SMALL POLYNOMIALS WITH INTEGER COEFFICIENTS

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Abstract. We study the problem of minimizing the supremum norm, on a segment of the real line or on a compact set in the plane, by polynomials with integer coefficients. The extremal polynomials are naturally called integer Chebyshev polynomials. Their factors, zero distribution and asymptotics are the main subjects of this paper. In particular, we show that the integer Chebyshev polynomials for any infinite subset of the real line must have infinitely many distinct factors, which answers a question of Borwein and Erdélyi. Furthermore, it is proved that the accumulation set for their zeros must be of positive capacity in this case.

We also find the first nontrivial examples of explicit integer Chebyshev constants for certain classes of lemniscates. Since it is rarely possible to obtain an exact value of integer Chebyshev constant, good estimates are of special importance.

Introducing the methods of weighted potential theory, we generalize and improve the Hilbert-Fekete upper bound for integer Chebyshev constant. These methods also give bounds for the multiplicities of factors of integer Chebyshev polynomials, and lower bounds for integer Chebyshev constant. Moreover, all the mentioned bounds can be found numerically, by using various extremal point techniques, such as weighted Leja points algorithm. Applying our results in the classical case of the segment $[0,1]$, we improve the known bounds for the integer Chebyshev constant and the multiplicities of factors of the integer Chebyshev polynomials.

1. Integer Chebyshev problem: History and new results

Define the uniform (sup) norm on a compact set $E \subset \mathbb{C}$ by

$$\|f\|_E := \sup_{z \in E} |f(z)|.$$ 

The primary goal of this paper is the study of polynomials with integer coefficients that minimize the sup norm on the set $E$. In particular, we consider the asymptotic behavior of these polynomials and of their zeros. Let $\mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{Z})$ be the classes of algebraic polynomials of degree at most $n$, respectively with complex and with integer coefficients. The problem of minimizing the uniform norm on $E$ by
monic polynomials from $P_n(\mathbb{C})$ is well known as the Chebyshev problem (see [4], [31], [13], [10], etc.) In the classical case $E = [-1, 1]$, the explicit solution of this problem is given by the monic Chebyshev polynomial of degree $n$:

$$T_n(x) := 2^{1-n} \cos(n \arccos x), \quad n \in \mathbb{N}.$$

Using a change of variable, we can immediately extend this to an arbitrary interval $[a, b] \subset \mathbb{R}$, so that

$$t_n(x) := \left( \frac{b - a}{2} \right)^n T_n \left( \frac{2x - a - b}{b - a} \right)$$

is a monic polynomial with real coefficients and the smallest uniform norm on $[a, b]$ among all monic polynomials from $P_n(\mathbb{C})$. In fact,

$$(1.1) \quad \|t_n\|_{[a,b]} = 2 \left( \frac{b-a}{4} \right)^n, \quad n \in \mathbb{N},$$

and we find that the Chebyshev constant for $[a, b]$ is given by

$$(1.2) \quad t_C([a,b]) := \lim_{n \to \infty} \|t_n\|_{[a,b]}^{1/n} = \frac{b-a}{4}.$$

The Chebyshev constant of an arbitrary compact set $E \subset \mathbb{C}$ is defined in a similar fashion:

$$(1.3) \quad t_C(E) := \lim_{n \to \infty} \|t_n\|^E_{1/n},$$

where $t_n$ is the Chebyshev polynomial of degree $n$ on $E$. It is known that $t_C(E)$ is equal to the transfinite diameter and the logarithmic capacity $cap(E)$ of the set $E$ (cf. [13], pp. 71-75), [10] and [30] for the definitions and background material).

One may notice that the Chebyshev polynomials on the interval $[-2,2]$ have integer coefficients. The roots of the $n$-th Chebyshev polynomial on $[-2,2]$ are

$$(1.4) \quad x_k = 2 \cos \left( \frac{2k-1}{2n} \pi \right), \quad k = 1, \ldots, n.$$

A remarkable result of Kronecker [21] states that any complete set of conjugate algebraic integers, i.e., roots of a monic irreducible polynomial over $\mathbb{Z}$, all contained in $[-2,2]$, must belong to one of the sets (1.4) for some $n \in \mathbb{N}$. Thus we have an exhaustive description of all complete sets of conjugate algebraic integers in $[-2,2]$, which indicates that there are infinitely many such sets in this interval. In fact, Kronecker first proved in [21] that any complete set of conjugates on the unit circle $\{|z| = 1\}$ must be a subset of the roots of unity, and then deduced the above result by using the transformation $x = z + 1/z$. It is difficult to obtain such a complete characterization when $[-2,2]$ is replaced by a more general set, but one can extract substantial amount of interesting information from the study of integer Chebyshev problem.
An integer Chebyshev polynomial $Q_n \in \mathcal{P}_n(\mathbb{Z})$ for a compact set $E \subset \mathbb{C}$ is defined by

$$\|Q_n\|_E = \inf_{0 \not\equiv P_n \in \mathcal{P}_n(\mathbb{Z})} \|P_n\|_E,$$

where the inf is taken over all polynomials from $\mathcal{P}_n(\mathbb{Z})$, which are not identically zero. Further, the integer Chebyshev constant (or integer transfinite diameter) for $E$ is given by

$$t_{\mathbb{Z}}(E) := \lim_{n \to \infty} \|Q_n\|_E^{1/n}.$$

The existence of the limit in (1.6) follows by the same argument as for (1.3), which may be found in [16] or [43]. Note that, for any $P_n \in \mathcal{P}_n(\mathbb{Z})$,

$$\|P_n\|_E = \|P_n\|_{E^*},$$

where $E^* := E \cup \{z : \bar{z} \in E\}$, because $P_n$ has real coefficients. Thus the integer Chebyshev problem on a compact set $E$ is equivalent to that on $E^*$, and we can assume that $E$ is symmetric with respect to the real axis ($\mathbb{R}$-symmetric) without any loss of generality.

One may readily observe that if $E = [a, b]$ and $b - a \geq 4$, then $Q_n(x) \equiv 1$, $n \in \mathbb{N}$, by (1.1) and (1.6), so that

$$t_{\mathbb{Z}}([a, b]) = 1, \quad b - a \geq 4.$$

On the other hand, we obtain directly from the definition and (1.2) that

$$\frac{b - a}{4} = t_{\mathbb{C}}([a, b]) \leq t_{\mathbb{Z}}([a, b]), \quad b - a < 4.$$

Hilbert [19] proved an important upper bound

$$t_{\mathbb{Z}}([a, b]) \leq \sqrt[4]{\frac{b - a}{4}},$$

by using Legendre polynomials and Minkowski theorem on the integer lattice points in a convex body. Actually, he worked with $L_2$ norm on $[a, b]$, but this gives the same $n$-th root behavior as for $L_\infty$ norm in (1.4).

With the help of Hilbert’s result [19], Schur and Polya (see [39]) showed that any interval $[a, b] \subset \mathbb{R}$, of length less than 4, can contain only finitely many complete sets of conjugate algebraic integers. Thus one may be able to explicitly find those polynomials with integer coefficients and all roots in $[a, b]$, $b - a < 4$. These results were generalized to the case of an arbitrary compact set $E \subset \mathbb{C}$ by Fekete [9], who developed a new analytic setting for the problem, by introducing the transfinite diameter of $E$ and showing that it is equal to $t_{\mathbb{C}}(E)$. Both quantities were later proved to be equal to the logarithmic capacity $\text{cap}(E)$, by Szegő [41]. Therefore we state the result of Fekete as follows:

$$t_{\mathbb{Z}}(E) \leq \sqrt{t_{\mathbb{C}}(E)} = \sqrt{\text{cap}(E)},$$
where $E$ is $\mathbb{R}$-symmetric. It contains Hilbert’s estimate (1.9) as a special case, since $t,(a,b) = (b-a)/4$ by (1.2). Using the same argument as in [39], Fekete concluded by (1.10) that there are only finitely many complete sets of conjugate algebraic integers in any compact set $E$, satisfying $\text{cap}(E) < 1$. These ideas found many applications, but we only discuss here the developments that are closely related to the subject of this paper. Fekete and Szegő [10] showed that any open neighborhood of the set $E$, which is symmetric in real axis and has $\text{cap}(E) = 1$, must contain infinitely many complete sets of conjugates. Robinson [32] proved that any interval of length greater than 4 carries infinitely many complete sets of conjugates. But the case of intervals of length exactly 4, or sets of capacity 1, in general, remains open (for further references, see [33], [35], etc.).

The following useful observation on the asymptotic sharpness for the estimates (1.9) of Hilbert and (1.10) of Fekete is due to Trigub [42].

**Remark 1.1.** For the sequence of the intervals $I_m := [1/(m + 4), 1/m]$, we have

$$t_Z(I_m) > \frac{1}{m + 2},$$

so that

$$\lim_{m \to \infty} \left( t_Z(I_m) - \sqrt{\frac{|I_m|}{4}} \right) = 0.$$

We include a proof of this fact, due to a relative inaccessibility of the original paper [42].

The value $t_Z([a,b])$ is not known for any segment $[a,b]$, $b-a < 4$. This represents a difficult open problem, as can be seen from the study of the classical case $E = [0,1]$, which is considered below. From a more general point of view, we are able to find the exact value of $t_Z(E)$ only for a special class of compact sets, namely for lemniscates. Note that if $\text{cap}(E) \geq 1$ then the problem is trivial, because $\|P_n\|_E \geq (\text{cap}(E))^n$ for any $P_n \in \mathcal{P}_n(\mathbb{Z})$ of exact degree $n$ (cf. [30, p. 155]). This implies that

$$t_Z(E) = 1, \quad \text{if } \text{cap}(E) \geq 1.$$

**Proposition 1.2.** Let

$$V_m(z) := a_m z^m + \ldots + a_0 \in \mathcal{P}_m(\mathbb{Z}), \ a_m \neq 0.$$

Then we have for the lemniscate

$$L_r := \{ z : |V_m(z)| = r \}, \quad 0 \leq r < 1,$$

that

$$\left( \frac{r}{|a_m|} \right)^{1/m} \leq t_Z(L_r) \leq r^{1/m}.$$

This gives an immediate corollary.
Corollary 1.3. If $V_m(z)$ of $[(1.11)]$ is monic, then

$$t_Z(L_r) = r^{1/m},$$

where $L_r$ is defined in $[(1.12)]$. Furthermore, $(V_m)^k$ is an integer Chebyshev polynomial of degree $km$, $k \in \mathbb{N}$.

One may notice that $t_Z(L_r) = t_C(L_r) = \text{cap}(L_r)$ (see [30, p. 135]) in Corollary 1.3. However, the following result is more interesting.

Theorem 1.4. Suppose that the polynomial $V_m(z)$ of $[(1.11)]$ is irreducible over integers and that $L_r$ of $[(1.12)]$ satisfies $0 \leq r \leq 1/|a_m|$. Then

$$t_Z(L_r) = r^{1/m},$$

and $(V_m)^k$ is an integer Chebyshev polynomial of degree $km$, $k \in \mathbb{N}$.

Observe that $t_Z(L_r) \neq t_C(L_r) = \text{cap}(L_r) = (r/|a_m|)^{1/m}$ in this case (cf. [30, p. 135]).

A deeper insight into the nature of integer Chebyshev constant and properties of the asymptotically extremal polynomials for integer Chebyshev problem can be found in the study of this problem for $E = [0,1]$. It was initiated by Gelfond and Schnirelman, who discovered an elegant connection with the distribution of prime numbers (see [15] and Gelfond’s comments in [7, pp. 285–288]). Their argument shows that if $t_Z([0,1]) = 1/e$, then the Prime Number Theorem follows. Unfortunately, $t_Z([0,1]) > 1/e$, as we shall see below. One can find a nice exposition of this and related topics in Montgomery [22, Ch. 10] (also see Chudnovsky [8]). Let $F_n \subset P_n(\mathbb{Z})$ be the set of irreducible over $\mathbb{Z}$ polynomials, of exact degree $n$, that have all their zeros in $[0,1]$. Define

$$s := \liminf_{n \to \infty} \frac{c_1}{c_n},$$

where $F_n = c_n x^n + \ldots$. Then

$$t_Z([0,1]) \geq 1/s,$$

which is the content of Theorem 2 in [22, p. 182]. In fact, Montgomery conjectured that equality holds in $[(1.16)]$, but this remains open (essentially the same conjecture was also made in [8, p. 90]). One may try to construct various sequences of polynomials $F_n \in F_n$, $n \in \mathbb{N}$, to obtain lower bounds for $t_Z([0,1])$ from $[(1.17)]$. A few of such sequences have been devised (cf. [22] and [8]), with the best known being the Gorshkov sequence of polynomials. It was originally found by Gorshkov in [17], and rediscovered by Wirsing [22] and others. These polynomials arise as the numerators in the sequence of iterates of the rational function

$$u(x) = \frac{x(1-x)}{1-3x(1-x)},$$
and they give the following lower bound:

\[
t_Z([0,1]) \geq 1/s_0 = 0.420726 \ldots
\]

(see \cite{22} pp. 183-188).

