A unified formula of the optimal portfolio for piecewise hyperbolic absolute risk aversion utilities

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We propose a general family of piecewise hyperbolic absolute risk aversion (PHARA) utilities, including many classic and non-standard utilities as examples. A typical application is the composition of a HARA preference and a piecewise linear payoff in asset allocation. We derive a unified closed-form formula of the optimal portfolio, which is a four-term division. The formula has clear economic meanings, reflecting the behavior of risk aversion, risk seeking, loss aversion and first-order risk aversion. We conduct a general asymptotic analysis to the optimal portfolio, which directly serves as an analytical tool for financial analysis. We compare this PHARA portfolio with those of other utility families both analytically and numerically. One main finding is that risk-taking behaviors are greatly increased by non-concavity and reduced by non-differentiability of the PHARA utility. Finally, we use financial data to test the performance of the PHARA portfolio in the market.

Keywords: Utility theory; Portfolio selection; Non-differentiability; Asymptotic analysis; Empirical study

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1. Introduction

In the field of continuous-time portfolio selection, after the seminal work of Merton (1969) investigates hyperbolic absolute risk aversion (HARA) utilities, more and more non-standard (e.g. non-concave, non-differentiable) utility functions appear. These utility functions come from the following optimization problems (where \( U \) is the utility function, \( \pi \in V \) is the portfolio process to be selected and \( X_T \) is the corresponding terminal wealth):

1. The decision-maker solves a utility maximization problem with the basic HARA utility:

\[
\max_{\pi \in V} E[U(X_T)],
\]

where \( U \) is a classic HARA utility function (Merton 1969). This is revisited in example 1.

2. In the field of asset allocation (e.g. the hedge fund), \( \hat{U} \) is a HARA preference function and \( \Theta \) is usually a (non-concave) piecewise linear payoff function. The decision-maker solves a utility maximization problem with the objective being a composition function \( U := \hat{U} \circ \Theta \):

\[
\max_{\pi \in V} E[U(X_T)] = \max_{\pi \in V} E[\hat{U}(\Theta(X_T))].
\]

We take (Carpenter 2000) as an example, which is revisited in example 2. See also Berkelaar et al. (2004), Larsen (2005), Hodder and Jackwerth (2007), Bichuch and Sturm (2014) and He et al. (2019).

3. In the setting of behavioral finance, the decision-maker solves a utility maximization problem with a non-concave utility (e.g. S-shaped):

\[
\max_{\pi \in V} E[U(X_T)],
\]

where \( U \) is an S-shaped function or a composition function related to an S-shaped function. In the above category (2), some papers also use an S-shaped utility. These papers may also be included in this category. We take (Lin et al. 2017) as an example and
revisit it in example 3. See also Kouwenberg and Ziemba (2007), Dong and Zheng (2020) and Liang and Liu (2020).

(4) In the setting of risk management, the decision-maker solves the utility maximization problem with other objectives and constraints (e.g. the constraints of liquidation, the performance ratio and the Value-at-Risk):

$$\max_{\pi \in \mathcal{V}} \mathbb{E}[U(X_T^\pi)],$$

s.t. \( \mathcal{R}(X_T) \geq 0, \) \( \text{(4)} \)

where \( \mathcal{R} \) is a risk constraint. We take He and Kou (2018) as an example and revisit it in example 4. See also Bernard et al. (2019), Chen et al. (2019), Bergk et al. (2021) and Guan et al. (2023).

A specific utility model may be classified in multiple categories above. Although fortunately, the problems of these utilities admit closed-form solutions in many cases, the optimal portfolios are expressed in different forms in the literature, making the analytical analysis difficult to conduct consistently. In this paper, we propose a large family of utility, 'piecewise HARA (PHARA) utility', to unify and contain these classic and non-standard utility functions. Intuitively, the PHARA utility is a direct generalization of the HARA utility: it reduces to a HARA utility on each part of the domain. On its whole domain, the utility may not be concave or differentiable. We establish the definition of the PHARA utility in section 2.2. In examples 1–4 in section 2.3, Problems (1)–(4) can be represented by a PHARA utility maximization problem (10), up to some necessary transformations (e.g. converting to an unconstrained problem); the corresponding contexts and motivations are further demonstrated.

Throughout, we use the standard Black-Scholes model in a complete market, which is introduced in section 2. We provide a comprehensive overview of the model setting in appendix 1. Our contributions are fourfold.

First, we provide a unified formula of the optimal portfolio choice for the general PHARA utility (section 3). The portfolio is expressed in a feedback form (theorem 1), i.e. an explicit function of the wealth level, given a Lagrangian multiplier \( \gamma^* \) later. This unified formula includes optimal portfolios in the preceding literature as typical examples. As a direct formula, our result can be applied to quickly obtain the optimal portfolio in utility maximization. We show the usage of our formula, how to specify parameters of the PHARA utility and how to directly write down the optimal portfolio, at examples 1–5 in section 4.

Second, we illustrate clear economic meanings for the formula (section 5). The unified formula enjoys an advantage of analytical tractability and consists of four terms: Merton relative risk aversion term, risk-seeking term (due to non-concavity), loss aversion term (due to benchmark levels) and first-order risk aversion term (due to non-differentiability).

The division of the closed-form portfolio illustrates the risky behavior of the decision-maker. Further, we propose an asymptotic analysis to our portfolio and the corresponding wealth process (theorem 3). It is a generalization of the asymptotic approach in Liang and Liu (2023). The main technique is asymptotic analysis and we use a more detailed analysis of the limiting of various terms in theorem 3. One can directly apply theorem 3 for asymptotic financial analysis given a specific PHARA utility. As a summary, the optimal portfolio has a pattern of ‘multiple-peak-multiple-valley’ with the tail trend to the famous (Merton 1969)’s constant risky percentage. We find that not only non-concavity in the utility function causes great risk taking, but also non-differentiability greatly reduces risk-taking. The former finding of non-concavity is studied in Carpenter (2000) and Kouwenberg and Ziemba (2007), while the latter finding of non-differentiability is new.

Third, we compare the portfolios of the PHARA utility, the HARA utility and the SAHARA utility (symmetric asymptotic HARA; proposed by Chen et al. (2011) analytically and numerically in sections 5.3 and 5.4. We find that all these three corresponding portfolios approach to the Merton percentage when the market state is good. The HARA and PHARA portfolios become a pure risk-free investment when the market state is bad, while the SAHARA portfolio gambles more in a recession state. Different from the HARA portfolio’s simple increasing structure with respect to the wealth, the PHARA portfolio has a relatively sophisticated structure: gamble (peak) at the non-concavity and become conservative (valley) at the non-differentiability of the utility.

Fourth, we adopt the proposed PHARA utility and implement the portfolio using financial data for an empirical study (section 6). We set up our study by estimating our strategy parameters in an eight-year in-sample period and testing our strategy performance in a two-year out-of-sample period. We run a total of 1000 simulations in the out-of-sample period.

We find that the proposed PHARA portfolio performs with high returns but high volatility, demonstrating positive alpha and a positive Sharpe Ratio. The majority of the calculated Sharpe Ratio values fall between 0 and 1.0, demonstrating that one of the drawbacks of the PHARA portfolio is high volatility. On the other hand, our simple returns are clustered around 100% and the mean alpha value is greater than 0. This shows us that the PHARA portfolio can provide an advantage to the market.

In terms of methodology in deriving the optimal portfolio (theorems 1 and 2), the basic approach is the martingale and duality method and the concavification technique. In a word, by duality arguments and concavification techniques, one can get the optimal terminal wealth, and hence obtain the optimal wealth process and the optimal portfolio by the martingale representation theorem and Itô’s formula. The martingale and duality method dates back to Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989). The concavification technique is established by Carpenter (2000) and developed by Kouwenberg and Ziemba (2007), Bichuch and Sturm (2014), Liang and Liu (2020) and so on. This combined approach is frequently adopted in the literature, including He and Kou (2018), He et al. (2020), Dong and Zheng (2020) and other papers above. Throughout, we implement this approach multiple times. We emphasize that our main theoretical novelty is the generality of the PHARA utility family, the delicate structure of the unified formula and the technical details of general asymptotic analysis. For a reference, we provide a
basic theory on the concavification (also known as the concave envelope) in appendix 2. Finally, section 7 concludes. All proofs are relegated to appendix 3.

2. Model setting

2.1. Market model

In the modern theory of investment, the Black-Scholes model is usually adopted to describe the financial market. Basically, there are a risk-free asset and multiple risky assets in the setting. A risk-free asset (e.g. bond) has a deterministic return rate and a zero volatility, while a risky asset (e.g. stock) has a higher expected return rate and a positive volatility. The key assumptions include that the price process of the risky asset is characterized by a geometric Brownian motion; see appendix 1 for a comprehensive overview.

In this paper, we consider a standard Black-Scholes model in a complete market, consisting of one risk-free asset (i.e. a bond) and m risky assets (i.e. stocks) where $m \geq 1$ is an integer. We introduce the notation as follows. Throughout, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ describes the financial market. The equipped filtration $\mathcal{F} = \{ \mathcal{F}_t \}_{0 \leq t \leq T}$ is the one generated by an $m$-dimensional standard Brownian motion $(\mathbf{W}_t)_{0 \leq t \leq T} = \{ (W_{1,t}, \ldots, W_{m,t})' \}_{0 \leq t \leq T}$ and further augmented by all the $\mathbb{P}$-null sets. For the risk-free asset, the interest rate is a constant $\rho$, and thus the price process $S_0$ is a geometric Brownian motion satisfying

$$dS_0_t = rS_0_t \, dt, \quad 0 \leq t \leq T.$$  

(5)

For $m$ risky assets, we denote a vector of expected return rates by $\mu = (\mu_1, \ldots, \mu_m)'$ and an $m \times m$ matrix of volatility by $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq m}$. For $i = 1, \ldots, m$, the expected return rate of the $i$th risky asset is $\mu_i > r$. The price process of the $i$th risky asset $(S_{i,t})_{0 \leq t \leq T}$ is a geometric Brownian motion satisfying the following stochastic differential equation:

$$dS_{i,t} = S_{i,t} \left( \mu_i \, dt + \sum_{j=1}^{m} \sigma_{ij} \, dW_{j,t} \right), \quad 0 \leq t \leq T. \quad (6)$$

We assume that there exists some $\varepsilon > 0$ such that

$$\eta^T (\sigma \sigma^T) \eta \geq \varepsilon ||\eta||_2^2$$

for any $\eta \in \mathbb{R}^m$,  

(7)

where $||\eta||_2 := (\sum_{i=1}^{m} \eta_i^2)^{1/2}$. Under the assumption (7), the matrices $\sigma^T$ and $\sigma$ are invertible; see lemma 2.1 of Karatzas et al. (1987). We denote an $m$-dimensional vector by $1_m = (1, \ldots, 1)'$. We further define a vector $\theta := \sigma^{-1}(\mu - r1_m)$, which actually means the market price of risk. Finally, we define the pricing kernel process $(\xi_t)_{0 \leq t \leq T}$ as follows:

$$\xi_t := \exp \left\{ - \left( r + \frac{1}{2} ||\theta||_2^2 \right) t - \theta^T \mathbf{W}_t \right\}, \quad 0 \leq t \leq T. \quad (8)$$

The pricing kernel process is an important quantity in portfolio selection, and it is also known as the stochastic discount factor. The terminal pricing kernel variable $\xi_T$ plays a direct role in solving the terminal utility optimization problem.

For any $i = 1, \ldots, m$, for any time $t \in [0, T]$, we denote by $\pi_{i,t}$ the dollar amount invested in the $i$th risky asset and denote by $\pi_t = (\pi_{1,t}, \ldots, \pi_{m,t})'$ the $m$-dimensional portfolio vector. The process of the asset value $\{ X_t \}_{0 \leq t \leq T}$ is uniquely determined by the $m$-dimensional portfolio process $\{ \pi_t \}_{0 \leq t \leq T}$ and an initial value $x_0 \in \mathbb{R}$:

$$dX_t = \sum_{i=1}^{m} \pi_{i,t} \left( \mu_i \, dt + \sum_{j=1}^{m} \sigma_{ij} \, dW_{j,t} \right) + \left( X_t - \sum_{i=1}^{m} \pi_{i,t} \right) r \, dt$$

$$= (rx_t + \pi_t^T (\mu - r1_m)) \, dt + \pi_t^T \sigma \, dW_t, \quad X_0 = x_0. \quad (9)$$

A portfolio $\pi = \{ \pi_t \}_{0 \leq t \leq T}$ is called admissible if and only if $\pi$ is an $(\mathcal{F}_t)_{0 \leq t \leq T}$-progressively measurable $\mathbb{R}^m$-valued process and $\int_{0}^{T} ||\pi_t||_2^2 \, dt < \infty$ almost surely. Hence $\pi$ is allowed to do short-selling, i.e. it may happen that $\pi_{i,t} < 0$ for some $i = 1, \ldots, m$ and $t \in [0, T]$. The decision-maker solves an expected utility optimization problem to conduct portfolio selection:

$$\max_{\pi \in \mathcal{P}} \mathbb{E}[U(X_T)], \quad (10)$$

where $U$ is the utility function and $\mathcal{P}$ is the collection of all the admissible portfolios.

