NEWTON SLOPES FOR TWISTED ARTIN–SCHREIER–WITT TOWERS

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ABSTRACT. We fix a monic polynomial \( f(x) \in \mathbb{F}_q[x] \) over a finite field of characteristic \( p \) of degree relatively prime to \( p \). Let \( \alpha \mapsto \omega(\alpha) \) be the Teichmüller lift of \( \mathbb{F}_q \), and let \( \chi : \mathbb{Z} \to \mathbb{C}_p^\times \) be a finite character of \( \mathbb{Z}_p \). The \( L \)-function associated to the polynomial \( f \) and the so-called twisted character \( \omega^m \times \chi \) is denoted by \( L_f(\omega^m, \chi, s) \) (see Definition 1.1). We prove that, when the conductor of the character is large enough, the \( p \)-adic Newton slopes of this \( L \)-function form arithmetic progressions.

Contents

1. Introduction 1
2. Notation 3
3. The \( T \)-adic Dwork’s Trace Formula 5
4. Proof of Theorem 1.4 and Theorem 1.5 9
References 14

1. Introduction

In [DWX], Davis, Wan, and Xiao studied the \( L \)-function \( L_f(\chi, s) \) associated to a polynomial \( f(x) \in \mathbb{F}_q[x] \) and a finite character \( \chi : \mathbb{Z}_p \to \mathbb{C}_p^\times \). They proved that

- If \( \chi_1 \) and \( \chi_2 \) are two finite characters with the same conductor \( p^m \geq \frac{a p(d-1)^2}{8d} \), then \( L_f(\chi_1, s) \) and \( L_f(\chi_2, s) \) have the same Newton polygons.

- Let \( \chi_0 : \mathbb{Z}_p \to \mathbb{C}_p^\times \) be a fixed character with conductor \( p^{\lceil \log p \frac{a p(d-1)^2}{8d} \rceil} \). Then the \( p \)-adic Newton slopes of \( L_f(\chi_1, s) \) (see Definition 1.3) form a disjoint union of arithmetic progressions determined by the \( p \)-adic Newton slopes of \( L_f(\chi_0, s) \).

In [BFZ], Blache, Ferard, and Zhu studied the so-called twisted \( L \)-functions (see Definition 1.1), whose \( p \)-adic Newton polygons satisfy a universal lower bound proved by C.Liu and W.Liu in [LL]. This lower bound is similar to the one given in [DWX]. Therefore, it is of interest to ask if the \( p \)-adic Newton slopes of the twisted \( L \)-functions also form arithmetic progressions. In this paper, we give an upper bound for the twisted \( L \)-function and prove that it coincides with the lower bound at \( x = kd \) for any integer \( k \geq 0 \). As a consequence, we prove that its \( p \)-adic Newton slopes indeed form arithmetic progressions.

We start with introducing some basic setups. We fix a prime number \( p \). Let \( \mathbb{F}_q \) be the finite field of \( q = p^n \) elements. Let

\[
\omega : \mathbb{F}_q \to \mathbb{Z}_q \\
\alpha \mapsto \omega(\alpha) := \hat{\alpha}
\]

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be the Teichmüller lift of $\mathbb{F}_q$. For any $u \in \{0, 1, \ldots, q - 1\}$, we put
\[ \omega^u : \mathbb{F}_q \rightarrow \mathbb{Z}_q, \quad \alpha \mapsto \omega^u(\alpha) = \hat{\alpha}^u. \]

We fix a monic polynomial $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{F}_q[x]$ of degree $d$ which is coprime to $p$. Set $a_d = 1$ and put $\hat{\alpha}_i := \omega(a_i)$ for $i = 0, \ldots, d$. The Teichmüller lift of the polynomial $f(x)$ is defined by $\hat{f}(x) := x^d + \hat{a}_{d-1}x^{d-1} + \cdots + \hat{a}_0 \in \mathbb{Z}_q[x].$

Let $u$ be an integer in the set $\{0, 1, \ldots, q - 2\}$ and put
\[ u = \sum_{i=0}^{a-1} u(i)p^i, \]
where $0 \leq u(i) \leq p - 1$ for any $0 \leq i \leq a - 1$.

**Definition 1.1.** Let $\chi : \mathbb{Z}_p \rightarrow \mathbb{C}_p^\times$ be a finite character with conductor $p^{m_x}$. The twisted $L$-function associated to the characters $\chi$ and $\omega^u$ is defined by (1.1.1)
\[ L_f(\omega^u, \chi, s) = \prod_{x \in \mathbb{G}_{m,\mathbb{F}_q}} \frac{1}{1 - \omega^u \circ \text{Norm}_{\mathbb{F}_q(\hat{x})/\mathbb{F}_q}(x) \cdot \chi \left( \frac{\text{Tr}_{\mathbb{Q}_{\hat{x}}/\mathbb{Q}_p}(\hat{f}(x))}{\text{deg}(x)} \right)^s}, \]
where $\mathbb{G}_{m,\mathbb{F}_q}$ is the one-dimensional torus over $\mathbb{F}_q$ and $\text{deg}(x)$ stands for the degree of $x$. By [Lin-Wei], the $L$-function $L_f(\omega^u, \chi, s)$ is a polynomial of degree $dp^{m_x - 1}$.

**Notation 1.2.** For simplicity of notations, we denote
\[ y_u(k) := \frac{ak(k-1)(p-1)}{2d} + \frac{k}{d} \sum_{i=0}^{a-1} u(i). \]

**Definition 1.3.** We call the slopes of the line segments of the $p$-adic Newton polygon of $L_f(\omega^u, \chi, s)$ the $p$-adic Newton slopes of $L_f(\omega^u, \chi, s)$.

In this paper, we prove the following.

