Relative Bott-Chern Secondary Characteristic Classes

Lin Weng

Department of Mathematics, Graduate School of Science, Osaka University,
Toyonaka, Osaka 560, Japan

(October 8, 1998)

Abstract. In this paper, we introduce six axioms for relative Bott-Chern secondary characteristic classes and prove the uniqueness and existence theorem for them. Such a work provides us a natural way to understand and hence to prove the arithmetic Grothendieck-Riemann-Roch theorem.

Content

Introduction
1. Classical Bott-Chern secondary characteristic classes
2. Axioms of relative Bott-Chern secondary characteristic classes
3. Deformation to the normal cone
4. Uniqueness of relative Bott-Chern secondary characteristic classes
5. Some intermediate results
6. Proof of the uniqueness of relative Bott-Chern secondary characteristic classes
7. Existence of relative Bott-Chern secondary characteristic classes

Acknowledgement

References
Introduction

This paper comes partly as a result of our renewed attempt to simplify and clarify our previous work on the so-called relative Bott-Chern secondary characteristic classes and an arithmetic Grothendieck-Riemann-Roch theorem for local complete intersection morphisms done in 1991 (see e.g., [We1,2]). Our (initial and final) aim is to expose the natural structure of the arithmetic Grothendieck-Riemann-Roch theorem via a theory for relative Bott-Chern secondary characteristic classes.

Yet, the works done in [We1,2] are far from being perfect. To make things worse, as what we found recently, the original uniqueness assertion for the relative Bott-Chern secondary characteristic classes stated in [We1,2] is not entirely correct. Fortunately, this mistake is by no means a fatal one. It is our main task here to correct this mistake and to fix some other flaws (appeared in [We1,2]) so as to make the theory of relative Bott-Chern secondary characteristic classes and our arithmetic Grothendieck-Riemann-Roch theorem for local complete intersection morphisms revive. In this way, we then hope that the reader can understand the arithmetic Grothendieck-Riemann-Roch theorem at its own disposal.

This paper is organized as follows. We begin with a review of a theory of the classical Bott-Chern secondary characteristic classes in Chapter 1. Then, in Chapter 2, we introduce five of six key axioms for our relative Bott-Chern secondary characteristic classes, which consist of the downstairs rule, the projection rule, the functorial rule, the uniqueness rule with respect to hermitian vector bundles, and the uniqueness rule with respect to metrized morphisms. The final axiom, the deformation to the normal cone rule, for relative Bott-Chern secondary characteristic classes is introduced in Chapter 3. After this, we state two uniqueness theorems for the relative Bott-Chern secondary characteristic classes in Chapter 4. In Chapter 5, we prove a few intermediate results as a preparation to prove the uniqueness theorems. In Chapter 6, we complete the proof of the uniqueness theorems for relative Bott-Chern secondary characteristic classes. Finally, in Chapter 7, we prove a weak version of the existence theorem for relative Bott-Chern secondary characteristic classes by an effective construction, which is sufficient and necessary for the application to the arithmetic Grothendieck-Riemann-Roch theorem.
1. Classical Bott-Chern secondary characteristic classes

A) Axioms for classical Bott-Chern classes

B) Uniqueness of classical Bott-Chern classes

C) Existence of classical Bott-Chern classes

Associated to a hermitian vector bundle $(E, \rho)$ on a complex manifold $M$ are Chern forms $\text{ch}(E, \rho)$. It is well-known that the differential forms $\text{ch}(E, \rho)$ are $d$-closed on $M$. Hence they define de Rham cohomology classes $[\text{ch}(E, \rho)] \in H^*(M, \mathbb{R})$. Moreover, the classes $[\text{ch}(E, \rho)]$ do not depend on the choice of the metric $\rho$.

On the other hand, if we view $\text{ch}(E, \rho)$ as genuine differential forms, they do depend on the metric $\rho$. Hence a natural question is to understand how Chern forms depend on metrics. This problem is solved by Bott and Chern in [BC], via the so-called classical Bott-Chern secondary characteristic classes. We in this chapter review the associated theory.

A. Axioms for classical Bott-Chern classes

(A.1) Let $(E, \rho)$ be a hermitian vector bundle of rank $r$ on a complex manifold $M$. It is well-known that there exists a unique connection $\nabla_{E,\rho}$ on $E$ which is compatible with the complex structure $\bar{\partial}$ of $E$ and preserves the metric $\rho$. The associated curvature is then defined to be $\nabla^2_{E,\rho}$.

With respect to a local frame $s_U$ over an open trivialization chart $U$ for $E$, the connection $\nabla_{E,\rho}$ (resp. the curvature $\nabla^2_{E,\rho}$) may be represented by an $r \times r$-matrix $\omega_U$ (resp. $\Omega_U$) of differential 1-forms (resp. differential (1,1)-forms) on $U$. Furthermore, $\Omega_U$ satisfies the Bianchi identity

$$d\Omega = \Omega \wedge \omega - \omega \wedge \omega.$$  

(1.1)

Moreover, if $V$ is another open trivialization chart for $E$ with $s_V$ a local frame of $E$ on $V$, $g_{UV} : U \cap V \to \text{GL}(r, \mathbb{C})$, the associated holomorphic transformation, then on $U \cap V$, $s_U = s_V \cdot g_{UV}$, and

$$\Omega_U = g_{UV}^{-1} \cdot \Omega_V \cdot g_{UV}.$$  

(1.2)

Here $\Omega_V$ denotes the curvature forms of $(E, \rho)$ on $V$ with respect to the local frame $s_V$.

(A.2) To facilitate the ensuing discussion, we now recall a result from algebra. Let $B \subset \mathbb{R}$ be a subring, and let $\phi \in B[[T_1, \ldots, T_r]]$ be a symmetric power series. For
every $k \geq 0$, denote by $\phi[k]$ the degree $k$ homogeneous component of $\phi$. Then there exists a unique polynomial map

$$\Phi[k] : M_r(\mathbb{C}) \to \mathbb{C}$$

from the ring of $r \times r$ complex matrices such that

1. $\Phi[k]$ is invariant under the conjugation of $\text{GL}(n, \mathbb{C})$;
2. $\Phi[k](\text{diag}(a_1, \ldots, a_r)) = \phi[k](a_1, \ldots, a_r)$.

Thus, more generally, for any $B$-algebra $A$, we have a well-defined map

$$\Phi = \oplus_{k \geq 0} \Phi[k] : M_r(A) \to A.$$ 

This then implies that for a nilpotent subalgebra $I$ of $A$, we may also have a well-defined map

$$\Phi = \oplus_{k \geq 0} \Phi[k] : M_r(I) \to A.$$ 

(A.3) We apply (A.1) and (A.2) as follows. With the same notation as above, if $(E, \rho)$ is a hermitian vector bundle of rank $r$ on a complex manifold $M$, and $\phi \in B[[T_1, \ldots, T_r]]$ is a symmetric power series, then on a local trivialization chart $U$ of $E$, define differential forms $\phi(E, \rho; U)$ on $U$ by

$$\phi(E, \rho; U) := \Phi(\frac{1}{2\pi \sqrt{-1}} \cdot \Omega_U).$$

Here we set $A := A(U) := \oplus_{p \geq 0} \text{A}^{p,p}(U)$ be the space of all $(p, p)$-differential forms on $U$ and $I := \oplus_{p \geq 1} A^{p,p}(U)$.

Then by (1.2), and A.2.1, $\{(U, \phi(E, \rho; U))\}$ defines global differential forms on $M$. Denote these resulting forms by $\phi(E, \rho)$ and call them the characteristic forms associated to $(E, \rho)$ with respect to $\phi$.

**Theorem.** With the same notation as above,

1. $\phi(E, \rho)$ are $d$-closed differential forms on $M$, i.e. $d(\phi(E, \rho)) = 0$.
2. For any map $f : N \to M$ of complex manifolds, $f^*(\phi(E, \rho)) = \phi(f^*E, f^*\rho)$.
3. If $(E, \rho)$ is a hermitian line bundle $(L, \rho)$, then $\phi(L, \rho) = \phi(c_1(L, \rho))$. Here $c_1(L, \rho)$ is the so-called first Chern form of $(L, \rho)$ defined to be the differential form $dd^c( - \log \|s\|_\rho^2)$.
4. The de Rham cohomology classes $[\phi(E, \rho)] \in H^*(M, \mathbb{R})$ of $\phi(E, \rho)$ do not depend on the choice of $\rho$, but the forms $\phi(E, \rho)$ themselves do depend on $\rho$. 

The proof of this theorem may be found in standard textbooks which contain the theory of characteristic forms. (In fact, (1) is a direct consequence of the Bianchi identity (1.1).) So we omit it. Instead, we point out that first, (1), (2), (3) and (4) characterize the characteristic classes associated to \( \phi \) uniquely; and secondly, (4) is the starting point of our story.

(A.4) Now a natural question is how characteristic forms depend on hermitian metrics. This problem is solved by Bott and Chern via the so-called the classical Bott-Chern secondary characteristic classes [BC].

To state their fundamental result, following [GS2], we start with the following axioms for the classical Bott-Chern secondary characteristic classes, \( \phi_{BC}(E., \rho.) \), with values in \( \tilde{A}(M) := A(M)/(\text{Im}\partial + \text{Im}\bar{\partial}) \), associated to a symmetric power series \( \phi \), a short exact sequence of vector bundles \( E.: 0 \to E_1 \to E_2 \to E_3 \to 0 \) on \( M \), and hermitian metrics \( \rho_j \) on \( E_j \) for \( j = 1, 2, 3 \).

Axiom 1. (Downstairs Rule) In \( A(M) \),

\[
\ddc \phi_{BC}(E., \rho.) = \phi(E_2, \rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3).
\]

Axiom 2. (Functorial Rule) For any morphism \( f : N \to M \) of complex manifolds,

\[
f^*(\phi_{BC}(E., \rho.)) = \phi_{BC}(f^*E., f^*\rho.).
\]

Axiom 3. (Uniqueness Rule) If \( (E., \rho.) \) splits, i.e. \( (E_2, \rho_2) = (E_1 \oplus E_3, \rho_1 \oplus \rho_3) \), then

\[
\phi_{BC}(E., \rho.) = 0.
\]

(A.5) We have the following

**Theorem.** (Existence and Uniqueness of Classical Bott-Chern Secondary Characteristic Classes) ([BC] & [GS2]) For any symmetric power series \( \phi \), there exists a unique construction \( \phi_{BC} \) such that associated to each short exact sequence of vector bundles on a complex manifold \( M \)

\[
E.: 0 \to E_1 \to E_2 \to E_3 \to 0
\]

and hermitian metrics \( \rho_j \) on \( E_j \) for \( j = 1, 2, 3 \), is a unique element \( \phi_{BC}(E., \rho.) \in \tilde{A}(M) \) which satisfies Axioms 1, 2, and 3.

**Remark 1.1.** We will call the unique element \( \phi_{BC}(E., \rho.) \in \tilde{A}(M) \) the classical Bott-Chern secondary characteristic classes, or classical Bott-Chern classes or simply
**Bott-Chern classes**, associated to \((E., \rho.)\) and \(\phi\). Useful examples are given when \(\phi\) corresponds to \(\text{ch}\), the Chern characteristic class, or \(\text{td}\), the Todd characteristic class. In these cases, we denote them by \(\text{ch}_{\text{BC}}\) and \(\text{td}_{\text{BC}}\) respectively.

**Remark 1.2.** \(\phi_{\text{BC}}(E., \rho.)\)'s are not differential forms, rather they are in \(\tilde{A}\), the quotient space of differential forms modulo the exact forms.

**Remark 1.3.** When \(E_3 = 0\), so that \(E_1 = E_2 =: E\), we write \(\phi_{\text{BC}}(E., \rho.)\) simply as \(\phi_{\text{BC}}(E; \rho_2, \rho_1)\). Thus, by Axiom 1,

\[
\ddc \phi_{\text{BC}}(E; \rho_2, \rho_1) = \phi(E, \rho_2) - \phi(E, \rho_1).
\]

So \(\phi_{\text{BC}}(E; \rho_2, \rho_1)\) measures how \(\phi(E, \rho)\) depends on the metric \(\rho\).

**B. Uniqueness of classical Bott-Chern classes**

(B.1) For the time being, assume that there exist elements \(\phi_{\text{BC}}(E., \rho.) \in \tilde{A}(M)\) satisfying Axioms 1, 2 and 3. Our aim in this section is to prove the uniqueness part of Theorem A.5 by using a \(\mathbb{P}^1\)-deformation following [GS2].

Let \(1_\infty\) be a section of the bundle \(\mathcal{O}_{\mathbb{P}^1}(1)\), which vanishes at \(\infty\) and has the value 1 at 0. So we have an inclusion \(\text{Id}_{E_1} \otimes 1_\infty : E_1 \to E_1(1)\). Set

\[
DE_1 := E_1(1) := E_1 \otimes \mathcal{O}_{\mathbb{P}^1}(1), \quad DE_2 := (E_2 \oplus E_1(1))/E_1, \quad DE_3 := E_3.
\]

Then we obtain an exact sequence of vector bundles on \(M \times \mathbb{P}^1:\)

\[
DE. : \quad 0 \to DE_1 \to DE_2 \to DE_3 \to 0.
\]

Easily we see that the restriction of \(DE.\) on \(M \times \{0\}\) gives \(E.\) on \(M\) while the restriction of \(DE.\) on \(M \times \{\infty\}\) gives the splitting exact sequence \(0 \to E_1 \to E_1 \oplus E_3 \to E_3 \to 0\) on \(M\). (So we deform \(E.\) to a split exact sequence.) By a partition of unity, we may further choose hermitian metrics \(D\rho_i\) on \(DE_i, i = 1, 2, 3\), such that the induced metrics on \(M \times \{0\}\) (resp. on \(M \times \{\infty\}\)) via restrictions coincide with the original metrics \(\rho_i\) (resp. splits). Now define an element in \(A(M)\) by setting

\[
\phi'_\text{BC}(E., \rho.): = \int_{\mathbb{P}^1} \log |z|^2 \cdot \left(\phi(DE_2, D\rho_2)\right).
\]

By an abuse of notation, we use \(\phi'_\text{BC}(E., \rho.)\) to denote its image in \(\tilde{A}(M)\) as well.

**Claim.** With the same notation as above,
(1) $\phi_{BC}(E_\cdot, \rho) \in \tilde{A}(M)$ does not depend on $D\rho_2$;

(2) $\phi_{BC}(E_\cdot, \rho) = \phi_{BC}(E_\cdot, \rho)$ in $\tilde{A}(M)$.

(B.2) To prove Claim B.1.1, we use a $\mathbb{P}^1$-deformation again. Suppose that $D\rho_2$ and $D\rho'_2$ are metrics on $DE_2$ such that they induce the same metrics on $M \times \{0\}$ as well as on $M \times \{\infty\}$. Consider the product $M \times \mathbb{P}^1 \times \mathbb{P}^1$ with points $(y, z, u)$. We have the following natural maps:

$$M \times \mathbb{P}^1 \xrightarrow{i_{12}} M \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_{12}} M \times \mathbb{P}^1, \quad M \times \mathbb{P}^1 \xrightarrow{i_{13}} M \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_{13}} M \times \mathbb{P}^1,$$

defined by

$$i_{12}(y, z) := (y, z, u), \quad p_{12}(y, z, u) := (y, z), \quad i_{13}(y, u) := (y, z, u), \quad p_{13}(y, z, u) := (y, u).$$

Also let $p_1 : M \times \mathbb{P}^1 \to M$ be the projection to the first factor. Then by a partition of unity, we may find a metric $\tau$ on the bundle $p_{12}^*DE_2$ such that

(i) $(i_{12}^*)^*(p_{12}^*DE_2, \tau) \simeq (DE_2, D\rho_2)$ and $(i_{12}^*)^*(p_{12}^*DE_2, \tau) \simeq (DE_2, D\rho'_2)$;

(ii) $(i_{13}^*)^*(p_{12}^*DE_2, \tau) \simeq p_1^*(E_2, \rho_2)$ and $(i_{13}^*)^*(p_{12}^*DE_2, \tau) \simeq p_1^*(E_1 \oplus E_3, \rho_1 \oplus \rho_3)$.

Hence,

$$\int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho_2) - \int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho'_2)$$

$$= \int_{\mathbb{P}^1} [\log |z|^2] \left( \phi(DE_2, D\rho_2) - \phi(DE_2, D\rho'_2) \right)$$

$$= \int_{\mathbb{P}^1} [\log |z|^2] \left( \phi((i_{12}^*)^*(p_{12}^*DE_2, \tau)) - \phi((i_{12}^*)^*(p_{12}^*DE_2, \tau)) \right)$$

$$= \int_{\mathbb{P}^1 \times \mathbb{P}^1} [\log |z|^2] [\log |u|^2] \left( d_u d_u^c (\phi(p_{12}^*DE_2, \tau)) \right) \quad \text{(by Stokes' formula)}.$$

But if we let $\partial = \partial_M + \partial_z + \partial_u$ and $\bar{\partial} = \bar{\partial}_M + \bar{\partial}_z + \bar{\partial}_u$ be the differentials on $M \times \mathbb{P}^1 \times \mathbb{P}^1$, then by the fact that characteristic forms are $d$-closed, we have

$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} [\log |z|^2] [\log |u|^2] \left( d_u d_u^c (\phi(p_{12}^*DE_2, \tau)) \right)$$

$$= \int_{\mathbb{P}^1 \times \mathbb{P}^1} [\log |z|^2] [\log |u|^2] \left( d_z d_z^c (\phi(p_{12}^*DE_2, \tau)) \right).$$
Thus, using Stokes’ formula again, we have
\[
\int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho_2) - \int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho'_2)
\]
\[
= \int_{\mathbb{P}^1} [\log |u|^2] \left( \phi((i_0^{13})^*(p_{12}^*DE_2, \tau)) - \phi((i_0^{13})^*(p_{12}^*DE_2, \tau)) \right)
\]
\[
= \int_{\mathbb{P}^1} [\log |u|^2] \left( p_1^*(\phi(E_2, \rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3)) \right)
\]
\[
= \int_{\mathbb{P}^1} [\log |u|^2] \left( p_1^*(\phi(E_2, \rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3)) \right)
\]
\[= 0.
\]
Here, in the last step, we use the fact that \( p_1^*(\phi(E_2, \rho_2) - \phi(E_1 \oplus E_3, \rho_1 \oplus \rho_3)) \) is a constant form with respect to \( \mathbb{P}^1 \) and hence its integration with respect to \( \log |u|^2 \) over \( \mathbb{P}^1 \) is identically zero. This proves Claim B.1.1.

(B.3) Next we prove Claim B.1.2. By Axiom 1, \( \int_{\mathbb{P}^1} [\log |z|^2] d_{\mathbb{P}^1} \phi'_{BC}(DE_2, D\rho_2) \)
\[= \int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho_2) - \int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_1 \oplus DE_3, D\rho_1 \oplus D\rho_3) \quad (1.3)
\]
\[= \int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho_2).
\]
Here, in the last step, we use the fact that \( \phi(DE_1 \oplus DE_3, D\rho_1 \oplus D\rho_3) \) remains the same if we make a change from \( z \) to \( z^{-1} \). Therefore, in \( \mathcal{A}(M) \),
\[
\phi'_{BC}(E_2, \rho_2)
\]
\[
= \int_{\mathbb{P}^1} [\log |z|^2] \phi(DE_2, D\rho_2)
\]
\[
= \int_{\mathbb{P}^1} [\log |z|^2] d_{\mathbb{P}^1} \phi'_{BC}(DE_2, D\rho_2) \quad \text{(by (1.3))}
\]
\[
= i_0^* \phi_{BC}(DE_2, D\rho_2) - i_\infty^* \phi_{BC}(DE_2, D\rho_2)
\]
\[
= \phi_{BC}(E_2, \rho_2) - i_\infty^* \phi_{BC}(DE_2, D\rho_2) \quad \text{(by Axiom 2 as } i_0^*(DE_2, D\rho_2) = (E_2, \rho_2))
\]
\[
= \phi_{BC}(E_2, \rho_2) - 0 \quad \text{(by Axiom 3 as } i_\infty^*(DE_2, D\rho_2) \text{ splits)}
\]
\[
= \phi_{BC}(E_2, \rho_2).
\]
This completes the proof of Claim B.1.2 and hence the uniqueness of the classical Bott-Chern secondary characteristic classes.

C. The existence of classical Bott-Chern classes

(C.1) In this section, we show that there exists a construction \( \phi_{BC} \) which satisfies Axioms 1, 2 and 3 in (A.4).
To do so, as above, set

$$\phi_{BC}(E., \rho.) := \int_{\mathbb{P}^1} [\log |z|^2] \cdot \left( \phi(DE_2, D\rho_2) \right).$$

By the fact that $dd^c[\log |z|^2] = \delta_0 - \delta_\infty$, and characteristic forms are $d$-closed and functorial, (see e.g., Theorem A.3(1) and (2),) $\phi_{BC}(E., \rho.)$ satisfies Axiom 1.

(C.2) By the fact that $\phi_{BC}(E., \rho.)$ does not depend on $D\rho_2$, i.e., Claim B.1.1, and characteristic forms are functorial, i.e., Theorem A.1.3, $\phi_{BC}(E., \rho.)$ satisfies Axiom 2 as well.

(C.3) If $(E., \rho.)$ splits, we may take $(DE., D\rho.)$ as the pull-back of $(E., \rho.)$ via the natural projection to $M \times \mathbb{P}^1$. In this case, $\phi(DE_2, D\rho_2)$ does not depend on $z$. Hence

$$\phi_{BC}(E., \rho.) := \int_{\mathbb{P}^1} [\log |z|^2] \cdot \left( \phi(DE_2, D\rho_2) \right) = 0.$$ 

This shows that $\phi_{BC}(E., \rho.)$ satisfies Axiom 3 too. This then ends the proof of the uniqueness and the existence of classical Bott-Chern secondary characteristic classes.

(C.4) We end this chapter with the following remarks. At the very beginning, it is not quite clear why the secondary characteristic classes should be in $\tilde{A}(M)$, a quotient of $A(M)$. One possible explanation is that with the limitation of the methods employed, the uniqueness can only be established after modulo exact forms. On the other hand, we may equally use other spaces as well, e.g., $\hat{A}(M) := A(M)/(\text{exact forms} + d$–$\text{closed forms})$. If we wish, we may justify this later choice of $\hat{A}(M)$ by arguing that, after all, $\phi(E, \rho)$ is equal to $\phi(E, c \cdot \rho)$ for any positive constant $c$, yet, in general, $\phi_{BC}(E; \rho, c \cdot \rho) \neq 0$ in $\hat{A}(M)$, (so it is too sensitive,) and the $d$-closed forms have already been taken care of in the theory of characteristic classes (so in the theory of secondary characteristic classes, $d$-closed forms should be simply viewed as zero). As a matter of fact, later in Section 7.B, we use yet a third space to understand classical Bott-Chern classes.
2. Axioms for relative Bott-Chern Secondary Characteristic Classes

A) Downstairs Rule
B) Projection Rule
C) Functorial Rule
D) Uniqueness Rule w.r.t. Hermitian Bundles
E) Uniqueness Rule w.r.t. Metrized Morphisms

In this chapter, we naturally expose first five of six key axioms for the so-called relative Bott-Chern secondary characteristic classes associated to smooth proper morphisms of compact Kähler manifolds.

A. Downstairs Rule

(A.1) Let $X$ and $Y$ be two compact Kähler manifolds. Let $f : X \to Y$ be a smooth, proper morphism between $X$ and $Y$. For any $y \in Y$, denote by $X_y$ the fiber of $f$ at $y$. Fix a hermitian metric $\tau_f$ on $T_f$, the relative tangent vector bundle of $f$. Moreover, we assume that the metric on $X_y$ induced by the restriction of $(T_f, \tau_f)$ is Kähler for all $y \in Y$.

Let $(E, \rho)$ be a hermitian vector bundle on $X$. Assume that $E$ is $f$-acyclic, i.e., all higher direct images $R^i f_* E$ vanish, where $i > 0$. As a direct consequence of this assumption, the direct image $f_* E$ of $E$ is a vector bundle on $Y$. Moreover, $f_* E$ admits a natural metric $L^2(\rho, \tau)$, the $L^2$-metric induced by taking integration of $\rho$ along $f$ with respect to $\tau_f$. In the sequel, we call $(f : X \to Y; E, \rho; T_f, \tau_f)$, or simply $(E, \rho; f, \tau_f)$ a properly metrized datum if all the above conditions are satisfied.

Associated to a properly metrized datum $(E, \rho; f, \tau_f)$ is the so-called relative Bott-Chern secondary characteristic classes $\text{ch}_{BC}(E, \rho; f, \tau_f) \in \tilde{A}(Y)$ which may be essentially characterized by the following axioms.

