On the non-resistive limit and the magnetic boundary-layer for one-dimensional compressible magnetohydrodynamics

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Received 21 September 2016, revised 4 July 2017
Accepted for publication 28 July 2017
Published 17 August 2017

Abstract

We consider an initial-boundary value problem for the one-dimensional equations of compressible isentropic magnetohydrodynamic (MHD) flows. The non-resistive limit of the global solutions with large data is justified. As a by-product, the global well-posedness of the compressible non-resistive MHD equations is established. Moreover, the thickness of the magnetic boundary-layer of the value $O(\nu^{\alpha})$ with $0 < \alpha < 1/2$ is proved, where $\nu > 0$ is the resistivity coefficient. The proofs of these results are based on a full use of the so-called ‘effective viscous flux’, the material derivative and the structure of the equations.

Keywords: compressible MHD equations, non-resistive limit, magnetic boundary-layer, global well-posedness, initial-boundary value problem

Mathematics Subject Classification numbers: 35M10, 35Q60, 76N10, 76N17, 76N20

1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of a conducting fluid in an electromagnetic field and has a very wide range of applications in astrophysics, plasma, and so on. Because the dynamic motion of the fluid and the magnetic field interact on each other, both the hydrodynamic and electrodynamic effects are strongly coupled, and thus, the mathematical
structure and the physical mechanism of MHD flows are considerably complicated. The threedimensional equations for compressible isentropic magnetohydrodynamic flows, derived from fluid mechanics with appropriate modifications to account for electrical forces, read as follows (see [3, 20]):

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \nabla (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) &= \mu \Delta \mathbf{u} + (\mu + \lambda') \nabla \nabla \cdot \mathbf{u} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) &= -\nu \nabla \times (\nabla \times \mathbf{B}), \\
\text{div} \mathbf{B} &= 0
\end{align*}
\]  

(1.1)

with \( \mathbf{x} \in \Omega \subset \mathbb{R}^3 \) and \( t \geq 0 \). Here, the unknown functions \( \rho, \mathbf{u} \in \mathbb{R}^3, P \) and \( \mathbf{B} \in \mathbb{R}^3 \) are the density of the fluid, the velocity, the pressure, and the magnetic field, respectively. The viscosity coefficients \( \mu \) and \( \lambda' \) satisfy the physical conditions \( \mu > 0, 3\lambda' + 2\mu > 0 \). The constant \( \nu > 0 \) is the resistivity coefficient which is inversely proportional to the electrical conductivity constant (the magnetic Reynolds number) and acts as the diffusivity coefficient of the magnetic fields. The pressure \( P(\rho) \) is determined through the equation of state (the so-called \( \gamma \)-law):

\[
P(\rho) \equiv A \rho^\gamma \quad \text{with} \quad A > 0, \gamma > 1.
\]  

(1.2)

Equations (1.1) and (1.2) describe the macroscopic behavior of the electrically conducting compressible (isentropic) fluid in a magnetic field. From equation (1.1), it is clear that the time rate of change of the magnetic field (i.e. \( \mathbf{B}(t) \)) is dominated by both the advection term \( \nabla \times (\mathbf{u} \times \mathbf{B}) \) and the resistive term \( \nu \nabla \times (\nabla \times \mathbf{B}) \). It is well known that the resistivity \( \nu \) is inversely proportional to the electrical conductivity \( \sigma \), and therefore, it is more reasonable to assume that there is no magnetic diffusion (i.e. \( \nu = 0 \)) in many cosmical and geophysical problems where the conducting fluid is of extremely high conductivity (ideal conductors) (see [4, 10]). So, instead of (1.1), the induction equation for the magnetic field in such situations takes the form:

\[
\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0,
\]

which particularly implies that in a highly conducting fluid, the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. This type of behavior is physically expressed as that the magnetic field lines are frozen into the fluid. In effect, the fluid can flow freely along the magnetic field lines, but any motion of the conducting fluid, perpendicular to the field lines, carries them with the fluid. The ‘frozen-in’ nature of magnetic fields plays very important roles and has a very wide range of applications in both astrophysics and nuclear fusion theory, where the magnetic Reynolds number \( R_m \sim 1/\nu \) is usually very high. A typical illustration of the ‘frozen-in’ behavior is the phenomenon of sunspots. For more details of its physical background and applications, we refer to [1–4, 10, 14, 20, 21].

Formally, when \( \nu = 0 \), system (1.1) turns into

\[
\begin{align*}
\bar{\rho}_t + \nabla \cdot (\bar{\rho} \mathbf{u}) &= 0, \\
(\bar{\rho} \mathbf{u})_t + \nabla (\bar{\rho} \mathbf{u} \otimes \mathbf{u}) + \nabla P(\bar{\rho}) &= \mu \Delta \bar{\mathbf{u}} + (\mu + \lambda') \nabla \nabla \cdot \bar{\mathbf{u}} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\mathbf{B}_t - \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}) &= 0, \\
\text{div} \mathbf{B} &= 0
\end{align*}
\]  

(1.3)

where the pressure \( P(\bar{\rho}) \) satisfies the \( \gamma \)-law (1.2). This is often called the compressible, isentropic, viscous and non-resistive MHD equations.

Because of the tight interaction between the dynamic motion and the magnetic field, the presence of strong nonlinearities and the lack of dissipation mechanism of the magnetic field, many physically important and mathematically fundamental problems of system (1.3) are still open. For example, to the authors’ knowledge, there is not any rigorously mathematical
literature on the global well-posedness of the initial (boundary) value problem of (1.3), even when the initial data are close to a constant non-vacuum equilibrium, though the same problem was successfully solved for the compressible Navier–Stokes equations (i.e. $B = 0$) by Matsumura and Nishida [25].

Due to the mathematically complicate structure of the multi-dimensional equations, in this paper we shall consider the simpler one-dimensional equations (see [13]) on a bounded spatial domain $\Omega \equiv (0, 1)$ (without loss of generality):

\[
\begin{aligned}
\rho_t + \rho u_x &= 0, \\
\rho u_t + \rho u u_x + p(u) + \frac{1}{2} b^2 &= \lambda_\text{ext}, \\
b_t + (ub)_x &= \nu b_\text{ext},
\end{aligned}
\]

(1.4)

with the following initial and boundary conditions:

\[
\begin{aligned}
(\rho, u, b)(x, 0) &= (\rho_0, u_0, b_0)(x), & x &\in [0, 1], \\
u(0, t) &= u(1, t) = 0, & b(0, t) &= b_1(t), & b(1, t) &= b_2(t), & t &\geq 0,
\end{aligned}
\]

(1.5)

where the pressure $P(\rho)$ obeys the $\gamma$-law (1.2) and $\lambda = 2\mu + \lambda'$.

Indeed, assume that the MHD flow is moving only in the longitudinal direction $x$ and uniform in the transverse directions ($y, z$). Then it is easy to derive (1.4) from (1.1), based on the specific choice of dependent variables:

\[
\rho = \rho(x, t), \quad u = (u(x, t), 0, 0), \quad B = (0, 0, b(x, t)).
\]

We mention here that the one-dimensional equations for the compressible, heat-conducting, viscous, resistive MHD flows in a form similar to that in (1.4) were studied by Kazhikhov–Smagulov [19], where the global well-posedness of solutions was announced.

Analogously, the compressible, viscous, non-resistive MHD equations (1.3) in dimension one can be written as follows:

\[
\begin{aligned}
\rho_t + \rho u_x &= 0, \\
\rho u_t + \rho u u_x + p(\rho) + \frac{1}{2} b^2 &= \lambda_\text{ext}, \\
b_t + (ub)_x &= \nu b_\text{ext},
\end{aligned}
\]

(1.6)

which is completed with the following initial and boundary conditions:

\[
\begin{aligned}
(\rho, u, b)(x, 0) &= (\rho_0, u_0, b_0)(x), & x &\in [0, 1], \\
u(0, t) &= u(1, t) = 0, & b(0, t) &= b_1(t), & b(1, t) &= b_2(t), & t &\geq 0.
\end{aligned}
\]

(1.7)

**Remark 1.1.** It is worth pointing out that system (1.6) looks similar to the compressible model for gas and liquid two-phase fluids (see, for example, [6, 30]). However, to prove the global well-posedness of the two-phase model, it is technically assumed in [6, 30] that the proportion between the mass of gas and liquid must be bounded, in analogy to the assumptions that $\dot{\rho}, \dot{b} \geq 0$ and $0 \leq b/\rho < \infty$ for (1.6) (see [7]), which implies that the magnetic field $b$ is bounded, provided the density $\rho$ is bounded. Of course, this is somewhat reasonable for the two-phase model, but not physical and realistic in magnetohydrodynamics. To our best knowledge, the global well-posedness of (1.6) and (1.7) with generally large data is still unknown.

The first purpose of this paper is to show the global well-posedness of strong solutions of the initial-boundary value problem (1.6) and (1.7) with generally large data, and to justify the non-resistive limit (i.e. $\nu \to 0$) from the problem (1.4) and (1.5) to the problem (1.6) and (1.7) rigorously.
Theorem 1.1.

(i) Assume that \( P(\rho) = A\rho^\gamma \) with \( A > 0 \) and \( \gamma > 1 \), and that the initial data \((\rho_0, u_0, b_0)\) satisfy
\[
\inf_{0 \leq s \leq 1} \rho_0(x) > 0, \quad (\rho_0, b_0) \in H^1, \quad u_0 \in H^2_0, \quad u_0(0) = u_0(1) = 0.
\]
(1.8)

Then for any \( 0 < T < \infty \), there exist a positive constant \( C \) and a unique global strong solution \((\bar{\rho}, \bar{u}, \bar{b})\) to the initial-boundary value problem (1.6) and (1.7) on \((0, 1) \times [0, T]\), such that
\[
0 < C^{-1} \leq \bar{\rho}(x, t) \leq C < \infty, \quad \forall (x, t) \in [0, 1] \times [0, T],
\]
(1.9)
and
\[
\begin{cases}
(\bar{\rho}, \bar{b}) \in L^\infty(0, T; H^1), & (\bar{\rho}_x, \bar{b}_t) \in L^\infty(0, T; L^2),
\\
\bar{u} \in L^\infty(0, T; H^0_0), & (\bar{u}_x, \bar{u}_{xx}) \in L^2(0, T; L^2).
\end{cases}
\]
(1.10)

(ii) Assume that \( P(\rho) = A\rho^\gamma \) with \( A > 0 \) and \( \gamma > 1 \). Moreover, in addition to (1.8), suppose that for any given \( T \in (0, \infty) \),
\[
(b_1, b_2) \in C^1([0, T]), \quad b_0(0) = b_1(0), \quad b_0(1) = b_2(0).
\]
(1.11)

Then for each fixed \( \nu > 0 \), there exist a positive constant \( C \) and a unique global strong solution \((\rho, u, b)\) to the initial-boundary value problem (1.4) and (1.5) on \((0, 1) \times [0, T]\), such that
\[
0 < C^{-1} \leq \rho(x, t) \leq C < \infty, \quad \forall (x, t) \in [0, 1] \times [0, T],
\]
(1.12)
and
\[
\sup_{0 \leq s \leq T} \left( \|u_s\|^2_{L^2} + \|b\|^2_{L^\infty} + \nu^{1/2}\|\rho_x\|^2_{L^2} + \nu^{1/2}\|b_x\|^2_{L^2} \right) (t)
+ \int_0^T \left( \|u_s\|^2_{L^2} + \nu^{1/2}\|u_{xx}\|^2_{L^2} + \nu^{3/2}\|b_{xx}\|^2_{L^2} \right) dt \leq C.
\]
(1.13)

Moreover, as \( \nu \to 0 \), one has
\[
\begin{cases}
(\rho, u, b) \to (\bar{\rho}, \bar{u}, \bar{b}) \quad \text{strongly in} \quad L^\infty(0, T; L^2),
\\
\nu b_x \to 0, \quad u_x \to \bar{u}_x \quad \text{strongly in} \quad L^2(0, T; L^2),
\end{cases}
\]
and
\[
\sup_{0 \leq s \leq T} \left( \|\rho - \bar{\rho}\|^2_{L^2} + \|b - \bar{b}\|^2_{L^2} + \|u - \bar{u}\|^2_{L^2} \right) (t)
+ \int_0^T \|(u - \bar{u})_x\|^2_{L^2} dt \leq C\nu^{1/2}.
\]
(1.14)

Here, \( C > 0 \) is a positive constant independent of \( \nu \).

