Threshold of primordial black hole formation in modified theories of gravity

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It is believed that primordial black holes (PBHs), if they exist, can serve as a powerful tool to probe the early stage of the cosmic history. Essentially, in the radiation dominated universe, PBHs could form by the gravitational collapse of overdense primordial perturbations produced during inflation. In this picture, one important ingredient is the threshold of density contrast, which defines the onset of PBH formation. In the literature, most of the estimations of threshold, no matter numerically or analytically, are implemented in the framework of general relativity. In this paper, by performing analytic estimations, we point out that the threshold for PBH formation depends on the gravitational theory under consideration. In GR, the analytic estimations adopted in this paper give a constant value of the formation threshold, assuming a fixed equation of state. If the theory is characterized by additional mass scales other than the Planck mass, the estimated threshold of density contrast may depend on the energy scale of the universe at the time of PBH formation. In this paper, we consider the Eddington-inspired-Born-Infeld gravity as an example. However, this conclusion is expected to be valid for any gravitational theory characterized by additional mass scales, suggesting the possibility of testing gravitational theories with PBHs.

I. INTRODUCTION

The scrutiny of primordial black holes (PBHs) and their cosmological implications have been an intensive field of research over the last decades. Generically, PBHs would form from the gravitational collapse of overdense primordial density perturbations generated quantum mechanically during inflation. Essentially, quantum fluctuations generated during inflation exit the horizon and become classical before the end of inflation. Then, the comoving horizon shrinks and these density perturbations reenter the horizon. After horizon reentry, if the overdensity of the overdense region is too large, exceeding a certain threshold, the overdense region would collapse and eventually form PBHs. It can be expected that the whole process of PBH formation and its relevant observables would bring several important information about the history of the early universe [1–3], including the imprints of the inflationary model under consideration. Therefore, PBHs can be an important tool to probe the early universe, such as the extremely small scale perturbations which are not accessible by cosmic microwave background observations and the high energy physics up to grand unified scales [4, 5]. In addition, PBHs with a certain range of mass could be a possible dark matter candidate [6–9]. Also, the merger events of binary black hole systems observed by LIGO/Virgo collaborations indicate a population of black holes which are slightly too massive such that their origin cannot be well-explained by the current model of stellar evolution. In this regard and considering the possibility of late-time successive merging of PBHs [10], these massive black holes may originate from PBHs [11]. The possibility that PBHs may be created from cosmological phase transitions was proposed in Refs. [12–16]. See Ref. [17] for a nice review of PBH physics.

Based on the above picture of PBH formation, one can determine the observables such as the PBH mass function and abundance, which in principle can be constrained by observations. In determining these observables, the threshold of the density contrast over which the overdense region would collapse to form PBHs plays a very important role. In the literature, several numerical calculations of the PBH formation threshold have been carried out [18–25]. In Ref. [26], a pioneering analytic estimation of the threshold has been performed by using a very simple argument that the threshold is defined by comparing the proper size of the overdense region and its Jeans length. In [27], a more refined analytic estimation has been proposed and it has been shown that the refined analytic estimation agrees with numerical simulations qualitatively as compared with the estimation given in Ref. [26]. These analytic estimates are feasible because they are based on the assumption that the overdensity is uniform, i.e., the top-hat profile. Generically, the threshold would depend on the shape of the density profile under consideration [28–33], while in these cases only numerical calculations are feasible. Recently, the threshold of density contrast has been estimated when PBHs have spins [34, 35]. It has been shown that due to the repulsive property of the centrifugal force, the threshold would increase in the presence of PBH spins. This result suppresses the existence of high spin PBHs, which can be confirmed or falsified if the spin distribution of PBHs is observationally available in the future. The instability and the threshold corresponding to collapsing scalar fields have been studied in Refs. [36–37].

It should be emphasized that most works about the PBH formation mechanism and the calculations of the formation threshold are mainly within the framework of Einstein’s general relativity (GR). However, there are a
lot of modified theories of gravity which change the description of the early universe from the standard big bang cosmology. Some of them are motivated by the ambition to ameliorate the initial big bang singularity. One may also treat these modified theories of gravity as effective theories of a fundamental yet unknown quantum theory of gravity. See Refs. [38, 39] for reviews on modified theories of gravity.

