A PROPERTY OF ALGEBRAIC UNIVOQUE NUMBERS

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Abstract. Consider the set $U$ of real numbers $q \geq 1$ for which only one sequence $(c_i)$ of integers $0 \leq c_i \leq q$ satisfies the equality $\sum_{i=1}^{\infty} c_i q^{-i} = 1$. In this note we show that the set of algebraic numbers in $U$ is dense in the closure $\overline{U}$ of $U$.

1. Introduction

Given a real number $q \geq 1$, a $q$-expansion (or simply expansion) is a sequence $(c_i) = c_1 c_2 \ldots$ of integers satisfying $0 \leq c_i \leq q$ for all $i \geq 1$ such that
\[
\frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \cdots = 1.
\]

One such expansion, denoted by $(\gamma_i(q)) = (\gamma_i)$, is obtained by performing the greedy algorithm of Rényi ([11]): if $\gamma_i$ is already defined for $i < n$, then $\gamma_n$ is the largest integer satisfying
\[
\sum_{i=1}^{n} \gamma_i q^{-i} \leq 1.
\]

Equivalently, $(\gamma_i)$ is the largest expansion in lexicographical order.

If $q > 1$, then another such expansion, denoted by $(\alpha_i(q)) = (\alpha_i)$, is obtained by performing the quasi-greedy algorithm: if $\alpha_i$ is already defined for $i < n$, then $\alpha_n$ is the largest integer satisfying
\[
\sum_{i=1}^{n} \alpha_i q^{-i} < 1.
\]

An expansion is called infinite if it contains infinitely many nonzero terms; otherwise it is called finite. Observe that there are no infinite expansions if $q = 1$: the only 1-expansions are given by $10^\infty, 010^\infty, 0010^\infty, \ldots$. On the other hand, if $q > 1$, then $(\alpha_i)$ is the largest infinite expansion in lexicographical order.

For any given $q > 1$, the following relations between the quasi-greedy expansion and the greedy expansion are straightforward. The greedy expansion is finite if and only if $(\alpha_i)$ is periodic. If $(\gamma_i)$ is finite and $\gamma_m$ is its last nonzero term, then $m$ is the smallest period of $(\alpha_i)$, and
\[
\alpha_i = \gamma_i \quad \text{for } i = 1, \ldots, m-1, \quad \text{and } \alpha_m = \gamma_m - 1.
\]

Erdős, Horváth and Joó ([4]) discovered that for some real numbers $q > 1$ there exists only one $q$-expansion. Subsequently, the set $U$ of such univoque numbers was characterized in [5], [6], [9] (see Theorem 2.1). Using this characterization, Komornik and Loreti showed in [7] that $U$ has a smallest element $q' \approx 1.787$ and
the corresponding expansion \((\tau_i)\) is given by the truncated Thue-Morse sequence, defined by setting 
\[\tau_{2^N+i} = 1 - \tau_i \quad \text{for} \quad 1 \leq i < 2^N, \quad N = 1, 2, \ldots.\]

Allouche and Cosnard \([1]\) proved that the number \(q'\) is transcendental. This raised the question whether there exists a smallest algebraic univoque number. Komornik, Loreti and Pethő \([8]\) answered this question in the negative by constructing a decreasing sequence \((q_n)\) of algebraic univoque numbers converging to \(q'\).

It is the aim of this note to show that for each \(q \in \mathcal{U}\) there exists a sequence of algebraic univoque numbers converging to \(q\):

**Theorem 1.1.** The set \(\mathcal{A}\) consisting of all algebraic univoque numbers is dense in \(\mathcal{U}\).

Our proof of Theorem 1.1 relies on a characterization of the closure \(\overline{\mathcal{U}}\) of \(\mathcal{U}\), recently obtained by Komornik and Loreti in \([9]\) (see Theorem 2.2).

2. **Proof of Theorem 1.1**

In the sequel, a sequence always means a sequence of nonnegative integers. We use systematically the lexicographical order between sequences; we write \((a_i) < (b_i)\) if there exists an index \(n \geq 1\) such that \(a_i = b_i\) for \(i < n\) and \(a_n < b_n\). This definition extends in the obvious way to sequences of finite length.

