Strings in Discrete and Continuous Target Spaces: A Comparison

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We find the precise relationship between the loop gas method and the matrix quantum mechanics approach to two-dimensional string theory. The two systems are distinguished by different target spaces (\(\mathbb{Z}\) and \(\mathbb{R}\), respectively) as far as observables are concerned. We argue that target space loop correlators should coincide in the two models and demonstrate this for a number of examples. As a consequence some interesting generic observations about the structure of two-dimensional string theory may be made: Restricting to a discrete target space leads to factorization of amplitudes and thus to very simple sewing rules. It is also demonstrated that the restriction to the discrete target space still allows to calculate the correlation functions of tachyon operators in the unrestricted theory.

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1. Introduction

Over the past few years dramatic progress towards formulating a theory of low-dimensional bosonic strings has been achieved. In fact, for non-critical strings living in space-times of dimensions less or equal than one the theory could be considered solved: One is now able to calculate almost all physical quantities using lattice models which are exactly solvable through matrix models or combinatorial techniques. However, possessing the solution does not necessarily mean one understands it. Consider for example the KPZ scaling relation [1]: It is certainly confirmed by all exact solutions but in order to give a universal and simple argument for its validity one should use continuum reasoning [2]. Now there clearly exists a host of non-trivial results beyond mere scaling laws which simply “come out” of the exact solutions but are not yet derivable from a continuum theory. It is in precisely this sense that the lattice model solutions have been termed “experimental” by some authors. Thus, the ultimate theory explaining and unifying all approaches is still missing.

In this situation it is important to carefully analyze all the available data. In the present work we aim to contribute to this program by relating two rather different lattice models designed to discretize the non-critical string in one dimension. The first model is formulated as the quantum mechanics of a large $N$ hermitian matrix [3]. It can be rewritten as an intriguingly simple system of free fermions which upon bosonization turns into a rather unusual string field theory with only one interaction vertex [4]. The model has been exhaustively solved in references [3], [4], [5], [6], [7], [8] (and references therein). The method of solution appears however rather removed from more traditional approaches to string field theory which uses factorization, sewing and infinitely many interactions. It is precisely these concepts which appear naturally in the second model we are discussing: The SOS string [10], [11], [12]. Here the integrable models where the target space is an extended $\hat{A}\hat{D}\hat{E}$ Dynkin diagram are adapted to dynamical lattices. These extended

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1 The techniques apply equally to the case of the non-extended diagrams ADE corresponding to $C < 1$ systems [13]. In fact, one of the strenghts of the loop gas approach is a unified description of all $C \leq$ noncritical strings. In the present work we restrict the discussion to $C = 1$. 

diagrams correspond to special points in the space of $C = 1$ theories. In particular the $n \to \infty$ limit of the $\hat{A}_n$ model (i.e. $\hat{A}_n \to \mathbb{Z}$) turns into the non-compact $C = 1$ system. It was therefore quite dissatisfying to observe that not only the methods, but also the results of the SOS approach seemed to be incompatible with the matrix quantum mechanics; e.g. the correlator of two macroscopic loops (the propagator) seemed to be distinct in the two systems. The main result of this paper is a reconciliation of these differences. It will be argued that both models indeed describe the same continuum string theory, albeit in a rather different manner.

The outline of the paper is as follows: In the next section we give a short description of the matrix quantum mechanics method as well as the SOS string models which we also call $\mathbb{R}$ and $\mathbb{Z}$ strings, respectively. In section 3 we compare the two systems using both known results as well as some novel calculations. Our conclusions and some speculations are presented in section 4.

