CHARACTERISTIC 2 APPROACH TO BIVARIATE INTERPOLATION PROBLEMS

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Abstract
We investigate bivariate interpolation problems in characteristic 2. Given a nonnegative integer \( t \), we describe all the sub-linear systems generated by monomials, in which there is no curve passing through a general point with multiplicity at least \( 2^t \). As an application, we show that a certain linear system of plane curves with 10 base points is non-special.

1 Introduction

We deal with bivariate interpolation problems in an infinite field \( K \) of characteristic 2. Characteristic 2 condition is not a very restrictive assumption because solvability of an interpolation problem in characteristic 2 implies solvability of the same problem in characteristic 0. Moreover working in characteristic 2 has many advantages. For instance, we do not need to take care of signs when we compute the determinants of matrices.

For the reader of this paper, an acquaintance with the notions and methods in \([1], [7]\) would be very useful. We more or less follow the notations of \([8]\). Given a fixed set \( S \) of lattice points \((i, j), i, j \geq 0\), the sub-linear system \( P(S) \) with respect to \( S \) consists of

\[
P(x, y) = \sum_{(i, j) \in S} a_{i,j}x^i y^j \in \mathbb{K}[x, y].
\]

Notice that unlike in \([8]\), we do not necessarily assume \( S \) to be a lower set.

Throughout this note, the coordinates of lattice points are always non-negative. Let \( T_m \) be the triangle of all \((i, j)\) with \( i + j \leq m - 1 \). \( T_m \) contains \(|T_m| = \frac{1}{2}m(m + 1)\) lattice points.

For a set of \( n \) distinct interpolation knots \( Z = \{z_q := (x_q, y_q)\}_{q=1}^{n} \) in \( \mathbb{K}^2 \), it is interesting to study (sub-)linear systems of plane curves passing through \( Z \) with multiplicity \( \geq m_q \) at each point \( z_q \) (for example, see \([1], [2], [3], [6], [8], [9], [10], [11], [12]\) and references therein). To put it in another way, we are interested in solving the interpolation problem

\[
\frac{1}{\alpha!\beta!} \left. \frac{\partial^{\alpha+\beta} P}{\partial x^\alpha \partial y^\beta} \right|_{z_q} = 0, \quad (\alpha, \beta) \in T_{m_q}, \quad q = 1, \cdots, n.
\]

Note that we do not necessarily require \(|S| = \sum_{q=1}^{n} |T_{m_q}|\). As in \([8]\), we say that an interpolation scheme is almost surely solvable or almost regular if \((*)\)
is solvable for almost all $Z \in (\mathbb{K}^2)^n$. Since the right hand sides in (*) are 0, our interpolation problem is almost regular if and only if (*) has only trivial solution for almost all $Z$. We remark that if $S$ has a double element $(i, j)$ then a nontrivial solution $P = x^i y^j + x^i y^j$ exists hence the interpolation problem is never almost regular.

Since it is natural to ask which (sub-)linear systems are almost regular, there has been some interest in trying to understand it. But up to now, even in the case $n = |Z| = 1$ there have been no explicit criterions in positive characteristic, and no other criterions in characteristic 0 than Bezout-Dunning lemma [3] Lemma 20 which gives a sufficient but not necessary condition for a (sub-)linear system to be almost regular.

In this note, we completely solve the interpolation problem in characteristic 2 in the case when $n = |Z| = 1$ and $m = m_1 = 2^t$ ($t \in \mathbb{N}$). In other words, given $t \in \mathbb{N}$, we describe all the sub-linear systems generated by monomials, in which there is no curve passing through a general point with multiplicity $\geq 2^t$. This case is already interesting in its own right, and is indispensable for dealing with the cases of $n \geq 2$ knots (c.f. [3] Proposition 12).

