Unitary scheme model calculations of the ground and excited state characteristics of $^3$H and $^4$He

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Abstract
The ground and excited states of the $^3$H and $^4$He nuclei are studied in the framework of group-theoretical methods. Basis functions of the unitary scheme model corresponding to even numbers of quanta of excitation in the range $0 \leq N \leq 20$ are constructed for the even-parity states of these two nuclei, and bases with $1 \leq N \leq 19$ are constructed for the odd-parity states of the $^4$He nucleus. The ground-state and first-excited-state energies and wave functions, the ground S-, P- and D-state probabilities, the root-mean-square radius and the magnetic dipole moment of triton are calculated. Furthermore, for the $^4$He nucleus, the spectrum and the wave functions, the ground S-, P- and D-state probabilities, the root-mean-square radius and the total integral cross section of the dipole electric transition accompanying the photoabsorption of $\gamma$ quanta by this nucleus are calculated. The GPT and AV8' two-body interactions and the UIX three-nucleon interaction are used in these investigations. Moreover, the convergence of the calculations is examined by incrementally extrapolating the nuclear characteristics calculated for $N \leq 20$ to $N = 30$ for both nuclei.

1. Introduction

Group-theoretical methods provide us with accurate techniques for calculating the different matrix elements of the operators associated with nuclear characteristics, as can be seen from the success of the nuclear shell model [1] and the unitary scheme model (USM) [2–4]. Group-theoretical methods of expanding the total antisymmetric nuclear wave function in terms of a complete set of orthonormal functions, i.e., basis functions, have been extensively used, especially for nuclei with $3 \leq A \leq 6$ [2]. The USM, also known as the translational invariant shell model (TISM) [2–4], has yielded good results for the structure of light nuclei with $A \leq 7$ through the use of nucleon-nucleon interactions [5–10]. In the USM, a nucleus is regarded as a system of non-interacting quasi-particles; this treatment enables us to apply algebraic methods to study the general features of the matrix elements of operators that correspond to physical quantities. The bases of the USM for a nucleus with mass number $A$, total angular momentum $J$ and isotopic spin $T$ are expanded in the form of products of two types of functions, one corresponding to the set of $A'$ nucleons and the other corresponding to the set of $A''$ nucleons, where $A' + A'' = A$, by means of many-particle fractional parentage coefficients (FPCs) [2, 5, 6, 10]. Using two- and three-body operators, it is then possible to calculate the corresponding matrix elements with respect to the different nuclear states by means of two- and three-particle FPCs. The expectation values of on-body operators can also be used by means of one-particle FPCs.

On the other hand, recent experimental results for three-body systems have unambiguously shown that calculations based only on nucleon–nucleon forces fail to accurately describe many experimental observables; instead, one needs to include effects that extend beyond the realm of two-body potentials. In addition, microscopic calculations for light nuclei and nuclear matter [11] have indicated that it is difficult to explain the observed binding energies and densities if we assume a non-relativistic nuclear Hamiltonian containing
only two-nucleon interactions, consistent with the nucleon-nucleon scattering data at low energies \( (E_{lab} < \sim 400 \text{ MeV}) \).

Marsden et al \[12\] studied the \(^3\text{H}\) and \(^3\text{He}\) nuclei with a realistic nucleon-nucleon potential and the Tucson-Melbourne (TM) three-body interaction using a translational invariant harmonic oscillator (HO) basis. They replaced the nucleon-nucleon potential with the no-core--shell model two-body effective interaction, and the three-nucleon interaction was added without any renormalization. Also, the effects of three-nucleon interactions have been recently studied in \[13\], where the effects of different three-nucleon interactions in \(p-^3\text{He}\) elastic scattering at low energies were calculated for four-nucleon scattering observables by using the Kohn variational principle and the technique of hyperspherical harmonics. Meanwhile, the effects of two- and three-body hyperon-nucleon interactions in \(\Lambda\) hypernuclei have been studied by assessing the relative importance of the two- and three-body hyperon-nucleon forces and by studying the effect of the hyperon-nucleon-nucleon interaction in closed-shell \(\Lambda\) hypernuclei for \(A = 5\) to \(91\) \[14, 15\]. Moreover, Cipollone et al \[16\] extended the formalism of self-consistent Green’s function theory to include three-body interactions and applied it to isotopic chains around oxygen for the first time. Furthermore, Wirting et al \[17\] used the realistic Argonne \(v_{18}\) two-nucleon potential and Urbana three-nucleon potentials to generate accurate variational Monte Carlo (VMC) wave functions for the \(A \leq 12\) nuclei.

