On a constrained 2-D Navier-Stokes Equation

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Abstract

The planar Navier-Stokes equation exhibits, in absence of external forces, a trivial asymptotics in time. Nevertheless the appearence of coherent structures suggests non-trivial intermediate asymptotics which should be explained in terms of the equation itself. Motivated by the separation of the different time scales observed in the dynamics of the Navier-Stokes equation, we study the well-posedness and asymptotic behaviour of a constrained equation which neglects the variation of the energy and moment of inertia.

1 Introduction

Consider the two-dimensional Euler equation in vorticity form

\( \left( \partial_t + u \cdot \nabla \right) \omega(x,t) = 0, \quad x \in \mathbb{R}^2 \) (1.1)

where the divergence free velocity field \( u \) is given by \( u = \nabla^\perp \psi, \psi = -\Delta^{-1} \omega \). Explicitely, we can write:

\[ u = K * \omega, \quad K(x) = \nabla^\perp g = -\frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad g(x) = -\frac{1}{2\pi} \log |x|. \] (1.2)

The rigorous justification of the formation of coherent structures in two-dimensional fluid-dynamics, which is observed in real and numerical experiments (see e.g. [22]) remains a widely open problem. An attempt to justify the appearance of these coherent structure is due to Onsager [27], see also [17], [25], and [11] for a recent review. The main idea is to replace the incompressible Euler equation by the system of \( N \) point vortices and to study the Statistical Mechanics of these point vortices. In the mean field limit \( N \to +\infty \), the Gibbs measure associated to the point vortices concentrates to some special stationary solutions of the Euler equation (called mean field solutions), we refer to [7], [8], [15], [16] for the rigorous justification. These states are under the form:

\[ \omega = \frac{e^{b\psi + a|x|^2}}{Z}, \quad Z = \int_{\mathbb{R}^2} e^{b\psi + a|x|^2} dx. \] (1.3)

In this last expression, \( Z \) is a normalization factor to have \( \int \omega = 1 \) and \( b \) real and \( a < 0 \) are parameters. From a mathematical point of view this equation enters in the framework of a general class of nonlinear elliptic equations given by

\[ -\Delta \psi = \frac{e^{b\psi + a|x|^2}}{Z} \] (1.4)
which has been studied in [7], [8]. Nevertheless, there is no justification of the fact that, among the infinite number of stationary stable solutions of the Euler equation, the mean field solutions play indeed a special role in the dynamics.

Another justification of (1.3) could come from the intermediate asymptotic behaviour of the two-dimensional Navier-Stokes equation which, in vorticity form, reads

\[(\partial_t + u \cdot \nabla)\omega(x, t) = \nu \Delta \omega(x, t), \quad x \in \mathbb{R}^2,\]  

(1.5)

where \(\nu > 0\) is the viscosity coefficient. Indeed, due to the dissipation term in the right hand side of eqn (1.5), the asymptotic behaviour of the solutions is trivial, namely, when \(t \to +\infty\), \(\omega(x, t) \to 0\) pointwise and in the \(L^p\) sense, for \(p > 1\). Consequently, the states (1.3) could play a part only in the intermediate behaviour of the equation before the dissipation scale. To give a more quantitative description of this idea, it is useful to recall that the solutions of (1.3) can be studied through a variational principle: the radial solutions are obtained as minimizers of the Boltzmann entropy (this is proven in the Appendix)

\[S(\omega) = \int_{\mathbb{R}^2} \omega \log \omega \, dx\]

under the constraints

\[E(\omega) = E, \quad M(\omega) = 0, \quad I(\omega) = I, \quad \omega \geq 0, \quad \int_{\mathbb{R}^2} \omega \, dx = 1,\]

for some fixed \(E\) and \(I\), where the energy \(E(\omega)\), the center of vorticity \(M(\omega)\) and the moment of inertia \(I(\omega)\) are respectively given by

\[E(\omega) = \frac{1}{2} \int_{\mathbb{R}^2} \psi \omega \, dx, \quad M(\omega) = \int_{\mathbb{R}^2} x \omega \, dx, \quad I(\omega) = \frac{1}{2} \int_{\mathbb{R}^2} |x - M|^2 \omega.\]

Note that it is always possible to choose the coordinates so that \(M(\omega) = 0\).

It is thus interesting to study how these quantities, which are conserved by the Euler equation (1.1), evolve under the Navier-Stokes flow. At first, it is well known that the Navier-Stokes equation preserves the nonnegativity and that \(\int \omega\) and \(M\) are conserved. Consequently, throughout this paper, we will focus on non-negative solutions which are normalized such that \(\int \omega = 1\) and \(M(\omega) = 0\). Next, we also observe that

\[\dot{I}(\omega) = 2\nu, \quad \dot{E} = -\nu \int_{\mathbb{R}^2} \omega^2, \quad \dot{S}(\omega) = -\nu \int_{\mathbb{R}^2} |\nabla \omega|^2 / \omega.\]

It is easy to see that \(I\) can be considered as constant for times \(t << \nu^{-1}\). Moreover, it is likely that in certain cases (see for instance [22]) the energy dissipation rate is much smaller than the entropy dissipation rate. Coming back to an attempt to justify eq. (1.4) in terms of the Navier-Stokes evolution, the first naive remark is that, if the energy and the moment of inertia are assumed to vary on a long time scale, they can be considered as constant in a first approximation. On such a time scale, the motion should be governed by a master equation, which modifies the Navier-Stokes equation leaving constant both energy and moment of inertia, but retaining all the other features of the Navier-Stokes dynamics. Therefore such a master equation, dissipating the entropy at constant energy and moment of inertia, would lead to the solution to eqn (1.4) as \(t \to \infty\). By using a recent geometric gradient flow characterization of the Navier-Stokes equation (see [34] for example)
connected with the mass transport problem and the associated differential calculus introduced in [29] (see also [1]) we have derived such an equation in [10]. In this framework the Navier-Stokes equation can be written as a differential equation for a vector field which is the sum of a dissipative part, which is the gradient flow of the entropy, and a conservative part, corresponding to the Euler equation, which is the orthogonal gradient of the energy. The equation we were looking for was obtained by keeping only the orthogonal projection of the vector field in the tangent space of the manifold \( I = \text{const}\) and \( E = \text{const}\). Due to the nature of this decomposition, this procedure modifies the dissipative part while leaving invariant the conservative part. Thus we have found the equation:

\[
\partial_t \omega + u \cdot \nabla \omega = \nu \text{div}(\nabla \omega - b\omega \nabla \psi - a\omega x)
\]

(1.6)

where the Lagrange multipliers \( b \) and \( a \) are given by

\[
b = b(\omega) = \frac{2I \int \omega^2 + 2V}{2I \int |\nabla \psi|^2 - V^2}; \quad a = a(\omega) = -\frac{2}{2I \int |\omega|^2} \frac{\omega}{\omega}.
\]

(1.7)

and

\[
V = \int \omega x \cdot \nabla \psi = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x,t)\omega(y,t) x \cdot \nabla g(x - y) \, dx \, dy = \frac{-1}{4\pi}.
\]

(1.8)

A way to validate this approach is to test it in the following simpler and well understood case. A special self-similar solution to eqn (1.5) is the so called Oseen vortex:

\[
\omega(x,t) = \frac{1}{4\pi\nu(t+1)} e^{-\frac{|x|^2}{4\nu(t+1)}}.
\]

Note that this is also a solution to the heat equation. It was shown by Gallay and Wayne [12] that this solution describes the long time asymptotic of the Navier-Stokes equation in \( L^1 \). Indeed, with the change of variables

\[
\xi = \frac{x}{\sqrt{1 + t}}; \quad \tau = \log(1 + t), \quad \omega(x,t) = (1 + t)^{-1} w(\xi, \tau),
\]

the Navier-Stokes equation in the new variables reads:

\[
\partial_\tau w + v \cdot \nabla \xi w = \nu \Delta w + \nu \nabla \xi \cdot \left( \frac{1}{2} \nabla w \right).
\]

(1.9)

It is possible to show that \( w \to W \) in \( L^1 \) as \( \tau \to \infty \), where \( W(\xi) \) is the rescaled Oseen vortex. As a consequence the Oseen vortex can be thought as characterizing an intermediate asymptotics for times \( \nu t << 1 \).

This analysis enters perfectly in the context of the projected gradient flows. Indeed imposing the constance of \( I \) in the Navier-Stokes equation we find

\[
\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \nu \frac{1}{I} \nabla \cdot (\omega x),
\]

(1.10)

that is eqn (1.5) for \( I = 2 \), as well as the rescaled Oseen vortex \( W(\xi) \) is a Mean Field solution for \( \beta = 0 \). This suggest to impose also the constance of \( E \) in the attempt of outlining what happens
before the occurrence of the Oseen vortex. Indeed one could argue that \( I \) is more robust than \( E \) in many interesting physical situations. If so eqn (1.6) should be more appropriate on the time scale when \( E \) is practically constant, while eqn (1.9) should describe the fluid when \( E \) start to be dissipated at constant \( I \).

The aim of this paper is the mathematical study of equations (1.6), (1.7). For more details on the derivation of these equations and of the physical motivations, we refer to [10]. As explained in [10], the procedure of constraining a diffusion equation is highly non unique. Nevertheless, an interesting feature of eqn (1.6) is that it can be obtained by many different methods. As already explained, it appears naturally by using the geometric structure of the Navier-Stokes equation. Moreover, it was noticed in [10] that eqn (1.6) can also be obtained by constraining the stochastic vortex dynamics which is a finite dimensional approximation to the Navier-Stokes equation (see [18], [19], [28]...). Indeed (at a formal level), it is shown that the stochastic process for a system of \( N \) stochastic vortices, once constrained on the \( E=\text{const} \) manifold, produces, in the mean-field limit, exactly eqn (1.6) which turns out to be compatible with this particle approximation.

