Abstract In this paper, we focus on the weak cosmic censorship conjecture (WCCC) in the Einstein–Maxwell gravity sourced by a conformally coupled scalar field, where the black hole solution can carry a scalar hair apart from the conserved charges of the electromagnetic field. Motivated by the new version of the gedanken experiments proposed by Sorce and Wald, we derive the first two order perturbation inequalities in the scalar-hair Reissner–Nordström (shRN) black holes based on the Iyer–Wald formalism under the stability consumption. In the collision process, we assume that the scalar hair can also be changed. As a result, we find that the nearly extremal shRN black holes cannot be destroyed by the above collision process under the second-order approximation after the two perturbation inequalities are considered. Our result at some level implies that the WCCC can also be restored in the black holes with some scalar hairs.

1 Introduction

General relativity is the most successful and beautiful theory to describe gravity. It predicts the existence of black holes. There exists a singularity in the center for most of the spacetimes. However, the geometry of the spacetime becomes ill-defined and all the physical quantities diverge at the singularity. Then, the predictive power will be lost and the law of causality will be destroyed near the singularity. Therefore, to ensure the well-definition of the spacetime, Penrose proposed the weak cosmic censorship conjecture (WCCC) [1] to avoid the naked singularity, which states that the spacetime singularity must be hidden inside the event horizon and the law of causality out of the horizon will not be influenced by the singularity. This conjecture also indicates that the spacetime always ends up in a black hole rather than a naked singularity. In 1974, Wald suggested a gedanken experiment to test the WCCC [2], in which he considered a process by dropping a test particle to destroy the extremal Kerr–Newman (KN) black hole. His result showed that the extremal KN black holes cannot be overcharged or overspun under the first-order approximation. Since then, Hubeny extended the discussion into the nearly extremal case and found that the nearly extremal KN black hole can be destroyed by dropping the test particle after we consider the second-order correction. Based on this setup, many researchers investigated the WCCC in a variety of theories [3–18].

In their investigations, the spacetime is only treated as a background geometry when considering the motion of the test particle. This is true in the first-order approximation of perturbation. If we need to consider the second-order correction, the interaction between the spacetime and test particle must be taken into account. For this story to be truly consistent, Sorce and Wald proposed a new version of the gedanken experiments to examine the WCCC in KN black holes [17]. Unlike the test particle version, they straightly consider a complete dynamical collision process of the matter fields under the second-order approximation of perturbation. Based on the Iyer–Wald formalism [19] as well as the null energy condition, the first two order perturbation inequalities are derived. Their results showed that the nearly extremal black holes cannot be destroyed by the collision process after these two inequalities are taken into consideration.

Motivated by their result, this new version was also extended to some other theories in Einstein gravity [20–24]. Although all of them indicated the validity of the WCCC in the new version of gedanken experiments, general proof is still lacking to show its correctness. Therefore, Examining it for case by case is also necessary for us to show its validity. We can see that all of the matter fields in the above theories are only minimal coupled to the Einstein gravity. As shown in [25], when the Einstein gravity is sourced by a conformally coupled scalar field, the spacetime admits a scalar hair apart from the charges of the gauge fields. Therefore, in the fol-
lowing, we would like to investigate whether the WCCC is still true for the spacetime with the scalar hair by performing the new version of the gedanken experiments proposed by Sorce and Wald.

This paper is outlined as follows. In Sect. 2, the spacetime geometry of scalar-hair Reissner–Nordstrom (shRN) black holes perturbed by the spherically matter collision is discussed. In Sect. 3, based on the Iyer–Wald formalism as well as the null energy condition, we derive the first two perturbation inequalities of the matter fields. In Sect. 4, we employ the new version of the gedanken experiment to destroy the near extremal shRN black holes under the second-order approximation of perturbation. Section 5 is devoted to our conclusions.