Since physics of PBHs highly depends on the early history of our universe, it is expected that the mechanism of PBH formation also depends on the gravitational theories under consideration. This is similar to the fact that different inflationary models naturally give rise to distinct PBH observables. For example, the PBH formation and the threshold has been investigated under the framework of ad hoc bouncing cosmologies [40]. In Ref. [41], the mass function has been analyzed in non-standard cosmologies in which an intermediate stage with a stiff equation of state appears between the end of inflation and the radiation dominated era. The pair creation rate of PBHs in the \( f(R) \) gravity was investigated in Ref. [42]. Very recently, the primordial curvature perturbation and its influence on PBH formation has been studied in a subclass of scalar-tensor theories [43].

In this paper, we will relax the GR assumption and adopt the analytic approaches developed in Refs. [26, 27] to estimate the threshold of density contrast for PBH formation. We will particularly choose the Eddington-inspired-Born-Infeld (EiBI) gravity [44, 45] as an example. This theory reduces to GR in vacuum but deviates from it in the presence of matter fields. Most importantly, the EiBI gravity predicts a completely different history of the early universe as compared with that in GR. The initial big bang singularity can be avoided in different manners according to the sign of the Born-Infeld coupling constant [46]. In GR, the threshold obtained from the analytic estimations is constant, given a fixed equation of state in the background universe. In EiBI gravity, on the other hand, we will assume a radiation dominated universe and show that the threshold not only deviates from its GR counterpart, but also depends on the energy scale of the universe at the time of PBH formation. This energy scale dependence results from the existence of an additional mass scale characterizing the theory. We expect that such a dependence would happen for any theory with additional mass scales other than the Planck mass.

This paper is outlined as follows. In Sec. [II] we will start with a brief review of the EiBI gravity. Then we will adopt two analytic approaches to estimate the threshold of the density contrast within the EiBI scenario. In Sec. [III], we will discuss how the PBH mass function and abundance are affected through the changes of the threshold in different gravitational theories. We finally draw our conclusions in Sec. [IV].

II. THRESHOLD OF PBH FORMATION IN EIBI GRAVITY

In order to estimate the threshold \( \delta_{th} \) of the density contrast for PBH formation, we first assume that the overdense region has uniform overdensity and can be approximately described by a part of the closed Friedmann universe [26, 27]. The metric describing the overdense region reads

\[
\text{ds}^2 = -dt^2 + a^2(t) \left( d\chi^2 + \sin^2 \chi d\Omega_2^2 \right),
\]  

(2.1)

where the scale factor \( a(t) \) is a function of the cosmic time \( t \). In this setup, the overdense region corresponds to the region where \( 0 \leq \chi \leq \chi_a \). The areal radius of the overdense region is given by

\[
R_a \equiv a \sin \chi_a.
\]  

(2.2)

The proper size of the overdense region is assumed to be on super-Hubble scales initially. With the expansion of the universe, the overdense region would reenter the Hubble horizon at the time when the areal radius of the overdense region is equal to the Hubble horizon \( 1/H \) (in the uniform Hubble slicing):

\[
\frac{1}{H_{hc}} = a_{hc} \sin \chi_a,
\]  

(2.3)

where the subscript “hc” stands for quantities at the time of horizon reentry. In addition, the overdense region itself would undergo a transition from the expansion phase to the contraction phase. The scale factor reaches \( a_m \) at the maximum expansion time. From now on, we will use the subscript “m” to represent quantities at the maximum expansion time.

In Refs. [26, 27], the threshold \( \delta_{th} \) for PBH formation in the GR framework was estimated based on two seemingly similar, but conceptually different arguments, respectively. In Ref. [26], the threshold was obtained from a simple argument that the proper size of the overdense region at the maximum expansion time should be larger than the Jeans length in order to initiate the gravitational collapse. Assuming that the universe is dominated by a perfect fluid with a constant equation of state \( w \), the estimated threshold is \( \delta_{th} = w \) [26]. On the other hand, a more refined approach was proposed in [27] and the threshold was estimated by requiring that the sound crossing distance by the maximum expansion time should be smaller than the proper size of the overdense region for the onset of the collapse. The threshold was estimated as \( \delta_{th} = \sin^2 \left[ \pi \sqrt{w/(1 + 3w)} \right] \) and it was shown in [27] that this estimation is in much better agreement with the numerical simulations as compared with the estimation given in Ref. [26]. In the following subsections, we will use the two analytic estimations mentioned above in different gravitational theories. As a specific example, we will consider the EiBI gravity and show how the threshold value is affected by different descriptions of the early universe.
A. EiBI gravity: A brief review

