A Logical Theory for Conditional Weak Ontic Necessity Based on Context Update

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Abstract
Weak ontic necessity is the ontic necessity expressed by “should/ought to” in English. An example of it is “I should be dead by now”. A feature of this necessity is that whether it holds at the present world is irrelevant to whether its prejacent holds at the present world. In this paper, by combining premise semantics and update semantics for conditionals, we present a logical theory for conditional weak ontic necessity based on context update. A context is a set of ordered defaults, determining expected possible states of the present world. Sentences are evaluated with respect to contexts. When evaluating the conditional weak ontic necessity with respect to a context, we first update the context with the antecedent, then check whether the consequent holds with respect to the updated context. We compare this theory to some related works. The logic’s expressivity is studied.

Keywords Conditional weak ontic necessity · Ordered defaults · Expected possible states · Update

1 Introduction
1.1 Modalities
The world can be in different possible states, and every possible state is called a possibility. There are different kinds of possibilities: deontic, epistemic, and so forth.
Modalities locate their *prejacents* in spaces of possibilities (von Fintel, 2006). Different modalities locate their prejacents in different kinds of spaces, that is, different modalities have different *flavors* (Matthewson, 2016). Different modalities locate their prejacents in different ways: *universal, existential*, and so forth, that is, different modalities have different *forces* (Matthewson, 2016). The modalities related to universal quantifications are called *necessities*, and those related to existential quantifications are called *possibilities*. Consider two examples:

1. *The man talking aloud might be drunk.*
2. *Bob must go to school.*

The first sentence says that “*The man talking aloud is drunk*” is true at some *epistemic* possibility, and the second says that “*Bob goes to school*” is true at all *deontic* possibilities.

Modalities are complex, and there are many theories of them in the literature. We refer to von Fintel (2006) and Matthewson (2016) for some general discussions about the existing works. The most influential theory in linguistics was proposed by Kratzer in Kratzer (1991), among her other works. Its framework consists of a set of possible worlds and two functions, respectively called a *modal base* and an *ordering source*. For every possible world, the modal base specifies a set of propositions, determining the accessible worlds to the world, and the ordering source also specifies a set of propositions, ordering possible worlds from the perspective of the world. Many modalities can be interpreted in this framework.

### 1.2 Conditional Modalities

Conditional modalities are those sentences claiming that a modalized sentence is the case in a proposed scenario, which may or may not be actual (von Fintel, 2011). Here are two examples:

1. *If the baby crying is not hungry, then he must be angry.*
2. *If I were there now, then I should help the victims as well.*

What is called *conditionals* in the literature are always conditional modalities, and there are no genuinely bare conditionals (Kratzer, 1986). Those seemly bare conditionals have implicit modalities. For example, the following sentence has an implicit epistemic necessity:

1. *If the man approaching is not Jack, then he is Zack.*

Note that it is not always clear what the implicit modality is in seemly bare conditionals. Consider an example:

1. *If the match were struck, it would light.*

Leitgeb (2012) contends that the implicit modality in this sentence is “necessary” or “very likely”, but Wawer and Wroński (2015) disagree with the former reading¹.

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¹ Leitgeb (2012) did not give many arguments for his claim. An argument from Wawer and Wroński (2015) is as follows. Suppose that I did not toss the coin. Here are two conditionals: (1) *If I had flipped the coin, it
Conditional modalities can be classified into two classes from a semantic perspective: those whose antecedent is compatible with the speaker’s information state and those whose antecedent is contrary to it. The latter are commonly called *counterfactual* conditionals. No name for the former has been widely established in the literature. Following some sources, such as Rott (1999), we call them *open* conditionals. For example, Sentence 3 is an open conditional and Sentence 4 is a counterfactual conditional in typical situations. There is a syntactic distinction between conditionals in English: *indicative* and *subjunctive*. For example, Sentence 3 is an indicative conditional and Sentence 4 is a subjunctive conditional. The relation between the two distinctions is complicated. Simply speaking, they are not identical. We refer to Anderson (1951) and von Fintel (2011) for some counter-examples.

How do we formally make sense of conditionals? There have been many studies on that. Generally speaking, the main research lines include *ordering semantics*, *premise semantics*, *probability semantics*, and *belief revision approach*. All these lines are greatly influenced by Ramsey’s Test, proposed in Ramsey (1990), and there are many connections among them. We refer to Egré and Rott (2021) for general discussions.

Here, we briefly mention some main references in ordering semantics and premise semantics closely related to this work. We will discuss some of them in detail later. Ordering semantics was proposed by Stalnaker (1968) and Lewis (1973), which is the most commonly accepted theory for counterfactual conditionals. Its core idea is that a counterfactual is true in the present state of the world if its consequent is true in all the alternative states satisfying the antecedent, which are most similar to the present state. The ordering of *being more similar* plays a crucial role in this theory, which is how it gets its name. Premise semantics was proposed by Veltman (1976) and Kratzer (1979). Its core idea is that evaluation of a conditional in a situation involves evaluation of its consequent in situations where not only its antecedent but also some *additional* premises are satisfied. This idea was already discussed in Chisholm (1946) and Goodman (1947), among others. The two approaches influenced many works. Here, we mention two important ones. As said above, Kratzer (1991) proposed a general framework to address modalities. This work also contains a way to handle conditionals, which combines the two approaches: Premises are used to induce orderings among possible worlds. Veltman (2005) provided a semantics for counterfactual conditionals based on an update mechanism and premise semantics.

### 1.3 Weak Ontic Necessity

*Ontic possibilities* are possible states in which our world could be if things had gone differently in the past. Ontic possibilities can be easily confused with *epistemic* possibilities. Epistemic possibilities are possible states in which our world could be in the *epistemic* sense: That they are possible is due to our ignorance about the present state of the world.

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1. *would have landed heads*; (2) *If I had flipped the coin, it would have necessarily landed heads*. Wawer and Wroński thought that our degree of belief is about 0 in the latter, but about 0.5 in the former.
Weak ontic necessity is the ontic necessity expressed by “should/ought to” in English, which is identified explicitly and called metaphysical necessity by Copley (2006). Here are some examples:

(7) Suppose Jones is in a crowded office building when a severe earthquake hits. The building topples. By sheer accident, nothing falls upon Jones; the building just happens to crumble in such a way as not to touch the place where he is standing. He emerges from the rubble as the only survivor. Talking to the media, Jones says the following:

I ought to be dead right now. (From Yalcin (2016))

(8) Our guests ought to be home by now. They left half an hour ago, have a fast car, and live only a few miles away. However, they are not home yet. (Adapted from Leech (1971))

(9) The beer should be cold by now. I put it into the refrigerator one hour ago. But I have absolutely no idea whether it is. (Adapted from Copley (2006))

(10) Consider Rasputin. He was hard to kill. First his assassins poisoned him, then they shot him, then they finally drowned him. Let us imagine that we were there. Let us suppose that the assassins fed him pastries dosed with a powerful, fast-acting poison, and then left him alone for a while, telling him they would be back in half an hour. Half an hour later, one of the assassins said to the others, confidently, “He ought to be dead by now.” The others agreed, and they went to look. Rasputin opened his eyes and glared at them. “He ought to be dead by now!” they said, astonished. (From Thomson (2008))

It can be seen whether the necessity holds has nothing to do with whether its prejacent is true. This implies that it is not epistemic but ontic.

After Copley (2006), weak ontic necessity was discussed by Swanson (2008), Thomson (2008), Finlay (2009), Yalcin (2016) and Ju (2023). Yalcin (2016) gave a formal theory for weak ontic necessity influenced by Veltman (1996). Ju (2023) presented a formal theory for weak ontic necessity adopting a temporal perspective. Generally speaking, this modality has not received enough attention yet.

Remarks In general, necessities expressed by “should/ought to” and necessities expressed by “must/have to” have different properties (McNamara, 1996; Copley, 2006; von Fintel & Iatridou, 2008). Here, we mention some of them. First, “should φ” is weaker than “must φ”. This is how necessities expressed by “should/ought to” are called “weak necessities”. Second, “should φ” can coexist with “not φ” but “must φ” cannot. Third, “should φ” is gradable, but “must φ” is not.

1.4 Conditional Weak Ontic Necessity

Conditional weak ontic necessity is common in reality. Here are some examples:

(11) In early morning, a group of soldiers came to a villa to arrest a man. They entered this man’s bedroom but did not find him. The leader said the following:

If his quilt is still warm, then he should not be far yet.
(12) If her mother had taken a metro, then she \textit{should} be home by now.

(13) If the alarm had sounded yesterday, I \textit{should} have ignored it. (From Dudman (1984))

(14) My car \textit{should} be parked on the street outside, but if it was stolen last night, it \textit{should} be in a chop shop by now. (Adapted from Yalcin (2016))

How do we formally address conditional weak ontic necessity? This is not a trivial question. To see this, note that conditional weak ontic necessity is clearly not monotonic, as shown by Sentence 14. As far as we read, no work in the literature explicitly handles conditional weak ontic necessity yet.

1.5 Our Work

In this work, we present a logical theory for conditional weak ontic necessity. Our approach is as follows. The agent has a system of defaults, called a context, that determines which possible states of the present world are expected. Weak ontic necessity quantifies over the set of expected possible states. A conditional weak ontic necessity is true with respect to a context if its consequent is true with respect to the result of updating the context with its antecedent.

This approach combines three research lines. First, following premise semantics, we think that additional propositions play a role in evaluating conditionals. Second, we think that when evaluating a conditional, we should first update something with its antecedent and then evaluate its consequent with respect to the update result, which follows the update semantics proposed by Veltman (2005). Third, our understanding of weak ontic necessity follows Ju (2023).

Conceptually, our logic differs from ordering semantics. However, technically, its flat fragment is equivalent to the flat fragment of the conditional logic $VN$ proposed by Lewis (1973), which is based on ordering semantics.

The rest of the paper is structured as follows. In Sect. 2, we state our approach to conditional weak ontic necessity in detail. In Sect. 3, we present a logic $\text{ConWON}$ for conditional weak ontic necessity. In Sect. 4, we compare our theory to some closely related theories on conditionals. In Sect. 5, we study the expressivity of $\text{ConWON}$ with respect to the regular notion of validity. In Sect. 6, we study the expressivity of $\text{ConWON}$ with respect to a special notion of validity. In Sect. 7, we summarize our work and mention some future directions. The Appendix contains proofs of some results.

2 Our Approach to Conditional Weak Ontic Necessity

We assume an agent. The world is in a state. It could have evolved differently in the past, and the actual state has alternatives. However, not all alternatives are expected for the agent. She has some defaults concerning which possible states are expected. The defaults can be of many kinds: natural laws (Light is faster than sound), common natural phenomena (It is cold during the winter), or simply customs (Jones wears...
The system of defaults as a whole determines which possible states are expected and which are not. According to it, the actual state might not be expected.

The order among defaults results from many factors. Here, we mention two of them. First, some kinds of defaults usually have higher priority than others. For example, natural laws usually have higher priority than customs. Second, special defaults usually have higher priority than general defaults. Here is an example. Suppose that there are two defaults: (1) The higher the altitude is, the colder the weather is; (2) In winter in Yili Valley, the higher the altitude is, the warmer the weather is. Suppose that you are climbing a mountain in winter in Yili Valley. In this case, you would expect that the weather is warmer in higher positions.

The sentence “If φ is the case now, then ψ should be the case now” is true at the actual state of the world with respect to the system of defaults if and only if ψ is true at all expected possible states determined by the result of putting φ to the system of defaults with the highest priority.

The proposition φ can be true or not at the actual state, and the agent might know its truth value or not. Assume that she does not know. Then, “If φ is the case now, then ψ should be the case now” is an open conditional for her. Assume that φ is true, and she knows it. Then, “If φ is the case now, then ψ should be the case now” is an open conditional for her. Assume that φ is false and she knows it. Then, “If φ is the case now, then ψ should be the case now” is a counterfactual conditional for her. Whether the agent knows the truth value of φ does not matter for whether she accepts the sentence “If φ is the case now, then ψ should be the case now”.

The following example can illustrate our understanding of conditional weak ontic necessity. This will be our running example.

