Mc-Donald modified Burr-III distribution: properties and applications

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ABSTRACT

We propose herein a six-parameter McDonald modified Burr-III distribution. This distribution contains several special cases, including the modified Burr-III, Burr-III, Beta Burr-III, and Kum Burr-III distributions. Accordingly, we obtained its hazard function, survival function, moments, Renyi’s entropy, β-entropy, and mean deviation. We also used the method of maximum likelihood to estimate the parameters of our proposed distribution. Simulation studies were also performed for different values of sample sizes. The proposed model is applied on real-life datasets and shows that our proposed distribution yields better fits than other existing models.

1. Introduction

1.1. Burr–III distribution

Burr [1] introduced a system based on twelve types of cumulative distribution functions. These cumulative distribution functions are based on the Pearson system of distributions which have a variety of density shapes. The Burr system is valid for a large range of applications. Based on the system of Burr distributions, the Burr–XIII distribution is a commonly used model. The Burr–III distribution has also been used in various settings for statistical modelling. Gove et al. [2] and Lindsay et al. [3] used the Burr–III distribution in forestry. The Burr–III model was used by Nadarajah [4] and Kotz [5] in fracture roughness, and by Wingo [6,7] in life testing. Additionally, the model was also used by Chernobai et al. [8] for operational risks and by Sherrick et al. [9] for option market price distributions. In turn, Mielke and Paul [10] used it in meteorology, Tejeda and Goodwin used it to model rice crops [11], and Abdel-Ghaly et al. [12] used it in reliability analyses.

The Burr–III distribution is the inverse of the BX–II distribution. The Burr–III distribution is used in various fields of science. In studies of wages, wealth, and income, the Burr–III distribution is called the Degum distribution [13]. The Burr–III distribution is either equal to the inverse of the Burr distribution [14] or the inverse of the kappa distribution [10] in the respective literatures of actuarial and meteorological sciences. The extended Burr–III distribution was also used in low-flow frequency analyses by Shao et al. [15].

Ali et al. [16] defined MB–III as

\[
F(x) = [1 + γx^{-β}]^{-\frac{γ}{γ}} , \alpha, \beta, γ > 0, x > 0
\]  

(1)

Accordingly, the pdf of the MB–III is \( f(x) \)

\[
= αβx^{-β} \left[1 + γx^{-β}\right]^{-\frac{γ}{γ} - 1},
\]

(2)

where \( α, β, \) and \( γ, \) are the shape parameters of the MB–III distribution. The limiting distribution for \( γ \to 0 \) of the modified Burr–III distribution becomes a generalized inverse Weibull distribution as indicated by Gusmao et al. [17]. If \( γ = 1, \) the MB–III distribution becomes the Burr–III distribution [1]. The modified Burr–III distribution reduces to a log–logistic distribution if \( α = γ = 1 \) and if a scale parameter is added, as indicated by Shoukri et al. [18].

2. McDonald modified Burr–III (McMB–III) distribution

2.1. McMB–III distribution

If the random variable \( x \) follows an McMB–III distribution, then it is defined as

\[
F(x) = \frac{1}{B(\lambda, \eta)} \sum_{j=0}^{∞} (-1)^j χ^n \frac{1}{(κ - j)!} \left( λ + j \right) \\
\times (1 + γx^{-β})^{-\frac{αβ}{γ} - \frac{1}{η}} , \alpha, β, γ, λ, η, κ > 0, x > 0
\]

(3)
where the probability density function is

\[
f(x; \alpha, \beta, \gamma, \lambda, \eta, \xi) = \frac{\xi \alpha \beta}{B(\lambda, \eta)} x^{-\beta - 1} \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha \xi}{\eta} - 1} \times \left(1 - (1 + \gamma x^{-\beta})^{-\frac{\alpha \xi}{\eta}}\right)^{\eta - 1}
\]

where \(\alpha, \beta, \gamma, \lambda, \eta, \xi\) are the shape parameters.

