On the complexity of colouring antiprismatic graphs

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Abstract

A graph $G$ is prismatic if for every triangle $T$ of $G$, every vertex of $G$ not in $T$ has a unique neighbour in $T$. The complement of a prismatic graph is called antiprismatic. The complexity of colouring antiprismatic graphs is still unknown. Equivalently, the complexity of the clique cover problem in prismatic graphs is not known.

Chudnovsky and Seymour gave a full structural description of prismatic graphs. They showed that the class can be divided into two subclasses: the orientable prismatic graphs, and the non-orientable prismatic graphs. We give a polynomial time algorithm that solves the clique cover problem in every non-orientable prismatic graph. It relies on the structural description and on later work of Javadi and Hajebi. We give a polynomial time algorithm which solves the vertex-disjoint triangles problem for every prismatic graph. It does not rely on the structural description.

1 Introduction

We consider undirected simple graphs (no multiple edge, no loop). A $k$-colouring of a graph $G$ is a function $f$ from $V(G)$ to $\{1,\ldots,k\}$ such that for all $uv \in E(G)$, $f(u) \neq f(v)$. The colouring problem is the problem whose input is a graph $G$ and whose output is a $k$-colouring of $G$ such that

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k is minimum. The colouring problem is NP-hard and is the object of much attention. In particular, its complexity is known for several classes of graphs.

When \( H \) is a graph, a graph \( G \) is \( H \)-free if it does not contain any induced subgraph isomorphic to \( H \). When \( \mathcal{H} \) is a set of graphs, \( G \) is \( \mathcal{H} \)-free if \( G \) is \( H \)-free for all \( H \in \mathcal{H} \). We denote by \( P_k \) the path on \( k \) vertices and by \( C_k \) the cycle on \( k \) vertices. When \( G \) and \( H \) are graphs, we denote by \( G + H \) their disjoint union. When \( G \) is a graph and \( k \) an integer, we denote by \( kG \) the disjoint union of \( k \) copies of \( G \). We denote by \( K_k \) the complete graph on \( k \) vertices. We denote by \( K_{k,l} \) the complete bipartite graph with one side of the bipartition of size \( k \) and the other side of size \( l \). A stable set is a set of vertices that are pairwise nonadjacent. A clique is a set of vertices that are pairwise adjacent.

Kráľ', Kratochvíl, Tuza and Woeginger [7] proved the following dichotomy: the coloring problem for \( H \)-free graphs is polynomial time solvable if \( H \) is an induced subgraph of \( P_4 \) or an induced subgraph of \( K_1 + P_3 \), and NP-hard otherwise. This motivated the systematic study of the colouring problem restricted to \( \{H_1, H_2\} \)-free graphs for all possible pairs of graphs \( H_1, H_2 \), or even to \( H \)-free graph in general. The complexity has been shown to be polynomial or NP-hard for all sets of graphs on at most four vertices (see Figure 1), apart from the following four cases that are still open (see [8] and [3]).

- \( \mathcal{H} = \{C_4, 4K_1\} \)
- \( \mathcal{H} = \{K_{1,3}, 4K_1\} \)
- \( \mathcal{H} = \{K_{1,3}, 2P_1 + P_2\} \)
- \( \mathcal{H} = \{K_{1,3}, 2P_1 + P_2, 4K_1\} \)

Lozin and Malyshev [8] noted that a \( \{K_{1,3}, 2P_1 + P_2\} \)-free graph is either \( 4K_1 \)-free or has no edges. Therefore, \( \{K_{1,3}, 2P_1 + P_2, 4K_1\} \)-free graphs are essentially equivalent to \( \{K_{1,3}, 2P_1 + P_2\} \)-free graphs, in the sense that the complexity of the coloring problem is the same for both classes.

In this paper, we study \( \{K_{1,3}, 2P_1 + P_2, 4K_1\} \)-free graphs. They have been introduced in a different context. Chudnovsky and Seymour (see [1]) described the structure of claw-free graphs in a series of articles (the claw is another name for \( K_{1,3} \)). The two first articles of the series are about the so-called prismatic graphs. Let us define them. A triangle in a graph is a set of three pairwise adjacent vertices. A graph \( G \) is prismatic if for every triangle \( T \) of \( G \), every vertex of \( G \) not in \( T \) has a unique neighbour in \( T \). A graph is antiprismatic if its complement is prismatic. It is straightforward
to check that antiprismatic graphs are precisely \( \{K_{1,3}, 2P_1 + P_2, 4K_1\} \)-free graphs.

Observe that if \( \{s_1, s_2, s_3\} \) and \( \{t_1, t_2, t_3\} \) are two vertex-disjoint triangles in a prismatic graph \( G \), then there is a perfect matching in \( G \) between \( \{s_1, s_2, s_3\} \) and \( \{t_1, t_2, t_3\} \), so that \( \{s_1, s_3, t_1, t_2, t_3\} \) induces the complement of a \( C_6 \), commonly called prism, see Figure 2. This is where the name prismatic comes from.

Chudnovsky and Seymour gave a full structural description of prismatic graphs (and therefore of their complement). They showed that the class can be divided into two subclasses, to be defined later: the orientable prismatic graphs, and the non-orientable prismatic graphs. They described the structure of the two subclasses: orientable in [1] and non-orientable in [2]. Javadi and Hajebi [6] discovered a flaw in [2], that they could fix at the expense of adding a basic class in the structural description. It is natural to ask whether this description yields a polynomial time algorithm to color antiprismatic graphs.

The clique cover problem is the problem of finding, in an input graph \( G \), a minimum number of cliques that partition \( V(G) \). It is equivalent to the coloring problem for the complement. It is therefore NP-complete in
the general case. Our work is about the coloring problem for antiprismatic graphs. However, it is more convenient to view it as a study of the clique cover problem for prismatic graphs. Hence, from here on, we focus on the prismatic graphs and the clique cover problem.

Our main result is that there exists an $O(n^{7.5})$-time algorithm to solve the clique cover problem in non-orientable prismatic graphs. Our proof is based on the existence, in any non-orientable prismatic graph $G$ with more than 27 vertices, of a set of at most 5 vertices that intersects all triangles of $G$ (see Theorem 4.1). This follows directly from the structural description, but needs careful verifications. For the orientable case, we could not settle the complexity of the clique cover problem, but we prove that a related problem can be solved in polynomial time: the vertex-disjoint triangles problem. It consists in finding a maximum number of disjoint triangles in an input graph. This problem is known to be NP-hard in the general case [4].

Our algorithm for the clique cover problem in the non-orientable case relies on the existence of a hitting set of the triangles of bounded size. Sepehr Hajebi [5] observed that the existence of such a set can be proved with a short argument that relies on several lemmas of [2]. This argument gives a hitting set of size at most 15. The way we use hitting sets then provides an algorithm for the clique cover problem of complexity $O(n^{17.5})$.

Outline

In Section 2 we give the main definitions that we need and several results about prismatic graphs.

In Section 3, we give the structural description of non-orientable prismatic graphs from [2] and show that it implies the existence of a set of bounded number of vertices that intersects all triangles. Since our proof mostly relies on the structural description of Chudnovsky and Seymour, we have to give many long definitions extracted from their work.

In Section 4 we show that this yields a polynomial time algorithm for the clique cover problem in non-orientable prismatic graphs.

In Section 5 we prove that the vertex-disjoint triangles problem can be solved in polynomial time for prismatic graphs since by Section 4. Our proof does not rely on the structural description from [1].

In Section 6 we describe Hajebi’s approach.

Section 7 is devoted to concluding remarks.
2 Definition, notation and prismatic graphs

Let $G$ be a graph. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$. For $v \in V(G)$, we denote by $N_G(v)$ the set of neighbours of $v$ in $G$ (note that $v \notin N_G(v)$); when there is no risk of ambiguity, we may write $N(v)$. The line graph of $G$, denoted by $L(G)$, is the graph whose vertices are the edges of $G$, and such that two vertices are adjacent if and only if they are adjacent edges of $G$.

