HEIGHT ESTIMATES FOR BIANCHI GROUPS

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Abstract. We study the action of Bianchi groups on the hyperbolic 3-space \( \mathbb{H}^3 \). Given the standard fundamental domain for this action and any point in \( \mathbb{H}^3 \), we give an upper bound for the height of the unique element in the group which sends the point into the fundamental domain. The height is bounded by a polynomial function on some coordinates of the point whose degree does not depend on the Bianchi group. This generalizes a similar result of Habegger and Pila for the action of the Modular group on the hyperbolic plane. Our main theorem can be applied in the reduction theory of binary Hermitian forms with entries in the ring of integers of quadratic imaginary fields.

1. Introduction

The theory of lattices in the Lie group \( \text{PSL}(2, \mathbb{C}) \) is a source of important problems in many branches of mathematic, e.g. complex variables, dynamics systems, group theory, hyperbolic geometry and number theory. Among such lattices, the class of arithmetic groups attracts a special attention because of their remarkable properties. A hyperbolic 3-orbifold is called arithmetic if it is the quotient of the hyperbolic 3-space by an arithmetic lattice. We recall that the closed hyperbolic manifold of minimal volume, the Weeks manifold, and the cusped hyperbolic 3-manifold of minimal volume are arithmetic manifolds.

An important subclass of arithmetic lattices of \( \text{PSL}(2, \mathbb{C}) \) is that of the Bianchi groups. A basic fact about such groups is that any non-uniform arithmetic lattice of \( \text{PSL}(2, \mathbb{C}) \) is commensurable to some Bianchi group [10, Thm. 8.2.3]. These groups have been studied since the seminal work of Bianchi [2] as the natural generalization of the modular group \( \text{PSL}(2, \mathbb{Z}) \) and until today there are some deep open questions about them. For example, there exist explicit formulae for the covolume of Bianchi groups, but it is not known whether the quotient of the covolumes of any two Bianchi groups is a rational number [3]. Another important question [5, Section 7.6] is a particular case for a conjecture raised by Selberg which asks if the spectral gap in the discrete spectrum of any Bianchi orbifold is at least 1.

For any lattice in \( \text{PSL}(2, \mathbb{C}) \), there exists a measurable fundamental domain in the hyperbolic 3-space. The usefulness of a fundamental domain is not limited to giving the covolume. There is a lot of geometric and algebraic information which can be

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extracted from these sets. For example, finite presentation for any lattice can be obtained from its fundamental domain [11, paragraph 4.7]. Although the existence of these sets is always guaranteed, there are two issues involving them. Firstly, it is not easy in general to find explicit fundamental domains for arbitrary lattices, and secondly, even if some fundamental domain is given, it may not be the most useful for particular applications. Nevertheless, we can approximate the fundamental domain by fundamental sets, which are measurable sets whose projection on the quotient space is surjective and finite-to-one.

This paper is part of a project aimed to describe fundamental sets for Bianchi groups with good geometric properties which are more convenient for some applications than the corresponding fundamental domains.

An example of application of fundamental sets was given in the proof of the Selberg conjecture for Bianchi orbifolds of low volume [6]. In the proof, two key properties are the facts that the fundamental set which they constructed is a product of a rectangle with a segment in the hyperbolic 3-space and that it is covered by few copies of a specific fundamental domain. More generally, let $D$ be a fundamental domain for a lattice $\Gamma$ and let $\Sigma \supset D$ be a fundamental set, the set $\{ \gamma \in \Gamma \mid \gamma D \cap \Sigma \neq \emptyset \}$ has finite cardinality $N \geq 1$. We say that $\Sigma$ is a good approximation of $D$ if $N$ is small.

In the general theory of lattices in real points of semisimple Lie Groups, Siegel sets are fundamental sets with a simple geometric description. On the other hand, in the paper of the second author [13] it was shown that in general the approximation by Siegel sets may not be so good from the point of view of the discussion above.

Inspired by the good geometric shape of Siegel sets we are interested in, the natural candidates for fundamental sets in $\mathbb{H}^3$, which are given by finite unions of sets of the form $h\Sigma$, where $h \in \text{PSL}(2, \mathbb{C})$ and $\Sigma = R \times [t, +\infty)$, $R$ is a polygon in $\mathbb{C}$ and $t > 0$, both depending on the group. Since we need to count elements in the Bianchi group, there exists a useful tool in number theory, the height of a matrix with entries in a number field. For any real positive constant $L$, the number of elements in the Bianchi group of height at most $L$ is finite. The naive idea is therefore to estimate from above in terms of our fundamental set the height of any element $\gamma$ in the Bianchi group satisfying $\gamma D \cap \Sigma \neq \emptyset$, where $\Sigma$ is our candidate for fundamental set and $D$ is an explicit fundamental domain for the Bianchi group. After this, we need to be able to count the number of matrices with bounded height.

It is worth to mention that this idea of giving an upper bound for the height of matrices with the property of finite intersection for fundamental sets appeared in Orr’s paper [12] motivated by a previous result of Habegger and Pila [7]. Our main result is a natural generalization of their height estimate for the similar problem in $\mathbb{H}^2$.

This paper is organized as follows. In Section 2 we define the main objects of study along this work. In Section 3 we present the main result, where we give an estimate for the height of the group element sending a point in $\mathbb{H}^3$ into the fundamental domain of the Bianchi group in terms of special coordinates of this point. In Section 4 we give
an application of the main theorem in the theory of reduction of binary Hermitian forms with coefficients in the ring of integers of a quadratic imaginary number field. For the sake of completeness, in Section 5 we prove an analogue of Duke, Rudinick and Sarnak’s result [4] on giving the asymptotic growth of the cardinality of the set of matrices with height smaller than a given constant.

2. Background

2.1. Geometry of the hyperbolic 3-space. The hyperbolic 3-space $H^3$ is the unique 3-dimensional connected and simply connected Riemannian manifold with constant sectional curvature equal to -1. We use the upper half-space model of $H^3$, which in its properties closely resembles the well-known upper half-plane model $H^2$ of plane hyperbolic geometry.

$$H^3 = \{(z,t); z = x + iy \in \mathbb{C}, t > 0\}.$$  

We can think of $H^3$ as a subset of Hamilton’s quaternions, by writing $(z,t) = z + tj$, where $z = x + iy$ and $1, i, j, k$ is the usual basis of the quaternion space.

