Enhancing Branch-and-Bound for Multi-Objective 0-1 Programming

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Abstract:
In the bi-objective branch-and-bound literature, a key ingredient is objective branching, i.e. to create smaller and disjoint sub-problems in the objective space, obtained from the partial dominance of the lower bound set by the upper bound set. When considering three or more objective functions, however, applying objective branching becomes more complex, and its benefit has so far been unclear. In this paper, we investigate several ingredients which allow to better exploit objective branching in a multi-objective setting. We extend the idea of probing to multiple objectives, enhance it in several ways, and show that when coupled with objective branching, it results in significant speed-ups in terms of CPU times. We also investigate cut generation based on the objective branching constraints. Besides, we generalize the best-bound idea for node selection to multiple objectives and we show that the proposed rules outperform the, in the multi-objective literature, commonly employed depth-first and breadth-first strategies. We also analyze problem specific branching rules. We test the proposed ideas on available benchmark instances for three problem classes with three and four objectives, namely the capacitated facility location problem, the uncapacitated facility location problem, and the knapsack problem. Our enhanced multi-objective branch-and-bound algorithm outperforms the best existing branch-and-bound based approach and is the first to obtain competitive and even slightly better results than a state-of-the-art objective space search method on a subset of the problem classes.

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1 Introduction

In many real-world problem situations, decision makers have to consider several different objectives simultaneously, such as, e.g., travel times, costs, and CO2 emissions. These objectives are often conflicting, which means that the optimal solution for one of the objectives is often not optimal for the others. Instead, one may search for all the optimal trade-off solutions. For this purpose, a multi-objective optimization problem is solved. We focus here on solving Multi-Objective Integer Linear Problems (MOILP).

A MOILP can be solved using an Objective Space Search (OSS) algorithm, which consists of solving a series of single-objective problems obtained by scalarizing the objective functions so that all optimal trade-offs are enumerated (Ehrgott, 2005). The main advantage of this methodology is that the power of single-objective solvers can be used. Consequently, objective space search algorithms have received much attention over the past decades (see e.g., Ulungu and Teghem (1995); Visée et al. (1995); Sylva and Crema (2004); Ozlen et al. (2014); Kirlik and Sayın (2014); Boland et al. (2017); Boland and Savelsbergh (2016); Tamby and Vanderpooten (2021)).

Alternatively, a MOILP can be solved using a Decision Space Search (DSS) algorithm, typically a Multi-Objective Branch & Bound (MOBB) algorithm. Almost all recent contributions in this area address the bi-objective case (Adelgren and Gupte, 2022; Gadegaard et al., 2019; Parragh and Tricoire, 2019; Stidsen et al., 2014), and they all rely on an efficient branching scheme that creates sub-problems in the objective space. Forget et al. (2020b) have recently generalized this scheme to problems with more than two objectives. However, in its straightforward form, the speed-ups observed for the two-objective case did not translate to the three or more objective case. In this paper, we take Forget et al. (2020b)’s work as the starting point and of we propose an enhanced MOBB framework designed to solve MOILP with three or more objective functions. In order to improve the performance of the MOBB, we generalize the idea of probing to the multi-objective case, which allows us to better exploit the constraints generated by the branching scheme. Probing is a technique successfully employed in single objective branch-and-bound to locally reduce the domains of the decision variables. In a 0-1 integer context, this results in fixing variables to either 0 or 1 (Savelsbergh, 1994). To the best of our knowledge, probing has not been generalized to more than two objectives. Moreover, we investigate whether a decrease in CPU times can be achieved by deriving stronger cuts from the constraints generated by the objective branching scheme. Then, we investigate new node selection rules based on the best-bound principle, and compare them to the traditional breadth and depth-first strategies typically used in the MOBB literature. Finally, we show through a computational study that the suggested improvements lead to a significant speed-up for the framework in terms of CPU time, and that our algorithm is competitive with recent OSS algorithms from the literature on some of the problem classes considered in this paper.
The paper is organized as follows. In Section 2, we present the notation and definitions used throughout the paper. In Section 3, we discuss related work, and in Section 4, we describe the basic MOBB framework used here. Sections 5 and 6 are dedicated to the main novelties of our framework, namely variable fixing, cut generation, and node selection rules. Finally, in Section 7, we present the computational study, and our conclusions in Section 8.

2 Definitions and notation

A MOILP with \( n \) variables, \( p \) objectives, and \( m \) constraints is written as follows:

\[
P : \quad \min \{ z(x) : x \in X \}
\]

where the \( p \) objective functions \( z(x) = Cx \) are defined by a \( p \times n \) matrix of objective coefficients \( C \). The feasible set \( X = \{ x \in \{0, 1\} : Ax \geq b \} \) is given by a \( m \times n \) matrix of constraint coefficients \( A \), and a right-hand-side vector \( b \) of size \( m \). The image of the feasible set in the objective space is \( Y := CX = \{Cx : x \in X\} \).

Since the objective function is vector-valued, the following operators are introduced to compare solutions. Let \( y^1, y^2 \in Y \). \( y^1 \) weakly dominates \( y^2 \) (\( y^1 \leq y^2 \)) if \( y^1_k \leq y^2_k \) for all \( k = 1, \ldots, p \). Besides, \( y^1 \) dominates \( y^2 \) (\( y^1 \leq y^2 \) and \( y^1 \neq y^2 \)). These relations also extend to sets of points. Let \( S^1, S^2 \subset \mathbb{R}^p \), we say that \( S^1 \) dominates \( S^2 \) if for all \( s^2 \in S^2 \), there exists \( s^1 \in S^1 \) such that \( s^1 \leq s^2 \). The set \( S^1 \) partially dominates \( S^2 \) if \( S^1 \) does not dominate \( S^2 \), but there is at least one \( s^2 \in S^2 \) such that there exists \( s^1 \in S^1 \) such that \( s^1 \leq s^2 \).

Derived from the dominance relations, the set of non-dominated points is defined as \( N = \{ y \in Y : \nexists y' \in Y, y' \leq y \} \). This notation can be extended to any set \( S \subset \mathbb{R}^p \), i.e. \( S_N = \{ y \in S : \nexists y' \in S, y' \leq y \} \). Moreover, we define the set \( \mathbb{R}^p_+ = \{ y \in \mathbb{R}^p : y \geq 0 \} \).

Ehrgott and Gandibleux (2007) introduced the notions of lower and upper bound sets for \( S_N \), \( S \subset \mathbb{R}^p \), and extended the concept of lower and upper bound to the multi-objective case. Let \( S^1, S^2 \subset \mathbb{R}^p_+ \), we define the operation \( S^1 + S^2 \) as the Minkowski sum, i.e., \( S^1 + S^2 = \{ s^1 + s^2 : s^1 \in S^1, s^2 \in S^2 \} \). Moreover, \( S^1 \) is \( \mathbb{R}^p_+ \)-closed if \( S^1 + \mathbb{R}^p_+ \) is closed, and \( \mathbb{R}^p_+ \)-bounded if there exists \( s \in \mathbb{R}^p \) such that \( S^1 \subset \{s\} + \mathbb{R}^p_+ \). The definition of Ehrgott and Gandibleux (2007) is recalled in Definition 1.

**Definition 1.** (Ehrgott and Gandibleux, 2007) Let \( S \subset \mathbb{R}^p \).

- A lower bound set \( L \) for \( S_N \) is an \( \mathbb{R}^p_+ \)-closed and \( \mathbb{R}^p_+ \)-bounded set such that \( S_N \subset L + \mathbb{R}^p_+ \) and \( L = L_N \).
- An upper bound set \( U \) for \( S_N \) is an \( \mathbb{R}^p_+ \)-closed and \( \mathbb{R}^p_+ \)-bounded set such that \( S_N \subset cl[\mathbb{R}^p_+ \backslash (U + \mathbb{R}^p_+)] \) and \( U = U_N \), where \( cl(\cdot) \) denotes the closure operator.
A particular lower bound set is the singleton \( \{ y^I \} \), where \( y^I \), called the ideal point, is defined by \( y^I_k = \min_{y \in Y} \{ y_k \} \). Similarly, one can define the upper bound set \( \{ y^N \} \) where \( y^N \), the nadir point, is such that \( y^N_k = \max_{y \in Y} \{ y_k \} \).

The linear relaxation of a MOILP \( P \) is the problem \( P^{LP} : \min \{ \{ z(x) : x \in X^{LP} \} \), where \( X^{LP} = \{ x \in [0, 1] : Ax \geq b \} \). The problem \( P^{LP} \) belongs to the class of Multi-Objective Continuous Linear Problem (MOCLP).

Ehrgott and Gandibleux (2007) showed that solving the linear relaxation yields a valid lower bound set.

Given an upper bound set \( U \), Klamroth et al. (2015) proposed an alternative description of the region \( \text{cl}[\mathbb{R}P \setminus (U + \mathbb{R}_P^n)] \) using the set of local upper bounds \( N(U) \). Let \( u \in \mathbb{R}P \), we define \( C(u) = \{ z \in \mathbb{R}P : z \leq u \} \). Using the definition of Klamroth et al. (2015), the set of local upper bounds \( N(U) \) is the set such that \( \bigcup_{u \in N(U)} C(u) = \text{cl}[\mathbb{R}P \setminus (U + \mathbb{R}_P^n)] \), and for all \( u^1, u^2 \in N(U), C(u^1) \nsubseteq C(u^2) \). The first condition makes sure that \( N(U) \) describes properly \( \text{cl}[\mathbb{R}P \setminus (U + \mathbb{R}_P^n)] \), whereas the second condition implies that there is no pair of local upper bounds such that one dominates the other, i.e., \( N(U) \) is of minimal size.

Given a lower bound set \( L \) and an upper bound set \( U \), we define the search region as the set \( L + \mathbb{R}_P^n \cap \text{cl}[\mathbb{R}P \setminus (U + \mathbb{R}_P^n)] \). The search region can be interpreted as the region of the objective space where non-dominated points are possibly located.

A weighted-sum scalarization \( P_\lambda \) of a MOILP \( P \) is a single-objective optimization problem where the objective function is a weighted sum of the objective functions of \( P \). Hence, given a weight vector \( \lambda \in \mathbb{R}_n \), the problem \( P_\lambda \) is written as \( P_\lambda : \min \{ \lambda z(x) : x \in \mathcal{X} \} \).

**Property 1.** (Ehrgott, 2005) Let \( P \) be a MOCLP. Its non-dominated set \( Y_N \) corresponds to the non-dominated part of a polyhedron, and for any weighted-sum scalarization \( P_\lambda \) with weight \( \lambda \in \mathbb{R}_n \), there is an extreme point of \( Y_N \) that is optimal for \( P_\lambda \).

