NUMERICAL SOLUTIONS FOR A TIMOSHENKO-TYPE SYSTEM WITH THERMOELASTICITY WITH SECOND SOUND

Makram Hamouda*1, Ahmed Bchatnia2 and Mohamed Ali Ayadi3

1Department of Basic Sciences, Deanship of Preparatory Year and Supporting Studies
Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 34212, Saudi Arabia

2UR ANALYSE NON-LINÉAIRE ET GÉOMÉTRIE, UR13ES32
Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El-Manar
2092 El Manar II, Tunisia

3UR ANALYSE NON-LINÉAIRE ET GÉOMÉTRIE, UR13ES32
ESPRIT School of Engineering, 1, 2 rue André Ampère
2083 - Pôle Technologique, El Ghazala

Dedicated to the memory of Professor Ezzedine Zahrouni

Abstract. We consider in this article a nonlinear vibrating Timoshenko system with thermoelasticity with second sound. We first recall the results obtained in [2] concerning the well-posedness, the regularity of the solutions and the asymptotic behavior of the associated energy. Then, we use a fourth-order finite difference scheme to compute the numerical solutions and we prove its convergence. The energy decay in several cases, depending on the stability number $\mu$, are numerically and theoretically studied.

1. Introduction. Historically, the first model of Timoshenko system was introduced by Stephen Timoshenko (1921) in the absence of dissipative term. This system describes the transverse vibration of the beam. More precisely, Timoshenko considered the following hyperbolic system:

$$
\begin{align*}
\rho \varphi_{tt} &= (k(\varphi_x + \psi))_x, \\
I_\rho \psi_{tt} &= (EI\psi_x)_x + k(\varphi_t + \psi),
\end{align*}
$$

where $\rho$, $k$, $I_\rho$ and $EI$ are positive constants, $\varphi = \varphi(x,t)$ is the displacement vector and $\psi = \psi(x,t)$ is the rotation angle of the filament.

In several previous works, the authors used the classical model for the propagation of heat, that is the well-known relationship between the temperature $\theta$ and the heat flux vector $q$, which reads as follows:

$$
\theta_t + \beta \text{div} q = 0,
$$

and

$$
q + \kappa \nabla \theta = 0,
$$

where $\beta$ and $\kappa$ are given positive constants.
Substituting (3) (Fourier’s law) into (2), yields the following parabolic heat equation
\[ \theta_t - \beta \kappa \Delta q = 0. \] (4)

Using the Fourier’s law, Rivera and Racke [15] investigated the following system:
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_3 \theta_t - \kappa \beta \theta_{xx} + \delta \psi_{xt} &= 0, & \text{in } (0, L) \times \mathbb{R}_+,
\end{align*}
\] (5)

where \( \rho_1, \rho_2, \rho_3, k, b, \kappa, \beta, \) and \( \delta \) are positive constants. The authors proved in [15] several exponential decay results for the system (5) in the case of equal wave speeds \( (\frac{k}{\rho_1} = \frac{k}{\rho_2}) \) and non-exponential (polynomial) stability results when the wave speeds are different \( (\frac{k}{\rho_1} \neq \frac{k}{\rho_2}) \).

However, the use of Fourier’s law suffers from a lack of consistency with respect to the Einstein’s relativity theory. Hence, Fernández Sare and Racke considered in [5] the following system:
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_3 \theta_t + q_x + \delta \psi_{xt} &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \theta_x(0, t) = \theta_x(L, t) = 0, & \forall \ t \in \mathbb{R}_+,
\end{align*}
\] (6)

and proved that the coupling via Cattaneo’s law (6) does not make the energy decays exponentially which is usually obtained for the coupling via Fourier’s law (system (5)).

Numerically, Raposo et al [18] considered the following Timoshenko system with a delay term in the feedback:
\[
\begin{align*}
\rho_1 \varphi_{tt} (x, t) - (\varphi_x + \psi)(x, t) + \mu_1 \varphi_t (x, t) + \mu_2 \varphi_t (x, t - \tau) &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} (x, t) - b \psi_{xx}(x, t) + k(\varphi_x + \psi)(x, t) + \mu_3 \varphi_t(x, t - \tau) + \mu_4 \psi_t(x, t - \tau) &= 0, & \text{in } (0, L) \times \mathbb{R}_+, \\
\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, & \forall \ t > 0,
\end{align*}
\] (7)

and they gave different numerical tests in relationship with the decay results of the solutions of the above system.

Recently, Ayadi et al. [2] considered the following coupled Timoshenko system:
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x + \alpha(t) h(\psi_t) &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\rho_3 \theta_t + q_x + \delta \psi_{xt} &= 0, & \text{in } (0, 1) \times \mathbb{R}_+, \\
\tau q_t + \beta q + \theta_x &= 0, & \text{in } (0, 1) \times \mathbb{R}_+.
\end{align*}
\] (8)

In order to study the stability properties of the solution of the system (8), it is necessary to introduce a stability number defined as follows:
\[ \mu = \left( \tau - \frac{\rho_1}{k \rho_3} \right) \left( \frac{\rho_2}{b} - \frac{\rho_1}{k} \right) - \frac{\tau \delta^2 \rho_1}{b k \rho_3}. \]

This number \( \mu \) is crucial in determining the asymptotic behavior of the energy associated with the system (8).

For more results concerning the well-posedness and controllability of Timoshenko systems, we refer the reader to [7, 8, 10, 12, 13, 20] and [11, 14]. See also [1] for other upper estimates of some dissipative evolution systems.
This article is organized as follows. Section 2 aims to recall the theoretical results related to the existence and asymptotic behavior of the solutions of the system (8). These results are proven in [2]. Several examples for the damping term are given. In Section 3, which is composed itself of three subsections, we first present the discrete scheme using a finite difference method, then we prove some stability and convergence results related to the numerical scheme, and finally several numerical tests are given with a linear damping term and for both cases when the stability number equals zero or not (corresponding thus to exponential and polynomial stability, respectively).

2. Existence and asymptotic behavior results. The aim of this section is to recall the results obtained in [2]. Indeed, the main objective of the present article is the numerical validation of the theoretical results. One of the advantages of this numerical study, besides its intrinsic interest, is to look for the optimality of the damping term which is useful in the stabilization process. For that purpose and in order to make the article self-reading, we recall the theoretical results and we state the different cases of linear and nonlinear damping. Let us rewrite the system (8) which reads as follows:

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, & \text{in } (0,1) \times \mathbb{R}_+,
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x + \alpha(t) h(\psi_t) &= 0, & \text{in } (0,1) \times \mathbb{R}_+,
\rho_3 \theta_t + q_x + \delta \psi_{xt} &= 0, & \text{in } (0,1) \times \mathbb{R}_+,
\tau q_t + \beta q + \theta_x &= 0, & \text{in } (0,1) \times \mathbb{R}_+.
\end{aligned}
\]

where, \(\rho_1, \rho_2, \rho_3, b, k, \delta, \beta\) are positive constants, \(\varphi = \varphi(x,t)\) is the displacement vector, \(\psi = \psi(x,t)\) is the rotation angle of the filament, \(\theta = \theta(x,t)\) is the temperature difference and \(q = q(x,t)\) is the heat flux vector.

We assume that the damping functions \(\alpha\) and \(h\) verify the following assumptions:

(A1) \(\alpha: \mathbb{R}_+ \to \mathbb{R}_+\) is a differentiable and decreasing function.

