Correspondence between stochastic Wigner trajectories and individual experimental runs

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We examine the interpretation of individual phase-space trajectories of the Wigner function as corresponding to possible outcomes of single experimental trials. To this end, we investigate the relation between the true (measured) particle number distribution $P_n$ for a single-mode state and that obtained by discretely binning the individual stochastic realizations of squared mode amplitudes $|\alpha|^2$ of the sampled Wigner distribution $W(\alpha)$, which we denote via $P_\alpha$. We provide an operational definition of $P_n$ in terms of the underlying Wigner function, which allows us to explicitly calculate the overlap between the two number distributions and hence quantify the statistical distance between them. We find that there is indeed a close quantitative correspondence between $P_n$ and $P_\alpha$ for a wide range of states, justifying the broadly accepted view that, for highly occupied modes, individual stochastic realizations of Wigner trajectories should approximately correspond to outcomes of single experiments. However, we also find counterexamples for which high mode occupation may not be sufficient for such an interpretation; we find instead that a more relevant and sufficient requirement is the smoothness and broadness of the Wigner function $W(\alpha)$ for the state of interest relative to the scale of oscillations of the Wigner functions for the relevant Fock states.

I. INTRODUCTION

The Wigner function, or the Wigner quasi-probability distribution [1,5], has proven to be a versatile tool in understanding quantum mechanics. Firstly, by providing a complete representation of the quantum mechanical density operator in phase space, the Wigner function serves as the quantum moment-generating functional that allows the calculation of quantum mechanical expectation values of operators in the spirit of classical statistical physics. Secondly, the Wigner function has been extensively used in the so-called truncated Wigner approximation as a calculation technique for quantum dynamical simulations, most notably in the fields of quantum optics and ultracold atoms [6–15]. This latter utility follows from the possibility of converting the master equation for the quantum density operator into a generalised Fockker-Planck equation, which itself – for dissipationless systems and after truncation of third- and higher-order derivative terms (if any) [16] – acquires the form of a classical Liouville equation and can be cast as an equivalent set of (stochastic) $\omega$-number differential equations for the phase-space variables.

Despite the formal analogy of the evolution equation for the Wigner function to the Liouville equation for the classical probability distribution, the strict interpretation of the Wigner function as a true probability distribution fails as it attains negative values for certain quantum states. Furthermore, even when the Wigner function is strictly non-negative, its difference from a classical probability distribution stems from the fact that it is still constrained by the quantum mechanical uncertainty principle: it is a joint probability distribution for quantum mechanically incompatible observables and, therefore, cannot be regarded as having direct physical significance. In the truncated Wigner approximation, this constraint manifests itself through the fact that even though the $\omega$-number differential equations formally coincide with their classical deterministic counterparts, the quantum mechanical uncertainties are mimicked via random initial conditions that are sampled stochastically from the Wigner-function representation of the initial density matrix.

Given this understanding and constraining ourselves to problems involving a non-negative initial Wigner function – such that its non-negativity throughout the ensuing dynamics is either intrinsically preserved (such as for systems described by Hamiltonians that depend no-higher-than quadratically on creation or annihilation field operators) or enforced by the truncated Wigner approximation [16,17] – we address the question of whether and when the individual stochastic trajectories can be thought of as a faithful representation of the outcomes of individual experimental runs. We clarify our terminology here by noting that evolution of stochastic trajectories for a phase-space variable from some initial state (defined appropriately by a corresponding initial Wigner function) under a particular Hamiltonian, is completely equivalent to directly sampling this variable from the known Wigner function of the final state after said evolution. Even though this question has been discussed in the literature previously [8–9,15,18–23], the answer appears to be far from trivial. For example, Blakie et al. make the remark that for highly occupied ‘classical’ states, such as those near the critical transition to a Bose-Einstein condensate, “it is plausible that single realizations of Wigner trajectories should approximately correspond to a possible outcome of a given experiment”. Furthermore, the question seems to be heuristically posed; instead, we seek to address it in an operationally defined manner.

