LEADING-ORDER TEMPORAL ASYMPTOTICS OF
THE FOKAS-LENELLS EQUATION WITHOUT
SOLITONS

JIAN XU AND ENGUI FAN*

ABSTRACT. We use the Deift-Zhou method to obtain, in the solit-
onless sector, the leading order asymptotic of the solution to the
Cauchy problem of the Fokas-Lenells equation as \( t \rightarrow +\infty \) on the
full-line.

1. INTRODUCTION

The Fokas-Lenells equation (FL equation shortly) is a completely in-
tegrable nonlinear partial differential equation which has been derived
as an integrable generalization of the nonlinear Schrödinger equation
(NLS equation) using bi-Hamiltonian methods [1]. In the context of
nonlinear optics, the FL equation models the propagation of nonlinear
light pulses in monomode optical fibers when certain higher-order
nonlinear effects are taken into account [2]. The FL equation is related
to the NLS equation in the same way as the Camassa-Holm equation
associated with the KdV equation. The soliton solutions of the FL
equation have been constructed via the Riemann-Hilbert method in
[4]. And The initial-boundary value problem for the FL equation on
the half-line was studied in [5]. A simple N-bright-soliton solution was
given by Lenells [3] and the N-dark soliton solution was obtained by
means of Bäcklund transformation [6]. And Matsuno get the bright
and dark soliton solutions for the FL equation in [7] and [8] by a direct
method.

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tial value problem, Deift-Zhou method.
In this paper, we use the Riemann-Hilbert problem showed in [4] to get the long-time asymptotics behavior of the solution of the FL equation (2.9) by the nonlinear steepest descent method or Deift-Zhou method. The nonlinear steepest descent method is introduced by Deift and Zhou in [9] in 1993, the history of the long-time asymptotics problem also can be found in [9], and it is the first time to obtain the long-time asymptotics behavior of the solution rigorously, for the MKdV equation. Then it becomes a most power tool for the long-time asymptotics of the nonlinear evolution equations in complete integrable system, for example, the non-focusing NLS equation [10], the Sine-Gordon equation [15], the KdV equation [19], the Cammasa-Holm equation [20], and so on. Deift and his collaborators extend this method to analyse the small-dispersion problem for the KdV equation and the semiclassical problem of the focusing NLS equation. And this is also a very useful tool in the asymptotics problem in orthogonal polynomials and large \( n \) limit problem in random matrix theory.

For several soliton-bearing equations, for example, KdV, Landau-Lifshitz, and NLS, and the reduced Maxwell-Bloch system, it is well known that the dominant \( O(1) \) asymptotic \( t \to \infty \) effect of the continuous spectra on the multisoliton solutions is a shift in phase and position of their constituent solitons [12]. The purpose of our studies is to derive an explicit functional form for the next-to-leading-order \( O(t^{-\frac{1}{2}}) \) term of the effect of this interaction for the Fokas-Lenells equation. An asymptotic investigation of the solution can be divided into two stages: (i) the investigation of the continuum (solitonless) component of the solution [13]; and (ii) the inclusion of the soliton component via the application of a dressing procedure [14] to the continuum background. The purpose of this paper is to carry out, systematically, stage (i) of the abovementioned asymptotic paradigm (since this phase of the asymptotic procedure is rather technical and long in itself, the completed results for stage (ii) are the subject of a forthcoming article). The results obtained in this paper are formulated as theorems [3,18].
The outline of this paper is as follows: In Section 2 we recall some classic definition of Riemann-Hilbert problem and then, we write down the Riemann-Hilbert problem of the Fokas-Lenells equation. In Section 3 we analyse the leading order asymptotics of the solution of the Fokas-Lenells equation as $t \to +\infty$ via the Deift-Zhou method.

2. The Riemann-Hilbert problem for the Fokas-Lenells equation

2.1. What a Riemann-Hilbert problem is. In this subsection, we first explain what a Riemann-Hilbert problem is

Definition 2.1. Let the contour $\Gamma$ be the union of a finite number of smooth and oriented curves (orientation means that each arc of $\Gamma$ has a positive side and a negative side: the positive (respectively, negative) side lies to the left (respectively, right) as one traverses the contour in the direction of the arrow) on the Riemann sphere $\bar{\mathbb{C}}$ (i.e. the complex plane with the point at infinity) such that $\bar{\mathbb{C}} \setminus \Gamma$ has only a finite number of connected components. Let $V(k)$ be an $2 \times 2$ matrix defined on the contour $\Gamma$. The Riemann-Hilbert problem $(\Gamma, V)$ is the problem of finding an $2 \times 2$ matrix-valued function $M(k)$ that satisfies

(i) $M(k)$ is analytic for $k \in \bar{\mathbb{C}} \setminus \Gamma$ and extends continuously to the contour $\Gamma$.

(ii) $M_+(k) = M_-(k)V(k)$, $k \in \Gamma$.

(iii) $M(k) \to I$, as $k \to \infty$.

The Riemann-Hilbert problem can be solved as follows (see, [II]). Assume that $V(k)$ admits some factorization

$$V(k) = b_+(k)b_-(k),$$

where

$$b_+(k) = \omega_+(k) - I, \quad b_-(k) = I - \omega_-(k).$$

And define

$$\omega(k) = \omega_+(k) + \omega_-(k).$$
Let
\[
(C_{\pm} f)(k) = \int_{\Gamma} \frac{f(\xi)}{\xi - k \pm 2\pi i} \, d\xi, \quad k \in \Gamma, f \in L^2(\Gamma),
\] (2.4)
denote the Cauchy operator on $\Gamma$. As is well known, the operator $C_{\pm}$ are bounded from $L^2(\Gamma)$ to $L^2(\Gamma)$, and $C_+ - C_- = I$, here $I$ denote the identify operator.

Define
\[
C_\omega f = C_+(f \omega_{-}) + C_-(f \omega_{+})
\] (2.5)
for $2 \times 2$ matrix-valued functions $f$. Let $\mu$ be the solution of the basic inverse equation
\[
\mu = I + C_\omega \mu.
\] (2.6)

Then
\[
M(k) = I + \int_{\Gamma} \frac{\mu(\xi) \omega(\xi)}{\xi - k} \, d\xi, \quad k \in \mathbb{C}\setminus\Gamma,
\] (2.7)
is the solution of the Riemann-Hilbert problem. (See [9], P.322).

2.2. Riemann-Hilbert problem for FL equation. The Fokas-Lenells equation is
\[
iu_t - \nu u_{tx} + \gamma u_{xx} + \sigma|u|^2(u + i\nu u_x) = 0, \quad \sigma = \pm 1.
\] (2.8)
where $\nu$ and $\gamma$ are constants.

If we replaced $u(x, t)$ by $u(-x, t)$, we can see the sign of $\nu$ is the same as the $\gamma$'s. Hence, we can assume that $\alpha = \frac{\nu}{\gamma} > 0$ and $\beta = \frac{1}{\nu}$. Then we change the variable as follows:
\[
u u_x \rightarrow \beta \sqrt{\alpha} e^{ikx} u, \quad \sigma \rightarrow -\sigma
\]
the equation (2.8) can be changed into the desired form:
\[
u_{tx} + \alpha \beta^2 u - 2i\alpha \beta u_x - \alpha u_{xx} + \sigma i\alpha \beta^2 |u|^2 u_x = 0, \quad \sigma = \pm 1.
\] (2.9)

This equation admits Lax pair
\[
\begin{cases}
\Phi_x + ik^2 \sigma_3 \Phi = kU_x \Phi \\
\Phi_t + i\eta^2 \sigma_3 \Phi = [\alpha kU_x + i\alpha \beta^2 \sigma_3(kU - U^2)]\Phi.
\end{cases}
\] (2.10)
where $U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, $\eta = \sqrt{\alpha (k - \frac{\beta}{2k})}$ with $v = \sigma \bar{u}$. And in the following of the paper we just consider $\sigma = 1$.

According to the paper [4], we can get the Riemann-Hilbert problem of the Fokas-Lenells equation (2.9) as follows:

\[
\begin{cases}
M_+(x, t, k) = M_-(x, t, k)J(x, t, k), & k \in \mathbb{R} \cup i\mathbb{R}, \\
M(x, t, k) \to I, & k \to \infty.
\end{cases}
\]  

where the function $M(x, t, k)$ is defined by (4.24) in [4] and the jump matrix $J(x, t, k)$ is defined by

\[
J(x, t, k) = e^{(-ik^2x - i\eta^2t)\sigma_3} \begin{pmatrix} \frac{1}{a(k)a(k)} & \frac{b(k)}{a(k)} \\ \frac{b(k)}{a(k)} & 1 \end{pmatrix}
\]

with $\eta = \sqrt{\alpha (k - \frac{\beta}{2k})}$ and $a(k), b(k)$ are defined by (4.26) in [4]. And $e^{\sigma_3 A} = e^{\sigma_3 A}e^{-\sigma_3}$, here $A$ is a $2 \times 2$ matrix. We can also know that $a(k)a(k) - b(k)b(k) = 1$ and $a(k) = \frac{b(k)}{a(k)}$, $b(k) = -\frac{a(k)}{b(k)}$ from [4].

We introduce $r(k) = \frac{b(k)}{a(k)}$, then the jump matrix $J(x, t, k)$ can be transformed into the following form:

\[
J(x, t, k) = e^{(-ik^2x - i\eta^2t)\sigma_3} \begin{pmatrix} 1 - r(k)r(k) & r(k) \\ -r(k) & 1 \end{pmatrix}
\]
The solution of Fokas-Lenells equation (2.9) can be expressed by

\[
    u_x(x, t) = 2im(x, t)e^{4i \int_{-\infty}^{x} |m|^2(x', t)dx'}
\]

(2.14)

where

\[
    m(x, t) = \lim_{k \to \infty} (kM(x, t, k))_{12}
\]

(2.15)

with \( M(x, t, k) \) is the unique solution of the Riemann-Hilbert problem (2.11).

**Remark 2.1.** In this paper, we consider the case when \( a(k) \) has no zeros, that is without solitons, the unique solvability of the Riemann-Hilbert problem in (2.11) is a consequence of a vanishing lemma 4.2 in [4].

3. The Long-time asymptotics for the Fokas-Lenells equation

In this section, we get the asymptotics behavior of the solution of the Fokas-Lenells equation (2.9) as \( t \to \infty \) by the Deift-Zhou method [9].

Let \( F(x, t, k) = k^2x + \eta^2t \) and \( \theta(k) = k^2x + \eta^2 \), then \( F = t\theta \).

3.1. Case 1: \( \frac{\alpha}{t} + \alpha < 0 \). In this case, the real part of \( i\theta(k) \) has the signature

\[
    \text{Re}i\theta(k) \begin{cases} 
    > 0, & \text{if } \text{Im}k^2 > 0, \\
    < 0, & \text{if } \text{Im}k^2 < 0.
    \end{cases}
\]

(3.1)

showed in Figure 2.

The jump matrix \( J(x, t, k) \), i.e. (2.13), has an factorization

\[
    J(x, t, k) = \begin{pmatrix}
    1 & 0 \\
    -r(k)e^{2i\theta(k)} & 1
    \end{pmatrix}
    \begin{pmatrix}
    1 - r(k)r(\bar{k}) & 0 \\
    0 & 1
    \end{pmatrix}
    \begin{pmatrix}
    1 & r(k) \frac{e^{-2i\theta(k)}}{1-r(k)r(\bar{k})} \\
    0 & 1
    \end{pmatrix}
\]

(3.2)

We find that the transformation

\[
    \tilde{M}(x, t, k) = M(x, t, k)\tilde{\delta}^{-\sigma_3},
\]

(3.3)
Figure 2. The signature table of $\text{Re}(i\theta)$ in the case 1.

leads to the Riemann-Hilbert problem

$$
\begin{align*}
\tilde{M}_+(x, t, k) &= \tilde{M}_- (x, t, k) \tilde{J} (x, t, k), \quad \text{Im} k^2 = 0, \\
\tilde{M} &\to I, \quad k \to \infty.
\end{align*}
$$

(3.4)

with jump matrix $\tilde{J}(x, t, k)$ that admits the lower/upper factorization

$$
\tilde{J}(x, t, k) = \begin{pmatrix}
-\frac{r(k)}{1-r(k)\bar{r}(k)} & \frac{1}{\text{det}(k)} e^{2it\theta(k)} \\
\frac{1}{1-r(k)\bar{r}(k)} & \frac{1}{\text{det}(k)} e^{-2it\theta(k)}
\end{pmatrix} = \tilde{J}_1^{-1}\tilde{J}_2
$$

(3.5)

if the function $\tilde{\delta}(k)$ solves the scalar Riemann-Hilbert problem

$$
\begin{align*}
\tilde{\delta}_+(k) &= \tilde{\delta}_-(k)(1 - r(k)\bar{r}(k)), \quad \text{Im} k^2 = 0, \\
\tilde{\delta}(k) &\to 1, \quad k \to \infty.
\end{align*}
$$

(3.6)

The solution for the Riemann-Hilbert problem for $\tilde{\delta}$ has the explicit form

$$
\tilde{\delta}(k) = e^{\frac{1}{2\pi i} \int_{\mathbb{H} \cup \mathbb{R}} \frac{\log(1-r(k')\bar{r}(k'))}{k' - k} dk'}.
$$

(3.7)

Without loss of generality, we may assume that the left factor of (3.5) extends analytically to the region $\text{Im} k^2 < 0$ and continuous in the closure of the region. Then the right factor extends the region $\text{Im} k^2 > 0$. 

Our Riemann-Hilbert problem on \( \mathbb{R} \cup i\mathbb{R} \) is equivalent to a new Riemann-Hilbert problem on the contour

\[
\tilde{\Sigma} = e^{i\pi/6} \mathbb{R} \cup e^{-i\pi/6} \mathbb{R} \cup e^{i\pi/3} \mathbb{R} \cup e^{-i\pi/3} \mathbb{R},
\]

where the orientation of the contour \( \tilde{\Sigma} \) and the new function \( \tilde{M}(x, t, k) \) are given in the following

\[
\tilde{M}(x, t, k) = \begin{cases} 
\tilde{M}(x, t, k), & k \in \hat{D}_2 \cup \hat{D}_5 \cup \hat{D}_8 \cup \hat{D}_{11}, \\
\tilde{M}(x, t, k)\tilde{J}_2^{-1}, & k \in \hat{D}_1 \cup \hat{D}_3 \cup \hat{D}_7 \cup \hat{D}_9, \\
\tilde{M}(x, t, k)\tilde{J}_1^{-1}, & k \in \hat{D}_4 \cup \hat{D}_6 \cup \hat{D}_{10} \cup \hat{D}_{12}.
\end{cases}
\]

where the domains \( \{\hat{D}_j\}_{12}^1 \) are showed in Figure 3. Then one can verify

\[
\begin{aligned}
\hat{J}(x, t, k) &= \begin{cases}
1 - \frac{\bar{r}(k)}{1-r(k)r(k)} \tilde{\delta}^2(k) e^{-2it\theta(k)} & , k \in \tilde{\Sigma} \cap \{\text{Im} k^2 > 0\}, \\
0 & , k \in \tilde{\Sigma} \cap \{\text{Im} k^2 < 0\}.
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\hat{M}(x, t, k) &= \tilde{M}(x, t, k) \tilde{\delta}_k^{-1} , k \in \tilde{\Sigma}. \\
\hat{M}(x, t, k) &\to I , k \to \infty.
\end{aligned}
\]

Figure 3. The contour \( \tilde{\Sigma} \) and regions in case 1.
Theorem 3.1. As $t \to \infty$,

$$||\hat{M}_\pm(x, t, k) - II||_{L^2(\tilde{\Sigma})} \to 0, \text{ rapidly.} \quad (3.12)$$

$u_x(x, t)$ and therefore $u$ decay rapidly as $t \to \infty$.

Proof. Since the Riemann-Hilbert problem for $M$ and the Riemann-Hilbert problem for $\hat{M}$ are equivalent, the existence of the solution of $M$ implies the existence of the solution of $\hat{M}$.

