SYMPLECTIC AND COSYMPLECTIC REDUCTION FOR SIMPLE HYBRID FORCED MECHANICAL SYSTEMS WITH SYMMETRIES

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ABSTRACT. This paper discusses symplectic and cosymplectic reduction for autonomous and non-autonomous simple hybrid forced mechanical systems, respectively. We give general conditions on whether it is possible to perform symmetry reduction for simple hybrid Hamiltonian and Lagrangian systems subject to non-conservative external forces, as well as time-dependent external forces. We illustrate the applicability of the symmetry reduction procedure with examples and numerical simulations.

1. INTRODUCTION

Hybrid systems are dynamical systems with continuous-time and discrete-time components on its dynamics. This class of dynamical systems are capable of modeling several physical systems, such as multiple UAV (unmanned aerial vehicles) systems [42] and bipedal robots among many others [51] [26], [50]. Simple hybrid systems are a class of hybrid system introduced in [35], denoted as such because of their simple structure. A simple hybrid system is characterized by a tuple...
\( \mathcal{H} = (D, X, S, \Delta) \) where \( D \) is a smooth manifold, \( X \) is a smooth vector field on \( D \), \( S \) is an embedded submanifold of \( D \) with co-dimension 1, and \( \Delta : S \to D \) is a smooth embedding. This type of hybrid system has been mainly employed for the understanding of locomotion gaits in bipeds and insects [5], [31], [51]. In the situation where the vector field \( X \) is associated with a mechanical system (Lagrangian or Hamiltonian), alternative approaches for mechanical systems with nonholonomic and unilateral constraints have been considered in [11], [15], [16], [32], [34].

The reduction of mechanical systems with symmetries plays a fundamental role in understanding the many important and interesting properties of these systems. Given a Hamiltonian on a symplectic manifold on which a Lie group acts symplectically, Marsden-Weinstein Reduction Theorem [46] states that under certain conditions one can reduce the phase space to another symplectic manifold by “dividing out” by the symmetries. In addition, the trajectories of the Hamiltonian on the original phase space determine the corresponding trajectories on the reduced space. Furthermore, one can reconstruct the trajectories on the original phase space from the trajectories on the reduced phase space by choosing a connection. The key idea is that when a dynamical system exhibits a symmetry, it produces a conserved quantity for the system, and one can reduce the degrees of freedom in the dynamics by making use of these conserved quantity. One of the classical reduction by symmetry procedures in mechanics is the Routh reduction method [27], [1]. During the last few years there has been a growing interest in Routh reduction, mainly motivated by physical applications –see [24], [39], [40] and references therein. Furthermore, Routh reduction for hybrid systems has been introduced and applied in the field of bipedal locomotion [5]. The reduced simple hybrid system is called simple hybrid Routhian system [4]. A hybrid scheme for Routh reduction for simple hybrid Lagrangian systems with cyclic variables can be found in [4] and [12], with the motivation of gaining a better understanding of bipedal walking models (see also [5] and references therein). Symplectic reduction for hybrid Hamiltonian systems has been introduced in [3], and extended to Poisson reduction in [23] and to time-dependent systems in [13]. However, to the best of our knowledge, the hybrid analogue for symmetry reduction in mechanical systems subject to external forces has not been explored in the literature. This is important in practice compared with the previous mentioned approaches since external forces allow to describe friction, dissipation, as well as control forces or certain non-holonomic constraints. Additionally, the hybrid systems whose reduction has been studied are restricted to those for which the momentum map is preserved in the impact.

This paper considers symmetry reduction of simple hybrid mechanical systems, both time-independent and time-dependent, via Routh reduction. As it was studied in [18] (see also [19, 43]), the reduction of a forced continuous system requires considering a group of symmetries which leaves invariant both the Lagrangian (or Hamiltonian) function and the external force. Regarding the reduction of simple hybrid systems, Ames and Sastry [3, 4] had considered the so-called hybrid momentum maps, i.e., a momentum map which is preserved both by the continuous and the discrete dynamics. Here, we consider a more general class of hybrid systems for which the reset map can change the value of the momentum map (see
Example 2). This will lead to the existence of a reduced space for each interval of time between to subsequent impacts. Additionally, we considered hybrid systems which are forced, which requires characterizing the (sub)group of symmetries which preserves both the Lagrangian (or Hamiltonian) function and the external force.

The paper is organized as follows. Section 2 presents the necessary background on the geometry of forced mechanical systems and Routh reduction. The existence of symmetries and their associated conserved quantities are also presented. Section 3 introduces the class of hybrid Lagrangian and Hamiltonian systems under consideration and the correspondence relation between both formalisms. In Section 4, the notion of generalized hybrid momentum map is introduced, and the symplectic reduction for simple hybrid forced Hamiltonian and Lagrangian systems is developed. Section 5 extends the symplectic reduction scheme to cosymplectic manifolds, in order to derive sufficient conditions for the reduction of time-dependent simple hybrid forced Lagrangian and Hamiltonian systems. In Sections 4 and 5 we present an illustrative example: the forced mechanical system described by a rough billiard with and without moving walls, respectively. In addition, a hybrid system with a generalized hybrid momentum map which is not a hybrid momentum map is illustrated by Example 2. Finally, in Section 6, we give some conclusions and suggest topics for future work.

Notation and conventions. Throughout this paper, let \( Q \) be an \( n \)-dimensional differentiable manifold, which represents the configuration space of a dynamical system. Let \( T_q Q \) and \( T^*_q Q \) denote the tangent and cotangent spaces of \( Q \) at the point \( q \in Q \). Let \( \tau_Q : TQ \to Q \) and \( \pi_Q : T^*Q \to Q \) be its tangent bundle and its cotangent bundle, respectively; namely, \( TQ = \cup_{q \in Q} T_q Q \) and \( T^*Q = \cup_{q \in Q} T^*_q Q \), with the canonical projections \( \tau_Q : (q^i, \dot{q}^i) \mapsto (q^i) \) and \( \pi_Q : (q^i, p_i) \mapsto (q^i) \). Here \( (q^i) (1 \leq i \leq n) \) are local coordinates in \( Q \), and \( (q^i, \dot{q}^i) \) and \( (q^i, p_i) \) are their induced coordinates in \( TQ \) and \( T^*Q \), respectively.

Let \( M \) and \( N \) be smooth manifolds. For each \( p \)-form \( \alpha \) and each vector field \( X \) on \( M \), \( \iota_X \alpha \) denotes the interior product of \( \alpha \) by \( X \), and \( \mathcal{L}_X \alpha \) denotes the Lie derivative of \( \alpha \) with respect to \( X \). For a smooth map \( F : M \to N \), its tangent map \( TF : TM \to N \) will indistinctly be called its pushforward and denoted by \( F_* \). Unless otherwise stated, sum over paired covariant and contravariant indices will be understood.

2. Geometry of Mechanical Systems with External Forces and Routh Reduction

We start by recalling some basic facts about mechanical systems subject to external forces and Routh reduction.

2.1. Geometric formulation of Lagrangian and Hamiltonian systems. The dynamics of a mechanical system can be determined by the Euler-Lagrange equations associated with a Lagrangian function \( L : TQ \to \mathbb{R} \). A mechanical Lagrangian is given by \( L(q, \dot{q}) = K(q, \dot{q}) - V(q) \), where \( K : TQ \to \mathbb{R} \) is the kinetic energy and \( V : Q \to \mathbb{R} \) the potential energy. The kinetic energy is given by \( K(q, \dot{q}) = \frac{1}{2} \| \dot{q} \|_q^2 \).
where \( ||\cdot||_g \) denotes the norm at \( T_qQ \) defined by some (pseudo)Riemannian metric on \( Q \). In particular, a mechanical Lagrangian will be called \textit{kinetic} if \( V = 0 \).

A Lagrangian \( L \) is said to be \textit{regular} if \( \det M \neq 0 \), where \( M = (M_{ij}) := \left( \frac{\partial^2 L}{\partial q^i \partial q^j} \right) \) for all \( i, j \) with \( 1 \leq i, j \leq n \). The equations describing the dynamics of the system are given by the Euler-Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad \text{with } i = 1, \ldots, n;
\]
a system of \( n \) second-order ordinary differential equations. If \( L \) is regular, the Euler-Lagrange equations induce a vector field \( X_L : TQ \to T(TQ) \) describing the dynamics of the Lagrangian system, given by
\[
X_L(q^i, \dot{q}^i) = \left( q^i, \dot{q}^i, M_{ij}^{-1} \left( \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \dot{q}^j \right) \right).
\]

For the Lagrangian \( L : TQ \to \mathbb{R} \), let us denote by \( \mathbb{F}L : TQ \to T^*Q \) the Legendre transformation associated with \( L \); that is, \( \mathbb{F}L : (q,p) \mapsto (q, \dot{q} := \partial L / \partial \dot{q}) \). The map \( \mathbb{F}L : TQ \to T^*Q \) relates velocities and momenta. In fact, the Legendre transformation connects Lagrangian and Hamiltonian formulations of mechanics. We say that the Lagrangian is \textit{hyperregular} if \( \mathbb{F}L \) is a diffeomorphism between \( TQ \) and \( T^*Q \) (this is always the case for mechanical Lagrangians). If \( L \) is hyperregular, one can work out the velocities \( \dot{q} = \dot{q}(q,p) \) in terms of \((q,p)\) and define the Hamiltonian function (the “total energy”) \( H : T^*Q \to \mathbb{R} \) as \( H(q,p) = p^T \dot{q}(q,p) - L(q, \dot{q}(q,p)) \), where we have used the inverse of the Legendre transformation to express \( \dot{q} = \dot{q}(q,p) \). The evolution vector field corresponding to the Hamiltonian \( H \), denoted by \( X_H \), is defined by \( X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \), and its integral curves are solutions of Hamilton’s equations \( \dot{q}^i = \frac{\partial H}{\partial p_i} \) and \( \dot{p}_i = -\frac{\partial H}{\partial q^i} \). A Hamiltonian is said to be \textit{mechanical} (resp. \textit{kinetic}) if its associated Lagrangian is mechanical (resp. kinetic).

### 2.2. Geometry of mechanical systems subject to external forces

An external force is geometrically interpreted as a semibasic 1-form on \( T^*Q \) (see [18] and [25] for instance). A Hamiltonian system with external forces, so called \textit{forced Hamiltonian system}, is given by the pair \((H,F)\) determined by a Hamiltonian function \( H : T^*Q \to \mathbb{R} \) and a semibasic 1-form \( F \) on \( T^*Q \) locally described as \( F = F_i(q,p) dq^i \).

Let \( \omega_Q = -d\theta_Q \) be the canonical symplectic form of \( T^*Q \), where locally \( \theta_Q = p_i dq^i \) and \( \omega_Q = dq^i \wedge dp_i \). The dynamics of the forced Hamiltonian system is given by the vector field \( X_{H,F} \), defined by
\[
\iota_{X_{H,F}} \omega_Q = dH + F.
\]
If \( X_H \) is the Hamiltonian vector field for \( H \), that is, \( \iota_{X_H} \omega_Q = dH \) and \( Z_F \) is the vector field defined by \( \iota_{Z_F} \omega_Q = F \), then we have \( X_{H,F} = X_H + Z_F \). Locally,
these vector fields can be written as

\[ X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i}, \]

\[ Z_F = -F_i \frac{\partial}{\partial p_i}, \quad F = F_i dq^i, \]

\[ X_{H,F} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial H}{\partial q^i} + F_i \right) \frac{\partial}{\partial p_i}. \]

The Poincaré-Cartan 1-form on \( TQ \) associated with the Lagrangian function \( L: TQ \to \mathbb{R} \) is defined by \( \theta_L = S^*(dL) \) where \( S^* \) is the adjoint operator of the vertical endomorphism on \( TQ \), which is locally defined by \( S = dq^i \otimes \frac{\partial}{\partial q^i} \). The Poincaré-Cartan 2-form is \( \omega_L = -d\theta_L \), so locally \( \omega_L = dq^i \wedge d \left( \frac{\partial L}{\partial q^i} \right) \). One can easily verify that \( \omega_L \) is symplectic if and only if \( L \) is regular (see [1]). The energy of the system is given by \( E_L = \frac{1}{2} \frac{\partial^2 L}{\partial q^i \partial q^j} \). On the tangent bundle an external force is also represented by a semibasic 1-form \( F^L \) on \( TQ \), locally given by \( F^L = F^L(q, \dot{q})dq^i \). A forced Lagrangian system is determined by the pair \( (L, F^L) \) and its dynamics is given by

\[ t_{X_{L,F^L}} \omega_L = dE_L + F^L. \]

The forced Euler-Lagrange vector field \( X_{L,F^L} \) is a SODE and its integral curves satisfy the forced Euler-Lagrange equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = -F^L_i(q, \dot{q}), \quad i = 1, \ldots, n. \]

If \( L \) is regular, the forced Euler-Lagrange vector field is given by

\[ X_{L,F^L}(q^i, \dot{q}^i) = \left( \dot{q}^i, \ddot{q}^i, q^i \frac{\partial^2 L}{\partial q^j \partial q^i} - \frac{\partial^2 L}{\partial q^i \partial q^j} \right). \]

Let \((L, F^L)\) be a forced hyperregular Lagrangian system and let \((H, F)\) be its associated forced Hamiltonian system, i.e., \( E_L = H \circ F^L \) and \( F^L = F^L \circ F \). As in the un-forced case we can relate \( X_{L,F^L} \) and \( X_{H,F} \) as follows.

**Proposition 1.** The tangent map of \( F^L \) maps \( X_{L,F^L} \) onto \( X_{H,F} \). In other words

\[ (TF^L)X_{L,F^L} = X_{H,F}, \]

where \((TF^L): T(TQ) \to T(T^*Q)\). In particular, the flow of \( X_{L,F^L} \) is mapped onto the flow of \( X_{H,F} \).

**Proof.** The evolution vector field \( X_{H,F} \) is characterized by \( t_{X_{H,F}} \omega_Q = dH + F \). Observe that

\[ (FL)^*(t_{X_{H,F}} \omega_Q) = (FL)^*(dH + F) = (FL)^*(dH) + (FL)^*(F) = d((FL)^*H) + (FL)^*F = d(E_L) + F^L = t_{X_{L,F^L}} \omega_L. \]

This means that

\[ t_{X_{L,F^L}} \omega_L = (FL)^*(t_{X_{H,F}} \omega_Q) = t_{(FL)^{-1}}(X_{H,F})(FL^* \omega_Q) = t_{(FL)^{-1}}(X_{H,F} \omega_L) \]

This last implies \( X_{L,F^L} = (FL^{-1})X_{H,F} \), that is, \((FL)_* X_{L,F^L} = X_{H,F} \). \( \square \)
2.3. Types of symmetries of forced mechanical systems. Let us briefly recall the different types of symmetries a forced Hamiltonian or Lagrangian system can exhibit. See [18, 43] for more details.

Let \((L, F^L)\) be a forced Lagrangian system on \(TQ\). A function \(f\) on \(TQ\) is called a constant of the motion (or a conserved quantity) for \((L, F^L)\) if it takes a constant value along the trajectories of the system or, in other words, \(X_{L,F^L}(f) = 0\).

