WILLMORE SPHERES IN QUATERNIONIC PROJECTIVE SPACE

K. LESCHKE

Abstract. The Willmore energy for Frenet curves in quaternionic projective space $\mathbb{HP}^n$ is the generalization of the Willmore functional for immersions into $S^4$. Critical points of the Willmore energy are called Willmore curves in $\mathbb{HP}^n$.

Using a Bäcklund transformation on Willmore curves, we generalize Bryant’s result on Willmore spheres in 3-space: a Willmore sphere in $\mathbb{HP}^n$ has integer Willmore energy, and is given by complex holomorphic data.

1. Introduction

This work is part of a project where “quaternionified” complex analysis is used to study old and new questions in surface theory. The first accounts of this program are presented in [PP98], [BFL02] and [FLPP01]. An important feature in the quaternionic setup is that conformal maps from a Riemann surface into $S^4 = \mathbb{HP}^1$ play the role of the meromorphic functions in complex analysis. More generally, if we consider holomorphic curves $f : M \to \mathbb{HP}^n$ then the components of $f$ are branched conformal immersions into $S^4 = \mathbb{HP}^1$. Thus we can think of a holomorphic curve in $\mathbb{HP}^n$ as a family of branched conformal immersions into $S^4$.

Basic constructions of complex Riemann surface theory, such as holomorphic line bundles, the Kodaira embedding, the Plücker relations and the Riemann–Roch theorem, carry over to the quaternionic setting. There is an important new invariant of the quaternionic holomorphic theory distinguishing it from its complex counterpart: the Willmore energy is defined for holomorphic curves in $\mathbb{HP}^n$. For immersions $f : M \to S^4$ into $S^4 = \mathbb{HP}^1$ we obtain the classical Willmore functional $\int_M (|H|^2 - K - K^\perp) |df|^2$ where $H$ is the mean curvature vector, $K$ the Gaussian curvature, and $K^\perp$ the curvature of the normal bundle. Willmore surfaces $f : M \to S^4$ are critical points of the Willmore functional [Wil93]. It is a well-known fact, that Willmore surfaces in $S^4$ are characterized by the harmonicity of the conformal Gauss map.

To generalize the notion of Willmore surfaces to the case of a holomorphic curve $f : M \to \mathbb{HP}^n$, we use the the analogue of the conformal Gauss map of an immersion into $S^4$, the so-called canonical complex structure of $f$. In general, the canonical complex structure exists only away from a discrete set of $M$. In view of the relation between the Willmore condition and harmonicity, we restrict to the case of Frenet curves. These are holomorphic curves for which the canonical complex structure exists smoothly on $M$. A Frenet curve in $\mathbb{HP}^n$ is called Willmore if it is a critical point of the Willmore energy under compactly supported variations by Frenet curves.

Similar to the $\bar{\partial}$ and $\partial$ transforms of harmonic maps into complex projective space [Wol88], we define a Bäcklund transform of a Willmore curve by using the $(1, 0)$–part of the derivative of the canonical complex structure. This generalizes the Bäcklund transformation in [BFL02] for Willmore surfaces in $\mathbb{HP}^1$ to Willmore curves in $\mathbb{HP}^n$. 

MSC–class: 53Axx, 53Cxx, 30Fxx

Partially supported by SFB 288 and by NSF-grant DMS-9626804.
The Bäcklund transform \( \tilde{f} \) of a Willmore sphere \( f : S^2 \to \mathbb{H} \mathbb{P}^n \) is a Willmore sphere in \( \mathbb{H} \mathbb{P}^k \), \( k < n \), or a constant point in \( \mathbb{H} \mathbb{P}^n \). More precisely, the Bäcklund transform \( \tilde{f} : S^2 \to \mathbb{H} \mathbb{P}^k \) is given by a twistor projection of a holomorphic curve \( g : S^2 \to \mathbb{C} \mathbb{P}^{2k+1} \). Using the special form of the Bäcklund transform, we prove a generalization of the results of Bryant \[Bry84\] and Ejiri, \[Eji88\]: every Willmore sphere \( f \) with planar ends or is given by a twistor projection of a holomorphic curve in some complex projective space. Moreover, a Willmore sphere has Willmore energy in \( 4\pi \mathbb{N} \).

## 2. Holomorphic curves and holomorphic bundles

We set up some basic notation used throughout the paper. For more details of the underlying quaternionic theory, we refer to \[BFL+02\], \[FLPP01\] and \[PP98\].

We view a Riemann surface \( M \) as a 2–dimensional, real manifold with an endomorphism field \( J \in \Gamma(\text{End}(TM)) \) satisfying \( J^2 = -1 \). If \( V \) is a vector bundle over \( M \), we denote the space of \( V \) valued quaternionic \( k \)–forms by \( \Omega^k(V) \). If \( \omega \in \Omega^1(V) \), we set

\[
\ast \omega := \omega \circ J.
\]

Moreover, we will identify \( \omega \in \Omega^2(V) \) with the induced quadratic form \( \omega(X) := \omega(X, JX) \).

In particular for pairings \( V_1 \times V_2 \to V_3 \) we will identify

\[
\omega \land \eta = \omega \ast \eta - \ast \omega \eta, \ \omega \in \Omega^1(V_1), \eta \in \Omega^1(V_2),
\]

where the wedge product is defined over the pairing.

Most of the vector bundles occurring will be \emph{quaternionic} vector bundles, i.e., the fibers are quaternionic vector spaces and the local trivializations are quaternionic linear on each fiber. We adopt the convention that all quaternionic vector spaces are \emph{right} vector spaces. A \emph{quaternionic connection} on a quaternionic vector bundle satisfies the usual Leibniz rule over quaternionic valued functions.

If \( V_1 \) and \( V_2 \) are quaternionic vector bundles, we denote by \( \text{Hom}(V_1, V_2) \) the bundle of quaternionic linear homomorphisms. As usual, \( \text{End}(V) = \text{Hom}(V, V) \) denotes the quaternionic linear endomorphisms. Notice that \( \text{Hom}(V_1, V_2) \) is \emph{not} a quaternionic bundle.

Let \( V \) be a quaternionic vector bundle with complex structure \( S \in \Gamma(\text{End}(V)), \ S^2 = -1 \). We phrase this as \( (V, S) \) is a complex quaternionic vector bundle.

Given a connection \( \nabla \) on \( V \), we can decompose \( \nabla = \nabla' + \nabla'' \) into \((1, 0)\) and \((0, 1)\) parts with respect to \( S \), where

\[
\nabla' := \frac{1}{2}(\nabla - S \ast \nabla) \ \text{and} \ \nabla'' := \frac{1}{2}(\nabla + S \ast \nabla).
\]

Let \( \text{Hom}_+(V, W) = \{ B \in \text{Hom}(V, W) \mid S_W B = \pm BS_V \} \) where \( (V, S_V) \) and \( (W, S_W) \) are complex quaternionic vector bundles. We denote by \( B_{\pm} = \frac{1}{2}(B \mp \text{S}_W \text{S}_V) \in \text{Hom}_\pm(V, W) \) the \pm–part of \( B \in \text{Hom}(V, W) \). We can decompose \( \nabla' \) and \( \nabla'' \) further into

\[
\nabla'' = \bar{\partial} + Q, \ \nabla' = \partial + A
\]

where

\[
\bar{\partial} S = S \bar{\partial}, \ \partial S = S \partial
\]

and

\[
Q = \nabla''_\ast \in \Gamma(K\text{End}_-(V)), \ A = \nabla'_\ast \in \Gamma(K\text{End}_-(V)).
\]

Here we denote by

\[
KE := \{ \omega \in \Lambda^1(TM) \otimes E \mid \ast \omega = S \omega \}
\]
and
\[ \nabla = [\nabla, S] = [Q + A, S] = 2(*Q - A). \]
In particular, if \( \nabla \) is flat, then
\[ d\nabla * A = d\nabla * Q \]
and
\[ 4 * A = S * \nabla S - \nabla S, \quad 4 * Q = S * \nabla S + \nabla S. \]

We denote by \( V \) the trivial \( \mathbb{H}^{n+1} \)-bundle over \( M \). Let \( \Sigma \to G_{k+1}(V) \) be the tautological \((k + 1)\)-plane bundle whose fiber over \( V_k \in G_{k+1}(\mathbb{H}^{n+1}) \) is \( \Sigma V_k = V_k \subset V \). A map \( f : M \to G_{k+1}(V) \) can be identified with a rank \( k + 1 \) subbundle \( V_k \subset V \) via \( V_k = f^* \Sigma \), i.e., \((V_k)_p = \Sigma f(p) = f(p)\) for \( p \in M \). From now on, we will make no distinction between a map \( f \) into the Grassmannian \( G_{k+1}(V) \) and the corresponding subbundle \( V_k \subset V \).

The derivative of \( V_k \subset V \) is given by the \( \text{Hom}(V_k, V/V_k) \) valued 1–form
\[ \delta = \pi_{V_k} \nabla |_{V_k}, \]
where \( \pi_{V_k} : V \to V/V_k \) is the canonical projection. Under the identification \( TG_{k+1}(V) = \text{Hom}(\Sigma, V/\Sigma) \) the 1–form \( \delta \) is the derivative \( df \) of \( f : M \to G_{k+1}(V) \).

**Definition 2.1.** Let \( V = \mathbb{H}^{n+1} \) be the trivial quaternionic \( n+1 \)-plane bundle over a Riemann surface \( M \).

1. A rank \( k + 1 \) subbundle \( V_k \subset V \) is a **holomorphic curve** in \( V \) if there exists a complex structure \( J \in \Gamma(\text{End}(V_k)), J^2 = -1 \), such that
   \[ *\delta = \delta J. \]
2. The **Frenet flag** of a holomorphic curve \( f : M \to \mathbb{H}^n \) is a full flag
   \[ L = f^* \Sigma = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V \]
   of quaternionic subbundles of rank \( V_k = k + 1 \) together with complex structures \( J_k \) on the quotient bundles \( V_k/V_{k-1}, J_0 = J \), such that
   (a) \( \nabla \Gamma(V_k) \subset \Omega^1(V_{k+1}) \).
   (b) The derivatives \( \delta_k = \pi_{V_k} \nabla : V_k/V_{k-1} \to T^* M \otimes V_{k+1}/V_k \) satisfy
   \[ *\delta_k = J_{k+1} \delta_k = \delta_k J_k. \]

**Example 2.2.** Consider the line bundle \( L = f^* \Sigma \subset \mathbb{H}^2 \) induced by \( f : M \to S^1 = \mathbb{H}^1 \). The corresponding flag is \( L \subset V_1 = \mathbb{H}^2 \). The line bundle \( L \) is a holomorphic curve if there exists a complex structure \( J \) on \( L \) such that \( *\delta = \delta J \). We obtain an equivalent condition written in terms of \( f \): \( L \) is a holomorphic curve if and only if there exists \( J \in \Gamma(\text{End}(L)) \) such that \( *df = df \circ J \), i.e., \( f : M \to \mathbb{H}^1 \) is a branched conformal immersion (for a detailed development of conformal surface theory using quaternionic valued functions see [BFL+02]). If the derivative of \( L \) has no zeros, then \( f \) is an immersion. In this case, we can define the conformal Gauss map which can be identified with a complex structure \( S \) on \( \mathbb{H}^2 \) which induces \( J \) on \( L \). Moreover, \( S \) satisfies \( *\delta = S \delta = \delta S \) and the second order tangency condition...
$Q_{|L} = 0$. The immersion $f$ is Willmore if and only if the conformal Gauss map is harmonic \cite{Ebi88, Rig87}, which is equivalent \cite{BFL02} Thm. 3 to
\[ d\nabla \ast A = 0. \]

Recall that a complex structure $S \in \Gamma(\text{End}(V))$ is called adapted to the Frenet flag $L = V_0 \subset V_1 \subset \ldots \subset V$ of a holomorphic curve $f : M \to \mathbb{H}P^n$ if $S$ induces the complex structures given by the Frenet flag, i.e., if
\[ *\delta_k = S\delta_k = \delta_k S \]
for $k = 0, \ldots, n - 1$. The analogue of the conformal Gauss map of a conformal immersion $f : M \to S^4$ is an adapted complex structure which satisfies a certain second order condition.

