Some remarks on nonnil-coherent rings and $\phi$-IF rings

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Abstract

Let $R$ be a commutative ring. If the nilpotent radical $\text{Nil}(R)$ of $R$ is a divided prime ideal, then $R$ is called a $\phi$-ring. In this paper, we first distinguish the classes of nonnil-coherent rings and $\phi$-coherent rings introduced by Bacem and Ali [10], and then characterize nonnil-coherent rings in terms of $\phi$-flat modules, nonnil-injective modules and nonnil-FP-injective modules. A $\phi$-ring $R$ is called a $\phi$-IF ring if any nonnil-injective module is $\phi$-flat. We obtain some module-theoretic characterizations of $\phi$-IF rings. Two examples are given to distinguish $\phi$-IF rings and IF $\phi$-rings.

Key Words: nonnil-coherent rings; $\phi$-IF rings; $\phi$-flat modules; nonnil-injective modules; nonnil-FP-injective modules.

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Throughout this paper, all rings are commutative with identity and all modules are unital. Recall from [6] that a ring $R$ is called an NP-ring provided that $\text{Nil}(R)$ is a prime ideal, and a ZN-ring provided that $\text{Z}(R) = \text{Nil}(R)$ where $\text{Z}(R)$ is the set of all zero-divisors of $R$. A prime ideal $P$ of $R$ is called divided prime if $P \subseteq (x)$, for every $x \in R - P$. A ring $R$ is called a $\phi$-ring, denoted by $R \in \mathcal{H}$, if $\text{Nil}(R)$ is a divided prime ideal of $R$. A $\phi$-ring is called a strongly $\phi$-ring if it is also a ZN-ring. Recall from [2] that for a $\phi$-ring $R$ with total quotient ring $\text{T}(R)$, the map $\phi : T(R) \to R_{\text{Nil}(R)}$ such that $\phi(\frac{a}{b}) = \frac{a}{b}$ is a ring homomorphism, and the image of $R$, denoted by $\phi(R)$, is a strongly $\phi$-ring. The classes of $\phi$-rings and strongly $\phi$-rings are good extensions of integral domains to commutative rings with zero-divisors.

In 2002, Badawi [8] generalized the concept of Noetherian rings to that of nonnil-Noetherian rings in which all nonnil ideals are finitely generated. They showed that a $\phi$-ring $R$ is nonnil-Noetherian if and only if $\phi(R)$ is nonnil-Noetherian, if and only if $R/\text{Nil}(R)$ is a Noetherian domain. Generalizations of Dedekind domains, Prüfer domains, Bezout domains, Pseudo-valuation domains and Krull domains to
the context of rings that are in the class $\mathcal{H}$ are also introduced and studied (see [2, 3, 7, 9]).

In 2016, Bacem and Ali [10] introduced two versions of coherent rings that are in the class $\mathcal{H}$. A $\phi$-ring $R$ is called nonnil-coherent provided that each finitely generated nonnil ideal of $R$ is finitely presented, and $R$ is called $\phi$-coherent provided that $\phi(R)$ is a nonnil-coherent ring. However, they neither showed these two versions of coherent rings coincide for all $\phi$-rings nor presented any example to distinguish them. In Section 1 of this article, we show that any nonnil-coherent ring is $\phi$-coherent (see Proposition 1.3), and give an example to show the converse does not hold (see Example 1.5). Bacem and Ali [10] obtained the Chase Theorem for nonnil-coherent rings using $\phi$-flat modules. We continue to characterize nonnil-coherent rings in terms of nonnil-injective modules and nonnil-FP-injective modules. Actually, we generalized Stenström’s, Dai-Ding’s and Chen’s results on coherent rings to these of nonnil-coherent rings (see Theorem 1.11 and Theorem 1.12).

Recall from [20] that a ring $R$ is called an IF ring (also is called semi-regular in [1, 23]) if any injective $R$-module is flat. The class of IF rings can be seen as a natural extension of that of QF rings to coherent rings and has many module-theoretic characterizations (see [15, 22, 23]). Recently, Wang and Kim [28] introduced and studied a new version of IF rings under $w$-operations. In Section 2 of this article, we generalize the class of IF rings to that of rings in $\mathcal{H}$. A $\phi$-ring $R$ is called a $\phi$-IF ring if any nonnil-injective $R$-module is $\phi$-flat. We obtain that a $\phi$-ring $R$ is a $\phi$-IF ring if and only if $R$ is nonnil-coherent and $R_m$ is a $\phi$-IF ring for any $m \in \text{Max}(R)$, if and only if the class of $\phi$-flat modules is equal to that of nonnil-FP-injective modules, if and only if any $\phi$-torsion $R$-module is $\phi$-copure flat, if and only if $R/\text{Nil}(R)$ is a field (see Theorem 2.5). We show that IF rings are $\phi$-IF rings for all strongly $\phi$-rings (see Proposition 2.6) and give two examples to distinguish IF $\phi$-rings and $\phi$-IF rings (see Example 2.7 and Example 2.8). Finally, we give an analogue of Matlis’ result to characterize $\phi$-Prüfer rings (see Corollary 2.9).

1. ON NONNIL-COHERENT RINGS

Recall that a ring $R$ is coherent if any finitely generated ideal is finitely presented. Bacem and Ali [10] generalized the notion of coherent rings to two classes of rings in $\mathcal{H}$: nonnil-coherent rings and $\phi$-coherent rings. Let $R$ be a $\phi$-ring. Then

(1) $R$ is called nonnil-coherent, provided that any finitely generated nonnil ideal of $R$ is finitely presented.