In this paper, we investigate the connection between the outcomes of Wigner trajectories and experimental runs by comparing the respective particle number distributions; for simplicity we focus on treating single-mode problems. Experimentally the particle number distribution is measured by counting shots in which $n$ quanta are detected, for instance photons hitting a detector, and corresponds to the true particle number distribution defined strictly via $P_n = |\langle \alpha|\psi\rangle|^2$ where $n = 0, 1, 2..., \ldots$ for a pure state $|\psi\rangle$. Similarly, for all positive Wigner functions $W_\psi(\alpha)$, where $\alpha$ is the complex
field amplitude, we can formally introduce an operationally well-defined binned number distribution \( P_n \) by calculating \( n_i = |\alpha_i|^2 - 1/2 \), where the index \( i \) indicates an individual trajectory (or equivalently individual samples appropriately taken from a known Wigner function), and sorting the continuous values into discrete bins such that \( \tilde{P}_n \) is the probability to find \( n - 1/2 \leq n_i < n + 1/2 \). The subtraction of 1/2 in the calculation of \( n_i = |\alpha_i|^2 - 1/2 \) can be thought of as representing the subtraction on average of half a quantum of noise (that has been added to the initial state to mimic quantum fluctuations), which is required in the calculation of the average mode occupation \( \langle \langle \hat{n} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle \equiv \langle \alpha^+ \alpha \rangle_W - 1/2 \), where \( \hat{n} \) is the particle number operator, while \( \hat{a}^\dagger \) and \( \hat{a} \) are the creation and annihilation operators) using the Wigner function due to its correspondence to expectation values of symmetrically ordered operator products.

We find that the defining feature governing the interpretation of \( \tilde{P}_n \) as a valid approximation to the true \( P_n \) is the smoothness and the broadness of the Wigner function relative to the oscillatory structure in \( W_{\langle |n\rangle} (\alpha) \). For thermal states, this criterion is in fact equivalent to high mean occupation of the mode, and therefore our findings confirm the heuristic assertion of Blakie et al. \cite{13} that such an interpretation is valid for highly occupied ‘classical’ states. However, we also show – using an explicit counterexample which is for a highly squeezed coherent state (the Wigner function of which is always positive and smooth) – that high mode occupation alone is not always sufficient for such an interpretation and cannot be generally used to assert the ‘classical’-like nature of the mode in question. The broadness of the Wigner distribution for the squeezed coherent states can, on the other hand, still serve as the sufficient condition.

The article is organized such that in Sec. \textsection II we demonstrate formally the underlying mathematical relation between \( P_n \) and \( \tilde{P}_n \) in the Wigner representation and the conditions on \( W_{\langle |\psi\rangle} (\alpha) \) for \( \tilde{P}_n \) to approximately correspond to \( P_n \). In Sec. \textsection III we investigate quantitatively the legitimacy of the method by applying it to the thermal and squeezed coherent states. Finally, in Sec. \textsection IV we examine under what conditions we expect the method to fail, and how such a failure would manifest in calculations.

\section{Formal Derivation}

To formally evaluate the particle number distribution \( P_n \) of a single-mode state \( |\psi\rangle \), one may calculate the overlap of the state \( |\psi\rangle \) with the Fock state \( |n\rangle \), which in the Wigner representation is given by \cite{3}:

\[ P_n \equiv |\langle \psi | n \rangle|^2 = \pi \int d^2 \alpha W_{|\psi\rangle} (\alpha) W_{|n\rangle} (\alpha), \quad (1) \]

where \( W_{|\psi\rangle} (\alpha) \) and \( W_{|n\rangle} (\alpha) \) are the respective Wigner functions, with \( W_{|n\rangle} (\alpha) \) given by \cite{3}:

\[ W_{|n\rangle} (\alpha) = \frac{2}{\pi} (-1)^n e^{-2|\alpha|^2} L_n (|\alpha|^2), \quad (2) \]

where \( L_n (x) \) is the \( n \)th-order Laguerre polynomial. With knowledge of the explicit form of \( W_{|\psi\rangle} (\alpha) \) one may then analytically or numerically evaluate the integral in Eq. \( (1) \) to derive the number distribution of the state exactly. In dynamical simulations one may numerically solve the integral \( (1) \) by first reconstructing the Wigner function \( W_{|\psi\rangle} (\alpha) \) itself, or by noting that the RHS of Eq. \( (1) \) is formally equivalent to:

\[ P_n = \pi \langle W_{|n\rangle} (\alpha) \rangle_W, \quad (3) \]

where the subscript refers to averaging over many stochastic trajectories which provide samples of \( \alpha \) according to the distribution \( W_{|\psi\rangle} (\alpha) \). Such a computation is in general non-trivial for highly occupied states or those with a sufficiently broad number distribution as it requires evaluation of high-order Laguerre polynomials with large arguments. Usually, computational techniques such as quadruple precision will be required to overcome numerical issues for \( n \gtrsim 256 \). Our analysis of \( \tilde{P}_n \) thus has interest beyond the interpretation of individual stochastic trajectories of the Wigner function as the underlying binning formalism overcomes such computational issues, inherent to the exact method, and offers instead a much simpler method to implement numerically.

