DICHOTOMY AND PERIODIC SOLUTIONS TO PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We establish a sufficient condition for existence and uniqueness of periodic solutions to partial functional differential equations of the form
\[ \dot{u} = A(t)u + F(t)(u_t) + g(t,u_t), \]
where the operator-valued functions \( t \mapsto A(t) \) and \( t \mapsto F(t) \) are 1-periodic, the nonlinear operator \( g(t,\phi) \) is 1-periodic with respect to \( t \) for each fixed \( \phi \in C := C([-r,0],X) \), and satisfying
\[ \| g(t,\phi_1) - g(t,\phi_2) \| \leq \varphi(t)\| \phi_1 - \phi_2 \|_C \]
for \( \phi_1, \phi_2 \in C \) with \( \varphi \) being a positive function such that \( \sup_{t \geq 0} \int_{t}^{t+1} \varphi(\tau)d\tau < \infty \). We then apply the results to study the existence, uniqueness, and conditional stability of periodic solutions to the above equation in the case that the family \( (A(t))_{t \geq 0} \) generates an evolution family having an exponential dichotomy. We also prove the existence of a local stable manifold near the periodic solution in that case.

1. Introduction and preliminaries. Consider the abstract partial functional differential equation
\[ \dot{u} = A(t)u + F(t)(u_t) + g(t,u_t), \quad t \in \mathbb{R}_+, \]
where for each \( t \in \mathbb{R}_+ \), \( A(t) \) is a possibly unbounded operator on a Banach space \( X \) such that \( (A(t))_{t \geq 0} \) generates an evolution family \( (U(t,s))_{t \geq s \geq 0} \) on \( X; F(t) \in \mathcal{L}(C,X) \) with \( C := C([-r,0],X) \), and \( g : \mathbb{R}_+ \times C \to X \) is continuous and locally Lipschitz; \( u_t \) is the history function defined by \( u_t(\theta) = u(t + \theta) \) for \( \theta \in [-r,0] \).

One of the important research directions related to the asymptotic behavior of the solutions to the above equation is to find conditions for the existence and stability of a periodic solution to (1.1) in case that \( F \) and \( g \) are 1-periodic functions with respect to \( t \). There are several approaches to handle this problem such as Tikhonov’s fixed point method [21], Lyapunov functionals [24], Fredholm operator and translation...
formulae [17, 23], and the most popular approach is the use of ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings (see [1, 11, 20, 21, 22, 24] and references therein). However, in some concrete applications, e.g., to partial differential equations in unbounded domains or to equations that have unbounded solutions, such compact embeddings are no longer valid, and the existence of bounded solutions is not easy to obtain since one has to carefully choose an appropriate initial vector (or conditions) to guarantee the boundedness of the solution emanating from that vector.

Recently, in [9], for the case of partial differential equations without delay we have proposed a new approach to overcome such difficulties. Namely, we start with the linear equation

$$\dot{u} = A(t)u + f(t), \quad t \geq 0$$  \hspace{1cm} (1.2)  

and use a Cesàro sum to prove the existence of a periodic solution through the existence of bounded solution whose sup-norm can be controlled by the sup-norm of the input function $f$. Then, we use the fixed point argument to prove counterpart results for the corresponding semi-linear problem. We refer to [7] for the use of an ergodic approach for the case of Stokes and Navier-Stokes equations around rotating obstacles. Such an approach has also been extended to more general fluid flow problems in [4].

In the present paper, we will consider the existence and uniqueness of periodic solutions to partial functional differential equations (PFDE) with a $\varphi$-Lipschitz nonlinear term $g$, i.e., $\|g(t, \phi_1) - g(t, \phi_2)\| \leq \varphi(t) \|\phi_1 - \phi_2\|_C$ for $\phi_1, \phi_2 \in C$ where $\varphi$ is a real and positive function such that $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau < \infty$. The difficulties we face when passing to this case of PFDE are the following two features: Firstly, since the nonlinear delay $g$ is $\varphi$-Lipschitz, the standard method for construction of bounded solutions relevant for uniform Lipschitz continuous functions is no longer valid. Secondly, the evolution family generated by $(A(t))_{t \geq 0}$ does not act on the same Banach space as that the initial functions belong to (in fact, the former acts on $X$, and the latter belong to $C$).

To overcome such difficulties, we combine the methods and results in [9] with the use of admissible spaces and appropriate choices of nonlinear operators to prove the existence and uniqueness of the periodic solution to Equation (1.1) without using the uniform boundedness and smallness of Lipschitz constants of the nonlinear terms. Instead, the smallness is now understood as the sufficient smallness of $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau$.

It is worth noting that our framework fits perfectly to the situation of exponentially dichotomic linear parts, i.e., the case when family $(A(t))_{t \geq 0}$ generates an evolution family $(U(t, s))_{t \geq s \geq 0}$ having an exponential dichotomy (see Definition 4.1 below), since in this case we can choose the initial vector from that emanates a bounded solution. Moreover, we can also prove the conditional stability of periodic solutions as well as the existence of a local stable manifold around the periodic solution. Our main results are contained in Theorems 2.3 and 3.1. The applications of our abstract results to semi-linear delay PFDE with the exponentially dichotomic linear parts are given in Subsection 4.1. Finally, in Subsection 4.2, we also prove the existence of a local stable manifold around the periodic solution.

We now recall some notions for latter use. Firstly, as in [10] we denote by

$$M = M(\mathbb{R}^+):= \left\{ f \in L_{t+1,t}(\mathbb{R}^+): \sup_{t \geq 0} \int_t^{t+1} |f(\tau)| d\tau < \infty \right\} \hspace{1cm} (1.3)$$
endowed with the norm \( \|f\|_M := \sup_{t \geq 0} \int_t^{t+1} |f(\tau)| d\tau \). Clearly, \( M \) is a Banach space and it is an admissible Banach function space in the sense of [10, Def. 1.2, Examp. 1.3]. We have also properties of \( M \) in the following proposition which is a consequence of [5, Proposition 2.6].

**Proposition 1.1.** The following assertions hold.

(a) \( M \) is \( T_1^+\)-invariant, where \( T_1^+ \) is defined by

\[
[T_1^+ \varphi](t) := \begin{cases} 
\varphi(t-1) & \text{for } t \geq 1 \\
0 & \text{for } 0 \leq t \leq 1.
\end{cases}
\]

(b) For \( \varphi \in M \) and \( \sigma > 0 \) we define functions \( \Lambda_\sigma^\prime \varphi \) and \( \Lambda_\sigma^\prime\prime \varphi \) by

\[
\Lambda_\sigma^\prime \varphi(t) = \int_0^t e^{-\sigma(t-s)} \varphi(s) ds \quad \text{and} \quad \Lambda_\sigma^\prime\prime \varphi(t) = \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds.
\]

Then, \( \Lambda_\sigma^\prime \varphi \) and \( \Lambda_\sigma^\prime\prime \varphi \) belong to \( M \), and they are bounded. Moreover, denoted by \( \|\cdot\|_\infty \) the esssup-norm, we have

\[
\|\Lambda_\sigma^\prime \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|T_1^+ \varphi\|_M \quad \text{and} \quad \|\Lambda_\sigma^\prime\prime \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\varphi\|_M.
\]

(1.4)

**Proof.** See [5, Proposition 2.6].

Next, in space \( M \) we consider the following subset consisting of 1-periodic functions denoted by

\[
P := \{f \in M : f \text{ is 1-periodic}\}.
\]

Let now \( \varphi \) be a positive function belonging to \( P \). Then, we have the following inequalities which are taken from [10, Ineq. (1.8)]

\[
\|\Lambda_\sigma^\prime \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\varphi\|_M \quad \text{and} \quad \|\Lambda_\sigma^\prime\prime \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\varphi\|_M \quad \text{for all positive function } \varphi \in P.
\]

(1.6)

We now recall the cone-inequality theorem which will be used to prove the conditional stability of solutions. Given a cone \( K \) in a Banach space \( W \) (see [2, Chapt. I] for the notion of a cone), we will write \( x \leq y \) if \( y - x \in K \). The following cone-inequality theorem is taken from ([2], Theorem I.9.3).

**Theorem 1.2** (Cone Inequality). Let \( K \) be a cone given in a Banach space \( W \) such that \( K \) is invariant under a bounded linear operator \( A \in \mathcal{L}(W) \) having spectral radius \( r < 1 \). If a vector \( x \in W \) satisfies the inequality

\[x \leq Ax + z \quad \text{for some given } z \in W,\]

then it also satisfies the estimate \( x \leq y \), where \( y \in W \) is the solution of the equation \( y = Ay + z \).

