The insulating phases and superfluid-insulator transition of disordered boson chains

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Using a strong disorder real-space renormalization group (RG), we study the phase diagram of a fully disordered chain of interacting bosons. Since this approach does not suffer from runaway flows, it allows a direct study of the insulating phases, which are not accessible in a weak disorder perturbative treatment. We find that the universal properties of the insulating phase are determined by the details and symmetries of the onsite chemical-potential disorder. Three insulating phases are possible: (i) an incompressible Mott glass with a finite superfluid susceptibility, (ii) a random-singlet glass with diverging compressibility and superfluid susceptibility, (iii) a Bose glass with a finite compressibility but diverging superfluid susceptibility. In addition to characterizing the insulating phases, we show that the superfluid-insulator transition is always of the Kosterlitz-Thouless universality class.

Bose systems can be driven into an insulating phase by quantum fluctuations due to strong repulsive interactions and lattice effects. The impact of a disordered potential on this superfluid-insulator transition and on the nature of the insulating phases is a long standing question [1]. In weakly disordered one dimensional systems, the momentum-shell renormalization group (RG) afforded much progress [2, 3]. But recently, an analysis using a real space RG suggested that strong disorder can have very different effects on a one dimensional Bose system [4]. In particular, it was found that a certain type of disorder, that is perturbatively irrelevant at weak disorder, can actually induce a transition when sufficiently strong, and lead to a new kind of an insulator termed the Mott glass [4, 5]. The existence of this phase transition and the Mott glass were confirmed numerically [6, 7]. The disorder considered in Refs. [4, 6], however, had a very special particle-hole symmetry properties that would not be easy to realize in actual experiments (e.g., [8, 9]).

In this paper we extend the real-space RG of Ref. [4] to treat strong and general disorder potentials, not confined to the commensurability requirement in Ref. [4]. Our starting point is the disordered quantum-rotor model:

\[ \mathcal{H} = \sum_j \frac{U_j}{2} (\hat{n}_j - \overline{\pi}_j)^2 - \sum_j J_j \cos (\phi_{j+1} - \phi_j), \] (1)

This Hamiltonian describes a chain of superfluid grains that are connected by a random Josephson coupling \( J_j \) (see Fig. 1(a)). Each grain has a random charging energy \( U_j \), and offset charge \( \overline{\pi}_j \), which represents an excess screening charge on the site or in its environment. The offset charge parameterizes a random on-site chemical potential \( \mu_j = U_j \overline{\pi}_j \). We point out that the lattice model (1) can also be derived as a coarse-grained description of continuum bosons, where the grain size or lattice spacing is set by the healing length of the condensate.

Using a real-space RG, we show that the system can undergo a transition from a superfluid to three possible insulating phases, whose nature depends on the symmetry properties of the distribution of offset charges, \( \overline{\pi}_j \). We characterize the insulating phases using the charging gap \( \Delta \), the compressibility \( \kappa = \frac{\partial \mu}{\partial n} \) and the superfluid susceptibility \( \chi_s \). The latter corresponds to the linear response of the order parameter \( \langle e^{i \phi_j} \rangle \) to the coupling \( \frac{J_j}{\overline{\Delta}} \sum_i \cos \phi_i \), where the angular brackets and overline denote a quantum expectation value and disorder average in that order. The three insulating phases we find are illustrated schematically in Fig. 1. They include: (i) an incompressible Mott Glass arising for the case of zero offset charges \( \overline{\pi}_j = 0 \), (ii) a glass phase with a diverging compressibility which arises if \( \overline{\pi}_j \) can only take the values of 0 or 1/2, and which we term a Random Singlet Glass, and (iii) a Bose Glass phase characterized by a finite compressibility and a diverging superfluid susceptibility in the case of a generic distribution of the offset charges, i.e., a non-singular distribution in the range \(-1/2 < \overline{\pi}_j \leq 1/2 \). We argue that the superfluid phase and the nature of the transition are insensitive to the disorder properties of the offset charges.

Our real-space RG analysis, as in random spin chains where it was first applied [10–12], eliminates the highest energy scale in the system at each stage through a local decimation step. The Hamiltonian then maintains its form (Eq. 1), but with renormalized distributions of \( J_j, U_j \) and \( \overline{\pi}_j \). From the flow of the coupling distributions we can learn about both the phases of the system and its critical points. In particular, the RG provides quantitative predictions for the properties of the insulating phases and the transition into them, on which we will concentrate.

