Abstract

Generalized prolate spheroidal functions (GPSFs) arise naturally in the study of bandlimited functions as the eigenfunctions of a certain truncated Fourier transform. In one dimension, the theory of GPSFs (typically referred to as prolate spheroidal wave functions) has a long history and is fairly complete. Furthermore, more recent work has led to the development of numerical algorithms for their computation and use in applications. In this paper we consider the more general problem, extending the one dimensional analysis and algorithms to the case of arbitrary dimension. Specifically, we introduce algorithms for efficient evaluation of GPSFs and their corresponding eigenvalues, quadrature rules for bandlimited functions, formulae for interpolation via GPSF expansion, and various analytical properties of GPSFs. We illustrate the numerical and analytical results with several numerical examples.

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1 Introduction

Prolate spheroidal wave functions (PSWFs) are the natural basis for representing bandlimited functions on the interval. Much of the theory and numerical machinery for PSWFs in one dimension is fairly complete (see, for example, [13] and [10]). Slepian et al. showed in [12] that the so-called Generalized Prolate Spheroidal Functions (GPSFs) are the natural extension of PSWFs in higher dimensions. GPSFs are functions \( \psi_j: \mathbb{R}^n \rightarrow \mathbb{C} \) satisfying

\[
\lambda_j \psi_j(x) = \int_B \psi_j(t) e^{ic(x,t)} dt
\]

for some \( \lambda_j \in \mathbb{C} \) where \( B \) denotes the unit ball in \( \mathbb{R}^n \). A function \( f: \mathbb{R}^n \rightarrow \mathbb{C} \) is referred to as bandlimited with bandlimit \( c > 0 \) if

\[
f(x) = \int_B \sigma(t) e^{ic(x,t)} dt
\]

where \( B \) denotes the unit ball in \( \mathbb{R}^n \) and \( \sigma \) is a square-integrable function defined on \( B \). Bandlimited functions are encountered in a variety of applications including in signal processing, antenna design, radar, etc.
Much of the theory and numerical machinery of GPSFs in two dimensions is described in \cite{11}. In this report, we provide analytical and numerical tools for GPSFs in $\mathbb{R}^n$. We introduce algorithms for evaluating GPSFs, quadrature rules for integrating bandlimited functions, and numerical interpolation schemes for expanding bandlimited functions into GPSF expansions. We also provide numerical machinery for efficient evaluation of eigenvalues $\lambda_j$ (see \cite{11}).

The structure of this paper is as follows. In Section 2, we provide basic mathematical background that will be used throughout the remainder of the paper. Section 3 contains analytical facts related to the numerical evaluation of GPSFs that will be used in subsequent sections. In Section 4, we describe a numerical scheme for evaluating GPSFs. Section 5 contains a quadrature rule for integrating bandlimited functions. Section 6 includes a numerical scheme for expanding bandlimited functions into GPSFs. In Section 7, we provide the numerical results of implementing the quadrature and interpolation schemes as well as plots of GPSFs and their eigenvalues. In the appendix, we include miscellaneous technical lemmas relating to GPSFs.

2 Mathematical and numerical preliminaries

In this section, we introduce notation and elementary mathematical and numerical facts which will be used in subsequent sections.

In accordance with standard practice, we define the Gamma function, $\Gamma(x)$, by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

where $e$ will denote the base of the natural logarithm. We will be denoting by $\delta_{i,j}$ the function defined by the formula

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The following is a well-known technical lemma that will be used in Section 3.2.

**Lemma 2.1** For any real number $a > 0$ and for any integer $n > ae$,

$$\frac{a^n\sqrt{n}}{\Gamma(n+1)} < 1$$

where $\Gamma(n)$ is defined in (3).

The following lemma follows immediately from Formula 9.1.10 in \cite{1}.

**Lemma 2.2** For all real numbers $x \in [0, 1]$, and for all real numbers $\nu \geq -1/2$,

$$|J_\nu(x)| \leq \frac{|x/2|^\nu}{\Gamma(\nu + 1)}$$

where $J_\nu$ is a Bessel function of the first kind and $\Gamma(\nu)$ is defined in (3).
2.1 Jacobi polynomials

In this section, we summarize some properties Jacobi polynomials. Jacobi Polynomials, denoted \( P^{(\alpha,\beta)}_n \), are orthogonal polynomials on the interval \((-1,1)\) with respect to weight function

\[
w(x) = (1 - x)^\alpha (1 + x)^\beta.
\]  

(7)

Specifically, for all non-negative integers \( n, m \) with \( n \neq m \) and real numbers \( \alpha, \beta > -1 \),

\[
\int_{-1}^{1} P^{(\alpha,\beta)}_n(x) P^{(\alpha,\beta)}_m(x)(1 - x)^\alpha (1 + x)^\beta \, dx = 0
\]  

(8)

The following lemma, provides a stable recurrence relation that can be used to evaluate a particular class of Jacobi Polynomials (see, for example, [1]).

**Lemma 2.3** For any integer \( n \geq 1 \) and \( \alpha > -1 \),

\[
P^{(\alpha,0)}_{n+1}(x) = \frac{(2n + \alpha + 1)\alpha^2 + (2n + \alpha)(2n + \alpha + 1)(2n + \alpha + 2)x}{2(n+1)(n+\alpha+1)(2n+\alpha)} P^{(\alpha,0)}_{n}(x)
\]

\[
-\frac{2(n+\alpha)(n)(2n+\alpha+2)}{2(n+1)(n+\alpha+1)(2n+\alpha)} P^{(\alpha,0)}_{n-1}(x),
\]  

(9)

where

\[
P^{(\alpha,0)}_{0}(x) = 1
\]  

(10)

and

\[
P^{(\alpha,0)}_{1}(x) = \frac{\alpha + (\alpha + 2)x}{2}.
\]  

(11)

The Jacobi Polynomial \( P^{(\alpha,0)}_n \) is defined in (8).

The following lemma provides a stable recurrence relation that can be used to evaluate derivatives of a certain class of Jacobi Polynomials. It is readily obtained by differentiating (9) with respect to \( x \),

**Lemma 2.4** For any integer \( n \geq 1 \) and \( \alpha > -1 \),

\[
P^{(\alpha,0)'}_{n+1}(x) = \frac{(2n + \alpha + 1)\alpha^2 + (2n + \alpha)(2n + \alpha + 1)(2n + \alpha + 2)x}{2(n+1)(n+\alpha+1)(2n+\alpha)} P^{(\alpha,0)'}_{n}(x)
\]

\[
-\frac{2(n+\alpha)(n)(2n+\alpha+2)}{2(n+1)(n+\alpha+1)(2n+\alpha)} P^{(\alpha,0)'}_{n-1}(x)
\]

\[+\frac{(2n+\alpha)(2n+\alpha+1)(2n+\alpha+2)}{2(n+1)(n+\alpha+1)(2n+\alpha)} P^{(\alpha,0)}_{n}(x),
\]  

(12)

where

\[
P^{(\alpha,0)'}_{0}(x) = 0
\]  

(13)
and
\[ P_1^{(\alpha,0)}(x) = \frac{\alpha + 2}{2}. \]  

(14)

The Jacobi Polynomial \( P_n^{(\alpha,0)} \) is defined in (3) and \( P_n^{(\alpha,0)'}(x) \) denotes the derivative of \( P_n^{(\alpha,0)}(x) \) with respect to \( x \).

The following two lemmas, which provide a differential equation and a recurrence relation for Jacobi polynomials, can be found in, for example, [1].

Lemma 2.5 For any integer \( n \geq 2 \) and \( \alpha > -1 \),
\[ (1 - x^2)P_n^{(\alpha,0)''}(x) + (-\alpha - (\alpha + 2)x)P_n^{(\alpha,0)'}(x) + n(n + \alpha + 1)P_n^{(\alpha,0)}(x) = 0 \]  

(15)

for all \( x \in [0,1] \) where \( P_n^{(\alpha,0)} \) is defined in (3).

Lemma 2.6 For all \( \alpha > -1 \), \( x \in (0,1) \), and any integer \( n \geq 2 \),
\[ a_{1n}P_n^{(\alpha,0)}(x) + 1 \]  

(16)

where
\[ a_{1n} = 2(n + 1)(n + \alpha + 1)(2n + \alpha) \]
\[ a_{2n} = (2n + \alpha + 1)\alpha^2 \]
\[ a_{3n} = (2n + \alpha)(2n + \alpha + 1)(2n + \alpha + 2) \]
\[ a_{4n} = 2(n + \alpha)(n)(2n + \alpha + 2) \]  

(17)

and
\[ P_0^{(\alpha,0)}(x) = 1 \]
\[ P_1^{(\alpha,0)}(x) = \frac{\alpha + (\alpha + 2)x}{2}. \]  

(18)

2.2 Zernike polynomials

In this section, we describe properties of Zernike polynomials, which are a family of orthogonal polynomials on the unit ball in \( \mathbb{R}^{p+2} \). They are the natural basis for representing GPFS.

Zernike polynomials are defined via the formula
\[ Z_{N,n}^\ell(x) = R_{N,n}(\|x\|)S_N^\ell(x/\|x\|), \]  

(19)

for all \( x \in \mathbb{R}^{p+2} \) such that \( \|x\| \leq 1 \), where \( N \) and \( n \) are nonnegative integers, \( S_N^\ell \) are the orthonormal surface harmonics of degree \( N \) (see Section 2.7), and \( R_{N,n} \) are polynomials of degree \( 2n + N \) defined via the formula
\[ R_{N,n}(r) = r^N \sum_{m=0}^{n} (-1)^m \binom{n + N + \frac{p}{2}}{m} \binom{n}{m} (r^2)^{n-m} (1 - r^2)^m, \]  

(20)
for all $0 \leq r \leq 1$. The polynomials $R_{N,n}$ satisfy the relation
\[ R_{N,n}(1) = 1, \tag{21} \]
and are orthogonal with respect to the weight function $w(r) = r^{p+1}$, so that
\[ \int_0^1 R_{N,n}(r) R_{N,m}(r) r^{p+1} \, dr = \frac{\delta_{n,m}}{2(2n + N + \frac{p}{2} + 1)}. \tag{22} \]

We define the polynomials $\overline{R}_{N,n}$ via the formula
\[ \overline{R}_{N,n}(r) = \sqrt{2(2n + N + p/2 + 1)} R_{N,n}(r), \tag{23} \]
so that
\[ \int_0^1 (\overline{R}_{N,n}(r))^2 r^{p+1} \, dx = 1, \tag{24} \]
where $N$ and $n$ are nonnegative integers. In an abuse of notation, we refer to both the polynomials $Z_{N,n}^\ell$ and the radial polynomials $R_{N,n}$ as Zernike polynomials where the meaning is obvious.

Remark 2.1 When $p = -1$, Zernike polynomials take the form
\[
\begin{align*}
Z_{1,n}^1(x) &= R_{0,n}(|x|) = P_{2n}(x), \\
Z_{1,n}^2(x) &= \text{sgn}(x) \cdot R_{1,n}(|x|) = P_{2n+1}(x),
\end{align*}
\tag{25}
\]
for $-1 \leq x \leq 1$ and nonnegative integer $n$, where $P_n$ denotes the Legendre polynomial of degree $n$ and
\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-1 & \text{if } x < 0,
\end{cases}
\tag{26}
\]
for all real $x$.

Remark 2.2 When $p = 0$, Zernike polynomials take the form
\[
\begin{align*}
Z_{N,n}^1(x_1, x_2) &= R_{N,n}(r) \cos(N\theta), \\
Z_{N,n}^2(x_1, x_2) &= R_{N,n}(r) \sin(N\theta),
\end{align*}
\tag{27, 28}
\]
where $x_1 = r \cos(\theta)$, $x_2 = r \sin(\theta)$, and $N$ and $n$ are nonnegative integers.

The following lemma, which can be found in, for example, [1], shows how Zernike polynomials are related to Jacobi polynomials.

Lemma 2.7 For all non-negative integers $N, n$,
\[ R_{N,n}(r) = (-1)^n r^N P_n^{(N+\frac{p}{2}, 0)}(1 - 2r^2), \tag{29} \]
where $0 \leq r \leq 1$, and $P_n^{(\alpha, 0)}$, $\alpha > -1$, is defined in (20).
2.3 Numerical evaluation of Zernike polynomials

In this section, we provide a stable recurrence relation (see Lemma 2.8) that can be used to evaluate Zernike Polynomials.

The following lemma follows immediately from applying Lemma 2.7 to (16).

Lemma 2.8 The polynomials $R_{N,n}$, defined in (20) satisfy the recurrence relation

$$R_{N,n+1}(r) =$$

$$-((2n+N+1)N^2 + (2n+N)(2n+N+1)(2n+N+2)(1-2r^2))R_{N,n}(r)$$

$$-\frac{2(n+N)(n)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)}R_{N,n-1}(r)$$

(30)

where $0 \leq r \leq 1$, $N$ is a non-negative integer, $n$ is a positive integer, and

$$R_{N,0}(r) = r^N$$

(31)

and

$$R_{N,1}(r) = -r^N \frac{N + (N+2)(1-2r^2)}{2}.$$  

(32)

Remark 2.3 The algorithm for evaluating Zernike polynomials using the recurrence relation in Lemma 2.8 is known as Kintner’s method (see [7] and, for example, [3]).

2.4 Modified Zernike polynomials, $T_{N,n}$

In this section, we define the modified Zernike polynomials, $T_{N,n}$ and provide some of their properties. This family of functions will be used in Section 4 for the numerical evaluation of GPSFs.

We define the function $T_{N,n}$ by the formula

$$T_{N,n}(r) = r^{n+1 \over 2} R_{N,n}(r)$$

(33)

where $N,n$ are non-negative integers. We define $T_{N,n} : [0, 1] \to \mathbb{R}$ by the formula,

$$T_{N,n}(r) = r^{n+1 \over 2} R_{N,n}(r)$$

(34)

where $N,n$ are non-negative integers and $R_{N,n}$ is a normalized Zernike polynomial defined in (23), so that

$$\int_0^1 (T_{N,n}(r))^2 dr = 1.$$  

(35)

Lemma 2.9 The functions $T_{N,n}$ are orthonormal on the interval $(0, 1)$ with respect to weight function $w(x) = 1$. That is,

$$\int_0^1 T_{N,n}(r)T_{N,m}(r) dr = \delta_{n,m}.$$  

(36)
Proof. Using (34), (22) and (24), for all non-negative integers \( N, n, m \),

\[
\int_0^1 T_{N,n}(r)T_{N,m}(r)dr = \int_0^1 r^{p+1} R_{N,n}(r) r^{p+1} R_{N,m}(r)dr = \int_0^1 R_{N,n}(r) R_{N,m}(r) r^{p+1} dr = \delta_{n,m}
\]

(37)

The following identity follows immediately from the combination of (34), (29), and (23).

Lemma 2.10 For all non-negative integers \( N, n \),

\[
T_{N,n}(r) = P_n^{(N+p/2,0)}(1-2r^2)(-1)^n \sqrt{2(2n+N+p/2+1)} r^{p+1/2}
\]

(38)

where \( T_{N,n} \) is defined in (34) and \( P_n^{(N+p/2,0)} \) is a Zernike polynomial defined in (8).

The following lemma, which provides a differential equation for \( T_{N,n} \), follows immediately from substituting (38) into Lemma 15.