When $n = 1$, our interpolation scheme $\langle S, T_m \rangle$ becomes

$$
\frac{1}{\alpha! \beta!} \frac{\partial^{\alpha+\beta} P}{\partial x^\alpha \partial y^\beta} \bigg|_{z_1} = 0, \quad (\alpha, \beta) \in T_m \quad (**).
$$

Our main theorem shows that the interpolation problem $\langle S, T_{2^t} \rangle$ is inductive on $t$, in other words, almost regularity of $\langle S, T_{2^{t+1}} \rangle$ can be determined by some interpolation problems with $T_{2^t}$.

**Theorem 1.1.** Let $\mathbb{K}$ be an infinite field of characteristic 2. The following statements are equivalent.

(i) The interpolation problem $\langle S, T_{2^{t+1}} \rangle$ is almost regular.

(ii) There is no triple $(U, V, W) \subset S_{h,v}, S_{v,d}, S_{d,h}$ (see Definition 2.3) of subsets such that

- each of $\sum_{(i,j) \in U} x^i y^j, \sum_{(i,j) \in V} x^i y^j, \sum_{(i,j) \in W} x^i y^j$ is a solution of (**) for $m = 2^t$,
- at most one of $U, V, W$ is empty, \quad (\dagger)
- and $(U - V) \cup (V - U) = W$
  
  $(V - W) \cup (W - V) = U$
  
  $(W - U) \cup (U - W) = V$.

The following two corollaries are often useful in practice.

**Corollary 1.2.** If the interpolation problem $\langle S, T_{2^t} \rangle$ is almost regular, then the three of $\langle S_{\text{hori}}, T_{2^t} \rangle$, $\langle S_{\text{vert}}, T_{2^t} \rangle$, and $\langle S_{\text{diag}}, T_{2^t} \rangle$ are all almost regular.

**Corollary 1.3.** If at least two of the three interpolation problems $\langle S_{\text{h,v}}, T_{2^t} \rangle$, $\langle S_{\text{v,d}}, T_{2^t} \rangle$, and $\langle S_{\text{d,h}}, T_{2^t} \rangle$ are almost regular, then the interpolation problem $\langle S, T_{2^{t+1}} \rangle$ is almost regular.
For example, if \( S = \{(0,0),(0,1), (1,0),(0,2), (1,1),(2,0), (1,2),(2,1), (3,0),(1,3)\} \) and \( m = 2^2 \), then \( \langle S,T_{2^2} \rangle \) (see the preceding sentence of Definition 2.3) is almost regular because \( \langle S_1,d,T_2 \rangle \)$ $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \) and \( \langle S_{[d,h],T_2} \rangle \) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \) are almost regular.

As an application of Theorem 1.1 we show that, without the aid of a computer, the linear system of plane curves of degree 26 passing through 10 general base points with \( m_1 = m_2 = 9, m_3 = ... = m_{10} = 8 \) is empty. Our future project is to generalize this to bigger multiplicity cases.

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2 Definitions

Throughout this note, for each lattice point \((i,j)\), the vector whose \((a,b)\)-th component \(\langle i-j \rangle \) \((a,b) \in T_{2^v}\) is \(\binom{i}{a} \binom{j}{b} \mod 2\) will be denoted by \(v_{i,j}\). This is nothing but the vector consisting of the coefficients of a column in an interpolation matrix (c.f. [8] p.670). We always arrange \((a,b)\)-components with respect to the total degree order, that is, \((0,0) < (0,1) < (1,0) < (0,2) < (1,1) < (2,0) < (0,3) < \cdots \). For example, \(v_{0,0}^0 = (1)\),

\[
v_{0,0}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_{0,1}^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_{1,0}^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_{1,1}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\]

\[
v_{0,0}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{0,1}^2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{1,0}^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{1,1}^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

\[
v_{1,2}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{3,3}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

**Definition 2.1.** We say that \( S \) is \(2^f\)-independent if \( \{v_{i,j}^f\} (i,j) \in S \) are linearly independent. This is equivalent to saying that \( \langle S,T_{2^f} \rangle \) is almost regular.
The problem of deciding whether $S$ is $2^t$-independent can be reduced modulo $2^t$. Let $B_t := \{(x, y) | x, y = 0, 1, \cdots, 2^t - 1 \}$ and consider the natural projection $\rho : S \longrightarrow B_t$ defined by $(i, j) \in S \longmapsto (x, y) \in B_t$ where $x \equiv i (\text{mod } 2^t), y \equiv j (\text{mod } 2^t)$.

