Parabolic twists for algebras $sl(n)$.

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Abstract

New solutions of twist equations for universal enveloping algebras $U(A_{n-1})$ are found. They can be presented as products of full chains of extended Jordanian twists $\mathcal{F}_{\hat{\Delta}}$, Abelian factors ("rotations") $\mathcal{F}_R$ and sets of quasi-Jordanian twists $\mathcal{F}_{\hat{J}}$. The latter are the generalizations of Jordanian twists (with carrier $b^2$) for special deformed extensions of the Hopf algebra $U(b^2)$. The carrier subalgebra $g_P$ for the composition $\mathcal{F}_P = \mathcal{F}_{\hat{J}} \mathcal{F}_R \mathcal{F}_{\hat{\Delta}}$ is a nonminimal parabolic subalgebra in $A_{n-1}$, $g_P \cap N^*_g = \emptyset$. The parabolic twisting elements $\mathcal{F}_P$ are obtained in the explicit form. The details of the construction are illustrated by considering the examples $n = 4$ and $n = 11$.

1 Introduction

The parabolic subgroups play an essential role in group theory and applications. Parabolic subgroups of a real simple Lie groups can be treated as the groups of motion of noncompact symmetric spaces with the special properties of geodesics [1]. The quantized versions of such subgroups are important for the models of noncommutative space-time [2, 3]. Quantizations of antisymmetric $r$-matrices, (solutions of classical Yang-Baxter equation) form an important class of deformations producing the quantized groups. The quantum duality principle [4, 5] provides the possibility to describe quantum groups with dual Lie algebras in terms of quantized universal enveloping algebras. Thus to construct a quantized parabolic subgroup of a simple Lie group it is sufficient to consider a parabolic subalgebra $g_P$ in a simple Lie algebra $g$ and to develope for the universal enveloping algebra $U(g)$ a quantization where the parabolic properties are selfdual. The important class of such quantum deformations is known as twisting [6].

If we supply the elements of Lie algebra $g$ with primitive coproducts $\Delta^{(0)}$ and consider $U(g)$ as a Hopf algebra with the structure generated by $\Delta^{(0)}$.
then the invertible solution $\mathcal{F} \in U(g)^{\otimes 2}$ of the twist equations \[ (\mathcal{F})_{12} (\Delta^{(0)} \otimes \text{id}) \mathcal{F} = (\mathcal{F})_{23} (\text{id} \otimes \Delta^{(0)}) \mathcal{F}, \]
\[ (\epsilon \otimes \text{id}) \mathcal{F} = (\text{id} \otimes \epsilon) \mathcal{F} = 1. \] (1)
allows to transform $U(g)$ into the Hopf algebra $U_{\mathcal{F}}(g)$,
\[ \mathcal{F} : U(g) \rightarrow U_{\mathcal{F}}(g). \] (2)
The twisted Hopf algebra $U_{\mathcal{F}}(g)$ has the same multiplication, unit and counit as $U(g)$ but the twisted coproduct and antipode given by:
\[ \Delta_{\mathcal{F}}(a) = \mathcal{F} \Delta^{(0)}(a) \mathcal{F}^{-1}, \quad S_{\mathcal{F}}(a) = v S(a) v^{-1}, \] (3)
\[ v = \sum f_i^{(1)} S(f_i^{(2)}), \quad a \in U(g). \]
The minimal subalgebra $g_c \subset g$ necessary for the definition of the twist $\mathcal{F}$ is called the carrier of the twist \[7\].

To investigate the deformation quantization problems and in particular to apply the twisted parabolic groups to this study explicit solutions of twist equations are necessary \[8\]. Only certain classes of twists $\mathcal{F}$ are known in the explicit form \[9\] \[10\] \[11\] \[12\] \[13\]. All such solutions are factorizable, i.e. can be presented as a product of the so called basic twisting factors \[14\]. One of the twisting factors is the jordanian twist \[10\] defined on the Borel subalgebra $b^2 = \{H,E \mid [H,E] = E\}$, it has the twisting element
\[ \mathcal{F}_J = \exp\{H \otimes \sigma\}, \]
where $\sigma = \ln(1 + E)$. This twist generates the solution of the Yang-Baxter equation: $R = (\mathcal{F}_J)_{21} \mathcal{F}_J^{-1}$. The classical $r$-matrix associated with it is $r = H \wedge E$.

The quantization problem for parabolic subgroups of a simple Lie group is reduced to the problem of constructing the explicit solutions $\mathcal{F}$ (of equations \[11\]) whose carriers $g_c(\mathcal{F})$ are parabolic subalgebras in $g$.

In the general case the solutions of the twist equation do not form an algebra with respect to (Hopf algebra) multiplication: the product of twists is not a twist. However under certain conditions the compositions of twists constitute the solutions of the twist equations. For infinite series of simple Lie algebras the twists (called "chains") were found \[12\]. They were constructed as products of factors each being itself a twisting element for the previously twisted algebra. Certain groups of factors ("links" of the chain) are twisting elements for the initial Hopf algebra $U(g)$. The structure of these chains is determined by the fundamental symmetry properties of the corresponding root system and is common for all simple Lie algebras \[13\]. The possibility to compose a chain of twists is based on the existence of sequences of regular injections
\[ g_m \subset g_{m-1} \ldots \subset g_2 \subset g_1 = g \] (4)
where the subalgebras $g_k$ belong to the same series of simple algebras and $m$ is the length of the chain. The sequence $\mathbf{H}$ is formed according to the following rule. Among the long roots of the root system $\Lambda(g_k)$ the initial root $\lambda_k^0$ is chosen. Let the subspace $V_{\lambda_k^0}^\perp$ in the root space of $g_k$ be orthogonal to the initial root $\lambda_k^0$ then the root subsystem $\Lambda(g_{k+1}) = \Lambda(g_k) \cap V_{\lambda_k^0}^\perp$ defines the subalgebra $g_{k+1}$. The sets $\pi_k$ are called the constituent roots for $\lambda_k^0$:

$$
\pi_k = \{ \lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_k^0 \}, \quad \pi_k' = \{ \lambda' \}, \quad \pi_k'' = \{ \lambda'' \}.
$$

It was proved [12, 13] that for each sequence (4) corresponds the chain of twists containing $m$ links of the standard form. For series $A_{n-1}$ consider the Cartan-Weil basis and let $E_{\lambda} \in g$ be the vector corresponding to the root $\lambda \in \Lambda(g)$. For each initial root $\lambda_k^0$ define the element $\sigma_{\lambda_k^0} = \ln (1 + E_{\lambda_k^0})$ and the Cartan element $H_{\lambda_k^0}$ so that $\text{ad}_{H_{\lambda_k^0}} \circ E_{\lambda_k^0} = E_{\lambda_k^0}$ and $\text{ad}_{H_{\lambda_k^0}} \circ E_\lambda = \frac{1}{2} E_\lambda$. Then the product of twisting factors

$$
F^k_{\text{link}} = \prod_{\lambda' \in \pi_k'} \exp \left\{ E_{\lambda'} \otimes E_{\lambda_k^0 - \lambda'} e^{-\frac{1}{2} \sigma_{\lambda_k^0}} \right\} \cdot \exp \{ H_{\lambda_k^0} \otimes \sigma_{\lambda_k^0} \}
$$

forms the link and the product of links – the chain:

$$
F_{\text{ch}(1 \prec m)} = \prod_k F^k_{\text{link}}.
$$

$F_{\text{ch}(1 \prec m)}$ is a solution of equations (1) for the Hopf algebra $U(g)$.

Chains permit to construct twists with large carriers. For any universal enveloping algebra $U(g)$ with simple $g$ chains of extended jordanian twists give the possibility to construct the deformations whose carrier subalgebras $g_{ch} \subset g$ contain maximal nilpotent subalgebras, $g_{ch} \supset N^\pm + g$. Such chains are called full. Even in the case of full chains the Cartan subalgebra $H(g)$ is not always contained in $g_{ch}$. The structure of chain is based on the solvability of its carrier and thus the only possibility for a chain carrier $g_{ch}$ to be parabolic is to coincide with the Borel subalgebra $B^+(g)$.

The complement subalgebra in $H(g)$ orthogonal to all the initial roots of the chain is denoted by $H^\perp$,

$$
H(g) = \left( H(g) \cap g_{ch} \right) \oplus H^\perp.
$$

It follows from the structure of the sequence $\mathbf{H}$ and the chain $\mathbf{7}$ that the number $m$ of links in the chain is equal to the dimension of the Cartan subalgebra in the carrier,

$$
\dim (H(g) \cap g_{ch}) = m,
$$

Obviously the number of links is not greater than the rank $r$ of $g$. It was shown in [13, 15] that for $g = A_1, B_n, C_n$ and $D_{n=2k}$ these two characteristics coincide: $m = r$. 