2.2. PHARA utility

In our context, the utility function $U$ may not be concave and cause difficulties in the solving procedure. A highly related concept is the concave envelope. Let $D \subseteq \mathbb{R}$ be a convex set. Denote a continuous function by $U : D \rightarrow \mathbb{R}$, where the domain of $U$ is denoted by dom $U = D$. The concave envelope of $U$ (denoted by $U^{**}$) is defined as the smallest continuous concave function larger than $U$. That is, for $x \in D$,

$$U^{**}(x) := \inf \{ h(x) : h \text{ maps } D \rightarrow \mathbb{R}, h \text{ is a concave and continuous function on } D \text{ and } h \geq U \}. \quad (11)$$

Hence, for a continuous utility function $U$, the concave envelope $U^{**}$ is concave, continuous and larger than $U$, and $U^{**}$ is the smallest function satisfying these three conditions; see later figure 1 as an example for graphical illustration. We introduce the basic theory on the non-concave function and its concave envelope in appendix 2, where we provide a procedure for generating the concave envelope from the original function.

The concave envelope plays an important role in portfolio selection problems with non-concave utilities; see appendix 2. Indeed, if $\xi_T$ in (8) has a continuous distribution, Problem (10) has the same optimal solution as the following problem with the utility $U$ replaced by the concave envelope $U^{**}$:

$$\max_{\pi \in \mathcal{P}} \mathbb{E}\left[U^{**}(X_T)\right]. \quad (12)$$

It is shown in theorem 2.5 in Bichuch and Sturm (2014), theorem 3.3 in Liang and Liu (2020) and also the proof of
propose 2 later. Thus, we can study $U^{**}$ instead of $U$ and later obtain the same optimal portfolio for both Problems (10) and (12).

Next, we define the main object in this paper, piecewise HARA (PHARA) utility. Intuitively, it means being a HARA function on each part of its domain.

**Definition 1 (PHARA utility)** Define the function $U : \mathbb{R} \rightarrow \mathbb{R}$ with relative risk aversion parameter $R \in [0, \infty]$, benchmark level $A \in \mathbb{R}$, partition point $\hat{x} \in \mathbb{R}$, utility value $u \in \mathbb{R}$, slope $\gamma \in (0, \infty)$, absolute risk aversion parameter $\alpha \in (0, \infty)$ as

$$U(x; R, A, \hat{x}, u, \gamma, \alpha) = \begin{cases} 
\gamma x + u, & \text{if } R = 0; \\
\gamma \log(x - A) + u, & \text{if } R = 1; \\
\gamma \left(\frac{x - A}{1 - R} - 1\right) + u, & \text{if } R \in (0, 1), \ \hat{x} \neq A; \\
\frac{\gamma}{\alpha} \left(e^{-\alpha x} - 1\right) + u, & \text{if } R = \infty, \ A = -\infty, \ \alpha \in (0, \infty). 
\end{cases}$$

(13)

Set the domain $D := [a_0, \infty)$ or $(a_0, \infty)$, where $a_0 := \inf\{a \in \mathbb{R} | U(a) > -\infty\}$. A function $U : D \rightarrow \mathbb{R}$ is a piecewise HARA utility if and only if there exists a partition $\{[a_k, b_k]\}_{k=0}^{n+1}$ and a family of parameter pairs $\{(R_k, A_k, \gamma_k, \alpha_k)\}_{k=0}^{n}$ (note that it is no need to specify $A_k$ for $k$ satisfying $R_k = 0$, i.e. on the linear part) such that

(i) $n \geq 0$, $a_0 < a_1 < \cdots < a_{n+1}$, $a_1, \ldots, a_n \in \mathbb{R}$, $a_{n+1} = \infty$;

(ii) $U$ is increasing and continuous on $D$;

(iii) (a) If $n = 0$ and $D = [a_0, \infty)$, or $n = 0$ and $D = (a_0, \infty)$ and $A_0 = a_0$, then $U(x) = \hat{U}(x; R, A, u, \gamma, \alpha)$ for any $x \in (a_0, \infty)$, where

$$\hat{U}(x; R, A, u, \gamma, \alpha) = \begin{cases} 
k_1(x - a_0)^{1-R} - k_1(a_1 - a_0)^{1-R} - \lambda(a_2 - a_1)^{1-R}, & \text{if } a_0 \leq x < a_1; \\
- \lambda(a_2 - a_1,2)^{1-R}, & \text{if } a_1 \leq x < a_{2}; \\
- \lambda(a_2 - x)^{1-R}, & \text{if } a_1,2 \leq x < a_{3}; \\
0, & \text{if } a_2 \leq x < A_3; \\
k_1(x - A_3)^{1-R}, & \text{if } A_3 \leq x < a_4; \\
k_2(x - A_4)^{1-R} + k_1A_3^{1-R} - k_2(a_4 - A_4)^{1-R}, & \text{if } x \geq a_4. 
\end{cases}$$

(b) If $n = 0$ and $D = [a_0, \infty)$, then $U(x) = \hat{U}(x; R_0, A_0, u_0, \gamma_0, \alpha_0)$ for any $x \in [a_0, \infty)$.

(c) If $n \geq 1$, for $k = 0$, $U(x) = \hat{U}(x; R_0, A_0, a_1, U(a_1), U(a_1), \gamma_0)$ for any $x \in (a_0, a_1)$; for any $k \in \{1, 2, \ldots, n\}$, $U(x) = \hat{U}(x; R_k, A_k, a_k, U(a_k), U(a_k), \gamma_0, \alpha_0)$ for any $x \in (a_k, a_{k+1})$.

(iv) For $n, R_n \neq 0$ (i.e. $U$ is not linear on the last interval $(a_n, \infty)$).

**Remark 1** Technically, definition 1 is classified in categories (a)–(c) because of the different settings under $n = 0$ or $n \geq 1$ and $D = [a_0, \infty)$ or $D = (a_0, \infty)$. In (b)–(c), one could use information of partition points $a_0, a_1$, etc., to describe the expression of the utility. In (a), there is no specific partition point with information of utility values and slopes, so one should specify the parameters directly.

We show in the following proposition that the concave envelope preserves the property of HARA.

**Proposition 1** If $U$ is a PHARA utility function, then $U^{**}$ is also a PHARA utility function. Conversely, if $U^{**}$ is PHARA, $U$ may not be PHARA.

Hence, the PHARA utility family has a high level of generality and includes many explicit utility functions, such as (piecewise) power/log/exponential and S-shaped ones. In fact, our result only requires that the concave envelope $U^{**}$ is PHARA. This increases the generality of the choice of utility functions, because: (i) $U^{**}$ is PHARA if $U$ is PHARA; (ii)
$U^{* *}$ is possible to be PHARA even if $U$ is not PHARA; (iii) PHARA utilities can be used as building blocks to approximate/characterize an unknown preference. In many cases, the explicit expression of a utility function is unknown, while only information on non-concavity and differentiability of some intervals is available; see, e.g. Liang and Liu (2020). The PHARA utilities can hence serve as basic elements for approximation. Finally, for simplicity, we use the notation: $\gamma^+_k = U'(a_k +)$, $\gamma^-_k = U'(a_k -)$, where $\gamma_0 = \infty$ and $\gamma_{\alpha + 1} = 0$.

**Remark 2** There exist some utility functions such that $U(a_0) = -\infty$; e.g. $U(x) = \log(x)$ or $U(x) = -\frac{1}{2}x^2$, $x \in (a_0, a_1)$, where $a_0 = 0$ and $a_1 \in (0, \infty)$. These cases are also included in the PHARA utility family. For these cases, it is only possible to appear on the first interval of the domain ($k = 0$) as $U(a_0) = -\infty$. According to definition 1, we can write

$$U(x) = \hat{U}(x; R_0 = 1, A_0 = a_0 = 0, a_1, U(a_1) = 3 \log(a_1), \quad \begin{cases} U'(a_1) = 3a_1^{-1}, & \text{for } U(x) = 3 \log(x), \\ x \in (a_0, a_1); \end{cases}$$

$$U(x) = \hat{U}(x; R_0 = 3, A_0 = a_0 = 0, a_1, U(a_1) = -\frac{1}{2}a_1^2, \quad \begin{cases} U'(a_1) = a_1^{-2}, & \text{for } U(x) = -\frac{1}{2}x^2, \\ x \in (a_0, a_1). \end{cases}$$

### 2.3. Examples

In this section, we show that plenty of models in the literature, either classic or non-standard, become examples of our PHARA utility in definition 1. Later in section 4 we establish the parameter tables in terms of a PHARA utility and use a unified formula to write down the optimal portfolio of these examples.

**Example 1** In the context of Merton (1969), the decisionmaker (i.e. a single investor) selects the optimal portfolio using the above standard Black-Scholes model with $m$ risky assets and one risk-free asset. As discussed above, the initial wealth for investment is $x_0 > 0$ and the investment period is $T > 0$. We use the same notation in the model setting (5)–(10).

- The decision-maker has the following CRRA utility with $R > 0$:

$$U(x) = \begin{cases} \frac{1}{1 - R}x^{1-R}, & x \in [0, \infty), \quad \text{if } R \neq 1; \\ \log(x), & x \in (0, \infty), \quad \text{if } R = 1. \end{cases}$$

Hence, $U$ is PHARA and there is only one part in the domain $(n + 1 = 1)$; see parameters in table 1. In addition, implied by this utility, the condition of nonnegative wealth holds:

$$X_t \geq 0, \quad \text{for any } t \in [0, T).$$

| $k$ | $a_k$ | $\gamma^-_k$ | $\gamma^+_k$ | $R_k$ | $A_k$ | $q_k$ | $p_k$ |
|-----|------|-------------|-------------|------|------|------|------|
| $k = 0$ | -\infty | -\infty | -\infty | $R$ | 0 | 1 | 0 |
| $k = 1$ | $0$ | NA | NA | NA | NA | NA | NA |

**Table 1.** Parameter specification for the PHARA utility function (13) and theorem 1 ($U$ is PHARA: $n + 1 = 1$) according to the notation of the CRRA utility in Merton (1969).

**Example 2** In the context of Carpenter (2000), a hedge fund manager makes the investment decision. The financial market also follows a multi-asset Black-Scholes model (5)–(10) and the condition of nonnegative wealth:

$$X_t \geq 0, \quad \text{for any } t \in [0, T).$$

According to the notation of Carpenter (2000), there is an option compensation scheme under which the decision-maker (i.e. the manager) has the following payoff function:

$$\Theta(x) = \alpha(x - B_T)^+ + K, \quad x \in [0, \infty),$$

where $B_T := B_0e^{\rho T}$ is a riskless benchmark, $B_0$ is a constant, $\alpha > 0$ is the number of options and $K > 0$ is a constant amount of wealth. The option compensation scheme means that the decision-maker has his/her own payoff $\Theta(x)$ if the terminal fund wealth is $X_T$, where the function $\Theta$ is actually an option. The decision-maker has a HARA utility $\hat{U}(W)$ on his/her own payoff $W$:

$$\hat{U}(W) = \frac{1 - \gamma}{\gamma} \left( \frac{A(W - \omega)}{1 - \gamma} \right)^\gamma, \quad W \in [K, \infty),$$

where parameters include $\gamma < 1$ (as above, $\hat{U}$ is log if $\gamma = 0$), $\omega < K$ and $A > 0$.