The survival function of the McMB–III distribution is given by

\[
S(x) = 1 - \left(\frac{1}{B(\lambda, \eta)} \sum_{j=0}^{\infty} (-1)^j \Gamma(\eta - j) B(\lambda + j) (1 - (1 + \gamma x^{-\beta})^{-\frac{\alpha \xi}{\eta}})^{\eta - 1}\right)
\]

The Hazard function of the McMB–III distribution can be defined as

\[
h(x) = \frac{\xi \alpha \beta}{B(\lambda, \eta)} x^{-\beta - 1} \left(1 + \gamma x^{-\beta}\right)^{-\frac{\alpha \xi}{\eta} - 1} \times \left(1 - (1 + \gamma x^{-\beta})^{-\frac{\alpha \xi}{\eta}}\right)^{\eta - 1}
\]

Plots of the pdf, CDF, survival function, and hazard function for various values of \(\alpha, \beta, \gamma, \lambda, \eta, \xi\), are given below Figures 1 and 2.

The graphs of (3) for various values of \(\alpha, \beta, \gamma, \lambda, \eta, \xi\) are plotted in Figure 3 and Table 1.

3. Properties of McMB–III distribution

In this subsection we discuss the \(k\)th moments for the McMB–III distribution. Moments are necessary and important in any statistical analyses, especially in different types of applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness, and kurtosis).

The \(k\)th moments of the McMB–III distribution are obtained as

\[
\mu' k = \frac{\xi \alpha \gamma k/\beta}{\gamma B(\lambda, \eta)} \sum_{j=1}^{\infty} (-1)^j \Gamma(\eta - j) B(A_k, B_k)
\]

where \(B(A_k, B_k)\) is a beta function,

\[
A_k = \frac{k}{\beta} + \frac{\alpha \xi}{\gamma} (\lambda + j), ~ B_k = 1 - \frac{k}{\beta}
\]

and \(k < \beta, 1 < k/\beta\) and \(k/\beta\) should not be integers.

The negative moments for the McMB–III distribution are expressed according to

\[
\mu'' k = \frac{\xi \alpha \gamma r/\beta}{\gamma B(\lambda, \eta)} \sum_{j=1}^{\infty} (-1)^j \Gamma(\eta - j) B(A_k, B_k)
\]

\[
\times \left(\frac{\alpha \xi}{\gamma} (\lambda + j) - \frac{r}{\beta} + 1 - \frac{r}{\beta}\right)
\]

where

\[
\frac{\alpha \xi}{\gamma} (\lambda + j) > \frac{r}{\beta}, \quad \frac{\alpha \xi}{\gamma} (\lambda + j) < \frac{r}{\beta}
\]

and \(r/\beta\) should not be integers.

The factorial moments of the McMB–III distribution are expressed by

\[
\mu''(k) = E[x^k] = E[(x - 1)(x - 2) \cdots (x - k + 1)]
\]

\[
= \sum_{k=0}^{n} S(n, k) E(x^k)
\]

Substitution of (7) in (9) leads to,

\[
\mu''(k) = \sum_{k=0}^{n} S(n, k) \frac{\xi \alpha \gamma k/\beta}{\gamma B(\lambda, \eta)} \sum_{j=1}^{\infty} (-1)^j \Gamma(\eta - j) B(A_k, B_k)
\]

\[
\times \left(\frac{k}{\beta} + \frac{\alpha \xi}{\gamma} (\lambda + j), 1 - \frac{k}{\beta}\right)
\]
Figure 3. (a) Effect of $\alpha$ for fixed $\beta, \gamma, \lambda, \eta,$ and $\xi$ values. (b) Effect of $\beta$ for fixed $\alpha, \gamma, \lambda, \eta,$ and $\xi$ values. (c) Effect of $\gamma$ for fixed $\alpha, \beta, \lambda, \eta,$ and $\xi$ values. (d) Effect of $\lambda$ for fixed $\alpha, \beta, \gamma, \eta,$ and $\xi$ values. (e) Effect of $\eta$ for fixed $\alpha, \beta, \gamma, \lambda,$ and $\xi$ values. (f) Effect of $\xi$ for fixed $\alpha, \beta, \gamma, \lambda,$ and $\eta$ values.