Let \(X = X^i \partial / \partial q^i\) be a vector field on \(Q\). It has two associated vector fields on \(TQ\), namely its vertical lift \(X^v = X^i \partial / \partial q^i\) and its complete lift \(X^c = X^i \partial / \partial q^i + \dot{q}^i \partial X^i / \partial \dot{q}^i \partial / \partial q^i\) (see [18, 43, 52] for an intrinsic definition). Then, \(X\) is called a

(i) symmetry of the forced Lagrangian if \(X^c(L) = F^L(X^c)\),
(ii) Lie symmetry if \([X^c, X_{L,F^L}] = 0\),
(iii) Noether symmetry if \(X^c(E_L) + F^L(X^c) = 0\) and \(\mathcal{L}_{X^c} \theta_L = df\) for some function \(f\) on \(TQ\).

Similarly, let \(\hat{X}\) be a vector field on \(T^*Q\). Then, \(\hat{X}\) is called a

(i) dynamical symmetry if \([\hat{X}, X_{L,F^L}] = 0\),
(ii) Noether symmetry if \(\hat{X}(E_L) + F^L(\hat{X}) = 0\) and \(\mathcal{L}_{\hat{X}} \theta_L = df\) for some function \(f\) on \(TQ\).

Moreover, the following relations between symmetries and constants of the motion hold:

(i) \(X\) is a symmetry of the forced Lagrangian if and only if \(X^v(L)\) is a constant of the motion.
(ii) If \(X\) satisfies that \(\mathcal{L}_{X^c} \theta_L = df\), then \(X\) is a Noether symmetry if and only if \(f = X^v(L)\) is a constant of the motion.
(iii) If \(X\) is a Noether symmetry, it is also a Lie symmetry if and only if \(\iota_{X^c} d\beta = 0\).
(iv) \(X\) is a Lie symmetry if and only if \(X^c\) is a dynamical symmetry.
(v) \(X\) is a Noether symmetry if and only if \(X^c\) is a Cartan symmetry.
(vi) If \(\hat{X}\) satisfies that \(\mathcal{L}_{\hat{X}} \theta_L = df\), then \(\hat{X}\) is a Cartan symmetry if and only if \(f = (S_{\hat{X}})(L)\) is a constant of the motion. Here \(S\) is the vertical endomorphism.
(vii) If \(\hat{X}\) is a Cartan symmetry, it is also a dynamical symmetry if and only if \(\iota_{\hat{X}} d\beta = 0\).

Let now \((H, F)\) be a forced Hamiltonian system on \(T^*Q\). A function \(f\) on \(T^*Q\) is called a constant of the motion (or a conserved quantity) for \((H, F)\) if it takes a constant value along the trajectories of the system or, in other words, \(X_{H,F}(f) = 0\). A vector field \(\hat{X}\) on \(T^*Q\) is called a symmetry of the forced Hamiltonian if \(\hat{X}(H) + F(\hat{X}) = 0\) and \(\mathcal{L}_{\hat{X}} \theta_Q = df\) for some \(f\) on \(T^*Q\). If \(\hat{X}\) is a symmetry of the forced Hamiltonian, then \(f = \theta_Q(\hat{X})\) is a constant of the motion.

In addition, the Hamiltonian and Lagrangian symmetries are related as follows. Suppose that \((L, F^L)\) is the Lagrangian counterpart of \((H, F)\), namely, \(H \circ F^L = E_L\) and \(F^L(\dot{F}) = F^L\). Let \(\hat{X}\) be a vector field on \(TQ\) and let \(\hat{X}\) be a \(F^L\)-related vector field on \(T^*Q\), i.e., \(F^L_\ast \circ \hat{X} = \hat{X} \circ F^L\). Then, the following relations hold:

(i) \([\hat{X}, X_{H,F}] = 0\) if and only if \(\hat{X}\) is a dynamical symmetry of \((L, F^L)\).
(ii) $\mathcal{L}_X \theta_Q = df$ if and only if $\mathcal{L}_{\tilde{X}} \theta_L = d(f \circ \mathbb{F} L)$.

(iii) If $\mathcal{L}_X \theta_Q = df$, then the following assertions are equivalent:

(a) $\dot{X}(H) + F(\dot{X}) = 0$,

(b) $f - \theta_Q(\dot{X})$ is a constant of the motion,

(c) $\tilde{X}(E_L) + F^L(\dot{X}) = 0$,

(d) $f \circ \mathbb{F} L - \theta_L(\dot{X})$ is a constant of the motion.

When the set of transformations that leave invariant a forced Lagrangian (or Hamiltonian) system form a Lie group (so the vector fields generating the infinitesimal symmetries close a Lie subalgebra), we can introduce a momentum map, which associates an independent constant of the motion to each of the generators of the Lie algebra (see Section 4). This, when the group action “behaves well”, allows to project the dynamics of the system to a reduced space of less dimensions. As a matter of fact, if we know that an (unforced) Hamiltonian system $(H, F)$ it suffices to consider the Lie subgroup $G_F$ whose infinitesimal generators act as symmetries of the forced Hamiltonian (see Remark 4).

2.4. Routh reduction for forced mechanical systems. There exists a large class of systems for which the Lagrangian (or Hamiltonian) does not depend on some of the generalized coordinates. Such coordinates are called cyclic or ignorable, and the corresponding generalized momenta are easily checked to be constants of the motion - see [1], [27]. Routh’s reduction procedure is a classical reduction technique which takes advantage of the conservation laws to define a reduced Lagrangian function, the so-called Routhian function, such that, when the conservation of momenta is taken into account, the solutions of the Euler-Lagrange equations for the Routhian are in correspondence with the solutions of the Euler-Lagrange equations for the original Lagrangian.

Routh’s reduction can be extended to forced systems as follows [27]. Suppose that the configuration space is of the form $Q = Q_1 \times Q_2$, where we denote an element $q \in Q$ by $q = (q^1, q^j)$, with $q^1 \in Q_1$ and $q^j \in Q_2$, for $j = 2, \ldots, n$.

Let $L(q^1, \dot{q}^1, q^j, \dot{q}^j)$ be a hyperregular Lagrangian with cyclic coordinate $q^1$, that is, $\frac{\partial L}{\partial \dot{q}^1} = 0$ and let $F_i$ be a non-conservative force such that $F_i$ is independent of $q^1$ for all $i = 1, \ldots, n$ and $F_i(q^2, \ldots, q^n) = 0$. Fundamental to reduction is the notion of a momentum map $J_L : TQ \to g^*$, which makes explicit the conserved quantities in the system. Here $g$ is the Lie algebra associated with the Lie group of symmetries $G$, and $g^*$ denotes its dual as vector space. In the framework we are considering here, $J_L(q^1, \dot{q}^1, q^j, \dot{q}^j) = \frac{\partial L}{\partial \dot{q}^1}$. Fix a value of the momentum $\mu = \frac{\partial L}{\partial \dot{q}^1}$.

Since $L$ is hyperregular, the last equation admits an inverse, and allows us to write $\dot{q}^1 = f(q^2, \ldots, q^n, \dot{q}^2, \ldots, \dot{q}^n, \mu)$. Consider the function

$$R_F^\mu(q^j, \dot{q}^j) = (L - \dot{q}^1 \mu) \bigg|_\mu,$$
where the notation \( |_\mu \) means that we have used the relation \( \mu = \frac{\partial L}{\partial \dot{q}^1} \) to replace all the appearances of \( \dot{q}^1 \) in terms of \( (q^j, \dot{q}^j) \) and the parameter \( \mu \). The function \( R^\mu_F \) is called Routhian function. Similarly we define the reduced force \( F^\mu \) as \( F^\mu := F |_\mu \).

This definition of Routhian function is the classical one as in [27], [48]. A more general notion of Routhian can be given in terms of the mechanical connection (see [45]), when \( L \) is kinetic, or even more general by using any connection on the principal bundle \( Q \to Q/G \). Indeed, let \( \mathcal{A} : TQ \to g \) be a connection one form, and let us denote by \( \mathcal{A}_\mu(\cdot) = \langle \mu, \mathcal{A}(\cdot) \rangle \) the 1-form on \( Q \) obtained by contraction with \( \mu \in g^* \). Then one can define the Routhian function as \( R^\mu = L - \mathcal{A}_\mu : TQ \to \mathbb{R} \). Similarly one can define the reduced force \( F^\mu \) by contraction with \( \mu \in g^* \). A review of the Routh method in these more general settings can be found in [45] (see also [39]).

**Remark 1.** Note that it may happen, even in the hyperregular case, that there is no Routhian function as one function generating the reduced dynamics but there is a family of functions much as it happens with the Hamiltonian as one try to employ the Legendre transformation for singular Lagrangians. An example of such a case can be found in [30] (see Example 3.4). Nevertheless in this paper we shall restrict to the case of mechanical Lagrangians (i.e., kinetic minus potential energy), where the last situation cannot happen.

If we regard the pair \((R^\mu_F, F^\mu)\) as a new forced Lagrangian system in the variables \((q^j, \dot{q}^j)\), then the solutions of the forced Euler-Lagrange equations for \((R^\mu_F, F^\mu)\) are in correspondence with those for \((L, F)\) when one takes into account the relation \( \mu = \frac{\partial L}{\partial \dot{q}^1} \). More precisely:

(a) Any solution of the forced Euler-Lagrange equations for \((L, F)\) with momentum \( \mu = \frac{\partial L}{\partial \dot{q}^1} \) projects onto a solution of the forced Euler-Lagrange equations for \((R^\mu_F, F^\mu)\).

(b) Conversely, any solution of forced Routh equations for \((R^\mu_F, F^\mu)\) can be lifted to a solution of the forced Euler-Lagrange equations for \((L, F)\) with \( \mu = \frac{\partial L}{\partial \dot{q}^1} \).

3. SIMPLE HYBRID FORCED HAMILTONIAN SYSTEMS

Roughly speaking, the term hybrid system refers to a dynamical system which exhibits both continuous and discrete time behaviours. In the literature, one finds slightly different definitions of hybrid system depending on the specific class of applications of interest. For simplicity, and following [35] and [14], we will restrict ourselves to the so-called simple hybrid mechanical systems in Hamiltonian form.
Simple hybrid systems [35] (see also [51]) are characterized by the 4-tuple $\mathcal{H} = (D, X, S, \Delta)$, where $D$ is a smooth manifold called the domain, $X$ is a smooth vector field on $D$, $S$ is an embedded submanifold of $D$ with co-dimension 1 called the switching surface, and $\Delta : S \to D$ is a smooth embedding called the impact map. $S$ and $\Delta$ are also referred to as the guard and the reset map, respectively. The triple $(D, S, \Delta)$ is called a hybrid manifold.

The dynamics associated with a simple hybrid system is described by an autonomous system with impulse effects as in [51]. We denote by $\Sigma_{\mathcal{H}}$ the simple hybrid dynamical system generated by $\mathcal{H}$, that is,

$$\Sigma_{\mathcal{H}} : \left\{ \begin{array}{ll}
\dot{\gamma}(t) = X(\gamma(t)), & \text{if } \gamma^-(t) \notin S, \\
\gamma^+(t) = \Delta(\gamma^-(t)), & \text{if } \gamma^-(t) \in S,
\end{array} \right.$$ 

where $\gamma : I \subset \mathbb{R} \to D$, and $\gamma^-, \gamma^+$ denote the states immediately before and after the times when integral curves of $X$ intersect $S$ (i.e., pre and post impact of the solution $\gamma(t)$ with $S$), namely $\gamma^-(t) := \lim_{\tau \to t^-} x(\tau)$, $\gamma^+(t) := \lim_{\tau \to t^+} x(\tau)$ are the left and right limits of the state trajectory $\gamma(t)$.

A solution of a simple hybrid system may experience a Zeno state if infinitely many impacts occur in a finite amount of time. It is particularly problematic in applications where numerical work is used, as computation time grows infinitely large at these Zeno points. There are two primary modes through which Zeno behaviour can occur: (i) A trajectory is reset back onto the guard, prompting additional resets. To exclude this behaviour, we require that $S \cap \overline{\Delta}(S) = \emptyset$, where $\overline{\Delta}(S)$ denotes the closure as a set of $\Delta(S)$. This ensures that the trajectory will always be reset to a point with positive distance from the guard. (ii) The set of times where a solution of the system reaches the guard (and is correspondingly reset) has a limit point. This happens, for example, in the case of the bouncing ball with coefficient of restitution $1/2$ – see [7] and [28] for instance. Another example of this bouncing effect is the laser cooling of an atom by means of a laser beam\(^1\) (see [49] and references therein). To exclude these types of situations, we require the set of impact times to be closed and discrete, as in [51], so we will assume implicitly throughout the remainder of the paper that $\overline{\Delta}(S) \cap S = \emptyset$ and that the set of impact times is closed and discrete.

**Definition 1.** A simple hybrid system $\mathcal{H} = (D, X, S, \Delta)$ is said to be a simple hybrid forced Hamiltonian system if it is determined by $\mathcal{H}_F := (T^*Q, X_{H,F}, S_H, \Delta_H)$, where $X_{H,F} : T^*Q \to T(T^*Q)$ is the Hamiltonian forced vector field associated with the forced Hamiltonian system $(H, F)$ (see Subsection 2.1), $S_H$ is the switching surface, a submanifold of $T^*Q$ with co-dimension one, and $\Delta_H : S_H \to T^*Q$ is the impact map, a smooth embedding.

The simple hybrid forced dynamical system generated by $\mathcal{H}_F$ is given by

$$\Sigma_{\mathcal{H}_F} : \left\{ \begin{array}{ll}
\dot{\gamma}(t) = X_{H,F}(\gamma(t)), & \text{if } \gamma^-(t) \notin S_H, \\
\gamma^+(t) = \Delta_H(\gamma^-(t)), & \text{if } \gamma^-(t) \in S_H,
\end{array} \right.$$ 

where $\gamma(t) = (q(t), p(t)) \in T^*Q$.

\(^1\)Of course, this is actually a quantum phenomena, but one can consider a simple classical model as an atom impacting the surface of the laser beam.
Alternatively, $\Delta_H$ could be described by an impulsive external force appearing only on the instant on the impact (see [15, 32–34] and references therein).

**Remark 2.** Note that it is possible to consider more general definitions of simple hybrid forced Hamiltonian system. Indeed, it suffices to take $D$ a manifold with a certain geometric structure (e.g., a symplectic, Poisson, Jacobi or contact structure), and then take $X_H$ the Hamiltonian vector field that this structure defines for a Hamiltonian function $H$ on $D$, and take $F$ a semibasic 1-form on $D$.

**Example 1.** Suppose that $(Q, g)$ is a Riemannian manifold. Then, the switching map could be the tangent sphere bundle $S = \{(q, \dot{q}) \in TQ \mid g_q(\dot{q}, \dot{q}) = 1\}$ (see [36] for instance).

