Implications of Advertising Lag on the Dynamic Optimal Decisions in an O2O Supply Chain

Xiuxian Li,1 Ling Li,2 Xinyu Wang,3 and Shuhua Zhang3

1College of Science, Tianjin University of Commerce, Tianjin 300134, China
2Business School, Tianjin University of Finance and Economics, Tianjin 300222, China
3Coordinated Innovation Center for Computable Modeling in Management Science, Tianjin University of Finance and Economics, Tianjin 300222, China

Correspondence should be addressed to Xiuxian Li; lxxcaptain@126.com

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1. Introduction

An O2O (online to offline) business is a business model that integrates online and offline markets, such as Suning, Leyou, Ele, and Ctrip. Consumers can search for and buy services or products online and then consume them in offline stores. In addition, they can visit offline stores to check the product quality, then receive coupons, and place orders online. Group buying models, such as Amazon, the initial O2O business model that appeared, have been widely accepted by customers [1, 2]. Scholars have pointed that [3] the retailer’s O2O pattern (such as Meituan, Leyou, Walmart, and Suning) typically receives more attention than the manufacturer’s O2O pattern (such as Huawei, Nike, and Apple) in the literature, as the latter is similar to that of the centralized dual-channel supply chain, which has been extensively studied in the past. According to the data of the Electronic Commerce Research Center (100EC.CN), the amount of takeout transactions on the Meituan Dianping platform reached 127.7 billion yuan ($18.23 billion) in January-June 2018, up 97% from 62.3 billion yuan ($8.9 billion) in the same period the prior year. In our paper, we concentrate on the setting of an O2O retailer supply chain.

An outstanding difference between the offline and the online sales under an O2O environment is that the offline consumers can observe the product quality before purchase, which it is not the case under the online channel. This impacts the offline consumer’s purchase behavior and is formulated as the reference quality effect [4]. This effect is conceptualized as the reference quality expectation, which is based on the consumer’s memory about the past quality [5]. Gavious and Lowengart [6] reported that the reference quality increases consumers’ elasticities to both price and quality compared with the case of no reference quality. The
existence of the reference quality effect under an online/offline environment arouses the interest of scholars [4, 5, 7]. The reference quality comprises positive and negative effects, which are decided by the current product quality being better or worse than the reference quality. A positive reference quality effect promotes market demand, while a negative one reduces it. In our paper, we design a positive reference quality effect to consider its impact on the supply chain.

It is a phenomenon that firms advertise to create present and future demand for the firm’s products and, hence, to create present and future revenues for the firm [8]. In recent decades, supply chain management has been devoted to the study of the immediate effects of advertising [9–14]. Indeed, a more plausible assumption would be that the brand goodwill reacts to advertising gradually. It was pointed out that when companies do not advertise, their sales will not drop to zero [15]. Aravindakshan and Naik [16] reported that, in the milk industry, when advertising is stopped, sales can remain steady for a year and then drastically decline. This indicates that some advertising carryover effect must exist. In our paper, we call this effect the advertising lag effect. The advertising lag effect means that an advertising investment often does not produce the expected advertising effect within the expected time, which does not mean that the advertising investment has no effect at all but that the effect of the advertising investment has a certain delay. In fact, a large number of studies in the supply chain research on the advertising lag effect exist [8, 16–18]. However, its impact on the supply chain management under an O2O environment has not been studied till now. Our paper would fill this gap.

It is recognized that a manufacturer’s advertising (national advertising) tends to promote product awareness, the brand image, brand preference, and product purchasing [11, 19]. Expenditures on this type of advertising represent a significant portion of the marketing budgets in many companies. Advertising age reported that GM spent $3.5 billion on advertising in 2015, then its net earnings rose sharply in the fourth quarter of 2015. Ford’s overall advertising spending in the United States was $2.68 billion in 2015, up 8.5% year-on-year. The revenue at Ford was $40.3 billion in the fourth quarter of 2015, up 12% from a year earlier. A vertical advertising cooperation refers to companies operating in a manufacturer-retailer supply chain, which is widely discussed in the literature [9, 10, 13]. In our paper, we study the lag effect of national advertising on the supply chain and consider whether vertical advertising cooperation is efficient in improving the channel performance.

Although the advertising lag effect is an intrinsic property of the market, the operating environment of the O2O business model is in full swing, and the reference quality effect is a key factor that affects offline consumers’ purchasing behavior; the academic marketing literature has not, to the best of our knowledge, dealt with the decision questions incorporating the three topics. This paper is researched based on the above deficiency.

We confine our interest to an O2O retailer supply chain with one manufacturer and one retailer, where the manufacturer wholesales product to the retailer and the retailer sells them through online and offline (a real store) channels to the market. The consumer’s buying behavior may be affected by the brand goodwill and the retail price; it may also be affected by the consumers’ reference quality effect through the offline channel. In our model, the supply chain members play a Stackelberg game. The manufacturer, the Stackelberg leader, decides the wholesale price, advertising effort level, and product quality level. The retailer, the follower, decides the retail price. We assume that the supply chain members are all individually rational. That is, as long as income increases, cooperation will not be rejected. Because the dual-channels are operated by the retailer, we set the same retail price under the two channels to avoid channel competition [4]. We give the optimal decision in the following two situations: with and without vertical advertising cooperation modes. For convenience, we use “he” when referring to the retailer, and “she” when referring to the manufacturer; additionally, “non-cooperation mode” and “cooperation mode” are used to denote the non-vertical-advertising-cooperation mode and the vertical-advertising-cooperation mode, respectively.

Our research focuses on the following four main questions: (1) when faced with the advertising lag effect and the consumers’ reference quality effect, what are the manufacturer’s optimal wholesale price, advertising effort level, and product quality level under two modes? What is the optimal retail price for the retailer? (2) How do the advertising lag effect and reference quality effect impact supply chain performance? (3) Is vertical advertising cooperation efficient in improving channel performance under our setting? (4) How should the manufacturer balance the investment in advertising and product quality when faced with different advertising lag times, differentiated market sensitivity to quality, and diverse advertising cost sharing ratios? The answers to these questions are not obvious and need careful study.

Through comparison and sensitivity analysis, some management opinions are given as follows: (1) the equilibrium brand goodwill (advertising effort level, product quality effort level, retail price, wholesale price, reference quality effect, and sales) increases or decreases exponentially to its steady-state value based on whether the initial goodwill is lower or higher than its static value under the two modes. (2) The vertical advertising cooperation strategy is efficient in improving channel performance, i.e., the supply chain members’ profits and product competitiveness are all improved after advertising cooperation. (3) The advertising lag effect negatively impacts the supply chain under the two modes, and the size of negative impact is larger under cooperation than under the non-cooperation mode. (4) The reference quality effect has a positive impact on the supply chain under the two modes, and the size of the positive impact is larger under cooperation than under the non-cooperation mode. (5) The manufacturer pays more attention to product quality improvements (advertising) with a longer (shorter) advertising lag time or a larger (smaller) sensitivity factor to the reference quality effect under the two modes. They pay more attention to brand advertising relative to quality improvement under cooperation than under
the non-cooperation mode. Along with the increasing of the advertising cost sharing ratio, both the advertising and product quality improvement expenditures increase, but the manufacturer pays more attention to advertising.

This paper is organized as follows. We review the related literature in Section 2. In Section 3, we describe the problem and modeling assumptions. In Section 4, we study two advertising models in the O2O retailer supply chain, i.e., with and without advertising cooperation modes, and derive the optimal dynamic and static equilibrium solutions for the supply chain members. Section 5 gives a comparison analysis between the two modes. Some important conclusions are obtained in this section. Section 6 concludes by identifying the direction for further research. All figures in our paper are plotted by Matlab 2016a. To make the paper more readable, all proofs are presented in Appendix.

2. Literature Review

Our paper is closely related to four topics, which are the O2O environment, advertising lag effect, reference quality effect, and vertical advertising cooperation.

Operation management under an O2O environment is an important topic. The existing literature on the O2O environment contains research on the coordination mechanism, advertising cooperation, and carbon emission reduction decisions. Regarding the coordination mechanism, Govindan and Malomfalean [20] considered the deterministic demand and stochastic demand under the following three coordination mechanisms: revenue-sharing, buy-back, and quantity flexibility contracts. They demonstrated that the best outcome and the highest profit are achieved with the O2O deterministic demand under the quantity flexibility agreement. Xie et al. [12] investigated the coordination contracts of centralized and decentralized dual-channel closed-loop supply chains under an O2O environment. They concluded that the optimized online and offline prices will be increased with an increase in the advertisement input. From the cooperation advertising side, Li et al. [10] considered a cooperative advertising strategy for the O2O supply chain. They investigated an integration model, a unilateral co-op advertising model, and a bilateral co-op advertising model. In the view of a firm’s emission reduction behaviors, Ji et al. [3] focused on the carbon allowance allocation rules in the retailer’s O2O supply chain with the following three models: without a cap-and-trade regulation, a cap-and-trade regulation based on the grandfathering mechanism, and a cap-and-trade regulation based on the benchmarking mechanism. Additionally, the emission reduction strategies of the supply chain members in the three cases are discussed. We extend the above literatures by adding the advertising lag effect and consumers’ reference quality effect, to consider their impacts to the supply chain.

Elsadany and Matouk [21] pointed that the delayed system has the same Nash equilibrium point and the delay can increase the local stability region. A significant portion of the published works used differential games to study the advertising strategies that depend on the evolution of the brand goodwill with the advertising lag effect. Pauwels [17] generalized the NerloveArrow model of optimal dynamic advertising policies; they incorporated a continuously distributed lag between advertising expenditures and increases of goodwill using the integro-differential equation. Luhta and Virtanen [8] analyzed the linear, nonlinear, and the bounded effects of advertising to present a time delayed feedback model and derived the stability conditions of equilibrium with the help of the goodwill elasticity of advertising at the target. Aravindakshan and Naik [16] extended the advertising models by incorporating the memory of advertising and captured it using delay differential equations; they derived 90% duration of advertising effects under various scenarios. They pointed out that if we ignore the consumer’s memory, we will overstate the forgetting rate by 39%. Aravindakshan and Naik [18] used delay differential equations to understand the evolution of awareness in the presence of ad memorability. Optimal advertising policies were generated that include an even spending policy, blitz policy, and various cyclic pulsing policies depending on whether the consumer memorability exceeds a critical threshold. Unlike the present paper, these studies only explore the impact of the advertising lag effect on the advertising strategy, and little discussion has been given to the decision of product quality and price, even though it is significant in supply chain management.

The reference quality effect is an important factor that influences the consumer purchasing behavior, and there are studies on this topic in reference to the supply chain. Chenavaz [5] studied the dynamic quality policy of a firm whose consumers use a reference point in their decision-making. The main result proposed a general rule linking the dynamics of quality to both the dynamics of the reference quality and its adjustment process. Different from our paper, this paper never considered the advertising operation. He et al. [4] formulated the reference quality effect with a modified NerloveArrow model to study the supply chain decisions with the reference quality effect under the O2O environment. They concluded that as the reference quality effect becomes greater, the supplier will concentrate more efforts on quality improvement, which results in a higher ratio of quality to price. Unlike in our paper, this literature considered advertising to have instantaneous effect instead of a lag effect.