Table 1. Summary of some special cases of the McMB–III distribution.

| Parameters | Distribution Name                        |
|------------|----------------------------------------|
| $\alpha\beta\gamma\lambda\eta\xi$ | Modified Burr–III distribution       |
| $\alpha\beta\gamma\lambda\eta\xi$ | Burr–III distribution                |
| $\alpha\beta\gamma\lambda\eta\xi$ | McDonald Burr–III distribution       |
| $\alpha\beta\gamma\lambda\eta\xi$ | Beta modified Burr–III distribution  |
| $\alpha\beta\gamma\lambda\eta\xi$ | Beta Burr–III distribution           |
| $\alpha\beta\gamma\lambda\eta\xi$ | Kumarasamy modified Burr–III distribution |
| $\alpha\beta\gamma\lambda\eta\xi$ | Kumarasamy Burr–III distribution     |
| $\alpha\beta\gamma\lambda\eta\xi$ | Exponential modified Burr–III distribution |
| $\alpha\beta\gamma\lambda\eta\xi$ | Exponential Burr–III distribution    |
| $\alpha\beta\gamma\lambda\eta\xi$ | Kumarasamy Log–Logistic distribution |
| $\alpha\beta\gamma\lambda\eta\xi$ | Beta Log–Logistic distribution       |
| $\alpha\beta\gamma\lambda\eta\xi$ | Log–Logistic distribution            |

where $0, \beta > k, \quad 1 - \frac{k}{\beta} < 0, \quad \frac{k}{\beta} > \frac{1}{r\beta}$ should not be integers.

The fractional moments for the McMB–III distribution are defined as

$$E\left(x^r\right) = \frac{\xi \alpha^r}{\gamma^r B(\lambda, \eta)} \sum_{j=1}^{\infty} \left(-\frac{1}{\Gamma(\eta-j)r}\right) \left(\frac{\alpha \xi}{\gamma} (\lambda+j) - \frac{1}{r\beta} \frac{1}{1 + \frac{1}{r\beta}}\right)$$

where

$$\frac{\alpha \xi}{\gamma} (\lambda+j) > \frac{1}{r\beta}, \quad \frac{\alpha \xi}{\gamma} (\lambda+j) < \frac{1}{r\beta}, \quad 1/r\beta \text{ should not be integers.}$$

is the Stirling number of the first kind and estimates the number of ways to permute a list of $r$ items into $k$ cycles.
The mean and variance of the McMB–III distribution are given as

\[
\mu = E(x) = \frac{\xi \alpha \gamma^{1/\beta}}{\gamma B(\lambda, \eta)} \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(\eta - j)}{j!} B(\lambda, \eta) \times \left( \frac{1}{\beta} + \frac{\alpha \xi}{\gamma} \right) (\lambda + j, 1 - \frac{1}{\beta}) \tag{12}
\]

\[
\text{Var}(x) = \frac{\xi \alpha \gamma^{2/\beta}}{(\gamma B(\lambda, \eta))^2} \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(\eta - j)}{j!} B(\lambda, \eta) \times \left[ B \left( \frac{2}{\beta} + \frac{\alpha \xi}{\gamma} (\lambda + j), 1 - \frac{2}{\beta} \right) - B^2 \left( \frac{1}{\beta} + \frac{\alpha \xi}{\gamma} (\lambda + j), 1 - \frac{1}{\beta} \right) \right] \tag{13}
\]

The central moments are

\[
\mu_r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left( \frac{\xi \alpha \gamma^{1/\beta}}{\gamma B(\lambda, \eta)} \right)^{i+1} \gamma^{(i-\beta)} \times \left( \frac{1}{\beta} + \frac{\alpha \xi}{\gamma} \right) (\lambda + j, 1 - \frac{1}{\beta}) \tag{14}
\]

Correspondingly, the cumulants are

\[
\kappa_r = \frac{\xi \alpha \gamma^{1/\beta}}{\gamma B(\lambda, \eta)} \left( \frac{1}{\beta} - \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(\eta - j)}{j!} B(\lambda, \eta) \right) \times \left( \frac{1}{\beta} + \frac{\alpha \xi}{\gamma} \right) (\lambda + j, 1 - \frac{1}{\beta}) \tag{15}
\]

where \( \kappa_1 = \mu_1 \), then \( \kappa_2 = \mu_2 - \mu_1^2, \kappa_3 = \mu_3 - 3\mu_1 \mu_2 + 2\mu_1^3, \kappa_4 = \mu_4 - 4\mu_1 \mu_3 + 3\mu_1^2 \mu_2 - 6\mu_1^3 \), etc.

