TROPICALIZED QUARTICS AND CANONICAL EMBEDDINGS
FOR TROPICAL CURVES OF GENUS 3

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ABSTRACT. In [9], it was shown that not all abstract tropical curves of genus 3 can be realized as a tropicalization of a quartic in $\mathbb{R}^2$. In this article, we focus on the interior of the maximal cones in the moduli space and classify all curves which can be realized as a faithful tropicalization in a tropical plane. Reflecting the algebro-geometric world, we show that these are exactly those which are not realizably hyperelliptic.

Our approach is constructive: For any not realizably hyperelliptic curve, we explicitly construct a realizable model of the tropical plane and a faithfully tropicalized quartic in it. These constructions rely on modifications resp. tropical refinements. Conversely, we prove that any realizably hyperelliptic curve cannot be embedded in such a fashion. For that, we rely on the theory of tropical divisors and embeddings from linear systems [25, 3], and recent advances in the realizability of sections of the tropical canonical divisor [34].

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1. INTRODUCTION

1.1. Background and context. Tropical geometry can be viewed as a piece-wise linear shadow of algebraic geometry. In good cases, many geometric properties of an algebraic variety are encoded in its tropicalization as long as the tropicalization is sufficiently "fine". A problem with this philosophy is that the naive tropicalization depends on the embedding of the algebraic variety, and it is not always clear how to choose the right embedding to work with.

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Berkovich analytic geometry offers a way to overcome this problem: Berkovich analytification, which can be viewed as the inverse limit of all tropicalization [35], does not depend on an embedding and contains complete geometric information about the algebraic variety. The price to pay is that Berkovich analytic spaces are complicated objects, while in tropical geometry we hope to replace an algebraic variety with a simpler object in order to study the algebraic variety by means of combinatorics. A faithful tropicalization can be viewed as a hands-on compromise between a naive tropicalization and Berkovich analytification. For curves, it offers a way to avoid infinite graphs by working with a concrete tropicalization, which is combinatorially much easier to digest. On the other hand, it is a tropicalization which captures the wanted geometric properties.

In applications of tropical geometry to algebraic geometry, the question how to capture geometric properties in a tropicalization is foundational, and consequently the question how to construct faithful tropicalizations has attracted lots of attention in the recent years [16, 15, 19, 29, 38, 17]. In particular, modifications and re-embeddings have already been used to construct faithful tropicalizations [16, 30, 17]. The tropicalization of a re-embedded variety on a modification is sometimes called a tropical refinement [27, 31].

1.2. The question. It is well-known that any smooth projective non-hyperelliptic curve of genus 3 can be embedded as a smooth quartic in the plane. Moreover the embedding is given by the complete canonical linear system, and any smooth plane quartic is obtained this way.

In [9], Brodsky, Joswig, Morrison and Sturmfels observed that the naive tropical analogue of the above statement does not hold. Namely, not any non-hyperelliptic curve of genus 3 admits an embedding as a tropical quartic in $\mathbb{R}^2$. More precisely, they computed the locus of embeddable tropical curves, which is far from being the union of all open top-dimensional cones in $M_3^{\text{trop}} \setminus M_3^{\text{hyp}}$ (see [9, Theorem 5.1]). In fact, a whole top-dimensional cone of $M_3^{\text{trop}}$ (corresponding to hyperelliptic, but not realizably hyperelliptic tropical curves) is not in the locus. With this computational project — which offers interesting perspectives on computations with secondary fans and tropical moduli spaces beyond this result — they thus uncovered a discrepancy between the algebraic and tropical world, which as commonly suspected arises due to the use of naive tropicalization of plane curves.

In this paper, we propose a natural setting in which the tropical analogue of the above statement does hold true. Before presenting our setting, let us mention that using Payne’s theorem [35] and basic deformation theory (cf. [37]) one can prove that any (even hyperelliptic) tropical curve of genus 3 can be realized faithfully as a quartic in some tropical model of the plane. However, since abstract tropical models of the plane are very complicated such a tautological tropical analogue is not very satisfactory. In this paper we consider the class of tropical planes, which are tropicalizations of linear subplanes of projective spaces. Thus, embeddings of
genus 3 curves as quartics in such planes correspond to different choices of spanning systems of canonical sections, and on the tropical side, it is natural to ask: What tropical curves $\Gamma \in M_3^{\text{trop}}$ admit embeddings in $\mathbb{R}^n$ given by an $n$-tuple of canonical sections? Notice that the embedded images of such curves can be obtained as linear modifications of tropical quartics in the usual tropical plane $\mathbb{R}^2$.

1.3. The results. To formulate our two main results we need the following notions: an (embedded) hyperelliptic tropical curve is called realizable hyperelliptic if it is the (embedded) tropicalization of an algebraic hyperelliptic curve (see Definition 2.2); a tropical curve is called maximal if it belongs to the open interior of a cone of maximal dimension in the moduli space of tropical curves (see Definition 2.1).

**Theorem 1.1** Any maximal tropical curve of genus 3, which is not realizable hyperelliptic, can be embedded as a faithfully tropicalized quartic in a linear modification of the tropical plane $\mathbb{R}^2$ (after attaching unbounded edges appropriately).

Our proof of this result is constructive. Namely, for a given maximal not realizable hyperelliptic tropical curve $\Gamma$ of genus 3, we explicitly construct a map from $\Gamma$ to $\mathbb{R}^2$ and a series of linear modifications yielding an embedding of $\Gamma$ into the modified plane as a faithful tropicalized quartic. Our construction can be viewed as a canonical embedding of $\Gamma$ into $\mathbb{R}^n$ via an $n$-tuple of canonical divisors. In addition, it provides a way to realize algebraically a curve of genus 3 together with an $n$-tuple of canonical divisors. An exhaustive set of Examples can be found on [https://software.mis.mpg.de](https://software.mis.mpg.de).

**Theorem 1.2** No realizable hyperelliptic maximal tropical curve of genus 3 can be embedded faithfully and realizable into a linearly modified tropical plane.

The proof of this theorem is based on the theory of tropical divisors (cf. [25, 3]), and uses the recent description of the locus of realizable sections of the tropical canonical divisor of Möller, Ulirsch, and Werner [34]. We shall emphasize that although for some curves there are purely tropical obstructions to the existence of an embedding as faithful tropicalized quartic, there also exist hyperelliptic tropical curves that can be embedded in a linearly-modified tropical plane, but which do not come as tropicalizations of plane quartics.

In this paper, we restrict our study to maximal tropical curves. It would be interesting to extend this study to lower-dimensional cones, and our methods are suitable for doing so. We leave this task for further research.

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2. Preliminaries

2.1. The moduli spaces of tropical curves of genus 3. An abstract tropical curve is a connected metric graph $\Gamma$ with unbounded edges called ends, together with a function associating a genus $g(V)$ to each vertex $V$. The ends are considered to have infinite length. The genus of an abstract tropical curve $\Gamma$ equals $g(\Gamma) := b_1(\Gamma) + \sum_V g(V)$, where $b_1(\Gamma)$ is the first Betti number. An isomorphism of a tropical curve is an automorphism of the underlying graph that respects the lengths and the genus at vertices. The combinatorial type of a tropical curve is obtained by disregarding the metric structure. Given a combinatorial type, the set of all tropical curves with this type can be parameterized by the quotient of an open orthant in a real vector space under the action of automorphisms of the underlying graph that preserve the genus at vertices. Cones corresponding to different combinatorial types can be glued together by keeping track of possible degenerations under the poset of combinatorial types. In this way, the tropical moduli space $M_{g}^{\text{trop}}$ of curves of genus $g$ inherits the structure of an abstract cone complex. For more details on tropical moduli spaces of curves, see e.g. [1, 12, 13, 21, 33].

