**Abstract**

We investigate some $L^s$-rate optimality properties of dilated/contracted $L^r$-optimal quantizers and $L^r$-greedy quantization sequences $(a^n)_{n \geq 1}$ of a random variable $X$. Based on the results established in [10] for $L^r$-optimal quantizers, we show, for a larger class of distributions, that the dilatation $(\alpha^n_{\theta, \mu})_{n \geq 1}$ of an $L^r$-optimal quantizer, defined by $\alpha^n_{\theta, \mu} = \{\mu + \theta(\alpha_i - \mu), \alpha_i \in a^n \}$ is $L^s$-rate optimal for $s < r + d$. We establish $L^s$-rate optimality results for $L^r$-optimal greedy quantization sequences for different values of $s$ and obtain both asymptotic and non-asymptotic results. We lead a specific study for $L^r$-optimal greedy quantization sequences of radial density distributions and show that they are $L^s$-rate optimal for $s \in (r, r + d)$. We show, for various probability distributions, that there exists a parameter $\theta^*$ minimizing the $L^s$-quantization error induced by the dilated quantization sequence and present an application of this approach to numerical integration.

**Keywords**: Optimal quantization; greedy quantization sequence; rate optimality; radial density; Zador theorem; Pierce Lemma; numerical integration.

1 **Introduction**

The aim of this paper is, on the one hand, to extend some “robustness” results of optimal quantizers to a much wider class of distributions and, on the other hand to establish similar results for greedy quantization sequences introduced in [5] and developed in [1]. Let $L^r_{\mathbb{P}}(\mathbb{R}^d)$ (or simply $L^r(\mathbb{P})$), $r \in (0, +\infty)$, denote the set of $d$-dimensional random vectors $X$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $P = \mathbb{P}_X$ and such that $\mathbb{E}\|X\|^r < +\infty$ (for any norm $\| \cdot \|$ on $\mathbb{R}^d$). Optimal vector quantization consists in finding the best approximation of a multidimensional random vector $X$ by a random variable $Y$ taking at most a finite number $n$ of values. Consider $\Gamma = \{x_1, \ldots, x_n\}$ a $d$-dimensional grid of size $n$. The principle is to approximate $X$ by $\pi_\Gamma(X)$ where $\pi_\Gamma : \mathbb{R}^d \rightarrow \Gamma$ is a nearest neighbor projection defined by

$$\pi_\Gamma(\xi) = \sum_{i=1}^{n} x_i \mathbb{1}_{W_i(\Gamma)}(\xi)$$

where $(W_i(\Gamma))_{1 \leq i \leq n}$ is a so-called Voronoï partition of $\mathbb{R}^d$ induced by $\Gamma$ i.e. a Borel partition satisfying

$$W_i(\Gamma) \subset \{\xi \in \mathbb{R}^d : \|\xi - x_i\| \leq \min_{j \neq i} \|\xi - x_j\|\}, \quad i = 1, \ldots, n. \quad (1)$$
Then,

$$\hat{X}_\Gamma = \pi_\Gamma(X) := \sum_{i=1}^n x_i \mathbb{1}_{W_i(\Gamma)}(X) \tag{2}$$

is called the Voronoi quantization of $X$. The $L^r$-quantization error induced when replacing $X$ by its quantization $\hat{X}_\Gamma$ is naturally defined by

$$e_r(\Gamma, X) = \|X - \pi_\Gamma(X)\|_r = \|X - \hat{X}_\Gamma\|_r = \left\| \min_{1 \leq i \leq n} |X - x_i| \right\|_r \tag{3}$$

where $\| \cdot \|_r$ denotes the $L^r(\mathbb{P})$-norm (or quasi-norm if $0 < r < 1$). Consequently, the optimal quantization problem at level $n$ boils down to finding the grid $\Gamma^n$ of size $n$ that minimizes this error, i.e.

$$e_{r,n}(X) = \inf_{\Gamma, \text{card}(\Gamma) \leq n} e_r(\Gamma, X). \tag{4}$$

where card($\Gamma$) denotes the cardinality of $\Gamma$. The existence of a solution to this problem and the convergence of $e_{r,n}(X)$ to 0 at an $O(n^{-\frac{3}{2}})$-rate of convergence when the level (or size) $n$ goes to $+\infty$ have been shown (see [2, 6, 7] for example). The convergence to 0 of such a sequence $(\Gamma^n)_{n \geq 1}$ of $L^r$-optimal quantizers of (the distribution of) $X$ is an easy consequence of the separability of $\mathbb{R}^d$. Its rate of convergence to 0 is a much more challenging problem that has been solved in several steps over between 1950’s and the early 2000’s and the main results in their final form are summed up in Section 2.

However, numerical implementation of multidimensional $L^r$-optimal quantizers requires to optimize grids of size $n \times d$ which becomes computationally too costly when $n$ or $d$ increase. So, a greedy version of optimal vector quantization (which is easier to handle) has been introduced in [5] as a sub-optimal solution to the quantization problem. It consists in building a sequence of points $(a_n)_{n \geq 1}$ in $\mathbb{R}^d$ which is recursively $L^r$-optimized level by level, in the sense that it minimizes the $L^r$-quantization error at each iteration in a greedy way. This means that, having the first $n$ points $a^{(n)} = \{a_1, \ldots, a_n\}$ for $n \geq 1$, we add, at the $(n+1)$-th step, the point $a_{n+1}$ solution to

$$a_{n+1} \in \arg\min_{\xi \in \mathbb{R}^d} e_r(a^{(n)} \cup \{\xi\}, X), \tag{5}$$

noting that $a^{(0)} = \emptyset$, so that $a_1$ is simply the $L^r$-median of the distribution $P$ of $X$. The sequence $(a_n)_{n \geq 1}$ is called an $L^r$-optimal greedy quantization sequence for $X$ or its distribution $P$. It is proved in [5] that the problem (5) admits, as soon as $X$ lies in $L_{\mathbb{R}^d}(\mathbb{P})$, a solution $(a_n)_{n \geq 1}$ which may be not unique due to the dependence of greedy quantization on the symmetry of the distribution $P$. The corresponding $L^r$-quantization error $e_r(a^{(n)}, X)$ is decreasing w.r.t $n$ and converges to 0 when $n$ goes to $+\infty$. Greedy quantization sequences have an optimal convergence rate to 0 compared to optimal quantizers, in the sense that the grids $\{a_1, \ldots, a_n\}$ are $L^r$-rate optimal. This was established first in [5] for a rather wide family of absolutely continuous distribution using some maximal functions approximating the density $f$ of $P$. Then, it has been extended in [1] to a much larger class of probability density functions where the authors relied on an exogenous auxiliary probability distribution $\nu$ on ($\mathbb{R}^d, \mathfrak{B}or(\mathbb{R}^d)$) satisfying a certain control on balls, the result is recalled in Section 2.

A very important field of applications is quantization-based numerical integration where we approximate an expectation $\mathbb{E}h(X)$ of a function $h$ on $\mathbb{R}^d$ by some cubature formulas. The error bounds induced by such numerical schemes always involve the $L^s$-quantization error induced by the approximation of $X$ by its (optimal or greedy) quantizer usually with $s \geq r$. This problem also appears when we use optimal quantization as a space discretization scheme of ARCH models, namely the Euler scheme of a diffusion devised to solve stochastic control, optimal stopping or filtering problems (see
Another approach to this problem was considered in [10] where the author was interested in the fact that an appropriate dilatation or contraction of a (sequence of) $L^s$-optimal quantizer(s) $(\Gamma_n)_{n \geq 1}$ remains $L^s$-rate optimal. This study was also motivated by its application to the algorithms of designing $L^s$-optimal quantizers for $s \neq 2$. In fact, several stochastic procedures, like Lloyd’s algorithm or the Competitive Learning Vector Quantization algorithm (CLVQ), are based on the stationarity property satisfied by optimal quadratic quantizers and designed for $s = 2$. However, when $s > 2$, these procedures become unstable and difficult and their convergence is very dependent on the initialization. So, in order to design $L^s$-optimal quantizers, $s > 2$, one can use the $L^2$-dilated quantizers to initialize the algorithms and speed their convergence.

In this paper, we extend the original results established on the $L^s$-rate optimality of dilatations/contractions of $L^r$-optimal quantizers in [10] to a larger class of distributions taking advantage of new tools developed in [1] to analyze quantization errors. These tools are based on auxiliary probability distributions with a certain property of control on balls. Then, with the same motivations as in [10], we establish similar rate optimality results to greedy quantization sequences. In other words, if $(\alpha^n)_{n \geq 1}$ is a sequence of $L^r$-optimal quantizers or an $L^r$-optimal greedy quantization sequence, then the sequence $(\alpha^n_{\theta,\mu})_{n \geq 1}$ defined, for every $\theta > 0$ and $\mu \in \mathbb{R}^d$, by $\alpha^n_{\theta,\mu} = \{\mu + \theta (a_i - \mu), a_i \in \alpha^n\}$, is $L^s$-rate optimal for $s \neq r$. A lower bound was given in [10] for $L^r$-optimal quantizers and it also holds for greedy quantization sequences: If $P = f.\lambda_d$, then for every $\theta > 0$, $\mu \in \mathbb{R}^d$ and $n \geq 1$,

$$\liminf_{n \to +\infty} n^{\frac{s}{r}} \varepsilon_s(\alpha^n_{\theta,\mu}, P)^s \geq Q_{r,s}(P, \theta) := \theta^{s+d} \int_{\mathbb{R}^d} f_{\theta,\mu}(x) \frac{d\theta + \mu}{d\mu} d\lambda_d$$

where $f_{\theta,\mu}$ denotes the function $f_{\theta,\mu}(x) = f(\mu + \theta(x - \mu))$. Likewise, if $X \sim P = f.\lambda_d$, then $P_{\theta,\mu}$ will denote the probability distribution of the random variable $X_{\theta,\mu} + \mu$ and $dP_{\theta,\mu} = \theta^d f_{\theta,\mu} d\lambda_d$. Our goal is then to estimate upper bounds. Two results are already given in [10]: one showing that an asymptotically $L^r$-optimal sequence of quantizers is $L^s$-rate optimal and another restricted to a sequence of (exactly) $L^r$-optimal quantizers and showing that it is $L^s$-rate optimal for $s \in (0, +\infty)$. In this paper, we will change the approach and use auxiliary probability distributions having a certain criterion to extend these results to a larger class of distributions for $L^r$-optimal quantizers. Then, relying on a greedy version of this criterion, we establish upper estimates for greedy quantization sequences and give two results depending on the values of $s$. For this latter case, we obtain Pierce type universal non-asymptotic results of $L^s$-rate optimality of $(\alpha^n_{\theta,\mu})_{n \geq 1}$ by considering particular auxiliary distributions. An interesting study is to consider a particular class of distributions, the radial density probability distributions, and show that the corresponding $L^r$-greedy quantization sequences are $L^s$-rate optimal for $s \in (r, d+r)$. At this stage, one wonders whether there exists an optimal dilatation of the sequence, i.e. whether one can find a parameter $\theta^*$ that gives the lowest possible value of the upper bounds. This prompts us to consider several particular probability distributions and find the optimal $\theta^*$ for each distribution. The dilated optimal quantizer $\Gamma^\theta_{\theta,\mu}$ turns out to satisfy the so-called $L^s$-empirical measure theorem. Finally, the application of this study to numerical integration, introduced in [10], is detailed and illustrated by numerical examples, for optimal and greedy quantization.
This paper will be organized as follows: We start, in Section 2, with some results and tools, mostly from [1], that will be useful in the whole paper. In Section 3, we give upper bounds for dilated (or contracted) sequences of $L^r$-optimal quantizers. Such bounds are given for greedy quantization sequences in Section 4. A specific study for greedy quantization sequences of radial density distributions is established in Section 5. In Section 6, we find an optimal parameter $\theta^*$ that minimizes the upper bounds for different particular probability distributions. Finally, Section 7 is devoted to an application to numerical integration.

2 Main tools

In this section, we will present some useful results and inequalities which constitute essential tools needed to achieve the desired results in the rest of the paper. Let $X$ be an $\mathbb{R}^d$-valued random variable with distribution $P$ such that $\mathbb{E}\|X\|^r < +\infty$ for $r > 0$. Let $(\Gamma^n)_{n \geq 0}$ be a sequence of $L^r$-optimal quantizers of $X$. Let $(a_n)_{n \geq 0}$ be the greedy quantization sequence. We start by giving the result concerning the rate of convergence to 0 of a sequence of $L^r$-optimal quantizers. The first part of the following theorem is an asymptotic result and the second part is universal non-asymptotic.

**Theorem 2.1.** (a) Zador’s Theorem (see [11]): Let $X \in L^{r+\eta}(\mathbb{R}^d)$, $\eta > 0$, with distribution $P$ such that $dP(\xi) = \varphi(\xi)d\lambda_d(\xi) + d\nu(\xi)$. Then,

$$
\lim_{n \to +\infty} n^{\frac{1}{d}} e_{r,n}(X) = Q_r(P) = \bar{J}_{r,d}\|\varphi\|_{L^{r+\eta}(\lambda_d)}^{\frac{1}{r}}
$$

where $\bar{J}_{r,d} = \inf_{n \geq 1} n^{\frac{1}{d}} e_{r,n}(U([0,1]^d)) \in (0, +\infty)$.

(b) Extended Pierce’s Lemma (see [4, 7]): Let $r, \eta > 0$. There exists a constant $\kappa_{d,r,\eta} \in (0, +\infty)$ such that, for any random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$,

$$
\forall n \geq 1, \quad e_{r,n}(X) \leq \kappa_{d,r,\eta} \sigma_r(X) n^{-\frac{1}{d}}
$$

where, for every $r \in (0, +\infty)$, $\sigma_r(X) = \inf_{a \in \mathbb{R}^d} \|X - a\|_r \leq +\infty$.