**Example 1** A tribe captured many animals and put them in isolated cages in a cave with a door. Every evening, the priest randomly opens two nonempty cages and leaves the cave with its door locked. The next morning, people come and release the surviving one of the two freed animals. Yesterday evening, three animals remained: a tiger, a dog, and a goat. This morning, they come to the front of the door. The chief says:

(15) If the goat is still alive now, then the dog should be dead.

They open the door and see that the tiger killed the goat. The chief says:

(16) If the goat were still alive now, then the dog should be dead.

Intuitively, the two sentences are true. How? It seems natural to assume the following with the chief’s default system. There are three defaults: Tigers kill dogs, Tigers kill goats, and Dogs kill goats. The second default has the highest priority, and the first and third are incomparable. First, we put The goat is still alive now to the default system with the highest priority. According to the new default system, those possible states

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2 In this case, the sentence seems odd. We do not have clear ideas about the reason. It is possible that “should” can be both an ontic modal and an epistemic modal, and the oddness is related to the ambiguity. It is actually a controversial issue in the literature whether “should” can be an epistemic modal. We refer to Copley (2006) and Yalcin (2016) for some discussions.
of the world where the tiger is dead are not expected. Then, *the dog should be dead now.*

### 3 A Logic for Conditional Weak Ontic Necessity

#### 3.1 Languages

**Definition 1** (The languages $\Phi_{PL}$ and $\Phi_{ConWON}$) Let $AP$ be a countable set of atomic propositions. The language $\Phi_{PL}$ of the Propositional Logic (PL) is defined as follows, where $p$ ranges over $AP$:

$$\alpha ::= p \mid \perp \mid \neg \alpha \mid (\alpha \land \alpha)$$

The language $\Phi_{ConWON}$ of the Logic for Conditional Weak Ontic Necessity (ConWON) is defined as follows, where $\alpha \in \Phi_{PL}$:

$$\phi ::= p \mid \perp \mid \neg \phi \mid (\phi \land \phi) \mid [\alpha] \phi$$

The intuitive reading of $[\alpha] \phi$ is *if $\alpha$, then $\phi$ should be true.* Note that we do not allow for nesting of conditionals in the antecedent. The reason will be given in Sect. 3.4, after we present the semantics.

What follows are some derivative expressions:

- The propositional connectives $\top, \lor, \rightarrow$ and $\leftrightarrow$ are defined as usual.
- Define the dual $\langle \alpha \rangle \phi$ of $[\alpha] \phi$ as $\neg [\alpha] \neg \phi$. This operator does not seem to have a natural meaning but we introduce it due to technical reasons.
- Define $\Box \phi$ as $[\top] \phi$, meaning $\phi$ *should be true*.
- Define the dual $\Diamond \phi$ of $\Box \phi$ as $\neg \Box \neg \phi$. This operator does not seem to have a natural meaning, too.
- We define $E \alpha$, where $\alpha \in \Phi_{PL}$, as $\langle \alpha \rangle \top$, meaning $\alpha$ *is possible*. Its dual $A \alpha$ is defined as $\neg E \neg \alpha$, meaning $\alpha$ *is necessary*.

The following two fragments of $\Phi_{ConWON}$ will be used later, where $\alpha \in \Phi_{PL}$:

**Definition 2** (The languages $\Phi_{ConWON-1}$ and $\Phi_{A-1}$)

$$\Phi_{ConWON-1}: \quad \phi ::= p \mid \perp \mid \neg \phi \mid (\phi \land \phi) \mid [\alpha] \phi$$

$$\Phi_{A-1}: \quad \phi ::= p \mid \perp \mid \neg \phi \mid (\phi \land \phi) \mid A \alpha$$

The language $\Phi_{ConWON-1}$ is the flat fragment of $\Phi_{ConWON}$, containing no nested conditionals. The language $\Phi_{A-1}$ is the flat fragment of $\Phi_A$, the language of the modal logic S5.
3.2 Semantic Settings

3.2.1 Models

**Definition 3 (Models)** A model is a tuple $M = (W, V)$, where $W$ is a nonempty set of states and $V : AP \rightarrow \mathcal{P}(W)$ is a valuation.

Intuitively, $W$ consists of all possible states of the world at an instant.

3.2.2 Contexts

**Definition 4 (Contexts)** Let $M = (W, V)$ be a model. A pair $C = (D, \succ)$ is called a context for $M$ if $D$ is a finite (possibly empty) set of (possibly empty) subsets of $W$ and $\succ$ is an irreflexive and transitive relation on $D$. The elements of $D$ are called defaults.

Intuitively, $D_1 \succ D_2$ means that $D_1$ has higher priority than $D_2$. We use $\emptyset$ to indicate the special context $(\emptyset, \emptyset)$.

3.2.3 Expected States by Contexts

**Definition 5 (Hierarchy of defaults in contexts)** Let $C = (D, \succ)$ be a context for a model $M$. Define $HI(C)$, the hierarchy of defaults in $C$, as a sequence $(D_0, \ldots, D_n)$ constructed as follows:

- Let $D_0 = \{ D \in D | D$ is a maximal element of $D \}$;
- $\ldots$
- If $D_0 \cup \cdots \cup D_k \neq D$, let $D_{k+1} = \{ D \in D | D$ is a maximal element of $D \setminus (D_0 \cup \cdots \cup D_k) \}$, or else stop.

Here are some observations about $HI(C)$. Firstly, $HI(C)$ cannot be an empty sequence. Secondly, if $D_0 = \emptyset$, then $n = 0$. Thirdly, $D_0, \ldots, D_n$ are pairwise disjoint and their union is $D$.

**Example 2 (Hierarchy of defaults in contexts)**

- $HI(\emptyset) = \emptyset$. Here, $\emptyset$ is not the empty sequence but the sequence with the empty set as its only element.
- Let $C = (D, \succ)$ be a context, where $D = \{ D_1, D_2, D_3 \}$, $D_1 \succ D_2$ and $D_1 \succ D_3$. Then, $HI(C) = (\{ D_1 \}, \{ D_2, D_3 \})$.

Similar ways of defining hierarchies with respect to an ordered set can also be found in some literature on social choice theory, such as (Jiang et al., 2018, Definition 7).

**Definition 6 (Expected states by contexts)** Let $M = (W, V)$ be a model, $C = (D, \succ)$ be a context, and $HI(C) = (D_0, \ldots, D_n)$. Define the set $\|C\|$ of expected states by $C$, as follows:

- Suppose that $\bigcap D_0 = \emptyset$. Then, $\|C\| := \bigcap D_0$. 

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• Suppose that $\bigcap D_0 \neq \emptyset$. Then, $\|C\| := \bigcap D_0 \cap \cdots \cap \bigcap D_k$, where $(D_0, \ldots, D_k)$ is the longest initial segment of $(D_0, \ldots, D_n)$ such that $\bigcap D_0 \cap \cdots \cap \bigcap D_k \neq \emptyset$.

Note that specially $\bigcap \emptyset = W$. This definition follows the following idea: From the top level of the hierarchy, consider as many levels as possible.

**Example 3** (Expected states by contexts) Let $M = (W, V)$ be a model, where $W = \{w_1, w_2, w_3, w_4\}$.

- $\|0\| = \bigcap \emptyset = W$.
- Let $C = (\mathbb{D}, \succ)$ be a context such that $\text{HI}(C) = (\{\emptyset\}, \{W\})$. Then, $\|C\| = \bigcap \{\emptyset\} = \emptyset$.
- Let $C$ be a context such that $\text{HI}(C) = (\{D_1, D_2\}, \{D_3\})$, where $D_1 = \{w_1, w_2\}$, $D_2 = \{w_2, w_3\}$ and $D_3 = \{w_3, w_4\}$. Then, $\|C\| = \bigcap \{D_1, D_2\} = D_1 \cap D_2 = \{w_2\}$.

### 3.2.4 Context Update

**Definition 7** (Update of contexts) Let $C = (\mathbb{D}, \succ)$ be a context for a model $M = (W, V)$. Let $D$ be a subset of $W$. We define the **update of $C$ with $D$** as the context $C \oplus D = (\mathbb{D}', \succ')$, where

- $\mathbb{D}' = \mathbb{D} \cup \{D\}$;
- For all $D_1$ and $D_2$ in $\mathbb{D}'$, $D_1 \succ' D_2$ if and only if one of the following conditions is met:
  - $D_1 \neq D, D_2 \neq D$ and $D_1 \succ D_2$;
  - $D_1 = D$ and $D_2 \neq D$.

Intuitively, $C \oplus D$ is got in the following way. First, we eliminate $D$ from $C$, if $D$ occurs in $C$. Second, we add $D$ to $C$ as the greatest element.

### 3.2.5 Contextualized Pointed Models

**Definition 8** (Contextualized pointed models) For every model $M$, context $C$ and state $w$, $(M, C, w)$ is called a **contextualized pointed model**.

**Example 4** (Contextualized pointed models) Fig. 1 indicates a contextualized pointed model $(M, C, w_1)$, where

- $\mathbb{D}$ has three defaults: $D_1 = \{w_1, w_2\}$, $D_2 = \{w_1, w_2, w_3\}$ and $D_3 = \{w_4\}$;
- $D_1 \succ D_2 \succ D_3$.

It can be verified that $\text{HI}(C) = (\{D_1\}, \{D_2\}, \{D_3\})$ and $\|C\| = \{w_1, w_2\}$.

The intuitive reading of $(M, C, w_1)$ is as follows. The present state is $w_1$, which has three alternatives: $w_2, w_3, w_4$. By the context $C$, neither $w_3$ nor $w_4$ is expected.
3.3 Semantics

Definition 9 (Semantics for $\Phi_{\text{ConWON}}$) Consider a model $M$ and a context $C$.

- For every $\alpha \in \Phi_{\text{PL}}$, we define the default generated by $\alpha$ as $|\alpha| = \{w \mid M, C, w \models \alpha\}$.
- For every $\alpha \in \Phi_{\text{PL}}$, we define the update of $C$ with $\alpha$ as $C + \alpha = C \oplus |\alpha|$.
- Truth conditions for formulas of $\Phi_{\text{ConWON}}$ at contextualized pointed models are defined as follows:

\[
\begin{align*}
M, C, w &\models p \iff w \in V(p) \\
M, C, w &\not\models \bot \\
M, C, w &\models \neg \phi \iff M, C, w \not\models \phi \\
M, C, w &\models \phi \land \psi \iff M, C, w \models \phi \text{ and } M, C, w \models \psi \\
M, C, w &\models [\alpha] \phi \iff M, C + \alpha, u \models \phi \text{ for every } u \in \|C + \alpha\| \\
M, C, w &\models (\alpha) \phi \iff M, C + \alpha, u \models \phi \text{ for some } u \in \|C + \alpha\| \\
M, C, w &\models \Box \phi \iff M, C, u \models \phi \text{ for every } u \in \|C\| \\
M, C, w &\models \Diamond \phi \iff M, C, u \models \phi \text{ for some } u \in \|C\| \\
M, C, w &\models E \alpha \iff M, C, u \models \alpha \text{ for some } u \\
M, C, w &\models A \alpha \iff M, C, u \models \alpha \text{ for every } u
\end{align*}
\]

It can be verified that

\[
\begin{align*}
M, C, w &\models (\alpha) \phi \iff M, C + \alpha, u \models \phi \text{ for some } u \in \|C + \alpha\| \\
M, C, w &\models \Box \phi \iff M, C, u \models \phi \text{ for every } u \in \|C\| \\
M, C, w &\models \Diamond \phi \iff M, C, u \models \phi \text{ for some } u \in \|C\| \\
M, C, w &\models E \alpha \iff M, C, u \models \alpha \text{ for some } u \\
M, C, w &\models A \alpha \iff M, C, u \models \alpha \text{ for every } u
\end{align*}
\]

We say that a formula $\phi$ is valid ($\models_{\text{ConWON}} \phi$) if $M, C, w \models \phi$ for every contextualized pointed model $(M, C, w)$. We say that $\phi$ is 0-valid ($\models_{0-\text{ConWON}} \phi$) if $M, 0, w \models \phi$ for every contextualized pointed model $(M, 0, w)$.