(A2) \(h: \mathbb{R} \to \mathbb{R}\) is a continuous non-decreasing function with \(h(0) = 0\), and there exists a continuous strictly increasing odd function \(h_0 \in C([0, +\infty))\), continuously differentiable in a neighborhood of 0, satisfying \(h_0(0) = 0\) and such that

\[
\begin{aligned}
\{ h_0(s) \leq |h(s)| \leq h_0^{-1}(s), & \text{ for all } |s| \leq \varepsilon, \\
c_1 |s| \leq |h(s)| \leq c_2 |s|, & \text{ for all } |s| \geq \varepsilon,
\end{aligned}
\]

where \(c_i > 0\) for \(i = 1, 2\).

We associate with (9) the boundary conditions given by

\[
\varphi_x(0,t) = \varphi_x(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0, \quad \forall \ t \geq 0,
\]

and the initial conditions

\[
\begin{aligned}
\varphi(x,0) &= \varphi_0(x), & \varphi_t(x,0) &= \varphi_1(x), & \forall \ x \in (0,1),
\psi(x,0) &= \psi_0(x), & \psi_t(x,0) &= \psi_1(x), & \forall \ x \in (0,1),
\theta(x,0) &= \theta_0(x), & q(x,0) &= q_0(x), & \forall \ x \in (0,1).
\end{aligned}
\]

Thus, we now have a complete system composed of (9), (10) and (11) for which we state the existence and regularity results in the next section.

Remark 1. For further use we introduce the following notation:

\[
f_n(x) = \partial^n f(x,0), \quad \forall x \in (0,1), \forall \ n \geq 0,
\]

and \(f_{n,x}\) denotes the first derivative of \(f_n\) with respect to \(x\), and so on for the higher derivatives.
In addition, in order to ensure the regularity up to the boundary of the solution of (9), we assume that the initial data satisfy the boundary conditions. These are called the compatibility conditions. Otherwise, we suppose for example that \(\varphi_0(0) = \psi_0(0) = q_0(0) = 0\), etc.

2.1. **Well-posedness and regularity.** Before stating the existence and regularity results of the solutions of (9)–(11) we introduce some space functions’ notations as below:

\[
L^2_\ast(0,1) = \left\{ v \in L^2(0,1) \mid \int_0^1 v(s)ds = 0 \right\},
\]

and

\[
H^1_\ast(0,1) = H^1(0,1) \cap L^2_\ast(0,1),
\]

Here, we state the existence and uniqueness results of the solutions of the Timoshenko system (9)–(11).

**Theorem 2.1** ([2]). Assume that (A1) and (A2) are satisfied. Then, for all initial data \((\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0) \in (H^2_\ast(0,1) \cap H^1_\ast(0,1)) \times H^1_\ast(0,1) \times (H^2(0,1) \cap H^1_\ast(0,1)) \times H^1_\ast(0,1) \times H^1_\ast(0,1) \times H^1_\ast(0,1)\), the system (9)–(11) has a unique solution \((\varphi, \psi, \theta, q)\) verifying

\[
(\varphi, \psi) \in C^0(\mathbb{R}_+, (H^2_\ast(0,1) \cap H^1_\ast(0,1)) \times (H^2(0,1) \cap H^1_\ast(0,1))) \cap C^1(\mathbb{R}_+, H^1_\ast(0,1) \times H^1_\ast(0,1)) \cap C^2(\mathbb{R}_+, L^2_\ast(0,1) \times L^2(0,1)),
\]

and

\[
(\theta, q) \in C^0(\mathbb{R}_+, H^1_\ast(0,1) \times H^1_\ast(0,1)) \cap C^1(\mathbb{R}_+, L^2_\ast(0,1) \times L^2(0,1)).
\]

**Remark 2.** For the numerical part, since we consider the system of equations (9) with a simple linear damping term \((h(s) = s\) and \(v(t) = 1\)), it suffices to reinforce the regularity of the initial data to enhance the regularity of the solutions \((\varphi, \psi, \theta, q)\). This is somehow necessary for our numerical study; see (38)–(40) below. Therefore, this question will be considered in Corollary 1 below and consequently we assume that the solutions of (9)–(11), in this particular case, are as regular as necessary (at least \(C^1(C^1_\ast) \cap C^1(L^2_\ast) \cap C^1(H^2_\ast) \cap C^1(H^1_\ast)\)) ensuring thus the quantities defined in (38)–(40) below.

**Corollary 1.** Assume that the initial data \(U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0)\) in (11) satisfies, for all \(m \in \mathbb{N}\),

\[
(H_m) \quad \frac{d^m U_0}{dt^m} \in (H^2_\ast(0,1) \cap H^1_\ast(0,1)) \times H^1_\ast(0,1) \times (H^2(0,1) \cap H^1_\ast(0,1)) \cap H^1_\ast(0,1) \times H^1_\ast(0,1).
\]

Then, the solution of (9)–(11) with \(h(s) = s\) and \(v(t) = 1\) verifies, for all \(m \in \mathbb{N}\),

\[
(C_m) \quad (\varphi, \psi) \in C^m(\mathbb{R}_+, H^{m+2}(0,1) \times H^{m+2}(0,1)) \cap C^{m+1}(\mathbb{R}_+, H^{m+1}(0,1) \times H^m(0,1)),
\]

\[
\quad \cap C^{m+2}(\mathbb{R}_+, H^m(0,1) \times H^m(0,1)),
\]

\[
(\theta, q) \in C^m(\mathbb{R}_+, H^{m+1}(0,1) \times H^{m+1}(0,1)) \cap C^{m+1}(\mathbb{R}_+, H^m(0,1) \times H^m(0,1)).
\]

**Proof.** We prove the result by induction on \(m\). First, the case \(m = 0\) is shown in the proof of Theorem 2.1; see [2]. Then, in order to prove Corollary 1 for the case \(m = 1\), we differentiate the linear system corresponding to (9) with respect to the time variable (we recall here that \(h(\psi_t) = \psi_t\) and \(v(t) = 1\)). The obtained
equations constitute, hence, a third-order system with respect to the time derivative. Therefore, in addition to the initial conditions (11), we need to express the new initial conditions in terms of the second-order time derivatives of \((\varphi, \psi)\) and the first-order time derivatives of \((\theta, q)\), which can be inherited from the system (9) and the initial conditions (11) as follows:

\[
\begin{align*}
\varphi_{tt} &= k\rho_1^{-1}(\varphi_{0,xx} + \psi_{0,x}), & \text{at } t = 0, \\
\psi_{tt} &= \rho_2^{-1}(b\psi_{0,xx} - k(\varphi_{0,x} + \psi_0) - \delta\theta_{0,x} - \psi_1), & \text{at } t = 0, \\
\theta_t &= -\rho_3^{-1}(q_{0,x} + \delta\psi_{1,x}), & \text{at } t = 0, \\
q_t &= -\tau^{-1}(\beta q_0 + \theta_{0,x}), & \text{at } t = 0.
\end{align*}
\]

(13)

We aim to prove that the solution \(U = (\varphi, \psi, \theta, q)\) fulfills the conclusion \((C_m)\) for \(m = 1\). Thus, using (13), the hypothesis \((H_m)\) for \(m = 1\) and remember the notation (12), we deduce that \(\varphi_2\) and \(\psi_2\) belong to \(H^1(0, 1)\). Similarly we prove an analogous result for \(\theta_1\) and \(q_1\). Otherwise, using again (13) and more precisely its third and fourth equations, we obtain \(\theta_1\) and \(q_1\). Hence, we can now apply Theorem 2.1 for \(U\) replaced by \(U_t\). Hence, we infer that \(e.g. (\varphi, \psi) \in C^1(\mathbb{R}_+, H^2(0, 1) \times H^2(0, 1)) \cap C^2(\mathbb{R}_+, H^1(0, 1) \times H^1(0, 1)) \cap C^3(\mathbb{R}_+, L^2(0, 1) \times L^2(0, 1))\). In particular, we have on one hand \(\varphi_{tt} \in C^0(\mathbb{R}_+, H^1(0, 1)) \cap C^1(\mathbb{R}_+, L^2(0, 1))\) and on the other hand \(\psi_x \in C^1(\mathbb{R}_+, H^1(0, 1)) \cap C^2(\mathbb{R}_+, L^2(0, 1))\). Now, using equation (9)1, we deduce that \(\varphi_x \in C^0(\mathbb{R}_+, H^2(0, 1)) \cap C^1(\mathbb{R}_+, H^1(0, 1))\). In a similar way we prove the other affirmations in \((C_1)\).