To characterize the connection of \( \tilde{P}_n \) to this formal definition of \( P_n \) we can mathematically define the binned probability distribution as:

\[ \tilde{P}_n = \int_n^{n+1} d(|\alpha|^2) \mathcal{P}(|\alpha|^2), \quad (4) \]

where \( \mathcal{P}(|\alpha|^2) \) is the probability density of sampling \( |\alpha|^2 \) from an ensemble of stochastic trajectories. In terms of the Wigner function, this is equivalent to the probability of sampling \( \alpha \) from within an annulus in phase-space with inner and outer radii of \( \sqrt{n} \) and \( \sqrt{n+1} \) respectively. Thus we may rewrite Eq. \( (1) \) using the Heaviside step function \( \theta(x) \), as:

\[ \tilde{P}_n = \pi \int d^2 \alpha \left[ \frac{1}{\pi} \theta(|\alpha| - \sqrt{n}) \theta(\sqrt{n+1} - |\alpha|) \right] W_{|\psi\rangle} (\alpha). \quad (5) \]

Comparing the result of Eq. \( (5) \) to Eq. \( (1) \) we see that the binning procedure is mathematically equivalent to replacing \( W_{|n\rangle} (\alpha) \) by a radially symmetric boxcar function in phase-space defined as:

\[ \tilde{W}_{|n\rangle} (\alpha) = \frac{1}{\pi} \theta(|\alpha| - \sqrt{n}) \theta(\sqrt{n+1} - |\alpha|). \quad (6) \]

This representation of the Fock state Wigner function is known as a Planck-Bohr-Sommerfeld band \cite{4}, and is equivalent to a smearing out of the classical (Kramers) trajectory of a Fock state in phase-space, which is a ring along \( |\alpha| = \sqrt{n + 1/2} \). The binning procedure as characterized by Eq. \( (5) \) is then similar to the area-of-overlap formalism developed previously by Schleich \cite{4}, wherein the number distribution of a state can be approximated by the overlap of the phase-space distribution with a band in phase-space, representing the number state. We point out the subtle difference that Schleich’s formalism can account for interference
between probability amplitudes, which is equivalent to retaining negative contributions in Eq. (1), whereas the binning procedure rules this out as Eq. (5) is a sum of contributions from a strictly non-negative Wigner function.

One can also justify the approximation of $\tilde{W}[n](\alpha)$ by a more practical argument by noting that low-order moments of $\alpha$ with respect to $W[n](\alpha)$ are dominated by contributions of the final ‘crest’ in the highly-oscillatory Wigner distribution (see Fig. 1 for illustration), whilst earlier contributions effectively cancel out. Such an approach is similar to the approximations applied by Gardiner et al. in Ref. [10], wherein the authors observed that the Wigner function of the Fock state could be approximated as a radially symmetric Gaussian ring,

$$W[n](\alpha) = A e^{-2(|\alpha|^2-n-1/2)^2},$$

which is strictly positive ($A$ being the normalization constant). In Refs. [23,25] Olsen et al. demonstrated explicitly that sampling of $W[n](\alpha)$ indeed produced all moments $\langle |\alpha|^m \rangle_W$ of the exact Wigner distribution up to $O(1/n^2)$ relative to the leading order, implying that the contribution of all but the final oscillation in $W[n](\alpha)$ can be considered approximately negligible. In light of this, one could also regard $\tilde{W}[n](\alpha)$, Eq. (6), as a further crude approximation to $W[n](\alpha)$.

Following the reasoning of Gardiner et al. [10], we thus intuitively expect the replacement of $W[n](\alpha)$ by $W[n](\alpha)$ in Eq. (5) to only be a good approximation when $W[\psi](\alpha)$ is a sufficiently smooth function of $\alpha$. Qualitatively, this means that we require $W[\psi](\alpha)$ to be slowly varying on the order of the characteristic length scale of oscillations in $W[n](\alpha)$, which, using the properties of the Laguerre polynomial $L_n(4|\alpha|^2)$, can be estimated to be $\sim 1/\sqrt{n}$. There are two complementary properties of $W[\psi](\alpha)$ which achieve this outcome. Firstly, for states localized near the origin in phase-space – such as the thermal state – one requires that the Wigner function has a characteristic width $\sigma \gg 1$. This implies that $\tilde{P}_n$ will approximate $P_n$ well even for small $n \sim 1$.

Secondly, for states of fixed width – such as the coherent or squeezed coherent states – one requires a large coherent displacement $|\beta|$ from the origin. As the overlap between $W[\psi](\alpha)$ and $W[n](\alpha)$ will generally be greatest for $n \sim |\beta|^2$, the length-scale of the oscillations in $W[n](\alpha)$ in the relevant regions of $W[\psi](\alpha)$ will scale as $1/|\beta|$. The width of $W[\psi](\alpha)$ relative to the scale of these oscillations thus increases as $|\beta|$ increases, improving the validity of replacing $W[n](\alpha)$ with $\tilde{W}[n](\alpha)$. In the following section we illustrate these arguments both qualitatively and quantitatively for the thermal and squeezed coherent states.

