Axiomatizations for the Shapley–Shubik power index for games with several levels of approval in the input and output

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Received: 28 January 2020 / Accepted: 6 October 2020 / Published online: 12 October 2020
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Abstract
The Shapley–Shubik index is a specialization of the Shapley value and is widely applied to evaluate the power distribution in committees drawing binary decisions. It was generalized to decisions with more than two levels of approval both in the input and the output. The corresponding games are called \((j, k)\) simple games. Here we present a new axiomatization for the Shapley–Shubik index for \((j, k)\) simple games as well as for a continuous variant, which may be considered as the limit case.

1 Introduction

Shapley (1953) introduced a function that could be interpreted as the expected utility of a game from each of its positions via the axiomatic approach—the so-called Shapley value. A bit later, see Shapley and Shubik (1954), it was restricted to games with binary decisions, i.e., simple games. An axiomatization of this so-called Shapley–Shubik index was given quite a few years later by Dubey (1975). Nowadays, the Shapley–Shubik index is one of the most established power indices for committees drawing binary decisions. However, not all decisions are binary. Abstaining from a vote might be seen as a third option for the committee members. In general, there might also be any number \(j \geq 2\) of alternatives that can be chosen from.
To this end, simple games were generalized to \((j, k)\) simple games, see Freixas and Zwicker (2003), where \(j\) is the number of ordered alternatives in the input, i.e., the voting possibilities, and \(k\) the number of ordered alternatives for the group decision.\(^1\) A Shapley–Shubik index for these \((j, k)\) simple games was introduced in Freixas (2005) generalizing earlier attempts for special cases, see e.g. (Felsenthal and Machover 1998), pp. 291–293). However, also other variants have been introduced in the literature, see e.g. Amer et al. (1998); Friedman and Parker (2018); Hsiao and Raghavan (1993). Here, we will only consider the variant from Freixas (2005). A corresponding axiomatization is given in Freixas (2019) for \((j, 2)\) simple games and for \(j\)-cooperative games (when outputs are real numbers).

If we normalize the input and output levels to numbers between zero and one, we can consider the limit if \(j\) and \(k\) tend to infinity for \((j, k)\) simple games. More precisely we can consider the input levels \(i/(j - 1)\) for \(0 \leq i \leq j - 1\) and the output levels \(i/(k - 1)\) for \(0 \leq i \leq k - 1\). Then those games are discrete approximations for games with input and output levels freely chosen from the real interval \([0, 1]\). The latter games were called simple aggregation functions in Kurz (2018), linking to the literature on aggregation functions in Grabisch et al. (2009), and interval simple games in Kurz et al. (2019). A Shapley–Shubik like index for those games was motivated and introduced in Kurz (2014); an axiomatization is given in Kurz et al. (2019). So, results for interval simple games might be obtained from results for \((j, k)\) simple games via a suitable limit argument. We do not use this approach in our paper.

The success story of the Shapley–Shubik index for simple games, initiated by Shapley (1953) and Shapley and Shubik (1954), triggered a huge amount of modifications and generalizations to different types of games, see e.g. Algaba et al. (2019) for some current research directions. We think that the variants from Freixas (2005), for \((j, k)\) simple games, and from Kurz (2014), for interval simple games, form one consistent way to generalize the Shapley–Shubik index for simple games. Here we mainly focus on an axiomatic justification. This is the object of our main result in Theorem 2. More precisely, we replace the axiom of additivity and the recent axiom of level change effect on unanimity games introduced in Freixas (2019) by the new axiom of average convexity. The underlying idea is to associate a TU game \(\tilde{v}\), called the average game, to every \((j, k)\) simple game \(v\), see Definition 7. The axiom then says that if the convex combinations of average games coincide then also the power distributions for the same convex combinations of the underlying \((j, k)\) simple games should coincide. The average game itself seems to be a very natural object on its own and has some nice properties. Indeed, the Shapley value of the TU game \(\tilde{v}\) coincides with the Shapley–Shubik index of the \((j, k)\) simple game \(v\). Moreover, we present another formula for the Shapley–Shubik index for \((j, k)\) simple games which is better suited for computation issues, see Lemma 1 and Theorem 1. For a generalization of the Banzhaf index a similar result was obtained in Pongou et al. (2012). As

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\(^1\) In our definition of a \((j, k)\) simple game we will deviate from Freixas and Zwicker (2003) by numbering the input and output levels starting from 0 instead of 1 and assuming that lower numbers correspond to a lower level of approval.
the title of the preface of Algaba et al. (2019) names it, the idea of the Shapley value is the root of a still ongoing research agenda.

The remaining part of this paper is organized as follows. In Sect. 2 we introduce the necessary preliminaries and present the first few basic results. A Shapley–Shubik index $\Phi$ for general $(j, k)$ simple games is introduced in Sect. 3. Moreover, we study the first basic properties of $\Phi$. In Sect. 4 we introduce the average game, which is a TU game associated to each $(j, k)$ simple game. This notion is then used to formulate the new axiom of average convexity, which culminates in an axiomatic characterization of $\Phi$ in Sect. 5. In Sect. 6 we transfer all notions and the axiomatic characterization to interval simple games. We close with a brief conclusion in Sect. 7.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a finite set of voters. Any subset $S$ of $N$ is called a coalition and the set of all coalitions of $N$ is denoted by the power set $2^N$. For given integers $j, k \geq 2$ we denote by $J = \{0, \ldots, j - 1\}$ the possible input levels and by $K = \{0, \ldots, k - 1\}$ the possible output levels, respectively. We write $x \leq y$ for $x, y \in \mathbb{R}^n$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For each $\emptyset \subseteq S \subseteq N$ we write $x_S$ for the restriction of $x \in \mathbb{R}^n$ to $\{i\} \in S$. As an abbreviation, we write $x_{-S} = x_{N\setminus S}$. Instead of $x_{\{i\}}$ and $x_{\{-i\}}$ we write $x_i$ and $x_{-i}$, respectively. Slightly abusing notation we write $a \in \mathbb{R}^n$, for the vector that entirely consists of $a$’s, e.g., $0$ for the zero vector.

A TU game with player set $N$ is a mapping $\nu$ from the set of all subsets $N$ to $\mathbb{R}$ such that $\nu(\emptyset) = 0$. In particular, a simple game with player set $N$ is a $\{0, 1\}$-valued TU game $\nu$ such that $\nu(N) = 1$, and $\nu(S) \leq \nu(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$. A coalition $S \subseteq N$ is said to be a winning coalition if $\nu(S) = 1$ and a losing coalition if $\nu(S) = 0$. The interpretation in the voting context is as follows. Those elements $i \in N$, called voters or players, that are contained in a coalition $S$ are those that are in favor of a certain proposal. The other voters, i.e., those in $N \setminus S$, are against the proposal. If $\nu(S) = 1$ then the proposal is implemented and otherwise the status quo persists. A simple game $\nu$ is weighted if there exists a quota $q \in \mathbb{R}_{\geq 0}$ and weights $w_i \in \mathbb{R}_{\geq 0}$ for all $i \in N$ such that $\nu(S) = 1$ iff $w(S) := \sum_{i \in S} w_i \geq q$. As notation we use $[w_1, \ldots, w_n]$ for a weighted (simple) game. An example is given by $\nu = [4; 3, 2, 1, 1]$, also called “My Aunt and I”, see e.g. (Osborne and Rubinstein (1994), page 297), with winning coalitions $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$, and $\{2, 3, 4\}$. As an example, the coalition $\{1, 3\}$ is winning since player 1 has a weight of $w_1 = 3$ and player 3 has a weight of $w_3 = 1$, so that $w(\{1, 3\}) = w_1 + w_3 = 4 \geq q$. A simple game $\nu$ is a unanimity game if there exists a coalition $\emptyset \neq T \subseteq N$ such that $\nu(S) = 1$ iff $T \subseteq S$. As an abbreviation we use the notation $\gamma_T$ for a unanimity game with defining coalition $T$. It is well known that each simple game admits a representation as disjunctions of a finite list of unanimity games. Calling a winning coalition minimal if all proper subsets are losing; such a list is given by the minimal winning coalitions, i.e., by $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{2, 3, 4\}$ in the above example.
If being part of a coalition is modeled as voting ‘‘yes’’ and ‘‘no’’ otherwise, represented as 1 and 0, respectively, then one can easily reformulate and generalize the definition of a simple game:\(^2\):

**Definition 1** A \((j, k)\) simple game for \(n\) players, where \(j, k \geq 2\) and \(n \geq 1\) are integers, is a mapping \(v : J^n \to K\) with \(v(0) = 0, v(j - 1) = k - 1,\) and \(v(x) \leq v(y)\) for all \(x, y \in J^n\) with \(x \leq y\). The set of all \((j, k)\) simple games on \(N\) is denoted by \(U_n^k\) or by \(U_n\), whenever \(j\) and \(k\) are clear from the context.

So, \((2, 2)\) simple games are in one-to-one correspondence to simple games. We use the usual ordering of \(J\) (and \(K\)) as a set of integers, i.e., \(0 < 1 < \cdots < j - 1\). In words, in the input set, 0 is the lowest level of approval, followed by 1 and so on. In general, we call a function \(f : \mathbb{R}^n \supseteq U \to \mathbb{R}\) monotone if we have \(f(x) \leq f(y)\) for all \(x, y \in U\) with \(x \leq y\). We remark that Freixas (2005) considers a more general definition of a \((j, k)\) simple game than we have presented here. Additionally the \(j\) input levels and the \(k\) output levels are given by a so-called numeric evaluation. Our case is called uniform numeric evaluation there, which motivated the notation \(U_n\) for \((j, k)\) simple games for \(n\) players. We also call a vector \(x \in J^n\) a profile. In words, we assume that the input and output levels are cardinal quantities and not ordinal quantities that are qualitatively ordered. To be more precise, we assume that the distance between two input levels is equally sized. This is in contrast to the more general interpretation of \((j, k)\) simple games as introduced in Freixas and Zwicker (2003).

**Definition 2** Given a \((j, k)\) simple game \(v\) with player set \(N\), we call a player \(i \in N\) a null player iff \(v(x) = v(x_{-i}, y_i)\) for all \(x \in J^n\) and all \(y_i \in J\).\(^3\) Two players \(i, h \in N\) are called equivalent if \(v(x) = v(x')\) for all \(x, x' \in J^n\) with \(x_i = x'_i\) for all \(i \in N \setminus \{i, h\}\), \(x_i = x'_h\), and \(x_h = x'_i\).