2. A brief description of the models

Both models can be considered as discretizations of the Polyakov path integral

$$Z = \int \mathcal{D}x \mathcal{D}g \ e^{-\mathcal{A}[x,g]}$$

$$\mathcal{A}[x,g] = \frac{1}{4\pi} \int d^2 \xi \sqrt{\det g(\xi)} \ g^{ab}(\xi) \partial_a x(\xi) \partial_b x(\xi) + \mu \int d^2 \xi \sqrt{\det g(\xi)}$$

where the parameter $\mu$ coupled to the intrinsic area of the world sheet is the cosmological constant. The measure over the metrics $g_{ab}(\xi)$ is discretized in both cases by planar graphs but the embedding into the $x$-space is constructed differently.

a) $\mathbb{R}$-string (Matrix quantum mechanics)

The path integral of the $\mathbb{R}$-string is given by the sum of all $\varphi^3$ planar graphs embedded in the continuous line $\mathbb{R}$. Each point $s$ of the graph $\mathcal{S}$ has a coordinate $x(s)$ and the partition function is obtained by summing over all possible graphs and integration over
the coordinates of the points. The weight of an embedded graph \( S \rightarrow \mathbb{R} \) is composed by a topology-dependent factor \( N^\chi \) where \( \chi[S] \) is the Euler characteristic of the graph, and a product of local factors \( \Omega_{x(s),x(s')} \) associated with the edges \(<ss'>\rangle \) of the graph

\[
\Omega_{x,x'} = e^{-\beta - |x-x'|}
\]  

The partition function reads

\[
Z_{\mathbb{R}} = \sum_{S} N^{\chi[S]} \prod_{s \in S} \int_{-\infty}^{\infty} dx(s) \prod_{<ss'> \in S} \Omega_{x(s),x(s')}
\]  

The sum over graphs with given topology is convergent for \( \beta \) larger than some critical \( \beta^* \); the difference \( \beta - \beta^* \) is proportional to the renormalized cosmological constant \( \mu \).

b) \( \mathbb{Z} \)-string (SOS model on a random lattice)

The SOS string has as a target space the discretized line \( \mathbb{Z} \). Its path integral is defined as the sum of triangulated surfaces embedded in \( \mathbb{Z} \). An embedded surface is described by a triangulation (the world sheet) and an integer valued local field variable (height) \( x(s) \in \mathbb{Z} \) associated with each site \( s \) of the triangulation. The rules of embedding are such that the the heights of the endpoints of each bond \(<ss'>\rangle \) either coincide or differ by 1. The weight of an embedded surface \( S \rightarrow \mathbb{Z} \) depends on its Euler characteristics \( \chi[S] \) through the factor \( N^\chi[S] \). Apart of this it is a product of factors \( \Omega_{x(s),x(s')} \) associated with the bonds \(<ss'>\rangle \) of the triangulation \( S \)

\[
\Omega_{xx'} = e^{-\beta \delta_{x,x'} + e^{-\kappa} [\delta_{x,x'+1} + \delta_{x,x'-1}]}
\]  

The partition function is defined therefore as

\[
Z_{\mathbb{Z}} = \sum_{S} N^{\chi[S]} \sum_{\{x(s) \in \mathbb{Z} | s \in S\}} \prod_{<ss'> \in S} \Omega_{x(s),x(s')}
\]  

The continuum limit is achieved along a critical line in the \( \beta, \kappa \) space \( [12] \).
3. Comparison

3.1. Torus

The torus diagram is of prime importance since it gives information about the states of the theory [14], [15]. Moreover, it has been calculated in continuum Liouville theory [15]. The result for \( C = 1 \) matter compactified on a circle of radius \( R \) is

\[
Z_{\text{torus}}(R) = -\frac{1}{24} (R + \frac{1}{R}) \log \mu
\]  

(3.1)

This is precisely the result obtained from the \( \mathbb{R} \)-string with compact target space [16].

In the \( \mathbb{Z} \)-string the radius of the circle can take only integer values \( R = h; \ h = 1, 2, \ldots \). (We took into account the scale factor \( \pi \) between the two \( x \)-spaces.) The corresponding compactified target space \( \mathbb{Z}_{2h} \) is constructed as a closed chain of \( 2h \) points and is identical to the \( \hat{A}_{2h-1} \) Dynkin diagram \(^2\). The calculation is done in [12] and one again finds eq.(3.1). We therefore conclude that \( \mathbb{R} \) and \( \mathbb{Z} \) strings describe the same continuum string theories in the bulk. The reader may thus wonder whether the name \( \mathbb{Z} \) string is really appropriate since we just argued that the discreteness of the target space vanishes in the continuum limit: The Dynkin diagram turns into a continuous space in much the same way an Ising model renormalizes onto a continuous field. We will however see shortly that a remnant of the discreteness of the target space of the lattice models remains in the continuum once we introduce boundaries into the manifolds. Before turning to the cylinder where this effect appears let us first review the simplest case of a diagram with boundary: The disc.