Thus, the decision-maker’s actual utility is defined by a composition function on the fund wealth $x \geq 0$:

$$U(x) := \hat{U} \circ \Theta(x)$$
The concave envelope of all constants given in Lin et al. (2017) while the policyholder’s payoff is behavioral preference (i.e. S-shaped preference):

\[
U(x) := \tilde{U}(\Psi_1(x)) = \begin{cases} 
(1 - \gamma)\gamma^{y}(x - \delta L_T^p)\gamma, & \text{if } x > \frac{L_T^p}{\alpha}; \\
(x - L_T^p)\gamma, & \text{if } \frac{L_T^p}{1 - \gamma} \leq x < \frac{L_T^p}{\alpha}; \\
0, & \text{if } 0 \leq x < \frac{L_T^p}{1 - \gamma}.
\end{cases}
\]

(26)

It can be checked straightforward that \( U \) is PHARA. We define \( U^{**} \) as the concave envelope of \( U \). In the Case A1 \((1 - \gamma > \alpha)\), according to the notation of Lin et al. (2017), we have

\[
U^{**}(x) = \begin{cases} 
(1 - \delta \alpha)\gamma^y \left(x - \frac{1 - \delta}{1 - \delta \alpha} L_T^p\right)^\gamma, & \text{if } x > \frac{L_T^p}{\alpha}; \\
(x - L_T^p)^\gamma, & \text{if } \frac{L_T^p}{1 - \gamma} \leq x < \frac{L_T^p}{\alpha}; \\
0, & \text{if } 0 \leq x < \frac{L_T^p}{1 - \gamma}.
\end{cases}
\]

(27)

where \( \frac{L_T^p}{1 - \gamma} \) is a tangent point. Hence, both \( U \) and \( U^{**} \) are PHARA; see parameters of \( U^{**} \) in Table 3.

Example 3 In the context of Lin et al. (2017), the decision-maker is an insurer with participating insurance contracts and aims to solve the optimal asset allocation. The market setting is a standard one-dimensional Black-Scholes model (5)–(10) with \( m = 1 \). In this example, the decision-maker has a behavioral preference (i.e. S-shaped preference):

\[
\tilde{U}(W) = \begin{cases} 
W^\gamma, & \text{if } W \geq 0; \\
-\lambda(-W)^\gamma, & \text{if } W < 0.
\end{cases}
\]

(24)

It means that the decision-maker shows risk aversion when exceeding the reference point 0 and loss aversion when falling below. Moreover, there is a so-called participating contract such that according to different levels, the total wealth \( X_T \) is shared between the insurer \( \Psi(X_T) \) and the policyholder \( X_T - \Psi(X_T) \) with different proportions. The contract hence provides a minimal guarantee for the policyholder and a high-reward incentive for the insurer. Mathematically, the contract results in a piecewise linear payoff for the insurer (in the following discussion, \( \gamma, \delta, \alpha \in (0, 1) \) and \( L_T^p > 0 \) are all constants given in Lin et al. 2017):

\[
\Psi(X_T) = \begin{cases} 
(1 - \delta \alpha)X_T - (1 - \delta)L_T^p, & \text{if } X_T \geq \frac{L_T^p}{\alpha}; \\
X_T - L_T^p, & \text{if } \frac{L_T^p}{1 - \gamma} \leq X_T < \frac{L_T^p}{\alpha}; \\
0, & \text{if } 0 \leq X_T < \frac{L_T^p}{1 - \gamma}.
\end{cases}
\]

(25)

while the policyholder’s payoff is \( X_T - \Psi(X_T) \). Hence, the optimization problem of the insurer is to maximize the expected utility of the corresponding payoff, which is exactly Problem (10) with the objective function defined as \( U := \tilde{U} \circ \Psi \). Specifically,

\[
U(x) := \tilde{U}(\Psi_1(x)) = \begin{cases} 
(1 - \gamma)\gamma^y \left(x - \frac{1 - \delta}{1 - \delta \alpha} L_T^p\right)^\gamma, & \text{if } x > \frac{L_T^p}{\alpha}; \\
(x - L_T^p)^\gamma, & \text{if } \frac{L_T^p}{1 - \gamma} \leq x < \frac{L_T^p}{\alpha}; \\
0, & \text{if } 0 \leq x < \frac{L_T^p}{1 - \gamma}.
\end{cases}
\]

(26)

Table 3. Parameter specification for the PHARA utility function (13) and theorem 1 (\( U^{**} \) is PHARA: \( n + 1 = 2 \), non-differentiable points: \( a_0 \), tangent point: \( a_1 \)) according to the notation in Carpenter (2000).

| \( a_k \) | \( \gamma_k^- \) | \( \gamma_k^+ \) | \( R_k \) | \( A_k \) | \( q_k \) | \( p_k \) |
|---|---|---|---|---|---|---|
| 0 | 0 | \infty | \( U'(\hat{x}) \) | 0 | NA | 0 |
| 1 | \hat{x} | \( U'(\hat{x}) \) | 1 - \gamma | \( B_T - \frac{K - \omega}{\alpha} \) | \Phi \left( d_1 \left( \frac{U'(\hat{x})}{y^\gamma 6\gamma} \right) \right) | 0 |
| 2 | \infty | 0 | NA | NA | NA | NA |
In addition, there is a lower-bound liquidation constraint for the wealth:

$$X_T \geq bx_0, \text{ a.s.}$$  \hspace{1cm} (30)

Here $bx_0 > 0$ means a threshold that the terminal fund wealth has to exceed in order to avoid liquidation. By multiplying the discount factor $e^{r(T-t)}$, one obtains a liquidation constraint $X_t \geq bx_0 e^{-r(T-t)}$ for any $t \in [0,T]$. As a result, the decision-maker’s objective is defined by a composition function on the fund wealth $x \geq bx_0$:

$$U(x) := \tilde{U} \circ \Theta(x)$$  \hspace{1cm} (31)

and the concave envelope becomes

$$U^{**}(x) := \begin{cases} 
U(a_2) + U'(a_2) & x \in [bx_0, (1+c)x_0); \\
(x - a_2)^p & x \in [(1+c)x_0, \infty). 
\end{cases}$$  \hspace{1cm} (32)

where $(1+c)x_0 > x_0$ is the solution of

$$\frac{U((1+c)x_0) - U(bx_0)}{(1+c)x_0 - bx_0} = U'(1+c)x_0).$$  \hspace{1cm} (33)

Hence, both $U$ and $U^{**}$ are PHARA; see parameters of $U^{**}$ in Table 5.

**Example 5** We provide another example with a graph for the PHARA utility family to show its generality. Figure 1 shows the contour of a general (highly non-concave and non-differentiable) utility. The concave envelope is the smallest concave and continuous function dominating the original utility function, and hence involves a lot of tangent lines and linear connections; see appendix 2 for more details. Later we will show that, using the unified formula in theorem 1, we can directly compute the closed-form optimal portfolio for this example and know the risk-taking behavior on each part of the utility’s domain. This example is revisited in section 5.2.

### 3. A unified formula of the optimal portfolio

We proceed to show the optimal terminal wealth, the wealth process and the portfolio vector of PHARA utilities for Problem (10). The basic approach is the martingale method and the concavification technique. Above all, proposition 2 proposes a five-term division of the optimal wealth process. The proof is included in appendix 3.

**Proposition 2** Suppose that the concave envelope $U^{**}$ has the form in definition 1 and $R_k \in [0, \infty]$ for each $k \in \{0, 1, \ldots, n\}$. For Problem (10),

(1) the optimal terminal wealth is given by

$$X^*_T = \sum_{k=0}^{n} \left[ a_k \mathbb{I}[\gamma^1, \gamma^2] \right]$$

$$+ \left( A_k + \frac{\gamma^1}{\gamma^2} \right)^{-1} \left( a_k - A_k \right) \mathbb{I}[\gamma^1, \gamma^2] \right) \times \mathbb{I}[R_k \leq 0, \infty) + \left( a_k + \frac{1}{\alpha_k} - a_k \right) \mathbb{I}[\gamma^1, \gamma^2] \right) \times \mathbb{I}[\gamma^1, \gamma^2] \right) \times \mathbb{I}[R_k \leq 0, \infty) + \left( a_k + \frac{1}{\alpha_k} - a_k \right) \mathbb{I}[\gamma^1, \gamma^2] \right), \text{ a.s.,}$$  \hspace{1cm} (34)

**Table 5.** Parameter specification for the PHARA utility function (32) and theorem 1 ($U^{**}$ is PHARA; $n + 1 = 2$, non-differentiable points: $a_0, a_2$, tangent point: $a_1$) according to the notation in He and Kou (2018).
where $y^*$ is a unique positive number that satisfies
\[
\mathbb{E} [\xi_7 X_T^*] = x_0. \tag{35}
\]

(2) The optimal wealth at time $t \in [0, T]$ is given by
\[
X_t^* = X_t^D + X_t^A + X_t^R + X_t^R
\]
\[
= \sum_{k=0}^n \left( X_t^{D_k} + X_t^{A_k} + X_t^{R_k} + X_t^{R_k} \right), \tag{36}
\]
where the terms are given by
\[
X_t^{D_k} := e^{-r(T-t)} a_k
\]
\[
\times \left[ \Phi \left( d_1 \left( \frac{y_k^+}{y^*_k} \right) \right) - \Phi \left( d_1 \left( \frac{y_k^-}{y^*_k} \right) \right) \right],
\]
\[
X_t^{A_k} := e^{-r(T-t)} A_k
\]
\[
\times \left[ \Phi \left( d_1 \left( \frac{y_k^{-1}}{y^*_k} \right) \right) - \Phi \left( d_1 \left( \frac{y_k^+}{y^*_k} \right) \right) \right]
\]
\[
\times 1_{\{R_k = 0, \infty \}},
\]
\[
X_t^{R_k} := e^{-r(T-t)} (a_k - A_k)
\]
\[
\times \left[ \Phi \left( d_1 \left( \frac{y_k^{-1}}{y^*_k} \right) \right) - \Phi \left( d_1 \left( \frac{y_k^+}{y^*_k} \right) \right) \right]
\]
\[
\times 1_{\{R_k = 0, \infty \}},
\]
\[
X_t^{R_k} := e^{-r(T-t)} (a_k - A_k)
\]
\[
\times \left[ \Phi \left( d_1 \left( \frac{y_k^{-1}}{y^*_k} \right) \right) - \Phi \left( d_1 \left( \frac{y_k^+}{y^*_k} \right) \right) \right]
\]
\[
\times 1_{\{R_k = 0, \infty \}},
\]
\[
and a transformation function is denoted by $d(z, h) := \frac{1}{-||\theta||^2 \sqrt{T-t}} \left( \log(z) + (r + ||\theta||^2 (T-t)) + h ||\theta||^2 \sqrt{T-t} T \right)$. We further set $d_0(z) := d(z, 0), d_1(z) := d(z, 1), d_k(z) := d(z, 1 - R_k), k = 0, \ldots, n$.

The optimal terminal wealth (34) contains three terms, while the optimal wealth process (36) contains five terms. The term $X_t^D$ comes from the non-differentiable point $a_k$ of the utility function (the left- and right- derivatives are not equal: $y_k^+ \neq y_k^-$. We call $X_t^D$ the first-order risk aversion term, and later in theorem 1 it leads to the term $\pi_i^{(4)}$ with the same name in the optimal portfolio (38). The name is because compared to Pratt (1964)’s classic ‘second-order’ risk aversion caused by concavity, the non-differentiability causes ‘first-order’ risk aversion; see Segal and Spivak (1997). We call the terms $X_t^A$ and $X_t^R$ the loss aversion terms, as they originate from a loss-averse benchmark (e.g. $A_k$). They lead to the term $\pi_i^{(3)}$ with the same name in the portfolio. We call the terms $X_t^R$ and $X_t^R$ the risk-seeking terms and they lead to the term $\pi_i^{(2)}$ with a same name in the portfolio. These names are explained in detail after theorem 1. Lastly, the terms $X_t^A$ and $X_t^R$ come from a CRRA utility on the part ($a_k$, $a_{k+1}$), while $X_t^A$ and $X_t^R$ come from a CARA utility on the part.

Now we present the unified formula of the optimal portfolio of PHARA utilities for Problem (10). The proof is included in appendix 3. Basically, we apply the martingale method on the optimal wealth $X_t^*$ of proposition 2 and hence obtain the optimal portfolio vector.