3
If dim $H^\perp$ is even one can always construct a nontrivial Abelian twist $F_A$ with the carrier $g_A = H^\perp$. It can be easily seen that the product $F_A F_{ch}$ is again a twist and in this case its carrier $g_{Ach}$ coincides with $B^+(g)$. This gives the desired parabolic carrier but its Levy factor is still Abelian. The new carrier is an extension of $g_{ch}$ by an Abelian subalgebra and thus the twists $F_A F_{ch}$ are trivial extensions of the ordinary full chains $F_{ch}$.

In this paper we shall study the special types of chains $F^R_{ch}$ for algebras $A_{n-1}$. It will be demonstrated that for such chains there exist such nontrivial additional factors $F^J$ that the compositions $F^J F^R_{ch}$ are twists and their carriers $g_P$ are nonminimal parabolic subalgebras of $A_{n-1}$. These parabolic carriers cannot be reduced to an Abelian extensions of $g_{ch}$. They contain $B^+(g)$ and their intersection with $N^{-}_g$ is nontrivial, $g_P \cap N^{-}_g = \emptyset$. The corresponding parabolic twisting elements $F_P = F^J F^R_{ch}$ will be explicitly constructed and their properties will be illustrated by examples. In the case of $A_2$ the proposed construction degenerates into the so called elementary parabolic twist presented in [20].

2 What parabolic subalgebras are we interested in?

In the previous Section it was mentioned that for algebras $B_n$, $C_n$ and $D_{n=2k}$ there exist the twist deformations (full chains of extended twists) whose carrier subalgebras are minimal parabolic: $g_{ch} = B^+(g)$. In the case $g = A_{n-1}$ the situation is different. Here only for $sl(2)$ the number of links coincides with the rank. In all other cases $r$ is greater than $m$:

$$m = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} & \text{for even } n \end{cases}$$

Thus the full chains themselves cannot have the parabolic carriers for the algebras $A_{n-1}$ with $n > 2$.

Our aim is to construct for $U(A_{n-1})$ the nonminimal parabolic twists $F_P$ whose carriers $g_P$ contain nontrivial semisimple factors. We shall search such carriers in the form

$$g_P = s \triangleright \left( g_{ch} \setminus \sum_{i=1}^{p} g^{\gamma_i} \right),$$

$$s = \bigoplus_{i=1}^{p} sl(2), \quad p = r_{A_{n-1}} - m,$$

$$sl(2)_i = g^{-\gamma_i} + H^+_i + g^{\gamma_i}.$$  

Thus in the space $H^\perp$ there must exists a basis whose elements $H^+_i$ generate the Cartan subalgebras in $s = \bigoplus_{i=1}^{p} sl(2)_i$.

The parabolic subalgebra $g_P$ of a simple Lie algebra $g$ with the root system $\Lambda$ is defined by the subset $\Psi$ of the set $\Sigma = \{\alpha_k\}$ of basic roots. The roots
of \( g_P \) (in their standard decomposition) contain \( \alpha_j \in \Psi \) only with nonnegative coefficients. Let \( \Phi (\Psi) \subset \Lambda \) be the set of roots with such properties. Then the Cartan decomposition of \( g_P \) can be written as follows,

\[
g_P = H(g) + \sum_{\lambda \in \Phi(\Psi)} g^\lambda. \tag{13}
\]

Let us define the sets \( \Psi_P \) so that the factor \( s \) in the semidirect sum is equivalent to the direct sum \( \oplus_{s=1}^p sl(2)_s \). Let the basic roots of \( A_{n-1} \) be canonically expressed in the orthonormal basis \( \{e_j\} \) of \( \mathbb{R}^n \)

\[
\alpha_k := e_k - e_{k+1}. \tag{14}
\]

We shall use the function

\[
\chi(l) = \frac{1}{2} \left( n + (n - 2l)(-1)^{l+1} \right) \tag{15}
\]

and fix the set \( \Psi_P \) as follows

\[
\Psi_P = \begin{cases} 
\{ \alpha_{\chi(s)} | s = 1, 2, \ldots, \frac{n-1}{2} = p \} & \text{for odd } n, \\
\{ \alpha_{\chi(s)}, \alpha_{\frac{n}{2}} | s = 1, 2, \ldots, \frac{n-2}{2} = p \} & \text{for even } n.
\end{cases} \tag{16}
\]

The Cartan decomposition for the parabolic subalgebra \( g_P \) is thus defined

\[
g_P = H + \sum_{\lambda \in \Lambda^+} g^\lambda + \sum_{\gamma \in \Gamma_P} g^{-\gamma}, \tag{17}
\]

where

\[
\Gamma_P = \{ \alpha_j | j(s) = n - \chi(s), s = 1, \ldots, p \}. \tag{18}
\]

We know how to construct full chains \( \mathcal{F}_{ch} \) with the carriers \( g_{ch} \) (see Section 1). Consider the space \( g_P \setminus g_{ch} \equiv g_J \). This is the space of an even dimensional algebra that can be presented as a direct sum of Borel subalgebras,

\[
g_J = \oplus b_s, \tag{19}
\]

with

\[
b_s = \left\{ H_k^{\perp_s} g^{-\alpha_r} \right\}_{s=n-\chi(i), i=1,\ldots,p}. \tag{20}
\]

In Section 5 we shall demonstrate that for the subalgebra \( U_{ch}(g_P) \) twisted by the full chain \( \mathcal{F}_{ch} \) we can construct the product of \( p \) quasi-Jordanian twists whose integral carrier contains the subalgebra \( g_J \). Thus the new twist

\[
\mathcal{F}_P = \mathcal{F}^J \mathcal{F}_{ch} \tag{21}
\]

will have the parabolic carrier

\[
g_P = \left( \bigoplus_{s=1}^p sl(2)_s \right) \triangleright \left( g_{ch} \setminus \sum_{s=1}^p g_{\gamma_s} \right). \tag{22}
\]
3 Full chain and dual coordinates.

We start by constructing the special type $\mathcal{F}_{ch}$ of full chain of extended jordanian twists that differs from (6), (7) by the choice of the Cartan elements and correspondingly by the normalization of the extension factors. In [16] and [17] it was proved that such a solution of the twist equations (1) exists:

$$\mathcal{F}_{ch} = \prod_{l=1}^{m} \mathcal{F}_l^{\text{link}} = \prod_{l=1}^{m} \left( e^{-\sum_{k=1}^{l} E_{k,n} \otimes E_{k,n} e^{-\frac{1}{2}(1-(-1)^l)}} e^{H_l \otimes \sigma_{l,n-l+1}} \right)$$

(23)

Remember that $m$ is the number of links in the full chain (see (10)) and $\sigma_{l,n-l+1} = \ln(1 + E_{l,n-l+1})$. The set \{\(\hat{H}_l\)\} of Cartan elements is chosen as follows:

$$\hat{H}_1 = +\frac{1}{n} I - E_{n,n}; \quad \hat{H}_2 = -\frac{3}{n} I + E_{1,1} + E_{2,2} + E_{n,n};$$

$$\vdots$$

$$\hat{H}_l = +\frac{2l-1}{n} I - E_{l-1,l-1} - E_{l-1,l+1} - E_{n-l+1,n-l+1} - \ldots - E_{n,n}; \quad l = 2k - 1;$$

$$\hat{H}_l = -\frac{2l-1}{n} I + E_{1,1} + \ldots + E_{l-1,l-1} + E_{l+1,l+1} + E_{n-l+2,n-l+2} + \ldots + E_{n,n}; \quad l = 2k;$$

$$\vdots$$

(24)

These elements form the Cartan subalgebra $H_{ch}^{\hat{}}$ and the Cartan decomposition of the carrier $g_{ch}^{\hat{}}$ of the chain (23) looks like

$$g_{ch}^{\hat{}} = H_{ch}^{\hat{}} \triangleright \left( \sum_{\lambda \in \Lambda^+} g_{\lambda} \right).$$

The full chain bears the set of $m$ free parameters [12] but here for simplicity we shall omit them.