The hyperbolic metric in $H^3$ is $ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$. In the sense of Riemannian geometry this metric gives rise to the hyperbolic volume measure $\nu$ with corresponding volume element

$$d\nu = \frac{dxdydt}{t^3}.$$  

Topology in $H^3$ is induced from $\mathbb{R}^3$, but geometry is hyperbolic. The geodesic lines are half-circles or half-lines in $H^3$ which are orthogonal to the boundary plane $\mathbb{C}$ in the Euclidean sense. The geodesic surfaces are Euclidean hemispheres or half-planes which again are orthogonal to the boundary $\mathbb{C}$.

The group $G = SL(2, \mathbb{C})$ acts by isometries on the upper half space $H^3$. This action can be described as follows: if we represent a point $P \in H^3$ as a quaternion $P = (z,t) = z + tj$, then the action of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ on $H^3$ is given by

$$P \mapsto MP := (aP + b)(cP + d)^{-1},$$  

where the inverse is taken in the skew field of quaternions. A direct computation in quaternionic coordinates shows that this defines an action of $SL(2, \mathbb{C})$ on $H^3$. More explicitly, Equation (1) may be written in the form

$$M(z,t) = \left(\frac{(az + b)(\bar{c}z + \bar{d}) + ac\bar{t}^2}{|cz + d|^2 + |c|^2 t^2}, \frac{t}{|cz + d|^2 + |c|^2 t^2}\right).$$  

The action of $G$ on $H^3$ extends to an action on $H^3 \cup \partial H^3$, where $\partial H^3 = \mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$. On $\partial H^3$ the action may be described by a simple formula. We represent
an element of \( \mathbb{P}^1 \mathbb{C} \) by \([x : y]\), where \((x, y) \in \mathbb{C}^2 - \{(0, 0)\}\), and represent \( \infty \in \mathbb{P}^1 \mathbb{C} \) by \([1 : 0]\). Then the action of \( M \) on \( \mathbb{P}^1 \mathbb{C} \) is given by

\[
[x : y] \mapsto M[x : y] := [ax + by : cx + dy].
\]

For any \( M \in G \), the matrix \(-M \in G\) and \( M(P) = -M(P) \) for all \( P \in \mathbb{H}^3 \cup \partial \mathbb{H}^3\). Therefore we can consider the action of the group \( \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{I, -I\} \), where \( I \) is the identity matrix. We will denote the class of a matrix \( M \in \text{SL}(2, \mathbb{C}) \) in \( \text{PSL}(2, \mathbb{C}) \) by \([M]\).

2.2. Bianchi groups. Given a squarefree integer \( d > 0 \), consider the imaginary quadratic number field \( \mathbb{Q}(\sqrt{-d}) \) and let \( \mathcal{O}_d \) be the ring of integers of \( K_d \),

\[
\mathcal{O}_d = \mathbb{Z}[\omega] = \{a + b\omega; a, b \in \mathbb{Z}\},
\]

where \( \omega = \begin{cases} \frac{-1 + \sqrt{-d}}{2} & \text{if } d \equiv 3(\text{ mod } 4); \\ \frac{-1 - \sqrt{-d}}{2} & \text{otherwise.} \end{cases} \)

**Definition 2.1.** For each \( d \), we define the Bianchi group:

\[
\Gamma_d := \text{PSL}(2, \mathcal{O}_d) = \{M \in \text{SL}(2, \mathbb{C}); M_{ij} \in \mathcal{O}_d\}/\{I, -I\}.
\]

**Definition 2.2.** The cusps \( C_{\Gamma_d} \) of \( \Gamma_d \) are the elements \([x, y] \in \mathbb{P}^1 \mathbb{C}\), for which the stabilizer in \( \Gamma_d \) of the action in Equation (3) contains a free abelian group of rank 2.

The set of cusps of the Bianchi group \( \Gamma_d \) is given by \( C_{\Gamma_d} = \mathbb{P}^1 K_d \subset \mathbb{P}^1 \mathbb{C} \) as it can be seen in [5, Proposition 2.2, Chapter 7].

The quotient of the induced action of \( \Gamma_d \) on \( C_{\Gamma_d} \) is a finite set, which has exactly \( h(d) \) elements, where \( h(d) \) is the class number of the field \( K_d \), i.e., the cardinality of the ideal class group \( \mathcal{I}_d \) of \( K_d \) (see [5, Theorem 2.4., Chapter 7]).

2.3. The fundamental domains for Bianchi groups. In [5, Section 7.3] a complete geometric description of the construction of fundamental domains for the action of Bianchi groups on \( \mathbb{H}^3 \) is given.

We define the sets

\[
\mathcal{B}_d = \{(z, t) \in \mathbb{H}^3; |cz + d| + |c|^2t^2 \geq 1 \text{ for all } c, d \in \mathcal{O}_d, \langle c, d \rangle = \mathcal{O}_d\}
\]

\[
\mathcal{P}_1 = \left\{x + iy \mid |x| \leq \frac{1}{2}, \quad 0 \leq y \leq \frac{1}{2}\right\},
\]

\[
\mathcal{P}_3 = \left\{x + iy \mid 0 \leq x \leq \sqrt{3} y, \quad x + \sqrt{3} y \leq 1 \right\} \cup \left\{x + iy \mid 0 \leq x \leq \frac{1}{2}, \quad \frac{-x}{\sqrt{3}} \leq y \leq \frac{x}{\sqrt{3}}\right\}.
\]

For any \( d \not\in \{1, 3\} \), we have

\[
\mathcal{P}_d = \left\{x + iy \mid 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{d}\right\} \quad \text{if } d \equiv 1, 2 \text{ modulo } 4,
\]

\[
\mathcal{P}_d = \left\{x + iy \mid 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{d/2}\right\} \quad \text{if } d \equiv 3 \text{ modulo } 4.
\]
Note that $\text{PSL}(2, \mathcal{O}_d)_{\infty} = \{ \gamma \in \text{PSL}(2, \mathcal{O}_d); \gamma(\infty) = \infty \}$ acts by isometries in $\mathbb{C}$ and this action is cocompact. The sets $\mathcal{P}_d$ above are fundamental domains for this induced action which contain the origin.

In analogy with the action of the modular group in the hyperbolic plane, we have the following theorem (see [5]).

