An extension to Lorentz spaces of the Prodi-Serrin weak-strong uniqueness criterion for the Navier-Stokes equations

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Abstract

For initial data \( f \in L^2(\mathbb{R}^n) \) (\( n \geq 2 \)), we prove that if \( p \in (n, \infty) \), any solution \( u \in L^{p,s}_t L^{p/3}_x \cap L^2_t \mathcal{H}^1 \cap L^p_t L^{p,\infty} \) to the Navier-Stokes equations satisfies the energy equality, and that such a solution \( u \) is unique among all solutions \( v \in L^{p,s}_t L^{p/3}_x \cap L^2_t \mathcal{H}^1 \) satisfying the energy inequality. This extends well-known results due to G. Prodi (1959) and J. Serrin (1963), which treated the Lebesgue space \( L^p_x \) rather than the larger Lorentz (and ‘weak Lebesgue’) space \( L^{p,\infty}_x \). In doing so, we also prove the equivalence of various notions of solutions in \( L^{p,\infty}_x \), generalizing in particular a result proved for the Lebesgue setting in Fabes-Jones-Riviere (1972).

1. Introduction

It has been known since the pioneering work of J. Leray \cite{leray1934} that certain weak solutions to the Navier-Stokes equations with initial data in the natural energy space \( L^2(\mathbb{R}^n) \) always exist for all time. These solutions (now known as ‘Leray-Hopf’ solutions due to the later contribution of E. Hopf \cite{hopf1958}) moreover satisfy an energy inequality which implies that they belong to the space \( L^\infty_t L^2_x \cap L^2_t \mathcal{H}^1 \). When \( n = 2 \), such solutions are known to be unique (see \cite{prodi1959}), while for \( n \geq 3 \), uniqueness of such solutions is not known without additional assumptions. One has, for example, the well-known early results of G. Prodi \cite{prodi1959} and J. Serrin \cite{serrin1963} for any \( n \geq 2 \) that if, for some fixed initial data, there exists a Leray-Hopf solution which belongs moreover to the space \( L^{\frac{9}{2},s}_t \cap L^p_x \) for some \( p > n \), then it is the only Leray-Hopf solution for that data. (This is now known as well for the difficult endpoint \( p = n \), see \cite{prodi1959} for \( n = 3 \) and its generalization in \cite{prodi1959} to \( n > 3 \).

In this paper, we extend the uniqueness results of \cite{prodi1959, serrin1963} to the so-called ‘weak Lebesgue’ setting, namely the Lorentz spaces \( L^{p,\infty}(\mathbb{R}^n) \). In order to do so, we work with various notions of ‘solution’ (including in particular the Leray-Hopf type of weak solution) which we show in Theorem 1.1 below to be equivalent under our assumptions. Such equivalences are well-known in the Lebesgue setting – for example, see \cite[Theorem 2.1]{fabes1972} in the work of Fabes-Jones-Riviere which relates the notions of ‘weak’ and ‘mild’ solutions in such settings, which we generalize in Theorem 2.7 (along with Theorem 2.6) to similar weak Lebesgue settings. The full set of equivalent notions which we address in the weak Lebesgue setting is described in Theorem 1.3 below.

We point out here the recent work of T. Barker \cite{barker2021} which establishes a similar type of uniqueness result (see \cite[Proposition 1.6]{barker2021}) for \( n = 3 \) if there exists a solution in the mixed space-time Lorentz space \( L^{\frac{9}{2},s}_t \cap L^p_x \) for some \( p > 3 \) and \( s < \infty \), and for \( s = \infty \) but only if its norm in that space is sufficiently small. In contrast, by considering the Lebesgue setting in time only, we are able to remove the requirement of smallness under the assumption of existence in \( L^{\frac{9}{2},s}_t \cap L^p_x \) (\( p > n \)), for any \( n \geq 2 \). Moreover, the result in \cite{barker2021} is established as a by-product of the more general result \cite[Theorem 1.3]{barker2021} in the setting of certain larger Besov spaces, which is proved using more sophisticated tools than those in \cite{prodi1959, serrin1963}. Our proof is more direct, and largely follows the method in \cite{barker2021}.

1.1. General discussion. On the space-time domain \((0,T) \times \mathbb{R}^n\), we consider the linearised Navier-Stokes equations

\[
\begin{align*}
\frac{\partial u_j}{\partial t} - \Delta u_j + \nabla_k F_{jk} + \nabla_j P &= 0 \quad (1 \leq j \leq n), \\
\nabla \cdot u &= 0, \\
\n\text{and } u(0) &= f,
\end{align*}
\]

(1.1)

where the Navier-Stokes equations are realised by taking \( F_{jk} = u_j u_k \). At the formal level, we can eliminate the pressure term \( P \) by applying the Leray projection

\[
P_{ij} = \delta_{ij} - \frac{\nabla_i \nabla_j}{\Delta}
\]

(1.2)

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onto solenoidal vector fields, which transforms (1.1) into the heat equation
\[
\begin{align*}
&\begin{cases}
   u'_i - \Delta u_i + \nabla k F_{jk} = 0 & (1 \leq i \leq n), \\
   \nabla \cdot u = 0, \\
   u(0) = f,
\end{cases}
\end{align*}
\]
for which we have the corresponding integral equation
\[
u parcel\]
\[
\begin{align*}
u parcel\]
for $1 \leq i \leq n$, where the heat kernel $\Phi$ is defined by
\[
\Phi(t, x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} = \frac{1}{(2\pi t)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi - t|\xi|^2} d\xi \text{ for } (t, x) \in (0, \infty) \times \mathbb{R}^n.
\end{align*}
\]
We will work here in the context of \textit{Lorentz spaces} $L^{p,q}(\mathbb{R}^n)$, where for a measurable function $f$

\[
\lambda_f(y) := \{|x \in \mathbb{R}^n : |f(x)| > y\} \text{ for } y \in (0, \infty),
\end{align*}
\]
\[
\nu parcel\]

and the quasinorms $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ satisfy
\[
\begin{align*}
\|f\|_{L^{p,q}(\mathbb{R}^n)}^* &= \|f\|_{L^p(\mathbb{R}^n)} \text{ for } p \in [1, \infty],
\end{align*}
\]
\[
\nu parcel\]

and
\[
\begin{align*}
\|f\|_{L^{p,q}(\mathbb{R}^n)}^* \leq \|f\|_{L^{p,q}(\mathbb{R}^n)}^* \text{ for } p \in [1, \infty) \text{ and } 1 \leq q_1 \leq q_2 \leq \infty.
\end{align*}
\]
In order to understand how (1.3) relates to (1.1) and (1.3), we require an understanding of the \textit{Leray projection} $\mathcal{P}$. We will achieve this by defining $\mathcal{P}_{ij} := \delta_{ij} + \mathcal{R}_i \mathcal{R}_j$, where the \textit{Riesz transform} $\mathcal{R}$ is the unique linear map
\[
\begin{align*}
\mathcal{R} : \bigcup_{p \in (1, \infty)} \left(L^1(\mathbb{R}^n) + L^{p,\infty}(\mathbb{R}^n)\right) \rightarrow \bigcup_{p \in (1, \infty)} \left(L^1(\mathbb{R}^n) + L^{p,\infty}(\mathbb{R}^n)\right) \text{ (1.11)}
\end{align*}
\]
which satisfies
\[
\begin{align*}
\mathcal{R} f = \mathcal{F}^{-1} \left[ \xi \mapsto \frac{-i\xi}{|\xi|^2} \mathcal{F} f(\xi) \right] \text{ for all } f \in L^2(\mathbb{R}^n),
\end{align*}
\]
\[
\nu parcel\]

and
\[
\begin{align*}
\begin{cases}
   \|\mathcal{R} f\|_{L^{p,q}(\mathbb{R}^n)} \lesssim_{n,p,q} \|f\|_{L^{p,q}(\mathbb{R}^n)} \text{ and } \langle \mathcal{R} f, g \rangle = -(f, \mathcal{R} g)
   \\
   \text{for all } p \in (1, \infty), \ q \in [1, \infty], \ f \in L^{p,q}(\mathbb{R}^n) \text{ and } g \in L^{p,q}(\mathbb{R}^n).
\end{cases}
\end{align*}
\]

When attempting to relate the integral equation (1.1) to the differential equations (1.1) and (1.3), the following subtlety arises regarding measurability: if $p \in (1, \infty)$, $(E, E)$ is a measurable space, and $u$ is a measurable function on $E \times \mathbb{R}^n$ satisfying $u(t) \in L^{p,\infty}(\mathbb{R}^n)$ for all $t \in E$, then it is not immediately obvious that the Riesz transform defines a measurable function $\mathcal{R} u$ on $E \times \mathbb{R}^n$. We therefore define a weaker notion of measurability, by saying that a function $u : E \rightarrow L^{p,\infty}(\mathbb{R}^n)$ is \textit{weakly* measurable} if the map $t \mapsto \langle u(t), \phi \rangle$ is measurable for each $\phi \in C_0^\infty(\mathbb{R}^n)$. For $p, q \in [1, \infty)$ satisfying $(p = 1 \Rightarrow q = 1)$, we can prove the following:

\begin{itemize}
   \item If $u : E \rightarrow L^{p',q'}(\mathbb{R}^n)$ is weakly* measurable, then $t \mapsto \langle u(t), \phi \rangle$ is measurable for each $\phi \in L^{p,q}(\mathbb{R}^n)$, and the map $t \mapsto \|u(t)(\cdot)\|_{(L^{p,q}(\mathbb{R}^n))^{\ast}} := \sup_{\|\phi\|_{L^{p,q}(\mathbb{R}^n)} \leq 1} \langle u(t), \phi \rangle$ is measurable, where the norm $\|\cdot\|_{(L^{p,q}(\mathbb{R}^n))^{\ast}}$ is equivalent to the quasinorm $\|\cdot\|_{L^{p',q'}(\mathbb{R}^n)}$ on $L^{p',q'}(\mathbb{R}^n)$.
   \item If $u : E \rightarrow L^{p,q}(\mathbb{R}^n)$ is weakly* measurable, then $t \mapsto \langle u(t), \phi \rangle$ is measurable for each $\phi \in L^{p',q'}(\mathbb{R}^n)$, and the map $t \mapsto \|u(t)(\cdot)\|_{(L^{p,q}(\mathbb{R}^n))^{\ast}} := \sup_{\|\phi\|_{L^{p,q}(\mathbb{R}^n)} \leq 1} \langle u(t), \phi \rangle$ is measurable, where the norm $\|\cdot\|_{(L^{p,q}(\mathbb{R}^n))^{\ast}}$ is equivalent to the quasinorm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ on $L^{p,q}(\mathbb{R}^n)$.
\end{itemize}

\[ In [1], we develop local well-posedness (with $F = u \circ u$) of (1.1) in this context, but essentially with the projection $\mathcal{P}$ on the kernel $\Phi$ rather than on $\mathcal{F}$; this small discrepancy can be resolved using estimates in [3]. \]
• **Pettis’ theorem:** If \( u : E \to L^{p,q}(\mathbb{R}^n) \) is weakly* measurable, then there exist measurable functions \( u_m : E \to L^{p,q}(\mathbb{R}^n) \) with finite image, satisfying \( \|u_m(t)\|_{L^{p,q}(\mathbb{R}^n)} \leq \|u(t)\|_{L^{p,q}(\mathbb{R}^n)} \) and \( \|u_m(t) - u(t)\|_{L^{p,q}(\mathbb{R}^n)} \xrightarrow{m \to \infty} 0 \) pointwise.

For \( T \in (0, \infty) \), and \( p, q, \bar{p}, \bar{q} \in [1, \infty] \) satisfying \((p = 1 \Rightarrow q = 1) \) and \((p = \infty \Rightarrow q = \infty) \), we write \( L^{p,q}_T(T) \) to denote (equivalence classes of) weakly* measurable functions \( u : (0, T) \to L^{p,q}(\mathbb{R}^n) \) satisfying \( \|u\|_{L^{p,q}(\mathbb{R}^n)} \) is equivalent to a norm which is a measurable function of \( t \). We write \( L^{p,q}_T(T) : = \cap_{T \in (0, T]} L^{p,q}_\bar{t}(T) \), we remove the \( q \) from the notation in the case \( p = q \), and we remove the * from the notation if \( u \) defines a measurable function on \((0, T) \times \mathbb{R}^n\).

We can now state the following theorem, which relates various formulations of the linearised Navier-Stokes equations.

**Theorem 1.1.** Assume that \( T \in (0, \infty) \), \( p_0, p, \bar{p} \in (1, \infty) \), \( f \in L^{p_0, \infty}(\mathbb{R}^n) \) satisfies \( (f, \nabla \psi) = 0 \) for all \( \psi \in C_c^\infty(\mathbb{R}^n) \), and then the equivalence result of Theorem 1.1.

- **Weak formulation:** \( \int_0^T \langle u_j(t), \nabla \psi(t) \rangle \, dt \) for all \( \psi \in C_c^\infty((0, T) \times \mathbb{R}^n) \), and there exists some \( P \in L^{p, \infty}_T(T) \) such that

\[
\langle f_j, \phi_j(0) \rangle + \int_0^T \left( \langle u_j(t), \phi_j'(t) + \Delta \phi_j(t) \rangle + \langle F_{jk}(t), \nabla_k \phi_j(t) \rangle + \langle P(t), \nabla_j \phi_j(t) \rangle \right) \, dt = 0
\]

for all \( \phi \in C_c^\infty((0, T) \times \mathbb{R}^n) \), \( P(t) = R_j R_j F(t) \) for almost every \( t \in (0, T) \).

- **Projected formulation:** \( \int_0^T \langle u_j(t), \nabla \psi(t) \rangle \, dt \) for all \( \psi \in C_c^\infty((0, T) \times \mathbb{R}^n) \), and

\[
\langle f_j, \phi_i(0) \rangle + \int_0^T \left( \langle u_j(t), \phi_i'(t) + \Delta \phi_i(t) \rangle + \langle F_{jk}(t), \nabla_k \phi_i(t) \rangle \right) \, dt = 0
\]

for all \( \phi \in C_c^\infty((0, T) \times \mathbb{R}^n) \).

- **Mild formulation:** \( u(t)(x) = v(t, x) \) for almost every \( (t, x) \in (0, T) \times \mathbb{R}^n \), where \( v \) is the (almost everywhere defined) measurable function on \((0, T) \times \mathbb{R}^n \) given by

\[
v_i(t, x) := \int_{\mathbb{R}^n} \Phi(t, x - y) f_i(y) \, dy - \int_0^t \int_{\mathbb{R}^n} \nabla_k \Phi(t - s, x - y) [P_{ij} F_{jk}(s)](y) \, dy \, ds
\]

for almost every \( (t, x) \in (0, T) \times \mathbb{R}^n \).