**Theorem 1.4.**

(a) The $p$-adic Newton polygon of $L_f(\omega^u, \chi, s)$ passes through the points
\[ (kd, \frac{y_u(kd)}{p^{m_x - 1}}) \text{ for any } 0 \leq k \leq p^{m_x - 1}. \]

(b) The $p$-adic Newton polygon of $L_f(\omega^u, \chi, s)$ has slopes (in increasing order)
\[ \bigcup_{k=1}^{p^{m_x - 1}} \{ \alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kd} \}, \]
where
\[ \frac{a(k-1)}{p^{m_x - 1}} + \frac{\sum_{i=0}^{a-1} u(i)}{d(p-1)p^{m_x - 1}} \leq \alpha_{kj} \leq \frac{a(k-1)}{p^{m_x - 1}} + \frac{\sum_{i=0}^{a-1} u(i)}{d(p-1)p^{m_x - 1}} + \frac{a(d-1)}{dp^{m_x - 1}} \]
for any $1 \leq j \leq d$.

When the conductor $p^{m_x}$ of $L_f(\omega^u, \chi, s)$ is large enough, the $p$-adic Newton slopes of $L_f(\omega^u, \chi, s)$ have the following property.
Theorem 1.5 (Main theorem). Let $m_0$ be the minimal positive integer such that $p^{m_0} > \frac{\text{adp}}{\text{S}(p-1)}$ and let $0 \leq \alpha_1, \alpha_2, \ldots, \alpha_{m_0-1} < a$ denote the slopes of the $p$-adic Newton polygon of $L_f(\omega^u, \chi_0, s)$ for a finite character $\chi_0 : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $m_{\chi_0} = m_0$. Then for any finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $m_\chi \geq m_0$, the $p$-adic Newton polygon of $L_f(\omega^u, \chi, s)$ has slopes
\begin{equation}
\bigcup_{i=0}^{p^{m_\chi-m_0}-1} \left\{ \frac{\alpha_1 + ai}{p^{m_\chi-m_0}}, \frac{\alpha_2 + ai}{p^{m_\chi-m_0}}, \ldots, \frac{\alpha_{adp^{m_0-1} + ai}}{p^{m_\chi-m_0}} \right\}.
\end{equation}

Theorem[1.5] says that when $m_\chi$ is large enough, the $p$-adic Newton slopes of $L_f(\omega^u, \chi, s)$ form a disjoint union of arithmetic progressions determined by the $p$-adic Newton slopes of $L_f(\omega^u, \chi_0, s)$.

This paper is inspired by the $p$-adic Newton slopes of $L_f(\chi, s)$ in arithmetic progressions (proved in [DWX]), the twisted decomposition of $L_f(x^{q-1})(\chi, s) := \prod_{u=0}^{q-2} L_f(\omega^u, \chi, s)$ in [BEZ], and the lower bound for the Newton polygon of $L_f(\omega^u, \chi, s)$ given in [LL]. Let
$$C_0 \to C_1 \to \cdots \to C_m \to \cdots$$
be the Artin–Schreier–Witt curve tower associated to the polynomial $f(x^{q-1})$, and let $Z(C_m, s)$ be the zeta function of the curve $C_m$. It is known that
$$\begin{cases}
L_f(\omega^u, \chi, s) & \text{if } u \neq 0 \\
L_f(\omega^0, \chi, s) & \text{if } u = 0
\end{cases}$$
are factors of $Z(C_m, s)$, and the degree of $L_f(\omega^u, \chi_0, s)$ is $\frac{1}{q-1}$ of the degree of $L_f(\chi_0, s)$. Therefore, as a corollary of Theorem [1.5], we give a more precise description of zeros of $Z(C_m, s)$ than the one given in [DWX].

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2. Notation

In this section, we introduce some notations that we will use throughout the paper.

Notation 2.1. We write $v_T(\cdot)$ for the $T$-adic valuation of elements in $\mathbb{C}_p[T]$ and $v_p(\cdot)$ for the $p$-adic valuation of elements in $\mathbb{C}_p$.

Definition 2.2. Given a set $S := \{(k, d_k) \mid 0 \leq k \leq n\}$. The Newton polygon of $S$, denoted by NP($S$), is the lower convex hull of points in $S$. We call $n$ the length of NP($S$).

For a power series $F(s) = \sum_{i=0}^{n} u_i(T)s^i$, we put
$$\text{NP}_T(F) := \text{NP} \left( \left\{ (k, v_T(u_k)) \mid 0 \leq k \leq n \right\} \right),$$
where $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Definition 2.3. For a Newton polygon NP, we write $R(\text{NP})$ for the multiset of slopes in NP.

It has an inverse, denoted by $R^{-1}$, mapping a multiset $\mathbb{S}$ to the lower convex whose slopes coincide with this multiset.

Notation 2.4.
(a) Let $S_1$ and $S_2$ be two multisets in $\mathbb{Q}$. We denote by
$$S_1 \uplus S_2$$
the union of $S_1$ and $S_2$ as multisets.

(b) For any two Newton polygons $NP_1$ and $NP_2$, we write
$$NP_1 \oplus NP_2$$
for the Newton polygon whose slopes are the union of the slopes of $NP_1$ and $NP_2$.

(c) We denote by $NP(i)$ the height of $NP$ at $x = i$.

(d) For any $t \in \mathbb{Q}$, we denote $t + NP$ be the Newton polygon such that
$$(t + NP)(k) = NP(k) + t$$
for any $0 \leq k \leq n$, where $n$ is the length of $NP$.

**Definition 2.5.** Let $NP_1$ and $NP_2$ be two polygons of same length $n$. If
$$NP_1(k) \geq NP_2(k)$$
holds for any $0 \leq k \leq n$, then we call that $NP_1$ is above $NP_2$ and denote this by $NP_1 \geq NP_2$.