**Definition 2.3.** Let $f : M \to \mathbb{H}P^n$ be a holomorphic curve and $L = V_0 \subset V_1 \subset \ldots \subset V$ be the Frenet flag of $L$. The unique adapted complex structure $S \in \Gamma(\text{End}(V))$ with
\[ Q|_{V_{n-1}} = 0 \text{ or, equivalently } A(V) \subset TM^* \otimes L, \]
is called the canonical complex structure of $L$. Here $\nabla = \bar{\partial} + Q + \partial + A$ is the decomposition of $\nabla$ with respect to $S$.

In general, the Frenet flag and the canonical complex structure of a holomorphic curve $f : M \to \mathbb{H}P^n$ only exist away from a discrete set $D$, the Weierstrass points of $L$. These are the zeros of the derivatives $\delta_k$ of the flag bundles $V_k$. In case of a holomorphic curve $f$ in $\mathbb{H}P^1$ the Weierstrass points are the branch points of the map $f : M \to S^4$.

Whereas the Frenet flag of a holomorphic curve always extends continuously into the Weierstrass points \cite[Lemma 4.10]{FLPP01}, the canonical complex structure may become singular as the following example \cite{Pet04} shows: If $f : M \to \mathbb{H}P^1$ is the twistor projection of a complex holomorphic curve $h : M \to \mathbb{C}P^3$ then the canonical complex structure is given by the tangent line $W_1 \subset V$ of $h$, namely $S|_{W_1} = i$ and $S|_{W_1} = -i$. But the tangent $W_1 \subset V$ of $h$ can become quaternionic, i.e., $W_1 = W_1j$ at some $p \in M$. In this case the canonical complex structure $S$ degenerates to a point at $p \in M$ and thus $S$ cannot be extended into $p \in M$. To avoid these difficulties, we will only consider holomorphic curves $f : M \to \mathbb{H}P^n$ which have a smooth canonical complex structure. For conformal maps $f : M \to \mathbb{H}P^1$ this means that the mean curvature sphere congruence extends smoothly across the branch points.

**Definition 2.4.** A Frenet curve $f : M \to \mathbb{H}P^n$ is a holomorphic curve which has a smooth canonical complex structure on $M$.

**Remark 2.5.** Note that this definition is more general then the one given in \cite{FLPP01}. In contrast to \cite[Def. 4.3]{FLPP01} we allow $f$ to have Weierstrass points. However, the smoothness of the canonical complex structure guarantees, as we will see below, the existence of the Frenet flag on $M$.

**Example 2.6.** A trivial example for a Frenet curve is an unramified curve $f : M \to \mathbb{H}P^n$, i.e., a holomorphic curve without Weierstrass points. In the case $n = 1$ an unramified curve is a conformal immersion $f : M \to S^4$.

In what follows, the example of the dual curve of a Frenet curve also will play an important role. For any subbundle $V_k$ of $V$ let
\[ V_k^\perp := \{ \alpha \in V^{-1} \mid <\alpha, \psi >= 0 \text{ for all } \psi \in V_k \}, \]
where $V^{-1}$ is the dual bundle of $V$. The dual curve $L^*$ of a Frenet curve $L \subset V$ with Frenet flag $L = V_0 \subset V_1 \subset \ldots \subset V$ is the holomorphic curve in $V^{-1}$ defined by
\[ L^* := V_{n-1}^\perp. \]
The Frenet flag of the dual curve \( f^* \) is given by
\[
(2.5) \quad V^*_k = V^\perp_{n-1-k}.
\]
An adapted complex structure \( S \) of \( L \) induces an adapted complex structure on \( L^* \) by the dual map \( S^* \). If \( \nabla = \bar{\partial} + \partial + A + Q \) is a connection on \( V \) then the induced dual connection \( \nabla^* \) on \( V^{-1} \) is decomposed as
\[
\nabla^* = \bar{\nabla}^* + Q^\dagger + A^\dagger,
\]
where
\[
(2.6) \quad A^\dagger = -Q^* \in \Gamma(K\operatorname{End}_-(V^{-1})) \quad \text{and} \quad Q^\dagger = -A^* \in \Gamma(K\operatorname{End}_-(V^{-1})).
\]
The decomposition of the dual connection \( \bar{\nabla}^* = \bar{\partial}^* + \partial^* \) is given by
\[
(2.7) \quad <\bar{\partial}^* \alpha, \psi > + <\alpha, \bar{\partial} \psi >= \frac{1}{2}(d <\alpha, \psi > + *d <\alpha, S \psi >)
\]
(and a corresponding equation for \( \partial^* \)). If \( S \) is the canonical complex structure of \( L \) then
\[
Q^\dagger|_{V^*_{n-1}} = -A^*|_{L^\perp} = 0,
\]
and \( S^* \) is the canonical complex structure of the dual curve \( L^* \). In particular, the dual curve of a Frenet curve is Frenet. In the case of a Frenet curve \( f : M \to S^4 \), the dual curve is given by the antipodal map since \( L^* = V^\perp_{n-1} = L^\perp \).

A holomorphic curve \( f : M \to \mathbb{HP}^n \) induces a holomorphic structure \( D \) on the dual bundle \( L^{-1} \) of \( L = f^* \Sigma \). Recall the definition of a (quaternionic) holomorphic structure:

**Definition 2.7** (see [PP98]). A holomorphic structure on a complex quaternionic vector bundle \((V,S)\) is a real linear map \(D : \Gamma(V) \to \Gamma(KV)\) satisfying
\[
D(\psi \lambda) = (D \psi) \lambda + \frac{1}{2}(\psi d\lambda + S \psi * d\lambda), \quad \lambda : M \to \mathbb{H}.
\]
We denote by \( \mathcal{H}^0(V) = \ker D \subset \Gamma(V) \) the space of holomorphic sections and call \((V,S,D)\) a holomorphic quaternionic vector bundle. A subbundle \( W \subset V \) is called a holomorphic subbundle of \((V,S,D)\) if \( \Gamma(W) \) is \( D \) stable. In this case \((W,S|_W, D|_W)\) is a holomorphic quaternionic vector bundle.

We decompose a holomorphic structure into
\[
D = \bar{\partial} + Q,
\]
where \( \bar{\partial} S = S \bar{\partial} \) and \( Q \in \Gamma(K\operatorname{End}_-(V)) \). The case when \( Q = 0 \) gives the usual theory of (doubles) of complex holomorphic vector bundles. We call \( D = \bar{\partial} \) a complex holomorphic structure.

**Examples 2.8.**

1. If \( \nabla \) is a connection on \( V \) then \( \nabla'' \) is a holomorphic structure on \((V,S)\).
2. We denote by \( \bar{V} \) the complex vector bundle \((V,-S)\) of the complex quaternionic vector bundle \((V,S)\). Then \( \nabla' \) is the \((0,1)\)-part of the connection \( \nabla \) with respect to \(-S\) and \((\bar{V}, \nabla')\) is a holomorphic vector bundle. We call \( \nabla' \) an antiholomorphic structure on \((V,S)\).
3. If \( V_k \) is an \( S \) stable subbundle of \((V,S)\) which is also a holomorphic curve with respect to \( S|_{V_k} \), i.e.,
\[
*\delta = \delta S, \quad \text{where} \quad \delta = \pi_{V_k} \nabla|_{V_k} : V_k \to V/V_k,
\]
then \( V_k \) is in general not a holomorphic subbundle of \( V \) with respect to \( \nabla'' \). The condition for \( V_k \) being a holomorphic subbundle is exactly
\[
*\delta = S \delta,
\]
which is equivalent to the condition that $V_k^\perp$ is a holomorphic curve.

In particular, if $f : M \rightarrow \mathbb{CP}^N$ is a Frenet curve and $S$ is an adapted complex structure of $f$ on $M$, then $S$ stabilizes the flag spaces $V_k$ and satisfies $*\delta_k = S\delta_k = \delta_k S$. Thus the $V_k$’s are holomorphic subbundles of $V$ with respect to $\nabla''$ and holomorphic curves with respect to $S$.

(4) However, the holomorphic curve $f : M \rightarrow \mathbb{CP}^n$ induces a canonical holomorphic structure on the dual bundle $L^{-1}$ of $L$: it is given by the requirement that restrictions of forms $\alpha \in (\mathbb{H}^{n+1})^*$ to $L$ are holomorphic sections. The space $\{\alpha|_L \mid \alpha \in (\mathbb{H}^{n+1})^*\} \subset H^0(L^{-1})$ is a basepoint free linear system [FLPP01, Sec. 2.6].

In fact, the correspondence between holomorphic curves and basepoint free linear systems $H \subset H^0(L^{-1})$ of a holomorphic line bundle $L^{-1}$ is one to one: by the Kodaira correspondence one can consider the bundle $L$ as a subbundle of $V = H^{-1}$, see [FLPP01, Thm. 2.8].

**Lemma 2.9** (see [BFL+02, Lemma 2.4]). Let $S$ be a complex structure on $V$ and $\nabla = \bar{\partial} + \partial + Q + A$ the decomposition of the trivial connection on $V$ with respect to $S$. Then

$$R^{\bar{\partial} + \partial} = -(Q \wedge Q + A \wedge A) = 2S(A^2 - Q^2)$$

and, for $Z \in H^0(TM)$,

$$R^{\bar{\partial} + \partial}_{Z,Z} = 2S(\bar{\partial}_Z \partial_Z - \partial_Z \bar{\partial}_Z).$$

If $V_1$ and $V_2$ are two complex holomorphic vector bundles with complex holomorphic structures $\bar{\partial}_k$, then $\text{Hom}_+(V_1, V_2)$ inherits a complex holomorphic structure $\bar{\partial}$ via

$$\bar{\partial}(A)\psi := \bar{\partial}_2(A\psi) - A(\bar{\partial}_1 \psi).$$

The usual tensor product construction for complex holomorphic structures induces a complex holomorphic structure on $K\text{Hom}_+(V_1, V_2)$.

**Lemma 2.10.** Let $S$ be a complex structure on $V$ and let $V_k \subset V$ be an $S$ stable subbundle. Then the following statements are equivalent:

1. $V_k$ is $A, Q$ and $\bar{\partial}$ stable.
2. $*\delta_k = S\delta_k = \delta_k S$ where $\delta_k = \pi_{V_k} \nabla|_{V_k} : V_k \rightarrow V/V_k$.

In this case

$$\delta_k = \pi_{V_k} \bar{\partial} \in H^0(K\text{Hom}_+(V_k, V/V_k))$$

is a holomorphic section. Here the holomorphic structure on $V_k$ is $\bar{\partial}$ and the holomorphic structure on $V/V_k$ is defined by

$$\bar{\partial} \pi_{V_k} = \pi_{V_k} \bar{\partial}.$$ 

**Proof.** By a type consideration, a vector bundle is $\bar{\partial} + A + Q$ stable if and only if it is stable under $\bar{\partial}$, $A$ and $Q$. Since

$$\delta_k = \pi_{V_k} (\bar{\partial} + \bar{\partial} + Q + A),$$

we see that $V_k$ is $\bar{\partial} + A + Q$ stable if and only if $\delta_k = \pi_{V_k} \bar{\partial}$. But this is equivalent to $*\delta_k = S\delta_k = \delta_k S$ again by type considerations $*A = -AS, *Q = -SQ$, and $*\bar{\partial} = -S \bar{\partial}$.