Lastly, although this derivation has focused on the single-mode case it may be trivially generalized to a multi-mode state and an equivalent form of $P[n_1,n_2...]$ may be found. The same generalized conditions regarding the relative width of the Wigner function may be applied. However, in the following section we will continue to focus our analysis on the single-mode case as it allows us to illustrate the correspondence between the two number distributions in a transparent manner. More specifically, we will use two particular states to analyse the similarity between $P_n$ and $\tilde{P}_n$: (i) thermal and (ii) squeezed coherent states.

III. SIMILARITY OF $P_n$ AND $\tilde{P}_n$

A. Thermal state

The first state we consider is the thermal state, which is a mixed state defined by the density matrix

$$\hat{\rho}_\text{th} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|,$$

where the number distribution is given by [5]

$$P_n = \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}},$$

and is characterized solely by the mean occupation $\langle \hat{n} \rangle = \bar{n}$.

The corresponding Wigner function is [5]

$$W_{\text{th}}(\alpha) = \frac{1}{\pi(\bar{n} + 1/2)} \exp \left( -\frac{|\alpha|^2}{\bar{n} + 1/2} \right).$$

The rms width of this distribution is then $\sigma = \sqrt{(\bar{n} + 1/2)/2}$ and, therefore, according to our criterion, the sufficient requirement ($\sigma \gg 1$) for $\tilde{P}_n$ to agree well with the physical $P_n$ is equivalent in this case to high mean mode occupation $\bar{n} \gg 1$.

Substituting $W_{\text{th}}(\alpha)$ into Eq. (5) leads to

$$\tilde{P}_n = e^{-n/(\bar{n}+1/2)} \left[ 1 - e^{-n/(\bar{n}+1/2)} \right].$$
Although this form of \( \tilde{P}_n \) clearly differs from \( P_n \), a keen eye will note that in fact
\[
\tilde{P}_n = \frac{(n)_\text{bin}^\alpha}{(\langle n \rangle_{\text{bin}} + 1)^{n+1}},
\]
where
\[
(n)_\text{bin} \equiv \sum_{n=0}^\infty n \tilde{P}_n = \frac{1}{e^{1/(n+1/2)} - 1}.
\]
Hence while both distributions may be written solely in terms of their respective means, \( \tilde{P}_n \neq P_n \) explicitly as \( (n)_\text{bin} \neq \bar{n} \).

As a quantitative measure of how well the binned particle number distribution \( \tilde{P}_n \) approximates the true distribution \( P_n \), we introduce the Bhattacharyya statistical distance, which is defined as \[26\]
\[
D_B = -\ln[B(P, \tilde{P})],
\]
where the Bhattacharyya coefficient is given by
\[
B(P, \tilde{P}) = \sum_{n=0}^\infty \sqrt{P_n \tilde{P}_n}.
\]
For \( \tilde{P}_n \rightarrow P_n \) the Bhattacharyya coefficient becomes
\[
B(P, \tilde{P}) \rightarrow \sum_{n=0}^\infty P_n = 1
\]
due to the normalization condition and hence \( D_B \rightarrow 0 \), indicating complete overlap of the distributions.

For the thermal state the Bhattacharyya coefficient can be calculated exactly to give
\[
B(P, \tilde{P}) = \frac{\left[1 - e^{-2/(2\bar{n}+1)}\right]^{1/2}}{(\bar{n}+1)^{1/2} - \bar{n}^{1/2}e^{-1/(2\bar{n}+1)}},
\]
and thus the Bhattacharyya distance is
\[
D_B = -\frac{1}{2} \ln \left[1 - e^{-2/(2\bar{n}+1)}\right] + \ln \left[\sqrt{\bar{n}+1} - \sqrt{\bar{n}}e^{-1/(2\bar{n}+1)}\right].
\]
In the limit of \( \bar{n} \gg 1 \) we find the behaviour
\[
D_B \propto \bar{n}^{-4},
\]
which indicates that for large mean occupation \( \tilde{P}_n \) rapidly approaches the true \( P_n \). To illustrate this strong correspondence between \( P_n \) and \( \tilde{P}_n \), we plot a comparison of the distributions for a thermal state with \( \bar{n} = 10 \) in Fig. 2(a); as we see, even only moderately large mean occupations, such as in this example, render the two distributions visually identical, with a Bhattacharyya distance of \( D_B = 6.63 \times 10^{-5} \).

