Entanglement versus gap, quantum teleportation, and the AKLT model

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Abstract
Quantum-mechanical entanglement is notoriously volatile because of its susceptibility to external disturbances. However, entanglement can be stabilized if it is present in the non-degenerate ground state of a gapped, time-independent Hamiltonian. In this paper, we devise a spin-chain Hamiltonian whose ground state contains a Bell pair, with one member of the pair at each end of the chain. We study the Hamiltonian numerically, using full numerical diagonalization and a carefully tailored mean-field theory, to show that it is gapped. Whenever the Hamiltonian is tuned to increase its gap, the fidelity of its Bell pair decreases, manifesting a fundamental contention. The form of the Hamiltonian is motivated by quantum teleportation. Comparing it to the canonical Affleck, Kennedy, Lieb, and Tasaki (AKLT) model, we find that the AKLT model exhibits a sort of ‘failed quantum teleportation’.

Keywords: Affleck, Kennedy, Lieb, and Tasaki model, quantum teleportation, spin chain, entanglement

(Some figures may appear in colour only in the online journal)

1. Introduction

Generically, quantum-mechanical entanglement is vulnerable to environmental noise. However, there are exceptions. For example, two neighboring localized spins get robustly locked into a spin-singlet when coupled by a strong antiferromagnetic interaction. It is natural to ask whether an approach like this can be employed broadly to stabilize entangled particles, even if the entangled particles are far apart. One might envisage a system that provides a resource of entangled pairs without requiring active intervention. Here, we formulate a specific Hamiltonian whose non-degenerate ground state contains a long-range Bell pair. This model offers a particularly clear account of the trade-offs associated with protecting long-range entanglement passively in the ground state of a gapped Hamiltonian. The Hamiltonian has a parameter that can be tuned to increase the fidelity of the ground state entanglement at the cost of decreasing the size of the gap.

We refer to the model as the ‘teleportation’ Hamiltonian since its structure is motivated by quantum teleportation. Quantum teleportation [1] is an essential protocol in quantum information theory. It is directly tied to a number of other protocols such as a ‘superdense’ encoding scheme for data transmission [2], a universal quantum computation method [3], and a strategy for fault-tolerant quantum error correction [4]. The ‘teleportation’ Hamiltonian presented here has a ground state designed to provide a time-independent emulation [5–7] of the standard time-dependent quantum teleportation protocol.

Section 2 of this paper describes the teleportation Hamiltonian and its non-degenerate ground state. The ground state possesses a Bell pair with one member at each end of the chain; the fidelity of this pair can be computed analytically. In section 3, we analyze the teleportation Hamiltonian using a carefully designed mean field theory inspired by full numerical diagonalization results. We conclude that the Hamiltonian is gapped. As expected from general arguments [8, 9], the gap of the spin
chain cannot be increased without decreasing the fidelity of the Bell pair in the ground state. This dependence is calculated quantitatively in section 4. Section 5 investigates the relationship between this teleportation Hamiltonian and the canonical Affleck, Kennedy, Lieb, and Tasaki (AKLT) model [10–12]. In a sense, the AKLT model can be regarded as exhibiting ‘failed quantum teleportation’.

2. Hamiltonian and ground state

2.1. Short chain

A diagram of a spin chain appears in figure 1(a). It has a Hilbert space of dimension $2 \otimes (3 \otimes 3)^{2\ell} \otimes 2 = 2 \otimes 3^{2\ell} \otimes 2$. We will refer to the length of the chain as $\ell$ since our construction presumes the three-dimensional spaces come in pairs; our construction does not permit a chain of Hilbert space dimension $2 \otimes 3^{2\ell+1} \otimes 2$. The chain can be regarded as a system composed of qutrits with a qubit capping each end or as a chain of spins of magnitude 1 with a spin of magnitude 1/2 capping each end.

Before studying the full chain, we consider the right-most unit, which has a 3 \otimes 3 \otimes 2 dimensional Hilbert space describing two qutrits and a qubit. It is depicted in figure 1(b). Its basis is defined by $\{|0\rangle, |1\rangle, |\text{idle}\rangle\} \otimes \{|0\rangle, |1\rangle, |\text{idle}\rangle\} \otimes \{|\downarrow\rangle, |\uparrow\rangle\}$; the motivation behind the choice of ket label \text{idle} will become clear. The notation $|\downarrow\rangle, |\uparrow\rangle$ is introduced to distinguish clearly between the states of a qubit and the states $|0\rangle, |1\rangle$ of a qubit. We take the Hamiltonian of the rightmost unit of the chain to be

$$H^R(\theta) = I \otimes H^{BR} + H^I(\theta) \otimes I_2,$$

(1)

where $I$ is the identity operator for a qutrit, $I_2$ is the identity operator for a qubit,

$$H^{BR} =$$

$$\epsilon \left((1) \otimes |\downarrow\rangle - |0\rangle \otimes |\uparrow\rangle\right)(1) \otimes |\downarrow\rangle - |0\rangle \otimes |\uparrow\rangle$$

$$+ (1) \otimes |\downarrow\rangle + |0\rangle \otimes |\uparrow\rangle)(1) \otimes |\downarrow\rangle + |0\rangle \otimes |\uparrow\rangle$$

$$+ (|0\rangle \otimes |\downarrow\rangle - |1\rangle \otimes |\uparrow\rangle)(|0\rangle \otimes |\downarrow\rangle - |1\rangle \otimes |\uparrow\rangle) / 2,$$

(2)

and

$$H^I(\theta) =$$

$$\epsilon \left( \sin \theta \sqrt{\frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{2}} - \cos \theta |\text{idle}\rangle \otimes |\text{idle}\rangle \right)$$

$$+ \sum_{b=0,1} (|\text{idle}\rangle \otimes |b\rangle)(|\text{idle}\rangle \otimes \langle b|)$$

$$+ \sum_{b=0,1} (|b\rangle \otimes |\text{idle}\rangle)(|b\rangle \otimes \langle \text{idle}|),$$

(3)

The quantity $\epsilon$ has units of energy. In (1), the term $I \otimes H^{BR}$ acts on the right qutrit and its neighboring qubit. The term $H^I(\theta) \otimes I_2$ acts on the two qutrits.