Additionally, the coefficient of skewness is

\[
\sqrt{\beta_1} = \frac{3}{\sum_{i=0}^{3} \binom{3}{i} (-1)^i \left( \frac{\xi \alpha \gamma^{1/\beta}}{\gamma B(\lambda, \eta)} \right)^{i+1} \gamma^{(3-i-\beta)}} \times \left( \frac{1}{\beta} + \frac{\alpha \xi}{\gamma} \right) (\lambda + j, 1 - \frac{1}{\beta}) \tag{16}
\]

The variance of the McMB–III distribution is approximately symmetrical. In Figure 3.2, in incomplete moments of the McMB–III distribution, we plot \( \beta_1 \) and \( \beta_2 \) values for the values of \( \alpha, \beta, \gamma, \lambda, \eta, \xi, \) and vary \( \beta \). Both \( \beta_1 \) and \( \beta_2 \) values which are close to the x-axis. Thus, at \( \alpha = 9.24, \beta = 4.43, \gamma = 1.5, \lambda = 1, \eta = 1, \xi = 0.53, \) the McMB–III distribution is approximately symmetrical. Similarly, in Figure 4(c), we fix \( \alpha, \beta, \gamma, \lambda, \eta, \) and vary \( \gamma \). Both \( \beta_1 \) and \( \beta_2 \) values which are close to the x-axis. Thus, at \( \alpha = 9.24, \beta = 4.43, \gamma = 1.5, \lambda = 1, \eta = 1, \xi = 0.53, \) the McMB–III distribution is approximately symmetrical. Similarly, in Figure 4(d), we fix \( \alpha, \beta, \gamma, \lambda, \eta, \xi, \) and vary \( \lambda \). Both \( \beta_1 \) and \( \beta_2 \) values which are close to the x-axis. Thus, at \( \alpha = 9.24, \beta = 4.43, \gamma = 1.5, \lambda = 1, \eta = 1, \xi = 0.53, \) the McMB–III distribution is approximately symmetrical. Similarly, in Figure 4(e), we fix \( \alpha, \beta, \gamma, \lambda, \eta, \xi, \) and vary \( \eta \). Both \( \beta_1 \) and \( \beta_2 \) values which are close to the x-axis. Thus, at \( \alpha = 9.24, \beta = 4.43, \gamma = 1.5, \lambda = 1, \eta = 1, \xi = 0.53, \) the McMB–III distribution is approximately symmetrical. Similarly, in Figure 4(f), we set the values of \( \alpha, \beta, \gamma, \lambda, \eta, \xi, \) at 10, 4.43, 1.5, 1.15, and 0.53, respectively, and vary \( \xi \). For \( \xi = 0.5, \) both \( \beta_1 \) and \( \beta_2 \) values which are close to the x-axis. Thus, at \( \alpha = 9.24, \beta = 4.43, \gamma = 1.5, \lambda = 1, \eta = 1, \xi = 0.53, \) the McMB–III distribution is approximately symmetrical.

### 3.2. Incomplete moments of the McMB–III distribution

Assume that \( f(x) \) is the probability density function of the McMB–III distribution of \( X \) defined on \((0, \infty)\). The incomplete moments of the McMB–III distribution are given as

\[
m_r = \frac{\xi \alpha \gamma^{r/\beta}}{\gamma B(\lambda, \eta)} \sum_{i=1}^{\infty} \frac{(-1)^i \Gamma(r \eta - ji)}{\Gamma(r \eta - ji) \Gamma(\eta - ji)} B(A_{1, i}, 1, B_{-1, i}) \times \left( 1 - \frac{(1 + \gamma x^{-\beta})^{(b-1)}}{\beta(a, b)} \right) \left( \sum_{i=0}^{\infty} (-1)^i \right) \tag{17}
\]
3.3. Doubly truncated moments of McMB–III distribution

Assume that \( f(x) \) is the probability density function of the McMB–III distribution of \( X \) defined on \((0, \infty)\). Doubly truncated moments of McMB–III are expressed according to

\[
E \left( x^i \bigg| \frac{X}{1} \leq x \leq \frac{x}{2} \right) = \frac{(a - 1)\left(1 - \left(1 + \beta x^{-\alpha}\right)^{-1}\right)}{b + i}
\]  

where

\[
a = \frac{r}{\beta} + \frac{\lambda \xi \alpha}{\gamma} + \frac{\alpha \xi j}{\gamma}, \quad b = 1 - \frac{r}{\beta},
\]

and

\[
\frac{\alpha \xi}{\gamma} (\lambda + j) \geq \frac{r}{\beta}, \quad \beta \geq r
\]

3.4. Mean residual life (Moments of residual life)

The moments of mean residual life of the McMB–III are given as

\[
M_n(t) = \frac{1}{1 - F(t)} \frac{\xi \alpha \beta}{B(\lambda, \eta)} \sum_{n=0}^{\infty} \frac{\eta}{(n - i)!} \xi^i (1 + \gamma n - i)^{1 - \frac{i}{\beta} - 1 - \frac{\gamma}{\eta} - \frac{\xi}{\eta} - \frac{\alpha \xi j}{\gamma} + \frac{\alpha \xi j}{\gamma} (1 - \frac{r}{\beta})}
\]

where

\[1 - \frac{i}{\beta} > 0, \quad \beta > i \quad 1 - \frac{i}{\beta} < 0, \quad \frac{i}{\beta}\]

should not be integers.