Definition 2.1 We say that an abstract tropical curve is maximal if it appears in the interior of a top-dimensional cone of $M_{3}^{\text{trop}}$.

The combinatorial types of maximal tropical curves of genus 3 are (for the full poset of combinatorial types and cones, see [13, Figure 1]):

\[
\Delta \ (000), \quad \square \ (020), \quad \circlearrowleft \ (111), \quad \circlearrowright (212), \quad \diamondsuit \ (303).
\]

A tropical curve is called hyperelliptic, if it has a divisor of degree 2 and rank 1. For an introduction to the theory of tropical divisors and their linear systems, see e.g. [5, 25]. Equivalently, a tropical curve is hyperelliptic, if it admits a degree 2 cover of a tree (see [14, Theorem 3.13]). The cover must be balanced resp. harmonic, but does not have to be realizable. Hyperelliptic maximal tropical curves of genus 3 have types $\square$, $\circlearrowleft$, $\circlearrowright$ or $\diamondsuit$, where in the first three pictures edges which form a 2-edge cut need to have the same lengths. A picture with the poset of all types of hyperelliptic curves of genus 3 is shown in [14, Figure 2].

Definition 2.2 We say that a tropical curve is realizably hyperelliptic, if it is hyperelliptic and the corresponding degree 2 cover is realizable.

Equivalently, we cannot have a 3-valent vertex mapping to a 3-valent vertex with each adjacent edge having weight 2 — such a local picture is excluded by the Riemann-Hurwitz condition for realizable tropical covers [4, 11]. Realizably hyperelliptic maximal tropical curves of genus 3 have types $\square$, $\circlearrowleft$ or $\circlearrowright$ where edges which form a 2-edge cut need to have the same lengths.
2.2. **Tropicalized quartics in $\mathbb{R}^2$.** We assume that the reader is familiar with basics in tropical geometry, in particular with tropical curves in $\mathbb{R}^2$ and their dual Newton subdivisions. For an introduction to these topics, see e.g. [36, 20, 10].

For the sake of explicitness, we pick as ground field the field $K$ of generalized Puiseux series in $t$ (with real exponents), with valuation $\text{val}$ sending a series to its least exponent [32]. We use the max-convention, i.e. the tropicalization map sends a point $(x_1, \ldots, x_n) \in (K^*)^n$ to $(-\text{val}(x_1), \ldots, -\text{val}(x_n)) \in \mathbb{Q}^n$. By $\text{Trop}(X)$ we denote the tropicalization of an algebraic variety $X \subset (K^*)^n$. A picture of a tropicalized quartic can be found in Figure 6.

2.3. **Modifications, re-embeddings and coordinate changes.** For our purposes, it is sufficient to consider modifications along tropical lines without a vertex. Here, we assume without restriction that we modify at a tropical line defined by the tropical polynomial $L = \max\{0, Y\}$. The graph of $L$ considered as a function on $\mathbb{R}^2$ consists of two linear pieces. At the break line, we attach a two-dimensional cell spanned in addition by the vector $(0, 0, -1)$ (see e.g. [2, Construction 3.3]). We assign multiplicity 1 to each cell and obtain a balanced fan in $\mathbb{R}^3$. It is called the modification of $\mathbb{R}^2$ along $L$. If $\Gamma \subset \mathbb{R}^2$ is a plane tropical curve, we bend it analogously and attach downward ends to get the modification of $\Gamma$ along $L$, which now is a tropical curve in the modification of $\mathbb{R}^2$ along $L$.

Let $\ell = m + y \in K[x, y]$ be a lift of $L$, i.e. $-\text{val}(m) = 0$. We fix an irreducible polynomial $q \in K[x, y]$ defining a curve in the torus $(K^*)^2$. The tropicalization of the variety defined by $I_{q, \ell} = \langle q, z - \ell \rangle \subset K[x, y, z]$ is a tropical curve in the modification of $\mathbb{R}^2$ along $L$. We call it the linear re-embedding of the tropical curve $\text{Trop}(V(q))$ with respect to $\ell$.

For almost all lifts $\ell$, the linear re-embedding equals the modification of $\text{Trop}(V(q))$ along $L$, i.e. we only bend $\text{Trop}(V(q))$ so that it fits on the graph of $L$ and attach downward ends. However, for some choices of lifts $\ell$, the part of $\text{Trop}(V(I_{q,\ell}))$ in the cell of the modification attached to the graph of $L$ contains more attractive features. We are most interested in these special linear re-embeddings. We describe $\text{Trop}(V(I_{q,\ell}))$ by means of two projections (see Figure 1):

1. The projection $\pi_{XY}$ to the coordinates $(X, Y)$ produces the original tropical curve $\text{Trop}(V(q))$.
2. The projection $\pi_{XZ}$ gives a new tropical plane curve $\text{Trop}(V(\tilde{q}))$ inside the projections of the cells $\{Y \geq 0, Z = Y\}$ and $\{Y = 0, Z \leq 0\}$, where $\tilde{q} = q(x, z - m)$.

The polynomial $\tilde{q}$ generates the elimination ideal $I_{q,\ell} \cap K[x, z]$.

By [16, Lemma 2.2], the projections above define $\text{Trop}(V(I_{q,\ell}))$. The content of the lemma is to recover the parts of $\text{Trop}(V(I_{q,\ell}))$ which are not contained in the interior of top-dimensional cells — for the images of the lower codimension cell the preimage under the projection is not unique. The projections are given by linear coordinate changes of the original curve. We can study the Newton subdivision of
the projected curve in terms of these coordinate changes: A term $a \cdot x^iy^j$ of $q$ is replaced by $a \cdot x^i(z - m)^j$, and so it contributes to all terms of the form $x^iz^k$ for $0 \leq k \leq j$. This is called the “feeding process” and is visualized in Figure 2 of [30]. From the feeding, we can deduce expected valuations of the coefficients. The subdivision corresponding to the expected valuations is dual to the projection of the modified curve. We care for cases in which there is cancellation and the expected valuation is not taken, these are the special re-embeddings that will make hidden geometric properties of $\text{Trop}(V(q))$ visible.