The Hamiltonian (1) is motivated by the teleportation circuit [1, 2] shown in figure 2. $H^{BR}$ produces the Bell pair needed for teleportation; the superscript BR indicates that this Bell pair is at the right end of the chain. The projector Hamiltonian $H^I(\theta)$ performs a Hamiltonian analogue of the basis measurement. Using them in conjunction, $H^I(\theta)$ emulates teleportation of a qubit of information from left to right. To see this, note that the unit shown in figure 1(b) has a two-dimensional zero-energy ground state space spanned by

$$|\psi_0(0)\rangle = \frac{\cos \theta |0\rangle \otimes (|0\rangle \otimes |\downarrow\rangle + |1\rangle \otimes |\uparrow\rangle)}{\sqrt{\cos^2 \theta + (1/4) \sin^2 \theta}}.$$

(4)

and
In $|\psi_0(0)\rangle$, the first term (proportional to $\cos \theta$) includes an initial state $|0\rangle$ alongside an initial Bell pair. The form of this initial Bell pair is dictated by $H^{B\cdot L}$. The second term (proportional to $\sin \theta$) corresponds to successful teleportation of $|0\rangle$ to $|\downarrow\rangle$ after the Bell measurement. This term arises because $H^B(\theta)$ uses an extra ‘post-measurement’ state $|\text{idle}\rangle \otimes |\text{idle}\rangle$ to amplify the part of the first term that would result from a basis measurement with outcome $(0) \otimes (0) + (1) \otimes (1)$. inspecting the form of (4), one sees that $\theta \approx 0$ produces negligible amplification while $\theta$ close to $\pi/2$ produces strong amplification and therefore successful teleportation. Note that the two sums in $H^B(\theta)$ impose an energy penalty unless two particles transition to the post-measurement state concomitantly. The form of $|\psi_0(1)\rangle$ is clearly analogous to that of $|\psi_0(0)\rangle$. More generally, the zero energy state $\alpha |\psi_0(0)\rangle + \beta |\psi_0(1)\rangle$ describes teleportation of $\alpha |\downarrow\rangle + \beta |\uparrow\rangle$ into $|\downarrow\rangle + |\uparrow\rangle$. The chain depicted in figure 1(b) has a two-dimensional ground state space spanned by the state vectors (4) and (5), but our goal is to devise a chain with a non-degenerate ground state containing a long-range entangled pair. To achieve this, we add a qubit and couple it to the left qutrit with

$$H^{B\cdot L} = \epsilon \left[ |\uparrow\rangle \otimes |0\rangle - |\downarrow\rangle \otimes |1\rangle \right] \left( |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle \right) / \sqrt{2} + \cos \theta |\uparrow\rangle \otimes |\downarrow\rangle + \sin \theta |\text{idle}\rangle \otimes |\text{idle}\rangle / \sqrt{2}.$$ 

Here, the superscript B, L refers to the creation of a Bell pair at the left end of the chain. The Hilbert space is then 2 \otimes 3 \otimes 2 dimensional, realizing the spin chain in figure 1(a) in the case $\ell = 1$. Its Hamiltonian is $H^B(\theta) \otimes I + I \otimes H^B(\theta)$. Its unique ground state is $(|\downarrow\rangle \otimes |\psi_0(0)\rangle + |\uparrow\rangle \otimes |\psi_0(1)\rangle)/\sqrt{2}$.  

2.2. Long chain

For longer chains with length $\ell > 1$, the system shown in figure 1(a) emulates repeated teleportation down the chain. The total Hamiltonian is

$$H(\theta) = H^{B\cdot L} \otimes I^{\otimes (2\ell-1)} \otimes I_2 + I_2 \otimes \sum_{j=0}^{\ell-2} I^{(2j)} \otimes H(\theta) \otimes I^{(2\ell-3-2j)} \otimes I_2 + I_2 \otimes I^{(2\ell-2)} \otimes H^B(\theta),$$

where $H(\theta) = I \otimes H^B + H^B(\theta) \otimes I$ and

$$H^B = \epsilon \left[ |1\rangle \otimes (|0\rangle \otimes |\downarrow\rangle + |1\rangle \otimes |\uparrow\rangle) / \sqrt{2} + \cos \theta |\text{idle}\rangle \otimes |\text{idle}\rangle \otimes |\mathcal{I}\rangle \right] / \sqrt{2}.$$ 

This creates a Bell pair.