A vertex $v$ is complete to $A \subseteq V(G)$ if $v \notin A$ and $v$ is adjacent to all vertices of $A$. When $A, B$ are two disjoint subsets of $V(G)$, $A$ is complete to $B$ if every vertex in $A$ is complete to $B$. A vertex $v$ is anticomplete to $A \subseteq V(G)$ if $v \notin A$ and $v$ is adjacent to no vertex in $A$. When $A, B$ are two disjoint subsets of $V(G)$, $A$ is anticomplete to $B$ if every vertex in $A$ is anticomplete to $B$.

A triangle in a graph $G$ is covered by a set $S$ of vertices if at least one vertex of the triangle is in $S$. A set $S \subseteq V(G)$ is a hitting set of the triangles of $G$ if every triangle in $G$ is covered by $S$. We often write hitting set instead of hitting set of the triangles.

We call clique cover of $G$ a set of disjoint cliques of $G$ that partition $V(G)$.

Orientable and non-orientable prismatic graphs

Let $T = \{a, b, c\}$ be a triangle in a graph $G$. There are two cyclic permutations of $T$, and we use the notation $a \rightarrow b \rightarrow c \rightarrow a$ to denote the cyclic permutation mapping $a$ to $b$, $b$ to $c$ and $c$ to $a$. Thus $a \rightarrow b \rightarrow c \rightarrow a$ and $b \rightarrow c \rightarrow a \rightarrow b$ mean the same permutation.

A prismatic graph $G$ is orientable if there is a choice of a cyclic permutation $O(T)$ for every triangle $T$ of $G$, such that if $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are disjoint triangles and $s_it_i$ is an edge for $1 \leq i \leq 3$, then $O(S)$ is $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_1$ if and only if $O(T)$ is $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_1$. In that case, the set of permutations containing $O(T)$ for every triangle $T$ of $G$ is called a correct orientation of $G$.

A graph $G$ is non-orientable if there exists no correct orientation of $G$.

Orientable and non-orientable prismatic graphs have very different structures. By Theorem 4.1, a non-orientable prismatic graph contains at most 9 disjoint triangles. It might seem surprising that having a tenth triangle in a prismatic graph implies the existence of an orientation. This is because having a large number of disjoint triangles in a prismatic graph entails so many constraints that the only way to satisfy them all is in an orientable prismatic graph.
There is a nice characterisation of orientable prismatic graphs. The *rotator* and the *twister* are the graphs represented on Figure 3. In the rotator there exists one triangle that intersects all triangles, we call it the *center of the rotator*.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{rotator_twister.png}
\caption{The rotator with center \{a, b, c\} and the twister}
\end{figure}

**Theorem 2.1** (6.1 in [2]). A prismatic graph is orientable if and only if it is \{rotator, twister\}-free.

The structure of non-orientable prismatic graphs presented in [2] can be seen as a description of how the rotator and the twister can be completed in order to obtain all non-orientable prismatic graphs.

### 3 Non-orientable prismatic graphs

Our goal in this section is to prove the following.

**Lemma 3.1.** Every prismatic non-orientable graph contains a hitting set of the triangles of cardinality smaller or equal to 10.

The *core* of a graph $G$ is the union of all triangles of $G$. Clearly, in a prismatic graph, deleting an edge between two vertices that are not in the core yields a prismatic graph. It follows that vertices not in the core are less structured than vertices in the core. Clearly, to prove Lemma 3.1 we may restrict our attention to the cores of the graphs in the class.

A prismatic graph $G$ with core $W$ is *rigid* if
• there do not exist distinct \( u, v \in V(G) \setminus W \) adjacent to precisely the same vertices in \( W \), and

• every two non-adjacent vertices of \( G \) have a common neighbour in \( W \).

Replicating a vertex \( v \) in a graph \( G \) means replacing \( v \) by a stable set \( S \) that is complete to \( N(v) \) and anticomplete to \( V(G) \setminus (N(v) \cup \{v\}) \). We need the following.

**Theorem 3.2** (2.2 from [2]). Every non-orientable prismatic graph can be obtained from a rigid non-orientable prismatic graph by replicating vertices not in the core, and then deleting edges between vertices not in the core.

It follows, from this result, that to prove Lemma 3.1 it is enough to prove it for rigid non-orientable prismatic graphs.

Theorem 4.1 in [2] states that the class of rigid non-orientable prismatic graphs is included in the union of 13 classes. Javadi and Haji-Bi [6] discovered that one class is missing in Theorem 4.1, the so-called class \( \mathcal{F}_0 \). We describe these 14 classes whose union is called the *menagerie*.

In the definition of the menagerie, two operations are sometimes needed, the so-called *multiplication* and *exponentiation*.

The rest of the section is therefore organized as follows. The first two subsections describe the multiplication and exponentiation together with a proof that applying them under some specific hypotheses preserves the existence of a hitting set. The next 14 subsections each presents one class of the menagerie, together with a proof of the existence of a small hitting set. These subsections with Theorem 3.2 therefore form the proof of Lemma 3.1.

Before we start, we state the following lemma which is a direct consequence of the definition of prismatic graphs.

**Lemma 3.3.** If \( v \) be a vertex of a prismatic graph \( G \) then \( N_G(v) \) is a hitting set of \( G \).

### 3.1 Multiplication

Let \( H \) be a prismatic graph and \( X \) be a subset of vertices of \( H \). For each vertex \( x \in X \), let \( A_x \) be a set of vertices not in \( V(H) \) such that for all distinct \( x, x' \in X \), \( A_x \cap A_{x'} = \emptyset \). Let \( A = \cup_{x \in X} A_x \) and let \( \varphi \) be a map from \( A \) to the set of integers such that for all \( x \in X \), \( \varphi \) is injective on \( A_x \).

Let now \( G \) be the graph defined as follows:

- \( V(G) = (V(H) \setminus X) \cup A \).

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Let $v$ and $v'$ be two distinct vertices of $G$.

- If there is an $x \in X$ such that both $v$ and $v'$ are in $A_x$ then $v$ and $v'$ are not adjacent. $A_x$ is a stable set of $G$.
- If $v$ and $v'$ are in $V(H) \setminus X$ then $vv' \in E(G)$ if and only if $vv' \in E(H)$.
- If $v \in V(H) \setminus X$ and $v' \in A_x$ for some $x \in X$ then $vv' \in E(G)$ if and only if $vx \in E(H)$.
- If $v \in A_x$ and $v' \in A_{x'}$ where $x, x' \in X$ are distinct and adjacent in $H$, then $vv' \in E(G)$ if and only if $\varphi(v) = \varphi(v')$.
- If $v \in A_x$ and $v' \in A_{x'}$ where $x, x' \in X$ are distinct and nonadjacent in $H$, then $vv' \notin E(G)$ if and only if $\varphi(v) = \varphi(v')$.

The graph $G$ is obtained from $H$ by multiplying $X$. For $x \in X$, the set $A_x$ is the set of new vertices corresponding to $x$ and $\varphi$ is the corresponding integer map. As noted in [2], the multiplication does not preserve being prismatic in general, but it is used only in situations where it does.

**Lemma 3.4.** If $H[X]$ is an induced subgraph of $C_4$ and non-adjacent vertices of $X$ have no common neighbours in $V(H) \setminus X$ then any hitting set of $H$ disjoint from $X$ is also a hitting set of $G$.

**Proof.** Let $S_H$ be a hitting set of $S_H$. We prove that every triangle $\{u, v, w\}$ in $G$ is covered by $S_H$. If $\{u, v, w\} \subseteq V(H)$, then it is covered by $S_H$, so we may assume that $|\{u, v, w\} \cap A| > 0$.