**Lemma 2.2.** $S$ is $2^t$-independent if and only if the image $\rho(S)$, counting multiple elements, is $2^t$-independent.

**Proof.** It follows from the fact that if $u \equiv v (\text{mod } 2^t)$ and $0 \leq z \leq 2^t - 1$ then $\binom{u}{z} \equiv \binom{v}{z} (\text{mod } 2)$. \hfill $\Box$

Thanks to Lemma 2.2 we can and will assume that $S$ is a subset (possibly counting multiple elements) of $B_t$. Since we use induction on $t$ in the proof of Theorem 1.1 we will sometimes write a pair $(B_t, S)$ in place of $S \subset B_t$ to avoid confusion. The visualization of $(B_t, S)$ will be used frequently, for instance,

$$S = \{(0, 0), (1, 2), (1, 2), (3, 3)\} \subset B_2$$

**Definition 2.3.** Suppose that $S \subset B_{t+1}$ does not have multiple points. Then we define $S|_{h,v} \subset B_t$ as follows. Given an element $(i, j) \in B_t$, the element $(i, j)$ belongs to $S|_{h,v}$ if and only if one of the four subsets

\begin{align*}
\{&(i, j), (i + 2^t, j)\}, \\
\{&(i, j + 2^t), (i + 2^t, j + 2^t)\}, \\
\{&(i, j), (i, j + 2^t)\}, \\
\{&(i + 2^t, j), (i + 2^t, j + 2^t)\} \quad (2.1)
\end{align*}

is contained in $S$. If all the four are contained in $S$ then we call $(i, j)$ a double element. We notice that $h$ stands for horizontal, $v$ vertical, and $d$ diagonal. Illustrations are given below.

![Illustration](image-url)
In the above definition we replace \((2.1)\) by
\[
\{(i, j), (i, j + 2^t), (i + 2^t, j), (i + 2^t, j + 2^t)\},
\{(i, j), (i + 2^t, j + 2^t), (i, j + 2^t), (i + 2^t, j)\}
\]
(respectively \(\{(i, j), (i + 2^t, j + 2^t), (i, j + 2^t), (i + 2^t, j)\} \)).

To define \(S_{\text{hor}}, S_{\text{vert}}, S_{\text{diag}}\) respectively, we replace \((2.1)\) by
\[
\{(i, j), (i + 2^t, j), (i, j + 2^t), (i + 2^t, j + 2^t)\},
\{(i, j), (i + 2^t, j), (i, j + 2^t), (i + 2^t, j + 2^t)\},
\{(i, j), (i + 2^t, j + 2^t), (i, j + 2^t), (i + 2^t, j)\}.
\]

respectively.

3 Proofs

Proof of Theorem 1.1.

\[(i) \implies (ii)\]:

Before giving a proof, we introduce some notations. Let \(V^t\) be the \((2^{t+1})\)-dimensional vector space over \(\mathbb{F}_2\) consisting of \((a, b)\)-components \(((a, b) \in T_{2^t})\), so that \(v_{i,j}^t \in V^t\). We decompose \(V^{t+1}\) into \(V^t_w \oplus V^t_y \oplus V^t_z\), where \(V^t_w\) consists of
\[(a, b)\text{-components with } (a, b) \in B_t, V^t_y \text{ with } b \geq 2^t, \text{ and } V^t_z \text{ with } a \geq 2^t. \text{ As a matter of fact, we consider the following isomorphism of vector spaces:}
\[
\tau : V^{t+1} \simeq V^t_w \oplus V^t_y \oplus V^t_z
\]