**2.4. Faithful tropicalization and the forgetful map.** The relation of Berkovich analytic spaces [8] and tropical geometry [35] hands us a way to overcome the challenge that naive tropicalization depends on the chosen coordinates: a faithful tropicalization is the best candidate to reflect relevant geometric properties of the algebraic curves [7]. An embedding of an algebraic curve $X \subset (K^*)^n$ induces a **faithful tropicalization** if $\text{Trop}(X)$ contains an isometric copy of the minimal Berkovich skeleton of $X^{an}$ under the tropicalization map $\text{trop}: X^{an} \rightarrow \text{Trop}(X)$. The latter can be obtained from a given (extended) skeleton by contracting it to its minimal expression [6]. Readers not familiar with Berkovich analytic curves, skeleta and faithful tropicalizations can nevertheless follow our constructions in Section 3 to prove Theorem 1.1: they can take our construction as a way to produce a tropicalization on a realizable model of the tropical plane with a prescribed image under the tropical forgetful map (that, essentially, forgets the embedding and shrinks ends). Notice that a smooth tropicalized curve in $\mathbb{R}^2$ (i.e. one dual to a unimodular triangulation) is faithful. Following [7, Theorem 5.24] we can in general obtain faithfulness by ensuring that the tropical multiplicities of all vertices and edges on the skeleton equal one. For vertices, this means that the corresponding initial degeneration has to be
irreducible, for edges, the weight has to be one. Since in all our constructions, only the first summands of the Puiseux series coefficients of a defining polynomial for a plane quartic are important, we can easily achieve this.

If we send an algebraic curve $X$ of genus $g$ to its minimal skeleton, we obtain an element in $M^\text{trop}_g$. If we have a faithful tropicalization for $X$, we can construct this element in $M^\text{trop}_g$ from $\text{Trop}(X)$: we equip the graph $\Gamma = \text{Trop}(X) \subset \mathbb{R}^n$ with a genus function on its vertices, and a length function on its edges. For the genus function, we use an extended Berkovich skeleton $\Sigma(X)$ coming from a semistable model of $X$ with a horizontal divisor that is compatible with $\Gamma$ [24, 23]. To each vertex $V$ in $\Gamma$ we assign the sum of the genera of all semistable vertices of $\Sigma(X)$ mapping to $V$ under $\text{trop}: \Sigma(X) \to \text{Trop}(X)$. The semistable vertices correspond exactly to the components of the central fiber $X_0$ [6], so we equip them with the genus of the associated component. For a tropicalization of a smooth curve in $\mathbb{R}^2$, where $\Gamma$ is dual to a Newton subdivision, we assign to each vertex $V$ of $\Gamma$ the number of interior lattice points of its dual polygon. For the length function, note that every edge $e$ of a tropicalized curve comes with a natural direction vector $v(e)$ (defined up to sign). In the case of tropicalized curves in $\mathbb{R}^2$, it is orthogonal to the dual edge in the Newton subdivision, and of the same length. The balancing condition ensures that the sum of the direction vectors of the edges adjacent to a vertex $V$ (with appropriate signs) is zero. The weight of an edge equals the greatest common divisor of the coordinates of its direction vector. We define the length of an edge $e$ to be the Euclidean length in $\mathbb{R}^n$ divided by the Euclidean norm of $v(e)$.

The tropical forgetful map $\text{ftrop}$ shrinks all ends and leaf edges of a faithful tropicalized curve, and equips the remaining graph with the genus function on its vertices and length function on its edges as above. If we start with a faithful tropicalized quartic curve, we obtain an element in $M^\text{trop}_3$ as image under the forgetful map.

3. Constructing faithful tropicalized quartics

3.1. Modification techniques. In this subsection, we present different scenarios which allow us to unfold certain edges using suitable linear modifications. We will employ them in the constructive proof of Theorem 1.1 (see Section 3.2).

3.1.1. Enlarging edges from trapezoids. Let $V(q), q = \sum_{i+j \leq 4} a_{i,j} x^i y^j \in K[x, y]$, be a plane curve whose Newton subdivision contains a trapezoid with a simplex adjacent to its shorter edge as in Figure 2. For each of the three types, we will introduce conditions under which the shorter edge hides an edge of prescribed length.

Figure 2. Trapezoids which allow to enlarge edge lengths.
Proposition 3.1 Suppose the Newton subdivision of $V(q)$ contains a trapezoid and triangle as on the left in Figure 2. Let $f'$ be the length of the edge $E$ dual to the red edge. Without restriction, assume that the upper vertex of $E$ is at $(0,0)$, and the valuations of the coefficients corresponding to the triangle dual to $(0,0)$ are 0 (see Figure 3 top). Let $f > f'$ and suppose the coefficients $a_{ij}$ of $q$ satisfy the following conditions for some $l_1, l_2$ with $l_2 < f'$ and $f = 2l_1 + l_2 + f'$:

\[
\begin{align*}
&\text{val} \left( \sum_{k \geq i-1} (-1)^k a_{k,j-1} \right) \geq f + 1, \quad \text{val} \left( \sum_{k \geq i} (-1)^k a_{k,j} \right) = l_1 + l_2, \\
&\text{val} \left( \sum_{k \geq i} (-1)^k a_{k,j+1} \right) = 0, \quad \text{val} \left( \sum_{k \geq i-1} (-1)^{k-1} \cdot k \cdot a_{k,j-1} \right) = l_1 + f', \\
&\text{val} \left( \sum_{k \geq i+1} (-1)^{k-1} \cdot k \cdot a_{k,j} \right) = 0, \quad \text{val} \left( \sum_{k \geq i-1} (-1)^{k-2} \cdot \left( \frac{k}{2} \right) \cdot a_{k,j-1} \right) = f'.
\end{align*}
\]

Then the modification $x = z - 1$ unfolds an edge of length $f$ from $E$ (see Figure 3 bottom).

**Proof.** We add the equation $x = z - 1$ to $q$ and project to the $yz$-coordinates to study the newly attached part of the re-embedded curve $\text{Trop}(V(q, x-z+1))$. The conditions given above are precisely the conditions on the $y^{i-1}_{z-}, y^i_{z-}, y^{i+1}_{z-}, y^{j-1}z_-$, $y^jz_-$ and $y^{j-1}z^2_-$-terms in $q(z-1,y)$ as imposed in the lower left of Figure 3. The sums above are taken over all terms of $q$ which “feed” to the term in question. The valuations determine the positions of the vertices of $\text{Trop}(V(q(z-1,y)))$, as in the lower right of Figure 3. By adding the lengths of the edges projecting to $E$, it follows that the length $f$ is unfolded. $\square$
Remark 3.2 Note that the conditions in Proposition 3.1 imply cancellation behaviour for the coefficients, which requires more than one coefficient of the same valuation. This explains the trapezoid in the subdivision.

Proposition 3.3 Suppose the Newton subdivision of $V(q)$ contains a trapezoid and triangle as in the middle of Figure 2. Let $d'$ be the length of the edge $E$ dual to the red edge. Without restriction, assume that the upper vertex of $E$ is at $(0,0)$, and the valuations of the coefficients corresponding to the triangle dual to $(0,0)$ are zero. Let $d > d'$ and suppose the coefficients $a_{ij}$ of $q$ satisfy the following conditions for some $l_1, l_2$ with $l_2 < d'$ and $f = 2l_1 + l_2 + d'$:

$$\text{val} \left( \sum_{k \geq 0} (-1)^k a_{i+j-1-k,k} \right) = 0, \quad \text{val} \left( \sum_{k \geq 0} (-1)^k a_{i+j-k,k} \right) = l_1 + l_2,$$

$$\text{val} \left( \sum_{k \geq 0} (-1)^k a_{i+j+1-k,k} \right) \geq d + 1, \quad \text{val} \left( \sum_{k \geq 0} (-1)^{k-1} \cdot k \cdot a_{i+j-k,k} \right) = 0,$$

$$\text{val} \left( \sum_{k \geq 0} (-1)^{k-1} a_{i+j+1-k,k} \right) = l_1 + d', \quad \text{val} \left( \sum_{k \geq 0} (-1)^{k-2} \left( \frac{k}{2} \right) a_{i+j+1-k,k} \right) = d'.$$

Then the modification $y = z - x$ unfolds an edge of length $d$ from $E$.

