DIOPHANTINE APPROXIMATION IN SMALL DEGREE

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1. Introduction

This paper (partly a survey) deals with the problem of finding optimal exponents in Diophantine estimates involving one real number \( \xi \). The prototype of such an estimate is the fact, known at least since Euler, that, for any given irrational real number \( \xi \), there exist infinitely many rational numbers \( \frac{p}{q} \) with

\[
|\xi - \frac{p}{q}| \leq q^{-2}.
\]

Here, the exponent of \( q \) in the upper bound is optimal because, when \( \xi \) has bounded partial quotients, there is also a constant \( c > 0 \) such that \( |\xi - p/q| \geq cq^{-2} \) for all rational numbers \( p/q \) (see Chapter I of [14]).

Define the height \( H(P) \) of a polynomial \( P \in \mathbb{R}[T] \) as the largest absolute value of its coefficients, and the height \( H(\alpha) \) of an algebraic number \( \alpha \) as the height of its irreducible polynomial over \( \mathbb{Z} \). Then the above estimate may be generalized in the following two ways related respectively with Mahler’s and Koksma’s classifications of numbers.

Consider a real number \( \xi \) which, for a fixed integer \( n \geq 1 \), is not algebraic over \( \mathbb{Q} \) of degree \( \leq n \). On one hand, an application of Dirichlet’s box principle shows that there exist infinitely many non-zero polynomials \( P \in \mathbb{Z}[T] \) such that

\[
|P(\xi)| \ll H(P)^{-n}
\]

where, as the sequel, the implied constant depends only on \( n \) and \( \xi \). On the other hand, Wirsing showed in [17] that there exist infinitely many algebraic numbers \( \alpha \) of degree at most \( n \) with

\[
|\xi - \alpha| \ll H(\alpha)^{-(n+3)/2}.
\]

For \( n = 1 \), both estimates are equivalent to (1), up to the values of the implied constants. For general \( n \), Spind\'zuk proved in [15] that the exponent of \( H(P) \) in the upper bound (2) is optimal by showing that, for \( \xi \) outside of a set of Lebesgue measure zero and for each \( \epsilon > 0 \), there are only finitely many non-zero integer polynomials of degree at most \( n \) with \( |P(\xi)| \leq H(P)^{-n-\epsilon} \). However, in the second estimate (3), it was conjectured by Wirsing (p. 1991 Mathematics Subject Classification. Primary 11J13; Secondary 11J04, 11J82.

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69 of [17] and Schmidt (p. 258 of [14]) that the optimal exponent for $H(\alpha)^{-1}$ is $n+1$ instead of $(n+3)/2$. Aside from the case $n = 1$, this is known to be true only for $n = 2$ thanks to work of Davenport and Schmidt [5]. Despite of several refinements by Bernik, Tishchenko and Wirsing, the optimal exponent remains unknown for any $n \geq 3$ (see Chapter 3 of [3] for more details and references).

In 1969, Davenport and Schmidt [6] devised a new method based on geometry of numbers to study the second type of estimate. Its flexibility is such that it allowed them to treat approximation by algebraic integers. Assuming, for a fixed $n \geq 2$, that $\xi$ is not algebraic over $\mathbb{Q}$ of degree $\leq n - 1$, they proved that there exists infinitely many algebraic integers $\alpha$ of degree at most $n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-[(n+1)/2]}.$$  

They showed that, for $n = 2$, the optimal exponent of approximation is 2 in agreement with the natural conjecture that it should, in general, be $n$. They also provided sharper estimates for $n = 3, 4$.

Their approach which we will describe in the next section conveys them to establish first another Diophantine approximation result. In the case $n = 3$, it is the following statement (Theorem 1a of [6]) where

$$\gamma = \frac{1 + \sqrt{5}}{2}$$

denotes the golden number.

**Theorem 1.1** (Davenport-Schmidt). Suppose that $\xi \in \mathbb{R}$ is neither rational nor quadratic over $\mathbb{Q}$. Then there are arbitrarily large values of $X$ such that the inequalities

$$|x_0| \leq X, \quad |x_0 \xi - x_1| \leq cX^{-1/\gamma}, \quad |x_0 \xi^2 - x_2| \leq cX^{-1/\gamma},$$

where $c$ is a suitable positive number depending on $\xi$, have no solution in integers $x_0, x_1, x_2$, not all 0.

Note that, an application of Dirichlet’s box principle shows that, for any $X \geq 1$, there exists a non-zero point $(x_0, x_1, x_2)$ in $\mathbb{Z}^3$ with $|x_0| \leq X$ and $|x_0 \xi^j - x_j| \leq \lfloor \sqrt{X} \rfloor^{-1}$ for $j = 1, 2$ (see Theorem 1A in Chapter II of [14]). Since $1/\gamma \approx 0.618 > 1/2$, the condition (5) is a far stronger requirement. Nevertheless, it is shown in [10] [11] that the exponent $1/\gamma$ is best possible for this problem. More precisely, there are countably many real numbers $\xi$ which are neither rational nor quadratic over $\mathbb{Q}$ such that, for a different choice of $c > 0$ (depending on $\xi$), the inequalities [14] admit a non-zero integer solution for any $X \geq 1$ (Theorem 1.1 of [11]). Because of this property, we shall call these numbers extremal.

In Section 3 below, we will sketch a proof of Theorem [11] and of a criterion for a real number to be extremal. This criterion attaches to an extremal real number a sequence of approximation triples which we will show in Section 4 to be essentially unique and to satisfy
a certain recurrence relation. This will allow us in Section 5 to derive a construction of extremal real numbers which generalizes that of Section 6 of [11].

Theorem 1.1 concerns simultaneous approximations of a real number and its square by rational numbers with the same denominator. Davenport and Schmidt looked more generally at simultaneous approximations of the first $n - 1$ powers $\xi, \ldots, \xi^{n-1}$ of a real number $\xi$ by rational numbers with the same denominator (Theorem 2a of [6]), and their result was recently improved by M. Laurent [7]. However, the optimal exponent for this problem is unknown for $n \geq 4$.

**Corollary 1.2** (Davenport-Schmidt). Let $\xi$ be as above. There exist infinitely many algebraic integers $\alpha$ of degree at most 3 with

$$|\xi - \alpha| \ll H(\alpha)^{-\gamma}.\$$

It is shown in [12] that here also the exponent $\gamma + 1 \simeq 2.618$ is best possible, against the natural conjecture that the optimal exponent would be 3. More precisely, there are real numbers for which the above corollary is optimal up to the value of the implied constant (Theorem 1.1 of [12]). Such numbers have to be extremal and it would be interesting to know if this property extends to all extremal real numbers.

Considering the irreducible polynomials in $\mathbb{Z}[T]$ of the approximations $\alpha$ provided by the above corollary, we readily deduce:

**Corollary 1.3.** Let $\xi$ be as above. There exist infinitely many monic polynomials $P \in \mathbb{Z}[T]$ of degree at most 3 with

$$|P(\xi)| \ll H(P)^{-\gamma}.\$$

We will prove in Section 6 that the exponent $\gamma$ in this statement is also best possible. Note that an argument of Bugeaud and Teulié [4, 16] shows more precisely that the inequality [4] has infinitely many solutions in algebraic integers $\alpha$ of degree exactly $n$ under the same assumption that $\xi$ is not algebraic of degree $\leq n - 1$. Therefore, one may require that the algebraic integers of Corollary 1.2 have degree 3 and that the polynomials of Corollary 1.3 also have degree 3.

Denote by $\mathbb{Z}[T]_{\leq n}$ the group of integer polynomials of degree at most $n$. In another direction, we have the following Gel’fond type criterion in degree two [11] which is, in a sense, dual to Theorem 1.1.