The upper bounds for \(t_Z([0,1])\) can be obtained from the very definition of integer Chebyshev constant \((1.5)-(1.6)\). One may even try to find some low degree integer Chebyshev polynomials and compute their norms, to find out that this is quite a nontrivial exercise. It was noticed in many papers that small polynomials from \(\mathcal{P}_n(\mathbb{Z})\), \(n \in \mathbb{N}\), arise as products of powers of polynomials from \(\mathcal{F}_k\), \(k < n\). Aparicio was the first to prove this in the following strong form (cf. Theorem 3 in \cite{2}):

If a sequence \(Q_n \in \mathcal{P}_n(\mathbb{Z})\), \(n \in \mathbb{N}\), satisfies

\[
\lim_{n \to \infty} \frac{\|Q_n\|_{[0,1]}}{n} = t_Z([0,1]),
\]

then

\[
Q_n(x) = (x(1-x))^{[\alpha_1 n]}(2x-1)^{[\alpha_2 n]}(5x^2-5x+1)^{[\alpha_3 n]} R_n(x), \quad \text{as } n \to \infty,
\]

where

\[
\alpha_1 \geq 0.1456, \quad \alpha_2 \geq 0.0166 \quad \text{and} \quad \alpha_3 \geq 0.0037,
\]

and \(R_n \in \mathcal{P}_n(\mathbb{Z})\), \(n \in \mathbb{N}\).

This gives a good indication of what might be the asymptotic structure of the integer Chebyshev polynomials on \([0,1]\) and other sets. Thus Amoroso \cite{1} considered intervals with rational endpoints, and applied a refinement of Hilbert's approach in \cite{19} to the polynomials vanishing with high multiplicities at the endpoints, to improve upon \((1.9)\). Essentially the same ideas were used by Kashin \cite{20} for dealing with the symmetric intervals \([-a,a]\), in which case one should consider polynomials with factors \(x^k\).

Borwein and Erdélyi \cite{5} used numerical optimization techniques to find small polynomials of the form

\[
Q_n(x) = \prod_{i=1}^{k} Q_{m_i,n}^{[\alpha_i n]}(x), \quad 0 < \alpha_i < 1, \quad i = 1, \ldots, k,
\]

where \(Q_{m_i,n} \in \mathcal{P}_{m_i}(\mathbb{Z})\) and \(\sum_{i=1}^{k} \alpha_i m_i = 1\). They improved the upper bound for \(t_Z([0,1])\), which triggered a number of numerical studies on the integer Chebyshev polynomials for \([0,1]\) and other intervals. Borwein and Erdélyi also improved the result of Aparicio \((1.19)-(1.21)\):

\[
\alpha_1 \geq 0.26,
\]

and used this to show that the strict inequality holds in \((1.18)\). Hence the Gorshkov polynomials do not give the exact value of \(t_Z([0,1])\).

The ideas of Borwein and Erdélyi have been developed in the papers by Flamman \cite{13}, by Flamman, Rhin and Smyth \cite{14}, and by Habsieger and Salvy \cite{18}, to obtain further numerical improvements in the upper bounds for \(t_Z\) on \([0,1]\) and
on Farey intervals. In particular, Habsieger and Salvy computed 75 first integer Chebyshev polynomials for $[0, 1]$ and found the best known upper bound

$\alpha_1 \geq 0.264151, \quad \alpha_2 \geq 0.021963 \quad \text{and} \quad \alpha_3 \geq 0.005285,$

as well as bounds for six additional factors of the integer Chebyshev polynomials on $[0, 1]$. They also extended the Gorshkov polynomials technique to the Farey intervals $[p/q, r/s]$, with $qr - ps = 1$, and obtained an interesting generalization of (1.18).

From the above discussion, it is natural to expect that the integer Chebyshev polynomials for $[0, 1]$ are built out of the factors as in (1.22), which is suggested in Montgomery [22, p. 182]. In addition, Montgomery proposed to study the zero distribution of these polynomials, associated measures and extremal potentials. Potential theory indeed provides powerful methods for dealing with various extremal problems for polynomials, which proved to be very effective for classical Chebyshev polynomials, orthogonal polynomials, etc. It is clear that the study of zeros for integer Chebyshev polynomials is essentially equivalent to the study of their factors and asymptotic behavior. We should note that not all of the zeros of the integer Chebyshev polynomials for $[0, 1]$ actually lie on $[0, 1]$. This was discovered by Habsieger and Salvy [18], who found a factor of an integer Chebyshev polynomial of degree 70, with two pairs of complex conjugate roots.

One might hope that the sequence of the integer Chebyshev polynomials for $[0, 1]$ is composed from products of powers of a finite number of irreducible polynomials over $\mathbb{Z}$. Unfortunately, this is not true as we show by the following result, answering a question of Borwein and Erdélyi (see [5], Q7).

**Theorem 1.5.** Let $E \subset \mathbb{R}$ be a compact set, $\text{cap}(E) < 1$, consisting of infinitely many points. The integer Chebyshev polynomials $Q_n$ for $E$, $n \in \mathbb{N}$, have infinitely many distinct factors with integer coefficients, as $n \to \infty$.

It is obvious from the known results that integer Chebyshev polynomials are completely different from the classical companions in their “discrete” nature. However, their zeros cannot be so isolated, as it might appear.

**Theorem 1.6.** Let $Z$ be the set of accumulation points for the zeros of the integer Chebyshev polynomials for a compact set $E \subset \mathbb{R}$, $0 < \text{cap}(E) < 1$. Then

$\text{cap}(Z) > 0.$
This immediately implies that \( Z \) cannot be too small, e.g., it cannot be a countable set. One might conjecture that the zeros of the integer Chebyshev polynomials on \([0, 1]\) are dense in a Cantor-type set of positive capacity.

Since the nature of the unknown factors of the integer Chebyshev polynomials for \([0, 1]\) is rather obscure, we may view the integer Chebyshev polynomials as being of the form

\[
Q_n(x) = \left( \prod_{i=1}^{k} Q_{m_i,i}(x) \right) R_n(x), \quad n \in \mathbb{N},
\]

where \( l_i(n) \in \mathbb{N}, Q_{m_i,i}(x) \) is the known irreducible factor of degree \( m_i, i = 1, \ldots, k \), and \( R_n(x) \) is the remainder. Assuming that the limits

\[
\lim_{n \to \infty} \frac{l_i(n)}{n} =: \alpha_i > 0, \quad i = 1, \ldots, k,
\]

exist, at least along a subsequence, we observe that the \( n \)-th root of the absolute value of the product in (1.25) converges to a fixed “weight” function, as \( n \to \infty \), locally uniformly in \( \mathbb{C} \):

\[
\lim_{n \to \infty} \left( \prod_{i=1}^{k} |Q_{m_i,i}(x)|^{l_i(n)} \right)^{1/n} = \prod_{i=1}^{k} \left| Q_{m_i,i}(x) \right|^{\alpha_i},
\]

where \( \sum_{i=1}^{k} \alpha_i m_i \leq 1 \). Hence, for the purposes of studying the asymptotic behavior, as \( n \to \infty \), we may regard \( Q_n(x) \) of (1.25) as a “weighted polynomial” and use the methods of weighted potential theory \[37\]. Following this idea, we generalize the Hilbert-Fekete upper bound for \( t_Z \) and find new lower bounds. We also prove various results on the multiplicities of factors and zeros of integer Chebyshev polynomials in the next section. Then we apply the general theory to the integer Chebyshev problem on \([0, 1]\) and obtain substantial improvements over the previously known results in Section 3. Section 4 contains a brief outline of the basic facts of weighted potential theory, used in this paper. All proofs are given in Section 5.

It must be mentioned that the history of the problem as sketched here is far from being complete. Integer Chebyshev problem is closely connected to approximation by polynomials with integer coefficients (see Ferguson \[11\] and Trigub \[42\] for surveys), which has interesting history of its own. Further related topics are entire functions with integer coefficients (or integer valued) (cf. Pólya \[26\], \[27\] and \[28\], Pisot \[23\], \[24\] and \[25\], and Robinson \[34\], \[36\], etc.), integer moment problem (see Barnsley, Bessis and Moussa \[3\]), Schur-Siegel trace problem (cf. Schur \[39\], Siegel \[38\], Smyth \[40\], Borwein and Erdélyi \[5\], etc.) and many others.

2. Upper and lower bounds for integer Chebyshev constant

Motivated by the known results on the asymptotic structure of the integer Chebyshev polynomials, we study the weighted polynomials \( w^n(z) P_n(z) \), where \( w(z) \) is a continuous nonnegative function on a compact \( \mathbb{R} \)-symmetric set \( E \subset \mathbb{C} \) and
Let \( P_n \in P_n(\mathbb{Z}) \). By analogy with (1.5)-(1.6), consider the weighted integer Chebyshev polynomials \( q_n \in P_n(\mathbb{Z}) \), \( n \in \mathbb{N} \), such that

\[
v_n(E, w) := \|w^n q_n\|_E = \inf_{0 \neq P_n \in P_n(\mathbb{Z})} \|w^n P_n\|_E,
\]

and define the weighted integer Chebyshev constant by

\[
t_Z(E, w) := \lim_{n \to \infty} (v_n(E, w))^{1/n}.
\]

The limit in (2.1) exists by the following standard argument. Note that

\[
v_{k+m}(E, w) \leq \|w^{k+m} q_k q_m\|_E \leq \|w^k q_k\|_E \|w^m q_m\|_E = v_k(E, w) v_m(E, w).
\]

If we set \( a_n = \log v_n(E, w) \), then

\[
a_{k+m} \leq a_k + a_m, \quad k, m \in \mathbb{N}.
\]

Hence

\[
\lim_{n \to \infty} \frac{a_n}{n} = \lim_{n \to \infty} \log (v_n(E, w))^{1/n}
\]

exists by Lemma on page 73 of [13].

Our first goal is to give an upper bound for \( t_Z(E, w) \). It is possible to generalize the Hilbert-Fekete method for this purpose, but we also need the concept of the weighted capacity of \( E \), denoted by \( \text{cap}(E, w) \) (see [37] and a brief overview of the weighted potential theory in Section 4).

**Theorem 2.1.** Let \( E \subset \mathbb{R} \) be a compact set and let \( w : E \to [0, +\infty) \) be a continuous function. Then

\[
t_Z(E, w) \leq \sqrt{\text{cap}(E, w)}.
\]

**Remark 2.2.** If \( w(z) \equiv 1 \) on \( E \) then \( \text{cap}(E, 1) = \text{cap}(E) \), i.e., (2.2) reduces to the result of Fekete [14,10].

It is clear from Section 1 that our main applications are related to the weights of the following type:

\[
w(z) = \left( \prod_{i=1}^{k} |Q_{m,i}(z)|^{\alpha_i} \right)^{1/(1-\alpha)},
\]

where factors \( Q_{m,i} \in P_{m_i}(\mathbb{Z}) \) have the form

\[
Q_{m,i}(z) = a_i \prod_{j=1}^{m_i} (z - z_{j,i}), \quad a_i \neq 0, \quad i = 1, \ldots, k,
\]

and

\[
\alpha := \sum_{i=1}^{k} \alpha_i m_i < 1,
\]

with \( 0 < \alpha_i < 1, \quad i = 1, \ldots, k \). Thus we immediately obtain an upper bound for the classical (not weighted) integer Chebyshev constant.
Theorem 2.3. Suppose that $E \subset \mathbb{R}$ is a compact set, and that the weight $w(z)$ satisfies (2.3)-(2.5). Then

$$t_{z}(E) \leq (\text{cap}(E, w))^{(1-\alpha)/2}. \quad (2.6)$$

Theorem 2.3 suggests that we may be able to improve the results of Hilbert (1.9) and of Fekete (1.10), by using (2.6) with a proper choice of factors $Q_{m_i}$, $i = 1, \ldots, k$, for the weight $w$. It is natural to utilize the known factors of integer Chebyshev polynomials for that purpose. We shall carry out this program in the next section, and obtain an improvement of the upper bound (1.23).

