Spherical and non-spherical bubbles in cosmological phase transitions

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Abstract

The cosmological remnants of a first-order phase transition generally depend on the perturbations that the walls of expanding bubbles originate in the plasma. Several of the formation mechanisms occur when bubbles collide and lose their spherical symmetry. However, spherical bubbles are often considered in the literature, in particular for the calculation of gravitational waves. We study the steady state motion of bubble walls for different bubble symmetries. Using the bag equation of state, we discuss the propagation of phase transition fronts as detonations and subsonic or supersonic deflagrations. We consider the cases of spherical, cylindrical and planar walls, and compare the energy transferred to bulk motions of the relativistic fluid. We find that the different wall geometries give similar perturbations of the plasma. For the case of planar walls, we obtain analytical expressions for the kinetic energy in the bulk motions. As an application, we discuss the generation of gravitational waves.

1 Introduction

Cosmological phase transitions generically produce cosmic relics such as topological defects [1], magnetic fields [2], baryon number asymmetries [3], inhomogeneities [4, 5], and gravitational waves [6, 7]. The walls of expanding bubbles usually play a relevant role in the mechanisms which generate these relics. Indeed, most of them depend on the disturbance produced by the motion of bubble walls in the surrounding plasma. Moreover, the wall velocity itself depends on such hydrodynamics [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] as well as on the friction with the plasma [18]. The friction is determined by microphysics, i.e., the interactions of particles with the wall. The hydrodynamics is determined by the relativistic fluid equations. The latent heat that is injected at the phase transition fronts spreads out, causing reheating and bulk motions of the plasma. There are essentially

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three propagation modes for the phase transition front. A detonation, which is supersonic and is followed by a rarefaction wave, a subsonic deflagration, which is preceded by a shock front, and a supersonic deflagration, which is preceded by a shock and followed by a rarefaction wave.

Several of the mechanisms which generate the aforementioned cosmological relics operate when bubbles meet. For instance, in order to produce gravity waves (GWs), bubbles must collide and lose their spherical symmetry, since a spherical source cannot generate gravitational radiation. The hydrodynamics of colliding bubble walls is complicated. The fluid velocity and temperature profiles during bubble collisions were studied using numerical simulations in Ref. [12]. In Ref. [13], a few configurations for the collision of shock fronts were considered analytically. Both studies were performed in 1+1 dimensions. In applications, further simplifications are needed. For instance, in the case of gravitational waves, the relevant quantity is the energy that is injected into bulk motions of the plasma. This is parameterized by the efficiency factor \( \kappa \), which is defined as the ratio of the kinetic energy in bulk motions to the released vacuum energy. In calculations of \( \kappa \), the interaction between bubbles is neglected, and spherical bubbles are assumed. This treatment implies two hypotheses, namely, that the motion of a bubble wall is not affected by the perturbations other bubbles caused in the plasma, and that the deformation of the bubble wall does not affect the energy transfer from the wall to the plasma. The first assumption should be correct for supersonic walls, which either do not have shock fronts preceding them or the shock fronts are very close to the walls. For subsonic walls, the main influence from other bubbles is the reheating of the plasma. This effect must be taken into account in a complete calculation of the phase transition [5, 19, 20]. Regarding the dependence on the wall geometry, some progress can be made by comparing different bubble symmetries. Of course, after a few collisions bubbles will take arbitrary forms. Nevertheless, considering a few specific symmetries will be useful to tell whether the disturbance of the fluid has a strong dependence on the wall geometry or not. Furthermore, it is particularly important to study the case of planar walls, for which analytical results can be obtained.

The growth of plane, cylindrical, and spherical bubbles (equivalently, bubbles in 1, 2, and 3 spatial dimensions, respectively) was considered in Ref. [10] for the case of deflagrations. The fluid velocity profiles are quite different. In particular, for planar walls the shock wave preceding the phase transition front has a constant fluid velocity, whereas in the cylindrical and spherical cases the fluid velocity falls quickly in front of the wall. This is because in higher dimensions there is more room for the shock wave to carry away the injected energy. Nevertheless, the total amount of kinetic energy of the fluid must be a fraction of the released latent heat (another fraction goes into thermal energy), and there is no reason for this fraction (and for the ratio \( \kappa \)) to have a strong dependence on the bubble wall geometry. For planar walls, the velocity profile of the fluid can be obtained analytically (see, e.g., [10]), and one expects to find analytical formulas for quantities such as the efficiency factor as well. An analytical approximation for \( \kappa \) was obtained in Ref. [21] for small wall velocities. Recently, the efficiency factor was calculated numerically for spherical bubbles [22] in the whole wall velocity range.

In this paper we address the issue of the effect of the wall geometry on the disturbance caused by the motion of the walls. Thus, we study the hydrodynamics of spherical, cylindrical, and planar bubble walls. In particular, we show that the efficiency factor does
not differ significantly between the different wall geometries. For planar walls, we obtain the efficiency factor analytically for any wall velocity. We also discuss the generation of GWs in the electroweak phase transition. The plan of the paper is the following. In the next section we review the dynamics of the first-order phase transition. In section 3 we calculate the fluid profiles and the efficiency factor for the three bubble symmetries. In section 4 we study in detail the case of planar walls. We discuss the different hydrodynamical modes and we derive analytical formulas for $\kappa$. In section 5 we apply our results to the estimation of the gravitational wave signal from the electroweak phase transition. The conclusions are in section 6.

2 Phase transition dynamics

A cosmological phase transition occurs when the free energy of the model (i.e., the finite-temperature effective potential) depends on an order parameter $\phi$ (e.g., a Higgs field), such that the free energy density $F(\phi, T)$ has a minimum $\phi_+(T)$ at high temperatures, and a different minimum $\phi_-(T)$ at low temperatures (we shall use a “+” index for variables in the high-temperature phase and a “−” index for the low-temperature phase). For a first-order phase transition, there is a temperature range in which the two minima coexist and a barrier in the free energy separates them. The critical temperature $T_c$ is that at which the two minima have the same free energy. Below $T_c$, bubbles of the stable phase nucleate and grow. The nucleated bubble is a configuration $\phi = \phi(r, t)$ with spherical symmetry, such that at the center of the bubble the system is in the low-$T$ phase, whereas far from this point the system is in the high-$T$ phase. Hence, we have $\phi(r = 0) = \phi_-(T)$ and $\phi(r = \infty) = \phi_+(T)$. There is a region, the “bubble wall”, in which $\phi$ varies continuously from $\phi_-$ to $\phi_+$. In this work, we shall assume for simplicity an infinitely thin wall separating the two phases.

We are interested in the energy in bulk motions of the fluid which is caused by the moving walls. We assume that the plasma is a perfect relativistic fluid with four-velocity field $u^\mu = (\gamma, \gamma v)$, with $\gamma = 1/\sqrt{1 - v^2}$. The energy-momentum tensor is of the form

$$T^{\mu\nu} = (e + p) u^\mu u^\nu - pg^{\mu\nu},$$

where $e$ and $p$ are the energy density and pressure in the proper system of the fluid element [23]. The energy density is given by $T^{00} = w\gamma^2 - p = (e + pv^2)\gamma^2$, and the kinetic energy density is defined as $e_{\text{kin}} = T^{00}(v) - T^{00}(0)$. Therefore, we have $e_{\text{kin}} = T^{00} - e = wv^2\gamma^2$.

All the thermodynamical quantities can be derived from the free energy densities $F_+(T) \equiv F(\phi_+(T), T)$ and $F_-(T) \equiv F(\phi_-(T), T)$ for each phase. Thus, the pressure is given by $p = -F$, the entropy density by $s = dp/dT$, the energy density by $e = Ts - p$, and the enthalpy by $w = e + p = Ts$. At the critical temperature, the pressure in the two phases is the same, i.e., $p_+(T_c) = p_-(T_c)$. However, other quantities such as the energy, entropy, and enthalpy are different even at $T = T_c$. The latent heat is defined as the energy density difference $L = \Delta e(T_c) = \Delta w(T_c) = T_c\Delta s(T_c)$.

