§0. Introduction.

A quantum Kac-Moody algebra $U_q(g)$ introduced in [D1] and [Ji] is a $q$-deformation of the universal enveloping algebra $U(g)$ of a Kac-Moody Lie algebra $g$. Lusztig [L] (see also [R]) has shown that the character formula for dominant highest weight representations is preserved when $U_q(g)$ is deformed to $U(g)$.

This paper deals with the quantum affine algebra $U_q(\widehat{gl}_N)$, the quantum toroidal algebra $U_q(sl_N,tor)$, and vertex operators. Our purpose is to give an irreducible vertex representation for the newly developed quantum toroidal algebra $U_q(sl_N,tor)$. This leads an interesting phenomena: the irreducible basic $\widehat{gl}_N$-module allows an irreducible $U_q(\widehat{gl}_N)$ action. Moreover, each weight space as both $U_q(\widehat{gl}_N)$-module and $\widehat{gl}_N$-module coincides. Therefore, this fact enhances the above mentioned Lusztig theorem in the special case $g = \widehat{gl}_N$.

Quantum toroidal algebras were introduced by Ginzburg-Kapranov-Vasserot [GKV] in the study of the Langlands reciprocity for algebraic surfaces. These algebras are quantized analogues for toroidal Lie algebras of Moody-Rao-Yokonuma [MRY]. A Schur type duality between representations of the quantum toroidal algebra (of type $A$) and the double affine Hecke algebra was established by Varagnolo-Vasserot in [VV1]. They further obtained in [VV2] a nice representation for the quantum toroidal algebra by gluing the standard module of the quantum affine algebra together with a level-0 module arising from solvable lattice models. Some other interesting representations were constructed in [FJW], [S] and [TU] from various point of view.

Vertex representations for quantum affine algebras were developed by Frenkel-Jing [FJ] and Jing [J], which are $q$-analogues of Frenkel-Kac [FK] and Segal [Se] construction for affine Lie algebras. Vertex representations of the affine Lie algebra $\widehat{gl}_N$ was given in

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Frenkel [F]. Representations for \( U_q(\hat{\mathfrak{gl}}_N) \) or \( U_q(\hat{\mathfrak{sl}}_N) \) have been studied by many people (for example, [H], [JKKMP] and [KMS]).

We shall use \( N \) copies of orthogonal (or independent) Fock spaces to construct a family of vertex operators as was done in [G]. More precisely, given a non-zero complex number \( p \), we define vertex operators \( X_{ij}(r, z) \) depending on the parameter \( p \), for \( i, j, r \in \mathbb{Z} \). In the non-quantum case (i.e. \( q = 1 \)), there are only finite such operators for a fixed \( r \) as \( X_{ij}(r, z) \) is doubly periodic with respect to indices \( i \) and \( j \). However, in the quantum case, due to the \( q \)-twisting, \( X_{ij}(r, z) \) is no longer doubly periodic so there are infinite such operators for a fixed \( r \). The algebra generated by some of those operators will give an irreducible representation of the quantum toroidal algebra \( U_q(\hat{\mathfrak{sl}}_{N,tor}) \). As one by-product, we obtain another realization of the quantum affine algebra \( U_q(\hat{\mathfrak{sl}}_N) \) and recover the vertex representation of [FJ] for the quantum affine algebra \( U_q(\hat{\mathfrak{sl}}_N) \). The other by-product is the fact of level-0 \( U_q(\hat{\mathfrak{gl}}_N) \) action on level-1 \( U_q(\hat{\mathfrak{sl}}_{N,tor}) \)-module which appeared in [JJKMP].

Our results are based on a new realization of the quantum affine algebra \( U_q(\hat{\mathfrak{gl}}_N) \). In [DF] the quantum affine algebra \( U_q(\hat{\mathfrak{gl}}_N) \) was defined in the context of the quantum Yang-Baxter equation, which contains the quantum affine algebra \( U_q(\hat{\mathfrak{sl}}_N) \) in a canonical way. We construct a new set of generators within the quantum affine algebra \( U_q(\hat{\mathfrak{gl}}_N) \) with the advantage that they are orthogonal (mutually commutative) in the usual sense, and this immediately establishes the isomorphism between the classical enveloping algebra of \( \hat{\mathfrak{gl}}_N \) and its quantum counterpart over the field of functions in \( q^{c/2} \). Drinfeld [D2] showed that in an appropriate completion the quantum enveloping algebra of the simple Lie algebra is isomorphic to the enveloping algebra of the simple Lie algebra up to a twisting of the Hopf algebra structure. Our result adds another example to this general statement and also facilitates the construction of Weyl bases in the algebra.

The main results in sections 1 to 3 were announced in [GJ]. The paper is organized as follows. In Sect. 1 we recall the notion of the quantum toroidal algebra. Sect. 2 lays the foundation for the Fock space representation of \( U_q(\hat{\mathfrak{sl}}_{N,tor}) \) where we emphasize the role of the affine general linear algebra. Sect. 3 is devoted to the proof of our construction. In the last section we construct a new set of generators for the quantum affine algebra \( U_q(\hat{\mathfrak{gl}}_N) \) and through the new basis we show that the module provides both actions for \( U_q(\hat{\mathfrak{gl}}_N) \) and \( U_q(\hat{\mathfrak{sl}}_{N,tor}) \).

§1. Quantum toroidal algebra of type \( A_{N-1} \).

We always assume that the complex number \( q \) is generic and \( N \) is a positive integer with \( N \geq 3 \). Let \( d \) be a nonzero complex number. The quantum toroidal algebra \( U_q(\hat{\mathfrak{sl}}_{N,tor}) \) is the unital associative algebra over \( \mathbb{C} \) generated by \( e_{i,k}, f_{i,k}, h_{i,l}, k^\pm_1 \), where \( i = 0, 1, \ldots, N - 1, k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\} \), and the central elements \( c^\pm_1 \). The relations are
expressed in terms of the formal series:

\[
e_i(z) = \sum_{k \in \mathbb{Z}} e_{i,k} z^{-k}, \quad f_i(z) = \sum_{k \in \mathbb{Z}} f_{i,k} z^{-k},
\]

and

\[
k_i^{\pm}(z) = k_i^{\pm 1} \exp(\pm(q - q^{-1}) \sum_{k=1}^{\infty} h_{i,\pm k} z^{\mp k}),
\]
as follows

\[
k_i k_i^{-1} = k_i^{-1} k_i = cc^{-1} = 1,
\]

\[
[k_i^{\pm}(z), k_j^{\pm}(w)] = 0,
\]

\[
\theta_{-a_{ij}}(c^2 d^{m_{ij}} w z) k_i^{\pm}(z) k_j^{-}(w) = \theta_{-a_{ij}}(c^{-2} d^{-m_{ij}} w z) k_j^{-}(w) k_i^{\pm}(z),
\]

\[
k_i^{\pm}(z) e_j(w) = \theta_{a_{ij}}(c^{-1} d^{m_{ij}} (w z)^{\pm 1}) e_j(w) k_i^{\pm}(z),
\]

\[
k_i^{\pm}(z) f_j(w) = \theta_{\pm a_{ij}}(c d^{m_{ij}} (w z)^{\pm 1}) f_j(w) k_i^{\pm}(z),
\]

\[
[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} (\delta(c^{-2} \frac{z}{w}) k_i^{+}(cw) - \delta(c^2 \frac{z}{w}) k_i^{-}(cw)),
\]

\[
(d^{m_{ij}} z - q^{a_{ij}} w) e_i(z) e_j(w) = (q^{a_{ij}} d^{m_{ij}} z - w) e_j(w) e_i(z),
\]

\[
(q^{a_{ij}} d^{m_{ij}} z - w) f_i(z) f_j(w) = (d^{m_{ij}} z - q^{a_{ij}} w) f_j(w) f_i(z),
\]

\[
\{e_i(z_1) e_i(z_2) e_j(w) - (q + q^{-1}) e_i(z_1) e_j(w) e_i(z_2) + e_j(w) e_i(z_1) e_i(z_2)\}
+ \{z_1 \leftrightarrow z_2\} = 0, \quad \text{if } a_{ij} = -1,
\]

\[
\{f_i(z_1) f_i(z_2) f_j(w) - (q + q^{-1}) f_i(z_1) f_j(w) f_i(z_2) + f_j(w) f_i(z_1) f_i(z_2)\}
+ \{z_1 \leftrightarrow z_2\} = 0, \quad \text{if } a_{ij} = -1,
\]

\[
[e_i(z), e_j(w)] = [f_i(z), f_j(w)] = 0, \quad \text{if } a_{ij} = 0,
\]

where

\[
\theta_m(z) = \frac{q^m z - 1}{z - q^m} \in \mathbb{C}[[z]]
\]
is understood as the Taylor series expansion,

\[
A = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}
\]

is the Cartan matrix of affine type \(A^{(1)}_{N-1}\) and

\[
M = (m_{ij}) = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}
\]

is a skew symmetric matrix (i.e., \(m_{ij} = \delta_{i,j+1} - \delta_{j,i+1}\) for \(0 \leq i, j \leq N - 1\), where \(\delta_{0,N} = \delta_{N0} = 1\).