**Lemma 2.6.** If $\{NP_1,i \mid 1 \leq i \leq m\}$ and $\{NP_2,i \mid 1 \leq i \leq m\}$ are two sets of Newton polygons such that for any $0 \leq i \leq m$,
- $NP_1,i$ and $NP_2,i$ have the same length, and
- $NP_1,i \geq NP_2,i$,
then
$$\bigoplus_{i=1}^{m} NP_1,i \geq \bigoplus_{i=1}^{m} NP_2,i.$$

**Proof.** It follows directly from the definition of “$\oplus$”.

**Definition 2.7.** For any positive integer $\ell$, the sum
$$S_{f,\omega,u,\chi}(\ell, s) = \sum_{x \in F_{q^{\ell}}^*} \omega^u \circ \text{Norm}_{F_{q^\ell}/F_q}(x) \cdot (1 + T)^{\left(\text{Tr}_{Q_{q^\ell}/Q_{q^2}}(f(x))\right)} \in \mathbb{Z}_q[T]$$
is called a twisted $T$-adic exponential sum of $f(x)$.

The $L$-function $L_f(\omega^u, T, s)$ (see Definition 1.1) satisfies
$$L_f(\omega^u, T, s) = \exp\left(\sum_{\ell=1}^{\infty} S_{f,\omega,u,\chi}(\ell, s) \frac{\ell}{\ell} \right).$$

It is easy to check
$$L_f(\omega^u, T, s)|_{T=\chi(1)-1} = L_f(\omega^u, \chi, s).$$

**Lemma 2.8.** If we put $g(x) := f(x^{q-1})$, then
$$L_g(\omega^0, T, s) = \prod_{u=0}^{q-2} L_f(\omega^u, T, s).$$

**Proof.** Put
$$(1 + T)^{\left(\text{Tr}_{Q_{q^\ell}/Q_p}(f(x))\right)} := \sum_{n=0}^{\infty} b_{\ell,n}(T)\hat{x}^n \in \mathbb{Z}_p[T][\hat{x}],$$
where $b_{\ell,n}(T) \in \mathbb{Z}_p[T]$ for all $n \geq 0$ and $\ell \geq 1$. 

Notice that for each \( u \in \{0, 1, \ldots, q - 2\} \), we have

\[
S_{f, \omega^u, \chi}(\ell, s) = (q^\ell - 1) \sum_{n=1}^{\infty} b_{\ell, (q^\ell - 1)(n - \frac{u}{q-1})} (T).
\]

Therefore, by taking the sum of \( u \) over the set \( \{0, 1, \ldots, q - 2\} \), we get

\[
\sum_{u=0}^{q-2} S_{f, \omega^u, T}(\ell, s) = (q^\ell - 1) \sum_{u=0}^{q-2} \sum_{n=1}^{\infty} b_{\ell, (q^\ell - 1)(n - \frac{u}{q-1})} (T)
\]

\[
= (q^\ell - 1) \sum_{n=1}^{\infty} b_{\ell, \frac{n(q^\ell - 1)}{q-1}} (T).
\]

On the other hand, by definition, it is easy to check that

\[
S_{g, \omega^0, T}(\ell, s) = (q^\ell - 1) \sum_{n=1}^{\infty} b_{\ell, \frac{n(q^\ell - 1)}{q-1}} (T).
\]

Therefore, we have

\[
S_{g, \omega^u, T}(\ell, s) = \sum_{u=0}^{q-2} S_{f, \omega^u, T}(\ell, s),
\]

for all \( \ell \geq 1 \), which implies

\[
L_g(\omega^0, T, s) = \prod_{u=0}^{q-2} L_f(\omega^u, T, s).
\]

**Definition 2.9.** The characteristic power series of \( f \) is given by

\[
(2.9.1) \quad C_f(\omega^u, T, s) := \prod_{i=0}^{\infty} L_f(\omega^u, T, q^i s),
\]

which is known as a \( p \)-adic entire power series.

By Lemma 2.8, we know that

\[
C_g(\omega^0, T, s) = \prod_{u=0}^{q-2} C_f(\omega^u, T, s).
\]

**Notation 2.10.** We denote by \( \text{NP}_f(L, \omega^u, T) \) (resp. \( \text{NP}_f(L, \omega^u, \chi) \)) the \( T \)-adic Newton polygon (resp. \( p \)-adic Newton polygon) of \( L_f(\omega^u, T, s) \) (resp. \( L_f(\omega^u, \chi, s) \)).

Similarly, we write \( \text{NP}_f(C, \omega^u, T) \) and \( \text{NP}_f(C, \omega^u, \chi) \) for the \( T \)-adic Newton polygon (resp. \( p \)-adic Newton polygon) of \( C_f(\omega^u, T, s) \) and \( C_f(\omega^u, \chi, s) \) respectively.

### 3. The \( T \)-adic Dwork’s Trace Formula

In this section, we recall properties of the \( L \)-function associated to a \( T \)-adic exponential sum as considered by Liu and Wan in [LW]. Its specializations to appropriate values of \( T \) interpolate the \( L \)-functions considered above.

**Notation 3.1.** We first recall that the Artin–Hasse exponential series is defined by

\[
(3.1.1) \quad E(\pi) = \exp \left( \sum_{i=0}^{\infty} \frac{\pi^p^i}{p^i} \right) = \prod_{p \mid i, i \geq 1} (1 - \pi^i)^{-\mu(i)/i} \in 1 + \pi + \pi^2 \mathbb{Z}_p[\pi].
\]

Setting \( T = E(\pi) - 1 \) defines an isomorphism \( \mathbb{Z}_p[\pi] \cong \mathbb{Z}_p[T] \).
**Notation 3.2.** For our given polynomial \( f(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}_q[x] \), we put

\[
E_f(x) := \prod_{i=0}^{d} E(a_i \pi x^i) \in \mathbb{Z}_q[[\pi]][x].
\]

We follow the notation of [LL]. Set

\[
C_u := \{ v \in \mathbb{Z}_{\geq 0} \mid v \equiv u \}
\]

and

\[
B_u := \left\{ \sum_{v \in C_u} b_v T^{\frac{v}{d(q-1)}} x^{\frac{v}{q-1}} \mid b_v \in \mathbb{Z}_q[T^{\frac{1}{q-1}}] \text{ and } v_T(b_v) \to \infty \right\}.
\]

**Notation 3.3.**

(a) For two integers \( n \) and \( m \), we denote by \( n \% m \) the residue class of \( n \) modulo \( m \) in \( \{0, 1, \ldots, m - 1\} \).