We could recast the result of Eq. (18) in terms of the width of the distribution, \( \sigma \equiv \sqrt{\bar{n}/2} \) for \( \bar{n} \gg 1 \), as
\[
D_B \propto \sigma^{-8}.
\]
This strong scaling is a key result, particularly given that the statement of Blakie et. al. (quoted in the Introduction section) pertains directly to the interpretation of c-field methods for Bose gases, for which this interpretation is applied directly to thermally populated states above the condensate mode. As we have shown, the heuristic link between Wigner trajectories and individual experimental runs, thought to be plausible for highly occupied states, can indeed be justified and quantified in terms of the similarity of \( P_n \) and \( \tilde{P}_n \). While for a thermal state, high mean occupation is actually equivalent to our sufficient requirement of having a broad Wigner distribution for this interpretation to be valid, there are situations (see next Section) in which high mode occupation alone may not suffice for such an interpretation.

\section*{B. Squeezed coherent state}

The second state which we consider is the squeezed coherent state, defined as
\[
|\beta, \eta \rangle = \hat{D}(\beta) \hat{S}(\eta)|0\rangle,
\]
where \( \hat{D}(\beta) = \exp(\beta \hat{a}^\dagger - \beta^* \hat{a}) \) is the displacement operator and the squeezing operator is \( \hat{S} = \exp \left[ \eta \left( \hat{a}^\dagger \hat{a} - \eta (\hat{a}^\dagger)^2 / 2 \right) \right] / 2 \) with \( \eta = se^{i\theta} \) for \( s \geq 0 \) \[5, 27\]. In Fig. 3 we illustrate the actions of these operators in phase-space. Firstly the squeezing operator ‘squeezes’ the Gaussian Wigner distribution of the vacuum by an amount \( e^{-s} \) along an axis defined by the squeezing angle \( \theta \), whilst the perpendicular axis is stretched by \( e^s \). The displacement operator then shifts the distribution in phase space by \( \beta = |\beta|e^{i\theta} \). There exist two special sub-cases of the squeezed coherent state: (i) the coherent state \( |\beta \rangle \) where \( \beta \neq 0 \) and \( s = 0 \); and (ii) the squeezed vacuum state \( |0,\eta \rangle \) where \( \beta = 0 \) and \( s \neq 0 \).

The Wigner function of the general squeezed coherent state can be written in a simple form \[28\]
\[
W_{|\beta,\eta \rangle}(\gamma) = \frac{2}{\pi} \exp \left( -\frac{\gamma^2}{2\sigma^2} - \frac{\gamma^2}{2\sigma^2} \right),
\]

![Figure 2](image_url)

\begin{center}
FIG. 2. (a) Example of the true particle number distribution \( P_n \) (grey bars) for a thermal state with \( \bar{n} = 10 \), compared with the binned number distribution \( \tilde{P}_n \) (red markers). (b) Statistical distance \( D_B \) between the two distributions, calculated from Eq. (17) for a range of mean occupations \( \bar{n} \), which scales as \( \propto 1/\bar{n}^4 \).\end{center}
where

\[ \gamma_x = (\alpha_x - \beta_x) \cos \left(\frac{\theta}{2}\right) + (\alpha_y - \beta_y) \sin \left(\frac{\theta}{2}\right), \tag{22} \]

\[ \gamma_y = - (\alpha_x - \beta_x) \sin \left(\frac{\theta}{2}\right) + (\alpha_y - \beta_y) \cos \left(\frac{\theta}{2}\right), \tag{23} \]

for \( \alpha = \alpha_x + i\alpha_y \) and \( \beta = \beta_x + i\beta_y \). The rms widths along the squeezed and anti-squeezed axes are given by \( \sigma_x = e^{-\theta/2} \) and \( \sigma_y = e^{\theta/2} \), respectively. Independent control over the parameters \( \beta \) and \( \eta \) allows us to quantitatively probe the similarity of \( \tilde{P}_n \) and \( P_n^s \), as a function of the width of the Wigner distribution.

The number distribution of the squeezed state is nontrivial,

\[ P_n = \frac{1}{n! \cosh(s)^n} e^{-|\beta|^2 [1 + \cos(2\varphi - \theta) \tanh(s)]} \times |H_n(\beta + \beta^* e^{i\theta} \tanh(s))|^2, \tag{24} \]

with mean occupation \( \langle \hat{n} \rangle = |\beta|^2 + \sinh^2(s) \) \[27, 29\]. For large coherent displacement such that \( |\beta|^2 \gg e^{2s} \), this \( P_n \) can be approximated by a simple Gaussian \[27\]

\[ P_n \approx \frac{1}{\sqrt{2\pi (\Delta^2 \hat{n})}} \exp \left[ -\frac{(n - |\beta|^2)^2}{2(\Delta^2 \hat{n})} \right], \tag{25} \]

whose rms width is given by \( \sigma = \sqrt{\langle \Delta^2 \hat{n} \rangle} \), where

\[ \langle \Delta^2 \hat{n} \rangle = |\beta|^2 \left[ e^{-2s} \cos^2 \left(\varphi - \frac{\theta}{2}\right) + e^{2s} \sin^2 \left(\varphi - \frac{\theta}{2}\right) \right]. \tag{26} \]

This form demonstrates how the squeezing operator stretches or squeezes the probability distribution \( P_n \) according to the relative orientation of the squeezing and coherent displacement. In this section, our analysis will be limited to a range of squeezing such that the above approximation for \( P_n \) is valid.

