On Vaughan’s approximation in restricted sets of arithmetic progressions

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Abstract

We investigate the approximation to the number of primes in arithmetic progressions given by Vaughan [7]. Instead of averaging the expected error term over all residue classes to modules in a given range, here we only consider subsets of arithmetic progressions that satisfy additional congruence conditions and provide asymptotic approximations.

1 Introduction

The distribution of primes in arithmetic progressions is a long standing topic in analytic number theory. Defining

$$\vartheta(x, q, b) = \sum_{\substack{p \leq x \backmod d \equiv b}} \log p,$$

the approximation $$\vartheta(x, q, b) \sim \frac{x}{\phi(q)}$$ is known to be true for small $$q$$ and $$(a, q) = 1$$. For larger $$q$$, in [1], [4], and [5] the quantity

$$\sum_{d \leq Q} \sum_{\substack{b \leq 1 \backmod d \equiv 1}} E_2(x, d, b), \tag{1.1}$$

$$E(x, d, b) = \vartheta(x, d, b) - \frac{x}{\phi(d)},$$

has been investigated for $$Q \leq x$$. The most accurate asymptotic expression for (1.1) was given in [3] as follows:

For $$Q < x$$, any $$A > 0$$, and a constant $$C$$,

$$\sum_{d \leq Q} \sum_{\substack{b \leq 1 \backmod d \equiv 1}} E^2(x, d, b) = Qx \log Q + CQx + O \left( Q^{3/2}x^{1/2} + x^2(\log x)^{-A} \right). \tag{1.2}$$
In [7], it was shown that for large $q$ the approximation $\frac{f}{\varphi(q)}$ used in (1.1) is not best possible. Setting

$$F_R(n) = \sum_{r \leq R} \frac{\mu(r)}{\varphi(r)} \sum_{\substack{b \leq R \atop (b,r)=1}} e\left(\frac{bn}{r}\right), \quad \Delta(n) = \Lambda(n) - F_R(n),$$

$$\rho(x,d,b) = \sum_{n \leq x, n \equiv b (\text{mod} d)} F_R(n),$$

the following approximation was shown in [7]:

**Theorem 1.1** For any positive integer $A$ and $R = L^A$, there is

$$\sum_{d \leq Q} d \sum_{d \leq Q} (\vartheta(x,d,b) - \rho(x,d,b))^2 = Qx \log(x/R) - c_0 Qx + O \left( Qx R^{-1/2} + x^2 (\log x)^2 R^{-1} \right), \quad (1.3)$$

where $c_0 = 1 + \gamma + \sum_{p \geq 2} \log p \cdot \frac{1}{p(p-1)}$.

For $x/R < Q$, the main term in (1.3) is smaller than the main term in (1.2).

We notice that in Theorem 1.1 the average is taken over all reduced residue classes $b \text{ mod } d$. It is of interest to understand if one can get a better asymptotic when limiting the summation to reduced rest classes only. Theorem 1.2 shows that this is indeed the case.

**Theorem 1.2** For a real number $x$, a real number $R$ satisfying $R = L^G$ for some $G > 10$ and $Q = x (\log x)^{-B}$ for some $B > 0$, there is

$$\sum_{d \leq Q} d \sum_{d \leq Q} (\vartheta(x,d,b) - \rho(x,d,b))^2$$

$$= xQ \left( \log \left( \frac{x}{R^{2-\zeta^{-1}(2)}} \right) \right) + xQ \left( -c_1 + 1 - \prod_{p \geq 2} \left( 1 - \frac{1}{p(p-1)} \right) + \prod_{p \geq 2} \left( 1 - \frac{1}{p^2} \right) c_2 \right)$$

$$+ O \left( \frac{xQ}{R^{1/2}} + \frac{x^2 (\log x)^2}{R} \right),$$

where

$$c_1 = 1 + 2\gamma + 2 \sum_{p \geq 2} \frac{\log p}{p(p-1)},$$

$$c_2 = \gamma + \sum_{p \geq 2} \frac{\log p}{p(p-1)}.$$
Comparing Theorems 1.1 and 1.2, we see that the non-reduced residue classes with \( b \) with \((r, b) > 1\) indeed make a quantifiable contribution to the considered variance. This is different from the classical approach as in (1.2) where the contribution of the non-reduced rest-classed can be neglected. This observation is in alignment with the findings from [2]. In [2, Theorem 1.5], for a fixed \( b \), the average over \( q \) of \( \vartheta(x, d, b) - \rho(x, d, b) \) is considered. As in our case, it is shown that the non-reduced rest-classes make a non-negligible contribution.