We make the trivial factorization

$$\hat{J}(x, t, k) = b_{-1} b_+, \quad b_- = I, b_+ = \hat{J}.$$ 

and define $\hat{\omega}$ as (2.3). Then as section 2 (also see, [11] or [9]) , we obtain the solution of the Riemann-Hilbert problem for $\hat{M}$,

$$\hat{M}(x, t, k) = I + \int_{\hat{\Sigma}} \hat{\mu}(x, t, \xi) \hat{\omega}(x, t, \xi) \frac{d\xi}{\xi - k} \frac{2i}{2\pi}, \quad k \in \mathbb{C} \setminus \hat{\Sigma}. \quad (3.13)$$

where $\hat{\mu}$ is the solution of the singular integral equation $\hat{\mu} = I + C_{\hat{\omega}} \hat{\mu}$, where $C_{\hat{\omega}}$ defined as (2.5) with $\omega$ replaced by $\hat{\omega}$. Since

$$||\hat{J}(x, t, k) - I||_{L^2(\tilde{\Sigma}) \cap L^\infty(\tilde{\Sigma})} \to 0, \text{ exponentially, as } t \to \infty,$$

by (3.13),

$$||\hat{M} - I||_{L^2(\tilde{\Sigma})} \to 0, \text{ rapidly, as } t \to \infty.$$ 

Then, by (2.14) we get $u_x$ decays rapidly, and then $u$ decays rapidly, as $t \to \infty$. □

3.2. Case 2: $\frac{\alpha}{\pi} + \alpha > 0$. In this case, the real part of $i\theta(k)$ has the signature as the Figure 3. And we set $k_0 = (\frac{\alpha \beta^2}{\pi(\pi + \alpha)})^{\frac{1}{4}}$.

The jump matrix $J(x, t, k)$ has the following factorization

$$J(x, t, k) = \begin{cases}
\left(\begin{array}{cc}
1 & \frac{r(\bar{k})e^{-2it\theta(k)}}{1 - r(k)r(\bar{k})}
\\
0 & 1
\end{array}\right)
\left(\begin{array}{cc}
1 & 0
\\
-r(k)e^{2it\theta(k)} & 1
\end{array}\right),
\\
\left(\begin{array}{cc}
1 & 0
\\
-r(k)e^{2it\theta(k)} & 1
\end{array}\right)
\left(\begin{array}{cc}
1 - r(k)r(\bar{k}) & 0
\\
0 & 1 - r(k)r(\bar{k})
\end{array}\right)
\left(\begin{array}{cc}
1 & \frac{r(k)e^{-2it\theta(k)}}{1 - r(k)r(\bar{k})}
\\
0 & 1
\end{array}\right).
\end{cases} \quad (3.14)$$
3.2.1. The conjugate transform. Introducing a scalar function $\delta(k)$ which solves the Riemann-Hilbert problem

$$\begin{cases}
\delta(k)_{+} = \delta(k)_{-}(1 - r(k)r(k)) & k \in \Sigma = (-k_0, k_0) \cup i(-k_0, k_0), \\
\delta(k)_{-} = \delta(k) & k \in \{\text{Im}k^2 = 0\} \setminus \Sigma, \\
\delta(k) \to 1 & k \to \infty.
\end{cases}$$

The solution of this Riemann-Hilbert problem is given by

$$\delta(k) = \left(\frac{k - k_0}{k}\right)\left(\frac{k + k_0}{k}\right)^{i\vartheta}e^{\chi_{+}(k)e^{\chi_{-}(k)}}\left(\frac{k}{k - i k_0}\right)\left(\frac{k}{k + i k_0}\right)^{i\tilde{\vartheta}}e^{\tilde{\chi}_{+}(k)e^{\tilde{\chi}_{-}(k)}},$$

where

$$\vartheta = -\frac{1}{2\pi} \ln (1 - |r(k_0)|^2),$$

$$\tilde{\vartheta} = -\frac{1}{2\pi} \ln (1 + |r(ik_0)|^2),$$

$$\chi_{\pm}(k) = \frac{1}{2\pi i} \int_{0}^{\pm k_0} \ln \left(\frac{1 - |r(k')|^2}{1 - |r(k_0)|^2}\right) \frac{dk'}{k' - k},$$

$$\tilde{\chi}_{\pm}(k) = \frac{1}{2\pi i} \int_{0}^{\pm ik_0} \ln \left(\frac{1 - r(k')r(k')}{1 + |r(ik_0)|^2}\right) \frac{dk'}{k' - k}.$$  

Moreover, for all $k \in \mathbb{C}$, $|\delta|$ and $|\delta^{-1}|$ are bounded.
The conjugate transform is that

\[ M^{(1)}(x, t, k) = M(x, t, k)\delta(k)^{-\sigma_3}. \]  

(3.18)

Then we can get the Riemann-Hilbert problem of \( M^{(1)}(x, t, k) \)

\[
\begin{cases}
  M^{(1)}(x, t, k)_+ = M^{(1)}(x, t, k)_- J^{(1)}(x, t, k), & k \in \mathbb{R} \cup i\mathbb{R} \\
  M^{(1)}(x, t, k) \to \mathbb{I}, & k \to \infty.
\end{cases}
\]

(3.19)

where

\[
J^{(1)}(x, t, k) = \begin{cases}
  \begin{pmatrix}
    1 & 0 \\
    -r(k) & 1
  \end{pmatrix}, & k \in \Sigma^{(1)} = \Sigma, \\
  \begin{pmatrix}
    1 & 0 \\
    -r(k) & 1
  \end{pmatrix}, & k \in \{\text{Im}k^2 = 0\} \setminus \Sigma^{(1)}.
\end{cases}
\]

(3.20)

Then we reverse the direction of the part of \( \{\text{Im}k^2 = 0\} \setminus \Sigma^{(1)} \), we have

\[
\begin{cases}
  M^{(1)}(x, t, k)_+ = M^{(1)}(x, t, k)_- \tilde{J}^{(1)}(x, t, k), & k \in \mathbb{R} \cup i\mathbb{R} \\
  M^{(1)}(x, t, k) \to \mathbb{I}, & k \to \infty.
\end{cases}
\]

(3.21)
Figure 6. The jump contour $\tilde{\Sigma}^{(1)}$ for $\tilde{\mathcal{M}}^{(1)}(x, t, k)$.

$$\tilde{\mathcal{M}}^{(1)}(x, t, k) = \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
1-r(k) & 1 \\
r(k) & 1 \\
\end{array} \right) \left( \begin{array}{cc} \delta^2(k) & e^{-2it\theta(k)} \\
0 & 1 \\
0 & 1 \\
\end{array} \right), & k \in \tilde{\Sigma}^{(1)} = \Sigma \\
\left( \begin{array}{cc} -r(k) & 1+r(k) \delta^2(k) \\
1-r(k) & 1 \\
r(k) & 1 \\
\end{array} \right) \left( \begin{array}{cc} \delta^2(k) & e^{-2it\theta(k)} \\
0 & 1 \\
0 & 1 \\
\end{array} \right), & k \in \{\text{Im}k^2 = 0\} \setminus \tilde{\Sigma}^{(1)}. 
\end{cases} \quad (3.22)$$

3.2.2. The second transform. The main purpose of this section is to reformulate the original Riemann-Hilbert problem (3.21) as an equivalent Riemann-Hilbert problem on the augmented contour $\Sigma^{(2)}$ (see Figure 7),

$$\Sigma^{(2)} = L \cup L_0 \cup \tilde{L} \cup \tilde{L}_0 \cup \mathbb{R} \cup i\mathbb{R}. \quad (3.23)$$

where $L = L_1 \cup \tilde{L}_1 \cup L_2 \cup \tilde{L}_2$,

Denote the contour

$$L_1 = \{k = k_0 + uk_0e^{i\frac{\pi}{4}}, \quad u \in (-\infty, \frac{1}{\sqrt{2}}]\}, \quad \tilde{L}_1 = \{k = i k_0 + uk_0e^{-i\frac{\pi}{4}}, \quad u \in (-\infty, \frac{1}{\sqrt{2}}]\}$$

$$L_2 = \{k = -k_0 + uk_0e^{-i\frac{\pi}{4}}, \quad u \in (-\infty, \frac{1}{\sqrt{2}}]\}, \quad \tilde{L}_2 = \{k = i k_0 + uk_0e^{i\frac{\pi}{4}}, \quad u \in (-\infty, \frac{1}{\sqrt{2}}]\} \quad (3.24)$$

Denote the contour

$$L_0 = \{uk_0e^{i\frac{\pi}{4}}, \quad u \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]\}. \quad (3.25)$$
Denote the contour

\[ L_\varepsilon = L_{1\varepsilon} \cup \tilde{L}_{1\varepsilon} \cup L_{2\varepsilon} \cup \tilde{L}_{2\varepsilon} \]

\[ = \{ k = k_0 + uk_0 e^{i\frac{3\pi}{4}}, \quad \varepsilon < u \leq \frac{1}{\sqrt{2}} \} \cup \{ k = ik_0 + uk_0 e^{-i\frac{\pi}{4}}, \quad u \in (\varepsilon, \frac{1}{\sqrt{2}}] \} \]

\[ \cup \{ k = -k_0 + uk_0 e^{-i\frac{\pi}{4}}, \quad u \in (\varepsilon, \frac{1}{\sqrt{2}}] \} \cup \{ k = ik_0 + uk_0 e^{i\frac{\pi}{4}}, \quad u \in (\varepsilon, \frac{1}{\sqrt{2}}] \} \]

(3.26)

Following the method in [9], we can have

**Proposition 3.2.** Let

\[ \rho(k) = \begin{cases} 
\rho_1(k) = \frac{r(\bar{k})}{1-r(k)r(k)}, & k \in \Sigma \\
\rho_2(k) = -r(k), & k \in \{\text{Im}k = 0\} \backslash \Sigma. 
\end{cases} \] (3.27)

Then \( \rho \) has a decomposition

\[ \rho(k) = h_I(k) + (h_{II}(k) + R(k)), \] (3.28)

where \( h_I(k) \) is small and \( h_{II}(k) \) has an analytic continuation to \( L \) and \( L_0 \).

For example, if \( \rho(k) = r(k) \) as \( k > k_0 \), \( h_{II}(k) \) of this function \( \rho(k) \) has an analytic continuation to the first quadrant. And \( R(k) \) is piecewise rational \( (R(k) = 0, \text{ if } k \in L_0) \) function.

And \( R, h_I, h_{II} \) satisfy

\[ |e^{-2it\theta(k)}h_I(k)| \leq \frac{c}{(1+|k|^2)t^l}, \text{ for } z \in \mathbb{R} \cup i\mathbb{R}, \quad \text{for } k \in L \cup iL, \] (3.29a)

\[ |e^{-2it\theta(k)}h_{II}(k)| \leq \frac{c}{(1+|k|^2)t^l}, \quad k \in L, \quad k_0 < M. \] (3.29b)

\[ |e^{-2it\theta(k)}h_{II}(k)| \leq ce^{\frac{-4\alpha^2}{4k_0}}, \quad k \in L_0, \quad k_0 < M. \] (3.29c)

and

\[ |e^{-2it\theta(k)}R(k)| \leq ce^{-\frac{\varepsilon^2\alpha^2}{M^2}}, \quad k \in L_\varepsilon. \] (3.29d)

for arbitrary natural number \( l \), for sufficiently large constants \( c \), for some fixed positive constant \( M \).

**Proof.** See appendix. \[ \square \]

**Remark 3.3.** Taking conjugate \( \rho(k) = h_I(k) + h_{II}(k) + R(k) \) leads to the same estimates for \( e^{2it\theta(k)}h_I(k), e^{2it\theta(k)}h_{II}(k) \) and \( e^{2it\theta(k)}R(k) \) on \( \mathbb{R} \cup i\mathbb{R} \cup \tilde{L} \cup \tilde{L}_0 \).
From the Riemann-Hilbert problem (3.21) and formula (3.22), the Riemann-Hilbert problem across $\mathbb{R} \cup i\mathbb{R}$ oriented as Figure 6 is given by

\[
\begin{align*}
M^{(1)}(x, t, k)_+ &= M^{(1)}(x, t, k)_-(b_-)^{-1}b_+, \quad k \in \mathbb{R} \cup i\mathbb{R} \\
M^{(1)}(x, t, k) &\to \mathbb{I}, \quad k \to \infty.
\end{align*}
\]  
(3.30)

where

\[
b_+ = \mathbb{I} + \omega_+ = \delta^+_\omega e^{-i\theta(k)}\sigma_3 \left( \begin{array}{cc} 1 & \rho(k) \\ 0 & 1 \end{array} \right),
\]  
(3.31)

\[
b_- = \mathbb{I} - \omega_- = \delta^-\omega e^{-i\theta(k)}\sigma_3 \left( \begin{array}{cc} 1 & 0 \\ \rho(k) & 1 \end{array} \right),
\]  
(3.32)

and $\rho$ is given by (3.27).

We write

\[
b_+ = b_+^o b_+^a = (\mathbb{I} + \omega_+^o)(\mathbb{I} + \omega_+^a) = \left( \begin{array}{cc} 1 & h_1(k)\delta^2 e^{-2it\theta} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & (h_1(k) + R(k))\delta^2 e^{-2it\theta} \\ 0 & 1 \end{array} \right),
\]  
(3.33a)

\[
b_- = b_-^o b_-^a = (\mathbb{I} - \omega_-^o)(\mathbb{I} - \omega_-^a) = \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{h_1(k)}\delta e^{2it\theta} & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{(h_1(k) + R(k))}\delta e^{2it\theta} & 1 \end{array} \right).
\]  
(3.33b)

Now we can use the signature table of $\Re e^{i\theta}$ showed in Figure 3 to open the jump contour for the Riemann-Hilbert problem of $M^{(1)}$ to the contours in Figure 7. Introducing $M^{(2)}(x, t, k) = M^{(1)}(x, t, k)\phi$, where $\phi$ is defined as follows:

\[
\phi = \begin{cases} 
\mathbb{I}, \quad &k \in D_2, D_5, D_{16}, D_{19} \\
(b_+^o)^{-1}, \quad &k \in D_1, D_3, D_{15}, D_{17}, D_9, D_{10}, D_{11}, D_{12} \\
(b_+^a)^{-1}, \quad &k \in D_4, D_6, D_{18}, D_{20}, D_7, D_8, D_{13}, D_{14}
\end{cases}
\]  
(3.34)

where the regions $\{D_j\}_{j=1}^{20}$ are showed in Figure 7.

Then the Riemann-Hilbert problem of $M^{(2)}(x, t, k)$ is defined

\[
M^{(2)}_+(x, t, k) = M^{(2)}_-(x, t, k)J^{(2)}(x, t, k)
\]  
(3.35)

with

\[
J^{(2)}(x, t, k) = \begin{cases} 
(b_+^o)^{-1}(b_+^a), \quad &k \in \mathbb{R} \cup i\mathbb{R} \\
\mathbb{I}^{-1}(b_+^a), \quad &k \in L \cup L_0 \\
(b_+^a)^{-1}\mathbb{I}, \quad &k \in \bar{L} \cup \bar{L}_0
\end{cases}
\]  
(3.36)
Using the symbol $J^{(2)}(x, t, k) = b^{-1}(x, t, k) b_±(x, t, k)$, and set $\omega_\pm(x, t, k) = \pm(b_\pm(x, t, k) - \mathbb{I})$, $\omega(x, t, k) = \omega_+(x, t, k) + \omega_-(x, t, k)$. From section 2, we have

$$M^{(2)}(x, t, k) = \mathbb{I} + \int_{\Sigma^{(2)}} \frac{\mu(x, t, \xi) \omega(x, t, \xi)}{\xi - k} \frac{d\xi}{2\pi i}, \quad k \in \mathbb{C}\setminus\Sigma^{(2)}. \quad (3.37)$$

And substituting (3.37) into (2.15), we learn that

$$m(x, t) = \frac{1}{2} \lim_{k \to \infty} (k[\sigma_3, M^{(2)}(x, t, k)])_{12},$$

$$= -\frac{1}{2}(\sigma_3, \int_{\Sigma^{(2)}} \mu(x, t, \xi) \omega(x, t, \xi) \frac{d\xi}{2\pi i})_{12}, \quad (3.38)$$

$$= -\frac{1}{2}(\sigma_3, \int_{\Sigma^{(2)}} ((\mathbb{I} - C_\omega)^{-1}(\xi) \omega(x, t, \xi) \frac{d\xi}{2\pi i})_{12}. \quad (3.38)$$

### 3.2.3. Transform to the Riemann-Hilbert problem of $M^{(3)}(x, t, k)$.

Follow the method of [9] P.323-329, we can reduce the Riemann-Hilbert problem of $M^{(2)}(x, t, k)$ to the Riemann-Hilbert problem of $M^{(3)}(x, t, k)$.