The following algebraic characterization of the set \(\mathcal{U}\) can be found in \([5]\), \([6]\), \([9]\):

**Theorem 2.1.** The map \(q \mapsto (\gamma_i(q))\) is a strictly increasing bijection between the set \(\mathcal{U}\) and the set of all sequences \((\gamma_i)\) satisfying

\[
\gamma_{j+1} \gamma_{j+2} \ldots < \gamma_1 \gamma_2 \ldots \quad \text{for all} \quad j \geq 1
\]

and

\[
\tau_{j+1} \tau_{j+2} \ldots < \gamma_1 \gamma_2 \ldots \quad \text{for all} \quad j \geq 1
\]

where we use the notation \(\gamma_n := \gamma_1 - \gamma_n\).

**Remark.** It was essentially shown by Parry (see \([10]\)) that a sequence \((\gamma_i)\) is the greedy \(q\)-expansion for some \(q \geq 1\) if and only if \((\gamma_i)\) satisfies the condition (2.1).

Using the above result, Komornik and Loreti \([9]\) investigated the topological structure of the set \(\mathcal{U}\). In particular they showed that \(\overline{\mathcal{U}} \setminus \mathcal{U}\) is dense in \(\overline{\mathcal{U}}\). Hence the set \(\overline{\mathcal{U}}\) is a perfect set. Moreover, they established an analogous characterization of the closure \(\overline{\mathcal{U}}\) of \(\mathcal{U}\):

**Theorem 2.2.** The map \(q \mapsto (\alpha_i(q))\) is a strictly increasing bijection between the set \(\overline{\mathcal{U}}\) and the set of all sequences \((\alpha_i)\) satisfying

\[
\alpha_{j+1} \alpha_{j+2} \ldots \leq \alpha_1 \alpha_2 \ldots \quad \text{for all} \quad j \geq 1
\]

and

\[
\overline{\alpha_{j+1}} \overline{\alpha_{j+2}} \ldots < \alpha_1 \alpha_2 \ldots \quad \text{for all} \quad j \geq 1
\]

where we use the notation \(\overline{\alpha_n} := \alpha_1 - \alpha_n\).

**Remarks.**
• It was shown in [3] that a sequence \((\alpha_i)\) is the quasi-greedy \(q\)-expansion for some \(q > 1\) if and only if \((\alpha_i)\) is infinite and satisfies (2.3). Note also that a sequence satisfying (2.3) and (2.4) is automatically infinite.

• If \(q \in \overline{U} \setminus U\), then we must have equality in (2.3) for some \(j \geq 1\), i.e., the greedy \(q\)-expansion is finite for each \(q \in \overline{U} \setminus U\). On the other hand, it follows from Theorems 2.1 and 2.2 that a sequence of the form \((1^n0)^\infty\) \((n \geq 2)\) is the quasi-greedy \(q\)-expansion for some \(q \in \overline{U} \setminus U\). Hence the set \(\overline{U} \setminus U\) is countably infinite.

The following technical lemma is a direct consequence of Theorem 2.2 and Lemmas 3.4 and 4.1 in [9]:

**Lemma 2.3.** Let \((\alpha_i)\) be a sequence satisfying (2.3) and (2.4). Then

(i) there exist arbitrary large integers \(m\) such that

\[
\alpha_{j+1} \ldots \alpha_m < \alpha_1 \ldots \alpha_{m-j} \quad \text{for all } 0 \leq j < m;
\]

(ii) for all positive integers \(m \geq 1\),

\[
\alpha_1 \ldots \alpha_m < \alpha_{m+1} \ldots \alpha_{2m}.
\]

**Proof of Theorem 1.1.** Since the set \(\overline{U} \setminus U\) is dense in \(U\), it is sufficient to show that \(A \supset \overline{U} \setminus U\). In order to do so, fix \(q \in \overline{U} \setminus U\). Then, according to Theorem 2.2, the quasi-greedy \(q\)-expansion \((\alpha_i)\) satisfies (2.3) and (2.4). Let \(k\) be a positive integer for which equality holds in (2.3), i.e.,

\[
(\alpha_i) = (\alpha_1 \ldots \alpha_k)^\infty.
\]

According to Lemma 2.3 there exists an integer \(m \geq k\) such that (2.5) is satisfied. Let \(N\) be a positive integer such that \(kN \geq m\) and consider the sequence

\[
(\gamma_i) = (\gamma_i^N) = (\alpha_1 \ldots \alpha_k)^N(\alpha_1 \ldots \alpha_m)^{\gamma_1 \alpha_1 \ldots \alpha_m})^\infty.
\]

For ease of exposition we suppress the dependence of \((\gamma_i)\) on \(N\). Note that \(\gamma_i = \alpha_i\) for \(1 \leq i \leq m + kN\). In particular, we have

\[
\gamma_i = \alpha_i \quad \text{for } 1 \leq i \leq 2m.
\]

Since \((\gamma_i)\) has a periodic tail, the number \(qN\) determined by

\[
1 = \sum_{i=1}^{\infty} \gamma_i / q_N
\]

is an algebraic number and \(qN \rightarrow q\) as \(N \rightarrow \infty\).