2 Perturbed geometry of scalar-hairy RN black holes

In this paper, we would like to consider the WCCC in the four-dimensional Einstein–Maxwell gravitational theory conformally coupled to a scalar field. The Lagrangian four-form of this theory can be expressed as

\[ L = \frac{1}{16\pi} \left[ R - F_{ab} F^{ab} - \left( \nabla_a \psi \nabla^a \psi + \frac{R}{6} \right) \psi^2 \right] + L_{\text{mt}}, \]

where \( F = dA \) is the strength of the electromagnetic field with the vector potential \( A \), \( R \) is the Ricci scalar, \( \psi \) is a real scalar field, and \( L_{\text{mt}} \) denotes the Lagrangian four-form of the extra matter fields. The equation of motion can be written as

\[ G_{ab} = 8\pi \left( T_{ab}^{\text{EM}} + T_{ab}^{S} + T_{ab}^{\text{mt}} \right), \]

\[ \Box \psi = \frac{1}{6} R \psi + \psi, \quad \nabla_a F^{ab} = 4\pi j^b, \]

in which we have denoted the derivative operator \( \Box = \nabla_c \nabla^c \), \( j^a \) and \( \psi \) are corresponding to the electromagnetic charge current and the scalar field source of the extra matter fields, \( T_{ab}^{\text{EM}} \), \( T_{ab}^{S} \), and \( T_{ab}^{\text{mt}} \) are the stress-energy tensors of the electromagnetic field, scalar field and extra matter fields, separately. Among them, \( T_{ab}^{\text{EM}} \) and \( T_{ab}^{S} \) can be expressed as

\[ T_{ab}^{\text{EM}} = \frac{1}{4\pi} \left[ F_{ac} F_b^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right], \]

\[ T_{ab}^{S} = \frac{1}{8\pi} \left( \nabla_a \psi \nabla_b \psi - \frac{1}{2} g_{ab} \nabla_c \psi \nabla^c \psi \right) + \frac{1}{48\pi} \left( g_{ab} \nabla_c \nabla^c - \nabla_a \nabla_b + G_{ab} \right) \psi^2. \]

\[ ds^2 = -f(r) dv^2 + 2dvdr + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]

\[ A = -\frac{Q}{r} dv, \quad \psi = \pm \sqrt{\frac{6\alpha}{\alpha + Q^2}} \]

with the blackening factor

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2 + \alpha}{r^2}. \]

The parameters \( M \) and \( Q \) are the physical mass and electric charge of the black hole. \( \alpha \) is a parameter to describe the scalar hair of the black hole. Although the scalar field is constant, its contribution to the geometry is non-trivial.

The radius of the event horizon \( r_h \) is corresponding to the largest root of the equation \( f(r) = 0 \), which gives

\[ r_h = M + \sqrt{M^2 - Q^2 - \alpha}. \]

Furthermore, the surface gravity, area, and electric potential for the event horizon are given as follows:

\[ \kappa = \frac{f'(r_h)}{2}, \quad A_H = 4\pi r_h^2, \quad \Phi_H = \frac{Q}{r_h}. \]

The shRN black holes become extremal when

\[ M^2 - Q^2 - \alpha = 0. \]

If the parameters satisfy \( M^2 - Q^2 - \alpha < 0 \), the solution describes a naked singularity.

In this paper, we only consider the case where all of the matter fields satisfy the null energy condition, which means that for any null vector \( l^a \), we must have \( T_{ab} l^a l^b \geq 0 \). According to the line element in (4), the stress-energy tensor of the scalar field can be straightly obtained,

\[ T_{ab}^{S} = \frac{\alpha f(r)}{r^4} (dv)_a (dv)_b - \frac{2\alpha}{r^4} (dv)_a (dr)_b + \frac{\alpha}{r^2} [(d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b]. \]

From above expression, it is not difficult to check that in order to ensure the null energy condition of the scalar field, we must have \( \alpha \geq 0 \).

Next, we consider a one-parameter family spherically symmetric matter perturbation in the shRN black hole. Here we assume that the matter collision only occurs in a finite region of the spacetime and the perturbation vanishes on the bifurcation surface \( B \). For simplification, we denote \( \phi(\lambda) \) to the dynamical fields in this family, where \( \phi \) is the collection of \( g_{ab}, A, \psi \) as well as some extra matter fields. In this family, the equation of motion can be written as
\[ G_{\alpha\beta}(\lambda) = 8\pi \left[ T^{EM}_{\alpha\beta}(\lambda) + T^{S}_{\alpha\beta}(\lambda) + T^{m}_{\alpha\beta}(\lambda) \right], \]