We proceed to construct the generalized decimation procedure. Let us define the global energy scale \( \Omega = \max_j (\Delta_j, J_j) \), where \( \Delta_j = U_j (1 - 2|\overline{\pi}_j|) \) is the charging energy of the site \( j \). For the Hamiltonian in Eq. (1) three types of decimation steps are possible. \textit{Type 1: site decimation.} If \( \Omega = \Delta_j \) for some \( j \), we freeze the charge on the site \( j \), thus eliminating this degree of free-
FIG. 1: The three insulating phases that emerge for different classes of disorder. (a) The Mott glass is realized when only the Josephson couplings and charging energies are disordered, and with no offset charge, $\pi = 0$. At large scales it consists of effectively disconnected superfluid clusters of random size. (b) The "Random singlet" glass appears when the random offset charge is restricted to $\pi = 0, 1/2$ in terms of the basic boson charge. In this phase bosons are delocalized on random pairs of remote clusters at all scales. (c) The Bose glass is realized for a generic offset-charge distribution. It consists of large superfluid clusters acting effectively as weakly coupled spin-1/2's in a uniformly-distributed random $z$-field, given by the onsite gap times the sign of $\pi$: $\Delta \text{sgn}(\pi) = U(1 - 2|\pi|)\text{sgn}(\pi)$. 

A Josephson coupling $J_{j-1} \approx J_{j} J_{j+1}/\Omega$ between the sites $j - 1$ and $j + 1$ is generated by a virtual tunneling process through the eliminated site (see Ref. [4]).

Type 2: Bond decimation. If $\Omega = J_{j}$ for some $j$, sites $j$ and $j + 1$ merge into a superfluid cluster with an effective interaction parameter: $1/U_j = 1/U_j + 1/U_{j+1}$ (this corresponds to additivity of two capacitances connected in parallel). The offset charges of the two sites simply add up $\pi_j = \pi_j + \pi_{j+1}$. Type 3: Doublet formation. A special RG step is introduced for sites with the offset charge $\pi_j = 1/2$. In this case, if $U_j > \Omega$ the site $j$ is frozen to its two lowest-lying degenerate charge states. Then we can set $U_j \to \infty$ and proceed to treat the site as a spin-1/2 degree of freedom (spin-site) with $s_j = (n_j - \pi_j)$ and, similarly, $\exp(\pm i \phi_j) \to s_j \pm i$. The existence of spin sites requires revisiting rules for the bond decimation step (Type 2). If the strong bond connects a spin site to a regular site, then the spin site can be simply treated as a half integer site with infinite $U$. But when two spin sites are strongly coupled by large $J$ corresponding to $xy$-coupling: $-J_{j} (s_j^x s_{j+1}^x + s_j^y s_{j+1}^y)$, they freeze into the triplet state with the total spin $s = 1$ and the projection $m_s = 0$: $1/2(|\downarrow_j \uparrow_{j+1}) + |\uparrow_j \downarrow_{j+1})$. This represents a boson resonating in a symmetric superposition of the two sites, with a "charging" gap $J_{j}$ to the excited (anti-symmetric) state. Quantum fluctuations now allow tunneling between sites $j - 1$ and $j + 2$ with strength $J_{j-1} J_{j+1}/J_{j}$. The last step is identical to the real-space RG in the random $xx$ spin chain[12].

For a given system, the RG flow, parameterized by probability distributions of the couplings, can lead to either a superfluid or an insulating phase. In the superfluid the system coalesces to one large superfluid cluster, while in the insulator, it breaks down to clusters with large effective charging gaps connected by weak tunneling. Quantitative analysis of the RG flow requires in general the solution of integro-differential equations for the disorder distributions [12]. Remarkably, for the disorder type we consider, the distributions of $U_i$, $J_i$, and $\pi_i$, are universal in a large vicinity around the fixed points of the RG, which govern the superfluid-insulator transitions. This greatly simplifies the flow equations, and in several cases allows an analytical solution. Below we discuss the three classes of disorder corresponding to different symmetry properties of the offset-charge distribution.