3.2. One loop (Disc)

The disc amplitude can be formally considered as the mean value of the operator \( w(\ell, x) \) creating on the world sheet a boundary of length \( \ell \) and position \( x \) in the target space

\(^2\) Generally it makes sense to consider only target spaces with even number of points; otherwise winding modes are kinematically impossible. There is however one exception: the space \( \mathbb{Z}_1 \) which corresponds to \( R = 1/2 \). The \( \mathbb{Z}_1 \) string is identical to the \( O(2) \) model on a random lattice [17].
space. In the continuum approach the disc amplitude should be given by a product of three separate path integrals over matter, ghost and Liouville sectors:

\[ \langle w(\ell, x) \rangle = Z_{\text{ghost}} Z_{\text{matter}}(x) Z_{\text{Liouville}}(\ell) \] (3.2)

The reason for this “factorization” is that there are no moduli on the disc. The path integral for \( Z_{\text{matter}}(x) \) with a Dirichlet boundary condition for the position field actually does not depend on its boundary value \( x \) since our target space is translationary invariant. It is not known yet how to calculate the path-integral for \( Z_{\text{Liouville}}(\ell) \). Indeed it is not clear how to properly treat boundaries in Liouville theory. Aside from the technical problem of carrying out the integration one first has to understand the correct boundary conditions on the Liouville and ghost fields. In \([18]\) qualitative arguments were given that at \( C = 1 \), and for Dirichlet boundary conditions on the matter field \( w(\ell) \) should satisfy the Wheeler-deWitt equation \[ -((\partial_\ell)^2 + \mu \ell^2 + 1)[\ell w_\ell(\ell)] = 0 \] which is solved by

\[ \langle w(\ell, x) \rangle = \frac{\sqrt{\pi}}{\ell} K_1(\sqrt{\mu \ell}) \] (3.3)

where \( K_1 \) is a modified Bessel function. This is indeed what one finds in both the IR \([18],[7]\) and the \( \mathbb{Z} \) string \([19],[12]\).

3.3. Two loops (Cylinder)

The cylinder amplitude is the tree level propagator of the two-dimensional string theory. It should be given in the continuum by the formal path integral

\[ \langle w(\ell_1, x_1) w(\ell_2, x_2) \rangle = \ell_1 \ell_2 \int_0^\infty \frac{d\tau}{\tau} Z_{\text{matter}}(\tau) Z_{\text{ghosts}}(\tau) Z_{\text{Liouville}}(\tau, \ell_1, \ell_2) \] (3.4)

where \( \tau \) is the modular parameter of the cylinder playing the rôle of a proper time for the closed string and \( \ell_1, \ell_2 \) are the lengths of the two boundaries. Again this path-integral has not been directly calculated but may be computed with relative ease in the discrete approach. Let us now discuss the IR and \( \mathbb{Z} \) strings separately.
In the IR case we may place the boundaries at arbitrary positions \(x_1, x_2\) in target space. Then one Fourier-transforms to momentum space

\[
\delta(q_1 + q_2) \langle w_{q_1}(\ell_1) w_{-q_1}(\ell_2) \rangle = \int_{-\infty}^{\infty} dx_1 e^{i q_1 x_1} \int_{-\infty}^{\infty} dx_2 e^{i q_2 x_2} \langle w(\ell_1, x_1) w(\ell_2, x_2) \rangle \tag{3.5}
\]

and obtains \([7], [18]\)

\[
\langle w_q(\ell_1) w_{-q}(\ell_2) \rangle = \frac{\pi q}{\sin \pi q} I_q(\sqrt{\mu \ell_1}) K_q(\sqrt{\mu \ell_2}) + \sum_{r=1}^{\infty} \frac{2(-1)^r r^2}{r^2 - q^2} I_r(\sqrt{\mu \ell_1}) K_r(\sqrt{\mu \ell_2}) \tag{3.6}
\]

