Large-$N_c$ QCD meets Regge theory:
the example of spin-one two-point functions

Maarten Golterman$^a$ and Santiago Peris$^b$

$^a$Department of Physics, Washington University, St. Louis, MO 63130, USA
  e-mail: maarten@aapje.wustl.edu

$^b$Grup de Física Teòrica and IFAE
  Universitat Autònoma de Barcelona, 08193 Barcelona, Spain
  e-mail: peris@ifae.es

Abstract

We discuss the phenomenological implications of assuming a Veneziano-type spectrum for the vector and axial-vector two-point functions in QCD at large $N_c$. We also compare the phenomenological results with those of Lowest-Meson Dominance, and find that they agree.
1 Introduction

Quantum Chromodynamics simplifies when we take the number of colors, $N_c$, to infinity; we will denote this limit by $QCD_{\infty}$. Assuming confinement, it is generally believed that the large-$N_c$ solution qualitatively resembles the real world enough to be taken as a serious starting point for a systematic approximation by expanding in inverse powers of $N_c$.

Unfortunately, no solution to $QCD_{\infty}$ has been found to date, except in the two-dimensional case. We know that even in the large-$N_c$ limit this solution has to be very complex. Large $N_c$, together with confinement and asymptotic freedom, predicts the existence of an infinite set of zero-width meson resonances in any channel with given quantum numbers. These resonances contribute to any Green’s function made of quark bilinear operators corresponding to those quantum numbers. At very large energies a simplification occurs, as the infinite tower of resonances has to become equivalent to the free-quark loop of the parton model obtained in leading-order perturbative QCD (pQCD), with the connection provided by the operator product expansion (OPE). However, how this happens is not known, except in the two-dimensional case. We would therefore like to propose a simple model for this equivalence, which despite its simplicity is phenomenologically quite successful.

In this work we will deal only with two-point functions in the chiral limit. We limit ourselves to this choice because these are the simplest Green’s functions which obey non-trivial chiral symmetry and OPE constraints. The model is based on the observation that, although the equivalence between the resonance description of $QCD_{\infty}$ and the quark-gluon description of pQCD is valid at asymptotically large energy scales, phenomenology suggests that asymptotic behavior may set in already at values of $s = Q^2 \sim (1-2)$ GeV$^2$. In the vector channel, the $\rho$ meson is clearly lighter than this scale $s$, but all higher resonances ($\rho', \rho'', \rho''', etc.$) appear to lie already in this asymptotic region. Therefore, it is reasonable to consider this channel as consisting of a separate $\rho$ resonance of mass $M_\rho$, plus an infinite tower labeled by an index $n$, of masses $M_V(n)$, which corresponds to the perturbative description in a way to be described below. In the axial-vector channel, on the contrary, the $a_1$ may already be heavy enough to group all resonances together in a corresponding tower, $M_A(n)$. Here the exception is of course the Goldstone meson which, because of chiral-symmetry breakdown, is massless and stays away from the tower.

How these resonances are spaced as a function of the index $n$ is not known in $QCD_{\infty}$. Instead, we will rely on Regge theory. It has been long suspected that there is a connection between large-$N_c$ and Regge theory, but the precise link has never been found. One of the natural features of Regge theory is the existence of so-called daughter trajectories. Their most important feature for us is that they give rise to towers of resonances with a given spin obeying an equal-spacing rule with respect to the index $n$, i.e. the mass squared is given by $M^2(n) = a + bn$. In two dimensions, where the solution to $QCD_{\infty}$ is known, it can be shown that indeed $M^2(n) \sim n$ for large $n$. Here, in four dimensions, we will conjecture that this linear behavior in $n$ is
also valid for both the vector and axial-vector towers, and assume that
\[ M_{V,A}^2(n) = m_{V,A}^2 + n \Lambda_{V,A}^2, \]  
(1.1)
is a reasonable approximation for all mesons except the \( \rho \) meson and the pion.

For simplicity, we will content ourselves with this equal-spacing (ES) ansatz producing only the leading term in pQCD, the parton-model logarithm. A possible remedy to this is to delay the onset of the equally spaced tower by including more and more individual resonances which are not part of the tower, and thus move the “onset” of this tower to a higher energy, where the parton-model logarithm is a better approximation. In this way the description would become more accurate. Alternatively, one could take the \( n \) dependence in eq. (1.1) to be more complicated such that non-leading perturbative corrections are reproduced from our ansatz as well. The price to pay, of course, is that this introduces more unknown parameters into our ansatz, which would have to be fixed by matching to the OPE and chiral perturbation theory (ChPT) to higher orders in a way similar to what we will describe for our simple ansatz in section 2.

Much of this has been said before, although perhaps not within the unifying framework of the large-\( N_c \) expansion. For instance, in the pioneering work by Bramon, Etim and Greco \[6\] and Sakurai \[7\], it was already observed that a resonance tower including \( \rho \) as the first state the \( \rho \), and obeying
\[ M_{V}^2(n) = M_{\rho}^2 (1 + \alpha \ n), \]  
(1.2)
reproduces reasonably well the parton-model logarithm of the two-point vector correlation function if \( \alpha = 2 \) and the \( \rho \) decay constant is taken at its experimental value. This was an exciting result because the Veneziano model \[8\], which can be considered as the model paradigm of Regge theory, predicted exactly this value. We will later on see that, if the distribution of eq. (1.2) is assumed, the value \( \alpha = 2 \) is actually an exact algebraic consequence of the OPE, independent of the value of the \( \rho \) decay constant. However, in that case eq. (1.2) also leads to a negative value for the gluon condensate, a result which is clearly disfavored phenomenologically \[9\]. In our approach we will not take the \( \rho \) to be part of the equally spaced tower.