The second and main purpose of this paper is to study the effects of magnetic boundary layer as the resistivity coefficient \( \nu \to 0 \). In fact, when the resistivity goes to zero, the parabolic equation (1.4) turns into the hyperbolic equation (1.6), and moreover, the boundaries become characteristic due to the non-slip boundary conditions. Thus, it follows from the
classical theory in [26] that the boundary conditions of the magnetic field for (1.6) (see (1.5)) should be dropped. So, due to the disparity of boundary conditions, we cannot expect that as \( \nu \to 0 \), the solution of the problem (1.4) and (1.5) will tend to the one of the problem (1.6) and (1.7) uniformly up to the boundaries \( x = 0,1 \). In other words, the phenomena of (magnetic) boundary layer appears near the boundaries.

Similar to the relations among the Euler, Navier–Stokes and Prandtl equations (see, for example, [22, 27, 28]), it is expected that as \( \nu \to 0 \), the solution of the problem (1.4) and (1.5) converges uniformly to the solution of the problem (1.6) and (1.7) away from the boundaries, while there is a sharp change of gradient in a thin layer near the boundary. Inspired by this, we introduce the following concept of magnetic boundary-layer thickness (MBL-thickness).

\[ \begin{align*}
\lim_{\nu \to 0} \| (\rho - \bar{\rho}, u - \bar{u}, b - \bar{b}) \|_{L^\infty(0,T;C(\overline{\Omega})))} &= 0, \\
\lim \inf_{\nu \to 0} \| (\rho - \bar{\rho}, u - \bar{u}, b - \bar{b}) \|_{L^\infty(0,T;C(\overline{\Omega}))} &> 0,
\end{align*} \]  
(1.15)

where \((\rho, u, b)\) and \((\bar{\rho}, \bar{u}, \bar{b})\) are the solutions of the problems (1.4)–(1.7), respectively, and \( \Omega_\delta = \{ x \in \Omega | \delta < x < 1 - \delta \} \).

The concept of the boundary-layer thickness (BL-thickness) has been introduced in [11] for the one-dimensional cylindrical compressible isentropic Navier–Stokes equations at small shear viscosity, and later in [17] for the cylindrical compressible heat-conducting Navier–Stokes equations. The BL-thickness for the scalar conservation laws and the 2D Boussinesq equations with vanishing viscosity/diffusivity was also studied in [12] and [18], respectively. It is worth mentioning that definition 1.1 does not determine the MBL-thickness uniquely, since any function \( \delta_1(\nu) \), satisfying \( \delta_1(\nu) \geq \delta(\nu) \) and \( \delta_1(\nu) \downarrow 0 \) as \( \nu \downarrow 0 \), is also a MBL-thickness. Thus, there should exist a minimal MBL-thickness \( \delta_*(\nu) \) which may be viewed as the true MBL-thickness.

In this paper, we shall prove that a function \( \delta_n(\nu) = \nu^{1/2 - 1/n} \) with \( n > 2 \) is a MBL-thickness in the sense of definition 1.1. This is somewhat in agreement with the famous Stokes–Blasius law in the laminar boundary layer theory (see [28]), since \( \lim \inf_{n \to \infty} \delta_n(\nu) = \nu^{1/2} \). More precisely, we shall prove

**Theorem 1.2.** In addition to (1.8) and (1.11), assume further that \((\rho_0, u_0, b_0) \in H^2\). Then any function \( \delta(\nu) \in (0,1/2) \), satisfying

\[ \delta(\nu) \to 0 \quad \text{and} \quad \frac{\delta(\nu)}{\nu^{1/2}} \to \infty, \quad \text{as} \quad \nu \to 0, \]  
(1.16)

is a MBL-thickness in the sense of definition 1.1 such that

\[ \lim_{\nu \to 0} \left( \| \rho - \bar{\rho} \|_{L^\infty(0,T;C(\overline{\Omega})))}^2 + \| b - \bar{b} \|_{L^\infty(0,T;C(\overline{\Omega})))}^2 \right) = 0 \]  
(1.17)

and

\[ \lim \inf_{\nu \to 0} \left( \| \rho - \bar{\rho} \|_{L^\infty(0,T;C(\overline{\Omega})))}^2 + \| b - \bar{b} \|_{L^\infty(0,T;C(\overline{\Omega})))}^2 \right) > 0, \]  
(1.18)

provided \( b_i(t) \neq \bar{b}_i(t) \) \((i = 1, 2)\), where \( \bar{b}_1(t) \) and \( \bar{b}_2(t) \) denote the boundary values of \( \bar{b}(x,t) \) on the boundaries \( x = 0,1 \), respectively.
Remark 1.2. It readily follows from (1.17) and (1.18) that there is no boundary layer effect on the velocity in the sense of definition 1.1. Indeed, due to (1.10), (1.13) and (1.14) and Sobolev’s inequality, one has as $\nu \to 0$,
\[
\| (u - \bar{u}(t)) \|_{L^2}^2 \leq C \left( \| (u - \bar{u}(t)) \|_{L^2}^2 + \| (u - \bar{u}) \|_{L^2}^2 \right) \\
\leq C \left( \nu^{1/2} + \nu^{1/4} \right) \leq C \nu^{1/4} \to 0, \quad \forall t \in [0, T]. \tag{1.19}
\]

The proofs of theorems 1.1 and 1.2 will be done respectively in sections 3 and 4, based on the global (uniform-in-$\nu$) a priori estimates of the solutions established in section 2. The key issue here is to exclude the presence of vacuum. To do this, as it was indicated in [29] that the key step is to prove the global (uniform) estimate $\| u_0 \|_{L^1(0, T)}$, which is difficult to achieve directly, due to the lack of smooth mechanism of the magnetic field.

Motivated by the theory of the multi-dimensional Navier–Stokes/MHD equations (see, for example, [9, 15, 16, 23, 24]), to prove the global $\nu$-independent estimates, it is more convenient to work with the so-called ‘effective viscous flux’ $F$ and the material derivative $\dot{u}$ defined as follows:
\[
F = \lambda u_x - P(\rho) - \frac{b^2}{2} \quad \text{and} \quad \dot{u} = u_t + u u_x. \tag{1.20}
\]

It turns out that $F = F(x, t)$ possesses more regularity than the $(x, t)$-derivatives of the velocity. Indeed, noting that $F_x = \rho \dot{u}$ due to (1.4), we can utilize (1.20) to improve the integrability of the magnetic field and to obtain the key estimates of $\| u_x \|_{L^\infty(0, T, L^2)}$ and $\| \rho^{1/2} \dot{u} \|_{L^1(0, T, L^2)}$ (see lemma 2.4), which particularly imply that $\| F \|_{L^\infty} \in L^2(0, T)$. This, together with the fact that $u_x = \lambda^{-1}(F + P(\rho) + b^2/2)$, enables us to prove the boundedness of the magnetic field and the lower bound of the density (see lemma 2.5).

The justification of the non-resistive limit and the analysis of magnetic boundary layer need more elaborate estimates. However, unlike that in [11, 17], due to the presence of magnetic boundary layer, it is difficult to derive the uniform estimates of the first-order derivative of the density and the second-order derivative of the velocity in some $L^p$-norm with $p \geq 1$. Indeed, we only have
\[
\nu^{1/2} \sup_{0 \leq x \leq 1} \left( \| \rho_x \|_{L^2}^2 + \| b_x \|_{L^2}^2 \right) + \nu^{3/2} \int_0^T \| b_{xx} \|_{L^2}^2 \, dt \leq C, \tag{1.21}
\]
since the hydrodynamic motion and the magnetic filed interact strongly on each other. Moreover, for some technical reasons we shall make use of the weighted $L^2$-method to deal with the MBL-thickness, instead of the weighted $L^1$-method used for the compressible Navier–Stokes equations (see [11, 17]). In fact, based on (1.21), we can prove that for all $t \in [0, T]$,
\[
\| \eta(\rho - \bar{\rho})_x \|_{L^2}^2 + \| \eta(b - \bar{b})_x \|_{L^2}^2 \leq C \nu^{1/2} \quad \text{with} \quad \eta(x) \equiv x(1-x),
\]
which, together with (1.14), yields the desired result stated in (1.17). This particularly indicates that the strong $H^1$-convergence will take place in the inner domain strictly away from the boundary layer, whose width is of the order $O(\nu^\alpha)$ with $0 < \alpha < 1/2$.

2. Global $\nu$-independent estimates for (1.4) and (1.5)

The global existence of strong solutions of (1.4) and (1.5) can be shown in a similar manner (indeed much easier) as that used in [5, 31] by combining the standard local existence result and the global a priori estimates. Thus, the main purpose of this section is to derive the global
ν-independent estimates of the solution \((ρ, u, b)\) to the problem \((1.4) and (1.5)\), which will be used to justify the non-resistive limit. To do this, we assume that \((ρ, u, b)\) is a smooth solution of \((1.4) and (1.5)\) defined on \((0, 1) \times [0, T]\) with \(0 < T < \infty\). For simplicity, we also denote by \(C_i (i = 1, 2, \ldots)\) the generic positive constant, which may depend on \(λ, A, γ, T\), the norms of the initial data \((ρ₀, u₀, b₀)(x)\) and boundary data \((b₁, b₂)(t)\), but is independent of \(ν\).

First, to deal with the boundary terms induced by integration by parts, we need the following formulas of \(b₁\) on the boundaries \(x = 0, 1\).

**Lemma 2.1.** Let \((ρ, u, b)\) be a smooth solution of \((1.4) and (1.5)\) on \((0, 1) \times [0, T]\). Then,

\[
νb_x(0, t) = ν[b₂(t) − b₁(t)] − ∂_xb_0^x b(ξ, t)dξ − ∫_{0}^{1} (ub)(x, t)dx, \tag{2.1}
\]

\[
νb_x(1, t) = ν[b₂(t) − b₁(t)] + ∂_x∫_{0}^{1} (b(ξ, t)dξ − ∫_{0}^{1} (ub)(x, t)dx. \tag{2.2}
\]

**Proof.** On one hand, integrating \((1.4)_{3}\) over \((0, x)\) and using the non-slip boundary condition \(u|_{x=0} = 0\), we deduce

\[
νb_x(0, t) = νb_x − ∂_x∫_{0}^{x} b(ξ, t)dξ − ub, \tag{2.3}
\]

and hence, after integrating \((2.3)\) with respect to \(x\) over \((0, 1)\), we obtain \((2.1)\).

On the other hand, integrating \((1.4)_{3}\) over \((x, 1)\), we have from the non-slip boundary condition \(u|_{x=1} = 0\) that

\[
νb_x(1, t) = νb_x + ∂_x∫_{x}^{1} b(ξ, t)dξ − ub. \tag{2.4}
\]

Thus, integrating \((2.4)\) with respect to \(x\) over \((0, 1)\) immediately leads to \((2.2)\). \(□\)

With the help of lemma 2.1, we can prove the following elementary energy estimates.

