Group Theoretic Bases for Tribimaximal Mixing

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Abstract

Present data on neutrino masses and mixing favor the highly symmetric tribimaximal neutrino mixing matrix which suggests an underlying flavor symmetry. A systematic study of non-abelian finite groups of order $g \leq 31$ reveals that tribimaximal mixing can be derived not only from the well known flavor symmetry $T \equiv A_4$, the tetrahedral group, but also by using the alternative flavor symmetry $X(24) \equiv SL_2(F_3) \equiv Q_4 \times Z_3$. $X(24)$ does not contain the tetrahedral group as a subgroup, and has the advantage over it as a flavor symmetry that it can not only underwrite bitrimaximal mixing for neutrinos, equally as well, but also provide a first step to understanding the quark mass hierarchy.
Progress in our knowledge of the three neutrino masses and mixings has been remarkable since SuperKamiokande found the first convincing evidence of non zero neutrino mass in 1998 [1], and the Sudbury Neutrino Observatory (SNO) exceeded all expectations by abruptly solving the solar neutrino puzzle in 2001 [2] thereby resolving the 35-year old conundrum set up by the persistent, and correct, experiments by Davis [3]. It is probably fair to say that previously the majority of colleagues believed the data of Davis were explicable by suspected inaccuracies of the Standard Solar Model (SSM), but as we now know the SSM is a description of our Sun which is accurate to much better than a factor three, actually to within ten per cent [2]. It is fair to say that our present knowledge of neutrino flavor is at least comparable to that of quark flavor despite the fact that the theory for quark flavor goes back to the 1963 article by Cabibbo [4] (pre saged by a footnote in the 1960 paper by Gell-Mann and Lévy [5]) and the paper by Gatto et al. in 1968 [6]. No complete understanding of the quark masses and mixings has subsequently emerged and the prediction of CP violation in [7] provides no insight into its magnitude.

We shall consider only three left-handed neutrinos at first, so avoiding any encounter with the see-saw mechanism [8]. The Majorana mass matrix $\mathcal{M}$ is a $3 \times 3$ unitary symmetric matrix and without CP violation has six real parameters. Let write the diagonal form as $\mathcal{M} = \text{diag}(m_1, m_2, m_3)$, related to the flavor basis $\mathcal{M}$ by $\mathcal{M} = U^T \mathcal{M} U$ where $U$ is orthogonal. It is the form of $\mathcal{M} = U \mathcal{M} U^T$ and $U$ which are the targets of lepton flavor physics. One technique for analysis of $\mathcal{M}$ is to assume texture zeros [9–11] in $\mathcal{M}$ and this gives rise to relationships between the mass eigenvalues $m_i$ and the mixing angles $\theta_{ij}$. For example, it was shown in [10] that $\mathcal{M}$ cannot have as many as three texture zeros out of a possible six but can have two. A quite different interesting philosophy is that neutrino masses may arise from breaking of lorentz invariance [12]. Clearly, a wide range of approaches is being aimed at the problem.

In the present study we focus on a symmetric texture for $\mathcal{M}$ with only four independent parameters, of the form

$$\mathcal{M} = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix} \quad (1)$$

The $\mathcal{M}$ can be reached from a diagonal $\mathcal{M}$ by the orthogonal transformation

$$U = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \theta_{12}/\sqrt{2} & \cos \theta_{12}/\sqrt{2} & -1/\sqrt{2} \\ -\sin \theta_{12}/\sqrt{2} & \cos \theta_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad (2)$$

where one commits to a relationship between $\theta_{12}$ and the four parameters in Eq. (1), namely

$$\tan 2\theta_{12} = 2\sqrt{2B(A - C - D)^{-1}} \quad (3)$$
Written in the standard PMNS form \[13\]

\[
U = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & \sin \theta_{23} \\
0 & -\sin \theta_{23} & \cos \theta_{23}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{13} & 0 & \sin \theta_{13} e^{i\delta} \\
0 & 1 & 0 \\
-\sin \theta_{13} e^{-i\delta} & 0 & \cos \theta_{13}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{12} & \sin \theta_{12} & 0 \\
-\sin \theta_{12} & \cos \theta_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (4)

this ansatz requires that \(\theta_{23} = \pi/4\) and \(\theta_{13} = 0\), both of which are consistent with present data. These values of maximal \(\theta_{23}\) and vanishing \(\theta_{13}\) are presumably only approximate but departures therefrom, if they show up in future experiments, could be accommodated by higher order corrections.