Also Geshkenbein considered \[10\] the problem of an infinite set of resonances in the vector channel and how this would match onto the OPE; this time with a general function \( M_{V}^2(n) \). However, nobody seems to have considered an infinite set of resonances in the vector and axial-vector correlation functions on the same footing, together with their properties under chiral symmetry \[11\]. In addition, the experimental knowledge of the vector spectrum has improved considerably recently, so that an updated comparison is interesting.

In this paper, we will study the phenomenological consequences of our ansatz, in which we take equally spaced towers in both the vector and axial-vector channels with separate resonances for the \( \rho \) and the pion, to leading order in \( N_c \), and in the chiral limit. Furthermore, although we will refer explicitly to the \( \rho \) and \( a_1 \) mesons for simplicity, it should be clear that our results also apply in the \( U(3) \times U(3) \) chiral limit.

\[ ^1\text{In principle one could also play with the } n \text{ dependence of the resonances’ decay constants (see below).} \]
The rest of the paper is organized as follows. Section 2 is devoted to the presentation of our model. Section 3 contains the phenomenological consequences of this model. In section 4 we make a comparison with the Lowest-Meson Dominance approximation of ref. [12], and we conclude in section 5.

2 The model

Let us start by defining the vector two-point function as

$$\Pi_{V,A}^{\mu\nu} = i \int d^4 x \ e^{i qx} \langle J_{V,A}^{\mu}(x) J_{V,A}^{\nu}(0) \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi_{V,A}(q^2) \ , \quad (2.3)$$

with $J_{V}^{\mu}(x) = \bar{d}(x) \gamma^\mu u(x)$ and $J_{A}^{\mu}(x) = \bar{d}(x) \gamma^\mu \gamma^5 u(x)$. Both functions $\Pi_{V,A}(Q^2 \equiv -q^2)$ satisfy the dispersion relation (up to one subtraction)

$$\Pi_{V,A}(Q^2) = \int_0^\infty dt \ \frac{1}{t + Q^2} \frac{1}{\pi} \text{Im} \Pi_{V,A}(t) \ . \quad (2.4)$$

Following the discussion in the introduction we will assume that

$$\frac{1}{\pi} \text{Im} \Pi_{V}(t) = 2 F_{\rho}^2 \delta(t - M_{\rho}^2) + 2 \sum_{n=0}^\infty F_{V}^2 \delta(t - M_{V}^2(n)) \ , \quad (2.5)$$

$$\frac{1}{\pi} \text{Im} \Pi_{A}(t) = 2 F_{\pi}^2 \delta(t) + 2 \sum_{n=0}^\infty F_{A}^2 \delta(t - M_{A}^2(n)) \ . \quad (2.6)$$

In these expressions, $F_{\rho}$ is the electromagnetic decay constant of the $\rho$, and $F_{\pi} = 93$ MeV is the pion decay constant. $F_{V}$ and $F_{A} = F_{a_1}$ are the corresponding constants for the towers of vector and axial-vector mesons. Note that, along with the ansatz of eq. (1.1) for $M_{V,A}^2(n)$, we take $F_{V,A}$ both to be independent of $n$.

Since QCD has not been solved, it is unclear what values to take for all these parameters even in the large-$N_c$ limit. We estimate that $1/N_c$ subleading corrections are roughly of the order of $\Gamma/M \sim 20 - 30\%$ so that decay constants and masses are uncertain by this amount. In addition, we will give results for both $F_{\pi} \simeq 87$ MeV and $F_{\pi} \simeq 93$ MeV as a rough estimate of how much our results change as a consequence of chiral corrections.

The analysis of this ansatz is just a straightforward application of the Euler-Maclaurin summation formula. For instance, in the vector case one obtains

$$\sum_{n=0}^{N_{V}} \frac{F_{V}^2}{Q^2 + n\Lambda_{V}^2 + m_{V}^2} = \int_0^{N_{V}+1} dn \ \frac{F_{V}^2}{Q^2 + n\Lambda_{V}^2 + m_{V}^2} + \frac{1}{2} \frac{F_{V}^2}{Q^2 + m_{V}^2} + \frac{1}{12} \frac{F_{V}^2 \Lambda_{V}^2}{(Q^2 + m_{V}^2)^2} + \cdots \ , \quad (2.7)$$

where $N_{V} \gg 1$, with the condition that $\Lambda_{\text{cutoff}}^2 = (N_{V}+1)\Lambda_{V}^2 + m_{V}^2$ is a momentum cutoff which regularizes the theory and is taken equal to the momentum cutoff in the axial.
channel in order to avoid spurious chiral symmetry breaking in the deep ultraviolet. Performing the integral and expanding in inverse powers of $Q^2$, one finds

$$\Pi_V(Q^2) = \frac{2F^2_V}{\Lambda^2_V} \log \frac{\Lambda^2_{\text{cutoff}}}{Q^2} + \frac{1}{Q^2} \left[ 2F^2_\rho - 2F^2_V \left( \frac{m^2_V}{\Lambda^2_V} - \frac{1}{2} \right) \right]$$

$$+ \frac{1}{Q^4} \left[ -2F^2_\rho M^2_\rho + F^2_V \Lambda^4_V \left( \frac{m^4_V}{\Lambda^4_V} - \frac{m^2_V}{\Lambda^2_V} + \frac{1}{6} \right) \right]$$

and in the axial channel

$$\Pi_A(Q^2) = \frac{2F^2_A}{\Lambda^2_A} \log \frac{\Lambda^2_{\text{cutoff}}}{Q^2} + \frac{1}{Q^2} \left[ 2F^2_\pi - 2F^2_A \left( \frac{m^2_A}{\Lambda^2_A} - \frac{1}{2} \right) \right]$$

$$+ \frac{1}{Q^4} \left[ F^2_A \Lambda^2_A \left( \frac{m^4_A}{\Lambda^4_A} - \frac{m^2_A}{\Lambda^2_A} + \frac{1}{6} \right) \right]$$