**Definition 2.** A simple hybrid forced Lagrangian system is a simple hybrid system determined by $\mathcal{Z}_F := (TQ, X_{L_F}, S_L, \Delta_L)$, where $X_{L_F} : TQ \to T(TQ)$ is the forced Lagrangian vector field associated with the forced Lagrangian system $(L, \mathcal{F}_L)$, $S_L$ the switching surface, a submanifold of $TQ$ with co-dimension one, and $\Delta_L : S_L \to TQ$ the impact map as defined before.

**Definition 3.** A hybrid flow for $\mathcal{H}_F$ is a tuple $\chi_{\mathcal{H}_F} = (\Lambda, \mathcal{J}, \mathcal{C})$, where

- $\Lambda = \{0, 1, 2, \ldots \} \subseteq \mathbb{N}$ is a finite (or infinite) indexing set,
- $\mathcal{J} = \{I_i\}_{i \in \Lambda}$ a set of intervals, called hybrid intervals, where $I_i = [\tau_i, \tau_{i+1}]$ if $i, i + 1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, $N$ finite, with $\tau_i, \tau_{i+1}, \tau_N \in \mathbb{R}$ and $\tau_i \leq \tau_{i+1}$,
- $\mathcal{C} = \{c_i\}_{i \in \Lambda}$ is a collection of solutions for the vector field $X_{H,F}$ specifying the continuous-time dynamics, i.e., $\tilde{c}_i = X_{H,F}(c_i(t))$ for all $i \in \Lambda$, and such that for each $i, i + 1 \in \Lambda$, (i) $c_i(\tau_{i+1}) \in S_H$, and (ii) $\Delta_H(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1})$.

Analogously, one can introduce the notion of hybrid flow $\chi_{\mathcal{Z}_F}$ for a simple hybrid forced Lagrangian system $\mathcal{Z}_F$. The relation between both hybrid flows is given by the following result, based on the equivalence between the Lagrangian and Hamiltonian dynamics in the hyperregular case achieved via the fiber derivative $\mathcal{F}_H$ that we will define below.

**Definition 4.** The fiber derivative of a Hamiltonian is the map $\mathcal{F}_H : T^*Q \to TQ$ defined by $\alpha_q : \mathcal{F}_H(\beta_q) = \frac{d}{de} \bigg|_{e=0} H(\beta_q + e\alpha_q)$, $\alpha_q, \beta_q \in T_q^*Q$ which in local coordinates is $\mathcal{F}_H(q, p) = (q, \dot{q}) = (q, \frac{\partial H}{\partial p}(q, p))$. We say that $H$ is regular if $\mathcal{F}_H$ is a local diffeomorphism and that $H$ is hyperregular if $\mathcal{F}_H$ is a (global) diffeomorphism. Equivalently, $H$ is regular (resp. hyperregular) if $\mathcal{F}_H$ is a local (resp. global) isomorphism of fibre bundles.

**Proposition 2.** If $\chi_{\mathcal{H}_F} = (\Lambda, \mathcal{J}, \mathcal{C})$ is a hybrid flow for $\mathcal{H}_F$, $S_L = \mathcal{F}_H(S_H)$, and $\Delta_L$ is defined in such a way that $\mathcal{F}_H \circ \Delta_H = \Delta_L \circ \mathcal{F}_H \mid_S$, then $\chi_{\mathcal{Z}_F} = (\Lambda, \mathcal{J}, (\mathcal{F}_H)(\mathcal{C}))$ with $(\mathcal{F}_H)(\mathcal{C}) = \{(\mathcal{F}_H)(c_i)\}_{i \in \Lambda}$.

**Proof:** If $c_i(t)$ is an integral curve of $X_{H,F}$, $\tilde{c}_i(t) = (\mathcal{F}_H \circ c_i)(t)$ is an integral curve for $X_{L,F}$. In this way, if we consider a solution $c_0(t)$ with initial value $c_0 = (q_0, p_0)$ defined on $[\tau_0, \tau_1]$, then $\tilde{c}_0(t)$ is a solution with initial value
\( \tilde{c}_0 = (q_0, \dot{q}_0) \) defined on \([\tau_0, \tau_1]\). Likewise for a solution \( c_1(t) \) defined on \([\tau_1, \tau_2]\), we get a corresponding solution \( \tilde{c}_1(t) \) defined on the same hybrid interval \([\tau_1, \tau_2]\). Proceeding inductively, one finds \( \tilde{c}_i(t) \) defined on \([\tau_i, \tau_{i+1}]\). It only remains to check that \( \tilde{c}_i(t) \) satisfies \( \tilde{c}_i(\tau_{i+1}) \in S_L \) and \( \Delta_L(\tilde{c}_i(\tau_{i+1})) = \tilde{c}_{i+1}(\tau_{i+1}) \), but using the properties of \( F H \),

(i) \( \tilde{c}_i(\tau_{i+1}) = (F H \circ c_i)(\tau_{i+1}) = F H(c_i(\tau_{i+1})) \) and given that \( c_i(\tau_{i+1}) \in S_H \) then \( \tilde{c}_i(\tau_{i+1}) \in S_L \).

(ii) \( \Delta_L(\tilde{c}_i(\tau_{i+1})) = \Delta_L \circ F H \circ c_i(\tau_{i+1}) = F H \circ H \circ c_i(\tau_{i+1}) = F H \circ c_{i+1}(\tau_{i+1}) = \tilde{c}_{i+1}(\tau_{i+1}) \).

\[\Box\]

**Definition 5.** Let \((H, F)\) be a forced Hamiltonian system on \( T^*Q \), with Hamiltonian forced vector field \( X_{H.F} \). Let \((T^*Q, X_{H.F}, S_H, \Delta_H)\) be a simple hybrid forced Hamiltonian system. Then, a function \( f \) on \( T^*Q \) is called a hybrid constant of the motion if

(i) it is a constant of the motion for \((H, F)\), i.e., \( X_{H.F}(f) = 0 \),

(ii) it is left invariant by the impact map, namely, \( f \circ \Delta_H = f \circ i \), where \( i : S_H \hookrightarrow T^*Q \) denotes the canonical inclusion.

The definition of hybrid constant of the motion for a simple hybrid forced Lagrangian system is completely analogous.

### 4. Symplectic Reduction of Simple Hybrid Forced Mechanical Systems with Symmetries

The starting point for symmetry reduction is a Lie group action \( \psi : G \times Q \to Q \) of some Lie group \( G \) on the manifold \( Q \). One needs to assume that all the actions considered satisfy some regularity conditions in order to do reduction. Indeed, from now on, we will consider free and proper actions [1]. These conditions ensure that the quotient of a smooth manifold by the action is a smooth manifold [41].

Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( \mathfrak{g}^* \) its dual as vector space. There is a natural lift \( \psi^{T^*Q} \) of the action \( \psi \) to \( T^*Q \), the cotangent lift, defined by \((g, (q,p)) \mapsto (T^*\psi_{g^{-1}}(q,p))\). The lifted action \( \psi^{T^*Q} \) enjoys the following properties [1], [18]:

(i) It preserves the canonical 1-form, meaning that \( (\psi^{T^*Q}_g)^*\theta_Q = \theta_Q \) for all \( g \in G \). Therefore, it is a symplectic action, i.e., \( (\psi^{T^*Q}_g)^*\omega_Q = \omega_Q \) for all \( g \in G \).

(ii) It admits an \( \text{Ad}^* \)-equivariant momentum map \( J : T^*Q \to \mathfrak{g}^* \) given by \( J(\alpha_g)(\xi) = \left( t_{\xi_Q}(\theta_Q) \right) (\alpha_g) = \alpha_g(\xi_Q(q)) \) for each \( \xi \in \mathfrak{g} \). Here \( \xi_Q \) is the infinitesimal generator of the action of \( \xi \in \mathfrak{g} \) on \( Q \) and \( \xi_Q^c \) is the generator of the lifted action on \( T^*Q \).

**Remark 3.** One could consider a general action \( \Phi : G \times T^*Q \to T^*Q \) of \( G \) on \( T^*Q \), not necessarily lifted from an action of \( G \) on \( Q \). However, in order for the map \( J \), defined as above, to be an \( \text{Ad}^* \)-equivariant momentum map, it is required that \( \Phi \) preserves \( \theta_Q \) (see [1, Theorem 4.2.10]). Then, it is easy to show (e.g., by direct computation in local coordinates) that \( \Phi \) is necessarily a lifted action. ◊
Recall that (see [19, 43, 52]) by denoting \( \{ \phi_t^X \} \) the flow of a vector field \( X \) on \( Q \), we can define the complete lift \( X^c \) of \( X \) as the vector field on \( T^*Q \) whose flow is the cotangent lift of \( \{ \phi_t^X \} \). In local coordinates, it is given by \( X^c = X^1 \frac{\partial}{\partial q^1} - p_j \frac{\partial X^1}{\partial p_j} \).

Let us first introduce the symplectic reduction for the forced Hamiltonian systems \( (H, F) \). If \( H \) is known to be \( G \)-invariant, the subgroup \( G_F \) of \( G \) such that \( H \) and \( F \) are both \( G_F \)-invariant can be described as follows. For each \( \xi \in g \), consider the real-valued function \( J^\xi : T^*Q \to \mathbb{R} \) given by \( J^\xi(\alpha, \xi) = \langle J(\alpha), \xi \rangle \), that is \( J^\xi = \iota_{\xi^Q} \theta_Q \). Let \( \xi \in g \), then \( J^\xi \) is a conserved quantity for \( X_{H,F} \) if and only if \( F(\xi^Q) = 0 \) (see [18, 43]) and \( \xi \) leaves \( F \) invariant if and only if \( \iota_{\xi^Q} dF = 0 \). In addition, the vector subspace of \( g \) given by \( g_F = \{ \xi \in g : F(\xi^Q) = 0, \ i_{\xi^Q} dF = 0 \} \) is a Lie subalgebra of \( g \). Observe that, for each \( \xi \in g_F \), \( \xi^Q \) is a symmetry of the forced Hamiltonian system \( (H, F) \) (see Subsection 2.3).

Let \( G_F \subset G \) be the Lie subgroup generated by \( g_F \), let \( J_F : T^*Q \to g_F^* \) be the reduced momentum map with \( \mu \in g_F^* \) a regular value of \( J_F \), and let us denote by \( (G_F)_\mu \) the isotropy subgroup in \( \mu \). Since the \( G \)-action on \( T^*Q \) is free and proper by hypothesis and \( \mu \) is a regular value, obviously the \( (G_F)_\mu \)-action on \( J^{-1}(\mu) \) is free and proper as well, and thus \( J^{-1}(\mu)/(G_F)_\mu \) is a smooth manifold [41]. We have that [18, 43]:

(i) \( J_F^{-1}(\mu) \) is a submanifold of \( T^*Q \) and \( X_{H,F} \) is tangent to it.

(ii) The reduced space \( M_\mu := J_F^{-1}(\mu)/(G_F)_\mu \) is a symplectic manifold, whose symplectic structure \( \omega_\mu \) is uniquely determined by \( \pi_\mu^* \omega_\mu = \iota_{\mu^Q}^* \omega_Q \), where \( \pi_\mu : J_F^{-1}(\mu) \to M_\mu \) and \( \iota_\mu : J_F^{-1}(\mu) \to T^*Q \) denote the canonical projection and the canonical inclusion, respectively.

(iii) \( H \) induces a reduced function \( H_\mu : M_\mu \to \mathbb{R} \) defined by \( H_\mu \circ \pi_\mu = H \circ \iota_\mu \).

(iv) \( F \) induces a reduced 1-form \( F_\mu \) on \( M_\mu \), uniquely determined by \( \pi_\mu^* F_\mu = \iota_{\mu^Q}^* F \).

(v) The forced Hamiltonian vector field \( X_{H,F} \) projects onto \( X_{H,F,\mu} \).

**Remark 4.** In order to obtain a reduced Hamiltonian function \( H_\mu \) and a reduced external force \( F_\mu \), both \( H \) and \( F \) need to be, independently, \( G_F \)-invariant. The conditions that each \( \xi^Q \) has to satisfy for this to occur are stronger than the ones required for being a symmetry of the forced Hamiltonian (see Subsection 2.3). As a matter of fact, we can weaken this requirement and reduce \( \alpha_{H,F} := dH + F \) instead of \( H \) and \( F \) separately. Suppose that \( \alpha_{H,F} \) is \( \psi_{T^*Q} \)-invariant, i.e., \( \alpha_{H,F}(\xi^Q) = \xi^Q(H) + F(\xi^Q) = 0 \) for every \( \xi \in g \). In other words, \( \xi_Q \) is a symmetry of the forced Hamiltonian for every \( \xi \in g \) (since the action \( \psi_{T^*Q} \) leaves \( \theta_Q \) invariant, \( \mathcal{L}_{\xi^Q} \theta_Q = 0 \)). Let \( G_\mu \) be the isotropy group of \( G \) in \( \mu \), where \( \mu \in g^* \) is a regular value of \( J \). Then,

(i) \( J_F^{-1}(\mu) \) is a submanifold of \( T^*Q \) and \( X_{H,F} \) is tangent to it.

(ii) The reduced space \( M_\mu := J_F^{-1}(\mu)/(G_F)_\mu \) is a symplectic manifold, and its symplectic structure \( \omega_\mu \) is uniquely determined by \( \pi_\mu^* \omega_\mu = \iota_{\mu^Q}^* \omega_Q \).

(iii) \( \alpha_{H,F} \) induces a reduced 1-form \( \alpha_{H,F,\mu} \) on \( M_\mu \), uniquely determined by \( \pi_\mu^* \alpha_{H,F} = \iota_{\mu^Q}^* \alpha_{H,F} \).
(iv) The forced Hamiltonian vector field $X_{H,F} = X_{\alpha_{H,F}}$ projects onto $X_{\mu_{H,F}}$, where $\xi_{X_{\mu_{H,F}}} \omega_{\mu} = \alpha_{H,F}^I$.

Next, we extend the symplectic reduction for forced Hamiltonian systems to simple hybrid forced Hamiltonian systems with symmetries. Consider a simple hybrid forced Hamiltonian system $H_F = (T^*Q, X_{H,F}, S_H, \Delta H)$. To perform a hybrid reduction one needs to impose some compatibility conditions between the action and the hybrid system (see [3] and [4]). By a hybrid action on the simple hybrid forced Hamiltonian system $H_F$ we mean a Lie group action $\psi: G \times Q \rightarrow Q$ such that

- $H$ is invariant under $\psi^{T*Q}$, i.e. $H \circ \psi^{T*Q} = H$,
- $\psi^{T*Q}$ restricts to an action of $G$ on $S_H$,
- $\Delta_H$ is equivariant with respect to the previous action, namely $\Delta_H \circ \psi_g^{T*Q}|_{S_H} = \psi_g^{T*Q} \circ \Delta_H$.

**Definition 6.** A momentum map $J$ will be called a generalized hybrid momentum map for $H_F$ if, for each regular value $\mu_-$ of $J$,

$$\Delta_H \left(J^{-1}_{S_H}(\mu_-)\right) \subset J^{-1}(\mu_+)$$

(1)

for some regular value $\mu_+$. In other words, for every point in the switching surface such that the momentum before the impact takes a value of $\mu_-$, the momentum will take a value $\mu_+$ after the impact. That is, the switching map translates the dynamics from one level set of the momentum map into another.

