Conformal mass in Einstein-Gauss-Bonnet AdS gravity

Dileep P. Jatkar\textsuperscript{1}\textsuperscript{*}, Georgios Kofinas \textsuperscript{2}†, Olivera Miskovic\textsuperscript{3}‡, Rodrigo Olea\textsuperscript{4}§

\textsuperscript{1}Harish-Chandra Research Institute,  
Chhatnag Road, Jhunsi, Allahabad 211019, India  
\textsuperscript{2}Research Group of Geometry, Dynamical Systems and Cosmology  
Department of Information and Communication Systems Engineering  
University of the Aegean, Karlovassi 83200, Samos, Greece  
\textsuperscript{3}Instituto de Física, Pontificia Universidad Católica de Valparaíso,  
Casilla 4059, Valparaíso, Chile  
\textsuperscript{4}Departamento de Ciencias Físicas,  
Universidad Andres Bello, República 220, Santiago, Chile

January 28, 2015

ABSTRACT

In this paper, we show that the physical information given by conserved charges for asymptotically AdS spacetimes in Einstein-Gauss-Bonnet AdS gravity is encoded in the electric part of the Weyl tensor. This result generalizes the conformal mass definition by Ashtekar-Magnon-Das (AMD) to a gravity theory with a Gauss-Bonnet term. This proof makes use of the Noether charges obtained from an action renormalized by the addition of counterterms which depend on the extrinsic curvature (Kounterterms). If the asymptotic fall-off behaviour of the Weyl tensor is same as the one considered in the AMD method, then the Kounterterm charges and the AMD charges agree in any dimension.
1 Introduction

The AdS/CFT correspondence has been studied in great detail over the years [1]. It is one of the most striking relations between a theory of gravity and a theory without gravity [2–4]. The central premise of this correspondence is that all the information stored in the theory of gravity is contained in the information stored in the boundary theory, and vice versa. As far as the theory of gravity is concerned, this fact was known in a different guise much before the holographic relationship was proposed in the AdS/CFT correspondence. For example, it was known that one cannot define local observables in the theory of gravity, which in turn implies that one cannot define local energy-momentum tensor out of the metric. Only sensible definition of the energy-momentum tensor is the quasi-local energy-momentum tensor defined on the boundary.

Asymptotic symmetries and conserved charges do depend on the bulk geometry and there are various ways one can define conserved charges by studying asymptotic symmetries. Among them, the Ashtekar-Magnon-Das [5, 6] (AMD) method applies Penrose’s conformal transformation to determine conserved charges in the asymptotically AdS (AAdS) spaces. A defining feature of this method is that all information about the charge is contained in the electric part of the Weyl tensor,

\[ E^i_j = \frac{1}{D-3} W^i_{j\mu} n_\mu n_\nu. \] (1.1)

Here, \( n_\mu \) is the normal vector to the boundary, whose local coordinates are \( x^i \).
In Einstein gravity, AMD charges are obtained by conformal completion techniques applied to AAdS spaces. The conformal mapping between the physical metric $g_{\mu\nu}$ (a given solution of the Einstein equations) to an unphysical one, $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, serves the purpose of defining a regular boundary, as long as the conformal factor $\Omega$ vanishes on the boundary and its derivative is finite. In doing so, the conformal mass is encoded in the electric part of the Weyl tensor for $\tilde{g}_{\mu\nu}$. Suitable rescaling of tensorial quantities make the charge formula expressible in terms of the tensor defined in Eq. (1.1), instead.

In Ref. [7], it was shown that in arbitrary dimensions the AMD conserved charges for asymptotically AdS spaces can be derived from the extrinsic curvature counterterms (Kounterterms [8, 9]) when the bulk gravity action was given by the Einstein-Hilbert action. In this gravity theory, the on-shell Weyl tensor $W_{\alpha\beta}^{\mu\nu}$ is equal to the AdS curvature tensor $F_{\alpha\beta}^{\mu\nu} = \frac{1}{2} \delta^{[\mu\nu]}_{[\alpha\beta]}$, and the asymptotic behaviour of the Weyl tensor in the radial direction is $W_{\alpha\beta}^{\mu\nu} = O(1/r^{D-1})$. These features of the theory in addition to the fact that the Kounterterm charge factorizes over the AdS curvature projected to the boundary, $F_{\mu\nu}^{ij}$, are the key-ingredients in proving the equivalence of the Kounterterm mass and the conformal mass.

In this paper, we will take up the problem of computing the conformal mass in the Einstein-Gauss-Bonnet (EGB) AdS gravity using the Kounterterm method in arbitrary dimensions.

In the EGB-AdS case, the on-shell Weyl tensor is not equal to the AdS curvature tensor, i.e. $W_{\alpha\beta}^{\mu\nu} \neq F_{\alpha\beta}^{\mu\nu}$. The difference, however, is subleading in $r$. As the first step in deriving the conformal mass, we will show that in the EGB theory the difference between these two tensors is $W_{\alpha\beta}^{\mu\nu} = F_{\alpha\beta}^{\mu\nu} + O(1/r^{D-2})$. This is true provided we define the AdS curvature $F_{\alpha\beta}^{\mu\nu}$ in terms of the effective AdS radius.

We show the asymptotic relation between $W_{\alpha\beta}^{\mu\nu}$ and $F_{\alpha\beta}^{\mu\nu}$ only on a class of solutions. However, we expect that it holds generically for localized configurations in AAdS spacetimes. We then prove that the Kounterterm charges in EGB AdS gravity in all dimensions also factorize by the boundary AdS curvature. After these steps, the calculation of the conformal mass is the same as in Einstein-Hilbert case. The obtained result matches the one of Pang [10], up to replacing the AdS radius by the effective one.

## 2 Fall-off of the Weyl tensor in EGB AdS gravity

Let us consider the $D$-dimensional spacetime $\mathcal{M}$ endowed with the metric $g_{\mu\nu}(x)$ whose dynamics is described by the gravity action with the Einstein-Hilbert term, the cosmological constant term and the Gauss-Bonnet term. The Gauss-Bonnet term is a specific quadratic combination of the Riemann tensor $R_{\mu\nu\lambda\rho}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar $R$,

$$I_{\text{EGB}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^Dx \sqrt{-g} \left[ R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}) \right], \quad (2.1)$$

where the cosmological constant term is related to the AdS radius $\ell$ by the relation $\Lambda = -\frac{(D-1)(D-2)}{2\ell^2}$ and $G$ is the $D$-dimensional gravitational constant. The Gauss-Bonnet coupling
\( \alpha \) is a dimensionfull parameter with spatial scaling dimension 2. If this parameter is related to the string tension, it takes only nonegative values.

The Gauss-Bonnet term is important when the spacetime dimension is \( D \geq 5 \). For \( D = 4 \), the Gauss-Bonnet term is a topological invariant and therefore does not contribute to the equations of motion. It can, however, modify the boundary dynamics owing to the fact that it is a total derivative term in four dimensions. In dimensions less than four, the Gauss-Bonnet term vanishes identically. For later convenience, we will use the notation \( \mathcal{L} \) for the Lagrangian density to denote the term in the square brackets in (2.1).

The equation of motion derived from the action (2.1) takes the form

\[
\mathcal{E}^\mu_\nu \equiv \mathcal{R}^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu + \Lambda \delta^\mu_\nu + \alpha H^\mu_\nu = 0,
\]  

(2.2)

where the Lanczos tensor \( H^\mu_\nu \) is the contribution coming from the variation of the Gauss-Bonnet term. This requires giving a prescription for the variational principle for the manifolds with boundary. For the Einstein-Gauss-Bonnet theory we need to provide a generalized Gibbons-Hawking term at the boundary for consistency of the variational principle. This has already been done in the literature \([11, 12]\), what we will discuss in the next section. Since the Gauss-Bonnet term contains Riemann tensor, it is useful to write variation of the Riemann tensor in terms of variation of the Riemann-Christoffel connection

\[
\delta \mathcal{R}^\mu_\nu \lambda \rho = \nabla^\lambda (\delta \Gamma^\mu_\nu \rho) - \nabla^\rho (\delta \Gamma^\mu_\nu \lambda) .
\]  

(2.3)

Using this expression for variation of the Riemann tensor, the Lanczos tensor can be written as

\[
H^\mu_\nu = -\frac{1}{8} \delta^{[\mu_1 \cdots \mu_p]}_{[\nu_1 \cdots \nu_p]} R^\nu_1 \nu_2 R^\nu_3 \nu_4 .
\]  

(2.4)

The tensor \( \delta^{[\mu_1 \cdots \mu_p]}_{[\nu_1 \cdots \nu_p]} \) used in the definition of the Lanczos tensor in (2.4) is the totally antisymmetric product of \( p \) Kronecker delta symbols (see Appendix A).

