Intrinsic Localized Modes in a Two-Dimensional Checkerboard Ferromagnetic Lattice

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Received: 1 March 2023 / Accepted: 29 June 2023 / Published online: 31 July 2023
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Abstract
An analytical work on intrinsic localized modes in a two-dimensional Heisenberg ferromagnet on the checkerboard lattice is presented. Taking advantage of an asymptotic method, the governing lattice dynamical equations are reduced to one (2 + 1)-dimensional nonlinear Schrödinger equation. In our work, we obtain two types of nonlinear localized mode solutions, namely, Brillouin zone center modes and Brillouin zone corner modes. The occurrence conditions for these intrinsic localized modes are given in detail. Especially, we find that the competition between the Dzialozinskii-Moriya interaction and the next-nearest neighbor interaction of the checkerboard ferromagnet has an effect on the local structure of the Brillouin zone corner acoustic mode.

Keywords Intrinsic localized modes · The two-dimensional ferromagnet · The checkerboard lattice · Soliton

1 Introduction
Intrinsic localized modes (also referred to as discrete breathers) were for the first time found by Taken and Sievers in a 1D β-Fermi–Pasta–Ulam lattice chain [1]. Soon afterwards, numerous works on the intrinsic localized mode in 1D discrete nonlinear lattice systems have been reported [2]. Especially, the existences of the intrinsic localized mode in those one dimensional nonlinear lattice systems have been rigorously proven via using an anti-continuous limit [3–5]. It has been shown that both the nonlinearity and the discreteness are of vital importance to the emergence of such intrinsic localized modes. Experimentally, the intrinsic localized mode has been detected in the antiferromagnetic material [6], the Josephson array [7], and the diatomic electrical lattice [8].

Over the past decades, more and more researchers also have paid attention to the intrinsic localized mode in those higher dimensional lattices [9–20]. In the early paper, Takeno has put forward lattice Green's functions to seek for localized mode solutions in 1D, 2D and 3D nonlinear lattices [9]. By use of a similar method, he has studied the characteristic and profile of the localized mode in general d-dimensional nonlinear lattices [10]. Afterwards, Wattis
et al. have obtained analytical forms of the intrinsic localized mode in some two-dimensional Fermi–Pasta–Ulam lattices via using an asymptotic method based on the small amplitude ansatz [11–13]. In their works, two-dimensional square, honeycomb, and triangular lattice structures have been respectively considered. On the other hand, some authors have used numerical methods to obtain intrinsic localized mode solutions in various two-dimensional nonlinear lattice systems [14–20]. For example, Marin et al. have performed extensive numerical works on intrinsic localized modes in two-dimensional nonlinear lattices [16–18], and English et al. have showed that numerical simulations is in good agreement with experimental results for the intrinsic localized mode in 2D electrical square and honeycomb electrical lattices [19]. Recently, Watanabe and Izumi have obtained numerical solutions of intrinsic localized modes in a 2D hexagonal Fermi–Pasta–Ulam nonlinear lattice system by using a generalized minimal residual method [20].

In this article, we analytically investigate existences and properties of intrinsic localized modes in a two-dimensional checkerboard Heisenberg ferromagnetic spin lattice by using an asymptotic method developed by Butt and Wattis [13]. In our previous works, we have applied this analytic scheme to obtain some intrinsic localized mode solutions in the two-dimensional square, hexagonal, and honeycomb Heisenberg ferromagnetic lattice systems [23–25]. Theoretically, the checkerboard lattice structure can be regarded as the 2D counterpart of one 3D pyrochlore lattice structure [26]. In the linear spin wave approximation, topological properties of magnons have been well studied in the two-dimensional checkerboard Heisenberg ferromagnet [27, 28]. The goal of the present research is to explore that the interaction between magnons may cause a nonlinear localized excitation, i.e., the intrinsic localized mode. More particulars will be exhibited in the following sections.

2 The Lattice Model Hamiltonian and its Quantization

In the present research, we take into account an anisotropy Heisenberg ferromagnet placed on the 2D checkerboard lattice. Under an external magnetic field, the spin Hamiltonian of this 2D checkerboard ferromagnet reads

\[
H = -J_1 \sum_{\langle i,j \rangle} S_i \cdot S_j - J_2 \sum_{\langle\langle i,j \rangle\rangle} S_i \cdot S_j + D \sum_{\langle i,j \rangle} v_{ij} \left( S_i^x S_j^y - S_i^y S_j^x \right) - A \sum_i \left( S_i^z \right)^2 - g \mu_B H_{\text{ext},z} \sum_i S_i^z, \tag{1}
\]