Consider $\mathcal{H}_F = (T^*Q, X_{H,F}, S_H, \Delta_H)$ equipped with a hybrid action $\psi$ such that $H$ and $F$ are $G_F$-invariant. We can apply a hybrid analogue of the symplectic reduction for forced Hamiltonian systems to the simple hybrid forced Hamiltonian system $\mathcal{H}_F$ as follows. Let $G_F \subset G$ be the Lie subgroup generated by $g_F$ and $J_F : T^*Q \rightarrow g_F^*$ the reduced momentum map, which is also assumed to be a generalized hybrid momentum map. Then, for each $\xi \in g$, $J_F^\xi = \langle J_F, \xi \rangle$ is a hybrid constant of the motion.

Let $\mu_-, \mu_+ \in g_F^*$ be two regular hybrid values of $J_F$, which means that they are regular values of both $J_F$ and $J_F|_{S_H}$. When we combine this definition with the condition (1), we obtain that the following diagram

$$
\begin{array}{ccc}
J_F^{-1}(\mu_-) & \xrightarrow{\quad \Delta_H \mid_{J_F^{-1}(\mu_-)} \quad} & J_F^{-1}(\mu_+) \\
\uparrow & & \uparrow \\
T^*Q & \xrightarrow{\Delta_H} & T^*Q \\
\end{array}
$$

commutes, where $J_F^{-1}(\mu)$ and $J_F^{-1}_{S_H}(\mu)$ are embedded submanifolds of $T^*Q$ and $S_H$, respectively. The hook arrows $\hookrightarrow$ in the diagram denote the corresponding canonical inclusions.
Let \( \psi : G \times Q \to Q \) be a Lie group action of a connected Lie group \( G \) on \( Q \). If \( \Delta_H \) is equivariant with respect to \( \psi^{T^*Q} \), and \( \mu_- \), \( \mu_+ \) are regular values of \( J \) such that \( \Delta_H \left( J|_{S_H}^{-1}(\mu_-) \right) \subset J^{-1}(\mu_+) \), then \( G_{\mu_-} = G_{\mu_+} \).

Proof. Let \( g \in G_{\mu_-} \). Then,

\[
J \circ \Delta_H \left( J|_{S_H}^{-1}(\mu_-) \right) = J \circ \Delta_H \circ \psi^{T^*Q} \left( J|_{S_H}^{-1}(\mu_-) \right)
\]

\[
= J \circ \psi^{T^*Q} \circ \Delta_H \left( J|_{S_H}^{-1}(\mu_-) \right)
\]

\[
= \text{Ad}_{\mu_-}^* \circ J \circ \Delta_H \left( J|_{S_H}^{-1}(\mu_-) \right),
\]

where we have used the equivariance of \( J \) and \( \Delta_H \), so \( g \in G_{\mu_+} \), and hence \( G_{\mu_-} \) is a Lie subgroup of \( G_{\mu_+} \).

Now, observe that \( G_{\mu} \) has the same dimension, for each \( \mu \in g^* \). Therefore, the identity components of \( G_{\mu_-} \) and \( G_{\mu_+} \) coincide. If we assume that \( G \) is connected, \( G_{\mu_-} \) and \( G_{\mu_+} \) are equal to their identity components, so \( G_{\mu_-} = G_{\mu_+} \).

\[\Box\]

**Theorem 4.** Let \( \mathcal{H}_F = (T^*Q, X_{H,F}, S_H, \Delta_H) \) be a hybrid forced Hamiltonian system. Let \( \psi : G \times Q \to Q \) be a hybrid action of a connected Lie group \( G \) on \( Q \). Suppose that \( H \) and \( F \) are \( G \)-invariant and assume that \( J_F \) is a generalized hybrid momentum map. Consider a sequence \( \{\mu_i\} \) of regular values of \( J_F \), such that \( \Delta_H \left( J|_{S_H}^{-1}(\mu_i) \right) \subset J^{-1}(\mu_{i+1}) \). Let \( (G_F)_{\mu_i} = (G_F)_{\mu_0} \) be the isotropy subgroup in \( \mu_i \) under the co-adjoint action. Then,

(i) \( J_F^{-1}(\mu_i) \) is a submanifold of \( T^*Q \) and \( X_{H,F} \) is tangent to it.

(ii) The reduced space \( M_{\mu_i} := J_F^{-1}(\mu_i)/(G_F)_{\mu_0} \) is a symplectic manifold, whose symplectic structure \( \omega_{\mu_i} \) is uniquely determined by \( \pi_{\mu_i}^* \omega_{\mu_i} = \varpi_{\mu_i} \omega_{\mu_i} \), where \( \pi_{\mu_i} : J_F^{-1}(\mu_i) \to M_{\mu_i} \) and \( \varpi_{\mu_i} : J_F^{-1}(\mu_i) \to T^*Q \) denote the canonical projection and the canonical inclusion, respectively.

(iii) \((H,F)\) induces a reduced forced Hamiltonian system \((H_{\mu_i}, F_{\mu_i})\) on \( M_{\mu_i} \), given by \( H_{\mu_i} \circ \pi_{\mu_i} = H \circ \varpi_{\mu_i} \) and \( \pi_{\mu_i}^* F_{\mu_i} = \varpi_{\mu_i}^* F \). Moreover, the forced Hamiltonian vector field \( X_{H,F} \) projects onto \( X_{H_{\mu_i}, F_{\mu_i}} \).

(iv) \( J_F \left|_{S_H}^{-1}(\mu_i) \right. \subset S_H \) reduces to a submanifold of the reduced space \( (S_H)_{\mu_i} \subset J_F^{-1}(\mu_i)/(G_F)_{\mu_0} \).

(v) \( \Delta_H|_{J^{-1}(\mu_i)} \) reduces to a map \( (\Delta_H)_{\mu_i} : (S_H)_{\mu_i} \to J_F^{-1}(\mu_{i+1})/(G_F)_{\mu_0} \).

Therefore, after the reduction procedure, we get a sequence of reduced simple hybrid forced Hamiltonian systems \( \{\mathcal{H}_{F_{\mu_i}}\} \), where \( \mathcal{H}_{F_{\mu_i}} = (J_F^{-1}(\mu_i)/(G_F)_{\mu_0}, X_{H_{\mu_i}, F_{\mu_i}}, (S_H)_{\mu_i}, (\Delta_H)_{\mu_i}) \).
\[ \cdots \to J_F^{-1}(\mu_i) \to J_F^{-1}(\mu_i) \to J_F^{-1}(\mu_i+1) \to \cdots \]

**Proof.** See [1, 47] for a proof of the first two assertions. The third statement was proven in [18] (see also [43]).

Since the \((G_F)_{\mu_0}\)-action restricts to a free and proper action on \(S_H\), \((S_H)_{\mu_i} = J_F^{-1}\Delta_H^{-1}(\mu_i), (G_F)_{\mu_0}\) is a smooth manifold. Clearly, it is a submanifold of \(J_F^{-1}(\mu_i)/(G_F)_{\mu_0}\). Since \(\Delta_H\) is equivariant, it induces an embedding \((\Delta_H)_{\mu_i} : (S_H)_{\mu_i} \to J_F^{-1}(\mu_i+1)/(G_F)_{\mu_0}\).

The reduction picture in the Lagrangian side can now be obtained from the Hamiltonian one by adapting the scheme developed in [39]. In the same fashion as in the Hamiltonian side, by a hybrid action on the simple hybrid Lagrangian system \(\mathcal{L}_F = (TQ, X_L, F, S_L, \Delta_L)\) we mean a Lie group action \(\psi : G \times Q \to Q\) such that

- \(L\) is invariant under \(\psi^TQ\), i.e. \(L \circ \psi^TQ = L\).
- \(\psi^TQ\) restricts to an action of \(G\) on \(S_L\).
- \(\Delta_L\) is equivariant with respect to the previous action, namely \(\Delta_L \circ \psi^TQ|_{S_L} = \psi^TQ \circ \Delta_L\).

where \(\psi^TQ\) is the tangent lift of the action \(\psi\) to \(TQ\), defined by \((g, (q, \dot{q})) \mapsto (T\psi g(q, \dot{q}))\).

The key idea is that, since \(\psi^TQ\) is a hybrid action under which \(L\) is invariant, the Legendre transformation \(FL\) is a diffeomorphism such that:

- It is equivariant with respect to \(\psi^TQ\) and \(\psi^TQ\),
- It preserves the level sets of the momentum map, that is, \(FL((J_L)_{F^{-1}}(\mu_i)) = J_F^{-1}(\mu_i)\),
- It relates the symplectic structures, that is, \((FL)^*\omega_Q = \omega_L\), meaning that, \(FL\) is a symplectomorphism.

It follows that the map \(FL\) reduces to a symplectomorphism \((FL)_{\text{red}}\) between the reduced spaces. Consider the Lie subalgebra \(\mathfrak{g}_{FL} = \{\xi \in \mathfrak{g} : F^L(\xi^L) = 0, \ i_{\xi^L}dF^L = 0\}\) of \(\mathfrak{g}\), and let \(G_{FL}\) be the Lie subgroup it generates. Then, the following diagram commutes:

\[
\begin{array}{ccc}
(TQ, S_L, \Delta_L) & \xrightarrow{FL} & (T^*Q, S_H, \Delta_H) \\
\text{Red.} \downarrow & & \text{Red.} \downarrow \\
((J_L)_{F^{-1}}(\mu_i)/(G_{FL})_{\mu_0}, (S_L)_{\mu_i}, (\Delta_L)_{\mu_i}) & \xrightarrow{(FL)_{\text{red}}} & (J_F^{-1}(\mu_i)/(G_F)_{\mu_0}, (S_H)_{\mu_i}, (\Delta_H)_{\mu_i})
\end{array}
\]
If \((L, F^L)\) is the Lagrangian counterpart of \((H, F)\) (i.e., \(E_L = H \circ F^L\) and \(F^L = \nabla L^* F\)), then \(G_{FL} = G_F\). Indeed, for any \(g \in G_F\),

\[
\left(\psi_g^{\text{T}Q}\right)^* F^L = \left(\psi_g^{\text{T}Q}\right)^* \circ F^L = \left(F_L \circ \psi_g^{\text{T}Q}\right)^* F = \left(\psi_g^{\text{T}Q} \circ \nabla F\right)^* F
\]

where we have used the equivariance of \(\nabla F\) with the symplectic structure \(pr\).

This identification is a symplectomorphism when we endow the space on the right-hand side of (2) with the symplectic structure \(J\). Let \(\mathcal{A}: TQ \to \mathfrak{g}\) be a connection one form, and let us denote by \(\mathcal{A}_{\mu_i}(\cdot) = (\mu_i, \mathcal{A}(\cdot))\) the 1-form on \(Q\) obtained by contraction with \(\mu_i \in \mathfrak{g}^*\). Building on the well-known results on cotangent bundle reduction [44] it is possible to show that there is an identification

\[
J_{FL}^{-1}(\mu_i)/(G_F)_{\mu_0} \simeq \left(T^*(Q/G) \times_{Q/G} Q/(G_F)_{\mu_0}\right).
\]  

This identification is a symplectomorphism when we endow the space on the right-hand side of (2) with the symplectic structure \(pr_1^*\omega_{Q/G} + pr_2^*\mu_i\), where \(pr_1\) and \(pr_2\) are the canonical projections, and \(\mu_i\) is the so-called magnetic term, obtained from the reduction of \(d\mathcal{A}_{\mu_i}\) to \(Q/(G_F)_{\mu_0}\). For details, see [39, 44].

For the Lagrangian side, one needs a further regularity condition, sometimes referred to as \(G\)-regularity. Precisely, one has the following definition [40].

**Definition 7.** Let \((L, F^L)\) be an \(G_{FL}\)-invariant forced Lagrangian system on \(TQ\) (i.e., \(L \circ \psi_g^{\text{T}Q} = L\) and \((\psi_g^{\text{T}Q})^* F^L = F^L\) for every \(g \in G\)) and denote by \(\xi_Q^c\) the infinitesimal generator for the associated lifted action. We say that \((L, F^L)\) is \(G_{FL}\)-regular if, for each \(v_q \in TQ\), the map \((J_L)^{v_q}_{FL} : \mathfrak{g}_F \to \mathfrak{g}_F, \xi \mapsto (J_L)^{v_q}_{FL} (\xi + \xi_Q^c(q))\) is a diffeomorphism.

That is, \(G_{FL}\)-regularity amounts to regularity “with respect to the subgroup variables”. From now on we will assume that the pair \((L, F^L)\) is \(G_{FL}\)-regular. In this case, there is an identification \((J_L)^{-1}_{FL}(\mu_i)/(G_{FL})_{\mu_0} \simeq \left(T(Q/G) \times_{Q/G} Q/(G_{FL})_{\mu_0}\right)\) (see [24] for instance).

It is possible to interpret the reduced dynamics on this space as being the Lagrangian dynamics of some regular Lagrangian subjected to a gyroscopic force (arising from the magnetic term) if one works in the more general class of *magnetic Lagrangians* [38], which in the present situation should be extended to include forced Lagrangians. The so-called magnetic Lagrangian systems are a wide class of Lagrangian systems on which the Lagrangian function might not depend on some of the velocities, and which may as well include a force term given by a 2-form. The framework of magnetic Lagrangian systems is very convenient when carrying out Routh reduction, since the reduced system is not, in general, a standard Lagrangian system. Routh reduction has been extended to magnetic Lagrangian systems, and this permits to carry out Routh reduction by stages [40]. The function...
which plays the role of the reduced Lagrangian is the \textit{Routhian}², and it is defined as (the reduction of) the \((G_F)_{\mu_0}\)-invariant function \(R_{F}^{\mu_i} = L - \mathcal{A}_{\mu_i}\) restricted to \((J_L)_{F}^{-1}(\mu_i)\). The next diagram summarizes the situation:

\[
\begin{array}{ccc}
TQ & \overset{\mathcal{F}_L}{\longrightarrow} & T^*Q \\
\text{Red.} & & \text{Red.} \\
(T(Q/G) \times_{Q/G} Q/(G_{F})_{\mu_0}) & \overset{\mathcal{F}R_{F}^{\mu_i}}{\longrightarrow} & (T^*(Q/G) \times_{Q/G} Q/(G_{F})_{\mu_0})
\end{array}
\]

\textbf{Remark 5.} The case \(G = S^1\) corresponds to the notion of cyclic coordinates (the case \(G = \mathbb{R}\) is analogous; if \(G\) is a product of \(S^1\) or \(\mathbb{R}\) one can iterate the procedure). Since \(G\) is abelian, \(G_{\mu_i} = G\) for every \(\mu_i \in \mathfrak{g}^*\). The reduced space \((J_L)_{F}^{-1}(\mu_i)/(G_{F})_{\mu_0}\) can be identified with \(T(Q/S^1)\) and the reduced dynamics is Lagrangian with respect to the reduced Lagrangian (i.e., the Routhian) \(R_{F}^{\mu_i} = L - \mathcal{A}_{\mu_i}\) on \(T(Q/S^1)\). The reduced switching \((S_L)_{\mu_i}\) can be identified with a submanifold of \(T(Q/S^1)\) and the reset map is identified with a map \((\Delta_L)_{\mu_i} : (S_L)_{\mu_i} \rightarrow T(Q/S^1)\). We will use the same notations for both of them.

