Infinitesimal Variation Functions for Families of Smooth Varieties

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Abstract. In this paper we introduce some variation functions associated to the rank of the Infinitesimal Variations of Hodge Structure for a family of smooth projective complex curves. We give some bounds and inequalities and, in particular, we prove that if $X$ is a smooth plane curve, then, there exists a first order deformation $\xi \in H^1(T_X)$ which deforms $X$ as plane curve and such that $\xi : H^0(\omega_X) \to H^1(O_X)$ is an isomorphism. We also generalize the notions of variation functions to higher dimensional case and we analyze the link between IVHS and the weak and strong Lefschetz properties of the Jacobian ring of a smooth hypersurface.

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Introduction

The period mapping associates a variation of Hodge structure (VHS) (see [15, 16]) to a family of complex algebraic varieties. This operation can be seen as a partially linearization functor. The periods of an algebraic variety are objects of transcendental nature, whereas its differential is of algebraic nature. This leads to the infinitesimal variation of Hodge structure (IVHS) (see [5, 6, 13, 17, 18]), which is a complete linearization functor.

The IVHS has been proved to be an important tool to tackle several interesting problems and in dealing with lot of applications. For several families of varieties, e.g. many smooth projective hypersurfaces, IVHS was proved powerful enough to give a complete reconstruction of the algebraic variety, a Torelli-type theorem (see [7, 8, 19, 30], for example, but also [26, 27]).

Recently the IVHS of families of curves has been studied in connection with the Fujita decomposition (see [14]), the Xiao conjecture (see [1, 2, 10]) and holomorphic forms on some moduli spaces (see [11]). In these articles, very roughly, the authors tried to use the rank of differential of the period mapping in a systematic way.
Accordingly, in this paper we introduce some functions, associated to families of smooth curves (which we generalise then to the case of families of arbitrary smooth varieties) that give a quantitative measure of the variation of the family. We can call them \textit{variation functions}. If the simplest one is the dimension of the image of the modular map (this has been studied in many papers—see [20, Section 2.3], for example—and again in the recent article [9]), we consider some variation functions which take into account the IVHS.

To be more concrete, if \( \pi : X \to B \) is a family of algebraic smooth curves and \( 0 \in B \) let \( X = \pi^{-1}(0) \) be an element of this family. The differential \( d_0 \mathcal{P} \) of the period map \( \mathcal{P} \) in 0 is the composition of two maps. The first one is the Kodaira–Spencer map \( KS : T_B,0 \to H^1(T_X) \) and second is the IVHS map \( \phi : H^1(T_X) \to \text{Hom}(H^0(\omega_X),H^1(O_X)) \), which sends \( \eta \in H^1(T_X) \) to the cup product by \( \eta \) and it is the dual of the multiplication map between canonical sections \( H^0(\omega_X) \otimes H^0(\omega_X) \to H^0(\omega_X^2) \). Inspired by the importance of the study of the ranks of these maps in many of the papers cited above, we introduce two functions \( d_M \) and \( d_m \) which compute, respectively the maximal rank and the minimal rank of the cup product by elements in \( KS(T_B,0)\{0\} \) (for detail see Definition 1.1). Then we use \( d_M \) and \( d_m \) to define the variation functions \( \delta_M, \delta_m, \delta'_M \) and \( \delta'_m \) associated to \( \pi : X \to B \), which take into account all the elements of the family (see Definition 1.4). We say that a family \( \pi \) has \textit{I-maximal variation} if \( \delta'_M(\pi) = \min_{b \in B} d_M(KS(T_{B,b})) \) is equal to the genus of the curves of \( \pi \). In particular, a family \( \pi : X \to B \) has I-maximal variation if for all \( b \in B \) there exists \( \xi \in KS(T_{B,b}) \) such that \( \xi \) is an isomorphism.

The structure of the paper is as follows: first of all, in Sect. 1, we introduce the variation functions and prove some general bounds on them. We also propose one of the possible generalizations to the case of (families of) smooth varieties of dimension \( n \): the Yukawa-Coupling map (see [19, Construction 1, pag. 53] but also [22,29])

\[ \varphi : \text{Sym}^n(H^1(T_X)) \to \text{Hom}(H^0(\omega_X),H^n(O_X)) \]

sending \( \bigotimes_{i=1}^n \xi_i \) to the cup product by \( \prod_{i=1}^n \xi_i \). In Sect. 2, we introduce some notations and prove some technical results that are needed in the following sections. In Sect. 3 we focus on the case of families of smooth plane curves and prove:

\textbf{Theorem.} (Theorem 3.1) \textit{Smooth plane curves of degree} \( d \text{ have I-maximal variations as smooth curves, i.e. if} \ \pi \text{ is the family of smooth curves induced by} \ |O_{\mathbb{P}^2}(d)|, \text{we have that} \ \delta'_M(\pi) = (d-1)(d-2)/2.} 

The techniques involved are a mix of results on Jacobian rings, geometric constructions and the Castelnuovo Uniform Position Theorem (see Proposition 3.2), used to bound the dimension of certain decomposable elements contained in the Jacobian ideal, which could be interesting on its own.

We notice that our main Theorem is complementary to a result in [10], where the minimum of the variation function for a family of plane curves of degree \( d \geq 5 \) was computed (see Proposition 1.9). Finally, in Sect. 4, we analyze the higher
dimensional case: we prove that the general hypersurface of degree $d$ in $\mathbb{P}^n$ has $I$-maximal variation as hypersurface of $\mathbb{P}^n$ (see Proposition 4.2).

When $X$ is a smooth hypersurface in $\mathbb{P}^n$, the cohomology of $X$ is codified in its jacobian ring $R$, which is an Artinian graded standard algebra. For such algebras there are two properties that have been studied a lot and whose interest is still very appealing nowadays: the Weak and Strong Lefschetz Properties (see Sect. 2 for a description of these properties and some related results). There is a clear link between SLP and our variation functions: indeed, it is easy to see that if the jacobian ring of an hypersurface $X$ of $\mathbb{P}^n$ has the SLP, then $X$, as hypersurface in $\mathbb{P}^n$, has $I$-maximal variation (see Lemma 2.8).

Among the various results about these topics we would like to highlight the one which implies that Fermat hypersurfaces of degree $d$ in $\mathbb{P}^n$ have jacobian rings which satisfy the SLP. These results can be used to prove also our Proposition 4.2. Nevertheless, we decided to leave our proof since in the literature there are several and very different proofs of the above result (see Remark 2.6 for the precise statement). Our proof relies on an induction argument performed by using partial derivatives and the Euler identity.

Finally, we would like to remark that our main result, i.e. the fact that plane curves have maximal variation (as plane curves), is not a consequence of any known result regarding either the WLP or the SLP. The SLP is conjectured to hold (see Remark 2.6) for standard graded Artinian algebras of codimension $3$. Hence the main result of this article (Theorem 3.1), gives an evidence for the validity of this important conjecture.

1. Variation Functions

Let $X$ be a smooth projective curve over $\mathbb{C}$. Let

$$\varphi : H^1(T_X) \to \text{Hom}(H^0(\omega_X), H^1(O_X))$$

be the map induced by the infinitesimal variation of the periods. It is the dual map of the multiplication map $H^0(\omega_X)^{\otimes 2} \to H^0(\omega_X^{\otimes 2})$. We give two numbers to measure the variation on subspaces of $H^1(T_X)$.

**Definition 1.1.** Let $U \neq \{0\}$ be a vector subspace of $H^1(T_X)$. We set

$$d_M(U) = \max_{\xi \in U} \dim \varphi(\xi)(H^0(\omega_X)) = \max_{\xi \in U} \text{Rk}(\varphi(\xi))$$

and

$$d_m(U) = \min_{\xi \in U, \xi \neq 0} \dim \varphi(\xi)(H^0(\omega_X)) = \min_{\xi \in U, \xi \neq 0} \text{Rk}(\varphi(\xi))$$

and call them variations of $U$. We say that $U$ has $I$-maximal variation if $d_M(U) = g$, i.e. there exists $\xi \in U$ such that $\varphi(\xi)$ is an isomorphism.