We show how Example 1 is analyzed in our formalization.

We use $f_x$ to indicate $x$ is free and use $a_x$ to indicate $x$ is alive, where $x$ can be $t$ (the tiger), $d$ (the dog) or $g$ (the goat).

Figure 2 indicates a contextualized pointed model $(M, C, w_3)$ for Example 1, where $C = (\mathbb{D}, >)$, where

- $\mathbb{D}$ has three defaults: $D_1 = \{w_1, w_3, w_4, w_5, w_6\}$, $D_2 = \{w_1, w_2, w_3, w_5, w_6\}$, and $D_3 = \{w_1, w_2, w_3, w_4, w_5\}$;
- $D_2 \succ D_1$ and $D_2 \succ D_3$.

The default $D_1$ indicates that if the tiger and the dog are free, then the tiger is alive. The defaults $D_2$ and $D_3$ are understood similarly.

The formula $[a_g] \neg a_d$ means if the goat is still alive now, then the dog should be dead. It is true at $(M, C, w_3)$. How?
3.4 Remarks

As mentioned above, nesting of conditionals in the antecedent is not allowed in this logic. The reason is as follows. We think that the function of the antecedent of conditionals is to change the domain of possibilities in consideration by its truth value at possibilities. This requires that its truth value at a possibility be determined by the information of the possibility itself. Conditionals have modalities and their truth value at a possibility might be dependent on the information of other possibilities. Therefore, conditionals cannot be nested in the antecedent.

It can be seen that the truth value of $[\alpha] \phi$ at $(M, C, w)$ is not dependent on $w$. Thus, our semantics is global. This results from that contexts are not dependent on states. This coincides with an important feature of weak ontic necessity: Whether it holds at a state is not dependent on the state.

Conditionals are not monotonic in the semantics. Here is a counter-example. Let $(M, C, w_1)$ be a contextualized pointed model, where

- $M = (W, V)$, where $W = \{w_1, w_2\}$, $V(p) = \{w_1, w_2\}$ and $V(q) = \{w_2\}$;
- $C = (D, >)$, where $D = \{\{w_2\}\}$.

It can be verified $M, C, w_1 \models [p]q$ but $M, C, w_1 \not\models [p \land \neg q]q$.

By closely observing this example, we can see that the failure of monotonicity is due to the following reason: When evaluating $[p]q$, the default in $C$ plays a role, but when evaluating $[p \land \neg q]q$, it is defeated by the default $[p \land \neg q]$.

3.5 Equivalent Variants of Some Semantic Settings

3.5.1 Sequential Contexts

Technically, we can define contexts as nonempty sequences of sets of states without changing the set of valid formulas.
Definition 10 (Sequential contexts) Let $M = (W, V)$ be a model. A nonempty finite sequence $C = (D_0, \ldots, D_n)$ of defaults is called a sequential context for $M$.

Definition 11 (Expected states by sequential contexts) Let $M = (W, V)$ be a model, $C = (D_0, \ldots, D_n)$ be a sequential context. Define the set $\|C\|$ of expected states by $C$ as follows:

- Suppose that $D_0 = \emptyset$. Then, $\|C\| := D_0$.
- Suppose that $D_0 \neq \emptyset$. Then, $\|C\| := D_0 \cap \cdots \cap D_k$, where $(D_0, \ldots, D_k)$ is the longest initial segment of $(D_0, \ldots, D_n)$ such that $D_0 \cap \cdots \cap D_k \neq \emptyset$.

We use $C_1 \cdot C_2$ to indicate the concatenation of two sequential contexts $C_1$ and $C_2$.

Definition 12 (Update of sequential contexts) Let $C$ be a sequential context for a model $M = (W, V)$. Let $D$ be a subset of $W$. We define the update of $C$ with $D$ as the sequential context $C \cup D = D; C$.

Other ingredients of the semantics are given as before.

Definition 13 (Cores of sequential contexts) Let $C = (D_0, \ldots, D_n)$ be a sequential context for a model $M$. Recursively define $C^\tau$, the core of $C$, as follows:

- Let $D_0^\tau = D_0$.
- $\vdots$
- If $D_{k+1}$ does not occur in $(D_0, \ldots, D_k)^\tau$, let $(D_0, \ldots, D_k, D_{k+1})^\tau = (D_0, \ldots, D_k)^\tau, D_{k+1}$, or else let $(D_0, \ldots, D_k, D_{k+1})^\tau = (D_0, \ldots, D_k)^\tau$.

Here is an example for $C^\tau$: If $C = (D_0, D_1, D_0, D_2, D_1)$, then $C^\tau = (D_0, D_1, D_2)$. One default can occur in a sequential context more than one times. Intuitively, the function $\tau$ just keeps the occurrence of a default in a sequential context that is closest to the beginning of the sequential context.

Lemma 1 Let $C$ be a sequential context for a model $M = (W, V)$ and $D \subseteq W$. Then:

1. $\|C\| = \|C^\tau\|$;
2. $(C \cup D)^\tau = (C^\tau \cup D)^\tau$.

Proof

1. Let $C = (D_0, \ldots, D_n)$ and $C^\tau = (D_{i_0}, \ldots, D_{i_m})$. Assume that $D_0 \cap \cdots \cap D_n = \emptyset$. Then, $\|C\| = \emptyset = \|C^\tau\|$. Assume that $D_0 \cap \cdots \cap D_n \neq \emptyset$. Let $l$ be the greatest number such that $D_0 \cap \cdots \cap D_l \neq \emptyset$. Assume that $l = n$. Then, $\|C\| = D_0 \cap \cdots \cap D_n = D_{i_0} \cap \cdots \cap D_{i_m} = \|C^\tau\|$. Assume that $l < n$. Then, $D_{l+1}$ does not occur in $(D_0, \ldots, D_l)$. Then, $(D_0, \ldots, D_l)^\tau$ is a proper initial segment of $C^\tau$. Let $(D_0, \ldots, D_l)^\tau = (D_{i_0}, \ldots, D_{i_k})$. Then, $D_0 \cap \cdots \cap D_l = D_{i_0} \cap \cdots \cap D_{i_k}$ and $D_{i_{k+1}} = D_{i_{k+1}}$. Then, $D_{i_0} \cap \cdots \cap D_{i_k} \neq \emptyset$ and $D_{i_0} \cap \cdots \cap D_{i_k} \cap D_{i_{k+1}} = \emptyset$. Then, $\|C\| = D_0 \cap \cdots \cap D_l = D_{i_0} \cap \cdots \cap D_{i_k} = \|C^\tau\|$.

2. Assume that $D$ does not occur in $C$. Then, $(C \cup D)^\tau = (D; C)^\tau = D; C^\tau = (D; C^\tau)^\tau$. Assume that $D$ occurs in $C$. Then, $(C \cup D)^\tau = (D; C)^\tau = D; (C - D)^\tau = D; (C^\tau - D) = (D; C^\tau)^\tau$.
The equivalence $(\phi)$. We have the following equivalences:

**Theorem 1** For every $\phi \in \Phi_{\text{ConWON}}$, pointed model $(M, w)$, and sequential contexts $C_1$ and $C_2$ such that $C_1 = C_2$, $M, C_1, w \models \phi$ if and only if $M, C_2, w \models \phi$.

**Proof** We put an induction on $\phi$. We only consider the case $\phi = \alpha \psi$ and skip others. Assume that $M, C_1, w \models \alpha \psi$. Then, for every $u \models C_1 + \alpha$, $M, C_1 + \alpha, u \models \psi$. By Item 2 in Lemma 1, $(C_1 + \alpha)^T = (C_2 + \alpha)^T$. By Item 1 in Lemma 1, $\|C_1 + \alpha\| = \|C_2 + \alpha\|$. Then, for every $u \models C_2 + \alpha$, $M, C_2 + \alpha, u \models \psi$. Then, $M, C_2, w \models \alpha \psi$. The other direction is similar. □

The following two results can be easily shown and we omit their proofs.

**Lemma 2** Let $\phi$ be a formula in $\Phi_{\text{ConWON}}$. Let $M$ be a model, $C$ be a context, and $w$ be a state of $M$. Let $\mathbf{HI}(C) = (D_0, \ldots, D_n)$. Define a sequential context $f(C) = (\bigcap D_0, \ldots, \bigcap D_n)$. Then, $M, C, w \models \phi$ if and only if $M, f(C), w \models \phi$.

**Proof** It can be easily shown that (1) $\|C\| = \|f(C)\|$, and (2) for every $\alpha \in \Phi_{\text{PL}}$, $(f(C + \alpha))^T = (f(C) + \alpha)^T$.

We put an induction on $\phi$. We consider only the case $\phi = \alpha \psi$ and skip others. We have the following equivalences:

$$M, C, w \models [\alpha] \psi$$

$\iff$ $M, C + \alpha, u \models \psi$ for every $u \models C + \alpha$

$\iff^a$ $M, f(C + \alpha), u \models \psi$ for every $u \models f(C + \alpha)$

$\iff^b$ $M, f(C) + \alpha, u \models \psi$ for every $u \models f(C) + \alpha$

$\iff$ $M, f(C), w \models [\alpha] \psi$.

The equivalence $(a)$ holds by the fact (1) mentioned above and the inductive hypothesis. The equivalence $(b)$ holds by the fact (2) mentioned above, the first item of Lemma 1 and Theorem 1. □

**Lemma 3** Let $\phi$ be a formula in $\Phi_{\text{ConWON}}$. Let $M$ be a model, $C$ be a sequential context, and $w$ be a state of $M$. Let $C^\ast = (D_0, \ldots, D_n)$. Define a context $g(C) = (\mathbb{D}, >)$ where $\mathbb{D} = \{D_0, \ldots, D_n\}$ and $D_0 > \cdots > D_n$. Then, $M, C, w \models \phi$ if and only if $M, g(C), w \models \phi$.

**Proof** The following facts can be easily shown: (1) $\|C\| = \|g(C)\|$, and (2) for every $\alpha \in \Phi_{\text{PL}}$, $g(C + \alpha) = g(C) + \alpha$.

We put an induction on $\phi$. We consider only the case $\phi = \alpha \psi$ and skip others. We have the following equivalences:

$$M, C, w \models [\alpha] \psi$$

$\iff$ $M, C + \alpha, u \models \psi$ for every $u \models C + \alpha$

$\iff^a$ $M, g(C + \alpha), u \models \psi$ for every $u \models g(C + \alpha)$

$\iff^b$ $M, g(C) + \alpha, u \models \psi$ for every $u \models g(C) + \alpha$

$\iff$ $M, g(C), w \models [\alpha] \psi$.

The equivalence $(a)$ holds by the fact (1) mentioned above and the inductive hypothesis. The equivalence $(b)$ holds by the fact (2) mentioned above. □

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From Lemma 2 and 3, we can get the following result:

**Theorem 2** The class of contextualized pointed models \((\mathcal{M}, C, w)\), where \(C\) is a context, and the class of contextualized pointed models \((\mathcal{M}, C, w)\), where \(C\) is a sequential context, determine the same set of valid formulas.

In the sequel, we will use this semantics when technical points are involved. We will use \(\theta\) to indicate the special sequential context \(W\) for a model \(\mathcal{M} = (W, V)\).

### 3.5.2 Stronger Orders Among Defaults

In Definition 4 for contexts \((\mathcal{D}, >)\), \(>\) is only required to be irreflexive and transitive. In fact, we can require it to be a strict linear order without changing the set of valid formulas.

**Theorem 3** The class of contextualized pointed models \((\mathcal{M}, \mathcal{D}, >, w)\), where \(>\) is a strict linear order, and the class of contextualized pointed models \((\mathcal{M}, \mathcal{D}, >, w)\), where \(>\) is irreflexive and transitive, determines the same set of valid formulas.

**Proof** It suffices to show that for every formula \(\phi\) in \(\Phi_{\text{ConWON}}\) and contextualized pointed model \((\mathcal{M}, \mathcal{D}, >, w)\), where \(>\) is irreflexive and transitive, such that \(\mathcal{M}, \mathcal{D}, >, w \models \phi\), there is a contextualized pointed model \((\mathcal{M}, \mathcal{D}', >', w)\), where \(>\) is a strict linear order, such that \(\mathcal{M}, \mathcal{D}', >', w \models \phi\).