In words, a player \(i\) is a null player if its input \(y_i\) does not alter the output \(v(x)\). If interchanging the input \(x_i\) and \(x_h\) of two players does never alter the output \(v(x)\), then players \(i\) and \(h\) are equivalent. By \(\pi_{ih}\) we denote the transposition on \(N\) interchanging \(i\) and \(h\), so that the previous condition reads \(v(x) = v(\pi_{ih} x)\) for all \(x \in J^n\). By \(S_n\) we denote the set of permutations of length \(n\), i.e., the bijections on \(N\).

Now let us introduce a subclass of \((j, k)\) simple games with the property that for each profile \(x\), the collective decision \(v(x)\) is either 0 (the lowest level of approval) or it is \(k - 1\) (the highest level of approval) depending on whether some given voters report some minimum approval levels. For example, when any full support of the proposal necessitates a full support of each voter in a given coalition \(S\), players in \(S\) are each empowered with a veto. One may require from each player in \(S\) only a

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\(^2\) Note that we slightly deviate from the original definition of a \((j, k)\) simple game in Freixas and Zwicker (2003), see Footnote 1. With this, we ensure that \((2, 2)\) simple games are in one-to-one correspondence to simple games encoding ‘‘no’’ as 0 and ‘‘yes’’ as 1.

\(^3\) Using the notation introduced at the beginning of Sect. 2, we have \(v(x_{-i}, y_i) = v(x_{N \setminus \{i\}}, y_i) = v(x_1, \ldots, x_{-1}, y_i, x_{i+1}, \ldots, x_{j}).\)
certain level of approval for a full support of the proposal. All such games will be called \((j, k)\) simple games with point-veto.

**Definition 3** A \((j, k)\) simple game with a point-veto is a \((j, k)\) simple game \(v\) such that there exists some \(a \in J^n \setminus \{0\}\) satisfying \(v(x) = k - 1\) if \(a \leq x\) and \(v(x) = 0\) otherwise for all \(x \in J^n\). In this case, \(a\) is the veto and the game \(v\) is denoted by \(u^a\). For each coalition \(S \subseteq 2^N\) we use the abbreviation \(w^S = u^a\) for a game, where \(a_i = j - 1\) for all \(i \in S\) and \(a_i = 0\) otherwise.

We remark that \((2, 2)\) simple games with a point veto are in one-to-one correspondence to the subclass of unanimity games within simple games.

The set of all players who report a non-null approval level is denoted by \(N^a\), i.e., \(N^a = \{i \in N : 0 < a_i \leq j - 1\}\). Every player in \(N^a\) will be called a vetoer of the game \(u^a\). Note that for the vector \(a\) defined via \(w^S = u^a\) we have \(N^a = S\).

Null players as well as equivalent players can be identified easily in a given \((j, k)\) simple game with point-veto:

**Proposition 1** Let \(a \in J^n \setminus \{0\}\). A player \(i \in N\) is a null player of \(u^a\) iff \(i \in N \setminus N^a\). Two players \(i, h \in N\) are equivalent in \(u^a\) iff \(a_i = a_h\).

**Proof** For every \(a \in J^n \setminus \{0\}\) and every \(i \in N \setminus N^a\) we have \(a_i = 0\) by the definition of \(N^a\). Now let \(i \in N \setminus N^a\). For every \(x \in J^n\) and every \(y_i \in J\) we have \(a \leq x\) iff \(a \leq (x_{-i}, y_i)\). Thus, \(u^a(x) = u^a(x_{-i}, y_i)\) and \(i\) is a null player in \(u^a\). Now let \(i \in N^a\), i.e., \(a_i > 0\). Since \(v(a) = k - 1 \neq 0 = v(a_{-i}, 0)\), player \(i\) is not a null player in \(u^a\).

Assume that \(a_i = a_h\) and consider an arbitrary \(x \in J^n\). Then we have \(a \leq x\) if and only if \(a \leq \pi_{jh}x\). The definition of \(u^a\) directly gives \(u^a(x) = u^a(\pi_{jh}x)\), so that the players \(i\) and \(h\) are equivalent in \(u^a\). Now suppose that the players \(i\) and \(h\) are equivalent in \(u^a\). Since \(a \leq a\), we obtain \(u^a(a) = u^a(\pi_{jh}a) = k - 1\). This implies that \(a \leq \pi_{jh}a\). Therefore \(a_i \leq a_h\) and \(a_h \leq a_j\), that is \(a_i = a_h\). \(\square\)

Note that \((j, k)\) simple games can be combined using the disjunction \((\lor)\) or the conjunction \((\land)\) operations to obtain new games. This generalizes the union and intersection operations for simple games.

**Definition 4** Let \(v'\) and \(v''\) be two \((j, k)\) simple games with player set \(N\). By \(v' \lor v''\) we denote the \((j, k)\) simple game \(v\) defined by \(v(x) = \max\{v'(x), v''(x)\}\) for all \(x \in J^n\). Similarly, by \(v' \land v''\) we denote the \((j, k)\) simple game \(v\) defined by \(v(x) = \min\{v'(x), v''(x)\}\) for all \(x \in J^n\).

We remark that the defining properties of a \((j, k)\) simple game, i.e., monotonicity, domain, and codomain, can be easily checked. This can be specialized to the subclass of \((j, k)\) simple games with point veto, i.e., \((j, k)\) simple games with point-veto can be combined using the disjunction \((\lor)\) or the conjunction \((\land)\) operations to obtain new games. To see this, consider a non-empty subset \(E\) of \(J^n \setminus \{0\}\) and define the \((j, k)\) simple game denoted by \(u^E\) by \(u^E(x) = k - 1\) if \(a \leq x\) for some \(a \in E\) and
\(u^E(x) = 0\) otherwise, where \(x \in \mathcal{N}\) is arbitrary. Note that the notational simplification \(u^{[a]} = u^a\), where \(a \in \mathcal{N} \setminus \{0\}\), goes in line with Definition 3.

**Proposition 2** Let \(E\) and \(E'\) be two non-empty subsets of \(\mathcal{N} \setminus \{0\}\). Then, we have \(u^E \lor u^{E'} = u^{E \cup E'}\) and \(u^E \land u^{E'} = u^{E''}\), where \(E'' = \{ c \in \mathcal{N} : c_i = \max(a_i, b_i) \text{ for all } i \in \mathcal{N} \text{ and some } a \in E, b \in E' \}\).

**Proof** In order to prove \(u^E \lor u^{E'} = u^{E \cup E'}\) we consider an arbitrary \(x \in \mathcal{N}\). If \(u^{E \cup E'}(x) = k - 1\), then there exists \(a \in E \cup E'\) such that, \(a \leq x\). Therefore \(u^E(x) = k - 1\) or \(u^{E'}(x) = k - 1\) and \((u^E \lor u^{E'})(x) = k - 1\). Now suppose that \(u^{E \cup E'}(x) = 0\). Then, for all \(a \in E \cup E'\) we have \(a \not\leq x\). Since \(E \subseteq E \cup E'\) and \(E' \subseteq E \cup E'\) we have \(b \not\leq x\) and \(c \not\leq x\) for all \(b \in E\) and all \(c \in E'\). This implies that \(u^E(x) = u^{E'}(x) = 0\) and \((u^E \lor u^{E'})(x) = 0\). Thus, \(u^E \lor u^{E'} = u^{E \cup E'}\).

Similarly, in order to prove \(u^E \land u^{E'} = u^{E''}\) we consider an arbitrary \(x \in \mathcal{N}\). If \(u^{E''}(x) = k - 1\), then there exists \(c \in E''\) such that \(c \leq x\). But, by definition of \(E''\), \(c = \max(a, b)\) for some \(a \in E\) and \(b \in E'\), that is \(a \leq c \leq x\) and \(b \leq c \leq x\). Hence, \(u^E(x) = u^{E'}(x) = k - 1\) and \((u^E \land u^{E'})(x) = k - 1\). Now assume that \(u^{E \cup E'}(x) = 0\) and \((u^E \land u^{E'})(x) \neq 0\). By definition of \(u^E\) and \(u^{E'}\), we have \((u^E \land u^{E'})(x) = k - 1\). Thus, there exists \(a \in E\) and \(b \in E'\) such that \(a \leq x\) and \(b \leq x\). It follows that \(c = \max(a, b) \leq x\), which is a contradiction to \(u^{E''}(x) = 0\). This proves \(u^E \land u^{E'} = u^{E''}\).

For \((j, k) = (5, 3)\) and \(n = 3\) an example is given by \(E = \{(1, 2, 3), (2, 1, 2)\}\), \(E' = \{(4, 1, 1), (1, 1, 3)\}\). With this, \(E'' = \{(4, 2, 3), (1, 2, 3), (2, 1, 3), (4, 1, 2)\}\). Note that we may remove \((4, 2, 3)\) from that list since \((4, 2, 3) \geq (1, 2, 3)\) (or \((4, 2, 3) \geq (4, 1, 2)\)).

Especially, Proposition 2 yields that every \((j, k)\) simple game of the form \(u^E\) is a disjunction of some \((j, k)\) simple games with point-veto. So, each \((j, k)\) simple game of the form \(u^E\) will be called a \((j, k)\) simple game with veto. In the game \(u^E\), \(E\) can be viewed as some minimum requirements (or thresholds) on the approval levels of voters’ inputs for the full support of the proposal. It is worth noticing that \(u^E\) is \(\{0, k - 1\}\)-valued; the final decision at all profiles is either a no-support or a full-support. The set of all veto \((j, k)\) simple games on \(\mathcal{N}\) is denoted \(\mathcal{V}_n\). Note that Proposition 2 shows that \(\mathcal{V}_n\) is a lattice.

The sum of two \((j, k)\) simple games cannot be a \((j, k)\) simple game itself. However, we will show that each \((j, k)\) simple game is a convex combination of \((j, k)\) simple games with veto.

**Definition 5** A convex combination of the games \(v_1, v_2, \ldots , v_p \in \mathcal{U}_n\) is given by \(v = \sum_{t=1}^{p} a_t v_t\) for some non-negative numbers \(a_t\), where \(t = 1, 2, \ldots , p\), that sum to 1.