**Theorem 1** Suppose that the concave envelope $U^*$ has the form in definition 1 and $R_k \in [R, 0)$ for each $k \in \{0, 1, \ldots, n\}$ (where $R \in (0, \infty)$). For Problem (10), the optimal portfolio vector at time $t \in [0, T)$ is
\[
\pi_t = \frac{(\sigma^{-1})^{-1} \theta}{R} X_t^*
\]
\[
- e^{-r(T-t)} (\sigma^{-1})^{-1} \theta \sum_{k=0}^n A_k q_k 1_{\{R_k \neq 0\}}
\]
\[
- e^{-r(T-t)} (\sigma^{-1})^{-1} \theta \sum_{k=0}^n a_k p_k
\]
\[
\pi_t^{(1)} + \pi_t^{(2)} + \pi_t^{(3)} + \pi_t^{(4)}, \tag{38}
\]
where $X_t^*$ is the optimal wealth amount at time $t$, $y^*$ is a unique positive number that satisfies (35), $\Phi(\cdot)$ is the standard normal cumulative distribution function and we define
\[
d_1(z) := \frac{1}{-||\theta||^2 \sqrt{T-t}} \left( \log(z) + \left( r - \frac{||\theta||^2}{2} \right) (T-t) \right),
\]
\[
z > 0,
\]
\[
p_k := \Phi \left( d_1 \left( \frac{y_k^+}{y^*_k} \right) \right) - \Phi \left( d_1 \left( \frac{y_k^-}{y^*_k} \right) \right),
\]
\[
q_k := \Phi \left( d_1 \left( \frac{y_k^{-1}}{y^*_k} \right) \right) - \Phi \left( d_1 \left( \frac{y_k^+}{y^*_k} \right) \right),
\]
\[
k = 0, 1, \ldots, n. \tag{40}
\]

**Remark 3** (i) As $\sum_{k=0}^n p_k + \sum_{k=0}^n q_k = 1$, $p_k$ and $q_k$ are interpreted to be probabilities.
(ii) By definition and computation, we have $q_k = 0$ if $R_k = 0$; in this case, there is no need to specify $A_k$.
(iii) We have $p_k = 0$ if $a_k$ is differentiable.
Under a general PHARA utility, the optimal portfolio is expressed in a unified formula consisting of four terms. The first term \(\pi^{(1)}\) is called the \textit{Merton relative risk aversion term} (RA), i.e. the classic constant percentage strategy in Merton (1969). The second term \(\pi^{(2)}\) is called the \textit{risk-seeking term} (RS), which only appears on the linear part of the concave envelope \((R_0 = 0)\). As Carpenter (2000) shows, the non-concave parts in the utility (or, the linear parts in the concave envelope) lead to a strong increase in the investment of the risky asset. The risk-seeking term acts as the increase of the risky investment. Such an increase \(\pi^{(2)}\) is proportional to the length \((a_{k+1} - a_k)\) of the linear parts in the concave envelope. The optimal percentage in the risky asset exceeds 100% when time \(t\) approaches the terminal time \(T\) and the wealth amount lies in the linear parts. In figure 1, it happens when \(X^*_t\) lies in \((a_1, a_2) \cup (a_2, a_3)\) and \(t \to T\).

The third term \(\pi^{(3)}\) is called the \textit{loss aversion term} (LA). There are some benchmarks in the utility, e.g. a wealth level \(A \in \mathbb{R}\) for the HARA utility \(U(x) = \frac{1}{\gamma} (x - A)^{1-\gamma}\) and a reference point for the S-shaped utility \(U(x) = \frac{1}{\gamma} (x - A)^{1-\gamma} + (-\lambda) \frac{1}{\gamma} (A - x)^{1-\gamma}\). These benchmarks are regarded as deposit wealth and induce a decrease of the risky investment to avoid loss. \(\pi^{(3)}\) is actually a weighted sum of all discounted benchmarks (in figure 1, they are \(A_0, A_1, A_2\)). The weight is given by \(q_k\).

The fourth term \(\pi^{(4)}\) is called the \textit{first-order risk aversion term} (First-order RA). Pratt (1964) and Segal and Spivak (1997) discuss the different orders of risk aversion. Segal and Spivak (1997) show that a non-differentiable point leads to a ‘first-order’ risk premium. Here we find that non-differentiability also causes a decrease \(\pi^{(4)}\) in the risky position. \(\pi^{(4)}\) is actually a weighted sum of all discounted non-differentiable points (in figure 1, they are \(a_0, a_1, a_2, a_3\)). The weight is given by \(p_k\) (if \(a_k\) is differentiable, \(p_k = 0\)). As a comparison, the Merton term \(\pi^{(1)}\) is actually caused by ‘second-order’ risk aversion; see Pratt (1964) for discussion on risk premium. Later in section 5.2 we will find the decreasing effect of \(\pi^{(4)}\) is much greater than that of \(\pi^{(3)}\).

Finally, we show a formula of the optimal portfolio for general case \((R_0 \in [0, \infty))\) in theorem 2.

**Theorem 2** (General case) Suppose that the concave envelope \(U^{**}\) has the form in definition 1 and \(R_k \in [0, \infty)\) for each \(k \in \{0, 1, \ldots, n\}\). For Problem (10), the optimal portfolio vector at time \(t \in [0, T]\) is given by

\[
\pi^*_t = (\sigma^\top)^{-1} \theta \sum_{k=0}^{n} R_k^{-1} X^R_k I_{\{R_k \neq 0, \infty\}} - e^{-r(T-t)} a_k \Phi\left[d_1 \left(\frac{y^+}{\nu^+} \right) \right] I_{\{R_0 = 0\}} + e^{-r(T-t)} \frac{a_{k+1} - a_k}{\Vert \theta \Vert_2 \sqrt{T - t}} \Phi\left(d_1 \left(\frac{y^+}{\nu^+} \right) \right) I_{\{R_0 = 0\}} + e^{-r(T-t)} \frac{1}{a_k} \Phi\left(d_1 \left(\frac{y^+}{\nu^+} \right) \right) - \Phi\left(d_1 \left(\frac{y^+}{\nu^+} \right) \right) \right] \times I_{\{R_0 = \infty, A_1 = -\infty, A_2 > 0\}}. \tag{41}
\]

This theorem shows a formula (2) of the optimal portfolio for the general case \((R_0 \in [0, \infty))\), while \(\pi^*_t\) relies on \(X^R_k\) defined in (37). Here \(X^R_k\) is a part of \(X^*_t\). Hence, for the more general case, this portfolio (41) is not a feedback form of the wealth \(X^*_t\).

**4. Examples revisiting**

The unified formula (38) includes various closed-form portfolios in the literature as examples. We illustrate how to use the formula to directly write down the optimal portfolio by examples 1–4.

**Example 1** (cont.). Now we apply our formula (38) to solve Problem (10) in example 1.

- According to the expression of the CRRA utility \(U\) in (15), we directly have parameters in table 1. Thus, according to (38) in theorem 1, the optimal portfolio of Problem (10) is given by

\[
\pi^*_t = (\sigma^\top)^{-1} \theta \frac{X^*_t}{R}, \quad t \in [0, T],
\]

which coincides with Equation (43) in Merton (1969). It is the well-known Merton portfolio with the percentage \((\sigma^\top)^{-1} \theta \frac{1}{\alpha}\). We will discuss other portfolios with the Merton portfolio in section 5.

- According to the expression of the CARA utility \(U\) in (17), we directly have parameters in table 2. Thus, according to (41) in theorem 2, the optimal portfolio of Problem (10) is given by

\[
\pi^*_t = (\sigma^\top)^{-1} \theta e^{-r(T-t)} \frac{a_1 - a_0}{\alpha} \Phi\left(d_1 \left(\frac{y^+}{\nu^+} \right) \right) - e^{-r(T-t)} \frac{a_0 p_{\text{los}}}{\alpha} \Phi\left(d_1 \left(\frac{y^+}{\nu^+} \right) \right) \tag{44}
\]

where we use the market parameters of our current paper (section 2) as notation. We can see \(\pi^*_t\) in (44) is exactly the optimal portfolio given by Equation (25) in Carpenter (2000).
EXAMPLE 3 (cont.). Now we apply our formula (38) to solve Problem (10) in example 3. According to the expression of $U^{**}$ in (27), we directly have parameters in Table 4.

Thus, according to theorem 1, we directly have the optimal portfolio for both Problems (10) and (12):

$$
\pi^*_t = \frac{\theta}{\sigma(1-p)} X^*_t + \frac{e^{-r(T-t)}}{\sqrt{T-t}} (a_1 - a_0) \Phi' \left( d_1 \left( \frac{\gamma^*_0}{\gamma^* \xi^*_t} \right) \right)
$$

where we use the market parameters of our current paper (section 2) as notation.

It is checked by tedious computation that $\pi^*_t$ in (45) is exactly the same as the optimal portfolio given by (34) in Lin et al. (2017), where they use a totally different expression. Compared to their portfolio (34), the portfolio (45) is directly given by our formula without complicated derivation and has a four-term division with clear economic meanings.

EXAMPLE 4 (cont.). Now we apply our formula (38) to solve Problem (10) in example 4. According to the expression of $U^{**}$ in (32), we directly have parameters in Table 5.

Thus, according to (38) in theorem 1, we directly have the optimal portfolio for both Problems (10) and (12):

$$
\pi^*_t = \frac{\theta}{\sigma(1-p)} X^*_t + \frac{e^{-r(T-t)}}{\sqrt{T-t}} (a_1 - a_0) \Phi' \left( d_1 \left( \frac{\gamma^*_0}{\gamma^* \xi^*_t} \right) \right)
$$

where we use the market parameters of our current paper (section 2) as notation. We can see $\pi^*_t$ in (46) is exactly the optimal portfolio given by Equation (3.4) in He and Kou (2018).

Table 6 summarizes the portfolios above and in the literature in terms of four-term division. In the above and other literature of continuous-time portfolio selection, the type of optimal investment problems involve long computation and complex expression in the portfolio. In the following section 5, we will see that our formula (38) not only serves to ease computation, but also provides analytical tractability for financial analysis with a delicate four-term division.

5. Economic meanings

We illustrate the economic meanings of our optimal portfolio formula for PHARA utilities in three aspects. First, we conduct an asymptotic analysis to the optimal portfolio $\pi^*_t$ and the optimal wealth $X^*_t$ in terms of the pricing kernel $\xi_t$ for $t \in (0, T)$. Second, we numerically plot and illustrate the meaning of the PHARA utility. Finally, we compare the optimal portfolios under the PHARA utility (using example 5) with other utility families analytically and numerically. We elaborate on special features of the PHARA portfolio.

5.1. Asymptotic analysis

We conduct an asymptotic analysis for the general PHARA utility to illustrate financial insights in theorem 3, following the ideas of Carpenter (2000) and Liang and Liu (2023). As $\xi_t$ is an indicator of the market state, by asymptotic analysis we know the risk-taking behavior of the portfolio. Theorem 3 serves as a general version of asymptotic analysis and adopts detailed limiting techniques. One can directly apply theorem 3 for asymptotic financial analysis given a specific PHARA utility. The proof is stated in appendix 3.

THEOREM 3 Suppose the setting in theorem 1 holds. Fix $t \in (0, T)$ and $y^* \in (0, \infty)$.

(i) As $\xi_t \to 0$, we have

$$
X^*_t \to \infty, \quad \pi^*_t \to \infty, \quad \frac{\pi^*_t}{X^*_t} \to \frac{1}{R} (\sigma^*)^{-1} \theta, \quad (47)
$$

and more detailed results for different terms are given in tables 7–8.

(ii) As $\xi_t \to \infty$, we have

$$
\frac{X^*_t}{a_0 e^{-r(T-t)}} \to 1, \quad \pi^*_t \to 0, \quad \frac{\pi^*_t}{X^*_t} \to 0, \quad (48)
$$

and more detailed results for different terms are given in tables 7–8.

Table 6. Unifying the literature.

| Literature                  | Context             | RA | RS | LA | First-order RA |
|-----------------------------|---------------------|----|----|----|----------------|
| Merton (1969)               | CRRA and CARA       |    |    |    |                |
| Carpenter (2000)            | Option payoff with HARA utilities |    |    |    |                |
| Berkelaar et al. (2004)     | Loss aversion case  |    |    |    |                |
| Berkelaar et al. (2004)     | Kinked power utility case |   |    |    |                |
| Lin et al. (2017)           | Participating insurance contracts |   |    |    |                |
| He and Kou (2018)           | First-Loss schemes in hedge funds |    |    |    |                |
| He et al. (2019)            | Incentive schemes in pension funds |    |    |    |                |
| Liang and Liu (2020)        | Principal’s constraint |    |    |    |                |
We first analyze the wealth process. When the market state is good (\( \xi_t \rightarrow 0 \)), we see that the wealth is going to infinity, which means that the portfolio performs well and leads to an increasing wealth process. The increase is mainly due to \( X_t^R \), caused by the strictly concave CRRA parts in the utility. The CRRA part on \( (a_0, \infty) \) also provides an amount of wealth \( X_t^A \), which approaches a positive term \( a_0 e^{-\gamma_0(t-T)} \). In this scenario, the other wealth terms \( X_t^{A_k}, k = 0, \ldots, n-1 \) and \( X_t^D \) are negligible and approach 0. When the market state is bad (\( \xi_t \rightarrow \infty \)), we see that the wealth is going to a lower bound \( (a_0 e^{-\gamma_0(t-T)}) \), which means that the portfolio performs badly and lies at the least possible level (otherwise the utility value will be negative infinity). This lower bound level is also known as the liquidation boundary; see He and Kou (2018).