**Case I:** $|\{u, v, w\} \cap A| = 1$

Suppose up to symmetry that there exists $x \in X$ such that $u, v, w \in V(H) \setminus X$. Then $\{x, v, w\}$ is a triangle in $H$ and it has to be covered by $S_H$. Since $X \cap S_H = \emptyset$, $u$ or $w$ belongs to $S_H$. Hence, $S_H$ covers $\{u, v, w\}$.

**Case II:** $|\{u, v, w\} \cap A| = 2$

Since for every $x \in X$, $A_x$ is a stable set, we may assume up to symmetry that there exist distinct $x, x' \in X$ such that $u \in A_x$, $v \in A_{x'}$ and $w \in V(H) \setminus X$. Since $w$ is a common neighbour of $u$ and $v$ in $G$, $w$ is a common neighbour of $x$ and $x'$ in $H$. From our assumptions, it follows that $x$ and $x'$ are adjacent in $H$. Hence $\{x, x', w\}$ is a triangle in $H$ and it has to be covered by $S_H$. Since $X \cap S_H = \emptyset$, $w \in S_H$, we have $S_H$ covers $\{u, v, w\}$.

**Case III:** $|\{u, v, w\} \cap A| = 3$

Since for every $x \in X$, $A_x$ is a stable set, there exist then distinct $x, y, z \in X$ such that $u \in A_x$, $v \in A_y$ and $w \in A_z$. Because of the hypothesis on $X$, $H[\{x, y, z\}]$ induces a $P_3$. Without loss of generality, suppose $xz \notin E[H]$.
Since $x$ and $y$ are adjacent in $H$, in order to have $u$ and $v$ also adjacent in $G$, we have $\varphi(u) = \varphi(v)$. Similarly, $\varphi(v) = \varphi(w)$. Hence $\varphi(u) = \varphi(w)$.

But since $x$ and $z$ are not adjacent, in order to have $u$ and $v$ adjacent in $G$, we need $\varphi(u) \neq \varphi(w)$, a contradiction. 

\section*{3.2 Exponentiation}

A triangle $T = \{a, b, c\}$ of a graph $G$ is a \textit{leaf triangle at} $c$ if every triangle of $G$ distinct from $T$ contains neither $a$ nor $b$.

Let $T = \{a, b, c\}$ be a leaf triangle at $c$ of a prismatic graph $H$. We define a partition of the neighbours of $c$, distinct from $a$ and $b$, into three disjoint sets: $D_1$, $D_2$, and $D_3$ as follows (see Figure 4). Let $v \neq a, b$ be a vertex adjacent to $c$ in $H$, then:

- $v \in D_1$ if $v$ belongs to a triangle that does not contain $c$.
- $v \in D_2$ if $v \notin D_1$ and $v$ belongs to a triangle (then this triangle is unique and contains $c$).
- $v \in D_3$ if $v$ does not belong to any triangle.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Neighbourhood of $c$}
\end{figure}

Let $A$, $B$ and $C$ be three pairwise disjoint sets of vertices. The graph $G$ is defined as follows:
\[ V(G) = (V(H) \setminus \{a, b\}) \cup A \cup B \cup C \]

with the following adjacencies:

- Vertices in \( V(H) \setminus \{a, b\} \) are adjacent in \( G \) if and only if there are adjacent in \( H \).
- \( A, B \) and \( C \) are stable sets.
- Every vertex in \( A \) has at most one neighbour in \( B \) and vice versa.
- Every vertex in \( V(H) \setminus \{a, b\} \) adjacent (resp. non-adjacent) to \( a \) in \( H \) is complete (resp. anticomplete) to \( A \) in \( G \).
- Every vertex in \( V(H) \setminus \{a, b\} \) adjacent (resp. non-adjacent) to \( b \) in \( H \) is complete (resp. anticomplete) to \( B \) in \( G \).
- \( C \) is complete to \( D_1 \cup D_3 \) and anticomplete to \( V(H) \setminus (\{a, b\} \cup D_1 \cup D_3) \).
- Every vertex in \( C \) is adjacent to exactly one end of every edge between \( A \) and \( B \) and adjacent to every vertex in \( A \cup B \) with no neighbour in \( A \cup B \).

The graph \( G \) is obtained from \( H \) by exponentiating the leaf triangle \( \{a, b, c\} \).

Before proving that the exponentiation preserves hitting sets, note that every prismatic graph \( H \) with a leaf triangle \( T = \{a, b, c\} \) at \( c \) has a hitting set \( S_H \) that contains \( c \) but neither \( a \) nor \( b \).

**Lemma 3.5.** If \( G \) is prismatic and \( S_H \) is a hitting set of \( H \) containing \( c \) but neither \( a \) nor \( b \), then \( S_H \) is also a hitting set of \( G \).

**Proof.** Let \( \{u, v, w\} \) be a triangle in \( G \). We show that one of \( u, v \) or \( w \) belongs to \( S_H \). Since \( u, v, w \in V(G) \), none of them is \( a \) or \( b \).

**Case I:** \( |\{u, v, w\} \cap (V(H) \setminus \{a, b\})| = 3 \)

This case is trivial because then \( \{u, v, w\} \) is a triangle in \( H \) and has to be covered by \( S_H \).

**Case II:** \( |\{u, v, w\} \cap (V(H) \setminus \{a, b\})| = 2 \)

Without loss of generality suppose \( u \notin V(H) \) and \( v, w \in V(H) \).

If \( u \) belongs to \( A \), that means that \( \{a, v, w\} \) is a triangle in \( H \). By our hypothesis this triangle should be \( T \) and \( c \in \{u, v, w\} \). So \( S_H \) covers \( \{u, v, w\} \).

The case where \( u \) belongs to \( B \) is similar.
If \( u \) belongs to \( C \), then \( v \) and \( w \) have to belong to \( D_1 \cup D_3 \) by definition of the neighbourhood of \( C \). Then \( \{c, v, w\} \) is a triangle in \( G \). So \( \{u, v, w, c\} \) is a diamond in \( G \), a contradiction to \( G \) being prismatic.

**Case III:** \( |(\{u, v, w\} \cap V(H) \setminus \{a, b\})| = 1 \)

Without loss of generality suppose \( w \in V(H) \setminus \{a \cup b\} \). Note that \( A, B \) and \( C \) are stable sets so \( \{u, v, w\} \) contains at most one vertex of each.

Suppose that one of \( u, v \) is in \( C \) (so the other one is in \( A \cup B \)). Up to symmetry, we may assume that \( u \) belongs to \( A \) and \( v \) belongs to \( C \). Then \( aw \in E(H) \) and \( cw \in E(H) \). This means that \( \{a, w, c\} \) is a triangle in \( H \). This contradicts \( \{a, b, c\} \) being a leaf triangle at \( c \).

Suppose that none of \( u, v \) belong to \( C \). Up to symmetry, suppose \( u \in A \) and \( v \in B \). So, \( \{a, b, w\} \) is a triangle of \( H \) and this triangle can only be \( \{a, b, c\} \). So \( w = c \in S_H \).

**Case IV:** \( |(\{u, v, w\} \cap H \setminus \{a, b\})| = 0 \)

This case cannot happen because \( A, B \) and \( C \) are stable sets and every vertex of \( C \) has a unique neighbour in any edge of \( G[A \cup B] \).

### 3.3 Schlafli-prismatic graphs

We have to define the Schlafli graph, and it is more convenient to work in the complement. The complement of the Schlafli graph has 27 vertices \( r_{ij}, s_{ij}, t_{ij}, 1 \leq i, j \leq 3 \) with adjacencies as follows. For \( 1 \leq i, i', j, j' \leq 3 \):

- If \( i \neq i' \) and \( j \neq j' \), then \( r_{ij} \) is adjacent to \( r_{i'j'} \), \( s_{ij} \) is adjacent to \( s_{i'j'} \), and \( t_{ij} \) is adjacent to \( t_{i'j'} \).
- If \( j = i' \), then \( r_{ij} \) is adjacent to \( s_{ij} \), \( s_{ij} \) is adjacent to \( t_{ij} \), and \( t_{ij} \) is adjacent to \( r_{ij} \).

