A NOTE ON FELLER SEMIGROUPS AND RESOLVENTS

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ABSTRACT. Various equivalent conditions for a semigroup or a resolvent generated by a Markov process to be of Feller type are given.

The Feller property of the semigroup generated by a Markov process plays a prominent role in the theory of stochastic processes. This is mainly due to the fact that if the Feller property holds true, then — under the additional assumption of right continuity of the paths — the simple Markov property implies the strong Markov property (e.g., [3, Theorem III.3.1] or [4, Theorem III.15.3]).

However, in many instances it is of advantage to consider the associated resolvent instead of the semigroup. Therefore we present in this note a result which states various forms of the equivalence of the Feller property as expressed in terms of the semigroup or of the resolvent. The material here seems to be quite well-known, and our presentation of it owes very much to [2] — most notably the inversion formula for the Laplace transform, equation (3) in connection with lemma 5. On the other hand, we were not able to locate a reference where the results are collected and stated in the form of the theorem given below.

Assume that $(E, d)$ is a locally compact separable metric space with Borel $\sigma$–algebra denoted by $\mathcal{B}(E)$. $\mathcal{B}(E)$ denotes the space of bounded measurable real valued functions on $E$, $C_0(E)$ the subspace of continuous functions vanishing at infinity. $\mathcal{B}(E)$ and $C_0(E)$ are equipped with the sup-norm $\| \cdot \|$.

The following definition is as in [3]:

**Definition 1.** A Feller semigroup is a family $U = (U_t, t \geq 0)$ of positive linear operators on $C_0(E)$ such that

- (a) $U_0 = id$ and $\|U_t\| \leq 1$ for every $t \geq 0$;
- (b) $U_{t+s} = U_t \circ U_s$ for every pair $s, t \geq 0$;
- (c) $\lim_{t \downarrow 0} \|U_t f - f\| = 0$ for every $f \in C_0(E)$.

Analogously we define

**Definition 2.** A Feller resolvent is a family $R = (R_\lambda, \lambda > 0)$ of positive linear operators on $C_0(E)$ such that

- (a) $\|R_\lambda\| \leq \lambda^{-1}$ for every $\lambda > 0$;
- (b) $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda \circ R_\mu$ for every pair $\lambda, \mu > 0$.

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Lemma 3. Let \( \text{Lemma 3.} \) nated convergence theorem:

Lemma 4. The semigroup \( \lambda P \) with respect to \( X \) in definition 1, follows from the Markov property of Property (a) of Definitions 1 and 2 is obviously satisfied. The semigroup property, (2) \( X \) with \( B \) is measurable from \( \mathbb{R} \) is independent of the choice of \( U \) or \( R \) map \( C_0(E) \) into itself, nor the strong continuity property (c) in Definitions 1, 2 hold true on \( X \) property of \( U \) and resolvents \( R \) follows from the Markov property of \( X \), and this in turn implies the resolvent equation, (b) of definition 2. Moreover, it follows also from the Markov property of \( X \) that the semigroup and the resolvent commute. On the other hand, in general neither the property that \( U \) or \( R \) map \( C_0(E) \) into itself, nor the strong continuity property (c) in Definitions 1, 2 hold true on \( B(E) \) or on \( C_0(E) \).

If \( W \) is a subspace of \( B(E) \) the resolvent equation shows that the image of \( W \) under \( R \) is independent of the choice of \( \lambda > 0 \), and in the sequel we shall denote the image by \( RW \). Furthermore, for simplicity we shall write \( UC_0(E) \subset C_0(E) \), if \( U_t f \in C_0(E) \) for all \( t \geq 0 \), \( f \in C_0(E) \).

**Theorem.** The following statements are equivalent:

(a) \( U \) is Feller.
(b) \( R \) is Feller.
(c) \( UC_0(E) \subset C_0(E) \), and for all \( f \in C_0(E), x \in E \), \( \lim_{t \downarrow 0} U_t f(x) = f(x) \).
(d) \( UC_0(E) \subset C_0(E) \), and for all \( f \in C_0(E), x \in E \), \( \lim_{\lambda \to \infty} \lambda R \lambda f(x) = f(x) \).
(e) \( RC_0(E) \subset C_0(E) \), and for all \( f \in C_0(E), x \in E \), \( \lim_{t \downarrow 0} U_t f(x) = f(x) \).
(f) \( RC_0(E) \subset C_0(E) \), and for all \( f \in C_0(E), x \in E \), \( \lim_{\lambda \to \infty} \lambda R \lambda f(x) = f(x) \).

We prepare a sequence of lemmas. The first one follows directly from the dominated convergence theorem:

**Lemma 3.** Assume that for \( f \in B(E) \), \( U_t f \to f \) as \( t \downarrow 0 \). Then \( \lambda R \lambda f \to f \) as \( \lambda \to +\infty \).

**Lemma 4.** The semigroup \( U \) is strongly continuous on \( R \).
Proof. If strong continuity at $t = 0$ has been shown, strong continuity at $t > 0$ follows from the semigroup property of $U$, and the fact that $U$ and $R$ commute. Therefore it is enough to show strong continuity at $t = 0$.