The \(ab\ initio\) no-core--shell model (NCSM) is a well-established theoretical framework with the aim of providing an exact description of nuclear structure starting from highly precise descriptions of the interactions between nucleons. Barrett, Navrátil, and Vary \[18\] discussed, in detail, the extension of the \(ab\ initio\) NCSM to nuclear reactions and outlined some of the promising future directions for research emerging from the foundation provided by the NCSM, including a microscopic non-perturbative framework for a theory with a core. In the NCSM, Forssén, Navrátil and Quaglioni \[19\] considered a system of point-like non-relativistic nucleons that interact through realistic inter-nucleon interactions. They considered two-nucleon interactions that reproduce nucleon-nucleon phase shifts with high precision, typically up to 350 MeV in laboratory energy. They also included three-nucleon interactions with terms related to, e.g., two-pion exchanges with an intermediate delta excitation. Both semi-phenomenological potentials (based on meson-exchange models) and modern chiral interactions were considered.

Calculations within a three-body translational invariant HO basis using realistic two- and three-nucleon forces have previously been performed for the three-nucleon system (see, e.g., \[12\]) and the four-nucleon system (see, e.g., \[20, 21\]). These calculations were based on the translationally invariant form of the NCSM, which is equivalent to the TISM except that the antisymmetrization of the wave function is achieved not by means of group theory but rather by diagonalizing the antisymmetrization operator and retaining the antisymmetric eigenstates as bases.

Three- and four-nucleon systems have also been prominently studied by means of numerically exact few-body approaches (such as the Faddeev, Faddeev-Yakubovsky and hyperspherical harmonics approaches starting from realistic two- and three-nucleon forces; see, for example, \[22-24\]). A review can be found in \[25\].

Furthermore, the effects of the two-body short range correlation and the occupation probability of higher states on the elastic electron scattering longitudinal form factors \(F(q)\) are investigated for the nucleus \(^4\text{He}\) \[26\]. Also, the inelastic longitudinal form factors for \(^4\text{He}\) were studied \[27\] where the deformation in nuclear collective modes and the shell-model transition density are taken into consideration. The core polarization transition density was also evaluated by adopting the shape of Tassie model together with the form of the ground state two-body charge density distributions and the effect of two body short range correlation function.

In previous papers, Doma et al \[10\] applied the USM with a number of quanta of excitation satisfying \(0 \leq N \leq 8\) to investigate the ground-state wave function, the binding energy, the first-excited-state energy and the root-mean-square radius of \(^3\text{H}\) by using the Gogny, Pires and de Tourreil (GPT) potential \[28\], the Hu and Massey (HM) potential \[29\] and an effective potential proposed by Vanagas \[2\]. Furthermore, for the \(^4\text{He}\) nucleus, Doma \[30\] investigated the binding energy, the structure of the wave functions, the excitation spectrum, the root-mean-square radius and the total integral cross section of the photoabsorption of \(\gamma\) quanta by this nucleus by using the USM with basis functions corresponding to \(N \leq 8\) and the GPT potential. Moreover, Doma et al \[6\] investigated the binding energy, the structure of the wave functions, the excitation spectrum, the root-mean-square radius, the magnetic dipole moment, the \(ft\) value for the allowed \(\beta^-\) transition, and the partial and integral cross sections of the dipole electric transition accompanying \(\gamma\) absorption for the \(A = 6\) nuclei by using the USM with the GPT and HM potentials. It was concluded in the cited paper that the GPT potential, which was the most suitable potential for the nucleon-nucleon interaction at small distances, yielded the best fits to the calculated characteristics of the \(^6\text{Li}\) nucleus and that the agreement between the theory and the experimental data was considerably improved by including higher configurations in the analysis.