It is clear that we need an effective method of finding weighted capacity, in order to make the estimate (2.6) practical. For the “polynomial-type” weights we are considering here, one can express $\text{cap}(E, w)$ through the regular logarithmic capacity and Green functions.

Theorem 2.4. Let $E \subset \mathbb{R}$ be a compact set, $\text{cap}(E) > 0$, and let $w(z)$ be as in (2.3)-(2.5). Then there exists a compact set $S_{w} \subset E \setminus \cup_{i=1}^{k} \{ z_{j,i} \}_{j=1}^{m_i}$, such that (2.6) holds with

$$\text{cap}(E, w) = \exp \left( \int \log w \, d\mu_w - F_w \right), \quad (2.7)$$

where

$$F_w = \frac{1}{\alpha - 1} \left( \log \text{cap}(S_w) + \sum_{i=1}^{k} \alpha_i \log |a_i| + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \alpha_i g_{\Omega}(z_{j,i}, \infty) \right) \quad (2.8)$$

and

$$\mu_w = \frac{1}{1 - \alpha} \left( \omega(\infty, \cdot, \Omega) - \sum_{i=1}^{k} \sum_{j=1}^{m_i} \alpha_i \omega(z_{j,i}, \cdot, \Omega) \right) \quad (2.9)$$

is the unit positive measure supported on $S_w$. Alternatively,

$$\text{cap}(E, w) = \text{cap}(S_w) \exp \left( \int \log w \, d(\omega(\infty, \cdot, \Omega) + \mu_w) \right). \quad (2.10)$$

Here, $\Omega := \mathbb{C} \setminus S_w$, $g_{\Omega}(z, \xi)$ is the Green function of $\Omega$ with pole at $\xi \in \Omega$, and $\omega(\xi, \cdot, \Omega)$ is the harmonic measure at $\xi \in \Omega$ with respect to $\Omega$.

Note that $\mu_w$ arises as the equilibrium measure in the weighted energy problem associated with the weight $w$ of (2.3)-(2.5), and $F_w$ is the modified Robin constant for that energy problem (cf. [37] and Section 4 of this paper for the details). The measure $\omega(\infty, \cdot, \Omega)$ is the classical equilibrium distribution on $S_w$, in the sense of logarithmic potential theory (see [38], [30], etc.)

Using certain information on the asymptotic behavior of integer Chebyshev polynomials, we can find lower bounds for integer Chebyshev constant, as below.
Theorem 2.5. Suppose that the integer Chebyshev polynomials of a compact set $E \subset \mathbb{C}$, $\text{cap}(E) > 0$, satisfy, along a subsequence of $n \to \infty$, 

\begin{equation}
Q_n(z) = \left( \prod_{i=1}^{k} Q_{m_i}^{l_i(n)}(z) \right) R_n(z), \quad \deg Q_n = n,
\end{equation}

where $Q_{m_i}(z) \in \mathcal{P}_{m_i}(\mathbb{Z})$, $l_i(n) \in \mathbb{N}$, and the limits 

\begin{equation}
\lim_{n \to \infty} \frac{l_i(n)}{n} =: \alpha_i > 0, \quad i = 1, \ldots, k,
\end{equation}

exist. Then 

\begin{equation}
t_Z(E) \geq e^{(\alpha - 1)F_w},
\end{equation}

where $F_w$ is the modified Robin constant for the weight $w$ of (2.3) and $\alpha$ is given by (2.5). 

Moreover, if $E \subset \mathbb{R}$ then 

\begin{equation}
t_Z(E) \geq \text{cap}(S_w) \prod_{i=1}^{k} |a_i|^{\alpha_i} \exp \left( \sum_{i=1}^{k} \sum_{j=1}^{m_i} \alpha_i g_\Omega(z_{j,i}, \infty) \right),
\end{equation}

in the notations of Theorem 2.4.

Theorem 2.5 is an easy consequence of the results in weighted potential theory and a simple fact that the leading coefficient of $R_n(z)$ is at least 1 in absolute value, being a nonzero integer. It turns out that we can obtain better lower bounds for $t_Z(E)$, by using rational points. One of the possible results in this direction is given below. Recall that the logarithmic potential of a Borel measure $\mu$ is defined by 

$$U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu(t).$$

Theorem 2.6. Assume that the integer Chebyshev polynomials of $E$, $\text{cap}(E) > 0$, satisfy (2.3) and (2.5), where $R_n(\zeta) \neq 0$ for a point $\zeta \in \mathbb{C}$, along a subsequence of $n \to \infty$. If $\zeta = (p_1 + ip_2)/q$ is a complex rational number in reduced form, i.e., $\gcd(p_1, p_2, q) = 1$, then 

\begin{equation}
t_Z(E) \geq q^{\alpha - 1} \exp ((\alpha - 1)(F_w - U^\mu_w(\zeta))),
\end{equation}

where $F_w$ is the modified Robin constant and $\mu_w$ is the weighted equilibrium distribution associated with the weight $w$ of (2.3)-(2.5). For $\zeta = 0$, we set $q = 1$ in (2.16).

Estimate (2.15) has interesting applications in the “opposite” direction, as it can be used to improve the bounds for the multiplicities of the known factors of integer Chebyshev polynomials. Thus we can deduce the “asymptotic structure” result from the upper bound for $t_Z(E)$, as an immediate corollary of Theorem 2.6.
Corollary 2.7. Suppose that the assumptions of Theorem 2.6 are satisfied and that $t_Z(E) \leq M$.

Then the set of multiplicities $\{\alpha_i\}_{i=1}^k$ must satisfy

\begin{equation}
q^{\alpha - 1} \exp((\alpha - 1)(F_w - U^\mu(\zeta))) \leq M.
\end{equation}

This inequality defines a domain for the possible values of $\alpha_i$, $i = 1, \ldots, k$, which allows to significantly improve the known bounds for $\alpha_i$'s.

Another immediate, but nontrivial, consequence of the weighted potential theory is the following fact.

Proposition 2.8. Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$. Suppose that the integer Chebyshev polynomials for $E$ satisfy (2.11) and (2.12), along a subsequence of $n \to \infty$. Then there exists $\varepsilon > 0$, so that

\begin{equation}
t_Z(E) = t_Z(E \cup H_\varepsilon),
\end{equation}

where

\[ H_\varepsilon = \bigcup_{i=1}^k \bigcup_{j=1}^{m_i} \{z : |z - z_{j,i}| \leq \varepsilon\}. \]

Perhaps, the most interesting application of our general results, developed in this section, is the classical case $E = [0, 1]$. Therefore, we concentrate on its study below, to demonstrate the strength of the method.

3. INTEGER CHEBYSHEV PROBLEM ON $[0, 1]$

We remind that the best known bounds for $t_Z([0, 1])$, as mentioned in (1.18) and (1.23), are as follows:

\[ 0.42072638... < t_Z([0, 1]) \leq 0.42347945. \]

The above lower bound was believed to be the precise value of $t_Z([0, 1])$, but Borwein and Erdélyi showed that there must be the strict inequality. However, they did not give a numerical value for the improvement in the lower bound. Using the general methods of Section 2, based on weighted potential theory, we show here that

Theorem 3.1.

\[ 0.4213 < t_Z([0, 1]) < 0.4232. \]

It is convenient for technical reasons to use the symmetry of $[0, 1]$ and the standard change of variable $x(1-x) \to z$, which reduces the integer Chebyshev problem on $[0, 1]$ to that on $[0, 1/4]$:

\begin{equation}
(t_Z([0, 1]))^2 = t_Z([0, 1/4]).
\end{equation}
Furthermore, we have by Lemmas 1-2 of [18] that the integer Chebyshev polynomials for \([0,1]\) and \([0,1/4]\) are related by

\[
Q_{2k}(x) = q_k(x(1 - x))
\]

and

\[
Q_{2k+1}(x) = (1 - 2x)q_k(x(1 - x)).
\]

Hence we can study the integer Chebyshev problem on \([0,1/4]\), and then return to \([0,1]\) without any loss of information.

Habsieger and Salvy [18] give the following list of known factors of the integer Chebyshev polynomials for \([0,1]\):

\[
\begin{align*}
A_1(x) &= x(1 - x), & A_2(x) &= 2x - 1, & A_3(x) &= 5x^2 - 5x + 1, \\
A_4(x) &= 6x^2 - 6x + 1, & A_5(x) &= 29x^4 - 58x^3 + 40x^2 - 11x + 1, \\
A_6(x) &= (13x^3 - 20x^2 + 9x - 1)(13x^3 - 19x^2 + 8x - 1), \\
A_7(x) &= (31x^4 - 63x^3 + 44x^2 - 12x + 1)(31x^4 - 61x^3 + 41x^2 - 11x + 1), \\
A_8(x) &= 4921x^{10} - 24605x^9 + 53804x^8 - 67586x^7 + 53866x^6 - 28388x^5 \\
&\quad + 9995x^4 - 2317x^3 + 338x^2 - 28x + 1.
\end{align*}
\]

Incidentally, \(A_8(x)\) is the “surprise factor” with four non-real zeros. Changing the variable to \(z = x(1 - x)\), we obtain the following factors for \([0,1/4]\):

\[
\begin{align*}
Q_{1,1}(z) &= z, & Q_{1,2}(z) &= A_2^2(z) = 4z - 1, & Q_{1,3}(z) &= 5z - 1, \\
Q_{1,4}(z) &= 6z - 1, & Q_{2,5}(z) &= 29z^2 - 11z + 1, \\
Q_{3,6}(z) &= 169z^3 - 94z^2 + 17z - 1, \\
Q_{4,7}(z) &= 961z^4 - 712z^3 + 194z^2 - 23z + 1, \\
Q_{5,8}(z) &= 4921z^5 - 4594z^4 + 1697z^3 - 310z^2 + 28z - 1.
\end{align*}
\]

Exactly these factors will be used in the definition of the weight \(w\) of (2.3) for the applications of the results from Section 2. We start with the case of two factors \(Q_{1,1}\) and \(Q_{1,2}\), vanishing at the endpoints of \([0,1/4]\), where all the parameters of the corresponding weighted potential theory can be found explicitly.

### 3.1. Two factors on \([0,1/4]\).

Note that if the relative multiplicities for the factors \(A_1(x) = x(1 - x)\) and \(A_2(x) = 2x - 1\) in the integer Chebyshev polynomial on \([0,1]\), are \(\alpha_1\) and \(\alpha_2\), then the relative multiplicities for the corresponding factors \(Q_{1,1}(z) = z\) and \(Q_{1,2}(z) = 4z - 1\) on \([0,1/4]\) are \(2\alpha_1\) and \(\alpha_2\) (see [18]). Thus we define the weight \(w\) according to (2.3):

\[
w(x) = (x^{2\alpha_1}(1 - 4x)^{\alpha_2})^{1/(1 - 2\alpha_1 - \alpha_2)}, \quad x \in [0,1/4],
\]

where \(\alpha_1, \alpha_2 > 0\) and \(2\alpha_1 + \alpha_2 < 1\). The needed quantities of the weighted potential theory are contained in the following lemma.
Lemma 3.2. For the weight $w$ of (3.7), we have that $S_w = [a, b] \subset (0, 1/4)$, with

\[(3.6) \quad a := (4\alpha_2^2 - \alpha_2^2 - \sqrt{\Delta} + 1)/8 \quad \text{and} \quad b := (4\alpha_2^2 - \alpha_2^2 + \sqrt{\Delta} + 1)/8,
\]

where $\Delta := (1 - (2\alpha_1 + \alpha_2)^2)(1 - (2\alpha_1 - \alpha_2)^2)$. Furthermore,

\[(3.7) \quad F_w = \frac{1 - \alpha_2}{1 - 2\alpha_1 - \alpha_2} \log 4 - \log(b - a) - \frac{4\alpha_1}{1 - 2\alpha_1 - \alpha_2} \log(\sqrt{a} + \sqrt{b})
\]
\[\quad - \frac{2\alpha_2}{1 - 2\alpha_1 - \alpha_2} \log(\sqrt{1/4 - a} + \sqrt{1/4 - b}),\]

\[(3.8) \quad d\mu_w(x) = \frac{\sqrt{(x - a)(b - x)}}{x(1 - 2\alpha_1 - \alpha_2)x(1/4 - x)} \, dx, \quad x \in [a, b],
\]

and

\[(3.9) \quad F_w - U_{w}^\mu(z) = (g_2(z, \infty) - 2\alpha_1 (\log |z| + g_3(z, 0))
\]
\[\quad - \alpha_2 (\log |4z - 1| + g_3(z, 1/4)))/(1 - 2\alpha_1 - \alpha_2),
\]

where we use the notation of Theorem 2.3.