Before starting the estimation, we will briefly review the EiBI gravity and introduce the modified Friedmann equation describing a closed universe, which is interpreted as the overdense region embedded in a background flat universe. The action of the EiBI gravity [44, 45] is given by

$$S = \frac{1}{\kappa} \int d^4x \left( \sqrt{|g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|} - \sqrt{-g} \right) + S_m,$$

(2.4)

where $\kappa$ is the Born-Infeld coupling constant (we have assumed $8\pi G = c = 1$). Note that $\kappa$ is dimensional and it can be either positive or negative. The EiBI theory is formulated with the Palatini variational principle in which the metric and the affine connection are treated independently at the Lagrangian level [44]. The Ricci tensor is constructed from the affine connection and only the symmetric part of $R_{\mu\nu}$ is considered. Due to the Born-Infeld structure (the square root) inherent in the theory, the big bang singularity can be avoided for a radiation dominated universe. Therefore, the description of the early universe is significantly different from that in GR. We expect that this would give rise to different physics of PBH formation.

Assuming that the universe is dominated by a perfect fluid with energy density $\rho$ and a constant equation of state $w$, the modified Friedmann equation describing a closed universe can be written as

$$6\kappa H^2 = X(w, \bar{\rho}) \frac{6\kappa}{a^2} Y(w, \bar{\rho}),$$

(2.5)

where $H$ is the Hubble rate and

$$X(w, \bar{\rho}) = \left[ 1 + \frac{2(1-w\bar{\rho})}{1+\bar{\rho}} - 3 \left( \frac{1-w\bar{\rho}}{1+\bar{\rho}} \right)^3 \right],$$

$$Y(w, \bar{\rho}) = \left[ 1 + \frac{3\bar{\rho}(1+w)(2w\bar{\rho}+w-1)}{(1-w\bar{\rho})(1+\bar{\rho})} \right]^2.$$  

(2.6)

Here $\bar{\rho} \equiv \kappa \rho$ stands for the dimensionless energy density. Note that $\bar{\rho}$ can also be negative because $\kappa$ can be negative as well. When the energy density is small, i.e., $|\bar{\rho}| \ll 1$, the Friedmann equation in GR is recovered:

$$H^2 \approx \frac{\rho}{3} - \frac{1}{a^2} \quad \text{when } |\bar{\rho}| \ll 1.$$  

(2.7)

According to Eq. (2.5), it can be shown that the evolution of the early universe in the EiBI gravity is quite different from that in GR. More explicitly, when the energy density is large, i.e., $|\bar{\rho}| = O(1)$, the Born-Infeld corrections become significant. One of the important features in the EiBI gravity is its capability of removing the big bang singularity in a purely classical level, without any violation of energy conditions.

In the picture of PBH formation, some mechanisms, such as inflation, are needed to generate initial quantum fluctuations seeding the density perturbations in the universe. The possibility of regarding the EiBI theory as an alternative to inflation has been discussed in Ref. [44], in which the authors showed that the fundamental problems in standard cosmology, such as the flatness problem and the horizon problem may be solved in the EiBI gravity. Additionally, the quadratic inflationary paradigm in the EiBI gravity coupled with a scalar field (inflaton) has been proposed in [48] and studied in Refs. [49–53]. This inflationary model allows not only the generation of primordial quantum fluctuations we need for PBH formation, but also allows a well-defined pre-inflationary stage without the need of quantum gravity. Other interesting models based on the Born-Infeld gravity which allow viable inflationary scenarios can be found in Refs. [54, 55].

