A discrete form of the Beckman-Quarles theorem for two-dimensional strictly convex normed spaces

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Abstract

Let $X$ be a real normed vector space and $\dim X \geq 2$. Let $\rho > 0$ be a fixed real number. We prove that if $x, y \in X$ and $||x - y||/\rho$ is a rational number then there exists a finite set $\{x, y\} \subseteq S_{xy} \subseteq X$ with the following property: for each strictly convex $Y$ of dimension 2 each map from $S_{xy}$ to $Y$ preserving the distance $\rho$ preserves the distance between $x$ and $y$. It implies that each map from $X$ to $Y$ that preserves the distance $\rho$ is an isometry.

Let $\mathbb{Q}$ denote the field of rational numbers. All vector spaces mentioned in this article are assumed to be real. A normed vector space $E$ is called strictly convex ([5]), if for each pair $a, b$ of nonzero elements in $E$ such that $||a + b|| = ||a|| + ||b||$, it follows that $a = \gamma b$ for some $\gamma > 0$. It is known ([15]) that two-dimensional strictly convex normed spaces satisfy the following condition ($\ast$):

$(\ast)$ for any $a \neq b$ on line $L$ and any $c, d$ on the same side of $L$, if $||a - c|| = ||a - d||$ and $||b - c|| = ||b - d||$, then $c = d$.

Conversely ([15]), for any two-dimensional normed space the condition $(\ast)$ implies that the space is strictly convex.
The classical Beckman-Quarles theorem states that any map from $\mathbb{R}^n$ to $\mathbb{R}^n$ ($2 \leq n < \infty$) preserving unit distance is an isometry, see [1], [2] and [6]. Various unanswered questions and counterexamples concerning the Beckman-Quarles theorem and isometries are discussed by Ciesielski and Rassias [4]. For more open problems and new results on isometric mappings the reader is referred to [7]-[13]. The Theorem below may be viewed as a discrete form of the Beckman-Quarles theorem for two-dimensional strictly convex normed spaces.

**Theorem.** Let $X$ and $Y$ be normed vector spaces such that $\dim X \geq \dim Y = 2$ and $Y$ is strictly convex. Let $\rho > 0$ be a fixed real number.

1. If $x, y \in X$ and $||x - y||/\rho$ is a rational number then there exists a finite set $S_{xy} \subseteq X$ containing $x$ and $y$ such that each injective map $f : S_{xy} \to Y$ preserving the distance $\rho$ preserves the distance between $x$ and $y$.

2. If $x, y \in X$ and $\varepsilon > 0$ then there exists a finite set $T_{xy}(\varepsilon) \subseteq X$ containing $x$ and $y$ such that each injective map $f : T_{xy}(\varepsilon) \to Y$ preserving the distance $\rho$ preserves the distance between $x$ and $y$ to within $\varepsilon$ i.e.

$$||f(x) - f(y)|| - ||x - y|| \leq \varepsilon.$$ 

3. Let $X = \mathbb{R}^n$ ($2 \leq n < \infty$) be equipped with euclidean norm. Then the assumption of injectivity is unnecessary in items 1 and 2.

4. More generally (cf. item 3), for each normed space $X$ the assumption of injectivity is unnecessary in items 1 and 2.

**Proof of item 1.** Let $D$ denote the set of all non-negative numbers $d$ with the following property (**):

(**) if $x, y \in X$ and $||x - y|| = d$ then there exists a finite set $S_{xy} \subseteq X$ such that $x, y \in S_{xy}$ and any injective map $f : S_{xy} \to Y$ that preserves the distance $\rho$ also preserves the distance between $x$ and $y$. 

Obviously $0, \rho \in D$. We first prove that if $d \in D$, then $2 \cdot d \in D$. Assume that $d \in D$, $d > 0$, $x, y \in X$, $\|x - y\| = 2 \cdot d$. Using the notation of Figure 1

\[
\begin{align*}
\|x - y\| &= 2 \cdot d \\
 z &:= \frac{x + y}{2} \\
\|x - z\| &= \|x - y\| = \|z - y\| = d \\
x_1 := y_1 + (z - x)
\end{align*}
\]

we show that

\[
S_{xy} := S_{xz} \cup S_{zy} \cup S_{y_1x_1} \cup S_{x_1y_1} \cup S_{zx_1} \cup S_{zy_1} \cup S_{yx_1}
\]

satisfies the condition (**). Let an injective $f : S_{xy} \rightarrow Y$ preserves the distance $\rho$. By the injectivity of $f$: $f(x) \neq f(x_1)$ and $f(y) \neq f(y_1)$. According to (*): $f(y_1) - f(x_1) = f(x) - f(z)$ and $f(y_1) - f(x_1) = f(z) - f(y)$. Hence $f(x) - f(z) = f(z) - f(y)$. Therefore $\|f(x) - f(y)\| = 2 \cdot \|f(x) - f(z)\| = 2 \cdot \|x - z\| = 2 \cdot d = \|x - y\|$.