Cooperative advertising is an important instrument for aligning manufacturer and retailer decisions in a supply chain, and there are abundant works studying this theme. For example, He et al. [13] designed a model in which the manufacturer paid a certain percentage of the retailer’s advertising expenditure. They obtained the condition when offering co-op advertising is optimal for the manufacturer and provided in feedback in regard to the optimal advertising and pricing policies for the manufacturer and the retailer. Xiao et al. [9] considered a two-echelon supply chain consisting of one manufacturer and multiple retailers whose demands are affected by the manufacturer’s brand advertising and the retailers’ local advertising strategies. They found that after the manufacturer joins the coalition, the advertising expenditures for all participants may be increased. Jorgensen et al. [22] concluded that a cooperative program, in which
the manufacturer reimburses some of the retailer’s promotion costs, is Pareto improving and implementable in a situation where retailer promotions are harmful to the image of the manufacturer’s brand. Different from the above cooperation mode, Zhang et al. [19] proposed a new mechanism to coordinate a supply chain in which both the manufacturer and the retailer share each other’s advertising costs. They concluded that it is necessary for a retailer to share a part of the manufacturer’s national advertising costs if the retailer occupies most of the supply chain profit. In our paper, we apply vertical advertising cooperation in an O2O retailer supply chain, where the cooperation is carried out by the retailer affording a ratio of the manufacturer’s advertising expenditure.

Among the literature listed above, the most similar to our paper is He et al.’s one [4]. Just as the literature, we no longer emphasize the price competition of the two channels and combine the online and offline channels into one unit to compete with the manufacturer. On the other hand, we analyze the impacts of the advertising lag effect and reference quality effect on the supply chain, which is not considered in [4].

3. Model Description

In this section, we attempt to formulate a revised NerloveArrow (N-A) model considering an advertising lag time, given the demand function under an O2O environment. For simplicity, we summarize the model parameters and decision variables in Table 1.

3.1. Modified NerloveArrow Model. Nerlove and Arrow [23] developed the classic NerloveArrow (N-A) model, where the goodwill evolves over time according to the following dynamic equation:

\[ \dot{G}(t) = u(t) - \psi G(t), \] (1)

where \( u(t) \) is the instantaneous advertising effort level and \( \psi \) is a constant proportional rate at which depreciation occurs. This implies that advertising effort leads to an instantaneous increasing of \( G(t) \). In reality, the brand goodwill reacts to advertising gradually. It is stored for a while and then reactivates a period of time after product exposure. We call this phenomenon the advertising lag effect. It is determined by the behavior process and cycle of the consumer from receiving advertising information to remembering the brand. In our paper, we use a modified N-A dynamic goodwill evolving model to capture the leftover effect of the advertising effort to goodwill growth [8, 18] as follows:

\[
\begin{cases}
  \dot{G}(t) = \beta u(t-d) - \psi G(t), & t \geq 0, \\
  G(t) = G_0, & -d \leq t \leq 0.
\end{cases}
\] (2)

Here, \(-d\) refers to the number of periods before the product enters the market. A larger \( d \) means a higher advertising lag effect. The first equation in (2) is a delay differential equation. It states that the brand goodwill growth depends on not only the goodwill level at time \( t \) but also the advertising effort \( d \) periods ago. The second equation shows that, during the \( d \) periods, before the product goes into the market, the manufacturer invests in advertising, establishes the product image, and generates the initial brand goodwill \( G_0 \) for the market. Regarding the advertising lag time, Wansink and Ray [24] suggested a time delay of three months. Aravindakshan and Naik [16] set the advertising lag time at 52 weeks. Leone [25] found that, for 90% of the products, the interval of advertising lag time is 8.8 months. In our paper, we take \( d \) in \([0, 20]\) weeks.

3.2. The Demand Function. In our model, we consider the supply chain scene with one manufacturer and one retailer in a dynamic setting of infinite horizon. For convenience, we assume that the market demand is equal to the sales quantity and adopts an additive separable demand function format without price competition. First, at time \( t \), the manufacturer decides the wholesale price \( (w(t)) \) and wholesales the products to the retailer. As a response, the retailer sets the retail price \( (p(t)) \) to maximize his profit. Then, the retailer sells his product through the offline channel (physical store) and online channel (network sale) to the market. To obtain the optimal response to the retail price, the manufacturer decides on the instantaneous national advertising effort level \( u(t) \) and product quality level \( q(t) \). Our scenario operates in multiple modes in reality. For example, Haier wholesales its products to retailers, such as Suning and Guomei, and the retailers sell the products to the market through online and offline channels. Mead Johnson wholesales milk powder to Leyou, and Leyou sells it to the market through Leyou’s physical stores and online stores.

We formulate the latent market demand as \( S = a + \theta G \), where \( a \) denotes the basic market demand when the brand goodwill is zero. The parameter \( \theta > 0 \) captures the positive impact of brand goodwill on the latent demand [4, 17]. In the O2O retailer supply chain, once a consumer visits some offline store, regardless of whether he/she eventually buys the product through the online or offline channel, his/her buying behavior is impacted by the reference quality effect. We state that he/she belongs to offline demand consumers. Denote by \( \lambda \) a latent consumer’s propensity level to reach offline physical stores. Then, the latent demands for the offline and online channels are \( S_{\text{off}} = \lambda S \) and \( S_{\text{on}} = (1-\lambda)S \). This latent online and offline demand function appeared in the dual channel or O2O environment literatures, such as that by Ji et al. [3]; Xie et al. [12]; and Govindan and Malomfalean [20].

Regardless of whether the retailer sells a product through the offline or online channel, the sales quantity decreases with the increasing of the retail price. Suppose the impact factor is \( b/2 \) under the two channels. On the other hand, the offline sales volume is impacted by the consumers’ reference quality effect [4–6]. We give the sales under offline channel as follows:

\[ S_{\text{off}} = S_{\text{off}}^* - \frac{b}{2}p + k(q-rq) = \lambda (a + \theta G) - \frac{b}{2}p + k(q-rq). \] (3)
Table 1: Notations.

| Symbol | Description |
|--------|-------------|
| a > 0  | Parameter, baseline market demand |
| t ≥ 0  | Parameter, time t |
| β > 0  | Parameter, advertising efficiency to goodwill accumulation |
| γ > 0  | Parameter, goodwill decay proportional rate |
| q > 0  | Parameter, advertising lag time |
| ρ > 0  | Parameter, discount rate |
| θ > 0  | Parameter, effectiveness of goodwill to sales |
| k > 0  | Parameter, the offline consumer’s sensitivity factor to reference quality effect |
| ξ > 0  | Parameter, goodwill’s effectiveness to the reference quality |
| λ ∈ [0, 1] | Parameter, a latent consumer’s propensity to reach offline physical stores |
| b > 0  | Parameter, the influencing factor of retail price on sales |
| σ ∈ [0, 1] | Exogenous variable, advertising cost sharing ratio |
| S_{O2O}(t) | Variant, the sales volume under O2O model at time t |
| r_q(t) | Variant, the consumer’s reference quality at time t |
| G(t) | State variant, the brand goodwill at time t |
| p(t) | Control variant, the retail price at time t |
| w(t) | Control variant, the wholesale price at time t |
| u(t) | Control variant, manufacturer’s advertising effort level at time t |
| q(t) | Control variant, product quality level at time t |
| R(t) | The reference quality effect at time t |
| M = θ − kξ | Parameter |

Denoted by $R = k(q - r_q)$, it represents the consumers’ reference quality effect on the offline channel, that is, the differential offline sale at quality $q$ due to the reference quality being $r_q$ instead of the actual quality $q$. That is, $R = S_{\text{off}}(q, r_q) - S_{\text{off}}(q, q)$. When a product’s current observed quality $q$ is larger (less) than its reference quality $r_q$, $S_{\text{off}}$ is positively (negatively) related to the reference quality effect, and we call it the positive (negative) reference quality effect. $k > 0$ refers to the consumers’ sensitivity factor to the reference quality effect. The higher the $k$ value, the more sensitive the offline consumers are to the reference quality effect. In this paper, we assume $(\partial R/\partial G) > 0$, i.e., the real product quality increases more quickly than the reference quality along with the increasing of the brand goodwill. In fact, a larger branded firm has the ability to improve the product quality along with the increasing of the brand goodwill. In this paper, we assume $s > 0$ and the consumers’ sensitivity factor to the reference quality effect would not too large. In the Section 4, we determine the Stackelberg game equilibrium solutions under the following different modes: with and without vertical advertising cooperation.

4. Model Solutions and Analysis

Equations (2) and (6) tell us that the sales volume can be improved (increasing channel members’ profit) through the increasing of the advertising effort and product quality levels. However, increasing the investments in quality and advertising also augment the cost (reducing profit) for the manufacturer. Therefore, the manufacturer’s choices regarding the product quality level, advertising effort level, and wholesale price need to be weighed. To respond to the manufacturer’s decision, the retailer needs to give the optimal retail price to optimize his profit. A well-designed model helps to obtain the best decisions for the supply chain members. We assume that all the demand and cost information is known to both the manufacturer and the retailer. For convenience, we use the subscripts “ss” and the superscript “*” to denote the manufacturer and the retailer, respectively, and we use the subscript “on” and the superscript “*” to denote the steady-state values and equilibrium solutions of the variables, respectively.
4.1. Non-Advertising-Cooperative Mode. In this section, we consider the optimal strategies taken by the manufacturer (the Stackelberg leader) and the retailer (the Stackelberg follower) in the case that both sides seek to answer the profit maximum control problem without advertising cooperation. We use the superscript "nc" to denote the non-cooperative mode, and the manufacturer’s and retailer’s instantaneous profits are given by the following equation:

\[ \pi_{m}^{nc}(t) = wS_{O2O} - \frac{u^2}{2} - q^2, \]

\[ \pi_{r}^{nc}(t) = (p - w)S_{O2O}. \]  

(7)

The manufacturer and the retailer in the Stackelberg game must solve the following infinite horizon problems to maximize their profits. For the retailer, the optimal problem is as follows:

\[ V_r^{nc} = \max_{\rho} \int_{0}^{\infty} e^{-\rho t} \pi_{r}^{nc}(t)dt \]

\[ = \max_{\rho} \int_{0}^{\infty} e^{-\rho t} \left( (p - w)(a - bp + kq + MG) \right) dt, \]

subject to (2), where \( \rho > 0 \) denotes the discount rate, which is an exogenous constant and is determined by the cost of capital. For the manufacturer, the optimal problem is as follows:

\[ V_m^{nc} = \max_{w,q,u} \int_{0}^{\infty} e^{-\rho t} \pi_{m}^{nc}(t)dt \]

\[ = \max_{w,q,u} \int_{0}^{\infty} e^{-\rho t} \left( w[a - bp + kq + MG] - \frac{u^2}{2} - q^2 \right) dt, \]

subject to equations (2) and (8). We use backward induction to solve the corresponding dynamic game between the manufacturer and the retailer. Specifically, we first solve the retailer’s problem for the given wholesale price and then solve the manufacturer’s problem by taking the retailer’s reaction function into consideration.