The AdS space is a solution to the Eq. (2.2), but with a different than \( \ell \) radius when \( \alpha \neq 0 \). We will call it the effective AdS radius, \( \ell_{\text{eff}} \), which can be derived from the following condition

\[
\frac{\alpha^*}{\ell_{\text{eff}}^4} - \frac{1}{\ell_{\text{eff}}^2} + \frac{1}{\ell^2} = 0 , \quad \alpha^* = \alpha (D - 3)(D - 4) .
\]  

(2.5)

Since the equation is quadratic in \( \ell_{\text{eff}}^2 \), there are two AdS solutions with different effective radii when \( 0 \neq \alpha^* < \ell^2/4 \). The case where both AdS radii become the same \((\alpha^* = \ell^2/4)\) represents an isolated point of the theory, as the asymptotic behavior of the curvature is radically different. In five dimensions, this particular value of the GB coupling is referred to as the Chern-Simons point, with black holes solutions which are almost of constant curvature \([13]\). In higher dimensions, it corresponds to a particular case of a large family of theories known as Lovelock Unique Vacuum \([14]\).
After manipulating the equation of motion (2.2) and denoting $H = H_\mu^\mu$, we can write the following two on-shell conditions for the Ricci tensor and the Ricci scalar,

$$R_\nu^\mu = \frac{\alpha}{D - 2} H \delta_\nu^\mu - \alpha H_\nu^\mu - \frac{D - 1}{\ell^2} \delta_\nu^\mu , \quad (2.6)$$

$$R = \frac{2}{D - 2} \left( \alpha H - \frac{D(D - 1)(D - 2)}{2\ell^2} \right) . \quad (2.7)$$

Using Eq.(2.6) and (2.7), the on-shell Weyl tensor can be written as

$$W_\alpha^\mu_\beta = R_\alpha^\mu_\beta - \frac{1}{D - 2} \delta_\alpha^\mu_\beta R_\beta^\nu + \frac{R}{(D - 1)(D - 2)} \delta_\alpha^\mu_\beta = R_\alpha^\mu_\beta + \frac{1}{\ell^2} \delta_\alpha^\mu_\beta + \frac{\alpha}{D - 2} \left( \delta_\alpha^\mu_\beta H_\beta^\nu - \frac{2}{D - 1} H \delta_\alpha^\mu_\beta \right) , \quad (2.8)$$

where $\delta_\alpha^\mu_\beta R_\beta^\nu = \delta_\alpha^\mu R_\beta^\nu - \delta_\beta^\nu R_\alpha^\mu + \delta_\beta^\nu R_\alpha^\mu$.

The AdS space is a maximally symmetric solution to the equations of motion of the Einstein-Hilbert action with negative cosmological constant. It is conformally flat space and, as a result, the on-shell Weyl tensor $W_\alpha^\mu_\beta = R_\alpha^\mu_\beta + \frac{1}{\ell^2} \delta_\alpha^\mu_\beta$ in this background vanishes. In what follows, we will refer to this tensor as the AdS tensor with appropriate AdS radius.

In the Einstein-Gauss-Bonnet theory, however, the AdS solution is a bit different, because it has a different effective cosmological constant (equivalently, different AdS radius). As a result, the AdS tensor $F_\alpha^\mu_\beta = R_\alpha^\mu_\beta + \frac{1}{\ell^2} \delta_\alpha^\mu_\beta$ in the Einstein-Gauss-Bonnet theory differs from the Weyl tensor. We can write the Weyl tensor (2.8) as

$$W_\alpha^\mu_\beta = R_\alpha^\mu_\beta + \frac{1}{\ell^2} \delta_\alpha^\mu_\beta + X_\alpha^\mu_\beta , \quad (2.9)$$

$$X_\alpha^\mu_\beta = \frac{\alpha}{D - 2} \delta_\alpha^\mu_\beta H_\beta^\nu - \left( \frac{2\alpha H}{(D - 1)(D - 2)} + \frac{\alpha (D - 3)(D - 4)}{\ell^2 \ell^\text{eff}} \right) \delta_\alpha^\mu_\beta . \quad (2.10)$$

Clearly, when $\alpha = 0$, the difference between the AdS curvature and Weyl tensors vanishes as expected, $X_\alpha^\mu_\beta = 0$. A question is what is the behaviour of $X_\alpha^\mu_\beta$ for $\alpha \neq 0$.

We will now use a heuristic argument to deduce a fall-off of the tensor $X_\alpha^\mu_\beta$. In the original discussion by Ashtekar and Das [6], the fall-off of the Weyl tensor in EH AdS gravity was based on dimensional analysis. Namely, the Weyl tensor vanishes identically for the global AdS space, so the difference between $W|_{\text{AdS}}$ and $W$ is due to a non-vacuum state with total mass $M$, which should satisfy $W - W|_{\text{AdS}} \sim GM$ asymptotically. Since the only other dimensionful quantity is the radial coordinate, dimensional analysis leads to the asymptotic relation $W_\alpha^\mu_\beta \sim GM/r^{D-1}$.

On the other hand, the Weyl tensor is on-shell equal to the AdS tensor in EH gravity, $X|_{\text{EH}} = 0$. Thus, in EGB AdS gravity, the difference between the Weyl and AdS tensors is proportional to the GB coupling and a quadratic mass correction of a non-vacuum state, that is $X - X|_{\text{EH}} \sim \alpha (GM)^2$. This, in turn, implies the units (length)^{-2} of the tensor $X_\alpha^\mu_\beta$ when its fall-off is of the form

$$X_\alpha^\mu_\beta \sim \alpha \frac{(GM)^2}{r^{2D-2}} . \quad (2.11)$$
We will prove that this asymptotic behavior is suitable for static black holes in EGB theory.

Let us consider the Einstein-Gauss-Bonnet AdS-Schwarzschild solution \[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \gamma_{mn} d\Omega^m d\Omega^n, \]
where \( \gamma_{mn} \) is the metric on a \((D-2)\)-dimensional constant curvature Riemannian space with \( \Omega^m \) being the local coordinates, and \( k = 0, 1, \) or \(-1\) represents flat, spherical and hyperbolic metric on the \((D-2)\)-dimensional space transverse to \( r \) and \( t \). There are two branches of the theory and both of them have AdS asymptotics, with the effective radius given by Eq.\((2.5)\), that is,
\[
\frac{1}{\ell_{\text{eff}}^2} = \frac{1}{2\alpha^*} \left( 1 \pm \sqrt{1 - 4\alpha^* \left( \frac{1}{\ell^2} - \frac{\mu}{r^{D-1}} \right)} \right), \quad \alpha^* < \frac{\ell^2}{4}. \tag{2.13}
\]
Without loss of generality of the results, we shall choose one particular (non-degenerate) AdS vacuum with radius \( \ell_{\text{eff}} \).