Since $\bar{\partial}$ maps sections of $V_k$ to one-forms in $V_k$, we can define a holomorphic structure $\bar{\partial}$ on $V/V_k$ by

$$\bar{\partial} \pi_{V_k} = \pi_{V_k} \bar{\partial}.$$ 

By (2.8) we see that $R^S$ stabilizes $V_k$ and we obtain for any local holomorphic sections $\psi \in H^0(V_k)$ and $Z \in H^0(TM)$:

$$\bar{\partial}_Z(\delta_k(Z, \psi)) = \bar{\partial}_Z(\pi_{V_k}\partial_Z\psi) = \pi_{V_k}(\bar{\partial}_Z(\partial_Z\psi)) = \pi_{V_k}(\partial_Z \bar{\partial}_Z \psi) = 0.$$
Thus, $\delta_k$ is holomorphic because it maps holomorphic sections $\psi \in H^0(V_k)$ and $Z \in H^0(TM)$ to a holomorphic section $\delta_k(Z, \psi) \in H^0(V/V_k)$.

Since the flag derivatives of a Frenet curve are holomorphic sections by the previous lemma, we see that a Frenet curve has a smooth Frenet flag.

**Corollary 2.11** (see [LP03]). The Frenet flag of a Frenet curve $f : M \to \mathbb{HP}^n$ is smooth on $M$.

**Proof.** The canonical complex structure $S$ of the Frenet curve $f$ exists smoothly on $M$. In particular, $*\delta_0 = S\delta_0 = \delta_0 S$ and the previous Lemma shows that $\delta_0 \in H^0(K\text{Hom}_+(L, V/L))$ is a holomorphic section. Therefore, the image of $\delta_0$ defines a smooth subbundle $V_1 \subset V$. Proceeding inductively, the Frenet flag exists smoothly on $M$.

Using again a type argument as in the proof of Lemma 2.10 we derive a criterion to decide whether a given complex structure is the canonical complex structure of a Frenet curve:

**Lemma 2.12.** Let $S$ be a complex structure on $V$. Assume that $L \subset V$ is a holomorphic curve with respect to $S$ and that $L$ has a Frenet flag $L \subset V_1 \subset \ldots \subset V_n = V$. Denote the derivative of $V_k$ by $\delta_k$ and define

$$\delta^k := \delta_{k-1} \circ \ldots \circ \delta_0$$

for $k = 1, \ldots, n$, where $\delta^0 = \text{id}|_L$ denotes the identity map of $V$ restricted to $L$. With the usual decomposition $\nabla = \bar{\partial} + \partial + Q + A$ of $\nabla$ with respect to the complex structure $S$, the following are equivalent

1. $Q\delta^k = 0$ for all $k = 0, \ldots, n-1$.
2. $L$ is a Frenet curve and $S$ is the canonical complex structure of $L$.

**Proof.** Since the canonical complex structure $S$ of a Frenet curve satisfies $Q|_{V_{n-1}} = 0$ and $\text{Im} \, \delta_j \subset V_{j+1}/V_j \subset V_{n-1}/V_j$, we get $Q|_{\text{Im} \, \delta_j} = 0$ for all $j = 0, \ldots, n-2$.

For the converse, observe that $Q\delta^i = 0$ for all $i \leq k$ implies that $V_k$ is $Q$ stable, i.e., the derivative of $V_k$ is given by

$$(2.11) \quad \delta_k = \pi_{V_k}(\partial + \bar{\partial} + A)|_{V_k}.$$ 

We proceed by induction. Since $L$ is a holomorphic curve with respect to $S$, we have $*\delta_0 = \delta_0 S$. But then (2.11) shows that $V_0 = L$ is $A$ and $\bar{\partial}$ stable and we get

$$*\delta_0 = S\delta_0$$

since $\delta_0 = \pi_L(\partial)$. In particular, the complex structure of the Frenet flag, see Definition 2.11, is given by $J_1 = S$ on $V_1/L$, and $*\delta_1 = \delta_1 S$. Proceeding inductively, we see $J_k = S$ on $V_k/V_{k-1}$ and $\delta_k = \pi_{V_k}\partial|_{V_k}$. Hence $S$ is adapted to the flag, and satisfies $Q|_{V_{n-1}} = 0$.

We finish this section by a fact on the Hopf fields $A$ and $Q$ of a Frenet curve which will allow later on to describe Willmore curves with vanishing Willmore energy.

**Lemma 2.13.** Let $f : M \to \mathbb{HP}^n$ be a holomorphic curve and $U \subset M$ an open subset of $M$ so that both the canonical complex structure $S$ and the Frenet flag $L \subset V_1 \subset \ldots \subset V_{n-1} \subset V$ are smooth on $U$. Then

1. If the restriction of $A$ to $L$ vanishes on $U$ then $A$ vanishes on $U$.
2. If $Q$ has image in $V_{n-1}$ on $U$ then $Q$ vanishes on $U$. 

Proof. Note first, that \( Q|_{V_{n-1}} = 0 \) implies for \( \varphi \in \Gamma(V_{n-1}) \) that \((d\nabla \ast A)\varphi = (d\nabla \ast Q)\varphi = d\nabla(\ast Q \varphi) + \ast Q \wedge \nabla \varphi = \ast Q \wedge \delta_n \varphi = 0\) by type.

If we assume that \( A|_{V_k} = 0 \) on \( U \) for some \( k \in \{0, \ldots, n-1\} \), then for \( X \in \Gamma(TU) \) and for \( \psi \in \Gamma(V_{k+1}) \) we obtain

\[
2A_X(\delta_k)_X \psi = (\ast A \wedge \delta_k)_X,_{JX} \psi = (d\nabla \ast A)_X,_{JX} \psi - d\nabla(\ast A)_X,_{JX} = 0,
\]

and hence \( A|_{V_{k+1}} = 0 \) on \( U \). By induction we see that \( A|_{L} = 0 \) on \( U \) implies \( A = 0 \) on \( U \).

Moreover, \( \text{Im} \ Q \subset \ker \ Q \) if and only if \( \text{Im} \ A^\dagger \subset \ker \ A^\dagger \) because \( Q^\ast = -A^\dagger \),

\[
\ker A^\dagger = \ker Q^\ast = (\text{Im} \ Q)^\perp, \quad \text{and} \quad \text{Im} \ A^\dagger = \text{Im} Q^\ast = (\ker Q)^\perp.
\]

Since \( \text{Im} A^\dagger = L^\ast \), we see \( A^\dagger|_{L^\ast} = 0 \) and conclude \( A^\dagger = 0 \) on \( U \) by part (1). Therefore, \( Q = -(A^\dagger)^\ast = 0 \) on \( U \).

3. Willmore curves in \( \mathbb{HP}^n \)

The Willmore functional of an immersion to \( S^4 \) can be generalized to the Willmore energy of a holomorphic curve \([LPP01]\). We define Willmore curves in \( \mathbb{HP}^n \) as Frenet curves which are critical points of the Willmore functional under compactly supported variations by Frenet curves, \([LP03]\). As in the case of Willmore surfaces in \( \mathbb{R}^3 \) the Willmore condition is related to harmonicity: the canonical complex structure of a Willmore curve in \( \mathbb{HP}^n \) is harmonic.

Recall that a holomorphic curve \( f : M \to \mathbb{HP}^n \) induces a holomorphic structure on the dual \( L^{-1} \) of the line bundle \( L = f^\ast \Sigma \).

**Definition 3.1.** Let \( f : M \to \mathbb{HP}^n \) be a holomorphic curve from a compact Riemann surface \( M \) into quaternionic projective space. The Willmore energy of \( f \) is given by

\[
W(f) := 2 \int_M < Q_{L^{-1}} \wedge \ast Q_{L^{-1}} >,
\]

where \( Q_{L^{-1}} \) is given by the holomorphic structure \( D = \bar{\partial} + Q_{L^{-1}} \) on \( L^{-1} \), and \( < B > := \frac{1}{2} \text{tr} \ B \) for \( B \in \text{End}(V) \).

**Remark 3.2.** Note that the definition of the Willmore energy is invariant under projective transformations of \( f \). In the case \( n = 1 \), we obtain \([BFL02]\) Prop. 13] the usual Willmore functional

\[
W(f) = \int_M (|H|^2 - K - K^\perp)|df|^2,
\]

of an immersion \( f : M \to S^4 \). Here \( H \) is the mean curvature vector of \( f \), \( K \) the Gaussian curvature, and \( K^\perp \) the curvature of the normal bundle.

From now on, \( M \) will always denote a compact Riemann surface.

For a Frenet curve the Willmore energy can be computed in terms of the canonical complex structure:

**Lemma 3.3.** For a Frenet curve \( f : M \to \mathbb{HP}^n \) the Willmore energy is given by

\[
W(f) = 2 \int_M < A \wedge \ast A >.
\]

**Proof.** Since \( L \) is a Frenet curve the mixed structure \( \frac{1}{2}(\nabla + \ast \nabla \text{S}) \) stabilizes \( L \). But \( \tilde{D} = \frac{1}{2}(\nabla + \ast \nabla \text{S})|_L \) has \( \tilde{D} = \nabla,_{JL} = A|_L \) and satisfies for \( \alpha \in \Gamma(L^{-1}) \), \( \psi \in \Gamma(L) \) the product rule

\[
< D\alpha, \psi > + < \alpha, \tilde{D}\psi > = \frac{1}{2}(d < \alpha, \psi > + \ast d < \alpha, S\psi >),
\]
where $D$ is the holomorphic structure on $L^{-1}$. This equation and (2.7) imply $\hat{D} = \bar{\partial} - Q_{L^{-1}}$ and hence $A|_L = -Q_{L^{-1}}$. Since $f$ is a Frenet curve, $A$ has image in $L$ so that

$$<Q_{L^{-1}} \wedge *Q_{L^{-1}} >= <A|_L \wedge *A|_L >= A \wedge *A > .$$

Definition 3.4 (see [LP03]). A Frenet curve $f : M \to \mathbb{HP}^n$ is called Willmore if $f$ is a critical point of the Willmore energy under compactly supported variations of $f$ by Frenet curves where we allow the conformal structure on $M$ to vary.

Definition 3.5. The energy functional of $S : M \to Z := \{S \in \text{End}(V) \mid S^2 = -I\}$ is given by

\[(3.3) \quad E(S) = \frac{1}{2} \int_M <\nabla S \wedge *\nabla S > = 2 \int_M <Q \wedge *Q > + <A \wedge *A > .\]

A map $S : M \to Z$ is called harmonic if it is a critical point of the energy functional.

Let $\nabla = \bar{\partial} + \partial + Q + A$ the decomposition of the trivial connection $\nabla$ on $V$ with respect to a complex structure $S$. By changing the complex structure to $-S$ we get $K\text{End}_-(V) = K\text{Hom}_+(\bar{V},V)$ and $\partial$ and $\bar{\partial}$ on $V$ induce by (2.10) a complex holomorphic structure on $K\text{End}_-(V)$. If we change the complex structure on $\bar{K}\text{End}_-(V)$ to $-S$ then $\partial$ and $\bar{\partial}$ give a complex holomorphic structure $\bar{\partial}$ on $K\text{End}_-(V) = \bar{K}\text{End}_-(V)$, i.e., an antiholomorphic structure $\partial$ on $\bar{K}\text{End}_-(V)$.

As in [BFL+02, Prop. 5] one shows

Theorem 3.6. Let $S : M \to Z$. Then following are equivalent

1. $S$ is harmonic.
2. $*Q$ is closed which due to (2.2) is the same as $*A$ is closed.
3. $Q$ is antiholomorphic, i.e., $\bar{\partial}Q = 0$.
4. $A$ is holomorphic, i.e., $\bar{\partial}A = 0$.