There are no other edges.

This graph will be denoted by \( \Sigma \) throughout the rest of the paper. We will often rely on the fact that \( \Sigma \) is vertex-transitive.

We introduce more notation. We set \( R = \{r_{ij} : 1 \leq i, j \leq 3\} \), \( S = \{s_{ij} : 1 \leq i, j \leq 3\} \) and \( T = \{t_{ij} : 1 \leq i, j \leq 3\} \) and call tile each of the sets \( R, S, T \). We call line \( i \) of \( R \) the set \( \{r_{ij} : 1 \leq j \leq 3\} \) and column \( j \) of \( R \) the set \( \{r_{ij} : 1 \leq i \leq 3\} \). We use a similar notation for \( S \) and \( T \).

By definition an edge between \( u \) and \( v \) in a same tile exists if and only if \( u \) and \( v \) are in different lines and columns. Edges between tiles are conveniently described as follows: for every \( i = 1, 2, 3 \), column \( i \) of \( R \) is complete to line \( i \)
of $S$, column $i$ of $S$ is complete to line $i$ of $T$ and column $i$ of $T$ is complete to line $i$ of $R$. There are no other edges.

A triangle in $\Sigma$ is internal if it is included in a tile, and external otherwise. The observations above show that an internal triangle is made of three vertices that are in three different lines, and also in three different columns of the tile. An external triangle $\{u, v, w\}$ satisfies $\{u, v, w\} = \{r_i^j, s^j_k, t^k_i\}$ for some $1 \leq i, j, k \leq 3$.

This shows that there exist 6 internal triangles in each tile and 27 external triangles, that gives 45 triangles in total. Each vertex lies in two internal triangles and three external triangles. Every edge is contained in exactly one triangle. See Figure 5.

Figure 5: $\Sigma$, the complement of the Schläfli graph. (Only the 10 edges and the 5 triangles that contain $r_{3}^{3}$ are represented.)

We call Schläfli-prismatic graph every induced subgraph of $\Sigma$. It is easy to see that they are prismatic.

**Lemma 3.6.** A smallest hitting set of $\Sigma$ has cardinality 10.

**Proof.** Every vertex $v$ in $\Sigma$ has degree 10, so by Lemma 3.3 $N_{\Sigma}(v)$ is a hitting set of $\Sigma$ of size 10.

Suppose for a contradiction that $W$ is a hitting set of $\Sigma$ and $|W| = 9$. Since $\Sigma$ contains 45 triangles and every vertex of $\Sigma$ is contained in exactly
5 triangles (2 internal and 3 external), no two vertices of \(W\) hit the same triangle. Since every edge of \(\Sigma\) is contained in a triangle, it follows that \(W\) is a stable set.

For each tile \(X\), a maximum stable set in \(\Sigma\) has cardinality 3 and is a line or a column. It follows that for \(X \in \{R, S, T\}\), \(W \cap X\) is a line or a column of \(X\).

By the pigeon hole principle, \(W\) contains either two lines or two columns (of different tiles). This contradicts the fact that \(W\) is a stable set, because between two lines (or two columns) of different tiles, there exists at least one edge.

### 3.4 Fuzzily Schl"{a}fli-prismatic graphs

Let \(\{a, b, c\}\) be a leaf triangle at \(c\) in a Schl"{a}fli-prismatic graph \(H\). If a prismatic graph \(G\) can be obtained from \(H\) by multiplying \(\{a, b\}\), and \(A, B\) are the two sets of new vertices corresponding to \(a, b\) respectively, the graph \(G\) is Schl"{a}fli-prismatic. Note that this operation is not iterated.

**Lemma 3.7.** Every fuzzily Schl"{a}fli-prismatic graph has a hitting set of cardinality smaller or equal to 5.

**Proof.** Let \(G, H\) and \(\{a, b, c\}\) as in the definition.

Since \(a\) belongs to exactly one triangle of \(H\) and to exactly 5 triangles in \(\Sigma\), we have \(|N_H(a)| \leq |N_\Sigma(a)| - 4 = 6\).

By Lemma 3.3 \(N_H(a)\) is a hitting set of \(H\).

Since \(b \in N_H(a)\) and since the unique triangle containing \(b\) is \(\{a, b, c\}\) which is already covered by \(c\), we have that \(N_H(a) \setminus \{b\}\) is a hitting set of \(H\) of cardinality at most 5.

By Lemma 3.4 it is also a hitting set of \(G\).

### 3.5 Graphs of parallel-square type

Let \(X\) be the edge-set of some \(C_4\) of the complete bipartite graph \(K_{3,3}\), and let \(z\) be the edge of \(K_{3,3}\) disjoint from all edges in \(X\). Thus \(X\) induces a \(C_4\) of the line graph \(H\) of \(K_{3,3}\). Any graph \(G\) obtained from \(H\) by multiplying \(X\), and possibly deleting \(z\), is prismatic and is called a graph of parallel-square type.
Lemma 3.8. Every prismatic graph of parallel square type admits a hitting set of cardinality smaller or equal to 4.

Proof. Let $H$, $X$, $z$ and $G$ as in the definition.

Let $S_H = V(H) \setminus (X \cup \{z\})$. Obviously $S_H$ is a hitting set of $H$.

The set $X$ induces a $C_4$ in $H$ and no two non-adjacent vertices of $X$ have common neighbours in $V(H) \setminus X$. By Lemma 3.4, $S_H$ is a hitting set of $G$.

Note that the deletion of $z$ does not change the result because $z$ is not in $S_H$. Since $|S_H| = 4$, the proof is completed.

3.6 Graphs of skew-square type

Let $K$ be a graph with five vertices $a$, $b$, $c$, $s$, $t$, where $\{s, a, c\}$ and $\{t, b, c\}$ are triangles and there is no more edge. Let $H$ be obtained from $K$ by multiplying $\{a, b, c\}$, let $A$, $B$, $C$ be the sets of new vertices corresponding to $a$, $b$, $c$ respectively, and let $\varphi$ be the corresponding integer map. Add three more vertices $d_1$, $d_2$, $d_3$ to $H$, with adjacency as follows:

- $d_1$, $d_2$, $d_3$, $s$, $t$ are pairwise non-adjacent,
- for $1 \leq i \leq 3$ and $v \in A \cup B$, $d_i$ is adjacent to $v$ if and only if $1 \leq \varphi(v) \leq 3$ and $\varphi(v) \neq i$,
- for $1 \leq i \leq 3$ and $v \in C$, $d_i$ is non-adjacent to $v$ if and only if $1 \leq \varphi(v) \leq 3$ and $\varphi(v) \neq i$.

Any graph obtained by this way is prismatic, and is called a graph of skew-square type.

Lemma 3.9. Every prismatic graph of skew-square type admits a hitting set of cardinality smaller or equal to 5.
Proof. Let $G$ be a graph of skew-square type.

Let $S_K = \{s, t\}$. We first show that $S_K$ is a hitting set of $H$ and then prove that $S_K \cup \{d_1, d_2, d_3\}$ is a hitting set of $G$.

First it is obvious that $S_K$ is a hitting set of $K$. Furthermore, in $K$, $\{a, b, c\}$ induces a $P_3$ which is a induced subgraph of $C_4$ and vertices $a$ and $b$ do not have a common neighbour in $V(K) \setminus \{a, b, c\}$. We may therefore apply Lemma 3.4, showing that $S_K$ is a hitting set of $H$.

Since we just add three vertices $d_1$, $d_2$ and $d_3$ to construct $G$ from $H$, every triangle in $G$ either contains one vertex of $\{d_1, d_2, d_3\}$ or is a triangle in $H$ that is covered by $S_K$.

This shows that $S_K \cup \{d_1, d_2, d_3\}$ is a hitting set of $G$ of size 5. 

3.7 The class $\mathcal{F}_0$

Note that this class is defined in [6].

