Stability of spinor Fermi gases in tight waveguides

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The two and three-body correlation functions of the ground state of an optically trapped ultracold spin-$\frac{1}{2}$ Fermi gas (SFG) in a tight waveguide (1D regime) are calculated in the plane of even and odd-wave coupling constants, assuming a 1D attractive zero-range odd-wave interaction induced by a 3D $p$-wave Feshbach resonance, as well as the usual repulsive zero-range even-wave interaction stemming from 3D s-wave scattering. The calculations are based on the exact mapping from the SFG to a “Lieb-Liniger-Heisenberg” model with delta-function repulsions depending on isotropic Heisenberg spin-spin interactions, and indicate that the SFG should be stable against three-body recombination in a large region of the coupling constant plane encompassing parts of both the ferromagnetic and antiferromagnetic phases. However, the limiting case of the fermionic Tonks-Girardeau gas (FTG), a spin-aligned 1D Fermi gas with infinitely attractive p-wave interactions, is unstable in this sense. Effects due to the dipolar interaction and a Zeeman term due to a resonance-generating magnetic field do not lead to shrinkage of the region of stability of the SFG.

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I. INTRODUCTION

At the low densities of ultracold bosonic atomic vapors, the interatomic interactions are accurately parametrized by the 3D s-wave scattering length $a_s$. It was shown by Olshanii [1–3] that when such a vapor is confined in a de Broglie wave guide with transverse trapping so tight and temperature so low that the transverse vibrational de Broglie wave guide with transverse trapping so tight $\hbar \omega_{\perp}$ is smaller than that of the available longitudinal zero point and thermal energies, the effective dynamics becomes one-dimensional (1D) and accurately described by a 1D Hamiltonian with delta-function interactions $g_{1D}(x_j - x_k)$, where $x_j$ and $x_k$ are 1D longitudinal position variables. This is a famous integrable model, the Lieb-Liniger (LL) model, exactly solved in 1963 by a Bethe ansatz method [4]. The value of $g_{1D}$ is a known function [1, 2] of $a_s$ exhibiting a confinement-induced 1D Feshbach resonance (CIR), such that the value of the 1D scattering length $a_{1D}$ can be tuned from $-\infty$ to $+\infty$ by shifting the position of a 3D Feshbach scattering resonance (hence the value of $a_s$) via an external magnetic field [5]. Such CIRs also occur in spin-aligned Fermi gases in tight waveguides [6]. Near the CIRs the ground states have strong short-range correlations not representable by effective field theories, and such systems have become the subject of extensive theoretical and experimental investigations [7–11], particularly since the recent experimental realization [9] of the 1D gas of impenetrable point bosons ($g_{1D} \to \infty$ limit of the LL model), solved exactly in 1960 by Fermi-Bose (FB) mapping to the ideal Fermi gas, and now known as the Tonks-Girardeau (TG) gas.

The “fermionic Tonks-Girardeau” (FTG) gas is the “mirror image” of the TG gas; instead of 1D bosons with infinitely strong zero-range repulsions, one has 1D spin-aligned fermions with infinitely strongly attractive zero-range odd-wave interactions. The exact ground state of this system was determined recently [6, 14–17] by FB mapping to the ideal Bose gas; this system models a magnetically trapped ultracold gas of spin-$\frac{1}{2}$ fermionic atoms with infinitely strong odd-wave interactions induced by a $p$-wave Feshbach resonance. More generally, by considering a spin-aligned Fermi gas with a very strong but finite interaction modelled by a very deep and narrow square well interaction of depth $V_0$ and width $2x_0$ and carrying out the zero-range limit such that $V_0 x_0^2$ is held constant as $V_0 \to \infty$ and $x_0 \to 0$, one obtains a model with a finite and negative 1D scattering length $a_{1D}$ determined by the value of $V_0 x_0^2$, which is exactly soluble [6, 14–16] by FB mapping to the LL 1D Bose gas, and models a spin-aligned Fermi gas in a tight waveguide in the neighborhood of a 3D $p$-wave Feshbach resonance with an associated 1D odd-wave CIR.