**Theorem 1.4** (Arbour-Roy). Let $\xi \in \mathbb{C}$. Assume that for any sufficiently large positive real number $X$ there exists a non-zero polynomial $P \in \mathbb{Z}[T]_{\leq 2}$ of height at most $X$ such that

$$|P(\xi)| \leq \frac{1}{4}X^{-\gamma-1}.\$$ (6)

Then $\xi$ is algebraic over $\mathbb{Q}$ of degree at most 2.
Theorem 1.2 of [11] shows that the exponent of $X$ in (6) is best possible. Analog statements involving polynomials of degree at most $n$ for a fixed integer $n \geq 3$ are well-known but the corresponding optimal exponent is not known (see Theorem 1 of [2] for a general setting, and Theorem 2b of [6] for a sharper estimate in the present context).

Finally, we will show in Section 7 that the extremal real numbers (associated with Theorem 1.1) are also characterized as those real numbers $\xi$ for which Theorem 1.4 is optimal up to the value multiplicative constant.

**Theorem 1.5.** Let $\xi$ be an real number which is not rational nor quadratic over $\mathbb{Q}$. The following conditions are equivalent:

(a) there exists a constant $c > 0$ such that, for any real number $X \geq 1$, there is a non-zero point $x = (x_0, x_1, x_2) \in \mathbb{Z}^3$ satisfying the condition (3);

(b) there exists a constant $c > 0$ such that, for any real number $X \geq 1$, there is a non-zero polynomial $P \in \mathbb{Z}[T]_{\leq 2}$ of height at most $X$ satisfying $|P(\xi)| \leq cX^{-\gamma-1}$.

It would be interesting to know if a similar property holds in higher degree. Note that any real number satisfying one of the above conditions (a) or (b) is transcendental over $\mathbb{Q}$ by virtue of Schmidt’s subspace theorem.

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**2. The method of Davenport and Schmidt**

Let $n$ be a fixed positive integer and let $\xi$ be a fixed real number which is not algebraic over $\mathbb{Q}$ of degree at most $n$. The natural approach to construct algebraic approximations of $\xi$ of degree at most $n$ is to produce non-zero polynomials of $\mathbb{Z}[T]_{\leq n}$ with “small” value and “large” derivative at $\xi$. One then concludes using the fact that any non-zero polynomial $P \in \mathbb{R}[T]$ of degree at most $n$ with $P'(\xi) \neq 0$ has at least one root $\alpha$ with

\[ |\xi - \alpha| \leq n \frac{|P(\xi)|}{|P'(\xi)|}. \]  

(7)

Define a convex body of $\mathbb{R}^n$ to be a compact, convex, neighborhood $C$ of 0 which is symmetric with respect to 0 (i.e. $C = -C$). According to a well-known result of Minkowski, if such a convex body $C$ has volume at least $2^n$, then it contains a non-zero point of $\mathbb{Z}^n$. Applying
this result to the convex body of $\mathbb{R}^{n+1}$ defined, for a real number $X \geq 1$, by
\[
\begin{cases}
|x_0 + x_1 \xi + \cdots + x_n \xi^n| \leq X^{-n} \\
|x_1| \leq X \\
\vdots \\
|x_n| \leq X
\end{cases}
\]
and noting that its volume is $2^{n+1}$, we deduce that there exists a non-zero polynomial $P \in \mathbb{Z}[T]$ with
\[
\deg(P) \leq n, \quad H(P) \leq (1 + |\xi| + \cdots + |\xi|^n)X \quad \text{and} \quad |P(\xi)| \leq X^{-n}.
\]
The difficulty is to control the derivative of $P$ at $\xi$. The best one can hope is, for arbitrarily large values of $X$, to have $|P'(\xi)| \gg X$. Then, using (7), one finds that there is at least one root $\alpha$ of $P$ with
\[
\deg(\alpha) \leq n, \quad H(\alpha) \ll X \quad \text{and} \quad |\xi - \alpha| \ll X^{-n-1} \ll H(\alpha)^{-n-1}.
\]
This motivates the conjecture mentioned in the introduction. In general, one has recourse to resultants to establish lower bounds on $|P'(\xi)|$.

The approach of Davenport and Schmidt in [6] is different as they require a set $\{P_1, \ldots, P_n\}$ of $n$ linearly independent polynomials of $\mathbb{Z}[T]_{\leq n-1}$, all having small absolute value at $\xi$. Then, taking any monic polynomial $Q \in \mathbb{R}[T]$ of degree $n$ with $Q(\xi) = 0$ and writing it as a linear combination
\[
Q = T^n + \theta_1 P_1 + \cdots + \theta_n P_n
\]
with $\theta_1, \ldots, \theta_n \in \mathbb{R}$, one sees that the polynomial $P \in \mathbb{Z}[T]$ given by
\[
P = T^n + [\theta_1]P_1 + \cdots + [\theta_n]P_n
\]
is monic of degree $n$, has height $H(P) \leq H(Q) + \sum_{i=1}^{n} H(P_i)$, and satisfies
\[
|P(\xi)| \leq \sum_{i=1}^{n} |P_i(\xi)| \quad \text{as well as} \quad |P'(\xi)| \geq |Q'(\xi)| - \sum_{i=1}^{n} |P'_i(\xi)|.
\]
So, if $X$ denotes an upper bound for $\sum_{i=1}^{n} H(P_i)$, an appropriate choice of $Q$ (with $H(Q) \ll X$ and $|Q'(\xi)| \gg X$) produces a polynomial $P$ with
\[
H(P) \ll X, \quad |P(\xi)| \leq \sum_{i=1}^{n} |P_i(\xi)| \quad \text{and} \quad |P'(\xi)| \geq X.
\]
The roots of such a polynomial are algebraic integers of degree $\leq n$ and height $\ll X$ and, by (7), at least one of them, say $\alpha$, satisfies
\[
|\xi - \alpha| \leq \frac{n}{X} \sum_{i=1}^{n} |P_i(\xi)|.
\]
To construct appropriate sets of polynomials \(\{P_1, \ldots, P_n\}\), Davenport and Schmidt apply a result of duality of Mahler [3]. To state this result or rather a consequence of it, let \(C\) be a convex body of \(\mathbb{R}^n\) and let \(C^*\) denote the set of points \((y_1, \ldots, y_n) \in \mathbb{R}^n\) satisfying
\[
|x_1y_1 + \cdots + x_ny_n| \leq 1
\]
for all \((x_1, \ldots, x_n) \in C\). Then, \(C^*\) is again a convex body of \(\mathbb{R}^n\), called the dual (or polar) convex body to \(C\) (the bi-dual \(C^{**}\) being \(C\) itself) and, if \(C\) contains no non-zero integral point, then \(n!C^*\) contains \(n\) linearly independent points of \(\mathbb{Z}^n\).

For example, given real numbers \(X, Y \geq 1\), the convex bodies of \(\mathbb{R}^n\) defined by
\[
C : \begin{cases}
|x_0| \leq X \\
|x_0\xi - x_1| \leq Y^{-1} \\
\vdots \\
|x_0\xi^{n-1} - x_{n-1}| \leq Y^{-1}
\end{cases}
\]
and
\[
K : \begin{cases}
|y_0 + y_1\xi + \cdots + y_{n-1}\xi^{n-1}| \leq X^{-1} \\
|y_1| \leq Y \\
\vdots \\
|y_{n-1}| \leq Y
\end{cases}
\]
are essentially dual to each other in the sense that
\[n^{-1}K \subseteq C^* \subseteq K.\]

So, if \(C\) contains no non-zero integral point, then there are \(n\) linearly independent polynomials of \(\mathbb{Z}[T]_{\leq n-1}\) of height \(\ll Y\) whose absolute values at \(\xi\) are \(\ll X^{-1}\) and therefore there exists an algebraic integer of degree \(\leq n\) and height \(\ll Y\) whose distance to \(\xi\) is \(\ll (XY)^{-1}\).