- **Very weak formulation:** \( \int_0^T \langle u_j(t), \nabla \psi(t) \rangle \, dt \) for all \( \psi \in C_c^\infty((0, T) \times \mathbb{R}^n) \), and

\[
\langle f_j, \theta(0) \phi_i \rangle + \int_0^T \left( \langle u_j(t), \theta'(t) \phi_i + \theta(t) \Delta \phi_i \rangle + \langle F_{jk}(t), \theta(t) \nabla_k \phi_j \rangle \right) \, dt = 0
\]

for all \( \theta \in C_c^\infty([0, T]) \) and \( \phi \in C_c^\infty(\mathbb{R}^n) \), where the subscript \( \sigma \) means that \( \nabla \cdot \phi = 0 \).

Measurability issues in the above formulations can be addressed by approximating the test functions using Pettis’ theorem. The weak/projected/mild formulations correspond to equations [13], [13] and [13], respectively.

Of particular interest are solutions to the Navier-Stokes equations which belong to the energy class \( L^\infty \cap L^2_\sigma H^1 \). More precisely, for \( n \geq 2 \) and \( T \in (0, \infty) \), we define \( \mathcal{H}_T \) to be the space of (equivalence classes of) weakly* measurable functions \( u : (0, T) \to H^1(\mathbb{R}^n) \subseteq L^1_{loc}(\mathbb{R}^n) \) satisfying \( u \in L^2_\sigma((0, T)_\mathbb{R}^n) \) and \( \nabla u \in L^2_\sigma((0, T)_\mathbb{R}^n) \), where we observe that weak* measurability of \( u \) implies weak* measurability of \( \nabla u \). By virtue of the identification \( L^2_\sigma((0, T)_\mathbb{R}^n) \cong (L^2((0, T) \times \mathbb{R}^n))^\ast \cong L^2((0, T) \times \mathbb{R}^n), \) we may identify \( u \) and \( \nabla u \) with measurable functions \( u \in L^2_\sigma((0, T) \times \mathbb{R}^n) \) and \( \nabla u \in L^2_\sigma((0, T) \times \mathbb{R}^n) \). For each \( p \in [n, \infty] \) we have the Sobolev inequality \( \|u\|_{L^p_\sigma(\mathbb{R}^n)} \leq C_{p, n} \|u\|_{L^2_\sigma(\mathbb{R}^n)} \|\nabla u\|_{L^2_\sigma(\mathbb{R}^n)}, \) from which we deduce that \( \mathcal{H}_T \subseteq C_{p, n}^\infty(\mathbb{R}^n) \). In particular, if \( u \in \mathcal{H}_T \) and \( F_{jk} = u_j u_k \) then the equivalence result of Theorem 1.1 holds. Independently of Theorem 1.1, we will prove the following.
Theorem 1.2. Let $T \in (0, \infty]$, and assume that $f \in L^2(\mathbb{R}^n)$ satisfies $\{f_t, \nabla \phi \} = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.

(i) If $u \in \mathcal{H}_T$ satisfies the mild formulation of the Navier-Stokes equations with initial data $f$, then $u$ satisfies the continuity condition

\[
\left\{ \begin{array}{l}
\text{there exists a subset } \Omega \subseteq (0, T) \text{ of total measure such that} \\
\langle u(t), \phi \rangle \to \langle f, \phi \rangle \text{ for all } \phi \in L^2(\mathbb{R}^n) \text{ as } t \to 0 \text{ along } \Omega.
\end{array} \right.
\]

(ii) Assume that $p \in (n, \infty]$, and that $u \in \mathcal{H}_T \cap L^{2p/(p-n)}(T_-)$ and $v \in \mathcal{H}_T$ satisfy the projected formulation of the Navier-Stokes equations with initial data $f$, with $u$ and $v$ both satisfying the continuity condition \((1.18)\). Then for almost every $t \in (0, T)$ we have

\[
\langle u_t(t), v_t(t) \rangle = \|f\|_{L^2(\mathbb{R}^n)}^2 - 2 \int_0^t \langle \nabla u(s), \nabla v(s) \rangle \, ds - \int_0^t \langle (u \cdot \nabla)v(s), v(s) \rangle \, ds - \int_0^t \langle u(s), [(v \cdot \nabla)v](s) \rangle \, ds.
\]

In particular, $u$ satisfies the energy equality

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 = \|u(t)\|_{L^2(\mathbb{R}^n)}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}^n)}^2 \, ds \quad \text{for a.e. } t \in (0, T).
\]

If we make the additional assumption that $v$ satisfies the energy inequality

\[
\|f\|_{L^2(\mathbb{R}^n)}^2 \geq \|v(t)\|_{L^2(\mathbb{R}^n)}^2 + 2 \int_0^t \|\nabla v(s)\|_{L^2(\mathbb{R}^n)}^2 \, ds \quad \text{for a.e. } t \in (0, T),
\]

then $u(t, x) = v(t, x)$ for almost every $(t, x) \in (0, T) \times \mathbb{R}^n$.

Part (ii) of the previous theorem generalises the work of Prodi \[18\] and Serrin \[20\], in which the assumption $u \in L^{2p/(p-n)}(T_-)$ is replaced by the stronger assumption $u \in L^{2p/(p-n)}(T_-)$ by the Sobolev inequality. For initial data $f \in J(\mathbb{R}^n) := \overline{C_c^\infty(\mathbb{R}^n)}_{L^2(\mathbb{R}^n)}$, Hopf \[11\] constructs a solution $u \in \mathcal{H}_\infty$ to the weak formulation of the Navier-Stokes equations which satisfies the energy inequality.

1.2. Lorentz spaces. For a measurable function $f$ on $\mathbb{R}^n$ we define the auxiliary functions

\[
\lambda_f(y) := |\{ x \in \mathbb{R}^n : |f(x)| > y \}| \quad \text{for } y \in (0, \infty),
\]

\[
f^*(\tau) := \sup\{ y \in (0, \infty) : \lambda_f(y) > \tau \} \quad \text{for } \tau \in (0, \infty),
\]

\[
f^{**}(\tau) := \sup_{|A| \geq \tau} \frac{1}{|A|} \int_A |f(x)| \, dx = \frac{1}{\tau} \int_0^\tau f^*(\eta) \, d\eta \quad \text{for } \tau \in (0, \infty).
\]

The Lorentz quasinorms $\|f\|_{L^p,q(\mathbb{R}^n)}$ and the Lorentz norms $\|f\|_{L^p(\mathbb{R}^n)}$ are then defined by

\[
\|f\|_{L^p,q(\mathbb{R}^n)} := \left( \frac{1}{q} \int_0^\infty \left[ \tau^{1/p} f^*(\tau) \right]^q \frac{d\tau}{\tau} \right)^{1/q} \quad p \in [1, \infty), q \in [1, \infty),
\]

\[
\|f\|_{L^p(\mathbb{R}^n)} := \left( \sup_{\tau \in (0, \infty)} \tau^{1/p} f^*(\tau) \right)^{1/q} \quad p \in [1, \infty), q = \infty,
\]

\[
\|f\|_{L^p,q(\mathbb{R}^n)} := \left( \frac{1}{q} \int_0^\infty \left[ \tau^{1/p} f^{**}(\tau) \right]^q \frac{d\tau}{\tau} \right)^{1/q} \quad p \in (1, \infty), q \in (1, \infty),
\]

\[
\|f\|_{L^p(\mathbb{R}^n)} := \left( \sup_{\tau \in (0, \infty)} \tau^{1/p} f^{**}(\tau) \right)^{1/q} \quad p \in (1, \infty), q = \infty.
\]

We have the following properties, many of which generalise familiar properties of Lebesgue spaces. As we discuss in \[5\], the majority of these can be found in (or derived easily from statements in) \[12\], while part of property (viii) follows by arguing for example along the lines of \((16) \text{ Theorem } 1.26\).
If \( \|f\|_{L^p(X)} = \|f\|_{L^p(Y)} \) for \( p \in [1, \infty] \).

(ii) \( \|f\|_{L^p(X)} \leq \|f\|_{L^q(X)} \) for \( p \in [1, \infty) \) and \( 1 \leq q_1 \leq q_2 \leq \infty \).

(iii) \( f^*(\tau) \leq f^{**}(\tau) \), and \( \|f\|_{L^p(X)} \leq \|f\|_{L^q(X)} \leq p\|f\|_{L^q(X)} \) for \( p \in (1, \infty] \) and \( q \in [1, \infty] \) with \( (p = \infty \Rightarrow q = \infty) \).

(iv) \( \int_{\mathbb{R}^d} |f(t, \cdot)| \, dt \leq \|f\|_{L^p(X)} \) for a measurable function \( f \) on \( \mathbb{R} \times \mathbb{R}^n \), with \( p \in (1, \infty) \) and \( q \in [1, \infty] \) satisfying \( (p = \infty \Rightarrow q = \infty) \).

(v) \( \int_{\mathbb{R}^d} |f(x)| \, dx \leq \int_{0}^{\infty} f^*(\tau) \, d\tau \), \( (fg)^*(\tau) \leq f^{**}(\tau)g^{**}(\tau) \), and \( (fg)^*(\tau) \leq \|f\|_{L^p(X)}g^{**}(\tau) \).

(vi) \( f^* \) is defined to be the space of linear maps \( F: C^0_b(\mathbb{R}^n) \rightarrow \mathbb{R}^m \) with \( \int_{\mathbb{R}^d} |f(t, \cdot)| \, dt \leq \|f\|_{L^p(X)} \) for \( p \in (1, \infty) \) and \( q \in [1, \infty] \) satisfying \( (p = \infty \Rightarrow q = \infty) \).

(vii) \( \parallel f \parallel_{L^p(X)} \leq \parallel f \parallel_{L^p(Y)} \) \( \parallel f \parallel_{L^p(X)} \) and \( \parallel f \parallel_{L^p(Y)} \) satisfying \( (p = \infty \Rightarrow q = \infty) \).

(viii) \( f \) is defined to be the space of linear maps \( F: C^0_b(\mathbb{R}^n) \rightarrow \mathbb{R}^m \) with \( \int_{\mathbb{R}^d} |f(t, \cdot)| \, dt \leq \|f\|_{L^p(X)} \) for \( p \in (1, \infty) \) and \( q \in [1, \infty] \) satisfying \( (p = \infty \Rightarrow q = \infty) \).

(ix) \( f \) is defined to be the space of linear maps \( F: C^0_b(\mathbb{R}^n) \rightarrow \mathbb{R}^m \) with \( \int_{\mathbb{R}^d} |f(t, \cdot)| \, dt \leq \|f\|_{L^p(X)} \) for \( p \in (1, \infty) \) and \( q \in [1, \infty] \) satisfying \( (p = \infty \Rightarrow q = \infty) \).

We also recall Hardy’s inequalities ([12], p. 256), which state that for \( p \in (0, \infty) \) and \( q \in [1, \infty) \) we have

\[
\left( \int_{0}^{\infty} \left[ \int_{0}^{t} t^{-\frac{s}{q}} \phi(s) \frac{ds}{s} \right]^{q} \, dt \right)^{\frac{1}{q}} \leq p \left( \int_{0}^{\infty} \left[ \int_{0}^{s} s^{-\frac{t}{q}} \phi(s) \frac{ds}{s} \right] \right)^{\frac{1}{q}},
\]

(1.29)

For \( p, q \in [1, \infty] \) satisfying \( (p = 1 \Rightarrow q = 1) \) and \( (p = \infty \Rightarrow q = \infty) \), the normed space \( (L^p(X), \| \cdot \|_{L^p(X)}) \) is isomorphic to the space of linear maps \( F: X \rightarrow \mathbb{R} \) for which the dual norm \( \|F\|_{X^*} := \sup_{\|\phi\|_{X} \leq 1} |F(\phi)| \) is finite. If \( Y \) is a linear subspace of \( X \), and \( F: Y \rightarrow \mathbb{R} \) is a linear map satisfying \( |F(\phi)| \leq \|\phi\|_{X} \) for \( \phi \in Y \), then by the Hahn-Banach theorem there exists \( \hat{F} \in X^* \) with \( \|\hat{F}\|_{X^*} \leq \|F\|_{Y} \) and \( \hat{F}|_{Y} = F \). It follows that a normable quasinorm \( \| \cdot \|_{X} \) is equivalent to its \textit{bidual norm} \( \| \cdot \|_{X^*} := \sup_{\|\phi\|_{X^*} \leq 1} |F(\phi)| \) for \( \phi \in X \).

To prove completeness, we prove the following standard result concerning the dual space of \( L^p(X) \) for \( p < \infty \).

