Non-associative and projective linear logics

Daniel Lehmann
School of Computer Science and Engineering,
Hebrew University,
Jerusalem 91904, Israel
lehmann@cs.huji.ac.il

January 2022

Abstract

A non-commutative, non-associative weakening of Girard’s linear logic is developed for multiplicative and additive connectives. Additional assumptions capture the logic of quantic measurements.

1 Introduction

The novelty of Quantum Mechanics seemed to require a new Logic and such a Quantum Logic was proposed in [1] and gave rise to sustained activity on non-distributive lattices. The starting point of this approach is that the atomic propositions of Quantum Logic denote (closed) subspaces of a Hilbert space and that the connectives are interpreted as orthogonal complement, intersection and closure of the union.

In [2] Jean-Yves Girard proposed Linear Logic as the logic that could express many logics and stressed a parallel between some of the features of Linear Logic and some properties of quantum systems, expressing his hope that Linear Logic could be more successful in explaining the oddities of Quantum Physics. It contains more connectives than the three connectives just mentioned. To go quantic linear logic must go non-commutative since quantic measurements are represented by self-adjoint operators that do not always commute. A non-commutative, but still associative, version of Linear Logic has been developed, surprisingly easily,
in [3, 5], but no direct connection to the logics of quantum measurements has been put in evidence.

This paper claims that the three operations considered by [1] need to be complemented by a fourth operation between subspaces already studied in [4]. This operation expresses the temporal composition of measurements. It is not associative. This paper develops a non-associative linear logic capable of expressing this operation as a connective.

Section 2 presents the motivation for the paper: a very limited logic for describing the possible states of a quantum system after a sequence of measurements based on a non-associative operation. Sections 3 to 10 present non-associative phase semantics for the multiplicative and the additive linear connectives, and a sound and complete set of proof rules. The rules are the rules presented in [2] except for the exchange rule that is replaced by two limited exchange rules. Section 11 presents a restriction on phase semantics, projective structures, that validates a limited Weakening rule and seems to fit quantum reasoning. A sound and complete set of rules for multiplicative and additive connectives in projective structures is presented.

2 A baby quantum logic

We shall develop a very simple set of quantic propositions and propose that the connective \( \text{times} \) of linear logic express the temporal succession of measurements. This has a double purpose. On one hand, it introduces linear logic to those readers interested in the quantum world and is intended to show them the power of the language of linear logic, before proceeding to its formal mathematical presentation and, on the other hand, it is intended to put in evidence before the logicians, both the need for generalizing linear logic and the specific properties of quantum logic.

We want to talk about the state of a quantic system, say that certain propositions hold in the system, that other propositions do not hold and describe what follows from what. One gets information about a physical system by performing measurements on it. The simplest such piece of information is of the type: \( I \text{ measured a certain variable and I found it has value } x \). In Quantum Physics a variable is a self-adjoint operator in some Hilbert space \( \mathcal{H} \), a value is an eigenvalue of the operator and finding value \( x \) means: \( the \text{ state of the system is in the eigen subspace corresponding to the eigenvalue } x \). In this first effort we shall assume that any subspace of \( \mathcal{H} \) can be the eigen subspace of some operator. To justify this as-
sumption we shall assume that the Hilbert space is finite-dimensional (otherwise
we should probably consider only closed subspaces) and we shall assume that no
superselection rules have to be considered.

We shall identify basic propositions with subspaces of $\mathcal{H}$. Let $\mathcal{P}$ be the set of
all subspaces of $\mathcal{H}$.

The operation “.” allows us to describe sequences of propositions, i.e., se-
quences of measurements. Let $A, B \in \mathcal{P}$ be subspaces of $\mathcal{H}$. The subspace $A.B$
contains the projections on $B$ of the elements of $A$, in other terms $A.B$ is the pro-
jection of $A$ on $B$. The subspace $A.B$ subsumes the proposition: the system has
been measured in subspace $A$ and then measured in subspace $B$. Note that the
operation “.”, already studied in $[4]$, is neither associative nor commutative. The
space $\mathcal{H}$ is a neutral element for “.”: for any subspace $A$, $A.H = H.A = A$
and

Another operation is available on subspaces: to any subspace $A$ corresponds
its orthogonal complement $A^\perp$ and, for any subspace $A$, $A = A^{\perp\perp}$. The calculus
involving “.” and $\perp$ has beautiful properties that will be developed in a general-
ization of Girard’s linear logic starting in Section 3.

We shall now prove two properties of the operation “.” that are crucial for the
generalization of linear logic to be presented.

**Lemma 1** *For any subspaces $A, B, C$:

1. $A.B = \{\vec{0}\}$ iff $B.A = \{\vec{0}\}$,

2. $(A.B).C = \{\vec{0}\}$ iff $A.(C.B) = \{\vec{0}\}$.

**Proof:**

1. $A.B = \{\vec{0}\}$ iff $A$ is orthogonal to $B$ and the orthogonality relation is symmet-
ric.

2. Notice, first, that if $x \in A$, then $x$ is orthogonal to $B$ iff it is orthogonal to $B.A$. Indeed, for any $y \in B$, one has $y = z + w$ where $z$ is the projection of $y$
on $A$ and $w$ is the projection of $y$ on $A^\perp$. Therefore $x$ is orthogonal to $y$ iff $x$
is orthogonal to $z$.

   Let $(A.B).C = \{\vec{0}\}$, i.e. $A.B$ is orthogonal to $C$. Assume $x \neq \vec{0}$, $x \in A.(C.B)$.
   Clearly
   \begin{itemize}
   \item $x$ is not orthogonal to $A$ and
   \item $x \in C.B$ and therefore $x \in B$ and $x$ is not orthogonal to $C$.
   \end{itemize}
By the remark above, $x \in A.B$ and $x$ is not orthogonal to $C$. A contradiction.

Suppose now that $x \in A.(B.C)$, $x \neq \vec{0}$. There are non-null vectors $a$ and $b$ such that $a \in A$, $b$ is the projection of $a$ onto $B$ and $x$ is the projection of $b$ onto $C$. By the remark above, since $b \in B$ and $b$ is not orthogonal to $C$, $b$ is not orthogonal to $C.B$. Therefore the projection, say $y$, of $b$ on $C.B$ is not null. But $y$ is the projection on $C.B$ of the projection of $a$ onto $B$, and therefore $y$ is the projection of $a$ onto $C.B$. We have shown that $A.(C.B)$ is not empty.