To achieve the proof of Corollary 1, it suffices to differentiate (in time) \((m + 1)\) times the system (9) and repeat the above steps for \(m = 1\).

This ends the proof of Corollary 1.

\(\square\)

2.2. Asymptotic behavior. In this subsection, we give the general decay results for a wide class of relaxation functions (the function \(h\) as given in (9)).

Definition 2.2. We recall here the energy associated with the system (9) by

\[
E(t) := \frac{1}{2} \int_0^1 \left( \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3 \theta^2 + \tau q^2 \right) dx.
\]

(14)

Theorem 2.3 ([2]). Suppose that \((A_1)\) and \((A_2)\) are satisfied, then for \(\mu = 0\) there exist positive constants \(k_1, k_2, k_3\) and \(\varepsilon_0\) such that the energy \(E(t)\) associated with (9)–(11) satisfies

\[
E(t) \leq k_3 H_1^{-1}\left( k_1 \int_0^t \alpha(s) \, ds + k_2 \right), \quad \text{for all } t \geq 0,
\]

(15)

where

\[
H_1(t) = \int_t^1 \frac{1}{H_2(s)} \, ds, \quad \text{and} \quad H_2(t) = t H'(\varepsilon_0 t).
\]

Here \(H_1\) is a strictly decreasing and convex function on \((0, 1)\), with \(\lim_{t \to 0} H_1(t) = +\infty\).

We show in the following some explicit stability results related to the asymptotic profiles (in time) of the energy \(E(t)\). For that purpose, we consider several explicit examples for the function \(h_0\).

Example 1. For \(h_0(s) = cs^p\), we have

- If \(p = 1\), then \(E(t) \leq k_3 \exp(-c(k_1 \int_0^t \alpha(s) \, ds + k_2))\).
- If \(p > 1\), then \(E(t) \leq c(k_1 \int_0^t \alpha(s) \, ds + k_2)^{-\frac{2}{p-1}}\).
Example 2. For \( h_0(s) = \exp(-\frac{1}{s}) \), we have
\[
E(t) \leq k_3 e^{-\varepsilon_0 \left( \ln \left( k_1 \int_0^t \alpha(s) ds + k_2 + c \exp\left( \frac{1}{\sqrt{\varepsilon_0}} \right) \right) \right)}^{-2}.
\]

Example 3. For \( h_0(s) = \frac{1}{s} \exp(-\frac{1}{s^2}) \), we have
\[
E(t) \leq \varepsilon \left( \ln \left( k_1 \int_0^t \alpha(s) ds + k_2 + c \exp\left( \frac{1}{\sqrt{\varepsilon_0}} \right) \right) \right)^{-1}.
\]

Next, we will consider the case where the stability number \( \mu \neq 0 \).

**Theorem 2.4 ([2])**. Let us suppose that the derivative of the function \( h \) is bounded and that the assumptions \((A_1)\) and \((A_2)\) hold. Then, for \( \mu \neq 0 \), the energy solution of \((9)-(11)\) satisfies
\[
E(t) \leq E(0) H_2^{-1}(\frac{C}{t}),
\]
where
\[
H_2(t) = tH'(\varepsilon_0 t) \text{ with } \lim_{t \to 0} H_2(t) = 0.
\]

In the following, we give some examples to illustrate the energy decay rates given by Theorem 3.

**Example 1.** For \( h_0(s) = cs^p \), then
- If \( p=1 \), we have \( E(t) \leq \frac{c}{\varepsilon} \).
- If \( p > 1 \) we have \( E(t) \leq ct^{-\frac{p}{p-1}} \).

**Example 2.** Let \( h \) be given by \( h(x) = \frac{1}{x^3} \exp(-\frac{1}{x^2}) \) and we choose \( h_0(x) = \frac{1+x^2}{x^3} \exp(-\frac{1}{x^2}) \), we obtain
\[
E(t) \leq c(\ln(t))^{-1}.
\]

**Remark 3.** In fact the results in Theorems 2.3 and 2.4 are now reinforced by lower bounds for the energy obtained in [3].

### 3. Numerical solution.

We will start making use of Finite Difference Method (FDM) to derive a discrete representation of the solution of the Timoshenko system \((9)-(11)\) in the particular case \( \alpha(t) = 1 \) and \( h(s) = s \), see e.g. [22] for more details about FDM for the wave equations.

More precisely, we use the classical finite difference discretization for the time variable and the Implicit Compact Finite Difference Method of fourth-order for discretization of the space variable. The full nonlinear case and the comparison between the different Difference Finite Methods (implicit, explicit and semi-implicit) will be considered in a subsequent work. We notice here also that our approach is based on the study of the solutions of the Timoshenko system which is different than what we considered in [4] where the authors considered the numerical study of the energy of the Timoshenko system with Finite Elements Method.

#### 3.1. Discrete formulation.

Consider the discrete domain \( \Omega_h \) of \( \Omega = (0, 1) \) with uniform grid \( x_i = ih \), \( i = 0, 1, ..., I(I \geq 1); h = \frac{1}{I} \). The time discretization of the interval \( T_n = (0, T) \) is given by \( t_n = nk \), \( n = 0, 1, ..., N(N \geq 1); \kappa = ch \), where \( c \) is a positive constant and \( I \) and \( N \) are two positive integers. Denote by \( \omega(x_i, t_n) = \omega^n_i \) the value of the function \( \omega \) evaluated at the point \( x_i \) and the instant \( t_n \).
In Figure 1, we show the pattern mesh of $\omega$ using the discretization of the intervals $(0,1)$ and $(0,T)$, where the classification of nodes is as follows: internal (circles), boundaries (stars), initials (squares) and ghost (diamonds).

Now, we define the following approximation of the derivatives of $\omega$

\[
\begin{align*}
(\omega_t)_i^n & \simeq \frac{\omega_i^{n+1} - \omega_i^{n-1}}{2\kappa}, \\
(\omega_{tt})_i^n & \simeq \frac{\omega_i^{n+1} - 2\omega_i^n + \omega_i^{n-1}}{\kappa^2},
\end{align*}
\]

\[
(\omega_x)_i^n \simeq \frac{\omega_{i+1}^n - \omega_{i-1}^n}{2h},
\]

\[
(\omega_{xx})_i^n \simeq \frac{1}{h^2} \left( 1 + \frac{1}{12} \delta^2_x \right)^{-1} \delta^2_x \omega_i^n,
\]

with

\[
\delta^2_x \omega_i^n = \omega_{i+1}^n - 2\omega_i^n + \omega_{i-1}^n \quad \text{and} \quad \left[ 1 + \frac{1}{12} \delta^2_x \right] \omega_i^n = \frac{1}{12} \omega_{i+1}^n + \frac{5}{6} \omega_i^n + \frac{1}{12} \omega_{i-1}^n.
\]