\textbf{Lemma 1.3.} For \( p, q \in [1, \infty] \) satisfying \( (p = 1 \Rightarrow q = 1) \), the map \( u \mapsto F_u \) given by \( F_u(\phi) = \langle u, \phi \rangle \) defines a Banach space isomorphism \( L^{p'}(\mathbb{R}^n) \rightarrow (L^{p,q}(\mathbb{R}^n))^* \).
Proof. By property (v) of Lorentz spaces, the map \( u \mapsto F_u \) is well-defined, linear, injective and bounded. It remains to show that this map is surjective and that its inverse is bounded, so for each \( F \in (L^{p,q}(\mathbb{R}^n))^* \) we seek \( u \in L^{p',q'}(\mathbb{R}^n) \) satisfying \( \|u\|_{L^{p',q'}(\mathbb{R}^n)} \lesssim_{p,q} \|F\|_{(L^{p,q}(\mathbb{R}^n))^*} \) and

\[
F(\phi) = \int_{\mathbb{R}^n} u(x) \phi(x) \, dx \quad \text{for all } \phi \in L^{p,q}(\mathbb{R}^n). \tag{1.30}
\]

The case \( q = 1 \) is proved in ([12], p. 261), so we assume that \( p, q \in (1, \infty) \). If \( p, q \in (1, \infty) \) and \( F \in (L^{p,q}(\mathbb{R}^n))^* \), then \( |F(\phi)| \leq \|F\|_{(L^{p,q}(\mathbb{R}^n))^*} \|\phi\|_{L^{p,q}(\mathbb{R}^n)} \leq \|F\|_{(L^{p,q}(\mathbb{R}^n))^*} \|\phi\|^*_{L^{p,q}(\mathbb{R}^n)} \) for all \( \phi \in L^{p,1}(\mathbb{R}^n) \), so by the case \( q = 1 \) there exists \( u \in L^{p,\infty}(\mathbb{R}^n) \) satisfying

\[
F(\phi) = \int_{\mathbb{R}^n} u(x) \phi(x) \, dx \quad \text{for all } \phi \in L^{p,1}(\mathbb{R}^n). \tag{1.31}
\]

We make the claim (stated without proof in [12]) that \( \|u\|_{L^{p',q'}(\mathbb{R}^n)} \lesssim_{p,q} \|F\|_{(L^{p,q}(\mathbb{R}^n))^*} \), which implies (1.30). We argue along the lines of ([2], Theorem IV.4.7). Consider the simple functions \( u_m = \text{sign}(u) \min\{m, 2^{-m} |2^m |u| \} \), so that \( \|u_m\|_{L^{p',q'}(\mathbb{R}^n)} \to \|u\|_{L^{p',q'}(\mathbb{R}^n)} \) by monotone convergence. Since \( u'_m \) is a decreasing function, we have

\[
\left( \|u_m\|_{L^{p',q'}(\mathbb{R}^n)} \right)^{1/q'} \lesssim_{p,q} \int_0^\infty \left( \int_0^t s^{q'-1} u'_m(s) \, ds \right)^{1/q'} \, dt
\]

\[
= \int_0^\infty \left( \int_0^t \left( \int_0^s \frac{ds}{s} \right)^{q'-1} u'_m(s) \, ds \right) u'_m(t) \, dt
\]

\[
\lesssim \int_0^\infty \left( \int_0^t \frac{ds}{s} \right)^{q'-1} u'_m(t) \, dt
\]

\[
= \int_0^\infty \psi_m(t) u'_m(t) \, dt,
\]

where \( \psi_m(t) := \int_t^\infty s^{q'-1} u'_m(s) \, ds \) defines a decreasing continuous function on \((0, \infty)\). Since \( u_m \) is a simple function, the map

\[
\tau_m(x) := \{|z \in \mathbb{R}^n : |u_m(x)| > |u_m(x)|\} + \{|z \in \mathbb{R}^n : |z| < |x|, |u_m(z)| = |u_m(x)|\}
\]

defines a measure preserving transformation \( \tau_m : (\mathbb{R}^n, dx) \to ((0, \infty), dt) \) satisfying \( |u_m| = u_m^\star \circ \tau_m \) almost everywhere. Writing \( \phi_m = \text{sign}(u) \psi_m \circ \tau_m \), we consider the simple functions \( \phi_{m,j} = \text{sign}(u) \min\{j, 2^{-j} |2^j |\phi_m| \} \), so by monotone convergence we have

\[
\int_0^\infty \psi_m(t) u'_m(t) \, dt = \int_{\mathbb{R}^n} \phi_m(x) u_m(x) \, dx
\]

\[
= \lim_{j \to \infty} \int_{\mathbb{R}^n} \phi_{m,j}(x) u_m(x) \, dx
\]

\[
\leq \lim_{j \to \infty} \int_{\mathbb{R}^n} \phi_{m,j}(x) u(x) \, dx
\]

\[
= \lim_{j \to \infty} F(\phi_{m,j})
\]

\[
\leq \lim_{j \to \infty} \|F\|_{(L^{p,q}(\mathbb{R}^n))^*} \|\phi_{m,j}\|_{L^{p,q}(\mathbb{R}^n)}^*
\]

\[
= \|F\|_{(L^{p,q}(\mathbb{R}^n))^*} \|\phi_m\|_{L^{p,q}(\mathbb{R}^n)}^*.
\]
We use \( \phi_m^* = \psi_m \), the substitution \( r = es \) and Hardy’s inequality to estimate

\[
\|\phi_m\|_{L^p,q(R^n)} \lesssim_{p,q} \left( \int_0^\infty \left[ \frac{t^\frac{1}{q} \psi_m(t)}{t} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
= \left( \int_0^\infty \left[ \frac{1}{t} \int_{\frac{s}{t}}^\infty \psi_m(s) \frac{ds}{s} \right]^{q-1} \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\lesssim_{p,q} \left( \int_0^\infty \left[ \frac{1}{t} \int_{\frac{s}{t}}^\infty \psi_m(s) \frac{ds}{s} \right]^{q-1} \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
\lesssim_{p,q} \left( \int_0^\infty \left[ \frac{1}{t} \int_{\frac{s}{t}}^\infty \psi_m(s) \frac{ds}{s} \right]^{q-1} \frac{dt}{t} \right)^{\frac{1}{q}}
\]

\[
= \left( \int_0^\infty \left[ \frac{1}{s} \psi_m(s) \right]^{q-1} \frac{ds}{s} \right)^{\frac{1}{q}}
\]

\[
\lesssim_{p,q} \left( \|u_m\|_{L^p,q(R^n)} \right)^{\frac{1}{q}}
\]

We conclude that

\[
\left( \|u_m\|_{L^p,q(R^n)} \right)^{\frac{1}{q}} \lesssim_{p,q} \int_0^\infty \psi_m(t) u_m(t) dt \lesssim \|F\|_{L^p,q(R^n)}, \|\phi_m\|_{L^p,q(R^n)}
\]

\[
\lesssim_{p,q} \|F\|_{L^p,q(R^n)} \left( \|u_m\|_{L^p,q(R^n)} \right)^{\frac{1}{q}},
\]

so \( \|u_m\|_{L^p,q(R^n)} \lesssim_{p,q} \|F\|_{L^p,q(R^n)} \) and hence \( \|u\|_{L^p,q(R^n)} \lesssim_{p,q} \|F\|_{L^p,q(R^n)} \). \( \Box \)

2. Formulations of the linearised Navier-Stokes equations

On the space-time domain \( (0, T) \times \mathbb{R}^n \), we will discuss various formulations of the linearised Navier-Stokes equations

\[
\begin{cases}
    u_t' - \Delta u_j + \nabla_k F_{jk} + \nabla_j P = 0, \\
    \nabla \cdot u = 0, \\
    u(0) = f,
\end{cases}
\]

(2.1)

where the Navier-Stokes equations are realised by taking \( F_{jk} = u_j u_k \). Each formulation will be preceded by a discussion of some of the technical preliminaries relevant to that formulation.

2.1. Weak* measurability and the weak formulation. For a measurable space \((E, \mathcal{E})\), a function \( u : E \to L^1_{loc}(\mathbb{R}^n) \) is said to be weakly* measurable if the map \( t \mapsto \langle u(t), \phi \rangle \) is measurable for all \( \phi \in C_c^\infty(\mathbb{R}^n) \). The following results make particular use of property (viii) from our discussion of Lorentz spaces.

**Lemma 2.1.** For \( p, q \in [1, \infty) \) with \( (p = 1 \Rightarrow q = 1) \) and \( (p = \infty \Rightarrow q = \infty) \), a function \( u : E \to L^{p,q}(\mathbb{R}^n) \) is weakly* measurable if and only if the map \( t \mapsto \langle u(t), \phi \rangle \) is measurable for all \( \phi \in L^{p',q'}(\mathbb{R}^n) \).

**Proof.** Define the cutoff function \( \rho_R(x) = \rho(x/R) \) for \( R \in \mathbb{Q}_{>0} \), where \( \rho \in C_c^\infty(\mathbb{R}^n) \) satisfies \( \rho(x) = 1 \) for \( |x| < 1 \) and \( \rho(x) = 0 \) for \( |x| > 2 \). Define also the approximate identity \( \eta_\epsilon(x) = \epsilon^{-n} \eta(x/\epsilon) \) for \( \epsilon \in \mathbb{Q}_{>0} \), where \( \eta \in C_c^\infty(\mathbb{R}^n) \) satisfies \( \int_{\mathbb{R}^n} \eta(x) dx = 1 \). Then the function

\[
\langle u(t), \phi \rangle = \lim_{\epsilon \to 0} \langle u(t), \eta_\epsilon * \phi \rangle = \lim_{\epsilon \to 0} \lim_{R \to \infty} \langle u(t), \rho_R * (\eta_\epsilon * \phi) \rangle
\]

(2.2)

is measurable, where the limit \( R \to \infty \) follows from dominated convergence, and the limit \( \epsilon \to 0 \) follows from approximation of identity (if necessary, use Fubini's theorem to throw the convolution onto whichever one of \( u(t), \phi \) belongs to a separable Lorentz space).

\( \Box \)
Lemma 2.2. Let \( p, q \in [1, \infty) \) satisfy \( p = 1 \Rightarrow q = 1 \).

(i) If \( u : E \rightarrow L^{p,q}(\mathbb{R}^n) \) is weakly* measurable, then \( t \mapsto \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \) is measurable.

(ii) If \( u : E \rightarrow L^{p,q}(\mathbb{R}^n) \) is weakly* measurable, then \( t \mapsto \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \) is measurable.

Proof. (i) Let \( A \) be a countable dense subset of \( \{ \phi \in L^{p,q}(\mathbb{R}^n) : \|\phi\|_{L^{p,q}(\mathbb{R}^n)} \leq 1 \} \), so the function \( \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} = \sup_{\phi \in A} \langle u(t), \phi \rangle \) is measurable.

(ii) If there exists a countable family \( A \subseteq B := \{ F \in L^{p,q}(\mathbb{R}^n) : \|F\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \leq 1 \} \), such that for each \( F \in B \) there exists a sequence \( (F_m)_{m \geq 1} \) in \( A \) satisfying \( (F_m, \phi) \rightarrow (F, \phi) \) for all \( \phi \in L^{p,q}(\mathbb{R}^n) \), then the function \( \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} = \sup_{F \in A} \langle F, u(t) \rangle \) is measurable. We construct \( A \) as follows.

Let \( (\phi_m)_{m \geq 1} \) be dense in \( L^{p,q}(\mathbb{R}^n) \), and observe that each family \( \{((F_k, \phi_l))_{l \leq m} : k \geq 1\} \) has a dense subset \( \{((F_k, \phi_l))_{l \leq m} : k \geq 1\} \). We claim that \( A = (F_k, \phi_l))_{k,m \geq 1} \) is as required. Given \( F \in B \), for each \( m \geq 1 \) there exists \( k(m) \) such that \( \|F_k(m), F, \phi_l) \| < \frac{1}{m} \) for all \( l \leq m \), so for any \( \phi \in L^{p,q}(\mathbb{R}^n) \) and \( \epsilon > 0 \) we can choose \( l(\epsilon) \) with \( \|\phi_l(\epsilon) - \phi\|_{L^{p,q}(\mathbb{R}^n)} < \epsilon \) to obtain

\[
\limsup_{m \rightarrow \infty} \left| \langle (F_k(m), m - F, \phi_l) \rangle \right| \leq \limsup_{m \rightarrow \infty} \left( \|F_k(m), m - F, \phi_l(\epsilon)\| + \|F_k(m), m, \phi_l(\epsilon) - \phi\| \right) \leq 2\epsilon.
\]

(2.3)

Lemma 2.3. (Pettis’ theorem). Let \( p, q \in [1, \infty) \) satisfy \( p = 1 \Rightarrow q = 1 \). If the function \( u : E \rightarrow L^{p,q}(\mathbb{R}^n) \) is weakly* measurable, then there exist measurable functions \( u_m : E \rightarrow L^{p,q}(\mathbb{R}^n) \) with finite image, satisfying \( \|u_m(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \leq \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \) and \( \|u_m(t) - u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \rightarrow 0 \) pointwise.

Proof. Let \( A \) be a countable dense subset of \( L^{p,q}(\mathbb{R}^n) \), let \( QA = \{ qa : q \in \mathbb{Q}, a \in A \} \), and let \( (f_j)_{j \geq 0} \) be an enumeration of \( QA \) with \( f_0 = 0 \). For each \( m \geq 0 \) and \( t \in E \), define the non-empty set

\[
K(m, t) := \left\{ j \in \{0, \ldots, m\} : \|f_j\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \leq \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \right\}
\]

and the integer

\[
k(m, t) := \min \left\{ j \in K(m, t) : \|f_j - u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} = \min_{i \in K(m, t)} \|f_i - u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \right\}.
\]

(2.4)

(2.5)

Then \( u_m(t) := f_{k(m, t)} \) defines a measurable function \( u_m : E \rightarrow L^{p,q}(\mathbb{R}^n) \) with finite image, satisfying \( \|u_m(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \leq \|u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \) and \( \|u_m(t) - u(t)\|_{(L^{p,q}(\mathbb{R}^n))^{**}} \rightarrow 0 \) pointwise.

For \( T \in (0, \infty] \), and \( p, q \in [1, \infty) \) satisfying \( p = 1 \Rightarrow q = 1 \) and \( p = \infty \Rightarrow q = \infty \), we write \( L_{p,q}^{\infty}(T) \) to denote (equivalence classes of) weakly* measurable functions \( u : (0, T) \rightarrow L^{p,q}(\mathbb{R}^n) \) satisfying \( \|u\|_{L^{p,q}(\mathbb{R}^n)} \in L^\infty(0, T) \), understood in the sense that the quasinorm \( \|u(t)\|_{L^{p,q}(\mathbb{R}^n)} \) is equivalent to a norm which is a measurable function of \( t \). If \( p, q \in [1, \infty) \) satisfy \( p = 1 \Rightarrow q = 1 \), and \( u \in L_{p,q}^{\infty}(T) \), then the approximating functions from Pettis’ theorem are simple (supported on sets of finite measure), and satisfy \( \|t \mapsto u_m(t) - u(t)\|_{L^{p,q}(\mathbb{R}^n)} \rightarrow 0 \) by dominated convergence.

Sometimes we will want to replace the interval \((0, T)\) by the interval \((\infty, T)\); in such situations, we will write \( \mathcal{E} \) instead of \( \mathcal{L} \). One such situation is the following approximation lemma, which will play an important role in establishing energy estimates for the Navier-Stokes equations.
Lemma 2.4. Let $\alpha \in (1, \infty)$, let $p, q \in [1, \infty)$ satisfy $(p = 1 \Rightarrow q = 1)$, and let $\eta_\varepsilon(t) = e^{-1/\varepsilon} \eta(t/\varepsilon)$ for some $\eta \in L^1((-1,1))$ satisfying $\int_{-1}^1 \eta(t) \, dt = 1$. For $T \in (-\infty, \infty]$, $u \in \tilde{L}_p^\alpha(\bar{T}_-)$ and $U \in \tilde{L}_p^\alpha(T_-)$, define the mollified functions

$$u^\varepsilon(t, x) := \int_{-\infty}^T \eta_\varepsilon(t-s) u(s, x) \, ds, \quad U^\varepsilon(t, x) := \int_{-\infty}^T \eta_\varepsilon(t-s) U(s, x) \, ds$$

(2.6)

for almost every $(t, x) \in (-\infty, T - \varepsilon) \times \mathbb{R}^n$. Then for all $T' \in (-\infty, T)$ we have

(i) $u^\varepsilon \to u$ in $\tilde{L}_p^\alpha(T')$ as $\varepsilon \to 0$,

(ii) $(U^\varepsilon, u^\varepsilon) \to (U, u)$ in $L^1((-\infty, T'))$ as $\varepsilon \to 0$.