The amount of wealth is contributed by the liquidation boundary; see He and Kou (2018). The CRRA part on \( (a_0, \infty) \) also provides an amount of wealth \( X_t^A \), which approaches a positive term \( a_0 e^{-\gamma_0(t-T)} \). In this scenario, the other wealth terms \( X_t^{A_k}, k = 0, \ldots, n-1 \) and \( X_t^D \) are negligible and approach 0. When the market state is good (\( \xi_t \rightarrow 0 \)), we see that the wealth is going to a lower bound \( (a_0 e^{-\gamma_0(t-T)}) \), which means that the portfolio performs badly and lies at the least possible level (otherwise the utility value will be negative infinity). This lower bound level is also known as the liquidation boundary; see He and Kou (2018).

5.2. Numerical illustration

We illustrate the economic meaning of the PHARA utility \( U \) by investigating the optimal portfolio in example 5 and figure 1. Note that here \( m = 1 \), and there is no bold symbol. We denote the one-dimensional optimal portfolio by \( \pi^* \). First, we can directly write the optimal portfolio from (38):

\[
\pi^*_t = \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+.
\]

In figure 2, we numerically demonstrate the optimal portfolio (49) for the PHARA utility \( U \) of figure 1. The optimal portfolios under a HARA utility and a SAHARA utility (symmetric asymptotic HARA; proposed by Chen et al. 2011) are also plotted. We will later discuss and compare these portfolios in section 5.3.

By numerical illustration and analytical analysis to (49), we obtain the following economic meanings, some of which is also shown in asymptotic analysis in section 5.1. Here the effect of non-differentiable points is our main novel finding.

(a) Generally, the contour of the optimal portfolio is "multiple-peak-multiple-valley" with the tail trend to the Merton term.

(b) Peak on linearity: The optimal portfolio gambles dramatically on the linear parts (here, \( (a_1, a_2), (a_2, a_3) \) of the concave envelope \( U^\pi \). Especially, the risky investment percentage exceeds 100% if it approaches to the terminal time. From the formula (49), we know

\[
\pi^*_t = \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+ + \frac{\theta}{\sigma R} X_t^+.
\]
utility family is also a generalization of the HARA family. Analytically, the SAHARA (symmetric asymptotic HARA) utility and the SAHARA utility proposed by Chen et al. (2011) is a special case of non-monotone absolute risk aversion (ARA) functions, where the utility $U$ satisfies

$$\text{ARA}(x) := -\frac{U''(x)}{U'(x)} = \frac{1}{\sqrt{\frac{b^2}{x^2} + \frac{(x-c)^2}{v^2}}}.$$

where $\psi > 0$, $\beta > 0$ and $v \in \mathbb{R}$ are parameters. According to proposition 2.2 of Chen et al. (2011), if $v = 0$, the expression of a SAHARA utility has a domain $\mathbb{R}$ and is given by

$$U(x) = c_1 + c_2 \tilde{U}(x),$$

where

$$\tilde{U}(x) := \begin{cases} 
-\frac{1}{\psi^2 - 1} \left( x + \sqrt{\beta^2 + x^2} \right)^{-\psi} & \text{if } \psi \neq 1; \\
\frac{1}{2} \log \left( x + \sqrt{\beta^2 + x^2} \right) & \text{if } \psi = 1.
\end{cases}$$

for $x \in \mathbb{R}$, (52)

that the percentage actually tends to infinity with a rate $(T - t)^{-1/2}$ when $t \to T$. The gambling effect is because of the risk-seeking term $\pi^{(2)}$.

(c) **Valley at non-differentiability**: The optimal portfolio becomes extremely conservative at the non-differentiable points (here, $a_0, a_1, a_2, a_4$) of the concave envelope $U^\ast$. We see that the percentage actually tends to zero at non-differentiable points. The conservative effect is because of the first-order risk aversion term $\pi^{(1)}$.

(d) **Merton trend**: The optimal risky investment percentage tends to the Merton percentage (here, $\frac{\mu}{\sigma^2} = 0.8$) if the wealth is large enough. In this case, for $i = 2, 3, 4$, we have $\pi^{(i)} / X^\ast \to 0$ ($X^\ast \to \infty$). As a result, the trend is due to $\pi^{(1)}$.

(e) The decreasing effect of the loss aversion term $\pi^{(3)}$ at the benchmark points (here, $A_1, A_4$) is slight. At these points, surprisingly, the decision-maker is still taking great risk.

### 5.3 Portfolio comparison

In figure 2, we also plot the strategies under the HARA utility and the SAHARA utility proposed by Chen et al. (2011). Analytically, the SAHARA (symmetric asymptotic HARA) utility family is also a generalization of the HARA family as the PHARA utility. The SAHARA utility characterizes a special case of non-monotone absolute risk aversion (ARA) functions, where the utility $U$ satisfies

$$\text{ARA}(x) := -\frac{U''(x)}{U'(x)} = \frac{1}{\sqrt{\frac{b^2}{x^2} + \frac{(x-c)^2}{v^2}}}.$$

where $\psi > 0$, $\beta > 0$ and $v \in \mathbb{R}$ are parameters. According to proposition 2.2 of Chen et al. (2011), if $v = 0$, the expression of a SAHARA utility has a domain $\mathbb{R}$ and is given by

$$U(x) = c_1 + c_2 \tilde{U}(x),$$

where

$$\tilde{U}(x) := \begin{cases} 
-\frac{1}{\psi^2 - 1} \left( x + \sqrt{\beta^2 + x^2} \right)^{-\psi} & \text{if } \psi \neq 1; \\
\frac{1}{2} \log \left( x + \sqrt{\beta^2 + x^2} \right) & \text{if } \psi = 1.
\end{cases}$$

for $x \in \mathbb{R}$, (52)

where $c_1, c_2 \in \mathbb{R}$ are constants; we simply set $c_1 = 0$ and $c_2 = 1$. If $\beta = 0$, the SAHARA utility reduces to the HARA
utility. In our paper, on the contrary, the PHARA utility focuses on the different risk aversion on different parts of the domain. For each part, the ARA function is hyperbolic, i.e. for \( k \in \{0, 1, \ldots, n\} \), for any \( x \in (a_k, a_{k+1}) \),

\[
\text{ARA}(x) := \frac{U''(x)}{U'(x)} = \frac{R_k}{x-A_k}
\]

\[
= \begin{cases} 
0, & \text{if } R_k = 0; \\
\alpha_k, & \text{if } R_k = \infty, A_k = -\infty, \alpha_k \in (0, \infty); \\
\frac{1}{R_k} x - \frac{\alpha_k}{R_k}, & \text{if } R_k \in (0, \infty),
\end{cases}
\]

(53)

where \( A_k, R_k, \alpha_k \) are given in definition 1. It is clear that the ARA function (53) is locally monotone (decreasing) but not globally monotone. As a result, the two families of PHARA and SAHARA utility functions lead to very different optimal portfolios.

In the one-dimensional Black-Scholes model \((m = 1)\), recalling the CRRA portfolio (42) in example 1, Merton (1969)'s optimal portfolio for

\[
U(x) = \frac{1}{1-R} x^{1-R}, \quad x \in (0, \infty),
\]

(54)
is given by

\[
\pi_t^{\text{Merton}} = \frac{\theta}{\sigma R} X_t^*, \quad t \in [0, T],
\]

(55)

which is proportional to the corresponding optimal wealth \( X_t^* \). For the HARA utility, using the convention in definition 1, we write

\[
U(x) = \frac{1}{1-R} (x - a_0)^{1-R}, \quad x \in (a_0, \infty),
\]

(56)

where \( a_0 \) is a constant in \( \mathbb{R} \). According to theorem 1, the portfolio for (56) is directly given by

\[
\pi_t^{\text{HARA}} = \frac{\theta}{\sigma R} \left( X_t^* - a_0 e^{-(T-t)} \right), \quad t \in [0, T].
\]

(57)
The optimal portfolio (57) is a ratio of the optimal wealth (Merton term) plus an LA correction. For the PHARA utility, the optimal portfolio (38) is a ratio of the optimal wealth (Merton term) plus three corrections (RS, L.A, First-order RA) based on different parts of the utility function. Finally, the SAHARA utility leads to the optimal portfolio

\[
\pi_t^{\text{SAHARA}} = \frac{\theta}{\sigma \psi} \sqrt{(X_t^* - ve^{-(T-t)})^2 + b_t^2}, \quad t \in [0, T],
\]

(58)

where \( b_t := \beta e^{-(r + \frac{\psi}{2}) (T-t)} \). This portfolio does not contain a proportional Merton term anymore. It is due to the non-monotone feature of the ARA function (50). Therefore, there is an essential discrepancy between PHARA and SAHARA utility families.

5.4. Numerical comparison

In figure 2, the portfolios (57) and (58) under the HARA utility and the SAHARA utility are plotted and compared with the proposed portfolio (49).

Above all, all these portfolios share the Merton trend if the wealth is large enough. The HARA portfolio is a simplified version of the PHARA portfolio. When the market state is good (\( \xi_t \to 0 \)), the asymptotic behavior of the HARA portfolio is to invest in a Merton term. When the market state is bad (\( \xi_t \to \infty \)), that of the HARA portfolio is to invest all in the risk-free asset. These two features are the same in the PHARA portfolio. In between, the HARA portfolio is to simply increase the risky investment with respect to the wealth. The PHARA portfolio has a sophisticated behavior: peak on linearity (here, \( (a_1, a_2) \)) and valley at non-differentiability (here, \( a_0, a_1, a_2, a_3 \)) as discussed in section 5.2.

For the SAHARA utility, the key parameter is \( \beta > 0 \). A feature of the SAHARA portfolio is to invest more in risky assets when the market state is bad and the wealth is low, i.e. to gamble back at a recession state. When the market state is good, the SAHARA portfolio becomes conservative and decreases the risky investment. We additionally plot figure 3 with the only difference \( \beta = 5 \). As \( \beta \) is smaller is figure 3, the SAHARA portfolio becomes more similar to the HARA portfolio. This gambling effect at a bad state is mitigated in figure 2 when \( \beta \) is small.

In conclusion, the PHARA portfolio is more subtle than the HARA portfolio and the latter is more subtle than the Merton portfolio, while the SAHARA portfolio holds an investment logic opposite to the HARA portfolio to some degree.

6. Real-data study

We provide simulation results for the PHARA portfolio based on parameters collected empirically from the real-world financial data. We set \( m = 1 \) in the following setting.

We analyze the PHARA strategy (49) over a time span of 10 years from 10/04/2013 to 10/04/2023. The first 8 years serve as the in-sample period. The latest two years serve as the out-of-sample period. Our investment time horizon, \( T \), is set to be equivalent to the out-of-sample period of two years. The initial wealth, \( x_0 \), is arbitrarily set to $10, and the risk-free interest rate, \( r \), is set to 0.05 to capture the current two-year treasury rate. We select the SPDR S&P 500 ETF Trust (SPY) to be our single asset. The S&P 500 is a widely-accepted benchmark index and so we choose to trade an ETF that tracks this index. The estimated standard deviation of daily returns for the asset is denoted by \( \sigma \). The mean of daily returns, the expected return rate, is denoted by \( \mu \). We use the in-sample data to estimate \( \mu = 0.27 \) and \( \sigma = 0.24 \). Both \( \mu \) and \( \sigma \) are annualized to the two-year time horizon. This means that the average daily return is multiplied by 504 and the standard deviation of daily returns is multiplied by \( \sqrt{504} \); here 504 is the number of trading days in two years.

Figure 4 provides a snapshot example of one simulation of the strategy (49)'s performance for the one-dimensional
Figure 3. Comparison among portfolios (y-axis: ‘Risky Investment Percentage’ means the optimal percentage invested in the risky asset $\pi^*_t/X^*_t$, x-axis: ‘Wealth’ means $X^*_t$). All the setting is the same as figure 2 expect that here $\beta = 5$ in the SAHARA utility (50).