To arrive at tribimaximal mixing \[14–19\], one more parameter \(\theta_{12}\) in Eq. (2) is assigned such that the entries of the second column are equal, \(i.e.\ \sin \theta_{12} = \cos \theta_{12}/\sqrt{2}\) which implies that \(\tan^2 \theta_{12} = 1/2\). Experimentally \(\theta_{12}\) is non-zero and over 5\(\sigma\) from a maximal \(\pi/4\). The present value \[20\] is \(\tan^2 \theta_{12} = 0.452_{+0.088}^{−0.070}\), so the tribimaximal value is within the allowed range. With this identification Eq. (2) becomes \[18\]

\[
U_{HPS} = \begin{pmatrix}
\sqrt{2/3} & \sqrt{1/3} & 0 \\
-\sqrt{1/6} & \sqrt{1/3} & -1/\sqrt{2} \\
-\sqrt{1/6} & \sqrt{1/3} & 1/\sqrt{2}
\end{pmatrix}
\] (5)

This ensures that the three mixing angles \(\theta_{ij}\) are consistent with present data, although more accurate experiments may require corrections to \(U_{HPS}\).

The data allow the normal hierarchy which occurs most often from models with \(|m_3| \gg |m_{2,1}|\). In the normal hierarchy one expects \(|m_3| \sim \sqrt{\Delta_{23}} \sim 0.05\text{ eV},\ |m_2| \sim \sqrt{\Delta_{12}} \sim 0.009\text{ eV}\) and \(|m_1|\) essentially zero, as is the prediction for the eigenvalues in the FGY model \[11\]. The data also allow for an inverted hierarchy with \(|m_1| \sim |m_2| \gg |m_3|\). A third possible pattern is the degenerate case \(|m_1| \sim |m_2| \sim |m_3| \gg |(m_3 - m_2)|\). The tribimaximal mixing, \(U_{HPS}\) of Eq. (5), can accommodate all three of these neutrino mass patterns and so makes no prediction in that regard.

The success of \(U_{HPS}\) tribimaximal neutrino mixing has precipitated many studies of its group theoretic basis \[16,17,19\] and the tetrahedral group \(A_4\) has emerged. was prompted by earlier work of one of the present authors (PHF) with Kephart in systematically studying all non-abelian finite groups of order \(g \leq 31\) both for a quark flavor group \[22\] and for orbifold compactification in string theory \[23\]. Our question is whether or not \(A_4\) is singled out from these as the neutrino flavor symmetry?

**Character Table of \(T\)**

| \(\omega\) | \(\omega^2\) | \(\omega^3\) | \(\omega^4\) |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | -1 |
| 1 | \(\omega\) | \(\omega^2\) | 0 |
| 1 | \(\omega^2\) | \(\omega\) | 0 | 2
Kronecker Products for Irreducible Representations of \( T \)

\[
\begin{array}{|c|c|c|c|}
\hline
& 1_1 & 1_2 & 1_3 \\
\hline
1_1 & 1_1 & 1_2 & 1_3 \\
\hline
1_2 & 1_2 & 1_3 & 1_1 \\
\hline
1_3 & 1_3 & 1_1 & 1_2 \\
\hline
3 & 3 & 3 & 3 \text{ or } 1_1 + 1_2 + 1_3 + 3 + 3 \\
\hline
\end{array}
\]

The Kronecker products for irreducible representations for all the forty-five non-abelian finite groups with order \( g \leq 31 \) are explicitly tabulated in the Appendix of [23], where the presentation is adapted to a style aimed at model builders in theoretical physics rather than at mathematicians as in [21].