If optically trapped instead, such a spinor Fermi gas (SFG) has richer properties since its spins are unconstrained, and there are both spatial even-wave interactions associated with spin singlet scattering and odd-wave interactions associated with spin triplet scattering. One can vary the ratio of the two coupling constants by Feshbach resonance tuning of the odd-wave one, leading to a rich phase diagram of ground state spin, with both ferromagnetic and antiferromagnetic phases [16, 18, 19].

In [19] the ground and low excited states of the SFG were calculated via a generalized FB mapping to a 1D Lieb-Liniger-Heisenberg (LLH) model of particles with delta-function repulsions depending on Heisenberg spin-spin interactions, allowing exact determination of the ground state energy in the ferromagnetic phase in terms of the LL Bethe ansatz solution [4], and a variational

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II. SFG AND LLH MODELS

The SFG Hamiltonian in LLH units $h = 2m = 1$ \cite{4} is \cite{19}
\[ \hat{H}_{\text{SFG}} = - \sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < \ell \leq N} \left[ g_{1D}^j \delta(x_j) \hat{P}^\ell_j + v_{1D}^j(x_j) \hat{P}^\ell_j \right]. \] (1)

Here $x_j = x_j - x_\ell$, $\hat{P}^\ell_j = \frac{1}{2} - \hat{S}_j \cdot \hat{S}_\ell$, and $\hat{P}^\ell_j = \frac{1}{2} + \hat{S}_j \cdot \hat{S}_\ell$ are the projectors onto the subspaces of singlet and triplet functions of the spin arguments ($\sigma_j, \sigma_\ell$) for fixed values of all other arguments, and $v_{1D}^j$ is a strong, attractive, zero-range, odd-wave interaction (1D analog of 3D p-wave interaction) which is the zero-range limit of a deep and narrow square well of depth $V_0$ and width $2x_0$, where the zero-range limit $x_0 \to 0^+$ and $V_0 \to \infty$ is taken such that $\sqrt{V_0/2} = (\pi/2x_0)[1 + (2/\pi)^2(x_0/a_{1D}^e)]$, thus generating a relative wave-function with a contact discontinuity in the zero-range limit \cite{14,15,20} and satisfying the contact condition
\[ \psi_F(x_j = 0+) = - \psi_F(x_j = 0-) = -a_{1D}^o \psi'_F(x_j = 0 \pm), \]
where $a_{1D}^o < 0$ is the 1D odd-wave scattering length and the prime denotes differentiation. The even-wave 1D coupling constant $g_{1D}^j$ in (1) is related to the even-wave scattering length $a_{1D}^e$ derived \cite{1} from 3D s-wave scattering by
\[ g_{1D}^j = -4/a_{1D}^e \] and the even-wave contact condition is the usual LL one \cite{4}
\[ \psi_F(x_j = 0+) = - \psi'_F(x_j = 0-) = - \frac{\psi_F(x_j = 0 \pm)}{a_{1D}^e}. \]

Assume that the system is contained in a ring trap of circumference $L$, with periodic boundary conditions of periodicity length $L$, the ring circumference. The eigenfunctions of $\hat{H}_{\text{SFG}}$ can be mapped to those of the LL model by multiplying the part of the SFG wave function which is odd in $x_j$ by $e(x_j)$, where $e(x) = +1$ ($-1$) for $x > 0$ ($x < 0$), and $e(0) = 0$, while leaving the even part unchanged \cite{19}; this generalizes the original FB mapping of \cite{12} to the spin-dependent case here, by converting the odd-wave contact discontinuities as $x_j \to 0$ into contact cusps of the usual LL form \cite{4}. Moreover, the complete energy spectrum is unchanged in terms of a transformed LL Hamiltonian, with usual LL delta function interactions supplemented by spin-spin interactions of the usual Heisenberg form arising from the singlet and triplet spin projectors $\hat{P}^s_j$ and $\hat{P}^t_j$. One finds \cite{19}
\[ \hat{H}_{\text{LL}} = - \sum_{j=1}^{N} \partial_{x_j}^2 + \sum_{1 \leq j < \ell \leq N} \left[ \frac{3c_o + c_e}{2} + 2(c_o - c_e)\hat{S}_j \cdot \hat{S}_\ell \right] \delta(x_j) \] (2)