A case of special interest with regards to applications is when \(Q = S^1 \times M\), where \(M\) is called the shape space and the action is simply \((\theta, x) \mapsto (\theta + \alpha, x)\). This is often the situation when dealing with simple models of bipedal walkers, see e.g. [5]. In what follows, we will assume we work in this setting. While this is indeed a strong assumption, it is always the case locally, so, as long as it applies to the domain of interest of a specific problem, the procedure applies.

If the forced Lagrangian system \((L, F^L)\) has a cyclic coordinate \(\theta\), i.e., \(L\) is a function of the form \(L(\theta, x, \dot{x})\), and \(F^L\) is of the form \(F^L(\theta, x, \dot{x}) = F_\theta(\theta, x, \dot{x}) \, dx\). The conservation of the momentum map \((J_L)_F = \mu_i\) reads \(\frac{\partial L}{\partial \dot{\theta}} = \mu_i\), and one can use this relation to express \(\dot{\theta}\) as a function of the remaining –non cyclic– coordinates and their velocities, and the prescribed regular value of the momentum map \(\mu_i\) (i.e., \(\dot{\theta} = \dot{\theta}(x, \dot{x}, \mu_i)\)) in the Lagrangian \(L\) and external force \(F^L\). It is worth noting that this is the stage where the \(G_F\)-regularity of \((L, F^L)\) is used: it guarantees that \(\dot{\theta}\) can be worked out in terms of \(x, \dot{x}\) and \(\mu_i\). If one chooses the canonical flat connection on \(Q \rightarrow Q/S^1 = M\), then the Routhian and the reduced external force can be computed as

\[
R_{F}^{\mu_i}(x, \dot{x}) = \left[ L(\dot{\theta}, x, \dot{x}) - \mu_i \dot{\theta} \right]_{\theta = \theta(x, \dot{x}, \mu_i)} = F_{\mu_i}(x, \dot{x}) = F^L(\dot{\theta}, x, \dot{x})_{\theta = \theta(x, \dot{x}, \mu_i)}
\]

where the notation means that we have everywhere expressed \(\dot{\theta}\) as a function of \((x, \dot{x}, \mu_i)\). Note that (3) is the classical definition of the Routhian [48]. Besides, since the connection is flat, one has no magnetic term in the reduced dynamics.

Collisions with the switching surface will, in general, modify the value of the momentum map (nonelastic impacts). Therefore, if \(\mathcal{J} = \{I_i\}_{i \in \Lambda}\) is the hybrid interval, the reduced Hamiltonian \(H_{\mu_i}\) and the reduced external force \(F_{\mu_i}\) have to

²Note the difference with the Hamiltonian reduction.
be defined in each $I_i$, taking into account the value of the momentum $\mu_i$ after the collision at time $\tau_i$. Note that this also has influence in the way the reset map $\Delta_H$ is reduced.

Let us denote: (1) $\mu_i$ the momentum of the system in $I_i = [\tau_i, \tau_{i+1}]$, (2) $(\Delta_H)_{\mu_i}$ the reduction of $\Delta_H|_{\{\mu_i\}}$, and (3) $(S_H)_{\mu_i}$ the reduction of $S_H$, so there is a sequence of reduced simple hybrid forced Hamiltonian systems:

$$
[\tau_0, \tau_1] \xrightarrow{\text{Red.}} (T^*(Q/\mathbb{S}^1) \times Q/\mathbb{S}^1, Q/(G_F)_{\mu_0}, X_{H_{\mu_0}}, F_{\mu_0}, (S_H)_{\mu_0}, (\Delta_H)_{\mu_0})
$$

Coll. \hline
$[\tau_1, \tau_2] \xrightarrow{\text{Red.}} (T^*(Q/\mathbb{S}^1) \times Q/\mathbb{S}^1, Q/(G_F)_{\mu_1}, X_{H_{\mu_1}}, F_{\mu_1}, (S_H)_{\mu_1}, (\Delta_H)_{\mu_1})$

Coll. \hline
$\cdots \xrightarrow{\text{Red.}} \cdots$

Here "Coll." and "Red." stand for collision and reduction, respectively.

The fact that the momentum will, in general, change with the collisions makes the reconstruction procedure more challenging. If one wishes, as usual, to use a reduced solution to reconstruct the original dynamics, one needs to compute the reduced hybrid data after each collision. This means that once the reduced solution has been obtained between two collision events, say at $t = \tau_i$ and $t = \tau_{i+1}$, one has to reconstruct this solution to obtain the new momentum after the collision at $\tau_{i+1}$, and use this new momentum to build a new reduced hybrid system whose solution should be obtained until the next collision event at $\tau_{i+2}$ and so on. As usual, the reconstruction procedure from the reduced hybrid flow to the hybrid flow involves an integration of the regular value at each stage in the previous diagram using the solution of the reduced simple hybrid forced Hamiltonian system. Essentially, this accounts to imposing the momentum constraint on the reconstructed solution. More precisely, suppose that $\chi^{\mathcal{L}_F^\mu_i}(c_0) = (\Lambda, \mathcal{J}, \mathcal{C})$ is a hybrid flow of $\mathcal{L}_F^\mu_i$. Then we can construct a hybrid flow $\mathcal{L}_F(c_0(\tau_0)) = (\Lambda, \mathcal{J}, \mathcal{C})$ of $\mathcal{L}_F$ by constructing the flow recursively between two collisions. Writing $c_i(t) = (x_{\mu_i}, \dot{x}_{\mu_i}, \theta_{\mu_i}, \hat{\theta}_{\mu_i})$ define recursively as follows.

Assume that we have a mechanical Lagrangian of the form $L(x, \dot{x}, \dot{\theta}) = \dot{q}^T M \dot{q} - V(q)$, where $\dot{q} = \left(\frac{\dot{x}}{\dot{\theta}}\right)$ and the mass matrix is $M = \begin{pmatrix} M_x(x) & M_{\dot{\theta},x}(x) \\ M_{\theta,x}(x) & M_{\theta}(x) \end{pmatrix}$. First note that $(J_L)_f(x, \dot{x}, \theta, \dot{\theta}) = \frac{\partial L}{\partial \dot{q}}(x, \dot{x}, \theta, \dot{\theta}) = M_{\theta,x}(x) \dot{x} + M_{\theta}(x) \dot{\theta}$. Then it is easy to see that

$$
\dot{\mu}_i(t) = M_{\theta}^{-1} \left(\theta_{\mu_i}(t)\right)(\mu_i) - M_{\theta,x}(x_{\mu_i}) \dot{x}_{\mu_i}(t), \quad (4)
$$

$$
\theta_{\mu_i}(t) = (\Delta_H)_{\mu_i}(e^{\mu_i-1}(\tau_i)) + \int_{\tau_i}^{t-\tau_i} \hat{\theta}_{\mu_i}(s) ds, \quad (5)
$$

where $t \in [\tau_i, \tau_{i+1}]$ and $(\Delta_H)_{\mu_i}(e^{\mu_i-1}(\tau_i))$ is the $\theta$-component of $(\Delta_H)_{\mu_i}(e^{\mu_i-1}(\tau_i))$. Note that, at each step, one has to reconstruct with the corresponding momenta $\mu_i$ in...
equation (4) and reduce again the dynamics after the collision with a new momenta \( \mu_{i+1} \) as conserved quantity.

\[ \diamond \]

**Remark 6.** A generalized hybrid momentum map is called a *hybrid momentum map* if \( \Delta_H \) preserves the momentum map. In other words, \( J \) is a hybrid momentum map if the diagram

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{J} & T^*Q \\
S_H & \xrightarrow{\Delta_H} & T^*Q
\end{array}
\]

commutes (see [4]). If the momentum map is hybrid, the reduction of a simple hybrid forced Hamiltonian system \( \mathcal{H}_F = (T^*Q, X_{H_F}, S_H, \Delta_H) \), with initial value of the momentum map \( \mu_0 \), yields a single reduced simple hybrid forced Hamiltonian system \( H_{\mu_0}^F = (J^{-1}_F(\mu_0)/(G_F)_{\mu_0}, X_{H_{\mu_0}}F_{\mu_0}, (S_H)_{\mu_0}, (\Delta_H)_{\mu_0}) \). Additionally, \( J^*_F \) is a hybrid constant of the motion for each \( \xi \in g_F \).

\[ \diamond \]

**Example 2.** Consider a homogeneous circular disk of radius \( R \) and mass \( m \) moving in the vertical plane \( xOy \) (see [32, Example 8.2], see also [34, Example 3.7]). Let \( (x, y) \) be the coordinates of the centre of the disk and \( \theta \) the angle between a point of the disk and the axis \( Oy \). The dynamics of the system is determined by the Lagrangian \( L \) on \( T(R^2 \times \mathbb{S}^1) \) given by

\[
L = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 + k^2 \dot{\theta}^2 \right),
\]

subject to the nonholonomic constraint of rolling without sliding, namely, \( \dot{x} = R \dot{\theta} \).

Note that the phase space is thus 5-dimensional. Suppose that the system hits a wall determined by the axis \( Ox \), and the impact map is given by

\[
\left( \dot{x}^-, \dot{y}^-, \dot{\theta}^- \right) \mapsto \left( \frac{R^2 \dot{x}^- + k^2 R \dot{\theta}^-}{k^2 + R^2}, -\dot{y}^-, \frac{R \dot{\theta}^- + k^2 \dot{\theta}^-}{k^2 + R^2} \right).
\]

Consider the Lie group action of \( SO(2) \cong \mathbb{S}^1 \) on \( Q \) by rotations on \( \mathbb{R}^2 \) and translations on \( \mathbb{S}^1 \), namely, \( (x, y, \theta) \mapsto (\cos \alpha x - \sin \alpha y, \sin \alpha x + \cos \alpha y, \theta + \alpha) \).

Clearly, the Lagrangian is invariant under the lifted action on \( TQ \). The corresponding momentum map is \( J_L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = (m x \dot{y} - m y \dot{x}, mk \dot{\theta}) \).

One can check that \( J_L \) is a generalized hybrid momentum map but not a hybrid momentum map.

\[ \diamond \]

**Remark 7.** In a simple hybrid Hamiltonian system with a mechanical Hamiltonian function the impact can be obtained from the Newtonian impact equation (see [7] for instance). That is, \( \Delta_H(q, p) = (q, P_q(p)) \), where \( P_q : T^*_qQ \to T^*_qQ \) is given by

\[
P_q(p) = p - (1 + e) \frac{\langle p, dh_q \rangle}{||dh_q||_q^2} dh_q,
\]

with \( || \cdot ||_q \) denoting the corresponding norm on \( T^*_qQ \), and \( \langle \cdot, \cdot \rangle_q \) is the inner-product on the vector space \( T^*_qQ \) defined through the kinetic energy of the system.
as \(\langle \alpha, \beta \rangle_q = \sum_{i,j=1}^{\dim(Q)} \alpha_i \beta_j M_{ij}^{-1}(q)\), being \(M(q)\) the inertia matrix associated with the mechanical system under study. The parameter \(0 \leq e \leq 1\) is the coefficient of restitution (for instance, \(e = 1\) corresponds with elastic impacts and \(e = 0\) with inelastic impacts). The switching surface is also defined through the inner product as

\[
S_H = \{(q, p) \in T^*Q : h(q) = 0\} \text{ and } \langle \langle p, dh_q \rangle \rangle_q \leq 0\}.
\]

Note that the analytical expressions for the switching surface and the impact map depend on the chosen metric. Hence, by choosing different metrics one can obtain different expressions for the impact map and switching surface, which could help to obtain invariant expressions for \(S_H\) and \(\Delta_H\) for a given action.

Similarly, in a simple hybrid Lagrangian system with a mechanical Lagrangian function \(L = \dot{q}^T M(q) \dot{q} - V(q)\), the impact can be obtained from the Newtonian impact equation \(P : TQ \rightarrow TQ\) given by

\[
P(q, \dot{q}) = \dot{q} - (1 + e) \frac{dh_q \dot{q}}{dh_q M(q)^{-1} dh_q^T} M(q)^{-1} dh_q T,
\]

where \(M(q)\) is the inertial matrix for the Lagrangian system, \(h\) a function describing the switching surface as a submanifold of \(Q\) and \(e\) the coefficient of restitution. The switching surface is

\[
S_L = \{(q, \dot{q}) : TQ | h(q) = 0 \text{ and } dh_q \dot{q} \leq 0\}.
\]

Example 3 (Billiard with dissipation). Consider a particle of mass \(m\) in the plane which is free to move inside the surface defined by \(x^2 + y^2 = 1\). The surface of the “billiard” is assumed to be rough in such a way that the friction is non-linear on the velocities.