In the case \(n = 3\), this explains why Theorem 14 implies the existence of infinitely many algebraic integers \(\alpha\) of degree \(\leq 3\) with \(|\xi - \alpha| \ll H(\alpha)^{-7/4}\) as stated in Corollary 12.

In general, the fact that (14) has infinitely many solutions in algebraic integers of degree \(\leq n\) follows from a similar statement (Theorem 2a of [3]) showing that for \(X = cY^n\) with \(\nu = [(n-1)/2]\) and an appropriate constant \(c > 0\), there are arbitrarily large values of \(Y\) for which the convex body \(C\) contains no non-zero integral point.

A modification of the method produces approximation by algebraic numbers or algebraic integers of degree \(n\) or even by algebraic units of degree \(n\) (see [14, 16]). A more general choice of convex bodies involving several derivatives still produces simultaneous approximations of a real number by conjugate algebraic integers [13].

### 3. Extremal real numbers

In this section, we present a sketch of proof of Theorem 1.1 and establish some properties of the corresponding “extremal” real numbers.

Let \(\xi\) be a fixed real number. For each point \(x = (x_0, x_1, x_2) \in \mathbb{Z}^3\), we define
\[
\|x\| = \max\{|x_0|, |x_1|, |x_2|\} \quad \text{and} \quad L(x) = L_\xi(x) = \max\{|x_1 - \xi x_0|, |x_2 - \xi^2 x_0|\}.
\]
Identifying any such point with the corresponding symmetric matrix
\[
x = \begin{pmatrix}
x_0 & x_1 \\
x_1 & x_2
\end{pmatrix},
\]

we define
\[ \det(\mathbf{x}) = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} = x_0x_2 - x_1^2. \]

Then, using the multilinearity of the determinant, one finds
\[ |\det(\mathbf{x})| = \begin{vmatrix} x_0 & x_1 - \xi x_0 \\ x_1 & x_2 - \xi x_1 \end{vmatrix} \ll |x|L(x). \]

Similarly, if \( \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) \) denotes the determinant of the \( 3 \times 3 \) matrix whose rows are points \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^3 \), one finds,
\[ |\det(\mathbf{x}, \mathbf{y}, \mathbf{z})| = \begin{vmatrix} x_0 & x_1 - \xi x_0 & x_2 - \xi^2 x_0 \\ y_0 & y_1 - \xi y_0 & y_2 - \xi^2 y_0 \\ z_0 & z_1 - \xi z_0 & z_2 - \xi^2 z_0 \end{vmatrix} \ll |x|L(y)L(z) + |y|L(x)L(z) + |z|L(x)L(y). \]

We are now ready to present a sketch of proof of Theorem 1. To this end, assume that \( \xi \) is neither rational nor quadratic over \( \mathbb{Q} \) and that there exists a positive real number \( c \) such that the inequalities \( 5 \) have a non-zero solution \( \mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3 \) for any sufficiently large real number \( X \). We need to show that \( c \) is bounded below by some positive constant depending only on \( \xi \).

First note that there is a sequence of points \( (\mathbf{x}_i)_{i \geq 1} \) in \( \mathbb{Z}^3 \) with the following three properties:
- \( 1 \leq \|x_1\| < \|x_2\| < \|x_3\| < \ldots \)
- \( L(x_1) > L(x_2) > L(x_3) > \ldots \)
- if \( \mathbf{y} \in \mathbb{Z}^3 \) has \( 1 \leq \|y\| < \|x_{i+1}\| \), then \( L(y) \geq L(x_i) \).

Although this differs slightly from the construction in §3 of [6], we say that \( (\mathbf{x}_i)_{i \geq 1} \) is a sequence of minimal points for \( \xi \). In such a sequence, any point \( \mathbf{x}_i \) with \( i \geq 2 \) is primitive, i.e. has relatively prime coordinates. Moreover any two consecutive points \( \mathbf{x}_i \) and \( \mathbf{x}_{i+1} \) with \( i \geq 2 \) are linearly independent. Define
\[ X_i = \|x_i\| \quad \text{and} \quad L_i = L(x_i) \]
for each \( i \geq 1 \). Then, for any sufficiently large \( i \), the hypotheses imply the existence of a non-zero point \( \mathbf{x} \in \mathbb{Z}^3 \) with \( \|x\| < X_{i+1} \) and \( L(x) \ll cX_{i+1}^{-1/\gamma} \). This gives
\[ L_i \ll cX_{i+1}^{-1/\gamma}. \]

Davenport and Schmidt show that, for any sufficiently large \( i \), we have \( \det(\mathbf{x}_i) \neq 0 \) (Lemma 2 of [6]). Since \( \det(\mathbf{x}_i) \) is an integer, its absolute value is then bounded below by 1 and, using the estimate \( 9 \) combined with \( 11 \), we find
\[ 1 \leq |\det(\mathbf{x}_i)| \ll X_iL_i \ll cX_iX_{i+1}^{-1/\gamma}. \]
They also show that, for infinitely many \( i \), the points \( x_{i-1}, x_i \) and \( x_{i+1} \) are linearly independent (Lemma 5 of [3]). For these \( i \), the same argument based on (10) and (11) then gives

\[
1 \leq |\det(x_{i-1}, x_i, x_{i+1})| \ll X_{i+1} L_{i-1} \ll c^2 X_{i+1}^{-1/\gamma} X_i^{-1/\gamma}.
\]

The required lower bound on \( c \) then follows by comparing (12) and (13), upon noting that

\[
1 - \frac{1}{\gamma} = \frac{1}{\gamma^2}.
\]

The above considerations apply in particular to any extremal real number \( \xi \). Combining (12) and (13) then shows that, for all indices \( i \) such that \( x_{i-1}, x_i \) and \( x_{i+1} \) are linearly independent, we have

\[
\|x_{i+1}\| \sim \|x_i\|^{\gamma}, \quad L_\xi(x_i) \sim \|x_i\|^{-1}, \quad |\det(x_i)| \sim 1, \quad |\det(x_{i-1}, x_i, x_{i+1})| \sim 1
\]

writing \( a \sim b \) to mean \( a \ll b \) and \( a \gg b \). A further analysis shows that, by going to a subsequence \((y_i)_{i \geq 1}\) of \((x_i)_{i \geq 1}\), one may assume these estimates to hold for all \( i \geq 1 \). More precisely, we have the following equivalence (Theorem 5.1 of [11]):

**Theorem 3.1.** A real number \( \xi \) is extremal if and only if there exists a constant \( c \geq 1 \) and an unbounded sequence of non-zero primitive points \((y_i)_{i \geq 1}\) of \( \mathbb{Z}^3 \) satisfying, for all \( i \geq 1 \),

\[
\begin{align*}
c^{-1}\|y_i\|^\gamma & \leq \|y_{i+1}\| \leq c\|y_i\|^\gamma, \\
c^{-1}\|y_i\|^{-1} & \leq L_\xi(y_i) \leq c\|y_i\|^{-1}, \\
1 & \leq |\det(y_i)| \leq c, \\
1 & \leq |\det(y_i, y_{i+1}, y_{i+2})| \leq c.
\end{align*}
\]

In the next section, we show that the sequence \((y_i)_{i \geq 1}\) is essentially uniquely determined by \( \xi \).