Proof. We add the equation $y = z - x$ and project to the $yz$-coordinates to study the newly attached part of the re-embedded curve $\text{Trop}(V(q, y - z + x))$ in terms of $\text{Trop}(V(q(x, z - x)))$. The conditions given above are conditions on the $x^{i+j-1}$, $x^{i+j}$, $x^{i+j+1}$, $x^{i+j-1-z}$, $x^{i+j-1-z}$, and $x^{i+j+1-z^2}$-terms of $q(x, z - x)$ which determine the position of the relevant vertices of $\text{Trop}(V(q(x, z - x)))$. The positions imply that the length $d$ is unfolded from the edge $E$. \hfill $\square$

Proposition 3.4 Suppose the Newton subdivision of $V(q)$ contains a trapezoid and triangle as on the right in Figure 2. Let $e'$ be the length of the edge $E$ dual to the red edge. Without restriction, assume that the upper vertex of $E$ is at $(0,0)$, and the valuations of the coefficients corresponding to the triangle dual to $(0,0)$ are zero. Let $e > e'$ and suppose the coefficients $a_{ij}$ of $q$ satisfy the following conditions for some $l_1, l_2$ with $l_2 < e'$ and $f = 2l_1 + l_2 + e'$:

$$\text{val} \left( \sum_{k \geq 0} (-1)^k a_{i-1,k} \right) \geq e + 1, \quad \text{val} \left( \sum_{k \geq j} (-1)^k a_{i,k} \right) = l_1 + l_2,$$

$$\text{val} \left( \sum_{k \geq j+1} (-1)^k a_{i+1,k} \right) = 0, \quad \text{val} \left( \sum_{k \geq j+1} (-1)^{k-1} \cdot k \cdot a_{i-1,k} \right) = l_1 + e',$$

$$\text{val} \left( \sum_{k \geq j} (-1)^{k-1} \cdot k \cdot a_{i,k} \right) = 0, \quad \text{val} \left( \sum_{k \geq j+1} (-1)^{k-2} \left( \frac{k}{2} \right) \cdot a_{i-1,k} \right) = e'.$$

Then the modification $y = z - 1$ unfolds an edge of length $d$ from $E$.

Proof. We add the equation $y = z - 1$ and project to the $xz$-coordinates to study the newly attached part of the re-embedded curve $\text{Trop}(V(q, y - z + 1))$ in terms...
of $\text{Trop}(V(q(x, z - 1)))$. The conditions given above are conditions on the $x^{i-1}$, $x^{i}$, $x^{i+1}$, $x^{i-1}z^{1}$, $x^{i}z^{1}$, and $x^{i-1}z^{2}$-terms of $q(x, z - 1)$ which determine the position of the relevant vertices of $\text{Trop}(V(q(x, z - 1)))$. The positions imply that the length $e$ is unfolded from the edge $E$. 

3.1.2. Unfolding lollis from weight 2 edges. Let $V(q) = \sum_{i+j\leq 4} a_{i,j}x^iy^j \in \mathbb{C}[x, y]$, be a plane curve whose Newton subdivision contains either the triangle with vertices $(0, 2)$, $(2, 2)$ and $(0, 4)$ or the triangle with vertices $(2, 0)$, $(2, 2)$ and $(4, 0)$ or the triangle with vertices $(2, 0)$, $(0, 2)$ and $(0, 0)$. We will introduce conditions under which the corresponding bounded weight 2 edges hide a lolli of prescribed edge and cycle length. For the sake of brevity, we consider the first case only.

Let $V$ be the dual vertex of the triangle $V$ and let $E$ be the weight 2 edge of $\text{Trop}(V(q))$ dual to $E^\vee := \text{conv}((0, 2), (2, 2))$. We call $V'$ the other vertex of $E$ (see Figure 4 top). Without restriction, we can assume that the vertex $V'$ is at $(0, 0)$, that the valuations of the coefficients of $y^2$, $xy^2$ and $x^2y^2$ are 0, and that the coefficient of $x^2y^2$ is 1.

**Proposition 3.5** For a tropicalized quartic as above, assume that

1. $q$ restricted to $E^\vee$ is the square of $y$ times a linear form $L$ in $x$ up to order $2a$,
2. the $t^{2a}$-contribution of $|q|_{E^\vee}$ is not divisible by the $t^0$-contribution of $L$,
3. the length of $E$ is $3a + \frac{b}{2}$,
4. the sum of the $t^0$-terms of $q$ corresponding to monomials $y \cdot x^i$ for some $i$ is not divisible by the $t^0$-contribution of $L$.

Then we can use one linear modification to unfold a lolli with edge length $a$ and cycle length $b$ from the edge $E$ (see Figure 4 bottom right).

**Proof.** First, we unfold a bridge edge and a loop in two steps. Then we combine the two steps to one. By condition (1), $q$ restricted to $E^\vee$ equals

$$(y \cdot (x + A))^2 + t^{2a} \cdot y^2 \cdot (\alpha x + \beta) + O(t^{2a+1}),$$

where $A = A_0 + A_1t + A_2t^2 + \ldots + A_{2a-1}t^{2a-1} \in K$ is of valuation 0, and $\alpha, \beta \in \mathbb{C}$ satisfy either $\alpha = 0, \beta \neq 0$ or $0 \neq \frac{\alpha}{\beta} \neq A_0$ by condition (2). By assumption (3), $V$ is at $(0, 6a + b)$, and the valuation of the $y^4$-term is $6a + b$.

We start by adding the equation $z_1 = x + A$. As described in Section 2.3, we can make the new part of $\text{Trop}(V(q, z_1 - x - A))$ visible using the projection $\pi_{y, z_1}$ defined by $\tilde{q} := q(z_1 - A, y)$.

Condition (4) ensures that there is no cancellation in $\tilde{q}$ involving the terms $y \cdot x^i$. Thus, the $y$-term of $\tilde{q}$ has valuation 0. For the $y^2$- and $z_1y^2$-term however, there is cancellation. The respective terms of $\tilde{q}$ are:

$$(y \cdot z_1)^2 + t^{2a} \cdot y^2 \cdot (\alpha(z_1 - A) + \beta) + O(t^{2a+1}),$$
By condition (2), the \(y^2\)-term has valuation \(2a\), and the \(y^2z_1\) term has valuation bigger or equal to \(2a\).

The Newton subdivision of newly attached part is depicted below in the lower right of Figure 4. Notice that the bounded edge of weight 2 is dual to the edge with initial \(y^2z_1^2 + t^{2a} \cdot (-A_0\alpha + \beta) \cdot y^2 = y^2 \cdot (z_1 - t^a \cdot \sqrt{-A_0\alpha + \beta})(z_1 + t^a \cdot \sqrt{-A_0\alpha + \beta})\). As this is not a square, it follows from Theorem 3.4 in [16] that we can use the linear modification induced by the equation \(z_2 = z_1 - t^a \cdot \sqrt{-A_0\alpha + \beta}\) to unfold a cycle of length \(2 \cdot \frac{b}{2} = b\). This cycle is attached to the vertex \((0, 0)\) via an edge of length \(a\).