**Lemma 1.** When the retailer sells a product via the O2O model without advertising cooperation, for a given goodwill at time \( t \), the equilibrium wholesale price, retail price, product quality, and reference quality effect are as follows:

\[ w^{nc}(G) = \frac{4(a + MG)}{8b - k^2}, \]

\[ p^{nc}(G) = \frac{6(a + MG)}{8b - k^2}, \]

\[ q^{nc}(G) = \frac{(a + MG)k}{8b - k^2}, \]

\[ R^{nc}(G) = k \frac{ak + G(8k - 8b\zeta)}{8b - k^2}. \]

(10)

According to our parameter assumption \( 8b - k^2 > 0 \) and \( \theta k - 8b\zeta > 0 \), we have \( R(t) > 0 \), i.e., the product quality level is larger than the consumers’ reference quality in our paper. Lemma 1 illustrates the following results:

(1) \( (\partial (w^{nc}(G))/\partial G > 0, (\partial (q^{nc}(G))/\partial G > 0, (\partial (p^{nc}(G))/\partial G > 0, \text{ and } (\partial (R^{nc}(G))/\partial G > 0 \text{ indicate that the optimal wholesale price, product quality, retail price, and reference quality effect increase as the brand goodwill increases. In fact, a larger band goodwill means that the consumer will have a higher expected quality. This urges the manufacturer to invest more in product quality to improve the reference quality effect, which stimulates the market demand. The increasing market demand induces the manufacturer to raise the wholesale price. As a response, the retailer sets at a higher retail price.} \)

(2) \( (q^{nc}(G)/p^{nc}(G)) = (k/6) \) means when consumers are more sensitive to the reference quality effect, they will get cost-effective product. For consumers with the same reference quality sensitivity factor \( k \), the higher the quality, the higher the price. This conclusion is the same as that stated in Gavious and Lowengart [6]. \( (p^{nc}(G)/w^{nc}(G)) = (3/2) \) indicates that the optimal ratio for the retail price to the wholesale price is 3/2.

Taking \( w^{nc}(G), q^{nc}(G), \) and \( p^{nc}(G) \) into (9), we have the following equation:

\[ V_m^{nc} = \max_{u} \int_{0}^{\infty} e^{-\rho t} \left( \frac{(a + MG)^2}{8b - k^2} - \frac{u^2}{2} \right) dt, \]

s.t. (2) and (8).

The current Hamiltonian value function related to the optimal control (12) can be constructed as follows:

\[ H_m(t) = \frac{(a + MG)^2}{8b - k^2} - \frac{u^2}{2} + \nu(t)(\beta u(t - d) - \psi G). \]

(13)

Here, \( \nu(t) \) is a co-state variable (shadow variable) associated with the evolution of the state equation (2).

Set \( G^{nc}_{ss} = ((2aMG^2)/(-2M^2bp + (p + \psi)w^2d(8b - k^2))), \)

\( \lambda^{nc}_{2} = (p - \sqrt{\Delta^{nc}})/2, \) and \( \Delta^{nc} = p^2 - 4((2M^2bp)/((8b - k^2)w^2d)) - \psi(p + \psi); \) then, we have equation (14):

**Lemma 2.** Under the retailer’s O2O supply chain without advertising cooperation, the advertising lag time satisfies the following equation:

\[ d > \frac{1}{\rho} \ln \frac{2M^2b^2}{(8b - k^2)\psi(p + \psi)}. \]

(14)

(1) The manufacturer’s optimal dynamic equilibrium wholesale price, product quality level, and advertising effort level at time \( t \) are as follows:
\[ w^{\ast \text{nc}}(t) = \frac{4M(G_0 - G_{\text{nc}}^\text{ss})}{8b - k^2} e^{\lambda_2^\text{nc} t} + \frac{4a + 4MG_{\text{ss}}}{8b - k^2}, \]  
(15)

\[ q^{\ast \text{nc}}(t) = \frac{kM(G_0 - G_{\text{nc}}^\text{ss})}{8b - k^2} e^{\lambda_2^\text{nc} t} + \frac{k(a + MG_{\text{ss}})}{8b - k^2}, \]  
(16)

\[ u^{\ast \text{nc}}(t) = \frac{G_{\text{ss}}^\text{nc} \psi}{\beta} + \frac{G_{\text{ss}}^\text{nc} \psi}{\beta} t \geq 0, \]  
(17)

\[ u^{\ast \text{nc}}(t) = \frac{G_{\text{ss}}^\text{nc} \psi}{\beta} + \frac{G_{\text{ss}}^\text{nc} \psi}{\beta t} \leq 0, \]  
(18)

Under these optimal control variables, the goodwill moving trajectory is positive with the following formulation:

\[ G_{\text{nc}}^\text{ss}(t) = (G_0 - G_{\text{nc}}^\text{ss}^\ast) e^{\lambda_2^\text{nc} t} + G_{\text{nc}}^\text{ss}. \]  
(19)

Substituting \( G_{\text{nc}}^\ast(t) \) into \( R_{\text{nc}}^\ast(G) \), we have the equilibrium reference quality effect as follows:

\[ R_{\text{nc}}^\ast(t) = k \left( \theta k - 8b \xi \right) (G_0 - G_{\text{nc}}^\text{ss}^\ast) e^{\lambda_2^\text{nc} t} \]  
+ \frac{a k^2 + (\theta k - 8b \xi) k G_{\text{ss}}^\text{nc} \psi}{8b - k^2}. \]  
(20)

Substituting equations (17) and (19) into (12), we obtain the manufacturer’s maximum profit as follows:

\[ V_{\text{nc}}^\ast = \left( M^2 (G_0 - G_{\text{nc}}^\text{ss})^2 \right) + \frac{8 \left( G_0 - G_{\text{nc}}^\text{ss} \right)^2 M^4 \beta^2 e^{\left( \lambda_2^\text{nc} - \rho \right)^2 d} \left( 2 \psi + \rho + \sqrt{\Delta_{\text{nc}}} \right)^2}{8b - k^2} \]  
\[ + \frac{1}{\rho - 2 \lambda_2^\text{nc}} \]  
\[ + \left( 2M (G_0 - G_{\text{nc}}^\text{ss}) \left( a + MG_{\text{ss}}^\text{nc} \right) \right) \right) \frac{1}{\rho - \lambda_2^\text{nc}} \]  
\[ + \left( \frac{a + MG_{\text{ss}}^\text{nc}}{8b - k^2} - \frac{\psi G_{\text{ss}}^\text{nc}}{8b - k^2} \right) \frac{1}{\rho - \lambda_2^\text{nc}}. \]  
(23)
The dynamic investment into advertising can be formulated as follows. Before the product enters into market, the manufacturer makes an effort to advertise at a constant level $u_0$. This evolves into the initial brand goodwill $G_0$. When the product enters to the market, if $G_0$ is lower (higher), the manufacturer increases (decreases) the advertising investment dynamically. This leads to an increase (decrease) in the brand goodwill to its steady state $G^c_{ss}$. Then, the manufacturer adjusts the advertising effort to its steady state $(\psi G^c_{ss}/\beta)$.

4.2. Vertical Cooperative Advertising Mode. From literature, we know that channel cooperation may be an effective method to improve the channel performance in most instances. Does it still work in our scenario? Based on this problem, we consider the vertical advertising cooperation mode to give the optimal decision variables of the supply chain members.

In this subsection, vertical cooperative advertising is carried out as follows: the retailer provides a ratio $\sigma$ ($\sigma \in (0,1)$) of manufacturer’s advertising expenditure. We use superscript “cc” to denote the cooperation mode. The manufacturer and the retailer’s instantaneous profits are given by the following equation:

$$\pi^c_m(t) = wS_{o2o} - (1 - \sigma)\frac{u^2}{2} - q^2,$$

$$\pi^c_r(t) = (p - w)S_{o2o} - \sigma\frac{u^2}{2}.$$  

(24)

We specify the optimal control problem of the retailer with the infinity horizon as follows:

$$V^c = \max_p \int_0^{\infty} e^{-\rho t} \pi^c_r(t) \, dt$$

$$= \max_p \int_0^{\infty} e^{-\rho t} \left( (p - w)(a - bp + kq + MG) - \frac{\sigma u^2}{2} \right) \, dt,$$

(25)

subject to (2). The optimal control problem for the manufacturer is as follows:

$$V^m = \max_{w,p,u} \int_0^{\infty} e^{-\rho t} \pi^c_m(t) \, dt$$

$$= \max_{w,p,u} \int_0^{\infty} e^{-\rho t} \left( w(a - bp + kq + MG) - \frac{(1 - \sigma)u^2}{2} - q^2 \right) \, dt,$$

(26)

subject to (2) and (25).

**Lemma 3.** When the retailer sells a product via the O2O model with advertising cooperation, for a given goodwill at time $t$, the equilibrium retail price, wholesale price, product quality level, and reference quality effect are as follows:

$$p^c(G) = \frac{6(a + MG^c)}{8b - k^2},$$

$$q^c(G) = \frac{(a + MG^c)k}{8b - k^2},$$

$$w^c(G) = \frac{4(a + MG^c)}{8b - k^2},$$

$$R^c(G) = \frac{ak^2 + k(\theta b - 8b^2)G^c}{8b - k^2}.$$  

(27, 28)

According to our parameters assumption $8b - k^2 > 0$ and $\theta b - 8b^2 > 0$ in Section 4.1, the above variables are all positive. Substituting (27) into (26), we obtain the following equation:

$$V^c_m = \max_{\sigma} \int_0^{\infty} e^{-\rho t} \left( \frac{(a + MG^c)^2}{8b - k^2} - \frac{(1 - \sigma)u^2}{2} \right) \, dt,$$

(29)

subject to (2) and (25).

The current Hamiltonian value function related to the optimal control (29) can be constructed as follows:

$$H^c_m = \frac{(a + MG^c)^2}{8b - k^2} - \frac{(1 - \sigma)u^2}{2} + v(t)(\beta u^c(t - d) - \psi G^c).$$

(30)

We set $G^c = ((2aM\beta^2)/(-2M^2\beta^2 + (\rho + \psi)(1 - \sigma)\psi\omega + (8b - k^2)^2)), \lambda^c = (\rho - \sqrt{\Delta^c})/2$, and $\Delta^c = \rho^2 + 4 ((2M^2\beta^2)/(a - 1)(8b - k^2))\omega + (\rho + \psi)$. Just as the reason in non-cooperative mode (see the proof of Lemma 2), to ensure positive variables, we suppose that $d > \Delta^c = (1/\rho)\ln((2M^2\beta^2)/(8b - k^2)(1 - \sigma)\psi(\rho + \psi)))$. We obtain the following conclusions.