Note that not all convex combinations of \((j, k)\) simple games are \((j, k)\) simple games.
Proposition 3 For each \((j, k)\) simple game \(v\) there exists a collection of positive numbers \(\alpha_t\), where \(t = 1, 2, \ldots, p\), that sum to 1 and a collection \(F_t(v)\), where \(t = 1, 2, \ldots, p\), of non-empty subsets of \(J^n\) such that \(v = \sum_{t=1}^{p} \alpha_t u_{F_t(v)}\).

Proof Let \(v \in \mathcal{U}_n\) and \(F(v) = \{x \in J^n, v(x) > 0\}\). Since \(J^n\) is finite and \(v\) is monotone, the elements of \(F(v)\) can be labeled in such a way that \(F(v) = \{x^1, x^2, \ldots, x^p\}\), where \(x^j = j - 1\), \(v(x^j) \leq v(x^{j+1})\) for all \(1 \leq t < p\), and \(t \leq s\) whenever \(x^j \leq x^s\). Now, set \(x^0 = 0\) and \(F_t(v) = \{x^t, t \leq s \leq p\}\), \(\alpha_t = \frac{v(x^t) - v(x^t-1)}{k-1}\) for all \(1 \leq t \leq p\). By our assumption on \(x^j\) we have \(\alpha_t \geq 0\) for all \(1 \leq t \leq p\). Moreover, it can be easily checked that \(\sum_{t=1}^{p} \alpha_t = 1\). Set \(u = \sum_{t=1}^{p} \alpha_t u_{F_t(v)}\).

In order to prove that \(v = u\), we consider an arbitrary \(x \in J^n\). First suppose that \(x \notin F(v)\). Since \(v\) is monotone, there is no \(a \in F(v)\) such that \(a \leq x\). By definition, it follows that \(v_{F_t(v)}(x) = 0\) for all \(t = 1, 2, \ldots, p\). Therefore \(v(x) = u(x) = 0\). Now suppose that \(x \in F(v)\). Then \(x = x^s\) for some \(s = 1, 2, \ldots, p\). It follows that for all \(t = 1, 2, \ldots, p\) we have \(v_{F_t(v)}(x) = k - 1\) if \(1 \leq t \leq s\) and \(v_{F_t(v)}(x) = 0\) otherwise. Therefore

\[
    u(x) = \sum_{t=1}^{s} \alpha_t = \sum_{t=1}^{s} \left( \frac{v(x^t) - v(x^{t-1})}{k-1} \cdot (k - 1) \right) = v(x^t) = v(x).
\]

Clearly, the game \(v\) is a convex combination of the games \(u_{F_t(v)}\), where \(t = 1, 2, \ldots, p\).\(\square\)

Proposition 3 underlines the importance of \((j, k)\) simple games with veto, i.e., every \((j, k)\) simple game can be obtained from \((j, k)\) simple games with veto as a convex combination.

Now let us consider a continuous version of \((j, k)\) simple games normalized to the real interval \(I := [0, 1]\) for the input as well as the output levels. Following Kurz (2018) and using the name from Kurz et al. (2019), we call a mapping \(v : [0, 1]^n \to [0, 1]\) an interval simple game if \(v(0) = 0\), \(v(1) = 1\), and \(v(x) \leq v(y)\) for all \(x, y \in [0, 1]^n\) with \(x \leq y\). The set of all interval simple games on \(N\) is denoted by \(\mathcal{ISG}_n\). Replacing both \(J\) and \(K\) by \([0, 1]\) in Definition 2 we can transfer the concept of a null player and that of equivalent players to interval simple games.

3 The Shapley–Shubik index for simple and \((j, k)\) simple games

Since in a typical simple game \(v\) not all players are equivalent, the question of influence of a single player \(i\) on the final group decision \(v(S)\) arises. Even if \(v\) can be represented as a weighted game, i.e., \(v = [q; w]\), the relative individual influence is not always reasonably reflected by the weights \(w_i\). This fact is well-known and triggered the invention of power indices, i.e., mappings from a simple game on \(n\) players to \(\mathbb{R}^n\) reflecting the influence of a player on the final group decision. One of the most established power indices is the Shapley–Shubik index, see Shapley and Shubik (1954). It can be defined via
\[
\text{SSI}_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot [v(S) - v(S\setminus\{i\})]
\] (1)

for all players \(i \in N\), where \(s = |S|\). If \(v(S) - v(S\setminus\{i\}) = 1\), then we have \(v(S) = 1\) and \(v(S\setminus\{i\}) = 0\) in a simple game and voter \(i\) is called a swing voter.

We remark that the Shapley–Shubik index is a restriction of the Shapley value to simple games. Both, the Shapley value and the Shapley–Shubik index have compelling axiomatizations. Besides that, Shapley and Shubik (1954) motivate the Shapley–Shubik index by the following interpretation. Assume that the proposal can be put through is then called pivotal. Considering all \(n!\) orderings \(\pi \in S_n\) of the players with equal probability then gives a probability for being pivotal for a given player \(i \in N\) that equals its Shapley–Shubik index. So we can rewrite Eq. (1) as

\[
\text{SSI}_i(v) = \frac{1}{n!} \cdot \sum_{\pi \in S_n} \left( v(\{j \in N : \pi(j) \leq \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\}) \right).
\] (2)

Setting \(S^i_\pi := \{ j \in N : \pi(j) \leq \pi(i)\}\) we have \(S^i_\pi = S\) for exactly \((s-1)!(n-s)!\) permutations \(\pi \in S_n\) and an arbitrary set \(\{i\} \subseteq S \subseteq N\), so that Eq. (1) is just a simplification of Eq. (2).

Instead of assuming that all players vote ‘‘yes’’, one can also assume that all players vote ‘‘no’’. Mann and Shapley (1964) mention that the model also yields the same result if we assume that all players independently vote ‘‘yes’’ with a fixed probability \(p \in [0, 1]\). This was further generalized to probability measures \(p\) on \(\{0, 1\}^n\) where vote vectors with the same number of ‘‘yes’’ votes have the same probability, see Hu (2006). In other words, individual votes may be independent but must be exchangeable. That no further probability measures lead to the Shapley–Shubik index was finally shown in Kurz and Napel (2018). For the most symmetric case \(p = \frac{1}{2}\) we have

\[
\text{SSI}_i(v) = \frac{1}{n! \cdot 2^n} \cdot \sum_{(\pi,x) \in S_n \times \{0,1\}^n} M(v, (\pi,x), i),
\] (3)

where \(M(v, (\pi,x), i)\) is one if player \(i\) is pivotal for ordering \(\pi\) and vote vector \(x\) in \(v\), see Kurz and Napel (2018), and zero otherwise. To be more precise, consider a given ordering \(\pi\) of the players and a vector \(x \in \{0,1\}^n\) encoding a sequence of ‘‘yes’’ and ‘‘no’’ votes. Then, we call a player \(i\) pivotal, with respect to \(\pi\) and \(x\), if player \(i\) is the first player such that the votes of player \(i\) and her predecessors uniquely determine the outcome \(v(x)\) of the group decision. As an example we again consider the weighted game \(v = [4;3,2,1,1]\). For ordering \(\pi = (2,1,3,4)\) and vote vector \(x = (1,1,0,1)\) player 1 is the pivotal player that determines the outcome \(v(x) = 1\). If \(x \in \{0,1\}^4\) coincides with \(x\) for player 1 and her predecessors, then we have \(v(x) \geq v(1,1,0,0) = 1\), i.e., the outcome is determined to be 1 in any case. Player 1 is the first such player since for player 2 we may consider the continued
vote vector $\bar{x} = (0, 1, 0, 0)$ with $v(\bar{x}) = 0$. For ordering $\pi = (2, 1, 4, 3)$ and vote vector $x = (0, 1, 1, 0)$ player 4 is the pivotal player that determines the outcome $v(x) = 0$.

This line of reasoning can be used to motivate a definition of a Shapley–Shubik index for $(j, k)$ simple games as defined by Freixas (2005), c.f. Kurz (2014). Suppose that voters successively and independently each choose a level of approval in $J$ with equal probability. Such a vote scenario is modeled by a roll-call $(\pi, x)$ that consists in a permutation $\pi$ of the voters and a profile $x \in J^n$ such for all $i \in N$; the integer $\pi(i) \in \{1, 2, \ldots, n\}$ is the entry position of voter $i$ and $x_i$ is his approval level. Given an index $h \in \{1, \ldots, k - 1\}$, a voter $i$ is an $h$-pivotal voter if the vote of player $i$, according to the ordering $\pi$ and the approval levels of his predecessors, pushes the outcome to at least $h$ or pushes the outcome to at most $h - 1$.

**Example 1** Let $v$ be the $(3, 3)$ simple game $v$ for 2 players defined by $v(0, 0) = v(1, 0) = 0$, $v(1, 1) = v(0, 1) = 1$, and $v(2, 0) = v(0, 2) = v(2, 1) = v(1, 2) = v(2, 2) = 2$. As an example, consider the ordering $\pi = (2, 1)$, i.e., player 2 is first, and the vote vector $x = (2, 1)$. Before player 2 announces his vote $x_2 = 1$ all outcomes in $K = \{0, 1, 2\}$ are possible. After the announcement the outcome 0 is impossible, since $v(2, 1) \geq v(1, 1) \geq v(0, 1) = 1$, while the outcomes 1 and 2 are still possible. Thus, player 2 is the 1-pivotal voter. Finally, after the announcement of $x_1 = 2$, the outcome is determined to be $v(2, 1) = 2$, so that player 1 is the 2-pivotal voter.

Going in line with the above motivation and the definition from Freixas (2005), the Shapley–Shubik index for $(j, k)$ simple games is defined for all $v \in U_n$ and for all $i \in N$ by:

$$
\Phi_i(v) = \frac{1}{n! \cdot j^n \cdot (k - 1)} \times \sum_{h=1}^{k-1} \left| \{ (\pi, x) \in S_n \times J^n : \text{i is an h-pivot for} \pi \text{ and x in} v \} \right|.
$$

(4)

Since several different definitions of a Shapley–Shubik index for $(j, k)$ simple games have been introduced in the literature, we prefer to use the notation $\Phi_i(v)$ instead of the more suggestive notation $SSI_i(v)$. For the $(j, k)$ simple game $v$ from Example 1 we have

$$
\Phi(v) = \left( \Phi_1(v), \Phi_2(v) \right) = \left( \frac{5}{12}, \frac{7}{12} \right).
$$

Hereafter, some properties of $\Phi$ are explored. To achieve this, we introduce further definitions and axioms for power indices on $(j, k)$ simple games. First of all, we simplify Eq. (4) to a more handy formula that will allow us to associate a TU game $\hat{v}$, called the average game, to each $(j, k)$ simple game $v$ in Sect. 4 such that the Shapley value of $\hat{v}$ coincides with $\Phi(v)$.