Let \(\phi\) be a formula in \(\Phi_{\text{ConWON}}\) and \((\mathcal{M}, \mathcal{D}, >, w)\) be a contextualized pointed model, where \(>\) is irreflexive and transitive, such that \(\mathcal{M}, \mathcal{D}, >, w \models \phi\). Let \((\mathcal{D}', >') = g(f(\mathcal{D}, >))\). We can see that \(>\) is a strict linear order. By Lemma 2 and 3, \(\mathcal{M}, \mathcal{D}, >, w \models \phi\) if and only if \(\mathcal{M}, \mathcal{D}', >', w \models \phi\). Then, \(\mathcal{M}, \mathcal{D}', >', w \models \phi\). \(\Box\)

By this result, every class of contextualized pointed model \((\mathcal{M}, \mathcal{D}, >, w)\) in between the two classes determines the same set of valid formulas. For example, we can require \(>\) to be irreflexive, transitive, and almost connected without changing the set of valid formulas.\(^3\)

### 4 Comparisons

As mentioned before, there are no theories explicitly handling conditional weak ontic necessity in the literature yet. In this section, we compare our theory to the following works on general conditionals, which are closely related to our theory: Veltman’s update semantics for counterfactual conditionals, Kratzer’s premise semantics for counterfactual conditionals, Kratzer’s semantics for conditional modalities, Stalnaker and Lewis’s ordering semantics for counterfactual conditionals, and Lewis’s conditional logic VN. The comparisons are mainly from the following perspective: Where do they differ in handling conditional weak ontic necessity?

Conceptually, our theory is close to Veltman’s update semantics and Kratzer’s premise semantics but different from Stalnaker and Lewis’s ordering semantics. Technically, the flat fragment of our logic is identical to the flat fragment of the logic VN.

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\(^3\) A binary relation \(<\) is almost connected if for all \(w, u\) and \(v\), if \(w < u\), then \(w < v\) or \(v < u\).
Our logic and the logic VN are different when nested conditionals are involved. As mentioned above, our semantics is global, and this coincides with whether weak ontic necessity holds at a state is not dependent on the state. All the related theories are local.

4.1 To Veltman’s Update Semantics for Counterfactual Conditionals

Veltman (2005) provided an update semantics for counterfactual conditionals, which combines the approaches from his earlier work (Veltman, 1976) and from Kratzer (1981). Veltman (2005) focused on the following question: When assessing a counterfactual conditional at the actual world, which facts about the actual world matter? Indeed, this is the central question of much previous literature, including (Lewis, 1979) and Kratzer (1981).

This work assumes an agent, who accepts some general laws, including natural laws, customs, etc. Veltman thought that general laws play a special role in interpreting counterfactual conditionals: “In making a counterfactual assumption we are not prepared to give up propositions we consider to be general laws.”

This work’s core idea is as follows: “Facts stand and fall together. In making a counterfactual assumption, we are prepared to give up everything that depends on something that we must give up to maintain consistency. But we want to keep in as many independent facts as we can.”

The key definition of this work is the following: A cognitive state $S$ supports a counterfactual conditional “If $\phi$ then $\psi$” ($S \models If \phi then \psi$) if and only if the cognitive state $S[If\phi]$, the result of updating $S$ with “If $\phi$”, supports $\psi$ ($S[If\phi] \models \psi$).

A cognitive state $S$ is defined as a binary tuple $(U, F)$, where $U$ and $F$ are sets of possible worlds such that $F \subseteq U$. A possible world $w$ in $U$ means that $w$ meets all the general laws accepted by the agent. $U$ is called the universe of $S$. A world $w$ is in $F$ if, for all the agent knows, $w$ might be the actual world. $F$ represents the agent’s information state about the actual world. There are three types of updates of a cognitive state $S = (U, F)$.

- The update of $S$ with a formula $\phi$ in the Propositional Logic ($S[\phi]$), is defined as $(U', F')$, where $U' = U$, and $F'$ consists of the possible worlds in $F$ meeting $\phi$. This means that $\phi$ only changes the agent’s information state about the actual world.
- The update of $S$ with a general law $\Box\phi$ ($S[\Box\phi]$), where $\phi$ is a formula in the Propositional Logic, is defined as $(U', F')$, where $U'$ consists of the possible worlds in $U$ meeting $\phi$, and $F'$ consists of the possible worlds in $F$ meeting $\phi$.
- The result of retracting the information of $\neg\phi$ from $S$ ($S \downarrow [\neg\phi]$), where $\phi$ is a formula in the Propositional Logic, is defined as a cognitive state $(U', F')$, where $U' = U$. The conditions that $F'$ meets are a bit complicated, and we do not go into the details.

$S[If\phi]$ is defined as $S \downarrow [\neg\phi][\phi]$.

Our semantics follows this work in the following respects. First, when evaluating a conditional, general laws accepted by the agent play an important role. Second,
when evaluating a conditional, we first update something with the antecedent and then evaluate its consequent against the update result.

Despite technical differences, the main difference between our work and Veltman’s work is as follows. The focus of Veltman’s work is which facts about the actual world matter for the evaluation of counterfactual conditionals, and in it, what “If φ” updates is the information state of the agent about the actual world. Our work focuses on conditional weak ontic necessity. In our work, what “If φ” updates is an ordered set of defaults concerning which alternatives of the actual world are expected. The difference cause different consequences. First, what the actual world is like does not matter for the evaluation of conditionals at it in our work. However, it matters in Veltman’s work. Second, the antecedent of a conditional is allowed to update accepted laws in our work but not allowed in Veltman’s work.

4.2 To Kratzer’s Premise Semantics for Counterfactual Conditionals

Kratzer (1979) presented a premise semantics for counterfactual conditionals, which was developed further in Kratzer (1981). The following words, which are from Kratzer (1981), show the core idea of this semantics: “The truth of counterfactuals depends on everything which is the case in the world under consideration: in assessing them, we have to consider all the possibilities of adding as many facts to the antecedent as consistency permits. If the consequent follows from every such possibility, then (and only then), the whole counterfactual is true.”

In this theory, the evaluation context for formulas is a tuple \((W, \Gamma, V, w)\), where \(W\) and \(V\) are as usual, \(w\) is a state in \(W\), and for every \(w \in W\), \(\Gamma(w)\) is a set of subsets of \(W\). Intuitively, the elements of \(\Gamma(w)\) are propositions, indicating premises for evaluating counterfactual conditionals at \(w\). Constraints can be introduced to \(\Gamma\). For example, it can be required that for every \(X \in \Gamma(w)\), \(w \in X\), which means that all the propositions in \(\Gamma(w)\) are real at \(w\).

Let \(\Delta\) be a set of sets of states and \(X\) be a set of states. Define \(K(\Delta, X)\) as \(\{\Delta' \cup \{X\} | \Delta' \text{ is a maximal subset of } \Delta \text{ such that } \bigcap(\Delta' \cup \{X\}) \neq \emptyset\}\).

A conditional “If φ then □ψ” is true at \((W, \Gamma, V, w)\) if and only if for every \(\Lambda\) in \(K(\Delta, |\phi|)\), \(\bigcap \Lambda \subseteq |\psi|\), where \(|\phi|\) and \(|\psi|\) are sets of states respectively satisfying \(\phi\) and \(ψ\).

Lewis (1981) showed that technically, premise semantics is equivalent to ordering semantics: Every set of premises at a state \(w\) can induce an equivalent ordering at \(w\), and for every ordering at \(w\), there is an equivalent set of premises at \(w\).

In our theory, the evaluation context is \((W, V, C, w)\). It shares similar ideas with \((W, \Gamma, V, w)\): Evaluation of conditionals involves additional premises. The main difference is that we introduce an order among premises.

Our semantics is dynamic in the following sense: When evaluating conditionals, states’ premises might change. Kratzer’s semantics is not dynamic in this sense. As a consequence, nested conditionals are handled differently in the two semantics. We look at an example. As mentioned below, the formula \(E(p \land q) \rightarrow \{p\}|q|(p \land q)\) is valid in our semantics. However, the following sentence is not valid in Kratzer’s theory: Given \(p\) and \(p\) could be true at the same time, if \(p\), then if \(q\), then \(p\) and \(q\).
The invalidity of this sentence can be shown by a pointed premise model transformed from the pointed ordering model given in Fact 1.

4.3 To Kratzer’s Semantics for Conditional Modalities

Influenced by Lewis (1981), Kratzer presented a general approach to deal with modalities in Kratzer (1991), which contains a way to handle conditional modalities.

In this work, the evaluation context for formulas is a tuple \((W, f, g, V, w)\), where \(W\) and \(V\) are as usual, \(w\) is a state in \(W\), and \(f\) and \(g\) are functions from \(W\) to \(P(P(W))\), respectively called the modal base and the ordering source. To ease the comparisons, we assume that \(W\) is finite.

For every \(w\) of \(W\), \(\bigcap f(w)\) is a set of states, intuitively understood as the set of accessible states to \(w\). For every \(w\) of \(W\), \(g(w)\) is a set of sets, which induces a binary relation \(\leq_w\) on \(W\) in the following way: For all \(u\) and \(v\), \(u \leq_w v\) if and only if for all \(X \in g(w)\), if \(v \in X\), then \(u \in X\). Intuitively, \(u \leq_w v\) means that \(u\) is at least as optimal as \(v\).

A necessity \(\Box \phi\) is true at \((W, f, g, V, w)\) if and only if for all \(\leq_w\)-minimal elements \(u\) of \(\bigcap f(w)\), \(\phi\) is true at \((W, f, g, V, u)\). A conditional necessity “If \(\phi\) then \(\Box \psi\)” is true at \((W, f, g, V, w)\) if and only if for all \(\leq_w\)-minimal elements \(u\) of \(|\phi| \cap \bigcap f(w)\), \(\psi\) is true at \((W, f, g, V, u)\), where \(|\phi|\) is the set of possibilities meeting \(\phi\).

Our work differs from Kratzer’s semantics for conditional necessities in the following respects. First, Kratzer’s theory uses two things to deal with conditional necessities, that is, a modal base and an ordering source. Our theory just uses one thing, that is, a context. Second, minimal elements are directly used in Kratzer’s theory but not in ours. Third, in Kratzer’s theory, the if-clause of a conditional necessity restricts the domain of accessible states. Consequently, conditionals are monotonic. In our theory, the if-clause of conditional weak ontic necessity updates contexts. Conditionals are not monotonic in our theory.

4.4 To Stalnaker and Lewis’s Ordering Semantics for Counterfactual Conditionals

Stalnaker and Lewis’s ordering semantics was proposed in Stalnaker (1968) and Lewis (1973). By the semantics, a conditional “If \(\phi\) then \(\Box \psi\)” is true at a possibility \(x\) if and only if \(\psi\) is true at all the possibilities that are most similar to \(x\) among the alternatives of \(x\) where \(\phi\) is true. In our theory, a conditional “If \(\phi\) then \(\Box \psi\)” is true at a possibility \(x\) relative to a context \(C\) if and only if \(\psi\) is true at all the possibilities in \(\parallel C + \phi \parallel\) relative to the context \(C + \phi\). Conceptually, our work is differs from Stalnaker and Lewis’s theory in two aspects. First, comparison among possible worlds plays an important role in Stalnaker and Lewis’s theory but does not in ours. Second, our theory uses contexts, while Stalnaker and Lewis’s theory does not.

We illustrate the first difference by an example adapted from Fine (1975).

**Example 5** Suppose that it is the time of Nixon as the president of the United States, and there has been no nuclear holocaust. Look at the following sentence:

\[\square \phi\]
If Nixon had pressed the button yesterday, then there should have been a nuclear holocaust.

Intuitively, this sentence is true.

By Stalnaker and Lewis’s theory, Sentence 17 is analyzed as follows. First, we get the set of alternatives to the present state where Nixon pressed the button yesterday. Second, we decide which states in the set resemble the present state the most. Third, we check whether there have been a nuclear holocaust at these states. The second step is hard and causes much controversy in the literature. In a natural sense, among the alternatives to the present state where Nixon pressed the button yesterday, those where there has been no nuclear holocaust resemble the present state the most. However, if so, Sentence 17 would be false. To avoid this, Lewis (1979) proposed some criteria to compare possible worlds.