For the proof of Theorem 1.2 we will need the following Theorem that describes the distribution of the term \( \Delta^2(n) \) in arithmetic progressions:

**Theorem 1.3** For a real number \( x \), a non-negative integer \( N \), a positive integers \( F \) and a squarefree integer \( v \), where \( v \ll (\log x)^F \), and a real number \( R \leq x^{1/3} \),

\[
\sum_{n \leq x \atop n \equiv N, (n,v)} \Delta^2(n) = \delta(N,v) \frac{x}{\phi(v)} (\log x - 2 \log R - c_1) + \frac{x}{v} (\log R + c_2) + \delta(N,v) \frac{v}{\phi^2(v)} - \frac{x}{\phi(v)} + O \left( \frac{x \exp \left( -cL^{1/2} \right) + \frac{x \tau(v)}{vR^{1/2}} + \frac{x}{\phi(v)R^{1/2}} + R^2 \log R + \tau(v)R + \frac{x(\log v + \tau(v))}{v} \sum_{r|v \atop r>R} 1 }{ } \right),
\]

where \( \delta(N,v) = 1 \) if either \( N > 0 \) and \((N,v) = 1\) or \( N = 0 \) and \( v = 1 \), and \( \delta(N,v) = 0 \), otherwise, \( c \) is a positive constant, and \( c_1 \) and \( c_2 \) are as defined in Theorem 1.2.

Further applying Theorem 1.3 we consider another variant of Theorem 1.1 where only average over such arithmetic progressions satisfying specific congruence conditions. We prove the following result:

**Theorem 1.4** For a real number \( x \), a real number \( R \) satisfying \( R = L^G \) for some \( G > 10 \) and \( Q = x(\log x)^{-B} \) for some \( B > 0 \), there is

\[
\sum_{d \leq Q} \sum_{(N-d,b)=1}^d (\vartheta(x, d, b) - \rho(x, d, b))^2
= \begin{align*}
xQ & \prod_{p \geq 2 \atop (p,N)=1} \left( 1 - \frac{1}{p(p-1)} \right) \left( \log \left( \frac{x}{R^{4(N)}} \right) \right) \\
+ xQ \left( - \prod_{p \geq 2 \atop (p,N)=1} \left( 1 - \frac{1}{p(p-1)} \right) c_1 + \prod_{p \geq 2 \atop (p,N)=1} \left( 1 - \frac{1}{(p-1)^2} \right) + \prod_{p \geq 2} \left( 1 - \frac{1}{p^2} \right) c_2 \\
& - \prod_{p \geq 2} \left( 1 - \frac{1}{p(p-1)} \right) + O \left( \frac{xQ}{R^{1/2}} + \frac{x^2(\log x)^2}{R} \right) \right),
\end{align*}
\]
where $c_1$ and $c_2$ are as defined in Theorem 1.2 and

\[
t(N) = \left(2 - \prod_{p \geq 2, p \nmid N} \left(1 - \frac{1}{p(p-1)}\right)^{-1} \prod_{p \geq 2} \left(1 - \frac{1}{p^2}\right)\right)^{-1} \geq 1.
\]

As the proofs of Theorem 1.2 and 1.4 are very similar, we will give the detailed proof of Theorems 1.4, but subsequently only provide a shortened version of the proof of Theorem 1.2. We will use the abbreviation $L = \log x$ throughout this paper. $c$ denotes a positive absolute constant that can take different values at different occasions. $\epsilon$ denotes an arbitrarily small positive number and $\tau$ denotes the divisor function. We use the common abbreviations $e(x) = e^{2\pi ix}$ and $\exp(x) = e^x$.

2 Auxiliary Lemmas

**Lemma 2.1** There is a positive constant $c$ such that whenever $A$ is a fixed positive number, $R \leq (\log x)^A$, $\sqrt{x} \leq y \leq x$, and $q \leq R$, we have

\[
\vartheta(y, q, a) - \rho(y, q, a) \ll y \exp\left(-cL^{1/2}\right).
\]

*Proof:* [7, Theorem 2].