Equation (1.6) however is not new. It was previously derived by Chavanis in [5] and [6] following a completely different approach based on the kinetic theory of (deterministic) point vortices.

We finally note that the projection on the manifold \( I=\text{const} \) according to the gradient notion used here, has been considered by Carlen and Gangbo in [4] for a different class of equations.

A different approach to the one of Onsager [27] in order to understand the coherent structures arising in 2D flows was proposed by Robert and Sommeria [31] and Miller [24]. The equilibrium solutions are obtained by using the maximum entropy principle over a state space formed by a selected family of possible values of the vorticity. Note that this procedure preserves all the Euler invariants so that, as far as the equilibrium is concerned, the Robert-Sommeria-Miller theory is quite different from the approaches, as the present one, based on the mean field equation. As regards the dynamics, a class of master equations leading to such equilibria has been introduced in [32]. One of them, which has some formal similarities with our model, has been systematically investigated from a mathematical point of view in [23]. We remark that such an equation exhibits a maximum principle leading to useful a priori estimates, for instance the \( L^\infty \) norm of the vorticity is uniformly bounded while, in our case, we do not have such a priori control.

Our aim is to establish global existence results for (1.6) and to study the asymptotic behaviour of global solutions. There are two main difficulties. The first one is that eqn (1.6) makes no sense whenever the denominator in the definition of \( a \) and \( b \) (see eqn (1.7)) vanishes. Note that by the Cauchy-Schwarz inequality we have

\[
V^2 = \left( \int \omega x \cdot \nabla \psi \right)^2 \leq 2I \int |\omega| \nabla \psi|^2
\]

and thus this denominator is always non-negative. Nevertheless, when \( \omega \nabla \psi \) and \( \omega x \) are collinear, it vanishes. This happens for the one-dimensional family of circular vortex patches:

\[
\omega = \frac{1}{\pi R^2} \chi_{B(0,R)}
\]

where \( \chi_{B(0,R)} \) is the characteristic function of \( B(0, R) \), the disk of center 0 and radius \( R \). Indeed, we have:

\[
\omega \nabla \psi = -\frac{\omega x}{2\pi R^2} \chi_{B(0,R)}.
\]

The other difficulty is that \( b \) is well-defined if \( \omega \in L^2 \) but there is no a priori estimates available for the \( L^2 \) norm of the vorticity. The only a priori information we have at our disposal are that \( E \)
and $I$ are conserved (note that in this setting $E$ is not very useful since the energy has no sign) and that the entropy $S$ decays. Indeed, we formally have

$$\frac{dS(\omega)}{dt} = -\nu \int \omega \nabla \log \omega \cdot \nabla \left( \log \omega - b\psi - a\frac{|x|^2}{2} \right)$$

$$= -\nu \int \omega |\nabla \left( \log \omega - a\frac{|x|^2}{2} - b\psi \right)|^2.$$  

This identity can be checked by direct computation and it is obvious by using the geometric interpretation of the equation (see [10]). Note that this identity is also useful to guess that asymptotic states should be given by the mean field solutions (1.3) since the entropy dissipation vanishes precisely on these states.

Because of these difficulties, we will be able to get global existence results only for data sufficiently close to a mean field solution. Nevertheless, we point out that our smallness constraint is independent of the viscosity parameter $\nu$. It remains an open problem to establish if eq. (1.6) can produce a singularity in a finite time and in particular if the $L^2$ norm of $\omega$ can blow up. Note that if we consider (1.6) without (1.7) i.e. we consider the equation with some given parameters $a < 0$ and $b$ fixed, it is easy to establish the existence of solution which blow-up. Indeed, since the inertial term does not play any part in the estimates, all the result established in [3] for the Keller-Segel equation remains true for this equation. In particular we have that the evolution of $I$ which is nonnegative is given by

$$\dot{I} = aI + \left(2 - \frac{b}{4\pi} \right)$$

and hence, if $b$ is larger than $8\pi$, the solution must blow up in finite time. It would be very interesting to know if there is a nonlinear stabilization for (1.6), (1.7).

The paper is organized as follows: the global existence proof is presented in Sect. 3 after that some preliminary steps are discussed in Sect. 2. Sect. 4 is devoted to the proof of $L^p$ estimates necessary both for the existence part and the asymptotic behavior discussed in Sect. 5. Finally, the Appendix is devoted to the study of equation (1.3) and the connected variational principles. The main ideas are in [7], [8], but the adaptation of these results to the $\mathbb{R}^2$ case requires some care.

## 2 Preliminaries

Let us introduce the submanifold of probability densities

$$\mathcal{M}(E, I) = \left\{ \omega, \ 0 \leq \omega \leq 1, \ E(\omega) = E, \ I(\omega) = I \right\}$$

for some fixed $E$ and $I > 0$. Next, by using Theorems 12, 14 in the Appendix, we denote the unique minimizer of the entropy functional $S(\omega)$ on $\mathcal{M}(E, I)$ by $\omega_{MF}$. Note that $\omega_{MF}$ is a (radial) solution to eq. (1.3) with parameters $b$ and $a$ that we denote by $b_{MF}$ and $a_{MF}$ respectively.

### 2.1 A stability property

We first establish a crucial result asserting the continuity of the $L^1$ norm with respect to variation of the entropy, in a neighbourhood of $\omega_{MF}$, in the manifold $\mathcal{M}(E, I)$.
**Theorem 1** For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\omega \in \mathcal{M}(E,I)$ for which

\[ S(\omega) - S(\omega_{MF}) \leq \delta, \quad (2.2) \]

then

\[ \|\omega - \omega_{MF}\|_{L^1} \leq \varepsilon. \quad (2.3) \]

We remark that Thm 1 provides a proof of the Lyapounov stability of $\omega_{MF}$ with respect to the Euler flow (by virtue of the time invariance of $S(\omega)$). The stability of $\omega_{MF}$ was also proved in [7] by using the arguments of [20] in which all the conserved quantities of the Euler equation are used.

**Proof of Theorem 1.** Assume, by contradiction, the existence of a sequence $\omega_n \in \mathcal{M}(E,I)$ and $\varepsilon > 0$ such that

\[ \lim_n S(\omega_n) = S(\omega_{MF}) \quad \text{and} \quad \|\omega_n - \omega_{MF}\|_{L^1} \geq \varepsilon. \quad (2.4) \]

Thanks to the entropy bound, we can find a probability distribution $\omega \in L^1$ such that, up to the extraction of a subsequence, $\lim_n \omega_n = \omega$ in the sense of the weak convergence of measures. Next, we also have that

\[ \lim_n E(\omega_n) = E(\omega) = E \]

(see the proof of [A,11] in the Appendix) and that

\[ \lim_n I(\omega_n) = I \geq I(\omega), \quad \lim_n S(\omega_n) \geq S(\omega) \]

by convexity. Let now $a_{MF}, b_{MF}$ be the multipliers for which $\omega_{MF}$ solves eq.n (1.16) with those values of parameters, and $F(a_{MF}, b_{MF})$ the free energy functional (see (A,1) for the definition). We get, since $a_{MF} < 0$, that

\[ F(a_{MF}, b_{MF})(\omega) \leq \lim_n \left( S(\omega_n) - b_{MF}E(\omega_n) - a_{MF}I(\omega_n) \right) = \left( S(\omega_{MF}) - b_{MF}E(\omega_{MF}) - a_{MF}I(\omega_{MF}) \right) = F(a_{MF}, b_{MF})(\omega_{MF}). \]

Since $\omega_{MF}$ is the unique minimizer of $F(b_{MF}, a_{MF})$ (see Theorem 12), it follows that $\omega = \omega_{MF}$. As a consequence, we also get that

\[ \lim_n I(\omega_n) = I(\omega) = I(\omega_{MF}). \quad (2.5) \]

Finally, we consider the relative entropy

\[ S(\omega_n|\omega_{MF}) = \int \omega_n \log \left( \frac{\omega_n}{\omega_{MF}} \right) \]

\[ = S(\omega_n) - S(\omega_{MF}) + b_{MF} \int \left( \omega_{MF} - \omega_n \right) \psi_{MF} + a_{MF} \left( I(\omega_{MF}) - I(\omega) \right). \]

Now, we observe that $S(\omega_n) - S(\omega)$ goes to zero thanks to (2.4) and that

\[ b_{MF} \int \left( \omega_{MF} - \omega_n \right) \psi_{MF} + a_{MF} \left( I(\omega_{MF}) - I(\omega) \right) \]
also goes to zero by weak convergence and (2.5). Consequently, the relative entropy \( S(\omega_n|\omega_{MF}) \) goes to zero. Thus we conclude by using the Csiszar-Kullback inequality

\[
\|\omega_n - \omega_{MF}\|_{L^1}^2 \leq 2S(\omega_n|\omega_{MF}) \to 0
\]

which yields the desired contradiction. ■

2.2 Properties of the coefficients \( a(\omega), b(\omega) \).

Our next step is the study of the properties of \( a(\omega) \) and \( b(\omega) \). We shall use the notation

\[
b(\omega) = \frac{2I \int \omega^2 + 2V}{D(\omega)}, \quad a(\omega) = -\frac{2 \int \omega |\nabla \psi|^2 + V \int \omega^2}{D(\omega)},
\]

where

\[
D(\omega) = 2I \int \omega |\nabla \psi|^2 - V^2, \quad V = -\frac{1}{4\pi}.
\]

As we have seen in the introduction, one of the main difficulty is that the denominator \( D(\omega) \) may vanish for a vortex patch.