B. PBH formation in EiBI gravity: Simple estimation

In order to estimate the threshold of density contrast for PBH formation, we assume that in the overdense region, the overdensity is uniform and is given by $\bar{\rho}(1+\delta)$, such that $\bar{\rho}$ stands for the background energy density. Replacing all the $\bar{\rho}$ in the modified Friedmann equation [2.5] with $\bar{\rho}(1+\delta)$ and considering the contributions from $\delta$ up to the first order, the modified Friedmann equation can be expanded as follows

$$6\kappa H^2 = X(w, \bar{\rho}) \frac{6\kappa}{a^2} Y(w, \bar{\rho})$$

$$+ \left[ \bar{X}(w, \bar{\rho}) - \frac{6\kappa}{a^2} \bar{Y}(w, \bar{\rho}) \right] \delta + O(\delta^2),$$

(2.8)

where $X$ and $Y$ are some functions of $w$ and $\bar{\rho}$. Due to their lengthy expressions, we are not going to include their expressions explicitly in this paper.

Considering the time of horizon reentry and using the uniform Hubble slicing in Eq. (2.8), one can obtain

$$\frac{Y_{\text{hc}}}{a_{\text{hc}}^2 H_{\text{hc}}^2} = \left( \frac{\bar{X}_{\text{hc}}}{X_{\text{hc}}} - \frac{\bar{Y}_{\text{hc}}}{a_{\text{hc}}^2 H_{\text{hc}}^2} \right) \delta_{\text{UH}}^\text{UH},$$

(2.9)

where $\delta_{\text{UH}}^\text{UH}$ represents the density contrast at the horizon reentry in the uniform Hubble slicing. Combining with

1 In the original EiBI proposal, there is a dimensionless parameter $\lambda = \kappa \Lambda + 1$ quantifying the effective cosmological constant at low curvature scales. Here we will assume $\Lambda = 0$ since its contribution can be neglected in the early universe.

2 In general, the density contrast $\delta$, in terms of PBH formation, is not necessary to be small. We consider the contribution of $\delta$ only up to the first order just for the sake of simplicity.
As mentioned in Eq. \((2.11)\), the estimation of the threshold \(\delta_{th}\) for PBH formation is based on the requirement that the proper size of the overdense region at the time of maximum expansion is larger than the Jeans length \(R_J\). In Ref. \([26]\), the Jeans length was approximated as \(R_J \approx c_s \sqrt{3}/\rho_m \approx 3\sqrt{3}\rho/\rho_m\). In this regard, the formation of PBHs can happen when

\[
3\omega\rho_m \approx a_m^2 \sin^2 \chi_a = a_m^2 \left(\frac{\dot{X}_{hc}}{X_{hc} Y_{hc}}\right) \delta_{hc}^{\text{th}}, \quad (2.12)
\]

where Eq. \((2.10)\) has been used in the last equality. If we assume that PBHs form in a regime where \(\rho \ll 1\), we can expand the equation in terms of \(\rho\) up to quadratic order. In a radiation dominated universe where \(w = 1/3\), we obtain

\[
\delta_{hc}^{\text{th}} > \delta_{th} = \left(1 + \frac{4}{27}\rho_m^2\right) \left(1 + \frac{28}{27}\rho_{nc}\right) \frac{1}{3}. \quad (2.12)
\]

It can be seen that the threshold deviates from its GR counterpart \(\delta_{th} = \omega = 1/3\). Furthermore, one can also see that the threshold depends on the background energy density at the time of PBH formation. This is due to the presence of the additional dimensional coupling constant \(\kappa\) in the theory.

Apparently, the estimation of \(\delta_{th}\) here highly depends on the approximation of the Jeans length in Eq. \((2.11)\). In fact, as has been pointed out in Ref. \([27]\), the choice of the Jeans length here could lead to some ambiguities about a numerical factor of order one. In GR, this is one of the reasons why the estimation of the threshold \(\delta_{th} = \omega = 1/3\) is far consistent with that obtained from numerical simulations. Therefore, the threshold obtained in Eq. \((2.12)\) in the EiBI gravity is not expected to be accurate either. Actually, according to Eq. \((2.12)\) one may naively conclude that the threshold of density contrast for PBH formation seems to be enhanced in the EiBI gravity, irrespective of the sign of \(\kappa\). However, as we have just mentioned, this result is based on several assumptions and approximations, including an ambiguous choice of the Jeans length. This means that whether the threshold would be enhanced or not may depend on the assumptions one has made during the estimation. What we can really conclude in this stage is that the estimated threshold for PBH formation depends on the description of early universe, as well as on the underlying gravitational theories. Specifically, the threshold would depend on the energy scale at the time of PBH formation.