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1The first known proof [3] is of combinatorial nature, whereas Stanley [28] proposed a geometric, concise and elegant proof which works when $\mathbb{K} = \mathbb{C}$. There are also algebraic proofs (like [25,33] and the recent [24]).
Remark 1.2. If $X$ is a smooth projective curve of genus $g \geq 3$ we have $d_m(H^1(T_X)) \in \{0, 1\}$ with $d_m(H^1(T_X)) = 1$ if and only if $X$ is not hyperelliptic. If $g \geq 2$ we have $d_m(H^1(T_X)) = 1$.

Using a result in [23, Lemma 2.3] one can prove that for the very general non-hyperelliptic curve, $H^1(T_X)$ has maximal variation. The following proposition strengthens this result by extending it to any smooth curve.

**Lemma 1.3.** Let $X$ be a smooth curve of genus $g$. Then $d_M(H^1(T_X)) = g$, i.e. $H^1(T_X)$ has $I$-maximal variation.

**Proof.** The aim is to prove that there exists $\xi \in H^1(T_X)$ such that the cup product $\xi : H^0(\omega_X) \to H^1(\mathcal{O}_X)$ is an isomorphism or, equivalently, injective. Assume, by contradiction, that this is not the case, so $\xi$ is never injective. Take $\xi$ such that $\xi$ is of maximal rank and let $\alpha \in H^0(\omega_X) \setminus \{0\}$ be such that $\xi \cdot \alpha = 0$. We will prove that $\xi'\alpha^2 = 0$ for all $\xi' \in H^1(T_X)$ and then see that this yields a contradiction.

Let $\xi' \in H^1(T_X)$. If $\xi'\alpha = 0$ then we have also $\xi'\alpha^2 = 0$ and there is nothing to prove. If, instead, $\xi'\alpha \neq 0$ consider $h_t = \xi + t\xi'$. Since $\xi$ is such that $\varphi(\xi)$ has maximal rank and is not injective we have also that $\varphi(\xi)$ has kernel of minimal dimension. In particular, $h_t$ has non trivial kernel for all $t$. Let $\gamma$ such that $\gamma(t) \in \text{Ker}(h_t)$ for all $t$ and $\gamma(0) = \alpha$. By considering the expansion of $\gamma(t)$, we can assume that the first non-zero term of the expansion of $\gamma$ has order $k \geq 1$. Then $\gamma(t) = \alpha + \alpha^t t^k + o(k + 1)$ and we have

$$0 \equiv h_t(\gamma(t)) = (\alpha + \alpha^t t^k + o(k + 1)) + t\xi'(\alpha + \alpha^t t^k + o(k + 1))$$

$$= t\xi'\alpha + t^k \xi' \alpha^t + o(k + 1)$$

which is impossible unless $k = 1$ since $\xi'\alpha \neq 0$. Then $k = 1$ and we get $\xi'\alpha + \xi'\alpha' = 0$. Then, by multiplying by $\alpha$ we have

$$0 = \xi'\alpha^2 + \xi'\alpha' = \xi'\alpha^2$$

as claimed.

Since $\alpha^2 \in H^0(\omega_X^2)$ and $\xi' \in H^1(T_X) = H^0(\omega_X^2)^*$, having $\xi'\alpha^2 = 0$ for all $\xi'$ is possible only if $\alpha^2 = 0$. But this is impossible since we have assumed $\alpha \neq 0$. Therefore, there exists $\xi \in H^1(T_X)$ such that $\xi : H^0(\omega_X) \to H^1(\mathcal{O}_X)$ is an isomorphism. □

We now specialize the notion of variations just introduced to families of curves. If $\pi : \mathcal{X} \to B$ is a family of smooth curves of genus $g$ over a smooth base $B$, then for any $b \in B$ we have the Kodaira–Spencer map $KS : T_{B, b} \to H^1(T_{X_b})$ where $X_b = \pi^{-1}(b)$.

We define now the variation functions discussed informally in the introduction.

**Definition 1.4.** We set

$$\delta_M(\pi) = \max_{b \in B} d_M(KS(T_{B, b})) \quad \delta_M'(\pi) = \min_{b \in B} d_M(KS(T_{B, b}))$$

$$\delta_m(\pi) = \max_{b \in B} d_m(KS(T_{B, b})) \quad \delta_m'(\pi) = \min_{b \in B} d_m(KS(T_{B, b}))$$

and call them variations functions related to $\pi$. We will say that the family has $I$-maximal variation if $\delta_M(\pi) = g$. Given a smooth projective surface $S$ and a smooth
curve $X$ in $|L|$ where $L$ is a line bundle on $S$, we define $\delta_M(L), \delta'_M(L), \delta_m(L)$ and $\delta'_m(L)$ when we consider the family of smooth curves defined by the sections of $L$.

We have the following consequence of Lemma 1.3.

**Corollary 1.5.** For a family $\pi : X \to B$ for which the moduli map is dominant (e.g. for a versal family), we have $\delta_M(\pi) = \delta'_M(\pi) = g$ by Lemma 1.3. We have $\delta_m(\pi) = \delta'_m(\pi) = 1$ with possible exceptions when the family contains hyperelliptic curves.

**Remark 1.6.** If $S$ is a surface, $L$ is a line bundle and $f \in H^0(L) \setminus \{0\}$ is smooth (i.e. $f$ is such that $X = Z(f)$ is smooth), the family induced by $|L|$ has base $B$ which is an open set in $\mathbb{P}(H^0(O_S(X)))$ and $T_{B,X} \cong H^0(O_S(X))/\langle f \rangle$. If we consider the restriction $|X : H^0(O_S(X)) \to H^0(N_{X/S})$ induced by

$$0 \to O_S \to O_X(X) \to O_X(X) = N_{X/S} \to 0$$

and the coboundary map $\partial : H^0(N_{X/S}) \to H^1(T_X)$ of the tangent sequence, the Kodaira–Spencer map is described as the composition $\partial \circ |X = KS'$.

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(O_S) \\
\downarrow & & \downarrow f \\
H^0(O_S(X)) & \longrightarrow & H^0(N_{X/S}) \\
\downarrow & & \downarrow \partial \\
H^1(T_X) & \longrightarrow & H^1(O_S)
\end{array}
$$

We will say that $X$ has **I-maximal variation in** $S$ if the subspace $U = KS'(H^0(O_S(X))) = \text{Im}(KS)$ has maximal variation for $X$.

**Proposition 1.7.** Let $S$ be a surface and $L$ an ample line bundle with a smooth section. Then $\delta_M(L) \leq g - q(S)$ where $g$ is the genus of the general curve in $|L|$.

**Proof.** Let $Z(f) = X \in |L|$ be a smooth element and fix the notation as in Remark 1.6. By Kodaira vanishing, we have the inclusions $|X : H^0(\Omega^1_S) \hookrightarrow H^0(\Omega^1_{S,X})$ and $j : H^0(\Omega^1_S|X) \to H^0(\omega_X)$ (the latter is induced by the cotangent sequence). We have then two commutative diagrams

$$
\begin{array}{ccc}
H^0(O_S) & \xrightarrow{f} & H^0(O_S(X)) \\
\downarrow & & \downarrow |X \\
H^0(N_{X/S}) & \longrightarrow & H^1(T_X) \\
\downarrow & & \downarrow \partial \\
H^0(\Omega^1_S) & \xrightarrow{\gamma} & H^0(\omega_X)
\end{array}
$$

where $\gamma$ is simply the restriction $H^0(\Omega^1_S) \to H^0(\omega_X)$ of 1-forms on $S$ to $X$. Hence, by varying $X \in |L|$, we have that $H^0(\omega_X)$ contains a constant part which comes from $H^0(\Omega^1_S)$. We want to show that these 1-forms annihilate the elements in the image of the Kodaira–Spencer map. Consider $\omega \in H^0(\Omega^1_S)$ and $\eta \in H^0(O_S(X))$. The cup product $KS'(\eta) \cdot \gamma(\omega) \in H^1(O_X)$ can be computed by factorising $KS'$ and $\gamma$ as $\partial \circ |X$ and $j \circ |X$ respectively and by using the commutativity of Diagram (3).
Remark 1.8. Let $f : S \to B$ be a non-isotrivial fibration with $S$ and $B$ smooth (of dimension 2 and 1 respectively) and let $F_b = f^{-1}(b)$ be a smooth fiber with genus $g \geq 2$. If $q_f$ is the relative irregularity of $f$ and $\xi_b \in H^1(T_{F_b})$ is the first order deformation of $F_b$ induced by $f$, then

$$q_f \leq g - \text{Rk}(\xi_b : H^0(\omega_{F_b}) \to H^1(\mathcal{O}_{F_b}))$$

(see, for instance, [1]). Notice that the above inequality can be strict (see, for example, [12,14]). If $f' = f|_U : U \to B_0$ where $U = f^{-1}(B^o)$ and $B^o \subset B$ is the open set which parametrizes smooth fibers, the above inequality implies

$$q_f \leq g - \max_{b \in U} \text{Rk}(\xi_b) = g - \max_{b \in U} d_m(KS(T_{B,b})) = g - d_m(f') \leq g - \delta_m(f').$$