Its non-degenerate ground state is most easily described by defining an operator $g_0 = |\psi_0(0)\rangle \langle \downarrow| + |\psi_0(1)\rangle \langle \uparrow|$ in terms of (4) and (5). This operator is a map from a two-dimensional qubit Hilbert space to a $2 \otimes 3 \otimes 2$ dimensional Hilbert space. In this paper, it always acts on the qubit at the right end of the spin chain. In terms of $g_0$, the ground state has the form

$$|\Psi\rangle = (I_2 \otimes I^{\otimes (2\ell-2)} \otimes g_0) \ldots (I_2 \otimes I^{(2)} \otimes g_0)(I_2 \otimes g_0)|\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\uparrow\rangle / \sqrt{2}.$$ 

In the case $\ell = 1$, this reduces as expected to

$$\left( I_2 \otimes g_0 \right) |\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\uparrow\rangle / \sqrt{2} = |\downarrow\rangle \otimes |\psi_0(0)\rangle + |\uparrow\rangle \otimes |\psi_0(1)\rangle / \sqrt{2}.$$ 

Using the forms (4) and (5), we obtain

$$\cos \theta \left[ |1\rangle \otimes |0\rangle + |\uparrow\rangle \otimes \sqrt{2} |\text{idle}\rangle \otimes |\text{idle}\rangle \otimes |\mathcal{I}\rangle \right] / \sqrt{2} + \sin \theta \left[ |\text{idle}\rangle \otimes |\text{idle}\rangle \otimes |\mathcal{I}\rangle \right] / \sqrt{2}$$

a state inhabiting a Hilbert space of dimension $2 \otimes 3 \otimes 2$. We see that the second term (proportional to $\sin \theta$) is associated with the successful creation of a Bell pair linking the ends of the chain while the qutrits reside in $|\text{idle}\rangle \otimes |\text{idle}\rangle$. In agreement with our analysis of (4) and (5), this term dominates for $\theta \approx \pi/2$. In the case $\ell = 2$, we have

$$\left( I_2 \otimes I^{(2)} \otimes g_0 \right) (I_2 \otimes g_0)|\downarrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\uparrow\rangle / \sqrt{2}.$$ 

Applying the operator $(I_2 \otimes I^{(2)} \otimes g_0)$ to (10), we obtain

$$\cos \theta \left[ |1\rangle \otimes |0\rangle + |\uparrow\rangle \otimes \sqrt{2} |\text{idle}\rangle \otimes |\text{idle}\rangle \otimes |\mathcal{I}\rangle \right] / \sqrt{2} + \sin \theta \left[ |\text{idle}\rangle \otimes |\text{idle}\rangle \otimes |\mathcal{I}\rangle \right] / \sqrt{2}$$

The $\ell = 2$ state inhabits a Hilbert space of dimension $2 \otimes 3 \otimes 3 \otimes 2 = 2$ as expected.

2.3. Fidelity of ground state entanglement

We can verify that, in the ground state, the qubits capturing the ends of the chain are entangled into a Bell pair. To do this, we trace out the qutrits from the density matrix, computing $\text{Tr}_{3\otimes 3} \ldots \text{Tr}_{3\otimes 3} |\Psi\rangle \langle \Psi|$. Evaluating the trace is simplified...
by defining a superoperator that acts on a $2 \times 2$ dimensional density matrix $\rho$

$$g_0(\rho) \equiv \text{Tr}_{3:5} \tilde{g}_0 \rho \tilde{g}_0 = \frac{4(\text{Tr}_\rho \cos^2 \theta I_2/2 + \sin^2 \theta \rho)}{4 \cos^2 \theta + \sin^2 \theta}. \quad (11)$$

The second equality is obtained by inserting the definition of $\tilde{g}_0$. The superoperator $g_0(\rho)$ is a depolarizing channel that approaches perfect transmission as $\theta$ approaches $\pi/2$. We find that

$$\text{Tr}_{3:5} \ldots \text{Tr}_{3:5} |\Psi\rangle \langle \Psi| = \frac{1}{7} \sum_{b,b'=L,1} \sum_{b',b''} |b\rangle \langle b'\rangle \otimes g_0(\ldots g_0(|b\rangle \langle b'|) \ldots)$$

$$= \frac{1}{2} \frac{\sin^2 \theta}{4 \cos^2 \theta + \sin^2 \theta} \sum_{b,b'=L,1} |b\rangle \langle b'\rangle \otimes |b\rangle \langle b'|$$

$$+ \left(1 - \frac{\sin^2 \theta}{4 \cos^2 \theta + \sin^2 \theta}\right) \frac{I_2}{2} \otimes \frac{I_2}{2}. \quad (12)$$

Thus, the qubits ending up the ends are entangled into a Bell pair of density matrix $\sum_{b,b'=L,1} |b\rangle \langle b'\rangle \otimes |b\rangle \langle b'|$ with a fidelity that decreases as $(1/2)f(\theta)^2 + (1 - f(\theta)^2)/4 = (1 + 3f(\theta^2))/4$ with the length of the chain $L$. Here, we have defined

$$f(\theta) = \frac{\sin^2 \theta}{4 \cos^2 \theta + \sin^2 \theta}. \quad (13)$$

It approaches 1 when $\theta$ approaches $\pi/2$, in which case the qubits at the end of the chain are entangled into a nearly perfect Bell pair, while each qubit mostly resides in state $|\text{idle}\rangle$. Note that $(1 + 3f(\theta^2))/4$ is the same fidelity that would be obtained by forming a pair on adjacent qubits and then swapping the quantum information in one member of the pair $\ell$ times down a chain, undergoing the depolarizing channel (11) each swap.