4. **The sequence of approximation triples**

In this section, we fix an extremal real number \( \xi \in \mathbb{R} \) and a sequence of approximation triples \((y_i)_{i \geq 1}\) as in Theorem 3.1. We first prove:

**Proposition 4.1.** There exists a constant \( c_3 > 0 \) such that any non-zero primitive point \( y \in \mathbb{Z}^3 \) with

\[
L_\xi(y) \leq c_3\|y\|^{-1/\gamma}
\]

is of the form \( y = \pm y_i \) for some index \( i \geq 1 \).

**Proof.** Fix a constant \( c_3 \) with \( 0 < c_3 \leq \|y_1\|^{-1} \). Then, for any non-zero point \( y \in \mathbb{Z}^3 \), there exists an index \( i \geq 1 \) such that

\[
c_3\|y_i\| \leq \|y\| < c_3\|y_{i+1}\|.
\]
If (14) holds, we then find
\[ \det(y, y_i, y_{i+1}) \leq \|y\|\|y_i\|^{-1}\|y_{i+1}\|^{-1} + \|y_{i+1}\|\|y_i\|^{-1}L_\xi(y) \leq c_3 + c_3^{1/2}, \]
\[ \det(y, y_i, y_{i-1}) \leq \|y\|\|y_i\|^{-1}\|y_{i-1}\|^{-1} + \|y_i\|\|y_{i-1}\|^{-1}L_\xi(y) \leq c_3 + c_3^{1/2}. \]
So, provided that \( c_3 \) is sufficiently small, these determinants vanish and, since \( y_{i-1}, y_i \) and \( y_{i+1} \) are linearly independent with \( y_i \) primitive, we conclude that \( y = \pm y_i \). \[ \square \]

Since, for all sufficiently large values of \( i \), the point \( y = y_i \) satisfies the condition (14), we deduce from this proposition that the sequence \((y_i)_{i \geq 1}\) is uniquely determined by \( \xi \) up to its first terms and up to multiplication of its terms by \( \pm 1 \).

In proving the inequalities (9) and (10), we used the multi-linearity of the determinant. Equivalently, we could have looked at the Taylor series expansion of \( \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) \) at the points \( \mathbf{x} = (x_0, x_0\xi, x_0\xi^2), \ldots, \mathbf{z} = (z_0, z_0\xi, z_0\xi^2) \). Proposition A.1 of the appendix generalizes this idea and, through a computer search, provided the following relations.

**Proposition 4.2.** For any sufficiently large index \( i \), we have
\[
\det(y_i, y_{i+1}, [y_{i+3}, y_{i+3}, y_4]) = 0 \quad \text{and} \quad \det(y_{i+1}, y_{i+2}, [y_{i+3}, y_{i+3}, y_{i+4}]) = 0,
\]
where, upon identifying points \( \mathbf{x}, \mathbf{z} \in \mathbb{Z}^3 \) with the corresponding symmetric matrices as in (8) and upon denoting by \( \text{Adj}(\mathbf{z}) \) the adjoint matrix of \( \mathbf{z} \), we define
\[
[x, x, z] = x \text{Adj}(z)x.
\]

A direct proof of these relations can be found in [11], as part of the proof of Corollary 5.2 of [11]. It uses the estimates of Lemma 3.1 of [11] to show that the above determinants (15) have absolute values tending to zero as \( i \) tends to infinity. As in Proposition 2.3 of [12], we deduce:

**Corollary 4.3.** There exists a \( 2 \times 2 \) matrix \( M \) with relatively prime integer coefficients and an index \( i_0 \) such that the symmetric matrix corresponding to \( y_{i+2} \) is a rational multiple of \( y_{i+1}My_i \) when \( i \geq i_0 \) is odd, and a rational multiple of \( y_{i+1}^tMy_i \) when \( i \geq i_0 \) is even. Such a matrix \( M \) is non-singular, non-symmetric and non-skew-symmetric.

**Proof.** Choose \( i_0 \geq 2 \) so that the relations (15) hold for \( i \geq i_0 - 1 \). Since \( y_i, y_{i+1} \) and \( y_{i+2} \) are linearly independent, these relations imply that \([y_{i+3}, y_{i+3}, y_{i+4}]\) is a rational multiple of \( y_{i+1} \) and thus, by definition of the latter symbol, since all these matrices are invertible, that \( y_{i+4} \) is a rational multiple of \( y_{i+3}^{-1}y_{i+1}y_{i+3} \) for \( i \geq i_0 - 1 \). Therefore, assuming that \( y_{i+3} \) is a rational multiple of \( y_{i+2}Sy_{i+1} \) for some integer matrix \( S \) and some index \( i \geq i_0 - 1 \), we find that \( y_{i+4} \) is a rational multiple of \( y_{i+2}Sy_{i+3} \), and so, by taking transpose, that \( y_{i+4} \) is a rational multiple of \( y_{i+3}^tSy_{i+2} \). The first assertion of the corollary then follows by induction on \( i \), upon choosing \( M \) so that it holds for \( i = i_0 \). The matrix \( M \) is clearly non-singular. It
is not symmetric since a simple computation based for example on the formulas (2.1) and (2.2) of [11] gives

\begin{equation}
\det(y_i, y_{i+1}, y_{i+1}M y_i) = \det(y_i) \det(y_{i+1}) \text{trace}(MJ)
\end{equation}

where

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

while, for odd \(i \geq i_0\), the above determinant is non-zero. Finally, it is not skew-symmetric, otherwise in the above notation we would have \(M = \pm J\) which, for \(i \geq i_0\), would imply proportionality relations

\[
y_{i+3} \propto y_{i+2} y_{i+1} \propto y_i J y_{i+1} y_{i+1} \propto y_i
\]

and thus \(y_{i+3} = \pm y_i\), against the fact that the norms of the \(y_i\)'s are unbounded. \(\square\)

As the sequence \((y_i)_{i \geq 1}\) is uniquely determined by \(\xi\) up to its first terms and up to multiplication of its terms by \(\pm 1\), we deduce that the matrix \(M\) of the corollary is uniquely determined by \(\xi\) up to multiplication by \(\pm 1\) and up to transposition. We say that \(M\) is the matrix associated with \(\xi\) and, for a given \(M\), we denote by \(E(M)\) the set of extremal real numbers with associated matrix \(M\). In the next section, we present a criterion for showing that \(E(M)\) is not empty.

Remark that, in the notation of the corollary, the sequence of matrices \((M_i)_{i \geq 1}\) given by \(M_i = y_i M\) for \(i\) even and by \(M_i = y_i^t M\) for \(i\) odd satisfies

\[
M_{i+2} = M_{i+1} M_i
\]

for all \(i \geq i_0\). It can therefore be viewed as a Fibonacci sequence of matrices in \(\text{GL}_2(\mathbb{Q})\).

5. CONSTRUCTION OF EXTREMAL REAL NUMBERS

**Proposition 5.1.** Let \(M\) be a non-singular, non-symmetric \(2 \times 2\) matrix with relatively prime integer coefficients. Assume that there exist non-singular \(2 \times 2\) symmetric matrices \(y_1, y_2, y_3\) with relatively prime integer coefficients such that \(y_3\) is a rational multiple of \(y_2 M y_1\). Extend the definition of \(y_i\) coherently for \(i \geq 4\) by asking that \(y_i\) has relatively prime integer coefficients and that it is a rational multiple of \(y_{i-1} M y_{i-2}\) for odd \(i \geq 3\) and a rational multiple of \(y_{i-1}^t M y_{i-2}\) for even \(i \geq 4\). Assume further that the \(y_i\)'s are unbounded and that there exist positive constants \(c_4\) and \(c_5\) such that

\[
|\det(y_i)| \leq c_4 \quad \text{and} \quad \|y_{i+2}\| \geq c_5 \|y_{i+1}\| \|y_i\|
\]

for all \(i \geq 1\). Then \((y_i)_{i \geq 1}\) is a sequence of approximation triples associated with an extremal real number \(\xi \in E(M)\).

