Urysohn operators as adaptive filters

A. Polar and M. Poluektov

1Independent Software Consultant, Duluth, GA, USA
2International Institute for Nanocomposites Manufacturing, WMG, University of Warwick, Coventry
CV4 7AL, UK

DRAFT: January 15, 2020

Abstract

Adaptive filtering is a common approach to a system modeling by recursively adjusting model parameters. The algorithms for adaptive filtering have common concept but designed individually for each pre-selected model such as linear regression, Volterra series, kernel least mean squares or other. This article provides the new version of adaptive filtering algorithms capable of constructing models in a form of multiple sequential and parallel Urysohn operators. It is shown that multiple interconnected discrete Urysohn operators may in fact form Kolmogorov-Arnold representation for the generic multivariable continuous functions and that suggested approach allows to build such generic representation.

Keywords: Nonlinear adaptive filtering, discrete Urysohn models, non-linear system identification, Kolmogorov-Arnold representation.

1 Introduction

In a wide variety of dynamic models three of them may be united into one class. They are: finite impulse response (FIR)

\[ y_i = \sum_{j=1}^{m} w_j \cdot x_{i-j+1}, \]  

(1)

discrete Hammerstein model

\[ y_i = \sum_{j=1}^{m} w_j \cdot h(x_{i-j+1}), \]  

(2)

and discrete Urysohn model

\[ y_i = \sum_{j=1}^{m} f_j(x_{i-j+1}), \]  

(3)

where \( x_k \) are input values, \( y_k \) are output values, \( w_j \) are weight coefficients, \( h \) is input nonlinearity of Hammerstein model and \( f_j \) is set of functions that discretely form kernel \( U[\cdot, \cdot] \) of Urysohn model

\[ y(t) = \int_{0}^{T} U[x(t-s), s] ds. \]  

(4)

The common property for 1, 2, 3 is input superposition principle, which means that each discrete input value \( x_k \), after certain transformation according to particular model, makes an additive contribution to corresponding output value.
Any output nonlinearity, even static, destroys this input superposition property. The common block-oriented models of that kind are Wiener, Wiener-Hammerstein and Hammerstein-Wiener. While Urysohn (3) is more generic than either of (1) or (2), it can’t accurately replace Wiener, which is FIR followed by static nonlinearity.

If to bring into consideration the generic deterministic model

\[ y_i = F(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-m+1}), \]  

and its Kolmogorov-Arnold representation \[ \tag{4} \]

\[ y_i = F(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-m+1}) = \sum_{k=0}^{2m} \Phi_k \left( \sum_{j=1}^{m} f_{k,j}(x_{i-j+1}) \right), \]  

than it can be seen that right-hand side of (4) is actually a sum of \( 2m + 1 \) parallel discrete Urysohns with individual static nonlinearity followed by each of them. Function \( F \) should be continuous, functions \( f_{k,j} \) are independent of \( F \) and functions \( \Phi_k \) depend on \( F \).

This article provides nonlinear adaptive filtering algorithm for building the model (6) by iterative adjusting right-hand side concurrently with reading input/output data.

Comparing other popular models of dynamic systems to above representation, it can be noted that Volterra series, when limited by few terms, is a particular case of (3). Neuron networks, kernel least mean squares (KLMS) [2] and nonparametric (or so called point-cloud) models [3] are simply different models and are neither generalization nor particular case of (5) (nonparametric modeling is a technique of mapping inputs to outputs by selection of similar data samples from a large collection of previously seen data).

2 The basic concept of identification

The answer to the question “How many discrete Urysohns you can recognize in formula (6)?” is \( 2m + 2 \), not \( 2m + 1 \), because functions \( \Phi_k \) also form a discrete Urysohn operator and the model is actually a tree. The identification of representation (4) is based on already completed research [1] of modeling the single operator. For the convenience of reading it is briefly repeated below in a form allowing to understand the rest part of the article, the omitted details can be found in referred paper.