The reversed residual life moments can be obtained from (20)

\[
M_n(t) = \int_{1}^{\infty} (x - t)^{n} f(x) \, dx
\]

\[
M_n(t) = \frac{\xi \alpha \beta}{B(\lambda, \eta)} \sum_{n=0}^{\infty} \frac{\eta}{(n - i)!} \frac{\xi^i}{(1 + \gamma n - i)^{1 - \frac{i}{\beta} - 1 - \frac{\gamma}{\eta} - \frac{\xi}{\eta} - \frac{\alpha \xi j}{\gamma} + \frac{\alpha \xi j}{\gamma} (1 - \frac{r}{\beta})}}
\]

\[
\times \left(\frac{i}{\beta} + \frac{\lambda \xi \alpha}{\gamma} + \frac{\xi \alpha}{\gamma} (1 - \frac{i}{\beta})\right)
\]
3.5. Mean deviation of the McMMB–III distribution

The mean deviation about the mean and median can be used to measure the spread in a population. Accordingly, it can be expressed as $\delta_1(x) = E( - \mu_1 | -\mu_1 ) = 2\mu_1 F(\mu_1 ) - 2T_1(\mu_1 )$ and $\delta_2(x) = E( -m | -m ) = \mu_1 - 2T_1(m)$, where $T_1(.)$ is a doubly truncated moments in Eq. 19 and $F(.)$ is distribution function in Eq. 3. The mean deviations about the mean and median are given below,

\[
\delta_1(x) = \frac{2\mu_1 \xi \alpha \beta}{\gamma B(\lambda, \eta) \frac{\alpha \xi}{\beta} (\lambda + j)} + 1 \\
\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta - j + 1)}{\Gamma(\eta - j)!} A_1^{-\frac{\alpha \xi}{\beta} (\lambda + j) - 1} - \frac{2\gamma^{\frac{1}{\beta}}}{\beta} \\
\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta - j)}{\Gamma(\eta - j)!} A_2 \\
\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta - j + 1)}{\Gamma(\eta - j)!} A_1^{-\frac{\alpha \xi}{\beta} (\lambda + j) - 1}
\]

Where $A_1 = 1 + \gamma \mu_1^{-\frac{1}{\beta}}$,

\[
A_2 = \sum_{\lambda = 0}^{\infty} \frac{\Gamma\left(\frac{1}{\beta} + j\right)}{\Gamma\left(\frac{1}{\beta} - j\right)!} \\
\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta - j + 1)}{\Gamma(\eta - j)!} A_1^{-\frac{\alpha \xi}{\beta} (\lambda + j) - 1} \\
\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta - j)}{\Gamma(\eta - j)!} A_2 \\
\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\eta - j + 1)}{\Gamma(\eta - j)!} A_1^{-\frac{\alpha \xi}{\beta} (\lambda + j) - 1}
\]

3.6. Measures of uncertainty

In this section, we derive some expressions for the McMB–III distributions of Viz Rényi’s, $\tilde{\beta}$, and Shannon’s entropies. These measures of uncertainty have a broad range of applications in probability theory, science, and engineering. They have also been used in different situations like the measures of variations for uncertainty.

The entropy of a random variable $x$ is defined in terms of its probability distribution, and it can be shown to be a good measure of uncertainty or randomness.

3.6.1. Rényi and Shannon entropy

We derive Rényi’s entropy, which is used to compare the shapes of various densities. It is also used to measure the heaviness of tails. Rényi defined entropy in 1961 as

\[
H_\theta(f) = \log \frac{1}{1 - \theta} \left( \int f^\theta(x) dx \right) = \log \frac{1}{1 - \theta} (I(\theta))
\]

where $\theta > 0, \theta \neq 1$, and $\theta$ is a real noninteger.