It is useful to consider a simplified model which exhibits the general features of a phase transition. This allows in particular to obtain results which depend on a few parameters. Then, the results can be applied to realistic phase transitions by calculating
these parameters in specific models. Therefore, we shall use the bag equation of state (EOS),

\[ \begin{align*}
    e_{+} &= a_{+} T^4 + \varepsilon, \\
    p_{+} &= \frac{1}{3} a_{+} T^4 - \varepsilon \\
    e_{-} &= a_{-} T^4, \\
    p_{-} &= \frac{1}{3} a_{-} T^4,
\end{align*} \]

(2)

with \( \varepsilon > 0 \) and \( a_{+} > a_{-} > 0 \). This system has two components, namely, a vacuum energy density \( \varepsilon \) and a radiation energy density \( a T^4 \). The condition \( a_{+} > a_{-} \) implies that some of the radiation degrees of freedom disappear after the phase transition. The vacuum energy density is positive in the high-temperature phase and vanishes in the low-temperature one.

In this model, the critical temperature is given by

\[ T_{c} = \left( \frac{3 \varepsilon}{\Delta a} \right)^{1/4}, \]

where \( \Delta a = a_{+} - a_{-} \), and the latent heat is related to the vacuum energy density by \( L = 4 \varepsilon \). In both phases, the speed of sound is given by

\[ c_{s}^2 \equiv \frac{\partial p}{\partial e} = 1/3. \]

(3)

We shall assume for simplicity a stationary state in which the wall moves with a constant velocity \( v_w \). We aim to consider three kinds of bubble wall geometry, namely, a spherical, a cylindrical and a plane wall. In this approximation, at time \( t \) the wall is at a distance \( R_b = v_w t \) from a point, axis, or plane, and the volume of the bubble is of the form \( V_b = c_j R_b^{j+1} / (j+1) \), where \( j = 2, 1, \) or 0 for the spherical, cylindrical or planar case, respectively (the factor \( c_j \) will cancel out in our calculations). The kinetic energy in bulk motions of the fluid is given by

\[ E_{kin} = c_j \int_{0}^{\infty} w v^2 \gamma^2 R^j dR. \]

The efficiency factor is defined as the ratio of the kinetic energy to the released vacuum energy,

\[ \kappa = \frac{E_{kin}}{(\varepsilon V_b)}. \]

(4)

Thus, we can write

\[ \kappa = \frac{j + 1}{\varepsilon \xi_w^{j+1}} \int_{0}^{\infty} w v^2 \gamma^2 j \xi^j d\xi, \]

(5)

where \( \xi = R/t \) and \( \xi_w = R_b/t = v_w \).

3 Fluid profiles and kinetic energy

3.1 The fluid equations

The fluid equations are obtained from the conservation of the energy-momentum tensor (1),

\[ \partial_{\mu} T^{\mu\nu} = 0. \]

If we denote by \( r \) the distance from the symmetry point, axis or plane, and \( t \) the time from nucleation, Eqs. (6) become

\[ \begin{align*}
    \partial_r \left[ (e + p v^2) \gamma^2 \right] + \partial_r \left[ (e + p) \gamma^2 v \right] &= -\frac{j}{r} \left[ (e + p) \gamma^2 v \right], \\
    \partial_r \left[ (e + p) \gamma^2 v \right] + \partial_r \left[ (e v^2 + p) \gamma^2 \right] &= -\frac{j}{r} \left[ (e + p) \gamma^2 v^2 \right].
\end{align*} \]

(7)

\(^1\)In a real phase transition, even after reaching the stationary state, the wall velocity may vary due to the adiabatic cooling of the universe and the release of latent heat (see, e.g., [5 19 24]).
Since there is no characteristic distance scale in the problem, it is usual to assume the similarity condition, namely, that \( e, p \) and \( v \) depend only on the combination \( \xi = r/t \). Thus, we have

\[
(\xi - v) \frac{e'}{w} = \frac{v}{\xi} + \gamma^2 (1 - v \xi) v',
\]

\[
(1 - v \xi) \frac{p'}{w} = \gamma^2 (\xi - v) v',
\]

where a prime indicates derivative with respect to \( \xi \). The pressure and energy density are further related by the equation of state. According to Eq. (3), we have \( p' = c_s^2 e' \). Using this relation, Eqs. (8) can be combined to obtain the central equation for the velocity profile [8, 10, 7]. We obtain

\[
\frac{v}{\xi} = \gamma^2 (1 - v \xi) \left[ \frac{\mu^2}{c_s^2} - 1 \right] v',
\]

where

\[
\mu (\xi, v) = \frac{\xi - v}{1 - \xi v}.
\]

From Eqs. (8) we also obtain the equation for the enthalpy profile,

\[
\frac{w'}{w} = \left( \frac{1}{c_s^2} + 1 \right) \mu \gamma^2 v',
\]

which is readily integrated [7]. For \( c_s^2 = 1/3 \) we have

\[
\frac{w_b}{w_a} = \exp \left[ \int_{\xi_a}^{\xi_b} 4 \gamma^2 \mu (\xi, v) v' d\xi \right].
\]

Notice that, in this model, the equations for the velocity and enthalpy profiles are the same in both phases.

### 3.2 Discontinuities

#### 3.2.1 The phase transition front

In the \( \xi \) axis, the bubble wall is at \( \xi_w = v_w \). The enthalpy and other quantities are discontinuous at \( \xi_w \), and so will be the fluid velocity. In the reference frame of the wall, the fluid comes from the high-T phase with a velocity \( v_+ \), and goes out into the low-T phase with a velocity \( v_- \). The incoming and outgoing flow velocities are related by the conservation of \( T^{\mu \nu} \) across the wall [23],

\[
\begin{align*}
  w_- v_- \gamma_-^2 + p_- &= w_+ v_+^2 \gamma_+^2 + p_+, \\
  w_- v_- \gamma_-^2 &= w_+ v_+ \gamma_+^2.
\end{align*}
\]

Equivalently,

\[
\begin{align*}
  v_+ v_- &= \frac{p_+ - p_-}{e_+ - e_-}, & v_+ &= \frac{e_- + p_+}{e_+ + p_-},
\end{align*}
\]
Notice that these equations do not depend on $j$. This is because the surface of discontinuity is locally planar. In the reference frame of the bubble center, the fluid velocities on each side of the wall are given by $\dot{v}_\pm = \mu (\xi, |v_\pm|)$.

According to Eq. (13), $v_+$ and $v_-$ have the same sign. Indeed, the fluid velocities in the system of the wall must be both negative. Using the bag EOS, Eqs. (15) can be combined to obtain a relation between $v_+$, $v_-$ and the parameter

$$\alpha_+ \equiv \frac{\varepsilon}{a_+ T_+^4}. \quad (16)$$

We can solve, e.g., for $v_+$ as a function of $v_-$ and $\alpha_+$ [8],

$$v_+ = \left( \frac{v_-}{2} + \frac{1}{6v_-} \right) \pm \frac{\sqrt{\left( \frac{v_-}{2} + \frac{1}{6v_-} \right)^2 + (1 + \alpha_+) (\alpha_+ - 1/3)}}{1 + \alpha_+}. \quad (17)$$

Two kinds of hydrodynamical processes may occur at the phase transition front, corresponding to the + and − signs in Eq. (17), namely, a detonation, for which the incoming flow is supersonic ($|v_+| > c_s$) and faster than the outgoing flow ($|v_-| < |v_+|$), and a deflagration, with $|v_+| < c_s$ and $|v_-| > |v_+|$. In either case, the incoming velocity $|v_+|$ has an extremum at $|v_-| = c_s$, namely, a minimum for detonations and a maximum for deflagrations. A process with $|v_-| = c_s$ is called a Jouguet detonation or deflagration. In this case we have $|v_+| = v_{J} (\alpha_+)$, with

$$v_{J}^{det} (\alpha_+) = \frac{1 \pm \sqrt{\alpha_+ (2 + 3\alpha_+)}}{\sqrt{3} (1 + \alpha_+)}. \quad (18)$$

Hence, for detonations we have $c_s < v_{J}^{det} (\alpha_+) \leq |v_+|$, and for deflagrations $|v_+| \leq v_{J}^{def} (\alpha_+) < c_s$. The hydrodynamical process is called weak if the velocities $v_+$ and $v_-$ are either both supersonic, or both subsonic. In such a case, $|v_-|$ lies between $|v_+|$ and $c_s$. Otherwise, one of the two velocities is subsonic and the other one is supersonic. Such hydrodynamical process is called strong.