The reason to call \(U_q(\mathfrak{sl}_N, \text{tor})\) the quantum toroidal algebra is that it is a two-parameter deformation of the enveloping algebra of the toroidal Lie algebra \(\hat{\mathfrak{sl}}_N(\mathbb{C}[x^\pm 1, y^\pm 1])\) (the universal central extension of the double loop algebra \(\mathfrak{sl}_N(\mathbb{C}[x^\pm 1, y^\pm 1])\), see \([MRY]\)). Actually, \(U_q(\mathfrak{sl}_N, \text{tor})\) has been proven to be a one-parameter deformation of the enveloping algebra of a Lie algebra over a quantum torus, see Section 13 in \([VV2]\).

As pointed out in \([GKV]\), the quantum toroidal algebra \(U_q(\mathfrak{sl}_N, \text{tor})\) contains two remarkable subalgebras, the horizontal subalgebra \(\mathcal{U}_h\) and the vertical subalgebra \(\mathcal{U}_v\), where \(\mathcal{U}_h\) is generated by

\[
e_{i,0}, f_{i,0}, k_i^{\pm 1}, \quad i = 0, 1, 2, \ldots, N - 1,
\]

while \(\mathcal{U}_v\) is generated by

\[
e_{i,n}, f_{i,n}, h_{i,l}, k_i^{\pm 1}, \quad i = 1, 2, \ldots, N - 1, n \in \mathbb{Z} \text{ and } l \in \mathbb{Z}^\times.
\]

The central elements of \(\mathcal{U}_h\) and \(\mathcal{U}_v\) are \(k_i^{\pm 1} = \prod_{i=0}^{N-1} k_i^{\pm 1}\) and \(c^{\pm 1}\) respectively. Both \(\mathcal{U}_h\) and \(\mathcal{U}_v\) are isomorphic to the quantum affine algebra \(U_q(\widehat{\mathfrak{sl}}_N)\).

\section{Fock space and vertex operators.}

In this section, we shall set up our Fock space and construct a family of vertex operators indexed by \(\mathbb{Z} \times \mathbb{Z}\).
Let
\[ P = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_N \]
be a rank \( N \) free abelian group provided with a \( \mathbb{Z} \)-bilinear form \( (\cdot, \cdot) \) defined by \( (\varepsilon_i, \varepsilon_j) = \delta_{ij}, 1 \leq i, j \leq N \). Let
\[ Q = \mathbb{Z}(\varepsilon_1 - \varepsilon_2) \oplus \cdots \oplus \mathbb{Z}(\varepsilon_{N-1} - \varepsilon_N) \]
be the rank \((N-1)\) free subgroup of \( P \). Then
\[ \Delta = \{ \alpha \in Q : (\alpha, \alpha) = 2 \} = \{ \varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq N \} \]
is the root system of type \( A_{N-1} \).

Let \( \varepsilon : Q \times Q \to \{ \pm 1 \} \) be a bimultiplicative function such that
\[ \varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma), \quad \varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma) \]
(2.4)
\[ \varepsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)} , \]
(2.5)
for \( \alpha, \beta, \gamma \in Q \). The formula (2.5) immediately implies
\[ \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}, \quad \alpha, \beta \in Q. \]

Let \( C[Q] = \sum \oplus Ce^\alpha \) be the group algebra of \( Q \). For \( \beta \in Q \), define \( e_\beta \in \text{End}C[Q] \) by
\[ e_\beta e^\alpha = \varepsilon(\beta, \alpha)e^{\alpha + \beta}, \quad \text{for } \alpha \in Q. \]
(2.7)
It follows that
\[ e_\alpha e_\beta = (-1)^{(\alpha, \beta)}e^\beta e_\alpha \]
(2.8)
for \( \alpha, \beta \in Q \). Also, for \( \beta \in H = Q \otimes \mathbb{C} \), define \( \beta(0) \in \text{End}C[Q] \) by
\[ \beta(0)e^\alpha = (\beta, \alpha)e^\alpha, \quad \text{for } \alpha \in Q. \]
(2.9)
Next let \( \epsilon_i(n) \) and \( C \) be the generators of the Heisenberg algebra \( \mathcal{H} \), \( 1 \leq i \leq N, n \in \mathbb{Z} \setminus \{0\} \), subject to relations that \( C \) is central and
\[ [\epsilon_i(m), \epsilon_j(n)] = m\delta_{ij}\delta_{m+n,0}C. \]
(2.10)
Define
\[ \epsilon_{ij}(n) = q^{(j-i)|n|^2/2}\epsilon_i(n) - q^{(i-j)|n|^2/2}\epsilon_j(n). \]
(2.11)
Here we observe the relation

\[ \epsilon_{ik}(n) = q^{(k-j)|n|/2} \epsilon_{ij}(n) + q^{(i-j)|n|/2} \epsilon_{jk}(n). \]

We have

\[ [\epsilon_{ij}(m), \epsilon_{ij}(n)] = m \delta_{m+n,0} (q^{(j-i)m} + q^{(i-j)m}) C, \quad j > i, \]

and it is zero if \( i = j \).

Let

\[ \mathcal{S}(\mathcal{H}^-) = \mathbb{C}[\epsilon_i(n) : 1 \leq i \leq N, n \in -\mathbb{Z}_+] \]

denote the symmetric algebra of \( \mathcal{H}^- \), which is the algebra of polynomials in infinitely many variables \( \epsilon_i(n), 1 \leq i \leq N, n \in -\mathbb{Z}_+ \), where \( \mathbb{Z}_+ = \{ n \in \mathbb{Z} : n > 0 \} \). \( \mathcal{S}(\mathcal{H}^-) \) is an \( \mathcal{H} \)-module in which \( C = 1 \), \( \epsilon_i(n) \) acts as the multiplication operator for \( n \in -\mathbb{Z}_+ \), and \( \epsilon_i(n) \) acts as the partial differential operator for \( n \in \mathbb{Z}_+ \).

For \( \alpha \in \{ \pm \epsilon_1, \ldots, \pm \epsilon_N \} \) we introduce the operators \( E_\pm(r, z) \) as follows

\[ E_\pm(\alpha, z) = \exp(\mp \sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{n} z^{\mp n}). \]

It follows that for \( \alpha, \beta \in \{ \pm \epsilon_1, \ldots, \pm \epsilon_N \} \)

\[ E_+^{(\alpha)}(\beta, w) E_+^{(\beta)}(\alpha, z) = E_+^{(\beta)}(\alpha, z) E_+^{(\alpha)}(\beta, w)(1 - \frac{w}{z})^{(\alpha, \beta)}. \]

Set

\[ V_Q = \mathcal{S}(\mathcal{H}^-) \otimes \mathbb{C}[Q]. \]

The operator \( z^\alpha \in (\text{End}\mathbb{C}[Q])[z, z^{-1}] \) is defined as \( z^\alpha = \exp(\alpha(0) \ln z) \).

\[ z^\alpha e^\beta = z^{(\alpha, \beta)} e^\beta \]

for \( \alpha, \beta \in Q \). Thus, in \( (\text{End}\mathbb{C}[Q])[z] \), we have

\[ [\alpha(0), z^\beta] = 0, \quad \text{and} \quad z^\alpha e_\beta = e_\beta z^\alpha \]

for \( \alpha, \beta \in Q \). It is clear that the formula (2.18) expresses \( z^\alpha \) as an operator from \( \mathbb{C}[Q] \) to \( \mathbb{C}[Q][z, z^{-1}] \).
Let $\mu$ be any non-zero complex number. Consider the valuation $\mu^\alpha$ of the operator $z^\alpha$. Namely, $\mu^\alpha$ is the operator $C[Q] \to C[Q]$ given by

$$\mu^{\alpha} e^\beta = \mu^{(\alpha, \beta)} e^\beta, \text{ for } \alpha, \beta \in Q.$$  

(2.20)

Most notations related to formal series and the Fock space used in this section can be found in [FLM] and [J].

