Article

On Geometric Properties of Bessel–Struve Kernel Functions in Unit Disc

Najla M. Alarifi 1,* and Saiful R. Mondal 1,†

1 Department of Mathematics, Imam Abdulrahman Bin Faisal University, Dammam 31113, Saudi Arabia
2 Department of Mathematics and Statistics, College of Science, King Faisal University, Al Hasa 31982, Saudi Arabia; smondal@kfupm.edu.sa
* Correspondence: nalareefi@iau.edu.sa
† These authors contributed equally to this work.

Abstract: The Bessel–Struve kernel function defined in the unit disc is used in this study. The Bessel–Struve kernel functions are generalized in this article, and differential equations are derived. We found conditions under which the generalized Bessel–Struve function is Lemniscate convex by using a subordination technique. The relation between the Janowski class and exponential class is also derived.

Keywords: Bessel–Struve kernel functions; subordination; Lemniscate convex; Janowski class; exponential class

MSC: 30C40; 33C10; 30C45; 30C80

1. Introduction

This article focused on the Bessel–Struve Kernel function. This study explores a range of possible geometric features, including Lemniscate and exponential Carathéodory properties, and Lemniscate convexity. The details of these particular functions, as well as the geometric properties required, are explained further below.

1.1. Bessel–Struve Kernel Functions

Consider the Bessel–Struve kernel function $B_\nu$ defined on the unit disk $D = \{ z : |z| < 1 \}$ as

$$B_\nu(z) := j_\nu(iz) - ih_\nu(iz), \quad \nu > -\frac{1}{2},$$

where $j_\nu(z) := 2^{\nu}z^{-\nu}\Gamma(\nu+1)J_\nu(z)$ and $h_\nu(z) := 2^{\nu}z^{-\nu}\Gamma(\nu+1)H_\nu(z)$ are, respectively, known as the normalized Bessel functions and the normalized Struve functions of the first kind of index $\nu$. More information about the Bessel and Struve functions can be found in [1,2]. The Bessel–Struve transformation and the Bessel–Struve kernel functions have appeared in many articles [3–7]. In [6], Hamem et al. studied an analog of the Cowling–Price theorem for the Bessel–Struve transform defined on a real domain and also provide Hardy’s type theorem associated with this transform. The Bessel–Struve intertwining operator on $\mathbb{C}$ is considered in [4], and $\mathbb{R}$ is studied in [7]. The Fock space of the Bessel–Struve kernel functions is discussed in [5]. The monotonicity and log-convexity properties for the Bessel–Struve kernel and the ratio of the Bessel–Struve kernel and the Kummer confluent hypergeometric function are investigated in [3].

The kernel $z \mapsto B_\nu(\lambda z)$, $\nu \in \mathbb{C}$ is the unique solution of the initial value problem

$$L_\nu u(z) = \lambda^2 u(z), \quad u(0) = 1, u'(0) = \frac{\nu\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})},$$

where

$$L_\nu u(z) := z^\nu J_\nu(z) - iz^{\nu+1}J_\nu(iz).$$

Citation: Alarifi, N.M.; Mondal, S.R. On Geometric Properties of Bessel–Struve Kernel Functions in Unit Disc. Mathematics 2022, 10, 2516. https://doi.org/10.3390/math10142516
Here, $L_v$, $v > -1/2$ is the Bessel–Struve operator given by

$$L_v(u(z)) := \frac{d^2 u}{dz^2}(z) + \frac{2v + 1}{z} \left( \frac{du}{dz}(z) - \frac{du}{dz}(0) \right).$$

Now, the Bessel functions and the Struve functions of order $v$, respectively, have the power series

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n+v}}{n! \Gamma(v+n+1)} \quad \text{and} \quad H_v(z) := \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n+v+1}}{n! \Gamma(n+v+\frac{3}{2}) \Gamma(n+\frac{1}{2})}.$$  

This implies that $B_v$ (taking $\lambda = 1$) possesses the power series

$$B_v(z) := \sum_{n=0}^{\infty} \frac{\Gamma(v+1) \Gamma(n+1)}{\sqrt{\pi} n! \Gamma(\frac{n+1}{2})} z^n.$$  

The kernel $B_v$ also have the integral representation

$$B_v(z) := \frac{2\Gamma(v+1)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \int_0^1 (1-t^2)^{v-\frac{1}{2}} e^{zt} dt.$$  