Lemma 2.11 For all \( r \in [0,1] \), non-negative integers \( N, n \) and real \( p \geq -1 \),

\[
(1-r^2)\ddot{T}_{N,n}(r) - 2rT_{N,n}'(r) + \left( \chi_{N,n} + \frac{1}{r^2} \right) T_{N,n}(r) = 0
\]

(39)

where \( \chi_{N,n} \) is defined by the formula

\[
\chi_{N,n} = (N+p/2+2n+1/2)(N+p/2+2n+3/2).
\]

(40)

The following lemma provides a recurrence relation satisfied by \( T_{N,n} \). It follows immediately from the combination of Lemma 2.10 and (9).

Lemma 2.12 For any non-negative integers \( N, n \) and for all \( r \in [0,1] \),

\[
r^2 T_{N,n}(r) = \frac{\sqrt{2(2n+N+p/2+1)}}{\sqrt{2(2(n-1)+N+p/2+1)}} a_{4n} T_{N,n-1}(r) + \frac{a_{2n} + a_{3n}}{2a_{3n}} T_{N,n}(r) + \frac{\sqrt{2(2n+N+p/2+1)}}{\sqrt{2(2(n+1)+N+p/2+1)}} a_{1n} T_{N,n+1}(r)
\]

(41)

where \( T_{N,n} \) is defined in (34) and

\[
a_{1n} = 2(n+1)(n+N+p/2+1)(2n+N+p/2)
\]
\[
a_{2n} = (2n+N+p/2+1)N+p/2^2
\]
\[
a_{3n} = (2n+N+p/2)(2n+N+p/2+1)(2n+N+p/2+2)
\]
\[
a_{4n} = 2(n+N+p/2)(n)(2n+N+p/2+2).
\]

(42)
Proof. Applying the change of variables \(1 - 2r^2 = x\) to (16) and setting \(\alpha = N + p/2\), we obtain

\[
\begin{align*}
 r^2 P_n^{(N+p/2,0)}(1 - 2r^2) &= \frac{a_{2n}}{2a_3} P_n^{(N+p/2,0)}(1 - 2r^2) + \frac{1}{2} P_n^{(N+p/2,0)}(1 - 2r^2) \\
 &\quad - \frac{a_{4n}}{2a_3} P_{n-1}^{(N+p/2,0)}(1 - 2r^2) - \frac{a_{4n}}{2a_3} P_{n+1}^{(N+p/2,0)}(1 - 2r^2).
\end{align*}
\]

(43)

Identity (44) follows immediately from the combination of (43) with Lemma 2.10.

The following observation provides a scheme for computing \(T_{N,n}\).

Observation 2.4 Combining (34), Lemma 2.8, and (23), we observe that the modified Zernike polynomial \(T_{N,n}(r)\) can be evaluated by first computing

\[
P_n^{(N+p/2,0)}(1 - 2r^2)
\]

via recurrence relation (16) and then multiplying the resulting number by

\[
r^N (-1)^n \sqrt{2(2n + N + p/2 + 1)} r^{\frac{p+1}{2}}.
\]

(45)

We define the function \(T^*_n(r)\) by the formula

\[
T^*_n(r) = \frac{T_{N,n}(r)}{r^{N+p+1}.}
\]

(46)

where \(N, n\) are non-negative integers and \(r \in (0, 1)\). The following technical lemma involving \(T^*_n\) will be used in Section 3.3.

Lemma 2.13 For all non-negative integers \(N, n\),

\[
T^*_n(0) = \sqrt{2(2n + N + p/2 + 1)} (-1)^n \binom{n + N + p/2}{n}.
\]

(47)

Proof. Combining (34) and (20), we observe that

\[
T_{N,n}(r) = \sum_{k=0}^{n} a_{N+k} r^{N+p+1} + 2k
\]

(48)

where \(a_{N+k}\) is some real number for \(k = 0, 1, ..., n\). In particular, using (20),

\[
a_N = \sqrt{2(2n + N + p/2 + 1)} (-1)^n \binom{n + N + p/2}{n}.
\]

(49)

Combining (46) and (49), we obtain (47).

The following lemma follows immediately from (48) and provides a relation that will be used in Section 4.1 for the evaluation of certain eigenvalues.
Lemma 2.14 Suppose that $N$ is a nonnegative integer and that $n \geq 1$ is an integer. Then

$$
\tilde{a}_n r T_{N,n-1}(r) - \tilde{b}_n r T_{N,n}'(r) + \tilde{c}_n r T_{N,n+1}'(r) \\
= a_n T_{N,n-1}(r) - b_n T_{N,n}(r) + c_n T_{N,n+1}(r),
$$

(50)

for all $0 \leq r \leq 1$, where

$$
\tilde{a}_n = -2n(2n + N + p/2 + 2),
$$

$$
\tilde{b}_n = 2(N + p/2)(2n + N + p/2 + 1),
$$

$$
\tilde{c}_n = 2(2(n + N + p/2 + 1)(2n + N + p/2) + p/2 + 2),
$$

$$
a_n = n(2(N + p/2 + 4n - 1)(2n + N + p/2 + 2),
$$

$$
b_n = (N + p/2)(2n + N + p/2 + 1) - 2(2n + N + p/2)3,
$$

$$
c_n = (2(N + p/2 + 4n + 5)(n + N + p/2 + 1)(2n + N + p/2),
$$

with $(\cdot)_k$ denoting the Pochhammer symbol or rising factorial.

2.5 Prüfer transform

In this section, we describe the Prüfer Transform, which will be used in Section 5.1 in an algorithm for finding the roots of GPSFs. A more detailed description of the Prüfer Transform can be found in [4].

Lemma 2.15 (Prüfer Transform) Suppose that the function $\phi : [a, b] \to \mathbb{R}$ satisfies the differential equation

$$
\phi''(x) + \alpha(x)\phi'(x) + \beta(x)\phi(x) = 0,
$$

(52)

where $\alpha, \beta : (a, b) \to \mathbb{R}$ are differentiable functions. Then,

$$
\frac{d\theta}{dx} = -\sqrt{\beta(x)} - \left( \frac{\beta'(x) + \alpha(x)}{4\beta(x)} \right) \sin(2\theta),
$$

(53)

where the function $\theta : [a, b] \to \mathbb{R}$ is defined by the formula,

$$
\frac{\phi'(x)}{\phi(x)} = \sqrt{\beta(x)} \tan(\theta(x)).
$$

(54)

Proof. Introducing the notation

$$
z(x) = \frac{\phi'(x)}{\phi(x)}
$$

(55)

for all $x \in [a, b]$, and differentiating (55) with respect to $x$, we obtain the identity

$$
\frac{\phi''}{\phi} = \frac{dz}{dx} + z^2(x).
$$

(56)
Substituting (56) and (55) into (52), we obtain,

$$\frac{dz}{dx} = -(z^2(x) + \alpha(x)z(x) + \beta(x)).$$  \tag{57}$$

Introducing the notation,

$$z(x) = \gamma(x) \tan(\theta(x)), \tag{58}$$

with $\theta, \gamma$ two unknown functions, we differentiate (58) and observe that,

$$\frac{dz}{dx} = \gamma(x) \frac{\theta'(x)}{\cos^2(\theta)} + \gamma'(x) \tan(\theta(x)) \tag{59}$$

and squaring both sides of (58), we obtain

$$z(x)^2 = \tan^2(\theta(x)) \gamma(x)^2. \tag{60}$$

Substituting (59) and (60) into (57) and choosing

$$\gamma(x) = \sqrt{\beta(x)} \tag{61}$$

we obtain

$$\frac{d\theta}{dx} = -\sqrt{\beta(x)} - \left( \frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2} \right) \sin(2\theta). \tag{62}$$

\[\blacksquare\]

**Remark 2.5** The Prüfer Transform is often used in algorithms for finding the roots of oscillatory special functions. For instance, suppose that $\phi : [a, b] \to \mathbb{R}$ is a special function satisfying differential equation (34). It turns out that in most cases, coefficient

$$\beta(x) \tag{63}$$

in (52) is significantly larger than

$$\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2} \tag{64}$$

on the interval $[a, b]$, where $\alpha$ and $\beta$ are defined in (52).

Under these conditions, the function $\theta$ (see (54)), is monotone and its derivative neither approaches infinity nor 0. Furthermore, finding the roots of $\phi$ is equivalent to finding $x \in [a, b]$ such that

$$\theta(x) = \pi/2 + k\pi \tag{65}$$

for some integer $k$. Consequently, we can find the roots of $\varphi$ by solving (62), a well-behaved differential equation.
Remark 2.6 If for all $x \in [a, b]$, the function $\sqrt{\beta(x)}$ satisfies
\[
\sqrt{\beta(x)} > \frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2},
\]  
then, for all $x \in [a, b]$, we have $\frac{d\theta}{dx} < 0$ (see (63)) and we can view $x : [-\pi, \pi] \to \mathbb{R}$ as a function of $\theta$ where $x$ satisfies the first order differential equation
\[
\frac{dx}{d\theta} = \left(-\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}\right) \sin(2\theta)\right)^{-1}.
\]  

2.6 Miscellaneous analytical facts

In this section, we provide several facts from analysis that will by used in subsequent sections.

The following theorem is an identity involving the incomplete beta function.

**Theorem 2.16** Suppose that $a, b > 0$ are real numbers and $n$ is a nonnegative integer. Then
\[
B_x(a + n, b) = \frac{\Gamma(a + n)}{\Gamma(a + b + n)} \left(\frac{\Gamma(a + b)}{\Gamma(a)} B_x(a, b) - (1 - x)^b \sum_{k=1}^{n} \frac{\Gamma(a + b + k - 1)}{\Gamma(a + k)} x^{a+k-1}\right)
\]  
for all $0 \leq x \leq 1$, where $B_x(a, b)$ denotes the incomplete beta function.

The following lemma is an identity involving the gamma function.

**Lemma 2.17** Suppose that $n$ is a nonnegative integer. Then
\[
\sqrt{\pi} + \sum_{k=1}^{n} \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k + 1)} = \frac{2\Gamma(n + \frac{3}{2})}{\Gamma(n + 1)}.
\]  

The following two lemmas are identities involving the incomplete beta function.

**Lemma 2.18** Suppose that $0 \leq r \leq 1$. Then
\[
B_{1-r^2}(1, \frac{1}{2}) = 2(1 - r).
\]  

**Lemma 2.19** Suppose that $0 \leq r \leq 1$. Then
\[
B_{1-r^2}(\frac{1}{2}, \frac{1}{2}) = 2 \arccos(r).
\]  

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2.6.1 Bessel functions

The primary analytical tool of this subsection is Theorem 2.25. The following lemmas, 2.20, 2.21, 2.22, 2.23, 2.24 describe the limiting behavior of certain integrals involving Bessel functions.

Lemma 2.20 Suppose that $\nu > 0$. Then

$$\int_0^1 (J_\nu(2cr))^2 \frac{1}{r} dr = \frac{1}{2\nu} + O\left(\frac{1}{c}\right),$$  \hspace{1cm} (72)

as $c \to \infty$.

Lemma 2.21 Suppose that $\nu > 0$. Then

$$\int_0^1 (J_\nu(2cr))^2 dr = \frac{1}{2\pi} \log\left(\frac{c}{\log(c)}\right),$$  \hspace{1cm} (73)

as $c \to \infty$.

Lemma 2.22 Suppose that $\nu > 0$ is real and $k$ is a positive integer. Then

$$\int_0^1 (J_\nu(2cr))^2 r^k dr = O\left(\frac{1}{c}\right),$$  \hspace{1cm} (74)

as $c \to \infty$.

Lemma 2.23 Suppose that $n$ is a positive integer. Then

$$\int_0^1 \frac{(J_n(2cr))^2}{r} \arccos(r) dr = \frac{\pi}{4n} - \frac{1}{2\pi} \frac{\log(c)}{c} + o\left(\frac{\log(c)}{c}\right),$$  \hspace{1cm} (75)

as $c \to \infty$.

Lemma 2.24 Suppose that $n$ and $k$ are positive integers. Then

$$\int_0^1 (J_n(2cr))^2 (1 - r^2)^{k-\frac{1}{2}} dr = \frac{1}{2\pi} \frac{\log(c)}{c} + o\left(\frac{\log(c)}{c}\right),$$  \hspace{1cm} (76)

as $c \to \infty$.

The following theorem describes the limiting behavior of a certain integral involving a Bessel function and the incomplete beta function.

Theorem 2.25 Suppose that $p \geq -1$ is an integer. Then

$$\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2}(\frac{p}{2} + \frac{3}{2}, \frac{1}{2}) dr = \frac{\sqrt{\pi} \Gamma\left(\frac{p}{2} + \frac{3}{2}\right)}{(p+2) \Gamma\left(\frac{p}{2} + 2\right)} - \frac{1}{\pi} \frac{\log(c)}{c} + o\left(\frac{\log(c)}{c}\right)$$  \hspace{1cm} (77)

as $c \to \infty$, where $B_x(a,b)$ denotes the incomplete beta function.
\textbf{Proof.} Suppose that }p \geq -1\text{ is an odd integer, and let } n = \frac{p}{2} + \frac{1}{2}.\text{ Then}

\[
\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2} \left( \frac{p}{2} + \frac{3}{2}, \frac{1}{2} \right) \, dr = \int_0^1 \frac{(J_{n+1/2}(2cr))^2}{r} B_{1-r^2} \left( 1 + n, \frac{1}{2} \right) \, dr. \tag{78}
\]

By Theorem 2.16 and Lemma 2.18, we observe that

\[
\int_0^1 \frac{(J_{n+1/2}(2cr))^2}{r} B_{1-r^2} \left( 1 + n, \frac{1}{2} \right) \, dr
= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \int_0^1 \frac{(J_{n+1/2}(2cr))^2}{r} \left( \frac{\sqrt{\pi}}{2} B_{1-r^2} \left( 1, \frac{1}{2} \right) - r \sum_{k=1}^n \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k+1)} (1 - r^2)^k \right) \, dr
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \int_0^1 \frac{(J_{n+1/2}(2cr))^2}{r} \left( \sqrt{\pi} (1 - r) - r \sum_{k=1}^n \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k+1)} (1 - r^2)^k \right) \, dr, \tag{79}
\]

where }0 \leq r \leq 1\text{ and } n \text{ is a nonnegative integer. By lemmas 2.20, 2.21 and 2.22 it follows that

\[
\int_0^1 \frac{(J_{n+1/2}(2cr))^2}{r} B_{1-r^2} \left( 1 + n, \frac{1}{2} \right) \, dr
= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \left( \frac{\sqrt{\pi}}{2n+1} - \frac{1}{2\pi} \left( \sqrt{\pi} + \sum_{k=1}^n \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k+1)} \log(c) \right) + o \left( \frac{\log(c)}{c} \right) \right), \tag{80}
\]

as }c \to \infty\text{, where }0 \leq r \leq 1\text{ and } n \text{ is a nonnegative integer. Applying Lemma 2.17,}

\[
\int_0^1 \frac{(J_{n+1/2}(2cr))^2}{r} B_{1-r^2} \left( 1 + n, \frac{1}{2} \right) \, dr
= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \left( \frac{\sqrt{\pi}}{2n+1} - \frac{1}{\pi} \frac{\Gamma(n + \frac{3}{2}) \log(c)}{c} \right) + o \left( \frac{\log(c)}{c} \right)
\]

\[
= \frac{\sqrt{\pi} \Gamma(n+1)}{2(n+\frac{3}{2})\Gamma(n+\frac{3}{2})} - \frac{1}{\pi} \log(c) + o \left( \frac{\log(c)}{c} \right), \tag{81}
\]

as }c \to \infty\text{, where }0 \leq r \leq 1\text{ and } n \text{ is a nonnegative integer. Therefore,

\[
\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2} \left( \frac{p}{2} + \frac{3}{2}, \frac{1}{2} \right) \, dr = \frac{\sqrt{\pi} \Gamma(\frac{p}{2} + \frac{3}{2})}{(p+2)\Gamma(\frac{p}{2} + 2)} - \frac{1}{\pi} \log(c) \frac{\log(c)}{c} + o \left( \frac{\log(c)}{c} \right), \tag{82}
\]

as }c \to \infty\text{, for all }0 \leq r \leq 1\text{ and odd integers } p \geq -1.\text{ The proof in the case when } p \geq 0\text{ is an even integer is essentially identical.}

