EMBEDDINGS OF GRAPH INVERSE SEMIGROUPS INTO COMPACT-LIKE TOPOLOGICAL SEMIGROUPS

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Abstract. In this paper we investigate graph inverse semigroups which are subsemigroups of compact-like topological semigroups. More precisely, we characterise graph inverse semigroups which admit compact semigroup topology and describe graph inverse semigroups which can be embedded densely into d-compact topological semigroups.

1. Preliminaries

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [16, 19, 29]. By \( \omega \) we denote the first infinite ordinal. Put \( \mathbb{N} = \omega \setminus \{0\} \). The cardinality of a set \( X \) is denoted by \( |X| \). A semigroup \( S \) is called an inverse semigroup if for each element \( a \in S \) there exists a unique element \( a^{-1} \in S \) such that \( aa^{-1}a = a \) and \( a^{-1}aa^{-1} = a^{-1} \).

The map which associates every element of an inverse semigroup to its inverse is called an inversion.

For a subset \( A \) of a topological space \( X \) by \( \overline{A} \) we denote the closure of the set \( A \) in \( X \). The interior of a subset \( A \) of a topological space \( X \) is denoted by \( \text{int}(A) \). For each non-zero cardinal \( k \) by \( d_k \) we denote the set \( k \) endowed with a discrete topology. Topological space \( X \) is said to be:

- compact, if each open cover of \( X \) contains a finite subcover;
- countably compact, if each countable open cover of \( X \) contains a finite subcover;
- countably compact at a subset \( A \subset X \), if each infinite subset of \( B \subset A \) has an accumulation point in \( X \);
- countably pracompact, if there exists a dense subset \( A \) of \( X \) such that \( X \) is countably compact at \( A \);
- feebly compact, if each locally finite open cover of \( X \) is finite;
- pseudocompact, if \( X \) is Tychonoff and each continuous real-valued function on \( X \) is bounded;
- \( d \)-compact, if for each continuous map \( f : X \to d_\omega \) the image \( f(X) \) is finite.

By [23], for a topological space \( X \) the following implications hold: \( X \) is compact \( \Rightarrow \) \( X \) is countably compact \( \Rightarrow \) \( X \) is countably pracompact \( \Rightarrow \) \( X \) is feebly compact \( \Rightarrow \) \( X \) is \( d \)-compact. Also, a feebly compact topological space \( X \) is pseudocompact iff \( X \) is Tychonoff.

A topological (inverse) semigroup is a Hausdorff topological space together with a continuous semigroup operation (and an inversion, respectively). If \( S \) is a semigroup (an inverse semigroup) and \( \tau \) is a topology on \( S \) such that \( (S, \tau) \) is a topological (inverse) semigroup, then we shall call \( \tau \) a (inverse) semigroup topology on \( S \). A semitopological semigroup is a Hausdorff topological space together with a separately continuous semigroup operation.

Let \( X \) be a non-empty set. By \( B_X \) we denote the set \( X \times X \cup \{0\} \) where \( 0 \not\in X \times X \) endowed with the following semigroup operation:

\[
(a, b) \cdot (c, d) = \begin{cases} 
(a, d), & \text{if } b = c; \\
0, & \text{if } b \neq c,
\end{cases}
\]

and \( (a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0 \), for each \( a, b, c, d \in X \).

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The semigroup $B_X$ is called the semigroup of $X \times X$-matrix units. Observe that semigroups $B_X$ and $B_Y$ are isomorphic iff $|X| = |Y|$.

If a set $X$ is infinite then the semigroup of $X \times X$-matrix units can not be embedded into a compact topological semigroup (see [21, Theorem 3]). In [22, Theorem 5] the above result was extended over the class of countably compact topological semigroups. In [13, Theorem 4.4] it was showed that for an infinite set $X$ the semigroup $B_X$ can not be embedded as a dense subsemigroup into a feebly compact topological semigroup.

A bicyclic monoid $C(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subject to the condition $pq = 1$. Topologization of the bicyclic semigroup was investigated in [18] and [20]. Neither stable nor $\Gamma$-compact topological semigroups can contain a copy of the bicyclic monoid (see [3, 25]). In [24] it was proved that the bicyclic monoid does not embed into a countably compact topological inverse semigroup. A topological semigroup which has a pseudocompact square can not contain the bicyclic monoid [6]. However, in [6, Theorem 6.6] it was proved that under Martin’s Axiom there exists a Tychonoff countably compact semigroup $S$ containing a bicyclic monoid. The existence of a ZFC-example of a countably compact topological semigroup which contains the bicyclic monoid is still an open problem (see [6, Problem 7.1]).

One of the generalizations of the bicyclic monoid is a polycyclic monoid. For a cardinal $\lambda$ a polycyclic monoid $P_{\lambda}$ is the semigroup with identity 1 and zero 0 given by the presentation:

$$P_{\lambda} = \langle 0, 1, \{p_i \}_{i \in \lambda}, \{p_i^{-1} \}_{i \in \lambda} | p_i^{-1}p_i = 1, p_j^{-1}p_i = 0 \text{ for } i \neq j \rangle.$$

Polycyclic monoid $P_k$ over a finite non-zero cardinal $k$ was introduced in [33]. Observe that the bicyclic semigroup with adjoined zero is isomorphic to the polycyclic monoid $P_1$ and the polycyclic monoid $P_0$ is isomorphic to the semilattice $\langle \{0, 1\}, \min \rangle$. An embedding of the polycyclic monoid into compact-like topological semigroups was investigated in [14]. More precisely, it was proved that for each cardinal $\lambda > 1$ polycyclic monoid $P_{\lambda}$ does not embed as a dense subsemigroup into a feebly compact topological semigroup.