**Theorem 2.3.** The set $\mathcal{F}_d = \{(z, t) \in \mathcal{B}_d \mid z \in \mathcal{P}_d \}$ is a fundamental domain for the action of $\Gamma_d$ on $\mathbb{H}^3$.

### 3. Height estimates for Bianchi groups

We refer to [14, Section 7.1] for the description of the absolute logarithmic height $H(x)$ of an algebraic number $x$. In the special case when $x \in \mathcal{O}_d$, the height of $x$ is given by $H(x) = |x|$, the complex norm of $x$. For an element $T \in \text{SL}(2, \mathcal{O}_d)$, the height $H(T)$ of $T$ is the maximum of the heights of its entries. If $[M] \in \Gamma_d$ with $M \in \text{SL}(2, \mathcal{O}_d)$, then we can define $H([M]) := H(M)$, since $H(T) = H(-T)$ for any $T \in \text{SL}(2, \mathcal{O}_d)$. We note that for $A, B \in \Gamma_d$, we have $\text{H}(AB) \leq 2\text{H}(A)\text{H}(B)$.

In [16], Pila defines the notion of intricacy of a point of a symmetric space with respect to a fundamental domain of a lattice acting on the space. If we consider for each $d$ as above, the fundamental domain $\mathcal{F}_d$ of the Bianchi group, we can extend this notion by defining the $\Gamma_d$-intricacy of a point $(z, t) \in \mathbb{H}^3$ with respect to $\mathcal{F}_d$ by

$$I_{\mathcal{F}_d}(z, t) = H(\sigma),$$

where $\sigma \in \Gamma_d$ is the unique element which takes $(z, t)$ into $\mathcal{F}_d$.

Motivated by the analogue problem solved by Habegger and Pila in [7] we define the function $D(z, t) = \max\{1, |z|, t^{-1}\}$ on $\mathbb{H}^3$, and prove our main theorem.

**Theorem 3.1.** Let $d > 0$ be a squarefree integer. There exist constants $c = c(d) > 0$ and $m = m(d) > 0$ such that

$$I_{\mathcal{F}_d}(z, t) \leq cD(z, t)^m$$

for any $(z, t) \in \mathbb{H}^3$. Furthermore, $c(d)$ and $m(d)$ satisfy:

1. $m(d) = \frac{5}{2}$, if the class number $h(d) = 1$ and $m(d) = \frac{9}{2}$ otherwise;

2. $c(d) \leq Cd^{\frac{3}{2}}$ for some universal constant $C > 0$.

Before proving Theorem 3.1 we will prove some auxiliary results.

**Lemma 3.2.** Let $d > 0$ be a squarefree integer. Let $\alpha, \beta \in \mathcal{O}_d \setminus \{0\}$. Suppose that $\langle \alpha, \beta \rangle = \mathcal{O}_d$. There exist a constant $C_d \geq 1$ and $x, y \in \mathcal{O}_d$ such that

$$1 = \alpha x + \beta y \quad \text{with} \quad |x| \leq C_d|\beta| \quad \text{and} \quad |y| \leq C_d|\alpha|.$$
is another ball, centered in $-C$.

The lemma is then proven if we take $C$ such that $d = 2C$. Consider $\varepsilon_d = \frac{\text{diam}(F_d)}{2} > 0$.

If $(x_0, y_0) \in O^2_d$ is a solution of the equation $1 = \alpha x_0 + \beta y_0$, then for any $\lambda \in O_d$ the pair $(x_\lambda, y_\lambda) = (x_0 - \lambda \beta, y_0 + \lambda \alpha)$ is another solution.

Consider now the complex number $\frac{x_0}{\beta}$. There must exist $z_0 \in F_d$ and $\lambda_0 \in O_d$ such that $\frac{x_0}{\beta} = z_0 + \lambda_0$ and also a vertex $\mu$ in the boundary of $F_d$ with $|z_0 - \mu| \leq \varepsilon_d$.

Therefore, $\lambda = \lambda_0 + \mu \in O_d$ satisfies $\left| \frac{x_0}{\beta} - \lambda \right| \leq \varepsilon_d$, which is equivalent to $|x_0 - \lambda \beta| \leq \varepsilon_d|\beta|$.

For the pair $(x_\lambda, y_\lambda)$, we have that $1 = \alpha x_\lambda + \beta y_\lambda$ and the following inequalities hold:

$$|x_\lambda| \leq \varepsilon_d|\beta| \text{ and } |y_\lambda| = \frac{|1 - \alpha x_\lambda|}{|\beta|} \leq 1 + |\alpha|\varepsilon_d \leq (1 + \varepsilon_d)|\alpha|.$$ 

The lemma is then proven if we take $C_d = 1 + \varepsilon_d$. \hfill \square

**Remark 3.3.** If $O_d$ is norm-Euclidean, then the constant $\varepsilon_d < 1$. Therefore we can always assume that $C_d = 2$ for $d = 1, 2, 3, 7, 11$. Moreover, there exists a universal constant $c_1 > 0$ such that $C_d \leq c_1 \sqrt{d}$ for all $d$.

**Proposition 3.4.** Let $d$ be as before and consider the function $\mu_d : \mathbb{H}^3 \to \mathbb{R}$ given by

$$\mu_d(z, t) = \min_{(\gamma, \delta) \in O^2_d \setminus \{0\}} \frac{|\gamma z + \delta t|^2 + |\gamma|^2t^2}{t}.$$ 

Then

$$\mu_d(z, t) \leq \frac{4\sqrt{2} \text{vol}(\mathbb{C}/O_d)}{\pi}, \text{ for all } (z, t) \in \mathbb{H}^3.$$ 

**Proof.** We look at $O^2_d$ as a lattice in $\mathbb{C}^2$. For each $\lambda > 0$, we define the region

$$R(\lambda, z, t) = \{(u, v) \in \mathbb{C}^2; |uz + v|^2 + |u|^2t^2 \leq \lambda t\}.$$ 

Note that $R(\lambda, z, t)$ is symmetric with respect to the origin $(0, 0)$ and convex.