To the best of our knowledge, the introduction of the discrete operator $1 + \frac{1}{12} \delta^2_x$ is due to the well-known Numerov Method ([16, 17]). In fact, it was proven in many articles that the accuracy is enhanced while dealing; this somehow justifies the appeal to this method in several works in the literature, see e.g. [6, 9, 21] among many other references. In addition to the fact that this approximation gives more accuracy to the scheme, it can be also understood as a correction of the second derivative terms in the original equations.
Now, using (17) and (19), we obtain the discrete formulation of the system (9) as follows:

\[
\begin{align*}
\rho_1 \psi_{i+1}^{n+1} - \frac{k}{h^2} \left( 1 + \frac{1}{12} \delta^2 \right) \delta_i^2 \psi_i^{n+1} - \frac{k}{2h} (\psi_{i+1}^n - \psi_{i-1}^n) &= 0, \\
\rho_2 \psi_i^{n+1} - \frac{k}{h^2} \left( 1 + \frac{1}{12} \delta^2 \right) \delta_i^2 \psi_i^n + \frac{k}{2h} (\psi_{i+1}^n - \varphi_{i-1}^n) + k\psi_i^n &= 0, \\
\rho_3 \left( \frac{\theta_{i+1}^n - \theta_{i-1}^n}{2h} \right) + \frac{1}{2h} (q_{i+1}^n - q_{i-1}^n) + \frac{\delta}{4\pi n} (\psi_{i+1}^{n+1} - \psi_{i-1}^{n+1}) - \frac{\delta}{4\pi n} (\psi_{i+1}^n - \psi_{i-1}^n) &= 0,
\end{align*}
\]

(21)

Applying the discrete (matrix) operator $\kappa^2[1 + \frac{1}{12} \delta_0^2]$ to (21)\_1 and (21)\_2, and using (20), we obtain

\[
\begin{align*}
\rho_1 \left( \frac{\varphi_{i+1}^{n+1} + \frac{5}{6} \varphi_i^{n+1} + \frac{1}{12} \varphi_{i+1}^{n+1}}{h^2} - \frac{1}{6} (5\rho_1 + 6a_1) \varphi_i^{n+1} - \frac{1}{6} (5\rho_1 + 6a_1) \varphi_{i+1}^{n+1} \right) + \rho_1 \left( \frac{\varphi_{i+1}^{n-1} + \frac{5}{6} \varphi_i^{n-1} + \frac{1}{12} \varphi_{i+1}^{n-1}}{h^2} - \frac{1}{6} (5\rho_1 + 6a_1) \varphi_i^{n-1} - \frac{1}{6} (5\rho_1 + 6a_1) \varphi_{i+1}^{n-1} \right) - a_2 \left( \frac{\varphi_{i+1}^{n+2} + \frac{5}{6} \varphi_i^{n+2} - \frac{1}{12} \varphi_{i+1}^{n+2}}{h^2} - \frac{1}{6} (5\rho_1 + 6a_1) \varphi_i^{n+2} - \frac{1}{6} (5\rho_1 + 6a_1) \varphi_{i+1}^{n+2} \right) &= 0, \\
(\rho_2 + b_0) \left( \frac{\psi_{i+1}^{n+1} + \frac{5}{6} \psi_i^{n+1} + \frac{1}{12} \psi_{i+1}^{n+1}}{h^2} - \frac{1}{6} (2\rho_2 + 12b_1 - b_0) \psi_i^{n+1} - \frac{1}{6} (10\rho_2 - 12b_1 - 5b_0) \psi_{i+1}^{n+1} \right) - \frac{1}{12} (2\rho_2 + 12b_1 - b_0) \psi_{i+1}^{n+1} + b_2 \left( \frac{\psi_{i+1}^{n+2} + \frac{5}{6} \psi_i^{n+2} - \frac{1}{12} \psi_{i+1}^{n+2}}{h^2} - \frac{1}{6} (2\rho_2 + 12b_1 - b_0) \psi_i^{n+2} - \frac{1}{6} (10\rho_2 - 12b_1 - 5b_0) \psi_{i+1}^{n+2} \right) + b_4 \left( \frac{\theta_{i+1}^n + \frac{5}{6} \theta_i^n + \frac{1}{12} \theta_{i+1}^n - \frac{1}{6} \theta_i^n} {h^2} - \frac{1}{6} (2\rho_2 + 12b_1 - b_0) \theta_i^n - \frac{1}{6} (10\rho_2 - 12b_1 - 5b_0) \theta_{i+1}^n \right) - b_5 \left( \frac{\psi_{i+1}^{n+1} - \psi_{i+1}^{n+1}}{h^2} - \frac{1}{6} \psi_{i+1}^{n+1} - \psi_{i+1}^{n+1} \right) - \frac{\delta}{4\pi n} (\psi_{i+1}^{n+1} - \psi_{i+1}^{n+1}) &= 0,
\end{align*}
\]

(22)

where the coefficients in the above system are given by

\[
\begin{align*}
a_1 &= \frac{kn^2}{h^2}, \quad a_2 = b_2 = \frac{kn^2}{2h^2}, \quad b_0 = kn^2, \quad b_1 = \frac{kn^2}{h^2}, \quad b_3 = \frac{\kappa}{2} \quad \text{and} \quad b_4 = \frac{\delta k^2}{2h^2}.
\end{align*}
\]

The discrete formulation of the initial conditions (11) reads as follows:

\[
\begin{align*}
\varphi(x_i, 0) &= \varphi_0^n, \quad \psi(x_i, 0) = \psi_0^n, \quad \text{for all } x_i \in \Omega_h, \\
\varphi_1(x_i, 0) &= \varphi_1^n, \quad \psi_1(x_i, 0) = \psi_1^n, \quad \text{for all } x_i \in \Omega_h, \\
\theta(x_i, 0) &= \theta_0^n, \quad q(x_i, 0) = q_0^n, \quad \text{for all } x_i \in \Omega_h,
\end{align*}
\]

(23)

whereas the discrete boundary conditions corresponding to (10) are simply given by

\[
(\varphi_{n+1}^0) = (\varphi_{n+1}^0), \quad (\psi_{n+1}^0) = (\psi_{n+1}^0), \quad (\psi_{n+1}^0) = (\psi_{n+1}^0), \quad (\theta_{n+1}^\gamma) = (\theta_{n+1}^\gamma), \quad (q_{n+1}^\gamma) = (q_{n+1}^\gamma), \quad \text{for all } t_n \in T_n.
\]

In addition, it is natural to assume that $\varphi_{n+1}^0 = \varphi_0^n, \quad \varphi_{n+1}^0 = \varphi_0^n, \quad \varphi_{n+1}^0 = \varphi_0^n, \quad \varphi_{n+1}^0 = \varphi_0^n, \quad \theta_{n+1}^0 = \theta_0^n \quad \text{and} \quad \theta_{n+1}^0 = \theta_0^n, \quad \text{since we have } (\theta_{n+1}^0) = (\theta_{n+1}^0), \quad \text{thanks to } (9)\_4. \quad \text{Hence, we obtain the following linear algebraic system:}

\[
\begin{align*}
A_1 \Phi_{n+1}^+ &= B_1 \Phi^n + C_1 \Psi^n + D_1 \Phi_{n-1}^-, \\
A_2 \Psi_{n+1}^+ &= B_2 \Psi^n + C_2 \Phi^n + D_2 \Psi_{n-1}^- + F_2 \Theta^n, \\
A_3 \Theta_{n+1}^+ + L_3 \Psi_{n+1}^+ &= B_3 \Theta_{n-1}^- - C_3 Q^n + D_3 \Psi_{n-1}^- - F_2 \Theta^n, \\
A_4 Q_{n+1}^+ &= B_4 Q_{n-1}^- - C_4 Q^n - D_4 \Theta^n,
\end{align*}
\]