Let $\omega^{c}$ be a sum of three terms

$$\omega^{c} = \omega^{a} + \omega^{b} + \omega^{c} + \omega^{d}. \quad (3.39)$$
We then have the following:

\[ \omega^a = \omega \text{ is supported on the } \mathbb{R} \cup i\mathbb{R} \text{ and consists of terms of type } h_I(k) \text{ and } \overline{h_I(k)}. \]
\[ \omega^b = \omega \text{ is supported on the } L \cup \overline{L} \text{ and consists of terms of type } h_{II}(k) \text{ and } \overline{h_{II}(k)}. \]
\[ \omega^c = \omega \text{ is supported on the } L_\varepsilon \cup \overline{L_\varepsilon} \text{ and consists of terms of type } R(k) \text{ and } \overline{R(k)}. \]
\[ \omega^d = \omega \text{ is supported on the } L_0 \cup \overline{L_0}. \]

(3.40)

Set \( \omega' = \omega - \omega^e \). Then, \( \omega' = 0 \) on \( \Sigma^{(2)} \setminus \Sigma^{(3)} \). Thus, \( \omega' \) is supported on \( \Sigma^{(3)} \) with contribution to \( \omega \) from rational terms \( R \) and \( \overline{R} \).

**Proposition 3.4.** For \( 0 < k_0 < M \), we have

\[ \|\omega^a\|_{L^1(L \cup \overline{L}) \cap L^2(L \cup \overline{L}) \cap L^\infty(L \cup \overline{L})} \leq \frac{c}{t^4}, \]  
\[ (3.41a) \]
\[ \|\omega^b\|_{L^1(L \cup \overline{L}) \cap L^2(L \cup \overline{L}) \cap L^\infty(L \cup \overline{L})} \leq \frac{c}{t}, \]  
\[ (3.41b) \]
\[ \|\omega^c\|_{L^1(L_\varepsilon \cup \overline{L_\varepsilon}) \cap L^2(L_\varepsilon \cup \overline{L_\varepsilon}) \cap L^\infty(L_\varepsilon \cup \overline{L_\varepsilon})} \leq ce^{-\frac{a^2 t}{k_0^2}}, \]  
\[ (3.41c) \]
\[ \|\omega^d\|_{L^1(L_0 \cup \overline{L_0}) \cap L^2(L_0 \cup \overline{L_0}) \cap L^\infty(L_0 \cup \overline{L_0})} \leq ce^{-\frac{a^2 t}{4k_0^2}}, \]  
\[ (3.41d) \]

Moreover,

\[ \|\omega'\|_{L^2(\Sigma^{(3)})} \leq \frac{c}{t^4}, \quad \|\omega'\|_{L^1(\Sigma^{(3)})} \leq \frac{c}{t^2}, \]  
\[ (3.42) \]

**Proof.** Consequence of proposition 3.2, and analogous calculations as in lemma 2.13 of [9]. Let us show equation (3.42).

From the appendix, we have

\[ |R(k)| \leq C(k_0)(1 + |k|^5)^{-1} \]  
\[ (3.43) \]
on the contour $k = \{k_0 + uk_0e^{\frac{i3\pi}{4}}, -\infty < u \leq \varepsilon\}$, $\varepsilon \leq \frac{1}{\sqrt{2}}$.

Since
\[\text{Re}i\theta(k) = \frac{\alpha\beta^2 (u^2 - \sqrt{2}u) (u^2 - \sqrt{2}u + 2)}{4k_0^2} \geq \frac{\alpha\beta^2}{4k_0^2} Ku^2\] (3.44)
on the contour $k = \{k_0 + uk_0e^{i3\pi/4}, -\infty < u \leq \varepsilon\}$, and
\[\text{Re}i\theta(k) \geq -\frac{\alpha\beta^2}{4k_0^2} K'u\] (3.45)
on the contour $k = \{k_0 + uk_0e^{i3\pi/4}, -\infty < u \leq -\varepsilon\}$, where $K$ and $K'$ are positive constants.

We have the similar estimates on the other parts of the contour $\Sigma^{(3)}$.

Moreover,
\[||\omega'||^2_{L^2(\Sigma^{(2)})} = ||\omega'||^2_{L^2(\Sigma^{(3)})} \leq C_1(k_0) \int_{\Sigma^{(3)}} e^{-u^2K_1\frac{\alpha\beta^2}{k_0^2}} e^{-uK_2t\frac{\alpha\beta^2}{k_0^2}} (1 + |k|^5 - 2|dk|) \]
\[\leq C_2(k_0) \left(\int_{\mathbb{R}} e^{-u^2K_1\frac{\alpha\beta^2}{k_0^2}} k_0du + \int_{\mathbb{R}} e^{-uK_2t\frac{\alpha\beta^2}{k_0^2}} k_0du \right) \leq C_3(k_0) \left(\frac{k_0^2}{\alpha\beta^2}t\right)\] (3.46)
where $K_1, K_2$ are constants.

**Proposition 3.5.** As $t \to \infty$ and $0 < k_0 < M$, \(||(1 - C_\omega)^{-1}||_{L^2(\Sigma^{(2)})} \leq C\) is equivalent to \(||(1 - C_{\omega'})^{-1}||_{L^2(\Sigma^{(2)})} \leq C\).

**Proof.** Consequence of the following inequality, \(||C_{\omega'} - C_\omega||_{L^2(\Sigma^{(2)})} \leq \varepsilon||\omega'||_{L^2(\Sigma^{(2)})}\), the fact that \(||\omega'||_{L^2(\Sigma^{(2)})} \leq \frac{C}{\pi}\), and the second resolvent identity. \(\square\)

**Proposition 3.6.** If \(||(1 - C_\omega)^{-1}||_{L^2(\Sigma^{(2)})} \leq C\), then for arbitrary positive integer $l$, as $t \to \infty$ such that $0 < k_0 < M$,
\[m(x, t) = \frac{1}{2}(\sigma_3, \int_{\Sigma^{(2)}} ((1 - C_{\omega'})^{-1}I)(\xi)\omega'(x, t, \xi) \frac{d\xi}{2\pi i})_{12} + O\left(\frac{C}{t^l}\right).\] (3.47)
Proof. From the second resolvent identity, one can derive the following expression (see equation (2.27) in [9]),

\[
\int_{\Sigma(2)} ((1 - C_{\omega})^{-1} I) \omega \frac{d\xi}{2\pi i} = \int_{\Sigma(2)} ((1 - C_{\omega'})^{-1} I) \omega' \frac{d\xi}{2\pi i} + \int_{\Sigma(2)} ((1 - C_{\omega'})^{-1} (C_{\omega} I)) \omega \frac{d\xi}{2\pi i} + \int_{\Sigma(2)} ((1 - C_{\omega'})^{-1} (C_{\omega} I)) \omega' \frac{d\xi}{2\pi i} + \int_{\Sigma(2)} ((1 - C_{\omega'})^{-1} C_{\omega'}(1 - C_{\omega})^{-1} (C_{\omega} I)) \omega \frac{d\xi}{2\pi i} = \int_{\Sigma(2)} ((1 - C_{\omega'})^{-1} I) \omega' \frac{d\xi}{2\pi i} + I + II + III + IV.
\]

(3.48)

For \(0 < k_0 < M\), from Proposition \(3.4\) it follows that,

\[
|I| \leq ||\omega^e||_{L^1(\partial_{\omega} J)} + ||\omega^d||_{L^1(\partial_{\omega} L)} + ||\omega^c||_{L^1(\partial_{\omega} L)} + ||\omega^d||_{L^1(L_0 \cup L_0)} \\
\leq ct^{-l},
\]

(3.49)

\[
|II| \leq ||(1 - C_{\omega'})^{-1} I||_{L^2(\Sigma(2))} ||(C_{\omega} I)||_{L^2(\Sigma(2))} ||\omega||_{L^2(\Sigma(2))} \\
\leq c ||\omega^e||_{L^2(\Sigma(2))} (||\omega^e||_{L^2(\Sigma(2))} + ||\omega^c||_{L^2(\Sigma(2))}) \\
\leq ct^{-l}(ct^{-l} + c) \leq ct^{-l},
\]

(3.50)

\[
|III| \leq ||(1 - C_{\omega'})^{-1} I||_{L^2(\Sigma(2))} ||(C_{\omega} I)||_{L^2(\Sigma(2))} ||\omega^e||_{L^2(\Sigma(2))} \\
\leq ct^{-l}.
\]

(3.51)

\[
|IV| \leq ||(1 - C_{\omega'})^{-1} C_{\omega'}(1 - C_{\omega})^{-1} (C_{\omega} I)||_{L^2(\Sigma(2))} ||\omega||_{L^2(\Sigma(2))} \\
\leq ||(1 - C_{\omega'})^{-1} I||_{L^2(\Sigma(2))} ||C_{\omega} I||_{L^2(\Sigma(2))} ||(1 - C_{\omega})^{-1}||_{L^2(\Sigma(2))} ||(C_{\omega} I)||_{L^2(\Sigma(2))} ||\omega||_{L^2(\Sigma(2))} \\
\leq c ||\omega^e||_{L^2(\Sigma(2))} ||(C_{\omega} I)||_{L^2(\Sigma(2))} ||\omega||_{L^2(\Sigma(2))} \\
\leq c ||\omega^e||_{L^2(\Sigma(2))} ||\omega||_{L^2(\Sigma(2))}^2 \\
\leq ct^{-l}.
\]

(3.52)

Hence,

\[
|I + II + III + IV| \leq ct^{-l}.
\]

(3.53)

Applying these estimates to equation (3.35), we can obtain equation (3.37). \(\square\)

Let us now show that, in the sense of appropriately defined operator norms, one may always choose to delete (or add) a portion of a contour(s) on which the jump is \(I\), without altering the Riemann-Hilbert problem in the operator sense.

Suppose that \(\Sigma_1\) and \(\Sigma_2\) are two oriented skeletons in \(\mathbb{C}\) with

\[
\text{card}(\Sigma_1 \cap \Sigma_2) < \infty;
\]

(3.54)
let $u = u(\lambda) = u_+^{(\lambda)} + u_-(\lambda)$ be a $2 \times 2$ matrix-valued function on

$$\Sigma_{12} = \Sigma_1 \cup \Sigma_2$$

(3.55)

with entries in $L^2(\Sigma_{12}) \cap L^\infty(\Sigma_{12})$ and suppose that

$$u = 0 \quad \text{on } \Sigma_2.$$  

(3.56)

Let

$$R_{\Sigma_1} \text{ denote the restriction map } L^2(\Sigma_{12}) \to L^2(\Sigma_1),$$

(3.57)

$$\mathbb{I}_{\Sigma_1 \to \Sigma^{(12)}} \text{ denote the embedding } L^2(\Sigma_1) \to L^2(\Sigma_{12}),$$

(3.58)

$C_{u}^{12} : L^2(\Sigma_{12}) \to L^2(\Sigma_{12})$ denote the operator in (2.5) with $u \leftrightarrow \omega$,  

(3.59)

$C_{u}^{1} : L^2(\Sigma_1) \to L^2(\Sigma_1)$ denote the operator in (2.5) with $u \uparrow \Sigma_1 \leftrightarrow \omega$, 

(3.60)

$C_{u}^{E} : L^2(\Sigma_1) \to L^2(\Sigma_{12})$ denote the restriction of $C_{u}^{12}$ to $L^2(\Sigma_1)$.  

(3.61)

And, finally, let

$$\left\{ \begin{array}{l}
\mathbb{I}_{\Sigma_1} \text{ and } \mathbb{I}_{\Sigma_{12}} \text{ denote the identity operators on } \\
L^2(\Sigma_1) \text{ and } L^2(\Sigma_{12}), \text{ respectively.} 
\end{array} \right.$$  

(3.62)

We then have the next lemma:

**Lemma 3.7.**

$$C_{u}^{12}C_{u}^{E} = C_{u}^{E}C_{u}^{12},$$

(3.63)

$$(\mathbb{I}_{\Sigma_1} - C_{u}^{1})^{-1} = R_{\Sigma_1}(\mathbb{I}_{\Sigma_{12}} - C_{u}^{12})^{-1}\mathbb{I}_{\Sigma_1 \to \Sigma_{12}},$$

(3.64)

$$(\mathbb{I}_{\Sigma_{12}} - C_{u}^{12})^{-1} = \mathbb{I}_{\Sigma_{12}} + C_{u}^{E}(\mathbb{I}_{\Sigma_1} - C_{u}^{1})^{-1}R_{\Sigma_1},$$

(3.65)

in the sense that if the right-hand side of (3.64), resp. (3.65), exists, then the left-hand side exists and identity (3.64), resp. (3.65), holds true.

**Proof.** See Lemma 2.56 in [9].  

We apply this lemma to the case $u = \omega'$, $\Sigma_{12} = \Sigma^{(2)}$ and $\Sigma_1 = \Sigma^{(3)}$. From identity (3.64), we get the following proposition, which is the main result of this subsection.

**Proposition 3.8.**

$$m(x, t) = -\frac{1}{2}([\sigma_3, \int_{\Sigma^{(3)}} (\mathbb{I} - C_{\omega'})^{-1}(\xi)\omega'(x, t, \xi)\frac{d\xi}{2\pi i}]_{12}.  $$

(3.66)
Set
\[ L' = L \setminus L_\xi \]
. Then, \( \Sigma^{(3)} = L' \cup L' \). On \( \Sigma^{(3)} \), set \( \mu' = (1^{\Sigma^{(3)}} - C^{\Sigma^{(3)}})^{-1}I \). Then,

\[
M^{(3)}(x, t, k) = I + \int_{\Sigma^{(3)}} \frac{\mu'(\xi)\omega'(\xi) d\xi}{\xi - k} \tag{3.67}
\]
solves the Riemann-Hilbert problem

\[
\begin{cases}
M^+_+(x, t, k) = M^-_-(x, t, k) J^{(3)}(x, t, k), & k \in \Sigma^{(3)}, \\
M^{(3)} \to I, & k \to \infty.
\end{cases}
\tag{3.68}
\]

where

\[
\omega' = \omega'_+ + \omega'_, \tag{3.69}
\]

\[
b'_+ = I \pm \omega'_+, \tag{3.70}
\]

\[
J^{(3)}(x, t, k) = (b'_+)^{-1}b'_- \tag{3.71}
\]

### 3.2.4. The Scaling operators

In this subsection, we make a further simplification of the Riemann-Hilbert problem on the truncated contour \( \Sigma^{(3)} \) by reducing it to the one which is stated on the four disjoint crosses, \( \Sigma_{A'}, \Sigma_{B'}, \Sigma_{C'} \) and \( \Sigma_{D'} \), and prove that the leading term of the asymptotic expansion for \( m(x, t) \) (proposition 3.8, (3.66)) can be written as the sum of four terms corresponding to the solutions of four auxiliary Riemann-Hilbert problems, each of which is set on one of the crosses; moreover, the solution of the latter Riemann-Hilbert problem can be presented in terms of an exactly solvable model matrix Riemann-Hilbert problem, which is studied in the next subsection.

Let us prepare the notations which are needed for exact formulations. Write \( \Sigma^{(3)} \) as the disjoint union of the four crosses, \( \Sigma_{A'}, \Sigma_{B'}, \Sigma_{C'} \) and \( \Sigma_{D'} \), extend the contours \( \Sigma_{A'}, \Sigma_{B'}, \Sigma_{C'} \) and \( \Sigma_{D'} \) (with orientations unchanged) to the following ones,

\[
\hat{\Sigma}_{A'} = \{ k = k_0 + uk_0 e^{\frac{3\pi i}{4}}, u \in \mathbb{R} \},
\]

\[
\hat{\Sigma}_{B'} = \{ k = -k_0 + uk_0 e^{\frac{\pi i}{4}}, u \in \mathbb{R} \},
\]

\[
\hat{\Sigma}_{C'} = \{ k = ik_0 + uk_0 e^{\frac{3\pi i}{4}}, u \in \mathbb{R} \} \cup \{ k = ik_0 + uk_0 e^{-\frac{3\pi i}{4}}, u \in \mathbb{R} \},
\]

\[
\hat{\Sigma}_{D'} = \{ k = -ik_0 + uk_0 e^{\frac{3\pi i}{4}}, u \in \mathbb{R} \} \cup \{ k = -ik_0 + uk_0 e^{-\frac{3\pi i}{4}}, u \in \mathbb{R} \}.
\]
and define by $\Sigma_A, \Sigma_B, \Sigma_C$, and $\Sigma_D$, respectively, the contours \( \{ k = u k_0 e^{\pm i \theta}, u \in \mathbb{R} \} \) oriented inward as in $\Sigma_A$ and $\Sigma_A'$, inward as in $\Sigma_B'$ and $\Sigma_B''$, outward as in $\Sigma_C'$ and $\Sigma_C''$, and outward as in $\Sigma_D'$ and $\Sigma_D''$, respectively.