The Lanczos tensor can now be evaluated in this black hole background. More details of the Lanczos tensor in this background are given in Appendix B. It is convenient to define a function \( P(r) = \left( \frac{\ell^2}{\ell_{\text{eff}}^2} \right)^2 \), so that the Lanczos tensor can be written in a simpler form. In addition, it will be helpful in the study of the asymptotic behaviour of the solution. Let us first rewrite the Lanczos tensor \((B.2), (B.3)\) in terms of \( P(r) \),
\[
H^r_r = H^t_t = -\frac{(D-2)(D-3)(D-4)}{2r^{D-2}} (r^{D-1} P)', \quad H^m_m = -\frac{(D-3)(D-4)}{2} \left[ r^2 P'' + 2(D-1)rP' + (D-1)(D-2)P \right] \delta^m_m. \tag{2.14}
\]
The function \( P(r) \) evaluated on the solution reads
\[
P = \frac{1}{4\alpha^*} \left[ 1 + \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right)^2 + \frac{4\alpha^* \mu}{r^{D-1}} \pm 2 \sqrt{ \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right)^2 + \frac{4\alpha^* \mu}{r^{D-1}} } \right]. \tag{2.15}
\]
The asymptotic expansion of \( P(r) \) will be convenient for analyzing the asymptotic behavior of the solution. The small parameter of the expansion is a dimensionless constant \( \varepsilon = 4\alpha^* \mu / r^{D-1} \ll 1 \). We note that the constant \( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \) appearing under the root can be either positive or negative since, according to \((2.13)\), we have
\[
1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} = \mp \sqrt{1 - \frac{4\alpha^*}{\ell^2}}. \tag{2.16}
\]
Thus, the expansion will not depend on the choice of a branch. We find that all mass terms
cancel out in the asymptotic expansion and we obtain

\[ P = \frac{1}{\ell_{\text{eff}}^4} + \mathcal{O}(1/r^{D-1}), \]

\[ rP' = \mathcal{O}(1/r^{D-1}), \]

\[ r^2P'' = \mathcal{O}(1/r^{D-1}), \]

\[ \frac{(r^{D-1}P)'}{r^{D-2}} = \frac{D-1}{\ell_{\text{eff}}^4} + \mathcal{O}(1/r^{2D-2}). \]

(2.17)

Using this leading behavior of \( P(r) \), the asymptotic behavior of the Lanczos tensor evaluated on this solution is

\[ H^\nu_{\mu} = -\frac{(D-1)(D-2)(D-3)(D-4)}{2\ell_{\text{eff}}^4} \delta^\nu_{\mu} + \mathcal{O}(1/r^{2D-2}). \]

(2.18)

Substituting this behaviour of \( H^\nu_{\mu} \) in Eqs. (2.9), (B.4), a fall-off of the tensor \( X^{\mu\nu}_{\alpha\beta} \) that describes the difference between the Weyl and the AdS curvature tensors can be evaluated as

\[ X^{\mu\nu}_{\alpha\beta} = \mathcal{O}(1/r^{2D-2}), \]

(2.19)

in agreement with eq. (2.11). Also note that the asymptotic expansion of the function \( P(r) \) defined in (2.15) is in the small quantity \( 4\alpha^* \mu/r^{D-1} \), that is \( \alpha GM/r^{D-1} \), and the first non-vanishing term is quadratic in mass, in accordance with (2.11).

In the next section we will utilize this leading behavior of these tensors to determine conserved charges in this theory. We will derive this result using the Kounterterm method.

### 3 Conformal charge for EGB-AdS gravity

In this section we determine the conserved charges in Einstein-Gauss-Bonnet AdS gravity. In order to derive equations of motion, we need to define a variational principle for a theory of gravity. If the theory is defined on a manifold with boundary, we need to impose boundary conditions on the metric so that the action has an extremum on-shell. The Dirichlet boundary condition on the metric can be imposed by adding (generalized) Gibbons-Hawking terms to the action which cancel variations of the extrinsic curvature coming from the variation of the bulk action. In case of AdS-like geometries where the boundary is defined along the radial direction, it is convenient to write the metric in the Gauss-normal coordinates,

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(r)dr^2 + h_{ij}(r, x)dx^idx^j. \]

(3.1)

The Gibbons-Hawking term is then written in terms of the extrinsic curvature \( K_{ij}(h) \) and the intrinsic curvature of the boundary \( R^{ij}_{kl}(h) \). The extrinsic curvature is given by the Lie derivative of the induced metric \( h_{ij}(r, x) \) on the boundary along the outward pointing normal vector \( n_\mu = (n_r, n_i) = (N, \vec{0}) \),

\[ K_{ij} = -\frac{1}{2} L_n h_{ij} = -\frac{1}{2N} \partial_r h_{ij}. \]

(3.2)
The intrinsic curvature of the boundary can be reexpressed in terms of the bulk Riemann curvature tensor and the extrinsic curvature of the boundary,

\[ R^{ij}_{kl}(h) = R^{ij}_{kl}(g) + K^i_k K^j_l - K^i_l K^j_k. \] (3.3)

Then, the generalized Gibbons-Hawking term \([11, 12]\) for the Gauss-Bonnet theory has the form

\[
I_{\text{GGH}} = \int_{\partial M} d^{D-1}x \sqrt{-h} \left[ \frac{\delta^{ij}_{[ijkl]} K^i_{ji}}{8\pi G} \left( \frac{\delta^{i_2}_{j_2} \delta^{i_3}_{j_3}}{(D-2)(D-3)} + 2\alpha \left( \frac{1}{2} R_{j_2j_3} - \frac{1}{3} K^i_{j_2} K^{i_3}_{j_3} \right) \right) \right]
\]

\[
= \int_{\partial M} d^{D-1}x \sqrt{-h} \left[ K + 2\alpha K \left( K^{ij} K_{ij} - \frac{1}{3} K^2 \right) - \frac{2\alpha}{3} K^i_k K^j_l K^k_i \right] , \quad (3.4)
\]

where \(G_{ij} = R_{ij} - \frac{1}{2} R h_{ij}\) is the Einstein tensor derived from the induced metric \(h_{ij}\) on the boundary.

In addition to the boundary Gibbons-Hawking term, we also need to introduce local counterterms at the boundary to take care of divergences. In the AdS/CFT correspondence, these terms are derived using holographic renormalization technique. However, this method becomes cumbersome as one departs from the Einstein gravity and includes higher-derivative terms. There are various proposals to handle higher-derivative terms, but all of them seem to work in specific dimensions. The Kounterterm regularization scheme \([8, 9, 16]\), where the boundary term explicitly depends on the extrinsic curvature \(K_{ij}\), comes to rescue at this point. In this scheme, depending on even or odd dimensions, the counterterms are either written in terms of topological terms, or in terms of the Chern-Simons form in those dimensions. In the Kounterterm regularization method, the generalized Gibbons-Hawking term \((3.4)\) is not added to the action, but the total action can be written as

\[
I_{\text{ren}} = I_{\text{EGB}} + c_{D-1} \int_{\partial M} d^{D-1}x B_{D-1} . \quad (3.5)
\]

Here, \(c_{D-1}\) is a specific dimension dependent constant and \(B_{D-1}\) is the Kounterterm action which is written in terms of both the boundary extrinsic curvature and the boundary intrinsic curvature. The form of \(B_{D-1}\) depends on whether \(D\) is even or odd. The explicit form of the Kounterterm will be discussed later for \(D\) even, as well as \(D\) odd.

Eq.\((3.5)\) gives a well defined prescription for variation of the metric. Extremization of the total action is then given by

\[
\delta I_{\text{ren}} = \int_{\partial M} d^{D-1}x \left( \Theta_{\text{EGB}} + c_{D-1} \delta B_{D-1} \right) , \quad (3.6)
\]

where \(\Theta_{\text{EGB}}\) is the boundary contribution coming from variation of the action \((2.1)\).