### 3.2.2 Shock fronts

Discontinuities in the same phase may also be needed to satisfy the boundary conditions. Such discontinuities are called shock fronts. We shall use the index 1 for fluid variables behind the shock front at $\xi = \xi_{sh}$, and the index 2 for variables in front of the shock. The EOS is the same on both sides of the discontinuity, and Eqs. (15) trivially give

$$v_1 v_2 = \frac{1}{3}, \quad \frac{v_1}{v_2} = \frac{3T_1^4 + T_2^4}{3T_1^4 + T_2^4}, \quad (19)$$

where $v_1$ and $v_2$ are the (negative) fluid velocities in the shock frame.

When dealing with discontinuities, entropy considerations can be useful for discarding possible processes (see, e.g., [9]). Consider a portion of the fluid which passes through
a discontinuity surface. Requiring the entropy of the fluid to increase, one obtains the condition
\[ s_1v_1 \gamma_1 \geq s_2v_2 \gamma_2. \] Using \[ s = w/T \] and Eq. (14), this condition becomes
\[ T_2/T_1 \geq \gamma_1/\gamma_2. \] (20)
For the bag EOS, we have \[ w_2/w_1 = (T_2/T_1)^4, \] and we may insert the inequality (20) back in Eq. (14) to obtain
\[ v_2(1 - v_2^2) \leq v_1(1 - v_1^2). \] (21)
Using the relation \[ v_1 v_2 = 1/3, \] this condition becomes \( (v_1^2 - 1/3)^3 \leq 0, \) which implies
\[ |v_1| < \frac{1}{\sqrt{3}} < |v_2|. \] (22)
In the frame of the bubble center, the fluid velocities on each side of the shock front are given by \( \tilde{v}_{1,2} = \mu (\xi_{sh}, |v_{1,2}|). \) According to the condition (22), we have
\[ \tilde{v}_1 > \tilde{v}_2. \] (23)
Hence, the fluid velocity must have a negative jump. As a consequence, \( \tilde{v}_1 \) cannot vanish (otherwise, we would have \( \tilde{v}_2 < 0, \) but, in this reference frame, the fluid velocities are positive or zero). On the other hand, we may have \( \tilde{v}_2 = 0. \) In this case, the velocity of the shock front is given by \( \xi_{sh} = -v_2, \) and the first of Eqs. (19) gives
\[ \tilde{v}_1 = \frac{3\xi_{sh}^2 - 1}{2\xi_{sh}}. \] (24)
Equivalently,
\[ \xi_{sh} = \frac{\tilde{v}_1}{3} + \sqrt{\left(\frac{\tilde{v}_1}{3}\right)^2 + \frac{1}{3}}, \] which implies that the shock is supersonic.

### 3.3 Kinds of solutions

Equation (9) can be solved numerically for the spherical and cylindrical cases, and analytically for the planar case. General solution curves for the three symmetries can be found in Ref. [10] (for the planar case, see section 4 below). The velocity profile of the fluid must fulfill the discontinuity conditions at the bubble wall, and is constructed by matching different solutions of Eq. (9). The boundary conditions are that the fluid is at rest far ahead of the phase transition surface (where no information of the phase transition has arrived yet) and far behind that surface (near the center of the bubble). Far in front of the phase transition front, the temperature is still \( T_N. \) Therefore, the boundary condition for the enthalpy density is that it takes the value
\[ w_N = \frac{4}{3} a_+ T_N^4 \] (26)
\[ \text{Similarly, for the phase transition front we obtain } v_+(1 - v^2_+) \leq (a_-/a_+) v_-(1 - v^2_+). \]
far in front of the bubble wall.

Not all of the aforementioned hydrodynamical processes will be realized in a phase transition. It is known that strong detonations are not possible, since they cannot satisfy the boundary conditions [23] [8] [14]. It has been argued in Ref. [8] that weak detonations are also impossible, like in the case of chemical burning [23]. As a consequence, we would only have Jouguet detonations, and the velocity \( v_w = v_J \) would be completely determined by hydrodynamics and would not depend on microphysics. However, it has been shown [14] that this is not true in the case of phase transitions, where the situation is similar to that of condensation discontinuities [23] rather than chemical burning. Consequently, only strong detonations are forbidden.

Strong deflagrations do not seem to be realized either. In Ref. [14] it was argued that, like in the case of chemical burning, they are forbidden by entropy considerations [23]. However, in Ref. [15], it was shown that this proof is not valid for cosmological phase transitions. Nevertheless, a mechanical instability argument against strong deflagrations [23] seems to be valid also for cosmological phase transitions [15]. Numerical calculations [12] [15] support this assertion. On the other hand, in Ref. [15] supersonic Jouguet deflagrations were shown to exist.

As a consequence, three kinds of solutions seem to be realized in nature, namely, weak detonations, subsonic weak deflagrations, and supersonic Jouguet deflagrations. In section 4, for the planar case, we argue that these are the only possible solutions (the argument is simpler for the planar case, but can be straightforwardly generalized to the other symmetries). In the rest of the present section we describe these solutions and calculate their profiles and the efficiency factor (the spherical case has been studied recently in Ref. [22]).

### 3.3.1 Detonations

For detonations we have \( |v_+| \geq v_{J}^{\text{det}} \), and the wall moves supersonically with respect to the fluid in front of it. Thus, outside a detonation bubble the fluid has not yet been perturbed. Hence the fluid velocity vanishes and the temperature is still that at which the bubble nucleated, i.e., \( \tilde{v}_+ = 0 \) and \( \tilde{T}_+ = T_N \). Therefore, we have

\[
v_+ = -\xi_w, \quad \alpha_+ = \alpha_N, \tag{27}
\]

where

\[
\alpha_N \equiv \frac{\varepsilon}{a_+ T_N^4}. \tag{28}
\]

As a consequence, we have \( \xi_w \geq v_{J}^{\text{det}} \). The velocity profile of a detonation is shown in Fig. II (left) for the three wall geometries. The bubble wall is followed by a rarefaction wave which ends at \( \xi = c_s \). The velocity profile is determined by the boundary condition \( v (\xi_w) = \tilde{v}_- \), with

\[
\tilde{v}_- = \mu (\xi_w, |v_-|). \tag{29}
\]

The velocity \( v_- \) is given by the inverse of relation (17),

\[
|v_-| = \left( \frac{|v_+| (1 + \alpha_+)}{2} + \frac{\frac{3}{2} - \alpha_+}{2 |v_+|} \right) \pm \sqrt{\left( \frac{|v_+| (1 + \alpha_+)}{2} + \frac{\frac{3}{2} - \alpha_+}{2 |v_+|} \right)^2 - \frac{1}{3}}. \tag{30}
\]
Weak detonations (i.e., \(|v_-| > c_s\)) correspond to the + sign in Eq (30) and strong detonations (\(|v_-| < c_s\)) correspond to the − sign. The two branches match at the Jouguet point \(|v_+| = v^\text{det}_J, |v_-| = c_s\). As we mentioned, the process must be a weak (or, at most, Jouguet) detonation for compatibility with the boundary conditions.