The effects of stronger squeezing and its implications for both \( P_n \) and \( \tilde{P}_n \) will be discussed in Sec. IV.

An analytic form of \( \tilde{P}_n \) can be found by substituting Eq. (21) into the definition of Eq. (5), however, the result is not particularly insightful. We point the interested reader to Ref. [30] as a guide to the general form of the calculation. Instead, we numerically evaluate \( \tilde{P}_n \) by stochastically sampling \( W_{[\beta, \eta]}(\alpha) \) according to the prescription of Ref. [24] and binning the calculated occupation of each sample. Such a construction is equivalent to obtaining the same state and results via a dynamical simulation of stochastic equations (trajectories) in the Wigner representation, as the phenomenological squeezed vacuum state can be generated from a Hamiltonian for spontaneous parametric down-conversion (in the undepleted pump approximation) \( \hat{H} = i\hbar [g^* \hat{a}^2 - g(\hat{a}^\dagger)^2] \), in which case the squeezing parameter \( \eta \) is actually given by \( \eta \equiv g \). The subsequent coherent displacement of the squeezed state is achieved by coupling the mode \( \hat{a} \) to a classical field of amplitude \( \varepsilon \), equivalent to evolution under the Hamiltonian \( \hat{H} = i\hbar \kappa [\varepsilon^* \hat{a} - \varepsilon \hat{a}^\dagger] \) where \( \kappa \) is the coupling strength and hence the resulting displacement is related as \( \beta \equiv \kappa t \).

We numerically evaluate the Bhattacharyya distance as a function of coherent displacement for some example squeezed coherent states with \( \varphi = 0, s = 0.4 \) and squeezing angles of \( \theta = 0 \) and \( \theta = \pi \), which are referred to as amplitude- and phase-squeezing respectively. The results are plotted in the inset to Fig. 3. Also plotted is the simple case of the coherent state for which \( s = 0 \), whilst other parameters are kept
identical. We find a generic scaling independent of \(s\),

\[ D_B \propto |\beta|^{-2}, \tag{27} \]

in the regime where \(|\beta|^2 \gg e^{2s}\) and the approximate form of Eq. (25) is valid. This result predicts a rapid convergence of \(P_n\) to \(P_n\) with increasing occupation \(\langle n \rangle \approx |\beta|^2\). For \(|\beta| = 50\) and for the three cases of squeezing, the calculated distributions \(P_n\) and \(P_n\) are plotted in the main panel of Fig. 4 and are visually indistinguishable from each other.

Beyond the scaling with coherent displacement, we may also examine how the absolute width of the Wigner function affects the agreement of \(P_n\) with \(P_n\) by manipulation of the squeezing strength \(s\) and angle \(\theta\). The relevant length scale will be the effective width \(\sigma_{\text{eff}}\) of the distribution (see Fig. 3) with respect to the radially directed oscillations in \(W_{|n\rangle\langle\alpha|}\),

\[ \sigma_{\text{eff}} = \sqrt{\sigma_s^2 \cos^2 \left( \varphi - \frac{\theta}{2} \right) + \sigma_s^2 \sin^2 \left( \varphi - \frac{\theta}{2} \right)}. \tag{28} \]

We plot the dependence of the Bhattacharyya distance as a function of this parameter in Fig. 5(a) and find it scales as

\[ D_B \propto \sigma_{\text{eff}}^{-6}, \tag{29} \]

independently of coherent displacement \(\beta\). This strong scaling again agrees with our intuitive argument, indicating that \(P_n\) rapidly approaches \(P_n\) as the Wigner function becomes increasingly smooth on the length scale of oscillations in \(W_{|n\rangle\langle\alpha|}\). We note the difference to the scaling of the thermal state \(P_n\) as the radially directed oscillations in \(W_{|n\rangle\langle\alpha|}\) are offset by a decrease in the perpendicular axis (\(\sigma_{\perp}\)) such that \(\sigma_{\perp}\) is preserved. This is in contrast to the thermal state which has a radially symmetric rms width which increases with average occupation.