**Proof.** For each \(i \geq 1\), we have

\begin{equation}
y_{i+2} = \rho_i y_{i+1} S y_i
\end{equation}

for some appropriate choice of $S = M$ or $^t M$ and some non-zero rational number $\rho_i$ with $|\rho_i| \leq 1$. Assuming that $y_i$ and $y_{i+1}$ are non-singular, this shows that $y_{i+2}$ is also non-singular. Thus, by induction, all $y_i$'s are non-singular, and so

$$1 \leq |\det(y_i)| \leq c_4$$

for $i \geq 1$. We also deduce that

$$y_{i+3} = \rho_{i+1}y_{i+2}^tSy_{i+1} = \rho_i\rho_{i+1}y_{i+1}^tSy_i^tSy_{i+1}$$

which, starting from the fact that $y_1$, $y_2$ and $y_3$ are symmetric, implies by induction that all $y_i$'s are symmetric. The formula (17) with $M$ replaced by $S$ also gives

$$\det(y_i, y_{i+1}, y_{i+2}) = \rho_i \det(y_i) \det(y_{i+1}) \text{trace}(SJ).$$

Since $M$ is non-symmetric, we have $\text{trace}(SJ) = \pm \text{trace}(MJ) \neq 0$ and therefore, the above determinant being an integer, it satisfies

$$1 \leq |\det(y_i, y_{i+1}, y_{i+2})| \leq c_4^2|\text{trace}(MJ)|.$$  

The relation (18) also implies that

$$\|y_{i+2}\| \leq c_6\|y_{i+1}\|\|y_i\|$$

where $c_6$ denotes the sum of the absolute values of the coefficients of $M$. Defining $q_i = \|y_{i+1}\|\|y_i\|^{-\gamma}$ and using the similar lower bound for $\|y_{i+2}\|$ from the hypotheses of the theorem, we deduce that

$$c_5q_i^{-1/\gamma} \leq q_{i+1} \leq c_6q_i^{-1/\gamma},$$

which, by induction, implies $c_7^{-1} \leq q_i \leq c_7$ with $c_7 = \max\{q_i, q_i^{-1}, c_5^{-\gamma}, c_6^2\}$ and so

$$c_7^{-1}\|y_i\|^{\gamma} \leq \|y_{i+1}\| \leq c_7\|y_i\|^{\gamma},$$

for all $i \geq 1$.

Denote by $[x]$ the image of a non-zero point $x$ of $\mathbb{R}^3$ in the projective space $\mathbb{P}^2(\mathbb{R})$, and, for any non-zero point $y \in \mathbb{R}^3$, define

$$d([x], [y]) = d(x, y) = \frac{\|x \wedge y\|}{\|x\|\|y\|}$$

where $x \wedge y$ denotes the vector product of $x$ and $y$. This distance function defines the usual topology on $\mathbb{P}^2(\mathbb{R})$ (see for example Lemma 1.16 of [9]) and it is easily proved to satisfy

$$d(x, z) \leq d(x, y) + 2d(y, z) \quad (19)$$

for any three non-zero points $x, y, z$ of $\mathbb{R}^3$. Since $\mathbb{P}^2(\mathbb{R})$ is compact, the sequence $([y_i])_{i \geq 1}$ has an accumulation point $[y]$ for some non-zero $y \in \mathbb{R}^3$. Since the points $y_i$ have bounded determinant and norm tending to infinity with $i$, we deduce that, by continuity,

$$\det(y) = 0.$$
In order to estimate the distance between two consecutive points of this sequence, we note that

\[ y_{i+2} Jy_{i+1} = \rho_i y_i^t S y_{i+1} Jy_{i+1} = \rho_i \det(y_{i+1}) y_i^t S J, \]

thus

\[ \max_{k, \ell = 0, 1} |y_{i+2,k}y_{i+1,\ell+1} - y_{i+2,k+1}y_{i+1,\ell}| \leq c_4 c_6 \|y_i\|, \]

and so

\[ d(y_{i+1}, y_{i+2}) \leq 2c_4 c_6 \frac{\|y_i\|}{\|y_{i+1}\| \|y_{i+2}\|} \leq c_8 \|y_{i+1}\|^{-2} \]

with \( c_8 = 2c_4 c_6 / c_5 \). Using (19), (21) and the fact that the norms of the \( y_i \)'s grow faster than any geometric series, we deduce that, for \( k > i \geq 2 \), we have

\[ d(y_i, y_k) \leq \sum_{j=i}^{k-1} 2^{j-i} d(y_j, y_{j+1}) \leq c_8 \sum_{j=i}^{k-1} 2^{j-i} \|y_j\|^{-2} \leq c_9 \|y_i\|^{-2} \]

for some constant \( c_9 > 0 \). As \( d(y_k, y) \) can be made arbitrarily small for a suitable choice of \( k > i \), this implies

\[ d(y_i, y) \leq c_9 \|y_i\|^{-2}, \]

showing in particular that the sequence \( ([y_i])_{i \geq 1} \) converges to \([y] \).

We claim that the point \( y = (y_0, y_1, y_2) \) has \( y_0 \neq 0 \). Otherwise, upon denoting by \( k \) a fixed index for which \( y_k \neq 0 \), we would have, for \( i \geq 2 \),

\[ |y_k y_{i,0}| = |y_k y_{i,0} - y_0 y_{i,k}| \leq \|y\| \|y_i\| d(y_i, y) \leq c_9 \|y\| \|y_i\|^{-1}. \]

As this upper bound tends to zero for \( i \to \infty \), this would force the integer \( y_{i,0} \) to be zero for all sufficiently large values of \( i \), against the fact that the determinant of any three consecutive \( y_i \)'s is non-zero.

Since \( y_0 \neq 0 \), we may assume without loss of generality that \( y_0 = 1 \). Writing \( \xi = y_1 \), we then deduce, by virtue of (20), that

\[ y = (1, \xi, \xi^2). \]

and so

\[ L_\xi(y_i) \leq \|y \wedge y_i\| \leq \|y_i\| d(y_i, y) \leq c_9 \max\{1, \xi^2\} \|y_i\|^{-1} \]

for any \( i \geq 2 \). We also get a lower bound of the same type for \( L_\xi(y_i) \) by combining the estimate (9) with the lower bound \( |\det(y_i)| \geq 1 \). Therefore, by Theorem 3.1, the number \( \xi \) is extremal, and \( (y_i)_{i \geq 1} \) is an associated sequence of approximation triples. \( \square \)

To apply the above proposition for a given skew-symmetric matrix \( M \), one has to choose \( y_1 \) and \( y_2 \) so that

- the product \( y_2 M y_1 \) is symmetric,
- the \( y_i \)'s have bounded non-zero determinants,
• the $y_i$’s are unbounded and the ratios $\|y_{i+2}\|/(\|y_{i+1}\|\|y_i\|)$ are bounded below by some positive constant.

The second condition is automatically fulfilled if $M, y_1$ and $y_2$ have determinant $\pm 1$, because then all $y_i$’s have determinant $\pm 1$. The third condition is also fulfilled if, for example, the coefficients of $M$ are positive while those of $y_1$ and $y_2$ are non-negative.