Proof. Approximating $u$ via Pettis’ theorem, it suffices to consider the case where the image of $u : (-\infty, T) \to L^{p,q}(\mathbb{R}^n)$ is contained within a finite dimensional subspace $X \subseteq L^{p,q}(\mathbb{R}^n)$. Let $(f_i)_{i=1}^m$ be a basis for $X$, and let $(\theta_j)_{j=1}^n$ be the basis for $X^*$ satisfying $\theta_i(f_j) = \delta_{ij}$. By the Hahn-Banach theorem, there exist $F_i \in L^{p,q}(\mathbb{R}^n)$ satisfying $\langle F_i, f_j \rangle = \theta_i(f_j)$ for all $f_j \in X$. Then $u = \sum_{i=1}^m u_i f_i$, and $\langle U, f_i \rangle = \sum_{i=1}^m U_i F_i, f_i$ for all $f_i \in X$, where the functions $u_i := \langle F_i, u \rangle$ and $U_i := \langle F_i, U \rangle$ are measurable on $(-\infty, T)$. It therefore suffices to consider the functions $u = (u_i)_{i=1}^m \in \cap_{T' \in (-\infty, T)} L^{p,q}((\cdot, T'); \mathbb{R}^m)$ and $U = (U_i)_{i=1}^m \in \cap_{T' \in (-\infty, T)} L^{p,q}((\cdot, T'); \mathbb{R}^m)$. For $T' \in (-\infty, T - \varepsilon)$, we use Hölder’s inequality and continuity of translation to estimate

$$\|u^\varepsilon - u\|_{L^p((\cdot, T'); \mathbb{R}^m)} = \left( \int_{-\infty}^T \left| \int_{-\varepsilon}^\varepsilon \eta_\varepsilon(s) (u(t-s) - u(t)) \, ds \right|^p \, dt \right)^{1/p} \leq \left( \int_{-\infty}^T \left( \int_{-\varepsilon}^\varepsilon \eta_\varepsilon(s) \, ds \right)^{p-1} \left( \int_{-\varepsilon}^\varepsilon |\eta_\varepsilon(s)| (|u(t-s) - u(t)|)^{p-1} \, ds \right) \, dt \right)^{1/p} \leq \|u\|_{L^p(\mathbb{R})} \left( \int_{-\varepsilon}^\varepsilon \eta_\varepsilon(s) \, ds \right)^{p-1} \left( \int_{-\varepsilon}^\varepsilon |u(t-s) - u(t)| \, ds \right)^{1/p} \left( \int_{-\varepsilon}^\varepsilon |\eta_\varepsilon(s)| \, ds \right)^{1/p-1} \to 0 \quad (2.7)$$

and analogously $\|U^\varepsilon - U\|_{L^p((\cdot, T'); \mathbb{R}^m)} \to 0$, so $U^\varepsilon \cdot u^\varepsilon \to u$ in $L^1((-\infty, T'); \mathbb{R})$. $\square$

We are now in a position to define the weak formulation of the linearised Navier-Stokes equations. For $T \in (0, \infty]$, $p_0, p, \bar{p} \in (1, \infty)$, $f \in L^{p_0, \infty}(\mathbb{R}^n)$ satisfying $(f_j, \nabla_i \phi)$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and $F \in L^{p_0, \infty}_c(T_-)$, we say that $u \in L^{1, \infty}_c(T_-)$ is a weak solution to the linearised Navier-Stokes equations if the existence $P \in L^{1, \infty}_c(T_-)$ such that

$$\begin{cases} \langle F_j, \phi_j(0) \rangle + \int_0^T \left( \langle u_j, \phi_j' + \Delta \phi_j \rangle + \langle F_jk, \nabla_k \phi_j \rangle + \langle P, \nabla_j \phi_j \rangle \right) \, dt = 0 & \forall \phi \in C_c^\infty([0, T) \times \mathbb{R}^n) \\ \int_0^T \langle u_j, \nabla_j \psi \rangle \, dt = 0 & \forall \psi \in C_c^\infty([0, T) \times \mathbb{R}^n) \end{cases}$$

(2.8)

where measurability issues are addressed by approximating the test functions using Pettis’ theorem. This definition depends a priori on the index $\bar{p}$, since we seek a pressure term $P \in L^{1, \infty}_c(T_-)$. However, we will eventually prove that the weak formulation is equivalent to another formulation which does not depend on $\bar{p}$.

For $T \in (0, \infty]$, $p_0, p \in (1, \infty)$, and $f \in L^{p_0, \infty}(\mathbb{R}^n)$ satisfying $(f_i, \nabla_i \phi)$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, we say that $u \in L^{p_0, \infty}_c(T_-)$ is a weak solution to the linearised Navier-Stokes equations if $u$ is a weak solution to the linearised Navier-Stokes equations with $F_{jk} = u_j u_k$. To show that $u_j u_k$ is weakly measurable, we write $\langle u_j(0)u_k(0), \phi \rangle \to (u_j(t), u_k(t))\phi$ for $\phi \in C_c^\infty(\mathbb{R}^n)$, and we approximate the weakly measurable function $u_k \phi : (0, T) \to L^{p_0, \infty}_c(\mathbb{R}^n)$ using Pettis’ theorem.

9
2.2. The Riesz transform and the projected formulation. The Riesz transform $\mathcal{R}$ gives a precise meaning to the pseudo-differential operator $-\nabla^\alpha$.

Lemma 2.5. There exists a unique linear map

$$\mathcal{R} : \cup_{p \in (1, \infty)} \left( L^p(\mathbb{R}^n) + L^{p,\infty}(\mathbb{R}^n) \right) \to \cup_{p \in (1, \infty)} \left( L^1(\mathbb{R}^n) + L^{1,\infty}(\mathbb{R}^n) \right)$$

(2.9)

which satisfies

$$\mathcal{R} f = F^{-1} \left[ \xi \mapsto \frac{-i}{|\xi|} Ff(\xi) \right] \quad \text{for all } f \in L^2(\mathbb{R}^n),$$

(2.10)

$$\left\{ \begin{array}{l}
\|\mathcal{R} f\|^p_{L^p,\eta}(\mathbb{R}^n) \lesssim_{\eta, p, q} \|f\|^p_{L^p,\eta}(\mathbb{R}^n) \text{ and } \langle \mathcal{R} f, g \rangle = -\langle f, \mathcal{R} g \rangle
\end{array} \right.$$ (2.11)

Assuming Lemma 2.5, we can define the Leray projection $\mathbb{P}_{ij} := \delta_{ij} + \mathcal{R}_i \mathcal{R}_j$ onto divergence-free vector fields, which allows us to reduce the weak formulation of the linearised Navier-Stokes equations to a projected formulation, which is independent of the pressure $P$.

Theorem 2.6. Let $T \in (0, \infty]$, $p_0, p, \tilde{p} \in (1, \infty)$, $f \in L^{p_0,\infty}(\mathbb{R}^n)$ with $(f_i, \nabla_i \phi)$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and $F \in L^{\tilde{l}_i,\infty}(T_-)$. Then $u \in L^{\tilde{l}_i,\infty}(T_-)$ is a weak solution to the linearised Navier-Stokes equations if and only if $u$ satisfies the projected formulation

$$\left\{ \begin{array}{l}
\langle f_i, \phi_t(0) \rangle + \int_0^T \left( (u_i, \phi_t^* + \Delta \phi_t) + (F_{jk}, \mathbb{P}_{ij} \nabla_k \phi_t) \right) \, dt = 0 \quad \forall \phi \in C_c^\infty((0, T) \times \mathbb{R}^n),
\int_0^T (u_i, \nabla_i \psi) \, dt = 0 \quad \forall \psi \in C_c^\infty((0, T) \times \mathbb{R}^n),
\end{array} \right.$$ (2.12)

where $\mathbb{P}_{ij} F_{jk} \in L^{\tilde{l}_i,\infty}(T_-)$ by properties of the Riesz transform. Moreover, the function $P$ in the weak formulation is uniquely determined by $P = \mathcal{R}_j \mathcal{R}_k F_{jk}$.

Proof. Given $F \in L^{\tilde{l}_i,\infty}(T_-)$, the expression $P := \mathcal{R}_j \mathcal{R}_k F_{jk}$ defines a weakly* measurable function $P \in L^{\tilde{l}_i,\infty}(T_-)$ satisfying

$$\langle P(t), \nabla_i \psi_i \rangle = (F_{jk}(t), \mathcal{R}_j \mathcal{R}_k \nabla_i \phi_i) = (F_{jk}, \mathcal{R}_j \mathcal{R}_k \nabla_i \psi_i)$$ (2.13)

for almost every $t \in (0, T)$, so it solves the projected problem then $u$ is a weak solution.

Conversely, by considering $\phi_t(x) = \theta(t) \nabla \psi(x)$, where $\psi \in C_c^\infty(\mathbb{R}^n)$ and $\theta \in C_c^\infty((0, T))$ is fixed and $\theta \in C_c^\infty((0, T))$ is allowed to vary, if we have a weak solution with pressure $P$ then

$$0 = \langle F_{jk}(t), \nabla_j \nabla_k \psi \rangle + \langle P(t), \nabla \psi \rangle = -\langle F_{jk}(t), \mathcal{R}_j \mathcal{R}_k \nabla \psi \rangle + \langle P(t), \nabla \psi \rangle$$ (2.14)

for all $\psi \in C_c^\infty(\mathbb{R}^n)$, so $\langle P(t), \chi \rangle = \langle \mathcal{R}_j \mathcal{R}_k F_{jk}(t), \chi \rangle$ for almost every $t \in (0, T)$ for all $\chi \in \Delta C_c^\infty(\mathbb{R}^n)$. By considering a countable dense subset of $\Delta C_c^\infty(\mathbb{R}^n)$, it follows that $\langle P(t), \chi \rangle = \langle \mathcal{R}_j \mathcal{R}_k F_{jk}(t), \chi \rangle$ for all $\chi \in \Delta C_c^\infty(\mathbb{R}^n)$ for almost every $t \in (0, T)$. To conclude that $P(t) = \mathcal{R}_j \mathcal{R}_k F_{jk}(t)$ for almost every $t \in (0, T)$, we make the following use of the lemma:

If $P \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies $\langle P, \nabla \psi \rangle$ for all $\psi \in C_c^\infty(\mathbb{R}^n)$, then $P$ is almost everywhere equal to some $\tilde{P} \in L^2(\mathbb{R}^n)$ satisfying $\Delta \tilde{P} = 0$, so if $P \in L^2(\mathbb{R}^n)$ for $\tilde{P} \in (1, \infty)$ then $P = 0$ almost everywhere.

Proof of lemma. Let $\eta(x) = e^{-\nu(x)}\eta(x/e)$ for some $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \eta(x) \, dx = 1$. Then $P_\varepsilon := P * \eta_\varepsilon$ is a smooth function satisfying $\langle \Delta P_\varepsilon, \psi \rangle$ for all $\psi \in C_c^\infty(\mathbb{R}^n)$, so $\Delta P_\varepsilon = 0$. By the mean value property (10, Theorem 2.7), we therefore have $P_\varepsilon(x) = \frac{1}{|B(x, \varepsilon)|} \int_{B(x, \varepsilon)} P_\varepsilon(y) \, dy$ for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$. By continuity of translation in $L^1(\mathbb{R}^n)$ we have

$$\left\| \int_{B(x, \varepsilon)} (P_\varepsilon(y) - P(y)) \, dy \right\| = \int_{B(x, \varepsilon)} \left( \int_{\mathbb{R}^n} |P(y-z) - P(y)| \eta_\varepsilon(z) \, dz \right) \, dy$$ (2.15)

$$\leq \int_{\mathbb{R}^n} \left( \int_{B(x, \varepsilon)} |P(y-z) - P(y)| \, dy \right) |\eta_\varepsilon(z)| \, dz \xrightarrow{\varepsilon \to 0} 0$$

locally uniformly in $x \in \mathbb{R}^n$. Therefore, as $\varepsilon \to 0$, $P_\varepsilon$ converges to $P$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, and $P_\varepsilon$ converges locally uniformly to a continuous function $P$ satisfying $P(x) = \frac{1}{|B(x, \varepsilon)|} \int_{B(x, \varepsilon)} P(y) \, dy$. Then $P = \tilde{P}$ almost everywhere, so $\tilde{P}(x) = \frac{1}{|B(x, \varepsilon)|} \int_{B(x, \varepsilon)} P(y) \, dy$ and by the mean value property we deduce that $\tilde{P} \in C^2(\mathbb{R}^n)$ with $\Delta \tilde{P} = 0$. By Liouville’s theorem, if $\tilde{P}$ is not constant then it is unbounded, so for any $R > 0$ we can choose $x \in \mathbb{R}^n$ to make $\tilde{P}(x) = \frac{1}{|B(x, \varepsilon)|} \int_{B(x, \varepsilon)} P(y) \, dy$ arbitrarily large, which implies that the maximal function $P^{**}(\tau) = \sup_{|A| > \tau} \frac{1}{|A|} \int_A |P(y)| \, dy$ is infinite for all $\tau \in (0, \infty)$, so $P$ cannot belong to a Lorentz space.
For the sake of completeness, we give a construction of the Riesz transform. For $\epsilon \in (0, \infty)$ and $n \geq 2$ (a simplified version of the argument exists for $n = 1$), we define the truncated Riesz transform

$$\mathcal{R}_\epsilon f(x) := \frac{1}{\pi \omega_{n-1}} \int_{|y| < \epsilon} \frac{y}{|y|^{n+1}} f(x - y) \, dy \quad \text{for } f \in L^2(\mathbb{R}^n), \ x \in \mathbb{R}^n,$$

(2.16)

where $\omega_{n-1}$ is the volume of the unit ball in $\mathbb{R}^{n-1}$. By Hölder’s inequality and continuity of translation in $L^2(\mathbb{R}^n)$, $\mathcal{R}_\epsilon$ maps $L^2(\mathbb{R}^n)$ to the space $C_0^\infty(\mathbb{R}^n)$ of bounded, uniformly continuous functions. Then $\langle \mathcal{R}_\epsilon f, g \rangle = - \langle f, \mathcal{R}_\epsilon g \rangle$ for all $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by Fubini’s theorem. For $f \in L^1(\mathbb{R}^n) \cap FL^1(\mathbb{R}^n)$ and $0 < \epsilon < R < \infty$, we use Fourier inversion and Fubini’s theorem to write

$$\int_{|x| < |y| < R} f(x - y) \, dy = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \int_{|x| < |y| < R} e^{-i\xi \cdot y} \frac{y}{|y|^{n+1}} \, dy \right) \mathcal{F}f(\xi) \, d\xi,$$

where the improper integral $\int_{|y| = \infty} e^{-iy \cdot \xi} \frac{y}{|y|^{n+1}} \, dy = 0$ exists, we have that $\int_{|y| = \infty} \frac{y}{|y|^{n+1}} \, dy$ defines a bounded continuous function on $\{ (a, b) : 0 \leq a \leq b \leq \infty \}$. By dominated convergence, and using the identity $|S^{n-2}| f_0^\xi \cos \theta \sin^{n-2} \theta \, d\theta = \frac{1}{n-1} |S^{n-2}| = \omega_{n-1}$, we deduce that

$$\mathcal{R}_\epsilon f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{\xi}{|\xi|^2} \lambda_n(\epsilon|\xi|) \mathcal{F}f(\xi) \, d\xi \quad \text{for } f \in L^1(\mathbb{R}^n) \cap FL^1(\mathbb{R}^n), \ x \in \mathbb{R}^n,$$

(2.19)

where $\lambda_n : [0, \infty) \to \mathbb{R}$ is a bounded continuous function with $\lambda_n(0) = 1$.