**Lemma 2.2** For $(a, q) = 1$, $q \leq x$, $q \leq R$,

\[
\rho(x, q, a) \ll R + \frac{x\tau(q)}{q}.
\]

*Proof:* [7, Theorem 1, (1.12)].

**Lemma 2.3** Let $v_r$ be any set of complex numbers and let $x_r$ be any set of real numbers distinct modulo 1. If $0 < \delta = \min_{r \neq s} ||x_r - x_s||$, then for any real $t$,

\[
\left|\sum_{r \neq s} v_r v_s \frac{\sin t(x_r - x_s)}{\sin \pi(x_r - x_s)}\right| \leq \sum_r |v_r|^2 \delta^{-1}.
\]

*Proof:* [6, Corollary 2].

3 Proof of Theorem 1.3

As in [7, Proof of Theorem 3], we square out the term $\Delta^2(n)$:

\[
\sum_{n \leq x} \Delta^2(n) = \sum_{n \equiv N \pmod{v}} \Lambda^2(n) - 2 \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{(b, r) = 1} \sum_{n \equiv N \pmod{v}} \Lambda(n) e\left(\frac{bn}{r}\right) + \sum_{n \leq x} |F_R(n)|^2.
\]

(3.1)
Using the prime number theorem in arithmetic progressions, we obtain
\[
\sum_{n \leq x} \Lambda^2(n) = \delta(N, v) \frac{xL - x}{\phi(v)} + O \left( x \exp \left( -cL^{-1/2} \right) \right). \tag{3.2}
\]

Applying the prime number theorem in arithmetic progressions once more, we see
\[
\sum_{(b, r)\equiv 1 \atop n \leq x} (b, r) = \sum_{n \leq x} \Lambda(n) \sum_{r \leq x} e(bn/r) + O(RL^2)
\]
\[
= \sum_{n \leq x} \Lambda(n) \mu(r) + O(RL^2) = \mu(r) \sum_{n \leq x} \Lambda(n) + O(RL^2)
\]
\[
= \delta(N, v) \frac{2\mu(r)}{\phi(v)} + O(x \exp(-cL^{1/2})),
\]
which implies that
\[
\sum_{r \leq R} \mu(r) \sum_{(b, r)\equiv 1 \atop n \leq x} \Lambda(n) e \left( \frac{bn}{r} \right) = \delta(N, v) \frac{x}{\phi(v)} \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} + O(x \exp(-cL^{1/2}/2)). \tag{3.3}
\]

We know from [7, Proof of Theorem 3] that
\[
\sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} = \log R + \gamma + \sum_{p \geq 2} \frac{\log p}{p(p-1)} + O \left( R^{-1/2} \right). \tag{3.4}
\]
Inserting (3.4) into (3.3), we obtain
\[
\sum_{r \leq R} \mu(r) \sum_{(b, r)\equiv 1 \atop n \leq x} \Lambda(n) e \left( \frac{bn}{r} \right)
\]
\[
= \delta(N, v) \frac{x}{\phi(v)} \left( \log R + \gamma + \sum_{p \geq 2} \frac{\log p}{p(p-1)} \right) + O \left( \frac{x}{\phi(v)R^{1/2}} + x \exp(-cL^{1/2}/2) \right). \tag{3.5}
\]

We now estimate the third, most complicated term in (3.1):
\[
\sum_{n \leq x} |F_R(n)|^2 = \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{(b, r)\equiv 1 \atop n \leq x} \sum_{r_1 \leq R} \frac{\mu(r_1)}{\phi(r_1)} \sum_{b_1 \equiv 1 \atop (b_1, r_1)\equiv 1} \sum_{n \leq x} e \left( \frac{bn - b_1n}{r} \right) + A_N + 2B_N + C_N, \tag{3.6}
\]
where

\[
A_N = \frac{x}{\varphi(N)} \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{\substack{b = 1 \atop (b, r) = 1 \atop r \leq R}}^{r} \sum_{\substack{r_1 = 1 \atop (b_1, r_1) = 1 \atop r_1 \leq R}}^{r_1} e\left(\frac{bN}{r} - \frac{b_1N}{r_1}\right)
\]

