VIRTUAL OPERAD ALGEBRAS AND REALIZATION OF HOMOTOPY TYPES

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1. Introduction

1.1. Let $k$ be a base commutative ring, $C(k)$ be the category of complexes of $k$-modules. The category of operads $\operatorname{Op}(k)$ in $C(k)$ admits a closed model category (CMC) structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations (see [H], Sect. 6 and also Section 2 below).

Let now $\mathcal{O}$ be a cofibrant operad. The main result of this note (see Theorem 3.1) claims that the category of $\mathcal{O}$-algebras admits as well a CMC structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations. This allows one, following the pattern of [H], 5.4, to construct the homotopy category of virtual $\mathcal{O}$-algebras for any operad $\mathcal{O}$ over $C(k)$ as the homotopy category of $P$-algebras for a cofibrant resolution $P \to \mathcal{O}$ of the operad $\mathcal{O}$.

The main motivation of the note was to understand the following main result of Mandell’s recent paper [Man].

1.2. Theorem. The singular cochain functor with coefficients in $\mathbb{F}_p$ induces a contravariant equivalence from the homotopy category of connected $p$-complete nilpotent spaces of finite $p$-type to a full subcategory of the homotopy category of $E_\infty \mathbb{F}_p$-algebras.

In his approach, Mandell realizes the homotopy category of $E_\infty$-algebras as a localization of the category of algebras over a “particular but unspecified” operad $\mathcal{E}$. One of major technical problems was that the category of $\mathcal{E}$-algebras did not seem to admit a CMC structure.

We suggest to choose $\mathcal{E}$ to be a cofibrant resolution of the Eilenberg-Zilber operad. Then according to Theorem 3.1, the category of $\mathcal{E}$-algebras admits a CMC structure. This considerably simplifies the proof of Theorem 1.2.

1.3. Content of Sections. The main body of the note (Sections 2 – 4) can be considered as a complement to [H] where some general homology theory of operad algebras is presented.

In Section 2 we recall some results of [H] we need in the sequel. In Section 3 we prove the Main theorem 3.1. In Section 4 we present, using Theorem 3.1, a construction of the homotopy category Viral$(\mathcal{O})$ of virtual $\mathcal{O}$-algebras.

In Section 5 we review the proof of Mandell’s theorem [Man], stressing the simplifications due to our Theorem 3.1.
1.4. **Acknowledgement.** This work was made during my stay at the Max-Planck Institut für Mathematik at Bonn. I express my gratitude to the Institute for the hospitality. I am also grateful to P. Salvatore for a useful discussion.

2. Homotopical algebra of operads: a digest of \[H\]

In this Section we recall some results from \[H\] and give some definitions we will be using in the sequel.

2.1. **Category of operads.** Let \(k\) be a commutative ring and let \(C(k)\) denote the category of complexes of \(k\)-modules.

Recall (\[H\], 6.1.1) that the category \(\text{Op}(k)\) of operads in \(C(k)\) admits a closed model category (CMC) structure in which weak equivalences are componentwise quasi-isomorphisms and fibrations are componentwise surjective maps.

Cofibrations in \(\text{Op}(k)\) are retractions of standard cofibrations; a map \(O \to O'\) is a standard cofibration if \(O' = \lim_{s \in \mathbb{N}} O_s\) with \(O_0 = O\) and each \(O_{s+1}\) is obtained from \(O_s\) by adding a set of free generators \(g_i\) with prescribed values of \(d(g_i) \in O_s\).

2.2. **Algebras over an operad.** Let \(O \in \text{Op}(k)\).

The category of \(O\)-algebras is denoted by \(\text{Alg}(O)\). For \(X \in C(k)\) we denote by \(F(O,X)\) the free \(O\)-algebra generated by \(X\).

For any \(d \in \mathbb{Z}\) denote by \(W_d \in C(k)\) the contractible complex

\[
0 \to k = k \to 0
\]

concentrated in degrees \(d, d + 1\).

2.2.1. **Definition.** An operad \(O \in \text{Op}(k)\) is called \(H_1\)-operad if for any \(A \in \text{Alg}(O)\) the natural map

\[
A \to A \sqcup F(O,W_d)
\]

is a quasi-isomorphism.

2.2.2. **Proposition.** (see \[H\], Thm. 2.2.1) Let \(O\) be an \(H_1\)-operad. Then the category of \(O\)-algebras admits a CMC structure with quasi-isomorphisms as weak equivalences and surjective maps as fibrations.

2.3. **Examples.**

2.3.1. First of all, not all operads are \(H_1\)-operads. In fact, let \(k = \mathbb{F}_p\), \(O = \text{COM}\) (the operad of commutative algebras). Then the symmetric algebra of \(W_d\) fails to be contractible in degree \(p\).