Study of [23] shows that a promising flavor group is \( X(24) \equiv SL_2(F_3) \equiv Z_3 \times Q \). The Kronecker products are identical to those of \( T \equiv A_4 \) if the doublet representations are omitted and so the group \( X(24) \) can reproduce successes of \( A_4 \) model building. The use of \( X(24) \) as a flavor group first appeared in [22] and then analysed in more details in [24].

\( X(24) \) has an advantage over \( T \) in extension to the quark sector because the doublets of \( X(24) \), absent in \( T \), allow the implementation of a \((2 + 1)\) structure to the three quark families, thus permitting the third heavy family to be treated differently as espoused in [22, 25, 26]

### Character Table of \( X(24) \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
& 1_1 & 1_2 & 1_3 & 2_1 & 2_2 & 3 \\
\hline
C_1 & 1 & 1 & 1 & 2 & 2 & 3 \\
\hline
C_2 & 1 & 1 & 1 & -2 & -2 & 3 \\
\hline
C_3 & 1 & \omega^2 & \omega^4 & -1 & \omega^8 & 0 \\
\hline
C_4 & 1 & \omega^* & \omega^2 & -1 & \omega & 0 \\
\hline
C_5 & 1 & 1 & 1 & 0 & 0 & -1 \\
\hline
C_6 & 1 & \omega^2 & \omega^4 & -1 & \omega^2 & 0 \\
\hline
C_7 & 1 & \omega^4 & \omega^2 & 1 & \omega^4 & 0 \\
\hline
\end{array}
\]

\( \omega = \exp(2\pi i/6) \)
Kronecker Products for Irreducible Representations of X(24)

|   | 1_1 | 1_2 | 1_3 | 2_1 | 2_2 | 2_3 | 3  |
|---|-----|-----|-----|-----|-----|-----|----|
| 1_1 | 1_1 | 1_2 | 1_3 | 2_1 | 2_2 | 2_3 | 3  |
| 1_2 | 1_2 | 1_3 | 1_1 | 2_2 | 2_3 | 2_1 | 3  |
| 1_3 | 1_3 | 1_1 | 1_2 | 2_3 | 2_1 | 2_2 | 3  |
| 2_1 | 2_1 | 2_2 | 2_3 | 1_1 + 3 | 1_2 + 3 | 1_3 + 3 | 2_1 + 2_2 + 2_3 |
| 2_2 | 2_2 | 2_3 | 2_1 | 1_2 + 3 | 1_3 + 3 | 1_1 + 3 | 2_1 + 2_2 + 2_3 |
| 2_3 | 2_3 | 2_1 | 2_2 | 1_3 + 3 | 1_1 + 3 | 1_2 + 3 | 2_1 + 2_2 + 2_3 |
| 3  | 3  | 3  | 3  | 2_1 + 2_2 + 2_3 | 2_1 + 2_2 + 2_3 | 2_1 + 2_2 + 2_3 | 1_1 + 1_2 + 1_3 + 3 + 3 |

It is important to remark that X(24) does not contain $T$ as a subgroup [21] so our discussion about quark masses does not merely extend $T$, but replaces it.

The philosophy used for X(24) is reminiscent of much earlier work in [25,26] where the dicyclic group $Q_6$ was used with doublets and singlets for the (1st, 2nd) and (3rd) families to transform as $(2 + 1)$ respectively. On the other hand, $Q_6$ is not suited for tribimaximal neutrino mixing because like all dicyclic groups $Q_{2n}$ it has no triplet representation. Recall that when the work on $Q_6$ was done, experiments had not established neutrino mixing for the reason explained in our first paragraph.

For the quark sector, the X(24) assignments are most naturally chosen using the set of model building steps for a quark flavor group $G$ introduced in [22]. The main purpose is to understand why the third family of quarks and leptons is heavy, and especially why the top quark is very heavy. The steps are:

(A) The $t$ quark mass (and only the $t$) transforms as a $1$ of $G$.