Given a MOILP \( P \), since its linear relaxation \( P^{LP} \) is a MOCLP, Property 1 implies that the optimal solution of a weighted-sum scalarization of \( P^{LP} \) can be obtained by searching for the extreme point of the lower bound set that has the minimal weighted-sum of its objective values. This property will be exploited in Section 6.1.

3 Related work

To our knowledge, the first MOBB was proposed by Klein and Hannan (1982). In their algorithm, the authors use a single branching tree to solve a series of single-objective problems to generate all desired solutions. Later, Kiziltan and Yucaoglu (1983) proposed a framework that uses the minimal completion, which consists of setting variables to 0 or 1 depending on their objective coefficients, to generate lower bounds. The resulting solution is integer but is not necessarily feasible for the initial problem.
In the following decade, a lot of attention was paid to DSS approaches tailored to specific problems. We refer the reader to Ramos et al. (1998) and Visée et al. (1998) for studies on the minimum spanning tree problem and the knapsack problem, respectively. In the latter, the novelty lies in the fact that they use a branch-and-bound algorithm in the second phase of a two-phase method, a well-known OSS algorithm proposed by Ulungu and Teghem (1995). In other words, they embedded a DSS algorithm into an OSS algorithm, resulting in the first hybrid method.

In multi-objective optimization, the ideal point provides a straightforward lower bound set. The first to introduce more complex lower bounds in a DSS algorithm are Sourd and Spanjaard (2008). In their paper, the authors use a surface as a lower bound set, namely the convex relaxation. Thereafter, the linear relaxation, weighted-sum scalarizations, and the linear relaxations of weighted sum scalarizations have been widely used in a similar way (see e.g., Vincent et al. (2013), Stidsen et al. (2014), Belotti et al. (2016), Stidsen and Andersen (2018), Parragh and Tricoire (2019), Gadegaard et al. (2019), Adelgren and Gupte (2022)). Although all these studies focus on the bi-objective case, the separating hypersurface principle from Sourd and Spanjaard (2008) is also applicable in higher dimensions. Recently, Santis et al. (2020) generated hyperplanes to obtain lower bound sets for multi-objective convex optimization problems (thus including MOILP) with three or more objective functions, whereas Forget et al. (2022) proposed to solve the linear relaxation for MOILP using more than two objectives. In the latter, the authors emphasized the difficulties raised by adding a third objective function. Indeed, if a simple dichotomic search is sufficient to calculate the linear relaxation with two objectives, things become more difficult when more dimensions are considered. In their paper, the authors suggest using Benson’s outer approximation algorithm (Benson, 1998; Hamel et al., 2013; Löhne and Weißing, 2020) to compute the linear relaxation based lower bound set.

**Multi-Objective Mixed-Integer Linear Problems (MOMILP),** i.e., problems with both continuous and integer variables, have also received some attention in the MOBB literature. Mavrotas and Diakoulaki (1998) proposed a branch-and-bound framework that can handle MOMILP, as well as an improved version of their algorithm in Mavrotas and Diakoulaki (2005). Later, Vincent et al. (2013) proposed a refined version of their framework for the bi-objective 0-1 case. The use of MOBB to solve bi-objective MOMILP was further studied by Belotti et al. (2016) and Adelgren and Gupte (2022).

In their paper, Vincent et al. (2013) also conducted a study of different node selection rules. They tested depth-first and breadth-first strategies on randomly generated instances, and depth-first appeared to be the most efficient. Similarly, Parragh and Tricoire (2019) tested both approaches on a different set of instances, but breadth-first performed the best. This suggests that the performance of the two classical node selection rules used in the literature, namely depth-first and breadth-first, are, in fact, dependent on the problem class. A similar observation was made in the preliminary study of Forget et al. (2022), where both rules resulted in very different
CPU times depending on the problem class of the instance solved. This issue is addressed in Sections 6 and 7.

In the past decade, a lot of attention has been paid to methods that hybridize DSS and OSS algorithms. For the bi-objective case, Stidsen et al. (2014) proposed partitioning the objective space into multiple slices, leading to stronger upper bound sets. This also opened the door to parallelization, which was exploited in (Stidsen and Andersen, 2018), and resulted in promising improvements in performance. The authors also developed the concept of Pareto branching (or objective (space) branching): when the upper bound set partially dominates the lower bound set, it is possible to create disjoint sub-problems in the objective space by adding upper bounds on the objective functions to discard dominated regions from the search. This principle was further explored and improved independently by Gadegaard et al. (2019) and Parragh and Tricoire (2019). In both papers, their experiments showed the great efficiency of this technique for the bi-objective case. Later, Adelgren and Gupte (2022) also showed promising results using objective branching in MOBB applied to bi-objective MOMILP.

Forget et al. (2020a) extended objective branching to the multi-objective case. In their paper, the authors highlighted several challenges that arise when three or more objective functions are considered. In particular, they established that generating sub-problems without redundancies is a much more complex task compared to the bi-objective case, and they proposed a new method to overcome these difficulties. As a consequence, although still beneficial, objective branching did not appear to be as efficient as in the bi-objective case in their computational study. In this paper, we improve this result by showing that combining objective branching with probing results in a significant speed-up.

Recently, Adelgren and Gupte (2022) have proposed to use probing to enhance their bi-objective branch-and-bound framework. The probing procedure of Adelgren and Gupte (2022) relies on solving the bi-objective linear relaxation based bound set, and showed promising results in their experiments. However, the impact of probing for problems with three or more objectives is unclear, as bound sets are more complex to compute. Furthermore, objective branching cannot be applied as often and easily as in the bi-objective case, which may also have an impact on the performance of probing.

4 Branch-and-bound framework

The branch-and-bound framework developed in this paper is based on the framework of Forget et al. (2022), and is presented in this section. Similarly to the single-objective case, the principle is to divide a problem that is too hard to be solved into easier sub-problems. Each sub-problem is stored in a node, and the nodes are grouped together into a tree data structure $T$. For each node $\eta$, the sub-problem contained in $\eta$ is called $P(\eta)$, and each subproblem of $P(\eta)$ is stored in a child node of $\eta$. Instead of single numerical values, lower and upper bounds sets are used to determine whether a given sub-problem potentially contains new non-dominated
Algorithm 1 Branch-and-bound algorithm for MOCOs

1: Create the root node $\eta^0$; set $\mathcal{T} \leftarrow \{\eta^0\}$ and $\mathcal{U} \leftarrow \emptyset$
2: while $\mathcal{T} \neq \emptyset$ do
3: Select a node $\eta$ from $\mathcal{T}$; Set $\mathcal{T} \leftarrow \mathcal{T} \setminus \{\eta\}$
4: Compute a local lower bound set for $P(\eta)$
5: If possible, update the upper bound set $\mathcal{U}$
6: if $\eta$ cannot be fathomed then
7: Split $P(\eta)$ into disjoint subproblems $P(\eta^1), \ldots, P(\eta^b)$, and store each in a unique child node of $\eta$ and add them to $\mathcal{T}$.
8: end if
9: end while
10: return $\mathcal{U}$

solutions, which are feasible for the initial problem. If not, the corresponding node is fathomed. Otherwise, it is divided into disjoint sub-problems. A general outline of our framework is presented in Algorithm 1.

The branch-and-bound is initialized with an empty upper bound set $\mathcal{U}$, and a list of non-explored nodes, denoted by $\mathcal{T}$, that contains the initial problem (at the root node of the tree). At each iteration, a node $\eta$ is selected and removed from $\mathcal{T}$ (Line 3 of Algorithm 1). The classical tree exploration strategies from the literature are depth-first (last in first out) and breadth-first (first in first out), and are further discussed in Section 7. If there is no non-explored node remaining, the algorithm stops and $\mathcal{U} = \mathcal{Y}_N$.

When a node $\eta$ is selected, a lower bound set $\mathcal{L}(\eta)$ for $P(\eta)$ is computed (Line 4 of Algorithm 1). In this framework, the linear relaxation $P^{LP}(\eta)$ is solved, and the result yields a valid lower bound set (Ehrgott and Gandibleux, 2007). A Benson-type algorithm is used for this purpose (see, e.g., Hamel et al. (2013)). As a result, a description of $\mathcal{L}(\eta)$ in terms of its extreme points is obtained, as well as a description of $\mathcal{L}(\eta) + \mathbb{R}^P_\mathcal{E}$ in terms of its hyperplanes.

Once the lower bound set is obtained, if possible, new non-dominated points are harvested (Line 5 of Algorithm 1). Indeed, Benson’s algorithm returns a pre-image for each extreme point of $\mathcal{L}(\eta)$. Hence, any extreme point $l$ with an integer pre-image that is not dominated by any existing point in the upper bound set will be added to $\mathcal{U}$; and all points $y \in \mathcal{U}$ that are dominated by $l$ ($l \geq y$) are removed from $\mathcal{U}$.

We distinguish three cases in which a node can be fathomed:

(i) If $P^{LP}(\eta)$ is infeasible, no new non-dominated points are searched (i.e., Line 5 is skipped), and the node is fathomed by infeasibility. Indeed, similarly to the single-objective case, if $P^{LP}(\eta)$ is infeasible, $P(\eta)$ is also infeasible.

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(ii) If $\mathcal{L}(\eta)$ is made of a unique extreme point $l$ with an integer pre-image, all new points found in $P(\eta)$ will be dominated by the integer solution $l$ and consequently, $\eta$ is fathomed by optimality.

(iii) Finally, a third way to fathom a node exists: fathoming by dominance. This case happens when the lower bound set $\mathcal{L}(\eta)$ is dominated by the upper bound set $\mathcal{U}$. From the definitions, this situation is equivalent to saying that each feasible point of $P_L^P(\eta)$, and thus of $P(\eta)$, is dominated by at least one already known integer point $u \in \mathcal{U}$. Consequently, no new non-dominated point can be found in $P(\eta)$. In practice, if there exists no local upper bound $u \in N(\mathcal{U})$ such that $u \in \mathcal{L}(\eta) + \mathbb{R}_+^P$, then the node is fathomed by dominance. This dominance test was first introduced by Sourd and Spanjaard (2008), and used multiple times in the literature (see, e.g., Forget et al. (2022); Gadegaard et al. (2019)).