(A.2) Axiom 1. (Downstairs Rule) In $A(Y)$,

$$dd^c \text{ch}_{BC}(E, \rho; f, \tau_f) = f_* \left( \text{ch}(E, \rho) \cdot \text{td}(T_f, \tau_f) \right) - \text{ch}(f_* E, L^2(\rho, \tau)).$$
B. Projection Rule

(B.1) Let \((F, \rho_F)\) be a hermitian vector bundle on \(Y\). Then \(E \otimes f^* F\) is \(f\)-acyclic as well. (Indeed, we know that \(f_*(E \otimes f^* F) = f_* E \otimes F\).) So, \((E \otimes f^* F, \rho \otimes f^* \rho_F; f, \tau_f)\) is again a properly metrized datum. Thus, we also have the relative Bott-Chern secondary characteristic classes \(ch_{BC}(E \otimes f^* F, \rho \otimes f^* \rho_F; f, \tau_f) \in \tilde{A}(Y)\).

(B.2) Axiom 2. (Projection Rule) In \(\tilde{A}(Y)\),

\[
ch_{BC}(E \otimes f^* F, \rho \otimes f^* \rho_F; f, \tau_f) = ch_{BC}(E, \rho; f, \tau_f) \cdot ch(F, \rho_F).
\]

C. Functorial Rule

(C.1) Let \(g : Y' \to Y\) be a morphism between two compact Kähler manifolds. Then we have the Cartesian product

\[
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{g_f} & X \\
f_g & \downarrow & \downarrow f \\
Y' & \xrightarrow{g} & Y.
\end{array}
\]

Define the pull-back of \((E, \rho; f, \tau_f)\) via \(g_f\) by

\[
g_f^*(E, \rho; f, \tau_f) = (g_f^*(E, \rho); f_g^*f, g_f^*\tau_f).
\]

Easily, we see that

(i) by the \(f\)-acyclic condition on \(E\), \(g_f^*E\) is \(f_g\)-acyclic;

(ii) \(g_f^*\tau_f\) gives a hermitian metric on the relative tangent bundle \(T_{f_g}\) for the smooth, proper morphism \(f_g\) which induces Kähler metrics on its fibers.

Hence, \(g_f^*(E, \rho; f, \tau_f)\) is again a properly metrized datum, and we may have the corresponding relative Bott-Chern secondary characteristic classes \(ch_{BC}(g_f^*(E, \rho; f, \tau_f)) \in \tilde{A}(Y')\) as well.

(C.2) Axiom 3. (Functorial Rule) In \(\tilde{A}(Y')\),

\[
g_f^*(ch_{BC}(E, \rho; f, \tau_f)) = ch_{BC}(g_f^*(E, \rho; f, \tau_f)).
\]
D. Uniqueness Rule w.r.t. Hermitian Bundles

(D.1) Let \( E : 0 \to E_1 \to E_2 \to E_3 \to 0 \) be an exact sequence of \( f \)-acyclic vector bundles on \( X \). Then the direct image of \( E \) gives an exact sequence of vector bundles on \( Y \) as follows;

\[
f_\ast E : 0 \to f_\ast E_1 \to f_\ast E_2 \to f_\ast E_3 \to 0.
\]

If we put hermitian metrics \( \rho_i \) on \( E_i \), \( i = 1, 2, 3 \), we then get naturally \( L^2 \)-metrics \( L^2(\rho_i, \tau_f) \) on \( f_\ast E_i \). By Theorem 1.A.5, there exists the so-called (classical) Bott-Chern secondary characteristic classes associated to both \((E., \rho.)\) and \((f_\ast E., L^2(\rho., \tau_f))\). Denote these two classical Bott-Chern secondary characteristic classes by \( \text{ch}_{\text{BC}}(E., \rho.) \in \tilde{A}(X) \) and \( \text{ch}_{\text{BC}}(f_\ast E., L^2(\rho., \tau_f)) \in \tilde{A}(Y) \) respectively.

Surely, for each \((E_i, \rho_i)\), \( i = 1, 2, 3 \), we have the corresponding properly metrized datum \((E_i, \rho_i; f, \tau_f)\), and hence the relative Bott-Chern secondary characteristic classes \( \text{ch}_{\text{BC}}(E_i, \rho_i; f, \tau_f) \in \tilde{A}(Y) \).

(E.1) Let \( Z \) be a compact Kähler manifold and \( g : Y \to Z \) be a smooth, proper morphism. So we have a triangle of smooth fibrations of compact Kähler manifolds among compact Kähler manifolds as follows;

\[
\begin{array}{ccc}
X & f & Y \\
& g \circ f \searrow & \downarrow g \\
& & Z
\end{array}
\]

Naturally, we have the short exact sequence of relative tangent bundles

\[
T : 0 \to T_f \to T_{g \circ f} \to f^* T_g \to 0,
\]

where \( T_f, T_g \) and \( T_{g \circ f} \) stand for the relative tangent bundles associated to the smooth, proper morphisms \( f, g \) and \( g \circ f \) respectively. Put hermitian metrics \( \tau_f, \tau_g \) and \( \tau_{g \circ f} \) on \( T_f, T_g \) and \( T_{g \circ f} \) respectively such that the induced metrics on all corresponding fibers are Kähler. By Theorem 1.A.5, we have the corresponding
(classical) Bott-Chern secondary characteristic classes $\text{td}_{BC}(T., \tau.) \in \tilde{A}(X)$ with respect to the Todd characteristic class.

(E.2) Recall that $E$ is $f$-acyclic so that $f_*E$ is a vector bundle on $Y$. In addition, assume that $E$ is $g\circ f$-acyclic and $f_*E$ is $g$-acyclic. As a direct consequence, $(g\circ f)_*E$ is a vector bundle on $Z$ and is equal to $g_*(f_*E)$. All this gives us the properly metrized data $(E, \rho; f, \tau_f)$, $(f_*E, L^2(\rho, \tau_f); g, \tau_g)$, and $(E, \rho; g\circ f, \tau_{g\circ f})$, and hence their associated relative Bott-Chern secondary characteristic classes $\text{ch}_{BC}(E, \rho; f, \tau_f) \in \tilde{A}(Y)$, $\text{ch}_{BC}(f_*E, L^2(\rho, \tau_f); g, \tau_g) \in \tilde{A}(Z)$ and $\text{ch}_{BC}(E, \rho; g \circ f, \tau_{g\circ f}) \in \tilde{A}(Z)$.

Moreover, on $Z$, we have a short exact sequence of vector bundles

$$0 \to g_*(f_*E) \to (g \circ f)_*E \to 0.$$ 

Put the $L^2$-metrics $L^2(\rho, \tau_{g\circ f})$ and $L^2(L^2(\rho, \tau_f), \tau_g)$ on $(g \circ f)_*E$ and on $g_*(f_*E)$ respectively, we then have the associated (classical) Bott-Chern secondary characteristic classes, which, as in Remark 1.1.3 in 1.A.5, we denote by

$$\text{ch}_{BC}((g \circ f)_*E; L^2(\rho, \tau_{g\circ f}), L^2(L^2(\rho, \tau_f), \tau_g)) \in \tilde{A}(Z).$$

(E.3) Axiom 5. (Uniqueness Rule w.r.t. Metrized Morphisms) In $\tilde{A}(Z)$,

$$\text{ch}_{BC}(E, \rho; g \circ f, \tau_{g\circ f}) - g_*\left(\text{ch}_{BC}(E, \rho; f, \tau_f) \cdot \text{td}(T_g, \tau_g)\right) - \text{ch}_{BC}\left(f_*E, L^2(\rho, \tau_f); g, \tau_g\right)$$

$$=(g \circ f)_*\left(\text{ch}(E, \rho) \cdot \text{td}_{BC}(T., \tau.)\right) - \text{ch}_{BC}\left((g \circ f)_*E; L^2(\rho, \tau_{g\circ f}), L^2(L^2(\rho, \tau_f), \tau_g)\right).$$

(E.4) To state the last axiom, the deformation to the normal cone rule, we need to make a lengthy discussion. So we delay it untill the next chapter.
3. Deformation to the Normal Cone Rule

A) Deformation to the normal cone: an algebraic construction

B) Deformation to the normal cone: the associated metrics

C) Deformation to the normal cone rule

In this chapter, we give the final axiom for the relative Bott-Chern secondary characteristic classes, the deformation to the normal cone rule.

A. Deformation to the normal cone: an algebraic construction

(A.1) Let \( f : X \rightarrow Y \) and \( g : Z \rightarrow Y \) be two smooth, proper morphisms of compact Kähler manifolds and \( i : X \rightarrow Z \) be a codimension one closed immersion over \( Y \), i.e., \( i \) is a closed immersion of codimension one such that \( f = g \circ i \). Then we have the following standard construction of the deformation to the normal cone.

Denote by \( \pi : W := B_{X \times \{\infty\}}Z \times \mathbb{P}^1 \rightarrow Z \times \mathbb{P}^1 \), the natural projection, where \( B_{X \times \{\infty\}} \) denotes the blowing-up of \( Z \times \mathbb{P}^1 \) along \( X \times \{\infty\} \). Denote the exceptional divisor of \( \pi \) by \( \mathbb{P} \). It is well-known that the map \( q_W : W \rightarrow \mathbb{P}^1 \), obtained by composing \( \pi \) with the projection \( q : Z \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \), is flat, and that for \( t \in \mathbb{P}^1 \):

\[
q_{W}^{-1}(t) = \begin{cases} \{Z \times \{t\} \}, & \text{for } t \neq \infty, \\ \mathbb{P} \cup B_X Z, & \text{for } t = \infty. \end{cases}
\]

Here \( B_X Z \) denotes the blowing-up of \( Z \) along \( X \). Moreover, by the construction, \( \mathbb{P} \) and \( B_X Z \) intersect transversally, and \( \mathbb{P} \cap B_X Z \) is the exceptional divisor \( X \) on \( B_X Z(= Z \times \{\infty\}, \text{due to the dimensional reason}) \).

Denote by \( I : X \times \mathbb{P}^1 \hookrightarrow W \) the induced codimension one closed embedding. Easily we see that the image of \( I \) does not intersect with \( B_X Z \), and the image \( X \times \{\infty\} \) in \( W \) is a section of \( \mathbb{P} \). Denote the induced fibration \( Z \times \{t\} \rightarrow Y \times \{t\} \) by \( g_t \) for \( t \neq \infty \) and set \( g_\infty \) to be the composition of the projection of \( \mathbb{P} \) on \( X \) with \( (X =) X \times \{\infty\} \rightarrow Y \times \{\infty\}(= Y) \). Denote by \( f_t : X \times \{t\} \rightarrow Y \times \{t\} \) the smooth morphisms induced from \( f \) for all \( t \in \mathbb{P}^1 \), by \( i'_\infty \) the inclusion of \( W_\infty = \mathbb{P} + B_X Z \) into \( W \), and by \( k \) (resp. \( l \)) the inclusion of \( \mathbb{P} \) (resp. \( B_X Z \)) to \( W \). So we have the
following commutative diagram;

(3.2) Now let $E$ be a $g$-acyclic vector bundle on $Z$ such that $E(-X) := E \otimes O_Z(-X)$ is $g$-acyclic as well. In particular, we have the short exact sequence

$$0 \to E(-X) \to E \to i_* i^* E \to 0$$

(3.1) on $Z$. We want to introduce a $\mathbb{P}^1$-deformation for (3.1) via $W \to \mathbb{P}^1$.

First, pulling it back to $Z \times \mathbb{P}^1$, we have the short exact sequence $0 \to p_2^* E(-X \times \mathbb{P}^1) \to p_2^* E \to I_*(I')^* E \to 0$ on $Z \times \mathbb{P}^1$. Here $p_2 : Z \times \mathbb{P}^1 \to Z$ denotes the natural projection and $I' : X \times \mathbb{P}^1 \hookrightarrow Z \times \mathbb{P}^1$ denotes the induced injection from $i$.

To get a $\mathbb{P}^1$-deformation for (3.1) on $W$, we should take, instead of simply taking $\pi^*(p_2^* E(-X \times \mathbb{P}^1))$, the vector bundle $\pi^*(p_2^* E(Z \times \{\infty\} - X \times \mathbb{P}^1))$. (For a possible motivation, see e.g., the definition $DE_1$ in 1.B.1.) Then, we know that

$$\pi^*(p_2^* E(Z \times \{\infty\} - X \times \mathbb{P}^1))$$

$$= (\pi \circ p_2)^* E \otimes O_{Z \times \mathbb{P}^1}(Z \times \{\infty\} - X \times \mathbb{P}^1)$$

$$= (\pi \circ p_2)^* E \otimes O_W(\mathbb{P} + B_X Z - X \times \mathbb{P}^1 - \mathbb{P})$$

$$= (\pi \circ p_2)^* E \otimes O_W(B_X Z - X \times \mathbb{P}^1)$$

$$= (\pi \circ p_2)^* E(B_X Z - X \times \mathbb{P}^1).$$

So the correct choice of the $\mathbb{P}^1$-deformation of (3.1) on $W$ we seek is the following exact sequence of coherent sheaves on $W$;

$$0 \to (\pi \circ p_2)^* E(B_X Z - X \times \mathbb{P}^1) \to (\pi \circ p_2)^* E(B_X Z) \to I_* I^* ((\pi \circ p_2)^* E(B_X Z)) \to 0.$$
Indeed, by the flatness of \( q_W : W \to \mathbb{P}^1 \), the restrictions of (3.2) to the fibers \( W_t \) of \( q_W \) for all \( t \in \mathbb{P}^1 \) are exact. Thus, in particular, for each \( t \neq \infty \) in \( \mathbb{P}^1 \), from (3.2), we have the induced exact sequence

\[
0 \to E_t(-X) \to E_t \to (i_t)_*i_t^*E_t \to 0 \quad \text{over } \mathbb{P} \times \{t\} \tag{3.3}
\]

as \((\pi \circ p_Z)^*E(B_XZ)|_{\mathbb{P}\times\{t\}} = E_t, (\pi \circ p_Z)^*E(B_XZ-X\times\mathbb{P}^1)|_{\mathbb{P}\times\{t\}} = E_t(-X)\) with \( E_t \) the pull-back of \( E \) under the canonical identity \( X \times \{t\} \simeq X \). Similarly, for the fiber at \( \infty \), if we set \( E(B_XZ)|_{\mathbb{P}} := E_\infty, E(B_XZ)|_{B_XZ} =: E'_\infty \) and \( E(B_XZ-X\times\mathbb{P}^1)|_{B_XZ} =: E''_\infty \). Then \( E(B_XZ-X\times\mathbb{P}^1)|_{\mathbb{P}} = E_\infty(-X) \), and (3.2) splits into two exact sequences

\[
0 \to E_\infty(-X) \to E_\infty \to (i_\infty)_*i_\infty^*E_\infty \to 0 \quad \text{over } \mathbb{P} \tag{3.4}
\]

and

\[
0 \to E'_\infty \to E''_\infty \to 0 \to 0 \quad \text{over } B_XZ. \tag{3.5}
\]

Here in the last statement, we use that fact that \( I(X \times \mathbb{P}) \) is away from \( B_XZ \) in \( W \). Thus, in particular, on \( B_XZ \), \( E'_\infty = E''_\infty \). Now we further assume that \( E \) on \( Z \) is chosen so that \( E_\infty \) and \( E_\infty(-X) \) are all \( g_\infty \)-acyclic.

Note that \( X \times \mathbb{P}^1 \) is away from \( B_XZ \), the restriction of \( E_\infty - E_\infty(-X) \) to \( B_XZ \) gives a zero element in the \( K \)-group of \( \mathbb{P} \cap B_XZ \). As a direct consequence, in \( K(Y) \), we get, for all \( t \in \mathbb{P}^1 \),

\[
(g_t)_*(E_t) - (g_t)_*(E_t(-X)) = (g_\infty)_*(E_\infty) - (g_\infty)_*(E_\infty(-X)). \tag{3.6}
\]

**B. Deformation to the normal cone: the associated metrics**

(B.1) With the same notation as in the previous section, choose a Kähler metric \( \tau_W \) on \( W \). Then \( \tau_W \) naturally induces Kähler metrics \( \tau_{g_t} \) and \( \tau_{g_\infty} \) on \( Z_t := Z \times \{t\} \) and \( \mathbb{P} \) respectively for all \( t \neq \infty \). Note that for a smooth morphism, the relative tangent bundle is a subbundle of the tangent bundle of the total space, by taking restriction again, we then get hermitian metrics \( \tau_t \) on \( T_{g_t} \) for all \( t \in \mathbb{P}^1 \). Easily, we see that the induced metrics for fibers of \( g_t \) from \( \tau_t \) are all Kähler.

Now let \( E \) be a \( g \)-acyclic vector bundles on \( Z \) such that \( E_t \) and \( E_t(-X) \) are \( g_t \)-acyclic as well for all \( t \in \mathbb{P}^1 \). Fix hermitian metrics \( \rho \) and \( \rho' \) on \( E \) and on \( E(-X) \) respectively. Use the same notation to denote the pull-back of \( (E, \rho) \) onto \( W \). Choose the Fubini-Study metric on \( \mathcal{O}_{\mathbb{P}^1}(\infty) \) and a metric on \( \mathcal{O}_W(-X \times \mathbb{P}^1) \) such that in a neighborhood \( U \) of \( B_XZ \), which is away from \( X \times \mathbb{P}^1 \), the natural isomorphism
\(O_W(-X \times \mathbb{P}^1) \simeq O_W\) induces an isometry, once we put the trivial metric on \(O_W\). Denote these final induced metrics on \(E(B_X Z)\) and \(E(B_X Z - X \times \mathbb{P}^1)\) by \(D\rho\) and \(D\rho'\) respectively. (Question: Figure out how the metric on \(O_W(B_X Z)\) is defined?)

Denote the induced metrics via restriction to \(E_t\) and \(E_t(-X)\) (resp. to \(E'_\infty\) and \(E'_\infty\) on \(B_X E\)) by \(\rho_t\) and \(\rho'_t\) respectively for all \(t \in \mathbb{P}^1\), (resp. \(\rho''_\infty\) and \(\rho'''_\infty\)). Easily, we see that \(\rho_0 = \rho\) and \(\rho'_0 = \rho'\), and \((E'_\infty, \rho''_\infty)\) is isomorphic to \((E''_\infty, \rho'''_\infty)\) by the construction. In this way,

\[(E(-X), \rho') \mapsto (E, \rho) \quad \text{on} \quad Z\]

is deformed to

\[(E_\infty(-X), \rho'_\infty) \mapsto (E_\infty, \rho_\infty) \quad \text{on} \quad \mathbb{P}\]

(and

\[(E'_\infty, \rho''_\infty) \simeq (E''_\infty, \rho'''_\infty) \quad \text{on} \quad B_X Z).\]

In particular, we have, for all \(t \in \mathbb{P}^1\),

(i) smooth, proper morphisms \(g_t\) for compact Kähler manifolds, together with hermitian metrics \(\tau_t\) on relative tangent bundles \(T_t := T_{g_t}\), whose induced metrics on all fibers of \(g_t\) are Kähler as well;

(ii) \(g_t\)-acyclic hermitian vector bundles \((E_t, \rho_t)\) and \((E_t(-X), \rho'_t)\).

Thus, as a direct consequence, we have the properly metrized data \((E_t, \rho_t; g_t, \tau_{g_t})\) and \((E_t(-X), \rho'_t; g_t, \tau_{g_t})\) for all \(t \in \mathbb{P}^1\), and their associated relative Bott-Chern secondary characteristic classes \(\text{ch}_\text{BC}(E_t, \rho_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)\) and \(\text{ch}_\text{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)\) for all \(t \in \mathbb{P}^1\). At this point, we may say that the final axiom for the relative Bott-Chern secondary characteristic classes is the one to understand the relation between

\[
\text{ch}_\text{BC}(E_0, \rho_0; g_0, \tau_{g_0}) - \text{ch}_\text{BC}(E_0(-X), \rho'_0; g_0, \tau_{g_0})
\]

and

\[
\text{ch}_\text{BC}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \text{ch}_\text{BC}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty})
\]

in \(\tilde{A}(Y)\).

(B.2) To facilitate the ensuring discussion, we now find out how \(\text{ch}_\text{BC}(E_t, \rho_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)\) and \(\text{ch}_\text{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)\) change with respect to \(t \in \mathbb{A}^1\).

By the construction, for \(t \neq \infty\), we may simply view \(g_t : Z \times \{t\} \to Y \times \{t\}\) as the original \(g : Z \to Y\). In this way, \(\text{ch}_\text{BC}(E_t, \rho_t; g_t, \tau_{g_t}) \in \tilde{A}(Y)\) may be interpreted as
the relative Bott-Chern secondary characteristic classes for the original morphism \( g \) but with metrics \( \rho_t \) on \( E \) and \( \tau_t \) on \( T_g \). Moreover, we may get this type of changing metrics in two steps: First, from \( (\rho_0; \tau_0) \) to \( (\rho_t; \tau_t) \); then from \( (\rho_0; \tau_t) \) to \( (\rho_t; \tau_t) \). In this way, by applying Axiom 5 in step 1 and Axiom 4 in step 2, we have the following relations:

\[
\begin{align*}
\text{ch}_{BC}(E_0, \rho_0; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_0, \rho_0; g_0, \tau_{g_0}) &= g_* \left( \text{ch}(E; \rho_0) \cdot \text{td}_{BC}(T_g; \tau_t, \tau_0) \right) - \text{ch}_{BC} \left( g_* E; L^2(\rho_0, \tau_t), L^2(\rho_0, \tau_0) \right), \\
\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_0, \rho_0; g_t, \tau_{g_t}) &= g_* \left( \text{ch}_{BC}(E; \rho_t, \rho_0) \cdot \text{td}(T_g, \tau_t) + \text{ch}(E; \rho_0) \cdot \text{td}_{BC}(T_g; \tau_t, \tau_0) \right) - \text{ch}_{BC} \left( g_* E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0) \right) \\
\end{align*}
\]

That is to say,

\[
\begin{align*}
\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_0, \rho_0; g_0, \tau_{g_0}) &= g_* \left( \text{ch}_{BC}(E; \rho_t, \rho_0) \cdot \text{td}(T_g, \tau_t) + \text{ch}(E; \rho_0) \cdot \text{td}_{BC}(T_g; \tau_t, \tau_0) \right) - \text{ch}_{BC} \left( g_* E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0) \right). \\
\end{align*}
\]

Similarly,

\[
\begin{align*}
\text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_0(-X), \rho'_0; g_0, \tau_{g_0}) &= g_* \left( \text{ch}_{BC}(E(-X); \rho'_t, \rho'_0) \cdot \text{td}(T_g, \tau_t) + \text{ch}(E(-X); \rho'_0) \cdot \text{td}_{BC}(T_g; \tau_t, \tau_0) \right) - \text{ch}_{BC}(g_* (E(-X)); L^2(\rho'_t, \tau_t), L^2(\rho'_0, \tau_0)). \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) &= \left( \text{ch}_{BC}(E_0, \rho_0; g_0, \tau_{g_0}) - \text{ch}_{BC}(E_0(-X), \rho'_0; g_0, \tau_{g_0}) \right) \\
&= g_* \left( \text{ch}_{BC}(E; \rho_t, \rho_0) - \text{ch}_{BC}(E(-X); \rho'_t, \rho'_0) \right) \cdot \text{td}(T_g, \tau_t) \\
&\quad + (\text{ch}(E; \rho_0) - \text{ch}(E(-X); \rho'_0)) \cdot \text{td}_{BC}(T_g; \tau_t, \tau_0) \\
&\quad - \text{ch}_{BC}(g_* (E(-X)); L^2(\rho'_t, \tau_t), L^2(\rho'_0, \tau_0)) - \text{ch}_{BC}(g_* E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0)) \\
&\quad - \text{ch}_{BC}(g_* (E(-X)); L^2(\rho'_t, \tau_t), L^2(\rho'_0, \tau_0)) - \text{ch}_{BC}(g_* E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0)). \\
\end{align*}
\]

So if \( q_W : W \to \mathbb{P}^1 \) was smooth, we would then have found that the relation between

\[
\begin{align*}
\text{ch}_{BC}(E_{\infty}, \rho_{\infty}; g_{\infty}, \tau_{g_{\infty}}) - \text{ch}_{BC}(E_{\infty}(-X), \rho'_{\infty}; g_{\infty}, \tau_{g_{\infty}}) \\
\text{ch}_{BC}(E_0, \rho_0; g_0, \tau_{g_0}) - \text{ch}_{BC}(E_0(-X), \rho'_0; g_0, \tau_{g_0})
\end{align*}
\]
can be deduced from Axioms 4 and 5.

