Weighted Endpoint Estimates for Multilinear Commutators of Marcinkiewicz Integrals

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Abstract: Let \(\mu_{\Omega,b}\) be the multilinear commutator generalized by \(\mu_\Omega\), the \(n\)-dimensional Marcinkiewicz integral, with \(\text{Osc}_{\text{exp}} L^\tau(\mathbb{R}^n)\) functions for \(\tau \geq 1\), where \(\text{Osc}_{\text{exp}} L^\tau(\mathbb{R}^n)\) is a space of Orlicz type satisfying that \(\text{Osc}_{\text{exp}} L^\tau(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)\) if \(\tau = 1\) and \(\text{Osc}_{\text{exp}} L^\tau(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n)\) if \(\tau > 1\). The authors establish the weighted weak \(L \log L\)-type estimates for \(\mu_{\Omega,b}\) when \(\Omega\) satisfies a kind of Dini conditions.

Keywords: Marcinkiewicz integral; multilinear commutator; \(A_p\) weight; Orlicz space

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1 Introduction and Main Result

Denote by \(S^{n-1}\) the unit sphere in \(\mathbb{R}^n\) \((n \geq 2)\) equipped with the normalized Lebesgue measure \(d\sigma(x')\). Let \(\Omega(x) \in L^1(S^{n-1})\) be homogeneous function of degree zero in \(\mathbb{R}^n\) satisfying

\[
\int_{S^{n-1}} \Omega(x')dx' = 0, \tag{1.1}
\]

where \(x' = x/|x|\) \((x \neq 0)\).

The \(n\)-dimensional Marcinkiewicz integral introduced by Stein \(^1\) is defined by

\[
\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y)dy \right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad x \in \mathbb{R}^n.
\]

A weight will always mean a positive locally integrable function. As usual, we denote by \(A_p\) \((1 \leq p \leq \infty)\) the Muckenhoupt weights classes (see \(^2\) and \(^3\) for details). For a weight \(\omega\) on \(\mathbb{R}^n\), we write \(\|f\|_{L_p^\omega(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx)^{1/p}\) and \(\omega(E) = \int_E \omega(x)dx\).

In 2004, Ding, Lu and Zhang \(^4\) studied the weighted weak \(L \log L\)-type estimates for the commutators \(\mu_{\Omega,b}\) of the Marcinkiewicz integral, which is defined by

\[
\mu_{\Omega,b}^m(f)(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{(b(x) - b(y))^m \Omega(x-y)}{|x-y|^{n+1}} f(y)dy \right|^2 \frac{dt}{t^3}\right)^{1/2}, \quad m \in \mathbb{Z}^+, \ b \in \text{BMO}(\mathbb{R}^n),
\]

when the kernel \(\Omega\) satisfies the \(\text{Lip}_\alpha(0 < \alpha \leq 1)\) condition, that is, there exists a constant \(C > 0\) such that

\[
|\Omega(x') - \Omega(y')| \leq C|x' - y'|^\alpha, \quad \forall \ x', y' \in S^{n-1}. \tag{1.2}
\]

In 2008, Zhang \(^5\) established the weighted weak \(L(\log L)^{1/r}\)-type estimates for the multilinear commutators of the Marcinkiewicz integral when \(\omega \in A_1\), and \(\Omega\) satisfies \(^4\) and \(^6\).
Let $\Omega \in L^{r}(S^{n-1})$ ($r \geq 1$), the integral modulus of continuity of order $r$ of $\Omega$ is defined by

$$\omega_r(\delta) = \sup_{|\rho| < \delta} \left( \int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^r dx' \right)^{1/r},$$

where $\rho$ is a rotation in $\mathbb{R}^n$ with $|\rho| = \sup_{x' \in S^{n-1}} |\rho x' - x'|$.

We say $\Omega \in L^{r}(S^{n-1})$ ($r \geq 1$) satisfies the $L^{r}$-Dini condition if $\int_0^1 \omega_r(\delta)\delta^{-1}d\delta < \infty$.

Recently, Zhang also considered the following result.

**Theorem A.** Let $b \in \text{BMO}(\mathbb{R}^n), \Omega \in L^{r}(S^{n-1})$ for some $r > 1$, and $\omega^{\ast}_{r'} \in A_1$. If $\Omega$ satisfies and

$$\int_0^1 \omega_{r'}(\delta) \left( \log \frac{1}{\delta} \right)^m d\delta < \infty,$$

then for all $\lambda > 0$, there has

$$\omega(\{x \in \mathbb{R}^n : \mu^{\ast}_{\Omega, b}(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left( 1 + \log^+ \frac{|f(y)|}{\lambda} \right)^m \omega(y)dy,$$

where $C$ is a positive constant independent of $f$ and $\lambda$.

In this paper, by applying the calderón-Zygmund decomposition theory, we will study the weighted weak $L \log L$-type estimates for the multilinear commutators generated by $\mu^{\ast}_{\Omega}$ and $\text{Osc}_{\exp L}^{\ast}(\mathbb{R}^n)$ functions, in analogy with the results established by Pérez and Trujillo-González in for the multilinear commutators of Calderón-Zygmund operators. Before stating our results, we first recall some notation.