The Rényi entropy for the McMB–III distribution is

\[
H_\tilde{\beta}(f) = \frac{1}{\tilde{\beta} - 1} \left( \frac{\xi \alpha \beta}{\gamma} \right)^{\frac{\tilde{\beta}}{\gamma} (\lambda - \beta + 1)} \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(\tilde{\beta} - j + 1)}{\Gamma(\tilde{\beta} - j)!} B \\
\left( \frac{\lambda \xi \alpha \beta}{\gamma} - \frac{\tilde{\beta}}{\beta} + 1 + \frac{\alpha \xi j}{\gamma} \right)^{\tilde{\beta} - 1} \beta
\]

The $\tilde{\beta}$ entropy for the McMB–III distribution is defined as

\[
H_\tilde{\beta}(f) = \frac{1}{\tilde{\beta} - 1} \left( \frac{\xi \alpha \beta}{\gamma} \right)^{\frac{\tilde{\beta}}{\gamma} (\lambda - \beta + 1)} \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma(\tilde{\beta} - j + 1)}{\Gamma(\tilde{\beta} - j)!} B \\
\left( \frac{\lambda \xi \alpha \beta}{\gamma} - \frac{\tilde{\beta}}{\beta} + 1 + \frac{\alpha \xi j}{\gamma} \right)^{\tilde{\beta} - 1} \beta
\]

4. Maximum likelihood estimation

In this section, we estimate the model parameters for the McMB–III distribution $(\alpha, \beta, \gamma, \lambda, \eta, \xi)$ using maximum likelihood estimation. We assume that $x \sim McMB-III(\alpha, \beta, \gamma, \lambda, \eta, \xi)$ and we let the parameter vector be $\Theta = (\alpha, \beta, \gamma, \lambda, \eta, \xi)$. The LL function for $\Theta$ is

\[
I(\Theta) = \frac{\ln(\xi \alpha \beta) - \ln(\beta(\lambda, \eta)) - (\beta - 1) \ln x - (1 + \gamma x^{-\beta}) + (\gamma - 1)}{\ln (1 + \gamma x^{-\beta}) - \frac{\alpha \xi}{\gamma}}
\]

We partially differentiate (Eq. 22) with respect to $\alpha, \beta, \gamma, \lambda, \eta, \xi$, and then equate it to zero

\[
\gamma + \left[ \frac{\alpha \xi (\eta - 1)(1 + \gamma x^{-\beta})^{-\alpha \xi /\gamma} (\ln (1 + \gamma x^{-\beta}))}{1 - (1 + \gamma x^{-\beta})^{-\alpha \xi /\gamma}} \right] - \alpha \lambda \xi (\ln (1 + \gamma x^{-\beta})) = 0
\]
4.225, 4.395, 5.020.

\[ \begin{align*}
3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, \\
3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, \\
3.562, 3.628, 3.871, 3.886, 3.971, 4.024, 4.027, \\
2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, \\
2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.614, 2.616, \\
2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.614, 2.616, \\
2.618, 2.624, 2.659, 2.675, 2.532, 2.575, 2.454, 2.454, \\
2.474, 2.518, 2.522, 2.525, 2.738, 2.420, 2.856, 2.917, 2.928, \\
2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, \\
3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.279, 3.332, \\
3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, \\
3.562, 3.628, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.
\end{align*} \]

\[ \begin{align*}
\beta x^{-\beta} &\ln \left( \frac{\alpha \lambda + \gamma}{1 + \gamma x^{-\beta}} \right) - \beta x^{-\beta} = 0
\end{align*} \]

\[ \begin{align*}
\frac{\alpha \xi}{\gamma} \left( \lambda (1 + \gamma x^{-\beta}) + \frac{(\eta - 1) (1 + \gamma x^{-\beta})}{1 - (1 + \gamma x^{-\beta})} \right)
\times (\alpha \lambda x - (\ln(1 + \gamma x^{-\beta}) + 1))
\end{align*} \]

\[ \begin{align*}
- (\alpha \lambda x + \gamma) x^{-\beta}
\end{align*} \]

\[ \begin{align*}
\alpha \beta B(\lambda, \eta) \ln(1 + \gamma x^{-\beta}) - \gamma B'(\lambda, \eta) = 0
\end{align*} \]

\[ \begin{align*}
B(\lambda, \eta) \ln \left[ 1 - (1 + \gamma x^{-\beta}) \frac{-1}{\gamma} \right] - B' (\lambda, \eta) = 0
\end{align*} \]

\[ \begin{align*}
\frac{\alpha \xi (\eta - 1) (1 + \gamma x^{-\beta}) (1 + \gamma x^{-\beta})}{1 - (1 + \gamma x^{-\beta})} + \gamma
\end{align*} \]

\[ \begin{align*}
- \alpha \lambda \xi (\ln(1 + \gamma x^{-\beta})) = 0
\end{align*} \]