Note that the function in (3.9) is continuous in $\mathbb{C}$ and harmonic in $\mathbb{C}\setminus [a, b]$. The weighted capacity $cap([0, 1/4], w)$ is found from (2.7) or (2.10), with the help of (3.4)-(3.8). Obviously, $cap([0, 1/4], w)$ is a function of two variables $\alpha_1$ and $\alpha_2$, defined on the triangle $T := \{\alpha_1, \alpha_2 > 0 : 2\alpha_1 + \alpha_2 < 1\}$. Thus we obtain from Theorem 2.8 via computation, that

\[t_z([0, 1/4]) \leq \inf_{\alpha_1, \alpha_2 \in T} (cap([0, 1/4], w))^{(1 - 2\alpha_1 - \alpha_2)/2} \approx 0.18043338.
\]

The bound is attained for $\alpha_1 \approx 0.290447$ and $\alpha_2 \approx 0.09$, which matches the result of [P] p. 906. But this upper bound is greater than the one in [1229], by 333, so that it is not interesting for us.

We can also apply Theorem 2.3 here, with $\zeta_1 = 0$ and $\zeta_2 = 1/4$, because these zeros are absorbed by the weight $w$. Hence we have two simultaneous lower bounds

\[t_z([0, 1/4]) > l_1(\alpha_1, \alpha_2) := \exp((2\alpha_1 + \alpha_2 - 1)(F_w - U_{w}^\mu(0)))
\]

and

\[t_z([0, 1/4]) > l_2(\alpha_1, \alpha_2) := 4^{2\alpha_1 + \alpha_2 - 1} \exp((2\alpha_1 + \alpha_2 - 1)(F_w - U_{w}^\mu(1/4))),
\]

for $\alpha_1, \alpha_2 \in T$. It follows that

\[t_z([0, 1/4]) \geq \inf_{\alpha_1, \alpha_2 \in T} \max(l_1(\alpha_1, \alpha_2), l_2(\alpha_1, \alpha_2)) \approx 0.1760565,
\]

where the numerical value, attained for $\alpha_1 \approx 0.330333$ and $\alpha_2 \approx 0.128$, is found by using (3.9) and computations. Again, this lower bound is weaker than (1.18).

However, the application of Corollary 2.7 with the upper bound $M$ obtained from (1.23) and (3.1), gives an interesting new result (also see [29]). We translate it to $[0, 1]$ setting here.
Theorem 3.3. The integer Chebyshev polynomials $\{Q_n\}_{n=1}^{\infty}$ on $[0, 1]$ satisfy
\begin{equation}
Q_n(x) = (x(1-x))^{[\alpha_1 n]}(2x - 1)^{[\alpha_2 n]}R_n(x), \quad \text{as } n \to \infty,
\end{equation}
where
\begin{equation}
0.2961 \leq \alpha_1 \leq 0.3634 \quad \text{and} \quad 0.0952 \leq \alpha_2 \leq 0.1767,
\end{equation}
and $R_n \in P_n(\mathbb{Z})$, $n \in \mathbb{N}$. Furthermore, the pair $(\alpha_1, \alpha_2)$ must belong to the region $G$ pictured below in Figure 1, which is determined by the inequalities
\[
\exp \left( (2\alpha_1 + \alpha_2 - 1)(F_w - U^\mu(0)) \right) < 0.179335
\]
and
\[
4^{2\alpha_1 + \alpha_2 - 1} \exp \left( (2\alpha_1 + \alpha_2 - 1)(F_w - U^\mu(1/4)) \right) < 0.179335.
\]

Figure 1. Region $G$ for $\alpha_1$ and $\alpha_2$.

Note that, in addition to improving the previous lower bounds obtained in [2], [5] and [14], (3.11) also gives the upper bounds for $\alpha_1$ and $\alpha_2$.

3.2. Three and more factors on $[0, 1/4]$; Numerical approach. It is natural to expect improvements in the bounds for $t_Z([0, 1/4])$ and for $\alpha_i$'s, if we use three or more known factors from (3.4). There is, however, a substantial difficulty arising on our way. Although Theorem 2.4 can still produce the needed quantities of weighted
potential theory, it assumes the knowledge of the set \( S_w \). In fact, when \( w \) is defined by (2.3)-(2.5), with the help of the factors (3.12)

\[
Q_{1,1}(z) = z, \quad Q_{1,2}(z) = 4z - 1 \quad \text{and} \quad Q_{1,3}(z) = 5z - 1,
\]

we have that \( S_w = [a_1, b_1] \cup [a_2, b_2] \), where \([a_1, b_1] \subset (0, 1/5)\) and \([a_2, b_2] \subset (1/5, 1/4)\). But the endpoints of the intervals \([a_1, b_1]\) and \([a_2, b_2]\) are unknown functions of the multiplicities \((\alpha_1, \alpha_2, \alpha_3)\). The problem becomes even more complicated, if we consider further factors listed in (3.4). Fortunately, we can apply the numerical methods for finding \( S_w \) and the weighted equilibrium measure \( \mu_w \), based on weighted Leja points (see Section V.1 of [37]). Weighted Leja points are easy to generate numerically, as they are defined by the following simple recursive procedure. For a general compact set \( E \) and \( w \) of (2.3)-(2.5), let \( a_0 \in E \) be a point such that

(3.13)

\[|a_0| w(a_0) = \|zw(z)\|_E.\]

Given the points \( \{a_i\}_{i=0}^{n-1} \), we define the weighted Leja polynomial

(3.14)

\[L_n(z) = \prod_{i=0}^{n-1}(z - a_i),\]

so that \( a_n \) is found as a point satisfying

(3.15)

\[w^n(a_n) |L_n(a_n)| = ||w^nL_n||_E, \quad a_n \in E.\]

Of course, the choice of \( a_n \) might not be unique. The fundamental property of weighted Leja points is that they give a discrete approximation to the weighted equilibrium measure \( \mu_w \), corresponding to the weight \( w \) on \( E \), \( \text{cap}(E) > 0 \). This is stated in the best way by using the weak* convergence of measures:

(3.16)

\[
\tau_n := \frac{1}{n+1} \sum_{i=0}^{n} \delta_{a_i} \rightharpoonup \mu_w, \quad \text{as} \quad n \to \infty,
\]

where \( \delta_{a_i} \) is the unit point mass at \( a_i, \ i = 0, 1, 2, \ldots \) (see Theorem V.1.1 in [37]). Furthermore, Theorem V.1.2 of [37] gives that

(3.17)

\[
\lim_{n \to \infty} w(a_n)|L_n(a_n)|^{1/n} = e^{-F_w}.
\]

It follows from Theorem III.2.1 and Remark III.2.2 of [37] (also cf. [44]) that \( \{a_i\}_{i=0}^{\infty} \subset S_w^* \), where \( S_w^* \supset S_w \) is a compact set defined by

(3.18)

\[S_w^* := \{z \in E : U^\mu_w(z) - \log w(z) \leq F_w\}.
\]

One can immediately see from (3.18) and the form of \( w \) in (2.3)-(2.5), that

(3.19)

\[S_w^* \subset E \setminus \bigcup_{i=1}^{k} \{z_{j,i} \}_{j=1}^{m_i}.
\]
Thus we obtain from (3.10), (3.17) and the definition of weak* convergence that

\[
\exp(F_w - U^{w}(\zeta)) = \lim_{n \to \infty} \frac{1}{w(a_n)} \left| \frac{L_n(\zeta)}{L_n(a_n)} \right|^{1/n}, \quad \zeta \in \mathbb{C} \setminus S_w^*.
\]

Similarly, we have from (2.7), (3.16), (3.17) and (3.19) that

\[
\text{cap}(E, w) = \lim_{n \to \infty} w(a_n) \left( \left| L_n(a_n) \right| \prod_{i=0}^{n-1} w(a_i) \right)^{1/n}.
\]

Equations (3.20) and (3.21) give a straightforward way of computing all quantities of weighted potential theory, necessary for the applications of our results from Section 2.

We now proceed in the same fashion as in the case of two factors, by defining the weight

\[
w(x) = \left( x^{2\alpha_1} |4x - 1|^{2\alpha_2} |5x - 1|^{2\alpha_3} \right)^{1/(1 - 2\alpha_1 - 2\alpha_2 - 2\alpha_3)}, \quad x \in [0, 1/4],
\]

where

\[
(\alpha_1, \alpha_2, \alpha_3) \in T := \{2\alpha_1 + \alpha_2 + 2\alpha_3 < 1\} \cap \mathbb{R}_+^3.
\]

Theorem 2.6 applies here with \(\zeta_1 = 0\), \(\zeta_2 = 1/4\) and \(\zeta_3 = 1/5\), so that we have the corresponding values \(q_1 = 1\), \(q_2 = 4\) and \(q_3 = 5\) for (3.10). Each \(q_i\) gives a lower bound

\[
l_i(\alpha_1, \alpha_2, \alpha_3) := q_i^{2\alpha_1 + \alpha_2 + 2\alpha_3 - 1} e^{(2\alpha_1 + \alpha_2 + 2\alpha_3 - 1) (F_w - U^{w}(\zeta_i))}, \quad i = 1, 2, 3,
\]

by (2.15), so that

\[
t_z([0, 1/4]) \geq \inf_T \max_i l_i(\alpha_1, \alpha_2, \alpha_3) > 0.1775.
\]

This numerical lower bound was found by using (3.20) and a simple C code for generating weighted Leja points. It should be noted that the most time consuming part of the computation is finding of the above inf, which is done by a search over a discrete lattice in \(T\). The lower bound of Theorem 3.1 follows at once from (3.1).

Using the upper bound \(M = 0.179335\) in Corollary 2.7, as in the two-factor case, and taking advantage of the ready numerical results on computing (3.22), we find the following improved bounds for \(\alpha_1\), \(\alpha_2\) and \(\alpha_3\).

**Theorem 3.4.** The integer Chebyshev polynomials \(\{Q_n\}_{n=1}^{\infty}\) on \([0, 1]\) satisfy

\[
Q_n(x) = (x(1-x))^{\lfloor \alpha_1 n \rfloor} (2x - 1)^{\lfloor \alpha_2 n \rfloor} (5x^2 - 5x + 1)^{\lfloor \alpha_3 n \rfloor} R_n(x), \quad \text{as } n \to \infty,
\]

where

\[
0.31 \leq \alpha_1 \leq 0.34, \quad 0.11 \leq \alpha_2 \leq 0.14 \quad \text{and} \quad 0.035 \leq \alpha_3 \leq 0.057,
\]

and \(R_n \in \mathcal{P}_n(\mathbb{Z}), \quad n \in \mathbb{N}\).
The upper bound in Theorem 3.1 was obtained by using all eight factors of (3.4) to find an upper bound for $t_Z([0, 1/4])$, with the help of the weight

$$w(x) = \left( \prod_{i=1}^{8} |Q_{m_{i,j}}(z)|^{\beta_i} \right)^{1/(1-\sum_{i=1}^{8} \beta_i)}.$$ 

If we choose the following set of values for $\beta_i$'s

$$(0.625, 0.11, 0.07, 0.0032, 0.0302, 0.0112, 0.0048, 0.00094),$$

then the upper bound

$$t_Z([0, 1/4]) < 0.1791$$

is easily found from Theorem 2.3 and (3.21) via another computation, involving weighted Leja points. This implies by (3.1) that

$$t_Z([0, 1]) < 0.4232,$$

as claimed in Theorem 3.1.