Applying the twist $\mathcal{F}_{ch}$ to the Hopf algebra $U(A_{n-1})$ we obtain the deformation

$$\mathcal{F}_{ch} : U(A_{n-1}) \rightarrow U_{ch}(A_{n-1}).$$

(25)

According to the arguments developed in the previous Section we are to find the solutions $\mathcal{F}^J$ of the twist equations for $\Delta_{ch}$. The carrier of such solutions is to contain the direct sum of 2-dimensional Borel subalgebras: $g_J = \oplus_b ba$.

Usually the Hopf structure of the deformed algebras such as $U_{ch}(A_{n-1})$ is explicitly described in terms of matrix coordinate functions $E_{ij}$ and the normalized Cartan generators

$$H_{ij} := \frac{1}{2} (E_{ii} - E_{jj}).$$

(26)
Despite the property of self-duality of twisted algebras $U_F(A_{n-1})$ the coordinates $E_{ij}$ are not convenient for the description of the costructure. For example, the Jordanian deformation $U_J(A_1)$ is always written in terms of $\sigma_{12}$ but not $E_{12}$. For the deformed Hopf algebra $U_F(A_{n-1})$ the most appropriate is the basis consisting of the dual group coordinates. The dual group basis for twisted universal enveloping algebras was studied in [18], there the explicit procedure was proposed to find the relations between the algebraic coordinates (of the type $E_{ij}$) and the dual coordinates $\hat{E}_{ij}$. As we shall see below the Borel subalgebras $b_s$ in $gP \subset U^{ch}_{\hat{c}}(A_{n-1})$ with the necessary Hopf structure could be defined only in the deformed (dual group) basis.

As far as the general form of the link factor $F_{\text{link}}$ of the proposed twists (21) is known (up to the Abelian "rotation" that will be fixed later) we shall focus our attention on the dual coordinates for the subalgebra $U^{ch}_{\hat{c}}(g_J)$.

Let us remind [12] that the full chain does not change the costructure on the subspace $H \perp$ (see (8)). This means that on this space the dual group coordinates coincide with the algebraic ones. We denote them by $H_{\hat{s}} \perp$ and choose the following set as a basis in $H \perp$:

$$H_{\hat{s}} \perp = \left(-1\right)^s \left(-\frac{2s}{n} + \sum_{u=1}^{s} (E_{u,u} + E_{n-u+1,n-u+1})\right); \quad (27)$$

For the generators of the subspaces $g^{-\alpha_j}$, $j(s) = n - \chi(s)$, $s = 1, \ldots, p$, we apply the algorithms developed in [18] and obtain the following expressions for the dual coordinates $\hat{E}_s$.

$$\hat{E}_s = \left\{ \begin{array}{ll}
E_{j(s)+1,j(s)} + E_{\chi(s),\chi(s)} + 1 & s = 2k - 1; \\
E_{j(s)+1,j(s)} + \left(\hat{H}_{\chi(s)} + 1 - \hat{H}_{\chi(s)}\right)E_{\chi(s),j(s)} + E_{\chi(s),\chi(s)} + 1 & s = 2k. \end{array} \right. \quad (28)$$

When $n$ is odd the form of the "last" external coordinate $\hat{E}_p$ degenerates:

$$\hat{E}_p = \left\{ \begin{array}{ll}
E_{j(p)+1,j(p)}; & \text{for odd } n \text{ and odd } p \\
E_{j(p)+1,j(p)} - \hat{H}_{\chi(p)}E_{\chi(p),j(p)}; & \text{for odd } n \text{ and even } p. \end{array} \right. \quad (29)$$

As far as

$$\begin{array}{l}
\text{for odd } s \quad j(s) = s, \\
\text{for even } s \quad \chi(s) = s,
\end{array} \quad (29)$$
these expressions can be rewritten in a more compact way:

\[
\hat{E}_s = \begin{cases} 
E_{s+1,s} + E_{n-s,n-s+1} e^{H_{s+1,n-s-s,n-s+1} + \hat{H}_s} & s = 2k - 1; \\
E_{n-s+1,n-s} + \left( \hat{H}_{s+1} - \hat{H}_s \right) E_{s,n-s} + E_{s,s+1} & s = 2k; 
\end{cases} 
\]

(30)

\[
s = 1, \ldots, \begin{cases} 
p - 1 & \text{for odd } n \\
p & \text{for even } n 
\end{cases}
\]

\[
\hat{E}_p = \begin{cases} 
E_{p+1,p} & \text{for odd } n \text{ and odd } p \\
E_{n-p+1,n-p} - \hat{H}_p E_{p,n-p} & \text{for odd } n \text{ and even } p 
\end{cases} 
\]

4 Properties of external coordinates

Conjecture 1 Let \( \mathcal{L} \left( \hat{E}_s \right) \) be the space spanned by the set of external coordinates. Then \( H^\perp \oplus \mathcal{L} \left( \hat{E}_s \right) \) is the space of the direct sum \( g_J = \oplus b_s \). Each 2-dimensional Borel subalgebra \( b_s \) is generated by the corresponding pair \( \{ H^\perp_s, \hat{E}_s \} \).

Proof. This can be checked by direct computations. All the external coordinates commute,

\[
\left[ \hat{E}_s, \hat{E}_t \right] = 0; \quad s, t = 1, \ldots, p. 
\]

(31)

Notice that the subsets of odd external coordinates and of even coordinates are trivially commutative. This obviously follows from the form of their expressions in terms of the elements \( E_{i,j} \) (see (30)): the ranges of indices \( i, j \) in different \( \hat{E}_s \) in these two subsets do not intersect. Thus only the vanishing of odd-even commutators is to be checked. It also can be verified that the Cartan elements \( H^\perp_s \) act on the external coordinates as follows

\[
\left[ H^\perp_s, \hat{E}_t \right] = \delta_{s,t} \hat{E}_t. 
\]

(32)

There exists an important relation connecting the Cartan generators \( \hat{H}_s \) in the Jordanian factors \( e^{H_s \otimes \sigma_{l,n-l+1}} \) of the chain \( \mathcal{F}_{ch} \) and the basic elements \( H^\perp_s \):

\[
-2H^\perp_s = \begin{cases} 
2H_{s,s+1} + \hat{H}_{s+1} - \hat{H}_s & s = 2k - 1, \\
2H_{n-s,n-s+1} + \hat{H}_{s+1} - \hat{H}_s & s = 2k. 
\end{cases} 
\]

(33)

Now consider the costructure induced by the twist deformation \( \mathcal{F}_{ch}^- \) (see (23)) on the subalgebra generated by

\[
\left\{ H^\perp_s, \hat{E}_t, \sigma_{l,n-l+1} \mid s, t = 1, \ldots, p; l = 1, \ldots, m \right\}. 
\]
Conjecture 2 The only nontrivial cocommutator in $\mathcal{L}(\hat{E}_s)$ is a map $(g^{\alpha(s)})^\# \otimes (H_s^\#)^\# \longrightarrow (\hat{E}_s)^\#$. The corresponding coproducts are

$$\Delta_{ch}(\hat{E}_s) = \begin{cases} \hat{E}_s \otimes 1 + 1 \otimes \hat{E}_s - 2H^1_\# \otimes E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}; & s = 2k - 1; \\ \hat{E}_s \otimes 1 + 1 \otimes \hat{E}_s - E_{s,n-s} \otimes 2H^1_s; & s = 2k. \end{cases}$$ (34)

Proof. From (23) it can be observed that for any odd indicae $s$ the element $E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}$ is dual to $E_{s,s+1}$. The functional corresponding to $E_{s+1,n-s+1}e^{-\sigma_{s,n-s+1}}$ acts inversely with respect to $E_{s,s+1}$ and thus can shift the elements from $H^1$ to $\mathcal{L}(\hat{E}_s)$. Correspondingly for even $s$ the dual for $E_{s,n-s}$ is $E_{n-s,n-s+1}$ with the similar shifting properties. Notice that the appearance of the factor $e^{-\sigma_{s,n-s+1}}$ in the odd case is in complete correspondence with the form of odd and even links in the chain $\mathcal{F}_{ch}$.

Let us check the relation (34).