(2) $R$ is called $\phi$-coherent, provided that $\phi(R)$ is nonnil-coherent.
Obviously, if a \( \phi \)-ring \( R \) is coherent then \( R \) is nonnil-coherent, and a strongly \( \phi \)-ring is nonnil-coherent if and only if it is \( \phi \)-coherent.

**Proposition 1.1.** Let \( R \) be a \( \phi \)-ring, then the following assertions are equivalent:

1. \( R \) is nonnil-coherent;
2. \((0 :_R r)\) is a finitely generated ideal for any non-nilpotent element \( r \in R \), and the intersection of two finitely generated nonnil ideals of \( R \) is a finitely generated nonnil ideal of \( R \);
3. \((I :_R b)\) is a finitely generated ideal for any non-nilpotent element \( b \in R \) and any finitely generated ideal \( I \) of \( R \).

**Proof.** (1) \( \iff \) (2): See [10, Theorem 2.1].

(1) \( \implies \) (3): Let \( I \) be a finitely generated ideal of \( R \) and \( b \) a non-nilpotent element in \( R \). Consider the following pull-back diagram:

\[
0 \longrightarrow (I :_R b) \longrightarrow R \longrightarrow (Rb + I)/I \longrightarrow 0
\]

\[
0 \longrightarrow I \longrightarrow Rb + I \longrightarrow (Rb + I)/I \longrightarrow 0.
\]

Since \( R \) is nonnil-coherent, \( Rb + I \) is finitely presented. Since \( I \) is finitely generated, \((Rb + I)/I\) is finitely presented by [17, Theorem 2.1.2(2)]. Thus \((I :_R b)\) is finitely generated by [17, Theorem 2.1.2(3)].

(3) \( \implies \) (1): Let \( I \) be a finitely generated nonnil ideal of \( R \) generated by \( \{a_1, \ldots, a_n\} \) where each \( a_i \) is non-nilpotent. We will show \( I \) is finitely presented by induction on \( n \). The case \( n = 1 \) follows from the exact sequence \( 0 \to (0 :_R a_1) \to R \to Ra_1 \to 0 \).

For \( n \geq 2 \), let \( L = \langle a_1, \ldots, a_{n-1} \rangle \). Consider the exact sequence \( 0 \to (L :_R a_n) \to R \to (Ra_n + L)/L \to 0 \). Then \((Ra_n + L)/L = I/L\) is finitely presented by (3) and [17, Theorem 2.1.2(2)]. Consider the exact sequence \( 0 \to L \to I \to I/L \to 0 \). Since \( L \) is finitely presented by induction and \( I/L \) is finitely presented, \( I \) is also finitely presented by [17, Theorem 2.1.2(1)]. \( \square \)

**Proposition 1.2.** [10, Corollary 3.1] Let \( R \) be a \( \phi \)-ring, then \( R \) is \( \phi \)-coherent if and only if \( R/\text{Nil}(R) \) is a coherent domain.

**Proposition 1.3.** A \( \phi \)-ring \( R \) is nonnil-coherent if and only if \( R \) is \( \phi \)-coherent and \((0 :_R r)\) is a finitely generated ideal for any non-nilpotent element \( r \in R \).

**Proof.** Let \( R \) be a nonnil-coherent ring and \( r \) a non-nilpotent element in \( R \), then \((0 :_R r)\) is finitely generated by Proposition [11]. Let \( I/\text{Nil}(R) \) and \( J/\text{Nil}(R) \) be finitely generated non-zero ideals of \( R/\text{Nil}(R) \). By [2, Lemma 2.4], \( I \) and \( J \) are finitely generated nonnil ideals of \( R \). Thus by Proposition [11] \( I \cap J \) is a finitely generated
nonnil ideal. By [2, Lemma 2.4] again, \( I/\text{Nil}(R) \cap J/\text{Nil}(R) = (I \cap J)/\text{Nil}(R) \) is a finitely generated non-zero ideal of \( R/\text{Nil}(R) \). Then \( R/\text{Nil}(R) \) is a coherent domain by [27, Theorem 3.7.6]. Thus \( R \) is \( \phi \)-coherent by Proposition 1.2.

On the other hand, suppose \( R \) is a \( \phi \)-coherent ring. Then \( R/\text{Nil}(R) \) is a coherent domain. Let \( I \) and \( J \) be finitely generated nonnil ideals of \( R \). Then \( I/\text{Nil}(R) \) and \( J/\text{Nil}(R) \) are finitely generated non-zero ideals of \( R/\text{Nil}(R) \). Thus \( (I \cap J)/\text{Nil}(R) = I/\text{Nil}(R) \cap J/\text{Nil}(R) \) is a finitely generated non-zero ideal of \( R/\text{Nil}(R) \). Consequently, \( I \cap J \) is a finitely generated nonnil ideal of \( R \) by [2, Lemma 2.4]. Since \((0 :_R r)\) is a finitely generated ideal for any non-nilpotent element \( r \in R \), \( R \) is nonnil-coherent by Proposition 1.1.

□

By Proposition 1.3, any nonnil-coherent ring is \( \phi \)-coherent. In order to give a \( \phi \)-coherent ring that is not nonnil-coherent, we recall from [19] the idealization construction \( R(+)M \) where \( M \) is an \( R \)-module. Let \( R(+)M = R \oplus M \) as an \( R \)-module, and define

\[
\begin{align*}
(1) \quad (r, m) + (s, n) &= (r + s, m + n), \\
(2) \quad (r, m)(s, n) &= (rs, sm + rn).
\end{align*}
\]

Under this construction, \( R(+)M \) becomes a commutative ring with identity \((1, 0)\).