In the present paper, we applied the USM with bases corresponding to even numbers of quanta of excitation in the range \(0 \leq N \leq 20\) to investigate the ground-state and first-excited-state wave functions and energies, the \(S-, P-\) and \(D-\)state probabilities, the root-mean-square radius, and the magnetic dipole moment of triton. In
addition, for the $^4$He nucleus, the bases of the USM with numbers of quanta of excitation in the range $0 \leq N \leq 20$ were used to calculate the wave functions and energies of the ground state and the even- and odd-parity excited states; the $S$, $P$-, and $D$-state probabilities; the root-mean-square radius, and the total integral cross section of the dipole electric transition accompanying the photoabsorption of $\gamma$ quanta by this nucleus. To perform these calculations, we used two nucleon-nucleon interactions and a three-nucleon interaction. The first nucleon-nucleon interaction is the GPT potential [28], which is a smooth, realistic local nucleon-nucleon force. It fits the two-body data and the deuteron binding energy, quadrupole moments and magnetic moments and is suitable for nuclear Hartree–Fock calculations. The second is the AV8' nucleon-nucleon interaction [31]. For the three-body interaction, we used the Urbana IX model (UIX) interaction [32]. Moreover, the convergence behaviour of the USM calculations was examined by incrementally extrapolating the results for $N \leq 20$ to $N = 30$ for both nuclei.

2. The Hamiltonian and the total nuclear wave function

The Hamiltonian $\mathcal{H}$ of a nucleus consisting of $A$ nucleons, interacting via two-body potentials, can be written in terms of the relative coordinates of the nucleons in the following form [30, 33]:

$$\mathcal{H} = \frac{1}{2m} \sum_{i=1}^{A} p_i^2 + \frac{1}{2} \sum_{i=1}^{A} \sum_{j=1}^{A} V(|r_i - r_j|).$$  

(2.1)

The translational invariance of the Hamiltonian $\mathcal{H}$ permits the separation of the centre-of-mass motion, and consequently, the Hamiltonian corresponding to the internal motion becomes

$$H = \mathcal{H} - \frac{1}{2mA} \left( \sum_{i=1}^{A} p_i \right)^2.$$  

(2.2)

By adding and subtracting an oscillator potential referred to the centre of mass, the internal Hamiltonian $H$ can be rewritten in terms of the relative coordinates of the nucleons in the form

$$H = H^{(0)} + V',$$

(2.3)

where

$$H^{(0)} = \frac{1}{A} \sum_{i<j}^{A} \left[ \frac{1}{2m} (p_i - p_j)^2 + \frac{1}{2} m\omega^2 (r_i - r_j)^2 \right]$$

(2.4)

is the well-known USM Hamiltonian, also known as the TISM Hamiltonian, and

$$V' = \sum_{i<j}^{A} \left[ V(|r_i - r_j|) - \frac{m\omega^2}{2A} (r_i - r_j)^2 \right]$$

(2.5)

is the residual two-body interaction.

The energy eigenfunctions and eigenvalues of the Hamiltonian $H^{(0)}$ are given by [30]

$$|A \Gamma; M_l M_S M_T\rangle \equiv |A \{ N \{ \rho \} \{ \alpha \{ f \} \} LS; M_l M_S M_T\rangle,$$

$$E^{(0)}_N = \left\{ N + \frac{3}{2} (A - 1) \right\} \hbar \omega,$$

(2.6)

(2.7)

The functions expressed in (2.6) form a complete set of functions, or bases. It is easy to construct bases that have a definite total momentum $J$ in the following form [8, 9, 30, 33]:

$$|A \Gamma J M_l M_S M_T\rangle = \sum_{M_{l'} + M_{s'} = M_l} (LM_{l'}; SM_{S'}|JM_l) |A \Gamma; M_l M_S M_T\rangle,$$

(2.8)

where the $(LM_{l'}, SM_{S'}|JM_l)$ are the Clebsch–Gordan coefficients of the rotational group $SO_3$. The nuclear wave function of a state with total momentum $J$, isospin $T$, and parity $\pi$ can be constructed as follows [8, 9, 30, 33]:

$$|A \Gamma J^z T M_l M_T\rangle = \sum_{\Gamma} C_{\Gamma J^z T}^{T\pi} |A \Gamma J M_l T M_T\rangle,$$

(2.9)

where the $C_{\Gamma J^z T}^{T\pi}$ are the state-expansion coefficients. In the sum on the right-hand side of (2.9), the number of quanta of excitation $N$ can be either an even or odd integer depending on the parity of the state $\pi$. It is obvious that the USM Hamiltonian is free of spurious states. The spurious states that must be eliminated correspond to the non-zero motion of the centre of mass of the entire nucleus.