For any fixed $(z, t) \in \mathbb{H}^3$ and $\lambda > 0$ we will compute the volume of $R(\lambda, z, t)$. The set $\{u \in \mathbb{C}; \text{there exists } v \text{ with } (u, v) \in R(\lambda, z, t)\}$ is an open ball in $\mathbb{C}$ with center 0 and radius $\sqrt{\lambda t}$. If we fix $u_0$ in this ball, the set $\{v \in \mathbb{C}; |u_0z + v|^2 + |u_0|^2t^2 < \lambda t\}$ is another ball, centered in $-u_0z$ and with radius $\sqrt{\lambda t - |u_0|^2t^2}$. Hence,

$$\text{vol}(R(\lambda, z, t)) = \int_{\mathbb{C}} \int_{\mathbb{C}} \pi(\lambda t - |u|^2t^2) \, dvdu = \int_{\mathbb{C}} \pi(\lambda t - |u|^2t^2) \, du$$ 

$$= \int_{\mathbb{C}} \pi(\lambda t - |u|^2t^2) \, du$$ 

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By using polar coordinates, we get
\[
\text{vol}(\mathcal{R}(\lambda, z, t)) = \pi \int_{0}^{2\pi} \int_{0}^{\sqrt{\lambda t - 1}} (\lambda t - r^2 t^2) r dr d\theta = 2\pi^2 \int_{0}^{\sqrt{\lambda t - 1}} (\lambda t - r^2 t^2) r dr.
\]
Finally,
\[
\text{vol}(\mathcal{R}(\lambda, z, t)) = 2\pi^2 \left[ \frac{\lambda^2}{2} - \frac{\lambda^2}{4} \right] = \frac{\lambda^2 \pi^2}{2}.
\]
Since for any \( \lambda < \mu_d(z, t) \) the intersection \( \mathcal{O}_d^2 \cap \mathcal{R}(\lambda, z, t) \) must be trivial, by Minkowski’s theorem we obtain that
\[
\text{vol}(\mathcal{R}(\lambda, z, t)) = \frac{\lambda^2 \pi^2}{2} \leq 16\text{vol}(\mathcal{C}/\mathcal{O}_d)^2, \text{ for all } \lambda < \mu_d(z, t).
\]
Therefore, \( \mu_d(z, t) \leq \frac{4\sqrt{2}\text{vol}(\mathcal{C}/\mathcal{O}_d)}{\pi} \).

**Remark 3.5.** Note that \( \left| uz + v \right|^2 + |u|^2 t^2 = H[(u, v)] \) where \( H \) is a positive definite Hermitian binary quadratic form of discriminant 1. It follows from [9, Theorem 1] that \( \mu_d(z, t) \leq \sqrt{2\text{vol}(\mathcal{C}/\mathcal{O}_d)} \). Although Oppenheim’s Theorem gives a better estimate, Proposition 3.4 is easier and it is sufficient for our purpose.

We can now prove our main theorem.

**Proof of Theorem 3.1.** We fix \( d > 0 \). Our goal is to give information about the unique element which sends an arbitrary element \((z, t) \in \mathbb{H}^3 \) into the fundamental domain \( \mathcal{F}_d \). By [5, Lemma 3.3], this can be done in two steps. In the first step, we need to construct \( \tau \in \text{PSL}(2, \mathcal{O}_d) \) such that \( \tau(z, t) = (z', t') \in \mathcal{B}_d \). After it, we use the invariance of the action of \( \text{PSL}(2, \mathcal{O}_d) \) in \( \mathcal{B}_d \) in order to construct \( \delta \in \text{PSL}(2, \mathcal{O}_d) \) such that \( \delta \tau(z, t) = \delta(z', t') = (z'', t') \) with \( (z'', t') \in \mathcal{F}_d \). In this second step it is easy to control the height of \( \delta \) in terms of \( z' \).

The difficult part is to bound the height of \( \tau \) in terms of \((z, t)\). Define
\[
\mu'_d(z, t) = \min_{(\gamma, \delta) = \mathcal{O}_d} \frac{|\gamma z + \delta|^2 + |\gamma|^2 t^2}{t}.
\]
By [5, Lemma 3.3(1) Chapter 7], an element \( \tau = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} \) of \( \text{PSL}(2, \mathcal{O}_d) \) satisfies \( \tau(z, t) = (z', t') \in \mathcal{B}_d \) if and only if
\[
(4) \quad \mu'_d(z, t) t = |\gamma_0 z + \delta_0|^2 + |\gamma_0|^2 t^2.
\]
Note that \( \mu_d(z, t) \leq \mu'_d(z, t) \) for any \((z, t) \in \mathbb{H}^3 \).

Consider a pair \((\gamma_0, \delta_0) \in \mathcal{O}_d^2 \) with \((\gamma_0, \delta_0) = \mathcal{O}_d \) satisfying Equation (4).
If \( \gamma_0 = 0 \) or \( \delta_0 = 0 \), then either \( \delta_0 \) or \( \gamma_0 \) must be a unit of \( \mathcal{O}_d \). It is then easy to find some \( \tau = \left( \begin{array}{c} \alpha_0 \\ \beta_0 \\ \gamma_0 \\ \delta_0 \end{array} \right) \) \in \text{PSL}(2, \mathcal{O}_d) \) with \( H(\tau) = 1 \) and \( \tau(z, t) \in B_d \). In this case we have the trivial inequality \( H(A) \leq D(z, t) \).

Suppose now that \( \gamma_0, \delta_0 \in \mathcal{O}_d \setminus \{0\} \). By Lemma 3.2, there exist a constant \( C_d \) and integers \( \alpha_0, \beta_0 \in \mathcal{O}_d \) such that \( \tau = \left( \begin{array}{c} \alpha_0 \\ \beta_0 \\ \gamma_0 \\ \delta_0 \end{array} \right) \) \in \text{PSL}(2, \mathcal{O}_d) \) satisfies \( |\alpha_0| \leq C_d |\gamma_0| \) and \( |\beta_0| \leq C_d |\delta_0| \), i.e.

\[
H(\tau) \leq C_d \max\{|\gamma_0|, |\delta_0|\}.
\]

By construction, \( \tau(z, t) = (z', t') \in B_d \) with

\[
t' = \frac{t}{|\gamma_0 z + \delta_0|^2 + |\gamma_0|^2 t^2} = \frac{1}{\mu'_d(z, t)} \geq t.
\]