(24)

with $\Phi^n = (\varphi_1^n, \varphi_2^n, \ldots, \varphi_{n-1}^n)^t, \Psi^n = (\psi_1^n, \psi_2^n, \ldots, \psi_{n-1}^n)^t, \Theta^n = (\theta_1^n, \theta_2^n, \ldots, \theta_{n-1}^n)^t, \quad Q^n = (q_1^n, q_2^n, \ldots, q_{n-1}^n)^t, \quad \text{for all } n \in \{0, 1, \ldots, N-1\} \quad \text{and} \quad A_p, B_p, C_p, D_p, F_2 \text{ and } L_3$.
are $(I - 1)$ square matrices for $p = 1, \ldots, 4$ which will be defined below. First, we have
\[
A_1 = \begin{pmatrix}
\frac{11}{12} \rho_1 & \frac{\rho_1}{12} & 0 & \cdots & \cdots & 0 \\
\frac{\rho_1}{12} & \frac{5}{6} \rho_1 & \ddots & \cdots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \frac{5}{6} \rho_1 & \frac{\rho_1}{12} \\
\frac{\rho_1}{12} & \frac{5}{6} \rho_1 & \frac{\rho_1}{12} & \frac{11}{12} \rho_1
\end{pmatrix} = -D_1.
\]
It is clear that $A_1$ is a tridiagonal matrix with a constant coefficient in each diagonal except in the first and last main diagonal terms where the coefficient is $\frac{11}{12} \rho_1$. Similarly, the matrix $B_1$ is a tridiagonal matrix given as follows:
\[
B_1 = \begin{pmatrix}
\beta_4 & \beta_1 & 0 & \cdots & \cdots & 0 \\
\beta_4 & \beta_2 & \beta_1 & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \beta_2 & \beta_1 & \beta_3
\end{pmatrix},
\]
where $\beta_1 = \frac{1}{6} (\rho_1 + 6a_1)$, $\beta_2 = \frac{1}{3} (5\rho_1 - 6a_1)$ and $\beta_3 = \frac{1}{6} (11\rho_1 - 6a_1)$.
Moreover, the matrices $C_1, A_2, B_2$ and $D_2$ are given by:
\[
C_1 = \text{pentadiag}(-\frac{1}{12} a_2, -\frac{5}{6} a_2, 0, \frac{5}{6} a_2, \frac{1}{12} a_2),
\]
\[
A_2 = \text{tridiag}(\frac{1}{12} (\rho_2 + b_3), \frac{5}{6} (\rho_2 + b_3), \frac{1}{12} (\rho_2 + b_3)),
\]
\[
B_2 = \text{tridiag}(\frac{1}{12} (2\rho_2 + 12b_1 - b_0), \frac{1}{6} (10\rho_2 - 12b_1 - 5b_0), \frac{1}{12} (2\rho_2 + 12b_1 - b_0)),
\]
\[
D_2 = \text{tridiag}(\frac{1}{12} (-\rho_2 + b_3), \frac{5}{6} (-\rho_2 + b_3), \frac{1}{12} (-\rho_2 + b_3)).
\]
The matrices $C_2$ and $F_2$ are given as follows:
\[
C_2 = \begin{pmatrix}
\frac{11b_2}{12} & -\frac{5b_2}{6} & -\frac{b_2}{12} & 0 & \cdots & \cdots & 0 \\
\frac{11b_2}{12} & 0 & -\frac{5b_2}{6} & -\frac{b_2}{12} & \ddots & \vdots & \vdots \\
\frac{b_2}{12} & \frac{5b_2}{6} & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \frac{5b_2}{6} & -\frac{b_2}{12} & \frac{11b_2}{12}
\end{pmatrix},
\]
Finally, we denote by $I_d$ the identity matrix of size $(I - 1)$ and we introduce the following parameters:

$$\tau_1 = \frac{\rho_3}{2\kappa}, \quad \tau_2 = \frac{1}{2h}, \quad \tau_3 = \frac{\delta}{4\kappa h}, \quad r_1 = \frac{\tau}{2\kappa}, \quad r_2 = \beta \quad \text{and} \quad r_3 = \frac{1}{2h}.$$  

Finally, we define the remaining matrices as below:

$$A_3 = B_3 = \tau_1 I_d,$$

$$L_3 = D_3 = \text{tridiag}(-\tau_3, 0, \tau_3),$$

$$C_3 = \text{tridiag}(-\tau_2, 0, \tau_2),$$

$$A_4 = B_4 = r_1 I_d,$$

$$C_4 = r_2 I_d,$$

$$D_4 = \begin{pmatrix} -r_3 & r_3 & 0 & \cdots & \cdots & \cdots & 0 \\ -r_3 & 0 & r_3 & \cdots & \cdots & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & r_3 \\ 0 & \cdots & \cdots & 0 & -r_3 & r_3 \end{pmatrix}.$$  

In the following, we reformulate the system (24) in abstract form as follows:

$$AU^{n+1} = BU^n + CU^{n-1}, \quad \text{for all } n \geq 1, \quad \text{for all } n \geq 1,$$  

where we denoted by $U = (\Phi, \Psi, \Theta, Q)^t \in \mathbb{R}^{4I-4}$, and

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & L_3 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{pmatrix} \in \mathcal{M}_{4I-4}(\mathbb{R}),$$  

$$B = \begin{pmatrix} B_1 & C_1 & 0 & 0 \\ C_2 & B_2 & F_2 & 0 \\ 0 & 0 & 0 & -C_3 \\ 0 & 0 & -D_4 & -C_4 \end{pmatrix} \in \mathcal{M}_{4I-4}(\mathbb{R}).$$  

$^1$The initial data for the sequence $(U^n)_{n \geq 0}$, $U^0$ and $U^1$, are given by (23) and (13), at the discrete level for the latter one.
and
\[
C = \begin{pmatrix}
D_1 & 0 & 0 & 0 \\
0 & D_2 & 0 & 0 \\
0 & D_3 & B_3 & 0 \\
0 & 0 & 0 & B_4
\end{pmatrix} \in \mathcal{M}_{4\times 4}(\mathbb{R}).
\] (28)

3.2. Convergence. In this section, we will show that the approximate solution of the system (21) converges to the exact solution of (9). Indeed, we prove in the following that the scheme (21) is consistent and stable, and thus it is convergent.

We first start by proving the consistency of the proposed scheme. Hence, for all \(1 \leq i \leq I - 1\), we define the consistency error by
\[
(\varepsilon^n_i)_j = (\epsilon^n_i)_j - (\varepsilon^n_i)_j,
\] (29)
where \((\varepsilon^n_i)_j\) is the LHS of (9), for \(j = 1, \ldots, 4\), applied at the point \((x_i, t_n)\), and the \((\epsilon^n_i)_j\), for \(j = 1, \ldots, 4\), are given by
\[
\begin{align*}
(\epsilon^n_i)_1 &= \frac{1}{\kappa^2} \left( 1 + \frac{1}{12} \delta_x^2 \right)^{-1} \delta_x^2 \varphi(x_i, t_n) - \frac{k}{\kappa^2} \left( 1 + \frac{1}{12} \delta_x^2 \right)^{-1} \delta_x^2 \varphi(x_i, t_n) - \frac{h}{\kappa} (\psi(x_{i+1}, t_n) - \psi(x_{i-1}, t_n)), \\
(\epsilon^n_i)_2 &= \frac{1}{\kappa^2} \left( 1 + \frac{1}{12} \delta_x^2 \right)^{-1} \delta_x^2 \varphi(x_i, t_n) + \frac{\kappa}{h^2} (\varphi(x_{i+1}, t_n) - \varphi(x_{i-1}, t_n)) + 2 h (\theta(x_{i+1}, t_n) - \theta(x_{i-1}, t_n)), \\
(\epsilon^n_i)_3 &= \frac{1}{\kappa^2} \left( 1 + \frac{1}{12} \delta_x^2 \right)^{-1} \delta_x^2 \varphi(x_i, t_n) + \frac{\kappa}{h^2} (\varphi(x_{i+1}, t_n) - \varphi(x_{i-1}, t_n)) + 2 h (\theta(x_{i+1}, t_n) - \theta(x_{i-1}, t_n)), \\
(\epsilon^n_i)_4 &= \frac{1}{\kappa^2} \left( 1 + \frac{1}{12} \delta_x^2 \right)^{-1} \delta_x^2 \varphi(x_i, t_n) - \frac{h}{\kappa} (\psi(x_{i+1}, t_n) - \psi(x_{i-1}, t_n)).
\end{align*}
\] (30)

We make here use of the regularity results proven in Corollary 1 to assume the necessary initial data regularity, and consequently that of the solution of (9), to prove the following lemma.