2.3.2. **Proposition.** (see \[H\], Thm. 4.1.1) Any \(\Sigma\)-split operad (see \[H\], 4.2) is \(H_1\)-operad.

In particular, all operads over \(k \supseteq \mathbb{Q}\) are \(H_1\)-operads. Also, all operads of form \(T^\Sigma\) where \(T\) is an asymmetric operad, in particular, \(\text{ASS}\) (see \[H\], 4.2.5), are \(H_1\)-operads.
2.3.3. The main result of this note claims that any cofibrant operad is an $H_1$-operad.

2.4. Base change and equivalence. Let $f : O \to O'$ be a map of operads. Then a pair of adjoint functors

$$f^* : \text{Alg}(O) \to \text{Alg}(O') : f_*$$

is defined in a standard way.

2.4.1. Proposition. (see [H], 4.6.4.) Let $f : O \to O'$ be a map of $H_1$-operads. The inverse and direct image functors induce the adjoint functors

$$Lf^* : \text{Hoalg}(O) \to \text{Hoalg}(O') : Rf_* = f_*$$

between the corresponding homotopy categories.

2.4.2. Definition. A map $f : O \to O'$ of operads is called strong equivalence if for each $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ the induced map

$$O(|d|) \otimes_{\Sigma_d} k \to O'(|d|) \otimes_{\Sigma_d} k$$

is a quasi-isomorphism.

Here $|d| = \sum d_i$ and $\Sigma_d = \Sigma_{d_1} \times \ldots \times \Sigma_{d_n} \subseteq \Sigma_{|d|}$.

2.4.3. Proposition. Let $f : O \to O'$ be a strong equivalence of $H_1$-operads. Then the functors $Lf^*, f_*$ are equivalences.

In Section 5 we will be using the following version of Proposition 2.4.3.

2.4.4. Proposition. Let $f : O \to O'$ be a strong equivalence of operads. Suppose $O$ is $H_1$-operad. Then for each cofibrant $O$-algebra $A$ the natural map

$$A \to f_*(f^*(A))$$

is an equivalence.

2.4.5. Remark. A quasi-isomorphism of $\Sigma$-split operads compatible with the $\Sigma$-splittings is necessarily a strong equivalence.

Theorem 4.7.4 of [H] actually proves Proposition 2.4.4 and Proposition 2.4.3 together with the last Remark.

3. Main theorem

3.1. Theorem. Any cofibrant operad $O \in \text{Op}(k)$ is an $H_1$-operad.

In particular, the category of algebras $\text{Alg}(O)$ over a cofibrant operad $O$ admits a CMC structure with quasi-isomorphisms as weak equivalences and epimorphisms as fibrations.
3.2. **Proof of the theorem.** First of all, we can easily reduce the claim to the case $\mathcal{O}$ is standard cofibrant. In fact, since $\mathcal{O}$ is cofibrant, it is a retraction of a standard cofibrant operad $\mathcal{O}'$. Let

$$\mathcal{O} \xrightarrow{\alpha} \mathcal{O}' \xrightarrow{\pi} \mathcal{O}$$

be a retraction. Let $A$ be a $\mathcal{O}$-algebra. We can consider $A$ as a $\mathcal{O}'$-algebra via $\pi$. Then the map $A \to A \sqcup F(\mathcal{O}, M)$ is a retraction of the map $A \to A \sqcup F(\mathcal{O}', M)$. This reduces the theorem to the case $\mathcal{O}$ is standard cofibrant.

3.3. **Standard cofibrant case.** Let $\mathcal{O} = \lim_{s \in \mathbb{N}} \mathcal{O}_s$ (see notation of 2.1, $\mathcal{O}_0 = 0$) be a standard cofibrant operad. Let $\{g_i\}, i \in I$ be a set of free (homogeneous) generators of $\mathcal{O}$. Let a function $s : I \to \mathbb{N}$ be given so that $\mathcal{O}_s$ is freely generated as a graded operad by $g_i$ with $s(i) \leq s$ and, of course, $dg_i \in \mathcal{O}_{s(i)-1}$.

Let, finally, $\text{val} : I \to \mathbb{N}$ and $d : I \to \mathbb{Z}$ be the valency and the degree functions defined by the condition $g_i \in \mathcal{O}(\text{val}(i))^d(i)$.

The collection $I = (I, s, \text{val}, d)$ will be called a type of $\mathcal{O}$.

Since we deal with free operads and free algebras, it is worthwhile to have an appropriate notion of tree. Fix a type $I = (I, s, \text{val}, d)$.

Put $I^+ = I \cup \{a, m\}$ ($a$ and $m$ will be special marks on some terminal vertices of our trees) and extend the functions $\text{val} : I \to \mathbb{N}$ and $d : I \to \mathbb{Z}$ to $I^+$ by setting $\text{val}(a) = \text{val}(m) = d(a) = d(m) = 0$.

