The Cut-and-Play Algorithm: Computing Nash Equilibria via Outer Approximations

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Abstract

We introduce Cut-and-Play, a practically-efficient algorithm for computing Nash equilibria in simultaneous non-cooperative games where players decide via nonconvex and possibly unbounded optimization problems with separable payoff functions. Our algorithm exploits an intrinsic relationship between the equilibria of the original nonconvex game and the ones of a convexified counterpart. In practice, Cut-and-Play formulates a series of convex approximations of the game and iteratively refines them with cutting planes and branching operations. Our algorithm does not require convexity or continuity of the player's optimization problems and can be integrated with existing optimization software. We test Cut-and-Play on two families of challenging nonconvex games involving discrete decisions and bilevel problems, and we empirically demonstrate that it efficiently computes equilibria while outperforming existing game-specific algorithms.

1 Introduction

Decision-making is hardly an individual task; instead, it often involves the mutual interaction of several self-driven decision-makers, or players, and their heterogeneous preferences. The most natural framework to model each player's decision problem is often an optimization problem whose solutions, or strategies, provide prescriptive recommendations on the best course of action. However, in contrast to optimization, the solution to the game involves the concept of stability, i.e., a condition ensuring that the players are playing mutually-optimal strategies. In two seminal papers, Nash [38, 39] formalized a solution concept for finite non-cooperative games, namely, the concept of Nash equilibrium. Nash equilibria are stable solutions, as no rational and self-driven decision-maker can unilaterally and profitably defect them.

Historically, convexity played a central role in shedding light on the existence and computation of Nash equilibria, e.g., see Daskalakis [18], Facchinei and Pang [27] and the references therein. Indeed, von Neumann [54] proved that 2-player zero-sum games always admit a Nash equilibrium if the players’ cost (payoff) functions are convex (concave) in their strategies, and the conditions yielding Nash equilibria are essentially equivalent to linear-programming duality [17]. However, the plausibility of the Nash equilibrium also stems from the availability of algorithms to compute it. As Roth [47] argued in his claim “economists as engineers”, computing Nash equilibria plays a central role in designing markets and deriving pragmatic insights. For instance, from the perspective of an external regulator, Nash equilibria are invaluable tools for understanding, designing and intervening in markets where the agents’ interests conflict with broader societal objectives. However, markets and their mathematical models rarely satisfy well-structured convexity assumptions: nonconvexities often model complex operational requirements and are vital to representing reality and extracting valid conclusions. Although there is a wealth of methodologies to compute Nash equilibria for finite and convex games, e.g., normal-form games, little is known about the computation of equilibria in nonconvex settings [18]. This gap represents the core motivation of this work.
This paper presents a practically-efficient method for computing equilibria in non-cooperative games where players solve nonconvex optimization problems. The nonconvexities may stem from integer variables modeling indivisible quantities and logical conditions, bilevel constraints rendering hierarchical relationships among decision-makers, or from physical phenomena, for instance, water distribution and signal processing [30, 43]. Because of their extreme practical interest, several methodologies in optimization have made certain nonconvexities tractable, at least from a computational perspective. Although some recent papers focused on games with specific nonconvexities, e.g., integer variables as in Carvalho et al. [8, 10], Crönert and Minner [16], Dragotto and Scatamacchia [22], Fuller and Pirnia [30], Köppe et al. [35], Sagratella [48], Schwarz and Stein [51] and bilevel constraints as in Carvalho et al. [9], Hu and Ralph [33], Pang and Fukushima [42], Sherali [52], to date, there is no general-purpose algorithm to compute equilibria in games where different nonconvexities arise.

Our Contributions. This paper presents a practically-efficient algorithm to compute equilibria for a large class of games where players decide by solving optimization problems with nonconvex feasible regions. Specifically, we assume each player optimizes a separable function in its variables, i.e., a function expressed as a weighted sum-of-products function in the player’s variables [25, 26, 53]. We summarize our contributions as follows:

(i.) We prove that when the player’s objective functions (i.e., payoffs) are separable, the game admits an equivalent convex representation, even if the players’ feasible sets are inherently nonconvex. We present this equivalence in terms of a novel correspondence between the Nash equilibria of the original game and those of a convexified game where each player's feasible set gets replaced by its convex hull. Furthermore, unlike similar results in optimization, we highlight a fundamental difference between formulating the game over the convex hulls and the closed convex hulls of the players’ feasible sets. First, if an equilibrium does not exist in the game formulated over the closed convex hulls, it also does not exist in the original game. Second, the existence of an equilibrium in the game formulated over the closed convex hulls may not necessarily lead to a corresponding equilibrium in the original game.

(ii.) We introduce Cut-and-Play (CnP), a cutting plane algorithm to compute Nash equilibria for a large family of simultaneous and non-cooperative \( n \)-player nonconvex games. In essence, CnP exploits a sequence of polyhedral convexifications of the original nonconvex game and involves a well-crafted mix of discrete, e.g., Mixed-Integer Optimization (MIP), and continuous optimization techniques. For instance, it blends concepts such as relaxation (approximation), cutting planes, branching, and complementarity problems. The algorithm finds an exact Nash equilibrium (up to a machine epsilon) or proves its non-existence. To our knowledge, this is also the first approach exploiting outer approximations to compute Nash equilibria.

(iii.) CnP overcomes some well-known algorithmic limitations. Specifically, our algorithm does not:
(a.) require that the players’ optimization problems are convex or continuous (b.) compute only pure (i.e., deterministic) equilibria (c.) rely on alternative or weaker concepts of equilibria. Finally, our algorithm does not require the players to have a bounded set of strategies. On the contrary, it supports unbounded strategy sets and can also certify the non-existence of equilibria. Our primary assumption is that the convex hull of each player’s feasible set and its payoff are polyhedral and separable, respectively.

(iv.) We present an extensive set of computational results on two important families of challenging nonconvex games: Integer Programming Games (IPGs) and Nash games among Stackelberg Players (NASPs), i.e., a class of simultaneous games among players solving mixed-integer and bilevel optimization problems, respectively. In both cases, our algorithm outperforms the baselines in terms of computing times and social welfare (i.e., the sum of the players’ payoffs).
Outline. We structure the paper as follows. Section 2 provides a literature review, and Section 3 formalizes the background definitions and our notation. Sections 4 and 5 present the principles behind our approach and introduce the CnP algorithm and its practical implementation. Section 6 details how to customize CnP when some of the game’s structure is known, showcasing a comprehensive set of computational results. Finally, we present our conclusions in Section 7.

2 Literature Review

Nash [38, 39] formalized a concept of stability for non-cooperative simultaneous games through the Nash equilibrium. We distinguish between two types of equilibria: Pure Nash Equilibria (PNEs), where players employ deterministic strategies, and Mixed Nash Equilibria (MNEs), where players randomize over their strategies. If the game is finite, i.e., there are finitely many players and strategies, Nash proved that an MNE always exists. Glicksberg [31] extended the result, proving that an MNE always exists when players have continuous payoff functions and compact strategy sets. Although MNEs are guaranteed to exist under some assumptions, an equilibrium may not exist in the general case. Moreover, in IPGs and other nonconvex games, deciding if an MNE exists is \( \Sigma^P_2 \)-complete [6, 8, 9], i.e., it is at the second level of the polynomial hierarchy of complexity [56]. Besides existence, computing equilibria or certifying their non-existence pose significant algorithmic challenges [19].

Finite Games. Most algorithmic approaches for computing Nash equilibria deal with finite games represented in normal form, i.e., through payoff matrices describing the outcome under any combination of the players’ strategies. Besides duality for 2-player zero-sum games [54], the first algorithm for computing equilibria in 2-player normal-form games is the Lemke-Howson algorithm [37]. Although Wilson [55] and Rosenmüller [46] extended the Lemke-Howson algorithm to \( n \)-player normal-form games, their methods often require the solution of a series of nonlinear systems. More recently, Sandholm et al. [50] and Porter et al. [45] proposed two algorithms for 2-player normal-form games exploiting the idea of support enumeration, i.e., the idea of computing an equilibrium by guessing the strategies played with strictly-positive probabilities in an equilibrium. Porter et al. [45] solve a system of inequalities (nonlinear for more than 2 players) to determine if a given support (i.e., a subset of pure strategies for each player) leads to an equilibrium. Sandholm et al. [50] avoid explicit support enumeration by modeling the same idea via a MIP problem. In contrast to the above works, we focus on a broader class of games (containing 2-player normal-form games) where the number of pure strategies can be exponential, perhaps uncountable, in the input size of the game.

Continuous Games. If each player’s optimization problem is continuous and convex, equilibrium programming methods can often determine a Nash equilibrium by: (i.) reformulating the game as a complementarity or variational inequality problem [14], and (ii.) employing globally-convergent Jacobi or Gauss-Seidel algorithms [27]. On the one hand, these reformulations require restrictive convexity assumptions on the players’ optimization problems and may not otherwise guarantee convergence. Besides a few exceptions (e.g., Sagratella [48]), to date, the majority of equilibrium programming methods require convexity and, otherwise, develop weaker concepts of equilibrium, for instance, quasi Nash equilibria [40, 43]. On the other hand, they generally are extremely scalable and efficient under convexity. CnP exploits this efficiency by solving, at each step, a convexified game via a complementarity problem.

Integer Nonconvexities. In the particular case of integer nonconvexities, Köppe et al. [35] introduced the taxonomy of IPGs, i.e., non-cooperative games where each player solves a parametrized mixed-integer optimization problem. IPGs generalize any finite game and implicitly describe the set of strategies via constraints, as opposed to the explicit description of, for instance, normal-form games. Arguably, IPGs represent the most prominent emerging family of nonconvex games and have several
3 The Problem and Our Assumptions

3.1 Separable-Payoff Games

As a standard game-theory notation, let the operator $(\cdot)^{-1}$ represent the elements of $(\cdot)$ but the $i$-th element. We focus on nonconvex Separable-Payoff Games (SPGs), a large family of games where each player’s objective takes a sum-of-products form as in Definition 1.

**Definition 1** (Separable-Payoff Game). An SPG $G = (P^1, \ldots, P^n)$ is a non-cooperative, complete-information, and simultaneous game among $n$ players where each player $i$ solves

$$\min_{x^i} \{ f^i(x^i; x^{-i}) := (c^i)^\top x^i + \sum_{j=1}^{m_i} g^i_j(x^{-i}) x^i_j \} \quad \text{subject to} \quad x^i \in \mathcal{X}^i \subseteq \mathbb{R}^{m_i}. \quad (P^i)$$

For each player $i$, $\mathcal{X}^i \neq \emptyset$ is the set of player $i$’s strategies, $c^i$ is a real-valued vector, and $g^i_j(x^{-i})$ is of the form $g^i_j(x^{-i}) = \prod_{k=1,k \neq i}^n h^i_{jk}(x^k)$ with $h^i_{jk}$ being an affine function. An SPG is polyhedrally representable if, for each $i$, $\text{conv}(\mathcal{X}^i)$ (i.e., the convex hull of $\mathcal{X}^i$) is a polyhedron.

From Definition 1, the optimization problem $P^i$ of player $i$ is parametrized in its opponent choices $x^{-i}$. For each player $i$, we call $x^i \in \mathcal{X}^i$ a pure strategy, $\mathcal{X}^i$ the feasible set (or set of strategies), and $f^i(x^i; x^{-i})$ the payoff of $i$ under $x = (x^i, x^{-i}).$

**Remark 1** (Linear Form). Without loss of generality, in Definition 1, we present SPGs in a linear (objective) form, i.e., we let $c^i$ be a vector and $h^i_{jk}(x^k)$ be affine functions. If, for any player $i$, $f^i(x^i, x^{-i}) = s^i(x^i) + \sum_{j=1}^{m_i} g^i_j(x^{-i}) q^i_j(x^i)$, where $s^i$ and $g^i_j$-s are nonlinear functions, and $g^i_j(x^{-i})$ is the product of nonlinear functions $h^i_{jk}(x^k)$ for $k = 1, 2, \ldots, (i - 1), (i + 1), \ldots, n$, we can always reformulate the game so that each payoff has the linear form of Definition 1. To this end, we can introduce (i) the auxiliary variables $\gamma^i$, $\psi^i_j$ and the constraints $\gamma^i = s^i(x^i)$, $\psi^i_j = g^i_j(x^i)$ in $P^i$, and (ii) for each $j = 1, \ldots, m_i$ and $k$, an auxiliary variable $\delta^i_{jk}$ and a constraint $\delta^i_{jk} = h^i_{jk}(x^k)$ in $P^k$. Thus, the payoff of player $i$ becomes $\gamma^i + \sum_{j=1}^{m_i} \prod_{k \neq i}^{\delta^i_{jk}} \psi^i_j$, which is in linear form. Furthermore, if $s^i(x^i)$ is convex, we can write the convex inequality $\gamma^i \geq s^i(x^i)$ instead of the equality. Finally, we remark that SPGs can represent broad classes of games, for instance, any normal-form game and separable game (which, in addition to separable payoffs, requires $\mathcal{X}^i$ to be a compact set).