where the term in brackets plays the role of $2c$ in the LL model. The coupling constants are related to the odd and even-wave SFG scattering lengths $a_{1D}^o$ and $a_{1D}^e$ by
\[ c_o = \frac{2}{|a_{1D}^o|}, \quad c_e = \frac{1}{2} \frac{|a_{1D}^e|}{|a_{1D}^o|}. \] (3)

In a real optically trapped ultracold allali metal vapor there are also dipolar interactions between the valence electron spins as well as their Zeeman interactions with an external magnetic field required to generate the p-wave Feshbach resonance. The experimental results of Ticknor et al. for $^{40}$K \cite{21} show that the dipolar interaction causes a very small ($\sim 1\%$) shift and splitting of the resonance, thus shifting the 1D CIR and hence the value of $c_o$. However, $c_o$ is an adjustable parameter in (2), so we shall not include the dipolar interaction explicitly. The effect of the Zeeman interaction is more complicated and will be considered after we have evaluated the correlation functions of the ground state of (2).

The ground state of (2) and hence of (1) is ferromagnetic with total spin $S = N/2$ for $c_o < c_e$ and antiferromagnetic with $S = 0$ for $c_o > c_e$ \cite{16,19}. The ferromagnetic ground state is of space-spin product form $\psi_{\text{space}}\psi_{\text{spin}}$ and is a simultaneous eigenstate of the spin interaction operators $\hat{S}_j \cdot \hat{S}_\ell$ with eigenvalue $\frac{N}{2}$ for each, reducing the Hamiltonian (2) to the usual LL form with interactions $2c_o\delta(x_j)$ independent of $c_e$ and with the spatial ground state $\psi_{\text{space}}$ given exactly for all $c_o$ by the LL Bethe ansatz \cite{4}. The antiferromagnetic ground state is not known exactly, but a natural variational approximation is given by a space-spin product state $\psi_{\text{space}}\psi_{\text{spin}}$ with $\psi_{\text{spin}}$ the exact Bethe ansatz ground state of the antiferromagnetic Heisenberg model. Ordering the particle coordinates $x_1 < x_2 < \ldots < x_N$ and replacing $\hat{S}_j \cdot \hat{S}_{j\pm 1}$ in (2) by its expectation value $\frac{1}{2} - \ln 2$ in the antiferromagnetic Heisenberg ground state \cite{22} again reduces the spatial Hamiltonian to LL form, but now with the coupling constant $c$ of LL replaced by $c = c_o(1 - \ln 2) + c_e\ln 2$ \cite{19}.
FIG. 1: Plot of the two-body correlation function $g_2(\gamma_o, \gamma_e)/n^2$ of the SFG in the $\gamma_o, \gamma_e$-plane where the different phases are identified. The quantum phase transition lies along the diagonal dashed line $\gamma_o = \gamma_e$.

III. TWO-BODY CORRELATIONS

The two-body correlation function of the SFG, which determines the rate of photoassociation to excited diatomic molecules [10], is equal to that of the LL model, $g_2(\gamma) = \langle \hat{\Psi}^\dagger(x)\hat{\Psi}(x) \rangle_0$, where $\gamma = e/n$ is the dimensionless coupling constant, $n = N/L$ is the 1D density, $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ are the LL boson creation and annihilation operators, and $\langle \cdot \cdot \cdot \rangle_0$ is the ground-state expectation value. It can be evaluated exactly in the thermodynamic limit using the Hellmann-Feynman theorem [23, 24]:

$$g_2(\gamma) = (\gamma/\pi) \frac{d}{d\gamma} \left[ (1 + \gamma/2) \epsilon_2 \right]$$