We introduce the scaling operators:

\[
N_A : f(k) \to (N_A f)(k) = f(\frac{k^2}{2\sqrt{\alpha \beta} \sqrt{t}} k + k_0) \quad (3.72a)
\]

\[
N_B : f(k) \to (N_B f)(k) = f(\frac{k^2}{2\sqrt{\alpha \beta} \sqrt{t}} k - k_0) \quad (3.72b)
\]

\[
N_C : f(k) \to (N_C f)(k) = f(\frac{-k^2}{2\sqrt{\alpha \beta} \sqrt{t}} k + i k_0) \quad (3.72c)
\]

\[
N_D : f(k) \to (N_D f)(k) = f(\frac{-k^2}{2\sqrt{\alpha \beta} \sqrt{t}} k - i k_0) \quad (3.72d)
\]

Considering the action of the operators $N_k, k \in \{ A, B, C, D \}$ on $\delta(k)e^{-it\theta(k)}$, we find that,

\[
(N_A \delta e^{-it\theta})(k) = \delta^0_A(k) \delta^1_A(k) \quad (3.73)
\]

where

\[
\delta^0_A(k) = \frac{k^3_{\nu - 2i\nu}}{(\sqrt{\alpha \beta})^{2i\nu}} 2^{-i\nu} e^{i\alpha \beta \nu - i\frac{\alpha \beta^2}{2k_0^2}} e^{\chi_\pm(k_0)} e^{\tilde{\chi}'_\pm(k_0)} \quad (3.74a)
\]

\[
\delta^1_A(k) = k^{i\nu} e^{-i\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}}} \frac{k_{\nu + i\nu}}{2^{i\nu}} (\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k + 2k_0)^{i\nu} (\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k + k_0)^{2i\nu} (\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k + k_0 - ik_0)^{-i\nu} e^{\chi_\pm(\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k + k_0) - \chi_\pm(k_0)} e^{\tilde{\chi}'_\pm(\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k + k_0) - \tilde{\chi}'_\pm(k_0)} \quad (3.74b)
\]

with

\[
\tilde{\chi}'_\pm(k) = e^{-\frac{1}{2t} \int_{k_0}^{0} \ln |k - ik'| d\ln(1 + |r(ik')|^2)} \quad (3.75)
\]

And

\[
(N_B \delta e^{-it\theta})(k) = \delta^0_B(k) \delta^1_B(k) \quad (3.76)
\]

where

\[
\delta^0_B(k) = \frac{k^3_{\nu - 2i\nu}}{(\sqrt{\alpha \beta})^{2i\nu}} 2^{-i\nu} e^{i\alpha \beta \nu - i\frac{\alpha \beta^2}{2k_0^2}} e^{\chi_\pm(-k_0)} e^{\tilde{\chi}'_\pm(-k_0)} \quad (3.77a)
\]

\[
\delta^1_B(k) = (-k)^{i\nu} e^{-i\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}}} \frac{k^{\nu + i\nu}}{2^{i\nu}} (\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k - 2k_0)^{i\nu} (\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k - k_0)^{2i\nu} (\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k - k_0 - ik_0)^{-i\nu} e^{\chi_\pm(\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k - k_0) - \chi_\pm(-k_0)} e^{\tilde{\chi}'_\pm(\frac{k^2_{\nu + 2i\nu}}{2\sqrt{\alpha \beta}} k - k_0) - \tilde{\chi}'_\pm(-k_0)} \quad (3.77b)
\]
with $\tilde{\chi}_\pm^l(k)$ defined by (3.75).

For $N_C$, 

$$(N_C\delta e^{-it\theta})(k) = \delta^0_C(k)\delta^1_C(k)$$

(3.78)

where

$$\delta^0_C(k) = \frac{k_0^{2i\nu-i\vartheta}}{(\sqrt{\alpha\beta})^{-i\vartheta}} 2^{i\nu} e^{i\alpha\beta t + i\frac{k_0^2}{2k_0^2} e^{2i\nu}(ik_0)} e^{\chi_\pm^l(ik_0)} e^{\tilde{\chi}_\pm^l(ik_0)}$$

(3.79a)

$$\delta^1_C(k) = (ik)^{-i\vartheta} e^{-k_0^{2i\nu-i\vartheta}} e^{i\frac{k^2}{2k_0^2} e^{2i\nu}(ik_0) - 2i(\frac{-k^2}{2\sqrt{\alpha\beta}} k + ik_0))}^2}$$

(3.79b)

$$e^{\chi_\pm^l(\frac{-k^2}{2\sqrt{\alpha\beta}} k + ik_0) - e^{\chi_\pm^l(ik_0) - \tilde{\chi}_\pm^l(ik_0)}}$$

with

$$\chi_\pm^l(k) = e^{-\frac{1}{2\sigma} k^2 \ln |k-k'|dln(1-|r(k')|^2)}$$

(3.80)

For $N_D$

$$(N_D\delta e^{-it\theta})(k) = \delta^0_D(k)\delta^1_D(k)$$

(3.81)

where

$$\delta^0_D(k) = \frac{k_0^{2i\nu-i\vartheta}}{(\sqrt{\alpha\beta})^{-i\vartheta}} 2^{i\nu} e^{i\alpha\beta t + i\frac{k_0^2}{2k_0^2} e^{2i\nu}(ik_0)} e^{\chi_\pm^l(-ik_0)} e^{\tilde{\chi}_\pm^l(-ik_0)}$$

(3.82a)

$$\delta^1_D(k) = (ik)^{-i\vartheta} e^{-k_0^{2i\nu-i\vartheta}} e^{i\frac{k^2}{2k_0^2} e^{2i\nu}(ik_0) - 2i(\frac{-k^2}{2\sqrt{\alpha\beta}} k - ik_0))}^2}$$

(3.82b)

$$e^{\chi_\pm^l(\frac{-k^2}{2\sqrt{\alpha\beta}} k - ik_0) - e^{\chi_\pm^l(ik_0) - \tilde{\chi}_\pm^l(ik_0)}}$$

Set

$$\Delta_l^0 = (\delta^0_l(k))^{\sigma^3}, \quad l \in \{A, B, C, D\}$$

(3.83)

and let $\tilde{\Delta}_l^0$ denote right multiplication by $\Delta_l^0$,

$$\tilde{\Delta}_l^0 \phi = \phi \Delta_l^0.$$  

(3.84)

Denote

$$\omega' = \begin{cases} \omega', & k \in \Sigma' \\ 0, & k \in \Sigma^{(3)} \setminus \Sigma' \end{cases} \quad \text{and} \quad \tilde{\omega}' = \begin{cases} \omega', & k \in \tilde{\Sigma}' \\ 0, & k \in \tilde{\Sigma}' \setminus \Sigma' \end{cases}$$

(3.85)
According to this,
\[
\omega' = \sum_{l \in \{A,B,C,D\}} \omega'_l, \quad C^{\Sigma(3)}_{\omega'} = \sum_{l \in \{A,B,C,D\}} C^{\Sigma(3)}_{\omega'_l} = \sum_{l \in \{A,B,C,D\}} C^{\Sigma(3)}_{\omega'_l}.
\] (3.86)

**Proposition 3.9.** For \( l, \iota = \{A,B,C,D\}, \ l \neq \iota \) we have
\[
||C^{\Sigma(3)}_{\omega'_l} C^{\Sigma(3)}_{\omega'_\iota}||_{L^2(\Sigma(3))} \leq C(k_0) t^{-\frac{1}{2}}, \quad (3.87a)
\]
\[
||C^{\Sigma(3)}_{\omega'_l} C^{\Sigma(3)}_{\omega'_\iota}||_{L^{\infty}(\Sigma(3)) \to L^2(\Sigma(3))} \leq C(k_0) t^{-\frac{3}{4}}. \quad (3.87b)
\]

**Proof.** Analogous to lemma 3.5 in [9]. □

Let us prove some technical results concerning the operators \( C^{\Sigma(3)}_{\omega'_l} \) and \( \hat{C}^{\Sigma(3)}_{\omega'_l} \)

**Proposition 3.10.** For \( l \in \{A,B,C,D\} \),
\[
C^{\Sigma(3)}_{\omega'_l} = (N_l)^{-1} \Delta^0_l^{-1} C^{\Sigma(3)}_{\omega'_l} (\Delta^0_l)N_l, \quad \omega^l = (\Delta^0_l)^{-1} (N_l \hat{\omega}'_l) \Delta^0_l.
\] (3.88)

where
\[
C^{\Sigma(3)}_{\omega'_l} \big|_{L_l} = - C_\cdot \left( \begin{array}{cc} 0 & 0 \\ \frac{R((N_l k))}{(\delta^1_l)^2} & 0 \end{array} \right), \quad (3.89a)
\]
\[
C^{\Sigma(3)}_{\omega'_l} \big|_{L_l} = C_\cdot \left( \begin{array}{cc} 0 & \frac{R((N_l k))}{(\delta^1_l)^2} \\ 0 & 0 \end{array} \right). \quad (3.89b)
\]

here
\[
L_e = \{ k = \frac{2u \sqrt{\alpha t \beta}}{k_0} e^{-\frac{iu}{4}}, \ -\varepsilon < u < \infty \}, \quad e = A, B, \quad (3.90a)
\]
\[
L_n = \{ k = -\frac{2u \sqrt{\alpha t \beta}}{k_0} e^{\frac{iu}{4}}, \ -\varepsilon < u < \infty \}, \quad n = C, D. \quad (3.90b)
\]

**Proof.** We consider the case \( l = A \), the cases \( l = B, l = C \) and \( l = D \) follow in an analogous manner. Since from (3.74a), \( |\delta^0_A| = 1 \), it follows from the definition of the operator \( \hat{\Delta}_A^0 \) in (3.83) that \( \hat{\Delta}_A^0 \) is a unitary operator. Then the equation (3.88) is a simple change-of-variables argument.

We note that
\[
((\Delta^0_A)^{-1} (N_A \hat{\omega}'^A) \Delta^0_A)(k) = \left( \begin{array}{cc} 0 & \frac{R((N_A k))}{(\delta^1_A)^2} \\ 0 & 0 \end{array} \right) \quad (3.91)
\]
on \( L_A \), otherwise \( ((\Delta^0_A)^{-1} (N_A \hat{\omega}'^A) \Delta^0_A)(k) = 0 \). Similarly,
\[
((\Delta^0_A)^{-1} (N_A \hat{\omega}'^A) \Delta^0_A)(k) = \left( \begin{array}{cc} 0 & 0 \\ \frac{R((N_A k))}{(\delta^1_A)^{-2}} & 0 \end{array} \right) \quad (3.92)
\]
on $L_A$, otherwise $((\Delta^0_A)^{-1}(N_A\omega A')\Delta^0_J)(k) = 0$. \hfill \Box

From definitions of $R(k)$, we know that (for case $A$)
\begin{align}
R(k_0^+ &= \lim_{\text{Re}k > k_0} R(k) = -\frac{r(k_0)}{\sqrt{|r(k_0)|}}, \quad (3.93a)
\end{align}
\begin{align}
R(k_0^- &= \lim_{\text{Re}k < k_0} R(k) = \frac{r(k_0)}{1 - |r(k_0)|^2}. \quad (3.93b)
\end{align}

As $t \to \infty$,
$$
\hat{R}\left(\frac{k_0^2}{2\sqrt{\alpha k \beta t}}k + k_0\right)(\delta^1_A(k))^{-2} - \hat{R}(k_0\pm)k^{-2i\nu}e^{i\frac{k_0^2}{t}} \to 0. \quad (3.94)
$$

We obtain the following estimate on the rate of convergence:

**Proposition 3.11.** Let $\kappa$ be a fixed small number with $0 < \kappa < \frac{1}{2}$. Then, for $k \in \tilde{L}_A$,
$$
\left|\hat{R}\left(\frac{k_0^2}{2\sqrt{\alpha k \beta t}}k + k_0\right)(\delta^1_A(k))^{-2} - \hat{R}(k_0\pm)k^{-2i\nu}e^{i\frac{k_0^2}{t}}\right| \leq C(k_0)|e^{\frac{\kappa k^2}{t}}|\left(\frac{\log t}{\sqrt{t}}\right) \quad (3.95)
$$

**Proposition 3.12.** (see Proposition 6.2 in [16]) For general operators $C_{\omega',l}^\Sigma^\ell$, $l \in \{1,2,\ldots,N\}$, if $(1' - C_{\omega',l}^\Sigma^\ell)^{-1}$ exist, then
\begin{align}
(1' - \sum_{1 \leq X \leq N} C_{\omega', l}^\Sigma^\ell)(1' + \sum_{1 \leq Y \leq N} C_{\omega', l}^\Sigma^\ell(1' - C_{\omega', l}^\Sigma^\ell)^{-1}) &= 1' - \sum_{1 \leq Y \leq N} \sum_{1 \leq X \leq N} (1 - \delta_{XY})C_{\omega', l}^\Sigma^\ell C_{\omega', l}^\Sigma^\ell (1' - C_{\omega', l}^\Sigma^\ell)^{-1} \quad (3.96a)
\end{align}

and
\begin{align}
(1' + \sum_{1 \leq Y \leq N} C_{\omega', l}^\Sigma^\ell(1' - C_{\omega', l}^\Sigma^\ell)^{-1})(1' - \sum_{1 \leq X \leq N} C_{\omega', l}^\Sigma^\ell) &= 1' - \sum_{1 \leq Y \leq N} \sum_{1 \leq X \leq N} (1 - \delta_{XY})(1' - C_{\omega', l}^\Sigma^\ell)^{-1}C_{\omega', l}^\Sigma^\ell C_{\omega', l}^\Sigma^\ell \quad (3.96b)
\end{align}

where $\delta_{XY}$ is the Kronecker delta.

**Proof.** Assumption the existence of general operators $(1' - C_{\omega', l}^\Sigma^\ell)^{-1}, l \in \{1,2,\ldots,N\}$, induction, and a straightforward application of the second resolvent identity. \hfill \Box

**Lemma 3.13.** If, for $l \in \{A,B,C,D\}$, $(1_{\Sigma, l}' - C_{\omega', l}^\Sigma^\ell)^{-1}$ bounded, then as $t \to \infty$,
\begin{align}
m(x,t) &= \frac{1}{2} \sum_{l \in \{A,B,C,D\}} \left(\int_{\Sigma, l}' \sigma_3, ((1_{\Sigma, l}' - C_{\omega', l}^\Sigma^\ell)^{-1}\Pi(\xi)\omega'\xi)\frac{d\xi}{2\pi i}\right)^{12} + O\left(\frac{C}{t}\right) \quad (3.97)
\end{align}
Proof. Analogous to the proof of Lemma 6.2 in [16]. □

**Lemma 3.14.** For \( l \in \{A, B, C, D\} \), \( \| (1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1} \|_{L^2} \leq C \)

**Proof.** Consider the case \( l = A \), the case \( l = B, C \) and \( l = D \) follow in an analogous manner. From Lemma [3.7], the boundedness of \( (1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1} \) follows from the boundedness of \( (1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1} \). From formula (3.88) we have

\[
(1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1} = (N_A)^{-1}(\Delta_0 A)^{-1}(1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1}(\Delta_0 A),
\]

and then the boundedness of \( (1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1} \) follows from the boundedness of \( (1_{\Sigma_l'} - C_{\Sigma_{l'}}^{\Sigma_l'})^{-1} \).