**Lemma 1** For each $(j, k)$ simple game $v \in U_n$ and each player $i \in N$ we have
\[ \Phi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot [C(v, S) - C(v, S \setminus \{i\})], \tag{5} \]

where \( s = |S| \) and

\[ C(v, T) = \frac{1}{j^n(k-1)} \cdot \sum_{i \in J^n} \left( v((j-1)_T, x_{-T}) - v(0_T, x_{-T}) \right) \tag{6} \]

for all \( T \subseteq N \).

**Proof** For a given permutation \( \pi \in S_n \) and a player \( i \in N \), we set
\[
\pi_{<i} = \{ h \in N : \pi(h) < \pi(i) \}, \quad \pi_{\leq i} = \{ h \in N : \pi(h) \leq \pi(i) \}, \quad \pi_{>i} = \{ h \in N : \pi(h) > \pi(i) \}, \quad \text{and} \quad \pi_{\geq i} = \{ h \in N : \pi(h) \geq \pi(i) \}.
\]

With this, we can rewrite \( n! \cdot j^n \cdot (k-1) \) times the right hand side of Eq. (4) to

\[
\sum_{(\pi, i) \in S_n \times J^n} \left[ v(x_{\pi_{<i}}, (j-1)_{\pi_{>i}}) - v(x_{\pi_{<i}}, 0_{\pi_{>i}}) \right] - \left[ v(x_{\pi_{\geq i}}, (j-1)_{\pi_{<i}}) - v(x_{\pi_{\geq i}}, 0_{\pi_{<i}}) \right]. \tag{7}
\]

The interpretation is as follows. Since \( v \) is monotone, before the vote of player \( i \) exactly the values in \( \{ v(x_{\pi_{<i}}, 0_{\pi_{>i}}), \ldots, v(x_{\pi_{<i}}, (j-1)_{\pi_{>i}}) \} \) are still possible as final group decision. After the vote of player \( i \) this interval eventually shrinks to \( \{ v(x_{\pi_{\geq i}}, 0_{\pi_{<i}}), \ldots, v(x_{\pi_{\geq i}}, (j-1)_{\pi_{<i}}) \} \). The difference in (7) just computes the difference between the lengths of both intervals, i.e., the number of previously possible outputs that can be excluded for sure after the vote of player \( i \).

As in the situation where we simplified the Shapley–Shubik index of a simple game given by Eq. (2) to Eq. (1), we observe that it is sufficient to know the sets \( \pi_{>i} \) and \( \pi_{\geq i} \) for every permutation \( \pi \in S_n \). So we can condense all permutations that lead to the same set and can simplify the expression in (7) and obtain Eq. (5). \( \square \)

For the \((3, 3)\) simple game \( v \) from Example 1 we obtain \( C(v, \emptyset) = 0, C(v, \{1\}) = \frac{1}{2}, C(v, \{2\}) = \frac{2}{3} \), and \( C(v, \{1, 2\}) = 1 \), so that \( \Phi_1(v) = \frac{1}{2} \cdot \left( \frac{2}{3} - 0 \right) + \frac{1}{2} \cdot \left( 1 - \frac{2}{3} \right) = \frac{3}{12} \), and \( \Phi_2(v) = \frac{1}{2} \cdot \left( \frac{2}{3} - 0 \right) + \frac{1}{2} \cdot \left( 1 - \frac{1}{2} \right) = \frac{7}{12} \).

While we think that the roll-call motivation stated above for Eq. (4) is a valid justification on its own, we also want to pursue the more rigorous path to characterize power indices, i.e., we want to give an axiomatization. A set of properties that are satisfied by the Shapley–Shubik index for simple games and uniquely characterize the index is given, e.g., in Dubey (1975). In order to obtain a similar result for \((j, k)\) simple games, we consider a power index \( F \) as a map form \( v \) to \( \mathbb{R}^n \) for all \((j, k)\) simple games \( v \in U_n \).

**Definition 6** A power index \( F \) for \((j, k)\) simple games satisfies

- Positivity (P) if \( F(v) \neq 0 \) and \( F_i(v) \geq 0 \) for all \( i \in N \) and all \( v \in U_n \);
– Anonymity (A) if $F_{\pi(i)}(\pi v) = F_i(v)$ for all permutations $\pi$ of $N$, $i \in N$, and $v \in \mathcal{U}_n$, where $\pi v(x) = v(\pi(x))$ and $\pi(x) = (x_{\pi(i)})_{i \in N}$;
– Symmetry (S) if $F_i(v) = F_j(v)$ for all $v \in \mathcal{U}_n$ and all voters $i, j \in N$ that are equivalent in $v$;
– Efficiency (E) if $\sum_{i \in N} F_i(v) = 1$ for all $v \in \mathcal{U}_n$;
– the Null player property (NP) if $F_i(v) = 0$ for every null voter $i$ of an arbitrary game $v \in \mathcal{U}_n$;
– the transfer property (T) if for all $u, v \in \mathcal{U}_n$ and all $i \in N$ we have $F_i(u) + F_i(v) = F_i(u \vee v) + F_i(u \wedge v)$, where $(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$ for all $x \in J^n$, see Definition 4 and Proposition 2;
– Convexity (C) if $F(w) = \alpha F(u) + \beta F(v)$ for all $u, v \in \mathcal{U}_n$ and all $\alpha, \beta \in \mathbb{R}_{\geq 0}$ with $\alpha + \beta = 1$, where $w = \alpha u + \beta v \in \mathcal{U}_n$;
– Linearity (L) if $F(w) = \alpha F(u) + \beta F(v)$ for all $u, v \in \mathcal{U}_n$ and all $\alpha, \beta \in \mathbb{R}$, where $w = \alpha u + \beta v \in \mathcal{U}_n$.

Note that $\alpha \cdot u + \beta \cdot v$ does not need to be a $(j, k)$ simple game for $u, v \in \mathcal{U}_n$, where $\alpha \cdot u$ is defined via $(\alpha \cdot u)(x) = \alpha \cdot u(x)$ for all $x \in J^n$ and all $\alpha \in \mathbb{R}$. This is already true for $(j, k) = (2, 2)$, i.e. simple games, so that (T) was introduced in Dubey (1975). We remark that, obviously, (L) implies (C) and (L) implies (T). Also (S) is implied by (A) since it is a restriction. Some of the properties of Definition 6 have been proven to be valid for $\Phi$ in Freixas (2005). However, for the convenience of the reader we give an extended result and a full proof next:

**Proposition 4** The power index $\Phi$, defined in Eq. (4), satisfies the axioms (P), (A), (S), (E), (NP), (T), (C), and (L).

**Proof** We use the notation from the proof of Lemma 1 and let $v$ be an arbitrary $(j, k)$ simple game with $n$ players.

For each $x \in J^n$, $\pi \in \mathcal{S}_n$, and $i \in N$, we have $v(x_{\pi(i)}, (j - 1)_{\pi(j)}) \geq v(x_{\pi(i)}, (j - 1)_{\pi(j)})$ and $v(x_{\pi(i)}, 0_{\pi(j)}) \geq v(x_{\pi(i)}, 0_{\pi(j)})$, so that $\Phi_i(v) \geq 0$ due to Eq. (7). Since we will show that $\Phi$ is efficient, we especially have $\Phi_i(v) \neq 0$, so that $\Phi$ is positive.

For any permutation $\pi \in \mathcal{S}_n$ and any $0 \leq h \leq n$ let $\pi[h] := \{\pi(i) : 1 \leq i \leq h\}$, i.e., the first $h$ players in ordering $\pi$. Then, for any profile $x \in J^n$, we have

\[
\sum_{i=1}^{n} \left( v(x_{\pi(i)}, (j - 1)_{\pi(i)}) - v(x_{\pi(i)}, (j - 1)_{\pi(i)}) + v(x_{\pi(i)}, 0_{\pi(i)}) - v(x_{\pi(i)}, 0_{\pi(i)}) \right)
\]

\[
= \sum_{h=1}^{n} \left( v(x_{\pi[h-1]}, (j - 1)_{\pi[h-1]} - v(x_{\pi[h-1]}, (j - 1)_{\pi[h-1]}) \right) + \sum_{h=1}^{n} \left( v(x_{\pi[h]}, 0_{\pi[h]} - v(x_{\pi[h-1]}, 0_{\pi[h-1]}) \right)
\]

\[
= v((j - 1)) - v(0) + v(0) - v(0) = k - 1 - 0 = k - 1,
\]

so that Eq. (7) gives $\sum_{i=1}^{n} \Phi_i(v) = 1$, i.e., $\Phi$ is efficient.
The definition of \( \Phi \) is obviously anonymous, so that it is also symmetric. If player \( i \in N \) is a null player and \( x_{\pi_{S_i}}(j-1) \in S \) arbitrary, then \( v(x_{\pi_{S_i}}, 0, x_{\pi_{S_i}}) = v(x_{\pi_{S_i}}, 0, x_{\pi_{S_i}}) \) and \( v(x_{\pi_{S_i}}, (j-1)_{\pi_{S_i}}) = v(x_{\pi_{S_i}}, (j-1)_{\pi_{S_i}}) \), so that \( \Phi_i(v) = 0 \), i.e., \( \Phi \) satisfies the null player property. Since Eq. (7) is linear in the involved \((j, k)\) simple game, \( \Phi \) satisfies \((L)\) as well as \((C)\), which is only a relaxation. Since \( x + y = \max \{x, y\} + \min \{x, y\} \) for all \( x, y \in \mathbb{R} \), \( \Phi \) also satisfies the transfer axiom \((T)\). \( \square \)

Actually the proof of Proposition 4 is valid for a larger class of power indices for \((j, k)\) simple games. To this end we associate each vector \( a \in J^n \) with the function \( v_a \) defined by

\[
v_a(S) = \frac{1}{k-1} \cdot [v((j-1)_S, a_{-S}) - v(0_S, a_{-S})]
\]

for all \( S \subseteq N \). With this, we define the mapping \( \Phi^a \) on \( U_n \) by

\[
\Phi^a_i(v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v_a(S) - v_a(S\setminus\{i\})]
\]

for all \( i \in N \). We remark that it can be easily checked that \( v_a \) is a TU game, c.f. Sect. 2.