5. Applications

In this section we fit our purposed McMB-III distribution on real life data set. We fit the density functions of the McMB-III, MBIII, BIII, KMBIII and BMBIII distributions.

The first data set from Hussaini and Hussein [19] represents 64 observations for breaking strengths of single carbon fibres of length 10 are 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.532, 2.575, 2.454, 2.454, 2.474, 2.518, 2.522, 2.525, 2.738, 2.420, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.279, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

The second data set of Average Annual Percent Change in Private Health Insurance Premiums from Kirby and Shakil [20] are 14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5, 14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1, 1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7, 5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8

Estimates of the parameters of McMB-III distribution, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Hannan-Quinn Information Criterion are given in Table 2 for the first data set and in Table 3 for the second data set.

Figure 5 fitted densities for breaking strengths of single carbon fibres.

Figure 6 fitted densities for Average Annual Percent Change in Private Health Insurance Premiums.

The McMBIII distribution gives the best fit than MBIII and BIII distributions. In Table 3 W is -2loglikelihood. The new sub-models, BMIII and KMBIII, give less -2loglikelihood and information criterion as compared to our proposed model and old models. There are larger values of AIC, CAIC and HQIC for MBIII and BIII distribution as compared to McMB-III distribution. The McMBIII distribution gives the best fit than B III distribution. In Table 3 W is -2loglikelihood. The new sub-model,

| Table 2. Maximum Likelihood Estimator and Information Criterion for breaking strengths of single carbon fibres. |
|---|---|---|---|---|---|
| Model | MLE | Information Criterion | W | AIC | CAIC | HQIC |
| McMBIII | 3.106 | 1.558 | 0.00 | 3.046 | 13.889 | 3.164 |
| MBIII | 2043.21 | 7.198 | 1025.1 | 1 | 1 | 1 |
| BIII | 233.77 | 5.44 | 1 | 1 | 1 | 1 |
| BMIII | 5.92 | 1.14 | 0.0001 | 6.05 | 13.10 | 1 |
| KMBIII | 4.988 | 1.775 | 0.00 | 1 | 17.17 | 4.57 |

| Table 3. Maximum Likelihood Estimator and Information Criterion Average Annual Percent Change in Private Health Insurance Premiums. |
|---|---|---|---|---|---|
| Model | MLE | Information Criterion | W | AIC | CAIC | HQIC |
| McMBIII | 22.73 | 1.46 | 28.002 | 0.244 | 23.40 | 11.68 |
| BIII | 29.05 | 1.69 | 1 | 1 | 1 | 1 |
| KMBIII | 26.14 | 1.13 | 53.19 | 1 | 46.91 | 5.37 |
KMBIII, give less -2loglikelihood and information criterion as compared to our proposed model and old models. There are larger values of AIC, BIC, CAIC and HQIC for BIII distribution as compared to McMB-III distribution. The McMB-III distribution provides the close fit to a real data set. Thus our new distribution gives a better fit as compared to other sub models like Modified Burr-III and Burr-III distribution etc.

6. Conclusions

We have presented and developed the mathematical properties of a new class of distributions known as the McDonald modified Burr–III, including the hazard functions, moments, entropy, mean deviations, and maximum likelihood estimates. Two real data paradigms are presented to prove the application of the proposed model. The results show that our new model McMB–III is more flexible and yields the best fit compared to all the other submodels.

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Appendix

Representing mode of MMB-III for different values of $\alpha$, $\beta$, $\gamma$, $\lambda$, $\eta$, $\xi$.