3 Q-structures

We shall now define structures into which our propositions will be interpreted. Such structures are a generalization of the phase spaces of Girard’s [2]. We shall define multiplicative and additive connectives on such structures, but no exponentials. We shall provide sound and complete axiomatization for the logic of such structures.

**Definition 1** A Q-structure is a 4-tuple $\langle \mathcal{P}, \mathcal{Z}, \cdot, 1 \rangle$ such that

1. $\mathcal{P}$ is a set,
2. $\mathcal{Z} \subseteq \mathcal{P}$ is a subset of $\mathcal{P}$, the garbage set,
3. $\cdot$ is a binary operation on $\mathcal{P}$ that satisfies, for any $x, y, z \in \mathcal{P}$,
   
   (a) $x \cdot y \in \mathcal{Z}$ iff $y \cdot x \in \mathcal{Z}$,
   (b) $(x \cdot y) \cdot z \in \mathcal{Z}$ iff $x \cdot (y \cdot z) \in \mathcal{Z}$,
4. $1 \in \mathcal{P}$ is a neutral element for $\cdot$, i.e. $1 \cdot x = x \cdot 1 = x$ for any $x \in \mathcal{P}$.

Note that

- the operation $\cdot$ is not assumed to be associative or commutative,
- condition $[3]$ is the suitable weakening of the assumption that the operation $\cdot$ is commutative proposed in $[3][5]$. 

4
• condition \ref{3b} is automatically satisfied if the operation “.” is both commutative and associative, but not if it is only associative,

• the reason condition \ref{3b} has been preferred to the condition \((x.y).z \in \mathcal{Z} \text{ iff } x.(y.z) \in \mathcal{Z}\) is purely circumstantial: we prefer to consider the information gathered by measurements on the final state and not the information gathered on the initial state. A structure satisfying this latter condition instead of condition \ref{3b} satisfies mirror images of the properties of Q-structures,

• in the presence of \ref{3a} condition \ref{3b} is equivalent to: \((x.y).z \in \mathcal{Z} \text{ iff } (z.y).x \in \mathcal{Z}\),

• we use \(\mathcal{Z}\) where Girard uses \(\bot\) because the latter is already heavily overloaded,

The garbage set \(\mathcal{Z}\) and the orthogonality relation, so basic to quantum physics, define each other.

**Definition 2** The orthogonality relation on \(\mathcal{P}\), denoted \(\perp\) is defined by: \(x \perp y \text{ iff } x.y \in \mathcal{Z}\). Then \(\mathcal{Z} = \{x.y \mid x \perp y\}\).

The following is obvious.

**Lemma 2** Condition \ref{3a} is equivalent to the requirement that the relation \(\perp\) be symmetric.

The set \(\mathcal{P}\) should be understood as the set of all imaginable, possible and impossible, situations, possible and impossible worlds. The set \(\mathcal{Z}\) is the set of all impossible, contradictory situations. The operation “.” composes two situations. Two situations are orthogonal iff their composition is impossible. Condition \ref{3a} requires that orthogonality be symmetric. The interpretation of Condition \ref{3b} is less obvious.

We noticed that any phase space, as defined in [2], is a Q-structure. Let us describe two more examples of Q-structures.

Our first example is a presentation of the phase semantics of classical propositional logic, which can throw light on the differences between classical and quantum logic. Let \(\mathcal{V}\) be a set of propositional variables, \(\mathcal{M} = 2^\mathcal{V}\) the set of models for \(\mathcal{V}\) and let \(\mathcal{P} = \mathcal{M} \cup \{0, 1\}\). Define “.” by: for any \(x, y \in \mathcal{M}\) \(x.x = x\) and, if \(x \neq y\), \(x.y = y.x = 0\), 1 is a neutral element and 0 is a zero for “.”: \(0.x = x.0 = 0\). The “.” operation is both commutative and associative.

Our second example is a presentation of the baby quantum logic of Section [2] and is to be compared to the previous example. Given a Hilbert space \(\mathcal{H}\), the set \(\mathcal{P}\)
includes all one-dimensional subspaces of $\mathcal{H}$, its zero-dimensional subspace $\{\vec{0}\}$ and the space $\mathcal{H}$ itself. Note that not all subspaces of $\mathcal{H}$ are elements of $\mathcal{P}$. The set $\mathcal{Z}$ is the singleton that contains the zero-dimensional subspace. The element $1$ is $\mathcal{H}$. The operation "·" is defined by: for any one-dimensional subspaces $x, y$, $x \cdot y = y$ if $x$ and $y$ are not orthogonal and $x \cdot y = \vec{0}$ if $x \perp y$. The set $\mathcal{H}$ is a neutral element and the $\{\vec{0}\}$ is a zero element for "·". Lemma 1 shows that items 3a and 3b of Definition 1 hold.

Here is a summary of this paper’s claims.

1. Q-structures provide the natural extension of Linear Logic to the non-associative, non-commutative case. Their logic exhibits most of the beautiful symmetries of Linear Logic.

2. Associative structures are not fit for Quantum Logic since the basic operation of quantum logic is not associative as already noticed in [4].

3. Q-structures in which the garbage set satisfies an additional property are a suitable framework for quantum logics as will be shown in Section 11.

4 Facts

If the elements of $\mathcal{P}$ are the possible situations, the subsets of $\mathcal{P}$ represent the possible states of information about the situation. In quantum logic, not all subsets of $\mathcal{P}$ represent bona fide information states. For example, an information state that contains $x \in \mathcal{P}$ and also $y \in \mathcal{P}$ must, at least in the absence of a superselection rule, contain all linear combinations of $x$ and $y$. This requirement can be formalized in terms of the orthogonality relation. Girard calls the subsets that represent information states facts and we shall stick with his terminology.

Definition 3 Let $A \subseteq \mathcal{P}$.

$$A^\perp = \{b \in \mathcal{P} \mid b \perp a, \ \forall a \in A\}.$$ (1)

The set $A$ is a fact iff $A = A^{\perp\perp}$.