**Lemma 4.** Under the vertical advertising cooperation mode, the following are studied:

(1) The optimal dynamic equilibrium advertising effort level, wholesale price, and product quality level for the manufacturer at time $t$ are as follows:
\[ u^*_{cc}(t) = \frac{4(G_0 - G_{ss}^{cc})M^2 \sigma e^{\psi} e^{\beta(t+d)}}{(8b - k^2) (2\psi + \rho + \sqrt{\Delta^{cc}}) (1 - \sigma)} + \frac{\psi G_{ss}^{cc}}{\beta}, \quad t \geq 0, \]

\[ u^*_{cc}(t) = \frac{4(G_0 - G_{ss}^{cc})M^2 \sigma e^{\psi} e^{\beta d}}{(8b - k^2) (2\psi + \rho + \sqrt{\Delta^{cc}}) (1 - \sigma)} + \frac{\psi G_{ss}^{cc}}{\beta}, \quad -d \leq t \leq 0, \]

\[ w^*_{cc}(t) = \frac{4M(G_0 - G_{ss}^{cc})}{8b - k^2} e^{\beta t} + \frac{4a + 4MG_{ss}^{cc}}{8b - k^2}, \]

\[ q^*_{ss}(t) = \frac{kM(G_0 - G_{ss}^{cc})}{8b - k^2} e^{\beta t} + \frac{k(a + MG_{ss}^{cc})}{8b - k^2}, \]

where \( \lambda_{2}^{cc} < 0 \). The optimal dynamic equilibrium retail price for the retailer at time \( t \) is as follows:

\[ p^*_{cc}(t) = \frac{6M(G_0 - G_{ss}^{cc})}{8b - k^2} e^{\beta t} + \frac{6a + 6MG_{ss}^{cc}}{8b - k^2}. \]  \tag{32}

Under these optimal control variables, the trajectory of the brand goodwill is as follows:

\[ G^{cc}(t) = (G_0 - G_{ss}^{cc}) e^{\beta t} + G_{ss}^{cc}. \]  \tag{33}

Substituting \( G^{cc}(t) \) into \( R^{cc}(G) \), we have the equilibrium reference quality effect, as follows:

\[ R^{cc}(t) = \frac{k(\theta k - 8b\xi)(G_0 - G_{ss}^{cc})e^{\beta t}}{8b - k^2} \]

\[ + \frac{ak^2 + k(\theta k - 8b\xi)G_{ss}^{cc}}{8b - k^2}. \]  \tag{34}

(2) The optimal steady-state wholesale price, retail price, advertising effort level, and product quality level for the supply chain members are as follows:

\[ w_{ss}^{cc} = \frac{4a + 4MG_{ss}^{cc}}{8b - k^2}, \]

\[ p_{ss}^{cc} = \frac{6a + 6MG_{ss}^{cc}}{8b - k^2}, \]

\[ u_{ss}^{cc} = \frac{\psi G_{ss}^{cc}}{\beta}, \]

\[ q_{ss}^{cc} = \frac{k(a + MG_{ss}^{cc})}{8b - k^2}. \]

Under this steady-state solution, the goodwill of the product is stable to \( G_{ss}^{cc} \) and the steady-state reference quality effect and market sales are \( R_{ss}^{cc} = ((ak^2 + k(\theta k - 8b\xi)G_{ss}^{cc})/(8b - k^2)) \) and \( S_{ss}^{cc}(t) = ((2b(a + MG_{ss}^{cc}))/(8b - k^2)) \), respectively.

**Proof.** See the proof in Proof A5. \( \square \)

The retailer’s maximum profit under the cooperation mode is as follows (see the proof in Proof A6):
Substituting equations (31) and (33) into (29), we obtain the manufacturer’s maximum profit under the cooperation mode, as follows:

\[
V_{ss}^{cc} = \left(\frac{a + M\epsilon_{ss}^{cc}}{8b - k^2} \right)^2 \frac{1}{\rho} \left(1 - \sigma\right)\psi \frac{\sigma_{ss}^{cc} G_{ss}^{cc}}{2\beta^2} + \frac{2M \left(G_0 - G_{ss}^{cc}\right) \left(a + M\epsilon_{ss}^{cc}\right)}{8b - k^2} - 4\left(G_0 - G_{ss}^{cc}\right) M^2 e^{\left(\frac{1}{2} - \rho\right) d} \frac{\sigma_{ss}^{cc} G_{ss}^{cc}}{2\beta^2} \frac{1}{\rho - \lambda_2^2}
\]

When \( \sigma = 0 \), the above results degenerate to those under the non-cooperative mode. To ensure that the static goodwill is positive, the range of the advertising cost sharing ratio is as follows (for the proof, see Proof A7):

\[
0 < \sigma < 1 - \frac{2M^2 \beta^2}{\left(\rho + \psi\right) \psi e^d \left(8b - k^2\right)} = \sigma < 1.
\] (38)

That is, the maximum advertising cost sharing ratio is less than 1.

5. Model Comparison

In this section, we provide a comparison of the supply chain under the two modes through analytic calculation and numerical analysis. Through comparison, we answer the following questions. Is vertical advertising cooperation an effective method in channel performance improvement? How do the advertising lag effect and reference quality effect impact the channel performance under the two modes? How can the manufacturer balance the investments in advertising and product quality when faced with different advertising lag times, differentiated market sensitivities to the reference quality effect, and various advertising cost sharing ratios?

First, we show our choice of parameters. According to our assumption \( 8b - k^2 > 0 \), \( \theta = k\xi > 0 \) and \( k\theta - 8b\xi > 0 \), the range of \( k \) is \( k \in (3.2, 3.45) \). Other parameters can be taken as \( \theta = 0.8, b = 1.6, \) and \( \xi = 0.2 \). The bank discounted rate is taken as \( \rho = 0.03 \) according to common sense. In [18], the goodwill decay proportional rate \( \psi \in [0.05, 0.2] \). In our paper, we take \( \psi = 0.15 \). To make \( G_{ss} > 0 \), we set \( \beta = 0.5 \). The baseline market demand is \( a = 20 \) [27].

We take \( \sigma = 0.3 \) and \( d \in (0, 20) \) with a step width of 3; the values of \( G_{ss}^{nc} \) and \( G_{ss}^{cc} \) are listed as in Table 2. It can be observed that the static goodwill is improved after using advertising cooperation, and the higher the value of \( d \), the smaller the value of \( G_{ss} \). For illustration, we take \( G_0 = 20, k = 3.3 \), and the motion trajectory of \( G_{ss}^{nc} (t) \) with \( d = 4, 7, 10, 13, \) and 16 are depicted in Figure 1. It shows that the brand goodwill increases or decreases exponentially in relation to its stationary value when the initial goodwill \( G_0 \) is smaller or larger than the static goodwill \( G_{ss}^{nc} \). The motion tendency of \( G_{ss}^{cc} (t) \) is similar to \( G_{ss}^{nc} (t) \), so we omit it.

5.1. Motion Tendency of the Variables corresponding to \( t \). For the static valuables under the two modes, we have \( G_{ss}^{cc} > G_{ss}^{nc}, u_{ss}^{cc} > u_{ss}^{nc}, q_{ss}^{cc} > q_{ss}^{nc}, P_{ss}^{cc} > P_{ss}^{nc}, u_{ss}^{cc} > u_{ss}^{nc}, r_{ss}^{cc} > r_{ss}^{nc} \) and \( R_{ss}^{nc} - R_{ss}^{cc} = (k(k\theta - 8b\xi)/(8b - k^2))(G_{ss}^{nc} - G_{ss}^{cc}) < 0 \). These can be explained as follows: under the advertising cooperation mode, the manufacturer increases the advertising investment, then the brand goodwill is enlarged and the consumers’ reference quality is improved. The manufacturer increases the product quality investment to promote the reference quality effect. This induces higher sales and higher prices. We take \( d = 10 \), and obtain the following conclusions through numerical simulation.

Lemma 5

(1) When \( G_{ss}^{nc} < G_0 < G_{ss}^{cc} \), the goodwill (retail price, product quality level, advertising effort level, wholesale price, sales, and reference quality effect) decreases along with time under the non-cooperation mode, but there is a reverse trend in the cooperation mode. For an illustration, see Figure 2.

(2) When \( G_0 > G_{ss}^{cc} > G_{ss}^{nc} \), the goodwill (retail price, quality level, advertising effort level, wholesale price, sales, and reference quality effect) decreases in time under the two modes, and the decline rate in the non-cooperative mode is much faster than that in the cooperative mode (for an illustration, see Figure 3). When \( G_0 < G_{ss}^{nc} < G_{ss}^{cc} \), the results are opposite (for an illustration, see Figure 4).

Proof. See the proof in Proof B1. □

When \( G_{ss}^{nc} < G_0 < G_{ss}^{cc} \), because the initial goodwill is smaller than its static value under the cooperation mode, the manufacturer invests more in advertising to boost it, leading to a higher consumer reference quality. Then, she invests more in the product quality to improve the reference quality...
effect. This expands the market demand, inducing higher prices. However, the results under the non-cooperation mode are opposite. When \(G_0 > G_{ss} > G_{nc}\), the initial goodwill is higher, the manufacturer invests less in advertising, leading to the static goodwill decreasing. However, under the cooperation mode, the manufacturer would like to invest more than under the non-cooperation mode, because the retailer affords part of the advertising fee.

From Lemma 5, we conclude the following:

1. When \(G_0\) is lower (that is \(G_0 < G_{ss}\)), the goodwill (retail price, quality level, advertising effort level, wholesale price, sales, and reference quality effect) increases with time to its steady-state value under the two modes. However, the growth rate is faster under the cooperative than under the non-cooperative mode.

2. When \(G_0\) is higher (that is \(G_0 > G_{ss}\)), the goodwill (retail price, quality level, advertising effort level, wholesale price, sales, and reference quality effect) decreases with time to its steady-state value under the two modes. However, the decline rate is slower under the cooperative than under the non-cooperative mode.

From the above analyses, we obtain the following conclusion.

**Proposition 1.** Whether the initial goodwill is smaller or larger than its static goodwill, the stationary values of the state and control variables and the reference quality effect are all improved after advertising cooperation. This illustrates that the supply chain performance is improved and the competitiveness of the product is enhanced after cooperation.

5.2. Sensitivity Analysis of the Variables corresponding to \(\sigma\). Now, we give the variant rates of the static variables relative to the advertising cost sharing ratio (for the proof, see Proof B2).

**Proposition 2.** \((\partial G_{nc}^{ss}/\partial \sigma) > 0, (\partial p_{nc}^{ss}/\partial \sigma) > 0, (\partial w_{nc}^{ss}/\partial \sigma) > 0, (\partial q_{nc}^{ss}/\partial \sigma) > 0, \text{and} (\partial r_{nc}^{ss}/\partial \sigma) > 0,\) along with the increase in the participate ratio, imply that we obtain a higher static brand goodwill (advertising effort level, product quality level, reference quality level, wholesale price, and retail price).

This is because when the advertising cost sharing ratio is larger, the manufacturer increases the advertising investment. This expands the brand goodwill and improves the consumers’ reference quality. The manufacturer invests more into the product quality to cater to the consumers’ psychological quality, thus causing the sales to improve. The increased sales lead to higher wholesale and retail prices.

Strategically, it is only beneficial to the manufacturer benefits in the cooperative mode (Figure 5(a) the curve of \(v_{m}^{cc}\)). Does larger advertising cost sharing ratio mean better? Let us use numerical simulation to examine it from the perspective of the retailer.
It can be observed from Figure 5(a) that when the advertising cost share ratio $\sigma \in (0, 0.823)$, the retailer’s profit is larger than that when $\sigma = 0$. The retailer’s profit obtains its maximum value at $\sigma \approx 0.787$. When $\sigma > 0.823$, the retailer’s profit is lower than that under the non-cooperation mode and is decreasing sharply. This can be explained by the fact that, along with the increase of the advertising cost share ratio, the manufacturer invests more into advertising. This results in a higher advertising participation fee to the retailer. When $\sigma$ is large enough, the increased sales revenue cannot offset the additional advertising participation fee; then, the retailer’s profit decreases and may even become negative.

5.3. The Sensitivity Analysis of the Variables corresponding to $d$. What is the impact of the advertising lag effect on the supply chain? Under the non-cooperation mode, we obtain the following conclusion.