where \(I\) and \(K\) are modified Bessel functions. Introducing the complete system of \(\delta\)-function normalized eigenstates of the kernel \((3.6)\)

\[
\langle \ell | E \rangle = \frac{2}{\pi} \sqrt{E \sinh(\pi E)} \ K_{1E}(\sqrt{\mu \ell}), \quad E > 0 \tag{3.7}
\]

one can represent it as an integral

\[
\langle w_q(\ell_1) w_{-q}(\ell_2) \rangle = \int_{0}^{\infty} dE \ \langle \ell_1 | E \rangle \frac{1}{E^2 + q^2} \frac{E}{\sinh(\pi E)} \langle E | \ell_2 \rangle \tag{3.8}
\]

with \(E\) playing the role of the momentum associated with the Liouville-mode. The representation as a discrete sum \((3.6)\) is to be interpreted as a sum over on-shell (microscopic) states of the closed string.

Let us turn to the case of the ZZ-string. We argue that in view of \((3.4)\) and the Gaussian nature of the matter field one should obtain the same target space correlator \(\langle w(\ell_1, x_1) w(\ell_2, x_2) \rangle\) as in the IR case. However, there is one important difference: In the loop gas formulation it is impossible to choose \(x_1, x_2\) arbitrary: The target space distance \(|x_2 - x_1|\) does not renormalize. Since \(x_1, x_2\) take on integer values in the Dynkin diagram they have to remain integers even after taking the continuum limit. This “nonrenormalization” effect in SOS models (which is a particularity of the statistical model and has nothing to do with 2D gravity) is e.g. explained in \([20]\). It immediately follows that we cannot transform to momentum space using \((3.3)\). What remains possible is to transform
to a compact momentum space $-1 < p < 1$ dual to our discrete target space. The analog of eq. (3.5) is then\footnote{We will only discuss the noncompact case $\mathbb{Z}_{2h}$, $h \to \infty$ when the target space becomes the set of integers $\mathbb{Z}$. For compact target space ($h$ finite) the $p$-momentum space is compact and discrete: $p = 0, \pm \frac{1}{h}, \pm \frac{2}{h}, 1$.}

$$\delta^{(2)}(p_1 + p_2) \langle w_{p_1}(\ell_1) w_{p_2}(\ell_2) \rangle = \sum_{x_1 \in \mathbb{Z}} e^{i\pi p_1 x_1} \sum_{x_2 \in \mathbb{Z}} e^{i\pi p_2 x_2} \langle w(\ell_1, x_1) w(\ell_2, x_2) \rangle$$ (3.9)

Here $\delta^{(2)}$ is a periodic delta function of period 2. In this $p$-momentum space it has been shown shown \cite{12} that

$$\langle w_p(\ell_1) w_{-p}(\ell_2) \rangle = \int_0^\infty dE \langle \ell_1 | E \rangle \frac{1}{\cosh \pi E - \cos \pi p} \langle E | \ell_2 \rangle$$

$$= \frac{1}{\sinh \pi |p|} \left[ \sum_{n = -\infty}^{\infty} (|p| + 2n) I_{|p|+2n}(\sqrt{\mu} \ell_1) K_{|p|+2n}(\sqrt{\mu} \ell_2) \right]$$ (3.10)

Now let us demonstrate that the two propagators (3.6), (3.10) indeed coincide in $x$-space. To prove this we simply calculate the inverse Fourier transforms; first for the $\mathbb{Z}$ case

$$\frac{1}{2} \int_{-1}^{1} dp \frac{e^{i\pi px}}{\cosh \pi E - \cos \pi p} = \frac{1}{\sinh \pi E} e^{-E\pi |x|} \quad x \in \mathbb{Z}; \quad E > 0$$ (3.11)

and subsequently for the $\mathbb{R}$ case:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dq \frac{e^{iqx}}{E^2 + q^2} \frac{E}{\sinh \pi E} = \frac{1}{\sinh \pi E} e^{-E\pi |x|} \quad x \in \mathbb{R}; \quad E > 0$$ (3.12)