Next, for a Banach space \( X \) (with a norm \( \|\cdot\| \)) and a given \( r > 0 \) we denote by \( C := C([-r, 0], X) \) the Banach space of all continuous functions from \([-r, 0]\) into \( X \), equipped with the norm \( \|\phi\|_C = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\| \) for \( \phi \in C \). For a continuous function \( v : [-r, \infty) \to X \) the history function \( v_t \in C \) is defined by \( v_t(\theta) = v(t + \theta) \) for all \( \theta \in [-r, 0] \).
Definition 1.3 (Local $\varphi$-Lipschitz functions). Consider a positive function $\varphi \in M$ and let $B_\rho$ be the ball with radius $\rho$ in $\mathcal{C}$ i.e. $B_\rho := \{ \phi \in \mathcal{C} : \| \phi \|_\mathcal{C} \leq \rho \}$. A function $g : [0, \infty) \times B_\rho \to X$ is said to belong to the class $(L, \varphi, \rho)$ for some positive constants $L, \rho$ if $g$ satisfies,

(i) $\| g(t, \phi) \| \leq L \varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and $\phi \in B_\rho$,
(ii) $\| g(t, \phi_1) - g(t, \phi_2) \| \leq \varphi(t) \| \phi_1 - \phi_2 \|$ for a.e $t \in \mathbb{R}_+$ and all $\phi_1, \phi_2 \in B_\rho$.

Now, for the space $M$ defined as in (1.3) we denote by

$$\mathfrak{M} := \{ f : \mathbb{R}_+ \to X \mid \| f(\cdot) \| \in M \}$$

(1.7)

endowed with the norm $\| f \|_{\mathfrak{M}} := \| \| f(\cdot) \| \|_M$. Clearly $\mathfrak{M}$ is a Banach Space.

We also need the following space of bounded, continuous functions

$$C_b(\mathbb{R}_+, X) := \{ v : \mathbb{R}_+ \to X \mid v \text{ is continuous and } \sup_{t \in \mathbb{R}_+} \| v(t) \| < \infty \}$$

(1.8)

endowed with the norm $\| v \|_{C_b} := \sup_{t \in \mathbb{R}_+} \| v(t) \|$.

2. Bounded and periodic solutions to linear evolution equations. Given a function $f$ taking values in a Banach space $X$ having a separable predual $Y$ (i.e., $X = Y'$ for a separable Banach space $Y$) we consider the non-homogeneous linear problem for the unknown function $u(t)$

$$\begin{cases}
\frac{du}{dt} = A(t)u(t) + f(t) \text{ for } t > 0, \\
u(0) = x \in X,
\end{cases}$$

(2.1)

where the family of partial differential operators $(A(t))_{t \geq 0}$ is given such that the homogeneous Cauchy problem

$$\begin{cases}
\frac{du}{dt} = A(t)u(t) \text{ for } t > s \geq 0 \\
u(s) = y \in X
\end{cases}$$

(2.2)

is well-posed. By this we mean that there exists an evolution family $(U(t, s))_{t \geq s \geq 0}$ such that the solution of the Cauchy problem (2.2) is given by $u(t) = U(t, s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [19] (see also Nagel and Nickel [18] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$). We next give the precise concept of an evolution family in the following definition.

Definition 2.1. A family of bounded linear operators $(U(t, s))_{t \geq s \geq 0}$ on a Banach space $X$ is a (strongly continuous, exponentially bounded) evolution family if

(i) $U(t, t) = \text{Id}$ and $U(t, r)U(r, s) = U(t, s)$ for all $t \geq r \geq s \geq 0$,
(ii) the map $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in X$, where $(t, s) \in \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$,
(iii) there are constants $K, \alpha \geq 0$ such that $\| U(t, s)x \| \leq Ke^{\alpha(t-s)}\| x \|$ for all $t \geq s \geq 0$ and $x \in X$.

The existence of the evolution family $(U(t, s))_{t \geq s \geq 0}$ allows us to define a notion of mild solutions as follows. By the mild solution to (2.1) we mean a function $u$
satisfying the following integral equation
\[ u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau)f(\tau)d\tau \text{ for all } t \geq 0. \]

(2.3)

We refer the reader to Pazy [19] for more detailed treatments on the relations between classical and mild solutions of evolution equations of the form \((2.1)\).

We now state an assumption that will be used in the rest of the paper.

Assumption 2.2. We assume that \(A(t)\) is 1-periodic, i.e., \(A(t + 1) = A(t)\) for all \(t \in \mathbb{R}_+\). Then \((U(t, s))_{t \geq s \geq 0}\) becomes 1-periodic in the sense that
\[ U(t + 1, s + 1) = U(t, s) \text{ for all } t, s \geq 0. \]

(2.4)

We also assume that the space \(Y\) considered as a subspace of \(Y''\) (through the canonical embedding) is invariant under the operator \(U''(1, 0)\) which is the dual of \(U(1, 0)\).

We now recall a Massera type’s Theorem for existence and uniqueness of a periodic solution whose proof can be found in [10, Theorem 2.3].

Theorem 2.3. Let \(M\) be defined as in \((1.7)\). For the Banach space \(X\) possessing a separable predual \(Y\) let the following condition holds true: For \(f \in M\) there exists \(u_0 \in X\) such that the mild solution \(u\) of \((2.4)\) with \(u(0) = u_0\) (i.e., \(u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s)ds \text{ for } t \geq 0\)) satisfies \(u \in C_b(\mathbb{R}_+, X)\) and
\[ \|u\|_{C_b} \leq M\|f\|_{M}. \]

(2.5)

Then, under the Assumption 2.2, if \(f\) is 1-periodic, then Equation \((2.1)\) has an 1-periodic mild solution \(\hat{u}\) satisfying
\[ \|\hat{u}\|_{C_b} \leq (M + 1)Ke^\alpha\|f\|_{M}. \]

(2.6)

Furthermore, if the evolution family \((U(t, s))_{t \geq s \geq 0}\) satisfies:
\[ \lim_{t \to \infty} \|U(t, 0)x\| = 0 \text{ for } x \in X \text{ such that } U(t, 0)x \text{ is bounded in } \mathbb{R}_+, \]

(2.7)

then the 1-periodic mild solution of \((2.1)\) is unique.

3. Bounded and periodic solutions to semi-linear problems. For a Banach space \(X\) with a separable predual \(Y\) as in the previous section, we now consider the following partial functional differential equation
\[
\begin{align*}
\frac{du}{dt} &= A(t)u(t) + F(t)(u_t) + g(t, u_t), \quad t \geq 0, \\
u(s) &= \phi(s) \text{ for all } s \in [-r, 0], \text{ with } \phi \in \mathcal{C}.
\end{align*}
\]

(3.1)

where the linear operators \(A(t), t \geq 0\), act on \(X\) and satisfy the hypotheses of Theorem 2.3 and the linear term \(F : [0, \infty) \to \mathcal{L}(\mathcal{C}, X)\) and the nonlinear term \(g : [0, \infty) \times \mathcal{C} \to X\) satisfies:

(1) the map \(t \mapsto F(t)(u_t)\) is 1-periodic for each 1-periodic function \(v \in C_b([-r, \infty), X)\),

(2) the map \(t \mapsto \|F(t)\| \) belongs to \(\mathbf{P}\),

(3) \(g\) belongs to class \([L, \varphi, \rho]\) for some \(L, \rho > 0\) and \(0 < \varphi \in \mathbf{P}\),

(4) the map \(t \mapsto g(t, v_t)\) is 1-periodic for each 1-periodic function \(v \in C_b([-r, \infty), X)\),

(3.2)
Furthermore, by the mild solution to (3.1), we mean the function $u$ satisfying the following equation

$$
\begin{aligned}
&u(t) = U(t, 0)u(0) + \int_0^t U(t, \tau)(F(\tau)(u_\tau) + g(\tau, u_\tau))d\tau \quad \text{for all } t \geq 0 \\
u_0 = \phi \in C.
\end{aligned}
$$

(3.3)

We then come to our next result on the existence and uniqueness of the periodic mild solution to Equation (3.1).