3.3.1. **Definition.** A $I$-tree is a finite connected directed graph such that any vertex has $\leq 1$ ingoing arrows; each vertex is marked by an element $i \in I^+$ so that $\text{val}(i)$ equals the number of outgoing arrows which are numbered by $1, \ldots, \text{val}(i)$.

The set of vertices of a tree $T$ will be denoted by $V(T)$. Terminal vertices of a $I$-tree are the ones having no outgoing arrows. In particular, all vertices marked by $a$ or by $m$ are terminal.

3.3.2. **Definition.** A $I$-tree $T$ is called proper if the following property (P) is satisfied.

(P) For any vertex $v$ of $T$ one of the possibilities (a)–(c) below occurs:

(a) $v$ is terminal;

(b) $v$ admits an outgoing arrow to a non-terminal vertex;

(c) $v$ admits an outgoing arrow to a vertex marked by $m$.

We denote by $\mathcal{P}(I)$ the set of isomorphism classes of proper $I$-trees. The following obvious result justifies the notion of proper tree.

3.3.3. **Proposition.** Let $\mathcal{O}$ be a standard cofibrant operad of type $I = (I, s, \text{val}, d)$, $A$ be a $\mathcal{O}$-algebra and $M \in C(k)$. Then the coproduct $B := A \sqcup F(M)$ is given, as a graded $k$-module, by the formula

$$B = \bigoplus_{T \in \mathcal{P}(I)} A^{\otimes a(T)} \otimes M^{\otimes m(T)}[d(T)]$$

(3)
where $a(T)$ (resp., $m(T)$) is the number of vertices of type $a$ (resp., of type $m$) in $T$ and $d(T) = \sum_{v \in V(T)} d(v)$.

3.3.4. Let $\mathcal{W}$ be the set of maps $\mathbb{N} \to \mathbb{N}$ having finite support. Endow $\mathcal{W}$ with the following lexicographic order. For $f, g \in \mathcal{W}$ we will say that $f > g$ if there exists a $s \in \mathbb{N}$ such that $f(s) > g(s)$ and $f(t) = g(t)$ for all $t > s$.

The set $\mathcal{W}$ well-ordered.

Our next step is to define a filtration of $B = A \sqcup F(M)$ indexed by $\mathcal{W}$.

3.3.5. **Definition.** Let $T \in \mathcal{P}(I)$. The weight of $T$, $w(T) \in \mathcal{W}$ is the function $\mathbb{N} \to \mathbb{N}$ which assigns to any $s \in \mathbb{N}$ the number of vertices $v$ of $T$ whose mark $i \in I$ satisfies $s(i) = s$.

Now we are able to define a filtration on $B$.

3.3.6. Let $A, M, B = A \sqcup F(M)$ be as above. For each $f \in \mathcal{W}$ define

$$F_f(B) = \bigoplus_{T : w(T) \leq f} A^{\otimes a(T)} \otimes M^{\otimes m(T)}[d(T)].$$

The homogeneous components of the associated graded complex are defined as

$$\text{gr}^F_f(B) = F_f(B) / \bigoplus_{g < f} F_g(B).$$

3.3.7. **Proposition.** 1. For each $f \in \mathcal{W}$ the graded submodule $F_f$ is a subcomplex of $B$.

2. One has $F_0 = A$.

3. Suppose $M$ is a contractible complex. Then for each $f > 0$ the homogeneous components $\text{gr}^F_f$ are contractible.

**Proof.** Obvious.

3.3.8. **Corollary.** The natural map $A \to B = A \sqcup F(O, M)$ is a quasi-isomorphism of complexes. This implies Main Theorem 3.1.

**Proof.** Obvious.

4. Virtual algebras

4.1. Theorem 3.1 suggests the following definition.

Let $O \in \text{Op}(k)$. The homotopy category of virtual $O$-algebras $\text{Vir}(O)$ is defined as $\text{Hoalg}(P)$ where $P \to O$ is a cofibrant resolution of $O$ in the category of operads.

One should, however, do some work, to ensure the definition above makes sense.
4.2. **Base change.** Any morphism \( f : P \to Q \) of operads induces a pair of adjoint functors

\[
\begin{align*}
f^* : \text{Alg}(P) & \rightleftharpoons \text{Alg}(Q) : f_*.
\end{align*}
\]

(4)

Theorem 3.1 together with 2.4.1 give immediately the following

4.2.1. **Proposition.** For any morphism \( f : P \to Q \) of cofibrant operads the adjoint functors (4) induce a pair of adjoint functors

\[
\begin{align*}
Lf^* : \text{Hoalg}(P) & \rightleftharpoons \text{Hoalg}(Q) : Rf_* = f_*
\end{align*}
\]

between the homotopy categories.