The variable \(z_1\) is not needed to produce a faithful tropicalization, we can eliminate it and combine the two steps to one. That is, we only add one equation, namely \(z_2 = x + A - t^a \cdot \sqrt{-A_0\alpha + \beta}\). \(\square\)

### 3.2. Proof of Theorem 1.1

This sections forms a constructive proof of Theorem 1.1. Examples for all constructions can be found on https://software.mis.mpg.de.

For a maximal, not realizably hyperelliptic abstract tropical curve \(C\) of genus 3 we explicitly construct a faithful tropicalized quartic in a realizable model of the tropical plane such that its image under \(f(t)\) (resp. its minimal skeleton) is \(C\). We obtain the model of the tropical plane and the tropicalized quartic by a series of linear modifications, starting with a plane quartic defined over \(K\). We built on [9, Theorem 5.1]: we only construct faithful tropicalized quartics in a realizable model of the tropical plane for abstract tropical curves which are not embeddable in \(\mathbb{R}^2\) as tropical quartics.
3.2.1. Type □. Let $a, b, c, d, e, f$ be the edge lengths of an abstract curve $C$ as in Figure 5. Due to symmetry, we may assume $a \leq b$, $c \leq d$, $e \leq f$ and $a \leq e$. By [9, Theorem 5.1], $C$ is embeddable as a quartic in $\mathbb{R}^2$ if $c + e \leq d$. Additionally, if the inequality holds as an equality and $a = b$, then necessarily $a < e < f$.

![Figure 5](image-url) Figure 5. An abstract and plane tropical curve of type □.

To show that any curve $C$ is embeddable as a quartic in a tropical plane, let $a, b, c, d, e, f$ be any lengths satisfying our assumptions due to symmetry. Note that curves of type □ with $c = d$ are realizably hyperelliptic, which is why they are excluded in our considerations. Hence we will focus on the case that $e \geq d - c > 0$. Let $e' := d - c > 0$ and pick $l_1, l_2 \geq 0$ such that $2l_1 + 2 + e' = e$. Consider the quartic polynomial $g = \sum_{i,j} a_{ij}x^iy^j$ where

- all $a_{ij}$ except $a_{11}, a_{10}, a_{00}$ are of the form $t^{\lambda_{ij}}$ for suitable valuations $\lambda_{ij}$ such that the tropical curve of $g$ is as depicted in Figure 6, provided the valuations of $a_{00}, a_{10}, a_{11}$ are 0 (this is possibly by [9, Theorem 5.1]),
- $a_{11} = 1 + a_{01} - a_{31} - t^{l_1 + l_2}$,
- $a_{10} = 2 - 3a_{30} + 4a_{40} + t^{l_1}$,
- $a_{00} = 1 - 2a_{30} + 3a_{40} + t^{l_1} + t^{2l_1 + l_2 + 1}$.

![Figure 6](image-url) Figure 6. A degenerate tropical plane quartic of type □ whose edge $e'$ can be prolonged.

This is a tilted version of Proposition 3.4, i.e. the modification $x = z - 1$ will reveal an bounded edge of length $e$. 
3.2.2. **Type \( \Delta \).** Let \( a, b, c, d, e, f \) be the edge lengths of an abstract tropical curve \( C \) as in Figure 7. Due to symmetry, we may assume \( b \leq c \leq a \). By [9, Theorem 5.1], \( C \) is embeddable as quartic in \( \mathbb{R}^2 \) if \( \max(d, e) \leq a, \max(e, f) \leq b \) and \( \max(d, f) \leq c \), such that

- at most two of the inequalities may hold as equalities,
- if two hold as equalities, then either \( d, e, f \) are distinct and the edge \( a, b \) or \( c \) that connects the shortest two of \( d, e, f \) attains equality or \( \max(d, e, f) \) is attained exactly twice and the edge connecting the longest two does not attain equality.

![Figure 7. An abstract and plane tropical curve of type \( \Delta \).](image)

To show that any curve \( C \) is embeddable as a quartic in a tropical plane, let \( a, b, c, d, e, f \) be any lengths satisfying our assumptions due to symmetry. Pick \( d' \leq d, e' \leq e, f' \leq f \) such that \( a, b, c, d', e', f' \) is realizable as a plane tropical quartic. We distinguish between the following cases:

\[
\begin{align*}
(===): & \quad d' = d, e' = e, f' = f, \\
(<==): & \quad d' \leq d, e' = e, f' = f, \\
(==<): & \quad d' = d, e' \leq e, f' = f, \\
(==<=): & \quad d' = d, e' = e, f' \leq f, \\
(==<=): & \quad d' = d, e' \leq e, f' < f, \\
(==<==): & \quad d' = d, e' < e, f' = f, \\
(==<<): & \quad d' < d, e' < e, f' < f,
\end{align*}
\]

In the case \((===)\) the curve \( C \) is even embeddable as a quartic in \( \mathbb{R}^2 \). Cases \((+-+), (-++) \) and \((-+-)\) can be obtained via modifications from the subdivisions in Figure 8, if the coefficients are chosen according to Propositions 3.1, 3.4 and 3.3.

![Figure 8. Subdivisions of type \( \Delta \) that allow to prolong an edge.](image)

For the subdivision depicted on the left, consider a quartic \( g = \sum_{i,j} a_{ij}x^iy^j \) where all \( a_{ij} \) except \( a_{11}, a_{10}, a_{00} \) are of the form \( t^{\lambda_{ij}} \) for suitable valuations \( \lambda_{ij} \) and

\[
\begin{align*}
a_{21} &= a_{12} + a_{30} - a_{03} - t^{l_1+l_2}, \quad a_{31} = 2a_{40} - t^{d'+l_1} - 2t^{2l_1+l_2+d'+1} + a_{13} - 2a_{04}, \\
a_{22} &= a_{40} - t^{d'+l_1} - t^{2l_1+l_2+d'+1} + 2a_{13} - 3a_{04},
\end{align*}
\]
Then the coefficients satisfy the equations in Proposition 3.3, and \( C \) is embeddable in the tropical curve given by \( \langle g, y - z + x \rangle \) in the tropical plane given by \( y - z + x \). For the subdivision in the middle of Figure 8, consider a quartic \( g = \sum_{i,j} a_{ij} x^i y^j \) where all \( a_{ij} \) except \( a_{01}, a_{02}, a_{12} \) are of the form \( t^{\lambda_{ij}} \) for suitable valuations \( \lambda_{ij} \) and

\[
\begin{align*}
    a_{02} &= a_{00} - t^{l_1+e} - 2 t^{l_1+l_2+e'+1} + 2a_{03} - 3a_{04}, & a_{12} &= -a_{10} + a_{11} + a_{13} + t^{l_1+l_2}, \\
    a_{01} &= 2a_{00} - t^{l_1+e'} - 2 t^{l_1+l_2+e'+1} + a_{03} - 2a_{04}.
\end{align*}
\]

Then the coefficients satisfy the equations in Proposition 3.1, and \( C \) is embeddable in the tropical curve given by \( \langle g, x - z + 1 \rangle \) in the tropical plane given by \( x - z + 1 \).