Our theory deals with this sentence in the following way. It seems reasonable to think that the context of the sentence contains the following default: The command system of the United States works well. Add Nixon pressed the button yesterday as a default to the context with the highest priority. At all the expected states by the new context, there has been a nuclear holocaust. Thus, the sentence is true.

The second difference makes a difference for nested conditionals. We consider an example adapted from McGee (1985).

Example 6 Suppose that it is just after the 1980 election of the United States, and Reagan has won. Before the election, Reagan and Anderson were the only two Republic candidates, and Carter was a Democratic candidate. Opinion polls showed that Reagan was decisively ahead of the other candidates, and Carter was far ahead of the other candidates except Reagan. Here is a nested conditional:

If Reagan hadn’t won the election, then if a Republican had won, it should have been Reagan.

This sentence is clearly false.

By Stalnaker and Lewis’s theory, this sentence is true. We cite an argument for this from McGee (1985).

However, the possible world most similar to the actual world in which Reagan did not win the election will be a world in which Carter finished first and Reagan second, with Anderson again a distant third, and so a world in which “If a Republican had won it would have been Reagan” is true. Thus Stalnaker’s theory wrongly predicts that, in the actual world, “If Reagan hadn’t won the election, then if a Republican had won, it would have been Reagan” will be true.

This example is analyzed in our formalization as follows. Note that the result of opinion polls was that Reagan was decisively ahead of the other candidates, and Carter was far ahead of the other candidates except Reagan. Two defaults can represent the result: Reagan or Carter will win and Reagan will win, where the former is prior to the latter. We use r, a and c to respectively indicate “Reagan wins the election”, “Anderson wins the election” and “Carter wins the election”. Figure 3 indicates a
contextualized pointed model \((M, C, w_1)\) for this example, where \(C = (\mathbb{D}, \succ)\), where \(\mathbb{D}\) has two defaults: \(D_1 = \{w_1, w_3\}\) and \(D_2 = \{w_1\}\), and \(D_1 \succ D_2\).

The nested conditional is translated as \([\neg r] [r \lor a] r\). It is false at \((M, C, w_1)\). How?

- \(C + \neg r\), the update of \(C\) with \(\neg r\), is such that \(\| C + \neg r \|\), the set of expected states by \(C + \neg r\), equals to \(\{w_3\}\).
- \((C + \neg r) + (r \lor a)\), the update of \(C + \neg r\) with \(r \lor a\), is such that \(\|(C + \neg r) + (r \lor a)\|\), the set of expected states by \((C + \neg r) + (r \lor a)\), equals to \(\{w_2\}\).
- \(M, (C + \neg r) + (r \lor a), w_2 \not\models r\). Then, \(M, C, w_1 \not\models [\neg r] [r \lor a] r\).

4.5 To Lewis’s Conditional Logic VN

Lewis (1973) presented a class of conditional logics based on ordering semantics, and one of them is called VN. In what follows, we first present this logic and then compare it with ours.

**Definition 14** (Language \(\Phi_{VN}\)) The language \(\Phi_{VN}\) is defined as follows:

\[
\phi ::= p \mid \bot \mid \neg \phi \mid (\phi \land \phi) \mid (\phi \triangleright \phi)
\]

The formula \(\phi \triangleright \psi\) is for conditionals. Define \(A \phi\) as \(\neg \phi \triangleright \bot\), meaning \(\phi\) is necessary. Define \(E \phi\) as \(\neg A \neg \phi\), meaning \(\phi\) is possible.

**Definition 15** (Relational models for \(\Phi_{VN}\)) A tuple \(M = (W, \Gamma, V)\) is a relational model for \(\Phi_{VN}\) if

- \(W\) and \(V\) are as usual;
- For every \(w \in W\), \(\Gamma(w)\) is a tuple \((W_w, <_w)\), where \(W_w\) is a nonempty subset of \(W\), and \(<_w\) is an irreflexive, transitive, and almost connected binary relation on \(W_w\).

**Definition 16** (Relational semantics for \(\Phi_{VN}\)) Let \(M = (W, \Gamma, V)\) be a relational model.

\[M, w \models \phi \triangleright \psi \iff \text{for every } u \in W_w \cap |\phi|, \text{there is } v \in |\phi| \text{ such that } v \leq_w u \text{ and for every } z \in |\phi|, \text{if } z \leq_w v, \text{then } M, z \models \psi\]

The notion of validity of VN is defined as usual.

The logic VN has the finite model property (Burgess, 1981). Then, we can assume \(<_w\) to be well-founded without changing the set of valid formulas. Then, the semantics can be simplified as follows:
M, w \models \phi \wedge \psi \iff M, u \models \psi \text{ for every } <_w\text{-minimal element } u \text{ of } |\phi| \\
where |\phi| = \{x \mid M, x \models \phi\}.

There is a natural translation tr from \(\Phi_{\text{ConWON}}\) to \(\Phi_{\text{VN}}\), whose definition is omitted. Note that tr is not surjective, as nesting of conditionals in the antecedent is not allowed in \(\Phi_{\text{ConWON}}\) but is allowed in \(\Phi_{\text{VN}}\).

The flat fragment \(\Phi_{\text{VN}-1}\) of \(\Phi_{\text{VN}}\) is defined as follows, where \(\alpha \in \Phi_{\text{PL}}\):

**Definition 17** (The flat fragment \(\Phi_{\text{VN}-1}\) of \(\Phi_{\text{VN}}\))

\[\phi ::= p \mid \bot \mid \neg \phi \mid (\phi \wedge \phi) \mid \alpha \triangleright \alpha\]

It is easy to see that tr restricted to \(\Phi_{\text{ConWON}-1}\), the flat fragment of \(\Phi_{\text{ConWON}}\), is a bijective translation from \(\Phi_{\text{ConWON}-1}\) to \(\Phi_{\text{VN}-1}\).

The logics ConWON and VN share the same flat fragments up to the translation:

**Theorem 4** For every \(\Phi_{\text{ConWON}-1}\), \(\phi\) is valid in ConWON if and only if \(\text{tr}(\phi)\) is valid in VN.

The proof for this result can be found in Section A in the Appendix.

The logics ConWON and VN are different when nested conditionals are involved, as the following fact shows:

**Fact 1**

1. \(E(p \wedge q) \rightarrow [p][q](p \wedge q)\) is valid in ConWON.
2. \(E(p \wedge q) \rightarrow (p \triangleright (q \triangleright (p \wedge q)))\) is invalid in VN.

**Proof** 1. Assume that \(M, C, w \models E(p \wedge q)\) but \(M, C, w \not\models [p][q](p \wedge q)\). Then, there is \(u \in \|C + p\|\) such that \(M, C + p, u \not\models [q](p \wedge q)\). Then, there is \(v \in \|(C + p) + q\|\) such that \(M, (C + p) + q, v \not\models p \wedge q\). Then, \(v \not\models p \wedge q\). By Items 2. and 4. in Lemma 12, \(\|(C + p) + q\| \leq |p \wedge q|\). Then, \(v \not\models |p \wedge q|\). We have a contradiction.

2. Let \(M = (W, \Gamma, V)\) be a relational model for \(\Phi_{\text{VN}}\), where \(W = \{w_1, w_2\}, \Gamma(w_1) = \Gamma(w_2) = (W, <)\), where \(w_2 < w_1\), \(V(p) = \{w_1\}\) and \(V(q) = \{w_1, w_2\}\). We can see \(M, w_1 \models E(p \wedge q)\). Note that \(w_2\) is a \(<\)-minimal \(q\)-state but \(M, w_2 \not\models p \wedge q\). Then, \(M, w_1 \not\models q \triangleright (p \wedge q)\). Note that \(w_1\) is a minimal \(p\)-state. Then, \(M, w_1 \not\models p \triangleright (q \triangleright (p \wedge q))\).

\(\square\)

**5 Expressivity with Respect to Validity**

The language \(\Phi_{\text{ConWON}}\) is as expressive as its fragment \(\Phi_{\text{ConWON}-1}\) with respect to validity.

**Definition 18** (Closed formulas) Closed formulas of \(\Phi_{\text{ConWON}}\) are defined as follows, where \(\alpha \in \Phi_{\text{PL}}\) and \(\phi \in \Phi_{\text{ConWON}}\):

\[\chi ::= [\alpha]\phi \mid \neg \chi \mid (\chi \wedge \chi)\]
By the following fact, which is easy to verify, the truth value of a closed formula at a contextualized pointed model \((M, C, w)\) is independent of \(w\).

**Fact 2** Let \(\chi\) be a closed formula. Let \(M\) be a model and \(C\) be a context. Then, \(M, C, w \models \chi\) if and only if \(M, C, u \models \chi\) for all \(w\) and \(u\).

**Lemma 4** The following formulas are valid, where \(\alpha, \beta\) and \(\gamma\) are in \(\Phi_{PL}\):

1. \([\alpha](\phi \land \psi) \iff ([\alpha]\phi \land [\alpha]\psi)\)
2. \([\alpha](\phi \lor \chi) \iff ([\alpha]\phi \lor [\alpha]\chi),\) where \(\chi\) is a closed formula
3. \([\alpha][\beta]\gamma \iff (E\alpha \rightarrow ((E(\alpha \land \beta) \land [\alpha \land \beta]\gamma) \lor (\neg E(\alpha \land \beta) \land A(\beta \rightarrow \gamma))))\)
4. \([\alpha]\langle\beta\rangle\gamma \iff (E\alpha \rightarrow ((E(\alpha \land \beta) \land \langle \alpha \land \beta \rangle\gamma) \lor (\neg E(\alpha \land \beta) \land E(\beta \land \gamma))))\)

The proof for this result can be found in Section B in the Appendix.

**Theorem 5** There is an effective function \(\sigma\) from \(\Phi_{ConWON}\) to \(\Phi_{ConWON-1}\) such that for every \(\phi \in \Phi_{ConWON}, \phi \iff \sigma(\phi)\) is valid.

The proof for this result can be found in Section B in the Appendix.

**Remarks on Axiomatization of the Set of Valid Formulas**

We leave axiomatization of the set of valid formulas for another occasion. By Theorem 5, to axiomatize the set of valid formulas of \(\Phi_{ConWON}\) with respect to \(ConWON\), it suffices to axiomatize the set of valid formulas of \(\Phi_{ConWON-1}\) with respect to \(ConWON\). By Theorem 4, the set of valid formulas of \(\Phi_{ConWON-1}\) with respect to \(ConWON\) equals to the set of valid formulas of \(\Phi_{VN-1}\) with respect to \(VN\) up to the straightforward translation. Thus, it suffices to axiomatize the set of valid formulas of \(\Phi_{VN-1}\) with respect to \(VN\). Lewis (1973) provided a complete axiomatic system for the set of valid formulas of \(\Phi_{VN}\) with respect to \(VN\). It seems natural to guess that the result of restricting the system to \(\Phi_{VN-1}\) is complete for the set of valid formulas of \(\Phi_{VN-1}\) with respect to \(VN\).

**6 Expressivity with Respect to 0-Validity**

The language \(\Phi_{ConWON}\) is as expressive as its fragment \(\Phi_{A-1}\) with respect to 0-validity.

We first introduce some auxiliary notations. Let \(\delta\) be a nonempty finite sequence of formulas of \(\Phi_{PL}\) with \(n\) elements and \(1 \leq i \leq n\). We use \(\delta[0, i]\) to denote the initial segment of \(\delta\) with \(i\) elements. For every nonempty finite sequence of formulas \(\delta\), we use \(\bigwedge \delta\) to denote the conjunction of all the elements of \(\delta\). For every formula \(\phi\) and nonempty finite sequence of formulas \(\delta\), we use \(\phi; \delta\) to denote the result of prefixing \(\delta\) with \(\phi\).

