Stability of transverse vibration of rod under longitudinal step-wise loading

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Abstract. The problem of dynamic stability of a simply supported rod subjected to axial jump loading is considered. A systematic application of the method of expansion in terms of the normal axial and bending vibration modes is utilised. Longitudinal vibrations give rise to periodic longitudinal forces which in turn causes unstable bending vibrations. Application of the Galerkin approach results in a system of ordinary differential equations with periodic coefficients which are reduced to Mathieu equation. The instability regions whose form depends on the spectral properties of the longitudinal and flexural vibrations, damping values and longitudinal force are obtained. An example of unusual shapes of the instability regions is shown: the twelfth transverse mode caused by the first longitudinal mode turns out to be unstable for some parameters of the rod. The critical value of the jump load leading to instability of the considered transverse vibration modes is derived.

1. Introduction. The case history

The problem of the static buckling of the simply supported beam of length $l$ loaded by the longitudinal force $P$ was first set by L. Euler. This problem is now referred to as Euler elastica and the boundary-value problem is as follows

$$EI \frac{d^2 w}{dx^2} + Pw = 0, \quad w(0) = w(l) = 0,$$

where $EI$ denoted the bending stiffness and $w$ stands for the beam deflection. The solution is given by $w(x) = A \sin(\lambda x)$, $\lambda = \sqrt{P/EI}$ and the bifurcation takes place at $P_E = \pi^2 EI/l^2$ known as the Euler (buckling) force. The straight form of the beam ($A = 0$) remains stable if $P < P_E$. More realistic is the non-linear formulation of the problem in terms of the angle of rotation of the beam cross-section $\psi = dw/dx$. In this case the boundary-value problem is given by

$$EI \frac{d^2 \psi}{dx^2} + P \sin \psi = 0, \quad \frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(l) = 0,$$

whose solution gives the laterally deformed forms of equilibrium for each value of the longitudinal...
force $P > P_e$ in addition to the straight form. It should be noted that this formulation of the problem has nothing in common with the problem of transition from unstable equilibrium to a stable equilibrium because this approach operates only with the concept of bifurcation of solutions and the existence of the alternative forms of static equilibrium. From a perspective of the theory of motion stability the transition from the straight form to a laterally deformed form corresponds to the divergent loss of stability, also called the static instability, cf. [1].

For the first time the problem of dynamic buckling of a simply supported beam subjected to the axial jump loading was addressed in the article by Lavrentiev and Ishlinsky [2]. It was assumed that the beam has some initial curvature and the external compressive force applied to one end immediately covers the entire beam and then remains constant. This particular problem statement is inherently contradictory since the beam is assumed to behave statically in axial direction however stability analysis in transverse direction is carried out by means of dynamic approach. It is interesting that this specific dynamic formulation also predicts a divergent loss of stability of the original pre-curved form. It is shown that there is no one-to-one correspondence between the number of the Euler force and the number of the buckling mode with the highest rate of increase in the deflection. For example, if the rod is suddenly loaded by the third Euler force ($9P_e$) then the second buckling shape will have the highest rate of instability. When the fourth Euler force ($16P_e$) is applied, the fastest instability will develop in the third form, etc. It is easy to see that these assumptions are quite restrictive:

1. The result is sensitive to the shape of the initial deflection. The highest rate of increase in a particular buckling form only implies that this form will have a larger deflection at $t \to \infty$ regardless of its contribution to the initial deflection shape. For example, if the initial shape is orthogonal, say, to the second buckling form, i.e. this form does not represented in the initial deflection, then the conclusion of paper [2] that this buckling form possesses the highest rate of instability is useless for stability analysis as this particular form will not be excited at all. Another shortcoming is as follows.
2. The conclusions are valid only for the higher buckling modes,
3. Axial dynamics of the rod is not considered.

Let us also notice paper [3] where the authors carried out a study of the lateral stability of the rod taking into account the longitudinal waves generated by a jump of axial force.