(b) Recall \( u \in \{0, 1, \ldots, q - 2\} \). We write \( b_u | a \) for the minimal positive integer such that \( u^{b_u} \equiv u \pmod{q} \).

(c) Denote

\[
u_i := (up^i)\% (q - 1) \text{ for } i = 0, \ldots, b_u - 1 \]

and put

\[
\tilde{B}_u = \bigoplus_{i=0}^{b_u-1} B_{u_i}
\]

to be the total Banach space associated to \( u \).

(d) Choose a permutation \((i_1, i_2, \ldots, i_{b_u})\) of \( \{1, 2, \ldots, b_u\} \) such that the sequence \( \{u_{i_n}\} \) is non-decreasing. Put

\[
\bigcup_{i=0}^{b_u-1} C_{u_i} := (c_{u,n})_{n \in \mathbb{Z}_{\geq 0}}
\]

to be a non-decreasing sequence.

It is easy to check that

\[
c_{u,n} = (q - 1) \lfloor \frac{n}{b_u} \rfloor + u_{i(n \% b_u)}.
\]

Let \( \psi_p \) denote the operator on \( \tilde{B}_u \) given by

\[
\psi_p \left( \sum_{n \geq 0} d_n(T)x^n \right) := \sum_{n \geq 0} d_{pn}(T)x^n;
\]

and let \( \psi \) be the composite linear operator

\[
\psi := \sigma \circ \psi_p \circ E_f(x) : \tilde{B}_u \longrightarrow \tilde{B}_u,
\]

where \( \sigma \) is the Frobenius automorphism of \( \mathbb{Z}_q \), and \( E_f(x) \) acts on \( g \in \tilde{B}_u \) by

\[
E_f(x)(g) := E_f(x) \cdot g.
\]

By Dwork’s trace formula, we have

**Lemma 3.4.** The characteristic power series \( C_f(\omega^u, T, s) \) satisfies

\[
C_f(\omega^u, T, s) = \det \left( 1 - \psi^a s \mid B_u/\mathbb{Z}_q[T^{\frac{1}{q-1}}] \right).
\]
By [LL] Lemma 4.2, we have

\[ C_f(\omega^u, T, s)^b = \det \left( 1 - \psi^a s \mid \hat{B}_u / \mathbb{Z}_q[T^{\frac{1}{q-1}}] \right). \]

We write

\[ B = (T^{\frac{v}{d(q-1)}} T^{\frac{c_{u,n}}{(q-1)}})_{n \in \mathbb{Z}_{\geq 0}} \]

for a basis of \( \hat{B}_u \) over \( \mathbb{Z}_q[T^{\frac{1}{q-1}}] \) and denote by \( N \) the standard matrix of \( \psi \) associated to the basis \( B \).

It is not hard to check that \( N \) is an infinite dimensional nuclear matrix of the form

\[ N = \begin{pmatrix}
T^{\frac{(p-1)c_{u,0}}{d(q-1)}} & T^{\frac{(p-1)c_{u,0}}{d(q-1)}} & \cdots & T^{\frac{(p-1)c_{u,0}}{d(q-1)}} \\
T^{\frac{(p-1)c_{u,1}}{d(q-1)}} & T^{\frac{(p-1)c_{u,1}}{d(q-1)}} & \cdots & T^{\frac{(p-1)c_{u,1}}{d(q-1)}} \\
\vdots & \vdots & \ddots & \vdots \\
T^{\frac{(p-1)c_{u,n}}{d(q-1)}} & T^{\frac{(p-1)c_{u,n}}{d(q-1)}} & \cdots & T^{\frac{(p-1)c_{u,n}}{d(q-1)}}
\end{pmatrix}. \]

By [RWXY] Corollary 3.9, we know

\[ \det \left( 1 - \psi^a s \mid \hat{B}_u / \mathbb{Z}_q[T^{\frac{1}{q-1}}] \right) = \det \left( I - s\sigma^{-1}(N) \cdots \sigma(N)N \right). \]

**Notation 3.5.** For a matrix \( M \), we write

\[ \begin{pmatrix}
m_1 & m_2 & \cdots & m_k \\
n_1 & n_2 & \cdots & n_k
\end{pmatrix}_M \]

for the \( k \times k \) submatrix formed by elements whose row indices belong to \( \{m_1, \ldots, m_k\} \) and whose column indices belong to \( \{n_1, \ldots, n_k\} \).

**Lemma 3.6.** Let \( (t_{ij})_{j \in \mathbb{Z}_{\geq 0}} \) be \( n \) non-decreasing sequences, and let \( M_1, M_2, \ldots, M_n \) be \( n \) nuclear matrices such that

\[ M_i = \text{Diag}(T^{t_{i1}}, T^{t_{i2}}, \ldots) \cdot M'_i \quad \text{for any } 1 \leq i \leq n, \]

where \( M'_i \) are infinite matrix whose entries belong to \( \mathbb{Z}_q[T^{\frac{1}{q-1}}] \). Then the \( T \)-adic Newton polygon

\[ \text{NP}_T \left( \det(1 - M_n \cdots M_2 M_1 s) \right) \geq \text{NP} \left( \left\{ (k, \sum_{i=1}^n \sum_{j=1}^k (t_{ij})) \mid k \geq 0 \right\} \right). \]