Let $m$ be a positive integer and $\vec{b} = (b_1, b_2, \cdots, b_m)$, we define the multilinear commutators $\mu^{\ast}_{\Omega, \vec{b}}$ by

$$\mu^{\ast}_{\Omega, \vec{b}}(f)(x) = \left( \int_0^\infty \left[ \int_{|x-y| \leq t} \frac{\Omega(x-y)f(y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x) - b_j(y)) dy \right]^2 \frac{dt}{t^s} \right)^{\frac{1}{2}}.$$

It is easy to see, when $m = 1$, $\mu^{\ast}_{\Omega, \vec{b}}$ is the commutator of Marcinkiewicz integral and when $b_1 = \cdots = b_m$, $\mu^{\ast}_{\Omega, \vec{b}}$ is the higher commutator of Marcinkiewicz integral.

To state the weak type estimate for the multilinear commutator $\mu^{\ast}_{\Omega, \vec{b}}$, we need to introduce the following notation. As in for any positive integer $m$, for all $1 \leq j \leq m$, we denote by $\mathcal{C}^m_j$ the family of all finite subsets $\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(j)\}$ of $\{1, 2, \cdots, m\}$ with $j$ different elements. For any $\sigma \in \mathcal{C}_j^m$, we define the complementary sequence $\sigma' = \{1, 2, \cdots, m\} \setminus \sigma$.

In the following, we will always assume that $\Omega$ be homogeneous function of degree 0, and let $\vec{b} = (b_1, b_2, \cdots, b_m)$ be a finite family of locally integrable functions. For all $1 \leq j \leq m$ and $\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(j)\} \in \mathcal{C}_j^m$, we write for any $i$-tuple $(\tau_1, \tau_2, \cdots, \tau_m)$ with $\tau_j \geq 1$ for all $1 \leq j \leq m$, $1/\tau_\sigma = 1/\tau_{\sigma(1)} + \cdots + 1/\tau_{\sigma(j)}$ and $1/\tau_{\sigma'j} = 1/\tau_1 + \cdots + 1/\tau_m$, where $1/\tau_1 = 1/\tau_1 + \cdots + 1/\tau_m$, we will denote $\vec{b}_\sigma = (b_{\sigma(1)}, b_{\sigma(2)}, \cdots, b_{\sigma(j)})$ and the product $b_{\sigma} = b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(j)}$. With this notation, we write

$$\|b_{\sigma}\|_{\text{Osc}_{\exp L}^{\ast}(\mathbb{R}^n)} = \|b_{\sigma(1)}\|_{\text{Osc}_{\exp L}^{\ast}(\mathbb{R}^n)} \cdots \|b_{\sigma(j)}\|_{\text{Osc}_{\exp L}^{\ast}(\mathbb{R}^n)}.$$ 

In particular, we write

$$\langle b(x) - b(y) \rangle_\sigma = \langle b_{\sigma(1)}(x) - b_{\sigma(1)}(y) \rangle \cdots \langle b_{\sigma(j)}(x) - b_{\sigma(j)}(y) \rangle,$$

and

$$\langle b - b(y) \rangle_\sigma = \langle (b_{\sigma(1)})B - b_{\sigma(1)}(y) \rangle \cdots \langle b_{\sigma(j)}(y) - b_{\sigma(j)}(y) \rangle,$$

where $B$ is any ball in $\mathbb{R}^n$, $x, y \in \mathbb{R}^n$, and $f_B = \frac{1}{|B|} \int_B f(y)dy$. For any $\sigma \in \mathcal{C}_j^m$, we set

$$\mu^{\ast}_{\Omega, \vec{b}}(f)(x) = \left( \int_0^\infty \left[ \int_{|x-y| \leq t} \frac{\Omega(x-y)f(y)}{|x-y|^{n-1}} \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(y)) dy \right]^2 \frac{dt}{t^s} \right)^{\frac{1}{2}}.$$
If \(\sigma = \{1, 2, \cdots, m\}\), then \(\sigma'\) is an empty set, we understand \(\mu_{\Omega, b_\sigma} = \mu_{\Omega, b}\) and \(\mu_{\Omega, b_{\sigma'}} = \mu_{\Omega}\).

Our result can be stated as follows.

**Theorem 1.1.** Let \(b_j \in \text{Osc}_{\exp L^r}, \tau_j \geq 1 (1 \leq j \leq m), \Omega \in L^r(S^{n-1})\) for some \(r > 1\), and \(\omega' \in A_1\). If \(\Omega\) satisfies (1.1) and (1.3), then for all \(\lambda > 0\), there has

\[
\omega(\{x \in \mathbb{R}^n : \mu_{\Omega, b}(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log \frac{|f(y)|}{\lambda}\right)^m \omega(y)dy,
\]

where \(C\) is a positive constant independent of \(f\) and \(\lambda\).

**Remark 1.** Noting that \(\text{Osc}_{\exp L^1} = \text{BMO}\) and \(\text{Osc}_{\exp L^r} \subset \text{BMO}\) for \(r > 1\). For more information on Orlicz space see [10].

Obviously, condition (1.3) is slightly stronger than the \(L^r\)-Dini condition, but much more weaker than the Lip_\(\alpha\) condition. Noting that \(\mu_{\Omega, b}\) coincides with \(\mu_{\Omega, b}^m\) when \(b_j = b\) for \(j = 1, 2, \cdots, m\). So, Theorem 1.1 improves the main results in [5] and [6].