**Lemma 1.4.** Let \( R(+)M \) be the idealization construction defined as above, \( N \) an \( R \)-submodule of \( M \). Then \( 0(+)N \) is an ideal of \( R(+)M \). Moreover, \( 0(+)N \) is a finitely generated ideal of \( R(+)M \) if and only if \( N \) is finitely generated over \( R \).

**Proof.** It follows from [4, Theorem 3.1] that \( 0(+)N \) is an ideal of \( R(+)M \). It is easy to verify that the \( R(+)M \)-ideal \( 0(+)N \) is generated by \( \{(0, n_1), ..., (0, n_t)\} \) if and only if \( N \) is generated by \( \{n_1, ..., n_t\} \) over \( R \). □

The following example shows that the condition “\((0 :_R r)\) is a finitely generated ideal for any non-nilpotent element \( r \in R \)” in Proposition 1.3 cannot be removed.

**Example 1.5.** Let \( D \) be a coherent domain not a field, \( Q \) its quotient field and \( E = \bigoplus_{i=1}^{\infty} Q/D \). Let \( R = D(+)E \) be the idealization construction. Since \( E \) is divisible and \( \text{Nil}(R) = 0(+)E \) is a prime ideal, \( R \) is a \( \phi \)-ring by [4, Corollary 3.4]. Note that \( R/\text{Nil}(R) \cong D \), and thus \( R \) is \( \phi \)-coherent by Proposition 1.2. Let \( d \) be a non-zero non-unit element of \( D \). Then \((d, 0)\) is a non-nilpotent element in \( R \). Then one can easily check that \((0 :_R (d, 0)) = 0(+)\text{Ann}_E d \), where \( \text{Ann}_E d = \bigoplus_{i=1}^{\infty} (\frac{1}{d} + D) \). Since \( \text{Ann}_E d \) is an infinitely generated \( D \)-module, \((0 :_R (d, 0)) \) is an infinitely generated \( R \)-ideal by Lemma 1.4. Thus \( R \) is not nonnil-coherent by Proposition 1.3.
Remark 1.6. Following from Badawi [8], a $\phi$-ring $R$ is called a nonnil-Noetherian ring provided that each nonnil ideal of $R$ is finitely generated. They showed that a $\phi$-ring $R$ is nonnil-Noetherian if and only if $\phi(R)$ is nonnil-Noetherian (see [8, Theorem 2.4]). However, Example 1.5 shows the analogue of Badawi’s result is not true for nonnil-coherent. By Proposition 1.2, any nonnil-Noetherian ring is $\phi$-coherent. However, nonnil-Noetherian rings are not always nonnil-coherent. Indeed, let $D$ be a Noetherian domain not a field in Example 1.5, then $R$ is a nonnil-Noetherian ring but not nonnil-coherent.

Let $R$ be an NP-ring and $M$ an $R$-module. Set
\[ \phi\text{-tor}(M) = \{ x \in M \mid Ix = 0 \text{ for some nonnil ideal } I \text{ of } R \} . \]

An $R$-module $M$ is said to be $\phi$-torsion (resp., $\phi$-torsion free) provided that $\phi\text{-tor}(M) = M$ (resp., $\phi\text{-tor}(M) = 0$). The classes of $\phi$-torsion modules and $\phi$-torsion free modules constitute a hereditary torsion theory of finite type (see [26]). The rest of this section will give some module-theoretic characterization of nonnil-coherent rings. Recall that an $R$-module $M$ is called $\phi$-flat provided that $\text{Tor}_1^R(T, M) = 0$ for any $\phi$-torsion module $T$ (see [35]), and is called nonnil-injective provided that $\text{Ext}_R^1(T, M) = 0$ for any $\phi$-torsion module $T$ (see [36]). Certainly, the class of $\phi$-flat modules is closed under pure sub-modules, extensions and direct limits; the class of nonnil-injective modules is closed under direct products and extensions.

Definition 1.7. Let $R$ be an NP-ring. An $R$-module $M$ is called nonnil-$FP$-injective provided that $\text{Ext}_R^1(T, M) = 0$ for any finitely presented $\phi$-torsion module $T$.

Note that the class of nonnil-$FP$-injective modules is closed under direct sums, direct products, extensions and pure sub-modules.

Proposition 1.8. Let $R$ be an NP-ring, then the following assertions are equivalent:

1. $M$ is $\phi$-flat;
2. $\text{Hom}_R(M, E)$ is nonnil-injective for any injective module $E$;
3. $\text{Hom}_R(M, E)$ is nonnil-$FP$-injective for any injective module $E$;
4. if $E$ is an injective cogenerator, then $\text{Hom}_R(M, E)$ is nonnil-injective.
5. if $E$ is an injective cogenerator, then $\text{Hom}_R(M, E)$ is nonnil-$FP$-injective.

Proof. (1) $\Rightarrow$ (2): Let $T$ be a $\phi$-torsion $R$-module and $E$ an injective $R$-module. Since $M$ is $\phi$-flat, $\text{Ext}_R^1(T, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Tor}_R^1(T, M), E) = 0$. Thus $\text{Hom}_R(M, E)$ is nonnil-injective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5): Trivial.

(5) $\Rightarrow$ (1): Let $I$ be a finitely generated nonnil ideal of $R$ and $E$ an injective cogenerator. Since $\text{Hom}_R(M, E)$ is nonnil-$FP$-injective, $\text{Hom}_R(\text{Tor}_R^1(R/I, M), E) \cong \text{Hom}_R(M, E)$.
Ext$_R^1(R/I, \text{Hom}_R(M, E)) = 0$. Since $E$ is an injective cogenerator, Tor$_1^R(R/I, M) = 0$. Thus $M$ is $\phi$-flat by [35, Theorem 3.2].