**Theorem 3.1.** Assume that there exists a constant $M$ such that for each $f \in \mathcal{M}$ there is a mild solution $u$ of (2.7) satisfying $u \in C_b(\mathbb{R}_+, X)$ and

$$
||u||_{C_b} \leq M||f||_{\mathcal{M}},
$$

and that the evolution family $U(t, s)_{t \geq s \geq 0}$ satisfies:

$$
\lim_{t \to \infty} ||U(t, 0)x|| = 0 \text{ for } x \in X \text{ such that } U(t, 0)x \text{ is bounded in } \mathbb{R}_+.
$$

Let $F$ and $g$ satisfy the conditions in (3.2). Then, if $\gamma := ||F(\cdot)||_M + ||\varphi||_M$ is small enough, Equation (3.1) has one and only one 1-periodic mild solution $\tilde{u}$ in $C_b([-r, \infty), X)$.

**Proof.** Consider the following closed set $\mathcal{B}_\rho^1 \subset C_b([-r, \infty), X)$ defined by

$$
\mathcal{B}_\rho^1 := \{v \in C_b([-r, \infty), X) : v \text{ is 1-periodic and } ||v||_{C_b([-r, \infty), X)} \leq \rho\},
$$

(3.4)

where $||v||_{C_b([-r, \infty), X)} := \sup_{t \geq r} ||v(t)||$ for all $v \in C_b([-r, \infty), X)$.

We then define the following transformation $\Phi$ given as follows: Consider the equation for given $v \in C_b([-r, \infty), X)$ with $u$ being the solution:

$$
u(t) = U(t, 0)u(0) + \int_0^t U(t, \tau)(F(\tau)(v_\tau) + g(\tau, v_\tau))d\tau \quad \text{for all } t \geq 0.
$$

(3.5)

Then, for $v \in \mathcal{B}_\rho^1$ we set

$$
\Phi(v)(t) := \begin{cases} 
u(t) & \text{for } t \geq 0 \\ \tilde{u}(t) & \text{for } -r \leq t < 0. \end{cases}
$$

(3.6)

where $u \in C_b(\mathbb{R}_+, X)$ is the unique 1-periodic solution to (3.5) (the existence and uniqueness of such a $u$ is guaranteed by Theorem 2.3, and $\tilde{u}(t)$, $-r \leq t < 0$, is the 1-periodic extension of $u$ on the interval $[-r, 0)$.

We will prove that if $\gamma$ is small enough, then the transformation $\Phi$ acts from $\mathcal{B}_\rho^1$ into itself and is a contraction. To do this, fixing any $v \in \mathcal{B}_\rho^1$, since $F$ and $g$ satisfy the conditions in (3.2) we have

$$
||F(\tau)(v_\tau) + g(\tau, v_\tau)||_M = \sup_{t \geq 0} \int_t^{t+1} ||F(\tau)(v_\tau) + g(\tau, v_\tau)||d\tau
\leq (\rho + L) (\sup_{t \geq 0} \int_t^{t+1} ||F(\tau)||d\tau + \sup_{t \geq 0} \int_t^{t+1} ||\varphi(\tau)||d\tau)
\leq (\rho + L)(||F(\cdot)||_M + ||\varphi||_M) = (\rho + L)\gamma.
$$

(3.7)
Applying Theorem 2.3 for the right-hand side \( F(\tau)(v_\tau) + g(\tau, v_\tau) \) instead of \( f(\tau) \) (in formula of the mild solution) we obtain that for \( v \in \mathcal{B}_\rho^1 \) there exists a unique 1-periodic solution \( u \) to (5.5) satisfying
\[
\|\Phi(v)\|_{C_b([-r, \infty), X)} = \|u\|_{C_b} \leq (M + 1)Ke^\alpha\|F(\tau)(v_\tau) + g(\tau, v_\tau)\|_M \leq (M + 1)K(\rho + L)\gamma e^\alpha. \tag{3.8}
\]

Here, the first equality in the above formulas holds true since \( \Phi(v) \), \( t \in [-r, 0) \), is the 1-periodic extension of \( u \) to the interval \([-r, 0)\).

Therefore, we obtain that if \( \gamma \) is small enough, then the map \( \Phi \) acts from \( \mathcal{B}_\rho^1 \) into itself.

Now, by Formula (3.5) we have the following representation of \( \Phi \)
\[
\Phi(v)(t) = \begin{cases} 
U(t, 0)u(0) + \int_0^t U(t, \tau)(F(\tau)(v_\tau) + g(\tau, v_\tau))d\tau & \text{for } t \geq 0 \\
\hat{u}(t) & \text{for } -r \leq t < 0,
\end{cases} \tag{3.9}
\]
where, as above, the function \( \hat{u}(t) \) is the 1-periodic extension to interval \([-r, 0)\) of the periodic function
\[
u(t) = U(t, 0)u(0) + \int_0^t U(t, \tau)(F(\tau)(v_\tau) + g(\tau, v_\tau))d\tau \quad \text{for } t \geq 0.
\]
Furthermore, for \( v, w \in \mathcal{B}_\rho^1 \) and \( u_1 = \Phi(v), u_2 = \Phi(w) \) by the representation (3.9) we obtain that \( u = u_1 - u_2 = \Phi(v) - \Phi(w) \) is the unique 1-periodic mild solution to the equation
\[
\begin{cases}
u(t) = U(t, 0)u(0) + \int_0^t U(t, \tau)(F(\tau)(v_\tau) + g(\tau, v_\tau) - F(\tau)(w_\tau) - g(\tau, w_\tau))d\tau \\
u(t) = \hat{u}(t) = \hat{u}_1(t) - \hat{u}_2(t) & \text{for } -r \leq t < 0.
\end{cases}
\]
Since \( u(t), t \geq 0 \), is 1-periodic, and for \(-r \leq t < 0\) the function \( \hat{u}(t) \) is an 1-periodic extension of \( u \) to interval \([-r, 0)\), we have that
\[
\|\Phi(v) - \Phi(w)\|_{C_b([-r, \infty), X)} = \sup_{t \geq -r} \|u(t)\| = \sup_{t \geq 0} \|u(t)\|.
\]
 Thus, from Theorem 2.3 and the fact that \( g \) belongs to class \((L, \varphi, \rho)\) we arrive at
\[
\begin{align*}
\|\Phi(v) - \Phi(w)\|_{C_b([-r, \infty), X)} & \leq (M + 1)Ke^\alpha \sup_{t \geq 0} \int_0^{t+1} \|F(\tau)(v_\tau) + g(\tau, v_\tau) - F(\tau)(w_\tau) - g(\tau, w_\tau)\|d\tau \\
& \leq (M + 1)Ke^\alpha \sup_{t \geq 0} \int_0^{t+1} \|F(\tau)(v_\tau - w_\tau)\| + \|g(\tau, v_\tau) - g(\tau, w_\tau)\|d\tau \\
& \leq 2(M + 1)Ke^\alpha \left( \sup_{t \geq 0} \int_0^{t+1} \|F(\tau)\|d\tau + \sup_{t \geq 0} \int_0^{t+1} |\varphi(\tau)|d\tau \right) \|v - w\|_{C_b([-r, \infty), X)} \\
& \leq 2(M + 1)Ke^\alpha \|v - w\|_{C_b([-r, \infty), X)}.
\end{align*}
\]
We thus obtain that if \( \gamma = \|F(\cdot)\|_M + \|\varphi\|_M \) is small enough, then \( \Phi : \mathcal{B}_\rho^1 \to \mathcal{B}_\rho^1 \) is a contraction. Therefore, for such a \( \gamma \) there exists a unique fixed point \( \hat{u} \) in \( \mathcal{B}_\rho^1 \).
4. Periodic solutions in the case of dichotomic evolution families.

4.1. Existence, uniqueness and conditional stability. In this subsection, we will consider equations (2.3) and (3.3) in the case that the evolution family \((U(t, s))_{t \geq s \geq 0}\) has an exponential dichotomy. In this case, the existence of bounded solutions to (2.3) (i.e., bounded mild solutions to (2.1)) is convenient to prove. Therefore, the existence and uniqueness of periodic solutions to (2.3) and hence to (3.3) easily follow. Moreover, using the cone-inequality Theorem 1.2, we will show the conditional stability of such periodic solutions. To do so, we start with the definitions of exponential dichotomy and stability of an evolution family.

Definition 4.1. Let \(\mathcal{U} := (U(t, s))_{t \geq s \geq 0}\) be an evolution family on Banach space \(X\).