In the present paper we propose an alternative approach to the problem by Lavrentiev and Ishlinsky [2]. In contrast to [2] and [3], we carry out a systematic application of the method of series expansion in terms of both axial and bending vibration modes. The longitudinal vibrations give rise to periodic longitudinal forces, which in turn can cause unstable bending vibrations. This phenomenon is known as the parametric resonance, cf. [4].

### 2. Regions of dynamical instability

Let us consider a simply supported beam with a constant cross-sectional area loaded at the end $x = l$ by an axial force $P(t) = P_0 H(t)$ where $H(t)$ is the Heaviside step function. Elastic axial vibrations are assumed, that is, we are allowed to expand the axial displacement in series in terms of the normal modes of axial vibration $u_k(x)$

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x)q_k(t), \quad u_k(x) = \sin \frac{\omega_k}{a} x, \quad \omega_k = \frac{\pi a}{2l} (2k - 1), \quad a = \sqrt{\frac{E}{\rho}}, \quad k = 1, 2, \ldots \quad (1)$$

where $\rho$ is the mass density of the rod and $\omega_k$ denotes the natural frequency of the axial vibration. Accounting for zero initial conditions one obtains the following expression for the generalized coordinate

$$q_k(t) = (-1)^{k+1} \frac{2a^2 P_0}{lEF \omega_k^2} (1 - \cos \omega_k t) \quad (2)$$

and the equation or the axial force in the form of a series
\[ P(x,t) = EF \frac{\partial u}{\partial x} = \frac{4}{\pi} P_0 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} (1 - \cos \omega_x t) \cos \left( k - \frac{1}{2} \right) \frac{x}{l}, \quad (3) \]

where \( F \) stands for the cross-sectional area of the rod.

The inner damping is introduced by means of the Kelvin-Voigt rheological visco-elastic model [5], then the governing equation for the transverse vibration of the model accounting for the varying axial force \( P(x,t) \) is given by

\[
E \left( 1 + \frac{\gamma}{\partial t} \right) I \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial x} \left( P(x,t) \frac{\partial w}{\partial x} \right) + \rho F \frac{\partial^2 w}{\partial t^2} = 0 \quad (4)
\]

where \( \gamma \) is a non-dimensional viscosity factor. The necessity of introducing the viscosity is due to the fact that the damping has a crucial influence on the shapes of the regions of dynamic instability at small magnitudes of the axial force, cf. [4]. The transverse displacement \( w(x,t) \) is represented as a series in terms of the normal modes of bending vibration

\[ w(x,t) = \sum_{m=1}^{\infty} \frac{m \pi x}{l} Q_m(t) \]

The normal modes of bending vibration \( \sin \frac{m \pi x}{l}, m = 1,2,3\ldots \) are also the buckling form of the simply supported beam. It means that both the beam vibration and beam buckling are accurately modeled.

Substituting the expression (3) for the axial force into equation (4) and applying Galerkin method we arrive at the infinite system of coupled ordinary differential equations for the generalised coordinates \( Q_m(t) \):

\[
\rho F \frac{1}{2} \dot{Q}_j(t) + E \gamma I \frac{\pi^4}{l^4} \frac{1}{2} \dot{Q}_j(t) + EI \frac{\pi^4}{l^4} \frac{1}{2} \dot{Q}_j(t) - \frac{EF \pi^2}{8l^2} \left[ \sum_m \sum_k (2k-1) \alpha_{mk} q_k(t) Q_m(t) \right] = 0 \quad (5)
\]

where

\[
\alpha_{mk} = \left( k - \frac{1}{2} + m \right) \frac{2j}{(k - \frac{1}{2} + m)^2 - j^2} (-1)^{k+m+j+1} + \left( k - \frac{1}{2} - m \right) \frac{2j}{(k - \frac{1}{2} - m)^2 - j^2} (-1)^{k-m-j+1},
\]

\( j \) and \( m \) are the numbers of the bending modes whereas \( k \) denotes the number of the axial mode. One can see that the generalized coordinates of bending vibration \( Q_j(t) \) depend on the coefficients of the generalized coordinates of axial vibration \( q_k(t) \) as well as on the other generalized coordinates of bending vibration \( Q_m(t) \). This coupling reflects the interaction of the generalized coordinates of bending vibration.