Hence, \( t \mu'_d(z, t) \leq 1 \). Moreover, from the definition of \( D(z, t) \) and from Equation (4), we obtain

\[
H(\gamma_0) = |\gamma_0| \leq \sqrt{\frac{|\mu'_d(z, t)|}{t}} \leq \sqrt{\mu'_d(z, t) D(z, t)};
\]

\[
H(\delta_0) = |\delta_0| \leq |\gamma_0 z| + |\gamma_0 z + \delta_0| \leq |\gamma_0||z| + 1 \leq \left( \sqrt{\mu'_d(z, t) + 1} \right) D(z, t)^{\frac{3}{2}}.
\]

Therefore,

\[
H(\tau) \leq C_d \left( \sqrt{\mu'_d(z, t) + 1} \right) D(z, t)^{\frac{3}{2}}.
\]

We now estimate the norm of

\[
z' = \left( \frac{\alpha_0 z + \beta_0 (\gamma_0 z + \delta_0) + \alpha_0 \gamma_0 t^2}{|\gamma_0 z + \delta_0|^2 + |\gamma_0|^2 t^2} \right).
\]

We have \( |\alpha_0| \leq C_d |\gamma_0| \) and

\[
|\alpha_0 z + \beta_0| |\gamma_0 z + \delta_0| \leq |\alpha_0 \left( z + \frac{\delta_0}{\gamma_0} \right) - \frac{1}{\gamma_0} |\gamma_0 z + \delta_0| \leq |\alpha_0| \frac{|\gamma_0 z + \delta_0|^2}{|\gamma_0|} + \frac{|\gamma_0 z + \delta_0|}{|\gamma_0|}.
\]

Therefore,

\[
z' \leq \frac{|\alpha_0| |\gamma_0 z + \delta_0|^2}{|\gamma_0| |\gamma_0 z + \delta_0|^2 + t^2} + \frac{|\gamma_0 z + \delta_0|}{|\gamma_0 z + \delta_0|^2 + t^2} + \frac{|\alpha_0|}{|\gamma_0|} \leq 2C_d + \frac{1}{2t} \leq 3C_d D(z, t).
\]

If we repeat the argument of the proof of Lemma 3.2 we can show that there exists a constant \( L_d \leq c_2 \sqrt{d} \) (for some universal constant \( c_2 > 0 \)) such that for any \( w \in \mathbb{C} \), we can find \( \delta_w \in \text{PSL}(2, \mathcal{O}_d) \) with \( \delta_w(w) \in \mathcal{P}_d \) and

\[
H(\delta_w) \leq L_d |w|.
\]
In particular, if $\delta_{z'} \in \text{PSL}(2, \mathcal{O}_d)$ sends $(z', t')$ into $\mathcal{F}_d$, then $\delta_{z'} \tau(z, t) \in \mathcal{F}_d$. Therefore, the following inequality follows from (7), (5) and from the fact that $H(\delta_{z'} \tau) \leq 2H(\delta_{z'})H(\tau)$:

(8) $I_{\mathcal{F}_d}(z, t) \leq 6C_d^2 L_d \left( \sqrt{\mu_d'(z, t) + 1} \right) D(z, t)^{\frac{5}{2}}.$

In order to finish the proof, we give estimates for $\mu_d'(z, t)$ in terms of $d$ and $D(z, t)$. Let $(\gamma_1, \delta_1) \in \mathcal{O}_d^2 \setminus \{0\}$ be a solution of

$$|\gamma_1 z + \delta_1|^2 + |\gamma_1|^2 t^2 = \mu_d(z, t)t.$$

Since $\langle \gamma_1, 1 \rangle = \mathcal{O}_d$, by the definition of $\mu_d'(z, t)$ we have

$$\mu_d'(z, t) \leq |\gamma_1 z + 1|^2 + |\gamma_1|^2 t^2.$$

Note that

$$|\gamma_1 z + 1|^2 + |\gamma_1|^2 t^2 \leq t \left[ t^{-1} \max\{|\gamma_1|^2|z|^2, 1\} + |\gamma_1|^2 t \right].$$

From the definition of $\gamma_1$ and $D(z, t)$ we get

$$|\gamma_1|^2 t \leq \mu_d(z, t) \quad \text{and} \quad \max\{|\gamma_1|^2|z|^2, 1\} \leq (1 + \mu_d(z, t)) D(z, t)^3.$$

Hence

$$\mu_d'(z, t) \leq [4 + 5\mu_d(z, t)] D(z, t)^4.$$

By Proposition 3.4,

(9) $\mu_d'(z, t) \leq \left( \frac{4\pi + 20\sqrt{2} \text{vol}(\mathcal{C}/\mathcal{O}_d)}{\pi} \right) D(z, t)^4.$

From (8) and (9) we finally obtain

$$I_{\mathcal{F}_d}(z, t) \leq 6L_d C_d^2 \left( \sqrt{\left( \frac{4\pi + 20\sqrt{2} \text{vol}(\mathcal{C}/\mathcal{O}_d)}{\pi} \right) + 1} \right) D(z, t)^{\frac{9}{2}}.$$

If we define $c(d) = 6L_d C_d^2 \left( \sqrt{\left( \frac{4\pi + 20\sqrt{2} \text{vol}(\mathcal{C}/\mathcal{O}_d)}{\pi} \right) + 1} \right)$, then the theorem is proved for any $d$ with $m = \frac{9}{2}$.

We can improve this exponent if $h(d) = 1$. Indeed, if $h(d) = 1$, the ring $\mathcal{O}_d$ is a Unique Factorization Domain. Hence

$$\mu_d(z, t) = \mu_d'(z, t).$$

Therefore we can apply Proposition 3.4 directly in (8). Thus for $h(d) = 1$ we have

$$m = \frac{5}{2} \quad \text{and} \quad c(d) = 6L_d C_d^2 \left( \sqrt{\frac{4\sqrt{2} \text{vol}(\mathcal{C}/\mathcal{O}_d)}{\pi} + 1} \right).$$
Moreover, for any \( d \) we have
\[
C_d \leq c_1 d^{\frac{5}{2}}, \quad L_d \leq c_2 d^{\frac{7}{2}} \quad \text{and} \quad \text{vol}(\mathbb{C}/\mathcal{O}_d) \leq c_3 d^{\frac{3}{2}},
\]
for universal constants \( c_1, c_2, c_3 > 0 \). Hence, \( c(d) \leq C d^{\frac{5}{2}} \), where \( C > 0 \) is a universal constant. \( \square \)

**Remark 3.6.** The exponent is not necessarily sharp. However, we can see that \( m(d) \geq 2 \) by looking at the following example.