Mixed Strategies and Equilibria. For each player $i$, $\sigma^i$ is a mixed strategy, or simply a strategy, if it is a probability distribution over the pure strategies $\mathcal{X}^i$. Let $\Delta^i$ be the space of atomic probability distributions over $\mathcal{X}^i$ such that $\sigma^i \in \Delta^i$. Let $\text{supp}(\sigma^i) := \{ x^i \in \mathcal{X}^i : \sigma^i(x^i) > 0 \}$ be the support of the strategy $\sigma^i$, where $\sigma^i(x^i)$ is the probability of playing $x^i$ in $\sigma^i$. If a mixed strategy $\sigma^i$ has singleton support, i.e., $|\text{supp}(\sigma^i)| = 1$, then it is also a pure strategy. We denote by $\sigma^{-i} \in \prod_{j=1,j \neq i}^n \Delta^j$ the other players’ strategies, a probability distribution over the strategies of $i$’s opponents. If $\mathcal{X}^i$ is a compact set, any mixed-strategy $\sigma^i$ of an SPG has an equivalent finitely-supported mixed-strategy $\tilde{\sigma}^i$ [53, Theorem...
2.8]. The latter equivalence means that each player $i$’s expected payoff under $\sigma^i$ equals that under $\tilde{\sigma}^i$. This result extends to SPGs, with $\lambda^i$ being possibly non-compact, as long as the support of $\sigma^i$ is a compact set. This is why, w.l.o.g., $\Delta^i$ is an atomic distribution over $\lambda^i$. The expected payoff $\mathbb{E}_{X \sim \sigma} [f(X^i, X^-i)]$ for $i$ under $\sigma = (\sigma^1, \ldots, \sigma^n)$ is

$$f^i(\sigma^i; \sigma^{-i}) = \sum_{x^i \in \text{supp}(\sigma^i)} (c^i)^\top x^i \sigma^i(x^i) + \sum_{x \in \text{supp}(\sigma)} \sigma^i(x^i) \sum_{j=1}^{m_i} g^i_j(\sigma^{-i}(x^{-i})x^{-i})x^j.$$  

(1)

For simplicity, we refer to (1) as $f^i(\sigma^i; \sigma^{-i})$. A strategy $\sigma^i$ is a best response for player $i$ given its opponents’ strategies $\sigma^{-i}$ if $f^i(\sigma^i; \sigma^{-i}) \leq f^i(\tilde{\sigma}^i; \sigma^{-i})$ for any possible deviation $\tilde{\sigma}^i \in \Delta^i$. In practice, we can restrict the search of deviations $\tilde{\sigma}^i$ to pure strategies. A strategy profile $\sigma = (\sigma^1, \ldots, \sigma^n)$ is an MNE if, for each player $i$ and strategy $\tilde{\sigma}^i \in \Delta^i$, then $f^i(\sigma^i; \sigma^{-i}) \leq f^i(\tilde{\sigma}^i; \sigma^{-i})$.

### 3.2 Polyhedral Representability

Our algorithmic framework hinges on the assumption of polyhedral representability. In other words, for any player $i$, we assume that $\text{conv}(\lambda^i)$, i.e., the convex hull of the set of feasible strategies $\lambda^i$, is a polyhedron. Whenever $\text{conv}(\lambda^i)$ is polyhedral for every player $i$, we prove that our algorithm terminates with either an MNE or a proof of its non-existence. The set $\text{conv}(\lambda^i)$ is a polyhedron if $\lambda^i$ is, for example, a union of finitely many polytopes, the set of mixed-integer points in a polyhedron, or even the union of finitely many polyhedra sharing their set of recession directions. Among the polyhedrally-representable games, we mention the class of linear IPGs [35], where each player $i$ solves a mixed-integer linear optimization problem. Besides the assumption of polyhedral representability, our approach is general as it does not leverage any game-specific structure, and it computes an MNE in any polyhedrally-representable SPG.

### 4 Algorithmic Scheme

This section outlines CnP and its conceptual components. The underlying idea behind our algorithm is to compute an MNE by solving a series of convex (outer) approximations of the original game. In principle, solving these convex approximations is computationally more tractable than solving the original nonconvex game. Nevertheless, building an efficient and convergent algorithm poses several algorithmic-design challenges that we discuss in this section.

**Notation.** Let $K \subseteq \mathbb{R}^k$ be a closed convex set, and $\text{rec}(K)$ and $\text{ext}(K)$ be the set of its recession directions and extreme points, respectively. An inequality $\pi^\top x \leq \pi_0$ is valid for $K$ if it holds for any $x \in K$. Given $K$ and a point $\pi \notin K$, we say $\pi^\top x \leq \pi_0$ is a cut if it is valid for $K$ and $\pi^\top \pi > \pi_0$. We say $O \subseteq \mathbb{R}^k$ is an outer approximation of $K$ if $K \subseteq O$ and $O$ is polyhedral. Conversely, a (polyhedral) set $I \subseteq \mathbb{R}^k$ is a (polyhedral) inner approximation of $K$ if $I \subseteq K$.

### 4.1 Convex Reformulation

If the players’ objectives are separable, we prove the game admits an equivalent convex representation. Specifically, in **Theorem 1**, we establish a correspondence between the MNEs of an SPG instance $G$ and the PNEs of a convexified instance $\tilde{G}$ where the feasible set for each player’s optimization problem is $\text{conv}(\lambda^i)$ instead of $\lambda^i$. Our result generalizes Carvalho et al. [9, Theorem 4] by letting players have separable payoff functions and arbitrary nonconvex strategy sets. In what follows, we assume that each player’s payoff $f^i$ is in the linear form mentioned in **Remark 1**.
Theorem 1. Let $G$ be an SPG where each player $i$ solves $\min_{x^i} \{ f^i(x^i, x^{\cdot i}) : x^i \in X^i \}$. Let $\overline{G}$ be a convexified version of $G$ where each player $i$ solves $\min_{x^i} \{ f^i(x^i, x^{\cdot i}) : x^i \in \text{conv}(X^i) \}$. For any PNE $\overline{\pi}$ of $\overline{G}$, there exists an MNE $\bar{\sigma}$ of $G$ such that, for any player $i$, $f^i(\overline{\pi}; \pi^{-i}) = f^i(\bar{\sigma}^i; \pi^{-i})$. Conversely, if $\overline{G}$ has no PNE, then $G$ has no MNE.

Proof of Theorem 1. First, we show that if $G$ has an MNE $\bar{\sigma} = (\bar{\sigma}^1, \ldots, \bar{\sigma}^n)$, then the convexified game $\overline{G}$ has a PNE where each player $i$ plays a strategy in $\text{conv}(X^i)$. For each player $i$, we interpret $\bar{\sigma}^i$ as the mixed (equilibrium) strategy of playing $x^i$ with probability $p^i$ for $\ell = 1, \ldots, \kappa_i$. We claim that the strategy $\overline{\pi} := \sum_{\ell=1}^{\kappa_i} p^i \pi^\ell$ is feasible in $\overline{G}$ and a PNE of $\overline{G}$. Feasibility follows from the fact that $\overline{\pi}^i$ is a convex combination of points in $X^i$, and hence $\overline{\pi}^i \in \text{conv}(X^i)$. Furthermore, the corresponding strategy profile $\pi = (\pi^1, \ldots, \pi^n)$ is a PNE of $\overline{G}$ because, for each player $i$, $\pi$ and $\bar{\sigma}^i$ induce the same payoffs in $\overline{G}$ and $G$, respectively. This is because

$$
\mathbb{E}_{X \sim \bar{\sigma}} \left( f^i(X^i, X^{-i}) \right) = \mathbb{E}_{X \sim \bar{\sigma}^i} \left( (c^i)^{\top} X^i + \sum_{j=1}^{\kappa_i} \left( \prod_{k \neq i} h^i_{jk}(X^k) \right) X^i_j \right) \tag{2a}
$$

$$
= \mathbb{E}_{X \sim \overline{\pi}} \left( (c^i)^{\top} X^i \right) + \mathbb{E}_{X \sim \bar{\sigma}^i} \left( \sum_{j=1}^{\kappa_i} \left( \prod_{k \neq i} h^i_{jk}(X^k) \right) X^i_j \right) \tag{2b}
$$

$$
= (c^i)^{\top} \mathbb{E}_{X \sim \overline{\pi}} \left( X^i \right) + \sum_{j=1}^{\kappa_i} \left( \prod_{k \neq i} \mathbb{E}_{X \sim \bar{\sigma}^k} [h^i_{jk}(X^k)] \right) \mathbb{E}_{X \sim \bar{\sigma}^i} \left( X^i_j \right) \tag{2c}
$$

$$
= (c^i)^{\top} \mathbb{E}_{X \sim \overline{\pi}} \left( X^i \right) + \sum_{j=1}^{\kappa_i} \left( \prod_{k \neq i} h^i_{jk}( \mathbb{E}_{X \sim \bar{\sigma}^k}(X^k) ) \right) \mathbb{E}_{X \sim \bar{\sigma}^i} \left( X^i_j \right) \tag{2d}
$$

$$
= c^i \pi^i + \sum_{j=1}^{\kappa_i} \left( \prod_{k \neq i} h^i_{jk} \left( \pi^k \right) \right) \pi^i_j. \tag{2e}
$$

Equation (2a) holds because the SPG is in linear form (see Remark 1), and (2b) holds because of the linearity of the expectations. Equation (2c) holds because $\bar{\sigma}^i$ and $\bar{\sigma}^k$ are independent probability distributions of two distinct players $i$ and $k$, and the expectation of the product is the product of the expectations of the random variables. Equation (2d) holds due to the linearity of $h^i_{jk}$, and Equation (2e) holds by the definition of $\pi^i$. Thus, for any MNE of $G$, a PNE of $\overline{G}$ exists.

Second, we show that any PNE $\pi$ of $\overline{G}$ induces an MNE $\bar{\sigma}$ for $G$. Because, for each player $i$, $\pi^i \in \text{conv}(X^i)$, we rewrite $\pi^i$ as $\sum_{\ell=1}^{\kappa_i} p^i \pi^\ell$ for some $p^i \geq 0$, $\sum_{\ell=1}^{\kappa_i} p^i = 1$ and $x^i \in X^i$ for $\ell = 1, \ldots, \kappa_i$. We construct a mixed strategy $\bar{\sigma}^i$ by letting $i$ select the strategy $x^i$ with probability $p^i$ for $\ell = 1, \ldots, \kappa_i$. Then, the payoff of $i$ under $\pi$ is the expression in (2e). By following the equalities (2) in the reverse direction, the payoff of $i$ under $\pi$ is the same as the payoff of $i$ under the mixed strategy $\bar{\sigma}^i$. We claim that $\bar{\sigma}$ is an MNE for $G$ because, for any player $i$ and for each unilateral profitable deviation from $\bar{\sigma}$ in $G$, there exists a unilateral profitable deviation from $\pi$ in $\overline{G}$ for $i$. Without loss of generality, let the pure strategy $\sigma^i$ be such deviation for $i$ given $\bar{\sigma}$ in $G$. Because $\bar{\sigma}^i \in \text{conv}(X^i)$ and (2), $\bar{\sigma}^i$ is a deviation for player $i$ given $\pi$ in $\overline{G}$, contradicting the assumption that $\pi$ is a PNE for $\overline{G}$. Therefore, any pure strategy in $\overline{G}$ induces a mixed strategy in $G$ with the same payoff. Then, if $\overline{G}$ has no PNE, then $G$ has no MNE (and vice versa).

### 4.2 Building Convex Approximations

Intuitively, Theorem 1 proves that any SPG has an equivalent convex representation where each player $i$ optimizes $f^i(x^i, x^{\cdot i})$ over $\text{conv}(X^i)$. Therefore, simultaneously satisfying the optimality conditions of each player’s optimization problem in $\overline{G}$ yields an MNE for $G$ and a PNE for $\overline{G}$. Finding a PNE in
G (if any) is then equivalent to solving an Nonlinear Complementarity Problem (NCP) expressing the optimality conditions in the form of complementarity conditions.