**Proof.** Put

\[ \det(I - s M_n \cdots M_2 M_1) := \sum_{k=0}^{\infty} (-1)^k \tau_k(T)s^k. \]
From the definition of characteristic power series, we get
\[
r_k(T) = \sum_{0 \leq m_1 < m_2 < \cdots < m_k < \infty} \det \begin{bmatrix} m_1 & m_2 & \cdots & m_k \\
 m_1 & m_2 & \cdots & m_k \\
 \vdots & \vdots & \ddots & \vdots \\
 M_n & M_{n-1} & \cdots & M_1 \end{bmatrix}
\]
\[= \sum_{0 \leq m_1,1 < m_1,2 < \cdots < m_k,k < \infty} \prod_{i=1}^{n} \det \begin{bmatrix} m_{i+1,1} & m_{i+1,2} & \cdots & m_{i+1,k} \\
 m_{i,1} & m_{i,2} & \cdots & m_{i,k} \\
 \vdots & \vdots & \ddots & \vdots \\
 M_{i+1} & M_i & \cdots & M_1 \end{bmatrix} \]  
(3.6.1)

\[
\sum_{i=1}^{n} \prod_{i=1}^{n} \det \begin{bmatrix} m_{i+1,1} & m_{i+1,2} & \cdots & m_{i+1,k} \\
 m_{i,1} & m_{i,2} & \cdots & m_{i,k} \\
 \vdots & \vdots & \ddots & \vdots \\
 M_{i+1} & M_i & \cdots & M_1 \end{bmatrix} \]
\]

Here and after, we set \( m_{n+1,i} = m_{1,i} \) for all \( 1 \leq i \leq k \). Since
\[
v_T \left( \det \begin{bmatrix} m_{i+1,1} & m_{i+1,2} & \cdots & m_{i+1,k} \\
 m_{i,1} & m_{i,2} & \cdots & m_{i,k} \\
 \vdots & \vdots & \ddots & \vdots \\
 M_{i+1} & M_i & \cdots & M_1 \end{bmatrix} \right) \geq \sum_{j=1}^{k} t_{ij},
\]
we complete the proof. \( \square \)

**Definition 3.7.** The *Hodge polygon* of \( C_f(\omega^u, T, s) \), denoted by \( \text{HP}(d, \omega^u, T) \), is the lower convex hull of set
\[
\{ \left( k, \frac{a(p-1)}{db_u(q-1)} \sum_{j=0}^{kb_u-1} c_{u,j} \right) \mid k \geq 0 \}.
\]

**Lemma 3.8.** Each point in \( \{ \left( k, \frac{a(p-1)}{db_u(q-1)} \sum_{j=0}^{kb_u-1} c_{u,j} \right) \} \) is a vertex of \( \text{HP}(d, \omega^u, T) \).

**Proof.** It follows that sequence \( \left( \frac{a(p-1)}{db_u(q-1)} \sum_{j=(k-1)b_u}^{kb_u-1} c_{u,j} \right)_{k \in \mathbb{Z}_{\geq 0}} \) is strictly increasing in \( k \). \( \square \)

Recall
\[
u = \sum_{j=0}^{a-1} u(j)p^j \quad \text{and} \quad y_u(k) = \frac{ak(k-1)(p-1)}{2d} + \frac{k \sum_{j=0}^{a-1} u(j)}{d}.
\]

**Lemma 3.9.** We have
\[
y_u(k) = \frac{a(p-1)}{db_u(q-1)} \sum_{j=1}^{kb_u} c_{u,j}.
\]

**Proof.** From (3.3.1), we know
\[
\sum_{j=0}^{kb_u-1} c_{u,j} = \sum_{j=0}^{k-1} \sum_{\ell=0}^{b_u-1} c_{u,jb_u+\ell}
\]
\[
= \sum_{j=0}^{k-1} [j b_u(q-1) + \sum_{i=0}^{b_u-1} u_i]
\]
\[
= \frac{k(k-1)b_u(q-1)}{2} + k \sum_{i=0}^{b_u-1} u_i.
\]
Since
\[
\sum_{i=0}^{b_u-1} u_i = \frac{b_u}{a} \sum_{i=0}^{a-1} u^i p^{i(q-1)} = \frac{b_u}{a} \sum_{j=0}^{a-1} u(j) p^{j(q-1)}/p-1,
\]
we know
\[
\frac{a(p-1)}{db_u(q-1)} \sum_{j=1}^{k b_u} c_{u,j} = \frac{a(p-1)}{db_u(q-1)} \left( \frac{k(k-1)b_u(q-1)}{2} + \frac{k(q-1)b_u}{a(p-1)} \sum_{i=0}^{b_u-1} u_i \right)
\]
\[
= \frac{ak(k-1)(p-1)}{2d} + \frac{k}{d} \sum_{j=0}^{a-1} u(j) = y_u(k).
\]

**Corollary 3.10.** The Hodge polygon \( \text{HP}(d, \omega^u, T) \) passes through the points \((k, y_u(k))\) for any \(k \geq 0\).

**Proposition 3.11.** The polygons \( \text{NP}_f(C, \omega^u, T) \) and \( \text{HP}(d, \omega^u, T) \) satisfy
\[
\text{NP}_f(C, \omega^u, T) \geq \text{HP}(d, \omega^u, T).
\]

**Proof.** Since matrix \( N \) is nuclear as in (3.4.3), its conjugates \( \sigma^i(N) \) are also nuclear matrices of the same form. Therefore, applying Lemma 3.6 to the product of these matrices yields
\[
\text{NP}_T \left( \det \left( I - s \sigma^{a-1}(N) \cdots \sigma(N)N \right) \right) \geq \text{NP} \left( \left\{ (k, \frac{a(p-1)}{d(q-1)} \sum_{j=1}^{k} c_{u,j} ) \mid k \geq 0 \right\} \right).
\]