From Figure 2, the previous step and the property that defines strictly convex normed spaces it is clear that if $d \in D$, then all distances $k \cdot d$ ($k$ a positive integer) belong to $D$.

\[
\begin{align*}
\|x - y\| &= k \cdot d \\
S_{xy} &= \cup \{S_{ab} : a, b \in \{w_0, w_1, ..., w_k\}, \|a - b\| = d \lor \|a - b\| = 2 \cdot d\}
\end{align*}
\]

From Figure 3, the previous step and the property that defines strictly convex normed spaces it is clear that if $d \in D$, then all distances $d/k$ ($k$ a
positive integer) belong to \( D \). Hence \( D/\rho := \{d/\rho : d \in D \} \supseteq Q \cap [0, \infty) \).

This completes the proof of item 1.

\[ \tilde{y} \]
\[ (k - 1) \cdot d \]
\[ d \]
\[ y \]
\[ z \]

\[ \tilde{x} \]
\[ (k - 1) \cdot d \]
\[ d \]
\[ x \]

Figure 3

\[ ||x - y|| = d/k \]
\[ \tilde{x} := x + (k - 1)(x - z) \]
\[ \tilde{y} := y + (k - 1)(y - z) \]
\[ \tilde{x} - \tilde{y} = x - y + (k - 1)((x - z) - (y - z)) = k(x - y) \]
\[ S_{xy} = S_{\tilde{xy}} \cup S_{xz} \cup S_{xz} \cup S_{yx} \cup S_{yz} \cup S_{yz} \]

**Proof of item 2.** It follows from Figure 4.

\[ x \]
\[ y \]

Figure 4

\[ |x - z|/\rho, |z - y|/\rho \in Q \cap [0, \infty), |z - y| \leq \varepsilon/2 \]

\[ T_{xy}(\varepsilon) = S_{xz} \cup S_{zy} \]

**Proof of item 3.** In proofs of items 1 and 2 the assumption of injectivity is necessary only in the first step for distances \( 2 \cdot d, d \in D \). Let \( X = \mathbb{R}^n \) \((2 \leq n < \infty)\) be equipped with euclidean norm and \( D \) is defined without the assumption of injectivity. Let \( d \in D, d > 0 \). We need to prove that \( 2 \cdot d \in D \). Let us see at configuration from Figure 5 below, all segments have the length \( d \).
Assume that \( f : S_{xy} \rightarrow Y \) preserves the distance \( \rho \). It is sufficient to prove that \( f(x) \neq f(x_1) \) and similarly \( f(y) \neq f(y_1) \). Suppose, on the contrary, that \( f(x) = f(x_1) \), the proof of \( f(y) \neq f(y_1) \) is similar. Hence four points: \( f(\bar{x}), \ f(z_y), \ f(\bar{y}_1), \ f(x_1) \) have the distance \( d \) from each other. We prove that it is impossible in two-dimensional strictly convex normed spaces. Suppose, on the contrary, that \( a_1, a_2, a_3, a_4 \in Y \) and \( ||a_1 - a_2|| = ||a_1 - a_3|| = ||a_1 - a_4|| = ||a_2 - a_3|| = ||a_2 - a_4|| = ||a_3 - a_4|| = d > 0 \). Let us consider the segment
exists a unique $h$. Obviously the following function is continuous. We have:

For each $x, y \in X$, $x \neq y$ there exist points forming the configuration from Figure 5 where all segments have the length $||x - y||/2$. Let us consider $x, y \in X$, $x \neq y$. We choose two-dimensional subspace $\bar{X} \subseteq X$ containing $x$ and $y$.

**First case:** the norm induced on $\bar{X}$ is strictly convex. Obviously $\bar{X}$ is isomorphic to $\mathbb{R}^2$ as a linear space. Let us consider $\mathbb{R}^2$ with a strictly convex norm $|| \cdot ||$. It suffices to prove that for each $a, b \in \mathbb{R}^2$ satisfying $||a|| = ||b|| = ||a - b|| = d > 0$ there exist $\tilde{a}, \tilde{b} \in \mathbb{R}^2$ satisfying $||\tilde{a}|| = ||\tilde{b}|| = ||\tilde{a} - \tilde{b}|| = ||(\tilde{a} + \tilde{b}) - (a + b)|| = d$. We fix $a = (a_x, a_y)$ and $b = (b_x, b_y)$. Let $S := \{x \in \mathbb{R}^2 : ||x|| = d\}$. According to (*) for each $u = (u_x, u_y) \in S$ there exists a unique $h(u) = (h(u)_x, h(u)_y) \in S$ such that $||u - h(u)|| = d$ and

$$\det \begin{bmatrix} u_x & u_y \\ h(u)_x & h(u)_y \end{bmatrix} \cdot \det \begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix} > 0.$$ 