Now $L^1(\mathbb{R}^n) \cap FL^1(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, so we deduce that $\mathcal{R}_\epsilon f = F^{-1} \left[ \xi \mapsto \frac{\xi}{|\xi|^2} \lambda_n(\epsilon|\xi|) \mathcal{F}f(\xi) \right]$ for all $f \in L^2(\mathbb{R}^n)$. By dominated convergence and continuity of $F^{-1} : L^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, it follows that $\mathcal{R}_\epsilon f \to \mathcal{R}_0 f$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$, where the $L^2$ Riesz transform is defined by $\mathcal{R}_0 f := \mathcal{R}^{-1} f = - (f, \mathcal{R}_0 g)$ for all $f, g \in L^2(\mathbb{R}^n)$. By density of $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, we have $\langle \mathcal{R}_0 f, g \rangle = - (f, \mathcal{R}_0 g)$ for all $f, g \in L^2(\mathbb{R}^n)$.

If we can establish the estimate

$$|\{ x \in \mathbb{R}^n : |\mathcal{R}_0 f(x)| > \alpha \} | \lesssim_n \frac{\| f \|_{L^1(\mathbb{R}^n)}}{\alpha} \quad \text{for all } f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \text{ and } \alpha \in (0, \infty),$$

(2.20)

then the $L^2$ Riesz transform extends by density to $\mathcal{R}_0 : L^1(\mathbb{R}^n) \to L^{1, \infty}(\mathbb{R}^n)$. By interpolation (property (iv) from our discussion of Lorentz spaces), it follows that $\mathcal{R}_0 : L^{p,q}(\mathbb{R}^n) \to L^{p,q}(\mathbb{R}^n)$ for $p \in (1, 2)$ and $q \in [1, \infty]$, so by duality the expression $\langle \mathcal{R}_0 f, g \rangle := \langle f, \mathcal{R}_0 g \rangle$ defines $\mathcal{R}_0 : L^{p,q}(\mathbb{R}^n) \to L^{p,q}(\mathbb{R}^n)$ for $p \in (2, \infty)$. For $p_0 \in (1, 2)$ and $p_1 \in (2, \infty)$, we have $\mathcal{R}_0 : L^{p_0, \infty}(\mathbb{R}^n) \to L^{p_0, \infty}(\mathbb{R}^n)$ and $\mathcal{R}_0 : L^{p_1, \infty}(\mathbb{R}^n) \to L^{p_1, \infty}(\mathbb{R}^n)$, where $\mathcal{R}_0 : L^{p, \infty}(\mathbb{R}^n) \cap L^{p, \infty}(\mathbb{R}^n) \to L^{2, \infty}(\mathbb{R}^n)$ (this inclusion follows from property (vii) from our discussion of Lorentz spaces), so the expression $\mathcal{R}(f_0 + f_1) := \mathcal{R}(f_0) - \mathcal{R}(f_1)$ defines a well-defined operator $\mathcal{R} : L^{p, \infty}(\mathbb{R}^n) \to L^{p, \infty}(\mathbb{R}^n) \to L^{p, \infty}(\mathbb{R}^n) + L^{p, \infty}(\mathbb{R}^n)$ which extends $\mathcal{R}_0$ and $- \mathcal{R}_0$. By interpolation, we have $\mathcal{R} : L^{p,q}(\mathbb{R}^n) \to L^{p,q}(\mathbb{R}^n)$ for $p \in (1, \infty)$ and $q \in [1, \infty]$. The operator $\mathcal{R}$ is called the Riesz transform, and satisfies $\langle \mathcal{R} f, g \rangle = - (f, \mathcal{R} g)$ for $f \in L^{p,q}(\mathbb{R}^n)$ and $g \in L^{p,q}(\mathbb{R}^n)$.

We now prove (2.20), arguing along the lines of (21), sections II.2.4.1, II.2.4.2 and II.3.1. Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\alpha \in (0, \infty)$. Since $f \in L^p(\mathbb{R}^n)$, we can use the Calderón-Zygmund cube decomposition (21, Theorem I.4) to construct a countable collection $\{ Q_k \}$ of closed hypercubes in $\mathbb{R}^n$ (with sides parallel to the coordinate axes) with disjoint interiors such that

$$|f(x)| \leq \alpha \quad \text{for almost every } x \in \mathbb{R}^n \setminus (\cup_k Q_k),$$

(2.21)

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} |f(x)| \, dx \leq 2^n \alpha \quad \text{for each hypercube } Q_k.$$

(2.22)

Writing $\Omega = \cup_k Q_k$, we use (2.22) to estimate

$$|\Omega| \leq \sum_k |Q_k| \leq \sum_k \frac{1}{\alpha} \int_{Q_k} |f(x)| \, dx \leq \frac{\| f \|_{L^1(\mathbb{R}^n)}}{\alpha}.$$

(2.23)
We write \( f = g + h \), where
\[
g(x) := \begin{cases} 
  f(x) & \text{if } x \in \mathbb{R}^n \setminus \Omega, \\
  \frac{1}{|Q_k|} \int_{Q_k} f(y) \, dy & \text{if } x \in \text{int}(Q_k).
\end{cases}
\]

(2.24)

Here the “good” part \( g \) is bounded by estimates (2.21) and (2.22), while the “bad” part \( h \) is supported on the set \( \Omega \) of bounded measure (2.23), with \( b \) having zero average on each hypercube \( Q_k \). For \( p \in \{1, 2\} \) we have
\[
\|g\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n \setminus \Omega} |g(x)|^p \, dx + \int_{\Omega} |g(x)|^p \, dx \\
\leq \int_{\mathbb{R}^n \setminus \Omega} \alpha^{p-1} |f(x)| \, dx + |\Omega| 2^{np} \alpha^p \\
\leq (1 + 2^{np}) \alpha^{p-1} \|f\|_{L^1(\mathbb{R}^n)}.
\]

(2.25)

Therefore \( g \) and \( b = f - g \) belong to \( L^2(\mathbb{R}^n) \), so we can write \( R_0 f = R_0 g + R_0 b \) and we have
\[
|\{x \in \mathbb{R}^n : |R_0 f(x)| > \alpha\}| \leq |\{x \in \mathbb{R}^n : |R_0 g(x)| > \alpha/2\}| + |\{x \in \mathbb{R}^n : |R_0 b(x)| > \alpha/2\}|.
\]

(2.26)

By Chebychev’s inequality and (2.25) with \( p = 2 \), we have
\[
|\{x \in \mathbb{R}^n : |R_0 g(x)| > \alpha/2\}| \leq \frac{4 \|R_0 g\|_{L^2(\mathbb{R}^n)}^2}{\alpha^2} = \frac{4 \|g\|_{L^2(\mathbb{R}^n)}^2}{\alpha^2} \leq \frac{4 (1 + 2^n) \|f\|_{L^1(\mathbb{R}^n)}}{\alpha}.
\]

(2.27)

To estimate \( |\{x \in \mathbb{R}^n : |R_0 b(x)| > \alpha/2\}| \), we make the following definition: if the hypercube \( Q_k \) has centre \( y^{(k)} \) and side length \( r \), then we let \( Q^*_k \) be the hypercube with centre \( y^{(k)} \) and side length \( 2\sqrt{n}r \). The scaling factor \( 2\sqrt{n} \) is chosen to ensure that
\[
|x - y^{(k)}| \geq 2|y - y^{(k)}| \quad \text{for all } x \in \mathbb{R}^n \setminus Q^*_k \text{ and } y \in Q_k.
\]

(2.28)

Writing \( \Omega^* = \bigcup_k Q^*_k \), we have
\[
|\Omega^*| \leq \sum_k |Q^*_k| \leq n \sum |Q_k| \leq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\alpha},
\]

(2.29)

so it remains for us to estimate \( |\{x \in \mathbb{R}^n \setminus \Omega^* : |R_0 b(x)| > \alpha/2\}| \). Writing \( b_k(x) = 1_{Q_k}(x) b(x) \), we note that the series \( b = \sum b_k \) converges absolutely in \( L^2(\mathbb{R}^n) \), so the series \( R_0 b = \sum R_0 b_k \) converges absolutely in \( L^2(\mathbb{R}^n) \), so \( R_0 b(x) = \sum R_0 b_k(x) \) for almost every \( x \in \mathbb{R}^n \). By our analysis of truncated Riesz transforms, we have \( R_0 b_k = \lim_{r \to 0} R_0 b_k \) in \( L^2(\mathbb{R}^n) \). For all \( x \in \mathbb{R}^n \setminus Q^*_k \) we have
\[
R_0 b_k(x) := \frac{1}{\pi \omega_{n-1}} \int_{y \in Q_k, |x - y| > r} \frac{x - y}{|x - y|^{n+1}} b(y) \, dy \\
\overset{r \to 0}{\to} \frac{1}{\pi \omega_{n-1}} \int_{Q_k} \frac{x - y}{|x - y|^{n+1}} b(y) \, dy.
\]

(2.30)

Writing \( K(z) = \frac{1}{\pi \omega_{n-1} |z|^{n+1}} \), for almost every \( x \in \mathbb{R}^n \setminus Q^*_k \) we deduce that
\[
R_0 b_k(x) = \int_{Q_k} K(x-y) b(y) \, dy = \int_{Q_k} (K(x-y) - K(x-y^{(k)})) b(y) \, dy.
\]

(2.31)

so for almost every \( x \in \mathbb{R}^n \setminus \Omega^* \) we have
\[
R_0 b(x) = \sum_k \int_{Q_k} (K(x-y) - K(x-y^{(k)})) b(y) \, dy.
\]

(2.32)
\[
\int_{\mathbb{R}^n \setminus \Omega^*} |\mathcal{R}_0 b(x)| \, dx \leq \sum_k \int_{\mathbb{R}^n \setminus Q_k} \left( \int_{Q_k} \left| K(x-y) - K(x-y^{(k)}) \right| |b(y)| \, dy \right) \, dx \\
\leq \sum_k \int_{Q_k} \left( \int_{|x-y^{(k)}| \geq 2|y-y^{(k)}|} \left| K(x-y) - K(x-y^{(k)}) \right| \, dx \right) |b(y)| \, dy \\
= \sum_k \int_{Q_k} \left( \int_{|\tilde{x}| \geq 2|\tilde{y}|} |K(\tilde{x} - \tilde{y}) - K(\tilde{x})| \, d\tilde{x} \right) |b(y)| \, dy \\
\leq \sum_k \int_{Q_k} \left( \int_{|\tilde{x}| \geq 2|\tilde{y}|} |\tilde{y}| \sup_{|z| \geq |\tilde{x}|/2} |\nabla K(z)| \, d\tilde{x} \right) |b(y)| \, dy \\
\lesssim n \sum_k \int_{Q_k} \left( \int_{|\tilde{x}| \geq 2|\tilde{y}|} |\tilde{y}| |\tilde{x}|^{-n+1} \, d\tilde{x} \right) |b(y)| \, dy \\
\lesssim n \sum_k \int_{Q_k} |b(y)| \, dy \\
\lesssim_n \|f\|_{L^1(\mathbb{R}^n)},
\]
where we use (2.28) in the second line, the mean value theorem in the fourth line, and (2.29) with \( p = 1 \) in the last line. By Chebychev’s inequality, we conclude that
\[
\{ x \in \mathbb{R}^n \setminus \Omega^* : |\mathcal{R}_0 b(x)| > \alpha/2 \} \leq \frac{2}{\alpha} \int_{\mathbb{R}^n \setminus \Omega^*} |\mathcal{R}_0 b(x)| \, dx \lesssim_n \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\alpha}.
\tag{2.34}
\]

2.3. The heat kernel and the mild formulation. The heat kernel \( \Phi \) is defined by
\[
\Phi(t, x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi - t|\xi|^2} \, d\xi \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}^n.
\tag{2.35}
\]
One easily verifies that \( \Phi \) is a smooth solution to the heat equation \( \Phi_t = \Delta \Phi \) on \( (0, \infty) \times \mathbb{R}^n \). The heat kernel also satisfies \( \|\nabla^\alpha \Phi(t)\|_{L^p(\mathbb{R}^n)} \lesssim_{\alpha, n} t^{-\frac{\alpha}{2} + (\alpha + 1)} \) and \( \nabla^\alpha \Phi(s + t) = \Phi(s) \ast \nabla^\alpha \Phi(t) \) for every \( \alpha \in \mathbb{Z}_{\geq 0}^n, p \in (1, \infty) \) and \( s, t \in (0, \infty) \) (these well-known results are discussed in [2]).