Moreover, if $f : M \to \mathbb{HP}^n$ is a Frenet curve and $S : M \to Z$ its canonical complex structure, then $S$ is conformal, i.e.,

$$<*\nabla S, *\nabla S >= <\nabla S, \nabla S > .$$

The degree of a complex quaternionic vector bundle $V = E \oplus E$ is defined by the degree of the complex vector bundle $E$ which is given by the $+i$–eigenspace of $S$, see [FLPP01, Sec. 2.1]. Since $\bar{\partial} + \partial$ is a complex connection on $V$, the degree of $V$ can be computed by

\[(3.4) \quad 2\pi \deg(V,S) = \int_M <SR^{\bar{\partial} + \partial} > = \int_M <A \wedge *A > - <Q \wedge *Q > .\]

Combining (3.2), (3.3), and (3.4), we obtain:

Corollary 3.7. Let $f : M \to \mathbb{HP}^n$ be a Frenet curve with canonical complex structure $S$. Then

$$E(S) + 4\pi \deg(V,S) = 2W(L) .$$

Similar techniques as used in the $S^4$–case [BFL+02, Thm. 3] give the usual relation between the Willmore condition and harmonicity.

Theorem 3.8 (see [LP03]). A Frenet curve $f : M \to \mathbb{HP}^n$ is Willmore if and only if the canonical complex structure of $f$ is harmonic, i.e.,

$$d^\nabla *A = 0 .$$
We have the Kodaira correspondence between holomorphic curves \( f : M \to \mathbb{HP}^n \) and base point free linear systems \( H \subset H^0(L^{-1}) \). For a Willmore curve \( f : M \to \mathbb{HP}^n \), it is natural to ask for which choices of basepoint free linear systems \( \hat{H} \subset H^0(L^{-1}) \) the induced holomorphic curve \( \hat{L} \subset \hat{H}^{-1} \) is again Willmore.

**Proposition 3.9.** Let \( f : M \to \mathbb{HP}^n \) be a Willmore curve. Let \( L \subset V \) and \( H \subset H^0(L^{-1}) \) be the corresponding line bundle and basepoint free linear system. Let \( \hat{H} \subset H^0(L^{-1}) \) be a linear system with \( H = V^{-1} \subset \hat{H} \) so that the map \( \hat{f} : M \to \mathbb{HP}^n \) given by the Kodaira correspondence has a canonical complex structure which extends continuously into the Weierstrass points. Then \( \hat{f} \) a Willmore curve in \( \mathbb{HP}^n \) where \( m = \dim \hat{H} \).

**Proof.** Let \( \hat{f}_t : M \to \mathbb{HP}^n \) be a variation of \( \hat{f} \) so that the compact support \( K \) does not contain Weierstrass points. Without loss of generality, we can assume that \( \hat{f}_t \) is unramified on \( K \). Then \( \pi : \hat{V} = \hat{H}^{-1} \to V \) defines a variation of \( f \) by Frenet curves \( f_t : M \to \mathbb{HP}^n \) by \( \pi(\hat{L}_t) = L_t \). Since the Willmore energy only depends on the holomorphic structure on \( L^{-1} \) and not on the linear system, we see that

\[
\frac{\partial}{\partial t} W(\hat{f}_t) = \frac{\partial}{\partial t} W(f_t) = 0.
\]

The usual arguments, see [LP03], show that the canonical complex structure of \( \hat{f} \) is harmonic on \( K \), i.e.,

\[
d^\nabla \ast A = 0
\]

away from the Weierstrass points. A recent result on the removability of singularities of harmonic maps [He04] shows that the canonical complex structure extends smoothly into the Weierstrass points, and therefore \( \hat{f} \) is a Frenet curve. \( \Box \)

**Example 3.10.** In [Pet04] this construction is used to show that Willmore spheres \( f : S^2 \to S^4 \) are soliton spheres. More precisely, there exists a 3-dimensional linear system \( H \subset H^0(L^{-1}) \) such that the Kodaira embedding of \( L \) into \( H^{-1} \) is the dual curve of a twistor projection of a holomorphic curve in \( \mathbb{CP}^5 \).

In general, projections of Willmore curves \( L \subset V \) into flat subbundles \( \hat{V} \subset V \) fail to be Willmore, see [Pet04].

**Proposition 3.11.** Let \( f : M \to \mathbb{HP}^n \) be a Willmore curve with canonical complex structure \( S \) and let \( H \subset H^0(L^{-1}) \) be the corresponding linear system. Let \( \hat{H} \) be an \( S \) stable basepoint free linear system \( \hat{H} \subset H \subset H^0(L^{-1}) \) with \( m = \dim \hat{H} \geq 2 \). Then

\[
\hat{L} = \pi(L) \subset \hat{V} = \hat{H}^{-1}
\]

defines a Willmore curve \( \hat{f} : M \to \mathbb{HP}^n \). Here \( \pi : V = H^{-1} \to \hat{V} \) is the canonical projection.

**Proof.** Since \( f \) is a holomorphic curve, the line bundle \( L \) is full, i.e., \( L \) is not contained in a lower dimensional flat subbundle of \( V \). The kernel \( \hat{H}^\perp = \ker \pi \) of \( \pi \) is \( \nabla \) stable which shows that \( \pi|_L \neq 0 \). Since \( \hat{H} \) is a linear system the induced connection \( \hat{\nabla} \) on \( \hat{V} \) satisfies \( \pi \nabla = \hat{\nabla} \pi \). Moreover,

\[
\pi S =: \hat{S} \pi
\]

defines a complex structure on \( \hat{V} \) since \( \ker \pi = \hat{H}^\perp \) is \( S \) stable. The complex holomorphic structures \( \nabla''_+ \) and \( \nabla''_+ \) on \( V \) and \( \hat{V} \) given by the complex structures \( \hat{S} \) and \( S \) are related by

\[
\hat{\nabla''}_+ \pi = \pi \nabla''_+.
\]

Since \( \nabla''_+ \) and \( \hat{\nabla''}_+ \) stabilize \( L \) and \( \pi L \) respectively, the map \( \pi|_L \) is a complex holomorphic map. In particular, the zeros of \( \pi|_L \) are isolated and the complex bundle \( \text{Im} \pi|_L \) can be
Let $L$ be the Frenet flag of $f$. Since $\pi\pi_L = \pi_L\pi$ we see

$$\delta_0\pi_L = \pi\delta_0.$$  

If $\delta_0 = 0$ then $V_1$ is contained in the flat bundle $L + \ker \pi$ which has rank $\leq n$ since $\dim \ker \pi = \text{rank } V - \text{rank } V \leq n - 1$. This contradicts the assumption that $L$ is a full curve in $V$, i.e., the assumption that $\delta_k \neq 0$ for $k = 0, \ldots, n - 1$. Thus the map $\delta_0 \neq 0$ is complex holomorphic since

$$\ast \delta_0 = S\tilde{\delta}_0 = \tilde{\delta}_0 S,$$

and defines a vector bundle $\tilde{V}_1$. Clearly, $\tilde{V}_1$ extends $\pi V_1$.

Proceeding inductively, we see that $\tilde{\delta}_k\pi|_{V_k} = \pi\delta_k$ and $\tilde{\delta}_k \neq 0$ for all $0 \leq k \leq \text{rank } V - 2$. In particular, $\tilde{L}$ is a full curve in $\tilde{V}$ with Frenet flag $\tilde{V}_k = \pi V_k$. Moreover, $\ast \delta_k = S\tilde{\delta}_k = \tilde{\delta}_k \tilde{S}$ yields that $\tilde{S}$ is an adapted complex structure.

By construction $\tilde{A} = \frac{1}{2} \ast (\tilde{\nabla} \tilde{S})'$ and $A = \frac{1}{2} \ast (\nabla S)'$ satisfy $\tilde{A}\pi = \pi A$, hence $\tilde{S}$ is the canonical complex structure of $\tilde{f}$. In particular $\tilde{f}$ is a Frenet curve, and

$$d\nabla \ast \tilde{A}\pi = \pi d\nabla \ast A = 0.$$

shows that $\tilde{f}$ is Willmore.

**Remark 3.12.** If $\dim \tilde{H} = 1$ the same arguments as in the proof above show that $(\pi(L), \pi S, \pi \nabla)$ defines a flat complex quaternionic line bundle.

Since the Hopf fields $A$ and $Q$ of a Willmore curve are holomorphic, the zeros of $A$ and $Q$ are isolated. Therefore, Lemma 2.13 implies:

**Corollary 3.13.** Let $S$ be the canonical complex structure of a Willmore curve $f : M \to \mathbb{H}P^n$.

1. If $A \neq 0$ then the set

$$\tilde{M} := \{ p \in M \mid L_p \subset \ker A_p \}$$

has no inner points.

2. If $Q \neq 0$ then the set

$$\tilde{M} := \{ p \in M \mid \Im Q_p \subset (V_{n-1})_p \}$$

has no inner points.

We collect some examples of Willmore curves in $\mathbb{H}P^n$ and methods to construct new Willmore curves out of given ones.

**Examples 3.14.**

1. Let $h : M \to \mathbb{C}P^{2n+1}$ be a (complex) holomorphic curve whose $n^{\text{th}}$ osculating space $W_n$ does not contain a quaternionic subspace, i.e., $W_n \oplus W_{n+1} = \mathbb{C}^{2n+2} = \mathbb{H}P^{n+1}$. It is shown in [FLPP01, Lemma 2.7] that the twistor projection $f : M \to \mathbb{H}P^n$ of $h$ has the smooth canonical complex structure $S$ given by $S|_W = i$. Moreover, it is shown that $A|_L = 0$ so that Lemma 2.13 gives that $f$ is Willmore since $A = 0$. Moreover, $f$ has Willmore energy $W(f) = 0$.

   Conversely, every Willmore surface $f : M \to \mathbb{H}P^n$ with $W(f) = 0$ is given as a twistor projection of a holomorphic curve in $\mathbb{C}P^{2n+1}$.

2. Let $f : M \to \mathbb{H}P^n$ be a Willmore curve, and $L \subset V$ the corresponding line bundle. The flat connection $\nabla^*$ on $V^{-1}$ decomposes as $\nabla^* = \bar{\partial}^* + \partial^* - A^* - Q^*$ with respect to the canonical complex structure $S^*$ of the dual curve $f^* : M \to \mathbb{H}P^n$. Therefore, we compute

$$(d\nabla^* \ast A^*)^* = d\nabla \ast A$$
Lemma 4.1. Let \( f : M \to \mathbb{H}P^n \) be a Frenet curve. We compute the degrees of various complex bundles involved in the Plücker relation:

\[ \deg f \text{ is Willmore if and only if } \deg f^* : M \to \mathbb{H}P^n \text{ is Willmore.} \]

(3) A flat connection \( \nabla \) on a complex quaternionic vector bundle \( (V, S) \) is called Willmore connection [FLPP01, Sec. 6.1] if \( S \) is harmonic, i.e., \( d\nabla * A = 0 \). In general, the harmonic complex structure \( S \) will not be the canonical complex structure of a Frenet curve. But if \( \text{rank} A = 1 \) then \( \bar{\partial} A = 0 \) implies that the image of \( A \) defines a \( \bar{\partial} \) holomorphic line bundle. Thus, \( L = \text{Im} A \subset V \) is a Willmore curve if \( Q\delta^k = 0 \) for all \( k = 0, \ldots, n - 1 \) see Lemma 2.12.

Moreover, if \( L \) is Willmore then the connections \( \nabla^\lambda = \nabla + (\lambda - 1)A \) are flat for all \( \lambda = \alpha + \beta S, \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1 \). Denote by \( L^\lambda \) the line bundle \( L \) considered as subbundle in \( (V, \nabla^\lambda) \). If we decompose \( \nabla^\lambda \) with respect to the complex structure \( S \) of \( L \) then \( Q^\lambda = Q \). Thus \( S \) is the canonical complex structure of \( L^\lambda \) and \( L^\lambda \) is a Willmore curve. Its Willmore energy is given by \( W(L^\lambda) = W(L) \).