Before stating the result, we shall recall a useful set of inequalities which will be used throughout the paper (see e.g. [33]...).

Lemma 2 (Useful inequalities in \( \mathbb{R}^2 \))

\[\text{i)}\ \text{Sobolev-Gagliardo-Nirenberg} : \quad \text{The following inequalities hold for some } C > 0:\]

\[
||\omega||_{L^2} \leq C||\nabla \omega||_{L^1} \tag{2.8}
\]

\[
||\omega||_{L^2}^2 \leq C ||\omega||_{L^1} ||\nabla \omega||_{L^2}. \tag{2.9}
\]

\[\text{ii)}\ \text{Biot et Savart law} : \quad \text{Let } u = K \ast \omega \text{ with } K \text{ defined by (1.2), then we have}\]

\[
1 < p < 2, \ 2 < q < +\infty, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{2}, \quad ||u||_{L^q} \leq C||\omega||_{L^p}, \tag{2.10}
\]

\[
1 \leq p \leq 2, \ 2 < q \leq \infty, \quad \frac{1}{2} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad ||u||_{L^\infty} \leq C||\omega||_{L^p}^{\alpha} ||\omega||_{L^q}^{1-\alpha}, \tag{2.11}
\]

\[
1 < p < +\infty, \quad ||\nabla u||_{L^p} \leq C||\omega||_{L^p}. \tag{2.12}
\]

\[\text{iii)}\ \text{Interpolation in } L^p \text{ spaces} : \]

\[
1 \leq p < r < q \leq +\infty, \quad ||\omega||_{L^r} \leq 2||\omega||_{L^p}^\alpha ||\omega||_{L^q}^{1-\alpha}, \quad \alpha = \frac{p}{r} \frac{1 - \frac{r}{q}}{1 - \frac{r}{q}} \tag{2.13}
\]

We shall prove the following result:

Theorem 3 Suppose \( \omega \in L^p \cap M(E,I) \) for some \( p \geq 2 \).

1. We have:

\[
b(\omega_{MF}) = b_{MF}, \quad a(\omega_{MF}) = a_{MF} \tag{2.14}
\]

and \( D(\omega_{MF}) > 0 \).
2. $D(\omega), a(\omega)$ and $b(\omega)$ are continuous on $L^2$

3. Assume that $p > 2$, then if $S(\omega) - S(\omega_{MF})$ is sufficiently small, we have

$$|D(\omega) - D(\omega_{MF})| + |a(\omega) - a(\omega_{MF})| + |b(\omega) - b(\omega_{MF})| \leq C\|\omega - \omega_{MF}\|_{L^1}^{\alpha},$$  \hspace{1cm} (2.15)

for some $\alpha < 1$. Here $C$ depends on $E, I$ and $\|\omega\|_{L^p}$.

**Proof of Theorem 3.**

We first prove 1. Since $\omega_{MF}$ is a solution of the mean field equation (1.3), we have

$$\nabla \omega_{MF} = b_{MF} \omega_{MF} \nabla \psi_{MF} + a_{MF} \omega_{MF} x.$$

Taking the scalar product of this equation by $\nabla \psi_{MF}$ and $x$ we find

$$b_{MF} \int \omega_{MF} |\nabla \psi_{MF}|^2 + a_{MF} V = \int \omega_{MF}^2, \quad b_{MF} V + 2a_{MF} I = -2.$$  \hspace{1cm} (2.16)

The resolution of this two by two linear system precisely gives that

$$b(\omega_{MF}) = b_{MF}, \quad a(\omega_{MF}) = a_{MF}.$$  \hspace{1cm} (2.17)

Consequently, since the numerator in the definition of $b(\omega_{MF})$ and $a(\omega_{MF})$ is finite, we find that

$$D(\omega_{MF}) > 0.$$  \hspace{1cm} (2.18)

Next, we shall estimate the differences

$$\int \omega^2 - \int \omega_{MF}^2,$$  \hspace{1cm} (2.19)

$$\int \omega |\nabla \psi|^2 - \int \omega_{MF} |\nabla \psi_{MF}|^2.$$  \hspace{1cm} (2.20)

By Cauchy-Schwarz, we obtain

$$|\text{(2.19)}| \leq \|\omega - \omega_{MF}\|_{L^2} (\|\omega\|_{L^2} + \|\omega_{MF}\|_{L^2})$$

and hence we find

$$|\text{(2.17)}| \leq C \|\omega - \omega_{MF}\|_{L^1}^{\alpha},$$  \hspace{1cm} (2.21)

for some $\alpha > 0$, by using (2.13) with $r = 2$ and $q = 1$. Next, we split (2.18) as

$$\text{(2.18)} \leq \int (\omega - \omega_{MF}) |\nabla \psi|^2 + \int \omega_{MF} \nabla (\psi - \psi_{MF}) \cdot \nabla (\psi + \psi_{MF})$$

$$\leq \|\omega - \omega_{MF}\|_{L^2} \|u\|_{L^2}^2$$

$$+ \|\omega_{MF}\|_{L^2} (\|u\|_{L^4} + \|u_{MF}\|_{L^4}) \|u - u_{MF}\|_{L^4}.$$  \hspace{1cm} (2.22)

We notice that thanks to (2.10), the $L^4$ norm of the velocity is bounded in term of the $L^4$ norm of the vorticity. Therefore, by a new use of (2.13), we find

$$|\text{(2.18)}| \leq C \left(\|\omega - \omega_{MF}\|_{L^2} + \|\omega - \omega_{MF}\|_{L^4}\right) \leq C \|\omega - \omega_{MF}\|_{L^1}^{\alpha}.$$  \hspace{1cm} (2.23)
Next, by using \( (2.19), (2.20) \), we get that
\[
|D(\omega) - D(\omega_{MF})| \leq C\|\omega - \omega_{MF}\|_{L^1}^\alpha.
\]
Consequently, we can use Theorem 1 to get that
\[
D(\omega) \geq D(\omega_{MF}) - |D(\omega) - D(\omega_{MF})| \geq \frac{1}{2}D(\omega_{MF}) \tag{2.21}
\]
provided \( S(\omega) - S(\omega_{MF}) \) is sufficiently small. Finally, by using \( (2.21), (2.19), (2.20) \) and \( (2.16) \), we easily conclude the proof. ■

3 Global Existence and uniqueness

We start with a brief explanation about the construction of a classical local solution. Let \( \omega_0 \in L^p \) with \( p > 2 \) be the initial condition such that \( \omega_0 \in M(E,I) \). Let \( \omega_{MF} \) the unique Mean-Field solution associated to \( M(E,I) \) as above. We assume that \( S(\omega_0) - S(\omega_{MF}) \) is small enough so that, by virtue of Theorems 1, 3, we have
\[
|D(\omega_0) - D(\omega_{MF})| \leq \frac{1}{2}D(\omega_{MF})
\]
and also
\[
|a(\omega_0) - a(\omega_{MF})| \leq \frac{1}{2}|a(\omega_{MF})|, \quad |b(\omega_0) - b(\omega_{MF})| \leq \frac{1}{2}|b(\omega_{MF})|.
\]
This implies that we have the upper bound
\[
|a(\omega_0)| + |b(\omega_0)| \leq 2(|a(\omega_{MF})| + |b(\omega_{MF})|)
\]
and also that
\[
D(\omega_0) \geq \frac{1}{2}D(\omega_{MF}) > 0.
\]
Note that the positivity of \( D \) is important in order to stay away from the singularity.

For every \( p \geq 2 \), by using a standard iterative scheme, we can easily establish, a local existence and uniqueness result for a classical solution \( \omega \in C([0,T],L^p) \cap L^\infty([0,T],L^1((1+|x|^2)dx) \), for which
\[
\|\omega(t)\|_{L^p} \leq 2(\|\omega_0\|_{L^p} + \|\omega_{MF}\|_{L^p}), \tag{3.1}
\]
and
\[
D(\omega(t)) \geq \frac{1}{4}D(\omega_{MF}), \quad \forall t \in [0,T]. \tag{3.2}
\]
Moreover, we can continue the solution as long as the \( L^p \) norm of \( \omega \) remains finite and the denominator \( D(\omega) \) remains positive.

Note that \( L^2 \) seems the natural space for our equation in order to have \( a \) and \( b \) well-defined. Note also that the condition \( (3.2) \) allows to avoid the singularity of \( D(\omega) \) by Theorem 3.

Let \( T > 0 \) be the maximal time for which, the estimates \( (3.1), (3.2) \) are verified. Our purpose is to prove, by a priori estimates, that \( T = +\infty \).