Let us now define conserved quantities. They are associated with certain global symmetries. In the theory of gravity, these symmetries are asymptotic isometries of the background.
In our case we will be interested in the diffeomorphism invariance,

\[ \delta_\epsilon g_{\mu\nu} = \mathcal{L}_\epsilon g_{\mu\nu} = -\left( \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \right) . \]  

(3.7)

The variation of the total action \( I_{\text{ren}} \), which contains the Einstein-Gauss-Bonnet action as well as the Kounterterm action, can be succinctly written as

\[ \delta_\epsilon I_{\text{ren}} = \int_{\partial M} d^{D-1}x \, n_\mu \left[ \frac{1}{N} \Theta^\mu(\epsilon) + \sqrt{-h} \, \epsilon^\mu \mathcal{L} + c_{D-1} \, n^\mu \partial_i (\epsilon^i B_{D-1}) \right] - \frac{1}{16\pi G} \int_M d^Dx \, \sqrt{-g} \, \mathcal{L} g_{\mu\nu} \, \mathcal{L}_\epsilon g_{\mu\nu} , \]  

(3.8)

where \( n_\mu \Theta^\mu \) is the boundary contribution coming from the variation of the Einstein-Gauss-Bonnet action. It is now easy to read the conserved current from the variation of the action,

\[ J^\mu(\epsilon) = \Theta^\mu(\epsilon) + \sqrt{-g} \, \epsilon^\mu \mathcal{L} + c_{D-1} \, N \, n^\mu \partial_i (\epsilon^i B_{D-1}) . \]  

(3.9)

The conserved charge \( Q[\epsilon] \) is in fact the boundary term appearing in the variation (3.8),

\[ Q[\epsilon] = \int_{\partial M} d^{D-1}x \, \frac{1}{N} n_\mu J^\mu(\epsilon) . \]  

(3.10)

The conserved charge is typically computed by integrating the density over a codimension two hypersurface. The Noether charge defined above is not an integral over a codimension two space and, in general, it is not obvious that the boundary integral will reduce to an integral over a codimension one subspace of the boundary. In the case of Einstein-Gauss-Bonnet AdS theory, the conserved current projected along the outward pointing normal can be written as a total derivative. In the Gauss normal coordinates, the boundary integral can be further restricted to a codimension one subspace \( \Sigma \) of the boundary. The radial component of the conserved current can be written as

\[ J^r = \frac{1}{N} n_\mu J^\mu = \partial_j \left( \sqrt{-h} \, \epsilon^i (q^j_i + q^j_{(0)i}) \right) , \]  

(3.11)

where \( q^j_{(0)i} \) is the contribution of vacuum energy which is non-vanishing in odd dimensions only. The charge density tensor \( q^j_i \), in general, contains information about all the charges that a black hole can carry. In our case, however, this charge would be the mass of the black hole. This charge vanishes when evaluated in vacuum. Using the fact that \( J^r \) can be written as a total derivative, we can re-express the Noether charge as an integral over a codimension one subspace of the boundary,

\[ Q[\epsilon] = \int_\Sigma d^{D-2}y \sqrt{\sigma} u_j \epsilon^i \left( q^j_i + q^j_{(0)i} \right) , \]  

(3.12)

where \( u_i \) is the unit time-like normal to the boundary of the spatial section described by the metric \( \sigma \).
For global AdS space, in particular $F_{ij}^{kl} = 0$, the charge density $q_i^j$ vanishes so that $q_{(0)i}^j$ can be interpreted as the vacuum energy tensor of the spacetime (which cannot be observed in perturbative, background-dependent methods in EGB theory \[17–19\]). The fact that $q_i^j = 0$ in the vacuum suggests that $q_i^j$ should be factorizable by $F$, making this vanishing explicit. Indeed, denoting $n = [D/2]$, it can be shown (see Appendix C.1 and C.2) that the tensor $q_i^j$ can always be written as

$$q_i^j = a_n \delta_{[j_1…i_{2n-1}]1}^{[j_2…j_{2n-1}]} K^{i_1}_{j_1} F^{i_2j_3}_{j_2j_3} J^{i_4…i_{2n-1}}_{j_4…j_{2n-1}} (F),$$

(3.13)

where $a_n$ is a constant that depends on dimension and the GB coupling, and $J^{i_4…i_{2n-1}}_{j_4…j_{2n-1}} (F)$ is a given polynomial of the AdS curvature in integral representation.

To study the conserved charges further, we will have to consider two cases: (i) $D$ is even and (ii) $D$ is odd. This distinction is required for two reasons. The conserved charges are derived from the Kounterterms, and latter have one form in even dimensions and a different form in odd dimensions. This implies that the form of the charge density tensor also differs between odd and even dimensions. Secondly, the term $q_{(0)i}^j$ vanishes in even dimensions but is non zero in odd dimensions. As a result, it is convenient to split the discussion and study even-dimensional and odd-dimensional cases separately.

### 3.1 Even dimensions, $D = 2n$

The relevant Kounterterms in even dimensions are \[9, 16\]

$$B_{2n-1} = 2n\sqrt{\hbar} \int_0^1 dt \delta^{[j_1…j_{2n-1}]}_{[i_1…i_{2n-1}]} K^{i_1}_{j_1} \left( \frac{1}{2} R^{i_2j_3}_{j_2j_3} - t^2 K^{i_2j_2} K^{i_3j_3} \right) \times$$

$$\cdots \times \left( \frac{1}{2} R^{i_{2n-2}j_{2n-2}}_{j_{2n-2}j_{2n-2}} - t^2 K^{i_{2n-2}j_{2n-2}} K^{i_{2n-1}j_{2n-1}} \right),$$

(3.14)

and the coupling constant in (3.5), fixed by the variational principle, is

$$c_{2n-1} = -\frac{(-\ell_{\text{eff}}^2)^{n-1}}{16\pi G n(2n-2)!}.$$  

(3.15)

The regularized charge density tensor (3.12) in even dimensions has the form

$$q_i^j = \frac{1}{16\pi G (2n-2)!2^{n-2} \delta_{[j_1…j_{2n-1}]}^{[i_1…i_{2n-1}]} K^{i_1}_{j_1} \times}$$

$$\left[ \left( \delta^{i_2j_3}_{j_2j_3} + 2\alpha(2n-2)(2n-3) R^{i_2j_3}_{j_2j_3} \right) \delta_{[i_4j_5]}^{j_4j_5} \cdots \delta_{[i_{2n-2}j_{2n-2}]}^{j_{2n-2}j_{2n-2}} \right]$$

$$- (-\ell_{\text{eff}}^2)^{n-1} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} (2n-2)(2n-3) R^{i_2j_3}_{j_2j_3} \cdots R^{i_{2n-2}j_{2n-2}}_{j_{2n-2}j_{2n-2}} \right).$$

(3.16)

As discussed before, the tensor $q_{(0)i}^j$ vanishes. The tensor $q_i^j$ can be factorized by $F_{ik}^j$ in the form (3.13), in terms of an integral (C.6) involving only the AdS curvature tensor with the constant $a_n = -\frac{n c_{2n-1}}{2^{n-1}}$. The details of this calculation are given in Appendix C.1.
Recall that the Weyl tensor decreases asymptotically as

\[ W_{\alpha\beta}^{\mu\nu} = \mathcal{O}(1/r^{D-1}) \, . \]  

(3.17)

Using the behaviour of \( X_{\alpha\beta}^{\mu\nu} \) in Eq.(2.19) and substituting it in the Eq.(2.9), we see that \( F_{ij}^{kl} = W_{ij}^{kl} + \mathcal{O}(1/r^{2D-2}) \). This allows us to trade the AdS tensor \( F_{ij}^{kl} \) off in terms of the Weyl tensor \( W_{ij}^{kl} \) in Eq.(3.13) because, according to (3.17), it decreases milder than \( r^{-2D-2} \) for large distances. From Eq.(3.2), it is easy to see that

\[ K_j^i = -\frac{1}{\ell_{\text{eff}}} \delta_j^i + \mathcal{O}(1/r^2) \, . \]

(3.18)

On account of the fall-off of the Weyl tensor and the behaviour of \( K_j^i \), the expression of \( q_j^i \) is equivalent (up to a relevant order) to

\[ q_j^i = -\frac{(\ell_{\text{eff}}^2)^{n-1}}{16\pi G \ell_{\text{eff}} (2n - 2)! 2^{n-2}} \delta_{[i_2 \ldots i_{2n-1}]}^{[j_2 \ldots j_{2n-1}]} W_{i_2 j_2}^{i_3 j_3} W_{i_4 j_4}^{i_5 j_5} \ldots W_{i_{2n-2} j_{2n-2}}^{i_{2n-1} j_{2n-1}} (0) \, . \]

(3.19)

We have substituted the AdS tensor \( F \) in the expression of \( J \) by zero because of its fast fall-off and the fact that higher powers of the Weyl tensor do not contribute to the conserved charge.