Set $\varepsilon_{i+N} = \varepsilon_i$, for $i \in \mathbb{Z}$. Accordingly,

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij} = \delta_{\overline{i}, \overline{j}}, \text{ for } \overline{i}, \overline{j} \in \mathbb{Z}/N\mathbb{Z}.$$  

This implies that $a_{ij} = (\varepsilon_i - \varepsilon_{i+1}, \varepsilon_j - \varepsilon_{j+1})$ and $m_{ij} = (\varepsilon_i, \varepsilon_{j+1}) - (\varepsilon_j, \varepsilon_{i+1})$ which are used in Section 1.

Let $p$ be a non-zero complex number. For $r, i, j \in \mathbb{Z}$, we define the vertex operator $X_{ij}(r, z)$ as follows.

$$X_{ij}(r, z) = \exp(-\sum_{n \neq 0} \frac{(\varepsilon_i(n) - p^{-rn}q^{(i-j)|n|} \varepsilon_j(n))}{n} z^{-n}),$$  

(2.21)
and \( V_Q = \sum_{\mu \in P} \oplus V_\mu \), where \( P = \{ \omega_0 + \alpha + n\tau : \alpha \in Q, n \in \mathbb{Z}_{\leq 0} \} \) is the set of weights, \( V_\mu = \{ v \in V : h.v = \mu(h)v \) for \( h \in \mathfrak{h} \}. More precisely, we have \( V_{\omega_0 + \alpha + n\tau_0} = W \otimes e^{\alpha} \), where \( W \) is the homogeneous subspace of \( S(\mathcal{H}^-) \) of degree \( n + \frac{1}{2}(\alpha, \alpha) \), for \( \alpha \in Q, n \in \mathbb{Z}_{\leq 0} \), and

\[
ch V_Q = \left( \sum_{\alpha \in Q} e^{\omega_0 + \alpha - \frac{1}{2}(\alpha, \alpha) \tau} \varphi(e^{-\tau})^{-N}, \right)
\]

where \( \omega_0|_{H \oplus CD} = 0, \omega_0(C) = 1, \tau|_{H \oplus CC} = 0, \tau(D) = 1. \)

Next, for \( r, i, j \in \mathbb{Z} \), and \( i \neq j \), we define

\[
u_{ij}(r, z) = q^{(j-i)(\epsilon_i - \epsilon_j)}
\cdot \exp\left( \sum_{n \geq 1} \frac{q^{(j-i)n} - q^{(i-j)n}}{n} (q^{\frac{j+i}{2}n} \epsilon_i(n) - p^{-nr} q^{\frac{j+i}{2}n} \epsilon_j(n) z^{-n}) \right)
\]

\[
u_{ij}(r, z) = q^{(i-j)(\epsilon_i - \epsilon_j)}
\cdot \exp\left( \sum_{n \geq 1} \frac{q^{(i-j)n} - q^{(j-i)n}}{n} (q^{\frac{j+i}{2}n} \epsilon_i(-n) - p^{nr} q^{\frac{j+i}{2}n} \epsilon_j(-n) z^{n}) \right).
\]

The normal ordering can be defined as usual. For instance we have

\[
X_{ij}(r_1, z_1) X_{kl}(r_2, z_2) : = E_i(\epsilon_i, z_1) E_j(-\epsilon_j, z_1 q^{i-j} p^{r_1}) E_k(\epsilon_k, z_2) E_j(-\epsilon_j, z_2 q^{k-l} p^{r_2}) E_k(\epsilon_k, z_2) E_i(\epsilon_i, z_1) E_j(-\epsilon_j, z_2 q^{l-k} p^{r_2})
\cdot e_{\epsilon_i - \epsilon_j} (z_1)\frac{1}{(z_1 - z_2)^{\frac{1}{2}}} (z_2 - \frac{1}{2}) \frac{1}{r_1 - \frac{1}{2} r_2}.
\]

Moreover, we have

\[
X_{ij}(r_1, z_1) X_{kl}(r_2, z_2) : = (-1)^{(\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l)} X_{kl}(r_2, z_2) X_{ij}(r_1, z_1) :. \]

where \( r_1, r_2, i, j, k, l \in \mathbb{Z} \).

The following fact is straightforward.

**Lemma 2.29.** For \( r_1, r_2, i, j, k, l \in \mathbb{Z} \),

\[
X_{ij}(r_1, z_1) X_{kl}(r_2, z_2) : = X_{ij}(r_1, z_1) X_{kl}(r_2, z_2) :
\]

\[
\cdot \left( \frac{z_1}{z_2} \right)^{(\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l)} (1 - \frac{z_2}{z_1}) \delta_{ik} (1 - \frac{p^{r_2} q^{k-l} z_2}{p^{r_1} q^{l-i} z_1}) \delta_{jl}
\cdot (1 - \frac{p^{r_2} q^{k-l} z_2}{z_1})^{-\delta_{il}} (1 - \frac{z_2}{p^{r_1} q^{l-i} z_1})^{-\delta_{jk}} p^{-\delta_{jk} r_1 + \delta_{jl} r_1}. \]
Remark 2.31. Note that \( \delta_{ij} \) used above is doubly periodic with respect to \( i \) and \( j \) as \( \delta_{ij} = (\epsilon_i, \epsilon_j) = \delta_{\bar{i}, \bar{j}} \), where \( \bar{i}, \bar{j} \in \mathbb{Z}/N\mathbb{Z} \).

To calculate the commutators of vertex operators, we need some more notations and identities. Set

\[
F_{ij}^{kl}(r_1, r_2, z_1, z_2) = X_{ij}(r_1, z_1)X_{kl}(r_2, z_2) : 
\]

\[
(2.32) \quad (\frac{z_1}{z_2})^{\epsilon_i - \epsilon_j} p^{-\delta_{jk}r_1 + \delta_{ij}r_1} \left(1 - \frac{z_2}{z_1}\right)^{\delta_{ik}} \left(1 - \frac{p^{r_2q^{-l}z_2}}{p^{r_1q^{j-i}z_1}}\right)^{\delta_{jl}} 
\]

\[
\cdot (1 - \frac{p^{r_2q^{-l}z_2}}{z_1})^{1-\delta_{il}} \left(1 - \frac{z_2}{p^{r_1q^{j-i}z_1}}\right)^{-1} \frac{z_2}{p^{r_1z_1}}. 
\]

In particular we have \( F_{ji}^{ij}(r_1, r_2, z_1, z_2) = F_{ij}^{ij}(r_2, r_1, z_2, z_1) =: X_{ij}(r_1, z_1)X_{ji}(r_2, z_2) : \).

Then we can rewrite Lemma 2.29 as follows.

\[
X_{ij}(r_1, z_1)X_{kl}(r_2, z_2) = F_{ij}^{kl}(r_1, r_2, z_1, z_2)(1 - \frac{p^{r_2q^{-l}z_2}}{z_1})^{-1} \left(1 - \frac{z_2}{p^{r_1q^{j-i}z_1}}\right)^{-1} \frac{z_2}{p^{r_1z_1}}. 
\]

(2.33)

The next lemma can be checked directly.

Lemma 2.34. For \( r_1, r_2, i, j \in \mathbb{Z} \) with \( r_1 + r_2 = 0 \) and \( i \neq j \),

\[
\lim_{z_1 \to p^{r_2q^{-l}z_2}} F_{ji}^{ij}(r_1, r_2, z_1, z_2) =: X_{ij}(r_1, p^{r_2q^{j-i}z_2})X_{ji}(r_2, z_2) : 
\]

\[
= u_{ij}(r_1, p^{r_2q^{j-i}z_2}) 
\]

\[
\lim_{z_2 \to p^{r_1q^{j-i}z_1}} F_{ji}^{ij}(r_1, r_2, z_1, z_2) =: X_{ij}(r_1, z_1)X_{ji}(r_2, p^{r_1q^{j-i}z_1}) : 
\]

\[
= v_{ij}(r_1, q^{j-i}z_1) 
\]

The following basic result is similar to (4.16) in [J] whose proof is straightforward.