For the subdivision on the right of Figure 8, consider a quartic \( g = \sum_{i,j} a_{ij} x^i y^j \) where all \( a_{ij} \) except \( a_{10}, a_{20}, a_{11} \) are of the form \( t^{\lambda_{ij}} \) for suitable valuations \( \lambda_{ij} \) and

\[
\begin{align*}
    a_{10} &= 2a_{00} - t^{l_1+f'} - 2 t^{l_1+l_2+f'+1} + a_{30} - 2a_{40}, & a_{11} &= a_{01} + a_{21} - a_{31} - t^{l_1+l_2}, \\
    a_{20} &= a_{00} - t^{l_1+f'} - 2 t^{l_1+l_2+f'+1} + 2a_{30} - 3a_{40}.
\end{align*}
\]

Then the coefficients satisfy the equations of Proposition 3.4, and \( C \) is embeddable in the tropical curve given by \( \langle g, y - z + 1 \rangle \) in the tropical plane given by \( y - z + 1 \).

Cases \((<<), (<==), (==<))\) can be obtained via two independent modifications from the subdivisions Figure 9, combining the two respective choices for the coefficients from above.

![Figure 9. Subdivisions of type \( \Delta \) that allow to prolong two edges.](image)

Finally, we consider the case \((<<<)\). The previous cases were covered with up to two modifications which were independent of each other. This is no longer possible in the current case, in which we will start with a subdivision as in the center of Figure 10 and require three modifications. The central coefficients \( a_{11}, a_{12} \) and \( a_{21} \) are critical in the sense that they play a role for each of the three pentagons which we use to prolong edges.

Recall that, we assumed \( b \leq c \leq a \). For the current case, we may also assume \( d' = c, e' = b, f' = b \). Consider a quartic \( g = \sum_{i,j} a_{ij} x^i y^j \) where \( a_{03} = a_{13} = 0 \) and
Figure 10. A subdivision that allows to prolong three edges.

the remaining $a_{ij}$ are chosen as follows:

\[
\begin{align*}
    a_{00} &= -t^e - 3t^d - t^f + 3t^a + 2t^b, \\
    a_{10} &= -4t^d - 2t^f - 2t^c + 5t^b, \\
    a_{11} &= -4t^d - 2t^f - 2t^c + 6t^b + 1, \\
    a_{12} &= t^b + 1, \\n    a_{20} &= 2t^e - 4t^d + t^a + 2t^b + 1, \\
    a_{31} &= -2t^d + 2t^c, \\n    a_{21} &= 2t^e - 4t^d + t^a + t^b, \\n    a_{30} &= -2t^a + t^c. 
\end{align*}
\]
Then the modification $z_1 = x + 1$ yields the following coefficients $a'_{ij}$ in $g(z_1 - 1, y)$:

$$
a'_{00} = a_{00} - a_{10} + a_{20} - a_{30} + a_{40} = t^f,
$$

$$
a'_{10} = a_{10} - 2a_{20} + 3a_{30} - 4a_{40} = -2t^f,
$$

$$
a'_{20} = a_{20} - 3a_{30} + 6a_{40} = t^b + 3t^e + 10t^a - 6t^d - 3t^e
$$

$$
a'_{01} = a_{01} - a_{11} + a_{21} - a_{31} = 0.
$$

Moreover, $\text{val}(a'_{02}) = \text{val}(a'_{12}) = \text{val}(a'_{21}) = \text{val}(a'_{21}) = 0$. Thus, the Newton subdivision is as in the left branch of Figure 10, and the length of the prolonged edge is as desired: $f/2 + b/2 + (f - b)/2 = f$.

Furthermore, the modifications $z_2 = y + 1$ resp. $z_0 = x + y$ will yield edges of desired lengths $d$ resp. $e$ in the tropical curves given by $q(x, z_2 - 1)$ resp. $q(z_3 - y, y)$, see the bottom resp. top branches of Figure 10.

### 3.2.3. Type $\bigtriangleup$.

Let $a, b, c, d, e, f$ be the edge lengths of an abstract tropical curve $C$ as in Figure 11. Due to symmetry, we may assume $a \geq c \geq e$. By [9, Theorem 5.1], no curve of type $\bigtriangleup$ is embeddable as a quartic in $\mathbb{R}^2$.

To show that any curve $C$ is embeddable as a quartic in a tropical plane, let $a, b, c, d, e, f$ be any lengths satisfying our assumptions due to symmetry. Consider the following plane quartic, whose Newton subdivision and tropical curve is depicted in Figure 12:

$$
p = t^{6a+b} \cdot y^4 + t^{6c+d} \cdot x^4 + t^{6e+f} + x^2y^2 + 2 \cdot xy^2 + (1 - t^{2a})y^2 + 2 \cdot x^2y + \left(1 - t^{2e}\right) \cdot x^2 - 2 \cdot (1 + t^{2e}) \cdot xy.
$$

We use Proposition 3.5 to unfold three lollis from the edges of weight 2. For the polygon $\text{conv}\{(0,2),(2,2),(0,4)\}$, we can read off immediately that the requirements of Proposition 3.5 are satisfied: $p$ restricted to the corresponding weight 2 edge equals

$$
x^2y^2 + 2xy^2 + (1 - t^{2a}) \cdot y^2 = (y(x + 1))^2 + y^2 \cdot t^{2a} \cdot (-1).
$$

This is a square of $y$ times the linear form $L = x + 1$ up to order $t^{2a}$, and the $t^{2a}$-contribution, $-y^2$, is not divisible by $x + 1$. Thus conditions (1) and (2) of Proposition 3.5 are satisfied. The vertex $V$ is at $(0, 6a + b)$ as requested in condition (3). Finally, the $t^0$-terms of $xy$ and $x^2y$ are $2 \cdot (-1 + x)xy$ which is again not divisible by $x + 1$, so (4) is satisfied. The reasoning for the polygon $\text{conv}\{(2,0),(2,2),(4,0)\}$

![Figure 11. An abstract tropical curve of type $\bigtriangleup$.](image-url)
is symmetric. For the polygon \( \text{conv}\{(2,0),(0,2),(0,0)\} \), we have to divide by 
\((1 - t^{2a}) = \sum_{k \geq 0}(t^{2a})^k \) first to express the polynomial in the form used in Proposition 3.5. To simplify the computation, we assume \( e \leq c \leq a \). If \( e < c \), \( p \) restricted to the corresponding weight 2 edge equals 
\[ y^2 + x^2 - 2 \cdot (1 + t^{2e}) \cdot xy + O(t^{2c}) = (x - y)^2 + t^{2e} \cdot x \cdot (-2y) + O(t^{2c}). \]
If \( e = c < a \), we have 
\[ y^2 + (1 - t^{2e})x^2 - 2 \cdot (1 + t^{2e}) \cdot xy + O(t^{2a}) = (x - y)^2 + t^{2e} \cdot x \cdot (-2y - x) + O(t^{2a}). \]
If \( e = c = a \), we have 
\[ y^2 + x^2 - 2 \cdot (1 + t^{2e})/(1 - t^{2e}) \cdot xy + O(t^{2a+1}) = y^2 + x^2 - 2 \cdot (1 + 2t^{2e}) \cdot xy + O(t^{2a+1}) = (x - y)^2 + t^{2e} \cdot x \cdot (-4y) + O(t^{2a+1}). \]
In any case, the \( t^{2e} \)-contribution is not divisible by \( x - y \). The \( t^0 \)-contribution of \( p \) restricted to \( \text{conv}\{(1,2),(2,1)\} \) equals \( 2 \cdot xy \cdot (y + x) \) which is not divisible by \( x - y \). With three linear modifications in total, we arrive at a model of the tropical plane and a tropicalized quartic defined by \( p \) and the additional linear equations which is faithful on the skeleton and realizes the desired abstract curve.