In the next subsection, we will estimate the threshold \(\delta_{th}\) by using the refined method proposed in Ref. \([27]\). This new analytic method has been shown to be more consistent qualitatively with the results obtained from numerical simulations. We will show that in this refined approach, the threshold indeed not only differs from its GR counterpart, but also changes with respect to the energy scale at the time of PBH formation.

### C. PBH formation in EiBI gravity: Refined estimation

As has been mentioned above, the estimation of the threshold \(\delta_{th}\) for PBH formation carried out in \([26]\), and that in the previous subsection, relies on the choices of the Jeans length and on several additional assumptions. In Ref. \([27]\), a more refined estimation of the threshold was proposed based on the comparison between the sound crossing distance and the proper size of the overdense region at the maximum expansion time. It was shown that for a spherically symmetric overdense region with uniform oversensitivity, the threshold is estimated as \(\delta_{th} = \sin^2 [\pi \sqrt{\omega/(1 + 3\omega)}]\) and it fits well qualitatively with that obtained from numerical simulations.

In this subsection, we will apply this refined approach to estimate the threshold \(\delta_{th}\) in the EiBI gravity. It should be noticed that the early universe in the EiBI gravity evolves very differently as compared with the standard big bang scenario in the sense that the big bang singularity can be replaced either with a primordial bouncing phase or a loitering phase, depending on the sign of the Born-Infeld coupling constant \(\kappa\). When calculating the sound crossing distance, one has to integrate the infinitesimal distance from the beginning of the universe to the maximum expansion time. Therefore, the sound crossing distance could be significantly different in different cosmological scenarios, hence leading to an estimated threshold \(\delta_{th}\) completely distinct from its GR counterpart.

Similar to what we have done in the previous subsection, we first assume that the overdense region has uniform overdensity and can be described by a part of the closed Friedmann universe \((2.1)\). The modified Friedmann equation in the EiBI gravity is given by Eq. \((2.5)\).

In a radiation dominated universe where \(w = \omega = 1/3\), the comoving sound crossing distance is given by

\[
\int_0^{\chi_s} d\chi = \chi_s = \sqrt{w} \int_{t_i}^{t_s} \frac{dt}{a} = \frac{\sqrt{w}}{3} \int_{\rho_m}^{\rho} \frac{d\rho}{\rho(1+w)\sqrt{(\frac{\rho}{\rho_m})^{\omega+1} \frac{Y_{X_{hc}}}{X_{hc}} - Y}}. \quad (2.13)
\]

where \(X = X(w, \rho)\) and \(Y = Y(w, \rho)\) are defined by Eq. \((2.6)\). The subscript “i” represents the quantities at the time when the sound wave starts propagating.
If the universe starts from a big bang singularity, the initial stage corresponds to \( t_i = 0 \) and \( \rho_i = \infty \). However, in the EiBI gravity, the energy density is bounded from above and the big bang singularity is replaced with either a bouncing scenario when \( \kappa < 0 \) or a loitering stage when \( \kappa > 0 \). For the bouncing scenario, the bounce happens when \( \bar{\rho} = -1 \) and it connects the contracting phase and the expanding phase of the universe. For the loitering scenario, the universe starts with a minimum size at the infinite past and the energy density is bounded by a maximum value \( \bar{\rho} = 1/\bar{w} \). The integral in Eq. (2.13) does not allow an analytic expression. Therefore, we will assume \( w = 1/3 \) and obtain \( \chi_\delta \) numerically. For the bouncing scenario, we will assume that the sound wave starts propagating at the bounce where \( \bar{\rho}_i = -1 \). As for the loitering scenario, on the other hand, we will assume that the sound wave starts propagating at a finite cosmic time in the past. In fact, the integration in Eq. (2.13) does not converge if we assume \( \bar{\rho}_i = 1/\bar{w} = 3 \). This means that the sound wave starts propagating in the infinite past and the wave can cross through an infinitely long distance. This does not make physical sense for a standard mechanism of PBH formation. Therefore, we will assume \( \bar{\rho}_i = 3 - \Delta \), where \( \Delta \gg 0 \), when numerically calculating \( \chi_\delta \).