To do this we shall use the \( L^p \) estimates given by the following Theorem which will be proven in the next section.
Theorem 4  Consider a local solution as above of (1.6) such that
\[ |a(\omega(t))| + |b(\omega(t))| \leq C_0, \quad \forall t \in [0, T]. \]  
Assume also that \( \omega_0 \in L^p, \ p \in [2, +\infty) \) is a probability density. Then there exists \( C_p \) which depends only on \( \omega_0, C_0 \) and \( p \) (and hence does not depend on \( \nu \) and \( T \) if \( C_0 \) does not) such that
\[ ||\omega(t)||_{L^p} \leq C_p, \quad \forall t \in [0, T]. \]  

We are now in position to get a global existence result by showing that \( T = +\infty \). Indeed by the H-Theorem (see (1.22)), we have
\[ S(\omega(t)) - S(\omega_{MF}) \leq S(\omega_0) - S(\omega_{MF}), \quad \forall t \in [0, T] \]  
and thus by Theorem 1, we get that \( ||\omega(t)||_{L^1} \leq \varepsilon, \forall t \in [0, T] \) (with \( \varepsilon \) independent of \( T \)), provided that \( S(\omega_0) - S(\omega_{MF}) \) is sufficiently small. Next, we can use Theorem 4. Indeed, because of (3.1), (3.2) we have the bound \( |a| + |b| \leq C_0 \) on \( [0, T] \) for some \( C_0 > 0 \). This yields a control of the \( L^p \) norm of \( \omega \) with \( p > 2 \), depending only on \( C_0 \). In particular, thanks to (3.2), we get that
\[ |D(\omega(t)) - D(\omega_{MF})| \leq C\varepsilon^n, \quad ||\omega(t) - \omega_{MF}||_{L^p} \leq C\varepsilon^n, \quad \forall t \in [0, T] \]  
where \( C \) depends on \( C_0 \) only. Consequently, we can choose \( \varepsilon \) sufficiently small to have
\[ ||\omega(t)||_{L^p} < 2(||\omega_0||_{L^p} + ||\omega_{MF}||_{L^p}), \quad D(\omega(t)) > \frac{1}{4} D(\omega_{MF}), \quad \forall t \in [0, T]. \]
and hence \( T = +\infty. \) ■

Therefore we have proven the following global existence result:

Theorem 5  There exists \( \delta_0 > 0 \) (independent of \( \nu > 0 \)) such that, for any initial datum \( \omega_0 \in L^p \cap M(E, I) \) with \( p > 2 \), close to \( \omega_{MF} \) in the sense that
\[ S(\omega_0) - S(\omega_{MF}) \leq \delta_0, \]  
there exists a unique classical solution
\[ \omega(t) \in C([0, +\infty[, L^p) \cap L^\infty([0, T], M(E, I)) \]
to eq.n (1.6) with initial datum \( \omega_0 \).

Moreover, we have the Lyapounov stability of \( \omega_{MF} \), namely, for any \( \varepsilon > 0 \), there exists \( \delta, 0 < \delta \leq \delta_0 \) such that if \( S(\omega_0) - S(\omega_{MF}) \leq \delta \), then
\[ ||\omega(t) - \omega_{MF}||_{L^1} \leq \varepsilon, \quad \forall t \geq 0. \]

As noticed after Theorem 1, \( \omega_{MF} \) is also Lyapounov stable as a stationary solution of the Euler equation (1.1). Consequently, we also have the following global stability result between the flows of (1.6) and (1.1) in the vicinity of \( \omega_{MF} \):

Corollary 6  For every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \omega_0 \in L^\infty \cap M(E, I) \) verifies
\[ S(\omega_0) - S(\omega_{MF}) \leq \delta \]
then the global solution \( \omega^E \) of Euler equation (1.1) and the global solution \( \omega^\nu \) of (1.6) for \( \nu > 0 \) with the same initial datum \( \omega_0 \) satisfy
\[ ||\omega^\nu(t) - \omega^E(t)||_{L^1} \leq \varepsilon, \quad \forall t \geq 0. \]

Note that this global approximation property of the Euler evolution (even for \( \nu \) large) is of course false for the Navier-Stokes evolution.

10
4 Propagation of $L^p$ regularity

In this section we prove Theorem 4. Our strategy in proving Theorem 4 will be based on weighted energy estimates because in this way we can use the fact that the inertial term $u \cdot \nabla \omega$ does not contribute. This is crucial in order to find estimates independent of $\nu$.

We first focus on the $L^2$ estimate. We shall use the notation

$$\overline{B} = \sup_{t \in [0,T]} (|a(t)| + |b(t)|).$$

The standard $L^2$ energy estimate for (1.6) gives:

$$\frac{d}{dt} \left( \frac{1}{2} \|\omega(t)\|_{L^2}^2 \right) + \nu \| \nabla \omega(t) \|_{L^2}^2 = \nu \left( b(t) \int \omega(t)^3 - a(t) \int \omega(t)^2 \right). \tag{4.1}$$

Next, as in [3], [14], we can use the Sobolev inequality (2.9) to get (we recall that $\int \omega = 1$)

$$\int \omega^3 = \int (\omega^{\frac{3}{2}})^2 \leq \frac{9}{4} C \left( \int |\nabla \omega| \omega^{\frac{1}{2}} \right)^2 \leq \frac{9}{4} C \| \nabla \omega \|_{L^2}^2$$

where $C$ is the best constant in the Gagliardo-Nirenberg-Sobolev inequality. Consequently we get

$$\frac{d}{dt} \left( \frac{1}{2} \|\omega(t)\|_{L^2}^2 \right) + \nu \| \nabla \omega(t) \|_{L^2}^2 \leq \nu C \overline{B} \| \nabla \omega(t) \|_{L^2}^2 + \nu \overline{B} \| \omega(t) \|_{L^2}^2$$

where $C$ is an explicit harmless number. Now, let us assume for the moment that $C \overline{B}$ is sufficiently small (less than $1/2$ for example), then we can deduce that

$$\frac{d}{dt} \left( \frac{1}{2} \|\omega(t)\|_{L^2}^2 \right) + \frac{1}{2} \| \nabla \omega(t) \|_{L^2}^2 \leq \frac{1}{2} \nu \overline{B} \| \omega(t) \|_{L^2}^2. \tag{4.2}$$

If we directly integrate this differential inequality, we still cannot conclude. Indeed we shall find that $\| \omega(t) \|_{L^2}$ grows exponentially in time and this does not allow us to get a uniform in time estimate. Note however that the bad term in the right hand-side of (4.7) comes from the linear term $\nabla \cdot (x \omega)$ in eqn (1.6). The explanation for the bad behaviour we get is simple: the semigroup generated by the linear Fokker-Planck operator

$$L \omega = \Delta \omega + \overline{B} \nabla \cdot (x \omega),$$

with $\overline{B} > 0$, is not uniformly bounded in time as an operator in $L(L^2)$. Nevertheless, it is bounded as an operator in $L(L^1 \cap L^2, L^2)$. A very simple way to see this property is to use a weighted energy estimate. Indeed, multiplying (4.1) by $e^{\nu \overline{B} t}$, we find

$$\frac{d}{dt} \left( \frac{e^{\nu \overline{B} t}}{2} \|\omega(t)\|_{L^2}^2 \right) + \nu e^{\nu \overline{B} t} \| \nabla \omega(t) \|_{L^2}^2 \leq \frac{3}{2} \nu \overline{B} e^{\nu \overline{B} t} \| \omega(t) \|_{L^2}^2, \quad \forall t \in [0,T]. \tag{4.3}$$

Now, we can use (2.9) and the Young inequality to get, for some harmless explicit number $C$ (independent of $\nu$) which changes from line to line,

$$\frac{3}{2} \nu \overline{B} e^{\nu \overline{B} t} \| \omega(t) \|_{L^2}^2 \leq C \nu \overline{B} e^{\nu \overline{B} t} \| \nabla \omega(t) \|_{L^2}^2 \leq \frac{1}{4} \nu e^{\nu \overline{B} t} \| \nabla \omega(t) \|_{L^2}^2 + C \nu \overline{B}^2 e^{\nu \overline{B} t}.$$
Consequently, we can plug this last inequality in (4.3) to get
\[ \frac{d}{dt} \left( \frac{e^{\nu \|\omega\|^2}}{2} \right) \leq C \nu e^{\nu \|\omega\|^2}. \]
The integration finally gives
\[ \|\omega(t)\|^2 \leq \|\omega_0\|^2 + C B, \quad \forall t \in [0, T]. \]

We now remove the assumption on the smallness of $B$ and prove also the propagation of the $L^p$ regularity in the general case $p \geq 2$.

The starting point is to use the idea of [14], [3] in the study of the Keller-Segel equation. For $K > 1$, a parameter which will be fixed later, we define
\[ m_K(t) = \int (\omega(t) - K)^+ \, dx. \]
We note that
\[ m_K(t) \leq \int_{\omega(t) \geq K} \omega(t) \leq \frac{1}{\log K} \int \omega(t) \log(\omega(t)) \, dx \leq \frac{1}{\log K} \int \omega(t) |\log \omega(t)|. \quad (4.4) \]
Now we can use the following useful inequality:

**Lemma 7** There exists $C > 0$, such that, for all probability distribution $\omega$, we have
\[ \int \omega |\log \omega| \leq S(\omega) + C(1 + I(\omega)) \quad (4.5) \]
We postpone the proof of the lemma to the end of the section.