The enthalpy profile is given by Eq. (12) with the condition \(w(\xi_w) = w_-\). The value \(w_-\) just behind the wall is related to \(w_+ = w_N\) through Eq. (14),

\[
|v_+| = \left| v^\text{det}_J \right| \leq c_s \quad \text{and} \quad \left| v_- \right| \leq c_s.
\]

The efficiency factor is obtained by integrating the kinetic energy density \(e_{\text{kin}} = \frac{w v^2}{\gamma^2}\). Figure 1 (right) shows the kinetic energy density profile for the three wall geometries.

![Figure 1](image)

Figure 1: Left: the fluid velocity profile of a detonation with \(v_w = 0.8\) and \(\alpha_N = 0.1\) for a spherical wall (solid), a cylindrical wall (dashed), and a planar wall (dotted). Right: the corresponding kinetic energy density profiles.

### 3.3.2 Subsonic deflagrations

For \(\xi_w < c_s\), the wall is preceded by a shock front at \(\xi_{sh} > c_s\). Behind the wall the fluid velocity vanishes. Hence, we have \(\hat{v}_- = 0\) and (since \(\hat{v}_+ > 0\)) \(|v_+| < |v_-| = \xi_w < c_s\), i.e., the hydrodynamical process is a weak deflagration (in the limiting case \(\xi_w = c_s\), we have a Jouguet deflagration). The fluid velocity vanishes also beyond the shock front. Thus, we have \(\hat{v}_2 = 0\) and \(\alpha_2 = \alpha_N\).

The velocity profile of the shock wave between \(\xi_w\) and \(\xi_{sh}\) is given by a solution of Eq. (9) (see Fig. 2). One can choose as boundary condition the value of the velocity \(\hat{v}_+\) in front of the wall, or the value \(\hat{v}_1\) behind the shock. The velocity \(\hat{v}_+\) is given by \(\mu (\xi_w, |v_+|)\), where \(v_+\) is given by Eq. (17) as a function of \(\alpha_+\) and of \(v_- = -v_w\). However, in the present case \(\alpha_+\) does not equal \(\alpha_N\), since the shock wave reheats the fluid in front of the wall. The temperatures at each end of the shock wave are related by

\[
\alpha_+ = \frac{w_1}{w_+} \alpha_1,
\]

where \(w_1 = \beta \gamma w_+, \alpha_1 = \alpha_N\), and \(\beta = \frac{|v_-|}{\gamma^2 c_s}\).

3In Eq. (30), \(|v_-|\) is real only for \(|v_+| \geq v^\text{det}_J\) (corresponding to detonations) or \(|v_+| \leq v^\text{def}_J\) (corresponding to deflagrations).
with \( w_1/w_+ \) given by Eq. (12),

\[
\frac{w_1}{w_+} = \exp \left[ \int_{\tilde{v}_+}^{\tilde{v}_1} 4\gamma^2 \mu (\xi, v) \, dv \right].
\]  

(33)

The fluid velocities \( \tilde{v}_+ \) and \( \tilde{v}_1 \) are related by the fluid equation (2). Let us denote \( v(\xi; \tilde{v}_1) \) the solution with boundary condition \( v(\xi_{sh}) = \tilde{v}_1 \), with \( \xi_{sh} \) depending on \( \tilde{v}_1 \) through Eq. (25). Then, evaluating at \( \xi = \xi_w \), we obtain \( \tilde{v}_+ \) as a function of \( \tilde{v}_1 \) and \( \xi_w \),

\[
\tilde{v}_+ = v(\xi_w; \tilde{v}_1).
\]  

(34)

Equations (32-34) give \( \alpha_+ \) as a function of \( \alpha_1, \xi_w \) and \( \tilde{v}_1 \). The variables \( \alpha_1 \) and \( \alpha_2 = \alpha_N \) on each side of the shock discontinuity are related by Eqs. (19) with \( v_2 = -\xi_{sh} \). We obtain

\[
\alpha_1 = \frac{3 (1 - \xi_{sh}^2)}{9\xi_{sh}^2 - 1} \alpha_N.
\]  

(35)

Equivalently, using Eq. (25), we have

\[
\alpha_1 = \frac{\tilde{\gamma}_2}{3} \left( 3 + 5\tilde{v}_1^2 - 4\tilde{v}_1 \sqrt{3 + \tilde{v}_1^2} \right) \alpha_N.
\]  

(36)

Thus, we have \( \alpha_+ \) as a function of \( \alpha_N, \xi_w \), and either \( \tilde{v}_1 \) or \( \xi_{sh} \). On the other hand, the variables at the wall discontinuity are related by Eq. (17) or Eq. (30) or, equivalently, by

\[
\alpha_+ = \gamma_+^2 \left( v_+^2 + \frac{1}{3} - v_+ v_- - \frac{1}{3} v_+/v_- \right).
\]  

(37)

In the present case, we have \( v_- = -\xi_w, v_+ = (\tilde{v}_+ - \xi_w) / (1 - \tilde{v}_+ \xi_w) \), and we obtain

\[
\alpha_+ = \frac{\tilde{\gamma}_2}{3} \tilde{v}_+ \left( 2v_w \tilde{v}_+ + 1 - 3v_w^2 \right).
\]  

(38)

Inserting Eq. (38) in Eq. (32), we eliminate \( \alpha_+ \) and we can solve for \( \tilde{v}_1 \) as a function of \( \xi_w \) and \( \alpha_N \). Searching for \( \tilde{v}_1 \) numerically, implies the evaluation of all the quantities which appear in Eqs. (32-38), for several values of \( \tilde{v}_1 \). Such evaluation involves numerically solving the differential equation which gives \( \tilde{v}_+ \) as a function of \( \tilde{v}_1 \) and then performing numerically the integral in Eq. (33). As we shall see in section 4, in the planar case Eqs. (32-38) reduce to a single, algebraic equation, which can be solved analytically.

Once the value of \( \tilde{v}_1 \) is found, one can compute the velocity and enthalpy profiles and perform the integral of the kinetic energy density to obtain the efficiency factor. The enthalpy profile is given by Eq. (12) and determined by the condition \( w(\xi_{sh}) = w_1 \), where \( w_1 = (\alpha_N/\alpha_1) w_N \), with \( \alpha_N/\alpha_1 \) given by Eq. (35). Thus, we have

\[
w_1 = \frac{9\xi_{sh}^2 - 1}{3(1 - \xi_{sh}^2)} w_N.
\]  

(39)

Figure 2 shows the velocity and kinetic energy density profiles for a subsonic deflagration.
3.3.3 Supersonic deflagrations

For \( c_s < \xi_w < v_{j,\text{det}} \), the phase transition front is preceded by a shock front at \( \xi_{sh} > \xi_w \), and is followed by a rarefaction wave solution which vanishes at \( \xi = c_s \). Therefore, both velocities \( \tilde{v}_+ \) and \( \tilde{v}_- \) are non-vanishing (see Fig. 3). The hydrodynamic process is a Jouguet deflagration, i.e., \( |v_+| < |v_-| = c_s \) [15]. The wall velocity \( \xi_w = (\tilde{v}_- + c_s) / (1 + \tilde{v}_- c_s) \) is supersonic and depends on the value of \( \tilde{v}_- \). For \( \tilde{v}_- = 0 \) this solution matches the “ordinary” deflagration considered before. As \( \xi_w \) approaches \( v_{j,\text{det}} \), the shock wave gets shorter, i.e., \( \xi_{sh} \rightarrow \xi_w \), and the profile matches that of the detonation considered before.