C. Deformation to the normal rule

(C.1) Unfortunately, as indicated before, $q_W : W \to \mathbb{P}^1$ is far from being smooth. So we cannot simply deduce a relation between the relative Bott-Chern secondary characteristic classes for the fibers at 0 and $\infty$ directly by only using previous 5 axioms for relative Bott-Chern secondary characteristic classes with respect to smooth morphisms. Nevertheless, it seems most likely that with a suitable generalization of relative Bott-Chern classes, say for proper and flat morphisms, one then can understand how the relative Bott-Chern classes change even across $\infty$. All this is indeed the motivation for introducing the so-called ternary characteristic classes in [We2], which themselves need an additional work to justify. So here, we take an alternative approach by sticking on smooth, proper morphisms.

(C.2) The idea is also quite simple. That is, instead of trying to find out what is exactly the difference between

$$\text{ch}_{BC}(E_\infty, \rho_\infty; g_\infty, \tau_\infty) - \text{ch}_{BC}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_\infty),$$

and

$$\text{ch}_{BC}(E_0, \rho_0; g_0, \tau_\infty) - \text{ch}_{BC}(E_0(-X), \rho'_0; g_0, \tau_\infty),$$

we view $\{\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{gt}) - \text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{gt})\}_{t \in \mathbb{P}^1}$ as a family on $\mathbb{P}^1$ and try to establish the continuity of such a family of forms, or better of classes.

It looks that we are now in a position to state the final axiom. No, it is not so. There is yet another difficulty: It is well-known that for the family $W \to Y \times \mathbb{P}^1$, usually the induced $L^2$-metrics on the direct images for $g_t$ when $t \to \infty$ have singularities. So we need to take care of such singularities as well. (Recall that by Axiom 1 in 2.A.2 for relative Bott-Chern secondary characteristic classes,

$$dd^c\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{gt}) = (g_t)_*\left(\text{ch}(E_t, \rho_t) \cdot \text{td}(T_t, \tau_{gt})\right) - \text{ch}\left((g_t)_* E, L^2(\rho_t, \tau_{gt})\right),$$

and

$$dd^c\text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{gt})$$

$$= (g_t)_*\left(\text{ch}(E_t(-X), \rho_t) \cdot \text{td}(T_t, \tau_{gt})\right) - \text{ch}\left((g_t)_* E(-X), L^2(\rho'_t, \tau_{gt})\right).$$

For this purpose, consider the virtual vector bundle $(g_t)_*(E_t) - (g_t)_*(E_t(-X))$ on $Y$ for $t \in \mathbb{P}^1$. By the discussion in (B.1), in particular, (3.6), we know that for
\[ t \neq \infty, (g_t)_*(E_t) = (g_0)_*(E_0), (g_t)_*(E_t(-X)) = (g_0)_*(E_0(-X)) \] once we identify \( Y \times \{0\} \) with \( Y \times \{t\} \), and as virtual vector bundles on \( Y \),

\[
(g_\infty)_*(E_\infty) - (g_\infty)_*(E_\infty(-X)) = (g_t)_*(E_t) - (g_t)_*(E_t(-X)).
\]

So in particular, there exists a natural number \( r \) such that, locally over \( Y \), as holomorphic vector bundles, we have the local isomorphism

\[
O_Y^{\oplus r} \oplus (g_\infty)_*(E_\infty) \oplus (g_t)_*(E_t(-X)) = O_Y^{\oplus r} \oplus (g_t)_*(E_t) \oplus (g_\infty)_*(E_\infty(-X)).
\]

Hence, by the fact that Chern form depends only on the \( C^\infty \)-structure of a vector bundle, we may choose hermitian metrics \( \gamma_t \) and \( \gamma'_t \) on \( (g_t)_*E \) and \( (g_t)_*(E_t(-X)) \) respectively for all \( t \in \mathbb{P}^1 \) such that

\[
\text{ch}(E_t, \gamma_t) - \text{ch}(E_t(-X), \gamma'_t) = \text{ch}(E_\infty, \gamma_\infty) - \text{ch}(E_\infty(-X), \gamma'_\infty). \quad (3.10)
\]

With this, via \( Y \times \{0\} = Y \times \{\infty\} \), we have the classical Bott-Chern secondary characteristic classes \( \text{ch}_{BC}((g_t)_*(E_t); L^2(\rho_t, \tau_t), \gamma_t) \) and \( \text{ch}_{BC}((g_\infty)_*(E_\infty(-X)); L^2(\rho'_t, \tau_t), \gamma'_t) \).

Now we may consider the family

\[
\{(\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}))

+ (\text{ch}_{BC}((g_t)_*(E_t); L^2(\rho_t, \tau_t), \gamma_t) - \text{ch}_{BC}((g_t)_*(E_t(-X)); L^2(\rho'_t, \tau_t), \gamma'_t)) \}_{t \in \mathbb{P}^1}.
\]

As a matter of fact, the final axiom for the relative Bott-Chern secondary characteristic classes requires that this family is continuous.

(C.3) **Axiom 6. (Deformation to the normal cone rule)** In \( \tilde{A}(Y) \),

\[
\lim_{t \to \infty} \left( (\text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}))

+ (\text{ch}_{BC}((g_t)_*(E_t); L^2(\rho_t, \tau_t), \gamma_t) - \text{ch}_{BC}((g_t)_*(E_t(-X)); L^2(\rho'_t, \tau_t), \gamma'_t)) \right)

= (\text{ch}_{BC}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - \text{ch}_{BC}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty})) +

(\text{ch}_{BC}((g_\infty)_*(E_\infty); L^2(\rho_\infty, \tau_\infty), \gamma_\infty) - \text{ch}_{BC}((g_\infty)_*(E_\infty(-X)); L^2(\rho'_\infty, \tau_\infty), \gamma'_\infty)).
\]

(C.4) All this then gives us the six key axioms for the relative Bott-Chern secondary characteristic classes. The reader may see that Axioms 2 and 3 (resp. Axioms 4 and 5) may be grouped together naturally. So there are essentially four groups of axioms. The first group, the downstairs rule, relates the secondary theory with its origin; the second group tells us how to deal with the changes from the base; the third group tells us how to deal with the changes from the total space; while the last tells us how to relate the theory for different relative dimensions naturally.
4. Uniqueness of Relative Bott-Chern Secondary Characteristic Classes

A) Weak Uniqueness Theorem

B) Strong Uniqueness Theorem

In this chapter, we will state the uniqueness theorem for relative Bott-Chern secondary characteristic classes. There are two versions of it. Roughly speaking, the first uniqueness theorem claims that after modulo $d$-closed forms, the classes of the relative Bott-Chern secondary characteristic classes are unique, while the second uniqueness theorem claims that after modulo exact forms, the relative Bott-Chern secondary characteristic classes are essentially unique, i.e., the difference of any two possible relative Bott-Chern secondary characteristic classes can be understood precisely.

A. Weak Uniqueness Theorem

(A.1) We have already stated the axioms for relative Bott-Chern secondary characteristic classes associated to smooth, proper metrized morphisms and relative acyclic hermitian vector bundles in Chapter 2. Now we state the first uniqueness result for them. Recall that $(f : X \to Y; E, \rho; T_f, \tau_f)$ is called a properly metrized datum if $f : X \to Y$ is a smooth morphism of compact Kähler manifolds, $(E, \rho)$ is an $f$-acyclic hermitian vector bundle, and $\tau_f$ is a hermitian metric on the relative tangent bundle $T_f$ of $f$ such that the induced metrics on all fibers of $f$ are Kähler.

(A.2) Theorem. (The Weak Uniqueness for Relative Bott-Chern Secondary Characteristic Classes) Suppose that there are two constructions $\text{ch}_{\text{BC}}$ and $\text{ch}'_{\text{BC}}$ which satisfy six axioms for relative Bott-Chern secondary characteristic classes, then, for all properly metrized data $(f : X \to Y; E, \rho; T_f, \tau_f)$, the images of $\text{ch}_{\text{BC}}(E, \rho; f, \tau_f)$ and $\text{ch}'_{\text{BC}}(E, \rho; f, \tau_f)$ in $\hat{A}(Y) := A(Y)/(\text{exact forms} + d - \text{closed forms})$ are the same.

B. Strong Uniqueness Theorem

(B.1) To state the second uniqueness theorem, we need a preparation.

Let $B$ be a subring of $\mathbb{R}$, and let $P(x) \in B[[x]]$ be any symmetric power series. Then for any vector bundle $E$ on $Y$, by the splitting principle for vector bundles, there exists a unique additive characteristic class $P(E) \in H^*(X, \mathbb{R})$, the de Rham cohomology of $X$.

Now assume that there exists a construction $\text{ch}_{\text{BC}}$ such that it satisfies the six axioms for relative Bott-Chern secondary characteristic classes. Then, for
any properly metrized datum \((f : X \to Y; E, \rho; T_f, \tau_f)\), we have an element \(\text{ch}_{BC}(E, \rho; f, \tau_f) \in \tilde{A}(Y)\).

With such an existence, then we claim that \(\text{ch}_{BC}(E, \rho; f, \tau_f) + f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)\right) \in \tilde{A}(Y)\) for any fixed additive characteristic class \(P\) as above also satisfies the listed six axioms for relative Bott-Chern secondary characteristic classes. Indeed, easily, by the fact that \(f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)\right)\) is in \(H^*(Y, \mathbb{R})\), we see that \(\text{ch}_{BC}(E, \rho; f, \tau_f) + f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)\right) \in \tilde{A}(Y)\) satisfies the first five axioms. Moreover note that the deformation to the normal cone does not change the cohomology classes, which is the key property used in the proof of Grothendieck-Riemann-Roch theorem in algebraic geometry, we equally can check the axiom 6 for \(\text{ch}_{BC}(E, \rho; f, \tau_f) + f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)\right)\). (See e.g., Lemma 5.B.7.4 below.) In this sense, we may say that there is no uniqueness for relative Bott-Chern secondary characteristic classes.

Even the latest statement is absolutely correct, we still can understand precisely the structure of relative Bott-Chern secondary characteristic classes. Roughly speaking, we may say that the twisting by an additive characteristic class stated above is the only flaw for establishing the uniqueness for relative Bott-Chern secondary characteristic classes.

(B.2) Theorem. (The Strong Uniqueness for Relative Bott-Chern Secondary Characteristic Classes) Suppose that there are two constructions \(\text{ch}_{BC}\) and \(\text{ch}'_{BC}\) which satisfy six axioms for relative Bott-Chern secondary characteristic classes, then, there exists an additive characteristic class \(R\) such that, for all properly metrized data \((f : X \to Y; E, \rho; T_f, \tau_f)\), in \(\tilde{A}(Y)\),

\[
\text{ch}'_{BC}(E, \rho; f, \tau_f) = \text{ch}_{BC}(E, \rho; f, \tau_f) + f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot R(T_f)\right).
\]

Remark 4.1. The weak uniqueness theorem says that the relative Bott-Chern classes are unique modulo \(d\)-closed forms, while the strong uniqueness theorem says that the relative Bott-Chern classes are unique up to a certain well-structured topological term.

Obviously, the weak uniqueness is a direct consequence of the strong uniqueness, so it is sufficient to prove the later one. We will do it in the following two chapters.
5. Some intermediate results

A) Statements of intermediate results

B) The proofs

In this chapter, we prove some intermediate results for relative Bott-Chern secondary characteristic classes, which will be used in the proof of the uniqueness theorems. So in Chapter 5 and Chapter 6, we will assume that there is a construction \( \text{ch}_{\text{BC}} \) which satisfies all six axioms for the relative Bott-Chern secondary characteristic classes.

A. Statements of intermediate results

(A.1) Suppose that for any properly metrized datum \((f : X \to Y; E, \rho; T_f, \tau_f)\), there exist elements \( \text{ch}_{\text{BC}}(E, \rho; f, \tau_f) \in \tilde{A}(Y) \) and \( \text{ch}'_{\text{BC}}(E, \rho; f, \tau_f) \in \tilde{A}(Y) \) which satisfy the six axioms stated in Chapters 2 and 3. Let \( P \) be an additive characteristic class and set

\[
\text{Err}(E, \rho; f, \tau_f; P) := \text{ch}_{\text{BC}}(E, \rho; f, \tau_f) - \text{ch}'_{\text{BC}}(E, \rho; f, \tau_f) + f_*(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)).
\]

We want to show that there exists a unique universal additive characteristic class \( R \) such that \( \text{Err}(E, \rho; f, \tau_f; R) = 0 \) for all properly metrized data \((E, \rho; f, \tau_f)\). For this purpose, let us state some intermediate results.

First we study how \( \text{Err} \) depends on hermitian metrics on vector bundles.

**Proposition.** Let \( f : X \to Y \) be a smooth, proper morphism of smooth complex Kähler manifolds. Fix a hermitian metric \( \tau_f \) on the relative tangent vector bundle \( T_f \) of \( f \) such that the induced metrics on all fibers of \( f \) are Kähler. Then for any short exact sequence of \( f \)-acyclic hermitian vector bundles

\[
E. : 0 \to E_1 \to E_2 \to E_3 \to 0,
\]

with hermitian metrics \( \rho_i \) on \( E_i \) for \( i = 1, 2, 3 \),

\[
\text{Err}(E_1, \rho_1; f, \rho_f; P) + \text{Err}(E_3, \rho_3; f, \rho_f; P) = \text{Err}(E_2, \rho_2; f, \rho_f; P).
\]

In particular, \( \text{Err}(E, \rho; f, \rho_f; P) \) does not depend on the choice of the metric \( \rho \). Furthermore, \( \text{Err}(E, \rho; f, \rho_f; P) \) lies in the classes of \( \ddbar \)-closed forms.

(A.2) Now we study how \( \text{Err} \) depends on hermitian metrics on manifolds.

**Proposition.** Let \( f : X \to Y \) and \( g : Y \to Z \) be two smooth, proper morphisms of compact Kähler manifolds which admit hermitian metrics \( \tau_f, \tau_g \) and \( \tau_{gof} \) on the
relative tangent vector bundles of \( f \), \( g \) and \( g \circ f \) respectively, such that the induced metrics on all fibers are Kähler. Let \((E, \rho)\) be an \( f \)- and \( g \circ f \)-acyclic hermitian vector bundle on \( X \) such that \( f_\ast E \) is \( g \)-acyclic as well. Then

\[
\text{Err}(E, \rho; g \circ f, \tau_{g \circ f}; P) = \text{Err}(f_\ast E, f_\ast \rho; g, \tau_g; P) + g_\ast (\text{Err}(E, \rho; f, \tau_f; P) \cdot \text{td}(T_g, \tau_g)).
\]

In particular, \( \text{Err}(E, \rho; f, \tau_f; P) \) does not depend on the metric \( \tau_f \).

Remark 5.1. Because of these two propositions, we from now on in this section denote \( \text{Err}(E, \rho; f, \tau_f; P) \) simply by \( \text{Err}(E; f; P) \).

(A.3) We may go slight further. Indeed, we have the following

**Proposition.** Let \( f : X \to Y \) be a smooth, proper morphism of compact Kähler manifolds. There is a natural morphism

\[
\text{Err}(\cdot; f, P) : K(X) \to H^\ast(Y, \mathbb{R}),
\]

such that \( \text{Err}(E; f, P) = \text{Err}(E; f; P) \) for all \( f \)-acyclic vector bundles \( E \) on \( X \).

Remark 5.2. Because of this proposition, we may equally write \( \text{Err}(\cdot; f, P) \) as \( \text{Err}(\cdot; f; P) \).

(A.4) We have a functorial property as well.

**Proposition.** Let \( f : X \to Y \) be a smooth, proper morphism of compact Kähler manifolds. Then, for the Cartesian diagram

\[
\begin{array}{ccc}
Y'' \times_Y X & \xrightarrow{9f} & X \\
f_g \downarrow & & \downarrow f \\
Y' & \xrightarrow{9g} & Y
\end{array}
\]

induced from a base change \( g : Y' \to Y \) of compact Kähler manifolds,

\[
g^\ast \text{Err}(E; f; P) = \text{Err}(g^\ast E; f_g; P).
\]

(A.5) We now give a projection formula for \( \text{Err} \).

**Proposition.** Let \( f : X \to Y \) be a smooth, proper morphism of compact Kähler manifolds. Let \( E \) and \( F \) be vector bundles on \( X \) and \( Y \) respectively. Then

\[
\text{Err}(E \otimes f^\ast F; f; P) = \text{Err}(E; f; P) \cdot \text{ch}(F).
\]

(A.6) We now study what happens when \( f \) is a \( \mathbb{P}^1 \)-bundle on \( Y \).
**Proposition.** There is a unique characteristic class $R$ for vector bundles of rank $\leq 2$ such that for any $\mathbb{P}^1$-bundle $f : X = \mathbb{P}_Y(F) \to Y$, $\text{Err}(\cdot ; f; R) \equiv 0$.

(A.7) Finally, we consider Err for closed immersions. For doing so, we introduce a new Err term: Let $i : X \hookrightarrow Z$ be a closed immersion with the smooth structure morphisms $f : X \to Y$ and $g : Z \to Y$ of compact Kähler manifolds, set

$$\text{Err}(E; i; P) := \text{Err}(E; f; P) - \text{Err}(i_*E; g; P).$$

By Proposition A.3, this definition makes sense, even though $i_*E$ is usually a coherent sheaf only.

**Proposition.** Let $i : X \hookrightarrow Z$ be a codimension-one regular closed immersion of compact Kähler manifolds over a compact Kähler manifold $Y$ with smooth structure morphisms $f : X \to Y$ and $g : Z \to Y$. Then for any vector bundle $E$ on $X$,

$$\text{Err}(E; i; P) = 0,$$

for any additive characteristic class $P$, which, for rank two vector bundles, coincides with $R$ in Proposition A.6.

**B. The Proofs**

(B.1) **Proof of Proposition (A.1).** In fact, in $\tilde{A}(Y)$,

$$\text{Err}(E_2, \rho_2; f, \tau_f; P) - \left(\text{Err}(E_1, \rho_1; f, \tau_f; P) + \text{Err}(E_3, \rho_3; f, \tau_f; P)\right)$$

$$= \text{ch}_{BC}(E_2, \rho_2; f, \tau_f) - \text{ch}_{BC}(E_2, \rho_2; f, \tau_f) + f_*\left(\text{ch}(E_2) \cdot \text{td}(T_f) \cdot P(T_f)\right)$$

$$- \left(\text{ch}_{BC}(E_1, \rho_1; f, \tau_f) - \text{ch}_{BC}(E_1, \rho_1; f, \tau_f) + f_*\left(\text{ch}(E_1) \cdot \text{td}(T_f) \cdot P(T_f)\right)\right)$$

$$+ \text{ch}_{BC}(E_3, \rho_3; f, \tau_f) - \text{ch}_{BC}(E_3, \rho_3; f, \tau_f) + f_*\left(\text{ch}(E_3) \cdot \text{td}(T_f) \cdot P(T_f)\right)$$

(by definition)

$$= \text{ch}_{BC}(E_2, \rho_2; f, \tau_f) - \left(\text{ch}_{BC}(E_1, \rho_1; f, \tau_f) + \text{ch}_{BC}(E_3, \rho_3; f, \tau_f)\right)$$

$$- \left(\text{ch}_{BC}(E_2, \rho_2; f, \tau_f) - \left(\text{ch}_{BC}(E_1, \rho_1; f, \tau_f) + \text{ch}_{BC}(E_3, \rho_3; f, \tau_f)\right)\right)$$

$$+ f_*\left(\left(\text{ch}(E_2) - \text{ch}(E_1) - \text{ch}(E_3)\right) \cdot \text{td}(T_f) \cdot P(T_f)\right)$$

$$= f_*\left(\text{ch}_{BC}(E, \rho) \cdot \text{td}(T_f, \tau_f)\right) - \text{ch}_{BC}(f_*(E), L^2(\rho; \tau_f))$$

$$- \left(f_*\left(\text{ch}_{BC}(E, \rho) \cdot \text{td}(T_f, \tau_f)\right) - \text{ch}_{BC}(f_*(E), L^2(\rho; \tau_f))\right)$$

$$+ f_*\left(0 \cdot \text{td}(T_f) \cdot P(T_f)\right)$$

(by Axiom 4 in 2.D.2)

$$= 0.$$

This is simply the first statement of the proposition. With this, by taking \( E_3 \) to be zero, we get the second statement. Finally,

\[
\begin{align*}
\text{dd}^c \text{Err}(E, \rho; f, \tau_f; P) &= \text{dd}^c \left( \text{ch}_{BC}(E, \rho; f, \tau_f) - \text{ch}'_{BC}(E, \rho; f, \tau_f) + f_*(\text{ch}(E_2) \cdot \text{td}(T_f) \cdot P(T_f)) \right) \\
&= \text{dd}^c \left( \text{ch}_{BC}(E, \rho; f, \tau_f) \right) - \text{dd}^c \left( \text{ch}'_{BC}(E, \rho; f, \tau_f) \right) + \text{dd}^c \left( f_*(\text{ch}(E_2) \cdot \text{td}(T_f) \cdot P(T_f)) \right) \\
&= f_*(\text{ch}(E, \rho) \cdot \text{td}(T_f, \tau_f)) - \text{ch}(f_*E, L^2(\rho; \tau_f)) \\
&\quad - \left( f_*(\text{ch}(E, \rho) \cdot \text{td}(T_f, \tau_f)) - \text{ch}(f_*E, L^2(\rho; \tau_f)) \right) + 0 \quad \text{(by Axiom 1 in 2.A.2)} \\
&= 0.
\end{align*}
\]

This completes the proof of Proposition A.1.

(B.2) \textit{Proof of Proposition A.2.} Setting \( g = \text{Id}_Y \), the identity map of \( Y \), we find that the second statement of this proposition is a consequence of the first one.

Now we prove the first statement.

\[
\begin{align*}
\text{Err}(E, \rho; g \circ f, \tau_{g \circ f}; P) &= \left( \text{Err}(f_*E, f_*\rho; g, \tau_g; P) + g_*(\text{Err}(E, \rho; f, \tau_f; P) \cdot \text{td}(T_g, \tau_g)) \right) \\
&= \text{ch}_{BC}(E, \rho; g \circ f, \tau_{g \circ f}) - \text{ch}'_{BC}(E, \rho; g \circ f, \tau_{g \circ f}) \\
&\quad + (g \circ f)_*\left( \text{ch}(E) \cdot \text{td}(T_{g \circ f}) \cdot P(T_{g \circ f}) \right) \\
&\quad - \left( \text{ch}_{BC}(f_*E, L^2(\rho, \tau_f); g, \tau_g) - \text{ch}'_{BC}(f_*E, L^2(\rho, \tau_f); g, \tau_g) \\
&\quad + f_* \left( \text{ch}(f_*E) \cdot \text{td}(T_g) \cdot P(T_g) \right) \right) \\
&\quad - g_* \left( \left( \text{ch}_{BC}(E, \rho; f, \tau_f) - \text{ch}'_{BC}(E, \rho; f, \tau_f) \\
&\quad + f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f) \right) \right) \cdot \text{td}(T_g, \tau_g) \right) \quad \text{(by definition)} \\
&= 0.
\end{align*}
\]
\[= \left( \mathrm{ch}_{BC}(E, \rho; g \circ f, \tau_{g \circ f}) - \mathrm{ch}_{BC}(f_* E, L^2(\rho, \tau_f); g, \tau_g) \right. \\
- g_* \left( \mathrm{ch}_{BC}(E, \rho; f, \tau_f) \cdot \mathrm{td}(T_g, \tau_g) \right) \\
- \left. \left( \mathrm{ch}'_{BC}(E, \rho; g \circ f, \tau_{g \circ f}) - \mathrm{ch}'_{BC}(f_* E, L^2(\rho, \tau_f); g, \tau_g) \right) \\
- g_* \left( \mathrm{ch}'_{BC}(E, \rho; f, \tau_f) \cdot \mathrm{td}(T_g, \tau_g) \right) \\
+ \left( (g \circ f)_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_{g \circ f}) \cdot P(T_{g \circ f}) \right) - g_* \left( \mathrm{ch}(f_* E) \cdot \mathrm{td}(T_g) \cdot P(T_g) \right) \right. \\
- g_* \left( f_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f) \right) \cdot \mathrm{td}(T_g, \tau_g) \right) \\
= (g \circ f)_* \left( \mathrm{ch}(E, \rho) \cdot \mathrm{td}_{BC}(T, \tau) \right) \\
- \left( (g \circ f)_* \left( \mathrm{ch}(E, \rho) \cdot \mathrm{td}_{BC}(T, \tau) \right) \right) \\
- \left. \left( (g \circ f)_* \left( \mathrm{ch}(E, \rho) \cdot \mathrm{td}_{BC}(T, \tau) \right) \right) \\
+ \left( (g \circ f)_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_{g \circ f}) \cdot P(T_{g \circ f}) \right) - g_* \left( \mathrm{ch}(f_* E) \cdot \mathrm{td}(T_g) \cdot P(T_g) \right) \right. \\
- g_* \left( f_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f) \right) \cdot \mathrm{td}(T_g, \tau_g) \right) \\
\right) \\
\text{(by Axiom 4 in 2.D.2)} \\
= \left( (g \circ f)_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_{g \circ f}) \cdot P(T_{g \circ f}) \right) - g_* \left( \mathrm{ch}(f_* E) \cdot \mathrm{td}(T_g) \cdot P(T_g) \right) \right. \\
- g_* \left( f_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f) \right) \cdot \mathrm{td}(T_g) \right) \right). \\
\]

From here it is sufficient to prove the following:

**Lemma.** With the same notation as above,

\[ (g \circ f)_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_{g \circ f}) \cdot P(T_{g \circ f}) \right) = g_* \left( \mathrm{ch}(f_* E) \cdot \mathrm{td}(T_g) \cdot P(T_g) \right) + g_* \left( f_* \left( \mathrm{ch}(E) \cdot \mathrm{td}(T_f) \cdot P(T_f) \right) \cdot \mathrm{td}(T_g) \right). \]

**Proof.** By definition, \( P \) is additive and \( \mathrm{td} \) is multiplicative. Hence applying them to the exact sequence of relative tangent bundles

\[0 \to T_f \to T_{g \circ f} \to f^* T_g \to 0,\]

we have \( P(T_{g \circ f}) = P(T_f) + f^* P(T_g), \) \( \mathrm{td}(T_{g \circ f}) = \mathrm{td}(T_f) \cdot f^* \mathrm{td}(T_g). \) Here we use
the fact that as characteristic classes, $P$ and $\text{td}$ have the functorial property. Hence

\[(g \circ f)_* \left( \text{ch}(E) \cdot \text{td}(T_{g \circ f}) \cdot P(T_{g \circ f}) \right) \]

\[= (g \circ f)_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot f^*\text{td}(T_g) \cdot \left( P(T_f) + f^*P(T_g) \right) \right) \]

\[= (g \circ f)_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot f^*\text{td}(T_g) \cdot P(T_f) \right) \]

\[+ (g \circ f)_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot f^*\text{td}(T_g) \cdot f^*\left( P(T_g) \right) \right) \]

\[= g_* \left( f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f) \cdot f^*\text{td}(T_g) \right) \right) \]

\[+ g_* \left( f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot f^*\left( \text{td}(T_g) \cdot P(T_g) \right) \right) \right) \]

\[= g_* \left( f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f) \right) \cdot \text{td}(T_g) \right) \]

\[+ g_* \left( f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \right) \cdot \text{td}(T_g) \cdot P(T_g) \right) \]

(by the projection formula)

\[= g_* \left( f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f) \right) \cdot \text{td}(T_g) \right) + g_* \left( \text{ch}(f_* (E)) \cdot \text{td}(T_g) \cdot P(T_g) \right) \]

(by the Grothendieck-Riemann-Roch Theorem in Algebraic Geometry).