Thanks to (4.5), we find that $\int \omega |\log \omega|$ is bounded in terms of the initial datum because the entropy is decreasing and $I(\omega)$ is constant. Thus
\[ m_K(t) \leq \frac{1}{\log K} C(\omega_0), \quad \forall t \geq 0. \quad (4.6) \]

Next, we can perform a modified $L^p$ energy estimate for the solution of (1.6). After a few integrations by parts, we find
\[ \frac{d}{dt} \left( \frac{1}{p} \int (\omega - K)^p_+ \right) + \nu(p - 1) \int (\omega - K)^{p-2}_+ |\nabla(\omega - K)_+|^2 \]
\[ = \nu B \left( \int (\omega - K)^{p+1}_+ + 2K \frac{2}{p} + (K^2 + 2K) \int (\omega - K)^{p-1}_+ \right). \quad (4.7) \]
To estimate the first term in the right hand side of (4.7), we use the Sobolev-Gagliardo-Niremberg inequality [2,8] and Cauchy-Schwarz. We have
\[ \int (\omega - K)^{p+1}_+ = \int \left( (\omega - K)^{\frac{p+1}{2}}_+ \right)^2 \leq C \frac{(p+1)^2}{4} \int (\omega - K)^{\frac{p+1}{2}}_+ |\nabla(\omega - K)_+|^2 \]
\[ \leq C \frac{(p+1)^2}{4} m_K(t) \int (\omega - K)^{p-2}_+ |\nabla(\omega - K)_+|^2. \]
This yields, thanks to our assumption (3.3) and (4.6),
\[ \nu \int B (\omega - K)_+^{p+1} \leq \nu \frac{C_0 S_0 (p + 1)^2}{4 \log K} \int (\omega - K)_+^{p+2} |\nabla (\omega - K)_+|^2. \]

Hence, by choosing \( K \) such that
\[ \frac{C_0 S_0 (p + 1)^2}{4 \log K} = \frac{1}{2} (p - 1), \]
we obtain
\[ \nu \int B (\omega - K)_+^{p+1} \leq \nu \frac{(p - 1)}{2} \int (\omega - K)_+^{p-2} |\nabla (\omega - K)_+|^2. \]  
(4.8)

Note that \( K \) depends only on \( C_0, S_0, p \) and is diverging with \( p \).

To estimate the last term in (4.7), we write
\[ \int (\omega - K)_+^{p-1} \leq \int_{K \leq \omega \leq K+1} (\omega - K)_+^{p-1} + \int_{\omega \geq K+1} (\omega - K)_+^{p-1} \]
\[ \leq 1 + \int (\omega - K)_+^{p} \]  
(4.9)

By plugging (4.9) and (4.8) in (4.7), we find
\[ \frac{d}{dt} \left( \frac{1}{p} \int (\omega - K)_+^{p} \right) + \frac{\nu (p - 1)}{2} \int (\omega - K)_+^{p-2} |\nabla (\omega - K)_+|^2 \]
\[ \leq \nu C_0 \left( K^2 + 4K + \frac{2}{p} \int (\omega - K)_+^{p} + K^2 + 2K \right) \]
\[ \leq C_0 C \nu \left( \int (\omega - K)_+^{p} + 1 \right) \]

where, from now on, \( C \) is a harmless number which depends only on \( K \) and \( p \). Again, we note that we cannot directly conclude by using the Gronwall Lemma in the last differential inequality because it gives an estimate which is not uniform in time. We now use the technique that we have explained in the beginning. We find
\[ \frac{d}{dt} \left( e^{\nu t} \frac{1}{p} \int (\omega - K)_+^{p} \right) + \frac{\nu (p - 1)}{2} e^{\nu t} \int (\omega - K)_+^{p-2} |\nabla (\omega - K)_+|^2 \]
\[ \leq C(1 + C_0) \nu e^{\nu t} \left( \int (\omega - K)_+^{p} + 1 \right). \]  
(4.10)

Next, we can use the inequality (2.9) to get
\[ \int (\omega - K)_+^{p} = \int \left( (\omega - K)_+^{p} \right)^2 \leq C \int (\omega - K)_+^{p} \left( \int (\omega - K)_+^{p-2} |\nabla (\omega - K)_+|^2 \right)^{\frac{1}{2}}. \]  
(4.11)

By using the interpolation inequality (2.13) of \( L^{p/2} \) between \( L^1 \) and \( L^p \), we have
\[ \int (\omega - K)_+^{p} \leq C \left( \int (\omega - K)_+^{p} \right)^{\frac{p-2}{2(p-1)}} \]
and hence, we deduce from (4.11) that
\[
\int (\omega - K)^p_+ \leq C \left( \int (\omega - K)^{p-2}_+ |\nabla (\omega - K)_+|^2 \right)^{\frac{1}{q}}
\]
where \( q \) is such that \( p^{-1} + q^{-1} = 1 \). Thanks to the inequality
\[
ab \leq a^{p^{-1}}b^{q^{-1}} \quad a \geq 0, \ b \geq 0,
\]
we finally obtain
\[
C(1 + C_0) \int (\omega - K)^p_+ \leq \frac{p-1}{4} \int (\omega - K)^{p-2}_+ |\nabla (\omega - K)_+|^2 + C_p,
\]
where \( C_p \) will now stand for a number which depends only on \( \omega_0, C_0 \) and \( p \). By using this last inequality in (4.10), we finally arrive to
\[
\frac{d}{dt} \left( e^{\nu t} \frac{1}{p} \int (\omega - K)^p_+ \right) \leq C_p \nu e^{\nu t}.
\]
The integration gives
\[
\int (\omega - K)^p_+ \leq C_p.
\]
Now we can conclude as in [14], [3]. By using the inequality
\[
x^p \leq \left( \frac{\lambda}{\lambda - 1} \right)^{p-1} (x - 1)^p
\]
for every \( x \geq \lambda > 1 \), we find
\[
\int \omega^p \leq \int_{\omega \leq K} \omega^p + \int_{\omega > K} \omega^p \leq K^{p-1} + \int_{K < \omega \leq \lambda K} \omega^p + \int_{\omega \geq \lambda K} \omega^p \leq K^{p-1} + (\lambda K)^{p-1} + \frac{\lambda}{\lambda - 1} \int_{\omega \geq \lambda K} \left( \frac{\omega}{k} - 1 \right)^p \leq K^{p-1} + (\lambda K)^{p-1} + \frac{\lambda}{\lambda - 1} \int (\omega - K)^p_+.
\]
This ends the proof of Theorem 4.\( \blacksquare \).

It remains to prove Lemma 7 which is a classical estimate we present for completeness. Define \( \overline{\omega} = \omega 1_{|\omega| \leq 1} \). Since we have
\[
\int |\omega| \log |\omega| = S(\omega) - 2 \int \overline{\omega} \log \overline{\omega},
\]
(4.14)
it suffices to find a bound from below of \( \int \overline{\omega} \log \overline{\omega} \). By using the fact that the relative entropy between two probability measures is non negative (this is an easy consequence of the Jenssen inequality) we get
\[
\int_{\mathbb{R}^2} (\overline{\omega}/\overline{m}) \log \left( \frac{\overline{\omega}/\overline{m}}{2\pi e^{\frac{|x|^2}{2}}} \right) \geq 0
\]
where $\overline{m} = \int_{\mathbb{R}^2} \omega \leq 1$. Then we get

$$\int \omega \log \omega \geq -\overline{m} I(\omega) + \overline{m} \log \frac{1}{2\pi} - \overline{m} \log \overline{m} \geq -I(\omega) + \log \frac{1}{2\pi} - \frac{1}{e}$$

and hence we get (4.5) by using this last estimate and (4.14). ■

5 Asymptotic behaviour

In this section we investigate the asymptotic behaviour of the global solutions given by Theorem 5. More generally, one can consider a global solution $\omega(t)$ of (1.6) such that $\omega \in C([0, +\infty[; L^2 \cap L^1((1 + |x|^2)dx)$ and such that $\omega(t) \in \mathcal{M}(E, I)$ and which satisfies for some $C > 0$, the uniform estimates

$$||\omega(t)||_{L^2} \leq C, \quad |b(\omega(t))| + |a(\omega(t))| \leq C, \quad \forall t \geq 0.$$  \hfill (5.1)

The main result of this section is given by the following Theorem.

**Theorem 8** Let $\omega(t)$ a global solution of (1.6) as above which satisfies (5.1). Then $\omega(t)$ converges in $L^1$, as $t \to \infty$, to the unique solution $\omega_{MF} \in \mathcal{M}(E, I)$ of the associated microcanonical variational problem.

Note that the solutions constructed in Theorem 5 satisfy the estimate (5.1) and hence their asymptotic behaviour is given by Theorem 8.

**Proof of Theorem 8**

The first step consists in proving that the orbit $\{\omega(t)\}_{t \geq 0}$ is relatively compact in $L^1$ and uniformly bounded in $L^\infty$. Before, we need to study the evolution operator generated by the non-autonomous Fokker-Planck type operator

$$L_\gamma \omega = \nu (\Delta \omega + \gamma(t) \nabla \cdot (x \omega))$$

where $\gamma(t)$ is a given continuous curve. Denoting by $S_\gamma(t, \tau) \omega_0$ the solution of

$$\partial_t \omega = L_\gamma \omega, \quad t > \tau, \quad \omega(\tau) = \omega_0,$$

we have the following estimates

**Lemma 9** Suppose that for all $t \geq 0$, $|\gamma(t)| \leq K_0$, for some $K_0 > 0$. Then, there exists $C > 0$ independent of $\nu > 0$ and such that for $(p, q, r) \in [1, +\infty]^3$ we have:

$$||S_\gamma(t, \tau)\omega||_{L^p} \leq CK_0^{1 - \frac{1}{r}} \frac{e^{2\nu K_0 (1 - \frac{1}{r})(t-\tau)}}{(1 - e^{-2\nu K_0 (t-\tau)})^{1 - \frac{1}{r}}} ||\omega||_{L^q}, \quad \frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{p}, \quad \hfill (5.2)$$

$$||\nabla S_\gamma(t, \tau)\omega||_{L^p} \leq CK_0^{\frac{3}{2} - \frac{1}{r}} \frac{e^{2\nu K_0 (1 - \frac{1}{r})(t-\tau)}}{(1 - e^{-2\nu K_0 (t-\tau)})^{\frac{3}{2} - \frac{1}{r}}} ||\omega||_{L^q}, \quad \frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{p}, \quad \hfill (5.3)$$

$$||S_\gamma(t, \tau)\nabla \omega||_{L^p} \leq CK_0^{\frac{3}{2} - \frac{1}{r}} \frac{e^{\nu K_0 (1 - \frac{2}{q})(t-\tau)}}{(1 - e^{-2\nu K_0 (t-\tau)})^{\frac{3}{2} - \frac{1}{q}}} ||\omega||_{L^q}, \quad \frac{1}{r} + \frac{1}{q} = 1 + \frac{1}{p}, \quad \hfill (5.4)$$

for all $p \in [1, +\infty]$.
Proof of Lemma 9

A simple computation in Fourier space allows to find the explicit representation

\[ S_\gamma(t, \tau) \omega(x) = e^{2\nu(B(t) - B(\tau))} \int \frac{e^{-\frac{|x-y|^2}{4 \nu \int_t^{\tau} e^{2\nu(B(s) - B(t))} ds}}}{\pi \nu} \omega_0(e^{\nu(B(t) - B(\tau))} y) dy \tag{5.5} \]

where \( B(t) = \int_0^t \gamma(s) \, ds \). The result of Lemma 9 then follows by standard convolution estimates.

