Atomic coherent state in Schwinger bosonic realization for optical Raman coherent effect

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Abstract

For optical Raman coherent effect we introduce the atomic coherent state (or the angular momentum coherent state with various angular momentum values) in Schwinger bosonic realization, they are the eigenvectors of the Hamiltonian describing the Raman effect. Similar to the fact that the photon coherent state describes laser light, the atomic coherent state is related to Raman process.

1 Introduction

Atomic coherent states (or the angular momentum coherent state with various angular momentum values) are sometimes referred to in the literature as spin coherent states or Bloch states [1–6]. They have been successfully applied to many branches of physics [7–10]. For example, Arecchi et al. applied atomic coherent states to describe interactions between radiation field and an assembly of two-level atoms [4]. Narducci, Bowden, Bluemel, Garrahan and Tuft [7] used atomic coherent state to study multitime correlation function for systems with observables satisfying an angular momentum algebra, which suggested a convenient classical-quantum correspondence rule for angular momentum degrees of freedom. Takahashi and Shibata [9] transformed some equation of motion for density matrix of a damped spin system into that of a quasi-distribution. Gerry and Ben-moussa [10] have studied the generation of spin squeezing by the repeated action of the angular momentum Dicke lowering operator on an atomic coherent state. In this work we shall introduce the atomic coherent state in Schwinger bosonic realization to study Raman coherent effect in the context of quantum optics.

It is known that the Raman coherent effect, a monochromatic light wave incident on a Raman active medium gives rise to a parametric coupling between an optical vibrational mode and the mode of the radiation field, the so-called Stocks mode. (In the case of Brillouin scattering, there is a similar coupling, where the vibrations are at acoustical, rather than optical frequencies.) The simplest Hamiltonian model for describing Raman coherent effect is

\[ H = \omega_1 a^\dagger a + \omega_2 b^\dagger b - i\lambda (a^\dagger b - ab^\dagger), \]

which is a two coupled oscillator model. In this work we shall show that the atomic coherent state (some assembly of angular momentum states, so named angular momentum coherent state) expressed in terms of Schwinger bosonic realization of angular momentum [11] has its obvious physical background, i.e., a set of energy eigenstates of two coupled bosonic oscillators with the Hamiltonian can be classified as the atomic coherent state \(|\tau\rangle\), according to the angular momentum...
value of $j$, where $\tau$ is determined by the dynamic parameters $\omega_1, \omega_2, \lambda$. Thus the Raman coherent effect is closely related to atomic coherent state theory, while the laser is described by the coherent state theoretically.

2 Brief review of the atomic coherent state (ACS) in Schwinger bosonic realization

The atomic coherent state with angular momentum value $j$ is defined as \[ |\tau\rangle = \exp(\mu J_+ - \mu^* J_-) |j, -j\rangle = (1 + |\tau|^2)^{-j} e^{\tau J_+} |j, -j\rangle, \] (2)

where $J_+$ is the raising operator of the angular momentum state $|j, m\rangle$, $|j, -j\rangle$ is the lowest weight state annihilated by $J_-$, and

\[ \mu = \frac{\theta}{2} e^{-\frac{i}{4} \phi}, \tau = e^{-\frac{i}{4} \phi} \tan\left(\frac{\theta}{2}\right). \] (3)

In the $j$-subspace the completeness relation for $|\tau\rangle$ is

\[ \int \frac{d\Omega}{4\pi} |\tau\rangle \langle\tau| = \sum_{m=-j}^{j} |j, m\rangle \langle j, m| = 1, \] (4)

where $d\Omega = \sin \theta d\theta d\phi$, and

\[ \langle\tau' |\tau\rangle = \frac{(1 + \tau' \tau^*)^{2j}}{(1 + |\tau|^2)^j(1 + |\tau'|^2)^j}. \] (5)

Using $[J_+, J_-] = 2J_z$, $[J_\pm, J_z] = \pm J_\pm$, one can show that $|\tau\rangle$ obeys the following eigenvector equations,

\[ (J_- + \tau^2 J_+) |\tau\rangle = 2j\tau |\tau\rangle, \]
\[ (J_- + \tau J_z) |\tau\rangle = j\tau |\tau\rangle, \]
\[ (\tau J_+ - J_z) |\tau\rangle = j |\tau\rangle. \] (6)