1. The evolution family \(\mathcal{U}\) is said to have an exponential dichotomy on \([0, \infty)\) if there exist bounded linear projections \(P(t)\), \(t \geq 0\), on \(X\) and positive constants \(N, \nu\) such that
   
   \(U(t, s)P(s) = P(t)U(t, s), \quad t \geq s \geq 0,\)

2. The restriction \(U(t, s) : \text{Ker} P(s) \rightarrow \text{Ker} P(t), \ t \geq s \geq 0,\) is an isomorphism, and we denote its inverse by \(U(s, t) := (U(t, s))^{-1}, \ 0 \leq s \leq t,\)

3. \(|U(t, s)x| \leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in P(s)X, \ t \geq s \geq 0,\)

4. \(|U(s, t)x| \leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in \text{Ker} P(t), \ t \geq s \geq 0.\)

The projections \(P(t), \ t \geq 0,\) are called the dichotomy projections, and the constants \(N, \nu\) the dichotomy constants.

2. The evolution family \(\mathcal{U}\) is called exponentially stable if it has an exponential dichotomy with the dichotomy projections \(P(t) = Id\) for all \(t \geq 0\). In other words, \(\mathcal{U}\) is exponentially stable if there exist positive constants \(N, \nu > 0\) such that

\[\|U(t, s)\| \leq Ne^{-\nu(t-s)} \text{ for all } t \geq s \geq 0.\]  

We remark that properties (a)–(d) of dichotomy projections \(P(t)\) already imply that

i) \(H := \sup_{t \geq 0} \|P(t)\| < \infty,\)

ii) \(t \mapsto P(t)\) is strongly continuous

(see [16 Lemm. 4.2]). We refer the reader to [3] for characterizations of exponential dichotomies of evolution families in general admissible spaces.

If \((U(t, s))_{t \geq s \geq 0}\) has an exponential dichotomy with dichotomy projections \((P(t))_{t \geq 0}\) and constants \(N, \nu > 0\), then we can define the Green’s function on a half-line as follows:

\[G(t, \tau) := \begin{cases} 
P(t)U(t, \tau) & \text{for } t > \tau \geq 0, \\
-U(t, \tau)(I - P(\tau)) & \text{for } 0 \leq t < \tau. \end{cases}\]  

Also, \(G(t, \tau)\) satisfies the estimate

\[\|G(t, \tau)\| \leq (1 + H)Ne^{-\nu|t-\tau|} \text{ for } t \neq \tau, \ t, \tau \geq 0.\]  

Using the projections \((P(t), t \geq 0)\) on \(X\), we can define the family of operators \(\tilde{P}(t), t \geq 0\) on \(C\) as follows:

\[\tilde{P}(t) : C \rightarrow C, \ (\tilde{P}(t)\phi)(\theta) = U(t - \theta, t)P(t)\phi(0) \text{ for all } \theta \in [-r, 0].\]
Then, we have that \((\hat{P}(t))^2 = \hat{P}(t)\), and therefore the operators \(\hat{P}(t), t \geq 0\), are projections on \(C\). Moreover, \(\text{Im}\hat{P}(t) = \{ \phi \in C : \phi(0) = U(t, 0)\eta \text{ for all } t \in [-r, 0] \text{ for some } \eta \in \text{Im} \hat{P}(t) \} \)

The following lemma gives the form of bounded solutions of equations (2.3) and (3.3).

**Lemma 4.2.** Let the evolution family \((U(t, s))_{t \geq s \geq 0}\) have an exponential dichotomy with the corresponding dichotomy projections \((P(t))_{t \geq 0}\) and dichotomy constants \(N, \nu > 0\). Let \(f \in \mathcal{M}\), and let \(F\) and \(g\) satisfy conditions given in (3.2). Then, the following assertions hold true.

(a) Let \(v \in C_b(\mathbb{R}_+, X)\) be the solution to equation (2.3). Then, \(v\) can be rewritten in the form

\[
v(t) = U(t, 0)\zeta + \int_0^\infty G(t, \tau)f(\tau)d\tau \text{ for some } \zeta \in X_0 := P(0)X, \tag{4.5}\]

where \(G(t, \tau)\) is the Green’s function defined by equality (4.2).

(b) Let \(u \in C_b([-r, \infty), X)\) be a solution to equation (3.3) with \(\sup_{t \geq -r} \|u(t)\| \leq \rho\) for a fixed \(\rho > 0\). Then, for \(t \geq 0\) this function \(u(t)\) can be rewritten in the form

\[
\begin{cases} u(t) = U(t, 0)\eta + \int_0^\infty G(t, \tau)(F(\tau)(u(\tau) + g(\tau, u(\tau)))d\tau \\ u_0 = \phi \in \mathcal{C} \end{cases} \tag{4.6}
\]

for some \(\eta = P(0)\phi(0) \in X_0\) where \(G\) and \(X_0\) are determined as in Item (a).

**Proof.** (a): Put \(y(t) := \int_0^\infty G(t, \tau)f(\tau)d\tau\) for \(t \geq 0\). Since \(f \in \mathcal{M}\), using estimates (4.3) and (1.4) we obtain that

\[
\|y(t)\| \leq (1 + H)N \int_0^\infty e^{-\nu|\tau-t|}\|f(\tau)\|d\tau 
\leq \frac{N(1 + H)(N_1 + N_2)(\|f(\cdot)\|_\infty + N_2\|f(\cdot)\|_\infty)}{1 - e^{-\nu}} \text{ for all } t \geq 0.
\]

Moreover, it is straightforward to see that \(y(\cdot)\) satisfies the equation

\[
y(t) = U(t, 0)y(0) + \int_0^t U(t, \tau)f(\tau)d\tau \text{ for } t \geq 0.
\]

Since \(v(t)\) is a solution of the equation (2.3) we obtain that \(v(t) - y(t) = U(t, 0)(v(0) - y(0))\) for \(t \geq 0\). Put now \(\zeta = v(0) - y(0)\). The boundedness of \(v(\cdot)\) and \(y(\cdot)\) on \([0, \infty)\) implies that \(\zeta \in X_0\). Finally, since \(v(t) = U(t, 0)\zeta + y(t)\) for \(t \geq 0\), the equality (4.5) follows.

(b): Similarly as in the Item (a) we put \(y(t) := \int_0^\infty G(t, \tau)(F(\tau)(u(\tau) + g(\tau, u(\tau)))d\tau\) for \(t \geq 0\). Since \(F, g\) satisfies the conditions in (3.2) and using estimates (4.3) and (1.6) we obtain that

\[
\|y(t)\| \leq (1 + H)N \int_0^\infty e^{-\nu|\tau-t|}\|(F(\tau)(u(\tau) + g(\tau, u(\tau)))d\tau 
\leq (1 + H)N \int_0^\infty e^{-\nu|\tau-t|}(\rho\|F(\tau)\| + L\varphi(\tau))d\tau 
\leq \frac{(1 + H)N(N_1 + N_2)}{1 - e^{-\nu}}(\rho\|F(\cdot)\|_M + L\|\varphi\|_M) \text{ for } t \geq 0.
\]
Also, it is straightforward to see that \( y(\cdot) \) satisfies the equation

\[
y(t) = U(t, 0)y(0) + \int_0^t U(t, \tau)(F(\tau)(u_\tau) + g(\tau, u_\tau))d\tau \quad \text{for } t \geq 0.
\]

Since \( u(t) \) is a solution of the equation (3.3) we obtain that \( u(t) - y(t) = U(t, 0)(u(0) - y(0)) \) for \( t \geq 0 \). Put now \( \eta = u(0) - y(0) \). The boundedness of \( u(\cdot) \) and \( y(\cdot) \) on \([0, \infty)\) implies that \( \eta \in X_0 \) and \( P(0)u(0) = P(0)\phi(0) = \eta \). Finally, the relation \( u(t) = U(t, 0)\eta + y(t) \) for \( t \geq 0 \) yields the equality (4.6).

**Remark 4.3.** By straightforward computations we can prove that the converses of statements (a) and (b) are also true, i.e., a solution of Equation (4.5) satisfies Equation (2.3) for \( t \geq 0 \), and that of Equation (4.6) satisfies Equation (3.3) for \( t \geq 0 \).

We next prove the existence of bounded solutions to Equation (2.3) and (3.3) (i.e., bounded mild solutions to (2.1) and (3.1)) and hence that of periodic solutions in the following theorem.