A directed graph $E = (E^0, E^1, r, s)$ consists of sets $E^0, E^1$ of vertices and edges, respectively, together with functions $s, r : E^1 \to E^0$ which are called source and range, respectively. In this paper we refer to directed graph simply as “graph”. We consider each vertex being a path of length zero. A path of a non-zero length $x = e_1 \ldots e_n$ in a graph $E$ is a finite sequence of edges $e_1, \ldots, e_n$ such that $r(e_i) = s(e_{i+1})$ for each positive integer $i < n$. We extend functions $s$ and $r$ on the set $\text{Path}(E)$ of all pathes in graph $E$ as follows: for each vertex $e \in E^0$ put $s(e) = r(e) = e$ and for each path of a non-zero length $x = e_1 \ldots e_n \in \text{Path}(E)$ put $s(x) = s(e_1)$ and $r(x) = r(e_n)$. By $|x|$ we denote the length of a path $x$. Let $a = e_1 \ldots e_n$ and $b = f_1 \ldots f_m$ be two pathes such that $r(a) = s(b)$. Then by $ab$ we denote the path $e_1 \ldots e_n f_1 \ldots f_m$. A path $x$ is called a prefix of a path $y$ if there exists a path $z$ such that $y = zx$. An edge $e$ is called a loop if $s(e) = r(e)$. A path $x$ is called a cycle if $s(x) = r(x)$ and $|x| > 0$. A graph $E$ is called cyclic if it contains no cycles.

For a given directed graph $E = (E^0, E^1, r, s)$ a graph inverse semigroup (or simply GIS) $G(E)$ over a graph $E$ is a semigroup with zero generated by the sets $E^0, E^1$ together with a set $E^{-1} = \{e^{-1} | e \in E^1\}$ satisfying the following relations for all $a, b \in E^0$ and $e, f \in E^1$:

(i) $a \cdot b = a$ if $a = b$ and $a \cdot b = 0$ if $a \neq b$;

(ii) $s(e) \cdot e = e \cdot r(e) = e$;

(iii) $e^{-1} \cdot s(e) = r(e) \cdot e^{-1} = e^{-1}$;

(iv) $e^{-1} \cdot f = r(e)$ if $e = f$ and $e^{-1} \cdot f = 0$ if $e \neq f$.

Graph inverse semigroups are a generalization of the polycyclic monoids. In particular, for every cardinal $\lambda$ polycyclic monoid $P_{\lambda}$ is isomorphic to the graph inverse semigroup over the graph which consists of one vertex and $\lambda$ distinct loops. However, by [10, Theorem 1], each graph inverse semigroup $G(E)$ is isomorphic to a subsemigroup of the polycyclic monoid $P_{G(E)}$.

According to [26, Chapter 3.1], each non-zero element of a graph inverse semigroup $G(E)$ is of the form $uv^{-1}$ where $u, v \in \text{Path}(E)$ and $r(u) = r(v)$. A semigroup operation in $G(E)$ is defined by the
Let $u_1v_1^{-1} \cdot u_2v_2^{-1} = \begin{cases} u_1wv_2^{-1}, & \text{if } u_2 = v_1w \text{ for some } w \in \text{Path}(E); \\ u_1(v_2w)^{-1}, & \text{if } v_1 = u_2w \text{ for some } w \in \text{Path}(E); \\ 0, & \text{otherwise}, \end{cases}

\text{and } uv^{-1} \cdot 0 = 0 \cdot uv^{-1} = 0 \cdot 0 = 0.

Simple verifications show that $G(E)$ is an inverse semigroup and $(uv^{-1})^{-1} = vu^{-1}$.

Graph inverse semigroups play an important role in the study of rings and $C^*$-algebras (see [11, 17, 28, 33]). Algebraic theory of graph inverse semigroups is well developed (see [2, 8, 10, 26, 27, 30, 32]). Topological properties of graph inverse semigroups were investigated in [9, 11, 12, 13, 33].

In this paper we investigate graph inverse semigroups which are subsemigroups of compact-like topological semigroups. More precisely, we characterise graph inverse semigroups which admit compact semigroup topology and describe graph inverse semigroups which can be embedded densely into d-compact topological semigroups.

2. Graph inverse semigroups which admit compact semigroup topology

**Lemma 1.** Let $X$ be a d-compact space with a dense discrete subspace $Y$. Then for each subset $Z \subseteq Y$ the set $\overline{Z}$ is countably compact at $Z$. Moreover, if $\overline{Z} \setminus Z$ is compact then $\overline{Z}$ is compact.