**Approximate Game.** From a practical perspective, however, the description of each \( \text{conv}(X^i) \) may be challenging to characterize explicitly. For instance, the description of \( \text{conv}(X^i) \) may often be exponentially large in the number of variables or constraints (e.g., mixed-integer sets). Therefore, formulating the game \( \bar{G} \) by employing \( \text{conv}(X^i) \) for each \( i \) is practically prohibitive. Starting from this observation, we devise the concept of approximate game, a more tractable convex game where each player \( i \)'s feasible set is an outer approximation of \( \text{conv}(X^i) \).

**Definition 2** (Approximate Game). Let \( G \) be an SPG where each player \( i \) solves \( \min_{x^i} \{ f^i(x^i; x^{-i}) : x^i \in X^i \} \). Then, \( \bar{G} \) is an approximate SPG of \( G \) if each player \( i \) solves \( \min_{\xi^i} \{ f^i(x^i; x^{-i}) : x^i \in \bar{X}^i \} \) with \( \bar{X}^i \supseteq X^i \). Furthermore, \( \bar{G} \) is a Polyhedrally-Approximated Game (PAG) of \( G \) if \( \bar{X}^i \) is a polyhedron for each player \( i \).

### 4.2.1 Optimization, Relaxations, and Games.

In optimization, a feasible solution to the original problem is feasible for its relaxations. However, this relationship may not hold when dealing with games and Nash equilibria. For instance, a game’s approximation may admit an MNE, whereas the original game may not, or vice versa. We illustrate this phenomenon in **Example 1**.

**Example 1.** Consider an SPG \( G \) with \( n = 2 \), where player 1 solves \( \min_{x} \{ \xi x : x \in \mathbb{R}, x \geq 1 \} \) and player 2 solves \( \min_{\xi} \{ \xi \xi : \xi \in \mathbb{R}, \xi \in [1, 2] \} \). This game admits the MNE (which is also a PNE) \( (x, \xi) = (1, 1) \).

Let \( G \) be PAG where the players’ feasible regions are \( \bar{X}^1 = \text{conv}(X^1) \) and \( \bar{X}^2 = \{ \xi \in \mathbb{R} : \xi \in [-1, 2] \} \), respectively. Although \( G \) has no MNE, \( G \) admits an MNE. If player 1’s objective changes to \( -x\xi \), then \( G \) does not have an MNE, whereas \( \bar{G} \) admits the MNE \( (x, \xi) = (1, -1) \).

An MNE for \( G \) may not be an MNE for one of its PAGs as the latter can introduce a destabilizing strategy for \( i \) in the associated approximation \( \bar{X}^i \), i.e., a strategy that does not belong to \( \text{conv}(X^i) \) but prevents the existence of that equilibrium in \( \bar{G} \). This issue is critical when \( X^i \) is unbounded or uncountable, as an MNE to the original game may not exist.

### 4.2.2 Computing Equilibria for the Approximation

In **Definition 2**, we let \( \bar{X}^i \) be an outer approximation of \( \text{conv}(X^i) \), namely, \( \bar{X}^i \) enlarges the feasible set of player \( i \). Suppose the approximate game \( \bar{G} \) is a PAG. We can formulate an NCP encompassing the optimality conditions associated with each player’s optimization problem in \( \bar{G} \) to determine a PNE for \( \bar{G} \). At some step \( t \) and for some PAG \( \bar{G} \), let \( \bar{X}^i_t = \{ x^i : \bar{A}_i^i x^i \leq \bar{b}_i^i, x^i \geq 0 \} \) be the increasingly-accurate polyhedral approximation of \( \text{conv}(X^i) \) for player \( i \). Let \( \sigma^i \) and \( \mu^i \) be the primal and dual variables of each player’s problem in \( \bar{G} \), respectively. We can compute a PNE of \( \bar{G} \) by solving the NCP

\[
0 \leq \sigma^i \perp (c^i + \sum_{j=1}^{m} g^i_j(\sigma^{-i}) + \bar{A}_i^i \mu^i) \geq 0, \quad 0 \leq \mu^i \perp (\bar{b}_i^i - \bar{A}_i^i \sigma^i) \geq 0 \quad i = 1, 2, \ldots, n,
\]  

(3)

where \( \perp \) is equivalent to \( \perp^T \perp = 0 \). Any solution \( \sigma = (\sigma^1, \ldots, \sigma^n) \) of (3) is a PNE for the PAG \( \bar{G} \) at step \( t \). Although a solution to (3) includes both \( \sigma \) and \( \mu \), we omit \( \mu \) as we are interested in \( \sigma \). If \( X^i = \text{conv}(X^i) \) for any \( i \), \( G \) is the exact convex representation of \( G \) (i.e., \( G = \bar{G} \)), and the solutions to (3) are all the MNEs of \( G \).

**Remark 2.** The NCP (3) carries most of the computational burden of CnP, as we empirically show in **Section 6**. If \( g^i_j(\sigma^{-i}) \) is linear in \( \sigma^{-i} \) for any \( i \) and \( j \), the NCP becomes a Linear Complementarity
The workhorse of CnP is the NCP problem (3) associated with each PAG. Starting from a game \( G \), CnP computes the PNEs for a finite sequence \( t = 0, 1, 2, \ldots \) of PAGs by repeatedly solving (3), and determines whether their solutions are MNEs for the original game \( G \). If not, the algorithm refines \( X^i_t \) for some player \( i \) via cutting or branching on general disjunctions. The algorithm evokes the same scheme one would use to solve a MIP via a branch-and-cut algorithm [41], where, instead of a game and an approximate game, one considers a MIP and its relaxation.

We emphasize that the approximation of the feasible sets \( X^i_t \) is refined via branching and cutting. Branching refers to rewriting the feasible set \( X^i_t \) as the convex hull of the union of two disjoint sets, such that the union contains every point in \( X^i_t \). Cutting refers to adding a valid inequality to \( X^i_t \). Both branching and cutting provide a new outer approximation \( X^i_{t+1} \) that is closed and convex.

However, in some cases, \( \text{conv}(X^i) \) may not be a closed set, while its topological closure \( \text{cl } \text{conv}(X^i) \) is a polyhedron. In this case, the best possible refinement is to get the outer approximation to converge to \( \text{cl } \text{conv}(X^i) \) but not further. In optimization, the issue of non-closedness is minor, as we can often recover \( \varepsilon \)-optimal (i.e., approximate) solutions by optimizing over the feasible set’s topological closure. A natural question is if we can use a similar argument for SPGs and approximate equilibria, i.e., equilibria where each player’s strategy is an \( \varepsilon \)-optimal best response. In Example 2, we show that, in contrast to optimization, we cannot find an approximate equilibrium for the original game for any approximation constant if \( \text{conv}(X^i) \) is not closed.

Example 2. Consider a 2-player SPG where the players solve

\[
\text{Player 1: } \max_{x \in \mathbb{R}^2} : x_2 \quad \text{s.t. } x \in \{(x_1, x_2) : x_2 = 0\} \cup \{(0, 1)\}
\]

\[
\text{Player 2: } \max_{y \in \mathbb{R}^2} : (1-x_1)y_1 + (1-x_2)y_2
\]

Although \( X^1 \) is a union of two polyhedra, \( \text{conv}(X^1) := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 < 1\} \cup \{(0, 1)\} \) is not closed. All points along the line \( x_2 = 1 \) are accumulation points of \( \text{conv}(X^1) \), but only the point \( (0, 1) \) belongs to \( \text{conv}(X^1) \). Consider the convexified version of the game. If player 1 chooses the (infeasible) strategy \((1, 1)\), then, every point in \( \mathbb{R}^2 \) is feasible and optimal for player 2 as its objective is 0. For any other choice of player 1, player 2’s objective can be arbitrarily large for an appropriate choice of \( y \). However, \((1, 1)\) is not a feasible point for player 1. So, for any feasible strategy by player 1, player 2 can make its objective arbitrarily large. Thus, the convexified game has no PNE and Theorem 1 implies that the original game has no MNE. For a similar reasoning, the game does not even admit an \( \varepsilon \)-approximate PNE for any \( \varepsilon > 0 \). However, \((1, 1)\) is a point in the closure of the convex hull of player 1’s feasible set. Thus, if our convexification procedure involves cutting and branching, \((1, 1)\) is always included in the approximate game, and the PNE where player 1 plays \((1, 1)\) is the output equilibrium. Unlike optimization, there is no sequence of feasible strategies for player 1, which converges to \((1, 1)\), and is part of any reasonable definition of approximate (pure or mixed-strategy) Nash equilibrium.

A natural question is whether it is possible that the game where each player \( i \) plays over \( \text{cl } \text{conv}(X^i) \) does not admit an equilibrium while the convex game \( \bar{G} \), i.e., where each player \( i \) plays over \( \text{conv}(X^i) \), admits an equilibrium. In Theorem 2, we prove this is not possible.

Theorem 2. Let \( \bar{G} \) be a convexified SPG where each player \( i \) solves \( \min_{x^i} \{ f^i(x^i ; x^{-i}) : x^i \in \text{conv}(X^i) \} \) given some set \( X^i \). Let \( \tilde{G} \) be the SPG where the feasible set of each player in \( \bar{G} \) is replaced with its topological closure, i.e., player \( i \) solves \( \min_{x^i} \{ f^i(x^i ; x^{-i}) : x^i \in \text{cl } \text{conv}(X^i) \} \). If \( \tilde{G} \) has no PNE, then \( \bar{G} \) has no PNE.
Proof. We prove the contrapositive of the statement, i.e., if \( G \) has a PNE, then \( \tilde{G} \) has a PNE. In particular, we claim that if \((\tilde{\pi}^1, \ldots, \tilde{\pi}^n)\) is a PNE of \( G \), then it is also a PNE of \( \tilde{G} \). We show this by contradiction. Suppose \((\tilde{\pi}^1, \ldots, \tilde{\pi}^n)\) is a PNE of \( G \), and there exists a player \( i \) and a feasible profitable deviation \( \tilde{\pi}^i \in \text{cl conv}(\mathcal{X}^i) \) such that \( f^i(\tilde{\pi}^i, \pi^{-i}) - f^i(\tilde{\pi}^i, \pi^{-i}) = \epsilon > 0 \). Because \( \tilde{\pi}^i \in \text{cl conv}(\mathcal{X}^i) \), there exists a sequence of points in \( \text{conv}(\mathcal{X}^i) \) that converges to \( \tilde{\pi}^i \). By definition, every payoff function \( f^i \) is in linear form and thus continuous. Therefore, there exists a point \( \tilde{\pi}^i \in \text{conv}(\mathcal{X}^i) \) sufficiently close to \( \tilde{\pi}^i \) such that \( |f^i(\tilde{\pi}^i, \pi^{-i}) - f^i(\tilde{\pi}^i, \pi^{-i})| < \epsilon/2 \), which is equivalent to \( f^i(\tilde{\pi}^i, \pi^{-i}) - f^i(\tilde{\pi}^i, \pi^{-i}) < \epsilon/2 \) and \( f^i(\tilde{\pi}^i, \pi^{-i}) - f^i(\tilde{\pi}^i, \pi^{-i}) > \epsilon/2 \). Combining these two inequalities with the definition of \( \epsilon \) implies that \( f^i(\pi^i, \pi^{-i}) - f^i(\pi^i, \pi^{-i}) > \epsilon/2 \). However, \( \tilde{\pi}^i \) is now a feasible profitable deviation in \( G \), providing the contradiction that \((\tilde{\pi}^1, \ldots, \tilde{\pi}^n)\) is not a PNE of \( G \), and thus completing the proof. \( \square \)

Given the currently available computational tools, we can only optimize, and thus solve games, \( G \). In \( \text{Algorithm 1} \), we present a procedure that precisely works with \( \text{cl conv}(\mathcal{X}^i) \). \( \text{Example 2} \) shows that, unlike optimization problems, working with the closure could lead to important issues. We will discuss these issues and possible solutions in \( \text{Section 4.3.2} \).

### 4.3.1 The Algorithm

**Algorithm 1** presents the general version of CnP for polyhedrally-representable SPG. The input of **Algorithm 1** is a polyhedrally-representable SPG \( G \) whereas the output is either an MNE \( \hat{\pi} \) or a certificate of its non-existence. We will employ an Enhanced Separation Oracle (ESO) associated with \( P^i \), for each \( i \), to separate infeasible equilibria and refine \( G \) via cutting planes, as we will detail in **Section 5**. We assume to have access to an initial PAG \( \hat{G} \), where, for each player \( i \), \( X_0^i \) is the starting approximation of the feasible region of player \( i \) at step \( t = 0 \). For instance, if \( i \) solves a parametrized mixed-integer optimization problem, \( X_0^i \) can be its linear relaxation (i.e., the problem without the integrality requirements). We determine if \( \hat{G} \) has PNEs \( \hat{\pi} \) by solving the NCP (3) induced by \( \hat{G} \) at step \( t \).