\section{2.6.2 The area and volume of a hypersphere}

The following theorem provides well-known formulas for the volume and area of a \((p+2)\)-dimensional hypersphere. The formulas can be found in, for example, [8].
Theorem 2.26 Suppose that $S^{p+2}(r) = \{x \in \mathbb{R}^{p+2} : ||x|| = r\}$ denotes the $(p + 2)$-dimensional hypersphere of radius $r > 0$. Suppose further that $A_{p+2}(r)$ denotes the area of $S^{p+2}(r)$ and $V_{p+2}(r)$ denotes the volume enclosed by $S^{p+2}(r)$. Then

$$A_{p+2}(r) = \frac{2\pi^{p/2+1}}{\Gamma(\frac{p}{2} + 1)} r^{p+1},$$

and

$$V_{p+2}(r) = \frac{\pi^{p/2+1}}{\Gamma(\frac{p}{2} + 2)} r^{p+2}.$$ 

Theorem 2.27 Suppose that $p \geq -1$ is an integer, let $B$ denote the closed unit ball in $\mathbb{R}^{p+2}$, and let $B(c)$ denote the set $\{x \in \mathbb{R}^{p+2} : ||x|| \leq c\}$, where $c > 0$. Then

$$\int_{\mathbb{R}^D} 1_B(u - t)1_B(t) \, dt = V_{p+2}(1) \frac{B_{1-||u||^2/4}(\frac{p}{2} + \frac{3}{2}, \frac{1}{2})}{B(\frac{p}{2} + \frac{3}{2}, \frac{1}{2})},$$

for all $u \in B(2)$, where $B(a,b)$ denotes the beta function, $B_x(a,b)$ denotes the incomplete beta function, $V_{p+2}$ is defined by (84), and $1_A$ is defined via the formula

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

2.7 Spherical harmonics in $\mathbb{R}^{p+2}$

Suppose that $S^{p+1}$ denotes the unit sphere in $\mathbb{R}^{p+2}$. Spherical harmonics are a set of real-valued continuous functions on $S^{p+1}$, which are orthonormal and complete in $L^2(S^{p+1})$. The spherical harmonics of degree $N \geq 0$ are denoted by $S_1^1, S_2^1, \ldots, S_N^1, \ldots, S_N^{h(N)} : S^{p+1} \to \mathbb{R}$, where

$$h(N) = (2N + p)(N + p - 1)! \frac{p! \, N!}{},$$

for all nonnegative integers $N$.

The following theorem defines the spherical harmonics as the values of certain harmonic, homogeneous polynomials on the sphere (see, for example, [8]).

Theorem 2.28 For each spherical harmonic $S_\ell^N$, where $N \geq 0$ and $1 \leq \ell \leq h(N)$ are integers, there exists a polynomial $K_\ell^N : \mathbb{R}^{p+2} \to \mathbb{R}$ which is harmonic, i.e.

$$\nabla^2 K_\ell^N(x) = 0,$$

for all $x \in \mathbb{R}^{p+2}$, and homogenous of degree $N$, i.e.

$$K_\ell^N(\lambda x) = \lambda^N K_\ell^N(x),$$

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for all $x \in \mathbb{R}^{p+2}$ and $\lambda \in \mathbb{R}$, such that
\[ S_N^\ell(\xi) = K_N^\ell(\xi), \tag{90} \]
for all $\xi \in S^{p+1}$.

The following lemma follows immediately from the orthonormality of spherical harmonics and Theorem 2.28.

**Lemma 2.29** For all $N > 0$ and for all $1 \leq \ell \leq h(N)$,
\[ \int_{S^{p+1}} S_N^\ell(x) dx = 0. \tag{91} \]

For $N = 0$ and $\ell = 1$, $S_N^1$ is the constant function defined by the formula
\[ S_0^1(x) = A_{p+2}(1)^{(-1/2)} \tag{92} \]
where $A_{p+2}$ is defined in (83).

The following theorem is proved in, for example, [2].

**Theorem 2.30** Suppose that $N$ is a nonnegative integer. Then there are exactly
\[ (2N + p)(N + p - 1)! \over p! N! \tag{93} \]
linearly independent, harmonic, homogenous polynomials of degree $N$ in $\mathbb{R}^{p+2}$.

The following theorem states that for any orthogonal matrix $U$, the function $S_N^\ell(U\xi)$ is expressible as a linear combination of $S_N^1(\xi), S_N^2(\xi), \ldots, S_N^{h(N)}(\xi)$ (see, for example, [2]).

**Theorem 2.31** Suppose that $N$ is a nonnegative integer, and that
\[ S_N^1, S_N^2, \ldots, S_N^{h(N)} : S^{p+1} \to \mathbb{R} \tag{94} \]
are a complete set of orthonormal spherical harmonics of degree $N$. Suppose further that $U$ is a real orthogonal matrix of dimension $p + 2 \times p + 2$. Then, for each integer $1 \leq \ell \leq h(N)$, there exist real numbers $v_{\ell,1}, v_{\ell,2}, \ldots, v_{\ell,h(N)}$ such that
\[ S_N^\ell(U\xi) = \sum_{k=1}^{h(N)} v_{\ell,k} S_N^k(\xi), \tag{95} \]
for all $\xi \in S^{p+1}$. Furthermore, if $V$ is the $h(N) \times h(N)$ real matrix with elements $v_{i,j}$ for all $1 \leq i, j \leq h(N)$, then $V$ is also orthogonal.

**Remark 2.7** From Theorem 2.31, we observe that the space of linear combinations of functions $S_N^\ell$ is invariant under all rotations and reflections of $S^{p+1}$. 

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The following theorem states that if an integral operator acting on the space of functions \( S^{p+1} \rightarrow \mathbb{R} \) has a kernel depending only on the inner product, then spherical harmonics are eigenfunctions of that operator (see, for example, [2]).

**Theorem 2.32 (Funk-Hecke)** Suppose that \( F : [-1, 1] \rightarrow \mathbb{R} \) is a continuous function, and that \( S_N : S^{p+1} \rightarrow \mathbb{R} \) is any spherical harmonic of degree \( N \). Then

\[
\int_{\Omega} F(\langle \xi, \eta \rangle) S_N(\xi) \, d\Omega(\xi) = \lambda_N S_N(\eta),
\]

(96)

for all \( \eta \in S^{p+1} \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^{p+2} \), the integral is taken over the whole area of the hypersphere \( \Omega \), and \( \lambda_N \) depends only on the function \( F \).

### 2.8 Generalized prolate spheroidal functions

#### 2.8.1 Basic facts

In this section, we summarize several facts about generalized prolate spheroidal functions (GPSFs). Let \( B \) denote the closed unit ball in \( \mathbb{R}^{p+2} \). Given a real number \( c > 0 \), we define the operator \( F_c : L^2(B) \rightarrow L^2(B) \) via the formula

\[
F_c[\psi](x) = \int_B \psi(t)e^{ic\langle x, t \rangle} \, dt,
\]

(97)

for all \( x \in B \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( \mathbb{R}^{p+2} \). Clearly, \( F_c \) is compact. Obviously, \( F_c \) is also normal, but not self-adjoint. We denote the eigenvalues of \( F_c \) by \( \lambda_0, \lambda_1, \ldots, \lambda_n, \ldots \), and assume that \( |\lambda_j| \geq |\lambda_{j+1}| \) for each non-negative integer \( j \). For each non-negative integer \( j \), we denote by \( \psi_j \) the eigenfunction corresponding to \( \lambda_j \), so that

\[
\lambda_j \psi_j(x) = \int_B \psi_j(t)e^{ic\langle x, t \rangle} \, dt,
\]

(98)

for all \( x \in B \). We assume that \( \|\psi_j\|_{L^2(B)} = 1 \) for each \( j \). The following theorem is proved in [12] and describes the eigenfunctions and eigenvalues of \( F_c \).

**Theorem 2.33** Suppose that \( c > 0 \) is a real number and that \( F_c \) is defined by (97). Then the eigenfunctions \( \psi_0, \psi_1, \ldots, \psi_n, \ldots \) of \( F_c \) are real, orthonormal, and complete in \( L^2(B) \). For each \( j \), the eigenfunction \( \psi_j \) is either even, in the sense that \( \psi_j(-x) = \psi_j(x) \) for all \( x \in B \), or odd, in the sense that \( \psi_j(-x) = -\psi_j(x) \) for all \( x \in B \). The eigenvalues corresponding to even eigenfunctions are real, and the eigenvalues corresponding to odd eigenfunctions are purely imaginary. The domain on which the eigenfunctions are defined can be extended from \( B \) to \( \mathbb{R}^{p+2} \) by requiring that (98) hold for all \( x \in \mathbb{R}^{p+2} \); the eigenfunctions will then be orthogonal on \( \mathbb{R}^{p+2} \) and complete in the class of band-limited functions with bandlimit \( c \).

We define the self-adjoint operator \( Q_c : L^2(B) \rightarrow L^2(B) \) via the formula

\[
Q_c = \left( \frac{c}{2\pi} \right)^{p+2} F_c^* \cdot F_c.
\]

(99)
Since $F_c$ is normal, it follows that $Q_c$ has the same eigenfunctions as $F_c$, and that the $j$th eigenvalue $\mu_j$ of $Q_c$ is connected to $\lambda_j$ via the formula
\[
\mu_j = \left(\frac{c}{2\pi}\right)^{p+2} |\lambda_j|^2.
\]  
(100)

We also observe that
\[
Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_B J_{p/2+1}(c\|x-t\|) \psi(t) \, dt,
\]  
(101)
for all $x \in \mathbb{R}^{p+2}$, where $J_\nu$ denotes the Bessel functions of the first kind and $\| \cdot \|$ denotes Euclidean distance in $\mathbb{R}^{p+2}$ (see Appendix A for a proof).

We observe that
\[
Q_c[\psi](x) = 1_B(x) \cdot F^{-1} \left[ 1_{B(c)}(t) \cdot F[\psi](t) \right](x),
\]  
(102)
where $F: L^2(\mathbb{R}^{p+2}) \to L^2(\mathbb{R}^{p+2})$ is the $(p + 2)$-dimensional Fourier transform, $B(c)$ denotes the set \{ $x \in \mathbb{R}^{p+2}$ : $\|x\| \leq c$ \}, and $1_A$ is defined via the formula
\[
1_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \not\in A.
\end{cases}
\]  
(103)
From (102) it follows that $\mu_j < 1$ for all $j$.

We observe further that $Q_c$ is closely related to the operator $P_c: L^2(\mathbb{R}^{p+2}) \to L^2(\mathbb{R}^{p+2})$, defined via the formula
\[
P_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_{\mathbb{R}^{p+2}} J_{p/2+1}(c\|x-t\|) \psi(t) \, dt,
\]  
(104)
which is the orthogonal projection onto the space of bandlimited functions on $\mathbb{R}^{p+2}$ with bandlimit $c > 0$.

2.8.2 Eigenfunctions and eigenvalues of $F_c$

In this section we describe the eigenvectors and eigenvalues of the operator $F_c$, defined in (97). Suppose that $\psi$ is some eigenfunction of the integral operator $F_c$, with corresponding complex eigenvalue $\lambda$, so that
\[
\lambda \psi(x) = \int_B \psi(t) e^{ic \langle x, t \rangle} \, dt,
\]  
(105)
for all $x \in B$ (see Theorem 2.33).

**Observation 2.8** The operator $F_c$, defined by (97), is spherically symmetric in the sense that, for any $(p+2) \times (p+2)$ orthogonal matrix $U$, $F_c$ commutes with the operator $\hat{U}: L^2(B) \to L^2(B)$, defined via the formula
\[
\hat{U}[\psi](x) = \psi(Ux),
\]  
(106)
for all $x \in B$. Hence, the problem of finding the eigenfunctions and eigenvalues of $F_c$ is amenable to the separation of variables.
Suppose that
\[
\psi(x) = \Phi_N^\ell(\|x\|)S_N^\ell(x/\|x\|),
\]
where \( S_N^\ell, \ell = 0, 1, \ldots, h(N,p) \) denotes the spherical harmonics of degree \( N \) (see Section 2.7), and \( \Phi_N^\ell(r) \) is a real-valued function defined on the interval \([0,1]\). We observe that
\[
e^{ic(x,t)} = \sum_{N=0}^{\infty} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|)S_N^\ell(t/\|t\|),
\]
where \( x, t \in B \), and where \( J_\ell \) denotes the Bessel functions of the first kind (see Section VII of [12] for a proof). Substituting (107) and (108) into (105), we find that
\[
\lambda \Phi_N^\ell(r) = i^N (2\pi)^{p/2+1} \int_0^1 \frac{J_{N+p/2}(c\rho)}{(c\rho)^{p/2}} \Phi_N^\ell(\rho) \rho^{p+1} d\rho,
\]
for all \( 0 \leq r \leq 1 \). We define the operator \( H_{N,c} : L^2([0,1], \rho^{p+1} d\rho) \rightarrow L^2([0,1], \rho^{p+1} d\rho) \) via the formula
\[
H_{N,c}[\Phi](r) = \int_0^1 \frac{J_{N+p/2}(c\rho)}{(c\rho)^{p/2}} \Phi(\rho) \rho^{p+1} d\rho,
\]
where \( 0 \leq r \leq 1 \), and observe that \( H_{N,c} \) is clearly compact and self-adjoint, and does not depend on \( \ell \). Dropping the index \( \ell \), we denote by \( \beta_{N,0}, \beta_{N,1}, \ldots, \beta_{N,n}, \ldots \) the eigenvalues of \( H_{N,c} \), and assume that \( |\beta_{N,n}| \geq |\beta_{N,n+1}| \) for each nonnegative integer \( n \). For each nonnegative integer \( n \), we let \( \Phi_{N,n} \) denote the eigenvector corresponding to eigenvalue \( \beta_{N,n} \), so that
\[
\beta_{N,n} \Phi_{N,n}(r) = \int_0^1 \frac{J_{N+p/2}(c\rho)}{(c\rho)^{p/2}} \Phi_{N,n}(\rho) \rho^{p+1} d\rho,
\]
for all \( 0 \leq r \leq 1 \). Clearly, the eigenfunctions \( \Phi_{N,n} \) are purely real. We assume that \( \|\Phi_{N,n}\|_{L^2([0,1], \rho^{p+1} d\rho)} = 1 \) and that \( \Phi_{N,n}(1) > 0 \) for each nonnegative integer \( N \) and \( n \) (see Theorem 9.6). It follows from (111) and (109) that the eigenvectors and eigenvalues of \( F_c \) are given by the formulas
\[
\psi_{N,n}^\ell(x) = \Phi_{N,n}(\|x\|)S_N^\ell(x/\|x\|),
\]
and
\[
\lambda_{N,n}^\ell = i^N (2\pi)^{p/2+1} \beta_{N,n},
\]
respectively, where \( x \in B, N \) and \( n \) are nonnegative integers, and \( \ell \) is an integer so that \( 1 \leq \ell \leq h(N,p) \) (see Section 2.7). We note in formula (113) the expected degeneracy of eigenvalues due to the spherical symmetry of the integral operator \( F_c \) (see Observation 2.8); we denote \( \lambda_{N,n}^\ell \) by \( \lambda_{N,n} \) where the meaning is clear.