In the presentation of classical propositional logic of Section 3 $\mathcal{P}$ is a fact and it is the only fact that contains 1, a subset of $Q$ that does not contain 1 is a fact iff it contains 0.
In baby quantum logic, $\mathcal{P}$ is a fact and it is the only fact that contains 1, a subset of $\mathcal{Q}$ that does not contain 1 is a fact iff it contains 0 and all the one-dimensional subspaces of a certain subspace. The facts are in one-to-one correspondence with the subspaces of $\mathcal{H}$, as expected.

The following lemma is proved as in [2]. The use of commutativity is replaced by that of Condition 3a of Definition 1.

**Lemma 3** For any $A, B \subseteq \mathcal{P}$,

1. $A \subseteq A^\perp$,
2. if $B \subseteq A \subseteq \mathcal{P}$, then $A^\perp \subseteq B^\perp$,
3. $A^\perp = A^{\perp \perp}$.

Some basic results about facts will be presented now. They parallel the material in [2], slightly streamlined and replacing the commutativity and associativity of “.” by the conditions in Definition 1.

**Lemma 4**

1. A subset $F \subseteq \mathcal{P}$ is a fact iff there is some $A \subseteq \mathcal{P}$ such that $F = A^\perp$.
2. If $\{F_i\}, i \in I$ is a collection of facts, then its intersection $\bigcap_{i \in I} F_i$ is a fact.
3. $\mathcal{Z} = \{1\}^\perp$. Let $1 \overset{\text{def}}{=} \mathcal{Z}^\perp$. 1 is a fact, 1 $\in 1$. If $x, y \in 1$, then $x, y \in 1$.
4. Let $\mathbf{0} \overset{\text{def}}{=} \mathcal{P}^\perp$. $\mathbf{0}$ is the intersection of all facts.

**Proof:**

1. The only if part is obvious. For the if part, assume $F = A^\perp$. By Lemma 3, we have $F^{\perp \perp} = A^{\perp \perp} = A^\perp = F$.

2. By Lemma 3, $\bigcap_{i \in I} F_i \subseteq (\bigcap_{i \in I} F_i)^{\perp \perp}$. Then, $\bigcap_{i \in I} F_i \subseteq F_j$ for any $j \in I$ implies that $(\bigcap_{i \in I} F_i)^{\perp \perp} \subseteq F_j^{\perp \perp} = F_j$ for any $j \in I$.

3. The first claim is obvious. By item 1 above, 1 is a fact. For any $z \in \mathcal{Z}$, 1 $z = z \in \mathcal{Z}$. This proves that $1 \in \mathcal{Z}^\perp$. For any $x, y \in 1 = \mathcal{Z}^\perp$ we have, for any $z \in \mathcal{Z}$, by Definition 1, item 3a, $z, y \in \mathcal{Z}$, $x, (z, y) \in \mathcal{Z}$ and therefore, by item 3b, $(x, y).z \in \mathcal{Z}$ for any such $z$. We conclude that $x, y \in 1$.

4. For any $A \subseteq \mathcal{Q}$, $\mathbf{0} \subseteq A^\perp$. Therefore $\mathbf{0} \subseteq F$ for any fact $F$. But $\mathbf{0}$ is a fact by Lemma 1.
The operation "." can be applied to subsets of $Q$: $A.B = \{ x \mid x = a.b, a \in A, b \in B \}$.

In the presentation of classical logic proposed above, for any facts $F, G$, $F.G = F \cap G$.

In baby quantum logic, since facts are subspaces, for any facts $F, G$, $F.G$ is the projection of $F$ onto $G$.

## 5 Multiplicative Connectives

We shall now introduce a number of multiplicative connectives. Connectives transform facts into facts. It is important to remember that the arguments of a connective must be facts, not arbitrary subsets of $P$ and that the result must also be a fact.

### 5.1 Linear negation

Our first connective, linear negation is unary.

**Definition 4** For any fact $F$, linear negation is defined by $\sim F = F^\perp$. It is a fact by Lemma 4.

**Lemma 5** Linear negation is involutive: for any fact $F$, $\sim \sim F = F$.

**Proof:** By Definition 3.

### 5.2 The times connective

The multiplicative conjunction, the times connective will be introduced now. It is denoted $\otimes$.

**Definition 5** For any facts $F, G$, $F \otimes G \overset{\text{def}}{=} (F.G)^\perp$. By Lemma 4 it is a fact.

We shall study, now, the properties of the connective times.

**Lemma 6** In any $Q$-structure

1. $1$ is a left-neutral element: for any fact $F$ one has $F = 1 \otimes F$, and
2. *1 is only half a right-neutral element: for any fact *F* one has *F* ⊆ *F* ⊗ *1*. The consideration of baby quantum logic shows that *F* ⊗ *I* is not, in general, equal to *F*.

Proof:

1. Let *A* ⊆ *P*. Let *x* ∈ *A* ⊥. For any *y* ∈ *A* and any *z* ∈ *1* = *Z* ⊥, we have *x*y* ∈ *Z*, *(x.y)z* ∈ *Z* and *x.(z.y)* ∈ *Z*. We have shown that *A* ⊥ ⊆ (*1.A*) ⊥ and therefore (*1.A*) ⊥ ⊆ *A* ⊥ ⊥. For any fact *F*, then *1* ⊗ *F* ⊆ *F*. But, since *1* ∈ *1*, *F* ⊆ *F* ⊗ *1*.

2. Since *1* ∈ *1*, *F* ⊆ *F* ⊗ *1* and *F* = *F* ⊗ *1* ⊆ *F* ⊗ *1*.

The connective ⊗ is not associative, but one can show the following.

**Lemma 7** In a *Q*-structure, for any *A*, *B* ⊆ *P*, one has (*A.B*) ⊥ ⊆ (*A* ⊗ *B*) ⊥.

Proof: Suppose *x* ∈ (*A.B*) ⊥. For any *a* ∈ *A* and any *b* ∈ *B* we have (*a.b)*, *x* ∈ *Z*. Therefore *a.(x.b)* ∈ *Z* and *x.b* ∈ *A* ⊥. Consider any *c* ∈ *A* ⊥ ⊥. We have (*c.b)*, *x* = *c.(x.b)* ∈ *Z* and we see that *x* ∈ (*A* ⊗ *B*) ⊥.

