SCHWINGER’S FORMULA AND THE PARTITION FUNCTION
FOR THE BOSONIC AND FERMIONIC HARMONIC OSCILLATOR

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Abstract

We use Schwinger’s formula, introduced by himself in the early fifties to compute effective actions for QED, and recently applied to the Casimir effect, to obtain the partition functions for both the bosonic and fermionic harmonic oscillator.

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The computation of the partition function of the usual harmonic oscillator is probably one of the most elementary exercises in a Statistical Mechanics course. There are many ways of making this calculation, and undoubtedly the easiest is the direct one, that is,

\[
Z(\beta) := \text{Tr} e^{-\beta \hat{H}} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\omega} = \frac{1}{1 - e^{-\beta\omega}} = \frac{1}{2 \sinh(\frac{\beta\omega}{2})},
\]

where we simply used that the eigenvalues of the Hamiltonian operator for the harmonic oscillator are given by \((n + \frac{1}{2})\omega\), with \(n = 0, 1, 2, \ldots\) (we are using \(\hbar = 1\)) and summed the infinite terms of a geometric series.

However, it is exactly the simplicity of handling with this example that makes it a perfect “laboratory” to test or develop other methods of computation, as for instance, the path integral method\(^{(1)}\), the Green function method\(^{(2)}\), etc.

Our purpose in this letter is to apply a formula invented by Schwinger in 1951\(^{(3)}\) to compute effective actions for QED, and recently applied with success in the computation of the Casimir energy for both the massless and massive scalar field\(^{(4-6)}\), to obtain not only the result of equation (1), but also the partition function for a fermionic harmonic oscillator. Curious as it may seem, this approach has never appeared in the literature.

Let us start with the bosonic case. It is well known that the corresponding partition function can be written as\(^{(7)}\)

\[
Z(\beta) = \det -\frac{1}{2} (\omega^2 - \partial^2_\tau)|_{F_p},
\]

where the subscript \(F_p\) means that the operator \(\omega^2 - \partial^2_\tau\) acts only on a set of functions which are periodic, with period \(\beta\). In ref. [7], Gibbons used the generalized \(\zeta\)-function method to compute such a determinant. Here, we shall use Schwinger’s
formula (deduced in the Appendix A):

\[
\ln Z(\beta) = \frac{1}{2} Tr \int_0^\infty ds \, s^{\nu-1} e^{-i s \hat{L}_\omega},
\]

(3)

where \( \hat{L}_\omega := \omega^2 - \partial_t^2 \) and we chose the regularization based on the analytical continuation, instead of Schwinger’s one.

For periodic boundary conditions we have

\[
Tr e^{-i s \hat{L}_\omega} = \sum_{n=-\infty}^{\infty} e^{-i s [\omega^2 + n^2 (\frac{2\pi}{\beta})^2]},
\]

(4)

so that

\[
\ln Z(\beta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int_0^\infty ds \, s^{\nu-1} e^{-i s [\omega^2 + n^2 (\frac{2\pi}{\beta})^2]} \\
= \frac{1}{2} \Gamma(\nu) \sum_{n=-\infty}^{\infty} \left[ \omega^2 + \left( \frac{n 2\pi}{\beta} \right)^2 \right]^{-\nu} \\
= \frac{1}{2} \omega^{-2\nu} \Gamma(\nu) + \left( \frac{\beta}{2\pi} \right)^{2\nu} \Gamma(\nu) E_{1}^{\mu_2}(\nu, 1),
\]

(5)

where we introduced the one-dimensional inhomogeneous Epstein function

\[
E_{1}^{\mu_2}(\nu, 1) := \sum_{n=1}^{\infty} (n^2 + \mu^2)^{-\nu} \quad \text{Re} \nu > \frac{1}{2},
\]

(6)

defined \( \mu = \frac{\beta \omega}{2\pi} \) and used the well known integral representation of the Euler Gamma function \( \int_0^\infty dt \, t^{\nu-1} e^{-\alpha t} = \alpha^{-\nu} \Gamma(\nu) \).