**Lemma 3.1.** Let \((\varphi, \psi, \theta, q)\) be the solution of (9) verifying
\[
(\varphi, \psi) \in C^4(\mathbb{R}^+, H^6(0, 1) \times H^6(0, 1)) \cap C^5(\mathbb{R}^+, H^5(0, 1) \times H^5(0, 1)) \cap C^6(\mathbb{R}^+, H^4(0, 1) \times H^4(0, 1)),
\]
\[
(\theta, q) \in C^4(\mathbb{R}^+, H^5(0, 1) \times H^5(0, 1)) \cap C^5(\mathbb{R}^+, H^4(0, 1) \times H^4(0, 1)) \cap C^5(\mathbb{R}^+, H^4(0, 1) \times H^4(0, 1)).
\]
and \((\varphi^n, \psi^n, \theta^n, q^n)\) be the solution of (21). We define the following error
\[
E^n_j = \max_{1 \leq i \leq I - 1} |(\varepsilon^n_i)_j| \quad \text{for } n = 0, 1, \ldots, N \text{ and } j = 1, \ldots, 4.
\]

Then, for any \(T > 0\), there exists a positive constant \(C\), which depends on \(T\) and \(U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0)\), such that
\[
\max_{2 \leq i \leq I - 1} E^n_j \leq C(T, U_0)(\kappa^2 + h^2), \tag{30}
\]
for all \(t_n \leq T\).
Proof. First, thanks to the Taylor formula, we have

\[
\frac{w(x_i, t_{n + 1}) - w(x_i, t_{n - 1})}{2\kappa} = \frac{\partial w}{\partial t}(x_i, t_n) + \kappa^2 \frac{\partial^3 w}{\partial t^3}(x_i, t_n + \kappa \sigma_1) + \kappa^2 \frac{\partial^3 w}{\partial t^3}(x_i, t_n - \kappa \sigma_0),
\]

(32)

\[
w(x_i, t_{n + 1}) - 2w(x_i, t_n) + w(x_i, t_{n - 1}) = \frac{\partial^2 w}{\partial t^2}(x_i, t_n) + \kappa^2 \frac{\partial^4 w}{\partial t^4}(x_i, t_n + \kappa \sigma_1) + \kappa^2 \frac{\partial^4 w}{\partial t^4}(x_i, t_n - \kappa \sigma_0),
\]

(33)

Second, to evaluate the term \( \left(1 + \frac{1}{12} \delta_x^2\right)^{-1} \delta_x^2 w(x_i, t_n) \), we use the following approximations:

\[
\left[1 + \frac{1}{12} \delta_x^2\right] \frac{\partial^2 w}{\partial x^2}(x_i, t_n) = \frac{\partial^2 w}{\partial x^2}(x_i+1, t_n) + \frac{5}{6} \frac{\partial^2 w}{\partial x^2}(x_i, t_n) + \frac{1}{12} \frac{\partial^2 w}{\partial x^2}(x_i-1, t_n)
\]

\[
= \frac{\partial^2 w}{\partial x^2}(x_i, t_n) + \frac{h^2}{12} \frac{\partial^4 w}{\partial x^4}(x_i, t_n) + \frac{1}{12} \frac{\partial^4 w}{\partial x^4}(x_i - \sigma_1, t_n) + \frac{1}{12} \frac{\partial^4 w}{\partial x^4}(x_i + \sigma_1, t_n) + \frac{1}{12} \frac{\partial^4 w}{\partial x^4}(x_i - \sigma_1, t_n),
\]

and

\[
\frac{1}{h^2} \delta_x^2 w(x_i, t_n) = \frac{\partial^2 w}{\partial x^2}(x_i, t_n) + \frac{h^2}{12} \frac{\partial^4 w}{\partial x^4}(x_i, t_n) + \frac{h^2}{6} \frac{\partial^6 w}{\partial x^6}(x_i + \sigma_1, t_n) + \frac{h^2}{6} \frac{\partial^6 w}{\partial x^6}(x_i - \sigma_1, t_n).
\]

Hence, we obtain

\[
\frac{1}{h^2} \left(1 + \frac{1}{12} \delta_x^2\right)^{-1} \delta_x^2 w(x_i, t_n) - \left(1 + \frac{1}{12} \delta_x^2\right)^{-1} \left(-\frac{h^4}{12} \frac{\partial^6 w}{\partial x^6}(x_i + \sigma_1, t_n) + \frac{h^4}{12} \frac{\partial^6 w}{\partial x^6}(x_i - \sigma_1, t_n) + \frac{h^4}{12} \frac{\partial^6 w}{\partial x^6}(x_i + \sigma_1, t_n) + \frac{h^4}{12} \frac{\partial^6 w}{\partial x^6}(x_i - \sigma_1, t_n)\right).
\]

(35)

Using (32)–(34), we arrive at

\[
\frac{w(x_{i+1}, t_{n+1}) - w(x_{i-1}, t_{n+1}) - w(x_{i+1}, t_{n-1}) + w(x_{i-1}, t_{n-1})}{4\kappa h}
\]

\[
= \frac{\partial^2 w}{\partial x^2}(x_i, t_n) + \kappa^2 \frac{\partial^4 w}{\partial x^4}(x_i + \kappa \sigma_0, t_n + \kappa \sigma_1) + \kappa^2 \frac{\partial^4 w}{\partial x^4}(x_i - \kappa \sigma_0, t_n + \kappa \sigma_1)
\]

\[
+ \kappa^2 \frac{\partial^4 w}{\partial x^4}(x_i + \kappa \sigma_0, t_n - \kappa \sigma_0) + \kappa^2 \frac{\partial^4 w}{\partial x^4}(x_i - \kappa \sigma_0, t_n - \kappa \sigma_0)
\]