Set \( \omega^A = (\Delta_0 A)^{-1}(N_A^0 \omega^A) \Delta_0 A \),

so that

\[
C_{\omega_A} = C_+ (\omega^A) + C_- (\omega^A).
\]

On \( \Sigma_A \), we have the diagram in Figure 9. Set \( J^{A^0} = (b_{A^0}^{-1} b_{A^0}^{-1}) = (I - \)

\[
\begin{pmatrix}
1 & \frac{r(k_0)}{1 - |r(k_0)|} k^{-2iv} e^{i k^2} \\
0 & 1
\end{pmatrix}, \Sigma_A^1, \\
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \Sigma_A^2, \\
\begin{pmatrix}
\frac{r(k_0)}{1 - |r(k_0)|} k^{-2iv} e^{-i k^2} \\
0 & 1
\end{pmatrix}, \Sigma_A^3, \\
\begin{pmatrix}
1 & \frac{-r(k_0)}{1 - |r(k_0)|} k^{2iv} e^{-i k^2} \\
0 & 1
\end{pmatrix}, \Sigma_A^4
\]

**Figure 9.** The jump condition of cross \( k_0 \) by scaling.

\( \omega_1^{A^0} \)). Defining as usual \( \omega^A = \omega_+^A + \omega_-^A \), and using Proposition 3.11 one finds that

\[
\| \omega^A - \omega_+^A \|_{L^\infty(\Sigma_A) \cap L^1(\Sigma_A) \cap L^2(\Sigma_A)} \leq C(k_0) t^{-\frac{1}{2}}.
\]

Hence, as \( t \to \infty \),

\[
\| C_{\omega_A} - C_{\omega_0} \|_{L^2(\Sigma_A)} \leq C(k_0) t^{-\frac{1}{2}},
\]

(3.102)
and consequently, one sees that the boundedness of \((1_{\Sigma_A} - C^{\Sigma_A})^{-1}\) follows
from the boundedness of \((1_{\Sigma_A} - C^{\Sigma_A})^{-1}\) as \(t \to \infty\).

Then reorient \(\Sigma_A\) to \(\Sigma_{A,r}\) as Figure 10. A simple computation shows that

\[
J^{A,r} = (b^{A,r}_-)^{-1}(b^{A,r}_+)(\mathbb{I} - \omega^{A,r}_-)^{-1}(\mathbb{I} + \omega^{A,r}_+) \text{ on } \Sigma_{A,r}
\]

is determined by

\[
\omega^{A,r}_\pm(k) = -\omega^{A_0}_\pm(k), \quad \text{for } \text{Re} \ k > 0, \quad (3.103a)
\]

and

\[
\omega^{A,r}_\pm(k) = \omega^{A_0}_\pm(k), \quad \text{for } \text{Re} \ k < 0. \quad (3.103b)
\]

The third step is that extending \(\Sigma_{A,r} \to \Sigma_e = \Sigma_{A,r} \cup \mathbb{R}\) with the orientation on \(\Sigma_{A,r}\) as Figure 10 and the orientation on \(\mathbb{R}\) from \(-\infty\) to \(\infty\). And the jump \(J^e = (b^e_-)^{-1}b^e_+ = (\mathbb{I} - \omega^e_-)^{-1}(\mathbb{I} + \omega^e_+)\) with

\[
\omega^e(k) = \omega^{A,r}(k), \quad k \in \Sigma_{A,r}, \quad (3.104a)
\]

\[
\omega^e(k) = 0, \quad k \in \mathbb{R}. \quad (3.104b)
\]

Set \(C_{\omega^e}\) on \(\Sigma_e\). Once again, by Lemma \(3.7\) it is sufficient to bound \((1_{\Sigma_e} - C_{\omega^e})^{-1}\) on \(L^2(\Sigma_e)\).
Then define a piecewise-analytic matrix function $\phi$ as follows:

$$
\hat{M}^{(k_0)} = M^{(k_0)} \phi,
$$

where

$$
\phi = \begin{cases} 
    k^{iv_3}, & k \in \Omega_2^e, \Omega_5^e, \\
    k^{iv_3} \begin{pmatrix} 1 & 0 \\ -r(k_0)e^{-i k^2/2} & 1 \end{pmatrix}, & k \in \Omega_4^e, \\
    k^{iv_3} \begin{pmatrix} 1 & -r(k_0)e^{i k^2/2} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_6^e, \\
    k^{iv_3} \begin{pmatrix} 1 & \frac{r(k_0)}{1-|r(k_0)|}e^{i k^2/2} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_3^e, \\
    k^{iv_3} \begin{pmatrix} 1 & \frac{r(k_0)}{1-|r(k_0)|^2}e^{-i k^2/2} \\ 0 & 1 \end{pmatrix}, & k \in \Omega_4^e.
\end{cases}
$$

Thus, we can get the Riemann-Hilbert problem of $\hat{M}^{(k_0)}$

$$
\hat{M}^{(k_0)}(x, t, k) = \hat{M}^{(k_0)}(x, t, k)J^{e, \phi},
$$

$$
\hat{M}^{(k_0)}(x, t, k) = (I + \frac{M^{(0)}}{k} + O(\frac{1}{k^2}))k^{iv_3}, \ k \to \infty.
$$

where

$$
J^{e, \phi} = \begin{cases} 
    \begin{pmatrix} 1 - |r(k_0)|^2 & r(k_0) e^{-i k^2/2} \\ -r(k_0) e^{i k^2/2} & 1 \end{pmatrix}, & k \in \mathbb{R}, \\
    I, & k \in \Sigma_{A,r}.
\end{cases}
$$

On $\mathbb{R}$ we have

$$
J^{e, \phi} = (b_-^{e, \phi})^{-1} b_+^{e, \phi} = (I - \omega_-^{e, \phi})^{-1} (I + \omega_+^{e, \phi}) = \begin{pmatrix} 1 & e^{-ik^2/2} \hat{r}(k_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{ik^2/2} \hat{r}(k_0) & 1 \end{pmatrix}.
$$

Set $C_{e, \phi} = C_{\omega, e, \phi} \equiv C_{\pm}(\omega_-^{e, \phi}) + C_{-}(\omega_+^{e, \phi})$ as the associated operator on $\Sigma_\phi^e$, with $\omega_+^{e, \phi} = \omega_+^{e, \phi} + \omega_-^{e, \phi}$. By Lemma 3.7, the boundedness of $C_{e, \phi}$ follows from the boundedness of the operator $C_{\omega, e, \phi}^0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ associated with the restriction of $\omega_+^{e, \phi}$ to $\mathbb{R}$. However, $||C_{\omega, e, \phi}^0||_{L^2(\mathbb{R})} \leq \sup_{k \in \mathbb{R}} |e^{-ik^2/2} \hat{r}(k_0)| \leq ||r||_{L^\infty(\mathbb{R})} < 1$, and hence, $||\hat{r} - C_{\omega, e, \phi}^0||_{L^2(\mathbb{R})} \leq (1 - ||r||_{L^\infty(\mathbb{R})})^{-1} < \infty$ for all $k_0$, which in turn implies that $(1_{\Sigma_\phi} - C_{e, \phi})^{-1}$ is bounded. □
3.3. Model Riemann-Hilbert Problem. In this subsection, we reduce the evaluation of the integrals in Lemma 3.13 to four Riemann-Hilbert problems on \( \mathbb{R} \) which can be solved explicitly.

For \( l \in \{A, B, C, D\} \), define
\[
M_l^1(k) = \mathbb{I} + \int_{\Sigma_l} \frac{(1 - C_{\omega^0_l})^{-1}(\xi)\omega^0(\xi)}{\xi - k} \frac{d\xi}{2\pi i}, \quad k \in \mathbb{C}\setminus \Sigma_l. \tag{3.108}
\]
Then, \( M_l^1(k) \) solves the Riemann-Hilbert problem
\[
\begin{cases}
M_l^1(k) = M_l^1(k)J^l(k) = M_l^1(k)(\mathbb{I} - \omega^0_l)^{-1}(\mathbb{I} + \omega^0_l), & k \in \Sigma_l, \\
M_l^1(k) \to \mathbb{I}, & k \to \infty.
\end{cases} \tag{3.109}
\]
In particular we see that if
\[
M_l^1(k) = \mathbb{I} + \frac{M_l^1(k)}{k} + O(k^{-2}), \quad k \to \infty, \tag{3.110}
\]
then
\[
M_l^1(k) = -\int_{\Sigma_l} ((1 - C_{\omega^0_l})^{-1}\mathbb{I})(\xi)\omega^0(\xi) \frac{d\xi}{2\pi i}. \tag{3.111}
\]
Substituting into (3.97), we obtain
\[
m(x, t) = \frac{k^2_0}{2\sqrt{\alpha t \beta}}((\delta^0_A)^2(M^A_1)_{12} + (\delta^0_B)^2(M^B_1)_{12} - (\delta^0_C)^2(M^C_1) - (\delta^0_D)^2(M^D_1)_{12}) + O\left(\frac{C}{t}\right). \tag{3.112}
\]

We consider in detail only case \( A \). Write
\[
\Psi = \hat{M}^{(k_0)}e^{-\frac{k^2}{4}\sigma_3} = \hat{\Psi}_k\exp{i\sigma_3 e^{-\frac{1}{4}k^2}}. \tag{3.113}
\]
From formula (3.105),
\[
\Psi_+(k) = \Psi_-(k)J^{(k_0)}, \quad k \in \mathbb{R}. \tag{3.114}
\]
where
\[
J^{(k_0)} = \begin{pmatrix}
1 - |r(k_0)|^2 & r(k_0) \\
-r(k_0) & 1
\end{pmatrix}
\]
By differentiation with respect to \( k \) and Liouville theorem we can get
\[
\frac{d\Psi}{dk} + \frac{1}{2}ik\sigma_3 \Psi = \beta \Psi, \tag{3.115}
\]
where
\[
\beta = \frac{i}{2}[\sigma_3, M^A_1] = \begin{pmatrix}
o & \beta_{12} \\
\beta_{21} & 0
\end{pmatrix}.
\]
Following [9](P.350-352), we have
\[
\beta_{12} = -e^{-\frac{\pi}{2} \nu} \sqrt{2\pi e^{\frac{i\pi}{4}}} \frac{r(k_0)}{\Gamma(-i\nu)}.
\] (3.116)

Hence,
\[
(M_{A}^{0})_{12} = -i\beta_{12} = ie^{-\frac{\pi}{2} \nu} \sqrt{2\pi e^{\frac{i\pi}{4}}} \frac{r(k_0)}{\Gamma(-i\nu)}.
\] (3.117)

From the symmetry reduction for \( M(k) \), i.e., \( M(-k) = \sigma_3 M(k) \sigma_3 \), we have that
\[
(M_{A}^{0})_{12} = (M_{B}^{0})_{12}.
\] (3.118)

For \( C \),
\[
\beta_{12} = e^{\frac{\pi}{2} \nu} \sqrt{2\pi e^{\frac{i\pi}{4}}} \frac{r(i k_0)}{\Gamma(i\nu)}.
\] (3.119)

And similarly \( (M_{C}^{0})_{12} = (M_{D}^{0})_{12} \).

Thus, we have

**Theorem 3.15.** As \( t \to \infty \), such that \( k_0 < M \),
\[
m(x, t) = \frac{k_0^2}{\beta} \sqrt{\frac{|\nu|}{\alpha t}} e^{i(2\alpha \beta t + 2\nu \ln \frac{k_0}{\sqrt{\alpha t}} + \frac{\pi}{4} + \frac{2\pi^2 t - 2i\nu \ln 2k_0^2 - 2i\chi_\pm(k_0) - 2i\chi_\mp(k_0) - \arg r(k_0) - \arg \Gamma(-i\nu)}{2\pi^2}}}
\]
\[
- \frac{k_0^2}{\beta} \sqrt{\frac{|\nu|}{\alpha t}} e^{i(2\alpha \beta t - 2\nu \ln \frac{k_0}{\sqrt{\alpha t}} + \frac{\pi}{4} + \frac{2\pi^2 t + 2i\nu \ln 2k_0^2 - 2i\chi_\pm(k_0) - 2i\chi_\mp(k_0) - \arg r(i k_0) - \arg \Gamma(i\nu))}{2\pi^2}}
\]
\[
+ O\left(\frac{1}{t}\right).
\] (3.120)

**Proposition 3.16.**
\[
\|2m\|_{L^2(\mathbb{R})}^2 = \frac{2}{\pi} \left( \int_0^{+\infty} \log \frac{1 + |r(i\mu)|^2}{\mu} d\mu - \int_0^{+\infty} \log \frac{1 - |r(\mu)|^2}{\mu} d\mu \right)
\] (3.121)

**Proof.** Analogous to Proposition 8.2 in [10]. \( \square \)

**Lemma 3.17.** As \( t \to \infty \),
\[
e^{4i \int_{-\infty}^{+\infty} |m(x, t)|^2 dx'} = e^{\frac{2i}{\pi} \left( \int_0^{+\infty} \ln(1 + |r(k')|^2) dk' - \int_0^{+\infty} \ln(1 - |r(k')|^2) dk' - \tilde{\psi} \right)} + O\left(\frac{C}{t^\pi}\right),
\] (3.122)

where
\[
\tilde{\psi} = \sqrt{\int_0^{k_0} \frac{\ln(1 + |r(k')|^2) \ln(1 - |r(k')|^2)}{k'} \cos(\xi_1 - \xi_2) dk'}
\] (3.123)
Remark 3.19. Although, Fokas-Lenells equation (2.8) is an evolution equation of the solution to the Fokas-Lenells equation has order $\frac{1}{t}$ with $\ln k_0$ and $\frac{\alpha^2}{k_0}$. Thus, the solution of the Fokas-Lenells equation $u(x,t)$ can be obtained by integration with respect to $x$. This implies that the leading order asymptotic of the solution to the Fokas-Lenells equation has order $t^{-\frac{3}{2}}$.

\[\tilde{\xi}_1 = 2\alpha\beta t + 2\nu \ln \frac{k_0}{\sqrt{\alpha\beta}} + \frac{\pi}{4} \frac{\alpha^2}{k_0} t - 2\nu \ln 2k_0^2 - 2i\chi_{\pm}(k_0) - 2i\tilde{\chi}_{\pm}(k_0) - \arg r(k_0) - \arg \Gamma(-i\nu)\]

and

\[\tilde{\xi}_2 = 2\alpha\beta t - 2\nu \ln \frac{k_0}{\sqrt{\alpha\beta}} + \frac{\pi}{4} \frac{\alpha^2}{k_0} t + 2\nu \ln 2k_0^2 - 2i\chi_{\pm}(ik_0) - 2i\tilde{\chi}_{\pm}(ik_0) - \arg r(ik_0) - \arg \Gamma(i\nu)\]

**Theorem 3.18.** As $t \to \infty$,

\[u(x,t) = 2\sqrt{\frac{2\nu}{\alpha\beta}} e^{i(2\alpha\beta t + 2\nu \ln \frac{k_0}{\sqrt{\alpha\beta}} + \frac{3\pi}{4} - \frac{\alpha^2}{k_0} t - 2\nu \ln 2k_0^2 - 2i\chi_{\pm}(k_0) - 2i\tilde{\chi}_{\pm}(k_0) - \arg r(k_0) - \arg \Gamma(-i\nu)) + \phi} + O\left(t^{-\frac{1}{2}}\right)\]

where $\phi = \frac{2}{\pi} \left( \int_{k_0}^{+\infty} \frac{\ln(1+|r(k')|^2)}{k'} dk' - \int_{k_0}^{+\infty} \frac{\ln(1+|r(k')|^2)}{k'} dk' - \psi \right)$.

Thus, the solution of the Fokas-Lenells equation $u(x,t)$ can be obtained by integration with respect to $x$. This implies that the leading order asymptotic of the solution to the Fokas-Lenells equation has order $t^{-\frac{3}{2}}$.

**Remark 3.19.** Although, Fokas-Lenells equation (2.8) is an evolution equation in $u_x$ and that any solution $u(x,t)$ is undetermined up to $u(x,t) \to u(x,t) + h(t)$ for an arbitrary function $h(t)$, the requirement that $u$ goes to zero as $|x| \to \infty$ removes this non-uniqueness.

**Remark 3.20.** It is not normal that we get the solution $u_x(x,t)$ in terms of the solution of the Riemann-Hilbert problem (2.14). And we find that if we use the asymptotic behavior of the M($x$,t,$k$) as $k \to 0$, we can get the solution of $u(x,t)$ from the $t-$part of Lax pair (2.10). We will use this to deal with general initial value problem case in another paper [21].