**Theorem 4.4.** Consider equations (2.3) and (3.3). Let the evolution family \((U(t, s))_{t \geq s \geq 0}\) satisfy (4.4) and have an exponential dichotomy with the dichotomy projections \(P(t), t \geq 0\), and constants \(N, \nu\). Let \( f \in \mathfrak{M} \) be 1-periodic and suppose that \( F \) and \( g \) satisfies the conditions in (3.2) with given positive constants \( \rho, L \) and function \( \varphi \in \mathcal{P} \). Then, the following assertions hold true.

(a) Equation (2.3) has a unique 1-periodic solution in \( C_b(\mathbb{R}_+, X) \).

(b) If \( \|\varphi\|_M + \|F(\cdot)\|_M \) is sufficiently small, then Equation (3.3) has a unique 1-periodic solution in \( C_b([-\tau, \infty), X) \).

**Proof.** (a): For a given \( f \in \mathfrak{M} \) taking \( \zeta = 0 \in X_0 \) in (4.5) we have that equation (2.3) has a bounded solution

\[
u(t) = \int_0^\infty \mathcal{G}(t, \tau)f(\tau)d\tau.
\]

and this solution can be estimated using the inequalities (4.3) and (1.6) by

\[
\|\nu\|_{C_b} \leq (1 + H)N \int_0^\infty e^{-\nu(t-\tau)}\|f(\tau)\|d\tau \\
\leq (1 + H)N(N_1 + N_2)\|f\|_{\mathfrak{M}} \text{ for all } t \geq 0.
\]

Applying Theorem 2.3 we obtain that for the 1-periodic function \( f \in \mathfrak{M} \) there exists an 1-periodic solution \( \hat{u} \) of (2.3) satisfying

\[
\|\hat{u}\|_{C_b} \leq \left(\frac{(1 + H)(N_1 + N_2)}{1 - e^{-\nu}} + 1\right)Ke^\alpha\|f\|_{\mathfrak{M}}.
\]

The uniqueness of the 1-periodic solution follows from the fact that for two 1-periodic and continuous (hence bounded on \( \mathbb{R}_+ \)) solutions \( \hat{u} \) and \( \hat{v} \) (with the corresponding initial values \( \zeta, \eta \in X_0 \)) we obtain by using the form for bounded solutions (4.5) that \( \|\hat{u}(t) - \hat{v}(t)\| = \|U(t, 0)(\zeta - \eta)\| \leq Ne^{-\nu t}\|\zeta - \eta\| \to 0 \) as \( t \to \infty \) since \( \eta, \zeta \in X_0 \). This, together with the periodicity, implies \( \hat{u}(t) = \hat{v}(t) \) for all \( t \geq 0 \), finishing the proof of Assertion (a).

(b): By Assertion (a), for each 1-periodic input function \( f \), the linear problem (2.3) has a unique 1-periodic solution \( \hat{u} \in C_b(\mathbb{R}_+, X) \) satisfying inequality (4.8). Therefore, Assertion (b) then follows from Theorem 3.1.\]
We now prove the conditional stability of periodic solutions to (3.3). To do this, for $x \in X, \phi \in C$, and $\tilde{\phi} \in C_b((-r, \infty), X)$ denote by $B_\alpha(x) := \{y \in X : \|x - y\| \leq \alpha, x \in X\}$, by $B_\alpha(\tilde{\phi}) := \{\phi \in C : \|\phi - \tilde{\phi}\| \leq \alpha\}$, and by $B_\alpha(\hat{v}) := \{v \in C_b((-r, \infty), X) : \|v - \hat{v}\|_{C_b} \leq \alpha\}$, respectively. Let $B_\rho(0)$ be the ball containing $\hat{u}$ as in Assertion (b) of Theorem 4.4.

Suppose further that there exists a positive function $\tilde{\phi} \in P$ such that:

$$\|g(t, \phi_1) - g(t, \phi_2)\| \leq \tilde{\phi}(t)\|\phi_1 - \phi_2\|_C$$

for all $\phi_1, \phi_2 \in \B_2(0)$, and all $t \geq 0$.  (4.9)

**Theorem 4.5.** Let the assumptions of the Theorem [4.4] hold, and let $\hat{u}$ be the 1-periodic solution of (3.3) obtained in Assertion (b) of Theorem 4.4. Let $F$ and $g$ satisfy conditions given in (3.2) and (4.9), respectively. Then, if $\sup P(0) + \|P(0)\|X$ is small enough, then to each $\zeta \in C$ with $\|\zeta - \hat{u}\|_C \leq \rho/2$ and $P(0)\zeta(0) \in B_{\rho}(P(0)\hat{u}(0)) \cap P(0)X$ there corresponds one and only one solution $u(\cdot)$ of Equation (3.3) on $[-r, \infty)$ satisfying the conditions $u_0 = \zeta$ and $u \in B_\rho(\hat{u})$. Moreover, the following estimate is valid for $u(t)$ and $\hat{u}(t)$:

$$\|u(t) - \hat{u}(t)|C \leq C_\mu \rho \pi^{-\mu t} \text{ for } t \geq 0,$$  (4.10)

for some positive constants $C_\mu$ and $\mu$ independent of $u$, $\hat{u}$ and $\rho$.

**Proof.** Putting $w = u - \hat{u}$ we have that $u$ is a solution to Equation (3.3) in $B_\rho(\hat{u})$ with $u_0 = \zeta$ if and only if $w$ is the solution in $B_\rho(0)$ of the equation

$$w(t) = \begin{cases} U(t,0)(\zeta(0) - \hat{u}(0)) \\ + \int_0^t U(t,\tau)[F(\tau)(w_\tau) + g(\tau, w_\tau + \hat{u}_\tau) - g(\tau, \hat{u}_\tau)]d\tau & \text{for } t \geq 0 \\ \zeta(t) - \hat{u}(t) & \text{for } t \leq 0 \end{cases}$$

(4.11)

We now prove that Equation (4.11) has a unique solution in $B_\rho(0)$. To do this, putting $\tilde{g}(t, w_\tau) = g(t, w_\tau + \hat{u}_\tau) - g(t, \hat{u}_\tau)$ we obtain that $\tilde{g}(t, 0) = 0$ and $\tilde{g}$ belongs to class $(\rho, \tilde{\phi}, 2\rho)$.

Setting $\xi = P(0)\zeta(0) - P(0)\hat{u}(0)$ we prove that the transformation $K$ defined by

$$(Kw)(t) = \begin{cases} U(t,0)\xi + \int_0^\infty \tilde{g}(t,\tau)[F(\tau)(w_\tau) + \tilde{g}(\tau, w_\tau)]d\tau & \text{for } t \geq 0 \\ \zeta(t) - \hat{u}(t) & \text{for } -r \leq t \leq 0 \end{cases}$$

acts from $B_\rho(0)$ into itself and is a contraction. In fact, we have

$$||(Kw)(t)|| \leq \begin{cases} Ne^{-\nu t}\|\xi\| \\ + (1 + H)N \int_0^{\infty} e^{-\nu|t-\tau|}\|w_\tau\|\|F(\tau)\| + \tilde{\phi}(\tau)d\tau & \text{for } t \geq 0 \\ \sup_{\theta \in [-r, 0]} \|\zeta(\theta) - \hat{u}(\theta)\| & \text{for } -r \leq t \leq 0 \end{cases}$$

$$\leq \begin{cases} Ne^{-\nu t}\|\xi\| + (1 + H)N \rho \int_0^{\infty} e^{-\nu|t-\tau|}\|F(\tau)\| + \tilde{\phi}(\tau)d\tau & \text{for } t \geq 0 \\ \rho/2 & \text{for } -r \leq t < 0 \end{cases}$$

$$\leq \rho/2 + (1 + H)N \rho (N_1 + N_2)e^{\nu t}(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M)$$

for $t \geq -r$

since $\|\xi\| = \|P(0)\zeta(0) - P(0)\hat{u}(0)\| \leq \rho/2$. Therefore,

$$\|Kw\|_{C_b((-r, \infty), X)} \leq \rho/2 + (1 + H)N \rho (N_1 + N_2)e^{\nu t}(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M)$$