It is well known that three-body forces are important for describing the properties of finite nuclei. The parameters in the nucleon-nucleon potential may not be unique, or there may be some redundant parameters necessary to reproduce the deuteron properties. To investigate these possibilities, we consider the following Hamiltonian operator, which includes three-body forces:
Here the pion mass \( A \) and a shorter-range phenomenological part, with a two-pion exchange contribution as

\[
V_{\pi} = A \sum_{\text{cyc}} \left\{ x_{ij}, x_{jk} \right\} \left[ \tau_i \tau_j, \tau_j \tau_k \right] + \frac{1}{4} [x_{ij}, x_{jk}] \left[ \tau_i, \tau_j \right] [\tau_j, \tau_k],
\]

(2.16)

and a shorter-range phenomenological part,

\[
V_{\text{fr}} = \sum_{\text{cyc}} U \left( m_\pi r_{ij} \right) T^2 (m_\pi r_{jk}).
\]

(2.17)

A and \( U \) are adjustable parameters and the \( T(m_\pi r) \) and \( Y(m_\pi r) \) are radial functions associated with the tensor and Yukawa parts of the one pion-exchange interaction:

\[
Y(r) = \frac{e^{-mr}}{\mu r} (1 - e^{-br}),
\]

(2.18)

\[
T(r) = \left[ 1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right] (1 - e^{-br}).
\]

(2.19)

Here the pion mass \( \mu = 0.7 \text{ fm}^{-1} \) and \( b = 2 \text{ fm}^{-2} \). The \( V_{\pi} \) is the familiar Fujita-Miyazawa two-pion exchange operator, and is attractive while the \( V_{\text{fr}} \) is repulsive.

The matrix elements of the residual two-body interaction \( V' \) (equation (2.5)) with respect to the bases (equation (2.6)) are given in detail in [8, 9, 30, 33] by using two-particle FPCs; specifically, they are products of orbital and spin–isospin two-particle FPCs. Similarly, the matrix elements of the three-body interaction are calculated by using three-particle FPCs. The ground- and excited-state nuclear wave functions, which are obtained via the diagonalization of the energy matrices, are used to calculate the root-mean-square radius, the magnetic dipole moment and the total integral cross section for the photoabsorption of \( \gamma \) quanta by the nucleus.

The methods of calculating the one-, two-, three- and four-particle FPC in the USM are given in [34]. Also, recurrence relations for the two-particle orbital FPC and tables of these coefficients for \( 3 \leq A \leq 6 \) and \( N \leq 3 \) are given by Vanagas [2]. General and direct method for calculating the two-particle orbital FPC and tables of these coefficients for \( A = 6 \) and \( 2 \leq N \leq 4 \) are given by Doma and Machabeli [35]. Furthermore, this direct method has been used to calculate the two-particle orbital FPC for nuclei with \( A = 3 \) and \( 0 \leq N \leq 10 \) in [36]. Finally, we calculated in the present paper the necessary orbital FPC for nuclei with \( A = 3 \) and 4.
3. The Root-mean-square radius and the magnetic dipole moment

The root-mean-square radius is defined as

\[ R = \sqrt{\frac{r_p^2}{2} + \langle R_{\text{Nuc}}^2 \rangle}, \]  

where \( r_p = 0.85 \text{ fm} \) is the proton radius and the second term in the sum is the mean value of the following operator [30]:

\[ R_{\text{Nuc}}^2 = \frac{1}{A^2} \sum_{i<j} r_{ij}^2, \]  

This operator does not depend on the spin-isospin variables of the nuclear wave function, and its calculation is straightforward [30].