Bacem and Ali [10] generalized the Chase Theorem for coherent rings to that for nonnil-coherent rings.

**Theorem 1.9.** [10, Theorem 2.4] Let $R$ be a $\phi$-ring. The following statements are equivalent:

1. $R$ is nonnil-coherent;
2. any direct product of $\phi$-flat $R$-modules is $\phi$-flat;
3. for any indexing set $I$, any $R$-module $R^I$ is $\phi$-flat.

**Lemma 1.10.** Let $R$ be a nonnil-coherent ring. Let $T$ be a finitely presented $\phi$-torsion module generated by $\{t_1, ..., t_k, t_{k+1}\}$ with $k \geq 1$ and $T_k$ the submodule of $T$ generated by $\{t_1, ..., t_k\}$. Then $T_k$ is finitely presented.

**Proof.** Note $T/T_k = (T_k + Rt_k)/T_k \cong Rt_k/(T_k \cap Rt_k) \cong R/I$ where $I = (0 :_R t_k + T_k \cap Rt_k)$ is an ideal of $R$. Since $T$ is finitely presented $\phi$-torsion and $T_k$ is finitely generated, then $I$ is a finitely generated nonnil ideal of $R$ by [17, Theorem 2.1.2]. Since $R$ is nonnil-coherent, then $I$ is finitely presented. Consider the following Pull-back diagram:

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & & \downarrow \\
K & & K \\
\downarrow & & \downarrow \\
0 & X & F & R/I & 0 \\
\downarrow & & & & \downarrow \\
0 & T_k & T & R/I & 0 \\
\downarrow & & & & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

where $F$ is finitely generated free. Then $K$ is finitely generated by [17, Theorem 2.1.2(3)]. Since $I$ is finitely presented, $X$ is finitely presented by [17, Theorem 2.1.2(3)] again. Thus $T_k$ is finitely presented by [17, Theorem 2.1.2(2)].

In 1970, Stenström [25] obtained that a ring $R$ is coherent if and only if any direct limit of FP-injective modules is FP-injective. In 2008, Pinzon [24] showed that if $R$ is coherent, the class of FP-injective modules is (pre)covering. Recently, Dai and Ding [13, 14] showed that the converse of Pinzon’s result also hold. The next result generalizes these results to nonnil-coherent rings.
Theorem 1.11. Let $R$ be a $\phi$-ring. The following statements are equivalent:

(1) $R$ is nonnil-coherent.
(2) The class of nonnil-FP-injective $R$-modules is closed under pure quotients.
(3) The class of nonnil-FP-injective $R$-modules is closed under direct limits.
(4) The class of nonnil-FP-injective $R$-modules is precovering.
(5) The class of nonnil-FP-injective $R$-modules is covering.

Proof. (1) $\Rightarrow$ (3): Let $\{M_i\}_{i \in \Gamma}$ be a direct system of nonnil-FP-injective $R$-modules. Let $T$ be a finitely presented $\phi$-torsion module generated by $n$ elements $\{t_1, ..., t_n\}$. We will prove $\text{Ext}^1_R(T, \lim_{\rightarrow} M_i) = 0$ by induction on $n$. Denote $M = \lim_{\rightarrow} M_i$. If $n = 1$, $T = R t_1$. Then there exists an exact sequence $0 \to (0 : R t_1) \to R \to R t_1 \to 0$. Since $T$ is finitely presented $\phi$-torsion, then $(0 : R t_1)$ is a finitely generated nonnil-ideal of $R$. Consider the following commutative diagrams with exact rows.

\[
\begin{array}{ccc}
\lim_{\rightarrow} \text{Hom}_R(R, M_i) & \longrightarrow & \lim_{\rightarrow} \text{Hom}_R((0 : R t_1), M_i) \\
\varphi_R & & \varphi_{(0 : R t_1)} \\
\text{Hom}_R(R, \lim_{\rightarrow} M_i) & \longrightarrow & \lim_{\rightarrow} \text{Ext}^1_R(R t_1, M_i) \\
\varphi_{R t_1} & & 0
\end{array}
\]

Since $R$ is nonnil-coherent, $(0 : R t_1)$ is finitely presented. Then $\varphi_{(0 : R t_1)}$ is an isomorphism, and thus $\varphi_{R t_1}$ is an isomorphism as $\varphi_R$ is an isomorphism. Consequently, $\text{Ext}^1_R(R t_1, \lim_{\rightarrow} M_i) = 0$. Let $T$ be a finitely presented $\phi$-torsion module generated by $\{t_1, ..., t_k, t_{k+1}\}$ and $T_k$ the $\phi$-torsion submodule of $T$ generated by $\{t_1, ..., t_k\}$. Then $T_k$ is finitely presented $\phi$-torsion by Lemma 1.10. Consider the exact sequence $0 \to T_k \to T \to R/I \to 0$ where $I = (0 : R t_k + T_k \cap R t_k)$. By the proof of Lemma 1.10, $R/I$ is finitely presented $\phi$-torsion. We have a long exact sequence

\[
\text{Ext}^1_R(R/I, \lim_{\rightarrow} M_i) \to \text{Ext}^1_R(T, \lim_{\rightarrow} M_i) \to \text{Ext}^1_R(T_k, \lim_{\rightarrow} M_i).
\]

By induction, $\text{Ext}^1_R(T_k, \lim_{\rightarrow} M_i) = \text{Ext}^1_R(R/I, \lim_{\rightarrow} M_i) = 0$, thus $\text{Ext}^1_R(T, \lim_{\rightarrow} M_i) = 0$. Consequently, $\lim_{\rightarrow} M_i$ is nonnil-FP-injective.