**Proposition 3**

1. $(\partial G_{nc}^{pc}/\partial d) < 0$, $(\partial u_{nc}^{pc}/\partial d) < 0$, $(\partial w_{nc}^{pc}/\partial d) < 0$, $(\partial p_{nc}^{pc}/\partial d) < 0$, and $(\partial S_{O2O}^{nc}/\partial d) < 0$ implies the static brand goodwill, advertising effort level, wholesale price, retail price, and market sales are all decreasing with respect to $d$ under the non-cooperation mode.
\( (\partial q / \partial d) < 0, (\partial q / \partial d) < 0, \) and \( (\partial R_v / \partial d) < 0 \) mean that the product quality and consumers’ reference quality are all decreasing along with the increasing of the advertising lag time. However, the former decreases more quickly than the latter under the non-cooperation mode.

\( (\partial V_m / \partial d) < 0 \) (see Proof B3) and \( (\partial V_m / \partial d) < 0 \) (see the curve \( V_m \) in Figure 6(b)) mean that the manufacturer’s and the retailer’s profits decrease along with the increasing of the advertising lag time under the non-cooperation mode.

Proposition 3 illustrates that the supply chain members tend to the immediate effect of advertising. In fact, the longer advertising lag time hinders the manufacturer’s enthusiasm for investing in advertising, thus reducing the brand goodwill and the consumers’ reference quality. With the decline of the reference quality, the product quality decreases more rapidly. These lead to sales dropping off. To increase the sales, the manufacturer sets a lower wholesale price. In response, the retailer sets a lower retail price. The vicious circle continues, leading to a decline in the earnings of both sides.
For the cooperation mode, we take \( \sigma = 0.2, 0.4, \) and 0.6 (which are all in its feasible field) to show the change of profits and variables corresponding to \( d \) (Figures 6(a)–6(f)). Because \( q_{ss} = (k/6)p_{ss} \) and \( w_{ss} = (2/3)p_{ss} \), the motion trends of \( q_{ss} \) and \( w_{ss} \) are similar to those of \( p_{ss} \), and we omit them.

**Proposition 4**

1. **Corresponding to the same lag time, the supply chain members' profits, the static brand goodwill, advertising effort level, product quality level, reference quality effect, and prices are all increased after advertising cooperation, and the higher the advertising cost sharing ratio, the faster the growth.**

(2) **Whether under the non-cooperation or cooperation modes, the supply chain members’ profits (the static advertising effort level, product quality level, price, reference quality effect, and brand goodwill) are all decreasing along with \( d \), and the descent is more rapid under the cooperation than under the non-cooperation mode. Furthermore, the higher the advertising cost sharing ratio, the more rapid the descent.**

Proposition 4 illustrates the advertising lag effect has a negative impact on the supply chain under the two modes, and the negative impact increases after advertising cooperation; i.e., supply chain members’ profits (the static advertising effort level, product quality level, price, reference quality effect, and brand goodwill) are all decreasing along with \( d \), and the descent is more rapid under the cooperation than under the non-cooperation mode. Furthermore, the higher the advertising cost sharing ratio, the more rapid the descent.
quality effect, and brand goodwill) decline more quickly with than without vertical advertising cooperation. Additionally, the higher the advertising cost sharing ratio, the stronger the negative impact. This illustrates that the vertical advertising cooperation strategy would not alleviate but would worsen the negative impact of the advertising lag effect on the supply chain.

5.4. The Sensitivity Analysis of the Variables corresponding to \( k \). What is the impact of the reference quality effect on the supply chain? That is, what are the changes of the profits and variables corresponding to the differentiated sensitivity factor (\( k \)) to the reference quality effect? For the non-cooperation mode, we obtain the following conclusion (see the proof in Proof B4).

**Proposition 5**

1. \((\partial G_{nc}^{w_s}/\partial k) > 0, (\partial u_{nc}^{w_s}/\partial k) > 0, (\partial p_{nc}^{w_s}/\partial k) > 0, \) and \((\partial R_{nc}^{w_s}/\partial k) > 0\) imply that the static brand goodwill, advertising effort level, wholesale price, and retail price increase with respect to \( k \) under the non-cooperation mode. \((\partial q_{nc}^{w_s}/\partial k) > 0, (\partial v_{nc}^{w_s}/\partial k) > 0, \) and \((\partial R_{nc}^{w_s}/\partial k) > 0\) mean that the product quality and reference quality are both increasing along with \( k \). However, the former increases more quickly than the latter.

2. The curves of \( v_{nc}^{w_s} \) and \( v_{mc}^{w_s} \) in Figure 5(b) indicate that the retailer’s profit increases and the manufacturer’s profit is concave along with the increasing of the consumers sensitivity factor to the reference quality effect.

The above result states that when consumers are more sensitive to reference quality effect, the manufacturer invests more to improve the product quality. At the same time, he increases the product advertising. This increases the brand goodwill and the consumers’ reference quality. This advertising strategy is in line with the result of Zhang et al. [19] in which they proved that the larger the reference price, the more advertising should be invested. However, the real product quality grows more quickly than the consumers’ reference quality (for an illustration, see Figure 7(d)). This is because when the manufacturer observes an increasing of consumers’ sensitivity to the reference quality effect, she invests much more into product quality improvements; thus, the reference quality effect increases. These result in a growth of sales. Then, the wholesale price and retail price increase, and the retailer earns more. However, the manufacturer has to afford the quality improvement cost, so her profit decreases first and then increases.

For the cooperation mode, we take \( \sigma = 0.2, 0.4, \) and \( 0.6 \) to show the changes of profits and variables corresponding to \( k \) (see Figures 7(a)–7(f)). Because \( w_{ss} = (2/3)p_{ss} \), the motion trend of \( w_{ss} \) is similar to that of \( p_{ss} \), and we omit it.

**Proposition 6**

1. Corresponding to the same sensitivity factor \( k \), the supply chain members’ profits, the static brand goodwill, advertising effort level, reference quality effect, price, and product quality level are all increased after advertising cooperation, and the higher the advertising cost sharing ratio, the faster the growth.

2. Whether under the non-cooperation or cooperation mode, the supply chain members’ profits (the static advertising effort level, price, reference quality effect,
Figure 6: Continued.
Figure 6: Variables corresponding to \( d \) with different \( \sigma \): \( \beta = 0.5, \psi = 0.15, G_0 = 25, \rho = 0.03, \theta = 0.8, b = 1.6, a = 20, \xi = 0.2, \) and \( k = 3.3 \). 
(a) \( v_r \leftrightarrow d \). (b) \( v_m \leftrightarrow d \). (c) \( G_{ss} \leftrightarrow d \). (d) \( R_{ss} \leftrightarrow d \). (e) \( u_{ss} \leftrightarrow d \). (f) \( p_{ss} \leftrightarrow d \).

Figure 7: Continued.
Figure 7: Continued.
brand goodwill, and product quality level) are increasing along with $k$, and the growth is faster under the cooperation than under non-cooperation mode. Additionally, the higher the advertising cost sharing ratio, the faster the growth.

The above analysis shows a different result than with the advertising lag effect, i.e., the reference quality effect has a positive impact on the supply chain.

5.5. The Optimal Investment Strategy for the Manufacturer. Denote by $\Omega = (c_u/c_q) = ((u^2/2)/q^2)$ the ratio of the advertising expenditure to the quality improvement expenditure. A higher $\Omega$ means that the manufacturer pays more attention to brand advertising relative to quality improvement. It is easy to obtain the following equation:

$$
\Omega_{ss}^{nc} = \frac{c_u^{nc}}{c_q^{nc}} = \frac{u_{ss}^{nc}/2}{d_{ss}^{nc}} = \frac{1}{2} \left( \psi G_{ss}^{nc} (8b - k^2) / \beta k (a + MG_{ss}^{nc}) \right)^2,
$$

$$
\Omega_{ss}^{cc} = \frac{c_u^{cc}}{c_q^{cc}} = \frac{u_{ss}^{cc}/2}{d_{ss}^{cc}} = \frac{1}{2} \left( \psi G_{ss}^{cc} (8b - k^2) / \beta k (a + MG_{ss}^{cc}) \right)^2.
$$

Differentiating $\Omega_{ss}$ with respect to $d$, $k$, and $\sigma$, we have the following conclusion (for the proof, see in Proof B5).

Proposition 7

(1) $(\partial \Omega_{ss}^{nc}/\partial d) < 0$ and $(\partial \Omega_{ss}^{cc}/\partial d) < 0$ imply that with a longer lag time, both the advertising and product quality expenditures decrease, but the manufacturer pays more attention to the product quality improvement under the two modes.

(2) $(\partial \Omega_{ss}^{nc}/\partial \sigma) > 0$ means that along with the increasing of the advertising cost sharing ratio, both the advertising and product quality expenditures increase, but the manufacturer pays more attention to the advertising investment.

(3) $(\partial \Omega_{ss}^{nc}/\partial k) < 0$ and $(\partial \Omega_{ss}^{cc}/\partial k) < 0$ mean that along with the increasing of the consumers sensitivity to the reference quality effect, both the advertising and product quality expenditures increase, but the manufacturer pays more attention to the product quality improvement under the two modes.

(4) $(\partial \Omega_{ss}^{nc}/\partial \sigma) = ((aG_{ss}^{nc} + MG_{ss}^{nc}G_{ss}^{cc})/(aG_{ss}^{cc} + MG_{ss}^{cc}G_{ss}^{nc}))^2 < 1$ implies the manufacturer pays more attention to the brand advertising investment relative to quality improvement investment under the cooperation mode than under the non-cooperation mode.

6. Conclusions

The advertising lag effect is a crucial factor that influences the brand goodwill and the decisions of the supply chain members. In the O2O scene, the consumer’s buying behavior may be affected by the brand goodwill and the retail price, and it also is affected by the reference quality effect during the offline channel. In this paper, we have examined the decision processes faced with the advertising lag effect and reference quality effect in an O2O retailer supply chain in an infinity horizon. Specifically, we develop a Stackelberg differential game that consists of one manufacturer and one retailer. Thereinto, the
manufacturer wholesales products to the retailer, and the retailer sells them to the market through an online store and physical store. Both the online and offline retail prices are assumed to be uniform. To expand the brand goodwill and improve sales, the manufacturer makes efforts toward improving the national advertising and product quality. The main purpose of this paper is to study the impacts of the advertising lag effect and reference quality effect on supply chain and decision-making for supply chain members. We design two scenarios, one is without advertising cooperation; another is with vertical advertising cooperation mode, where the retailer affords a proportion of the advertising expenditure. The main findings of this paper are summarized as follows:

(1) We obtain the optimal equilibrium decisions in two modes. The advertising effort level (product quality level, wholesale price, retail price, reference quality effect, and brand goodwill) increases or decreases exponentially to its steady state decided by whether the initial goodwill is lower or higher than its static value, whether under the cooperative or non-cooperative mode.

(2) The advertising lag effect has a negative impact on supply chain, and the negative impact will be intensified after advertising cooperation. The consumers' reference quality effect has a positive impact on the supply chain, and the positive impact will be intensified after advertising cooperation.

(3) From the analysis, we find that vertical advertising cooperation is effective in improving channel performance. That is, the product competitiveness increases and the supply chain performance is improved after cooperation.

(4) It is optimal for the manufacturer to pay more attention to product quality improvement (advertising) than to advertising (product quality improvement) when the lag time is longer (shorter) or consumers are more (less) sensitive to quality. When the advertising cost sharing ratio is higher, it is optimal for the manufacturer to pay more attention to advertising. The manufacturer pays more attention to advertising investment relative to quality improvement investment under advertising cooperation than under the non-cooperation mode.