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Appendix A. Prove Proposition 3.2 and 3.11.

Prove Proposition 3.2.

For the convenience of reader, we show the details of the procedure of the analytic continuation.

1. $|\frac{k_0}{2}| < |k| < k_0, k \in \mathbb{R}$.

We just consider $|\frac{k_0}{2}| < k < k_0$, the case for $-k_0 < k < -|\frac{k_0}{2}|$ is similarly.

Set

$$\rho(k) = \frac{-r(k)}{1 - r(k)r(k)} = \frac{-r(k)}{1 - |r(k)|^2}.$$  \hspace{1cm} (A.1)

We split $\rho(k)$ into even and odd parts, $\rho(k) = H_e(k^2) + kH_o(k^2)$, where $H_e(\cdot)$ and $H_o(\cdot)$ are of the Schwartz class.

For any positive integer $m$,

$$H_e(k^2) = \mu_0^e + \mu_1^e(k^2 - k_0^2) + \cdots + \mu_m^e(k^2 - k_0^2)^m + \frac{1}{m!} \int_{k_0^2}^{k^2} H_e^{(m+1)}(\gamma)(k^2 - \gamma)^m d\gamma$$  \hspace{1cm} (A.2)

and

$$H_o(k^2) = \mu_0^o + \mu_1^o(k^2 - k_0^2) + \cdots + \mu_m^o(k^2 - k_0^2)^m + \frac{1}{m!} \int_{k_0^2}^{k^2} H_o^{(m+1)}(\gamma)(k^2 - \gamma)^m d\gamma.$$

(A.3)

Set

$$R(k) = R_m(k) = \sum_{i=0}^{m} \mu_i^e(k^2 - k_0^2)^i + k \sum_{i=0}^{m} \mu_i^o(k^2 - k_0^2)^i.$$  \hspace{1cm} (A.4)

Assume $m = 4q + 1$, where $q$ is a positive integer. Write

$$\rho(k) = h(k) + R(k), \quad \frac{k_0}{2} < k < k_0, k \in \mathbb{R}.$$  \hspace{1cm} (A.5)

Then

$$\frac{d^j h(k)}{dk^j} \bigg|_{\pm k_0} = 0, \quad 0 \leq j \leq m.$$  \hspace{1cm} (A.6)

And we have

$$h(k) = \frac{(k^2 - k_0^2)^{m+1}}{m!} g(k, k_0)$$  \hspace{1cm} (A.7)

where

$$g(k, k_0) = \left( \int_{0}^{1} H_e^{(m+1)}(k^2_0 + u(k^2 - k_0^2))(1 - u)^m du + k \int_{0}^{1} H_o^{(m+1)}(k_0^2 + u(k^2 - k_0^2))(1 - u)^m du \right)$$  \hspace{1cm} (A.8)

and

$$\frac{|d^j g(k, k_0)|}{dk^j} \leq C, \quad \frac{k_0}{2} \leq k \leq k_0.$$  \hspace{1cm} (A.9)
We will split \( h(k) = h_I(k) + h_{II}(k) \), where \( h_I \) is small and \( h_{II} \) has an analytic continuation to \( \text{Im}k > 0 \). Thus

\[
\rho = h_I + (h_{II} + R). \tag{A.10}
\]

Set \( p(k) = (k^2 - k_0^2)^q \). Recall

\[
\theta(k) = k^2 (\frac{q}{r} + \alpha) + \frac{\alpha^2}{4kr} - \alpha \beta \nonumber
\]

\[
= \frac{\alpha^2}{4k^3} k^2 + \frac{\alpha^2}{4kr} - \alpha \beta. \tag{A.11}
\]

We define

\[
\left\{ \begin{array}{l}
\frac{h}{p}(\theta) = \frac{h(k(\theta))}{p(k(\theta))}, \quad \theta(k_0) < \theta < \theta\left(\frac{k_0}{2}\right), \\
0, \quad \theta \leq \theta(k_0) \text{ or } \theta \geq \theta\left(\frac{k_0}{2}\right). \tag{A.12}
\end{array} \right.
\]

As \(|\theta| \to \theta(k_0) = \frac{\alpha^2}{2k^3} - \alpha \beta\) and \(|\theta| > \frac{\alpha^2}{2k^3} - \alpha \beta\), we have \( \frac{h}{p}(\theta) = O((k^2(\theta) - k_0^2)^{m + 1 - q}) \) and

\[
\frac{d\theta}{dk} = \frac{\alpha^2}{2k^4 k^3}. \tag{A.13}
\]

We claim that \( \frac{h}{p} \in H^j(-\infty < \theta < \infty) \) for \( 0 \leq j \leq \frac{3q + 2}{2} \). As by Fourier inversion,

\[
\frac{h}{p}(k) = \int_{-\infty}^{\infty} e^{is\theta(k)} \frac{\overline{h}}{p}(s) \overline{ds}, \quad \frac{k_0}{2} < k < k_0, \tag{A.14}
\]

where

\[
\left( \frac{h}{p} \right)(s) = \int_{\theta(k_0)}^{\theta(k)} e^{-is\theta(k)} \frac{\overline{h}}{p}(\theta(k)) d\theta(k), \quad s \in \mathbb{R}. \tag{A.15}
\]

Thus,

\[
\int_{\theta(k_0)}^{\theta(k)} \left| \frac{d}{d\theta} \left( \frac{h}{p}(\theta(k)) \right) \right|^2 |\overline{d\theta}(k)| \nonumber
\]

\[
= \int_{\frac{k_0}{2}}^{k_0} \left( \frac{2k_0^2 k^3}{\alpha^2 k^4 - k_0^4} \frac{d}{dk} \right)^j \left| \frac{h}{p}(k) \right|^2 \left| \frac{\alpha^2}{2k_0^2 k^3} \right| |dk| \leq C < \infty, \tag{A.16}
\]

for \( 0 < k_0 < M, 0 \leq j \leq \frac{3q + 2}{2} \). Hence,

\[
\int_{-\infty}^{\infty} (1 + s^2)^j |(h/p)(s)|^2 ds \leq C < \infty \tag{A.17}
\]

for \( 0 < k_0 < M, 0 \leq j \leq \frac{3q + 2}{2} \).

Split

\[
h(k) = p(k) \int_{t}^{\infty} e^{is\theta(k)} \frac{\overline{h}}{p}(s) \overline{ds} + p(k) \int_{-\infty}^{t} e^{is\theta(k)} \frac{\overline{h}}{p}(s) \overline{ds} \nonumber
\]

\[
= h_I(k) + h_{II}(k). \tag{A.18}
\]
Thus, for $\frac{k_0}{2} < k < k_0 \leq M$ and any positive integer $n \leq \frac{3q+2}{2}$.

$$\left| e^{-2it\theta}(\kappa_I(k)) \right| \leq |p(k)| \int_{t}^{\infty} |(h/p)(s)|ds \leq |p(k)| \int_{t}^{\infty} (1 + s^2)^{-n}ds \leq c \left( \frac{2}{\theta^2} \right).$$  

(A.19)

Consider the contour $l_1 : k(u) = k_0 + u k_0 e^{-\frac{3\pi}{4}}, 0 \leq u \leq \frac{1}{\sqrt{2}}$. Since $\text{Re} \theta(k)$ is positive on this contour, $\kappa_I(k)$ has an analytic continuation to contours $l_1$.

On the contour $l_1$,

$$\left| e^{-2it\theta}(\kappa_I(k)) \right| \leq |k + k_0|^q (k_0 u)^q e^{-t \text{Re} \theta(k)} \int_{\infty}^{t} e(s-t) \text{Re} \theta(k)(h/p)(s)ds \leq c k_0^q u^q e^{-t \text{Re} \theta(k)} \int_{\infty}^{t} (1 + s^2)^{-1}ds \leq c k_0^q u^q e^{-t \text{Re} \theta(k)}.$$  

(A.20)

Recall $\theta(k) = \frac{\alpha^2}{4k_0} k^2 + \frac{\alpha \beta}{4k_0} - \frac{\alpha}{k}$, and set $k = k_1 + ik_2$, thus

$$\text{Re} \theta(k) = -2 \frac{\alpha \beta^2}{4k_0} k_1 k_2 \frac{(k_1^2 + k_2^2)^2 - k_0^4}{4k_0^2 (k_1^2 + k_2^2)} = \frac{\alpha \beta^2}{4k_0} \frac{(u^2 - \sqrt{2} \alpha)^2 (u^2 - \sqrt{2} \alpha + 2)}{(u^2 - \sqrt{2} \alpha + 1)^2} \geq \frac{\alpha \beta^2 u^2}{2k_0^2},$$

for $0 \leq u \leq \frac{1}{\sqrt{2}}$.

Thus, on the contour $l_1$

$$\left| e^{-2it\theta}(\kappa_I(k)) \right| \leq c k_0^q u^q e^{-t \frac{\alpha \beta^2 u^2}{2k_0^2}} \leq c k_0^q u^q e^{-t \frac{\alpha \beta^2 u^2}{2M^2}}$$  

(A.22)

for $k_0 < M$.

Fix $\varepsilon$, $0 < \varepsilon < \frac{1}{\sqrt{2}}$. If $k(u)$ is on the contour $l_1$, $\varepsilon < u < \frac{1}{\sqrt{2}}$, then we obtain

$$\left| e^{-2it\theta}(R(k)) \right| \leq c e^{-\varepsilon \frac{\alpha \beta^2 u^2}{k_0^2}} \leq c e^{-\varepsilon \frac{\alpha \beta^2 u^2}{M^2 t}}$$  

(A.23)

$2.0 < |k| < \frac{k_0}{2}, k \in \mathbb{R}$.

We consider $0 < k < \frac{k_0}{2}$, the case for $-\frac{k_0}{2} < k < 0$ is similarly.

Define

$$\rho(\theta) = \rho(k(\theta)), \quad \theta > \theta(k_0), \quad 0, \quad \theta \leq \theta(k_0).$$  

(A.24)

We claim that $\rho(\theta) \in H^j(-\infty < \theta < \infty)$ for any nonnegative integer $j$. 

By Fourier inversion,
\[ \rho(\theta(k)) = \int_{-\infty}^{\infty} e^{is\theta(k)} \hat{\rho}(s) \overline{d(k)}, \quad 0 < k < \frac{k_0}{2}, \]  
(A.25)
where
\[ \hat{\rho}(s) = \int_{\theta(k_0)}^{\infty} e^{-is\theta(k)} \rho(\theta(k)) d\theta(k). \]  
(A.26)

Then,
\[ \int_{\theta(k_0)}^{\infty} \left| \frac{d}{d\theta} \rho(\theta(k)) \right|^2 |\overline{d(k)}| \]
\[ = \int_{0}^{\frac{k_0}{2}} \left| \left( \frac{2k_0^3k^3}{\alpha^2(4k_0^2-\hat{k}_0^2)} \frac{d}{dk} \right)^j \rho(k) \right|^2 \left| \frac{\alpha^2(k_0^4-k_1^2k_2^2)}{2k_0^3} \right| |\overline{d(k)}| \leq C < \infty, \]  
(A.27)
for any nonnegative integer \( j, 0 < k_0 < M, \) since \( r(k) \to 0 \) rapidly, as \( k \to 0. \)

Hence
\[ \int_{-\infty}^{\infty} (1 + s^2)^j |\hat{\rho}(s)|^2 ds \leq C, \]  
(A.28)
for any nonnegative integer \( j. \)

Split
\[ \rho(k) = \int_{t}^{\infty} e^{is\theta(k)} \hat{\rho}(s) \overline{d(s)} + \int_{-\infty}^{t} e^{is\theta(k)} \hat{\rho}(s) \overline{d(s)} = h_I(k) + h_{II}(k). \]  
(A.29)

Then, for \( 0 < k \) and any positive integer \( j, \) we obtain,
\[ |e^{-2it\theta(k)}h_I(k)| \leq \int_{t}^{\infty} |\hat{\rho}| \overline{d(s)} \]
\[ \leq (\int_{t}^{\infty} (1 + s^2)\overline{d(s)}\frac{1}{2})^\frac{1}{2} (\int_{t}^{\infty} (1 + s^2)|\hat{\rho}(s)|^2 ds)^\frac{1}{2} \]  
(A.30)
Consider the contour \( l_2: k(u) = uk_0 e^{i\frac{\pi}{4}}, 0 < u < \frac{1}{\sqrt{2}}. \) Since \( \Re e^{i\theta(k)} \) is positive on this contour, \( h_{II} \) has an analytic continuation to contour \( l_2. \)

On the contour \( l_2, \)
\[ |e^{-2it\theta(k)}h_{II}(k)| \leq e^{-t\Re e^{i\theta(k)}} \int_{-\infty}^{\infty} e^{(s-t)\Re e^{i\theta(k)}} |\hat{\rho}(k)| \overline{d(s)} \]
\[ \leq e^{-t\Re e^{i\theta(k)}} \left( \int_{-\infty}^{\infty} (1 + s^2)^{-\frac{1}{2}} \overline{d(s)} \right)^\frac{1}{2} (\int_{-\infty}^{\infty} (1 + s^2)|\hat{\rho}(k)|^2 ds)^\frac{1}{2}, \]  
(A.31)
where
\[ \Re e^{i\theta(k)} = -2\alpha \beta^2 k_1 k_2 \frac{(k_1^2 + k_2^2)^2 - k_0^4}{4k_0^3(k_1^2 + k_2^2)^2} \]
\[ = -\frac{2\alpha \beta^2 u^4 - 1}{4k_0^3} \]
\[ \leq \frac{\alpha \beta^2}{4k_0^3} \]
(A.32)
for \( 0 < u \leq \frac{1}{\sqrt{2}}. \)
Thus, we obtain,
\[ |e^{-2it\theta(k)} h_{II}(k)| \leq c e^{-\frac{t\alpha^2}{4t_0}}. \] (A.33)

3. \(|k| > k_0, k \in \mathbb{R}\)
We consider \(k > k_0\), the case for \(k < -k_0\) is similarly.
Set
\[ \rho(k) = r(k). \] (A.34)
We write
\[ (k-i)^{m+5} \rho(k) = \mu_0 + \mu_1(k-k_0) + \cdots + \mu_m(k-k_0)^m + \frac{1}{m!} \int_{k_0}^{k} \rho(\gamma)(m+1)(\gamma)(k-\gamma)^m d\gamma. \] (A.35)
Define
\[ R(k) = \sum_{i=0}^{m} \mu_i(k-k_0)^i \] (k-i)^{m+5} \] (A.36)
and write \(\rho(k) = h(k) + R(k)\). We have
\[ \frac{d^j h(k)}{dk^j} \bigg|_{k_0} = 0, \quad 0 \leq j \leq m. \] (A.37)
For \(0 < k_0 < M\), set
\[ v(k) = \frac{(k-k_0)^q}{(k-i)^{q+2}}. \] (A.38)
Let
\[ \begin{aligned}
\frac{h}{v}(\theta) &= \frac{h}{v}(k(\theta)), \quad \theta > \theta(k_0), \\
&= 0, \quad \theta \leq \theta(k_0).
\end{aligned} \] (A.39)
Then
\[ \frac{h}{v}(\theta(k)) = \int_{-\infty}^{\infty} e^{is\theta(k)} \frac{\hat{h}}{v}(s) ds, \quad k \geq k_0, \] (A.40)
where
\[ \frac{\hat{h}}{v}(s) = \int_{\theta(k_0)}^{\infty} e^{-is\theta(k)} \frac{h}{v}(\theta(k)) d\theta(k). \] (A.41)
Moreover, we have
\[ \frac{h}{v}(\theta(k)) = \frac{(k-k_0)^{3q+2}}{(k-i)^{3q+4}} g(k, k_0), \] (A.42)
where
\[ g(k, k_0) = \frac{1}{m!} \int_{0}^{1} ((-i)^{m+5} \rho(\gamma))(m+1)(k_0 + u(k-k_0))(1-u)^k du \] (A.43)
and
\[ \left| \frac{d^j g(k, k_0)}{dk^j} \right| \leq C, \quad k \geq k_0. \] (A.44)
Using the identity \( \left| \frac{k-k_0}{k+k_0} \right| \leq 1 \) for \( k \geq k_0 \), we have

\[
\int_{\theta(k_0)}^{\infty} \left| \frac{1}{\theta(k)} \frac{d}{d\theta} \left( \frac{\beta}{\theta(k)} \right) \right|^2 d\theta(k) = \int_{\theta(k_0)}^{\infty} \left| \frac{2k_0^4}{\alpha^2} \frac{d}{dk} \left( \frac{\beta}{\theta(k)} \right) \right|^2 \frac{k_0^2 \beta^2 (1-k_0^2)}{2k^6} |dk|
\leq c \int_{\theta(k_0)}^{\infty} \left| \frac{2(k-k_0)^2 + 2 - 3k^2}{(k-k_0)^4 + 4} \right|^2 k_0^4 |dk|
\leq C_1, \quad 0 \leq j \leq \frac{3q+2}{3}.
\]