From (3.4.2) and (3.4.4), we have
\[
\text{NP}_f(C, \omega^u, T) \geq \text{NP} \left( \left\{ (k, \frac{a(p-1)}{d(q-1)} \sum_{j=1}^{k} c_{u,j} ) \mid k \geq 0 \right\} \right)
\]
\[
= \text{HP}(d, \omega^u, T).
\]

**Corollary 3.12.** For any character \( \chi : \mathbb{Z}_p \to \mathbb{C}_p^\times \) with conductor \( p^{m_x} \), we have
\[
\text{NP}_f(C, \omega^u, \chi) \geq \frac{1}{p^{m_x-1}(p-1)} \text{HP}(d, \omega^u, T).
\]

**Proof.** It simply follows
\[
\text{NP}_f(C, \omega^u, \chi) \geq \frac{1}{p^{m_x-1}(p-1)} \text{NP}_f(C, \omega^u, T)
\]
\[
\geq \frac{1}{p^{m_x-1}(p-1)} \text{HP}(d, \omega^u, T).
\]

4. **Proof of Theorem 1.4 and Theorem 1.5**

In this section, we prove the main theorems.

**Proposition 4.1.**
(a) The Newton polygon \( \text{NP}_f(C, \omega^u, T) \) passes through the points \((kd, y_u(kd))\) for all \(k \geq 0\).
(b) If we write

\[(4.1.1) \quad C_f(\omega^u, T, s) = \sum_{k=0}^{\infty} r_{u,k}(T)s^k,\]

then for any \(k \geq 0\) and \(0 \leq u \leq q - 2\), the leading term of \(r_{u,kd}\) is of the form

\(*T^{y_u(kd)},\)

where \(*\) represents for a \(p\)-adic unit.

Notation 4.2. We denote by \(\text{UP}(d, \omega^u, T)\) the lower convex hull of the points in

\[\{(kd, y_u(kd)) \mid k \geq 0\}.\]

Corollary 4.3. The polygon \(\text{UP}(d, \omega^u, T)\) forms an upper bound of \(\text{NP}_f(C, \omega^u, T)\).

Proof. This follows directly from Proposition 4.1 (a). \(\Box\)

Corollary 4.4. Any finite character \(\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times\) with conductor \(p^{m_\chi}\) satisfies

\[(4.4.1) \quad \text{NP}_f(C, \omega^u, \chi) \leq \frac{1}{(p - 1)p^{m_\chi} - 1}\text{UP}(d, \omega^u, T).\]

Proof. It follows from Theorem 4.1 (b). \(\Box\)

We will give the proof of Proposition 4.1 later.

Lemma 4.5. Let \(\text{NP}_1, \text{NP}_2, \ldots, \text{NP}_n\) be \(n\) Newton polygons. Assume for each \(1 \leq i \leq n\) there is a rational number \(c\) and a vertex \((k_i, y_i)\) of \(\text{NP}_i\) such that all segments of \(\text{NP}_i\) before this point have slopes strictly less than \(c\), while all segments after that point have slopes greater than \(c\). Then \(\bigoplus_{i=1}^{n} \text{NP}_i\) passes thorough the point

\[\left(\sum_{i=1}^{n} k_i, \sum_{i=1}^{n} y_i\right).\]

Proof. The proof follows from the definition of direct sum “\(\oplus\)” of polygons. \(\Box\)

Lemma 4.6. Any finite character \(\chi\) with conductor \(p^{m_\chi}\) satisfies

\[(4.6.1) \quad (p - 1)p^{m_\chi} \text{NP}_g(C, \omega^0, \chi) \geq \text{NP}_g(C, \omega^0, T).\]

Proof. It is enough to show each monomial \(aT^i \in \mathbb{Z}_q[T]\) satisfy

\[v_p(a(\chi(1) - 1)^i) \geq v_T(aT^i),\]

which follows

\[\begin{align*}
(p - 1)p^{m_\chi}v_p(a(\chi(1) - 1)^i) &= (p - 1)p^{m_\chi}(v_p(a) + iv_p(\chi(1) - 1)) \\
&= (p - 1)p^{m_\chi}v_p(a) + i \\
&\geq v_T(aT^i). \quad \Box
\end{align*}\]

Proof of Proposition 4.1. Proof of (a). Fix a finite character \(\chi_1 : \mathbb{Z}_p \to \mathbb{C}_p^\times\) with conductor \(p\). By Lemma 4.6 we have

\[(4.6.2) \quad (p - 1)\text{NP}_g(C, \omega^0, \chi_1) \geq \text{NP}_g(C, \omega^0, T).\]

By [DWX] Proposition 3.2, the \(p\)-adic Newton polygon \(\text{NP}_g(C, \omega^0, \chi_1)\) passes through the points

\[\left(kd(q - 1), \frac{ak((q - 1)kd - 1)(p - 1)}{2}\right) \quad \text{for any } k \geq 0.\]
Hence, we know that $NP_g(C, \omega^0, T)$ is not above point
\[
\left( kd(q - 1), \frac{ak((q - 1)kd - 1)(p - 1)}{2} \right) \text{ for any } k \geq 0.
\]

On the other hand, by Definition 3.7 and Lemma 3.8 we have

(1) For any $0 \leq u \leq q - 2$ and $k \geq 0$, the point
\[
\left( kd, y_u(kd) \right)
\]
is a vertex of $HP(d, \omega^u, T)$.

(2) All segments of $HP(d, \omega^u, T)$ before this point have slopes strictly less than $ak(p-1)$, while all segments after this point have slopes greater than $ak(p-1)$.

By checking the conditions in Lemma 4.5 we prove $q^{-2} \bigoplus_{u=0}^{q-2} HP(d, \omega^u, T)$ passes through
\[
\left( kd(q - 1), \sum_{u=0}^{q-2} y_u(kd) \right)
\]
for any $k \geq 0$.