We signify the first-nearest neighbor spins with interactions \( J_1 \) and \( D \) between spins at lattice sites A and B via \( \langle i,j \rangle \). What is more, \( \langle\langle i,j \rangle\rangle \) stands for second-nearest neighbor exchange interactions between lattice sites A and A, and B and B, with \( J_2 \) along those solid lines. The third term corresponds to the first-nearest neighbor (Dzyaloshinskii-Moriya) DM interaction, where \( D_{ij} \) is referred to as the DM interaction vector between nearest neighbor lattice sites \( i \) and \( j \). According to the Moriya rule, one can write down \( D_{ij} = v_{ij} \mathbf{D}_e \), where \( v_{ij} \) is in fact an orientation-dependent coefficient [26]. The last term represents the Zeeman coupling with one applied magnetic field \( \mathbf{H} = H_{\text{ext},z} \mathbf{e}_z \). As can be clearly seen in Fig. 1, this 2D checkerboard lattice possesses two inequivalent lattice sites A and B, which lie on a
subset of one underlying square lattice. For the sake of depicting the position of the \((m, n)\) lattice site for this square lattice, one can use a pair orthonormal basis vectors \(\mathbf{e}_x\) and \(\mathbf{e}_y\) so that those position vectors can be represented as \(m\mathbf{e}_x + n\mathbf{e}_y\). In order to specify this two-dimensional checkerboard ferromagnetic lattice, we need to reserve only those lattice sites \((m, n)\), which meet the conditions \(m + n = \text{even}\). In Fig. 1, those color solid circles stand for the sites reserved in this two-dimensional checkerboard lattice that satisfy these conditions, and those black open circles hold up all the rest of lattice sites in the square lattice.

Here, lattice sites need to be reindexed via considering a rectangular lattice, whose basis vector is \(\mathbf{R} = \{\mathbf{e'_x}, \mathbf{e'_y}\}\) with \(\mathbf{e'_x} = [2, 0]^T\) and \(\mathbf{e'_y} = [0, 2]^T\), as shown in Fig. 2. We only consider half of the \((m, n)\) indices so as to ensure that the sum \(m + n\) is even. Let us set an origin with the coordinate \((0, 0)\), thus the position vector for the lattice site \((m, n)\) is written as \(\mathbf{r}_{m,n} = m\mathbf{e'_x} + n\mathbf{e'_y}\).

To bosonize the checkerboard Heisenberg ferromagnet Hamiltonian (1), we use the Dyson-Maleev transformations [29, 30] for the two sublattices A and B, which are given by

![Fig. 1](Color online) Diagrammatic drawing of the checkerboard lattice arranged on the x–y plane. Color solid circles are those lattice sites in the 2D checkerboard lattice, black open circles correspond to the unused lattice sites in the fundamental rectangular lattice. The dotted lines stand for those unit cells, each of which owns two non-equivalent lattice sites A and B.

![Fig. 2](Labelling of sites in the two-dimensional checkerboard lattice with the basis vector \(\mathbf{R} = \{\mathbf{e'_x}, \mathbf{e'_y}\}\): a Arrangement 1, \(S_{m,n}^A\) in centre, four neighbouring site are \(S_{m-1,n-1}^B\), \(S_{m-1,n+1}^B\), \(S_{m,n-1}^B\), and \(S_{m+1,n-1}^B\). b Arrangement 2, in centre, four neighbouring sites are \(S_{m-1,n-1}^A\), \(S_{m-1,n+1}^A\), \(S_{m,n-1}^A\), \(S_{m,n+1}^A\), and \(S_{m+1,n-1}^A\).)

\(\frac{\text{3}}{\text{1}}\)
\[ S^{A+} = S^{kv} + iS^{kv} = \sqrt{2S(1 - \frac{a^+a}{4S})}a, \] (2a)

\[ S^{A-} = S^{kv} - iS^{kv} = \sqrt{2Sa^+}, \] (2b)

\[ S^{kz} = S - a^+a \] (2c)

(similarly for sublattice B).

When Glauber coherent-state presentation \([31]\) is adopted, we can suppose that the system state vector \(|\Psi(t)\rangle\) has the following form

\[
|\Psi(t)\rangle = \left( \prod_{A\text{- sublattice}} |\alpha_{m,n}\rangle \right) \left( \prod_{B\text{- sublattice}} |\beta_{m,n}\rangle \right).
\] (3)

By utilizing the variational method, one can derive the equations of motion for coherent-state amplitudes \(\alpha_{m,n}\) and \(\beta_{m,n}\). Through some simple calculations, we find that the equations of motion for A sites in arrangement 1 is

\[
i\frac{\partial \alpha_{m,n}}{\partial t} = \omega_0 \alpha_{m,n} - J_1 S(\alpha_{m+1,n+1} + \alpha_{m+1,n-1} + \alpha_{m-1,n+1} + \alpha_{m-1,n-1}) - J_2 S(\alpha_{m,n+2} + \alpha_{m,n-2})
\]

\[
+i DS(\beta_{m+1,n+1} - \beta_{m-1,n+1} + \beta_{m+1,n-1} - \beta_{m-1,n-1}) + \frac{J_1}{4} (\beta^+_{m+1,n+1} \alpha_{m,n}^+ + \beta_{m+1,n+1} \alpha_{m,n})^2 + (\beta_{m+1,n+1}^* \beta_{m+1,n+1})^2 \beta_{m+1,n+1}
\]