Let us insert expression (2) for \( q_k(t) \) into equation (5) and neglect the mutual interaction of the different modes of bending vibration, then only the term with \( m=j \) remains in the double series (5). This assumption is widely used in the vibration analysis and is known to deliver a good approximation. The result is a system of uncoupled equations, each describing the influence of the axial vibration mode (with number \( k \)) on the bending mode under consideration (with number \( m \)):

\[
\dot{Q}_m + \frac{E \gamma \pi^4 m^4}{\rho F l^4} \dot{Q}_m + \left\{ \frac{EI \pi^4}{\rho F l^4} m^4 + \frac{(-1)^j 2m \alpha_{mk}}{2k-1} \frac{P_0}{\rho Fl^2} (1 - \cos \omega_x t) \right\} Q_m = 0 \quad (6)
\]
Here the subscripts at \( Q_{mk} \) \((k = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots)\) correspond to number \( m \) of the bending vibration and number \( k \) of the axial vibration. This equation provides one with the answer to the question of the axial load magnitude \( P_0 \) leading to the instability of  \( m \)-th bending vibration caused by \( k \)-th axial vibration mode.

Let us rewrite equation (6) in the standard form of the problem of parametric instability, cf. [5,6]

\[
\ddot{Q}_{mk} + 2\delta_{mk} \Omega_{mk} \dot{Q}_{mk} + \Omega_{mk}^2 (1 - 2\mu_{mk} \cos \theta t) Q_{mk} = 0
\]  

(7)

where the coefficients are given by

\[
\Omega_{mk}^2 = \frac{EI\pi^4 m^4}{\rho F l^4} + \frac{(-1)^k 2m\alpha}{2k-1} \frac{P_0}{\rho F l^2}, \quad \mu_{mk} = \frac{(-1)^{k+1} \alpha P_0}{2k-1} \frac{EI\pi^4 m^4}{\rho F l^4} + \frac{(-1)^k 2\alpha}{2k-1} \frac{P_0}{\rho F l^2}
\]  

(8)

Before we begin to determine the instability regions let us state that these regions exist only under the condition \( \Omega_{mk}^2 > 0 \). If \( \Omega_{mk}^2 \leq 0 \) then the solution of equation (7) will be unbounded even for \( \mu_{mk} = 0 \), i.e. the corresponding bending mode will be unstable. Therefore the critical value of the axial force affected by \( k \)-th axial vibration mode can be determined from the condition

\[
\frac{EI\pi^4 m^4}{\rho F l^4} + \frac{(-1)^k 2m\alpha}{2k-1} \frac{P_0}{\rho F l^2} \leq 0
\]  

(9)

Provided that \( \Omega_{mk}^2 > 0 \), equation (7) is nothing else than the Mathieu equation with damping. It is known, cf. [4], that the damping does not considerably affect the main instability region and makes the side regions not feasible in practical engineering. Based on this remark, we restrict ourselves to the main region of dynamic instability.

In the above notation, the main region of instability of transverse vibration modes is determined by the systems of inequalities, cf. [4],

\[
2\Omega \sqrt{1 - \sqrt{\mu^2 - 4\delta^2}} \leq \theta \leq 2\Omega \sqrt{1 + \sqrt{\mu^2 - 4\delta^2}}
\]  

(10)

where in what follows the subscripts are omitted for simplicity. These inequalities are formally identical with the inequalities of the classical Mathieu equation however it should be borne in mind that the parameters \( \Omega, \mu, \delta \) are functions of \( P_0 \), i.e. the axial vibration changes fundamentally the formula for each parameter, see equation (8), thus drastically changing the dynamic instability regions. This system of inequalities describes implicitly the critical value of the load at which \( m \)-th transverse vibration mode will be unstable under the impact of \( k \)-th axial mode.

By constructing the dynamic instability regions defined by the inequalities (10) for all possible axial – bending pairs we can obtain the critical load values which are smaller than the values obtained from equation (9).