**Example 1.** If we define

$$A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M = AB = \begin{pmatrix} ab + 1 & a \\ b & 1 \end{pmatrix}$$

for a choice of distinct positive integers $a$ and $b$, then

$$y_1 = A, \quad y_2 = ABA \quad \text{and} \quad y_3 = y_2My_1 = ABAABA$$

are symmetric matrices of determinant $\pm 1$ with non-negative entries while $M$ has positive entries. The resulting sequence $(y_i)_{i \geq 1}$ thus fulfills all requirements of Proposition 5.1. It can be shown that the corresponding extremal real number has continued fraction expansion

$$\xi = [0, a, b, a, a, b, a, \ldots]$$

given by the Fibonacci word on $\{a, b\}$ (see Theorem 2.2 of [10] or Corollary 6.3 of [11]).

**Example 2.** Take

$$y_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} a^3 + 2a & a^3 - a^2 + 2a - 1 \\ a^3 - a^2 + 2a - 1 & a^3 - 2a^2 + 3a - 2 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

for a fixed positive integer $a$. One readily checks that $y_2My_1$ is symmetric, that $\det(y_1) = \det(y_2) = -1$ and that $\det(M) = 1$. Thus, to ensure that the corresponding sequence $(y_i)_{i \geq 1}$ defines an extremal real number, it remains only to verify the growth condition on the norms of these points. To this end, we note that if $x, y \in \mathbb{Z}^3$ have coordinates satisfying $x_0 \geq x_1 \geq x_2 \geq 0$ and $y_0 \geq y_1 \geq y_2 \geq 0$ and if the product $z = yMx$ is symmetric, then we also have $z_0 \geq z_1 \geq z_2 \geq 0$ and moreover $z_0 \geq (a-1)y_0x_0$. By recurrence on $i$, using $y_{i+2} = y_{i+1}My_i$ for odd $i$ and $y_{i+2} = y_iMy_{i+1}$ for even $i$, we deduce that $\|y_i\| = y_{i,0}$ and that $\|y_{i+2}\| \geq (a-1)\|y_{i+1}\|\|y_i\|$ for all $i \geq 1$. So, for $a \geq 2$, the required growth condition is satisfied (we have $\lim_{i \to \infty} \|y_i\| = \infty$ since $\|y_2\| > \|y_1\| = 1$). In the case where $a = 1$, one finds that the points $x = y_3$ and $y = y_2$ satisfy the stronger conditions $x_0 \geq 2x_1 \geq 4x_2 \geq 0$ and $y_0 \geq 2y_1 \geq 4y_2 \geq 0$ and that, for such points $x, y \in \mathbb{Z}^3$, when the product $z = yMx$ is symmetric, we also have $z_0 \geq 2z_1 \geq 4z_2 \geq 0$ and $z_0 \geq y_0x_0/2$. By recurrence on $i$, this gives $\|y_i\| = y_{i,0}$ and $\|y_{i+2}\| \geq (1/2)^i\|y_{i+1}\|\|y_i\|$ for all $i \geq 2$. In particular, since $\|y_2\| = 3$ and $\|y_3\| = 5$, we deduce that $\|y_{i+1}\| > \|y_i\| > 2$ for $i \geq 1$ and so $\lim_{i \to \infty} \|y_i\| = \infty$. Thus, in all cases, the sequence $(y_i)_{i \geq 1}$ defines an extremal real number in $\mathcal{E}(M)$. This proves the remark at the end of §3 of [12].
6. Approximation by cubic algebraic integers

In order to show that the exponent in Corollary 1.3 is best possible, we apply the following criterion where, for a real number $x$, the symbol $\{x\}$ denotes the distance from $x$ to a closest integer (compare with Proposition 9.1 of [11]).

**Lemma 6.1.** Let $\xi$ be an extremal real number and let $(y_i)_{i \geq 1}$ be a corresponding sequence of approximation triples. Assume that there exists a constant $c_1 > 0$ such that

$$\{y_{i,0}\xi^3\} \geq c_1$$

for any sufficiently large index $i$. Then there exists a constant $c_2 > 0$ such that, for any monic polynomial $P \in \mathbb{Z}[T]_{\leq 3}$, we have

$$|P(\xi)| \geq c_2 H(P)^{-\gamma}.$$  

**Proof.** Choose an index $i_0 \geq 1$ such that (22) holds for each $i \geq i_0$. Multiplying $P$ by a suitable power of $T$ if necessary, we may assume without loss of generality that $P$ has degree three. Writing $P(T) = T^3 + pT^2 + qT + r$, we find, for any $i \geq i_0$,

$$\{y_{i,0}\xi^3\} \leq |y_{i,0}P(\xi)| + |p| \{y_{i,0}\xi^2\} + |q| \{y_{i,0}\xi\} \leq \|y_i\| |P(\xi)| + 2H(P)L_\xi(y_i) \leq \|y_i\| |P(\xi)| + c_3 H(P) \|y_i\|^{-1}$$

with a constant $c_3 > 0$ depending only on $\xi$. Choosing $i$ to be the smallest integer $i \geq i_0$ for which

$$H(P) \leq \frac{c_1}{2c_3}\|y_i\|,$$

and using (22) this implies

$$|P(\xi)| \geq \frac{c_1}{2}\|y_i\|^{-1}.$$  

The conclusion follows as the above choice of $i$ implies $\|y_i\| \ll H(P)^\gamma$. \Box

Let $a \geq 1$ be an integer and, for shortness, define $E_a = E\left(\begin{array}{cc} a \\ -1 \\ 0 \end{array}\right)$. Example 2 of the preceding section shows that this set is not empty. Moreover, it is proved in §4 of [12] that any extremal real number $\xi$ in $E_a$ satisfies the hypotheses of the preceding lemma (more precisely it is shown that $\lim_{j \to \infty} \{y_{i+3j,0}\xi^3\}$ exists and is a positive real number for $i = 0, 1, 2$). So, we deduce:

**Theorem 6.2.** Let $a \geq 1$ be an integer and let $\xi \in E_a$. There exists a constant $c > 0$ such that, for any monic polynomial $P \in \mathbb{Z}[T]_{\leq 3}$, we have

$$|P(\xi)| \geq cH(P)^{-\gamma}.$$  

**Corollary 6.3.** With $\xi$ as above, there exists a constant $c' > 0$ such that, for any algebraic integer $\alpha$ of degree at most 3 over $\mathbb{Q}$, we have

$$|\xi - \alpha| \geq c'H(\alpha)^{-\gamma-1}.$$
7. Proof of Theorem 1.5

Theorem 8.1 of [11] provides a sequence of polynomials in $\mathbb{Z}[T]_{\leq 2}$ with small absolute value at a given extremal real number $\xi$. These polynomials are constructed by taking exterior products of consecutive approximation triples of $\xi$. As a corollary, this result implies Theorem 1.2 of [11] which in turn tells us that (a) implies (b) in Theorem 1.5. Our proof that (b) implies (a) will be similar. We will first prove that, for any given real number $\xi$ which satisfies this condition (b), there is a sequence of polynomials in $\mathbb{Z}[T]_{\leq 2}$ with properties parallel to those stated in Theorem 3.1. Then, by taking exterior products of consecutive polynomials in this sequence, we will get points $x$ in $\mathbb{Z}^3$ for which $L_\xi(x)$ is small. This will require the following lemma.

Lemma 7.1. Let $\xi$ be a real number and let $P = p_0 + p_1T + p_2T^2$ and $Q = q_0 + q_1T + q_2T^2$ be polynomials of $\mathbb{Z}[T]_{\leq 2}$ with $2H(P)|Q(\xi)| \leq H(Q)|P(\xi)|$.