**Lemma 8** In a *Q*-structure, for any facts *F*, *G* and *H* one has (*F* ⊗ *G*) ⊗ *H* = ((*F.G*) ⊗ *H*) ⊥ ⊥.

Proof: Let *F*, *G* and *H* be facts. By Lemma 7, ((*F.G*) ⊗ *H*) ⊥ ⊆ ((*F.G*) ⊥ ⊥ ⊗ *H*) ⊥ ⊥ and therefore, by Lemma 3, (*F* ⊗ *G*) ⊗ *H* ⊆ ((*F.G*) ⊥ ⊥ ⊗ *H*) ⊥ ⊥. But, (*F.G*) ⊗ *H* ⊆ (*F* ⊗ *G*) ⊗ *H* and therefore (*F* ⊗ *G*) ⊗ *H* = ((*F.G*) ⊗ *H*) ⊥ ⊥.

### 5.3 The parallelization connective

The parallelization connective, denoted ⊗ and called *par* is defined as expected.

**Definition 6** *F* ⊗ *G* = (*F* ⊥ ⊥ ⊗ *G* ⊥ ⊥) ⊥.

Clearly, *F* ⊗ *G* is a fact. One easily sees that ⊗ and ⊗ are dual connectives: *F* ⊗ *G* = ~(*F* ⊗ ~*G*) and *F* ⊗ *G* = ~(*F* ⊗ ~*G*). It follows that *Z* is a left-neutral element for ⊗: *Z* ⊗ *F* = *F*, and half a right-neutral element: *F* ⊗ *F* ⊆ *F*. 
5.4 Linear implication

The linear implication will be denoted by $\sim$.

**Definition 7** For any facts $F, G$, one defines $F \sim G = (F \otimes G) \perp$.

Clearly, $F \sim G$ is a fact. Contrary to the commutative case $\sim G \sim F$ is not equal to $F \sim G$. One sees that, as in the commutative case, $F \sim G = \sim (F \otimes \sim G) = \sim F \otimes G, F \otimes G = \sim (F \sim \sim G)$ and $F \otimes G = \sim F \sim G$.

**Lemma 9** For any facts $F, G$, $x \in F \sim G$ iff $x.h \in F \perp$ for every $h \in G \perp$.

**Proof:** $x \in (F \otimes G) \perp$ iff $x.(f.h) \in \mathcal{Z}$ for every $f \in F$ and every $h \in G \perp$ iff $(x.h).f \in \mathcal{Z}$ for every $f \in F$ and every $h \in G \perp$ iff $x.h \in F \perp$ for every $h \in G \perp$.

**Corollary 1** If $F, G$ are facts, then $1 \subseteq F \otimes G$ iff $1 \in F \otimes G$ iff $G \perp \subseteq F$ iff $F \perp \subseteq G$ iff $1 \in G \otimes F$.

**Proof:**

1. $1 \in \{1\} \perp = 1$ and $F \otimes G$ is a fact.

2. $1 \in F \otimes G$ iff $1 \in F \sim G$ iff, by Lemma 9, $G \perp \subseteq F$.

3. Since $F$ and $G$ are facts.

4. As above.

6 Validity

**Definition 8** A fact $F$ is said to be valid in a $Q$-structure if one of the following, equivalent, properties hold:

- $1 \in F$,
- $1 \subseteq F$,
- $F \perp \subseteq \mathcal{Z}$.
The equivalence of the conditions above is obvious.

The next lemma shows that linear implication expresses deduction.

**Lemma 10** Let $F, G$ be facts in a $Q$-structure: $F \rightarrow G$ is valid iff $F \subseteq G$.

**Proof:** Suppose, first, that $1 \in F \rightarrow G$. By Lemma 9, $G \subseteq F^\bot$ and, therefore, $F \subseteq G$.

Suppose, now, that $F \subseteq G$. We have $F.G^\bot \subseteq Z$ and therefore $1 \in F \rightarrow G$ since $Z = \{1\}$.

The next lemma shows that the linear negation allows a jump over the turnstile in both directions. Note that $G$ jumps from the rightmost position to the rightmost position.

**Lemma 11** In a $Q$-structure, for any facts $F, G, H$, $(F \otimes G) \rightarrow H$ is valid iff $F \rightarrow (H \otimes \sim G)$ is valid.

**Proof:** By Lemma 10 we must show that $(F.G)^{\bot \bot} \subseteq H$ iff $F \subseteq (H^\bot \cdot G)^{\bot \bot}$. Assume the former. We have $H^\bot \subseteq (F.G)^{\bot \bot}$. For any $f \in F, g \in G$ and $d \in H^\bot$, we have $(f.g).d \in Z$ and $f.(d.g) \in Z$ and we see that $f \in (H^\bot \cdot G)^{\bot \bot}$.

Assume, now, that $F \subseteq (H^\bot \cdot G)^{\bot \bot}$. For any $f \in F, g \in G$ and $d \in H^\bot$, we have $f.(d.g) \in Z$ and therefore $f.g \in H^{\bot \bot} = H$. We conclude that $F.G \subseteq H$ and therefore $(F.G)^{\bot \bot} \subseteq H^{\bot \bot} = H$.

## 7 Additive connectives

### 7.1 with, the additive conjunction

**Definition 9** If $F, G \subseteq \mathcal{P}$ are facts, $F \& G = F \cap G$.

By Lemma 4 part 2 $F \& G$ is indeed a fact. One sees that with, i.e. $\&$ is associative, commutative and that $\mathcal{P} \& F = F$.

**Lemma 12** The connective par distributes over with. For any facts $F, G, H$, $F \otimes (G \& H) = (F \otimes G) \& (F \otimes H)$ and $(G \& H) \otimes F = (G \otimes F) \& (H \otimes f)$. 
Proof: One easily sees that for any \( A, B \subseteq \mathcal{P} \), one has \((A \cup B)^\perp = A^\perp \cap B^\perp\). Therefore \((F \otimes G) \& (F \otimes H) = ((F^\perp \otimes G^\perp) \cup (F^\perp \otimes H^\perp))^\perp = (F^\perp \otimes (G^\perp \cup H^\perp))^\perp\). But now, \((G^\perp \cup H^\perp)^\perp = G^\perp \cap H^\perp = G \cap H\) and therefore \((G^\perp \cup H^\perp)^\perp = (G \cap H)^\perp\) and \((F \otimes G) \& (F \otimes H) = (F^\perp \otimes (G \cap H)^\perp)^\perp = F \otimes (G \& H)\). The second claim is proved similarly. 