**Algorithm 1:** Cut-and-Play for SPGs

**Data:** A polyhedrally-representable SPG instance \( G \)

**Result:** An MNE \( \hat{\pi} = (\hat{\pi}^1, \ldots, \hat{\pi}^n) \) for \( G \) or \( \emptyset \) (i.e., no MNE exists)

1. \( t \leftarrow 0 \), \( \hat{X}^i_0 \leftarrow \{x^i: A^i x^i \leq b^i, x^i \geq 0\} \) for each player \( i = 1, \ldots, n \)
2. repeat
3. \( t \leftarrow t + 1 \), \( \hat{G} \leftarrow \text{PAG} \) where each player \( i \) solves \( \min_{x^i} \{f^i(x^i; x^{-i}) : x^i \in \hat{X}^i_t\} \)
4. \( \hat{\pi} \leftarrow \text{PNE deriving from the NCP (3)} \)
5. if no PNE in \( \hat{G} \) then
6. if \( \hat{X}^i_t = \text{conv}(\mathcal{X}^i) \) for every \( i \) then return \( \emptyset \)
7. else \( \text{BRANCH-OR-CUT: refine} \ \hat{X}^i_t \) for some \( i \)
8. else if there exists a PNE \( \hat{\pi} \) for \( \hat{G} \) then
9. for each player \( i = 1, 2, \ldots, n \) do
10. \( A \leftarrow \text{ESO}(\hat{\pi}^i, \mathcal{X}^i, f^i(\hat{\pi}^i; \pi^{-i})) \)
11. if \( A \) is no then \( \hat{X}^i_{t+1} \leftarrow \hat{X}^i_t \cap \{x^i: \pi^T x^i \leq \pi_0\} \) // \( \pi, \pi_0 \) from the ESO
12. if \( A \) returned yes for every player \( i \) then return \( \hat{\pi} \)

\( \hat{G} \) has no PNE. If \( \hat{G} \) has no PNE, we cannot infer that \( G \) has no MNE (see, e.g., **Example 1**), unless \( \mathcal{X}^i_t = \text{conv}(\mathcal{X}^i) \) for every \( i \). This non-existence can happen when at least one \( \mathcal{X}^i_t \) is unbounded for some player \( i \). The only viable option is to improve \( \hat{G} \) by refining at least one \( \mathcal{X}^i_t \) (Step 7). Unfortunately,
because \( \tilde{G} \) has no PNE, we do not have any information on which \( \tilde{X}_i \) to refine, nor how to refine it. Thus, in Step 7 we resort to a Branch-or-Cut subroutine that refines, for some player \( i \), \( \tilde{X}_i \) by either cutting or branching. If, at step \( t \), we need to refine \( \tilde{X}_i \) via Step 7, then there exists a \( \tilde{\sigma}^i \in \tilde{X}_i \setminus \text{conv}(\lambda^i) \). In a branching refinement, we find two sets \( Y^i_{t+1} \subseteq \tilde{X}_i \) and \( Z^i_{t+1} \subseteq \tilde{X}_i \) such that \( \tilde{\sigma}^i \notin \tilde{X}_{i+1} = \text{cl} \text{conv}(Y^i_{t+1} \cup Z^i_{t+1}) \), with \( \tilde{X}_{i+1} \subseteq \tilde{X}_i \). This is equivalent to the computation of \( \tilde{X}_{i+1} \) through Balas' theorem for the union of polyhedra [1, 2]. Besides MIP, branching can handle several other nonconvex problems (e.g., LCPs). In addition (or instead) of branching, Step 7 can add to \( \tilde{X}_{i+1} \) a valid inequality for \( \text{conv}(\lambda^i) \) such that \( \tilde{X}_{i+1} \subseteq \tilde{X}_i \).

\( \tilde{G} \) has a PNE. If \( \tilde{G} \) admits a PNE \( \tilde{\sigma} = (\tilde{\sigma}^1, \ldots, \tilde{\sigma}^n) \) (Step 8), then the ESO will determine whether \( \tilde{\sigma} \) is an MNE for \( G \). Specifically, given \( \tilde{\sigma}^i \) and \( \lambda^i \), the ESO answers yes if \( \tilde{\sigma}^i \in \text{conv}(\lambda^i) \) or no and an hyperplane separating \( \tilde{\sigma}^i \) from \( \text{conv}(\lambda^i) \). On the one hand, if it outputs at least one no for a given player \( i \), the oracle certifies that the strategy \( \tilde{\sigma}^i \) is infeasible or it is not a best-response to \( \tilde{\sigma}^{-i} \). Then, there exists a valid inequality for \( \text{conv}(\lambda^i) \) that does not hold for \( \tilde{\sigma}^i \), i.e., an inequality \( \pi^i x^i \leq \pi_0 \) that refines \( \tilde{X}_i \) to \( \tilde{X}_{i+1} \). On the other hand, if the ESO outputs yes for every player, then \( \tilde{\sigma} \) is an MNE for \( G \) (Step 12). Figure 1 illustrates the flow of CnP.

**4.3.2 Convergence and Practical Requirements**

Keeping Example 2 in mind, let us assume that for each player \( \text{conv}(\lambda^i) = \text{cl} \text{conv}(\lambda^i) \); we will discuss the case where this does not hold later. To guarantee the convergence of CnP, we need to be able to computationally retrieve, in finite time, the description of \( \text{conv}(\lambda^i) \) for any \( i \). We formalize this idea with the concept of computational convexifiability of Definition 3.

**Definition 3** (Computational Convexifiability). A set \( \lambda \subseteq \mathbb{R}^k \) is computationally convexifiable if (i.) \( \lambda \) is closed and convex, or (ii.) the ESO terminates in a finite number of steps, and given any initial \( X_0 \supseteq \lambda \), we can obtain \( \text{conv}(\lambda) \) with a finite sequence of refinements in Step 7.

Naturally, any implementation of an ESO handling computationally-convexifiable \( \text{conv}(\lambda^i) \) requires the input data of \( G \) to be rational. Additionally, branching and cutting should be able to refine the approximations to \( \text{conv}(\lambda^i) \) in a finite number of steps. Finally, in Theorem 3, we prove that if \( \lambda^i \) is computationally convexifiable for every \( i \), then CnP converges; we provide the full proof in the electronic companion.

**Theorem 3.** Let \( G \) be a polyhedrally-representable SPG. If \( \lambda^i \) is computationally convexifiable for each player \( i \), then, Algorithm 1 terminates in a finite number of steps and (i.) if it returns \( \tilde{\sigma} = (\tilde{\sigma}^1, \ldots, \tilde{\sigma}^n) \), then \( \tilde{\sigma} \) is an MNE for \( G \), and (ii.) if it returns \( \emptyset \), then \( G \) has no MNE.

The assumption of computational convexifiability plays a fundamental role in the convergence of CnP (Theorem 3). Specifically, a set is not computationally convexifiable for two main reasons. First,
no finite sequence of branching and cutting can lead to the convex hull, but we could have an infinite sequence of sets, which converges to the convex hull (more formally, the intersection of all these sets is the convex hull). Second, the convex hull might not be closed as shown in Example 2. Hence, no sequence of refinements converging to the convex hull exists. In either case, Theorem 3 does not apply. The first case is generally due to the fundamental properties of the set \( X \) itself. For example, if \( \text{cl conv}(X) \) is not a polyhedron, one might need infinitely many halfspaces to describe it. This is less of an issue, as refinement can continue indefinitely.

The second case, however, is more problematic. Once we obtain \( \text{cl conv}(X) \), we cannot refine the outer approximation any further, i.e., to \( \text{conv}(X) \). Nevertheless, we can still verify whether a PNE of the outer approximation is feasible, and thus an MNE for the original game. A downside is that there might be infinitely many such equilibria to enumerate. While we acknowledge these issues, we also provide reassuring computational evidence. In the NASPs of Section 6.2, the players’ feasible sets do not necessarily satisfy \( \text{conv}(X) = \text{cl conv}(X) \). However, we observe that in every instance, the ESO terminates in a finite number of steps with a PNE in the convex hull (and not just in the closure), and the algorithm returns an MNE for the original game. We reiterate that, despite considering \( \text{cl conv}(X) \) (as opposed to \( \text{conv}(X) \)), if we obtain a PNE that is in the \( \text{conv}(X) \), then we can interpret it as an MNE to the original game. If, however, \( \text{conv}(X) \neq \text{cl conv}(X) \) or there is no computational convexifiability, we cannot guarantee that an obtained PNE will be in \( \text{conv}(X) \).

5 The Enhanced Separation Oracle

Let \( X \subseteq \mathbb{R}^d \) and \( \pi \in \mathbb{R}^d \), and assume to have access to an oracle to optimize a linear function over \( X \) in a computationally-tractable manner. The ESO is an algorithm that, given a point \( \pi \), the set \( X \), and a vector \( c \in \mathbb{R}^d \):

(i.) outputs yes and \((V, \alpha)\) if \( \pi \in \text{conv}(X) \), with \( V \subseteq X \), and \( \alpha \in \mathbb{R}^{|V|} \) being the coefficients of the convex combination of elements in \( V \) (i.e., \( \pi \in \text{conv}(V) \subseteq \text{conv}(X) \)), or

(ii.) outputs no and a tuple \((\pi, \pi_0)\) so that \( \pi^\top x \leq \pi_0 \) for any \( x \in \text{conv}(X) \), \( \pi \in \text{conv}(X) \), and \( \pi^\top \pi > \pi_0 \).

Whenever the ESO outputs no, we separate \( \pi \) from \( \text{conv}(X) \) via a cutting plane. Due to Theorem 1, this separation task also has the following game-theoretic interpretation if applied to SPGs; given a set of pure strategies \( X \) and a point \( \bar{\sigma} \), if the ESO returns yes, then \( \bar{\sigma} \) is a mixed strategy, \( \text{supp}(\bar{\sigma}) = V \), and \( \alpha \) is the vector of probabilities associated with the strategies in \( V \). A theoretical version of this ESO would include polynomially-many runs of the ellipsoid algorithm, which is theoretically viable yet impractical. Furthermore, compared to a standard separation oracle, the yes answer also describes \( \pi \) as a convex combination of points in \( \text{conv}(X) \). This last requirement is a hard task and further motivates the definition of the ESO. Finally, to improve the ESO’s applicability to SPGs, we optionally require a vector \( c \) to perform an optimization test that provides a sufficient condition for the ESO to return a no and a value cut.

5.1 Value Cuts

We start from the concept of equality of payoffs [38, 39], i.e., the concept that, for each player, the payoff of any single pure strategy in the support of an MNE strategy must be equal to the MNE’s payoff. Formally, let \( \sigma^i \) be a (mixed) best-response for player \( i \) given \( \sigma^{-i} \). Then, \( f^i(\sigma^i; \sigma^{-i}) = f^i(x^i; \sigma^{-i}) \) for any \( x^i \in \text{supp}(\sigma^i) \). We develop an optimization-based test to diagnose the infeasibility of a given strategy in \( \tilde{G} \) with respect to the original SPG \( G \). Let \( \tilde{\sigma} = (\tilde{\sigma}^1, \ldots, \tilde{\sigma}^n) \) for \( \tilde{G} \) be the solution to \( \tilde{G} \) at a given CnP step. Let \( \min_{x^i} \{ f^i(x^i; \tilde{\sigma}^{-i}) : x^i \in X^i \} \) be the best-response problem of \( i \) given \( \tilde{\sigma}^{-i} \), and let \( \pi^i \) be its optimal value. If \( \pi^i > f^i(\tilde{\sigma}^i; \tilde{\sigma}^{-i}) \), then \( \tilde{\sigma}^i \notin \text{conv}(X^i) \), and a valid separating hyperplane for \( \text{conv}(X^i) \) and \( \tilde{\sigma}^i \) is \( f^i(x^i; \tilde{\sigma}^{-i}) \geq \pi^i \). This follows from the equality of payoffs and that \( \pi^i \) is the
best payoff \( i \) can achieve among any pure strategy in \( \mathcal{X}^i \). We call these separating hyperplanes value cuts. In Proposition 1, we prove such inequalities are valid for \( \text{conv}(\mathcal{X}^i) \); we provide the proof in the electronic companion.

**Proposition 1.** Consider an SPG \( G \) and an arbitrary game approximation \( \bar{G} \) of \( G \). Then, for each player \( i \) and feasible strategy \( \bar{\sigma} = (\bar{\sigma}^1, \ldots, \bar{\sigma}^n) \) in \( \bar{G} \), \( f^i(x^i; \bar{\sigma}^i) \geq \inf_{\bar{x}^i} \{ f^i(\bar{x}^i; \bar{\sigma}^i) : \bar{x}^i \in \mathcal{X}^i \} \) is a valid inequality for \( \text{conv}(\mathcal{X}^i) \) if \( \inf_{\bar{x}^i} \{ f^i(\bar{x}^i; \bar{\sigma}^i) : \bar{x}^i \in \mathcal{X}^i \} = z^i < \infty \). If \( z^i > f^i(\bar{\sigma}^i; \bar{\sigma}^i) \), we call the inequality a value cut for \( \text{conv}(\mathcal{X}^i) \) and \( \bar{\sigma}^i \).