Setting \( F = 0 \) in Eq.(C.6), we can evaluate the integral because it essentially becomes trivial, \( \int_0^1 du = 1 \),

\[ J(0) = \frac{\gamma (n - 2) - (n - 1)}{(\ell_{\text{eff}}^2)^{n-2}} (\delta^{[2]}_{[2]})^{n-2} \]

\[ = -\frac{n - 1}{(\ell_{\text{eff}}^2)^{n-2}} \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right) (\delta^{[2]}_{[2]})^{n-2} \, . \]

(3.20)

As a result of this simplification, the charge density tensor also simplifies to

\[ q_j^i = -\frac{\ell_{\text{eff}} (n - 1)}{16\pi G (2n - 2)! 2^{n-2}} \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right) \delta_{[i_2 \ldots i_{2n-1}]}^{[j_2 \ldots j_{2n-1}]} W_{i_2 j_2}^{i_3 j_3} W_{i_4 j_4}^{i_5 j_5} \ldots W_{i_{2n-2} j_{2n-2}}^{i_{2n-1} j_{2n-1}} \, . \]

(3.21)

The last expression can be calculated using the identity (A.3) given in Appendix A. For simplification of Eq.(3.21) the relevant values of the variables are \( N = 2n - 1, \ k = 2n - 4 \) and \( p = 2n - 1 \). This gives the relation

\[ \delta_{[i_2 \ldots i_{2n-1}]}^{[j_2 \ldots j_{2n-1}]} \delta_{j_4}^{i_4} \ldots \delta_{j_{2n-1}}^{i_{2n-1}} = (2n - 4)! \delta_{[i_2 j_3]}^{[j_2 j_3]} \, . \]

(3.22)

This further simplifies the expression of the charge density tensor and allows us to write it in a compact form,

\[ q_j^i = -\frac{\ell_{\text{eff}}}{32\pi G (2n - 3)} \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right) \delta_{[i_2 j_3]}^{[j_2 j_3]} W_{i_2 j_2}^{i_3 j_3} \, . \]

(3.23)

Expansion of the Kronecker delta implies \( \delta_{[i_2 j_3]}^{[j_2 j_3]} W_{i_2 j_2}^{i_3 j_3} = 2\delta_i^j (W_{kl}^{jk} - 2W_{kl}^{kj}) \). Furthermore, since the Weyl tensor is traceless \( (W_{\mu\nu}^{\mu\nu} = 0) \), it enables to write the charge density tensor in terms of the electric part of the Weyl tensor, that is

\[ q_j^i = -\frac{\ell_{\text{eff}}}{8\pi G (2n - 3)} \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right) W_{ri}^{\mu\nu} \, . \]

(3.24)
Recall that in the even spacetime dimensions, the tensor \( q^{(0)i} \) vanishes identically, \( q^{(0)i} = 0 \). We can then use the result (3.23) and substitute it in Eq. (3.12), which in a straightforward manner (as in [7]) leads to

\[
Q_{\text{EGB}}[\epsilon] = -\frac{\ell_{\text{eff}}}{8\pi G (D - 3)} \left( 1 - \frac{2\alpha^*}{\ell_{\text{eff}}^2} \right) \int_{\Sigma_{\infty}} d\Sigma W^{ijr} e^i u_j. \tag{3.25}
\]

Our results are in agreement with those of [10], where the conformal mass was calculated in general quadratic curvature gravity using a different method. Restricting the coupling constants to EGB gravity in AdS space with the radius \( \ell_{\text{eff}} \), this charge matches Eq. (3.25).

### 3.2 Odd dimensions, \( D = 2n + 1 \)

The Kounterterms in odd dimensions are [9, 16]

\[
B_{2n} = 2n\sqrt{-h} \int_0^1 dt \int_0^t ds \delta^{[i_1 \cdots i_{2n}]}_1 K_{i_1}^i \delta^{i_2}_{j_2} \left( \frac{1}{2} R^{i_3 i_4}_{j_3 j_4} - t^2 K_{j_3}^{i_3} K_{j_4}^{i_4} + \frac{s^2}{\ell_{\text{eff}}^2} \delta^{i_3}_{j_3} \delta^{i_4}_{j_4} \right) \times 
\]

\[
\cdots \times \left( \frac{1}{2} R^{i_{2n-1} i_{2n}}_{j_{2n-1} j_{2n}} - t^2 K^{i_{2n-1}}_{j_{2n-1}} K^{i_{2n}}_{j_{2n}} + \frac{s^2}{\ell_{\text{eff}}^2} \delta^{i_{2n-1}}_{j_{2n-1}} \delta^{i_{2n}}_{j_{2n}} \right). \tag{3.26}
\]

The value of the coupling \( c_{2n} \) which appears in the action (3.5) is

\[
c_{2n} = -\frac{1}{16\pi G 2^{2n-2} n(n - 1)!^2} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} \left( 2n - 1 \right) \left( 2n - 2 \right) \right). \tag{3.27}
\]

The expression for the charge density tensor \( q^j_i \), similar to the even-dimensional one (3.16), is given by Eq. (C.8) in Appendix C.2. This appendix also contains various technical details and identities required to simplify the form of \( q^j_i \).

After an integral parametrization of the constant \( c_{2n} \) and factorization of \( q^j_i \) by \( F^i_{kl} \), the charge density tensor \( q^j_i \) takes the form (3.13), with the constant \( a_n = -\frac{n c_{2n}}{\left( 1 - \gamma \right) 2^{n-1/2}} \), where \( \gamma = \frac{2\alpha}{\ell_{\text{eff}}^2} \left( 2n - 1 \right) \left( 2n - 2 \right) \).

For completeness, let us write the expression of \( q^j_i \) with all indices in place,

\[
q^j_i = \frac{n c_{2n} \ell_{\text{eff}}}{\ell_{\text{eff}}^2 (n-1)!} \delta^{i_{j_1 j_2 j_3 \cdots i_{2n}}}_{[i_1 i_2 i_3 \cdots i_{2n}]} F_{j_1 j_2}^{i_3 i_4 \cdots i_{2n}} (F). \tag{3.28}
\]

As in the even-dimensional case, we use the asymptotic behavior of the Weyl tensor (3.17) and the relation \( F^i_{kl} = W^r_{kl} + O(1/r^{2D-2}) \) to conclude that the AdS tensor and Weyl tensor have the same fall-off at the leading order, and

\[
K^i_j = -\frac{1}{\ell_{\text{eff}}} \delta^i_j + O(1/r^2). \tag{3.29}
\]
This allows us to replace $F$ by $W$ in Eq.(3.28). The tensor $J$ depends on $F$, but since higher powers of $F$ would fall off faster, they do not contribute to the conserved charge. We will therefore substitute $F = 0$ in the expression of $J$ and the explicit AdS tensor in the expression of $q^i_j$ will be replaced by the Weyl tensor $W$. Thus,

$$q^i_j = \frac{n_c 2 n_{\text{eff}}}{\ell_{\text{eff}}^{2(n-1)} (1 - \gamma)} \delta^{[j_2 j_3 \cdots j_{2n}]} \epsilon_{[i_{2} i_{3} \cdots i_{2n}]} W^{i_{2} i_{3} \cdots i_{2n}} (0).$$

(3.30)

The integral $J$, after setting $F = 0$, reads

$$J(0) = \int_0^1 du \left[ - (n - 1) (u^2 - 1)^{n-2} + \gamma u^{2(n-1)} - (n - 1) (n - 2) \gamma \int_0^1 ds \left( u^2 - s \right)^{n-3} \right].$$

(3.31)

We can carry out the integral involving the variable $s$ using

$$\int_0^1 ds \left( u^2 - s \right)^{n-3} = \frac{u^{2(n-1)} - (u^2 - 1)^{n-2} (u^2 + n - 2)}{(n - 1) (n - 2)}.$$  

(3.32)

Substituting Eq.(3.32) in Eq.(3.31), we get

$$J(0) = \int_0^1 du \left[ - (n - 1) + \gamma (u^2 + n - 2) \right] (u^2 - 1)^{n-2}.$$  

(3.33)

It is now easy to use the identity Eq.(C.10) to write

$$J(0) = (-1)^{n-1} \frac{(n - 1)! (n - 2)! 2^{2n-4}}{(2n - 3)!} \left( 1 - \gamma \frac{2n - 3}{2n - 1} \right).$$

(3.34)

This form of $J(0)$ simplifies the expression of the charge density tensor, which can now be written as

$$q^i_j = -\frac{\ell_{\text{eff}}}{32 \pi G (2n - 2)} \left( 1 - \frac{2 \alpha^*}{\ell_{\text{eff}}^2} \right) \delta^{[j_2 j_3 \cdots j_{2n}]} \epsilon_{[i_{2} i_{3} \cdots i_{2n}]} W^{i_{2} i_{3} \cdots i_{2n}}.$$  

(3.35)

In the end, we see that both in even dimensions (3.23) and odd dimensions (3.35), the expression of the charge density tensor is of the form

$$q^i_j = -\frac{\ell_{\text{eff}}}{32 \pi G (D - 3)} \left( 1 - \frac{2 \alpha^*}{\ell_{\text{eff}}^2} \right) \delta^{[j_2 j_3 \cdots j_{2n}]} \epsilon_{[i_{2} i_{3} \cdots i_{2n}]} W^{i_{2} i_{3} \cdots i_{2n}}.$$  

(3.36)

Thus, the conserved charge $Q_{\text{EGB}}[\xi]$ in odd dimensional is also given by the expression (3.25).