\[
= \frac{x}{\varphi(N)} \left(\sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \right)^2,
\]

\[
B_N = \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{\substack{b = 1 \atop (b, r) = 1 \atop r \leq R}}^{r} \sum_{\substack{r_1 = 1 \atop (b_1, r_1) = 1 \atop r_1 \leq R}}^{r_1} \sum_{n \leq x} e\left(\frac{bn}{r} - \frac{b_1n}{r_1}\right)
\]

\[
= \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} C_r(N) \rho^* (x, v, N) \ll \tau(v) |\rho^* (x, v, N)|,
\]

where \(C_r(N)\) denotes the Ramanujan sum and

\[
\rho^* (x, v, N) = \sum_{r_1 \leq R} \frac{\mu(r_1)}{\phi(r_1)} \sum_{\substack{r \leq R \atop r_1 \leq R}} e\left(\frac{b_1n}{r_1}\right).
\]

\[
C_N = \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{\substack{b = 1 \atop (b, r) = 1 \atop r \leq R}}^{r} \sum_{\substack{r_1 = 1 \atop (b_1, r_1) = 1 \atop r_1 \leq R}}^{r_1} \sum_{n \leq x} e\left(\frac{bn}{r} - \frac{b_1n}{r_1}\right)
\]

\[
= C_{N,1} + C_{N,2},
\]

where

\[
C_{N,1} = \frac{x}{\varphi(N)} \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} = \frac{x}{\varphi(N)} \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} - \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} - \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} + \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)}
\]

\[
= \left(\frac{x}{\varphi(N)} \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} - \frac{x}{\phi(v)}\right) + \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} - \frac{x}{\phi(v)} + O\left(\frac{x \tau(v) \log R}{vR}\right).
\]

\[
C_{N,2} = \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{\substack{b = 1 \atop (b, r) = 1 \atop r \leq R}}^{r} \sum_{\substack{r_1 = 1 \atop (b_1, r_1) = 1 \atop r_1 \leq R}}^{r_1} e\left(\frac{-Nb_1}{r_1}\right) \sum_{n \leq x} e\left(\frac{sv(b_1r_1 - b_1r)}{rr_1}\right)
\]

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Using Lemma 2.3 with $v_r := v_{r,b} = \frac{\mu(r)}{\phi(r)} e\left(\frac{(N+v/2)b}{r}\right) e\left(\left\lfloor \frac{(x-N)/v}{2r} \right\rfloor \frac{vb}{r} \right)$, $x_r := x_{r,b} = \frac{vb}{r}$, and $\delta \gg R^{-2}$, we obtain

$$C_{N,2} \ll R^2 \log R. \quad (3.12)$$

Combining (3.10), (3.11) and (3.12), we see

$$C_N = \frac{x_{\tau(v)\log R}}{vR} + R^2 \log R. \quad (3.13)$$

To evaluate $A_N$, we see that for squarefree $v$ and $N \geq 1$,

$$\sum_{r \leq R} \frac{\mu(r)C_r(N)}{\phi(r)} = \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \sum_{d \mid (N,v)} d \mu\left(\frac{r}{d}\right) = \sum_{d \mid (N,v)} d \sum_{r \leq R} \frac{\mu(r)}{\phi(r)} \mu\left(\frac{r}{d}\right)
$$

$$= \sum_{d \mid (N,v)} d \sum_{h \mid \frac{v}{d}} \frac{\mu(hd)}{\phi(hd)} \mu(h) = \sum_{d \mid (N,v)} \frac{\mu(d)\mu(h)d}{\phi(h)\phi(d)} = \sum_{d \mid (N,v)} \frac{\mu(d)\mu(h)d}{\phi(h)\phi(d)} \frac{v}{\phi(d)} \delta(N,v). \quad (3.14)$$