(3) $\Rightarrow$ (1): Let $I$ be a finitely generated nonnil ideal, $\{M_i\}_{i \in \Gamma}$ a direct system of $R$-modules. Let $\alpha : I \to \lim_{\rightarrow} M_i$ be a homomorphism. For any $i \in \Gamma$, $E(M_i)$ is the injective envelope of $M_i$. Then $E(M_i)$ is nonnil-FP-injective. By (3), there exists an
\( R \)-homomorphism \( \beta : R \to \lim E(M_i) \) such that the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \rightarrow & I & \rightarrow & R & \rightarrow & R/I & \rightarrow & 0 \\
\downarrow & & \alpha & & & & \downarrow & & \\
0 & \rightarrow & \lim M_i & \rightarrow & \lim E(M_i) & \rightarrow & \lim E(M_i)/M_i & \rightarrow & 0.
\end{array}
\]

Thus, by [18, Lemma 2.13], there exists \( j \in \Gamma \), such that \( \beta \) can factor through \( R \xrightarrow{\beta_j} E(M_j) \). Consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & I & \rightarrow & R & \rightarrow & R/I & \rightarrow & 0 \\
\downarrow & & \alpha_j & & & & \downarrow & & \\
0 & \rightarrow & M_j & \rightarrow & E(M_j) & \rightarrow & E(M_j)/M_j & \rightarrow & 0.
\end{array}
\]

Since the composition \( I \to R \to E(M_j) \to E(M_j)/M_j \) becomes to be 0 in the direct limit, we can assume \( I \to R \to E(M_j) \) can factor through some \( I \xrightarrow{\alpha_j} M_j \). Thus \( \alpha \) can factor through \( M_j \). Consequently, the natural homomorphism \( \lim \text{Hom}_R(I, M_i) \xrightarrow{\phi} \text{Hom}_R(I, \lim M_i) \) is an epimorphism. Now suppose \( \{M_i\}_{i \in \Gamma} \) is a direct system of finitely generated \( R \)-modules such that \( \lim M_i = I \). Then there exists \( f \in \text{Hom}_R(I, M_j) \) with \( j \in \Gamma \) such that the identity map \( \text{Id}_I = \phi(u_j(f)) \) where \( u_j \) is the natural homomorphism \( \text{Hom}_R(I, M_j) \to \lim \text{Hom}_R(I, M_i) \). Then \( I \) is a direct summand of \( M_j \), and thus \( I \) is finitely generated. It follows from [29, Section 24.9, Section 24.10] that \( I \) is finitely presented.

(2) \( \iff \) (3) \( \iff \) (4) \( \iff \) (5): Follows from [30, Lemma 3.4]. \( \square \)

In 1993, Chen and Ding in [12] showed that a ring \( R \) is coherent if and only if \( \text{Hom}_R(M, E) \) is flat for any absolutely pure \( R \)-module \( M \) and injective \( R \)-module \( E \) if and only if \( \text{Hom}_R(M, E) \) is flat for any injective \( R \)-modules \( M \) and \( E \). We also generalize this result to nonnil-coherent rings.

**Theorem 1.12.** Let \( R \) be a \( \phi \)-ring. The following statements are equivalent:

1. \( R \) is nonnil-coherent;
2. \( \text{Hom}_R(M, E) \) is \( \phi \)-flat for any nonnil-FP-injective module \( M \) and any injective module \( E \);
3. \( \text{Hom}_R(M, E) \) is \( \phi \)-flat for any nonnil-injective module \( M \) and any injective module \( E \);
4. \( \text{Hom}_R(\text{Hom}_R(M, E_1), E_2) \) is \( \phi \)-flat for any \( \phi \)-flat module \( M \) and any injective modules \( E_1, E_2 \);
5. if \( E_1 \) and \( E_2 \) are injective cogenerators, then \( \text{Hom}_R(\text{Hom}_R(M, E_1), E_2) \) is \( \phi \)-flat for any \( \phi \)-flat module \( M \).
Proof. (2) ⇒ (3) and (4) ⇒ (5): Trivial.
(3) ⇔ (4): Follows from Proposition 1.8.
(1) ⇒ (2): Let $I$ be a finitely generated nonnil ideal of $R$. Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Tor}_1^R((M, E), R/I) & \longrightarrow & (M, E) \otimes_R I & \longrightarrow & (M, E) \otimes_R R & \longrightarrow & (M, E) \otimes_R R/I & \longrightarrow & 0 \\
& & \downarrow \psi_{R/I} & & \downarrow \psi_I & & \downarrow \psi_R & & \downarrow \psi_{R/I} & \\
0 & \longrightarrow & \text{Ext}_1^R(R/I, M) & \longrightarrow & ((I, M), E) & \longrightarrow & ((R, M), E) & \longrightarrow & ((R/I, M), E) & \longrightarrow & 0.
\end{array}
\]

Since $I$ and $R$ are finitely presented, then $\psi_I$ and $\psi_R$ are isomorphisms by [5, Proposition 8.14(1)] and [21, Theorem 2]. Thus $\psi_{R/I}$ is an isomorphism by the Five Lemma.

Since $M$ is nonnil-FP-injective, $\text{Ext}_1^R(R/I, M) = 0$. Then $\text{Tor}_1^R(\text{Hom}_R(M, E), R/I) = 0$, and thus $\text{Hom}_R(M, E)$ is $\phi$-flat.