**Observation 2.9** The domain on which the functions \( \Phi_{N,n} \) are defined may be extended from the interval \([0,1] \) to the complex plane \( \mathbb{C} \) by requiring that (105) hold for all \( r \in \mathbb{C} \). Moreover, the functions \( \Phi_{N,n} \), extended in this way, are entire.
2.8.3 The dual nature of GPSFs

In this section, we observe that the eigenfunctions $\Phi_{N,0}, \Phi_{N,1}, \ldots, \Phi_{N,n}, \ldots$ of the integral operator $H_{N,c}$, defined in (110), are also the eigenfunctions of a certain differential operator. Let $\beta_{N,n}$ denote the eigenvalue corresponding to the eigenfunction $\Phi_{N,n}$, for all nonnegative integers $N$ and $n$, so that

$$\beta_{N,n} \Phi_{N,n}(r) = \int_0^1 J_{N+p/2}(cr\rho) \Phi_{N,n}(\rho) \rho^{p+1} d\rho,$$

where $0 \leq r \leq 1$, $N$ and $n$ are nonnegative integers, and $J_{\nu}$ denotes the Bessel functions of the first kind (see (111)). Making the substitutions

$$\varphi_{N,n}(r) = r^{(p+1)/2} \Phi_{N,n}(r),$$

and

$$\gamma_{N,n} = c^{(p+1)/2} \beta_{N,n},$$

we observe that

$$\gamma_{N,n} \varphi_{N,n}(r) = \int_0^1 J_{N+p/2}(cr\rho) \sqrt{cr\rho} \varphi_{N,n}(\rho) d\rho,$$

where $0 \leq r \leq 1$, and $N$ and $n$ are arbitrary nonnegative integers. We define the operator $M_{N,c}: L^2([0,1]) \to L^2([0,1])$ via the formula

$$M_{N,c}[\varphi](r) = \int_0^1 J_{N+p/2}(cr\rho) \sqrt{cr\rho} \varphi(\rho) d\rho,$$

where $0 \leq r \leq 1$, and $N$ is an arbitrary nonnegative integer. Obviously, $M_{N,c}$ is compact and self-adjoint. Clearly, the eigenvalues of $M_{N,c}$ are $\gamma_{N,0}, \gamma_{N,1}, \ldots, \gamma_{N,n}, \ldots$, and $\varphi_{N,n}$ is the eigenfunction corresponding to eigenvalue $\gamma_{N,n}$, for each nonnegative integer $n$.

We define the differential operator $L_{N,c}$ via the formula

$$L_{N,c}[\varphi](x) = \frac{d}{dx} \left( (1 - x^2) \frac{d\varphi}{dx}(x) \right) + \left( \frac{1}{x} - \frac{N + p}{x^2} - c^2 x^2 \right) \varphi(x),$$

where $0 < x < 1$, $N$ is a nonnegative integer, and $\varphi$ is twice continuously differentiable. Let $C$ be the class of functions $\varphi$ which are bounded and twice continuously differentiable on the interval $(0,1)$, such that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise. Then it is easy to show that, operating on functions in class $C$, $L_{N,c}$ is self-adjoint. From Sturmian theory we obtain the following theorem (see [12]).

**Theorem 2.34** Suppose that $c > 0$, $N$ is a nonnegative integer, and $L_{N,c}$ is defined via (119). Then there exists a strictly increasing unbounded sequence of positive numbers $\chi_{N,0} < \chi_{N,1} < \ldots$ such that for each nonnegative integer $n$, the differential equation

$$L_{N,c}[\varphi](x) + \chi_{N,n} \varphi(x) = 0$$

has a solution which is bounded and twice continuously differentiable on the interval $(0,1)$, so that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise.
The following theorem is proved in [12].

**Theorem 2.35** Suppose that $c > 0$, $N$ is a nonnegative integer, and the operators $M_{N,c}$ and $L_{N,c}$ are defined via (118) and (119) respectively. Suppose also that $\varphi: (0,1) \to \mathbb{R}$ is in $L^2([0,1])$, is twice differentiable, and that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise. Then

$$L_{N,c}[M_{N,c}[\varphi]](x) = M_{N,c}[L_{N,c}[\varphi]](x),$$

for all $0 < x < 1$.

**Remark 2.10** Since Theorem 2.34 shows that the eigenvalues of $L_{N,c}$ are not degenerate, Theorem 2.35 implies that $L_{N,c}$ and $M_{N,c}$ have the same eigenfunctions.

### 2.8.4 Zernike polynomials and GPSFs

In this section we describe the relationship between Zernike polynomials and GPSFs. We use $\varphi_{N,n}^c$, where $c > 0$ and $N$ and $n$ are arbitrary nonnegative integers, to denote the $n$th eigenfunction of $L_{N,c}$, defined in (119); we denote by $\chi_{N,n}(c)$ the eigenvalue corresponding to eigenfunction $\varphi_{N,n}^c$.

For $c = 0$, the eigenfunctions and eigenvalues of the differential operator $L_{N,c}$, defined in (119), are

$$\overline{T}_{N,n}(r)$$

and

$$\chi_{N,n}(0) = (N + \frac{p}{2} + 2n + \frac{1}{2})(N + \frac{p}{2} + 2n + \frac{3}{2})$$

respectively, where $0 \leq r \leq 1$, $N$ and $n$ are arbitrary nonnegative integers, and $\overline{T}_{N,n}$ is defined in (34).

For small $c > 0$, the connection between Zernike polynomials and GPSFs is given by the formulas

$$\varphi_{N,n}^c(r) = \overline{T}_{N,n}(r) + o(c^2),$$

and

$$\chi_{N,n}(c) = \chi_{N,n}(0) + o(c^2),$$

as $c \to 0$, where $0 \leq r \leq 1$ and $N$ and $n$ are arbitrary nonnegative integers (see [12]).

For $c > 0$, the functions $T_{N,n}$ are also related to the integral operator $M_{N,c}$, defined in (118), via the formula

$$M_{N,c}[T_{N,n}](x) = \int_0^1 J_{N+p/2}(cxy) \sqrt{cxy} T_{N,n}(y) dy = \frac{(-1)^n J_{N+p/2+2n+1}(cx)}{\sqrt{cx}},$$

where $x \geq 0$ and $N$ and $n$ are arbitrary nonnegative integers (see Equation (85) in [6]).
3 Analytical apparatus

In this section, we provide analytical apparatus relating to GPSFs that will be used in numerical schemes in subsequent sections.

3.1 Properties of GPSFs

The following theorem provides a formula for the ratios of eigenvalues $\beta_{N,n}$ (see (111)), and is used in the numerical evaluation of $\beta_{N,n}$. A proof follows immediately from Theorem 7.1 of [10].

**Theorem 3.1** Suppose that $N$ is a nonnegative integer. Then

$$\frac{\beta_{N,m}}{\beta_{N,n}} = \int_0^1 x\Phi_{N,n}'(x)\Phi_{N,m}(x)x^{p+1} \, dx \int_0^1 x\Phi_{N,m}'(x)\Phi_{N,n}(x)x^{p+1} \, dx,$$

(127)

for each nonnegative integers $n$ and $m$.

3.2 Decay of the expansion coefficients of GPSFs in Zernike polynomials

Since the functions $\Phi_{N,n}$ are analytic on $\mathbb{C}$ for all nonnegative integers $N$ and $n$ (see Observation 2.9), and $\Phi_{N,n}^{(k)}(0) = 0$ for $k = 0, 1, \ldots, N - 1$ (see Theorem 9.5), the functions $\Phi_{N,n}$ are representable by a series of Zernike polynomials of the form

$$\Phi_{N,n}(r) = \sum_{k=0}^{\infty} a_{n,k} R_{N,k}(r),$$

(128)

for all $0 \leq r \leq 1$, where $a_{n,0}, a_{n,1}, \ldots$ satisfy

$$a_{n,k} = \int_0^1 R_{N,k}(r)\Phi_{N,n}(r)dr$$

(129)

where $R_{N,n}$ is defined in (23). The following technical lemma will be used in the proof of Theorem 3.3.

**Lemma 3.2** For any integer $p \geq -1$, for all $c > 0$, and for all $\rho \in [0, 1],

$$\left| \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} R_{N,k}(r)r^{p+1} \, dr \right| < \left( \frac{1}{2} \right)^{N+p/2+2k+1}$$

(130)

for any non-negative integers $N, k$ such that $N + 2k \geq ec$ where $R_{N,n}$ is defined in (23) and $J_{N+p/2}$ is a Bessel function of the first kind.
Proof. According to equation (85) in [6],
\[ \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} R_{N,k}(r)^{p+1}dr = \frac{(-1)^n J_{N+p/2+2k+1}(cp)}{(cp)^{p/2+1}}, \]  
where \( J_{N+p/2} \) is a Bessel function of the first kind. Applying Lemma 2.2 to (131), we obtain
\[ \left| \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} R_{N,k}(r)^{p+1}dr \right| \leq \frac{(cp/2)^{N+p/2+2k+1}}{(cp)^{p/2+1}} \sqrt{2(N + p/2 + 2k + 1)} \cdot \frac{\sqrt{2(N + p/2 + 2k + 1)}}{\Gamma(N + p/2 + 2k + 2)}. \]  
Combining Lemma 2.1 and (132), we have
\[ \left| \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} R_{N,k}(r)^{p+1}dr \right| \leq \frac{1}{2} \frac{N+p/2+2k+1}{(p/2)^{N/2+2k+1}} \frac{\sqrt{2(N + p/2 + 2k + 1)}}{\Gamma(2k + N + 1)}. \]  
for \( N + 2k \geq ec \).  

The following theorem shows that the coefficients \( a_{N,k} \) of GPSFs in a Zernike polynomial basis decay exponentially and establishes a bound for the decay rate.

**Theorem 3.3** For all non-negative integers \( N, n, k \) and for all \( c > 0 \),
\[ \int_0^1 \Phi_{N,n}(r) R_{N,k}(r)^{p+1}dr < (p + 2)^{-1/2}(\beta_{N,n})^{-1} \left( \frac{1}{2} \right)^{N+p/2+2k+1} \]  
where \( N + 2k \geq ec \).

**Proof.** Combining (114) and (129), we have
\[ \int_0^1 \Phi_{N,n}(r) R_{N,k}(r)^{p+1}dr = \int_0^1 (\beta_{N,n})^{-1} \left( \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_{N,n}(\rho)^{p+1}d\rho \right) R_{N,k}(r)^{p+1}dr. \]  
Changing the order of integration of (135),
\[ \int_0^1 \Phi_{N,n}(r) R_{N,k}(r)^{p+1}dr = (\beta_{N,n})^{-1} \int_0^1 \Phi_{N,n}(\rho)^{p+1} \left( \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} R_{N,k}(r)^{p+1}dr d\rho \right). \]  
Applying Lemma 3.2 to (136) and applying Cauchy-Schwarz, we obtain
\[ \int_0^1 \Phi_{N,n}(r) R_{N,k}(r)^{p+1}dr \leq (\beta_{N,n})^{-1} \left( \frac{1}{2} \right)^{N+p/2+2k+1} \int_0^1 \Phi_{N,n}(\rho)^{p+1}d\rho \right) \left( \int_0^1 \Phi_{N,n}(r)^{p+1}dr \right) \]  
for \( N + 2k \geq ec \).  

\[ \Box \]
3.3 Tridiagonal nature of $L_{N,c}$

In this section, we show that in the basis of $T_{N,n}$ (see (34)), the matrix representing differential operator $L_{N,c}$ (see (119)) is symmetric and tridiagonal.

The following lemma provides an identity relating the differential operator $L_{N,c}$ to $T_{N,n}$.

**Lemma 3.4** For all non-negative integers $N, n$ and real numbers $c > 0$

$$L_{N,c}[T_{N,n}] = -\chi_{N,n}T_{N,n}(x) - c^2 x^2 T_{N,n}(x)$$

(138)

for all $x \in [0, 1]$ where $\chi_{N,n}$ is defined in (40) and $L_{N,c}$ is defined in (119).

**Proof.** Applying $L_{N,c}$ to $T_{N,n}$, we obtain

$$L_{N,c}[T_{N,n}](x) = (1 - x^2)T_{N,n}''(x) - 2x T_{N,n}'(x) + \left(\frac{1}{4} - \frac{(N + \frac{p}{2})^2}{x^2} - c^2 x^2\right)T_{N,n}(x).$$

(139)

Identity (138) follows immediately from the combination of (39) and (139).

The following theorem follows readily from the combination of Lemma 3.4 and Lemma 2.12.

**Theorem 3.5** For any non-negative integer $N$, any integer $n \geq 1$, and for all $r \in (0, 1)$,

$$L_{N,c}[T_{N,n}] = a_n T_{N,n-1}(r) + b_n T_{N,n}(r) + c_n T_{N,n+1}(r)$$

(140)

where

$$a_n = \frac{-c^2(n + N + p/2)n}{(2n + N + p/2)\sqrt{2n + N + p/2} + 1}\sqrt{2n + N + p/2 - 1}$$

$$b_n = \frac{-c^2(N + p/2)^2}{2(2n + N + p/2)(2n + N + p/2 + 2)} - \frac{c^2}{2} + \chi_{N,n}$$

(141)

$$c_n = \frac{-c^2(n + 1 + N + p/2)(n + 1)}{(2n + N + p/2 + 2)\sqrt{2n + N + p/2} + 3}\sqrt{2n + N + p/2 + 1}$$

and $\chi_{N,n}$ is defined in (40).

**Observation 3.1** It follows immediately from Theorem 3.5 that the matrix corresponding to the differential operator $L_{N,c}$ acting on the $T_{N,n}$ basis is symmetric and tridiagonal. Specifically, for any positive integer $n$ and for all $r \in (0, 1)$,

$$
\begin{bmatrix}
  b_0 & c_0 & 0 \\
  c_0 & b_1 & c_1 \\
  c_1 & b_2 & c_2 \\
  \vdots & \ldots & \ldots \\
  c_{n-2} & b_{n-1} & c_{n-1} \\
  0 & c_{n-1} & b_n
\end{bmatrix}
\begin{bmatrix}
  T_{N,0}(r) \\
  \vdots \\
  T_{N,n}(r)
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  \vdots \\
  0
\end{bmatrix}
= 
\begin{bmatrix}
  T_{N,0}(r) \\
  \vdots \\
  T_{N,n}(r)
\end{bmatrix}
$$

(142)

where $b_k$ and $c_k$ are defined in (141) and $T_{N,k}$ is defined in (34).
Observation 3.2 Let $A$ be the infinite symmetric tridiagonal matrix satisfying $A_{1,1} = b_0$, $A_{1,2} = c_0$ and for all integers $k \geq 2$,

\begin{align*}
A_{k,k-1} &= c_{k-1} \\
A_{k,k} &= b_k \\
A_{k,k+1} &= c_k,
\end{align*}

(143)

where $b_k, c_k$ are defined in (141). That is,

\[ A = \begin{bmatrix}
  b_0 & c_0 & 0 & 0 & \cdots \\
  c_0 & b_1 & c_1 & 0 & \cdots \\
  c_1 & b_2 & c_2 & 0 & \cdots \\
  & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}. \]

(144)

Suppose further that we define the infinite vector $\alpha_n$ by the equation

\[ a_n = (a_{n,0}, a_{n,1}, \ldots)^T, \]

(145)

where $a_{n,k}$ is defined in (122). By the combination of Theorem 2.35 and Remark 23, we know that $\varphi_{N,n}$ is the eigenfunction corresponding to $\chi_{N,n}(c)$, the $n$th smallest eigenvalue of differential operator $L_{N,c}$. Therefore,

\[ A\alpha_n = \chi_{N,n}(c)\alpha_n. \]

(146)

Furthermore, the $a_{n,k}$ decay exponentially fast in $k$ (see Theorem 3.3).