This completes the proof of the lemma ane hence Proposition A.2.

(B.3) **Proof of Proposition A.3.** The first point we need to address is that by Proposition A.1, $\text{Err}$ is only $dd^c$-closed, but in this proposition, we instead choose the de Rham cohomology groups as the range. This may be solved as follows.

For a compact Kähler manifold $Y$, denote the space of $dd^c$-closed forms by $A^*_{dd^c}(Y)$ and set $H^*_{dd^c}(Y, \mathbb{R})$ to be the quotient space of $A^*_{dd^c}(Y)$ modulo the $\partial$- and $\bar{\partial}$-forms. It is well-known that for a compact Kähler manifold, a form $\omega$ is $d$-closed and is either $d-$, or $\partial-$, or $\bar{\partial}$-exact if and only if there exists a form $\gamma$ such that $dd^c\gamma = \omega$. (See e.g. 7.A.1 later.) Hence if $\omega$ is $dd^c$-closed, then we may choose $\gamma$ to be $\phi + \partial\alpha + \bar{\partial}\beta$ with $\phi$ harmonic. Therefore, we see that $H^*_{dd^c}(Y, \mathbb{R})$ is isomorphic to the cohomology of $Y$.

The second point here is that we should construct a map $\text{Err}$ from the Grothendieck $K$-group of coherent sheaves of $X$ to $H^*_{dd^c}(Y, \mathbb{R})$ which coincides with the previous $\text{Err}$ for $f$-acyclic vector bundles. But this is rather formal.

(i) For complex Kähler manifolds, the Grothendieck $K$ group is isomorphic to the $K$ group generated by vector bundles. So we will write both of these $K$-groups as $K(X)$;

(ii) for $X$, a compact Kähler manifold, smooth and proper over $Y$, $K(X)$ is generated by $f$-acyclic vector bundles. Indeed, for any coherent sheaf $F$ on $X$, there
exists an vector bundle resolution $0 \to F \to E_0 \to E_2 \to \cdots \to E_m \to 0$ such that all $E_i$’s are $f$-acyclic;

(iii) any two $f$-acyclic vector bundle resolutions of a fixed coherent sheaf in (ii) are dominated by a common third one.

Now, we are ready to define a morphism $\text{Err}(\cdot; f, P) : K(X) \to H^*(Y, \mathbb{R})$ as follows: For any coherent sheaf $F$, let $0 \to F \to E_0 \to E_1 \to \cdots \to E_m \to 0$ be a resolution of $F$ by $f$-acyclic vector bundles. Set

$$\text{Err}(F; f; P) := \sum_{i=0}^{m} (-1)^i \text{Err}(E_i; f; P).$$

Err is well-defined, i.e., it does not depend on the $f$-acyclic resolutions of vector bundles we choose. Indeed, if $F$ is a vector bundle, then by Proposition A.2 and (ii) and (iii), we see Err is well-defined. All this, together with (i), implies that the same is true for coherent sheaves, which then completes the proof of Proposition A.3.

(B.4) Proof of Proposition A.4. It is clear that we only need to show this for $f$-acyclic vector bundles. But then

$$\text{Err}(E, \rho; f, \tau_f; P) = \text{ch}_{\text{BC}}(E, \rho; f, \tau_f) - \text{ch}_{\text{BC}}^1(E, \rho; f, \tau_f) + f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)\right).$$

Now, by Axiom 3 for relative Bott-Chern secondary characteristic classes in 2.C.2, we see that $\text{ch}_{\text{BC}}(E, \rho; f, \tau_f)$ and $\text{ch}_{\text{BC}}^1(E, \rho; f, \tau_f)$ are compactible with base change. Obviously, so does the term $f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f)\right)$. This then proves Proposition A.4.

Remark 5.3. In the past, the functorial rule for relative Bott-Chern secondary characteristic classes are systematically denied by a group of very ill-motivated mathematicians, who simply claim that the functorial rule I used is completely wrong. Unfortunately, they stand on the wrong side. Later we will give more consequences for the functorial rule to indicate its absolute importance.

(B.5) Proof of Proposition A.5. Obviously we only need to show it for $f$-acyclic vector bundles $E$ on $X$. Choose hermitian metrics $\rho_E$ and $\rho_F$ on $E$ and $F$ respectively, and a hermitian metric $\tau_f$ on $T_f$ such that all the induced metrics on the
fibers of $f$ are Kähler. Then, by Proposition A.1,

\[
\text{Err}(E \otimes f^*F; f; P) \\
= \text{Err}(E \otimes f^*F; \rho \otimes f^*F; f, \tau_f; P) \\
= \text{ch}_{BC}(E \otimes f^*F; \rho \otimes f^*F; f, \tau_f) - \text{ch}_{BC}'(E \otimes f^*F; \rho \otimes f^*F; f, \tau_f) \\
+ f_* \left( \text{ch}(E \otimes f^*F) \cdot \text{td}(T_f) \cdot P(T_f) \right) \quad \text{(by definition)} \\
= \left( \text{ch}_{BC}(E; \rho; f, \tau_f) - \text{ch}'_{BC}(E; \rho; f, \tau_f) \right) \cdot \text{ch}(F, \rho_F) \\
+ f_* \left( \text{ch}(E) \cdot f^*\text{ch}(F) \cdot \text{td}(T_f) \cdot P(T_f) \right) \quad \text{(by Axiom 2 in 2.B.2)} \\
= \text{Err}(E; f; 0) \cdot \text{ch}(F, \rho_F) + f_* \left( \text{ch}(E) \cdot \text{td}(T_f) \cdot P(T_f) \right) \cdot \text{ch}(F) \\
\quad \text{(by definition and the projection formula)} \\
= \text{Err}(E; f; P) \cdot \text{ch}(F).
\]

This completes the proof of the proposition.

(B.6) Proof of Proposition A.6. In the proof of this proposition, all axioms but the last one, i.e., the rule of the deformation to the normal cone, will be used.

First, it is well-known that, as a $K(Y)$-module, $K(X)$ is generated by line bundles $O_X$ and $O_X(-1)$. Obviously, $O_X$, and $O_X(-1)$ are both $f$-acyclic. So by Propositions A.3 and A.5, it is sufficient to show that there exists a unique additive characteristic class $R$ such that

\[
\text{Err}(O_X; f; R) = 0, \quad \text{and} \quad \text{Err}(O_X(-1); f; R) = 0.
\]

We prove this by a direct calculation and Proposition A.4, i.e., the functorial rule.

For the time being, we will assume that $F$ is generated by global sections. Such a condition can be automatically granted if we tensor $F$ by a very ample line bundle on $Y$. Moreover, it is well-known that such a change will not change the structure $f$, nor the sheaves $O$ and $O(-1)$.

However, with this latest condition, then, we may find a certain natural classifying map $\phi : Y \to G(2, 2 + n)$, where $G(2, 2 + n)$ denotes the Grassmannian of dimension two subspaces in a complex $(2 + n)$-dimensional space. Moreover, $F$ can be reconstructed as the pull-back of the canonical rank two subbundle $S_n$ of the rank $(2 + n)$ trivial bundle on $G(2, 2 + n)$. By the functorial rule, or better, Proposition A.4, we can then without loss of generality work on the canonical $\mathbb{P}^1$ bundle $p_n : \mathbb{P}(S_n) \to G(2, 2 + n)$.
Note that in a natural way, \( \{p_n\}_{n \geq 0} \) forms an inverse limit system: We indeed has the Cartesian product

\[
P(S_n) \xrightarrow{i_{n,m}} P(S_{n+m})
\]
\[
p_n \downarrow \downarrow p_{n+m}
\]

\[G(2, 2 + n) \xrightarrow{j_n} G(2, 2 + (n + m))\]
such that \( i_{n,m}^* \mathcal{O}_{P(S_{n+m})}(k) = \mathcal{O}_{P(S_n)}(k) \). Hence, by the functorial rule, i.e., Proposition A.5, we see that Err for each \( p_n \) for \( \mathcal{O}_{P(S_n)} \) or \( \mathcal{O}_{P(S_n)}(-1) \) forms an inverse limit system in \( \{H^*(G(2, 2 + n), \mathbb{R})\}_{n \geq 0} \).

But it is well-known that

\[H^*(G(2, 2 + n), \mathbb{R}) = \mathbb{R}[c_1(S_n), c_2(S_n), c_1(Q_n), \ldots, c_n(Q_n)]/(c(S_n) \cdot c(Q_n) = 1)\]

where \( Q_n \) is the canonical quotient bundle on \( G(2, 2 + n) \) which may be defined via the canonical exact sequence

\[0 \to S_n \to G(2, 2 + n) \times \mathbb{C}^{2+n} \to Q_n \to 0,\]

\[c(S_n) = 1 + c_1(S_n) + c_2(S_n) \text{ and } c(Q_n) = 1 + c_1(Q_n) + c_2(Q_n) + \cdots + c_n(Q_n).\] (See e.g., [BT, §23].) Note that the relation above says that \( c_j(Q_n) \) can be written as a polynomial of \( c_1(S_n) \) and \( c_2(S_n) \) for \( j = 1, \ldots, n \). More precisely, for \( k = 2, \ldots, n \), we have the recurrence formula \( c_k(Q_n) = -c_1(S_n)c_{k-1}(Q_n) - c_2(S_n)c_{k-2}(Q_n) \) with \( c_0(Q_n) = 1, c_1(Q_n) = -c_1(S_n) \). Hence, in particular, \( \lim_{n \to \infty} H^*(G, \mathbb{R}) = \mathbb{R}[[c_1, c_2]] \).

So, Err for \( \mathcal{O} \) and \( \mathcal{O}(-1) \) can be written as some universal classes in a ring of power series of \( c_1 \) and \( c_2 \). Set \( Err(\mathcal{O}; \pi; P) := P_1(c_1, c_2) \) and \( Err(\mathcal{O}(-1); \pi; P) := P_2(c_1, c_2) \) be the two power series.

Moreover, it is clear that if we change \( S_n \) by tensoring an ample line bundle \( L \) on \( S_n, \mathcal{O}, \pi \) and \( P \) will not be changed. But \( c_1 \) and \( c_2 \) are changed to \( c_1 + 2c_1(L) \) and \( c_2 + c_1(L)(c_1 + c_1(L)) \) respectively. This implies that

\[c_2(S_n) - \frac{1}{4}c_1^2(S_n) = c_2(S_n \otimes L) - \frac{1}{4}c_1^2(S_n \otimes L)\]

remains the same. From this, we see that \( P_1(c_1, c_2) \) is actually a power series in \( c_2 - \frac{1}{4}c_1^2 \), which we denote by \( P_1 \) as well by an abuse of notation.

On the other hand, we have the exact sequence

\[0 \to \mathcal{O}_X \to p^* S^\vee \otimes \mathcal{O}_X(1) \to T_p \to 0.\]
Hence, $T_p \simeq p^*(\text{det} S)^\vee \otimes \mathcal{O}_X(2)$. So $\mathcal{O}(-1) \otimes \frac{1}{2}p^*(\text{det} S) = \frac{1}{2}T_p \vee$. But if we tensor $S$ by a line bundle, $T_p$, associated to $p$ which remains the same, will not be changed. Thus, by Proposition A.5, we see that if we multiply the Err by $\exp(\frac{1}{2}c_1)$, which takes care of the part of $\frac{1}{2}p^*(\text{det} S)$ in $\frac{1}{2}T_p \vee = \mathcal{O}(-1) \otimes \frac{1}{2}p^*(\text{det} S)$, $P_2(c_1, c_2) = \text{Err}(\mathcal{O}(-1); p; R)$ becomes a power series in $c_2 - \frac{1}{4}c_1^2$ as well, which we still denote by $P_2$ by an abuse of notation.

With all this, now we are ready to determine Err precisely. For this purpose, first, let $A = c_1(\mathcal{O}(1)) - \frac{1}{2}p^*c_1(S)$. Then, we have

$$c_1(T_p, \tau_p) = 2A, \quad \text{and} \quad A^2 = c_1(\mathcal{O}(1))^2 - c_1(\mathcal{O}(1)) \cdot p^*c_1(S) = p^*(-c_2(F) + \frac{1}{4}c_1(S)^2).$$

Therefore, by the fact that the direct image of the structure sheaf is the structure sheaf on the base, $p_*A = 1$,

$$p_*A^{2m} = 0, \quad \text{and} \quad p_*(A^{2m+1}) = (-c_2(F) + \frac{1}{4}c_1(F)^2)^m p_*A = (-c_2(F) + \frac{1}{4}c_1(F)^2)^m$$

for all positive integer $m$.

Now, we want to see how Err depends on $P$. Obviously, if we change $P$ by adding $P'$, then the corresponding Err for $\mathcal{O}_X$ changes by

$$p_*(\frac{2A}{1 - e^{-2A}}P'(2A))$$

and similarly, for $\mathcal{O}_X(-1)$, (up to the factor $\exp(\frac{1}{2}c_1)$ as discussed above,) the error changes by

$$p_*(\frac{2Ae^{-A}}{1 - e^{-2A}}P'(2A)).$$

In particular, for $\mathcal{O}(-1)$, the factor before $P'$ is an even function in $A$, and hence is a series in even powers. So, by (5.1), we may choose a unique odd power series $R_{\text{odd}}$, such that for any even power series $P'$,

$$\text{Err}(\mathcal{O}_X(-1); p; R_{\text{odd}} + P') = \text{Err}(\mathcal{O}_X(-1); p; R_{\text{odd}}) = 0.$$  

Similarly, as for $\mathcal{O}$, once the odd part of $P$ is fixed, by (5.1) again, there is one and only one even power series $R_{\text{even}}$ such that

$$\text{Err}(\mathcal{O}_X; p; R_{\text{odd}} + R_{\text{even}}) = 0,$$

as now only the odd terms of $\frac{2A}{1 - e^{-2A}}(R_{\text{odd}}(2A) + P'(2A))$ matters.
Therefore, finally, we see that there exists a unique power series $R = R^{\text{even}} + R^{\text{odd}}$ such that $\text{Err}(\cdot; p; R) = 0$ for all $\mathbb{P}^1$-bundles. This completes the proof of Proposition A.6.

(B.7) Proof of Proposition A.7. We divide the proof of this proposition into two steps;

(i) Consider the special situation where codimension-one closed immersions are sections of $\mathbb{P}^1$-bundles. In this case, we have the following

**Lemma 1.** Let $f : X \to Y$ be a smooth, proper morphism of compact Kähler manifolds, and $\pi : \mathbb{P}^1_X \to X$ be a $\mathbb{P}^1$-bundle over $X$. Assume that $i : X \to \mathbb{P}^1_X$ is a section, so that $\pi \circ i = \text{Id}_X$. Then for the closed immersion $i$ over $Y$, and any vector bundle $E$ on $X$, we have

$$\text{Err}(E; i; P) = 0.$$ 

(ii) Deduce a general codimension-one closed immersion to a section of a $\mathbb{P}^1$-bundle by the deformation to the normal cone technique. More precise, we have the following

**Lemma 2.** With the same notation as in 3.A.1, i.e., suppose we have the diagram

$$
\begin{array}{ccc}
\mathbb{P}^1_X & \xrightarrow{i_\infty} & X & \xrightarrow{i_0 = i} & Z \\
\downarrow g_\infty & & \downarrow f_\infty & \downarrow f_0 & \downarrow g_0 \\
& & Y,
\end{array}
$$

then for any vector bundle $E$ on $Z$, we have

$$\text{Err}((i_0)_*(i^*E); g_0; P) = \text{Err}((i_\infty)_*(i^*E); g_\infty; P).$$

Before proving these two lemmas, let us show how Proposition A.7 follows from them.

First of all, by Lemma 2, we see that for all coherent sheaves $F$ on $Z$,

$$\text{Err}((i_0)_*i^*F; g_0; P) = \text{Err}((i_\infty)_*i^*F; g_\infty; P).$$

 Obviously, all coherent sheaves on $X$ can be realized as the pull-back of coherent sheaves from $Z$, so, for all coherent sheaves $F'$ on $X$,

$$\text{Err}((i_0)_*F'; g_0; P) = \text{Err}((i_\infty)_*F'; g_\infty; P).$$
This then certainly implies that for all vector bundles $E'$ on $X$,

$$\text{Err}((i_0)_*E'; g_0; P) = \text{Err}((i_\infty)_*E'; g_\infty; P). \quad (5.2)$$

On the other hand, by definition, $f_0 = f_\infty$, we have

$$\text{Err}(E'; f_0; P) = \text{Err}(E'; f_\infty; P). \quad (5.3)$$

Therefore,

$$\text{Err}(E'; i_0; P) = \text{Err}(E'; i_\infty; P) \quad (by \ Definition)$$

Now we go back to show Lemma 1 and Lemma 2.

(i) Proof of Lemma 1. Here that $i$ is a section of a $\mathbb{P}^1$-bundle over $X$ plays a very important rule. Indeed, if $g$ denotes the natural composition $\mathbb{P}^1_X \xrightarrow{\pi} X \xrightarrow{f} Y$, then by definition,

$$\text{Err}(E; i; P) = \text{Err}(E; f; P) \quad (by \ Definition)$$

$$= \text{Err}(E; f; P) - \text{Err}(i_* E; g; P)$$

$$= \text{Err}(E; f; P) - \text{Err}(i_* E; f \circ \pi; P)$$

$$= \text{Err}(E; f; P) - \left( \text{Err}(\pi_* i_* E; f; P) + f_*(\text{Err}(i_* E; \pi; P) \cdot \text{td}(T_f)) \right) \quad (by \ Proposition \ A.2)$$

$$= \text{Err}(E; f; P) - \left( \text{Err}((\text{Id}_X)_* E; f; P) + f_*(\text{Err}(i_* E; \pi; R) \cdot \text{td}(T_f)) \right) \quad (as \ \pi \circ i = \text{id}_X \ and \ by \ our \ assumption \ on \ P)$$

$$= - f_*(\text{Err}(i_* E; \pi; R) \cdot \text{td}(T_f))$$

$$= - f_*(0 \cdot \text{td}(T_f)) \quad (by \ Proposition \ A.6)$$

$$= 0.$$

Remark 5.4. Note that in this proof, we only use the fact that $i$ is a section of a $\mathbb{P}^1$-bundle and for $\mathbb{P}^1$-bundles, Err is simply zero. Thus if $i$ is a section of a $\mathbb{P}^n$-bundle
and, for all $\mathbb{P}^n$-bundles, $\text{Err}$ is zero as well, we may have a similar discussion. We will use this remark in the next chapter.

(ii) Proof of Lemma 2. This is where we should use the deformation to the normal cone rule. So we recall the following commutative diagram.

\[
\begin{array}{cccc}
X \times \{t\} & \hookrightarrow & X \times \mathbb{P}^1 & \hookrightarrow & X \times \{\infty\} \\
{\downarrow i_t} & & {\downarrow I} & & {\downarrow i_\infty} \\
(t \neq \infty) \; Z \times \{t\} & \hookrightarrow & B_{X \times \{\infty\}}Z \times \mathbb{P}^1 & \hookrightarrow & + \; \mathbb{P} \\
{\downarrow f_t} & & {\downarrow \pi} & & {\downarrow f_\infty} \\
{\downarrow g_t} & & Z \times \mathbb{P}^1 & & \\
Y \times \{t\} & \hookrightarrow & Y \times \mathbb{P}^1 & \hookrightarrow & Y \times \{\infty\} \\
\end{array}
\]

By pulling back $E$ from $Z$ onto $W = B_{X \times \{\infty\}}Z \times \mathbb{P}^1$ via the composition of natural maps $W \xrightarrow{\pi} Z \times \mathbb{P}^1 \rightarrow Z$, we obtain a vector bundle on $W$, denoted by $E$ as well by an abuse of notation. Twisted by $B_{X}Z$, we then get the exact sequence of coherent sheaves on $W$:

\[0 \rightarrow E(B_{X}Z - X \times \mathbb{P}^1) \rightarrow E(B_{X}Z) \rightarrow I_* I^*(E(B_{X}Z)) \rightarrow 0. \quad (5.4)\]

Note that here in particular $E(B_{X}Z - X \times \mathbb{P}^1)$ and $E(B_{X}Z)$ are vector bundles on $W$. Easily, we have

\[I_* I^*(E(B_{X}Z))|_{X \times \{t\}} = (i_t)_*(i^*E), \quad \text{if } t \neq \infty;\]

\[I_* I^*(E(B_{X}Z))|_{\mathbb{P}} = (i_\infty)_*(i^*E),\]

and

\[E(B_{X}Z - X \times \mathbb{P}^1) = \pi^*(E(-X \times \mathbb{P}^1) \otimes O_{Z \times \mathbb{P}^1}(Z \times \{\infty\})) = \pi^*(p_Z^*(E(-X)) \otimes p_{\mathbb{P}^1}^*(O_{\mathbb{P}^1}(\{\infty\})).\]

Here $p_Z$ and $p_{\mathbb{P}^1}$ denote the projection of $Z \times \mathbb{P}^1$ to $Z$ and $\mathbb{P}^1$ respectively.

On the other hand, by the fact that $W \rightarrow \mathbb{P}^1$ is flat, the restrictions of (5.4) to $Z \times \{t\}$ and $\mathbb{P} \cup B_{X}Z$ are again exact. In particular, with the same notation as in 3.A.2, we have the short exact sequences

\[0 \rightarrow E_t(-X) \rightarrow E_t \rightarrow (i_t)_* i_t^* E_t \rightarrow 0 \quad \text{on } Z \times \{t\};\]
\[ 0 \to E_\infty(-X) \to E_\infty \to (i_\infty)_* i_\infty^* E_\infty \to 0 \quad \text{on} \quad \mathbb{P}; \]

and

\[ 0 \to E'_\infty \to E''_\infty \to 0 \to 0 \quad \text{on} \quad B_XZ. \]

Thus, by Proposition A.3, it is sufficient to show that

\[
\text{Err}\left(E(B_XZ - X \times \mathbb{P}^1)|_{Z \times \{0\}}; g_0; P\right) - \text{Err}\left(E(B_XZ)|_{Z \times \{0\}}; g_0; P\right)
\]

\[
= \text{Err}\left(E(B_XZ - X \times \mathbb{P}^1)|_P; g_\infty; P\right) - \text{Err}\left(E(B_XZ)|_P; g_\infty; P\right),
\]

or better, to show that

\[
\text{Err}\left(E(-X); g_0; P\right) - \text{Err}\left(E; g_0; P\right) = \text{Err}\left(E_\infty(-X); g_\infty; P\right) - \text{Err}\left(E_\infty; g_\infty; P\right).
\]

Here \( E_\infty \) denotes \( E(B_XZ)|_P \). From here, by Proposition A.3 again, without loss of generality, we may assume that \( E_t \) and \( E_t(-X) \) are all \( g_t \)-acyclic.