Throughout this paper, \(C\) denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index \(p \in [1, \infty]\), we denote by \(p'\) its conjugate index, namely, \(1/p + 1/p' = 1\). For \(A \sim B\), we mean that there is a constant \(C > 0\) such that \(C^{-1}B \leq A \leq CB\).

## 2 Preliminaries and Lemmas

In this section, we will formulate some lemmas and preliminaries.

**Lemma 2.1.** Suppose that \(0 < \alpha < n, r > 1\) and \(\Omega\) satisfies the \(L^r\)-Dini condition. If there is a constant \(C_0\) with \(0 < C_0 < 1/2\) such that \(|y| < C_0K\), then

\[
\left(\int_{K < |x| < 2K} \frac{\Omega(x - y)}{|x - y|^{n-\alpha}} - \frac{\Omega(x)}{|x|^{n-\alpha}}|dx\right)^{1/r} \leq CK^{n/r - \alpha}\left(\frac{|y|}{K} + \int_{|y|/(2K) < \delta < |y|/K} \frac{\omega_{\tau}(\delta)}{\delta}d\delta\right).
\]

**Lemma 2.2.** Suppose \(\Omega \in L^r(S^{n-1})\) for some \(r > 1\) and \(\omega' \in A_1\). Then for any \(\lambda > 0\), there is a constant \(C > 0\) independent of \(f\) and \(\lambda\), such that

\[
\omega(\{x \in \mathbb{R}^n : \mu_{\Omega}(f)(x) > \lambda\}) \leq C\lambda^{-1}\|f\|_{L^r_0(\mathbb{R}^n)}.
\]

**Lemma 2.3.** Let \(\omega \in A_1, 1 < p < \infty, b_j \in \text{Osc}_{\exp L^r}, \tau_j \geq 1 (1 \leq j \leq m), \Omega \in L^r(S^{n-1})\) for some \(r > 1\) and satisfies (1.1) and (1.3). Then, there is a constant \(C > 0\) independent of \(f\), such that

\[
\|\mu_{\Omega, b}(f)\|_{L^p_0(\mathbb{R}^n)} \leq C\|b\|_{\text{Osc}_{\exp L^r}}\|f\|_{L^p_0(\mathbb{R}^n)}.
\]

The idea of the proof of Lemma 2.3 comes from the corollary 1.3 in [5]. We omit the details.

We also need a few facts of Orlicz spaces, see [10] for more information.

A function \(\varphi : [0, +\infty) \to [0, +\infty)\) is called a Young function if \(\varphi\) is continuous, convex and increasing with \(\varphi(0) = 0\) and \(\varphi(t) \to +\infty\) as \(t \to +\infty\). We defined the \(\varphi\)-average of a function \(f\) over a ball \(B\) by means of the Luxemburg norm

\[
\|f\|_{\varphi, B} = \inf \left\{\lambda > 0 : \frac{1}{|B|} \int_B \varphi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\},
\]

which satisfies the following inequalities (see [10], P.69 or formula (7) in [11])

\[
\|f\|_{\varphi, B} \leq \inf \left\{\eta + \frac{\eta}{|B|} \int_B \varphi\left(\frac{|f(y)|}{\eta}\right)dy \leq 1\right\} \leq 2\|f\|_{\varphi, B}. \quad (2.1)
\]
The Young function that we are going to be using is \( \Phi_\alpha(t) = t(1 + \log^+ t)^\alpha \) \((\alpha > 0)\) with its complementary Young function \( \Phi_\alpha(t) \approx \exp(t/\alpha) \). Denote by \( \|f\|_{L(\log L)^\alpha, B} = \|f\|_{\Phi_\alpha, B} \) and \( \|f\|_{\exp L^{1/\alpha}, B} = \|f\|_{\tilde{\Phi}_\alpha, B} \). When \( \alpha = 1 \), we simply denote by \( \Phi(t) = t(1 + \log^+ t) \) and \( \tilde{\Phi}(t) \approx e^t \), and by \( \|f\|_{L(\log L), B} = \|f\|_{\Phi, B} \) and \( \|f\|_{\exp L, B} = \|f\|_{\tilde{\Phi}, B} \). By the generalized Hölder’s inequality (see \([12]\)), we have

\[
\frac{1}{|B|} \int_B |f(y)|g(y)|dy \leq 2\|f\|_{L(\log L)^\alpha, B}\|g\|_{\exp L^{1/\alpha}, B}. \tag{2.2}
\]

As usual, for a locally integrable function \( f \) and a ball \( B \), we denote \( f_B = \frac{1}{|B|} \int_B f(y)dy \). Let \( b \in \text{BMO}(\mathbb{R}^n) \), for any ball \( B \) and integer \( k \geq 0 \), there has (see \([2]\), p.141)

\[
|b_{2^{k+1}B} - b_B| \leq C(k + 1)\|b\|_*, \tag{2.3}
\]

where \( \ell B \) denotes the \( \ell \)-times concentric expansion of \( B \) and \( \|b\|_* \) denotes the BMO norm of \( b \).