Let $H$ be a subgraph of $\Sigma$ induced by:

$$\{r_j^i : (i, j) \in I_1\} \cup \{s_j^i : (i, j) \in I_2\} \cup \{t_j^i : (i, j) \in I_3\}$$

where $I = \{(i, j) : 1 \leq i, j \leq 3\}$ and $I_1$, $I_2$, $I_3$ are subset of $I$ such that:

- $(1, 1), (1, 3), (2, 2), (2, 3), (3, 1)(3, 2) \in I_1$ and $(3, 3) \notin I_1$,
- $(1, 1), (2, 1), (3, 2) \in I_2$ and $(1, 2), (1, 3), (2, 2), (2, 3) \notin I_2$,
- $(1, 3), (2, 1), (2, 2) \in I_3$ and $(1, 1), (1, 2), (3, 1), (3, 2) \notin I_3$.

Let $G$ be the graph obtained from $H$ by adding the edges $s_3^1t_3^2$, $s_1^3t_3^1$ and $s_3^2t_3^2$ if the corresponding vertices are in $H$. We define $\mathcal{F}_0$ to be the class of all such graphs $G$.

Lemma 3.10. Every graph of the class $\mathcal{F}_0$ admits a hitting set of cardinality smaller or equal to 3.

Proof. By Lemma 3.3, $N_2(s_1^3) \cap V(H) = S_H = \{r_3^1, r_3^3, t_3^2\}$ is a hitting set of $H$. Note that $s_1^3$ and $t_3^2$ do not have common neighbours in $H$ so as $s_1^1$ and $t_3^3$, and $s_3^3$ and $t_3^2$. Since $G[s_1^1, s_3^1, t_3^2, t_3^3]$ induces a $C_4$, we have that the addition of the new edges does not add any triangle in $G$. Therefore $S_H$ is also a hitting set of $G$. 

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3.8 The class $\mathcal{F}_1$

Let $G$ be a graph with vertex set the disjoint union of sets $\{s, t\}, R, A, B$, where $|R| \leq 1$, and with edges as follows:

- $s, t$ are adjacent, both are complete to $R$, and $s$ is complete to $A$; $t$ is complete to $B$;
- every vertex in $A$ has at most one neighbour in $A$, and every vertex in $B$ has at most one neighbour in $B$;
- if $a, a' \in A$ are adjacent and $b, b' \in B$ are adjacent, then the subgraph induced by $\{a, a', b, b'\}$ is a cycle;
- if $a, a' \in A$ are adjacent and $b \in B$ has no neighbour in $B$, then $b$ is adjacent to exactly one of $a, a'$;
- if $b, b' \in B$ are adjacent and $a \in A$ has no neighbour in $A$, then $a$ is adjacent to exactly one of $b, b'$;
- if $a \in A$ has no neighbour in $A$, and $b \in B$ has no neighbour in $B$, then $a, b$ are adjacent.

We define $\mathcal{F}_1$ to be the class of all such graphs $G$.

**Lemma 3.11.** Every prismatic graph of the class $\mathcal{F}_1$ admits a hitting set of cardinality smaller or equal to 2.

**Proof.** We claim that $\{s, t\}$ is a hitting set of $G$. This is equivalent to the fact that in $G \setminus \{s, t\}$, the neighborhood of any vertex $v$ is a stable set. And this follows directly from the definition in all cases ($v = r$, $v$ in $A$ with no neighbor in $A$, $v$ in $A$ with one neighbor in $A$, symmetric cases with $v \in B$).

Note that graphs in $\mathcal{F}_1$ can have arbitrarily large minimum degree.

3.9 The class $\mathcal{F}_2$

Let $K$ be the line graph of $K_{3,3}$ with vertices numbered $s^i_j$ $(1 \leq i, j \leq 3)$, where $s^i_j$ and $s^{i'}_{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. Note that this is how usually the complement of the line graph of $K_{3,3}$ is defined, but since it is a self-complementary graph, it makes no difference.

Let $H$ be a graph obtained from this by multiplying $\{s^1_2, s^1_3, s^2_1, s^3_1\}$, thus, $H$ is of parallel-square type. Let $A^1_2, A^1_3, A^2_1, A^3_1$ be the sets of new vertices corresponding to $\{s^1_2, s^1_3, s^2_1, s^3_1\}$ respectively, and let $\varphi$ be the corresponding integer map. Suppose that:
Proof.\ Let \( K \subseteq F_3 \). We prove that \( K \) be obtained from \( K \) be the class of all such graphs \( G \).

Lemma 3.12. Every prismatic graph of the class \( \mathcal{F}_2 \) admits a hitting set of cardinality smaller or equal to 4.

Proof. Let \( S_K = \{s_1^3, s_2^3, s_2^1, s_3^3\} \). We can easily see that \( S_K \) is a hitting set of \( K \). We prove that \( S_K \) is a hitting set of \( G \).

By definition \( H \) is obtained from \( K \) by multiplying \( \kappa = \{s_3^1, s_2^1, s_1^3, s_3^3\} \). Note that, in \( K \), \( \kappa \) induces a \( C_4 \), \( \kappa \cap S_K = \emptyset \). \( s_3^1, s_2^1 \) do not have a common neighbours outside of \( \kappa \) and \( s_3^3, s_2^3 \) do not have a common neighbours outside of \( \kappa \). We may now apply Lemma 3.5, and conclude that \( S_K \) is a hitting set of \( H \).

We may apply Lemma 3.5, and we obtain that \( S_K \) is a hitting set of \( G \) (note that the fact that \( H \) is a prismatic graph is not used in the proof of Lemma 3.5). \( \square \)

3.10 The class \( \mathcal{F}_3 \)

Let \( K \) be the line graph of \( K_{3,3} \), with vertices numbered \( s_i^j \) (\( 1 \leq i, j \leq 3 \)), where \( s_i^j \) and \( s_{i'}^{j'} \) are adjacent if and only if \( i' \neq i \) and \( j' \neq j \). Let \( H \) be obtained from \( K \) by deleting the vertex \( s_2^2 \) and possibly \( s_1^3 \), and then multiplying \( \{s_1^3, s_2^3, s_2^1, s_3^3\} \). Let \( A_2^1, A_3, A_1^2, A_3^3 \) be the sets of new vertices corresponding to \( s_2^1, s_3^1, s_1^3, s_3^3 \) respectively, and let \( \varphi \) be the corresponding integer map. Suppose that

- there exist \( a_2^1 \in A_2^1 \) and \( a_3^3 \in A_3^3 \) such that \( \varphi(a_2^1) = \varphi(a_3^3) = 1 \);
- \( \varphi(v) \neq 1 \) for all \( v \in A_2^1 \cup A_3^3 \);
- there exist \( a_3^1 \in A_3^1 \) and \( a_2^3 \in A_2^3 \) such that \( \varphi(a_3^1) = \varphi(a_2^3) = 2 \);
- \( \varphi(v) \neq 2 \) for all \( v \in A_2^1 \cup A_3^3 \).

Let \( G \) be obtained from \( H \) by exponentiating \( \{a_3^1, a_1^3, s_2^3\} \) and \( \{a_3^1, a_1^3, s_2^3\} \), leaf triangles respectively at \( s_2^1 \) and \( s_3^2 \). We define \( \mathcal{F}_3 \) to be the class of all such graphs \( G \).
Lemma 3.13. Every prismatic graph of the class $F_3$ admits a hitting set of cardinality smaller or equal to 3.

Proof. Let $S_H = \{s_3^3, s_2^3, s_3^2\}$. Note that in $K$ minus vertex $s_3^2$, $S_H$ is a hitting set, $X = \{s_3^1, s_3^1, s_2^1, s_1^2\}$ induces a $C_4$, $s_3^1, s_2^1$ do not share a common neighbour in $V(H) \setminus X$ so as $s_2^1, s_1^3$. We may apply Lemma 3.4. It follows that $S_H$ is a hitting set of $H$.

We may apply Lemma 3.5, and we conclude that $S_H$ is a hitting set of $G$.