$$- 2 \epsilon_2 \frac{\epsilon_2}{\gamma} + 9 \frac{\epsilon_2^2}{\gamma^2},$$

where

$$\epsilon_m = \left( \frac{\gamma}{\alpha} \right)^m \int_{-1}^1 dk k^m \sigma(k) \quad (m = 2, 4)$$

are the moments of the quasi-momentum distribution [28] which obeys the usual LL linear integral equation [4],

$$\sigma(k) = C \frac{1}{2\pi} \int_{-1}^1 dq \frac{2\alpha \sigma(q)}{\alpha^2 + (k - q)^2} = \frac{1}{2\pi},$$

with $\alpha = \gamma \int_{-1}^1 dk \sigma(k)$, and primes denote derivative with respect to $\gamma$. Approximate expressions for $g_3(\gamma)$ with a relative error smaller than $2 \times 10^{-3}$ were provided for $\gamma \in [0, 30]$ [26]. Note that such results cover both asymptotics. For $\gamma \ll 1$ the Bogoliubov result holds, $g_3(\gamma)/n^3 \approx 1 - 2\sqrt{\gamma}/\pi$ which in particular, points out the instability of the FTG gas with respect to three-body losses. As $\gamma \to \infty$, $g_3(\gamma)/n^3 \approx 16\pi^6/15\gamma^6$. Such limiting cases are known for the correlation functions of all orders [29]. Fig. 2 shows the decay of $g_3$ as a function of both $\gamma_o$ and $\gamma_e$ in both phases. Decay processes due to three-body collisions make the gas unstable for high attractive interactions but become negligible as soon as $\gamma \gtrsim 1$. 

![Three-body correlation function](image)

FIG. 2: Plot of the three-body correlation function $g_3(\gamma_o, \gamma_e)/n^3$ in the $\gamma_o, \gamma_e$-plane.

IV. THREE-BODY CORRELATIONS

The rate coefficient of three-body losses is proportional to the three-body correlation function $g_3(\gamma) = \langle \hat{\Psi}^\dagger(x)\hat{\Psi}(x) \rangle_0$ as demonstrated experimentally [7]. The exact expression for $g_3(\gamma)$ was recently derived using an integrable lattice model, the $g$-boson hopping model, which reduces to the LL model in the thermodynamic limit [26, 27],

$$\frac{g_3(\gamma)}{n^3} = \frac{3}{2\gamma} \epsilon_4 - \frac{5\epsilon_4}{\gamma^2} + \left( 1 + \frac{\gamma}{2} \right) \epsilon_2'$$

$$- 2 \epsilon_2 \frac{\epsilon_2}{\gamma} + 9 \frac{\epsilon_2^2}{\gamma^2},$$

in the ferromagnetic phase and

$$\gamma = \gamma_o (1 - \ln 2) + \gamma_e \ln 2$$

in the antiferromagnetic phase with $\gamma_o = e_o/n = 2/(n|a_{1D}^2|)$ and $\gamma_e = e_e/n = 2/(n|a_{1D}^2|)$. The results are shown in Fig. 1, where the two-body correlation function is shown to reach values near unity in both ferromagnetic and antiferromagnetic phases when the p-wave interactions are highly attractive, which corresponds to small values of $\gamma_o$. Note that this concerns the limiting case of the FTG gas, defined as a spin-aligned 1D Fermi gas (hence, not subjected to the s-wave pseudopotential) with infinitely attractive odd-wave interactions ($\gamma_o = 0$). Indeed, for $\gamma \ll 1$, $g_2(\gamma)/n^2 = 1 - 2\sqrt{\gamma}/\pi$ whereas for $\gamma \gg 1$, $g_2(\gamma)/n^2 = 4\pi^2/3\gamma^2$ [23]. In the TG limit of the LLH model, $\gamma \to \infty$, all local correlation functions vanish.
Actually, the magnetization can be changed to an arbitrary value using the microwave technique and remains constant in a given experiment. Indeed, experiments often employ the lowest Zeeman sublevel of the atom, say $|F = 9/2, m_F = -9/2 \rangle$ for $^{40}$K, in combination with the next lowest state $(m_F = -7/2)$; for which spin-changing collisions are energetically disfavoured. When tuning the p-wave interactions, the atoms are prepared in the $|F = 9/2, m_F = -7/2 \rangle$ state from which depolarization into $m_F = -9/2, -5/2$ also is suppressed by the second order Zeeman effect [31, 32].