(A.45)

Thus,

\[
\int_{-\infty}^{\infty} (1 + s^2)^j \left| \left( \frac{\beta}{\theta} \right) (s) \right|^2 \tilde{d}s \leq C < \infty.
\]

(A.46)

We write

\[
h(k) = v(k) \int_{k}^{\infty} e^{i\theta(k)} \left( \frac{\beta}{\theta}\right)(s) \tilde{d}s + v(k) \int_{-\infty}^{k} e^{i\theta(k)} \left( \frac{\beta}{\theta}\right)(s) \tilde{d}s
= h_I(k) + h_{II}(k).
\]

(A.47)

For \( k \geq k_0, 0 < k_0 < M \), and any positive integer \( e \leq \frac{3q+2}{3} \), we obtain,

\[
|e^{-2it\theta(k)}h_I(k)| \leq \frac{|k-k_0|^q}{|k-k_0|^q+2} \int_{k}^{\infty} |(\frac{\beta}{\theta})(s)| \tilde{d}s
\leq \frac{|k-k_0|^q}{|k-k_0|^q} \left( \int_{k}^{\infty} (1 + s^2)^{-e} \tilde{d}s \right)^{1/2} \left( \int_{k}^{\infty} (1 + s^2)^e |(\frac{\beta}{\theta})(s)|^2 \tilde{d}s \right)^{1/2}
\leq \frac{c}{(1+|k|^2)^e}.
\]

(A.48)

And \( h_{II}(k) \) has an analytic continuation to the lower half-plane, where \( \text{Re} \theta(k) \) is positive. We estimate \( e^{-2it\theta(k)}h_{II}(k) \) on the contour \( k(u) = k_0 + uk_0 e^{-it\theta}, u \geq 0 \).

If \( 0 < u \leq M_1 \),

\[
|e^{-2it\theta(k)}h_{II}(k)| \leq \frac{k_0^q u^q e^{-t\text{Re} \theta(k)}}{|k-k_0|^q+2},
\]

(A.49)

where

\[
\text{Re} \theta(k) = -2\alpha \beta^2 k_1 k_2 \left( \frac{k_0^4 + k^2}{4k_0^2 (k_0^2 + k^2)} \right)
= \frac{\alpha \beta^2 u^2 + \sqrt{2u} (u^2 + \sqrt{2u})}{4u^2 + \sqrt{2u} (u^2 + \sqrt{2u})}
\geq \frac{\alpha \beta^2 u^2}{4u^2 + \sqrt{2u} (u^2 + \sqrt{2u})}.
\]

(A.50)

Then

\[
|e^{-2it\theta(k)}h_{II}(k)| \leq \frac{k_0^q u^q e^{-t\text{Re} \theta(k)}}{|k-k_0|^q+2} \leq \frac{k_0^q u^q e^{-t\text{Re} \theta(k)}}{|k-k_0|^q+2} \leq \frac{\alpha \beta^2 u^2}{4u^2 + \sqrt{2u} (u^2 + \sqrt{2u})}
\leq \frac{c}{(1+|k|^2)^{e+1/2}} \leq \frac{c}{(1+|k|^2)^{e+1/2}}.
\]

(A.51)
If \( u > M_1 \), then
\[
\text{Re}i\theta(k) \geq \frac{\alpha \beta^2}{4k_0^2} \frac{u^6}{(u^2 + \sqrt{2u + 1})^2}
\] (A.52)
and
\[
|e^{-2it\theta(k)}h_{II}(k)| \leq c_3\frac{k_0^6}{|k-\gamma|^{q+2}} e^{-\frac{t\alpha \beta^2}{4k_0^2} \frac{u^6}{(u^2 + \sqrt{2u + 1})^2}}
\] (A.53)
Hence, for \( u > 0 \), we obtain
\[
|e^{-2it\theta(k)}h_{II}(k)| \leq c_5 \frac{(1 + |k|^2) t^\frac{q}{2}}{(1 + |k|^2) t^\frac{q}{2}}.
\] (A.54)

4. \( \frac{k_0}{2} < |k| < k_0, k \in i\mathbb{R} \).
We just consider \( \frac{k_0}{2} < \text{Im} k < k_0 \), the case for \( -k_0 < \text{Im} k < -\frac{k_0}{2} \) is similarly.
Set
\[
\rho(k) = \frac{-r(k)}{1 + |r(k)|^2}, \quad \frac{k_0}{2} < \text{Im} k < k_0, k \in i\mathbb{R}.
\] (A.55)
The following process is similar as the case \( \frac{k_0}{2} < k < k_0 \). That is,
We split \( \rho(k) \) into even and odd parts, \( \rho(k) = H_e(k^2) + kH_o(k^2) \), where \( H_e(\cdot) \) and \( H_o(\cdot) \) are of the Schwartz class.
For any positive integer \( m \),
\[
H_e(k^2) = \mu_0^e + \mu_1^e(k^2 + k_0^2) + \cdots + \mu_m^e(k^2 + k_0^2)^m + \frac{1}{m!} \int_{k_0^2}^{k^2} H_e^{(m+1)}(\gamma)(k^2 - \gamma)^m d\gamma
\] (A.56)
and
\[
H_o(k^2) = \mu_0^o + \mu_1^o(k^2 + k_0^2) + \cdots + \mu_m^o(k^2 + k_0^2)^m + \frac{1}{m!} \int_{k_0^2}^{k^2} H_o^{(m+1)}(\gamma)(k^2 - \gamma)^m d\gamma.
\] (A.57)
Set
\[
R(k) = R_m(k) = \sum_{i=0}^{m} \mu_i^e(k^2 + k_0^2)^i + k\sum_{i=0}^{m} \mu_i^o(k^2 + k_0^2)^i.
\] (A.58)
Assume \( m = 4q + 1 \), where \( q \) is a positive integer. Write
\[
\rho(k) = h(k) + R(k), \quad \frac{k_0}{2} < \text{Im} k < k_0, k \in i\mathbb{R}.
\] (A.59)
Then
\[
\left. \frac{d^j h(k)}{dk^j} \right|_{\pm ik_0} = 0, \quad 0 \leq j \leq m.
\] (A.60)
And we have
\[ h(k) = \frac{(k^2 + k_0^2)^m + 1}{m!} g(k, k_0) \]  \hspace{1cm} (A.61)
where
\[ g(k, ik_0) = \left( \int_0^1 H_m^{(m+1)}(-k_0^2 + u(k^2 + k_0^2))(1 - u)m du + k \int_0^1 H_m^{(m+1)}(-k_0^2 + u(k^2 + k_0^2))(1 - u)m du \right) \]  \hspace{1cm} (A.62)
and
\[ \left| \frac{d^j g(k, ik_0)}{dk^j} \right| \leq C, \quad \frac{k_0}{2} \leq \text{Im} k \leq k_0. \]  \hspace{1cm} (A.63)
We will split \( h \) as \( h(k) = h_I(k) + h_{II}(k) \), where \( h_I \) is small and \( h_{II} \) has an analytic continuation to \( \text{Re} k > 0 \). Thus
\[ \rho = h_I + (h_{II} + R). \]  \hspace{1cm} (A.64)
Set \( p(k) = (k^2 + k_0^2)^q \). Recall
\[ \theta(k) = k^2 \left( \frac{q}{2} + \alpha \right) + \frac{\alpha \beta^2}{4k^2} - \alpha \beta = \frac{\alpha \beta^2}{4k_0^2} k^2 + \frac{\alpha \beta^2}{4k^2} - \alpha \beta. \]  \hspace{1cm} (A.65)
We define
\[ \{ \begin{array}{l}
\frac{h}{p}(\theta) = \frac{h(k(\theta))}{p(k(\theta))}, \quad \theta(ik_0) < \theta < \theta(ik_0/2), \\
0, \quad \theta \leq \theta(ik_0) \text{ or } \theta \geq \theta(ik_0/2).
\end{array} \]  \hspace{1cm} (A.66)
As \( |\theta| \rightarrow \theta(ik_0) = -\frac{\alpha \beta^2}{2k_0} - \alpha \beta \) and \( |\theta| > -\frac{\alpha \beta^2}{2k_0} - \alpha \beta \), we have \( \frac{h}{p}(\theta) = O((k^2(\theta) + k_0^2)^{m+1-q}) \) and
\[ \frac{d\theta}{dk} = \frac{\alpha \beta^2 (k^4 - k_0^4)}{2k_0^3 k^3}. \]  \hspace{1cm} (A.67)
We claim that \( \frac{h}{p} \in H^j(-\infty < \theta < \infty) \) for \( 0 \leq j < \frac{3d+2}{2} \). As by Fourier inversion,
\[ \frac{h}{p}(k) = \int_{-\infty}^{\infty} e^{i\theta(k)} \overline{\frac{h}{p}(s)} ds, \quad \frac{k_0}{2} < \text{Im} k < k_0, \]  \hspace{1cm} (A.68)
where
\[ \overline{\frac{h}{p}(s)} = \int_{\theta(ik_0)}^{\theta(ik_0/2)} e^{-i\theta(k)} \frac{h}{p}(\theta(k)) d\theta(k), \quad s \in \mathbb{R}. \]  \hspace{1cm} (A.69)
where \( ds = \frac{ds}{\sqrt{2\pi}} \) and \( d\theta(k) = \frac{d\theta(k)}{\sqrt{2\pi}} \).
Thus,

\[ \int_{\theta(k_0)}^{\theta(ik_0)} \left| \frac{d}{d\theta} \frac{e^{i\theta}}{p(k)} \right|^2 |\bar{d}\theta(k)| \]

\[ = \int_{\theta(k_0)}^{\theta(ik_0)} \left| \frac{\partial}{\partial\theta} \frac{e^{i\theta}}{p(k)} \right|^2 \frac{\partial \beta^2 (k^4-k^4)}{2k_0^2} \mid \bar{dk} \leq C < \infty, \]  

(A.70)

for 0 < k_0 < M, 0 \leq j \leq \frac{3a+2}{2}. Hence,

\[ \int_{-\infty}^{\infty} (1 + s^2)^j |(h/p)(s)|^2 ds \leq C < \infty \]  

(A.71)

for 0 < k_0 < M, 0 \leq j \leq \frac{3a+2}{2}.

Split

\[ h(k) = p(k) \int_{-\infty}^{\infty} e^{is\theta(k)}(h/p)(s)ds + p(k) \int_{-\infty}^{t} e^{is\theta(k)}(h/p)(s)ds \]

\[ = h_I(k) + h_{II}(k). \]  

(A.72)

Thus, for \( \frac{k_0}{2} < \text{Im}k < k_0 \leq M \) and any positive integer \( n \leq \frac{3a+2}{2} \).

\[ |e^{-2it\theta(k)}h_I(k)| \leq |p(k)| \int_{-\infty}^{\infty} |(h/p)(s)| ds \]

\[ \leq |p(k)| \left( \int_{-\infty}^{t} (1 + s^2)^{-n} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^{t} (1 + s^2)^{n} |(h/p)(s)|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq \frac{c}{\rho^{n+\frac{1}{2}}}. \]  

(A.73)

Consider the contour \( l_1' : k(u) = ik_0 + uk_0e^{-i\frac{\theta}{T}} \), 0 \leq u \leq \frac{1}{\sqrt{2}}. Since \( \text{Re} \theta(k) \) is positive on this contour, \( h_{II}(k) \) has an analytic continuation to contours \( l_1' \).

On the contour \( l_1' \),

\[ |e^{-2it\theta(k)}h_{II}(k)| \leq |k + ik_0|^n (k_0 u)^n e^{-i\text{Re} \theta(k)} \left( \int_{-\infty}^{t} e^{(s-t)\text{Re} \theta(k)} |(h/p)(s)| ds \right) \]

\[ \leq c k_0^{2n} u^2 e^{-i\text{Re} \theta(k)} \left( \int_{-\infty}^{t} (1 + s^2)^{-1} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^{t} (1 + s^2)^{1} |(h/p)(s)|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq c k_0^{2n} u^2 e^{-i\text{Re} \theta(k)}. \]  

(A.74)

Recall \( \theta(k) = \frac{\omega^2}{2k_0} k^2 + \frac{\alpha \beta^2}{4k^2} - \alpha \beta \), and set \( k = k_1 + ik_2 \), thus

\[ \text{Re} \theta(k) = -2\alpha \beta^2 k_1 k_2 (k_1^2 + k_2^2) - k_0 \]

\[ = \frac{\alpha \beta^2 (u^2 - \sqrt{2}u)^2 (u^2 - \sqrt{2}u + 2)}{4k_0^2 (u^2 - \sqrt{2}u + 1)^2} \]

\[ \geq \frac{\alpha \beta^2 u^2}{2k_0^2}, \]  

(A.75)

for 0 \leq u \leq \frac{1}{\sqrt{2}}.
Thus, on the contour $l_1'$

$$|e^{-2it\theta(k)}h_{11}(k)| \leq c k_0^{2q}u^q e^{-t\alpha^2\beta^2/2k_0} \leq c k_0^{2q}u^q e^{-t\alpha^2\beta^2/2M^2} \tag{A.76}$$

for $k_0 < M$.

Fix $\varepsilon$, $0 < \varepsilon < \frac{1}{\sqrt{2}}$. If $k(u)$ is on the contour $l_1'$, $\varepsilon < u < \frac{1}{\sqrt{2}}$, then we obtain

$$|e^{-2it\theta(k)}R(k)| \leq ce^{-t\alpha^2\beta^2/2k_0^2} \leq ce^{-\varepsilon^2M^2 t} \tag{A.77}$$

5. $0 < |k| < \frac{k_0}{2}, k \in i\mathbb{R}$.

We consider $0 < \text{Im}k < \frac{k_0}{2}, k \in i\mathbb{R}$, the case for $-\frac{k_0}{2} < \text{Im}k < 0$ is similarly.

Define

$$\rho(k) = \left\{ \begin{array}{ll}
\rho(k(\theta)), & \theta > \theta(i\frac{k_0}{2}), \\
0, & \theta \leq \theta(i\frac{k_0}{2}).
\end{array} \right. \tag{A.78}$$

We claim that $\rho(\theta) \in H^j(-\infty < \theta < \infty)$ for any nonnegative integer $j$.

By Fourier inversion,

$$\rho(\theta(k)) = \int_{-\infty}^{\infty} e^{is\theta(k)} \hat{\rho}(s) ds, \quad 0 < \text{Im}k < \frac{k_0}{2}. \tag{A.79}$$

where

$$\hat{\rho}(s) = \int_{\theta(i\frac{k_0}{2})}^{\infty} e^{-is\theta(k)} \rho(\theta(k)) d\theta(k). \tag{A.80}$$

Then,

$$\int_{\theta(i\frac{k_0}{2})}^{\infty} \left| \left( \frac{d}{d\theta} \right)^j \rho(\theta(k)) \right|^2 |d\theta(k)| = \int_0^{\frac{\pi}{2}} \left| \left( \frac{2k_0^4k^3}{\alpha\beta^2(k^4-k_0^4)} \frac{d}{dk} \right)^j \rho(k) \right|^2 \frac{\alpha\beta^2(k^4-k_0^4)}{2k_0^4k^3} |dk| \leq C < \infty, \tag{A.81}$$

for any nonnegative integer $j$, $0 < k_0 < M$, since $r(k) \to 0$ rapidly, as $k \to 0$.

Hence

$$\int_{-\infty}^{\infty} (1 + s^2)^j |\hat{\rho}(s)|^2 ds \leq C, \tag{A.82}$$

for any nonnegative integer $j$.