4 Conclusions

In this paper, we have shown that the information regarding mass and other conserved quantities for AAdS solutions in EGB theory is always contained in the electric part of the
Weyl tensor. This comparison has been carried out by direct expansion of the charge formulas obtained within the Kounterterm regularization for AdS gravity, just assuming a standard asymptotic fall-off of the metric.

The asymptotic charges are derived using the surface terms in the variation of the action, which includes boundary terms. A key role is played by the appearance of the Weyl tensor in this surface term, only for a fine-tuned coupling \( c_d \) given by Eq. (3.15) in even dimensions and Eq. (C.9) in odd dimensions.

This form of the surface term does not lend itself to a clear definition of the quasilocal stress tensor, because the counterterms added here are extrinsic and not purely intrinsic, as in the standard methods in EGB theory [20, 21]. However, the general expression of the variation of the action can shed some light on the problem of separability of the quasilocal stress tensor in odd dimensions. In particular, the quasilocal stress tensor should contain a piece which is written in terms of the electric part of the Weyl tensor and the rest should give rise to the terms which contribute to the vacuum energy for the AAdS space, as in Eq. (3.12).

Since the expression for the surface term is the same regardless of the type of variation performed, any transformation of the fields will preserve the vacuum configuration, as the Weyl tensor vanishes identically for maximally-symmetric spacetimes. At the same time, at least in even dimensions, the value of the action is identically zero for global AdS.

In the Kounterterm regularization method, the action is stationary under arbitrary variation of the fields, including the contributions that appear at the boundary, provided we impose specific asymptotic conditions on the curvature. This method can be used to study supersymmetric extensions of the AdS gravity action. The closure of supersymmetry not only in the bulk, but also at the boundary, might be the ultimate reason behind the success of the Kounterterm regularization method. In this regard, there is new evidence supporting this claim given in Ref. [22] that in four dimensions, demanding the vanishing of (super) AdS curvature at the boundary, one is able to fix the coupling of Gauss-Bonnet term. Recall that the addition of the GB term is equivalent to Holographic Renormalization in 4D AdS gravity [23]. It would be interesting to use this to extend the relation between renormalized AdS action and supersymmetry [22] to higher dimensions.

Acknowledgements

This work was funded by FONDECYT Grants No. 1131075 and No. 1110102. O.M. thanks DII-PUCV for support through the project No. 123.711. The work of R.O. is financed in part by the UNAB grant No. DI-551-14/R. The work of D.P.J. is partly supported by DAE project 12-R&D-HRI-5.02-0303.
### A Kronecker delta of rank $p$

In this appendix we will write down the notation for the rank $p$ Kronecker delta. This notation is taken from Ref.\[16\]. The totally-antisymmetric Kronecker delta of rank $p$ is defined as the determinant

\[
\delta^{v_1 \cdots v_p}_{\nu_1 \cdots \mu_p} := \begin{vmatrix}
\delta^{\nu_1}_{\mu_1} & \delta^{\nu_2}_{\mu_1} & \cdots & \delta^{v_p}_{\mu_1} \\
\delta^{\nu_1}_{\mu_2} & \delta^{\nu_2}_{\mu_2} & \cdots & \delta^{v_p}_{\mu_2} \\
\vdots & \vdots & \ddots & \vdots \\
\delta^{\nu_1}_{\mu_p} & \delta^{\nu_2}_{\mu_p} & \cdots & \delta^{v_p}_{\mu_p}
\end{vmatrix}.
\]

(A.1)

A contraction of $k \leq p$ indices in the Kronecker delta of rank $p$ produces a delta of rank $p - k$,

\[
\delta^{[v_1 \cdots v_k] \cdots v_p}_{[\nu_1 \cdots \mu_k \cdots \mu_p]} \delta^{\nu_1}_{v_1} \cdots \delta^{\nu_k}_{v_k} = \frac{(N - p + k)!}{(N - p)!} \delta^{[v_{k+1} \cdots v_p]}_{[\nu_{k+1} \cdots \mu_p]},
\]

(A.2)

where $N$ is the range of indices.

If $N$ is the range of indices, a contraction of $k$ indices in the Kronecker delta of rank $p$ produces a delta of rank $p - k$,

\[
\delta^{[v_1 \cdots v_k \cdots v_p]}_{[\nu_1 \cdots \mu_k \cdots \mu_p]} \delta^{\nu_1}_{v_1} \cdots \delta^{\nu_k}_{v_k} = \frac{(N - p + k)!}{(N - p)!} \delta^{[v_{k+1} \cdots v_p]}_{[\nu_{k+1} \cdots \mu_p]},
\]

(A.3)

### B The Lanczos Tensor in EGB AdS Schwarzschild background

Consider the Einstein-Gauss-Bonnet AdS-Schwarzschild solution \[15\]

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 g_{mn}d\Omega_m d\Omega^n,
\]

\[
f(r) = k + \frac{r^2}{2\alpha^*} \left[ 1 \pm \sqrt{1 - 4\alpha^* \left( \frac{1}{f^2} - \frac{\mu}{r^{D-1}} \right)} \right].
\]

(B.1)

The Lanczos tensor (2.4) for this solution is given by

\[
H^r_r = H^t_t = -(D - 2)(D - 3)(D - 4) \frac{f - k}{r^2} \left( \frac{f'}{r} + \frac{D - 5}{2} \frac{f - k}{r^2} \right),
\]

(B.2)

\[
H^m_m = -(D - 3)(D - 4) \delta^m_m \left[ \frac{f - k}{r^2} f'' + \left( \frac{f'}{r} \right)^2 + 2(D - 5) \frac{f - k}{r^2} \frac{f'}{r} \\
+ \frac{1}{2} (D - 5) (D - 6) \left( \frac{f - k}{r^2} \right)^2 \right].
\]

(B.3)

The Lanczos tensor can be defined in terms of an auxiliary function $P(r) = \left( \frac{f - k}{r} \right)^2$. The components of the tensor $X^\mu_{\alpha \beta}$ can be written in terms of the components of the Lanczos tensor.
C Useful identities in EGB AdS gravity

Here we present the detailed derivation of the charge density tensor in even and odd dimensions. We manipulate them to write in a simpler form which is quoted in the main text.