Lemma 2.35. For any \( a, b \in \mathbb{C} \) and \( a \neq b \), we have in \( \mathbb{C}[[z, z^{-1}]] \)

\[
(1 - az)^{-1}(1 - bz)^{-1} = \frac{z^{-1}}{a-b}((1 - az)^{-1} - (1 - bz)^{-1}) 
\]

Proposition 2.36. If \( r_1 + r_2 = 0 \), then as a formal series we have

\[
(1 - \frac{p^{r_2q^{j-i}z_2}}{z_1})^{-1}(1 - \frac{z_2}{p^{r_1q^{j-i}z_1}})^{-1} \frac{z_2}{p^{r_1z_1}} - (1 - \frac{p^{r_1q^{i-j}z_1}}{z_2})^{-1}(1 - \frac{z_1}{p^{r_2q^{j-i}z_2}})^{-1} \frac{z_1}{p^{r_2z_2}} 
\]

\[
= (q^{j-i} - q^{i-j})^{-1}(\delta(\frac{p^{r_2q^{j-i}z_2}}{z_1}) - \delta(\frac{p^{r_1q^{j-i}z_1}}{z_2})). 
\]
Proof. By Lemma 2.35, we see that the left hand side of the identity is equal to

\[
(q^{-i} - q^{j-i})^{-1}((1 - \frac{p^2 q^{j-i} z_2}{z_1})^{-1} - (1 - \frac{z_2}{p^r q^{j-i} z_1})^{-1})
- (q^{-j} - q^{i-j})^{-1}((1 - \frac{p q^{j-i} z_1}{z_2})^{-1} - (1 - \frac{z_1}{p^r q^{j-i} z_2})^{-1})
= (q^{-i} - q^{j-i})^{-1}\left(\frac{q^{j-i} p^2 z_2}{z_1} - \delta\left(\frac{q^{j-i} p^r z_1}{z_2}\right)\right)
\]

as needed. □

Now we are in the position to show our commutation relation:

**Proposition 2.37.** If \( r_1 + r_2 = 0 \), then

\[
[X_{ij}(r_1, z_1), X_{ji}(r_2, z_2)]
= -\frac{1}{q^{j-i} - q^{i-j}}(u_{ij}(r_1, q^{j-i} p^2 z_2)\delta\left(\frac{q^{j-i} p^2 z_2}{z_1}\right) - v_{ij}(r_1, q^{j-i} z_1)\delta\left(\frac{q^{j-i} p^r z_1}{z_2}\right))
\]

**Proof.** By (2.33), we have

\[
[X_{ij}(r_1, z_1), X_{ji}(r_2, z_2)]
= X_{ij}(r_1, z_1)X_{ji}(r_2, z_2) - X_{ji}(r_2, z_2)X_{ij}(r_1, z_1)
= F_{ji}(r_1, r_2, z_1, z_2)(1 - \frac{p^2 q^{j-i} z_2}{z_1})^{-1}(1 - \frac{z_2}{p^r q^{j-i} z_1})^{-1} \frac{z_2}{p^r z_1}
- F_{ij}(r_2, r_1, z_1, z_2)(1 - \frac{p q^{j-i} z_1}{z_2})^{-1}(1 - \frac{z_1}{p^r q^{j-i} z_2})^{-1} \frac{z_1}{p^r z_2}.
\]

From Proposition 2.36 and the normal ordering relations of the F’s, the above becomes

\[
F_{ji}(r_1, r_2, z_1, z_2)(q^{j-i} - q^{i-j})^{-1}\left(\delta\left(\frac{q^{j-i} p^2 z_2}{z_1}\right) - \delta\left(\frac{q^{j-i} p^r z_1}{z_2}\right)\right).
\]

Now (2.37) follows from (2.34). □

§3. **Representations for the quantum toroidal algebra.**

In this section, we shall use some of vertex operators constructed in the previous section to generate a unital associative algebra which turns out to be a homomorphic image of \( U_q(\mathfrak{sl}_{N,tor}) \). From now on we assume that

\[
p = d^{-N}.
\]
For simplicity, we let
\[
E_i(z) = X_{i,i+1}(0, p^\frac{i}{2} d^i z), \quad F_i(z) = X_{i+1,i}(0, p^\frac{i}{2} d^i z), \quad i = 1, 2, \ldots, N - 1,
\]
\[
E_0(z) = X_{01}(1, p^{-\frac{1}{2}} z), \quad F_0(z) = X_{10}(-1, p^\frac{1}{2} z),
\]
\[
K^+_i(z) = u_{i,i+1}(0, p^\frac{i}{2} d^i z), \quad K^-_i(z) = v_{i,i+1}(0, p^\frac{i}{2} d^i z), \quad i = 1, 2, \ldots, N - 1,
\]
\[
K^+_0(z) = u_{01}(1, p^{-\frac{1}{2}} z), \quad K^-_0(z) = v_{01}(1, p^{\frac{1}{2}} z), \quad K^\pm_i = q^{\pm (\epsilon_i - \epsilon_{i+1})}.
\]

Let \( \mathcal{A} \) be the associative algebra generated by the coefficients of \( E_i(z), F_i(z), K^\pm_i(z) \), for \( 0 \leq i \leq N - 1 \).

We now state our main result of this paper.

**Theorem 3.2.** The linear map \( \pi \) given by
\[
\pi(e_i(z)) = E_i(z), \quad \pi(f_i(z)) = F_i(z), \quad i = 1, 2, \ldots, N - 1,
\]
\[
\pi(e_0(z)) = E_0(z), \quad \pi(f_0(z)) = F_0(z),
\]
\[
\pi(k^+_i(z)) = K^+_i(z), \quad \pi(k^-_i(z)) = K^-_i(z), \quad i = 1, 2, \ldots, N - 1,
\]
\[
\pi(k^+_0(z)) = K^+_0(z), \quad \pi(k^-_0(z)) = K^-_0(z),
\]
\[
\pi(c) = q^\frac{i}{2}, \pi(k^\pm_i) = K^\pm_i, \quad i = 0, 1, \ldots, N - 1
\]
yields an algebra homomorphism from \( U_q(\mathfrak{sl}_{N,\text{tor}}) \) to \( \mathcal{A} \).

**Proof.** The proof will be carried out in several steps.

Step 1. First we have
\[
u_{i,i+1}(r_1, z)v_{j,j+1}(r_2, w) = v_{j,j+1}(r_2, w)u_{i,i+1}(r_1, z) \exp(-(q - q^{-1})^2 J)
\]
where
\[
J = \sum_{n \geq 1} \frac{q^{\frac{n}{2}} \epsilon_i(n) - p^{-nr_1} q^{-\frac{n}{2}} \epsilon_{i+1}(n)}{n} q^{\frac{n}{2}} \epsilon_j(-n) - p^{nr_2} q^{-\frac{n}{2}} \epsilon_{j+1}(-n) \left( \frac{w}{z} \right)^n
\]

Hence the above becomes
\[
= v_{j,j+1}(r_2, w)u_{i,i+1}(r_1, z)
\]
\[
\frac{(1 - q^2 w) \delta_{ij} (1 - q^{-1} w) \delta_{ij}}{(1 - q^2 w) 2 \delta_{ij}} \frac{(1 - p^{r_2 - r_1} q w) \delta_{ij} (1 - p^{r_2 - r_1} q^{-1} w) \delta_{ij}}{(1 - p^{r_2 - r_1} q^{-1} w) 2 \delta_{ij}}
\]
\[
\cdot \frac{(1 - p^{r_2} w) 2 \delta_{ij+1}}{(1 - q^{-2} p^{r_2} w) \delta_{ij+1}} \frac{(1 - p^{r_2} q^{r_1} w) 2 \delta_{ij+1}}{(1 - q^{2} p^{r_1} w) \delta_{ij+1}}
\]

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By restricting \(0 \leq i, j \leq N - 1\) and \(r_1, r_2 = 0, 1\), noting that \(p = d^{-N}\) which is only involved when \(K^+_0(z)\) presents, we thus obtain

\[
(3.3) \quad \theta_{-a_{ij}}(qd^{-m_{ij}} \frac{w}{z})K^+_i(z)K^-_j(w) = \theta_{-a_{ij}}(q^{-1}d^{-m_{ij}} \frac{w}{z})K^-_j(w)K^+_i(z),
\]

for \(0 \leq i, j \leq N - 1\).