By the John-Nirenberg’s inequality, it is not difficult to see that (c.f. \([13]\), p.169)

\[
\|b - b_B\|_{\exp L, B} \leq C\|b\|_. \tag{2.4}
\]

Let \( M_{L(\log L)^\alpha}(f)(x) = \sup_{B \ni x} \|f\|_{L(\log L)^\alpha, B} \). Denote by \( M \) the Hardy-Littlewood maximal function and \( M^k \) the \( k \)-times iterations of \( M \), then \( M_{L(\log L)^k}(f)(x) \approx M_{L(\log L)^k + 1}(f)(x) \) for \( k = 0, 1, 2, \cdots \). We also need the following estimates in the proof of Theorem \([14]\).

**Lemma 2.4.** Let \( 1 \leq p < \infty, \omega^p \in A_1 \) and \( B \) be a ball. Then for any \( y \in B \) and any positive integer \( m \), there has

\[
\left( \frac{1}{|2^kB|} \int_{2^kB} \omega^p(x) \prod_{j=1}^m |b_j(x) - (b_j)_B|^p dx \right)^{1/p} \leq C\|b\|_*(k + 1)^m \inf_{y \in B} \omega(y), \quad k = 0, 1, 2, \cdots.
\]

**Lemma 2.5.** Let \( 1 \leq p < \infty, \omega^p \in A_1 \) and \( B \) be a ball. Then for any \( y \in B \) and any positive integer \( m \), there has

\[
\left( \frac{1}{|2^kB|} \int_{2^kB} \omega^p(x) \prod_{j=1}^m |b_j(x) - (b_j)_B|^p dx \right)^{1/p} \leq C\|\tilde{b}\|_*(k + 1)^m \inf_{y \in B} \omega(y), \quad k = 0, 1, 2, \cdots.
\]

**Proof.** By the Hölder’s inequality and Lemma 2.4, we obtain

\[
\left( \frac{1}{|2^kB|} \int_{2^kB} \omega^p(x) \prod_{j=1}^m |b_j(x) - (b_j)_B|^p dx \right)^{1/p} \leq \prod_{j=1}^m \left( \frac{1}{|2^kB|} \int_{2^kB} \omega^p(x)|b_j(x) - (b_j)_B|^{\gamma_j} dx \right)^{\frac{1}{\gamma_j}}
\]

\[
\leq C \prod_{j=1}^m \left( \|b_j\|_{L, B}^{\gamma_j}(k + 1)^{\gamma_j} \inf_{y \in B} \omega(y) \right)^{\frac{1}{\gamma_j}}
\]

\[
\leq C\|\tilde{b}\|_*(k + 1)^m \inf_{y \in B} \omega(y),
\]

where \( 1 = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \cdots + \frac{1}{\gamma_m} \).

This completes the proof of Lemma 2.5. \( \square \)

We also need the following notations. For \( \omega \in A_\infty \) and a ball \( B \), denote by

\[
\|f\|_{L(\log L)^m, B, \omega} = \inf \left\{ \lambda > 0 : \frac{1}{\omega(B)} \int_B \Phi_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y)dy \leq 1 \right\}
\]

and

\[
\|f\|_{\exp L^{1/m}, B, \omega} = \inf \left\{ \lambda > 0 : \frac{1}{\omega(B)} \int_B \tilde{\Phi}_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y)dy \leq 1 \right\}.
\]
Noting that, we have (c.f. [10], p.69)

$$\|f\|_{L(\log L)^m, B, \omega} \approx \inf \left\{ \eta + \frac{\eta}{\omega(B)} \int_B \Phi_m \left( \frac{|f(y)|}{\eta} \right) \omega(y) dy \right\}. \quad (2.5)$$

By (2.2), there also holds the following generalized H"older’s inequality

$$\frac{1}{\omega(B)} \int_B |f_1(y) \cdots f_m(y)| \omega(y) dy \leq C \|g\|_{L(\log L)^m, B, \omega} \prod_{j=1}^m \|f_j\|_{\exp L, B, \omega}. \quad (2.6)$$

Furthermore, for any $b \in \text{BMO}(\mathbb{R}^n)$, any ball $B$ and any $\omega \in A_\infty$, there has

$$\|b - b_B\|_{\exp L, B, \omega} \leq C\|b\|_* \quad (2.7)$$

Indeed, by John-Nirenberg’s inequality, there exist positive constants $C_1$ and $C_2$, such that

$$|\{x \in B : |b(x) - b_B| > t\}| \leq C_1 |B| e^{-C_2 t/\|b\|_*}.$$

Noting that $\omega \in A_\infty$, from the proof of Theorem 5 in [14], there is a $\delta > 0$, such that

$$\omega(\{x \in B : |b(x) - b_B| > t\}) \leq C_1 \omega(B) e^{-C_2 \delta t/\|b\|_*}.$$

Similar to the proof of Corollary 7.1.7 in [3] (p.528), we have

$$\frac{1}{\omega(B)} \int_B \exp \left( \frac{|b(x) - b_B|}{C_3 \|b\|_*} \right) \omega(x) dx \leq C, \quad (2.8)$$

which implies (2.7).