In Fig. 5(b) we also plot the Bhattacharyya distance as a function of the squeezing angle. For states with a purely real coherent displacement (\(\varphi = 0\)), we find \(D_B\) is minimal for phase-squeezed states (\(\theta = \pi\)) and maximal for amplitude squeezed states (\(\theta = 0\)). For phase-squeezing, the anti-squeezed axis of the distribution is aligned radially, along the direction of the oscillations in \(W_{|n\rangle\langle\alpha|}\), and \(\sigma_{\text{eff}}\) is maximal. We thus expect for this scenario that our approximation \(W_{|n\rangle\langle\alpha|}\) should be the most valid as any oscillations will be averaged out in Eq. (1), leading to minimal \(D_B\). Conversely, for amplitude squeezing the squeezed axis of the distribution is aligned radially, minimizing \(\sigma_{\text{eff}}\) and thus we expect our approximation to be the least valid, leading to a larger \(D_B\).

IV. BREAKDOWN OF RELATIONSHIP

The analysis of the previous section has demonstrated how in general, \(P_n\) closely replicates \(P_n\) when the relative width of the Wigner distribution \(W_{|\psi\rangle\langle\alpha|}\) is large compared to the oscillation period of the Fock state Wigner function, \(W_{|n\rangle\langle\alpha|}\). However, one can also find a few simple counter-examples to demonstrate how the correspondence breaks down when the underlying approximations are no longer valid. In particular, we demonstrate this with states that are highly-occupied, showing that large occupation alone is not sufficient for approximating \(P_n\) by \(\tilde{P}_n\).

As an example, in Fig. 6 we plot \(P_n\) and \(\tilde{P}_n\) for \(|\beta|^2 = 20\), \(s = 1.5\) and for two squeezing angles: (a) \(\theta = 0\) and (b) \(\theta = \pi\). In both cases we see a region emerges wherein the probability of odd and even \(n\) oscillates strongly. For amplitude squeezing (\(\theta = 0\)) these oscillations arise for \(n \gtrsim |\beta|^2\) as predicted by Schleich and Wheeler [31]. In terms of the binning procedure it is clear that \(W_{|\psi\rangle\langle\alpha|}\) is sufficiently elongated that it is approximately the width of the oscillations in \(W_{|\psi\rangle\langle\alpha|}\) and multiple oscillations become important in the calculation of Eq. (1) as illustrated in Fig. 6(b). Similar arguments apply to the case of phase squeezing (\(\theta = \pi\)), illustrated in Figs. 6(c) and (d). In both cases, the narrowness of the Wigner distribution implies it is not valid to approximate \(W_{|n\rangle\langle\alpha|}\) with \(\tilde{W}_{|n\rangle\langle\alpha|}\) and thus \(\tilde{P}_n\) does not well approximate \(P_n\).

Related issues arise for the squeezed vacuum state, which can be considered an extreme case of the above examples wherein \(|\beta|^2 = 0\) and the Wigner distribution is centered at the phase-space origin. The state is notable for its even-odd oscillatory number distribution,

\[ P_{2m} = \frac{[\tanh(s)/2]^{2m}}{(2m)!\cosh(s)}|H_{2m}(0)|^2, \quad P_{2m+1} = 0. \tag{30} \]

Again, in terms of phase-space representation of the state, this property is an effect of the narrowness of \(W_{|n\rangle\langle\alpha|}\) com-
bined with the negativity of the true Fock state Wigner function $W_{|n\rangle}$($\alpha$). Setting $\theta = 0$ for definitiveness, the Wigner distribution has an rms width of $\sigma_x \leq 1/2$ in the $\alpha_x$ direction, and thus it will obviously be sufficiently narrow to probe the individual oscillations of $W_{|n\rangle}$($\alpha$), which have a period on the order of 1 for small $n$. Accordingly, the interpretation of $P_n \sim P_{\frac{\alpha}{\sigma}}$ is not valid as the replacement of $W_{|n\rangle}$($\alpha$) by $\tilde{W}_{|n\rangle}$($\alpha$) in Eq. (1) is a poor approximation.

In Fig. 7 we plot the Bhattacharyya distance for a broader range of squeezed coherent states, highlighting specifically the regimes in which $P_n$ replicates $P_{\frac{\alpha}{\sigma}}$ and where this breaks down. We find for $\theta = \pi$ ($\sigma_{\text{eff}} \geq 1/2$) the Bhattacharyya distance behaves according to the power-law of Eq. (29) (indicated by the linear regime on the log-log axes) until a turning point $\sigma_{\text{eff}} \approx 0.84|\beta|^2/3$, where the oscillations of $W_{|n\rangle}$($\alpha$) near the origin become important for small $n$. For $\theta = 0$ ($\sigma_{\text{eff}} \leq 1/2$) the statistical distance worsens due to the narrowing of the distribution according to Eq. (29) (again, the linear regime) until the emergence of oscillations. The transition from the linear relationship occurs in the vicinity $\sigma_{\text{eff}} \approx 1/(2|\beta|^1/3)$, which agrees with that regarding the emergence of oscillations in $P_n$, as previously studied in Ref. [31].