### 3.2.4. Type \( \infty \infty \infty \infty \).
Let \( a, b, c, d, e, f \) be the edge lengths of a curve \( C \) as in Figure 13. Due to symmetry, we may assume \( c \leq d \). By [9, Theorem 5.1], \( C \) is embeddable as quartic in \( \mathbb{R}^2 \) if \( c < d \leq 2c \).
To show that any curve $C$ is embeddable as a quartic in a tropical plane, let $a, b, c, d, e, f$ be any lengths satisfying our assumptions due to symmetry. Note that curves with $c = d$ are realizably hyperelliptic, which is why they are excluded in our considerations. To construct curves satisfying $d > 2c$, consider the following quartic polynomial whose Newton subdivision and tropical curve as depicted in Figure 14:

$$p = t^{a+6b+2c} \cdot x^4 + t^{6c+f+2c} \cdot y^4 + x^2 y^2 + 2 \cdot x^2 y + (1 - t^{2b}) y^2 + 2 \cdot x^2 y + (1 - t^{2c}) \cdot x^2 + t^{-c} xy + t^{-\frac{d+4c-1}{2}} y + t^{-\frac{d+4c-1}{2}} x + t^{-d+3c}.$$

Figure 14. A tropicalized quartic in $\mathbb{R}^2$ from which we can unfold two lollis.

As in the previous type $\mathfrak{A}$, Proposition 3.5 allows us to independently unfold lollis from the weight 2 edges, using the modification $z_1 = x + 1 - t^b$ and $z_2 = y + 1 - t^e$.

3.2.5. Type $\mathfrak{D} \gg \omega$. Let $(a, b, c, d, e, f)$ be the edge lengths of a curve $C$ as in Figure 15. Due to symmetry, we may assume $a \geq b$ and $c \leq d$. By [9, Theorem 5.1], $C$ is embeddable as quartic in $\mathbb{R}^2$ if $\mathbb{R}^2$ if $b + c < d < b + 3c$.

Figure 15. An abstract and plane tropical curve of type $\mathfrak{D} \gg \omega$.

To show that any curve $C$ is embeddable as a quartic in a tropical plane, let $a, b, c, d, e, f$ be any lengths satisfying our assumptions due to symmetry. Note that curves with $c = d$ are realizably hyperelliptic, which is why they are excluded in our considerations. We distinguish between three cases

1. $d \geq b + 3c$ and $a > b$,
2. $d \geq b + 3c$ and $a = b$,
3. $c < d \leq b + c$. 


If \( d \geq b+3c \) and \( a > b \), we can consider a tropical plane quartic as in Figure 16, where the coefficients responsible for the bounded edge of weight 2 are chosen according to Proposition 3.5. This allows us to unfold a lolli with stick \( e \) and cycle length \( f \).

![Figure 16](image)

**Figure 16.** A curve of type \( \triangleright \triangleright \circ \) with \( d \geq b+3c \), \( a < b \), and a hidden lolli with edge lengths \( e, f \).

If \( d \geq b+3c \) and \( a = b \), we can further degenerate the picture above, letting the edges \( a \) and \( b \) form a double edge of weight 2 which we can unfold for suitably chosen coefficients using [16, Theorem 3.4].

Now assume \( c < d \leq b+c \). Picking \( b' \) such that \( b' + c = d < b' + 3c \), we can consider a tropical plane quartic as in Figure 17, where the coefficients are chosen according to Proposition 3.4. This allows us to unfold an edge of length \( b \).

![Figure 17](image)

**Figure 17.** A curve of type \( \triangleright \triangleright \circ \) with \( c < d \leq b+c \).

### 4. Obstructions for realizably hyperelliptic curves

In this section, we prove Theorem 1.2.

**Lemma 4.1** Assume a faithfully tropicalized quartic \( C \) in a realizable model of the tropical plane satisfies \( \Gamma^{\text{trop}}(C) = \Gamma \in M^\text{trop}_a \). Then the tropical canonical divisor of \( \Gamma \) is very ample, and its sections obtained by intersecting \( C \) with a hyperplane are simultaneously realizable.
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Proof. Since $C$ is a tropicalized quartic in a realizable model of the tropical plane, there exists a smooth algebraic curve $C$ of genus 3 and an embedding $C \to \mathbb{P}^2$ tropicalizing to $C$ in the tropical plane. Here, the second arrow is a linear embedding. After compactifying, the first map yields $\overline{C} \to \mathbb{P}^2$, which is necessarily a canonical embedding. Composing with the map $(\mathbb{K}^*)^2 \to \mathbb{K}^n$ amounts to picking $n$ generators of the canonical linear system. The $n$ coordinate functions of $\overline{C} \subset \mathbb{R}^n$ are thus tropicalizations of sections of the canonical divisor. In particular, the tropical canonical divisor must be very ample, since by faithfulness the map $\text{ft}(C) \to C$ is injective and thus any two points are separated by a section. Also, the sections are realizable by sections of the canonical divisor of $C$, so they are simultaneously realizable. □

Let $\Gamma$ be a trivalent tropical curve and $\chi: \Gamma \to \mathbb{R}$ be a piece-wise linear function such that

$$K_\Gamma + \text{div}(\chi) \geq 0. \quad (1)$$

The following simple observations will be useful in the proofs below:

(a) $\chi$ restricted to any edge is convex, and hence achieves no maximum in the inner points of the edge unless it is constant on this edge;

(b) for any vertex where $\chi$ achieves its global maximum $M_\chi$, the slopes of $\chi$ vanish along at least two of the attached edges due to condition (1). Moreover, $\chi$ must be constant on any such edge by convexity.

For a piece-wise linear function $\chi: \Gamma \to \mathbb{R}$, we denote the slope of $\chi$ at a vertex $V$ along an attached edge $E$ by $\frac{\partial \chi}{\partial E}(V)$.

Lemma 4.2 The canonical divisor of an abstract tropical curve $\Gamma$ of type $\square$, where the edges forming the 2-cut have equal lengths, is not very ample.

Proof. Denote the vertices of $\Gamma$ by $A, B, C, D$ and the edges by $E_a, E_b, E_c, E_d, E_e, E_f$ as in Figure 5. The length of $E_i$ is $i$, and we have $c = d$. Let $\chi: \Gamma \to \mathbb{R}$ be a piece-wise linear function satisfying (1). We shall show that $\chi(A) = \chi(B)$ and $\chi(C) = \chi(D)$, and hence the canonical linear system does not distinguish points, and as a result is not very ample.