### 3 Proof of Theorem 1.1

**Proof.** Without loss of generality, we may assume that for $j = 1, \ldots, m, \|b_j\|_{\text{Osc}_{\exp L} \tau_j (\mathbb{R}^n)} = 1$. In fact, let

$$b_j = \frac{b_j}{\|b_j\|_{\text{Osc}_{\exp L} \tau_j (\mathbb{R}^n)}}$$

for $j = 1, \ldots, m$. The homogeneity tells us that for any $\lambda > 0$,

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega, \tilde{b}}(f)(x) > \lambda\}) = \omega(\{x \in \mathbb{R}^n : \mu_{\Omega, \tilde{b}}(f)(x) > \lambda/\|b\|_{\text{Osc}_{\exp L} \tau_j (\mathbb{R}^n)}\}) \quad (3.1)$$

Noting that $\|b_j\|_{\text{Osc}_{\exp L} \tau_j (\mathbb{R}^n)} = 1$ for $j = 1, \ldots, m$, if when $\|b_j\|_{\text{Osc}_{\exp L} \tau_j (\mathbb{R}^n)} = 1$ ($j = 1, \ldots, m$), the theorem is true. By (3.1) and the inequality

$$\Phi_s(t_1 t_2) \leq C \Phi_s(t_1) \Phi_s(t_2)$$

for any $s > 0, t_1, t_2 \geq 0$, we easily obtain that the theorem still holds for any $b_j \in \text{Osc}_{\exp L} \tau_j (\mathbb{R}^n)$ ($j = 1, \ldots, m$).

For a fixed $\lambda$, we consider the Calderón-Zygmund decomposition of $f$ at height $\lambda$ and get a sequence of balls $\{B_i\}$, where $B_i$ is a ball centered at $x_i$ with radius $r_i$, such that $|f(x)| \leq C\lambda$ for a.e. $x \in \mathbb{R}^n \setminus \cup_i B_i$ and

$$\lambda < \frac{1}{|B_i|} \int_{B_i} |f(y)| dy \leq 2^n \lambda. \quad (3.2)$$

Moreover, there is an integer $N \geq 1$, independent of $f$ and $\lambda$, such that for every point in $\mathbb{R}^n$ belongs to at most $N$ balls in $\{B_i\}$. 

5
We decompose \( f = g + h \), where
\[
g(x) = \begin{cases} 
  f(x), & x \in \mathbb{R}^n \setminus \bigcup_i B_i, \\
  f_{B_i}, & x \in B_i.
\end{cases}
\]

Then \( h(x) = f(x) - g(x) = \sum h_i(x) \) with \( h_i(x) = (f(x) - f_{B_i})\chi_{B_i}(x) \). Obviously, \( \text{supp } h_i \subset B_i \), \( \int_{B_i} h_i(y)dy = 0 \) and
\[
|g(x)| \leq 2^n \lambda, \quad \text{a.e. } x \in \mathbb{R}^n. \tag{3.3}
\]

Noting that if \( \omega' \in A_1 \) then \( \omega \in A_1 \), and then \( M(\omega)(x) \leq C\omega(x) \) for \( \text{a.e. } x \in \mathbb{R}^n \). By (3.2) and the fact that \( |B_i|^{-1}\omega(B_i) = |B_i|^{-1} \int_{B_i} \omega(x)dx \leq C \inf_{y \in B_i} \omega(y) \), we have
\[
\omega(B_i) \leq C|B_i| \inf_{y \in B_i} \omega(y) \leq C\lambda^{-1} \int_{B_i} |f(y)|dy \inf_{y \in B_i} \omega(y) \leq C\lambda^{-1} \int_{B_i} |f(y)|\omega(y)dy. \tag{3.4}
\]

Denote by \( E = \bigcup_i (4B_i) \), it follows from (3.4) that
\[
\omega(E) \leq C \sum_i \int_{B_i} \omega(x)dx = C \sum_i \omega(B_i) \leq C\lambda^{-1}\|f\|_{L^1_\lambda(\mathbb{R}^n)}.
\]

Write
\[
\omega(\{x \in \mathbb{R}^n : \mu_{\Omega, \delta}(f)(x) > \lambda\}) \leq \omega(\{x \in \mathbb{R}^n \setminus E : \mu_{\Omega, \delta}(f)(x) > \lambda\}) + \omega(E)
\leq \omega(\{x \in \mathbb{R}^n \setminus E : \mu_{\Omega, \delta}(g)(x) > \frac{\lambda}{2}\}) + \omega(\{x \in \mathbb{R}^n \setminus E : \mu_{\Omega, \delta}(h)(x) > \frac{\lambda}{2}\}) + \omega(E)
\leq I_1 + I_2 + C\lambda^{-1}\|f\|_{L^1_\lambda(\mathbb{R}^n)}.
\]

We consider \( I_1 \) first. For \( \omega' \in A_1 \) there has \( \omega \in A_1 \). Noting that \( A_1 \subset A_s \) \((s \geq 1)\), then for any \( p > r' \), we have \( \omega \in A_{p/r'} \). It follows from Lemma (2.3), (3.3) and (3.4) that
\[
I_1 \leq C \lambda^{-p} \int_{\mathbb{R}^n} \left(\mu_{\Omega, \delta}(g)(x)\right)^p \omega(x)dx \leq C \lambda^{-p} \int_{\mathbb{R}^n} |g(x)|^p \omega(x)dx \leq C \lambda^{-1} \int_{\mathbb{R}^n} |g(x)|\omega(x)dx
\leq C \lambda^{-1} \int_{\mathbb{R}^n} |g(x)|\omega(x)dx + \sum_i \int_{B_i} |f_{B_i}|\omega(x)dx
\leq C \lambda^{-1} \|f\|_{L^1_\lambda(\mathbb{R}^n)} + C \lambda^{-1} \sum_i \int_{B_i} \left(|B_i|^{-1} \int_{B_i} |f(y)|dy\right)\omega(x)dx
\leq C \lambda^{-1} \|f\|_{L^1_\lambda(\mathbb{R}^n)} + C \lambda^{-1} \sum_i \int_{B_i} |f(y)|dy \inf_{y \in B_i} \omega(y)
\leq C \lambda^{-1} \|f\|_{L^1_\lambda(\mathbb{R}^n)} + C \lambda^{-1} \sum_i \int_{B_i} |f(y)|\omega(y)dy
\leq C \lambda^{-1} \|f\|_{L^1_\lambda(\mathbb{R}^n)}. \tag{3.5}
\]