Lemma 13 Let us define \( 0 = \mathcal{P}^\perp \). \( 0 \) is a fact. For any fact \( F \), \( 0 \& F = F \& 0 = 0 \).

Proof: By Lemma 4 and since \( F^\perp \subseteq \mathcal{P} \) we have \( 0 \subseteq F^\perp \subseteq F \). 

Lemma 14 The connective \( \times \) semi-distributes over with, i.e., \( : F \otimes (G \& H) \subseteq (F \otimes G) \& (F \otimes H) \) and \( (G \& H) \otimes F \subseteq (G \otimes F) \& (H \otimes F) \).

Proof: \( F \otimes (G \cap H) \subseteq F \otimes G\) and \( (F \otimes (G \cap H))^\perp \subseteq (F \otimes G)^\perp = F \otimes G\). The reader will easily complete the proof.

7.2  plus, the additive disjunction

Definition 10 If \( F, G \subseteq \mathcal{P} \) are facts, \( F \oplus G = (F \cup G)^\perp \).

By Lemma 4, \( F \oplus G \) is a fact.

One easily sees that \( F \oplus G = \sim (\sim F \& \sim G) \) and \( F \& G = \sim (\sim F \oplus \sim G) \), that \( \oplus \) is associative and commutative, that \( \times \) distributes over \( \oplus \), that \( \mathcal{P} \) is a zero element for \( \oplus \) and that \( \mathcal{P} \) semi-distributes over \( \oplus \); i.e., \((F \otimes G) \oplus (F \otimes H) \subseteq F \otimes (G \oplus H)\).

8  Sequent

We consider the language proposed by Girard on p. 21 of [2], where negation can only be applied to atomic propositions, but limit the individual constants to \( \mathbf{1} \) and \( \mathbf{T} \) since \( \perp \) and \( \mathbf{0} \) can be defined as their linear negations respectively. Given a phase space, if every propositional variable is assigned a fact every formula defines a fact. The constant \( \mathbf{1} \) denotes the set \( \mathcal{Z}^\perp \) and \( \mathbf{T} \) denotes \( \mathcal{P} \).

Since our \( \times \) and \( \mathcal{P} \) connectives are not associative, we must decide how the sequences are interpreted: we choose association to the left. This is consistent with the remark after Definition 1 in [4] that \((F \& G) \cdot H\) has an immediate interpretation: first \( F \), then \( G \), finally \( H \), whereas \( F \cdot (G \& H) \) does not.
Definition 11 A sequent

\[ A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_m \]

where the A’s and the B’s are facts is valid in a Q-structure iff the fact

\[ ((\ldots (A_1 \otimes A_2) \otimes \ldots) \otimes A_n) \sim ((\ldots (B_1 \otimes B_2) \otimes \ldots) \otimes B_m) \]

is valid in the structure. A sequent is valid iff it is valid in any Q-structure.

9 Proof rules for multiplicative and additive connectives: soundness

We shall present sound proof rules for the logic of Q-structures. As in the commutative case, by Lemma 11, we can consider only sequents whose left-and side is empty. Note that the sequent \( \vdash A_1, \ldots, A_n \) is valid iff \( 1 \in (\ldots (A_1 \otimes A_2) \ldots) \otimes A_n \).

9.1 Logical axioms

A logical axiom: \( \vdash \neg A, A \).

Soundness: \( A \otimes A^\bot = (A^\bot . A)^\bot \) and \( A^\bot . A \subseteq Z \). Therefore \( Z^\bot \subseteq A \otimes A^\bot \). But \( 1 \in Z^\bot \).

9.2 Cut rule

\[
\begin{array}{c}
\vdash A, B \\
\vdash \neg A, C
\end{array}
\]

\[ \vdash B, C \]

Soundness: one easily sees, for example by Corollary 1, that \( 1 \in A \otimes B \) iff \( A^\bot \subseteq B \) and \( 1 \in \neg A \otimes C \) iff \( A \subseteq C \). If both assumptions hold, then we have \( A^\bot . A \subseteq B . C \) and \( A^\bot \otimes A \subseteq B \otimes C \). But we have seen in subsection 9.1 that \( 1 \in A^\bot \otimes A \).

9.3 Exchange rules

There is no sweeping exchange rule as in [2] but there are two limited exchange rules.

\[
\text{Exchange I} \quad \begin{array}{c}
\vdash A_1, A_2 \\
\vdash A_2, A_1
\end{array}
\]
Soundness: by Corollary 1, 1 ∈ $A_1 \otimes A_2$ iff $A_1^+ \subseteq A_2$ iff $A_2^+ \subseteq A_1$ iff 1 ∈ $B \otimes A$.

\[
\text{Exchange2} \quad \frac{\vdash A_1, A_2, A_3}{\vdash A_3, A_2, A_1}
\]

Let us fix a Q-structure. Assume $\vdash A_1, A_2, A_3$ is valid in the Q-structure. By Lemma 11 $\sim A_3 \vdash A_1, A_2$ is valid. By Lemma 10 $A_3^\perp \subseteq A_1 \otimes A_2$. Therefore $(A_1 \otimes A_2)^\perp \subseteq A_3$ and $A_1^+ \otimes A_2^+ \subseteq A_3$ and $\sim A_1 \otimes \sim A_2 \vdash A_3$. By Lemma 11 again: $\sim A_1 \vdash A_3, A_2$ and $\vdash A_3, A_2, A_1$ in the Q-structure.

### 9.4 Additive rules

- $\vdash \top, A$  
  **Axiom** $\top$.
  Soundness: $\mathcal{P} \otimes A = 0 \rightarrow A$ and, by Lemma 4, $0 \subseteq A$ and $1 \in 0 \rightarrow A$.

- $\vdash A, C \vdash B, C \vdash A \& B, C$ &

  Soundness follows from the distributivity of $\mathsf{par}$ over with: $(A \& B) \otimes C = (A \otimes C) \& (B \otimes C)$ and therefore, if $1 \in A \otimes C$ and $1 \in B \otimes C$, we have $1 \in (A \& B) \otimes C$.

- $\vdash A, C \vdash A \oplus B, C$ & $\vdash A, C \vdash B \oplus A, C$  
  **⊕1** & **⊕2**

  Soundness of $\oplus 1$ follows from the half-distributivity of $\mathsf{par}$ over plus: $(A \otimes C) \oplus (B \otimes C) \subseteq (A \oplus B) \otimes C$ and therefore $1 \in A \otimes C$ implies $1 \in (A \oplus B) \otimes C$.