Analyzing the explicit form of the external coordinates $\hat{E}_s$ one can see that each of them is defined on the subalgebra

$$M_s = \left( sl(\mathfrak{g}_{s,s+1}) + sl(\mathfrak{g}_{n-s,n-s+1}) \right) \supset J_s$$ (35)

with $J_s$ being an ideal corresponding to the roots: $e_s - e_{n-s}$, $e_s - e_{n-s+1}$, $e_{s+1} - e_{n-s}$ and $e_s - e_{n-s+1}$. Under the action of first $s - 1$ links of $\mathcal{F}_{ch}$ the coproducts in $M_s$ remain unchanged (primitive on the generators) due to the “matreshka” effect [12]. On the other hand the links $\mathcal{F}_{\text{link}}^{l}$ with $l > s + 1$ are defined on the subalgebras

$$sl^\vee(n - 4s)$$ (36)

with the roots $e_i - e_j; i, j = s + 2, \ldots, n - s - 1$. In $sl(n)$ these subalgebras, $M_s$ and $sl^\vee(n - 4s)$, form a direct sum:

$$sl(n) \supset M_s \oplus sl^\vee(n - 4s).$$ (37)

It follows that the links $\mathcal{F}_{l \geq s+1}$ also cannot deform the costructure of the coordinate $\hat{E}_s$. Thus only two links are to be considered for each $\hat{E}_s$, the $s$-th and the $(s + 1)$-th:

$$\mathcal{F}_s = \exp \left( \sum_{k_s=s+1}^{n-s} E_{s,k_s} \otimes E_{s,n-s+1}e^{-\sigma_{s,n-s+1}} \right) \exp \left( \hat{H}_s \otimes \sigma_{s,n-s+1} \right)$$ (38)

$$\mathcal{F}_{s+1} = \exp \left( \sum_{k_{s+1}=s+2}^{n-s-1} E_{s+1,k_{s+1}} \otimes E_{s+1,n-s} \right) \exp \left( \hat{H}_{s+1} \otimes \sigma_{s+1,n-s} \right)$$ (39)
Let us apply the corresponding adjoint operators to the initial primitive coproducts $\Delta^{(0)} (E_{s+1,s})$ and obtain for them the twisted expressions:

$$\Delta_{s,s+1} (E_{s+1,s}) = F_{s+1} \circ F_s \circ (E_{s+1,s} \otimes 1 + 1 \otimes E_{s+1,s}) =$$

$$= e^{\text{ad} \sum_{k=s+1}^{n-s-1} E_{s+1,k} \otimes E_{k,s+1} \otimes 1 + 1 \otimes E_{s+1,s} \otimes \text{ad} (H_{s+1} \otimes 1 + 1 \otimes \text{ad} (H_s \otimes 1 + 1 \otimes E_{s+1,s} \otimes 1 + 1 \otimes E_{s+1,s} + 1 + 1 \otimes E_{s+1,s} + \left( 2H_{s+1} + \tilde{H}_s \right) \otimes E_{s+1,n-s+1} e^{-s,s,n-s+1} +$$

$$- \sum_{k=s+1}^{n-s-1} E_{s+1,k} e^{s,s+1,n-s} \otimes E_{k,s+1,n-s+1} e^{s,s+1,n-s} +$$

$$+ \sum_{k=s+1}^{n-s-1} E_{k,s+1} e^{s,s+1,n-s} \otimes E_{s+1,k,s+1,n-s+1} e^{s,s+1,n-s} -$$

$$e^{s,s+1,n-s} \otimes E_{n-s,n-s+1} e^{s,s+1,n-s} +$$

$$+ 1 \otimes E_{s+1,n-s+1} e^{s,s+1,n-s} -$$

$$- \tilde{H}_s e^{s,s+1,n-s} \otimes E_{s+1,n-s+1} e^{s,s+1,n-s} +$$

$$= E_{s+1,s} \otimes 1 + 1 \otimes E_{s+1,s} +$$

$$+ \left( 2H_{s+1} + \tilde{H}_s \right) \otimes E_{s+1,n-s+1} e^{-s,s,n-s+1} +$$

$$= E_{s+1,n-s+1} e^{-s,s,n-s+1} \otimes E_{s+1,n-s+1} e^{-s,s,n-s+1}; \quad (40)$$

The result of the twist deformation by $F_{s+1} \circ F_s$ for the exponential factor $e^{s,s+1,n-s} - e^{s,s+1,n-s}$ in the second summand of $E_s$ is well known [11] — this factor is group-like:

$$\Delta_{s,s+1} (e^{s,s+1,n-s} - e^{s,s+1,n-s}) = e^{s,s+1,n-s} - e^{s,s+1,n-s} \otimes e^{s,s+1,n-s} +$$

$$= (s,s+1,n-s - e^{s,s+1,n-s}) \otimes e^{s,s+1,n-s} +$$

$$= (s,s+1,n-s) \otimes e^{s,s+1,n-s}; \quad (41)$$

Thus only the coproduct $\Delta_{s,s+1} (E_{n-s,n-s+1})$ remains to be constructed:

$$\Delta_{s,s+1} (E_{n-s,n-s+1}) = F_{s+1} \circ F_s \circ (E_{n-s,n-s+1} \otimes 1 + 1 \otimes E_{n-s,n-s+1} =$$

$$= E_{n-s,n-s+1} \otimes e^{s,s+1,n-s} - e^{s,s+1,n-s} \otimes E_{n-s,n-s+1} +$$

$$+ \sum_{k=s+1}^{n-s-1} E_{s+1,k,s+1} e^{s,s+1,n-s} \otimes E_{k,s+1,n-s+1} +$$

$$+ \tilde{H}_s e^{s,s+1,n-s} \otimes E_{s+1,n-s+1} e^{s,s+1,n-s} -$$

$$- \sum_{k=s+1}^{n-s-1} E_{s+1,k,s+1} e^{s,s+1,n-s} \otimes E_{k,s+1,n-s+1} e^{s,s+1,n-s} +$$

$$As we have expected the cumbersome parts of these coproducts cancel in the
integral expression $\Delta_{ch} \left( \hat{E}_s \right)$ and we obtain for $s = 2k - 1$

$$\Delta_{ch} \left( \hat{E}_s \right) = \Delta_{s,s+1} \left( \hat{E}_s \right) =$$

$$= E_s \otimes 1 + 1 \otimes E_s +$$

$$+ \left( 2H_{s+1} + \hat{H}_{s+1} - \hat{H}_s \right) \otimes E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}}. \quad (43)$$

For even coordinates $\hat{E}_{s=2k} = E_{n-s+1,n-s} + \left( \hat{H}_{s+1} - \hat{H}_s \right) E_{s,n-s} + E_{s,s+1}$

the nontrivial part of the deformation is performed by the links

$$\mathcal{F}_{s+1} \mathcal{F}_s = e^{\sum_{k=1}^{n-s} E_{s+1,k+1} \otimes E_{k+1,n-s} e^{-\sigma_{s+1,n-s}} e^{\hat{H}_{s+1} \otimes \sigma_{s+1,n-s}}} e^{\sum_{k=1}^{n-s} E_{s,k} \otimes E_{k,n-s+1} e^{\left( \hat{H}_s \otimes \sigma_{s,n-s+1} \right)}}, \quad (44)$$

$s = 2k$.

Here is the result of the chain deformation for even $s \left( \hat{E}_{s=2k} \right)$

$$\Delta_{ch} \left( \hat{E}_s \right) = \Delta_{s,s+1} \left( \hat{E}_s \right) =$$

$$= E_{n-s+1,n-s} + \left( \hat{H}_{s+1} - \hat{H}_s \right) E_{s,n-s} + E_{s,s+1} =$$

$$= E_s \otimes 1 + 1 \otimes E_s + E_{s,n-s} \otimes \left( 2H_{n-s,n-s+1} + \hat{H}_{s+1} - \hat{H}_s \right).$$

Taking into account the relations (43) we can rewrite the obtained coproducts:

$$\Delta_{ch} \left( \hat{E}_s \right) = \begin{cases} 
E_s \otimes 1 + 1 \otimes E_s - 2H_s \otimes E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}}, & s = 2k - 1; \\
E_s \otimes 1 + 1 \otimes E_s - E_{s,n-s} \otimes 2H_s, & s = 2k.
\end{cases} \quad (45)$$

5 Rotations for the chains of twists.

Now let us demonstrate that the twisted coproducts $\Delta_{ch} \left( \hat{E}_s \right)$ can be reduced further with the help of additional Abelian twists 9.

We have found out that the only nontrivial terms in $\Delta_{ch} \left( \hat{E}_s \right)$ are due to the coadjoint action of $\left( E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}} \right)^\#$ and $\left( E_{n-s,n-s+1} \right)^\#$. The corresponding coadjoint maps (dual to the adjoint action of $\text{ad} \left( E_{s,s+1} \right)$ and $\text{ad} \left( E_{s,n-s} \right)$) depend on the dualization $\left( \hat{H}_s \right)^\# \sim \sigma_{s,n-s+1}$. In the chain $\mathcal{F}_{ch}$

this dualization depends on the form of the Jordanian factors $e^{\hat{H}_i \otimes \sigma_{i,n-i+1}}$. 