According to Theorem 2.1 it remains to verify the inequalities (2.1) and (2.2). First we verify (2.1) and (2.2) for \(j \geq kN\). For those values of \(j\) the inequality (2.1) for \(j + m\) is equivalent to (2.2) for \(j\) and (2.2) for \(j + m\) is equivalent to (2.1) for \(j\). Therefore it suffices to verify the inequalities (2.1) and (2.2) for \(kN \leq j < kN + m\). Fix \(kN \leq j < kN + m\). From (2.3), (2.4) and (2.7) we have

\[
\gamma_{j+1} \ldots \gamma_{kN+2m} = \alpha_{j-kN+1} \ldots \alpha_m \alpha_1 \ldots \alpha_m < \alpha_{j-kN+1} \ldots \alpha_m \alpha_{m+1} \ldots \alpha_{2m} \leq \alpha_1 \ldots \alpha_{kN+2m-j} = \gamma_1 \ldots \gamma_{kN+2m-j}
\]
and from inequality (2.3) we have
\[ \gamma_{j+1} \cdots \gamma_{kN+m} = \alpha_{j-kN+1} \cdots \alpha_m < \alpha_1 \cdots \alpha_{kN+m-j} = \gamma_1 \cdots \gamma_{kN+j} \cdot \]
Now we verify (2.1) for \( j < kN \). If \( m \leq j < kN \), then by (2.3) and (2.6),
\[ \gamma_{j+1} \cdots \gamma_{kN+2m} < \alpha_{j+1} \cdots \alpha_{kN+2m} \leq \alpha_1 \cdots \alpha_{kN+2m-j} = \gamma_1 \cdots \gamma_{kN+2m-j} \cdot \]
If \( 1 \leq j < m \), then by (2.3) and (2.5),
\[ \gamma_{j+1} \cdots \gamma_{kN+m+j} = \alpha_{j+1} \cdots \alpha_{kN+m} \alpha_1 \cdots \alpha_j \leq \alpha_1 \cdots \alpha_{kN+m-j} \alpha_1 \cdots \alpha_j < \alpha_1 \cdots \alpha_{kN+m-j} \alpha_{m-j+1} \cdots \alpha_m = \gamma_1 \cdots \gamma_{kN+m} \cdot \]
Finally, we verify (2.2) for \( j < kN \). Write \( j = k\ell + i, 0 \leq \ell < N \) and \( 0 \leq i < k \). If \( i = 0 \), then (2.2) follows from the relation
\[ \gamma_{j+1} = \alpha_1 = 0 < \alpha_1 = \gamma_1 \cdot \]
If \( 1 \leq i < k \), then applying Lemma 2.3(ii) we get
\[ \alpha_{i+1} \cdots \alpha_{2i} < \alpha_1 \cdots \alpha_i \cdot \]
Hence
\[ \gamma_{j+1} \cdots \gamma_{j+k} = \alpha_{j+1} \cdots \alpha_{j+k} = \alpha_{i+1} \cdots \alpha_{i+k} < \alpha_1 \cdots \alpha_k = \gamma_1 \cdots \gamma_k \cdot \]
(In order for the first equality to hold in case \( \ell = N-1 \), we need the condition \( m \geq k \).)

Remarks.
• Since the set \( \mathbb{T} \) is a perfect set and \( \mathbb{T} \setminus \mathcal{U} \) is countable, each neighborhood of \( q \in \mathcal{U} \) contains uncountably many elements of \( \mathcal{U} \). Hence the set of transcendental univoque numbers is dense in \( \mathbb{T} \) as well.
• Recently, Allouche, Frougny and Hare ([2]) proved that there also exist univoque Pisot numbers. In particular they determined the smallest three univoque Pisot numbers.

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