Then the point

$$x = P \wedge Q = (p_2q_1 - p_1q_2, p_0q_2 - p_2q_0, p_1q_0 - p_0q_1) \in \mathbb{Z}^3$$

satisfies

$$(2 \max\{1, |\xi|, |\xi^2|\})^{-1}H(Q)|P(\xi)| \leq L_\xi(x) \leq \frac{3}{2}H(Q)|P(\xi)|.$$

Proof. The upper bound follows immediately from the relations

$$x_0\xi - x_1 = p_2Q(\xi) - q_2P(\xi) \quad \text{and} \quad x_0\xi^2 - x_2 = q_1P(\xi) - p_1Q(\xi).$$

For the lower bound, we simply note that

$$q_0P(\xi) - p_0Q(\xi) = -\xi(q_1P(\xi) - p_1Q(\xi)) - \xi^2(q_2P(\xi) - p_2Q(\xi))$$

implies

$$\max\{1, |\xi|, |\xi^2|\}L_\xi(x) \geq \max_{0 \leq i \leq 2} |q_iP(\xi) - p_iQ(\xi)| \geq H(Q)|P(\xi)| - H(P)|Q(\xi)|.$$

We now prove Theorem 1.5 by adding one more equivalent condition (compare with Theorem 5.1 of [11]):

Theorem 7.2. Let $\xi$ be a real number. The following conditions are equivalent:

(a) the number $\xi$ is extremal;

(b) the number $\xi$ is neither rational nor quadratic over $\mathbb{Q}$ and, for any real number $X \geq 1$, there is a non-zero polynomial $P \in \mathbb{Z}[T]_{\leq 2}$ of height at most $X$ satisfying $|P(\xi)| \leq c_1X^{-\gamma-1}$ with a constant $c_1 = c_1(\xi)$;
(c) There exists a constant $c_2 \geq 1$ and an unbounded sequence of non-zero polynomials $(Q_k)_{k \geq 1}$ of $\mathbb{Z}[T]_{\leq 2}$ with relatively prime coefficients satisfying, for all $k \geq 1$,
\begin{align*}
c_2^{-1}H(Q_k)^\gamma & \leq H(Q_{k+1}) \leq c_2H(Q_k)\gamma, \\
c_2^{-1}H(Q_k)^{-\gamma^3} & \leq |Q_k(\xi)| \leq c_2H(Q_k)^{\gamma^3}, \\
1 & \leq |\text{Res}(Q_k, Q_{k+1})| \leq c_2, \\
1 & \leq |\det(Q_k, Q_{k+1}, Q_{k+2})| \leq c_2.
\end{align*}

Proof. As mentioned earlier, Theorem 1.2 of [1] shows that (a) implies (b).

Assume now that (b) is satisfied. We prove (c) by going back to the arguments of [1]. First of all, we recall that there is a sequence of “minimal polynomials” $(P_i)_{i \geq 1}$ in $\mathbb{Z}[T]_{\leq 2}$ with the following three properties:
\begin{itemize}
  \item $1 \leq H(P_1) < H(P_2) < H(P_3) < \ldots$
  \item $|P_1(\xi)| > |P_2(\xi)| > |P_3(\xi)| > \ldots$
  \item if $P \in \mathbb{Z}[T]_{\leq 2}$ has $1 \leq H(P) < H(P_{i+1})$, then $|P(\xi)| \geq |P_i(\xi)|$
\end{itemize}
(see §3 of [5] or Lemma 5 of [1]). For each $i \geq 1$, let $V_i$ denote the sub-$\mathbb{Q}$-vector space of $\mathbb{Q}[T]$ generated by $P_i$ and $P_{i+1}$. Then, for $i \geq 2$, the polynomials $P_i$ and $P_{i+1}$ form a basis of the group $V_i \cap \mathbb{Z}[T]$ of integral polynomials in $V_i$ (see the proof of Lemma 2 of [5]). In particular, $P_i$ has relatively prime coefficients for $i \geq 2$. Moreover the condition (b) implies
\begin{equation}
|P_i(\xi)| \leq c_1H(P_{i+1})^{-\gamma-1}
\end{equation}
for any $i \geq 1$.

Let $I$ denote the set of indices $i \geq 2$ for which $P_{i-1}$, $P_i$ and $P_{i+1}$ are linearly independent. Lemma 6 of [1] shows that $I$ is an infinite set and the arguments in §3 of [1] show that there exists an index $i_1 \in I$ such that $\text{Res}(P_i, P_{i+1}) \neq 0$ for all $i \in I$ with $i \geq i_1$. We define a sub-sequence $(Q_k)_{k \geq 1}$ of $(P_i)_{i \geq 1}$ by putting $Q_k = P_{i_k}$ where $i_k$ denotes the $k$-th element of $I$ with $i \geq i_1$. We claim that this sequence enjoys all properties stated in (c).

Take $i = i_k$ for some $k \geq 1$. Using Lemmas 2 and 4 of [1] together with (23), we find
\begin{align*}
1 \leq |\det(P_{i-1}, P_i, P_{i+1})| & \leq 6H(P_i)H(P_{i+1})|P_{i-1}(\xi)| \leq 6c_1H(P_i)^{-\gamma}H(P_{i+1}) \\
and
1 \leq |\text{Res}(P_i, P_{i+1})| & \leq 12H(P_i)H(P_{i+1})^2|P_i(\xi)| \leq 12c_1H(P_i)H(P_{i+1})^{-1/\gamma}.
\end{align*}
Comparing these two sets of inequalities, we deduce
\begin{equation}
H(P_{i+1}) \sim H(P_i)^\gamma, \quad |P_{i-1}(\xi)| \sim H(P_i)^{-\gamma-1} \quad \text{and} \quad |P_i(\xi)| \sim H(P_{i+1})^{-\gamma-1}.
\end{equation}
In particular, the polynomial $Q_k = P_i$ satisfies
\begin{equation}
|Q_k(\xi)| \sim H(Q_k)^{-\gamma^3}.
\end{equation}
Moreover, if $i$ is large enough, the pairs $(P, Q) = (P_{i-1}, P_i)$ and $(P, Q) = (P_i, P_{i+1})$ both satisfy the hypotheses of Lemma 7.1 and so we get

$$L_\xi(P_{i-1} \wedge P_i) \sim H(P_i)|P_{i-1}(\xi)| \sim H(P_i)^{-\gamma},$$

(26)

$$L_\xi(P_i \wedge P_{i+1}) \sim H(P_{i+1})|P_i(\xi)| \sim H(P_{i+1})^{-\gamma}.$$  

Adjusting the implied constants if necessary, we may assume that these estimates hold for all $i \in I$.

Consider now the next element $j = i_{k+1}$ of $I$. By construction, we have $V_i = V_{j-1}$ and so $P_i \wedge P_{i+1} = \pm P_{j-1} \wedge P_j$. Using (26), we find

$$H(P_j)^{-\gamma} \sim L_\xi(P_{j-1} \wedge P_j) = L_\xi(P_i \wedge P_{i+1}) \sim H(P_{i+1})^{-\gamma}$$

and so, by (24),

$$H(Q_{k+1}) = H(P_j) \sim H(P_{i+1}) \sim H(Q_k)^{-\gamma}.$$  

(27)

Moreover, since $P_i$ and $P_j$ are linearly independent (being primary of distinct height), they form a basis of $V_j$. Since $V_i$ contains $P_i$ and $P_{i+1}$ whose resultant is non-zero, we deduce that $P_i$ and $P_j$ also have a non-zero resultant and using Lemma 2 of [1] together with (25) and (27), we get

$$1 \leq |\text{Res}(Q_k, Q_{k+1})| \ll H(Q_k)H(Q_{k+1})^2|Q_k(\xi)| \ll 1.$$  

Finally, putting $\ell = i_{k+2}$, we observe that the polynomials $P_i$, $P_j$ and $P_{\ell}$ are linearly independent as they span the same vector space over $\mathbb{Q}$ as $P_{j-1}$, $P_j$ and $P_{j+1}$. Using Lemma 4 of [1] together with (25) and (27), this gives

$$1 \leq |\text{det}(Q_k, Q_{k+1}, Q_{k+2})| \ll H(Q_{k+1})H(Q_{k+2})|Q_k(\xi)| \ll 1.$$  

The last three estimates together with (26) show that the sequence $(Q_k)_{k \geq 1}$ satisfies the condition (c).