The calculation of the boundary value \( \tilde{v}_1 \) for the shock wave profile is very similar to that of the ordinary deflagration. Indeed, Eqs. (32-37) hold in this case. Since the deflagration is now Jouguet, we have the condition \( v_- = -c_s \) instead of \( v_- = -v_w \). As a consequence, Eq. (37) becomes \( \alpha_+ = \gamma^2_+ (v_+ + 1/\sqrt{3})^2 \), and Eq. (38) gets replaced by

\[
\alpha_+ = \frac{\gamma^2_+ v^2_w}{3} \left(1 - \sqrt{3}v_w - \tilde{v}_+(v_w - \sqrt{3})\right)^2. \tag{40}
\]

From Eqs. (32) and (40) we eliminate \( \alpha_+ \) and we obtain \( \tilde{v}_1 \) as a function of \( \xi_w \) and \( \alpha_N \) as before. The enthalpy profile of the shock wave is determined by the value \( w_1 \) at the shock, given by Eq. (39).

The profile of the rarefaction wave is similar to that of the detonation, with the boundary condition \( v(\xi_w) = \tilde{v}_- \), with \( \tilde{v}_- \) now given by

\[
\tilde{v}_- = \frac{\xi_w - c_s}{1 - \xi_w c_s}. \tag{41}
\]

The enthalpy profile behind the wall is determined by the condition \( w(\xi_w) = w_- \). The value of the enthalpy just behind the wall is now given by

\[
w_- = (2/\sqrt{3})|v_+|\gamma^2_+ w_+. \tag{42}
\]
with \( v_+ \) given by
\[
v_+ = (\tilde{v}_+ - \xi_w) / (1 - \tilde{v}_+ \xi_w),
\]
and \( \tilde{v}_+ \) and \( w_+ \) are obtained from the shock wave profile. Figure 3 shows the profile of a supersonic deflagration for the three geometries.

\[\text{Figure 3: Left: the fluid velocity profile of a deflagration with } v_w = 0.7 \text{ and } \alpha_N = 0.1 \text{ for a spherical wall (solid), a cylindrical wall (dashed), and a planar wall (dotted). Right: the corresponding kinetic energy density profiles.}\]

### 3.4 Energy injected into the plasma

The latent heat released at the phase transition fronts spreads out in the plasma. Part of this energy causes reheating, and another part causes bulk motions. Some of the consequences of the phase transition will depend on the thickness of the plasma shell where the energy is concentrated. The issue of reheating was addressed, e.g., in Refs. [5, 19] for the electroweak phase transition. The size of the regions of reheated plasma affects the dynamics of the phase transition and, as a consequence, the baryogenesis mechanism. Here we shall focus on the energy in bulk motions of the plasma.

The bubble walls set the fluid moving forward with velocities \( \tilde{v}_+ \) and \( \tilde{v}_- \). In the left panels of Figs. 1, 2, and 3 we observe that these velocities are very similar for different wall geometries. However, away from the wall the fluid velocity decays faster for more symmetric bubbles, since in that case there is more room for the energy to get distributed. For illustrative purposes, we define the shell where kinetic energy is concentrated as the region around the wall where the kinetic energy density remains higher than half the maximum on each side of the wall. Thus, the thickness is given by \( \delta \xi = \xi_+ - \xi_- \), where \( \xi_+ \) and \( \xi_- \) are determined by the condition \( e_{\text{kin}}(\xi_{\pm}) = 0.5 w_{\pm} \tilde{v}_{\pm}^2 \gamma_{\pm}^2 \). We plot the value of \( \delta \xi \) in Fig. 4. The shape of the curves is different from those given in Ref. [22] for the spherical case, since we have defined \( \delta \xi \) differently. Nevertheless, the general structure is similar: the thickness is larger for subsonic deflagrations than for detonations, and is quite small for Jouguet solutions. We also see that the energy is more widely distributed in the planar case, especially for weak deflagrations.

Some of the cosmological remnants of the phase transition do not depend on the thickness of the region of perturbed fluid around the wall. This is the case, e.g., when the
Figure 4: The thickness $\delta \xi$ of the region around the wall inside which the kinetic energy density decreases to a half of its maximum value, for $\alpha_N = 0.1$ as a function of $v_w$. Solid lines correspond to spherical bubbles, dashed lines correspond to cylindrical bubbles, and dotted lines correspond to planar walls. Subsonic deflagrations are plotted in blue, supersonic deflagrations are in black, and detonations in red.

generating mechanism is based on the turbulence produced by the colliding bubble walls. Eddies are formed at all size scales up to the size of the largest bubbles.

It is usually assumed that detonation walls cause a stronger disturbance of the fluid than deflagration walls, since detonations have higher velocities. For weak detonations, however, as pointed out in Ref. [25], the higher the wall velocity, the smaller the disturbance of the fluid. Thus, the strongest disturbance is caused by the Jouguet detonation (which is the case usually considered in the GW literature). However, this is just a limiting case; real detonations are generally weak. Furthermore, as noticed in Ref. [21], ordinary deflagrations may cause important perturbations in the fluid if they are close to the Jouguet limit $v_w = c_s$. In Fig. 5 we give the efficiency factor $\kappa$ as a function of $v_w$ for several values of $\alpha_N$. We calculated $\kappa$ numerically for the spherical and cylindrical cases, and analytically for the planar case (see section 4). For the spherical case (solid lines), our results agree with the numerical fits provided in Ref. [22]. As expected, for fixed $\alpha_N$, the efficiency is larger for stronger solutions (i.e., solutions which are closer to Jouguet processes). Thus, for subsonic deflagrations the efficiency factor increases with the wall velocity, whereas for detonations $\kappa$ decreases with $v_w$. The efficiency peaks for supersonic deflagrations, which are Jouguet processes. It is interesting to notice that even subsonic deflagrations can give larger efficiency factors than detonations.

We see that the different geometries give in general similar values of $\kappa$, except for the case of small wall velocities ($v_w \lesssim 0.2$). This case will not be interesting in general for GW generation, since the efficiency factor is small. For faster walls, the three curves are quite close; they only separate somewhat for supersonic deflagrations (in Fig. 5 the differences at the top of the curves are less than a 20%). In any case, it is clear from Fig. 5 that the difference between geometries will always be at most an $\mathcal{O}(1)$ factor, except for uninterestingly small values of $\kappa$. 

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Figure 5: The efficiency factor \( \kappa \) as a function of \( v_w \) for \( \alpha_N = 0.01, 0.03, 0.1, 0.3 \). Solid lines correspond to the spherical case, dashed lines correspond to the cylindrical case, and dotted lines correspond to the planar case. Subsonic deflagrations are plotted in blue, supersonic deflagrations are in black, and detonations in red.

4 Planar walls: analytic results

In this section we calculate analytically the kinetic energy density and the efficiency coefficient for planar walls. In this case the “bubble” consists of two planar walls at positions \( x = v_w t \) and \( x = -v_w t \), and is equivalent to a bubble in 1+1 dimensions. The system is symmetric under reflection through a plane, and we need only consider the wall moving to the right. The absence of a length scale in the problem implies that the profiles depend only on \( \xi = x/t \). The solutions of the fluid velocity equation for the planar case are well known. For \( j = 0 \), Eq. (9) implies either that \( v'(\xi) \equiv 0 \) or that \( \mu(\xi, v) \equiv \pm c_s \). The latter implies that \( v = \mu(\xi, \pm c_s) \). These solutions are shown in Fig. 3 for \( \xi \geq 0 \) and \( v \geq 0 \). We need only consider that quadrant since the wall at \( \xi_w = v_w \) sets the fluid moving forward (the reflected profiles around the opposite wall \( \xi = -v_w \) are constructed with the solutions for \( \xi < 0 \) and \( v < 0 \)). As we shall see, the solution \( v = \mu(\xi, -c_s) \) will not take part in the fluid profile. Indeed, the possible values of the fluid velocity will be those below the curve \( v = \xi \) (in dots in Fig. 3) due to the fact that, at the wall, the fluid velocity fulfills \( \tilde{v}_\pm < \xi_w \) (since in the wall frame we have \( v_\pm < 0 \)). As a consequence, the physical solutions are either the constants or the “rarefaction”

\[
v_{\text{rar}}(\xi) = \frac{\xi - c_s}{1 - c_s \xi}.
\]

The solution \( v_{\text{rar}} \) is positive only for \( \xi \geq c_s \). Between \( \xi = c_s \) and \( \xi = 1 \), \( v_{\text{rar}}(\xi) \) grows monotonically from \( v = 0 \) to \( v = 1 \).