Step 2. The relations (1.6-1.7) are proved similarly. We only show one case in the following.

\[
u_i,i+1(r_1, z)X_{jk}(r_2, w) = X_{jk}(r_2, w)u_{i,i+1}(r_1, z)q^{(\epsilon_j - \epsilon_{i+1}, \epsilon_j - \epsilon_k)} \times \exp((-q^{-1} \sum_{n \geq 1} q^{n+1} \frac{w}{z} \epsilon_i(n) - p^{-nr_1} q^{-\frac{n}{2}} \epsilon_{i+1}(n)) \frac{1}{n} \frac{w}{z} n) \times \\
= X_{jk}(r_2, w)u_{i,i+1}(r_1, z)q^{(\epsilon_j - \epsilon_{i+1}, \epsilon_j - \epsilon_k)} (1 - q^{-1} \frac{w}{z}) \delta_{ij} (1 - q^{-k} \frac{1}{z} p^{r_2 - r_1} w) \delta_{i+1,k} \times \\
\left(1 - q^{-k-\frac{4}{3}} p^{r_2 - r_1} w \right) \delta_{i+1,j} \times \\
\left(1 - q^{-k-\frac{4}{3}} p^{r_2 - r_1} w \right) \delta_{i+1,j}
\]

By restricting \(0 \leq i, j, k \leq N, k = j \pm 1\) and \(r_1, r_2 = 0, 1\), noting that \(p = d^{-N}\) which is only involved when \(K^+_0(z)\), \(E_0(z)\) or \(F_0(z)\) presents, we get for \(0 \leq i, j \leq N - 1\)

\[
(3.4) \quad K^+_i(z)E_j(w) = \theta_{-a_{ij}}(q^{- \frac{1}{2}} d^{-m_{ij}} (\frac{w}{z})) E_j(w) K^+_i(z),
\]

\[
(3.5) \quad K^+_i(z)F_j(w) = \theta_{-a_{ij}}(q^{\frac{1}{2}} d^{-m_{ij}} (\frac{w}{z})) F_j(w) K^+_i(z).
\]

Step 3. Assume that \(0 \leq i \leq j \leq N - 1\). One has

\[
X_{i,i+1}(r_1, z)X_{j,j+1}(r_2, w) =: X_{i,i+1}(r_1, z)X_{j,j+1}(r_2, w) : \\
= (\frac{z}{w})^{\delta_{ij} - \frac{\delta_{i+1,j}}{2}} (1 - \frac{w}{z})^{\delta_{ij} (1 - \frac{p^{r_2} q^{-1} w}{p^{r_1} q z}) \delta_{ij} (1 - \frac{w}{p^{r_1} q z}) - \delta_{i+1,j} p^{-\delta_{i+1,j} r_1 + \delta_{ij} r_1} \\
X_{j,j+1}(r_2, w)X_{i,i+1}(r_1, z) =: X_{j,j+1}(r_2, w)X_{i,i+1}(r_1, z) : \\
= (\frac{w}{z})^{\delta_{ij} - \frac{\delta_{i+1,j}}{2}} (1 - \frac{z}{w})^{\delta_{ij} (1 - \frac{p^{r_1} q^{-1} z}{p^{r_2} q w}) \delta_{ij} (1 - \frac{p^{r_1} q^{-1} z}{w}) - \delta_{i+1,j} p^{\delta_{ij} r_2}.
\]

It follows that

\[
(3.6) \quad (d^{m_{ij}} z - q^{a_{ij}} w) E_i(w) E_j(w) = (q^{a_{ij}} d^{m_{ij}} z - w) E_j(w) E_i(z),
\]

\[
(3.7) \quad [E_i(z), E_j(w)] = 0, \text{ if } a_{ij} = 0.
\]
Note that $a_{ij} = 0$ if and only if $|i - j| \neq 0, 1$ and $p = d^{-N}$ which is only involved when $E_0(z)$ presents. We can prove relation (1.10) similarly.

Assume that $0 \leq i \neq j \leq N - 1$.

$$X_{i,i+1}(r_1, z)X_{j,j+1}(r_2, w) =: X_{i,i+1}(r_1, z)X_{j,j+1}(r_2, w) :$$

$$\cdot \left( \frac{z}{w} \right)^{\delta_{i,i+1} + \delta_{j,j+1}} \left( 1 - \frac{w}{z} \right)^{\delta_{i,j+1}} \left( 1 - \frac{p^{r_2} w}{p^{r_1} z} \right)^{\delta_{i+1,j}} p^{\delta_{i+1,j} r_1}$$

$$X_{j+1,j}(r_2, w)X_{i,i+1}(r_1, z) =: X_{j+1,j}(r_2, w)X_{i,i+1}(r_1, z) :$$

$$\cdot \left( \frac{w}{z} \right)^{\delta_{i,j+1} + \delta_{j,j+1}} \left( 1 - \frac{z}{w} \right)^{\delta_{i,j+1}} \left( 1 - \frac{p^{r_1} z}{p^{r_2} w} \right)^{\delta_{i+1,j}} p^{\delta_{i+1,j} r_2}$$

It immediately gives us

$$(3.8) \quad [E_i(z), F_j(w)] = 0,$$

for $0 \leq i \neq j \leq N - 1$.

Step 4. To prove the Serre relation, we set

$$I(z_1, z_2, w) = \frac{w}{(z_1 z_2)^{1/2}} \left( \frac{(z_1 - z_2)(z_1 - q^{-2} z_2)}{(z_1 - q^{-1} w)(z_2 - q^{-1} w)} \right) + (q + q^{-1}) \frac{(z_1 - z_2)(z_1 - q^{-2} z_2)}{(z_1 - q^{-1} w)(w - q^{-1} z_2)} + \frac{(z_1 - z_2)(z_1 - q^{-2} z_2)}{(w - q^{-1} z_1)(w - q^{-1} z_2)} \right).$$

Lemma 3.9. [J] Let $S(z_1, z_2, w) = I(z_1, z_2, w) + I(z_2, z_1, w)$, then

$$S(z_1, z_2, w) = 0.$$

It follows from the usual vertex operator computation that

$$X_{ij}(r_1, z_1)X_{ij}(r_2, z_2)X_{kl}(r_3, w)$$

$$=: X_{ij}(r_1, z_1)X_{ij}(r_2, z_2)X_{kl}(r_3, w) :$$

$$\cdot \left( 1 - \frac{z_1}{z_2} \right) \left( 1 - \frac{p^{r_2} z_2}{p^{r_1} z_1} \right) p^{r_1} \left( \frac{z_1}{w} \right)^{(\epsilon_{i-j} + \epsilon_{k-l})} \left( 1 - \frac{w}{z_1} \right) \delta_{ik}$$

$$(3.10) \quad \cdot \left( 1 - \frac{p^{r_3} q^{-l} w}{p^{r_1} q^{j-i} z_1} \right)^{\delta_{ij}} \left( 1 - \frac{p^{r_3} q^{-l} w}{p^{r_1} q^{j-i} z_1} \right)^{-\delta_{ik}} \left( 1 - \frac{w}{p^{r_1} q^{j-i} z_1} \right)^{-\delta_{jk}}$$

$$\cdot p^{-\delta_{jk} r_1 + \delta_{ji} r_1} \left( \frac{z_2}{w} \right)^{(\epsilon_{i-j} + \epsilon_{k-l})} \left( 1 - \frac{w}{z_2} \right)^{\delta_{ik}} \left( 1 - \frac{p^{r_3} q^{-l} w}{p^{r_2} q^{j-i} z_2} \right)^{\delta_{ji}}$$

$$\cdot \left( 1 - \frac{p^{r_3} q^{-l} w}{z_2} \right)^{-\delta_{ij}} \left( 1 - \frac{w}{p^{r_2} q^{j-i} z_2} \right)^{-\delta_{jk}} p^{-\delta_{jk} r_2 + \delta_{ji} r_2}. $$
Its associated products $X_{ij}(r_1, z_1)X_{kl}(r_3, w)X_{ij}(r_2, z_2), X_{kl}(r_3, w)X_{ij}(r_1, z_1)X_{ij}(r_2, z_2)$ are expressed in terms of their product expansions similarly by using Lemma 2.29. Note that for all $i, j, k, l \in \mathbb{Z}$

$$X_{ij}(r_1, z_1)X_{ij}(r_2, z_2)X_{kl}(r_3, w) := X_{ij}(r_1, z_1)X_{ij}(r_2, z_2) : (-1)^{(e_i - e_j, e_k - e_l)} : X_{ij}(r_1, z_1)X_{kl}(r_3, w)X_{ij}(r_2, z_2) : .$$

Case 1. For $0 \leq i \leq N - 2$, using (3.10) and its associates as well as Lemma 3.9 we have

$$\begin{align*}
\{E_i(z_1)E_i(z_2)E_{i+1}(w) - (q + q^{-1})E_i(z_1)E_{i+1}(w)E_i(z_2) & \\
+ E_{i+1}(w)E_i(z_1)E_i(z_2)\} + \{z_1 \leftrightarrow z_2\} & \\
= : X_{i,i+1}(0, p^{\frac{1}{2}}dz_1)X_{i,i+1}(0, p^{\frac{1}{2}}dz_2)X_{01}(0, p^{\frac{1}{2}}d^{i+1}w) : S(z_1, z_2, dw) = 0
\end{align*}$$

Case 2. It follows from (3.10) that

$$X_{12}(0, p^{\frac{1}{2}}dz_1)X_{12}(0, p^{\frac{1}{2}}dz_2)X_{01}(1, p^{-\frac{1}{2}}w) = : X_{12}(0, p^{\frac{1}{2}}dz_1)X_{12}(0, p^{\frac{1}{2}}dz_2)X_{01}(1, p^{-\frac{1}{2}}w) : \frac{p^{-1}w}{(z_1 - z_2)(z_1 - q^{-2}z_2)} \cdot \frac{(z_1 - z_2)(z_1 - q^{-2}z_2)}{(z_1 - z_2)z_2^{1/2}} \cdot (z_1 - q^{-1}d^{-1}w)(z_2 - q^{-1}d^{-1}w)$$