If the node $\eta$ cannot be fathomed, we resort to branching, and $P(\eta)$ is split into several sub-problems (Line 7 of Algorithm 1). To do so, objective branching is used first. This technique was initially introduced for the bi-objective case by Stidsen et al. (2014), further improved by Gadegaard et al. (2019) and Parragh and Tricoire (2019), and finally extended to the multi-objective case in Forget et al. (2020a). It consists of creating disjoint sub-problems in the objective space when the lower bound set is partially dominated by the upper bound set, with the purpose of discarding regions that cannot contain any new non-dominated points. The subproblems are created by adding constraints in the form $z(x) \leq s$, $s \in \mathbb{R}^P$, and the point $s$ is called super local upper bound. Objective branching will be further elaborated upon in Section 5.

Once objective branching is applied, a set of subproblems $\eta^1, \ldots, \eta^\gamma$ is obtained. Note that only one subproblem is obtained ($\gamma = 1$) if it is not possible to create two or more disjoint subproblems in the objective space. Then, for each of the $\gamma$ subproblems, decision space branching is performed. To do so, one variable $x_i$ is chosen, and two subproblems with the constraints $x_i = 0$ or $x_i = 1$ are created. The variable $x_i$ has to be a free variable, i.e., not fixed to a specific value in the current node by one of the previous branching decisions.

5 Objective branching induced enhancements

We now elaborate on the concept of objective branching and how it can be exploited to enhance our MOBB. Figure 1 depicts a situation where the lower bound set $\mathcal{L}(\eta)$ is partially dominated by the upper bound set $\mathcal{U}$, and the resulting search region is given by the hashed areas. Any part of the objective space that is not included in one of these areas cannot contain any feasible non-dominated point for $P(\eta)$. Objective branching consists in generating disjoint subproblems in a way such that as much of the region of the objective space dominated by $\mathcal{U}$ is discarded from all subproblems, without excluding any point of the search region from the sub-problems. In the example from Figure 1, this results in three subproblems, defined by three super local upper bounds depicted by the large circles.
Figure 1: The lower bound set $L(\eta)$, depicted by the solid line, is partially dominated by the upper bound set $U$, represented by the crosses. In this situation, it is possible to split the problem into three disjoint sub-problems in the objective space. Each sub-problem is highlighted by the hatched areas, and its corresponding super local upper bound is depicted by its closest large circle.

In the bi-objective case, multiple ways to compute the subproblems exist. Stidsen et al. (2014) and Gadegaard et al. (2019) generated new subproblems when the algorithm detected that one or several points of the upper bound set partially dominate the lower bound set, whereas Parraghand Tricoire (2019) kept track of the various non-dominated segments of the lower bound set and generated a subproblem for each. The two approaches are equivalent in the sense that exactly the same subproblems are generated with both methods.

Recently, Forget et al. (2020a) showed that the computation of the subproblems in the case where $p \geq 3$ was more complex but still possible. However, the increased complexity resulted in a less significant benefit of using objective branching when $p \geq 3$ compared to the case where $p = 2$. For some problem classes, it even resulted in worse computation times, which is in great contrast to the bi-objective case, where using objective branching systematically led to lower CPU times. As a result, it appears that when $p \geq 3$, the use of objective branching is not always sufficient by itself, and in this paper, we aim to study whether objective branching constraints can be further exploited to help reduce the total CPU time of the branch-and-bound framework.

5.1 Probing

In multi-objective optimization, an intuitive belief is that two points that are close to each other in the objective space are more likely to have similar pre-images than two points that are far away from each other. Figure 2 shows the set of non-dominated points of two tri-objective MOCO instances (one row for each instance). A blue point corresponds to a non-dominated point where the chosen variable $x_i$ takes value 0, whereas an orange
Figure 2: The first row depicts the set of non-dominated points for a tri-objective Uncapacitated Facility Location Problem with 4672 non-dominated points. The second row depicts the set of non-dominated points for a tri-objective Capacitated Facility Location Problem with 1912 non-dominated points. For each plot, a variable $x_i$ was chosen. A blue point is a non-dominated point where $x_i = 0$ whereas an orange point is a non-dominated point for which $x_i = 1$.

point corresponds to a non-dominated point such that $x_i = 1$. From these four pictures, it is clear that some problem classes have variables that take a particular value in certain parts of the objective space. Moreover, when applying objective branching, the algorithm reduces the search to particular regions of the objective space. Hence, based on the previous observation, it is possible that some variables cannot take specific values in certain subproblems. The process of identifying such values is commonly referred to as probing. In the following, we present our probing strategies.

5.1.1 A naive strategy

We explore at first a naive strategy. At node $\eta$, we define the set of variables fixed to 0 and 1 as $I^0(\eta) = \{i \in \{1, \ldots, n\} : x_i = 0\}$ and $I^1(\eta) = \{i \in \{1, \ldots, n\} : x_i = 1\}$ respectively. The set of free variables is $I^f(\eta) = \{1, \ldots, n\}\setminus (I^0(\eta) \cup I^1(\eta))$. With a little abuse of notation, we will consider that writing $x_i \in I^0(\eta)$ is equivalent to $i \in I^0(\eta)$. We will consider analogous statements for $I^1(\eta)$ and $I^f(\eta)$.

Let $x_i \in I^f(\eta)$ be a free variable at node $\eta$. Since we consider problems with binary variables only, the possible values for $x_i$ are 0 or 1. A first approach in order to check whether $x_i$ can take value $v \in \{0, 1\}$ in $P(\eta)$
is to solve the linear program $F(i, v) : \min \{0 \mid x \in X^P(\eta), \ x_i = v\}$. If $F(i, v)$ is not feasible, then $x_i$ cannot take value $v$. When both $F(i, 0)$ and $F(i, 1)$ are solved for $x_i$, there are four possible scenarios:

- Both $F(i, 0)$ and $F(i, 1)$ are feasible: nothing can be concluded about $x_i$, and thus, the variable remains a free variable;
- $F(i, 0)$ is feasible and $F(i, 1)$ is infeasible: $x_i$ is fixed to 0;
- $F(i, 0)$ is infeasible and $F(i, 1)$ is feasible: $x_i$ is fixed to 1;
- Both $F(i, 0)$ and $F(i, 1)$ are infeasible: there is no possible integer value for $x_i$. Thus, the node $\eta$ is fathomed by infeasibility.

When $x_i$ is fixed to 0, the set of free variables $I^f(\eta)$ is updated to $I^f(\eta) \setminus \{i\}$ since $x_i$ is not free anymore, and $I^0(\eta)$ becomes $I^0(\eta) \cup \{i\}$. Similarly, if $x_i$ is fixed to 1, then $I^f(\eta)$ and $I^1(\eta)$ become $I^f(\eta) \setminus \{i\}$ and $I^1(\eta) \cup \{i\}$ respectively. If all free variables are fixed to a particular value, the node is fathomed by optimality. Indeed, this situation implies that there is only one integer solution in $P(\eta)$, and no new non-dominated point can be reached in $P(\eta)$. The upper bound set is updated with the new point obtained by fixing all variables.

A first naive strategy is to solve $F(i, v)$ at each node $\eta$, for each free variable $x_i \in I^f(\eta)$, and for each possible value $v \in \{0, 1\}$. The approach is similar to Adelgren and Gupte (2022) in the sense that they also perform probing at each node. In their paper, the authors suggest to perform probing both before the computation of the linear relaxation, and when creating sub-problems. In the latter case, they apply probing after selecting a free variable to branch on, and change the branching variable if they conclude that the decision led to an infeasible problem. In this paper, we adopted a slightly different approach: we perform probing only after objective branching and before variable branching. In this way, we aim to reduce the set of branching candidates at each node, while still benefiting from the objective branching constraints. Indeed, we expect these to be the most constraining to the problem, since they restrict the search to a particular region of the objective space and thus, hopefully, reduce the possible values taken by the variables and help the algorithm to make an appropriate branching decision.

Another difference to Adelgren and Gupte (2022) is that when performing probing, they solve the bi-objective linear relaxation of the corresponding problem instead of solving a simple feasibility problem as we do. However, as we consider more objective functions, the linear relaxation becomes significantly more expensive to compute. Consequently, our approach requires at most one single-objective linear program to be solved for each variable and value, which, in the binary case, limits the maximum number of LPs to be solved to $2|I^f(\eta)|$ at each node.
5.1.2 An advanced strategy

The naive strategy can be improved. Indeed, it is possible in some cases to detect if \( F(i, v) \) is feasible without actually solving the linear program. For instance, if a solution that is feasible for \( F(i, v) \) is already known, there is no need to solve \( F(i, v) \). Such solutions can be collected, for example, from the extreme points of the lower bound set, or by keeping track of the solutions obtained from previously solved linear programs \( (F(j, v'), j \neq i, v' \in \{0, 1\}) \) in the current node.

Moreover, variables can be fixed by inspection. Let \( x_j \in \mathcal{I}^f(\eta) \) be a free variable, and \( \sum_{l=1}^{n} a_{il}x_l \leq b_i \) a constraint of the problem. By comparing \( a_{ij} \) to the maximal possible value of the right-hand side of the constraint, one may be able to conclude that \( x_i \) cannot take value 1. First, all variables fixed to 1 are considered as constants and will be used to adjust the right-hand side. Then, all free variables \( x_l \in \mathcal{I}^f(\eta) \) such that \( a_{il} \leq 0 \) will be temporarily fixed to 0 meanwhile, those where \( a_{il} < 0 \) will be temporarily fixed to 1. The constraint then becomes \( a_{ij}x_j \leq b - \sum_{l \in \mathcal{I}^f(\eta)} a_{il} + \sum_{l \in \mathcal{I}^f(\eta), a_{il} < 0, l \neq j} a_{il} \). Then, if \( a_{ij} > b - \sum_{l \in \mathcal{I}^f(\eta)} a_{il} - \sum_{l \in \mathcal{I}^f(\eta), a_{il} < 0, l \neq j} a_{il} \), the variable \( x_j \) can be fixed to 0. Similar rules can be used for constraints in the form \( \sum_{l=1}^{n} a_{il}x_l \geq b_i \) and \( \sum_{l=1}^{n} -a_{il}x_l \leq -b_i \). More complex rules could be used as well, but this is out-of-scope of this paper and thus, we will stick to these simple rules here.

We note that probing may be applied after variable branching as well as after objective branching. However, our experiments showed that it is most effective when used in conjunction with objective branching. A possible explanation relates to the example given in Figure 2: some variables may only take certain values in certain regions of the objective space and objective branching induces such regions. Hence, at node \( \eta \), we propose to apply probing only if an improvement in the objective branching constraints is observed compared to its parent node \( \hat{\eta} \). In other words, we perform probing only if \( s \leq \hat{s} \), where \( s \) and \( \hat{s} \) are the super local upper bounds defining the objective branching constraints in nodes \( \eta \) and \( \hat{\eta} \), respectively.