C.1 Even dimensions

In \( D = 2n \) dimensions, the regularized charge density tensor is given by the formula (3.16), which can be explicitly written down using the definition of the rank \( p \) Kronecker delta given in Eq. (A.1),

\[
q^j_i = \frac{1}{16\pi G (2n - 2)!2^{n-2}} \delta^{[2n-1]} K_i \left[ (\delta^{[2]} + 2\alpha (2n - 2) (2n - 3) R) (\delta^{[2]})^{n-2} - (-\ell^2_{\text{eff}})^{n-1} \left( 1 - \frac{2\alpha}{\ell^2_{\text{eff}}} (2n - 2) (2n - 3) \right) R^{n-1} \right],
\]

where \( \delta^{[2]} \) is a shorthand notation for \( \delta^{[ij]} \) and similarly \( \delta^{[2n-1]} \) is the Kronecker delta of rank \( 2n - 1 \) with \( j \) being the only unpaired index. This expression of \( q^j_i \) can further be simplified by defining

\[
\Delta = \frac{1}{\ell^2_{\text{eff}}} \delta^{[2]}, \quad \gamma = \frac{2\alpha}{\ell^2_{\text{eff}}} (2n - 2) (2n - 3).
\]

Using Eq. (C.2), we can write the charge density tensor given in Eq. (C.1) as

\[
q^j_i = \frac{(-\ell^2_{\text{eff}})^{n-1}}{16\pi G (2n - 2)!2^{n-2}} \delta^{[2n-1]} K_i \left\{ \gamma R \left[ R^{n-2} - (-\Delta)^{n-2} \right] - \left[ R^{n-1} - (-\Delta)^{n-1} \right] \right\}. \tag{C.3}
\]

This form of the charge density tensor can now be manipulated using the following identity,

\[
R^{n-1} - (-\Delta)^{n-1} = (n - 1) (R + \Delta) \int_0^1 du \left[ u (R + \Delta) - \Delta \right]^{n-2}. \tag{C.4}
\]

This identity allows us to write \( q^j_i \) as an integral formula,

\[
q^j_i = \frac{(-\ell^2_{\text{eff}})^{n-1}}{16\pi G (2n - 2)!2^{n-2}} \delta^{[2n-1]} K_i (R + \Delta) \mathcal{J}(R + \Delta), \tag{C.5}
\]
where the function $J(R + \Delta)$ contains the integral over the variable $u$,

$$J(R + \Delta) = \int_0^1 du \left\{ \gamma R (n - 2) [u (R + \Delta) - \Delta]^{n-3} - (n - 1) [u (R + \Delta) - \Delta]^{n-2} \right\}. \quad (C.6)$$

Although it is not obvious from the form of the integral, the quantity $J(R + \Delta)$ is an antisymmetric tensor with $2n - 4$ covariant and $2n - 4$ contravariant indices. To see this explicitly, let us substitute the form of $\Delta$ in the expression of $J(R + \Delta)$. Since $R + \Delta$ is exactly the AdS curvature $F_{ij} = R_{ij} + \frac{1}{\ell_{\text{eff}}^2} \delta_{[ij]}$, the charge density tensor $q_i^j$ can be written as

$$q_i^j = \frac{(-\ell_{\text{eff}}^2)^{n-1}}{16\pi G (2n - 2)! 2^{n-2}} \delta^{[ij]} \delta^{[j2n-1]} K_i^i F^i_{j2} J^i_{j2} J_{i2}^{j2} (F). \quad (C.7)$$

### C.2 Odd dimensions

In odd $D = 2n + 1$ dimensions, the formula for the charge density tensor reads [9, 16]

$$q_i^j = \frac{1}{16\pi G (2n - 1)! 2^{n-2}} \delta^{[ij]} \delta^{[j2n]} K_i^i \delta^{[j]} \times$$

$$\times \left[ \left( \delta^{[i2]} + 2\alpha (2n - 1) (2n - 2) R^{[i2]}_{[j2]} \right) \delta^{[i2]} \delta^{[j2]} \cdots \delta^{[i2n-1]} \delta^{[j2n]} \right] + 16\pi G (2n - 1)! nc_{2n} \int_0^1 du \left( R^{[i2]}_{[j2]} + \frac{u^2}{\ell_{\text{eff}}^2} \delta^{[i2]} \delta^{[j2]} \cdots \delta^{[i2n-1]} \delta^{[j2n]} \right) \right] \quad (C.8)$$

where the constant $c_{2n}$ has the form (3.27), or equivalently

$$c_{2n} = -\frac{1}{16\pi G n (2n - 1)!} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} (2n - 1) (2n - 2) \right) \left[ \int_0^1 du \left( u^2 - 1 \right)^{n-1} \right]^{-1}. \quad (C.9)$$

The equivalence between (3.27) and (C.9) is explicit after evaluating the integral,

$$\int_0^1 du \left( u^2 - 1 \right)^{n-1} = (-1)^{n-1} \frac{2^{2n-2} (n - 1)!^2}{(2n - 1)!}. \quad (C.10)$$

The charge (C.8) can be written in a more compact form by suppressing repeated indices,

$$q_i^j = \frac{1}{16\pi G (2n - 1)! 2^{n-2}} \delta^{[i2n-1]} K_i \delta \left[ \left( \delta^{[2]} + 2\alpha (2n - 1) (2n - 2) R \right) \delta^{[2]} \right]^{n-2}$$

$$+ 16\pi G (2n - 1)! nc_{2n} \int_0^1 du \left( R + \frac{u^2}{\ell_{\text{eff}}^2} \delta^{[2]} \right)^{n-1}. \quad (C.11)$$
This compact notation is same as the one used in Appendix C.1. However, instead of carrying out this integral, we will pull out the constant $c_{2n}$ so that the $u$-integral is introduced in the first line as well. With this rearrangement, the charge density tensor can be written in a compact form as an integral

$$q_i^j = -\frac{n c_{2n}}{2^{n-2}(1 - \gamma)} \delta[j^{2[n-1]} K_i \delta \int_0^1 du \mathcal{Q}(R, u),$$

where we have used the same notation as in the even-dimensional case but with $2n$ replaced by $2n + 1$,

$$\Delta = \frac{1}{\ell_{\text{eff}}^2} \delta[2], \quad \gamma = \frac{2\alpha}{\ell_{\text{eff}}^2} (2n - 1)(2n - 2).$$

The integrand is given by

$$\mathcal{Q}(R, u) = (u^2 - 1)^{n-1} \Delta^{n-1} - (R + u^2 \Delta)^{n-1} + \gamma \left[R\Delta^{n-2} (u^2 - 1)^{n-1} + (R + u^2 \Delta)^{n-1}\right].$$

As in the previous subsection C.1, we will repeatedly use the formula (C.4) to factorize the expression of $\mathcal{Q}$ so that a factor of $\Delta + R$ can be pulled out.

Let us consider the first line of Eq. (C.14). Using the formula (C.4), we get

$$(u^2 - 1)^{n-1} \Delta^{n-1} - (R + u^2 \Delta)^{n-1} = -(n - 1)(R + \Delta) \int_0^1 ds \left[(u^2 - s) \Delta + (1 - s) R\right]^{n-2}.$$

The second line of $\mathcal{Q}$ in Eq. (C.14) needs repeated application of the formula (C.4). Before we use Eq. (C.4), let us first write the term as

$$(R + u^2 \Delta)^{n-1} + R\Delta^{n-2} (u^2 - 1)^{n-1} = (R + \Delta) (\Delta^{n-2} u^{2(n-1)} + Y),$$

where the tensor $Y$ is determined by demanding that it satisfies

$$(R + \Delta) Y = (R + u^2 \Delta)^{n-1} - (u^2 \Delta)^{n-1} + R\Delta^{n-2} \left[(u^2 - 1)^{n-1} - (u^2)^{n-1}\right].$$

To extract the form of $Y$, we will manipulate the right hand side of (C.17). This is done by applying Eq. (C.4), which results in

$$(R + \Delta) Y = (n - 1) R \int_0^1 ds \left[(sR + u^2 \Delta)^{n-2} - \Delta^{n-2} (u^2 - s)^{n-2}\right].$$

The expression on the right of (C.18) is still not convenient for us to read out the formula for $Y$. We will apply the same formula for the second time to the expression in the square brackets in (C.18). This gives us the desired result. After stripping off the $R + \Delta$ factor, we obtain

$$Y = (n - 1)(n - 2) R \int_0^1 ds \int_0^1 dt \left[tsR + (u^2 - s + ts) \Delta\right]^{n-3}.$$
Using Eqs.(C.15), (C.19), we can finally write the integrand $Q(R, u)$ in the factorized form,

$$Q(R, u) = (R + \Delta) \left\{ -(n - 1) \int_0^1 ds \left[ (u^2 - s) \Delta + (1 - s) R \right]^{n-2} + \gamma \Delta^{n-2} u^{2(n-1)} 
+ (n - 1) (n - 2) \gamma R \int_0^1 ds \int_0^1 dt \left[ tsR + (u^2 - s + ts) \Delta \right]^{n-3} \right\}. \quad (C.20)$$