**Definition 19** (A reduction function) Let \(\Delta\) be the set of nonempty finite sequences of formulas of \(\Phi_{PL}\). Define a function \(\Gamma\) from \(\Delta \times \Phi_{ConWON-1}\) to \(\Phi_{A-1}\) as follows, where \(\delta\) has \(n\) elements:
\[ \Gamma(\delta, \phi) = \phi, \text{ where } \phi \in \Phi_{PL} \]
\[ \Gamma(\delta, \neg\phi) = \neg\Gamma(\delta, \phi) \]
\[ \Gamma(\delta, (\phi \land \psi)) = (\Gamma(\delta, \phi) \land \Gamma(\delta, \psi)) \]
\[ \Gamma(\delta, [\alpha]\beta) = \\
= (E \land \alpha; \delta[0, n + 1] \rightarrow A(\land \alpha; \delta[0, n + 1] \rightarrow \beta)) \]
\[ \land \\
= (\neg E \land \alpha; \delta[0, n + 1] \land E \land \alpha; \delta[0, n]) \rightarrow A(\land \alpha; \delta[0, n] \rightarrow \beta)) \]
\[ \land \\
= \vdots \\
\[ \land \\
= (\neg E \land \alpha; \delta[0, 2] \land E \land \alpha; \delta[0, 1]) \rightarrow A(\land \alpha; \delta[0, 1] \rightarrow \beta)) \]

Fix a model \( M = (W, V) \). For every nonempty finite sequence \( \delta \) of formulas in \( \Phi_{PL} \), define a sequential context \( C^\delta \) as \((|\alpha_0|, \ldots, |\alpha_n|)\), where \( \delta = (\alpha_0, \ldots, \alpha_n) \).

**Lemma 5** Let \( \delta \) be a nonempty finite sequence of formulas in \( \Phi_{PL} \) and \( \phi \in \Phi_{ConWON-1} \). Then, for every model \( M \) and state \( w \), \( M, C^\delta, w \not\models \phi \) if and only if \( M, w \not\models \Gamma(\delta, \phi) \).

**Proof** We put an induction on \( \phi \). We only consider the case \( \phi = [\alpha]\beta \) and skip others.

Let \( M \) be a model and \( w \) be a state of \( M \). Suppose that \( \delta \) has \( n \) elements. We can see that one and only one of the following cases holds:

1. \( M, 0, w \models E \land \alpha; \delta[0, n + 1] \)
2. \( M, 0, w \models \neg E \land \alpha; \delta[0, n + 1] \land E \land \alpha; \delta[0, n] \)

\( (n + 1) \) \( M, 0, w \not\models E \land \alpha; \delta[0, 2] \land E \land \alpha; \delta[0, 1] \)

\( (n + 2) \) \( M, 0, w \models \neg E \land \alpha; \delta[0, 1] \)

We respectively consider these cases. Note that \( \Gamma(\delta, [\alpha]\beta) \) is a conjunction of \( n + 1 \) implications.

1. **Sub-case** \( M, 0, w \models E \land \alpha; \delta[0, n + 1] \). Note that in this case, all the implications in \( \Gamma(\delta, [\alpha]\beta) \) except the first one hold trivially. Also note that \( C^\delta + \alpha = C^{\alpha; \delta} \) and \( \|C^{\alpha; \delta}\| = |\land \alpha; \delta[0, n + 1]|. \) Then, the following equivalences hold:

\[ M, C^\delta, w \models [\alpha]\beta \]
\[ \iff M, C^\delta + \alpha, u \models \beta \text{ for every } u \in \|C^\delta + \alpha\| \]
\[ \iff M, C^\delta + \alpha, u \models \beta \text{ for every } u \in \|\land \alpha; \delta[0, n + 1]\| \]
\[ \iff M, 0, w \models A(\land \alpha; \delta[0, n + 1] \rightarrow \beta) \]
\[ \iff M, 0, w \models E \land \alpha; \delta[0, n + 1] \rightarrow A(\land \alpha; \delta[0, n + 1] \rightarrow \beta) \]
\[ \iff M, 0, w \models \Gamma(\delta, [\alpha]\beta) \]

2. **Sub-case** \( M, 0, w \not\models \neg E \land \alpha; \delta[0, n + 1] \land E \land \alpha; \delta[0, n] \). Note that in this case, all the implications in \( \Gamma(\delta, [\alpha]\beta) \) except the second one hold trivially. Also note that \( C^\delta + \alpha = C^{\alpha; \delta} \) and \( \|C^{\alpha; \delta}\| = |\land \alpha; \delta[0, n]|. \) Then, the following equivalences hold:
it seems natural to guess that the result of restricting such a system to \( \Phi_1 \)

It is easy to see that the set of 0-valid formulas of \( \Phi_1 \) equals to the set of valid formulas of \( \Phi_1 \)

For every \( \phi \in \Phi_{\text{ConWON-1}} \), \( \models_{0-\text{ConWON}} \phi \iff \Gamma(T, \phi) \).

Proof Let \( \phi \) be in \( \Phi_{\text{ConWON-1}} \), \( M = (W, V) \) be a model, and \( w \) be a state of \( M \). By Lemma 5, \( M, C_T \), \( w \models \phi \) if and only if \( M, 0, w \models \Gamma(T, \phi) \). Note that \( C_T = W \). By Lemma 2, \( M, 0, w \models \phi \) if and only if \( M, C_T, w \models \phi \). Then, \( M, 0, w \models \phi \iff \Gamma(T, \phi) \).

By Theorems 5 and 6, we know that \( \Phi_{\text{ConWON}} \) is as expressive as \( \Phi_{A-1} \) with respect to 0-validity.

Remarks on Axiomatization of the Set of 0-Valid Formulas

We leave axiomatization of the set of 0-valid formulas for another occasion. By Theorem 6, to axiomatize the set of 0-valid formulas of \( \Phi_{\text{ConWON}} \) with respect to ConWON, it suffices to axiomatize the set of 0-valid formulas of \( \Phi_{A-1} \) with respect to ConWON. It is easy to see that the set of 0-valid formulas of \( \Phi_{A-1} \) with respect to ConWON equals to the set of valid formulas of \( \Phi_{A-1} \) with respect to S5. Therefore, it suffices to axiomatize the set of valid formulas of \( \Phi_{A-1} \) with respect to S5. Complete axiomatic systems for the set of valid formulas of \( \Phi_A \) with respect to S5 are well known. Again, it seems natural to guess that the result of restricting such a system to \( \Phi_{A-1} \) is complete for the set of valid formulas of \( \Phi_{A-1} \) with respect to S5.

7 Looking Backward and Forward

In this work, we presented a logical theory for conditional weak ontic necessity. It has the following features. It introduces contexts, which are sets of ordered defaults. Contexts determine which possible states of the world are expected. Conditionals are evaluated relative to contexts, and their if-clauses change contexts.

There is work worth doing in the future. Ontic possibilities are deeply related to time: They are possible states in which our world could be if things had gone differently in the past. Consequently, time plays an important role in conditional weak ontic necessity. However, our formalization is just a slice of the time flow and cannot explicitly handle the temporal dimension of conditional weak ontic necessity. For example, consider the scenario of Example 1 and the following three sentences:
(19) \textit{If the goat is still alive tomorrow, then the dog should be dead.}
(15) \textit{If the goat is still alive now, then the dog should be dead.}
(16) \textit{If the goat were still alive now, then the dog should be dead.}

There are clear connections between the utterance of the first sentence yesterday and the utterances of the last two today, but our formalization cannot capture them. We leave the introduction of temporality as future work.

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\textbf{Declarations}

\textbf{Ethical approval} This article does not contain any studies with human participants or animals performed by any of the authors.

\textbf{Conflict of interest} Fengkui Ju declares that he has no conflict of interest.

\section*{A Proofs About Comparisons to the Conditional Logic VN}

\subsection*{A.1 Finite Universal Relational Models for $\Phi_{VN}$}

Note that $\Phi_{VN-1}$ contains no nested conditionals. This implies that we can just consider universal relational models without changing the set of valid formulas in $\Phi_{VN-1}$.

\textbf{Definition 20} (Universal relational models for $\Phi_{VN}$) A tuple $M = (W, <, V)$ is a universal relational model for $\Phi_{VN}$ if

- $W$ and $V$ are as usual;
- $<$ is a well-founded, irreflexive, transitive, almost connected binary relation on $W$.

\textbf{Lemma 6} The class of universal relational models and the class of relational models determine the same set of valid formulas in $\Phi_{VN-1}$.

\subsection*{A.2 Rephrasing the Semantics for $\Phi_{VN}$}

\textbf{Fact 3} Let $M = (W, <, V)$ be a universal relational model for $\Phi_{VN}$. Define a relation $\equiv$ on $W$ as follows: For all $w$ and $u$, $w \equiv u$ if and only if $w \not< u$ and $u \not< w$. Then, $\equiv$ is an equivalence relation. Let $\Delta_W$ be the partition of $W$ under $\equiv$. Define a relation $\ll$ on $\Delta_W$ as follows: For all $X$ and $Y$, $X \ll Y$ if and only if for all $x \in X$ and $y \in Y$, $x < y$. Then, $\ll$ is a well-founded strict well-ordering on $\Delta_W$.

\textbf{Definition 21} (Sphere models for $\Phi_{VN}$) A tuple $M = (W, \Delta_W, \ll, V)$ is a sphere model for $\Phi_{VN}$ if

- $W$ and $V$ are as usual;
- $\Delta_W$ is a partition of $W$ and $\ll$ is a well-founded strict well-ordering on $\Delta_W$.
Definition 22 (Sphere semantics for $\Phi_{VN}$) Let $M = (W, \Delta_W, \ll, V)$ be a sphere model.

$M, w \models \phi \supset \psi \iff$ if there is $X \in \Delta_W$ such that $X \cap |\phi| \neq \emptyset$, then $X' \cap |\phi| \subseteq |\psi|$, where $X'$ is the $\ll$-least element in $\Delta_W$ such that $X' \cap |\phi| \neq \emptyset$

where $|\phi| = \{x \mid M, x \models \phi\}$ and $|\psi| = \{x \mid M, x \models \psi\}$.

It can be verified that the following result holds:

Lemma 7 Sphere semantics is equivalent to relational semantics for $\Phi_{VN}$.

A.3 Rephrasing the Semantics for $\Phi_{VN}$ Again

Definition 23 (Pseudo sphere models for $\Phi_{VN}$) A tuple $M = (W, \Pi, V)$ is a pseudo sphere model for $\Phi_{VN}$ if

- $W$ and $V$ are as usual;
- $\Pi = (X_0, \ldots, X_n, \ldots)$ is a sequence of pairwise disjoint (possibly empty) subsets of $W$ such that the union of them is $W$.

Definition 24 (Pseudo sphere semantics for $\Phi_{VN}$) Let $M = (W, \Pi, V)$ be a pseudo sphere model.

$M, w \models \phi \supset \psi \iff$ if there is $X$ in $\Pi$ such that $X \cap |\phi| \neq \emptyset$, then $X_l \cap |\phi| \subseteq |\psi|$, where $l$ is the least number such that $X_l \cap |\phi| \neq \emptyset$

where $|\phi| = \{x \mid M, x \models \phi\}$ and $|\psi| = \{x \mid M, x \models \psi\}$.

It can be verified that the following result holds:

Lemma 8 Pseudo sphere semantics is equivalent to sphere semantics for $\Phi_{VN}$.

By the following result, which is easy to show, empty elements in $\Pi$ do not matter in pseudo sphere semantics.

Lemma 9 Let $(M, w)$ and $(M', w)$ be two pointed pseudo sphere models, where $M = (W, \Pi, V)$ and $M' = (W, \Pi', V)$. Assume that $\Pi$ and $\Pi'$ are identical if we remove all the empty elements in them. Then, $(M, w)$ and $(M', w)$ are equivalent for $\Phi_{VN}$.

A.4 Two Transformation Lemmas

Lemma 10 Let $W$ be a nonempty set of states. Let $Y_0, \ldots, Y_k$ be a sequence of pairwise disjoint nonempty subsets of $W$ such that $Y_0 \cup \cdots \cup Y_k = W$. Define a sequence $X_0, \ldots, X_k$ as follows:

- $X_0 = Y_0 \cup \cdots \cup Y_k$
- $X_1 = Y_1 \cup \cdots \cup Y_k$
- $\vdots$
- $X_k = Y_k$

1. Let $Z \subseteq W$. Let $l$ be the greatest number such that $l \leq k$ and $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$. Then, $Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l$ and $l$ is the greatest number such that $Z \cap Y_l \neq \emptyset$. 