Thus, we obtain that if $(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M)$ is small enough, then the transformation $K$ acts form $B_\rho(0)$ into $B_\rho(0)$.
Now, for $x, z \in B_\rho(0)$ we estimate
\[
\|(Kx)(t) - (Kz)(t)\|
\leq \int_0^\infty \|G(t, \tau)\|\|F(\tau)(x, \tau) + \check{g}(\tau, x, \tau) - F(\tau)(z, \tau)\|d\tau
\leq (1 + H) N \int_0^\infty e^{-\nu|t-\tau|}(\|F(\tau)(x, \tau) - F(\tau)(z, \tau)\| + \|\check{g}(\tau, x, \tau) - \check{g}(\tau, z, \tau)\|)d\tau
\]
for $t \geq 0$ and $\|(Kx)(t) - (Kz)(t)\| = 0$ for all $-r \leq t \leq 0$. Hence,
\[
\|(Kx)(t) - (Kz)(t)\|
\leq (1 + H) N (N_1 + N_2) e^{\nu r}(\|F(\cdot)\|_M + \|\check{g}\|_M)\|x - z\|_{C_b([-r, \infty), X)}
\]
for $t \geq -r$. Thus,
\[
\|Kx - Kz\|_{C_b([-r, \infty), X)}
\leq (1 + H) N (N_1 + N_2) e^{\nu r}(\|F(\cdot)\|_M + \|\check{g}\|_M)\|x - z\|_{C_b([-r, \infty), X)}.
\]

Therefore, if $\|F(\cdot)\|_M + \|\check{g}\|_M$ is small enough, then the transformation $K : B_\rho(0) \to B_\rho(0)$ is a contraction. Thus, there exists a unique $w \in B_\rho(0)$ such that $Kw = w$.

By definition of $K$, Lemma 4.2 and Remark 4.3 we have that $w$ is the unique solution in $B_\rho(0)$ of Equation (4.11). Note that by Lemma 4.2 the above solution $w$ of Equation (4.11) can be written as
\[
w(t) = \begin{cases} 
U(t, 0) \xi + \int_0^\infty G(t, \tau) \left[F(\tau)(w_\tau) + \check{g}(\tau, w_\tau)\right]d\tau & \text{for } t \geq 0 \\
u(t) - \hat{u}(t) & \text{for } -r \leq t \leq 0.
\end{cases} \tag{4.12}
\]

Returning to the solution $u$ of (3.3) by replacing $w$ by $u - \hat{u}$ then, there exists a unique $u \in B_\rho(\hat{u})$ of Equation (3.3) with $u_0 = \zeta$.

Finally, we prove the estimate (4.10). To do this, putting as above $\xi := P(0)u(0) - P(0)\hat{u}(0)$, $w = u - \hat{u}$ with $u \in B_\rho(\hat{u})$, $\check{g}(t, w_t) = g(t, w_t + \hat{u}_t) - g(t, \hat{u}_t)$ we can use the formula (4.12) to write
\[
w(t) = \begin{cases} 
U(t, 0) \xi + \int_0^\infty G(t, \tau) \left[F(\tau)(w_\tau) + \check{g}(\tau, w_\tau)\right]d\tau & \text{for } t \geq 0 \\
u(t) - \hat{u}(t) & \text{for } -r \leq t \leq 0.
\end{cases}
\]

Using the facts that $\|\xi\| \leq \frac{\rho}{2N}$ and $\|u_0 - \hat{u}_0\|_c \leq \frac{\rho}{2}$ it follows that
\[
\|w(t)\| \leq \begin{cases} 
e^{-\nu t} & \text{if } -r \leq t < 0 \\
_N(1 + H) \int_0^\infty e^{-\nu|t-\tau|}(\|F(\tau)\|_M + \|\check{g}(\tau)\|)d\tau & \text{if } t \geq 0
\end{cases}
\]
Since $t + \theta \in [-r + t, t]$ for fixed $t \in [0, \infty)$ and $\theta \in [-r, 0]$, we obtain
\[
\|u_t\|_c \leq \frac{\rho}{2} e^{\nu r} e^{-\nu t} + (1 + H) Ne^{\nu r} \int_0^\infty e^{-\nu|t-\tau|}(\|F(\tau)\| + \|\check{g}(\tau)\|)d\tau & \text{for } t \geq 0.
\]

Put $\phi(t) = \|w_t\|_c$. Then $\sup_{t \geq 0} \phi(t) < \infty$ and
\[
\phi(t) \leq \frac{\rho}{2} e^{\nu r} e^{-\nu t} + (1 + H) Ne^{\nu r} \int_0^\infty e^{-\nu|t-\tau|}(\|F(\tau)\| + \|\check{g}(\tau)\|)d\tau & \text{for } t \geq 0.
\]

We will use the cone-inequality Theorem 1.2 applying to Banach space $W := L_\infty(\mathbb{R}_+)$ which is the space of real-valued functions defined and essentially bounded on $\mathbb{R}_+$ (endowed with the esssup-norm denoted by $\| \cdot \|_\infty$) with the cone $K$ being
the set of all (a.e.) nonnegative functions. We then consider the linear operator $B$ defined for $u \in W$ by

$$(Bu)(t) = (1 + H)Ne^{\nu t} \int_0^\infty e^{-\nu(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))u(\tau)d\tau$$

for $t \geq 0$.

By inequalities in \[1.6\] we have that

$$\sup_{t \geq 0}(Bu)(t) = \sup_{t \geq 0}(1 + H)Ne^{\nu t} \int_0^\infty e^{-\nu(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))u(\tau)d\tau \leq \frac{(1 + H)Ne^{\nu}}{1 - e^{-\nu}}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M)\|u\|_\infty.$$ 

Therefore, $B \in \mathcal{L}(W)$ and $\|B\| \leq \frac{(1 + H)Ne^{\nu}}{1 - e^{-\nu}}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M) < 1$. Obviously, $B$ leaves the cone $K$ invariant. The inequality \[4.13\] can now be rewritten as

$$\phi \leq B\phi + z$$

for $z(t) = \frac{\rho}{2}e^{\nu t}e^{-\nu t}$, $t \geq 0$.

Hence, by cone-inequality Theorem \[1.2\] we obtain that $\phi \leq \psi$, where $\psi$ is a solution in $W$ of the equation $\psi = B\psi + z$ which can be rewritten as

$$\psi(t) = \frac{\rho}{2}e^{\nu t}e^{-\nu t} + (1 + H)Ne^{\nu t} \int_0^\infty e^{-\nu(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))\psi(\tau)d\tau$$

for $t \geq 0$. \hspace{1cm} (4.14)

We now estimate $\psi$. To that purpose, for

$$0 < \mu < \nu + \ln(1 - (1 + H)Ne^{\nu t}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M))$$

we set $h(t) = e^{\mu t}\psi(t)$ for $t \geq 0$. Then, by \[4.14\] we obtain that

$$h(t) = \frac{\rho}{2}e^{\nu t}e^{-\nu t} + (1 + H)Ne^{\nu t} \int_0^\infty e^{-\nu(t-\tau)+\mu(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))h(\tau)d\tau$$

for $t \geq 0$. \hspace{1cm} (4.15)

We next consider the linear operator $D$ defined for $u \in W$ by

$$(Du)(t) = (1 + H)Ne^{\nu t} \int_0^\infty e^{-\nu(t-\tau)+\mu(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))u(\tau)d\tau$$

for $t \geq 0$.

By inequalities \[1.6\] we have that

$$\sup_{t \geq 0}(Du)(t) = \sup_{t \geq 0}(1 + H)Ne^{\nu t} \int_0^\infty e^{-\nu(t-\tau)+\mu(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))u(\tau)d\tau \leq \sup_{t \geq 0}(1 + H)Ne^{\nu t} \int_0^\infty e^{-(\nu-\mu)(t-\tau)}(\|F(\tau)\| + \tilde{\phi}(\tau))u(\tau)d\tau \leq \frac{(1 + H)Ne^{\nu}}{1 - e^{-(\nu-\mu)}}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M)\|u\|_\infty.$$ 

Therefore, $D \in \mathcal{L}(W)$ and $\|D\| \leq \frac{(1 + H)Ne^{\nu}}{1 - e^{-(\nu-\mu)}}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M)$. Equation \[4.15\] can now be rewritten as

$$h = Dh + \tilde{z}$$

for $\tilde{z}(t) = \frac{\rho}{2}e^{\nu t}e^{-(\nu-\mu)t}$, $t \geq 0$.