It is evident from (2) and (3) that $B_v$ satisfies the differential equation

$$z^2 B''_v(z) + (2v+1)z B'_v(z) - z^2 B_v(z) = zM,$$  

where $M = 2\Gamma(v+1) \left( \sqrt{\pi} \Gamma(v+\frac{1}{2}) \right)^{-1}.$

From (3) a computation yields that $B_v$ satisfies the recurrence relation

$$zB'_v(z) = 2vB_{v-1}(z) - 2vB_v(z).$$

This article considers the function defined by

$$B_{v,b,c}(z) := \Gamma(v+\frac{b+1}{2}) 2^v z^{-v} (J_{v,b,c}(iz) - i \sqrt{c} S_{v,b,c}(iz)), \quad v > -\frac{b+1}{2}, \quad i = \sqrt{-1}.$$  

Here, $J_{v,b,c}$ is the Generalized Bessel function and $S_{v,b,c}$ is the Generalized Struve function. A detailed study about the function $J_{v,b,c}$ can be seen in the book [8], while the function $S_{v,b,c}$ was first studied in [9]. There have been several articles where geometric properties such as close-to-convexity, starlikeness and convexity, radius of starlikeness and convexity of Bessel and Struve functions, along with their generalizations, were studied [9–19].

More development and properties about the Generalized Bessel–Struve kernel function $B_{v,b,c}$ along with the differential equation is discussed in Section 2. More specifically, the power series of $B_{v,b,c}$ is established, and it is shown that $B_{v,b,c}$ is a solution of a second-order differential equation.

Section 3 is devoted to the study of the geometric properties $B_{v,b,c}$. In particular, we derived the conditions on parameters $v, b, c$ for which $B_{v,b,c}$ belongs to specific classes of geometric function theory, namely Lemniscate, Exponential and Janowski class. Detailed notes about geometric classes and terminologies are given below.

1.2. Basic Concept of Geometric Properties and Require Lemmas

Let $A$ denote the class of normalized analytic functions $f$ in the open unit disk $D = \{ z : |z| < 1 \}$ satisfying $f(0) = 0 = f'(0) - 1$. Denote by $S^*$ and $C$, respectively, the widely studied subclasses of $A$ consisting of univalent (one-to-one) starlike and convex functions. Geometrically, $f \in S^*$ if the linear segment $tw$, $0 \leq t \leq 1$, lies com-
pletely in $f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$, while $f \in \mathcal{C}$ if $f(\mathbb{D})$ is a convex domain. Related to these subclasses is the Carathéodory class $\mathcal{P}$ consisting of analytic functions $p$ satisfying $p(0) = 1$ and $\Re p(z) > 0$ in $\mathbb{D}$. Analytically, $f \in \mathcal{S}^*$ if $zf''(z)/f'(z) \in \mathcal{P}$, while $f \in \mathcal{C}$ if $1 + zf''(z)/f'(z) \in \mathcal{P}$.

For two analytic functions $f$ and $g$ in $\mathbb{D}$, the function $f$ is subordinate to $g$, written $f \prec g$, or $f(z) \prec g(z)$, $z \in \mathbb{D}$, if there is an analytic self-map $\omega$ of $\mathbb{D}$ satisfying $\omega(0) = 0$ and $f(z) = g(\omega(z))$, $z \in \mathbb{D}$.

Consider now the class $\mathcal{P}[\varphi]$ of analytic functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec \varphi(z)$, where $\varphi$ is an analytic function with positive real part on $\mathbb{D}$, $\varphi(0) = 1$ and $\varphi'(0) > 0$. In a sequel, this article will consider three different $\varphi$, namely $\varphi(z) = (1 + Az)/(1 + Bz)$, $\varphi(z) = \sqrt{1 + z}$ and $\varphi(z) = e^z$.

For $-1 \leq B < A \leq 1$ and $\varphi(z) = (1 + Az)/(1 + Bz)$, denote the class as $\mathcal{P}[A, B]$. This family $\mathcal{P}[A, B]$ has been widely studied by several authors and most notably by Janowski in [20], and the class also refers to a Janowski class of functions. The class $\mathcal{P}[A, B]$ contains several known classes of functions for judicious choices of $A$ and $B$. For instance, if $0 \leq \beta < 1$, then $\mathcal{P}[1 - 2\beta, -1]$ is the class of functions $p(z) = 1 + c_1 z + \cdots$ satisfying $\Re p(z) > \beta$ in $\mathbb{D}$. In the limiting case $\beta = 0$, the class reduces to the classical Carathéodory class $\mathcal{P}$.