\[
-4 \alpha_{m,n}|\beta_{m+1,n+1}|^2 + \beta^+_{m+1,n+1} \alpha_{m,n}^+ + |\beta_{m+1,n+1}|^2 \beta_{m+1,n+1} - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2
\]

\[
+ \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2 + \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2
\]

\[
+ \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2 + \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2
\]

\[
- \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2 + \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2
\]

\[
- \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2 + \beta^+_{m-1,n-1} \alpha_{m,n}^+ - 4 \alpha_{m,n} |\beta_{m+1,n+1}|^2
\]

\[
\text{(4)}
\]

with \(\omega_0 = (2S - 1)A + g \mu_B H_{ext,z} + 4J_1 S + 2J_2 S\). In a similar way, it is easy to obtain the equations of motion for B sites in arrangement 2, namely,

\[
i\frac{\partial \beta_{m,n}}{\partial t} = \omega_0 \beta_{m,n} - J_1 S(\alpha_{m+1,n+1} + \alpha_{m+1,n-1} + \alpha_{m-1,n+1} + \alpha_{m-1,n-1}) - J_2 S(\beta_{m+2,n} + \beta_{m-2,n})
\]

\[
-i DS(\alpha_{m+1,n+1} - \alpha_{m-1,n+1} + \alpha_{m-1,n-1} - \alpha_{m-1,n-1}) + \frac{J_1}{4} (\alpha^+_{m+1,n+1} \alpha_{m,n}^+ + \alpha^+_{m+1,n+1} \alpha_{m,n})^2 + (\alpha^+_{m+1,n+1} \alpha_{m,n})^2 \alpha_{m+1,n+1}
\]

\[
-4 |\alpha_{m+1,n+1}|^2 \beta_{m,n} + \alpha^+_{m+1,n+1} \alpha_{m,n}^+ + |\alpha_{m+1,n+1}|^2 \beta_{m,n} - 4 |\alpha_{m+1,n+1}|^2 \beta_{m,n} + |\alpha_{m+1,n+1}|^2 \alpha_{m+1,n+1}
\]

\[
+ \alpha^+_{m-1,n-1} \beta_{m,n}^2 - 4 |\alpha_{m-1,n-1}|^2 \alpha_{m,n} + |\alpha_{m-1,n-1}|^2 \alpha_{m,n} + |\alpha_{m-1,n-1}|^2 \alpha_{m,n} - 4 |\alpha_{m-1,n-1}|^2 \beta_{m,n}
\]

\[
+ \frac{J_2}{4} (\alpha^+_{m+2,n} \alpha_{m,n}^+ - |\alpha_{m+2,n}|^2 \beta_{m,n} + 4 |\beta_{m+2,n}^2 | \beta_{m+2,n}^2 + |\beta_{m-2,n}^2 | \beta_{m-2,n}^2 + 4 |\beta_{m-2,n}^2 | \beta_{m-2,n}^2
\]

\[
+ \frac{J_2}{4} (\alpha^+_{m+2,n} \alpha_{m,n}^+ - |\alpha_{m+2,n}|^2 \beta_{m,n} + 4 |\beta_{m+2,n}^2 | \beta_{m+2,n}^2 + |\beta_{m-2,n}^2 | \beta_{m-2,n}^2 + 4 |\beta_{m-2,n}^2 | \beta_{m-2,n}^2
\]

\[
\text{(5)}
\]
3 Asymptotic Analysis and Nonlinear Amplitude Equation

Here, we shall make use of an asymptotic method developed by Butt and Wattis to seek the localized solution to the Eqs. (4) and (5). First, we need to rescale the present variables \( m, n \) and \( t \) via considering the following new variables

\[
x = \rho m, \quad y = \rho n, \quad \tau = \rho t, \quad T = \rho^2 t, \tag{6}
\]

where \( \rho << 1 \) is a small quantity.

In principle, different ansatzes should be respectively applied to the A and B sites. For A sites, we consider that their trial solutions possess the following form

\[
\alpha_{m,n}(t) = \rho e^{i\phi} f(X, Y, \tau, T) + \rho^2 [g_0(X, Y, \tau, T) + e^{i\phi} g_1(X, Y, \tau, T) + e^{2i\phi} g_2(X, Y, \tau, T)]
+ \rho^3 [h_0(X, Y, \tau, T) + e^{i\phi} h_1(X, Y, \tau, T) + e^{2i\phi} h_2(X, Y, \tau, T) + e^{3i\phi} h_3(X, Y, \tau, T)] + \ldots.
\tag{7}
\]

Here, \( \phi = k_x m + k_y n - \omega t \) signifies the phase of the carrier wave. Note that the wavevector \( \mathbf{k} \) has been written as \( \mathbf{k} = k_x \mathbf{e}_x + k_y \mathbf{e}_y \). Naturally, trial solutions for B sites are expressed as

\[
\beta_{m,n}(t) = \rho e^{i\phi} p(x, y, \tau, T) + \rho^2 [q_0(x, y, \tau, T) + e^{i\phi} q_1(x, y, \tau, T) + e^{2i\phi} q_2(x, y, \tau, T)]
+ \rho^3 [r_0(x, y, \tau, T) + e^{i\phi} r_1(x, y, \tau, T) + e^{2i\phi} r_2(x, y, \tau, T) + e^{3i\phi} r_3(x, y, \tau, T)] + \ldots.
\tag{8}
\]

By inserting ansatzes (7) and (8) into those equations of motion on coherent-state amplitudes and equating coefficients of each harmonic frequency at each order of \( \rho \), we can get two sets of equations.