**Lemma 3.** With the same notation as above, for all \( t \neq \infty \),

\[
\text{Err}\left(E(-X); g_0; P\right) - \text{Err}\left(E; g_0; P\right) = \text{Err}\left(E_t(-X); g_t; P\right) - \text{Err}\left(E_t; g_t; P\right).
\]

**Proof.** We first show that

\[
\text{Err}\left(E; g_0; P\right) = \text{Err}\left(E_t; g_t; P\right). \tag{5.5}
\]

Choose a metric \( \rho \) on \( E \) as a vector bundle on \( Z \). By Proposition A.1,

\[
\text{Err}\left(E; g_0; P\right) = \text{ch}_{BC}(E_0, \rho; g_0, \tau_{g_0}) - \text{ch}_{BC}(E_0, \rho; g_0, \tau_{g_0}) + (g_0)_* \left( \text{ch}(E_0) \cdot \text{td}(T_{g_0}) \cdot P(T_{g_0}) \right),
\]

and

\[
\text{Err}\left(E_t; g_t; P\right) = \text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) + (g_t)_* \left( \text{ch}(E_t) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t}) \right).
\]

Note that in the construction of the deformation to the normal cone, the base curve is a rational curve \( \mathbb{P}^1 \). As a direct consequence, the de Rham cohomology classes will not be changed for this deformation family. This then shows that in \( H^*(Y, \mathbb{R}) \), for all \( t \neq \infty \),

\[
(g_t)_* \left( \text{ch}(E_t) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t}) \right) = (g_0)_* \left( \text{ch}(E_0) \cdot \text{td}(T_{g_0}) \cdot P(T_{g_0}) \right). \tag{5.6}
\]
Note that \( E_0 \simeq E_t \) and \( T_{g_0} \simeq T_{g_t} \), so by changing from 0 to \( t \), we may understand that we are working on the same fibration \( g : Z \to Y \) for the same \( g \)-acyclic vector bundle \( E \). Hence, from \((E_0, \rho_0; g_0, \tau_0)\) to \((E_t, \rho_t; g_t, \tau_t)\), the only changes come from the metrics: for \( E \), the metric \( \rho \) is changed to \( \rho_t \) while for \( T_g \), the metric \( \tau_{g_0} \) is changed to \( \tau_{g_t} \). Thus by Axiom 4 and Axiom 5 for relative Bott-Chern secondary characteristic classes in 2.D.2 and 2.E.3 respectively, we see that the change of relative Bott-Chern classes from 0 to \( t \) for \( ch_{BC} \) is the same as the change of relative Bott-Chern classes from 0 to \( t \) for \( ch'_{BC} \), as described in 3.B.2. More precisely, by (3.3.7),

\[
\begin{align*}
ch_{BC}(E_0, \rho; g_0, \tau_{g_0}) - ch_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) &= -g_* \left( ch_{BC}(E; \rho_t, \rho_0) \cdot \text{td}(T_g; \tau_t) + ch(E; \rho_0) \cdot \text{td}_{BC}(T_g; \tau_t, \tau_0) \right) \\
&\quad - ch_{BC} \left( g_* E; L^2(\rho_t, \tau_t), L^2(\rho_0, \tau_0) \right) \\
&= ch'_{BC}(E_0, \rho; g_0, \tau_{g_0}) - ch'_{BC}(E_t, \rho; g_t, \tau_{g_t}).
\end{align*}
\]

This, together with (5.6), then certainly gives (5.5). Similarly, we have

\[
\text{Err}\left(E(-X); g_0; P\right) = \text{Err}\left(E_t(-X); g_t; P\right). \tag{5.7}
\]

So finally, by (5.5) and (5.7), we complete the proof of Lemma 3.

Now we are finally ready to complete the proof of Proposition A.7.
For \( t \neq \infty \) in the projective line \( \mathbb{P}^1 \), we have

\[
\begin{align*}
&\left( \text{Err}(E_0(-X); g_0; P) - \text{Err}(E_0; g_0; P) \right) \\
&\quad - \left( \text{Err}(E_{\infty}(-X); g_{\infty}; P) - \text{Err}(E_{\infty}; g_{\infty}; P) \right) \\
&= \left( \text{Err}(E_t(-X); g_t; P) - \text{Err}(E_t; g_t; P) \right) \\
&\quad - \left( \text{Err}(E_{\infty}(-X); g_{\infty}; P) - \text{Err}(E_{\infty}; g_{\infty}; P) \right)
\end{align*}
\]

(by Lemma 3)

\[
\begin{align*}
&\left( \text{ch}_{BC}(E_t(-X), \rho_t; g_t, \tau_{g_t}) - \text{ch'}_{BC}(E_t(-X), \rho_t; g_t, \tau_{g_t}) \\
&\quad + (g_t)_*(\text{ch}(E_t(-X)) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t})) \right) \\
&\quad - \left( \text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - \text{ch'}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) \\
&\quad + (g_t)_*(\text{ch}(E_t) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t})) \right) \\
&\quad - \left( \text{ch}_{BC}(E_{\infty}(-X), \rho_{\infty}; g_{\infty}, \tau_{g_{\infty}}) - \text{ch'}_{BC}(E_{\infty}(-X), \rho_{\infty}; g_{\infty}, \tau_{g_{\infty}}) \\
&\quad + (g_{\infty})_*(\text{ch}(E_{\infty}(-X)) \cdot \text{td}(T_{g_{\infty}}) \cdot P(T_{g_{\infty}})) \right)
\end{align*}
\]

(by definition of \( \text{Err} \))

\[
\begin{align*}
&\left( \text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \text{ch}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) \right) \\
&\quad - \left( \text{ch}_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}) - \text{ch'}_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) \right) \\
&\quad - \left( \text{ch}_{BC}(E_{\infty}(-X), \rho'_{\infty}; g_{\infty}, \tau_{g_{\infty}}) - \text{ch}_{BC}(E_{\infty}, \rho_{\infty}; g_{\infty}, \tau_{g_{\infty}}) \right) \\
&\quad - \left( \text{ch'}_{BC}(E_{\infty}(-X), \rho'_{\infty}; g_{\infty}, \tau_{g_{\infty}}) - \text{ch'}_{BC}(E_{\infty}, \rho_{\infty}; g_{\infty}, \tau_{g_{\infty}}) \right) \\
&\quad + \left( (g_t)_*(\text{ch}(E_t(-X) - E_t) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t})) \right) \\
&\quad - \left( (g_{\infty})_*(\text{ch}(E_{\infty}(-X) - E_{\infty}) \cdot \text{td}(T_{g_{\infty}}) \cdot P(T_{g_{\infty}})) \right)
\end{align*}
\]

(by (5.6)).
From here, if we use Axiom 6 of the deformation to the normal cone rule for the relative Bott-Chern secondary characteristic classes in 3.C.3, by taking the limit $t \to \infty$, we see that in the above expression, the contribution of $\text{ch}_{\text{BC}}$ exactly cancels out the contribution from $\text{ch}'_{\text{BC}}$. This then implies that

$$
\left( \text{Err}(E_0(-X); g_0; P) - \text{Err}(E_0; g_0; P) \right) - \left( \text{Err}(E_\infty(-X); g_\infty; P) - \text{Err}(E_\infty; g_\infty; P) \right) = (g_0)_* \left( \text{ch}(E_0(-X) - E_0) \cdot \text{td}(T_{g_0}) \cdot R(T_{g_0}) \right) - (g_\infty)_* \left( \text{ch}(E_\infty(-X) - E_\infty) \cdot \text{td}(T_{g_\infty}) \cdot P(T_{g_\infty}) \right).
$$

(5.8)

**Lemma 4.** With the same notation as above,

$$(g_0)_* \left( \text{ch}(E_0(-X) - E_0) \cdot \text{td}(T_{g_0}) \cdot R(T_{g_0}) \right) = (g_\infty)_* \left( \text{ch}(E_\infty(-X) - E_\infty) \cdot \text{td}(T_{g_\infty}) \cdot P(T_{g_\infty}) \right).$$

**Proof.** Let $N_{i_t}$ denote the normal bundle of $i_t$ for all $t \in \mathbb{P}^1$. Then we have the following exact sequence

$$0 \to T_f \to i_t^* T_{g_t} \to N_{i_t} \to 0.$$

Therefore, $i_t^* \text{td}(T_{g_t}) = \text{td}(T_f) \cdot \text{td}(N_{i_t})$ and $i_t^* P(T_{g_t}) = P(T_f) + P(N_{i_t})$. Hence, for all points $t$ in the projective line $\mathbb{P}^1$,

$$(g_t)_* \left( \text{ch}(E_t(-X) - E_t) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t}) \right) = (g_t)_* \left( \text{ch}((i_{t_t})_* i_{t}^* E) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t}) \right) = (g_t)_* \left( (i_{t_t})_* (\text{ch}(i_{t_t}^* E) \cdot \text{td}(N_{i_t})^{-1}) \cdot \text{td}(T_{g_t}) \cdot P(T_{g_t}) \right) \quad \text{(by Grothendieck-Riemann-Roch Theorem in Algebraic Geometry for } i_{t_t})
$$

$$= (g_t)_* \left( (i_{t_t})_* (\text{ch}(i_{t_t}^* E) \cdot \text{td}(N_{i_t})^{-1} \cdot i_{t_t}^* \text{td}(T_{g_t}) \cdot i_{t_t}^* P(T_{g_t})) \right) \quad \text{(by the projection formula)}
$$

$$= (g_t)_* \left( (i_{t_t})_* (\text{ch}(i_{t_t}^* E) \cdot \text{td}(N_{i_t})^{-1} \cdot \text{td}(T_f) \cdot \text{td}(N_{i_t}) \cdot (P(T_f) + P(N_{i_t}))) \right)
$$

$$= f_* \left( \text{ch}(i^* E) \cdot \text{td}(T_f) \cdot \text{td}(N_{i_t}) \cdot (P(T_f) + P(N_{i_t}))) \right)
$$

On the other hand, $P(N_{i_t}) = P(N_{i_t})$. Indeed, since $X \times \mathbb{P}^1$ does not meet $B_X Z$, and $W \to \mathbb{P}^1$ is flat, the restriction of the normal bundle $N_I$ of the closed immersion $I : X \times \mathbb{P}^1$ to $X \times \{t\}$ is simply $N_t$ for all $t \in \mathbb{P}^1$. Therefore, the cycles corresponding
to \( \text{ch}(N_t) \) and \( \text{ch}(N_\infty) \) are the same for all \( t \). So do the corresponding de Rham cohomology classes. This then implies \( P(N_\infty) = P(N_t) \) in \( H^*(Y, \mathbb{R}) \), and hence completes the proof of Lemma 4.

Thus, we finally have;

\[
\text{Err}(i_* i^* E; g; P) - \text{Err}((i_\infty)_* i_\infty^* E; g_\infty; P) = \left( \text{Err}(E_0(-X); g_0; P) - \text{Err}(E_0; g_0; P) \right) - \left( \text{Err}(E_\infty(-X); g_\infty; P) - \text{Err}(E_\infty; g_\infty; P) \right) \quad \text{(by definition)}
\]

\[
= (g_0)_* (\text{ch}(E_0(-X) - E_0) \cdot \text{td}(T_{g_0}) \cdot R(T_{g_0})) - (g_\infty)_* (\text{ch}(E_\infty(-X) - E_\infty) \cdot \text{td}(T_{g_\infty}) \cdot P(T_{g_\infty})) \quad \text{(by (5.8))}
\]

\[
= 0 \quad \text{(by Lemma 4)},
\]

which completes the proof of Proposition A.7.
6. Proof of the Uniqueness for Relative Bott-Chern Secondary Characteristic Classes

A) Rank $n$-Projective Bundles

B) Closed Immersions of Higher Codimension

C) Proof of Uniqueness Theorems

In this chapter, we will complete the proof of the uniqueness theorems for relative Bott-Chern secondary characteristic classes. For doing so, we need to study $Err$ in two cases: $\mathbb{P}^n$-bundles and general closed immersions of codimension $m$. We will use a trick of Bott to deduce $Err$ for a $\mathbb{P}^n$-bundle and use a trick of Faltings to deduce $Err$ for a closed immersion of higher codimension to the cases of $\mathbb{P}^1$-bundles and closed immersions of codimension 1. Once this is achieved, we then can use Propositions 5.A.6 and 5.A.7 to finish the proof.

A. Rank $n$-Projective Bundles

(A.1) For simplicity, in this section, we will use $p_n$ to denote the projection from any $\mathbb{P}^n$-bundle to its base, and use $i_1$ to denote any codimension one closed immersion.

In order to prove $Err$’s are zero for projections $p_n$ from $\mathbb{P}^n$-bundles, we use an induction on $n$. In a more practical term, we deduce the problems for $Err$ with respect to $p_n$ to these for just $p_1$ and $i_1$.

If $n = 1$, by Proposition 5.A.6, we know that there exists a characteristic class $R$ for rank two bundles such that

$$Err(\cdot; p_1; R) \equiv 0. \quad (6.1)$$

Assume now that for any $m < n$, we have a characteristic class $R_m$ for rank $m$ vector bundles such that

1. $R_m$ is additive and $R_m(E) = R_{m-1}(E)$ for all rank $m - 1$ vector bundles;
2. $Err(\cdot; p_m; R_m) \equiv 0$.

We then want to prove that there exists a characteristic class $R_n$ for rank $n$ vector bundles such that

1. $R_n$ is additive and $R_n(E) = R_{n-1}(E)$ for all rank $n - 1$ vector bundles;
2. $Err(\cdot; p_n; R_n) \equiv 0$. 

41
(A.2) In order to prove this, consider the generator of $K(X)$ for $X = \mathbb{P}_Y(F)$, where $F$ is a rank $n + 1$ vector bundle on $Y$. Note that by using a classifying map to Grassmannian, we may assume that $F$ has a rank 1 sub-line bundle $L$ such that $F/L$ is again a vector bundle. In particular, we have the following closed embedding of codimension one:

$$
\begin{array}{ccc}
\mathbb{P}_Y(F/L) & \xrightarrow{i=i_1} & \mathbb{P}_Y(F) \\
p_{n-1} & \mathcal{L} & p_n \\
Y.
\end{array}
$$

**Lemma 1.** As a $K(Y)$-module, $K(X)$ is generated by $O_X(-1)$ and the direct image of $i_*\left(K(\mathbb{P}_Y(F/L))\right)$.

**Proof.** This is a direct consequence of the facts that

(i) we have a short exact sequence

$$0 \to L \to F \to F/L \to 0$$

and hence $\text{ch}(F) = \text{ch}(F/L) + \text{ch}(L)$;

(ii) $K(\mathbb{P}_n^Y)$ is generated by $O(i)$, $i = 1, \ldots, n$ and $i_*(O(a)) = O(a) - O(a - 1)$ from the structure exact sequence

$$0 \to O(-1) \to O \to i_*O \to 0.$$

(A.3) With this lemma, in order to prove the uniqueness theorems, we only need to show that there exists an $R_n$ satisfies (1) and (2) for $O_X(-1)$ and all the elements in $i_*\left(K(\mathbb{P}_Y(F/L))\right)$.

We first deal with the elements in $i_*\left(K(\mathbb{P}_Y(F/L))\right)$. For this purpose, we use Proposition 5.A.7. In fact, since

$$i_1 : \mathbb{P}_Y(F/L) \hookrightarrow \mathbb{P}_Y(F)$$

is a codimension-one closed imbedding, we have $\text{Err}(\alpha; i_1; R_n) = 0$ for all $\alpha \in K(\mathbb{P}_Y(F/L))$. But by definition,

$$\text{Err}(\alpha; i_1; R_n) = \text{Err}(\alpha; p_{n-1}; R_n) - \text{Err}((i_1)_*\alpha; p_n; R_n).$$

Here $p_{n-1}$ (resp. $p_n$) denotes the natural projection from $\mathbb{P}_Y(F)$ (resp. $\mathbb{P}_Y(F/L)$) to $Y$. This implies that $\text{Err}(\alpha; p_{n-1}; R_n) = \text{Err}((i_1)_*\alpha; p_n; R_n).$
Now, by the induction hypothesis (1) and (2), $m \leq n - 1$,

$$\text{Err}(\alpha; p_{n-1}; R_n = R_{n-1}) \equiv 0,$$

hence we have

$$\text{Err}((i_1)_* \alpha; p_n; R_n) \equiv 0,$$

which exactly means that (1) and (2) are valid for the elements in the direct image of $K(\mathbb{P}_Y(F/L))$.

(A.4) Now let us study $\text{Err}(\mathcal{O}_X(-1), p_n; R_n)$. For this special purpose, we use a trick of Bott following Faltings [F].

Let $\text{Flag}_Y(F)$ be the Flag variety of $F$ on $Y$. That is, the variety which classifies complete filtrations of $F$:

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{n+1} = F,$$

where the successive vector bundle quotients are of rank 1. There is a natural morphism from $\text{Flag}_Y(F)$ to $X$ which is just the composition of the forgetting maps. Hence the morphism from $\text{Flag}_Y(F)$ to $X$ is a composition of $\mathbb{P}^m$-bundles with $m < n$. Therefore, by Proposition 5.A.2, and the induction hypothesis, $\text{Err}$ becomes zero for the morphism $\text{Flag}_Y(F) \to X$ and any additive $R$ such that $R = R_m$ for rank $m < n$ vector bundles.

On the other hand, consider the pull-back of the line bundle $\mathcal{O}_X(-1)$ over $\text{Flag}_Y(F)$. By a simple calculation using the projection formula and the fact that the direct image of the structure sheaf on the total space $\text{Flag}_Y(F)$ is simply the structure sheaf on the base $X$, we see that the push-forward to $X$ of this pull-back line bundle, denoted by $\mathcal{O}'(-1)$, on $\text{Flag}_Y(F)$ coincides with $\mathcal{O}_X(-1)$ itself. Note that $\mathcal{O}'(-1)$ is $(\text{Flag}_Y(F) \to X)$-acyclic and its direct image $\mathcal{O}_X(-1)$ is $(X \to Y)$-acyclic. Thus by Proposition 5.A.2, we have

$$\text{Err}(\mathcal{O}'(-1); \text{Flag}_Y(F) \to X \xrightarrow{p_3} Y; R)$$

$$= (p_n)_* \left( \text{Err}(\mathcal{O}'(-1); \text{Flag}_Y(F) \to X; R) \cdot \text{td}(T_{p_n}) \right) + \text{Err}(\mathcal{O}_X(-1); p_n; R)$$

$$= (p_n)_* \left( 0 \cdot \text{td}(T_{p_n}) \right) + \text{Err}(\mathcal{O}_X(-1); p_n; R) \quad \text{(by induction hypothesis)}$$

$$= \text{Err}(\mathcal{O}_X(-1); p_n; R).$$

Therefore, it is sufficient to show that, for the natural morphism $\text{Flag}_Y(F) \to Y$ that $\text{Err}$ for $\mathcal{O}'(-1)$, the pull-back of $\mathcal{O}_X(-1)$, is zero.
In order to deal with the morphism $\text{Flag}_Y(F) \to Y$, we introduce another decomposition: Let $\text{Flag}'_Y(F)$ be the flag variety which classifies the following partial filtrations of $F$:

$$0 = F_0 \subset F_2 \subset \ldots \subset F_{n+1} = F,$$

where the rank of $F_k$ is $k$. Then the composition of the natural morphism from $\text{Flag}_Y(F)$ to $\text{Flag}'_Y(F)$ with the natural morphism from $\text{Flag}'_Y(F)$ to $Y$ is just $\text{Flag}_Y(F) \to Y$. But, the morphism from $\text{Flag}_Y(F)$ to $\text{Flag}'_Y(F)$ is a $\mathbb{P}^1$-bundle. Therefore, $\text{Err}$ for $O'(−1)$ on $\text{Flag}_Y(F)$ (with respect to the morphism $\text{Flag}_Y(F) \to \text{Flag}'_X F$) vanishes, by our Proposition 5.A.7 for $\mathbb{P}^1$-bundles. On the other hand, the push-forward of $O'(−1)$ via the $\mathbb{P}^1$-bundle $\text{Flag}_Y(F) \to \text{Flag}'_X F$ to $\text{Flag}'_X F$ is the zero bundle, which certainly is $(\pi : \text{Flag}'_X F \to Y)$-acyclic and satisfies the trivial condition that $\text{Err}(0; \text{Flag}_X'(F) \to Y; R) = 0$ by Proposition 5.A.3. Therefore, by Proposition 5.A.2 again, we have

$$\text{Err}(O'(−1); \text{Flag}_X F \to Y; R)$$

$$= \pi_* \left( \text{Err}(O'(−1); \text{Flag}_X F \to \text{Flag}'_X F; R) \cdot \text{td}(T_{\text{Flag}'_X F \to Y}) \right)$$

$$+ \text{Err}(0; \text{Flag}_X F \to Y; R)$$

$$= \pi_* \left( 0 \cdot \text{td}(T_{\text{Flag}'_X F \to Y}) \right) + 0$$

(by the fact that $\text{Flag}_X F \to \text{Flag}'_X F$ is a $\mathbb{P}^1$-bundle)

$$= 0.$$

This, by (6.2), then gives $\text{Err}(O_X(−1); X \to Y; R) = 0$, and hence completes the proof of (2n). In this way, we see that there exists a unique additive characteristic class $R$ such that $\text{Err}(\cdot; p_n; R) = 0$ for all $\mathbb{P}^n$-bundles $p_n$.

**B. Closed Immersions of higher codimension**

(B.1) In this section, we deal with a regular closed immersion of higher codimension. Similarly, we use the deformation to the normal cone technique so as to deduce a general immersion to the following situation:

(i) a section of a projective bundle;

(ii) codimension-one closed immersions.

(B.2) Let

$$X \xrightarrow{i_n} Z$$

$$f \searrow g$$

$$Y$$

44
be a closed immersion $i_n : X \hookrightarrow Z$ of codimension $n$, smooth over $Y$ via $f$ and $g$. Similarly, as for $i_1$ before, for any vector bundle $E$ on $X$ with $R$ as in Section A, set

$$\text{Err}(E; i_n; R) := \text{Err}(E; f; R) - \text{Err}((i_n)_* E; g; R).$$

We here want to show that $\text{Err}(\cdot; p_n; R) = 0$ and $\text{Err}(\cdot; i_1; R) = 0$ implies $\text{Err}(\cdot; i_n; R) = 0$.

(B.3) Let us now suppose that we can reduce the problem for a general closed immersion $i_n$ to the case 1(i) and 1(ii) above, then by definition, in case (i),

$$\text{Err}(E; i_n; R) = \text{Err}(E; X \xrightarrow{f} Y; R) - \text{Err}((i_n)_* E; \mathbb{P}^n_X \to X \to Y; R).$$

Note that now $i_n$ is simply a section of $\mathbb{P}^n_X \to X$, hence with the same proof as for Lemma 4.B.7.1, (see e.g. Remark 5.4,) we have $\text{Err}(\cdot; i_n; R) = 0$ for $i_n$ a section of a projective bundle. Note also that by Proposition 5.A.7, $\text{Err}$ is zero for any closed immersion of codimension one, so we should find a way to deduce a general closed immersion to a section of a $\mathbb{P}^n$-bundle and codimension one closed immersions.

(B.4) To deduce the case of an arbitrary closed immersion to 1(i) and 1(ii) above, we use deformation to the normal cone theory as usual. For this, recall the following basic fact concerning the deformation to the normal cone.

Let $f : X \to Y$ and $g : Z \to Y$ be two smooth, proper morphisms of compact Kähler manifolds and $i : X \to Z$ be a codimension $n$ closed immersion over $Y$, i.e., $i$ is a closed immersion of codimension $n$ such that $f = g \circ i$. Then we have the following standard construction of the deformation to the normal cone.