The classic LMS method [5] modifies vector of weight coefficients \( w \) at each step for a given input vector \( x \) and scalar output \( y \) (model [1])

\[ w_{k+1} = w_k + \mu \cdot (y - w_k^T \cdot x) \cdot x, \]  

where \( \mu \) is noise suppression parameter that also provides the convergence and controls convergence rate, \( k \) is index in modification sequence, \( x \) and \( y \) is given data, that assumed to be different at each step. Model (5) can be identified by the same principle if the model functions are chosen piecewise linear.

When each function \( f_j \) from (5) represents sequence of connected straight lines, the known argument \( x_{i-j+1} \) falls into a particular linear block, and each addend in (5) is computed by simple linear interpolation

\[ f_j(x_{i-j+1}) = f_j^{(left)} \cdot (1 - p) + f_j^{(right)} \cdot p, \]  

where \( f_j^{(left)} \) and \( f_j^{(right)} \) are known function values at the nodes of the linear block and \( p \) is known relative distance to the argument from the left within the block (\( 0 \leq p \leq 1 \)). So the output \( y_i \) can be predicted having all function values at the nodes and all arguments.

When model (5) needs to be identified, the outputs and arguments are known, but since unknown function values are included as multipliers into linear expression, the identification step is reduced to the same operation as for any linear regression model similar to (7). Using weights, piecewise linearity and the concept of estimation function values in fixed nodes allows to construct linear expression where arguments are used to build known parameters and estimated function nodes are presented as multipliers. This approach can be expanded to Urysohn operator with two inputs with the following continuous and discrete forms:

\[ z(t) = \int_0^T U[x(t-s), y(t-s), s]ds, \]  

\[ z(k) = \sum_{s=0}^{\infty} U[x(k-s), y(k-s), s]. \]
\[ z_i = \sum_{j=1}^{m} g_j(x_{i-j+1}, y_{i-j+1}). \] (10)

The set of functions \( g_j(\cdot, \cdot) \) approximates 3D kernel \( U[\cdot, \cdot, \cdot] \) of continuous model. The functions of single input models (3) for identification purposes were limited to piecewise linear, the functions of two input models (10) are also limited to blockwise bilinear for the same reason. The entire rectangular field of definition \([x_{\min}, x_{\max}], [y_{\min}, y_{\max}]\) of each function \( g_j \) is divided into multiple rectangular blocks without overlapping and gaps, not necessarily of the same size. Each pair of values \( x_{i-j+1} \) and \( y_{i-j+1} \) falls into a particular rectangular block and, having four corner function values \( g_{NW}, g_{NE}, g_{SW}, g_{SE} \), each addend in (10) can be computed. The subscripts \( NW, NE, SW, SE \) denote points at the nodes of rectangular block using geographic notations for convenience (North-West, North-East, South-West, South-East).

If to introduce relative distance from the North as \( 0 \leq p \leq 1 \) and relative distance from the West as \( 0 \leq q \leq 1 \), an individual addend in (10) is computed by known bilinear formula

\[ g_{NW} + (g_{NE} - g_{NW}) \cdot p + (g_{SW} - g_{NW}) \cdot q + ((g_{SE} + g_{NW}) - (g_{NE} + g_{SW})) \cdot p \cdot q \] (11)

and in identification procedure each node in involved bilinear block can be corrected using weight coefficients, computed by known inputs

\[
\begin{align*}
    w_{NW} &= (1 - p) \cdot (1 - q), \\
    w_{NE} &= p \cdot (1 - q), \\
    w_{SW} &= q \cdot (1 - p), \\
    w_{SE} &= q \cdot p,
\end{align*}
\]

which reduces identification of two input model to correcting of specific node values at each step.

As it can be seen, by choosing of piecewise linear and blockwise bilinear functions and introduction of weight coefficients, it appeared to be possible to reduce strictly nonlinear models of dynamic systems into linear models relatively to estimated parameters without making assumptions about estimated functions and obtain even their shapes from identification. This is the novelty introduced in referred article [4], the estimated parameters are not cofactors for given functions but the function values itselfs. LMS, obviously, is not the only method for estimation of parameters of linear combinations, but it appeared to be very effective in case of Urysohn operators. The convergence is extremely fast. That is explained by the physical method laying in the foundation of LMS. The linear equation is considered as a hyperplane, the current model is a point in space, the target solution is a point that belong to the same hyperplane and the operation of model update is making a projection to a hyperplane [6,7]. The equation (7) is turned into pure projection descent if \( \mu \) is replaced by \( 1/||x||^2 \). The rate of convergence for projection descent depends on angles between hyperplanes. The hyperplanes built using (8) and (11) are near orthogonal, which provides very quick convergence.