Quod erat demonstrandum. A more elegant way of showing this result consists in identifying all momenta congruent zero modulo two; thus “periodizing” the $\mathbb{R}$ propagator results in the $\mathbb{Z}$ propagator:

$$\sum_{n = -\infty}^{\infty} \frac{E}{\sinh \pi E} \frac{1}{E^2 + (p + 2n)^2} = \frac{\pi}{2} \frac{1}{\cosh \pi E - \cos \pi p}$$ (3.13)

It is interesting to compare the different pole structure in the two propagators (3.8), (3.10). Each pole at some $E = iv$ corresponds to a on-shell (microscopic) state with wavefunction
\[ \mu^{-|q|} K_{\nu}(\sqrt{\mu l}) \] satisfying the Wheeler-DeWitt constraint. In (3.6), (3.8) the pole at \( i\nu = iq \) signals the “tachyon” while the poles at \( i\nu = ir \) are believed to be related to redundant operators for generic \( q \) and to the special states for integer \( q \) \[21\]. It is quite curious that these poles \textit{disappear} upon periodization and are thus no longer present in the \( \mathbb{Z} \) string, according to (3.13). In its place appear an infinite number of gravitational descendents of the same structure as those of the \( C < 1 \) string models.

We will end this section by recalling how to extract \( n \)-point functions from the correlation functions of \( n \) macroscopic loops \[4\]. One simply shrinks the macroscopic loops and extracts the leading non-analytic piece in the \( \ell_i \)’s. Each macroscopic loop turns into a local operator \( \mathcal{O}(\ell_i, p_i) \) regularized by the loop-length \( \ell_i \). In the case of the propagator (3.10) of the \( \mathbb{Z} \)-string this leads to the two-point function

\[ \langle \mathcal{O}(\ell_1, p_1) \mathcal{O}(\ell_2, p_2) \rangle = -\delta^{(2)}(p_1 + p_2) \frac{1}{|p|} [\Gamma(1 - |p|)]^2 \mu^{\frac{|p|}{1}} \ell_1^{|p|} \ell_2^{|p|} \]  \hspace{1cm} (3.14)

where \( |p| < 1 \). After changing \( p \rightarrow q \) and \( \delta^{(2)} \rightarrow \delta \) (3.14) turns into the same expression one obtains from the \( \mathbb{R} \)-string propagator (3.6), valid for \( q \in \mathbb{R} \). This is of course not accidental and will be discussed at the end of section 3.5.

3.4. Three loops

Considering the three-vertex we learn something about the interactions in the two-dimensional string-theory. Here we have even less hope to be able to perform the continuum path-integral (the moduli space becomes quite complicated) than in the previous cases. Let us turn to our two formulations of the string theory and see what we can understand by comparing them. For the \( \mathbb{R} \) string the diagram with three external loops was calculated in \[21\]:

\[ \langle w_{q_1}(\ell_1)w_{q_2}(\ell_2)w_{q_3}(\ell_3) \rangle = \delta(q_1 + q_2 + q_3) \frac{1}{\mu} \prod_{j=1}^{3} \int_{-\infty}^{\infty} dE_j \frac{E_j}{E_j - i\delta_j} K_{\nu_j} \left( \sqrt{\mu \ell_j} \right) \]

\[ (E_1 + E_2 + E_3) \coth \left[ \frac{\pi}{2} (E_1 + E_2 + E_3) \right] \]  \hspace{1cm} (3.15)
For the $\mathbb{Z}$ string the three-loop correlator in $p$-space (as discussed in the previous section) has been found in $[12]$ using the string theory Feynman rules:

$$
\langle w_{p_1}(\ell_1)w_{p_2}(\ell_2)w_{p_3}(\ell_3) \rangle = \delta^{(2)}(p_1 + p_2 + p_3) \prod_{j=1}^{3} \int_{-\infty}^{\infty} dE_j \frac{E_j}{\sinh[\frac{\pi}{2}(E_j - ip_j)]} K_i E_j(\sqrt{\mu \ell_j})
$$