**Proof.** Let $Y$ be a dense discrete subset of $X$ and $Z \subseteq Y$. To obtain the contradiction assume that there exists an infinite subset $A \subseteq Z$ which is closed in $\overline{Z}$. Observe that the set $A$ is closed in $X$. Fix an arbitrary surjection $f : A \rightarrow \omega$. Define the map $h : X \rightarrow \omega$ as follows: $h(x) = f(x)$ for each $x \in A$ and $h(X \setminus A) = 0$. Since the set $A$ is discrete and closed in $X$, the map $h$ is continuous which contradicts d-compactness of the space $X$. Hence each infinite subset of $Z$ has an accumulation point in $\overline{Z}$ which implies that $\overline{Z}$ is countably compact at $Z$.

Assume that the set $\overline{Z} \setminus Z$ is compact. Since the set $\overline{Z}$ is countably compact at the set $Z$ we obtain that for each infinite subset $A$ of $Z$ there exists an accumulation point $a \in \overline{Z} \setminus Z$. Let $F$ be an open cover of $\overline{Z}$. Since the set $\overline{Z} \setminus Z$ is compact there exists a finite subset $\{F_1, \ldots, F_n\} \subseteq F$ such that $\overline{Z} \setminus Z \subset \bigcup_{i=1}^{n} F_i \in F$. We claim that the set $A = \overline{Z} \setminus (\bigcup_{i=1}^{n} F_i)$ is finite. Indeed, if the set $A$ is infinite then, it has an accumulation point $a \in \overline{Z} \setminus Z$. However, the set $\bigcup_{i=1}^{n} F_i$ is an open neighbourhood of $a$ which does not intersect the set $A$. The obtained contradiction implies that the set $A$ is finite. Put $A = \{y_1, \ldots, y_m\}$. Then $G = \{F_1, \ldots, F_n, y_1, \ldots, y_m\}$ is a finite subcover of $F$. Hence the space $\overline{Z}$ is compact.

**Lemma 2.** Let $Y$ be an open and closed subset in a d-compact space $X$. Then the space $Y$ is d-compact.

**Proof.** To obtain a contradiction assume that $Y$ is not d-compact. Then there exists a continuous surjection $f : Y \rightarrow \omega$. Define a map $h : X \rightarrow \omega$ as follows: $h(y) = f(y)$ if $y \in Y$ and $h(X \setminus Y) = 0$. Obviously, $h$ is continuous and surjective which contradicts d-compactness of the space $X$.

Let $G(E)$ be an arbitrary semitopological GIS. Observe that each non-zero element of $G(E)$ is isolated (see [12, Theorem 4]). For an arbitrary GIS $G(E)$ by $\tau_c$, we denote the topology on $G(E)$ which is defined as follows: each non-zero element is isolated in $(G(E), \tau_c)$ and open neighbourhood base of the point 0 consists of cofinite subsets of $G(E)$ which contain 0. According to [12, Lemma 3], $\tau_c$ is the unique topology which makes $G(E)$ a compact semitopological semigroup. Lemma 1 implies the following:

**Corollary 3.** For an arbitrary semitopological GIS $G(E)$ the following conditions are equivalent:

1. $G(E)$ is compact;
2. $G(E)$ is countably compact;
3. $G(E)$ is feebly compact;
4. $G(E)$ is d-compact;
5. $G(E)$ is topologically isomorphic to the $(G(E), \tau_c)$. 
By Corollary 2, two non-zero elements $ab^{-1}$ and $cd^{-1}$ of a GIS $G(E)$ are $D$-equivalent iff $r(a) = r(b) = r(c) = r(d)$. Observe that each non-zero $D$-class contains exactly one vertex. By $D_e$ we denote the $D$-class which contains vertex $e$. The following theorem characterises graph inverse semigroups which admit compact (inverse) semigroup topology.

**Theorem 4.** Let $G(E)$ be a semitopological GIS. Then the following statements are equivalent:

1. the semigroup operation is jointly continuous in $(G(E), \tau_e)$;
2. for each element $uv^{-1} \in G(E) \setminus \{0\}$ the set $M_{uv^{-1}} = \{(ab^{-1}, cd^{-1}) \in G(E) \times G(E) \mid ab^{-1} \cdot cd^{-1} = uv^{-1}\}$ is finite;
3. for each vertex $e$ the set $I_e = \{u \in \text{Path}(E) \mid r(u) = e\}$ is finite;
4. $G(E)$ neither contains the bicyclic monoid nor an infinite semigroup of $X \times X$-matrix units;
5. each $D$-class in $G(E)$ is finite.

**Proof.** (1) ⇒ (2). Suppose that $(G(E), \tau_e)$ is a topological semigroup and fix an arbitrary non-zero element $uv^{-1} \in G(E)$. Observe that, by Lemma 1, for a fixed non-zero element $ab^{-1} \in G(E)$ the subsets

$$C_{ab^{-1}} = \{(ab^{-1}, xy^{-1}) \mid ab^{-1} \cdot xy^{-1} = uv^{-1}\} \subset G(E) \times G(E)$$

and

$$D_{ab^{-1}} = \{(xy^{-1}, ab^{-1}) \mid xy^{-1} \cdot ab^{-1} = uv^{-1}\} \subset G(E) \times G(E)$$

are finite. The continuity of the multiplication in $(G(E), \tau_e)$ yields an open neighbourhood $V = G(E) \setminus \{x_1, \ldots, x_n\}$ of $0$ such that $V \cdot V \subseteq G(E) \setminus \{uv^{-1}\}$. Observe that $M_{uv^{-1}} = (\bigcup_{i=1}^{n} (C_{x_i} \cup D_{x_i})) \cup (M_{uv^{-1}} \cap V \cdot V)$. Since $uv^{-1} \notin V \cdot V$ we obtain that $M_{uv^{-1}} \cap (V \cdot V) = \emptyset$. Since the sets $C_{x_i}$ and $D_{x_i}$ are finite the set $M_{uv^{-1}}$ is finite as well.