3.11 The class $F_4$

Take $\Sigma$, with vertices numbered $r_i^j, s_i^j, t_i^j$ as usual. Let $H$ be the subgraph induced on

$$Y \cup \{s_j^i : (i, j) \in I\} \cup \{t_1^i, t_2^i, t_3^i\}$$

where $\emptyset \neq Y \subseteq \{r_1^3, r_2^3, r_3^3\}$ and $I \subseteq \{(i, j) : 1 \leq i, j \leq 3\}$ with $|I| \geq 8$ and including $\{(i, j) : 1 \leq i \leq 3$ and $1 \leq j \leq 2\}$.

We consider $T = \{t_1^i, t_2^i, t_3^i\}$ which is a leaf triangle at $t_3^3$. Let $G$ be obtained from $H$ by exponentiating $T$. We define $F_4$ to be the class of all such graphs $G$.

Lemma 3.14. Every prismatic graph of the class $F_4$ admits a hitting set of cardinality smaller or equal to 4.

Proof. Let $K$ be the subgraph of $\Sigma$ induced by the vertices: $\{r_1^3, r_2^3, r_3^3\} \cup S \cup \{t_1^i, t_2^i, t_3^i\}$.

By Lemma 3.3 $N_\Sigma(r_3^3) \cap K = S_K = \{s_1^3, s_2^3, s_3^3, t_3^3\}$ is a hitting set of $K$.

Since $H$ is a subgraph of $K$, $S_H = S_K \cap V(H)$ is a hitting set of $H$ and $|S_H| \leq 4$.

Since $t_3^3 \in S_H$, we can apply Lemma 3.5 and conclude that $S_H$ is a hitting set of $G$.

3.12 The class $F_5$

Take $\Sigma$, with vertices numbered $r_i^j, s_i^j, t_i^j$ as usual. Let $H$ be the subgraph induced on

$$\{r_j^i : (i, j) \in I_1\} \cup \{s_j^i : (i, j) \in I_2\} \cup \{t_j^i : (i, j) \in I_3\}$$

where $I_1, I_2, I_3 \subseteq \{(i, j) : 1 \leq i, j \leq 3\}$ are chosen such that:

- $(1, 1), (3, 1), (3, 2), (3, 3) \in I_1$ and $(2, 2), (2, 3) \notin I_1$
\( (1, 1) \notin I_2 \)
\( (1, 2), (1, 3), (2, 3), (3, 3) \in I_3 \) and \( (2, 1), (3, 1) \notin I_3 \)

Let \( G \) be obtained from \( H \) by adding the edge \( r_1^1 t_2^1 \). We define \( \mathcal{F}_5 \) to be the class of all such graphs \( G \).

**Lemma 3.15.** Every prismatic graph in the class \( \mathcal{F}_5 \) admits a hitting set of cardinality smaller or equals to 5.

**Proof.** By Lemma 3.3, \( N_{\Sigma}(r_1^1) \cap H = S_H = \{r_2^3, r_3^3, s_2^1, s_3^1, t_1^1\} \) is a hitting set of \( H \).

In \( \Sigma \), vertices \( r_1^1 \) and \( t_2^1 \) have as common neighbours the following vertices: \( s_1^1, r_2^2, r_3^2, t_1^2 \) and \( r_3^3 \). Since none of them are in \( H \), the addition of the edge \( r_1^1 t_2^1 \) does not create another triangle. This proves that \( S_H \) is a hitting set of \( G \). \( \square \)

### 3.13 The class \( \mathcal{F}_6 \)

Take \( \Sigma \) with vertices numbered \( r_j^i, s_j^i, t_j^i \) as usual. Let \( H \) be the subgraph induced by

\[
\{r_j^i : (i, j) \in I_1\} \cup \{s_j^i : (i, j) \in I_2\} \cup \{t_j^i : (i, j) \in I_3\}
\]

where:

- \( I_1 = \{(1, 1), (1, 2), (3, 1), (3, 2), (3, 3)\} \)
- \( I_2 = \{(1, 2), (2, 1), (2, 2), (3, 3)\} \)
- \( I_3 = \{(1, 2), (2, 2), (1, 3), (2, 3), (3, 3)\} \)

Let \( G \) be obtained from \( H \) by adding the edge \( r_1^1 t_2^1 \) and then multiplying \( \{r_3^3, t_3^3\} \). We define \( \mathcal{F}_6 \) to be the class of all such graphs \( G \).

**Lemma 3.16.** Every prismatic graph in the class \( \mathcal{F}_6 \) admits a hitting set of cardinality smaller or equals to 3.

**Proof.** By Lemma 3.3, \( N_G(r_1^3) = \{r_1^2, r_2^3, s_3^3\} \) is a hitting set of \( G \). \( \square \)
The six-vertex prism is the graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and edges:

$$a_1 a_2, a_1 a_3, a_2 a_3, b_1 b_2, b_1 b_3, b_2 b_3, a_1 b_1, a_2 b_2, a_3 b_3$$

Let $K$ be a graph with six vertices, with the six-vertex prism as a subgraph. Construct a new graph $G$ as follows. The vertices of $G$ consist of $E(K)$ and some of the vertices of $K$, so $E(K) \subseteq V(G) \subseteq E(K) \cup V(K)$; two edges of $K$ are adjacent in $G$ if they have no common end in $K$; an edge and a vertex of $K$ are adjacent in $G$ if they are incident in $K$; and two vertices of $K$ are adjacent in $G$ if they are non-adjacent in $K$. The class of all such graphs $G$ is called $\mathcal{F}_7$ (they are all prismatic).

**Lemma 3.17.** Every prismatic graph in the class $\mathcal{F}_7$ admits a hitting set of cardinality at most 5.

**Proof.** Let $K$ and $G$ be as in the definition. Let us show that

$$S_G = E(K) \cap \{a_1 a_2, a_1 a_3, a_2 a_3, a_1 b_1, a_1 b_2, a_1 b_3, a_2 b_2, a_3 b_3\}$$

is a hitting set of $G$. Let $T$ be a triangle in $G$. We now break into 4 cases.

- $|T \cap V(K)| = 1$: Such a triangle does not exist. Because if two edges in $K$ are adjacent in $G$ then they do not share a common vertex in $K$ and then they do not have in $G$ a common neighbour in $V(G) \cap V(K)$.

- $|T \cap V(K)| = 2$: Such a triangle does not exist. Indeed, if in $G$, there are two vertices $u, v \in V(K)$, both adjacent to $e \in E(K)$, then $e = uv$. A contradiction to $u, v$ being adjacent in $G$.

- $|T \cap V(K)| = 3$: Such a triangle does not exist. Otherwise it would induce in $K$ a stable set of cardinality 3 and there is no such stable set in the prism and hence in $K$.

- $|T \cap V(K)| = 0$: Every vertex of $G$ not in $V(K)$ is in $E(K)$. If such a triangle $T = \{e_1, e_2, e_3\}$, $(e_1, e_2, e_3 \in E(K))$ exists, then $e_1, e_2$ and $e_3$ do not have common ends in $K$. It follows that at least one of $e_1$, $e_2$ or $e_3$ has $a_1$ as an end point and therefore is included in $S_G$.

Hence $S_G$ is a hitting set of $G$ and $|S_G| \leq 5$. 

\(\Box\)
3.15 The class $F_8$

Let $H$ be the graph with nine vertices $v_1, \ldots, v_9$ and with edges as follows: 

- $\{v_1, v_2, v_3\}$ is a triangle, 
- $\{v_4, v_5, v_6\}$ is complete to $\{v_7, v_8, v_9\}$, and 
- for $i = 1, 2, 3$, $v_i$ is adjacent to $v_{i+3}$, $v_{i+6}$. 

Note that $H$ is a rotator. Let $G$ be obtained from $H$ by multiplying $\{v_4, v_7\}$, $\{v_5, v_8\}$ and $\{v_6, v_9\}$. We define $F_8$ to be the class of all such graphs $G$.