Although the above series converges only for \( \text{Re} \nu > \frac{1}{2} \), it can be analytically continued to a meromorphic function in the whole complex plane given by\(^{8}\)

\[
E_{1}^{\mu_2}(\nu, 1) = -\frac{1}{2\mu^{2\nu}} + \frac{\sqrt{\pi} \Gamma(\nu - \frac{1}{2})}{2\Gamma(\nu)\mu^{2\nu-1}} \\
+ \frac{2\sqrt{\pi}}{\Gamma(\nu)} \sum_{n=1}^{\infty} \left( \frac{n\pi}{\mu} \right)^{\nu - \frac{1}{2}} K_{\nu - \frac{1}{2}} (2\pi n \mu).
\]

(7)

It is worth noting that the structure of poles of \( E_{1}^{\mu_2}(\nu, 1) \) is governed by the poles of \( \Gamma(\nu - \frac{1}{2}) \). Hence, they are located at \( \nu = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots \), and so on. As we see, \( E_{1}^{\mu_2}(\nu, 1) \) is analytic at the origin.
Substituting (7) into (5), taking the limit $\nu \to 0$ and observing that the divergent terms cancel without any further subtraction, we get
\[
\ln Z(\beta) = -\frac{\omega \beta}{2} + 2 \sqrt{\frac{\omega \beta}{2\pi}} \sum_{n=1}^{\infty} \sqrt{\frac{1}{n}} K_{-\frac{1}{2}}(n\omega \beta). \tag{8}
\]

In order to compute the summation on the r.h.s. of (8), we appeal to the formula (9)
\[
K_{-\frac{1}{2}}(n\omega \beta) = \sqrt{\frac{\pi}{2n\omega \beta}} e^{-n\omega \beta}. \tag{9}
\]
Inserting (9) into (8), and using equation (B.1) (see Appendix B), we obtain the final result
\[
\ln Z(\beta) = -\ln \left[ 2 \sinh \left( \frac{\omega \beta}{2} \right) \right] \implies Z(\beta) = \frac{1}{2 \sinh \left( \frac{\omega \beta}{2} \right)}, \tag{10}
\]
in perfect agreement with (1).

For the fermionic case, it can be shown that
\[
Z_f(\beta) = \det \left[ \omega^2 - \partial_x^2 \right]_{F_a}, \tag{11}
\]
where the subscript $F_a$ now means that the operator $\hat{L}_\omega$ acts only on a set of antiperiodic functions. As a consequence, the eigenvalues turn to be $\lambda_n = \omega^2 + \left( p \frac{\pi}{\beta} \right)^2$, with $p$ an odd integer. Then, using these eigenvalues in the analogue of equation (3), but remembering that for the fermionic case, instead of the factor $\frac{1}{2}$ we must write $(-1)$, we get
\[
\ln Z_f(\beta) = -\Gamma(\nu) \sum_{p=\text{odd}} \left[ \omega^2 + \left( \frac{p\pi}{\beta} \right)^2 \right]^{-\nu}. \tag{12}
\]
Adding and subtracting $\Gamma(\nu) \sum_{p=\text{even}} \left[ \omega^2 + \left( \frac{p\pi}{\beta} \right)^2 \right]^{-\nu}$ to the r.h.s. of last equation, we may write
\[
\ln Z_f(\beta) = 2\Gamma(\nu) \left[ \left( \frac{\beta}{2\pi} \right)^{2\nu} E_1^{\mu^2}(\nu, 1) - \left( \frac{\beta}{\pi} \right)^{2\nu} E_1^{\mu^2}(\nu, 1) \right], \tag{13}
\]
where now $\mu = \frac{\beta \omega}{\pi}$. Following exactly the same steps as before, that is, using the analytical continuation of the one-dimensional inhomogeneous Epstein function given by (7) and taking the limit $\nu \to 0$, we get

$$\ln Z^f(\beta) = \beta \omega + 4 \sqrt{\frac{\omega \beta}{2\pi}} \sum_{n=1}^{\infty} \sqrt{\frac{1}{n}} K_{-\frac{1}{2}}(n\omega \beta) - 4 \sqrt{\frac{\omega \beta}{\pi}} \sum_{n=1}^{\infty} \sqrt{\frac{1}{n}} K_{-\frac{1}{2}}(2n\omega \beta). \quad (14)$$

Using equations (9) and (B.1) we finally obtain,

$$\ln Z^f(\beta) = \ln \left[ \frac{\sinh(\omega \beta)}{\sinh(\frac{\omega \beta}{2})} \right]^2 \implies Z^f(\beta) = 4 \cosh^2 \left( \frac{\omega \beta}{2} \right). \quad (15)$$