Split

$$\rho(k) = \int_t^{\infty} e^{is\theta(k)} \hat{\rho}(s) ds + \int_{-\infty}^t e^{is\theta(k)} \hat{\rho}(s) ds = h_I(k) + h_{11}(k). \tag{A.83}$$
Then, for $0 < \text{Im} k < \frac{k_0}{2}$ and any positive integer $j$, we obtain,

$$
|e^{-2it\theta(k)}h_I(k)| \leq \int_t^\infty |\hat{\rho}| \, \tilde{d}s
\leq (\int_t^\infty (1 + s^2)^{-\frac{j}{2}} \, \tilde{d}s)^{\frac{1}{2}} (\int_t^\infty (1 + s^2)j|\hat{\rho}(s)|^2 \, \tilde{d}s)^{\frac{1}{2}}
$$

(A.84)

Consider the contour $l'_2 : k(u) = uk_0 e^{iu}, 0 < u < \frac{1}{\sqrt{2}}$. Since Re$i\theta(k)$ is positive on this contour, $h_{II}$ has an analytic continuation to contour $l'_2$.

On the contour $l'_2$,

$$
|e^{-2it\theta(k)}h_{II}(k)| \leq e^{-t \text{Re}i\theta(k)} \int_{-\infty}^t e^{(s-t)\text{Re}i\theta(k)} |\hat{\rho}(k)| \, \tilde{d}s
\leq e^{-t \text{Re}i\theta(k)} (\int_{-\infty}^t (1 + s^2)^{-\frac{j}{2}} \, \tilde{d}s)^{\frac{1}{2}} (\int_{-\infty}^t (1 + s^2)|\hat{\rho}(k)|^2 \, \tilde{d}s)^{\frac{1}{2}},
$$

(A.85)

where

$$
\text{Re}i\theta(k) = -2\alpha\beta k_1 k_2 \frac{(k_1^2 + k_2^2)^2 - k_0^4}{4k_0^4(k_1^2 + k_2^2)^2}
= -\frac{\alpha\beta^2}{4k_0^3} u^4 - \frac{\alpha\beta^2}{4k_0^3} u^2
\geq \frac{\alpha\beta^2}{4k_0^3}
$$

for $0 < u < \frac{1}{\sqrt{2}}$.

Thus, we obtain,

$$
|e^{-2it\theta(k)}h_{II}(k)| \leq ce^{-t \frac{\alpha\beta^2}{4k_0^3}}.
$$

(A.87)

6. $|k| > k_0, k \in i\mathbb{R}$.

We consider $\text{Im} k > k_0, k \in \mathbb{R}$, the case for $\text{Im} k < -k_0$ is similarly.

Set

$$
\rho(k) = r(k).
$$

(A.88)

We write

$$(k+1)^{m+5} \rho(k) = \mu_0 + \mu_1(k-ik_0) + \cdots + \mu_m(k-ik_0)^m + \frac{1}{m!} \int_{k_0}^k (\gamma) (k-\gamma)^m \, d\gamma.
$$

(A.89)

Define

$$
R(k) = \frac{\sum_{i=0}^m \mu_i (k-ik_0)^i}{(k+1)^{m+5}}
$$

(A.90)

and write $\rho(k) = h(k) + R(k)$. We have

$$
\frac{d^j h(k)}{dk^j} \bigg|_{ik_0} = 0, \quad 0 \leq j \leq m.
$$

(A.91)

For $0 < k_0 < M$, set

$$
v(k) = \frac{(k-ik_0)^q}{(k+1)^{q+2}}.
$$

(A.92)
Let
\[
\begin{cases}
\frac{h}{v}(\theta) = \frac{h}{v}(k(\theta)), & \theta > \theta(i k_0), \\
0, & \theta \leq \theta(i k_0).
& \tag{A.93}
\end{cases}
\]

Then
\[
\frac{h}{v}(\theta(k)) = \int_{-\infty}^{\infty} e^{is \theta(k)} \left(\frac{h}{v}(s)\right) ds, \quad k \geq k_0, \tag{A.94}
\]
where
\[
\left(\frac{h}{v}\right)(s) = \int_{\theta(i k_0)}^{\infty} e^{-is \theta(k)} \frac{h}{v}(\theta(k)) d\theta(k). \tag{A.95}
\]
Moreover, we have
\[
\frac{h}{v}(\theta(k)) = \frac{(k - i k_0)^{3q+2}}{(k + 1)^{3q+4}} g(k, i k_0), \tag{A.96}
\]
where
\[
g(k, i k_0) = \frac{1}{m} \int_0^1 ((\cdot - i)^{m+5} \rho(\cdot))^{(m+1)} (i k_0 + u(k - i k_0))(1 - u)^k du \tag{A.97}
\]
and
\[
\left| \frac{d^j g(k, i k_0)}{d k^j} \right| \leq C, \quad \text{Im} k \geq k_0. \tag{A.98}
\]
Using the identity \( \left| \frac{k - ik_0}{k + ik_0} \right| \leq 1 \) for \( \text{Im} k \geq k_0 \), we have
\[
\int_{\theta(i k_0)}^{\infty} \left| \left(\frac{d}{d\theta}\right)^j \left(\frac{h}{v}(\theta(k))\right) \right|^2 d\theta(k) = \int_{ik_0}^{\infty} \left| \frac{-2k_0^j \frac{d}{dk}}{k_0 \beta^2 (1 - \frac{k_0^4}{k^4})} \right|^2 \left(\frac{h}{v}(k)\right)^j \frac{k_0 \beta^2 (1 - \frac{k_0^4}{k^4})}{2k_0^3} dk \leq C \int_{ik_0}^{\infty} \left(\frac{(k - ik_0)^{3q+2}}{(k + 1)^{3q+4}}\right)^2 k^{6j-3} (k^4 - k_0^4) dk \leq C_1, \quad 0 \leq j \leq \frac{3q+2}{3}. \tag{A.99}
\]
Thus,
\[
\int_{-\infty}^{\infty} (1 + s^2)^j \left| \left(\frac{h}{v}\right)(s) \right|^2 ds \leq C < \infty. \tag{A.100}
\]
We write
\[
h(k) = v(k) \int_t^{\infty} e^{is \theta(k)} \left(\frac{h}{v}(s)\right) ds + v(k) \int_{-\infty}^t e^{is \theta(k)} \left(\frac{h}{v}(s)\right) ds
= h_I(k) + h_{II}(k). \tag{A.101}
\]
For $\text{Im}k \geq k_0, k \in i\mathbb{R}, 0 < k_0 < M$, and any positive integer $e \leq \frac{3q+2}{3}$, we obtain,

$$|e^{-2it\theta(k)}h_I(k)| \leq \frac{|k-i|^{q+2}}{2} \int_{-\infty}^{\infty} |(h/u)(s)| ds \leq \frac{|k-i|^{q+2}}{2} \int_{-\infty}^{\infty} (1 + s^2)^{-e} ds \leq c \frac{1}{(1+|k|^2) t^{\frac{q}{2}}}.$$  

(A.102)

And $h_{II}(k)$ has an analytic continuation to the left half-plane, where $\text{Re}i\theta(k)$ is positive. We estimate $e^{-2it\theta(k)}h_{II}(k)$ on the contour $k(u) = ik_0 + uk_0e^{\frac{3k}{2}}, u \geq 0$.

If $0 < u \leq M_1$,

$$|e^{-2it\theta(k)}h_{II}(k)| \leq c \frac{k_0^q u^q e^{-t\text{Re}i\theta(k)}}{|k-i|^{q+2}},$$  

(A.103)

where

$$\text{Re}i\theta(k) = -\frac{2\alpha\beta^2}{4k_0} \frac{k_2 (k_1^2 + k_2^2)^2 - k_1^4}{4k_0 (k_1^2 + k_2^2)^2} \geq \frac{\alpha\beta^2}{4k_0} (u^2 + \sqrt{2u} + 1)^2.$$  

(A.104)

Then

$$|e^{-2it\theta(k)}h_{II}(k)| \leq c \frac{k_0^q u^q e^{-t\text{Re}i\theta(k)}}{|k-i|^{q+2}} \leq c_1 \frac{k_0^q u^q e^{-t\text{Re}i\theta(k)}}{(1+|k|^2) t^{\frac{q}{2}}}. $$  

(A.105)

If $u > M_1$, then

$$\text{Re}i\theta(k) \geq \frac{\alpha\beta^2}{4k_0} \frac{u^6}{(u^2 + \sqrt{2u} + 1)^2}.$$  

(A.106)

and

$$|e^{-2it\theta(k)}h_{II}(k)| \leq c \frac{k_0^q u^q e^{-t\text{Re}i\theta(k)}}{|k-i|^{q+2}} \leq c_3 \frac{k_0^q u^q e^{-t\text{Re}i\theta(k)}}{(1+|k|^2) t^{\frac{q}{2}}}. $$  

(A.107)

Hence, for $u > 0$, we obtain

$$|e^{-2it\theta(k)}h_{II}(k)| \leq c_5 \frac{k_0^q u^q e^{-t\text{Re}i\theta(k)}}{(1+|k|^2) t^{\frac{q}{2}}}. $$  

(A.108)

Note that if $l$ is an arbitrary positive integer, we can choose $m$ large enough such that $\frac{3q+2}{3} - \frac{1}{2} > q - \frac{1}{2} > \frac{q}{2} > l$ and Proposition 3.2 holds.

Prove Proposition 3.11.
Proof. Write

\[
R \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) \left( \delta_A(k) \right)^{-2} - R(k_0 \pm) k^{-2i\nu} e^{\frac{k^2}{2}} = \\
e^{i\frac{k^2}{2}} k^{-2i\nu} R \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) k^{-2i\nu} e^{i(1-2\kappa)^2} (1 - \frac{4k^6}{k^2 (2\sqrt{\alpha \beta}) k + k_0 - i k_0})^{2i\nu} \\
- \frac{k_0^{-4i\nu-2i\nu}}{2 \sqrt{\alpha \beta}} \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right)^{-4i\nu} \left( \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 + i k_0 \right) \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 - i k_0 \right) \right)^{2i\nu} \\
e^{-2} (\chi_\pm (\frac{k^2}{2\sqrt{\alpha \beta}} k + k_0) - \chi_\pm (k_0)) - 2 \left( \hat{\chi}'_\pm (\frac{k^2}{2\sqrt{\alpha \beta}} k + k_0) - \hat{\chi}'_\pm (k_0) \right) \\
- e^{i\frac{k^2}{2}} k^{-2i\nu} R(k_0 \pm) k^{-2i\nu} e^{i(1-2\kappa)^2} (I+II+III+IV+V+VI)
\]  
(A.109)

and also divided it into six terms

\[
R \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) \left( \delta_A(k) \right)^{-2} - R(k_0 \pm) k^{-2i\nu} e^{\frac{k^2}{2}} = e^{i\kappa^2} (I+II+III+IV+V+VI)
\]  
(A.110)

where

\[
I = e^{i\kappa^2} k^{-2i\nu} [R \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) - R(k_0 \pm)] \\
II = e^{i\kappa^2} k^{-2i\nu} \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) e^{i(1-2\kappa)^2} (1 - \frac{4k^6}{k^2 (2\sqrt{\alpha \beta}) k + k_0 - i k_0})^{2i\nu} \\
III = e^{i\kappa^2} k^{-2i\nu} \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) e^{i(1-2\kappa)^2} (1 - \frac{4k^6}{k^2 (2\sqrt{\alpha \beta}) k + k_0 - i k_0})^{2i\nu} \\
IV = e^{i\kappa^2} k^{-2i\nu} \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) e^{i(1-2\kappa)^2} (1 - \frac{4k^6}{k^2 (2\sqrt{\alpha \beta}) k + k_0 - i k_0})^{2i\nu} \\
V = e^{i\kappa^2} k^{-2i\nu} \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) e^{i(1-2\kappa)^2} (1 - \frac{4k^6}{k^2 (2\sqrt{\alpha \beta}) k + k_0 - i k_0})^{2i\nu} \\
VI = e^{i\kappa^2} k^{-2i\nu} \left( \frac{k^2}{2\sqrt{\alpha \beta}} k + k_0 \right) e^{i(1-2\kappa)^2} (1 - \frac{4k^6}{k^2 (2\sqrt{\alpha \beta}) k + k_0 - i k_0})^{2i\nu} \\
\]
Note that $|e^{i\kappa^2 f^2}| = e^{-\kappa u^2 2x^2 k^2}$. The terms $I, II, III, IV, V$ and $VI$ can be estimated as follows.

$$|I| \leq |k^{-2i\nu}| |e^{i\kappa^2 f^2}| || \frac{k^2}{2\sqrt{\alpha \beta}} k||\partial_k \tilde{R}(k)||_{L^\infty(A)}$$

$$< \frac{C}{\sqrt{t}},$$

where $C$ is independent of $k$.

$$|II| \leq |k^{-2i\nu}| |e^{i\kappa^2 f^2}| ||\tilde{R}||_{L^\infty(A)} | \frac{d}{ds} e^{i(1-2\kappa)\frac{k^2}{2}(1-s\frac{4\kappa^6 k}{(1-2\kappa)\sqrt{2\alpha \beta}})}|, \quad 0 < s < 1$$

$$\leq \frac{C}{\sqrt{t}}$$

To estimate $III$, we write

$$|III| \leq |k^{-2i\nu}| |e^{i\kappa^2 f^2}| ||\tilde{R}||_{L^\infty(A)} | e^{i(1-2\kappa)\frac{k^2}{2}(1-\frac{4\kappa^6 k}{(1-2\kappa)\sqrt{2\alpha \beta}})}|(III_1 + III_2)$$

where

$$III_1 = \frac{2^{2i\nu} (k^2 \sqrt{2\alpha \beta} k + k_0)^{4i\nu}}{k_0^{2i\nu} (k^2 \sqrt{2\alpha \beta} k + k_0 + k_0)^{4i\nu}}$$

$$III_2 = \left( \frac{k_0^2 \sqrt{2\alpha \beta} k + k_0}{k_0^2 \sqrt{2\alpha \beta} k + 2k_0} \right)^{4i\nu} - 1$$

The estimate of $III_2$ is as follows,

$$|III_2| = \left| \int_1^{1 + \frac{k_0}{2\sqrt{\alpha \beta}}} 4i\nu \xi^{4i\nu - 1} d\xi \right|$$

$$\leq \frac{C}{\sqrt{t}},$$

as $|\xi^{4i\nu - 1}| \leq ce^{-4\nu \arg \xi} \leq c$ for $\xi = 1 + sk\frac{k_0}{2\sqrt{\alpha \beta}} = 1 + s\text{e}^i\frac{\pi}{2}, 0 < s < 1, -\varepsilon < u < \infty$. Since the first term on the right-hand side of the equation for $III_1$ is bounded, namely,

$$\frac{2^{2i\nu} (\frac{k^2}{2\sqrt{\alpha \beta}} k + k_0)^{4i\nu}}{k_0^{2i\nu} (\frac{k^2}{2\sqrt{\alpha \beta}} k + 2k_0)^{2i\nu}} \leq e^{\frac{\pi}{2}}$$

one obtains an analogous estimate for $III_1$. And the estimate for $IV$ is similar as $III$.

$$|V| \leq C \sup_{0 < s \leq 1} |e^{-2\pi x^2} \left( \chi_\pm(\frac{k^2}{2\sqrt{\alpha \beta}} k + k_0) - \chi_\pm(k_0) \right) | \left| 2e^{ik^2 f^2} (\chi_\pm(\frac{k^2}{2\sqrt{\alpha \beta}} k + k_0) - \chi_\pm(k_0)) \right|$$

using the Lipschitz property of the function $\log\left(\frac{(1-\nu(\xi))^2}{1-\nu(k_0)^2}\right), |\xi| \leq k_0$, integrating by parts shows that

$$\left| 2e^{ik^2 f^2} (\chi_\pm(\frac{k^2}{2\sqrt{\alpha \beta}} k + k_0) - \chi_\pm(k_0)) \right| \leq C \frac{\log t}{\sqrt{t}},$$
The analogous estimates for $VI$ can be also obtained.

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School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China
E-mail address: 11110180024@fudan.edu.cn

School of Mathematical Sciences, Institute of Mathematics and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, People’s Republic of China
E-mail address: correspondence author:faneg@fudan.edu.cn