Now we will focus on families of plane curves. First of all we will reinterpret a result of the authors (see [10]) in the framework of variation functions.

**Proposition 1.9.** For all $d \geq 5$ we have $\delta_m^d(\mathcal{O}_{\mathbb{P}^2}(d)) = d - 3$ and $\delta_{m}(\mathcal{O}_{\mathbb{P}^2}(d)) \geq d - 3$.

**Proof.** Consider a smooth curve of degree $d \geq 5$ on $S = \mathbb{P}^2$ and let $\pi : \mathcal{C} \to B$ be the family of smooth plane curves of degree $d$. In [10, Theorem 1.3], it is proved that the rank of the cup product by $\xi \in \text{Im}(KS)$ is at least $d - 3$, unless $\xi = 0$. Then $\delta_m(\mathcal{O}_{\mathbb{P}^2}(d)) \geq d - 3$ and $\delta_m^d(\mathcal{O}_{\mathbb{P}^2}(d)) = d - 3$ since for Fermat curve of degree $d$ one can easily write an infinitesimal deformation with rank $d - 3$.

One of the main result of this paper is that the family of curves in $\mathbb{P}^2$ has $I$-maximal variation (see Theorem 3.1). This will be stated and proved in Sect. 3.

We conclude this section by giving one of the possible generalizations (perhaps, the more extreme) of our definition of variations to the case of higher dimensional varieties.

**Definition 1.10.** Let $X$ be a smooth complete projective variety of dimension $n$. Let

$$\varphi : \text{Sym}^n(H^1(T_X)) \to \text{Hom}(H^0(\omega_X) \to H^n(\mathcal{O}_X))$$

be the Yukawa coupling mapping [19, Construction 1, pag. 53], i.e. the map sending $\otimes_{i=1}^n \xi_i$ to the cup product by $\Pi_{i=1}^n \xi_i$. For any $\xi \in H^1(T_X)$ write $\varphi(\xi^\otimes n) = \xi^n$. :
$H^0(\omega_X) \to H^n(\mathcal{O}_X)$. We say then that $\xi$ has I-maximal variation if $\xi^n$ induces an isomorphism. We set, for $U \subset H^1(T_X)$,

$$\delta_M(U) = \max_{\xi \in U} \dim \xi^n \cdot (H^0(\omega_X))$$

and

$$\delta_m(U) = \min_{\xi \in U; \xi \neq 0} \dim \xi^n \cdot (H^0(\omega_X)).$$

Accordingly, we can define the numbers $\delta_M, \delta'_M, \delta_m$ and $\delta'_m$ associated to a family of varieties and of line bundles.

2. SAGAs and Lefschetz Properties

In this section we fix some notations, we recall some well known facts and we prove two lemmas. Good references for Jacobian rings, their relation to cohomology of hypersurfaces and the IVHS of the latter are the books [4,21,31,32] or the original works [6,15–18].

Definition 2.1. Let $K$ be a field and consider a standard graded Artinian $K$-algebra $R = \bigoplus_{s=0}^N R^s$. Then $R$ is Gorenstein if the following holds:

- $R^s = 0$ if $s > N$ or $s < 0$ and $R^0 \simeq R^N \simeq K$ as vector spaces;
- the multiplication map $R^a \times R^{N-a} \to R^N$ is a perfect pairing for all $0 \leq a \leq N$.

The graded piece of degree $N$, namely $R^N$, is called socle of $R$ whereas we refer to the second property as “Gorenstein duality” (since it induces isomorphisms $R^{N-a} \simeq (R^a)^*$). For brevity, we will call SAGA a standard Artinian Gorenstein $K$-algebra.

Example 2.2. If $X = V(F)$ is a smooth hypersurface in $\mathbb{P}^n$ of degree $d$ and $J = (F_{x_0}, \ldots, F_{x_n})$ is the Jacobian ideal associated to $F$, we have that $R = K[x_0, \ldots, x_n]/J$ is a SAGA with socle in degree $N = (n+1)(d-2)$. It is called the Jacobian ring associated to $X$.

Let $R = \oplus R^s$ be a SAGA with socle in degree $N$. Take $\alpha \in R^e \setminus \{0\}$ and consider the multiplication map $\mu : R^e \to R$. Since $\alpha \in R^e$, we have that

$$\mu_{\alpha} : R^s \overset{\alpha}{\to} R^{s+e}$$

is a graded morphism and we can set $K_s(\alpha) = \ker(\alpha \cdot : R^s \to R^{s+e})$. The quotient ring $R_\alpha = R/(0 : \alpha)$ has then a natural graded structure with $(R_\alpha)^s = R^s/K_s(\alpha)$. Denote by $r_j$ and $k_j(\alpha)$ (or simply $k_j$, if no confusion arises) the dimension of $R_j^i$ and of $K_j^i$, respectively.

Lemma 2.3. Let $R = \oplus R^s$ be a SAGA with socle in degree $N$. For all $\alpha \in R^e \setminus \{0\}$ we have that $R_\alpha = R/(0 : \alpha)$ is a SAGA with socle in degree $N_\alpha = N - e$. Moreover

$$r_{N-e-s} - k_{N-e-s} = r_s - k_s$$

for all $s$ with $0 \leq s \leq N - e$.

This means that $R$ is generated, as $K$-algebra, by $R^1$, i.e. the vector space of elements of $R$ of degree 1.
Proof. First of all, notice that $R^{N-e}_\alpha$ is one dimensional. Indeed, the map $\alpha \cdot R^{N-e} \rightarrow R^N$ is surjective (by Gorenstein duality in $R$, as $\alpha \neq 0$). Then $k_{N-e} = r_{N-e} - 1$ and $\dim(R^{N-e}_\alpha) = r_{N-e} - k_{N-e} = 1$ as claimed. Now we show that Gorenstein duality holds for $R_\alpha$. Let $[\beta] \in R^\alpha_\L \setminus \{0\}$ with $0 \leq s \leq N_\alpha = N - e$. Since $[\beta] \neq [0]$ we have that $\alpha \beta \neq 0$ in $R^{s+e}$. Then, by Gorenstein duality on $R$, there exists $\gamma \in R^{N-s-e}$ such that $\alpha \beta \cdot \gamma \neq 0$ in $R^N$. In particular, $(\beta \gamma) \cdot \alpha \neq 0$ so $\beta \gamma \notin K_{N-e}(\alpha)$, i.e. $[\beta \gamma] = 0$ in $R^{N_\alpha}$ and Gorenstein duality holds for $R_\alpha$ as claimed. The relation among $r_j$ and $k_j$ simply follows from $\dim(R^\alpha_\alpha) = \dim(R^{N_\alpha-e}_\alpha)$.

Using Gorenstein duality instead of the duality between $H^1(T_X)$ and $H^0(\omega_X^2)$ we are able to prove the following lemma which is an analogue of Lemma 1.3.

