De Gustibus Disputandum

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Abstract

We propose a simple method to predict individuals’ expectations about products using a knowledge network. As a complementary result, we show that the method is able, under certain conditions, to extract hidden information at neural level from a customers’ choices database.

1 Introduction

Personal tastes are universally considered very difficult to be analyzed (there’s an old latin proverb stating “De gustibus non disputandum est”, i.e. “There’s no accounting for taste”), nevertheless, there is evidence of certain regularities in personal preferences, allowing people to successfully choose Christmas presents for friends and relatives.

Indeed, any modeling of market behavior assumes a rational behavior of agents, that choose among the possible options on quantitative basis. In a famous paper [1], Stigler and Becker argued that all people have fixed tastes except for small variations, and that the different patterns in taste investments (like buying new music disks) are computable from the expectations in revenues (i.e. the forecasted enjoy in future music exposure). Many people have argued that this purely economic point of view is ignoring the “enormous role of historical and cultural forces, education, and values, as the initial shapers of our preferences” [2].

Anyhow, in order to test any economic/psychologic/moral point of view about tastes, we need a quantitative model of tastes formation and, more important yet, of tastes anticipation based on past experience.

The starting point of our analysis is that the opinion of an agent on a given product is formed by the match between agent’s set of preferences/tastes and product’s qualities.

While many commercial studies are based on surveys about customer’s preferences, we assume that both preferences and qualities are hidden degrees of freedom, and that only the expressed opinion is observable. One of the goal of our study is to develop techniques able to extract information about the hidden parts from the correlations among agent’s opinions on products.

Let us suppose that there exists a database of agents’ opinions on a given set of products. This database can be seen as a sparse matrix, with holes corresponding to missing opinions (say, agents that have never been exposed to a given product).
In geometrical words, one represents agent’s preferences as a vector in an hypothetical taste space, whose dimension and base vectors are unknown. A product is represented by a similar vector (in dual space). Agent’s opinion on a given product is given by an operation analogous to the scalar product between preferences and properties. Therefore, products act like a basis, and opinions as agent’s coordinates on such a basis. However, differently from usual geometrical problems, we do not know what the basis is, if it is complete, etc.

As we shall recall in Section 3, S. Maslov and Y.C. Zhang have shown that it is possible, if we know the basis of agent’s preferences, to reconstruct the vectors of the individual tastes from the knowledge of a sparsely connected network of the overlaps (scalar products) among preferences. We want to extend this result to the more usual case in which basis information is not at our disposal, as discussed in Section 6.

One of the outcome of our analysis is the possibility of opinion anticipation, i.e. the possibility of exploiting the correlations in the database to forecast the missing opinions. Alternatively, we can obtain information about the overlaps of tastes between two individuals from the knowledge of their expressed opinions.

What we think is our main result, is the possibility of extracting information about the hidden degrees of freedom, and in particular the dimensionality of hidden space, from the opinion database. In this way customer’s commercial interests can be used as tools of cognitive psychology.

As we shall discuss in Section 4, the sparseness of data and a bias in the database can be included in the model.

The results of the comparisons between the theory and numerical simulations over randomly-generated data are presented in Section 5.

Finally, in section 6 we summarize our work and draw some conclusions.

2 The Model

We consider a population of $M$ individuals interacting with a set of $N$ products. We assume that each product is characterized by an $L$-dimensional array $a = (a^{(1)}, a^{(2)}, \ldots, a^{(L)})$ of features, while each individual has the corresponding list of $L$ personal tastes on the same features $b = (b^{(1)}, b^{(2)}, \ldots, b^{(L)})$. For numerical simulations we have chosen both $a^{(l)}$ and $b^{(l)}$ in the set $\{-1, 0, 1\}$.

The opinion of individual $m$ on product $n$, denoted by $s_{m,n}$, is defined proportional to the internal product between $b_m$ and $a_n$: $s_{m,n} = \lambda(L)b_m \cdot a_n$, where $\lambda(L)$ is a suitably chosen normalization factor. In general, $\lambda(L)$ should scale as $L^{-1}$ and depend on the ranges of $a$ and $b$. For our choice of hidden parameters, we use $\lambda(L) = 1/L$, so that $s_{m,n}$ lies in the interval $[-1, 1]$.

