we discuss some consequences of Theorem 1.1. The first one is concerned with the fixed-point set of a symplectic \( \mathbb{Z}_p \)-action (of prime order) on a Calabi-Yau homology K3 surface, while the second one gives some general constraints on the singular set of the orbifold \( X \). Section 3 is devoted to a detailed account of the successive blowing-down procedure, which, in particular, contains a proof of Theorem 1.3. In Section 4, we discuss some constraints on the singular set of \( X \) where \( b_1(X) = 0 \). These constraints are consequences of the Seiberg-Witten-Taubes theory.

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2. Theorems 1.1 and 1.2 and some corollaries

Recall that an orbifold complex line bundle $p : L \rightarrow X$ is said to be trivial if there is a collection of local trivializations of $L$ such that the associated transition functions are given by identity maps; in the orbifold case this in particular means that if a local trivialization of $L$ is over an uniformizing system $(U, G)$, then the $G$-action on $p^{-1}(L|_U) \cong U \times \mathbb{C}$ is trivial on the $\mathbb{C}$-factor. Note that this latter property is equivalent to the statement that $L$ descends to an ordinary complex line bundle over the underlying topological space $|X|$. It follows easily that if $L$ is a trivial orbifold complex line bundle, then the underlying total space of $L$, denoted by $|L|$, is given by the product $|X| \times \mathbb{C}$.

**Lemma 2.1.** Set $L := K_X$. Then the $n$-th tensor power $L^n$ is a trivial orbifold complex line bundle over $X$. Moreover, $n$ is the minimal positive integer having this property, i.e., if $L^m$ is a trivial orbifold complex line bundle for some $m > 0$, then $m$ must be divisible by $n$.

**Proof.** For each $i$, let $(U_i, \mathbb{Z}_{m_i})$ be an uniformizing system such that $U_i/\mathbb{Z}_{m_i}$ is a regular neighborhood of $\Sigma_i$ in $|X|$. Then note that $H^2(U_i)$ is torsion-free, so that $c_1(K_X) = 0$ in $H^2(|X|, \mathbb{Q})$ implies that $K_{U_i}$ is trivial. With this understood, it is easy to see that there is a trivialization $K_{U_i} \cong U_i \times \mathbb{C}$ such that the induced $\mathbb{Z}_{m_i}$-action on the trivialization is given by the multiplication of $\exp(2\pi i/m_i)$ on the $\mathbb{C}$-factor for some generator of $\mathbb{Z}_{m_i}$. On the other hand, for each $j$, if we let $(U_j, G_j)$ be an uniformizing system centered at $q_j$, where $U_j$ is a 4-ball, then $K_{U_j} \cong U_j \times \mathbb{C}$, and the induced $G_j$-action is given by the multiplication of $\exp(2\pi i/m_j)$ for a generator of $\Gamma_j = G_j/H_j$. With this understood, we see immediately that $L^n$ descends to an ordinary complex line bundle over $|X|$. Furthermore, it also follows easily that if $L^m$ is a trivial orbifold complex line bundle for some $m > 0$, then $m$ must be divisible by $n = \text{lcm}\{m_i, m_j\}$.

To show that $L^n$ is the trivial orbifold complex line bundle, it remains to prove that $L^n$ descends to a trivial ordinary complex line bundle over $|X|$. With this understood, we note that $c_1(L^n) = nc_1(L) = 0$ in $H^2(|X|, \mathbb{Q})$, and with $L^n$ as an ordinary complex line bundle over $|X|$, $c_1(L^n)$ admits a lift in $H^2(|X|)$ (still denoted by $c_1(L^n)$ for simplicity), which is torsion. The assertion that $L^n$ descends to a trivial ordinary complex line bundle over $|X|$ follows readily from the claim that $c_1(L^n) = 0$ in $H^2(|X|)$.

We shall prove that $H^2(|X|)$ is torsion-free, so that $c_1(L^n) = 0$ in $H^2(|X|)$ as claimed. To see this, we note that the symplectic resolution $\tilde{X}$ of $X$ is either rational or ruled. Moreover, note that $\pi_1(|X|) = \pi_1(\tilde{X})$, where $\pi_1(\tilde{X}) = 0$ when $\tilde{X}$ is rational, and $\pi_1(\tilde{X}) = \pi_1(\Sigma)$ when $\tilde{X}$ is a ruled surface over a Riemann surface $\Sigma$. In any event, $H_1(|X|)$ is torsion-free, so that $H^2(|X|) = Hom(H_2(|X|), \mathbb{Z})$, which implies that $H^2(|X|)$ is torsion-free as well. This finishes the proof of the lemma.

**Proof of Theorem 1.1:**

Let $t$ denote the tautological section of the pull-back bundle of $p : L \rightarrow X$ over the total space $L$, i.e., for each $x \in L$, $t(x) = x \in (p^*L)_x = L_{p(x)}$. Then consider $\xi := t^n$, the $n$-th tensor power of $t$, which is a section of the pull-back bundle of $L^n$ over $X$ to
the total space $L$. Since $L^n$ is trivial (as orbifold complex line bundle), we can fix a trivialization $[L] \times \mathbb{C}$ of the pull-back bundle $p^*L^n \to L$, and denote by 1 the constant section $[L] \times \{1\}$. With this understood, we set $Y := \xi^{-1}(1)$, as a subset of the total space $L$. The map $\pi : Y \to X$ is simply given by the restriction of $p : L \to X$ to $Y$. Let $\lambda$ be the generator of $\mathbb{Z}_n$ which acts on $L$ by fiber-wise complex multiplication by $\exp(2\pi i/n)$. Then it is clear that the tautological section $t$ is equivariant under the $\mathbb{Z}_n$-action, i.e., $t(\lambda \cdot x) = \lambda \cdot t(x)$. With this understood, note that $\xi(\lambda \cdot x) = \xi(x)$, which implies that the set $Y$ is invariant under the action of $\lambda$. Furthermore, note that the quotient space of $Y$ under the $\mathbb{Z}_n$-action is identified with $X$ under $\pi : Y \to X$.

With the preceding understood, we shall first show that $Y$ is a smooth 4-orbifold (in particular, it is connected), and $\pi : Y \to X$ is a smooth orbifold covering. With this at hand, we simply endow $Y$ with the pull-back symplectic structure, so that $\pi : Y \to X$ becomes a symplectic orbifold covering.

To see that $Y$ is a smooth orbifold, we note that the tautological section $t$ is given by an equivariant section for any local trivialization of the pull-back of $L$ over an uniformizing system, and the argument we give below is obviously equivariant. With this understood, let $v$ be any given direction along the fiber of $L^n$. Suppose $x \in \xi^{-1}(1) = Y$ be any point. We choose a direction $u$ along the fiber of $L$ such that $ux^{n-1} = \frac{1}{n}v$ holds as tensor product (this is possible because $x \neq 0$ in $L$). Then it is easy to check that

$$\frac{d}{ds}(\ell^n(x + su))|_{s=0} = nx^{n-1} = v,$$

which implies that the section $\xi$ intersects the constant section 1 transversely. It follows that $Y$ is a smooth orbifold, which is easily seen of dimension 4.

Next we show that $\pi : Y \to X$ is a smooth orbifold covering. We shall only be inspecting the situation near the singular set of $X$, as the matter is trivial over the smooth locus. To this end, consider the uniformizing system $(U_i, \mathbb{Z}_{m_i})$ near the singular component $\Sigma_i$. As we have seen before, $K_{U_i}$ is trivial, so we fix a trivialization $K_{U_i} \cong U_i \times \mathbb{C}$. Let $\delta_i \in \mathbb{Z}_{m_i}$ be the generator such that the action of $\delta_i$ on $U_i \times \mathbb{C}$ is given by the multiplication of $\exp(2\pi i/m_i)$ for the $\mathbb{C}$-factor. With this understood, note that $(U_i \times \mathbb{C}, \mathbb{Z}_{m_i})$ is an uniformizing system for the orbifold $L$, over which the pull-back bundle $p^*L \to L$ admits a natural trivialization $(U_i \times \mathbb{C}) \times \mathbb{C} \to U_i \times \mathbb{C}$, where $\delta_i$ also acts as multiplication by $\exp(2\pi i/m_i)$ on the last $\mathbb{C}$-factor. With this understood, we note that $Y = \xi^{-1}(1)$ is given, in the uniformizing system $(U_i \times \mathbb{C}, \mathbb{Z}_{m_i})$, by the subset

$$V_i := \{(y, z) \in U_i \times \mathbb{C} | y \in U_i, z^n = 1\}.$$

Furthermore, the action of $\lambda \in \mathbb{Z}_n$ is given by $(y, z) \mapsto (y, \exp(2\pi i/n)z)$, and the action of $\delta_i$ is given by $(y, z) \mapsto (\delta_i \cdot y, \exp(2\pi i/m_i)z)$. It follows easily that the quotient space of $V_i = \{(y, z) \in U_i \times \mathbb{C} | y \in U_i, z^n = 1\}$ under the $\mathbb{Z}_{m_i}$-action can be identified with $\{(y, z) \in U_i \times \mathbb{C} | y \in U_i, z^{n/m_i} = 1\}$, which is a disjoint union of $n/m_i$ many copies of $U_i$. This shows easily that over $\pi^{-1}(U_i/\mathbb{Z}_{m_i})$, $Y$ is smooth, and $\pi : Y \to X$ is given by the quotient map of the action of $\delta_i^{-1}\lambda^{n/m_i}$. In particular, $\pi : Y \to X$ is a smooth orbifold covering near each $\Sigma_i$. The situation near each $q_i$ is similar. If we let $(U_j, G_j)$ be the uniformizing system near $q_j$, then in $(U_j \times \mathbb{C}, G_j)$, $Y$ is given by the subset $V_j := \{(y, z) \in U_j \times \mathbb{C} | y \in U_j, z^n = 1\}$. Moreover, the quotient space by the
$G_j$-action can be identified with $\{(y,z) \in (U_j/H_j) \times \mathbb{C}| y \in U_j/H_j, z^{n/m_j} = 1\}$. It follows easily that over $\pi^{-1}(U_j/G_j)$, $Y$ is a smooth orbifold, uniformized by $(U_j, H_j)$. Furthermore, $\pi : Y \to X$ is a smooth orbifold covering near each $q_j$ as well. Note that in particular, the argument above proved part (1) and part (3) of Theorem 1.1.

If we endow $Y$ with the pull-back symplectic structure via $\pi : Y \to X$, then $K_Y = \pi^*K_X = \pi^*L = (p^*L)|_Y$, as $\pi = p|_Y$. With this understood, the restriction of the tautological section $s := t|_Y$ is a nowhere vanishing section of $K_Y$. Moreover, the action of $\lambda \in \mathbb{Z}_n$ is given by $s \mapsto \exp(2\pi i/n)s$. This proves part (2).

It remains to show that $Y$ is connected, which is a consequence of $n = \gcd\{m_i, m_j\}$ being minimal. To see this, suppose $Y$ is not connected, and let $Y_0$ be a connected component of $Y$. Then there is a factor $m > 1$ of $n$ such that $\lambda^m$ generates the subgroup of $\mathbb{Z}_n$ which leaves $Y_0$ invariant. With this understood, note that the action of $\lambda^m$ is trivial on $K_Y^{n/m} = K_Y^n|_{Y_0} = (\pi^*L)^{n/m}|_{Y_0}$. Since $(\pi^*L)^{n/m}|_{Y_0}$ is trivial, it follows that $L^{n/m}$ must be the trivial orbifold complex line bundle over $X$. But this contradicts Lemma 2.1, hence $Y$ is connected. The proof of Theorem 1.1 is complete.

**Proof of Theorem 1.2:**

First, consider the case where $b_1(X) > 0$. In this case, $\tilde{X}$ is irrational ruled. This implies that the resolution of $\tilde{Y}/\mathbb{Z}_n$, which is in the same birational equivalence class of $\tilde{X}$ (cf. [7], Theorem 1.5(3)), is also irrational ruled. In particular, it follows that $\tilde{Y}$ is a symplectic Calabi-Yau 4-manifold with $b_1 > 0$.