Similar as in the proof of Lemma 4, we conclude:

**Proposition 5** For every \( a \in J^n \) such that \( a_i = a_h \) for all \( i, h \in N \), the mapping \( \Phi^a \) satisfies the axioms \((P), (A), (S), (E), (NP), (T), (C), \) and \((L)\).

While the Shapley–Shubik index for simple games is the unique power index that is symmetric, efficient, satisfies both the null player property and the transfer property, see Dubey (1975), this result does not transfer to general \((j, k)\) simple games.

**Proposition 6** When \( j \geq 3 \), there exists some \( a \in J^n \) such that \( \Phi^a \neq \Phi \).

**Proof** For \( b = (1, j - 1, 0, \ldots, 0) \in J^n \) let \( u^b \) be the \((j, k)\) simple game with point-veto \( b \). From Equation (8) we conclude \( \Phi^b(u^b) = (0, 1, 0, \ldots, 0) \), where we set \( a = (j-2, j-2, \ldots, j-2, j-2) = j-2 \in J^n \). Using Eq. (5) we easily compute \( \Phi(u^b) = \left( \frac{1}{j}, \frac{j-1}{j}, 0, \ldots, 0 \right) \neq \Phi^a(u^b) \). \( \square \)

We remark that the condition \( j \geq 3 \) is necessary in Proposition 6, since for \((2, 2)\) simple games the roll-call interpretation of Mann and Shapley, see Mann and Shapley (1964), for the Shapley–Shubik index for simple games yields \( \Phi^0 = \Phi^1 = \Phi \).
4 The average game of a \((j, k)\) simple game

Equation \((5)\) in Lemma \(1\) has the important consequence that \(\Phi(v)\) equals the Shapley value of the TU game \(C(v, \cdot)\), where a TU game is a mapping \(v : 2^N \rightarrow \mathbb{R}\) with \(v(\emptyset) = 0\). To this end, we introduce an operator that associates each \((j, k)\) simple game \(v\) with a TU game \(\tilde{v}\) as follows.

**Definition 7** Let \(v \in \mathcal{U}_n\) be an arbitrary \((j, k)\) simple game. The *average game*, denoted by \(\tilde{v}\), associated to \(v\) is defined by

\[
\tilde{v}(S) = \frac{1}{j^n(k - 1)} \sum_{x \in J^n} \left[ v(j - 1, x) - v(0, x - s) \right]
\]

for all \(S \subseteq N\).

For the \((j, k)\) simple game \(v\) from Example \(1\) the average simple game is given by

\[
\tilde{v}(\emptyset) = 0, \tilde{v}({1}) = \frac{1}{2}, \tilde{v}({2}) = \frac{2}{3}, \text{ and } \tilde{v}(N) = 1.
\]

We remark that Definition \(7\) depends on our assumption that the gaps between the output levels of a \((j, k)\) simple game are equally sized; see our remark on the uniform numeric evaluation in the introduction.

With the notation of Definition \(7\), i.e., \(C(v, T) = \tilde{v}(T)\) our above remark on Eq. \((5)\) in Lemma \(1\) reads:

**Theorem 1** For every \((j, k)\) simple game \(v\) the vector \(\Phi(v)\) equals the Shapley value of \(\tilde{v}\).

Before giving some properties of the average game operator we note that two distinct \((j, k)\) simple games may have the same average game, as illustrated in the following example.

**Example 2** Consider the \((j, k)\) simple games \(u, v \in \mathcal{U}_n\) defined by

- \(u(x) = k - 1\) if \(x = j - 1\) and \(u(x) = 0\) otherwise;
- \(v(x) = k - 1\) if \(x \neq 0\) and \(v(0) = 0\)

for all \(x \in J^n\). Obviously, \(u \neq v\). A simple calculation, using Eq. \((9)\), gives

\[
\tilde{u}(S) = \tilde{v}(S) = \frac{1}{j^n-1} \text{ for all } S \in 2^N.
\]

In other words, the mapping from a \((j, k)\) simple game \(v\) to its average game \(\tilde{v}\) is not injective and it would be interesting to characterize inclusion maximal sets of \((j, k)\) simple games with the same average game. Observe that the mapping is not surjective since, e.g., \(\tilde{v}\) attains only rational values by construction.

The average game operator has some nice properties among which are the following:
Proposition 7 Given a \((j, k)\) simple game \(v \in \mathcal{U}_n\),

(a) \(\nu\) is a TU game on \(N\) that is \([0, 1]\)-valued and monotone;
(b) any null player in \(v\) is a null player in \(\nu\);
(c) any two equivalent players in \(v\) are equivalent in \(\nu\);
(d) if \(v = \sum_{i=1}^p a_i v_i\) is a convex combination for some \(v_1, \ldots, v_p \in \mathcal{U}_n\), then \(\nu = \sum_{i=1}^p a_i \nu_i\) is a TU game.

Proof Let \(v \in \mathcal{U}_n\). All mentioned properties of \(\nu\) are more or less transferred from the corresponding properties of \(v\) via Eq. (9). More precisely:

(a) Note that \(\nu(\emptyset) = \frac{1}{j!} (k-1) \sum_{x \in \mathcal{P}^n} [v(x) - v(x)] = 0\) and
\[
\nu(N) = \frac{1}{j!}(k-1) \sum_{x \in \mathcal{P}^n} [v(j - 1) - v(0)] = 1.
\]

Since \(v\) is monotone and \(0 \leq x \leq j - 1\) for all \(x \in J^n\), we have
\[
v(j - 1)_S, x_{-S} \leq v(j - 1)_T, x_{-T} \quad \text{and} \quad v(0, S, x_{-S}) \geq v(0, T, x_{-T})
\]
for all \(\emptyset \subseteq S \subseteq T \subseteq N\). Thus, we can conclude \(0 \leq \nu(S) \leq \nu(T) \leq 1\) from Eq. (9).

(b) Let \(i \in N\) be a null player in \(v\) and \(S \subseteq N \setminus \{i\}\) be a coalition, so that \(v(j - 1)_{S \cup \{i\}}, x_{-(S \cup \{i\})} = v(j - 1)_S, x_{-S}\) and \(v(0, S \cup \{i\}, x_{-(S \cup \{i\})}) = v(0, S, x_{-S})\). Thus, we have that \(\nu(S \cup \{i\}) = \nu(S)\), i.e., player \(i\) is a null player in \(\nu\).

(c) Let \(i, h \in N\) be two equivalent players in \(v, S \subseteq N \setminus \{i, h\}\), and \(\pi_{ih} \in S_n\) the transition that interchanges \(i\) and \(h\). Since \(v((j - 1)_{S \cup \{i\}}, x_{-(S \cup \{i\})}) = v(j - 1)_{S \cup \{h\}}, (\pi_{ih} x)_{-(S \cup \{h\})}\) and \(v(0, S \cup \{i\}, x_{-(S \cup \{i\})}) = v(0, S \cup \{h\}, (\pi_{ih} x)_{-(S \cup \{h\})})\), we have \(\nu(S \cup \{i\}) = \nu(S \cup \{h\})\), i.e., players \(i\) and \(h\) are equivalent in \(\nu\).

(d) Now suppose that \(v = \sum_{i=1}^p \alpha_i v_i\) is a convex combination for some games \(v_1, v_2, \ldots, v_p \in \mathcal{U}_n\). Since \(v((j - 1)_S, x_{-S}) = \sum_{i=1}^p \alpha_i v_i((j - 1)_S, x_{-S})\) and \(v(0, S, x_{-S}) = \sum_{i=1}^p \alpha_i v_i(0, S, x_{-S})\), applying Eq. (9) gives \(\nu(S) = \sum_{i=1}^p \alpha_i \nu(S)\) for all \(\emptyset \subseteq S \subseteq N\).

The operator that associates each \((j, k)\) simple game \(v\) with its average game \(\nu\) can be seen as a coalitional representation of \((j, k)\) simple games. Moreover, Proposition 7 suggests that this representation preserves some properties of the initial game. The average game of a \((j, k)\) simple game with a point-veto is provided by:

Proposition 8 Given \(a \in J^n \setminus \{0\}\), the average game \(\nu^a\) satisfies for every coalition \(S \neq N\).
Axiomatizations for the Shapley–Shubik power index for games...

Proof Let $a \in J^n \setminus \{0\}$ and $\emptyset \subsetneq S \subsetneq N$.

First suppose that $S \cap N^a = \emptyset$. Then, for all $x \in J^n$ we have $a \leq ((j-1)_S, x_S)$ iff $a \leq (0_S, x_S)$. Thus, $\tilde{w}^a((j-1)_S, x_S) = \tilde{w}^a(0_S, x_S)$. It then follows from (9) that $\tilde{w}^a(S) = 0$.

Now suppose that $S \cap N^a \neq \emptyset$. Then, for all $x \in J^n$ we have $a \nleq ((j-1)_S, x_S)$ iff $a \nleq (0_S, x_S)$. Thus, $\tilde{w}^a((0_S, x_S)) = 0$. Note that $a \leq ((j-1)_S, x_S)$ iff $a - S \leq x - S$. Hence,

$$
\tilde{w}^a(S) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} u^a((j-1)_S, x_S)
$$

$$
= \frac{1}{j^{n-1}(k-1)} \sum_{x_S \in J^n} u^a((j-1)_S, x_S)
$$

$$
= \frac{1}{j^{n-1}(k-1)} \sum_{x_S \in J^n} u^a((j-1)_S, x_S)
$$

$$
= \frac{1}{k-1} \cdot (k-1) \cdot \frac{|\{x_S \in J^n \setminus S, a - S \leq x - S\}|}{j^{n-1}} = \prod_{i \in N \setminus S} \left(\frac{j - a_i}{j}\right).
$$

In Proposition 10 we will show that the average game of each $((j, k)$ simple game can be written as the linear combination of the average games of $((j, k)$ simple games with a point-veto of the form $a \in \{0, j - 1\}^n$. Before we prove this, recall that the average game associated with each $((j, k)$ simple game is a TU game on $N$. The set of all TU games on $N$ is vector space and a commonly used basis consists in all unanimity games $(\gamma_S)_{S \in 2^N}$, where $\gamma_S(T) = 1$ if $S \subseteq T$ and $\gamma_S(T) = 0$ otherwise.\(^4\)

In Definition 3 we have introduced the notation $w^S = u^a$ for a coalition $S \in 2^N$, where $a \in J^n$ is specified by $a_i = j - 1$ if $i \in S$ and $a_i = 0$ otherwise.