In order to predict whether the person $j$ will like or dislike a certain product $a_n$, assuming to know $a_n$, it is sufficient to predict the individual tastes of person $j$, i.e. the vector $b_j$.

The similarity between tastes of two individuals $i$ and $j$ is defined by the overlap $\Omega_{ij} = b_i \cdot b_j$ between the individual tastes $b_i$ and $b_j$.

One can build a knowledge network among people, using the vectors $b_m$ as nodes and the overlaps $\Omega_{ij}$ as edges. Maslov and Zhang [3] (MZ) assume that a fraction $p$ of these overlaps are known. They show that there are two important thresholds for $p$ in order to be able to reconstruct the missing information.

The first one is a percolation threshold, reached when the fraction of edges $p$ is greater than $p_1 = 1/M - 1$ where $M$ is the number of people. This means that there must be at least one path between two randomly chosen nodes, in order to be able to predict the second node starting from the first one.

Since vectors $b_m$ lie in an $L$ dimensional space, and a single link “kills” only one degree of freedom, a reliable prediction need more than one path connecting two individuals. Maslov and Zhang show that there is a “rigidity” threshold $p_2$, of the order of $2L/M$, such that for $p > p_2$ the mutual orientation of vectors in the network are fixed, and the knowledge of preferences of just one person is sufficient to know those of all the rest of individuals.
3 Extracting information from hidden quantities

In general one does not have access to individual’s preferences. Nor one knows the dimensionality \( L \) of this space. In order to address this problem, let us define the opinion correlation matrix \( C \):

\[
C_{i,j} = \frac{\sum_{n=1}^{N}(s_{i,n} - \bar{s}_i)(s_{j,n} - \bar{s}_j)}{\sqrt{\sum_{n=1}^{N}(s_{i,n} - \bar{s}_i)^2 \sum_{n=1}^{N}(s_{j,n} - \bar{s}_j)^2}},
\]

where \( \bar{s}_i \) is the average of the opinion matrix \( S \) over column \( i \).

We show below that one can compute an accurate opinion anticipation \( \tilde{s}_{m,n} \) of a true value \( s_{m,n} \) using this formula:

\[
\tilde{s}_{m,n} = \frac{k}{M} \sum_{i=1}^{M} C_{m,i}s_{i,n}
\]

where \( k \) is a factor that in general depends on \( L \) and on the statistical properties of the hidden components. However, it will be shown that if the components of \( a_n \) and \( b_m \) are independent random variables, \( k \) is independent of \( n \) and \( m \), so it can be simply chosen in order to have \( \tilde{s}_{m,n} \) defined over the same interval as \( s_{m,n} \).

For instance, if we define

\[
\tilde{s}^*_m = \frac{1}{M} \sum_{i=1}^{M} C_{m,i}s_{i,n},
\]

then in order to keep estimations in the range \([-1,1]\), \( k = \frac{1}{\tilde{S}^{*\text{max}}_m} \) where \( \tilde{S}^{*\text{max}}_m = \max|\tilde{s}^*_{m,n}| \). As we shall illustrate in the following, from this estimation of \( k \) we can get information about the dimension of hidden space \( L \).

We now justify the proposed formulas for the case in which the components of \( a_n \), \( b_m \) are independent random variables distributed according to

\[
P(a_n^{(l)}, b_m^{(l)}) = P_n,l(a)P_m,l(b).
\]

Averages over \( P(a_n^{(l)}, b_m^{(l)}) \) of any function \( h(a_n^{(l)}, b_m^{(l)}) \) are given by

\[
\langle h \rangle = \sum_{m,n,l} h(a_n^{(l)}, b_m^{(l)})P_n,l(a)P_m,l(b).
\]

For a set of hidden components distributed according to \( \mathbf{1} \), the opinions are uncorrelated in the thermodynamic limit. However, the idea is that the system present fluctuations mainly because \( L \) is finite, so correlations between opinions arise and can be used to predict unknown opinions. In order to keep the algebra simple, the discussion will be made for the case in which the variables \( a_n^{(l)} \) and \( b_m^{(l)} \) have zero mean. At the end a generalization to biased components will be given.