To proceed further, let $\Sigma_i$ be any 2-dimensional singular component of $X$ with multiplicity $m_i > 1$, or $q_j$ be any singular point with $m_j > 1$. We pick a prime factor $p$ of $m_i$ or $m_j$, and let $H$ be the subgroup of order $p$ that fixes $\Sigma_i$ or $q_j$. Then it is easy to see that the resolution of $\tilde{Y}/H$ is also irrational ruled. By Lemma 2.2 of [8], the fixed-point set of $H$ consists of only tori of self-intersection zero. It follows immediately that each $\Sigma_i$ is a torus of self-intersection zero, and that $X$ has no singular points $q_j$ with $m_j > 1$. Furthermore, by [8] the resolution of $\tilde{Y}/H$, which is simply the underlying space $|\tilde{Y}/H|$, is a $S^2$-bundle over $T^2$, and $\tilde{Y}$ is a $T^2$-bundle over $T^2$. In particular, $\pi_2(\tilde{Y}) = 0$, which easily implies that there are also no singular points $q_j$ with $m_j = 1$. It follows that $|X| = \tilde{X}$ which is a smooth manifold. With this understood, note that $b_2(X) \leq b_2(\tilde{Y}/\mathbb{Z}_n) \leq b_2(\tilde{Y}/H) = 1$, which implies $b_2(X) = 1$ because $\tilde{X} = |X|$ is irrational ruled. It follows easily that $\tilde{X}$ is a $S^2$-bundle over $T^2$. Finally, note that $Y = \tilde{Y}$ as claimed.

Next, suppose $b_1(X) = 0$. In this case, $\tilde{X}$ is rational, so is the resolution of $\tilde{Y}/\mathbb{Z}_n$. We first assume $\tilde{Y}$ is a symplectic Calabi-Yau 4-manifold with $b_1 > 0$. We shall prove that $Y = \tilde{Y}$ and $b_1(\tilde{Y}) = 4$, and determine the singular set of $X$, using the relevant results from [8].

The key issue is to show that $X$ does not have any singular points $q_j$ with $m_j = 1$, and moreover, for all singular points $q_j$ with $m_j > 1$, the subgroup $H_j = G_j \cap SU(2)$ is trivial, i.e., the order of $G_j$ equals $m_j$. In this regard, a useful observation is that $b_2(\tilde{Y}) \neq 2$, or equivalently, $b_1(\tilde{Y}) \neq 3$, by the fact that the resolution of $\tilde{Y}/\mathbb{Z}_n$ is rational (cf. [8], Lemma 2.6 and Theorem 1.2(3)(i)).
First, suppose there is a singular point \( q_j \) with \( m_j = 1 \). We note that the minimal resolution of \( \pi^{-1}(q_j) \) in \( \tilde{Y} \) consists of symplectic \((-2)\)-spheres. By Theorem 1.1, it implies easily that \( b_2(\tilde{Y}) \geq n \geq 2 \). On the other hand, \( b_2(\tilde{Y}) \leq 3 \) as \( b_1(\tilde{Y}) > 0 \) by our assumption. Consequently, we must have \( b_2(\tilde{Y}) = 3 \) (and \( b_1(\tilde{Y}) = 4 \)). In this case, the cohomology ring \( H^*(\tilde{Y}, \mathbb{Q}) \) is isomorphic to the cohomology ring of \( T^4 \) (cf. [23]); in particular, the Hurwitz map \( \pi_2(\tilde{Y}) \to H_2(\tilde{Y}) \) must have trivial image. But this implies that there cannot be any \((-2)\)-spheres at all! Hence our claim that \( X \) does not have any singular points \( q_j \) with \( m_j = 1 \).

Next, suppose there is a singular point \( q_j \) with \( m_j > 1 \), such that the subgroup \( H_j \) is nontrivial. Then by the same reason as in the previous paragraph, \( b_2(\tilde{Y}) \neq 3 \), so that \( b_2(\tilde{Y}) = 1 \) must be true. In particular, there is exactly one symplectic \((-2)\)-sphere in the resolution of \( \pi^{-1}(q_j) \) in \( \tilde{Y} \), and \( n = m_j \). Note that the symplectic \((-2)\)-sphere must be invariant under the symplectic \( \mathbb{Z}_n \)-action on \( \tilde{Y} \). With this understood, we choose a subgroup \( H \) of prime order \( p \) and consider the induced action of \( H \) on \( \tilde{Y} \). There are several possibilities. First, consider the case where the resolution of \( \tilde{Y}/H \) is neither rational nor ruled. Then by Lemma 2.1 in [8], \( H = \mathbb{Z}_2 \) and the fixed-point set of \( H \) consists of 8 isolated points. From the construction of \( Y \), it follows easily that these 8 isolated fixed points must be all contained in the symplectic \((-2)\)-sphere that is invariant under \( H \), which is clearly impossible. Next, suppose the resolution of \( \tilde{Y}/H \) is irrational ruled. In this case the fixed-point set of \( H \) consists of a disjoint union of embedded tori, which also contradicts the fact that the symplectic \((-2)\)-sphere is invariant under the action of \( H \). Lastly, assume that the resolution of \( \tilde{Y}/H \) is rational. Then \( p = 2 \) or \( p = 3 \) must be true (cf. [8], Theorem 1.2(3)). Moreover, if \( p = 2 \), the fixed-point set of \( H \) must contain 8 isolated points, which is not possible as we have seen. Suppose \( p = 3 \). In this case the fixed-point set of \( H \) consists of 6 isolated points plus one possible torus with self-intersection zero, where among the 6 isolated points, exactly 3 are of type \((1,2)\) (i.e., of Du Val type). On the other hand, examining the action of \( H \) on the invariant symplectic \((-2)\)-sphere, there are 3 possibilities: (a) the symplectic \((-2)\)-sphere is fixed by \( H \), (b) the symplectic \((-2)\)-sphere contains an isolated fixed point of type \((1,1)\) and a fixed point which is contained in a 2-dimensional fixed component of \( H \), and (c) the symplectic \((-2)\)-sphere contains 2 isolated fixed points of type \((1,2)\). Case (a) and case (b) are clearly not possible. In case (c), one of the 3 isolated fixed points of \( H \) of type \((1,2)\) does not lie in the symplectic \((-2)\)-sphere, which is not allowed by the construction of \( Y \). This proves our claim that there is no singular point \( q_j \) with \( m_j > 1 \) where the subgroup \( H_j \) is nontrivial.

With the preceding understood, we observe that \( Y = \tilde{Y} \) is true, and moreover, for any nontrivial element of the symplectic \( \mathbb{Z}_n \)-action on \( Y \), its fixed points are all of non-Du Val type. In particular, for any subgroup \( H \) of prime order \( p \), the resolution of \( Y/H \) is either rational or ruled.

To proceed further, we first note that there must be a subgroup \( H \) of prime order such that the resolution of \( Y/H \) is rational. To see this, suppose to the contrary that for every subgroup of prime order, the resolution of the group action is irrational ruled. Since such a group action has only 2-dimensional fixed-point set, it follows that \( X = Y/\mathbb{Z}_n \) has no isolated singular points. We pick a subgroup \( \Gamma \) of prime
order. Then the resolution of $Y/\Gamma$ is simply the underlying space $|Y/\Gamma|$, which is a $S^2$-bundle over $T^2$. The fixed-point set of $\Gamma$ consists of tori of self-intersection zero whose images in $|Y/\Gamma|$ intersect transversely with the fibers of the $S^2$-bundle. To derive a contradiction, we recall from [7] that $|Y/\Gamma|$ has a symplectic structure such that the induced $\mathbb{Z}_n/\Gamma$-action on $|Y/\Gamma|$ is symplectic. We choose a $\mathbb{Z}_n/\Gamma$-invariant, compatible almost complex structure $J$ on $|Y/\Gamma|$. Then note that there is a $J$-holomorphic $S^2$-bundle structure on $|Y/\Gamma|$ with base $T^2$, which is invariant under the $\mathbb{Z}_n/\Gamma$-action.

Now since $b_1(X) = 0$, it follows easily that the $\mathbb{Z}_n/\Gamma$-action on $|Y/\Gamma|$ must induce a homologically nontrivial action on the base $T^2$. This implies that there must be an element $g \in \mathbb{Z}_n/\Gamma$ which leaves a $S^2$-fiber invariant. Since $X$ has no isolated singular points, the $S^2$-fiber must be fixed by $g$. But note that this $S^2$-fiber intersects with the descendants of the fixed-point set of $\Gamma$ in $|Y/\Gamma|$, which contradicts the fact that the 2-dimensional singular components of $X$ are disjoint. Hence the claim that there must be a subgroup $H$ of prime order such that the resolution of $Y/H$ is rational. As a consequence, since $Y/H$ has no Du Val singularities, we must have $b_1(Y) = 4$ by the classification of fixed-point sets in [8]. Furthermore, with $b_1(Y) = 4$, $H$ is either of order 3 with 9 isolated fixed points, or of order 5 with 5 isolated fixed points (cf. [8], Theorem 1.2(3)).

With this understood, we next show that there is no subgroup $\Gamma$ of prime order such that the resolution of $Y/\Gamma$ is irrational ruled. To see this, suppose to the contrary that there is such a subgroup $\Gamma$. Then the order of $\Gamma$ must be either 2 or 3 (cf. [8], Theorem 1.2(2)). If $H$ is of order 3, then $\Gamma$ cannot be of order 3, because $H$ and $\Gamma$ are distinct subgroups of a cyclic group. Hence $\Gamma$ must be an involution. Examining the induced action of $\Gamma$ on the fixed-point set of $H$, which consists of 9 isolated points, we see that $\Gamma$ must fix one of them. But this contradicts the fact that the fixed-point set of $H$ are disjoint from that of $\Gamma$. If $H$ is of order 5, then $\Gamma$ can be of order either 2 or 3. In any event, $\Gamma$ will fix one of the 5 fixed points of $H$, also a contradiction. Hence there is no subgroup $\Gamma$ of prime order such that the resolution of $Y/\Gamma$ is irrational ruled.

Finally, we claim that $n = 3$ or 5. This is because if otherwise, there must be an element $g \in \mathbb{Z}_n$ of order 15, whose action on $Y$ violates Lemma 2.7 in [8]. Now with $n = 3$ or 5, the singular set of $X = Y/\mathbb{Z}_n$ must be as in (i) or (ii) of Theorem 1.2 by the classification of fixed-point sets in [8], Theorem 1.2(3).

Conversely, if $X$ is given as in (i) or (ii), then it is clear that $n = 3$ or 5, and $Y = \tilde{Y}$. We claim that $\chi(Y) = 0$. To see this, we use the Lefschetz fixed point theorem. For example, for the case of $n = 3$, we note that the resolution $\tilde{X} = \mathbb{C}P^2 \# 12\mathbb{C}P^2$ because $c_1(K_{\tilde{X}})^2 = -3$. This implies that $\chi(\tilde{X}) = \chi(\tilde{X}) - 9 = 15 - 9 = 6$. By the Lefschetz fixed point theorem, $3\chi(Y/\mathbb{Z}_3) = \chi(Y) + (3 - 1) \cdot \#Y^{\mathbb{Z}_3}$. With $X = Y/\mathbb{Z}_3$ and $\#Y^{\mathbb{Z}_3} = 9$, we obtain $\chi(Y) = 0$. The case of $n = 5$ is similar. Hence our claim that $\chi(Y) = 0$. It follows that $Y$ has $b_1 > 0$. By the classification in [8], we must have $b_1(Y) = 4$. This finishes the proof of Theorem 1.2.

**Example 2.2.** We list a few examples of holomorphic $G$-actions on a hyperelliptic surface or complex torus $M$ such that $M/G$ is rational and the quotient orbifold $X = M/G$ does not have the singular set in (i) or (ii) of Theorem 1.2. Hence by Theorem 1.2, the corresponding Calabi-Yau cover $\tilde{Y}$ is a $K3$ surface. By the nature of construction, $\tilde{Y}$
comes with a non-symplectic automorphism. An interesting feature of these examples is that the $K3$ surface contains a large number of $(-2)$-curves appearing in various types of configurations in the complement of the fixed-point set of the non-symplectic automorphism, coming from the resolution of the Du Val singularities in $X$.