Finally, because we outer approximate \( \text{conv}(\mathcal{X}^i) \) for each \( i \) and \( \bar{\sigma}^i \) generally comes from the PAG \( \bar{G} \), it is not possible to have that \( z^i < f^i(\bar{\sigma}^i; \bar{\sigma}^i) \). In other words, \( \bar{\sigma}^i \) is always a best-response to \( \bar{\sigma}^{-i} \) in \( \bar{G} \), yet, it may be infeasible in \( G \).

### 5.2 Implementing the Enhanced Separation Oracle

We provide an implementation of the ESO where we require \( \text{conv}(\mathcal{X}) \) to be a polyhedron. Our implementation decomposes \( \text{conv}(\mathcal{X}) \) as a conic combination of its rays \( \text{rec}(\text{conv}(\mathcal{X})) \) and convex combination of its extreme points \( \text{ext}(\text{conv}(\mathcal{X})) \), i.e., it exploits the so-called V-polyhedral representation of \( \mathcal{X} \). The ESO iteratively builds an inner approximation of \( \text{conv}(\mathcal{X}) \) by keeping track of its rays and vertices. At each ESO’s call, if the input point \( \pi \) cannot be expressed by the incumbent inner approximation of \( \text{conv}(\mathcal{X}) \), the ESO either improves the inner approximation by including new vertices and rays or outputs a no. In the case of a yes, this implementation also returns the rays \( R \) and the associated conic multipliers \( \beta \). In Section 5.3, we also show how to eliminate the conic multipliers and write \( \pi \) exclusively as a convex combination of points \( \mathcal{X} \).

**Algorithm 2:** Enhanced Separation Oracle

```
Data: A point \( \pi \), a set \( \mathcal{X} \), a storage of \( V, R \), and optionally a vector \( c \)
Result: Either: (i.) yes and (\( V, R, \alpha, \beta \)) if \( \pi \in \text{conv}(\mathcal{X}) \), or (ii.) no and a separating hyperplane \( \pi^\top x \leq \pi_0 \) for \( \text{conv}(\mathcal{X}) \) and \( \pi \)

1 if \( c \) was provided then

2 \( \bar{x} \leftarrow \arg \min_x \{ c^\top x : x \in \mathcal{X} \} \), with \( \pi = c^\top \bar{x} \)

3 if \( c^\top \pi < \pi \) then return no and \( -c^\top \bar{x} \leq -\pi \)

4 if \( \pi = \bar{x} \) then return yes and \((\{\pi\}, \emptyset, (1), (1))\)

5 repeat

6 \( \mathcal{W} \leftarrow \text{conv}(V) + \text{cone}(R) \). Solve (6) to determine if \( \pi \in \mathcal{W} \)

7 if \( \pi \in \mathcal{W} \) then return yes and \((V, R, \alpha, \beta)\)

8 else // \( \pi^\top x \leq \pi_0 \) is a separating hyperplane for \( \pi \) and \( \mathcal{W} \)

9 Let \( \mathcal{G} \) be the optimization problem \( \max_x \{ \pi^\top x : x \in \mathcal{X} \} \)

10 if \( \mathcal{G} \) is unbounded then \( R \leftarrow R \cup \{r\} \), where \( r \) is an extreme ray of \( \mathcal{G} \)

11 else if \( \mathcal{G} \) admits an optimal solution \( \nu \) then

12 if \( \pi^\top \nu < \pi^\top \pi \) then return no and \( \pi^\top x \leq \pi^\top \nu \)

13 else \( \nu \leftarrow \arg \max_x \{ \pi^\top x : x \in \mathcal{X} \} \), and \( V \leftarrow V \cup \{\nu\} \)
```

**The Algorithm.** Algorithm 2 introduces the implementation of the ESO. This implementation may be warm started with a real-valued vector \( c \) to perform the test of Proposition 1; for instance, Step 10 of Algorithm 1 calls the ESO with \( c = c^i + (C^1)^\top \bar{\sigma}^{-i} \), the set \( \mathcal{X}^i \) of \( i \), and \( \bar{\sigma} = \bar{\sigma}^i \). As a first step, if \( c \) is provided, the algorithm checks if there is any violated value cut by solving the optimization problem \( \pi = \min_x \{ c^\top x : x \in \mathcal{X} \} \) of Step 2. Specifically, the ESO compares the value of \( c^\top \pi \) to that of \( \pi \). Let
\( \bar{\pi} \) be the minimizer yielding \( \pi \). If (i.) \( c^T \pi < \overline{\pi} \), then the ESO returns a value cut (Step 3), or (ii.) if the minimizer is \( \overline{\pi} \) then the ESO returns yes (Step 4). Otherwise, let \( V \) and \( R \) be a set of vertices and rays of \( \mathcal{X} \) that the algorithm can store across its steps. We define \( \mathcal{W} \) (Step 6) as the \( \nu \)-polyhedral inner approximation of \( \text{conv}(\mathcal{X}) \) such that \( \mathcal{W} = \text{conv}(V) + \text{cone}(R) \). The central question is then to determine if \( \pi \in \mathcal{W} \subseteq \text{conv}(\mathcal{X}) \).

**The Point-Ray Separator.** To decide if \( \pi \in \mathcal{W} \), we formulate a linear program expressing \( \pi \) as a convex combination of points in \( V \) plus a conic combination of rays in \( R \). Let \( \alpha \) (resp., \( \beta \)) be the convex (conic) coefficients for the elements in \( V \) (resp., \( R \)). By duality, (5) has no solution if there is a separating hyperplane for \( \pi \) and \( \mathcal{W} \), practically, we also maximize the violation \( \pi^T \bar{\pi} - \pi_0 \), and normalize \( \pi \) such that \( ||\pi||_1 = 1 \). We can equivalently formulate the above requirements as

\[
\begin{align}
\text{maximize} & \quad \pi^T \bar{\pi} - \pi_0 \\
\text{subject to} & \quad v^T \pi - \pi_0 \leq 0 \quad \forall v \in V, \quad (\alpha) \\
& \quad r^T \pi \leq 0 \quad \forall r \in R, \quad (\beta) \\
& \quad ||\pi||_1 = 1 \quad (\delta)
\end{align}
\]

Inspired from Perregaard and Balas [44] and Chvátal et al. [11], we define (6) as the Point-Ray Linear Problem (PRLP). Each vertex \( v \in V \) (resp., ray \( r \in R \)) requires a constraint as in (6b) (resp., (6c)). As the problem may be unbounded, the normalization constraint (6d) truncates the cone of the PRLP by requiring the L1-norm of \( \pi \) to be 1.

Let \( \pi, \pi_0 \) be the optimal values of (6). On the one hand, if the optimal objective value of PRLP is 0, the oracle returns yes (Step 7) as \( \overline{\pi} \in \mathcal{W} \subseteq \text{conv}(\mathcal{X}) \). The convex multipliers \( \alpha \) (resp., conic multipliers \( \beta \)) are the dual values of (6b) (resp., (6c)). On the other hand, if \( \pi^T \bar{\pi} - \pi_0 > 0 \), then \( \pi^T x \leq \pi_0 \) is a separating hyperplane for \( \mathcal{X} \) and \( \mathcal{W} \). To determine if \( \pi^T x \leq \pi_0 \) is also a separating hyperplane for \( \mathcal{X} \) and \( \mathcal{W} \), the ESO solves the problem \( \mathcal{G} = \max_x (\pi^T x : x \in \mathcal{X}) \) (Step 9). If \( \mathcal{G} \) is unbounded, then its extreme ray \( r \) is a new ray for the set \( R \). Conversely, if \( \mathcal{G} \) admits an optimal solution \( \nu \), the latter is a new vertex for the set \( V \) (Step 13). Furthermore, if \( \pi^T \nu < \pi^T \bar{\pi} \), then \( \overline{\pi} \) is infeasible. In practice, this means \( \overline{\pi} \) is separated from \( \text{conv}(\mathcal{X}) \) by \( \pi^T x \leq \pi^T \nu \), and the ESO returns no. If this is not the case, the ESO identified a new vertex (or ray), and the process restarts from Step 6. We represent Algorithm 2 in Figure 2.

**Figure 2:** A 2-dimensional example of Algorithm 2 separating \( \overline{\pi} \) from \( \text{conv}(\mathcal{X}) \). Here, \( \mathcal{X} = \{\text{conv}([v^2, \nu]) \} \cup \{\text{conv}([v^1, v^3]) + \text{cone}([r^1])\} \). The set \( \text{conv}(\mathcal{X}) \) is the light-blue region, whereas its inner approximation \( \mathcal{W} = \text{conv}([v^1, v^2, v^3]) \) is in purple.
Practical Considerations. First, similarly to Perregaard and Balas [44], we can modify Step 9 of Algorithm 2 to retrieve multiple vertices and rays violating $\pi^Tx \leq \pi_0$, and subsequently add them in Step 13 and Step 10. In this way, the inner approximation $W$ tends to build faster without significantly impacting the computational overhead. Second, the normalizations of the PRLP in (6) are practically pivotal as they affect the algorithm’s overall stability (and convergence) through the generated cutting planes. Because normalizations tend to significantly affect the generators’ performance [4, 20, 29, 44], we normalize (6) with (6d). Finally, in Theorem 4, we show that our implementation of the ESO terminates in a finite number of steps; we defer the proof to the electronic companion.

**Theorem 4.** The ESO terminates in a finite number of steps if $\text{conv}(\mathcal{X})$ is a polyhedron.

### 5.3 Eliminating the Conic Coefficients

In Theorem 1, we interpret any convex combination of strategies in $\mathcal{X}^i$ in a game-theoretic fashion, i.e., as a mixed strategy for player $i$, where each element in the convex combination is a pure strategy whose probability of being played is given by the associated convex combination’s coefficient. However, Algorithm 2 returns, whenever the answer is yes, a proof of inclusion $(V, R, \alpha, \beta)$ that also includes a conic combination of the extreme rays in $R$. As the game-theoretic interpretability of the solution given by CnP is critical, especially when players solve unbounded problems, we provide a simple algorithm to repair a proof of inclusion $(V, R, \alpha, \beta)$ to a proof of inclusion $(V, \alpha)$ that does not include any conic combination. We illustrate its intuition in Example 3.

**Example 3 (Conic Combinations).** Consider the example in Figure 3, and assume that Algorithm 2 returns yes, and a proof of inclusion $(V, R, \alpha, \beta)$. Let $V$ be $\{v^1, v^2\}$, $R = \{r^1\}$, and $W$ be $\text{conv}(V) + \text{cone}(R)$. Let $\mathcal{X}$ be made by the points $\{v^1, v^2, v^3\}$ and the sequence of points $v^{3,1}, v^{3,2}, \ldots$ along $r^1$. Although the proof of inclusion of $\pi$ employs the ray $r^1$, we can equivalently express $\pi$ as a convex combination of $v^1$, $v^2$ and $v^{3,1}$ without resorting to $r^1$.

**A Repair Algorithm.** If CnP terminates with an MNE that includes rays, Algorithm 3 provides the repairing routine that eliminates the rays from the proof of inclusion of each player $i$. Algorithm 3 requires the point $\tilde{\pi} \in \text{conv}(\mathcal{X})$, the set $\mathcal{X}$, an arbitrarily-large constant $B \in \mathbb{R}$ and the proof of inclusion $(\tilde{V}, \tilde{\alpha}, \tilde{\beta})$ from the ESO of Algorithm 2. Algorithm 3 iteratively attempts to express $\pi$ as a convex combination of points in $V$ by augmenting $V$ with some points (not necessarily extreme points) of $\mathcal{X}$. Initialize $\tilde{V}$ as $V$. At each step, the algorithm augments $\tilde{V}$ with the optimal solutions of $\max_x\{r^T x : x \in \mathcal{X}, r^T x \leq B\}$ for any $r \in R$ (Step 3). If $\pi \in \tilde{V}$, due to the PRLP of Step 4, then the algorithm returns $(\tilde{V}, \tilde{\alpha})$, where $\tilde{\alpha}$ are the dual variables of the PRLP. Otherwise, the algorithm increments $\alpha$ with a positive integer and keeps iterating. We remark that Algorithm 3 terminates in a finite number of steps, as $x \in \text{conv}(\mathcal{X})$, and there exists a $B'$ such that $r^T x \leq B'$ for any $r \in R$ and $x \in \mathcal{X}$.