Denote by $\pi : W := B_{X \times \{\infty\}} Z \times \mathbb{P}^1 \to Z \times \mathbb{P}^1$, where $B_{X \times \{\infty\}} Z \times \mathbb{P}^1$ denotes the blowing-up of $Z \times \mathbb{P}^1$ along $X \times \{\infty\}$. Denote the exceptional divisor by $\mathbb{P}$. It is well-known that the map $q_W : W \to \mathbb{P}^1$, obtained by composing $\pi$ with the projection $q : Z \times \mathbb{P}^1 \to \mathbb{P}^1$, is flat, and that for $z \in \mathbb{P}^1$:

$$q^{-1}(z) = \begin{cases} Z, & \text{for } z \neq \infty, \\ \mathbb{P} \cup B_X Z, & \text{for } z = \infty. \end{cases}$$

Here $B_X Z$ denotes the blowing-up of $Z$ along $X$. By the construction, $\mathbb{P} \cap B_X Z$ is the exceptional divisor of $B_X Z$. In particular, $\mathbb{P}^n = \mathbb{P}_X (N_{i_n} \oplus O_X)$ with $N_{i_n}$ the normal bundle corresponding to $i_n$.

Denote by $I : X \times \mathbb{P}^1 \hookrightarrow W$ the induced codimension $n$ closed embedding. Easily we see that the image of $I$ does not intersect with $B_X Z$, and the image $X \times \{\infty\}$ in $W$ is a section of $\mathbb{P}$.
Lemma. With the same notation as above, the following two morphisms

$$X \overset{i_\infty}{\hookrightarrow} \mathbb{P}_X(N_{i_n} \oplus O_X) \overset{j_\infty}{\hookrightarrow} W \quad \text{and} \quad X \overset{i_0}{\hookrightarrow} W_0 \overset{j_0}{\hookrightarrow} W$$

induce the same morphism for $K$-groups.

Proof. Clearly, as $W$ is flat over the rational curve $\mathbb{P}^1$, and $K$-group is compactible with rational equivalence on cycles by the fact that there is a $Q$-isomorphism between $K$-groups and Chow groups, we see that $X \hookrightarrow W_t$ for all $t \in \mathbb{P}^1$ induce the same map on $K$-groups.

Now let $0 \to E \to I_*F \to 0$ be any resolution of a vector bundle on $X$, (viewed as a vector bundle on $X \times \mathbb{P}^1$.) Then by the flatness of $W \to \mathbb{P}^1$, the restrictions of resolutions $0 \to E \to I_*F \to 0$ to $W_t$ for all $t \in \mathbb{P}^1$ are all exact. On the other hand, by the construction, $X \times \mathbb{P}^1$ is disjoint from the component $B_XZ$ in $W_\infty$, so we see that on $B_XZ$, such a restriction is acyclic. That is to say, it gives zero element in $K(B_XZ)$ and as well as in $K(\mathbb{P} \cap B_XZ)$. Thus by definition, we see that the $K$-push-forward for $X \to W_\infty \to W$ may be simply calculated by the one induced via $X \to \mathbb{P} \to W$. This completes the proof of the lemma.

In this way, by definition, we see that

$$\text{Err}(E; j_\infty \circ i_\infty; R) = \text{Err}(E; j_0 \circ i_0; R).$$

By definition again, we know that

$$\text{Err}(E; j_\infty \circ i_\infty; R) = \text{Err}(E; i_\infty; R) + \text{Err}(i_\infty E; j_\infty; R)$$

and

$$\text{Err}(E; j_0 \circ i_0; R) = \text{Err}(E; i_0; R) + \text{Err}(i_0 E; j_0; R).$$

Thus to complete the proof, it is sufficient to prove that

$$\text{Err}(\cdot; i_\infty; R) \equiv 0,$$

$$\text{Err}(\cdot; j_\infty; R) \equiv 0,$$

and

$$\text{Err}(\cdot; j_0; R) \equiv 0.$$
and 1(ii) above. So we see that $\text{Err}(:i_n; R) \equiv 0$ for any closed immersion $i_n$ of codimension $n$.

**C. The Proof of Uniqueness Theorems**

(C.1) Clearly, the weak uniqueness theorem is a direct consequence of the strong uniqueness theorem, so it is sufficient to prove the later one.

(C.2) For doing so, factor $f : X \to Y$ as a regular closed immersion $i_m : X \to Z$ followed by a projection $p_n : \mathbb{P}^n_Y \to Y$. Then by definition, for any vector bundle $E$ on $X$,

$$
\text{Err}(E; f; R) = \text{Err}(E; i_m; R) + \text{Err}(i_*E; p_n; R)
$$

$$
= 0 + \text{Err}(i_*E; p_n; R) \quad (\text{by } (B.4))
$$

$$
= 0 + 0 \quad (\text{by } (A.3) \text{ and } (A.4))
$$

$$
= 0.
$$

This together with Proposition 5.A.3 implies that

$$
\text{Err}(:: f; R) \equiv 0.
$$

Thus by definition, we have

$$
\text{ch}'_{\text{BC}}(E, \rho; f, \tau_f) = \text{ch}_{\text{BC}}(E, \rho; f, \tau_f) + f_*\left(\text{ch}(E) \cdot \text{td}(T_f) \cdot R(T_f)\right)
$$

for any smooth proper morphism $f$ and any $f$-acyclic vector bundle $E$. This then surely completes the proof of the uniqueness theorems.
7. Existence of Relative Bott-Chern Secondary Characteristic Classes

A) Arithmetic intersection and arithmetic characteristic classes

B) An Effective Construction of Relative Bott-Chern Secondary Characteristic Classes

In this chapter, we prove a weak version of the existence of relative Bott-Chern secondary characteristic classes by effectively constructing some classes of differential forms, which is sufficient and necessary for our application to the arithmetic Grothendieck-Riemann-Roch theorem.

A. Arithmetic intersection and arithmetic characteristic classes

(A.1) In this section, we first recall the theory of arithmetic intersection and arithmetic characteristic classes developed by Arakelov [Ar1,2], Deligne [De2] and Gillet-Soulé [GS1,2]. All results in this section are mainly due to [GS1,2]. The notable difference is that here we only work over \( \mathbb{C} \). So some modifications are needed.

Let \( X \) be a complex compact manifold of dimension \( d \). Denote the space of differential forms of degree \( n \) on \( X \) by \( A^n(X) := \bigoplus_{p+q=n} A^{p,q}(X) \). There are natural boundary morphisms \( \partial : A^{p,q}(X) \to A^{p+1,q}(X) \), \( \bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X) \), and the usual differential \( d : A^n(X) \to A^{n+1}(X) \). We say that a linear function \( T \) on \( A^n(X) \) is a current, if \( T \) is continuous in the sense of Schwartz: for any sequence \( \{\omega_r\} \subset A^n(X) \) with the supports contained in certain fixed compact subset \( K \), \( T(\omega_r) \to 0 \) if all the coefficients of \( \omega_r \) together with their derivatives tend uniformly to zero when \( r \to \infty \). The set of currents forms a topological dual space \( A(X)^* \) of \( A(X) \). Denote by \( D_n(X) := A^n(X)^* \). There is a natural decomposition \( D_n(X) = \bigoplus_{p+q=n} D_{p,q}(X) \), where \( D_{p,q}(X) \) is the dual of \( A^{p,q}(X) \). It is convenient to set \( D^{p,q}(X) := D_{d-p,d-q}(X) \). Then \( \partial, \bar{\partial}, \) and \( d \) induce morphisms \( \partial', \bar{\partial}', d' \) from \( D^{p,q} \) to \( D^{p+1,q}(X), D^{p,q+1}(X), \) and \( D^{p+1,q+1}(X) \) respectively, e.g., \( (\partial' T)(\alpha) := T(\partial \alpha) \).

Examples. (i) There is a natural inclusion

\[
A^{p,q}(X) \leftrightarrow D^{p,q}(X)
\]

\[
\omega \mapsto [\omega],
\]

where \([\omega](\alpha) := \int_X \omega \wedge \alpha \) for any \( \alpha \in A^{d-p,d-q}(X) \). We say that a current \( T \) is smooth if there exists a smooth form \( \omega \) such that \( T = [\omega] \). In particular, if \( p + q = n \), it follows by Stokes’ theorem that \([d\omega](\alpha) = \int_X d\omega \wedge \alpha = \int_X d(\omega \wedge \)
\[ \alpha - \int_X (-1)^n \omega \wedge d\alpha = (-1)^{n+1} \int_X \omega \wedge d\alpha = (-1)^{n+1}d(\omega)(\alpha). \]

Therefore, we let \( \partial, \bar{\partial}, d \) on the currents be \( (-1)^{n+1}\partial' \), \((-1)^{n+1}\partial'\) respectively, and let \( d^c := \frac{1}{4\pi i}(\partial - \bar{\partial}) \). Then \( dd^c = -\frac{1}{2\pi i}\partial\bar{\partial} \) is a real operator, and we have the following commutative diagram:

\[
\begin{array}{ccc}
A^{p,q}(X) & \hookrightarrow & D^{p,q}(X) \\
\partial \downarrow & & \downarrow \partial \\
A^{p+1,q}(X) & \hookrightarrow & D^{p+1,q}(X).
\end{array}
\]

(ii) Let \( i : Y \hookrightarrow X \) be an irreducible subvariety of codimension \( p \). We get a current \( \delta_Y \in D^{p,p}(X) \) by letting \( \delta_Y(\alpha) := \int_{\gamma_{ns}} i^*\alpha \) for any \( \alpha \in A^{d-p,d-p}(X) \). Here \( Y_{ns} \) denotes the non-singular locus of \( Y \). We call this current the Dirac symbol of \( Y \).

Concerning the relations between smooth differential forms and currents, we have the following well-known facts:

(i) With the boundary morphisms \( \partial, \bar{\partial}, d \), the cohomology groups of \( X \) for differential forms are isomorphic to these for currents.

(ii) Let \( \gamma \) be a current on \( X \) such that \( dd^c\gamma \) is smooth. Then there exist currents \( \omega, \alpha, \beta \) such that \( \gamma = \omega + \partial\alpha + \bar{\partial}\beta \), with \( \omega \) smooth.

(iii) As a current, if \( \omega \) is smooth and \( \omega = \partial u + \bar{\partial}v \), then there exist smooth currents \( \alpha, \beta \) such that \( \omega = \partial\alpha + \bar{\partial}\beta \).

(iv) If \( X \) is a Kähler manifold, and \( \eta \in D^{p,q}(X) \), \( p, q \geq 1 \), is \( d \)-closed and is either \( d \), \( \partial \), \( \bar{\partial} \) exact. Then there exists \( \gamma \in D^{p-1,q-1}(X) \) such that \( dd^c\gamma = \eta \). In particular, if \( \eta = 0 \), we may choose \( \gamma = \omega + \partial\alpha + \bar{\partial}\beta \) with \( \omega \) a harmonic form.

(A.2) Let \( Y \) be a codimension \( p \) analytic subvariety of \( X \). Following [GS1], we say that a current \( g \in D^{p-1,p-1}(X) \) is a Green’s current of \( Y \) if \( dd^c g = [\omega] - \delta_Y \) for some \( \omega \in A^{p,p}(X) \). It is well-known that if \( X \) is a Kähler manifold, then Green’s currents for analytic subvarieties of \( X \) exist. And if \( g_1 \) and \( g_2 \) are two Green’s currents for \( Y \), then, by A.2(ii), \( g_1 - g_2 = [\eta] + \partial S_1 + \bar{\partial}S_2 \), where \( \eta \in A^{p-1,p-1}(X) \).

**Example. (The Poincaré-Lelong equation)** Let \( (L, \rho) \) be a hermitian line bundle on \( X \) and \( s \) a non-zero meromorphic section of \( L \). Then \(-\log|s|_\rho^2 \in L^1(X)\), and hence induces a distribution \([-\log|s|_\rho^2] \in D^{0,0}(X)\). Moreover,

\[ dd^c[-\log|s|_\rho^2] = [c_1(L, \rho)] - \delta_{\text{div}(s)}. \]

So, \([-\log|s|_\rho^2]\) is a Green’s current of \( \text{div}(s) \), the divisor of \( s \).

(A.3) From now on, assume that \( X \) is a projective complex manifold. For any irreducible subvariety \( Y \), following [GS1], we say a smooth form \( \alpha \) on \( X - Y \) has
Logarithmic growth along $Y$, if there exists a proper morphism $\pi : \tilde{X} \to X$ such that $E := \pi^{-1}(Y)$ is a divisor with normal crossings, $\pi : \tilde{X} - E \simeq X - Y$ and $\alpha$ is the direct image of a form $\beta$ on $\tilde{X} - E$ by $\pi$ with the following property: Near each $x \in \tilde{X}$, let $z_1 \ldots z_k = 0$ be a local defining equation of $E$. Then, there exists $d$-closed smooth forms $\alpha_i$ and a smooth form $\gamma$ such that $\beta = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \gamma$.

Obviously, such an $\alpha$ is always locally integrable on $X$, and hence defines a current $[\alpha]$, which is the direct image by $\pi$ of the current $[\beta]$.

By [GS1], we know that for every irreducible subvariety $Y \subset X$, there exists a smooth form $g_Y$ on $X - Y$ with logarithm growth along $Y$ such that $[g_Y]$ is a Green’s current for $Y$. (This may be viewed as a generalization of the Poincaré-Lelong equation.) Moreover, if $\alpha$ is a form on $X - Y$ with logarithmic growth along $Y$, then

(i) if $f : X' \to X$ is a morphism of smooth projective varieties such that $f^{-1}(Y)$ does not contain any component of $X'$, then the form $f^*(\alpha)$ is of logarithmic growth along $f^{-1}(Y)$;

(ii) $d[\alpha] = [d\alpha]$.

(A.4) We introduce now the arithmetic Chow groups and their cohomological properties following [GS1].

Let $X$ be a regular projective variety over $\mathbb{C}$. Set

$$\tilde{A}^{p,p}(X) := A^{p,p}(X)/(\text{Im} \partial + \text{Im} \bar{\partial}); \quad \tilde{A}(X) := \oplus_p \tilde{A}^{p,p}(X);$$

$$\tilde{D}^{p,p}(X) := D^{p,p}(X)/(\text{Im} \partial + \text{Im} \bar{\partial}); \quad \tilde{D}(X) := \oplus_p \tilde{D}^{p,p}(X).$$

Denote by $Z^p(X)$ the free group generated by subvarieties of $X$. We say that an element $(Z, g_Z) \in Z^p(X) \oplus \tilde{D}^{p-1,p-1}(X)$ is an arithmetic $p$-cycle if $g_Z$ is a Green’s current of $Z$, i.e. $dd^c g_z = \omega(Z, g_Z) - \delta_Z$ for some $\omega_Z := \omega(Z, g_Z) \in A^{p,p}(X)$. Denote by $Z^p_{\text{Ar}}(X)$ the abelian group generated by arithmetic $p$-cycles.

Next, define arithmetic rational equivalence among arithmetic cycles. Let $i : Y \hookrightarrow X$ be a subvariety of codimension $p - 1$. Then, there is a resolution of singularities of $Y$, $\pi : \tilde{Y} \to Y$ with $\pi$ proper. Thus any rational function $f \in k(Y)^*$ defines a rational function $\tilde{f}$ on $\tilde{Y}$ such that $\log |\tilde{f}|^2$ is $L^1$ on $\tilde{Y}$. Hence $\tilde{f}$ is contained in $D^{0,0}(\tilde{Y})$. Let $i : \tilde{Y} \to X$ be the natural induced morphism, then $i_*[\log |\tilde{f}|^2] \in D^{p-1,p-1}(X)$, and is independent of the choice of $\tilde{Y}$. Hence we may simply denote
it by \(i_*|\log|f|^2|\). Moreover, by the Poincaré-Lelong equation in Example A.2,

\[
\div_{\text{Ar}}(f) := (\div(f), -i_*|\log|f|^2|) \in Z^p_{\text{Ar}}(X),
\]

and is defined to be \textit{arithmetically rationally equivalent to zero}. Let \(R^p_{\text{Ar}}(X)\) be the subgroup of \(Z^p_{\text{Ar}}(X)\) generated by \(\div_{\text{Ar}}(f)\) for \(f \in k(W)^*\), with \(W\) any codimension-\((p - 1)\) subvariety. Define the \textit{p-th arithmetic Chow group}, denoted by \(\text{CH}^p_{\text{Ar}}(X)\), to be the quotient group \(Z^p_{\text{Ar}}(X)/R^p_{\text{Ar}}(X)\). Set \(\text{CH}_{\text{Ar}}(X) := \bigoplus_p \text{CH}^p_{\text{Ar}}(X)\).

(A.5) Let \(X\) be a smooth projective variety over \(\mathbb{C}\), and \(Y \subset X\) a closed irreducible subvariety and \(f : Z \to X\) a proper morphism of irreducible projective varieties such that \(f(Z) \not\subset Y\). Let \(g_Y\) be a Green’s current of \(Y\) with logarithmic growth along \(Y\) associated to \((\tilde{X}, E)\) as in A.3. Denote the associated current by \([g_Y]\).

Then by resolving the singularities of \(Z\), we may construct a commutative diagram

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{j} & \tilde{X} \\
p \downarrow & \searrow q \downarrow \pi & \\
Z & \xrightarrow{f} & X,
\end{array}
\]

such that \(D = j^{-1}(E)\) is a divisor with normal crossings, \(\tilde{Z}\) is projective and smooth, and \(p\) is birational. By A.3(i), \(q^*g_Y\) is of logarithmic growth along \(q^{-1}(Y)\), so it is integrable and \([g_Y] \wedge \delta_Z := q_*[q^*g_Y]\) defines a current in \(X\). Furthermore, if \(g_Z\) is an arbitrary Green’s current of \(Z\), we define the \(*\)-product of \([g_Y]\) and \(g_Z\) by

\[
[g_Y] * g_Z := [g_Y] \wedge \delta_Z + [dd^c g_Y + \delta_Y] \wedge g_Z.
\]

One checks that the following holds;

(i) If \(Y\) and \(Z\) intersect properly, i.e., \(Y \cap F = \cup_i S_i\) with \(\text{codim}_X S_i = \text{codim}_X Y + \text{codim}_X Z\), and as algebraic cycles, \([Y][Z] = \sum_i \mu_i S_i\), then \(dd^c([g_Y] * g_Z) = [\omega_Y \wedge \omega_Z] - \sum_k \mu_k \delta_{S_k}\).

(ii) For any two Green’s currents \(g_Y, g_Y'\) of \(Y\) with logarithmic growth, as an element of \(\tilde{D}(X)\), \([g_Y] * g_Z = [g_Y'] * g_Z\) for any Green’s current \(g_Z\). Hence, we may also define the \(*\)-product among general Green’s currents. In particular, \(g_Y * g_Z = g_Z * g_Y\), and \((g_Y * g_Z) * g_W = g_Y * (g_Z * g_W)\) whenever the products make sense.

(A.6) Easily, we have the following morphisms involving \(\text{CH}_{\text{Ar}}(X)\);

(i) \(\zeta : \text{CH}^p_{\text{Ar}}(X) \to \text{CH}(X), \quad (Z, g_Z) \mapsto Z\).

(ii) \(a : \tilde{A}^{p-1,p-1}(X) \to \text{CH}^p_{\text{Ar}}(X), \quad \alpha \mapsto (0, \alpha)\).
(iii) \[ \omega : \text{CH}^p_{\text{Ar}}(X) \to A^{p,p}(X), \quad (Z, g_Z) \mapsto dd^c g_Z + \delta_Z. \]

One checks that we have the following exact exact sequence

\[ \tilde{A}^{p-1,p-1}(X) \xrightarrow{\omega} \text{CH}^p_{\text{Ar}}(X) \xrightarrow{\zeta} \text{CH}^p(X) \to 0. \]

(A.7) Now we are ready following [GS1] to introduce an arithmetic intersection theory for regular projective varieties over \( \mathbb{C} \).

**Theorem.** ([GS1]) Let \( X \) be a regular projective variety over \( \mathbb{C} \). Then

(i) for each pair of natural numbers \((p, q)\), there is a pairing

\[ \text{CH}^p_{\text{Ar}}(X) \otimes \text{CH}^q_{\text{Ar}}(X) \xrightarrow{\alpha \otimes \beta} \text{CH}^{p+q}_{\text{Ar}}(X)_\mathbb{Q}. \]

The pairing is uniquely determined by the following property: If \( Y \) and \( Z \) are sub-varieties of \( X \) which intersect properly, and \( g_Y \) and \( g_Z \) are Green’s currents for \( Y \) and \( Z \), then

\[ ([Y], g_Y)([Z], g_Z) := ([Y][Z], g_Y \ast g_Z); \]

(ii) the product above makes \( \text{CH}_{\text{Ar}}(X)_\mathbb{Q} := \oplus_p \text{CH}^p_{\text{Ar}}(X)_\mathbb{Q} \) a commutative, associative \( \mathbb{Q} \)-algebra;

(iii) the natural morphism

\[ (\zeta, \omega) : \oplus_p \text{CH}^p_{\text{Ar}}(X)_\mathbb{Q} \to \oplus_p (\text{CH}^p(X) \oplus Z^{p,p}(X))_\mathbb{Q} \]

is a \( \mathbb{Q} \)-algebra homomorphism, where \( Z^{p,p}(X) := \text{the closed forms in } A^{p,p}(X) \).

Moreover, \( a(\phi) \cdot (Z, g) = a(\phi \wedge \omega(Z, g)) \) for all \( \phi \in \tilde{A}(X) \) and \( (Z, g) \in \text{CH}_{\text{Ar}}(X) \), and in particular, \( a(\oplus_p H^{p,p}(X)) \) is a square zero ideal in \( \text{CH}_{\text{Ar}}(X)_\mathbb{Q} \).

(A.8) Let \( f : X \to Y \) be a smooth morphism of regular projective varieties over \( \mathbb{C} \). Then for any subvariety \( Z \) in \( Y \), if \( g_Z \) is a Green’s current of \( Z \), \( f^*g_Z \) is a Green’s current for \( f^{-1}(Z) \). One checks that the arithmetic rational equivalence is compatible with such a pull-back. Hence we have a well-defined pull-back morphism \( f^* : \text{CH}^p_{\text{Ar}}(Y) \to \text{CH}^p_{\text{Ar}}(X) \), which induces a \( \mathbb{Q} \)-algebra morphism \( \text{CH}_{\text{Ar}}(Y)_\mathbb{Q} \to \text{CH}_{\text{Ar}}(X)_\mathbb{Q} \).

We may also construct a push-forward morphism. This is done as follows. First construct a map from \( Z^p_{\text{Ar}}(X) \) to \( Z^{p-r}_{\text{Ar}}(Y) \) as follows, where \( r \) denotes the relative dimension of \( f \): Let \( (Z, g_Z) \in Z^p_{\text{Ar}}(X) \), with \( Z \) irreducible, i.e. \( Z = \{ z \} \) with \( z \) the generic point of \( Z \). Set

\[ f_*(Z) := \begin{cases} [k(z) : k(f(z))] \{ f(z) \}, & \text{if } \dim f(z) = \dim z; \\ 0, & \text{otherwise}. \end{cases} \]
Then, for Green’s currents, we know that for any \( \eta \in \tilde{A}_{\dim Y-p, \dim Y-p}(X) \),

\[
(f_\ast \delta_Z)(\eta) = \delta_Z(f^\ast \eta) = \int \hat{Z} f^\ast \eta = \int \hat{Z} f^\ast(\eta|_{f(Z)})
\]

\[
= \begin{cases} 
\deg(Z/f(Z)) \int_{f(Z)} \eta, & \text{if } Z \to f(Z) \text{ is finite;} \\
0, & \text{otherwise.}
\end{cases}
\]

Hence \( f_\ast \delta_Z = \delta_{f_\ast(Z)} \), and

\[
dd^c(f_\ast g_Z) = [f_\ast \omega_Z] - \delta_{f_\ast(Z)}.
\]

That is, \( f_\ast g_Z \) defines a Green’s current of \( f_\ast(Z) \). Therefore we may set

\[
f_\ast(Z, g_Z) := (f_\ast Z, f_\ast g_Z) \in Z^{k-r}_{Ar}(Y).
\]

Furthermore, it is not difficult to check that this definition is compatible with the arithmetic rational equivalence, and hence we get a push-out morphism \( f_\ast \) for arithmetic Chow groups \( f_\ast : CH^{p}_{Ar}(X) \to CH^{p-r}_{Ar}(Y) \), which induces a \( \mathbb{Q} \)-algebra morphism \( CH_{Ar}(X)_\mathbb{Q} \to CH_{Ar}(Y)_\mathbb{Q} \). Moreover, we have the projection formula

\[
f_\ast(f^\ast(\alpha) \cdot \beta) = \alpha \cdot f_\ast(\beta).
\]

One may also introduce a pull-back morphism for arithmetic Chow rings with respect to closed immersions \( i : X \hookrightarrow Z \) of regular projective varieties over \( \mathbb{C} \). In fact if \( W \) is a subvariety of \( Z \) which properly intersects with \( X \), then it is easily to check that for any Green’s current \( g_W \) of \( W \), \( i^\ast g_Z \) is a Green’s current of \( W \cap Z \). Hence, \( i^\ast(W, g_W) \in CH_{Ar}(Z) \). For general arithmetic cycles, one may use the arithmetic intersection to define their pull-back onto \( Z \). So a suitable Chow type moving lemma is needed. All in all, as what does in [GS1], for \( i \), we finally can define a \( \mathbb{Q} \)-algebra morphism \( i^\ast : CH_{Ar}(Z)_\mathbb{Q} \to CH_{Ar}(X)_\mathbb{Q} \) which coincides \( i^\ast \) defined above for arithmetic cycles whose corresponding algebraic cycles are properly intersect with \( Z \).