We come back to the proof of Theorem 8. We shall prove that \( ||\nabla \omega(t)||_{L^1} \) is uniformly bounded. We use the same idea as in [12]. Note that the solution of (1.6) can be written as

\[ \omega(t) = S_\alpha(t,0) \omega_0 + \int_0^t S_\alpha(t,\tau) \nabla \cdot \left( -u \omega + \nu bu^\perp \omega \right)(\tau) \, d\tau. \tag{5.6} \]

Moreover, thanks to (5.1), we have a uniform estimate on \( |\nabla \omega(t)| \) for all times : there exists \( C_0 > 0 \) such that

\[ ||\nabla \omega(t)||_{L^2} \leq C_0, \quad |a(t)| + |b(t)| \leq K_0, \quad \forall t \geq 0. \tag{5.7} \]

Consequently, in (5.6), we can consider \( a \) and \( b \) as known and we can use the estimates of Lemma 9. Let us define \( F(\omega) \) as the right-hand side of (5.6). We have

\[ ||F(\omega(t))||_{L^\infty} \leq C \left( \frac{C_0 e^{\nu K_0 t}}{a_\nu(t)^{\frac{3}{2}}} + C_0 (1 + \nu) \int_0^t \frac{e^{\nu K_0 (t-s)}}{a_\nu(t-s)^{\frac{3}{4}}} ||u \omega||_{L^4} \, ds \right), \]

where \( a_\nu(t) = 1 - e^{-2\nu K_0 t} \). Since by (2.10), (2.13) and the uniform \( L^2 \) bound, we have :

\[ ||u \omega||_{L^4} \leq ||\omega||_{L^\infty} ||u||_{L^4} \leq C ||\omega||_{L^\infty} ||\omega||_{L^4} \leq CC_0 ||\omega||_{L^\infty}, \]

we finally get

\[ ||F(\omega(t))||_{L^\infty} \leq C \left( \frac{C_0 e^{\nu K_0 t}}{a_\nu(t)^{\frac{3}{2}}} + C_0 (1 + \nu) \int_0^t \frac{e^{\nu K_0 (t-s)}}{a_\nu(t-s)^{\frac{3}{4}}} ||\omega(s)||_{L^\infty} \, ds \right). \]

Consequently, we can set

\[ z(T) = \sup_{[0,T]} \left( e^{-\nu K_0 t} a_\nu(t)^{\frac{3}{2}} ||\omega(t)||_{L^\infty} \right) \]

to get

\[ z(T) \leq CC_0 (1 + \nu) \left( 1 + a_\nu(T)^{\frac{3}{2}} \left( \int_0^T \frac{1}{a_\nu(T-s)^{\frac{3}{4}}} \frac{ds}{a_\nu(s)^{\frac{3}{2}}} \right) z(T) \right). \]

Next, we notice that

\[ \lim_{T \to 0} a_\nu(T)^{\frac{1}{2}} \int_0^T \frac{1}{a_\nu(T-s)^{\frac{3}{4}}} \frac{1}{a_\nu(s)^{\frac{3}{2}}} = 0, \]

16
therefore, there exists \( T(\nu, C_0) > 0 \) such that

\[
z(T(\nu, C_0)) \leq CC_0(1 + \nu) + \frac{1}{2}z(T(\nu, C_0))
\]

and hence, we get that

\[
\|\omega(t)\|_{L^\infty} \leq \frac{C(\nu, C_0)}{a_\nu(t)^{\frac{1}{2}}}, \quad \forall t \in [0, T(\nu, C_0)]. \tag{5.8}
\]

Next, since to establish (5.8) we have only used (5.7), we can consider for every \( \nu, C_0 \) with initial value \( \omega(nT(\nu, C_0)/2) \). By the above argument, we get that \( \tilde{\omega} \) satisfies the estimate (5.9). By uniqueness, we have

\[
\tilde{\omega}(t) = \omega(t + nT(\nu, C_0)/2), \quad \forall t \in [0, T(\nu, C_0)]
\]

and hence,

\[
\|\omega(t)\|_{L^\infty} \leq C(\nu, C_0) \left(1 + \frac{1}{a_\nu(t)^{\frac{1}{2}}}\right), \quad \forall t \geq 0. \tag{5.9}
\]

for some \( C(\nu, C_0) \).

In a similar way we have by Duhamel formula

\[
\nabla \omega(t) = \nabla S(t, 0) + \omega_0 \int_0^t \nabla S(t - \tau) \left(- u \cdot \nabla \omega + \nu b u \perp \cdot \nabla \omega + \nu b \omega^2 \right)(s) ds \tag{5.10}
\]

\[
||\nabla F(\omega)||_{L^1} \leq C \left(\frac{C_0}{a_\nu(t)^{\frac{1}{2}}} + (1 + \nu)C_0 \int_0^t \frac{1}{a_\nu(t - s)^{\frac{1}{2}}} (||u \nabla \omega(s)||_{L^1} + C_0) ds\right)
\]

and since we have by (2.11)

\[
||u \nabla \omega||_{L^1} \leq C ||u||_{L^\infty} ||\nabla \omega||_{L^1},
\]

we get, thanks to (5.9),

\[
||\nabla F(\omega)||_{L^1} \leq C \left(\frac{C_0}{a_\nu(t)^{\frac{1}{2}}} + C_0 \int_0^t \frac{1}{a_\nu(t - s)^{\frac{1}{2}}} ds + C(\nu, C_0) \int_0^t \frac{1}{a_\nu(t - s)^{\frac{1}{2}}} a_\nu(\tau)^{\frac{1}{2}} ||\nabla \omega(s)||_{L^1} ds\right).
\]

Consequently, by using the same method as before, we can easily obtain

\[
||\nabla \omega(t)||_{L^1} \leq C(\nu, C_0) \left(1 + \frac{1}{a_\nu(t)^{\frac{1}{2}}}\right), \quad \forall t \geq 0 \tag{5.11}
\]

for some \( C(\nu, C_0) \).

We now consider \( \Omega \), the omega limit set of the trajectory \( \{\omega(t)\}_{t \geq 0} \). We deduce from the previous estimates that the positive orbit \( \{\omega(t)\}_{t \geq 0} \) is relatively compact in \( X = L^1((1 + |x|^{\alpha}) \, dx) \cap L^2 \) for \( \alpha < 2 \). Indeed, since \( \omega(t) \in C([0, +\infty[, X) \), it suffices to prove that \( \{\omega(t)\}_{t \geq 1} \) is relatively compact. The compactness in \( L^1((1 + |x|^{\alpha}) \) follows immediately from the Riesz-Frechet-Kolmogorov criterion: \( \omega(t) \) is uniformly bounded in \( L^1 \), the uniform (for \( t \geq 1 \)) bound (5.11) gives the equi-integrability and we have a uniform bound on the moment of inertia to control the mass far away. Next, thanks
to the uniform $L^\infty$ estimate for $t \geq 1$ given by (5.9) and the relative compactness in $L^1$, we also get that $\{\omega(t)\}_{t \geq 0}$ is relatively compact in $L^p$ for every $p < +\infty$.

By the relative compactness properties that we have just proven, we get that $\Omega$ is non empty and actually made by smooth $L^p$ functions thanks to the smoothing effect of the parabolicity. Moreover, we also have that the elements of $\Omega$ are probability densities. Also, if $\omega \in \Omega$, since there exists an increasing sequence $t_n$ such that $\omega(t_n)$ tends to $\omega$ in $X$, we also have

$$E(\omega) = \lim_n E(\omega(t_n)) = E, \quad I(\omega) \leq \lim_n I(\omega(t_n)) = I.$$  \hfill (5.12)

The first equality is proven in the Appendix, see (A.11). Finally, we notice that the entropy $S$ is constant on $\Omega$. Indeed, if $\omega_1, \omega_2 \in \Omega$, we can construct an increasing sequence $t_n$ such that $\omega(t_{2n})$ tends to $\omega_1$ and $\omega(t_{2n+1})$ tends to $\omega_2$ almost everywhere and such that there exists $g_1, g_2 \in L^1((1 + |x|^2)dx) \cap L^2$ with

$$\omega(t_{2n}) \leq g_1, \quad \omega(t_{2n+1}) \leq g_2.$$

By using that

$$\omega |\log \omega| \leq C\left(\omega^2 + |\omega|^{\frac{3}{4}}\right) \leq C\left(\omega^2 + (1 + |x|)\omega + \frac{1}{(1 + |x|)^{\frac{3}{4}}}\right),$$

we find by Lebesgue Theorem that

$$S(\omega_1) = \lim_n S(\omega(t_{2n})), \quad S(\omega_2) = \lim_n S(\omega(t_{2n+1})).$$

But, since the entropy is decreasing, we also have $S(\omega(t_{2n})) \leq S(\omega(t_{2n+1})) \leq S(\omega(t_{2n+2}))$ so that passing to the limit, we get

$$S(\omega_1) \geq S(\omega_2) \geq S(\omega_1)$$

and hence $S(\omega_1) = S(\omega_2)$.