Notice, that though \( \nabla \) is trivial, the Willmore curves of this family may have holonomy.

4. Plücker relation of a Frenet curve

The quaternionic Plücker relation [FLPP01, Thm. 4.7] gives the Willmore energy of a Frenet curve in terms of the Willmore energy of its dual curve, the genus of the surface and the degree of the associated line bundle. We give a proof of the Plücker relation in the case when \( f : M \to \mathbb{H}P^n \) is a Frenet curve.

We compute the degrees of various complex bundles involved in the Plücker relation:

**Lemma 4.1.** Let \( f : M \to \mathbb{H}P^n \) be a Frenet curve, \( S \) its canonical complex structure and \( L \subset V_1 \subset \ldots \subset V_n = V \) its Frenet flag with corresponding derivatives \( \delta_i \). Then the degree of the bundle \( V_k/V_{k-1} \) with respect to \( S \) is given by

\[
\deg V_k/V_{k-1} = \sum_{i=0}^{k-1} \text{ord} \delta_i - k \deg K + \deg L, \ 0 \leq k \leq n.
\]

where we put \( V_{-1} := \{0\} \).

**Proof.** The degree of a complex holomorphic line bundle \( E \) is given by the vanishing order of any holomorphic section of \( E \). Since \( \ast \delta_i = S\delta_i = \delta_i S \), Lemma 2.10 implies that \( \delta_i \in H^0(K\text{Hom}_+(V_i/V_{i-1}, V_{i+1}/V_i)) \) is a holomorphic section, and thus

\[
\text{ord} \delta_i = \deg(K\text{Hom}_+(V_i/V_{i-1}, V_{i+1}/V_i)).
\]

If \( (V_1, S_1), (V_2, S_2) \) are two complex quaternionic vector bundles then \( \text{Hom}_+(V_1, V_2) \) is canonically isomorphic to \( \text{Hom}_C(E_1, E_2) \), where the \( E_k \) are again the \(+i\)-eigenspaces of \( S_k \). Therefore,

\[
\text{ord} \delta_i = \deg K + \deg V_{i+1}/V_i - \deg V_i/V_{i-1},
\]

and, telescoping this identity, we get

\[
\sum_{i=0}^{k-1} \text{ord} \delta_i = k \deg K + \deg V_k/V_{k-1} - \deg L.
\]

\[\square\]

**Remark 4.2.** The degree of the dual curve is given by

\[
\deg L^* = n \deg K - \deg L - \sum_{i=0}^{n-1} \text{ord} \delta_i,
\]
since \((V/V_{n-1})^{-1} = V_{n-1}^\perp = L^*\).

We now prove the quaternionic Plücker relation [FLPP01, Thm. 4.7] in the case of Frenet curves:

**Theorem 4.3.** Let \(f : M \to \mathbb{HP}^n\) be a Frenet curve with canonical complex structure \(S\). Let \(L \subset V_1 \subset \ldots \subset V_n = V\) be the Frenet flag of \(f\) and \(\delta_i\) the derivatives of \(V_i\). For a compact Riemann surface \(M\) of genus \(g\), the Plücker relation

\[
\deg(V, S) = \frac{1}{4\pi} (W(f) - W(f^*)) = (n + 1)(n(1 - g) + \deg L) + \ord H
\]

holds, where \(\ord H = \sum_{i=0}^{n-1} (n - i) \ord \delta_i\) is the order of the linear system \(H = V^{-1} \subset H^0(L^{-1})\).

**Remark 4.4.** If \(L \subset V\) is a holomorphic curve then \(H = V^{-1} \subset H^0(L^{-1})\) is a basepoint free linear system. The order of \(H\) is defined by [FLPP01, Def. 4.2]

\[
\ord(H) = \sum_{p \in M} \ord_p(H).
\]

where \(\ord_p(H) = \sum_{k=0}^n (n_k(p) - k)\) is the order of \(H\) at \(p\) and \(n_0(p) < \ldots < n_n(p)\) is the Weierstraß gap sequence of \(H\). In the case of a Frenet curve, the expression for the order of \(H\) simplifies to \(\ord H = \sum_{i=0}^{n-1} (n - i) \ord \delta_i\). In particular, if \(f\) is an unramified Frenet curve, then \(\ord H = 0\).

**Proof.** Since as complex vector bundles \(V = \bigoplus_{k=0}^n V_k/V_{k-1}\) we have

\[
\deg(V, S) = \sum_{k=0}^n \deg V_k/V_{k-1} = \sum_{k=0}^n \left( \sum_{i=0}^{k-1} \ord \delta_i - k \deg K + \deg L \right)
\]

\[
= \left( \sum_{k=0}^n \sum_{i=0}^{k-1} \ord \delta_i \right) - \frac{n(n+1)}{2} \deg K + (n+1) \deg L.
\]

Moreover,

\[
4\pi \deg(V, S) = 2 \int_M < A \wedge *A > - < Q \wedge *Q > = W(f) - W(f^*).
\]

**Remark 4.5.** Let \(h : M \to \mathbb{CP}^{2n+1}\) be a holomorphic curve in \(\mathbb{CP}^n\) such that the twistor projection \(f : M \to \mathbb{HP}^n\) of \(h\) is a Frenet curve, compare Examples 3.14 (1). Since \(W(f) = 0\) the Plücker relation shows that the Willmore energy of the dual curve \(f^*\) of \(f\) is given by

\[
W(f^*) = 4\pi \deg(V, S) \in 4\pi \mathbb{N}.
\]

5. Bäcklund transformation on Willmore curves

Using the harmonicity of the canonical complex structure, a similar construction to the \(\bar{\partial}\) and \(\partial\) transforms of a harmonic map into \(\mathbb{CP}^n\), [Wol88], gives the Bäcklund transformation on Willmore curves. We show that the Bäcklund transform \(\tilde{f} : M \to \mathbb{HP}^n\) of a Willmore curve \(f : M \to \mathbb{HP}^n\) is again Willmore provided \(\tilde{f}\) is a Frenet curve. The latter assumption will be void in case of Willmore spheres.

Due to the harmonicity of the canonical complex structure \(A\) is \(\bar{\partial}\)-holomorphic and \(Q\) is \(\partial\)-holomorphic. Thus their kernels and images define smooth subbundles of the trivial \(\mathbb{H}^{n+1}\)-bundle, and we get new maps into \(\mathbb{HP}^n\).
Lemma 5.1. Let \( f : M \to \mathbb{H}P^n \) be a Willmore curve and \( S \) its canonical complex structure. Decomposing the trivial connection \( \nabla = \partial + \bar{\partial} + Q + A \) on \( V = \mathbb{H}^{n+1} \) with respect to \( S \), we see

(1) For \( A \neq 0 \) there exists a rank \( n \) subbundle \( \tilde{W}_{n-1} \subset V \) which agrees with \( \ker A \) except at finitely many points and which satisfies \( \tilde{W}_{n-1} \subset \ker A \).

(2) For \( Q \neq 0 \) there exists a line bundle \( \tilde{L} \subset V \) which agrees with \( \text{Im} Q \) except at finitely many points and satisfies \( \text{Im} Q \subset \tilde{L} \).

In abuse of notation, we write \( \ker A = \tilde{W}_{n-1} \) and \( \text{Im} Q = \tilde{L} \).

The image of \( A \) and the kernel of \( Q \) also define smooth bundles. However, these are the already known bundles \( \text{Im} A = L \) and \( \ker Q = V_{n-1} \) since \( S \) is the canonical complex structure of \( f \). The harmonicity of \( S \) implies that \( \ker A \) is a holomorphic bundle and \( \text{Im} Q \) a holomorphic curve.

Lemma 5.2. Let \( f : M \to \mathbb{H}P^n \) be a Willmore curve with canonical complex structure \( S \).

(1) If \( A \neq 0 \) then \( \ker A \subset V \) is a holomorphic subbundle with respect to the holomorphic structure induced by the complex structure \( -S \) on \( V \).

(2) If \( Q \neq 0 \) then \( \text{Im} Q \subset V \) is a holomorphic curve with respect to the complex structure \( -S \) on \( V \).

Proof. For \( \varphi \in \Gamma(\ker A) \) we have

\[
0 = (d^\nabla * A)\varphi = d^\nabla (*A\varphi) + *A \wedge \nabla \varphi = *A * \nabla \varphi + A \nabla \varphi = A * \tilde{\delta}_{n-1} \varphi + A \tilde{\delta}_{n-1} \varphi = -A S (*\tilde{\delta}_{n-1} \varphi + S \tilde{\delta}_{n-1} \varphi) .
\]

which implies

\[ *\tilde{\delta}_{n-1} + S \tilde{\delta}_{n-1} = 0 , \]

since \( A \) is, interpreted as a map in \( \Omega^1(\text{Hom}(V/\ker A, V)) \), injective away from finitely many points. This shows that \( \ker A \subset V \) is a holomorphic subbundle, compare Example 2.8(3).

Consider the dual curve \( f^* \) of \( f \) which is again a Willmore curve with \( A^\dagger = -Q^* \). Assume that \( Q \neq 0 \), then by the above argument \( \ker A^\dagger \subset V^{-1} \) is a holomorphic subbundle. But then \( \text{Im} Q \subset V \) is a holomorphic curve, by Example 2.8(3) and

\[ (\ker A^\dagger)^\perp = (\ker Q^*)^\perp = \text{Im} Q . \]

□

To be able to deal with holomorphic curves only, we consider instead of the holomorphic bundle \( \ker A \) the holomorphic curve \( (\ker A)^\perp \subset V^{-1} \). One of the main difficulties of this construction is that \( \text{Im} Q \) and \( (\ker A)^\perp \) might fail to be Frenet curves. We will show below that at least in the case of Willmore spheres \( f : S^2 \to \mathbb{H}P^n \) both line bundles are Frenet curves. The general case is more difficult and is a topic to which we will return in a future paper.

Therefore, at least for the purposes of the present paper, we will assume that \( \tilde{L} \) and \( (\ker A)^\perp \) are Frenet curves. We define

\[ \hat{V} \subset V \quad \text{and} \quad \hat{V}^{-1} \subset V^{-1} \]

as the trivial subbundles of \( V \) and \( V^{-1} \) so that the holomorphic curves \( \tilde{L} \subset \hat{V} \) and \( (\ker A)^\perp \subset \hat{V}^{-1} \) are full curves in \( \hat{V} \) and \( \hat{V}^{-1} \) respectively.
Since \((\ker A)^\perp\) is a line subbundle of the dual bundle of \(V\), we will rather consider the dual curve \(\tilde{L} \subset \tilde{V}\) of \((\ker A)^\perp\) which is again a Frenet curve unless \((\ker A)^\perp\) is a constant in \(\mathbb{P}V\).

In this case, we define \(\hat{L} := ((\ker A)^\perp)^{-1}\) to be the dual bundle of \((\ker A)^\perp\).

**Definition 5.3.** Let \(f : M \rightarrow \mathbb{H}P^n\) be a Willmore curve. Then \(\tilde{L} \subset \tilde{V}\) is called the forward Bäcklund transform of \(L\), and \(\hat{L} \subset \hat{V}\) the backward Bäcklund transform of \(L\).

**Remark 5.4.** Note the similarity to the \(\tilde{\partial}\) and \(\partial\) transforms of harmonic maps into \(\mathbb{C}P^n\) \([\text{WolSS}]\): we use the \((0, 1)\)-part \(Q\) and the \((1, 0)\)-part \(A\) of the derivative \(\nabla S\) of the harmonic map \(S : M \rightarrow \mathbb{Z}\) to construct new holomorphic curves. However, our construction will give new Willmore curves rather than the associated harmonic maps. Moreover, to obtain sequences of Willmore surfaces, we will have to guarantee the smoothness of the canonical complex structure of a Bäcklund transform. This can be done, at least in the case of Willmore spheres in \(\mathbb{H}P^n\), and we will see below that in this case the resulting sequence is finite.