Proposition 9 For every coalition $C \in 2^N$, there exists a collection of real numbers $(\gamma_S)_{S \in 2^C}$ such that

$$
\tilde{w}^C = \sum_{S \in 2^C} \gamma_S,
$$

where

\(^4\) The definition of unanimity games has already been given in the second paragraph of Sect. 2.
Proof Note that \( \tilde{w}^C \) is a TU game on \( N \). Therefore, we have
\[
\tilde{w}^C = \sum_{S \in 2^N} y_S \gamma_S.
\]
where the coefficients
\[
y_S = \sum_{T \subseteq S} (-1)^{|S| - |T|} \tilde{w}^C(T)
\]
are the well-known Harsanyi dividends, see Harsanyi (1963). This proves the result for \( C = N \). Now, suppose that \( C \neq N \). Consider \( S \subseteq N \) such that \( S \not\subseteq C \). Thus \( S \) contains some voter \( i \) such that \( i \not\in C \). By Proposition 1 and Proposition 7, voter \( i \) is a null player in \( w^C \). Thus, by rewriting \( y_S \) from (12), one gets
\[
y_S = \sum_{T \subseteq S} (-1)^{|S| - |T|} \tilde{w}^C(T) = \sum_{i \in T \subseteq S} (-1)^{|S| - |T|} (\tilde{w}^C(T) - \tilde{w}^C(T \setminus \{i\})) = 0.
\]
Finally we can rewrite Eq. (11) as
\[
\tilde{w}^C = \sum_{S \in 2^N} y_S \gamma_S = \sum_{S \in 2^C} y_S \gamma_S + \sum_{S \not\subseteq C} y_S \gamma_S = \sum_{S \in 2^C} y_S \gamma_S.
\]
Moreover for all \( S \subseteq C \), \( y_S \) can be clearly determined using Proposition 8 as follows:
\[
y_S = \sum_{t=1}^{s} (-1)^{s-t} \binom{s}{t} \left( \frac{1}{j} \right)^{c-t} = \frac{(j-1)^s - (-1)^s}{j^c}.
\]
This completes the proof. \( \square \)

Proposition 10 For every \((j, k)\) simple game \( u \in \mathcal{U}_n\), there exists a collection of real numbers \((x_S)_{S \in 2^n}\) such that
\[
\tilde{u} = \sum_{S \in 2^N} x_S \tilde{w}^S.
\]
Proof The result is straightforward when \( j = 2 \) since \( J \) reduces to \( J = \{0, 1\} \). In the rest of the proof, we assume that \( j \geq 3 \). Note that all TU games on \( N \) can be written as a linear combination of unanimity games \((\gamma_S)_{S \in 2^N}\). It is then sufficient to only prove that each TU game \( \gamma_C \) for \( C \in 2^N \) is a linear combination of the TU games \((\tilde{w}^S)_{S \in 2^C}\). The proof is by induction on \( 1 \leq k = |C| \leq n \). More precisely, we prove the assertion \( A(k) \) that for all \( C \in 2^N \) such that \( |C| \leq k \), there exists a collection \((z_S)_{S \in 2^C}\) such that
\[
\gamma_C = \sum_{S \in 2^C} z_S \tilde{w}^S.
\]
First assume that \( k = 1 \). Using Proposition 8, it can be easily checked that we have \( \gamma_{[i]} = w_{[i]} \) for all \( i \in N \). Therefore \( A(1) \) holds. Now, consider a coalition \( C \) such that \( |C| = k \in \{2, \ldots, n\} \) and assume that \( A(l) \) holds for all \( l \) such that \( 1 \leq l < k \). By Proposition 9, there exists some real numbers \((\alpha_S)_{S \in 2^C} \) and \((\beta_S)_{S \in 2^C \setminus \{C\}} \) such that

\[
\tilde{w}^C = \sum_{S \in 2^C} \alpha_S \gamma_S = \alpha_C \gamma_C + \sum_{S \in 2^C \setminus \{C\}} \alpha_S \gamma_S = \alpha_C \gamma_C + \sum_{S \in 2^C \setminus \{C\}} \beta_S \tilde{w}^S.
\]

where the last equality holds by the induction hypothesis. Moreover, since \( j - 1 \geq 2 \), from Eq. (10), we have,

\[
\alpha_C = \frac{(j - 1)^c - (-1)^c}{j^c} \neq 0.
\]

Therefore we get

\[
\gamma_C = \sum_{S \in 2^C} z_S \tilde{w}^S
\]

where for all \( S \in 2^C \), \( z_S = -\frac{1}{\alpha_C} \) if \( S = C \) and \( z_S = -\frac{\beta_S}{\alpha_C} \) otherwise. This gives \( A(k) \). In summary, for each coalition \( S \in 2^N \) the game \( \gamma_S \) is a linear combination of the games \( w^C \), where \( C \in 2^N \). Thus, the proof is completed since \( \tilde{u} \) is a linear combination of the games \( \gamma_S \), where \( S \in 2^N \).

Before we continue, note that by Eq. (14), for \( C \in 2^N \) each TU game \( \gamma_C \) is a linear combination of the TU games \( \tilde{w}^S \). Since \( (\gamma_S)_{S \in 2^N} \) is a basis of the vector space of all TU games on \( N \), it follows that \( (\tilde{w}^S)_{S \in 2^N} \) is also a basis of the vector space of all TU games on \( N \).

5 A characterization of the Shapley–Shubik index for \((j, k)\) simple games

As shown in Proposition 5 the axioms of Definition 6 are not sufficient to uniquely characterize the power index \( \Phi \) for the class of \((j, k)\) simple games. Therefore we introduce an additional axiom.

**Definition 8** A power index \( F \) for \((j, k)\) simple games is **averagely convex (AC)** if we always have

\[
\sum_{i=1}^{n'} \alpha_i F(u_i) = \sum_{i=1}^{n'} \beta_i F(v_i)
\]

whenever
where \( u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q \in \mathcal{U} \) and \((\alpha_i)_{1 \leq i \leq p}, (\beta_i)_{1 \leq i \leq q}\) are non-negative numbers that sum to 1 each.

One may motivate the axiom (AC) as follows. In a game, the a priori strength of a coalition, given the profile of the other individuals, is the difference between the outputs observed when all of her members respectively give each her maximum support and her minimum support. The average strength game associates each coalition with her expected strength when the profile of other individuals uniformly varies. Average convexity for power indices is the requirement that whenever two convex combinations of average games coincide, the corresponding convex combinations of the power distributions also coincide.

We remark that the axiom of average convexity is much stronger than the axiom of convexity. A minor technical point is that \( \sum_{t=1}^{p} \alpha_t u_t \alpha \) as well as \( \sum_{t=1}^{q} \beta_t v_t \) do not need to be \((j, k)\) simple games. However, the more important issue is that i.e., Eq. (16), is far less restrictive than

\[
\sum_{t=1}^{p} \alpha_t u_t = \sum_{t=1}^{q} \beta_t v_t,
\]

i.e., Eq. (16), is far less restrictive than

\[
\sum_{t=1}^{p} \alpha_t u_t = \sum_{t=1}^{q} \beta_t v_t,
\]

since two different \((j, k)\) simple games may have the same average game, see Example 2. Further evidence is given by the fact that the parametric power indices \( \Phi^a \), defined in Eq. (8), do not all satisfy (AC).

**Proposition 11** When \( j \geq 3 \), there exists some \( a \in J^n \) such that \( \Phi^a \) does not satisfy (AC).

**Proof** As in the proof of Proposition 6, consider the \((j, k)\) simple game with point-veto \( b = (1, j-1, 0, \ldots, 0) \in J^n \) and let \( a = (j-2, j-2, \ldots, j-2) \). It can be easily checked that, for all subsets \( T \subseteq N \) we have

\[
\tilde{u}^b(T) = \begin{cases} 
1 & \text{if } 1, 2 \in T \\
(j-1)/j & \text{if } 2 \in T \subseteq N \setminus \{1\} \\
1/j & \text{if } 1 \in T \subseteq N \setminus \{2\} \\
0 & \text{if } T \subseteq N \setminus \{1, 2\}
\end{cases}
\]

and that
\[ \tilde{u}^b = \frac{1}{j} \cdot \tilde{w}^{[1]} + \frac{j-1}{j} \cdot \tilde{w}^{[2]} \]  

(17)

holds. Since \( \Phi^a \) satisfies (NP), (E), (S) we can easily compute \( \Phi^a(\tilde{w}^{[1]}) = (1, 0, \ldots, 0) \) and \( \Phi^a(\tilde{w}^{[2]}) = (0, 1, 0, \ldots, 0) \). Therefore,

\[
\frac{1}{j} \cdot \Phi^a(\tilde{w}^{[1]}) + \frac{j-1}{j} \cdot \Phi^a(\tilde{w}^{[2]}) = \left( \frac{1}{j}, \frac{j-1}{j}, 0, \ldots, 0 \right).
\]

(18)

Using (8), one gets \( \Phi^a(u^b) = (0, 1, 0, \ldots, 0) \). It then follows from Eqs. (17) and (18) that \( \Phi^a \) does not satisfy (AC).

We remark that proving that a power index does not satisfy (AC) can always be done by suitable examples. For the other direction we, unfortunately, do not know an algorithmic method aside from verifying Eq. (15) directly.

As a preliminary step to our characterization result in Theorem 2 we state:

**Lemma 2** If a power index \( F \) for the class \( \mathcal{U}_n \) of \((j, k)\) simple games satisfies (E), (S), and (NP), then we have \( F(w^C) = \Phi(w^C) \) for all \( C \in 2^N \).

**Proof** Let \( F \) be a power index on \( \mathcal{U}_n \) that satisfies (E), (S), (NP) and let \( C \in 2^N \) be arbitrary.