To prove the last implication that (c) implies (b), assume the existence of a sequence of polynomials $(Q_k)_{k \geq 1}$ as in condition (c). We first claim that $\xi$ is neither rational nor quadratic over $\mathbb{Q}$. To prove this, assume on the contrary that there exists a non-zero polynomial $Q \in \mathbb{Z}[T]_{\leq 2}$ which vanishes at $\xi$. Using Lemma 4 of [1], we then find

$$|\text{det}(Q, Q_k, Q_{k+1})| \ll H(Q)H(Q_{k+1})|Q_k(\xi)| \ll H(Q)H(Q_k)^{-\gamma^2},$$

which implies that the integer $\text{det}(Q, Q_k, Q_{k+1})$ is zero for all sufficiently large values of $k$, against the hypothesis that any three consecutive $Q_k$’s are linearly independent. Define

$$y_k = Q_k \wedge Q_{k+1}$$

for each index $k \geq 1$. We have

$$\|y_k\| \ll H(Q_k)H(Q_{k+1}) \ll H(Q_k)^{\gamma^2}$$
for all $k \geq 1$ and, applying Lemma 7.1 we deduce

$$L_\xi(y_k) \sim H(Q_{k+1}|Q_k(\xi)| \sim H(Q_{k+1})^{-\gamma} \ll \|y_{k+1}\|^{-1/\gamma}.$$  

This shows in particular that $L_\xi(y_k)$ tends to zero as $k$ tends to infinity and thus that the sequence $(y_k)_{k \geq 1}$ is unbounded. So, for any sufficiently large real number $X$, there exists an index $k$ such that $\|y_k\| \leq X \leq \|y_{k+1}\|$ and, for such a choice of $k$, the previous estimate gives $L_\xi(y_k) \ll X^{-1/\gamma}$. This shows that $\xi$ is extremal.  

\[ \square \]

**Appendix A. Finding new relations**

Let the notation be as in Section 4. Fix an integer $k \geq 1$, a $(k+1)$-tuple of non-negative integers $d = (d_0, \ldots, d_k)$, an integer $p \geq 0$ and triples of indeterminates $u_j = (u_{j,0}, u_{j,1}, u_{j,2})$ for $j = 0, 1, \ldots, k$. To each monomial

$$u_0^{e_0} \cdots u_k^{e_k} = \prod_{j=0}^{k} \prod_{\ell=0}^{2} u_{j,\ell}^{e_{j,\ell}} \in \mathbb{Q}[u_0, \ldots, u_k]$$

we associate a weight given by

$$\sum_{j=0}^{k} \sum_{\ell=0}^{2} e_{j,\ell}$$

and a multi-degree given by

$$([e_0], \ldots, [e_k]) = (e_{0,0} + e_{0,1} + e_{0,2}, \ldots, e_{k,0} + e_{k,1} + e_{k,2}).$$

We denote by $E(d,p)$ the sub-$\mathbb{Q}$-vector space of $\mathbb{Q}[u_0, \ldots, u_k]$ generated by all monomials of multi-degree $d$ and weight $p$. We also denote by $1$ the element of $\mathbb{Q}^{3k+3}$ all of whose coordinates are equal to 1.

**Proposition A.1.** Suppose that a polynomial $P \in E(d,p)$ satisfies

$$\left( \prod_{j=0}^{k} \prod_{\ell=1}^{2} \frac{\partial}{\partial u_{j,\ell}} f_{j,\ell} \right) P(1) = 0$$

for any choice of non-negative integers $f_{0,1}, f_{0,2}, \ldots, f_{k,1}, f_{k,2}$ with

$$2 \sum_{j=0}^{k} (f_{j,1} + f_{j,2}) \gamma_j \leq \sum_{j=0}^{k} d_j \gamma_j \quad \text{and} \quad f_{j,1} + f_{j,2} \leq d_j, \quad (0 \leq j \leq k).$$

Then, we have

$$P(y_i, y_{i+1}, \ldots, y_{i+k}) = 0$$

for all sufficiently large values of $i$.  

Proof. We find

$$P(y_i, \ldots, y_{i+k}) = \xi P_{y_i,0} \cdots y_{i+k,0} P\left(1, \frac{y_i,1}{y_{i,0}}, \frac{y_i,2}{y_{i,0}}, \ldots, 1, \frac{y_{i+k,1}}{y_{i+k,0}}, \frac{y_{i+k,2}}{y_{i+k,0}}\right)$$

$$= \xi P_{y_i,0} \cdots y_{i+k,0} \prod_{j=0}^{k} \prod_{\ell=1}^{2} f_{j,\ell}^{1} \left(1 - \left(\frac{y_{i+j,0}}{y_{i+j,0}}\right)^{\ell} \right) P(1),$$

where the sum in the second expression ranges over all choices of non-negative integers \(f_{0,1}, f_{0,2}, \ldots, f_{k,0}, f_{k,1}\) with \(f_{j,1} + f_{j,2} \leq d_j\) for \(j = 0, \ldots, k\). Since, for any such choice of integers, we have

$$\left| y_{i,0} \cdots y_{i+k,0} \prod_{j=0}^{k} \prod_{\ell=1}^{2} f_{j,\ell}^{1} \left(1 - \left(\frac{y_{i+j,0}}{y_{i+j,0}}\right)^{\ell} \right)\right| \ll \prod_{j=0}^{k} \left\| y_{i+j} \right\|^{d_j-2f_{j,1}-2f_{j,2}} \ll \left\| y_i \right\|^2 \sum_{j=0}^{k} (d_j-2f_{j,1}-2f_{j,2})^{\gamma_j},$$

the hypothesis implies

$$\left| P(y_i, \ldots, y_{i+k}) \right| \ll \left\| y_i \right\|^{-\epsilon}$$

for some positive real number \(\epsilon\). Since \(P(y_i, \ldots, y_{i+k})\) is a rational number with bounded denominator, it must therefore vanish for all sufficiently large values of \(i\). \(\square\)

Empirically it seems that, for a given multi-degree \(d\), the dimension of \(E(d, p)\) is maximal with \(p = |d| = d_0 + \cdots + d_k\). For values of \(p\) at equal distance from \(|d|\), that is for integers \(p, q \geq 0\) with \(p + q = 2|d|\), the dimensions of the corresponding vector spaces \(E(d, p)\) and \(E(d, q)\) are the same. So, it is natural to look first for polynomials in \(E(d, |d|)\).

A computer search based on the above proposition found two non-zero polynomials with the appropriate vanishing, namely

$$\det(u_0, u_1, [u_3, u_3, u_4]) \text{ and } \det(u_1, u_2, [u_3, u_3, u_4])$$

where the symbol \([u_3, u_3, u_4]\) is defined by \([15]\). The first polynomial has multi-degree \((1, 1, 0, 2, 1)\) and weight 5, while the other has multi-degree \((0, 1, 1, 2, 1)\) and weight 5. They provide the relations \([15]\) of Proposition 4.2.

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