\[ \square^{(\lambda)} \psi(\lambda) = \frac{1}{6} R(\lambda) \psi(\lambda) + \varphi(\lambda), \]

\[ \nabla^a F_{\alpha\beta}(\lambda) = 4\pi j^b(\lambda) \] (10)

with \( T^{m}_{\alpha\beta}(0) = j^a(0) = \varphi(0) = 0 \) for the background fields. The spacetime in this case can be described by

\[ ds^2 = -f(r, v, \lambda) dv^2 + 2\mu(r, v, \lambda) drdv + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \] (11)

which satisfies \( f(r, v, 0) = f(r) \) and \( \mu(r, v, 0) = 1 \). In this case, the background fields can be expressed by (4). However, in this dynamical process, the scalar field can no longer be treated as a constant. For late convenience, we impose the gauge condition of the electromagnetic field such that

\[ \xi^a A_\alpha(\lambda)\big|_{r=r_h} = 0, \] (12)

where \( \xi^a = (\partial/\partial v)^a \) and \( r_h \) are only the static Killing vector and the horizon radius of the background spacetime, which means that they are independent on the parameter \( \lambda \). With similar consideration as [17], we also assume that the spacetime satisfies the stability condition, which states that at the sufficiently late times, the spacetime is also described by the scalar-hairy RN solution with different parameters which can be labeled by \( \lambda \), i.e., we have

\[ f(r, v, \lambda) = f(r, \lambda) = 1 - \frac{2M(\lambda)}{r} + \frac{Q^2(\lambda) - \alpha(\lambda)}{r^2}, \]

\[ \mu(r, v, \lambda) = 1, \quad \psi = \pm \sqrt{\frac{6\alpha(\lambda)}{\alpha(\lambda) + Q^2(\lambda)}}, \]

\[ A = \left( \frac{Q(\lambda)}{r_h} - \frac{Q(\lambda)}{r} \right) dv \] (13)

at sufficiently late times.

Due to the stability condition, we can see that the stress-energy tensor \( T^{m}_{\alpha\beta}(\lambda) \), the electric charge current \( j^a(\lambda) \) as well as the scalar source \( \varphi(\lambda) \) are vanishing at sufficient late times. In above setup, we can introduce a hypersurface \( \Sigma = \Sigma_0 \cup \Sigma_1 \). Here \( \Sigma_0 \) is a 3-hypersurface with constant radius \( r = r_h \) and connects the bifurcation surface \( B \) as well as a two-dimensional surface \( B_1 \) at a sufficient late time. \( \Sigma_1 \) is a time-slice at \( v = v_1 \) which connects \( B_1 \) and an asymptotic sphere \( S_\infty \). This implies that the dynamical fields on \( \Sigma_1 \) can be described by the expressions in (13).

### 3 Perturbation inequalities

In this section, following the process in [17], we would like to derive the first two order perturbation inequalities for the shRN black holes as introduced in last section. Different with the Einstein–Maxwell gravity minimally coupled to the a scalar field, here the action of the scalar field contains the Ricci scalar. Therefore, the variational identities become more complicated than the Einstein–Maxwell gravity. However, because the scalar field should satisfy the null energy condition, we can only consider the off-shell variation of the Einstein–Maxwell part and group the scalar field into the extra matter fields. Therefore, the Lagrangian four-form we are going to consider is

\[ L = \frac{\epsilon}{16\pi} \left( R - F_{\alpha\beta}F^{\alpha\beta} \right). \] (14)

Following the notations in [17], we will denote

\[ \chi = \chi(0), \quad \delta \chi = \left. \frac{d\chi}{d\lambda} \right|_{\lambda=0}, \quad \delta^2 \chi = \left. \frac{d^2\chi}{d\lambda^2} \right|_{\lambda=0} \] (15)

for the quantity \( \chi(\lambda) \) in the configuration \( \phi(\lambda) \). Then, the off-shell variation of above action can be written as

\[ \delta L = E_{\phi}\delta \phi + \delta \Theta(\phi, \delta \phi), \] (16)

in which

\[ E_{\phi}\delta \phi = -\epsilon \left( \frac{1}{2} T^{ab}\delta g_{ab} + j^a\delta A_a \right), \]