Lemma 2.4. Let $R = \bigoplus_{k=0}^{N} R_k$ be a SAGA with socle in degree $N$ and fix $0 \leq d, e, \leq N$ with $d + 2e \leq N$. Then, for all $\eta \in R^d$ with $\eta : R^e \rightarrow R^{d+e}$ of maximal rank and for all $\alpha \in K_s(\eta)$ we have $\alpha^2 = 0$.

Proof. The proof is similar to the one of Lemma 1.3. Let $\eta \in R^d$ of maximal rank. If $\eta$ is injective the thesis holds trivially so we can assume that $\ker(\mu_\eta(\eta)) = K_e(\eta) \neq \{0\}$ and $d \neq 0, N$. Let $\alpha \in K_e(\eta) \setminus \{0\}$ and consider $\eta' \in R^d$. If $\eta' \alpha = 0$ then we have also $\eta' \alpha^2 = 0$. We want to show that $\eta' \alpha^2 = 0$ also if $\eta' \alpha \neq 0$. As in Lemma 1.3, one can find a curve $\gamma$ such that $\gamma(0) = \alpha$ and $\gamma(t) \in K_e((\eta + t\eta')') \setminus \{0\}$. Then we have $\eta \alpha' + \eta' \alpha = 0$. Then, by multiplying by $\alpha$ we would have

$$0 = \eta' \alpha^2 + \eta \alpha' \alpha = \eta' \alpha^2$$

since $\eta \alpha = 0$. Hence, as claimed, $\eta' \alpha^2 = 0$ for all $\eta' \in R^d$ and this proves that $\alpha^2 = 0$ by the non-degeneracy of the product $R^d \times R^2e \rightarrow R^{2d+e}$ (which follows from the Gorenstein duality $R^d \times R^{N-d} \rightarrow R^N$).

We conclude the section by recalling the definition of weak and strong Lefschetz properties. The reader can refer to [21] for a comprehensive text about the Lefschetz properties.

Definition 2.5. Let $\mathbb{K}$ be a field and consider a standard Artinian graded $\mathbb{K}$-algebra $R = \bigoplus_{k=0}^{N} R^k$. We say that $R$ satisfies the

- **Weak Lefschetz Property (WLP)** if there exists $L \in R^1$ such that $L : R^k \rightarrow R^{k+1}$ has maximal rank for all $k$;
- **Strong Lefschetz Property (SLP)** if there exists $L \in R^1$ such that $L^d : R^k \rightarrow R^{k+d}$ has maximal rank for all $k, d$.

The **codimension** of $R$ is, by definition, the number of generators of $R^1$ as $\mathbb{K}$-vector space.

We summarize here some results and conjectures relevant with respect to the topics of our article.

Remark 2.6. Let $R$ be a standard Artinian graded $\mathbb{K}$-algebra. Then:

- if $R$ has codimension 2 or less, then $R$ satisfies the SLP;
- if $R$ has codimension 3 and it is a complete intersection ring, then $R$ satisfies the WLP;
Lemma 2.8. Let the cohomology of hypersurfaces of \(\mathbb{P}^n\) and \(R\) be its Jacobian ring, then \(R\) is a SAGA with socle in degree \(N = (d-1)(n+1)\) of codimension \(n+1\) which is also a complete intersection ring (since it is a regular ring). In particular, it is conjectured (see Remark 2.6) that \(R\) satisfies SLP.

As specified in the introduction, several proofs of the latter statement exist in literature. Up to our knowledge, the first proof of the Theorem can be found in [3].

Remark 2.7. If \(X\) is a smooth hypersurface in \(\mathbb{P}^n\) and \(R\) is its Jacobian ring, then \(R\) is a SAGA with socle in degree \(N = (d-1)(n+1)\) of codimension \(n+1\) which is also a complete intersection ring (since it is a regular ring). In particular, it is conjectured (see Remark 2.6) that \(R\) satisfies SLP.

The following lemma states the link between the variation functions for hypersurfaces in \(\mathbb{P}^n\) and the SLP.

Lemma 2.8. Let \(X\) be a smooth hypersurface of \(\mathbb{P}^n\) and let \(R\) be its Jacobian ring. If \(R\) has the SLP then \(X\) has \(I\)-maximal variation as hypersurface of \(\mathbb{P}^n\).

Proof. If \(R\) satisfies the SLP then there exists \(L \in R^1 = H^0(\mathcal{O}_{\mathbb{P}^n}(1))\) such that \(L^{d(n+1)} : R^{d-n-1} \to R^{N-d-1}\) is an isomorphism. Since \(R^{d-n-1} = H^0(\mathcal{O}_{\mathbb{P}^n}(d-n-1)) \simeq H^0(\omega_X)\) by adjunction and by the results of Griffiths, Green and Donagi about the cohomology of hypersurfaces of \(\mathbb{P}^n\) and the IVHS of the latter, we have that the above multiplication map is simply the map \(\varphi(L^{d(n+1)}) = L^{d(n-1)} : H^0(\omega_X) \to H^{n-1}(\mathcal{O}_X)\). Then, we have that \(X\), as hypersurface, has \(I\)-maximal variation. \(\square\)

In particular, by Remark 2.7, it is conjectured that hypersurfaces in \(\mathbb{P}^n\) should have \(I\)-maximal variation as hypersurfaces. The main result of this article proves this conjecture for the case \(n = 2\), i.e. for plane curves (see Theorem 3.1).

3. Plane Curves

In this section we prove the main theorem of the article:

Theorem 3.1. Smooth plane curves of degree \(d \geq 3\) have \(I\)-maximal variation as plane curves. More precisely, \(\delta_M(\mathcal{O}_{\mathbb{P}^2}(d)) = \delta_M^0(\mathcal{O}_{\mathbb{P}^2}(d)) = \frac{(d-1)(d-2)}{2}\) for \(d \geq 3\).

Before the proof, we introduce some notation and prove two technical propositions. Let \(X = V(F)\) be a smooth plane curve of degree \(d\) and let \(g = (d-1)(d-2)/2\) be its genus. We will denote by \(S\) the algebra \(\bigoplus_{k \geq 0} S^k\) where \(S^k = H^0(\mathcal{O}_{\mathbb{P}^2}(k))\) and by \(J\) the Jacobian ideal associated to \(X\), i.e. the ideal spanned by partial derivatives of \(F\): \(J = (F_{x_0}, F_{x_1}, F_{x_2})\). As recalled in Example 2.2, the associated Jacobian ring \(R = S/J\) is a SAGA with socle in degree \(N = 3d - 6\). The graded pieces of \(R\) are \(R^k = S^k/J^k\) where \(J^k = S^k \cap J\).

If \(\alpha \in R^e\), we recall that we have set \(K_m(\alpha) = \ker(\alpha \cdot R^m \to R^{m+e})\) and \(k_m(\alpha) = \dim(K_m(\alpha))\). Since we defined \(r_k\) to be equal to \(\dim(R^k)\), it is convenient to set \(s_k = \dim(S^k) = h^0(\mathcal{O}_{\mathbb{P}^2}(k))\).
If \( p \geq 0 \), according to the parity of \( p \), we define \( \bar{s} \) to be one of the following Segre-like morphisms:

\[
\bar{s} = \begin{cases} 
  \text{if } p = 2k & ([\beta_1], [\beta_2]) \in \mathbb{P}(S^k) \times \mathbb{P}(S^k) \mapsto [\beta_1 \cdot \beta_2] \in \mathbb{P}(Sp) \\
  \text{if } p = 2k + 1 & ([\beta], [\gamma]) \in \mathbb{P}(S^k) \times \mathbb{P}(S^{k+1}) \mapsto [\beta \cdot \gamma] \in \mathbb{P}(Sp).
\end{cases}
\]

We also denote by \( D^p \) the image of \( \bar{s} \) and by \( D^p_j \) the intersection \( D^p \cap \mathbb{P}(J^p) \). In other words, elements of \( D^p_j \) are (particular) decomposable elements which are also in the Jacobian ideal.