One can further improve the numerical results on the lower bound for $t_Z([0, 1])$ and the bounds for $\alpha_i$'s, by considering four and more factors from (3.4). The upper bound of Theorem 3.1 can also be improved by optimizing the choice of $\beta_i$'s. All the details for computations and suggested improvements of numerical results will be published separately.

4. Weighted capacity and potentials

We give a brief description of the basic facts from the potential theory with external fields, or weighted potential theory, for the convenience of the reader. One should consult Saff and Totik [37] for a complete exposition including the history of this subject.

With $\mathcal{M}(E)$ denoting the class of all positive Borel measures $\mu$ on $\mathbb{C}$ such that $\mu(\mathbb{C}) = 1$ and supp $\mu \subseteq E$, consider the following weighted energy problem (cf. [37, Section I.1]):

For the weighted energy integral

$$(4.1) \quad I_w(\mu) := \int \int \log \frac{1}{|z - t|w(z)w(t)} \, d\mu(z) d\mu(t), \quad \mu \in \mathcal{M}(E),$$

find

$$(4.2) \quad V_w := \inf_{\mu \in \mathcal{M}(E)} I_w(\mu),$$

and identify the extremal measures, if the infimum in (4.2) is attained.

The following is a special case of Theorem I.1.3 in [37].

**Proposition 4.1.** Let $w : E \to [0, +\infty)$ be a continuous function on a compact set $E \subset \mathbb{C}$ such that $\text{cap}(\{z \in E : w(z) > 0\}) > 0$. Then

(a) $V_w$ of (4.2) is finite:
(b) there exists a unique $\mu_w \in \mathcal{M}(E)$ such that $I_w(\mu_w) = V_w$;

(c) $U^\mu_w(z) - \log w(z) \geq F_w$, for quasi every $z \in E$;

d) $U^\mu_w(z) - \log w(z) \leq F_w$, $z \in S_w$,

where $S_w := \text{supp} \mu_w$ and $F_w := V_w + \int \log w(t) d\mu_w(t)$.

By saying in (c) that a property holds quasi everywhere (q.e.), we mean that it holds everywhere, with the possible exception of a set of zero logarithmic capacity (cf. [37, Sec. I.1]). The weighted capacity of $E$ is then defined by

$$\text{cap}(E, w) := e^{-V_w}.$$  

In the case $\text{cap}(\{z \in E : w(z) > 0\}) = 0$, we set $\text{cap}(E, w) = 0$.

It will become clear from the proofs that the $n$-th root asymptotic behavior of integer Chebyshev polynomials is essentially equivalent to that of the weighted polynomials, for $w$ given by (2.3)-(2.5). Therefore, weighted potential theory provides useful tools for the study of integer Chebyshev problem, such as the following proposition (see Theorem III.2.1 and Corollary III.2.6 in [37]).

**Proposition 4.2.** Suppose that the assumptions of Proposition 4.1 are satisfied. Then, for any polynomial $P_n \in \mathcal{P}_n(\mathbb{C})$, we have

$$|w^n(z)P_n(z)| \leq \|w^n P_n\|_{S_w} \exp(n(F_w - U^\mu_w(z) + \log w(z))), \quad z \in \mathbb{C}. \quad (4.4)$$

Assume further that for every point $z \in E$, the set $\{t : |t - z| < \delta, t \in E\}$ has positive capacity for any $\delta > 0$. Then

$$\|w^n P_n\|_E = \|w^n P_n\|_{S_w}. \quad (4.5)$$

5. **Proofs**

The following proof is found in Trigub [42, p. 316].

**Proof of Remark 1.1** Consider the Chebyshev polynomials for $[-2, 2]$, given by

$$t_n(x) = 2 \cos(n \arccos(x/2)) = 2^{-n} \left((x + \sqrt{x^2 - 4})^n + (x - \sqrt{x^2 - 4})^n\right), \quad n \in \mathbb{N}. \quad (5.1)$$

We already observed in Section 1 that $t_n(x)$ is a monic polynomial with integer coefficients, whose roots are given by (1.3). Schur showed that if $n = p$ is a prime number, then $t_p(x)/x$ is irreducible over integers (see [31, p. 228]). Hence the numbers

$$2 \cos \frac{(2k - 1)\pi}{2p}, \quad k = 1, \ldots, p; \quad k \neq \frac{p + 1}{2},$$

form a complete set of $p - 1$ conjugate algebraic integers in $[-2, 2]$, for any prime $p$. It is clear that the corresponding roots $\{b_k\}_{k=1}^{p-1}$ of $F_{p-1}(x) = t_p(x - m - 2)/(x - m - 2)$, obtained by shifting the above set by $m + 2$, form a complete set of conjugates on
Let \( Q_{p-2} \) be an integer Chebyshev polynomial of degree \( p - 2 \) for \([1/(m + 4), 1/m]\). Note that

\[
Q_{p-2} \left( \frac{1}{b_k} \right) = \frac{\tilde{Q}_{p-2}(b_k)}{b_k^{p-2}} \neq 0, \quad k = 1, \ldots, p - 1,
\]

where \( \tilde{Q}_{p-2} \in P_{p-2}(\mathbb{Z}) \). Indeed, if \( \tilde{Q}_{p-2}(b_k) = 0 \) for just one \( k \), then this must be true for all \( k = 1, \ldots, p - 1 \), i.e., \( \tilde{Q}_{p-2} \equiv 0 \). Since the product \( \prod_{k=1}^{p-1} \tilde{Q}_{p-2}(b_k) \) is a symmetric form in \( b_k \)'s with integer coefficients, it may be written as a polynomial in the elementary symmetric functions of \( b_k \)'s, with integer coefficients, by the fundamental theorem on symmetric forms. Thus the above product must be a nonzero integer, so that we have

\[
\left| \prod_{k=1}^{p-1} Q_{p-2} \left( \frac{1}{b_k} \right) \right| = \left| \prod_{k=1}^{p-1} \tilde{Q}_{p-2}(b_k) \right| \geq \frac{1}{\prod_{k=1}^{p-1} b_k^{p-2}}
\]

and

\[
(5.2) \quad \|Q_{p-2}\|_{[1/(m+4), m]}^{p-1} \geq \frac{1}{\prod_{k=1}^{p-1} b_k^{p-2}}.
\]

Observe that

\[
\left| \prod_{k=1}^{p-1} b_k \right| = |F_{p-1}(0)| = \left| \frac{t_{p,m}(m + 2)}{m + 2} \right| \leq \frac{(m + 2 + \sqrt{(m + 2)^2 - 4})^p}{2^{p-1}(m + 2)},
\]

where the last inequality follows from (5.1). Combining (5.2) with the above estimate and (1.6), we obtain that

\[
t_Z([1/(m + 4), 1/m]) \geq \frac{2}{m + 2 + \sqrt{(m + 2)^2 - 4}} > \frac{1}{m + 2}.
\]

□

Proof of Proposition 1.2. For the sequence of polynomials \( V^k_m(z) \), \( k \in \mathbb{N} \), we have

\[
t_Z(L_r) \leq \lim_{k \to \infty} \|V^k_m\|_{L_r}^{km} = r^{1/m}.
\]

Thus the upper bound in (1.13) follows. Suppose that \( P_l \in P_l(\mathbb{Z}) \) has a leading coefficient \( b_l \neq 0 \). Then we estimate

\[
\|P_l\|_{L_r}^{1/l} = |b_l|^{1/l} \|z^l + \ldots\|_{L_r}^{1/l} \geq \text{cap}(L_r) = (r/|a_m|)^{1/m},
\]

by [30, p. 155 and p. 135], which gives the lower bound of (1.13). □
Proof of Corollary 1.3. Since $a_m = 1$, (1.14) follows at once from (1.13). Furthermore, if $P_l \in \mathcal{P}_{km}(\mathbb{Z})$ is of exact degree $l$, with the leading coefficient $b_l \neq 0$, then
\[
\|P_l\|_{L_r} = |b_l| \|z^l + \ldots\|_{L_r} \geq (\operatorname{cap}(L_r))^l = r^{l/m} \geq r^k = \|V_m^k\|_{L_r},
\]
where we used [30, p. 155]. Hence $V_m^k(z)$ is an integer Chebyshev polynomial of degree $km$ on $L_r$, $k \in \mathbb{N}$.

Proof of Theorem 1.4. If $V_m^k(z)$ is an integer Chebyshev polynomial of degree $km$ on $L_r$, for any $k \in \mathbb{N}$, then (1.15) is immediate from the definition (1.6). Therefore, we only need to prove the second statement of the theorem. It is trivial for $r = 0$, so that we assume $r \in (0, 1/|a_m|]$. Suppose to the contrary that there exists a polynomial $P_l \in \mathcal{P}_{km}(\mathbb{Z})$, of exact degree $l$, such that
\[
\|P_l\|_{L_r} < \|V_m^k\|_{L_r} = r^k.
\]
Let $z_i, i = 1, \ldots, m$, be the zeros of $V_m$. Clearly, all $z_i$’s are inside $L_r$, so that we have by the maximum principle
\[
|P_l(z_i)| \leq \|P_l\|_{L_r} < r^k, \quad i = 1, \ldots, m.
\]
Using a known argument based on the fundamental theorem of symmetric forms (see Lemma in [22, p. 181]), we obtain that
\[
N = a_m^l \prod_{i=1}^m P_l(z_i) \in \mathbb{Z}.
\]
On the other hand, we estimate
\[
\left| a_m^l \prod_{i=1}^m P_l(z_i) \right| \leq |a_m|^l \|P_l\|_{L_r}^m < |a_m|^{km} r^{km} \leq 1.
\]
Consequently, this integer $N$ is equal to zero, which means that $P_l(z_i) = 0$ for some $i$. But then the irreducible polynomial $V_m$ must divide $P_l$.

Assume that $P_l(z) = V_m^d(z)R(z)$, where $d \in \mathbb{N}$ and $R \in \mathcal{P}_{m(k-d)}(\mathbb{Z})$, of exact degree $l - md$, does not have $V_m$ as a factor. It follows that
\[
|R(z)| = |P_l(z)|/|V_m(z)|^d < r^{k-d}, \quad z \in L_r,
\]
and
\[
\|R\|_{L_r} < r^{k-d}.
\]
Hence we can use the same argument for $R$, to conclude that
\[
a_m^{l-md} \prod_{i=1}^m R(z_i) = 0.
\]
This implies that $V_m$ divides $R$, contradicting our assumption. \qed
Before giving the proof of Theorem 1.5, we need to state two lemmas. The first one shows that if a sequence of polynomials is composed of only finitely many factors, then the $n$-th root behavior of this sequence can be essentially described by a fixed “polynomial-power” function.

**Lemma 5.1.** Suppose that all polynomials $P_n \in \mathbb{P}_n(\mathbb{C})$, of exact degrees $n \in \mathbb{N}$, have finitely many distinct factors $P_{m,i} \in \mathbb{P}_{m}(\mathbb{C})$, $i = 1, \ldots, K$. If, for a compact set $E \subset \mathbb{C}$,

$$\lim_{n \to \infty} \|P_n\|_E^{1/n} = A,$$

then there exist $\alpha_i \in (0, 1]$, $i = 1, \ldots, k \leq K$, such that

$$\left\| \prod_{i=1}^k |P_{m,i}|^{\alpha_i} \right\|_E = A,$$

where $\sum_{i=1}^k \alpha_i m_i = 1$.