\[ \Theta(\phi, \delta \phi) = \Theta^{GR}(\phi, \delta \phi) + \Theta^{EM}(\phi, \delta \phi) \] (17)

with

\[ \Theta^{GR}_{abc}(\phi, \delta \phi) = \frac{1}{16\pi} \epsilon_{dabc} \delta g_{de} f_g \left( \nabla_g \delta g_{ef} - \nabla_e \delta g_{fg} \right), \]

\[ \Theta^{EM}_{abc}(\phi, \delta \phi) = -\frac{1}{4\pi} \epsilon_{dabc} F^{de} \delta A_e. \] (18)

Here we have denoted \( T_{\alpha\beta} = T^{S}_{\alpha\beta} + T^{m}_{\alpha\beta} \) to the total energy-tensors of the scalar field and extra matter fields. The symplectic current three-form is defined by

\[ \omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi), \] (19)

which can be divided into the Einstein part and electromagnetic part

\[ \omega(\phi, \delta_1 \phi, \delta_2 \phi) = \omega^{GR} + \omega^{EM}, \] (20)

According to (18), we have

\[ \omega^{GR}_{abc} = \frac{1}{16\pi} \epsilon_{dabc} w^d, \]

\[ \omega^{EM}_{abc} = \frac{1}{4\pi} \left[ \delta_2 \left( \epsilon_{eabc} F^{ed} \right) \delta_1 A_d - \delta_1 \left( \epsilon_{eabc} F^{ed} \right) \delta_2 A_d \right]. \] (21)
in which we have denoted

\[ u^a = P^{abcd ef} \left( \delta_{2} g_{be} \nabla_{d} \delta_{1} g_{ef} - \delta_{1} g_{be} \nabla_{d} \delta_{2} g_{ef} \right) \]  

(22)

with

\[ p^{abcd ef} = g^{ae} g^{db} g^{cf} - \frac{1}{2} g^{ae} g^{ad} g^{be} g^{cf} - \frac{1}{2} g^{ab} g^{cf} g^{de} + \frac{1}{2} g^{be} g^{ad} g^{cf} \cdot \]  

(23)

Next, we consider the variation corresponding to a diffeomorphism which is generated by the Killing vector \( \xi = (\partial/\partial v)^a \) of the background spacetime. Then, we can define a Noether current three-form related to this vector as

\[ J_{\xi} = \Theta(\phi, \mathcal{L}_{\xi} \phi) - \xi \cdot L \].

(24)

From the discussion in [2], this current can also be written as

\[ J_{\xi} = C_{\xi} + d Q_{\xi}. \]

(25)

According to Eqs. (16), (17) and (18), we can further obtain that \( C_{\xi} = \xi \cdot C \) with

\[ C_{dabc} = \epsilon_{eabcd}(T_{d} + A_{d} f^{e}) \]

(26)

and can be regarded as the constraint of the purely Einstein–Maxwell gravity, and the Noether current two-form

\[ Q_{\xi} = Q^{GR}_{\xi} + Q^{EM}_{\xi} \]

(27)

with

\[ \left( Q^{GR}_{\xi} \right)_{ab} = -\frac{1}{16\pi} \epsilon_{abcd} \nabla^{c} \xi^{d}, \]

\[ \left( Q^{EM}_{\xi} \right)_{ab} = -\frac{1}{8\pi} \epsilon_{abcd} F^{cd} A_{e} \xi^{e}. \]

(28)

Based on above results as well as the fact that \( j^{a} = \varphi = \mathcal{L}_{\xi} \phi = 0 \) for the background fields, the first-order and second-order variational identities can be further obtained and they can be shown as

\[ d[\delta Q_{\xi} - \xi \cdot \Theta(\phi, \delta \phi)] + \xi \cdot E_{\phi} \delta \phi + \delta C_{\xi} = 0, \]

\[ d[\delta^{2} Q_{\xi} - \xi \cdot \Theta(\phi, \delta \phi)] = \omega(\phi, \delta \phi, \mathcal{L}_{\xi} \delta \phi) - \delta[\xi \cdot E_{\phi} \delta \phi] - \delta^{2} C_{\xi}. \]

(29)