According to the \( O(\rho e^{i\phi}) \) terms of Eqs. (4)–(5), one can get two equations relating \( f \) and \( p \), i.e.,

\[
\mathbf{M}(f, p) = \begin{pmatrix} \omega_0 - \omega - 2J_2 S \cos(2k_y) & -4S(J_1\gamma_k + iDm_k) \\ 4S(-J_1\gamma_k + iDm_k) & \omega_0 - \omega - 2J_2 S \cos(2k_x) \end{pmatrix} \begin{pmatrix} f \\ p \end{pmatrix} = 0,
\tag{9}
\]

with

\[
\gamma_k = \cos(k_x) \cos(k_y), \quad m_k = \sin(k_x) \sin(k_y).
\tag{10}
\]

Considering that we pay attention only to untrivial solution, thus Eq. (9) is in fact a common eigenvalue problem, with the eigenvalue \( \omega \). Making use of the corresponding secular equation, it is not difficult to give the magnon dispersion relation with the following form

\[
\omega = \omega_0 - J_2 S \left[ \cos(2k_x) + \cos(2k_y) \right] \\
\pm S \sqrt{J_2^2 \left[ \cos(2k_x) - \cos(2k_y) \right]^2 + 16(J_1^2 \gamma_k^2 + D^2 m_k^2)},
\tag{11}
\]

where the minus sign is corresponding to an acoustic branch, i.e., surface in \( (k_1, k_2, \omega) \) space of lower frequencies, and the surface corresponding to the plus sign is referred to as the optical branch. When the present checkerboard ferromagnet does not possess the DM interaction, i.e., \( D = 0 \), the optical “up” and acoustic “down” frequency band meet at the corner of the Brillouin zone, as displayed in Fig. 3(a). Once the DM interaction is
introduced, the spatial inversion symmetry of the two-dimensional checkerboard ferromagnetic lattice shall be destroyed, which leads to open a magnon band gap \( \Delta = 8DS \) at the corner of the Brillouin zone, as shown in Fig. 3(b). For the two-dimensional honeycomb ferromagnetic lattice, however, the DM interaction opens the band gap at Dirac points [25].

It is not sufficient to get the frequency eigenvalue, we call for the form of the solution for \( f \) and \( p \) as well. Thus, we need to get the eigenvector of Eq. (9). Owing to \( \det(M) = 0 \), here solutions are expressed as \( (f, p)^T = f(1, C)^T \). By solving Eq. (9), one can get \( C(k_1, k_2) \), which has the following form

\[
C_{ac} = \frac{J_2S\cos(2k_x) - \cos(2k_y) + \sqrt{J_2^2S^2[\cos(2k_x) - \cos(2k_y)]^2 + |\beta|^2}}{|\beta|}e^{i\theta},
\]

\[
C_{opt} = -\frac{\sqrt{J_2^2S^2[\cos(2k_x) - \cos(2k_y)]^2 + |\beta|^2} - J_2S[\cos(2k_x) - \cos(2k_y)]}{|\beta|}e^{i\theta}.
\]

where \( \beta \equiv |\beta|e^{i\theta} = 4S(J_1y_1 + iDm_b) \).

From the \( O(\eta^2e^{i\theta}) \) and \( O(\eta^2e^{2i\theta}) \) terms, we can get

\[
g_0 = q_0 = 0, g_2 = q_2 = 0.
\]

Let us consider the governing equations at \( O(\eta^2e^{2i\theta}) \), which can be recast into

\[
M \begin{pmatrix} g_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} if_x + id_1f_x + ic_1p_x + id_2p_y \\ ip_x + id_2p_x + ic_2f_x + id_3f_y \end{pmatrix}
\]

with

\[
c_k = 4S[J_1\cos(k_y)\sin(k_x) - iD\cos(k_x)\sin(k_y)],
\]

\[
d_k = 4S[J_1\cos(k_y)\sin(k_x) - iD\cos(k_x)\sin(k_x)]
\]

where \( M \) is the matrix shown in Eq. (9).

Due to \( \det(M) = 0 \), an equation such as Eq. (14), which can be rewritten as \( M(g_1, q_1)^T = F \), either does not exist solutions, or have a whole family of solutions on
\((g_1, q_1)^T\). According to the Fredholm alternative [32], the presence of solutions is in connection with \(F\). What is more, solutions can exist only if \(F\) appears in the range of the matrix \(M\). It is not hard to find that \(n = (1/C^*, 1)^T\) satisfies \(n \cdot F = 0\) in both the optical branch and the acoustic branch.