In future research, how the brand goodwill evolves dynamically to a differential equation with two delays will be considered. One delay is due to the advertising lag effect and the second is by the consumers’ memory effect to goodwill. Goodwill decays with a lag time would impact the advertising modes, such as even spending policy, blitz policy, and various cyclic pulsing policies would lead to different market demand, decision-making, and profits. The corresponding channel coordination problem needs to be studied. In addition, how advertising lag effect and consumers’ memory effect to goodwill affect other types of supply chain needs to be investigated.

Appendix

A

Proof A1. Observing that the state equation (2) is independent with $p$, we use the first-order condition, differentiating $p$ from $V^{nc}$, then the optimal decision of $p$ for the retailer is as follows:

$$p^{nc}(G, w, q) = \frac{a + MG + kq + wb}{2b}.$$  \hfill (A.1)

Substituting it to the objective function (9) of the manufacturer, we obtain the following problem:

$$V^{nc}_{m} = \max_{w,q,u} \int_{0}^{\infty} e^{-\rho t} \left\{ w \frac{a - wb + kq + MG}{2} - \frac{u^2}{2} - q^2 \right\} dt$$

s.t. $\dot{G}(t) = \beta u(t-d) - \psi G, G(0) = G_0.$ \hfill (A.2)

Observing that the state equation (2) is independent with $w$ and $q$, we use the first-order condition, differentiating $w$ and $q$ from $V^{nc}_{m}$, then, we have the following equation:

$$w^{nc}(q, G) = \frac{a + kq + MG}{2b}, q^{nc}(w, G) = \frac{wk}{4} \Rightarrow w^{nc}(G) = \frac{4(a + MG)}{8b - k^2}, q^{nc}(G) = \frac{k(a + MG)}{8b - k^2}.$$ \hfill (A.3)

Substituting them into (A.1), we have

$$p^{nc}(G) = \frac{6(a + MG)}{8b - k^2}.$$ \hfill (A.4)

Substituting $q^{nc}(G)$ into

$$R^{nc}(q, r^G, G) = k(q^{nc}(G) - \xi G) = k\left(\frac{k(a + MG)}{8b - k^2} - \xi G\right)$$

$$= k\frac{ak + (\theta k - 8b\xi)G}{8b - k^2}.$$ \hfill (A.5)

The proof is completed. □

Proof A2. According to Basin and Rodriguez-Gonzalez [28], we obtain the optimal advertising effort level by solving the first-order condition as follows:

$$\frac{\partial H_m(t)}{\partial u(t)} + e^{-\rho d} \frac{\partial H_m(t)}{\partial u(t-d)} \big|_{t=t+d} = 0 \Rightarrow u(t) = \nu(t + d) = \nu(t + d)\beta e^{-\rho d},$$ \hfill (A.6)

and the co-state equation is as follows:

$$\dot{y}(t) = \rho y(t) - \frac{\partial H_m(t)}{\partial G} = \rho y(t) - \left(\frac{2M(a + MG)}{8b - k^2} - \nu(t)\psi\right).$$ \hfill (A.7)
(A.6) indicates that \( \nu(t + d) > 0 \).
Substituting (A.6) into the state equation (2), we obtain the following:
\[
\dot{G}(t) = \beta \nu(t) \beta e^{-\rho d} - \psi G, \quad (A.8)
\]
with accompanying set of transversality condition \( \lim_{t \to \infty} \nu(t) G(t) e^{-\rho d} = 0 \), and the initial condition \( G(0) = G_0 \). (A.7) combined with (A.8) is a system of binary first-order linear differential equation. Now, we formulate the solution of \( \nu(t) \) and \( G(t) \) as follows.

Let the coefficient matrix be as follows:
\[
A = \begin{pmatrix}
\rho + \psi & -2M^2 \\
8b - k^2 & \beta^2 e^{-\rho d} - \psi
\end{pmatrix}, \quad (A.9)
\]

Then, (A.7) and (A.8) can be written as follows:
\[
\begin{pmatrix}
\dot{\nu}(t) \\
\dot{G}(t)
\end{pmatrix} = A \begin{pmatrix}
\nu(t) \\
G(t)
\end{pmatrix} + \begin{pmatrix}
-2Ma/(8b - k^2) \\
0
\end{pmatrix}, \quad (A.10)
\]
The eigenvalues of \( A \) are \( \lambda_{1,2}^{nc} = (\rho \pm \sqrt{\Delta^{nc}})/2 \); here \( \Delta^{nc} = \rho^2 - 4((2M^2 \beta^2/(8b - k^2)e^{\rho d}) - \psi(\rho + \psi)). \)

To promise (1) \( \lim_{t \to \infty} \nu(t) G(t) e^{-\rho d} = 0 \); (2) \( G(t) > 0 \); (3) \( \lim_{t \to \infty} \mu(t), \lim_{t \to \infty} \omega(t), \lim_{t \to \infty} \varphi(t), \lim_{t \to \infty} \rho(t) \) to be finite, the parameters need satisfy the following: \( \Delta^{nc} > 0, \quad d > \frac{T^{nc}}{\rho} = (1/\rho) \ln (2M^2 \beta^2/(8b - k^2)\psi(\rho + \psi)). \) That is, \( \lambda_{1}^{nc} > 0 \) and \( \lambda_{2}^{nc} < 0 \). From linear algebra theory, we know the following:

\[
\alpha_1 = \left( \frac{2M^2}{\psi + ((\rho - \sqrt{\Delta^{nc}})/2), 8b - k^2} \right)^T, \quad (A.11)
\]
\[
\alpha_2 = \left( \frac{2M^2}{\psi + ((\rho + \sqrt{\Delta^{nc}})/2), 8b - k^2} \right)^T,
\]
are the corresponding eigenvector with respect to \( \lambda_{1,2}^{nc} = (\rho \pm \sqrt{\Delta^{nc}})/2 \) and \( \lambda_{1}^{nc} = (\rho - \sqrt{\Delta^{nc}})/2, \lambda_{2}^{nc} = (\rho + \sqrt{\Delta^{nc}})/2, \) respectively. According to the theory of ordinary differential equation, we have the following:
\[
\begin{pmatrix}
\nu(t) \\
G(t)
\end{pmatrix} = c_1 e^{\lambda_1^{nc} t} \alpha_1 + c_2 e^{\lambda_2^{nc} t} \alpha_2 + H, \quad (A.12)
\]
where \( H \) is a particular solution of (A.10), and it can be taken as the following form:

\[
H = \left( e^{\lambda_1^{nc} t} \alpha_1, e^{\lambda_2^{nc} t} \alpha_2 \right) \int_0^t \left( e^{\lambda_1^{nc} s} \alpha_1, e^{\lambda_2^{nc} s} \alpha_2 \right)^{-1} \begin{pmatrix}
-2Ma/(8b - k^2) \\
0
\end{pmatrix} ds
\]
\[
= \frac{2MaB^2}{(8b - k^2)^2 \Delta^{nc} e^{\rho d}} \begin{pmatrix}
\frac{4M^2 (1 - e^{\lambda_1^{nc} t})}{(2\psi + \rho - \sqrt{\Delta^{nc}}) \lambda_1^{nc}} + \frac{4M^2 (e^{\lambda_1^{nc} t} - 1)}{(2\psi + \rho + \sqrt{\Delta^{nc}}) \lambda_1^{nc}} \\
\frac{(8b - k^2)(1 - e^{\lambda_2^{nc} t})}{\lambda_2^{nc}} + \frac{(8b - k^2)(e^{\lambda_2^{nc} t} - 1)}{\lambda_2^{nc}}
\end{pmatrix}. \quad (A.13)
\]
Hence,\[
y(t) = \left[ c_1 \frac{4M^2}{2\psi + \rho - \sqrt{\Delta}} - \frac{8M^2a\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\left(2\psi + \rho - \sqrt{\Delta}\right)\lambda_1^c}\right] e^{\lambda_1^c t} + \left[ c_2 \frac{4M^2}{2\psi + \rho + \sqrt{\Delta}} + \frac{8M^2a\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\left(2\psi + \rho + \sqrt{\Delta}\right)\lambda_2^c}\right] e^{\lambda_2^c t} + \frac{8M^2a\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\left(2\psi + \rho - \sqrt{\Delta}\right)\lambda_1^c} + \frac{1}{\lambda_2^c - \lambda_1^c} e^{\lambda_2^c t} - \frac{1}{\lambda_2^c - \lambda_1^c} e^{\lambda_1^c t}, \quad (A.14)
\]

Substituting (A.14) into (A.6), because the advertising expenditure cannot be infinity when \( t \to \infty \), the coefficient of \( e^{\lambda_1^c t} \) in \( \nu(t) \) is zero. That is,
\[
c_1 = \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_1^c}, \quad (A.16)
\]

substituting \( c_1 \) into (A.15), we obtain the following:
\[
G(t) = \left[ c_2 \frac{4M^2}{2\psi + \rho - \sqrt{\Delta}} + \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_2^c}\right] e^{\lambda_2^c t} + \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_2^c} \left(1 - \frac{\lambda_2^c}{\lambda_1^c}\right) e^{\lambda_1^c t} \quad (A.17)
\]

By \( G_0 = G(0) \), we have the following:
\[
c_2 = \frac{G_0}{8b - k^2} - \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_2^c} \left(\frac{1}{\lambda_1^c} - \frac{1}{\lambda_2^c}\right)
\]

\[
G(t) = \left( G_0 + \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_2^c} \left(\frac{1}{\lambda_1^c} - \frac{1}{\lambda_2^c}\right)\right) e^{\lambda_2^c t} + \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_2^c} \left(\frac{1}{\lambda_1^c} - \frac{1}{\lambda_2^c}\right) e^{\lambda_1^c t} \quad (A.18)
\]

For convenience, we set
\[
G_{ss}^{nc} = \frac{2Ma\beta^2}{(8b - k^2)^2\sqrt{\Delta}c^d\lambda_1^c} \left(\frac{1}{\lambda_1^c} - \frac{1}{\lambda_2^c}\right) \quad (A.19)
\]

and
\[
G_{nc}(t) = (G_0 - G_{ss}^{nc}) e^{\lambda_2^c t} + G_{ss}^{nc} \quad (A.20)
\]

We take \( c_1 \) and \( c_2 \) into (A.14), then
\[
\nu(t) = \frac{G_0 - G_{ss}^{nc}}{8b - k^2} e^{\lambda_2^c t} + \frac{2M^2}{\psi + \left(\rho + \sqrt{\Delta}\right)/2} + \frac{G_{nc}^{nc} \psi d}{\beta^2} \quad (A.21)
\]

It is easy to check that \( \lim_{t \to \infty} \nu(t) e^{-\beta t} = 0 \) is satisfied.