Note that a sequence of $n$-quantizers $(\Gamma^n)_{n \geq 1}$ is said to be asymptotically $L^r$-optimal if

$$
\lim_{n \to +\infty} n^{\frac{1}{d}} e_{r}(\Gamma^n, X)^r = Q_r(P)
$$

and $L^r$-rate optimal if

$$
\lim_{n \to +\infty} \sup_{n \geq 1} n^{\frac{1}{d}} e_{r}(\Gamma^n, X) < +\infty \quad \text{or equivalently} \quad \forall n \geq 1, \quad e_{r}(\Gamma^n, X) \leq C_1 n^{-\frac{1}{d}}
$$

where $C_1$ is a constant not depending on $n$.

The $L'$-rate optimality of greedy quantization sequences has been recently extended in [1]. The authors relied on auxiliary probability distributions $\nu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying the following control on balls, with respect to an $L'$-median $a_1$ of $P$: for every $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 \in (0, 1]$, there exists a Borel function $g_{\varepsilon} : \mathbb{R}^d \to [0, +\infty)$ such that, for every $x \in \text{supp}(P)$ and every $t \in [0, \varepsilon \|x - a_1\|_1]$,

$$
\nu(B(x, t)) \geq g_{\varepsilon}(x)V_d t^d
$$

where $V_d$ denotes the volume of the hyper unit Ball. Of course, this condition is of interest only if the set $\{g_{\varepsilon} > 0\}$ is sufficiently large with respect to $\{f > 0\}$ (where $f$ is the density of $P$).
Theorem 2.2. (see [1]) Let $P$ be such that $\int_{\mathbb{R}^d} ||x||^r dP(x) < +\infty$. For any distribution $\nu$ and any Borel function $g_{\varepsilon} : \mathbb{R}^d \rightarrow \mathbb{R}_+$, $\varepsilon \in (0, \frac{1}{d})$, satisfying (10),

$$\forall n \geq 2, \quad e_r(a^{(n)}, P) \leq \varphi_r(\varepsilon)^{-\frac{1}{d}} V_d^{-\frac{1}{d}} \left( \frac{r}{d} \right)^{\frac{1}{d}} \left( \int g_{\varepsilon}^{r} dP \right)^{\frac{1}{r}} (n-1)^{-\frac{1}{d}}$$

where $\varphi_r(u) = \left( \frac{1}{3^r} - u^r \right) u^d$.

Considering appropriate auxiliary distributions $\nu$ and “companion” functions $g_{\varepsilon}$ satisfying (10) yields a Pierce type and a hybrid Zador-Pierce type $L^r$-rate optimality results as established in [1] (Zador type results are established in [5]).

Now, we give a micro-macro inequality established in [3] (see proof of Theorem 2) to estimate the increments $e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r$, where $(\Gamma^n)_{n \geq 1}$ is a sequence of $L^r$-optimal quantizers of $P$. For every $n \geq 1$,

$$e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r \leq \frac{4(2^r - 1)e_r(\Gamma^{n+1}, P)^r}{n+1} + \frac{4.2^r C_2 n^{-\frac{r}{d}}}{n+1}$$

where $C_2$ is a finite constant independent of $n$.

The following Proposition provides a micro-macro inequality established in [1] for any quantizer $\Gamma$ of $X$ with distribution $P$.

Proposition 2.3. Assume $\int ||x||^r dP(x) < +\infty$. Let $y \in \mathbb{R}^d$ and $\Gamma \subset \mathbb{R}^d$ be a finite quantizer of a random variable $X$ with distribution $P$ such that $|\Gamma| \geq 1$. Then, for every probability distribution $\nu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, every $c \in (0, \frac{1}{2})$

$$e_r(\Gamma, P)^r - e_r(\Gamma \cup \{y\}, P)^r \geq \frac{(1-c)^r - c^r}{(c+1)^r} \int \nu \left( B \left( x, \frac{c}{c+1} d(x, \Gamma) \right) \right) d(x, \Gamma)^r dP(x).$$

From this Proposition, one concludes the following either for $L^r$ optimal quantizers or greedy sequences:

$\triangleright$ Since any sequence of $L^r$-optimal quantizers $(\Gamma^n)_{n \geq 1}$ clearly satisfies $e_r(\Gamma^{n+1}, P) \leq e_r(\Gamma^n \cup \{y\}, P)$ for every $y \in \mathbb{R}^d$, then

$$e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r \geq e_r(\Gamma^n, P)^r - e_r(\Gamma^n \cup \{y\}, P)^r \geq \frac{(1-c)^r - c^r}{(c+1)^r} \int \nu \left( B \left( x, \frac{c}{c+1} d(x, \Gamma^n) \right) \right) d(x, \Gamma^n)^r dP(x).$$

$\triangleright$ Likewise, since the greedy quantization sequence $(a^{(n)})_{n \geq 1}$ satisfies $e_r(a^{(n+1)}, P) \leq e_r(a^{(n)} \cup \{y\}, P)$ for every $y \in \mathbb{R}^d$, then

$$e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq \frac{(1-c)^r - c^r}{(c+1)^r} \int \nu \left( B \left( x, \frac{c}{c+1} d(x, a^{(n)}) \right) \right) d(x, a^{(n)})^r dP(x).$$

3 Upper estimate for $L^r$-optimal quantizers

Let $r, s > 0$ and $(\Gamma^n)_{n \geq 1}$ a sequence of $L^r(\mathbb{R}^d)$-optimal quantizers of a random vector $X$ with probability distribution $P$. For every $\mu \in \mathbb{R}^d$ and $\theta > 0$, we denote $\Gamma^n_{\theta, \mu} = \mu + \theta(\Gamma^n - \mu) = \{\mu + \theta(x_i - \mu), x_i \in \mathbb{R}^d\}$.
Then, by applying the change of variables 

\[ x = \frac{z - \mu}{\theta} + \mu, \]

one obtains

\[
e_s(\Gamma^n, P) = \theta^{s+d} \int_{\mathbb{R}^d} d(x, \Gamma^n)^s f(\mu + \theta(x - \mu)) d\lambda_d(x) = \theta^s \int_{\mathbb{R}^d} d(x, \Gamma^n)^s d\nu(\mu) = \theta^s e_s(\Gamma^n, \nu) \quad (17)
\]
Now, let us study $e_s(\Gamma^n, P_{\theta, \mu})$. We will rely on the micro-macro inequality (13) and assume that $c \in (0, \frac{1}{c+1} \cap (0, \frac{1}{2})$ so $\frac{c}{c+1} \leq \varepsilon$. Moreover, $d(x, \Gamma^n) \leq \|x - a_1\|$ for an $a_1 \in \Gamma^n$. So, $\frac{c}{c+1} d(x, \Gamma^n) \leq \varepsilon \|x - a_1\|$ and, hence, $\nu$ satisfies (16) w.r.t. $a_1$ and consequently, there exists a Borel function $g_\varepsilon : \mathbb{R}^d \to (0, +\infty)$ such that

$$\nu \left( B \left( x, \frac{c}{c+1} d(x, \Gamma^n) \right) \right) \geq V_d \left( \frac{c}{c+1} \right)^d d(x, \Gamma^n) g_\varepsilon(x).$$

Then, noticing that $(\frac{1-c}{c+1} - c)^r \geq \frac{1}{3} r \cdot (\frac{c}{c+1})^r > 0$, since $c \in (0, \frac{1}{2})$, yields

$$e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r \geq V_d \varphi_r \left( \frac{c}{c+1} \right) \int g_\varepsilon(x) d(x, \Gamma^n)^{d+r} dP(x)$$

where $\varphi_r(u) = \left( 1 - u^r \right)^{\frac{1}{3}} - u$, $u \in (0, \frac{1}{3})$. Consequently, since $dP = \frac{f}{\varphi} dP_{\theta, \mu}$ ($f_{\theta, \mu} > 0$ $P_{\theta, \mu}$-a.e.),

$$e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r \geq V_d \varphi_r \left( \frac{c}{c+1} \right) \theta^{-d} \int_{\mathbb{R}^d} g_\varepsilon(x) d(x, \Gamma^n)^{d+r} f(x) f_{\theta, \mu}^{-1}(x) dP_{\theta, \mu}(x).$$

Now, applying reverse Hölder inequality with the conjugate exponents $p = \frac{s}{d+r} \in (0, 1)$ and $q = \frac{s-d}{d+r} < 0$ gives

$$e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r \geq V_d \varphi_r \left( \frac{c}{c+1} \right) \theta^{-d} \left( \int_{\mathbb{R}^d} g_\varepsilon(x) d(x, \Gamma^n)^{s} dP_{\theta, \mu}(x) \right)^{\frac{1}{s}} \times \left( \int_{\mathbb{R}^d} d(x, \Gamma^n)^{s} dP_{\theta, \mu}(x) \right)^{\frac{1}{s}}$$

$$\geq V_d \varphi_r \left( \frac{c}{c+1} \right) \theta^{-d} \left( \int_{\mathbb{R}^d} g_\varepsilon(x) d(x, \Gamma^n)^{s} dP_{\theta, \mu}(x) \right)^{\frac{1}{s}} e_s(\Gamma^n, P_{\theta, \mu})^{d+r}.$$ (19)

We apply reverse Hölder inequality a second time with the conjugate exponents $p' = \frac{d+r-s}{d+r} = \frac{1}{1-q} \in (0, 1) \text{ and } q' = \frac{s-d}{s} = \frac{1}{q} < 0$ to obtain

$$\int_{\mathbb{R}^d} g_\varepsilon^q(x) f^q(x) f_{\theta, \mu}^{1-q}(x) d\lambda_d(x) \geq \left( \int_{\mathbb{R}^d} g_\varepsilon d\lambda_d \right)^q \left( \int f^{-\frac{s}{d+r}} f_{\theta, \mu} d\lambda_d \right)^{1-q}.$$

Consequently, denoting $C = V_d \varphi_r \left( \frac{c}{c+1} \right) \theta^{-d} \left( \int_{\mathbb{R}^d} g_\varepsilon d\lambda_d \right)^q \left( \int f^{-\frac{s}{d+r}} f_{\theta, \mu} d\lambda_d \right)^{-\frac{1}{d+r}}$, we will have

$$e_r(\Gamma^n, P)^r - e_r(\Gamma^{n+1}, P)^r \geq C e_s(\Gamma^n, P_{\theta, \mu})^{d+r}.$$ (20)

At this stage, since $(\Gamma^n)_{n \geq 1}$ is a sequence of $L'$-optimal quantizers, we will use (12) to obtain

$$e_s(\Gamma^n, P_{\theta, \mu}) \leq C^{-\frac{1}{d+r}} \left( \frac{4(2^r - 1) e_r(\Gamma^{n+1}, P)^r}{n + 1} + 4.2^r C_2^r n^{-\frac{5}{3}} \right)^{\frac{1}{d+r}}$$

$$\leq \left( \frac{4(2^r - 1) e_r(\Gamma^{n+1}, P)^r}{n + 1} + 4.2^r C_2^r n^{-\frac{5}{3}} \right)^{\frac{1}{d+r}} V_d^{-\frac{1}{d+r}} \varphi_r \left( \frac{c}{c+1} \right)^{-\frac{s}{d+r}} \left( \int_{\mathbb{R}^d} g_\varepsilon d\lambda_d \right)^{-\frac{1}{d+r}}$$

$$\times \left( \int f^{-\frac{s}{d+r}} f_{\theta, \mu} d\lambda_d \right)^{-\frac{1}{s}}$$

$$\leq \left( \frac{4(2^r - 1) C_1^r + 4.2^r C_2^r n^{-\frac{5}{3}} \right)^{\frac{1}{d+r}} V_d^{-\frac{1}{d+r}} \varphi_r \left( \frac{c}{c+1} \right)^{-\frac{s}{d+r}} \theta^{-\frac{d}{s}} \left( \int_{\mathbb{R}^d} g_\varepsilon d\lambda_d \right)^{-\frac{1}{d+r}}$$

$$\times \left( \int f^{-\frac{s}{d+r}} f_{\theta, \mu} d\lambda_d \right)^{-\frac{1}{s}}.$$ (21)
where we used, in the last inequality, the definition of an $L^r$-optimal quantizer given by (9). Now, since in most applications $\varepsilon \mapsto \left(\int f g_\varepsilon^{-\frac{r}{r+1}} dP\right)^{\frac{1}{r+1}}$ is increasing on $(0,1)$, we are led to study $\varphi_r\left(\frac{c}{c+1}\right)^{-\frac{1}{r+1}}$ subject to the constraint $c \in (0, \frac{\varepsilon}{1-\varepsilon}] \cap (0, \frac{1}{2})$. $\varphi_r$ is increasing in the neighborhood of 0 and $\varphi_r(0)$, so, one has, for every $\varepsilon \in (0, \frac{1}{3})$ small enough, $\varphi_r\left(\frac{c}{c+1}\right) \leq \varphi_r(\varepsilon)$, for $c \in (0, \frac{\varepsilon}{1-\varepsilon}]$. This leads to specify $c$ as $c = \frac{\varepsilon}{1-\varepsilon}$, so that $\frac{c}{c+1} = \varepsilon$, which means that one can use

$$\varphi_r\left(\frac{c}{c+1}\right)^{-\frac{1}{r+1}} \leq \min_{\varepsilon \in (0, \frac{1}{3})} \varphi_r(\varepsilon)^{-\frac{1}{r+1}}$$

in (21) to obtain

$$e_s(\Gamma^n, P_{\theta, \mu}) \leq \left(4(2^r - 1)C_1^r + 4.2^r C_2^r\right)^{\frac{1}{r+1}} n^{-\frac{1}{4}} V_d^{-\frac{1}{r+1}} \min_{\varepsilon \in (0, \frac{1}{3})} \varphi_r(\varepsilon)^{-\frac{1}{r+1}} \theta^d \left(\int_{\mathbb{R}^d} g_\varepsilon d\lambda_d\right)^{-\frac{1}{r+1}} \times \left(\int f^{-\frac{r}{r+1}} f_{\theta, \mu} d\lambda_d\right)^{-\frac{1}{2}}.$$

Finally, one deduces the result by injecting this last inequality in (17).