Figure 4. A simulation of the evolution of total wealth $X_t$ over the two-year investing period.

We can see a high degree of volatility but an overall positive outcome in this particular simulation. The first measure of performance we utilize is the Sharpe Ratio; see, e.g. Sharpe (1994). Let $p_o$ denote the set of daily returns of the optimal portfolio and $R_{po}$ denote the average of this set. Additionally, let $R_{fr} = r_{504}$ denote the daily risk-free interest rate. Finally, let $e_x$ denote the set of excess daily returns of the optimal portfolio. The Sharpe Ratio $S_a$ is given by

$$S_a = \frac{R_{po} - R_{fr}}{\sigma_{ex}},$$

where $\sigma_{ex}$ is the standard deviation of the set of excess daily returns of the optimal strategy. Figure 5 provides a histogram of simple returns and Sharpe Ratios over 1000 simulations. Note that outliers (returns greater or less than three standard deviations from the mean) are removed from the simple returns histogram. Additionally, note that the final Sharpe Ratios are annualized to the two-year out-of-sample period.

The second measure of performance we utilize is alpha or the capital asset pricing model (60); see, e.g. Perold (2004). We use the S&P 500 as a benchmark index representative of the market and calculate daily returns for the out-of-sample period; this set of returns is denoted by $m_a$. We also utilize $R_{po}$ and $R_{fr}$ in the calculation of $\alpha$:

$$\alpha = R_{po} - R_{fr} - \beta(R_{ma} - R_{fr}),$$

where $R_{ma}$ is the mean of the set of returns $m_a$, and the systematic risk $\beta := \text{Cov}(p_o, m_a)/\sigma_{ma}^2$ is given by the covariance of daily returns between the portfolio strategy and the market benchmark divided by the variance of the benchmark daily returns. The final alpha values are also annualized to the two-year out-of-sample period.

Figure 6 shows the distribution of the calculated alpha values for the simulations. Note that outliers are also removed from the histogram of alphas. We can see the majority of the distribution is greater than 0%. This shows us that we generate positive alpha with our strategy. Table 9 displays the mean and standard deviation for simple returns and our two measures of performance. From the summary statistics of simple returns, we see that the current PHARA portfolio
is highly volatile yet clearly successful. From Figure 5, we can see modality around a simple return of 100%. Additionally, Figure 5 shows us that Sharpe Ratio returns are almost all positive but mostly situated below 1.0. The Sharpe Ratio histogram is also clearly right-skewed. Traditionally, a Sharpe Ratio greater than 1.0 is regarded as a good performance. Our lower mean Sharpe Ratio demonstrates the fact that high volatility is a drawback of the PHARA portfolio. On the other hand, the high mean simple returns and positive mean alpha show that the PHARA portfolio provides high returns and better performance than market benchmarks. Specifically, the positive mean alpha shows that our strategy can outperform the market. Given this evidence, we conclude that our strategy may provide incredibly high rewards for a high volatility or risk.

### 7. Conclusion

We derive a unified formula for the optimal portfolio of the proposed piecewise hyperbolic absolute risk aversion (PHARA) utility family. In the field of asset allocation (e.g. hedge fund management), this formula can be directly applied to the portfolio strategy of a given utility function. The terms of the portfolio are illustrated with clear economic meaning on the risky behavior. We propose a general asymptotic analysis and find that non-concavity in the utility function causes great risk-taking, while non-differentiability greatly reduces the risky behavior. The PHARA portfolio is featured with a contour of ‘multiple-peak-multiple-valley’ with a tail trend to the Merton percentage, compared to the portfolios of HARA and SAHARA utilities. As a financial application, we empirically observe the performance of the PHARA portfolio over a two-year period. We see that the PHARA portfolio is able to generate high returns and positive alpha, yet one of the drawbacks of the PHARA portfolio is high volatility. We can use the PHARA utilities as building blocks to approximate the optimal portfolio in the cases that only information on concavity and differentiability of some intervals is available, which is left for a future direction.

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Appendices

Appendix I. Further discussion on market models

In the classic literature (Merton 1971, Karatzas et al. 1991), one uses the following standard Black-Scholes model to characterize the prices of $m \geq 1$ risky assets (stocks): for $i = 1, \ldots, m$,

$$dS_{it} = S_{it} \left( \mu_{it} dt + \sum_{j=1}^{l} \sigma_{ij} dW_{jt} \right), \quad 0 \leq t \leq T. \quad (A1)$$

Here, $(S_{it})_{0 \leq t \leq T}$ is the price process for the $i$th risky asset. They are modeled by an $l$-dimensional standard Brownian motion $(W_{it})_{i=1,\ldots,l}$, $0 \leq t \leq T$. Each one-dimensional Brownian motion $(W_{jt})_{0 \leq t \leq T}$, $j = 1, \ldots, l$, is interpreted as a source of randomness (systematic risks) and $l$ is the number of sources. It is always assumed more than the number of risky assets, i.e. $l > m$.

Further, $\mu_{it}$ models the instantaneous return rate (or, appreciation rate) of the $i$th risky asset at time $t$. For $1 \leq i \leq m$, $1 \leq j \leq l$, the volatility coefficient (or, dispersion coefficient) $\sigma_{ij}$ models the instantaneous intensity of how the $j$th source of randomness affects the price of the $i$th risky asset at time $t$. Usually, one includes a risk-free asset with the price process modeled by

$$dS_{0t} = r_t S_{0t} dt, \quad 0 \leq t \leq T, \quad (A2)$$

where $r_t$ means the interest rate at time $t$.

Further, if $l = m$, the market is called complete, otherwise it is incomplete. In the incomplete market ($l > m$), one could not hedge all the randomness ($l$ sources) by constructing a portfolio with $m$ risky assets and one risk-free asset. As our focus in this paper is the PHARA utility, we start from the complete market with constant market parameters, which already covers a lot of essential features in the financial market. Under the setting of a complete market, we are able to derive the closed-form results and discuss financial insights.

Hence, we use the standard Black-Scholes model in a complete market: for any $i = 1, \ldots, m$,

$$dS_{it} = S_{it} \left( \mu_{it} dt + \sum_{j=1}^{m} \sigma_{ij} dW_{jt} \right), \quad 0 \leq t \leq T, \quad (A3)$$

where $\mu_{it}$ is interpreted as the expected return rate of the $i$th risky asset and $\sigma_{ij}$ is interpreted as a constant volatility coefficient between the $i$th risky asset and the Brownian motion $W_t$. This model is given by (6) in the main text. Hence, the matrix of volatility $\sigma = (\sigma_{ij})_{1 \leq i,j \leq m}$ is a square matrix. We assume that there exists some $\epsilon > 0$ such that

$$\eta^\top (\sigma \sigma^{-1} \eta) \geq \epsilon ||\eta||_2^2$$

for any $\eta \in \mathbb{R}^m$. \quad (A4)
between two Brownian motions is a one-dimensional Brownian motion and $$\zeta$$ where $$3(1_1)$$.

The model (A5). These relationships help understand the modeling in model (A3) with independent Brownian motions is a special case of model (A5), each source of randomness includes the most generality, but also has the most parameters to estimate. A setting of a complete market reduces some estimate cost, where the standard Black-Scholes model enjoys the case of fewer estimation. The model (A5) is in between. One may choose among these different models according to the data availability and the statistical capability.

### Appendix 2. Basic theory of concave envelope

Here we introduce the basic theory of the concave envelope, in terms of definition, derivation and application to portfolio selection. Interested readers may further refer to the theory of convex analysis (e.g. Rockafellar 1970, Hiriart-Urruty and Lemaréchal 2001).

#### 2.1. Convex and concave functions

A direct definition of the concave envelope $$U^{**}$$ is defined in (11) from a graphical perspective. The superscript notation of $$U^{**}$$ comes from the perspective of convex analysis, where the concave envelope of a function $$U$$ can be also be defined as:

$$U^*(y) := \sup_{x \in \text{dom } U} (U(x) - xy), \quad y > 0;$$

$$U^{**}(x) := \inf_{y \in \text{dom } U^*} (U^*(y) + xy), \quad x \in \text{dom } U.$$  \hspace{1cm} (A12)

Here we actually apply the Legendre-Fenchel transform with $$f(x) := -U(-x)$$:

$$f^*(y) := \sup_{x \in \text{dom } f^*} (yx - f(x)), \quad y > 0;$$

$$f^{**}(x) := \sup_{y \in \text{dom } f^{**}} (yx - f^*(y)), \quad x \in \text{dom } f.$$  \hspace{1cm} (A13)

where $$f^*$$ is known as the convex conjugate function and $$f^{**}$$ is known as the convex envelope of $$f$$ or the convex biconjugate function. As we focus on the convex envelope, with an abuse of notation, we still denote by $$U^{**}$$ the concave envelope. We further define the optimizer in $$U^{**}$$ by

$$I(y) := \arg \sup_{x \in \text{dom } U} (U(x) - xy), \quad y > 0.$$  \hspace{1cm} (A14)

As indicated by the graphical definition (11), we can see that if $$U$$ is concave on $$D = \text{dom } U$$, then $$U^{**} = U$$. To derive the convex envelope of a non-concave function $$U$$, we provide a procedure below and refer to a classic result lemma 1 for theoretical foundation. Lemma 1 essentially originates from lemma 2.3 of Wang et al. (2019) and lemma 5.1 of Brighi and Chipot (1994), and is highly related to lemma 6.3 of Bichuch and Sturm (2014). The proof is referred to lemma 5.1 of Brighi and Chipot (1994).
Based on lemma 1, if $U$ is not concave, we have a descriptive procedure to generate its concave envelope. We can use this procedure to establish the concave envelope in figure 1.

(i) Find out the non-concave parts of $U$:

$$B := \{x \in D : U \text{ is not concave at } x\}.$$ 

Write $B = \bigcup_{j=1}^{\infty} B_j$, where $B_j$ is an interval. Note that the set $B$ has plenty of common intervals as the set $A$ in lemma 1.

(ii) On each $B_j$, $j \geq 1$, use the tangent lines and linear connections which dominate $U$.

(iii) Combine the concave parts and the linear parts as a new function $\tilde{U}$ dominating $U$.

(iv) Modify the function $\tilde{U}$ with new linear connections such that the modified function is concave on the domain.

(v) Denote the modified function by $U^{**}$ and this function $U^{**}$ is the concave envelope of $U$.

The concave envelope and the Legendre-Fenchel transform have an important application to portfolio selection in non-concave utility optimization. Assume the market is complete. According to the martingale method, Problem (10) is equivalent to the terminal wealth optimization problem (A15):

$$\max_{x_j \in \mathcal{F}_T, y \in \ell_{i,j} X_j \in \text{dom } U} \mathbb{E}[U(X_T)]. \quad (A15)$$

Applying Lagrange duality methods and the above Equation (A12), the optimal terminal wealth is given by

$$X_T^* = \arg \sup_{x \in \text{dom } U} U(x) - y^* \xi_T = I(y^* \xi_T),$$

where the Lagrangian multiplier $y > 0$ satisfies $\mathbb{E}[\xi_T(y^* \xi_T)] = x_0$. For a complete procedure of solving the portfolio selection problem in non-concave utility optimization, we refer to Appendix A of Liang and Liu (2023).

Appendix 3. Proofs

A.1. PHARA utility: proof of proposition 1

Proof of Proposition 1. We prove the first statement. Let $A = \{x \in D : U^{**}(x) - U(x) \neq 0\}$. Then $A$ is an open subset of $D$ as the function $U^{**} - U$ is continuous. Write $A = \bigcup_{k=1}^{\infty} A_k$, where $A_k$ is an open interval. By lemma 1, we have $U^{**}$ is linear and hence HARA on each $A_k$, which implies $U^{**}$ is PHARA on the set $A_k$. Because $U^{**} = U$ on $D \setminus A$, $U^{**}$ is PHARA on the set $D \setminus A$. As a whole, $U^{**}$ is PHARA on the domain $D$.