If \( p \) is an integer such that \( d - 1 \leq p \leq 2d - 2 \) we have that \( Sp-d+1 \) and \( J^p \) are non-trivial (and \( Sp-d = 0 \) if and only if \( p = d - 1 \)). Then we can carry on the following construction. For any \( v = (v_0, v_1, v_2) \neq (0, 0, 0) \) let \( F_v \) be the directional derivative \( v \cdot \nabla F = v_0 F_{x_0} + v_1 F_{x_1} + v_2 F_{x_2} \). If one denotes by \( f_v \) the map

\[
f_v : Sp-d \oplus Sp-d+1 \rightarrow Sp, \quad f_v(A, B) = AF + BF_v
\]

we have that \( f_v \) is bilinear and injective. Indeed, having \( (A, B) \neq (0, 0) \) such that \( AF + BF_v = 0 \) would yield \( AF = -BF_v \) which is impossible. Indeed, \( F \) can not divide \( BF_v \) since it is irreducible and \( \deg(B) \leq \deg(F) \) as we are assuming \( p \leq 2d-2 \).

Notice that, by construction, we have \( \text{Im}(f_v) \subseteq J^p \). Let \( \bar{f}_v \) be the projectivization of \( f_v \) and denote by

\[
E^p_v = \{ [AF + BF_v] \mid A \in Sp-d, B \in Sp-d+1 \}
\]

its image in \( \mathbb{P}(J^p) \).

If \( p \geq d \) we have then the diagram

\[
\begin{array}{ccc}
D^p & \hookrightarrow & \mathbb{P}(Sp) \\
\downarrow & & \downarrow \\
D^p_j & \hookrightarrow & \mathbb{P}(J^p) \\
\downarrow & & \downarrow \bar{f}_v \\
& & \mathbb{P}(Sp-d \oplus Sp-d+1)
\end{array}
\]

where the vertical morphisms are induced by the inclusion \( J^p \subset Sp \).

**Proposition 3.2.** Assume that \( d - 1 \leq p \leq 2d - 4 \) and take \( v \) as above and general.

Then \( D^p_j \cap E^p_v \) is empty. In particular,

\[
\dim(D^p_j) = \dim(\mathbb{P}(J^p) \cap \text{Im}(\bar{s})) < \dim(\mathbb{P}(J^p)) - s_{p-d} - s_{p-d+1} + 1
\]

where \( \bar{s} \) is the Segre morphism defined above.

**Proof.** Assume that \( p = d - 1 \). Then \( J^p = J^{d-1} \) is a linear system without base points and, by Bertini’s Theorem, its general element is irreducible. On the other hand, if \( p = d - 1 \) we have that \( E^p_v = \{ [F_v] \} \) is a single point so for \( v \) general \( E^{d-1}_v \cap D^{d-1}_j = \emptyset \). Since \( s_{p-d} = 0 \) and \( s_{p-d+1} = 1 \), Inequality 5 holds.

Assume now that \( p = 2k \geq d \) so, by the hypothesis on \( p \), we have

\[
k \leq d - 2.
\]

Suppose, by contradiction, that there exists an element in \( E^{2k}_v \cap D^{2k}_j \). Then, there exist \( \beta_1, \beta_2 \in S^k \setminus \{0\} \) and \( A \in S^{2k-d}, B \in S^{2k-d+1} \) such that

\[
\beta_1 \beta_2 = AF + BF_v.
\]
The dual map of the curve $X$ fits into the diagram

$$
\begin{array}{ccc}
\mathbb{P}^2 & \nabla F & (\mathbb{P}^2)^* \\
\downarrow \nu & & \downarrow \nu^* \\
X & & X^*
\end{array}
$$

where $\nabla F$ is the gradient of $F$. It is the morphism induced by the subsystem $|J^{d-1}|$ of $|S^{d-1}|$. It is well known that $\nu$ is a birational morphism and that $X^*$ is a curve of degree $d(d-1)$. Any choice of $\nu = (v_0, v_1, v_2) \neq (0, 0, 0)$ corresponds to the choice of a line in $(\mathbb{P}^2)^*$, namely the line $L_v : v_0 z_0 + v_1 z_1 + v_2 z_2 = 0$, if the $z_i$‘s are projective coordinates on $(\mathbb{P}^2)^*$. If $\nu$ is general, the corresponding line is also general so it cuts $X^*$ in exactly $d(d-1)$ distinct points. Since $\nu$ is general, we can also assume that the preimages of these points are distinct. Since $(\nabla F)^*(L_v)$ is the curve with equation $F_v = 0$, we have produced a set $P$ of $d(d-1)$ distinct points on $X$, which, by construction, also annihilate the polynomial $AF + BF_v$. In particular, we have that the product $\beta_1 \beta_2$ vanishes on all these points. Notice that neither $\beta_1$ nor $\beta_2$ can vanish on all the $d(d-1)$ points of $P$ since $\beta_i \in S^k = H^0(O_{\mathbb{P}^2}(k))$ cuts on $X$ a divisor of degree $kd$ and by Inequality (6) we have $k \leq d - 2$. Hence one of them (say $\beta_1$, for example) vanishes on $m$ of these points with

$$m \geq d(d-1)/2$$

and does not vanish on at least one of the other $d(d-1) - m$ points.

Let $\{p_1, \ldots, p_m\}$ be the points of $P$ in the support of the divisor cut by $\beta_1$ on $X$ and let $P_{m+1}$ be another point chosen among the $d(d-1)$ points. By the Uniform Position Theorem, by following loops around $L_v$ we can permute the points $\{p_1, \ldots, p_m, p_{m+1}\}$ via monodromy in any way we would like (since the monodromy is the full symmetric group of the fiber). Then we are able to construct $m+1$ sections $\omega_1, \ldots, \omega_{m+1}$ of $O_X(k)$ such that $\omega_i(p_j) = \lambda_i \delta_{ij}$ where $\lambda_i \neq 0$ and $\omega_{m+1} = \beta_1|_X$. The above vanishing conditions ensure that these forms are also independent so we have $m+1 \leq h^0(O_X(k))$. By Inequality (6) and since $\{s_k\}$ is an increasing sequence, we have $h^0(O_X(k)) = h^0(O_{\mathbb{P}^2}(k)) = s_k \leq s_{d-2}$. Then, by Inequality (7) we have the condition

$$\frac{d(d-1)}{2} + 1 \leq m + 1 \leq s_k \leq \frac{d(d-1)}{2}$$

which leads easily to a contradiction. The proof for $p \geq d$ with $p$ odd is analogous.

Since we have proved that $D_j^p$ and $E_v^p$ are disjoint we have also

$$\dim(E_v^p) + \dim(D_j^p) < \dim(\mathbb{P}(J^p)).$$

The last claim of the proposition, i.e. formula (5), follows from the last inequality since, for $p \geq d$, we have $\dim(E_v^p) = s_{p-d} + s_{p-d+1} - 1$ by construction. \hfill $\square$

The next corollary won’t be used in what follow but, in our opinion is worth to be mentioned since it has a relevant geometric meaning.

**Corollary 3.3.** Assume that $d \geq 3$. Then, the intersection $D_j^{2d-4} = D^{2d-4} \cap \mathbb{P}(J^{2d-4})$ has the expected dimension in $\mathbb{P}(S^{2d-4})$. 
Proof. Since $d \geq 3$ we have $d - 1 \leq 2d - 4$ and we can apply Proposition 3.2 with $p = 2d - 4$ and $k = d - 2$. By Gorenstein duality we have $(R^{2d-4})^* \simeq R^{d-2} \simeq S^{d-2}$ so
\[
\dim(\mathbb{P}(J^{2d-4})) = s_{2d-4} - r_{2d-4} - 1 = s_{2d-4} - s_{d-2} - 1 = \frac{3d^2 - 9d + 4}{2}.
\]
On the other hand
\[
\dim(D^{2d-4}) = 2(s_{d-2} - 1) = (d - 2)(d + 1) \quad \text{and}
\]
\[
\dim(\mathbb{P}(S^{2d-4})) = s_{2d-4} - 1 = (d - 2)(2d - 1)
\]
so we have that the expected dimension $\text{Edim}(D^{2d-4})$ of the intersection $D^{2d-4} = D^{2d-4} \cap \mathbb{P}(J^{2d-4})$ is
\[
\text{Edim}(D^{2d-4}) = \dim(D^{2d-4}) + \dim(\mathbb{P}(J^{2d-4})) - \dim(\mathbb{P}(S^{2d-4}))
\]
\[
= \frac{(d - 2)(d + 1)}{2} - 1.
\]
Comparing this with the inequality given by Proposition 3.2 we get
\[
\dim(D^{2d-4}) \leq \frac{3d^2 - 9d + 4}{2} - s_{d-4} - s_{d-3} + 1 - 1 = \text{Edim}(D^{2d-4})
\]
so $D^{2d-4}$ has actually the expected dimension. \qed

The following proposition gives a criterion which we will use to prove the main theorem.