Remark 3.3 The eigenvalues $\chi_{N,n}$ of differential operator $L_{N,c}$ and the coefficients in the Zernike expansion of the eigenfunctions $\Phi_{N,n}$ can be computed numerically to high relative precision by the following process. First, we reduce the infinite dimensional matrix $A$ (see (144)) to $A_K$, its upper-left $K \times K$ submatrix where $K$ is chosen, using Theorem 3.3, so that $a_{n,K-1}$ is smaller than machine precision and is in the regime of exponential decay. We then use standard algorithms to find the eigenvalues and eigenvectors of matrix $A_K$. See Algorithm 4.1 for a more detailed description of the algorithm.

4 Numerical evaluation of GPSFs

In this section, we describe an algorithm (Algorithm 4.1) for the evaluation of $\Phi_{N,n}(r)$ (see (111)) for all $r \in [0,1]$.

Algorithm 4.1 (Evaluation of $\Phi_{N,n}$)

1. Use Theorem 3.3 to determine how many terms are needed in a Zernike expansion of $\Phi_{N,n}$ to achieve some desired precision. We assume that we want a $K$ term expansion.

2. Generate $A_K$, the symmetric, tri-diagonal, upper-left $K \times K$ sub-matrix of $A$ (see (144)).
3. Use an eigensolver to find the eigenvector, $\tilde{a}_n$, corresponding to the $n+1$th largest eigenvalue, $\tilde{\chi}_{N,n}$. That is, find $\tilde{a}_n$ and $\tilde{\chi}_{N,n}$ such that

$$A_K\tilde{a}_n = \tilde{\chi}_{N,n}\tilde{a}_n$$  \hspace{1cm} (147)

where we define the components of $\tilde{a}_{N,n}$ by the formula,

$$\tilde{a}_n = (a_{n,0}, a_{n,1}, \ldots, a_{n,K-1}).$$  \hspace{1cm} (148)

4. Evaluate $\Phi_{N,n}(r)$ by the expansion

$$\Phi_{N,n}(r) = \sum_{i=0}^{k} a_{n,i} \overline{R}_{N,i}(r)$$  \hspace{1cm} (149)

where, $\overline{R}_{N,i}$ is evaluated via Lemma 2.8 and $a_{n,i}$ are the components of eigenvector $\tilde{a}_n$ recovered in Step 3.

**Remark 4.1** It turns out that because of the structure of $A_K$, standard numerical algorithms will compute the components of eigenvector $\tilde{a}_n$ (see (148)), and thus the coefficients of a GPSF in a Zernike expansion, to high relative precision. In particular, the components of $\tilde{a}_n$ that are of magnitude far less than machine precision, are computed to high relative precision. For example, when using double-precision arithmetic, a component of $\tilde{a}_n$ of magnitude $10^{-100}$ will be computed in absolute precision to 116 digits. This fact is proved in a more general setting in [9].

### 4.1 Numerical evaluation of the single eigenvalue $\beta_{N,i}$

In this section, we describe a numerical method to evaluate the eigenvalue $\beta_{N,n}$ (see (111)) for fixed $n$ to high relative precision.

The following is a technical lemma will be used in the proof of Theorem 4.3.

**Lemma 4.1** For all non-negative integers $N,k$,

$$\int_0^1 r^N \varphi_{N,k}(r) r^{\frac{p+1}{2}} dr = \frac{a_{k,0}}{\sqrt{2N + p + 2}}$$  \hspace{1cm} (150)

where $\varphi_{N,k}$ is defined in (115) and $p \geq -1$ is an integer.

**Proof.** Using (20),

$$\int_0^1 r^N \varphi_{N,k}(r) r^{\frac{p+1}{2}} dr = \int_0^1 R_{N,0}(r) \varphi_{N,k}(r) r^{\frac{p+1}{2}} dr$$  \hspace{1cm} (151)

Applying (129) and (34) to (151), we obtain

$$\int_0^1 r^N \varphi_{N,k}(r) r^{\frac{p+1}{2}} dr = \frac{1}{\sqrt{2N + p + 2}} \int_0^1 T_{N,0}(r) \varphi_{N,k}(r) dr = \frac{a_{k,0}}{\sqrt{2N + p + 2}}.$$  \hspace{1cm} (152)
We will denote by \( \varphi_{N,n}^*(r) \) the function on \([0, 1]\) defined by the formula
\[
\varphi_{N,n}^*(r) = \frac{\varphi_{N,n}(r)}{r^{N+p/2}}
\]  
(153)
where \( N, n \) are non-negative integers.

The following identity will be used in the proof of Theorem 4.3.

**Lemma 4.2** For all non-negative integers \( N, k \),
\[
\varphi_{N,k}^*(0) = \sum_{i=0}^{\infty} a_{k,i} \sqrt{2(2i + N + p/2 + 1)}(-1)^i \binom{i + N + p/2}{i}.
\]  
(154)
where \( \varphi_{N,k}^* \) is defined in (153) and \( a_{k,i} \) is defined in (129).

**Proof.** Combining (153) and (46), we have
\[
\varphi_{N,k}^*(r) = \frac{\varphi_{N,k}(r)}{r^{N+p/2}} = \sum_{i=0}^{\infty} a_{k,i} \frac{T_{N,i}(r)}{r^{N+p/2}} = \sum_{i=0}^{\infty} a_{k,i} T_{N,i}^*(r)
\]  
(155)
where \( T_{N,n} \) is defined in (46) and \( T_{N,n} \) is defined in (34). Identity (154) follows immediately from applying Lemma 2.13 to (155) and setting \( r = 0 \). ■

The following theorem provides a formula that can be used to compute \( \beta_{N,n} \) (see (116)), an eigenvalue of integral operator \( H_{N,c} \) (see (110)).

**Theorem 4.3** For all non-negative integers \( N, k \),
\[
\beta_{N,k} = \frac{a_{k,0} c^N (2^{N+p/2} \Gamma(N + p/2 + 1) \sqrt{2N + p + 2})^{-1}}{\sum_{i=0}^{\infty} a_{k,i} \sqrt{2(2i + N + p/2 + 1)}(-1)^i \binom{i + N + p/2}{i}}
\]  
(156)
where \( \beta_{N,k} \) is defined in (114) and \( a_{k,i} \) are defined in (129).

**Proof.** It is well known that \( J_{N+p/2} \), a Bessel Function of the first kind, satisfies the identity
\[
J_{N+p/2} = \left( \frac{cr}{2} \right)^{N+p/2} \sum_{k=0}^{\infty} \frac{(- (cr)^2 / 4)^k}{k! \Gamma(N + p/2 + k)}
\]  
(157)
where \( \Gamma(n) \) is the gamma function. Dividing both sides of (117) by \( r^{N+(p+1)/2} \), we obtain the equation
\[
\gamma_{N,k} \varphi_{N,k}^*(r) = \int_0^1 \frac{J_{N+p/2}(cr \rho)}{r^{N+p/2}} \sqrt{cr} \varphi_{N,k}(\rho) d\rho
\]  
(158)
where \( \varphi_{N,k}^* \) is defined in (153). Setting \( r = 0 \), in (158) and subsituting in (154) and (157), we obtain
\[
\gamma_{N,k} = \int_0^1 \left( \frac{cr}{2} \right)^{N+p/2} \frac{(cr)^{1/2}}{\Gamma(N + p/2 + 1)} \varphi_{N,k}(r) dr
\]  
(159)
\[
\left( \sum_{i=0}^{\infty} a_{k,i} \sqrt{2(2i + N + p/2 + 1)}(-1)^i \binom{i + N + p/2}{i} \right)^{-1}.
\]
Equation (156) follows immediately from applying Lemma 4.1 and (116) to (159).

Remark 4.2 For any non-negative integers \( N,k \), the eigenvalue \( \beta_{N,k} \) can be evaluated stably by first using Algorithm 4.1 to compute the eigenvector \( \tilde{a}_k \) (see (148)), and then evaluating \( \beta_{N,k} \) via sum (156) where \( \tilde{a}_k \) are approximations to \( a_k \). In (156), the sum
\[
\sum_{i=0}^{\infty} a_{k,i} \sqrt{2(2i + N + p/2 + 1)}(-1)^i \left( i + N + p/2 \right)
\]
(160)
can be computed to high relative precision by truncating the sum at a point when the partial sum up to that point is a factor of machine precision larger than the next term.

4.2 Numerical evaluation of eigenvalues \( \beta_{N,0}, \beta_{N,1}, \ldots, \beta_{N,k} \)

In this section, we describe an algorithm (Algorithm 4.2) for numerically evaluating the eigenvalues \( \beta_{N,0}, \beta_{N,1}, \ldots, \beta_{N,k} \) (see (111)) for any non-negative integers \( N,k \).

In the following observation, we describe a stable numerical scheme for converting an expansion of the form
\[
\sum_{i=0}^{K} x_i r^T_{N,i}(r),
\]
(161)
for some real numbers \( x_0, \ldots, x_K \), into a an expansion in modified Zernike polynomials, \( T_{N,0}, \ldots, T_{N,K} \).

Observation 4.3 For all non-negative integers \( N,K \) and \( x_0, \ldots, x_K \in \mathbb{R} \), equation (50) can be used to stably and efficiently construct \( \alpha_0, \ldots, \alpha_K \) such that
\[
\sum_{i=0}^{K} \alpha_i T_{N,i}(r)
\]
(162)
is an accurate approximation of
\[
\sum_{i=0}^{K} x_i r^T_{N,i}(r)
\]
(163)
for all \( 0 \leq r \leq 1 \). The approximation can be constructed as follows. Fix \( \epsilon > 0 \) and let \( x_0, \ldots, x_K \) be a sequence of real numbers such that
\[
\sum_{i=K_1+1}^{K} |x_i| < \epsilon
\]
(164)
where \( 0 \leq K_1 \leq K \). Using (34) and (23), we have
\[
\sum_{i=0}^{K} x_i T_{N,n}(r) = \sum_{i=0}^{K} \alpha_i T_{N,n}(r)
\]
(165)
where \( x_0, ..., x_K \) are real numbers and \( \alpha_i \) is defined by the formula
\[
\alpha_i = x_i \sqrt{2(2i + N + p/2 + 1)}
\]  
(166)
for \( i = 0, 1, ..., K \). Scaling both sides of (160) by \( \alpha_0/\tilde{a}_1 \) and setting \( n = 1 \), we obtain
\[
\alpha_0 r T_{N,0}^I(r) - \frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} r T_{N,1}^I(r) + \frac{\alpha_0 \tilde{c}_1}{\tilde{a}_1} r T_{N,2}^I(r)
\]
\[=
\frac{\alpha_0 a_1}{a_1} T_{N,0}(r) - \frac{\alpha_0 b_1}{a_1} T_{N,1}(r) + \frac{\alpha_0 c_1}{a_1} T_{N,2}(r)
\]  
(167)
where \( a_i, b_i, c_i, \tilde{a}_i, \tilde{b}_i, \tilde{c}_i \) are defined in Lemma 2.14. Scaling (50) with setting \( n = 2 \) and adding the resulting equation to (167), we obtain
\[
\alpha_0 r T_{N,0}^I(r) + \alpha_1 r T_{N,1}^I(r) + \left( \frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} - \frac{\tilde{b}_2}{\tilde{a}_2} \left( \frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} + \alpha_1 \right) \right) r T_{N,2}^I(r)
\]
\[+
\left( \left( \frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} + \alpha_1 \right) \tilde{a}_2^{-1} \right) (\tilde{c}_2 r T_{N,3}^I(r)).
\]  
(168)
We note that the coefficients of the first two terms on the left-hand-side of (168) coincide with the coefficients of the first two terms of (163).

We continue by adding scaled versions of (50) to (168) until the expansion on the left hand side of (168) approximates (163). After \( K_1 + 1 \) steps, the resulting expansion will be accurate to approximately \( \epsilon \) precision. Specifically, at the start of step \( k \), for \( 2 \leq k \leq K_1 + 1 \), we have
\[
\alpha_0 r T_{N,0}^I(r) + \alpha_1 r T_{N,1}^I(r) + ... + \alpha_{k-1} r T_{N,k-1}(r) + c_k r T_{N,k}(r) + c_{k+1} r T_{N,k+1}(r)
\]
\[=
\alpha_0 T_{N,0}(r) + y_1 T_{N,1}(r) + ... + y_k T_{N,k}(r) + y_{k+1} T_{N,k+1}(r)
\]  
(169)
for some real numbers \( c_k, c_{k+1}, y_0, y_1, ..., y_{k+1} \). Scaling both sides of (50) and adding the resulting equation to (169), we obtain
\[
\alpha_0 r T_{N,0}^I(r) + \alpha_1 r T_{N,1}^I(r) + ... + \alpha_{k-2} r T_{N,k-2}(r) + \alpha_{k-1} r T_{N,k-1}(r)
\]
\[+
\left( \frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} (-\tilde{b}_k + x_k) \right) r T_{N,k}(r) + \left( \frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} \right) r T_{N,k+1}(r)
\]
\[=
\alpha_0 T_{N,0} + y_1 T_{N,1} + ... + \left( \frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} a_k + y_{k-1} \right) T_{N,k-1}(r)
\]
\[+
\left( \frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} (-\tilde{b}_k + y_k) \right) T_{N,k}(r) + \left( \frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} \right) T_{N,k+1}(r).
\]  
(170)
The following observation, when combined with Observation 4.3, provides a numerical scheme for evaluating integrals of the form

\[ \int_0^1 r \Phi'_N,n(r) \Phi_{N,m}(r) r^{p+1} dr. \]  

(171)

This scheme will be used in Algorithm 4.2.

**Observation 4.4** Suppose that

\[ r \Phi'_N,n(r) = \sum_{i=0}^{K} x_i \overline{R}_{N,i}(r) \]  

(172)

and

\[ \Phi_{N,m}(r) = \sum_{i=0}^{K} y_i \overline{R}_{N,i}(r). \]  

(173)

where \( x_i, y_i \) are real numbers. Then, substituting (172) and (173) into (22) and (24), we have,

\[ \int_0^1 r \Phi'_N,n(r) \Phi_{N,m}(r) r^{p+1} dr = \int_0^1 \sum_{i=0}^{K} x_i \overline{R}_{N,i}(r) \sum_{i=0}^{K} y_i \overline{R}_{N,i}(r) r^{p+1} dr = \sum_{i=0}^{K} x_i y_i. \]  

(174)

We now describe an algorithm for evaluating the eigenvalues \( \beta_{N,0}, \beta_{N,1}, ..., \beta_{N,k} \) for any non-negative integers \( N, k \).

**Algorithm 4.2 (Evaluation of eigenvalues \( \beta_{N,0}, \beta_{N,1}, ..., \beta_{N,k} \))**

1. Use Algorithm 4.1 to recover the Zernike expansions of the GPSFs

   \( \Phi_{N,0}, \Phi_{N,1}, ..., \Phi_{N,n} \).  