For the second statement, we show that there exists some $U$ which is not PHARA but $U^{**}$ is PHARA. We propose a case that $U$ is convex and not HARA on some interval $I$ and is concave and PHARA otherwise. On this interval $I$, we have $U^{**} \neq U$ and hence $I \subset A$. By lemma 1, $U^{**}$ is linear on the interval $I$ and hence $U^{**}$ is HARA. Thus, $U^{**}$ is HARA on each part of its domain. A particular example is as follows:

$$U(x) = \begin{cases} \sin(x) + 1, & \text{if } x \in \left[\frac{3}{2} \pi, 2 \pi\right]; \\ (x - 2 \pi)^{\frac{1}{2}} + 1, & \text{if } x \in (2 \pi, \infty), \end{cases} \quad (A16)$$

and

$$U^{**}(x) = \begin{cases} y_1 \left(\frac{x - 3}{2} \pi\right), & \text{if } x \in \left[\frac{3}{2} \pi, a_1\right]; \\ (x - 2 \pi)^{\frac{1}{2}} + 1, & \text{if } x \in (a_1, \infty), \end{cases} \quad (A17)$$

where the domains of $U$ and $U^{**}$ are the same $\left[\frac{1}{2} \pi, \infty\right)$, and $a_1 \in (2 \pi, \infty)$ is the unique solution of the tangent equation:

$$\frac{(a_1 - 2 \pi)^{\frac{1}{2}} + 1}{a_1 - \frac{3}{2} \pi} = \frac{1}{2} (a_1 - 2 \pi)^{-\frac{1}{2}}. \quad (A18)$$

and we define its slope $\gamma_1 := \frac{(a_1 - 2 \pi)^{\frac{1}{2}} + 1}{a_1 - \frac{3}{2} \pi}$. Hence, $U^{**}$ is PHARA on $\left[\frac{1}{2} \pi, \infty\right)$ while $U$ is not.

A.2. Optimal wealth process: proof of proposition 2

Proof of Proposition 2. (1) Based on the martingale and duality method, the optimal terminal wealth is obtained from $X_T = \arg \sup_{x \in \mathcal{D}} \mathbb{E}[U(x) - y^* \xi_T x]$ satisfying (35). According to definition 1 and equation (13), we solve for $k \in \{0, 1, \ldots, n\}$:

(i) if $y^* \xi_T \in (y_k^+, y_{k-1}^+)$, then

$$X_T^* = \arg \sup_{x \in \mathcal{D}} U(x) - y^* \xi_T x = a_k. \quad (A19)$$

(ii) if $y^* \xi_T \in (y_k^+, y_{k-1}^-)$, then

$$X_T^* = \arg \sup_{x \in \mathcal{D}} U(x) - y^* \xi_T x = (U')^{-1}(a_k, a_{k+1}) \left(y^* \xi_T \right) = \begin{cases} a_k + \frac{R_k}{y^* \xi_T}, & \text{if } R_k \neq 0, \infty; \\ a_k + \frac{1}{a_k} \log \left(\frac{y_k^+}{y_{k-1}^-}\right), & \text{if } R_k = \infty, \quad a_k = \infty, a_k > 0. \end{cases} \quad (A20)$$

(iii) if $y^* \xi_T = y_k^+$ where $y_k^+ \neq y_{k+1}^-$, we have

$$X_T^* = \arg \sup_{x \in \mathcal{D}} U(x) - y^* \xi_T x = a_{k+1}; \quad (A21)$$

if $y^* \xi_T = y_{k-1}^-$ where $y_{k-1}^- \neq y_k^+$, we have

$$X_T^* = \arg \sup_{x \in \mathcal{D}} U(x) - y^* \xi_T x = a_k; \quad (A22)$$

if $y^* \xi_T = y_k^+$ where $y_k^+ = y_{k-1}^-$ ($U^{**}$ is linear on $[a_k, a_{k+1}]$), we have $X_T^* = a_k$.

(A23)

We define the index set

$$\mathcal{K} := \{k \in \{0, 1, \ldots, n\} : y_k^+ = y_{k+1}^-\}. \quad (A24)$$

But as $\xi_T$ has a continuous distribution (i.e. $\mathbb{P}(\xi_T = y) = 0$ for any $y \in \mathbb{R}$), we have

$$\mathbb{P} \left(y^* \xi_T \in (y_k^+, y_{k-1}^-)_{k \in \mathcal{K}} \right) \leq \mathbb{P} \left(y^* \xi_T \in (y_k^+, y_{k-1}^-)_{k = 0}^{n} \cup (y_n^-) \right) = \sum_{k=0}^{n} \mathbb{P}(\xi_T = y_k^+/y_n^-) + \mathbb{P}(\xi_T = y_n^-/y_n^+) = 0. \quad (A25)$$

Thus, in the almost-sure sense, it does not matter what the value of $X_T^*$ in this case. Moreover, $U^{**}(x) \neq U(x)$ if and only if $x \in \cup_{k \in \mathcal{K}} (a_k, a_{k+1}]$. Further, based on (A25), we have

$$\mathbb{E}[U(X_T^*)] = \mathbb{E}[U(X_T')] \mathbb{I}_{\{X_T' \in \cup_{k \in \mathcal{K}} (a_k, a_{k+1})\}}$$

$$+ \mathbb{E}[U(X_T')] \mathbb{I}_{\{|X_T' \in \cup_{k \in \mathcal{K}} (a_k, a_{k+1})\]} = \mathbb{E}[U(X_T^*)] \mathbb{I}_{\{|X_T^* \in \cup_{k \in \mathcal{K}} (a_k, a_{k+1})\}}$$

$$+ \mathbb{E}[U(X_T^*)] \mathbb{I}_{\{|X_T^* \in \cup_{k \in \mathcal{K}} (a_k, a_{k+1})\}}$$
Above all, we give out facts that will be used in the following computation: for any
\[ \sum_{1}^{\infty} \]
by (34) and (35).
(A26)
Hence, we show that Problem (10) is equivalent to the concavification problem (12), i.e. Problems (10) and (12) have the same optimal portfolio \( \pi^* \) and the optimal terminal wealth \( X_T^* \). Summarizing (A20)–(A23), we have that the optimal wealth \( X_T^* \) is given by (34) and (35).

(2) Above all, we give out facts that will be used in the following computation: for any \( b > a > 0 \),
\[
\xi_t^{-1} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right] 
= e^{-r(T-t)} \left( \Phi \left( d_1 \left( \frac{a}{\sqrt{y_t}} \right) \right) - \Phi \left( d_1 \left( \frac{b}{\sqrt{y_t}} \right) \right) \right). 
\]
\[
\xi_t^{-1} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right] 
= e^{-r(T-t)} \left( \Phi \left( d_k \left( \frac{a}{\sqrt{y_t}} \right) \right) - \Phi \left( d_k \left( \frac{b}{\sqrt{y_t}} \right) \right) \right) \times \left[ \Phi \left( d_1 \left( \frac{a}{\sqrt{y_t}} \right) \right) - \Phi \left( d_1 \left( \frac{b}{\sqrt{y_t}} \right) \right) \right] + e^{-r(T-t)} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right]. 
\]

According to the martingale representation argument and the expression (34) of \( X_T^* \), we compute:
\[
X_T^* = \xi_t^{-1} \mathbb{E} \left[ \xi_T X_T^* | F_t \right] 
= \sum_{k=0}^{n} \left( e^{-r(T-t)} a_k \right) \times \left[ \Phi \left( d_1 \left( \frac{\gamma_k^+}{\sqrt{y_t}} \right) \right) - \Phi \left( d_1 \left( \frac{\gamma_k^-}{\sqrt{y_t}} \right) \right) \right] + e^{-r(T-t)} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right]. 
\]

A.3. Optimal portfolio (general case): proof of theorem 2

Proof of Theorem 2. According to the martingale method, we apply Itô’s formula to \( X_t = X_t^*(\xi_t) \) by using (8) and (9), and obtain the optimal portfolio vector
\[
\pi_t^* = - (\sigma^{-1}) \mathbb{E} \left[ \xi_T X_T^* | F_t \right] = - (\sigma^{-1}) \mathbb{E} \left[ \xi_T \Phi \left( d_1 \left( \frac{\gamma_T^+}{\sqrt{y_T}} \right) \right) - \Phi \left( d_1 \left( \frac{\gamma_T^-}{\sqrt{y_T}} \right) \right) \right].
\]

In the following computation, we use facts that
\[
\frac{\partial}{\partial x} \left( \log(x) + \left( -r + \frac{|\theta|^2}{2} \right) (T-t) \right) \times \left[ \Phi \left( d_1 \left( \frac{a}{\sqrt{y_t}} \right) \right) - \Phi \left( d_1 \left( \frac{b}{\sqrt{y_t}} \right) \right) \right] + e^{-r(T-t)} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right].
\]

For each \( k \in \{0, \ldots, n\} \), we compute as follows:
- if \( R_k = 0 \), we have \( \gamma_k^+ = \gamma_{k+1}^- \), and hence
\[
\frac{\partial}{\partial x} \left( \log(x) + \left( -r + \frac{|\theta|^2}{2} \right) (T-t) \right) \times \left[ \Phi \left( d_1 \left( \frac{a}{\sqrt{y_t}} \right) \right) - \Phi \left( d_1 \left( \frac{b}{\sqrt{y_t}} \right) \right) \right] + e^{-r(T-t)} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right].
\]

- if \( R_k = \infty, A_k = -\infty, a_k > 0 \), we have
\[
\frac{\partial}{\partial x} \left( \log(x) + \left( -r + \frac{|\theta|^2}{2} \right) (T-t) \right) \times \left[ \Phi \left( d_1 \left( \frac{a}{\sqrt{y_t}} \right) \right) - \Phi \left( d_1 \left( \frac{b}{\sqrt{y_t}} \right) \right) \right] + e^{-r(T-t)} \mathbb{E} \left[ \xi_T 1_{\{\xi_T \in (a,b)\}} | F_t \right].
\]
\[ + \frac{-\|\theta\|_2 \sqrt{T - t}}{a_k} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ = \frac{a_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{1}{a_k} \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{1}{a_k} \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ - d_1 \left( \frac{Y_{k}}{y^2 \xi_t} \right) \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \]
\[ = \frac{a_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{1}{a_k} \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{1}{a_k} \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ - d_1 \left( \frac{Y_{k}}{y^2 \xi_t} \right) \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \]
\[ = \frac{a_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{1}{a_k} \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{1}{a_k} \left[ \Phi\left( d_1 \left( \frac{Y_{k+1}^-}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ - d_1 \left( \frac{Y_{k}}{y^2 \xi_t} \right) \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \]

where in the second equality we use \( Y_{k+1}^- = Y_k^+ \) due to (13);

- if \( R_k \neq 0, \infty \), we have

\[ (-\xi_t)^\frac{\partial}{\partial \xi_t} \left\{ \begin{array}{c}
\left( \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right) \\
+ A_k \left[ \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \\
+ (a_k - A_k) \frac{\Phi'}{\Phi}\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) \\
\end{array} \right\} \]

\[ = \frac{a_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]
\[ + \frac{A_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]

where the second equality holds because \( \Phi'(d_1(\frac{Y_{k}^-}{y^2 \xi_t})) = \Phi'(d_1(\frac{Y_{k}^+}{y^2 \xi_t})) \) \( (x - \Phi'(\frac{d_1(y^2 \xi_t)}{y^2 \xi_t})) \) \( \Phi(d_1(\frac{Y_{k}^+}{y^2 \xi_t})) \) \( A_k \); 

\( a_{k+1} - A_k \) = \( \frac{\Phi(d_1(\frac{Y_{k}^+}{y^2 \xi_t}))}{\Phi'(\frac{d_1(\frac{Y_{k}^+}{y^2 \xi_t}))}} (a_k - A_k) \). Finally, adding up each term, we derive the optimal portfolio vector:

\[ \pi_t^* = - (\sigma ^\top)^{-1} \Theta \xi_t, \frac{\partial X_t^R}{\partial \xi_t} \]

\[ = (\sigma ^\top)^{-1} \Theta \sum_{k=0}^{n} (-\xi_t)^\frac{\partial}{\partial \xi_t} \left\{ \begin{array}{c}
\left( \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right) \\
+ A_k \left[ \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \\
+ (a_k - A_k) \frac{\Phi'}{\Phi}\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) \\
\end{array} \right\} \]

\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_k+1}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]

\[ = \frac{a_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]

\[ + \frac{A_k}{\|\theta\|_2 \sqrt{T - t}} \]
\[ \times \left[ \Phi\left( d_1 \left( \frac{Y_k^+}{y^2 \xi_t} \right) \right) - \Phi\left( d_1 \left( \frac{Y_{k}^-}{y^2 \xi_t} \right) \right) \right] \]

where we use facts that \( R_k \neq 0, Y_{n+1}^- = 0 \) and \( y_0^- = \infty \).
A.4. Optimal portfolio: proof of theorem 1