Using the Stoke’s theorem as well as the assumption that the perturbation vanishes on \( B \), integration of the first-order variational identity on \( \Sigma \) gives

\[ \int_{S_{\infty}} [\delta Q_{\xi} - \xi \cdot \Theta(\phi, \delta \phi)] + \int_{\Sigma_{1}} \xi \cdot E_{\phi} \delta \phi + \int_{\Sigma_{0}} \delta C_{\xi} = 0. \]

(30)

For the first term, according to the stability condition, we can calculate it by using the explicit expression (13) of the dynamical fields at sufficiently late times. Then, the gravity part can be straightly calculated and it gives

\[ \int_{S_{\infty}} [\delta Q^{GR}_{\xi} - \xi \cdot \Theta^{GR}(\phi, \delta \phi)] = \delta M. \]

(31)

For the EM part of first term, we have

\[ Q^{EM}_{\xi} = -\frac{Q^{2}(\lambda)}{4\pi r^{2}} \left( \frac{1}{r_{h}} - \frac{1}{r} \right) \hat{e}. \]

(32)

on the asymptotic sphere \( \Sigma_{1} \), in which

\[ \hat{e} = r^{2} \sin \theta d\theta \wedge d\phi. \]

(33)

is the volume element of this sphere. The above expression then gives

\[ \delta Q^{EM}_{\xi} = \frac{d Q^{EM}_{\xi}}{d \lambda} \bigg|_{\lambda=0} = -\frac{Q \delta Q}{2\pi r^{2}} \left( \frac{1}{r_{h}} - \frac{1}{r} \right) \hat{e}. \]

(34)

Its integration gives

\[ \int_{S_{\infty}} \delta Q^{EM}_{\xi} = -\frac{2 Q \delta Q}{r_{h}} = -2 \Phi_{H} \delta Q. \]

(35)

With a similar calculation, we can also obtain

\[ \int_{S_{\infty}} \xi \cdot \Theta^{EM}(\phi, \delta \phi) = -\Phi_{H} \delta Q. \]

(36)

Then, we have

\[ \int_{S_{\infty}} \left[ \delta Q^{EM}_{\xi} - \xi \cdot \Theta^{EM}(\phi, \delta \phi) \right] = -\Phi_{H} \delta Q. \]

(37)

Using the stability assumption that \( j^{a}(\lambda) = 0, \varphi(\lambda) = 0 \) and \( T^{\text{mat}}_{ab}(\lambda) = 0 \) on \( \Sigma_{1} \), we have

\[ T_{ab}(\lambda) = T^{S}_{ab}(\lambda) + \frac{\alpha(\lambda) f(r, \lambda)}{r^{4}} (d\nu_{a}) (d\nu_{b}) - \frac{2 \alpha(\lambda)}{r^{4}} (d\nu)(d\nu)(d\nu)(d\nu)(d\nu)(d\nu). \]

(38)
on $\Sigma_1$. Based on above expression, it is easy to verify that

$$ T^{ab}(\lambda) \frac{dg_{ab}(\lambda)}{d\lambda} = 0 $$

(39)

on $\Sigma_1$, which implies that the second term vanishes. For the third term, according to Eqs. (26) and (38), we can further obtain

$$ \int_{\Sigma_0} C_\xi(\lambda) = - \int_{r_h}^{\infty} \frac{\alpha(\lambda)}{2r^2} dr = - \frac{\alpha(\lambda)}{2r_h}. $$

(40)

Then, we have

$$ \int_{\Sigma_0} \delta C_\xi(\lambda) = - \frac{\delta \alpha}{2r_h}. $$

(41)

Combining above results, we have

$$ \delta M - \Phi_H \delta Q - \frac{\delta \alpha}{2r_h} = \delta \left[ \int_{\Sigma_0} \tilde{\epsilon} T_{ab}(dr)^a \xi^b \right]. $$

(42)

In the calculations, we have used the gauge condition (12) of the electromagnetic field, and denoted $\tilde{\epsilon} = dv \wedge \tilde{\epsilon}$ to the volume element on the hypersurface $\Sigma_0$. Next, we are going to investigate the connection between above variational identity (42) and the null energy condition of the matter fields. According to the expression (11) of the spacetime line element, here we can choose a null vector field as

$$ l^a(\lambda) = \xi^a + \beta(\lambda) r^a. $$

(43)

with

$$ r^a = \left( \frac{\partial}{\partial r} \right)^a, \quad \beta(\lambda) = \frac{f(r_h, v, \lambda)}{2\mu(r_h, v, \lambda)}, $$