By specifying the function $g_\varepsilon$, one can obtain universal non asymptotic bounds for the error $e_s(\Gamma^n_{\theta, \mu}, P)$. In the following corollary, we give one example.

**Corollary 3.2.** Let $s \in (0, d+r)$ and let $X$ be an $\mathbb{R}^d$-valued random vector with distribution $P = f.\lambda_d$. Assume $\mathbb{E}\|X\|^{r+\eta} < +\infty$ for some $\eta > 0$ and let $(\Gamma^n_{\theta, \mu})_{n \geq 1}$ be a sequence of $L^r(\mathbb{R}^d)$-optimal quantizers of $X$. Assume

$$\int f^{-\frac{r}{r+1}} f_{\theta, \mu} d\lambda_d < +\infty.$$

Then, for every $n \geq 1$,

$$e_s(\Gamma^n_{\theta, \mu}, P) \leq \tilde{c}_{\theta, \mu}^{\text{optimal}} \theta^{1+\frac{d}{2}} \left(\int f^{-\frac{r}{r+1}} f_{\theta, \mu} d\lambda_d\right)^{\frac{1}{2}} n^{-\frac{1}{4}}$$

where $\tilde{c}_{\theta, \mu}^{\text{optimal}} = \left(4(2^r - 1)C_1^r + 4.2^r C_2^r\right)^{\frac{1}{r+1}} V_d^{-\frac{1}{r+1}} \min_{\varepsilon \in (0, \frac{1}{3})} \varphi_r(\varepsilon)^{-\frac{1}{r+1}} (1+\varepsilon)^{-\frac{d(\varepsilon + \delta)}{2(1+\varepsilon)}}$ with $C_1$ and $C_2$ are finite constants not depending on $n, \theta$ and $\mu$ and $\varphi_r : u \mapsto \left(\frac{1}{\sqrt{\varepsilon}} - \frac{1}{\sqrt{t}}\right)^{-\frac{d}{2}}$.

**Proof.** We consider $\nu(dx) = \gamma_{r, \delta}(x) \lambda_d(dx)$ where

$$\gamma_{r, \delta}(x) = \frac{K_{\delta, r}}{(1 + \|x - a_1\|)^{d(1+\frac{r}{2})}} \quad \text{with} \quad K_{\delta, r} = \left(\int \frac{dx}{(1 + \|x\|)^{d(1+\frac{r}{2})}}\right)^{-1} < +\infty$$

is a probability density with respect to the Lebesgue measure on $\mathbb{R}^d$. Let $\varepsilon \in (0,1)$ and $t > 0$. For every $x \in \mathbb{R}^d$ such that $\varepsilon\|x - a_1\| \geq t$ and every $y \in B(x, t)$, one has $\|y - a_1\| \leq \|y - x\| + \|x - a_1\| \leq (1+\varepsilon)\|x - a_1\|$ so that

$$\nu(B(x, t)) \geq \frac{K_{\delta, r} V_d t^d}{(1 + [(1+\varepsilon)\|x - a_1\|])^{d(1+\frac{r}{2})}}.$$

Hence, (16) is verified with

$$g_\varepsilon(x) = \frac{K_{\delta, r}}{(1 + [(1+\varepsilon)\|x - a_1\|])^{d(1+\frac{r}{2})}},$$
so we can apply Theorem 3.1 where we notice

\[
\int_{\mathbb{R}^d} g_c(x) dx \geq K_{\delta,r} \int_{\mathbb{R}^d} (1 \lor [(1 + \varepsilon)\|x - a_1\|])^{-d(1 + \frac{r}{d})} dx \\
\geq K_{\delta,r} (1 + \varepsilon)^{-d(1 + \frac{r}{d})} \int_{\mathbb{R}^d} (1 \lor \|x - a_1\|)^{-d(1 + \frac{r}{d})} dx
\]

so that

\[
\left( \int_{\mathbb{R}^d} g_c(x) dx \right)^{-\frac{1}{d+1}} \leq K_{\delta,r}^{-\frac{1}{d+1}} (1 + \varepsilon)^{-\frac{d(r+\delta)}{d(d+1)}} \left( \int_{\mathbb{R}^d} (1 \lor \|x\|)^{-d(1 + \frac{r}{d})} dx \right)^{-\frac{1}{d+1}} \leq (1 + \varepsilon)^{-\frac{d(r+\delta)}{d(d+1)}}.
\]

\[\square\]

Remark 3.3. One checks that \(\varphi_r\) attains its maximum at \(\frac{1}{3} \left( \frac{d}{d+r} \right)^{\frac{2}{3}}\) on \((0, \frac{1}{3})\).

4 Upper estimates for greedy quantizers

Let \(r, s > 0\) and \((a_n)_{n \geq 1}\) an \(L^r(\mathbb{R}^d)\)-optimal greedy quantization sequence of a random variable \(X\) with probability distribution \(P\). We denote \(a^{(n)} = \{a_1, \ldots, a_n\}\) the first \(n\) terms of this sequence. For every \(\mu \in \mathbb{R}^d\) and \(\theta > 0\), we denote \(a_{\theta,\mu}^{(n)} = \mu + \theta (a^{(n)} - \mu) = \{\mu + \theta (a_i - \mu), 1 \leq i \leq n\}\). In this section, we will study the \(L^s\)-optimality of the sequence \(a_{\theta,\mu}^{(n)}\).

For this, we will consider auxiliary probability distributions \(\nu\) satisfying the following control on balls with respect to an \(L^r\)-median \(a_1\) of \(P\): for every \(\varepsilon \in (0, 1)\), there exists a Borel function \(g_c: \mathbb{R}^d \to (0, +\infty)\) \(\lambda_d\)-integrable such that, for every \(x \in \text{supp}(P)\) and every \(t \in [0, \varepsilon \|x - a_1\|]\),

\[
\nu(B(x,t)) \geq g_c(x)V_d t^d. \tag{22}
\]

Note that \(a_1 \in a^{(n)}\) for every \(n \geq 1\) by construction of the greedy quantization sequence so that \(d(x, a^{(n)}) \leq d(x, a_1)\) for every \(x \in \mathbb{R}^d\).

Theorem 4.1. Let \(s < d + r\) and let \(X\) be a random variable in \(\mathbb{R}^d\) with distribution \(P\). Let \(P \in \mathcal{P}_{r+\eta}(\mathbb{R}^d)\) be a distribution of the form \(P = f \lambda_d\). Let \(E\|X\|^{r+\eta} < +\infty\) for some \(\eta > 0\). Assume

\[
\int f^{-\frac{s}{d+r}} f_{\theta,\mu} d\lambda_d < +\infty.
\]

Then, for every Borel function \(g_c, \varepsilon \in (0, \frac{1}{3})\), satisfying (22) and every \(n \geq 3\),

\[
e_s(a_{\theta,\mu}^{(n)}, P) \leq e^{1+\frac{d}{s} s_{\theta,\mu}^{\text{greedy}}} \left( \int g_c^{-\frac{s}{d}} dP \right)^{\frac{1}{d+1}} \left( \int g_c d\lambda_d \right)^{-\frac{1}{d+1}} \left( \int f^{-\frac{s}{d+r}} f_{\theta,\mu} d\lambda_d \right)^{\frac{1}{2}} (n - 2)^{-\frac{1}{2}} \tag{23}
\]

where \(s_{\theta,\mu}^{\text{greedy}} = 2^2 \nu^{-\frac{1}{2}} \left( \frac{r}{d} \frac{d+r}{d+1} \min_{\varepsilon \in (0, \frac{1}{3})} \varphi_r(\varepsilon) \right)^{-\frac{1}{2}}\).

Proof. First, as in the proof of Theorem 3.1, we have for every \(n \geq 3\),

\[
e_s(a_{\theta,\mu}^{(n)}, P)^s = \theta^s e_s(a^{(n)}, P_{\theta,\mu})^s. \tag{24}
\]

Then, assume \(c \in (0, \frac{e}{c+1}] \cap (0, \frac{1}{3})\) so that \(\frac{c}{c+1} \leq \varepsilon\) so that, for any such \(c\), \(\frac{c}{c+1} d(x, a^{(n)}) \leq \varepsilon \|x - a_1\|\). Hence, by (22), there exists a function \(g_c\) such that

\[
\nu \left( B\left( x, \frac{c}{c+1} d(x, a^{(n)}) \right) \right) \geq V_d \left( \frac{c}{c+1} \right)^d d(x, a^{(n)})^d g_c(x).
\]
We apply again the reverse Hölder inequality this time with the conjugate exponents
\[ \varphi_r(u) = \left( \frac{1}{3^r} - u^r \right)^{\frac{1}{r}} u^d, \quad u \in (0, \frac{1}{3}). \]
Consequently,
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq V_d \varphi_r \left( \frac{c}{c+1} \right) \theta_d \int_{\mathbb{R}^d} g_r(x) d(x, a^{(n)})^{d+r} dP(x)\]
where \( \varphi_r(u) = \left( \frac{1}{3^r} - u^r \right)^{\frac{1}{r}} u^d, \quad u \in (0, \frac{1}{3}). \)

Now, applying the reverse Hölder inequality with conjugate exponents \( p = \frac{s}{d+r} \in (0, 1) \) and \( q = \frac{s-r}{d+r} < 0 \) gives
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq V_d \varphi_r \left( \frac{c}{c+1} \right) \theta_d \left( \int_{\mathbb{R}^d} (g_r(x) f(x) f_{\theta, \mu}^{-1}(x))^{q} dP_{\theta, \mu}(x) \right)^{\frac{1}{q}} \times \left( \int_{\mathbb{R}^d} d(x, a^{(n)})^s dP_{\theta, \mu}(x) \right)^{\frac{1}{s}} \geq V_d \varphi_r \left( \frac{c}{c+1} \right) \theta_d \left( \int_{\mathbb{R}^d} g_r^q(x) f^q(x) f_{\theta, \mu}^{1-q}(x) d\lambda_d(x) \right)^{\frac{1}{q}} e_s(a^{(n)}, P_{\theta, \mu})^{d+r}.\]

We apply again the reverse Hölder inequality this time with the conjugate exponents \( p' = \frac{d+r-s}{d+r} = \frac{1}{1-q} \in (0, 1) \) and \( q' = \frac{s-r-d}{s} = \frac{1}{q} < 0 \) to obtain
\[
\int_{\mathbb{R}^d} g_r^q(x) f^q(x) f_{\theta, \mu}^{1-q}(x) d\lambda_d(x) \geq \left( \int_{\mathbb{R}^d} g_r d\lambda_d \right)^q \left( \int f^{-\frac{s}{d+r}} f_{\theta, \mu} d\lambda_d \right)^{\frac{d+r}{s}}.
\]
Consequently, denoting \( C_1 = V_d \varphi_r \left( \frac{c}{c+1} \right) \theta_d \int_{\mathbb{R}^d} g_r d\lambda_d \left( \int f^{-\frac{s}{d+r}} f_{\theta, \mu} d\lambda_d \right)^{-\frac{d+r}{s}} \), we will have
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq C_1 e_s(a^{(n)}, P_{\theta, \mu})^{d+r}.\]
At this stage, we know that \( e_r(a^{(k)}, P) \) in decreasing w.r.t \( k \) and it is clear that it is the same for \( e_s(a^{(k)}, P_{\theta, \mu}) \), since
\[
e_s(a^{(k)}, P_{\theta, \mu}) = E \left[ \min_{1 \leq i \leq k} | a_i - \frac{X - \mu}{\theta} - \mu |^s \right]^{\frac{1}{s}} \geq E \left[ \min_{1 \leq i \leq k+1} | a_i - \frac{X - \mu}{\theta} - \mu |^s \right]^{\frac{1}{s}} = e_s(a^{(k+1)}, P_{\theta, \mu}),
\]
so, one has
\[
n e_s(a^{(2n-1)}, P_{\theta, \mu})^{d+r} \leq \sum_{k=n}^{2n-1} e_s(a^{(k)}, P_{\theta, \mu})^{d+r} \leq \frac{1}{C_1} \sum_{k=n}^{2n-1} e_r(a^{(k)}, P)^r - e_r(a^{(k+1)}, P)^r \leq \frac{1}{C_1} e_r(a^{(n)}, P)^r.
\]
and, since \( 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq n, \)
\[
\frac{n}{2} e_s(a^{(n)}, P_{\theta, \mu})^{d+r} \leq \left\lfloor \frac{n}{2} \right\rfloor e_s(a^{(n)}, P_{\theta, \mu})^{d+r} \leq \left\lfloor \frac{n}{2} \right\rfloor e_s \left( a^{2 \left\lfloor \frac{n}{2} \right\rfloor - 1}, P_{\theta, \mu} \right)^{d+r} \leq \frac{1}{C_1} e_r \left( a^{\left\lfloor \frac{n}{2} \right\rfloor}, P \right)^r.
\]
Consequently, using the result of Theorem 2.2
\[
e_s(a^{(n)}, P_{\theta, \mu}) \leq \left( \frac{2}{C_1} \right)^{\frac{1}{d+r_+}} (n-1)^{-\frac{1}{d+r_+}} e_r \left( a^{\left[ \frac{d}{r} \right]}, P \right)^{\frac{r}{d+r_+}}
\leq 2^{1+\frac{1}{3}} V_d^{-\frac{1}{3}} \left( \frac{r}{d} \right)^{\frac{1}{d+r_+}} \varphi_r \left( \frac{c_0}{c+1} \right)^{-\frac{1}{d}} \theta^\frac{d}{r} \left( \int g_{\epsilon}^{-\frac{r}{d}} dP \right)^{\frac{1}{d+r_+}} \left( \int_{\mathbb{R}^d} g_{\epsilon} d\lambda_d \right)^{-\frac{1}{d+r_+}}
\left( \int f^{-\frac{r}{d+r_+}} f_{\theta, \mu} d\lambda_d \right)^{\frac{1}{2}} (n-2)^{-\frac{1}{d}}.
\]

Since in most applications \( \varphi \rightarrow \left( \int g_{\epsilon}^{-\frac{r}{d}} dP \right)^{\frac{1}{2}} \) is increasing on \((0, 1)\), we are led to study \( \varphi_r \left( \frac{c_0}{c+r} \right)^{-\frac{1}{d}} \) subject to the constraint \( c \in \left( 0, \frac{r}{c+r} \right) \cap \left( 0, \frac{1}{2} \right) \). \( \varphi_r \) is increasing in the neighborhood of 0 and \( \varphi_r(0) \), so, one has, for every \( \epsilon \in \left( 0, \frac{1}{3} \right) \) small enough, \( \varphi_r \left( \frac{c_0}{c+r} \right) \leq \varphi_r(\epsilon) \), for \( c \in \left( 0, \frac{r}{c+r} \right) \). This leads to specify \( c \) as \( c = \frac{r}{c+r} \), so that \( \frac{r}{c+r} = \epsilon \), which yields
\[
e_s(a^{(n)}, P_{\theta, \mu}) \leq 2^{1+\frac{1}{3}} V_d^{-\frac{1}{3}} \left( \frac{r}{d} \right)^{\frac{1}{d+r_+}} \min_{\epsilon \in \left( 0, \frac{1}{3} \right)} \varphi_r(\epsilon)^{-\frac{1}{d}} \theta^\frac{d}{r} \left( \int g_{\epsilon}^{-\frac{r}{d}} dP \right)^{\frac{1}{d+r_+}} \left( \int_{\mathbb{R}^d} g_{\epsilon} d\lambda_d \right)^{-\frac{1}{d+r_+}}
\left( \int f^{-\frac{r}{d+r_+}} f_{\theta, \mu} d\lambda_d \right)^{\frac{1}{2}} (n-2)^{-\frac{1}{d}}.
\]
Finally, one concludes by merging this with (24).