### 3 The critical properties of Urysohn models

Contrary to other known models, such as FIR, Hammerstein, Wiener and others, Urysohn obviously has lack of attention from researchers. The authors of this article can’t explain the reason for this bias. Very persistent search reveilled few works prior to this research [8–11], while all other models are widely presented in scientific papers. For example, the recent observation article for identification of nonlinear systems [12] has 174 references in bibliography and nothing about Urysohn models.

Perhaps Urysohn model needs some demystification. In discrete form Urysohn is a collection of parallel Hammersteins [13]. The product of two Hammerstein model functions \( h(s) \cdot f(x) \) (from model (2)) is an approximation to Urysohn kernel \( U[x, s] \). Multiple parallel Hammersteins approximate kernel by the sum of their function products and on that reason provide more accurate model. They actually can accurately replace Urysohn in discrete form, but identification of multiple parallel Hammersteins is not easier than a single Urysohn. Both Urysohn and Hammerstein have descriptive capabilities to model some input hard nonlinearities. Unfortunately, not all hard nonlinearities can be modeled by Urysohns and Hammersteins, only reversible. Reversible nonlinearities return to original state after input increment is rolled back. Dead-zone and Hysteresis are not reversible, but relay, saturation and rectifier are.

In case of turning the piecewise linear and blockwise bilinear models into piecewise and blockwise constant the identification needs literaly few lines of code in any modern computer language [4]. The
accuracy of the model in this case does not suffer. When model (8) is piecewise linear the computing of the output is similar to application of trapezoidal rule in numerical integration and when it is piecewise constant it is similar to rectangular rule. For approximately known function (by points) the accuracy of trapezoidal rule may not be higher than for rectangular rule.

As it is proven in [4] all discrete Urysohn models are redundant. For any given input/output data there are infinite number of Urysohn operators converting inputs into outputs with the same accuracy. In case when initial approximation is all zeros, the suggested algorithms build unique model with minimum Euclidian norm. Redundancy of the models means that comparison of the models built on different data readings does not make sense. The difference in model data is not a measure of success, failure or accuracy. The accuracy should be validated by comparing predicted and actual outputs on so-called unseen data, which were not used in identification.

4 Identification of Kolmogorov-Arnold representation

Kolmogorov-Arnold representation (6) is a tree of multiple Urysohns in discrete form. The outputs of lower level Urysohns

\[ v_{k,i} = \sum_{j=1}^{m} f_{k,j}(x_{i-j+1}), \]  

are inputs for top level Urysohn

\[ y_{i} = \sum_{k=0}^{2m} \Phi_{k}(v_{k,i}). \]

The current approximation of top level block can be updated for new values \( y_{i} \) and \( v_{k,i} \). All current approximations of lower blocks can be updated having \( v_{k,i} \) and \( x_{i-j+1} \). That means having values of \( v_{k,i} \) can solve identification problem by reducing it to identification of multiple individual Urysohns. Unfortunately values \( v_{k,i} \) are not only unobserved but also exist as auxiliary mathematical variables. When lower level model functions \( f_{k,j} \) are initially assigned and inputs \( x_{i-j+1} \) are known, it is possible to compute auxiliary parameters \( v_{k,i} \) and introduce increments for them \( \Delta v_{k,i} \), which reduce the absolute error \( E \) for top level operator

\[ E = \left| y_{i} - \sum_{k=0}^{2m} \Phi_{k}(v_{k,i}) \right| > \left| y_{i} - \sum_{k=0}^{2m} \Phi_{k}(v_{k,i} + \Delta v_{k,i}) \right|. \]  

(13)

Considering these incremented values \( v_{k,i} + \Delta v_{k,i} \) as wanted intermediate parameters, it is possible to update all operators. These increments \( \Delta v_{k,i} \) are very easy to obtain on the reason that top level operator consists of piecewise linear blocks and each \( v_{k,i} \) belong to a linear block with known edge points.