If \( p \in (1, \infty) \) and \( f \in L^{p, \infty}(\mathbb{R}^n) \), then \( e^{\Delta t} f(x) := \int_{\mathbb{R}^n} \Phi(t, x-y) f(y) \, dy \) defines a smooth function on \( (0, \infty) \times \mathbb{R}^n \), satisfying \( \|\nabla^\alpha e^{\Delta t} f\|_{L^p(\mathbb{R}^n)} \lesssim_{\alpha, n} t^{-\frac{\alpha}{2} + (\alpha + 1)} \|f\|_{L^{p, \infty}(\mathbb{R}^n)} \) and \( \|\nabla^\alpha e^{\Delta t} f\|_{L^p(\mathbb{R}^n)} \lesssim_{\alpha, n} t^{-\alpha/2} \|f\|_{L^{p, \infty}(\mathbb{R}^n)} \). The identities \( \int_{\mathbb{R}^n} \Phi(t, x) \, dx = 1 \) and \( \Phi(t, x) = t^{-\frac{n}{2}} \Phi(1, \frac{x}{\sqrt{t}}) \) will allow us to approximate results (property (viii) of Lorentz spaces) in the limit \( t \to 0 \).

If \( T \in (0, \infty) \), \( p \in (1, \infty) \), and \( G : (0, T) \to L^{p, \infty}(\mathbb{R}^n) \) is a weak* measurable function, then \( e^{\Delta \cdot} G(s, x) := [e^{\Delta} G(s)](x) \) satisfies \( \|\nabla^\alpha e^{\Delta} G(s, x)\|_{L^p(\mathbb{R}^n)} \lesssim_{\alpha, n} t^{-\alpha/2} \|G(s)\|_{L^{p, \infty}(\mathbb{R}^n)} \), where the expression \( \Phi_\alpha(t, x) := \nabla^\alpha \Phi(t, x-y) \) defines a weak* measurable function \( \Phi_\alpha : (0, \infty) \times \mathbb{R}^n \to L^{p, \infty}(\mathbb{R}^n) \). Approximating \( \Phi_\alpha \) using Pettis’ theorem, it follows that the function \( (s, t, x) \mapsto \nabla^\alpha e^{\Delta} G(s, x) \) is measurable. In particular, if \( T \in (0, \infty) \), \( p \in (1, \infty) \), \( \alpha \in \mathbb{Z}_{\geq 0}^n \), \( |\alpha| \leq 1 \), and \( G \in L^{1, \infty}_p(T_-) \), then the expression \( \int_0^T \nabla^\alpha e^{-(t-s)\Delta} G(s, x) \, ds \) defines an element of \( L^1_{p, \infty}(T_-) \), satisfying
\[
\int_0^{T'} \left\| \int_0^t \nabla^\alpha e^{-(t-s)\Delta} G(s, \cdot) \, ds \right\|_{L^p(\mathbb{R}^n)} \, dt \leq \int_0^{T'} \left( \int_0^t \left\| \nabla^\alpha e^{-(t-s)\Delta} G(s) \right\|_{L^p(\mathbb{R}^n)} \, ds \right) \, dt \\
\lesssim_{\alpha, n, p} \int_0^{T'} \left( \int_0^t (t-s)^{-|\alpha|/2} \|G(s)\|_{L^{p, \infty}(\mathbb{R}^n)} \, ds \right) \, dt \tag{2.36} \\
= \int_0^{T'} \left( \int_s^T (t-s)^{-|\alpha|/2} \, dt \right) \|G(s)\|_{L^{p, \infty}(\mathbb{R}^n)} \, ds
\]
for every \( T' \in (0, T) \), where we use Minkowski’s inequality (property (iv) of Lorentz space) in the first line, the inequality \( \|\nabla^\alpha e^{\Delta t} f\|_{L^p(\mathbb{R}^n)} \lesssim_{\alpha, n} t^{-|\alpha|/2} \|f\|_{L^{p, \infty}(\mathbb{R}^n)} \) in the second, and Fubini’s theorem in the third (having established measurability of \( \|G(s)\|_{L^{p, \infty}(\mathbb{R}^n)} \) in Lemma 2.2).
We are now in a position to define the \textit{mild formulation} of the linearised Navier-Stokes equations. For \( T \in (0, \infty) \), \( p_0, p \in (1, \infty) \), \( f \in L^{p_0, \infty}(\mathbb{R}^n) \) satisfying \((f_i, \nabla_i \phi)\) for all \( \phi \in C_c^\infty(\mathbb{R}^n) \), and \( F \in L^{1, p}_{\mu, \infty}(T_-) \), the expression
\[
v_i(t, x) := e^{t \Delta} f_i(x) - \int_0^t \nabla_k e^{(t-s) \Delta} \mathbb{P}_{ij} F_{jk}(s, x) \, ds \quad \text{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^n
\]
defines an element \( v \in L^{p_0, \infty}(T) + L^1_{\mu, \infty}(T_-) \), which we refer to as the \textit{mild solution}.

\textbf{Theorem 2.7.} Let \( T \in (0, \infty) \), \( p_0, p \in (1, \infty) \), \( f \in L^{p_0, \infty}(\mathbb{R}^n) \) with \((f_i, \nabla_i \phi)\) for all \( \phi \in C_c^\infty(\mathbb{R}^n) \), and \( F \in L^{1, p}_{\mu, \infty}(T_-) \). Then \( u \in L^{1, p}_{\mu, \infty}(T_-) \) satisfies the projected formulation of the linearised Navier-Stokes equations if and only if \( [u(t)](x) = v(t, x) \) for almost every \((t, x) \in (0, T) \times \mathbb{R}^n \), where \( v \) is the mild solution.

\textbf{Proof.} For the purposes of our proof, we choose a fixed function \( \rho \in C_c^\infty(\mathbb{R}^n) \), which satisfies \( \rho(x) = 1 \) for \(|x| < 1\) and \( \rho(x) = 0 \) for \(|x| > 2\). We define \( \rho_R(x) := \rho(x/R) \) for \( R \in (0, \infty) \) and \( x \in \mathbb{R}^n \). We define \( G_{ik} := \mathbb{P}_{ij} F_{jk} \), \( v^0(t) := e^{t \Delta} f_i \), and \( v^1 := v - v^0 \).

We start by verifying that
\[
\int_0^T \langle v_i(t), \nabla_i \phi(t) \rangle \, dt = 0 \quad \forall \phi \in C_c^\infty((0, T) \times \mathbb{R}^n). \tag{2.38}
\]

By Fubini’s theorem, integration by parts, and dominated convergence, we have
\[
\langle e^{t \Delta} f_i, \nabla_i \phi \rangle = \langle f_i, \nabla_i e^{t \Delta} \phi \rangle = \lim_{R \to \infty} \langle f_i, \nabla_i \left[ \rho_R e^{t \Delta} \phi \right] \rangle = 0 \tag{2.39}
\]
for all \( \phi \in C_c^\infty(\mathbb{R}^n) \) and \( t \in (0, \infty) \), while
\[
\int_{\mathbb{R}^n} \left( \int_0^t \nabla_k e^{(t-s) \Delta} G_{ik}(s, x) \, ds \right) \nabla_i \phi(x) \, dx = - \lim_{R \to \infty} \int_0^t \langle G_{ik}(s), \nabla_i \left[ \rho_R \nabla_k e^{(t-s) \Delta} \phi \right] \rangle \, ds = 0 \tag{2.40}
\]
for all \( \phi \in C_c^\infty(\mathbb{R}^n) \) for almost every \( t \in (0, T) \), where we used the fact that \( \mathbb{P}_{ij} \nabla_i \theta = 0 \) for all \( \theta \in C_c^\infty(\mathbb{R}^n) \).

Therefore (2.38) is verified.

Next we verify that
\[
\langle f_i, \phi_0(0) \rangle + \int_0^T \langle v_i(t), \phi_i^0(t) + \Delta \phi_0(t) \rangle + \langle G_{ik}(t), \nabla_k \phi_0(t) \rangle \, dt = 0 \quad \forall \phi \in C_c^\infty([0, T) \times \mathbb{R}^n). \tag{2.41}
\]

By Fubini’s theorem, integration by parts, and the identity \( \Phi' = \Delta \Phi \), for \( \phi \in C_c^\infty([0, T) \times \mathbb{R}^n) \) and \((\epsilon, s, x) \in (0, \infty) \times [0, T) \times \mathbb{R}^n \) we have
\[
\int_s^T e^{(\epsilon + t - s) \Delta} \phi(t, x) \, dt = \int_s^T \int_{\mathbb{R}^n} \Phi(\epsilon + t - s, x - y) \phi(t, y) \, dy \, dt
\]
\[
= - \int_{\mathbb{R}^n} \Phi(\epsilon, x - y) \phi(s, y) \, dy - \int_s^T \int_{\mathbb{R}^n} \Phi(\epsilon + t - s, x - y) \Delta \phi(t, y) \, dy \, dt
\]
\[
= - e^{t \Delta} \phi(s, x) - \int_s^T e^{(\epsilon + t - s) \Delta} \Delta \phi(t, x) \, dt. \tag{2.42}
\]

By Fubini’s theorem and the identity \( \Phi(\epsilon + t - s) = \Phi(\epsilon) \ast \Phi(t - s) \), we may write \( e^{(\epsilon + t - s) \Delta} = e^{t \Delta} e^{(\epsilon - s) \Delta} \) and take the \( e^{t \Delta} \) outside the integrals to obtain
\[
e^{t \Delta} \phi(s, x) + e^{t \Delta} \int_s^T e^{(t - s) \Delta} \phi(t, x) \, dt + e^{t \Delta} \int_s^T e^{(t - s) \Delta} \Delta \phi(t, x) \, dt = 0 \tag{2.43}
\]
for \( \phi \in C_c^\infty([0, T) \times \mathbb{R}^n) \) and \((\epsilon, s, x) \in (0, \infty) \times [0, T) \times \mathbb{R}^n \). By approximation of identity in \( C^0_{b, u}(\mathbb{R}^n) \), we obtain the equality
\[
\phi(s, x) + \int_s^T e^{(t - s) \Delta} \phi(t, x) \, dt + \int_s^T e^{(t - s) \Delta} \Delta \phi(t, x) \, dt = 0 \tag{2.44}
\]
for all \( \phi \in C^\infty_c([0, T) \times \mathbb{R}^n) \) and \((s, x) \in [0, T) \times \mathbb{R}^n\). This last equality allows us to compute

\[
\int_0^T \langle v_i^0(t), \phi'_i(t) + \Delta \phi_i(t) \rangle \, dt = \int_0^T (e^{t \Delta} f_i, \phi'_i(t) + \Delta \phi_i(t)) \, dt
\]

\[
= \int_0^T (f_i, e^{t \Delta} \phi'_i(t) + e^{t \Delta} \Delta \phi_i(t)) \, dt
\]

\[
= -(f_i, \phi_i(0))
\]

and

\[
\int_0^T \langle v_i(t), \phi'_i(t) + \Delta \phi_i(t) \rangle \, dt = -\int_0^T \left( \int_0^t \langle \nabla_k e^{(t-s) \Delta} G_{ik}(s), \phi'_i(t) + \Delta \phi_i(t) \rangle \, ds \right) \, dt
\]

\[
= \int_0^T \left( \int_s^T \langle G_{ik}(s), \nabla_k e^{(t-s) \Delta} \phi'_i(t) + \nabla_k e^{(t-s) \Delta} \Delta \phi_i(t) \rangle \, ds \right) \, dt
\]

\[
= -(\int_0^T \langle G_{ik}(s), \nabla_k \phi_i(s) \rangle \, ds
\]

for \( \phi \in C^\infty_c([0, T) \times \mathbb{R}^n) \). Therefore (2.41) is verified.

Having established (2.38) and (2.41), we have that \( u \in \mathcal{L}_{t_R}^{1, \infty}(T_-) \) satisfies the projected formulation of the linearised Navier-Stokes equations if and only if

\[
\int_0^T \langle u(t), \phi'_i(t) + \Delta \phi_i(t) \rangle \, dt = \int_0^T \langle v(t), \phi'_i(t) + \Delta \phi_i(t) \rangle \, dt \quad \forall \phi \in C^\infty_c([0, T) \times \mathbb{R}^n).
\]

We claim that (2.47) occurs if and only if \([u(t)](x) = v(t, x)\) for almost every \((t, x) \in (0, T) \times \mathbb{R}^n\). One direction is obvious, so we prove the other direction. Assume that (2.47) holds, and let \( \psi \in C^\infty_c((0, T) \times \mathbb{R}^n) \) be arbitrary. For \( R, \epsilon \in (0, \infty) \), we consider the test function \( \phi_{R, \epsilon}(x) \in C^\infty_c([0, T) \times \mathbb{R}^n) \) given by

\[
\phi_{R, \epsilon}(t, x) := -\rho_R(x) \int_t^T e^{(s-t+\epsilon) \Delta} \psi(s, x) \, ds,
\]

which satisfies

\[
\phi_{R, \epsilon}(t, x) = \rho_m(x) e^{\epsilon \Delta} \psi(t, x) + \rho_R(x) \int_t^T \Delta e^{(s-t+\epsilon) \Delta} \psi(s, x) \, ds,
\]

\[
\Delta \phi_{R, \epsilon}(t, x) = -\rho_R(x) \int_t^T \Delta e^{(s-t+\epsilon) \Delta} \psi(s, x) \, ds
\]

\[
- 2 \nabla \rho_R(x) \cdot \int_t^T \nabla e^{(s-t+\epsilon) \Delta} \psi(s, x) \, ds
\]

\[
- \Delta \rho_R(x) \int_t^T e^{(s-t+\epsilon) \Delta} \psi(s, x) \, ds,
\]

so for \( w = u - v \) we have

\[
0 = \int_0^T \langle w(t), \phi_{R, \epsilon}'(t) + \Delta \phi_{R, \epsilon}(t) \rangle \, dt
\]

\[
= \int_0^T \langle w(t), \rho_R e^{\epsilon \Delta} \psi(t) \rangle \, dt - 2 \int_0^T \int_t^T \langle w(t), \nabla \rho_R \cdot e^{(s-t+\epsilon) \Delta} \nabla \psi(s) \rangle \, ds \, dt
\]

\[
+ \int_0^T \int_t^T \langle w(t), \Delta \rho_R e^{(s-t+\epsilon) \Delta} \psi(s) \rangle \, ds \, dt.
\]

In the limit \( R \to \infty \) we obtain

\[
\int_0^T \langle w(t), e^{\epsilon \Delta} \psi(t) \rangle \, dt = 0,
\]

so in the limit \( \epsilon \to 0 \) we obtain

\[
\int_0^T \langle w(t), \psi(t) \rangle \, dt = 0.
\]

Therefore \( \int_0^T \langle u(t), \psi(t) \rangle \, dt = \int_0^T \langle v(t), \psi(t) \rangle \, dt \) for all \( \psi \in C^\infty_c((0, T) \times \mathbb{R}^n) \), so \([u(t)](x) = v(t, x)\) for almost every \((t, x) \in (0, T) \times \mathbb{R}^n\).
2.4. Approximation of solenoidal vector fields and the very weak formulation. For $\phi \in C^\infty_c (\mathbb{R}^n; \mathbb{R}^n)$, the expression $\psi_j = P_j \phi$ defines a vector field $\psi$ which is smooth and divergence-free, but not necessarily compactly supported. The following lemma allows us to approximate certain smooth, divergence-free vector fields by smooth, divergence-free, compactly supported vector fields.