We remark that the proof of (3.5) implies the following fact, which will be used later.
\[
\sum_i \int_{B_i} |f_{B_i}|\omega(x)dx \leq C \|f\|_{L^1_\lambda(\mathbb{R}^n)}. \tag{3.6}
\]
Now, let us estimate $I_2$. By the definition of $\mu_\Omega$ and $\mu_{\Omega_i}$, with the aid of the formula

$$\prod_{j=1}^{m} (b_j(x) - b_j(y)) = \sum_{\sigma \in \mathcal{E}^m} \left( \frac{\Omega(x-y)h(y)}{|x-y|^{n-1}} \sum_{j=0}^{m} \sum_{\sigma \in \mathcal{E}^m} (b(x) - b_{B_i})_\sigma (b_{B_i} - b(y))_\sigma \right) dy \left( \frac{2 \, dt}{t^3} \right)^{\frac{1}{2}},$$

we have

$$\mu_{\Omega_i}(h)(x) = \left( \int_0^\infty \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)h(y)}{|x-y|^{n-1}} \prod_{j=1}^{m} (b_j(x) - (b_j)_B) \, dy \right) \left( \frac{2 \, dt}{t^3} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq \left( \int_0^\infty \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)h(y)}{|x-y|^{n-1}} \prod_{j=1}^{m} (b_j(x) - (b_j)_B) \, dy \right) \left( \frac{2 \, dt}{t^3} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

For $I_{21}$, using Chebychev’s inequality and Minkowski’s inequality, we have

$$I_{21} = \omega \left( \left\{ x \in \mathbb{R}^n \setminus E : \sum_{i} \prod_{j=1}^{m} |b_j(x) - (b_j)_B| \mu_{\Omega_i}(h_i)(x) > \frac{\lambda}{6} \right\} \right),$$

and

$$\omega \left( \left\{ x \in \mathbb{R}^n \setminus E : \sum_{i} \prod_{j=1}^{m} |b_j(x) - (b_j)_B| \mu_{\Omega_i}(h_i)(x) > \frac{\lambda}{6} \right\} \right).$$

So, we can write $I_2$ as

$$I_2 \leq \omega \left( \left\{ x \in \mathbb{R}^n \setminus E : \sum_{i} \prod_{j=1}^{m} |b_j(x) - (b_j)_B| \mu_{\Omega_i}(h_i)(x) > \frac{\lambda}{6} \right\} \right).$$

For $I_{21}$, using Chebychev’s inequality and Minkowski’s inequality, we have

$$I_{21} = \omega \left( \left\{ x \in \mathbb{R}^n \setminus E : \sum_{i} \prod_{j=1}^{m} |b_j(x) - (b_j)_B| \mu_{\Omega_i}(h_i)(x) > \frac{\lambda}{6} \right\} \right).$$

and

$$\omega \left( \left\{ x \in \mathbb{R}^n \setminus E : \sum_{i} \prod_{j=1}^{m} |b_j(x) - (b_j)_B| \mu_{\Omega_i}(h_i)(x) > \frac{\lambda}{6} \right\} \right).$$

For $x \in \mathbb{R}^n \setminus 4B_i$ and $y \in B_i$, there has $|x-y| \leq |x-x_i| + r_i$ and $|x-y| \sim |x-x_i| \sim |x-x_i| + 2r_i$, and then

$$\int_{|x-y|}^{\infty} \frac{dt}{t^3} = \frac{1}{2} \left( \frac{1}{|x-y|^2} - \frac{1}{(|x-x_i| + 2r_i)^2} \right) \leq \frac{Cr_i}{|x-y|^2}.$$
Noting that \( \text{supp}\, h_i \subset B_i \), it follows from the Minkowski’s inequality that

\[
I_{211} \leq C \lambda^{-1} \sum_i \frac{1}{r_i} \int_{\mathbb{R}^n \setminus 4B_i} \left( \sum_{j=1}^{m} |b_j(x) - (b_j)_{B_i}| \left( \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \left( \int_{[x - y]} \frac{dt}{t^3} \right)^{1/3} dy \right) \omega(x) dx \right)
\]

\[
\leq C \lambda^{-1} \sum_i \frac{1}{r_i} \int_{\mathbb{R}^n \setminus 4B_i} \left( \sum_{j=1}^{m} \left[ \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} dy \right] \omega(x) dx \right)
\]

\[
\leq C \lambda^{-1} \sum_i \frac{1}{r_i} \int_{\mathbb{R}^n \setminus 4B_i} \left( \sum_{j=1}^{m} \left[ \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \prod_{j=1}^{m} |b_j(x) - (b_j)_{B_i}| \omega(x) dx \right] \right)
\]