**Lemma 2.2.** Let \((ρ, u, b)\) be a smooth solution of \((1.4), (1.5)\) on \((0, 1) \times [0, T]\). Then,

\[
0 < ∫_{0}^{1} ρ(x, t)dx = ∫_{0}^{1} ρ₀(x)dx < ∞, \quad ∀t \in [0, T], \tag{2.5}
\]

and

\[
\sup_{0 ≤ t ≤ T} ∫_{0}^{1} \left( \frac{1}{2} ρu^2 + \frac{1}{2} b^2 + \frac{A}{γ - 1} ρ \right) (x, t)dx + ∫_{0}^{T} \left( λ||u_x||_{L²}^2 + ν||b_x||_{L²}^2 \right) dt ≤ C. \tag{2.6}
\]

**Proof.** First, the conservation of the mass stated in \((2.5)\) follows immediately form \((1.4)_{1}\) and the non-slip boundary conditions.

Next, to prove \((2.6)\), multiplying \((1.4)_{2}, (1.4)_{3}\) by \(u\) and \(b\) in \(L²\) respectively, and integrating by parts over \((0, 1) \times (0, t)\), by virtue of \((1.4)_{1}\) and lemma 2.1 we deduce...
\[
\begin{align*}
\int_0^t \left( \frac{1}{2} \rho u^2 + \frac{1}{2} b^2 + \frac{A}{\gamma - 1} \rho \right) (x, t) \, dx + \int_0^t \left( \lambda \|u_x\|^2_{L^2} + \nu \|b_x\|^2_{L^2} \right) \, ds \\
\leq C + \int_0^t \nu \left( b_2(s)b_x(1, s) - b_1(s)b_x(0, s) \right) \, ds \\
\leq C + C \int_0^t |b(x, t)| \, dx + C \int_0^t \left( |b_2'| |b| + |b'| |b| + |u| |b| \right) \, dx \, ds \\
\leq C + \frac{1}{4} \int_0^t |b(x, t)|^2 \, dx + C \int_0^t \left( 1 + \|u\|_{L^\infty} \right) \|b\|_{L^2} \, ds \\
\leq C + \frac{1}{4} \int_0^t |b(x, t)|^2 \, dx + \frac{\lambda}{2} \int_0^t \|u_x\|^2_{L^2} \, ds + C \int_0^t \|b\|^2_{L^2} \, ds,
\end{align*}
\]

where we have also used Hölder’s, Sobolev’s and Cauchy–Schwarz’s inequalities. Thus, an application of Gronwall’s inequality leads to (2.6).

The upper bound of the density can be shown in the same manner as that in [8].

**Lemma 2.3.** Let \((\rho, u, b)\) be a smooth solution of (1.4) and (1.5) on \((0, 1) \times [0, T]\). Then,

\[0 \leq \rho(x, t) \leq C, \quad \forall (x, t) \in [0, 1] \times [0, T].\]

**Proof.** The non-negativity of the density (i.e. \(\rho \geq 0\)) follows directly from the method of characteristics and the fact that \(\rho_0 > 0\). For completeness, we sketch the proof of the upper bound below. Define

\[
\psi(x, t) \triangleq \int_0^t \left( \lambda u_x - \rho u^2 - P(\rho) - \frac{b_x^2}{2} \right) (x, s) \, ds + \int_0^t (\rho_0 u_0)(\xi) \, d\xi.
\]

It follows from (1.4)1 and (1.4)2 that

\[
\psi_a = \rho u, \quad \psi_t = \lambda u_x - \rho u^2 - P(\rho) - \frac{b_x^2}{2}, \quad \psi|_{s=0} = \int_0^x (\rho_0 u_0)(\xi) \, d\xi,
\]

and hence, using lemma 2.2 and the non-slip boundary condition \(u|_{x=0, 1} = 0\), one has

\[
\left| \int_0^t \psi(x, t) \, dx \right| \leq C \quad \text{and} \quad \|\psi\|_{L^\infty(0, T; L^2)} \leq C,
\]

which particularly implies that

\[
\|\psi\|_{L^\infty(0, T; L^\infty)} \leq \left| \int_0^t \psi(x, t) \, dx \right| + \|\psi\|_{L^\infty(0, T; L^2)} \leq C.
\]

Let \(D_t \triangleq \partial_t + u \partial_x\) be the material derivative, and set

\[
\Phi(x, t) \triangleq \exp \left\{ \frac{\psi(x, t)}{\lambda} \right\}.
\]
Then, by straightforward calculations we have

\[ D_t(\rho\Phi) = \partial_t(\rho\Phi) + u\partial_x(\rho\Phi) = -\frac{1}{\lambda} \left( P(\rho) + \frac{b^2}{2} \right) \rho\Phi \leq 0, \]

and consequently,

\[ \|(\rho\Phi)(t)\|_{L^\infty} \leq \|(\rho\Phi)(0)\|_{L^\infty} \leq C. \]

This, combined with the fact that \( C^{-1} \leq \Phi(x,t) \leq C \) for all \((x,t) \in [0,1] \times [0,T]\) due to (2.10), immediately yields an upper bound of the density. The proof of (2.7) is thus complete.

The lower bound of the density, depending strongly on the \( L^\infty \)-estimates of both the velocity and the magnetic field, is more difficult to achieve, compared with the upper one. As aforementioned, to circumvent the difficulties induced by the lack of dissipation mechanism of the magnetic field and the strong coupling between the dynamic motion and the magnetic field, we introduce the so-called ‘effective viscous flux’ \( F \) and the material derivative \( \dot{u} \):

\[ F(x,t) = \left( \lambda u_t - P(\rho) - \frac{b^2}{2} \right)(x,t) \quad \text{and} \quad \dot{u}(x,t) = (u_t + uu_x)(x,t). \]  

(2.12)

Based on a full use of the ‘effective viscous flux’ \( F \), the material derivative \( \dot{u} \) and the mathematical structure of the equations, we can prove the following key estimates.

**Lemma 2.4.** Let \((\rho, u, b)\) be a smooth solution of (1.4) and (1.5) on \((0,1) \times [0,T]\). Then,

\[
\begin{align*}
\sup_{0 \leq t \leq T} \left( \|u_t(t)\|_{L^2}^2 + \|b(t)\|_{L^4}^4 + \nu\|b_x(t)\|_{L^2}^2 \right) \\
+ \int_0^T \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|b\|_{L^4}^6 + \nu\|bb_x\|_{L^2}^2 + \nu^2\|b_{xx}\|_{L^2}^2 \right) dt \leq C.
\end{align*}
\]

(2.13)

**Proof.** First, multiplying (1.4) by \( \dot{u} \) in \( L^2 \) and integrating by parts, we obtain

\[
\frac{\lambda}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 = -\int_0^1 \left( P(\rho) + \frac{b^2}{2} \right) \dot{u} dx - \frac{\lambda}{2} \int_0^1 u_t^2 dx.
\]

(2.14)

To deal with the first term on the right-hand side of (2.14), we infer from (1.4) that

\[
P(\rho)_t + uP(\rho)_x + \gamma P(\rho)u_x = 0 \quad \text{with} \quad P(\rho) = A\rho^\gamma,
\]

(2.15)

and hence, recalling the definition of \( \dot{u} \) and integrating by parts, we find

\[
-\int_0^1 P(\rho) \dot{u} dx = \int_0^1 (P(\rho)u_t - P(\rho)_x u_x) dx \\
= \frac{d}{dt} \int_0^1 P(\rho) u_x dx - \int_0^1 (P(\rho)_x u_t + P(\rho)_x u_x) dx \\
= \frac{d}{dt} \int_0^1 P(\rho) u_x dx + \gamma \int_0^1 P(\rho) u_x^2 dx.
\]

(2.16)

Similarly, due to (1.4)_3, one has
\[ (b^2)_t + u(b^2)_x + 2b^2 u_x = 2\nu bb_{xx}, \quad (2.17) \]

so that,

\[ -\frac{1}{2} \int_0^1 (b^2)_x \, dx = \frac{1}{2} \int_0^1 (b^2 u_x - (b^2)_x u_x) \, dx = \frac{1}{2} \int_0^1 \left( (b^2)_x u_x + (b^2)_x u_x \right) \, dx = -\frac{1}{2} \int_0^1 b^2 u_x \, dx - \frac{1}{2} \int_0^1 (\nu bb_{xx} u_x - b^2 u_x^2) \, dx. \quad (2.18) \]

Thus, putting (2.16) and (2.18) into (2.14) and using Cauchy–Schwarz’s inequality, we have

\[
\frac{\lambda}{2} \frac{d}{dt} \|u_k\|_{L^2}^2 + \|u^{1/2} u\|_{L^2}^2 - \frac{d}{dt} \int_0^1 \left( P(\rho) + \frac{b^2}{2} \right) u_x \, dx = \int_0^1 \left( \gamma P(\rho) u_x^2 + b^2 u_x^2 - \nu bb_{xx} u_x - \frac{\lambda}{2} u_x^4 \right) \, dx \leq \frac{\nu^2}{2} \|u_x\|_{L^2}^2 + C \left( 1 + \|u_k\|_{L^2}^2 + \|b\|_{L^2}^6 \right) \leq \frac{\nu^2}{2} \|u_x\|_{L^2}^2 + C \left( 1 + \|F\|_{L^2}^3 + \|b\|_{L^2}^6 \right), \quad (2.19) \]

where we have also used (2.7) and the fact that (due to (2.12)_1)

\[ \|u_k\|_{L^2}^2 \leq C \left( 1 + \|F\|_{L^2}^3 + \|b\|_{L^2}^6 \right). \quad (2.20) \]

Secondly, multiplying (1.4) by $4b^3$ and integrating by parts, by (2.12) we have

\[
\frac{d}{dt} \|b\|_{L^2}^4 + 12\nu \|bb_{x}\|_{L^2}^2 = -3 \int_0^1 u_x b^4 \, dx + 4\nu \left[ b_2^2(t) b_x(1, t) - b_1^2(t) b_x(0, t) \right] = -\frac{3}{\lambda} \int_0^1 \left( F + P(\rho) + \frac{b^2}{2} \right) b^4 \, dx + 4\nu \left[ b_2^2(t) b_x(1, t) - b_1^2(t) b_x(0, t) \right] = -\frac{3}{\lambda} \int_0^1 b^4 \, dx - \frac{3}{\lambda} \int_0^1 \left( F + P(\rho) \right) b^4 \, dx + 4\nu \left[ b_2^2(t) b_x(1, t) - b_1^2(t) b_x(0, t) \right], \]

and thus, it follows from (2.7) and Cauchy–Schwarz’s inequality that

\[
\frac{d}{dt} \|b\|_{L^2}^4 + \|b\|_{L^2}^6 + \nu \|bb_{xx}\|_{L^2}^2 \leq C \left( 1 + \|F\|_{L^2}^3 + |R_1(t)| \right), \quad (2.21) \]

where $R_1(t)$ denotes the boundary term:

\[ R_1(t) \triangleq 4\nu \left[ b_2^2(t) b_x(1, t) - b_1^2(t) b_x(0, t) \right]. \]

Next, multiplying (1.4) by $2\nu b_{xx}$ in $L^2$, integrating by parts, and using the non-slip boundary conditions $u|_{x=0} = 0$, we know that
\[ \nu \frac{d}{dt} \|b_x\|^2_{L^2} + 2\nu^2 \|b_{xx}\|^2_{L^2} = 2\nu \int_0^1 (u_x b + ub_x) b_{xx} \, dx + 2\nu [b^2_x(t)b_x(1,t) - b'_x(t)b_x(0,t)] \]
\[ = \nu \int_0^1 (2u_x b_{xx} - u_x b^2_x) \, dx + 2\nu [b^2_x(t)b_x(1,t) - b'_x(t)b_x(0,t)] . \] (2.22)