No offset charge. The case of $\pi_j = 0$, for which the Hamiltonian (1) is particle-hole symmetric, was analyzed in Ref. [4]. For completeness, we review the main results. The RG decimation steps (in this case only the decimations of $Types$ 1 and 2 are needed) translate into the flow of the coupling distributions $F(U/\Omega)$ and $G(J/\Omega)$. Near the superfluid-insulator fixed point they acquire the universal form:

$$F(x) \approx \frac{A}{x^2} \exp \left[-\frac{f_0(\Omega)}{x}\right], \quad G(x) \approx g_0(\Omega)x^{g_0(\Omega) - 1}. \quad (2)$$

Here $A$ is a normalization constant, and $x \leq 1$ [13]. Note that the typical strength of the Josephson coupling and site charging energy monotonically depend on the parameters $g_0$ and $f_0$ respectively. In [4] we derived the following flow equations for these parameters:

$$\frac{df_0}{d\Gamma} = f_0(1 - g_0), \quad \frac{dg_0}{d\Gamma} = -f_0, \quad (3)$$

where $\Gamma = \ln(\Omega_0/\Omega)$ and $\Omega_0$ is the initial cutoff energy scale. The solutions to these equations are parametrized by a single constant $C$: $f_0 \approx C + (1 - g_0)^2/2$. Negative $C$ corresponds to the superfluid state in which the charging energy is irrelevant ($f_0$ flows to zero) and $g_0$ flows to constant larger than unity. Positive $C$ describes the insulator in which $f_0$ is relevant ($f_0 \to \infty$) and $g_0$, indicative of the typical Josephson coupling strength, flows to zero. The value $C = 0$ corresponds to the critical point separating the two phases. At this point $g_0$ flows to 1 and $f_0$ flows to zero. We note that substituting $f_0 \to g_0^2$ makes Eqs. (3) assume a standard Kosterlitz-Thouless form.

The fixed point in the superfluid phase corresponds to a classical model with $U_t = 0$ and a power-law distribution of the Josephson couplings with an exponent larger than one. The fact that the fixed point is noninteracting, implies neither the vanishing of the compressibility, nor the formation of true long range order: since our analysis relies on the grand canonical ensemble, the lowest excitation in the superfluid phase corresponds to an addition of a particle and not to a phase twist or to a Bogoliubov excitation. Thus the vanishing of $U_t$ only implies that the energy for adding an extra particle vanishes with the inverse system size, as expected in the superfluid phase. Calculation of the compressibility or
stiffness of the superfluid requires a more detailed analysis, that keeps track of the internal Josephson couplings and phonon modes within renormalized clusters.

In this paper we concentrate on the properties of the insulating phases, which are particularly interesting since they are most drastically affected by the type of the disorder. In the insulating phases, the canonical and grand canonical pictures of the excitations are identical, and the real space RG correctly describes their properties. The insulating phases are best described by a chain consisting of nearly disconnected clusters, each with its own charging gap. The lowest gap corresponds to the energy scale $\Omega$ at which the last site is decimated. In Ref. [4] we showed that for the commensurate case with no offset charges this gap vanishes with system size and is governed by rare and anomalously large superfluid clusters. The compressibility vanishes as $(\ln L)/L$. In addition, the insulator is characterized by a finite superfluid susceptibility. We termed this gapless and incompressible phase a Mott-glass. This phase was also discussed in Refs. [5, 14], and confirmed numerically in Refs. [6, 7].

**Mixed offset:** $\bar{n} = (0, 1/2)$. Let us now allow sites to have either zero or half-integer offset charge maintaining particle-hole symmetry in the problem. Thus the Hamiltonian (1) is invariant under the transformation $n_j \rightarrow 1 - n_j$. Such a restriction naturally arises in an array of small superconducting grains with a pairing gap much larger than the charging energy. The random parity of the electron number in each grain would lead to an offset charge which is randomly either integer or half-integer in units of the cooper-pair charge.

In addition to distributions of $U$ and $J$, one must now follow $\bar{n}$'s distribution as well, which can be parametrized by the three probabilities $-q$, $p$, and $s$ - corresponding to relative densities of integer $(7_j = 0)$, half integer $(\bar{n}_j = 1/2)$, and doublet (spin-1/2) spin-sites respectively. On first glance, the problem becomes much more complicated since one has to also consider the flow equations for $p$ and $q$ (note that $p + q + s = 1$ due to normalization). Postponing a detailed description of these equations, let us here observe that whether the cluster in a renormalized chain is integer or half integer depends only on the parity of the number of the bare sites with $\bar{n} = 1/2$ contained in it. This implies that no matter what the fraction of the half-integer sites in the original chain was (as long as it is non-zero) large clusters have odd or even parity with equal probabilities. For this reason the distribution quickly flows to a fixed line with $p = q = (1 - s)/2$, provided $p > 0$ initially.