Finally, we can prove that the elements of $\Omega$ are solutions of the mean field equation. If $\omega \in \Omega$, consider $\omega(t)$ the solution of (1.6) with initial value $\omega$. By the strong parabolic principle, we have that $\omega(t)$ is smooth and strictly positive for $t > 0$. Since $\Omega$ is invariant and $S$ is constant on it, the entropy dissipation identity (1.22) gives that for $t > 0$

$$\nabla \left(\log \omega(t) - b\psi(t) - a \frac{|x|^2}{2}\right) = 0.$$

By continuity in time we get that $\omega$ actually solves the mean field equation

$$\omega = \frac{1}{Z} e^{b\psi + a |x|^2}$$ \hfill (5.13)

in $\mathbb{R}^2$. Finally, by using the result of [26] and Lemma 4.3 of [3], we get that $\omega$ is radially symmetric. Note that, we also necessarily have that $a < 0$ and $b < 8\pi$.

To summarize, we have proven that the omega limit set $\Omega$ of $\{\omega(t)\}_{t \geq 0}$ is made by probability densities which are radially symmetric solutions of the mean field equation (5.13) with finite energy equals to $E$ and finite moment of inertia. Since the entropy separates the radial mean field solutions (see Remark 15 in the Appendix), we conclude that $\Omega$ consists in a single point. \hfill ■
A The Mean-Field Equation and related variational problems

In this Appendix we collect some useful facts concerning the Mean-Field Equation (MFE) in $\mathbb{R}^2$. The main ideas are in [7], [8], here, we adapt the results to the $\mathbb{R}^2$ case. For the microcanonical problem, the strategy of the proof is slightly different, we do not prove directly the existence of a solution. We focus on the negative temperature case (which corresponds to $b > 0$) which is the most interesting case.

**Definition 10 (Canonical Variational Principle)** For $a < 0$, $b > 0$, consider the free-energy functional

$$F_{a,b}(\omega) = S(\omega) - bE(\omega) - aI(\omega)$$  \hspace{1cm} (A.1)

defined on the space $\Gamma$ of probability densities on $\mathbb{R}^2$ for which $E(\omega)$, $I(\omega)$ and $S(\omega)$ are finite. We set

$$F(a,b) = \inf_{\omega \in \Gamma} F_{a,b}(\omega)$$  \hspace{1cm} (A.2)

**Definition 11 (Microcanonical Variational Principle)** For $E \in \mathbb{R}$ and $I > 0$ let us define

$$\mathcal{M}(E,I) = \{ \omega \in \Gamma : E(\omega) = E, I(\omega) = I \} .$$

We set

$$S = \inf_{\omega \in \mathcal{M}(E,I)} S(\omega)$$  \hspace{1cm} (A.3)

The main results of this Appendix are the two following theorems. For the positive temperature case (i.e. $b < 0$), $F_{b,a}$ is a convex functional so that there is a unique minimizer. Moreover, there exists a unique solution to the MFE [12] and all the following results are obvious.

**Theorem 12 (Canonical Variational Principle)** For $a < 0$, and $0 < b < 8\pi$ :

i) there exists $\omega \in \Gamma$ such that $F_{a,b}(\omega) = F(a,b)$. Moreover $\omega$ is radially symmetric and solves the mean field equation [12].

ii) There is only one radially symmetric solution of the mean field equation [12].

iii) As a consequence, there exists a unique minimizer $\omega_{a,b}$ of $F(a,b)$ over $\Gamma$.

**Remark 13** It is easy to prove that when $b \to 8\pi$, the solutions to the MFE concentrates at the origin. Indeed by multiplying the equation by $x \cdot \nabla \psi$ and integrating by parts, we arrive to the identity (that is the same argument leading to the Pohozaev inequality):

$$1 - \frac{2\pi a}{b} I = \frac{8\pi}{b} .$$  \hspace{1cm} (A.4)

Hence when $b \to 8\pi$, $I \to 0$ and the concentration takes place. For $b > 8\pi$ we do not have solutions.

As a consequence, we can solve the microcanonical variational principle.

**Theorem 14 (Microcanonical Variational Principle)** For $a < 0$, and $0 < b < 8\pi$ :

i) $F(a, b)$ is a concave smooth function.
ii) \[
\frac{\partial F}{\partial a} = -I(\omega_{a,b}) < 0, \quad \frac{\partial F}{\partial b} = -E(\omega_{a,b})
\] (A.5)

iii) For \( I > 0, E \in \mathbb{R} \) let us define \( S^*(I, E) \) as
\[
S^*(I, E) = \sup_{a,b} \left( F(a, b) + bE + aI \right),
\] (A.6)

Denote by \( a(I, E), b(I, E) \) the unique maximizer of (A.6), then for any \( I > 0, E \in \mathbb{R} \), \( S(I, E) = S^*(I, E) \), and hence \( S \) is a smooth convex function. Moreover, the microcanonical variational principle admits a unique minimizer \( \bar{\omega}_{I,E} \) in \( \Gamma_{I,E} \). Finally \( \bar{\omega}_{I,E} = \omega_{a(I,E),b(I,E)} \) (equivalence of the ensembles).

**Remark 15**

We finally underline that the function \( I \mapsto S(E,I) \) is strictly decreasing (\( \partial S/\partial I = a < 0 \)) so that different radial solutions of the MFE with the same energy cannot have the same entropy.

**Proof of Theorem 12**

By the logarithmic Hardy-Littlewood-Sobolev inequality (see [2], [9]), we have
\[
S(\omega) - 8\pi E(\omega) \geq -(1 + \log \pi).
\] (A.7)

Note that we also have the inequality (see (A.9))
\[
E(\omega) \geq -\frac{1}{8\pi} \log(4I(\omega)).
\] (A.8)

Indeed, we can write
\[
E = -\frac{1}{8\pi} \int \log |x-y|^2 \omega(x)\omega(y) \geq -\frac{1}{8\pi} \log \int |x-y|^2 \omega(x)\omega(y)
\] (A.9)

and hence
\[
E \geq -\frac{1}{8\pi} \log \left( 4I - 2 \left( \int x\omega \right)^2 \right) \geq -\frac{1}{8\pi} \log(4I).
\]

Consequently, thanks to (A.7), (A.8), we get that
\[
F_{(a,b)}(\omega) \geq -\frac{8\pi - b}{8\pi} \log(4I) - aI - (1 + \log \pi)
\] (A.10)

and hence, we find that \( F_{(b,a)} \) is bounded from below.

Let \( \omega_n \) be a minimizing sequence in \( \Gamma \). Up to the extraction of a subsequence, \( \omega_n \) converges in the sense of weak convergence of measures. Moreover, thanks to (A.10), we get that \( I(\omega_n) \) is uniformly bounded and, by using again (A.7), we also have
\[
(1 - \frac{b}{8\pi})S(\omega_n) \leq F(\omega_n) + \frac{1 + \log \pi}{8\pi}
\]

and hence \( S \) is bounded from above. Therefore the uniform integrability given by the bounds on \( S \) and \( I \) (which yields a bound on \( \int \omega |\log \omega| \) thanks to Lemma 7) implies that \( \omega_n \) converges to
a nonegative function $\omega$. Moreover, the uniform estimate on the moment of inertia provides the tightness of the sequence $\omega_n$ so that we obtain
\[
\lim_n \int \omega_n = \int \omega
\]
i.e. $\omega \in \Gamma$. Next, by lower semi-continuity, we have
\[
S(\omega) \leq \lim_n S(\omega_n), \quad I(\omega) \leq \lim_n I(\omega_n)
\]
and we claim that
\[
E(\omega) = \lim_n E(\omega_n). \tag{A.11}
\]
This proves that $F(a,b)(\omega) \leq \lim_n F(a,b)(\omega_n)$ and hence that $\omega$ is a minimizer. It remains to prove (A.11). We write
\[
E(\omega_n) = -\frac{1}{4\pi} \int \log |x-y| \omega_n(x)\omega_n(y) \, dx \, dy = I(\varepsilon) + J(\varepsilon)
\]
where
\[
I(\varepsilon) = -\frac{1}{4\pi} \int_{|x-y| \leq \varepsilon} \log |x-y| \omega_n(x)\omega_n(y) \, dx \, dy, \quad J(\varepsilon) = -\frac{1}{4\pi} \int_{|x-y| \geq \varepsilon} \log |x-y| \omega_n(x)\omega_n(y) \, dx \, dy
\]
for every $\varepsilon \in (0,1)$. By splitting the integration domain in $\{\omega_n(x)\omega_n(y) \leq |x-y|^{-1}\}$ and its complementary, we easily get that
\[
I(\varepsilon) \leq -C \int_{|x-y| \leq \varepsilon} |x-y|^{-1} \log |x-y| \, dx \, dy + 2C \left( \int \omega_n |\log \omega_n| \right) \sup_x \int_{|x-y| \leq \varepsilon} \omega_n(y) \, dy
\]
\[
\leq -C \int_{|x-y| \leq \varepsilon} |x-y|^{-1} \log |x-y| \, dx \, dy + C \sup_x \int_{|x-y| \leq \varepsilon} \omega_n(y) \, dy.
\]
For the last line, we have used that the entropy and the moment of inertia are uniformly bounded in $n$ and thanks to Lemma 7 we also have that $\int \omega_n |\log \omega_n|$ is uniformly bounded. This yields the uniform integrability:
\[
\lim_{\varepsilon \to 0} \sup_x \int_{|x-y| \leq \varepsilon} \omega_n(y) \, dy = 0.
\]
Consequently, we get that $\lim_{\varepsilon \to 0} I(\varepsilon) = 0$ uniformly in $n$. Finally, by weak convergence, we have that
\[
\lim_n J(\varepsilon) = -\frac{1}{4\pi} \int_{|x-y| \geq \varepsilon} \log |x-y| \omega(x)\omega(y) \, dx \, dy
\]
and since
\[
\lim_{\varepsilon \to 0} -\frac{1}{4\pi} \int_{|x-y| \geq \varepsilon} \log |x-y| \omega(x)\omega(y) \, dx \, dy = E(\omega),
\]
the conclusion follows easily.