\[
\begin{pmatrix}
(0, 0)\text{-th component of } v \\
(0, 1)\text{-th component of } v \\
(1, 0)\text{-th component of } v \\
(0, 2)\text{-th component of } v \\
(1, 1)\text{-th component of } v \\
(2, 0)\text{-th component of } v \\
\vdots \\
\vdots \\
\vdots \\
(2^t+1 - 3, 2)\text{-th} \\
(2^t+1 - 2, 1)\text{-th} \\
(2^t+1 - 1, 0)\text{-th}
\end{pmatrix}
\begin{pmatrix}
(0, 0)\text{-th} \\
(0, 1)\text{-th} \\
(1, 0)\text{-th} \\
\vdots \\
(2^t, 0)\text{-th} \\
(2^t, 1)\text{-th} \\
\vdots \\
(2^t+1 - 1, 0)\text{-th}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
w \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
(0, 0)^t \text{-th} \\
(0, 0)\text{-th} \\
(1, 0)\text{-th} \\
\vdots \\
(2^t, 0)\text{-th} \\
(2^t, 1)\text{-th} \\
\vdots \\
(2^t+1 - 1, 0)\text{-th}
\end{pmatrix}
\]

where we have only rearranged the order of components. Note that \(\dim(V^t_w) = (2^t)^2, \dim(V^t_y) = \dim(V^t_z) = (2^{t+1})\) and that \(V^t_y, V^t_z, \text{ and } V^t\) are isomorphic as vector spaces.

Seeking contradiction, suppose that there is a triple \((U, V, W) \subset S_{|h,v}, S_{|v,d}, S_{|d,h}\) of subsets satisfying the three conditions in (i) in the statement of Theorem 1.1. Without loss of generality, we assume that \(U\) and \(V\) are nonempty.

By using the proof of Lemma 2.2 we observe that

\[
\begin{align*}
(v^t_{i,j} + v^t_{i+2^t,j}, & \text{ or } v^t_{i,j+2^t} + v^t_{i+2^t,j+2^t}) \\
& = \begin{cases} 
0 & \text{if } a < 2^t, \\
\binom{i}{a-2^t} \binom{j}{b} & \text{if } a \geq 2^t,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
(v^t_{i,j} + v^t_{i,j+2^t}, & \text{ or } v^t_{i+2^t,j} + v^t_{i+2^t,j+2^t}) \\
& = \begin{cases} 
0 & \text{if } b < 2^t, \\
\binom{j}{a} \binom{i}{b-2^t} & \text{if } b \geq 2^t.
\end{cases}
\end{align*}
\]
Then

\[ \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) = \vec{0} \oplus \vec{0} \oplus v_{i,j}^t \in V_w^t \oplus V_y^t \oplus V_z^t, \] (3-2)

and

\[ \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) = \vec{0} \oplus v_{i,j}^t \oplus \vec{0} \in V_w^t \oplus V_y^t \oplus V_z^t. \] (3-3)

**Lemma 3.1.** Suppose that there is a triple \((U,V,W) \subset S_{h,v}, S_{v,d}, S_{d,h}\) of subsets satisfying the three conditions in \((\dagger)\) in the statement of Theorem 1.1. Then we have

\[
\sum_{(i,j) \in U-V} \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) \\
+ \sum_{(i,j) \in U \cap V} \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) \\
+ \sum_{(i,j) \in V-U} \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) \\
= \vec{0}.
\]

**Proof of Lemma 3.1.** By (3-2) and (3-3), we have

\[
\sum_{(i,j) \in U-V} \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) \\
+ \sum_{(i,j) \in U \cap V} \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) \\
= \sum_{(i,j) \in U-V} \vec{0} \oplus \vec{0} \oplus v_{i,j}^t \\
+ \sum_{(i,j) \in U \cap V} \vec{0} \oplus v_{i,j}^t \oplus \vec{0}.
\] (3-4)