Combining \(p = Cf\) with \(n \cdot F = 0\), Eq. (14) yields

\[
\begin{align*}
\frac{|C|^2 d_z + c_k^* C^* + c_k C}{1 + |C|^2} f_x + \frac{d_1 + d_k C + C^* d_k^*}{1 + |C|^2} f_y &= 0
\end{align*}
\]

which means that there exist the travelling wave solutions for \(f\) and \(p\). Therefore, \(f\) and \(p\) can be written as

\[
\begin{align*}
f(x,y,\tau,T) &\equiv f(z,w,T), p(x,y,\tau,T) \equiv p(z,w,T),
\end{align*}
\]

where \(z = x - u\tau\) and \(w = y - v\tau\). From Eq. (16), we can obtain the group velocity \(v_g = u e_x + v e_y\), where

\[
\begin{align*}
u &= \frac{\partial \omega}{\partial k_x} = \frac{|C|^2 d_z + c_k^* C^* + c_k C}{1 + |C|^2}, v = \frac{\partial \omega}{\partial k_y} = \frac{d_1 + d_k C + C^* d_k^*}{1 + |C|^2}.
\end{align*}
\]

Obviously, the magnitude of the magnon group velocity \(v_g\) can take the following form

\[
\begin{align*}
v_g &= \sqrt{u^2 + v^2}.
\end{align*}
\]

Noting that the forms of \(v_g^{ac}\) and \(v_g^{opt}\) are same. Thus, one writes down \(v_g^{ac} = v_g^{opt} = v_g\) for any wave-vector \(k\). In Fig. 4, we display the value of the magnon group velocity magnitude \(v_g\) as function of \((k_x, k_y)\).From Eq. (18), one can deduce that \(u_g, u_{opt}, v_{ac}, \) and \(v_{opt}\) are equal to zero at the corner and center of the first Brillouin zone. Naturally, the magnitude of the magnon group velocity \(v_g\) should be equal to zero for these particular wave vectors, as shown in Fig. 4(b).

\[\text{Fig. 4 (Color online)}\] a The 3D surface plot on the group velocity magnitude \(v_g\); b The corresponding 2D density plot. The corresponding parameters are selected as \(J_1 = 1, J_2 = 0.2, D = 0.1, S = 10, H_{ext} = 10, A = 1, g = 1, \) and \(\mu_B = 1\)
Practically, the solution of Eq. (14) is degenerate, hence its single-parameter family of solutions can be written as
\[
\begin{pmatrix} g_1 \\ q_1 \end{pmatrix} = \bar{g}_1 \begin{pmatrix} 1 \\ C \end{pmatrix} + \hat{g}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{20}
\]

Here, \( \hat{g}_1 \) is confirmed by Eq. (14), and \( \bar{g}_1 \) is considered as one arbitrary function. Since two generation equations for \( \hat{g}_1 \) from Eq. (14) are identical, \( \hat{g}_1 \) can be expressed as \( \hat{g}_1 = i\hat{f}_x + i\hat{f}_y \), where
\[
\hat{u} = \frac{s(c_k C - \frac{c_i}{C} - d_1)}{2\sqrt{J^2_2S^2[\cos(2k_x) - \cos(2k_y)]^2 + |\beta_1|^2}}, \quad \hat{v} = \frac{s(d_1 + d_k C - \frac{d_i}{C})}{2\sqrt{J^2_2S^2[\cos(2k_x) - \cos(2k_y)]^2 + |\beta_1|^2}}.
\tag{21}
\]

Here, \( s = \pm d_1 = 4SJ_2 \sin(2k_y) \) and \( d_2 = 4SJ_2 \sin(2k_y) \). It is pointed out that \( s = +1 \) and \( -1 \) correspond to the optical and acoustic branches, respectively.

By considering terms at \( O(\eta^3 e^{i\omega p}) \), one can obtain
\[
h_0 = r_0 = 0. \tag{22}
\]

By analyzing from terms of \( O(\eta^3 e^{i\omega p}) \), we can get the final equation, which is given by
\[
M \begin{pmatrix} h_1 \\ r_1 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{23}
\]

Here, the form of the coefficient matrix \( M \) is given in Eq. (9), and the RHS modules read
\[
C_1 = i(g_{1r} + f_T) + 4J_1 pf_T^2 - \rho_0 \rho^2 (J_1 \gamma_k + i\delta m_k) - \left[ -2(A + J_2) + J_2 \cos(k_x) \right] f^2 - 4J_1 \cos(k_x) f_T q_{1x} - 2iS_m k \delta_{px} + 2iS_m k \delta_{py} - 2S \gamma_k J_1 \delta_{px} + 2S \gamma_k J_1 \delta_{py}
\]
\[
-4S \left[ -iJ_1 \cos \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) - D \cos \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) \right] q_{1x},
\]
\[
C_2 = i(g_{1r} + f_T) + 4J_1 pf_T^2 - \left( J_1 \gamma_k - i\delta m_k \right) f^2 - \left[ -2(A + J_2) + J_2 \cos(k_x) \right] p^2
\]
\[
-2iS_m k \delta_{px} + 2iS_m k \delta_{py} + 4iS \gamma_k J_1 \delta_{px} - 2S \gamma_k J_1 \delta_{py}
\]
\[
-4S \left[ -iJ_1 \cos \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) + D \cos \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) \right] g_{1x},
\]
\[
-4S \left[ -D \cos \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) - iJ_1 \cos \left( \frac{k_x}{2} \right) \sin \left( \frac{k_y}{2} \right) \right] g_{1y},
\tag{24}
\]