Combining it with Proposition 3.11 yields that $NP_g(C, \omega^0, T)$ is not above the points
\[
\left( kd(q - 1), \sum_{u=0}^{q-2} y_u(kd) \right) \text{ for any } k \geq 0.
\]

Thus,
\[
\sum_{u=0}^{q-2} \left( \frac{ak(kd - 1)(p - 1)}{2} + k \sum_{i=0}^{a-1} u(i) \right) = \sum_{u=0}^{q-2} y_u(kd) \leq \frac{ak((q - 1)kd - 1)(p - 1)}{2}. \tag{4.6.3}
\]

Now we show that (4.6.3) is actually an equality.

Consider
\[
\sum_{u=0}^{a-1} \sum_{i=0}^{q-1} u(i) = -a(p - 1) + \sum_{i=0}^{a-1} \sum_{u=0}^{q-1} u(i) = -a(p - 1) + \sum_{i=0}^{a-1} q \frac{p - 1}{2} = \frac{aq(p - 1)}{2} - a(p - 1).
\]

Then we simplify the left-hand side of (4.6.3) by
\[
\sum_{u=0}^{q-2} \left( \frac{ak(kd - 1)(p - 1)}{2} + k \sum_{i=0}^{a-1} u(i) \right) = (p - 1) \left( \frac{aqk}{2} - ak + \frac{(q - 1)ak(kd - 1)}{2} \right) = \frac{ak((q - 1)kd - 1)(p - 1)}{2},
\]
which is equal to its right-hand side. It implies for any $u \in \{0, 1, \ldots, q - 2\}$, the Newton polygon $NP_f(C, \omega^u, T)$ passes through the points
\[
\left( kd, y_u(kd) \right) \text{ for any } k \geq 0.
\]
Proof of (b). From (a), we are able to write
\[(4.6.4)\]
\[r_{u,kd} (T) := \sum_{i = y_u(kd)}^{\infty} r_{u,kd,i} T^i,\]
where \(r_{u,kd,i}\) belongs to \(\mathcal{O}_{C_p}\).

Put \(C_g(\omega^0, T, s) = \sum_{n=0}^{\infty} w_n(T) s^n\). From [DWX], we know that the leading term of \(w_{kd}(T)\) has the form
\[\ast_k T^{\frac{a(k-1)(kd-1)(p-1)}{2}},\]
where \(\ast_k\) is a \(p\)-adic unit. It is easy to show that \(q-2 \prod_{u=0}^{a-2} r_{u,kd,y_u(kd)} = \ast_k\), which implies that \(r_{u,kd,i}\) are all \(p\)-adic units. \(\square\)

Now we are ready to prove our main theorems of this paper.

Proof of Theorem 1.4. (a) From (2.7.1), we obtain
\[C_f(\omega^u, T, s)|_{T=\chi(1)-1} = C_f(\omega^u, \chi, s) = \sum_{k=0}^{\infty} r_{u,k}(\chi(1) - 1)s^k.\]
Therefore, by Proposition 4.1 (b), the Newton polygon \((p - 1)p^{m_\chi - 1} \text{NP}_f(C, \omega^u, \chi)\) is not above point \((kd, y_u(kd))\) for all \(k \geq 0\).

On the other hand, the Hodge polygon \(\text{HP}(d, \omega^u, T)\) forms a lower bound of \((p - 1)p^{m_\chi - 1} \text{NP}_f(C, \omega^u, \chi)\) and for all \(k \geq 0\) the points \((kd, y_u(kd))\) are also vertices of \(\text{HP}(d, \omega^u, T)\).

Therefore, the points \((kd, y_u(kd))\) are forced to be the vertices of \((p - 1)p^{m_\chi - 1} \text{NP}_f(C, \omega^u, \chi)\).

A simple argument about the relation between roots of a power series and its \(p\)-adic Newton polygon completes the proof.

(b) Since the slopes of segments of \(\text{HP}(d, \omega^u, T)\) between \(x = d(k-1)\) and \(x = dk\) are in the interval
\[a(k-1)(p-1) + \frac{1}{d} \sum_{i=0}^{a-1} u(i), a(k-1)(p-1) + \frac{a}{d}(d-1)(p-1) + \frac{1}{d} \sum_{i=0}^{a-1} u(i),\]
by simply applying (a), we know that the slopes of segments of \((p - 1)p^{m_\chi - 1} \text{NP}_f(C, \omega^u, \chi)\) between \(x = d(k-1)\) and \(x = dk\) also in this interval, which completes the proof of (b). \(\square\)

Recall \(\text{UP}(d, \omega^u, T)\) is the upper bound of \(\text{NP}_f(C, \omega^u, T)\) defined in Notation 4.2.

Lemma 4.7. The vertical distance between points in \(\text{UP}(d, \omega^u, T)\) and \(\text{NP}_f(C, \omega^u, T)\) is bounded above by \(\frac{ad(p-1)}{8}\).
Proof. By Corollary 4.3 and Proposition 3.11, we know
\[ \text{UP}(d, \omega^u, T) \geq N_{f_j}(C, \omega^u, T) \geq \text{HP}(d, \omega^u, T). \]
By Corollary 3.10 the polygon \( \text{HP}(d, \omega^u, T) \) above the parabola \( P \) defined by
\[ P(x) := \frac{ax(x - 1)(p - 1)}{2d} + \frac{x \sum_{i=0}^{a-1} u(i)}{d}. \]
Since all vertices \((kd, y_u(kd))\) of \( \text{UP}(d, \omega^u, T) \) coincide with the parabola \( P \), by simple calculation, the maximal vertical distance of \( \text{UP}(d, \omega^u, T) \) and \( P \) is equal to
\[ \max_{k \in \mathbb{Z}_{\geq 0}} \left\{ \frac{P(d(k + 1)) + P(d(k))}{2} - P(d(k + 1/2)) \right\} = \max_{k \in \mathbb{Z}_{\geq 0}} \left\{ \frac{ad(p - 1)}{8} \right\} = \frac{ad(p - 1)}{8}. \]

Proposition 4.8. Let \( \chi \) be a finite character with conductor \( p^{m_x} > \frac{adp}{8} \). Then the Newton polygon \( p^{m_x} \text{NP}_f(C, \omega^u, \chi) \) is independent of \( \chi \).