Assume without loss of generality that $\chi(A) = M_\chi$. Then $\chi(B) = \chi(A)$ by (b). If $\chi$ is constant on either $E_c$ or $E_d$ then by the same argument $\chi(C) = M_\chi = \chi(D)$. Otherwise $\frac{\partial \chi}{\partial E_c}(A) = \frac{\partial \chi}{\partial E_d}(B) = -1$ by (b) and (1). Assume to the contrary that $\chi(C) \neq \chi(D)$, say $\chi(C) > \chi(D)$. By (a), $\frac{\partial \chi}{\partial E_c}(C), \frac{\partial \chi}{\partial E_f}(C) < 0$, and hence $\frac{\partial \chi}{\partial E_e}(C) \geq 1$ by condition (1). But $\chi$ is convex on $E_c$ and hence linear. Thus, by convexity of $\chi$ on $E_d$, we obtain $\chi(D) \geq \chi(B) - d = \chi(A) - c = \chi(C)$, which is a contradiction. □

Lemma 4.3 The canonical divisor of an abstract tropical curve $\Gamma$ of type $\diamond$, where the edges forming the 2-cut have equal lengths, is not very ample.
Proof. The argument is similar to the previous proof. Let \( \Gamma \) have vertices \( A, B, C \) and \( D \) and edges \( E_i \) of length \( i \) for \( i = a, \ldots, f \) as in Figure 15. Let \( \chi: \Gamma \to \mathbb{R} \) be a piece-wise linear function satisfying (1). We shall show that \( \chi(A) = \chi(B) \), and hence the canonical system is not very ample. First, notice that by convexity, the slopes of \( \chi \) along the loop at \( D \) are non-positive, and hence \( \frac{\partial \chi}{\partial E_e}(D) \geq -1 \). Thus, again by convexity, \( \frac{\partial \chi}{\partial E_e}(C) \leq 1 \). Second, assume to the contrary that \( \chi(A) \neq \chi(B) \), say \( \chi(A) < \chi(B) \). Then \( \frac{\partial \chi}{\partial E_a}(B), \frac{\partial \chi}{\partial E_b}(B) < 0 \) by (a), and hence \( \frac{\partial \chi}{\partial E_d}(B) \geq 1 \). By convexity this implies \( \chi(C) > \chi(B) > \chi(A) \). Thus, again by convexity, \( \frac{\partial \chi}{\partial E_c}(C), \frac{\partial \chi}{\partial E_d}(C) \leq -1 \), and hence \( \frac{\partial \chi}{\partial E_d}(C) = \frac{\partial \chi}{\partial E_c}(C) = -1 \) since the sum of the three slopes at \( C \) must be at least \(-1\). We conclude that \( \chi \) is linear along \( E_d \). Finally, \( \chi(B) = \chi(C) - d = \chi(C) - c \leq \chi(A) \) by convexity of \( \chi \) restricted to \( E_c \), which is a contradiction. \( \square \)

**Remark 4.4** For \( \Gamma \) of type \( \circ \circ \circ \), the canonical divisor is very ample, even if the lengths of the banana edges are equal. It is clear that we can separate points which are not on the banana edges, of the same distance to the vertices. For such two points, we can use functions as depicted in Figure 18.

![Figure 18](image)

**Figure 18.** A function in the linear system of the canonical divisor, separating two points \( P \) and \( Q \). The numbers above indicate edge slopes while the numbers below indicate edge lengths.

Notice that [3, Theorem 10] classifies all graphs with a not very ample canonical divisor, but this result seems to contain a small gap. For that reason, we include the arguments used in genus 3 here.

**Lemma 4.5** Let \( \Gamma = \circ \circ \circ \) with the lengths of the banana edges equal. Then any realizable canonical divisor \( K_\Gamma + \text{div}(\chi) \) satisfies \( \chi|_{E_c} = \chi|_{E_d} \). In particular, points on the banana edges cannot be separated with realizable sections.
Proof. Without loss of generality $\chi(B) \geq \chi(C)$. Thus, $\frac{\partial \chi}{\partial E_c}(B), \frac{\partial \chi}{\partial E_d}(B) \leq 0$ by convexity. If one of the slopes vanishes then $\chi$ is constant on both edges by convexity and condition (ii) in [34, Theorem 6.3]. Assume now that both slopes are negative. An argument identical to the one used in the proof of Lemma 4.3, shows that $\frac{\partial \chi}{\partial E_c}(B) + \frac{\partial \chi}{\partial E_d}(B) \geq -2$, and hence $\frac{\partial \chi}{\partial E_c}(B) = \frac{\partial \chi}{\partial E_d}(B) = -1$. Similarly, $\frac{\partial \chi}{\partial E_c}(C) + \frac{\partial \chi}{\partial E_d}(C) \geq -2$. (2)

Let us identify $E_c \simeq [0, c]$ and $E_d \simeq [0, d]$ such that $B$ is identified with 0. Then there exist $0 = t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 = c$ and $0 = s_0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 = d$ such that $\frac{\partial \chi}{\partial x}|(t_i,t_{i+1}) = \frac{\partial \chi}{\partial x}|(s_i,s_{i+1}) = i-1$. Thus, $\chi(C) = 2t_4 - t_1 - t_2 - t_3 = 2s_4 - s_1 - s_2 - s_3$, and since $c = d$, we obtain

$$t_1 + t_2 + t_3 = s_1 + s_2 + s_3.$$ (3)

By the symmetry of $\Gamma$ we may assume that $t_3 \leq s_3$. If $t_3 < s_3$ then $\frac{\partial \chi}{\partial E_c}(C) = -2$ and $\frac{\partial \chi}{\partial E_d}(C) \geq 0$ by (2). This implies that $s_2 = s_3 = s_4 = d$, and hence $t_4 > s_1$ by (3). But this contradicts condition (ii) in [34, Theorem 6.3]. Thus, $t_3 = s_3$.

Similarly we may assume that $t_1 \leq s_1$. If $t_1 < s_1$ then $t_2 > s_2 \geq s_1 > t_1$ by (3), which again contradicts condition (ii) in [34, Theorem 6.3]. Thus, $t_1 = s_1$, and hence also $t_2 = s_2$ by (3), which completes the proof. □

Proof of Theorem 1.2. Let $\Gamma$ be realizably hyperelliptic. Then, as described in Section 2.1, $\Gamma$ is of type $\varnothing$, $\varnothing \triangleright \varnothing$ or $\varnothing \otimes \varnothing$, where in each case the edges forming the 2-cut have equal lengths. Assume to the contrary that there exists a faithfully tropicalized quartic $C$ in a realizable model of the tropical plane with $\text{ft}(C) = \Gamma$. Then the canonical divisor on $\Gamma$ is very ample by Lemma 4.1. If $\Gamma$ is either of type $\varnothing$ or $\varnothing \triangleright \varnothing$ we get a contradiction to Lemmas 4.2 and 4.3. Thus, $\Gamma$ must be of type $\varnothing \otimes \varnothing$. Although in this case the tropical canonical divisor on $\Gamma$ is tropically very ample, it admits no realizable sections separating the two banana edges by Lemma 4.5. We again obtain a contradiction, which completes the proof. □

Remark 4.6 Our constructive proof of Theorem 1.1 can be viewed as a way to construct simultaneous lifts of sections of the canonical divisor.

Remark 4.7 As a side-product, we can list all maximal abstract tropical curves of genus 3 for which the canonical divisor is not very ample. In [25], the question how to characterize an abstract tropical curve whose canonical divisor is not very ample was posed. In [3], a list classifying all such tropical curves is presented, but there is a small gap in the classification, which is why we repeat the arguments in genus 3 here.
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