### 9.5 Multiplicative rules

- $\vdash 1$  
  **Axiom** $1$
  Soundness: $1 \in \{1\}^{\perp}$.

- $\vdash A \vdash \sim 1, A$  
  \[ \perp \]

  Soundness follows from $\mathcal{Z} \otimes A = A$. 

14
Soundness is proved by:
The assumptions are equivalent to $C \sqsubseteq A$ and $D \sqsubseteq B$. Therefore $C \sqcup D \sqsubseteq A \cdot B$ and $(C \otimes D) \sqsubseteq A \otimes B$. We see, by Lemma 10 that $1 \in (C \otimes D) \rightarrow (A \otimes B)$. We conclude that $1 \in (C \otimes D) \otimes (A \otimes B)$.

where $\sigma$ is any sequence of formulas. Soundness follows from our interpretation of the commas in a sequent as a left-associative par connective.

It is now clear that a sequent that is provable from the rules described above is valid in any Q-structure.

**Theorem 1 (Soundness)** Any sequent provable from the axioms and the rules above is valid in any Q-structure.

**Proof:** By induction on the length of the proof. 

---

**10 Completeness**

The proof that the rules above are complete for Q-structures follows the line of the corresponding proof in [2], but the differences require attention. Since the original exchange rule has been replaced by much weaker rules, the side of the sequents must be considered as sequences and not as multi-sets. Since the connectives $\otimes$ and $\otimes$ are not associative, the elements of the universal phase structure central in the completeness proof cannot be sequences and concatenation, they have to be formulas and the composition must be $\otimes$.

We shall define a suitable Q-structure. Let $\mathcal{L}$ be the propositional language defined in Section 8. The carrier of our Q-structure, $M \overset{\text{def}}{=} \mathcal{L} \cup \{\epsilon\}$ contains the propositions and a distinguished element $\epsilon$. 

15
The "•" operation is defined by \( x \cdot y \) if \( x, y \in \mathcal{L} \) and \( x \cdot \epsilon = x \cdot x = x \) for any \( x \in M \). The distinguished element \( \epsilon \) is a neutral element for "•".

To define the garbage set \( \mathcal{Z} \) we need some notation. Let \( \sigma \in \mathcal{L}^* \) be a sequence of propositions: \( \sigma = A_1, \ldots, A_n \) with \( n \geq 2 \). The sequence \( \sigma \) defines a proposition \( \overline{\sigma} = (\ldots (A_1 \otimes A_2) \ldots) \otimes A_n \) where the propositions are connected with the \( \text{par} \) connective \( \otimes \) in a left-associative way. We extend the definition by setting \( \overline{A} = A \) for any \( A \in \mathcal{L} \) and \( \overline{\epsilon} = \sim 1 \). The garbage set \( \mathcal{Z} \) can now be defined by: \( \mathcal{Z} = \{ \overline{\sigma} \mid \sigma \in \mathcal{L}^* \text{ such that } \vdash \sigma \} \).

We shall now verify that the conditions 3a and 3b of Definition 1 are satisfied. Cases involving \( \epsilon \) are easily treated and we may assume \( x, y \in \mathcal{L} \). For 3a, note that \( x \cdot y \in \mathcal{Z} \) iff \( x \cdot x \in \mathcal{Z} \) iff \( \vdash x \cdot y \) by our Exchange1 rule, and \( y \cdot x \in \mathcal{Z} \) by 3a.

For 3b, \( (x \cdot y) \cdot z \in \mathcal{Z} \) iff \( x \cdot z \in \mathcal{Z} \) iff \( \vdash x \cdot y \cdot z \) which is equivalent to \( (z \cdot y) \cdot x \) which is equivalent to \( (z \cdot y) \cdot x \in \mathcal{Z} \) and to \( x \cdot (z \cdot y) \in \mathcal{Z} \) by 3a.

We have just defined a Q-structure, that we shall call \( M \), as its carrier. Note that the definition of \( \mathcal{Z} \) implies that for any \( x, y \in \mathcal{L}, x \perp y \iff \vdash x, y \).

To any formula \( x \in \mathcal{L} \) we shall associate a subset of \( M, S(x) \). We intend \( S(x) \) to be a fact in the Q-structure \( M \) for any \( x \). The definition of \( S(x) \) proceeds by induction on the size of \( x \).

1. \( S(1) = \mathcal{Z}^\perp \),
2. \( S(\top) = M \),
3. \( S(\sim x) = (S(x))^\perp \),
4. \( S(x \& y) = S(x) \cap S(y) \),
5. \( S(x \oplus y) = (S(x) \cup S(y))^\perp \),
6. \( S(x \otimes y) = (S(x) \cdot S(y))^\perp \),
7. \( S(x \oslash y) = (S(x)^\perp \cdot S(y)^\perp)^\perp \),
8. for every propositional letter \( a \), \( S(a) = Pr(a) \) as defined in Definition 12 just below.

**Definition 12** For any \( x \in \mathcal{L} \), we let

\[
Pr(x) \overset{\text{def}}{=} \{ \perp \}. 
\]

For every \( x \in \mathcal{L} \), \( Pr(x) \) is a fact of the Q-structure \( M \).
An equivalent definition is:

\[ Pr(x) = \{ \sigma | \sigma \in \mathcal{L}^* \text{ such that } \vdash \sigma, x \}. \]

**Lemma 15** For any formula \( x \), \( (Pr(x))^\perp = Pr(\neg x) \).

**Proof:** Let \( y \in Pr(x) \) and \( z \in Pr(\neg x) \). We have \( y \in \{x\}^\perp \) and \( z \in \{\neg x\}^\perp \). Therefore \( \vdash x, y \) and \( \vdash \neg x, z \). By Cut we conclude that \( \vdash y, z \), and therefore, \( z \in \{y\}^\perp \). We have shown that \( Pr(\neg x) \subseteq Pr(x)^\perp \).

Conversely, since \( \vdash \neg x, x \) is an axiom \( \neg x \in Pr(x) \) and \( Pr(x)^\perp \subseteq (\neg x)^\perp = Pr(\neg x) \). \( \square \)

We want, now, to show that the interpretation \( S(x) \) in the Q-structure we defined for any formula \( x \) is exactly \( Pr(x) \).