Proof. Recall in (4.1.1) we denote
\[ C_f(\omega^u, T, s) = \sum_{k=0}^{\infty} r_{u,k}(T)s^k. \]
By Proposition 3.11 and Corollary 3.10 we are able to write \( r_{u,k}(T) \) of the form
\[ r_{u,k}(T) = \sum_{j=y_u(k)}^{\infty} r_{u,k,i}T^j. \]
Assume that \( i(k) \) is the smallest integer such that
- \( i(k) \leq \text{UP}(d, \omega^u, T)(k) \), where \( \text{UP}(d, \omega^u, T)(k) \) is the height of \( \text{UP}(d, \omega^u, T) \) at \( x = k \).
- The corresponding coefficient \( r_{u,k,i(k)} \) is a \( p \)-adic unit.
If such \( i(k) \) does not exist, we simply put \( i(k) = \infty \).
Then we will show that for any \( \chi \) satisfying
\[ p^{m_x} > \frac{adp}{8}, \]
the Newton polygon \( p^{m_x-1}(p - 1) \text{NP}_f(C, \omega^u, \chi) \) is same as \( \text{NP} \left( \left\{ (k, i(k)) \mid k \geq 0 \right\} \right) \).
Since \( C_f(\omega^u, T, s) \in \mathbb{Z}_p[T] \), for any \( \ell < i(k) \) we have
\[ v_p \left( r_{u,k,\ell}(\chi(1) - 1)^\ell \right) \geq 1 + \frac{\ell}{p^{m_x-1}(p - 1)} \]
\[ = \frac{p^{m_x-1}(p - 1) + \ell - i(k) + i(k)}{p^{m_x-1}(p - 1)}. \]
By Lemma 4.7 and the definition of \( i(k) \), we know that
\[ i(k) - \ell \leq \frac{ad(p - 1)}{8}. \]
It follows from the inequalities (4.8.3), (4.8.1) and (4.8.2) that
\[ v_p \left( r_{u,k,\ell} \cdot (\chi(1) - 1)^\ell \right) \geq \frac{p^{m_x-1}(p - 1) - \frac{ad(p - 1)}{8} + i(k)}{p^{m_x-1}(p - 1)} \]
\[ > \frac{i(k)}{p^{m_x-1}(p - 1)} \]
\[ = v_p \left( r_{u,k,i(k)} \cdot (\chi(1) - 1)^{i(k)} \right). \]
The inequality above implies that $v_p(u_{u,k} \cdot (\chi(1) - 1))$ is
- either equal to $\frac{i(k)}{p^{m_\chi-1}(p-1)}$
- or greater than $\frac{1}{(p-1)p^{m_\chi-1}}\text{UP}(d, \omega^u, T)(k)$.

Then this proposition follows directly from Corollary 4.4. □

For a Newton polygon NP and a rational number $t$ recall the definition of Newton polygon $t + NP$ in Notation 2.4 (d).

**Lemma 4.9.** Let $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ be a finite character with conductor $p^{m_\chi}$. Then we have

\[
\{ \alpha \in R(NP_f(C, \omega^u, \chi)) \mid \alpha < ak \} = \bigcup_{i=0}^{k-1} R\left( ai + NP(L_f(\omega^u, \chi, q^i s)) \right),
\]

where $R$ is defined in Definition 2.3.

**Proof.** Since

\[
C_f(\omega^u, \chi, s) = \prod_{i=0}^{\infty} L_f(\omega^u, \chi, q^i s)
\]

and $R(NP_f(\omega^u, \chi, s)) \subset [0, a)$, we know that

\[
\{ \alpha \in R(NP_f(C, \omega^u, \chi)) \mid \alpha < ak \} = \bigcup_{i=0}^{k-1} \{ p\text{-adic Newton slopes of } L_f(\omega^u, \chi, q^i s) \}
\]

\[
= \bigcup_{i=0}^{k-1} R\left( ai + NP(L, \omega^u, \chi) \right).
\]

□

**Proof of Theorem 1.5.** By Proposition 4.8, we have

\[
R\left( \text{NP}(L_f(\omega^u, \chi, s)) \right)
\]

\[
= \left\{ \alpha \in R\left( \text{NP}(C, \omega^u, \chi) \right) \mid \alpha < a \right\}
\]

\[
= \left\{ \frac{\alpha}{p^{m_\chi}} \left| \alpha \in p^{m_\chi}R\left( \text{NP}(C, \omega^u, \chi) \right) \text{ and } \alpha < ap^{m_\chi} \right\} \right. 
\]

\[
= \left\{ \frac{\alpha}{p^{m_\chi}} \left| \alpha \in p^{m_0}R\left( \text{NP}(C, \omega^u, \chi) \right) \text{ and } \alpha < ap^{m_\chi} \right\} \right. 
\]

\[
= \left\{ \frac{\alpha p^{m_0}}{p^{m_\chi}} \left| \alpha \in R\left( \text{NP}(C, \omega^u, \chi) \right) \text{ and } \alpha < ap^{m_\chi-m_0} \right\} \right. 
\]

\[
= \bigcup_{i=0}^{p^{m_\chi-m_0}-1} R\left( \frac{1}{p^{m_\chi-m_0}}(ai + NP(L, \omega^u, \chi)) \right)
\]

\[
= \bigcup_{i=0}^{p^{m_\chi-m_0}-1} \left\{ \frac{\alpha_1 + ai}{p^{m_\chi-m_0}}, \frac{\alpha_2 + ai}{p^{m_\chi-m_0}}, \ldots, \frac{\alpha_{dp^{m_0}-1} + ai}{p^{m_\chi-m_0}} \right\}. 
\]

□

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