Employing the Schwinger Bose operator realization of angular momentum

\[ J_+ = a^\dagger b, \quad J_- = ab^\dagger, \quad J_z = \frac{1}{2} (a^\dagger a - b^\dagger b), \] (7)

where $[a, a^\dagger] = 1, [b, b^\dagger] = 1$ and $|j, m\rangle$ is realized as

\[ |j, m\rangle = \frac{a^{ij+m} b^{ij-m}}{\sqrt{(j+m)! (j-m)!}} |00\rangle = |j+m\rangle \otimes |j-m\rangle, \quad (m = -j, \cdots, j), \] (8)

note that the last ket is written in two-mode Fock space, then $|j, -j\rangle = |0\rangle \otimes |2j\rangle$, and the atomic coherent state $|\tau\rangle$ is expressed as

\[ |\tau\rangle = e^{\mu J_+ - \mu^* J_-} |0\rangle \otimes |2j\rangle \]
\[ = \frac{1}{\sqrt{(2j)!}} |b^\dagger \cos(\frac{\theta}{2}) + a^\dagger e^{-\frac{i}{4} \phi} \sin(\frac{\theta}{2})|^{2j} |00\rangle \]
\[ = \frac{1}{(1 + |\tau|^2)^j} \sum_{l=0}^{2j} \sqrt{(2j)! l!(2j-l)!} |2j-l\rangle \otimes |l\rangle \] (9)
Especially when \( j = 0 \), \(|\tau\rangle = |00\rangle\) is just the two-mode vacuum state in Fock space. Using the normal ordering form of the two-mode vacuum projector \(|00\rangle \langle 00| = : e^{-a^\dagger a - b^\dagger b :}\), we can use the technique of integration within an ordered product of operators \([12, 13]\) to prove in \( j\)-subspace,

\[ \int \frac{d\Omega}{4\pi} |\tau\rangle \langle \tau| = \frac{1}{(2j)!} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \left( b^\dagger \cos \frac{\theta}{2} + a^\dagger e^{-i\phi} \sin \frac{\theta}{2} \right)^{2j} \times \left( b \cos \frac{\theta}{2} + ae^{i\phi} \sin \frac{\theta}{2} \right)^{2j} \exp \left( -a^\dagger a - b^\dagger b \right) : \]

\[ =: \frac{(a^\dagger a + b^\dagger b)^{2j}}{(2j)!} e^{-a^\dagger a - b^\dagger b :}, \quad (10) \]

the completeness relation of \(|\tau\rangle\) in the whole two-mode Fock space can be obtained after summing over \( j \):

\[ \sum_{2j=0}^\infty (2j+1) \int \frac{d\Omega}{4\pi} |\tau\rangle \langle \tau| = \sum_{2j=0}^\infty \frac{(a^\dagger a + b^\dagger b)^{2j}}{(2j)!} e^{-a^\dagger a - b^\dagger b :} = 1, \quad (11) \]

which means that atomic coherent states in Schwinger bosonic realization with all values of \( j \) forms a complete set.

### 3 Atomic coherent state as energy eigenstates of \( H \)

Now we inquire whether the atomic coherent state with a definite angular momentum value \( j \) is the solution of the stationary Schrodinger equation

\[ H |\tau\rangle = E |\tau\rangle. \quad (12) \]

In order to solve Eq. (12) we directly use Eq. (9) and the relation

\[ a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle, \quad (13) \]

to calculate

\[ H |\tau\rangle = \frac{1}{(1 + |\tau|^2)^j} \sum_{l=0}^{2j} \sqrt{\frac{(2j)!}{l!(2j-l)!}} \left[ \omega_1 (2j-l) + \omega_2 l \right] \tau^{2j-l} |2j-l\rangle \otimes |l\rangle \]

\[ - i\lambda \frac{1}{(1 + |\tau|^2)^j} \sum_{l=1}^{2j} \sqrt{\frac{(2j)!}{(l-1)!(2j-l+1)!}} (2j-l+1) \tau^{2j-l} |2j-l+1\rangle \otimes |l-1\rangle \]

\[ + i\lambda \frac{1}{(1 + |\tau|^2)^j} \sum_{l=0}^{2j-1} \sqrt{\frac{(2j)!}{(l+1)!(2j-l-1)!}} \tau^{2j-l} (l+1) |2j-l-1\rangle \otimes |l+1\rangle. \quad (14) \]
Let \( l \rightarrow 1 \rightarrow l \) in the second and third term of the r.h.s. of Eq. (14), respectively, we have