(3.16)

The above expression slightly differs from the one obtained in $[12]$. It is easy to see that if the propagator in the Feynman rules of $[12]$ (eqs. (4.51), (4.53)) is replaced by its chiral part

$$
\frac{E \sinh(\pi E) \cos(\pi p)}{\cosh(\pi E)[\cosh(\pi E) - \cos(\pi p)]} \rightarrow \frac{E}{\sinh(E - ip)}
$$

(3.17)

and the integration over $E$ is extended over the whole real $E$-axis, the final result does not change. Here and below we used the fact that the propagator is determined completely by its poles and residues and one has the freedom to neglect a factor which takes value 1 at all poles.

Now we may argue as in the last section that the three loop correlators of the two string models should coincide in target space. It is not difficult to transform (3.15),(3.16) back to $x$-space; the resulting expressions are indeed identical. Again the simplest way to demonstrate this is to periodize the function (3.15) with respect to the three momenta: $q_j = p_j + 2n_j; \ -1 \leq p_j \leq 1, \ n_j \in \mathbb{Z}; \ j = 1, 2, 3$. After that we introduce a Lagrange multiplier to write the momentum-conservation $\delta$-function as

$$
\delta[\sum_{j=1}^{3}(p_j + 2n_j)] = \sum_{n \in \mathbb{Z}} \int_{0}^{2\pi} \frac{d\alpha}{2\pi} e^{i\alpha(n_{-n_{1-n_{2-n_{3}}}})\delta[p_1 + p_2 + p_3 - 2n]} \delta
$$

(3.18)

and apply the formula

$$
\sum_{n \in \mathbb{Z}} \frac{e^{-i\alpha n}}{E - i(p + 2n)} = \frac{\pi}{2} \frac{e^{(\pi - \alpha)(E - ip)/2}}{\sinh[\pi(E - ip)/2]}, \ 0 < \alpha < 2\pi
$$

(3.19)

obtaining the r.h.s.of (3.16) up to a factor $\cos \frac{\pi}{2}(p_1 + p_2 + p_3) \cosh \frac{\pi}{2}(E_1 + E_2 + E_3)$ in the integrand which is equal to one at all poles, q.e.d.
Let us discuss the pole structure of the two correlators (3.15),(3.16). We see in the case of the \( \mathbb{R} \) string, aside from the tachyon poles at \( iq_j \), infinitely many further, momentum independent poles in (3.15). They were attributed in [21] to various contact terms of the macroscopic loops. Once we restrict our target space to \( \mathbb{Z} \) these contact terms disappear and we obtain (3.16). A rather dramatic consequence is that the removal of these contact terms leads to a factorization of the interaction which may be traced back to the fact that this interaction takes place at a single point in \( x \)-space.

We conclude the section by comparing the three-point functions obtained by shrinking the lengths of the macroscopic loops to zero. One easily finds from (3.16) for the \( \mathbb{Z} \)-string

\[
\langle 3 \prod_{i=1}^{3} O(\ell_i, p_i) \rangle = \delta^{(2)}(p_1 + p_2 + p_3) \frac{1}{\mu} \prod_{i=1}^{3} \Gamma(1 - |p_i|) \left( \frac{1}{2} \sqrt{\mu \ell_i} \right)^{|p_i|} \tag{3.20}
\]

Just as for the case of the two–point function this is, upon replacing \( p \rightarrow q \) and \( \delta^{(2)} \rightarrow \delta \) the same analytical expression one obtains for the \( \mathbb{R} \)-string (e.g. from eq. (3.15)).