   (175)

2. Compute eigenvalue \( \beta_{N,0} \) (see (111)) using Remark 4.2.

3. Use Observation 4.3 to evaluate the \( \overline{R}_{N,n} \) expansion of \( r \Phi'_N,0 \) and \( r \Phi'_N,1 \).

4. Use Observation 4.4 to compute the integrals

   \[ \int_0^1 r \Phi'_N,1(r) \Phi_{N,0}(r) r^{p+1} dr \]  

   (176)

   and

   \[ \int_0^1 r \Phi'_N,0(r) \Phi_{N,1}(r) r^{p+1} dr \]  

   (177)

   where the Zernike expansions of \( \Phi_{N,0}(r), \Phi_{N,1}(r) \) were computed in Step 1 and the Zernike expansions of \( r \Phi'_N,0(r), r \Phi'_N,1(r) \) were computed in Step 3.
5. Using Theorem 3.1, evaluate $\beta_{N,1}$ using the formula

$$
\beta_{N,1} = \frac{\int_0^1 r \Phi'_{N,1}(r) \Phi_{N,0}(r)r^{p+1} dr}{\int_0^1 r \Phi'_{N,0}(r) \Phi_{N,1}(r)r^{p+1} dr}.
$$

(178)

where $\beta_{N,0}$ was obtained in Step 2 and the numerator and denominator of (178) were evaluated in Step 4.

6. Repeat Steps 3-5 $k$ times, each time computing the next eigenvalue, $\beta_{N,i+1}$ via the formula

$$
\beta_{N,i+1} = \frac{\int_0^1 r \Phi'_{N,i+1}(r) \Phi_{N,i}(r)r^{p+1} dr}{\int_0^1 r \Phi'_{N,i}(r) \Phi_{N,i+1}(r)r^{p+1} dr}.
$$

(179)

5 Quadratures for band-limited functions

In this section, we describe a quadrature scheme for bandlimited functions using nodes that are a tensor product of roots of GPSFs in the radial direction and nodes that integrate spherical harmonics in the angular direction.

The following lemma shows that a quadrature rule that accurately integrates complex exponentials, also integrates bandlimited functions accurately.

**Lemma 5.1** Let $\xi_1, \ldots, \xi_n \in B$ and $w_1, \ldots, w_n \in \mathbb{R}$ satisfy

$$
\left| \int_B e^{ic(x,t)} dt - \sum_{i=1}^{n} w_i e^{ic(x,\xi_i)} \right| < \epsilon
$$

(180)

for all $x \in B$ where $B$ denotes the unit ball in $\mathbb{R}^n$ for any non-negative integer $n$ and $\epsilon > 0$ is fixed. Then, for all $f : B \to \mathbb{C}$ such that

$$
f(x) = \int_B \sigma(t)e^{ic(x,t)} dt
$$

(181)

where $\sigma \in L^2(B)$, we have

$$
\left| \int_B f(x)dx - \sum_{i=1}^{n} w_i f(\xi_i) \right| < \epsilon \int_B |\sigma(t)|dt
$$

(182)

**Proof.** Clearly,

$$
\left| \int_B f(t)dt - \sum_{i=1}^{n} w_i f(\xi_i) \right| = \left| \int_B \int_B \sigma(t)e^{ic(x,t)} dx dt - \sum_{i=0}^{n} w_i \int_B \sigma(t)e^{ic(\xi_i,t)} dt \right|
$$

(183)

$$
= \left| \int_B \sigma(t) \left( \int_B e^{ic(x,t)} dx - \sum_{i=0}^{n} w_i e^{ic(\xi_i,t)} \right) dt \right|.
$$
Applying (180) to (183), we obtain
\[
\left| \int_B e^{i\xi(t)} dt - \sum_{i=1}^{n} w_i f(\xi_i) \right| \leq \int_B |\sigma(t)| \left| \int_B e^{i\xi(t)} dx - \sum_{i=1}^{n} w_i e^{i\xi(\xi_i,t)} \right| dt
\]
\[
< \epsilon \int_B |\sigma(t)| dt.
\]
(184)

The following technical lemma will be used in the construction of quadratures for bandlimited functions.

**Lemma 5.2** For any positive integer \(K\) and any integer \(p \geq -1\),
\[
\left| \int_B e^{i\xi(t)} dt - \int_B \sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c||x||,||t||)}{(c||x||,||t||)^{p/2}} S^\ell_N(x/||x||) S^\ell_N(t/||t||) dt \right|
\[
\leq (2\pi)^{p/2+1} \sum_{N=K+1}^{\infty} \frac{c^{2N}(1/2)^{2N+p}}{\Gamma(N+p/2 + 1)^2} \sum_{\ell=1}^{h(N,p)} |S^\ell_N(x/||x||)| \left( \int_B \left| J_{N+p/2}(c||x||,||t||) \right|^2 \right)^{1/2} \right)
\]
(185)

for all \(x \in B\) and \(c > 0\), where \(S^\ell_N\) for \(\ell = 0, 1, \ldots, h(N,p)\) denote the spherical harmonics of degree \(N\).

**Proof.** It follows immediately from (108) that for any integer \(p \geq -1\) and for all \(x \in \mathbb{R}^{p+2}\),
\[
\left| \int_B e^{i\xi(t)} dt - \int_B \sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c||x||,||t||)}{(c||x||,||t||)^{p/2}} S^\ell_N(x/||x||) S^\ell_N(t/||t||) dt \right|
\[
\leq (2\pi)^{p/2+1} \sum_{N=K+1}^{\infty} \sum_{\ell=1}^{h(N,p)} \left| S^\ell_N(x/||x||) \right| \left( \int_B \left| J_{N+p/2}(c||x||,||t||) \right|^2 \right)^{1/2} \right) dt.
\]
(186)

where \(r = ||x||\), \(B\) denotes the unit ball in \(\mathbb{R}^{p+2}\), and \(S^\ell_N\) is defined in (90). Applying Cauchy-Schwarz and Lemma 2.2 to (186) and using the fact that spherical harmonics have \(L^2\) norm of 1, we obtain,
\[
\left| \int_B e^{i\xi(t)} dt - \int_B \sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c||x||,||t||)}{(c||x||,||t||)^{p/2}} S^\ell_N(x/||x||) S^\ell_N(t/||t||) dt \right|
\[
\leq (2\pi)^{p/2+1} \sum_{N=K+1}^{\infty} \sum_{\ell=1}^{h(N,p)} \left| S^\ell_N(x/||x||) \right| \left( \int_B \left| J_{N+p/2}(c||x||,||t||) \right|^2 \right)^{1/2} \right)^2 dt.
\]
(187)

Equation (185) follows immediately from applying (84) and (87) to (187). 

\[\blacksquare\]
Remark 5.1 Lemma 5.2 shows that a complex exponential on the unit ball is well approximated by a function of the form

\[
\sum_{N=0}^{K} \sum_{\ell=1}^{\h(N,p)} i^{N} (2\pi)^{p/2+1} J_{N+p/2}(c \|x\| \|t\|) \left( \frac{c}{\|x\| \|t\|} \right)^{p/2} S_{N}^{\ell}(x/\|x\|) S_{N}^{\ell}(t/\|t\|) dt
\]

where the error of the approximation decays super-exponentially in \(K\). Furthermore, the spherical harmonics \(S_{N}^{\ell}\) integrate to 0 for all \(N \geq 1\) (see Lemma 2.29). Combining these facts, we observe that in order to integrate a complex exponential on the unit ball, it is sufficient to use a quadrature rule that is the tensor product of nodes that integrate all spherical harmonics \(S_{N}^{\ell}\) for sufficiently large \(N\) and nodes in the radial direction that integrate functions of the form

\[
\frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1}.
\]

We will show in Remark 5.2 that accurately computing functions of the form of (189) can be done using a quadrature rule that integrates GPSFs.

The following lemma shows that (189) is well represented by an expansion in GPSFs. This lemma will be used to construct quadrature nodes for integrating bandlimited functions.

Lemma 5.3 For all real numbers \(r, \rho \in (0, 1)\),

\[
\frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1} = \sum_{i=0}^{\infty} \beta_{0,i} \Phi_{0,i}(r) \Phi_{0,i}(\rho)
\]

where \(J_{p/2}\) is a Bessel function, \(\Phi_{0,n}\) is defined in (111) and \(\beta_{0,i}\) is defined in (114).

Proof. Since \(\Phi_{0,i}\) are complete in \(L^{2}[0, 1]_{r^{p+1}}\),

\[
\frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{i,j} \Phi_{0,i}(r) \Phi_{0,j}(\rho)
\]

where \(\alpha_{i,j}\) is defined by the formula

\[
\alpha_{i,j} = \int_{0}^{1} \int_{0}^{1} \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1} \Phi_{0,i}(r) \Phi_{0,j}(\rho) dr \rho^{p+1} d\rho.
\]

Changing the order of integration of (192) and substituting in (114), we obtain,

\[
\alpha_{i,j} = \int_{0}^{1} \Phi_{0,j}(r) \int_{0}^{1} \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1} \Phi_{0,i}(\rho) d\rho \rho^{p+1} dr
\]

\[
= \beta_{0,i} \int_{0}^{1} \Phi_{0,i}(r) r^{p+1} dr
\]

\[
= \delta_{i,j} \beta_{0,i}
\]

where \(\beta_{0,i}\) is defined in (114). Identity (190) follows immediately from the combination of (191) and (193). ■

The following remark shows that a quadrature rule that correctly integrates certain GPSFs also integrates certain Bessel functions.
Remark 5.2 Let $\rho_1, ..., \rho_n$ be the $n$ roots of $\Phi_{0,n}$ and $w_1, ..., w_n \in \mathbb{R}$ the $n$ weights such that

$$\int_0^1 \Phi_{0,k}(r)r^{p+1}dr = \sum_{i=0}^{n} \Phi_{0,k}(\rho_i)w_i$$

(194)

for $k = 0, 1, ..., K$. By Lemma 5.3,

$$\left| \int_0^1 \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1}dr - \sum_{i=1}^{n} \frac{J_{p/2}(cr\rho_i)}{(cr\rho_i)^{p/2}} \rho_i^{p+1}w_i \right|$$

$$= \left| \int_0^1 \left( \sum_{j=0}^{\infty} \beta_{0,j}\Phi_{0,j}(r)\Phi_{0,j}(\rho) \right) dr - \sum_{i=1}^{n} w_i \left( \sum_{j=0}^{\infty} \beta_{0,j}\Phi_{0,j}(r)\Phi_{0,j}(\rho_i) \right) \right|$$

(195)

where $\beta_{0,j}$ is defined in (114). Applying (194) to (195), we obtain

$$\left| \int_0^1 \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1}dr - \sum_{i=1}^{n} \frac{J_{p/2}(cr\rho_i)}{(cr\rho_i)^{p/2}} \rho_i^{p+1}w_i \right|$$

$$= \left| \int_0^1 \left( \sum_{j=K+1}^{\infty} \beta_{0,j}\Phi_{0,j}(r)\Phi_{0,j}(\rho) \right) dr - \sum_{i=1}^{n} w_i \left( \sum_{j=K+1}^{\infty} \beta_{0,j}\Phi_{0,j}(r)\Phi_{0,j}(\rho_i) \right) \right|$$

(196)

Clearly, as long as $\beta_{0,K+1}$ is in the regime of exponential decay, (196) is of magnitude approximately $\beta_{0,K+1}$.

We now describe a quadrature rule that correctly integrates functions of the form of (188). This quadrature rule uses nodes that are a tensor product of roots of $\Phi_{0,n}$ in the radial direction and nodes that integrate spherical harmonics in the angular direction.

Observation 5.3 Suppose that $r_1, ..., r_n \in (0, 1)$ and weights $w_1, ..., w_n \in \mathbb{R}$ satisfy

$$\int_0^1 \Phi_{0,k}(r)r^{p+1}dr = \sum_{i=1}^{n} w_i\Phi_{0,k}(r_i)$$

(197)

for $k = 0, 1, ..., K_1$. Suppose further that $x_1, ..., x_m \in S^{p+1}$ are nodes and $v_1, ..., v_m \in \mathbb{R}$ are weights such that

$$\int_{S^{p+1}} S_N^\ell(x)dx = \sum_{i=1}^{m} v_i S_N^\ell(x_i)$$

(198)

for all $N \leq K_2$ and for all $\ell \in \{1, 2, ..., h(N,p)\}$. Then it follows immediately from Remark 5.1 and Remark 5.2 that

$$\left| \int_B e^{ic(x,t)}dt - \sum_{i=0}^{m} v_i \sum_{j=1}^{n} w_j e^{ic(x,r_jx_i)} \right|$$

(199)

will be exponentially small for large enough $n,m$. Lemma 5.1 shows that quadrature (199) will also accurately integrate functions of the form

$$\int_B \sigma(t) e^{ic(x,t)}dt$$

(200)

where $\sigma$ is in $L^2(B)$. 

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Remark 5.4 A generalized Chebyshev quadrature of the form (197) can be generated by first computing the $n$ roots of $\Phi_{0,n}$ (see Section 5.1) and then solving the $n \times n$ linear system of equations

$$\int_0^1 \Phi_{0,k}(r) r^{p+1} dr = \sum_{i=1}^n w_i \Phi_{0,k}(r_i)$$

for $w_1, ..., w_n$ where $r_1, ..., r_n$ are the $n$ roots of $\Phi_{0,n}$. Section 5.2 contains a description of an algorithm for generating generalized Gaussian quadratures for GPsFs.

5.1 Roots of $\Phi_{0,n}$

In this section, we describe an algorithm for finding the roots of $\Phi_{N,n}$. These roots will be used in the design of quadratures for GPsFs.

The following lemma provides a differential equation satisfied by $\phi_{0,n}$. It will be used in the evaluation of roots of $\phi_{0,n}$ later in this section.

**Lemma 5.4** For all non-negative integers $n$,

$$\phi_{0,n}''(r) + \alpha(r) \phi_{0,n}'(r) + \beta(r) \phi_{0,n}(r) = 0,$$

where

$$\alpha(r) = \frac{-2r}{1 - r^2}$$

and

$$\beta(r) = \frac{1/4 - (N + p/2)^2}{(1 - r^2)r^2} - \frac{c^2 r^2 + \chi_{N,n}}{1 - r^2}$$

where $\phi_{0,n}$ is defined in (115) and $\chi_{N,n}$ is defined in (120).

The following lemma is obtained by applying the Prufer Transform (see Lemma 2.15) to (202).

**Lemma 5.5** For all non-negative integers $n$, real numbers $k > -1$, and $r \in (0, 1)$,

$$\frac{d\theta}{dr} = -\sqrt{\beta(r)} - \left( \frac{\beta'(r)}{4\beta(r)} + \frac{\alpha(r)}{2} \right) \sin(2\theta(r)),$$

where the function $\theta : (0, 1) \to \mathbb{R}$ is defined by the formula

$$\frac{\phi_{N,n}(r)}{\phi_{N,n}'(r)} = \sqrt{\beta(r)} \tan(\theta(r)),$$

and $\beta'(r)$, the derivative of $\beta(r)$ with respect to $r$, is defined by the formula

$$\beta'(r) = \frac{-2(1/4 - (N + p/2)^2)(1 - 2r^2)}{(1 - r^2)r^3} + \frac{-2rc^2(1 - r^2) + 2r(-c^2 r^2 - \chi_{N,n})}{(1 - r^2)^2}$$

and where $\alpha(r)$, $\beta(r)$ are defined in (203) and (204), $\phi_{N,n}$ is defined in (115) and $\chi_{N,n}$ is defined in (120).
Remark 5.5 For any non-negative integer $n$,

\[ \frac{d\theta}{dr} < 0 \]  

for all $r \in (r_1, r_n)$ where $r_1$ and $r_n$ are the smallest and largest roots of $\varphi_{N,n}$ respectively. Therefore, applying Remark 2.6 to (205), we can view $r$ as a function of $\theta$ where $r$ satisfies the differential equation

\[ \frac{dr}{d\theta} = \left( -\sqrt{\beta(r)} - \left( \frac{\beta'(r)}{4\beta(r)} + \frac{\alpha(r)}{2} \right) \sin(2\theta(r)) \right)^{-1} \]

where $\alpha$, $\beta$, and $\beta'$ are defined in (203), (204) and (207).