The nuclear magnetic dipole moment is defined as the mean value of the operator

\[ \hat{\mu} = \sum_{i=1}^{A} \left[ (\mu_p + \mu_n) + 2(\mu_p - \mu_n) t_{0i} \right] s_{0i}, \]  

where \( \mu_p \) and \( \mu_n \) are the proton and neutron magnetic moments, respectively, and \( t_{0i}, s_{0i} \) and \( \ell_{0i} \) are the z components of the isospin, spin and orbital momenta, respectively, of the \( i \)th nucleon. By writing each of the two operators \( \hat{\mu}_p \) and \( \hat{\mu}_n \) as a sum of symmetric and antisymmetric operators of symmetry types \([A] \) and \([A - 1, 1]\), in the forms

\[ \hat{\mu}_p = \frac{\hbar}{E_c} \left[ t_{0i}^{[A]} + t_{0i}^{[A - 1, 1]} \right], \hat{\mu}_n = \frac{\hbar}{E_c} \left[ t_{0i}^{[A]} + t_{0i}^{[A - 1, 1]} \right], \]  

the mean value of the magnetic dipole moment can be transformed into an algebraic expression that depends on the orbital and spin-isospin quantum numbers of the \( A \)-nucleon state, and the calculations are then straightforward [2].

4. The partial integral cross section of \( \gamma \)-quanta photoabsorption

The partial integral cross section of the dipole electric transition that accompanies \( \gamma \)-quanta photoabsorption is calculated by using the well-known line integral [6, 37]

\[ \sigma_{i \rightarrow f} = \int \sigma dE_f = \frac{(2\pi)^2 E^2}{hc} \sum_{j, \mu} \left| \langle f | J'M' \bar{T}_j^{\text{elec}} | i \rangle \right|^2, \]  

where \( E_f = E_f - E_i \) is the energy of the \( \gamma \) quanta and \( |i\rangle \) and \( |f\rangle \) are the wave functions of the initial and final states, respectively. The operator for the dipole electric \( \gamma \) transition, \( T_\gamma^{\text{elec}} \), is defined as [6, 37]

\[ T_\gamma^{\text{elec}} = \frac{\hbar}{E_c} \left[ \sum_{i=1}^{A} t_{0i}(i) r(i) y_{\mu}(i) - \sum_{i<j} \frac{1}{A - 1} \sum_{\mu} \frac{3}{4\pi} [t_{0i}(i) n_{\mu}(i) + t_{0j}(j) n_{\mu}(j)] \right], \]  

where \( k \) is the angular wave number, \( k = \frac{E_f}{E_i} \). In equation (4.2), we have transformed the Cartesian components of the vectors into their corresponding spherical components, in the usual manner, as follows:

\[ A_\mu = \frac{4\pi}{3} A Y_{\mu}(\hat{A}), \mu = 0, 1, 2. \]  

If one-particle FPCs are not allowed for higher configuration spaces, i.e., higher values of \( N \), one can use the following transformations:

1. Introduce the centre-of-mass coordinates and the relative coordinates in the forms

\[ R_{ij} = \frac{1}{2} (r_i + r_j) \text{ and } r_{ij} = r_i - r_j, \]  

where \( r_i \) and \( r_j \) are the coordinates of the \( i \)th and \( j \)th nucleons, respectively.
Table 1. The S-, D- and P-state probabilities ($P_S$, $P_D$ and $P_P$, respectively) for the ground-state wave function of triton and the probabilities of the bases having irreducible representations [$f$] = [3, 2]1 and [111] of the symmetry group $S_3$. Previous results [22] obtained by using the Faddeev equation with the Argonne v18 and UIX three-nucleon interactions are shown. In addition, previous results [38] obtained by using the hyperspherical harmonics method with the effective AV18 + UIX potentials are also shown, along with previous results [40] obtained by using the ab initio few-body method with the effective AV8′ + 3NF potentials.