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When the carrier of the chain does not cover the Cartan subalgebra the relation
\( \widehat{H}_s \# \sim \sigma_{s,n-s+1} \) can be transformed by applying the Abelian twists
of the type \( e^{\gamma_s \hat{H}_s \# \otimes \sigma_{s,n-s+1}} \). Such a transformation takes place for example
when a canonical chain is changed into a peripheric one \[16\]. Varying the
parameters \( \gamma_s \) one can alter the operators \( \text{coad} (E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}}) \# \)
and \( \text{coad} (E_{n-s,n-s+1}) \# \) and under some additional conditions trivialize their action
on the space \( \left( \hat{H}_s \right)^\# \). Thus studying the dual algebra action we see that
there can exist a possibility to annihilate the nontrivial terms in \( \Delta_{\hat{g}} \hat{E}_s \) with
the help of Abelian twists.

When the Abelian twists are of the pure form \( e^{\gamma_s \hat{H}_s \# \otimes \sigma_{s,n-s+1}} \) they can be
incorporated into the initial Jordanian factors of the chain and their application
reduces to the transition from one type of the chain to the other \[16\]. In the
present case this will be impossible because the necessary Abelian twists have
more general configuration.

Let us introduce the set
\[ C_s = e^{\sum_{i=1}^{s} \sigma_{i,n-i+1}} \]
with the commutation properties
\[ (-1)^s \left[ \ln (C_s), \hat{E}_s \right] = 2\delta_{s,t} \cdot \left\{ \begin{array}{ll}
E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}} & \text{for odd } s \\
E_{s,n-s} & \text{for even } s
\end{array} \right. \]
\[ \left[ \ln (C_t), E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}} \right] = 0; \]
\[ \left[ \ln (C_t), E_{s,n-s} \right] = 0; \]
\[ [C_s,C_t] = 0; \]
and consider the rotation twists:
\[ F^R_s = e^{-\hat{H}_s^\# \otimes \ln (C_s)}; \text{ for odd } s, \]
\[ F^R_s = e^{\ln (C_s) \otimes \hat{H}_s^\#}; \text{ for even } s. \]

Due to the relations \[49\] and the obvious property
\[ \left[ \hat{H}_s^\#, \ln (C_s) \right] = 0 \]
the factors \( F^R_s \) commute with each other. According to the relation \[47\] we also have
\[ \left[ F^R_s, \Delta_{\hat{g}} \hat{E}_t \right] \mid_{s \neq t} = 0. \]

It follows that the factors \( F^R_s \) can be applied to the Hopf algebra \( U_{\hat{g}} (A_{n-1}) \)
and the rotation twist can be composed as a product:
\[ F^R = \prod_{s=1}^P F^R_s = \exp \left( \sum_{\text{even } s} \ln (C_s) \otimes \hat{H}_s^\# - \sum_{\text{odd } s} H_s^\# \otimes \ln (C_s) \right). \]
Each factor $F^R_s$ in $F^R$ acts nontrivially only on the coproduct of the corresponding external coordinate $\Delta_{ch} (\hat{E}_s)$.

Taking into account the relations (32), (48) and (52) we can construct the deformation of the coproduct $\Delta_{ch} (\hat{E}_s)$ induced by the rotation $F^R$:

$$\Delta^R_{ch} (\hat{E}_s) = F^R \circ \Delta_{ch} (\hat{E}_s) = \begin{cases} \hat{E}_s \otimes C_s^{-1} + 1 \otimes \hat{E}_s, & s = 2k - 1, \\ \hat{E}_s \otimes 1 + C_s \otimes \hat{E}_s, & s = 2k. \end{cases} \quad (55)$$

For the coproducts $\Delta_{ch} (H^\perp_s)$ the result of twisting by $F^R$ is trivial:

$$\Delta^R_{ch} (H^\perp_s) = F^R \circ \Delta (0) (H^\perp_s) = \Delta (0) (H^\perp_s). \quad (56)$$

Thus the rotated chain

$$F^R_{ch} = F^R \Delta_{ch} \quad (57)$$

performs the Hopf algebra deformation

$$F^R_{ch} : U (A_{n-1}) \rightarrow U^R_{ch} (A_{n-1}). \quad (58)$$

6 The quasi-Jordanian factors and the parabolic twists

The twisted algebra $U^R_{ch} (A_{n-1})$ contains the subalgebra generated by the primitive $H^\perp_s$, group-like $C_s$ and quasiprimitive external coordinates $\hat{E}_s$. At this point it is convenient to redefine the external coordinates. Let us introduce

$$D_s = \begin{cases} \hat{E}_s C_s, & s = 2k - 1, \\ \hat{E}_s, & s = 2k, \end{cases} \quad (59)$$

and consider in $U^R_{ch} (A_{n-1})$ the subalgebras $\mathfrak{B}_s$ generated by the triples $\{H^\perp_s, D_s, C_s\}$ and presented by the relations

$$\left[ H^\perp_s, D_t \right] = \delta_{st} D_t; \quad \left[ H^\perp_s, C_t \right] = 0; \quad s, t = 1, \ldots, p; \quad (60)$$

$$\Delta^R_{ch} (H^\perp_s) = \Delta (0) (H^\perp_s) = H^\perp_s \otimes 1 + 1 \otimes H^\perp_s; \quad (61)$$

$$\Delta^R_{ch} (D_s) = F^R \circ \Delta_{ch} (D_s) = D_s \otimes 1 + C_s \otimes D_s; \quad (62)$$

$$\Delta^R_{ch} (C_s) = C_s \otimes C_s. \quad (63)$$

Notice that for $s \neq t$ the generators $C_s$ and $D_t$ commute (as a consequence of (47)). It follows that

$$[D_s, D_t] = 0; \quad [D_s + C_s, D_t + C_t] = 0; \quad \text{for any } s, t. \quad (64)$$
The factor $C_s$ in (62) does not allow the canonical Jordanian twist to be a solution to the twist equations for $\Delta_{ch}^R$. To overcome this difficulty the following lemma will be proved.

**Lemma 3** Let $\mathcal{B}$ be a Hopf algebra generated by the elements $H, C, D$ with the properties:

1. \[ \Delta (H) = \Delta^{(0)} (H) \] (65)

2. \[ \Delta (C + D) = D \otimes 1 + C \otimes (C + D) . \] (66)

3. \[ \{H, D\} = D, \quad \{H, C\} = 0, \] (67)

Then
\[ F = e^{H \otimes \ln (C + D)} \]
is a solution to the twist equations for $\mathcal{B}$.

**Proof.** Consider
\[ \Delta_F (C + D) = F \circ (\Delta (C + D)) = \]
\[ F \circ (D \otimes 1 + C \otimes (C + D)) = D \otimes (C + D) + C \otimes (C + D) = \]
\[ = (C + D) \otimes (C + D); \]

This means that for \[ \omega = \ln (C + D) \]
we have
\[ \Delta_F (\omega) = \Delta^{(0)} (\omega) \] (68)
and taking into account the primitivity of $\Delta^{(0)} (H)$ the factorizable twist equations
\[ (\text{id} \otimes \Delta_F) F = (F)_{12} (F)_{13}; \]
\[ (\Delta \otimes \text{id}) F = (F)_{13} (F)_{23}. \]

are valid for $F$. \[ \blacksquare \]

**Remark 4** In what follows we shall consider the group-like $C$. But in the general case the class of algebras that meet the conditions of the Lemma has no restrictions on $\Delta (C)$ and the composition $[C, D]$.