Then we have

$$\begin{align*}
\{E_1(z_1)E_1(z_2)E_0(w) - (q + q^{-1})E_1(z_1)E_0(w)E_1(z_2) & + E_0(w)E_1(z_1)E_1(z_2)\} + \{z_1 \leftrightarrow z_2\} & \\
= : X_{12}(0, p^{\frac{1}{2}}dz_1)X_{12}(0, p^{\frac{1}{2}}dz_2)X_{01}(1, p^{-\frac{1}{2}}w) : S(z_1, z_2, d^{-1}w)p^{-1} = 0
\end{align*}$$

Case 3. In the case of $(i, i, j) = (0, 0, 1)$ we have as above

$$\begin{align*}
\{E_1(z_1)E_1(z_2)E_0(w) - (q + q^{-1})E_0(z_1)E_1(w)E_0(z_2) & + E_1(w)E_0(z_1)E_0(z_2)\} + \{z_1 \leftrightarrow z_2\} & \\
= : X_{01}(1, p^{-\frac{1}{2}}z_1)X_{01}(1, p^{-\frac{1}{2}}z_2)X_{12}(0, p^{\frac{1}{2}}dw) : S(z_1, z_2, dw) = 0
\end{align*}$$
Case 4. As above we have
\[
\{E_{N-1}(z_1)E_{N-1}(z_2)E_0(w) - (q + q^{-1})E_{N-1}(z_1)E_0(w)E_{N-1}(z_2) \\
+ E_0(w)E_{N-1}(z_1)E_{N-1}(z_2)\} + \{z_1 \leftrightarrow z_2\} = : X_{N-1,N}(0, p^{\frac{1}{2}} d^{N-1} z_1)X_{N-1,N}(0, p^{\frac{1}{2}} d^{N-1} z_2)X_{01}(1, p^{-\frac{1}{2}} w) : \cdot S(z_1, z_2, dw) = 0,
\]
where we have used \( p^{-1} d^{1-N} = d \).

Case 5. Finally we have
\[
\{E_0(z_1)E_0(z_2)E_{N-1}(w) - (q + q^{-1})E_0(z_1)E_{N-1}(w)E_0(z_2) \\
+ E_{N-1}(w)E_0(z_1)E_0(z_2)\} + \{z_1 \leftrightarrow z_2\} = : X_{01}(1, p^{\frac{1}{2}} z_1)X_{01}(1, p^{\frac{1}{2}} z_2)X_{N-1,N}(0, p^{\frac{1}{2}} d^{N-1} w) : \cdot S(z_1, z_2, pd^{N-1} w) = 0.
\]

The quantum Serre relations involving \( F_i \)'s are shown in a similar way. Let \( \alpha_i \) and \( \alpha_j \) be adjacent simple roots in the affine Dynkin diagram of type \( A_{N-1}^{(1)} \). Corresponding to the five cases we have considered above, we have
\[
\{F_i(z_1)F_i(z_2)F_j(w) - (q + q^{-1})F_i(z_1)F_j(w)F_i(z_2) + F_j(w)F_i(z_1)F_i(z_2)\} + \{z_1 \leftrightarrow z_2\} = : F_i(z_1)F_i(z_2)F_j(w) : \begin{cases} \\
S(z_1, z_2, dw) & j = i + 1 \leq N - 1 \\
S(z_1, z_2, d^{-1} w) & i = 1, \quad j = 0 \\
S(z_1, z_2, dw)p^{-1} & i = 0, \quad j = 1 \\
S(z_1, z_2, d^2 w)p & i = N - 1, \quad j = 0 \\
S(z_1, z_2, d^{-1} w)p & i = 0, \quad j = N - 1
\end{cases}
\]
\[
= 0.
\]

Therefore the proof is completed. \( \square \)

**Theorem 3.11.** If \( pq^{\pm N} \) is not a root of unity, then \( V_Q \) is an irreducible \( A \) module. Therefore \( V_Q \) is an irreducible \( U_q(\mathfrak{sl}_{N,\text{tor}}) \) module.

**Proof.** Let \( U \) be a nonzero submodule of
\[
(3.12) \quad V_Q = S(\mathcal{H}^-) \otimes_{\mathbb{C}} \mathbb{C}[Q].
\]
Consider the subalgebra $\mathcal{M}$ generated by the coefficient operators of $k_i^\pm(z)$. Since

$$(3.13) \quad \exp\left(\sum_{n=1}^{\infty} x_n z^n\right) = \sum_{n=0}^{\infty} y_n z^n$$

where

$$y_0 = 1, \quad y_1 = x_1, \quad y_2 = x_2 + \frac{x_1^2}{2}, \quad y_3 = x_3 + x_2 x_1 + \frac{x_1^3}{6},$$

$$y_4 = x_4 + x_3 x_1 + \frac{x_2 x_1^2}{2} + \frac{x_3^2}{2} + \frac{x_1^4}{24}, \ldots,$$

one can easily see that the algebra $\mathcal{M}$ is the same as the algebra generated by $K_i^\pm = q^{\pm(\epsilon_i - \epsilon_{i+1})}$ and

$$(3.14) \quad H_{i,n} = q^{\frac{1}{2}|n|} \epsilon_i(n) - q^{-\frac{1}{2}|n|} \epsilon_j(n), \quad 1 \leq i \leq N - 1, n \in \mathbb{Z} \setminus \{0\},$$

$$(3.15) \quad H_{0,n} = q^{\frac{1}{2}|n|} \epsilon_N(n) - p^n q^{-\frac{1}{2}|n|} \epsilon_1(n), \quad n \in \mathbb{Z} \setminus \{0\}.$$ 

From (2.11)-(2.12) we get

$$(3.16) \quad \epsilon_{1N}(n) = q^{\frac{N-1}{2}|n|} \epsilon_1(n) - q^{\frac{1-N}{2}|n|} \epsilon_N(n) \in \mathcal{M}$$

It follows from (3.15), (3.16) and our assumption on $p$ that

$$\epsilon_1(n), \epsilon_N(n) \in \mathcal{M}, \text{ for } n \in \mathbb{Z} \setminus \{0\}.$$ 

Therefore $\mathcal{M}$ contains the Heisenberg algebra $\mathcal{H}$. Lemma 9.13 in [K] (or Theorem 1.7.3 in [FLM]) implies that $U$ is completely reducible as $\mathcal{H}$-module and so

$$U = S(\hat{\mathcal{H}}^-) \otimes \Omega$$

for some subspaces $\Omega$ of $\mathbb{C}[Q]$. Suppose that $f = \sum_{i=1}^{m} s_i e^{\gamma_i} \in \Omega$, where $s_i \in \mathbb{C}, s_i \neq 0$, $\gamma_i \in Q$, for $1 \leq i \leq m$, and $\gamma_i \neq \gamma_j$ if $i \neq j$.

We claim that $e^{\gamma_k} \in \Omega$ for some $k$.

Since $K_i^\pm = q^{\pm(\epsilon_i - \epsilon_{i+1})}$ lie in $\mathcal{M}$, clearly, $\mathcal{M}$ contains $q^Q$. Pick $\alpha \in Q$ such that $(\alpha, \gamma_{m-1} - \gamma_m) \neq 0$. Then

$$q^\alpha f - q^{(\alpha,\gamma_m)} f = \sum_{i=1}^{m-1} s_i (q^{(\alpha,\gamma_i)} - q^{(\alpha,\gamma_m)}) e^{\gamma_i} = \sum_{i=1}^{m-1} s'_i e^{\gamma_i} \in \Omega,$$
where $s'_i = s_i(q^{(\alpha,\gamma_i)} - q^{(\alpha,\gamma_m)})$, $1 \leq i \leq m - 1$, and $q^\alpha e^\gamma = q^{(\alpha,\gamma)}e^\gamma$. Since $s'_{m-1} \neq 0$ (thanks to the generic $q$), we may continue this process and get some $e^{\gamma_k} \in \Omega$.

Also, from (2.21), we have

$$e_{\epsilon_i - \epsilon_{i+1}} = \exp\left(\sum_{n \in \mathbb{Z}_+} \frac{\epsilon_i(n) - q^n \epsilon_{i+1}(n)}{n} z^{-n}\right)X_{i,i+1}(0,z) \cdot \exp\left(\sum_{n \in \mathbb{Z}_+} \frac{\epsilon_i(n) - q^{-n} \epsilon_{i+1}(n)}{n} z^{-n}\right)z^{\epsilon_{i+1} - \epsilon_i - 1}$$

for $1 \leq i \leq N - 1$. Similarly, for $e_{\epsilon_{i+1} - \epsilon_i}$. It then follows that $e^{\gamma_k + Q} \subseteq \Omega$ and so $\Omega = \mathbb{C}[Q]$. □

**Corollary 3.17.** $V_Q$ is a level-1 $\tilde{U}_v = U_q(\hat{\mathfrak{sl}}_N)$ module and also a level-0 $\tilde{U}_h = U_q(\hat{\mathfrak{sl}}_N)$ module.