\[
\leq C \lambda^{-1} \sum_i \frac{1}{r_i} \int_{\mathbb{R}^n \setminus 4B_i} \left( \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \prod_{j=1}^{m} |b_j(x) - (b_j)_{B_i}| \omega(x) dx \right)
\]

\[
\frac{1}{r_i} \int_{2^{k+1} B_i \setminus 2^k B_i} \frac{\omega''(x)}{|x - y|^{n+1/2}} \prod_{j=1}^{m} |b_j(x) - (b_j)_{B_i}| \right)^{1/r_i}
\]

\[
\leq C \lambda^{-1} \int_{B_i} \left( \sum_{j=1}^{m} (k + 1)^m \inf_{y \in B_i} \omega(y) \right)
\]

This, together with (3.9) and (3.10), gives

\[
I_{211} \leq C \lambda^{-1} \sum_i \frac{1}{r_i} \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \omega(y) dy
\]

\[
\leq C \lambda^{-1} \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \omega(y) dy \leq C \lambda^{-1} \int_{B_i} \frac{\Omega(x - y)||h_i(y)||}{|x - y|^{n+1/2}} \omega(y) dy.
\]

Next, let us consider \( I_{212} \). Write \( K(x, y, y_i) = \frac{\Omega(x - y)}{|x - y|^{n+1/2}} - \frac{\Omega(x - y_i)}{|x - y_i|^{n+1/2}} \) for simplicity. Noting that for any \( y \in B_i \), any \( x \in \mathbb{R}^n \setminus 4B_i \) and \( t \) with \( |x - y_i| + 2r_i \leq t \), there has \( |x - y| \leq |x - y_i| + r_i < t \), then by the
cancellation condition of \( h_i \), we have

\[
I_{212} \leq C \lambda^{-1} \sum_i \int_{R^n \setminus 4B_1} \prod_{j=1}^m |b_j(x) - (b_j)_{B_i}| \left( \int_{B_i} |K(x, y, x_i)||h_i(y)| \left( \int_{|x-x_i|+2r_i}^{\infty} \frac{dt}{t^{3\beta+\frac{2}{r_i}}} \right)^{\frac{1}{2}} \right) \omega(x) dx
\]

\[
\leq C \lambda^{-1} \sum_i \int_{R^n \setminus 4B_1} \prod_{j=1}^m |b_j(x) - (b_j)_{B_i}| \left( \int_{B_i} |K(x, y, x_i)||h_i(y)| \frac{dy}{|x-x_i|} \right) \omega(x) dx
\]

\[
\leq C \lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \sum_{k=1}^{\infty} (2^{k} r_i)^{-1} \left( \int_{2^{k+1}B \setminus 2^{k}B_i} |K(x, y, x_i)| \prod_{j=1}^m |b_j(x) - (b_j)_{B_i}| \omega(x) dx \right) dy
\]

By the Hölder’s inequality, Lemma 2.1 and Lemma 2.5, there has

\[
\int_{2^{k+1}B \setminus 2^{k}B_i} |K(x, y, x_i)| \prod_{j=1}^m |b_j(x) - (b_j)_{B_i}| \omega(x) dx
\]

\[
\leq \left( \int_{2^{k+1}B \setminus 2^{k}B_i} |K(x, y, x_i)| \omega(r) dx \right)^{1/r} \left( \sum_{j=1}^m |b_j(x) - (b_j)_{B_i}| \omega'(r) dx \right)^{1/r'}
\]

\[
\leq C(k+1)^m 2^k r_i \left( 2^{-k} + \int_{|y-x_i| \leq 2^k r_i} \frac{\omega_r(\delta)}{\delta} d\delta \right) \inf_{y \in B} \omega(y).
\]

Therefore,

\[
I_{212} \leq C \lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \omega(y) \prod_{k=1}^{\infty} (k+1)^m \left( 2^{-k} + \int_{|y-x_i| \leq 2^k r_i} \frac{\omega_r(\delta)}{\delta} d\delta \right) dy
\]

\[
\leq C \lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \omega(y) \left( \sum_{k=1}^{\infty} (k+1)^m 2^{-k} + \int_0^1 \frac{\omega_r(\delta)}{\delta} \left( \log \frac{1}{\delta} \right)^m d\delta \right) dy
\]

(3.13)

\[
\leq C \lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \omega(y) dy.
\]

Note that \( h_i(y) = f(y) + f_{B_i} \) when \( y \in B_i \), it follows from (3.6), (3.8), (3.12) and (3.13) that

\[
I_{21} \leq C \lambda^{-1} \sum_i \int_{B_i} |h_i(y)| \omega(y) dy \leq C \lambda^{-1} \sum_i \int_{B_i} (|f(y)| + |f_{B_i}|) \omega(y) dy \leq C \lambda^{-1} \|f\|_{L^r(S^{n-1})}.
\]