Obviously $h(a) = b$. The mapping $h : S \to S$ is continuous. For each $u \in S$ $h(-u) = -h(u)$ and $||u + h(u)|| = ||2u - (u - h(u))|| \geq ||2u|| - ||u - h(u)|| = d$. The following function

$$S \ni x \mapsto ||x + h(x) - a - h(a)|| \in [0, \infty)$$

is continuous. We have:

$$g(a) = 0,$$

$$g(-a) = ||-a + h(-a) - a - h(a)|| = 2 \cdot ||a + h(a)|| \geq 2 \cdot d.$$ 

Since $g$ is continuous there exists $\tilde{a} \in S$ such that $g(\tilde{a}) = d$. From this $\tilde{a}$ and $\tilde{b} := h(\tilde{a})$ satisfy $||\tilde{a}|| = ||\tilde{b}|| = ||\tilde{a} - \tilde{b}|| = ||(\tilde{a} + \tilde{b}) - (a + b)|| = d$. This
A completes the proof of item 4 in the case where the norm induced on \( \tilde{X} \) is strictly convex.

**Second case:** we assume only that \( \| \| \) is a norm on \( \tilde{X} \). The graph \( \Gamma \) from Figure 5 (11 vertices, 19 edges) has the following matrix representation:

|   | \( v_0 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) | \( v_7 \) | \( v_8 \) | \( v_9 \) | \( v_{10} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( v_0 := x \) | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| \( v_1 := y \) | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| \( v_2 := \frac{x+y}{2} \) | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| \( v_3 := \tilde{x} \) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| \( v_4 := x_1 \) | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| \( v_5 := \tilde{x}_1 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| \( v_6 := \tilde{y} \) | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| \( v_7 := y_1 \) | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| \( v_8 := y_1 \) | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( v_9 := z_x \) | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| \( v_{10} := z_y \) | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |

Let \( u_0 := v_0 = x, u_1 := v_1 = y, u_2 := v_2 = \frac{x+y}{2} \). We define the following function \( \psi \):

\[ \tilde{X}^8 \ni (u_3, ..., u_{10}) \xrightarrow{\psi} (\| u_i - u_j \| : 0 \leq i < j \leq 10, (v_i, v_j) \in \Gamma) \in \mathbb{R}^{19}. \]

The image of \( \psi \) is a closed subset of \( \mathbb{R}^{19} \). For each \( \epsilon > 0 \) and each bounded \( B \subseteq \tilde{X} \) the norm \( \| \| \) may be approximate on \( B \) with \( \epsilon \)-accuracy by a strictly convex norm on \( \tilde{X} \). Therefore according to the first case for each \( x, y \in X, x \neq y \) and each \( \epsilon > 0 \) there exist points forming the configuration from Figure 5 where all segments have \( \| \| \)-lengths belonging to the interval \( (\| x-y \| / 2 - \epsilon, \| x-y \| / 2 + \epsilon) \). Therefore:

\( (\| x-y \| / 2, ..., \| x-y \| / 2) \in \overline{\psi(\tilde{X}^8)} \) (the closure of \( \psi(\tilde{X}^8) \)).
Since $\psi(\tilde{X}^8)$ is closed we conclude that
\[
(\|x - y\|/2, ..., \|x - y\|/2) \in \psi(\tilde{X}^8).
\]
This completes the proof of item 4.

**Corollary.** Let $X$ and $Y$ be normed vector spaces such that $\dim X \geq \dim Y = 2$ and $Y$ is strictly convex. From item 2 of the Theorem follows that an injective map $f : X \to Y$ that preserves a fixed distance $\rho > 0$ is an isometry. According to item 4 of the Theorem the assumption of injectivity is unnecessary in the above statement.

**Remark 1.** The set $S_{xy}$ constructed in the proof does not depend on $Y$.

**Remark 2.** Instead of injectivity in the Theorem and Corollary we may assume that
\[
\forall u, v \in \text{dom}(f)(\|u - v\|/\rho \in \mathbb{Q} \cap (0, \infty) \Rightarrow \|f(u) - f(v)\| \neq \|u - v\|/2)
\]
It follows from Figure 1.

**Remark 3.** W. Benz and H. Berens proved ([3], see also [2] and [10]) the following theorem: Let $X$ and $Y$ be normed vector spaces such that $Y$ is strictly convex and such that the dimension of $X$ is at least 2. Let $\rho > 0$ be a fixed real number and let $N > 1$ be a fixed integer. Suppose that $f : X \to Y$ is a mapping satisfying:
\[
\|a - b\| = \rho \Rightarrow \|f(a) - f(b)\| \leq \rho
\]
\[
\|a - b\| = N\rho \Rightarrow \|f(a) - f(b)\| \geq N\rho
\]
for all $a, b \in X$. Then $f$ is an affine isometry.

**Remark 4.** A. Tyszka proved ([14]) the following theorem: if $x, y \in \mathbb{R}^n$ ($2 \leq n < \infty$) and $|x - y|$ is an algebraic number then there exists a finite set $S_{xy} \subseteq \mathbb{R}^n$ containing $x$ and $y$ such that each map from $S_{xy}$ to $\mathbb{R}^n$ preserving unit distance preserves the distance between $x$ and $y$. 
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