Based upon the length dependence of $f(\theta)$, we can define a length scale $\ell_c(\theta)$ over which the coherence of the ground state decays. Setting $e^{-\ell_c}/\ell \approx f(\theta)^2$, we have

$$\ell_c(\theta) = \frac{1}{\ln(4 \cos^2 \theta + \sin^2 \theta)} \ln(\sin^2 \theta). \quad (14)$$

If we choose $\theta$ such that $\ell_c(\theta) \gg \ell$, then the ground state will include a Bell pair of reasonable fidelity entangling the qubits.

### 3. Energy gap

We now analyze the teleportation Hamiltonian (7) to show that it is ‘gapped’—that the energy gap between its ground state and first excited state is bounded below by a positive constant even as the length $L$ goes to infinity. The presence of a gap suppresses excitations, thereby stabilizing the Bell pair in the non-degenerate ground state. For the special point $\theta = 0$, the Hamiltonian decouples into independent units, and one sees from inspection of $\mathcal{H}(\theta)$ that there is a gap of size $\epsilon$ for all values of $\ell$. For the special point $\theta = \pi/2$, the system acquires a highly degenerate ground state and is no longer able to stabilize a Bell pair. What happens in the interesting region $0 < \theta < \pi/2$ in which there is a non-degenerate ground state with an approximate Bell pair entangling the ends of the chain?

To address this question, we study $\mathcal{H}(\theta)$ numerically. Figures 3(a) and (b) show gaps obtained from direct numerical diagonalization for small $\ell$, for various values of $\theta$. It certainly seems plausible that the system is gapped, but it is hard to be sure since the gaps have not completely leveled off as a function of $\ell$. One suspects that the continued $\ell$ dependence is caused by finite-size effects that would go away if we could simulate larger $\ell$ chains. Unfortunately, our numerical diagonalizations are limited to chains with Hilbert space dimension no larger than $2 \otimes (3 \otimes 3)^{\otimes 2}$. To overcome this issue, we devise a mean field, variational framework. This framework is tailored to suppress boundary effects in the numerical gap data and therefore to provide a better description of the system in the large $\ell$ limit.

To find an appropriate form for the variational state, we inspect our direct numerical diagonalization results for $\ell = 1$ to $\ell = 5$. For all these $\ell$, the first excited states comprise a degenerate triplet. (The ground state of the chain is always found to take the form (9), as expected.) We find that this excited triplet can be well described by states of the form

$$|\Psi_f\rangle \propto \left[ (I_2 \otimes I_2 \otimes 2 \otimes \tilde{g}_f) \ldots (I_2 \otimes I_2 \otimes \tilde{g}_f)(I_2 \otimes \tilde{g}_f) \right] \ldots$$

$$+ \left[ (I_2 \otimes I_2 \otimes 2 \otimes \tilde{g}_f) \ldots (I_2 \otimes I_2 \otimes \tilde{g}_f)(I_2 \otimes \tilde{g}_f) \right] \ldots$$

$$\cdots$$

$$\left[ (I_2 \otimes I_2 \otimes \tilde{g}_f)(I_2 \otimes \tilde{g}_f) \right] \left[ 1 \langle \downarrow \rangle \langle \uparrow | + | \uparrow \rangle \langle \downarrow | \right] \sqrt{2}, \quad (14)$$

where $\tilde{g}_f = |\psi_f(0)\rangle \langle \downarrow | + |\psi_f(1)\rangle \langle \uparrow |$ is defined in analogy to $\tilde{g}_0$. For instance, in the case $\ell = 3$ and $\theta = 1.56$, we can choose three variational states $|\Psi_1\rangle$, $|\Psi_2\rangle$, and $|\Psi_3\rangle$ of the form (14) that overlap with the three exact triplet states to within $10^{-3}$ of unity. To obtain this high overlap, for $|\Psi_1\rangle$, we set $\tilde{g}_f = \tilde{g}_1 = |\psi_1(1)\rangle \langle \downarrow | + |\psi_0(0)\rangle \langle \uparrow |$; for $|\Psi_2\rangle$, we set $\tilde{g}_f = \tilde{g}_2 = |\psi_1(1)\rangle \langle \uparrow | - |\psi_0(0)\rangle \langle \downarrow |$; and, for $|\Psi_3\rangle$, we set $\tilde{g}_f = \tilde{g}_3 = |\psi_0(0)\rangle \langle \downarrow | - |\psi_1(1)\rangle \langle \uparrow |$. Equation (14) can be regarded as a ‘spin-wave’ in which the excitation $\tilde{g}_f$ is traveling down the chain.

Motivated by the spin-wave form (14), we choose the following ansatz for excited states of a long chain

$$|\Psi_f^{(j)}\rangle = \frac{1}{\sqrt{\ell}} \sum_{j=0}^{\ell-1} \cos \frac{2\pi nj}{\ell} (I_2 \otimes I_2 \otimes 2 \otimes \tilde{g}_f) \ldots$$

$$+ \left[ (I_2 \otimes I_2 \otimes \tilde{g}_f)(I_2 \otimes \tilde{g}_f) \right] \left[ |\uparrow \rangle \langle \downarrow | + | \downarrow \rangle \langle \uparrow | \right] \sqrt{2}, \quad (15)$$