If we specify the function \( g_{\epsilon} \) in the previous theorem, one can deduce a kind of avatar of the Pierce Lemma.

**Proposition 4.2.** Assume \( \int \|x\|^r dP(x) < +\infty \). Let \( s < d + r \) and \( \delta > 0 \) and let \( P \in \mathcal{P}_{r+\delta}(\mathbb{R}^d) \) be of the form \( P = f_{\lambda_d} \). Assume
\[
\int f^{-\frac{r}{d+r_+}} f_{\theta, \mu} d\lambda_d < +\infty.
\]
Then \( e_{r+\delta}(a^{(1)}, P) = \sigma_{r+\delta}(P) < +\infty \) and, for every \( n \geq 3 \),
\[
e_s(a^{(n)}, P_{\theta, \mu}) \leq \kappa_{\theta, \mu}^{\text{Greedy,Pierce}} \theta^{1+\frac{d}{r}} \left( \int f^{-\frac{r}{d+r_+}} f_{\theta, \mu} d\lambda_d \right)^{\frac{1}{2}} \sigma_{r+\delta}(P)(n-2)^{-\frac{1}{d}}
\]
where
\[
\kappa_{\theta, \mu}^{\text{Greedy,Pierce}} = 2^{1+\frac{1}{3}+\frac{r+d}{r_+}} V_d^{-\frac{1}{3}} \left( \frac{r}{d} \right)^{\frac{1}{d+r_+}} \min_{\epsilon \in \left( 0, \frac{1}{3} \right)} \varphi_r(\epsilon)^{-\frac{1}{d}} (1+\epsilon)^{1+\frac{d(r+\delta)}{r(d+r_+)}} \left( \int (1+\|x\|)^{-\frac{r}{d(r+d+\delta)}} dx \right)^{-\frac{1}{d+r_+}}.
\]

**Proof.** We consider the particular auxiliary probability distribution \( \nu \) considered in Corollary 3.2 defined by \( \nu(dx) = \gamma_{r, \delta}(x) \lambda_d(dx) \) where
\[
\gamma_{r, \delta}(x) = \frac{K_{\delta,r}}{\left( 1+\|x-a_1\| \right)^{d(1+\frac{\delta}{r})}} \quad \text{with} \quad K_{\delta,r} = \left( \int \frac{dx}{\left( 1+\|x\| \right)^{d(1+\frac{\delta}{r})}} \right)^{-1} < +\infty
\]
is a probability density with respect to the Lebesgue measure on \( \mathbb{R}^d \). Similarly as in the proof of Corollary 3.2, (22) is satisfied with
\[
g_{\epsilon}(x) = \frac{K_{\delta,r}}{\left( 1+\left( \|x-a_1\| \right)^{d(1+\frac{\delta}{r})} \right)}.
\]
so we can apply Theorem 4.1. Likewise, we have
\[
\left(\int_{\mathbb{R}^d} g_{\varepsilon}(x)dx\right)^{-\frac{1}{\alpha+r}} \leq K_{\delta,r}^{-\frac{1}{\alpha+r}} (1 + \varepsilon)^{-\frac{d(r+\delta)}{r(d+r)}} \left(\int_{\mathbb{R}^d} (1 \vee \|x\|)^{-d(1+\frac{\delta}{r})} dx\right)^{-\frac{1}{\alpha+r}} \leq (1 + \varepsilon)^{-\frac{d(r+\delta)}{r(d+r)}}
\]
and
\[
\left(\int_{\mathbb{R}^d} g_{\varepsilon}^{r+\delta}(x)dP\right)^{-\frac{1}{\alpha+r}} \leq K_{\delta,r}^{-\frac{r}{\alpha+d+\delta}} \left(\int (1 \vee (1 + \varepsilon)\|x - a_1\|)^{r+\delta} dP\right)^{-\frac{1}{\alpha+r}}.
\]
So, applying the $L^{r+\delta}$-Minkowski inequality yields
\[
\left(\int g_{\varepsilon}(x)^{r+\delta}dP(x)\right)^{\frac{1}{r+\delta}} \leq K_{\delta,r}^{-\frac{r}{\alpha+d+\delta}} (1 + (1 + \varepsilon)\sigma_{r+\delta})^{\frac{r+\delta}{r+d}}.
\]
Consequently,
\[
e_s(a_{\theta,\mu}^{(n)}, P) \leq \theta^{1+d} \kappa_{\theta,\mu}^{Greedy} (1 + \varepsilon)^{-\frac{d(r+\delta)}{r(d+r)}} K_{\delta,r}^{-\frac{r}{\alpha+d+\delta}} (1 + (1 + \varepsilon)\sigma_{r+\delta})^{\frac{r+\delta}{r+d}} \left(\int f_{r,\mu}^{r+\delta} f_{\theta,\mu} d\lambda_d\right)^{\frac{1}{r}} (n - 1)^{-\frac{1}{r}}
\]
(30)
Now, we introduce an equivariance argument. For $\lambda > 0$, let $X_\lambda := \lambda(X - a_1) + a_1$ and $(\lambda r, n)_{n \geq 1} := (\lambda(a_n - a_1) + a_1)_{n \geq 1}$. It is clear that $e_r(a^{(n)}), X = \frac{1}{r} e_r(a^{(n)}, X_\lambda)$. Plugging this in inequality (30) yields
\[
e_s(a_{\theta,\mu}^{(n)}, P) \leq \theta^{1+d} \kappa_{\theta,\mu}^{Greedy} (1 + \varepsilon)^{-\frac{d(r+\delta)}{r(d+r)}} K_{\delta,r}^{-\frac{r}{\alpha+d+\delta}} \frac{1}{\lambda} (1 + (1 + \varepsilon)\lambda \sigma_{r+\delta})^{\frac{r+\delta}{r+d}} \left(\int f_{r,\mu}^{r+\delta} f_{\theta,\mu} d\lambda_d\right)^{\frac{1}{r}} (n - 1)^{-\frac{1}{r}}
\]
Finally, one deduces the result by setting $\lambda = \frac{1}{(1 + \varepsilon)\sigma_{r+\delta}}$.

The next theorem provides a similar upper bound as before but for $s < r$. We consider probability distributions satisfying (22) but this result does not require the function $g_{\varepsilon}$ to be $\lambda_{d'}$-integrable. Instead, we will need another condition made precise in the following theorem.

**Theorem 4.3.** Let $s < r$ and $X$ a random variable in $\mathbb{R}^d$ with distribution $P = f_{\lambda,d}$. Let $E\|X\|^{r+\eta} < +\infty$ for some $\eta > 0$. Assume
\[
\int g_{\varepsilon}^{-\frac{r}{d}} dP < +\infty \quad \text{and} \quad \int f_{r,\mu}^{r+\eta} f_{\theta,\mu}^{r} d\lambda_d < +\infty.
\]
then, for every distribution $\nu$, every function $g_{\varepsilon}$ satisfying (22) and every $n \geq 3$,
\[
e_s(a_{\theta,\mu}^{(n)}, P) \leq \theta^{1+d} \kappa_{\theta,\mu}^{Greedy} \left(\int g_{\varepsilon}^{r+\delta} dP\right)^{\frac{1}{r}} \left(\int f_{r,\mu}^{r+\delta} f_{\theta,\mu}^{r} d\lambda_d\right)^{\frac{r+\delta}{r}} (n - 1)^{-\frac{1}{r}}
\]
(31)
where $\kappa_{\theta,\mu}^{Greedy} = 2^{1+\frac{1}{2}} V_d^{-\frac{r}{2}} \lambda_{d'}^{-\frac{r}{2}} \min_{\varepsilon \in (0,\frac{1}{2})} \varphi_{r}(\varepsilon)^{-\frac{1}{2}}$.

**Proof.** We start from Equation (26) in the proof of Theorem 4.1 recalled below
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq C_1 e_s(a^{(n)}, P_{\theta,\mu})^{d+r}
\]
where $C_1 = \varphi_r \left( \frac{e^{c_r}}{c^{r+1}} \right) \theta^{-d + \frac{d}{q}} \left( \int_{\mathbb{R}^d} g^q \left( x \right) f^q \left( x \right) f^{1-q} \left( x \right) d\lambda_d \left( x \right) \right)^\frac{1}{q}$ and $q = -\frac{s}{d+r-s}$. At this stage, follow the lines of the proof of Theorem 4.1 to get, for $n \geq 3$,

$$e_s(a^{(n)}, P_{\theta,\mu}) \leq \left( \frac{2}{C_1} \right)^{\frac{d}{d+r}} \left( n - 1 \right)^{-\frac{1}{q + r}} e_r \left( a \left[ \frac{q}{r} \right], P \right)^{\frac{d}{q + r}}$$

$$\leq \kappa_{\theta,\mu}^{\text{Greedy}} \left( \int g^q \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} g^q \left( x \right) f^q \left( x \right) f^{1-q} \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} g^q \left( x \right) f^q \left( x \right) f^{1-q} \left( x \right) d\lambda_d \right)^{n - 2 \frac{1}{q}}.$$

where $\kappa_{\theta,\mu}^{\text{Greedy}} = 2^{1 + \frac{1}{q}} V_d^{-\frac{1}{q}} \left( \frac{e^{c_r}}{c^{r+1}} \right) \min_{\varepsilon \in \left( 0, \frac{1}{3} \right)} \varphi_r \left( \varepsilon \right)^{-\frac{1}{q}}.$

Now, since $s < r$, one can apply Hölder inequality with the conjugate exponents $p' = \frac{r(d+r-s)}{r(d+r-s) - ds} > 1$ and $q' = -\frac{r}{ds} = \frac{r(d+r-s)}{ds} > 1$ which yields

$$\int_{\mathbb{R}^d} g^q \left( x \right) f^q \left( x \right) f^{1-q} \left( x \right) d\lambda_d = \int_{\mathbb{R}^d} g^q \left( x \right) f^q \left( x \right) f^{1-q} \left( x \right) d\lambda_d \leq \left( \int g^q \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \left( \int f^q \left( x \right) d\lambda_d \right)^{\frac{1}{q}}$$

so

$$\left( \int_{\mathbb{R}^d} g^q \left( x \right) f^q \left( x \right) f^{1-q} \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \leq \left( \int g^q \left( x \right) d\lambda_d \right)^{\frac{d}{q + r}} \left( \int f^q \left( x \right) d\lambda_d \right)^{\frac{r}{q + r}}$$

and

$$e_s(a^{(n)}, P_{\theta,\mu}) \leq \kappa_{\theta,\mu}^{\text{Greedy}} \left( \int g^q \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \left( \int f^q \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \left( \int f^q \left( x \right) d\lambda_d \right)^{n - 2 \frac{1}{q}}.$$

and one deduces the result just as in the proof of Theorem 4.1. \qed

Likewise, one can obtain a Pierce type universal non-asymptotic upper bound for this error by specifying the function $g_\varepsilon$.