4.2.2. **Proposition.**

1. Let \( f : P \to Q \) be a weak equivalence of cofibrant operads. Then \( f \) is a strong equivalence. In particular, the derived functors of inverse and direct image \( f_* \) establish an equivalence of the homotopy categories.

2. Let \( f, g : P \to Q \) be homotopic maps between cofibrant operads. Then there is an isomorphism of functors

\[
\begin{align*}
f_*, g_* : \text{Hoalg}(Q) & \to \text{Hoalg}(P).
\end{align*}
\]

This isomorphism depends only on the homotopy class of the homotopy connecting \( f \) with \( g \).

**Proof.**

1. Let \( d = (d_1, \ldots, d_n), |d| = \sum d_i \) and let \( \Sigma_d = \prod \Sigma d_i \subseteq \Sigma |d| \).

We have to check that the map

\[
\begin{align*}
P(|d|) \otimes \Sigma_d k & \to Q(|d|) \otimes \Sigma_d k,
\end{align*}
\]

induced by \( f \), is a quasi-isomorphism.

Since \( P \) and \( Q \) are cofibrant operads, \( P(|d|) \) and \( Q(|d|) \) are cofibrant as complexes of \( k(\Sigma |d|) \)-modules. Therefore, their quasi-isomorphism is a homotopy equivalence of \( k(\Sigma |d|) \)-modules and therefore is preserved after tensoring by \( k \).

2. We present here a proof which is identical to the proof of Lemma 5.4.3(2) of \([\mathbb{H}]\).

Let \( Q \xrightarrow{\alpha} Q' \xrightarrow{p_0, p_1} Q \) be a path diagram for \( Q \) (see \([\mathbb{Q}]\), ch. 1) so that \( \alpha \) is an acyclic cofibration. Since the functors \( p_{0*} \) and \( p_{1*} \) are both quasi-inverse to an equivalence \( \alpha_* : \text{Hoalg}(Q') \to \text{Hoalg}(Q) \), they are naturally isomorphic. Therefore, any homotopy \( F : P \to Q' \) between \( f \) and \( g \) defines an isomorphism \( \theta_F \) between \( f_* \) and \( g_* \). Let now \( F_0, F_1 : P \to Q' \) be homotopic. The homotopy can be realized by a map \( h : P \to R \) where \( R \) is taken from a path diagram

\[
\begin{align*}
Q' & \xrightarrow{\beta} R \xrightarrow{q_0 \times q_1} Q' \times Q, \quad Q' \times Q \to Q'
\end{align*}
\]

(6)

where \( \beta \) is an acyclic cofibration, \( q_0 \times q_1 \) is a fibration, \( q_i \circ h = F_i, i = 0, 1 \). Passing to the corresponding homotopy categories we get the functors \( g_* \circ p_{i*} : \text{Hoalg}(Q) \to \text{Hoalg}(R) \) which are quasi-inverse to \( \alpha_* \circ \beta_* : \text{Hoalg}(R) \to \text{Hoalg}(Q) \). This implies that \( \theta_{F_0} = \theta_{F_1} \). \( \Box \)
4.3. **Virtual operad algebras.** Our construction of the category of virtual \(O\)-algebras follows the construction of virtual modules in [1], 5.4.

Let \(\mathcal{Op}^c(k)\) denote the category of cofibrant operads in \(C(k)\). For each \(\mathcal{P} \in \mathcal{Op}^c(k)\) let \(\mathcal{Hoalg}(\mathcal{P})\) be the homotopy category of \(\mathcal{P}\)-algebras. These categories form a fibred category \(\mathcal{Hoalg}\) over \(\mathcal{Op}^c(k)\), with the functors \(Rf_* = f_*\) playing the role of “inverse image functors”.

Let \(\mathcal{O} \in \mathcal{Op}(k)\). Let \(\mathcal{Op}^c(k)/\mathcal{O}\) be the category of maps \(\mathcal{P} \to \mathcal{O}\) of operads with cofibrant \(\mathcal{P}\). The obvious functor

\[c_\mathcal{O} : \mathcal{Op}^c(k)/\mathcal{O} \to \mathcal{Op}^c(k)\]

assigns the cofibrant operad \(\mathcal{P}\) to an arrow \(\mathcal{P} \to \mathcal{O}\).

4.3.1. **Definition.** The (homotopy) category \(\text{Viral}(\mathcal{O})\) of virtual \(\mathcal{O}\)-algebras is the fibre of \(\mathcal{Hoalg}\) at \(c_\mathcal{O}\). In other words, an object of \(\text{Viral}(\mathcal{O})\) consists of a collection \(A_a \in \mathcal{Hoalg}(\mathcal{P}_a)\) for each \(a : \mathcal{P}_a \to \mathcal{O}\) in \(\mathcal{Op}^c(k)/\mathcal{P}\) and of compatible collection of isomorphisms \(\phi_f : A_a \to f_*(A_b)\) given for every \(f : \mathcal{P}_a \to \mathcal{P}_b\) in \(\mathcal{Op}^c(k)/\mathcal{O}\).