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2. Let $Z \subseteq W$. Let $l$ be the greatest number such that $l \leq k$ and $Z \cap Y_l \neq \emptyset$. Then, $Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l$ and $l$ is the greatest number such that $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$.

Proof 1. Assume that $l = k$. Note that $X_0 \cap \cdots \cap X_k = X_k = Y_k$. It is easy to see that the result holds.

Assume that $l < k$.
We first show that $l$ is the greatest number such that $Z \cap Y_l \neq \emptyset$.
Note that $X_0 \cap \cdots \cap X_l = X_l$. Then, $Z \cap X_l \neq \emptyset$. Note that $Z \cap X_0 \cap \cdots \cap X_{l+1} = \emptyset$ and $X_0 \cap \cdots \cap X_{l+1} = X_{l+1}$. Then, $Z \cap X_{l+1} = \emptyset$. Note that $X_l = Y_l \cup \cdots \cup Y_k$ and $X_{l+1} = Y_{l+1} \cup \cdots \cup Y_k$. Then, $Z \cap Y_{l+1} = \emptyset, \ldots, Z \cap Y_k = \emptyset$. Then, $Z \cap Y_l \neq \emptyset$. Then, $l$ is the greatest number such that $Z \cap Y_l \neq \emptyset$.
Second, we show $Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l$.
Let $a \in Z \cap X_0 \cap \cdots \cap X_l$. Then, $a \in Z \cap X_l$. Note that $X_l = Y_l \cup \cdots \cup Y_k$. Then, $a \in Z \cap Y_l \cup \cdots \cup Y_k$. We claim $a \notin Y_{l+1}, \ldots, a \notin Y_k$. Why? Suppose that $a \in Y_{l+1}$.
Note that $X_{l+1} = Y_{l+1} \cup \cdots \cup Y_k$. Then, $a \in X_{l+1}$. Then, $a \in Z \cap X_0 \cap \cdots \cap X_{l+1}$. Then, $l$ is not the greatest number such that $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$. We have a contradiction.
Similarly, we know $a \notin Y_{l+1}, \ldots, a \notin Y_k$. Then, $a \in Y_l$. Then, $a \in Z \cap Y_l$.
Let $a \in Z \cap Y_l$. By the definitions of $X_0, \ldots, X_l$, we know $a \in X_0, \ldots, a \in X_l$. Then, $a \in Z \cap X_0 \cap \cdots \cap X_l$.

2. Note that $X_0 \cap \cdots \cap X_l = X_l = Y_l \cup \cdots \cup Y_k$ and $X_0 \cap \cdots \cap X_{l+1} = X_{l+1} = Y_{l+1} \cup \cdots \cup Y_k$. Also note that $Z \cap Y_{l+1} = \emptyset, \ldots, Z \cap Y_k = \emptyset$.
We first show that $l$ is the greatest number such that $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$.
Note that $l$ is the greatest number such that $Z \cap Y_l \neq \emptyset$. Then, $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$.
Then, $Z \cap X_{l+1} = \emptyset$. Then, $Z \cap X_0 \cap \cdots \cap X_l \cap X_{l+1} = \emptyset$. Then, $l$ is the greatest number such that $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$.
Second, we show $Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l$.
Let $a \in Z \cap X_0 \cap \cdots \cap X_l$. Then, $a \in Z \cap X_l$. Then, $a \in Z \cap (Y_l \cup \cdots \cup Y_k)$. Note that $l$ is the greatest number such that $Z \cap Y_l \neq \emptyset$. Then, $a \in Z \cap Y_l$.
Let $a \in Z \cap Y_l$. Then, $a \in Y_l \cup \cdots \cup Y_k = X_l = X_0 \cap \cdots \cap X_l$. Then, $a \in Z \cap X_0 \cap \cdots \cap X_l$.

Lemma 11 Let $W$ be a nonempty set of states. Let $X_0, \ldots, X_k$ be a sequence of subsets of $W$, where $X_0 = W$. Define a sequence $Y_0, \ldots, Y_k$ as follows:

- $Y_0 = X_0 - X_1$
- $Y_1 = X_0 \cap X_1 - X_2$
  ...
- $Y_k = X_0 \cap \cdots \cap X_k$

1. Then, $Y_0, \ldots, Y_k$ are pairwise disjoint and $Y_0 \cup \cdots \cup Y_k = W$.

2. Let $Z \subseteq W$. Let $l$ be the greatest number such that $l \leq k$ and $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$. Then, $Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l$ and $l$ is the greatest number such that $Z \cap Y_l \neq \emptyset$.

3. Let $Z \subseteq W$. Let $l$ be the greatest number such that $l \leq k$ and $Z \cap Y_l \neq \emptyset$. Then, $Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l$ and $l$ is the greatest number such that $Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset$. 

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Proof 1. Let \( i, j \leq k \) be such that \( i \neq j \). We want to show \( Y_i \cap Y_j = \emptyset \). Without loss of any generality, assume that \( i < j \). Assume that \( Y_i \cap Y_j \neq \emptyset \). Then, there is \( a \) such that \( a \in Y_i \) and \( a \in Y_j \). Note that \( Y_i = X_0 \cap \cdots \cap X_i - X_{i+1} \) and \( Y_j = X_0 \cap \cdots \cap X_j - X_{j+1} \). Then, \( a \notin X_{i+1} \) and \( a \in X_{i+1} \). There is a contradiction. Let \( a \in W \). Let \( l \) be the greatest number such that \( a \in X_0 \cap \cdots \cap X_l \). Note that \( l \) exists, as \( X_0 = W \). Suppose that \( l = k \). Then, \( a \in Y_k \). Suppose that \( l < k \). Then, \( a \notin X_{l+1} \). Then, by the definition of \( Y_l \), \( a \in Y_l \).

2. We first show \( Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l \). Let \( a \in Z \cap X_0 \cap \cdots \cap X_l \). As \( l \) is the greatest number such that \( Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset \), \( a \notin X_{l+1} \). Note that \( Y_l = X_0 \cap \cdots \cap X_l - X_{l+1} \). Then, \( a \notin Y_l \). Then, \( a \in Z \cap Y_l \). Let \( a \in Z \cap Y_l \). By the definition of \( Y_l \), \( a \in X_0 \cap \cdots \cap X_l \). Then, \( a \in Z \cap X_0 \cap \cdots \cap X_l \). Second, we show that \( l \) is the greatest number such that \( Z \cap Y_l \neq \emptyset \). Assume that there is a natural number \( l' > l \) such that \( l' \leq k \) and \( Z \cap Y_{l'} \neq \emptyset \). Let \( a \in Z \cap Y_{l'} \). By the definition of \( Y_{l'} \), \( a \in Z \cap X_0 \cap \cdots \cap X_{l'} \). Then, \( l \) is not the greatest number such that \( Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset \). There is a contradiction.

3. We first show that \( l \) is the greatest number such that \( Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset \). Assume that there is a natural number \( l' > l \) such that \( l' \leq k \) and \( l' \) is the greatest number such that \( Z \cap X_0 \cap \cdots \cap X_l' \neq \emptyset \). Assume that \( l' = k \). Note that \( Y_k = X_0 \cap \cdots \cap X_k \). Then, \( Z \cap Y_k \neq \emptyset \). Then, \( l \) is not the greatest number such that \( Z \cap Y_l \neq \emptyset \). We have a contradiction. Assume that \( l' < k \). Let \( a \in Z \cap X_0 \cap \cdots \cap X_{l'} \). As \( l' \) is the greatest number such that \( Z \cap X_0 \cap \cdots \cap X_{l'} \neq \emptyset \), we know \( a \notin X_{l'+1} \). By the definition of \( Y_{l'} \), \( a \in Z \cap Y_{l'} \). Then, \( l \) is not the greatest number such that \( Z \cap Y_l \neq \emptyset \). We have a contradiction. Second, we show that \( Z \cap X_0 \cap \cdots \cap X_l = Z \cap Y_l \). Let \( a \in Z \cap X_0 \cap \cdots \cap X_l \). As \( l \) is the greatest number such that \( Z \cap X_0 \cap \cdots \cap X_l \neq \emptyset \), \( a \notin X_{l+1} \). By the definition of \( Y_l \), \( a \in Y_l \). Then, \( a \in Z \cap Y_l \). Let \( a \in Z \cap Y_l \). By the definition of \( Y_l \), \( a \in X_0 \cap \cdots \cap X_l \). Then, \( a \in Z \cap X_0 \cap \cdots \cap X_l \).

A.5 The Equivalence Theorem

Theorem 4 For every \( \Phi_{\text{ConWON-1}} \), \( \phi \) is valid in ConWON if and only if \( \text{tr}(\phi) \) is valid in \( VN \).

Proof Let \( \alpha \) and \( \beta \) be in \( \Phi_{\text{PL}} \). It suffices to show that \( \alpha \models \beta \) is satisfiable in \( VN \) if and only if \([\alpha]\beta \) is satisfiable in ConWON.

Assume that \( \alpha \models \beta \) is satisfiable in \( VN \). Then, \( \alpha \models \beta \) is true at a pointed sphere model for \( \Phi_{VN} \). As \( \Phi_{VN} \) has the finite model property, \( \alpha \models \beta \) is true at a pointed finite sphere model \( (M, w) \) for \( \Phi_{VN} \). Let \( M = (W, \Pi, V) \), where \( \Pi = (Y_k, \ldots, Y_0) \).

Define \( M' = (W, V) \), which is a model for ConWON. From the sequence \((Y_0, \ldots, K_k)\), define a context \( C = (X_0, \ldots, X_k) \) for \( M' \) as follows:

- \( X_0 = Y_0 \cup \cdots \cup Y_k \)
- \( X_1 = Y_1 \cup \cdots \cup Y_k \)
- \( \vdots \)
- \( X_k = Y_k \)
Assume that $|\alpha| = \emptyset$. Let $x \in W$. Then, $M', W, x \models [\alpha] \beta$. Here, $W$ indicates a special context. Then, $[\alpha] \beta$ is satisfiable in ConWON.

Assume that $|\alpha| \neq \emptyset$. As $Y_0 \cup \cdots \cup Y_k = W$, there is $Y_l$ such that $|\alpha| \cap Y_l \neq \emptyset$. Let $l$ be the greatest number such that $l \leq k$ and $|\alpha| \cap Y_l \neq \emptyset$. Note that $M, w \models \alpha \triangleright \beta$. Then, $[\alpha] \cap Y_l \subseteq [\beta]$. By Lemma 10, $|\alpha| \cap X_0 \cap \cdots \cap X_l = |\alpha| \cap Y_l$ and $l$ is the greatest number such that $|\alpha| \cap X_1 \cap \cdots \cap X_l \neq \emptyset$. Then, $\|C + \alpha\| = |\alpha| \cap X_1 \cap \cdots \cap X_l$. Then, $\|C + \alpha\| \subseteq |\beta|$. Let $x \in \|C + \alpha\|$. Then, $M', C, x \models [\alpha] \beta$. Then, $[\alpha] \beta$ is satisfiable in ConWON.

Assume that $[\alpha] \beta$ is satisfiable in ConWON. Then, $[\alpha] \beta$ is true at a contextualized pointed model $(M, C, w)$ for ConWON. Then, $\|C + \alpha\| \subseteq |\beta|$. Let $M = (W, V)$. Define $C' = C \ominus W$. Let $C' = (X_0, \ldots, X_k)$. Note that $X_0 = W$. It can be verified that $\|C + \alpha\| = \|C' + \alpha\|$. Then, $\|C + \alpha\| \subseteq |\beta|$. Define a sequence $(Y_0, \ldots, Y_k)$ as follows:

- $Y_0 = X_0 - X_1$
- $Y_1 = X_0 \cap X_1 - X_2$
- ...
- $Y_k = X_0 \cap \cdots \cap X_k$

By Lemma 11, $Y_0, \ldots, Y_k$ are pairwise disjoint and $Y_0 \cup \cdots \cup Y_k = W$. Let $\Pi = (Y_k, \ldots, Y_0)$. Define $M' = (W, \Pi, V)$, which is a pseudo sphere model for VN.