Let $f \in B(E)$, $\lambda > 0$, $t > 0$, and consider for $x \in E$ the following computation

$$U_t R_\lambda f(x) - R_\lambda f(x)$$

where we used Fubini’s theorem and the Markov property of $X$. Thus we get the following estimation

$$\|U_t R_\lambda f - R_\lambda f\| \leq \left( (e^{\lambda t} - 1) \int_t^\infty e^{-\lambda s} ds + \int_0^t e^{-\lambda s} ds \right) \|f\|$$

which converges to zero as $t$ decreases to zero.

For $\lambda > 0$, $t \geq 0$, $f \in B(E)$, $x \in E$ set

$$U_1^t f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} n \lambda e^{n \lambda t} R_n \lambda f(x).$$

Observe that, because of $n \lambda \|R_n \lambda f\| \leq \|f\|$, the last sum converges in $B(E)$.

For the proof of the next lemma we refer the reader to [2, p. 477 f]:

Lemma 5. For all $t \geq 0$, $f \in RB(E)$, $U_1^t f$ converges in $B(E)$ to $U_t f$ as $\lambda$ tends to infinity.

Lemma 6. If $U_t C_0(E) \subset C_0(E)$ for all $t \geq 0$, then $R_\lambda C_0(E) \subset C_0(E)$, for all $\lambda > 0$. If $R_\lambda C_0(E) \subset C_0(E)$, for some $\lambda > 0$, and $R_\lambda C_0(E)$ is dense in $C_0(E)$, then $U_t C_0(E) \subset C_0(E)$ for all $t \geq 0$.

Proof. Assume that $U_t C_0(E) \subset C_0(E)$ for all $t \geq 0$, let $f \in C_0(E)$, $x \in E$, and suppose that $(x_n, n \in \mathbb{N})$ is a sequence converging in $(E, d)$ to $x$. Then a straightforward application of the dominated convergence theorem shows that for every $\lambda > 0$, $R_\lambda f(x_n)$ converges to $R_\lambda f(x)$. Hence $R_\lambda f \in C_0(E)$.

Now assume that $R_\lambda C_0(E) \subset C_0(E)$, for some and therefore for all $\lambda > 0$, and that $R_\lambda C_0(E)$ is dense in $C_0(E)$. Consider $f \in RC_0(E)$, $t > 0$, and for $\lambda > 0$ define $U_1^\lambda f$ as in equation (3). Because $R_n \lambda f \in C_0(E)$ and the series in formula (3) converges uniformly in $x \in E$, we get $U_1^\lambda f \in C_0(E)$. By lemma 5, we find that $U_1^\lambda f$ converges uniformly to $U_t f$ as $\lambda \to +\infty$. Hence $U_t f \in C_0(E)$. Since $RC_0(E)$...
is dense in $C_0(E)$, $U_t$ is a contraction and $C_0(E)$ is closed, we get that $U_tC_0(E) \subset C_0(E)$ for every $t \geq 0$. \hfill \square

The following lemma is proved as a part of Theorem 17.4 in [1] (cf. also the proof of Proposition 2.4 in [3]).

**Lemma 7.** Assume that $RC_0(E) \subset C_0(E)$, and that for all $x \in E$, $f \in C_0(E)$, \lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$. Then $RC_0(E)$ is dense in $C_0(E)$.

If for all $f \in C_0(E)$, $x \in E$, $U_tf(x)$ converges to $f(x)$ as $t$ decreases to zero, then similarly as in the proof of lemma 3, we get that $\lambda R_\lambda f(x)$ converges to $f(x)$ as $\lambda \to +\infty$. Thus we obtain the following

**Corollary 8.** Assume that $RC_0(E) \subset C_0(E)$, and that for all $x \in E$, $f \in C_0(E)$, \lim_{t\downarrow 0} U_tf(x) = f(x)$. Then $RC_0(E)$ is dense in $C_0(E)$.

Now we can come to the

**Proof of the theorem.** We begin by proving the equivalence of statements (a), (b), (d), and (f):

“(a) ⇒ (b)” Assume that $U$ is Feller. From lemma 6 it follows that $R_\lambda C_0(E) \subset C_0(E)$, $\lambda > 0$. Let $f \in C_0(E)$. Since $U$ is strongly continuous on $C_0(E)$, lemma 3 implies that $\lambda R_\lambda f$ converges to $f$ as $\lambda$ tends to $+\infty$. Hence $R$ is Feller.

“(b) ⇒ (f)” This is trivial.

“(f) ⇒ (d)” By lemma 7 $RC_0(E)$ is dense in $C_0(E)$, and therefore lemma 6 entails that $UC_0(E) \subset C_0(E)$.

“(d) ⇒ (a)” By lemmas 6 and 7 $RC_0(E)$ is dense in $C_0(E)$, and therefore by lemma 4 $U$ is strongly continuous on $C_0(E)$. Thus $U$ is Feller.

Now we prove the equivalence of (a), (c), and (e):

“(a) ⇒ (c)” This is trivial.

“(c) ⇒ (e)” This follows directly from Lemma 6.

“(e) ⇒ (a)” By corollary 8 $RC_0(E)$ is dense in $C_0(E)$, hence it follows from lemma 8 that $UC_0(E) \subset C_0(E)$. Furthermore, lemma 4 implies the strong continuity of $U$ on $RC_0(E)$, and by density therefore on $C_0(E)$. (a) follows. \hfill \square

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