To estimate \( I_{23} \), noting that \( \Omega \in L^r(S^{n-1}) \) for some \( r > 1 \) and \( \omega'' \in A_1 \), using Lemma 2.2, 2.4, 2.7, Lemma 2.5, 2.5 and 3.4, we have

\[
I_{23} \leq \omega \left( \left\{ x \in \mathbb{R}^n : \mu_{\Omega} \left( \sum_i h_i \prod_{j=1}^m ((b_j)_{B_i} - b_j) \right)(x) > \frac{\lambda}{6} \right\} \right)
\]

\[
\leq C \lambda^{-1} \int_{\mathbb{R}^n} \sum_i |h_i(x)| \omega(x) \prod_{j=1}^m |(b_j)_{B_i} - b_j(x)| dx
\]

\[
\leq C \lambda^{-1} \sum_i \left( \int_{B_i} |f(x)| \omega(x) \prod_{j=1}^m |(b_j)_{B_i} - b_j(x)| dx + \int_{B_i} \omega(x) \prod_{j=1}^m |(b_j)_{B_i} - b_j(x)| dx \right)
\]

\[
\leq C \lambda^{-1} \sum_i \omega(B_i) \|f\|_{L^r(S^{n-1})} \prod_{j=1}^m |(b_j)_{B_i} - b_j(x)|_{L^r(S^{n-1})} + C \lambda^{-1} \sum_i \int_{B_i} |f(y)| dy \int_{B_i} \omega(x) \prod_{j=1}^m |(b_j)_{B_i} - b_j(x)| dx
\]
\[
\begin{align*}
&\leq C \lambda^{-1} \sum_i \left( \omega(B_i) \|f\|_{L(\log L)^m, B_i, \omega} + \int_{B_i} |f(y)| \, dy \inf_{y \in B_i} \omega(y) \right) \\
&\leq C \lambda^{-1} \sum_i \left( \omega(B_i) \inf \left\{ \lambda + \frac{\lambda}{\omega(B_i)} \int_{B_i} \Phi_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy \right\} + \int_{B_i} |f(y)| \omega(y) \, dy \right) \\
&\leq C \sum_i \left( \omega(B_i) + \int_{B_i} \Phi_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy \right) + C \lambda^{-1} \int_{\mathbb{R}^n} |f(y)| \omega(y) \, dy \\
&\leq C \sum_i \left( \lambda^{-1} \int_{B_i} |f(y)| \omega(y) \, dy + \int_{B_i} \frac{|f(y)|}{\lambda} \left( 1 + \log^+ \frac{|f(y)|}{\lambda} \right)^m \omega(y) \, dy \right) + C \lambda^{-1} \int_{\mathbb{R}^n} |f(y)| \omega(y) \, dy \\
&\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left( 1 + \log^+ \frac{|f(y)|}{\lambda} \right)^m \omega(y) \, dy.
\end{align*}
\]

Now, let us turn to estimate for \( I_{22} \). Using the Minkowski’s inequality, we have

\[
I_{22} = \omega \left\{ \left\{ x \in \mathbb{R}^n \setminus E : \sum_{j=1}^{m-1} \sum_{\sigma \in \mathcal{E}_j} \sum_{i} \left| (b(x) - b_{B_i, \sigma}) \mu_{\Omega}(h_i(b_{B_i, \sigma}))(x) > \frac{\lambda}{6} \right) \right\} \right\}
\]

\[
\begin{align*}
&\leq C \lambda^{-1} \sum_i \sum_{j=1}^{m-1} \sum_{\sigma \in \mathcal{E}_j} \int_{\mathbb{R}^n \setminus B_i} \left| (b(x) - b_{B_i, \sigma}) \mu_{\Omega}(h_i(b_{B_i, \sigma}))(x) \omega(x) \, dx \right| \\
&\leq C \lambda^{-1} \sum_i \sum_{j=1}^{m-1} \sum_{\sigma \in \mathcal{E}_j} \int_{\mathbb{R}^n \setminus B_i} \left| (b(x) - b_{B_i, \sigma}) \right| \omega(x) \, dx \\
&\quad \times \left( \int_{\mathbb{R}^n \setminus B_i} \left( 1 + \log^+ \frac{|f(y)|}{\lambda} \right)^m \omega(y) \, dy \right)
\end{align*}
\]

For \( I_{221} \) and \( I_{222} \), similar to the estimates for \( I_{21} \) and \( I_{23} \), we can get

\[
I_{221} \leq C \left( \omega(B_i) \inf \left\{ \lambda + \frac{\lambda}{\omega(B_i)} \int_{B_i} \Phi_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy \right\} + \int_{B_i} |f(y)| \omega(y) \, dy \right)
\]

\[
I_{222} \leq C \left( \omega(B_i) \inf \left\{ \lambda + \frac{\lambda}{\omega(B_i)} \int_{B_i} \Phi_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy \right\} + \int_{B_i} |f(y)| \omega(y) \, dy \right)
\]

Thus, we have

\[
I_{22} \leq C \lambda^{-1} \sum_i \left( \omega(B_i) \inf \left\{ \lambda + \frac{\lambda}{\omega(B_i)} \int_{B_i} \Phi_m \left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy \right\} + \int_{B_i} |f(y)| \omega(y) \, dy \right)
\]

\[
\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left( 1 + \log^+ \frac{|f(y)|}{\lambda} \right)^m \omega(y) \, dy.
\]
From (3.7) and the above estimates for $I_{21}, I_{22}$ and $I_{23}$, we have

$$I_2 \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log \frac{|f(y)|}{\lambda}\right)^m \omega(y) dy.$$

This finishes the proof of Theorem 1.1.

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