The $|\psi_f(b)|$ appearing in $\tilde{g}_f$ are to be determined by minimizing the expectation value of $\langle \Psi_f^{(j)} \mathcal{H}(\theta) |\Psi_f^{(j)}\rangle - E \left( \langle \Psi_f^{(j)} |\Psi_f^{(j)}\rangle - 1 \right) + \kappa(\sum_{b,b'}|\langle \psi_f(b)|\psi_f(b')|)^2$. The last term, which includes the Lagrange multiplier $\kappa$, is included
Figure 3. Energy gap as function of chain length in units of the Hamiltonian energy scale $\epsilon$. (a) Symbols show the gap computed by direct numerical diagonalization. Gaps for 5 different values $\theta = 0$, $\pi/12$, $\pi/6$, $\pi/4$, and $\pi/3$ are shown. Lines are guides to the eye. (b) Data from (a) plotted again with inverse length rather than length on abscissa. (c) Gap to lowest excited state energy as a function of unit cell length $\ell_u$ obtained by solving (16) for $|\Psi_f^{(2)}\rangle$ and analogous equations for $|\Psi_f^{(3)}\rangle$, $|\Psi_f^{(4)}\rangle$, and $|\Psi_f^{(5)}\rangle$. It is evident that the system is gapped. (d) Data from (c) plotted again with inverse length rather than length on abscissa.

as a means of imposing orthogonality between $|\psi_f(b)\rangle$ and $|\psi_0(b')\rangle$. This orthogonality ensures that $|\Psi_f^{(1)}\rangle$ is properly normalized by its $1/\sqrt{7}$ prefactor.

The minimization yields the variational equation

$$
\begin{align*}
\begin{bmatrix}
H_{0,0}(\theta) & H_{0,1}(\theta) \\
H_{1,0}(\theta) & H_{1,1}(\theta)
\end{bmatrix}
\begin{bmatrix}
|\psi_f(0)\rangle \\
|\psi_f(1)\rangle
\end{bmatrix}
= E_f \begin{bmatrix}
|\psi_f(0)\rangle \\
|\psi_f(1)\rangle
\end{bmatrix},
\end{align*}
$$

(16)

where we have ignored small corrections at the ends of the chain that vanish in the limit $\ell \gg 1$. In deriving this equation, we have defined operators $H_{0,0}(\theta)$, $H_{0,1}(\theta)$, $H_{1,0}(\theta)$, and $H_{1,1}(\theta)$, the form of which are given in the appendix. We solve (16) numerically. The lowest excited states comprise a degenerate triplet, the form of which is described in the appendix in the case of $\theta$ approaching $\pi/2$.

To check for gapped behavior, we double the size of the ‘unit cell’ in (15), writing

$$
|\Psi_f^{(2)}\rangle = \sqrt{\frac{2^{3(1/2) - 1}}{\ell}} \sum_{j=0}^{\ell} \cos \frac{4\pi n j}{\ell} \left(I_2 \otimes I^{2j - 2} \otimes \hat{g}_0 \right) \left(I_2 \otimes I^{2j + 2} \otimes \hat{g}_0 \right) \ldots
$$

(17)

where we still have $\hat{g}_f^{(2)} = |\psi_f^{(2)}(0)\rangle \langle 4 | + |\psi_f^{(2)}(1)\rangle \langle 5 |$, but now $|\Psi_f^{(2)}(b)\rangle$ is defined on a $3 \otimes 3 \otimes 3 \otimes 3 \otimes 2$ dimensional space. We can recover (15) by taking

$$
\cos \frac{4\pi n j}{\ell} \hat{g}_f^{(2)} = \frac{\cos \frac{2\pi n j}{\ell} (I^{02} \otimes \hat{g}_f) \hat{g}_0 + \cos \frac{4\pi n j}{\ell} (I^{02} \otimes \hat{g}_0) \hat{g}_f}{\sqrt{2}}.
$$

but the form (17) is more general than (15) because it allows arbitrary behavior within a doubled unit cell with a $3 \otimes 3 \otimes 3 \otimes 3 \otimes 2$ dimensional Hilbert space rather than within a single unit cell with a $3 \otimes 3 \otimes 2$ dimensional Hilbert space. We minimize the energy to derive an equation analogous to (16) for $|\psi_f^{(2)}(b)\rangle$.

We then repeat the calculation with a triple-size unit cell ($\ell_u = 3$), a quadruple-size unit cell ($\ell_u = 4$), and a quintuple-size unit cell ($\ell_u = 5$). The corresponding variational states, $|\Psi_f^{(3)}\rangle$, $|\Psi_f^{(4)}\rangle$, and $|\Psi_f^{(5)}\rangle$, respectively, are less and less constrained, and the corresponding numerical calculations are more and more demanding. To find the lowest excited energies of the system, we take $2\pi n/\ell = 0$ in (15).
In section 2, we determined that the ground state of the Hamiltonian contains a long-range Bell pair. Our numerical conjecture[13].

Hamiltonian is gapped then seems consistent with Haldane’s that selects out a single ground state. The fact that our spin particles are coupled by an antiferromagnetic-type interaction at each ‘site’ of our spin chain. These effective spin-2 par

\[|\psi_0\rangle|\text{idle}\rangle\]

|\text{idle}\rangle\text{ idle}

energetically penalized by the sums in (3). The remain-

\[5\text{ remain-}\]

the endsofthe chain, it assumes a finite unit cell of size \(\ell_u\). Our largest calculations have \(\ell_u = 5\). Thus, when the length (13) reaches \(\ell_u(\theta) \gtrsim 5\), some remaining finite-size effects can come into play that are not eliminated by our variational method.