(2) ⇒ (3). Suppose that there exists a vertex $e \in E^0$ such that the set $I_e = \{x \in \text{Path}(E) \mid r(x) = e\}$ is infinite. Then $\{(x_1, x) \mid x \in I_e\}$ is an infinite subset of $M_e$ which contradicts condition (2).

(3) ⇒ (4). Suppose that $G(E)$ contains an isomorphic copy of the bicyclic monoid. Then there exists an element $uv^{-1} \in G(E)$ such that $uv^{-1} \cdot uv^{-1} \neq 0$. The definition of the semigroup operation in $G(E)$ implies that either $v = uv$ or $u = vw$. Then in both cases $s(w) = r(u) = r(v)$ and $r(w) = r(u) = r(v)$ which implies that $w$ is a cycle. Then the set $I_{r(w)} = \{u \in \text{Path}(E) \mid r(u) = r(w)\}$ is infinite, because it contains the set $\{w^n \mid n \in \mathbb{N}\}$, which contradicts condition (3).

Suppose that $G(E)$ contains an infinite semigroup of $X \times X$-matrix units. Observe that for each non-zero idempotent $e \in B_X$ the set $\{x \in B_X \mid x \cdot x^{-1} = e\}$ is infinite. Then there exists an idempotent $vv^{-1} \in G(E)$ and an infinite subset $A = \{a_i b_i^{-1} \mid i \in \omega\}$ of $G(E) \setminus \{0\}$ such that $a_i b_i^{-1} \cdot (a_i b_i^{-1})^{-1} = vv^{-1}$ for each $i \in \omega$. The above equation implies that $a_i = v$ for each $i \in \omega$. Since the set $A$ is infinite we obtain that the set $B = \{b_i \mid i \in \omega\}$ is infinite. Observe that $r(b_i) = r(a_i) = r(v)$ for each $i \in \omega$. Then the set $I_{r(v)} = \{u \in \text{Path}(E) \mid r(u) = r(v)\}$ is infinite, because it contains the set $B$, which contradicts condition (3).

(4) ⇒ (5). Suppose that there exists a vertex $e$ such that the $D$-class $D_e$ is finite. Then one of the following two cases holds:

(i) there exists a path $u$ such that $s(u) = r(u) = e$;
(ii) The set $I_e$ contains no cycles.

Consider case (i). Fix an arbitrary cycle $u \in D_e$. By $S$ we denote a subsemigroup of $G(E)$ which is generated by two elements $u$ and $u^{-1}$. Routine verifications show that the semigroup $S$ is isomorphic to the bicyclic monoid ($e = u^{-1}u$ is the identity of $S$) which contradicts condition (4).

Consider case (ii). By Corollary 5, $D_e \cup \{0\}$ is a subsemigroup of $G(E)$ which is isomorphic to the semigroup of $I_e \times I_e$-matrix units which contradicts condition (4).

(5) ⇒ (1). Suppose that each $D$-class of a GIS $G(E)$ is finite. By Lemma 3, $(G(E), \tau_e)$ is a semitopological semigroup. Since each non-zero element of $(G(E), \tau_e)$ is isolated the semigroup operation in $(G(E), \tau_e)$ is continuous if it is continuous at the point $(0,0) \in G(E) \times G(E)$. Fix an arbitrary open neighbourhood $U$ of $0$. Let $G(E) \setminus U = \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1}\}$. Put $V = G(E) \setminus (\bigcup_{i=1}^{n} D_{r(u_i)})$. Since each $D$-class is finite $V$ is an open neighbourhood of $0$. The inclusion $V \cdot V \subseteq U$ follows from
Lemma 1] which states that \((D_e \cup \{0\}) \cdot (D_f \cup \{0\}) \subseteq D_e \cup D_f \cup \{0\}\). Hence \((G(E), \tau_e)\) is a topological semigroup.

\[\square\]

3. Main theorem

For an arbitrary semigroup \(S\) by \(E(S)\) we denote the set of all idempotents of \(S\). By \(E_1 \sqcup E_2\) we denote the disjoint union of graphs \(E_1\) and \(E_2\). Recall that for a fixed vertex \(e \in E^0\), \(I_e = \{u \in Path(E) \mid r(u) = e\}\). Now we are going to formulate the main theorem of this paper.

**Theorem 5.** Let a graph inverse semigroup \(G(E)\) be a dense subsemigroup of a \(d\)-compact topological semigroup \(S\). Then the following statements hold:

1. there exists cardinal \(k\) such that \(E = (\sqcup_{\alpha \in k} E_\alpha) \sqcup F\) where graph \(F\) is acyclic, for each vertex \(f \in F^0\) the set \(I_f\) is finite and for each \(\alpha \in k\) the graph \(E_\alpha\) consists of one vertex and one loop;
2. if the graph \(F\) is non-empty, then the semigroup \(G(F)\) is a compact subset of \(G(E)\);
3. each open neighbourhood \(0\) contains all but finitely many subsets \(G(E_\alpha) \subseteq G(E_\alpha), \alpha \in k\).