In order make sure that these equations exist solutions, we should use the consistency condition \( \mathbf{n} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0 \). Taking \( g_{1r} q_{1f} \) as given via Eq. (20) with \( \bar{g}_1 = -\hat{g}_1 / (1 + |C|^2) \) and utilizing \( p = C^f \), we can get
\[ If_T + p_1 f_{zz} + p_2 f_{zw} + p_3 f_{ww} + q |f|^2 f = 0, \tag{25} \]

where

\[
p_1 = \frac{1}{2} \frac{\partial^2 \omega}{\partial k_z^2} = \frac{1}{1 + |C|^2} \left[ \frac{1}{2} \beta_C^* C^* + \frac{1}{2} \beta_C C + 2(\omega_0 + \omega_c)|C|^2 \right] - \frac{|C|^2 C^*}{(1 + |C|^2)^2} c_C^* \dot{u} + c_C \frac{C}{(1 + |C|^2)^2} \dot{u} + d_z \frac{|C|^2 \ddot{u}}{(1 + |C|^2)^2},
\]

\[
p_2 = \frac{\partial^\omega}{\partial k_x \partial k_y} = -\frac{1}{1 + |C|^2} \left[ 4 \mathcal{S}(m_a I - i \gamma_D) C^* + 4 \mathcal{S}(J_1 m_k + i \gamma_D C) \right] - \frac{C^* |C|^2}{(1 + |C|^2)^2} \dot{d}_y + \frac{C}{(1 + |C|^2)^2} d_i \ddot{u} - \frac{|C|^2}{(1 + |C|^2)^2} d_i \ddot{u} + d_z \frac{|C|^2 \ddot{v}}{(1 + |C|^2)^2},
\]

\[
p_3 = \frac{\partial \omega}{\partial k^2} = \frac{1}{2} \frac{\partial^2 \beta_C}{\partial k_x^2} = \frac{1}{1 + |C|^2} \left[ \frac{1}{2} \beta_C C^* + \frac{1}{2} \beta_C C + 2(\omega_0 + \omega_c)|C|^2 \right] - \frac{C^* |C|^2}{(1 + |C|^2)^2} d_i \dot{v} + \frac{C}{(1 + |C|^2)^2} d_i \dot{v} - \frac{|C|^2}{(1 + |C|^2)^2} d_i \ddot{v},
\]

\[ q = \frac{1}{1 + |C|^2} \{ 8 |C|^2 J_1 = [-2(A + J_2) + J_2 \cos(2k_z)](1 + |C|^2) - 2(J_1 \gamma_k - i \Delta m_k) C^* - 2(J_1 \gamma_k + i \Delta m_k)|C|^2 C \}. \tag{26} \]

Obviously, Eq. (25) is the famous (2+1)-dimensional nonlinear Schrödinger (NLS) equation.

4 Intrinsic Localized Modes

In our previous work [25], we have obtained bright and dark soliton solutions to the (2+1)-dimensional NLS equation via take advantage of the bilinear method. Here, we shall make use of these localized soliton solutions to construct the analytical expressions on intrinsic localized modes. In this study, we focus on two types of particular wave vectors: (A) \( \mathbf{k} = 0 \), (B) \( \mathbf{k} = (\pi/2, \pi/2) \).

A. The Brillouin zone center mode

At the Brillouin zone center, namely, \( \mathbf{k} = 0 \), the system may support an interesting nonlinear localized mode, namely, the Brillouin zone center mode. Since the magnon frequency band of the checkerboard ferromagnet possesses two branches, the acoustic- and optical mode at the Brillouin zone center shall be taken into account, respectively.

On the acoustic branch, one has \( P_{ac} = P_{3,ac} = 2J_1 S + 2J_2 S, \quad P_{2,ac} = 0, \) and \( Q_{ac} = 2J_1 + J_2 + 2A \). Furthermore, we find that \( \omega_{ac} = \omega_{min} = \omega_0 - 2J_2 S - 4J_1 S, \) \( u_{ac} = v_{ac} = 0, \) and \( C_{ac} = 1 \). Thus, Eq. (25) is recognized as a focusing (2+1)-dimensional NLS equation, which exists a bright soliton solution. In this case, it is not hard to get the analytical expression for the Brillouin zone center acoustic mode, which is

\[
\alpha_{m,n}(t) = \beta_{m,n}(t) \approx e^{-i \Omega^2 t} \frac{e^{i \Omega_1 (m + n) S}}{1 + \frac{2J_1 + J_2 + 2A}{16S(J_1 + J_2)(a_1^2 + a_2^2)} e^{2\epsilon(a_1 m + a_2 n)}}, \tag{27} \]
where $\Omega_1 = \omega_{\min} - 2\varepsilon^2 S(J_1 + J_2)(a_1^2 + a_2^2)$. We note that Eq. (27) is a 2D intrinsic localized mode possessing the bright localized structure. Its eigenfrequency $\Omega_1$ lies below the bottom of the magnon acoustic branch frequency band. Figure 5(a) displays the square modulus of the coherent-state amplitude of this 2D bright intrinsic localized mode, whose spatially localized structure can remain unchanged.