In view of the non-negativity of \( P(\rho) \) and \( b^2 \), by (2.12) we observe that
\[ - \int_0^1 u_x b_x^2 \, dx = -\frac{1}{\lambda} \int_0^1 \left( F + P(\rho) + \frac{b^2}{2} \right) b_x^2 \, dx \leq -\frac{1}{\lambda} \int_0^1 F b_x^2 \, dx , \]
which, inserted into (2.22) and combined with (2.20) and Cauchy–Schwarz’s inequality, yields
\[ \nu \frac{d}{dt} \|b_x\|^2_{L^2} + \nu^2 \|b_{xx}\|^2_{L^2} \leq C \left( \|u_x\|^3_{L^2} + \|b\|^6_{L^6} + \nu \|F\|_{L^\infty} \|b_x\|^2_{L^2} \right) + R_2(t) \]
\[ \leq C \left( 1 + \|F\|^3_{L^2} + \|b\|^6_{L^6} + \nu \|F\|_{L^\infty} \|b_x\|^2_{L^2} \right) + R_2(t) , \] (2.23)

where \( R_2(t) \) is the boundary term:
\[ R_2(t) \triangleq 2\nu [b^2_x(t)b_x(1,t) - b'_x(t)b_x(0,t)] . \]

By (2.7) and Cauchy–Schwarz’s inequality, we notice that
\[ \left| \int_0^1 \left( P(\rho) + \frac{b^2}{2} \right) u_x \, dx \right| \leq \frac{\lambda}{4} \|u_x\|^2_{L^2} + C \left( 1 + \|b\|^4_{L^2} \right) . \] (2.24)

Hence, taking (2.24) into account, multiplying (2.21) by a suitably large number \( K > 0 \), and adding the resulting inequality, (2.19) and (2.23) together, we obtain after integrating the resulting relation over \((0, t)\) that
\[ \left( \|u_x\|^2_{L^2} + \|b\|^4_{L^2} + \nu \|b_x\|^2_{L^2} \right) (t) \]
\[ + \int_0^t \left( \|\rho^{1/2} u\|^2_{L^2} + \|b\|^6_{L^6} + \nu \|b_{xx}\|^2_{L^2} + \nu^2 \|b_{xx}\|^2_{L^2} \right) \, ds \]
\[ \leq C + C \int_0^t (\|F\|^3_{L^2} + \nu \|F\|_{L^\infty} \|b_x\|^2_{L^2} + |R_1| + |R_2|) \, ds . \] (2.25)

For the boundary terms \( R_1, R_2 \), we can make use of the Sobolev’s and Young’s inequalities to get that for any \( 0 \leq t \leq T \),
\[ |R_1(t)| + |R_2(t)| \leq C \nu \|b_x\|_{L^\infty} \leq C \nu \left( \|b_x\|_{L^2} + \|b_x\|_{L^2}^{1/2} \|b_{xx}\|_{L^2}^{1/2} \right) \]
\[ \leq \varepsilon \nu^2 \|b_{xx}\|^2_{L^2} + C(\varepsilon) \left( \nu \|b_x\|^2_{L^2} + \nu^{1/2} \right) , \] (2.26)

which, inserted into (2.25), gives (choosing \( \varepsilon > 0 \) small enough)
\[
\left(\|u_t\|_{L^2}^2 + \|b\|_{L^2}^4 + \nu\|b_x\|_{L^2}^2\right)(t)
+ \int_0^t \left(\|\rho^{1/2}u\|_{L^2}^2 + \|b\|_{L^2}^6 + \nu\|bbx\|_{L^2}^2 + \nu^2\|b_{xx}\|_{L^2}^2\right)\,ds
\leq C + C\nu^2 \int_0^t \|b_{xx}\|_{L^2}^2\,ds + C \int_0^t (\|F\|_{L^2}^1 + \|F\|_{L^\infty}^2)\,ds,
\] (2.27)

where we have also used (2.6) and Cauchy–Schwarz’s inequality.

Due to (2.6) and (2.7), one has
\[
\|F\|_{L^2} \leq C (1 + \|u_t\|_{L^2} + \|b\|_{L^2}^2) \leq C (1 + \|u_t\|_{L^2}),
\] (2.28)
\[
\|F\|_{L^2} \leq C (1 + \|u_t\|_{L^2} + \|b^2\|_{L^2}) \leq C (1 + \|u_t\|_{L^2} + \|b\|_{L^2}^2).
\] (2.29)

Moreover, it follows from (1.4)\(2\) and (1.2) that \(F_x = \rho\dot{u}\), and hence, by (2.7) we have
\[
\|F_x\|_{L^2} \leq C\|\rho\dot{u}\|_{L^2} \leq C\|\rho^{1/2}u\|_{L^2}.
\] (2.30)

Using (2.28)–(2.30), Sobolev’s and Cauchy–Schwarz’s inequalities, we obtain
\[
\|F\|_{L^2}^3 + \|F\|_{L^\infty}^2 \leq C (1 + \|F\|_{L^2}) \left(\|F\|_{L^2}^2 + \|F\|_{L^2} \|F_x\|_{L^2}\right)
\leq C (1 + \|u_t\|_{L^2})(1 + \|u_t\|_{L^2} + \|b\|_{L^2}^2)
+ C (1 + \|u_t\|_{L^2})(1 + \|u_t\|_{L^2} + \|b\|_{L^2}^2)\|\rho^{1/2}u\|_{L^2}
\leq \varepsilon\|\rho^{1/2}u\|_{L^2}^2 + C(\varepsilon) \left(1 + \|u_t\|_{L^2}^2\right)(1 + \|u_t\|_{L^2}^2 + \|b\|_{L^2}^2).
\] (2.31)

Thus, substituting (2.31) into (2.27) and choosing \(\varepsilon > 0\) small enough, we get that
\[
\left(\|u_t\|_{L^2}^2 + \|b\|_{L^2}^4 + \nu\|b_x\|_{L^2}^2\right)(t)
+ \int_0^t \left(\|\rho^{1/2}u\|_{L^2}^2 + \|b\|_{L^2}^6 + \nu\|bbx\|_{L^2}^2 + \nu^2\|b_{xx}\|_{L^2}^2\right)\,ds
\leq C + C \int_0^t (1 + \|u_t\|_{L^2}^2 + \nu\|b_x\|_{L^2}^2)\left(1 + \|u_t\|_{L^2}^2 + \|b\|_{L^2}^4 + \nu\|b_x\|_{L^2}^2\right)\,ds.
\]

from which and Gronwall’s inequality, we immediately obtain the desired estimate (2.13), since it holds that \(\|u_t\|_{L^2} + \nu\|b_x\|_{L^2} \in L^1(0, T)\) due to (2.6).

With the help of (2.13), we can now prove the boundedness of the magnetic field and the lower bound of the density.

**Lemma 2.5.** Let \((\rho, u, b)\) be a smooth solution of (1.4) and (1.5) on \((0, 1) \times [0, T]\). Then,
\[
\sup_{0 \leq t \leq T} \left(\|b(t)\|_{L^\infty} + \|\rho^{-1}(t)\|_{L^\infty}\right) + \int_0^T \left(\|u_t\|_{L^2} + \|u_t\|_{L^\infty}^2\right)\,dt \leq C.
\] (2.32)

**Proof.** Multiplying (1.4) by \(2nb^2\) with \(1 \leq n \in \mathbb{N}\) and integrating by parts over \((0, 1)\), by (2.12) there exists a positive constant \(C\), independent of \(n\) and \(\nu\), such that (noting that \(P(\rho), b^2\) are non-negative)
\[
\frac{d}{dt} \int_0^1 b^{2n}(x, t) dx + 2n(2n-1)\nu \int_0^1 b^{2n-2}b_x^2 dx \\
= -(2n-1) \int_0^1 u_x b^{2n} dx + 2n\nu \left[ b_x^{2n-1}(t)b_x(1, t) - b_x^{2n-1}(t)b_x(0, t) \right] \\
\leq -\frac{2n-1}{\lambda} \int_0^1 \left( F + P(\rho) + \frac{b_{xx}^2}{2} \right) b^{2n} dx + CnC^{2n} \nu ||b_x||_{L^\infty} \\
\leq -\frac{2n-1}{\lambda} \int_0^1 Fb^{2n} dx + CnC^{2n}\nu \left( ||b_x||_{L^2} + ||b_x||_{L^\infty}^{1/2} ||b_{xx}||_{L^2}^{1/2} \right) \\
\leq Cn||F||_{L^\infty} ||b||_{L^{2n}}^{2n} + CnC^{2n} \left( 1 + \nu^2 ||b_{xx}||_{L^2}^2 \right), \tag{2.33}
\]

where we have also used (2.13), Sobolev’s and Cauchy–Schwarz’s inequalities.

It follows from (2.13), (2.29) and (2.30) and Sobolev’s inequality that

\[
||F||_{L^\infty} \leq C \left( 1 + ||\rho^{1/2}\bar{u}||_{L^2} \right) \in L^2(0, T), \tag{2.34}
\]

and hence, we conclude from (2.33) that

\[
||b(t)||_{L^\infty}^{2n} \leq CnC^{2n} \exp \left\{ Cn \int_0^T ||F||_{L^\infty} dt \right\} \left( 1 + \nu^2 \int_0^T ||b_{xx}||_{L^2}^2 dt \right) \\
\leq CnC^{2n} \exp \{ Cn \} .
\]

Thus, raising to the power \(1/(2n)\) to both sides and letting \(n \to \infty\), we arrive at

\[
||b(t)||_{L^\infty} \leq C, \quad \forall t \in [0, T]. \tag{2.35}
\]

Now, the lower bound of the density can be derived in the same way as in lemma 2.3, based on (2.7) and (2.35). Let \(\psi\) and \(\Phi\) be the same functions defined in (2.8) and (2.11), respectively. Then, analogously to the proof of lemma 2.3, we have

\[
D_t \left( \frac{1}{\rho \Phi} \right) = \frac{1}{\lambda} \left( P(\rho) + \frac{b_{xx}^2}{2} \right) \frac{1}{\rho \Phi} \quad \text{with} \quad D_t \triangleq \partial_t + \bar{u} \partial_x,
\]

so that, using (2.7), (2.10) and (2.35), we find

\[
|| (\rho \Phi)^{-1} ||_{L^\infty} \leq || (\rho \Phi)^{-1} (0) ||_{L^\infty} \exp \left\{ \frac{1}{\lambda} \int_0^T \left( ||P(\rho)||_{L^\infty} + \frac{1}{2} ||b||_{L^\infty}^2 \right) dt \right\} \leq C,
\]

which, combined with (2.10) again, shows that the density has a strictly positive lower bound.