Applying the same argument as in (3-2) or (3-3) for \((B_t, S_{v,d})\), we get

\[
\sum_{(i,j) \in V-U} \tau(v_{i,j}^{t+1} + v_{i,j+2t}^{t+1}) \text{ (or } v_{i,j+2t}^{t+1} + v_{i,j+2t+1}^{t+1}) \\
= \sum_{(i,j) \in V-U} \vec{0} \oplus v_{i,j}^t \oplus v_{i,j}^t.
\] (3-5)

On the other hand, since \((B_t, U) \subset (B_t, S_{h,v}), (B_t, V) \subset (B_t, S_{v,d})\) and \((B_t, W) \subset (B_t, S_{d,h})\) satisfy the first condition in \((\dagger)\), we have \(\sum_{(i,j) \in U} v_{i,j}^t = \vec{0}\), \(\sum_{(i,j) \in V} v_{i,j}^t = \vec{0}\) and \(\sum_{(i,j) \in W} v_{i,j}^t = \vec{0}\).
Adding (3.4) and (3.5) together gives

\[ \sum_{(i,j) \in U-V} \tau(v_{i,j}^{t+1} + v_{i+2',j}^{t+1}) + \sum_{(i,j) \in U \cap V} \tau(v_{i,j}^{t+1} + v_{i,j+2}^{t+1}) + \sum_{(i,j) \in U-V} \tau(v_{i,j}^{t+1} + v_{i+2',j+2}^{t+1}) + \sum_{(i,j) \in U} \tau(v_{i,j}^{t+1} + v_{i+2',j}^{t+1}) + \sum_{(i,j) \in V-U} \tau(v_{i,j}^{t+1} + v_{i+2',j+2}^{t+1}) \]

\[ = \sum_{(i,j) \in U-V} \bar{0} \oplus \bar{0} \oplus v_{i,j}^t + \sum_{(i,j) \in U \cap V} \bar{0} \oplus v_{i,j}^t \oplus \bar{0} + \sum_{(i,j) \in V-U} \bar{0} \oplus v_{i,j}^t \oplus v_{i,j}^t \]

\[ = \sum_{(i,j) \in V} \bar{0} \oplus v_{i,j}^t \oplus \bar{0} + \sum_{(i,j) \in (U-V) \cap (V-U)} \bar{0} \oplus v_{i,j}^t \]

\[ = \sum_{(i,j) \in V} \bar{0} \oplus v_{i,j}^t \oplus \bar{0} + \sum_{(i,j) \in W} \bar{0} \oplus \bar{0} \oplus v_{i,j}^t \]

\[ = \bar{0}. \]

We observe that \( U - V \subset U \subset S_{\text{hor}} \) and \( U - V \subset W \subset S_{\text{hor}} \), implying \( U - V \subset S_{\text{hor}} \). So, by Definition 2.3, if \((i,j) \in U - V\) then either \(\{(i,j), (i+2',j)\}\) or \(\{(i,j+2), (i+2',j+2')\}\) is contained in \(S\). In the same manner, \(V - U \subset S_{\text{hor}}\). If \((i,j) \in V - U\) then either \(\{(i,j), (i+2',j+2')\}\) or \(\{(i,j+2), (i+2',j)\}\) is contained in \(S\). It is obvious that if \((i,j) \in U \cap V \subset S_{\text{hor}} \cap S_{\text{hor}}\), then either \(\{(i,j), (i+2',j)\}\) or \(\{(i+2',j), (i+2',j+2')\}\) is contained in \(S\).

So Lemma 3.1 implies that there is a nonempty subset \(S'\) of \(S\) such that \(\sum_{(i,j) \in S'} v_{i,j}^{t+1} = \bar{0}\). Therefore \((B_{t+1}, S)\) is not \(2^{t+1}\)-independent.