This observation significantly simplifies the flow analysis, and reduces the number of flow equations to three. A detailed analysis of the integro-differential equations shows that the distributions of $\bar{U}$ and $\bar{J}$ flow to the same universal distributions as those described by Eqs. (2), and we obtain a simplified set of flow equations:

$$\frac{df_0}{dT} = f_0 [1 - g_0 (1 - s) (1 + f_0)],$$
$$\frac{dg_0}{dT} = -\frac{g_0}{2} [1 - s^2] f_0 + 2s^2 g_0,$$
$$\frac{ds}{dT} = \frac{f_0}{2} (1 - s^2) - g_0 s (1 - s).$$

These have two different families of solutions, corresponding to two different phases. In one family, the flow is to a stable fixed line with $s = 0$, $f_0 = 0$ and $g_0 > 1$: a superfluid phase identical to that of the no offset charge case. Close to this line, the flow equations reduce to those for the integer case Eqs. (3), except for an extra factor of $1/2$ in the equation for $g_0$. This factor enters because only integer sites with a charge gap can renormalize $J$. The fact that the random offset is not important for describing the superfluid is not surprising: large local particle number fluctuations in this phase completely overwhelm small fluctuations in the offset charge.

The superfluid-insulator transition is also very similar to the $\bar{n} = 0$ case, and it is described by essentially the same Kosterlitz-Thouless critical point with $g_0 = 1$, $f_0 = 0$ and $s = 0$, similar to Eqs. (3). In the insulating phase at $g_0 < 1$, however, $f_0$ becomes relevant, as well as the spin-site density $s$, which quickly flows to $s = 1$. This second family of solutions of (4) describes a completely different insulating phase than in the case of zero offset charges: it corresponds to an effective spin-1/2 chain with random $x - y$ ferromagnetic couplings, a model that was analyzed in detail in Ref. [12]. Its ground state consists of random non-crossing pairs of sites at varying distances, in which the spins form the $m_z = 0$ state, $\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$.

This phase is termed the Random-Singlet Glass, reaffirming the connection with anti-ferromagnetic random spin chains [15, 16]. In the bosonic language the ground state has bosons delocalized randomly between pairs of sites. Many properties of the Random-Singlet Glass can be found directly from the analysis of Ref. [12]. The energy scale associated with breaking a singlet between sites of distance $\ell$ is $\Omega_{\ell} = \Omega_0 \exp(-\sqrt{\ell})$. By setting $\ell = L$, the system size, we obtain the scaling of the gap vs. the system size. Following the identification $n_i = \frac{1}{2} + s_i^z$, the compressibility $\kappa$ and superfluid susceptibility $\chi_s$ in the insulating phase correspond to the susceptibilities of a random spin chain to a Zeeman field applied in the $z$ and $x$ directions respectively. From Ref. [12] we see that both susceptibilities diverge at the limit of small $\Omega$ as $\kappa(\Omega) \sim \chi_s(\Omega) \sim 1/\Omega \log^2(\Omega_0/\Omega)$. While the superfluid stiffness vanishes in the thermodynamic limit, unlike the stiffness of the Mott glass, it vanishes only sub-exponentially with system size: $\rho_s \propto e^{-\sqrt{\ell}}$.

In addition, note that $g_0$, indicative of the strength of the Josephson coupling, flows to zero as $g_0 \sim 1/\Gamma$ (as results from Eq. (4) with $s = 1$), much slower than $g_0 \sim \exp(e^{-1}) = \exp(-1/\Gamma)$ in the Mott glass.

**Generic chemical potential disorder.** When all possible offsets $-1/2 < n_j \leq 1/2$ are allowed, the rele-
vant energy scale for the RG site decimation is not $U$, but rather the local gap $\Delta_i = U_j(1 - 2|\mathbf{\pi}_j|)$. The interaction $U_i$ is allowed to exceed $\Omega$ by an amount which depends on the local charge offset:

$$U_j < \Omega/(1 - 2|\mathbf{\pi}_j|).$$

(5)

Thus we must consider a joint distribution for $U$ and $\mathbf{\pi}$, which makes the RG flow equations rather cumbersome. Their analysis, however, reveals a rather intuitive behavior. Below we describe the key steps of the derivation.