The following is a description of an algorithm for the evaluation of the $n$ roots of $\Phi_{N,n}$. We denote the $n$ roots of $\Phi_{N,n}$ by $r_1 < r_2 < \ldots < r_n$.

Algorithm 5.1 (Find roots of $\Phi_{N,n}$)

0. Compute the $T_{N,n}$ expansion of $\varphi_{N,n}$ using Algorithm 4.1.

1. Use bisection to find the largest root $x_0 \in (0,1)$ of $\beta(r)$ where $\beta(r)$ is defined in (204). If $\beta$ has no root on $(0,1)$, then set $x_0 = 1$.

2. If $\chi_{0,n}(c) > 1/\sqrt{c}$, place Chebyshev nodes (order $5n$, for example) on the interval $(0, x_0)$ and check, starting at $x_0$ and moving in the negative direction, for a sign change. Once a sign change has occurred, use Newton to find an accurate approximation to the root.

3. If $\chi_{0,n}(c) \leq 1/\sqrt{c}$, then use three steps of Mueller’s method starting at $x_0$, using the first and second derivatives of $\varphi_{0,n}$ followed by Newton’s method.

4. Defining $\theta_0$ by the formula

\[ \theta_0 = \theta(x_0), \]

where $\theta$ is defined in (139), solve the ordinary differential equation $\frac{dr}{d\theta}$ (see (201)) on the interval $(\pi/2, \theta_0)$, with initial condition $r(\theta_0) = x_0$. To solve the differential equation, it is sufficient to use, for example, second order Runge Kutta with 100 steps (independent of $n$). We denote by $\tilde{r}_n$ the approximation to $r(\pi/2)$ obtained by this process. It follows immediately from (65) that $\tilde{r}_n$ is an approximation to $r_n$, the largest root of $\varphi_{N,n}$.

5. Use Newton’s method with $\tilde{r}_n$ as an initial guess to find $r_n$ to high precision. The GPSF $\varphi_{N,n}$ and its derivative $\varphi'_{N,n}$ can be evaluated using the expansion computed in Step 0.

6. With initial condition

\[ x(\pi/2) = r_n, \]

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solve differential equation \( \frac{dr}{d\theta} \) (see (209)) on the interval \((-\pi/2, \pi/2)\) (212)

using, for example, second order Runge Kuta with 100 steps. We denote by \( \tilde{r}_{n-1} \) the approximation to

\[ r(-\pi/2) \]

obtained by this process.

6. Use Newton’s method, with initial guess \( \tilde{r}_{n-1} \), to find to high precision the second largest root, \( r_{n-1} \).

7. For \( k = \{1, 2, ..., n - 1\} \), repeat Step 4 on the interval

\[ (-\pi/2 - k\pi, -\pi/2 - (k - 1)\pi) \]

with initial condition

\[ x(-\pi/2 - (k - 1)\pi) = r_{n-k+1} \]

and repeat Step 5 on \( \tilde{r}_{n-k} \).

5.2 Generalized Gaussian quadratures for \( \Phi_{0,n} \)

In this section, we describe an algorithm for generating generalized Gaussian quadratures for the GPSFs \( \Phi_{0,0}, \Phi_{0,1}, ..., \Phi_{0,n} \).

Definition 5.1 A generalized Gaussian quadrature with respect to a set of functions

\[ f_1, ..., f_{2n-1} : [a, b] \to \mathbb{R} \] (216)

and non-negative weight function \( w : [a, b] \to \mathbb{R} \) is a set of \( n \) nodes, \( x_1, ..., x_n \in [a, b] \), and \( n \) weights, \( \omega_1, ..., \omega_n \in \mathbb{R} \), such that, for any integer \( j \leq 2n - 1 \),

\[ \int_a^b f_j(x)w(x)dx = \sum_{i=0}^{n} \omega_i f_j(x_i). \] (217)

A generalized Chebyshev quadrature consists of nodes \( x'_1, ..., x'_{2n-1} \in [a, b] \) and weights \( \omega'_1, ..., \omega'_{2n-1} \in \mathbb{R} \) such that

\[ \int_a^b f_j(x)w(x)dx = \sum_{i=0}^{2n-1} \omega'_i f_j(x'_i). \] (218)

Remark 5.6 In order to generate a generalized Gaussian quadrature for GPSFs with bandlimit \( c > 0 \), we first generate a generalized Chebyshev quadrature for GPSFs with bandlimit \( c/2 \) and then, using those nodes and weights as a starting point, we use Newton’s method with step-length control to find nodes and weights that form a generalized generalized Gaussian quadrature for GPSFs with bandlimit \( c \).
The following is a description of an algorithm for generating generalized Gaussian quadratures for the GPSFs

\[ \Phi_c^{0,0}, \ldots, \Phi_c^{0,2n-1}. \] (219)

**Algorithm 5.2 (Generalized Gaussian quadrature for \( \Phi_c^{0,0}, \ldots, \Phi_c^{0,2n-1} \))**

1. Using Algorithm 5.1, generate a generalized Chebyshev quadrature for the functions \( \Phi_c^{0,0}/2, \ldots, \Phi_c^{0,2n-1}/2 \).

That is, find, \( r_1, \ldots, r_n \), the \( n \) roots of \( \Phi_0^{0,n} \) and weights \( w_1, \ldots, w_n \) such that

\[ \int_0^1 \Phi_c^{0,k}/2(r) \, dr = \sum_{i=1}^n w_i \Phi_c^{0,k}/2(r_i) \] (221)

for \( k = 0, 1, \ldots, n-1 \).

2. Evaluate the vector \( d = (d_0, d_1, \ldots, d_{2n-1}) \) of discrepancies where \( d_k \) is defined by the formula

\[ d_k = \int_0^1 \Phi_c^{0,k}(r) \, dr - \sum_{i=1}^n w_i \Phi_c^{0,k}(r_i) \] (222)

for \( k = 0, 1, \ldots, 2n-1 \).

3. Generate \( A \), the \( 2n \times 2n \) matrix of partial derivatives of \( d \) with respect to the \( n \) nodes and \( n \) weights. Specifically, for \( i = 1, \ldots, 2n \), the matrix \( A \) is defined by the formula

\[ A_{i,j} = \begin{cases} \Phi_c^{0,j}(r_i) & \text{for } i = 1, \ldots, n, \\ w_i \Phi_c^{0,j}(r_i) & \text{for } i = n + 1, \ldots, 2n. \end{cases} \] (223)

where \( \Phi_c^{0,k}(r) \) denotes the derivative of \( \Phi_c^{0,k}(r) \) with respect to \( r \).

4. Solve for \( x \in \mathbb{R}^{2n} \) the \( 2n \times 2n \) linear system of equations

\[ Ax = -d \] (224)

where \( A \) is defined in (223) and \( d \) is defined in (222).

5. Update nodes and weights correspondingly. That is, defining \( r \in \mathbb{R}^{2n} \) to be the vector of nodes and weights

\[ r = (r_1, r_2, \ldots, r_n, w_1, w_2, \ldots, w_n)^T, \] (225)

we construct the updated vector of nodes and weights \( \tilde{r} \) so that

\[ \tilde{r} = r + (r, x)r \] (226)
6. If $\|\tilde{r}\|_2 < \|r\|_2$, continue to the next step. Otherwise, go back to Step 5 and divide the step length by 2. That is, define $\tilde{r}$ by the formula,

$$\tilde{r} = r + \frac{1}{2}(r, x)r.$$ (227)

Continue dividing the step length by 2 until $\|\tilde{r}\|_2 < \|r\|_2$.

7. Repeat steps 2-6 until the discrepancies, $d_i$ for $i = 0, 1, ..., 2n - 1$ (see (222)), are approximately machine precision.

6 Interpolation via GPSFs

In this section, we describe a numerical scheme for representing a bandlimited function as an expansion in GPSFs.

In general, the interpolation problem is formulated as follows. Suppose that $f$ is defined by the formula

$$f(x) = a_1g_1(x) + a_2g_2(x) + ... + a_ng_n(x)$$ (228)

where $g_1, ..., g_n$ are some fixed basis functions. Then interpolation problem is to recover the coefficients $a_1, ..., a_n$. This is generally done by solving the $n \times n$ linear system of equations obtained from evaluating $f$ at certain interpolation nodes. As long as $f$ is well-represented by the interpolation nodes, then the procedure is accurate.

As shown in the context of quadrature (see Section 5), GPSFs are a natural basis for representing bandlimited functions. We formulate the interpolation problem for GPSFs as recovering the coefficients of a bandlimited function $f$ in its GPSF expansion. That is, suppose that $f$ is of the form

$$f(x) = \int_B \sigma(t) e^{ic(x,t)} dt.$$ (229)

where $\sigma \in L^2(B)$. Then, $f$ is representable in the form

$$f(x) = \sum_{i=1}^{N} a_i \psi_i(x)$$ (230)

where $\psi_j(x)$ is a GPSF defined in (98) and $a_i$ satisfies

$$a_i = \int_B \psi_i(t) f(t) dt.$$ (231)

The interpolation problem we consider here is the evaluation of the coefficients $a_1, ..., a_N$.

The following lemma shows that a quadrature rule that recovers the coefficients of the expansion in GPSFs of a complex exponential will also recover the coefficients in a GPSF expansion of a bandlimited function.
Lemma 6.1 Suppose that for all \( t \in B \),
\[
\left| \int_B \psi_j(x)e^{ic(x,t)}dx - \sum_{k=1} w_k\psi_j(x_k)e^{ic(x_k,t)} \right| < \epsilon
\]
where \( B \) denotes the unit ball in \( \mathbb{R}^{p+2} \) and \( \psi_j \) is defined in (98). Then,
\[
\left| \int_B \psi_j(x)f(x)dx - \sum_{k=1} w_k\psi_j(x_k)f(x_k) \right| < \epsilon
\]
where
\[
f(x) = \int_B \sigma(t)e^{ic(x,t)}dt.
\]
The following lemma shows that the product of a complex exponential with a GPSF of bandlimit \( c > 0 \) is a bandlimited function with bandlimit \( 2c \). The proof is a slight modification of Lemma 5.3 in [11].

Lemma 6.2 For all \( x, \omega \in B \) where \( B \) denotes the unit ball in \( \mathbb{R}^{p+2} \) and for all \( c > 0 \),
\[
\psi_j(x)e^{ic(\omega,x)} = \int_B \sigma(\xi)e^{i2c(\xi,x)}d\xi
\]
where \( \psi_j \) is defined in (98) and \( \sigma \) satisfies
\[
\left| \int_B \sigma(t)^2dt \right| \leq 4/|\lambda_j|^2.
\]
where \( \lambda_j \) is defined in (98).

Proof. Using (98),
\[
\psi_j(x)e^{ic(\omega,x)} = \frac{1}{\lambda_j} \int_B e^{ic(\omega+t,x)}\psi_j(t)dt.
\]
Applying the change of variables \( \xi = (t + \omega)/2 \) to (237), we obtain
\[
\psi_j(x)e^{ic(\omega,x)} = \frac{1}{\lambda_j} \int_{B_{\omega}} e^{i2c(\xi,x)}2\psi_j(2\xi - \omega)d\xi
\]
where \( B_{\omega} \) is the ball of radius 1/2 centered at \( \omega/2 \). It follows immediately from (238) that
\[
\psi_j(x)e^{ic(\omega,x)} = \frac{1}{\lambda_j} \int_{B_{\omega}} e^{i2c(\xi,x)}\mu(\xi)d\xi.
\]
where
\[
\mu(\xi) = \begin{cases} 
\frac{2\psi_j(2\xi - \omega)}{\lambda_j} & \text{if } \xi \in B_{\omega}, \\
0 & \text{otherwise.}
\end{cases}
\]
Inequality (236) follows immediately from the combination of (240) with the fact that \( \psi_j \) is \( L^2 \) normalized.

The following observation provides a numerical scheme for recovering the coefficients in a GPSF expansion of a bandlimited function.
Observation 6.1 Suppose that $f$ is defined by the formula

$$f(x) = \int_B \sigma(t)e^{ix(t)}dt$$

(241)

where $\sigma$ is some function in $L^2(B)$. Then, $f$ is representable in the form

$$f(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$$

(242)

where

$$a_k = \int_B f(x)\psi_k(x)dx.$$ 

(243)

It follows immediately from the combination of Lemma 6.2 and Lemma 6.1 that using quadrature rule (199) with bandlimit $2c$ will integrate $a_k$ accurately. That is, following the notation of Observation 5.3,

$$|a_k - \sum_{i=0}^{n} w_i \sum_{j=1}^{m} v_j f(r_i x_j)\psi_k(r_i x_j)|$$

(244)

is exponentially small for large enough $m,n$.

Remark 6.2 When integrating a function of the form of (242) on the unit disk in $\mathbb{R}^2$, the $v_j$ in (244) are defined by the formula

$$v_j = j \frac{2\pi}{2m - 1}$$

(245)

for $j = 1, 2, ..., 2m - 1$ and the sums

$$\sum_{j=1}^{m} v_j f(r_i x_j)\psi_k(r_i x_j)$$

(246)

for each $i$ can be computed using an FFT.

The following lemma bounds the magnitudes of the coefficients of a GPSF expansion of a bandlimited function.

Lemma 6.3 Suppose that $f$ is defined by the formula

$$f(x) = \int_B \sigma(t)e^{ix(t)}dt$$

(247)

for all $x \in B$. Then,

$$f(x) = \sum_{i=1}^{\infty} a_i \psi_i(x)$$

(248)

where $\psi_i(x)$ is a GPSF defined in (98) and $a_i$ satisfies

$$|a_i| \leq |\lambda_i| \int_B |\sigma(t)|^2dt$$

(249)

where $\lambda_i$ is defined in (98).
Proof. Since $\psi_j$ form an orthonormal basis for $L^2[B]$, $f$ is representable in the form of (248) and for all positive integers $i$,

$$a_i = \int_B f(t)\psi_i(t)dt = \int_B \left( \int_B \sigma(\xi)e^{ic(t,\xi)}d\xi \right) \psi_i(t)dt. \quad (250)$$

Combining (250) and (98) and using Cauchy-Schwarz, we obtain

$$|a_i| = |\lambda_i\int_B \sigma(t)\psi_i(t)dt| \leq |\lambda_i|\int_B |\sigma(t)|^2dt \int_B |\psi_j(t)|^2dt = |\lambda_i|\int_B |\sigma(t)|^2dt. \quad (251)$$

Remark 6.3 Lemma 6.3 shows that in order to accurately represent a bandlimited function, $f$, it is sufficient to find the projection of $f$ onto all GPSFs with corresponding eigenvalue above machine precision.

7 Numerical experiments

The quadrature and interpolation formulas described in Sections 5 and 6 were implemented in Fortran 77. We used the Lahey/Fujitsu compiler on a 2.9 GHz Intel i7-3520M Lenovo laptop. All examples in this section were run in double precision arithmetic.

In Figure 1 and Figure 2 we plot the eigenvalues $|\lambda_{N,n}|$ of integral operator $F_c$ (see (97)) for different $N$ and different $c$.

Figures 3, 4, 5, and 6 include plots of the GPSFs $\Phi_{N,n}(r)$ (see (111)) for different $N, n, c$.