Let $\mathcal{B}$ be the unital associative algebra generated by the coefficients of $E_i(z), F_i(z)$ for $1 \leq i \leq N - 1$ (we may assume that $d = p = 1$) and $\epsilon_i(n), q^{\pm \epsilon_i}, 1 \leq i \leq N, n \in \mathbb{Z} \setminus \{0\}$. Note that $\mathcal{M}$ is a subalgebra of $\mathcal{B}$. Then it follows from the proof of Theorem 3.16, we have

**Proposition 3.18.** $V_Q$ is an irreducible $\mathcal{B}$ module.

Our next business is to show that $\mathcal{B}$ is a homomorphic image of $U_q(\hat{\mathfrak{g}}_N)$. Thus $V_Q$ is an irreducible $U_q(\hat{\mathfrak{g}}_N)$-module.

§4. **The quantum affine algebras $U_q(\hat{\mathfrak{g}}_N)$ and $U_q(\hat{\mathfrak{sl}}_N)$.**

In this section we will give a new realization for the quantum affine algebra $U_q(\hat{\mathfrak{g}}_N)$.

We define the quantum affine algebra $U_q(\hat{\mathfrak{g}}_N)$ to be the unital associative algebra generated by $\epsilon_{im}, q^{c/2}, k_{i0}^{\pm 1}, x_j^\pm (i = 1, \cdots, N; j = 1, \cdots, N - 1; n \in \mathbb{Z}, m \in \mathbb{Z}^*)$ subject
to the following relations that \(q^{c/2}\) is central and

\[
\begin{align*}
(4.1) \quad [\epsilon_{im}, \epsilon_{jn}] &= \frac{[m]}{m}[mc]\delta_{ij}\delta_{m,-n}, \\
(4.2) \quad [\epsilon_{im}, x_{jn}^\pm] &= 0, \quad j \neq i, i - 1, \\
(4.3) \quad [\epsilon_{im}, x_{im}^\pm] &= \pm \frac{[m]}{m}q^{\mp|m|c/2+|m|/2}x_{i,m+n}^\pm, \\
(4.4) \quad [\epsilon_{im}, x_{i-1,m}^\pm] &= \pm \frac{[m]}{m}q^{\mp|m|c/2-|m|/2}x_{i-1,m+n}^\pm, \\
(4.5) \quad x_{i,m+1,j}^\pm - q^{\pm a_{ij}}x_{j,n}^\pm x_{i,m+1}^\pm &= q^{\pm a_{ij}}x_{i,m+1,n}^\pm - x_{j,n+1}^\pm x_{i,m}^\pm, \\
(4.6) \quad [x_{im}, x_{jn}^\pm] &= 0, \quad |j - i| > 1, \\
(4.7) \quad x_{im_1}^\pm x_{im_2}^\pm x_{i,m+1,n}^\pm - (q + q^{-1})x_{im_1}^\pm x_{i+1,n}^\pm x_{im_2}^\pm + x_{i\pm,1,n}^\pm x_{im_1}^\pm + \{m_1 \leftrightarrow m_2\} &= 0,
\end{align*}
\]

where \((a_{ij})\) is the Cartan matrix of type \(A\) of rank \(N - 1\), \([m] = \frac{q^m - q^{-m}}{q - q^{-1}}\), \([mc] = \frac{q^{mc} - q^{-mc}}{q - q^{-1}}\), and

\[
(4.9) \quad k_i^\pm(z) = \sum_{n=0}^{\infty} k_{i,n}^\pm z^{-n} = k_{i,0}^\pm e^{\pm(q - q^{-1})\sum_{n>0}(q^{n/2}\epsilon_{in} - 2^{-n/2}\epsilon_{i+1,n})z^{\pm n}}. 
\]

Let \(\epsilon_i(m) = \frac{m}{[m]}\epsilon_{im}\) and \(c = 1\), then we have

\[
[\epsilon_i(m), \epsilon_j(n)] = m\delta_{ij}\delta_{m,-n}I,
\]

Thus the Fock module \(V_Q\) constructed in Sect. 3 is a vertex representation of the quantum affine algebra \(U_q(\widehat{\mathfrak{gl}}_N)\) at level one.

The following easily checked result shows that \(U_q(\widehat{\mathfrak{g}l}_N)\) contains the quantum affine algebra \(U_q(\widehat{\mathfrak{sl}}_N)\) canonically.

**Lemma 4.10.** Let \(h_{im} = q^{\pm|m|/2}\epsilon_{im} - q^{-|m|/2}\epsilon_{i+1,m}\), then the associative subalgebra generated by \(h_{im}, q^{c/2}, k_i^\pm, x_{jn}^\pm (i, j = 1, \ldots, N - 1; n \in \mathbb{Z}, m \in \mathbb{Z}^\times)\) is isomorphic to the quantum affine algebra \(U_q(\widehat{\mathfrak{sl}}_N)\). The commutation relations are as follows: Eqs. (4.5)-(4.9) and

\[
\begin{align*}
(4.11) \quad [h_{im}, h_{jn}] &= \frac{[a_{ij}m]}{m}[mc]\delta_{m,-n}, \\
(4.12) \quad [h_{im}, x_{jn}^\pm] &= \pm \frac{[a_{ij}m]}{m}q^{\mp|m|c/2}x_{j,m+n}^\pm,
\end{align*}
\]
Remark 4.13. In terms of generating functions $x_i^\pm(z) = \sum_n x_{in}^\pm z^{-n}$, $k_i^\pm(z)$, the commutation relations of the quantum affine algebra $U_q(\widehat{sl}_N)$ are given in Eqs. (1.2)-(1.13) with $e_i(z) = x_i^+(z)$, $f_i(z) = x_i^-(z)$ and $d = 1$.

We now introduce another set of basis in the Heisenberg subalgebra to establish isomorphism between the quantum affine algebra $U_q(\widehat{gl}_N)$ defined above and the Ding-Frenkel algebra $U_q(\widehat{gl}_N)$ [DF]. For $m \neq 0$ we define

$$a_{Nm} = q^{Nm+\lfloor m/2 \rfloor} \left( \frac{\lfloor m \rfloor \epsilon_{1m} + q^{\lfloor m/2 \rfloor} \epsilon_{2m} + \cdots + q^{(N-1)[m]} \epsilon_{Nm}}{1 + q^{2[m]} + \cdots + q^{2(N-1)[m]} \lfloor m \rfloor} + \epsilon_{Nm} \right).$$

(4.14)

For each $i = 1, \ldots, N-1$ let

$$a_{im} = \sum_{j=i}^{N-1} q^{jm}(q^{\lfloor m/2 \rfloor} \epsilon_{im} - q^{-\lfloor m/2 \rfloor} \epsilon_{i+1,m}) + a_{Nm}.$$  

(4.15)

Theorem 4.16. The quantum affine algebra $U_q(\widehat{gl}_N)$ is isomorphic to the associative algebra generated by the generators $x_i^\pm, a_{im}, q^{c/2}, k_i^\pm, x_j^\pm (i = 1, \ldots, N; j = 1, \ldots, N-1; n \in \mathbb{Z}, m \in \mathbb{Z}^\times)$ with the commutation relations (4.5-4.8) and:

$$[a_{im}, a_{in}] = 0,$$

(4.17)

$$[a_{im}, a_{jn}] = -\frac{\lfloor m \rfloor \lfloor m \rfloor c q^{-m} \delta_{m,-n}},$$

(4.18)

$$[a_{im}, x_{jn}^\pm] = 0, \quad j \neq i, i+1,$$

(4.19)

$$[a_{im}, x_{in}^\pm] = \pm \frac{\lfloor m \rfloor \lfloor m \rfloor q^{(i-1)m+\lfloor m/2 \rfloor} x_{i,m+n}^\pm},$$

(4.20)

$$[a_{i+1,m}, x_{in}^\pm] = \mp \frac{\lfloor m \rfloor \lfloor m \rfloor q^{(i+1)m+\lfloor m/2 \rfloor} x_{i,m+n}^\pm}.$$  

(4.21)