This result can be checked easily in the following way: the fermionic oscillator that is being considered here is the second order Grassmann oscillator studied by Finkestein and Villasante\(^{(10)}\) (in fact, we are dealing here with the particular case of $N = 2$ of their work). In this case, it can be shown that there are only three energies: 0 (double degenerated), $+\beta \omega$, $-\beta \omega$, so that, if we use directly the definition for $Z^f(\beta)$, we will obtain

$$Z^f(\beta) = Tr e^{-\beta H^f}$$

$$= 2 + e^{\beta \omega} + e^{-\beta \omega}$$

$$= 4 \cosh^2 \left( \frac{\beta \omega}{2} \right). \quad (16)$$

In this paper we applied the Schwinger’s formula for the one-loop effective action to the computation of the partition function for both the bosonic and Grassmann harmonic oscillator. Regularization by analytical continuation was adopted. We note that depending on the boundary condition choice (e.g., Dirichlet condition), new subtractions (renormalizations) can be needed. But this is easily done remembering that the Schwinger’s formula contains an integration constant, which can be used to subtract these divergent terms.

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Appendix A

Let
\[ e^{\Gamma(\omega)} = \det \hat{L}_\omega|_F = \exp Tr \ln \hat{L}_\omega|_F, \]  \hfill (A.1)

where \( \hat{L}_\omega = \omega^2 - \partial^2_\tau \), and the subscript \( F \) means that some boundary condition is assumed. Hence
\[ \Gamma(\omega) = Tr \ln(\omega^2 - \partial^2_\tau)|_F \]  \hfill (A.2)

Taking the variation with respect to \( \omega^2 \) (the subscript \( F \) will be omitted but the boundary condition is understood), we obtain
\[ \delta_{\omega^2} \Gamma(\omega) = Tr \hat{L}_\omega^{-1} \delta \hat{L}_\omega \]
\[ = iTr \int_0^\infty ds e^{-is(\hat{L}_\omega - i\epsilon)} \delta \hat{L}_\omega \]
\[ = \delta \left[ -Tr \int_0^\infty ds \frac{s}{s} e^{-i(\hat{L}_\omega - i\epsilon)} \right] \]  \hfill (A.3)

Apart from an additive constant to be fixed by normalization, integration leads to
\[ \Gamma(\omega) = -Tr \int_0^\infty ds \frac{s}{s} e^{-is\hat{L}_\omega}, \]  \hfill (A.4)

where the \( i\epsilon \) factor of convergence is tacitly assumed. Last equation is clearly ill-defined. One way to circumvent this problem was pointed out by Schwinger: one introduces a cut-off \( s_0 \), so that the integral can be performed. Then, subtracting the divergent terms for \( s_0 = 0 \), the remaining integral is finite. However, there is another possible choice\(^{(5)}\), which consists in replacing expression (A.4) by
\[ \Gamma(\omega) = -Tr \int_0^\infty ds \frac{\nu - 1}{s} e^{-is\hat{L}_\omega}, \]  \hfill (A.5)

with \( \nu \) big enough, such that integral (A.5) is well defined. Then, after the integral is made, we perform an analytical continuation to the whole complex plane. After subtracting the poles at \( \nu = 0 \) (when they exist), we take the limit \( \nu \to 0 \), and this way we get a finite prescription for the integral (A.4). In the main text we adopt this approach instead Schwinger’s one.
Appendix B

In this Appendix we shall prove that

\[ \sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha n} = \frac{\alpha}{2} - \ln \left[ 2 \sinh \left( \frac{\alpha}{2} \right) \right] \quad ; \quad \alpha > 0. \]  \hspace{2cm} (B.1)

With this purpose, we will define \( S(\alpha) \) such that

\[ S(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha n}. \]  \hspace{2cm} (B.2)

Diferentiating both sides of (B.2) with respect to \( \alpha \) and using the well known result of the sum of the infinite terms of a geometrical series, we get

\[ - \frac{dS(\alpha)}{d\alpha} = \frac{e^{-\frac{\alpha}{2}}}{2 \sinh \left( \frac{\alpha}{2} \right)}. \]  \hspace{2cm} (B.3)

Now, integrating in \( \alpha \) we readily get

\[ S(\alpha) = \frac{\alpha}{2} - \ln \left[ 2 \sinh \left( \frac{\alpha}{2} \right) \right], \]  \hspace{2cm} (B.4)

where we used that \( S(\infty) = 0 \). This completes the desired proof.

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