**Remark 5** For a particular case $C = 1$ the subalgebra $\mathcal{B}$ is reduced to $b^2$ and the element $F$ becomes the ordinary Jordanian twist.
For any \( s = 1, \ldots, p \) the triple \( \{ H_s, D_s, C_s \} \) comply with the conditions \( \text{(65-67)} \). Thus for any \( s \) we have a subalgebra \( \mathfrak{B}_s \) generated by \( \{ H_s, D_s, C_s \} \) and can construct the twisting element
\[
\mathcal{F}_s^J = e^{H_s^\perp \otimes \ln(C_s + D_s)} = e^{H_s^\perp \otimes \omega_s}
\]
that can be applied to \( U_{\text{ch}}^R (A_{n-1}) \)
\[
\mathcal{F}_s^J : U_{\text{ch}}^R (A_{n-1}) \longrightarrow U_{\text{ch}}^{2s} (A_{n-1}).
\]
In the twisted subalgebra \( \mathcal{F}_s^J \circ \mathfrak{B}_s = \mathfrak{B}_s^J \) the element \( \omega_s \) becomes primitive
\[
\Delta_{\text{ch}}^{2s} (\omega_s) = \omega_s \otimes 1 + 1 \otimes \omega_s.
\]
The Hopf algebra \( U_{\text{ch}}^{2s} (A_{n-1}) \) can be considered as the deformation of \( U (A_{n-1}) \) by the factorizable twist
\[
\mathcal{F}_s^J \mathcal{F}_{\text{ch}}^R = \mathcal{F}_s^J \mathcal{F}_{\text{ch}}^R \mathcal{F}_{\text{ch}}^R
\]
with the carrier
\[
g_{\text{ch}}^{2s} = (sl (2))_s \triangleright (g_{\text{ch}} \setminus g_{s}).
\]
The question arises whether the factors \( \mathcal{F}_t^J \) can be applied to \( U_{\text{ch}}^{2s} (A_{n-1}) \) with \( s \neq t \) and finally whether the same can be done with the product of all the factors \( \mathcal{F}_s^J \) to obtain finally the twist with the parabolic carrier \( g_{\text{p}} \).

Consider the action of \( \mathcal{F}_s^J \) on the subalgebra \( \mathfrak{B}_t \) with \( s \neq t \). The invariance of the coproducts \( \text{(65-67)} \) with respect to the twist \( \mathcal{F}_s^J \) is needed. The coproducts of \( H_t^\perp \mid t \neq s \) are invariant due to the relations \( \text{(60)} \). From \( \text{(60)} \) and \( \text{(64)} \) it follows that \( \Delta_{\text{ch}}^{2s} (C_t + D_t) \mid t \neq s \) are also invariant:
\[
\mathcal{F}_s^J \circ \left( \Delta_{\text{ch}}^{R} (C_t + D_t) \right) \mid s \neq t = \mathcal{F}_s^J \circ \left( e^{\text{ad} H_s^\perp \otimes \ln(C_s + D_s)} \circ (D_t \otimes 1 + C_t \otimes (C_t + D_t)) \right) = \mathcal{F}_s^J \circ \left( D_t \otimes 1 + C_t \otimes (C_t + D_t) \right).
\]
Thus the Hopf subalgebra \( \mathfrak{B}_t \mid t \neq s \) is invariant with respect to the twisting by \( \mathcal{F}_s^J \). The latter means that the product of all the factors \( \mathcal{F}_s^J \)
\[
\mathcal{F}^J = \prod_{s=1}^{p} \mathcal{F}_s^J = \prod_{s=1}^{p} e^{H_s^\perp \otimes \ln(C_s + D_s)} = \prod_{s=1}^{p} e^{H_s^\perp \otimes \omega_s}
\]
can be applied to \( U_{\text{ch}}^R (A_{n-1}) \)
\[
\mathcal{F}^J : U_{\text{ch}}^R (A_{n-1}) \longrightarrow U_{\text{p}} (A_{n-1}).
\]
The deformed algebra $U_P(A_{n-1})$ can be also presented as twisted by the product of the rotated chain and the quasi-Jordanian factors

$$F_P = F^\hat{J} F^R_{\text{ch}},$$  \hspace{1cm} (77)

$$F_P : U(A_{n-1}) \rightarrow U_P(A_{n-1}).$$  \hspace{1cm} (78)

The carrier of the twist $F_P$,

$$g_P = \left( \bigoplus_{s=1}^p \text{sl}(2) \right) \triangleright \left( g_{\text{ch}} \setminus \sum_{s=1}^p g^\gamma_s \right),$$  \hspace{1cm} (79)

is the parabolic subalgebra.

7 Examples.

7.1 $\text{sl}(4)$

Here the chain contains two links

$$F_{\text{ch}} = e^{\mathcal{H}_2 \sigma_2} e^{E_{1,2}} e^{E_{1,3}} e^{E_{1,4}} e^{E_{3,4}} e^{E_{3,3}},$$

with

$$\mathcal{H}_1 = \frac{1}{4} I - E_{4,4};$$  \hspace{1cm} (80)

$$\mathcal{H}_2 = -\frac{3}{4} I + E_{1,1} + E_{2,2} + E_{4,4}. $$  \hspace{1cm} (81)

The main parameters of this chain deformation are

$$m = 2; \hspace{1cm} r = 3;$$  \hspace{1cm} (82)

$$p = \dim H^\perp = 1;$$  \hspace{1cm} (83)

The generators included in the chain carrier subalgebra and the possible choice of external coordinates $E_{k+1,k}$,

$$\begin{bmatrix}
E_{1,1} & E_{1,2} & E_{1,3} & E_{1,4} \\
E_{2,1} & E_{2,2} & E_{2,3} & E_{2,4} \\
E_{3,3} & E_{3,4} & E_{4,4}
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
E_{1,1} & E_{1,2} & E_{1,3} & E_{1,4} \\
E_{2,2} & E_{2,3} & E_{2,4} \\
E_{3,3} & E_{3,4} & E_{4,4}
\end{bmatrix},$$  \hspace{1cm} (84)

show us that there are two minimal parabolic subalgebras of the type $g_P$ appropriate for our twist algorithm. They are defined by the sets

$$\Psi_{P_1} = \{ \alpha_2, \alpha_3 \},$$

$$\Psi_{P_3} = \{ \alpha_1, \alpha_2 \}, \hspace{1cm} \alpha_k := e_k - e_{k+1},$$  \hspace{1cm} (85)
\[ \Gamma_{P_1} = \{ \alpha_{j(k)} \mid k = 1, j(1) = 4 - \chi(1) = 1 \} = \{ \alpha_1 \}, \]
\[ \Gamma_{P_3} = \{ \alpha_{j(k)} \mid k = 3, j(3) = 4 - \chi(3) = 3 \} = \{ \alpha_3 \}, \]

and can be presented in the following form:

\[ \Phi (\Psi_{P_1}) = \Lambda^+ \cup \{-\alpha_1 \}, \]
\[ \Phi (\Psi_{P_3}) = \Lambda^+ \cup \{-\alpha_3 \}. \]

and can be presented in the following form:

\[ g_{P_1} = H + \sum_{\lambda \in \Lambda^+} g^\lambda + g^{-\alpha_1} = B^+ + g^{-\alpha_1}, \]
\[ g_{P_3} = H + \sum_{\lambda \in \Lambda^+} g^\lambda + g^{-\alpha_3} = B^+ + g^{-\alpha_3}. \]

First let us study the case \( g_{P_1} \). In the one-dimensional space \( H^\perp \) take the generator

\[ H^\perp_1 = \frac{1}{2} I - (E_{1,1} + E_{4,4}). \]

and consider the external coordinate:

\[ \hat{E}_1 = E_{2,1} + E_{3,4} e^{\sigma_{2,3} - \sigma_{3,4}}; \]

The peculiarity of the Cartan subalgebras in \( sl(4k) \), \( k \in \mathbb{N}^+ \), is that the following relation holds

\[ 2H_{n-3,n-2} + \overrightarrow{H}_{n-2} - \overleftarrow{H}_{n-3} = 0. \]

Thus in the twisted coproduct \( \Delta_{\hat{\text{ch}}} \left( \hat{E}_1 \right) \) the term connecting the space \( H^\perp \) and \( \hat{E}_1 \) is zero and both the Cartan generator \( H^\perp_1 \) and the external coordinate have primitive coproducts:

\[ D_1 = \hat{E}_1, \]
\[ \Delta_{\hat{\text{ch}}} (D_1) = D_1 \otimes 1 + 1 \otimes D_1; \]
\[ \Delta_{\hat{\text{ch}}} (H^\perp_1) = H^\perp_1 \otimes 1 + 1 \otimes H^\perp_1. \]

It is easy to check that they obey the \( b^2 \)-relations

\[ [H^\perp_1, D_1] = D_1. \]

These are the standard conditions guaranteeing that the ordinary Jordanian twist

\[ \mathcal{F}^{\mathcal{J}}_1 = e^{H^\perp_1 \otimes \ln(1 + D_1)} = e^{H^\perp_1 \otimes \ln(1 + \hat{E}_1)} = e^{H^\perp_1 \otimes \omega_1} \]

is a solution of the twist equations for \( \Delta_{\hat{\text{ch}}} \). As a consequence the product

\[ \mathcal{F}_{P_1} = \mathcal{F}^{\mathcal{J}}_1 \mathcal{F}_{\hat{\text{ch}}} \]