with \(0 < \sigma_0 < 1\).
Then, using (32)–(36) in \( c_i^n = (c_i^n)_1 + (c_i^n)_2 + (c_i^n)_3 + (c_i^n)_4 \), we obtain
\[
\begin{align*}
\epsilon_i^n &= \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^2 \varphi}{\partial x^2}(x_i, t_n + \kappa \sigma^1) + \frac{\partial^2 \varphi}{\partial t \partial x}(x_i, t_n - \kappa \sigma^1) \\
&\quad + k \left(1 + \frac{1}{12} \delta_x^2 \right)^{-1} \left( \frac{\rho_k \partial^2 \varphi}{\partial x^2}(x_i + \kappa \sigma^1, t_n) \right) \\
&\quad - \frac{k^4}{720} \frac{\partial^6 \varphi}{\partial x^6}(x_i, \kappa \sigma^1, t_n) + \frac{k^4}{288} \frac{\partial^6 \varphi}{\partial x^6}(x_i + \kappa \sigma^1, t_n) + \frac{k^4}{288} \frac{\partial^6 \varphi}{\partial x^6}(x_i, \kappa \sigma^1, t_n) \\
&\quad - \frac{k h^2}{12} \frac{\partial^6 \varphi}{\partial x^6}(x_i + \kappa \sigma^1, t_n) - \frac{k h^2}{12} \frac{\partial^6 \varphi}{\partial x^6}(x_i, \kappa \sigma^1, t_n) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x_i, t_n + \kappa \sigma^1) \\
&\quad + b \left(1 + \frac{1}{12} \delta_x^2 \right)^{-1} \left( \frac{\rho_k \partial^2 \varphi}{\partial x^2}(x_i + \kappa \sigma^1, t_n) - \frac{\rho_k \partial^2 \varphi}{\partial x^2}(x_i - \kappa \sigma^1, t_n) \right) \\
&\quad + \frac{k^4}{288} \frac{\partial^6 \varphi}{\partial x^6}(x_i + \kappa \sigma^1, t_n) + \frac{k^4}{288} \frac{\partial^6 \varphi}{\partial x^6}(x_i, \kappa \sigma^1, t_n) + \frac{k^4}{288} \frac{\partial^6 \varphi}{\partial x^6}(x_i, \kappa \sigma^1, t_n) \\
&\quad + \frac{k h^2}{12} \frac{\partial^6 \varphi}{\partial x^6}(x_i + \kappa \sigma^1, t_n) - \frac{k h^2}{12} \frac{\partial^6 \varphi}{\partial x^6}(x_i, \kappa \sigma^1, t_n) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x_i, t_n + \kappa \sigma^1) \\
&\quad + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x_i + \kappa \sigma^1, t_n) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x_i, \kappa \sigma^1, t_n) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x_i, \kappa \sigma^1, t_n) \\
&\quad + \frac{k^2}{12} \frac{\partial^4 \varphi}{\partial x^4}(x_i + \kappa \sigma^1, t_n) + \frac{k^2}{12} \frac{\partial^4 \varphi}{\partial x^4}(x_i, \kappa \sigma^1, t_n) + \frac{k^2}{12} \frac{\partial^4 \varphi}{\partial x^4}(x_i, \kappa \sigma^1, t_n) \\
&\quad + \frac{k^2}{12} \frac{\partial^4 \varphi}{\partial x^4}(x_i + \kappa \sigma^1, t_n) + \frac{k^2}{12} \frac{\partial^4 \varphi}{\partial x^4}(x_i, \kappa \sigma^1, t_n) + \frac{k^2}{12} \frac{\partial^4 \varphi}{\partial x^4}(x_i, \kappa \sigma^1, t_n).
\end{align*}
\]

Now, remembering the regularity results on the Timoshenko system (9), proven in Corollary 1, we denote by \( S_1, S_2 \) and \( S_3 \) the following quantities:
\[
S_1 = \max_{x,t} \left| \frac{1}{6} \frac{\partial^3 \varphi}{\partial t^3}(x,t) + \frac{\delta}{3} \frac{\partial^2 \varphi}{\partial t^2 \partial x}(x,t) + \frac{\delta}{6} \frac{\partial^2 \varphi}{\partial t \partial x^2}(x,t) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x,t) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x,t) \right|,
\]
\[
S_2 = \max_{x,t} \left[ \frac{k}{6} \frac{\partial^3 \varphi}{\partial t^3}(x,t) + \frac{\partial^3 \varphi}{\partial x^3}(x,t) + \frac{\delta}{6} \frac{\partial^2 \varphi}{\partial t^2 \partial x}(x,t) + \frac{\delta}{6} \frac{\partial^2 \varphi}{\partial t^2 \partial x}(x,t) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x,t) + \frac{\rho_k^2 \partial^4 \varphi}{\partial t^4}(x,t) \right],
\]
\[
S_3 = \max_{x,t} \left[ \frac{7k}{720} \frac{\partial^6 \varphi}{\partial x^6}(x,t) + \frac{7b}{720} \frac{\partial^6 \varphi}{\partial x^6}(x,t) \right].
\]

Then, using (32)–(36), we deduce that
\[
\max_{1 \leq i \leq n} c_i^n \leq S_1 \kappa^2 + S' h^2
\]
where \( S' = S_2 + h^2 S_3 \).
Remark 4. The constants $S_1, S_2$ and $S_3$, introduced in the proof of Lemma 3.1, are well-defined thanks to the asymptotic result on the energy $E$ proven in Theorem 2.4, and then applied in Example 1 (just after Theorem 2.4) for $p = 1$.

Now, we aim to prove the stability of the scheme (21). For that purpose, we use the abstract formulation of (25), which is equivalent to (24), that we rewrite in the following form:

$$V^{n+1} = MV^n,$$

where $V^n = (U^n, U^{n-1})^t$, and $M$ is a $(8I - 4)$ square matrix given by,

$$M = \left( \begin{array}{cc} \mathcal{A}^{-1}B & \mathcal{A}^{-1}C \\ \mathcal{I} & 0 \end{array} \right),$$

in which $\mathcal{I}$ denotes the identity matrix of size $(4I - 4)$.

Now, we state and prove the following theorem which shows the stability of (25) and consequently of the scheme (21).

**Theorem 3.2.** Let $\kappa$ and $h$ be very small and assume that the following CFL stability condition holds:

$$\frac{\kappa}{h} = \frac{c}{c_0} < 1, \quad \text{for a certain } c_0 > 0.$$  

Then, the spectral radius of the matrix $M$ verifies $\rho(M) \leq 1$.

**Proof.** We proceed by contradiction and we assume that $\rho(M) > 1$. Then, there exist $\lambda \neq 0$ and $X = (x_1, ..., x_{8I-8})^t \neq 0_{8I-8}$ such that

$$MX = \lambda X,$$

and $|\lambda| > 1$.

Now, we denote by $X_1 = (x_1, ..., x_{4I-4})^t$ and $X_2 = (x_{4I-3}, ..., x_{8I-8})^t$. Hence, the equation (44) can be written as

$$\left\{ \begin{array}{c} \mathcal{A}^{-1}B X_1 + \mathcal{A}^{-1}C X_2 = \lambda X_1, \\ X_1 = \lambda X_2. \end{array} \right.$$  

(45)

Combining (45)$_1$ and (45)$_2$, we obtain

$$\lambda \mathcal{A}^{-1}B X_2 + \mathcal{A}^{-1}C X_2 = \lambda^2 X_2,$$

or equivalently,

$$\lambda B X_2 + C X_2 = \lambda^2 A X_2.$$  

(46)

**First Case.** $X_2 \in \text{Ker}(C)$. Since we have $\|X_2\| \neq 0$ (thanks (45)$_2$), $\lambda \neq 0$ and the fact that $X \neq 0_{8I-8}$, then we deduce that

$$|\lambda| \|A\| \leq \|B\| + \frac{1}{|\lambda|} \|C\|,$$

(47)

where we denoted by $\|A\| = \|A\|_\infty = \max_{i \in \{1, ..., 8I-8\}} \sum_{j \in \{1, ..., 8I-8\}} |A_{i,j}|$.

$^2$The matrix $\mathcal{M}$ is well defined since the matrix $\mathcal{A}$ is invertible thanks to $\det(\mathcal{A}) = \prod_{l=1}^{4} \det(A_l)$.

This condition is a simple consequence of the fact that the blocks in $\mathcal{A}$ are pairwise commuting matrices; see e.g. [19].