For $N = 0$, we see similarly,

$$\sum_{r \mid v} \frac{\mu(r)C_r(N)}{\phi(r)} = \sum_{r \mid v} \mu(r) = \frac{v}{\phi(v)} \delta(N,v). \quad (3.15)$$
Further, we note that
\[
\sum_{r \mid v, \ r > R} \frac{\mu(r)C_r(N)}{\phi(r)} \leq \sum_{r \mid v, \ r > R} 1 \leq \tau(v). \tag{3.16}
\]
Combining (3.7) and (3.14) - (3.16), we see
\[
A_N = \left( \frac{x}{v} + O(1) \right) \left( \frac{v}{\phi(v)} \delta(N, v) + O \left( \sum_{r \mid v, \ r > R} \frac{\mu(r)C_r(N)}{\phi(r)} \right) \right)^2
\]
\[
= x \frac{v}{\phi^2(v)} \delta(N, v) + O \left( \frac{x(\log v + \tau(v))}{v} \sum_{r \mid v, \ r > R} 1 \right)
\]
\[
= x \frac{v}{\phi^2(v)} \delta(N, v) + O \left( \frac{x(\log v + \tau(v))}{v} \sum_{r \mid v, \ r > R} 1 \right). \tag{3.17}
\]
We know from [7, Proof of Theorem 1] that for \( \rho^*(x, v, N) \) as defined in (3.9):
\[
\rho^*(x, v, N) \ll R. \tag{3.18}
\]
Combining (3.8) and (3.18), we get
\[
B_N \ll \tau(v)R. \tag{3.19}
\]
Combining (3.9), (3.13), (3.17), and (3.19), and subsequently applying (3.4), we obtain
\[
\sum_{n \leq x, \ n \equiv N(\text{mod } v)} |F_R(n)|^2 = x \frac{v}{\phi^2(v)} \delta(N, v) + \frac{x}{v} \sum_{r \leq R} \frac{\mu^2(r)}{\phi(r)} - \frac{x}{\phi(v)}
\]
\[
+ O \left( \frac{x \tau(v) \log R}{vR} + R^2 \log R + \tau(v)R + \frac{x(\log v + \tau(v))}{v} \sum_{r \mid v, \ r > R} 1 \right)
\]
\[
= x \frac{v}{\phi^2(v)} \delta(N, v) + \frac{x}{v} \left( \log R + \gamma + \sum_{p \geq 1} \frac{\log p}{p(p-1)} \right) - \frac{x}{\phi(v)}
\]
\[
+ O \left( \frac{x \tau(v)}{vR^{1/2}} + R^2 \log R \tau(v)R + \frac{x(\log v + \tau(v))}{v} \sum_{r \mid v, \ r > R} 1 \right). \tag{3.20}
\]
Finally, we derive Theorem 1.3 from (3.1), (3.2), (3.5), and (3.20).
4 Proof of Theorem 1.4

For $Q_1 = x/R$, we see by arguing as in [7, Proof of Theorem 4] that

$$\sum_{d \leq Q_1} \sum_{b \leq 1 \atop (N - b, d) = 1} d (\vartheta(x, d, b) - \rho(x, d, b))^2 \ll x^2 R^{-1} L^2. \quad (4.1)$$

Thus, for $Q_1 < Q$ we only need to consider the expression

$$\sum_{Q_1 < d \leq Q} \sum_{b \leq 1 \atop v \mid N - b} d (\vartheta(x, d, b) - \rho(x, d, b))^2$$

$$= \sum_{v < Q} \mu(v) \sum_{Q_1 < d < Q} \sum_{b \leq 1 \atop v \mid N - b} (\vartheta(x, d, b) - \rho(x, d, b))^2. \quad (4.2)$$

We separately treat the cases $v \leq R^5$ and $v > R^5$. In the second case we see by using a trivial estimate for $\vartheta(x, q, a)$ and applying Lemma 2.2 to estimate $\rho(x, q, a)$,

$$\sum_{v > R^5} \mu(v) \sum_{Q_1 < d < Q} \sum_{b \leq 1 \atop v \mid N - b} (\vartheta(x, d, b) - \rho(x, d, b))^2$$

$$\ll \sum_{v > R^5} \sum_{Q_1 < d < Q} \sum_{b \leq 1 \atop v \mid N - b} \vartheta^2(x, d, b) + \rho^2(x, d, b)$$

$$\ll x^2 L^4 \sum_{v > R^5} \sum_{Q_1 < d < Q} \frac{\tau(d)}{d^2} \sum_{b \leq 1 \atop v \mid N - b} 1 + R^2 \sum_{v > R^5} \sum_{Q_1 < d < Q} \sum_{b \leq 1 \atop v \mid N - b} 1$$

$$\ll x^2 L^4 \sum_{v > R^5} \frac{1}{v^2} \sum_{Q_1 < d < Q} \frac{\tau(d)}{d} + R^2 \sum_{v > R^5} \frac{1}{v} \sum_{Q_1 < d < Q} d$$

$$\ll x^2 L^6 \sum_{v > R^5} \frac{1}{v^2} + R^2 Q^2 \sum_{v > R^5} \frac{1}{v^2} \ll x^2 R^{-3}. \quad (4.3)$$