Substituting (A.21) into (A.6), and (A.20) into (10) and (11), we obtain the optimal advertising effort, wholesale price, product quality level, and advertising effort level for the manufacturer at time \( t \) when \( t \geq 0 \) as follows:
\[
u^{nc} (t) = \left[ \frac{G_0 - G_{ss}^{nc}}{8b - k^2} e^{\lambda_2^c t} + \frac{2M^2}{\psi + \left(\rho + \sqrt{\Delta}\right)/2} + \frac{G_{nc}^{nc} \psi d}{\beta^2} \right] e^{-\beta t} \quad (A.22)
\]
When $-d \leq t \leq 0$, set
\[
 u^{\text{nc}}(t) = \left( G_0 - G^\text{nc}_{\text{ss}} \right) e^{- \rho d} \frac{4M^2}{(8b - k^2)} \left( 2\psi + T^\text{nc} \right) e^{\psi d} + \frac{G^\text{nc}_{\text{ss}} \psi}{\rho} = u^{\text{nc}}_0,
\]  
(A.23)
that is, a constant.
\[
w^{\text{nc}}(t) = \frac{4M (G_0 - G^\text{nc}_{\text{ss}})}{8b - k^2} e^{\psi d} t + \frac{4a + 4MG^\text{nc}_{\text{ss}}}{8b - k^2},
\]  
(A.24)
\[q^{\text{nc}}(t) = \frac{kM (G_0 - G^\text{nc}_{\text{ss}})}{8b - k^2} e^{\psi d} t + \frac{k(a + MG^\text{nc}_{\text{ss}})}{8b - k^2}.
\]  
We set
\[u^\text{nc}_{\text{ss}} = \frac{G^\text{nc}_{\text{ss}} \psi}{\rho},
\]
\[u^\text{nc}_{\text{ss}} = \frac{4a + 4MG^\text{nc}_{\text{ss}}}{8b - k^2}.
\]  
(A.26)
\[q^\text{nc}_{\text{ss}} = \frac{k(a + MG^\text{nc}_{\text{ss}})}{8b - k^2}.
\]  
\[p^\text{nc}_{\text{ss}} = \frac{6a + 6MG^\text{nc}_{\text{ss}}}{8b - k^2}.
\]  
It is easy to see that $u^{\text{nc}}(t) \rightarrow u^{\text{nc}}_{\text{ss}}, w^{\text{nc}}(t) \rightarrow u^{\text{nc}}_{\text{ss}}, q^{\text{nc}}(t) \rightarrow q^\text{nc}_{\text{ss}}, p^{\text{nc}}(t) \rightarrow p^\text{nc}_{\text{ss}}$, and $R^{\text{nc}}(t) \rightarrow R^\text{nc}_{\text{ss}}$ when $t \rightarrow \infty$. Take them into $S^\text{nc}_{O2O_o} = a - b p^\text{nc}_{\text{ss}} + kq^\text{nc}_{\text{ss}} + M C^\text{nc}_{\text{ss}}$, we have $S^\text{nc}_{O2O_o} = ((2b(a + MG^\text{nc}_{\text{ss}}))/(8b - k^2))$. The proof is completed. \(\square\)

**Proof A3.** In fact, when
\[
det(A) = -\psi (\rho + \psi) + \frac{2M^2 \beta^2 e^{-\rho d}}{8b - k^2} < 0
\]

\[
\Rightarrow \frac{2M^2 \beta^2}{(8b - k^2) (\psi (\rho + \psi)} < e^{\rho d}
\]

\[\Rightarrow G^\text{nc}_{\text{ss}} > 0 \Rightarrow \frac{2M^2 \beta^2}{(8b - k^2) (\psi (\rho + \psi)} < 1 \Rightarrow d^\text{nc}
\]

\[= \frac{1}{\rho} \ln \frac{2M^2 \beta^2}{(8b - k^2) (\psi (\rho + \psi)} < 0.
\]
This means our model includes $d \geq 0$ cases. \(\square\)

**Proof A4.** The elastic can be formulated as
\[K = \frac{(\Delta G^\text{nc}_{\text{ss}}/G^\text{nc}_{\text{ss}})}{(\Delta u^\text{nc}_{\text{ss}}/u^\text{nc}_{\text{ss}})} = \frac{\partial G^\text{nc}_{\text{ss}}}{\partial u^\text{nc}_{\text{ss}}} \frac{u^\text{nc}_{\text{ss}}}{G^\text{nc}_{\text{ss}}} = 1.
\]  
(A.28)

The Proof is completed. \(\square\)

**Proof A5.** The proof is similar to that in Proof A2, and we omitted. \(\square\)

**Proof A6.** Substituting (27), (28), and (33) into (25), we get
\[
V_r^{\text{nc}} = \int_0^\infty e^{-\rho t} \left\{ 4b (a + MG_{\text{ss}}) \right\} dt = \int_0^\infty e^{-\rho t} \frac{4b}{(8b - k^2)} \left[ a + M (G_0 - G_{\text{ss}}) e^{\psi d} + \frac{G^\text{nc}_{\text{ss}} \psi}{\rho} \right] dt
\]
\[= \frac{4b (a + MG_{\text{ss}}) \rho}{(8b - k^2)} - \frac{4b (G_0 - G_{\text{ss}}) e^{\psi d}}{(8b - k^2)} \left[ \frac{\rho}{2} \left[ 4(8b - k^2)(2\psi + \sqrt{\Delta^\text{nc}}) (1 - \sigma) \right] e^{\psi d} \right]
\]
\[+ \frac{8bM (G_0 - G_{\text{ss}}) (a + MG_{\text{ss}})}{(8b - k^2)} - \frac{4\sigma (G_0 - G_{\text{ss}}) M^2 e^{(\psi d)} \psi G^\text{nc}_{\text{ss}}}{(8b - k^2)(2\psi + \sqrt{\Delta^\text{nc}}) (1 - \sigma)} - \frac{1}{\rho \lambda^\text{nc}_{\text{ss}}}
\]
\[+ \frac{4b M^2 (G_0 - G_{\text{ss}}) (a + MG_{\text{ss}})}{(8b - k^2)} - \frac{8\sigma (G_0 - G_{\text{ss}}) M^2 e^{(\psi d)} \psi G^\text{nc}_{\text{ss}}}{(8b - k^2)(2\psi + \sqrt{\Delta^\text{nc}}) (1 - \sigma)} - \frac{1}{\rho - 2\lambda^\text{nc}_{\text{ss}}}
\]  
(A.29)
\]
Proof A7. By $G_{ss}^{cc} > 0$, we have

\[
G_{ss}^{cc} > 0 \Rightarrow (\rho + \psi)\psi e^\rho (8b - k^2) > 2M^2\beta^2 \Rightarrow \frac{2M^2\beta^2}{(\rho + \psi)\psi e^\rho (8b - k^2)} < 1 \Rightarrow 0 < \rho < 1 - \frac{2M^2\beta^2}{(\rho + \psi)\psi e^\rho (8b - k^2)} < 1. \quad (A.30)
\]

The proof is completed. \qed

B

Proof B1. Because $0 > \lambda_1^{cc} > \lambda_2^{nc} \Rightarrow 1 > e^{\lambda_1^{cc}} > e^{\lambda_2^{nc}}$,

\[
\frac{\partial G_{nc}^{cc}}{\partial t} - \frac{\partial G_{cc}^{cc}}{\partial t} = \left( G_0 - G_{ss}^{nc} \right) e^{\lambda_2^{nc}} - \left( G_0 - G_{ss}^{cc} \right) e^{\lambda_1^{cc}},
\]

(B.1)

\[
\begin{align*}
\frac{\partial G_{nc}^{cc}}{\partial t} & < \frac{\partial G_{cc}^{cc}}{\partial t} < 0, & G_0 > G_{ss}^{cc} > G_{ss}^{nc}, & (i), \\
0 < \frac{\partial G_{nc}^{cc}}{\partial t} & < \frac{\partial G_{cc}^{cc}}{\partial t}, & G_{ss}^{nc} < G_0 < G_{ss}^{cc}, & (ii), \\
0 < \frac{\partial G_{nc}^{cc}}{\partial t} & < \frac{\partial G_{cc}^{cc}}{\partial t}, & G_0 < G_{ss}^{nc} < G_{ss}^{cc}, & (iii).
\end{align*}
\]

(B.2)

\[
\frac{\partial G_{nc}^{cc}}{\partial t} - \frac{\partial G_{cc}^{cc}}{\partial t} = \frac{6M}{8b - k^2} \left( \frac{\partial G_{nc}^{cc}}{\partial t} - \frac{\partial G_{cc}^{cc}}{\partial t} \right).
\]

(B.3)

For the control variables,

(B.4)
It is hard to get the signal of $\partial u(t)/\partial t$; we obtain it from the numerical simulation (Figures 2(c), 3(c), and 4(c)). □

**Proof B2**

$$
\frac{\partial G_{ss}^{nc}}{\partial \sigma} = \frac{2aM\beta^2 (\rho + \psi)\psi e^{\theta}(8b - k^2)}{(-2M^2\beta^2 + (\rho + \psi)(1 - \sigma)e^{\theta}(8b - k^2))^2} = \frac{a(1 - \sigma)}{M(\sigma - \sigma)} > 0;
$$

$$
\frac{\partial \rho_{ss}^{nc}}{\partial \sigma} = \frac{6M}{8b - k^2} \frac{\partial G_{ss}^{nc}}{\partial \sigma} = \frac{6a(1 - \sigma)}{8b - k^2(\sigma - \sigma)} > 0;
$$

$$
\frac{\partial w_{ss}^{nc}}{\partial \sigma} = \frac{4M}{8b - k^2} \frac{\partial G_{ss}^{nc}}{\partial \sigma} = \frac{4a}{8b - k^2(\sigma - \sigma)} > 0;
$$

$$
\frac{\partial u_{ss}^{nc}}{\partial \sigma} = \frac{\psi}{\beta} \frac{\partial G_{ss}^{nc}}{\partial \sigma} = \frac{\psi a}{\beta M(\sigma - \sigma)} > 0;
$$

$$
\frac{\partial d_{ss}^{nc}}{\partial \sigma} = \frac{kM}{8b - k^2} \frac{\partial G_{ss}^{nc}}{\partial \sigma} = \frac{6a}{8b - k^2(\sigma - \sigma)} > 0.
$$

(B.5)

$$
\frac{\partial G_{ss}^{nc}}{\partial k} = \frac{-2a\beta^2 \left[ 2\beta^2 \xi (\theta - k\xi)^2 + e^{\theta} \psi (\rho + \psi) \right] - 2(\theta - k\xi)^2 \beta^2} \left[ -2\beta^2 M^2 + e^{\theta}(8b - k^2)^2 \psi (\rho + \psi)^2 \right].
$$

(B.7)

By $G_{ss}^{nc} > 0$, we get $(\rho + \psi)\psi e^{\theta} > (2(\theta - k\xi)^2 \beta^2)/(8b - k^2))$. While $-2k\theta + 8b\xi + k^2\xi = k(k\xi - \theta) + 8b\xi - k\theta < 0$. Hence,

$$
2\beta^2 \xi (\theta - k\xi)^2 + (\rho + \psi)\psi e^{\theta}(-2k\theta + 8b\xi + k^2\xi) < 2\beta^2 \xi (\theta - k\xi)^2 + \frac{2(\theta - k\xi)^2 \beta^2}{8b - k^2}(-2k\theta + 8b\xi + k^2\xi)
$$

$$
= 2\beta^2 (\theta - k\xi)^2 \left[ \xi + \frac{-2k\theta + 8b\xi + k^2\xi}{8b - k^2} \right]
$$

$$
= 2\beta^2 (\theta - k\xi)^2 \frac{(8b\xi - k\theta)}{8b - k^2} < 0.
$$

(B.8)
Hence, the numerator of \( \frac{\partial C_{ss}^{nc}}{\partial k} \) is positive. It is obvious that the denominator of \( \frac{\partial C_{ss}^{nc}}{\partial k} \) is positive. Therefore, \( \frac{\partial C_{ss}^{nc}}{\partial k} > 0 \) is proved. Because \( u_{s}^{nc} = (\psi/\beta)G_{s}^{nc} \) and \( r_{qss}^{nc} = \xi G_{s}^{nc} \), we get \( \frac{\partial u_{s}^{nc}}{\partial k} > 0 \) and \( \frac{\partial u_{qss}^{nc}}{\partial k} > 0 \):

\[
\frac{\partial u_{s}^{nc}}{\partial k} = \frac{8ae^d \psi (\rho + \psi) [2\beta^2 \xi (-\theta + k \xi) + ke^d \psi (\rho + \psi)]}{\left[ -2\beta^2 M^2 + e^d (8b - k^2) \psi (\rho + \psi) \right]^2},
\]