(i) \(\iff\) (ii):

Suppose that \((B_{t+1}, S)\) is not \(2^{t+1}\)-independent. Then there is a nonempty minimal subset \(S'\) of \(S\) such that \(\sum_{(i,j) \in S'} v_{i,j}^{t+1} = \bar{0}\).

**Lemma 3.2.** \(|S' \cap \{(i,j), (i,j+2'), (i+2',j), (i+2',j+2')\}| \equiv 0 \pmod{2}\) for any \((i,j) \in B_t\).

To prove this lemma, we need the following claims. Let \(w_{i,j} = \tau_1(v_{i,j}^{t+1})\) where \(\tau_1\) is the natural projection \(V^{t+1} = V_w \oplus V_y \oplus V_z \rightarrow V_w\).

**Claim 3.3.** \(w_{i,j} \ (i,j) \in B_t\) are linearly independent.

**Proof.** It is easy to see that the matrix in which columns \(w_{i,j}\) are arranged with respect to the total degree order is an upper triangle matrix with diagonal \((1, 1, ..., 1)\). \(\square\)
Claim 3.4. For any \((i, j) \in B_t\), the vectors \(w_{i,j}, w_{i,j+2t}, w_{i+2t,j}, w_{i+2t,j+2t}\) are identical.

Proof. This is essentially the same as the proof of Lemma 2.2.

Proof of Lemma 3.2. Since the vector \(\sum_{(i, j) \in S'} v_{i,j}^{t+1}\) is the zero vector, its image under \(\tau_1 : V^{t+1} \rightarrow V_w^t\) is also zero vector. So we have

\[
\bar{0} = \sum_{(i, j) \in S'} w_{i,j}
= \sum_{(i, j) \in B_t} |S'| \cap \{(i, j), (i, j + 2t), (i + 2t, j), (i + 2t, j + 2t)\}| w_{i,j},
\]

where we have used Claim 3.4. Then, by Claim 3.3, \(|S'| \cap \{(i, j), (i, j + 2t), (i + 2t, j), (i + 2t, j + 2t)\}| \equiv 0 \pmod{2}\) for every \((i, j) \in B_t\). This completes the proof of Lemma 3.2.

If \(|S' \cap \{(i, j), (i, j + 2t), (i + 2t, j), (i + 2t, j + 2t)\}| = 4\) for some \((i, j) \in B_t\), then \(|S' \cap \{(i, j), (i, j + 2t), (i + 2t, j), (i + 2t, j + 2t)\}| \leq 2\) for any \((i, j) \in B_t\). Of course we consider only \((i, j) \in B_t\) with \(|S' \cap \{(i, j), (i, j + 2t), (i + 2t, j), (i + 2t, j + 2t)\}| = 2\). For each of such \((i, j)\), depending on elements in \(S' \cap \{(i, j), (i, j + 2t), (i + 2t, j), (i + 2t, j + 2t)\}\), we consider one and only one of the three vectors

\[
\tau(v_{i,j}^{t+1} + v_{i+2t,j}^{t+1}) \quad \text{or} \quad v_{i,j}^{t+1} + v_{i+2t,j}^{t+1} + v_{i+2t,j+2t}^{t+1}
\]

or

\[
\tau(v_{i,j}^{t+1} + v_{i+2t,j+2t}^{t+1}) \quad \text{or} \quad v_{i,j}^{t+1} + v_{i+2t,j+2t}^{t+1} + v_{i,j}^{t+1}
\]

These give rise to three disjoint (possibly empty but not all empty) subsets \(S_1, S_2, S_3\) of \(B_t\) such that

\[
\sum_{(i, j) \in S_1} (\bar{0} \oplus \bar{0} \oplus v_{i,j}^t) + \sum_{(i, j) \in S_2} (\bar{0} \oplus v_{i,j}^t \oplus \bar{0}) + \sum_{(i, j) \in S_3} (\bar{0} \oplus v_{i,j}^t \oplus \bar{0}) = \bar{0}
\]

\[
in V_w^t \oplus V_y^t \oplus V_z^t.
\]