To explore this, we choose a very extreme value \(\theta = 1.56\) for which \(f(1.56) > 0.9995\) and coherence stretches over a length scale \(\ell_u(1.56) > 2000\). Figures 4(a) and (b) present direct numerical diagonalization calculations for \(\theta = 1.56\).

They exhibit a pronounced dependence of the gap on \(\ell\) that seems well-fit by a \(1/\ell^2\) curve for \(\ell > 1\). Applying our variational framework produces the data shown in figures 4(c) and (d). Suppressing boundary effects has reduced the energy many orders of magnitude below the direct numerical diagonalization results in figures 4(a) and (b). As anticipated given the large value of \(\ell_u\), some dependence on \(\ell_u\) is still present in the data. However, we emphasize that the dependence is very weak. To demonstrate this in an objective fashion (rather than just choosing the scale for the ordinate in figures 4(c) and (d)) we compare the data to curves that are proportional to \(1/\ell^2\) and to \(1/\ell_u^2\). The \(\ell_u\) dependence of the data is significantly weaker than either curve, and it thus seems quite convincing that the gap tends to a positive value rather than to zero as \(\ell_u\) gets large. We conclude that the system is gapped throughout \(0 \leq \theta < \pi/2\).

To put it another way: figures 3(c) and (d) showed that the system is gapped for a range of values \(\theta \geq 0\). Based on figures 4(c) and (d), it appears exceedingly unlikely that the system transitions at some \(\theta_0 < \pi/2\) to an ungapped phase for \(\pi/2 > \theta > \theta_0\).

It is worth noting that, within the 3 \(\otimes 3\) Hilbert spaces along the spin chain in figures 1 and 4 of 9 states have the form \(|\text{idle}\rangle \otimes |0\rangle, |\text{idle}\rangle \otimes |1\rangle, |0\rangle \otimes |\text{idle}\rangle, |1\rangle \otimes |\text{idle}\rangle\) and are energetically penalized by the sums in (3). The 5 remaining states \(|\text{idle}\rangle \otimes |\text{idle}\rangle, |0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\) can be regarded as forming an effective spin-2 particle at each ‘site’ of our spin chain. These effective spin-2 particles are coupled by an antiferromagnetic-type interaction that selects out a single ground state. The fact that our spin Hamiltonian is gapped then seems consistent with Haldane’s conjecture [13].

4. Entanglement versus gap

In section 2, we determined that the ground state of the Hamiltonian contains a long-range Bell pair. Our numerical

\[\text{Energy gap as function of chain length in units of the}\]

\(\text{Hamiltonian energy scale. (a) Symbols show the gap computed by}\]

\(\text{direct numerical diagonalization at } \theta = 1.56.\) Pronounced length dependence is evident; solid line is a fit to \(1/\ell^2\). A point at \(6.7 \times 10^{-5}\) for \(\ell = 1\) does not fit on figure. (b) Data from (a) plotted again with inverse length rather than length on abscissa. (c) Gap to lowest excited state energy as a function of unit cell length \(\ell_u\) obtained by solving (16) for \(|\psi_{n}\rangle\) and analogous equations for \(|\psi_{n}^{(2)}\rangle, \ldots, |\psi_{n}^{(5)}\rangle\). The dashed line, showing \(1/\ell_u\) behavior, and the solid line, showing \(1/\ell_u^2\) behavior, cannot account for the data. The \(\ell_u\) dependence is extremely weak, giving solid evidence for gapped behavior even at \(\theta = 1.56\). (d) Data from (c) plotted again with inverse length rather than length on abscissa.
calculations in section 3 indicated that the Hamiltonian is gapped. Thus, one might indeed imagine using this teleportation Hamiltonian, as described in section 1, to provide a stable Bell pair without active intervention. To achieve this, it would be desirable to maximize the fidelity of the Bell pair by choosing the value of \( \theta \) to be as close as possible to \( \pi/2 \). At the same time, we would like to have a large gap to stave off thermal excitations and weak environmental perturbations. Unfortunately, in accordance with general arguments [8, 9], it is not possible to simultaneously maximize both quantities. The antagonistic relationship between the two quantities is shown in figure 5, in which the value of \( \theta \) is tuned to change both \( f(\theta) \) and the gap. When one quantity goes up, the other quantity goes down; we are faced with a trade-off. It is necessary to choose \( \theta \) sufficiently below \( \pi/2 \) to keep the gap large enough to stave off thermal excitations. But then, the maximum practical length \( \ell \) of the chain is constrained to around \( \ell_c(\theta) \) to avoid an unacceptably low Bell pair fidelity.

5. AKLT and failed quantum teleportation

It is instructive to compare our parent spin Hamiltonian and ground state to those of the well-known AKLT model [10–12]. The AKLT model was introduced to investigate spin-1 Heisenberg antiferromagnetic chains. AKLT were able to prove that their Hamiltonian is gapped [10, 11], supporting Haldane’s conjecture [13] that integer spin Heisenberg antiferromagnet chains are gapped. Laboratory experiments on a number of such systems, such as NENP, have indeed found evidence of gapped behavior (for a review, see [14]).