Next, we turn our attention to the acoustic branch. For the acoustic branch, we have $P_{1,\text{opt}} = P_{3,\text{opt}} = -2S(J_1 - J_2) > 0$, $P_{2,\text{opt}} = 0$, and $Q_{\text{opt}} = 6J_1 + J_2 + 2A$. Then, Eq. (25) is recognized as a defocusing $(2 + 1)$-dimensional NLS equation, which supports a dark soliton solution. What is more, it is pointed out that $\omega_{\text{opt}} = \omega_{\max} = \omega_0 + 3JS \mu_{\text{opt}} = v_{\text{opt}} = 0$, and $C_{\text{opt}} = -1$. In this case, one can obtain the analytical expression for the optical Brillouin zone center mode, namely,

$$\alpha_{m,n}(t) = -\beta_{m,n}(t) \approx \varepsilon \eta e^{-i\Omega_2 t} \tanh \left( \frac{\sqrt{S(J_1 - J_2)}}{2S(J_1 - J_2)} \left[ \left( 6J_1 + J_2 + 2A \right) \eta^2 - S(J_1 - J_2) b_2^2 \right] \frac{\varepsilon m + b_2 \varepsilon n}{2} \right),$$

(28)

where $\Omega_2 = \omega_{\max} - \varepsilon^2 \eta^2 (6J_1 + J_2 + 2A)$. Obviously, Eq. (28) is a 2D intrinsic localized mode possessing a dark localized structure [see Fig. 5(b)]. In fact, this dark intrinsic localized mode is a resonant mode [33] since its vibration frequency lies within the optical magnon frequency band. Furthermore, this resonant mode is in resonance with the optical magnon, which causes the finite lifetime of the 2D dark intrinsic localized mode in real ferromagnetic materials.

B. The Brillouin zone corner mode

Now, let us consider nonlinear localized mode at the corner of the Brillouin zone. For the acoustic branch, we have $P_1,_{\text{ac}} = P_3,_{\text{ac}} = 2S(D - J_2), P_2,_{\text{ac}} = 0, q_{\text{ac}} = 4J_1 + 3J_2 + 2A - 2D, C_{\text{ac}} = -i, \mu_{\text{ac}} = v_{\text{ac}} = 0$, and $\omega_{\text{ac}} = \omega_0 + 2J_2 S - 4DS$. In this case, the $(2 + 1)$-dimensional NLS Eq. (25) becomes

$$i f_T + 2S(D - J_2) \left( f_{zz} + f_{ww} \right) + (4J_1 + 3J_2 + 2A - 2D) |f|^2 f = 0.$$  

(29)

Fig. 5 (Color online) Brillouin zone center modes in the checkerboard ferromagnet. a bright intrinsic localized mode: $\epsilon = 0.1, \alpha_1 = 1, \alpha_2 = 1$; b dark intrinsic localized mode: $\epsilon = 0.1, \eta = 2, b_2 = 0.6$
From the above equation, it can be seen that there is a competitive relationship between DM interaction and Heisenberg next-nearest neighbor exchange interaction, which causes that the Brillouin zone corner mode can have different local structures. In the case of $D < J_2$, Eq. (29) corresponds to a defocusing $(2 + 1)$-dimensional NLS equation, which has a dark soliton solution. Thus, we can get the analytical expression of the Brillouin zone corner acoustic mode,

$$a_{m,n}(t) = i\beta_{m,n}(t) \approx \varepsilon \eta e^{rac{i}{2}(m+n)} e^{-i\Omega_3 t} \times \tanh \left( \frac{\sqrt{S(D-D)} \left[ S(D-J) b_2 + (4J_1 + 3J_2 + 2A - 2D) \eta^2 \right]}{2S(D-J_2)} \varepsilon m + \frac{b_2 \varepsilon N}{2} \right)$$

with $\Omega_3 = \omega_{ac} - \eta^2 (4J_1 + 3J_2 + 2A - 2D) \varepsilon^2$, which is the 2D dark-type intrinsic localized mode. In Fig. 6(a), we show the square modulus of the coherent-state amplitude of this dark-type intrinsic localized mode, which has a finite lifetime due to due to resonance with the acoustic magnon.