After obtaining the sound crossing distance \( \chi_\delta \), the threshold \( \delta_{\text{th}} \) for PBH formation can be derived by requiring that the proper size of the overdense region at the time of maximum expansion is larger than the sound crossing distance, i.e., \( \chi_\alpha > \chi_\delta \). Using Eq. (2.10), we obtain

\[
\delta_{\text{th}} = \left( \frac{X_{hc}Y_{hc}}{X_{hc}} \right) \sin^2 \chi_\alpha > \delta_{\text{th}} = \left( \frac{X_{hc}Y_{hc}}{X_{hc}} \right) \sin^2 \chi_\delta . \tag{2.14}
\]

Finally, all the quantities at the time of horizon reentry, such as \( X_{hc}, Y_{hc}, \) and \( \bar{X}_{hc} \) appearing on the right-hand side of Eq. (2.14), can be expressed in terms of \( \sin \chi_\alpha \) and the quantities at the maximum expansion time. This can be done by using the following equation

\[
\sin^2 \chi_\alpha = \frac{X_m/Y_m}{\left( \frac{\rho_{hc}}{\rho_m} \right)^2 X_{hc} - Y_{hc} \left( \frac{\bar{X}_m}{\bar{Y}_m} \right)} . \tag{2.15}
\]

One can use Eq. (2.15), inserting the value of \( \chi_\delta \) into \( \chi_\alpha \), to solve the ratio \( \rho_{hc}/\rho_m \). The factor \( X_{hc}Y_{hc}/\bar{X}_{hc} \) in Eq. (2.14) can be expressed in terms of this ratio.

In FIG. 1, we show the threshold \( \delta_{\text{th}} \) of the density contrast for PBH formation estimated in the EiBI gravity. The dashed (dotted) curve corresponds to a negative (positive) Born-Infeld coupling constant \( \kappa \). For the case of positive \( \kappa \), we have assumed \( \Delta = 10^{-6} \). It can be shown that if the value of \( \Delta \) is smaller, the sound wave starts propagating earlier and the threshold for PBH formation would be larger. According to this figure, it can be seen that the threshold estimated in the EiBI gravity not only depends on the sign of the Born-Infeld coupling constant \( \kappa \), it also depends on the energy scale \( \bar{\rho}_m \) at the time of PBH formation. If the density contrast starts to collapse quite late such that the background energy density is low enough at the onset of PBH formation, the threshold value estimated in GR \cite{27} can be recovered. On the other hand, if the density contrast starts collapsing in an early time in which the deviation between GR and EiBI gravity is significant, the threshold could be significantly different from its GR counterpart.

### III. PBH Mass Function and Abundance

In the theory of PBH formation, the value of the threshold \( \delta_{\text{th}} \) plays a very important role in determining the mass function as well as the abundance of PBHs. One of the important physical quantities is the fraction \( \beta(M) \) of the universe, which forms PBHs at a certain time, with their initial mass larger than a chosen \( M \). In the literature, the estimation of this fraction is commonly carried out by applying the Press-Schechter formalism \cite{56}. In this formalism, the fraction \( \beta(M) \) can be written as

\[
\beta(M) = \int_{\delta_c(M)}^{\infty} P(\delta(M))d\delta(M) , \tag{3.1}
\]

where \( \delta(M) \) is the smoothed density contrast defined by the convolution between the density contrast and an appropriate window function. Usually, these quantities are defined on a comoving slicing. The lower bound of the integral \( \delta_c(M) \) corresponds to the formation threshold of the density contrast on the same slicing. Generically, the threshold \( \delta_c(M) \) can depend on the mass scale under consideration. Furthermore, \( P(\delta(M)) \) is the probability density distribution of the density contrast.