This advanced strategy is new compared to Adelgren and Gupte (2022), as we first aim at achieving the same results by solving less linear programs, and then suggest using probing only when it is expected to be the most relevant.

5.1.3 Combining probing and bounding

The linear program \( F(i, v) \) solved when applying the naive strategy does not have an objective function. However, using an objective function may provide us with additional information. For this purpose, we rely on adding a weighted sum objective function. Given a weight vector \( \lambda \in \mathbb{R}^p \), the linear program \( F(i, v, \lambda) : \min \{ \lambda z(x) \mid x \in \mathcal{X}^{LP}(\eta), x_i = v \} \) is solved, and the optimal value \( z^{v*} \) is obtained. By definition, when \( x_i \) is fixed to \( v \) in this sub-problem, all feasible solutions \( x \in \mathcal{X}(\eta) \cap \{ x_i = v \} \) are such that \( \lambda z(x) \geq z^{v*} \). Moreover, because of the objective branching constraints, all feasible solutions \( x \in \mathcal{X}(\eta) \cap \{ x_i = v \} \) are such that \( z(x) \leq s \).
(a) Fixing $x_i = 0$: the weighted-sum based LB set defined by $F(i, 0, \lambda)$ (dash-dotted line) and the objective branching constraints (OB) is only partially dominated by the current UB set $U$.

(b) Fixing $x_i = 1$: the weighted-sum based LB set defined by $F(i, 1, \lambda)$ (dash-dotted line) and the objective branching constraints (OB) is dominated by the current UB set $U$.

Figure 3: Both $F(i, 0, \lambda)$ and $F(i, 1, \lambda)$ are solved with $\lambda = (1, 1)$, resulting in the situations depicted in the left and right figures respectively. Given the objective branching constraints, it is concluded that $x_i$ cannot take value 1 and thus, $x_i$ is fixed to 0.

$s \in \mathbb{R}^p$. Hence, we can conclude that if there is no local upper bound $u \in N(U)$ such that $\lambda u \geq z^*v$ and $u \leq s$, then there is no feasible solution $x \in X(\eta) \cap \{x_i = v\}$ that can generate a new non-dominated point. In this case, $x_i$ is fixed to 1 if $v = 0$, or to 0 if $v = 1$. Note that the dominance test we employ here has, e.g., also been used by Stidsen and Andersen (2018); Stidsen et al. (2014) in a bi-objective context.

An example is given in Figure 3, where $\lambda = (1, 1)$ is used, and the programs $F(i, 0, \lambda)$ and $F(i, 1, \lambda)$ are solved. In the rightmost figure, $F(i, 0, \lambda)$ is feasible, and the weighted-sum resulted in a non-empty search region. On the contrary, in the leftmost figure, $F(i, 1, \lambda)$ is feasible but the weighted-sum is dominated by the upper bound set in the region considered. This implies that in this sub-problem, all integer solutions in which $x_i$ takes value 1 are dominated by at least one existing integer solution. Hence, there is no need to branch on $x_i$, and the variable can be fixed to 0.

This strategy is closer to the one proposed by Adelgren and Gupte (2022) for the bi-objective case in the sense that we also compute a lower bound set, namely the linear relaxation of a weighted-sum scalarization. However, our lower bound set is weaker than theirs (linear relaxation), but requires only one linear program to be solved.
5.2 Objective branching based cover inequalities

For an objective \( k \) that is minimized, an objective branching constraint has the form \( w^T x \leq b \), where \( w \) is given by the objective coefficients of \( z_k(x) \), and \( b \) is derived from the bound on objective \( k \) in the sub-problem at hand. One can observe that this constraint is in fact a knapsack constraint from which cover inequalities can be derived (see, e.g., Gu et al. (1998)).

Objective branching constraints are often expected to be binding constraints, as they are included to create disjoint sub-problems. Hence, for an objective \( k \) such that the objective branching constraint \( z_k^s \leq s^k \) is binding, there will be extreme points in the lower bound set whose \( k^{th} \) component will be equal to \( s^k \). In case such an extreme point is fractional, cover cuts can be generated to cut it from the lower bound set with the aim to move it closer to an integer point.

Let \( l \in \mathcal{L}(\eta) \) be an extreme point such that \( l_k = s_k \), and \( x^l \in X^{LP}(\eta) \) its pre-image. We define \( \mathcal{J}^{\text{max}}(l) = \{ j \in \{0, \ldots, n\} : x^l_j = 1 \} \) as the set of indices of the variables that take the maximum value for \( x^l \), i.e. value 1. Similarly, we define \( \mathcal{J}^{\text{mid}}(l) = \{ j \in \{0, \ldots, n\} : 0 < x^l_j < 1 \} \) as the set of indices of the variables that take a fractional value in \( x^l \), i.e. in the middle of the possible integer values. By definition, \( \sum_{j \in \mathcal{J}^{\text{max}}} c^k_j x^l_j + \sum_{j \in \mathcal{J}^{\text{mid}}} c^k_j x^l_j = s^k \) holds true. If \( x^l \) is fractional, i.e., \( \mathcal{J}^{\text{mid}} \neq \emptyset \), then \( \sum_{j \in \mathcal{J}^{\text{max}}} c^k_j + \sum_{j \in \mathcal{J}^{\text{mid}}} c^k_j > s^k \) also holds true because for all \( j \in \mathcal{J}^{\text{mid}} \), we have \( x^l_j < 1 \). Hence, all variables \( x^l_j \) such that \( j \in \mathcal{J}^{\text{max}} \cup \mathcal{J}^{\text{mid}} \) cannot simultaneously take value 1. Thus, \( \sum_{j \in \mathcal{J}^{\text{max}} \cup \mathcal{J}^{\text{mid}}} x^l_j \leq |\mathcal{J}^{\text{max}} \cup \mathcal{J}^{\text{mid}}| - 1 \) is an example of a cover inequality that can be generated from \( x^l \).

Of course, in many cases, different cover inequalities can be generated. These cuts can also be strengthened by using any of the available lifting procedures from the literature.

6 Node selection rules

In the MOBB literature, breadth-first and depth-first are the commonly employed node selection rules. Indeed, the fact that these rules are independent from the nature of the problem being solved constitutes a good reason to use such rules when expanding branch-and-bound methods to the multi-objective case. However, previous studies have shown that depth-first is significantly better for some problem classes, whereas breadth-first is better for others (see e.g., Forget et al. (2022); Parragh and Tricoire (2019); Vincent et al. (2013)). This inconsistency is problematic when building a generic solver as we do here, since it could easily lead to very poor performance in some cases.

In the single objective literature, the so-called best-bound strategy (and variations thereof) has shown to be of value (Linderoth and Savelsbergh, 1999). Its basic principle consists in exploring first the node that has the lowest lower bound value, as it constitutes the most promising area of the decision space. Unfortunately, in the
multi-objective case, it is often not a trivial task to determine which node has the best bound, since one may have a lower bound set that is better than the others in a particular region of the objective space, but worse in other regions.

In the following, we propose two rules based on the best-bound principle. In Section 6.1, we present a rule that searches for the best bound in a specific part of the objective space by using weighted-sum values. In Section 6.2, we define a rule that is based on gap measures between lower and upper bound sets.

6.1 Weighted-sum rule

A straightforward way to mimic the best-bound approach in a MOBB is to consider a weighted-sum scalarization, and to use the value of its linear relaxation as a measure of the quality of the lower bound set. Let \( \eta \) be a node of the tree, let \( \lambda \) be the weight vector used for the scalarization \( P_\lambda(\lambda) \), and \( z^* \) the optimal value of its linear relaxation \( P_{LP_\lambda}(\lambda) \). The score \( s(\eta) \) of the node \( \eta \) is then given by \( s(\eta) = z^* \), and the node with the lowest score is selected.

Note that this rule is equivalent to a best-bound strategy using a branch-and-bound to solve the problem \( P_\lambda \). Hence, translated into the context of MOBB, one can say that this strategy selects the node that is the most promising in direction \( \lambda \) first.

From a computational point of view, the score of a new node \( \eta \) has to be calculated at its creation, which, in the present framework, requires solving a single-objective linear program, namely \( P_{LP_\lambda}(\eta) \). However, using a simple re-ordering of the steps of Algorithm 1, it is possible to obtain the score of \( \eta \) without solving a linear program. Indeed, it is well known that all points of the non-dominated set of a multi-objective continuous linear problem correspond to an optimal solution of a weighted-sum scalarization (Ehrgott, 2005). In our context, this implies that at node \( \eta \), the score of \( \eta \) can be obtained by searching for the point \( l^* \in L(\eta) \) such that there is no other \( l \) for which \( \lambda l \leq \lambda l^* \). In other words, we search for the point of the lower bound set with the minimum weighted-sum value given the weight vector \( \lambda \). Fortunately, this point is given by an extreme point of \( L(\eta) \) (see Property 1), and only extreme points have to be checked. Hence, by computing the lower bound set at the creation of the node instead of when the node is selected, the score can be obtained at a very low cost. Note that whether the lower bound set is computed at the creation or at the selection of the node does not make a difference, as \( P_{LP_\lambda}(\eta) \) does not change. Furthermore, this also holds for the update of the upper bound set, that only depends on the solutions found in the lower bound set. However, this is not true for fathoming, and in particular, fathoming by dominance. Indeed, new feasible points may be found between the creation and the selection of a node, which may allow the node to be fathomed by dominance. Hence, Lines 4 and 5 are moved to after the creation of the node, and the computation of the score is performed as well, which results in the new framework given by Algorithm 2.
Algorithm 2 An alternative branch-and-bound algorithm for MOILPs using a best-bound strategy

1: Create the root node $\eta^0$; set $T \leftarrow \{\eta^0\}$ and $U \leftarrow \emptyset$
2: while $T \neq \emptyset$ do
3: Select the node $\eta$ with the best score from $T$; Set $T \leftarrow T\setminus \{\eta\}$
4: if $\eta$ cannot be fathomed then
5: Split $P(\eta)$ into disjoint subproblems $P(\eta^1), \ldots, P(\eta^h)$, and store each in a unique child node of $\eta$.
6: for $\hat{\eta} \in \{\eta^1, \ldots, \eta^h\}$ do
7: Compute a local lower bound set for $P(\hat{\eta})$
8: Update the upper bound set $U$
9: Compute the score $s(\hat{\eta})$ for $\hat{\eta}$
10: end for
11: end if
12: end while
13: return $U$

To conclude, this rule is very cheap and easy to compute. However, the drawback is that it is very representative in one direction only and neglects other regions of the objective space. For example, by using the weight vector $\lambda = (1, \ldots, 1)$, the rule is likely to find good solutions that are well balanced across all objectives more rapidly than good solutions that are very good in one of the objectives but bad for other objectives. However, the actual impact on the performance is unclear, and is further studied in Section 7.