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is a solution of the twist equations for the undeformed $U(\mathfrak{sl}(4)),$

$$\mathcal{F}_{P_1} : U(\mathfrak{sl}(4)) \rightarrow U_{P_1}(\mathfrak{sl}(4)),$$ (98)

with the parabolic carrier $g_{P_1}$. The alternative possibility is to choose $g_{P_3}$ as the carrier subalgebra. In this case taking into account (91) the external coordinate can be presented as

$$\hat{E}_3 = E_{1,3} - (\hat{H}_1 - \hat{H}_2) E_{1,3} + E_{1,2} =$$

$$= E_{1,3} - 2H_{3,4}E_{1,3} + E_{1,2}. (99)$$

The direct computation shows that the twisted coproduct $\Delta_{\hat{\mathfrak{ch}}} (\hat{E}_3)$ again has no terms like $H^\perp \wedge E$. This time it is quasiprimitive:

$$D_3 = \hat{E}_3 e^{\sigma_{1,4} - \sigma_{2,3}}, \quad C_3 = e^{\sigma_{1,4} - \sigma_{2,3}}$$

(100)

$$\Delta_{\hat{\mathfrak{ch}}} (D_3) = D_3 \otimes 1 + C_3 \otimes D_3; \quad (101)$$

$$\Delta_{\hat{\mathfrak{ch}}} (H^\perp_1) = H^\perp_1 \otimes 1 + 1 \otimes H^\perp_1. (102)$$

The pair $\{H^\perp_1, D_3\}$ forms a Borel subalgebra:

$$[H^\perp_1, D_3] = -D_3; \quad (103)$$

and according to Lemma 4 the factor

$$\mathcal{F}_3^\hat{J} = e^{-H^\perp_1 \otimes \ln(C_3 + D_3)} = e^{-H^\perp_1 \otimes \ln(1 + \hat{E}_3) e^{\sigma_{1,4} - \sigma_{2,3}}} = e^{-H^\perp_1 \otimes \omega_3} \quad (104)$$

is a solution to the twist equations for $\Delta_{\hat{\mathfrak{ch}}}$. The same is true for the product

$$\mathcal{F}_{P_3} = \mathcal{F}_3^\hat{J} \mathcal{F}_{\hat{\mathfrak{ch}}} \quad (105)$$

with respect to the undeformed $\Delta$ and thus $\mathcal{F}_{P_3}$ represents the parabolic twist with the carrier $g_{P_3}$.

Notice that these twists do not commute and we cannot apply both deformations simultaneously, no combinations of $\omega_1$ and $\omega_3$ are allowed.

### 7.2 $sl(11)$

For this case we have

$$m = 5; \quad r = 10; \quad (106)$$

$$p = \dim H^\perp = 5; \quad (107)$$

$$\Psi_P = \left\{ \alpha \chi(s) \mid \chi(s) = \frac{1}{2} \left( 11 + (11 - 2s) (-1)^{s+1} \right), s = 1, \ldots, 5 \right\} =$$

$$= \{ \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10} \}, \quad \alpha_k := e_k - e_{k+1}, \quad (108)$$
\[ \Phi(\Psi) = \Lambda^+ \cup \{-\alpha_j\}, \quad \alpha_j \in \Gamma_P, \]  
(109)

where

\[ \Gamma_P = \{\alpha_j \mid j(s) = 11 - \chi(s)\} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9\}. \]  
(110)

Consider the parabolic subalgebra

\[ g_P = H + \sum_{\lambda \in \Lambda^+} g^\lambda + \sum_{\gamma \in \Gamma_P} g^{-\gamma}, \]  
(111)

with the set of 2-dimensional Borel subalgebras

\[ b_s = \{\hat{H_s}, g^{-\alpha_j(s)}\}_{s=1,\ldots,5}, \]  
(112)

where

\[
\begin{align*}
\hat{H}_1^\perp &= \frac{2}{11}I - (E_{1,1} + E_{11,11}); \\
\hat{H}_2^\perp &= -\frac{4}{11}I + \sum_{u=1}^{2} (E_{u,u} + E_{12-u,12-u}); \\
\hat{H}_3^\perp &= \frac{6}{11}I - \sum_{u=1}^{3} (E_{u,u} + E_{12-u,12-u}); \\
\hat{H}_4^\perp &= -\frac{8}{11}I + \sum_{u=1}^{4} (E_{u,u} + E_{12-u,12-u}); \\
\hat{H}_5^\perp &= \frac{10}{11}I - \sum_{u=1}^{5} (E_{u,u} + E_{12-u,12-u}).
\end{align*}
\]  
(113)

The dual coordinates \( \hat{E}_s \) for the generators of the root subspaces \( g^{-\alpha_j(s)} \) are as follows:

\[
\begin{align*}
\hat{E}_1 &= E_{2,1} + E_{10,11} e^{\sigma_{2,10,11}}; \\
\hat{E}_2 &= E_{10,9} + (\hat{H}_3 - \hat{H}_2) E_{2,9} + E_{2,3}; \\
\hat{E}_3 &= E_{4,3} + E_{8,9} e^{\sigma_{4,8,9}}; \\
\hat{E}_4 &= E_{8,7} + (\hat{H}_5 - \hat{H}_4) E_{4,7} + E_{4,5}; \\
\hat{E}_5 &= E_{6,5}.
\end{align*}
\]  
(114)
After the application of the full chain twist

\[ F_{\text{ch}} = \prod_{l=1}^{5} F^l_{\text{link}} \]

\[ = e^{E_{5,6} \otimes E_{6,7} e^{-\sigma_{5,7}}} e^{E_{4,5} \otimes E_{5,6} e^{-\sigma_{4,5}}} \cdot e^{E_{3,4} \otimes E_{4,5} e^{-\sigma_{3,4}}} \cdot e^{E_{2,3} \otimes E_{3,4} e^{-\sigma_{2,3}}} \cdot e^{E_{1,2} \otimes E_{2,3} e^{-\sigma_{1,2}}} \]

with

\[ H_1 = +\frac{1}{14} I - E_{11,11}; \]
\[ H_2 = -\frac{1}{14} I + E_{1,1} + E_{2,2} + E_{11,11}; \]
\[ H_3 = +\frac{5}{11} I - E_{1,1} - E_{2,2} - E_{9,9} - E_{10,10} - E_{11,11}; \]
\[ H_4 = +\frac{7}{11} I + E_{1,1} + \ldots + E_{4,4} + E_{9,9} + E_{10,10} + E_{11,11}; \]
\[ H_5 = +\frac{9}{11} I - E_{1,1} - \ldots - E_{4,4} - E_{7,7} - E_{8,8} - \ldots - E_{11,11}; \]

(115)

to \( U(\text{sl}(11)) \) we obtain the following coproduct maps for the external coordinates

\[ \Delta_{\text{ch}} (\hat{E}_1) = \hat{E}_1 \otimes 1 + 1 \otimes \hat{E}_1 + (-2H_1^\perp) \otimes E_{2,11} e^{-\sigma_{1,11}}; \]
\[ \Delta_{\text{ch}} (\hat{E}_2) = \hat{E}_2 \otimes 1 + 1 \otimes \hat{E}_2 + E_{9,9} \otimes (-2H_2^\perp); \]
\[ \Delta_{\text{ch}} (\hat{E}_3) = \hat{E}_3 \otimes 1 + 1 \otimes \hat{E}_3 + (-2H_3^\perp) \otimes E_{4,9} e^{-\sigma_{3,9}}; \]
\[ \Delta_{\text{ch}} (\hat{E}_4) = \hat{E}_4 \otimes 1 + 1 \otimes \hat{E}_4 + E_{7,7} \otimes (-2H_4^\perp); \]
\[ \Delta_{\text{ch}} (\hat{E}_5) = \hat{E}_5 \otimes 1 + 1 \otimes \hat{E}_5 + (-2H_5^\perp) \otimes E_{6,7} e^{-\sigma_{5,7}}; \]

(116)

while the Cartan generators \( H_s^\perp \) remain primitive:

\[ \Delta_{\text{ch}} (H_s^\perp) = H_s^\perp \otimes 1 + 1 \otimes H_s^\perp. \]

(117)

Let us introduce the set

\[ C_s = e^{2\sum_{i=1}^{\gamma_i} \sigma_i n - i + 1}; \]

(118)