Of course the expansions of $\Pi_{V,A}(Q^2)$ in eqs. (2.8,2.9) have to match their respective OPE expansions [13]. This leads to the constraints

$$\frac{2}{3} \frac{N_c}{16\pi^2} = \frac{F^2_V}{\Lambda^2_V} = \frac{F^2_A}{\Lambda^2_A} ,$$

from the coefficient of the parton-model logarithm in front of the unit operator,

$$F^2_\rho = F^2_V \left( \frac{m^2_V}{\Lambda^2_V} - \frac{1}{2} \right) \quad \text{and} \quad F^2_\pi = F^2_A \left( \frac{m^2_A}{\Lambda^2_A} - \frac{1}{2} \right) ,$$

from the absence of a dimension-two operator in the OPE,

$$\frac{\alpha_s}{12\pi} \langle G_{\mu\nu} G^{\mu\nu} \rangle = -2F^2_\rho M^2_\rho + F^2_V \Lambda^4_V \left( \frac{m^4_V}{\Lambda^4_V} - \frac{m^2_V}{\Lambda^2_V} + \frac{1}{6} \right)$$

$$= F^2_A \Lambda^2_A \left( \frac{m^4_A}{\Lambda^4_A} - \frac{m^2_A}{\Lambda^2_A} + \frac{1}{6} \right) ,$$

from the contribution of the gluon condensate, and

$$- \frac{28}{9} \pi \alpha_s \overline{\langle \psi \psi \rangle}^2 = 2F^2_\rho M^4_\rho - \frac{2}{3} F^2_V \Lambda^4_V \left( \frac{m^6_V}{\Lambda^6_V} - \frac{3 m^4_V}{2 \Lambda^4_V} + \frac{1}{2} m^2_V \right) ,$$

$$\frac{44}{9} \pi \alpha_s \overline{\langle \psi \psi \rangle}^2 = - \frac{2}{3} F^2_A \Lambda^4_A \left( \frac{m^6_A}{\Lambda^6_A} - \frac{3 m^4_A}{2 \Lambda^4_A} + \frac{1}{2} m^2_A \right) ,$$

from the four-quark condensate.
There are two more constraints which we can impose using physical observables which depend only on $\Pi_{V,A}(Q^2)$. These are the coupling $L_{10}$ of the $O(p^4)$ chiral Lagrangian \[14\] and the electromagnetic pion-mass difference. They are expressed in terms of $\Pi_{V,A}(Q^2)$ as

$$L_{10} = -\frac{1}{4} dQ^2 \left( Q^2 \Pi_{LR}(Q^2) \right)_{Q^2=0},$$

and

$$m_{\pi^+} - m_{\pi^0} = -\frac{3\alpha}{8\pi m_{\pi} F_{\pi}^2} \int_0^\infty dQ^2 \, Q^2 \Pi_{LR}(Q^2),$$

where

$$\Pi_{LR}(Q^2) \equiv \frac{1}{2} \left( \Pi_V(Q^2) - \Pi_A(Q^2) \right).$$

Given $F_{\pi}$, the gluon condensate and the combination $\pi \alpha_s \langle \overline{\psi} \psi \rangle^2$, we can solve for the eight parameters $F_{\rho}, M_{\rho}, F_{V,A}, m_{V,A}$ and $\Lambda_{V,A}$ from eqs. (2.10–2.13), and from the solution determine $L_{10}$ and $m_{\pi^+} - m_{\pi^0}$. Thus, the system is overconstrained.

3 Phenomenological results

In principle, one could solve the system of equations eqs. (2.10–2.13) as indicated at the end of the previous section. However, the condensates are relatively poorly known, and we prefer to follow a different approach. We will use as inputs the values of $F_{\pi} = 93$ MeV (we will also consider the value $F_{\pi} = 87$ MeV, which is the value in the chiral limit \[14\]), $M_{\rho} = 770$ MeV\[1\], and $F_{\rho}$ ranging from about 120 to 150 MeV, the latter being the experimental value. (For larger values the combination $\pi \alpha_s \langle \overline{\psi} \psi \rangle^2$ becomes negative, in contradiction with the phenomenologically most favored value. See below.) Choosing a range like this is reasonable, because we are in the large-$N_c$ limit, and expect variations of order 20–30%. It allows us to eliminate the condensates from eqs. (2.12,2.13), and solve the remaining two equations along with eqs. (2.10,2.11) for $F_{V,A}$, $m_{V,A}$, and $\Lambda_{V,A}$, and then calculate the two condensates in terms of these.

The solutions for $F_{V,A}$ are shown in figure 1, those for $m_{V,A}$ in figure 2, and the condensates are shown in figures 3 and 4, as a function of $F_{\rho}$. Below $F_{\rho} \approx 123$ MeV the equations do not have a real solution. In the axial channel, $m_A = M_{a_1} = 1230$ MeV experimentally, while for $F_A = F_{a_1}$ a value of 135 MeV is reported in ref. \[16\]. $F_V$ is not known experimentally. Phenomenological estimates for the condensates are $\alpha_s \langle G^2 \rangle = 0.048 \pm 0.030$ GeV\(^4\) \[4\], and $\pi \alpha_s \langle \overline{\psi} \psi \rangle^2 = 9 \pm 2 \times 10^{-4}$ GeV\(^6\) \[7\]. Looking at figures 1 to 4, we see that very good agreement is obtained for values of $F_{\rho}$ of about 130 – 135 MeV. For $F_{\rho} = 130$ MeV we summarize both our solution along with the experimental values in table 1, in which we also give our results for $F_{\pi} = 87$ instead of 93 MeV.

\[2\]Since one should actually use large-$N_c$ values, one could try to vary the $\rho$ mass as well. However, since our choice $M_{\rho} \approx 770$ MeV already does a good job, we decided to keep the analysis as simple as possible and not to fiddle with $M_{\rho}$, while we wait for more data to come in (for instance on the spectrum).
Figure 1:
Decay constants $F_V$, $F_A$ in eqs. (2.3, 2.6) as a function of $F_\rho$ in eq. (2.3). All in GeV.