The proof of Theorem 5 could be found in Section 5 after some preparatory work made in Section 4.

4. Embeddings of graph inverse semigroups over acyclic graphs into compact-like topological semigroups

**Lemma 6.** Let \(E\) be an acyclic graph and \(G(E)\) be a dense subsemigroup of a topological semigroup \(S\). Then \(s^2 = 0\) for each non-idempotent element \(s \in S\).

**Proof.** By [3] Corollary 5, for each \(e \in E^0\) the set \(D_e \cup \{0\}\) is isomorphic to the semigroup of \(I_e \times I_e\)-matrix units. Hence \(x \cdot x = 0\) for each non-idempotent element \(x \in G(E)\). Fix an arbitrary element \(s \in S \setminus E(S)\). Since \(E(S)\) is closed in \(S\) and \(G(E)\) is dense in \(S\) we obtain that \(V \cap (G(E) \setminus E(S)) \neq \emptyset\) for each open neighbourhood \(V\) of \(s\). Hence \(0 \in V^2\) for each open neighbourhood \(V\) of \(s\) which provides that \(s^2 = 0\). \(\square\)

**Lemma 7.** Let \(E\) be an acyclic graph and \(G(E)\) be a topological GIS such that \(E(G(E))\) is compact. Then \(G(E)\) is closed in each topological semigroup \(S\) which contains \(G(E)\) as a subsemigroup.

**Proof.** Assuming the contrary let \(S\) be a topological semigroup which contains \(G(E)\) as a dense proper subsemigroup. Fix an arbitrary \(s \in S \setminus G(E)\) and disjoint open neighbourhoods \(U(s)\) and \(U(0)\) of \(s\) and 0, respectively. Since \(S\) is a topological semigroup there exist open neighbourhoods \(V(s)\) and \(V(0)\) of \(s\) and 0, respectively, such that \(V(s) \subseteq U(s), V(0) \subseteq U(0)\) and \(V(s) \cdot V(0) \cup V(0) \cdot V(s) \subseteq U(0)\). Observe that \(G(S)\) is combinatorial (each \(H\)-class is singleton) and the set \(V(s) \cap G(E)\) is infinite (for the description of Green’s relations on graph inverse semigroups see [22 Corollary 2]). Hence the set

\[A = \{uu^{-1} \mid uw^{-1} \in V(s) \cap G(E)\} \text{ or } vu^{-1} \in V(s) \cap G(E)\}\]

is infinite. Since the set \(E(G(E)) \setminus V(0)\) is finite there exist elements \(u, v \in Path(E)\) such that \(uu^{-1} \in V(0)\) and either \(uw^{-1} \in V(s)\) or \(vu^{-1} \in V(s)\). Hence either \(uu^{-1} = uu^{-1} \cdot uw^{-1} \in V(0) \cdot V(s) \subseteq U(0)\) or \(vu^{-1} = vu^{-1} \cdot uu^{-1} \in V(s) \cdot V(0) \subseteq U(0)\) which contradicts to the choice of the sets \(U(0)\) and \(U(s)\). \(\square\)

**Lemma 8.** Let \(E\) be an acyclic graph and \(G(E)\) be a dense subsemigroup of a \(d\)-compact topological semigroup \(S\). Then the following statements hold:

1. the set \(\overline{E(G(E))} \setminus \{0\}\) is open in \(S\);
2. the set \(\overline{E(G(E))}\) is countably compact at \(E(G(E))\).

**Proof.** Fix an arbitrary non-zero element \(x \in \overline{E(G(E))} \setminus \{0\}\). If \(x \in E(G(E))\) then \(x\) is isolated in \(S\) and hence \(x \in \text{int}(E(G(E)))\). Suppose that \(x \in \overline{E(G(E))} \setminus E(G(E))\) and each open neighbourhood \(U\) of \(x\) contains some element \(y \in S \setminus \overline{E(G(E))}\). Since \(G(E)\) is dense in \(S\) we obtain that each open neighbourhood \(U\) of \(x\) contains some element \(z \in G(E) \setminus \overline{E(G(E))}\). By Lemma 6 \(z \cdot z = 0\). Then \(0 \in V^2\) for each open neighbourhood \(V\) of \(x\). Hence \(x \cdot x = 0\) which contradicts to the choice of \(x\).
Consider statement (2). Observe that the set \( E(G(E)) \) is open and closed in \( S \). By Lemma 2 the set \( E(G(E)) \) is d-compact. Lemma 1 implies that the set \( E(G(E)) \) is countably compact at \( E(G(E)) \). 

Further we shall need properties of the semigroup \( G(T) \) where by \( T \) we denote the unary tree (see picture below).

![Unary Tree Diagram]

We enumerate vertices of graph \( E \) with non-negative integers and identify each edge \( x \) with a pair of integers \( (n, n + 1) \) where \( s(x) = n \) and \( r(x) = n + 1 \). For each positive integers \( k \leq p \) by \( (k, p) \) we denote the path \( u \) such that \( s(u) = k \) and \( r(u) = p \) (we identify vertex \( n \) with the pair \((n, n)\)). Observe that \((n, m)(n, m)^{-1} \leq (k, l)(k, l)^{-1} \) iff \( n = k \) and \( m \geq l \). Simply verifications show that each maximal chain in \( E(G(T)) \) coincides with the set \( L_n = \{(n, m)(n, m)^{-1} \mid m \geq n\} \cup \{0\} \) for some fixed \( n \in \omega \). By \( L_n \) we denote the set \( L_n \backslash \{0\} \).