For every integer \( n \geq 2 \), we take the sequence of points \( (z_n, t_n) := \left(\frac{2n^2 - 1}{2n}, \frac{1}{2n}\right) \in \mathbb{H}^3 \).

It is straightforward to check that \( \sigma_n := \left[\begin{pmatrix} n \end{pmatrix} \right] \in \Gamma_d \) and satisfy
\[
\sigma_n (z_n, t_n) = (0, n) \in \mathcal{F}_d,
\]
for every \( d \). We have \( D(z_n, t_n) = 2n \) and \( H(\sigma_n) = n^2 - 1 \). Therefore
\[
\mathcal{I}_{\mathcal{F}_d}(z_n, t_n) \geq \frac{1}{2}(D(z_n, t_n))^2.
\]

**Remark 3.7.** If \( h(d) = 1 \), then there exists an optimal constant \( t_d > 0 \) such that the set \( \Sigma_d = \mathcal{P}_d \times [t_d, +\infty) \) contains the fundamental domain \( \mathcal{F}_d \) and
\[
\{ \gamma \in \Gamma_d | \gamma \mathcal{F}_d \cap \Sigma_d \neq \emptyset \} \subset \{ \gamma \in \Gamma_d | H(\gamma) \leq c(d)s(d)^{\frac{5}{2}} \},
\]
where \( s(d) \) is the maximum of the function \( D(z, t) \) in \( \Sigma_d \).

We observe that the latter set is finite and this proves that \( \Sigma_d \) is a fundamental set for \( \Gamma_d \). We show in Section 5 how to estimate the cardinality of this finite set.

4. On the reduction theory of binary Hermitian forms

We start this section with a brief description of binary Hermitian forms. We are interested in the Reduction of binary Hermitian forms over \( \mathcal{O}_d \). For a more complete description of this subject, the reader can refer to [1].

For any subring \( R \) of \( \mathbb{C} \), we can consider the set of Hermitian matrices
\[
H(R) = \{ A \in M_2(R); A^* = A \}.
\]

Each \( f = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \in H(R) \) defines a Hermitian form in \( \mathbb{C} \) by the following
\[
f(X, Z) = aX\overline{X} + 2\Re(bX\overline{Z}) + dZ\overline{Z},
\]
and the discriminant of \( f \) is defined by \( \Delta(f) = ad - |b|^2 \).

If \( f(X, Z) > 0 \) for any \( (X, Z) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\} \), then \( f \) is said to be positive definite. We observe that if \( a \neq 0 \), then \( f(X, Z) = a\left( |X + \frac{bZ}{a}|^2 + \frac{\Delta}{a^2}|Z|^2 \right) \). Hence we conclude that \( f \) is positive definite if and only if \( a > 0 \) and \( \Delta > 0 \).

Let \( H^+(R) \) denote the set of positive definite Hermitian forms and consider its quotient by the action of \( \mathbb{R}^{>0} \) by scalar multiplication, which we denote by \( \tilde{H}^+(R) = \)
The following map gives a bijection between $\tilde{H}^+(\mathbb{C})$ and the hyperbolic 3-space $\mathbb{H}^3$:

$$\xi : \tilde{H}^+(\mathbb{C}) \rightarrow \mathbb{H}^3, \quad f \mapsto \left( -\frac{b}{a} \sqrt{\Delta(f)}, \frac{-z}{|z|^2 + t^2} \right).$$

This map is a bijection, with inverse map given by

$$\xi^{-1}(z,t) = \left\{ \lambda \left( \frac{1}{-\bar{z}}, \frac{-z}{|z|^2 + t^2} \right) \mid \lambda \in \mathbb{R}^>0 \right\}$$

and it is also $\text{SL}(2, \mathbb{C})$-equivariant. In other words, if we define the action of $\text{SL}(2, \mathbb{C})$ on $H^+(\mathbb{C})$ by $\rho(g)f(X,Z) = f(g^{-1}(X,Z))$, then $\xi(\rho(g)f) = g\xi(f)$, for every $g \in \text{SL}(2, \mathbb{C})$ and every $f \in H^+(\mathbb{C})$.

For $R = \mathcal{O}_d$, we say that $f \in H^+(\mathcal{O}_d)$ is a reduced form if $\xi(f) \in F_d$, the fundamental domain for the Bianchi group $\Gamma_d$.

The height of a positive definite binary Hermitian form is given by

$$H(f) = \max\{a, |b|, d\}.$$ 

For any $\Delta > 0$ and subring $R \subset \mathbb{C}$ we define $H^+(R, \Delta) = \{f \in H^+(R) \mid \Delta(f) = \Delta\}$.

**Lemma 4.1.** Let $D = D(z,t)$ be the function defined in Section 3. If $f \in H^+(R, \Delta)$, then

$$D(\xi(f)) \leq \frac{H(f)}{\sqrt{\Delta}}.$$ 

**Proof.** Consider $f = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in H^+(R, \Delta)$ and $D(\xi(f)) = \max\{1, \frac{|b|}{a}, \frac{a}{\sqrt{\Delta}}\}$. We look at the following cases:

- **$H(f) = |b|$;**
  This case cannot happen, because then we would have $a \leq |b|$ and $d \leq |b|$, what would imply $ad \leq |b|^2$ and then $\Delta \leq 0$, which is a contradiction.