| Case Character. | GPT | GPT + UIX | AV8′ | AV8′ + UIX | Faddeev, AV18 + UIX [22] | Hyperspherical harmonics AV18 + UIX [38] | ab initio AV8′ + 3NF [40] |
|----------------|-----|-----------|------|------------|--------------------------|---------------------------------|----------------------|
| $P_S$%         | 92.84 | 91.856 | 91.894 | 91.204 | 90.565 | 8.16 | — |
| $P_D$%         | 6.73  | 7.907 | 7.877 | 8.448 | 9.300 | 8.69 | — |
| $P_P$%         | 0.43  | 0.237 | 0.229 | 0.348 | 0.135 | 0.135 | — |
| $P_{1S0}$%     | 86.38 | 88.724 | 86.774 | 86.251 | — | — | — |
| $P_{1D1}$%     | 13.56 | 11.224 | 13.178 | 13.603 | — | — | — |
| $P_{1P1}$%     | 0.06  | 0.052 | 0.048 | 0.146 | — | — | — |

(2) Use similar relations for the isotopic spin components:

$$T_0(ij) = t_0(i) + t_0(j), \quad t_0(ij) = \frac{1}{2} \{ t_0(i) - t_0(j) \}. \tag{4.5}$$

By applying the graphical method for the addition of angular momenta [37], one can calculate $C_{12\rightarrow f}$. The total integral cross section of the dipole electric transition accompanying the photoabsorption of $\gamma$ quanta by $^4$He, $C$, is then the sum of the $C_{12\rightarrow f}$ for all possible final states $f$.

5. Results and discussions

In our investigation, the ground- and excited-state wave functions of $^3$H and $^4$He were expanded in series in terms of the bases of the USM. For $^3$H, bases corresponding to even numbers of quanta of excitation $N$ in the range $0 \leq N \leq 20$ were constructed. For $^4$He, bases corresponding to both even and odd numbers of quanta of excitation $N$ in the range $0 \leq N \leq 20$ were constructed. Each of these basis functions was expanded in terms of one-, two- and three-particle total FPCs, which are products of orbital and spin-isospin coefficients. As a result, the Hamiltonian matrices for the different states of $^3$H and $^4$He were constructed as functions of the oscillator parameter $\hbar \omega$. By diagonalizing these matrices for each value of $\hbar \omega$ which is allowed to vary in a wide energy range in order to obtain the best fit to the spectrum of the given nucleus, we obtained the nuclear energy eigenvalues and the corresponding eigenfunctions of $^3$H and $^4$He. Furthermore, the matrix elements of the different operators corresponding to different nuclear characteristics were calculated.

The ground state of triton has a total angular momentum of $J = \frac{1}{2}$, an isotopic spin of $T = \frac{1}{2}$, and even parity, i.e., $(J^T, T) = \left( \frac{1}{2}, \frac{1}{2} \right)$. The energy eigenvalues obtained via the diagonalization of the Hamiltonian matrices for the state $\left( \frac{1}{2}, \frac{1}{2} \right)$ of triton, for each value of the oscillator parameter $\hbar \omega$ ($8 \leq \hbar \omega \leq 20 \text{ MeV}$), showed two accepted values: the lower one belongs to the ground state and the next one corresponds to the first-excited-state energy $E^*$. Other higher values were also obtained, but we do not present them here because there is no experimental evidence for the existence of these excited states of triton. The obtained ground-state wave functions were used to calculate the root-mean-square radius and magnetic dipole moment of triton.

In table 1, we present various quantities that characterize the ground-state wave function of triton. For this purpose, we present the $S$, $D$- and $P$-state probabilities, denoted by $P_S$, $P_D$ and $P_P$, respectively, for the different potentials considered. The probabilities of the bases having irreducible representations [$f$] = [3, 2]1 and [111] of the symmetry group $S_3$ for the ground-state wave functions of triton are also given in this table. Obviously, $P_{S\frac{1}{2}} + P_{D\frac{3}{2}} + P_{P\frac{1}{2}} = P_S$. Previous results [22] obtained by using the Faddeev equation with the Argonne v18 nucleon-nucleon interaction plus the UIX three-nucleon interaction are shown. Furthermore, previous results [38] obtained by using the technique of hyperspherical harmonics [39] with the effective AV18 nucleon-nucleon potential plus the UIX three-nucleon interaction are also shown, along with previous results [40] obtained by using the ab initio few-body method with the effective AV8′ + 3NF potentials.