**Proposition 3.4.** Let $X$ be a smooth plane curve of degree $d \geq 5$ and genus $g$. If $\dim(D^{2d-6}_J) < g - 1$ we have that $X$ has I-maximal variation as plane curve.

Proof. Assume that $X$ is not of I-maximal variation. We will prove that $\dim(D^{2d-6}_J) \geq g - 1$. Since $d \geq 5$ we have $2d - 6 \geq d - 1$ and then $J^{2d-6}$ is not empty. Recall that, according to Equation (4), for $p = 2d - 6$, we have the Segre morphism $\tilde{s} : \mathbb{P}(S^{d-3}) \times \mathbb{P}(S^{d-3}) \to \mathbb{P}(S^{2d-6})$ such that $[\alpha], [\beta] \mapsto [\alpha\beta]$ and $D^{2d-6}_J$ is, by definition, the intersection of $\mathbb{P}(J^{2d-6})$ with the image of $\tilde{s}$.

In order to prove the desired inequality, set
\[
Y = \{[\alpha] \in \mathbb{P}(R^{d-3}) \mid \alpha^2 \in J^{2d-6}\}
\]
and consider the incidence correspondence
\[
Z = \{([\alpha], [\beta]) \in Y \times \mathbb{P}(R^{d-3}) \mid \alpha\beta \in J^{2d-6}\}
\]
with its projection $\pi_1$ and $\pi_2$. Then we have a diagram
\[
\begin{array}{ccc}
\mathbb{P}(J^{2d-6}) & \xrightarrow{\psi} & Z \quad \{([\alpha], [\beta]) \in Y \times \mathbb{P}(R^{d-3}) \mid \alpha\beta \in J^{2d-6}\} \xrightarrow{\pi_2} \mathbb{P}(R^{d-3}) \\
\downarrow \psi & & \downarrow \pi_1 \\
D^{2d-6}_J & \xrightarrow{\psi} & Y \quad \{[\alpha] \in \mathbb{P}(R^{d-3}) \mid \alpha^2 \in J^{2d-6}\}
\end{array}
\]
where $\psi : Z \to \mathbb{P}(J^{2d-6})$ is the multiplication morphism $\psi([\alpha], [\beta]) = [\alpha\beta]$. Note that it has image in $D^{2d-6}_J$ by construction and it is finite since it is the restriction of the Segre morphism $\tilde{s}$. Then, we have $\dim(D^{2d-6}_J) \geq \dim(\text{Im}(\psi)) = \dim Z$. 

By construction $\pi_1$ is surjective (indeed, if $[\alpha] \in Y$, we have $([\alpha], [\alpha]) \in Z$) and $\pi_1^{-1}([\alpha]) = [\alpha] \times \mathbb{P}(K_{d-3}(\alpha))$ so, for $\alpha$ general we have

$$\dim(D_j^{2d-6}) \geq \dim(Z) = \dim(Y) + k_{d-3}(\alpha) - 1.$$ 

In particular, in order to conclude the proof, it is enough to prove that

$$\dim(Y) + k_{d-3}(\alpha) \geq g. \quad (8)$$

Consider the incidence correspondence

$$\tilde{\mathcal{I}} = \{([\eta], [\alpha]) \in \mathbb{P}(R^d) \times \mathbb{P}(R^{d-3}) \mid \eta \alpha \in J^{2d-3}\}$$

with its projections $p_1$ and $p_2$. Since $X$ is not of $I$-maximal variation we have that $p_1$ is surjective. Then, there is an irreducible component $\mathcal{I}$ of $\tilde{\mathcal{I}}$ which dominates $\mathbb{P}(R^d)$ via $p_1$. We will denote by $p_i$ also the restriction of $p_i$ to $\mathcal{I}$ for brevity. Since $p_1$ is dominant we have

$$\dim(\mathcal{I}) \geq \dim \mathbb{P}(R^d) = r_d - 1. \quad (9)$$

Let $U$ be the open dense subset of $\mathcal{I}$ whose elements are exactly the pairs $([\eta], [\alpha])$ with $\eta \alpha \in J$ and $\eta : R^d-3 \to R^{2d-3}$ of maximal rank. Then, by Lemma 2.4, if $([\eta], [\alpha]) \in U$ we have $\alpha^2 \in J^{2d-6}$. Hence $p_2|_U : U \to Y$ and $\dim(p_2(U)) = \dim(p_2(\mathcal{I})) \leq \dim(Y)$. Then we can write

$$\dim(p_2^{-1}([\alpha])) \geq \dim(\mathcal{I}) - \dim(\text{Im}(p_2)).$$

By Inequality (9) and since the fiber over $[\alpha] \in p_2(\mathcal{I})$ is $p_2^{-1}([\alpha]) = \mathbb{P}(K_d(\alpha)) \times [\alpha]$, we have

$$r_d - 1 \leq \dim(\mathcal{I}) \leq k_d(\alpha) - 1 + \dim(Y). \quad (10)$$

In particular, by Lemma 2.3, it follows that

$$\dim(Y) \geq r_d - k_d(\alpha) = r_{d-3} - k_{d-3}(\alpha) = g - k_{d-3}(\alpha). \quad (11)$$

which yields Inequality (8), as claimed. \qed

We are now ready to prove our main theorem.

Proof. The Theorem is true for $d = 3$. Indeed, let $\pi : X \to B$ be the family of smooth plane curves of degree 3. Then, for all $b \in B$ we have that $K_{S_b} : T_{B,b} \to H^1(T_{X_b}) \simeq \mathbb{C}$ is not zero and so it is surjective. Then, by Lemma 1.3 we have that $X_b$ has $I$-maximal variation as plane curve.

Assume now $d \geq 4$ and proceed by contradiction by assuming also that $X$ is not of $I$-maximal variation.

If $d = 4$, for $\eta \in R^4$ general we have that $\eta : R^1 \to R^5$ has non trivial kernel. If $\alpha \in K_1(\eta)$, by Lemma 2.4, we would have $\alpha^2 \in J^2$. Since $J^2 = \{0\}$ (as $J$ is generated in degree $d - 1 = 3$) we have $\alpha = 0$ which gives a contradiction. One can also prove the thesis in this case by using the fact that quartic curves are canonical.

Let $d \geq 5$. Under this assumption we can apply Proposition 3.4. By doing so, we have the inequality

$$\frac{(d-1)(d-2)}{2} - 1 = g - 1 \leq \dim(D_j^{2d-6}). \quad (12)$$
Notice that, if we assume $d = 5$, this yields $\dim(D^g_j) \geq 5$ which is impossible since $\dim(D^g_j) \leq \dim(\mathbb{P}(J^4)) = 2$. Hence, we can assume that $d \geq 6$.