The first step is to note that $\mathbf{\pi}$ disorder width is a relevant variable due to the rule $\mathbf{\pi}_i \rightarrow \mathbf{\pi}_j + \mathbf{\pi}_{j+1}$ under the decimation of a strong bond. This implies that as the effective sites grow with the RG, their offset charges quickly become uniformly distributed between $-1/2$ and $1/2$, i.e., the largest disorder allowed. This observation significantly simplifies the analysis. In particular, it is straightforward to check that the distributions of $U$ and $J$ again approach the universal functions (2). In the superfluid regime and at the transition point the flow is also governed by the equations (3) with an extra factor of one half in the equation for $g_0$. Thus the system with generic disorder undergoes the same transition as in the two cases discussed above. Similarly, the elementary understanding of this result is that for small interactions the bounded disorder in the offset charge is overwhelmed by large particle number fluctuations. In the insulating case subject to a random Zeeman field. First, as implied by the gap distribution, in this phase the energy-length scaling is $\Omega \sim 1/L = \rho$. Next, the compressibility is given by the response to an external field in the $z$ direction: $\kappa = \partial^2 \langle \chi \rangle / \partial h^2_{\text{ext}} = 2H(0) = \rho/\Omega = \kappa_0$, a constant at low energies. The superfluid susceptibility is the response to a transverse field: $\chi_s = \rho \partial^2 \langle \chi \rangle / \partial h^2_{\text{ext}}$. We find this disorder average from the distribution of $\Delta_i$:

$$\chi_s = \kappa \int_{-\Omega}^{\Omega} \frac{d\Delta}{2\Omega} \frac{1}{\sqrt{(h^2_{\text{ext}} + \Delta^2)^2}} \approx \frac{\kappa_0}{2} \log \left( \frac{\Omega}{|h^2_{\text{ext}}|} \right),$$

(7)

which diverges as $h^2_{\text{ext}} \rightarrow 0$, with a functional form resulting from the non-singular behavior of $H(\Delta)$. In a finite chain this divergence is cut off by the smallest $|\Delta_i|$, and leads to $\chi_s \sim \log L$. These properties, namely finite compressibility and diverging superfluid susceptibility, coincide with the Bose Glass phase discussed in Ref. [1].

| $\mathbf{\pi}$ disorder | Glass type | $\Delta$ | $\kappa$ | $\chi_s$ |
|------------------------|------------|---------|---------|---------|
| 0                      | Mott       | $1/\log L$ | $\log L \rightarrow 0$ | const |
| 0, 1/2                 | Random-singlet | $e^{-cvT}$ | $1/L^{1/2} e^{-cvT}$ | $1/L^{1/2} e^{-cvT}$ |
| $-1/2 \leq \mathbf{\pi} < 1/2$ | Bose       | $1/2$ | $\kappa$ | $\frac{1}{2} \log L$ |

TABLE I: The gap $\Delta$, compressibility $\kappa$, and SF susceptibility $\chi_s$, of the insulating phases realized for the different classes of disorder in the offset charge $\mathbf{\pi}$. Here $L$ is the system size and $c$ denotes a nonuniversal constant.

In summary, we extended the real-space RG analysis of Ref. [4] to address non-commensurate random chemical potential. We found that the symmetry of the Hamiltonian and the details of the disorder, as encoded in the distribution of the offset $\mathbf{\pi}_j$, do not affect the universal properties of the superfluid phase and superfluid-insulator transition, but at the same time it completely determines the insulating phase of the system. In addition to the Mott-Glass of the integer filling case, we find the Random-Singlet Glass for the mixed $\mathbf{\pi} = 0, 1/2$ case, and for generic chemical-potential disorder, a phase we identify as the Bose-Glass[1]. We note that if we restrict $\mathbf{\pi}$ to a set of rational numbers, we would still obtain either the random-singlet glass, or the Mott-glass. All our results are supported by a numerical real-space RG study to be published separately.

Our focus here was the universal thermodynamic properties of the insulating phases, as summarized in Table I. An outstanding question, which we leave to a future publication, is the effective Luttinger parameter at the critical point, and how it compares to the low-disorder motivated results of Ref. [2], predicting $K = \sqrt{\kappa_0 d_e} = 3/2$. Our initial results, however, indicate that the critical value of the Luttinger parameter at sufficiently strong disorder is non-universal, and exceeds $3/2$; this possibility does not contradict a general thermodynamic statement of Ref. [6].

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