According to Proposition 1, all players \( i, h \in C \) are equivalent in \( w^C \) and those outside of \( C \) are null players in the game \( w^C \). Since both \( F \) and \( \Phi \) satisfy (E), (S), and (NP), we have \( F_i(w^C) = \Phi_i(w^C) = \frac{1}{|C|} \) if \( i \in C \) and \( F_i(w^C) = \Phi_i(w^C) = 0 \) otherwise.

\( \square \)

**Theorem 2** A power index \( F \) for the class \( \mathcal{U}_n \) of \((j, k)\) simple games satisfies (E), (S), (NP), and (AC) if and only if \( F = \Phi \).

**Proof** Necessity: As shown in Proposition 4, \( \Phi \) satisfies (E), (S), and (NP). For (AC) the proof follows from Theorem 1 since the average game operator is linear by Proposition 7.

Sufficiency: Consider a power index \( F \) for \((j, k)\) simple games that satisfies (E), (S), (NP), and (AC). Next, consider an arbitrary \((j, k)\) simple game \( \mu \in \mathcal{U}_n \). By Proposition 10, there exists a collection of real numbers \((x_S)_{S \in 2^N}\) such that

\[
\tilde{u} = \sum_{S \subseteq 2^N} x_S \tilde{w}^S = \sum_{S \in E_1} x_S \tilde{w}^S + \sum_{S \in E_2} x_S \tilde{w}^S,
\]

(19)

where \( E_1 = \{ S \in 2^N : x_S > 0 \} \) and \( E_2 = \{ S \in 2^N : x_S < 0 \} \). Note that \( E_1 \neq \emptyset \) since \( \tilde{u}(N) = 1 \). As an abbreviation we set

\[
\tilde{\nu} = \sum_{S \in E_1} x_S \tilde{w}^S(N) = \sum_{S \in E_1} x_S > 0.
\]

(20)
It follows that
\[
\frac{1}{v} \tilde{u} + \sum_{S \in E_2} \frac{-x_S}{v} \tilde{w}^S = \sum_{S \in E_1} \frac{x_S}{v} \tilde{w}^S. \tag{21}
\]
Since (21) is an equality among two convex combinations, axiom (AC) yields
\[
\frac{1}{v} F(u) + \sum_{S \in E_2} \frac{-x_S}{v} F(w^S) = \sum_{S \in E_1} \frac{x_S}{v} F(w^S).
\]
Therefore by Lemma 2,
\[
\frac{1}{v} F(u) + \sum_{S \in E_2} \frac{-x_S}{v} \Phi(w^S) = \sum_{S \in E_1} \frac{x_S}{v} \Phi(w^S). \tag{22}
\]
Since \( \Phi \) also satisfies (AC), we obtain
\[
\frac{1}{v} F(u) + \sum_{S \in E_2} \frac{-x_S}{v} \Phi(w^S) = \frac{1}{v} \Phi(u) + \sum_{S \in E_2} \frac{-x_S}{v} \Phi(w^S), \tag{23}
\]
so that \( F(u) = \Phi(u). \)

\textbf{Proposition 12} \ For \( j \geq 3 \), the four axioms in Theorem 2 are independent.

\textbf{Proof} \ For each of the four axioms in Theorem 2, we provide a power index on \( U_n \) that meets the three other axioms but not the chosen one.

- The power index \( 2 \cdot \Phi \) satisfies (NP), (S), and (AC) but not (E).
- Denote by ED the \textit{equal division} power index which assigns \( \frac{1}{n} \) to each player for every \((j, k)\) simple game \( v \). Then, the power index \( \frac{1}{2} \cdot \Phi + \frac{1}{2} \cdot \text{ED} \) satisfies (E), (S) and (AC), but not (NP).
- In Proposition 5 we have constructed a parametric series of power indices that satisfy (E), (S), and (NP). For \( j \geq 3 \), at least one example does not satisfy (AC), see Proposition 11.
- Recall that \( \left( \tilde{w}^S \right)_{S \in 2^N} \) is a basis of the vector space of all TU games on \( N \). Thus given a \((j, k)\) simple game \( u \), there exists a unique collection of real numbers \( \left( x^u_S \right)_{S \in 2^N} \) such that
\[
\tilde{u} = \sum_{S \in 2^N} x^u_S \tilde{w}^S. \tag{24}
\]
Consider some \( i_0 \in N \) and set
\[
F(u) = \sum_{S \in 2^N} x^u_S \cdot F(w^S). \tag{25}
\]
For each $S \in 2^N \setminus \{N\}$ we set $F_i(w^S) = \Phi_i(w^S)$. For $S = N$ we set $F_i(w^N) = \frac{2}{n + 1}$ if $i = i_0$ and $F_i(w^N) = \frac{1}{n + 1}$ otherwise. We can easily check that $F$ satisfies (E), (NP), (AC), but not (S).

This proves that the four axioms in Theorem 2 are independent.

If we compare the axiomatization given in Theorem 2 with the one from Freixas (2019) the only difference is our axiom (AC) and the axiom of level change effect on unanimity games introduced in Freixas (2019). In Freixas (2019, Lemma 1.3) it is made very transparent when the axioms (E), (A), and (NP) are sufficient to determine the power index on a unanimity game. The same statement is true for the axioms (E), (S), and (NP). Using axiom (AC) we can start from an arbitrary $(j, k)$ simple game $v$, write its average game $\bar{v}$ as a linear combination of unanimity TU games, and find for each unanimity TU game $\gamma_S$ a $(j, k)$ simple game $u_S$ which satisfies the conditions of Freixas (2019, Lemma 1.3) and has $\gamma_S$ as its average game.

6 Axiomatization of the Shapley–Shubik index for interval simple games

As with $(j, k)$ simple games, a Shapley–Shubik like index for interval simple games can be constructed from the idea of the roll-call model; see Definition 9. As introduced in Sect. 2, an interval simple game is a mapping $v : [0, 1]^n \rightarrow [0, 1]$ with $v(0) = 0$, $v(1) = 1$, and $v(x) \leq v(y)$ for all $0 \leq x \leq y \leq 1$. In Theorem 4 we will show that this index is uniquely characterized by the axioms (E), (S), (NP) and (AC), see Definition 10 for the definition of the average game. The technical details are rather similar to our considerations for $(j, k)$ simple games, so that we will mainly skip the proofs.

Definition 9 (cf. Kurz 2014, Definition 6.2)

Let $v$ be an interval simple game with player set $N$ and $i \in N$ an arbitrary player. We set

$$
\Psi_i(v) = \frac{1}{n!} \sum_{S \subseteq N} \int_0^1 \cdots \int_0^1 \left[ \sum_{x_{S_i} < i} v(x_{S_i}, 1_{x_{S_i}}) - v(x_{S_i}, 0_{x_{S_i}}) \right] \sum_{x_{S_i} > i} v(x_{S_i}, 1_{x_{S_i}}) - v(x_{S_i}, 0_{x_{S_i}}) \, dx_1 \cdots dx_n.
$$

(26)

In this section, we give a similar axiomatization for $\Psi$ (for interval simple games) as we did for $(j, k)$ simple games and $\Phi$. By a power index for interval simple games we understand a mapping from the set of interval simple games for $n$ players to $\mathbb{R}^n$. Replacing both $J$ and $K$ by $I = [0, 1]$ in Definition 6, allows us to directly transfer the properties of power indices for $(j, k)$ simple games to the present situation. Also Proposition 4 is valid for interval simple games and $\Psi$. More precisely, $\Psi$ satisfies (P), (A), (S), (E), (NP), and (T), see (Kurz (2018), Lemma 6.1). The proof for (C) and (L) goes along the same lines as the
proof of Proposition 4. Also the generalization of the power index to a parametric class can be done just as the one for \((j, k)\) simple games in Eq. (8).

**Proposition 13** For every \(\alpha \in [0, 1]\) the mapping \(\Psi^\alpha\), where \(\alpha = (\alpha, \ldots, \alpha) \in [0, 1]^n\), defined by

\[
\Psi^\alpha_i(v) = \frac{1}{n!} \sum_{\pi \in S_n} \left( [v(a_{\pi}, 1_{\pi})] - [v(a_{\pi}, 0_{\pi})] \right),
\]

for all \(i \in N\), satisfies (P), (A), (S), (E), (NP), (T), (C), and (L).

Again, there exist vectors \(\alpha \in [0, 1]^n\) and interval simple games \(v\) with \(\Psi^\alpha(v) \neq \Psi(v)\). Also the simplified formula for \(\Phi\) for \((j, k)\) simple games in Lemma 1 can be mimicked for interval simple games and \(\Psi\), see Kurz et al. (2019).

**Proposition 14** For every interval simple game \(v\) with player set \(N\) and every player \(i \in N\) we have

\[
\Psi_i(v) = \sum_{i \in S \subseteq N} \frac{(s - 1)! (n - s)!}{n!} \cdot [C(v, S) - C(v, S \setminus \{i\})],
\]

where \(C(v, T) = \int_{[0,1]} v(1_T, x_{-T}) - v(0_T, x_{-T}) dx\) for all \(T \subseteq N\).

This triggers:

**Definition 10** Let \(v\) be an interval simple game on \(N\). The average game associated with \(v\) and denoted by \(\hat{v}\) is defined via

\[
\forall S \subseteq N, \hat{v}(S) = \int_{P^1} [v(1_S, x_{-S}) - v(0_S, x_{-S})] dx.
\]

**Theorem 3** For all every interval simple game \(v\) on \(N\) and for all \(i \in N\),

\[
\Psi_i(v) = \sum_{i \in S \subseteq N} \frac{(s - 1)! (n - s)!}{n!} [\hat{v}(S) - \hat{v}(S \setminus \{i\})]
\]

In other words, for a given interval simple game \(v\) the power distribution \(\Psi(v)\) is given by the Shapley value of its average game \(\hat{v}\).

As with \((j, k)\) simple games, two distinct interval simple games may have the same average game as illustrated in the following example.

**Example 3** Consider the interval simple games \(u\) and \(v\) defined on \(N\) respectively for all \(x \in [0, 1]^n\) by \(u(x) = 1\) if \(x = 1\), and \(u(x) = 0\) otherwise; \(v(x) = 1\) if \(x \neq 0\), and \(v(0) = 0\). It is clear that, \(u \neq v\). But, Eq. (28) and a simple calculation give \(\hat{u}(S) = \hat{v}(S) = 1\) if \(S = N\) and \(\hat{u}(S) = \hat{v}(S) = 0\) otherwise.
We can also transfer Proposition 7, i.e., the average game operator preserves the following nice properties of interval simple games.