The components can be written in matrix form as

\[
A = \begin{pmatrix}
  a_{1,1} & \cdots & a_{N,1} \\
  \cdot & \ddots & \cdot \\
  \cdot & \cdot & \cdot \\
  a_{1,L} & \cdots & a_{N,L}
\end{pmatrix}, \quad B = \begin{pmatrix}
  b_{1,1} & \cdots & b_{M,1} \\
  \cdot & \ddots & \cdot \\
  \cdot & \cdot & \cdot \\
  b_{1,L} & \cdots & b_{M,L}
\end{pmatrix},
\]

so the opinion matrix is defined by

\[
S = \lambda(L)B^T A,
\]

where \( \lambda(L) \) is the normalization constant.
The opinion correlation matrix is essentially equivalent to

\[ C = \frac{SS^T}{N\langle s^2 \rangle}, \tag{8} \]

where \( \langle s^2 \rangle \) denotes the average of \( s^2 \) over \( P_{n,t}(a)P_{m,t}(b) \). Because of the finite size of the system, there are differences between the normalization factors in definitions (1) and (8). These differences are small for large \( N \) and \( L \), and we neglect them at this point because they give non dominant contributions to errors in the final expressions. An element of the opinion matrix \( S \) is expressed by the internal product

\[ s_{m,n} = \lambda(L) \sum_{l=1}^{L} b_{m,l} a_{n,l}, \tag{9} \]

so averaging over the distribution \( P_{n,t}(a)P_{m,t}(b) \)

\[ \langle s^2 \rangle = \lambda^2(L) L \langle a^2 \rangle \langle b^2 \rangle. \tag{10} \]

Using (10) the correlation matrix can be written as

\[ C = B^T A A^T B \frac{A}{L N \langle a^2 \rangle \langle b^2 \rangle}. \tag{11} \]

Let us now consider the expression

\[ \frac{1}{M} CS = \frac{\lambda(L) B^T A A^T B}{L N \langle a^2 \rangle \langle b^2 \rangle} \tag{12} \]

If \( N \) and \( M \) are large, the central limit theorem can be applied to the following matrix products

\[ AA^T = \langle a^2 \rangle \begin{pmatrix} [N + \mathcal{O}(\sqrt{N}) + ...] & \mathcal{O}(1) & ... \\ \mathcal{O}(1) & [N + \mathcal{O}(\sqrt{N}) + ...] & ... \\ ... & ... & ... \end{pmatrix}, \tag{13} \]

\[ BB^T = \langle b^2 \rangle \begin{pmatrix} [M + \mathcal{O}(\sqrt{M}) + ...] & \mathcal{O}(1) & ... \\ \mathcal{O}(1) & [M + \mathcal{O}(\sqrt{M}) + ...] & ... \\ ... & ... & ... \end{pmatrix}. \tag{14} \]

Introducing Eqs. (13) and (14) into Eq. (12) we obtain

\[ \frac{1}{M} CS = \frac{S}{L} + \mathcal{O} \left( \frac{L - 1}{L} \left[ \frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}} \right] + \frac{1}{L \sqrt{MN}} + ... \right) S. \tag{15} \]

For large values of \( N \) and \( M \), by comparing Eq. (15) with Eq. (2), we can identify the factor \( k \) with the number of components \( L \), and obtain an estimate for the average prediction error

\[ \varepsilon = \sqrt{\frac{1}{MN} \sum_{m,n} (s_{m,n} - \tilde{s}_{m,n})^2} \simeq \gamma L^{3/2} \sqrt{\frac{M + \sqrt{N}}{\sqrt{MN}}}, \tag{16} \]

where

\[ \gamma = \lambda(L) \sqrt{\langle a^2 \rangle \langle b^2 \rangle}. \tag{17} \]

Formula (16) implies that the predictive power of Eq. (2) grows with \( MN \) and diminishes with \( L \). This fact is a consequence of the decay of the correlations among opinions with \( L \), so that more amount of information is needed in order to perform a prediction as \( L \) grows. This condition can be compared with the “rigidity” threshold \( p_2 \) in the MZ analysis.
Sparse and biased data

In the real world one cannot expect to have at his disposal a fully connected opinion matrix. Indeed, one of the most important feature of an anticipation system is its hole-filling capability.