More complicated profiles will arise if one considers two bubbles nucleated at a distance \( d \) [13]. In particular, this separation introduces a new length scale in the problem.
The enthalpy is readily obtained as well. For the case $v' \equiv 0$ the enthalpy is a constant. For the case $\mu \equiv c_s$, the integral in Eq. (12) is simple; given the condition $w(\xi_a) = w_a$, we obtain

$$w = \frac{w_a \left( 1 - v_a \right)^2}{1 + v_a - v}$$

(45)

The velocity profile must be constructed by matching solutions in such a way that the discontinuity conditions at the bubble wall and the boundary conditions are satisfied (shock discontinuities may also be needed). Before obtaining the analytical results, we shall examine all the possible profiles and argue that only the three kinds of solutions considered in section 3 are acceptable. The argument is simpler for the planar case, but the generalization to the other wall geometries is straightforward.

### 4.1 Hydrodynamic processes and fluid profiles

The fluid velocity must vanish far in front of the wall, where no signal from the phase transition front has arrived yet. Since there are no decreasing solutions, we see that, no matter what the position $\xi_w$ of the wall is, the fluid velocity must have a jump from $v > 0$ to $v = 0$. This discontinuity can be either at $\xi = \xi_w$ or at some point $\xi_{sh}$ between $\xi_w$ and 1.

In the former case, we have $v = 0$ for all $\xi > \xi_w$. In particular, $\tilde{v}_+ = 0$ and, therefore, $\xi_w = |v_+|$. Besides, the fluid velocity behind the wall must be nonvanishing, i.e., $\tilde{v}_- > 0$. Thus, we have $\tilde{v}_- > \tilde{v}_+$ and, hence, $|v_+| > |v_-|$. Therefore, the hydrodynamical process is a detonation. As a consequence, the wall is supersonic, since $\xi_w = |v_+| \geq \nu^{\text{det}} > c_s$.

The fluid velocity must vanish also (by symmetry) at the bubble center. According to Eq. (23), the velocity cannot have positive jumps in the same phase. Therefore, $v(\xi)$ must grow continuously from 0 to $\tilde{v}_-$. The only possibility is that $v \equiv 0$ for $\xi < c_s$ and $v = v_{\text{rar}}(\xi)$ for $\xi \geq c_s$. At some point between $c_s$ and $\xi_w$, the solution $v_{\text{rar}}(\xi)$ may be matched again to a constant $v \equiv \tilde{v}_-$, or it may continue growing until $\xi = \xi_w$. In any case,
we have $\tilde{v}_- \leq v_{\text{rar}}(\xi_0) = \mu(\xi_w, c_s)$, which implies that $|v_-| \geq c_s$. Hence, the detonation can only be weak or Jouguet.

If the fluid velocity does not vanish in front of the wall (i.e., $\tilde{v}_+ > 0$), then it is clear from Fig. 6 that the profile must have a shock discontinuity at some point $\xi_{sh} > \xi_w$, so that $v = 0$ for $\xi > \xi_{sh}$. The fluid velocity $\tilde{v}_1$ at the shock front is given by Eq. (24) and shown in a dashed line in Fig. 6. We see that the solution $v_{\text{rar}}$ lays completely on the right of the curve of $\tilde{v}_1(\xi_{sh})$. Therefore, between $\xi_w$ and $\xi_{sh}$ the solution must be a constant, $v \equiv \tilde{v}_+ = \tilde{v}_1$.

Behind the wall, $v$ can grow continuously from 0 only if the wall is supersonic.

If the wall is subsonic, then the fluid velocity must be $v \equiv 0$ for $\xi < \xi_w$ (otherwise we would need a positive jump). In this case we have $\tilde{v}_- < \tilde{v}_+$, which implies $|v_+| < |v_-|$, and the process is a deflagration. Furthermore, since $\tilde{v}_- = 0$ we have $|v_-| = \xi_w \leq c_s$, i.e., the deflagration is weak or, at most, Jouguet.

If the wall is supersonic, we still have solutions for which $v \equiv 0$ behind the wall. In this case, the condition $\tilde{v}_- = 0$ implies a strong deflagration ($|v_-| = \xi_w > c_s$). However, numerical simulations indicate that strong deflagrations are unstable [12, 15]. Notice that, since $\xi_w$ is now $> c_s$, we may have a non-vanishing fluid velocity $\tilde{v}_-$ behind the wall, as in the case in which $\tilde{v}_+ = 0$. Like in that case, we have the condition $\tilde{v}_- \leq v_{\text{rar}}(\xi_w)$ but, instead of $\tilde{v}_+ = 0$, we now have $\tilde{v}_+ = \tilde{v}_1(\xi_{sh}) > v_{\text{rar}}(\xi_w)$ (see Fig. 6). Hence, we have $\tilde{v}_- < \tilde{v}_+$ and, thus, $|v_+| < |v_-|$. Therefore, the present case is again a deflagration, not a detonation. Now $\tilde{v}_-$ can take any value, with the only condition $\tilde{v}_- \geq c_s$, i.e., the deflagration must be strong or Jouguet. Of all these solutions, though, one expects that the stable one will be that which causes the smallest perturbation of the fluid, i.e., the Jouguet deflagration $|v_-| = c_s$. This is supported by numerical simulations. As we shall see, the supersonic deflagration matches the detonation solution at $\xi_w = v_{\text{jet}}$.

Thus, a subsonic phase transition front always propagates as a weak deflagration and is preceded by a shock wave. A supersonic phase transition front is always followed by a rarefaction wave, and may propagate either as a Jouguet deflagration preceded by a shock front, or as a detonation, without a shock wave.

### 4.2 Detonations

For detonations, the fluid velocity is given by $v = v_{\text{rar}}(\xi)$ between $\xi = c_s$ and a certain $\xi_0 \leq \xi_w$, and by $v \equiv \tilde{v}_-$ between $\xi_0$ and $\xi_w$ (see Fig. 11). The matching condition $v_{\text{rar}}(\xi_0) = \tilde{v}_-$ determines the value of $\xi_0$ as a function of $\tilde{v}_-$,

$$\xi_0 = \frac{\tilde{v}_- + c_s}{1 + \tilde{v}_- c_s}. \quad (46)$$

The velocity $\tilde{v}_-$ is given by Eqs. (29) and (30) as a function of $\alpha_+ = \alpha_N$ and $v_+ = -\xi_w$. The enthalpy is a constant $w \equiv w_-$ for $\xi_0 < \xi < \xi_w$, where $w_-$ is given by Eq. (31) as a function of $w_N$ and $\xi_w$. Between $c_s$ and $\xi_0$, the enthalpy is given by Eq. (43), with the condition $w(\xi_0) = w_-$. Inserting the velocity profile (44), we obtain