As an immediate result, we also infer from (2.7), (2.13), (2.34) and (2.35), we also see that

\[
\int_0^T (||u_t||_{L^2}^2 + ||u_x||_{L^\infty}^2) dt \leq C \int_0^T (||u_t||_{L^2}^2 + ||u_x||_{L^\infty}^2) dt \\
+ C \int_0^T (||F||_{L^\infty}^2 + ||b||_{L^\infty}^2 + ||P(\rho)||_{L^\infty}) dt \\
\leq C + C \int_0^T ||u_t||_{L^2}^2 dt \leq C.
\]

The proof of lemma 2.5 is therefore complete.  \( \square \)
\[ \nu^{1/2} \sup_{0 \leq t \leq T} \left( \|b_x(t)\|_{L^2}^2 + \|\rho_x(t)\|_{L^2}^2 \right) + \int_0^T \left( \nu^{3/2} \|b_{xx}\|_{L^2}^2 + \nu^{1/2} \|u_{xx}\|_{L^2}^2 \right) \, dt \leq C. \] (2.36)

**Proof.** Differentiating (1.4), with respect to \(x\), multiplying the resulting equation by \(2\rho_x\) in \(L^2\) and integrating by parts, by (2.7), (2.12), (2.13), (2.29), (2.30) and (2.32) we deduce

\[
\frac{d}{dt} \|\rho_x\|_{L^2}^2 \leq C \|u_x\|_{L^\infty} \|\rho_x\|_{L^2}^2 + C \|u_{xx}\|_{L^2} \|\rho_x\|_{L^2} \\
\leq C \|u_x\|_{L^\infty} \|\rho_x\|_{L^2}^2 + C \left( \|F_x\|_{L^2} + \|\rho_x\|_{L^2} + \|b_x\|_{L^2} \right) \|\rho_x\|_{L^2} \\
\leq C \left( 1 + \|u_x\|_{L^\infty} \right) \left( \|\rho_x\|_{L^2}^2 + \|b_x\|_{L^2}^2 \right) + C \left( 1 + \|\dot{u}\|_{L^2}^2 \right),
\]

(2.37)

where we have also used (2.7), (2.12), (2.30) and (2.32) to infer that

\[ \|u_{xx}\|_{L^2} \leq C \left( \|F_x\|_{L^2} + \|\rho_x\|_{L^2} + \|b_x\|_{L^2} \right) \leq C \left( \|\dot{u}\|_{L^2} + \|\rho_x\|_{L^2} + \|b_x\|_{L^2} \right). \]

(2.38)

Similarly, multiplying (1.4)_3 by \(2\nu b_{xx}\) in \(L^2\) and integrating by parts, we deduce

\[
\nu \frac{d}{dt} \|b_{xx}\|_{L^2}^2 + 2\nu^2 \|b_{xxx}\|_{L^2}^2 = 2\nu \int_0^1 (u_x b + ub_x) \, b_{xx} \, dx + R_2(t) \\
\quad = -3\nu \int_0^1 u_x b_{xx}^2 \, dx - 2\nu \int_0^1 u_{xx} b b_x \, dx + R_2(t) + R_3(t),
\]

(2.39)

where \(R_2(t)\) is the same one as in (2.23) and \(R_3(t)\) is also a boundary term given by

\[ R_3(t) = 2\nu \left[ b_2(t)(u_x b_x)(1, t) - b_1(t)(u_x b_x)(0, t) \right]. \]

By virtue of Sobolev’s and Young’s inequalities, we see that

\[ R_3(t) \leq C \nu \|u_x\|_{L^\infty} \|b_{xx}\|_{L^2} \leq C \nu \|u_x\|_{L^\infty} \left( \|b_x\|_{L^2} + \|b_{xx}\|_{L^2}^{1/2} \|b_{xxx}\|_{L^2}^{1/2} \right) \leq \frac{\nu}{2} \|b_{xx}\|_{L^2}^2 + C \nu \|b_x\|_{L^2}^2 + C \left( \nu + \nu^{1/2} \right) \left( 1 + \|u_x\|_{L^\infty}^2 \right). \]

(2.40)

Thus, substituting (2.26) (with \(\varepsilon \in (0, 1)\) small enough), (2.40) into (2.39), by (2.38) we deduce in a manner similar to the derivation of (2.37) that for any \(\nu \in (0, 1),\)

\[
\nu \frac{d}{dt} \|b_{xx}\|_{L^2}^2 + \nu^2 \|b_{xxx}\|_{L^2}^2 \leq C \nu \left( \|b_x\|_{L^2}^2 + \|\rho_x\|_{L^2}^2 \right) \left( 1 + \|u_x\|_{L^\infty} \right) \\
+ C \nu \left( 1 + \|\dot{u}\|_{L^2}^2 + \|u_x\|_{L^\infty}^2 \right).
\]

(2.41)

Now, multiplying (2.37) by \(\nu\) and adding the resulting inequality to (2.41), we obtain
This section is devoted to the proof of theorem 1.1. To do this, we still need the following

3. Proof of theorem 1.1

This section is devoted to the proof of theorem 1.1. To do this, we still need the following

global a priori estimates of the solution

\[ \nu \frac{d}{dr}(\|\rho(x)\|_{L^2}^2 + \|b(x)\|_{L^2}^2) + \nu^2\|b_{xx}\|_{L^2}^2 \leq C\nu(\|\rho(x)\|_{L^2}^2 + \|b(x)\|_{L^2}^2) (1 + \|a(x)\|_{L^\infty}) + C\nu^{1/2} (1 + \|\dot{u}\|_{L^2}^2 + \|u_t\|_{L^\infty}^2). \]

This, combined with (2.13) and (2.32) and Gronwall’s inequality, shows

\[ \nu \sup_{0 \leq t \leq T}(\|b(x)\|_{L^2}^2 + \|\rho(x)\|_{L^2}^2) + \nu^2 \int_0^T \|b_{xx}\|_{L^2}^2 dt \leq C\nu^{1/2}, \]

which, together with (2.13), (2.32) and (2.38), leads to the desired estimate of (2.36).

Remark 2.1. Note that if \( b_{x|x=0,1} = 0 \), then the boundary terms \( R_2, R_3 \) in (2.39) vanish. So, it is easy to see that

\[ \frac{d}{dr}\|b(x)\|_{L^2}^2 + \nu\|b_{xx}\|_{L^2}^2 \leq C \left( \|b(x)\|_{L^2}^2 + \|\rho(x)\|_{L^2}^2 \right) (1 + \|a(x)\|_{L^\infty}) + C \left( 1 + \|\dot{u}\|_{L^2}^2 \right), \]

which, together with (2.37) and Gronwall’s inequality, shows that

\[ \sup_{0 \leq t \leq T}(\|b(x)\|_{L^2}^2 + \|\rho(x)\|_{L^2}^2) + \nu \int_0^T \|b_{xx}\|_{L^2}^2 dt \leq C. \]

Thus, it follows from (1.14) and Sobolev’s inequality that \( \|\langle (\rho - \bar{\rho}, b - \bar{b}) \rangle \|_{C([0,T])} \to 0 \) as \( \nu \to 0 \), which particularly implies that in such situation (i.e. \( b_{x|x=0,1} = 0 \)), there is no boundary-layer effect in the sense of definition 1.1.

3. Proof of theorem 1.1

This section is devoted to the proof of theorem 1.1. To do this, we still need the following

Proposition 3.1. Let \((\bar{\rho}, \bar{u}, \bar{b})\) be a smooth solution of (1.6) and (1.7) on \((0,1) \times [0,T]\). Then,

\[ 0 < C^{-1} \leq \bar{\rho}(x,t) \leq C < \infty, \quad \forall \,(x,t) \in [0,1] \times [0,T], \quad \tag{3.1} \]

\[ \sup_{0 \leq t \leq T}(\|\bar{u}(x)\|_{L^2}^2 + \|\bar{b}(x)\|_{L^\infty}^2)(t) + \int_0^T (\|\bar{u}_t\|_{L^2}^2 + \|\bar{u}_t\|_{L^\infty}^2 + \|\bar{u}_t\|_{L^\infty}^2) dt \leq C. \tag{3.2} \]

and

\[ \sup_{0 \leq t \leq T}(\|\bar{\rho}(x)\|_{L^2}^2 + \|\bar{b}(x)\|_{L^2}^2 + \|\bar{b}(x)\|_{L^\infty}^2)(t) + \int_0^T \|\bar{u}_xx\|_{L^2}^2 dt \leq C. \tag{3.3} \]

where \( \bar{u} = \bar{u}_t + \bar{u} \).\]

Proof. Indeed, repeating the arguments in the proofs of lemmas 2.2–2.5 step by step, one easily obtains (3.1) and (3.2) (indeed, much more easily due to the lack of boundary effects).
To prove (3.3), differentiating (1.6)1, (1.6)3 with respect to \( t \), multiplying the resulting equations by \( \rho_t \) and \( b_t \) in \( L^2 \) respectively, and adding them together, by (3.1) and (3.2) we deduce in a manner similar to the derivation of (2.37) that
\[
\frac{d}{dt} \left( \| \rho_t \|_{L^2}^2 + \| b_t \|_{L^2}^2 \right) \leq C \left[ \| \rho_t \|_{L^\infty} + \| b_t \|_{L^2} + C(1 + \| \rho_t \|_{L^2} + \| b_t \|_{L^2}) \right] + C(1 + \| \rho_t \|_{L^2} + \| b_t \|_{L^2}),
\]
(3.4)
since it is easily seen from (1.6)2, (3.1) and (3.2) that
\[
\| u_{xx} \|_{L^2} \leq C \left[ \| \rho_t \|_{L^2} + \| \rho_t \|_{L^2} + \| b_t \|_{L^2} \right].
\]
(3.5)
Thus, using (3.2) and Gronwall’s inequality, we infer from (3.4) that
\[
\left( \| \rho_t \|_{L^2}^2 + \| b_t \|_{L^2}^2 \right) (t) \leq C, \quad \forall \ t \in [0, T],
\]
which, combined with (1.6)1, (3.1), (3.2) and (3.5), yields the desired estimates stated in (3.3). The proof of proposition 3.1 is therefore complete.

With these global (uniform) estimates at hand, we are now ready to prove theorem 1.1.

**Proof of theorem 1.1.** First, combining the standard local existence result with the global (uniform) estimates established in section 2 and proposition 3.1, we easily obtain the global well-posedness of strong solutions \((\rho, u, b)\) and \((\bar{\rho}, \bar{u}, \bar{b})\) to the problems (1.4)–(1.7), respectively.

In order to justify the non-resistive limit as \( \nu \to 0 \) and to derive the convergence rates, we consider the following equations of the differences \((\rho - \bar{\rho}, u - \bar{u}, b - \bar{b})\) which can be derived from (1.4) and (1.6):
\[
\begin{align*}
(\rho - \bar{\rho})_t + (\rho - \bar{\rho}) u_x + \bar{\rho} (u - \bar{u})_x + (\rho - \bar{\rho}) u + \bar{\rho}_x (u - \bar{u}) &= 0, \\
\rho (u - \bar{u})_t + \rho u (u - \bar{u})_x - \lambda (u - \bar{u})_{xx} &= - (\rho - \bar{\rho}) (u_t + \bar{u}_t) - \rho (u - \bar{u}) \bar{u}_x - (P' (\rho) - P'(\bar{\rho}))_x - \frac{1}{2} (b^2 - \bar{b}^2)_x,
\end{align*}
\]
(3.6)
(3.7)
and
\[
\begin{align*}
(b - \bar{b})_t + u_x (b - \bar{b}) + \bar{b} (u - \bar{u})_x + u (b - \bar{b})_x + (u - \bar{u}) \bar{b}_x &= \nu b_{xx}.
\end{align*}
\]
(3.8)