Substituting this expression of $Q(R, u)$ in the integral representation of the charge density tensor, we get

$$q^i_j = \frac{-nc_{2n}}{2^{n-2} (1 - \gamma)} \delta^{[2n-1]}_j K_i (R + \Delta) \delta \int_0^1 du \left[ (1 - n) \int_0^1 ds \left( \frac{u^2 - s}{\ell_{\text{eff}}^2} \delta^{[2]} + (1 - s) R \right)^{n-2} 
+ \frac{\gamma}{\ell_{\text{eff}}^{2(n-2)}} (\delta^{[2]})^{n-2} u^{2(n-1)} 
+ (n - 1) (n - 2) \gamma R \int_0^1 ds \int_0^1 dt \left( tsR + \frac{u^2 - s + ts}{\ell_{\text{eff}}^2} \delta^{[2]} \right)^{n-3} \right]. \quad (C.21)$$

Like in the even-dimensional case we will write the charge density tensor in a compact form using

$$J(R + \Delta) = -(n - 1) \int_0^1 ds \left[ (u^2 - s) \Delta + (1 - s) R \right]^{n-2} + \gamma \Delta^{n-2} u^{2(n-1)} 
+ (n - 1) (n - 2) \gamma R \int_0^1 ds \int_0^1 dt \left[ tsR + (u^2 - s + ts) \Delta \right]^{n-3}. \quad (C.22)$$

Here, to simplify the notation, we have suppressed indices on $J$. As mentioned in the previous subsection, $R + \Delta$ is equal to the AdS curvature $F_{\hat{k}\hat{l}} = R_{\hat{k}\hat{l}} + \frac{1}{\ell_{\text{eff}}^2} \delta^{[2]}_{[\hat{k}\hat{l}]}$. We can therefore replace $R + \Delta$ in the charge density expression by the AdS tensor $F$,

$$q^i_j = \frac{nc_{2n} \ell_{\text{eff}}}{2^{n-2} \ell_{\text{eff}}^{2(n-1)} (1 - \gamma)} \delta^{[2n-1]}_j \delta_i \delta (\delta^{[2]})^{n-2} F J(F), \quad (C.23)$$

where again we have suppressed repeated indices on the Kronecker deltas.

References

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity”. *Int.J.Theor.Phys.* 38 (1999), pp. 1113–1133. DOI: 10.1023/A:1026654312961. arXiv:hep-th/9711200 [hep-th].
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory”. *Phys.Lett.* B428 (1998), pp. 105–114. DOI: 10.1016/S0370-2693(98)00377-3. arXiv:hep-th/9802109 [hep-th].

[3] E. Witten, “Anti-de Sitter space and holography”. *Adv.Theor.Math.Phys.* 2 (1998), pp. 253–291. arXiv:hep-th/9802150 [hep-th].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity”. *Phys.Rept.* 323 (2000), pp. 183–386. DOI: 10.1016/S0370-1573(99)00084-7. arXiv:hep-th/9905111 [hep-th].

[5] A. Ashtekar and A. Magnon, “Asymptotically anti-de Sitter space-times”. *Class.Quant.Grav.* 1 (1984), pp. L39–L44. DOI: 10.1088/0264-9381/1/4/002.

[6] A. Ashtekar and S. Das, “Asymptotically Anti-de Sitter space-times: Conserved quantities”. *Class.Quant.Grav.* 17 (2000), pp. L17–L30. DOI: 10.1088/0264-9381/17/2/101. arXiv:hep-th/9912320 [hep-th].

[7] D. P. Jatkar, G. Kofinas, O. Miskovic, and R. Olea, “Conformal Mass in AdS gravity”. *Phys.Rev.* D89 (2014), p. 124010. DOI: 10.1103/PhysRevD.89.124010. arXiv:1404.1411 [hep-th].

[8] R. Olea, “Regularization of odd-dimensional AdS gravity: Kounterterms”. *JHEP* 0704 (2007), p. 073. DOI: 10.1088/1126-6708/2007/04/073. arXiv:hep-th/0610230 [hep-th].

[9] G. Kofinas and R. Olea, “Vacuum energy in Einstein-Gauss-Bonnet AdS gravity”. *Phys.Rev.* D74 (2006), p. 084035. DOI: 10.1103/PhysRevD.74.084035. arXiv:hep-th/0606253 [hep-th].

[10] Y. Pang, “Brief Note on AMD Conserved Quantities in Quadratic Curvature Theories”. *Phys.Rev.* D83 (2011), p. 087501. DOI: 10.1103/PhysRevD.83.087501. arXiv:1101.4267 [hep-th].

[11] R. C. Myers, “Higher Derivative Gravity, Surface Terms and String Theory”. *Phys.Rev.* D36 (1987), p. 392. DOI: 10.1103/PhysRevD.36.392.

[12] O. Miskovic and R. Olea, “Counterterms in Dimensionally Continued AdS Gravity”. *JHEP* 0710 (2007), p. 028. DOI: 10.1088/1126-6708/2007/10/028. arXiv:0706.4460 [hep-th].

[13] Y. Brihaye and E. Radu, “Black hole solutions in $d = 5$ Chern-Simons gravity”. *JHEP* 1311 (2013), p. 049. DOI: 10.1007/JHEP11(2013)049. arXiv:1305.3531 [gr-qc].

[14] D. Kastor and R. B. Mann, “On black strings and branes in Lovelock gravity”. *JHEP* 0604 (2006), p. 048. DOI: 10.1088/1126-6708/2006/04/048. arXiv:hep-th/0603168 [hep-th].

[15] D. G. Boulware and S. Deser, “String Generated Gravity Models”. *Phys.Rev.Lett.* 55 (1985), p. 2656. DOI: 10.1103/PhysRevLett.55.2656.

[16] O. Miskovic and R. Olea, “Conserved charges for black holes in Einstein-Gauss-Bonnet gravity coupled to nonlinear electrodynamics in AdS space”. *Phys.Rev.* D83 (2011), p. 024011. DOI: 10.1103/PhysRevD.83.024011. arXiv:1009.5763 [hep-th].

[17] S. Deser and B. Tekin, “Energy in generic higher curvature gravity theories”. *Phys.Rev.* D67 (2003), p. 084009. DOI: 10.1103/PhysRevD.67.084009. arXiv:hep-th/0212292 [hep-th].
[18] N. Deruelle, J. Katz, and S. Ogushi, “Conserved charges in Einstein Gauss-Bonnet theory”. Class. Quant. Grav. 21 (2004), pp. 1971–1985. DOI: 10.1088/0264-9381/21/8/004. arXiv:gr-qc/0310098 [gr-qc].

[19] A. N. Petrov, “Noether and Belinfante corrected types of currents for perturbations in the Einstein-Gauss-Bonnet gravity”. Class. Quant. Grav. 28 (2011), p. 215021. DOI: 10.1088/0264-9381/28/21/215021. arXiv:1102.5636 [gr-qc].

[20] Y. Brihaye and E. Radu, “Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method”. JHEP 0809 (2008), p. 006. DOI: 10.1088/1126-6708/2008/09/006. arXiv:0806.1396 [gr-qc].

[21] J. T. Liu and W. A. Sabra, “Hamilton-Jacobi Counterterms for Einstein-Gauss-Bonnet Gravity”. Class. Quant. Grav. 27 (2010), p. 175014. DOI: 10.1088/0264-9381/27/17/175014. arXiv:0807.1256 [hep-th].

[22] L. Andrianopoli and R. D’Auria, “N=1 and N=2 pure supergravities on a manifold with boundary”. JHEP 1408 (2014), p. 012. DOI: 10.1007/JHEP08(2014)012. arXiv:1405.2010 [hep-th].

[23] O. Miskovic and R. Olea, “Topological regularization and self-duality in four-dimensional anti-de Sitter gravity”. Phys. Rev. D79 (2009), p. 124020. DOI: 10.1103/PhysRevD.79.124020. arXiv:0902.2082 [hep-th].