(44)

on the hypersurface $\Sigma_0$. Then, we have

$$ T_{ab}(\lambda) l^a(\lambda) l^b(\lambda) = T_{ab}(\lambda) \xi^a (dr)^b + \beta^2(\lambda) T_{ab}(\lambda) r^a r^b. $$

(45)

Considering the fact that $\beta = 0$ for the background geometry, under the first-order approximation of perturbation, the null energy condition gives

$$ \int_{\Sigma_0} T_{ab}(\lambda) l^a(\lambda) l^b(\lambda) dv \wedge \tilde{\epsilon} \simeq \omega \tilde{\epsilon} T_{ab}(dr)^a \xi^b \geq 0. $$

(46)

Then, the first-order variational identity reduces to

$$ \delta M - \Phi_H \delta Q - \frac{\delta \alpha}{2r_h} \geq 0. $$

(47)

As introduced above, the main purpose of this paper is to investigate whether above collision process can destroy the event horizon of the black holes. According to the stability condition, it is equivalent to see whether the quantities at sufficiently late times can violate the condition $M^2 - Q^2 - s \geq 0$. It is not difficult to see that the optimal condition to destroy the black hole is to saturate the inequality (42), i.e.,

$$ \delta M - \Phi_H \delta Q - \frac{\delta \alpha}{2r_h} = 0. $$

(48)

This also means $\delta \left[ \sqrt{-g} T_{ab}(dr)^a \xi^b \right] = 0$ on $\Sigma_0$. According to the explicit expression (11) of the metric as well as the stress energy tensor of electromagnetic field in Eq. (3), we can easily check that $\delta \left[ \sqrt{-g} \tilde{\epsilon} T^{EM}(dr)^a \xi^b \right] = 0$ on $\Sigma_0$. Therefore, the optimal condition reduces to $\delta \left[ \sqrt{-g} G_{ab}(dr)^a \xi^b \right] = 0$. Straight calculation shows that the optimal condition is equivalent to $\delta_r f(r_h, v) = 0$, where we denoted

$$ \delta f(r, v) = \frac{\partial f(r, v, \lambda)}{\partial \lambda} \bigg|_{\lambda=0}. $$

(49)

For the second-order variational identity in (16), by integrating it on the hypersurface $\Sigma$ and using the Stokes’s theorem, we can further obtain

$$ \int_{S_{\infty}} \left[ \delta^2 Q_{\xi} - \delta \cdot \delta \Theta(\phi, \delta \phi) \right] + \int_{\Sigma_1} \delta [\xi \cdot E_{\phi} \delta \phi] + \int_{\Sigma_0} \delta^2 C_{\xi} - \delta_{\Sigma_1} - \delta_{\Sigma_0} = 0, $$

(50)

where we denote

$$ \delta \Sigma_i = \int_{\Sigma_i} \omega(\phi, \delta \phi, Z_{\xi} \delta \phi), $$

(51)

with $i = 0, 1$. According to (39) as well as the fact that $j^a(\lambda) = \mathcal{L}_{\xi} \phi(\lambda) = 0$ on $\Sigma_1$, the second and fifth terms of (50) vanish. For the gravity part of the first term, with a similar calculation of the first-order identity, we have

$$ \int_{S_{\infty}} \left[ \delta^2 Q_{\xi}^{GR} - \delta \cdot \delta \Theta^{GR}(\phi, \delta \phi) \right] = \delta^2 M. $$

(52)

According to (32), the electromagnetic part of first term in (50) reduces to

$$ \int_{S_{\infty}} \delta^2 Q_{\xi}^{EM} = - \frac{2\delta Q^2}{r_h} - 2\Phi_H \delta^2 Q. $$

(53)

A similar calculation can also give

$$ \int_{S_{\infty}} \xi \cdot \delta \Theta^{EM}(\phi, \delta \phi) = - \frac{\delta Q^2}{r_h} - \Phi_H \delta^2 Q. $$