### 6 Applications

In this section, we evaluate CnP on two challenging nonconvex games, demonstrate the algorithm’s effectiveness, and establish a solid computational benchmark against the literature. We consider IPGs and NASPs, two games among players solving integer and bilevel problems. In both cases, determining the existence of an equilibrium is generally $\sum_2$-hard. Thus, we expect the computation of an equilibrium, or the determination of its non-existence, to be challenging. We compare CnP against
Algorithm 3: Repairing Algorithm

Data: A point $\pi$, $\mathcal{X}$, a large $B \in \mathbb{R}$, and ($V, R, \alpha, \beta$) from Algorithm 2
Result: A proof of inclusion ($V, \alpha$)

1. $\tilde{V} \leftarrow V$
2. repeat
   3. for $r \in R$ do
      4. $\nu \leftarrow \arg \max_x \{ r^T x : x \in \mathcal{X}, r^T x \leq B \}$, and $\tilde{V} \leftarrow \tilde{V} \cup \{\nu\}$
   5. Solve the PRLP (6) to determine if $x \in \text{conv}(\tilde{V})$
   6. if $x \in \text{conv}(\tilde{V})$ then return yes and ($\tilde{V}, \tilde{\alpha}$) else increase $B$

the most advanced problem-specific algorithms and empirically demonstrate that our algorithm is scalable, practically efficient, and can exploit problem-specific structures.

Reciprocally-bilinear Games. In our tests, we focus on the Reciprocally-Bilinear Games (RBGs), a subclass of SPG in the form of Definition 4.

Definition 4 (Reciprocally-Bilinear Game). An RBG is an SPG where, for each player $i$, $f_i(x_i; x^{-i}) := (c_i)^T x_i + (x^{-i})^T C_i x_i$, where $C_i$ and $c_i$ are a matrix and vector with rational entries.

In RBGs, determining if a PAG $\hat{G}$ admits an MNE is equivalent to solving a LCP. Both specialized LCP solvers (e.g., PATH from Dirkse and Ferris [21], Ferris and Munson [28]), and MIP reformulations [34] can solve LCPs. Although a MIP reformulation does not exploit the underlying complementarity structure, MIP solvers can optimize several computationally-tractable functions over the set of the LCP’s solutions and, therefore, select an MNE in the PAG $\hat{G}$ that maximizes a given objective function. In this sense, CnP supports heuristic equilibria selection and enables the user to select the preferred balance between equilibrium quality (e.g., given a function that measures its quality) with the time required for its computation. In our tests, we employ Gurobi 9.2 as MIP solver and PATH as an LCP solver. We report all the time-related results as shifted geometric means with a shift of 10 seconds.

6.1 Integer Programming Games

We consider a class of IPGs where each player $i$ solves the mixed-integer optimization problem

$$\max\{ (c_i)^T x^i + (x^{-i})^T C_i x^i \} \text{ subject to } A^i x^i \leq b^i, \quad x^i \geq 0, \quad x^i_j \in \mathbb{Z} \quad \forall j \in \mathcal{I}^i. \quad (7)$$

The matrix $A^i$ and the vector $b^i$ have rational entries for any $i$, and $\mathcal{I}^i$ contains the indexes of the integer-constrained variables. Reciprocally-bilinear IPGs have applications in several domains, e.g., revenue management, healthcare, cybersecurity, and sustainability [5, 16, 24], and can also represent any 2-player normal-form game. We also note that for each player $\text{conv}(\mathcal{X}^i) = \text{cl} \text{conv}(\mathcal{X}^i)$ and the feasible sets are computationally convexifiable. Thus, the issues described in Example 2 do not apply to IPGs.

6.1.1 Customizing CnP.

We tailor Algorithm 1 as follows:

(i.) A set of inequalities describing $\text{conv}(\mathcal{X}^i)$ is a perfect formulation of $\mathcal{X}^i$ [13]. In Step 7 of Algorithm 1, we can refine each player’s approximation $\bar{x}^i$ with any family of MIP inequalities.

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2Our tests run on an Intel Xeon Gold 6142 with 128GB of RAM. The code and the instances are available in the package ZERO [23] at https://ds4dm.github.io/ZERO.
In our tests, we employ Gomory Mixed-Integer (GMIs), Mixed-Integer Rounding (MIRs), and Knapsack Cover (KPs) inequalities through the software CoinOR Cgl [12]. Furthermore, at step $t = 0$ of CnP, we let each $\tilde{X}^t_i$ be the associated linear relaxation.

(ii.) If Step 10 of Algorithm 1 returns no at step $t$, we add additional cuts (e.g., GMIs) to refine $\tilde{X}^t_{i+1}$. Furthermore, whenever the value cuts exhibit a numerically-ill behavior, we attempt to substitute it with a MIP cut with better numerical properties, e.g., we generate a MIP cut that cuts off the incumbent solution.

Remark 3 (Branching may not be enough). The ESO's cuts in Step 11 of Algorithm 1 are essential for the convergence of CnP. Specifically, a pure branching algorithm (i.e., without cutting) cannot guarantee the separation of an infeasible strategy $\tilde{\sigma}^i$ of $\tilde{G}$ from $\text{conv}(\tilde{X}^i)$. Consider, for instance, the incumbent PNE $\tilde{\sigma}$ of $\tilde{G}$ at step $t$: the associated strategy $\tilde{\sigma}^i$ of player $i$ may not be an extreme point of $\tilde{X}^i_0$ and, on the contrary, it can be in the interior of $\tilde{X}^i_0$. In this case, the branching operation may not be able to exclude $\tilde{\sigma}^i$ from the refined $\tilde{X}^i_{t+1}$ (Figure 4). Indeed, finding the branching strategy excluding infeasible strategies may be a hard problem. Thus, a pure and naive branching algorithm might not separate the incumbent PNE of $\tilde{G}$ from the players' approximations. Finally, whereas branching cannot always separate a strategy from the closure of the convex hull, neither cutting nor branching can separate a strategy in $\text{cl conv}(X^i)$ from $\text{conv}(X^i)$.

6.1.2 Computational Tests

Instances and Parameters. We compare CnP with the SGM algorithm of Carvalho et al. [8], currently the most efficient algorithm to compute an MNE in IPGs. We employ the popular instances of the knapsack game, where each player $i$ solves a knapsack problem with $m$ items; namely, each player $i$ has a set of strategies $X^i = \left\{ w^i x^i \leq w^i, \ x^i \in \{0,1\}^m \right\}$. The parameters $c^i \in \mathbb{Z}^m$ and $w^i \in \mathbb{Z}_{+}^m$ are integer vectors representing the profits and weights of player $i$, respectively; The parameter $w^i$ is the knapsack capacity, whereas $C^i \in \mathbb{Z}^{(n-1)m \times m}$ is the concatenation of $(n-1)$ diagonal matrices. The elements on the diagonals are the so-called interaction coefficients associated with each of the $(n-1)$ other players in the game and their $m$ decision items (in the lexicographic order given by each player's index). Thus, players interact only when selecting corresponding items, i.e., $x^i_j = x^j_i = 1$.
for $i, o \in \{1, \ldots, n\}$. We remark that $C^i$ are different among players, and their entries are integer-valued but not necessarily positive, i.e., the interaction for a given item can be positive or negative. Because $\text{conv}(X^i)$ and $\bar{X}^i$ are compact for every $i$, any PAG admits a PNE and CnP purely acts as a cutting plane algorithm. Furthermore, although an equilibrium always exists, computing it is at least $PPAD$-hard. We solve the LCPs in (3) with either: (i.) PATH, thus computing a feasible MNE for each PAG, or (ii.) MIP, by optimizing the quadratic social welfare function $SW(\sigma) = \sum_{i=1}^{n} f^i(\sigma^i; \sigma^{-i})$. When optimizing the social welfare, we aim to find equilibria exhibiting desirable properties from the perspective of a regulator, i.e., we aim to heuristically select the equilibria that favor the society the most. Finally, we set a numerical tolerance of $\varepsilon = 3 \times 10^{-5}$, a time limit of 300 seconds, and we employ the 70 instances from Carvalho et al. [8], where $n \in \{2, 3\}$ and $10 \leq m \leq 100$.

Results. Table 1 provides an overview of the results by categorizing the instances in two sets: the small instances (i.e., $mn \leq 80$) in rows 2 – 8, and the large ones (i.e., $mn > 80$) in rows 9 – 15. The first column reports the algorithm, where, for CnP, we specify whether we use Gurobi or PATH (resp., CnP-MIP and CnP-PATH). Column $O$ is the objective type, either $F$ for feasibility or $Q$ when CnP optimizes $SW$ while solving each PAG. Column $C$ reports the aggressiveness of the additional MIP inequalities generated. Specifically $C$ can be: (i.) $-1$ if CnP do not use MIP cuts, or (ii.) 0 if it adds MIP cuts to replace numerically-ill value cuts, or (iii.) 1 if it concurrently adds MIP cuts on top of numerically-stable value cuts. The columns $\text{Time}$ and $\#\text{TL}$ report the time (in seconds) and the number of time limit hits, respectively. Column $\text{MinMax}$ reports the absolute difference (in seconds) between the maximum and the minimum computing time. Column $\text{SW\%}$ reports the average social welfare improvement compared to the MNE computed by SGM. Finally, we report the average number of steps ($\#\text{It}$), cuts added ($\text{Cuts}$), and MIP cuts ($\text{MIPCuts}$). We remark that $\text{Cuts}$ includes any cut from the ESO (including value cuts) and the MIP cuts.

Discussion. CnP always computes equilibria with remarkably modest computing times. Furthermore, CnP improves the average social welfare compared to SGM. This is mainly due to the algorithm’s approximation structure: whereas SGM approximates the player’s strategy sets from the inside, CnP outer approximates them and thus has a larger search space. The social welfare improves when CnP leverages a MIP solver, with the average welfare values almost doubling. However, this improvement significantly increases the computing times as MIP solvers cannot exploit the structure of (3). When CnP uses PATH, there are dramatic computing time improvements over the whole set of instances. Furthermore, the more MIP cuts, the fewer steps are required to converge to an MNE. When combining CnP with MIP cuts, we generally observe a reduction in the computing time and a lower $\text{MinMax}$ value. Finally, we remark that although our instances are as large as the ones considered in Carvalho et al. [8], Crönert and Minner [16], Dragotto and Scatamacchia [22], CnP exhibits limited computing times.

6.2 Nash Games Among Stackelberg Players

NASPs [9] are SPGs where each player $i$ solves a bilevel program. This family of games is instrumental in energy and pricing, as it represents complex systems of hierarchical interaction and combines simultaneous and sequential interactions. In a NASP, each player $i$’s feasible region is

$$X^i := \{x^i : A^i x^i \leq b^i, \quad x^i \geq 0, \quad x^i = (w^i, \tilde{y}^i), \quad \tilde{y}^i \in \text{SOL}(w^i)\},$$

where $x^i$ is partitioned into the leader’s variables $w^i$, and the followers’ variables $y^i$. For each player $i$, there are $u_i \in \mathbb{Z}$ followers controlling the variables $y^{i,k}$ with $k = 1, \ldots, u_i$. Each follower $k$ solves a convex quadratic optimization problem in $y^{i,k}$ parametrized in $w^i$. The set $\text{SOL}(w^i)$ in (8a) represents the solutions $y^i = (y^{i,1}, \ldots, y^{i,u_i})$ to the $i$-th player lower-level simultaneous game. Therefore, the feasible set $X^i$ for each player is a set of linear constraints (8a) plus the optimality of the followers’
game (8a). Each player’s feasible region $\mathcal{X}^i$ is a union of finitely many polyhedra [3] and we can express $\mathcal{X}^i$ as
\[
\left\{(x^i, z^i) : A^i x^i \leq b^i, z^i = M^i x^i + q^i, x^i \geq 0, z^i \geq 0\right\} \cap \left\{(x^i, z^i) : z^i_j = 0 \right\} \cup \left\{(x^i, z^i) : x^i_j = 0\right\},
\]
where $C^i$ is a set of indexes for the complementarity equations that equivalently express SOL($w^i$). For any player $i$, we refer to $\mathcal{X}^i_0$ in (9) as its polyhedral relaxation. In other words, $\mathcal{X}^i_0$ is the polyhedron containing the leader constraints, the definitions for $z^i$, and the non-negativity constraints. Because $\text{conv}(\mathcal{X}^i)$ may be unbounded, an MNE in NASPs might not exist. Indeed, the problem of determining if a NASP admits an equilibrium is $\Sigma_p^p$-hard [9]. Finally, $\text{conv}(\mathcal{X}^i)$ is not necessarily closed because of the union of polyhedra appearing in (9). Therefore, CnP actually works with $\text{cl}\text{conv}(\mathcal{X}^i)$.