First, multiplying (3.8) by \( 2(b - \bar{b}) \) in \( L^2 \) and integrating by parts, we obtain
\[
\frac{d}{dt} \| b - \bar{b} \|_{L^2}^2 = - \int_0^1 u_x (b - \bar{b})^2 \ dx - 2 \int_0^1 \bar{b} (u - \bar{u})_x (b - \bar{b}) \ dx - 2 \int_0^1 (u - \bar{u}) b_x (b - \bar{b}) \ dx + 2 \nu \int_0^1 b_{xx} (b - \bar{b}) \ dx
\]
\[
\triangleq \text{R.H.S.}
\]
(3.9)
Using proposition 3.1, Sobolev’s and Cauchy–Schwarz’s inequalities, we find
R.H.S. ≤ C\|u\|_{L^\infty}(\|b - \bar{b}\|_{L^2}^2 + C\|\bar{b}\|_{L^\infty}(\|(u - \bar{u})\|_{L^2})\|b - \bar{b}\|_{L^2})
+ C\|b_x\|_{L^2}(\|u - \bar{u}\|_{L^\infty}\|b - \bar{b}\|_{L^2} + Cr\|b_x\|_{L^2}\|b - \bar{b}\|_{L^2})
\leq C(1 + \|u\|_{L^\infty})\|b - \bar{b}\|_{L^2}^2 + C(V\|b_x\|_{L^2} + \|(u - \bar{u})\|_{L^2})\|b - \bar{b}\|_{L^2}^2
\leq \varepsilon(\|(u - \bar{u})\|_{L^2}^2 + C\varepsilon^{-1}(\nu^2\|b_{xx}\|_{L^2}^2 + (1 + \|u\|_{L^\infty})\|b - \bar{b}\|_{L^2}^2)).

which, substituted into (3.9), gives (0 < \varepsilon < 1)
\frac{d}{dt}\|b - \bar{b}\|_{L^2}^2 \leq \varepsilon(\|(u - \bar{u})\|_{L^2}^2 + C\varepsilon^{-1}(1 + \|u\|_{L^\infty})\|b - \bar{b}\|_{L^2}^2). (3.10)

Similarly, multiplying (3.6) by 2(\rho - \bar{\rho}) in L^2 and integrating by parts, by proposition 3.1 we deduce
\frac{d}{dt}\|\rho - \bar{\rho}\|_{L^2}^2 \leq \varepsilon(\|(u - \bar{u})\|_{L^2}^2 + C\varepsilon^{-1}(1 + \|\bar{u}\|_{L^\infty})\|\rho - \bar{\rho}\|_{L^2}^2). (3.11)

Finally, multiplying (3.7) by 2(u - \bar{u}) in L^2 and integrating by parts, we infer from (2.7), (2.32) and proposition 3.1 that (noting that \hat{u} = \bar{u} + \bar{u}u_x)
\frac{d}{dt}(\|\nu^{1/2}(u - \bar{\nu})\|_{L^2}^2 + 2\lambda(\|u - \bar{u}\|_{L^2}^2)
\leq C\|\hat{u}\|_{L^2}(\|\rho - \bar{\rho}\|_{L^2} + \|\bar{u} - \bar{u}u_x\|_{L^\infty} + C\|\bar{u}\|_{L^\infty}\|\rho^{1/2}(u - \bar{\nu})\|_{L^2}^2)
+ C\|\|\nu^{1/2}(u - \bar{\nu})\|_{L^2}^2 + C\varepsilon^{-1}\|u\|_{L^\infty}\|\rho^{1/2}(u - \bar{\nu})\|_{L^2}^2
+ C\varepsilon^{-1}(1 + \|\bar{u}\|_{L^2}^2)\|\nu^{1/2}(\rho - \bar{\rho})\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2). (3.12)

Now, collecting (3.10)–(3.12) together, and choosing \varepsilon > 0 suitably small, we arrive at
\frac{d}{dt}(\|\rho - \bar{\rho}\|_{L^2}^2 + \|b - \bar{b}\|_{L^2}^2 + \|\nu^{1/2}(u - \bar{\nu})\|_{L^2}^2) + \|(u - \bar{u})\|_{L^2}^2
\leq C\mathcal{A}(t)(\|\rho - \bar{\rho}\|_{L^2}^2 + \|b - \bar{b}\|_{L^2}^2 + \|\rho^{1/2}(u - \bar{\nu})\|_{L^2}^2) + C\nu^2\|b_x\|_{L^2}^2,

where
\mathcal{A}(t) = (1 + \|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\bar{u}\|_{L^2}^2)(t) \in L^1(0, T),
due to (2.32) and proposition 3.1. Thus, it follows from (2.36) and Gronwall’s inequality that
\sup_{0 \leq t \leq T}(\|\rho^{1/2}(u - \bar{\nu})\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2 + \|b - \bar{b}\|_{L^2}^2)(t) + \int_0^T\|(u - \bar{u})\|_{L^2}^2 dt
\leq C\nu^2 \exp\left\{C\int_0^T\mathcal{A}(t) dt\right\}\int_0^T\|b_x\|_{L^2}^2 dt \leq C\nu^{1/2}.

This, together with the fact that \rho \geq C due to (2.32), finishes the proof of theorem 1.1. □
4. Proof of theorem 1.2

In this section, we aim to study of the thickness of MBL. To do this, we need more regularity of the solution \((\bar{\rho}, \bar{u}, \bar{b})\) of the problem (1.6) and (1.7).

**Lemma 4.1.** Let the conditions of theorem 1.2 be in force. Assume that \((\bar{\rho}, \bar{u}, \bar{b})\) is a solution of the problem (1.6) and (1.7) on \((0,1) \times [0, T]\). Then, in addition to (3.1)–(3.3), one has

\[
\sup_{0 \leq t \leq T} \left( ||\bar{u}_x||^2_{L^2} + ||\bar{u}_t||^2_{L^2} + ||\bar{\rho}_x||^2_{L^2} + ||\bar{b}_x||^2_{L^2} \right) (t) + \int_{0}^{T} \left( ||\bar{u}_x||^2_{L^2} + ||\bar{u}_{xxx}||^2_{L^2} \right) \, dt \leq C.
\]  

(4.1)

**Proof.** The proof of (4.1) is based on the standard \(L^2\)-method. First, differentiating (1.6)2 with respect to \(t\) gives

\[
\bar{\rho}_u u_t + \bar{\rho} u u_{xt} - \lambda u_{xxt} + \left( P(\bar{\rho}) + \frac{\bar{b}^2}{2} \right)_x = -\bar{\rho}_x u_x - \bar{\rho}_t (u_t + \bar{u}_x),
\]

which, multiplied by \(\bar{u}_x\) in \(L^2\) and integrated by parts, results in

\[
\frac{d}{dt} ||\bar{\rho}_x||^2_{L^2} + ||\bar{u}_x||^2_{L^2} \leq C \left( ||\bar{\rho}_x||^2_{L^2} + ||\bar{b}_x||^2_{L^2} \right) ||\bar{u}_x||^2_{L^2} + C ||\bar{u}_x||_{L^\infty} ||\bar{u}_x||^2_{L^2} + C ||\bar{u}_x||_{L^\infty} \left( ||\bar{u}_t||_{L^2} + ||\bar{u}_x||_{L^2} \right) ||\bar{u}_x||_{L^\infty} + \frac{1}{2} ||\bar{u}_x||^2_{L^2} + C (1 + ||\bar{u}_x||_{L^\infty}) ||\bar{u}_x||^2_{L^2},
\]

where we have used proposition 3.1, Sobolev’s and Cauchy–Schwarz’s inequalities. This, together with (3.1) and (3.2) and Gronwall’s inequality, shows

\[
\sup_{0 \leq t \leq T} ||\bar{u}_x(t)||^2_{L^2} + \int_{0}^{T} ||\bar{u}_x||^2_{L^2} \, dt \leq C.
\]  

(4.2)

As an immediate consequence of (4.2), it also follows from (1.6)2, proposition 3.1 and Sobolev’s inequality that

\[
\sup_{0 \leq t \leq T} ||\bar{u}_x(t)||_{L^2} + \sup_{0 \leq t \leq T} ||\bar{u}_x(t)||_{L^\infty} \leq C.
\]  

(4.3)

Next, differentiating (1.6)1, (1.6)3 with respect to \(x\) twice and multiplying the resulting equations by \(\bar{\rho}_xx\) and \(\bar{b}_xx\) in \(L^2\) respectively, and adding them together, by virtue of (4.3), proposition 3.1 and Sobolev’s inequality we obtain after integrating by parts that

\[
\frac{d}{dt} \left( ||\bar{\rho}_xx||^2_{L^2} + ||\bar{b}_xx||^2_{L^2} \right) \\
\leq C ||\bar{u}_x||_{L^\infty} \left( ||\bar{\rho}_x||^2_{L^2} + ||\bar{b}_x||^2_{L^2} \right) + C ||\bar{u}_{xxx}||_{L^2} \left( ||\bar{\rho}_x||_{L^2} + ||\bar{b}_x||_{L^2} \right) \\
+ C ||\bar{u}_{xxx}||_{L^\infty} \left( ||\bar{\rho}_x||^2_{L^2} + ||\bar{b}_x||^2_{L^2} \right) \left( ||\bar{\rho}_xx||_{L^2} + ||\bar{b}_xx||_{L^2} \right) \\
\leq C \left( ||\bar{\rho}_x||^2_{L^2} + ||\bar{b}_x||^2_{L^2} \right) + C \left( 1 + ||\bar{u}_{xxx}||^2_{L^2} \right).
\]

(4.4)

Moreover, it follows from (1.6)2, proposition 3.1, (4.2), (4.3) and Sobolev’s inequality that
\[ \|\bar{u}_{xx}\|_{L^2} \leq C \left( \|\bar{u}_{xx}\|_{L^2} + \|\bar{b}_{xx}\|_{L^2} \right) + C \left( \|\bar{b}_{x}\|_{L^\infty} + \|\bar{b}_{x}\|_{L^\infty} \right) \left( \|\bar{\rho}_{xx}\|_{L^2} + \|\bar{b}_{x}\|_{L^2} \right) \\
+ C \left( \|\bar{u}_{xx}\|_{L^2} + \|\bar{b}_{x}\|_{L^\infty} \right) + C \left( 1 + \|\bar{b}_{x}\|_{L^\infty} \right) \|\tilde{u}\|_{H^1}^2, \]
\[ \leq C \left( 1 + \|\bar{u}_{xx}\|_{L^2} + \|\bar{b}_{xx}\|_{L^2} + \|\bar{u}_{xx}\|_{L^2} \right), \]
(4.5)

which, inserted into (4.4), results in
\[ \frac{d}{dr} \|\bar{u}_{xx}\|_{L^2}^2 + \|\bar{b}_{xx}\|_{L^2}^2 \leq C \left( \|\bar{u}_{xx}\|_{L^2}^2 + \|\bar{b}_{xx}\|_{L^2}^2 \right) + C \left( 1 + \|\bar{u}_{xx}\|_{L^2}^2 \right), \]
so that, it follows from (4.2) and Gronwall’s inequality that
\[ \sup_{0 \leq t \leq T} \left( \|\bar{u}_{xx}\|_{L^2}^2 + \|\bar{b}_{xx}\|_{L^2}^2 \right) \leq C. \]
(4.6)

This, together with (4.5), also shows that
\[ \int_0^T \|\bar{u}_{xx}\|_{L^2}^2 \, dr \leq C, \]
which, together with (4.2), (4.3) and (4.6), finishes the proof of lemma 4.1.

The analysis of the thickness of MBL is based on the weighted \( L^2 \)-norms of \( (\rho - \bar{\rho}, b - \bar{b}) \).

**Lemma 4.2.** Let \( (\rho, u, b) \) and \( (\bar{\rho}, \bar{u}, \bar{b}) \) be the solutions of the problems (1.4)–(1.7) on \( (0, 1) \times [0, T] \) respectively. Then there exists a positive constant \( C \), independent of \( \nu \), such that
\[ \frac{d}{dr} \|\eta(u - \bar{u}) \|_{L^2}^2 + \|\eta_1(u - \bar{u}) \|_{L^2}^2 + \|\eta(u - \bar{u})_x \|_{L^2}^2 \]
\[ \leq C \left( \nu^{1/2} + \|u - \bar{u}\|_{L^2}^2 \right) + C \left( \|\eta(\rho - \bar{\rho})_x \|_{L^2}^2 + \|\eta(b - \bar{b})_x \|_{L^2}^2 \right), \]
(4.7)

where \( \eta(x) \triangleq x(1 - x) \) is the boundary cut-off function.