4.3.2. **Corollary.** Let \(\alpha : \mathcal{P} \to \mathcal{O}\) be a weak equivalence of operads with cofibrant \(\mathcal{P}\). Then the obvious functor

\[q_\alpha : \text{Viral}(\mathcal{O}) \to \mathcal{Hoalg}(\mathcal{P})\]

is an equivalence of categories.

**Proof.** We will construct a quasi-inverse functor \(q^\alpha : \mathcal{Hoalg}(\mathcal{P}) \to \text{Viral}(\mathcal{O})\). For this choose for any map \(\beta : \mathcal{Q} \to \mathcal{O}\) a map \(f_\beta : \mathcal{Q} \to \mathcal{P}\) making the corresponding triangle homotopy commutative. Then, for any \(A \in \mathcal{Hoalg}(\mathcal{P})\) we define \(q^\alpha(A)\) to be the collection of \(f_\beta_*(A) \in \mathcal{Hoalg}(\mathcal{Q})\). According to Proposition [4.2.2], the definition does not depend on the choice of \(f_\beta_\alpha\).

The corollary means that the homotopy category of virtual \(\mathcal{O}\)-algebras is really the category of algebras over a cofibrant resolution of \(\mathcal{O}\).

4.3.3. Any map \(f : \mathcal{O} \to \mathcal{O}'\) defines an obvious functor \(\mathcal{Op}^c(k)/\mathcal{O} \to \mathcal{Op}^c(k)/\mathcal{O}'\). This induces a direct image functor

\[f_* : \text{Viral}(\mathcal{O}') \to \text{Viral}(\mathcal{O}).\]

According to Corollary [3.3.2], this functor admits a left adjoint inverse image functor \(f^*\) which can be calculated using cofibrant resolutions for \(\mathcal{O}\) and \(\mathcal{O}'\).

4.4. **Comparing \text{Viral}(\mathcal{O}) with \mathcal{Hoalg}(\mathcal{O})\).**

4.4.1. Suppose \(k \supseteq \mathbb{Q}\). Let \(\mathcal{O} \in \mathcal{Op}(k)\) and let \(f : \mathcal{P} \to \mathcal{O}\) be a cofibrant resolution of \(\mathcal{O}\). Both \(\mathcal{O}\) and \(\mathcal{P}\) admit a \(\Sigma\)-splitting (see [1], 4.2.4 and 4.2.5.2.) Moreover, the quasi-isomorphism \(f\) preserves the \(\Sigma\)-splittings. Therefore, the categories \(\text{Viral}(\mathcal{O}) = \mathcal{Hoalg}(\mathcal{P})\) and \(\mathcal{Hoalg}(\mathcal{O})\) are equivalent by [1], 4.7.4.

Thus, in the case \(k \supseteq \mathbb{Q}\) virtual operad algebras give nothing new.
4.4.2. Let $T$ be an “asymmetric operad” i.e. a collection of complexes $T(n) \in C(k)$ (with no action of the symmetric group), associative multiplication
\[ T(n) \otimes T(m_1) \otimes \ldots \otimes T(m_n) \to T(\sum m_i) \]
and unit element $1 \in T(1)$ satisfying the standard properties.
Let $O = T^\Sigma$ be the operad induced by $T$ (see [1], 4.2.1).

**Lemma.** Suppose $T(n)$ are cofibrant in $C(k)$ (for example, $T(n) \in C^-(k)$ and consist of projective $k$-modules). Then the natural functor
\[ \text{Viral}(O) = \text{Hoalg}(P) \to \text{Hoalg}(O) \]
induced by a(ny) resolution $P \to O$, is an equivalence of categories.

**Proof.** It is enough to check that the map $P(n) \to O(n)$ is a homotopy equivalence of $k(\Sigma_n)$-complexes for each $n$. But $P(n)$ is cofibrant over $k(\Sigma_n)$ since $P$ is a cofibrant operad; $O(n) = T(n) \otimes k(\Sigma_n)$ is cofibrant over $k(\Sigma_n)$ since $T(n)$ is cofibrant over $k$. This proves the claim. \qed

4.4.3. Although the categories Viral($O$) and Hoalg($O$) turn out to be equivalent in all examples of $\Sigma$-split operads we know ([1], 4.2.5), we do not see any reason why this should always be the case. No doubt, the category Viral($O$) should always be used when it differs from Hoalg($O$).

5. Application: realization of homotopy $p$-types

Mandell’s theorem [Man] on the realization of homotopy $p$-types can be reformulated in terms of virtual commutative algebras. The advantage of this approach is that we can work with the category of operad algebras which has a CMC structure. This makes unnecessary a big part of [Man].