The formal steps in identification process may be listed as follows:

1. Make an initial approximation of functions \( f_{k,j} \) and \( \Phi_{k} \).
2. Compute values \( \hat{v}_{k,i} \) for known inputs.
3. Estimate error \( E \) by (13) without increments.
4. Find small increments \( \Delta \hat{v}_{k,i} \) which reduce error \( E \).
5. Replace \( v_{k,i} \) in (12) by \( \hat{v}_{k,i} + \Delta \hat{v}_{k,i} \) and update \( f_{k,j} \).
6. Recompute values \( \hat{v}_{k,i} \).
7. Update \( \Phi_{k} \).
8. Read new pair of input output values and return to 2.
The suggested algorithm is not backed up by strict mathematical proof in this paper. It may be
names as increment descent. The introduced auxiliary variables separate top level operator from lower
level. For each new pair of input/output values the small increments improving a discrepancy are found
and then all operators are updated according to found increments.

Since introduced method may be interleaved with data readings and executed at run-time, it falls
under category of adaptive filtering.

The other methods of identification of Kolmogorov-Arnold representations [14], found by authors, are
not based on discrete Urysohn models. The authors of this article did not even find the recognition that
representation itself is actually a tree of discrete Urysohn operators.

5 Numerical simulation

Since Kolmogorov-Arnold representation is more generic than input superposition models [1], [2], [3],
the data for the test must describe rather complex object, which can’t be accurately identified by them.
At the same time it should be data for some kind of realistic object, not specifically designed to back up
the article claim.

5.1 Wiener object test

Theoretically, any Wiener object must qualify for the test, because it does not support input superpo-
sition properties and can’t be accurately modeled by single Urysohn. However, Urysohns have certain
descriptive capabilities that may provide good enough approximation. It can be seen in the following test.
The model was generated as two FIR blocks and static nonlinearity that converts the sum of outputs
into observable signal

\[ r_i = \sum_{j=1}^{m} h_j \cdot x_{i-j+1} \]
\[ s_i = \sum_{j=1}^{m} w_j \cdot y_{i-j+1} \]
\[ p_i = r_i + s_i \]
\[ z_i = \text{sign}(p_i) \cdot p_i^2 + 0.2 \cdot p_i^3. \]

When this object was identified by two input Urysohn [10] the result was rather a success than failure.
The residual discrepancy estimated for unseen data was between 1% and 2%. The source code of the
program and data is publicly available [15].

The accuracy of the model was estimated by the cost function

\[ E = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (z_i - \hat{z}_i)^2}, \]
\[ z_{\text{max}} - z_{\text{min}}, \quad (14) \]

computed for the data not involved into identification process. This method of accuracy estimation
requires at least two data implementations, one for training and the other for validation. Notations \( z_i \)
and \( \hat{z}_i \) denote actual and modeled output values. The limits \( z_{\text{max}} \) and \( z_{\text{min}} \) are used to express residual
error in relative units.

5.2 Kolmogorov-Arnold representation test

Output static nonlinearity in the previous test did not turn linear model into complex enough object. In
order to make test more challenging two sequential Urysohns were tried. The chain of dynamic elements
where output of one element is an input of another is not unusual. For example, a car with automatic
transmission. The input is position of accelerator, the final output is the speed of the car and intermediate
signal is the angular velocity of the engine. If to assume that engine and transmission are not linear,
than we have that complex object that may need approximation as Kolmogorov-Arnold representation.
The data were generated by two sequential Urysohn operators, the output of the first operator was an
input of the second. In order to make sure that the object is challenging enough, it was first identified by
single Urysohn operator. The residual discrepancy, computed by the cost function [14], was near 12%. In
Figure 1: Piecewise-linear-kernel discrete-time Urysohn operators of the two-Urysohn model shown as elements of matrix $U$. The first and the second operators are shown in the left and the right subfigures.

case when Kolmogorov-Arnold representation was constructed according to section 4, the discrepancy for training data was between 1% and 2%, and for validation data near 2.5%. The generated data was quite challenging for identification. The kernels of two Urysohns used for data generation are shown in fig. 1. It can be seen that they have smooth but very irregular shape. If to make kernels not that complex, use, for example, simple bell shapes for both, the object still can be approximated by single Urysohn and advantage of the introduced algorithm would not be exposed.