- **$H(f) = a;$**
  Then we would get $D(\xi(f)) = \max\{1, \frac{a}{\sqrt{\Delta}}\}$, as $\frac{|b|}{a} \leq 1$. From $d \leq a$, we obtain $ad \leq a^2 \leq a^2 + |b|^2 \Rightarrow \Delta \leq a^2$,
  and thus
  $$D(\xi(f)) = \frac{a}{\sqrt{\Delta}} = \frac{H(f)}{\sqrt{\Delta}}.$$

- **$H(f) = d;$**
  In this case we look separately at the following options:
  If $|b| \leq a \leq d$, we get that $D(\xi(f)) = \max\{1, \frac{a}{\sqrt{\Delta}}\}$. We have then $\frac{a}{\sqrt{\Delta}} \leq \frac{H(f)}{\sqrt{\Delta}}$. On the other hand, $d^2 \geq ad \geq ad - |b|^2 = \Delta$. Hence $1 \leq \frac{d}{\sqrt{\Delta}} = \frac{H(f)}{\sqrt{\Delta}}$ and we conclude that $D(\xi(f)) \leq \frac{H(f)}{\sqrt{\Delta}}$ as stated.
It remains to consider $a \leq |b| \leq d$, which is equivalent to showing the inequality
\[
\max\{\sqrt{\Delta}, \frac{|b|\sqrt{\Delta}}{a}\} \leq d.
\]
By our assumption, we have
\[
\sqrt{\Delta} \leq \sqrt{ad} \leq d \quad \text{and} \quad a \leq d.
\]
We finish the proof if we show that $|b|\sqrt{\Delta} < ad$. Note that the real quadratic function $f(x) = x^2 - |b|^2x + |b|^4$ is strictly positive. In particular,
\[
f(ad) = (ad)^2 - |b|^2(ad) + |b|^4 > 0 \Leftrightarrow |b|\sqrt{\Delta} < ad.
\]
\[\square\]
As a consequence of our main result and lemma above, we get the following

**Corollary 4.2.** Let $d$ be a squarefree integer. Given a positive integer $\Delta > 0$ and $f \in H^+(O_d, \Delta)$, there exists $g \in \text{SL}(2, O_d)$ such that $\rho(g)f$ is a reduced form and which satisfies
\[
H(g) \leq c(d) D(\xi(f))^{m(d)} \leq \frac{c(d)H(f)^{m(d)}}{\sqrt{\Delta}},
\]
where $c(d)$ and $m(d)$ are given in Theorem 3.1.

5. Counting matrices with bounded height in Bianchi groups.

In what follows, the relation $f(t) \precsim g(t)$ for two positive functions $f, g$ means that
\[
\limsup_{t \to \infty} \frac{f(t)}{g(t)} \leq 1.
\]
We write $f(t) \sim g(t)$ if $f(t) \precsim g(t)$ and $g(t) \precsim f(t)$.

Given a linear algebraic group $G$ defined over $\mathbb{Z}$, a natural problem is to investigate the asymptotics as $T \to \infty$ of
\[
\#\{A \in G(R) \mid H(A) \leq T\},
\]
where $R$ is the ring of integers of a number field. In [4] this problem is studied in a more general context but they consider only the ring $R = \mathbb{Z}$. As an application of their main result, the authors consider $G = \text{SL}_n$ [op. cit., Example 1.6] and give an explicit constant $c(n) > 0$ such that
\[
\#\{A \in \text{SL}(n, \mathbb{Z}) \mid H(A) \leq T\} \sim c(n)T^{n^2-n}.
\]
We are interested in this problem for $G = \text{SL}_2$ and $R$ being the ring of integers of an imaginary quadratic extension of $\mathbb{Q}$. In fact, we will count elements in $\text{PSL}_2$, this is not a matter since the number of matrices of given height in $\text{SL}(2, R)$ is twice the number of elements in $\text{PSL}(2, R)$ with the same height.

For any Bianchi group $\Gamma_d$ and positive number $T > 0$ we consider the set $W_d(T) = \{A \in \Gamma_d \mid H(A) \leq T\}$. We define the function
\[
N_d(T) = \#W_d(T).
\]
It is known that $N_d(T) < \infty$ for any $T > 0$. Now we want to show that this function grows polynomially with degree 4. Consider the subgroup $\text{PSL}(2, \mathcal{O}_d)_\infty < \Gamma_d$ and define

$$N_d^\infty(T) = \# \{ B \in \text{PSL}(2, \mathcal{O}_d)_\infty \mid H(B) \leq T \} \quad \text{and} \quad N_d'(T) = \# \{ \alpha \in \mathcal{O}_d \mid H(\alpha) \leq T \}.$$ 

It follows from the definition of $\text{PSL}(2, \mathcal{O}_d)_\infty$ that

$$N_d^\infty(T) = \#(\mathcal{O}_d^*)N_d'(T) \quad \text{for all} \quad T > 0.$$ 

It is well known (see [8, Proposition 2.3] for example) that for any lattice $\Gamma < \mathbb{R}^n$ and for any norm $\| \cdot \|$ on $\mathbb{R}^n$, we have

$$\# \{ \gamma \in \Gamma \mid \|\gamma\| \leq T \} \sim cT^n,$$

for some constant $c > 0$ which depends only on $\Gamma$ and the norm $\| \cdot \|$.

All this information together prove the following lemma.

**Lemma 5.1.** Let $d > 0$ be a squarefree integer. There exists a constant $\sigma(d) > 0$ such that

$$N_d^\infty(T) \sim \sigma(d)T^2.$$ 

Before proving the main result of this section, we recall a classical result of Schanuel [15]. Let $\mathcal{I}_d$ be the ideal class group of $K_d$. The class $[P]$ of a point $P = (x : y) \in \mathbb{P}^1K_d$ is the class of the fractional ideal $\langle x, y \rangle$ in $\mathcal{I}_d$. Up to normalization, the definition of the height of $P$ is given by

$$H(P) := \frac{\max\{|x|, |y|\}}{\sqrt{N(\langle x, y \rangle)}}.$$ 

Let

$$X_d(T) = \{ P \in \mathbb{P}^1K_d \mid [P] = [(1 : 0)] \text{ and } H(P) \leq T \}.$$ 

It follows from [15, Theorem 1] that the cardinality of $X_d(T)$ has the following growth:

$$\#X_d(T) \sim \tau(d)T^4,$$

for some constant $\tau(d) > 0$ which depends only on $d$.

Consider the map $\phi_d : \Gamma_d \to \mathbb{P}^1K_d$ given by

$$\phi_d \left[ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] = (\alpha : \gamma).$$ 

Note that $[\phi_d(A)] = [(1 : 0)]$ for any $A \in \Gamma_d$ since $\langle \alpha, \gamma \rangle = \mathcal{O}_d$. If we consider the action of $\Gamma_d$ on $\mathbb{P}^1K_d$, then $\phi_d(A) = A(1 : 0)$. Hence, $\phi_d(A) = \phi_d(B)$ if, and only if, $B^{-1}A \in \text{PSL}(2, \mathcal{O}_d)_\infty$. Moreover, the map $\phi_d$ satisfies

$$H(\phi_d(A)) \leq H(A) \text{ for all } A \in \Gamma_d.$$
Theorem 5.2. Let $d > 0$ be a squarefree integer. There exist constants $0 < l_1(d) \leq l_2(d)$ such that

$$l_1(d)T^4 \preceq N_d(T) \preceq l_2(d)T^4,$$

for all sufficiently large $T$.