We may now use the values obtained for $m_{V,A}$ and $\Lambda_{V,A}$ in order to calculate the vector and axial-vector meson spectrum, from eq. (1.1). The results are shown in table 2. While experimental confirmation for the mass of the $a'_1$ (and higher resonances) is needed, table 2 shows a good agreement between the model and the real world.

Using these results, eq. (1.1) can now be rewritten as

$$M_{V'}^2(n) = M_{\rho'}^2(1 + an) \quad , \quad a \simeq 0.7 \ . \quad (3.17)$$

This is actually very close to the original proposal of ref. [6, 7] where the $\rho$ is not a separate resonance from the equally spaced tower but just another member. In order to make the comparison more clear, rewrite eq. (1.2) in terms of the $\rho'$ mass instead of the $\rho$. Equation (1.2) (with $\alpha = 2$) then reads $M_{V'}^2(n) = M_{\rho'}^2(1 + 2n/3)$ and $a = 2/3$, to be compared with eq. (3.17). However, although the spectrum is quite similar, other quantities are not. As a matter of fact, note that the result $\alpha = 2$ in eq. (1.2) (i.e. $m_{V'}^2/\Lambda_{V'}^2 = 1/2$), can also be derived as a consequence of the lack of dimension-two operator in the OPE: the solution $\alpha = 2$ follows from eq. (2.11) (after setting $F_\rho = 0$, since the $\rho$ meson belongs to the equally spaced tower of refs. [3, 7]). However, eq. (2.12) then yields a negative gluon condensate (the gluon condensate is very sensitive to variations in meson masses and decay constants).

From eq. (2.14), we find for the ES ansatz

$$L_{10} = -\frac{1}{4} \left( F_{\rho'}^2 M_{\rho'}^2 + \left( \frac{m_A^2}{\Lambda_A^2} - \frac{m_V^2}{\Lambda_V^2} \right) \sum_{n=0}^{\infty} \frac{1}{(n + m_A^2/\Lambda_A^2)(n + m_V^2/\Lambda_V^2)} + \frac{2}{3} \frac{N_c}{16\pi^2} \log \frac{F_A^2}{F_V^2} \right) . \quad (3.18)$$

For $F_\rho = 130$ MeV the values we obtain are recorded in table 1. The value of $L_{10}$ varies by less than 15% over the range of $F_\rho$ we considered.

For $m_{\pi^+} - m_{\pi^0}$ the expression following from eq. (2.15) is rather lengthy, and we relegate it to an appendix. Again, the values for $F_\rho = 130$ MeV are presented in
Table 1: Results obtained for two different values of $F_{\pi}$. $M_\rho = 770$ MeV is input, and we choose $F_\rho = 130$ MeV.

|               | $F_{\pi} = 87$ MeV | $F_{\pi} = 93$ MeV | Experiment       |
|---------------|--------------------|--------------------|------------------|
| $F_\rho$ (MeV)| 130                | 130                | 153 ± 4          |
| $M_\rho$ (GeV)| 0.77               | 0.77               | 0.7700 ± 0.0008  |
| $F_V$ (MeV)   | 140                | 130                | no available data|
| $m_V$ (GeV)   | 1.44               | 1.42               | 1.465 ± 0.025    |
| $\Lambda_V$ (GeV)| 1.21              | 1.17               | see Table 2      |
| $F_{a_1} = F_A$ (MeV)| 160              | 170                | 135 ± 30         |
| $M_{a_1} = m_A$ (GeV)| 1.27             | 1.35               | 1.230 ± 0.040    |
| $\Lambda_A$ (GeV)| 1.42              | 1.50               | see Table 2      |
| $\alpha_s(G^2)$ (GeV^4) | 0.01            | 0.02               | 0.048 ± 0.030    |
| $\pi\alpha_s(\bar{\psi}\psi)^2(10^{-4} \text{GeV}^6)$ | 7               | 9                  | (9 ± 2)          |
| $m_{\pi^+} - m_{\pi^0}$ (MeV) | 4.2             | 4.7                | 4.5936 ± 0.0005  |
| $L_{10}(\mu = M_\rho) \times 10^{-3}$ | −5.2           | −5.6               | −5.13 ± 0.19     |

Table 2: Vector and axial-vector mass spectra in GeV with the corresponding experimental values, where available.

|               | $F_{\pi} = 87$ MeV | $F_{\pi} = 93$ MeV | Exp.  |
|---------------|--------------------|--------------------|-------|
| $\rho'$       | 1.4                | 1.9                | 1.47  |
| $\rho''$      | 1.9                | 2.2                | 1.70  |
| $\rho'''$     | 2.2                | 2.5                | 2.15  |
| $\rho''''$    | 2.5                | 1.3                | no data|
| $a_1$         | 1.3                | 1.9                | 1.23  |
| $a_1'$        | 1.9                | 2.4                | 1.64  |
| $a_1''$       | 2.4                | 2.5                | 1.8   |

Frustrated
It is clear that the ES ansatz favors a lower value of $F_{\rho}$ of around 130 MeV. Since we find no solution for $F_{\rho} \leq 120$ MeV, the solution is quite constrained.