In this Section we review the proof Mandell’s theorem [12]

5.1. **Adjoint functors** $C^*$ and $U$.

5.1.1. Recall [HS] that the cochain complex $C^*(X)$ of an arbitrary simplicial set $X \in \Delta^{\text{op}}\text{Ens}$ admits a canonical structure of algebra over the Eilenberg-Zilber operad $Z$ which is weakly equivalent to the operad COM of commutative algebras. Choose any cofibrant resolution $\mathcal{E}$ of $Z$. The category of virtual commutative algebras Viral(COM) is canonically equivalent to Hoalg($\mathcal{E}$).

5.1.2. For each commutative ring $k$ define
\[ C^*(\_, k) : (\Delta^{\text{op}}\text{Ens})^{\text{op}} \to \text{Alg}(k \otimes \mathcal{E}) \]
(here and below $\otimes$ means tensoring over $\mathbb{Z}$) to be the functor of normalized $k$-valued cochains.
This functor admits an obvious left adjoint functor
\[ U_k : \text{Alg}(k \otimes \mathcal{E}) \to (\Delta^{\text{op}}\text{Ens})^{\text{op}} \]
given by the formula

\[ U_k(A)_n = \text{Hom}(A, C^*(\Delta^n, k)) \]  

(9)

The pair of functors \( C^*(\_ , k) \) and \( U_k \) satisfies the requirements of Quillen’s theorem \[Q\], §4, Theorem 3.

Since the functor \( C^*(\_ , k) \) preserves weak equivalences, one therefore obtains a pair of derived adjoint functors

\[ \mathbb{U}_k : \text{Viral}(\text{COM}) = \text{Hoalg}(k \otimes \mathcal{E}) \rightleftarrows \mathcal{H}o : C^*(\_ , k), \]  

(10)

\( \mathcal{H}o \) being the homotopy category of simplicial sets.

5.2. Following \[\text{Man}\], we call \( X \in \Delta^{\text{op}}\text{Ens} \) \( k \)-resolvable if the natural map

\[ u_X : X \to \mathbb{U}_k C^*(X, k) \]

is a weak equivalence.

The following two lemmas allow one to construct resolvable spaces.

5.2.1. **Lemma.** \[(\text{Man}, \text{Thm. 1.1})\] Let \( X \) be the limit of a tower of Kan fibrations

\[ \ldots \to X_n \to \ldots \to X_0. \]

Assume that the canonical map from \( H^*X \) to \( \text{colim} H^*X_n \) is an isomorphism. If each \( X_n \) is \( k \)-resolvable, then \( X \) is \( k \)-resolvable.

5.2.2. **Lemma.** \[(\text{Man, Thm. 1.2})\] Let \( X, Y \) and \( Z \) be connected simplicial sets of finite type, and assume that \( Z \) is simply connected. Let \( X \to Z \) and \( Y \to Z \) be given, so that \( Y \to Z \) is a Kan fibration. Then, if \( X, Y \) and \( Z \) are \( k \)-resolvable then so is the fibre product \( X \times_Z Y \).

Lemma 5.2.1 follows form the fact that the functor \( \mathbb{U} \) carries homotopy colimits in \( \text{Alg}(\mathcal{E}) \) into homotopy limits in \( \Delta^{\text{op}}\text{Ens} \). The proof of Lemma 5.2.2 is similar, but needs in addition Proposition 5.2.3 below which can be also easily deduced from Theorem 3.1.

Using the CMC structure on \( \text{Op}(k) \), one can embed the obvious map of operads \( \text{ASS} \to \text{COM} \) into the following commutative diagram

\[
\begin{array}{ccc}
\text{ASS}_\infty & \xrightarrow{\alpha} & \mathcal{E} \\
\downarrow & \tau & \downarrow \\
\text{ASS} & \xrightarrow{\pi} & \overline{\mathcal{E}} & \xrightarrow{\pi} & \text{COM}
\end{array}
\]

where \( \text{ASS}_\infty \) is the operad of \( A_\infty \)-algebras, \( \alpha \) is a cofibration, \( \pi \) is a weak equivalence and the square is cocartesian.
5.2.3. **Proposition.** (compare to [Man], Lemma 5.2). Let $A \rightarrow B$ and $A \rightarrow C$ be cofibrations of cofibrant $E$-algebras. Let $\overline{A} = \tau^*(A)$, and similarly for $\overline{B}, \overline{C}$. Then the natural maps
\[ B \sqcup^A C \rightarrow \overline{B} \sqcup \overline{A} \overline{C} \xrightarrow{\tau} \overline{B} \otimes \overline{C} \]
are quasi-isomorphisms in $C(k)$. Here $t$ is induced by $\tau$ and $r$ is induced by the composition
\[ \overline{B} \otimes \overline{C} \rightarrow (\overline{B} \sqcup \overline{A} \overline{C}) \otimes (\overline{B} \sqcup \overline{A} \overline{C}) \xrightarrow{\text{mult}} \overline{B} \sqcup \overline{A} \overline{C}. \]

**Proof.** 1. *$t$ is a quasi-isomorphism.* The functor $\tau^*$ commutes with colimits. Therefore, it is enough to prove that the natural map $A \rightarrow \tau_* \tau^*(A)$ is a weak equivalence for a cofibrant algebra $A$. According to [2, 4.4], it is enough to check that $\tau : E \rightarrow \overline{E}$ is a strong equivalence of operads.