One can extend the previous formalism to sparse datasets by considering the parameters $M_n$, $N_m$ as functions of the individual/product pair $(m, n)$ in the following way: $M_n$ represents the available number of opinions over product $n$ given by any agent and $N_m$ is the number of opinions expressed by agent $m$ about any product. Using formula (2) with the redefined parameters $M_n$, $N_m$, it follows from Eq. (15) that an unknown opinion $s_{m,n}$ can be estimated with an accuracy that scales as

$$|\tilde{s}_{m,n} - s_{m,n}| \sim \gamma L^2 \left( \frac{\sqrt{M_n} + \sqrt{N_m}}{\sqrt{M_n N_m}} \right)$$

for large values of $N_m$, $M_n$ and $L$.

The accuracy of our approach can be related with the “rigidity” threshold $p_2$. To illustrate this let us consider a situation in which $N_m = M_n$ and $N = M$. From formula (18) it turns out that the relative error in the estimation of an opinion will be

$$\frac{|\tilde{s}_{m,n} - s_{m,n}|}{|s_{m,n}|} \sim \frac{2L}{\sqrt{M_n}},$$

so in order to have relative errors order one or less, the inequality $M_n \gtrsim 4L^2$ must hold. This implies for the density of known opinions among all the elements of the opinion matrix that

$$p = \frac{\sum_{n=1}^{M} M_n}{M^2} \gtrsim 2Lp_2,$$

which means that our formulas work above the “rigidity” threshold $p_2$.

Our formalism is generalizable to systems with biased components, exploiting essentially the same arguments used to justify Eq. (2). It is found that in this case the factor $k$ that appears in the estimation formula (2) is given by

$$k = \left[ \frac{1}{L} + \left( \frac{L - 1}{L} \right) \frac{\langle b^2 \rangle - \langle b \rangle^2}{\langle b^2 \rangle} \right]^{-1}.$$  

(21)

Notice that $k$ does not depend on the $a^{(l)}_n$ variables, no matter if these variables are biased or not.

The existence of a constant value of $k$ independently of $n$ and $m$ justifies the previously proposed normalization approach $k = \frac{1}{s_{\max}}$. Moreover, $k$ can be interpreted as the effective number of components of the vector of internal preferences $b_m$. For instance, if the variance $\langle b^2 \rangle - \langle b \rangle^2$ is zero, then $b^{(l)}_m$ can take a unique value, so $b_m$ has only one effective degree of freedom, which is reflected by the value $k = 1$. On the other hand, the variance of $b^{(l)}_m$ is maximum when $\langle b \rangle = 0$, implying the value $k = L$ when all the $L$ degrees of freedom are relevant.

The behavior of the distance between the anticipated and actual values of opinions in the biased case is again given as in Eqs. (16) and (18), with

$$\gamma = \lambda \sqrt{\langle a^2 \rangle (\langle b^2 \rangle - \langle b \rangle^2)}$$

(22)

The asymmetry of formulas (21) and (22) with respect to variables $a^{(l)}_n$ and $b^{(l)}_m$ is related to the fact that the opinion correlation matrix $C$ basically reflects the overlap between the preferences of agents. To see this let us consider the following normalized overlap between $b_i$ and $b_j$

$$\Omega_{i,j} = \frac{\sum_{l=1}^{L} b_{i,l} b_{j,l}}{\sqrt{\sum_{l=1}^{L} b_{i,l}^2 \sum_{l=1}^{L} b_{j,l}^2}}.$$