In principle, our model may be used to predict hitherto unknown values of quantities of phenomenological interest. Since we have restricted our attention to vector and axial-vector two-point functions, there are not many such quantities. One such quantity is a $K \rightarrow \pi$ matrix element of the electro-magnetic penguin operator $Q_{7}$ which can be expressed solely in terms of $\Pi_{LR}$ [20]:

$$Q_{7}^{2/2}(\mu) \equiv \langle \pi^{+}|Q_{7}^{2/2}|K^{+}\rangle(\mu) = \frac{3}{8\pi^{2}f_{\pi}^{2}} \int_{0}^{\mu^{2}} dQ^{2} Q^{4} \Pi_{LR}(Q^{2}) . \quad (3.19)$$

Even though this quantity by itself is of limited phenomenological interest, it is interesting to compare the value we obtain with that obtained from Lowest-Meson Dominance [12] (LMD; cf. section 4 below). The difference in numerical values gives an indication of the robustness of this sum-rule approach with respect to variations in the ansatz taken for the spectral functions. In addition, lattice [21] and other phenomenological estimates [22, 23] exist [24]. Note that this quantity is scale dependent, because the integral is logarithmically divergent. This is because terms up to $1/Q^{4}$ cancel in $\Pi_{LR}$ (cf. eq. (2.16)), but not terms of order $1/Q^{6}$. We will return to this quantity in the next section.

---

3The physical $K \rightarrow (\pi\pi)_{I=2}$ matrix element is related to this simpler matrix element by ChPT.
4 Comparison with the Lowest-Meson Dominance approximation

It is interesting to compare the ES ansatz which we have been considering thus far with the Lowest-Meson Dominance (LMD) ansatz of ref. [12]. In the ES case, the (leading) perturbative values of $\Pi_{V,A}(Q^2)$ for large $Q^2$ come from the assumed Regge-like spectrum. One may also directly incorporate pQCD into an ansatz by approximating $\Pi_{V,A}(Q^2)$ with the expression one finds in pQCD which, to leading order in $\alpha_s$, reads

$$\frac{1}{\pi} \text{Im} \Pi_V(t)|_{pQCD} = \frac{2}{3} \frac{N_c}{16\pi^2} \theta(t - s_0).$$

(4.20)

Here a parameter $s_0$ is introduced which determines the onset of the perturbative continuum, above which we expect pQCD to give accurate results. For $t \geq s_0$ this expression replaces the sums in eqs. (2.5,2.6). All resonances for $t < s_0$ are kept; for an illustration, see figure 5. In the simplest version of this approach, one keeps one resonance each in the vector and axial-vector channels. Since, by definition, $s_0$ in eq. (4.20) marks the onset of the perturbative continuum, it is taken to be the same for both vector and axial-vector channels, because chiral symmetry is not broken in perturbation theory. This ansatz has recently been shown to yield a successful description of Aleph data for the moments of the $\Pi_{LR}$ function [23], it successfully predicts pseudoscalar decays [26] and, also recently, it has allowed an analytic calculation of the $K^0 - \bar{K}^0$ mixing parameter, $B_K$ [27].

Qualitatively one can understand the transition between ES and LMD in the following way. Low-energy observables are dominated by the lowest individual resonances to the extent that the effect of the continuum is suppressed by the fact that $s_0$ is larger than the mass (squared) of these individual resonances. For the LMD case, the continuum (4.20) produces a logarithm just like the integral in eq. (2.7) does,
with $s_0 = 24\pi^2 F_\rho^2 / N_c = m_V^2$ (see refs. [12, 28] for details on the LMD ansatz). Taking also into account the leading correction term in eq. (2.7) to order $1/Q^2$, we find $s_0 = m_V^2 \left(1 - \frac{1}{2} \Lambda_V^2 / m_V^2\right)$. Taking $\Lambda_V^2 / m_V^2$ smaller (keeping $F_\rho^2 / \Lambda_V^2$ fixed, cf. eq. (2.10)) clearly approaches the perturbative approximation of eq. (4.20), with $s_0 \to m_V^2$, and with the correction terms on the right-hand side of eq. (2.7) becoming smaller (for any value of $Q^2$). In this sense, the LMD approach can be seen as following from the ES approach in the limit $\Lambda_V^2 / m_V^2 \to 0$, at least in the vector channel. (In the axial channel, a similar argument would apply if we had taken one separate resonance apart from the tower in the ES approach. In our case, $m_A$ is relatively small, and $\frac{1}{2} \Lambda_A^2 / m_A^2$ is substantially bigger than $\frac{1}{2} \Lambda_V^2 / m_V^2$.)

Of course, it is easy to put $\alpha_s$ corrections into eq. (4.20) in the LMD approach, while this is less straightforward in the ES approach (it requires a more complicated $n$ dependence of $M_{V,A}^2$ and $F_{V,A}^2$). One can also improve the extent to which LMD approximates ES by keeping more individual resonances explicit in the LMD ansatz. This will move up $s_0$ or equivalently $m_V^2$, hence reducing the size of $\Lambda_V^2 / m_V^2$.

Using the same information from the OPE, and the relation $F_\rho^2 = 2 F_\pi^2$ [29], one finds that in the LMD approach $s_0$ is determined to be $s_0 = (4\pi F_\rho)^2$ from the vector channel, and then, from the axial-vector channel, that $M_{a1}$ is close to, but smaller than $\sqrt{s_0}$ [28, 3]. This implies that within LMD the $a_1$ resonance has to be kept, and cannot be folded into the perturbative continuum, eq. (4.20). Higher resonances are well above $\sqrt{s_0}$ (whether we take their experimental values, or those predicted from ES, cf. table 2). We will therefore compare the phenomenology of the ES ansatz, summarized in tables 1 and 2, with that of the LMD ansatz with one explicit resonance in each channel (in addition to the pion). The LMD results are presented in table 3. All quantities displayed in table 3 are predicted in terms of $F_\pi$ with the LMD ansatz [12, 28, 28].