The Hamiltonian function \(H : T^*\mathbb{R}^2 \rightarrow \mathbb{R}\) is given by

\[
H(x, y, p_x, p_y) = \frac{1}{2m}(p_x^2 + p_y^2)
\]

and \(F(x, y, p_x, p_y) = F_x dx + F_y dy\) is an external force given by \(F_x = -\frac{2c}{m}(p_x xy - p_y x^2)\), \(F_y = \frac{2c}{m}(p_y xy - p_x y^2)\), for some constant \(c > 0\). Hamilton equations of motion for the particle off the boundary are

\[
\dot{p}_x = \frac{2c}{m}(p_y x^2 - p_x xy), \quad \dot{p}_y = -\frac{2c}{m}(p_x y^2 - p_y xy), \quad \dot{x} = \frac{1}{m} p_x, \quad \dot{y} = \frac{1}{m} p_y.
\]

Using the Legendre transformation we can obtain the Lagrangian and external force \(F^L\). The Lagrangian function \(L : T\mathbb{R}^2 \rightarrow \mathbb{R}\) is given by

\[
L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)
\]

and \(F^L(x, y, \dot{x}, \dot{y}) = F^L_x dx + F^L_y dy\) is an external force given by \(F^L_x = -2c(\dot{x} xy - \dot{y} x^2)\), \(F^L_y = 2c(\dot{y} xy - \dot{x} y^2)\). The forced Euler-Lagrange equations for the particle...
off the boundary are

\[ m\ddot{x} = -2c(y\dot{x}^2 - \dot{x}xy), \]
\[ m\ddot{y} = 2c(\dot{x}y^2 - \dot{y}xy). \]

By introducing polar coordinates \( L \) and \( F^L \) become \( L(\theta, r, \dot{\theta}, \dot{r}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \) and \( F^L(\theta, r, \dot{\theta}, \dot{r}) = 2cr^3\dot{\theta}dr \), respectively. \( L \) is hyperregular and \( L \) and \( F^L \) are independent of \( \theta \). The forced Euler-Lagrange equations (in polar coordinates) are

\[
\dot{r} = \left( r - \frac{2c}{m}r^3 \right)\dot{\theta}, \quad mr^2\dot{\theta} = 0.
\]

Note that the momentum map \( J_L \) for \( \theta \), \( J_L (r, \dot{r}, \theta, \dot{\theta}) = mr^2\dot{\theta} \) is preserved, that is, by considering \( \mu = mr^2\dot{\theta} \) (i.e., \( \dot{\theta} = \frac{\mu}{mr^2} \)) the Routhian and the reduced force takes the form

\[
R^\mu_F (r, \dot{r}) = \frac{m}{2}\dot{r}^2 - \frac{\mu^2}{2mr^2}, \quad F^\mu_L = 2cr\frac{\mu}{m}dr
\]

and reduced forced Euler-Lagrange equations for the Routhian \( R^\mu_F \) are given by

\[
\dot{r} = \frac{\mu^2}{m^2r^3} - 2cr\frac{\mu}{m^2}.
\]

We consider the guard as the subset of \( T\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \) given by

\[
S = (T\mathbb{R}^2) \cap \{ x^2 + y^2 = 1, (\dot{x}, \dot{y}) \cdot (x, y) \geq 0 \}.
\]

Under the assumption of an elastic collision, using the Newtonian impact equation (7) with \( e = 1, h(x, y) = 1 - x^2 - y^2 \) and \( M = \text{id}_{\mathbb{R}^2} \), the reset map \((x, y, \dot{x}^-, \dot{y}^-) \mapsto (x, y, \dot{x}^+, \dot{y}^+)\), is given by

\[
\dot{x}^+ = \dot{x}^- - 2(x\dot{x}^- + y\dot{y}^-)x, \quad \dot{y}^+ = \dot{y}^- - 2(x\dot{x}^- + y\dot{y}^-)y.
\]

Therefore, the 4-tuple \( \mathcal{L}_F = (TQ, X_F^L, S, \Delta) \), is a simple hybrid forced Lagrangian system with \( Q = \mathbb{R}^2 \), and \( L \) and \( F \) as described before.

If we square both sides of (8), and noting that

\[ 2x\ddot{x} + 2y\ddot{y} = \frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(r^2) = 2r\dot{r}, \]

we have \((\dot{x}^+)^2 = (\dot{x}^-)^2 + (2r\dot{r}^-)^2x^2 - 4x\dot{x}^-r\dot{r}^-, \) and symmetrically to \( \dot{y}^+ \). Add \((\dot{x}^+)^2 + (\dot{y}^+)^2 \). We can conclude

\[ (\dot{r}^+)^2 = (\dot{r}^-)^2 + (2r\dot{r}^-)^2(x^2 + y^2) - 4(x\dot{x}^- + y\dot{y}^-)r\dot{r}^- \]
\[ = (\dot{r}^-)^2 + 4r^2(\dot{r}^-)(r^2 - 1). \]

This means that, since the collision occurs at \( r = 1 \), we have \( (\dot{r}^+)^2 = (\dot{r}^-)^2 \), then the solution that is obtained (physically) is \( \dot{r}^+ = -\dot{r}^- \). For \( \theta = \arctan(y/x) \), we have

\[ \dot{\theta}^+ = \frac{1}{1 + (y/x)^2} \left( \frac{\dot{y}^+x - \dot{x}^+y}{x^2} \right) = \frac{1}{r^2} (\dot{y}^-x - \dot{x}^-y) = \dot{\theta}^- . \]
where we have replaced the expression for $\dot{x}^+, \dot{y}^+$ and we used that $x^2 + y^2 = r^2$ and $(y\dot{x}^+ - \dot{y}^- x) = -r^2 \dot{\theta}^-$. It is understood that the “minus” square root is taken in $\dot{r}^+$ (the particle bounces on the boundary after the collision). The assumption of elastic collision implies, in particular, that the momentum map is preserved. This is clear since $r$ and $\dot{\theta}$ do not change with the collision.

The reduced reset map is determined by the expression for $\dot{r}^+$ (note that the expression drops to the quotient since it only involves $r$ and $\dot{r}$). The reduced switching surface is $S_\mu = \{r^2 = 1, \dot{r} > 0\}$. One obtains the simple hybrid forced Routhian system $L_{\beta}^F = (TQ_{\text{red}}, X_{R_{\mu}}^F, F, S_\mu, \Delta_\mu)$, with $Q_{\text{red}} \simeq \mathbb{R}^+$ parametrized by the radial coordinate $r$.

Figures 1 and 2 show numerical results using Python for two different values of the dissipation parameter $c$. The remaining parameters are the same for both simulations: $m = 1$, $r(0) = 0.5$, $\dot{r}(0) = 2$, $\theta(0) = 0$ (rad) and $\dot{\theta}(0) = 1$ (rad/s). The reduced dynamics is solved numerically (dashed purple line) and used to integrate (numerically) the reconstruction equation $\dot{\theta} = \frac{\mu}{mr}$, with $\mu$ determined from the initial conditions. Switching surfaces $S$ and $S_\mu$ are represented with a green solid line. Note also that the impact times on which the particle bounces are also obtained numerically.

Observe that the assumptions made to avoid the Zeno effect are satisfied. Indeed, $\dot{r}^+ < 0$, so $S \cap \Delta(S) = \emptyset$, and, at least with the numerical values chosen, no limit points in the set of impact times can be observed.

**Example 4.** This example illustrates the reduction of a simple hybrid Lagrangian system in which a non-flat connection appears. Let $\mathcal{M}$ be a 3-dimensional Riemannian manifold and let $\pi : Q \to \mathcal{M}$ be a circle bundle (that is, $S^1$ acts on $Q$ on the left and then $\pi : Q \to \mathcal{M}$ is a principal bundle, where $\mathcal{M} = Q/S^1$) with respect to left $SO(2)$ action. We will use the isomorphism (as Lie groups) of $SO(2)$ and $S^1$ to make our analysis consistent with the theory.
Let $A : TQ \to so(2) \simeq \mathbb{R}$ be a principal connection on $Q$ and consider the Lagrangian on $TQ$ given by $L(q, \dot{q}) = \frac{m}{2} \| T\pi(q, \dot{q}) \|^2_M + \frac{e}{c} \| A(q, \dot{q}) \|_{so(2)}$ where $e$ is the charge of the electron, $c$ is the speed of light, $\| \cdot \|_{so(2)} : so(2) \to \mathbb{R}$ the norm on $so(2)$, given by $\| \xi \|_{so(2)} = \langle \langle \xi, \xi \rangle \rangle = \text{tr}(\xi^T \xi)$. Note that in the absence of potential, $L$ is a Lagrangian of Kaluza-Klein type (see [9] for instance).

Note also that $\pi(\theta \cdot q) = \pi(q)$, for every $q \in Q$ and every $\theta \in S^1$. Thus, $L(\theta \cdot (q, \dot{q})) = \frac{m}{2} \| T\pi(\theta \cdot (q, \dot{q})) \|^2_M + \frac{e}{c} \| A(\theta \cdot (q, \dot{q})) \|_{so(2)}$

$= \frac{m}{2} \| T\pi(q, \dot{q}) \|^2_M + \frac{e}{c} \| Ad_\theta \cdot A(q, \dot{q}) \|_{so(2)}$

$= \frac{m}{2} \| T\pi(q, \dot{q}) \|^2_M + \frac{e}{c} \| A(q, \dot{q}) \|_{so(2)}$

$= L(q, \dot{q})$,

where $Ad_\theta = \text{Id}_{so(2)}$ since $SO(2)$ is Abelian. That is, $L$ is $SO(2)$-invariant and we may perform reduction by symmetries to get the equations of motion on the principal bundle $TQ/\text{SO}(2)$.

We consider the guard as the subset of $TQ$ given by $S_L = TQ \cap \{ x^2 + y^2 + z^2 = r^2, (\dot{x}, \dot{y}, \dot{z}) \cdot (x, y, z) \geq 0 \}$. Here $(x, y, z)$ are coordinates in $\mathcal{M}$. Under the assumption of an elastic collision, using the Newtonian impact equation (7) with elastic coefficient $e = 1$ (which should not be confused with the charge of the electron) and $h(x, y, z) = r^2 - x^2 - y^2 - z^2$, the reset map $(x, y, z, \dot{x}^-, \dot{y}^-, \dot{z}^-) \mapsto (x, y, z, \dot{x}^+, \dot{y}^+, \dot{z}^+)$ is given by

\[
\begin{align*}
\dot{x}^+ &= \dot{x}^- - 2(x\dot{x}^- + y\dot{y}^- + z\dot{z}^-)x \\
\dot{y}^+ &= \dot{y}^- - 2(x\dot{x}^- + y\dot{y}^- + z\dot{z}^-)y \\
\dot{z}^+ &= \dot{z}^- - 2(x\dot{x}^- + y\dot{y}^- + z\dot{z}^-)z
\end{align*}
\]
Therefore, the 4-tuple $\mathcal{H}_L = (TQ, X_L, S_L, \Delta_L)$ is a simple hybrid Lagrangian system with $L$ as described above.

Fixing the connection $\mathcal{A}$ on $Q$, we can use the principal connection $\mathcal{A}$ to get an isomorphism $\alpha_\mathcal{A} : TQ/SO(2) \to TM + \mathfrak{so}(2)$ which permits us to define the reduced Lagrangian

$$\mathcal{L}(x, \dot{x}, \xi) = \frac{m}{2} \| \dot{x} \|^2_M + \frac{c}{e} \| \xi \|_{\mathfrak{so}(2)}.$$ 

For the reduced Lagrangian $\mathcal{L}$, the dynamics is determined by

$$m \ddot{x}^2 = \langle \mu, \tilde{\mathcal{B}}(\dot{x}(t), \cdot) \rangle, \quad D_{\mathcal{L}} \mu = 0,$$

where $\mu = \frac{\partial L}{\partial \dot{\xi}}$ is the change of the particle. Here $\tilde{\mathcal{B}} : TM \wedge TM \to \mathfrak{so}(2)$ is the reduced curvature tensor associated with the connection form $\mathcal{A}$.

In the case $Q = \mathbb{R}^3 \times S^1$ the Lagrangian is

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{m}{2} \dot{x}^2 + e \frac{c}{\mathcal{A}(x, \dot{x})}.$$

In this case, we have that $TQ/SO(2) \simeq \mathbb{R}^3 \times \mathbb{R}$ where $AdQ = \mathbb{R}$ and the reduced Lagrangian is

$$\mathcal{L}(x, \dot{x}, \xi) = \frac{m}{2} \dot{x}^2 + \frac{c}{e} \xi.$$

The above equations reduce to Lorentz force law describing the motion of a charged particle of mass $m$ in a magnetic field

$$m \ddot{x} = \frac{e}{c} (\dot{x} \times \vec{B})$$

$$\dot{\mu} = 0,$$

where $\mu = \frac{\partial \mathcal{L}}{\partial \vec{B}} = \frac{e}{c}$ and $\vec{B} = (B_x, B_y, B_z) \in \mathfrak{X}(\mathbb{R}^3)$.

5. Cosymplectic Reduction for Time-Dependent Forced Mechanical Systems with Symmetries

Next we extend the reduction of simple hybrid forced Hamiltonian systems to the case in which the Hamiltonian, the external force and the switching surface depend explicitly on time. This is done in the context of cosymplectic reduction. First, we start by recalling some basic facts about time-dependent forced mechanical systems.

Our starting point is a time-dependent forced Lagrangian $L : \mathbb{R} \times TQ \to \mathbb{R}$ with a time-dependent external force denoted by $F^L$, a semibasic 1-form on $\mathbb{R} \times TQ$. Let us denote by $\mathbb{L} : \mathbb{R} \times TQ \to \mathbb{R} \times T^*Q$ the Legendre transformation associated with $L$, i.e., the map $(t, q, \dot{q}) \mapsto (t, q, p = \partial L/\partial \dot{q})$.

We will assume that the Lagrangian is hyperregular, i.e., that $\mathbb{L}$ is a diffeomorphism between $\mathbb{R} \times TQ$ and $\mathbb{R} \times T^*Q$. One can then work out the velocities

---

3This is always the case for mechanical Lagrangians.
\( \dot{q} \) in terms of \((t, q, p)\) using the inverse of \( FL \) and define the Hamiltonian function \( H : \mathbb{R} \times T^* \mathcal{Q} \rightarrow \mathbb{R} \) as \( H(t, q, p) = (p, \dot{q}(t, q, p)) - L(t, q, \dot{q}(t, q, p)) \), and the external force \( F \) such that \( FL(F^L) = F \), a semibasic 1-form on \( \mathbb{R} \times T^* \mathcal{Q} \).

Among the several geometric approaches to non-autonomous mechanics, we will be using the one based on the notion of cosymplectic manifold [8]. To describe the dynamics of a non-autonomous forced Hamiltonian system, one begins by considering the manifold \( \mathbb{R} \times T^* \mathcal{Q} \) equipped with the canonical cosymplectic structure \( \omega = dq^i \wedge dp_i, \eta = dt \), where \((q^i)\) are local coordinates on \( \mathcal{Q} \), and \((t, q^i, p_i)\) are the induced coordinates on \( \mathbb{R} \times T^* \mathcal{Q} \). There is an unique vector field \( R \) on \( \mathbb{R} \times T^* \mathcal{Q} \) such that \( \eta(R) = 1 \) and \( R \omega = 0 \), called the Reeb vector field.

In coordinates, \( R = \partial_t \). The cosymplectic structure defines an isomorphism between vector fields and 1-forms on \( \mathbb{R} \times T^* \mathcal{Q} \). With every smooth function \( f \) on \( \mathbb{R} \times T^* \mathcal{Q} \), one can associate a Hamiltonian vector field \( X_f \), such that \( \iota_{X_f} \eta = 0 \) and \( \iota_{X_f} \omega = df - R(f) \eta \). We can also define the evolution vector field \( X_{f,t} = X_f + R \).