\[
H |\tau\rangle = \frac{1}{(1 + |\tau|^2)} \sum_{l=0}^{2j} \sqrt{\frac{(2j)!}{l!(2j-l)!}} \tau^{2j-l} \left\{ [\omega_1 (2l-j-\frac{\lambda}{\tau}) + \omega_2 (2l-j-\frac{\lambda}{\tau}) - i\lambda (2l-j-\frac{\lambda}{\tau}) \frac{1}{\tau} + i\lambda \tau] \right\} |2j-l\rangle \otimes |l\rangle
\]

\[
= \frac{1}{(1 + |\tau|^2)} \sum_{l=0}^{2j} \sqrt{\frac{(2j)!}{l!(2j-l)!}} \tau^{2j-l} \left\{ 2\left(\omega_1 - i\frac{\lambda}{\tau}\right) j + \left[ (\omega_2 - \omega_1) + i\lambda \left( \tau + \frac{1}{\tau} \right) \right] l \right\} |2j-l\rangle \otimes |l\rangle
\]

\[
= 2\left(\omega_1 - i\frac{\lambda}{\tau}\right) j |\tau\rangle + \frac{1}{(1 + |\tau|^2)} \sum_{l=0}^{2j} \sqrt{\frac{(2j)!}{l!(2j-l)!}} \tau^{2j-l} \left[ (\omega_2 - \omega_1) + i\lambda \left( \tau + \frac{1}{\tau} \right) \right] l |2j-l\rangle \otimes |l\rangle.
\]

We see when the following condition is satisfied,

\[
i\lambda \tau^2 + \tau (\omega_2 - \omega_1) + i\lambda = 0 \Rightarrow \tau_{\pm} = \frac{(\omega_1 - \omega_2) \pm \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2}}{2i\lambda}.
\]

then \( |\tau_{\pm}\rangle \), expressed by Eq. (9), is the eigenstate of \( H \) with eigenvalue

\[
E = 2\left(\omega_1 - i\frac{\lambda}{\tau}\right) j = j \left[ (\omega_1 + \omega_2) \pm \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2} \right]
\]

(17)

Hence \( H \)'s eigenvectors are classifiable according to the angular momentum value \( j \). Especially, when \( \omega_1 = \omega_2 = \omega \), from Eqs. (16) - (17) we know \( \tau_{\pm} = \mp i \), \( E_{\pm} = 2j (\omega \pm \lambda) \).

4 Some fundamental atomic coherent states as \( H \)'s eigenstates

We now investigate some fundamental atomic coherent states as \( H \)'s eigenstates. In the case of \( j = 1/2 \), from Eq. (9) we know the eigenstate of \( H \) is

\[
|\tau_{\pm}\rangle_{j=1/2} = \frac{1}{(1 + |\tau|^2)^{1/2}} (\tau_{\pm} |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle)
\]

\[
= \frac{\sqrt{2}}{1 + |\tau|^2} \left( (1 + |\tau|^2) |1\rangle \otimes |0\rangle + (1 - |\tau|^2) |0\rangle \otimes |1\rangle \right).
\]

Indeed, one can check \( H |i_{\pm}\rangle_{j=1/2} = \frac{\sqrt{2}}{1 + |\tau|^2} \left( -i |1\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \right). \) In the case of \( j = 1 \),

\[
|\tau_{\pm}\rangle_{j=1} = \frac{1}{1 + |\tau|^2} \sum_{l=0}^{2j} \sqrt{\frac{(2j)!}{l!(2j-l)!}} \tau^{2j-l} |2l-j-\frac{\lambda}{\tau} \rangle \otimes |l\rangle
\]

\[
= \frac{1}{1 + |\tau|^2} \left( \tau_{\pm}^2 |2\rangle \otimes |0\rangle + \sqrt{2} \tau_{\pm} |1\rangle \otimes |1\rangle + |0\rangle \otimes |2\rangle \right)
\]

\[
= \frac{1}{2} \left( - |2\rangle \otimes |0\rangle \mp i\sqrt{2} |1\rangle \otimes |1\rangle + |0\rangle \otimes |2\rangle \right)
\]

(19)