**Lemma 3.18.** Every prismatic graph in the class $F_8$ admits a hitting set of cardinality smaller or equals to 3.

**Proof.** We can easily see that $S_H = \{v_1, v_2, v_3\}$ is a hitting set of $H$. Since $\{v_4, v_7\}$, $\{v_5, v_8\}$ and $\{v_6, v_9\}$ are each not in $S_H$ and are each edges, we can successively apply Lemma [3.4] for each one separately and $S_H$ stays a hitting set of $G$ (note that the fact that $H$ is a prismatic graph is not used in the proof of Lemma [3.4]).

3.16 The class $F_9$

Take $\Sigma$ with vertices numbered $r^i_j$, $s^i_j$, $t^i_j$ as usual. Let $H$ be the subgraph induced by

$$\{r^i_j : (i, j) \in I_1\} \cup \{s^i_j : (i, j) \in I_2\} \cup \{t^i_j : (i, j) \in I_3\}$$

where $I_1, I_2, I_3 \subseteq \{(i, j) : 1 \leq i, j \leq 3\}$ satisfy

- $(2, 1), (3, 1), (3, 2), (3, 3) \in I_1$ and $I_1$ contains at least one of $(1, 2), (1, 3)$ and $(1, 1), (2, 2), (2, 3) \notin I_1$,

- $(1, 1), (2, 2), (3, 3) \in I_2$ and $(1, 2), (1, 3) \notin I_2$,

- $(1, 3), (2, 3), (3, 3) \in I_3$, and $I_3$ contains at least one of $(1, 2), (2, 2), (3, 2)$, and $(1, 1), (2, 1), (3, 1) \notin I_3$,

- either $(1, 2), (1, 3) \in I_1$ or $I_3$ contains $(1, 2)$ and at least one of $(2, 2), (3, 2)$.

Let $G$ be obtained from $H$ by adding a new vertex $z$ adjacent to $r^3_2, r^3_3, s^1_1$, and to $t^3_2$ if $(2, 2) \in I_3$, and to $t^3_3$ if $(3, 2) \in I_3$. We define $F_9$ to be the class of all such graphs $G$.

**Lemma 3.19.** Every prismatic graph in the class $F_9$ admits a hitting set of cardinality smaller or equals to 3.

**Proof.** By Lemma [3.3] $N_G(r^1_1) = \{r^3_2, r^3_3, s^1_1\}$ is a hitting set of $G$.  

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4 Clique cover of non-orientable prismatic graphs

In this section we show how the presence of a hitting set of cardinality bounded by a constant can be used for solving the clique cover problem. We have seen in the previous section that every non-orientable prismatic graph admits a hitting set of size at most 10. The following is more useful for algorithmic purposes.

**Theorem 4.1.** If $G$ is a non-orientable prismatic graph then $G$ admits a hitting set of cardinality at most 5 or $G$ is a Schlafli-prismatic graph.

**Proof.** By Theorem 3.2, $G$ is obtained from a prismatic graph $H$ from the menagerie by replicating vertices not in the core of $H$ and then deleting edges between vertices not in the core.

It is easy to verify that $G$ and $H$ have exactly the same triangles. Therefore, it is obvious that if $S_H$ is a hitting set of $H$ then $S_H$ is a hitting set of $G$.

From Lemmas 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18 and 3.19 if $H$ is either Fuzzily Schlafli-prismatic, of parallel-square type, of skew-square type, or in $F_i$, $i \in \{0, \ldots, 9\}$, then $G$ admits a hitting set of size at most 5.

It remains to consider the case where $H$ is Schlafli-prismatic. Hence, $H$ is an induced subgraph of $\Sigma$.

If every vertex of $H$ is contained in a triangle of $H$, then no vertex of $H$ can be replicated, so $G = H$ and $G$ is Schlafli-prismatic. Hence, we may assume that some vertex $v$ of $H$ is contained in no triangle of $H$.

Since $N_\Sigma(v)$ induces a matching of 5 edges in $\Sigma$ and $v$ is contained in no triangle of $H$, we see that $N_H(v)$ contains at most 5 vertices. By Lemma 3.3, $N_H(v)$ is a hitting set of $H$. Hence, $H$ (and therefore $G$) contains a hitting set of size at most 5.

Observe that in a triangle-free graph, solving the clique cover problem is easily reducible to computing a matching of maximum cardinality by Edmonds’ algorithm. Furthermore, the clique cover problem is solvable in constant time when the number of vertices of the input graph is bounded. Note that Schlafli-prismatic graphs have at most 27 vertices.

We need the following notation: $T(G)$ is a variant of the adjacency matrix of $G$. For $v, w \in V(G)$, the entry $(v, w)$ of $T(G)$ is 0 if $v$ and $w$ are not adjacent, 1 if they are adjacent but without common neighbour, and $x$ if they are adjacent and have $x$ as a common neighbour. Note that in the last case, if $G$ is diamond-free and $K_4$-free, then $x$ is unique.
This matrix can be computed in time $O(n^3)$ at the beginning of an algorithm and used afterwards to find the triangles in $G$ or in any induced subgraph of $G$.

**Lemma 4.2.** Let $G \in \text{Free}\{\text{diamond}, K_4\}$. There is an algorithm that finds a hitting set of $G$ of cardinality at most 5 if such a set exists and answers 'no' otherwise. This algorithm has complexity $O(n^7)$.

**Proof.** First compute $T(G)$ in time $O(n^3)$ as above. Enumerate each set $X$ of vertices of $G$ of size at most 5 in time $O(n^5)$. For each $X$, test in time $O(n^2)$ if $G \setminus X$ is triangle-free by checking if every entry of $T(G)$ reduced to $G \setminus X$ is either 0, 1 or an element of $X$. If no such $X$ exists answer 'no' and otherwise output $X$. \hfill \Box

**Theorem 4.3.** The Clique Cover Problem for non-orientable prismatic graphs is solvable in time $O(n^{7.5})$.

**Proof.** Let $G$ be a prismatic non-orientable graph. The following method provides a minimum clique cover.

1. Compute the matrix $T(G)$ as previously defined. This can be done in time $O(n^3)$.

2. Use the method in time $O(n^7)$ described in Lemma 4.2. If the algorithm outputs a hitting set of $G$ of size at most 5 denoted by $S = \{s_1, \ldots, s_{i^*}\}$ ($i^* \leq 5$), then go to Step 4. Else by Theorem 4.1 $G$ is a Schläfli-prismatic graph and go to Step 3.

3. Since there is a bounded number of vertices in $G$ compute all possible clique covers of $G$ in constant time. Go to Step 5.

4. Enumerate all sets $X$ of at most 5 disjoint triangles of $G$, (this can trivially be done in time $O(n^{15})$). We can do it in time $O(n^5)$ as follows. Compute the set $T_i$ of triangles containing $s_i$ for each $1 \leq i \leq i^*$. This can be done in $O(n)$ by reading the line of $T(G)$ corresponding to $s_i$. Notice that there are at most $n/2$ triangles in each $T_i$. Then compute all subsets $\mathcal{T}$ of triangles of $G$ obtained by choosing at most one triangle in each $T_i$. For each such $\mathcal{T}$ which contains only pairwise vertex-disjoint triangles, compute by some classical algorithm a maximum matching $M_\mathcal{T}$ of $G \setminus (\cup_{T \in \mathcal{T}} T)$ and let $R_\mathcal{T}$ be the vertices of $G$ that are neither in $\mathcal{T}$ nor in $M_\mathcal{T}$. Notice that $\mathcal{T} \cup M_\mathcal{T} \cup R_\mathcal{T}$ is a clique cover of $G$. Go to Step 5.
5. Among all the clique covers generated by the previous steps, let $C^*$ be one of the smallest size. Return $C^*$.

**Correctness:**

If $G$ has no hitting set of size at most 5 then the algorithm will consider all possible clique covers of $G$. Therefore, the algorithm will give a clique cover of minimum size.