The AKLT ground state is a ‘valence-bond solid’ that can be obtained by imagining a chain of sites with two spin-1/2 particles on each site. At site \( i \) of the chain, we arbitrarily choose one of the spin-1/2 particles and form a singlet between it and a spin-1/2 particle at site \( i + 1 \); we then form a singlet between the remaining spin-1/2 particle at site \( i \) and a spin-1/2 particle at site \( i - 1 \). We symmetrize the resulting state over all such arbitrary choices at every site \( i \). Equivalently, instead of symmetrizing, we project out the singlet part of the state at each site \( i \), leaving only the symmetric spin triplet part. The result is a state with an effective spin-1 particle \( S_i \) at each site \( i \). The spin Hamiltonian is obtained by projecting adjacent sites \( i \) and \( i + 1 \) on to spin 2

\[
H_{\text{AKLT}} = \sum_i \left( S_i \cdot S_{i+1} \right)
\]

\[
\propto \sum_i S_i \cdot S_{i+1} + \frac{1}{3} (S_i \cdot S_{i+1})^2 + \text{const.} \quad (18)
\]

This Hamiltonian annihilates the AKLT ground state: of the 4 spin-1/2 particles at sites \( i \) and \( i + 1 \), two of them are bound into a spin singlet with total spin 0 and the remaining two spin-1/2 particles can produce a total spin of at most 1. Thus, the projection on to spin 2 two vanishes.

Suppose that we change our convention in equations (4) and (5) so that \( (|0\rangle \otimes |\uparrow\rangle) + (|1\rangle \otimes |\downarrow\rangle) / \sqrt{2} \) is replaced with the singlet part \( (|0\rangle \otimes |\uparrow\rangle - |1\rangle \otimes |\downarrow\rangle) / \sqrt{2} \). We change the Hamiltonians (2), (3), (6), and (8) accordingly. Examining equation (3), we then see that it causes the singlet part of two spin-1/2 particles to transition to the state \( |\text{IDLE} \rangle \otimes |\text{IDLE} \rangle \). This is reminiscent of the step during the construction of the AKLT ground state in which we project out the singlet part of the two spin-1/2 particles at each site \( i \). However, the effect of including equation (3) in our Hamiltonian is to amplify the relative contribution of the singlet state by a factor of \( \tan \theta \) rather than to project it out. As a result, our spin Hamiltonian emulates correct teleportation as \( \theta \) approaches \( \pi/2 \), and the function \( f(\theta) \) appearing in (12) falls off relatively slowly. In contrast, the AKLT ground state emulates failed teleportation in which projecting out the singlet state corresponds to Bell measurement of a triplet state. Then, since no Pauli operator gets applied to correct the teleported state as quantum teleportation requires, AKLT correlations fall off very rapidly [12].

There has been some interest in using spin chains as quantum information buses. In particular, we note that the AKLT Hamiltonian [15] has appeared in a proposal for a quantum channel in which measurement of the spin-1 particles in the AKLT chain and application of Pauli operation corrections allows teleportation of a quantum state along the chain. This is closely related to our statement above that the AKLT ground state enacts an time-independent emulation of failed teleportation. In this paper, we have considered using our Hamiltonian as a sort of quantum bus to passively serve up stable, but imperfect, Bell pairs. This bus would function quite differently than the channel of [15] since it would have limited fidelity but would require no active measurement to provide a Bell pair separated by the chain length.

6. Conclusion

We have proposed a spin chain Hamiltonian with a ground state that emulates teleportation. The ground state possesses
an approximate Bell pair between the qubits at the end of the chain. Guided by full numerical diagonalization results, we framed a mean field theory and concluded that the teleportation Hamiltonian is gapped. The Hamiltonian therefore stabilizes the approximate Bell pair. However, the fidelity of the Bell pair goes inversely with the gap.

One could imagine fabricating the Hamiltonian using a line of quantum dots, perhaps along the lines of the Hubbard goesinverselywith the gap. However, the fidelity of the Bell pair appears such an investigation.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

We specify here the form of the operators $H_{0,0}(\theta)$ and $H_{0,1}(\theta)$ appearing in (16).

\[
H_{0,0}(\theta) = H^0(\theta) \otimes I_2 + \frac{4 \cos^2 \theta}{4 \cos^2 \theta + \sin^2 \theta} (I \otimes H^{B,R} + \frac{\ket{0} \bra{0} + 2 \ket{1} \bra{1}}{2} \otimes I \otimes I_2 \\
+ \cos \frac{2 \pi n}{\ell} \left( \ket{0} \bra{1} \otimes \frac{\ket{0} \bra{1} \otimes \ket{1} \bra{1} + \ket{1} \bra{1} \otimes \ket{0} \bra{0}}{2} \right) + \ket{1} \bra{0} \otimes \frac{\ket{0} \bra{0} \otimes \ket{1} \bra{1} - \ket{1} \bra{1} \otimes \ket{0} \bra{0}}{4} \\
+ \frac{\cos \frac{2 \pi n}{\ell}}{ \ell} \left( \ket{0} \bra{0} \otimes \ket{0} \bra{0} + \ket{1} \bra{1} \otimes \ket{1} \bra{1} + \ket{0} \bra{0} \otimes \ket{1} \bra{1} - \ket{1} \bra{1} \otimes \ket{0} \bra{0} \right) \right) \\
+ \chi \left( \ket{\psi(0)} \bra{\psi(0)} + \ket{\psi(1)} \bra{\psi(1)} \right)
\]