(54)
Then, we have
\[ \int_{\Sigma_0} \left[ \delta^2 Q_{\xi} - \xi \cdot \delta \Theta(\phi, \delta \phi) \right] = -\frac{\delta Q^2}{r_h} - \Phi_H \delta^2 Q. \] (55)

For the third term, from Eq. (40), we can further obtain
\[ \int_{\Sigma_1} \delta^2 C_{\xi} = -\frac{\delta^2 \alpha}{2r_h}. \] (56)

The last term can be decomposed into the gravitational part and electromagnetic part, i.e.,
\[ \delta \Sigma_0 (\phi, \delta \phi) = \int_{\Sigma_0} \omega^{GR} + \int_{\Sigma_0} \omega^{EM}. \] (57)

For the gravitational part, based on the explicit expression of the metric (11), we can further obtain
\[ \int_{\Sigma_0} \omega^{GR} = -\frac{r_h}{2} \int_{\Sigma_0} \nu [\delta \mu(r_h, \nu) \partial_v \delta f(r_h, \nu)] - \frac{r_h}{2} \delta \mu(r_h, \nu, v) \delta f(r_h, v) = 0, \] (58)
where \( v = v_0 \) denotes the coordinate of the bifurcation surface \( B \), and
\[ \delta \mu(r_h, v) = \left. \frac{\partial \mu(r_h, v, \lambda)}{\partial \lambda} \right|_{\lambda=0} \] (59)
is the variation of function \( \mu(r, v, \lambda) \). In the second step we used the optimal condition of the first-order inequality, and in the last step we used the stability condition \( \mu(r_h, v_1, \lambda) = 1 \) on \( B_1 \). For the electromagnetic part, we first consider the symplectic current three-form on \( \Sigma_0 \). According to (21), we have
\[ \omega^{EM}_{abc} = \frac{1}{4\pi} \epsilon_{dabc} \left[ \delta A_c \delta F^{de} - \delta F^{de} \delta A_e \right] \]
\[ + \frac{1}{4\pi} \left[ (\delta \delta \epsilon_{dabc}) F^{de} \delta A_e - \delta \epsilon_{dabc} F^{de} \delta A_e \right]. \] (60)

Because we consider its integration on \( \Sigma_0 \), the index \( d \) must contribute a \( (dr)_d \). Using the explicit expression of the background fields, we can further obtain \( (dr)_d F^{de} \propto \xi^e \). Together with the gauge condition \( \xi^a A_{a} \lambda = 0 \) on \( \Sigma_0 \), the last two terms in (60) vanish and Eq. (60) becomes
\[ \omega^{EM}_{abc} = \frac{1}{4\pi} \delta \left( \delta \epsilon_{dabc} \delta A_e \delta F^{de} - \frac{1}{2\pi} \epsilon_{dabc} F^{de} \delta A_e \right). \] (61)

Integration of the first term in above equation only contributes to a boundary term. Using the explicit expression (13) of electromagnetic fields, this term will not contribute to \( \delta \Sigma_0 \). Then, we have
\[ \delta \Sigma_0 = \frac{1}{2\pi} \int_{\Sigma_0} \bar{\epsilon}(dr)_d \delta F^{de} \delta A_e \]
\[ - \frac{1}{2\pi} \int_{\Sigma_0} \left[ \bar{\epsilon}(dr)_a \xi^b \delta F^{ac} \delta F_{bc} + (dr)_a \xi^b \delta F^{ac} \nabla_c \xi^b \right] \]
\[ = \frac{1}{2\pi} \int_{\Sigma_0} \bar{\epsilon}(dr)_a \xi^b \delta F^{ac} \delta F_{bc}. \] (62)

In the last step we used the result \( \nabla_c \xi^b \delta A_b = 0 \) on \( \Sigma_0 \). According to Eq. (3), the last term of Eq. (50) can be expressed as
\[ \delta \Sigma_0 = \delta^2 \left( \int_{\Sigma_0} \bar{\epsilon}(dr)_a \xi^b T_{ab}^{EM} \right) \] (63)