1) Take a holomorphic involution on a hyperelliptic surface which fixes 2 tori and 8 isolated points. The orbifold $X$ has a singular set of 2 embedded tori and 8 isolated points of Du Val type. The Calabi-Yau cover $\tilde{Y}$ is a $K3$ surface with a non-symplectic involution, which fixes 2 tori, and in the complement of the fixed-point set, there are 16 disjoint $(-2)$-curves.

2) Take a holomorphic $Z_3$-action on a hyperelliptic surface, which has 6 isolated fixed points and either no fixed curve or a single fixed torus (both cases are possible), where exactly 3 of the isolated fixed points are Du Val. In this case, $\tilde{Y}$ comes with a non-symplectic automorphism of order 3, which has 3 isolated fixed points and either no fixed curve or a single fixed torus, such that in the complement of the fixed-point set, there are 9 pairs of $(-2)$-curves, each intersecting transversely in one point.

3) Consider a holomorphic $Z_4$-action on a hyperelliptic surface, which has 4 isolated fixed points where 2 of them are Du Val, and 4 isolated points of isotropy of order 2. The quotient orbifold $X$ has 6 singular points, of which 4 are Du Val. It is easy to see that $n = 2$ in this example, so the $K3$ surface $\tilde{Y}$ comes with a non-symplectic involution. Note that the orbifold $Y$ has 4 Du Val singularities of order 4, and 6 Du Val singularities of order 2, where 2 of the order 2 singularities are fixed by the $Z_2$ deck transformation. It follows easily that the non-symplectic involution on $\tilde{Y}$ has 2 fixed $(-2)$-curves, and in the complement there are 4 disjoint $(-2)$-curves and 4 linear chains of $(-2)$-curves, each containing 3 curves. (Totally, we see 18 $(-2)$-curves in $\tilde{Y}$.)

4) Finally, we consider a holomorphic $Z_8$-action on a complex torus. It has 2 isolated fixed points, all of type $(1,5)$, 2 isolated points of isotropy of order 4 of type $(1,1)$, and 12 isolated points of isotropy of order 2. The quotient orbifold $X$ has 6 singular points, of which 3 are Du Val singularities. It is easy to see that $n = 4$ in this example. Note that the orbifold $Y$ has 16 Du Val singularities of order 2, whose resolution gives 16 disjoint $(-2)$-curves in the $K3$ surface $\tilde{Y}$. Let $\tau$ be the non-symplectic automorphism of order 4. Then the action of $\tau$ on $\tilde{Y}$ is as follows: $\tau^2$ fixes 4 of the 16 disjoint $(-2)$-curves in $\tilde{Y}$, and furthermore, $\tau$ switches 2 of the 4 curves of isotropy of order 2, and leaves each of the remaining 2 curves invariant. In particular, note that $\tau$ has 4 fixed points, which are contained in the 2 invariant $(-2)$-curves.

The remaining part of this section is occupied with two theorems, both applications of Theorem 1.1. The first result is concerned with the fixed-point set of a symplectic $Z_p$-action (of prime order) on a symplectic Calabi-Yau 4-manifold with $b_1 = 0$. Note that by Theorem 1.2, concerning the classification of the topological type of the singular set of $X$, it remains to focus on the case where the Calabi-Yau cover $\tilde{Y}$ has $b_1 = 0$, and to understand the fixed-point set of the symplectic $Z_p$-action on $\tilde{Y}$.

A well-known property of holomorphic actions on a $K3$ surface is that the fixed-point set does not contain points of mixed types, i.e., of both Du Val and non-Du Val types. The reason is that the canonical line bundle of a $K3$ surface is holomorphically trivial, meaning that there is a nowhere vanishing holomorphic section. The aforementioned
property of the fixed-point set follows easily by examining the induced action on the nowhere vanishing holomorphic section. In the following theorem, we generalize this phenomenon to the symplectic category.

**Theorem 2.3.** Let $M$ be a symplectic Calabi-Yau 4-manifold with $b_1 = 0$, which is equipped with a symplectic $G$-action of prime order $p$. Let $X = M/G$ be the quotient orbifold, such that the resolution $M_G$ is rational. Then $M$ with the symplectic $G$-action is equivariantly symplectomorphic to the Calabi-Yau cover $Y$ equipped with the symplectic $\mathbb{Z}_p$-action of deck transformations (note that $n = p$ in this case). As a consequence, the canonical line bundle $K_M$ admits a nowhere vanishing section $s$, such that the induced action of $G$ on $K_M$ is given by multiplication of $\exp(2\pi i/p)$ for some generator $g \in G$. In particular, the fixed-point set $M^G$ does not contain any fixed points of Du Val type.

**Proof.** First of all, since $G$ is of prime order $p$, the singular set of $X = M/G$ consists of 2-dimensional components $\{\Sigma_i\}$ and isolated points $\{q_j\}$, where $m_i = p$ for each $i$ and $G_j = G$ for each $j$. If for some $q_j$, $m_j = 1$, then $H_j = G$, and if $m_j > 1$, then $H_j$ is trivial and $m_j = p$. It follows immediately that $n = p$, where $n := \text{lcm}\{m_i, m_j\}$.

We claim that the set $\{q_j | m_j = 1\}$ is empty, and $Y = \tilde{Y}$, which is a symplectic Calabi-Yau 4-manifold with $b_1 = 0$. To see this, we first note that the singularities of $Y$ are given by the pre-image $\pi^{-1}(q_j)$ where $q_j$ is a singular point of $X$ with $m_j = 1$, so the canonical symplectic $\mathbb{Z}_p$-action on $Y$ acts freely on the singular set of $Y$. With this understood, let $x$ be the number of singular points $q_j$ such that $m_j = 1$. Then

$$\chi(\tilde{Y}/\mathbb{Z}_p) = \chi(M/G) + x \cdot (p - 1)$$

On the other hand, the Lefschetz fixed point theorem implies that

$$p \cdot \chi(M/G) = \chi(M) + (p - 1) \cdot \chi(M^G), \quad p \cdot \chi(\tilde{Y}/\mathbb{Z}_p) = \chi(\tilde{Y}) + (p - 1) \cdot \chi(\tilde{Y}/\mathbb{Z}_p).$$

It follows easily that $\chi(\tilde{Y}) = \chi(M) + x(p^2 - 1)$. With $\chi(M) = 24$, it follows immediately that $\chi(\tilde{Y}) = 24$ and $x = 0$. Hence our claim.

It remains to show that $M$ is $G$-equivariantly symplectomorphic to $Y$ with the natural $\mathbb{Z}_p = G$ action. This part relies on a well-known property of $M$ that $\pi_1(M)$ has no subgroups of finite index. To finish the proof, we let $pr : M \to X = M/G$ be the quotient map. We claim that $pr$ can be lifted to a map $\psi : M \to Y$ under the orbifold covering $\pi : Y \to X$ from Theorem 1.1. To this end, we need to examine the image of $pr_* : \pi_1(M) \to \pi_1^{orb}(X)$, and show that $pr_*(\pi_1(M)) \subset \pi_*(\pi_1(Y))$. For this we observe that there is a surjective homomorphism $\rho : \pi_1^{orb}(X) \to \mathbb{Z}_p$ associated to the orbifold covering $\pi : Y \to X$ such that $\pi_*(\pi_1(Y))$ is identified with the kernel of $\rho$. With this understood, suppose to the contrary that $pr_*(\pi_1(M))$ is not contained in $\pi_*(\pi_1(Y))$. Then the homomorphism $\rho \circ pr_* : \pi_1(M) \to \mathbb{Z}_p$ must be surjective as $p$ is prime. The kernel of $\rho \circ pr_*$ is a subgroup of $\pi_1(M)$ of a finite index, which is a contradiction. Hence our claim that $pr$ can be lifted to a map $\psi : M \to Y$ under the orbifold covering $\pi : Y \to X$. The map $\psi : M \to Y$ is clearly an equivariant diffeomorphism, inducing the identity map on the orbifold $X$. Since the symplectic structure on $Y$ is the pull-back of the symplectic structure on $X$ via the orbifold...
covering \( \pi : Y \to X \), it follows that \( \psi \) is a symplectomorphism. This completes the proof of Theorem 2.3.

The second theorem is concerned with some general constraints on the singular set of \( X \) and the prime factors of the order \( n = \text{lcm}\{m_i, m_j\} \).

**Theorem 2.4.** Suppose the Calabi-Yau cover \( \tilde{Y} \) is an integral homology K3 surface. Then the number \( n := \text{lcm}\{m_i, m_j\} \) and the 2-dimensional components \( \{\Sigma_i\} \) of the singular set of \( X \) obey the following constraints.

1. If \( p \) is a prime factor of \( n \), then \( p \leq 19 \).
2. There can be at most one component in \( \{\Sigma_i\} \) which has genus greater than 1. If there is such a component in \( \{\Sigma_i\} \), then the remaining components must be all spheres. Moreover, \( n \) must equal the order of the isotropy group along the component of genus greater than 1, and if \( p \) is a prime factor of \( n \), then \( p \leq 5 \).
3. There can be at most two components in \( \{\Sigma_i\} \) which are torus, and if this happens, there are no other components in \( \{\Sigma_i\} \), and \( n = 2 \) must be true. If there is only one torus in \( \{\Sigma_i\} \), then \( n \) must equal the order of the isotropy group along the torus, and moreover, if \( p \) is a prime factor of \( n \), then \( p \leq 11 \).

We will prove the theorem by examining the prime order subgroup actions of the symplectic \( \mathbb{Z}_n \)-action on the Calabi-Yau cover \( \tilde{Y} \). To this end, we let \( M \) be a symplectic Calabi-Yau 4-manifold with \( b_1 = 0 \), equipped with a symplectic \( G \)-action of prime order \( p \). Note that \( M \) has the integral homology of \( K3 \) surface.

The induced action of \( G \) on \( H^2(M) \), as an integral \( \mathbb{Z}_p \)-representation, splits into a direct sum of 3 types of \( \mathbb{Z}_p \)-representations, i.e., the regular type of rank \( p \), the trivial representation of rank 1, and the representation of cyclotomic type of rank \( p - 1 \). If we let \( r, t, s \) be the number of summands of the above 3 types of \( \mathbb{Z}_p \)-representations in \( H^2(M) \), then we have the following identities:

\[
    b_2(M) = rp + t + s(p - 1), \quad \chi(M^G) = t - s + 2, \quad \text{and} \quad s = b_1(M^G).
\]

Note that the second identity is the Lefschetz fixed point theorem. As for the third one, i.e., \( s = b_1(M^G) \), it was proved in [10] under the assumption that \( M \) is simply connected. However, since its argument is purely cohomological, the identity continues to hold under the weaker condition \( H_1(M) = 0 \) (cf. [11]). As an immediate corollary, note that if \( p \) is a prime factor of \( n \), then there is an induced \( \mathbb{Z}_p \)-action on \( \tilde{Y} \). If \( p > 19 \), the \( \mathbb{Z}_p \)-representation on \( H^2(\tilde{Y}) \) can not have any summands of regular type or cyclotomic type, i.e., \( r = s = 0 \), because \( b_2(\tilde{Y}) = 22 \). In other words, the symplectic \( \mathbb{Z}_p \)-action on \( \tilde{Y} \) is homologically trivial. However, since \( c_1(K_{\tilde{Y}}) = 0 \), this is not possible (cf. [9]). Hence part (1) of Theorem 2.4 follows.