In the case $v \leq R^5$, we see for a fixed $v$,

$$\sum_{Q_1 < d < Q} \sum_{b \leq 1 \atop v \mid N - b} d (\vartheta(x, d, b) - \rho(x, d, b))^2 = \sum_{Q_1 < d < Q} \sum_{b \leq 1 \atop v \mid N - b} (\vartheta(x, mv, b) - \rho(x, mv, b))^2$$
Applying Theorem 1.3, we see from which we derive

\[
\Delta(n_1) \Delta(n_2),
\]

\[
\sum_{\phi^{-1} < m < v} \sum_{n_1, n_2 \leq \phi \mod m} \Delta^2(n) + 2 \sum_{\phi^{-1} < m < v} \sum_{n_1, n_2 \leq \phi \mod m} \Delta(n_1) \Delta(n_2)
\]

:= \ E_v + 2F_v. \quad (4.4)

Applying Theorem 1.3 we see

\[
E_v = \sum_{\phi^{-1} < m < v} \left( \delta(N, v) \frac{x}{\phi(v)} (\log x - 2 \log R - c_1) + \frac{x}{v} (\log R + c_2) + \delta(N, v) x \frac{v}{\phi^2(v)} - \frac{x}{\phi(v)} \right)
\]

\[
+ \ O \left( Q \left( \frac{x \exp(-cL^{1/2})}{v} + \frac{x \tau(v)}{v^2 R^{1/2}} + \frac{x}{v \phi(v) R^{1/2}} + \frac{R^2 \log R}{v} + \frac{\tau(v) R}{v} + \frac{x (\log v + \tau(v))}{v^2} \sum_{\mathbb{R} \setminus \mathbb{R}} 1 \right) \right)
\]

\[
= \delta(N, v) \frac{x Q}{v \phi(v)} (\log x - 2 \log R - c_1) + \frac{x Q}{v^2} (\log R + c_2) + \delta(N, v) \frac{x Q}{\phi^2(v)} - \frac{x Q}{v \phi(v)}
\]

\[
+ \ O \left( \frac{x Q}{v^{3/2} R^{1/2}} + \frac{x Q (\log v + \tau(v))}{v^2} \sum_{\mathbb{R} \setminus \mathbb{R}} 1 \right), \quad (4.5)
\]

from which we derive

\[
\sum_{v \leq R^5} \mu(v) E_v
\]

\[
= \sum_{v \leq R^5} \mu(v) \left( \frac{x Q}{v \phi(v)} (\log x - 2 \log R - c_1) + \frac{x Q}{\phi^2(v)} \right) + \sum_{v \leq R^5} \mu(v) \left( \frac{x Q}{v^2} (\log R + c_2) - \frac{x Q}{v \phi(v)} \right)
\]

\[
+ \ O \left( \sum_{v \leq R^5} \frac{x Q}{v^{3/2} R^{1/2}} + \sum_{v \leq R^5} \frac{x Q (\log v + \tau(v))}{v^2} \sum_{\mathbb{R} \setminus \mathbb{R}} 1 \right)
\]

\[
= \sum_{v \geq 1} \mu(v) \left( \frac{x Q}{v \phi(v)} (\log x - 2 \log R - c_1) + \frac{x Q}{\phi^2(v)} \right) + \sum_{v \geq 1} \mu(v) \left( \frac{x Q}{v^2} (\log R + c_2) - \frac{x Q}{v \phi(v)} \right)
\]

\[
+ \ O \left( \frac{x Q}{R^{1/2}} + \frac{x Q \sum_{R < r \leq R^4} \sum_{v \leq R^5} \frac{1}{v^{2-\epsilon}}} \right)
\]

\[
= xQ \prod_{\pi^2 \geq (p, N) = 1} \left( 1 - \frac{1}{p(p - 1)} \right) (\log x - 2 \log R - c_1) + xQ \prod_{\pi^2 \geq (p, N) = 1} \left( 1 - \frac{1}{p(p - 1)^2} \right)
\]