**Lemma 9.** Let \( G(E) \) be a GIS over an acyclic graph, and \( C \subset E(G(E)) \) be an infinite chain. Then there exists a subgraph \( T \subset E \) which is isomorphic to the unary three and \( L \subset E(G(T)) \).

**Proof.** Using the axiom of choice we can find a maximal chain \( L \) which contains \( C \). Easy to see that \( L = \{u_n u_n^{-1}\} \) where \( u_0 \) is some vertex \( e_0 \) and for each \( n \in \omega \) there exists an edge \( x_n \in E^1 \) such that \( u_{n+1} = u_n x_n \). Since graph \( E \) is acyclic for each \( n > 0 \) neither \( u_n \) nor \( x_n \) is a cycle. Then \( T = (T^0, T^1, s_T, r_T) \) is a desired unary tree, where \( T^0 = \{r(u_n) \mid n \in \omega\} \), \( T^1 = \{x_n \mid n \in \omega\} \) and the function \( s_T \) (resp. \( r_T \)) is a restriction of the source (resp. range) function \( s \) (resp. \( r \)) of the graph \( E \) on the set \( T^1 \).

**Lemma 10.** Let \( G(T) \) be a semitopological semigroup. If there exists an integer \( k \in \omega \) such that \( 0 \) is not an accumulation point of the set \( L_k^* \) then \( 0 \) is not an accumulation point of the set \( L_n^* \) for each \( n \in \omega \).

**Proof.** Let \( k \) be an integer such that \( 0 \notin \overline{L_k^*} \). Suppose to the contrary that \( 0 \in \overline{L_n^*} \) for some \( n \in \omega \). There are two cases to consider:

\[(i) \quad k < n; \]
\[(ii) \quad n < k.\]

Consider case (i). Fix an arbitrary open neighbourhood \( U \) of \( 0 \) such that \( U \cap L_k = \emptyset \). Since \((k, n) \cdot 0 \cdot (k, n)^{-1} = 0 \) the continuity of the semigroup operation in \( G(T) \) yields an open neighbourhood \( V \) of \( 0 \) such that \((k, n) \cdot V \cdot (k, n)^{-1} \subseteq U \). Since \( 0 \notin \overline{L_k^*} \) we obtain that the set \( V \) contains element \((n, m)(n, m)^{-1} \) for some \( m > n \). Hence

\[(k, n) \cdot (n, m)(n, m)^{-1} \cdot (k, n)^{-1} = (k, m)(k, m)^{-1} \in U \cap L_k^* \]

which contradicts to the choice of the set \( U \).

To obtain a contradiction in case (ii) we consider the product \((n, k)^{-1} \cdot 0 \cdot (n, k) = 0 \). Fix open neighbourhoods \( U \) and \( V \) of \( 0 \) such that \( U \) does not contain the set \( L_k^* \) and \((n, k)^{-1} \cdot V \cdot (n, k) \subseteq U \). Since \( 0 \notin \overline{L_k^*} \) there exists an element \((n, m)(n, m)^{-1} \in V \) such that \( m > k \). Then

\[(n, k)^{-1} \cdot (n, m)(n, m)^{-1} \cdot (n, k) = (k, m)(k, m)^{-1} \in U \cap L_k^* \]

which yields a contradiction.

**Lemma 11.** Let \( G(T) \) be a dense subsemigroup of a d-compact topological semigroup \( S \). Then for each open neighbourhood \( U \) of \( 0 \) there exists \( n \in \omega \) such that \( L_k \subset U \) for each \( k > n \).

**Proof.** Suppose to the contrary that there exists an open neighbourhood \( U \) of \( 0 \) such that the set \( A = \{k \in \omega \mid L_k \backslash U \neq \emptyset\} \) is infinite. Fix an arbitrary element \( x_k \in L_k \backslash U \), \( k \in A \). By Lemma 1 the semigroup \( S \) is countably compact at the dense discrete set \( G(T) \backslash \{0\} \). Hence there exists an element \( z \in S \) which is an accumulation point of the set \( X = \{x_k \mid k \in A\} \). Observe that \( x_k \cdot x_m = 0 \), whenever \( k \neq m \). Hence \( 0 \in V^2 \) for each open neighbourhood \( V \) of \( z \) which provides that \( z = 0 \). The contradiction.
Theorem 12. The GIS $G(T)$ embeds as a dense subsemigroup into a d-compact topological semigroup $S$ iff $G(T)$ is compact, i.e., $G(T) = (G(T), \tau_c)$.

Proof. Suppose that $G(T)$ is a dense subsemigroup of a d-compact topological semigroup $S$. There are two cases to consider:

1. there exists $n \in \omega$ such that $0$ is an accumulation point of $L^*_n$;
2. $0$ is not an accumulation point of $L^*_n$ for each $n \in \omega$.