In the case of \( j = 3/2 \),

\[
|\tau_{\pm}\rangle_{j=3/2} = \frac{1}{(1 + |\tau|^2)^{3/2}} \left( \tau_{\pm}^3 |3\rangle \otimes |0\rangle + \sqrt{3} \tau_{\pm}^2 |2\rangle \otimes |1\rangle + \sqrt{3} \tau_{\pm} |1\rangle \otimes |2\rangle + |0\rangle \otimes |3\rangle \right)
\]

\[
= \frac{1}{2^{3/2}} \left( (\pm i |3\rangle \otimes |0\rangle - \sqrt{3} |2\rangle \otimes |1\rangle \mp i\sqrt{3} |1\rangle \otimes |2\rangle + |0\rangle \otimes |3\rangle \right)
\]

(20)
In the case of \( j = 2 \)

\[
|\tau\rangle_{j=2} = \frac{1}{\left(1 + |\tau|\right)^2} \sum_{l=0}^{4} \sqrt{\frac{4!}{l!(4-l)!}} \tau^{l-1} |4-l\rangle \otimes |l\rangle
\]

\[
= \frac{1}{\left(1 + |\tau|\right)^2} \left( \tau^4 |4\rangle \otimes |0\rangle + 2\tau^3 |3\rangle \otimes |1\rangle + \sqrt{6} \tau^2 |2\rangle \otimes |2\rangle + 2\tau |1\rangle \otimes |3\rangle + |0\rangle \otimes |4\rangle \right)
\]

\[
\omega_1 = \omega_2 \rightarrow |i\rangle_{j=2} = \frac{1}{4} \left( |4\rangle \otimes |0\rangle + 2i |3\rangle \otimes |1\rangle - \sqrt{6} |2\rangle \otimes |2\rangle \mp 2i |1\rangle \otimes |3\rangle + |0\rangle \otimes |4\rangle \right)
\] (21)

Thus we know how the eigenstate of \( H \) is composed of the Fock states.

## 5 Partition function and the Internal energy for \( H \)

Knowing that \( H \) is diagonal in the basis of atomic coherent state \( |\tau\rangle \), we can directly calculate its partition function by virtue of its energy level.

\[
Z_+ (\beta) = \text{Tr}_+ \left(e^{-\beta H}\right) = \sum_{2j=0}^{\infty} j \langle \tau_+ | e^{-\beta H} | \tau_+ \rangle_j
\]

\[
= \sum_{2j=0}^{\infty} e^{-\beta A_2 j} = \frac{1}{e^{-\beta A} - 1} |_{\eta = -\beta A}
\]

\[
= \frac{1}{e^{-\beta A} - 1},
\] (22)

and

\[
Z_- (\beta) = \text{Tr}_- \left(e^{-\beta H}\right) = \sum_{2j=0}^{\infty} j \langle \tau_- | e^{-\beta H} | \tau_- \rangle_j
\]

\[
= \frac{1}{e^{-\beta B} - 1}
\] (23)

where

\[
A = \frac{(\omega_1 + \omega_2) + \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2}}{2},
\]

\[
B = \frac{(\omega_1 + \omega_2) - \sqrt{(\omega_1 - \omega_2)^2 + 4\lambda^2}}{2}.
\] (24)

satisfying \( H |\tau_\rangle = 2A_j |\tau_\rangle, H |\tau_-\rangle = 2B_j |\tau_-\rangle \). Thus the total partition function is

\[
Z (\beta) = Z_+ (\beta) Z_- (\beta) = \left(\frac{1}{e^{-\beta A} - 1}\right) \left(\frac{1}{e^{-\beta B} - 1}\right),
\] (25)

and the internal energy of system is

\[
\langle H \rangle_e = -\frac{\partial}{\partial \beta} \ln Z (\beta)
\]

\[
= -\frac{\partial}{\partial \beta} \left[ \ln \left(\frac{1}{e^{-\beta A} - 1}\right) + \ln \left(\frac{1}{e^{-\beta B} - 1}\right) \right]
\]

\[
= \frac{A}{e^{A\beta} - 1} + \frac{B}{e^{B\beta} - 1}.
\] (26)
In summary, similar to the fact that the photon coherent state describes laser light, the atomic coherent state is useful to classify the energy eigenstates of the Hamiltonian describing the Raman effect. This may be useful to further study stimulated Raman scattering since the scattered light behaves as laser light.

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