Thus, in particular, if \( f : X \to Y \) is a morphism between regular projective varieties over \( \mathbb{C} \), then write \( f \) as a composition of a closed immersions \( i \) and a smooth morphism \( p \). Set \( f^\ast := p^\ast \circ i^\ast : CH_{Ar}(Y) \to CH_{Ar}(X) \). One checks that such \( f^\ast \) does not depend on the decomposition \( f = p \circ i \).

Moreover, if \( f : X \to Y \) is smooth and \( g : Y' \to Y \) is proper morphism among regular projective varieties over \( \mathbb{C} \). Denote by \( g_f : X \times_Y Y' \to X \) and \( f_g : X \times_Y Y' \to Y' \) the projections as before. Then for any element \( (Z, g) \in CH_{Ar}(X) \), we have

\[
g^\ast f_\ast(Z, g) = (f_g)_\ast (g_f)^\ast(Z, g) \in CH_{Ar}(Y').
\]
Let $X$ be a regular projective variety over $\mathbb{C}$. Then, following [GS2], we define the arithmetic $K$-group $K_{\text{Ar}}(X)$ as the abelian group generated by hermitian vector bundles $(E, \rho)$ on $X$ and $\eta \in \tilde{A}(X)$ modulo the subgroup generated by the following relations: For any short exact sequence of vector bundles on $X$,

$$
E.: \quad 0 \to E_1 \to E_2 \to E_3 \to 0,
$$

let $\rho_i$ be hermitian metrics on $E_i$, then

$$
((E_1, \rho_1); \eta_1) + ((E_3, \rho_3); \eta_3) = ((E_2, \rho_2); -\text{ch}_{\text{BC}}(E., \rho_1, \rho_2, \rho_3) + \eta_1 + \eta_3).
$$

Here $\text{ch}_{\text{BC}}(E.; \rho.)$ denotes the classical Bott-Chern secondary characteristic classes associated with $(E., \rho.)$ on $X$ with respect to $\text{ch}$. (See e.g., Remark 1.1.1.)

To motivate the definition of arithmetic characteristic classes, we start with an example. Let $(L, \rho)$ be a hermitian line bundle on $X$. Then for any non-zero section $s$ of $L$, by the Poincaré-Lelong equation in Example A.2, $(\text{div}(s), -[\log |s|^2_\rho])$ is an element of $\text{CH}^1_{\text{Ar}}(X)$. Define $c_{\text{Ar}, 1}(L, \rho)$ as the class of this element in the arithmetic Chow ring. From here, as one may imagine, in general, we may use the splitting principle together with the classical Bott-Chern secondary characteristic classes to construct arithmetic characteristic classes for hermitian vector bundles.

Let $B$ be a subring of real number field $\mathbb{R}$, and let $\phi \in B[[T_1, \ldots, T_n]]$ be a symmetric power series. Then we have the following;

**Theorem.** ([GS2]) Associated to every hermitian vector bundle $(E, \rho)$ of rank $n$ on $X$ is an arithmetic characteristic class $\phi_{\text{Ar}}(E, \rho) \in CH_{\text{Ar}}(X)_\mathbb{Q}$ which satisfies the following properties:

(i) If $f : Y \to X$ is a morphism, $f^*(\phi_{\text{Ar}}(E, \rho)) = \phi_{\text{Ar}}(f^* E, f^* \rho)$.

(ii) If $(E, \rho) = (L_1, \rho_1) \oplus \cdots \oplus (L_n, \rho_n)$ is an orthogonal direct sum of hermitian line bundles, $\phi_{\text{Ar}}(E, \rho) = \phi(c_{\text{Ar}, 1}(L_1, \rho_1), \ldots, c_{\text{Ar}, 1}(L_n, \rho_n))$.

(iii) In $\tilde{A}(X)$, $\omega(\phi_{\text{Ar}}(E, \rho)) = \phi(E, \rho)$, and in $CH(X)_\mathbb{Q}$, $\zeta(\phi_{\text{Ar}}(E, \rho)) = \phi(E)$.

(iv) Let $E.: \quad 0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of vector bundles on $X$ together with hermitian metrics $\rho_i$ on $E_i$ for $i = 1, 2, 3$. Then

$$
\phi_{\text{Ar}}(E_2, \rho_2) = \phi_{\text{Ar}}(E_1 \oplus E_3, \rho_1 \oplus \rho_3) + a(\phi_{\text{BC}}(E., \rho_1, \rho_2, \rho_3)).
$$
In particular, we have a well-defined arithmetic Chern characteristic class

\[ \text{ch}_{\text{Ar}} : K_{\text{Ar}}(X)_\mathbb{Q} \to \text{CH}_{\text{Ar}}(X)_\mathbb{Q}. \]

It is a result of Gillet and Soulé that \( K_{\text{Ar}}(X)_\mathbb{Q} \) admits a so-called \( \lambda \)-ring structure, and \( \text{ch}_{\text{Ar}} \) indeed gives a ring isomorphism. We will not recall the details, instead, we state the following equivalent:

**Theorem'**. ([GS2]) Let \( X \) be a regular projective variety over \( \mathbb{C} \). Then

\[ \text{ch}_{\text{Ar}} : K_{\text{Ar}}(X)_\mathbb{Q} \to \text{CH}_{\text{Ar}}(X)_\mathbb{Q} \]

is a group isomorphism. Moreover,

\[ \text{ch}_{\text{Ar}} \left( (E, \rho) \otimes (E', \rho') \right) = (\text{ch}_{\text{Ar}}(E, \rho)) \cdot (\text{ch}_{\text{Ar}}(E', \rho')). \]

(A.11) We end this section by recalling a result due to Faltings.

Let \( f : X \to Y \) and \( g : Z \to Y \) be two smooth, proper morphisms of compact Kähler manifolds and \( i : X \to Z \) be a codimension one closed immersion over \( Y \), i.e., \( i \) is a closed immersion of codimension one such that \( f = g \circ i \). Denote by

\[ \pi : W := B_{X \times \{\infty\}} Z \times \mathbb{P}^1 \to Z \times \mathbb{P}^1, \]

the natural projection, where \( B_{X \times \{\infty\}} Z \times \mathbb{P}^1 \) denotes the blowing-up of \( Z \times \mathbb{P}^1 \) along \( X \times \{\infty\} \). Denote the exceptional divisor by \( \mathbb{P} \) of \( \pi \). We know that the map \( q_W : W \to \mathbb{P}^1 \), obtained by composing \( \pi \) with the projection \( q : Z \times \mathbb{P}^1 \to \mathbb{P}^1 \), is flat, and that for \( t \in \mathbb{P}^1 \):

\[ q_W^{-1}(t) = \begin{cases} Z \times \{t\}, & \text{for } t \neq \infty, \\ \mathbb{P} \cup B_{X} Z, & \text{for } t = \infty. \end{cases} \]

Here \( B_{X} Z \) denotes the blowing-up of \( Z \) along \( X \). Moreover, by the construction, \( \mathbb{P} \) and \( B_{X} Z \) intersect transversally, and \( \mathbb{P} \cap B_{X} Z \) is the exceptional divisor \( X \) on \( B_{X} Z \).

Denote by \( I : X \times \mathbb{P}^1 \hookrightarrow W \) the induced codimension one closed embedding. Then the image of \( I \) does not intersect with \( B_{X} Z \), and the image \( X \times \{\infty\} \) in \( W \) is a section of \( \mathbb{P} \). Denote the induced fibration \( Z \times \{t\} \to Y \times \{t\} \) by \( g_t \) for \( t \neq \infty \) and set \( g_\infty \) to be the composition of the projection of \( \mathbb{P} \) on \( X \) with \( (X =) X \times \{\infty\} \to Y \times \{\infty\}(= Y) \). Denote by \( f_t : X \times \{t\} \to Y \times \{t\} \) the smooth morphisms induced from \( f \) for all \( t \in \mathbb{P}^1 \).
Let $E$ be a vector bundles on $Z$. Then we have the following exact sequence of coherent sequences on $W$;

$$0 \to (\pi \circ p_Z)^* E(B_X Z - X \times \mathbb{P}^1) \to (\pi \circ p_Z)^* E(B_X Z) \to I_* I^* ((\pi \circ p_Z)^* E(B_X Z)) \to 0. \quad (7.1)$$

Thus, by the flatness of $q_W : W \to \mathbb{P}^1$, we know that the restriction of (7.1) to the fibers $W_t$ of $q_W$ for all $t \in \mathbb{P}^1$ are exact. Thus, in particular, for each $t \neq \infty$ in $\mathbb{P}^1$, from (7.1), we have the induced exact sequences

$$0 \to E_t(-X) \to E_t \to (i_t)_*i_t^* E_t \to 0 \quad \text{over } Z \times \{t\} \quad (7.2)$$

with $E_t$ the pull-back of $E$ under the canonical identity $X \times \{t\} \simeq X$. Similarly, for the fiber at $\infty$, if we set $E(B_X Z)|_\infty := E_\infty, E(B_X Z)|_{B_X Z} =: E'_\infty$ and $E(B_X Z - X \times \mathbb{P}^1)|_{B_X Z} =: E''_\infty$. Then $E(B_X Z - X \times \mathbb{P}^1)|_\infty = E_\infty(-X)$, and (7.1) splits into two exact sequences

$$0 \to E_\infty(-X) \to E_\infty \to (i_\infty)_*i_\infty^* E_\infty \to 0 \quad \text{over } \mathbb{P} \quad (7.3)$$

and

$$0 \to E'_\infty \to E''_\infty \to 0 \to 0 \quad \text{over } B_X Z. \quad (7.4)$$

In particular, we see that on $B_X Z$, $E'_\infty = E''_\infty$.

Choose a hermitian metric $\tau_W$ on $T_W$, the tangent bundle of $W$. Then $\tau_W$ naturally induces hermitian metrics $\tau_t$ on $T_{g_t}$ for all $t \in \mathbb{P}^1$. Similarly, $\tau_W$ induces a hermitian metric $\tau_G$ on $T_G(-\log \infty)$, the logarithmic relative tangent bundle associated to the morphism $G : W \to Y \times \mathbb{P}^1$, which may be naturally embedded in $T_W$ (see e.g. [De1]).

Fix hermitian metrics $\rho$ and $\rho'$ on $E$ and on $E(-X)$ respectively. Use the same notation to denote the pull-back of $(E, \rho)$ onto $W$. Choose the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^1}(\infty)$ and a metric on $\mathcal{O}_W(-X \times \mathbb{P}^1)$ such that in a neighborhood $U$ of $B_X Z$, which is away from $X \times \mathbb{P}^1$, the natural isomorphism $\mathcal{O}_W(-X \times \mathbb{P}^1) \simeq \mathcal{O}_W$ induces an isometry, once we put the trivial metric on $\mathcal{O}_W$. Denote these final induced metrics on $E(B_X Z)$ and $E(B_X Z - X \times \mathbb{P}^1)$ by $D \rho$ and $D \rho'$ respectively.

Denote the induced metrics via restriction to $E_t$ and $E_t(-X)$ (resp. to $E'_\infty$ and $E''_\infty$ on $B_X E$) by $\rho_t$ and $\rho'_t$ respectively for all $t \in \mathbb{P}^1$, (resp. $\rho'_\infty$ and $\rho''_\infty$). Easily, we see that $\rho_0 = \rho$ and $\rho'_0 = \rho'$, and $(E'_\infty, \rho'_\infty)$ is isomorphic to $(E''_\infty, \rho''_\infty)$ by the construction. In this way,

$$(E(-X), \rho') \leftrightarrow (E, \rho) \quad \text{on } Z$$
is deformed to

\[(E_{\infty}(-X), \rho'_{\infty}) \leftrightarrow (E_{\infty}, \rho_{\infty}) \quad \text{on} \quad \mathbb{P}\]

(and

\[(E'_{\infty}, \rho''_{\infty}) \simeq (E''_{\infty}, \rho'''_{\infty}) \quad \text{on} \quad B_XZ).\]

**Proposition.** ([F]) *With the same notation as above, for all* \( t \in \mathbb{P}^1, \)

\[
\left( \text{ch}_{Ar}(E_t, \rho_t) - \text{ch}_{Ar}(E_t(-X), \rho'_t) \right) \cdot \text{td}_{Ar}(T_t, \tau_{g_t}) = i^*_t \left( \left( \text{ch}_{Ar}(E, D\rho) - \text{ch}_{Ar}(E, D\rho) \right) \cdot \text{td}_{Ar}(T_G(-\log \infty), \tau_G) \right).
\]

**Proof.** Surely, by (A.8), \( i^*_t \) is well-defined. So at least all terms make perfect sense as they stand.

The proof is rather formal but standard. The key points are that \( E - E(-X \times \mathbb{P}^1) \)

is supported only on \( X \times \mathbb{P}^1; \) that \( X \times \mathbb{P}^1 \) does not intersect \( B_XZ \) where the hermitian exact sequence splits. As a direct consequence, in the calculation, we may only pay our attention on \( X \times \mathbb{P}^1, \) while pay no attention on \( B_XZ. \) With this in mind, certainly, we may also simply view \( (T_0, \tau_{g_0}) \) (resp. \( (T_{\infty}, \tau_{g_{\infty}}) \)) as the restriction of the \( (T_G(-\log \infty), \tau_G). \) (Recall that the induced metric on \( T_{\infty} \) from \( \tau_G \) is singular, but the singularity is concentrated on \( \mathbb{P} \cap B_XZ, \) which is away from \( X \times \mathbb{P}^1 \) by the construction.) To be more precise, in practice, we need the following preparation.

First, following [F], we give a generalization of arithmetic intersection introduced in A.7. Instead of working over a single regular projective variety over \( \mathbb{C}, \) we are now working over a triple \( (A; B; C). \) Here \( A \) and \( C \) are regular projective varieties over \( \mathbb{C}, \) \( A \subset B \subset C \) with \( B \) an open subset of \( C, \) i.e., \( B \) an open neighborhood of \( A \) in \( C. \) Define a relative arithmetic Chow group \( \text{CH}^{A,B}_{Ar}(C) \) by setting it to be the quotient of the group generated by arithmetic cycles \( (S, g_S) \) of \( C \) with \( S \subset A, \)

\( \text{Supp}(g_S) \subset B, \) modulo the subgroup generated by arithmetic cycles defined by rational functions on cycles in \( A, \) together with the forms \( \partial \alpha + \bar{\partial} \beta, \) where \( \alpha \) and \( \beta \) are currents with support in \( A. \) We point out that even by tensoring with \( \mathbb{Q}, \) the resulting space \( \text{CH}^{A,B}_{Ar}(C)_{\mathbb{Q}} \) only admits a group structure.

Nevertheless, we may introduce a natural \( \text{CH}^{A,B}_{Ar}(C)_{\mathbb{Q}} \)-module structure on \( \text{CH}^{A,B}_{Ar}(C)_{\mathbb{Q}}. \)

Indeed, for an element \( \omega \in \tilde{A}(C), \) it is clear that for any element \( (S, g_S) \in \text{CH}^{A,B}_{Ar}(C)_{\mathbb{Q}}, \) the standard arithmetic intersection on \( C \) gives an arithmetic cycle \( (0, \omega) \cdot (S, g_S) \) on \( C \) which then turns out to be an element in \( \text{CH}^{A,B}_{Ar}(C)_{\mathbb{Q}} \) as well. Similarly, for any hermitian line bundle \( (L, \rho) \) on \( C, \) Theorem A.7, in which the standard arithmetic intersection is introduced, shows that \( c_{1,Ar}(L, \rho) \cdot (S, g_S) := \)
(div(s), − log s∥ρ) · (S, gs) is again an element in CH^(A, B) Ar(C)_Q. Here s is a non-zero section of L. Easily, one checks that all the actions satisfies the module axioms. Thus in particular, by Theorem A.10’ and splitting principle, we have proved the first part of the following

**Lemma.** ([F]) With the same notation as above,

(i) CH^(A, B) Ar(C)_Q admits a natural CH Ar(C)_Q-module structure;

(ii) If (E_1, ρ_1) and (E_2, ρ_2) are two hermitian vector bundles on C such that over B, there is an isometry (E_1, ρ_1)|_B ≃ (E_2, ρ_2)|_B’, then for any arithmetic characteristic class φ Ar, and any element (S, gs) ∈ CH^(A, B) Ar(C), we have in CH^(A, B) Ar(C)_Q and hence in CH Ar(A),

\[ φ_{Ar}(E_1, ρ_1) · (S, gs) = φ_{Ar}(E_2, ρ_2) · (S, gs). \]

**Proof.** Let us start with the case for hermitian line bundles. In this case, we have the following

**Lemma.’** Let (A_1; B_1; C_1) and (A_2; B_2; C_2) be two triples as above such that there is an isomorphism a : B_1 ≃ B_2 which induces an isomorphism A_1 ≃ A_2. Then

(i) there is a natural induced isomorphism

\[ CH^{A_1, B_1}_{Ar}(C_1) ≃ CH^{A_2, B_2}_{Ar}(C_2); \]

(ii) If there are hermitian line bundles (L_1, ρ_1) and (L_2, ρ_2) on Z_1 and Z_2 respectively such that there is an isometry (L_1, ρ_1)|_B_1 ≃ (L_2, ρ_2)|_B_2, then we have the following commutative diagram:

\[
\begin{array}{ccc}
CH^{A_1, B_1}_{Ar}(C_1) & ≃ & CH^{A_2, B_2}_{Ar}(C_2) \\
\downarrow_{c_1, Ar(L_1, ρ_1) · (·)} & & \downarrow_{(·) · c_1, Ar(L_2, ρ_2)} \\
CH^{A_1, B_1}_{Ar}(C_1) & ≃ & CH^{A_2, B_2}_{Ar}(C_2).
\end{array}
\]

**Proof.** This comes directly from the definition.

Let us now go back to the proof of the lemma. For doing so, we use the flag varieties of E_1 and E_2. Denote by Flag_C E_1 (resp. Flag_C E_2) the complete flag variety of E_1 (resp. E_2), by π_i : Flag_C E_i → Z the natural projections, and B_i := π_i^−1(B), and A_i := π_i^−1(A). By our assumption on (E_i, ρ_i) and Lemma’(i), we have an isomorphism

\[ CH^{A_1, B_1}_{Ar}(Flag_C E_1) ≃ CH^{A_2, B_2}_{Ar}(Flag_C E_2). \]
with which the pull-back of arithmetic cycles $\pi_{i}^{\ast}(S, gs)$’s are compactible.

But for $i = 1, 2$, on $\text{Flag}_{C}E_{1}$, $\pi_{i}^{\ast}(E_{i})$ has a complete filtration by line bundles $L_{i, \alpha}$. Thus we can let any polynomial in the $c_{1, \text{Ar}}(L_{i, \alpha}, \rho_{i, \alpha})$ act on the arithmetic cycles in $\text{CH}_{\text{Ar}}^{A_{1}, B_{1}}(\text{Flag}_{C}E_{i})$. By Lemma’ (ii), at this level, the actions for $i = 1$ and for $i = 2$ correspond to each other in a unique way. Easily, one sees that so do the actions for elements in $\tilde{A}$, which in particular includes all classical Bott-Chern secondary characteristic classes $\text{ch}_{BC}$ measuring the change of arithmetic characteristic classes for hermitian vector bundles with respect to the extension of vector bundles and the change of their metrics. Thus, by Theorem A.10, for $\phi_{\text{Ar}}(E_{i}, \rho_{i})$, there exists an operator $Q(c_{1, \text{Ar}}(L_{i, \alpha}, \rho_{i, \alpha})) + \Omega_{BC}$ on $\text{CH}_{\text{Ar}}(\text{Flag}_{C}E_{i})_{Q}$, where $Q$ is a power series in $c_{1, \text{Ar}}(L_{i, \alpha}, \rho_{i, \alpha})$ and $\Omega_{BC}$ the multiplication by a classical Bott-Chern secondary characteristic class, such that for all $(S, gs) \in \text{CH}_{\text{Ar}}^{A_{1}, B_{1}}(C)$,

$$\phi_{\text{Ar}}(E_{i}, \rho_{i}) (S, gs) = \pi_{i}^{\ast}\left(\left((Q(c_{1, \text{Ar}}(L_{i, \alpha}, \rho_{i, \alpha})) + \Omega_{BC}\right) \pi_{i}^{\ast}(S, gs)\right).$$

But, by Lemma’(ii), we know that

$$\pi_{1}^{\ast}\left(\left((Q(c_{1, \text{Ar}}(L_{1, \alpha}, \rho_{1, \alpha})) + \Omega_{BC}\right) \pi_{1}^{\ast}(S, gs)\right) = \pi_{2}^{\ast}\left(\left((Q(c_{1, \text{Ar}}(L_{2, \alpha}, \rho_{2, \alpha})) + \Omega_{BC}\right) \pi_{2}^{\ast}(S, gs)\right),$$

via the isomorphism

$$\text{CH}_{\text{Ar}}^{A_{1}, B_{1}}(\text{Flag}_{C}E_{1}) \simeq \text{CH}_{\text{Ar}}^{A_{2}, B_{2}}(\text{Flag}_{C}E_{2}).$$

Thus in $\text{CH}_{\text{Ar}}^{A_{1}, B_{1}}(C)_{Q}$, and hence in $\text{CH}_{\text{Ar}}(C)_{Q}$, we have

$$\phi_{\text{Ar}}(E_{1}, \rho_{1}) (S, gs) = \phi_{\text{Ar}}(E_{2}, \rho_{2}) (S, gs)$$

for all elements $(S, gs) \in \text{CH}_{\text{Ar}}^{A_{1}, B_{1}}(C)$. This then completes the proof of the lemma.

With this lemma, now the proof of the proposition can be easily deduced. Indeed, first, since $E - E(-X \times \mathbb{P}^{1})$ is supported only on $X \times \mathbb{P}^{1}$ and $X \times \mathbb{P}^{1}$ does not intersect $B_{X}Z$ where the hermitian exact sequence splits, $\text{ch}_{\text{Ar}}(E - E(-X \times \mathbb{P}^{1}), D\rho - D\rho') \in \text{CH}_{\text{Ar}}^{X \times \mathbb{P}^{1}, \text{U} \cap \mathbb{P}^{1}}(W)_{Q}$. Secondly, we easily have the functorial properties for the relative arithmetic intersection defined above. Thus, we may apply the lemma as follows: At the fiber over $0$, we choose $(A; B; C)$ to be $(X; W \setminus \mathbb{U} \cap Z; Z), (S, gs)$ to be $\text{ch}_{\text{Ar}}(E - E(-X \times \mathbb{P}^{1}), D\rho - D\rho')|_{Z}, \phi_{\text{Ar}}$ to be $\text{td}_{\text{Ar}},$ and $(E_{1}, \rho_{1})$ to be $(T_{g}, \tau_{g}), (E_{2}, \rho_{2})$ to be $(T_{G}(- \log \infty), \tau_{G})|_{Z};$ while at the fiber over $\infty$ or better over $\mathbb{P}$, we choose $(A; B; C)$ to be $(X; W \setminus \mathbb{U} \cap \mathbb{P}; \mathbb{P}), (S, gs)$ to be $(E - E(-X \times \mathbb{P}^{1}), D\rho - D\rho')|_{\mathbb{P}},$
φ_{Ar} to be td_{Ar}, and (E_1, ρ_1) to be (T_∞, τ_g), (E_2, ρ_2) to be (T_G(−log ∞), τ_G)|_p.
All this then completes the proof of the proposition.

B. An Effective Construction of Relative Bott-Chern Secondary Characteristic Classes

(B.1) In (A.6), we see that there is an exact sequence

\[ \tilde{A}^{p-1,p-1}(X) \xrightarrow{a} CH^p_{Ar}(X) \xrightarrow{\zeta} CH^p(X) \to 0. \]  

(7.5)

Therefore the a-image of \( \tilde{A}^{p-1,p-1}(X) \) in \( CH^p_{Ar}(X) \) is a well-defined space. Due to the fact that if \( f: Z \to X \) is a morphism of regular projective varieties over \( \mathbb{C} \), then the maps appeared in the above exact sequence are compatible with all the pull-back \( f^* \) for the corresponding spaces, and hence \( a(\tilde{A}^{p-1,p-1}(X)) \), a quotient space of \( A^{p-1,p-1}(X) \), is as canonical as \( A^{p-1,p-1}(X) \) itself. In particular, we may equally use this space, denoted by \( \overline{A}^{p-1,p-1}(X) \), to develop the theory of relative Bott-Chern secondary characteristic classes. That is to say, in the axioms stated for relative Bott-Chern secondary characteristic classes, we may use \( \overline{A} \), instead of the original \( \tilde{A} \), e.g., we consider the classical and relative Bott-Chern secondary characteristic classes as elements in \( \overline{A} \) rather than as in \( \tilde{A} \). One checks easily that the previous uniqueness theorems and their proofs work exactly the same way as before.