### $\alpha = 1, \beta = 2, \gamma = 1$

| $\xi$ = 1 | $\xi$ = 2 | $\xi$ = 3 |
| --- | --- | --- |
| $\Lambda$ | $\eta = 2$ | $\eta = 3$ | $\eta = 4$ | $\eta = 2$ | $\eta = 3$ | $\eta = 4$ |
| 1 | 0.45 | 0.38 | 0.33 | 0.85 | 0.76 | 0.71 | 1.12 | 1.02 | 0.96 |
| 2 | 0.77 | 0.65 | 0.56 | 1.24 | 1.10 | 1.0 | 1.58 | 1.41 | 1.30 |
| 3 | 1.0 | 0.84 | 0.74 | 1.54 | 1.34 | 1.21 | 1.93 | 1.70 | 1.55 |
| 4 | 1.18 | 1.0 | 0.88 | 1.78 | 1.54 | 1.39 | 2.23 | 1.94 | 1.76 |
| 5 | 1.34 | 1.13 | 1.0 | 1.99 | 1.72 | 1.54 | 2.48 | 2.15 | 1.95 |
| 6 | 1.48 | 1.25 | 1.11 | 2.18 | 1.88 | 1.68 | 2.71 | 2.35 | 2.11 |

### $\alpha = 2, \beta = 1, \gamma = 1$

| $\xi$ = 1 | $\xi$ = 2 | $\xi$ = 3 |
| --- | --- | --- |
| $\Lambda$ | $\eta = 2$ | $\eta = 3$ | $\eta = 4$ | $\eta = 2$ | $\eta = 3$ | $\eta = 4$ |
| 1 | 0.41 | 0.35 | 0.32 | 1.28 | 1.15 | 1.05 | 2.16 | 1.95 | 1.81 |
| 2 | 1.12 | 0.91 | 0.78 | 2.71 | 2.28 | 2.01 | 4.30 | 3.66 | 3.25 |
| 3 | 1.80 | 1.44 | 1.21 | 4.08 | 3.34 | 2.88 | 6.36 | 5.25 | 4.56 |
| 4 | 2.47 | 1.95 | 1.63 | 5.43 | 4.37 | 3.72 | 8.39 | 6.80 | 5.83 |
| 5 | 3.14 | 2.46 | 2.04 | 6.78 | 5.39 | 4.55 | 10.41 | 8.34 | 7.07 |
| 6 | 3.81 | 2.96 | 2.45 | 8.12 | 6.14 | 5.37 | 12.43 | 9.86 | 8.30 |

### $\alpha = 1, \beta = 1, \gamma = 2$

| $\xi$ = 1 | $\xi$ = 2 | $\xi$ = 3 |
| --- | --- | --- |
| $\Lambda$ | $\eta = 2$ | $\eta = 3$ | $\eta = 4$ | $\eta = 2$ | $\eta = 3$ | $\eta = 4$ |
| 1 | – | – | – | 0.0 | 0.0 | 0.4 | 0.39 | 0.33 | 0.71 |
| 2 | – | – | – | 0.67 | 0.5 | 0.4 | 1.44 | 1.15 | 1.82 |
| 3 | 0.25 | 0.15 | 0.09 | 1.33 | 1.0 | 0.8 | 2.46 | 1.93 | 2.87 |
| 4 | 0.56 | 0.36 | 0.25 | 2.0 | 1.5 | 1.2 | 3.47 | 2.69 | 3.90 |
| 5 | 0.88 | 0.59 | 0.42 | 2.67 | 2.0 | 1.6 | 4.47 | 3.45 | 4.92 |
| 6 | 1.21 | 0.82 | 0.60 | 3.33 | 2.5 | 2.0 | 5.48 | 4.21 | 5.93 |

### Estimates, Log-Likelihood and $p$-values:

| Estimates parameters | $n = 50$ | $n = 100$ | $n = 150$ | $n = 200$ |
| --- | --- | --- | --- | --- |
| 0.5 | 1.1 | 0.4 | 0.5 | 0.6 |
| 1.3 | 0.9 | 1.7 | 1.1 | 1.2 |
| 1.7 | 0.5 | 0.6 | 1.7 | 1.75 |
| 1.4 | 0.5 | 0.8 | 1.2 | 1.3 |
| 0.65 | 0.6 | 0.4 | 0.7 | 0.6 |
| 0.6 | 0.6 | 0.3 | 0.5 | 0.6 |
| Log likelihood | –84 | –172 | –272 | –280 |
| $P$-values | Anderson-Darling | 0.5 | 0.2 | 0.7 | 0.8 |
| Kolmogorov-Smirnov | 0.6 | 0.4 | 0.6 | 0.7 |
| Pearson $\chi^2$ | 0.6 | 0.6 | 0.7 | 0.8 |