Summing above results, the second-order variational identity can be express as
\[ \delta M - \Phi_H \delta^2 Q - \frac{\delta Q^2}{r_h} - \frac{\delta^2 \alpha}{2r_h} = -\int_{\Sigma_0} \delta^2 C_{\xi} + \delta \Sigma_0 \]
\[ = \delta^2 \left( \int_{\Sigma_0} \bar{\epsilon} T_{ab}^{EM} (dr)^a \xi^b \right). \] (64)

where we have denoted \( T_{ab}^{EM} = T_{ab} + T_{ab}^{EM} \) as the total stress-energy tensor of the matter fields. According to (45) and the fact that \( T_{ab} (dr)^a \xi^b = 0 \) for the background fields, together with the optimal condition for the first-order inequality, the right hand of (64) also gives the null energy condition, i.e.,
\[ \int_{\Sigma_0} T_{ab} F^{ab} \nu = \bar{\epsilon} \approx \frac{1}{2} \delta^2 \left( \int_{\Sigma_0} \bar{\epsilon} T_{ab}^{EM} (dr)^a \xi^b \right). \] (65)

under the second-order approximation. Then, after taking the null energy condition into account, the second-order variational identity reduces to
\[ \delta^2 M - \Phi_H \delta^2 Q - \frac{\delta Q^2}{r_h} - \frac{\delta^2 \alpha}{2r_h} \geq 0. \] (66)

4 Gedanken experiments to destroy the scalar-hairy RN black holes

Now we shall investigate the WCCC in the scalar-hairy RN black holes based on the physical process introduced in the previous sections. Under the assumption of the stability condition, it is equivalent to check whether the spacetime geometry at sufficiently late times also describes a black hole. Therefore, we define
further obtain
as a function of $\lambda$. Then, the WCCC is violated if $h(\lambda) < 0$. Under the second-order approximation of $\lambda$, we have

$$h(\lambda) = M(\lambda)^2 - Q(\lambda)^2 - \alpha(\lambda),$$  \hspace{1cm} (67)

$$\text{with similar consideration of [17], we also focus on the case where the background geometry is a nearly extremal scalar-hairy RN black hole, which implies that}$$

$$\epsilon = \sqrt{M^2 - Q^2 - \alpha}$$  \hspace{1cm} (69)

$$\text{is small parameter. We choose it to be in agreement with the first-order perturbation of the matter source. That is to say, } \epsilon \text{ has the same order of } \lambda. \text{ According to the optimal first-order equality (48) and second-order inequality (66), we can further obtain}$$

$$h(\lambda) \geq \frac{(2Q\delta Q + \delta\alpha - 2M\epsilon)^2}{4M^2} \geq 0$$  \hspace{1cm} (70)

$$\text{under the second-order approximation of perturbation. This result indicates that the nearly extremal scalar-hairy RN black hole cannot be destroyed by above collision process under the second-order approximation, and there is no violation of WCCC around the Einstein–Maxwell gravity with scalar hair.}$$

5 Conclusion

Recently, Sorce and Wald proposed a new version of the gedanken experiments to examine the WCCC in Kerr–Newman black holes. Based on their method, the discussion was also extended to a variety of theories [20–24]. These results strongly indicate the validity of the WCCC for Einstein gravity. However, all of the cases considered above are only about the no-hair black holes. Whether is it also valid for the hairy black holes? Therefore, in this paper, we examined the WCCC in the Einstein gravity conformally coupled to a scalar field under the perturbation of the spherically matter collision. First of all, based on the Iyer–Wald formalism, we derived the first two order perturbation inequalities in shRN black holes after considering the null energy condition of the matter fields. Utilizing the optimal first-order equality as well as the second-order inequality, we showed that the nearly extremal shRN black holes cannot be destroyed by the collision process under the second-order approximation of perturbation. Moreover, in our investigation, we also assume that the scalar hair can also be changed by the collision process. Our result implies that the WCCC can also be restored in the black holes with some scalar hairs as long as all of the matter fields satisfy the null energy condition.

Acknowledgements Jie Jiang is supported by National Natural Science Foundation of China (NSFC) with Grant nos. 11775022 and 11873044. Ming Zhang is supported by the Initial Research Foundation of Jiangxi Normal University with Grant no. 12020023.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

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Funded by SCOAP3.

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