In table 2, we present for triton the binding energy in MeV, the root-mean-square radius in fm, the first-excited-state energy in MeV, and the magnetic dipole moment in N.M. calculated by using the two nucleon-nucleon potentials considered in this study (GPT and AV8′). The improved values that resulted from including the UIX three-nucleon interaction are also given. Moreover, the corresponding experimental values [41] and the values of the oscillator parameter $\hbar \omega$ that produced the minimum ground-state energy eigenvalues are provided. Previous results obtained by using the Faddeev equation together with the AV18 nucleon-nucleon...
 interaction plus the UIX three-body interaction are shown [22]. Previous results [12] obtained by using the NCSM with the AV18 nucleon-nucleon interaction plus the TM three-body interaction are also shown, along with results obtained by using the ab initio few-body method with the effective AV8′ + 3NF potentials [40].

Concerning the second \( \left( \frac{1}{2} \right) \) state of \(^3\)H, we note that the three- and four-nucleon states each have only one bound state, and all excited states are in the continuum. The use of a bound-state approach with square-integrable basis functions is only meaningful for bound states and narrow resonances. In contrast to \(^4\)He, which exhibits a broad \( 0^+ \) resonance, there are no resonances in the \( \frac{1}{2} \) channel of tritium. The reason why its energy increases with increasing HO frequency and does not exhibit a clear minimum is that this eigenstate and those above it represent a discretization of the energy continuum, and as such, they continuously move as the size of the model space is increased or as other parameters are varied.

The ground state of \(^4\)He is \((0^+,0)_0\), and its excited states are \((0^+,0)_{1}, (0^-,0), (2^-,0), (2^-,1), (1^-,1), (0^-,1), (1^-,1)_{2}, (2^+,0)\) and \((1^-,0)\). Another excited state, \((1^+,0)\), was predicted for this nucleus in [28]. The Hamiltonian matrices that belong to these states were constructed with respect to numbers of quanta of excitation \( N \) in the range \( 0 \leq N \leq 20 \) as functions of the oscillator parameter \( \hbar \omega \), which was allowed to vary over a large range of values, \( 8 \leq \hbar \omega \leq 28 \) MeV, to obtain the best fit to the spectrum of \(^4\)He. Among all levels of \(^4\)He, the dipole transition is allowed only for levels with \( J = 1 \) and \( T = 1 \), i.e., the levels \((1^-,1)_{2}\) and \((1^-,1)\). The eigenvalues that resulted from the diagonalization of the ground-state Hamiltonian matrices for \(^4\)He in this study showed two accepted values: the lower one belongs to the ground state, \((0^+,0)_0\), and the higher belongs to the first excited state, \((0^+,0)_{1}\). The obtained ground- and excited-state nuclear wave functions were used to calculate the spectrum, the root-mean-square radius and the total integral cross section of the dipole electric transition accompanying \( \gamma \)-quanta photoabsorption for the \(^3\)He nucleus with respect to each value of the oscillator parameter \( \hbar \omega \).

In table 3, we present different quantities that characterize the ground-state wave function of \(^4\)He. In this table, we present the probabilities \( P_0, P_2, P_4 \) for the ground-state wave function of \(^4\)He obtained by using the two nucleon-nucleon interactions alone as well as the improved values resulting from the inclusion of the three-nucleon interaction. The probabilities of bases having irreducible representations \( [f] = [4],[31],[22] \) and [211]
of the symmetric group $S_4$ for the ground-state wave functions of $^4\text{He}$ are also shown, as are previous results obtained by using the NCSM [42].