Next we prove part (2) of Theorem 2.4. Suppose \( \Sigma_i \) is a singular component of \( X \) whose genus \( g_i > 1 \). Then its descendant \( B_i \) in \( \tilde{X} \) has \( B_i^2 = 2m_i(g_i - 1) > 0 \). Since \( b_2^+(\tilde{X}) = 1 \), it follows immediately that one can have at most one such component. Moreover, suppose there is another component \( \Sigma_j \) which is a torus, then its descendant \( B_j \) has \( B_j^2 = 0 \). It is easy to see that \( \Sigma_i, \Sigma_j \) can not both exist, because \( (B_i + B_j)^2 > 0 \) and \( B_i \) and \( B_i + B_j \) are linearly independent. Hence if there is a singular component
\(\Sigma_i\) of genus \(g_i > 1\), then all other singular components must be spheres. To see that \(n = m_i\) in this case, we observe that the pre-image \(\pi^{-1}(\Sigma_i)\) in \(Y\) has \(n/m_i\) many components, each is fixed by a subgroup of \(\mathbb{Z}_n\) of order \(m_i\). The above argument on \(X\), if applied to the orbifold \(\tilde{Y}/\mathbb{Z}_{m_i}\), implies immediately that \(n/m_i = 1\) must be true. Finally, if \(p\) is a prime factor of \(n\), then there is a \(\mathbb{Z}_p\)-action on \(\tilde{Y}\) fixing \(\pi^{-1}(\Sigma_i)\). Now observe that in the identity \(b_2(\tilde{Y}) = rp + t + s(p - 1)\), \(s \geq b_1(\pi^{-1}(\Sigma_i)) = 2g_i \geq 4\), which implies that \(p \leq 22/4 + 1 < 7\). Hence part (2) of Theorem 2.4 is proved.

Finally, we consider part (3) of Theorem 2.4.

**Lemma 2.5.** Let \(G\) be a finite cyclic group of order \(m\). Suppose a symplectic \(G\)-action on \(M\) has at least two fixed components of torus. Then \(m = 2\), and there are no other fixed components besides the two tori.

**Proof.** Let \(p\) be any prime factor of \(m\). We shall first prove the lemma for the induced \(\mathbb{Z}_p\)-action on \(M\). To this end, we first note that in \(b_2(M) = rp + t + s(p - 1)\), \(s \geq 4\), so that \(p \leq 5\). To further analyze the \(\mathbb{Z}_p\)-action for these cases, we shall consider the resolution \(\tilde{X}\) of the quotient orbifold \(M/\mathbb{Z}_p\), which is a symplectic rational 4-manifold. Let \(H, E_1, \ldots, E_N\) be a reduced basis of \(H^2(\tilde{X})\). Recall that \(c_1(K_{\tilde{X}}) = -3H + E_1 + \cdots + E_N\).

Let \(\{\Sigma_i\}\) be the 2-dimensional fixed components and \(\{q_j\}\) the isolated fixed points of the \(\mathbb{Z}_p\)-action. Let \(B_i\) be the descendant of \(\Sigma_i\) in \(\tilde{X}\), and let \(D_j \subset \tilde{X}\) be the exceptional set of the minimal resolution of the singular point \([q_j] \in M/\mathbb{Z}_p\). Then we have

\[
c_1(K_{\tilde{X}}) = \frac{-p-1}{p} \sum_i B_i + \sum_j c_1(D_j).
\]

With this understood, we denote by \(B_1, B_2\) the two torus components in \(\{B_i\}\). Then by Lemma 4.2 in [8], the \(a\)-coefficients of both \(B_1, B_2\) (i.e., the coefficient \(a\) in the expression \(aH - \sum_{k=1}^N b_kE_k\) of \(B_1, B_2\)) are at least 3. Moreover, if the \(a\)-coefficient equals 3, then \(B_1\) or \(B_2\) must take the form \(B = 3H - E_{j_1} - E_{j_2} - \cdots - E_{j_p}\). On the other hand, if there is a symplectic sphere \(S\) in \(\{B_i\}\) or \(\{D_j\}\) whose \(a\)-coefficient is negative, then \(S\) must have the homological expression \(S = aH + (|a| + 1)E_1 - E_{i_1} - \cdots - E_{i_t}\) for some \(a < 0\) (cf. [8], Lemma 3.2). With this understood, note that \(A \cdot S \leq (|a| + 1) + 3a = 2a + 1 < 0\), which contradicts the fact that \(B_1, B_2\) are disjoint from \(S\). Hence if there is a symplectic sphere \(S\) in \(\{B_i\}\) or \(\{D_j\}\) whose \(a\)-coefficient is negative (such a component must be unique, see [8], Lemma 4.2), the \(a\)-coefficients of both \(B_1, B_2\) must be at least 4.

To derive a contradiction for the case where \(p = 3\) or 5, we first observe that the contribution of \(B_1, B_2\) to the \(a\)-coefficient of \(-p \cdot c_1(K_{\tilde{X}})\) is at least \(6(p - 1)\), which is greater than \(3p\) for \(p = 3\) or 5. Hence there must be a sphere \(S\) in \(\{B_i\}\) or \(\{D_j\}\) whose \(a\)-coefficient is negative. With this understood, the contribution of \(B_1, B_2\) to the \(a\)-coefficient of \(-p \cdot c_1(K_{\tilde{X}})\) is then at least \(8(p - 1)\). We will get a contradiction again if the contribution of \(S\) to the \(a\)-coefficient of \(-p \cdot c_1(K_{\tilde{X}})\) is greater than \(8 - 5p\).

Consider first the case of \(p = 3\). In this case, if \(S\) is a component of \(\{B_i\}\), then \(S\) is a \((-6)\)-sphere. The \(a\)-coefficient of \(S\) is no less than \(-2\) (cf. [8], Lemma 3.2), and the contribution to \(-p \cdot c_1(K_{\tilde{X}})\) is at least \(-2(p - 1) = -4 > 8 - 5p\). If \(S\) is a
component from \( \{D_j\} \), then \( S \) is a \((-3)\)-sphere, and its contribution to the \( a \)-coefficient of \(-p \cdot c_1(K_X)\) equals \( p \cdot 1/3 \cdot (-1) = -1 \). In either case, we arrive at a contradiction. Hence \( p = 3 \) is ruled out. For \( p = 5 \), the argument is similar. If \( S \) is a component of \( \{B_j\} \), then \( S \) is a \((-10)\)-sphere. In this case, the contribution of \( S \) to the \( a \)-coefficient of \(-p \cdot c_1(K_X)\) is at least \(-4(p - 1) = -16 > 8 - 5p \). If \( S \) is a component from \( \{D_j\} \), there are several possibilities. Note that \( D_j \) either consists of a single \((-5)\)-sphere, or a pair of \((-3)\)-sphere and \((-2)\)-sphere intersecting transversely at one point. With this understood, note that \( S \) cannot be a \((-2)\)-sphere as it has negative \( a \)-coefficient (cf. [8], Lemma 3.2). If \( S \) is a \((-5)\)-sphere, the contribution of \( S \) to the \( a \)-coefficient of \(-p \cdot c_1(K_X)\) is at least \(-6 \), and if \( S \) is a \((-3)\)-sphere, the contribution equals \(-2 \). In either case, we arrive at a contradiction. Hence \( p = 5 \) is also ruled out.

It remains to consider the case of \( p = 2 \). Note that by Theorem 2.3, there are no isolated fixed points, so \( \{D_j\} = \emptyset \). We first assume \( S \) exists. Then the contribution of \( B_1, B_2 \) to the \( a \)-coefficient of \(-p \cdot c_1(K_X)\) is at least \( 8(p - 1) = 8 \). On the other hand, \( S \) as a component in \( \{B_j\} \) must be a \((-4)\)-sphere. Its contribution to the \( a \)-coefficient of \(-p \cdot c_1(K_X)\) equals \(-1 \). This is a contradiction as the \( a \)-coefficient of \(-p \cdot c_1(K_X)\) equals \( 6 \) for \( p = 2 \). Hence \( S \) cannot exist, and both \( B_1, B_2 \) have \( a \)-coefficient equal to \( 3 \). Then it follows easily that \( B_1 = B_2 = 3H - E_{j_1} - E_{j_2} - \cdots - E_{j_9} \) for some classes \( E_{j_s}, s = 1, 2, \cdots, 9 \). On the other hand, \( c_1(K_X) = -3H + E_1 + \cdots + E_N \). By comparing with the equation \( c_1(K_X) = -1/2 \sum_i B_i \), it follows easily that there are no other components in \( \{B_j\} \) besides \( B_1, B_2 \) (and we must have \( N = 9 \)). This proves the lemma for the \( \mathbb{Z}_p \)-action.

It follows easily that \( m = 2^k \). With this understood, observe that if a point \( q \in M \) is fixed by some nontrivial element of \( G \), then it must be fixed by the subgroup of \( G \) of order \( 2 \). It follows easily that the singular set of the quotient orbifold \( M/G \) consists of only the two tori. If we continue to denote by \( B_1, B_2 \) the descendants of the fixed tori in the resolution \( M/G \) of \( M/G \), then we have

\[
c_1(K_M) = -\frac{m-1}{m} (B_1 + B_2).
\]

Again, the \( a \)-coefficients of \( B_1, B_2 \) are at least \( 3 \) (cf. [8], Lemma 4.2), from which the above equation implies that \( 3 \geq \frac{m-1}{m} (3 + 3) \) by comparing the \( a \)-coefficients of both sides. It follows immediately that \( m = 2 \), and the proof of the lemma is complete.

\[\square\]

Back to the proof of Theorem 2.4, suppose \( \Sigma_1, \Sigma_2 \) are two singular components of \( X \) which are torus, with \( m_1, m_2 \) being the order of the isotropy groups respectively. If \( m_1 \neq m_2 \), then one of \( n/m_1, n/m_2 \) must be greater than \( 1 \). Without loss of generality, assume \( n/m_1 > 1 \). Then there are at least two components in \( \pi^{-1}(\Sigma_1) \subset Y \), which is fixed by a subgroup of \( \mathbb{Z}_n \) of order \( m_1 \). By Lemma 2.5, we must have \( m_1 = 2 \) and \( n/m_1 = 2 \). It follows that we must have \( n = m_2 = 4 \) by the assumption that \( m_1 \neq m_2 \). But this implies that the \( \mathbb{Z}_n \)-action fixes \( \pi^{-1}(\Sigma_2) \subset Y \), so that the subgroup of order \( m_1 = 2 \) which fixes \( \pi^{-1}(\Sigma_1) \subset Y \) also fixes \( \pi^{-1}(\Sigma_2) \subset Y \). This is clearly a contradiction to Lemma 2.5. Hence \( m_1 = m_2 \). Then the above argument shows that we must have \( n/m_1 = n/m_2 = 1 \), and \( n = 2 \) by Lemma 2.5. Moreover, there are no other components in \( \{\Sigma_i\} \) besides \( \Sigma_1, \Sigma_2 \).
Finally, suppose there is only one component $\Sigma_1$ which is a torus, with $m_1$ being the order of the isotropy group along $\Sigma_1$. Then if $n > m_1$, there will be at least two components in $\pi^{-1}(\Sigma_1)$, which is fixed by a $\mathbb{Z}_{m_1}$-action on $\hat{Y}$. By Lemma 2.5, $m_1 = 2$ and $n/m_1 = 2$, so that $n = 4$. If there is a component in $\{\Sigma_i\}$ with $m_i = n = 4$, then this component is also fixed by the $\mathbb{Z}_{m_1}$-action, which contradicts Lemma 2.5. Hence there must be a singular point $q_j$ such that $m_j = n = 4$. Suppose first that the subgroup $H_j$ at $q_j$ is trivial. Then $\pi^{-1}(q_j)$, consists of one point, is a smooth point in $Y$, and is being fixed by the $\mathbb{Z}_n$-action on $\hat{Y}$. In particular, it is a fixed point of the subgroup of order $m_1 = 2$. But this contradicts Lemma 2.5. Suppose $H_j$ is nontrivial. Then $\pi^{-1}(q_j)$ is an orbifold point of $Y$. Let $D_j$ be the exceptional set of its minimal resolution in $\hat{Y}$. Then $D_j$ is invariant under the $\mathbb{Z}_n$-action on $\hat{Y}$. It is easy to see that the action of the subgroup of order $m_1 = 2$ has a fixed point contained in $D_j$, which is a contradiction to Lemma 2.5. This proves that $n$ must be equal to the order of the isotropy group along the unique torus component $\Sigma_1$. Finally, suppose $p$ is a prime factor of $n$. Then the action of the subgroup of $\mathbb{Z}_n$ of order $p$ on $\hat{Y}$ fixes the torus $\pi^{-1}(\Sigma_1)$. Now appealing to the identity $b_2(\hat{Y}) = rp + t + s(p - 1)$, we find that $s(p - 1) \leq 22$, where $s \geq b_1(\pi^{-1}(\Sigma_1)) = 2$. It follows easily that $p \leq 11$. This completes the proof of Theorem 2.4.