(B) The $b$ and $\tau$ masses appear as $G$ is broken to $G'$.

(C) After stage (B) first the $c$ mass ($G' \to G''$), then the $s$ and $\mu$ masses ($G'' \to G'''$) are generated. At stage (C) the $u$, $d$ and $e$ remain massless.

(D) No additional quarks and a minimal number of leptons be introduced beyond the usual three-family standard model.

We start by satisfying (A) through (D). We therefore assign the quarks as follows, with the charged leptons like down-type quarks.

\[
\begin{pmatrix}
  t \\ b \\
  c \\
  s \\
  u \\
  d
\end{pmatrix}_L \begin{pmatrix}
  1 \\
  1 \\
  1 \\
  1 \\
  1
\end{pmatrix} =
\begin{pmatrix}
  t_R \\
  c_R \\
  u_R \\
  b_R \\
  s_R \\
  d_R
\end{pmatrix}_R \begin{pmatrix}
  1 \\
  1 \\
  1 \\
  1 \\
  1 \\
  1
\end{pmatrix}
\]
whereupon the mass matrices are:

\[ U = \begin{pmatrix} <2_1> & <1> \\ <2_1> & <1> \end{pmatrix} \]

and

\[ D = L = \begin{pmatrix} <2_1 + 2_2 + 2_3> & <3> \\ <2_1 + 2_2 + 2_3> & <3> \end{pmatrix} \]

To implement the hierarchy complying with rules (A) to (D) above requires:

(A) A VEV to a \( SU(2)_L \) doublet which is a singlet of \( X(24) \) gives a heavy mass to \( t \) without breaking \( X(24) \).

(B) A VEV to a \( 3 \) of \( X(24) \) gives mass to \( b \) and \( \tau \).

(C) and (D) The \( c \) quark acquires mass radiatively through a VEV of \( (1, 2_1) \) via the diagram of Fig. (1) below. The \( s \) and \( \mu \) acquire mass at tree level through \( 2_1 \) or \( 2_2 \) VEVs, breaking \( G' \). The \( u, d \) and \( e \) are still massless.

In summary, while \( T \equiv A_4 \) is one candidate for a lepton flavor group which gives rise naturally to tribimaximal mixing it is not unique among the non abelian finite groups in this regard. The choice \( X(24) \equiv SL_2(F_3) \equiv Z_3 \times Q \) satisfies the requirement equally well, and because it has doublet representations can readily accommodate the quark mass spectrum, particularly the anomalously heavy third family.

As a flavor group to accommodate both quark and lepton masses \( X(24) \) emerges as a leading candidate and so a non-gravitational grand unified theory with asymptotic high energy symmetry

\[ G_{GUT} \times X(24) \] (6)

where \( G_{GUT} \) is the gauge group of a grand unified theory is strongly suggested.

In the present Letter we have shown how both the neutrino mixing angles and the quark masses are naturally fitted by the choice of flavor symmetry \( X(24) \); Nature chooses the triplet representations of \( X(24) \) for neutrinos and both doublet and triplet representations for quarks and charged leptons. The complementary goal of understanding the neutrino masses and the quark mixing angles undoubtedly requires dynamics associated with \( G_{GUT} \) in the overall symmetry (6). Time will tell whether (6) is the best selection of asymptotic symmetry but we believe the present article encourages such a choice.

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References

[1] Y. Fukuda et al. (Super-Kamiokande Collaboration) Phys. Rev. Lett. 81, 1562 (1998). [hep-ex/9807003]

[2] Q.R. Ahmad et al. (SNO Collaboration) Phys. Rev. Lett. 87, 071301 (2001). [nucl-ex/0106015]

[3] B.T. Cleveland et al. Astrophys. J. 496, 505 (1998).

[4] N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963).