When $D > J_2$, Eq. (25) is in fact a focusing $(2 + 1)$-dimensional NLS equation, possessing a bright soliton solution. Using the bright soliton solution obtained in Ref. [25], it is easy to write down the analytical expression for the Brillouin zone corner acoustic mode,

$$a_{m,n}(t) = i\beta_{m,n}(t) \approx \varepsilon e^{rac{i}{2}(m+n)} e^{-i\Omega_4 t} \frac{\phi_{\varepsilon(t)} + \phi_{\varepsilon(t)}^*}{1 + \frac{4J_1 + 3J_2 + 2A - 2D}{16S(-D+J_2)} \left( a_1^2 + a_2^2 \right)^2 \varepsilon^2 \varepsilon \varepsilon},$$

where $\Omega_4 = \omega_{ac} - 2S(D-J_2) \left( a_1^2 + a_2^2 \right) \varepsilon^2$ is within the magnon acoustic frequency band. It is evident that Eq. (31) is a 2D intrinsic localized resonant mode possessing a bright localized structure, as displayed in Fig. 6(b).

On account of the above analysis, one can clearly see that the competition between the DM interaction and the Heisenberg next-nearest neighbor exchange interaction of the checkerboard ferromagnet has a great influence on the property of the Brillouin zone corner acoustic mode. If $D < J_2$, then the present checkerboard ferromagnet supports the emergence of the dark-type intrinsic localized mode. When $D > J_2$, a bright-type intrinsic localized mode can emerge. In

![Fig. 6](Color online) The Brillouin zone corner acoustic modes in the checkerboard ferromagnet. a $D = 0.1$; b $D = 0.5$
addition, we note that these Brillouin zone corner acoustic modes are resonant modes so that their lifetimes are finite.

Finally, we focus on the Brillouin zone corner optical mode. For the optical branch, we have $p_{1,\text{opt}} = p_{3,\text{opt}} = -2S(J_2 + D)$, $p_{2,\text{opt}} = 0$, $q_{\text{opt}} = 0$, and $\omega_{\text{opt}} = \omega_0 + 2J_2S + 4DS$. Thus, the $(2+1)$-dimensional NLS Eq. (25) is reduced to a simplified form, namely,

$$i\frac{\partial f}{\partial t} = -2S(J_2 + D)(f_{xx} + f_{ww}) + (4J_1 + 3J_2 + 2A + 2D)|f|^2f = 0,$$

which is in fact a defocusing $(2+1)$-dimensional NLS equation supporting a dark soliton solution. With our dark soliton solution obtained in Ref. [24], one can get the analytical expression of the Brillouin zone corner optical mode, which is

$$|\alpha_{m,n}|^2 = |\beta_{m,n}|^2,$$

with $\Omega_5 = \omega_{\text{opt}} = \eta^2(4J_1 + 3J_2 + 2A + 2D)\epsilon^2$. Equation (33) is a dark-type 2D intrinsic localized mode, as exhibited in Fig. 7. When the DM interaction strength $D$ satisfies the following relation

$$\frac{\eta^2 (4J_1 + 3J_2 + 2A) \epsilon^2 - 4J_2 S}{4S - 2\eta^2 \epsilon^2} < D < \frac{\eta^2 (4J_1 + 3J_2 + 2A) \epsilon^2}{4S - 2\eta^2 \epsilon^2},$$

the eigenfrequency $\Omega_5$ lies in the gap of two magnon frequency bands. In this case, our Brillouin zone corner optical mode is a gap localized mode. In principle, it is stable for a long time.

Fig. 7 (Color online) The Brillouin zone corner optical mode in the checkerboard ferromagnet
5 Conclusions

In summary, the existence and property of the intrinsic localized mode in one 2D checkerboard Heisenberg ferromagnet have been investigated in the semiclassical limit. With the help of an asymptotic method developed by Butt and Wattis, the governing lattice dynamical equations have been simplified to a $(2 + 1)$–dimensional NLS equation. Using its soliton solutions obtained by us in the previous work, we have constructed different forms of intrinsic localized mode solutions in the current checkerboard Heisenberg ferromagnet. Analytical forms for the Brillouin zone center mode and corner mode have been obtained. Their existence conditions and properties have been confirmed. Especially, our results have shown that the local structure of the Brillouin zone corner acoustic mode depends on the competition between the DM interaction and the Heisenberg next-nearest neighbor exchange interaction. When the checkerboard ferromagnetic system is placed in an intense laser field with an oscillating electric field, a tunable DM interaction will appear due to the Aharanov Casher effect [34]. In future works, we will also try to study intrinsic localized modes in the laser-irradiated checkerboard ferromagnet.

Acknowledgements This theoretical study was supported by the National Natural Science Foundation of China under Grant Nos. 12064011 and 11964011, and the Natural Science Fund Project of Hunan Province under Grant No.2020JJ4498.

Author contributions Wenhui Feng and Bing Tang wrote the main manuscript text and Heng Zhu prepared figures 4-7. All authors reviewed the manuscript.

Data availability All data generated or analysed during this study are included in this published article.

Declarations

Competing interests The authors declare no competing interests.

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