**Proposition 4.4.** Assume $\int \|x\|^r dP(x) < +\infty$. Let $s < r$ and $\delta > 0$. Assume

$$\int f^{\frac{s}{r}} f^{\frac{r}{d}} d\lambda_d < +\infty.$$

Then $e_{\sigma + \delta}(a^{(1)}, P) = \sigma_{\sigma + \delta}(P) < +\infty$ and, for every $n \geq 3$,

$$e_s(a^{(n)}, P) \leq \kappa_{\theta,\mu}^{\text{Greedy, Pierce}} \left( \int f^q \left( x \right) d\lambda_d \right)^{\frac{1}{q}} \sigma_{\sigma + \delta}(P) (n - 2)^{-\frac{1}{q}} \quad (32)$$

where

$$\kappa_{\theta,\mu}^{\text{Greedy, Pierce}} = 2^{1 + \frac{1}{q}} + \frac{1}{q} V_d^{-\frac{1}{q}} \left( \frac{e^{c_r}}{c^{r+1}} \right) \min_{\varepsilon \in \left( 0, \frac{1}{3} \right)} \varphi_r \left( \varepsilon \right)^{-\frac{1}{q}} \left( \int \left( 1 + \|x\| \right)^{-d(1 + \frac{q}{r})} d\lambda_d \right)^{-\frac{1}{q}}.$$

**Proof.** We consider the function $g_\varepsilon$ defined by

$$g_\varepsilon \left( x \right) = \frac{K_{\delta,r}}{\left( 1 \vee \left( 1 + \varepsilon \right) \left( \|x - a_1\| \right) \right)^{d(1 + \frac{q}{r})}} \left( \int f^q \left( x \right) d\lambda_d \right)^{-\frac{1}{q}}$$

where $K_{\delta,r} = \left( \int \frac{dx}{\left( 1 \vee \left( 1 + \varepsilon \right) \left( \|x - a_1\| \right) \right)^{d(1 + \frac{q}{r})}} \right)^{-1} < +\infty$. Moreover, one has

$$\left( \int_{\mathbb{R}^d} g_\varepsilon \left( x \right) dP \right)^{\frac{1}{q}} \leq K_{\delta,r}^{-\frac{1}{q}} \left( \int \left( 1 \vee \left( 1 + \varepsilon \right) \left( \|x - a_1\| \right) \right)^{r + \delta} dP \right)^{\frac{1}{q}}.$$
so that, applying the $L^{r+\delta}$-Minkowski inequality, one obtains

$$\left( \int g_\delta(x)^{-\frac{r}{\delta}} dP(x) \right)^{\frac{\delta}{r}} \leq K_{\delta,r}^{-\frac{1}{2}} (1 + (1 + \varepsilon) \sigma_{r+\delta})^{1+\frac{\delta}{r}}.$$

Then, applying Theorem 4.3 yields, for every $n \geq 3$,

$$e_s(a_{\theta,\mu}^{(n)}, P) \leq \theta^{1+\frac{\delta}{\sigma_{\theta,\mu}^{\text{Greedy}}}} K_{\delta,r}^{-\frac{1}{2}} (1 + (1 + \varepsilon) \sigma_{r+\delta})^{1+\frac{\delta}{r}} \left( \int f_{\delta,\mu} \frac{f_{\theta,\mu}}{d\lambda_d} \right)^{\frac{r-\delta}{r}} (n - 2)^{-\frac{1}{2}} \tag{33}$$

Finally, using the equivariance argument introduced in the proof of Proposition 4.2, one deduces, in the same spirit, the result by considering $\lambda = \frac{1}{(1+\varepsilon)\sigma_{r+\delta}(P)}$.

5 Application to radial densities

In this section, we will consider $s \in (r, d+r)$ and we will consider probability distributions with radial densities and aim to obtain some hybrid Zador-Pierce upper bound for the error $e_s(a_{\theta,\mu}^{(n)}, P)$. In other words, if the random variable $X$ has distribution $P = h.\lambda_d$, we consider

$$\nu = \frac{h_{\frac{d}{d+r}}}{\int h_{\frac{d}{d+r}} d\lambda_d} \cdot \lambda_d := h_r.\lambda_d$$

where the density function $h$ is radial with non-increasing tails w.r.t. $a_1 \in A$ who is peakless w.r.t. $a_1$. These two terms are defined as follows

**Definition 5.1.** (a) Let $A \subset \mathbb{R}^d$. A function $f : \mathbb{R}^d \to \mathbb{R}_+$ is said to be almost radial non-increasing on $A$ w.r.t. $a \in A$ if there exists a norm $\| \|_0$ on $\mathbb{R}^d$ and real constant $M \in (0, 1]$ such that

$$\forall x \in A \setminus \{a\}, \quad f|_{B_{\| \|_0}(a,\|x-a\|_0)} \geq M f(x). \tag{34}$$

If (34) holds for $M = 1$, then $f$ is called radial non-increasing on $A$ w.r.t. $a$.

(b) A set $A$ is said to be star-shaped and peakless with respect to $a_1$ if

$$p(A, \| \cdot - a_1 \|) := \inf \left\{ \frac{\lambda_d(B(x,t) \cap A)}{\lambda_d(B(x,t))} : x \in A, 0 < t < \|x - a_1\| \right\} > 0. \tag{35}$$

**Remark 5.2.** (a) (34) reads $f(y) \geq M f(x)$ for all $x, y \in A \setminus \{a\}$ for which $\|y - a\|_0 \leq \|x - a\|_0$.

(b) If $f$ is radial non-increasing on $\mathbb{R}^d$ w.r.t. $a \in \mathbb{R}^d$ with parameter $\| \|_0$, then there exists a non-increasing measurable function $g : (0, +\infty) \to \mathbb{R}_+$ satisfying $f(x) = g(\|x-a\|_0)$ for every $x \neq a$.

(c) From a practical point of view, many classes of distributions satisfy (34), e.g. the d-dimensional normal distribution $\mathcal{N}(m, \sigma_d)$ for which one considers $h(y) = \frac{1}{(2\pi)^{d/2} \det(\sigma_d)^{1/2}} e^{-\frac{y^2}{2}}$ and density $f(x) = h(\|x-m\|_0)$ where $\|x\|_0 = \|\sigma_d^{-1/2} x\|$, and the family of distributions defined by $f(x) \propto \|x\|^c e^{-a\|x\|^b}$, for every $x \in \mathbb{R}^d, a, b > 0$ and $c > -d$, for which one considers $h(u) = u^c e^{-au^b}$. In the one dimensional case, we can mention the Gamma distribution, the Weibull distributions, the Pareto distributions and the log-normal distributions.

(d) If $A = \mathbb{R}^d$, then $p(A, \| \cdot - a \|) = 1$ for every $a \in \mathbb{R}^d$.  

14
(c) The most typical unbounded sets satisfying (35) are convex cones that is cones \( K \subset \mathbb{R}^d \) of vertex 0 with \( 0 \in K \) (\( K \neq \emptyset \)) and such that \( \lambda x \in K \) for every \( x \in K \) and \( \lambda \geq 0 \). For such convex cones \( K \) with \( \lambda_d(K) > 0 \), we even have that the lower bound
\[
p(K) := \inf \left\{ \frac{\lambda_d(B(x, t) \cap K)}{\lambda_d(B(x, t))}; x \in K, t > 0 \right\} = \frac{\lambda_d(B(0, 1) \cap K)}{V_d} > 0.
\]
Thus if \( K = \mathbb{R}^d \), then \( p(K) = 2^{-d} \).

**Theorem 5.3.** Let \( s \in (r, d + r) \). Assume \( P = h.\lambda_d \) with \( h \in L^{\frac{d}{r+s}}(\lambda_d) \) and \( \int_{\mathbb{R}^d} ||x||^r dP(x) < +\infty \). Let \( a_1 \) denote the \( L^r \)-median of \( P \). Assume that supp\((P) \subset A \) and \( a_1 \in A \) for some \( A \) star-shaped and peakless with respect to \( a_1 \) and that \( h \) is almost radial non-increasing with respect to \( a_1 \) in the sense of (34). Then, for every \( n \geq 3 \),
\[
e_s(a_{\theta,\mu}^{(n)}, P) \leq \kappa_{\theta,\mu}^{G,Z,P} \theta h \frac{1}{r+d} (n - 2)^{-\frac{1}{2}},
\]
where \( \kappa_{\theta,\mu}^{G,Z,P} \leq \frac{2^{1+\frac{4}{d}} C_0^2 r^{\frac{2}{d}}}{d^3 M \frac{1}{r+d} \|p(A,\|a_1\|)\|_1^{\frac{1}{2}}} \min_{\varepsilon \in (0,\frac{1}{4})} \varphi_r(\varepsilon)^{-\frac{1}{2}}. \)

For the proof of this theorem, we will use the following technical lemma (established in [1]).

**Lemma 5.4.** Let \( \nu = f.\lambda_d \) be a probability measure on \( \mathbb{R}^d \) where \( f \) is almost radial non-increasing on \( A \in B(\mathbb{R}^d) \) w.r.t. \( a_1 \in A \), \( A \) being star-shaped relative to \( a_1 \) and satisfying (35). Then, for every \( x \in A \) and \( t \in (0,\|x - a_1\|) \),
\[
\nu(B(x, t)) \geq M\nu(\|x - a_1\|)(2C_0^2)^{-d}V_d f(x)t^d
\]
where \( C_0 \in [1, +\infty) \) satisfies, for every \( x \in \mathbb{R}^d \), \( \frac{1}{C_0} \|x\|_0 \leq ||x|| \leq C_0\|x\|_0 \).

**Proof of theorem 5.3.** Let \( c \in (0,\frac{1}{2}) \). Since \( \frac{c}{c+1} < 1 \) and \( a_1 \in a^{(n)} \) then, for every \( x \in \mathbb{R}^d \),
\[
\frac{c}{c+1} d(x, a^{(n)}) \leq d(x, a^{(n)}) \leq ||x - a_1||.
\]
So, merging (14) with Lemma 5.4, one obtains
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq \varphi_r \left( \frac{c}{c+1} \right) M\nu(x, a^{(n+1)})(2C_0^2)^{-d}V_d \int h_r(x)d(x, a^{(n)})^d+r dP(x).
\]
Now, denoting \( C = \varphi_r \left( \frac{c}{c+1} \right) M\nu(x, a^{(n+1)})(2C_0^2)^{-d}V_d \) and having in mind that \( dP = h.\lambda_d \) and \( d\theta_{\mu} = \theta^d h_{\theta,\mu}.d\lambda_d \), yields
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq C \theta^{-d} \int h_r(x)h(x)h_{\theta,\mu}^{-1}(x)d(x, a^{(n)})^d+r dP_{\theta,\mu}(x).
\]
We recall that
\[
h_r = \frac{h^{\frac{d}{d+r}}}{\int h^{\frac{d}{d+r}} d\lambda_d} = K_{r,d} \frac{d}{d+r} \quad \text{where} \quad K_{r,d} = \left( \int h^{\frac{d}{d+r}} d\lambda_d \right)^{-1} < +\infty
\]
so that
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq C \theta^{-d} K_{r,d} \int h(x)^{1+\frac{d}{d+r}} h_{\theta,\mu}^{-1}(x)d(x, a^{(n)})^d+r dP_{\theta,\mu}(x).
\]
Applying the reverse Hölder inequality with the conjugate exponents $p = \frac{s}{d+r} \in (0, 1)$ and $q = \frac{d-s}{d+r-s} < 0$ yields
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq C \theta^{d} K_{r,d} \left( \int h(x)^{\frac{2d+r}{d+r}} h_{\theta,\mu}(x) dP_{\theta,\mu}(x) \right)^{\frac{1}{q}} \left( \int h(x)^{\frac{1}{d+r}} dP_{\theta,\mu}(x) \right)^{\frac{d+r}{q}}
\]
\[
\geq C \theta^{d} K_{r,d} \left( \int h(x)^{\frac{1}{d+r}} h_{\theta,\mu}^{-\frac{1}{q}}(x) d\lambda_d(x) \right)^{\frac{1}{q}} e_s(a^{(n)}, P_{\theta,\mu})^{d+r}.
\]

Reusing reverse Hölder inequality but this time with the exponents $p' = \frac{d}{(d+r-q)} < 0$ and $q' = \frac{d}{(d+r-q)}(d+r)(d+r-s) \in (0, 1)$ gives
\[
e_r(a^{(n)}, P)^r - e_r(a^{(n+1)}, P)^r \geq C \theta^{d} K_{r,d} \left( \int h_{\theta,\mu}^{\frac{d}{d+r}} d\lambda_d \right)^{\frac{2d+r}{d+r}} \left( \int h_{\theta,\mu}^{\frac{d}{d+r}} d\lambda_d \right)^{-\frac{(d+r)(d+r-s)}{a d}} e_s(a^{(n)}, P_{\theta,\mu})^{d+r}
\]
\[
\geq C \theta^{d} \|h\|_{\frac{d}{d+r}} \left( \int h_{\theta,\mu}^{\frac{d}{d+r}} d\lambda_d \right)^{-\frac{(d+r)(d+r-s)}{a d}} e_s(a^{(n)}, P_{\theta,\mu})^{d+r}.
\]

At this stage, we denote $C_1 = C \theta^{d} \|h\|_{\frac{d}{d+r}} \left( \int h_{\theta,\mu}^{\frac{d}{d+r}} d\lambda_d \right)^{-\frac{(d+r)(d+r-s)}{a d}}$ and follow the same steps as in the proof of Theorem 4.1 to obtain
\[
e_s(a^{(n)}, P_{\theta,\mu}) \leq \left( \frac{2}{C_1} \right)^{\frac{1}{d+r}} (n-1)^{-\frac{1}{d+r}} e_r\left(a^{\left[\frac{n}{2}\right]}, P\right)^{\frac{r}{d+r}}
\]
\[
\leq \frac{2^{1+\frac{1}{2}} C_0^2 r^\frac{1}{2}}{d^\frac{1}{2} M^\frac{1}{2} V_d^\frac{1}{2} (p(A, || - a_1||))^\frac{1}{2}} \min_{\varepsilon \in (0, \frac{1}{2})} \varphi_r(\varepsilon)^{-\frac{1}{2}} \|h_{\theta,\mu}\|^\frac{1}{d+r}.(n-2)^{-\frac{1}{2}}.
\]

Making the change of variables $z = \mu + \theta(x - \mu)$, one notices that
\[
\|h_{\theta,\mu}\|^\frac{d}{d+r} = \theta^{-d} \|h\|^\frac{d}{d+r} < +\infty
\]

since $\frac{d}{d+r} < \frac{d}{d+r}$ (s $\in (r, d + r)$) so that
\[
e_s(a^{(n)}, P_{\theta,\mu}) \leq \frac{2^{1+\frac{1}{2}} C_0^2 r^\frac{1}{2}}{d^\frac{1}{2} M^\frac{1}{2} V_d^\frac{1}{2} (p(A, || - a_1||))^\frac{1}{2}} \min_{\varepsilon \in (0, \frac{1}{2})} \varphi_r(\varepsilon)^{-\frac{1}{2}} \|h\|^\frac{1}{d+r}.(n-2)^{-\frac{1}{2}}.
\]

The result is deduced using exactly the same arguments as in the end of the proof of theorem 4.1. □

6 Examples

Let $X \sim P = f.\lambda_d$. The upper bounds established in Sections 3 and 4 induce that the quantizers $\Gamma_{\theta,\mu}^n$ and $a_{\theta,\mu}^{(n)}$ are $L^s(P)$-rate optimal under one of two necessary and sufficient conditions depending on the value of $s$, as follows

$\triangleright$ If $s < r$, then $a_{\theta,\mu}^{(n)}$ is $L^s(P)$-rate optimal iff
\[
\int f^\frac{s}{s+r} f_{\theta,\mu}^{\frac{s-r}{r}} d\lambda_d < +\infty.
\]
Note that it is the same condition for $\Gamma_{\theta,\mu}^n$ but this case is fully treated in [10].