Moreover, the players’ feasible sets in NASPs may not be computationally convexifiable. Since $\text{conv}(\mathcal{X}^i)$ may not be closed, we may encounter some of the issues illustrated in Example 2. This is an example of a problem class where all our assumptions for finite termination, as per Theorem 3, do not necessarily hold. However, we try Algorithm 1 on this problem, and assess the performance.

### 6.2.1 Customizing CnP.

We express each player’s optimization problem by reformulating $\mathcal{X}^i$ as in (9). In CnP, the initial relaxation of Step 1 is the polyhedral relaxation $\mathcal{X}^i_0$ for each player $i$, i.e., the feasible region where we omit all the complementarity constraints. As the description of (9) only needs a finite number of complementarity conditions in $C^i$, the branching steps account for finding the disjunction $j$ associated with the complementarity condition $z^i_{j^1} = 0$ or $x^i_{j^2} = 0$. Let $J^i_t$ be the set of disjunctions added at step $t$ of the algorithm. Then, Step 7 will include in $\mathcal{X}^i_{t+1}$ at least one complementarity $j$ for some player $i$ so that $j \in C^i \setminus J^i_t$. This boils down to the computation of $\mathcal{X}^i_{t+1}$ as the union of $\mathcal{X}^i_{t+1} \cap \{x^i, z^i : x^i_j \leq 0\}$ and $\mathcal{X}^i_t \cap \{x^i, z^i : z^i_{j^1} \leq 0\}$, where $x^i_{j^1}, z^i_{j^1}$ are the terms involved in the $j$-th complementarity of $i$.

If, at some step $t$, $J^i_t = C^i$ for all $i$, then $\text{conv}(\mathcal{X}^i) = \tilde{\mathcal{X}}^i_0$ and the algorithm terminates. Furthermore, we develop two custom branching rules for NASPs. Assume a PAG admits an MNE $\sigma$ at step $t - 1$ and needs to determine the best branching candidate in $C^i \setminus J^i_t$ at step $t$. Then, we branch via:

(i.) **Hybrid branching.** For any candidate set $C^i \setminus J^i_t$, we select the candidate $j$ that minimizes the distance between $\tilde{\sigma}^i$ and the set $\mathcal{X}^i_{t+1}$ that includes the $j$-th complementarity. Therefore, we select

### Table 1: IPGs results on the knapsack game. Extended results are available in the electronic companion.

| Algorithm | O | C | Time (s) | MinMax | #TL | SW% | #It | Cuts | MIPCuts |
|-----------|---|---|----------|--------|-----|-----|-----|------|---------|
| SGM       |   |   | 0.73     | 21.43  | 0   | 0.0%| 8.43|      |         |
| CnP-MIP   | Q | -1| 6.58     | 287.52 | 0   | 13.5%| 7.80| 9.57 | 0.00    |
| CnP-MIP   | Q | 0 | 6.13     | 287.01 | 0   | 12.9%| 5.73| 6.47 | 2.30    |
| CnP-MIP   | Q | 1 | 6.31     | 287.52 | 0   | 13.3%| 3.50| 9.60 | 7.47    |
| CnP-PATH  | F | -1| 0.36     | 10.54  | 0   | 1.8% | 7.60| 10.2 | 0.00    |
| CnP-PATH  | F | 0 | 0.05     | 0.19   | 0   | 2.9% | 5.27| 5.90 | 2.07    |
| CnP-PATH  | F | 1 | 0.04     | 0.19   | 0   | 4.9% | 3.23| 8.87 | 7.10    |
| SGM       |   |   | 20.86    | 300.00 | 6   | 0.0%| 18.58|      |         |
| CnP-MIP   | Q | -1| 61.08    | 294.50 | 0   | 22.5%| 13.70| 17.00| 0.00    |
| CnP-MIP   | Q | 0 | 57.85    | 299.45 | 1   | 19.6%| 11.62| 12.62| 3.45    |
| CnP-MIP   | Q | 1 | 68.20    | 299.04 | 0   | 22.3%| 9.48 | 16.80| 10.32   |
| CnP-PATH  | F | -1| 6.68     | 80.89  | 0   | 15.7%| 13.55| 16.35| 0.00    |
| CnP-PATH  | F | 0 | 4.48     | 74.37  | 0   | 15.7%| 9.62 | 10.25| 2.42    |
| CnP-PATH  | F | 1 | 4.32     | 75.88  | 0   | 15.9%| 8.22 | 14.35| 8.43    |
by solving \( \min_{\lambda_i} \{(\lambda^i)^2 : \sigma^i \in \text{cl conv} \left( \tilde{X}^i_i \cap \{(x^i, z^i) : x^i_j \leq 0\} \right) \cup \{\tilde{X}^i_i \cap \{(x^i, z^i) : z^i_j \leq 0\} \} \} \),

where \( \lambda \) is the vector of violations associated with each constraint in \( \tilde{X}^i_i \) plus the disjunctions on the \( j \)-th complementarity.

(ii.) Deviation branching. We solve the best-response problem of \( i \) given \( \sigma^{-i} \), and compute an optimal solution \( \tilde{x}^i \). We select the first (given an arbitrary order) candidate \( j \) that encodes the polyhedron containing \( \tilde{x}^i \).

6.2.2 Computational Tests

**Instance and Parameters.** We set a numerical tolerance of \( \varepsilon = 10^{-5} \), and consider a time limit of 300 seconds. We employ the 50 instances InstanceSet B from Carvalho et al. [9], where each instance has 7 players with up to 3 followers each, and compare against the problem-specific (sequential) inner approximation algorithm (Inn-S) from Carvalho et al. [9]. We also introduce 50 harder instances \( H7 \) with 7 players with 7 followers each. Large NASPs instances, such as the \( H7 \) set, are numerically badly scaled and thus helpful to perform stress tests on the numerical stability of the algorithms.

**Results and Discussion.** First, we note that, despite the fact that the convex hulls of the feasible sets may not be closed, in every instance, we obtained a PNE to the outer approximation that is in the convex hull of the feasible set, i.e., in the computational experiments we never encountered the type of issue described in Example 2. Thus, for every instance we either identified an MNE or proved its non-existence computationally. We report the aggregated results in Table 2. The first column reports the type of algorithm: the baseline (Inn-S) or CnP with either the hybrid (HB) or deviation (DB) branching. Furthermore, we test Inn-S-1 and Inn-S-3, two configurations of the inner approximation algorithm (we refer to the original paper for a detailed description). The second column reports the instance set (Inst), i.e., either B or \( H7 \). The three subsequent pairs of columns report the average computing time (Time) and the number of instances (#) for which the algorithm either: (i.) found an MNE (EQ), or (ii.) proved that no MNE exists (NO_EQ), or (iii.) terminated with either an MNE or a proof of its non-existence without exhibiting numerical issues (ALL). The last two columns report the number of numerical issues (#NI) and time limit hits (#TL) each algorithm encountered. The baseline Inn-S-1 systematically fails on the set \( H7 \) due to the size of the descriptions of \( \text{conv}(X^i) \). Indeed, Inn exhibits significant numerical issues in the set \( H7 \), even though the former is a problem-specific algorithm. On the contrary, CnP performs consistently and is especially effective in the hard set \( H7 \). The running times of both algorithms are comparable in the instance set B, and CnP is competitive with Inn while not being a problem-specific algorithm.

| Algo   | Inst | Time (s) | # | Time (s) | # | Time (s) | #N | #NI | #TL |
|--------|------|----------|---|----------|---|----------|----|-----|-----|
| Inn-S-1 | B    | 6.22     | 49 | 69.76    | 1 | 6.56     | 50 | 0   | 0   |
| Inn-S-3 | B    | 4.94     | 49 | 23.96    | 1 | 5.12     | 50 | 0   | 0   |
| CnP-HB  | B    | 7.47     | 46 | 29.37    | 1 | 7.71     | 47 | 3   | 0   |
| CnP-DB  | B    | 9.45     | 46 | 11.81    | 1 | 9.50     | 47 | 3   | 0   |
| Inn-S-1 | H7   | -        | 0  | -        | 0 | 300.00   | 46 | 4   | 46  |
| Inn-S-3 | H7   | -        | 0  | -        | 0 | -        | 0  | 50  | 0   |
| CnP-HB  | H7   | 53.79    | 41 | -        | 0 | 73.45    | 50 | 0   | 9   |
| CnP-DB  | H7   | 52.58    | 35 | -        | 0 | 88.92    | 50 | 0   | 15  |

Table 2: NASPs results.
7 Concluding Remarks

In this work, we presented CnP, a practically-efficient algorithm for computing Nash equilibria in SPGs, a large class of non-cooperative games where players solve nonconvex optimization problems. We showed that nonconvex SPGs admit equivalent convex formulations where the players’ feasible sets are the convex hulls of their original nonconvex feasible regions. Starting from this result, we designed CnP, a cutting-plane algorithm to compute MNEs for polyhedrally-representable SPGs or certify their non-existence. We defined the concept of game approximation, and we employed it through CnP by building an increasingly accurate sequence of convex approximations converging to an equilibrium or a certificate of its non-existence. Our algorithm is general and exhibits solid computational performance. Although CnP is a general-purpose algorithm, we also demonstrated how to tailor it to exploit the structure of specific classes of games. In addition, we also provided a series of insights into the relationships between equilibria, approximations, and approximated equilibria.

Given the generality of our algorithm, we prudently believe improvement opportunities lie ahead. We hope our contribution can inspire future methodological development in equilibria computation in nonconvex games. Among those, we foresee an extension of our methodology beyond the assumption of polyhedral representability. Finally, as Nash equilibria play a pivotal role in designing and regulating economic markets, we hope our algorithm will enable economists and optimizers to design complex markets stemming from sophisticated economic models.

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A Proof of Theorem 3.

Proof of Theorem 3. Finite Termination. We show that CnP terminates in a finite number of steps. The calls to solve the NCP in Step 4, and to the ESO in Step 10 terminate in a finite number of steps because we assume the players' feasible sets are computationally-convexifiable. The only loop that could potentially not terminate is the repeat loop starting in Step 2.

First, we restrict to the case where the set \( \lambda^i \) is bounded or finite for any player \( i \). Any PAG \( G \) necessarily admits a PNE if the approximations \( \lambda^i \) are compact. Therefore, the algorithm never triggers Step 5. Thus, Step 11 is the only step refining the sets \( \lambda^i \) for some player \( i \). As \( \lambda^i \) is computationally convexifiable for every player \( i \), the ESO can, in a finite number of steps, refine the approximations \( \lambda^i \) to \( \text{conv}(\lambda^i) \). As a consequence, the algorithm converges (in the worst case) to \( G \) (i.e., the exact convex approximation), and the correctness of the resulting MNE follows from Theorem 1.

Second, if \( \lambda^i \) is unbounded for some player \( i \), then a PNE for a given PAG \( G \) may not exist, and the algorithm may enter Step 5. However, because \( \lambda^i \) is computationally convexifiable for every player \( i \), the branching step and the ESO can refine, in a finite number of steps, the approximations \( \lambda^i \) to \( \text{conv}(\lambda^i) \). Therefore, even in the unbounded case, the algorithm correctly returns an MNE, similarly to the bounded case.

Proof of statements (i) and (ii). We show that \( \tilde{\sigma} \) is an MNE for \( G \). If the algorithm returns \( \tilde{\sigma} \), there exists an approximate game \( \tilde{G} \) in Step 3 with a PNE \( \tilde{\sigma} = \sigma \). Let the step associated with \( \tilde{G} \) having a PNE \( \tilde{\sigma} \) be denoted with \( t = \theta \), and, for each player \( i \), let \( \lambda^i_\theta \) be the associated feasible region \( i \).

Then, for any player \( i \) and \( \pi^i \in \lambda^i_\theta \), it follows that \( f^i(\tilde{\sigma}^i; \tilde{\sigma}^{-i}) \leq f^i(\sigma^i; \sigma^{-i}) \), i.e., no player \( i \) has the incentive to deviate from \( \tilde{\sigma}^i \) to any other strategy \( \pi^i \in \lambda^i_\theta \) in \( \tilde{G} \). Because \( \text{conv}(\lambda^i) \subseteq \lambda^i_\theta \) for any \( i \), then the previous inequality holds for any \( \pi^i \in \text{conv}(\lambda^i) \). Moreover, by construction, \( \tilde{\sigma}^i \in \text{conv}(\lambda^i) \); otherwise, the ESO would have returned no for player \( i \).

B Proof of Theorem 4.