(5) ⇒ (1): Let $\{M_i\}_{i \in \Gamma}$ be a family of $\phi$-flat modules and $E_1$ and $E_2$ be injective cogenerators. Then $\bigoplus_{i \in \Gamma} M_i$ is $\phi$-flat. Then

\[
\text{Hom}_R(\text{Hom}_R(\bigoplus_{i \in \Gamma} M_i, E_1), E_2) \cong \text{Hom}_R(\prod_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2)
\]

is $\phi$-flat by (5). Note that $\bigoplus_{i \in \Gamma} \text{Hom}_R(M_i, E_1)$ is the pure submodule of $\prod_{i \in \Gamma} \text{Hom}_R(M_i, E_1)$ by [11, Lemma 1(1)]. Thus the natural epimorphism

\[
\text{Hom}_R(\prod_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2) \twoheadrightarrow \text{Hom}_R(\bigoplus_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2)
\]

splits by [18, Lemma 2.19]. It follows that

\[
\prod_{i \in \Gamma} \text{Hom}_R(\text{Hom}_R(M_i, E_1), E_2) \cong \text{Hom}_R(\bigoplus_{i \in \Gamma} \text{Hom}_R(M_i, E_1), E_2)
\]

is $\phi$-flat. By [18, Corollary 2.21], $\prod_{i \in \Gamma} M_i$ is the pure submodule of $\prod_{i \in \Gamma} \text{Hom}_R(\text{Hom}_R(M_i, E_1), E_2)$. Thus $\prod_{i \in \Gamma} M_i$ is $\phi$-flat. By Theorem 1.9 $R$ is nonnil-coherent. \qed

2. ON $\phi$-IF RINGS

Recall from [20] that a ring $R$ is called an IF ring if any injective $R$-module is flat. We generalize the concept of IF rings to that of rings in $H$ using nonnil-injective modules and $\phi$-flat modules.

**Definition 2.1.** Let $R$ be a $\phi$-ring. Then $R$ is said to be a $\phi$-IF ring provided that any nonnil-injective $R$-module is $\phi$-flat.

**Lemma 2.2.** Let $R$ be a $\phi$-IF ring, then $R$ is nonnil-coherent.
Proof. Let $M$ be a nonnil-injective $R$-module and $E$ an injective $R$-module. Then $M$ is $\phi$-flat as $R$ is a $\phi$-IF ring. Thus $\text{Hom}_R(M, E)$ is nonnil-injective by Proposition 1.8. Consequently, $R$ is nonnil-coherent by Theorem 1.12. □

Proposition 2.3. Let $R$ be a nonnil-coherent ring, $M$ an $R$-module. Suppose $T$ is a finitely presented $\phi$-torsion module and $E$ is an injective $R$-module. Then

$$\text{Tor}^1_R(T, \text{Hom}_R(M, E)) \cong \text{Hom}_R(\text{Ext}^1_R(T, M), E).$$

Proof. Suppose $T$ is generated by $n$ elements, then there is a exact sequence $0 \to K \to P \to T \to 0$ with $P = R^n$ and $K$ finitely generated. We will show $K$ is finitely presented by induction on $n$. If $n = 1$, then $K$ is a finitely generated nonnil ideal of $R$. Thus $K$ is finitely presented as $R$ is nonnil-coherent. Suppose $n = k + 1$, then there is commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & K \cap R^k & \to & R^k & \to & R^k/K \cap R^k & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & K & \to & R^{k+1} & \to & R^{k+1}/K & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I & \to & R & \to & R/I & \to & 0 \\
\end{array}
\]

where $I = K/K \cap R^k$ is an ideal of $R$. Since $T = R^{k+1}/K$ is finitely presented $\phi$-torsion, then $R^k/K \cap R^k$ is finitely presented $\phi$-torsion by Lemma 1.10. Thus $R/I$ is finitely presented $\phi$-torsion by [17, Theorem 2.1.2(2)]. Since $R^k/K \cap R^k$ is generated by $k$ elements, $K \cap R^k$ and $I$ are finitely presented by induction. Thus $K$ is finitely presented by [17, Theorem 2.1.2(1)].

Let $F$ be a $\phi$-flat $R$-module and $E$ a injective $R$-module. Then there is a commutative diagram with exact rows($(\ldots, \ldots)$ is instead of $\text{Hom}_R(\ldots, \ldots)$):

\[
\begin{array}{ccccccccc}
0 & \to & \text{Tor}^1_R(T, (F, E)) & \to & (F, E) \otimes_R K & \to & (F, E) \otimes_R P & \to & (F, E) \otimes_R T & \to & 0 \\
\downarrow & & \psi_T & & \psi_K \cong & & \psi_P \cong & & \psi_T \cong & & \downarrow \\
0 & \to & ((\text{Ext}^1_R(T, F), E)) & \to & ((K, F), E) & \to & ((P, M), E) & \to & ((T, M), E) & \to & 0.
\end{array}
\]

Since $K, F$ and $T$ are finitely presented, $\psi_K, \psi_P$ and $\psi_T$ are isomorphisms by [27, Theorem 2.6.13(2)]. Thus $\psi_T$ is isomorphism by the Five Lemma. □

Let $M$ be an $R$-module and $N$ a submodule of $M$. Then $N \hookrightarrow M$ is said to be a $\phi$-embedding map provided that $M/N$ is a $\phi$-torsion module. Let $M$ be an $R$-module, then there is a nonnil-injective envelope, denoted by $E_{\phi}(M)$, of $M$ (see [36, Theorem 2.7]). Note that $M \hookrightarrow E_{\phi}(M)$ is a $\phi$-embedding map (see [36, Theorem 2.14]).
Recall from [22] that an $R$-module $M$ is said to be copure flat if $\text{Tor}_1^R(E, M) = 0$ for any injective module $E$. It was proved that a ring $R$ is an IF ring if and only if any $R$-module is copure flat (see [22, Proposition 2.14]). The following concepts give a “strong” version of copure flat modules.