Assume that $|\alpha| \neq \emptyset$. Then, there is no $Y_i$ in $\Pi$ such that $|\alpha| \cap Y_i \neq \emptyset$. Let $x \in W$. Then, $M', x \models \alpha \triangleright \beta$. Then, $\alpha \triangleright \beta$ is satisfiable in VN.

Assume that $|\alpha| \neq \emptyset$. Then, $\|C' + \alpha\| \neq \emptyset$. Let $\|C' + \alpha\| = |\alpha| \cap X_0 \cap \cdots \cap X_l$. By Lemma 11, $|\alpha| \cap X_0 \cap \cdots \cap X_l = |\alpha| \cap Y_l$ and $l$ is the greatest number such that $l \leq k$ and $|\alpha| \cap Y_l \neq \emptyset$. Note that $\|C + \alpha'\| \subseteq |\beta|$. Then, $|\alpha| \cap Y_l \subseteq |\beta|$. Let $x \in W$. Then, $M', x \models \alpha \triangleright \beta$. Then, $\alpha \triangleright \beta$ is satisfiable in VN.

**B Proofs About Expressivity with Respect to Validity**

**Lemma 12** Fix a model $M$.

1. $\|C + \alpha\| = \emptyset$ if and only if $|\alpha| = \emptyset$.
2. $\|(C + \alpha) + \beta\| = \|C + (\alpha \land \beta)\|$, where $|\alpha \land \beta| \neq \emptyset$.
3. $\|(C + \alpha) + \beta\| = \|\theta + \beta\|$, where $|\alpha \land \beta| = \emptyset$.
4. $\|C + \alpha\| \subseteq |\alpha|$.

This result is easy to show.

**Lemma 13** Let $\chi$ be a closed formula. Assume that $\|C + \alpha\| \neq \emptyset$. Then, for all $w$ and $u$ of $M, C, w \models [\alpha] \chi$ if and only if $M, C + \alpha, u \models \chi$.

**Proof** Assume that $M, C, w \not\models [\alpha] \chi$. Then, $M, C + \alpha, x \not\models \chi$ for some $x \in \|C + \alpha\|$. Then, $M, C + \alpha, u \not\models \chi$. Assume that $M, C + \alpha, u \not\models \chi$. Let $x \in \|C + \alpha\|$. Then, $M, C + \alpha, x \not\models \chi$. Then, $M, C, w \not\models [\alpha] \chi$.

**Lemma 4** The following formulas are valid, where $\alpha, \beta$ and $\gamma$ are in $\Phi_{PL}$:

\[\square\ Springer\]
1. \([\alpha](\phi \land \psi) \leftrightarrow ([\alpha]\phi \land [\alpha]\psi)\)
2. \([\alpha](\phi \lor \chi) \leftrightarrow ([\alpha]\phi \lor [\alpha]\chi)\), where \(\chi\) is a closed formula
3. \([\alpha](\beta)\gamma \leftrightarrow (E\alpha \rightarrow ((E(\alpha \land \beta) \land [\alpha \land \beta])\gamma) \lor (\neg (E(\alpha \land \beta) \land A(\beta \rightarrow \gamma))))\)
4. \([\alpha](\gamma)\beta \leftrightarrow (E\alpha \rightarrow ((E(\alpha \land \beta) \land [\alpha \land \beta])\gamma) \lor (\neg (E(\alpha \land \beta) \land E(\beta \land \gamma))))\)

**Proof** 1. This item is easy.

2. Assume that \(M, C, w \not\models [\alpha]\phi \lor [\alpha]\chi\). Then, \(M, C, w \not\models [\alpha]\phi\) and \(M, C, w \not\models [\alpha]\chi\). Then, there is \(u \in C + \alpha\) such that \(M, C + \alpha, u \not\models \phi\) and there is \(v \in C + \alpha\) such that \(M, C + \alpha, v \not\models \chi\). As \(\chi\) is a closed formula, \(M, C + \alpha, u \not\models \chi\). Then, \(M, C + \alpha, u \not\models \phi \lor \chi\). Then, \(M, C, w \not\models [\alpha](\phi \lor \chi)\). The other direction is easy.

3. Assume that \(M, C, w \not\models E\alpha\). Then, both sides of the equivalence hold at \((M, C, w)\) trivially.

Assume that \(M, C, w \models E\alpha\) and \(M, C, w \models E(\alpha \land \beta)\). Note that \((C + \alpha) \not= \emptyset\) by item 1. in Lemma 12. Also note that \(\|(C + \alpha) + \beta\| = \|(C + \alpha + \beta)\|\) by Item 2. in Lemma 12.

Assume that \(M, C, w \models [\alpha](\beta)\gamma\). Let \(u \in C + \alpha\). By Lemma 13, \(M, C + \alpha, u \models [\beta]\gamma\). Then, for every \(v \in C + \alpha\), \(M, C + \alpha, v \models \gamma\). Then, \(M, C, w \models [\alpha \land \beta]\gamma\). Then, for every \(u \in C + \alpha\), \(M, C + \alpha, u \models \gamma\). Then, \(M, C, w \models [\alpha \land \beta]\gamma\). Then, for every \(v \in C + \alpha\), \(M, C + \alpha, v \models \gamma\). Then, \(M, C, w \models [\alpha \land \beta]\gamma\). Then, for every \(u \in C + \alpha\), \(M, C + \alpha, u \models \gamma\). Then, \(M, C, w \models [\alpha\beta]\gamma\).

4. Assume that \(M, C, w \not\models E\alpha\). Then, both sides of the equivalence hold at \((M, C, w)\) trivially.

Assume that \(M, C, w \not\models E\alpha\) and \(M, C, w \not\models E(\alpha \land \beta)\). Note that \((C + \alpha) \not= \emptyset\) by item 1. in Lemma 12. Also note that \(\|(C + \alpha) + \beta\| = \|(C + \alpha + \beta)\|\) by Item 3. in Lemma 12.

Assume that \(M, C, w \models [\alpha\beta]\gamma\). Let \(u \in C + \alpha\). By Lemma 13, \(M, C + \alpha, u \models [\beta]\gamma\). Then, for every \(v \in \theta + \beta\), \(M, \theta + \beta, v \models \gamma\). Then, \(M, C, w \models A(\beta \rightarrow \gamma)\). Then, \(M, C, w \models E\alpha \rightarrow ((E(\alpha \land \beta) \land [\alpha \land \beta])\gamma) \lor (\neg (E(\alpha \land \beta) \land A(\beta \rightarrow \gamma)))\).

Assume that \(M, C, w \models [\alpha\beta]\gamma\). Then, for every \(u \in \theta + \beta\), \(M, \theta + \beta, u \models \gamma\). Then, \(M, C, w \models [\alpha\beta]\gamma\). Then, for every \(u \in \theta + \beta\), \(M, \theta + \beta, u \models \gamma\). Then, \(M, C, w \models [\alpha\beta]\gamma\).

4. Assume that \(M, C, w \not\models E\alpha\). Then, both sides of the equivalence hold at \((M, C, w)\) trivially.
Let $\gamma$. Then, $u \in \| (C + \alpha) + \beta \|$ and $M, (C + \alpha) + \beta, u \models \gamma$. Let $v \in \| C + \alpha \|$. Then, $M, C + \alpha, v \models (\beta)\gamma$. By Lemma 13, $M, C, w \models [\alpha](\beta)\gamma$.

Assume that $M, C, w \models E\alpha$ and $M, C, w \not\models E(\alpha \land \beta)$. Note that $\| C + \alpha \| \neq \emptyset$ by item 1. in Lemma 12. Also note that $\| (C + \alpha) + \beta \| = \| \theta + \beta \|$ by Item 3. in Lemma 12.

Assume that $M, C, w \models [\alpha](\beta)\gamma$. Let $u \in \| C + \alpha \|$. By Lemma 13, $M, C + \alpha, u \models (\beta)\gamma$. Then, there is $v \in \| (C + \alpha) + \beta \|$ such that $M, (C + \alpha) + \beta, v \models \gamma$. Then, $v \in \| \theta + \beta \|$ and $M, \theta + \beta, v \models \gamma$. Then, $M, C, w \models E(\beta \land \gamma)$. Then, $M, C, w \models E\alpha \rightarrow ((E(\alpha \land \beta) \land (\alpha \land \beta)\gamma) \lor (\neg E(\alpha \land \beta) \land E(\beta \land \gamma)))$. Assume that $M, C, w \models E\alpha \rightarrow ((E(\alpha \land \beta) \land (\alpha \land \beta)\gamma) \lor (\neg E(\alpha \land \beta) \land E(\beta \land \gamma)))$. Then, $M, C, w \models E(\beta \land \gamma)$. Then, there is $u \in \| \theta + \beta \|$ such that $M, \theta + \beta, u \models \gamma$. Then, $u \in \| (C + \alpha) + \beta \|$ and $M, (C + \alpha) + \beta, u \models \gamma$. Let $v \in \| C + \alpha \|$. Then, $M, C + \alpha, v \models (\beta)\gamma$. By Lemma 13, $M, C, w \models [\alpha](\beta)\gamma$. □

Theorem 5 There is an effective function $\sigma$ from $\Phi_{ConWON}$ to $\Phi_{ConWON-1}$ such that for every $\phi \in \Phi_{ConWON}$, $\phi \leftrightarrow \sigma(\phi)$ is valid.

Proof We define the modal depth of formulas of $\Phi_{ConWON}$ with respect to $[\cdot]$ in the usual way.

Pick a formula $\phi$ in $\Phi_{ConWON}$. Repeat the following steps until we cannot proceed.

- Pick a sub-formula $[\alpha]\psi$ of $\phi$ whose modal depth with respect to $[\cdot]$ is 2 if $\phi$ has such a sub-formula.
- Transform $\psi$ to $\chi_1 \land \cdots \land \chi_n$, where all $\chi_i$ is in the form of $(\beta_1 \lor \cdots \lor \beta_k) \lor ([\gamma_1]\lambda_1 \lor \cdots \lor [\gamma_l]\lambda_l) \lor ([\eta_1]\theta_1 \lor \cdots \lor [\eta_m]\theta_m)$, where all $\beta_i, \gamma_i, \lambda_i, \eta_i$ and $\theta_i$ are in $\Phi_{PL}$.

Note that $[\alpha]\psi \leftrightarrow ([\alpha]\chi_1 \land \cdots \land [\alpha]\chi_n)$ is valid by Item 1. in Lemma 4. Repeat the following steps until we cannot proceed:

- From $[\alpha]\chi_1 \land \cdots \land [\alpha]\chi_n$, pick a conjunct $[\alpha]\chi_i = [\alpha]((\beta_1 \lor \cdots \lor \beta_k) \lor ([\gamma_1]\lambda_1 \lor \cdots \lor [\gamma_l]\lambda_l) \lor ([\eta_1]\theta_1 \lor \cdots \lor [\eta_m]\theta_m))$.
- By Item 2. in Lemma 4, $[\alpha]\chi_i \leftrightarrow \xi$ is valid in $ConWON$, where $\xi = [\alpha](\beta_1 \lor \cdots \lor \beta_k) \lor ([\alpha][\gamma_1]\lambda_1 \lor \cdots \lor [\alpha][\gamma_l]\lambda_l) \lor ([\alpha][\eta_1]\theta_1 \lor \cdots \lor [\alpha][\eta_m]\theta_m)$. In the ways specified by Items 3. and 4. in Lemma 4, transform $\xi$ to $\xi'$, whose modal depth with respect to $[\cdot]$ is 1.
- Replace $[\alpha]\chi_i$ by $\xi'$ in $[\alpha]\chi_1 \land \cdots \land [\alpha]\chi_n$.

Define $\sigma(\phi)$ as the result. It is easy to see that $\sigma(\phi)$ is in $\Phi_{ConWON-1}$ and $\phi \leftrightarrow \sigma(\phi)$ is valid. □

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