$^3$This CFL condition is not optimal but it is sufficient to make the scheme convergent.
Now, computing the norms of the matrices $A$, $B$ and $C$. Then, under the hypothesis of Theorem 3.2, we obtain

$$\|A\| = \max(\rho_1, \rho_2 + b_3, 2\tau_3 + \tau_1, \tau_1) = 2\tau_3 + \tau_1,$$

$$\|B\| \leq 3\tau_3,$$

and

$$\|C\| \leq 2\tau_3 + \tau_1.$$ Consequently, the estimate (47) gives

$$|\lambda| \leq \frac{\|B\|}{\|A\|} + \frac{1}{|\lambda|} \frac{\|C\|}{\|A\|} \leq \frac{3\tau_3}{2\tau_3 + \tau_1} + \frac{1}{|\lambda|},$$

where we have

$$\frac{3\tau_3}{2\tau_3 + \tau_1} = \frac{\frac{3}{2\delta}}{\frac{\delta}{2\delta} + \frac{\rho_3}{2\kappa}} = \frac{3\kappa}{\delta + \rho_3 h} \leq \frac{3\kappa}{2\delta} \ll 1.$$ Hence, we end up with a contradiction. Indeed, using the inequality (48) rewritten differently, taken at the power $q$, and by letting $q \to +\infty$, we arrive at

$$\underbrace{(|\lambda| - \frac{1}{|\lambda|})^q}_{\downarrow +\infty} \leq \underbrace{(\frac{3\kappa}{\delta + \rho_3 h})^q}_{\downarrow 0}.$$ **Second Case.** $X_2 \notin Ker(C)$. In this case, writing the analogous estimate to (47) and using the above computations for the norms of $A$ and $B$, we infer that

$$1 < |\lambda| \leq \frac{\|B\|}{\|A\|} \ll 1,$$

which is again a contradiction.

This ends the proof of Theorem 3.2. \qed

### 3.3. Numerical tests.

To verify the asymptotic behavior of the solutions of the Timoshenko system (9), we consider the following data $I = 26$, $T = 35$, $c = 0.05$ and the initial conditions:

$$\begin{align*}
\varphi_0(x) &= \psi_0(x) = \theta_0(x) = q_0(x) = 0, \\
\varphi_1(x) &= \cos(\pi x), \quad \psi_1(x) = \sin(2\pi x), \\
\theta_1(x, 0) &= \theta_1(x) = \frac{2\delta}{\rho_3} \cos(2\pi x), \\
q_1(x, 0) &= q_1(x) = 0. 
\end{align*}$$

(50)

Note that in what follows the energy decay of the solution is proven by taking the maximum value of the function of the displacement $\varphi$.

#### 3.3.1. The case $\mu = 0$.

For the following numerical computations we will consider different values for the parameters $k, \rho_1, \rho_2, \rho_3, b, \beta$ and $\tau$. For example, in Figure 2 below, we take $k = \rho_1 = \rho_2 = 2, b = \rho_3 = \beta = 1, \delta = \sqrt{\frac{2}{3}}$ and $\tau = 3$. 
3.3.2. The case $\mu \neq 0$. Similarly as in the case $\mu = 0$, we take different values for the parameters $k, \rho_1, \rho_2, \rho_3, b, \beta$ and $\tau$. For example in Figure 3, we take $k = b = \rho_1 = \rho_2 = 2$ and $\rho_3 = \delta = \beta = \tau = 1$.

Here we have theoretically obtained a polynomial decay of the energy.

Finally, in Figure 4 we give the three dimensional pointwise numerical solution of the Timoshenko system (9). This proves again the energy decay of the transversal displacement $\varphi$, for $t$ large enough.
Acknowledgments. The authors would like to thank the referees for their attentive reading and valuable remarks and questions which improved this article.

REFERENCES

[1] K. Ammari, A. Bchatnia and K. El Mufti, Non-uniform decay of the energy of some dissipative evolution systems, *Z. Anal. Anwend.*, **36** (2017), 239–251.

[2] M. A. Ayadi, A. Bchatnia, M. Hamouda and S. Messaoudi, General decay in a Timoshenko-type system with thermoelasticity with second sound, *Adv. Nonlinear Anal.*, **4** (2015), 263–284.

[3] A. Bchatnia, S. Chebbi, M. Hamouda and A. Soufyane, Lower bound and optimality for a nonlinearly damped Timoshenko system with thermoelasticity, *Asymptot. Anal.*, **114** (2019), 73–91.

[4] S. Chebbi and M. Hamouda, Discrete energy behavior of a damped Timoshenko system, *Comput. Appl. Math.*, **39** (2020), Paper No. 4, 19 pp.

[5] H. D. Fernández Sare and R. Racke, On the stability of damped Timoshenko systems: Cattaneo versus Fourier law, *Arch. Ration. Mech. Anal.*, **194** (2009), 221–251.

[6] Z. Gao and S. Xie, Fourth-order alternating direction implicit compact finite difference schemes for two-dimensional Schrödinger equations, *Appl. Numer. Math.*, **61** (2011), 593–614.

[7] A. Guesmia and S. A. Messaoudi, General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping, *Math. Meth. Appl. Sci.*, **32** (2009), 2102–2122.

[8] A. Guesmia, and S. A. Messaoudi, On the control of a viscoelastic damped Timoshenko-type system, *Appl. Math. Compt.*, **206** (2008), 589–597.

[9] M. S. Ismail and F. Mosally, A fourth order finite difference method for the good Boussinesq equation, *Abstr. Appl. Anal.*, (2014), Art. ID 323260, 10 pp.

[10] J. U. Kim and Y. Renardy, Boundary control of the Timoshenko beam, *SIAM J. Control Optim.*, **25** (1987), 1417–1429.

[11] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[12] S. A. Messaoudi and M. I. Mustafa, On the stabilization of the Timoshenko system by a weak nonlinear dissipation, *Math. Meth. Appl. Sci.*, **32** (2009), 454–469.

[13] S. A. Messaoudi, M. Pokojovy and B. Said-Houari, Nonlinear damped Timoshenko systems with second sound–global existence and exponential stability, *Math. Meth. Appl. Sci.*, **32** (2009), 505–534.
[14] S. A. Messaoudi and A. Soufyane, Boundary stabilization of solutions of a nonlinear system of Timoshenko type, *Nonlinear Anal.*, 67 (2007), 2107–2121.

[15] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability, *J. Math. Anal. Appl.*, 276 (2002), 248–278.

[16] B. V. Numerov, A method of extrapolation of perturbations, *Monthly Notices of the Royal Astronomical Society.*, 84 (1924), 592–601.

[17] B. V. Numerov, Note on the numerical integration of $\frac{d^2x}{dt^2} = f(x, t)$, *Astronomische Nachrichten*, 230 (1927), 359–364.

[18] C. A. Raposo, J. A. D. Chuquipoma, J. A. J. Avila and M. L. Santos, Exponential decay and numerical solution for a Timoshenko system with delay term in the internal feedback, *International Journal of Analysis and Applications.*, 3 (2013), 1–13.

[19] D. M. Serre, Theory and applications, *Translated from the 2001 French original. Graduate Texts in Mathematics.*, 216. Springer-Verlag, New York, 2002.

[20] A. Soufyane and A. Wehbe, Uniform stabilization for the Timoshenko beam by a locally distributed damping, *Electron. J. Differential Equations.*, (2003), No. 29, 14 pp.

[21] B. Wang, T. Sun and D. Liang, The conservative and fourth-order compact finite difference schemes for regularized long wave equation, *J. Comput. Appl. Math.*, 356 (2019), 98–117.

[22] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, Pure and Applied Mathematics (New York), Wiley-Interscience, John Wiley & Sons, Hoboken, NJ., 2006.

Received June 2020; revised November 2020.

E-mail address: mmhamouda@lau.edu.sa
E-mail address: ahmed.bchatnia@fst.utm.tn
E-mail address: medali.ayadi@esprit.tn