**Lemma 2.8.** Let $p \in (1, \infty)$, $q \in [1, \infty)$ and $k \in \mathbb{Z}_{>0}$, and suppose that $\phi \in C^\infty (\mathbb{R}^n; \mathbb{R}^n)$ satisfies $\nabla \cdot \phi = 0$ and $\|\phi\|_{W^k_p (\mathbb{R}^n)} := \sum_{j=0}^k \|\nabla^j \phi\|_{L^p (\mathbb{R}^n)} < \infty$. Then there exist $\phi_R \in C^\infty_c (\mathbb{R}^n; \mathbb{R}^n)$ with $\nabla \cdot \phi_R = 0$ such that $\|\phi - \phi_R\|_{W^k_p (\mathbb{R}^n)} \to 0$ as $R \to \infty$. Moreover, the approximating functions $\phi_R$ can be chosen independently of $p, q, k$.

**Proof.** Let $\rho \in C^\infty (B(0, 2); \mathbb{R})$ be a fixed function satisfying $\rho(x) = 1$ for $|x| < 1$, and let $\rho_R(x) := \rho(x/R)$. Then $\rho_R \phi \in C^\infty_c (B(0, 2R); \mathbb{R}^n)$ approximates $\phi$ in $W^k_p (\mathbb{R}^n)$, since for $j \in \{0, \ldots, k\}$ we have

$$\|\nabla^j ((1 - \rho_R) \phi)\|_{L^{p,q}(\mathbb{R}^n)} \lesssim \sum_{i=0}^j \|\nabla^i (1 - \rho_R) \nabla^{j-i} \phi 1_{|x| > R}\|_{L^{p,q}(\mathbb{R}^n)}$$

(2.54)

We are not done yet, because $f_R := \nabla \cdot (\rho_R \phi) = \phi \cdot \nabla \rho_R$ need not be zero. We will resolve this issue by constructing $v_R \in C^\infty (\mathbb{R}^n; \mathbb{R}^n)$ satisfying $\nabla \cdot v_R = f_R$ and $\|v_R\|_{W^k_{p,q}(\mathbb{R}^n)} \to 0$ in $W^k_{p,q}(\mathbb{R}^n)$, so that $\phi_R = \rho_R \phi - v_R$ is as required. To this end, let $\omega \in C^\infty_c (B(0, 2); \mathbb{R})$ be a fixed function satisfying $\int_{\mathbb{R}^n} \omega(x) \, dx = 1$, and let $\omega_R(x) := R^{-n} \omega(x/R)$, so that

$$f_R(x) = \int_{\mathbb{R}^n} (\omega_R(x+y) f_R(x) - \omega_R(x) f_R(x-y)) \, dy$$

$$= - \int_{\mathbb{R}^n} \int_0^1 \frac{\partial}{\partial t} [\omega_R(x+y-t) y f_R(x-y)] \, dt \, dy$$

$$= \int_{\mathbb{R}^n} \int_0^1 y \frac{\partial}{\partial x} [\omega_R(x+y-t) f_R(x-y)] \, dt \, dy.$$ (2.55)

This last expression is the divergence of a smooth function if we can justify differentiation under the integral. For $\alpha \in \mathbb{Z}_{>0}^n$ with $|\alpha| = j$ we have

$$\int_{\mathbb{R}^n} \int_0^1 \left| y \frac{\partial^\alpha}{\partial x^\alpha} [\omega_R(x+y-t) f_R(x-y)] \right| \, dt \, dy$$

$$\lesssim \sum_{i=0}^j \int_{\mathbb{R}^n} \int_0^1 \left| y \nabla^i \omega_R(x+y-t) \nabla^{j-i} f_R(x-t) \right| \, dy \, dt$$

$$= \sum_{i=0}^j \int_{\mathbb{R}^n} \int_0^1 \left| z \nabla^i \omega_R(x+z-t) \nabla^{j-i} f_R(x-z) \right| t^{-n+1} \, dz \, dt$$

$$= \sum_{i=0}^j \int_0^\infty \int_{\mathbb{R}^n} \left| z \nabla^i \omega_R(x+z) \nabla^{j-i} f_R(x-z) \right| (1 + r)^{n-1} \, dz \, dr$$

$$= 1_{|x| < 2R} \sum_{i=0}^j \int_{|z| < 4R} \int_0^{4R/|z|} \left| z \nabla^i \omega_R(x+z) \nabla^{j-i} f_R(x-z) \right| (1 + r)^{n-1} \, dr \, dz$$

(2.56)

where

$$\int_{|z| < 4R} R^{-i} |z|^{-n} \, dz \lesssim R^{-i-1}.$$ (2.57)
We deduce from these estimates that
\[
v_R(x) := \int_{\mathbb{R}^n} \int_0^1 y \omega_R(x + y - ty) f_R(x - ty) \, dt \, dy
\]
defines \(v_R \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)\) satisfying \(\nabla \cdot v_R = f_R\) and
\[
|\nabla^j v_R(x)| \lesssim 1_{|x| < 2R} \sum_{i=0}^j \int |z| \lesssim |\nabla^{j-i} f_R(x - z)| \, dz.
\]
From the convolution inequality \(||K * f||_{L^p;\infty(\mathbb{R}^n)} \leq ||K||_{L^1(\mathbb{R}^n)} ||f||_{L^{p,q}(\mathbb{R}^n)}\), it follows for \(j \in \{0, \ldots, k\}\) that
\[
||\nabla^j v_R||_{L^{p,q}(\mathbb{R}^n)} \lesssim \sum_{i=0}^j R^{1-i} ||\nabla^{j-i} f_R||_{L^{p,q}(\mathbb{R}^n)}
\]
\[
\lesssim \sum_{i=0}^j R^{1-i} ||\nabla^{j-i} f_R||_{L^{p,q}(\mathbb{R}^n)}
\]
\[
\lesssim \sum_{i=0}^j R^{1-i} ||\nabla^{j-i} f_R||_{L^{p,q}(\mathbb{R}^n)}
\]
\[
\lesssim \sum_{m=0}^j R^{m} ||\nabla^{j-m} f_R||_{L^{p,q}(\mathbb{R}^n)} R^{-\infty} \to 0.
\]

For \(T \in (0, \infty), p_0, p, \tilde{p} \in (1, \infty), f \in L^{p_0, \infty}(\mathbb{R}^n)\) satisfying \(\langle f, \nabla \phi \rangle = 0\) for all \(\phi \in C_c^\infty(\mathbb{R}^n)\), and \(F \in L^{p_0, \infty}(T_-)\), we say that \(u \in L^{p_0, \infty}(T_-)\) satisfies the very weak formulation of the linearised Navier-Stokes equations if \(\int_0^T \langle u_j(t), \nabla_j \psi(t) \rangle \, dt\) for all \(\psi \in C_c^\infty((0, T) \times \mathbb{R}^n)\), and
\[
\langle f, \theta(0) \phi_j \rangle + \int_0^T \langle \nabla(t), \theta(t) \phi_j \rangle \, dt = 0
\]
for all \(\theta \in C_c^\infty([0, T])\) and \(\phi \in C_c^\infty(\mathbb{R}^n)\), where the subscript \(\sigma\) means that \(\nabla \phi = 0\).

**Theorem 2.9.** Let \(T \in (0, \infty), p_0, p, \tilde{p} \in (1, \infty), f \in L^{p_0, \infty}(\mathbb{R}^n)\) with \(\langle f, \nabla \phi \rangle = 0\) for all \(\phi \in C_c^\infty(\mathbb{R}^n)\), and \(F \in L^{p_0, \infty}(T_-)\). Then \(u \in L^{p_0, \infty}(T_-)\) satisfies the projected formulation of the linearised Navier-Stokes equations if and only if \(u\) satisfies the very weak formulation.

**Proof.** Clearly the projected formulation implies the very weak formulation, so we prove the other direction. Assume that \(u \in L^{p_0, \infty}(T_-)\) satisfies \(\int_0^T \langle u_j(t), \nabla_j \psi(t) \rangle \, dt\) for all \(\psi \in C_c^\infty((0, T) \times \mathbb{R}^n)\), and
\[
\langle f, \theta(0) \phi_j \rangle + \int_0^T \langle \nabla(t), \theta(t) \phi_j \rangle \, dt = 0
\]
for all \(\theta \in C_c^\infty([0, T])\) and \(\phi \in C_c^\infty(\mathbb{R}^n)\). We are then required to show that
\[
\langle f, \phi_j(0) \rangle + \int_0^T \langle \nabla(t), \phi_j(t) + \Delta \phi_j \rangle \, dt = 0
\]
for all \(\phi \in C_c^\infty((0, T) \times \mathbb{R}^n)\). Let \(\phi \in C_c^\infty([0, T) \times \mathbb{R}^n)\), and consider the smooth, divergence-free vector field \(\psi\) given by \(\psi_i = \mathbb{P}_{ij} \phi_j\). Since \(u\) and \(f\) are weakly divergence-free, we are required to show that
\[
\langle f, \psi(t) \rangle \in X := \|g\|_{L^{p_0,1}(\mathbb{R}^n)} \vee \|g\|_{W^{2,1}_p(\mathbb{R}^n)} \vee \|\nabla g\|_{L^{p_0,1}(\mathbb{R}^n)} (\text{where the norm } \| \cdot \|_{W^{2,1}_p(\mathbb{R}^n)} \text{ was defined in the previous lemma}), \text{ and let } \epsilon \in (0, \infty) \text{ be arbitrary. The map } \psi^\epsilon : [0, T) \to X \text{ is continuous}.
\]
and compactly supported, so there exists a compactly supported simple function \( \alpha : [0, T) \to X \), taking values in the range of \( \psi'' \), such that \( \| \psi''(t) - \alpha(t) \|_X < \epsilon \) for all \( t \in [0, T) \). By Lemma 2.8 we have a compactly supported simple function \( \beta : [0, T) \to C_{\text{c}}^\infty(\mathbb{R}^n) \) such that \( \| \alpha(t) \otimes \beta(t) \|_X < \epsilon \) for all \( t \in [0, T) \).

We then define \( \gamma(t) := -\int_t^T \beta(s, x) \, ds \) and \( \delta(t) := -\int_t^T \gamma(s, x) \, ds \), which satisfy

\[
\| \psi'(t) - \gamma(t) \|_X \leq \int_t^{T'} \| \psi''(s) - \beta(s) \|_X \, ds < 2\epsilon(T' - t),
\]

\[
\| \psi(t) - \delta(t) \|_X \leq \int_t^{T'} \| \psi'(s) - \gamma(s) \|_X \, ds < \epsilon(T' - t)^2,
\]

where \( T' \in (0, T) \) such that \( \psi, \alpha, \beta \) are compactly supported within \([0, T')\). Now \( \delta \) is a finite sum of functions of the form \( \theta(t)\chi(x) \), where \( \theta \in C_c^1([0, T]) \) and \( \chi \in C_{\text{c}}^\infty(\mathbb{R}^n) \). Approximating each of the functions \( \theta \) by a smooth function, and using the assumption that \( u \) satisfies the very weak formulation, we obtain

\[
\langle f_i, \delta_i(0) \rangle + \int_0^{T'} \langle u_i, \gamma_i + \Delta \delta_i \rangle \, dt = 0,
\]

where we note that \( \delta' = \gamma \). Using estimates (2.65) and (2.66), we deduce that the two sides of (2.64) differ by \( O(\epsilon) \), which shrinks to zero as \( \epsilon \to 0 \).

### 3. Energy estimates and weak-strong uniqueness

**3.1. The energy class.** The Sobolev space \( H^1(\mathbb{R}^n) \) consists of those \( u \in L^2(\mathbb{R}^n) \) for which there exists \( v \in L^2(\mathbb{R}^n) \) satisfying \( \langle u, \nabla \phi \rangle = -\langle v, \phi \rangle \) for all \( \phi \in C_0^\infty(\mathbb{R}^n) \). For \( u \in H^1(\mathbb{R}^n) \), we write \( \nabla u \) to denote the unique \( v \in L^2(\mathbb{R}^n) \) appearing in the definition. By considering \((u \ast \eta_k)\rho_R\), where \( \rho_R \) is a smooth cutoff function and \( \eta_k \) is a smooth approximate identity, we see that \( C_c^\infty(\mathbb{R}^n) \) is dense in \( H^1(\mathbb{R}^n) \), where \( \|u\|_{H^1(\mathbb{R}^n)} := \|(u, \nabla u)\|_{L^2(\mathbb{R}^n)} \). From (3.1) we have

\[
\|u\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{H^1(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)} \leq \frac{3}{2} \|u\|_{H^1(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)},
\]

from which we deduce the Sobolev inequality

\[
\|u\|_{L^\frac{2n}{n-2}(\mathbb{R}^n)} \leq \|u\|_{H^1(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)} \leq \frac{\sqrt{2}n}{n-2} \|u\|_{H^1(\mathbb{R}^n)} \|\nabla u\|_{L^2(\mathbb{R}^n)},
\]

(3.1)

(3.2)

For \( n \geq 2 \) and \( T \in (0, \infty) \), the energy class \( \mathcal{H}_T \) consists of those (equivalence classes of) weakly* measurable functions \( u : (0, T) \to H^1(\mathbb{R}^n) \subseteq L^2_{\text{loc}}(\mathbb{R}^n) \) satisfying \( u \in L^2_{2\times}\mathscr{X}(T_-) \) and \( \nabla u \in L^2_{2\times}(T_-) \), where we observe that weak* measurability of \( u \) implies weak* measurability of \( \nabla u \). By virtue of the identification \( L^2_{2\times}\mathcal{X}(T') \equiv \{L^2((0, T') \times \mathbb{R}^n)\} \cong L^2((0, T') \times \mathbb{R}^n) \), we may identify \( u \) and \( \nabla u \) with measurable functions \( u \in L^2_2(T_-) \) and \( \nabla u \in L^2_2(T_-) \). The Sobolev inequality implies that \( \mathcal{H}_T \subseteq \bigcap_{n \in (0, \infty]} L^2_{2p/n}(T_-) \), so we can apply our results from the previous section on the various formulations of the Navier-Stokes equations.