Taking into account the commutation properties of \( \sigma \)'s with the external coordinates \( \hat{E}_s \):

| \( \sigma \)     | \( \hat{E}_1 \) | \( \hat{E}_2 \) | \( \hat{E}_3 \) | \( \hat{E}_4 \) | \( \hat{E}_5 \) |
|-----------------|----------------|----------------|----------------|----------------|----------------|
| \( \sigma_{1,11} \) | \(-E_{2,11} e^{-\sigma_{1,11}}\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \( \sigma_{2,10} \) | \(+E_{2,11} e^{-\sigma_{1,11}}\) | \(+E_{2,9}\) | \(0\) | \(0\) | \(0\) |
| \( \sigma_{3,9} \) | \(0\) | \(-E_{2,9}\) | \(-E_{4,9} e^{-\sigma_{3,9}}\) | \(0\) | \(0\) |
| \( \sigma_{4,8} \) | \(0\) | \(+E_{4,9} e^{-\sigma_{2,3}}\) | \(+E_{4,7}\) | \(0\) | \(0\) |
| \( \sigma_{5,7} \) | \(0\) | \(0\) | \(0\) | \(-E_{4,7}\) | \(-E_{6,7} e^{-\sigma_{5,7}}\) |
we obtain the main set of commutation relations:

\[
(-1)^s \left[ \ln (C_s), \hat{E}_s \right] = 2\delta_{s,t} \cdot \begin{cases} 
E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}}; & \text{for odd } s \\
E_{s,n-s}; & \text{for even } s 
\end{cases}
\]

\[
[\ln (C_t), E_{s+1,n-s+1} e^{-\sigma_{s,n-s+1}}] = 0; \\
[\ln (C_t), E_{s,n-s}] = 0; \\
[C_s, C_t] = 0.
\]

Compose the rotation twist:

\[
F_R = \exp \left( -H_1^\perp \otimes \ln (C_1) + \ln (C_2) \otimes H_2^\perp - H_3^\perp \otimes \ln (C_3) - \right. \\
\left. \ln (C_4) \otimes H_4^\perp - H_5^\perp \otimes \ln (C_5) \right)
\]

(119)

and apply it to the algebra \( U^R_{ch} (sl(11)) \)

\[
F_R : U^R_{ch} (sl(11)) \rightarrow U^R_{ch} (sl(11))
\]

(120)

the coproducts of the external coordinates become quasiprimitive:

\[
\Delta^R_{ch} (\hat{E}_s) = \begin{cases} 
\hat{E}_s \otimes C_s^{-1} + 1 \otimes \hat{E}_s, & s = 2k-1, \\
\hat{E}_s \otimes 1 + C_s \otimes \hat{E}_s, & s = 2k.
\end{cases}
\]

(121)

while the coproducts \( \Delta^R_{ch} (H_s^\perp) \) remain primitive. In terms of redefined external coordinates

\[
D_s = \begin{cases} 
(E_{2,1} + E_{10,11} e^{\sigma_{2,10}-\sigma_{1,11}}) e^{2\sigma_{1,11}}, \\
E_{10,9} + (\hat{H}_3 - \hat{H}_2) E_{2,9} + E_{2,3}, \\
(E_{4,3} + E_{8,9} e^{\sigma_{4,8}-\sigma_{3,9}}) e^{2\sum_{i=1}^s \sigma_{i,n-i+1}}, \\
E_{8,7} + (\hat{H}_5 - \hat{H}_4) E_{4,7} + E_{4,5}, \\
E_{6,5} e^{2\sum_{i=1}^s \sigma_{i,n-i+1}}.
\end{cases}
\]

(122)

these coproducts are uninform:

\[
\Delta^R_{ch} (D_s) = D_s \otimes 1 + C_s \otimes D_s.
\]

(123)

For any \( s = 1, \ldots, p \) the subalgebras \( \mathfrak{B}_s \) generated by \( \{ H_s^\perp, D_s, C_s \} \) comply the conditions of Lemma 4 and the twisting elements

\[
F^\perp_s = e^{H_s^\perp \otimes \ln (C_s + D_s)} = e^{H_s^\perp \otimes \omega_s}
\]

(124)

with

\[
\omega_s = \ln (C_s + D_s)
\]
are the solutions to the Drinfeld equations \[1\]. The product
\[
\mathcal{F}^j = \prod_{s=1}^{5} \mathcal{F}^j_s = \prod_{s=1}^{5} e^{H^s \otimes \ln(C_s + D_s)} = \prod_{s=1}^{5} e^{H^s \otimes \omega_s};
\]
(125)
can be applied to \(U^R_{ch}(sl(11))\)
\[
\mathcal{F}^j : U^R_{ch}(sl(11)) \longrightarrow U_P(sl(11)).
\]
(126)
The product
\[
\mathcal{F}_P = \mathcal{F}^j \mathcal{F}^R_{ch},
\]
(127)
\[
\mathcal{F}_P : U(sl(11)) \longrightarrow U_P(sl(11))
\]
(128)
has the parabolic carrier
\[
g_P = \left( \bigoplus_{s=1}^{5} \text{sl}(2)_s \right) \triangleright \left( g_{ch} \setminus \sum_{s=1}^{5} g^s \right).
\]
(129)

8 Conclusions

We have demonstrated that for any algebra \(A_{n-1}\) \((n > 2)\) there exists a twist deformation \(\mathcal{F}_P : U(A_{n-1}) \longrightarrow U_P(A_{n-1})\) whose carrier \(g_P\) is a parabolic subalgebra with non-Abelian Levi factor. These carriers are the non-Abelian extensions of \(g_{ch}\) that contain \(B^+(g)\) and have nontrivial intersection \(g_P \cap N_g = \emptyset\).

The corresponding twisting elements \(\mathcal{F}_P\) are the products of deformed Jordanian twists \(\mathcal{F}^j\) and the rotated full chains \(\mathcal{F}^R_{ch}\). According to the quantum duality the Hopf subalgebras \(U_P(g_P)\) are the quantized algebras \(\text{Fun}_P(G_P)\) of coordinate functions on the universal covering groups \(G_P\) (with Lie algebras \(g_P\)). Notice that in this type of quantization the deformation of algebra \(\text{Fun}(G_P)\) is due to the fact that the coordinates are subject to the commutation relations of \(g_P\). Thus we have constructed quantum groups for a class of parabolic subgroups of linear groups \(SL(n)\).

It was mentioned in the Introduction that for series \(B_n, C_n\) and \(D_{n=2k}\) the full chains have (minimal) parabolic carriers. In the case of even-odd orthogonal algebras \(D_{n=2k+1}\) the chain carrier is not parabolic, \(\dim \left( B^+(g) \setminus g_{ch} \right) > 0\). Due to the canonical isomorphism one of such algebras has been already treated in the section Examples where we considered \(g = sl(4) \simeq so(6)\). Thus it is clear that the problem of constructing parabolic twists for \(D_{2k+1}\) can be solved using the same approach as in the above study.

It can be shown that the parabolic twists \(\mathcal{F}_P = \mathcal{F}^j \mathcal{F}^R_{ch}\) are completely factorizable. The factors in the decomposition \(\mathcal{F}_P = \prod_{s=1}^{P} \mathcal{F}^j_s \mathcal{F}^R_{ch} \prod_{l=1}^{m} \mathcal{F}^l_{link}\) can be supplied with the independent deformation parameters \(\{ \chi_i, \psi_j \mid i = 1, \ldots, m; \).
$j = 1, \ldots, p$} and any subset of the factors $\mathcal{F}^j_s$ and $\mathcal{F}^\ell_R$ can be switched off by tending the corresponding set of parameters to their limit values. The factors $\mathcal{F}^j_s$ depend on the parameters of the chain $\mathcal{F}^R_{ch}$. It is essential to mention that when the whole chain is switched off the factors $\mathcal{F}^j_s$ acquire the canonical Jordanian form. The important consequence of this property is that the parabolic twists $\mathcal{F}_P$ are products of the same set of basic twisting factors $[14]$ as used in different forms of chains. Thus parabolic twists can also be considered as chains of the special type.

When the variables $\{\chi_i, \psi_j\}$ are proportional to the overall deformation parameter $\tau$ ($\chi_i = \tau \xi_i, \psi_j = \tau \zeta_j$) the first order terms in the expansion of the $R$-matrices $R_P(\tau) = F_{P21}(\tau) F_{P1}^{-1}(\tau)$ give the parabolic classical $r$-matrices $r_{ech}$ constructed in [19]. Twisting by $\mathcal{F}_P$ performs the quantization of $r_{ech}$.

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