$$w = w_- \left( \frac{1 - c_s 1 - \tilde{v}_- 1 + \xi}{1 + c_s 1 + \tilde{v}_- 1 - \xi} \right)^{2/\sqrt{3}}. \quad (47)$$
Using Eqs. (47) and (44), the efficiency coefficient (5) is given by
\[
\kappa = \frac{w_+ - \xi w}{\xi_w \varepsilon} \left[ \tilde{v}_+^2 (\xi_w - \xi_0) + \frac{3}{2} \left( 2 - \sqrt{3} \right)^{2/\sqrt{3}} \left( \frac{1 - \tilde{v}_-}{1 + \tilde{v}_-} \right)^{2/\sqrt{3}} I \right],
\]
where \( I \) is the integral
\[
I = \int_{c_s}^{\xi_0} (\frac{1 + \xi}{1 - \xi})^{2/\sqrt{3}} (\xi - c_s)^2 \frac{1}{1 - \xi^2} d\xi.
\]
The change of variable \( x = (1 + \xi) / (1 - \xi) \) leads to the simpler expression
\[
I = \int_{1}^{2} x^{2/\sqrt{3} - 1} \left( 1 - c_s - \frac{2}{x + 1} \right)^2 dx.
\]
This integral can be expressed in terms of the hypergeometric function \( _2F_1 \) [26]. We obtain
\[
I = \frac{1}{2} [f(\xi_0) - f(c_s)],
\]
where
\[
f(\xi) = \left( \frac{1 + \xi}{1 - \xi} \right)^{2/\sqrt{3}} \left\{ \frac{2}{\sqrt{3}} - 1 + (1 - \xi) \left[ 2 - _2F_1(1,1, \frac{2}{\sqrt{3}}, 1, \frac{1 + \xi}{2}) \right] \right\}.
\]

### 4.3 Subsonic deflagrations

The profile for subsonic deflagrations is very simple in the planar case. The fluid velocity is a constant \( v \equiv \tilde{v}_+ = \tilde{v}_1 \) between \( \xi_w \) and \( \xi_{sh} \), and vanishes outside that region. Thus, we have \( w_+ = w_1 \) and \( \alpha_+ = \alpha_1 \), and Eqs. (32-38) give an algebraic equation for \( \tilde{v}_1 \). The equation is simpler in terms of \( \xi_{sh} \),
\[
(3 \xi_{sh}^2 - 1)^2 + \xi_{sh} \left( 3 \xi_{sh}^2 - 1 \right) \left( \frac{1 - 3 \xi_w^2}{\xi_w} \right) = \frac{9}{2} \alpha_N / \gamma_{sh}^4.
\]
Solving for \( \xi_{sh} \) as a function of \( \alpha_N \) and \( \nu_w \) amounts to finding the roots of a quartic polynomial. The algebraic expressions for the solutions are quite cumbersome and we shall not write them down. Only one of the four solutions gives \( \xi_{sh} \geq c_s \). The integral in Eq. (5) is trivial since \( v \) is a constant, and the efficiency factor is given by
\[
\kappa = \frac{1}{\xi_w \varepsilon} \tilde{v}_1^2 \gamma_1^2 (\xi_{sh} - \xi_w),
\]
where \( \tilde{v}_1 \) is given by Eq. (24) as a function of \( \xi_{sh} \), and \( w_1 / \varepsilon \) is given by Eq. (39) as a function of \( \xi_w \) and \( \alpha_N \).

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5 See Eqs. 3.194-1, 9.131-1 and 9.137-2 of Ref. [26].
4.4 Supersonic deflagrations

In this case the shock wave in front of the wall is similar to that of subsonic deflagrations. We have again \( \tilde{v}_+ = \tilde{v}_1, w_+ = w_1, \) and \( \alpha_+ = \alpha_1. \) From Eqs. (35), (40), and (24) we obtain

\[
\gamma_w^2 \left[ \xi_{sh}(1 - \sqrt{3}\xi_w) - \frac{3\xi_{sh}^2 - 1}{2}(\xi_w - \sqrt{3}) \right]^2 = \frac{9\alpha_N}{4\gamma_{sh}^4}, \tag{55}
\]

which trivially reduces to a quadratic equation. The solution is

\[
\xi_{sh} = \sqrt{\frac{1}{3} + 2\frac{\sqrt{\alpha_N}}{\xi_w} \frac{x + \left( \xi_w - \frac{1}{\sqrt{3}} \right)^2}{3x^2} + \frac{\xi_w - \frac{1}{\sqrt{3}}}{\sqrt{3}x}}, \tag{56}
\]

where \( x = \sqrt{3} - \xi_w + \sqrt{\alpha_N}/\gamma_w. \) The rarefaction wave behind the wall is given by the solution \( v_{rar}(\xi). \) In the Jouguet case, \( \tilde{v}_- \) is given by Eq. (41), which implies that \( \xi_0 = \xi_w. \) The efficiency factor is given by

\[
\kappa = \frac{w_1}{\xi_w \varepsilon} \left[ f(\xi_w) - f(c_s) \right] + \frac{w_1}{\xi_w \varepsilon} \tilde{v}_1^2 \tilde{v}_1 (\xi_{sh} - \xi_w). \tag{57}
\]

The value of \( w_- \) in the Jouguet deflagration case is given by Eq. (42) as a function of \( w_+ = w_1, \) and depends on \( v_+ = (\tilde{v}_1 - \xi_w)/(1 - \tilde{v}_1 \xi_w), \) with \( \tilde{v}_1 \) given by Eq. (24). The value of \( w_1/\varepsilon \) is again given by Eq. (39) as a function of \( \xi_w \) and \( \alpha_N. \)

For \( \xi_w = v_{det}^j(\alpha_N), \) Eq. (56) gives \( \xi_{sh} = \xi_w \) [this is more easily checked by setting \( \xi_{sh} = \xi_w \) in Eq. (55)]. Hence, at the Jouguet detonation velocity, the shock disappears and the profile for the supersonic deflagration matches the profile for the detonation.

5 Gravitational waves from real detonations and deflagrations

In section 3 we have discussed the disturbance that phase transition fronts cause on the plasma. In the present section we consider a particular consequence of such disturbance, namely, the generation of gravity waves. As we have already mentioned, gravitational radiation can only be produced once bubbles collide and lose their spherical symmetry. In fact, the “bubble collisions” mechanism [6, 7, 27, 28] is based on the envelope approximation [6], which consists of taking into account only the motion of the uncollided walls. The thickness of the shell in which the energy of the fluid is concentrated is relevant for this mechanism, and the envelope approximation assumes that the energy concentrations are infinitesimally thin. On the other hand, at a cosmological phase transition the Reynolds number is large enough for bubble collisions to cause the onset of turbulence [7]. Turbulence turns out to be a more effective source of gravitational radiation than bubble collisions [7, 29, 30]. In an electrically conducting fluid and in the presence of magnetic fields, turbulence develops in a completely different way. This gives a third mechanism for generation of GWs in a phase transition (see e.g. [30, 31]).
The energy density of gravitational radiation is usually expressed in terms of the quantity 
\[ h^2 \Omega_{GW}(f) = \frac{h^2}{\rho_c} \frac{d \rho_{GW}}{d \log f}, \] (58)
where \( \rho_{GW} \) is the energy density of the GWs, \( f \) is the frequency, and \( \rho_c \) is the critical energy density today, \( \rho_c = 3H_0^2/8\pi G \), with \( H_0 = 100 h \text{ km s}^{-1} \text{Mpc}^{-1} \). The GW spectrum depends on the details of the phase transition and the generating mechanism. Nevertheless, the peak frequency \( f_p \) is generally determined by the typical length scale of the source. In a first-order phase transition, the latter is the bubble size \( d \), which is proportional to the duration \( \Delta t \) of the phase transition, \( d \sim v_w \Delta t \). The time \( \Delta t \) is in turn a fraction of the Hubble time. Once redshifted to today, the peak frequency is roughly given by
\[ f_p \sim 10^{-2} \text{mHz} \frac{H_*^{-1}}{\Delta t} \frac{T_*}{100 \text{GeV}} \] (59)
where \( H_* \) and \( T_* \) are the Hubble rate and the temperature at the moment of the phase transition. The sensitivity peak of the space interferometer LISA is expected to be \( h^2 \Omega \sim 10^{-12} \) at a frequency \( f \sim 1 \text{mHz} \). Quite interestingly, GWs produced at the temperature scale of the electroweak phase transition, \( T_* \sim 100 \text{GeV} \), will have a characteristic frequency around \( f_p \sim 1 \text{mHz} \) for \( \Delta t \sim 10^{-2} H_*^{-1} \), which is a possible value for the duration of the phase transition. This motivated the investigation of the GW signal from the electroweak phase transition [25, 32, 33].