Figure 7 plots the magnitudes of coefficients of the GPSF expansion of the complex exponential $e^{ic(x,t)}$ for all $t$ on the unit disk where $x = (0.3, 0.4)$ and $c = 50$. The vertical axis, $|\alpha_{N,n}|$, is the magnitude of the coefficient of $\Phi_{N,n}(r)sin(\theta)$ in the GPSF expansion of (252). These coefficients were obtained via formula (244).

In Tables 1-6, we provide the accuracy of quadrature rule (199) in integrating the function $e^{ic(x,t)}$ (252) over the unit disk where $x = (0.9, 0.2)$. We provide results for $c = 20$ and $c = 100$ using both generalized Chebyshev and generalized Gaussian quadratures in the radial direction (see Remark 5.4) for various numbers of nodes in both the radial and angular directions.

In each table in this section, the column labeled “$c$” denotes the bandlimit $c$ in (252). The columns labeled “$N$” and “$n$” denote the $N$ and $n$ of $\Phi_{N,n}$ (see (111)). Relative errors of quadrature are denoted “relative error” and the true value of the integral was obtained by a calculation in extended precision using a large number of nodes.
Figure 1: Eigenvalues of $F_c$ (see (97)) for $c = 100$ and $p = 0$

Figure 2: Eigenvalues of $F_c$ (see (97)) for $c = 50$ and $p = 1$

| $c$ | radial nodes | angular nodes | relative error          |
|-----|---------------|---------------|-------------------------|
| 20  | 6             | 50            | $0.84109 \times 10^9$   |
| 8   |               |               | $0.70864 \times 10^{-3}$ |
| 10  |               |               | $0.15834 \times 10^{-7}$ |
| 12  |               |               | $0.75601 \times 10^{-13}$ |
| 14  |               |               | $0.68485 \times 10^{-14}$ |
| 16  |               |               | $0.29262 \times 10^{-14}$ |
| 18  |               |               | $0.75991 \times 10^{-14}$ |

Table 1: Quadratures for $e^{ic(x,t)}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 20$. Generalized Chebyshev quadratures are used in the radial direction.
Figure 3: Plots of GPSFs $\Phi_{0,n}$ (see (111)) with $c = 50$ and $p = 1$

Figure 4: Plots of GPSFs $\Phi_{0,n}$ (see (111)) with $c = 100$ and $p = 0$

Table 2: Quadratures for $e^{ic(x,t)}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of angular nodes for $c = 20$. Generalized Chebyshev quadratures are used in the radial direction.

| $c$ | radial nodes | angular nodes | relative error       |
|-----|--------------|---------------|----------------------|
| 20  | 14           | 20            | $0.46437 \times 10^6$ |
| 25  |              |               | $0.18500 \times 10^{-1}$ |
| 30  |              |               | $0.14547 \times 10^{-3}$ |
| 35  |              |               | $0.64949 \times 10^{-7}$ |
| 40  |              |               | $0.25015 \times 10^{-9}$ |
| 45  |              |               | $0.16653 \times 10^{-12}$ |
| 50  |              |               | $0.51483 \times 10^{-14}$ |
| 55  |              |               | $0.30672 \times 10^{-14}$ |
| 60  |              |               | $0.53592 \times 10^{-14}$ |
Figure 5: Plots of GPSFs $\Phi_{10,n}$ (see (111)) with $c = 50$ and $p = 1$

Figure 6: Plots of GPSFs $\Phi_{25,n}$ (see (111)) with $c = 100$ and $p = 0$

| $c$ | radial nodes | angular nodes | relative error |
|-----|---------------|---------------|----------------|
| 20  | 4             | 50            | $0.12603 \times 10^9$ |
| 6   |               |               | $0.36513 \times 10^{-6}$ |
| 8   |               |               | $0.41931 \times 10^{-12}$ |
| 10  |               |               | $0.15463 \times 10^{-14}$ |
| 12  |               |               | $0.35160 \times 10^{-14}$ |

Table 3: Quadratures for $e^{ic(x,t)}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 20$. Generalized Gaussian quadratures generated via Algorithm 5.2 are used in the radial direction.
Table 4: Quadratures for $e^{ic(x,t)}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 100$. Chebyshev quadratures are used in the radial direction.

| $c$   | radial nodes | angular nodes | relative error     |
|-------|---------------|---------------|--------------------|
| 100   | 30            | 140           | $0.10612 \times 10^2$ |
|       | 32            |               | $0.11305 \times 10^0$ |
|       | 34            |               | $0.45510 \times 10^{-4}$ |
|       | 36            |               | $0.63672 \times 10^{-6}$ |
|       | 38            |               | $0.54009 \times 10^{-9}$ |
|       | 40            |               | $0.94943 \times 10^{-13}$ |

Table 5: Quadratures for $e^{ic(x,t)}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of angular nodes for $c = 100$. Chebyshev quadratures are used in the radial direction.

| $c$   | radial nodes | angular nodes | relative error     |
|-------|---------------|---------------|--------------------|
| 100   | 40            | 115           | $0.12341 \times 10^{-3}$ |
|       | 120           |               | $0.12633 \times 10^{-5}$ |
|       | 125           |               | $0.28112 \times 10^{-7}$ |
|       | 130           |               | $0.60096 \times 10^{-9}$ |
|       | 135           |               | $0.13296 \times 10^{-11}$ |
|       | 140           |               | $0.94943 \times 10^{-13}$ |
|       | 145           |               | $0.23749 \times 10^{-12}$ |
|       | 150           |               | $0.16075 \times 10^{-12}$ |

Table 6: Quadratures for $e^{ic(x,t)}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 100$. Gaussian quadratures generated using Algorithm 5.2 are used in the radial direction.

| $c$   | radial nodes | angular nodes | relative error     |
|-------|---------------|---------------|--------------------|
| 100   | 20            | 150           | $0.77025 \times 10^{-5}$ |
|       | 22            |               | $0.20280 \times 10^{-9}$ |
|       | 24            |               | $0.28465 \times 10^{-12}$ |
|       | 26            |               | $0.50904 \times 10^{-13}$ |
|       | 28            |               | $0.35430 \times 10^{-12}$ |
|       | 30            |               | $0.39846 \times 10^{-12}$ |

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Figure 7: Coefficients, obtained via formula (244), of the GPSF expansion of the function on the unit disk $e^{ic\langle x,t \rangle}$ where $x = (0.3, 0.4)$ where $c = 50$. In the radial direction, 40 nodes were used and 140 nodes were used in the angular direction.

9 Appendix A

This appendix includes several technical results that were used in the main sections of this paper.

9.1 Properties of the derivatives of GPSFs

The following theorem follows immediately from (115) and (120).

**Theorem 9.1** Let $c > 0$. Then

$$
\frac{d}{dx}\left((x^{p+1} - x^{p+3}) \frac{d\Phi_{N,n}}{dx}(x)\right) + \left(\chi_{N,n}x^{p+1} - \frac{(p+1)(p+3)}{4}x^{p+1} - N(N + p)x^{p-1} - c^2x^{p+3}\right)\Phi_{N,n}(x) = 0, \tag{253}
$$

where $0 < x < 1$ and $N$ and $n$ are arbitrary nonnegative integers.

**Corollary 9.2** Let $c > 0$. Then

$$
x^2(1 - x^2)\Phi_{N,n}''(x) + ((p + 1)x - (p + 3)x^3)\Phi_{N,n}'(x)
+ \left(\chi_{N,n}x^2 - \frac{(p+1)(p+3)}{4}x^2 - N(N + p) - c^2x^4\right)\Phi_{N,n}(x) = 0, \tag{254}
$$

where $0 < x < 1$ and $N$ and $n$ are arbitrary nonnegative integers.

The following lemma connects the values of the $(k+2)$nd derivative of the function $\Phi_{N,n}$ with its derivatives of orders $k-4, k-3, \ldots, k+1$, and is obtained by repeated differentiation of (254).
Lemma 9.3 Let $c > 0$. Then

\[
(x^2 - x^4)\Phi_{N,n}^{(k+2)}(x) + ((2k + 1 + p)x - (4k + 3 + p)x^3)\Phi_{N,n}^{(k+1)}(x)
\]

\[
+ \left( k(k + p) - N(N + p) + \left[ \chi_{N,n} - \frac{1}{4}(p + 1)(p + 3) \right] \right)
\]

\[
- 3k(2k + 1 + p)]x^2 - c^2x^4 \right) \Phi_{N,n}^{(k)}(x)
\]

\[
+ \left( [2k(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - k(k - 1)(4k + 1 + 3p)]x - 4kc^2x^3 \right) \Phi_{N,n}^{(k-1)}(x)
\]

\[
+ \left( k(k - 1)(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - k(k - 1)(k - 2)(k + p) - 6k(k - 1)c^2x^2 \right) \Phi_{N,n}^{(k-2)}(x)
\]

\[
- 4k(k - 1)(k - 2)c^2x^2\Phi_{N,n}^{(k-3)}(x) - k(k - 1)(k - 2)(k - 3)c^2\Phi_{N,n}^{(k-4)}(x) = 0,
\]  

(255)

where $0 < x < 1$, $N$ and $n$ are arbitrary nonnegative integers, and $k$ is an arbitrary integer such that $k \geq 4$. Also,

\[
(x^2 - x^4)\Phi_{N,n}''(x) + ((p + 1)x - (p + 3)x^3)\Phi_{N,n}'(x)
\]

\[
+ \left( -N(N + p) + \left[ \chi_{N,n} - \frac{1}{4}(p + 1)(p + 2) \right] x^2 - c^2x^4 \right) \Phi_{N,n}(x) = 0,
\]

(256)

and

\[
(x^2 - x^4)\Phi_{N,n}^{(3)}(x) + ((p + 3)x - (p + 7)x^3)\Phi_{N,n}''(x)
\]

\[
+ \left( (p + 1) - N(N + p) + \left[ \chi_{N,n} - \frac{1}{4}(p + 1)(p + 3) - 3(p + 3) \right] x^2 - c^2x^4 \right) \Phi_{N,n}'(x)
 \]

\[
+ \left( 2[\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)]x - 4c^2x^3 \right) \Phi_{N,n}(x) = 0,
\]

(257)

and

\[
(x^2 - x^4)\Phi_{N,n}^{(4)}(x) + ((p + 5)x - (p + 11)x^3)\Phi_{N,n}^{(3)}(x)
\]

\[
+ \left( 2(p + 2) - N(N + p) + \left[ \chi_{N,n} - \frac{1}{4}(p + 1)(p + 3) - 6(p + 5) \right] x^2 - c^2x^4 \right) \Phi_{N,n}''(x)
 \]

\[
+ \left( [4(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - 6(p + 3)]x - 8c^2x^3 \right) \Phi_{N,n}'(x)
 \]

\[
+ \left( 2(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - 12c^2x^2 \right) \Phi_{N,n}(x) = 0,
\]

(258)

and

\[
(x^2 - x^4)\Phi_{N,n}^{(5)}(x) + ((p + 7)x - (p + 15)x^3)\Phi_{N,n}^{(4)}(x)
\]

\[
+ \left( 3(p + 3) - N(N + p) + \left[ \chi_{N,n} - \frac{1}{4}(p + 1)(p + 3) - 9(p + 7) \right] x^2 - c^2x^4 \right) \Phi_{N,n}^{(3)}(x)
 \]

\[
+ \left( [6(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - 6(3p + 13)]x - 12c^2x^3 \right) \Phi_{N,n}''(x)
 \]

\[
+ \left( 6(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - 6(p + 3) - 36c^2x^2 \right) \Phi_{N,n}'(x)
\]

\[
- 24c^2x\Phi_{N,n}(x) = 0,
\]

(259)

where $0 < x < 1$ and $N$ and $n$ are arbitrary nonnegative integers.
The following corollary and theorem are obtained immediately from Lemma 9.3.

**Corollary 9.4** Let \( c > 0 \). Then

\[
(k(k + p) - N(N + p))\Phi_{N,n}^{(k)}(0) \\
+ \left(k(k - 1)(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - k(k - 1)(k - 2)(k + p)\right)\Phi_{N,n}^{(k-2)}(0) \\
- k(k - 1)(k - 2)(k - 3)c^2\Phi_{N,n}^{(k-4)}(0) = 0,
\]

where \( N \) and \( n \) are arbitrary nonnegative integers, and \( k \) is an arbitrary integer so that \( k \geq 4 \). Also,

\[
N(N + p)\Phi_{N,n}(0) = 0,
\]

and

\[
((p + 1) - N(N + p))\Phi_{N,n}'(0) = 0,
\]

and

\[
(2(p + 2) - N(N + p))\Phi_{N,n}'(0) + 2(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3))\Phi_{N,n}(0) = 0,
\]

and

\[
(3(p + 3) - N(N + p))\Phi_{N,n}^{(3)}(0) \\
+ \left(6(\chi_{N,n} - \frac{1}{4}(p + 1)(p + 3)) - 6(p + 3)\right)\Phi_{N,n}'(0) = 0,
\]

where \( N \) and \( n \) are arbitrary nonnegative integers.

**Theorem 9.5** For all integers \( N \geq 1 \),

\[
\Phi_{N,n}^{(k)}(0) = 0 \quad \text{for } k = 0, 1, \ldots, N - 1,
\]

where \( n \) is an arbitrary nonnegative integer. where \( n \) is an arbitrary nonnegative integer.

**Theorem 9.6** Suppose that \( N \) and \( n \) are nonnegative integers. Then

\[
\Phi_{N,n}(1) \neq 0.
\]

**Proof.** Suppose that \( \Phi_{N,n}(1) = 0 \). Then using Lemma 9.3, we know \( \Phi_{N,n}^{(k)}(1) = 0 \) for all nonnegative \( k \). Since \( \Phi_{N,n} \) is analytic in the complex plane, we have \( \Phi_{N,n}(x) = 0 \) for all \( x \in \mathbb{R} \). ■

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9.2 Derivation of the integral operator $Q_c$

In this section we derive an explicit formula for the integral operator $Q_c$, defined in (99).

Suppose that $B$ denotes the closed unit ball in $\mathbb{R}^{p+2}$. From (99),

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p+2} \int_B \int_B e^{ic(x-t,u)} \psi(t) \, du \, dt,$$

(267)

for all $x \in B$. We observe that

$$e^{ic(v,u)} = \sum_{N=0}^{\infty} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} J_{N+p/2}(c\|u\|\|v\|) S_N^\ell(u/\|u\|) S_N^\ell(v/\|v\|),$$

(268)

for all $u, v \in B$, where $S_N^\ell$ denotes the spherical harmonics of degree $N$, and $J_{\nu}$ denotes Bessel functions of the first kind (see Section VII of [12]). Therefore,

$$\int_B e^{ic(v,u)} \, du = (2\pi)^{p/2+1} \int_0^1 \frac{J_{p/2}(c\|v\|\rho)}{(c\|v\|\rho)^{p/2}} \rho^{p+1} \, d\rho$$

$$= \frac{(2\pi)^{p/2+1}}{(c\|v\|)^{p/2}} \int_0^1 \rho^{p/2+1} J_{p/2}(c\|v\|\rho) \, d\rho$$

$$= \left(\frac{2\pi}{c}\right)^{p/2+1} \frac{J_{p/2+1}(c\|v\|)}{\|v\|^{p/2+1}},$$

(269)

for all $v \in \mathbb{R}^{p+2}$, where the last equality follows from formula 6.561(5) in [5]. Combining (267) and (269),

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_B \frac{J_{p/2+1}(c\|x-t\|)}{\|x-t\|^{p/2+1}} \psi(t) \, dt,$$

(270)

for all $x \in \mathbb{R}^{p+2}$.

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