---

4In comparing with ref. [28], note that $F_\rho = f_\rho M_\rho$, etc.
the relation $F_\rho^2 = 2F_\pi^2$, but (obviously) no predictions are made for excited states in both channels in this case. In the comparison with the results in table 1, one should keep in mind that the relation $F_\rho^2 = 2F_\pi^2$ was not used with the ES ansatz. However, for $F_\pi = 87$, resp. 93 MeV, $F_\rho = 130$ MeV satisfies this relation at the 10, resp. 2\% level.

Comparing the results of ES and LMD, it is clear that both approaches yield consistent results, especially if one takes into account that all results are in the large-$N_c$ and chiral limits, so that deviations of less than 30\% are not significant. Possible exceptions to this are $F_{a_1}$ and $\pi\alpha_s\langle \bar{\psi}\psi \rangle^2$, for which the LMD values are smaller, resp. larger, than the ES value. Unfortunately, the experimental values, while closer to the ES values, are rather poorly known. Possibly with these two exceptions, the differences between tables 1 and 3 give a good indication of the spread one can expect when one combines QCD techniques such as large $N_c$, OPE and pQCD with an ansatz for the spectrum in order to obtain explicit solutions.

Finally, we compare the ES prediction for $Q_7^{3/2}(\mu)$ with the LMD prediction for the same number. First, we show the function $Q_4^4\Pi_{LR}(Q^2)$ in figure 6, for both the ES and LMD ansätze. We evaluated the ES value of $Q_7^{3/2}(\mu)$ at $\mu = 2$ GeV from the area under the curve using $\mu$ as a sharp cutoff (choosing $F_\rho = 130$ MeV and $M_\rho = 770$ MeV), and quote the LMD value from ref. [20], finding,

$$Q_7^{3/2}(\mu = 2\text{ GeV}) = \begin{cases} -0.021 \text{ (GeV)}^4, & \text{ES, } F_\pi = 93 \text{ MeV}, \\ -0.019 \text{ (GeV)}^4, & \text{ES, } F_\pi = 87 \text{ MeV}, \\ -0.024 \text{ (GeV)}^4, & \text{LMD.} \end{cases}$$  

(4.21)
Table 3: LMD results obtained for two different values of $F_\pi$.

|                  | $F_\pi = 87$ MeV | $F_\pi = 93$ MeV |
|------------------|------------------|------------------|
| $F_\rho$ (MeV)   | 122              | 131              |
| $M_\rho$ (GeV)   | 0.77             | 0.82             |
| $F_{a_1}$ (MeV)  | 87               | 93               |
| $M_{a_1}$ (GeV)  | 1.08             | 1.16             |
| $\alpha_s \langle G^2 \rangle$ (GeV$^4$) | 0.01 | 0.02 |
| $\pi \alpha_s \langle \bar{\psi} \psi \rangle^2 (10^{-4}$GeV$^6)$ | 13 | 19 |
| $m_{\pi^+} - m_{\pi^0}$ (MeV) | 5.2 | 6.0 |
| $L_{10}(\mu = M_\rho)$ (10$^{-3}$) | $-4.8$ | $-4.8$ |

We took again $F_\rho^2 = 2F_\pi^2$ in the LMD case, in which case the LMD value is independent of $F_\pi$ (for a fixed $M_\rho$). We see that, as in the case of most other quantities, the predicted value we obtain for $Q_\pi^{3/2} (\mu = 2$ GeV$)$ is quite robust to the variation of the ansatz for the vector and axial-vector spectral functions we have been considering. The values we obtain are fully consistent with each other within the context of the large-$N_c$ and chiral limits.

![Figure 6:](image)

The function $Q^4 \Pi_{LR}(Q^2)$ (GeV$^4$) for our equal-spacing model (ES) (solid line) and its lowest meson dominance version (LMD) (dashed line) as a function of the euclidian momentum $Q^2$ (GeV$^2$).

5 Conclusion

In this paper we have considered the vector and axial-vector current two-point functions in QCD. The combination of taking the large-$N_c$ and chiral limits, the use of
asymptotic freedom and the OPE constrain the possible form of these two-point
functions. However, this is not enough to obtain concrete predictions: more knowledge, or
lacking that, an ansatz for the spectral functions in these two channels is still needed.

A particular ansatz which has been considered in the literature is the LMD ansatz,
in which the high-energy part of the spectral functions is calculated in pQCD, while
one (or a few) explicit resonances are kept at low to medium energy. This approach has
been quite successful in reproducing known phenomenology, and should therefore be
useful in estimating as yet unknown hadronic quantities, such as weak matrix elements.
However, although LMD yields a rather intuitive picture it is not a simple matter to
give it a fully systematic treatment. Such systematic treatment should, in particular,
produce a theory of the so-called quark-hadron duality violations [30], a problem which
is clearly beyond the scope of the present work. Consequently, determinations of
the systematic errors associated with this approach have to rely at some point on
phenomenological experience.

One way to address this issue is to vary the nature of the ansatz within the confines
of our general picture of the strong interactions, and this is what we have done in
this paper. Instead of LMD, we have based our ansatz on Regge theory, except for the
lowest states in each channel (the $\rho$ meson and the pion). For the precise form of the ES
ansatz, see eqs. (2.5, 2.6). The hadron phenomenology calculated with this assumption
is again very good, and agrees well with what is found with LMD (cf. tables 1 and
3). Possible exceptions perhaps are $F_{a_1}$ and $\pi \alpha_s \langle \bar{\psi} \psi \rangle^2$; experimentally these quantities
are not that well-known. In general, because of the use of large-$N_c$ and chiral limits,
agreement at the 20-30% level, between ES and LMD, or with experiment, should be
considered satisfactory.