For simplicity, we shall consider only GWs from turbulence, for which we shall use the analytic approximation obtained in Ref. [30] (we have checked, using the fit given in Ref. [28], that the intensity resulting from bubble collisions is an order of magnitude smaller). The GW energy density \( \Omega_p \) at the peak frequency depends on the length scale \( d \sim v_w \Delta t \). Thus, we have
\[ \Omega_p \approx \frac{9\Omega_R}{32\pi} v_w^2 \left( \frac{\Delta t}{H_*^{-1}} \right)^2 \kappa^2 \alpha_N^2 \begin{cases} 4 & \text{for } v \leq 1/2, \\ 1/v^2 & \text{for } v \geq 1/2, \end{cases} \] (60)
where \( \Omega_R \approx 5 \times 10^{-5} \) is the radiation energy density parameter \( \Omega_R = \rho_R/\rho_c \) today, and \( v \) is the characteristic eddy velocity, defined by \( v^2 = \frac{3}{2} \kappa \alpha_N \). Equation (60) should be valid both for detonations and deflagrations, although the time \( \Delta t \) must be calculated differently in each case.

Notice that \( \Omega_p \) is proportional to \( v_w^2 \) and to \( \kappa^2 \). The wall velocity is determined by hydrodynamics and by microphysics (see, e.g., [12]). As a consequence, \( v_w \) depends on a friction parameter \( \eta \) as well as on the nucleation temperature \( T_N \). Different approximations have been used for the friction force (see, e.g., [17, 22]). For generality, we shall leave the result expressed in terms of \( v_w \). We do not expect \( v_w \) to depend significantly on the geometry. In the case of deflagrations, the dependence of \( v_w \) on \( T_N \) is affected by the shock wave, which depends on the wall geometry. In the case of detonations, \( v_w \) is completely determined by the discontinuity equations (15) and the friction, and does not depend on the geometry at all. In any case, as explained before, there is no reason to assume any particular symmetry after bubbles collide, and we shall use the planar wall results.
In Fig. 7 we plot the peak amplitude $\Omega_p$ of GWs from turbulence at the electroweak phase transition for some values of $\alpha_N$, as a function of the wall velocity $v_w$. We chose $\Delta t H_* = 10^{-2}$, which gives $f_p \sim 1 \text{mHz}$. We see that the phase transition needs not be too strong, i.e., with $\alpha_N \gtrsim 0.3$ we obtain intensities above the peak sensitivity of LISA. This is important, since the value of $\alpha_N$ was found to be $\alpha_N \lesssim 1$ for several models of the electroweak phase transition [33]. For $\alpha_N \sim 1$ we get $h^2 \Omega_p$ as large as $\sim 10^{-9}$.

![Figure 7: The intensity of gravitational radiation from turbulence at a phase transition with $\Delta t H_* = 10^{-2}$ and $T_* = 100\text{GeV}$, for $\alpha_N = 0.03, 0.1, 0.3, 1, \text{and} 3$ (from bottom to top). The dashed line corresponds to the peak sensitivity of LISA.]

Since only Jouguet detonations are in general considered in the literature on GWs, the efficiency factor is in principle overestimated with respect to real detonations. However, as pointed out in Ref. [22], the value of the Jouguet detonation efficiency factor $\kappa_J(\alpha_N)$ is underestimated in the literature, due to a missing factor $v_w^3$ in the original paper [7]. This mistake compensates the fact that $\kappa_J(\alpha_N)$ is larger than the efficiency factor $\kappa(\alpha_N, v_w)$ for weak detonations. The effect of this compensation on the intensity of GWs is shown in Fig. 8, where we plotted $\Omega_p$ as a function of $v_w$ for $\alpha_N = 0.3$, together with the correct (upper dashed line) and the wrong (lower dashed line) values for the Jouguet detonation case. The efficiency factor for weak detonations lies between these two values. Notice, also, that supersonic deflagrations give the largest GW amplitudes, and that even subsonic deflagrations can give intensities comparable to those of detonations.

6 Conclusions

We have studied the motion of phase transition fronts in a first-order cosmological phase transition, focusing on the energy injected into bulk motions of the plasma, which is a relevant quantity for the generation of cosmological relics such as gravitational waves. This issue was recently addressed in Ref. [22] for the case of spherical bubbles. However, the GWs are generated once the bubbles (or the shock fronts) collide, so that the spherical symmetry is lost. Therefore, any bubble symmetry one may assume will be just an ap-
proximation. In order to study the dependence on the wall geometry, we have considered bubble walls with spherical, cylindrical, and plane symmetry, for all the possible hydrodynamic propagation modes, namely, subsonic deflagrations, supersonic deflagrations, and detonations.

We have seen that the region around the wall in which the energy spreads can be rather different for each wall geometry. In particular, for planar walls the region is larger, since the energy spreads in only one direction. For the strongest processes allowed, i.e., Jouguet deflagrations and Jouguet detonations, the kinetic energy of the fluid is, for the three geometries, concentrated in a thin region around the wall. This is because these processes produce a stronger disturbance of the plasma than weak processes; thus, the injected energy is larger and, hence, more difficult to distribute.

The efficiency factor $\kappa$, i.e., the part of the injected energy which goes into bulk motions (relative to the released vacuum energy), has a rather weak dependence on the wall geometry. This is an important result, since the walls can take arbitrary forms after colliding. The dependence on the wall geometry is stronger for small wall velocities, $\xi_w \lesssim 0.2$. For small velocities, however, the efficiency factor is small and will not play a relevant role in the cosmological consequences of the phase transition. Thus, it is clear that, for applications, it is convenient to consider planar walls, for which we have obtained exact analytical expressions for $\kappa$ (alternatively, one can use the numerical fits given in Ref. [22] for the spherical case).

The efficiency factor peaks for supersonic deflagrations, which are Jouguet processes. Thus, $\kappa$ is in general sizeable for fast (close to the speed of sound) subsonic deflagrations, and for the slowest detonations, i.e., those solutions which are close to the Jouguet velocity. On the other hand, $\kappa$ can decrease considerably for fast detonations, and vanishes for $v_w \to 0$. We have applied the results for the planar case to the estimation of the gravitational wave signal from turbulence at the electroweak phase transition. The GW amplitude peaks for supersonic deflagrations. It is interesting to notice that subsonic deflagrations

Figure 8: The same as Fig. 7 for $\alpha_N = 0.3$ alone. The dashed lines indicate the results obtained from the correct (upper) and the wrong (lower) values of $\kappa_J (\alpha_N)$. The two points indicate the cases $v_w = c_s$ and $v_w = \nu_{dJ} (\alpha_N)$, which separate subsonic deflagrations, supersonic deflagrations, and detonations.
can give efficiency factors larger than those given by detonations. Although the amplitude of the gravity waves may depend (according to the generation mechanism) on the wall velocity as well as on the efficiency factor, we have seen that subsonic deflagrations can produce GWs of intensity comparable to that of detonations.

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