**Proof.** We begin by choosing an increasing subsequence $n_j \in \mathbb{N}$, $j = 1, 2, \ldots$, such that

$$\lim_{j \to \infty} \frac{l_i(n_j)}{n_j} =: \alpha_i, \quad i = 1, \ldots, K,$$

where $l_i(n_j)$ is the power of the factor $P_{m,i}$ in $P_{n_j}$. Clearly, $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, K$, and $\sum_{i=1}^K \alpha_i m_i = 1$. We may assume that

$$0 < \alpha_i \leq 1, \quad i = 1, \ldots, k, \quad \text{and} \quad \alpha_i = 0, \quad i = k + 1, \ldots, K.$$

Our goal is to show that the factors with $\alpha_i = 0$ do not have influence on the $n$-th root behavior for the norms of the sequence. If $z$ is not a zero of $P_{m,i}$, $i = k + 1, \ldots, K$, then

$$\lim_{j \to \infty} |P_{n_j}(z)|^{1/n_j} = \lim_{j \to \infty} \prod_{i=1}^K |P_{m,i}(z)|^{l_i(n_j)/n_j} = \prod_{i=1}^k |P_{m,i}(z)|^{\alpha_i},$$

where convergence in the above equation is uniform on compact subsets of $\mathbb{C} \setminus \{z : P_{m,i}(z) = 0, \ i = k + 1, \ldots, K\}$. Hence

$$(5.3) \quad \prod_{i=1}^k |P_{m,i}(z)|^{\alpha_i} \leq A,$$

for any $z \in E$, with finitely many exceptions. But the function on the left of (5.3) is continuous, so that

$$\left\| \prod_{i=1}^k |P_{m,i}|^{\alpha_i} \right\|_E \leq A.$$
It is easy to obtain the opposite inequality from

\[ A = \lim_{j \to \infty} \left\| P_{n_j} \right\|_{E}^{1/n_j} \leq \lim_{j \to \infty} \left\| \prod_{i=1}^{k} |P_{m_{i,i}}|_{E}^{l_{i}(n_j)/n_j} \right\|_{E} \lim_{j \to \infty} \prod_{i=k+1}^{K} \left\| P_{m_{i,i}} \right\|_{E}^{l_{i}(n_j)/n_j} = \left\| \prod_{i=1}^{k} |P_{m_{i,i}}|^{\alpha_i} \right\|_{E}. \]

\[ \square \]

The following fact is intuitively obvious.

**Lemma 5.2.** Assume that \( P_{m_{i,i}} \in \mathcal{P}_{m_i}(\mathbb{Z}), \ i = 1, \ldots, k. \) For any \( A > 0 \) and any set of exponents \( \alpha_i > 0, \ i = 1, \ldots, k, \) the equation

\[ \prod_{i=1}^{k} |P_{m_{i,i}}(x)|^{\alpha_i} = A \]

has only finitely many solutions on the real line.

**Proof.** Suppose that this is not the case, and there are infinitely many real solutions of the above equation. Since the function \( f(x) \) on the left hand side grows indefinitely, as \( x \to \pm \infty, \) all these solutions are contained in a bounded open interval \( I. \) Note that in this case \( f(x) \) must have infinitely many points of local maximum in \( I \), with at least one point of accumulation \( x_0 \in I. \) Let \( x_n \in I, \ n \in \mathbb{N}, \) be the sequence of maxima for \( f(x) \) in \( I, \) such that \( \lim_{n \to \infty} x_n = x_0 \) and \( f(x_n) \geq A, \ n \in \mathbb{N}. \) Observing that \( f(x_0) \geq A > 0, \) we conclude that there exists a 2-dimensional neighborhood \( \Delta \) of \( x_0, \) free of zeros of \( P_{m_{i,i}}, \ i = 1, \ldots, k. \) Hence \( f(z) \) can be defined as a single valued analytic function in \( \Delta, \) which is real valued on \( \Delta \cap \mathbb{R}, \) by an appropriate choice of branches for the powers \( \alpha_i. \) It follows that \( f'(x_n) = 0, \ n \in \mathbb{N}, \) where \( f'(z) \) is also analytic in \( \Delta. \) Thus the zeros of \( f'(z) \) have a point of accumulation in its domain of analyticity, forcing this function to vanish identically in \( \Delta. \) This implies that \( f(z) \equiv A, \ z \in \Delta, \) which can be extended to the whole domain \( G \) of definition for \( f(z). \) But that gives an immediate contradiction, as \( G \) has the zeros of \( f \) on the boundary. \( \square \)

**Proof of Theorem 1.5.** We first note that the actual degrees of integer Chebyshev polynomials on \( E \) cannot be bounded, for our assumption that \( E \) has infinitely many points would give at once that \( \text{cap}(E) = 1, \) by [1.6 - 1.6]. Suppose to the contrary that there are only finitely many polynomials, with integer coefficients, that can be factors of \( Q_n, \ n \in \mathbb{N}. \) Then we have by Lemma 5.1 that, for a subsequence \( \{n_j\}_{j=1}^{\infty} \subset \mathbb{N}, \)

\[ t_{E}(E) = \lim_{n_j \to \infty} \left\| Q_{n_j} \right\|_{E}^{1/n_j} = \left\| \prod_{i=1}^{k} |Q_{m_{i,i}}|^{\alpha_i} \right\|_{E}, \]
where $Q_{m,i} \in \mathcal{P}_m(\mathbb{Z})$, $0 < \alpha_i \leq 1$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \alpha_i m_i = 1$. Observe that there are only finitely many points $x_j \in E$, where the function $\prod_{i=1}^{k} |Q_{m,i}(x)|^{\alpha_i}$ attains its norm on $E$:

$$\prod_{i=1}^{k} |Q_{m,i}(x_j)|^{\alpha_i} = t_Z(E), \quad j = 1, \ldots, M,$$

according to Lemma 5.2. Let $U(\delta) := \cup_{j=1}^{M} \{x \in E : |x - x_j| \leq \delta\}$. By choosing $\delta > 0$ sufficiently small, we can make the logarithmic capacity of $U(\delta)$ as small as we wish (see Theorem 5.1.4(a) in [30, p. 130]). This implies that $t_Z(U(\delta))$ can also be made arbitrarily small by (1.10). In particular, we can find an integer Chebyshev polynomial $P_l$ for $U(\delta)$ such that

$$\|P_l\|_{U(\delta)}^{1/l} < t_Z(E).$$

It follows that

$$\left\|P_l^{\varepsilon/l} \prod_{i=1}^{k} |Q_{m,i}(x)|^{(1-\varepsilon)\alpha_i}\right\|_{U(\delta)} \leq \|P_l\|_{U(\delta)}^{\varepsilon/l} (t_Z(E))^{1-\varepsilon} < t_Z(E),$$

for any $\varepsilon \in (0, 1)$. Note that

$$\prod_{i=1}^{k} |Q_{m,i}(x)|^{\alpha_i} < t_Z(E) - c(\delta), \quad x \in E \setminus U(\delta),$$

where $c(\delta) > 0$. Hence we can estimate for $x \in E \setminus U(\delta)$ that

$$|P_l(x)^{\varepsilon/l} \prod_{i=1}^{k} |Q_{m,i}(x)|^{(1-\varepsilon)\alpha_i}| < \|P_l\|_{E}^{\varepsilon/l} (t_Z(E) - c(\delta))^{1-\varepsilon} = \left(\frac{\|P_l\|_{E}^{1/l}}{t_Z(E) - c(\delta)}\right)^{\varepsilon} (t_Z(E) - c(\delta)).$$

It is clear that we can now choose $\varepsilon > 0$ sufficiently small, to insure that

$$\left\|P_l^{\varepsilon/l} \prod_{i=1}^{k} |Q_{m,i}(x)|^{(1-\varepsilon)\alpha_i}\right\|_{E} < t_Z(E).$$

But this immediately implies that the polynomials

$$P_l^{[n\varepsilon/l]} \prod_{i=1}^{k} Q_{m,i}^{(1-\varepsilon)\alpha_i}, \quad n \in \mathbb{N},$$

have smaller sup norms on $E$ than those of integer Chebyshev polynomials, as $n \to \infty$. This is an obvious contradiction.

One can generalize Theorem 1.5 to certain classes of compact sets $E \subset \mathbb{C}$. The major element needed in the proof is that the intersection of $E$ and any lemniscate defined by $\prod_{i=1}^{k} |Q_{m,i}(z)|^{\alpha_i} = t_Z(E)$ has integer Chebyshev constant less than $t_Z(E)$.

We are now passing to the proof of Theorem 1.6 and stating an auxiliary result.
Lemma 5.3. Let $E \subset \mathbb{C}$ be a compact set. Define

$$E_\delta := \{ z : |z - w| \leq \delta, \ w \in E \}.$$ 

Then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$t_Z(E_\delta) - t_Z(E) \leq \varepsilon.$$ 

Proof. Take $\varepsilon > 0$ and choose $n$ such that $\|Q_n\|_{E}^{1/n} \leq t_Z(E) + \varepsilon/2$. Clearly, $E \subset H := \{ z : |Q_n(z)|^{1/n} \leq t_Z(E) + \varepsilon/2 \}$. On the other hand, $H$ is inside the lemniscate $L_\varepsilon := \{ z : |Q_n(z)|^{1/n} = t_Z(E) + \varepsilon \}$, by the maximum principle, so that we can set $\delta := dist(H, L_\varepsilon) > 0$. Hence $E_\delta$ lies interior to $L_\varepsilon$, and

$$t_Z(E_\delta) \leq t_Z(L_\varepsilon) \leq t_Z(E) + \varepsilon.$$ 

The last inequality follows by considering a sequence of polynomials $(Q_n)^m$, $m \in \mathbb{N}$, on $L_\varepsilon$. □

The result of Lemma 5.3 can be quantified, provided we have some knowledge of the geometric properties for $E$. In fact, one can show that if $E$ consists of finitely many non-degenerate continua, then

$$t_Z(E_\delta) - t_Z(E) \leq C(E) \sqrt{\delta},$$ 

where $C(E) > 0$ depends only on $E$.

Proof of Theorem 1.6. We first assume that $\text{cap}(Z) = 0$, and then obtain a contradiction, to prove (1.24). Let $\tilde{Z}$ be the closure of all zeros of the integer Chebyshev polynomials $Q_n$, $n \in \mathbb{N}$, for $E$. Since the sets $Z$ and $\tilde{Z}$ differ only by countably many isolated points, we have that

$$\text{cap}(\tilde{Z}) = \text{cap}(Z) = 0$$

(cf. Theorem 5.1.4 in [30, p. 130]). Hence $\Omega := \mathbb{C} \setminus \tilde{Z}$ is a connected open set, which follows from Theorem 5.3.2(a) of [30, p. 138]. Consider a sequence of functions

$$u_n(z) := \frac{1}{n} \log |Q_n(z)|$$

that are subharmonic in $\mathbb{C}$ and harmonic in $\Omega$. It follows from Bernstein-Walsh lemma (see, e.g., Theorem 5.5.7(a) in [30, p. 156]) that this sequence is bounded on compact subsets of $\mathbb{C}$. Therefore, we can select a subsequence $u_{n_j}(z)$, converging to a harmonic function $u(z)$ locally uniformly in $\Omega$. Note that $u(z) = u(\tilde{z})$, $z \in \Omega$, which is inherited from the polynomials $Q_n$, $n \in \mathbb{N}$. Also,

$$u(z) \leq \log t_Z(E), \quad z \in E \setminus \tilde{Z}.$$ 

We want to show that all accumulation points for the solutions of the equation

$$u(x) = \log t_Z(E), \quad x \in E,$$

belong to $E \cap \tilde{Z}$. Indeed, if $x_0 \in E \setminus \tilde{Z}$ is such a point, then it must also be a point of accumulation for the local maxima of $u(x)$ on $\mathbb{R} \cap \Omega$. Let $x_k$, $k \in \mathbb{N}$, be those
maxima of \( u(x) \) such that \( \lim_{k \to \infty} x_k = x_0 \). Consider a 2-dimensional neighborhood \( \Delta \subset \Omega \) of \( x_0 \). We can define an analytic completion of \( u(z) \) in \( \Delta \), denoted by \( f(z) \), such that \( \Im f(x_0) = 0 \). It is easy to see from the Schwarz integral formula that 

\[
\int_{\Gamma} f(z) \, dz = f(z), \quad z \in \Delta,
\]

because \( u(z) = u(\bar{z}) \), \( z \in \Delta \). Hence \( \Im f(z) = 0, \ z \in \Delta \cap \mathbb{R} \), which means that

\[
u'(x_k) = 0 \Rightarrow f'(x_k) = 0, \quad k \in \mathbb{N},
\]

where by \( f'(z) \) we understand the complex derivative of \( f(z) \). It follows that \( f'(z) \) vanishes identically in \( \Delta \), so that \( f(z) \) and \( u(z) \) are identically constant in \( \Delta \). But then \( u(z) \) is identically constant in the whole domain \( \Omega \), which cannot be true, because \( \Omega \) contains compact sets \( \log \text{cap} H \) of arbitrarily large capacity and \( \|u_n\|_H \geq \log \text{cap}(H) \), for any \( n \in \mathbb{N} \) (cf. Theorem 5.5.4(a) of [30, p. 155]).