6.2 Gap measure rule

Another way to adapt the idea of best-bound strategies to the multi-objective case is to compute a measure of the gap between the upper and lower bound set in each node. In this case, the node with the largest gap is explored first, as it describes either a promising area, or a region where very few feasible points have been discovered, and possibly many more remain to be discovered.

An intuitive way to compute the gap in a given node is to compute the hypervolume of the search area. However, it is well known that it is a costly and difficult operation, particularly when three or more dimensions are considered. Hence, alternative measures are necessary. When Ehrgott and Gandibleux (2007) introduced the concept of lower and upper bound sets, they also proposed a number of measures to compare the quality of lower and upper bound set. One of these measures is similar to the Hausdorff distance, and consists in computing the minimal distance between the two points from each set respectively that are the furthest away. Recently, this measure has been used by Adelgren and Gupte (2022) to compute gaps between bound sets in the
In our context, at node $\eta$, the Hausdorff distance between the upper bound set and the lower bound set is given by $\max_{u \in U(\mathcal{U})} \min_{l \in L(\eta)} d(u, l)$, where $d(u, l)$ is the distance between $u$ and $l$. From an implementation point of view, only local upper bounds that are located above the lower bound set are considered, as they are the only ones that define the search region in $\eta$. If there is none, we consider that the node has a gap of 0. This approach is, in fact, analogous to the single objective case: as long as the lower and upper bound sets have not met entirely, the gap is strictly positive, and the node cannot be fathomed by dominance.

When multiple objectives are considered, and a new feasible point $u$ is added to the upper bound set $\mathcal{U}$, the gap in some nodes may change. In particular, a node $\eta^1$ with a smaller gap than $\eta^2$ may end up with a gap larger than that of $\eta^2$. This implies that in order to identify the node with the best score, the gaps have to be recomputed whenever $\mathcal{U}$ changes. Unfortunately, this may be computationally expensive, as it is not rare that many nodes are open at a given iteration of the MOBB.

To reduce the computational burden, we rely on two simple properties: First, whenever a new point is added to the upper bound set, the gaps at all nodes can only stay the same or decrease. The reason is that the lower bound sets remain unchanged and the upper bound set improves when a new feasible point is found. This implies that the search region shrinks: the upper bound set moves closer to the lower bound sets. Second, we are only interested in the node with the largest gap, as it corresponds to the next node being explored. Let $\eta^1 \in \mathcal{T}$ be the node with the largest gap, and $\eta^2 \in \mathcal{T}$ be the node with the second largest gap. Let $g^{old}(\eta)$ be the gap of a node $\eta$ before re-computation, and $g^{new}(\eta)$ be its gap after re-computation. By construction, we know that $g^{old}(\eta^2) \geq g^{new}(\eta^2)$. Furthermore, for all $\eta \in \mathcal{T}\setminus\{\eta^1, \eta^2\}$, we have $g^{old}(\eta^2) \geq g^{old}(\eta) \geq g^{new}(\eta)$. Hence, if $g^{new}(\eta^1) \geq g^{old}(\eta^2)$, only by re-computing the gap in $\eta^1$, we know that $\eta^1$ is the node with the largest gap after update of the upper bound set. If the condition is not satisfied, $\eta^1$ is put back into $\mathcal{T}$ and the process is repeated with $\eta^2$, the new potential node with the largest gap. The selection procedure is given in Algorithm 3.

7 Experiments

All algorithms are implemented in C++17, relying on Cplex 20.1 for solving single-objective linear programs, using a single thread. The experiments are carried out on Linux 10.3, on a Quad-core X5570 Xeon CPUs @2.93GHz processor and with 48GB of RAM. A time limit of one hour is set when running the algorithms.

Our computational study aims at answering the following research questions: (i) How does probing perform? In particular, how does it perform in combination with objective branching, and why? (ii) What is the impact of using an objective function in the linear programs used for performing probing? (iii) What is the impact of deriving cover cuts from the objective branching constraints on the performance of the algorithm? (iv) Can
Algorithm 3 Selection of the node with the largest gap

1: found ← FALSE
2: while !found do
3: Select the node \( \eta^1 \) with the largest gap from \( \mathcal{T} \); Set \( \mathcal{T} \leftarrow \mathcal{T} \setminus \{\eta\} \)
4: Compute \( g^{new}(\eta^1) \)
5: Select the node \( \eta^2 \) with the largest gap from \( \mathcal{T} \)
6: if \( g^{new}(\eta^1) \geq g^{old}(\eta^2) \) then
7: found ← TRUE
8: else
9: \( g^{old}(\eta^1) \leftarrow g^{new}(\eta^1) \)
10: \( \mathcal{T} \leftarrow \mathcal{T} \cup \{\eta^1\} \)
11: end if
12: end while
13: return \( \eta^1 \)

node selection rules based on the best-bound idea outperform the classical depth and breadth-first strategies often used in the literature? (v) Computing lower bound sets in the multi-objective case is expensive. Does resorting to pure enumeration at certain nodes in the tree improve the performance of the proposed MOBB? (vi) By fixing variables, the set of potential candidates for branching is reduced. What is the impact of the chosen variable selection rule? (vii) How does the proposed branch-and-bound framework perform in comparison to state-of-the-art objective space search algorithms?

We test our algorithms on the following three different types of problems and benchmark instances:

- Capacitated Facility Location Problem (CFLP). The instances are taken from An et al. (2022). Instances with 3 objectives 65, 230, and 495 variables are considered.

- Knapsack Problem (KP). The instances from Kirlik (2014) are used. Instances with 3 objectives and 40, 50, 60, 70, 80 variables are solved, as well as instances with 4 objectives and 20, 30, 40 variables.

- Uncapacitated Facility Location Problems (UFLP). The instances are extracted from Forget et al. (2022). For 3 objectives, instances with 56, 72, 90, 110 variables are used. For 4 objectives, instances with 42 and 56 variables are solved.

For each problem class, number of objectives, and number of variables, 10 instances are solved, leading to a total of 170 instances.

Unless specified otherwise, the framework uses the following parameters and heuristics:
• **Lower bound sets:** the linear relaxation is used as lower bound set. Its computation is warm-started by using the algorithm from Forget et al. (2022);

• **Local upper bounds:** the set of local upper bounds \( N(U) \) is updated whenever a new point is added to the upper bound set \( U \) by using the algorithm from Klamroth et al. (2015). Furthermore, all objective functions are expected to have integer coefficients, and all variables are binary. This implies that the non-dominated points can take integer values only. Consequently, when performing the dominance test and computing sub-problems through objective branching, each component of the local upper bound is shifted by \(-1\).

• **Variable selection rule:** At the creation of sub-problems in the decision space, a free variable is chosen for branching. First, this variable is chosen independently in each sub-problem obtained from objective branching. Let \( s \in \mathbb{R}^p \) be the super local upper bound defining the sub-problem in which a free variable has to be chosen. The variable that is the most often fractional among the extreme points \( l \) of the lower bound set \( L(\eta) \) that satisfies \( l \leq s \) is selected. In case of ties, the one whose average value is closest to 0.5 is selected and in case of equal average values, the one with the smallest index is chosen.

Furthermore, unless specified otherwise, cover cut generation is disabled. A number of different configurations are tested. The three main components evaluated in this study are the following:

• **Objective branching:** three options are considered: no objective branching (NOB); cone bounding (CB); and full objective branching (FOB), as presented in Section 4. Cone bounding is an alternative to objective branching proposed by Forget et al. (2020b). The idea is to derive upper bounds on the objective functions from the partial dominance of the lower bound set, but without splitting the objective space into sub-problems, i.e., only decision space branching is performed.

• **Probing/variable fixing:** three possibilities are considered: no variable fixing (NVF); variable fixing using the advanced strategy as presented in Section 5.1.2 (VF); and variable fixing using a weighted-sum objective function to allow for variable fixing by dominance (VFD) as explained in Section 5.1.3.

• **Node selection rule:** four configurations are considered: best of depth-first and breadth-first (DB); best-bound based on weighted sums (BBWS); a normalized version (BBWSN); and best-bound based on the gap measure presented in Section 6.2 (BBGAP). In the case of (DB), the best strategy per problem class is used. CFLP and UFLP use breadth-first, whereas KP uses depth-first. BBWS and BBWSN use \( \lambda = (1, \ldots, 1) \) (see Section 6.1). Normalization may be important for problems for which the coefficients of the different objective functions take values in very different ranges, such as the CFLP.
In the remainder of this section, each sub-section is designed to address one of the research questions raised earlier.

### 7.1 Probing and objective branching

In a first step, we investigate the effect of combining objective branching and probing. We fix the node selection rule to configuration DB. Six configurations are tested: the three objective branching strategies (NOB, CB, and FOB), in combination with (VF) and without (NVF) variable fixing. For FOB and CB, probing is performed only when an improvement is observed in the objective branching constraints, as suggested in Section 5.1.2. However, probing is performed at every node for NOB.

In Figure 4, the performance profiles of the different configurations are given. The x-axis represents the CPU time expressed in seconds, whereas the y-axis corresponds to the proportion of instances solved. From this figure, it is clear that for both CFLP and KP, the winning configuration is FOB–VF. For UFLP and $p = 3$, however,
although FOB-VF is the best configuration for smaller instances, CB-VF becomes the winning configuration for larger instances. For UFLP and $p = 4$, NOB-NVF is slightly faster than the other configurations.

It is interesting to note that enabling probing resulted in the greatest speed-ups for full objective branching (FOB). This is particularly striking for UFLP, where FOB-NVF is among the worst configurations, but FOB-VF is competitive with the best configurations. For CFLP, it is clear that FOB benefits more from probing than CB or NOB, since CB-NVF is faster than FOB-NVF, but FOB-VF is faster than CB-VF. This suggests that the branch-and-bound algorithm benefits the most from probing when tight objective branching constraints (i.e., small sub-problems) are generated.

Finally, it appears from Figure 4 that when no objective branching is used (NOB), probing (VF) can be slightly beneficial (KP) but also worsen the performance of the algorithm (CFLP, UFLP). This implies that probing is efficient mainly in combination with objective branching in MOBB.