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\[
+ xQ \prod_{p \geq 2} \left(1 - \frac{1}{p^2}\right) (\log R + c_2) - xQ \prod_{p \geq 2} \left(1 - \frac{1}{p(p-1)}\right) + O \left(\frac{xQ}{R^{1/2}} + xQ \sum_{R < r \leq R^\epsilon} \frac{1}{r^{d-1}}\right) \\
= xQ \prod_{p \geq 2, (p,N) = 1} \left(1 - \frac{1}{p(p-1)}\right) (\log x - (\log R)t(N)) \\
+ xQ \left(\prod_{p \geq 2, (p,N) = 1} \left(1 - \frac{1}{p(p-1)}\right) c_1\right) + \prod_{p \geq 2, (p,N) = 1} \left(1 - \frac{1}{p(p-1)^2}\right) c_2 \\
- \prod_{p \geq 2} \left(1 - \frac{1}{p(p-1)}\right) + O \left(\frac{xQ}{R^{1/2}}\right), \quad (4.6)
\]

where

\[
t(N) = \left(2 - \prod_{p \geq 2, (p,N) = 1} \left(1 - \frac{1}{p(p-1)}\right) \prod_{p \geq 2} \left(1 - \frac{1}{p^2}\right)\right) \geq 1.
\]

For \(F_v\), we see

\[
F_v = B_v, \quad \Omega_v - B_v, \quad (4.7)
\]

where

\[
B_{v,u} = \sum_{u < m \leq x} \sum_{u < m \leq x} \Delta(n_1) \Delta(n_2).
\]

To calculate \(F_{v,u}\), we first note that the summation condition \(m \leq x\) is superfluous as \(n_1, n_2 \leq x\). Using this fact and applying the divisor switching trick from [4], we see

\[
B_{v,u} = \sum_{u < m} \sum_{u < m} \sum_{n_2 \equiv n_1 (mod v)} \Delta(n_1) \Delta(n_2) \\
= \sum_{1 \leq x / vu} \sum_{u < m} \sum_{n_2 \equiv n_1 (mod v)} \Delta(n_1) \Delta(n_2) \\
= \sum_{1 \leq x / vu} \sum_{u \equiv n_1 (mod v)} \sum_{n_2 \equiv n_1 (mod v)} \Delta(n_1) \Delta(n_2) \\
= \sum_{l \leq x / vu} \sum_{u \equiv n_1 (mod v)} \sum_{n_2 \equiv n_1 (mod v)} \Delta(n_1) \Delta(n_2). \quad (4.8)
\]
Applying Lemma 2.1 to the sum over $n_2$, we see
\[ \sum_{l \leq n_2 \leq x \atop n_2 \equiv a \mod v} \Delta(n_2) \ll x \exp \left( -cL^{1/2} \right). \quad (4.9) \]

Applying Theorem 1.3 and the Cauchy inequality, we see
\[ \sum_{n_1 \leq x - lvu \atop n_1 \equiv a \mod v} |\Delta(n_1)| \ll xL. \quad (4.10) \]

Combining (4.8) - (4.10), we obtain
\[ B_{v,u,\psi} \ll x^2 L \exp \left( -cL^{1/2} \right) \sum_{l \leq x/vu \atop u \equiv N \mod v} 1 \ll x^2 L \exp \left( -cL^{1/2}/2 \right), \]

which together with (4.7) implies
\[ \sum_{v \leq R^5} \mu(v) F_v \ll x^2 \exp \left( -cL^{1/2}/4 \right). \quad (4.11) \]

Combining (4.11) - (4.10) and (4.11), we derive Theorem 1.4.