When we examine the aforementioned states for which our procedure does not reproduce $P_n$ accurately, we see that these are states for which the quantization of the field is important. The squeezed vacuum is a prime example of this, containing only even numbers of photons. As we coherently displace this state from the origin (squeezed coherent state), the displacement becomes more important than the squeezing and $P_n$ becomes more accurate. This is consistent with the fact that the truncated or a priori positive Wigner distribution is often described as equivalent to the classical theory of stochastic electrodynamics [72].

FIG. 7. Dependence of statistical distance $D_B$ on the effective width $\sigma_{\text{eff}}$ of the squeezed coherent state Wigner distribution for $|\beta|^2 = 20$ (blue solid line) and $|\beta|^2 = 40$ (red dashed line). Without loss of generality we arbitrarily set $\varphi = 0$ for all states. Identically to Fig. 5 (a) we calculate $\sigma_{\text{eff}} \leq 1/2$ by setting $\theta = 0$ and thus $\sigma_{\text{eff}} \equiv \sigma_s$, and similarly $\sigma_{\text{eff}} > 1/2$ by $\theta = \pi$ and $\sigma_{\text{eff}} \equiv \sigma_a$. Stochastic sampling error of one standard deviation is not indicated, however, it is restricted to less than 2% of calculated $D_B$ values. For relatively weak squeezing ($|\beta|^2 \gg e^{2S}$), the power-law scaling of Fig. 5 (a) is illustrated by the linear-regime ($0.2 \lesssim s \lesssim 0.9$) with the logarithmic scale. The deviation from the linear regime, indicating oscillatory structure in $P_n$, which is not replicated by $P_n$, occurs at $\sigma_{\text{eff}} \approx 1/(2|\beta|^1/3)$ for $\theta = 0$, whilst there is an obvious turning point in $D_B$ at $\sigma_{\text{eff}} \approx 0.84|\beta|^2/3$ for $\theta = \pi$. 

V. CONCLUSION

In summary, we have examined under which conditions a naive calculation of the binned number distribution from
individual (truncated) Wigner trajectories, $\tilde{P}_n$, can replicate closely the true particle number distribution $P_n$, hence justifying the interpretation of these trajectories as representing individual experimental outcomes. The sufficient requirement for this is that the Wigner function $W_{|\psi\rangle}(\alpha)$ of the state $|\psi\rangle$ varies sufficiently smoothly on the characteristic length scale of oscillations in the Wigner function $W_{|n\rangle}(\alpha)$ of the Fock state $|n\rangle$. This is, of course, in addition to the constraint that only positive Wigner functions $W_{|\psi\rangle}(\alpha)$ are being considered, which is the case in the truncated Wigner approximation or in model Hamiltonians that depend no-higher-than quadratically on creation or annihilation operators.

We have provided a rigorous operational definition of this seemingly heuristic binning procedure as one that corresponds to approximating the Wigner function of the Fock state (which appears in the definition of $P_n$ via an overlap integral with the Wigner function of the state of interest) as a boxcar function in phase space. For states localized around the phase-space origin (e.g., a thermal state), the requirement of smoothness of the Wigner function is satisfied by a broad distribution, having a characteristic width much larger than unity. In this case, the large width of the distribution is equivalent to having large mode occupation number. On the other hand, for states that have large coherent displacement $\beta$ (such as coherent and squeezed coherent states with $|\beta| \gg 1$), one can tolerate relatively narrow Wigner functions as long as its width remains much larger than $1/|\beta|$, which is the characteristic length scale of oscillations in $W_{|n\rangle}(\alpha)$ for the most relevant values of $n$ ($\sim |\beta|^2$). This condition is satisfied for coherent states and weakly squeezed states, but will break down for highly squeezed states when the width of the respective Wigner function in the narrow dimension becomes comparable to $1/|\beta|$, even though the mode occupation for such states can be very high. The latter case serves as a counterexample to the view that high mode occupation alone is sufficient to interpret individual Wigner trajectories as ‘samples’ of single experimental runs.

Although we have considered only a small subset of Wigner functions in this article, we expect that the relationship between $\tilde{P}_n$ and $P_n$ established on an analysis of the width and displacement of the Wigner function will allow an easy application to further states. Importantly, in the truncated Wigner formalism the reconstruction of an a priori unknown single-mode Wigner function from many stochastic trajectories is relatively trivial and allows one to extract the characteristic length scale of the quasidistribution and thus, according to our criterion, justify or reject the approximation $\tilde{P}_n$ with no knowledge of the exact $P_n$.

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