**Proof.** Multiplying (3.7) by \( \eta'(x)(u - \bar{u}) \), in \( L^2 \) and integrating by parts, we obtain
\[ \frac{\lambda}{2} \frac{d}{dr} \int_0^1 \eta^2(x)(u - \bar{u})_x^2 \, dx + \int_0^1 \eta^2(x)\rho((u - \bar{u})_x^2) \, dx \]
\[ = -2\lambda \int_0^1 \eta(x)\eta'(x)(u - \bar{u})_x(u - \bar{u})_x \, dx \]
\[ - \int_0^1 \eta^2(x) [\rho - \rho_\nu] \bar{u}_x + (\rho - \rho)\bar{u}_x \bar{u}_x \] \( (u - \bar{u})_x \, dx \)
\[ - \int_0^1 \eta^2(x) [\rho u(u - \bar{u})_x + \rho(u - \bar{u})\bar{u}_x] \] \( (u - \bar{u})_x \, dx \)
\[ - \int_0^1 \eta^2(x) \left( P(\rho) - P(\bar{\rho}) \right)_x \] \( (u - \bar{u})_x \, dx \)
\[ - \frac{1}{2} \int_0^1 \eta^2(x) \left( b^2 - \bar{b}^2 \right)_x \] \( (u - \bar{u})_x \, dx \]
\[ \triangleq \sum_{i=1}^5 I_i. \]
(4.8)
With the help of the global (uniform) estimates given in lemmas 2.1–2.6, proposition 3.1 and 4.1, we can estimate each term on the right-hand side of (4.8) as follows. First, it is easy to get that
\[ I_1 \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2} + C(\varepsilon) \| (u - \bar{u})_t \|_{L^2}, \quad \forall \varepsilon \in (0, 1). \] (4.9)

Using proposition 3.1, (4.1) and (1.14), we have
\[ I_2 \leq C\| \eta(\rho - \bar{\rho}) \|_{L^\infty} \left( \| u_x \|_{L^2} + \| \bar{u}_x \|_{L^2} + \| u \|_{L^\infty} \| \bar{u} \|_{L^2} \right) \| \eta(u - \bar{u}_t) \|_{L^2} \]
\[ \leq C \left( \| u - \bar{u} \|_{L^2} + \| \eta(\rho - \bar{\rho}) \|_{L^2} \right) \| \eta(u - \bar{u}_t) \|_{L^2} \]
\[ \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2}^2 + C(\varepsilon) \left( \nu^{1/2} + \| \eta(\rho - \bar{\rho}) \|_{L^2} \right), \] (4.10)

where we have also used the following Sobolev’s inequality \((f = f(x)):\)
\[ \| \eta f \|_{L^\infty} \leq \int_0^1 |(\eta f)'| \, dx \leq \int_0^1 \left( \| \eta(x) \|_{L^2} |f_x| + \| \eta'(x) \|_{L^2} |f| \right) \, dx \]
\[ \leq C \left( \| \eta f \|_{L^2} + \| f \|_{L^2} \right). \] (4.11)

Owing to (2.7), (2.13), (4.1) and (1.14), we see that
\[ I_3 \leq C \left( \left( \| (u - \bar{u})_t \|_{L^2} + \| \bar{u}_x \|_{L^\infty} \| u - \bar{u} \|_{L^2} \right) \| \eta(u - \bar{u}_t) \|_{L^2} \right) \]
\[ \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2}^2 + C(\varepsilon) \left( \nu^{1/2} + \| (u - \bar{u})_t \|_{L^2} \right). \] (4.12)

To deal with \(I_4\), we first rewrite it in the form:
\[ I_4 = - \int_0^1 \eta^2(x) \left| P'(\rho) \left( \rho_x - \bar{\rho}_x \right) + \left( \rho_x \left( \frac{\partial P}{\partial \rho} - \frac{P'}{\rho} \right) \right) \right| (u - \bar{u}) \, dx, \]
so that, it follows from (2.7), (4.1) and (1.14) that
\[ I_4 \leq C \left( \| \eta(\rho - \bar{\rho}) \|_{L^2} + \| \rho_x \|_{L^\infty} \| \rho - \bar{\rho} \|_{L^2} \right) \| \eta(u - \bar{u}_t) \|_{L^2} \]
\[ \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2}^2 + C(\varepsilon) \left( \| \eta(\rho - \bar{\rho}) \|_{L^2}^2 + \| \rho - \bar{\rho} \|_{L^2}^2 \right) \]
\[ \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2}^2 + C(\varepsilon) \left( \nu^{1/2} + \| \eta(\rho - \bar{\rho}) \|_{L^2} \right). \] (4.13)

In a similar manner, we also have
\[ I_5 = - \int_0^1 \eta^2(x) \left[ b(b_x - \bar{b}_x) + \bar{b}_x(b - \bar{b}) \right] (u - \bar{u}) \, dx \]
\[ \leq C \left( \| \eta(b - \bar{b}) \|_{L^2} + \| \bar{b}_x \|_{L^\infty} \| b - \bar{b} \|_{L^2} \right) \| \eta(u - \bar{u}_t) \|_{L^2} \]
\[ \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2}^2 + C(\varepsilon) \left( \| \eta(b - \bar{b}) \|_{L^2}^2 + \| b - \bar{b} \|_{L^2}^2 \right) \]
\[ \leq \varepsilon \| \eta(u - \bar{u}_t) \|_{L^2}^2 + C(\varepsilon) \left( \nu^{1/2} + \| \eta(b - \bar{b}) \|_{L^2} \right). \] (4.14)
Substituting (4.9), (4.10), (4.12)−(4.14) into (4.8) and choosing \( \varepsilon > 0 \) sufficiently small, by (2.32) we conclude that
\[
\frac{d}{dt}\|\eta(u - \bar{u})_x\|_{L^2}^2 + \|\eta(u - \bar{u})_t\|_{L^2}^2 \leq C \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right) + C \left( \nu^{1/2} + \|(u - \bar{u})_x\|_{L^2}^2 \right).
\]

Similarly to the derivation of (4.15), we also infer from (3.7) that
\[
\|\eta(u - \bar{u})_{xx}\|_{L^2}^2 \leq C \left( \|\eta(u - \bar{u})_x\|_{L^2}^2 + \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right) + C \left( \nu^{1/2} + \|(u - \bar{u})_x\|_{L^2}^2 \right).
\]

which, combined with (4.15), finishes the proof of (4.7).

Clearly, we still need to estimate \( \|\eta(\rho - \bar{\rho})_x\|_{L^2} \) and \( \|\eta(b - \bar{b})_x\|_{L^2} \).

**Lemma 4.3.** Let \( \eta(x) \triangleq x(1 - x) \) be in force. Assume that \((\rho, u, b)\) and \((\bar{\rho}, \bar{u}, \bar{b})\) are the solutions of the problems (1.4)−(1.7) on \((0, 1) \times [0, T]\), respectively. Then there exists a positive constant \( C \), independent of \( \nu \), such that
\[
\frac{d}{dt} \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right) + \nu \|\eta(b - \bar{b})_{xx}\|_{L^2}^2 \leq C \left( \nu^{1/2} + \|\eta(u - \bar{u})_{xx}\|_{L^2}^2 + \|(u - \bar{u})_x\|_{L^2}^2 \right) + C \left( 1 + \|u_\varepsilon\|_{L^\infty} \right) \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right).
\]

**Proof.** Differentiating (3.6) and (3.8) with respect to \( x \) gives
\[
(\rho - \bar{\rho})_{xt} = -\left[ \bar{\rho}(u - \bar{u}) + (\rho - \bar{\rho})u \right]_{xx} = -\left[ \bar{\rho}_x(u - \bar{u}) + 2\bar{\rho}_x(u - \bar{u}) + \bar{\rho}(u - \bar{u})_{xx} \right] + \left[ (\rho - \bar{\rho})_{xx}u + 2(\rho - \bar{\rho})_{x}u + (\rho - \bar{\rho})u_{xx} \right],
\]
and
\[
(b - \bar{b})_{xt} - \nu(b - \bar{b})_{xx} = \nu b_{xx} - \left[ b(u - \bar{u}) + (b - \bar{b})u \right]_{xx} = \nu b_{xx} - \left[ b_x(u - \bar{u}) + 2b_x(u - \bar{u}) + b(u - \bar{u})_{xx} \right] - \left[ (b - \bar{b})_{xx}u + 2(b - \bar{b})_{x}u + (b - \bar{b})u_{xx} \right].
\]

Then, multiplying (4.17) and (4.18) by \( \eta^2(x)(\rho - \bar{\rho})_x \) and \( \eta^2(x)(b - \bar{b})_x \) in \( L^2 \) respectively, adding them together, and integrating by parts, we deduce
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \eta^2(x)(\|(\rho - \bar{\rho})_x\|^2 + |(b - \bar{b})_x|^2) \, dx + \nu \int_0^1 \eta^2(x)|(b - \bar{b})_{xx}|^2 \, dx
\]
\[
= \nu \int_0^1 (\eta^2)''(x)|(b - \bar{b})_x|^2 \, dx - \int_0^1 \eta^2(x)\bar{\rho}_x(u - \bar{u})(\rho - \bar{\rho})_x \, dx - \int_0^1 \eta^2(x)\bar{b}_x(u - \bar{u})(b - \bar{b})_x \, dx
\]
\[
- 2 \int_0^1 \eta^2(x)\rho(u - \bar{u})_x(\rho - \bar{\rho})_x \, dx - \int_0^1 \eta^2(x)\bar{b}(u - \bar{u})_x(b - \bar{b})_x \, dx
\]
\[
- \int_0^1 \eta^2(x)u(\rho - \bar{\rho})_xx(\rho - \bar{\rho})_x \, dx - \int_0^1 \eta^2(x)\rho(u-b)_x(b-b)_x \, dx
\]
\[
- 2 \int_0^1 \eta^2(x)u_\varepsilon|(\rho - \bar{\rho})_x|^2 \, dx - 2 \int_0^1 \eta^2(x)u_\varepsilon|(b - \bar{b})_x|^2 \, dx
\]
\[
- \int_0^1 \eta^2(x)u_{xx}|(\rho - \bar{\rho})_x|(b - \bar{b})_x \, dx - \int_0^1 \eta^2(x)u_{xx}(b - \bar{b})_x(b - \bar{b})_x \, dx \equiv \sum_{i=1}^{14} J_i, \tag{4.19}
\]

where we have also used the fact that \(\eta^2(x) = (\eta^2)'(x) = 0\) on the boundaries \(x = 0, 1\).