To compare our definition of the Bäcklund transformation to the one given in \([\text{BFL}^+02]\) Prop. 17 for conformal immersions \(f : M \rightarrow S^4\), we contemplate the Frenet flag of \(\tilde{L}\).

Let \(k = \text{rank} \tilde{V}\) be the rank of the trivial bundle \(\tilde{V}\) and let \(\tilde{V}_i^*\) be the Frenet flag of the Frenet curve \((\ker A)^\perp \subset \tilde{V}^{-1}\). The Frenet flag of the Bäcklund transform \(\hat{L} = ((\ker A)^\perp)^*\) is thus given \([25]\) by

\[
\tilde{V}_i = (\tilde{V}_{i-1}^*)^\perp.
\]

In the case when \(\tilde{L}\) is a full curve in \(V\) then

\[
\tilde{V}_n = (\tilde{V}_0^*)^\perp = (\ker A)^\perp = \ker A.
\]

In particular, for Willmore surfaces \(f : M \rightarrow S^4\) the Bäcklund transform \(\tilde{f}\) is a full curve in \(\mathbb{H}^2\) unless it is a constant point in \(\mathbb{H}P^1\). Therefore, we obtain the (twofold) forward Bäcklund transform \(\tilde{L} = \tilde{V}_{n-1} = \ker A\) as defined in \([\text{BFL}^+02]\) Prop. 17.

We prove that Bäcklund transforms of Willmore curves are again Willmore curves:

**Theorem 5.5.** The forward and the backward Bäcklund transform of a Willmore curve are again Willmore curves.

**Proof.** Let \(f : M \rightarrow \mathbb{H}P^n\) be a Willmore curve, \(S\) its canonical complex structure and \(\tilde{f} : M \rightarrow \mathbb{H}P^k\) the forward Bäcklund transform of \(f\). The line bundle \((\ker A)^\perp =: \tilde{L}^*\) is a Frenet curve in some trivial quaternionic subbundle \(\tilde{V}^{-1} \subset V^{-1}\) of rank \(k+1\). We denote the induced projection by \(\pi : \tilde{V} \rightarrow \tilde{V}\). The trivial connection \(\nabla\) on \(\tilde{V}^{-1}\) induces a trivial connection \(\nabla\) on \(\tilde{V}\). Since \((\tilde{L}^*)^\perp\) equals \(\pi \ker A\), the Frenet flag of \(\tilde{f}\) is given by \(\tilde{L} \subset \tilde{V}_1 \subset \ldots \subset \tilde{V}_{k-1} \subset \tilde{V}\) where \(\tilde{V}_{k-1} = \pi \ker A\).

Since \(\ker A\) is a holomorphic vector bundle with respect to the holomorphic structure induced by \(-S\), the line bundle \(\tilde{L}^*\) is a holomorphic curve with respect to the complex structure \(-S^*|_{\tilde{V}^{-1}}\). In particular, the canonical complex structure of \(\tilde{f}^*\) is given by

\[
\tilde{S}^* = -S^*|_{\tilde{V}^{-1}} + \tilde{B}^*
\]

with \(\tilde{L}^* \subset \ker \tilde{B}^* = (\text{Im} \tilde{B})^\perp\). Therefore

\[
\tilde{S} = -\pi S + \tilde{B},
\]
defines the canonical complex structure of \( \tilde{f} \) where \( \tilde{B} \in \Gamma(\text{Hom}(V, \tilde{V}_{k-1})) \). The bundle \( \tilde{V}_{k-1} \) is \( \tilde{S} \) stable and for \( \varphi \in \Gamma(V) \) we calculate

\[
\pi_{\tilde{V}_{k-1}}(\ast \nabla \tilde{B} - \tilde{S} \nabla \tilde{B}) \varphi = \pi_{\tilde{V}_{k-1}}(\ast \nabla (\tilde{B} \varphi) - \tilde{B} \ast \nabla \varphi - \tilde{S} \nabla (\tilde{B} \varphi) + \tilde{S} \tilde{B} \nabla \varphi) = \pi_{\tilde{V}_{k-1}}(\ast \nabla (\tilde{B} \varphi) - \tilde{S} \nabla (\tilde{B} \varphi)) = (\ast \delta_{k-1} - \tilde{S} \delta_{k-1}) \tilde{B} \varphi = 0.
\]

This shows that \( \ast \nabla \tilde{B} - \tilde{S} \nabla \tilde{B} \) takes values in \( \tilde{V}_{k-1} \). Since \( Q|_{V_{n-1}} = 0 \) we also obtain

\[
4 \tilde{A} \pi|_{V_{n-1}} = (\tilde{S} \nabla \tilde{S} + \ast \nabla \tilde{S}) \pi|_{V_{n-1}} = (\pi(\tilde{S} \nabla S) - \ast \nabla S - \tilde{B} (\nabla S) + \tilde{S} (\nabla \tilde{B}) + \ast \nabla \tilde{B})|_{V_{n-1}} = (4 \pi Q - \tilde{B} (\nabla S) + \tilde{S} (\nabla \tilde{B}) + \ast \nabla \tilde{B})|_{V_{n-1}}.
\]

But \( \tilde{A} \) maps to \( \tilde{L} \subset \tilde{V}_{k-1} \), so we see that \( \ast \nabla \tilde{B} + \tilde{S} \nabla \tilde{B} \) restricted to \( V_{n-1} \) takes values in \( \tilde{V}_{k-1} \) and so does \( \nabla \tilde{B} \). For \( \psi \in V_{n-1} \) we get

\[
\delta_{k-1} \tilde{B} \psi = \pi_{\tilde{V}_{k-1}}(\nabla \tilde{B}) \psi = 0
\]

and hence

\[
\tilde{S} \pi = - \pi \bar{S} + \bar{B}, \text{ where Im } \bar{B} \subset \tilde{V}_{k-1} \text{ and } V_{n-1} \subset \ker \bar{B}.
\]

This yields

\[
\ast \nabla \pi \psi + (\nabla \tilde{S}) \pi \psi + \tilde{S} \nabla \pi \psi = \pi \ast \nabla \psi + \nabla (\tilde{S} \pi \psi) = \pi (\ast \nabla \psi - \nabla S \psi) \in \Omega^1(\pi L),
\]

for \( \psi \in \Gamma(V_{n-1}) \) since \( \pi_L(\ast \nabla \psi - \nabla S \psi) = \ast \delta_0 \psi - \delta_0 S \psi = 0 \). Moreover, \( \tilde{S} \) stabilizes \( \pi L \) by

\[
\tilde{S} \pi \varphi = - \pi \bar{S} \varphi + B \varphi = - \pi \bar{S} \varphi \in \pi L \text{ for } \varphi \in L.
\]

Thus we also get

\[
- \nabla \pi \psi + (\ast \nabla \tilde{S}) \pi \psi + \tilde{S} \ast \nabla \psi \in \Omega^1(\pi L)
\]

and

\[
\ast \nabla \pi \psi + (\ast \nabla \tilde{S}) \pi \psi + \tilde{S} \ast \nabla \psi \in \Omega^1(\pi L).
\]

Subtracting these equations, we find

\[
4 \tilde{Q} \pi \psi = (\tilde{S} (\nabla \tilde{S}) - \ast \nabla \tilde{S}) \pi \psi \in \Omega^1(\pi L) \text{ for } \psi \in \Gamma(V_{n-1}).
\]

Since \( \tilde{S} \) is the canonical complex structure of \( \tilde{f} \), i.e.,

\[
\tilde{Q}|_{\pi(\ker A)} = \tilde{Q}|_{\tilde{V}_{k-1}} = 0,
\]

this implies that \( \tilde{Q} \) takes values in \( \pi L \). Moreover,

\[
4 \tilde{Q} \pi = (\tilde{S} \nabla \tilde{S} - \ast \nabla \tilde{S}) \pi = (\tilde{S} \nabla \bar{B} - \bar{B} \nabla S - \ast \nabla \bar{B}) + \pi (S \nabla S + \ast \nabla S) = (\tilde{S} \nabla \bar{B} - \bar{B} \nabla S - \ast \nabla \bar{B}) + 4 \pi A.
\]

Since \( \bar{B} \) and \( \ast \nabla \bar{B} - \tilde{S} \nabla \bar{B} \) map to \( \tilde{V}_{k-1} \) while \( \pi A, \tilde{Q} \) have values in \( \pi L \), we obtain \((\tilde{S} \nabla \bar{B} - \bar{B} \nabla S - \ast \nabla \bar{B}) = 0 \) and

\[
(5.3) \quad \tilde{Q} \pi = \pi A.
\]

Now \((d^c \ast \tilde{Q}) \pi = \pi (d^c \ast A) = 0 \) yields by Theorem that \( \tilde{f} \) is Willmore.
Assume that $f : M \to \mathbb{H}P^n$ is a Willmore curve such that $Q \neq 0$ and such that $\hat{f}$ is a Frenet curve in some trivial quaternionic subbundle $\tilde{V} \subset V$. The dual curve $f^*$ of $f$ is Willmore, and by the above argument $\hat{f}^*$ is Willmore, too. Finally,

$$L^* = ((\ker A^\dagger)^\perp)^* = \hat{L}^*.$$

shows that $\hat{f}$ is the dual curve of $\hat{f}^*$ and therefore Willmore.

For later use we collect the information we have on the canonical complex structure $\hat{S}$ of $\hat{f}$:

The canonical complex structure $\hat{S}^*$ of $\hat{f}^*$ induces $\hat{S}$ via $\hat{S} = \hat{S}^*$, and thus

$$\hat{S} = -S|_{\hat{V}} + \hat{B},$$

where $\hat{L} \subset \ker \hat{B}$ and $\text{Im} \hat{B} \subset L$. Let $\pi : V^{-1} \to \hat{V}^{-1}$ be the canonical projection. Then

$$\pi A^\dagger = \tilde{Q}^\dagger \pi \quad \text{and} \quad \hat{A} = -(\tilde{Q}^\dagger \pi)^* = - (\pi A^\dagger)^* = Q|_{\hat{V}}.$$

As a consequence of (5.4), we get the relation between the forward Bäcklund transform of a Willmore curve and the backward Bäcklund transform of its dual curve. The dual statement follows similarly.

**Corollary 5.6.** Let $f : M \to \mathbb{H}P^n$ be a Willmore curve and $f^*$ its dual curve.

1. If the forward Bäcklund transform of $f^*$ exists, then the backward Bäcklund transform of $f$ exists and

$$\tilde{f}^* = \hat{f}^*.$$

2. If the backward Bäcklund transform of $f^*$ exists, then the forward Bäcklund transform of $f$ exists and

$$\tilde{f}^* = \hat{f}^*.$$

If the Bäcklund transform $\tilde{f}$ of a Willmore curve $f$ is a Frenet curve in $\mathbb{H}P^n$ then $\tilde{f}$ has $\tilde{Q} = A$ by (5.3). Hence $\text{Im} \tilde{Q} = \text{Im} A = L$ and the backward transform of $\tilde{f}$ exists.

**Corollary 5.7.** Let $f : M \to \mathbb{H}P^n$ be a Willmore curve.

1. Assume that the forward Bäcklund transform $\hat{f}$ is a Frenet curve in $\mathbb{H}P^n$. Then the backward Bäcklund transform of $\hat{f}$ exists and

$$\tilde{f} = f$$

2. Assume that the backward Bäcklund transform $\tilde{f}$ is a Frenet curve in $\mathbb{H}P^n$. Then the forward Bäcklund transform of $\tilde{f}$ exists and

$$\hat{f} = f.$$

We are now able to give the Willmore energy of a Bäcklund transform of $f$ in terms of the Willmore energy of $f$.