Thus the set \( M \) of solutions for (5.5) in \( \Omega \setminus \bar{Z} \) consists of isolated points, i.e., \( M \) is countable and \( \text{cap}(M) = 0 \). Furthermore,

\[
\text{cap} \left( (E \cap \bar{Z}) \cup M \right) = 0,
\]

by (5.11) and Theorem 5.1.4 of [30, p. 130]. Set

\[
U(\delta) := \{ y \in E : \|y - x\| = \delta, \ x \in (E \cap \bar{Z}) \cup M \}.
\]

We choose a sufficiently small \( \delta > 0 \), so that

\[
t_Z(U(\delta)) < t_Z(E),
\]

by Lemma 5.2 and (1.10). Hence there exists \( c_1(\delta) > 0 \), such that

\[
\|P_l\|_{L^1(U(\delta))} \leq t_Z(E) - c_1(\delta),
\]

for integer Chebyshev polynomials \( P_l \) on \( U(\delta) \) of degree \( l \geq l_0 \). Recall that for the integer Chebyshev polynomials \( Q_{n_j} \) on \( E \), we have

\[
\|Q_{n_j}\|_{L^1(E)}^{1/n_j} < t_Z(E) + \varepsilon_j, \quad j \in \mathbb{N},
\]

where \( \lim_{j \to \infty} \varepsilon_j = 0 \). We now let \( l_j = l_0 + [n_j \sqrt{c_j}] \) and consider sequences of polynomials \( \{P_{l_j}^{n_j}Q_{n_j}^m\}_{m=1}^{\infty}, \ j \in \mathbb{N} \). Using two preceding estimates and Young’s inequality, we obtain that

\[
\|P_{l_j}^{n_j}Q_{n_j}^m\|_{L^1(U(\delta))}^{1/(m(l_j+n_j))} \leq \|P_{l_j}^{1/(l_j+n_j)}\|_{L^1(U(\delta))} \|Q_{n_j}^{1/(l_j+n_j)}\|_{L^1(U(\delta))}
\]

\[
\leq (t_Z(E) - c_1(\delta))^{l_j/(l_j+n_j)}(t_Z(E) + \varepsilon_j)^{n_j/(l_j+n_j)}
\]

\[
\leq \frac{l_j}{l_j+n_j} (t_Z(E) - c_1(\delta)) + \frac{n_j}{l_j+n_j} (t_Z(E) + \varepsilon_j)
\]

\[
= t_Z(E) - \frac{l_j c_1(\delta) - n_j \varepsilon_j}{l_j+n_j} < t_Z(E),
\]

for all large \( j \in \mathbb{N} \).

Observe that we can find \( c_2(\delta) > 0 \), so that

\[
u(x) < \log (t_Z(E) - 2c_2(\delta)), \quad x \in E \setminus U(\delta),
\]
by our construction of the set $U(\delta)$. Therefore,
\[
\|Q_{n_j}\|_{E \setminus U(\delta)}^{1/n_j} < t_z(E) - c_2(\delta),
\]
for all sufficiently large $j \in \mathbb{N}$. This gives the following estimate
\[
\|P_{l_j}^m Q_{n_j}^m\|_{E \setminus U(\delta)}^{1/(m(l_j+n_j))} \leq \|P_{l_j}^m\|_E^{1/(l_j+n_j)} (t_z(E) - c_2(\delta))^{n_j/(l_j+n_j)}
\]
(5.7)
\[
= \left( \frac{\|P_{l_j}^m\|_E^{1/l_j}}{t_z(E) - c_2(\delta)} \right)^{l_j} (t_z(E) - c_2(\delta)) < t_z(E),
\]
where $j$ is selected to be sufficiently large. The last inequality in (5.7) follows because $\|P_{l_j}^m\|_E^{1/l_j} < c_3(\delta)$, $j \in \mathbb{N}$, by Bernstein-Walsh inequality, and because $\lim_{j \to \infty} l_j/n_j = 0$. Finally, we combine (5.6) and (5.7) to obtain the contradiction:
\[
t_z(E) \leq \lim_{m \to \infty} \|P_{l_j}^m Q_{n_j}^m\|_{E}^{1/(m(l_j+n_j))} < t_z(E).
\]

\[\square\]

Proof of Theorem 2.1. Observe that if the set $E' = \{ z \in E : w(z) > 0 \}$ is finite, then $t_z(E, w) = 0$. Indeed, we can use the regular integer Chebyshev polynomials $Q_n, n \in \mathbb{N}$, on $E'$, to find that
\[
t_z(E, w) \leq \limsup_{n \to \infty} \|w^n Q_n\|_E^{1/n} \leq \|w\|_{E'} \lim_{n \to \infty} \|Q_n\|_{E'}^{1/n} = \|w\|_{E'} t_z(E').
\]
But $\text{cap}(E') = 0$ in this case, so that $t_z(E') = 0$ by (5.4). Thus, (2.2) is trivially true when $E'$ is finite, and we assume that $E'$ has infinitely many points for the rest of this proof.

We need to find a sequence of polynomials
\[
P_n(z) = \sum_{k=0}^n a_k z^k \in \mathcal{P}_n(\mathbb{Z}), \quad n \in \mathbb{N},
\]
with small weighted norms $\|w^n P_n\|_E$. It is possible to use the Lagrange interpolation in weighted Fekete points for this purpose. The weighted Fekete points $\{\zeta_j\}_{j=0}^n \subset E$ are defined as a set of points maximizing the absolute value of the “weighted Vandermonde determinant” (cf. [37], p. 143)
\[
V_w(z_0, \ldots, z_n) := \prod_{0 \leq i < j \leq n} (z_i - z_j) w(z_i) w(z_j)
\]
among all $(n+1)$-tuples $\{z_j\}_{j=0}^n \subset E$. Note that $w(\zeta_j) \neq 0, j = 0, \ldots, n$, and we obtain from the Lagrange interpolation formula that
\[
w^n(z) P_n(z) = \sum_{i=0}^n w^n(\zeta_i) P_n(\zeta_i) \prod_{j \neq i} \frac{(z - \zeta_j) w(z)}{(\zeta_i - \zeta_j) w(\zeta_i)}.
\]
(5.8)

Since
\[
|V_w(\zeta_0, \ldots, z, \ldots, \zeta_n)| \leq |V_w(\zeta_0, \ldots, \zeta_i, \ldots, \zeta_n)|, \quad z \in E,
\]
for any \( i = 0, \ldots, n \), we have that
\[
\left| \prod_{j \neq i} (z - \zeta_j) w(z) \over (\zeta_i - \zeta_j) w(\zeta_j) \right| \leq 1, \quad z \in E.
\]

It follows at once from (5.8) that
\[
\| w^n P_n \|_E \leq \sum_{i=0}^n |w^n(\zeta_i)P_n(\zeta_i)| \leq (n + 1) \max_{0 \leq i \leq n} |w^n(\zeta_i)P_n(\zeta_i)|
\]
(also see Theorem III.1.12 in [37]). Observe that
\[
l_i := w^n(\zeta_i)P_n(\zeta_i) = \sum_{k=0}^n w^n(\zeta_i)\zeta_i^k a_k, \quad i = 0, \ldots, n,
\]
are linear forms in \( \{ a_k \}_{k=0}^n \) with real coefficients. Applying Minkowski’s theorem (see [6, p. 73]), we conclude that there exists a set of integers \( \{ a_k \}_{k=0}^n \), not all zero, such that
\[
|l_i| \leq | \det (w^n(\zeta_i)^k)_{0 \leq i, k \leq n} |^{1/(n+1)}.
\]

But
\[
\det (w^n(\zeta_i)^k)_{0 \leq i, k \leq n} = V_w(\zeta_0, \ldots, \zeta_n),
\]
so that we can find a sequence \( P_n(z) = \sum_{k=0}^n a_k z^k \neq 0 \), satisfying
\[
\| w^n P_n \|_E \leq (n + 1)|V_w(\zeta_0, \ldots, \zeta_n)|^{1/(n+1)}, \quad n \in \mathbb{N}.
\]

Hence
\[
\lim_{n \to \infty} \| w^n P_n \|_E^{1/n} \leq \lim_{n \to \infty} |V_w(\zeta_0, \ldots, \zeta_n)|^{1/(n+1)} = \sqrt{\text{cap}(E, w)},
\]
by Theorem III.1.3 of [37] p. 145.

**Proof of Theorem** Let \( P_n \in \mathcal{P}_n(\mathbb{Z}), \ n \in \mathbb{N} \), be a sequence polynomials satisfying
\[
\lim_{n \to \infty} \| w^n P_n \|_E^{1/n} = t_2(E, w),
\]
where \( w \) is defined in (2.8). We construct the following new sequence of polynomials with integer coefficients:
\[
P_n(z) \prod_{i=1}^k Q_{l_i(n)}^{a_i}(z), \quad n \in \mathbb{N},
\]
where \( l_i(n) \in \mathbb{N} \) are selected so that
\[
\frac{l_i(n)}{n} \to \frac{a_i}{1 - \alpha}, \quad \text{as} \ n \to \infty, \ i = 1, \ldots, k.
\]
Using (1.6), we obtain that

\[
\begin{align*}
t_Z(E) & \leq \limsup_{n \to \infty} \left\| P_n \prod_{i=1}^{k} Q_{m_i,i}^{l_i(n)} \right\|_E^{1/(n + \sum_{i=1}^k m_i l_i(n))} \\
& \leq \limsup_{n \to \infty} \left( \|w^n P_n\|_E \right)^{1/n} \prod_{i=1}^{k} \|Q_{m_i,i}^{l_i(n)}\|_E^{l_i(n) - \frac{\alpha_i}{1-\alpha}} \leq (t_Z(E, w))^{1-\alpha},
\end{align*}
\]

because

\[
\limsup_{n \to \infty} \|Q_{m_i,i}^{l_i(n)}\|_E^{l_i(n) - \frac{\alpha_i}{1-\alpha}} \leq 1, \quad i = 1, \ldots, k.
\]

It now follows from Theorem 2.1 that

\[
t_Z(E) \leq (t_Z(E, w))^{1-\alpha} \leq \left( \text{cap}(E, w) \right)^{(1-\alpha)/2}.
\]

Proof of Theorem 2.4. We need to find a solution of the weighted energy problem on \( E \), corresponding to the weight \( w \) of (2.3)-(2.5). It follows from Theorem I.1.3 of [37] that there exists a weighted equilibrium measure \( \mu_w \), whose support is a compact set \( S_w \subset E \cup \bigcup_{i=1}^k \{z_j,i\}_{j=1}^{m_i} \). Let \( \delta_z \) be a unit point mass at \( z \).

Observe that

\[
Q(z) = -\log w(z) = U^\nu(z) - \frac{1}{1-\alpha} \sum_{i=1}^k \alpha_i \log |a_i|,
\]

where \( U^\nu \) is the logarithmic potential of the measure

\[
\nu := \frac{1}{1-\alpha} \sum_{i=1}^k \sum_{j=1}^{m_i} \alpha_i \delta_{z_j,i}.
\]

It is clear that \( \nu \) is a positive Borel measure of total mass \( \nu(\mathbb{C}) = \alpha/(1-\alpha) \). Let \( \hat{\nu} \) be the balayage of \( \nu \) from \( \Omega \) onto \( S_w \) (see, e.g., Section II.4 of [37]). Then \( \hat{\nu} \) is a positive Borel measure of the same mass as \( \nu \), which is supported on \( S_w \). Furthermore, we can express \( \hat{\nu} \) via harmonic measures

\[
\hat{\nu} = \frac{1}{1-\alpha} \sum_{i=1}^k \sum_{j=1}^{m_i} \alpha_i \omega(z_j,i, \cdot, \Omega)
\]

(cf. Appendix A.3 of [37]). The potentials of \( \nu \) and \( \hat{\nu} \) are related by the equation

\[
U^\hat{\nu}(z) = U^\nu(z) + \int_\Omega g_\Omega(t, \infty) d\nu(t),
\]

which holds quasi everywhere on \( S_w \) (see Theorem II.4.4 of [37]). Hence the measure

\[
\mu := \frac{1}{1-\alpha} \omega(\infty, \cdot, \Omega) - \hat{\nu} = \frac{1}{1-\alpha} \left( \omega(\infty, \cdot, \Omega) - \sum_{i=1}^k \sum_{j=1}^{m_i} \alpha_i \omega(z_j,i, \cdot, \Omega) \right)
\]
is a probability measure on \( S_w \). Using (5.10) and (5.11), we obtain for quasi every \( z \in S_w \) that

\[
U^\mu(z) + Q(z) = \frac{1}{\alpha - 1} \log \text{cap}(S_w) - U^\nu(z) + \frac{1}{1 - \alpha} \sum_{i=1}^{k} \alpha_i \log |a_i|
\]

\[
= \frac{1}{\alpha - 1} \left( \log \text{cap}(S_w) + \sum_{i=1}^{k} \alpha_i \log |a_i| \right) - \int \nu(t, \infty) d\nu(t)
\]

\[
= \frac{1}{\alpha - 1} \left( \log \text{cap}(S_w) + \sum_{i=1}^{k} \alpha_i \log |a_i| + \sum_{i=1}^{k} \alpha_i g_\Omega(z_i, \infty) \right).
\]

Note that \( \mu \) has finite logarithmic energy, since it is composed of harmonic measures. Thus we can apply Theorem I.3.3 to prove that \( \mu \) is the weighted equilibrium measure \( \mu_w \) and that the associated modified Robin constant \( F_w \) is given by (2.3). Equation (2.7) expresses \( \text{cap}(E, w) \) through \( w, \mu_w \) and \( F_w \) (cf. Section I.6 of [37]).