(B.2) We now give an effective construction for relative Bott-Chern secondary characteristic classes \( ch_{BC} \) at the level of \( \overline{A} := \bigoplus_p \overline{A}^{p,p} \). That amounts to saying that for all smooth metrized morphisms and all relative acyclic hermitian vector bundles, we should effectively construct some classes in \( \overline{A} \), which satisfy the (corresponding modified) six axioms for relative Bott-Chern secondary characteristic classes.

Let \( (f: X \to Y; E, \rho; T_f, \tau_f) \) be a properly metrized datum. Moreover, from now on, we always assume that all manifolds are regular projective over \( \mathbb{C} \). Then on the direct image, we have a well-defined \( L^2 \)-metric \( L^2(\rho, \tau) \). (Please note that the Kähler conditions in the definition of a properly metrized datum for \( \tau_f \) guarantee that the \( L^2 \)-metric is well-defined via the Hodge decomposition.)

Lemma. With the same notation as above,

\[ f_*(ch_{Ar}(E, \rho) \cdot td_{Ar}(T_f, \tau_f)) - ch_{Ar}(f_*E, L^2(\rho, \tau_f)) \in \overline{A}(Y). \]
Proof. In fact,
\[ \zeta \left( f_\ast \left( \text{ch}_{\text{Ar}}(E, \rho) \cdot \text{td}_{\text{Ar}}(T_f, \tau_f) \right) \right) - \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \]
\[ = f_\ast \left( \zeta \text{ch}_{\text{Ar}}(E, \rho) \cdot \text{td}_{\text{Ar}}(T_f, \tau_f) \right) - \zeta \left( \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \right) \]
\[ = f_\ast \left( \zeta \text{ch}_{\text{Ar}}(E, \rho) \cdot \zeta \left( \text{td}_{\text{Ar}}(T_f, \tau_f) \right) - \zeta \left( \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \right) \right) \]
(by Theorem A.7(iii))
\[ = f_\ast \left( \text{ch}(E) \cdot \text{td}(T_f) \right) - \text{ch}(f_\ast E) \]
(by Theorem A.10(iii))
\[ = 0 \]
(by the Grothendieck-Riemann-Roch Theorem in Algebraic Geomtry).

Hence, by the exact sequence (7.5), we complete the proof of the lemma.

Now set
\[ \text{ch}'_{\text{BC}}(E, \rho; f, T_f) := f_\ast \left( \text{ch}_{\text{Ar}}(E, \rho) \cdot \text{td}_{\text{Ar}}(T_f, \tau_f) \right) - \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \in \overline{A}(Y). \]

We claim that \( \text{ch}'_{\text{BC}}(E, \rho; f, T_f) \) gives an effective construction of the relative Bott-Chern secondary characteristic classes. So it remains to check that \( \text{ch}'_{\text{BC}}(E, \rho; f, T_f) \) satisfies six axioms in Chapters 2 and 3.

(B.3) First, for Axiom 1, the downstairs rule, we have the following;

Lemma. With the same notation as above,
\[ \text{dd}^c \left( \text{ch}'_{\text{BC}}(E, \rho; f, T_f) \right) = f_\ast \left( \text{ch}(E, \rho) \cdot \text{td}(T_f, \tau_f) \right) - \text{ch}(f_\ast E, L^2(\rho, \tau_f)). \]

Proof. This may be obtained by the following calculation. Indeed,
\[ \text{dd}^c \left( \text{ch}'_{\text{BC}}(E, \rho; f, T_f) \right) \]
\[ = \omega \left( \text{ch}'_{\text{BC}}(E, \rho; f, T_f) \right) \quad \text{(as } \text{ch}'_{\text{BC}}(E, \rho; f, T_f) \in \overline{A}(Y) \text{)} \]
\[ = \omega \left( f_\ast \left( \text{ch}_{\text{Ar}}(E, \rho) \cdot \text{td}_{\text{Ar}}(T_f, \tau_f) \right) - \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \right) \]
(by definition)
\[ = f_\ast \left( \omega \left( \text{ch}_{\text{Ar}}(E, \rho) \cdot \text{td}_{\text{Ar}}(T_f, \tau_f) \right) \right) - \omega \left( \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \right) \]
(by Theorem A.7(iii))
\[ = f_\ast \left( \text{ch}(E, \rho) \cdot \omega \left( \text{td}_{\text{Ar}}(T_f, \tau_f) \right) \right) - \omega \left( \text{ch}_{\text{Ar}} \left( f_\ast E, L^2(\rho, \tau_f) \right) \right) \]
(by Theorem A.10(iii)).

This completes the proof.

(B.4) For Axiom 2, we should prove the following
Lemma. With the same notation as above, let \((F, \rho')\) be a hermitian vector bundle on \(Y\), then

\[
\text{ch}'_{BC}(E \otimes f^*F, \rho \otimes f^*\rho'; f, \tau_f) = \text{ch}'_{BC}(E, \rho; f, \tau_f) \land \text{ch}(F, \rho').
\]

Proof. In fact,

\[
\begin{align*}
\text{ch}'_{BC}(E \otimes f^*F, \rho \otimes f^*\rho'; f, \tau_f) &= f_* \left( \left( \text{ch}_A(E \otimes f^*F, \rho \otimes f^*\rho') \cdot \text{td}_A(T_f, \tau_f) \right) - \text{ch}_A \left( f_* (E \otimes f^*F), L^2(\rho \otimes f^*\rho', \tau_f) \right) \right) \\
&= f_* \left( \left( \text{ch}_A(E, \rho) \cdot f^*(\text{ch}_A(F, \rho')) \cdot \text{td}_A(T_f, \tau_f) \right) - \text{ch}_A \left( f_* (E, L^2(\rho, \tau_f)) \otimes (F, \rho') \right) \right) \\
&= f_* \left( \left( \text{ch}_A(E, \rho) \cdot \text{td}_A(T_f, \tau_f) \right) \cdot \text{ch}_A(F', \rho') - \text{ch}_A \left( f_* (E, L^2(\rho, \tau_f)) \right) \cdot \text{ch}_A(F, \rho') \right) \\
&= f_* \left( \left( \text{ch}_A(E, \rho) \cdot \text{td}_A(T_f, \tau_f) \right) - \text{ch}_A \left( f_* (E, L^2(\rho, \tau_f)) \right) \right) \cdot \text{ch}_A(F, \rho') \\
&= \text{ch}'_{BC}(E, \rho; f, \tau_f) \land \omega \left( \text{ch}_A(F, \rho') \right) \quad \text{(by Theorem A.7(iii))} \\
&= \text{ch}'_{BC}(E, \rho; f, \tau_f) \land \text{ch}(F, \rho') \quad \text{(by Theorem A.10(iii))}.
\end{align*}
\]

This completes the proof.

(B.5) For Axiom 3, we should prove the following

Lemma. With the same notation as above, if \(g : Y' \to Y\) is a morphism between regular projective varieties over \(\mathbb{C}\), then

\[
g_f^* \left( \text{ch}'_{BC}(E, \rho; f, \tau_f) \right) = \text{ch}'_{BC}(g_f^*(E, \rho; f, \tau_f)).
\]

Here as in 2.C.2, we have the base change diagram

\[
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{g_f^*} & X \\
\downarrow f_g & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

and define the pull-back of \((E, \rho; f, \tau_f)\) via \(g_f\) by

\[
g_f^*(E, \rho; f, \tau_f) = (g_f^*(E, \rho); f_g, g_f^*\tau_f).
\]

Proof. In fact, By Theorem A.10(i), the functorial property for arithmetic characteristic classes and A.8,

\[
\begin{align*}
g_f^* f_* \left( \text{ch}_A(E, \rho) \cdot \text{td}_A(T_f, \tau_f) \right) &= (f_g)_* (g_f)^* \left( \text{ch}_A(E, \rho) \cdot \text{td}_A(T_f, \tau_f) \right) \quad \text{(by A.8)} \\
&= (f_g)_* \left( \text{ch}_A((g_f)^*(E, \rho)) \cdot \text{td}_A((g_f)^*(T_f, \tau_f)) \right) \quad \text{(by Theorem A.10(i))} \\
&= (f_g)_* \left( \text{ch}_A((g_f)^*(E, \rho)) \cdot \text{td}_A(T_{f_g}, \tau_{f_g}) \right) \quad \text{(by definition)}.
\end{align*}
\]
So by definition and the functorial property for arithmetic characteristic classes, it suffices to show that
\[ g^* \left( f_* E, L^2(\rho, \tau_f) \right) \simeq \left( (f_g)_* g_f^* E, L^2(g_f^* \rho, \tau_{f_g}) \right). \]

But this is a direct consequence of our \( f \)-acyclic condition on \( E \). So Axiom 3 is checked.

(B.6) Now we check Axiom 4. This amounts to proving the following

**Lemma.** With the same notation as above, let \( 0 \to E_1 \to E_2 \to E_3 \to 0 \) be an exact sequence of \( f \)-acyclic vector bundles on \( X \). Then for hermitian metrics \( \rho_i \) on \( E_i, i = 1, 2, 3 \),
\[
ch_{BC}(E_2, \rho_2; f, \tau_f) - ch_{BC}(E_1, \rho_1; f, \tau_f) - ch_{BC}(E_3, \rho_3; f, \tau_f)
= f_* \left( ch_{BC}(E, \rho) \cdot td(T_f, \tau_f) \right) - ch_{BC}(f_* E, L^2(\rho, \tau_f)).
\]

**Proof.** By definition,
\[
ch'_{BC}(E_2, \rho_2; f, \tau_f) - ch'_{BC}(E_1, \rho_1; f, \tau_f) - ch'_{BC}(E_3, \rho_3; f, \tau_f)
= f_* \left( ch_{Ar}(E_2, \rho_2) \cdot td_{Ar}(T_f, \tau_f) \right) - ch_{Ar}(f_* E_2, L^2(\rho_2, \tau_f))
\]
\[
= f_* \left( f_* (ch_{Ar}(E_1, \rho_2) \cdot td_{Ar}(T_f, \tau_f)) - ch_{Ar}(f_* E_1, L^2(\rho_1, \tau_f)) \right)
\]
\[
= f_* \left( f_* (ch_{Ar}(E_2, \rho_2) \cdot td_{Ar}(T_f, \tau_f)) - ch_{Ar}(f_* E_1, L^2(\rho_1, \tau_f)) \right)
\]
\[
= f_* \left( ch_{Ar}(E_2, \rho_2) \cdot td_{Ar}(T_f, \tau_f) \right) - ch_{Ar}(f_* E_1, L^2(\rho_1, \tau_f))
\]
\[
= f_* \left( ch_{Ar}(E_2, \rho_2) - ch_{Ar}(E_1, \rho_1) - ch_{Ar}(E_3, \rho_3) \right) \cdot td_{Ar}(T_f, \tau_f)
\]
\[
= f_* \left( ch_{BC}(E, \rho) \cdot td_{Ar}(T_f, \tau_f) \right) - ch_{BC}(f_* E, L^2(\rho, \tau_f))
\]
(by Theorem 10(iv))
\[
= f_* \left( ch_{BC}(E, \rho) \cdot td(T_f, \tau_f) \right) - ch_{BC}(f_* E, L^2(\rho, \tau_f))
\]
(by Theorem A.7 and Theorem 10(iii))
\[
= f_* \left( ch_{BC}(E, \rho) \cdot td(T_f, \tau_f) \right) - ch_{BC}(f_* E, L^2(\rho, \tau_f))
\]
(by Theorem A.7(iii) and (A.8)).

This completes the proof of the lemma.

(B.7) Similarly, for Axiom 5, we need to show the following

**Lemma.** With the same notation as in section 2.E, then
\[
ch'_{BC}(E, \rho; g \circ f, \tau_{g\circ f}) - g_* \left( ch'_{BC}(E, \rho; f, \tau_f) \cdot td(T_g, \tau_g) \right) - ch'_{BC}(f_* E, L^2(\rho, \tau_f); g, \tau_g)
= (g \circ f)_* \left( ch(E, \rho) \cdot td_{BC}(T, \tau) \right) - ch_{BC}(g_* f_* E, L^2(\rho, \tau_{g\circ f}), L^2(\rho, \tau_f), \tau_g).
Proof. By definition,

\[ ch'_{BC}(E, \rho; g \circ f, \tau_{gof}) - g_*\left(ch'_{BC}(E, \rho; f, \tau_f) \cdot td(T_g, \tau_g)\right) - ch'_{BC}(f_*E, L^2(\rho, \tau_f); g, \tau_g) \]

\[ = (g \circ f)_*\left(ch_{Ar}(E, \rho) \cdot td_{Ar}(T_{gof}, \tau_{gof})\right) - ch_{Ar}\left((g \circ f)_*E, L^2(\rho, \tau_{gof})\right) \]

\[ - g_*\left(f_*\left(ch_{Ar}(E, \rho) \cdot td_{Ar}(T_f, \tau_f)\right) - ch_{Ar}\left(f_*E, L^2(\rho, \tau_f)\right)\right) \cdot td(T_g, \tau_g) \]

\[ - \left(g_*\left(ch_{Ar}(f_*E, L^2(\rho, \tau_f)) \cdot td_{Ar}(T_g, \tau_g)\right) - ch_{Ar}\left(g_*(f_*E), L^2(L^2(\rho, \tau_f), \tau_g)\right)\right) \]

(\text{by Theorem A.7(iii) and Theorem A.10(iii)}

\[ \text{as } f_*\left(ch_{Ar}(E, \rho) \cdot td_{Ar}(T_f, \tau_f)\right) - ch_{Ar}\left(f_*E, L^2(\rho, \tau_f)\right) \in \bar{A}(Y) \]

\[ = (g \circ f)_*\left(ch_{Ar}(E, \rho) \cdot \left(td_{Ar}(T_{gof}, \tau_{gof}) - td_{Ar}(T_f, \tau_f) \cdot f^*\left(td_{Ar}(T_g, \tau_g)\right)\right)\right) \]

\[ - \left(ch_{Ar}\left((g \circ f)_*E, L^2(\rho, \tau_{gof})\right) - ch_{Ar}\left(g_*(f_*E), L^2(L^2(\rho, \tau_f), \tau_g)\right)\right) \]

(by projection formula)

\[ = (g \circ f)_*\left(ch(E, \rho) \cdot td_{BC}(T, \tau)\right) - ch_{BC}\left((g \circ f)_*E; L^2(\rho, \tau_{gof}), L^2(L^2(\rho, \tau_f), \tau_g)\right) \]

(by Theorem A.10(iii)).

This completes the proof of the lemma.

(B.8) Finally, we are left with checking that \(ch'_{BC}\) satisfies Axiom 6 for relative Bott-Chern secondary characteristic classes.

Lemma. With the same notation as in B.2 and as in 3.C.3, we have

\[ \lim_{t \to \infty} \left((ch'_{BC}(E_t, \rho_t; g_t, \tau_{g_t}) - ch'_{BC}(E_t(-X), \rho'_t; g_t, \tau_{g_t}))\right) \]

\[ + \left(ch_{BC}((g_t)_*(E_t)); L^2(\rho_t, \tau_t), \gamma_t) - ch_{BC}((g_t)_*(E_t(-X)); L^2(\rho'_t, \tau_t), \gamma'_t)\right) \]

\[ = (ch'_{BC}(E_\infty, \rho_\infty; g_\infty, \tau_{g_\infty}) - ch'_{BC}(E_\infty(-X), \rho'_\infty; g_\infty, \tau_{g_\infty})) + \]

\[ (ch_{BC}((g_\infty)_*(E_\infty)); L^2(\rho_\infty, \tau_\infty), \gamma_\infty) - ch_{BC}((g_\infty)_*(E_\infty(-X)); L^2(\rho'_\infty, \tau_\infty), \gamma'_\infty)) \].

Proof. By definition, Theorem A.10(v), and the fact that the \(L^2\)-metrics are now replaced by a continuous family of metrics, it is sufficient to show that

\[ \lim_{t \to \infty} \left(ch_{Ar}(E_t, \rho_t) - ch_{Ar}(E_t(-X), \rho'_t)\right) \cdot td_{Ar}(T_t, \tau_{g_t}) \]

\[ = \left(ch_{Ar}(E_\infty, \rho_\infty) - ch_{Ar}(E_\infty(-X), \rho'_\infty)\right) \cdot td_{Ar}(T_\infty, \tau_{g_\infty}). \]
But this is a direct consequence of Proposition A.11. In fact, by Proposition A.11, for all \( t \in \mathbb{P}^1 \),
\[
\left( \text{ch}_A(E_t, \rho_t) - \text{ch}_A(E_t(-X), \rho'_t) \right) \cdot \text{td}_A(T_t, \tau_{g_t})
= i_t^* \left( \left( \text{ch}_A(E, D\rho) - \text{ch}_A(E, D\rho) \right) \cdot \text{td}_A(T_G(-\log \infty), \tau_G) \right).
\]

So, by the fact that there exists a well-defined pull-back morphisms \( \text{CH}_A(\mathbb{P}^1) \to \text{CH}_A(W) \) and that on \( \mathbb{P}^1 \), the arithmetic cycle \( \text{div}_A(z) := (0 - \infty, -\log |z|^2) \) is rational equivalent to zero, we have
\[
\left( \left( \text{ch}_A(E(B_XZ), D\rho) - \text{ch}_A(E(B_XZ - X \times \mathbb{P}^1), D\rho_0) \right) \cdot \text{td}_A(T_G(-\log \infty), \tau_G) \right)
\cdot (W_0 - W_\infty, -\log |z|^2) = 0.
\]

Here \( \log |z| \) denotes the pull-back of \( \log |z| \) on \( W \). Note that \( W_\infty \) has two components which intersects properly, the restriction of \( 0 \to E(B_XZ - X \times \mathbb{P}^1) \to E(B_XZ) \) together with metrics \( D\rho' \) and \( D\rho \) results an isometry \( (E'_\infty, \rho''_\infty) \simeq (E''_\infty, \rho'''_\infty), X \times \mathbb{P}^1 \) is away from \( B_XZ \), and \( E_\infty - E_\infty(-X) \) is supported on \( X \subset \mathbb{P} \),
\[
\left( \left( \text{ch}_A(E(B_XZ), D\rho) - \text{ch}_A(E(B_XZ - X \times \mathbb{P}^1), D\rho') \right) \right|_{W_\infty}
= \left( \left( \text{ch}_A(E(B_XZ), D\rho) - \text{ch}_A(E(B_XZ - X \times \mathbb{P}^1), D\rho') \right) \right|_{\mathbb{P}}.
\]

Hence, in particular,
\[
\left( \text{ch}_A(E_\infty, \rho_\infty) - \text{ch}_A(E_\infty(-X), \rho'_\infty) \right) \cdot \text{td}_A(T_\infty, \tau_{g_\infty})
- \left( \text{ch}_A(E_0, \rho_0) - \text{ch}_A(E_0(-X), \rho'_0) \right) \cdot \text{td}_A(T_0, \tau_{g_0})
= \int_{\mathbb{P}^1} \left( \left( \text{ch}_A(E(B_XZ), D\rho) - \text{ch}_A(E(B_XZ - X \times \mathbb{P}^1), D\rho') \right) \right.
\cdot \text{td}_A(T_G(-\log \infty), \tau_G) \left. \right) \cdot [\log |z|^2].
\]

Similarly,
\[
\left( \text{ch}_A(E_t, \rho_t) - \text{ch}_A(E_t(-X), \rho'_t) \right) \cdot \text{td}_A(T_t, \tau_{g_t})
- \left( \text{ch}_A(E_0, \rho_0) - \text{ch}_A(E_0(-X), \rho'_0) \right) \cdot \text{td}_A(T_0, \tau_{g_0})
= \int_{\mathbb{P}^1} \left( \left( \text{ch}_A(E(B_XZ), D\rho) - \text{ch}_A(E(B_XZ - X \times \mathbb{P}^1), D\rho') \right) \right.
\cdot \text{td}_A(T_G(-\log \infty), \tau_G) \left. \right) \cdot [\log \left| \frac{zt}{z - t} \right|^2].
\]

Easily, one sees that \( \lim_{t \to \infty} \left| \frac{zt}{z - t} \right|^2 = |z|^2 \). This completes the proof of the lemma.
Recall that \((f : X \to Y; E, \rho; T_f, \tau_f)\) is called a properly metrized datum if \(f : X \to Y\) is a smooth morphism of compact Kähler manifolds, \((E, \rho)\) is an \(f\)-acyclic hermitian vector bundle, and \(\tau_f\) is a hermitian metric on the relative tangent bundle \(T_f\) of \(f\) such that the induced metrics on all fibers of \(f\) are Kähler. Then all in all, with the space \(\tilde{A}\) defined in B.1, what we have already proved is the following

**Theorem.** (The Weak Existence and Strong Uniqueness for Relative Bott-Chern Secondary Characteristic Classes) There exists a construction \(ch_{\text{BC}}\) satisfies the six axioms for relative Bott-Chern secondary characteristic classes with values in \(\tilde{A}\). Moreover, if there are two constructions \(ch_{\text{BC}}\) and \(ch'_{\text{BC}}\) which satisfy these six axioms, then, there exists an additive characteristic classes \(R\) such that, for all properly metrized data \((f : X \to Y; E, \rho; T_f, \tau_f)\), in \(\tilde{A}(Y)\),

\[
ch'_{\text{BC}}(E, \rho; f, \tau_f) = ch_{\text{BC}}(E, \rho; f, \tau_f) + a(f_* (ch(E) \cdot td(T_f) \cdot R(T_f))).
\]

We end this paper by the following remark. There is in fact another effective construction for relative Bott-Chern secondary characteristic classes by using heat kernels and Mellin transform. Even though such an alternative construction involves only analysis, yet it offers us a strong existence of the relative Bott-Chern secondary characteristic classes, i.e., the relative Bott-Chern classes at the level of \(\tilde{A}\). Sure, once this is done, then the arithmetic Grothendieck-Riemann-Roch theorem may be viewed as a direct consequence of the uniqueness for relative Bott-Chern classes. We will study it carefully in another occasion.

**Acknowledgement.** This work was started when I was at MPI für Mathematik at Bonn and has been completed while I am at Mathematics Department of Osaka University. I would like thank both institutes for their hospitality and support. I also would like to thank Fujiki, Lang, and Ueno for their interests and encouragement. Obviously, the works done by Bott-Chern, Faltings and Gillet-Soulé provide the essential ingredients without which it is impossible to develop a theory of relative Bott-Chern classes.
REFERENCES

[Ar1] ARAKELOV, S.J.: Intersection theory of divisors on an arithmetic surface, Math. USSR Izvestija 8, 1974, pp. 1167-1180

[Ar2] ARAKELOV, S.J.: Theory of intersections on an arithmetic surface, Proc. Internl. Cong. of Mathematicians, Vancouver, 1975, Vol. 1, pp. 405-408

[BC] BOTT, R., CHERN, S.S.: Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. Acta Math. 114 (1965), pp. 71-112

[BT] BOTT, R., TU, L. W.: Differential forms in algebraic topology, Graduate Texts in Mathematics, 82, Springer-Verlag, New York-Berlin, 1982. xiv+331 pp.

[De1] DELIGNE, P.: Equations différentielles a points singuliers réguliers, Springer Lecture Notes in Mathematics, no. 163, 1970

[De2] DELIGNE, P.: Le déterminant de la cohomologie, in Current trends in Arithmetic Algebraic Geometry, Contemporary Math., 67 (1987), pp. 93-178

[Fa] FALTINGS, G.: Lectures on the arithmetic Riemann-Roch theorem, noted by S. W. Zhang, Ann. of Math. Study 127 (1992)

[GS1] GILLET, H., SOULÉ, CH.: Arithmetic Intersection Theory, Publ. Math. IHES 72 (1990), pp. 93-174

[GS2] GILLET, H., SOULÉ, CH.: Characteristic classes for algebraic vector bundles with hermitian metric, Annals of Math., 131, pp. 163-203, pp. 205-238 (1990)

[We1] WENG L.: Arithmetic Riemann-Roch Theorem for smooth morphism: An approach with relative Bott-Chern secondary characteristic forms, MPI preprint, 1991

[We2] WENG, L. Relative Bott-Chern secondary characteristic objects and an Arithmetic Riemann-Roch Theorem, monograph, 1992 (= MPI preprints, 1994)