Proof of Theorem 4. The ESO inner approximates the polyhedron \( \text{conv}(\lambda^i) \) with its \( V \)-representation, which is made of finitely many extreme rays and vertices. Hence, we have to prove that the ESO never finds, at any step, a vertex \( \nu \) (ray \( r \)) in Step 13 (Step 10) so that \( \nu \) is already in \( V \) (\( r \) is already in \( R \)).

This implies that the repeat loop in Algorithm 2 terminates.

The inequality after the else statement in Step 8 is valid for \( W \) if and only if \( \nu^\top \pi \leq \pi_0 \) for any \( \nu \in V \), and \( r^\top \pi \leq 0 \) for any \( r \in R \) as of (6b) and (6c). Also, because the latter inequality is a separating hyperplane between \( W \) and \( \pi \), then \( \pi^\top \pi > \pi_0 \). However, it may not necessarily be a valid inequality for any element in \( \text{ext}(\text{conv}(\lambda^i)) \) and \( \text{rec}(\text{conv}(\lambda^i)) \). Therefore, we must consider the optimization problem \( G \) in Step 9. On the one hand, if \( G \) is bounded, let \( \nu \) be its optimal solution. Then, either (i.) \( \pi^\top \nu < \pi^\top \pi \), with \( \pi^\top x \leq \pi^\top \nu \) being a separating hyperplane between \( \text{conv}(\lambda^i) \) and \( \pi \), and the algorithm terminates and returns no, or (ii.) \( \pi^\top \nu \geq \pi^\top \pi \), \( \nu \) is necessarily a vertex of \( \text{ext}(\text{conv}(\lambda^i)) \) \( \setminus V \) violating \( \pi^\top x \leq \pi_0 \), and the algorithm updates \( V \leftarrow V \cup \{\nu\} \). On the other hand, if \( G \) is unbounded, then there exists an extreme ray \( r \) so that \( r^\top \pi > 0 \). Then, \( r \) is necessarily in \( \text{rec}(\text{conv}(\lambda^i)) \setminus R \), \( \pi^\top r > \pi_0 \), and the algorithm updates \( R \leftarrow R \cup \{r\} \) and returns to Step 6. As there are finitely many extreme rays and vertices, the algorithm terminates.

C Proof of Proposition 1

Proof of Proposition 1. If the infimum is attained at a finite value \( z^i \), this implies that player \( i \) cannot achieve a payoff strictly less than \( z^i \) given the other players' strategies \( \tilde{\sigma}^{-i} \); in other words, \( z^i \) is the
payoff associated with the best response of $i$ to $\bar{\sigma}^{-i}$. Consider the problem

$$\inf_{\pi^i} \{ f^i(\pi^i; \bar{\sigma}^{-i}) : \pi^i \in \text{conv}(\lambda^i) \}. \quad (10)$$

The above problem attains a finite infimum because $z^i$ is finite. Let $\pi^i$ be an optimal solution, i.e., a best response, and $\bar{\pi}^i$ be the finite optimal value of (10), respectively. We claim that $z^i = \bar{z}^i$ for any $\pi^i \in \text{conv}(\lambda^i)$ that solves (10). On the one hand, assume that $z^i < \bar{z}^i$. Note that $\pi^i \in \text{conv}(\lambda^i)$ is a convex combination of points in $\lambda^i$, and $f^i(x^i; \bar{\sigma}^{-i})$ is linear in $x^i$. Then, any point $\pi^i$ involved in the convex combination resulting in $\pi^i$ belongs to $\lambda^i$; this also imply that $z^i$ is not the optimal value of $\inf_{\pi^i} \{ f^i(\pi^i; \bar{\sigma}^{-i}) : \pi^i \in \lambda^i \}$, resulting in a contradiction. On the other hand, assume that $\pi^i > \bar{z}^i$. Because the solutions to $\inf_{\pi^i} \{ f^i(\pi^i; \bar{\sigma}^{-i}) : \pi^i \in \lambda^i \}$ are also feasible for (10), $\pi^i$ cannot be a minimizer of (10).

D IPGs Results

Tables 3 and 4 presents the full computational results for our experiments. The column names are analogous to those of Table 1, with the addition of a few columns. Specifically, we report the value of the social welfare in SW and the average numbers of: (i.) cuts from the ESO excluding value cuts (ESOCuts), (ii.) value cuts from the ESO (VCuts). Finally, in the time column, we report in parenthesis the time the algorithm spent to compute the first MNE; this is relevant when CnP optimizes the social welfare function via a MIP solver.
Table 3: IPGs complete results, first set.

| Algorithm | O | C | Time (s) | #TL | SW | #It | Cuts | ESOCuts | VCuts | MIPCuts |
|-----------|---|---|----------|-----|----|-----|------|---------|-------|--------|
| **n=3 m=10** |
| SGM | - | - | 2.11 | 0 | 632.99 | 10.00 | - | - | - | - |
| CnP-MIP | Q | -1 | 0.47 (0.23) | 0 | 812.48 | 4.50 | 5.0 | 2.0 | 3.0 | 0.0 |
| CnP-MIP | Q | 0 | 0.31 (0.14) | 0 | 812.98 | 4.60 | 4.8 | 2.0 | 1.1 | 1.7 |
| CnP-MIP | Q | 1 | 0.20 (0.08) | 0 | 820.71 | 2.60 | 7.2 | 0.5 | 1.1 | 5.6 |
| CnP-PATH | F | -1 | 0.02 | 0 | 706.66 | 5.00 | 5.9 | 2.0 | 3.9 | 0.0 |
| CnP-PATH | F | 0 | 0.02 | 0 | 718.13 | 4.50 | 4.9 | 2.0 | 1.5 | 1.4 |
| CnP-PATH | F | 1 | 0.03 | 0 | 742.87 | 2.00 | 5.4 | 0.3 | 0.7 | 4.4 |

| **n=2 m=20** |
| SGM | - | - | 0.01 | 0 | 658.31 | 5.40 | - | - | - | - |
| CnP-MIP | Q | -1 | 0.96 (0.25) | 0 | 684.19 | 6.40 | 6.3 | 4.4 | 1.9 | 0.0 |
| CnP-MIP | Q | 0 | 0.93 (0.29) | 0 | 683.91 | 6.10 | 5.9 | 3.0 | 1.2 | 1.7 |
| CnP-MIP | Q | 1 | 0.75 (0.18) | 0 | 682.69 | 3.70 | 7.6 | 1.4 | 0.9 | 5.3 |
| CnP-PATH | F | -1 | 0.05 | 0 | 645.44 | 5.30 | 5.5 | 3.1 | 2.4 | 0.0 |
| CnP-PATH | F | 0 | 0.04 | 0 | 664.44 | 4.90 | 4.7 | 1.8 | 1.2 | 1.7 |
| CnP-PATH | F | 1 | 0.03 | 0 | 656.44 | 3.10 | 6.2 | 1.2 | 0.4 | 4.6 |

| **n=3 m=20** |
| SGM | - | - | 0.20 | 0 | 1339.98 | 9.90 | - | - | - | - |
| CnP-MIP | Q | -1 | 29.74 (1.49) | 0 | 1488.96 | 12.50 | 17.4 | 7.0 | 10.4 | 0.0 |
| CnP-MIP | Q | 0 | 27.22 (0.66) | 0 | 1473.46 | 6.50 | 8.7 | 4.0 | 1.2 | 3.5 |
| CnP-MIP | Q | 1 | 29.61 (0.61) | 0 | 1476.85 | 4.20 | 14.0 | 2.0 | 0.5 | 11.5 |
| CnP-PATH | F | -1 | 1.04 | 0 | 1327.47 | 12.50 | 19.2 | 6.3 | 12.9 | 0.0 |
| CnP-PATH | F | 0 | 0.08 | 0 | 1325.23 | 6.40 | 8.1 | 3.4 | 1.6 | 3.1 |
| CnP-PATH | F | 1 | 0.07 | 0 | 1361.74 | 4.60 | 15.0 | 2.2 | 0.5 | 12.3 |

| **n=2 m=40** |
| SGM | - | - | 1.25 | 0 | 1348.56 | 13.70 | - | - | - | - |
| CnP-MIP | Q | -1 | 27.87 (5.11) | 0 | 1433.13 | 16.70 | 21.9 | 11.1 | 10.8 | 0.0 |
| CnP-MIP | Q | 0 | 25.58 (3.53) | 0 | 1434.09 | 12.80 | 13.4 | 8.2 | 1.1 | 4.1 |
| CnP-MIP | Q | 1 | 29.72 (2.16) | 0 | 1405.30 | 10.50 | 18.7 | 6.4 | 0.7 | 11.6 |
| CnP-PATH | F | -1 | 0.89 | 0 | 1355.26 | 16.80 | 20.7 | 9.5 | 11.2 | 0.0 |
| CnP-PATH | F | 0 | 0.70 | 0 | 1355.01 | 10.00 | 9.9 | 7.1 | 0.8 | 2.0 |
| CnP-PATH | F | 1 | 0.62 | 0 | 1355.21 | 7.80 | 14.1 | 5.1 | 0.3 | 8.7 |
Table 4: IPGs complete results, second set.

| Algorithm | O | C | Time (s) | #TL | SW | #It | Cuts | ESOCuts | VCuts | MIPCuts |
|-----------|---|---|----------|-----|----|-----|------|---------|-------|---------|
| n=3 m=40  |   |   |          |     |    |     |      |         |       |         |
| SGM       | - | - | 27.04    | 2   | 2339.79 | 20.10 | - | - | - | - |
| CnP-MIP Q | -1|   | 140.33 (5.49) | 0 | 2991.76 | 20.20 | 28.5 | 13.2 | 15.3 | 0.0 |
| CnP-MIP Q | 0 |   | 128.74 (3.06) | 0 | 3016.22 | 11.60 | 15.6 | 8.9  | 1.9  | 4.8  |
| CnP-MIP Q | 1 |   | 162.20 (2.58) | 0 | 2980.69 | 9.30  | 21.9 | 6.7  | 0.9  | 14.3 |
| CnP-PATH F | -1|   | 2.35     | 0   | 2882.45 | 17.60 | 24.9 | 12.6 | 12.3 | 0.0  |
| CnP-PATH F | 0 |   | 0.87     | 0   | 2906.33 | 10.80 | 8.8  | 1.4  | 3.8  |      |
| CnP-PATH F | 1 |   | 0.79     | 0   | 2898.04 | 9.00  | 6.6  | 0.8  | 13.7 |      |
| n=2 m=80  |   |   |          |     |    |     |      |         |       |         |
| SGM       | - | - | 14.97    | 1   | 2676.52 | 19.40 | - | - | - | - |
| CnP-MIP Q | -1|   | 29.83 (11.47) | 0 | 3127.96 | 7.60  | 6.7  | 5.4  | 1.3  | 0.0  |
| CnP-MIP Q | 0 |   | 27.02 (7.27) | 0 | 3127.97 | 7.80  | 7.0  | 5.3  | 0.7  | 1.0  |
| CnP-MIP Q | 1 |   | 36.71 (10.06) | 0 | 3124.63 | 6.10  | 8.6  | 3.6  | 0.5  | 4.5  |
| CnP-PATH F | -1|   | 7.71     | 0   | 2914.36 | 8.80  | 8.1  | 6.7  | 1.4  | 0.0  |
| CnP-PATH F | 0 |   | 5.45     | 0   | 2926.82 | 7.00  | 6.1  | 4.5  | 0.4  | 1.2  |
| CnP-PATH F | 1 |   | 4.93     | 0   | 2936.52 | 5.80  | 7.4  | 3.4  | 0.4  | 3.6  |
| n=2 m=100 |   |   |          |     |    |     |      |         |       |         |
| SGM       | - | - | 77.13    | 3   | 2861.20 | 21.10 | - | - | - | - |
| CnP-MIP Q | -1|   | 102.57 (36.29) | 0 | 3750.38 | 10.30 | 10.9 | 7.4  | 3.5  | 0.0  |
| CnP-MIP Q | 0 |   | 105.97 (33.07) | 1 | 3454.41 | 14.30 | 14.5 | 9.4  | 1.2  | 3.9  |
| CnP-MIP Q | 1 |   | 107.04 (30.86) | 0 | 3771.62 | 12.00 | 18.0 | 6.3  | 0.8  | 10.9 |
| CnP-PATH F | -1|   | 23.02    | 0   | 3496.86 | 11.22 | 11.67 | 8.33 | 3.33 | 0.0  |
| CnP-PATH F | 0 |   | 14.46    | 0   | 3488.44 | 10.70 | 11.0 | 7.1  | 1.2  | 2.7  |
| CnP-PATH F | 1 |   | 14.56    | 0   | 3507.71 | 10.30 | 14.8 | 6.4  | 0.7  | 7.7  |