Otherwise, let $C$ be a minimum clique cover of $G$. Since $G$ is $K_4$-free, $C$ is the union of a set $\mathcal{T}$ of vertex-disjoint triangles, a set $\mathcal{E}$ of vertex-disjoint edges and a set $\mathcal{R}$ of vertices. Since $C$ is of minimum size, $\mathcal{E}$ should be a maximum matching in the subgraph of $G$ induced by the vertices not in any triangle of $\mathcal{T}$. Each triangle in $\mathcal{T}$ contains a vertex of the hitting set $S$ of $G$ obtained by Step 2 in the algorithm. Furthermore, each vertex of $S$ is contained in at most one triangle of $\mathcal{T}$. So the algorithm will consider $\mathcal{T}$ at some point in Step 4 and will compute a maximum matching in the remaining graph. Therefore it will return a clique cover $C^*$ of same size as $C$.

**Complexity:**

The procedure to enumerate all sets $\mathcal{T}$ takes time $O(n^5)$. For each such set, the maximum matching can be found by Micali and Vazirani’s algorithm in time $O(n^{2.5})$. Overall, a best clique cover is found in time $O(n^{7.5})$. 

### 5 Orientable prismatic graphs

In the non-orientable case, we have shown that the clique cover problem is polynomial time solvable. We do not know the complexity of this problem in the orientable case. In this section we show that the vertex-disjoint triangles problem (the problem of finding a maximum number of vertex-disjoint triangles) is polynomial time solvable in prismatic graphs. As noted in the introduction, this problem is NP-hard in the general case.

Remark that solving the vertex-disjoint triangle problem is not sufficient to solve the clique cover problem in orientable prismatic graph. See Figure 7.

The *derived graph* $D(G)$ of a graph $G$ is the intersection graph of the triangles of $G$. More formally, if $H = D(G)$, then $V(H) = \{T : T$ is a triangle in $G\}$. Two vertices of $H$ are adjacent if they are distinct triangles of $G$ sharing at least one vertex. Note that the class of derived graphs is not hereditary.
Figure 7: An orientable prismatic graph where the best clique cover is not obtained by first selecting the maximum number of disjoint triangles

Figure 8: $D(L(K_{3,3})) = K_{3,3}$

**Theorem 5.1.** Let $G$ an orientable prismatic graph. Every connected component of $D(G)$ is claw-free or is isomorphic to $K_{3,3}$.

*Proof.* Let $D$ be a connected component of $D(G)$ containing a claw. Hence, $G$ has to contain 4 triangles as represented on Figure 9 (not all edges of $G$ are represented). We will use the notation given there and we denote by $K$ the set of vertices $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$.

Figure 9: A graph whose derived graph is $K_{1,3}$
Since $G$ is prismatic there should be a matching between the extremities of the edge $a_2a_3$ and those of $b_2b_3$, $c_2c_3$.

Without loss of generality we may assume that these matching edges are: $a_2b_2$, $a_3b_3$ and $a_2c_2$, $a_3c_3$.

Now there are two possibilities for the matching between $b_2b_3$, $c_2c_3$. If $b_2c_3, b_3c_2 \in E(G)$, then $G[K]$ is a rotator of center $\{a_1, b_1, c_1\}$, a contradiction to $G$ being orientable (Theorem 2.1). We can now assume that $b_2c_2, b_3c_3 \in E(G)$. So $G[K]$ contains $L(K_{3,3})$.

It remains to show that there is no other vertex in $D$. Assume that it is not the case, then there is a triangle $T$ in $G$, that intersects $K$. Since in $G[K]$, every vertex and edge is in a triangle, $|T \cap K| = 1$.

Since $L(K_{3,3})$ is vertex transitive, we may assume up to symmetry that $T = \{u, v, b_2\}$. Since $T$ and $\{a_1, b_1, c_1\}$ form a prism, we may assume without loss of generality that $ua_1$ and $vc_1$ are edges of $G$. Now, because of triangles $T$ and $A$, $ua_3, vc_3 \in E(G)$ and because of triangles $T$ and $C$, $uc_3 \in E(G)$.

Then $\{u, v, b_2, a_2, a_3, a_1, c_2, c_3, c_1\}$ induces a rotator with center $\{a_2, b_2, c_2\}$, a contradiction to $G$ being orientable.

\[ \square \]

**Theorem 5.2.** The vertex-disjoint triangle problem is $O(n^5)$ time solvable in prismatic graphs.

\[ \text{Proof.} \text{ Let } G \text{ be a prismatic graph. If } G \text{ has at most 27 vertices, we solve the problem in constant time. Otherwise, we look by Lemma 4.2, for a hitting set of size at most 5 in } G. \text{ If one exists, then we know that at most 5 disjoint triangles exist in } G, \text{ and we find an optimal set of vertex-disjoint triangles of } G \text{ in time } O(n^5) \text{ as in the proof of Theorem 4.3. Hence, we may assume that no hitting set of size at most 5 exists. By Theorem 4.1, } G \text{ is orientable.}

A set $R$ is a stable set of $D(G)$ if and only if it is a set of vertex-disjoint triangles of $G$. Hence, it is enough to compute a maximum stable set in $D(G)$. Such a set can obviously be found by computing a maximum stable set in each connected component of $D(G)$. By Lemma 5.1 each such component is either isomorphic to $K_{3,3}$ or claw-free. The components that are isomorphic to $K_{3,3}$ are handled trivially. In the components that are claw-free, to find a maximum stable set, we may rely on the classical algorithm of Sbihi [10] ($O(n^3)$).

\[ \square \]

6 A shorter proof for a weaker result

Sepehr Hajebi [5] noted that the existence of a hitting set of bounded size (namely 15) in any non-orientable prismatic graph $G$ can be deduced from several parts of [2] with a small amount of additional work as follows.
Consider a non-orientable prismatic graph $G$. By Theorem 6.1 in [2], $G$ contains either a twister or a rotator as an induced subgraph.

If $G$ contains a twister $Z$ (so $|V(Z)| = 10$) but does not contain any rotator, then (5) in the proof of 7.2 from [2] shows that $V(Z)$ is a hitting set of $G$.

If $G$ contains a rotator, then Hajebi’s strategy is to rely on results from Section 10 of [2]. This section is about graphs that contain a rotator and no so-called “square-forcer” (we do not need the definition), but the argument relies only on 10.3, where the assumption that there is no square-forcer is not used.

From here on, we use notation from [2].

In 10.3, the set of all triangles of $G$ is described, and it is proved that they all fall in one of the following categories: subsets of $S$ (where $S$ is a set of cardinality at most 9), $R$-triangles, $T$-triangles, diagonal triangles and marginal triangles. The definition of these categories implies that the set of vertices $K = \{r_1^3, r_2^3, r_3^3, \, t_1^3, t_2^3, t_3^3\} \cup S$ is a hitting set of $G$. It has size at most 15 (because $|S| \leq 9$).

7 Concluding remarks

Despite our efforts to use the results of [1], the complexity of the clique cover problem remains unknown for orientable prismatic graphs. However, here is a simple remark. Suppose that $\{K_1, \ldots, K_l\}$ is an optimal clique cover of a prismatic graph $G$. If for some $1 \leq i, j \leq l$, $K_i$ is a triangle and $K_j$ an isolated vertex $v$, then by prismaticity, $v$ has a neighbour $u$ in $K_i$. Hence, we may replace $K_i$ and $K_j$ by $K_i \setminus \{u\}$ and $K_j \cup \{u\}$. We may iterate this until in the optimal cover, there is not simultaneously a triangle and an isolated vertex. It follows that for every prismatic graph, there exists an optimal cover that is either made only of triangles and edges, or made only of edges and isolated vertices. Observe that an optimal clique cover of the last kind is easily computable in polynomial time by some classical algorithm for the maximum matching. Hence, to solve the clique cover problem, it is enough to find an optimal clique cover of the first kind.

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