[5] M. Gell-Mann and M. Levy, Nuovo Cim. 16, 705 (1960).
[6] R. Gatto, G. Sartori and M. Tonin, Phys. Lett. B28, 128 (1968).

[7] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).

[8] P. Minkowski, Phys. Lett. B67, 421 (1977).
T. Yanagida, in Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, Editors: O. Sawada and A. Sugamoto. Tsukuba, Japan (1979). page 95.
P. Ramond. The Family Group in Grand Unified Theories. CALT-68-709 (Feb 1979). Senibel Symposium. hep-ph/9809459.
S.L. Glashow, NATO Adv. Study Inst. Ser. B. Phys. 59 687 (1979).
M. Gell-Mann, P. Ramond and R. Slansky, in Supergravity, Editors: P. van Nieuwenhuizen and D. Freedman. North-Holland (1979) page 315.
R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).

[9] P.H. Frampton and S.L. Glashow, Phys. Lett. B461, 95 (1999). hep-ph/9906375

[10] P.H. Frampton, S.L. Glashow and D. Marfatia, Phys. Lett. 536B, 79 (2002).
    hep-ph/0201008

[11] P.H. Frampton, S.L. Glashow and T. Yanagida, Phys. Lett. B548, 119 (2002).
    hep-ph/0208157

[12] A. Cohen and S.L. Glashow, Phys. Rev. Lett. 97, 021601 (2006). hep-ph/0601236
    hep-ph/0605036

[13] B. Pontecorvo, Sov. Phys. JETP 6, 429 (1957) [Zh. Eksp. Teor. Fiz. 33, 549 (1957)].
Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. 28, 870 (1962).

[14] N. Cabibbo, Phys. Lett. B72, 333 (1978).

[15] L. Wolfenstein, Phys. Rev. D18, 958 (1978).

[16] E. Ma and G. Rajasekaran, Phys. Rev. D64, 113012 (2001). hep-ph/0106291
K.S. Babu, E. Ma and J.W.F. Valle, Phys. Lett. B552, 207 (2003). hep-ph/0206292

[17] E. Ma, Mod. Phys. Lett. A20, 2601 (2005). hep-ph/0508099
    Phys. Lett. B632, 352 (2006). hep-ph/0508231
B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M.K. Parida, Phys. Lett. B638, 345 (2006). hep-ph/0603059
E. Ma, Phys. Rev. D73, 057304 (2006).
E. Ma, H. Sawanaka, and M. Tanimoto, Phys. Lett. B641, 301 (2006).
    hep-ph/0606103
E. Ma, Mod. Phys. Lett. A21, 1917 (2006). hep-ph/0607056.
E. Ma. hep-ph/0607142, hep-ph/0611181, hep-ph/0612013.
[18] P.F. Harrison, D.H. Perkins and W.G. Scott, Phys. Lett. B530, 167 (2002).

[19] G. Altarelli and F. Feruglio, Nucl. Phys. B720, 64 (2005). hep-ph/0504165

G. Altarelli, F. Feruglio and Y. Lin, hep-ph/0610165;
G. Altarelli. hep-ph/0508053, hep-ph/0610164, hep-ph/0611117.

[20] B.Aharmim, et al. (SNO Collaboration), Phys. Rev. C72, 055502 (2005).

[21] A.D. Thomas and G.V. Wood, Group Tables. Shiva Mathematical Series (1980).

[22] P.H. Frampton and T.W. Kephart, Int. J. Mod. Phys. 10A, 4689 (1995).

[23] P.H. Frampton and T.W. Kephart, Phys. Rev. D64, 086007 (2001).

[24] A. Aranda, C.R. Carone and R.F. Lebed, Phys. Lett. B474, 170 (2000).

[25] P.H. Frampton, The Third Family is Different. In Proceeding of the Fourth International Symposium on Particles, Strings and Cosmology. Editor: K.C. Wali. World Scientific (1995). page 63 hep-ph/9409331

[26] P.H. Frampton and T.W. Kephart, Phys. Rev. D51, R1 (1995). hep-ph/9409324