**Definition 2.4.** Let $R$ be an NP-ring. An $R$-module $M$ is called $\phi$-copure flat provided that $\text{Tor}_1^R(E, M) = 0$ for any nonnil-injective module $E$.

**Theorem 2.5.** Let $R$ be a $\phi$-ring. The following statements are equivalent:

1. $R$ is a $\phi$-IF ring;
2. $R$ is a nonnil-coherent ring and $R_p$ is a $\phi$-IF ring for any $p \in \text{Spec}(R)$;
3. $R$ is a nonnil-coherent ring and $R_m$ is a $\phi$-IF ring for any $m \in \text{Max}(R)$;
4. any $R$-module can be $\phi$-embedded into a $\phi$-flat module.
5. any nonnil-FP-injective module is $\phi$-flat;
6. $R$ is nonnil-coherent and any $\phi$-flat module is nonnil-FP-injective;
7. any $R$-module $M$ is $\phi$-flat if and only if $M$ is nonnil-FP-injective;
8. any $\phi$-torsion $R$-module is $\phi$-copure flat.
9. $R/\text{Nil}(R)$ is a field.

**Proof.** (1) $\Rightarrow$ (4): Let $M$ be an $R$-module, $M \hookrightarrow E_\phi(M)$ the $\phi$-embedding of $M$ into $E_\phi(M)$. Since $E_\phi(M)$ is a $\phi$-flat module by (1), then (4) holds naturally.

(4) $\Rightarrow$ (1): Let $M$ be a nonnil-injective and $M \subseteq F$ be the $\phi$-embedding of $M$ into a $\phi$-flat module $F$. Since $F/M$ is $\phi$-torsion, $\text{Ext}_1^R(F/M, M) = 0$. Then $M$ is a direct summand of the $\phi$-flat module $F$, and thus $M$ is $\phi$-flat.

(1) + (4) $\Rightarrow$ (2): By Lemma 2.2, $R$ is a nonnil-coherent ring. Let $p$ be a prime ideal of $R$, $A$ an $R_p$-module and $F$ a $\phi$-flat $R$-module containing $A$ such that $F/A$ is $\phi$-torsion. Then $A \cong A_p \subseteq F_p$. Note that $F_p/A_p$ is a $\phi$-torsion $R_p$-module (see [31 Proposition 2.12]) and $F_p$ is a $\phi$-flat $R_p$-module (see [35 Theorem 3.5]). Thus $R_p$ is a $\phi$-IF ring by (4).

(2) $\Rightarrow$ (3): Trivial.

(1) $\Rightarrow$ (6): Let $R$ be a $\phi$-IF ring, then $R$ is nonnil-coherent by Lemma 2.2. Suppose $T$ is a finitely presented $\phi$-torsion module. Let $F$ be a $\phi$-flat $R$-module and $E$ an injective $R$-module. By Proposition 2.3

$$\text{Tor}_1^R(T, \text{Hom}_R(F, E)) \cong \text{Hom}_R(\text{Ext}_1^R(T, F), E).$$

By Proposition 1.8, $\text{Hom}_R(F, E)$ is nonnil-injective. Since $R$ is a $\phi$-IF ring, $\text{Hom}_R(F, E)$ is $\phi$-flat. Since $T$ is finitely presented $\phi$-torsion, then $\text{Tor}_1^R(T, \text{Hom}_R(F, E)) = 0$, and thus $\text{Hom}_R(\text{Ext}_1^R(T, F), E) = 0$. It follows that $\text{Ext}_1^R(T, F) = 0$. Consequently, $F$ is nonnil-FP-injective.
(6) ⇒ (7): Let $M$ be a nonnil-FP-injective $R$-module and $E$ an injective cogenerator over $R$. Let $T$ be a finitely presented $\phi$-torsion module, then

$$\text{Hom}(\text{Tor}_1^R(T, M), E) \cong \text{Ext}_R^1(T, \text{Hom}_R(M, E)).$$

Since $R$ is nonnil-coherent, $\text{Hom}_R(M, E)$ is $\phi$-flat by Theorem 1.12. Then $\text{Hom}_R(M, E)$ is nonnil-FP-injective by (6). It follows that $\text{Ext}_R^1(T, \text{Hom}_R(M, E)) = 0$, and thus $\text{Hom}(\text{Tor}_1^R(T, M), E) = 0$. Consequently, $\text{Tor}_1^R(T, M) = 0$. So $M$ is $\phi$-flat.

(7) ⇒ (1): Note that nonnil-injective modules are nonnil-FP-injective. Thus any nonnil-injective module is $\phi$-flat by (7).

(1) ⇒ (8): Let $M$ be a nonnil-injective $R$-module and $T$ a $\phi$-torsion module. Then $M$ is $\phi$-flat, and thus $\text{Tor}_1^R(T, M) = 0$. Consequently, $T$ is $\phi$-copure flat.

(8) ⇒ (1): Let $M$ be a nonnil-injective $R$-module and $T$ a $\phi$-torsion module. Since $T$ is $\phi$-copure flat, then $\text{Tor}_1^R(T, M) = 0$. Thus $M$ is $\phi$-flat.