Inequality (12) gives us a bound for the dimension of $D^2_{2d-6}$ from below. We aim now to get a bound from above. Since $d \geq 6$, if we set $k = d - 3$ and $p = 2k$ we have $d \leq p \leq 2d - 4$ so we can apply Proposition 3.2 to obtain

$$\dim(D^g_j)^{2d-6} < \dim(\mathbb{P}(J^{2d-6})) - s_{d-6} - s_{d-5} + 1.$$ 

Since $(R^{2d-6})^* \simeq R^d$, by Gorenstein duality, we have

$$\dim(J^{2d-6}) = s_{2d-6} - r_{2d-6} = s_{2d-6} - r_d = s_{2d-6} - (s_d - 9)$$

so the above inequality yields

$$\dim(D^g_j)^{2d-6} < s_{2d-6} - s_{d-6} - s_{d-5} + 9 = \frac{(d - 1)(d - 4)}{2}. \tag{13}$$

We can conclude the proof of the Theorem by observing that Inequalities (12) and (13) lead to a contradiction. Indeed, from these inequalities we get

$$\frac{d(d - 3)}{2} = \frac{(d - 1)(d - 2)}{2} - 1 = g - 1 \leq \dim(\mathbb{P}(J^{2d-6})) < \frac{(d - 1)(d - 4)}{2}$$

and, consequently, $d(d - 3) < (d - 1)(d - 4)$ which is true if and only if $d < 2$. Then plane curves of degree $d \geq 3$ have $I$-maximal variation. \hfill \Box

4. Yukawa Coupling for Hypersurfaces

Let $n \geq 2$ and consider $S = \mathbb{K}[x_0, \ldots, x_n]$, the homogenous coordinate ring of $\mathbb{P}^n_\mathbb{K}$ with the standard graduation $S = \bigoplus_{m \geq 0} S^m$. Consider the Fermat polynomial of degree $d$, i.e. $F_d = \sum_{i=0}^n x_i^d$, and its Jacobian ideal

$$J_{F_d} = J_d = (x_0^{d-1}, \ldots, x_n^{d-1}).$$

We set

$$J^k_d = J_d \cap S^k, \quad H = F_1 = \sum_{i=1}^n x_i \quad \text{and} \quad \sigma_d = (\Pi_{i=0}^n x_i)^{d-2}$$

so that $\sigma_d$ is a generator for the socle of the Jacobian ring $R_d = S/J_d$.

For $d, k \geq 0$ consider the following property:

$$(*)_{d,k} : \text{ if } G \in S^k, G \neq 0 \implies G \cdot H^{d(n-1)} \notin J_d \quad \text{(more precisely, } J_d^{k+d(n-1)}).$$

Lemma 4.1. Property $(*)_{d,k}$ is true if $d \geq n + 1$ and $k \leq d - n - 1$.

Proof. First of all, notice that it is enough to prove the statement for $k = d - n - 1$. Indeed, if $(*)_{d,d-n-1}$ holds and if $G' \in S^k$ with $G' \neq 0$ and $G' \cdot H^{d(n-1)} \in J_d^{k+d(n-1)}$ for some $k < d - n - 1$, then we would have $G' \cdot H^{d(n-1)} \cdot H^e \in J_d^{k+d(n-1)+e}$ for all $e \geq 1$. In particular, for $e = d - n - 1 - k$,

$$G' \cdot H^{d(n-1)} \cdot H^{d-n-1-k} = (G' \cdot H^{d-n-1-k}) \cdot H^{d(n-1)} \in J_d$$

which is impossible since $G' \cdot H^{d-n-1-k} \in S^{d-n-1}\{0\}$ and we are assuming $(*)_{d,d-n-1}$. Hence $(*)_{d,k}$ holds also for $k < d - n - 1$. \hfill \Box
Set \( k = d - n - 1 \). We will prove now that \((*)_{d,k}\) holds by induction on \( d \geq n + 1 \).

First of all notice that if \( d = n + 1 \) we have \( k = 0 \) and the claim is equivalent to say that \( H^{d(n-1)} \notin J_d \). Since \( d = n + 1 \) we have

\[
H^{d(n-1)} = H^{(n+1)(d-2)} = \cdots + \frac{[d(d-2)]!}{(d-2)!d!} (\Pi_{i=0}^n x_i)^{d-2} + \cdots
\]

so \( H^{d(n-1)} \) is equal to \( \lambda \cdot \sigma_d \) in \( R_d \) with \( \lambda \neq 0 \). Since \( \sigma_d \) generates the socle of \( R_d \) we have that \( H^{d(n-1)} \notin J_d \), as claimed.

Assume now, as induction hypothesis, that \((*)_{l,l-n-1}\) is true for \( l \leq d - 1 \). Let \( G \in S^{d-n-1} \) and assume \( L = G \cdot H^{d(n-1)} \in J_d^{n-1} \). In order to conclude our proof we need to show that \( G = 0 \).

For simplicity we will introduce the following notations. Set \( I_n = \{0, 1, \ldots, n\} \). If \( I \subseteq I_n \) we set \( |I| \) to be the cardinality of \( I \). We will call it length of \( I \). For any polynomial \( g \in S^m \) and \( I = \{i_1, \ldots, i_r\} \subseteq I_n \) we denote by \( \partial_I(g) \) the derivative of \( g \) with respect to all the variables \( \{x_{i_1}, \ldots, x_{i_r}\} \). If \( I \subseteq I_n \) we set \( \hat{I} = I_n \setminus I \) and \( \hat{\partial}_I := \partial_{\hat{I}} \), i.e. the derivative with respect to all the variables with indices not in \( I \).

For brevity, we will write \( \partial_i \) and \( \hat{\partial}_i \) instead of \( \partial_{\{i\}} \) and \( \hat{\partial}_{\{i\}} \).

Since \( L = G \cdot H^{d(n-1)} \in J_d^{n(d-1)-1} = (x_0^{d-1}, \ldots, x_n^{d-1}) \) we can write it as

\[
L = \sum_{i=0}^{n} x_i^{d-1} f_i \tag{14}
\]

for suitable \( f_i \in S^{n(d-1)-n} \). We claim that \( \hat{\partial}_j(L) \in J_{d-1} \) for all \( j \). This follows easily, since \( J_{d-1} \) is generated by the monomials \( x_0^{d-2}, \ldots, x_n^{d-2} \) and since

\[
\hat{\partial}_j(L) = \sum_{i=0}^{n} \hat{\partial}_j(f_i x_i^{d-1}) = \hat{\partial}_j(f_j x_j^{d-1}) + \sum_{i \neq j} \hat{\partial}_j(f_i x_i^{d-1})
\]

\[
= x_j^{d-1} \hat{\partial}_j(f_j) + \sum_{i \neq j} \hat{\partial}_{\{i,j\}}(\partial_i(f_i x_i^{d-1}))
\]

\[
= x_j^{d-1} \hat{\partial}_j(f_j) + \sum_{i \neq j} \hat{\partial}_{\{i,j\}}(\partial_i(f_i) x_i^{d-1} + (d-1)f_i x_i^{d-2})
\]

\[
= \sum_{i=0}^{n} \hat{\partial}_{\{j\}}(f_i) x_i^{d-1} + (d-1) \sum_{i=0}^{n} f_i x_i^{d-2}.
\]

Now we want to express the difference \( \hat{\partial}_i(L) - \hat{\partial}_j(L) \) (which, as we have just shown, is an element of \( J_{d-1} \)). In order to do so, notice that, for \( I \subseteq \{0, \ldots, n\} \) and \( s \geq |I| \) we have

\[
\partial_I(GH^s) = \sum_{m=0}^{|I|} \frac{s!}{(s - |I| + m)!} H^{s-|I|+m} \left( \sum_{|J|=m \atop J \subseteq I} \partial_J \right) (G). \tag{15}
\]
In particular, by using Eq. (15), we can write

$$\hat{\sigma}_i(GH^{d(n-1)}) = \lambda_0 GH^{d(n-1) - n} + H^{(d-1)(n-1)} \sum_{m=1}^{n} \lambda_m H^{m-1} \left( \sum_{|J|=m} \sum_{i \notin J} \partial_i \right) (G)$$

(16)

where all the coefficients $\lambda_m$ are strictly positive for $m = 0, \ldots, n$ and $\lambda_n = 1$. Hence, if $i \neq j$, we have

$$J_{d-1} \ni (\hat{\sigma}_i - \hat{\sigma}_j)(GH^{d(n-1)})$$

$$= H^{(d-1)(n-1)} \left[ \sum_{m=1}^{n} \lambda_m H^{m-1} \left( \sum_{|I|=m} \sum_{i \notin I} \partial_i - \sum_{|J|=m} \sum_{j \notin J} \partial_j \right) \right] (G) .$$