3. The successive blowing-down procedure

In this section, we are given with $(X_N, \omega_N)$, where $N \geq 2$ and $\omega_N$ is odd. Recall that for each $l \geq 0$, we let $X_l = \mathbb{C}P^2 \# l\overline{\mathbb{C}P}^2$, and denote the relevant symplectic structure on $X_l$ by $\omega_l$. We begin with the following lemma.

**Lemma 3.1.** Let $H, E_1, \ldots, E_N$ be a reduced basis of $(X_N, \omega_N)$, and let $C_N$ be any symplectic $(-1)$-sphere in $(X_N, \omega_N)$ representing the class $E_N$. Denote by $(X_{N-1}, \omega_{N-1})$ the symplectic blowdown of $(X_N, \omega_N)$ along $C_N$. Then $H, E_1, \ldots, E_{N-1}$ naturally descend to a reduced basis $H', E'_1, \ldots, E'_{N-1}$ of $(X_{N-1}, \omega_{N-1})$, and $\omega_{N-1}$ is odd if $N \geq 3$.

**Proof.** It is clear that $H, E_1, \ldots, E_{N-1}$ naturally descend to a basis $H', E'_1, \ldots, E'_{N-1}$ of $H^2(X_{N-1})$. We need to show that it is a reduced basis of $(X_{N-1}, \omega_{N-1})$, and moreover, when $N \geq 3$, $\omega_{N-1}$ continues to be an odd symplectic structure.

First of all, we note that $H', E'_1, \ldots, E'_{N-1}$ has the standard intersection form, and the symplectic canonical class of $(X_{N-1}, \omega_{N-1})$ is given by

$$c_1(K_{\omega_{N-1}}) = -3H' + E'_1 + \cdots + E'_{N-1}.$$ 

It remains to verify that for each $i$, $E'_i \in \mathcal{E}_{\omega_{N-1}}$, and moreover, the following area conditions are satisfied: $\omega_{N-1}(E'_i) = \min_{E' \in \mathcal{E}_{\omega_{N-1}}} \omega_{N-1}(E')$, and for any $i < N-1$, $\omega_{N-1}(E'_i) = \min_{E' \in \mathcal{E}_{\omega_{N-1}}} \omega_{N-1}(E')$, where $\mathcal{E}'_{\omega_{N-1}} := \{E' \in \mathcal{E}_{\omega_{N-1}} | E' \cdot E'_j = 0, \forall j > i\}$.

The key step is to show that the set $\mathcal{E}_{N-1} = \{E \in \mathcal{E}_{\omega_N} | E \cdot E_N = 0\}$ may be identified with the set $\mathcal{E}_{\omega_{N-1}}$ by identifying the elements of $\mathcal{E}_{N-1}$ with their descendants in $H^2(X_{N-1})$, and moreover, under this identification the symplectic forms $\omega_N = \omega_{N-1}$. To see this, let $E \in \mathcal{E}_{N-1}$ be any class and let $E'$ be its descendant in $H^2(X_{N-1})$. We choose a $J_1$ such that $C_N$ is $J_1$-holomorphic. Then pick a generic $J_0$ and connect $J_0$
and $J_t$ through a smooth path $J_t$. Since $J_0$ is generic, $E$ can be represented by a $J_0$-holomorphic $(-1)$-sphere, denoted by $C_E$. On the other hand, since $E_N$ has minimal symplectic area, for each $t$, $E_N$ is represented by a $J_t$-holomorphic $(-1)$-sphere $C_t$, which depends on $t$ smoothly, with $C_1$ at $K = 1$ being the original $(-1)$-sphere $C_N$. Note also that the $J_0$-holomorphic $(-1)$-spheres $C_E$ and $C_0$ are disjoint because $E \cdot E_N = 0$.

With this understood, we note that the isotopy from $C_0$ to $C_1 = C_N$ is covered by an ambient isotopy $\psi_t : X_N \to X_N$, where each $\psi_t$ is a symplectomorphism (cf. Proposition 0.3 in [24]). It follows easily that $\text{E}_1$ is represented by the symplectic $(-1)$-sphere $\psi_1(C_E)$, which is disjoint from $C_N$. This shows that the descendant $E'$, which is represented by the symplectic $(-1)$-sphere $\psi(C_E)$ in $X_{N-1}$, lies in the set $\mathcal{E}_{\omega_{N-1}}$. Moreover, $\omega_N(E) = \omega_{N-1}(E')$. Finally, let $E'$ be any class in $\mathcal{E}_{\omega_{N-1}}$. Then $E'$ can be represented by a smooth $(-1)$-sphere, to be denoted by $S'$, and $E' \cdot c_1(K_{\omega_{N-1}}) = -1$.

Now recall that the 4-manifold $X_{N-1}$ is obtained from $X_N$ by removing the $(-1)$-sphere $C_N$ and then filling in a symplectic 4-ball $B$. Without loss of generality, we may assume $S'$ is lying outside $B$, because if otherwise, one can always apply an ambient isotopy to push $S'$ outside of $B$. With this understood, the smooth sphere $S'$ can be lifted to a smooth sphere $S$ in $X_N$. Let $E$ be the class of $S$. Then clearly $E \cdot E_N = 0$ and $E'$ is the descendant of $E$ in $H^2(X_{N-1})$. To see that $E \in \mathcal{E}_{\omega_{N-1}}$, we only need to verify that $E \cdot c_1(K_{\omega_N}) = -1$. But this follows easily from the fact that $c_1(K_{\omega_N}) = c_1(K_{\omega_{N-1}}) + E_N$ and $E' \cdot c_1(K_{\omega_{N-1}}) = -1$. Hence the claim that $\mathcal{E}_{N-1}$ and $\mathcal{E}_{\omega_{N-1}}$ are naturally identified and the symplectic forms $\omega_N$ and $\omega_{N-1}$ agree.

With the preceding understood, it follows easily that for each $i = 1, 2, \cdots, N - 1$, $E_i' \in \mathcal{E}_{\omega_{N-1}}$. Moreover, $\omega_{N-1}(E_i'_{N-1}) = \min_{E' \in \mathcal{E}_{\omega_{N-1}}} \omega_{N-1}(E')$. We further observe that for each $i < N - 1$, the subset $\mathcal{E}_i$ of $\mathcal{E}_{N-1}$ is identified with the subset $\mathcal{E}_i'$ of $\mathcal{E}_{\omega_{N-1}}$ under the identification between $\mathcal{E}_{N-1}$ and $\mathcal{E}_{\omega_{N-1}}$. With $\omega_N$ and $\omega_{N-1}$ agreeing with each other under the identification, it follows immediately that $H', E'_1, \cdots, E'_{N-1}$ is a reduced basis of $(X_{N-1}, \omega_{N-1})$, and $\omega_{N-1}$ is odd if $N \geq 3$. This finishes off the proof.

Lemma 3.1 makes it possible for a successive blowing-down procedure. For simplicity, we shall continue to use the notations $H, E_1, \cdots, E_{N-1}$ to denote the descendants in the symplectic blowdown $(X_{N-1}, \omega_{N-1})$, instead of the notations $H', E'_1, \cdots, E'_{N-1}$ in the lemma. We fix a reduced basis $H, E_1, \cdots, E_N$ of $(X_N, \omega_N)$.

Now suppose we are given with a union of symplectic surfaces $D = D_N = \cup_k F_k$ in $(X_N, \omega_N)$, where $D_N$ satisfies the following condition:

1. Any two symplectic surfaces $F_k, F_l$ are either disjoint, or intersect transversely and positively at one point, and no three distinct components of $D_N$ meet in one point.

3.1. Perturbing the $(-1)$-spheres to a general position. We shall first describe how to blow down $(X_N, \omega_N)$ along the class $E_N$ and how to define the descendants of the components $F_k$ of $D_N$ in $(X_{N-1}, \omega_{N-1})$. First of all, we slightly perturb the symplectic surfaces $F_k$ if necessary, so that the intersection of $F_k$ is $\omega_N$-orthogonal (cf. [13]). Furthermore, we choose an $\omega_N$-compatible almost complex structure $J_N$ which is integrable near each intersection point of the symplectic surfaces $F_k$ such that $D_N$ is
$J_N$-holomorphic. With this understood, since $N \geq 2$ and $\omega_N$ is odd, we may represent the class $E_N$ by an embedded $J_N$-holomorphic sphere $C_N$.

An important feature of the successive blowing-down procedure is that, before we blow down the $(-1)$-sphere $C_N$, we shall first put it in a general position, as long as $C_N$ is not part of $D_N$. We carry out this step as follows.

The intersection of $C_N$ with each $F_k$ is isolated, though not necessarily transverse, and furthermore, $C_N$ may contain the intersection points of the components $F_k$ in $D_N$. The local models for the intersection of $C_N$ with $D_N$ are as follows. If $p \in C_N \cap D_N$ is the intersection of $C_N$ with a single component $F_k$, then locally near $p$, $C_N$ and $F_k$ are given respectively by $z_2 = 0$ and $z_2 = z_1^n + \text{higher order terms}$. If $p \in C_N \cap D_N$ is the intersection of $C_N$ with more than one components of $D_N$, then near $p$ there is a standard holomorphic coordinate system such that the relevant components of $D_N$ are given by complex lines through the origin, and $C_N$ is given by an embedded holomorphic disc through the origin. With this understood, it is easy to see that one can always slightly perturb $C_N$ to a symplectic $(-1)$-sphere, still denoted by $C_N$ for simplicity, such that $C_N$ obeys the following **general position condition**:

$C_N$ intersects each $F_k$ transversely and positively, and $C_N$ does not contain any intersection points of the components of $D_N$. Furthermore, the intersection of $C_N$ with each $F_k$ is $\omega_N$-orthogonal (after a small perturbation if necessary, cf. [13]). (We should point out that when $C_N$ is part of $D_N$, there is no need to perturb $C_N$.)

By the Weinstein neighborhood theorem, a neighborhood $U$ of $C_N$ is symplectically modeled by a standard symplectic structure on a disc bundle associated to the Hopf fibration, where $C_N$ is identified with the zero-section. With this understood, for each $F_k$ which intersects $C_N$, we slightly perturb $F_k$ near the intersection points so that $F_k$ coincides with a fiber disc inside $U$. Now symplectically blowing down $(X_N, \omega_N)$ along $C_N$ amounts to cutting $X_N$ open along $C_N$ and then inserting a standard symplectic 4-ball of a certain radius back in (the radius of the 4-ball is determined by the area of $C_N$). We denote the resulting symplectic 4-manifold by $(X_{N-1}, \omega_{N-1})$. Then the descendant of $F_k$ in $X_{N-1}$ is defined to be the symplectic surface, to be denoted by $\tilde{F}_k$, which is obtained by adding a complex linear disc to $F_k \setminus C_N$ inside the standard symplectic 4-ball for each of the intersection points of $F_k$ with $C_N$. If $F_1$ is another symplectic surface intersecting $C_N$, then the descendant $\tilde{F}_1$ of $F_1$ in $X_{N-1}$ will intersect with $\tilde{F}_k$ at the origin of the standard symplectic 4-ball, which is the only new intersection point introduced to $F_k, F_1$ under the blowing down operation along $C_N$. We denote the origin of the standard symplectic 4-ball by $\tilde{E}_N \in X_{N-1}$. Note that under this construction, $\tilde{F}_k$ is immersed in general, where the (transverse) self-intersection at $\tilde{E}_N$ is introduced if $F_k$ intersects $C_N$ at more than one point. Finally, we denote by $B(\tilde{E}_N)$ a small 4-ball centered at $\tilde{E}_N$ such that $B(\tilde{E}_N) \cap (\bigcup_k \tilde{F}_k)$ consists of a union of (linear) complex discs through the origin. Note that for each $k$, the number of complex discs in $B(\tilde{E}_N) \cap \tilde{F}_k$ equals the intersection number $E_N \cdot F_k$.