Consider case (1). By Lemma 10, $0$ is an accumulation point of $L^*_n$ for each $n \in \omega$. Observe that for an arbitrary $n \in \omega$, $L^*_n$ is isomorphic to the semilattice $(\omega, \max)$. By [13] Theorem 2, zero is a unique accumulation point of $L^*_n$ for each $n \in \omega$. By Lemma 1 the set $L_n$ is compact. Lemma 11 provides that the semilattice $E(G(T)) = \bigcup_{n \in \omega} L_n$ is compact. Lemma 7 implies that $G(T)$ is closed in $S$ and hence $S = G(T)$. By Corollary 3, $G(T)$ is compact.

Consider case (2). By Lemma 8, the set $E(G(E))$ is countably compact at $E(G(E))$. By [15] Theorem 2, for each non-negative integer $n$, $L^*_n$ has only one accumulation point which we denote by $s_n$. Lemma 1 implies that the set $L^*_n \cup \{s_n\}$ is compact.

Consider the set $B = \{(0, 1), (0, 2), \ldots, (0, n), \ldots\}$. There exists $x \in B$. Since $(0, m)(0, m)^{-1} = (0, m)$ the continuity of the semigroup operation in $S$ implies that $s_0 \cdot x = x$. Fix an arbitrary open neighbourhood $U(x)$ of $x$. Since $s_0 \cdot x = x$ the continuity of the semigroup operation yields open neighbourhoods $U(s_0)$ and $V(x)$ of $s_0$ and $x$, respectively, such that $U(s_0) \cdot V(x) \subseteq U(x)$. Observe that there exists $k \in \omega$ such that for each $n > k (0, n)(0, n)^{-1} \in V(s_0)$. Since $x \in B$ there exists an infinite subset $C \in \omega$ such that $(0, m) \in V(x)$ for each $m \in C$. Hence $(0, n)(m, n)^{-1} = (0, n)(0, n)^{-1} \cdot (0, m) \in U(x)$, where $k < m < n$ and $m \in C$.

Fix an arbitrary disjoint open neighbourhoods $U(0)$ and $U(x)$ of 0 and $x$, respectively. Since $x \cdot 0 = 0$ the continuity of the semigroup operation yields open neighbourhoods $V(0) \subseteq U(0)$ and $V(x) \subseteq U(x)$ of 0 and $x$, respectively, such that $V(0) \cdot V(x) \subseteq U(0)$. By Lemma 11 there exists $k \in \omega$ such that $L^*_n \subset V(0)$ for each $n > k$. Hence there exist integers $m < n \in \omega$ such that $(m, n)(m, n)^{-1} \in V(0)$ and $(0, n)(m, n)^{-1} \in V(x)$. Then

$$(0, n)(m, n)^{-1} = (0, n)(m, n)^{-1} \cdot (m, n)(m, n)^{-1} \in V(x) \cdot V(0) \subseteq U(0)$$

which contradicts to the choice of the sets $U(0)$ and $U(x)$.

Theorem 13. Let $E$ be an acyclic graph. Then GIS $G(E)$ embeds as a dense subsemigroup into a d-compact topological semigroup $S$ iff $G(E)$ is compact, i.e., $G(E) = (G(E), \tau_c)$.

Proof. Let $S$ be a d-compact topological semigroup which contains $G(E)$ as a dense subsemigroup. There are two cases to consider:

1. $E(G(E))$ is closed in $E(S)$;
2. there exists element $s \in E(G(E)) \setminus E(G(E))$.

Consider case (1). By Lemma 1 the set $E(G(E))$ is compact. By Lemma 7 the set $G(E)$ is closed in $S$. Hence $G(E) = S$. By Corollary 3, the set $G(E)$ is compact and $G(E) = (G(E), \tau_c)$.

Consider case (2). Suppose that there exists an element $s \in E(G(E)) \setminus E(G(E))$. By Lemma 8 there exists an open neighbourhood $U$ of $s$ such that $U \cap G(E) \subseteq E(G(E))$. Since for each elements $a, b \in E(G(E))$, $a \cdot b \in \{a, b, 0\}$ there exists a linearly ordered set $L \in E(G(E))$ such that $U \cap G(E) \subseteq L$ (in the other case each open neighbourhood $V$ of $s$ will contain incomparable elements from $E(G(E))$ and hence $0 \in V^2$ which contradicts to the Hausdorffness of $S$). By Lemma 9 there exists an unary tree $T$ such that $L \subseteq G(T) \subseteq G(E)$. Lemma 11 implies that $G(T)$ is countably compact at the dense discrete subset $G(T) \setminus \{0\}$ which implies that the set $G(T)$ is d-compact. By Theorem 12, $G(T)$ is compact. Hence $G(T) = G(T)$ which contradicts to the choice of $s$.

The following remark shows that Theorem 13 does not hold for graphs which contain a cycle.
Remark 14. In [6] Theorem 6.1] it was proved that there exists a Tychonoff countably pracompact topological semigroup $S$ which densely contains the bicyclic monoid. Moreover, under Martin’s Axiom the semigroup $S$ is countably compact (see [6, Theorem 6.6 and Corollary 6.7]). Simply verifications show that the semigroup $S$ with adjoint isolated zero is countably pracompact and contains densely the discrete polycyclic monoid $P_1$.