$\triangleright$ If $s < r + d$, then $a^{(n)}_{\theta,\mu}$ or $\Gamma_{\theta,\mu}^n$ is $L^s(P)$-rate optimal iff

$$\int f^{-\frac{s}{s+r}} f_{\theta,\mu} d\lambda_d < +\infty. \quad (37)$$

Furthermore, it is interesting to determine the scaling factor $\theta = \theta^*$ that minimizes the upper bounds obtained for the different values of $s$. Doing so, we optimize the estimates already obtained.

Note that the terms depending on $\theta$ in the error bounds do not depend on the sequence itself but only depends on the value of $s$. So, we will lead the study distinguishing only the cases depending on the value of $s$ and we will denote, for the sake of simplicity, $\alpha^{(n)}_{\theta,\mu}$ both sequences $(\Gamma^n_{\theta,\mu})_{n\geq1}$ and $(a^{(n)}_{\theta,\mu})_{n\geq1}$.

The goal is to minimize, over $\theta$, the following quantity:

$$h(\theta) = \begin{cases} 
\theta^{1+\frac{d}{s}} \left( \int f^{-\frac{s}{s+r}} f_{\theta,\mu}^{\frac{r}{s}} d\lambda_d \right)^{\frac{r-s}{sr}} & \text{if } s < r, \\
\theta^{1+\frac{d}{s}} \left( \int f^{-\frac{s}{s+r}} f_{\theta,\mu} d\lambda_d \right)^{\frac{1}{s}} & \text{if } r < s < d + r.
\end{cases} \quad (38)$$

Since this problem entirely depends on the density function $f$, we will only carry out the optimization for specified families of distributions. In [10], always considering the case of $L^s$-optimal quantizers $(\Gamma^n)_{n\geq1}$, the author showed that, for the Gaussian, exponential and Gamma distributions, the conditions (36) and (37) are satisfied for a certain range of values of $\theta$ and that a minimum $\theta^*$ of $h(\theta)$ does exist. He also showed that the resulting sequence of $n$-quantizers $(\Gamma^n_{\theta,\mu})_{n\geq1}$ satisfies the so-called empirical measure theorem stated later (see Theorem 6.3). He made also a conjecture that this sequence is $L^s$-asymptotically optimal but could not show it because the bounds established did not allow to do so. We will recall briefly below the results obtained for the dilatation and contraction of the $L'$-optimal quantizers and note that they will be the same for greedy quantization sequences (since we have the same terms depending on $\theta$ in the upper bounds). We start with the upper bound established in [10] for $s < r$.

**Theorem 6.1.** Let $s < r$ and $X$ an $\mathbb{R}^d$-valued random vector with distribution $P = f.\lambda_d$. Let $(\Gamma^n)_{n\geq1}$ be a sequence of $L^s$-asymptotically quantizers of $P$. If

$$\int_{f > 0} f^{-\frac{s}{s+r}} f_{\theta,\mu}^{\frac{r}{s}} d\lambda_d < +\infty,$$

then,

$$\limsup_{n \to +\infty} n^{\frac{d}{s}} e_s(\Gamma^n_{\theta,\mu}, P) \leq \theta^{1+\frac{d}{s}} Q_r(P)^{\frac{1}{s}} \left( \int_{f > 0} f^{-\frac{s}{s+r}} f_{\theta,\mu} d\lambda_d \right)^{\frac{r-s}{sr}}.$$

$\triangleright$ The multivariate Gaussian distribution

Let $P = \mathcal{N}(m, \Sigma)$. We consider $\mu = m$ so that the distribution $P_{\theta,\mu}$ lies in the same family of distributions as $P$.

If $s < r$, then the sequence $\alpha^n_{\theta,m}$ is $L^s$-rate optimal iff $\theta \in (\sqrt{s^2}, +\infty)$ and, on this set,

$$\theta^* = \sqrt{s + d}$$

is the unique solution of (38) and does not depend on $\Sigma$ and $m$. Since $\theta^* < 1$, we check as expected that $\alpha^n_{\theta^*,m}$ is a contraction with scaling number $\theta^*$. 

17
If $r \leq s < d + r$, the sequence $\alpha^n_{\theta,m}$ is $L^s$-rate optimal iff $\theta \in \left( \frac{1}{d+r}, +\infty \right)$ and, on this set,

$$\theta^* = \frac{s + d}{r + d}$$

is the unique solution of (38). In this case, $\theta^* > 1$ so $\alpha^n_{\theta^*,m}$ is a dilatation of $\alpha^n$ with scaling number $\theta^*$. Furthermore, elementary computations show that

$$\int f^{\frac{d}{d+r}} d\lambda_d = \left( \frac{d}{d+r} \right) \frac{r}{d} \frac{d}{d+r} \int f^{\frac{d}{d+r}} f_{\theta^*,m} d\lambda_d = \left( \frac{d}{d+r} \right) \frac{r}{d} \frac{d}{d+r}$$

so the upper bounds of the quantization error of $P$ induced by the sequence $\Gamma^n_{\theta^*,m}$ for $\theta^* = \sqrt{\frac{d+r}{d+r}}$ is given by

$$Q^{\sup,\theta^*}_{r,s} \left\{ \begin{array}{ll} J^2 \left( \frac{1}{d} \right) \left( \frac{d}{d+r} \right) \frac{r}{d} \left( \frac{d}{d+r} \right) \frac{r}{d} & \text{if } s < r, \\
\kappa^2_{\theta^*,m} \left( \frac{d}{d+r} \right) & \text{if } r < s < d + r.
\end{array} \right.$$ 

**The Exponential Distribution**

Let $P = \mathcal{E}(\lambda)$. We consider $\mu = 0$ so that the distribution $P_{\theta,m}$ lies in the same family of distributions as $P$. If $s < r$, then the sequence $\alpha^n_{\theta,0}$ is $L^s$-rate optimal iff $\theta \in \left( \frac{s}{r}, +\infty \right)$ and, on this set,

$$\theta^* = \frac{s + 1}{r + 1}$$

is the unique solution of (38) and does not depend on $\lambda$. Since $\theta^* < 1$, $\alpha^n_{\theta^*,0}$ is a contraction with scaling number $\theta^*$. If $r < s < 1 + r$, the sequence $\alpha^n_{\theta,0}$ is $L^s$-rate optimal iff $\theta \in \left( \frac{s}{1+r}, +\infty \right)$ and, on this set,

$$\theta^* = \frac{s + 1}{r + 1}$$

is the unique solution of (38) and $\alpha^n_{\theta^*,0}$ is a dilatation with scaling number $\theta^*$. Furthermore, elementary computations show that the upper bounds of the quantization error of $P$ induced by the sequence $(\Gamma^n_{\theta,0})$, for $\theta^* = \frac{s+1}{r+1}$, are given by

$$Q^{\sup,\theta^*}_{r,s} \left\{ \begin{array}{ll} \frac{1}{d+r} \left( \frac{d}{d+r} \right) \left( 1 + s \right)^{\frac{1}{d+r}} & \text{if } s < r, \\
\kappa^2 \left( 1 + r \right)^{-1} \left( 1 + s \right)^{\frac{1}{d+r}} & \text{if } r < s < d + r.
\end{array} \right.$$ 

**The Gamma Distribution**

Let $P = \Gamma(a, \lambda)$, $a, \lambda > 0$, with density $f_{a,\lambda}(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}$ defined on $\mathbb{R}_+$ where $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$ and consider $\mu = 0$.

If $s < r$, then the sequence $\alpha^n_{\theta,0}$ is $L^s$-rate optimal iff $\theta \in \left( \frac{s}{r}, +\infty \right)$ and, on this set,

$$\theta^* = \frac{s + a}{r + a}$$

is the unique solution of (38) and $\alpha^n_{\theta^*,0}$ is a contraction with scaling number $\theta^*$ and translating number 0. If $s > r$, the sequence $\alpha^n_{\theta,0}$ is $L^s$-rate optimal iff $\theta \in \left( \frac{s}{1+r}, +\infty \right)$ and, on this set,

$$\theta^* = \frac{s + a}{r + a}$$
is the unique solution of (38) and \( \alpha^n_{\theta^*,0} \) is a dilatation with scaling number \( \theta^* \) and translating number 0. Since \( \theta^* \) depends on the parameter \( a \) of the distribution, it is clear that one cannot show that the sequence \((\Gamma^n_{\theta^*,0})_n \) satisfies the \( L^s \)-empirical measure theorem nor that it is a sequence of \( L^s \)-asymptotically optimal quantizers.

Looking at the constants \( Q^\sup_{r,s} \) in the upper bounds for the gaussian and exponential distributions that we obtained in this paper, we can conclude that our new upper bounds do not allow us to reach Zador’s sharp constant so we are not in position to demonstrate that the designed sequences are \( L^s \)-asymptotically optimal quantizers.

In the following, we consider examples not investigated in [10].

### 6.1 Example 1

Let \( X \sim P = f \lambda_d \) where \( f(x) = e^{-\lambda \|x\|^\alpha} \) for \( \alpha, \lambda > 0 \) and \( \|.| \) denotes a norm on \( \mathbb{R}^d \). We consider \( \mu = 0 \) so that the distribution \( P_{\theta,\mu} \) lies in the same family of distributions as \( P \). Note that if one considers the density function \( f(x) = e^{-\lambda \|x-m\|^\alpha} \) for \( m \in \mathbb{R} \), the study will be the same since the quantities considered are invariant by translation. In other words, if \( \Gamma \) is an optimal quantizer of \( X \), then \( \Gamma - m(1,\ldots,1) \) is an optimal quantizer for \( X - m \).

**Proposition 6.2.** (a) If \( s < r \), the sequence \( \alpha^n_{\theta,0} \) is \( L^s \)-rate optimal iff \( \theta \in \left( \left( \frac{s}{r} \right)^{\frac{1}{\alpha}}, +\infty \right) \) and, on this set,

\[
\theta^* = \left( \frac{s + d}{r + d} \right)^{\frac{1}{\alpha}}
\]

is the unique solution of (38) and does not depend on \( \lambda \).

(b) If \( s < d + r \), the sequence \( \alpha^n_{\theta,0} \) is \( L^s \)-rate optimal iff \( \theta \in \left( \left( \frac{s}{d+r} \right)^{\frac{1}{\alpha}}, +\infty \right) \) and, on this set,

\[
\theta^* = \left( \frac{s + d}{r + d} \right)^{\frac{1}{\alpha}}
\]

is the unique solution of (38).

**Proof.** First notice that considering the function \( H(\theta) = h(\theta)^s \) will not change the results but only make the calculations easier.

(a) If \( s < r \), one has

\[
\int f^{\frac{r}{s+r}}(x)f_{\theta,0}^{\frac{r}{s}}(x)dx = \int e^{-\lambda \|x\|^\alpha} e^{-\frac{r \theta \|x\|^\alpha}{s+r}} = \int e^{-\lambda \left( \frac{r}{s+r} + \frac{r \theta}{s} \right) \|x\|^\alpha}
\]

So \( \alpha^n_{\theta,0} \) is \( L^s \)-optimal iff (36) is satisfied which is clearly equivalent to

\[
\theta > \left( \frac{s}{r} \right)^{\frac{1}{\alpha}}.
\]

On the other hand, in order to solve (38), one considers \( \theta \in \left( \left( \frac{s}{r} \right)^{\frac{1}{\alpha}}, +\infty \right) \). Using the fact that

\[
\int_{\mathbb{R}^d} f(\|x\|)dx = V_d \int_0^{+\infty} f(r)r^{d-1}dr
\]

and then the fact that

\[
\int_0^{+\infty} x^ne^{-ax}ds = \frac{\Gamma(n+1)}{b^a(n+1)b},
\]

where \( V_d = V(B_d) \) is the volume of the hyper-unit ball on \( \mathbb{R}^d \) and \( \Gamma \) is the Gamma function, one obtains

\[
H(s) = \theta^{s+d} \left( \int e^{-\lambda \left( \frac{r}{s+r} + \frac{r \theta}{s} \right) \|x\|^\alpha} \right)^{\frac{r-s}{r}} = \theta^{s+d} \left( \frac{V_d \Gamma \left( \frac{d}{\alpha} \right)}{\lambda^{\frac{d}{\alpha}}} \right)^{\frac{r-s}{r}} \left( \frac{s}{s-r} + \frac{r}{r-s} \theta^\alpha \right)^{-\frac{d(r-s)}{\alpha}}.
\]

19
So, our aim is to minimize \( g(\theta) = \theta^{s+d} \left( \frac{s}{s-r} + \frac{r}{r-s} \theta^\alpha \right) \frac{d(r-s)}{r\xi} \). Its derivative is given by
\[
g'(\theta) = \theta^{s+d-1} \left( \frac{s}{s-r} + \frac{r}{r-s} \theta^\alpha \right)^{-\frac{d}{\alpha}} \left( \frac{s(s+r+d)}{r-s} \theta^\alpha + \frac{s(s+d)}{s-r} \right)
\]
which clearly attains its minimum at \( \theta^* = \left( \frac{s+d}{r+d} \right)^\frac{1}{\alpha} \) which is its unique minimum on \( \left( \left( \frac{s}{d+r} \right)^\frac{1}{\alpha}, +\infty \right) \).