The speed-ups can be largely explained by looking at the size of the branch-and-bound tree. Indeed, the number of nodes explored is positively correlated to the CPU time (see Table 6 in the appendix for detailed performances). For NOB, small differences are observed between NVF and VF in terms of the number of nodes, whereas larger gaps are observed for CB. Similarly to the CPU time, the most significant differences between NVF and VF are observed for FOB. Indeed, if we consider all instances for which both FOB-NVF and FOB-VF are solved, FOB-VF resulted in 12.78 times fewer nodes than FOB-NVF for $p = 3$. In many cases, FOB-NVF is the configuration with the largest tree size, whereas FOB-VF has the smallest. This indicates that probing strongly helps to reduce the size of the tree, both by improving lower bound sets and helping the algorithm to make better branching decisions. Note that a similar observation was made by Adelgren and Gupte (2022) for the bi-objective case.

Finally, one may notice that the speed-ups in terms of CPU times are smaller than the gains in terms of the number of nodes explored. This is due to the fact that probing has a significant cost: on average, 23.5% and 40.36% of the total CPU time for CB and FOB respectively, over all instances (see Table 7 in the appendix for more details).

### 7.2 Combining probing and bounding

We now analyze the impact of introducing a weighted-sum objective function when performing probing (setting VFD), as suggested in Section 5.1.3. It is important to test both cases (VF and VFD) because, on the one hand, introducing an objective function improves the fathoming potential, but, on the other hand, may also result in linear programs that are more difficult to solve. For this purpose, we now consider only two configurations: FOB-VF and FOB-VFD, using as weight vector $\lambda = (1, ..., 1)$ in the weighted-sum objective.

Table 1 gives the average CPU time of the two configurations per instance class. On average, it appears
Table 1: Average CPU time expressed in seconds over 10 instances for each problem class, number of objectives \(p\), and number of variables \(n\). Two configurations are tested: VF and VFD, both in combination with FOB.

that the objective function has a minimal impact on the CPU time. Except in rare cases (e.g., KP for \(p = 4\)), introducing the objective function is still slightly beneficial. A speed-up of 4.64\% is observed on average across all instances for which both FOB-VF and FOB-VFD are solved.

A potential explanation for the low gain is that the linear relaxation of a weighted-sum scalarization provides a rather weak lower bound set. This is true in particular in the case where \(p \geq 3\), as many local upper bounds can have infinite components. Such local upper bounds also have an infinite weighted-sum value and therefore, do not allow for variable fixing by dominance. In the bi-objective case, in comparison, at most, two local upper bounds can take infinite components. One could imagine a strategy where, if a local upper bound has an infinite value at component \(k\) in the considered node-problem, the weight of objective \(k\) is set to 0. This strategy was tested in preliminary experiments, but similarly to the weight vector \((1, \ldots, 1)\), it led to small speed-ups only.

### 7.3 Node selection rules

We now investigate the performance of the node selection rules proposed in Section 6. In this section, we use FOB-VF as the base configuration, and combine it with four node selection strategies: DB, BBWS, BBWSN, and BBGAP.

The results are given in Figure 5. The x-axis represents the time elapsed in seconds, and the y-axis
corresponds to the proportion of the instances solved. Hence, each curve represents the proportion of instances solved over time for each configuration. The first observation is that in most cases, BBWS performs better than DB, and similarly in the worst cases. As explained earlier, configuration DB uses depth-first for KP and breadth-first for CFLP and UFLP, and it is not rare to see either depth-first or breadth-first used in the literature. However, a problem dependent node selection rule is not desirable in the context of a generic solver such as MOBB. Using BBWS instead of depth- or breadth-first overcomes this significant drawback, and it even increases the efficiency of the MOBB framework: BBWS performs better on average than DB.

For KP and UFLP, the base version (BBWS) and the normalized version (BBWSN) perform similarly. This is expected, since all objective coefficients take values in the same range for each objective function. This is, however, not the case for CFLP, where an interesting difference in the performance of BBWS and BBWSN is observed: BBWSN is faster than BBWS, which suggests that normalization is important whenever heterogeneous objective functions are considered, as it is usually the case when dealing with real-world problems.
Finally, BBGAP is the worst configuration, despite exploring on average 45.2% fewer nodes than BBWSN over all instances solved for both configurations (see Table 8 in the Appendix for more details). This is due to the heavy computational cost of computing the gap measures in each node. Indeed, updating the gap measure when selecting a node (Algorithm 3) uses, on average, 17.37% of the total CPU time for \( p = 3 \), and 36.37% for \( p = 4 \) (see Table 8 in the Appendix for more details). Combined with the fact that the gap at each node has to be computed at its creation as well, the additional computation time required for computing the gap measure rule outweighs the benefit of exploring fewer nodes, overall leading to worse CPU times. Furthermore, the reduction in the number of nodes comes largely from those nodes which have a gap of 0 at creation. Those nodes are discarded immediately and not counted among the number of processed nodes. In all other settings, they are put into the list of open nodes \( T \) and are then fathomed by dominance once they are processed.

7.4 Objective branching and cover cuts

In this section, we investigate the incorporation of cover cuts derived from the objective branching constraints. We focus on UFLP, as all of its objectives are minimization objectives. They are constructed following the idea presented in Section 5.2, and lifted using the procedure from Letchford and Souli (2019).

In preliminary experiments, we tested to resolve the linear relaxation and regenerate cover cuts on the extreme points until no new cut could be generated. However, in the multi-objective case, cutting an extreme point in the objective space implies that one or several facets and extreme points are generated, which requires as many linear programs to be solved when using a Benson-type algorithm. In the end, the stronger lower bound sets clearly did not compensate for the larger number of linear programs solved. Hence, we opted for a less expensive strategy.

In this alternative cut-generation scheme, cover cuts are generated after objective branching is computed and before performing probing, with the expectation that the newly generated cuts will help fixing variables. Let \( s \) be the super local upper bound that defines the objective branching constraints, cover cuts are generated at each node for each objective bounded by a constraint, and for each extreme point \( l \) of the lower bound set that has non-integer pre-images. Moreover, a cut is generated only if it is violated, i.e. if it cuts the pre-image of an extreme point of the lower bound set from the feasible set. Naturally, cuts are more likely to be generated on extreme points \( l \) that have one component that is close to the objective branching constraint, i.e., for a \( k \in \{1, \ldots, p\} \) where \( l_k \) is close to \( s_k \).

Since the linear relaxation is not resolved, no additional LP is expected to be solved in the current node. However, the generated cuts are kept in the child nodes.

We compare the best previous configuration, namely FOB–VF–BBWSN, with (CC) and without cover cuts (NCC). The results of this experiment are given in Table 2. Configuration CC generally performs slightly worse
Table 2: Comparison of the setting FOB-VF-BBWSN with (CC) and without (NCC) cover cut generation for UFLP. For each number of objectives and number of variables, it reports the average CPU time (CPU), the average number of nodes explored (# nodes), the number of instances unsolved (# unsolved), and the average total number of linear programs solved to compute the linear relaxation (# LP (LP relax)).

The performance of CC can be understood by looking at the last column of Table 2, where the average total number of linear programs solved to compute the lower bound sets in an entire tree is reported. It tends to increase when cover cuts are used, which implies that after generating cuts, the lower bound sets in the child nodes are more complex, and require more linear programs to be solved for CC. This observation makes sense as the newly generated constraints are kept in the child nodes. This possibly results in more facets to generate and, consequently, more computational effort, even though the linear relaxation is not resolved after generating cuts. This phenomenon highlights a difficulty of cut generation in the multi-objective case: even simple cuts can become a computational burden, because they can easily complexify the lower bound set, possibly resulting in a considerable additional effort to obtain only small improvements.

7.5 Enumeration in MOBB

In the deepest parts of the tree, only a few variables are free. Consequently, the lower bound sets are also expected to be very simple, i.e., made of a few facets and extreme points only. According to Forget et al. (2022), each facet and extreme point possibly requires solving one linear program. Given the fact that the number of free variables is low, one may enumerate all possible solutions and update the upper bound set accordingly instead of keeping branching and computing lower bound sets until all nodes are fathomed. Here, after performing probing, if less than 14 variables are free, all $2^{14} = 16384$ solutions are enumerated. This value has been determined with preliminary experiments on a sub-set of instances.
| p  | pb | n   | CPU          | # LP         |
|----|----|-----|-------------|-------------|
| 65 |    | 10.0| BB 10.0     | BB-E 10.0   |
| 230|    | 1988.0| 1993.3 | 4159912.0 | 4179247.3 |
| 495|    | 3600.0| 3600.0 | 2795418.2 | 2920801.0 |
| 40 |    | 46.7 | BB 42.4    | BB-E 42.4  |
| 50 |    | 116.5| BB 108.2   | BB-E 108.2 |
| 60 |    | 457.8| BB 431.0   | BB-E 431.0 |
| 70 |    | 1414.1| 1341.1 | 7412250.4 | 7037987.6 |
| 80 |    | 2491.5| 2388.9 | 12508369.2| 12154795.8|
| 56 |    | 177.3| BB 134.1   | BB-E 134.1 |
| 72 |    | 564.4| BB 421.3   | BB-E 421.3 |
| 90 |    | 1590.5| 1229.2 | 4074781.8 | 3860124.8 |
| 110|    | 3381.7| 3132.8 | 7907290.8 | 7863352.6 |
| 20 |    | 7.0  | BB 2.4     | BB-E 2.4   |
| 30 |    | 42.0 | BB 36.0    | BB-E 36.0  |
| 40 |    | 854.8| BB 751.8   | BB-E 751.8 |
| 42 |    | 558.9| BB 306.4   | BB-E 306.4 |
| 56 |    | 3600.0| 3513.8 | 965511.4  | 542167.4  |

Table 3: Average CPU time expressed in seconds and average number of linear programs solved over 10 instances for each problem class, number of objectives, and number of variables. Two algorithms are tested here: BB and BB-E.

Results are reported in Table 3. Configuration BB corresponds to the best generic branch-and-bound so far, i.e., it uses full objective branching (FOB), variable fixing with an objective function (VFD), and the best-bound node selection rule based on the normalized weighted-sum idea (BBWSN). Configuration BB-E uses the same parameters, except that enumeration as described above is enabled.

In general, the enumeration procedure reduces the CPU time, even for larger instances. For CFLP, enumeration appears to be slightly slower (e.g., less than 0.5% for \( n = 230 \)). As expected, the proportion of CPU time spent in updating the upper bound set is greater in BB-E, but fewer linear programs are solved.