Assuming the probability density distribution to be

\[
\text{FIG. 1. The threshold } \delta_{\text{th}} \text{ of the density contrast for PBH formation estimated by using the refined approach} \text{. The dashed (dotted) curve corresponds to a negative (positive) } \kappa \text{. We have assumed } \Delta = 10^{-6} \text{ in the case of positive } \kappa \text{. The horizontal solid line represents the threshold value } \sin^2(\pi/2\sqrt{3}) = 0.62 \text{ obtained in GR. It can be seen that in the EiBI gravity, the threshold estimated based on this approach depends on the energy scale at the time of PBH formation.}
\]
gaussian, the fraction $\beta(M)$ can be written as

$$
\beta(M) = \int_{\delta_c(M)}^{\infty} \frac{2}{\sqrt{2\pi} \sigma^2(M)} \exp \left( -\frac{\delta^2(M)}{2\sigma^2(M)} \right) d\delta(M)
$$

$$
\approx \sqrt{\frac{2\sigma^2(M)}{\pi \delta_c^2(M)}} \exp \left( -\frac{\delta_c^2(M)}{2\sigma^2(M)} \right), \quad (3.2)
$$

where $\sigma(M)$ is the variance of $\delta(M)$. In the second line of Eq. (3.2), we have assumed $\delta_c(M) \gg \sigma(M)$ in the case of interest. In principle, the estimation would give different results if we consider different primordial curvature power spectra. Note that the primordial curvature power spectra depend on the variance. The point is that a few changes of the formation threshold $\delta_c$ could largely alter the estimation of the fraction $\beta(M)$. This can be seen, for example, in the Figure 4 of Ref. [57]. Therefore, the possibility that the threshold value depends on the underling gravitational theory should not be overlooked.

In fact, the estimated fraction $\beta(M)$ would also depend on which formalism we are choosing to estimate $\beta(M)$. If one considers the peaks theory [58], rather than the Press-Schecher formalism, to do the estimation, the result would be different, although it has been shown that the result is less sensitive to the use of different formalism, compared with the change of the threshold value [57, 59]. Also, it should be emphasized that the peaks theory and the Press-Schecher formalism have both assumed that PBHs with different masses are formed at the same time. However, in principle PBHs with different masses correspond to different moments of horizon reentry, hence different formation time. In modified theories of gravity, as we have shown, the threshold $\delta_c$ would even depend on the background energy scale at the time of PBH formation (even considering the top-hat profile). In order to have a more accurate estimation, therefore, some revised formalisms should be taken into account. See Ref. [60] for a recent related development.

IV. CONCLUSIONS

In this paper, we show explicitly that the threshold of density contrast for PBH formation depends on the gravitational theories under consideration. The estimated value of the threshold differs from its GR counterpart when considering gravitational theories which modify the early time description of the universe. The threshold estimations can be performed analytically by assuming that the overdensity is homogeneous in the overdense regions eventually collapsing to form PBHs. In GR, given a constant equation of state $w$ in the background universe, the analytic estimations give a constant threshold depending only on $w$. However, in modified theories of gravity with additional characteristic mass scales, the threshold would not only deviate from its GR counterpart, but also depend on the energy scale at the time of PBH formation. Although we have only considered the EiBI gravity in this paper, we expect that similar conclusions can be drawn in other gravitational theories with additional mass scales.

Since the threshold of density contrast for PBH formation plays a very crucial role in determining the mass function and abundance of PBHs, we can conclude that the changes of gravitational theories and inflationary models would affect the PBH mass function not just through the primordial power spectrum, they may modify the threshold value and largely change the predictions of PBH mass function. It can be seen that the prediction of PBH mass function depends on several physical mechanisms, which implies that there would be huge degeneracy in the parameter space. However, if there are other observational constraints which are able to break this degeneracy, it is expected that PBH physics could be another powerful tool to test gravitational theories in the future. This is similar to the ability of PBH physics in probing the dynamics of our early universe. For example, the Born-Infeld coupling constant $\kappa$ in the EiBI gravity is able to be constrained from solar system [61], cosmology [62], nuclear physics [63, 64], and recently from the speed of gravitational waves [65].

It should be emphasized that in this paper we only consider the analytic estimations of the threshold by assuming uniform overdensity, i.e., the top-hat profile. It is in fact well-known that the threshold value would also depend on the density profile under consideration [25–85]. In these scenarios, analytic estimation is not feasible anymore and one shall resort to numerical simulations. It would be interesting to implement a more rigorous threshold estimation in the framework of modified theories of gravity considering general overdensity profiles. We leave these issues for our future works.

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