Since $\mu < \nu + \ln(1 - (1 + H)Ne^{\nu t}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M))$ we obtain that

$$\|D\| \leq \frac{(1 + H)Ne^{\nu}}{1 - e^{-(\nu-\mu)}}(N_1 + N_2)(\|F(\cdot)\|_M + \|\tilde{\phi}\|_M) < 1.$$
Therefore, the equation \( h = Dh + \dot{z} \) is uniquely solvable in \( L_\infty(W) \), and its solution is \( h = (I - D)^{-1}\dot{z} \). Hence, we obtain that
\[
\|h\|_\infty = \|(I - D)^{-1}\dot{z}\|_\infty \leq \|(I - D)^{-1}\| \|\dot{z}\|_\infty \leq \frac{\|\dot{z}\|_\infty}{1 - \|D\|}.
\]
Therefore,
\[
\|h\|_\infty \leq C_\mu \rho \text{ for } C_\mu := \frac{e^{\mu t}}{2(1 - \frac{(1+H)Ne^{\nu r}}{1-e^{-(\nu r)/\mu}}(N_1 + N_2))(\|F(\cdot)\|_M + \|\ddot{\varphi}\|_M)}.
\]
This yields that
\[
h(t) \leq C_\mu \rho \text{ for } t \geq 0.
\]
Hence, \( \psi(t) = e^{-\mu t}h(t) \leq C_\mu e^{-\mu t} \). Since \( \|u_t\|_C = \phi(t) \leq \psi(t) \), we obtain that
\[
\|u_t\|_C \leq C_\mu e^{-\mu t}.
\]
Returning to the solution \( u \) of (3.3) by replacing \( w \) by \( u - \ddot{u} \) and \( \xi = P(0)u(0) - P(0)\ddot{u}(0) \) we have
\[
\|u_t - \ddot{u}_t\|_C \leq C_\mu e^{-\mu t}.
\]
finishing the proof of the theorem. \( \square \)

**Remark 4.6.** The assertion of the above theorem shows us the conditional stability of the periodic solution \( u \) in the sense that for any other bounded solution \( u \) such that \( \|u_0 - \ddot{u}_0\|_C \leq \rho/2 \) and \( P(0)u(0) \in B_{\frac{\rho}{2\mu}}(P(0)\ddot{u}(0)) \cap P(0)X \) we have \( \|u_t - \ddot{u}_t\| \rightarrow 0 \) exponentially as \( t \rightarrow \infty \) (see inequality (4.10)).

For an exponentially stable evolution family (see Definition 4.1 (2)) we have the following corollary which is a direct consequence of Theorem 4.5.

**Corollary 4.7.** Let the assumptions of the Theorem 4.4 hold, and let \( \ddot{u} \) be the periodic solution of (3.3) obtained in assertion (b) of Theorem 4.4. Let further the evolution family \( U(t,s) \) be exponentially stable. Then, the periodic solution \( \ddot{u} \) is exponentially stable in the sense that for any other solution \( u \) in \( C_b([-r, \infty), X) \) of (3.3) such that \( \|u_0 - \ddot{u}_0\|_C \) is small enough we have
\[
\|u_t - \ddot{u}_t\|_C \leq Ce^{-\mu t}\|u_0 - \ddot{u}_0\|_C \text{ for all } t \geq 0 \quad (4.16)
\]
for some positive constants \( C \) and \( \mu \) independent of \( u \) and \( \ddot{u} \).

**Proof.** We just apply Theorem 4.5 for \( P(t) = Id \) for all \( t \geq 0 \) to obtain the assertion of the theorem. \( \square \)

4.2. Local stable manifold around the periodic solution. In this subsection, under the same hypotheses as the previous subsection, we will prove the existence of a local stable manifold for Equation (3.3) near its periodic solution. As in the previous subsection, we denote by \( B_r(\phi) \) the ball in \( C \) centered at \( \phi \) with radius \( r \). We then give the definition of a local stable manifold for Equation (3.3) around \( \ddot{u} \) if for every \( t \in \mathbb{R}_+ \) the phase space \( C \) splits into a direct sum...
and if there exist positive constants \( \rho, \rho_0, \rho_1 \) and a family of Lipschitz continuous mappings

\[
h_t : \mathcal{B}_{\rho_0}(\hat{u}_t) \cap \bar{X}_0(t) \to \mathcal{B}_{\rho_1}(\hat{u}_t) \cap \bar{X}_1(t), \quad t \in \mathbb{R}_+
\]

with the Lipschitz constants being independent of \( t \) such that

(i) \( \mathbf{S} = \{(t, \psi + h_t(\psi)) \in \mathbb{R}_+ \times (\bar{X}_0(t) \oplus \bar{X}_1(t)) \mid t \in \mathbb{R}_+, \psi \in \mathcal{B}_{\rho_0}(\hat{u}_t) \cap \bar{X}_0(t)\}, \) and we denote by \( \mathbf{S}_t := \{\psi + h_t(\psi) \mid (t, \psi + h_t(\psi)) \in \mathbf{S}\}, t \geq 0, \)

(ii) \( \mathbf{S}_t \) is homeomorphic to \( \mathcal{B}_{\rho_0}(\hat{u}_t) \cap \bar{X}_0(t) := \{\psi \in \bar{X}_0(t) : \|\psi - \hat{u}_t\| \leq \rho_0\} \) for all \( t \geq 0, \)

(iii) to each \( \psi \in \mathbf{S}_{t_0} \) there corresponds one and only one solution \( u(t) \) of Equation (3.3) on \([t_0 - \tau, \infty)\) satisfying conditions \( u_{t_0} = \psi \) and \( \sup_{t \geq t_0} \|u_t\|_{\mathcal{C}} \leq \rho. \)

Moreover, every solution \( u(t) \) on the manifold \( \mathbf{S} \) is exponentially attracted to \( \hat{u}(t) \) in the sense that, there exist positive constants \( \mu \) and \( C_{\mu} \) independent of \( t_0 \geq 0 \) such that

\[
\|u_t - \hat{u}_t\|_{\mathcal{C}} \leq C_{\mu} e^{-\mu(t-t_0)} \|\bar{P}(t_0)u(t_0) - \bar{P}(t_0)\hat{u}(t_0)\| \quad \text{for all } t \geq t_0. \tag{4.17}
\]

Note that, if we identify \( \bar{X}_0(t) \oplus \bar{X}_1(t) \) with \( \bar{X}_0(t) \times \bar{X}_1(t) \), then we can write \( S_t = \text{graph}(h_t) \) where \( \text{graph}(h_t) \) is denoted for the graph of the mapping \( h_t \).

We now state and prove our last result on the existence of a stable manifold for solutions to Equation (3.3) around its periodic solution.

**Theorem 4.9.** Let the assumptions of Theorems 4.4 and 4.5 hold with the corresponding positive functions \( F, \varphi \) and \( \check{\varphi} \). Let \( \hat{u} \) be the 1-periodic solution of (3.3) obtained in Theorem 4.4 thanks to the sufficient smallness of \( \|F(\cdot)\|_{\mathcal{M}} + \|\varphi\|_{\mathcal{M}} \). Then, if \( \|F(\cdot)\|_{\mathcal{M}} + \|\varphi\|_{\mathcal{M}} \) is sufficiently small, then there exists a local stable manifold \( \mathbf{S} \) around the solution \( \hat{u} \).

**Proof.** We will apply our result obtained in [8, Theorem 3.7]. To this purpose, let \( u \) be a solution to Equation (3.3) and put \( w = u - \hat{u} \). Then, \( u \) satisfies Equation (3.3) if and only if \( w \) satisfies

\[
w(t) = U(t, 0)w(0) + \int_0^t U(t, \tau) \left[F(\tau)(w_\tau) + g(\tau, w_\tau + \hat{u}_\tau) - g(\tau, \hat{u}_\tau)\right]d\tau \quad \text{for } t \geq 0.
\tag{4.18}
\]

Putting now \( \hat{g}(t, w_1) = g(t, w_1 + \hat{u}_1) - g(t, \hat{u}_1) \) we obtain that \( \hat{g}(t, 0) = 0 \) and \( \hat{g} \) belongs to class \( (2\rho, \check{\varphi}, 2\rho) \) since \( g \) satisfies the assumption of Theorem 4.5. Therefore, by a similar way as in the proof of [8, Theorem 3.7] we obtain that if \( \|F(\cdot)\|_{\mathcal{M}} + \|\varphi\|_{\mathcal{M}} \) is small enough, then there exists a local stable manifold \( \mathbf{S} \) (near 0) for Equation (4.18). Returning to the solution \( u \) of Equation (3.3) by replacing \( w \) with \( u - \hat{u} \), we obtain that, this manifold \( \mathbf{S} \) is the local stable manifold for Equation (3.3) near the solution \( \hat{u} \).