The following result concerns continuity at the initial time for solutions in the energy class.

**Theorem 3.1.** If \( T \in (0, \infty) \), \( f \in L^2(\mathbb{R}^n) \) satisfies \( \langle f, \nabla \phi \rangle \) for all \( \phi \in C_c^\infty(\mathbb{R}^n) \), and \( u \in \mathcal{H}_T \) satisfies the mild formulation of the Navier-Stokes equations, then \( u \) satisfies the continuity condition

\[
\left\{ \begin{array}{l}
\text{there exists a subset } \Omega \subseteq (0, T) \text{ of total measure such that } \\
\langle u(t), \phi \rangle \to \langle f, \phi \rangle \text{ for all } \phi \in L^2(\mathbb{R}^n) \text{ as } t \to 0 \text{ along } \Omega.
\end{array} \right.
\]

(3.3)

**Proof.** The mild solution \( u \) satisfies

\[
u_i(t, x) = e^{t\Delta}f_i(x) - \int_0^t e^{(t-s)\Delta}P_{ij}[u_k(s)\nabla_k u_j(s)](x) \, ds \quad \text{for a.e. } (t, x) \in (0, T) \times \mathbb{R}^n,
\]

(3.4)
where the transfer of the derivative onto $u$ is justified using integration by parts, approximating $u(s)$ by smooth compactly supported functions, and using the fact that $\langle u_i(s), \nabla_i \phi \rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$ for almost every $s \in (0, T)$. Now $\|e^{t\Delta} f - f\|_{L^2(\mathbb{R}^n)} \rightarrow 0$ by approximation of identity, while by Minkowski’s inequality (property (iv) of Lorentz spaces), properties of the heat kernel and Riesz transform, and the Sobolev inequality, for $p \in (n, \infty)$ we have

$$\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}_{ij} [u_k(s) \nabla_k u_j(s)] (\cdot) \, ds \right\|_{L^2(\mathbb{R}^n)} \lesssim_{n,p} \int_0^t \|u(s)\|_{L^2(\mathbb{R}^n)} \|\nabla u(s)\|_{L^2(\mathbb{R}^n)} \, ds \xrightarrow{t \to 0} 0. \quad (3.5)$$

Therefore $\lim_{t \to 0} u(t, \phi) = \langle f, \phi \rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, where $t \in \Omega$ iff \[3.4\] holds for almost every $x \in \mathbb{R}^n$. We conclude by density of $C_c^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, and by the fact that $\|u(t)\|_{L^2(\mathbb{R}^n)}$ is essentially bounded near $t = 0$.

3.2. Energy estimates. For $p \in (n, \infty)$ and $T \in (0, \infty]$ we define $\mathcal{V}_T^p := \mathcal{H}_T \cap \mathcal{L}^{2p/(p-n)}_p(T_\cdot)$. In the particular case $n = 2$, we have $\mathcal{H}_T \subseteq \cap_{p \in (2, \infty)} \mathcal{V}_T^p$ by the Sobolev inequality.

**Theorem 3.2.** Let $T \in (0, \infty)$, and assume that $f \in L^2(\mathbb{R}^n)$ satisfies $\langle f_i, \nabla_i \phi \rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$. Assume that $p \in (n, \infty)$, and that $u \in \mathcal{H}_T \cap \mathcal{V}_T^p$ and $v \in \mathcal{H}_T$ satisfy the projected formulation of the Navier-Stokes equations with initial data $f$, with $u$ and $v$ both satisfying the continuity condition \[3.3\]. Then for almost every $t \in (0, T)$ we have

$$\langle u_i(t), v_i(t) \rangle = \|f\|^2_{L^2(\mathbb{R}^n)} - 2 \int_0^t \langle \nabla_j u_i(s), \nabla_j v_i(s) \rangle \, ds - \int_0^t \langle ((u \cdot \nabla) u)_i(s), v_i(s) \rangle \, ds - \int_0^t \langle u_i(s), ((v \cdot \nabla) v)_i(s) \rangle \, ds. \quad (3.6)$$

In particular, $u$ satisfies the energy equality

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \|u(t)\|^2_{L^2(\mathbb{R}^n)} + 2 \int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^n)} \, ds \text{ for a.e. } t \in (0, T). \quad (3.7)$$

**Proof.** By the Sobolev inequality, and by the assumptions $u \in \mathcal{V}_T^p$ and $v \in \mathcal{H}_T$, we have

$$u \in \mathcal{L}^\infty_T(T_\cdot) \cap \mathcal{L}^{2p/(p-n)}_p(T_\cdot), \quad \nabla u \in \mathcal{L}^2(T_\cdot), \quad (u \cdot \nabla) u \in \mathcal{L}^{2p/(2p-n)}_p(T_\cdot),$$

$$v \in \mathcal{L}^2_T(T_\cdot) \cap \mathcal{L}^{2p/n}_2(T_\cdot), \quad \nabla v \in \mathcal{L}^2(T_\cdot), \quad (v \cdot \nabla) v \in \mathcal{L}^{2p/(p+n)}_2(T_\cdot), \quad (3.8)$$

so \[3.6\] makes sense. To prove \[3.6\], let $\eta \in C_c^\infty((-1, 1))$ with $\int_{-1}^1 \eta(t) \, dt = 1$, let $\eta_\epsilon(t) = \epsilon^{-1} \eta(t/\epsilon)$, and define the regularised functions

$$u^\epsilon(t, x) := \int_0^T \eta_\epsilon(t-s) u(s, x) \, ds, \quad v^\epsilon(t, x) := \int_0^T \eta_\epsilon(t-s) v(s, x) \, ds \quad (3.9)$$

for all $t \in (-\infty, T-\epsilon)$ and almost every $x \in \mathbb{R}^n$ (by Minkowski’s inequality, the functions $u^\epsilon$ and $v^\epsilon$ are well-defined, with $u^\epsilon(t) \in H^1(\mathbb{R}^n) \cap L^{p/(p-1)}(\mathbb{R}^n)$ and $v^\epsilon(t) \in H^1(\mathbb{R}^n)$). Differentiating under the integral, for all $t \in (-\infty, T-\epsilon)$ we have

$$\frac{\partial}{\partial \epsilon} \langle u^\epsilon(t), v^\epsilon(t) \rangle = \int_0^T \eta_\epsilon'(t-s) \langle u(s), v^\epsilon(t) \rangle \, ds + \int_0^T \eta_\epsilon'(t-s) \langle u^\epsilon(t), v(s) \rangle \, ds. \quad (3.10)$$

Since $u$ satisfies the projected formulation of the Navier-Stokes equations, for $\theta \in C_c^\infty([0, T))$ and $\psi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\theta(0) \langle f_i, \psi_i \rangle + \int_0^T \langle (u_i(s), \theta'(s) \psi_i) - \langle \nabla_j u_i(s), \theta(s) \nabla_j \psi_i \rangle - \langle (u \cdot \nabla) u_i(s), \theta(s) \mathbb{P}_{ij} \psi_i \rangle \rangle \, ds = 0. \quad (3.11)$$

For $t \in (-\infty, T-\epsilon)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$, if we take $\theta(s) = \eta_\epsilon(t-s)$ in this last equality then we obtain

$$\int_0^T \eta_\epsilon'(t-s) \langle u(s), \psi_i \rangle \, ds = \eta_\epsilon(t) \langle f_i, \psi_i \rangle + \langle \nabla_j u_i(t), \nabla_j \psi_i \rangle - \langle (u \cdot \nabla) u_i(t), \mathbb{P}_{ij} \psi_i \rangle. \quad (3.12)$$
Analogously, for \( t \in (-\infty, T - \epsilon) \) and \( \psi \in C_c^\infty(\mathbb{R}^n) \) we have
\[
\int_0^T \eta_\epsilon'(t-s)(v_i(s), \psi_i) \, ds = \eta_\epsilon(t)(f_i, \psi_i) - \langle \nabla_j u_i'(t), \nabla_j \psi_i \rangle - \langle \|(v \cdot \nabla)v_i\|_s(t), \mathbb{P}_{ij} \psi_i \rangle.
\] (3.13)

Noting that \([(u \cdot \nabla)u_i]'(t) \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)\) and \([(v \cdot \nabla)v_i]'(t) \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)\) (with a minor modification in the case \( p = \infty \) to deal with the Leray projection), we see that (3.12) extends to all \( \psi \in H^1(\mathbb{R}^n) \subseteq L^{\frac{2n}{n+2}}(\mathbb{R}^n)\), while (3.13) extends to all \( \psi \in H^1(\mathbb{R}^n) \cap L^{p,\infty}(\mathbb{R}^n)\) (replace \( \psi \) by \((\phi \ast \psi)|_{\mathbb{R}^n}\), where \( \phi \) is an approximate identity and \( \phi \) is a smooth cutoff function, and take the limit \( R \to \infty \) then \( \delta \to 0 \)). By (3.10), we deduce that for \( t \in (-\infty, T - \epsilon) \) that
\[
\langle u_i'(t), v_i'(t) \rangle = \int_{-\epsilon}^t \eta_\epsilon(s) \langle f_i, u_i(s) + v_i'(s) \rangle \, ds - 2 \int_{-\epsilon}^t \langle \nabla_j u_i(s), \nabla_j v_i'(s) \rangle \, ds
\]
\[
- \int_{-\epsilon}^t \langle [(u \cdot \nabla)u_i]'(s), v_i'(s) \rangle \, ds - \int_{-\epsilon}^t \langle u_i'(s), [(v \cdot \nabla)v_i]'(s) \rangle \, ds.
\] (3.14)

By (3.3) and dominated convergence, for \( t \in (0, T - \epsilon) \) we have
\[
\int_{-\epsilon}^t \eta_\epsilon(s) \langle f_i, u_i(s) \rangle \, ds = \int_{-\epsilon}^1 \int_{-1}^\sigma \eta_\epsilon(\eta(\sigma)\eta(\rho) \langle f_i, u_i(\sigma - \rho) \rangle) \, d\rho \, d\sigma \to_0 \frac{1}{2} \| f \|_{L^2(\mathbb{R}^n)}^2,
\] (3.15)

so the first integral on the right hand side of (3.14) converges to \( \| f \|_{L^2(\mathbb{R}^n)}^2 \). By Lemma 2.1 (defining \( u(t) = v(t) = 0 \) for \( t < 0 \), and noting the regularity described in (3.3)), the last three integrals on the right hand side of (3.14) converge to the last three integrals on the right hand side of (3.6), while \( \langle u_i'(t), v_i'(t) \rangle \to \langle u(t), v(t) \rangle \) for almost every \( t \in (0, T) \) along some subsequence \( \epsilon_m \to 0 \). We conclude that (3.10) holds for almost every \( t \in (0, T) \).

### 3.3. Weak-strong uniqueness.

The estimates of Theorem 3.2 allow us to prove the following weak-strong uniqueness result.

**Theorem 3.3.** Let \( T \in (0, \infty] \), and assume that \( f \in L^2(\mathbb{R}^n) \) satisfies \( \langle f_i, \nabla_i \phi \rangle = 0 \) for all \( \phi \in C_c^\infty(\mathbb{R}^n) \). Assume that \( p \in (n, \infty) \), and that \( u \in \mathcal{H}_T \cap \mathcal{V}_p \) and \( v \in \mathcal{H}_T \) satisfy the projected formulation of the Navier-Stokes equations with initial data \( f \), with \( u \) and \( v \) both satisfying the continuity condition 3.3, and \( v \) satisfying the energy inequality
\[
\| f \|_{L^2(\mathbb{R}^n)}^2 \geq \| v(t) \|_{L^2(\mathbb{R}^n)}^2 + 2 \int_0^t \| \nabla v(s) \|_{L^2(\mathbb{R}^n)}^2 \, ds \quad \text{for a.e. } t \in (0, T).
\] (3.16)

Then \( u(t,x) = v(t,x) \) for almost every \((t,x) \in (0, T) \times \mathbb{R}^n \).

**Proof.** Let \( w = u - v \). By (3.6), (3.7) and (3.10), for almost every \( t \in (0, T) \) we have
\[
\| w(t) \|_{L^2(\mathbb{R}^n)}^2 + 2 \int_0^t \| \nabla w(s) \|_{L^2(\mathbb{R}^n)}^2 \, ds \leq 2 \int_0^t \langle [(u \cdot \nabla)u_i]'(s), v_i(s) \rangle \, ds + 2 \int_0^t \langle u_i(s), [(v \cdot \nabla)v_i]'(s) \rangle \, ds
\]
\[
= -2 \int_0^t \langle u_i(s), [(u \cdot \nabla)w_i]'(s) \rangle \, ds \leq n_p \int_0^t \| u(s) \|_{L^{p,\infty}(\mathbb{R}^n)} \| w(s) \|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \| \nabla w(s) \|_{L^2(\mathbb{R}^n)} \, ds
\]
\[
\leq n_p \int_0^t \| u(s) \|_{L^{p,\infty}(\mathbb{R}^n)} \| w(s) \|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \| \nabla w(s) \|_{L^2(\mathbb{R}^n)} \, ds
\]
\[
\leq n_p \frac{1}{2} \int_0^t \| u(s) \|_{L^{p,\infty}(\mathbb{R}^n)} \| w(s) \|_{L^2(\mathbb{R}^n)}^2 \, ds + \epsilon \int_0^t \| \nabla w(s) \|_{L^2(\mathbb{R}^n)}^2 \, ds.
\] (3.17)

Taking \( \epsilon > 0 \) sufficiently small, we deduce that
\[
\| w(t) \|_{L^2(\mathbb{R}^n)}^2 \leq n_p \int_0^t \| u(s) \|_{L^{p,\infty}(\mathbb{R}^n)} \| w(s) \|_{L^2(\mathbb{R}^n)}^2 \, ds \quad \text{for a.e. } t \in (0, T),
\] (3.18)
so by Grönwall’s inequality we have \( \| w(t) \|_{L^2(\mathbb{R}^n)} = 0 \) for almost every \( t \in (0, T) \).
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