5 Proof of Theorem 1.2

Arguing as in (4.1) and (4.2) in the proof of Theorem 1.4, we can limit ourselves to the expression
\[ \sum_{Q_1 \leq d \leq Q} \sum_{b=1}^d (\vartheta(x, d, b) - \rho(x, d, b))^2 \]
\[ = \sum_{v \leq Q} \mu(v) \sum_{Q_1 < d < Q} \sum_{b=1}^d (\vartheta(x, d, b) - \rho(x, d, b))^2. \quad (5.1) \]

We separately treat the cases $v \leq R^5$ and $v > R^5$. In the second case, we see by arguing as in (4.3),
\[ \sum_{v > R^5} \mu(v) \sum_{Q_1 < d < Q} \sum_{b=1}^d (\vartheta(x, d, b) - \rho(x, d, b))^2 \ll x^2 R^{-3}. \quad (5.2) \]
In the case $v \leq R^5$, we see by arguing as in (4.4) for a fixed $v$,

$$
\sum_{Q_1 < d < Q} \sum_{|b| \leq \frac{Q}{v}} (\vartheta(x, d, b) - \rho(x, d, b))^2 = \sum_{Q_1 < m < Q} \sum_{|b| \leq \frac{Q}{v}} (\vartheta(x, mv, b) - \rho(x, mv, b))^2
$$

$$
= \sum_{\frac{Q_1}{v} < m < \frac{Q}{v}} \sum_{n \leq x, n \equiv 0 \pmod{v}} \Delta^2(n) + 2 \sum_{\frac{Q_1}{v} < m < \frac{Q}{v}} \sum_{\frac{n_1 < n_2 \leq x}{n_1 \equiv n_2 \equiv 0 \pmod{mv}}} \Delta(n_1) \Delta(n_2)
$$

$$
= E_v + 2H_v, \quad (5.3)
$$

where $E_v$ is as given in (4.5) with $N \equiv 0$. Following the argument in (4.6) and recalling the definition of $\delta(N, v)$ in Theorem 1.3, we obtain

$$
\sum_{v \leq R^5} \mu(v) E_v
$$

$$
= xQ \left( \log x - 2 \log R - c_1 + 1 \right) + \sum_{v \geq 1} \left( \frac{xQ}{v^2} \left( \log R + c_2 \right) - \frac{xQ}{\phi(v)} \right) + O \left( \frac{xQ}{R^{1/2}} \right)
$$

$$
= xQ \left( \log x - 2 \log R - c_1 + 1 \right) + xQ \prod_{p \geq 2} \left( 1 - \frac{1}{p} \right) \left( \log R + c_2 \right) - xQ \prod_{p \geq 2} \left( 1 - \frac{1}{p(p-1)} \right)
$$

$$
+ O \left( \frac{xQ}{R^{1/2}} \right) + O \left( \frac{xQ}{R^{1/2}} \right) \left( \log x - (\log R) \left( 2 - \zeta^{-1}(2) \right) \right)
$$

$$
+ O \left( \frac{xQ}{R^{1/2}} \right), \quad (5.4)
$$

For $H_v$, we see by arguing as in (4.7) - (4.11),

$$
\sum_{v \leq R^5} \mu(v) F_v \ll x^2 \exp \left( -cL^{1/2}/4 \right). \quad (5.5)
$$

Combining (5.1) - (5.5), we derive Theorem 1.2.

References

[1] M. B. Barban., *The large sieve method and its applications in the theory of numbers*, Uspekhi Mat. Nauk, 21:1(127) (1966), 51–102; Russian Math. Surveys, 21:1 (1966), 49–103.

[2] D. Fiorilli, *On Vaughan’s approximation: The first moment.*, J. Lond. Math. Soc. (2) 95 (2017), no. 1, 305–322.
[3] D. A. Goldston and R. C. Vaughan, *On the Montgomery-Hooley asymptotic formula*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), pp.117–142, London Math. Soc. Lecture Note Ser., vol. 237, Cambridge Univ. Press, Cambridge, 1997.

[4] C. Hooley, *On the Barban-Davenport-Halberstam theorem I*, Journal für reine und angewandte Mathematik, 274/275 (1975), 206-223.

[5] H.L. Montgomery, *Primes in arithmetic progressions*, Michigan Math. J., Volume 17, Issue 1 (1970), 33-39.

[6] H.L. Montgomery *The analytic principle of the large sieve*, Bulletin of the American Mathematical Society, Volume 84, Number 4, July 1978.

[7] R.C. Vaughan, *Moments of primes in arithmetic progressions I*, Duke Mathematica Journal, Volume 120, No 2, 2003.