The source code and data are publicly available [16].

Opposite to identification of single Urysohn, the chain or tree of Urysohn objects need nonzero initial approximation. According to algorithm of section 4 either top level or lower level models must have nonzero elements at the start. In this test all lower level Urysohns were assigned prior to real-time identification procedure. Since each single Urysohn is redundant and its identified model depends on initial approximation, the same is true for Kolmogorov-Arnold representation built according to section 4. Increment descent leads to local minimum defined by initial approximation and data.

6 Conclusions

This article is very short observation of several year research conducted by authors in the field of identification of complex deterministic dynamic systems with multiple inputs. The methods were tested also on physical objects and results sometimes were much better expectations [17, 18]. In spite of all efforts by the authors in making this research public and noticed, there is no evidence up to this time of any other independent research in this field. The authors hope that by sharing their positive experience, they may wake up an interest of the other researchers.

References

[1] A. N. Kolmogorov. On the representation of continuous functions of several variables by superposition of continuous functions of one variable and addition. Dokl. Akad. Nauk SSSR, (114):369–373, 1957.
[2] Simon Haykin. Weifeng Liu, Jose C. Principle. Kernel Adaptive Filtering. John Wiley and Sons Inc., 2010.
[3] V.A. Epanechnikov. Non-parametric estimation of a multivariate probability density. Theory of Probability and its Applications, (14):4–20, 1969.
[4] M. Poluektov and A. Polar. Modelling non-linear control systems using the discrete urysohn operator. 2018. Submitted arXiv:1802.01700.
[5] Simon. Haykin. Adaptive Filter Theory. Prentice Hall, Upper Saddle River, New Jersey., 1996.
[6] S. Kaczmarz. Angenäherte auflösung von systemen linearer gleichungen. Bulletin international de l’Académie polonaise des sciences et des lettres. Classe des sciences mathématiques et naturelles. Série A, Sciences mathématiques, 35:355–357, 1937.

[7] R. P. Tewarson. Projection methods for solving sparse linear systems. The Computer Journal, 12(1):77–80, 1969.

[8] A. R. Poluektov. Development of automatic methods for identification of diesel engines as objects of automatic control and diagnostics. PhD dissertation, Leningrad State Technical University, 1990. In Russian.

[9] P. G. Gallman. Iterative method for identification of nonlinear systems using a Uryson model. IEEE Transactions on Automatic Control, 20(6):771–775, 1975.

[10] L. V. Makarov. An interpolation method for the solution of identification problems for a multidimensional functional system described by a Urysohn operator. Journal of Mathematical Sciences, 70(1):1508–1512, 1994.

[11] V. L. Makarov and I. I. Demkiv. Approximation of the Urysohn operator by operator polynomials of Stancu type. Ukrainian Mathematical Journal, 64(3):356–386, August 2012.

[12] J. Schoukens and L. Ljung. Nonlinear system identification: A user-oriented roadmap. 2019. Submitted arXiv:1902.00683.

[13] M. Schoukens, R. Pintelon, and Y. Rolain. Parametric identification of parallel Hammerstein systems. IEEE Transactions on Instrumentation and Measurement, 60(12):3931–3938, December 2011.

[14] Jonas Actor. Computation for the kolmogorov superposition theorem. Master’s thesis, Rice University, 2018.

[15] http://ezcodesample.com/naf/wiener.html.

[16] http://ezcodesample.com/naf/kolmogorov.html.

[17] http://ezcodesample.com/naf/reallife1.html.

[18] http://ezcodesample.com/naf/reallife2.html.