Since $\alpha$ is a cofibration, $\overline{\pi}$ is a cofibration as well. Therefore, both $E(n)$ and $\overline{E}(n)$ are cofibrant over $k\Sigma_n$. Then the strong equivalence of $E$ and $\overline{E}$ follows from their weak equivalence.

2. *$r$ is a quasi-isomorphism.*

Suppose $A$ is standard cofibrant and the maps $A \rightarrow B$, $A \rightarrow C$ are standard cofibrations. Let $\{e_i, i \in I\}$, $\{e_j, j \in I \cup J\}$, $\{e_k, k \in I \cup K\}$, be graded free bases of $A, B$ and $C$ respectively (the index sets $I, J, K$ are disjoint).

The sets $I, J$ and $K$ are well-ordered and the differential of $e_i$ is expressed through $e_{i'}$ with $i' < i$.

Put $S = I \cup J \cup K$ with the order given by $i < j < k$ for $i \in I, j \in J, k \in K$. Let $\tilde{S}$ be the set of maps $S \rightarrow \mathbb{N}$ with finite support and with the lexicographic order as in [3.3.4].

For $f \in \tilde{S}$ denote $|f| = \sum_{s \in S} f(s)$.

The algebra $\overline{B} \sqcup \overline{A} \overline{C}$ has an obvious increasing filtration by subcomplexes $\{F_f\}$ indexed by $f \in \tilde{S}$. The homogeneous component of the associated graded complex for $f \in \tilde{S}$ takes form
\[ \text{gr}_f(F) = \overline{E}(|f|) \otimes_{\Sigma_f} e^f \]
where $e^f = \prod_{s \in S} e_{s,f}$ and $\Sigma_f = \prod_{s \in S} \Sigma_{f,s}$.

Define a filtration $\{F'_f\}$ of $\overline{B} \otimes \overline{A} \overline{C}$ indexed by the same set $\tilde{S}$. It is given by the formula
\[ F'_f = \bigoplus_{g < f} \overline{E}(|g|_1) \otimes \overline{E}(|g|_2) \otimes_{\Sigma_g} e^g \]
where $|g|_1 = \sum_{s \in I \cup J} g(s)$ and $|g|_2 = \sum_{s \in K} g(s)$. The homogeneous component for $f \in \tilde{S}$ is given by
\[ \text{gr}_f(F'_f) = \overline{E}(|f|_1) \otimes \overline{E}(|f|_2) \otimes_{\Sigma_f} e^f. \]

The map $r : \overline{B} \otimes \overline{A} \overline{C} \rightarrow \overline{B} \sqcup \overline{A} \overline{C}$ is compatible with the filtrations. The corresponding map of the homogeneous components
\[ \text{gr}_f(r) : \overline{E}(|f|) \otimes_{\Sigma_f} e^f \rightarrow \overline{E}(|f|_1) \otimes \overline{E}(|f|_2) \otimes_{\Sigma_f} e^f \]
is induced by the map
\[ \mathcal{E}(|f_1|) \otimes \mathcal{E}(|f_2|) \to \mathcal{E}(|f|) \] (11)
which is obviously quasi-isomorphism. The assertion then follows from the observation that both the left and the right hand side of (11) are cofibrant over \( k(\Sigma f) \).

5.3. To construct \( k \)-resolvable spaces using 5.2.1 and 5.2.2 one needs a space “to start with”. This is the Eilenberg-Maclane space \( K(\mathbb{Z}/p, n) \). The key step in [Man] is the following

5.3.1. **Theorem.** (cf. [Man, Prop. A.7].) The space \( K(\mathbb{Z}/p, n) \) is \( k \)-resolvable iff \( k \supseteq \mathbb{F}_p \) and the frobenius \( F : k \to k \) gives rise to a short exact sequence of abelian groups
\[ 0 \to \mathbb{F}_p \to k \xrightarrow{1-F} k \to 0. \] (12)

**Proof.** This is the most important part of Mandell’s result and we cannot simplify the original Mandell’s proof.

1. The main step is to construct an explicit cofibrant resolution of \( C := C^*(K(\mathbb{Z}/p, n), \mathbb{F}_p) \) over \( k := \mathbb{F}_p \).