### 7.6 Problem specific variable selection rules:

the case of CFLP

By performing probing, the algorithm is able to reduce the set of branching candidates. This indirectly helps the algorithm to make better branching decisions, as variables that lead to redundant branches are discarded. In this section, we study whether a tailored variable selection rule could help to further improve the CPU time. For this purpose, we limit the analysis to the CFLP, for which a good branching rule is well known in the single
Table 4: Comparison of a problem specific branching rule PS and the general purpose rule MOF for CFLP. For each instance class given by the number of variables $n$, the average CPU time expressed in seconds (CPU) and the number of unsolved instances (# unsolved) is given.

| $n$ | MOF | PS | MOF | PS |
|-----|-----|----|-----|----|
| 65  | 10.0 | 12.3 | 0   | 0  |
| 230 | 1993.3 | 581.4 | 1   | 0  |
| 495 | 3600.0 | 3600.0 | 10  | 10 |

7.7 Comparison with objective space search algorithms

In this section, we compare our MOBB framework to a state-of-the-art objective space search algorithm. For this comparison, we use a C++ implementation of the redundancy avoidance method of Klamroth et al. (2015). The implementation was kindly shared with us by Dächert et al. (2021). The idea is to decompose the objective space based on the local upper bounds, and to explore each sub-region independently while avoiding redundant regions. In that aspect, the algorithm we use is similar to Tamby and Vanderpooten (2021), except that the implementation we use does not go as far as theirs in the fine-tuning of CPLEX’s parameters.

The OSS algorithm is compared to the best generic configuration of our MOBB, namely BB-E. No cover cuts are generated, and the generic variable selection rule is used for CFLP (MOF). Consequently, two configurations are tested here: our branch-and-bound framework (BB-E) and the objective space search algorithm (OSS). The results are reported in Table 5.

First, BB-E seems to be competitive with OSS for UFLP, meaning that the first milestone towards an efficient branch-and-bound framework for the multi-objective case has been reached. There is, however, a tendency for
Table 5: Average CPU time expressed in seconds and number of linear programs solved over 10 instances for each problem class, number of objectives, and number of variables. Two algorithms are tested here: BB-E and OSS. For UFLP, the performance of BB-E using cone objective branching (BB-E+CB) is reported in brackets, as it was shown to be the best configuration for this problem class (see Figure 4).

Larger instances to be solved faster by OSS, e.g., for UFLP, \( p = 3 \), and \( n = 110 \). This suggests that there are costs that do not occur for small and medium-sized instances but that become a burden for larger instances. In our opinion, this constitutes a direction for future research. Indeed, now that the branch-and-bound algorithm is competitive on medium-sized instances, the next step is to tackle larger problems. Note that in our experiments, for this specific problem class and problem size, some instances are solved faster using OSS whereas other are solved faster using BB-E (in particular with cone bounding enabled).

Regarding CFLP and KP with \( p = 3 \), OSS is significantly faster than BB-E. We have seen in Section 7.6 that major improvements could be obtained for CFLP by working on variable selection rules. Gadegaard et al. (2019) showed that cut generation at the root node is very effective for bi-objective CFLP, and this may apply to any number of objectives as well as to other problem classes. These two elements constitute very promising directions for future research.

Finally, it is interesting to note that BB-E is the most efficient on problems for which the number of non-dominated points per variable (\(|Y_N|/n\)) is the highest. Indeed, over all instances solved by at least one of the configurations, UFLP has 102.84 non-dominated points per variable, whereas KP and CFLP have 18.74 and
5.13, respectively. The results from this section confirm the intuition that OSS algorithms are more efficient on problems with fewer non-dominated points, and benefit a lot from the decades of progress embedded in single-objective solvers, which are particularly competitive on problems with a high number of variables.

8 Conclusion

In this paper, we first enhanced objective branching for MOBB with three or more objective functions. The experiments showed that combining probing and objective branching leads to significantly smaller search trees, resulting in lower CPU times.

Then, we proposed node selection rules based on the best-bound idea. Two variants were considered, depending on how the quality of a bound is measured: either based on the minimal value of a weighted-sum scalarization or on the smallest gap between upper and lower bound sets. The experiments showed that the former is the most efficient, and performs better than both the traditional depth-first and breadth-first strategies from the literature. Besides, we have observed that computing gaps during the resolution can be expensive. This opens the discussion on appropriate gap measures for multi-objective optimization, in particular for \( p \geq 3 \), and how to efficiently compute these gaps.

Moreover, other developments on additional features were explored. We first investigated cut generation based on the objective branching constraints, but this resulted in slower performances due to the increased complexity of the lower bound sets. Generating cuts only in the root node may be more beneficial, as done by Gadegaard et al. (2019) for the bi-objective case and constitutes a promising direction for future research. We also showed in our experiments that although variable fixing reduces the set of potential branching candidates when doing decision space branching, there are still possible improvements to be achieved by identifying appropriate variables to branch on. This constitutes another promising direction for future research. Then, enumeration techniques were tested and generated a speed-up in most cases.

Finally, our branch-and-bound framework is the first that proves to be competitive against a state-of-the-art objective space search algorithm on UFLP instances with three and four objectives. Future research should address the development of appropriate techniques for more efficiently solving large scale problem instances.

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Appendices

A Considered benchmark problems

A.1 CFLP

In this problem, there is a set of \( l \) locations where a facility can be opened, and a set of \( r \) customers that each have to be assigned to a location. Two decisions have to be made: which locations to open and which customers to assign to which facilities. Both opening a facility and assigning a customer to an open facility induces a cost. Moreover, the demand being handled in a facility \( j \) cannot exceed a threshold \( t_j \). Finally, the company may chose to ignore some of the customers to reduce their costs.

The first objective is to minimize the cost of assigning customers to facilities. The second objective consists in minimizing the opening cost of the facilities. The third objective aims to maximize the overall demand of the customers that is satisfied.

Let \( y_j = 1 \) if a facility is opened at location \( j \), and \( y_j = 0 \) otherwise, \( \forall j \in \{1, \ldots, l\} \). Furthermore, let \( x_{ij} = 1 \) if customer \( i \) is assigned to location \( j \), and \( x_{ij} = 0 \) otherwise, \( \forall i \in \{1, \ldots, r\}, \forall j \in \{1, \ldots, l\} \). Finally, \( z_i = 1 \) if customer \( i \) is served, 0 otherwise, \( \forall i \in \{1, \ldots, r\} \).

\[
\begin{align*}
\min & \sum_{i=1}^{r} \sum_{j=1}^{l} c_{ij} x_{ij} \\
\min & \sum_{j=1}^{l} f_j y_j \\
\max & \sum_{i=1}^{r} d_i z_i \\
\text{s.t.} & \sum_{j=1}^{l} x_{ij} = z_i & \forall i \in \{1, \ldots, r\} \\
x_{ij} \leq y_j & \forall i \in \{1, \ldots, r\}, j \in \{1, \ldots, l\} \\
\sum_{i=1}^{r} d_i x_{ij} \leq t_j y_j & \forall j \in \{1, \ldots, l\} \\
x_{ij} \in \{0, 1\} & \forall i \in \{1, \ldots, r\}, j \in \{1, \ldots, l\} \\
y_j \in \{0, 1\} & \forall j \in \{1, \ldots, l\} \\
z_i \in \{0, 1\} & \forall i \in \{1, \ldots, r\}
\end{align*}
\]

Instances are taken from An et al. (2022).
A.2 KP

In the Knapsack problem, a subset of items has to be selected from a set of $n$ items. Each item $i$ has a weight $w_i$, and there is a limit $b$ on the total weight of the items being selected. Moreover, each item $i$ has a utility $c_i^k$ in objective $k$, and the goal is to maximize the utility of the subset of items selected over all objective functions.

Let $x_i = 1$ if item $i$ is selected, 0 otherwise. The multi-objective Knapsack Problem (KP) with $p$ objectives can be formulated as follows:

$$\min \sum_{i=1}^{n} c_i^k x_i \quad \forall k \in \{1, \ldots, p\}$$

s.t. $\sum_{i=1}^{n} w_i x_i \leq b$

$x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, n\}$

Instances are taken from Kirlik and Sayın (2014).

A.3 UFLP

In this problem, there is a set of $l$ locations where a facility can be opened, and a set of $r$ customers that each have to be assigned to a location. Two decisions have to be made: which locations to open and which customers to assign to which facilities. Both opening a facility and assigning a customer to an open facility induces a cost, and the overall cost has to be minimized. Let $y_j = 1$ if a facility is opened at location $j$, and $y_j = 0$ otherwise, $\forall j \in \{1, \ldots, l\}$. Furthermore, let $x_{ij} = 1$ if customer $i$ is assigned to location $j$, and $x_{ij} = 0$ otherwise, $\forall i \in \{1, \ldots, r\}, \forall j \in \{1, \ldots, l\}$.

The multi-objective Uncapacitated Facility Location Problem (UFLP) with $p$ objectives can be formulated as the following MOCO problem

$$\min \sum_{i=1}^{r} \sum_{j=1}^{l} c_{ij}^k x_{ij} + \sum_{j=1}^{l} f_j^k y_j \quad \forall k \in \{1, \ldots, p\}$$

s.t. $\sum_{j=1}^{l} x_{ij} = 1 \quad \forall i \in \{1, \ldots, r\}$

$x_{ij} \leq y_j \quad \forall i \in \{1, \ldots, r\}, j \in \{1, \ldots, l\}$

$x_{ij} \in \{0, 1\} \quad \forall i \in \{1, \ldots, r\}, j \in \{1, \ldots, l\}$