Proof. We fix an arbitrary squarefree integer $d > 0$.

Let $U_d = \{ P \in \mathbb{P}K_d | [P] = [(1 : 0)] \}$. We have already observed that the image of $\phi_d$ is in $U_d$. Now we define a right inverse for $\phi_d$. For each $P \in U_d$ there exist $\alpha, \gamma \in O_d$ such that $\langle \alpha, \gamma \rangle = O_d$ and $P = (\alpha : \gamma)$. Indeed, if $P = (p : q : r : s)$, then $
abla = \frac{p}{q}s, \gamma' = \frac{r}{s}v, \xi = \frac{u}{s}q$ we get $P = (\alpha' : \beta')$ and $\langle \alpha', \gamma' \rangle = \xi O_d$. Finally, by writting $\alpha = \frac{\alpha'}{\xi}$ and $\gamma = \frac{\gamma'}{\xi}$, we get that $\alpha, \gamma \in O_d$, $P = (\alpha : \gamma)$ and $\langle \alpha, \gamma \rangle = O_d$.

Consider $P = (\alpha : \gamma) \in U_d$. If $\gamma = 0$ we define $\psi_d(P) = I \in \Gamma_d$, if $\alpha = 0$ we define $\psi_d(P) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \Gamma_d$. If $\alpha, \gamma \in O_d \{ 0 \}$ we can use Lemma 3.2 in order to obtain a constant $C_d \geq 1$ and elements $\beta, \delta \in O_d$ with $|\beta| \leq C_d|\alpha|, |\delta| \leq C_d|\gamma|$ and such that $\psi_d(P) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in PSL(2,O_d)$.

Therefore,

$$\phi_d(\psi_d(P)) = P \quad \text{for all} \quad P \in U_d.$$ $$H(\psi_d(P)) \leq C_d H(P) \quad \text{for all} \quad P \in U_d.$$

Hence, for any $T > 0$ the restriction $\psi_d^T$ of $\psi_d$ to $X_d \left( \frac{T}{C_d} \right)$ is an injective map whose image is contained in $W_d(T)$. Thus,

$$\#X_d \left( \frac{T}{C_d} \right) \leq N_d(T), \quad \text{for all} \quad T > 0.$$ 

By (10), if we define $l_1(d) = \frac{\tau(d)}{C_d^4}$, then

$$l_1(d)T^4 \preceq N_d(T).$$

On the other hand, for each $T > 0$ consider the restriction $\phi_d^T$ of $\phi_d$ to $W_d(T)$. Since $\phi_d(W_d(T)) \subset X_d(T)$ we have

$$N_d(T) = \#W_d(T) = \sum_{P \in X_d(T)} \#(\phi_d)^{-1}(P).$$
Let $P \in X_d(T)$ and suppose that there exists $A_P \in W_d(T)$ with $\phi_d(A_P) = P$, then we have an injection
\[
(\phi_d^T)^{-1}(P) \to \{B \in \text{PSL}(2,\mathcal{O}_d)_{\infty} \mid H(B) \leq 2T^2\}
\]
given by $A \mapsto A^{-1}A_P$.

Indeed, we are using the facts that $\phi_d(A) = \phi_d(A_P)$ if and only if $A^{-1}A_P \in \text{PSL}(2,\mathcal{O}_d)_{\infty}$ and that $H(A^{-1}A_P) \leq 2T^2$.

Therefore, by Lemma 5.1 and by (10) we have
\[
N_d(T) \lesssim (4\sigma(d) + \tau(d))T^4.
\]

Hence, if we let $l_2(d) = 4\sigma(d) + \tau(d)$, then
\[
N_d(T) \lesssim l_2(d)T^4.
\]

\[\square\]

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References

[1] L. Beshaj, Reduction theory of binary forms, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur. 41 (2015), 84–116.
[2] L. Bianchi, Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginar, Mathematische Annalen 40 (1892), 332–412.
[3] A. Borel, Commensurability classes and volumes of hyperbolic 3-manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 8 (1981), 1–33.
[4] W. Duke, Z. Rudinick, and P. Sarnak, Density of Integer Points on Affine Homogeneous Varieties, Duke Math. J. 71 (1993), 143–179.
[5] J. Elstrodt, F. Grunewald, and J. Mennicke, Groups acting on Hyperbolic Space - Harmonic Analysis and Number Theory, Springer Verlag, 1998.
[6] J. Elstrodt, F. Grunewald, and J. Mennicke, Some remarks on discrete subgroups of PSL(2, C), J. Soviet. Math. 46 (1989), 1760–1788.
[7] P. Habegger and J. Pila, Some unlikely intersections beyond André-Oort, Compos. Math. 148 (2012), 1–27.
[8] S. Lang, Survey of Diophantine Geometry. Springer, 1997.
[9] A. Oppenheim, The minima of positive definite Hermitian binary quadratic forms, Mathematische Zeitschrift 38 (1934), 538–545.
[10] C. Maclachlan and A. W. Reid, The arithmetic of hyperbolic 3-manifolds, Graduate Texts in Mathematics, 219, Springer, New York, 2003.
[11] D. W. Morris, Introduction to arithmetic groups, Deductive Press, 2015.
[12] M. Orr, Height bounds and the Siegel property, Algebra Number Theory 2 (2018), 455–478.
[13] G. T. Paula, Comparison of Volumes of Siegel Sets and Fundamental Domains for SL_n(Z), Geometriae Dedicata 199 (2019), 291–306.
[14] G. T. Paula, On geometry of Siegel set for lattices in SL_n, Ph.D thesis, Instituto Nacional de Matemática Pura e Aplicada (2018).
[15] S. Schanuel, On heights in number fields, *Bull. Amer. Math. Soc.* 70 (1964), 262–263.

[16] J. Pila, O-minimality and the André-Oort conjecture for $\mathbb{C}^n$, *Ann. of Math. (2)* 173 (2011), 1779–1840.

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