Let \( k = \mathbb{F}_p \). Let \( \mathcal{E} \) be a cofibrant resolution of the operad \( \text{COM} \) over \( \mathbb{Z} \). Recall that \( \mathcal{E} \)-algebra structure on \( A \) gives rise to the action of the generalized Steenrod algebra \( \mathcal{B} \) on \( H(A \otimes \mathbb{F}_p) \) — see [May].

Let \( A \) be a chain complex of a topological space and let the operad \( \mathcal{E} \) be endowed with a weak equivalence \( \mathcal{E} \to \mathcal{Z} \) to the Eilenberg-Zilber operad of [HS], so that \( A \) becomes an \( \mathcal{E} \)-algebra.

Then the action of \( fB \) on \( H(A \otimes \mathbb{F}_p) \) induces an action of the (conventional) Steenrod algebra \( \mathfrak{A} \) which is a quotient of \( \mathfrak{B} \) by the ideal generated by \( P^0 - 1 \), \( P^0 \) being the degree zero generalized Steenrod operation.

Choose a fundamental cycle \( e \in C^m \). This cycle defines a map \( \phi : \mathcal{E}_{\mathbb{F}_p}(x) \to C \) from the free \( \mathbb{F}_p \otimes \mathcal{E} \)-algebra with a generator \( x \) to \( C \) sending \( x \) to \( e \). Since \( P^0 \) acts trivially on \( H(C) \), the cohomology class \( P^0([x]) - [x] \) of \( \mathcal{E}_{\mathbb{F}_p}(x) \) (here \( [x] \) is the cohomology class of \( x \)), belongs to the kernel of \( H(\phi) \). Choose a representative \( z \) of the cohomology class \( P^0([x]) - [x] \) of \( \mathcal{E}_{\mathbb{F}_p}(x) \).

Finally, define \( B = \mathcal{E}_{\mathbb{F}_p}(x, y; dy = z) \). This is the \( \mathbb{F}_p \otimes \mathcal{E} \)-algebra obtained from the free algebra \( \mathcal{E}_{\mathbb{F}_p}(x) \) by adding a variable to kill the cycle \( z \) see also haha 2.2.2.

The map \( \phi \) can be obviously extended to a map \( \psi : B \to C \).

**Theorem.** (see [Man], Thm. 6.2). The map \( \psi \) is a quasi-isomorphism.

The proof of the theorem given in [Man], Sect. 12, is based on a study of free unstable modules over \( \mathfrak{B} \) and \( \mathfrak{A} \).

2. Once we have found a cofibrant resolution \( B \) of the algebra \( C \) of cochains of \( K(\mathbb{Z}/p, n) \), the life becomes very easy.

We have to study the map \( u_X : X \to U_k(C^*(X, k)) \) for \( X = K(\mathbb{Z}, n) \).
One has $U_k(C^*(X,k)) = U_k(B_k)$ where $B_k = k \otimes_{\mathbb{F}_p} B = \mathcal{E}_k(x,y; dy = z)$. Since the functor $U_k$ carries cofibrations to Kan fibrations and colimits to limits, one has a cartesian diagram of spaces

\[
\begin{array}{ccc}
U_k(B_k) & \longrightarrow & U_k(\mathcal{E}_k(z,y; dy = z)) \\
\downarrow & & \downarrow \\
U_k(\mathcal{E}_k(x)) & \overset{p}{\longrightarrow} & U_k(\mathcal{E}_k(t))
\end{array}
\]

The vertical maps are Kan fibrations and $U_k(\mathcal{E}_k(z,y; dy = z))$ is contractible since $\mathcal{E}_k(z,y; dy = z)$ is a contractible $k \otimes \mathcal{E}$-algebra.

Furthermore, $U_k(\mathcal{E}_k(t))$ identifies easily with the Eilenberg-Mac Lane space $K(k,n)$ and the map $p$ is induced by $1 - F : k \to k$ where $F : k \to k$ is the frobenius ([Man], prop. 6.4, 6.5).

Then the long exact sequence of the homotopy groups for the fibration $U_k(B_k) \to U_k(\mathcal{E}_k(x))$ gives the long exact sequence

\[
\cdots \to \pi_i(K(k,n+1)) \to \pi_i(U_k(B_k)) \to \pi_i(K(k,n)) \overset{p}{\longrightarrow} \pi_{i+1}(K(k,n+1)) \to \cdots
\]

where the map $p$ is induced by $1 - F$.

Now, if the condition on $k$ is not fulfilled, $U_k(B_k)$ is not an Eilenberg-Mac Lane space. If the sequence (12) is exact, the natural map $K(\mathbb{Z}/p,n) \to U_k(B_k)$ induces isomorphism of homotopy groups and this proves the assertion. \qed

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