**Proposition 15** Given an interval simple game \( v \in ISG_n \),

1. \( \hat{v} \) is a TU game on \( N \) that is \([0, 1]\)-valued and monotone;
2. any null voter in \( v \) is null player in \( \hat{v} \);
3. any two symmetric voters in \( v \) are symmetric players in \( \hat{v} \);
4. if \( v = \sum_{i=1}^{n} a_i v_i \) is a convex combination for some \( v_1, \ldots, v_p \in ISG_n \) then \( \hat{v} = \sum_{i=1}^{p} a_i \hat{v}_i \) is a TU game.

**Proof** Very similar to the one of Proposition 7.

From Theorem 3 we can directly conclude that \( \Psi \) also satisfies average convexity (AC), which is defined as in Definition 8.

For the remaining part of this section we introduce some further notation. For all \( x \in I^n \), let \( 1_x = \{ i \in N, x_i = 1 \} \); and given a coalition \( S \), let \( C^S \) be the interval simple game defined for all \( x \in I^n \) by \( C^S(x) = 1 \) if \( S \subseteq 1_x \) and \( C^S(x) = 0 \) otherwise.

**Proposition 16** For all \( T \in 2^N \), the average game \( \hat{C}^S \) (see Definition 10) equals \( \gamma_S \).

**Proof** Consider \( S, T \subseteq N \). If \( S \subseteq T \) then for all \( x \in [0, 1]^n \), \( S \subseteq T \subseteq \{ i \in N, (1_T \cdot x_{-T})_i = 1 \} \) and \( S \cap \{ i \in N, (0_T \cdot x_{-T})_i = 1 \} = \emptyset \). Then by definition of \( C^S \), \( C^S(1_T, x_{-T}) = 1 \) and \( C^S(0_T, x_{-T}) = 0 \). Therefore,

\[
\hat{C}^S(T) = \int_{[0, 1]^n} [C^S(1_T, x_{-T}) - C^S(0_T, x_{-T})] dx = 1 = \gamma_S(T).
\]

Now assume that \( S \not\subseteq T \). Let \( x \in [0, 1]^n \). Note that \( \{ i \in N, (1_T \cdot x_{-T})_i = 1 \} = T \) and \( \{ i \in N, (0_T \cdot x_{-T})_i = 1 \} = \emptyset \). Therefore, \( S \not\subseteq \{ i \in N, (1_T \cdot x_{-T})_i = 1 \} \) and \( S \not\subseteq \{ i \in N, (0_T \cdot x_{-T})_i = 1 \} \). By the definition of \( C^S \), it follows that \( C^S(1_T, x_{-T}) = C^S(0_T, x_{-T}) = 0 \). Hence

\[
\hat{C}^S(T) = \int_{[0, 1]^n} [C^S(1_T, x_{-T}) - C^S(0_T, x_{-T})] dx = 0 = \gamma_S(T).
\]

In both cases \( \hat{C}^S(T) = \gamma_S(T) \) for all \( T \in 2^N \); that is \( \hat{C}^S = \gamma_S \).

**Theorem 4** A power index \( \Psi' \) for interval simple games satisfies (E), (S), (NP) and (AC) if and only if \( \Psi' = \Psi \).

**Proof** **Necessity:** We have already remarked that \( \Psi \) satisfies (E), (S), (AC), and (NP).

**Sufficiency:** Let \( \Psi' \) be a power index for interval simple games on \( N \) that simultaneously satisfies (E), (S), (AC), and (NP). Consider an interval simple game \( u \). Note
that \( \hat{u} \) is a TU game by Proposition 15. Thus by Proposition 16, there exists a collection of real numbers \((\alpha_S)_{S \in 2^N}\) such that

\[
\hat{u} = \sum_{S \in 2^N} \alpha_S \cdot \widehat{\text{CS}} = \sum_{S \in E_1} \alpha_S \cdot \widehat{\text{CS}} + \sum_{S \in E_2} \alpha_S \cdot \widehat{\text{CS}}
\]

(30)

where \( E_1 = \{ S \in 2^N : \alpha_S > 0 \} \) and \( E_2 = \{ S \in 2^N : \alpha_S < 0 \} \). Moreover, \( E_1 \neq \emptyset \) since \( \hat{u}(N) = 1 \). We set

\[
\alpha = \sum_{S \in E_1} \alpha_S \cdot \widehat{\text{CS}}(N) = \sum_{S \in E_1} \alpha_S > 0.
\]

(31)

It follows that

\[
\frac{1}{\alpha} \hat{u} + \sum_{S \in E_2} \frac{-\alpha_S}{\alpha} \widehat{\text{CS}} = \sum_{S \in E_1} \frac{\alpha_S}{\alpha} \widehat{\text{CS}}.
\]

(32)

Since (32) is an equality among two convex combinations, then by (AC), we deduce that

\[
\frac{1}{\alpha} \psi'(u) + \sum_{S \in E_2} \frac{-\alpha_S}{\alpha} \psi'(\text{CS}) = \sum_{S \in E_1} \frac{\alpha_S}{\alpha} \psi'(\text{CS}).
\]

(33)

Note that given \( S \in 2^N \), all voters in \( S \) are equivalent in \( \text{CS} \) while all voters outside \( S \) are null players in \( \text{CS} \). Since \( \psi' \) and \( \psi \) satisfy (E), (S), and (NP), it follows that \( \psi'(\text{CS}) = \psi(\text{CS}) \). Thus,

\[
\frac{1}{\alpha} \psi'(u) + \sum_{S \in E_2} \frac{-\alpha_S}{\alpha} \psi(\text{CS}) = \sum_{S \in E_1} \frac{\alpha_S}{\alpha} \psi(\text{CS}).
\]

(34)

Since \( \psi \) also satisfies (AC), we get

\[
\frac{1}{\alpha} \psi'(u) + \sum_{S \in E_2} \frac{-\alpha_S}{\alpha} \psi(\text{CS}) = \frac{1}{\alpha} \psi(u) + \sum_{S \in E_2} \frac{-\alpha_S}{\alpha} \psi(\text{CS}).
\]

(35)

Hence \( \psi'(u) = \psi(u) \), which proves that \( \psi' = \psi \). \( \square \)

**Proposition 17** The four axioms in Theorem 4 are independent.

**Proof**

- The power index \( 2 \cdot \psi \) satisfies (NP), (S), (AC), but not (E).
- Denote by ED the equal division power index which assigns \( \frac{1}{n} \) to each player for every interval simple game. Then the power index \( \frac{1}{2} \cdot \psi + \frac{n}{2} \cdot \text{ED} \) satisfies (E), (S) and (AC), but not (NP).
- In Proposition 13 (c.f. (Kurz et al. (2019), Proposition 4)) we have stated a parametric classes of power indices for interval simple games that satisfy (E), (S), and (NP). In Kurz et al. (2019) it was also proved that there is at least one param-
eter \( a \in I^n \) for which the parameterized index \( \Psi^a \neq \Psi \). Thus, by Theorem 4 we can conclude that \( \Psi^a \) does not satisfies (AC). (Also Proposition 11 for \((j, k)\) simple games can be adjusted easily.)

- Note that by Proposition 16 the set \( \left( \widehat{C^S} \right)_{S \in 2^N} \) is a basis of the vector space of all TU games on \( N \). Thus, given an interval simple game \( u \), there exists a unique collection of real numbers \( \left( y_S^u \right)_{S \in 2^N} \) such that

\[
\widehat{u} = \sum_{S \in 2^N} y_S^u \widehat{C^S}. \tag{36}
\]

Consider some \( i_0 \in N \) and set

\[
F(u) = \sum_{S \in 2^N} y_S^u \cdot F(C^S). \tag{37}
\]

For each \( S \in 2^N \setminus \{ N \} \) we set \( F(C^S) = \emptyset(C^S) \). For \( S = N \) we set \( F(C^N) = \frac{2}{n+1} \) if \( i = i_0 \) and \( F(C^N) = \frac{1}{n+1} \) otherwise. We can easily check that \( F \) satisfies (E), (NP), (AC), but not (S).

This proves that the four axioms in Theorem 4 are independent. \( \square \)

In contrast with \((j, k)\) simple games, all convex combinations of interval simple games are also interval simple games. Thus, the axiom (AC) in Theorem 4 can be split into two easier axioms: the standard axiom of convexity (C) and the axiom of average equivalence (AE) stating that if \( F \) is a power index for interval simple games, then any two interval simple games that induce the same average game must have the same power distribution by \( F \).

7 Conclusion

Freixas (2005) extends the Shapley–Shubik index from simple games to some wider classes of interesting games with several levels of inputs and outputs. Related axiomatization results comprise Freixas (2019) for \((j, 2)\) simple games and for \( j \)-cooperative games (outputs are real numbers); and Kurz et al. (2019) for interval simple games. In this paper, we provide new axiomatizations of the Shapley–Shubik index for \((j, k)\) simple games as well as for interval simple games.

We introduce the notion of average game of a \((j, k)\) simple game and the axiom of average convexity. The average game allows us to give the Shapley–Shubik index of a \((j, k)\) simple game an explicit formula in terms of the characteristic function. More precisely, the Shapley–Shubik index of a \((j, k)\) simple game is simply the Shapley value of its average game. Theorem 2 differs from previous works essentially on the new axiom of average convexity. The axiom is the requirement that if two convex combinations of average games coincide, then do the corresponding convex combinations of the power distributions of the underlying games. Average convexity condition can be viewed as some form of linearity condition. It is an interesting open question whether this axiom can be decomposed into some weaker axioms.

In the case of interval simple games, it appears that average convexity is equivalent
to average equivalence and convexity. Whether this equivalence still holds for \((j, k)\) simple games remains an open issue since a convex combination of \((j, k)\) simple games may not be a \((j, k)\) simple game.

**Acknowledgements** Hilaire Touyem benefits from a financial support of the CETIC (Centre d’Excellence Africain en Technologies de l’Information et de la Communication) Project of the University of Yaounde I. We would like to thank the anonymous reviewers for their suggestions and comments.

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