(23)
For a large system size the opinion correlation matrix is written
\[
C = \frac{\langle \lambda(L) B^T A - \langle s \rangle 1 \rangle \langle \lambda(L) A^T B - \langle s \rangle 1 \rangle }{N [\langle s^2 \rangle - \langle s \rangle^2]}.
\] (24)

By introducing the product \(A A^T\) given in Eq. (13) on formula (24), it is found that
\[
C_{i,j} = \Omega_{i,j} \left[ 1 + O \left( \frac{1}{\sqrt{N}} \left( 1 + \frac{1}{\sqrt{L}} \right) \right) \right],
\] (25)
and the average error
\[
\sigma = \frac{1}{M} \sqrt{\sum_{m,m'} |C_{m,m'} - \Omega_{m,m'}|}
\] (26)
should grow like \(\sigma \sim N^{-1/2}\).

Eq. (26) states that for increasing \(N\) the correlation between the expressed opinions of agents \(i\) and \(j\) tends to be equivalent to the overlap \(\Omega_{i,j}\).

5 Numerical Results

In order to test the obtained relationships, we have performed simple simulations using random data.

The quantities \(L\), \(M\) and \(N\) are free parameters. We have used discrete components in the \([-1,1]\) set, randomly generated with variable average.

We have computed the opinion matrix \(S\) (Eq. (4)), the correlation matrix \(C\) (Eq. (1)) and the actual overlap matrix \(\Omega\) (Eq. (23)).

Then we have iterated over all the individuals’ opinions \(s_{m,n}\) computing \(\tilde{s}_{m,n}\) from Eq. (2), accumulating the average quadratic estimation error \(\varepsilon\), Eq. (16).

Figures 1, 2 and 3 show that the theoretical average errors, Eq. (16), are in good agreement with simulations.

Moreover, we show in Figure 4 that the distance \(\sigma\) between \(C\) and \(\Omega\) goes like \(N^{-1/2}\), as expected from Eq. (26).
Figure 2: Average estimation error $\varepsilon$ for $L = 10$ as a function of population size $M$ for two values of $N$: $N = 500$ (circles) and $N = 1000$ (crosses). The lines represent the best linear fit, with exponent $-0.515$ and $-0.530$, resp.

Figure 3: Average estimation error $\varepsilon$ as a function of $L$ for $N = M = 400$. The line represent the best linear fit, with exponent 0.523.
6 Discussion and conclusions

We assumed that an opinion is formed as a scalar product between individual preferences and products’ properties (both unobservable). This assumption relies on a kind of “universality” in cognitive processes, so that the opinion formation process should be analogous to other brain activity like the olfactory system, but honestly we do not have any rigorous justification. Individuals’ opinions are assumed to be stored in a database.

We have shown that, using central limit theorem (i.e. uncorrelated data) it is possible to anticipate an individual’s opinion, i.e. there is the possibility of exploiting the correlations in the database to forecast the missing opinions. Alternatively, we can obtain information about the overlaps of tastes between two individuals from the knowledge of their expressed opinions.

We have also shown that one can extract information about the dimensionality of the hidden taste space from the opinion database. We have also recovered the (almost trivial) expectation that the prediction error decreases when both the size of individual and product pools grow, and increases with the dimension of the hidden space.

We have not considered here the problem of coevolution of tastes and product qualities (which are produced in accordance to expectations about clients’ expectations). The coevolution of products’ features and individuals’ preferences induces correlations: people are not expected to blindly choose one movie from the available ones, but they tend to watch movies based on their anticipated opinion, thus filling the dataset with correlated data. On the other hand, movies are produced based on market expectation, reducing still more the variability.

The role of education emerges from this simple model: reliable opinion anticipations, that constitute an expectation of “revenues” from cultural investments, can come only from an assorted background of experiences both from a personal point of view, but also from the community’s one (due to the need of individuals’s correlations).

Finally, this model illustrate the value contained in personal information and the need for their protection.

Experimental verifications of the model are difficult, since personal data are jealously conserved. However, it is possible to identify similar “scalar product-like” mechanism in chemical or biological interactions [4], for which experimental data may be more easily available.
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