(3) + (1) ⇒ (6): Let $M$ be a $\phi$-flat module, then $M_m$ is a $\phi$-flat $R_m$-module for any maximal ideal $m$ of $R$. Thus $M_m$ is a nonnil-FP-injective $R_m$-module by (1) ⇒ (6) for the $\phi$-IF ring $R_m$. Let $T$ be a finitely presented $\phi$-torsion module. Then, by [27, Theorem 2.6.16], $\text{Ext}_R(T, M)_m \cong \text{Ext}_{R_m}^1(T_m, M_m) = 0$ as $T_m$ is a finitely presented $\phi$-torsion $R_m$-module by [31, Proposition 2.12]. Thus $\text{Ext}_R^1(T, M) = 0$. So $M$ is nonnil-FP-injective.

(7) ⇒ (5) ⇒ (1): Trivial.

(9) ⇒ (1): If $R/\text{Nil}(R)$ is a field, then any $R$-module is $\phi$-flat and nonnil-injective by [31, Theorem 1.7]. Thus $R$ is a $\phi$-IF ring.

(1) ⇒ (9): Suppose $R$ is a $\phi$-IF ring. Let $E$ be an injective $R/\text{Nil}(R)$-module, then $E$ is a nonnil-injective $R$-module by [31, Proposition 1.4]. Thus $E$ is $\phi$-flat over $R$. It follows from [32, Proposition 1.7] that $E$ is flat over $R/\text{Nil}(R)$. Consequently, $R/\text{Nil}(R)$ is an IF domain, and thus is a field by [20, Proposition 3.1].

Proposition 2.6. Let $R$ be a strongly $\phi$-ring. If $R$ is an IF ring, then $R$ is a $\phi$-IF ring.

Proof. Since $R$ is a strongly $\phi$-ring, any non-nilpotent element is regular. Because $R$ is an IF ring, any regular element is invertible by [23, Proposition 2.1(1)]. Then the Krull dimension of $R$ is 0, and thus $R/\text{Nil}(R)$ is a field. Consequently, $R$ is a $\phi$-IF ring by Theorem 2.5.

Note that every IF ring is not $\phi$-IF. For example, let $R$ be a von Neumann regular ring not a field. Then $R$ is an IF ring. Since $\text{Nil}(R) = 0$ is not a prime ideal, then $R$ is not a $\phi$-ring, and thus not a $\phi$-IF ring. The following example shows that IF rings are also not necessary $\phi$-IF rings for $\phi$-rings.
Example 2.7. Let $D$ be a Prufer domain not a field, and $Q$ its quotient field. Let $R = D(+)Q/D$ be the idealization construction. Then $\text{Nil}(R) = 0(+)Q/D$ is a prime ideal of $R$. Thus $R$ is a $\phi$-ring by [4, Corollary 3.4]. By [1, Example 2.12(1)] $R$ is an IF ring. However $R/\text{Nil}(R) \cong D$ is not a field. Thus $R$ is not a $\phi$-IF ring by Theorem 2.5.

It is also showed that $\phi$-IF rings are not necessary IF rings.

Example 2.8. Let $K$ be a field and $V = \bigoplus_{i=1}^{\infty} K$ an infinite dimensional vector space over $K$. Let $R = K(+)V$ be the idealization construction. Obviously, $R$ is a $\phi$-ring. Note that $\text{Nil}(R) = 0(+)V$. Since $K \cong R/\text{Nil}(R)$ is a field, $R$ is a $\phi$-IF ring by Theorem 2.5. Let $v$ be a vector in $V$ with each component equal to 1. Then \((0 :_{R} (0,v)) = 0(+)V\). Since $V$ is an infinite dimensional $K$-vector space, $0(+)V$ is not a finitely generated $R$-ideal by Lemma 1.4. Then $R$ is not coherent by [17, Theorem 2.3.2]. Thus $R$ is not an IF ring by [23, Proposition 3.3].

Recall from [2] that a $\phi$-ring $R$ is said to be a $\phi$-Pr"ufer ring if any finitely generated nonnil ideal of $R$ is $\phi$-invertible. They showed that a $\phi$-ring $R$ is a $\phi$-Pr"ufer ring if and only if $\phi(R)$ is a Prufer ring, if and only if $R/\text{Nil}(R)$ is a Prufer domain(see [2, Theorem 2.2, Theorem 2.6]). Matlis [23] obtained that an integral domain $R$ is a Prufer domain if and only if $R/I$ is an IF ring for any non-zero finitely generated $I$ of $R$. Now we give an analogue of Matlis’ result for $\phi$-rings.

Corollary 2.9. Let $R$ be a $\phi$-ring. Then $R$ is a $\phi$-Pr"ufer ring if and only if $R/I$ is an IF ring for any finitely generated nonnil ideal $I$ of $R$.

Proof. Let $R$ be a $\phi$-Pr"ufer ring and $I$ a finitely generated nonnil ideal of $R$. Then $R/\text{Nil}(R)$ is a Prufer domain. Since $I/\text{Nil}(R)$ is a finitely generated non-zero ideal of $R/\text{Nil}(R)$ by [2, Lemma 2.4], we have $R/I = \frac{R/\text{Nil}(R)}{I/\text{Nil}(R)}$ is an IF ring.

Let $I/\text{Nil}(R)$ be a finitely generated nonzero ideal of $R/\text{Nil}(R)$. Then $I$ is a finitely generated nonnil ideal of $R$ by [2, Lemma 2.4] again. Thus $\frac{R/\text{Nil}(R)}{I/\text{Nil}(R)} = R/I$ is an IF ring. Consequently, $R/\text{Nil}(R)$ is a Prufer domain. It follows that $R$ is a $\phi$-Pr"ufer ring.

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