**Corollary 5.8.** Let $f : M \to \mathbb{H}P^n$ be a Willmore curve, and let $\tilde{f}$ and $\hat{f}$ be the backward and forward Bäcklund transforms of $f$. Then the Willmore energies of $\tilde{f}$ and $\hat{f}$ are given by

$$W(\hat{f}) = W(f^*) \quad \text{and} \quad W(\tilde{f}^*) = W(f).$$

**Proof.**
1. Recall that by (2.6) and (3.2) the Willmore energy of the dual curve \( f^* \) is given by
\[
W(f^*) = \int_M <Q \wedge *Q >.
\]
By (5.5) we have \( \bar{A} = Q|_\tilde{\nu} \). Since \( \text{Im} \ Q = \tilde{L} \subset \tilde{V} \) we get
\[
W(f^*) = \int_M <Q \wedge *Q> = \int_M <Q|_\tilde{\nu} \wedge *Q|_\tilde{\nu}> = \int_M \bar{A} \wedge *\bar{A} = W(\hat{f}).
\]

2. Using Corollary 5.6 we see \( W(\hat{f}^*) = W(\dot{f}^*) \). \( \square \)

### 6. Bäcklund transforms with \(-S\) as the canonical complex structure

Given the forward and backward Bäcklund transforms \( \hat{f} \) and \( \ddot{f} \) of a Willmore curve \( f : M \to \mathbb{H}P^n \), we have seen that the negative \(-S\) of the canonical complex structure of \( f \) renders \( \hat{f}^* \) and \( \ddot{f}^* \) into holomorphic curves. We will now discuss the case when \(-S\) is in fact the canonical complex structure of \( \dddot{f} \) or \( \ddot{f} \). It turns out that in this case the Bäcklund transform comes from complex holomorphic data and \( f \) has integer Willmore energy.

Moreover, the Bäcklund transforms can be used to project \( f \) to a Willmore curve \( \dddot{f} : M \to \mathbb{H}P^{n-k} \) for some suitable \( \mathbb{H}P^{n-k} \subset \mathbb{H}P^n \) such that \( \dddot{f} \) is given by complex holomorphic data.

**Theorem 6.1.** Let \( f : M \to \mathbb{H}P^n \) be a Willmore curve.

1. If the backward Bäcklund transform \( \hat{f} : M \to \mathbb{H}P^k \) is a Frenet curve in \( \mathbb{H}P^k \) with \( k \leq n \) and has canonical complex structure \( \hat{S} = -S \), then \( \hat{f} \) is the dual curve of a twistor projection of a holomorphic curve \( h : M \to \mathbb{C}P^{2k+1} \).
2. If the forward Bäcklund transform \( \ddot{f} : M \to \mathbb{H}P^k \) is a Frenet curve in \( \mathbb{H}P^k \) with \( k \leq n \) and has canonical complex structure \( \ddot{S} = -S \) then \( \ddot{f} \) is the twistor projection of a holomorphic curve \( h : M \to \mathbb{C}P^{2k+1} \).

In both cases, \( f \) has Willmore energy \( W(f) \in 4\pi\mathbb{N} \).

**Proof.**

1. Let \( \tilde{V} \) be the trivial \( k+1 \) bundle so that \( \tilde{L} \subset \tilde{V} \) is a full curve. Since \( \hat{S} = -S|_{\tilde{\nu}} \) is the canonical complex structure of \( \hat{f} \), Lemma 2.12 shows that
\[
A\delta^k = 0,
\]
which implies \( A|_{\tilde{\nu}} = 0 \). Since \( \tilde{Q} = A|_{\tilde{\nu}} \) this yields that \( \hat{f} \) is the dual curve of the twistor projection of a holomorphic curve in \( \mathbb{C}P^{2k+1} \). By Remark 4.5 \( \hat{f} \) has integer Willmore energy, and the Plücker relation together with Corollary 5.8 gives
\[
W(f) = W(\ddot{f}) + 4\pi \deg(V,S) \in 4\pi\mathbb{N}.
\]

2. The dual curve \( f^* : M \to \mathbb{H}P^n \) of \( f \) is a Willmore curve. By Corollary 5.6 the backward Bäcklund transform of \( f^* \) is given by the dual
\[
\hat{f}^* = \hat{f}^*
\]
of the forward Bäcklund transform of \( f \). Since \( \hat{f} : M \to \mathbb{H}P^n \) has canonical complex structure \( -S \), the backward Bäcklund transform \( \hat{f}^* \) has canonical complex structure \( \hat{S} = -S^* \). Using the first part, \( \hat{f} = (f^*)^* \) is the twistor projection of a holomorphic curve \( h : M \to \mathbb{C}P^{2k+1} \), and \( W(f) = W(\hat{f}^*) \in 4\pi\mathbb{N} \).

\( \square \)
For Willmore curves $f : M \to \mathbb{HP}^n$ the backward Bäcklund transform $\hat{f} : M \to \mathbb{HP}^k$ is a Frenet curve in some trivial rank $k + 1$ subbundle $\hat{V} \subset V$ of $V$, $k \leq n$. If the canonical complex structure $\hat{\mathcal{S}}$ is given by $-S$ on $\hat{V}$, then $\hat{V}$ is in particular $S$ stable, too.

If rank $\nabla = n + 1$, i.e., $V = \hat{V}$, then $A\delta^i = 0$ for $i \leq n$ since $-S$ is the canonical complex structure of $\hat{f}$. Because $\text{Im} \delta^i = V_{i+1}/V_i$ except at finitely many points, this shows that $A = 0$. In other words, if the backward Bäcklund transform $\hat{f}$ is a full curve in $V$, then $f$ is the twistor projection of a holomorphic curve in $\mathbb{CP}^{2n+1}$.

If rank $\nabla = k + 1 < n$, the quotient bundle $\hat{V} = V/\nabla$ is a smooth trivial bundle of rank $n - k > 1$. The canonical projection $\pi : V \to \nabla$ has $S$ stable kernel $\ker \pi = \hat{V}$ so that we can project $L$ to a Willmore curve $\hat{L} = \pi(L) \subset \hat{V}$ by Proposition 3.11. In other words, $f : M \to \mathbb{HP}^n$ projects to a Willmore curve $\hat{f} : M \to \mathbb{HP}^{n-k}$.

Moreover, we know that the canonical complex structure of $\hat{f}$ is given by $\hat{S} \pi = \pi S$ so that

$$Q \pi = \pi Q.$$ 

The image $\hat{L}$ of $Q$ is a line subbundle of $\hat{V}$ and thus $\pi Q = 0$ and $Q \pi = 0$. Therefore, we have shown that $\hat{f}$ is the dual curve of a twistor projection of a holomorphic curve in $\mathbb{CP}^n$.

Dualizing this result, we obtain a similar result in case that the forward Bäcklund transform has $-S$ as complex structure.

**Proposition 6.2.** Let $f : M \to \mathbb{HP}^n$ be a Willmore curve so that $f$ and $f^*$ do not come from the twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^{2n+1}$.

1. If the backward Bäcklund transform $\hat{f} : M \to \mathbb{HP}^k$, $k < n$, has canonical complex structure $-S$ then the line bundle $L$ projects under the canonical projection $\pi : V \to \nabla$ to a Willmore curve $\hat{L} = \pi(L)$ in the rank $n - k$ trivial bundle $V = V/\nabla$ unless $\hat{L} = \nabla$ is 1-dimensional.

   More precisely, $\pi(f) = \hat{f} : M \to \mathbb{HP}^{n-k}$ is the dual curve of a twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^n$ unless $\hat{f}$ is a constant point.

2. If the forward Bäcklund transform $\hat{f} : M \to \mathbb{HP}^k$, $k < n$, has canonical complex structure $-S$ then the dual line bundle $L^*$ of $L$ projects under the canonical projection $\pi : V^{-1} \to (\nabla^*)$ to a Willmore curve $\hat{L}^* = \pi(L^*)$ in the rank $n - k$ trivial bundle $(\nabla^*) = V^{-1}/\nabla^{-1}$ unless $\hat{L}^* = (\nabla^*)$ is 1-dimensional.

   More precisely, $\pi(f^*) = (f^*) : M \to \mathbb{HP}^{n-k}$ is the dual curve of a twistor projection of a holomorphic curve $h : M \to \mathbb{CP}^n$ unless $(f^*)$ is a constant point.

**Remark 6.3.** The assumption that the canonical complex structure of a Bäcklund transform of a Willmore curve $f : M \to \mathbb{HP}^n$ is the negative of the canonical complex structure of $f$ does not hold in general: for example, there are Willmore tori $f : T^2 \to S^4$ which come from integrable system methods and do not have integer Willmore energy, c.f. [FP90].

7. Willmore spheres in $\mathbb{HP}^n$

We show that the forward and backward Bäcklund transforms of a Willmore sphere $f : S^2 \to \mathbb{HP}^n$ are Frenet curves whose canonical complex structures are the negative $-S$ of the canonical complex structure $S$ of $f$. In particular, combining the results of the previous section, we see that a Willmore sphere $f$ has integer Willmore energy $W(f) \in 4\pi\mathbb{N}$ and is either a minimal surface in $\mathbb{R}^4$ with planar ends, or $f$ or its dual curve $f^*$ is, at most after projection to a suitable $\mathbb{HP}^m$, a twistor projection of a holomorphic curve in complex projective space. In particular, a Willmore sphere is given by complex holomorphic data.
To show that a Bäcklund transform of a Willmore sphere is Frenet, we construct the Frenet flag and an adapted complex structure recursively.

**Lemma 7.1.** Let \( f : M \to \mathbb{H}^n \) be a Willmore curve with canonical complex structure \( S \) such that \( Q \neq 0 \) and let \( \hat{L} = \text{Im} \, Q \). Assume that there exists for \( 0 \leq k \leq n - 1 \) rank \( i + 1 \) bundles \( \hat{V}_i, i \leq k \), such that
\[
\nabla \Gamma(\hat{V}_i) \subset \Omega^1(\hat{V}_{i+1}), \quad 0 \leq i \leq k - 1.
\]
Define
\[
\hat{\delta}^i = \hat{\delta}_{i-1} \circ \hat{\delta}_{i-2} \circ \ldots \circ \hat{\delta}_0, \quad 1 \leq i \leq k,
\]
and \( \hat{\delta}^0 = \text{id} \mid \hat{L} \), where \( \hat{\delta}_i = \pi_{\hat{V}_i} \nabla \mid \hat{V}_i \) are the derivatives of the \( V_i \).

If \( -S \) is adapted to the flag, i.e.,
\[
*\hat{\delta}_i = -S\hat{\delta}_i = -\hat{\delta}_i S
\]
for \( 0 \leq i \leq k - 1 \), then the following statements hold:

1. For all \( i = 0, \ldots, k \),
\[
A\hat{\delta}^i Q \in H^0(K^{i+2} \text{Hom}_+(V/V_{n-1}, L))
\]
   is a holomorphic section. In particular, if \( f : S^2 \to \mathbb{H}^n \) is a Willmore sphere, then \( A\hat{\delta}^i Q = 0 \) for all \( i = 0, \ldots, k \).

2. If \( A\hat{\delta}^i Q = 0 \) for all \( 0 \leq i \leq k \), then the image of the derivative \( \hat{\delta}_k \) of \( V_k \) defines a rank \( k + 2 \) subbundle \( \hat{V}_{k+1} \subset V \) provided \( \hat{\delta}_k \neq 0 \). In this case, the derivative of \( V_k \) satisfies
\[
*\hat{\delta}_k = -S\hat{\delta}_k = -\hat{\delta}_k S.
\]

**Proof.**

1. Note that \( A\hat{\delta}^i Q \in \Gamma(K^{i+2} \text{Hom}_+(V/V_{n-1}, L)) \) since \( S \) is the canonical complex structure of \( f \) and thus ker \( Q = V_{n-1} \) and Im \( A = L \).