We show that the triple \((S_1 \cup S_3, S_2 \cup S_3, S_1 \cup S_2)\) satisfies the three conditions in (i) in the statement of Theorem 1.1. Recall that \(V_y^t \cong V_z^t \cong V^t\). This implies that

\[
\sum_{(i, j) \in S_1} v_{i,j} + \sum_{(i, j) \in S_3} v_{i,j} = \bar{0} \in V^t \cong V^t,
\]

\[
\sum_{(i, j) \in S_2} v_{i,j} + \sum_{(i, j) \in S_3} v_{i,j} = \bar{0} \in V_y^t \cong V^t,
\]

\[
\sum_{(i, j) \in S_1} v_{i,j} + \sum_{(i, j) \in S_2} v_{i,j} = \bar{0} \in V_z^t \cong V^t.
\]
hence
\[ \sum_{(i,j) \in S_1} v^t_{i,j} + \sum_{(i,j) \in S_2} v^t_{i,j} + 2 \sum_{(i,j) \in S_3} v^t_{i,j} = \sum_{(i,j) \in S_1} v^t_{i,j} + \sum_{(i,j) \in S_2} v^t_{i,j} = \vec{0} \in V_t. \]

Thus the first condition follows.

Since at least one of $S_1, S_2, S_3$ is nonempty, at most one of $(S_1 \cup S_3), (S_2 \cup S_3), (S_1 \cup S_2)$ is empty. The last condition follows from $S_1, S_2, S_3$ being disjoint. Proof of Theorem 1.1 is completed.

Proof of Corollary 1.2 If one, say $\langle S|_{\text{hori}}, T_{2^t} \rangle$, of the three is not almost regular, then there is a nontrivial solution of $(**)$.

Let $\sum_{(i,j) \in U} x^i y^j$ be one of the solutions. If $V = \emptyset$ and $W = U$ then the triple $(U, V, W)$ satisfies the three conditions in (†) in the statement of Theorem 1.1. So $\langle S, T_{2^t+1} \rangle$ is not almost regular.

Proof of Corollary 1.3 Without loss of generality, suppose that $\langle S|_{\text{v}, v}, T_{2^t} \rangle$ and $\langle S|_{\text{v}, d}, T_{2^t} \rangle$ are almost regular. If $\langle S, T_{2^t+1} \rangle$ were not almost regular, then there would be a triple $(U, V, W)$ satisfying the three conditions in (†). But since $U$ and $V$ are empty, $(U, V, W)$ cannot satisfy the third condition.

4 Application

We show that the linear system of plane curves of degree $d = 26$ passing through 10 general base points with $m_1 = m_2 = 9$, $m_3 = \ldots = m_{10} = 8$ is empty. In this case, $S = T_{d+1} = T_{27}$. We apply Dumnicki-Jarnicki reduction method [5] and Dumnicki’s cutting diagram method [4] to 8 points $z_1, \ldots, z_8$. Then we use Theorem 1.1 several times to check that the division below determines a unique nonzero (in characteristic 2) monomial of the form

\[ \prod_{q=1}^{10} \frac{x_q^{\sum_{(i,j) \in T_{mq}^{i}} y_q^{\sum_{(i,j) \in T_{mq}^{j}}} \in \text{the determinant of the interpolation matrix} (*)}}{x_q^{\sum_{(i,j) \in T_{mq}^{i}} y_q^{\sum_{(i,j) \in T_{mq}^{j}}} \in \text{the determinant of the interpolation matrix}}} \]

We remark that Bezout-Dumnicki lemma ([4], Lemma 20) does not work in positive characteristic and that even in characteristic zero, it cannot cover the central region corresponding to the point $z_{10}$. 

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