3. An example of calculation of the critical force

Let us determine the critical value of the force by constructing the regions of dynamic instability for the following parameters of the rod: \( E = 2.1 \cdot 10^{11} \text{N/m}^2 \), \( \rho = 7.8 \cdot 10^3 \text{kg/m}^3 \), \( l = 2 \text{m} \), \( h = 0.005 \text{m} \), \( b = 0.01 \text{m} \), \( \gamma = 10^{-4} \), where \( h \) and \( b \) are the height and width of the cross section respectively.
Figure 1 displays the effect of the first axial mode on the main instability region for the seventh-twelfth bending modes. In the standard problem of dynamic buckling of beams [4] the frequency of the axial force $\theta$ can be varied continuously. In the problem under consideration the frequency of the axial force $\theta$ belongs to the discrete spectrum of the natural frequencies of axial vibrations. The natural frequency of the first longitudinal mode of vibration is given by $\omega = \omega_1 = \pi a / 2l$. Figure 1 shows that the horizontal line $\theta = \omega_1$ does not cross the main instability regions for the bending modes with the number below 12, that is, the twelfth bending vibration mode becomes unstable and allows us to determine the minimum value of the load leading to instability twelfth transverse vibration mode which is denoted by $P_{cr}$. For the lower bending vibration modes the critical value of the force can be found from equation (9).

The estimates below are obtained with the help of the above criteria of dynamic instability. Euler’s force:

$$P_E = \frac{EI\pi^2}{l^2} \approx 53.97 \text{ N}.$$  

The condition of instability of the first bending mode:

$$P_0 \geq 1.32P_E \approx 71.3 \text{ N}.$$  

The condition of instability of the second bending mode:

$$P_0 \geq 5P_E \approx 270.6 \text{ N}.$$  

The condition of instability of the third bending mode:

$$P_0 \geq 11.2P_E \approx 603.44 \text{ N}$$ etc.

The condition of instability of the seventh bending mode:

$$P_0 \geq 3266.6 \text{ N}.$$  

For values of the load $P_0 \geq 3924.5 \text{ N}$ the twelfth bending mode can be theoretically unstable (in accordance with the results obtained from the consideration of equation (10)).
The condition of instability of the *eighth* bending mode:

\[ P_0 \geq 4265.3 \text{ N} \text{ etc.} \]

It should be noted that the location of the instability regions in the plane \((P_0, \theta)\) depends on the rod parameters (Young's modulus, coefficient of viscous damping etc.). Depending upon the rod geometry and the mechanical characteristics of the material the axial force can lead to dynamic instability of another bending mode rather than that of the twelfth bending mode.

4. Conclusions

Construction and analysis of the regions of dynamic instability lead to the following conclusions:
1. The standard approaches of static stability analysis predict the static (divergent) loss of stability. In contrast to them, the present study demonstrates that the dynamic (oscillatory) character of instability is also possible depending on the magnitude of the external load.
2. For the lower bending vibration modes the value of the critical load can be obtained from equation (9) whereas that for the higher modes - from conditions (10).
3. The dependence between the number of buckling mode and the axial load value is not directly proportional.
4. The rod buckling due to the highest modes is possible for values of the compressive force which are less than the values calculated by Euler's formula.
5. The smallest value of the critical load corresponds to the pair first axial mode – first bending mode. This value is greater than Euler’s force and is obtained from equation (9) to give \( P_{cr} = 1.3P_k \).
6. The formulae concerning the critical value of the axial force and the conclusion about instability due to the twelfth bending form are valid only for the particular geometric and mechanical parameters taken for the numerical work. If one takes other characteristics of the beam or frequency-dependent internal damping or other boundary conditions then one should expect that another bending form becomes unstable. Nevertheless, the general conclusions predicting both static and dynamic character of instability remain valid.
7. The result of the present analysis could be related to the practical and industrial applications, first of all, they are important from a perspective of the health of steel structural elements of buildings. The well-established standard methods of stability analysis rely on the static (linear or nonlinear) approaches. The present analysis has demonstrated that, in general, account for the dynamic effects of parametric resonance is required as well. This conclusion is of special importance for the structures subjected to intensive dynamic loading, such as those due to earthquake, explosion etc.

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