Now we are able to prove our main Theorem 5

5. The proof of Theorem 5

Proof. Assume that $G(E)$ is a dense subsemigroup of a d-compact topological semigroup $S$. Put $F^0 = \{ f \in E^0 \mid \text{there exists no cycle $u$ such that } s(u) = r(u) = f \}$. Denote $F^1 = \{ x \in E^1 \mid s(x) \in F^0 \text{ and } r(x) \in F^0 \}$. Let $F = (F^0, F^1, s_F, r_F)$ be a subgraph of $E$ where $s_F, r_F, \text{ resp.}$ is a restriction of the source (range, resp.) function $s, r, \text{ resp.}$ of the graph $E$ on the set $F^1$. Observe that graph $F$ is acyclic. If the graph $F$ is non-empty then $\overline{G(F)}$ is a d-compact subsemigroup of $S$. Theorem 13 implies that $G(E)$ is compact. By Theorem 4 the set $I_f$ is finite for each vertex $f \in F^0$.

Put

$$A = \{ e \in E^0 \mid \text{there exists a cycle } u \text{ such that } s(u) = r(u) = e \}$$

and $k = |A|$.

Assume that $k > 0$. Fix an arbitrary vertex $e \in A$ and a cycle $u \in \text{Path}(E)$ such that $s(u) = r(u) = e$. We claim that for each vertex $f \neq e$ both sets

$$B = \{ x \in E^1 \mid s(x) = e \text{ and } r(x) = f \} \quad \text{and} \quad C = \{ x \in E^1 \mid s(x) = f \text{ and } r(x) = e \}$$

are empty. Indeed, suppose that there exists $x \in B$. Let $T$ be an inverse subsemigroup of $G(E)$ which is generated by the set $\{ u^n x \mid n \in \mathbb{N} \}$. Observe that each non-zero element of $T$ is of the form $u^n x (u^m x)^{-1}$ and

$$u^n x (u^m x)^{-1} \cdot u^k x (u^l x)^{-1} = \begin{cases} u^n x (u^l x)^{-1}, & \text{if } m = k; \\ 0, & \text{if } m \neq k, \end{cases}$$

Simply verifications show that the semigroup $T$ is isomorphic to the semigroup of $\mathbb{N} \times \mathbb{N}$ matrix units (the isomorphism $h : T \to \mathcal{B}_\mathbb{N}$ can be defined as follows: $h(u^n x (u^m x)^{-1}) = (n, m)$ and $h(0) = 0$). Then $\overline{T}$ is countably compact at the dense discrete subspace $T \setminus \{ 0 \}$ which contradicts [13, Theorem 4.4] where it was proved that infinite semigroup of matrix units can not be embedded densely into a feebly compact topological semigroup (recall that countably pracompact topological spaces are feebly compact). Hence the set $B$ is empty.

Assume that there exists $x \in C$. Let $T$ be an inverse subsemigroup of $G(E)$ which is generated by the set $\{ xu^n \mid n \in \mathbb{N} \}$. Similar arguments imply that the semigroup $T$ is isomorphic to the semigroup of $\mathbb{N} \times \mathbb{N}$-matrix units and $\overline{T}$ is countably pracompact. This contradicts [13, Theorem 4.4]. Hence the set $C$ is empty as well.

For each $x \in A$ put $E^0_x = \{ x \}$, $E^1_x = \{ y \in E^1 \mid s(y) = r(y) = x \}$. Since the sets $B$ and $C$ are empty the cycle $u$ is a product of finitely many loops. Hence the set $E^1_x$ is non-empty. By $E_x$ we denote a subgraph of $E$ which contain one vertex $x$ and $E^1_x$ is the set of edges of $E_x$.

Observe that we already prove that $E = \bigsqcup_{a \in \mathbb{N}} E_a \sqcup F$. Now we are going to show that the set $E^1_x$ is singleton for each $x \in A$.

Assume that there exists a vertex $x \in A$ and two distinct loops $y, z \in E^1_x$. Then, by [12, Theorem 3], GIS $G(E)$ contains the polycyclic monoid $P_2$ which is generated by the elements $y, z, y^{-1}, z^{-1}$. By [13, Theorem 4.6], monoid $P_2$ can not embed as a dense subset into a feebly compact topological semigroup. However, the semigroup $\overline{P_2} \subset S$ is feebly compact which implies the contradiction. Hence for each vertex $x \in A$ the set $E^1_x$ is singleton. This completes the proof of the statements (1) and (2).

Consider statement (3). Suppose to the contrary that there exist an open neighbourhood $U$ of $0$ and an infinite subset $A$ of $k$ such that $E(G(E_a)) \setminus U \neq \emptyset$ for each $\alpha \in A$. Fix an arbitrary element $x_\alpha \in E(G(E_a)) \setminus U$. By Lemma 1, the semigroup $S$ is countably compact at the dense discrete subspace $G(E) \setminus \{ 0 \}$ which implies that the set $X = \{ x_\alpha \mid \alpha \in A \}$ has an accumulation point $y \in S$. Observe
that \( y \in E(G(E)) \) which provides that \( y \) is an idempotent. The continuity of the semigroup operation in \( S \) yields an open neighbourhood \( U \) of \( y \) such that \( U \cdot U \subset S \setminus \{0\} \). Since \( x_{\alpha} \cdot x_{\beta} = 0 \) if \( \alpha \neq \beta \) we obtain that \( 0 \in U \cdot U \subset S \setminus \{0\} \) which provides the contradiction. \( \square \)

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