**Lemma 16** For any \( x \in \mathcal{L} \), \( S(x) = Pr(x) \).

**Proof:** By induction on the length of the formula \( x \).

1. Let \( x = a \) for a propositional letter \( a \). By construction we have \( S(x) = Pr(x) \).

2. Let \( x = 1 \), by Axiom 1 we have \( 1 \in \mathcal{Z}, \{1\} \subseteq \mathcal{Z}, \mathcal{Z}^\perp \subseteq Pr(1) \). Assume, now, that \( y \in Pr(1) \). We have \( \vdash 1, y \). Let \( z \in \mathcal{Z} \). We have \( \vdash z \). By Rule \( \bot \) we have \( \bot, z \), i.e., \( \bot, 1, z \) and by the Cut rule we have \( \vdash z, y \) and \( y \in \mathcal{Z}^\perp \). We have now shown that \( Pr(1) \subseteq \mathcal{Z}^\perp \).

3. Let \( x = \top \). By Axiom \( \top \), \( Pr(\top) = M \).

4. Let \( x = \neg y \). We have \( S(\neg y) = S(x)^\perp = Pr(x)^\perp = Pr(\neg x) \) by Lemma 15.

5. Let \( x = y \& z \). We have \( S(y \& z) = S(y) \cap S(z) = Pr(y) \cap Pr(z) \subseteq Pr(y \& z) \) by Rule \&. For the converse inclusion, the proof is the classical one, see [2] for example: one only needs to check that Exchange is not used.

6. Let \( x = y \oplus z \). By duality with the previous case.

7. Let \( x = y \otimes z \). We have \( S(y \otimes z) = (S(y), S(z))^\perp = (Pr(y), Pr(z))^\perp \).

For any \( u \in Pr(y), t \in Pr(z) \), we have \( \vdash y, u \) and \( \vdash z, t \). By Rule \( \otimes \), then, we have \( \vdash u, t, y \otimes z, \vdash u \otimes t, y \otimes z \), by Exchange \( 1 \) \( \vdash y \otimes z, u \otimes t \) and \( u \otimes t \in Pr(y \otimes z) \). We have proved that \( Pr(y) \otimes Pr(z) \subseteq Pr(y \otimes z) \). But \( Pr(y), Pr(z) = Pr(y) \otimes Pr(z) \).
Therefore \((\text{Pr}(y).\text{Pr}(z))^\perp \subseteq \text{Pr}(y \otimes z)^\perp = \text{Pr}(y \otimes z)\). We have proved that 
\(S(y \otimes z) \subseteq \text{Pr}(y \otimes z)\).

Conversely, let \(t \in (\text{Pr}(y).\text{Pr}(z))^\perp\). For any \(u \in \text{Pr}(y)\), \(v \in \text{Pr}(z)\) we have 
\(\vdash u \otimes v, t\) and therefore \(\vdash \text{Pr}(y)\otimes \text{Pr}(z), t\). We see that \(\vdash \sim y \otimes z, t\) and \(\vdash \sim (y \otimes z), t\).

For any \(w \in \text{Pr}(y \otimes z)\), we have \(\vdash y \otimes z, w\). By Cut we get \(\vdash t, w\) and we conclude that \(\text{Pr}(y \otimes z) \subseteq \text{Pr}(y).\text{Pr}(z) \subseteq (\text{Pr}(y).\text{Pr}(z))^\perp = S(y \otimes z)\).

8. Let \(x = y \otimes z\). By duality with the previous case.

\[\]

We can now conclude.

**Theorem 2** Any formula valid in any Q-structure and any assignment of facts to propositional letters is provable in the system of axioms and rules presented in Section 9.

**Proof:** Any formula \(x\) valid in the Q-structure \(M\) satisfies \(\epsilon \in S(x)\) and therefore, by Lemma 16 \(\vdash x, \epsilon\). But \(\epsilon^\perp = 1\) and, by Axiom1 and Cut we get \(\vdash x\). \[\]

The present effort does not propose exponential connectives for Q-structures because no sound and complete rules were found for the natural generalization of topolinear structures to non-associative phase structures. In particular, the formula \(?A \otimes ?B \circ \circ (?A \otimes B)\), central in Girard’s treatment, is not valid in such structures: Girard’s use of the Exchange rule cannot be circumvented by Exchange1 and Exchange2.

### 11 Projective structures

The Q-structures we have described present a sub-structural logic without Contraction or Weakening and with a very limited Exchange rule. It generalizes the Linear Logic of Girard’s [2] and is satisfied by the baby quantum logic of Section 2. But some additional rules seem to be valid in Quantum Logic. We shall study limited forms of Contraction and Weakening.

If one performs a measurement on a quantic system, it is a fundamental principle that a second performance of the same measurement will not change the state of the system: the same value will be obtained with probability one. So it seems
that the rule:

\[
\frac{\vdash \sigma, A, \tau}{\vdash \sigma, A, A, \tau}
\]

should be valid. This is a limited form of Weakening.

The inverse Contraction rule also seems to be valid.

If we understand the sequent \(A_1, \ldots, A_n \vdash B_1, \ldots, B_m\) as meaning that any state resulting from the sequence of measurements on the left satisfies the condition described by the sequence on the right, we would expect that any extension on the left of the left-hand side can only restrict the set of final states and therefore we expect the rule

\[
\frac{\sigma \vdash \tau}{A, \sigma \vdash \tau}
\]

should be valid. This is a limited form of Weakening.

Note that Baby Quantum Logic and our presentation of classical logic satisfy the rules above.

**Definition 13** A Q-structure \(\langle P, Z, ., 1 \rangle\) is projective iff the operation “.” absorbs into \(Z\): for any \(x \in Z\) and any \(y \in P\) one has \(x.y \in Z\).

**Lemma 17** In a projective Q-structure, for any \(x, y, z \in P\):

1. \(x \perp y\) iff \(x.x \perp y\),
2. if \(x \perp z\) and \(y \perp z\), then \(x.y \perp z\).

Property 1 essentially means \(x.x = x\), and hence the term projective. It expresses the idea that combining something with itself leaves the situation essentially unchanged. Property 2 expresses the idea that if both \(x\) and \(y\) are incompatible with \(z\), the combination \(x.y\) must also be incompatible with \(z\). Note that our baby quantum logic is projective, and so is our presentation of classical logic in Section 3.