Proof. Let $\bar{h}_{im} = q^{im} h_{im} = q^{im}(q^{\lfloor m/2 \rfloor} \epsilon_{im} - q^{-\lfloor m/2 \rfloor} \epsilon_{i+1,m})$. First we compute that

$$[a_{Nm}, a_{Nm}] = \frac{\lfloor m \rfloor \lfloor m \rfloor c}{m} \delta_{m,-n}$$

$$- q^{\lfloor m \rfloor} + \left( \frac{\lfloor m \rfloor}{m} + \frac{|n|}{n} \right) \frac{1}{1 + q^{2|m|} + \cdots + q^{2(N-1)|m|} + q^{\lfloor m \rfloor}} = 0,$$

(4.22)
and we can also easily check that

\[(4.23) \quad [\bar{h}_{im}, a_{Nn}] = -\delta_{i,N-1} \frac{q^{-m}}{m}[m]mc\delta_{m,-n}.\]

It follows from Lemma 4.10 and Eq.(4.23) that

\[
[a_{im}, \bar{h}_{jn}] = [\bar{h}_{im} + \cdots + \bar{h}_{N-1,m} + a_{Nm}, \bar{h}_{jn}]
\begin{cases}
0, & i \neq j; j + 1 \\
-\frac{q^{-m}}{m}[m]mc\delta_{m,-n}, & i = j + 1 \\
\frac{q^{-m}}{m}[m]mc\delta_{m,-n}, & i = j 
\end{cases}
\]

\[(4.24)\]

We claim that the new set of generators of the Heisenberg subalgebra satisfy the following relations:

\[(4.25) \quad [a_{im}, a_{jn}] = \begin{cases} 0, & i = j \\
-\frac{q^{-m}}{m}[m]mc\delta_{m,-n}, & \text{for } i > j. \end{cases}\]

In fact for \(i > j\) it follows from Eqs. (4.22-23) that

\[
[a_{im}, a_{jn}] = [a_{i+1,m}, a_{j+1,n}]
\begin{cases}
0, & i = j + 1 \\
-\frac{q^{-m}}{m}[m]mc\delta_{m,-n}, & i = j \\
\frac{q^{-m}}{m}[m]mc\delta_{m,-n}, & \text{for } i > j. 
\end{cases}
\]

Similarly using Eq. (4.23) we obtain that

\[
[a_{im}, a_{in}] = [a_{i+1,m}, a_{i+1,n}] = \cdots = [a_{Nm}, a_{Nn}] = 0.
\]

Now let’s look at the commutation relations \([a_{im}, x_{jn}^{\pm}]\). For simplicity we write

\[
a_{Nm} = c_{1m}\varepsilon_{1m} + c_{2m}\varepsilon_{2m} + \cdots + c_{Nm}\varepsilon_{Nm}.
\]

Then it follows that \(c_{i+1,m} = q^{m|c_{im}|/2} c_{i,m}q^{m|c_{im}|/2} + q^{Nm}\). Using the latter equation and (4.2-4.3) we have that

\[
[a_{Nm}, x_{N-1,n}^{\pm}] = \pm \frac{m}{m} q^{m|c/2 + Nm} x_{N-1,n}^{\pm} (c_{N-1,m}q^{m|c/2} - c_{Nm}q^{-m|c/2})
\]

Similarly one proves that \([a_{Nm}, x_{in}^{\pm}] = 0\) for \(1 \leq i \leq N - 2\) by using \(c_{i+1,m} = q^{m|c_{im}|}.\)

The relations (4.18-19) are then immediate consequences of \(a_{im} = a_{i+1,m} + q^{im}\bar{h}_{im}\) and Lemma 4.10.

The other relations hold automatically, and the theorem is proved. □
Theorem 4.26. The new set of generators $a_{im}, q^{c/2}, k^\pm_{ij}, x^\pm_{jn}$ of $U_q(\widehat{\mathfrak{g}}_N)$ ($i = 1, \ldots, N; j = 1, \ldots, N - 1; n \in \mathbb{Z}, m \in \mathbb{Z}^\times$) satisfy the following commutation relations in terms of generating functions $v^\pm_i(z)$, $x^\pm_i(z)$ where

$$v^\pm_i(z) = v^\mp_{i0} \exp(\mp(q^{-1}) \sum_{n>0} a_{i, \mp n} z^{\pm n}),$$

and the relations are

$$v^\pm_i(z)v^\pm_j(w) = v^\pm_j(w)v^\pm_i(z), \quad v^\pm_j v^\mp_{j0} = 1,$$

$$v^\pm_i(z)v^\pm_i(w) = v^\mp_i(w) v^\pm_i(z),$$

$$\frac{z - wq^{\pm c}}{z - w q^{\pm c - 2}} v^\pm_i(z)v^\pm_j(w) = v^\pm_j(w)v^\mp_i(z) \frac{z - wq^{\mp c}}{z - wq^{c - 2}}, \quad j > i,$$

$$v^\pm_i(z)^{-1}x^\pm_j(w)v^\pm_i(z) = x^\pm_j(w), \quad i - j \leq -1,$$

$$v^\pm_i(z)^{-1}x^\mp_j(w)v^\pm_i(z) = x^\mp_j(w), \quad i - j \geq 2,$$

$$v^\pm_i(z)^{-1}x^\mp_j(w)v^\pm_i(z) = \frac{zq^{\mp c/2+1} - wq^{-i-1}}{zq^{\mp c/2} - wq^{-i}} x^\mp_j(w), \quad \varepsilon = + \text{ or } -$$

$$v^\pm_i(z)^{-1}x^\mp_j(w)v^\pm_i(z) = \frac{zq^{\mp c/2-1} - wq^{-i+1}}{zq^{\mp c/2} - wq^{-i}} x^\mp_j(w), \quad \varepsilon = + \text{ or } -$$

$$(z - wq^{\mp a_{ij}})x^\pm_i(z)x^\pm_j(z) = x^\pm_j(w)x^\pm_i(z)(zq^{\mp a_{ij}} - w),$$

$$[x^\pm_i(z), x^\pm_j(w)] = 0, \quad |i - j| > 1,$$

$$[x^\pm_i(z), x^\mp_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left\{ \delta(\frac{z}{wq^c}) v^\mp_{i+1}(wq^{c/2}) v^\pm_i(wq^{-c/2})^{-1} - \delta(\frac{wq^c}{z}) v^\pm_{i+1}(zq^{c/2}) v^\mp_i(zq^{-c/2})^{-1} \right\},$$

$$\{x^\pm_i(z_1)x^\pm_i(z_2)x^\pm_j(w) - (q + q^{-1}) x^\pm_i(z_1)x^\pm_j(w)x^\pm_i(z_2) + x^\pm_j(w)x^\pm_i(z_1)x^\pm_j(z_2) \} + \{ z_1 \leftrightarrow z_2 \} = 0, \quad i - j = \pm 1,$$

where the rational functions represent the corresponding Taylor expansions as usual.

Remark 4.38. One can also express the generator $\epsilon_{im}$ in terms of $a_{im}$ easily from Eqs. (4.14-4.15) to give another proof of the theorem. If we let

$$X^\pm_i(z) = x^\pm_i(q^{-i}z),$$
then the generators defined by $X_i^\pm(z), v_j^\pm(z), q^{i/2}$ are exactly the generators in Ding-Frenkel’s definition of $U_q(\hat{\mathfrak{g}l}_N)$ (with correction of some typos in [DF]). Our Heisenberg algebra is orthogonal and a direct deformation of the classical affine general linear algebra.

Under the presentation of Theorem 4.26 the subalgebra $U_q(\hat{\mathfrak{sl}}_N)$ is generated by $q^c$ and $x_i^\pm(z), k_i^\pm(z) = v_{i+1}^- (z q^i) v_i^- (z q^i)^{-1}, k_i^-(z) = v_{i+1}^+ (z q^i) v_i^+ (z q^i)^{-1}$.

In order to let $V_Q$ have a weight space decomposition, we need to add the operator $q^D$ to $U_q(\hat{\mathfrak{g}l}_N)$ and $\mathcal{B}$. As $U_q(\hat{\mathfrak{g}l}_N)$-module, we have $V_Q = \oplus V_\mu$, where

$$V_\mu = \{ v \in V_Q : q^h.v = q^{\mu(h)} v, \text{ for } h \in \mathfrak{h} \}.$$ 

It is easy to see that $V_\mu = V_\mu$, where $V_\mu = \{ v \in V_Q : h.v = \mu(h) v, \text{ for } h \in \mathfrak{h} \}$.

**Proposition 4.39.** $V_Q$ is an irreducible $U_q(\hat{\mathfrak{g}l}_N)$ module with the weight space decomposition $V_Q = \sum_{\mu \in \mathcal{P}} \oplus V_\mu$. Moreover, $V_Q$ is complete reducible as $U_q(\hat{\mathfrak{sl}}_N)$-module.

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