In the following, we shall estimate each term on the right-hand side of (4.19), using lemmas 2.1–2.6, proposition 3.1, lemma 4.1 and (1.14). First, since \((\eta^2)''(x)\) is bounded for all \(x \in [0, 1]\), it follows from (2.36) and (3.3) that
\[
J_1 \leq C \nu \left( \|b_\varepsilon\|_{L^2}^2 + \|\varepsilon b_\varepsilon\|_{L^2}^2 \right) \leq C \nu^{1/2}. \tag{4.20}
\]

Using (2.36), (3.3) and (4.1), and Cauchy–Schwarz’s inequality, we have
\[
J_2 \leq C \nu \|\bar{\rho}_x\|_{L^2} \left( \|\eta(b - \bar{b})_x\|_{L^2} + \|b_\varepsilon\|_{L^2} + \|\varepsilon b_\varepsilon\|_{L^2} \right)
\leq \nu \|\eta(b - \bar{b})_x\|_{L^2}^2 + C \nu^{1/2}. \tag{4.21}
\]

In view of lemma 4.1 and Sobolev’s inequality, we deduce
\[
J_3 + J_4 \leq C \|(u - \bar{u})_x\|_{L^2} \left( \|\bar{\rho}_x\|_{L^\infty} \|\eta(\rho - \bar{\rho})_x\|_{L^2} + \|\bar{b}_x\|_{L^2} \|\eta(b - \bar{b})_x\|_{L^2} \right)
\leq C \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right) + C \|(u - \bar{u})_x\|_{L^2}^2, \tag{4.22}
\]
and similarly,
\[
J_5 + J_6 \leq C \|(u - \bar{u})_x\|_{L^2} \left( \|\bar{\rho}_x\|_{L^\infty} \|\eta(\rho - \bar{\rho})_x\|_{L^2} + \|\bar{b}_x\|_{L^2} \|\eta(b - \bar{b})_x\|_{L^2} \right)
\leq C \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right) + C \|(u - \bar{u})_x\|_{L^2}^2. \tag{4.23}
\]

It is easily seen from proposition 3.1 and Cauchy–Schwarz’s inequality that
\[
J_7 + J_8 \leq C \|\eta(u - \bar{u})_x\|_{L^2} \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2} + \|\eta(b - \bar{b})_x\|_{L^2} \right)
\leq C \|\eta(u - \bar{u})_x\|_{L^2}^2 + C \left( \|\eta(\rho - \bar{\rho})_x\|_{L^2}^2 + \|\eta(b - \bar{b})_x\|_{L^2}^2 \right). \tag{4.24}
\]
To deal with $J_9$ and $J_{10}$, we first utilize the non-slip boundary conditions $u|_{x=0} = 0$ to get that for all $x \in [0, 1]$ and $t \in [0, T]$,
\[ |u(x, t)| = \left| \int_0^x u_\xi(\xi, t) \, d\xi \right| \leq x \|u_\xi(t)\|_{L^\infty}, \]
and
\[ |u(x, t)| = \left| \int_x^1 u_\xi(\xi, t) \, d\xi \right| \leq (1-x) \|u_\xi(t)\|_{L^\infty}. \]
Hence, it follows from the definition of $\eta(x)$ that for all $x \in [0, 1]$ and $t \in [0, T]$,
\[ |\eta'(x)u(x, t)| \leq (1-x)|u(x, t)| + x|u(x, t)| \leq 2(1-x)\|u_\xi(t)\|_{L^\infty} = 2\eta(x)\|u_\xi(t)\|_{L^\infty}. \]
so that, after integrating by parts we find
\[ J_9 + J_{10} = \frac{1}{2} \int_0^1 (\eta^2(x)u_x + 2\eta(x)\eta'(x)u \left[ \|\rho - \bar{\rho}\|_x^2 + \|b - \bar{b}\|_x^2 \right] ) \, dx \leq C\|u_x\|_{L^\infty} \left( \|\eta(\rho - \bar{\rho})\|_{L^2}^2 + \|\eta(b - \bar{b})\|_{L^2}^2 \right). \tag{4.25} \]

Similarly, it is easily seen that
\[ J_{11} + J_{12} \leq C\|u_x\|_{L^\infty} \left( \|\eta(\rho - \bar{\rho})\|_{L^2}^2 + \|\eta(b - \bar{b})\|_{L^2}^2 \right). \tag{4.26} \]

In order to estimate $J_{13}$ and $J_{14}$, we notice that
\[ J_{13} + J_{14} = -\int_0^1 [\eta^2(x)(u - \bar{u})_x \left[ (\rho - \bar{\rho})(\rho - \bar{\rho})_x + (b - \bar{b})(b - \bar{b})_x \right] ] \, dx - \int_0^1 [\eta^2(x)u_x \left[ (\rho - \bar{\rho})(\rho - \bar{\rho})_x + (b - \bar{b})(b - \bar{b})_x \right] ] \ \triangleq \overline{J}_{13} + \overline{J}_{14}. \tag{4.27} \]
Noting that $(\rho, b, \bar{\rho}, \bar{b})$ are bounded due to (2.7) and (2.32) and proposition 3.1, we have
\[ \overline{J}_{13} \leq C\|\eta(u - \bar{u})_x\|_{L^2} \left( \|\eta(\rho - \bar{\rho})\|_{L^2} + \|\eta(b - \bar{b})\|_{L^2} \right) \leq C\|\eta(u - \bar{u})_x\|_{L^2}^2 + C \left( \|\eta(\rho - \bar{\rho})\|_{L^2}^2 + \|\eta(b - \bar{b})\|_{L^2}^2 \right), \]
and moreover, using lemma 4.1, (4.11) and (1.14), we obtain
\[ \overline{J}_{14} \leq C\|u_x\|_{L^2} \left( \|\eta(\rho - \bar{\rho})\|_{L^\infty} \|\eta(\rho - \bar{\rho})\|_{L^2} + \|\eta(b - \bar{b})\|_{L^\infty} \|\eta(b - \bar{b})\|_{L^2} \right) \leq C \left( \|\eta(\rho - \bar{\rho})\|_{L^2}^2 + \|\eta(b - \bar{b})\|_{L^2}^2 \right) + C\nu^{1/2}, \]
which, inserted into (4.27), gives
\[ J_{13} + J_{14} \leq C \left( \nu^{1/2} + \|\eta(u - \bar{u})_x\|_{L^2}^2 \right) + C \left( \|\eta(\rho - \bar{\rho})\|_{L^2}^2 + \|\eta(b - \bar{b})\|_{L^2}^2 \right). \tag{4.28} \]
So, substituting (4.20)–(4.26) and (4.28) into (4.19), we immediately obtain (4.16).
Now, it readily follows from lemmas 4.2 and 4.3 that
\[
\frac{d}{dt} \left( \|\eta(u - \bar{u})_x\|^2_{L^2} + \|\eta(\rho - \bar{\rho})_x\|^2_{L^2} + \|\eta(b - \bar{b})_x\|^2_{L^2} \right)
+ \left( \|\eta(u - \bar{u})_{x\alpha}\|^2_{L^2} + \|\eta(u - \bar{u})_{x\alpha}\|^2_{L^2} + \nu \|\eta(b - \bar{b})_{x\alpha}\|^2_{L^2} \right)
\leq C \left( \nu^{1/2} + \|(u - \bar{u})_x\|^2_{L^2} \right) + C \left( 1 + \|u_0\|_{L^\infty} \right) \left( \|\eta(\rho - \bar{\rho})_x\|^2_{L^2} + \|\eta(b - \bar{b})_x\|^2_{L^2} \right).
\]

(4.29)

Owing to (2.32), it holds that \(\|u_0\|_{L^\infty} \in L^1(0, T)\). Thus, by (1.14) and Gronwall’s inequality we conclude from (4.29) that

**Lemma 4.4.** Let \(\eta(x) \triangleq x(1 - x)\) be in force. Assume that \((\rho, u, b)\) and \((\bar{\rho}, \bar{u}, \bar{b})\) are the solutions of the problems (1.4–1.7) on \((0, 1) \times [0, T]\) respectively. Then there exists a positive constant \(C\), independent of \(\nu\), such that

\[
\sup_{0 < t < T} \left( \|\eta(\rho - \bar{\rho})_x\|^2_{L^2} + \|\eta(b - \bar{b})_x\|^2_{L^2} + \|\eta(u - \bar{u})_x\|^2_{L^2} \right)(t)
+ \int_0^T \left( \|\eta(u - \bar{u})_{x\alpha}\|^2_{L^2} + \|\eta(u - \bar{u})_{x\alpha}\|^2_{L^2} + \nu \|\eta(b - \bar{b})_{x\alpha}\|^2_{L^2} \right) dt \leq C \nu^{1/2}.
\]

(4.30)

With lemma 4.4 at hand, we are now ready to prove theorem 1.2.

**Proof of theorem 1.2.** On one hand, it follows from (4.30) that for any \(\delta \in (0, 1/2)\),

\[
\delta^2 \int_0^{1-\delta} \left[ \|\eta(\rho - \bar{\rho})_x\|^2 + \|\eta(b - \bar{b})_x\|^2 \right] dx
= \delta^2 \left( \int_0^{1/2} + \int_0^{1-\delta} \right) \left[ \|\eta(\rho - \bar{\rho})_x\|^2 + \|\eta(b - \bar{b})_x\|^2 \right] dx
\leq \int_0^{1/2} x^2 \left[ \|\eta(\rho - \bar{\rho})_x\|^2 + \|\eta(b - \bar{b})_x\|^2 \right] dx
+ \int_0^{1-\delta} (1 - x)^2 \left[ \|\eta(\rho - \bar{\rho})_x\|^2 + \|\eta(b - \bar{b})_x\|^2 \right] dx
\leq 4 \left( \int_0^{1/2} + \int_0^{1-\delta} \right) x^2 (1 - x)^2 \left[ \|\eta(\rho - \bar{\rho})_x\|^2 + \|\eta(b - \bar{b})_x\|^2 \right] dx
\leq 4 \int_0^1 x^2 (1 - x)^2 \left[ \|\eta(\rho - \bar{\rho})_x\|^2 + \|\eta(b - \bar{b})_x\|^2 \right] dx \leq C \nu^{1/2},
\]

which implies that for any \(t \in [0, T]\),

\[
\|\eta(\rho - \bar{\rho})_x(t)\|_{L^2(\Omega_\delta)} + \|\eta(b - \bar{b})_x(t)\|_{L^2(\Omega_\delta)} \leq C \delta^{-1/4} \nu^{1/2} \quad \text{with} \quad \Omega_\delta \triangleq (\delta, 1 - \delta).
\]

(4.31)

Thus, it follows from (1.14) and (4.31) that as \(\nu \to 0\),

\[
\|\eta(\rho - \bar{\rho})_x(t)\|_{C^2(\Omega_\delta)} + \|\eta(b - \bar{b})_x(t)\|_{C^2(\Omega_\delta)}
\leq C \left( \|\eta(\rho - \bar{\rho})(t)\|_{L^2(\Omega_\delta)} + \|\eta(\rho - \bar{\rho})(t)\|_{L^2(\Omega_\delta)} \right)
+ \|\eta(\rho - \bar{\rho})_x(t)\|_{L^2(\Omega_\delta)} + \|\eta(b - \bar{b})_x(t)\|_{L^2(\Omega_\delta)}
\leq C \nu^{1/2} + C \nu^{1/2} \delta^{-1} \leq C \nu^{1/2} \delta^{-1} \to 0,
\]

(4.32)
provided \( \delta = \delta(\nu) \) satisfies

\[
\delta(\nu) \to 0 \quad \text{and} \quad \frac{\delta(\nu)}{\nu^{1/2}} \to \infty, \quad \text{as} \quad \nu \to 0.
\]

On the other hand, it is easy to see that

\[
\|(b - \bar{b})(t)\|_{C(\Omega)} > 0, \quad \forall \ t \in [0, T],
\]

provided \( b_i(t) \neq \bar{b}_i(t) \) with \( i = 1, 2 \), where \( \bar{b}_1(t) \triangleq \bar{b}(0, t) \) and \( \bar{b}_2(t) \triangleq \bar{b}(1, t) \) are the boundary values of \( \bar{b} \) on the boundaries \( x = 0, 1 \), respectively. This, together with (4.32), finishes the proof of theorem 1.2.

\[ \Box \]

Acknowledgments

Supported by NNSFC (Grant Nos. 11671333, 11271306, 11229101, 11371065), the Natural Science Foundation of Fujian Province of China (Grant No.2015J01023), the Fundamental Research Funds for the Central Universities (Grant No. 20720160012), the National Basic Research Program under the grant 2011CB309705, and the Beijing Center for Mathematics and Information Interdisciplinary Sciences.

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