$y_j \in \{0, 1\} \quad \forall j \in \{1, \ldots, l\}$

Instances are taken from Forget et al. (2022).
## B Additional computational results

| p  | pb | n   | CB        |       | FOB        |       | NOB        |       |
|----|----|-----|-----------|-------|------------|-------|------------|-------|
|    |    |     | NVF       | VF    | NVF        | VF    | NVF        | VF    |
|----|----|-----|-----------|-------|------------|-------|------------|-------|
| 65 |    | 2382.6 (0) | 11786.2 (0) | 1389.4 (0) | 15090.4 (0) | 15072.4 (0) |
| 230|    | 65861.8 (4) | 405597.4 (7) | 49365.9 (1) | 78613.1 (10) | 32593.7 (10) |
| 495|    | 4357.9 (10) | 33545.9 (10) | 4497.9 (10) | 4245.2 (10) | 2934.4 (10) |
| 40 |    | 15918.4 (0) | 136980.2 (0) | 13023.6 (0) | 137544.2 (0) | 71502.4 (0) |
| 50 |    | 34498.6 (0) | 385177 (0) | 27261.8 (0) | 385625.2 (0) | 174757.2 (0) |
| 60 |    | 121213.2 (0) | 1128878.7 (2) | 95023.4 (0) | 623624.4 (9) | 428664.6 (7) |
| 70 |    | 289369.3 (3) | 1839920.2 (7) | 226918 (2) | 617265.9 (10) | 456791.8 (10) |
| 80 |    | 377936.3 (5) | 1812818.1 (10) | 303356 (5) | 499527.9 (10) | 367138.9 (10) |
| 56 |    | 20885.2 (0) | 216010.6 (0) | 14512.6 (0) | 63239 (0) | 62944.2 (0) |
| 72 |    | 47018.2 (0) | 531514 (0) | 33598.6 (0) | 151994.2 (0) | 151525.4 (0) |
| 90 |    | 97638.6 (0) | 1103596 (5) | 72544.4 (0) | 275612.6 (6) | 235776.7 (9) |
| 110|    | 157547.9 (5) | 681660.1 (10) | 73505.6 (10) | 71013.9 (10) | 72963.3 (10) |
| 20 |    | 2103.6 (0) | 8442.4 (0) | 1899.4 (0) | 8998 (0) | 5043.2 (0) |
| 30 |    | 8469.2 (0) | 49641 (0) | 7589.4 (0) | 53294.2 (0) | 26693.4 (0) |
| 40 |    | 82242.8 (2) | 430534.1 (2) | 72322.6 (1) | 340657.3 (7) | 221146.5 (3) |
| 42 |    | 29925.4 (0) | 199163 (0) | 22728.8 (0) | 68111.2 (0) | 68070.2 (0) |
| 56 |    | 49605.7 (10) | 32013.5 (10) | 183316.4 (10) | 100648.1 (10) | 100648.1 (10) |

Table 6: The average number of nodes explored over 10 instances for each problem class, number of objectives, number of variables, and configuration. The number in brackets is the number of instances unsolved. Note that when the number of unsolved instances is high, the number of nodes explored may be low due to the fact that the algorithm could not explore a large number of nodes within the time limit of one hour.
| p | pb | n | % LB set | % Probing | % Other |
|---|---|---|---|---|---|
| CB | FOB | NOB | CB | FOB | NOB | CB | FOB | NOB | CB | FOB | NOB | CB | FOB | NOB | CB | FOB | NOB |
| 65 | 90.0 | 55.1 | 86.8 | 43.2 | 89.3 | 43.6 | 0.0 | 41.0 | 0.0 | 53.4 | 0.0 | 51.4 | 10.0 | 3.9 | 13.2 | 3.4 | 10.7 | 5.0 |
| 230 | 88.5 | 48.4 | 80.8 | 32.4 | 95.7 | 47.1 | 0.0 | 48.3 | 0.0 | 64.8 | 0.0 | 51.1 | 11.5 | 3.3 | 19.2 | 2.8 | 4.3 | 1.8 |
| 495 | 98.3 | 73.3 | 93.3 | 53.2 | 98.7 | 70.0 | 0.0 | 25.5 | 0.0 | 45.8 | 0.0 | 29.0 | 1.7 | 1.2 | 6.7 | 1.1 | 1.3 | 0.9 |
| 40 | 90.6 | 69.9 | 88.3 | 62.5 | 95.0 | 68.5 | 0.0 | 25.6 | 0.0 | 32.8 | 0.0 | 28.0 | 9.4 | 4.5 | 11.7 | 4.7 | 5.0 | 3.5 |
| 50 | 89.1 | 68.3 | 86.5 | 59.8 | 95.0 | 68.5 | 0.0 | 26.8 | 0.0 | 35.2 | 0.0 | 28.0 | 10.9 | 4.9 | 13.5 | 5.1 | 5.0 | 3.5 |
| 60 | 87.5 | 66.3 | 84.0 | 56.5 | 94.8 | 66.2 | 0.0 | 27.8 | 0.0 | 37.7 | 0.0 | 29.8 | 12.5 | 5.9 | 16.0 | 5.9 | 5.2 | 3.9 |
| 70 | 85.8 | 63.2 | 81.4 | 50.6 | 94.3 | 59.2 | 0.0 | 29.8 | 0.0 | 42.6 | 0.0 | 36.4 | 14.2 | 7.0 | 18.6 | 6.8 | 5.7 | 4.4 |
| 80 | 83.7 | 62.0 | 79.5 | 49.2 | 94.3 | 62.0 | 0.0 | 30.4 | 0.0 | 42.6 | 0.0 | 33.7 | 16.3 | 7.6 | 20.5 | 8.2 | 5.7 | 4.2 |
| 56 | 78.1 | 62.4 | 69.7 | 35.0 | 83.5 | 59.8 | 0.0 | 21.6 | 0.0 | 54.1 | 0.0 | 26.6 | 21.9 | 16.1 | 30.3 | 11.0 | 16.5 | 13.6 |
| 72 | 72.7 | 57.4 | 67.2 | 29.1 | 87.2 | 61.7 | 0.0 | 22.6 | 0.0 | 57.9 | 0.0 | 24.2 | 27.3 | 19.9 | 32.8 | 13.1 | 12.8 | 14.1 |
| 90 | 75.6 | 57.7 | 58.2 | 24.9 | 84.7 | 60.8 | 0.0 | 21.8 | 0.0 | 61.1 | 0.0 | 23.1 | 24.4 | 20.5 | 41.8 | 14.0 | 15.3 | 16.1 |
| 110 | 63.8 | 51.1 | 47.1 | 18.5 | 84.7 | 66.9 | 0.0 | 21.9 | 0.0 | 66.6 | 0.0 | 17.9 | 36.2 | 27.0 | 52.9 | 14.9 | 15.3 | 15.2 |
| 20 | 93.8 | 85.9 | 93.0 | 84.3 | 95.8 | 82.1 | 0.0 | 9.2 | 0.0 | 10.8 | 0.0 | 14.7 | 6.2 | 4.9 | 7.0 | 4.9 | 4.2 | 3.2 |
| 30 | 91.6 | 82.3 | 90.3 | 79.8 | 95.7 | 81.8 | 0.0 | 10.8 | 0.0 | 13.0 | 0.0 | 14.7 | 8.4 | 6.9 | 9.7 | 7.2 | 4.3 | 3.5 |
| 40 | 85.7 | 78.2 | 83.3 | 73.7 | 94.5 | 82.8 | 0.0 | 8.9 | 0.0 | 11.7 | 0.0 | 11.9 | 14.3 | 13.0 | 16.7 | 14.6 | 5.5 | 5.3 |
| 42 | 53.6 | 41.1 | 51.5 | 35.4 | 70.5 | 43.0 | 0.0 | 15.0 | 0.0 | 26.0 | 0.0 | 32.4 | 46.4 | 43.9 | 48.5 | 38.6 | 29.5 | 24.6 |
| 56 | 36.2 | 29.3 | 30.1 | 20.1 | 50.6 | 33.2 | 0.0 | 12.6 | 0.0 | 30.2 | 0.0 | 35.2 | 63.8 | 58.1 | 69.9 | 49.7 | 49.4 | 31.5 |

Table 7: Comparison of different objective branching settings (cone branching (CB), full objective branching (FOB), no objective branching (NOB)) in combination with (VF) and without (NVF) probing concerning the average percentage of CPU time spent in different parts of the algorithm over 10 instances for each problem class, number of objectives, and number of variables. % LB set represents the share of CPU time spent in the computation of lower bound sets. % Probing represents the proportion of CPU time dedicated to probing. % Other is the percentage of CPU time spent in other parts of the algorithms such as dominance test, creation of sub-problems, node selection, etc.
| p  | pb | n   | CPU BBGAP | BBWSN | CPU BBGAP | BBWSN | % CPU BBGAP |
|----|----|-----|----------|-------|----------|-------|-------------|
| 65 | 19.6 | 10.0 | 2064.6 (0) | 1278.8 (0) | 1.8 |
| 230 | 3718.7 | 1988.0 | 34494.4 (10) | 40765.4 (1) | 3.9 |
| 495 | 3602.3 | 3601.7 | 1918.1 (10) | 4496.5 (10) | 5.1 |
| 40 | 124.8 | 46.7 | 5826.8 (0) | 7405.8 (0) | 9.8 |
| 50 | 286.5 | 116.5 | 12459.3 (0) | 14972 (0) | 12.9 |
| 60 | 1034.9 | 457.8 | 33760 (0) | 43623 (0) | 22.0 |
| 495 | 3602.3 | 3601.7 | 1918.1 (10) | 4496.5 (10) | 5.1 |
| 40 | 124.8 | 46.7 | 5826.8 (0) | 7405.8 (0) | 9.8 |
| 50 | 286.5 | 116.5 | 12459.3 (0) | 14972 (0) | 12.9 |
| 60 | 1034.9 | 457.8 | 33760 (0) | 43623 (0) | 22.0 |
| 70 | 2348.9 | 1414.1 | 57832.4 (4) | 105508.2 (0) | 22.2 |
| 80 | 3260.9 | 2491.5 | 66489.8 (5) | 154884.5 (3) | 21.1 |
| 56 | 235.5 | 177.3 | 7524.1 (0) | 12282.8 (0) | 22.6 |
| 72 | 791.1 | 564.4 | 15755.7 (0) | 26574.4 (0) | 25.8 |
| 90 | 2539.7 | 1590.5 | 30022.4 (0) | 52141.6 (0) | 26.8 |
| 110 | 3601.1 | 3381.7 | 17823.2 (10) | 80985.7 (5) | 34.2 |
| 20 | 35.9 | 7.0 | 923.5 (0) | 1325.6 (0) | 8.8 |
| 30 | 150.5 | 42.0 | 3640.7 (0) | 5105.6 (0) | 25.8 |
| 40 | 2176.1 | 854.8 | 14339.3 (4) | 41642 (0) | 51.0 |
| 42 | 3530.1 | 558.9 | 7984.4 (7) | 22393.8 (0) | 48.3 |
| 56 | 3600.0 | 3600.0 | 350.25 (10) | 39079.1 (10) | 50.8 |

Table 8: Comparison of node selection rules BBWSN and BBGAP. Columns CPU give the average CPU time expressed in seconds over 10 instances for each problem class, number of objectives, and number of variables. Columns # Nodes provide the average number of nodes explored as well as the number of unsolved instances (indicated in brackets). Finally, Column % CPU represents the percentage of the total CPU time spent in updating gaps in configuration BBGAP.