Proof of Theorem 1. If $R_k \in (R, 0)$, then the optimal portfolio vector (38) is derived from the general case (A34) in the proof of theorem 2 by

\[ x_i^* = (\sigma^T)^{-1} \sum_{k=0}^{n} R^{-1} X_{i,k} \mathbb{I}_{(R_i \neq 0, \infty)} + e^{-r(T-t)} a_k - a_{k+1} - ||\theta||_2 \sqrt{T-t} \]

and

\[ \Phi' \left( d_1 \left( \frac{Y_{n+1}}{y^*_{\xi_t}} \right) \right) = 0, \quad \Phi \left( d_1 \left( \frac{Y_{n+1}}{y^*_{\xi_t}} \right) \right) = 1. \quad (A40) \]

Now we compute: for any $\gamma > 0$,

\[ \Phi' \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right) = \exp \left( (T-t) \left( \frac{\gamma}{R_k} + \frac{||\theta||^2}{2R_k^2} (1 - R_k) \right) \right) \]

\[ \times \left( \frac{y^*_{\xi_t}}{\gamma} \right)^{\frac{\gamma}{R_k}}. \quad (A41) \]

Based on the expression (37) of $X_i^*$, as $\xi_t \to 0$, for any $k \in \{0, 1, \ldots, n\}$, we have

\[ X_i^D = e^{-r(T-t)} a_k \]

\[ \times \left( \frac{y^*_{\xi_t}}{\gamma} \right)^{\frac{\gamma}{R_k}} \]

\[ \left( \frac{y^*_{\xi_t}}{\gamma} \right)^{\frac{\gamma}{R_k}} \leq e^{-r(T-t)} a_k, \quad \text{if } k = n; \]

\[ 0, \quad \text{if } k \neq n. \quad (A43) \]

For any $k \in \{0, 1, \ldots, n-1\}$ with $R_k \neq 0$, by the expression of the PHAR utility on $[a_k, a_{k+1}]$, we have $a_k > A_k$. We also have $y^*_{\xi_t} > y^*_{\xi_t}$ by strict concavity, which means $\Phi(\gamma^*_{\xi_t}) - \Phi(\gamma^*_{\xi_t}) > 0$. Hence, as $\xi_t \to 0$, we have

\[ x_{i,k}^R = e^{-r(T-t)} (a_k - A_k) \]

\[ \times \left( \frac{y^*_{\xi_t}}{\gamma} \right)^{\frac{\gamma}{R_k}} \]

\[ \times \mathbb{I}_{(R_{i,k} \neq 0, \infty)} \geq 0. \quad (A44) \]

For $k = n$, according to the expression of (A41), as $\xi_t \to 0$, we have

\[ \frac{\Phi' \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right)}{\Phi' \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right)} \to \infty, \quad (A45) \]

and

\[ \Phi \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right) \to 0, \quad \Phi \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right) = 1 - \Phi \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right) \to 1, \quad (A46) \]

which implies

\[ x_{i,k}^R = e^{-r(T-t)} (a_k - A_k) \]

\[ \times \left( \frac{y^*_{\xi_t}}{\gamma} \right)^{\frac{\gamma}{R_k}} \]

\[ \times \mathbb{I}_{(R_{i,k} \neq 0, \infty)} \to \infty. \quad (A47) \]

A.5. Asymptotic analysis: proof of theorem 3

Proof of Theorem 3. As $R_k \in (R, 0)$, according to proposition 2, we have

\[ X_i^* = \sum_{k=0}^{n} \left( X_{i,k}^R + X_{i,k}^A + X_{i,k}^B \right). \quad (A36) \]

Now we fix $t \in (0, T)$ and $\gamma^* > 0$ and conduct an asymptotic analysis.

(i) Using (39), for any $\gamma \in (0, \infty)$, as $\xi_t \to 0$, we have

\[ \gamma \to \infty, \quad d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \to -\infty, \quad (A37) \]

and

\[ \Phi' \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right) \to 0, \quad \Phi \left( d_1 \left( \frac{\gamma}{y^*_{\xi_t}} \right) \right) \to 0. \quad (A38) \]

Hence, for $\gamma \in \{\gamma_0, \gamma_0^*, \gamma_1^*, \gamma_2^*, \ldots, \gamma_n^*, y^*_{\xi_t}\}$, we have (A37)–(A38). For $\gamma = 0, \gamma_{n+1} = 0$, as $\xi_t \in (0, \infty)$, we have

\[ \gamma_{n+1}^* = 0, \quad d_1 \left( \frac{\gamma_{n+1}}{y^*_{\xi_t}} \right) = \infty. \quad (A39) \]
Hence, combining (A44) and (A47), we have
\[ X^R_k = \sum_{k=0}^{n} X^R_k \to \infty, \] (A48)
and
\[ X^*_k \to \infty. \] (A49)

Moreover, according to the expressions of (40), for any \( k \in \{0, \ldots, n-1\} \), as \( \xi_t \to 0 \), we have
\[ p_k \to 0, \quad q_k \to 0, \] (A50)
and
\[ p_n \to 0, \quad q_n \to 1, \] (A51)
where we use \( \gamma^{-}_n = 0 \). Thus, we obtain that if \( \xi_t \to 0 \), then
\[ \pi_t^{(2)} \to 0, \quad \pi_t^{(3)} \to -A_p e^{-(r-T)} \frac{R}{R} (\sigma^T)^{-1} \theta, \]
\[ \frac{\pi_t^2}{X^R_t} \to 1 R (\sigma^T)^{-1} \theta, \quad \pi_t \to \infty. \] (A52)

(ii) An important point throughout this part of proof is that (a) \( \gamma^{-}_0 = \infty \) always holds; and (b) \( \gamma^{-}_0 = \infty \) if and only if \( a_0 = A_0 \) and \( R_0 = 0 \). We can show (a)-(b) by definition 1 of the PHARLA utility.

Using (39), for any \( \gamma \in [0, \infty) \), as \( \xi_t \to \infty \), we have
\[ \frac{\gamma}{y^2 \xi_t} \to 0, \quad d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \to \infty, \] (A53)
and
\[ \phi' \left( d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \right) \to 0, \quad \phi \left( d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \right) \to 1. \] (A54)

Hence, for \( \gamma \in \{\gamma^{-}_1, \gamma^{+}_1, \ldots, \gamma^{-}_n, \gamma^{+}_n, \gamma^{-}_{n+1}\} \), we have (A53)–(A54). For \( \gamma = \infty \), for any \( \xi_t \in (0, \infty) \), we have
\[ \frac{\gamma}{y^2 \xi_t} = \infty, \quad d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) = -\infty, \] (A55)
\[ \phi' \left( d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \right) = 0, \quad \phi \left( d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \right) = 0. \] (A56)

For any \( \gamma \in [0, \infty) \), as \( \xi_t \to \infty \),
\[ \frac{\phi' \left( d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \right)}{\phi \left( d_1 \left( \frac{\gamma}{y^2 \xi_t} \right) \right)} = \exp \left\{ (T - \tau) \left( \frac{r}{R} + \frac{|\theta|}{2} \frac{1}{2} \frac{R}{R_0} (1 - R_k) \right) \right\} \]
\[ \times \left( \gamma^2 \xi_t \right)^{-1} \phi' \left( \frac{\gamma}{y^2 \xi_t} \right) \to 0. \] (A57)
Based on the expression (37) of \( X^*_k \), as \( \xi_t \to \infty \), for any \( k \in \{1, \ldots, n\} \), we have
\[ X^D_{k,0} = e^{-(r-T)} a_0 \left[ \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{-}_0}{y^2 \xi_t} \right) \right) \right] \]
\[ \to 0, \] (A58)
\[ X^A_{k,0} = e^{-(r-T)} A_0 \left[ \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{-}_0}{y^2 \xi_t} \right) \right) \right] \]
\[ \times I_{R_0 < 0} \to 0, \] (A59)
\[ X^R_{k,0} = e^{-(r-T)} (a_k - A_k) \left[ \phi' \left( \frac{\gamma^{+}_k}{y^2 \xi_t} \right) \right] \]
\[ \phi' \left( \frac{\gamma^{+}_k}{y^2 \xi_t} \right) \]
\[ \times 1_{R_0 \neq 0, \infty} \to 0. \] (A60)

For \( k = 0 \), if \( \gamma^{+}_0 \in (0, \infty) \), then \( a_0 > A_0 \) or \( R_0 = 0 \). Because \( \gamma^{-}_0 = \infty \), we have
\[ X^D_{k,0} = e^{-(r-T)} a_0 \left[ \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{-}_0}{y^2 \xi_t} \right) \right) \right] \]
\[ \to 0. \] (A61)
\[ X^A_{k,0} = e^{-(r-T)} A_0 \left[ \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) \right] \]
\[ \times I_{R_0 \neq 0, \infty} \to 0, \] (A62)
\[ X^R_{k,0} = e^{-(r-T)} (a_k - A_k) \left[ \phi' \left( \frac{\gamma^{+}_k}{y^2 \xi_t} \right) \right] \]
\[ \phi' \left( \frac{\gamma^{+}_k}{y^2 \xi_t} \right) \]
\[ \times 1_{R_0 \neq 0, \infty} \to 0. \] (A63)

Hence, in this case,
\[ X^*_n = \sum_{k=0}^{n} (X^D_{k,0} + X^A_{k,0} + X^R_{k,0}) \to e^{-r(T-t)} a_0. \] (A64)

If \( \gamma^{+}_0 = \infty \), then \( a_0 = A_0 \), which implies \( X^D_{k,0} = 0 \) and
\[ X^*_n = \sum_{k=0}^{n} (X^A_{k,0} + X^R_{k,0}), \] (A65)
As \( \xi_t \to \infty \), we have
\[ X^D_{k,0} = e^{-(r-T)} a_0 \left[ \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{-}_0}{y^2 \xi_t} \right) \right) \right] \]
\[ = 0 - 0 = 0, \] (A66)
\[ X^A_{k,0} = e^{-(r-T)} A_0 \left[ \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) \right] \]
\[ \times I_{R_0 \neq 0, \infty} \to e^{-r(T-t)} A_0 = e^{-r(T-t)} a_0. \] (A67)

Hence, in this case, similarly to (A64), we also have
\[ X^*_n = \sum_{k=0}^{n} (X^D_{k,0} + X^A_{k,0}) + \sum_{k=1}^{n} X^R_{k,0} \to e^{-r(T-t)} a_0. \] (A68)

Moreover, according to the expressions of (40), for any \( k \in \{1, \ldots, n\} \), as \( \xi_t \to \infty \), we have
\[ q_k \to 0, \quad p_k \to 0. \] (A69)
For \( k = 0 \), as \( \xi_t \to \infty \), because \( \gamma^{-}_0 = \infty \), we have
\[ q_0 = \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{+}_0}{y^2 \xi_t} \right) \right) \]
\[ \to 1 - 0 = 1, \quad \text{if } \gamma^{+}_0 = \infty; \]
\[ 1 - 1 = 0, \quad \text{if } \gamma^{+}_0 \in (0, \infty), \] (A70)
\[ p_0 = \phi \left( d_1 \left( \frac{\gamma^{-}_0}{y^2 \xi_t} \right) \right) - \phi \left( d_1 \left( \frac{\gamma^{-}_0}{y^2 \xi_t} \right) \right) \]
\[= \Phi \left( d_1 \left( \frac{y_0^+}{y^+} \right) \right) \rightarrow \begin{cases} 0, & \text{if } y_0^+ = \infty; \\ 1, & \text{if } y_0^+ \in [0, \infty). \end{cases} \] (A71)

Thus, we obtain that if \( \xi_t \to \infty \), then

\[
\pi_t^{(4)} \rightarrow \begin{cases} 0, & \text{if } y_0^+ = \infty; \\ -\frac{e^{-r(T-t)}}{R}(\sigma^\top)^{-1}\theta a_0, & \text{if } y_0^+ \in [0, \infty), \end{cases} \] (A72)

\[
\pi_t^{(3)} \rightarrow \begin{cases} -\frac{e^{-r(T-t)}}{R}(\sigma^\top)^{-1}\theta a_0, & \text{if } y_0^+ = \infty; \\ 0, & \text{if } y_0^+ \in [0, \infty), \end{cases} \] (A73)

\[
\pi_t^{(2)} \rightarrow 0, \quad \pi_t^{(1)} \rightarrow \frac{1}{R}(\sigma^\top)^{-1}\theta e^{-r(T-t)} a_0. \] (A74)

Hence, as \( \xi_t \to \infty \), we have

\[
\pi_t^* \to 0. \]