(b) If \( s < d + r \), one has
\[
\int f^{-\frac{1}{2\pi}}(x)f_{\theta,0}(x)dx = \int e^{\frac{s\lambda}{d+r}} e^{-\lambda\|x\|^\alpha} = \int e^{\left( \frac{s\lambda}{d+r} - \lambda\theta^\alpha \right)\|x\|^\alpha}
\]
So \( \alpha_{\theta,0}^n \) is \( L^s \)-optimal iff (37) is satisfied which is clearly equivalent to
\[
\theta > \left( \frac{s}{d+r} \right)^\frac{1}{\alpha}.
\]

On the other hand, in order to solve (38), one considers \( \theta \in \left( \left( \frac{s}{d+r} \right)^\frac{1}{\alpha}, +\infty \right) \). One has
\[
H(s) = \theta^{s+d} \int e^{\left( \frac{s\lambda}{d+r} - \lambda\theta^\alpha \right)\|x\|^\alpha} = \theta^{s+d} \left( \frac{d}{\alpha} \right) \lambda^{-\frac{d}{\alpha}} \left( \theta^\alpha - \frac{s}{r+d} \right)^{-\frac{d}{\alpha}}.
\]
So, our aim is to minimize \( g(\theta) = \theta^{s+d} \left( \theta^\alpha - \frac{s}{r+d} \right)^{-\frac{d}{\alpha}} \) which derivative is given by
\[
g'(\theta) = \theta^{s+d-1} \left( \theta^\alpha - \frac{s}{r+d} \right)^{-\frac{d}{\alpha}} \left( s\theta^\alpha - \frac{s(s+d)}{r+d} \right)
\]
which clearly attains its minimum at \( \theta^* = \left( \frac{s+d}{d+r} \right)^\frac{1}{\alpha} \) which is its unique minimum on \( \left( \left( \frac{s}{d+r} \right)^\frac{1}{\alpha}, +\infty \right) \).

\[\square\]

If \( s < r \), then \( \theta^* < 1 \) so \( \alpha_{\theta^*,0}^n \) is a contraction with scaling number \( \theta^* \) and, if \( s < r + d \), then \( \theta^* > 1 \) so that \( \alpha_{\theta^*,0}^n \) is a dilatation with scaling number \( \theta^* \). Note that \( \theta^* \) does not depend on the parameter \( \lambda \) of the distribution, only on \( \alpha \).

The following proposition shows that the sequence \( \Gamma_{0,0}^n \) satisfies the empirical measure theorem. Let us first recall this theorem.

**Theorem 6.3.** Let \( P \) be a \( L^r \)-Zador distribution, absolutely continuous w.r.t the Lebesgue measure on \( \mathbb{R}^d \) with density \( f \). Let \( \Gamma^n \) be an asymptotically optimal \( n \)-quantizer of \( P \). Then, denoting \( C_{f,r} = \int f^{-\frac{d}{2\pi}} d\lambda_d \), one has
\[
\frac{1}{n} \sum_{x_i \in \Gamma^n} \delta_{x_i} \Rightarrow P_r = \frac{1}{C_{f,r}} \int f^{-\frac{d}{r+d}} d\lambda_d,
\]
or, in other words, for every \( a, b \in \mathbb{R}^d \),
\[
\frac{1}{n} \text{card}\{x_i \in \Gamma^n \cap [a,b]\} \rightarrow \frac{1}{C_{f,r}} \int_{[a,b]} f^{-\frac{d}{r+d}} d\lambda_d.
\]
Proposition 6.4. Let $r,s > 0$ and $P = f,\lambda_d$ where $f(x) = e^{-\lambda|x|^\alpha}$ for $\alpha, \lambda > 0$. Assume $\Gamma^n$ is an asymptotically $L^r$-optimal quantizer of $P$. Then, the sequence $\Gamma^n_{\theta^*,0}$ satisfies the $L^s$-empirical measure theorem for $\theta^* = \left(\frac{s+d}{r+d}\right)^{\frac{1}{s}}$, i.e.

$$\frac{1}{n} \text{card} \{ x_i \in \Gamma^n_{\theta^*,0} \cap [a,b] \} \rightarrow \frac{1}{C_{f,s}} \int_{[a,b]} f_{\pi+\alpha}^d d\lambda_d.$$ 

**Proof.** For every $n \geq 1$, it is clear that

$$\{ x_i \in \Gamma^n_{\theta^*,0} \cap [a,b] \} = \{ x_i \in \Gamma^n \cap \left[ \frac{a}{\theta^*}, \frac{b}{\theta^*} \right] \}.$$ 

So, since $\Gamma^n$ satisfies the $L^r$-empirical measure theorem, then

$$\frac{1}{n} \text{card} \{ x_i \in \Gamma^n_{\theta^*,0} \cap [a,b] \} \rightarrow \frac{1}{C_{f,s}} \int_{[a,b]} f_{\pi+\alpha}^d d\lambda_d.$$ 

At this stage, knowing that $f(x) = e^{-\lambda|x|^\alpha}$ and $\theta^* = \left(\frac{s+d}{r+d}\right)^{\frac{1}{s}}$, one makes the change of variables $x = \theta^* z$ to obtain, via simple calculations, that

$$\frac{1}{C_{f,s}} \int_{[a,b]} f_{\pi+\alpha}^d d\lambda_d = \frac{1}{C_{f,s}} \int_{[a,b]} f_{\pi+\alpha}^d d\lambda_d.$$ 

The next proposition shows that the sequence $\alpha^n_{\theta^*,0}$ satisfies the lower bound (6).

Proposition 6.5. Let $r,s > 0$ and $P = f,\lambda_d$ where $f(x) = e^{-\lambda|x|^\alpha}$ for $\alpha, \lambda > 0$. Then, the asymptotic lower bound of the $L^s$-error of the sequence $\alpha^n_{\theta^*,0}$ satisfies

$$Q_{r,s}^{L^f}(P,\theta^*) = Q_s(P)$$

where $Q_{r,s}^{L^f}(P,\theta^*) = (\theta^*)^{s+d} \int_{S_d} \left( \int f_{\pi+\alpha}^d d\lambda_d \right)^{\frac{s}{\alpha}} \int f_{\pi+\alpha}^d (x) f_{\theta^*,0}(x) dx.$

**Proof.** Knowing that $f(x) = e^{-\lambda|x|^\alpha}$ and $\theta^* = \left(\frac{s+d}{r+d}\right)^{\frac{1}{s}}$, elementary computations show that

$$\int f_{\pi+\alpha}^d (x) f_{\theta^*,0}(x) dx = V_d \frac{\Gamma(d)}{\alpha} \lambda^{-\frac{d}{r+d}} \left( \frac{d}{r+d} \right)^{\frac{d}{\alpha}}$$

and

$$\int f_{\pi+\alpha}^d d\lambda_d = V_d \frac{\Gamma(d)}{\alpha} \lambda^{-\frac{d}{r+d}} \left( \frac{d}{r+d} \right)^{\frac{d}{\alpha}}$$

so that

$$(\theta^*)^{s+d} \left( \int f_{\pi+\alpha}^d d\lambda_d \right)^{\frac{s}{\alpha}} \int f_{\pi+\alpha}^d (x) f_{\theta^*,0}(x) dx = \left( V_d \frac{\Gamma(d)}{\alpha} \lambda^{-\frac{d}{r+d}} \left( \frac{d}{r+d} \right)^{\frac{d}{\alpha}} \right)^{1+\frac{d}{s}} \left( \frac{s+d}{d} \right)^{\frac{s+d}{\alpha}} = \left( \int f_{\pi+\alpha}^d d\lambda_d \right)^{\frac{d+s}{\alpha}}$$

and hence the result.  \□
Now, for the upper bound, we focus on the case where $\alpha^{(n)}$ is an $L^r$-optimal quantizer and see if the upper bounds in Corollary 3.2 for $\theta^*$ reach the sharp constant in Zador’s Theorem for the different values of $s$. Note that if $\alpha^{(n)}$ is a greedy quantization sequence, one cannot make any interesting conclusions since it is clear that the sharp Zador constant cannot be attained by our upper bounds.

Let $r, s > 0$ and $\Gamma_n$ an $L^r$-optimal quantizer of $P$. Elementary computations (similar to those in the previous propositions) show that the upper bounds of the quantization error of $P$ induced by $\Gamma_n$ for $\theta^*$, for $\theta^* = (s + d)^{\frac{1}{\alpha}}$, are given by

$$Q_{r,s}^{\sup, \theta^*} = \begin{cases} \frac{1}{2} J_{r,d}^\gamma \left( \int f \frac{\lambda}{2} d\lambda \right)^{\frac{d+\beta}{d}} & \text{if } s < r, \\ \frac{1}{2} \text{Optimal} \left( \frac{V_d \Gamma_0(\beta)}{\alpha \lambda^s} \right)^{\frac{1}{\alpha}} \left( s + d \right)^{\frac{d+\beta}{r+d}} & \text{if } r < s < d + r. \end{cases}$$

Furthermore, one can easily notice that, for the different values of $s$, $Q_s(P) \leq Q_{r,s}^{\sup, \theta^*}$. So, one cannot conclude whether the sequence $(\Gamma_n^{\theta^*,m})_{n \geq 0}$ is $L^s$-asymptotically optimal or not. However, one can affirm that, if we have $\tilde{J}_{s,d}^{\gamma}$ instead of $\tilde{J}_{r,d}^{\gamma}$, one can reach Zador’s sharp constant for $r < s$ and gets closer to it for $s \in (r, d + r)$.

### 6.2 Example 2

Let $X \sim P = f.\lambda_d$ where $f(x) = \|x\|^\beta e^{-\lambda\|x\|^\alpha}$ for $\alpha, \lambda > 0$ and $\beta > -d$. We consider $\mu = 0$ so that $P_{\theta,\mu}$ lies in the same family of distributions as $P$.

**Proposition 6.6.** (a) If $s < r$, the sequence $\alpha_{r,0}^n$ is $L^s$-rate optimal iff $\theta^* \in \left( \left( \frac{s}{r} \right)^{\frac{1}{\alpha}}, +\infty \right)$ and, on this set,

$$\theta^* = \left( \frac{s + d + \beta}{r + d + \beta} \right)^{\frac{1}{\alpha}}$$

is the unique solution of (38) and does not depend on $\lambda$.

(b) If $s < d + r$, the sequence $\alpha_{d,0}^n$ is $L^s$-rate optimal iff $\theta^* \in \left( \left( \frac{s}{d + r} \right)^{\frac{1}{\alpha}}, +\infty \right)$ and, on this set,

$$\theta^* = \left( \frac{s + d + \beta}{r + d + \beta} \right)^{\frac{1}{\alpha}}$$

is the unique solution of (38).

**Proof.** We consider the function $H(\theta) = h(\theta)^s$.

(a) If $s < r$, one has

$$\int f \frac{\gamma}{s} (x) f \frac{r}{s} (x) dx = \theta^{\frac{r}{s}} \int \|x\|^\beta e^{-\lambda \left( \frac{r}{s + r} \theta^* + \frac{r}{s + r} \theta^* \right)\|x\|^\alpha}$$

So $\alpha_{r,0}^n$ is $L^s$-optimal iff (36) is satisfied which is clearly equivalent to

$$\theta > \left( \frac{s}{r} \right)^{\frac{1}{\alpha}}.$$
On the other hand, in order to minimize (38), one considers \( \theta \in \left( \left( \frac{s}{d+r} \right)^{\frac{1}{\alpha}}, +\infty \right) \) and uses the same quantities as in the previous example to obtain

\[
H(s) = \theta^{s+d} \left( \frac{\theta^\frac{d}{r}}{r \alpha} \int \|x\|^\beta e^{-\lambda \left( \frac{x^\alpha}{s + r \theta^\alpha} \right)} \right)^{\frac{\alpha}{\beta}}
\]

\[
= \theta^{s+d+\beta} \left( \frac{V_d}{u^{\beta+d-1}} e^{-\lambda \left( \frac{r}{s-r} \right)} du \right)^{\frac{\alpha}{\beta}}
\]

\[
= \theta^{s+d+\beta} \left( \frac{V_d \Gamma \left( \frac{\beta+d}{\alpha} \right)}{\alpha \lambda \frac{\beta+d}{\alpha}} \right) \left( \frac{s}{s-r} + \frac{r}{r-s} \theta^\alpha \right)^{-\frac{(\beta+d)(r-s)}{\alpha r}}
\]

So, our aim is to minimize \( g(\theta) = \theta^{s+d+\beta} \left( \frac{s}{s-r} + \frac{r}{r-s} \theta^\alpha \right)^{-\frac{(\beta+d)(r-s)}{\alpha r}} \). Its derivative is given by

\[
g'(\theta) = \theta^{s+d+\beta-1} \left( \frac{s}{s-r} + \frac{r}{r-s} \theta^\alpha \right)^{-\frac{(\beta+d)(r-s)}{\alpha r}} \left( \frac{s(r+d+\beta)}{r-s} \theta^\alpha + \frac{s(s+d+\beta)}{s-r} \right)
\]

which clearly attains its minimum at \( \theta^* = \left( \frac{s+d+\beta}{r+d+\beta} \right)^{\frac{1}{\alpha}} \) which is its unique minimum on \( \left( \left( \frac{s}{d+r} \right)^{\frac{1}{\alpha}}, +\infty \right) \).