By symmetrizing $\omega$ (around the origin) we find that $F(a,b)$ is decreasing. Indeed $S$ and $I$ are unchanged and $E$ is increasing. Thus $\omega$ must be radially symmetric. It is not difficult to show that
\( \omega > 0 \) (otherwise one could find a better distribution as regards the minimization problem). Hence \( \omega \) satisfies the MFE.

Next we show that such a solution is also unique among all the radial solution to the MFE. Setting \( r = |x| \) and \( \psi(r) = \psi(x) \) (by an obvious notational abuse), we have

\[
\frac{1}{r} (r \psi')' = -e^{b \psi + 2 \frac{a}{2} r^2}.
\]  
(A.12)

We are assuming that \( Z = 2\pi \int_0^\infty dr \, re^{b \psi + \frac{a}{2} r^2} = 1 \), adding, if necessary, a constant to \( \psi \). After the change of variable \( t = \log r \), setting \( H = b \psi + 2t \) we readily arrive to the following equation:

\[
\ddot{H} = -F(t)e^H
\]  
(A.13)

where

\[
F(t) = be^{\frac{a}{2} 2t}.
\]  
(A.14)

We are looking for smooth solutions to eq.n (18) and hence

\[
\lim_{t \to -\infty} \dot{H} = 2
\]  
(A.15)

as a consequence of the fact that \( \lim_{r \to 0} r \psi'(r) \to 0 \). \( H(t) \) behaves as \( 2t + \chi \) as \( t \to -\infty \) and \( \chi \) must be chosen in such a way that \( Z = 1 \). In the new variables:

\[ Z = \frac{2\pi}{b} \int_{-\infty}^\infty dt \, e^H F(t) = \frac{2\pi}{b} (\dot{H}(-\infty) - \dot{H}(\infty)) \]  
(A.16)

It is convenient to change the time variable by setting \( 2t \to 2t - \chi \), so that the problem can be reformulated as

\[
\ddot{H} = -F(t - \frac{\chi}{2})e^H
\]  
(A.17)

\[
\dot{H}(-\infty) = 2, \, H(t) \approx 2t \quad \text{for} \quad t \to -\infty.
\]

Note that

\[ Z(\chi) \to 0 \quad \text{for} \quad \chi \to -\infty \]  
(A.18)

and

\[ Z(\chi) \to \frac{8\pi}{b} \quad \text{for} \quad \chi \to +\infty. \]  
(A.19)

Eq.n (A) is obvious, while eq.n (A.19) comes out by integrating the Hamiltonian system

\[
\ddot{H} = -e^H
\]  
(A.20)

for which, the energy conservation yields \( \dot{H}(\infty) = -2 \).

Since \( \frac{8\pi}{b} > 1 \) the value \( Z = 1 \) is certainly taken, at least once. In order to get uniqueness it remain to show that \( \chi \to Z(\chi) \) is a monotone function, actually it is not decreasing.

Defining

\[
G = H + \frac{a}{2} e^{2(t - \frac{\chi}{2})},
\]  
(A.21)
we find the following set of non-autonomous equations

\begin{align*}
\ddot{H} &= -be^G \\
\ddot{G} &= -be^G - 4(H - G). \quad (A.22)
\end{align*}

Note that \(H, \dot{H}\) and \(G, \dot{G}\) satisfy the same condition at \(t = -\infty\).

On the other hand the derivatives \(\partial_\chi H = h\) and \(\partial_\chi G = g\) satisfy

\begin{align*}
\ddot{h} &= -be^G g \\
\ddot{g} &= (4 - be^G)g - 4h. \quad (A.23)
\end{align*}

The conditions at \(t \to -\infty\) are vanishing for both \(h, \dot{h}\) and \(g, \dot{g}\).

Introducing the energy \(E = \frac{1}{2} \dot{H}^2 + be^G\) we get:

\[ \dot{E} = ba e^G e^{2(t - \frac{\chi}{2})} \leq 0 \]

and hence

\[ be^G(t) \leq E(t) \leq E(-\infty) = 2. \quad (A.24) \]

Therefore \(\ddot{g} \geq 0\) as far as \(h \leq 0\) and \(\ddot{h} \leq 0\) as far as \(g \geq 0\). This conditions are indeed verified for \(t \approx -\infty\) so that they are true for all the time. Then \(G\) is increasing as well as

\[ Z = \frac{2\pi}{b} \int_{-\infty}^{\infty} dte^G. \]

\[ \square \]

**Proof of Theorem [14]**

To prove the concavity of \(F\) we will prove that, for any \(a_1, b_1, a_2, b_2:\)

\[ F\left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}\right) > \frac{1}{2} (F(a_1, b_1) + F(a_2, b_2)). \]

Let \(a = \frac{a_1 + a_2}{2}\), and \(b = \frac{b_1 + b_2}{2}\). By the linearity of \(F_{a,b}(\omega)\) as a function of \(a, b\) at fixed \(\omega\), we get that

\[ F(a, b) = F_{a,b}(\omega_{a,b}) = \frac{1}{2} F_{a_1,b_1}(\omega_{a,b}) + \frac{1}{2} F_{a_2,b_2}(\omega_{a,b}) \]

\[ > \frac{1}{2} F_{a_1,b_1}(\omega_{a_1,b_1}) + \frac{1}{2} F_{a_2,b_2}(\omega_{a_2,b_2}) = \frac{1}{2} (F(a_1, b_1) + F(a_2, b_2)), \]

where we used the fact that \(\omega_{a_1,b_1}\) and \(\omega_{a_2,b_2}\) are the minimizer for \(F_{a_1,b_1}\) and \(F_{a_2,b_2}\) respectively.

The smoothness of \(F\) comes from the fact that the solution of the canonical variational principle depends smoothly upon \(a, b\). By taking the derivative of \(F\) with respect to \(a\) we get

\[ \frac{\partial F}{\partial a} = \frac{\partial F_{a,b}(\omega_{a,b})}{\partial a} = -I(\omega_{a,b}). \]
Here we have used the fact that the derivative of $F$ with respect to $\omega$ evaluated in $\omega_{a,b}$ vanishes, and the fact that the derivative of $F_{a,b}$ with respect to the parameter $a$ is given by $I$. In the same way we get $\partial F(a,b)/\partial b = -E(\omega_{a,b})$.

Finally, the concavity of $F_{a,b}$ implies that $\partial I/\partial a > 0$, and that $\partial E/\partial b > 0$ again with the notation $I(a,b) = I(\omega_{a,b}), E(a,b) = E(\omega_{a,b})$.

Now it remains to prove iii). Again, the concavity of $F(a,b)$ implies the existence of the convex function $S^*(I,E)$ defined in (A.6). Now we want to prove that $S(I,E) = S^*(I,E)$. First of all let us notice that

$$S^*(I,E) = \sup_{a,b} \left( F(a,b) + bE + aI \right) = F(\bar{a},\bar{b}) + \bar{b}E + \bar{a}I,$$

where $\bar{a}, \bar{b}$ is the unique maximum point for $S^*(I,E)$. Therefore, for any $a, b$

$$S^*(I,E) \geq F(a,b) + bE + aI = S(\omega_{a,b}) - b(E(\omega_{a,b}) - E) - a(I(\omega_{a,b}) - I).$$

Now, since $F$ is concave and smooth, we know that for any $I, E$, there exists unique $a, b$ such that $I(\omega_{a,b}) = I$, and $E(\omega_{a,b}) = E$. By choosing $a, b$ in this way in the previous equation we get

$$S^*(I,E) \geq S(\omega_{a,b}) \geq S(I,E). \quad (A.25)$$

On the other hand, let $\omega_k : k = 1, 2, ...$ be a minimizing sequence for $S(I,E)$, and $\omega$ a limit point for it. By lower semicontinuity of $S$ we know that $S(\omega) \leq S(I,E)$. Therefore, for any $a, b$

$$S(I,E) \geq S(\omega) = S(\omega) - bE(\omega) - aI(\omega) + bE(\omega) + aI(\omega) \geq F(a,b) + bE(\omega) + aI(\omega) \geq F(a,b) + bE + aI, \quad (A.26)$$

where we have used the continuity of $E$ from which $E(\omega) = E$, and the lower semicontinuity of $I$, from which $I(\omega) \leq I$.

Since $a, b$ are arbitrary in (A.26), we get $S(I,E) \geq S^*(I,E)$. Since we have already proven (A.25) this yields $S(I,E) = S^*(I,E)$.

Finally let us notice that

$$S(I,E) = S^*(I,E) = S(\omega_{\bar{a},\bar{b}}),$$

where $\bar{a}, \bar{b}$ is the unique minimun point for $S^*(I,E)$, where the relation between $a, b$ and $I, E$ is smooth and bijective (equivalence of the ensembles).

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