3.5. Four loops

For the \( \mathbb{R} \)-string the correlator of four macroscopic loops has not been calculated to our knowledge; in the SOS case the result is quickly derived using the Feynman rules of the string field theory [12]. There is a reducible \( s,u \) and \( t \) channel diagram and an irreducible diagram. The \( s \) channel diagram reads

\[
\delta^{(2)}(p_1 + p_2 + p_3 + p_4) \left( \frac{1}{4\mu} \right)^2 \left[ \prod_{j=1}^{4} \int_{-\infty}^{\infty} dE_j \frac{E_j}{\pi \sinh[\pi(E_j - ip_j)/2]} K_4(E_j) \left( \sqrt{\mu \ell_j} \right) \right] \\
\times \int_{-\infty}^{\infty} dE' \frac{E'}{\sinh[\frac{\pi}{2}(E' - i(p_1 + p_2))]} \tag{3.21}
\]

while the \( t \) and \( u \) channel diagrams are obtained by replacing \((p_1 + p_2)\) by \((p_1 + p_3)\) and \((p_1 + p_4)\), respectively. The irreducible diagram reads

\[
\delta^{(2)}(p_1 + p_2 + p_3 + p_4) \left( \frac{1}{4\mu} \right)^2 \left[ \prod_{i,j} \int_{-\infty}^{\infty} dE_j \frac{E_j}{\pi \sinh[\pi(E_j - ip_j)/2]} K_4(E_j) \left( \sqrt{\mu \ell_j} \right) \right] \\
\times \left( \frac{7}{4} + E_1^2 + E_2^2 + E_3^2 + E_4^2 \right) \ tag{3.22}
\]
After working out the integral in (3.21), shrinking the loops and combining the four diagrams one obtains

\[ \langle \prod_{i=1}^{4} O(\ell_i, p_i) \rangle = \delta^{(2)}(p_1 + p_2 + p_3 + p_4) \frac{1}{\mu^2} \prod_{i=1}^{4} \Gamma(1 - |p_i|)(\frac{1}{2}\sqrt{\mu \ell_i})^{p_i} \times (|p_1 + p_2| + |p_1 + p_3| + |p_1 + p_4| - 2) \] (3.23)

This is precisely (after passing from \( p \) to \( q \) space and exchanging \( \delta^{(2)} \) for \( \delta \)) the scattering amplitude obtained from the IR-string, valid for all \( q_j \in \mathbb{R} \). Note that the cuts in (3.23) stem entirely from the reducible diagrams.

An interesting issue is whether correlation functions of macroscopic loops as well as tachyonic microscopic operators in the IR theory may always (i.e. for any number of such insertions and for arbitrary genus) be reconstructed from the ZZ theory. We have argued that this should be possible for any given case by going through target space. A more subtle issue is whether the correlators of microscopic operators can always be obtained by the simple replacements \( p_j \to q_j \) and \( \delta^{(2)} \to \delta \). In fact it is straightforward to prove that this has to work in the case of an \( n \)-point function at least as long as \( |q_1| + \ldots + |q_n| < 2 \). We claim (but have not proven) that the resulting expression can always be unambiguously continued to the whole \( q \) space.

4. Discussion

The main purpose of the present work is to establish the connection between two alternative approaches to one dimensional non-critical string theory: matrix quantum mechanics ("IR strings") and the loop gas method ("ZZ strings"). We have argued that correlators of macroscopic loops situated at fixed points in the one dimensional target space ("punctual boundary conditions") should be identical. Note that the argument given should apply to any number of macroscopic loops as well as arbitrary genus. We

\[ ^4 \text{This is very reminiscent of the diagram technique of DiFrancesco and Kutasov} \overset{[22]}{\text{. Our diagram technique is not identical, but apparently closely related.}} \]
have presented some explicit examples at genus zero. We stressed the fact that in the case of the \( \mathbb{Z} \) string the macroscopic loops are confined to sit at integer points in the target space. The momentum space used to describe the scattering of \( \mathbb{Z} \) strings is therefore compact. This restricted scattering leads to rather dramatic effects: The “contact terms” in the interactions of the \( \mathbb{R} \) strings become inobservable and all amplitudes factorize into a simple set of elementary propagators and infinitely many vertices corresponding to interactions \( \textit{local} \) in the one-dimensional target. The amplitudes of the unrestricted target space may nevertheless be reconstructed.

It would be interesting to directly prove our assertion for arbitrary \( n \)-loop correlators and any genus. This might be difficult to do using the diagram technique; a possible way to proceed might make use of a matrix model formulation of the loop gas models which has recently been constructed \cite{23}.

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