(b) If \( s < d + r \), one has

\[
\int f^{-\frac{s}{d+r}}(x) f_{\theta,0}(x) dx = \theta^\beta \int \|x\|^\beta e^{-\lambda \left( \frac{x^\alpha}{s} - \frac{s}{d+r} \right)} \|x\|^\alpha \lambda_d(x).
\]

So \( \alpha_{\theta,0}^n \) is \( L^\alpha \)-optimal iff (37) is satisfied which is clearly equivalent to

\[
\theta > \left( \frac{s}{d+r} \right)^{\frac{1}{\alpha}}.
\]

Now, in order to minimize (38), one considers \( \theta \in \left( \left( \frac{s}{d+r} \right)^{\frac{1}{\alpha}}, +\infty \right) \) and has

\[
H(s) = \theta^{s+d+\beta} \left( \frac{V_d \Gamma(\gamma)}{\alpha \lambda \gamma} \left( \theta^\alpha - \frac{s}{r+d} \right)^{-\gamma} \right)
\]

where \( \gamma = \frac{1}{\alpha} (d+\beta(1-\frac{s}{d+r})) \). So, our aim is to minimize \( g(\theta) = \theta^{s+d+\beta} \left( \theta^\alpha - \frac{s}{r+d} \right)^{-\gamma} \) which derivative is given by

\[
g'(\theta) = \theta^{s+d+\beta-1} \left( \theta^\alpha - \frac{s}{r+d} \right)^{-\gamma-1} \left( \frac{r+d+\beta}{d+r} \theta^\alpha - \frac{s(s+d+\beta)}{(r+d)} \right).
\]

This function clearly attains its minimum at \( \theta^* = \left( \frac{s+d+\beta}{r+d+\beta} \right)^{\frac{1}{\alpha}} \) which is its unique minimum on \( \left( \left( \frac{s}{d+r} \right)^{\frac{1}{\alpha}}, +\infty \right) \).

If \( s < r \), then \( \theta^* < 1 \) so \( \alpha_{\theta,0}^n \) is a contraction with scaling number \( \theta^* \) and, if \( s < r + d \), then \( \theta^* > 1 \) so that \( \alpha_{\theta,0}^n \) is a dilatation with scaling number \( \theta^* \). Note that \( \theta^* \) does not depend on the parameter \( \lambda \) of the distribution but only on \( \alpha \) and \( \beta \). Moreover, note that one obtains the same results for the distribution with density \( \|x - m\|^\beta e^{-\lambda \|x-m\|^\alpha} \) since it is invariant by translation.
We have just designed a sequence $\alpha_{\theta^*, 0}^n$ with $\theta^* = \left(\frac{s + d + \beta}{r + d + \beta}\right)^{\frac{1}{2}}$. The question is whether it satisfies the empirical measure theorem 6.3. For $\beta = 0$, we obtain the same distribution as in the previous example and so the answer is yes, but for $\beta \neq 0$, elementary computations show that $\alpha_{\theta^*, 0}^n$ does not satisfy the $L^2$-empirical measure theorem. Likewise, one cannot make any conclusions on the $L^2$-rate optimality of this sequence, i.e. one cannot know whether the lower and upper bound of the $L^2$-quantization error of $\alpha_{\theta^*, 0}^n$ are equal or comparable to the sharp limiting constant $Q_x(P)$ in Zador’s Theorem. This is mainly due to the dependence of $\theta^*$ on the parameter $\beta \neq 0$ of the distribution.

### 6.3 Numerical observations

In [10], the author made a conjecture that the optimally $L^r$-dilated sequence $(\Gamma_{n, \theta^*, \mu}^r)$ of optimal quantizers is asymptotically $L^s$-optimal. In this section, we will implement numerical experiments, similar to those established in [10], to come to this type of conclusion for optimally $L^r$-dilated greedy quantization sequences. We denote $a^{r, (n)}$ the $L^r$-greedy quantization sequence.

**Normal distribution** We start with the Normal distribution $\mathcal{N}(0, 1)$ and compute the corresponding $L^3$-optimal greedy quantization sequence $a^{3, (n)}$ by a standard Newton Raphson algorithm on one hand, and the optimally $L^2$-dilated greedy quantization sequence $a_{\theta^*, \mu}^{2, (n)}$ with $\theta^* = \sqrt{\frac{s + d}{r + d}} = \sqrt{\frac{1}{3}}$ and $\mu = 0$, on the other hand. We make a linear regression of the two resulting sequences for different values of the size $n$ and expose, in Table 1, the corresponding regression coefficients.

**Exponential distribution** We consider the exponential distribution $\mathcal{E}(1)$ with parameter $\lambda = 1$ and realize the same study as for the Normal distribution. We compute the $L^3$-optimal greedy quantization sequence $a^{3, (n)}$ by a Newton Raphson algorithm and the optimally $L^2$-dilated greedy quantization sequence $a_{\theta^*, \mu}^{2, (n)}$ with $\theta^* = \sqrt{\frac{s + d}{r + d}} = \sqrt{\frac{4}{5}}$ and $\mu = 0$. The $L^2$-optimal greedy quantization sequence is obtained by a standard Lloyd’s algorithm. We expose, in Table 1, the regression coefficients obtained by regressing the $L^2$-dilated sequences on the $L^3$ greedy sequences.

**Distribution studied in Example 2** We consider the probability distribution studied in Example 2 (see Section 6.2). We consider $d = 1$, $\lambda = 1$ and $\alpha = \beta = 2$ so the density is given by

$$f(x) = x^2 e^{-x^2}.$$ We compute the $L^3$-optimal greedy quantization sequence $a^{3, (n)}$ by a Newton Raphson algorithm and the $L^2$-optimal greedy quantization sequence $a_{\theta^*, \mu}^{2, (n)}$ by a Lloyd’s algorithm. The optimally $L^2$-dilated greedy sequence is given by $a_{\theta^*, \mu}^{2, (n)}$ with $\theta^* = \left(\frac{s + d + \beta}{r + d + \beta}\right)^{\frac{1}{2}} = \sqrt{\frac{6}{5}}$ and $\mu = 0$. Table 1 shows the regression coefficients obtained by regressing the $L^2$-dilated sequences on the $L^3$ greedy sequences.

| Normal Distribution | Exponential distribution | $P = f.\lambda_d$ with $f(x) = x^2 e^{-x^2}$ |
|---------------------|-------------------------|---------------------------------------------|
| $n$ | Regression coefficient | $n$ | Regression coefficient | $n$ | Regression coefficient |
| 255 | 0.9818 | 373 | 0.981 | 255 | 0.9901 |
| 511 | 0.9855 | 745 | 0.988 | 511 | 0.9912 |
| 1023 | 0.9945 | 1489 | 0.990 | 1023 | 0.9914 |

Table 1: Regression coefficients of the optimally $L^2$-dilated greedy sequence on the $L^3$-optimal greedy sequence for $\mathcal{N}(0, 1)$, $\mathcal{E}(1)$ and $P = f.\lambda_d$ with $f(x) = x^2 e^{-x^2}$. 


Conjecture  For the three considered distributions, the regression coefficient converges to 1 for specific values of n. This leads us to conjecture that there exists a sub-sequence of the greedy quantization sequence for which the regression coefficient converges to 1, i.e. for which the sequence is asymptotically $L^s$-optimal.

In fact, this “subsequence” topic has already been investigated in [1] where the authors showed (numerically) that there exist sub-optimal greedy quantization sequences, in the sense that the graphs representing the weights of the Voronoï cells converge towards the density curve of the distribution for certain sizes n of the sequence. For example, the greedy quantization sequence of $\mathcal{N}(0,1)$, and more generally of symmetrical distributions around 0, is sub-optimal and the optimal sub-sequence is of the form $a^{(n)} = a^{(2^k - 1)}$ for $k \in \mathbb{N}^*$.

Hence, it is natural to conjecture that the optimally $L^r$-dilated sub-sequences of the same size are asymptotically $L^s$-optimal.

7 Application to numerical integration

Optimal quantizers and greedy quantization sequences are used in numerical probability where one relies on cubature formulas to approximate the exact value of $\mathbb{E}f(X)$, for a continuous bounded function $f$ and a random variable $X$ with distribution $P$, by

$$ \mathbb{E}f(X) \approx \mathbb{E}f(\hat{X}^{(n)}) = \sum_{i=1}^{n} p_i^n f(\alpha_i^n) $$

where $\alpha^{(n)}$ designates the optimal or greedy quantization sequence of the random variable $X$ and $p_i^n = P(X \in W_i(\alpha^{(n)}))$ represents the weight of the $i^{th}$ Voronoï cell corresponding to $\alpha^{(n)}$ for every $i \in \{1, \ldots, n\}$. A new iterative formula for the approximation of $\mathbb{E}f(X)$ using greedy quantization sequences is given in [1], based on the recursive character of greedy quantization. Upper error bounds of these approximations have been investigated repeatedly in the literature, in [1, 6, 7] for example.

In this section, we present what advantages the dilated quantization sequences bring to the numerical integration field. This application was first introduced in [10] by A. Sagna for optimal quantizers. Here, we briefly recall his idea and emphasize that it also works with dilated greedy quantization sequences as well.

Let $X \in L^\beta, \beta \in (2, +\infty)$ and let $f$ be a locally Lipschitz function, in the sense that, there exists a bounded constant $C > 0$ such that

$$ |f(x) - f(y)| \leq C|x-y|(1 + |x|^\beta - 1 + |y|^\beta - 1). \quad (41) $$

For every quantizer $\alpha^{(n)}$ (not necessarily stationary), one has, by applying Hölder’s inequality with the conjugate exponents $r$ and $r' = \frac{r}{r-1}$, that

$$ |\mathbb{E}f(X) - \mathbb{E}f(\hat{X}^{(n)})| \leq \mathbb{E}|f(X) - f(\hat{X}^{(n)})| \leq C \mathbb{E}
\left( |X - \hat{X}^{(n)}| (1 + |X|^\beta - 1 + |\hat{X}^{(n)}|^\beta - 1) \right) $$

$$ \leq C \|X - \hat{X}^{(n)}\|_r \left( 1 + \|X\|_{(\beta-1)r'} + \|\hat{X}^{(n)}\|_{(\beta-1)r'} \right). $$

In order for this upper bound to make sense, one should have

$$ (\beta - 1)r' = \frac{(\beta - 1)r}{r-1} \leq \beta \quad \iff \quad r \geq \beta > 2. \quad (43) $$
In practice, since most algorithms to optimize quantization (of \( n \)-tuples of greedy sequences) are much easier to implement in the quadratic case, it is more convenient to use such quadratic optimal or greedy quantizers in this type of applications to approximate expectations of the form \( \mathbb{E} f(X) \). However, if we use \( L^2 \)-quantizers \( \alpha^{(n)} \) in our case, we obtain upper bounds involving an \( L^r \)-quantization error for \( r > 2 \) (see (43)) which is not really optimal since the quantizer used is not \( L^r \)-optimal for \( r > 2 \). So, an idea is to use \( L^2 \)-dilated quantizers \( \alpha_{\theta, \mu}^{(n)} \) which is itself \( L^r \)-rate optimal for given values of \( \theta \) and \( \mu \) depending on the probability distribution \( P \). For example, if \( X \sim \mathcal{N}(m, I_d) \), then one chooses \( \mu = m \) and \( \theta = \sqrt{\frac{r+d}{2+d}} \).

Hence, one approximates \( \mathbb{E} f(X) \) by \( \mathbb{E} f(\hat{X}^{(n)}_{\theta, \mu}) \) rather than \( \mathbb{E} f(\hat{X}^{(n)}) \) via

\[
\mathbb{E} f(\hat{X}^{(n)}_{\theta, \mu}) = \sum_{i=1}^{n} p_{i, \mu} f(\alpha_{i}^{(n)}_{\theta, \mu})
\]

with \( p_{i, \mu} \) being the weight of the \( i \)-th Voronoi cell corresponding to the quantization sequence \( \alpha_{\theta, \mu}^{(n)} \) given by

\[
P(X \in W_{i}(\alpha_{\theta, \mu}^{(n)})) = \int_{W_{i}(\alpha_{\theta, \mu}^{(n)})} f(x) d\lambda_{d}(x) = \theta^{d} \int_{W_{i}(\alpha_{\theta, \mu}^{(n)})} f_{\alpha_{\theta, \mu}^{(n)}}(z) d\lambda_{d}(z) = P(\hat{X}^{(n)}_{\theta, \mu} \in W_{i}(\alpha^{(n)}))
\]

where we applied the change of variables \( z = \mu + \frac{x-\mu}{\theta} \). Then, since \( \|X - \hat{X}^{(n)}_{\theta, \mu}\|_{r} \) converges faster to 0 than \( \|X - \hat{X}^{(n)}\|_{r} \) for \( r > 2 \) if we consider an \( L^2 \)-quantizer \( \alpha^{(n)} \), one may expect to observe that

\[
|\mathbb{E} f(X) - \mathbb{E} f(\hat{X}^{(n)}_{\theta, \mu})| \leq |\mathbb{E} f(X) - \mathbb{E} f(\hat{X}^{(n)})|.
\]

To illustrate this numerically, we consider a one-dimensional example and approximate \( \mathbb{E} f(X) \), where \( X \) is a random variable with Normal distribution \( \mathcal{N}(0, 1) \) and \( f \) is defined on \( \mathbb{R} \) by \( f(x) = x^4 \sin(x) \) and satisfies (41) with \( \beta = 5 \). To satisfy (43), we choose \( r = 5 \) and implement the approximation by quadrature formulas based, on the one hand, on \( L^2 \)-optimal and greedy sequences \( \alpha^{(n)} \) and, on the other hand, on the \( L^2 \)-dilated optimal and greedy quantizer \( \alpha_{\theta, \mu}^{(n)} \) with \( \theta^{*} = \sqrt{\frac{5+d}{2+d}} = 2 \), which is \( L^r \)-rate optimal. The exact value of \( \mathbb{E} f(X) \) is 3. In figure 1, we illustrate the errors induced by these approximations and we observe that, for a same size \( n \) of the quantization sequence, the \( L^2 \)-dilated quantizers \( \alpha_{\theta, \mu}^{(n)} \) give more precise results than the standard sequences \( \alpha^{(n)} \) themselves.

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Figure 1: Errors of the approximation of $\mathbb{E} f(X)$, where $f(x) = x^4 + \sin(x)$, by quadrature formulas based on $L^2$ quantizers (blue) and dilated $L^2$ quantizers (red) for different sizes $n$.

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