To continue with the successive blowing-down procedure, we consider the union of the generally immersed symplectic surfaces $D_{N-1} := \bigcup \tilde{F}_k$ in $(X_{N-1}, \omega_{N-1})$. For simplicity, we shall continue to denote the descendant $\tilde{F}_k$ by the original notation $F_k$. By Lemma 3.1, the classes $H, E_1, \cdots, E_{N-1}$ descend to a reduced basis of $(X_{N-1}, \omega_{N-1})$,
continued to be denoted by the same notation (i.e., \(H, E_1, \ldots, E_{N-1}\)). However, the initial condition (†) concerning the intersections of the components \(F_k\) of \(D_N\) is replaced by the following condition:

(†) There exists an \(\omega_{N-1}\)-compatible almost complex structure \(J_{N-1}\) such that each component \(F_k\) in \(D_{N-1}\) is \(J_{N-1}\)-holomorphic, self-intersecting and intersecting with each other transversely. Moreover, \(J_{N-1}\) is integrable near the intersection points.

We shall continue this process if \(N - 1 \geq 2\). Now suppose we are at the stage of \((X_n, \omega_n)\) for some \(n < N\), with the descendant of \(D_N\) in \(X_n\) denoted by \(D_n\), which is \(J_n\)-holomorphic with respect to some \(\omega_n\)-compatible almost complex structure \(J_n\).

Suppose \(n \geq 2\) and we are trying to blow down the class \(E_n\) in the reduced basis of \((X_n, \omega_n)\), and to define the descendant of \(D_n\) under the blowing-down operation. To this end, we represent the class \(E_n\) by a \(J_n\)-holomorphic sphere \(C_n\). If \(C_n\) is not part of \(D_n\), then as we argued in the case of \(C_N\), one can slightly perturb \(C_n\) to a symplectic \((-1)\)-sphere, still denoted by \(C_n\), such that \(C_n\) obeys the general position condition. With this understood, we simply blow down \((X_n, \omega_n)\) along \(C_n\) in the same way as we blow down \((X_N, \omega_N)\) along \(C_N\), and move on to the next stage \((X_{n-1}, \omega_{n-1})\).

However, if \(C_n\) is part of \(D_n\), then we can no longer perturb \(C_n\) before blowing it down, in order to make the successive blowing-down procedure reversible. In the easy situation where \(C_n\) is one of the original symplectic surfaces in \(D_N\), we can simply blow it down without perturbing it. In general, \(C_n\) is the descendant of a symplectic sphere \(S \subset D_N\) to \(X_n\), where the \(a\)-coefficient of \(S\) is zero and the \(E_1\)-class \(E_n\) appears in \(S\) with a \((+1)\)-coefficient, i.e., \(S\) has the homological expression

\[ S = E_n - E_{l_1} - \cdots - E_{l_s}, \]

where \(n < l_s\) for all \(s\).

In this case, more care needs to be given in defining the descendant \(D_{n-1}\) of \(D_n\) in the next stage \((X_{n-1}, \omega_{n-1})\).

### 3.2. Tangency of higher orders and singularities

When \(C_n\) is part of \(D_n\), intersection of higher order tangency as well as singularities may occur in \(D_{n-1}\). In order to construct \(D_{n-1}\), we need the following technical lemma.

**Lemma 3.2.** Let \((M, \omega)\) be a symplectic 4-manifold and \(C\) be a symplectic \((-1)\)-sphere in \((M, \omega)\). Let \((M', \omega')\) be the symplectic blow-down of \((M, \omega)\) along \(C\), obtained by removing \(C\) and gluing back a standard symplectic 4-ball (with an appropriate size depending on the area of \(C\)). Note that the set of points on \(C\) corresponds naturally to the set of complex lines through the origin in the standard symplectic 4-ball in \((M', \omega')\).

With this understood, the following statements hold.

1. Let \(S_0, S_1, \ldots, S_k\) be symplectic surfaces in \((M, \omega)\), which intersect \(C\) at a point \(p\). Moreover, suppose there is a complex coordinate system \((w_1, w_2)\) centered at \(p\) in which the symplectic structure \(\omega\) is standard, such that \(C\) is defined by \(w_2 = 0\), \(S_0\) is defined by \(w_1 = 0\), and each \(S_i\), \(i > 0\), is defined by the complex line \(w_2 = a_i w_1\) for some distinct complex numbers \(a_i \neq 0\). Then the descendant \(S'_i\) of \(S_i\) in the blow-down \((M', \omega')\) can be defined as follows: let \((z_1, z_2)\) be the complex coordinates of the standard symplectic 4-ball in \((M', \omega')\), such that the complex line corresponding to the intersection point \(p \in C\) is given by \(z_1 = 0\),
then $S'_0$ is obtained by gluing a complex disc to $S_0 \setminus C$ contained in $z_1 = 0$, and for each $i > 0$, $S'_i$ is obtained by gluing a holomorphic disc to $S_i \setminus C$ defined by the equation $z_1 = b_i z_2^2$ for some distinct complex numbers $b_i \neq 0$.

(2) Let $S$ be a symplectic surface intersecting $C$ at $p$, such that there is a Darboux complex coordinate system $(w_1, w_2)$ centered at $p$, in which $C$ and $S$ are given by $w_2 = 0$ and $w_2^i = aw_1^m$ for some relative prime integers $m, n > 0$ and a complex number $a \neq 0$. Then the descendant $S'$ of $S$ in the blow-down $(M', \omega')$ can be defined as follows: let $(z_1, z_2)$ be the complex coordinates of the standard symplectic 4-ball $B^4 \setminus B^4(\delta)$, such that the complex line corresponding to the intersection point $p \in C$ is given by $z_1 = 0$, then $S'$ is obtained by gluing a holomorphic disc to $S \setminus C$ defined by the equation $z_1^m = b z_2^{m+n}$ for some complex number $b \neq 0$, which is explicitly determined by $a, m$ and $n$.

Proof. Let the symplectic area of $C$ be $\omega(C) = \pi \delta^2_0$ for some $\delta_0 > 0$. Then by the Weinstein neighborhood theorem, a neighborhood of $C$ in $(M, \omega)$ has a standard model which we describe below.

Let $(z_1, z_2)$ be the coordinates of $\mathbb{C}^2$ where the standard symplectic structure $\omega_0$ on $\mathbb{C}^2$ is given by $\omega_0 = \frac{1}{2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$. Let $B^4(\delta) = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < \delta^2\}$ denote the open ball of radius $\delta > 0$ in $\mathbb{C}^2$, and for any $\delta_1 > \delta_0$, let $W(\delta_1)$ be the symplectic 4-manifold which is obtained by collapsing the fibers of the Hopf fibration on the boundary of $B^4(\delta_1) \setminus B^4(\delta_0)$. Then a neighborhood of $C$ in $(M, \omega)$ is symplectomorphic to $W(\delta_1)$ for some $\delta_1$ where $\delta_1 - \delta_0$ is sufficiently small. With this understood, the symplectic blow-down $(M', \omega')$ is obtained by cutting $(M, \omega)$ open along $C$ and gluing in the standard symplectic 4-ball $B^4(\delta_0)$ after fixing an identification of a neighborhood of $C$ with $W(\delta_1)$. In the present situation, in order to extend the symplectic surfaces $S_i \setminus C$ or $S \setminus C$ across the 4-ball $B^4(\delta_0)$, we need to choose the identification of a neighborhood of $C$ with $W(\delta_1)$ more carefully.

To this end, we consider the following reparametrization of a neighborhood of the circle $\{z_1 = 0\} \cap S^3(\delta_0)$ in $\mathbb{C}^2$, where $S^3(\delta_0)$ is the sphere of radius $\delta_0$, by the map

$$(z_1, z_2) = \left( \frac{r \delta}{\sqrt{1 + r^2}} e^{i(\theta + \phi)}, \frac{\delta}{\sqrt{1 + r^2}} e^{i\phi} \right),$$

for $0 \leq r < r_0$, $\theta, \phi \in \mathbb{R}/2\pi \mathbb{Z}$, and $\delta$ lying in a small interval containing $\delta_0$. We note that $(r, \theta, \phi)$ gives a trivialization of the Hopf fibration near $z_1 = 0$ in $S^3(\delta_0)$, with $(r, \theta)$ for the base and $\phi$ for the fiber. In the new coordinates $(r, \theta, \delta, \phi)$, the standard symplectic structure on $\mathbb{C}^2$ takes the form

$$\omega_0 = \frac{r^2 \delta}{1 + r^2} d\delta \wedge d\theta + \frac{\delta^2 r}{(1 + r^2)^2} dr \wedge d\theta + \delta d\delta \wedge d\phi.$$  

Replacing $\delta^2$ by $\delta^2 + \delta_0^2$ and assuming $0 \leq \delta < \sqrt{\delta_1^2 - \delta_0^2}$, we obtain a description of the symplectic structure on $W(\delta_1)$ in a neighborhood of the image of $\{z_1 = 0\} \cap S^3(\delta_0)$ in $W(\delta_1)$ (where the image of $\{z_1 = 0\} \cap S^3(\delta_0)$ has coordinates $\lambda = \delta = 0$):

$$\omega_0 = \lambda d\lambda \wedge d\theta + \delta d\delta \wedge d\phi,$$

where $\lambda = \frac{r \sqrt{\delta^2 + \delta_0^2}}{\sqrt{1 + r^2}}$. 


With this understood, the map \((w_1, w_2) = (\lambda e^{i\theta}, \delta e^{i\phi})\) is a symplectomorphism which identifies a neighborhood of the image of \(\{z_1 = 0\} \cap S^3(\delta_0)\) in \(W(\delta_1)\) with a neighborhood of \(p \in C\) in \((M, \omega)\). Then by the relative version of the Weinstein neighborhood theorem, we may extend this symplectomorphism to a symplectomorphism which identifies \(W(\delta_1)\) with a neighborhood of \(C\) in \((M, \omega)\).

With the preceding understood, we now consider case (1) of the lemma. First, note that the symplectic surface \(S_0\) is given by \(w_1 = 0\) near the point \(p\). Hence under the symplectomorphism \((w_1, w_2) = (\lambda e^{i\theta}, \delta e^{i\phi})\) where \(\lambda = \frac{r\sqrt{\delta_0^2 + \delta^2}}{\sqrt{1 + r^2}}\), the part of \(S_0\) near \(p\) is parametrized in the \((r, \theta, \delta, \phi)\) coordinate system by the following equations (recall we have replaced \(W(\delta_1)\) with a neighborhood of \(C\) across the standard symplectic 4-ball in \((M', \omega')\) by gluing in a complex disc contained in the complex line \(z_1 = 0\). This is the descendant \(S'_0\) of \(S_0\) in \((M', \omega')\).