The gluon and four-quark condensates are rather sensitive to the values of other
quantities (masses and decay constants). Within our solution (cf. table 1), we find that
the $\rho$ comes out to be very close to lying on the daughter trajectory predicted by Regge
theory in that $M_{\rho}'^2 \approx M_{\rho}^2 + \Lambda_V^2$ and $F_{\rho} \approx F_V$. However, requiring the $\rho$ to be precisely
on this trajectory leads to a negative gluon condensate. Since large $N_c$ (supplemented
with Regge Theory) and pQCD suggest that asymptotically the tower of resonances
has to be equally spaced, it makes intuitive sense to model the resonance spectrum
with an equally spaced tower above 1 GeV, keeping those below 1 GeV separate. The
ES ansatz goes beyond LMD in that it yields a value for the spacing between the
resonances (the parameters $\Lambda_{V,A}$), and leads to a reasonable resonance pattern in the
vector and axial-vector channels (cf. table 2).

We have then used the ES ansatz in order to predict a weak matrix element, $Q_{\gamma}^{3/2} (\mu)$,
which is of interest in the context of non-leptonic kaon decays. We find a value rather
close to that predicted from LMD, thus lending support to the credibility of this pre-
diction.

Finally, we discussed the extent to which the two ansätze are equivalent. We qual-
itatively argued that in the limit in which we take the spacing between the resonances
in the ES tower to zero one recovers the LMD approach. However, with LMD, we kept
both the $\rho$ and the $a_1$ as separate resonances (because one finds that the perturbative
threshold $s_0$ is larger than $M_{a_1}^2$), while we did take the $a_1$ to be the first resonance of
the ES tower.  
After all these years we still do not have a real understanding of the large-$N_c$ expansion in QCD. Clearly the ES model is not meant to be a substitute for large-$N_c$ QCD at the theoretical level but, given its phenomenological success, it may help in our understanding of this limit of QCD.

Acknowledgments

MG and SP thank E. de Rafael for useful discussions and M. Knecht for reading the manuscript. SP thanks A. Bramon for interesting conversations on Regge Theory and F. Jegerlehner for drawing his attention to ref. [10] in the course of this work. MG and SP thank the IFAE at the Universitat Autònoma de Barcelona and the Department of Physics at Washington University for mutual hospitality. MG was supported in part by the US Dept. of Energy and by a fellowship of the Spanish Government SAB1998-0171. SP was supported by CICYT-AEN99-0766 and by TMR, EC-Contract No. ERBFMRX-CT980169 (EuroDaphne).

Appendix

The expression for the electromagnetic pion mass difference, following from the ES ansatz follows from eq. (2.15), is

\[
m_{\pi^+} - m_{\pi^0} = \frac{3\alpha}{8\pi m_{\pi} f_{\pi}^2} \left\{ -\frac{1}{2} F_{\rho}^2 M_{\rho}^2 + \frac{1}{2} F_{A}^2 m_{A}^2 
- \frac{N_c}{24\pi^2} \left[ \frac{11}{12} m_{\pi}^4 \log \frac{M_{V}^2}{m_{\pi}^2} - \frac{7}{12} (m_{V}^2 + \Lambda_{V}^2)^2 \log \frac{M_{V}^2}{m_{V}^2 + \Lambda_{V}^2} 
+ \frac{1}{6} (m_{V}^2 + 2\Lambda_{V}^2)^2 \log \frac{M_{\rho}^2}{m_{V}^2 + 2\Lambda_{V}^2} \right] 
+ \frac{N_c}{24\pi^2} \left[ \frac{11}{12} (m_{A}^2 + \Lambda_{A}^2)^2 \log \frac{m_{A}^2}{m_{A}^2 + \Lambda_{A}^2} 
- \frac{7}{12} (m_{A}^2 + 2\Lambda_{A}^2)^2 \log \frac{m_{A}^2}{m_{A}^2 + 2\Lambda_{A}^2} + \frac{1}{6} (m_{A}^2 + 3\Lambda_{A}^2)^2 \log \frac{m_{A}^2}{m_{A}^2 + 3\Lambda_{A}^2} \right] 
+ \frac{1}{2} \left[ 2F_{\rho}^2 M_{\rho}^2 - \frac{24\pi^2}{N_c} F_{\rho}^4 + \frac{1}{12} F_{V}^2 \Lambda_{V}^2 \right] \log \frac{m_{A}^2}{M_{\rho}^2} 
- 3F_{V}^2 \Lambda_{V}^2 G \left( \frac{m_{V}^2}{\Lambda_{V}^2} \right) + 3F_{A}^2 \Lambda_{A}^2 G \left( \frac{m_{A}^2}{\Lambda_{A}^2} \right) \right\}, \tag{5.22}
\]