**Proof:**

1. \(x \perp y\) iff \(y.x \in Z\) implies \((y.x).x \in Z\) since the structure is projective. But \((y.x).x \in Z\) implies \(y.(x.x) \in Z\) by Definition 1.
2. \(z.y \in Z\) implies \(x.(z.y) \in Z\) which implies \((x.y).z \in Z\).

Our next result presents basic properties of projective Q-structures.
Lemma 18 In a projective Q-structure:

1. for any $A \subseteq \mathcal{P}$, $A \subseteq (AA)_{\perp \perp}$,
2. for any $A \subseteq \mathcal{P}$, $A_{\perp}A_{\perp} \subseteq A_{\perp}$,
3. for any fact $F$, $F.F \subseteq F$, $F \otimes F = F$ and $F \bowtie F = F$,
4. $1 = \mathcal{P}$ and $\mathcal{Z} = 0$,
5. for any fact $F$, one has $\mathcal{Z} \subseteq F$.

Proof:

1. By property \red{1} of Lemma \red{17} for any $a \in A$, $a \in \{a\}_{\perp \perp}$ and $a.a \in \{a\}_{\perp \perp}$. We see that $A \subseteq (AA)_{\perp \perp}$.
2. By property \red{2} in Lemma \red{17}.
3. By item \red{2} $F_{\perp \perp}F_{\perp \perp} \subseteq F_{\perp \perp}$ and therefore $F.F \subseteq F$. We see that $F \otimes F \subseteq F_{\perp \perp} = F$. By item \red{1} $F \subseteq F \otimes F$. By duality, one easily sees that $F \bowtie F = F$.
4. By the absorption property $\mathcal{Z}_{\perp} = \mathcal{P}$.
5. For any $A \subseteq \mathcal{P}$, by the absorption property, one has $\mathcal{Z} \subseteq A_{\perp}$.

Lemma 19 In a projective Q-structure, for any $A, B \subseteq \mathcal{P}$, one has $A_{\perp} \subseteq (B.A)_{\perp}$ and $A \subseteq (B.A_{\perp})_{\perp}$.

Proof: Let $x \in A_{\perp}$, $y \in A, z \in B$. We have $x.y \in \mathcal{Z}$ and therefore, by absorption, $(x.y).z \in \mathcal{Z}$ and $x.(z.y) \in \mathcal{Z}$. We conclude that $A_{\perp} \subseteq (B.A)_{\perp}$. Similarly $y.x \in \mathcal{Z}$, $(y.x).z \in \mathcal{Z}$, $y.(z.x) \in \mathcal{Z}$ and $A \subseteq (B.A_{\perp})_{\perp}$.

Corollary 2 For any $A \subseteq \mathcal{P}$ and any facts $F$, $G$: $A.F \subseteq F$, $G \otimes F \subseteq F$ and $F \subseteq G \bowtie F$. 

20
Proof: By lemma \[19\] \( F \subseteq (A.F) \) and therefore \( (A.F) \subseteq F \) and \( G \otimes F \subseteq F \). Last claim is proved by duality. □

Our last result shows that the Contraction rule discussed above is valid.

**Lemma 20** In a projective Q-structure, for any facts \( F, G \), one has \( (G \otimes F) \otimes F \subseteq G \otimes F \).

**Proof:** By Corollary \[2\] \( F \subseteq G \otimes F \) and therefore \( (G \otimes F) \otimes F \subseteq (G \otimes F) \otimes (G \otimes F) \). Lemma \[18\] then, shows that \( (G \otimes F) \otimes F \subseteq G \otimes F \). □

### 12 Rule for the projective case

The added rule, that expresses the essence of our requirement that \( Z \) be absorbing is a Weakening rule.

\[
\frac{\vdash A}{\vdash A, B} \quad \text{WR (Right – Weakening)}
\]

Soundness follows from the following.

**Lemma 21** For any facts \( A, B \), if \( 1 \subseteq A \), then \( 1 \subseteq A \otimes B \).

**Proof:** \( 1 \subseteq A \) is equivalent to \( A \subseteq Z \), which implies \( A \otimes B \subseteq Z \otimes B \subseteq Z \) by the absorption property of Definition \[13\]. We conclude that \( 1 = Z \subseteq A \otimes B \). □

**Theorem 3** The system of the eleven rules and axioms in Section \[9\] and the added WR rule is sound and complete for projective Q-structures phase semantics.

**Proof:** Soundness has been proved on the way. For completeness, we shall use the technique used in Section \[10\].

Our only task is to show that, under the new rules, the Q-structure built is projective. This is guaranteed by the Right-Weakening rule WR. We want to show that, if \( A, B \in Z \), one has \( A \otimes B \in Z \), i.e., \( A \otimes B \in Z \). Assume \( \vdash A \), by WR we have \( \vdash A, B, \vdash A \otimes B \) and \( A \otimes B \in Z \).

We can now conclude.

**Theorem 4** Any formula valid in any projective Q-structure and any assignment of facts to propositional letters is provable in the system of axioms and rules presented in Sections \[9\] and \[12\].
\textbf{Proof:} Any formula $x$ valid in any projective Q-structure $M$ satisfies $\epsilon \in S(x)$ and therefore, by Lemma 3 $\vdash x, \epsilon$. But $\epsilon^\perp = 1$ and, by Axiom1 and Cut we get $\vdash x$. 

\section*{References}

[1] Garrett Birkhoff and John von Neumann. The logic of quantum mechanics. \textit{Annals of Mathematics}, 37:823–843, 1936.

[2] Jean-Yves Girard. Linear logic. \textit{Theoretical Computer Science}, 50:1–102, 1987.

[3] Jean-Yves Girard. Seminar lectures. Le Groupe Interuniversitaire en Etudes Categoriques, McGill University, Montreal, November 1987.

[4] Daniel Lehmann. A presentation of quantum logic based on an “and then” connective. \textit{Journal of Logic and Computation}, 18(1):59–76, February 2008. doi: 10.1093/logcom/exm054.

[5] David N. Yetter. Quantales and (noncommutative) linear logic. \textit{The Journal of Symbolic Logic}, 55(1):41–64, March 1990. https://www.jstor.org/stable/2274953.