For each \(i > 0\), \(S_i\) is given by the complex line \(w_2 = a_i w_1\) near the point \(p\). Writing \(a_i = \rho_i e^{i\kappa_i}\), we parametrize \(S_i\) near \(p\) by the equations \(w_1 = te^{is}\) and \(w_2 = t\rho_i e^{i(s + \kappa_i)}\). Under the symplectomorphism \((w_1, w_2) = (\lambda e^{i\theta}, \delta e^{i\phi})\) where \(\lambda = \frac{r\sqrt{\delta_0^2 + \delta^2}}{\sqrt{1 + r^2}}\), it is parametrized in the \((r, \theta, \delta, \phi)\) coordinate system by the following equations:

\[
 r = \frac{t}{\sqrt{\delta_0^2 + (\rho_i^2 - 1)t^2}}, \quad \theta = s, \quad \delta = t\rho_i, \quad \phi = s + \kappa_i.
\]

Now reviewing the part of \(S_i\) near \(p\) as a subset in \(\mathbb{C}^2\), it is parametrized in the coordinates \((z_1, z_2)\) by the following equations (recall we have replaced \(\delta^2\) by \(\delta_0^2 + \delta^2\)):

\[
 z_1 = \frac{r\sqrt{\delta_0^2 + \delta^2}}{\sqrt{1 + r^2}} e^{i(\theta + \phi)} = te^{i(2s + \kappa_i)}, \quad z_2 = \frac{\sqrt{\delta_0^2 + \delta^2}}{\sqrt{1 + r^2}} e^{i\phi} = \sqrt{\delta_0^2 + (\rho_i^2 - 1)t^2} \cdot e^{i(s + \kappa_i)}.
\]

With this understood, we observe that \(z_1, z_2\) satisfy the equation \(z_1 = b_i z_2^2\), where

\[
 b_i = \frac{te^{-i\kappa_i}}{\delta_0^2 + (\rho_i^2 - 1)t^2},
\]

for any \(t > 0\) which is sufficiently small. It is clear that \(b_i \neq 0\) for each \(i > 0\), and that \\{\(a_i\) being distinct implies that \\{\(b_i\) are also distinct (for each fixed \(t\)). Now we fix a value \(t_0 > 0\) which is sufficiently small, and remove the part \(\{t \leq t_0\}\) from \(S_i\) and glue onto it the holomorphic disc defined by the equation \(z_1 = b_i z_2^2\), where

\[
 b_i = \frac{t_0 e^{-i\kappa_i}}{\delta_0^2 + (\rho_i^2 - 1)t_0^2}.
\]

For \(t_0\) small, one can smooth off the corners near the gluing region to obtain a symplectic surface in \((M', \omega')\), which is defined to be the descendant \(S'_i\) of \(S_i\) in the symplectic blow-down. This finishes the proof for case (1).

The argument for case (2) is similar. The surface \(S\) near \(p\) is given by the equation \(w_2^n = aw_1^n\). Writing \(a = \rho e^{i\kappa}\), we parametrize \(S\) near \(p\) by the equations

\[
 w_1 = t^n e^{ins} \quad \text{and} \quad w_2 = t^m \rho^n e^{i(ms + \kappa)}.
\]
Under the symplectomorphism \((w_1, w_2) = (\lambda e^{i\theta}, \delta e^{i\phi})\) where \(\lambda = \frac{r\sqrt{\delta^2 + b^2}}{1+r^2}\), it is parametrized in the \((r, \theta, \delta, \phi)\) coordinate system by the following equations:

\[
r = \frac{t^n}{\sqrt{\delta_0^2 + \rho^2/n^2m - 2n}}, \quad \theta = ns, \quad \delta = \rho^{1/n}t^m, \quad \phi = ms + \kappa/n.
\]

In the coordinates \((z_1, z_2)\) on \(\mathbb{C}^2\), the part of \(S\) near \(p\) is parametrized by the following equations:

\[
z_1 = t^n e^{i\kappa/n} \cdot e^{i(m+n)s}, \quad z_2 = \sqrt{\delta_0^2 + \rho^2/n^2m - 2n} \cdot e^{i\kappa/n} e^{ims}.
\]

It follows easily that \(z_1, z_2\) satisfy the equation \(z_1^m = b z_2^{m+n}\), where

\[
b = \left(\frac{\delta_0^2 + \rho^2/n^2m - 2n}{\sqrt{\delta_0^2 + \rho^2/n^2m - 2n}}\right)^{\frac{m+n}{2}}
\]

for any \(t > 0\) which is sufficiently small. Clearly, \(b \neq 0\). As in case (1), we fix a value \(t_0 > 0\) sufficiently small, remove the part \(\{t \leq t_0\}\) from the surface \(S\) and glue onto it the holomorphic disc (singular in this case) defined by the equation \(z_1^m = b z_2^{m+n}\), where in \(b\) the variable \(t\) is evaluated at \(t_0\). The resulting surface (after smoothing off the corners) is the descendant \(S'\) of \(S\) in the symplectic blow-down \((M', \omega')\). This finishes the proof for case (2), and the proof of the lemma is complete.

With Lemma 3.2 at hand, we shall define the descendant \(D_{n-1}\) of \(D_n\) in the next stage \((X_{n-1}, \omega_{n-1})\) as follows. First, since \(n < l_s\) for each \(s\), the classes \(E_{l_s}\) all have been blown down in the earlier stages. We assume that for each \(s\), the class \(E_{l_s}\) does not show up in any of the components of \(D_N\) with a \((+1)\)-coefficient (i.e., this is the first time we cannot perturb the \((-1)\)-sphere to a general position). With this understood, for each \(s\), there is a point \(\hat{E}_{l_s}\) and a small, standard symplectic 4-ball \(B(\hat{E}_{l_s}) \subset X_n\) centered at \(\hat{E}_{l_s}\), such that \(\hat{E}_{l_s} \subset C_n\) for each \(s\), and the intersection \(B(\hat{E}_{l_s}) \cap C_n\) is a disc lying in a complex line (called a complex linear disc).

**Case (1):** Suppose the class \(E_n\) does not appear in any of the components of \(D_N\) which has zero \(a\)-coefficient. In this case, we can simply blow down \((X_n, \omega_n)\) along \(C_n\) to the next stage \((X_{n-1}, \omega_{n-1})\), which means that we will cut \(X_n\) open along \(C_n\) and then insert a standard symplectic 4-ball of appropriate size. For any component \(F_k\) in \(D_n\) which intersects with \(C_n\), there are two possibilities. If an intersection point of \(F_k\) with \(C_n\) is inherited from the original intersection in \(D_N\), then by the condition (†), there is no other component \(F_l\) passing through this intersection point. For such an intersection point on \(C_n\), we shall simply glue a disc to \(F_k \setminus C_n\) which is lying on a complex line in the standard symplectic 4-ball. Any other intersection point of \(F_k\) with \(C_n\) should occur at one of the points \(\hat{E}_{l_s}\). For any such intersection points, we shall define the descendant of \(F_k\) in \(X_{n-1}\) by extending the surface \(F_k \setminus C_n\) across the standard symplectic 4-ball according to Lemma 3.2(1). With this understood, we denote the center of the standard symplectic 4-ball by \(E_n\). Then it is easy to see that there is a small 4-ball \(B(\hat{E}_n)\) centered at \(\hat{E}_n\), such that each original intersection point
on $C_n$ from $D_N$ determines a linear complex disc in $B(\hat{E}_n)$ as part of the descendant $D_{n-1}$, and each point $\tilde{E}_{l_s} \in C_n$ determines a complex line in $B(\hat{E}_n)$ with the property that each linear complex disc in $B(\tilde{E}_{l_s}) \cap D_n$ which is not part of $C_n$ determines a holomorphic disc in $B(\hat{E}_n)$ as part of the descendant $D_{n-1}$, which has tangency of order 2 with the complex line determined by the point $\tilde{E}_{l_s}$. Finally, we remark that after shrinking the size, the 4-ball $B(\hat{E}_n)$, particularly the point $\hat{E}_n$, will survive to the last stage of the successive blowing-down.

Case (2): If the class $E_n$ appears in the expression of a symplectic sphere in $D_N$ whose $a$-coefficient is zero, then more care is needed in defining the descendant $D_{n-1}$. And here is the reason: suppose $E_n$ is contained in $S_1$ whose $a$-coefficient is zero, and let $E_m$ be the $E_i$-class in $S_i$ with $(+1)$-coefficient. Then $m < n$, and in a later stage of $(X_m, \omega_m)$ when we blow down the class $E_m$, we will be again in a situation where we cannot perturb the $(-1)$-sphere $C_m$ to a general position (because $C_m$ is the descendant of $S_1$ in $D_m$, so is part of $D_m$). In particular, we will have to apply Lemma 3.2 when blowing down the class $E_m$. With this understood, observe that in Lemma 3.2, near the point $p \in C$, the symplectic surfaces under consideration have to be in certain standard forms with respect to a complex coordinate system $(w_1, w_2)$ with standard symplectic structure, and in particular, the $(-1)$-sphere $C$ has to be given by a complex coordinate line $w_2 = 0$. This requires that, when we blow down the $(-1)$-sphere $C_n$, we need to arrange so that in the small 4-ball $B(\hat{E}_n) \subset X_{n-1}$, the holomorphic discs $B(\hat{E}_n) \cap D_{n-1}$ can be placed in the model required in Lemma 3.2.

With this understood, we first make the following observation.

Lemma 3.3. There are at most two components $F_k$ in $D_N$ such that (1) the $a$-coefficient of $F_k$ is zero, (2) the homological expression of $F_k$ contains the class $E_n$. Moreover, such a component $F_k$ can contain at most one of the classes $E_{l_s}$ in its homological expression, and the classes $E_{l_s}$ contained in two distinct such components $F_k$ must be distinct. (Recall $S = E_n = E_{l_1} - \cdots - E_{l_s}$ is the symplectic sphere in $D_N$ that is under consideration.)

Proof. Suppose $S_1$ is such a component in $D_N$, i.e., the $a$-coefficient of $S_1$ is zero and the homological expression of $S_1$ contains the class $E_n$. Let $E_{j_1}$ be the $E_i$-class in the expression of $S_1$ which has $(+1)$-coefficient. Then the fact that $E_n$ is contained in $S_1$ implies that $j_1 < n$ must be true. On the other hand, $S \cdot S_1 \geq 0$ implies that $S \cdot S_1$, in fact, equals either 0 or 1. In the former case, $S_1$ contains exactly one of the classes $E_{l_s}$, and in the latter case, $S_1$ contains none of the classes $E_{l_s}$.

Suppose $S_2$ is another such component in $D_N$, with $E_{j_2}$ being the $E_i$-class that has $(+1)$-coefficient in the homological expression of $S_2$. Without loss of generality, we assume $j_2 < j_1$. Then since $S_1, S_2$ both contain the class $E_n$, it follows easily from $S_1 \cdot S_2 \geq 0$ that $E_{j_1}$ must appear in the expression of $S_2$, the intersection $S_1 \cdot S_2 = 0$, and the classes $E_{l_s}$ which are contained in $S_1, S_2$ must be distinct. With this understood, suppose to the contrary that there are more than two such components, and let $S_3$ be a third such component. Then the same argument as in the case of $S_2$ implies that the expression of $S_3$ must contain both $E_{j_1}$ and $E_n$. But then this would imply $S_2 \cdot S_3 > 0$, which is a contradiction. The lemma follows easily from these considerations. $\square$
We shall consider separately according to the number of the symplectic spheres described in Lemma 3.3.

**Case (a):** Suppose there are two symplectic spheres $S_1, S_2 \subset D_N$ with zero $a$-coefficient whose homological expressions contain the class $E_n$ in $S$. We shall need to make some very specific identification of a neighborhood of $C_n$ in $(X_n, \omega_n)$ with the standard model, which is described below. Assume $\omega_n(C_n) = \pi_0$.

Fix a coordinate system $(z_1, z_2)$ of $\mathbb{C}^2$ such that the standard symplectic structure $\omega_0$ on $\mathbb{C}^2$ is given by $\omega_0 = \frac{1}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$. Let $B^4(\delta) = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 < \delta^2\}$ denote the open ball of radius $\delta > 0$ in $\mathbb{C}^2$, and for any $\delta_1 > \delta_0$, let $W(\delta_1)$ be the symplectic 4-manifold which is obtained by collapsing the fibers of the Hopf fibration on the boundary of $B^4(\delta_1) \setminus B^4(\delta_0)$. Then by the Weinstein neighborhood theorem, a neighborhood of $C_n$ in $(X_n, \omega_n)$ is symplectomorphic to $W(\delta_1)$ for some $\delta_1$ where $\delta_1 - \delta_0$ is sufficiently small. With this understood, the symplectic blow-down $(X_{n-1}, \omega_{n-1})$ is obtained by cutting $(X_n, \omega_n)$ open along $C_n$ and gluing in the standard symplectic 4-ball $B^4(\delta_0)$ after fixing an identification of a neighborhood of $C_n$ with $W(\delta_1)$.