and

\[
H_{0,1}(\theta) = \frac{4 \cos^2 \theta}{4 \cos^2 \theta + \sin^2 \theta} \left( -\frac{\ket{1} \bra{1}}{2} \otimes I \otimes I_2 \\
+ \cos \frac{2 \pi n}{\ell} \left( \ket{0} \bra{0} \otimes \frac{\ket{0} \bra{0} \otimes \ket{0} \bra{0} + \ket{1} \bra{1} \otimes \ket{1} \bra{1} + \ket{0} \bra{0} \otimes \ket{0} \bra{0} - \ket{1} \bra{1} \otimes \ket{1} \bra{1}}{2} \right) \right) \\
- \frac{\cos \frac{2 \pi n}{\ell}}{ \ell} \left( \ket{1} \bra{1} \otimes \ket{1} \bra{1} + \ket{0} \bra{0} \otimes \ket{0} \bra{0} + \ket{1} \bra{1} \otimes \ket{0} \bra{0} - \ket{0} \bra{0} \otimes \ket{1} \bra{1} \right) \right)
\]
The operators $H_{1,0}(\theta)$ and $H_{1,1}(\theta)$ are defined by taking $|1\rangle \leftrightarrow |0\rangle$ and $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$ in $H_{0,1}(\theta)$ and $H_{0,0}(\theta)$ respectively.

As stated in the text, the lowest excited states of the mean field calculation (16) form a triplet. For $\theta \to \pi/2$, the triplet of excited states has the form

$$|\psi_1(0)\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |\uparrow\rangle + |1\rangle \otimes |\downarrow\rangle \right),$$

$$|\psi_1(1)\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |\uparrow\rangle + |1\rangle \otimes |\downarrow\rangle \right),$$

$$|\psi_2(0)\rangle = -\frac{1}{\sqrt{2}} \left( |0\rangle \otimes |\uparrow\rangle - |1\rangle \otimes |\downarrow\rangle \right),$$

$$|\psi_2(1)\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |\uparrow\rangle - |1\rangle \otimes |\downarrow\rangle \right),$$

$$|\psi_3(0)\rangle = -\frac{1}{\sqrt{2}} \left( |0\rangle \otimes |\downarrow\rangle - |1\rangle \otimes |\uparrow\rangle \right),$$

$$|\psi_3(1)\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |\downarrow\rangle - |1\rangle \otimes |\uparrow\rangle \right),$$

corresponding to a bit-flip within the Bell pair such that $|\psi_1(b)\rangle = (I \otimes I \otimes X - X \otimes I \otimes I) |b\rangle \otimes (|0\rangle \otimes |\downarrow\rangle + |1\rangle \otimes |\uparrow\rangle)/2$, a phase flip such that $|\psi_3(b)\rangle = (I \otimes I \otimes Z - Z \otimes I \otimes I) |b\rangle \otimes (|0\rangle \otimes |\downarrow\rangle + |1\rangle \otimes |\uparrow\rangle)/2$, or a combined bit-and-phase flip such that $|\psi_2(b)\rangle = (I \otimes I \otimes XZ - XZ \otimes I \otimes I) |b\rangle \otimes (|0\rangle \otimes |\downarrow\rangle + |1\rangle \otimes |\uparrow\rangle)/2$. (For simplicity, we have focused on the single unit cell calculation rather than on the calculations with larger unit cells.)

Note that, for any $n$, equation (16) has 36 solutions, since $|\psi_1(0)\rangle$ and $|\psi_1(1)\rangle$ each occupies a $3 \otimes 3 \otimes 2$ dimensional Hilbert space.

In this sense, the basis is overcomplete; for many solutions $|\psi_2(b)\rangle$, the $(I_2 \otimes I^{2(2j-1)} \otimes \tilde{g}_j) (I_2 \otimes I^{2(2j-1)} \otimes \tilde{g}_0)$ part of (15) can be rewritten in terms of other solutions $|\psi_1(b)\rangle$ and $|\psi_3(b)\rangle$ in a form such as $(I_2 \otimes I^{2(2j-1)} \otimes \tilde{g}_j'(I_2 \otimes I^{2(2j-1)} \otimes \tilde{g}_0'))$.

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References

[1] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 Phys. Rev. Lett. 70 1895
[2] Bennett C H and Wiesner S J 1992 Phys. Rev. Lett. 69 2881
[3] Gottesman D and Chuang I L 1999 Nature 402 390
[4] Knill E 2005 Nature 434 39
[5] Mizel A, Mitchell M W and Cohen M L 2001 Phys. Rev. A 63 040302
[6] Mizel A, Mitchell M W and Cohen M L 2002 Phys. Rev. A 65 022315
[7] Mizel A 2004 Phys. Rev. A 70 012304
[8] Haselgrove H L, Nielsen M A and Osborne T J 2004 Phys. Rev. A 69 032303
[9] Hastings M B and Koma T 2006 Commun. Math. Phys. 265 781
[10] Affleck I, Kennedy T, Lieb E H and Tasaki H 1987 Phys. Rev. Lett. 59 799
[11] Affleck I, Kennedy T, Lieb E H and Tasaki H 1988 Commun. Math. Phys. 115 477
[12] Affleck I 1989 J. Phys.: Condens. Matter 1 3047
[13] Haldane F D M 1983 Phys. Lett. A 93 464
[14] Yamashita M, Ishii T and Matsuzaka H 2000 Coord. Chem. Rev. 198 347
[15] Verstraete F, Martin-Delgado M A and Cirac J I 2004 Phys. Rev. Lett. 92 087201
[16] Shim Y-P, Sharma A, Hsieh C-Y and Hawrylak P 2010 Solid State Commun. 150 2065