Given a Hamiltonian \( H(t, q, p) \) and an external force \( F(t, q, p) \), the forced Hamiltonian vector field \( X_{H,F} \) is the vector field on \( \mathbb{R} \times T^* \mathcal{Q} \) defined by \( X_{H,F} = X_H + Z_F \), where

\[
\iota_{X_H} \omega_Q = dH - R(H) \eta, \quad \iota_{X_H} \eta = 0, \quad \iota_{Z_F} \omega_Q = F - F(R) \eta, \quad \iota_{Z_F} \eta = 0.
\]

The forced evolution vector field corresponding to the forced Hamiltonian system \((H, F)\), denoted by \( X_{H,F,t} \), is defined by

\[
X_{H,F,t} = R + X_{H,F} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial H}{\partial q^i} + F_i \right) \frac{\partial}{\partial p_i}.
\]

A similar procedure can be used in the Lagrangian picture. Under the assumption of hyperregularity discussed above, the manifold \( \mathbb{R} \times T \mathcal{Q} \) can be endowed with a cosymplectic structure depending on the Lagrangian, namely, \( \omega_L = FL^* (dq_i \wedge dp_i) = dq^i \wedge \lambda \left( \frac{\partial L}{\partial \dot{q}^i} \right), \eta = dt \) (note that we will be using the same symbol \( \eta = dt \) for two different 1-forms on different manifolds). Let \( \mathcal{R}_L \) denote the associated Reeb vector field. One then constructs the energy function \( E_L : \mathbb{R} \times T \mathcal{Q} \rightarrow \mathbb{R} \) given by \( E_L(t, q, \dot{q}) = \langle FL(t, q, \dot{q}), \dot{q} \rangle - L(t, q, \dot{q}) \), and obtains the Hamiltonian forced vector field \( X_{L,F,t} \) associated to \( E_L \) and \( F^L \) via the Lagrangian cosymplectic structure. This leads to a forced evolution vector field \( X_{L,F^L,t} = X_{L,F,t} + \mathcal{R}_L \), locally given by

\[
X_{L,F^L,t}(t, q^i, \dot{q}^i) = \left( t, t, q^i, \dot{q}^i; \dot{q}^i, M^{-1} \left( -F^L_i + \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^i \partial q^j} \dot{q}^i \dot{q}^j \right) \right).
\]

Finally, the equivalence between the Lagrangian and Hamiltonian dynamics in the hyperregular case for forced time-dependent systems is achieved via \( FL \) as follows.

**Proposition 5.** The tangent map of \( FL \) maps \( X_{L,F^L,t} \) onto \( X_{H,F,t} \). In other words \((TFL) X_{L,F^L,t} = X_{H,F,t}, \) where \((TFL) : T(T \mathcal{Q}) \rightarrow T(T^* \mathcal{Q}) \). In particular, the flow of \( X_{L,F^L,t} \) is mapped onto the flow of \( X_{H,F,t} \).
The evolution vector field $X_{H,F,t}$ is characterized by $t_{X_t} \omega_Q = dH - \mathcal{R}(H)\eta$, $t_{X_t} \eta = 0$, $t_{Z_t} \omega_Q = F - \mathcal{R}(F)\eta$ and $t_{Z_t} \eta = 0$. Observe that

$$(\mathbb{F}L)^*(t_{X_t} \omega_Q) = (\mathbb{F}L)^*(dH - \mathcal{R}(H)\eta) = (\mathbb{F}L)^*(dH) - (\mathbb{F}L)^*(\mathcal{R}(H)\eta)$$

$$= d((\mathbb{F}L)^*H) - \mathcal{R}((\mathbb{F}L)^*H)\eta = d(E_L) - \mathcal{R}(E_L)\eta$$

$$= t_{X_t} \omega_L.$$

This means that

$$t_{X_t} \omega_L = (\mathbb{F}L)^*(t_{X_t} \omega_Q) = t_{(\mathbb{F}L^{-1})_*X_t}((\mathbb{F}L^*\omega_Q) = t_{(\mathbb{F}L^{-1})_*X_t} \omega_L.$$

In a similar way one can show that $t_{X_t} \eta = t_{(\mathbb{F}L^{-1})_*X_t} \eta$. This implies that $X_L = (\mathbb{F}L^{-1})_*X_H$, that is, $(\mathbb{F}L)_*X_L = X_H$.

On the other hand, observe that

$$(\mathbb{F}L)^*(t_{Z_t} \omega_Q) = (\mathbb{F}L)^*(F - \mathcal{R}(F)\eta) = (\mathbb{F}L)^*(F) - (\mathbb{F}L)^*(\mathcal{R}(F)\eta)$$

$$= (\mathbb{F}L)^*F - \mathcal{R}((\mathbb{F}L)^*F)\eta = F_L - \mathcal{R}(E_L)\eta$$

$$= t_{Z_t} \omega_L,$$

which implies that

$$t_{Z_t} \omega_L = (\mathbb{F}L)^*(t_{Z_t} \omega_Q) = t_{(\mathbb{F}L^{-1})_*Z_t}((\mathbb{F}L^*\omega_Q) = t_{(\mathbb{F}L^{-1})_*Z_t} \omega_L.$$ 

Similarly, one can see that $t_{Z_t} \eta = t_{(\mathbb{F}L^{-1})_*Z_t} \eta$. Hence, $Z_F = (\mathbb{F}L^{-1})_*Z_F$, that is, $(\mathbb{F}L)_*Z_F = Z_F$.

**Definition 8.** A simple hybrid time-dependent forced Lagrangian system is described by the tuple $\mathcal{L}_L^F = (\mathbb{R} \times TQ, X_{L,F,t}, S_L^I, \Delta_L^I)$, where $Q$ is a differentiable manifold, $X_{L,F,t}$ is the dynamical vector field associated with the time-dependent forced Lagrangian system $(L,F,t)$, $S_L^I$ is an embedded submanifold of $\mathbb{R} \times TQ$ with co-dimension one called the switching surface, and $\Delta_L^I : S_L^I \rightarrow \mathbb{R} \times TQ$ is a smooth embedding called the impact map.

Analogously, one can introduce the notion of simple hybrid time-dependent forced Hamiltonian system $\mathcal{H}_F^Q = (\mathbb{R} \times T^*Q, X_{H,F,t}, S_H^I, \Delta_H^I)$, where $X_{H,F,t}$ is the dynamical vector field associated with the time-dependent forced Hamiltonian system $(H,F)$. The relation between both hybrid flows is given by the following result, based on the equivalence between the Lagrangian and Hamiltonian dynamics in the hyperregular case can achieved via the fiber derivative $\mathbb{F}H$.

**Proposition 6.** If $\chi^{\mathcal{H}_F^Q} = (\Lambda, \mathcal{J}, \mathcal{C})$ is a hybrid flow for $\mathcal{H}_F^Q$, $S_L^I = \mathbb{F}H(S_H^I)$, and $\Delta_L^I$ is defined in such a way that $\mathbb{F}H \circ \Delta_H^I = \Delta_L^I \circ \mathbb{F}H |_{S_H^I}$, then $\chi^{\mathcal{L}_L^F} = (\Lambda, \mathcal{J}, (\mathbb{F}H)(\mathcal{C}))$ with $(\mathbb{F}H)(\mathcal{C}) = \{(\mathbb{F}H)(c_i)\}_{i \in A}$.

**Proof:** The proof follows straightforwardly from the Proof of Proposition 2. \qed

Let $\mathcal{L}_L^F = (\mathbb{R} \times TQ, X_{L,F,t}, S_L^I, \Delta_L^I)$ be a simple hybrid time-dependent forced Lagrangian system and let $\psi : G \times Q \rightarrow Q$ be a free and proper Lie group action with $\psi^{\mathbb{R} \times T^*Q}$ denoting its natural lift, namely, $G$ acts on $\mathbb{R}$ by the identity and on $T^*Q$ by $\psi^{T^*Q}$. The action $\psi^{\mathbb{R} \times T^*Q}$ enjoys the following properties [1], [18]:

- **Proposition 7.** \( \psi^{\mathbb{R} \times T^*Q} \) preserves the switching surface $\Delta_L^I$ and the fibration $\mathcal{H}_F^Q$. Moreover, \( \psi^{\mathbb{R} \times T^*Q} \) commutes with the actions of the Lie groups $G$ and $\mathbb{R}$ on $\mathbb{R} \times TQ$ and $T^*Q$, respectively.
is a cosymplectic action, meaning that \((\psi_{R \times T^*Q})^* \omega = \omega\) and \((\psi_{R \times T^*Q})^* \eta = \eta\) for every \(q \in G\).

- It admits an \(\text{Ad}^*\)-equivariant momentum map \(\tilde{J}: \mathbb{R} \times T^*Q \to \mathfrak{g}^*\) given by

\[
\langle \tilde{J}(t, q, p), \xi \rangle = \langle p, \xi_Q \rangle,
\]

for each \(\xi \in \mathfrak{g}\), where \(\xi_Q(q) = d(\psi_{\exp(t \xi)} q)/dt\) is the infinitesimal generator of \(\xi \in \mathfrak{g}\).

Likewise, \(\psi_{R \times TQ}\) denotes the natural lift action of \(G\) on \(\mathbb{R} \times TQ\), i.e., \(G\) acts on \(\mathbb{R}\) by the identity and on \(TQ\) by \(\psi_{TQ}\).

Let \(X\) be a vector field on \(Q\). The complete lift of \(X\) at \((t, \nu_q) \in \mathbb{R} \times T^*Q\) is given by \(X^c_{t, \nu_q} = (0_t, \check{X}^c(\nu_q))\), where \(\check{X}^c(\nu_q)\) denotes the complete lift of \(X\) to \(T^*Q\) at \(\nu_q\). Locally, \(X^c = X^i \frac{\partial}{\partial q^i} - p_j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial p^j}\). Analogously, one can define the complete lift of \(X\) to \(\mathbb{R} \times TQ\), locally, \(X^c = X^i \frac{\partial}{\partial q^i} + \dot{\nu}^j \frac{\partial X^i}{\partial \nu} \frac{\partial}{\partial \nu^j}\).

As in the autonomous case, to perform a hybrid reduction one needs to impose some compatibility conditions between the action and the hybrid system (see [3] and [4]). By a hybrid action on the simple hybrid system \(H^t = (\mathbb{R} \times T^*Q, X_{H,F,t}, S^t_H, \Delta^t_H)\) we mean a Lie group action \(\psi: G \times Q \to Q\) such that

- \(H\) is invariant under \(\psi_{R \times T^*Q}\), i.e., \(H \circ \psi_{R \times T^*Q} = H\).
- \(\psi_{R \times T^*Q}\) restricts to an action of \(G\) on \(S^t_H\).
- \(\Delta^t_H\) is equivariant with respect to the previous action, namely

\[
\Delta^t_H \circ \psi_{R \times T^*Q} \mid_{S^t_H} = \psi_{R \times T^*Q} \circ \Delta^t_H.
\]

From the Cosymplectic Reduction Theorem [2] (see also [21]), we can obtain the non-autonomous analogue of Theorem 4.

**Theorem 7.** Let \(H^t = (\mathbb{R} \times T^*Q, X_{H,F,t}, S^t_H, \Delta^t_H)\) be a time-dependent simple hybrid forced Hamiltonian system. Let \(\psi: G \times Q \to Q\) be a hybrid action of a connected Lie group \(G\) on \(Q\). Suppose that \(H\) and \(F\) are \(G_{F}\)-invariant and assume that \(J^t_F\) is a generalized hybrid momentum map. Consider a sequence \(\{\mu_i\}\) of regular values of \(J^t_F\), such that \(\Delta^t_H\left(J^t_F^{-1}(\mu_i)\right) \subset J^t_F^{-1}(\mu_{i+1})\). Let \((G_F)_{\mu_i} = (G_F)_{\mu_i}\) be the isotropy subgroup in \(\mu_i\) under the co-adjoint action. Then,

(i) \(J^t_F^{-1}(\mu_i)\) is a submanifold of \(\mathbb{R} \times T^*Q\) and \(X_{H,F,t}\) is tangent to it.

(ii) The reduced space \(M_{\mu_i} := J^t_F^{-1}(\mu_i)/(G_F)_{\mu_0}\) is a cosymplectic manifold, whose cosymplectic structure \((\omega_{\mu_i}, \eta_{\mu_i})\) is uniquely determined by \(\pi^*_{\mu_i} \omega_{\mu_i} = i^*_{\mu_i} \omega_Q\) and \(\pi^*_{\mu_i} \eta_{\mu_i} = i^*_{\mu_i} \eta\), where \(\pi_{\mu_i}: J^t_F^{-1}(\mu_i) \to M_{\mu_i}\) and \(i_{\mu_i}: J^t_F^{-1}(\mu_i) \hookrightarrow \mathbb{R} \times T^*Q\) denote the canonical projection and the canonical inclusion, respectively. In addition, the Reeb vector field \(\mathbb{R}\) projects onto \(\mathbb{R}_{\mu_i}\), the Reeb vector field defined by \(\omega_{\mu_i}\) and \(\eta_{\mu_i}\).
(iii) $(H, F)$ induces a reduced time-dependent forced Hamiltonian system $(H_{\mu_i}, F_{\mu_i})$ on $M_{\mu_i}$, given by $H_{\mu_i} \circ \pi_{\mu_i} = H \circ t_{\mu_i}$ and $\pi_{\mu_i}^* F_{\mu_i} = \bar{\gamma}_{\mu_i}^* F$. Moreover, the forced evolution vector field $X_{H,F,t}$ projects onto $X_{H_{\mu_i},F_{\mu_i},t}$.

(iv) $\bar{J}_F \mid S^1_H \subset S^1_H$ reduces to a submanifold of the reduced space $(S^1_H)_{\mu_i} \subset \bar{J}_F^{-1}(\mu_i)/(G_F)_{\mu_0}$.

(v) $\Delta^t_H|\bar{J}^{-1}(\mu_i)$ reduces to a map $(\Delta^t_H)_{\mu_i} : (S^1_H)_{\mu_i} \to \bar{J}_F^{-1}(\mu_i+1)/(G_F)_{\mu_0}$.

Therefore, after the reduction procedure, we get a sequence of reduced time-dependent simple hybrid forced Hamiltonian systems \{\mathcal{H}^\mu_F\}, where \mathcal{H}^\mu_F = (\bar{J}_F^{-1}(\mu_i)/(G_F)_{\mu_0}, X_{H_{\mu_i},F_{\mu_i}}, (S^1_H)_{\mu_i}, (\Delta^t_H)_{\mu_i}).

By a hybrid action on the simple hybrid system $\mathcal{L}^*_L = (TQ, X_L,F_L,t, S^1_L, \Delta^t_L)$ we mean a Lie group action $\psi : G \times Q \to Q$ such that

- $L$ is invariant under $\psi^{R \times TQ}$, i.e. $L \circ \psi^{R \times TQ} = L$.
- $\psi^{R \times TQ}$ restricts to an action of $G$ on $S^1_L$.
- $\Delta^t_L$ is equivariant with respect to the previous action, namely

\[ \Delta^t_L \circ \psi^{R \times TQ} |_{S^1_L} = \psi^{R \times TQ} \circ \Delta^t_L. \]

The reduction picture in the Lagrangian side can now be obtained from the Hamiltonian one by adapting the scheme developed in [39] to the cosymplectic setting. Suppose that $\mathcal{L}^*_L$ is equipped with a hybrid action $\psi$ and that $(L, F_L)$ is $G_{FL}$-regular. Since $L$ is invariant and hyperregular, the Legendre transformation $FL$ is a diffeomorphism such that:

- It is equivariant with respect to $\psi^{R \times TQ}$ and $\psi^{R \times T^*Q}$.
- It preserves the level sets of the momentum map, that is, $FL((\bar{J}_L)^{-1}(\mu_i)) = \bar{J}_F^{-1}(\mu_i)$.
- It relates both cosymplectic structures, that is, $(FL)^* \omega_Q = \omega_L$ and $(FL)^* \eta = \eta$, that is, $FL$ is a cosymplectomorphism.