   By Lemma 2.10 the derivatives \( \hat{\delta}_i \) are \( \partial \)-holomorphic. Since \( f \) is a Willmore curve, \( A \) and \( Q \) are holomorphic and antiholomorphic respectively, so that
\[
\partial(A\hat{\delta}^i Q) = (\partial A)\hat{\delta}^i Q + A(\partial(\hat{\delta}^i Q)) = 0
\]
which shows that \( A\hat{\delta}^i Q \in H^0(K^{i+2} \text{Hom}_+(V/V_{n-1}, L)) \) is a holomorphic section. Since the degree of a complex holomorphic bundle is given by the order of any non-vanishing holomorphic section, we see
\[
\text{ord}(A\hat{\delta}^i Q) = \text{deg} K^{i+2} + \text{deg} L - \text{deg} V/V_{n-1} = (i + 2 + n) \text{deg} K - \sum_{j=0}^{n-1} \text{ord} \delta_j,
\]
provided \( A\hat{\delta}^i Q \neq 0 \). If \( f : S^2 \to \mathbb{H}^n \) is a Willmore sphere, then \( \text{deg} K < 0 \) whereas the order of a holomorphic section is nonnegative. Thus, the above equation cannot hold, and \( A\hat{\delta}^i Q \) has to vanish.

2. The assumption \( A\hat{\delta}^i Q = 0 \) for all \( i \leq k \) implies that \( \hat{V}_k \subset \text{ker} \, A \), i.e. \( \hat{V}_k \) is \( A \)-stable. But \( V_k \) is \( Q \)-stable, too, since Im \( Q = \hat{L} \subset \hat{V}_k \). In view of Lemma 2.10 it remains to show that \( V_k \) is also \( \partial \)-stable. Lemma 2.10 then shows that \( -S \) is adapted and \( \hat{\delta}_k \) is \( \partial \)-holomorphic, so that the image of \( \hat{\delta}_k \) defines the smooth bundle \( \hat{V}_{k+1} \).

   If \( k = 0 \) then \( \hat{V}_k = \hat{L} = \text{Im} \, Q \), which is \( \partial \) stable since \( Q \) is antiholomorphic. If \( k > 0 \) then the flag \( \hat{L} \subset \ldots \subset \hat{V}_k \) has \( -S \) as an adapted complex structure. Hence \( \hat{\delta}_i \)
is $\partial$-holomorphic and $\text{Im} \hat{\delta}_i = \hat{V}_i / \hat{V}_{i-1}$ is $\partial$ stable for $0 \leq i \leq k - 1$. Thus $\hat{V}_k$ is again $\partial$ stable. 

Since the backward Bäcklund transform of a Willmore sphere $f : S^2 \to \mathbb{H}P^n$ is a holomorphic curve with respect to the complex structure $-S$, we can apply the previous Lemma successively as long as $\hat{\delta}_k \neq 0$ to construct the Frenet flag of $\hat{f}$. Since $-S$ is adapted to the flag and $A\delta^k Q = 0$, we see that $-S$ is the canonical complex structure of $\hat{f}$. Dualizing this result, we conclude:

**Corollary 7.2.** A Bäcklund transform of a Willmore sphere $f : S^2 \to \mathbb{H}P^n$ is a Frenet curve. Its canonical complex structure is the negative of the canonical complex structure of $f$.

**Remark 7.3.** The sequence of Willmore spheres obtained by applying successively forward (or backward) Bäcklund transformations breaks down after at most $n$ steps since the $i$th forward Bäcklund transform $f_i : S^2 \to \mathbb{H}P^n$, of $f$ maps to $\mathbb{H}P^n$ with $n_i \leq n - i$.

As we have seen before, a Willmore curve $f : M \to \mathbb{H}P^n$ whose backward Bäcklund transform is a full curve $\hat{f} : M \to \mathbb{H}P^n$ in $\mathbb{H}P^n$ and has negative canonical complex structure is a twistor projection of a holomorphic curve in $\mathbb{C}P^n$. Conversely, we show that the Bäcklund transforms of a twistor projection of a holomorphic curve in $\mathbb{C}P^n$ are Frenet curves:

**Corollary 7.4.** Let $h : M \to \mathbb{C}P^{2n+1}$ be a holomorphic curve in complex projective space such that the twistor projection $f : M \to \mathbb{H}P^n$ of $h$ is a Frenet curve.

1. If $f$ has $Q \neq 0$ then the backward Bäcklund transform $\hat{f} : M \to \mathbb{H}P^k$, $k \leq n$, of $f$ has $\hat{Q} = 0$.
2. If the dual curve $f^*$ of $f$ has $A^1 \neq 0$ then the forward Bäcklund transform $\hat{f}^* : M \to \mathbb{H}P^k$, $k \leq n$, of $f^*$ has $A^1 = 0$.

**Proof.** Since in the first case $A\delta^k Q = 0$ for all $0 \leq k \leq n$, we can construct with Lemma 7.1 successively flag spaces $\hat{V}_k$ as long as $\hat{\delta}_k \neq 0$. Let $\hat{V}$ be the $\nabla$ stable bundle so that $\hat{f}$ is a full curve in $\hat{V}$. By construction, the complex structure $\hat{S} = -S|_{\hat{V}}$ is adapted, and has $\hat{Q} = A|_{\hat{V}} = 0$. In particular, $\hat{f}$ is a Frenet curve with canonical complex structure $\hat{S}$.

The second part is the dual statement of (1). $\square$

We conclude the paper with a classification result for Willmore spheres in $\mathbb{H}P^n$. In the case of a Willmore sphere $f : S^2 \to S^4$ the forward and backward Bäcklund transform coincide and give a point $\infty$ in $\mathbb{H}P^1$ since $AQ = 0$. The canonical complex structure $S$ of $f$ stabilizes $\hat{L} = \hat{L}$. In this case the canonical complex structure gives the mean curvature congruence of the conformal map $f : M \to S^4$, and the fact that $S$ stabilizes $\hat{L} = \hat{L}$ translates to the property that the point $\infty$ lies on all mean curvature spheres of $f$. Using $\infty$ for a stereographic projection of $S^4$ to $\mathbb{R}^4$, we see that $f$ becomes a minimal surface in $\mathbb{R}^4$, see [BFL+02, Sec. 11.2] for the details of this argument. This yields an alternative proof of the result of Ejiri [Eji88], see also [Mon00], that a Willmore sphere $f : S^2 \to S^4$ is either a minimal surface in $\mathbb{R}^4$ with planar ends, or $f$ or its dual curve $f^*$ is a twistor projection of a holomorphic curve $h : S^2 \to \mathbb{C}P^3$.

In the case of Willmore spheres in $\mathbb{H}P^n$ the Bäcklund transforms are not necessary a constant point but can be Frenet curves in a lower dimensional $\mathbb{H}P^k$. Since the canonical complex structure of a Bäcklund transform of a Willmore sphere is $-S$, Proposition 5.2 allows to characterize all Willmore spheres.
Theorem 7.5. Every Willmore sphere \( f : S^2 \to \mathbb{H}P^n \) has Willmore energy
\[
W(f) \in 4\pi\mathbb{N},
\]
and is given by complex holomorphic data.

More precisely, \( f \) is either a minimal surface in \( \mathbb{R}^4 \) with planar ends, or \( f \) or its dual curve \( f^* \) is, at most after projection to a suitable \( \mathbb{H}P^m \), a twistor projection of a holomorphic curve in complex projective space.

Proof. If \( f : S^2 \to \mathbb{H}P^n \) or its dual curve \( f^* \) is a twistor projection of a holomorphic curve in \( \mathbb{C}P^{2n+1} \) then \( f \) has integer Willmore energy \( W(f) \in 4\pi\mathbb{N} \).

Let \( n > 1 \), \( A, Q \neq 0 \), and assume that one of the Bäcklund transforms, without loss of generality say \( \hat{f} \), is a full curve in a rank \( n - 1 \) vector bundle \( \hat{V} \). Then \( \hat{V} = V/\hat{V} \) has rank 1, so that the projection \( \pi : V \to \hat{V} \) only gives a constant point. However, in this case \( \ker A = \hat{V} \) is a \( \nabla \) parallel subbundle of \( V \), and \( (\ker A)^\perp \) is a constant point. In other words, \( \hat{V} \) has rank 1, and the dual curve \( f^* \) of \( f \) projects to a Willmore curve in \( \mathbb{H}P^{n-1} \).

In particular, Proposition 6.2 shows for \( n > 1 \) that at least one of \( f \) and \( f^* \) projects to the dual curve of a twistor projection of a holomorphic curve in complex projective space. Moreover, \( f : S^2 \to \mathbb{H}P^n \) has integer Willmore energy \( W(f) \in 4\pi\mathbb{N} \).

The remaining case \( n = 1 \), \( A, Q \neq 0 \), results in minimal spheres in \( \mathbb{R}^4 \) with planar ends as discussed above. \( \square \)

Acknowledgements. I would like to thank the members of the Department of Mathematics and Statistics at UMass for their hospitality during my stay in Amherst. I’m grateful to the members of GANG at UMass and SFB 288 at TU Berlin, in particular to Franz Pedit and to Ulrich Pinkall, for many fruitful and clarifying discussions on quaternionic holomorphic geometry. I also would like to thank the referee for many helpful suggestions.

References

[BFL+02] F. Burstall, D. Ferus, K. Leschke, F. Pedit, and U. Pinkall. Conformal Geometry of Surfaces in \( S^4 \) and Quaternions. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2002.

[Bry84] R. L. Bryant. A duality theorem for Willmore surfaces. J. Diff. Geom., Vol. 20, pages 23–53, 1984.

[Eji88] N. Ejiri. Willmore surfaces with a duality in \( S^n(1) \). Proc. Lond. Math. Soc., III Ser. 57, No.2, pages 383–416, 1988.

[FLPP01] D. Ferus, K. Leschke, F. Pedit, and U. Pinkall. Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori. Invent. math., Vol. 146, pages 507–593, 2001.

[FPP90] D. Ferus and F. Pedit. \( S^1 \)-equivariant minimal tori in \( S^4 \) and \( S^1 \)-equivariant Willmore tori in \( S^3 \). Math. Z., Vol. 204, pages 269–282, 1990.

[Hédl04] F. Hélein. Removability of singularities of harmonic maps into pseudo–Riemannian manifolds. Ann. Fac. Sci. Toulouse Math., Vol. 1, pages 45–71, 2004.

[LP03] K. Leschke and F. Pedit. Envelopes and Osculates of Willmore surfaces. arXiv: math.DG/0306150, 2003.

[Mon00] S. Montiel. Willmore two-spheres in the four sphere. Trans. Amer. Math. Soc., Vol. 352, pages 4449–4486, 2000.

[Pet04] P. Peters. Soliton Spheres. PhD thesis, Technische Universität Berlin, 2004.

[PP98] F. Pedit and U. Pinkall. Quaternionic analysis on Riemann surfaces and differential geometry. Doc. Math. J. DMV, Extra Volume ICM, Vol. II, pages 389–400, 1998.

[Rig87] M. Rigoli. The conformal Gauss map of submanifolds of the Moebius space. Ann. Global Anal. Geom., Vol. 5, No. 2, pages 97–116, 1987.

[Wil93] T. J. Willmore. Riemannian Geometry. Clarendon Press, Oxford, 1993.

[Wol88] J.G. Wolfson. Harmonic sequences and harmonic maps of surfaces into complex Grassman manifolds. J. Diff. Geom., Vol 27, pages 161–178, 1988.
Katrin Leschke, Institut für Mathematik, Lehrstuhl für Analysis und Geometrie, Universität Augsburg, D-86135 Augsburg

E-mail address: katrin.leschke@math.uni-augsburg.de