With the preceding understood, let $p_1, p_2$ be the intersection points of the descendants of $S_1, S_2$ in $D_n$ with $C_n$. Then by a relative version of the Weinstein neighborhood theorem, we can choose an identification of a neighborhood of $C_n$ with $W(\delta_1)$ such that $p_1$ and $p_2$ are identified with the images of the Hopf fibers at $z_1 = 0$ and $z_2 = 0$ respectively. With this understood, when we apply Lemma 3.2 to the points $p_1, p_2$, we can furthermore arrange the descendants of $S_1, S_2$ in $D_n$ to be the symplectic surface $S_0$ in Lemma 3.2, so that after applying Lemma 3.2, the descendants of $S_1, S_2$ in $D_{n-1} \setminus B^4(\delta_0)$ are given by the complex lines $z_1 = 0$ and $z_2 = 0$ respectively. Moreover, any other component of $D_n$ which intersects $C_n$ at either $p_1$ or $p_2$ will have its descendant in $D_{n-1}$ given by a holomorphic disc in $B^4(\delta_0)$ of the form $z_1 = bz_2^n$ or $z_2 = bz_1^n$ respectively (more generally, of the form $z_1^n = bz_2^m + n$ if before blowing down it is given by $w_2^n = aw_1^m$, etc.). It remains to deal with the intersection points $\tilde{E}_{l_i} \subset C_n$ which are not $p_1, p_2$. By the assumption (a) in Theorem 1.3, for any such an $\tilde{E}_{l_i}$, there is only one component in $D_n$ which intersects $C_n$ at $\tilde{E}_{l_i}$, with intersection number $+1$. (Equivalently, there is only one holomorphic disc in the small 4-ball $B(\tilde{E}_{l_i})$ which does not lie in $C_n$.) By a small perturbation, we can arrange this component to coincide with the fiber at $\tilde{E}_{l_i} \subset C_n$ in $W(\delta_1)$, so that it can be extended across the 4-ball $B^4(\delta_0)$ by a linear complex disc (given by equation $z_2 = n z_1$) when we blow down $C_n$. In summary, the holomorphic discs $B^4(\delta_0) \cap D_{n-1}$ can be placed in a model that is required in Lemma 3.2 before the blowing down, so that in a later stage, when we blow down the $(-1)$-sphere which is the descendant of $S_1$ or $S_2$, Lemma 3.2 can be applied in the process.

**Case (b):** Suppose there is only one symplectic sphere $S_1 \subset D_N$ with zero $a$-coefficient whose homological expression contains the class $E_n$ in $S$. Let $p_1$ be the intersection point of $C_n$ with the descendant of $S_1$ in $D_n$. Then by the assumption (b) in Theorem 1.3, there is at most one intersection point $\tilde{E}_{l_i} \neq p_1$ such that the small 4-ball $B(\tilde{E}_{l_i})$ contains more than one holomorphic discs which do not lie in $C_n$. With this understood, we shall choose an identification of a neighborhood of $C_n$ in $(X_n, \omega_n)$.
with \( W(\delta_1) \) such that \( p_1 \) and the intersection point \( \hat{E}_1 \) are identified with the images of the Hopf fibers at \( z_1 = 0 \) and \( z_2 = 0 \) respectively. Then by the same argument as in Case (a), we can arrange such that the holomorphic discs \( B^4(\delta_0) \cap D_{n-1} \) can be placed in an appropriate model, so that when we blow down the \((-1)\)-sphere which is the descendant of \( S_1 \) in a later stage, Lemma 3.2 can be applied in the process.

With the preceding understood, it follows easily that under assumptions (a) and (b), one can continue the process and successively blow down the classes \( E_1, E_{N-1}, \cdots, E_2 \) to reach to the stage \((X_1, \omega_1)\) (where \( X_1 = \mathbb{CP}^2 \# \mathbb{CP}^2 \)), obtaining a canonically constructed descendant \( D_1 \) of \( D_N \) in \((X_1, \omega_1)\). We remark that there is an \( \omega_1 \)-compatible almost complex structure \( J_1 \), such that \( D_1 \) is \( J_1 \)-holomorphic.

It remains to show that if any of the conditions (c), (d), (e) is satisfied, then one can further blow down the class \( E_1 \) to reach \( \mathbb{CP}^2 \) in the final stage. First, assume (c) is true. In this case, since \( \omega_1(E_1) = \omega_1(E_2) \), the class \( E_1 \) also has the minimal area in \((X_2, \omega_2)\), so that we can represent both \( E_1, E_2 \) by a \( J_2 \)-holomorphic sphere. It follows that we can blow down both \((-1)\)-classes at the same time.

Next, suppose condition (d) is satisfied. In this case, there is a symplectic sphere \( S \) in \( D_N \) such that \( E_1 \) appears in the expression of \( S \) with \((+1)\)-coefficient. We observe that the descendant of \( S \) in \( D_1 \) is a symplectic \((-1)\)-sphere representing the class \( E_1 \). We simply blow down \((X_1, \omega_1)\) along this \((-1)\)-sphere to reach the final stage \( \mathbb{CP}^2 \).

Finally, suppose condition (e) is satisfied. In this case, we appeal to Lemma 2.3 of [6], which says that either \( E_1 \) is represented by a \( J_1 \)-holomorphic sphere, or there is a \( J_1 \)-holomorphic sphere \( C \) such that \( E_1 = m(H - E_1) + C \) for some \( m \geq 1 \). In the former case, we can blow down the class \( E_1 \). In the latter case, we reach a contradiction as follows. By condition (e), there is a component \( F_k \) of \( D_N \) whose \( a \)-coefficient, \( a \), and the \( b \)-coefficient for \( E_1 \), \( b \), obeys \( 2b < a \). Let \( \hat{F}_k \) denote the descendant of \( F_k \) in \( D_1 \), which is \( J_1 \)-holomorphic and has class \( aH - bE_1 \). Then we have

\[
0 \leq C \cdot \hat{F}_k = (m + 1)b - ma,
\]

contradicting the assumption \( 2b < a \) and the fact \( m \geq 1 \). The proof of Theorem 1.3 is complete.

4. Constraints from Seiberg-Witten-Taubes theory

In this section, we derive some constraints on the singular set of the orbifold \( X \) using the Seiberg-Witten-Taubes theory. It is important to note that \( X \) is assumed to have \( b_1 = 0 \). (By Theorem 1.2, this is the only interesting case.)

The constraints are given in terms of certain numerical contributions of the singular set to the dimension of the moduli space of Seiberg-Witten equations. To describe them, we consider any orbifold complex line bundle \( L \) over \( X \) such that \( c_1(L) = 0 \in H^2(|X|, \mathbb{Q}) \). For each singular point \( q_j \) of \( X \), we denote by \( \rho_j^l : G_j \rightarrow \mathbb{C}^* \) the complex representation of the isotropy group \( G_j \) on the fiber of \( L \) at \( q_j \), and denote by \( \rho_{j,k}(g) \), for \( k = 1, 2 \), the eigenvalues of \( g \in G_j \) associated to the complex representation of \( G_j \) on the tangent space \( T_{q_j}X \). For each 2-dimensional singular component \( \Sigma_i \), we denote by \( G_i := \mathbb{Z}_{m_i} \) the isotropy group along \( \Sigma_i \), and let \( \rho_i^L : G_i \rightarrow \mathbb{C}^* \) be the complex representation of \( G_i \) on the fibers of \( L \) along \( \Sigma_i \), and let \( \rho_i : G_i \rightarrow \mathbb{C}^* \) be the complex
representation of $G_i$ on the normal bundle $ν_{Σ_i}$ of $Σ_i$. With this understood, we set

$$I_i(L) := \frac{1}{m_i} \sum_{g ∈ G_i \setminus \{ε\}} \frac{(1 + ρ_i(g^{-1}))(ρ_i(g) - 1)}{(1 - ρ_i(g^{-1}))^2},$$

and

$$I_j(L) := \frac{2}{|G_j|} \sum_{g ∈ G_j \setminus \{ε\}} \frac{2(ρ_j(g) - 1)}{(1 - ρ_j,1(g^{-1}))(1 - ρ_j,2(g^{-1}))}.$$ 

It is easy to check that $I_i(L) = I_i(K_X ⊗ L^{-1})$, $I_j(L) = I_j(K_X ⊗ L^{-1})$ for any $i, j$. Finally, we set

$$d(L) := \sum_i I_i(L) \chi(Σ_i) + \sum_j I_j(L).$$

One can easily check that, with $c_1(L) = 0$, and with

$$c_1(ν_{Σ_i})(Σ_i) = Σ_i^2 = 2g_i - 2 = -c_1(TΣ_i)[Σ_i] = -χ(Σ_i)$$

by the adjunction formula (here $g_i$ is the genus of $Σ_i$), $d(L)$ equals the dimension of the moduli space of Seiberg-Witten equations associated to the orbifold complex line bundle $L$ (cf. [5], Appendix A). With this understood, we note that $d(L) = d(K_X ⊗ L^{-1})$, and $d(L) = 0$ if $L$ is the trivial complex line bundle or $L = K_X$.

**Theorem 4.1.** Suppose $b_1(X) = 0$. Then for any orbifold complex line bundle $L$ such that $c_1(L) = 0 ∈ H^2(Σ_i, \mathbb{Q})$, one has

$$d(L) ≤ 0,$$

with “=” if and only if $L$ is the trivial complex line bundle or $L = K_X$.

**Proof.** Since $b_1^+(X) = 1$ and $b_1(X) = 0$, the wall-crossing number for the Seiberg-Witten invariant of $X$ equals ±1. With this understood, we denote by $SW_X(L)$ the Seiberg-Witten invariant of $X$ associated to an orbifold complex line bundle $L$ defined in the Taubes chamber. Then if the dimension $d(L) ≥ 0$, one has

$$|SW_X(L) − SW_X(K_X ⊗ L^{-1})| = 1.$$ 

Now we observe that if $SW_X(L) ≠ 0$ and $c_1(L) = 0$, $L$ must be the trivial complex line bundle. Since $K_X$ is torsion of order $n > 1$ (recall that $n = \text{lcm}\{m_i, m_j\}$ is the minimal number such that $K^n_X$ is trivial, cf. Lemma 2.1), it is clear that one of the orbifold complex line bundles, $L$ or $K_X ⊗ L^{-1}$, must be a non-trivial torsion bundle, hence has vanishing Seiberg-Witten invariant. It follows easily that one of $SW_X(L)$, $SW_X(K_X ⊗ L^{-1})$ must equal ±1. This implies that either $L$ or $K_X ⊗ L^{-1}$ must be the trivial orbifold complex line bundle. Theorem 4.1 follows easily.

Suppose $n = \text{lcm}\{m_i, m_j\} > 2$. Then $L = K^k_X$ is nontrivial and not equal to $K_X$ for $k = 2, 3, \cdots, n − 1$. By Theorem 4.1, $d(K^k_X) < 0$ for any $2 ≤ k ≤ n − 1$. On the other hand, note that a singular component $Σ_i$ makes zero contribution to $d(L)$ for any $L$ if $Σ_i$ is a torus, and one can check directly that $I_j(K^k_X) = 0$ for any $k$ if $q_j$ is a Du Val singularity (i.e., $m_j = 1$). It is easy to see that we have the following

□
Corollary 4.2. Suppose \( n = \text{lcm}\{m_i, m_j\} > 2 \). Then the following are true.

1. For each \( 2 \leq k \leq n - 1 \), \( d(K_k^X) \) is a negative, even integer.
2. Either there is a singular component \( \Sigma_i \) which is not a torus, or there is a singular point \( q_j \) which is non-Du Val.

Remarks: It is possible that for a singular point \( q_j \), the number \( I_j(K_k^X) > 0 \) for some \( 2 \leq k \leq n - 1 \); this depends on the isotropy type of \( q_j \). For example, if \( q_j \) is of isotropy of order 5 of type \((1, 1)\), then

\[
I_j(K_2^X) = \frac{1}{5} \sum_{1 \neq \lambda \in \mathbb{C}^*, \lambda^5 = 1} \frac{2(\lambda^4 - 1)}{(1 - \lambda)(1 - \lambda)} = \frac{2}{5}.
\]

So the conditions \( d(K_k^X) < 0 \) give rise to nontrivial constraints on the singular set.

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