We finally illustrate our results by the following example.
4.3. An example. We consider the problem
\[
\begin{aligned}
\frac{\partial w(x,t)}{\partial t} &= a(t)\left[\frac{\partial^2 w(x,t)}{\partial x^2} + \eta w(x,t)\right] \\
+ \psi(t)(w(x,t-1)(1 + (\int_0^t u^2(x,t)dx)^{\frac{1}{2}}) + h(x,t)) \\
\text{for } 0 < x < \pi, t \geq 0,
\end{aligned}
\]
\begin{align*}
w(0,t) &= w(\pi,t) = 0, \quad \text{for } t \geq 0, \\
w(x,\theta) &= \phi(x,\theta) \in C \quad \text{for } 0 < x < \pi, \theta \in [-1,0],
\end{align*}
(4.19)

here, \( \eta \in \mathbb{R} \) and \( \eta \neq n^2 \) for all \( n \in \mathbb{N} \); the function \( a(t) \in L_{1,loc}(\mathbb{R}_+) \) is 1-periodic and satisfies the condition \( 0 < \gamma_0 \leq a(t) \leq \gamma_1 \) for fixed \( \gamma_0, \gamma_1 \); the function \( h : [0,\pi] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous on \([0,\pi] \times \mathbb{R}_+\) and 1-periodic with respect to \( t \).

We next put \( X := L_2[0,\pi], \mathcal{C} := C([-1,0],X) \), and let \( A : X \supset D(A) \to X \) be defined by \( Ay = y'' + \eta y \), with the domain
\[ D(A) = \{ y \in X : y' \text{ and } y'' \text{ are absolutely continuous, } y'' \in X, y(0) = y(\pi) = 0 \}. \]

It can be seen (see [3]) that \( A \) is the generator of an analytic semigroup \( (\mathcal{T}(t))_{t \geq 0} \).
Since \( \sigma(A) = \{-n^2 + \eta : n = 1,2,3,...\} \) applying the spectral mapping theorem for analytic semigroups we get
\[ \sigma(\mathcal{T}(t)) = e^{t\sigma(A)} = \{ e^{t(-n^2+\eta)} : n = 1,2,3,... \} \]
and hence \( \sigma(\mathcal{T}(t)) \cap \Gamma = \emptyset \) for all \( t > 0 \),
(4.20)

where \( \Gamma := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \).

Putting now \( A(t) := a(t)A \) we have that \( A(t) \) is 1-periodic, and the family \((A(t))_{t \geq 0}\) generates an 1-periodic (in the sense of Assumption 2.2) evolution family \( U(t,s)_{t,s \geq 0} \) which is defined by the formula \( U(t,s) = \mathcal{T}(\int_s^t a(\tau)d\tau) \).

By [1.20] we have that the analytic semigroup \((\mathcal{T}(t))_{t \geq 0}\) is hyperbolic (or has an exponential dichotomy) with the projection \( P \) satisfying
\[ \begin{align*}
i) \quad &\|\mathcal{T}(t)x\| \leq Ne^{-\beta t}\|x\| \text{ for } x \in PX, t \geq 0 \\
ii) \quad &\|\mathcal{T}(-t)x\| = \|\mathcal{T}(t)^{-1}x\| \leq Ne^{-\beta t}\|x\| \text{ for } x \in \text{Ker}P, t \geq 0,
\end{align*} \]
where the invertible operator \( \mathcal{T}(t) \) is the restriction of \( T(t) \) to \( \text{Ker}P \), and \( N, \beta \) are positive constants.

Using the hyperbolicity of \((\mathcal{T}(t))_{t \geq 0}\) it is straightforward to check that the evolution family \( U(t,s)_{t,s \geq 0} \) has an exponential dichotomy with the projection \( P(t) = P \) for all \( t \geq 0 \) and the dichotomy constants \( N \) and \( \nu := \beta \gamma_0 \) by the following estimates:
\[ \begin{align*}
\|U(t,s)x\| &\leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in PX, t \geq s \geq 0, \\
\|U(s,t)x\| &\leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in \text{Ker}P, t \geq s \geq 0.
\end{align*} \]

We then define the function \( F : \mathbb{R}_+ \to \mathcal{L}(\mathcal{C},X) \) and \( g : \mathbb{R}_+ \times \mathcal{C} \to X \) by \( F(t)(\phi) := \psi(t)\delta_{-1}\phi, \quad g(t,\phi) := \psi(t)((\delta_{-1}\phi)\phi) + h(\cdot,t) \) for \( \phi \in \mathcal{C} := C([-1,0],X) \) where \( \delta_{-1} \) is the Dirac delta function concentrated at \(-1\), and the real function \( \psi(t) \) is defined for a fixed constant \( c > 0 \) by
\[ \psi(t) = \begin{cases}
t\quad &\text{if } t \in \left[\frac{2n+1}{2} - \frac{1}{2c}, \frac{2n+1}{2} + \frac{1}{2c}\right] \text{ for } n = 0,1,2,\cdots, \\
0 &\text{otherwise}.
\end{cases} \]
(4.21)
The equation [4.19] can now be rewritten as
\[
\begin{cases}
\frac{d}{dt}u(\cdot,t) = A(t)u(\cdot,t) + F(t)(u(\cdot,\theta)) + g(t,u(\cdot,\theta)), \\
u_0(\cdot,\theta) = \phi(\cdot,\theta) \in \mathcal{C}
\end{cases}
\]
where $F$ is linear and $\|F(t)(\phi)\| \leq \psi(t) \|\phi\|$ for $\phi \in \mathcal{B}_a$, since $h(\cdot, t)$ is 1–periodic, it follows that $g(t, \phi)$ is 1-periodic with respect to $t$ for each function $\phi \in \mathcal{B}_a$. Moreover, $\|g(t, 0)\| = \psi(t)\|h(\cdot, t)\| \leq \gamma \psi(t)$ for $\gamma := \sup_{t \in [0, \infty)} (\int_0^\pi |h(x, t)|^2 dx)^{1/2}$, and we have
\[
\|g(t, u_{t,}()) - g(t, v_{t,}())\| = \psi(t)\|u_{t,}()\|\|u_{t,}()\| - \psi(t)\|v_{t,}()\|\|v_{t,}()\| = \psi(t)\|u_{t,}()\|\|u_{t,}()\| + \psi(t)\|v_{t,}()\|\|v_{t,}()\| - \|v_{t,}()\|\|v_{t,}()\| \\
\leq \psi(t)\|u_{t,}()\|\|u_{t,}()\| + \psi(t)\|\psi(t)\|\|\|v_{t,}()\|\| - \|v_{t,}()\|\|v_{t,}()\| \\
\leq 2\psi(t)\sup_{\theta \in [-1, 0]}\|u_{t,} - v_{t,}\| \quad \text{for all } u_{t,}, v_{t,} \in \mathcal{B}_a, t \in \mathbb{R}^+.
\]
we have that
\[
\sup_{t \geq 0} \int_t^{t+1} |\psi(\tau)| d\tau \leq \sup_{n \in \mathbb{N}} \int_{2n+1}^{2n+2} \frac{1}{2c-1} (t - n) dt = \frac{1}{2c-1}.
\]
Hence, $\psi \in M(\mathbb{R}^+)$ and $\|\psi\|_M \leq \frac{1}{2c-1}$ in spite of the fact that the values of $\psi$ can be very large.

Therefore, $F$ and $g$ satisfies the hypotheses of Theorems 4.4 and 4.5 with $\rho = a$, $L := \rho + \frac{2}{2\tau}$, $\varphi(0) = 2\psi(t)$ and $\varphi(t) = 4\psi(t)$. By Theorem 4.4 and 4.5 we obtain that, if $c$ is large enough (consequently, $\|F(\cdot)\|_M + \|\varphi\|_M$ and $\|\varphi(\cdot)\|_M + \|\varphi(\cdot)\|_M$ are small enough), then Equation (4.19) has one and only one 1-periodic mild solution $\hat{u} \in \mathcal{B}_a$ and this solution $\hat{u}$ is conditionally stable in the sense of Remark 4.6. Moreover, by Theorem 4.9 there exists a local stable manifold for mild solutions to Equation (4.19) around the periodic solution $\hat{u}$.

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