The equivariance of $FL$ implies that $\psi^{R \times TQ}$ admits an $Ad^*$-equivariant momentum map $\bar{J}_L : \mathbb{R} \times TQ \to g^*$ given by $\bar{J}_L = \bar{J} \circ FL$.

It follows that the map $FL$ reduces to a cosymplectomorphism $(FL)_{\text{red}}$ between the reduced spaces. Therefore we get the following commutative diagram of hybrid
manifolds:

\[
(R \times TQ, S^I_L, \Delta^I_L) \xrightarrow{FL} (\mathbb{R} \times T^*Q, S^I_H, \Delta^I_H)
\]

\[\xrightarrow{\text{Red.}}\]

\[\left((\bar{J}_L)^{-1}_F(\mu_i)/(G_F)_{\mu_0}, (S^I_L)_{\mu_i}, (\Delta^I_L)_{\mu_i}\right) \xrightarrow{(FL)\text{red}} \left((\bar{J}_F^{-1}_L(\mu_i))/(G_F)_{\mu_0}, (S^I_H)_{\mu_i}, (\Delta^I_H)_{\mu_i}\right)\]

The \(G_F\)-regularity permits the identification

\[
(\bar{J}_L)_F^{-1}(\mu_i)/(G_F)_{\mu_0} \simeq \mathbb{R} \times \left(T(Q/G) \times_{Q/G} Q/(G_F)_{\mu_0}\right)
\]

It is possible to interpret the reduced dynamics on this space as being the Lagrangian dynamics of some regular time-dependent Lagrangian subjected to a time-dependent gyroscopic force (arising from the magnetic term) if one works in the class of magnetic Lagrangians [38], which in the present situation should be extended to include time-dependent Lagrangians and external forces. The Routhian is defined as (the reduction of) the \((G_F)_{\mu_0}\)-invariant function

\[
R_{FL}^{\mu_i} = L - \mathcal{A}_{\mu_i}
\]

restricted to \((\bar{J}_L)_F^{-1}(\mu_i)\). Then, the following diagram commutes:

\[
R \times TQ \xrightarrow{FL} R \times T^*Q
\]

\[\xrightarrow{\text{Red.}}\]

\[
R \times \left(T(Q/G) \times_{Q/G} Q/(G_F)_{\mu_0}\right) \xrightarrow{FR_{FL}^{\mu_i}} R \times \left(T^*(Q/G) \times_{Q/G} Q/(G_F)_{\mu_0}\right)
\]

**Remark 8.** Assume we work with \(Q = S^1 \times M\), where \(M\) is the shape space and the action is \((\theta, x) \mapsto (\theta + \alpha, x)\). The forced Lagrangian system has a cyclic coordinate \(\theta\), i.e., \(L\) is a function of the form \(L(t, \theta, x, \dot{x})\), and \(F\) is of the form \(F(\theta, x, \dot{x}) = F(x, \dot{x}) dx\). The conservation of the momentum map \((J_L)_F = \mu_i\) reads \(\frac{\partial L}{\partial \dot{\theta}} = \mu_i\), and one can use this relation to express \(\dot{\theta}\) as a function of the remaining –non cyclic– coordinates and their velocities, and the prescribed regular value of the momentum map \(\mu_i\), i.e., \(\dot{\theta} = \dot{\theta}(x, \dot{x}, \mu_i)\). Note that this is the stage at which the \(G_F\)-regularity of \(L\) and \(F^L\) is used: it guarantees that \(\dot{\theta}\) can be worked out in terms of \(x, \dot{x}\) and \(\mu_i\). If one chooses the canonical flat connection on \(Q \rightarrow Q/S^1 = M\), then the Routhian can be computed as

\[
R_{FL}^{\mu_i}(t, x, \dot{x}) = \left[L(t, \theta, x, \dot{x}) - \mu_i \dot{\theta}\right]_{\dot{\theta}=\dot{\theta}(t, x, \dot{x}, \mu_i)},
\]

where the notation means that we have everywhere expressed \(\dot{\theta}\) as a function of \((t, x, \dot{x}, \mu_i)\). Note that (10) is the classical definition of the Routhian [48]. Besides, since the connection is flat, one has no magnetic term in the reduced dynamics.

As in the non-autonomous case, collisions with the switching surface will, in general, modify the value of the momentum map (non-elastic case). Therefore, if \(\bar{I}_i\) is the hybrid interval (see Definition 3), the Routhian has to be defined in each \(I_i\) taking into account the value of the momentum \(\mu_i\) after the collision.
at time $\tau_i$. Note that this also has an influence in the way the reset map $\Delta^L_{\mu_i}$ and the switching $S_{\mu_i}$ are reduced. Let us denote: (1) $\mu_i$ the momentum of the system in $I_i = [\tau_i, \tau_{i+1})$, (2) $(\Delta^L_{\mu_i})_i$ the reduction of $\Delta^L_{\mid J^{-1}_{\mu_i}}$, and (3) $(S^L_{\mu_i})_i$ the reduction of $\tilde{J}_L \mid S^L_{\mu_i}$. There is a sequence of reduced simple hybrid time-dependent Lagrangian systems ("Coll." stands for collision and "Red." stands for reduction):

\[
\begin{align*}
&\tau_0, \tau_1 \xrightarrow{\text{Red.}} (T(Q/S^L_1), L_{\mu_0}, (S^L_{\mu_0})(\Delta^L_{\mu_0})) \\
&\tau_2, \tau_3 \xrightarrow{\text{Red.}} (T(Q/S^L_2), L_{\mu_1}, (S^L_{\mu_1})(\Delta^L_{\mu_1})) \\
&(\ldots) \xrightarrow{\text{Red.}} (\ldots)
\end{align*}
\]

As in the symplectic case, the fact that the momentum will, in general, change with the collisions makes the reconstruction procedure more challenging. If one wishes, as usual, to use a reduced solution to reconstruct the original dynamics, one needs to compute the reduced hybrid data after each collision. This means that once the reduced solution has been obtained between two collision events, say at $t = \tau_n$ and $t = \tau_{n+1}$, one has to reconstruct this solution to obtain the new momentum after the collision at $\tau_{n+1}$ and use this new momentum to build a new reduced hybrid system whose solution should be obtained until the next collision event at $\tau_{n+2}$, and so on. As usual, the reconstruction procedure from the reduced hybrid flow to the hybrid flow involves an integration of the cyclic variable at each stage in the previous diagram, using the solution of the reduced simple hybrid time-dependent Lagrangian system. Essentially, this accounts to imposing the momentum constraint on the reconstructed solution similarly as we exposed in equations (4) and (5).

\[\text{(Remark 9)}\]

If $Q$ is not a product, one needs to compute the Routhian using the general expression (9) and consider the reduced simple hybrid time-dependent Lagrangian system with a time-dependent magnetic term. If $G$ is non-Abelian one can use the class of magnetic Lagrangian systems, adapted to the time-dependent setting, to describe the reduced dynamics. Coordinate formulae for the reduced dynamics and reconstruction in this more general case might be obtained using the techniques in [17].

\[\text{(Remark 10)}\]

In a simple hybrid time-dependent Lagrangian system with a mechanical Lagrangian function $L = \dot{q}^T M(q) \dot{q} - V(q)$, one can consider a switching surface of the form

\[S_L = \{(q, \dot{q}) : TQ \mid h(q) + f(t) = 0 \text{ and } dh_q \dot{q} \leq 0\},\]  

for some function $h$ on $Q$ and some function $f$ on $\mathbb{R}$. The reset map is then given by Eq. (7). In the Hamiltonian framework, one could also consider the counterpart of Eq. (11) and a reset map given by Eq. (6).
Example 5 (Billiard with dissipation and moving walls). Consider a particle of mass \( m \) in the plane which is free to move inside the surface defined by circle whose radius varies in time according to a given function \( f(t) \), i.e. \( x^2 + y^2 = f(t) \). As in Example 3, the surface of the “billiard” is assumed to be rough in such a way that the friction is non-linear on the velocities.

The Lagrangian function \( L : \mathbb{R} \times T\mathbb{R}^2 \rightarrow \mathbb{R} \) is given by

\[
L(t, x, y, \dot{x}, \dot{y}) = \frac{m}{2} \dot{x}^2 + \dot{y}^2
\]

and \( F^L(t, x, y, \dot{x}, \dot{y}) = F_x^L dx + F_y^L dy \) is an external force given by \( F_x^L = -2c \dot{x} \dot{y} - \dot{f} \dot{x} \), \( F_y^L = 2c \dot{x} \dot{y} - \dot{f} \dot{y} \), for a constant \( c > 0 \). The equations of motion for the particle off the boundary are then

\[
c\dot{x} + m\ddot{x} = -2c(\dot{x}\dot{y} - \dot{f}\dot{x}), \quad c\dot{y} + m\ddot{y} = 2c(\dot{x}\dot{y} - \dot{f}\dot{y}).
\]

The guard is the subset of \( \mathbb{R} \times T\mathbb{R}^2 \simeq \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) given by

\[
S^L_t = (\mathbb{R} \times TQ) \cap \{ x^2 + y^2 = f(t), (\dot{x}, \dot{y}), (x, y) \geq 0 \}.
\]

This set \( S^L_t \) describes the situation in which the particle hits the moving boundary while heading “outwards” the billiard. For simplicity in the definition of the switching surface, we assume that \( f(t) \) is increasing: this guarantees the particle only hits the boundary when the boundary is also moving outwards. Under the assumption of an elastic collision, the reset map \( (t, x, y, \dot{x}^-, \dot{y}^-) \mapsto (t, x, y, \dot{x}^+, \dot{y}^+) \) is given by:

\[
\dot{x}^+ = \dot{x}^- + \frac{f(t) - 2(\dot{x}^- + \dot{y}^-)x}{f(t)}x, \quad \dot{y}^+ = \dot{y}^- + \frac{f(t) - 2(\dot{x}^- + \dot{y}^-)y}{f(t)}y.
\]

By introducing polar coordinates \( L \) and \( F^L \) become

\[
L(t, \theta, r, \dot{\theta}, \dot{r}) = \frac{m}{2} \dot{r}^2 + r^2 \dot{\theta}^2, \quad F^L(t, \theta, r, \dot{\theta}, \dot{r}) = 2e \dot{\theta} \dot{r} - cr\dot{\theta} dr,
\]

respectively. \( L \) is hyperregular and \( L \) and \( F^L \) are independent of \( \theta \). The forced Euler-Lagrange equations (in polar coordinates) are

\[
\dot{r} = \frac{2cr^3}{m} \dot{\theta} + r \dot{\theta}^2 - \frac{c}{m} \dot{r}, \quad \ddot{\theta} = -\frac{c}{m} \dot{\theta}.
\]

The reset map \( \Delta^L_t \), in polar coordinates, takes the form

\[
(\dot{r}^+) = (\dot{r}^-) + \frac{r}{f(t)}(\dot{f}(t) - 2r\dot{r}^-) \left( 2\dot{r}^- + \frac{\dot{f}(t) - 2r\dot{r}^-}{f(t)}r \right), \quad \dot{\theta}^+ = \dot{\theta}^-.
\]

It is understood that the “minus” square root is taken in \( \dot{r}^+ \) (the particle bounces on the boundary after the collision).

Note that the momentum map \( (J_L)_{\theta} \) for \( \theta \),

\[
(J_L)_{\theta}(t, r, \dot{r}, \dot{\theta}, \dot{\theta}) = mce \dot{\theta} \dot{r}^2
\]
is preserved, that is, by considering \( \mu = me^{\frac{c}{m}} \dot{t}^2 \) (i.e., \( \dot{\theta} = \frac{\mu}{mr^2 e^{-\frac{ct}{m}}} \)) the time-dependent Routhian and the reduced time-dependent force takes the form

\[
R^\mu_F(t, r, \dot{r}) = \frac{m}{2} e^{\frac{c}{m}} \dot{r}^2 - \frac{\mu^2}{2mr^2} e^{-\frac{ct}{m}}, \quad F^L_\mu = 2\dot{r} \frac{\mu}{m} dr,
\]

so, the time-dependent forced reduced Euler-Lagrange equations for the Routhian \( R^\mu_F \) are given by

\[
\ddot{r} = \frac{\mu^2}{m^2 r^3} e^{-\frac{ct}{m}} - \frac{2\dot{r} \mu}{m^2} e^{-\frac{ct}{m}} - \frac{\dot{r} c}{m}.
\]

The reduced reset map is determined by the expression (12) for \( i^+ \) (note that the expression drops to the quotient since it only involves \( r, \dot{r} \) and \( f(t) \)). The reduced switching surface is \( (S_L)_\mu = \{ r^2 = f(t), \dot{r} > 0 \} \). One then obtains the following simple hybrid time-dependent forced Lagrangian system

\[
\mathcal{L}_F = (Q_{\text{red}}, R^\mu_F, F_\mu, (S^L_L)_\mu, (\Delta^L_L)_\mu), \quad \text{with } Q_{\text{red}} \simeq \mathbb{R}^+ \text{ parametrized by the radial coordinate } r.
\]

**Figure 3.** Simulation for \( c = 0.005 \). The figure in the left corresponds with the reduced trajectory while the figure to the right corresponds with the reconstructed solution.

**Figures 3** and **4** show numerical results using **PYTHON** for two different values of the dissipation parameter \( c \). The remaining parameters are the same for both simulations: \( m = 1, r(0) = 0.5590, \dot{r}(0) = 2.8621, \theta(0) = 1.1071 \text{ (rad)}, \dot{\theta}(0) = -3.0400 \text{ (rad/s)}, \) and the function \( f(t) \) equals

\[
f(t) = 2 - \exp(t/10).
\]

The reduced dynamics corresponding to \( R^\mu_F \) is solved numerically (dashed black line) and used to integrate (numerically) the reconstruction equation

\[
\dot{\theta} = \exp \left( -\frac{c}{m} t \right) \frac{\mu}{mr^2},
\]

with \( \mu \) determined from the initial conditions.
6. Conclusions and future work

The celebrated symplectic reduction of (conservative) mechanical systems with symmetries, due to Marsden and Weinstein [1, 46, 47], was recently extended for forced autonomous Lagrangian [18, 43] as well as Hamiltonian [19] systems. In this paper we have gone a step further and considered simple hybrid forced mechanical systems, both autonomous and non-autonomous, with a generalized hybrid momentum map. The main difference in the reduction and reconstruction of these systems with respect to continuous systems, or hybrid systems with a hybrid momentum map, is that, since nonelastic collisions with the switching surface will modify the value of the momentum map (see Example 2), we will have a sequence of reduced simple hybrid forced reduced Hamiltonian systems. In particular, we consider $S^1$-invariant hybrid forced non-autonomous mechanical systems. In order to illustrate our results, we have reduced the dynamics of the billiard with dissipation, with fixed and moving walls (see Examples 3 and 5, respectively).

We plan to extend our results for more general settings. We would like to obtain a reduction procedure for dissipative hybrid systems in the framework of contact geometry. Moreover, we could consider the reduction of systems with both continuous and discrete time dynamics which are not simple hybrid systems. For instance, we could consider $D$ of co-dimension different from 1, or a system having several domains and switching surfaces that separate them [10, 15, 28, 29].

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