We now obtain (5.10) from (2.7) by the following simple manipulation. Recall that

\[
g_\Omega(z, \infty) = -\log \text{cap}(S_w) - U^{(\infty, \cdot)}(z) = \int \log |z - t| d\nu(\infty, t, \Omega) - \log \text{cap}(S_w)
\]

(see Chapter 4 of [30]). Substituting this relation into (2.3), we have that

\[
F_w = \frac{1}{\alpha - 1} \left( \log \text{cap}(S_w) + (1 - \alpha) \int \log w(t) d\nu(\infty, t, \Omega) - \alpha \log \text{cap}(S_w) \right)
\]

\[
= -\log \text{cap}(S_w) - \int \log w(t) d\nu(\infty, t, \Omega).
\]

Hence (5.10) is proved too.

\section*{Lemma 5.4.}
Assume that the integer Chebyshev polynomials of \( E, \text{cap}(E) > 0 \), satisfy (2.9) and (2.10), along a subsequence of \( n \to \infty \). Then

\[
\limsup_{n \to \infty} \left\| w^{m(n)} R_n \right\|_{S_w}^{1/n} \leq t_\overline{2}(E), \quad m(n) = \deg R_n,
\]

where \( w \) is given by (2.8) and \( S_w \) is the support of the weighted equilibrium measure \( \mu_w \), corresponding to the weight \( w \).

\section*{Proof.}
Observe that the actual degree of \( R_n \) is \( m(n) = n - \sum_{i=1}^{k} m_i l_i(n) \). It follows from Theorem I.1.3 of [37] that there exists a weighted equilibrium measure \( \mu_w \), whose support is a compact set \( S_w \subset E \setminus \bigcup_{i=1}^{k} \{ z_j, i \} \). Hence the factors \( Q_{m_i, i}, i = 1, \ldots, k \), do not vanish on \( S_w \). Estimating

\[
\left\| w^{m(n)} R_n \right\|_{S_w}^{1/n} \leq \left\| \prod_{i=1}^{k} Q_{m_i, i}^{l_i(n)} \right\|_{S_w}^{1/n} \left\| \prod_{i=1}^{k} Q_{m_i, i}^{(1 - \alpha) l_i(n) - \alpha m(n)} \right\|_{S_w}^{1/n}
\]

\[
\leq \| Q_n \|_{E}^{1/n} \prod_{i=1}^{k} \| Q_{m_i, i} \|_{S_w}^{(1 - \alpha) l_i(n) - \alpha m(n) / n},
\]
and noting that
\[ \lim_{n \to \infty} \frac{\|Q_{m,i}\|_{S_E}}{\|Q_m\|_{S_E}}^{ \frac{l_i(n)}{m(n)}} = 1, \quad i = 1, \ldots, k, \]
by (2.12), we obtain (5.12).

**Proof of Theorem 2.5.** Clearly, the degrees of integer Chebyshev polynomials on \(E\) must be unbounded, because \(\text{cap}(E) > 0\), so that the assumptions of this theorem are valid. Since the leading coefficient of \(R_n\) is at least 1 in absolute value and the degree of \(R_n\) is \(m(n) = n - \sum_{i=1}^{k} m_i l_i(n)\), we obtain from Theorem I.3.6 of [37] that
\[ \|w^{m(n)} R_n\|_{S_E} \geq e^{-m(n) F_w}. \]
Thus (2.13) follows by taking the \(n\)-th root in the above inequality, and using (5.12) together with (2.12), as \(n \to \infty\).

Finally, (2.14) is a direct consequence of (2.13) and (2.8), for \(E \subset \mathbb{R}\).

**Proof of Proposition 2.8.** Consider a sequence of integer Chebyshev polynomials \(Q_n, n \in \mathbb{N}\), satisfying (2.11) and (2.12). It is not difficult to see that we can assume
\[ \frac{l_i(n)}{n} \to \alpha_i, \quad \text{as } n \to \infty, \quad i = 1, \ldots, k, \]
while preserving the property
\[ \lim_{n \to \infty} \frac{\|Q_n\|_{E}}{\|Q_m\|_{E}}^{\frac{1}{n}} = t_{Z}(E). \]

Since
\[ t_{Z}(E) \leq t_{Z}(E \cup H_\varepsilon), \]
we only need to show that, for some \(\varepsilon > 0\),
\[ \limsup_{n \to \infty} \frac{\|Q_n\|_{H_\varepsilon}}{\|Q_m\|_{H_\varepsilon}}^{\frac{1}{n}} \leq t_{Z}(E). \]
Using the same notations as in the proofs of Theorems 2.5 and 2.6, we have that

\[
\limsup_{n \to \infty} \|Q^n\|_{H^s}^{1/n} \leq \limsup_{n \to \infty} \left( w^{m(n)} R_n \right)^{1/n} H_s \leq \limsup_{n \to \infty} \left( \prod_{i=1}^k \|Q_{m_i,i}\|_{H^s} \right)^{1/n} H_s.
\]

We next estimate, multiplying (5.13) by \(w^{m(n)}(z)\),

\[
|w^{m(n)}(z) R_n(z)| \leq \left\| w^{m(n)} R_n \right\|_{S_w} \exp \left( m(n) \left( F_w - U^{w} (z) + \log w(z) \right) \right), \quad z \in \mathbb{C}.
\]

Recall that \(S_w\) is a compact set, \(S_w \subset E \setminus \bigcup_{i=1}^k \{z_{j,i}\}_{j=1}^{m_i}\). Therefore, the potential \(U^{w}\) is harmonic and bounded on \(H_{\varepsilon}\), if \(\varepsilon > 0\) is sufficiently small. On the other hand, we can obviously make \(\log w\) smaller than any negative number on \(H_{\varepsilon}\), by choosing \(\varepsilon\) small. It follows that

\[
F_w - U^{w} (z) + \log w(z) \leq 0, \quad z \in H_{\varepsilon},
\]

for some \(\varepsilon\), which further implies that

\[
\left\| w^{m(n)} R_n \right\|_{H^s} \leq \left\| w^{m(n)} R_n \right\|_{S_w},
\]

by (5.14). Using Lemma 5.4 we now obtain that

\[
\limsup_{n \to \infty} \|Q^n\|_{H^s}^{1/n} \leq \limsup_{n \to \infty} \left( w^{m(n)} R_n \right)^{1/n} H_s \leq \limsup_{n \to \infty} \left( w^{m(n)} R_n \right)^{1/n} S_w \leq t_{\varepsilon}(E).
\]

**Proof of Lemma 3.2.** Note that the weight \(w\) of (3.5) is just a special case of the Jacobi weights of Examples IV.1.17 and IV.5.2 in [37]. Thus our problem on the interval \([0, 1/4]\) is easily reduced to that on the interval \([-1, 1]\) considered in [37], with the help of the change of variable \(x \to (x+1)/8\). We obtain from Example IV.1.17 of [37] that \(S_w = [a, b]\), with \(a\) and \(b\) given by (3.6) (see (1.27) and (1.28) in [37], p. 207]). Similarly, Example IV.5.2 gives (3.8) after this change of variable.

Consider the following natural extension for \(w(x)\) of (3.6):

\[
w(z) := |z|^{\frac{2\alpha_1 - 2\alpha_2 - 2}{1 - 2\alpha_1 - 2\alpha_2}} |1 - 4z|^{\frac{2}{1 - 2\alpha_1 - 2\alpha_2}}, \quad z \in \mathbb{C}.
\]

It follows from Theorem 1.1.3 of [37] that

\[
F_w - U^{w} (z) = - \log w(z),
\]

for quasi every \(z \in [a, b]\) (i.e., with the exception of a set of zero capacity). Denote the right hand side of (3.9) by \(h(z)\). Then \(F_w - U^{w} (z) - h(z)\) is a harmonic function in \(\Omega = \mathbb{C} \setminus [a, b]\), such that

\[
F_w - U^{w} (z) - h(z) = 0
\]
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for quasi every $z \in [a, b] = \partial \Omega$, by (5.15) and the basic properties of Green functions (see [43] p. 14). Using the uniqueness theorem for the solution of the Dirichlet problem in $\Omega$ (cf. Theorem III.28 and its Corollary in [43]), we conclude that

$$F_w - U^\mu_w(z) \equiv h(z), \quad z \in \mathbb{C}.$$ 

Thus $F_w$ can be found from

$$F_w = \lim_{z \to \infty} (U^\mu_w(z) + h(z)),$$

or from (2.8). We obtain the explicit representation of (3.7) by expressing the Green functions of (3.9) via the conformal mappings of $\Omega$ onto the exterior of the unit disk. Indeed, introducing these conformal mappings by

$$\Phi_\infty(z) := \frac{2z - a - b + 2\sqrt{(z - a)(z - b)}}{b - a}, \quad z \in \Omega,$$

$$\Phi_0(z) := \frac{2z^{-1} - b^{-1} - a^{-1} + 2\sqrt{(z^{-1} - b^{-1})(z^{-1} - a^{-1})}}{b^{-1} - a^{-1}}, \quad z \in \Omega,$$

and

$$\Phi_{1/4}(z) := \frac{2(z - 1/4)^{-1} - (b - 1/4)^{-1} - (a - 1/4)^{-1}}{(b - 1/4)^{-1} - (a - 1/4)^{-1}} + \frac{2\sqrt{((z - 1/4)^{-1} - (a - 1/4)^{-1})((z - 1/4)^{-1} - (b - 1/4)^{-1})}}{(b - 1/4)^{-1} - (a - 1/4)^{-1}}, \quad z \in \Omega,$$

we observe that

$$\Phi_\infty(\infty) = \infty, \quad \Phi_0(0) = \infty \quad \text{and} \quad \Phi_{1/4}(1/4) = \infty.$$ 

Hence

$$g_\Omega(z, \infty) = \log|\Phi_\infty(z)|, \quad g_\Omega(z, 0) = \log|\Phi_0(z)|$$

and

$$g_\Omega(z, 1/4) = \log|\Phi_{1/4}(z)|, \quad z \in \Omega,$$

by Theorem I.17 of [43] p. 18].

**Proof of Theorem 3.3**. This theorem is an immediate application of Corollary 2.7 to the integer Chebyshev polynomials on $[0, 1/4]$. Using (1.23) and (3.1), we obtain the upper bound

$$t_z([0, 1/4]) \leq 0.42347945^2 < 0.179335 = M.$$ 

Then we choose $\zeta_1 = 0$ and $\zeta_2 = 1/4$ to produce the inequalities, defining the region $G$ of Figure 11 by (2.16). The values of $F_w - U^\mu_w(\zeta)$ are readily found from (3.9) and the explicit formulas for the Green functions, obtained in the proof of Lemma 3.2. Figure 11 as well as the bounds for $\alpha_1$ and $\alpha_2$, is generated by Matlab. \qed
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