In table 4, we present the results obtained for $^4\text{He}$ by using the GPT and AV8' nucleon-nucleon interactions to calculate the binding energy in MeV, the root-mean-square radius in fm, and the total integral cross section of the dipole electric transition accompanying the photoabsorption of $\gamma$-quanta by this nucleus $(\ell)$ in MeV-mbarn. The improved values obtained by including the UIX three-nucleon interaction are also given. Moreover, the corresponding experimental values and the values of the oscillator parameter $h_\omega$ that resulted in the best fit between the calculated $^4\text{He}$ spectra and the experimental values are also given. Previous results obtained by using the Faddeev-Yakubovsky (FY) method [25] and the NCSM method [25] are also shown in this table.

|       | GPT         | GPT + UIX    | AV8'       | AV8' + UIX  | FY [25]     | NCSM [25]   | Exper.  |
|-------|-------------|--------------|------------|-------------|-------------|-------------|---------|
| B.E.  | 27.929      | 28.272       | 27.999     | 28.272      | 25.94       | 25.80       | 28.296  |
| $R$   | 1.670       | 1.520        | 1.666      | 1.511       | 1.485       | 1.485       | 1.46(4) |
| $\ell$| 70.48       | 68.995       | 69.464     | 69.321      | —           | —           | —       |
| $h_\omega$ | 17          | 17           | 16         | 16          | —           | —           | —       |

Table 4.

In figure 1, we present the spectra of $^4\text{He}$ that resulted from using the two nucleon-nucleon interactions alone and from including the three-nucleon interaction. The experimental spectrum [45] is also shown in this figure.

**Figure 1.** The $^4\text{He}$ spectra calculated by using the GPT, GPT + UIX, AV8', and AV8' + UIX potentials along with the experimental spectrum [45].
It is seen that the order of the calculated levels is correct and that the calculated spectra are in good agreement with the corresponding experimental data for both considered interactions. Interestingly, a new even-parity level for $^4$He is obtained near the threshold energy value. This is the level $(1^+, 0)$, with an energy equal to 32.61, 31.14, 33.39 or 32.10 MeV according to the GPT, GPT + UIX, AV8$'$ and AV8$'$ + UIX potentials, respectively. The main contributions to this level are due to bases with $|f| = |31\rangle$, whereas bases with $|f| = |221\rangle$ contribute little.

Moreover, to study the convergence properties of the USM approach with respect to the dimensionality of the adopted model space, the results of the USM calculations for $N \leq 20$ were incrementally extrapolated to $N = 30$ by using Stirling’s formula $[46]$ for both nuclei. In figures 2–5, we present the convergence behaviours of the binding energy, the root-mean-square radius, the first-excited-state energy and the magnetic dipole moment, respectively, for triton with respect to the number of quanta of excitation $N$. In figures 6–8, we present
the convergence behaviours of the binding energy, the root-mean-square radius and the first-excited-state energy, respectively, for $^4$He with respect to the number of quanta of excitation $N$. These figures show that the calculated characteristics of $^3$H and $^4$He converge to the corresponding experimental values as the value of $N$ increases.

6. Conclusion

The results obtained in this study show that three-body interactions play a very important role in the calculation of the different ground- and excited-state characteristics of $^3$H and $^4$He. Moreover, high configurations, with suitably chosen bases, also play a major part in determining the convergence behaviours of the nuclear characteristics. Calculations within a three-body translationally invariant HO basis using realistic two- and three-nucleon forces have previously been performed for the three-nucleon system (see, e.g., [47]) and for the spectrum and root-mean-square radius of the four-nucleon system (see, e.g., [20, 21]). These previous
calculations were based on the translationally invariant form of the no-core–shell model, which is completely equivalent to the USM except that the antisymmetrization of the wave function is achieved not by means of group theory but rather by diagonalizing the antisymmetrization operator and retaining the antisymmetric eigenstates as basis states. Furthermore, the spectrum and root-mean-square radius of the three- and four-nucleon systems have been studied by means of numerically exact few-body approaches (such as the Faddeev, Faddeev-Yakubovsky and hyperspherical harmonics approaches starting from realistic two- and three-nucleon forces; see, for example, [22, 23, 25]). A review can be found in [24].

By contrast, the calculations presented here, in addition to providing a complete picture of the nuclear structure based on the obtained nuclear wave functions of the different states of the investigated nuclei, also present a certain degree of novelty with respect to the translationally invariant no-core–shell model. In particular, the direct construction of the antisymmetric three- and four-body basis states by means of group theory is more elegant and may even prove to be computationally more advantageous, considering that all calculations of the nuclear characteristics reported in the present paper were performed by applying group-theoretical methods.
Figure 8. Convergence behaviour of the $^4$He characteristics.

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