The even-wave interactions are constrained by the traverse frequency of the waveguide and linear density and therefore $\gamma_e$ is considered as a parameter. The linear densities accessible in current experiments vary within the range $n = 0.2 - 2 \mu m^{-1}$, the transversal frequency $\omega_\perp \sim 100K$ and the background 3D even-wave scattering length of $^{40}K$ is $a_{bg} = 104a_0$. Using Olshanii’s relation $\sigma = \frac{a^2}{2a_{bg}} (1-1.4603 \frac{a_{bg}}{a_b})$ where $a_b = \sqrt{2\hbar/m\omega_\perp}$, it follows that the typical values of $\gamma_e \approx 0.1 - 10$. In Fig. 3 and 4 $g_2$, which determines the photoassociation rate, and $g_3$, which determines the rate of three-body recombination, are shown as a function of the magnetization $\sigma = S/N$ and $\gamma_0$ for different values of $\gamma_e$. It is clear from Fig. 4 that states of lower magnetization are more stable and for larger values of $\gamma_e$, that is, stronger s-wave interactions, the region of stability increases.

V. EFFECT OF THE ZEEMAN INTERACTION

We have assumed that the system is optically trapped (no trapping magnetic field). However, the external magnetic field $\mathcal{H}$ required to generate a p-wave Feshbach resonance adds a Zeeman term

$$\hat{H}_{\text{Zeeman}} = -g_\mu \mathcal{H} \hat{S}^z$$

(9)
to (1) and (2), where $\hat{S}^z = \sum_{j=1}^{N} \hat{S}_j^z$ is the total spin $z$-component operator. Such term only shifts the ground state energy without affecting the wavefunction. The stability of the gas for an arbitrary polarization can be analyzed by means of the same variational method used in [19], with a trial function $\psi_{\text{space}} \psi_{\text{spin}}$ where now $\psi_{\text{spin}}$ is taken to be the lowest state of the antiferromagnetic Heisenberg model with total spin $S$, with Hamiltonian $\sum_{j=1}^{N} \hat{S}_j \cdot \hat{S}_{j+1} + \hat{S}_N \cdot \hat{S}_1$. It has been determined exactly by Griffiths [33]. Denoting its ground state energy by $E_{0,\text{Heis}}(S)$, one has

$$\langle \hat{S}_j \cdot \hat{S}_{j+1} \rangle_0 = N^{-1} E_{0,\text{Heis}}(S),$$

(10)
which can be substituted into Eq. (2) to obtain a LL Hamiltonian with LL coupling constant

$$c = \frac{3c_a + c_c}{4} + (c_a - c_c) \frac{E_{0,\text{Heis}}(S)}{N}.$$  

(11)

Note that always $c \geq 0$ implying repulsive interactions and therefore excluding the possibility of pairing investigated in a 1D $\delta$-interacting Fermi gas in an external magnetic field [30].

VI. CONCLUSIONS

Using the mapping between a 1D spinor Fermi gas and the Lieb-Liniger-Heisenberg model, the two and three-body correlation functions $g_2$ and $g_3$ have been calculated. It is found that $g_3$ is small enough in a wide area of the $\gamma_0, \gamma_e$-plane, encompassing both ferromagnetic and antiferromagnetic phases, to ensure stability of this system against three-body recombination over experimental lifetimes. Due to the different decay scales of $g_2$ and $g_3$ it may be possible to perform photoassociation experiments in a range of p-wave interactions ($\gamma_0 \approx 2$ in Figs. 3 and 4) where $g_2$ is significant and the gas stable because of a negligible $g_3$. The limiting case of the fermionic Tonks-Girardeau gas, falls however within the region of instability. Finally, by means of a variational ansatz the results are extended for an arbitrary spin polarization.

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