(B.9)

Because \( k(\rho + \psi)\psi e^d > k((2(\theta - k \xi))^2 \beta^2)/(8b - k^2) \),

\[2\beta^2 \xi (-\theta + k \xi) + ke^d \psi (\rho + \psi) > 2\beta^2 \xi (-\theta + k \xi) + k \frac{2(\theta - k \xi)^2 \beta^2}{8b - k^2} \]

\[= 2\beta^2 (\theta - k \xi) \left( \frac{k(\theta - k \xi)}{8b - k^2} - \xi \right) \]

\[= 2\beta^2 M \frac{\theta k - 8b \xi}{8b - k^2} > 0. \]

Then, \( \frac{\partial u_{s}^{nc}}{\partial k} > 0 \) is proved. Because \( p_{ss}^{nc} = (3/2)\omega_{ss}^{nc} \), \( \frac{\partial q_{ss}^{nc}}{\partial k} > 0 \) is deserved:

\[
\frac{\partial q_{ss}^{nc}}{\partial k} = \frac{ae^d \psi (\rho + \psi) \left[ -2\beta^2 \left( \theta^2 - k^2 \xi^2 \right) + (8b + k^2) e^d \psi (\rho + \psi) \right]}{\left[ -2\beta^2 M^2 + e^d(8b - k^2) \psi (\rho + \psi) \right]^2},
\]

(B.11)

Because \( (8b + k^2) e^d \psi (\rho + \psi) > (8b + k^2)((2(\theta - k \xi))^2 \beta^2)/(8b - k^2) \),

\[-2\beta^2 \left( \theta^2 - k^2 \xi^2 \right) + (8b + k^2) e^d \psi (\rho + \psi) \]

\[> -2\beta^2 \left( \theta^2 - k^2 \xi^2 \right) + (8b + k^2) \frac{2(\theta - k \xi)^2 \beta^2}{8b - k^2} \]

\[= \frac{4k^2 \theta \beta^2 (\theta - k \xi)}{8b - k^2} > 0. \]

Then, \( \frac{\partial q_{ss}^{nc}}{\partial k} > 0 \) is proved.

\[
\frac{\partial R_{ss}^{nc}}{\partial k} = \frac{d_{ss}^{nc} - r_{qss}^{nc} + k \left( q_{ss}^{nc} - r_{qss}^{nc} \right)}{\partial k},
\]

(B.13)
It is obviously positive. Hence, \((\partial R^\text{nc}_{\text{as}}/\partial k) > 0\). The proof is completed.  

**Proof B5**

\[
\frac{\partial G_{\text{as}}^\text{nc}}{\partial d} = \frac{a \psi^2 (8b - k)^2}{\beta^2 k^2} \frac{C_{\text{as}}^\text{nc}}{(a + MG_{\text{as}}^\text{nc})^3} \frac{\partial G_{\text{as}}^\text{nc}}{\partial d} < 0,
\]

\[
\frac{\partial G_{\text{as}}^\text{cc}}{\partial k} = \frac{8e^{-2d\rho \beta \theta (-\theta + k\xi)}}{k^3 \psi^2 (\rho + \psi)^2} < 0,
\]

\[
\frac{\partial G_{\text{as}}^\text{cc}}{\partial \sigma} = \frac{a \psi^2 (8b - k)^2}{\beta^2 k^2} \frac{C_{\text{as}}^\text{cc}}{(a + MG_{\text{as}}^\text{cc})^3} \frac{\partial G_{\text{as}}^\text{cc}}{\partial \sigma} > 0.
\]

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**

[1] Y. Pan, D. Wu, C. Luo, and A. Dolgui, "User activity measurement in rating-based online-to-offline (O2O) service recommendation," *Information Sciences*, vol. 479, pp. 180–196, 2019.

[2] W. Zhong, "Service recommendation method on multiple dimension O2O," in *Proceedings of the International Conference on Intelligent Transportation*, pp. 713–716, IEEE, Halong Bay, Vietnam, April 2015.

[3] J. Ji, Z. Zhang, and L. Yang, "Comparisons of initial carbon allowance allocation rules in an O2O retail supply chain with the cap-and-trade regulation," *International Journal of Production Economics*, vol. 187, pp. 68–84, 2017.

[4] Y. He, J. Zhang, Q. Guo et al., "Supply chain decisions with reference quality effect under the O2O environment," *Annals of Operations Research*, vol. 268, no. 1-2, pp. 273–292, 2017.

[5] R. Chenavaz, "Dynamic quality policies with reference quality effects," *Applied Economics*, vol. 49, no. 32, pp. 3156–3162, 2016.

[6] A. Gavious and O. Lowengart, "Price-quality relationship in the presence of asymmetric dynamic reference quality effects," *Marketing Letters*, vol. 23, no. 1, pp. 137–161, 2012.

[7] K. Sivakumar, "Examining loss aversion for quality versus loss aversion for price," *Journal of Marketing Theory and Practice*, vol. 19, no. 3, pp. 317–324, 2011.

[8] I. Luhtan and A. Virtanen, "Non-linear advertising capital model with time delayed feedback between advertising and stock of goodwill," *Chaos, Solitons & Fractals*, vol. 7, no. 12, pp. 2083–2104, 1996.

[9] D. Xiao, Y.-W. Zhou, Y. Zhong, and W. Xie, "Optimal competitive advertising and ordering policies for a two-echelon supply chain," *Computers & Industrial Engineering*, vol. 127, pp. 511–519, 2019.

[10] X. Li, Y. Li, and W. Cao, "Cooperative advertising models in O2O supply chains," *International Journal of Production Economics*, vol. 215, pp. 144–152, 2017.

[11] M. H. Guimaraes and P. S. Simon, "Retailer and manufacturer advertising scheduling in a marketing channel," *Journal of Business Research*, vol. 78, pp. 93–100, 2017.

[12] J. Xie, L. Liang, L. Liu, and P. Jeromonachou, "Coordination contracts of dual-channel with cooperation advertising in closed-loop supply chains," *International Journal of Production Economics*, vol. 183, pp. 528–538, 2017.

[13] X. He, A. Prasad, and S. Sethi, "Cooperative advertising and pricing in a dynamic stochastic supply chain: feedback Stackelberg strategies," in *Proceedings of the Portland International Conference on Management of Engineering & Technology*, pp. 1634–1649, IEEE, Cape Town, South Africa, July 2008.

[14] T.-M. Choi and N. Liu, "Optimal advertisement budget allocation and coordination in luxury fashion supply chains with multiple brand-tier products," *Transportation Research Part E: Logistics and Transportation Review*, vol. 130, pp. 95–107, 2019.

[15] M. Freimer and D. Horsky, "Periodic advertising pulsing in a competitive market," *Marketing Science*, vol. 31, no. 4, pp. 637–648, 2012.

[16] A. Aravindakshan and P. A. Naik, "How does awareness evolve when advertising stops? The role of memory," *Marketing Letters*, vol. 22, no. 3, pp. 315–326, 2011.

[17] W. Pauwels, "Optimal dynamic advertising policies in the presence of continuously distributed time lags," *Journal of Optimization Theory and Applications*, vol. 22, no. 1, pp. 79–89, 1977.

[18] A. Aravindakshan and P. A. Naik, "Understanding the memory effects in pulsing advertising," *Operations Research*, vol. 63, no. 1, pp. 35–47, 2015.

[19] J. Zhang, Q. Guo, L. Liang, and Z. Huang, "Supply chain coordination through cooperative advertising with reference price effect," *Omega*, vol. 41, no. 2, pp. 345–353, 2013.

[20] K. Govindan and A. Malomfalean, "A framework for evaluation of supply chain coordination by contracts under O2O environment," *International Journal of Production Economics*, vol. 215, pp. 11–23, 2018.

[21] A. A. Elsadany and A. E. Matouk, "Dynamic courtnot duopoly game with delay," *Journal of Complex Systems*, vol. 20, no. 1, pp. 95–107, 2019.

[22] A. Aravindakshan and P. A. Naik, "Understanding the memory effects in pulsing advertising," *Operations Research*, vol. 63, no. 1, pp. 35–47, 2015.

[23] J. Zhang, Q. Guo, L. Liang, and Z. Huang, "Supply chain coordination through cooperative advertising with reference price effect," *Omega*, vol. 41, no. 2, pp. 345–353, 2013.

[24] K. Govindan and A. Malomfalean, "A framework for evaluation of supply chain coordination by contracts under O2O environment," *International Journal of Production Economics*, vol. 215, pp. 11–23, 2018.

[25] A. A. Elsadany and A. E. Matouk, "Dynamic courtnot duopoly game with delay," *Journal of Complex Systems*, vol. 20, no. 1, pp. 95–107, 2019.

[26] A. Aravindakshan and P. A. Naik, "Understanding the memory effects in pulsing advertising," *Operations Research*, vol. 63, no. 1, pp. 35–47, 2015.

[27] J. Zhang, Q. Guo, L. Liang, and Z. Huang, "Supply chain coordination through cooperative advertising with reference price effect," *Omega*, vol. 41, no. 2, pp. 345–353, 2013.

[28] K. Govindan and A. Malomfalean, "A framework for evaluation of supply chain coordination by contracts under O2O environment," *International Journal of Production Economics*, vol. 215, pp. 11–23, 2018.

[29] A. A. Elsadany and A. E. Matouk, "Dynamic courtnot duopoly game with delay," *Journal of Complex Systems*, vol. 20, no. 1, pp. 95–107, 2019.
consumption,” in Advances in Consumer Research, J. F. Sherry Jr. and B. Sternthal, Eds., vol. 19, pp. 806–812, Association for Consumer Research, Provo, UT, USA, 1992.

[25] R. P. Leone, “Generalizing what is known about temporal aggregation and advertising carryover,” Marketing Science, vol. 14, no. 3, pp. G141–G150, 1995.

[26] K. Weigelt and C. Camerer, “Reputation and corporate strategy: a review of recent theory and applications,” Strategic Management Journal, vol. 9, no. 5, pp. 443–454, 1988.

[27] L. Lu, Q. Gou, W. Tang, and J. Zhang, “Joint pricing and advertising strategy with reference price effect,” International Journal of Production Research, vol. 54, no. 17, pp. 5250–5270, 2016.

[28] M. Basin and J. Rodriguez-Gonzalez, “Optimal control for linear systems with multiple time delays in control input,” IEEE Transactions on Automatic Control, vol. 51, no. 1, pp. 91–97, 2006.