where

\[
G(x) = \sum_{n=0}^{\infty} \left( \frac{1}{(n+x)^2} + 4(n+1+x)^3 \log \frac{n+x}{n+1+x} - 3(n+2+x)^3 \log \frac{n+x}{n+2+x} 
+ \frac{4}{3} (n+3+x)^3 \log \frac{n+x}{n+3+x} - \frac{1}{4} (n+4+x)^3 \log \frac{n+x}{n+4+x} \right). \tag{5.23}
\]
This expression results from calculating the integral in eq. (2.15), which is finite, because \( \Pi_{LR}(Q^2) \sim 4\pi\alpha_s(\bar{\psi}\psi)^2/Q^6 \) for large \( Q^2 \). By partial integration
\[
\int_0^\infty dQ^2 Q^2 \Pi_{LR}(Q^2) = -\frac{1}{2} \int_0^\infty dQ^2 Q^4 \frac{d}{dQ^2} \Pi_{LR}(Q^2) .
\]
(5.24)
The sums over \( n \) inside \( \frac{d}{dQ^2} \Pi_{LR}(Q^2) \) now converge. Next, one would like to make use of the fact that terms up to order \( 1/Q^8 \) in \( \frac{d}{dQ^2} \Pi_{LR}(Q^2) \) vanish, as follows from eqs. (2.10–2.12). This can be done by writing
\[
\sum_{n=0}^{\infty} \frac{F_V^2}{(Q^2 + m_V^2 + n\Lambda_V^2)^2} = \sum_{n=0}^{\infty} \left( \frac{F_V^2}{(Q^2 + m_V^2 + n\Lambda_V^2)(Q^2 + m_V^2 + (n+1)\Lambda_V^2)} \right) + \frac{F_V^2}{(Q^2 + m_V^2 + n\Lambda_V^2)^2(Q^2 + m_V^2 + (n+1)\Lambda_V^2)} \right)
= \frac{F_V^2}{\Lambda_V^2(Q^2 + m_V^2)} + \sum_{n=0}^{\infty} \frac{F_V^2}{(Q^2 + m_V^2 + n\Lambda_V^2)^2(Q^2 + m_V^2 + (n+1)\Lambda_V^2)} ,
\]
(5.25)
using
\[
\sum_{n=0}^{N} \frac{1}{(p + nq)(p + (n + 1)q)\ldots(p + (n + \ell)q)} = \frac{1}{\ell q} \left( \frac{1}{p(p + q)\ldots(p + (\ell - 1)q)} - \frac{1}{(p + (N+1)q)\ldots(p + (N + \ell)q)} \right)
\]
(5.26)
with \( N = \infty \) and \( \ell = 1 \). This process can be repeated by employing this equation for higher values of \( \ell \) until the “left-over” sum on the right-hand side of eq. (5.25) is of order \( 1/Q^8 \), so that the integral over \( Q^2 \) in eq. (5.24) can be done term by term. The remaining terms combine with similar terms in the axial channel, and the \( Q^2 \) integral over these combined terms is finite as well, by virtue of eqs. (2.10–2.12). One ends up with the expression given in eqs. (5.22, 5.23).

References

[1] G. ‘t Hooft, Nucl. Phys. B72 (1974) 461.

[2] E. Witten, Nucl. Phys. B79 (1979) 57. See also A. Manohar, hep-ph/9802419, “Probing the Standard Model of Particle Interactions”, F. David and R. Gupta eds., Les Houches Summer School in Particle Physics, July 1997.

[3] G. ‘t Hooft, Nucl. Phys. B75 (1974)461; C.G. Callan et al., Phys. Rev. D13 (1976) 1649; M.B. Einhorn, Phys. Rev. D14 (1976) 3451.

[4] P.D.B. Collins, Phys. Reports C1 (1971) 103.
For a very recent proposal in this direction see A.D. Patel, hep-lat/0012004, presented at “Lattice 2000”, Bangalore, India.

A. Bramon et al., Phys. Lett. 41B (1972) 609. This work initially set $\alpha = 4$ but this was fixed later on (A. Bramon, private communication.); M. Greco, Nucl. Phys. B63 (1973) 398.

J.J. Sakurai, Phys. Lett. 46B (1973) 207.

G. Veneziano, Nuovo Cim. 57A (1968) 190.

See for instance, F.J. Yndurain, Phys. Rept. 320 (1999) 287.

B.V. Geshkenbein, Sov. J. Nucl. Phys. 49 (1989) 705.

For a very related analysis in which, however, a finite number of resonances is considered, see M. Knecht and E. de Rafael, Phys.Lett. B424 (1998) 335.

S. Peris, M. Perrottet and E. de Rafael, JHEP05 (1998) 011; M. Knecht, S. Peris and E. de Rafael, Nucl. Phys. (Proc. Suppl.) B86 (2000) 279.

M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B147 (1979) 385; 447.

J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465.

D.E. Groom et al., Eur. Phys. Jour. C15 (2000) 1 (particle data tables).

G. Ecker et al., Nucl. Phys. B321 (1989) 311.

M. Davier et al., Phys. Rev. D58 (1998) 096014.

C.A. Baker et al., Phys. Lett. B449 (1999) 114.

V. Dorofeev, VES Collaboration, hep-ex/9905002.

M. Knecht, S. Peris and E. de Rafael, Phys. Lett. B457 (1999) 227, Nucl. Phys. (Proc. Suppl.) B86 (2000) 279.

A. Donini et al., Phys. Lett. B470 (1999) 233, and references therein; R. Gupta, in Physics of Mass, Proceedings of International Conference Orbis Scientiae 1997 II, Miami Beach, Florida, 1997 (eds. B. Kursunoglu et al.), hep-ph/9801412, and references therein.

J. Donoghue and E. Golowich, Phys. Lett. B478 (2000) 172.

S. Narison, hep-ph/0004247.

A critical comparison between the different estimates is underway: M. Knecht, S. Peris and E. de Rafael, in preparation.

S. Peris, B. Phily and E. de Rafael, Phys. Rev. Lett. 86 (2001) 14; E. de Rafael, hep-ph/0010209; S. Narison, hep-ph/0012019 and references therein.
[26] M. Knecht, S. Peris, M. Perrottet and E. de Rafael, Phys. Rev. Lett. 83 (1999) 5230.

[27] S. Peris and E. de Rafael, Phys. Lett. B490 (2000) 213; S. Peris, hep-ph/0010162, presented at the International Euroconference in Quantum Chromodynamics: 15 Years of QCD, Montpellier, France, July 2000.

[28] M. Golterman and S. Peris, Phys. Rev. D61 (2000) 034018.

[29] G. Ecker et al., Phys. Lett. B223 (1989) 425.

[30] M. Shifman, hep-ph/0009131, to be published in “At the Frontier of Particle Physics/Handbook of QCD,” edited by M. Shifman, World Scientific, 2001.