Then, the sum in the square bracket is 0 by the induction hypothesis $(*)_{d-1,(d-1)-n-1}$ since it is an element of $S^{(d-1)-n-1}$ which falls into in $J_{d-1}$ if multiplied by $H^{(d-1)(n-1)}$. Hence we can write:

$$\sum_{m=1}^{n} \lambda_m H^{m-1} \Delta_{i,j}^m (G) = 0 \quad \text{with} \quad \Delta_{i,j}^m = \left( \sum_{|I|=m} \sum_{i \notin I} \partial_i - \sum_{|J|=m} \sum_{j \notin J} \partial_j \right) .$$

If $|J| = m$ and $i, j \in J$ then $|J|$ does not give contribution to the above sum. On the other hand the same is true if both $i$ and $j$ are not in $J$ since the contributions cancel out. If $J$ give a contribution to the sum then either $i \notin J$ and $J = J' \cup \{j\}$ or $j \notin J$ and $J = J' \cup \{j\}$. Then we can write

$$\Delta_{i,j}^m = \sum_{|J|=m} \partial_j - \sum_{|J|=m} \partial_J$$

$$= \sum_{|J|=m-1} \partial_j \partial_j - \sum_{|J|=m-1} \partial_J \partial_i = \left( \sum_{|J|=m-1} \partial_J \right) (\partial_j - \partial_i)$$

(17)

and we have proven that for all $i \neq j$, if we set $G_{i,j} = (\partial_j - \partial_i)(G)$, $G'_{i,j}$ satisfies the differential equation

$$\sum_{m=1}^{n} \lambda_m H^{m-1} \Gamma_{i,j}^{m-1} (G_{i,j}) = 0 \quad \text{where} \quad \Gamma_{i,j}^{m-1} = \sum_{|J|=m-1} \partial_J$$

(18)

and with $\lambda_m$ strictly positive and $\Gamma_{i,j}^0 = \text{id}$.

We claim now that $G_{i,j} = 0$. Notice that $\lambda_1 \neq 0$ and $\Gamma_{i,j}^0 = \text{id}$ imply that $G_{i,j}$ is divisible by $H$ so we can write $G_{i,j} = HG'_{i,j}$. We claim that $G'_{i,j}$ satisfies a differential equation like the one in (18) (with different coefficients $\lambda_m$ but always strictly positive). If $I = \{i_1, \ldots, i_m\}$ then
\[ \partial_t(HK) = \sum_{k=1}^{m} \partial_{i_k \setminus \{i_k\}}(K) + H \partial_I(K) \text{ so } \Gamma_{i,j}^{m-1}(H \cdot K) = \Gamma_{i,j}^{m-2}(K) + H \Gamma_{i,j}^{m-1}(K). \]

Hence, from Eq. (18) we have

\[
0 = \sum_{m=1}^{n} \lambda_m H^{m-1} \Gamma_{i,j}^{m-1}(HG'_{i,j}) = \lambda_1 HG'_{i,j}
\]

\[+ \sum_{m=2}^{n} \lambda_m H^{m-1} \left( \Gamma_{i,j}^{m-2}(G'_{i,j}) + H \Gamma_{i,j}^{m-1}(G'_{i,j}) \right) \]

\[= \lambda_1 HG'_{i,j} + \sum_{m=2}^{n} \lambda_m H^{m-1} \Gamma_{i,j}^{m-2}(G'_{i,j}) + \sum_{m=2}^{n} \lambda_m H^{m-1} \Gamma_{i,j}^{m-1}(G'_{i,j}) \]

\[= \lambda_1 HG'_{i,j} + \lambda_2 H G'_{i,j} + \sum_{m=2}^{n-1} (\lambda_m + \lambda_{m+1}) H^{m-1} \Gamma_{i,j}^{m-1}(G'_{i,j}) + \lambda_n H^{n-1} \Gamma_{i,j}^{n-1}(G'_{i,j}). \]

(19)

By dividing by \( H \) we get

\[ \sum_{m=1}^{n} \lambda'_m H^{m-1} \Gamma_{i,j}^{m-1}(G'_{i,j}) = 0 \text{ where } \lambda'_m = \lambda_m + \lambda_{m+1} \text{ for } m \leq n-1 \text{ and } \lambda'_n = \lambda_n. \]

Since \( \lambda_m > 0 \) we have, as claimed, that the coefficients of the differential equation are strictly positive. In particular, \( \lambda'_1 \neq 0 \) and so we obtain, as before, that \( G'_{i,j} \) is divisible by \( H \) and we can iterate this process. After a finite number of iterations of this process we have \( G_{i,j} = H^{d-n-3} \cdot G''_{i,j} \) with \( G''_{i,j} \in S^1 \) that satisfies an equation like Eq. (18) with coefficients \( \lambda''_m > 0 \) for all \( m \). Since \( G''_{i,j} \in S^1 \) we have \( \Gamma_{i,j}^{m-1}(G''_{i,j}) = 0 \) as soon as \( m \geq 2 \). Then the differential equation satisfied by \( G''_{i,j} \) is simply \( \lambda''_1 G''_{i,j} = 0 \) which yields \( G''_{i,j} = 0 \) and then, finally, \( G_{i,j} = H^{d-n-3} \cdot 0 = 0 \) as claimed.

Since \( G_{i,j} = (\partial_i - \partial_j)(G) = 0 \) for all \( i,j \) we have \( \partial_i(G) = \partial_j(G) \) for all \( i,j \). We claim that \( K \in S^m \) with \( \partial_0(K) = \cdots = \partial_n(K) \) can be written as \( \alpha H^m \) for suitable \( \alpha \). If \( m = 1 \) this is clear. If we assume that the claim holds till \( m - 1 \) and \( K \in S^m \) with \( \partial_0(K) = \cdots = \partial_n(K) = D \) we have that \( D \in S^{m-1} \) and it satisfies the same condition. Indeed, if \( i,j \) are two different indices, we have \( \partial_i(D) = \partial_i(\partial_j(K)) = \partial_j(\partial_i(K)) = \partial_j(D) \). Then, by induction, we obtain \( D = \alpha' H^{m-1} \).

Using Euler relation we get

\[ K = \frac{1}{m} \sum_{i=0}^{n} x_i \partial_i K = \frac{1}{m} \sum_{i=0}^{n} x_i \alpha' H^{m-1} = \frac{1}{m} \alpha' H^m, \]

as claimed and also \( G = \alpha H^{d-n-1} \).

Since \( G = \alpha H^{d-n-1} \) and \( G \cdot H^{d(n-1)} \in J_d \) we have that \( \alpha H^{d(n-1)+d-n-1} = 0 \) in the Jacobian ring \( R_d \). Hence, \( \alpha H^{d(n-1)+d-n-1} \cdot H^{d-n-1} = \alpha H^{(n+1)(d-2)} \) is also 0 in the Jacobian ring. As \( H^{(d-2)(n-1)} = \lambda \cdot \sigma_d \) in \( R \) with \( \lambda \neq 0 \) as observed at the beginning of this proof, we have \( 0 = \alpha \lambda \sigma_d \) in \( R \) which is only possible if and only if \( \alpha = 0 \), i.e. if and only if \( G = 0 \). □
Proposition 4.2. The general hypersurface $Y \subseteq \mathbb{P}^n$ of degree $d \geq n+1$ has I-maximal variation, i.e. $\delta_M(\mathcal{O}_{\mathbb{P}^n}(d)) = h^0(\omega_Y) = h^0(\mathcal{O}_{\mathbb{P}^n}(d-n-1))$.

Proof. Let $Y$ be a Fermat hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$. Then the Yukawa coupling associated to $Y$ is generically an isomorphism by Lemma 4.1. By semicontinuity, this holds also for the general hypersurface in $\mathbb{P}^n$ of degree $d \geq n + 1$. In particular, $\delta_M(\mathcal{O}_{\mathbb{P}^n}(d))$ is maximal. \qed

As already said in the introduction, the above proposition can be also seen as a consequence of the Theorem recalled in Remark 2.6 and the discussions in the Remark 2.7 and Lemma 2.8.

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