The class of Janowski starlike functions $\mathcal{S}^*[A, B]$ consists of $f \in \mathcal{A}$ satisfying

$$zf''(z)/f'(z) \in \mathcal{P}[A, B],$$

while the Janowski convex functions $\mathcal{C}[A, B]$ are functions $f \in \mathcal{A}$ satisfying

$$1 + zf''(z)/f'(z) \in \mathcal{P}[A, B].$$

For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] := \mathcal{S}^*(\beta)$ is the classical class of starlike functions of order $\beta$; $\mathcal{S}^*[1 - \beta, 0] := \mathcal{S}_1^*(\beta) = \{f \in \mathcal{A} : |zf''(z)/f'(z) - 1| < 1 - \beta\}$, and $\mathcal{S}^*[\beta, -\beta] := \mathcal{S}^*[\beta] = \{f \in \mathcal{A} : |zf''(z)/f'(z) - 1| < \beta |zf''(z)/f'(z) + 1|\}$. These are all classes that have been widely studied; see, for example, in the works of [20–22].

The next important class is related to the right half of the lemniscate of Bernoulli given by $\{w : |w^2 - 1| = 1\}$. The functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec \sqrt{1 + z}$ are known as lemniscate Carathéodory functions, and the corresponding class is denoted by $\mathcal{P}_L$. A lemniscate Carathéodory function is also a Carathéodory function and, hence, univalent. The class $\mathcal{S}_L$, known as lemniscate starlike, consists of functions $f \in \mathcal{A}$ such that $zf''(z)/f'(z) \prec \sqrt{1 + z}$. The class $\mathcal{K}_L = \{f \in \mathcal{A} : 1 + (zf''(z))/f'(z) \prec \sqrt{1 + z}\}$ is known as a class of lemniscate convex functions.

The third important class that is considered in the sequel relates to the exponential functions $e^z$. The functions $p(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec e^z$ are known as exponential Carathéodory function, and the corresponding class is denoted by $\mathcal{P}_E$. The class $\mathcal{S}_E$, known as exponential starlike, consists of functions $f \in \mathcal{A}$ such that $zf''(z)/f'(z) \prec e^z$. The class $\mathcal{K}_E = \{f \in \mathcal{A} : 1 + (zf''(z))/f'(z) \prec e^z\}$ is known as the class of exponential convex functions.

The principle of differential subordination [23,24] provides an important tool in the investigation of various classes of analytic functions. The following results are useful in a sequel.

**Lemma 1** ([23,24]). Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfy

$$\Psi(i \rho, \sigma; z) \notin \Omega$$

for $z \in \mathbb{D}$, and real $\rho, \sigma$ such that $\sigma \leq - (1 + \rho^2)/2$. If $p$ is analytic in $\mathbb{D}$ with $p(0) = 1$, and $\Psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\Re p(z) > 0$ in $\mathbb{D}$.
Lemma 2 ([25]). Let $Ω ⊂ ℂ$, and $Ψ : ℂ × ℍ → ℂ$ satisfy

$$Ψ(r,s,t;z) \notin Ω$$

whenever $z \in ℍ$, and for $m \geq n \geq 1$, $−π/4 \leq θ \leq π/4$,

$$r = \sqrt{2 \cos(2θ)} e^{iθ}, \quad s = \frac{m e^{3iθ}}{2\sqrt{2 \cos(2θ)}} \quad \text{and} \quad \text{Re}\left((t+s)e^{-3iθ}\right) \geq \frac{3m^2}{8\sqrt{2\cos(2θ)}}. \quad (9)$$

If $Ψ(p(z),zp'(z),z^2p''(z);z) ∈ Ω$ for $z ∈ ℍ$, then $p(z) < \sqrt{1+z}$ in $ℍ$.

Lemma 3 ([26]). Let $Ω ⊂ ℂ$, and $Ψ : ℂ^3 × ℍ → ℂ$ satisfy $Ψ(r,s,t;z) \notin Ω$ whenever $z \in ℍ$, and for $m \geq 1$, $θ \in (0,2π)$,

$$r = e^{iθ}, \quad s = me^{iθ} e^{iθ} \quad \text{and} \quad \text{Re}\left(1 + \frac{t}{s}\right) \geq m(1 + \cos(θ)). \quad (10)$$

If $Ψ(p(z),zp'(z),z^2p''(z);z) ∈ Ω$ for $z ∈ ℍ$, then $p(z) < e^z$ in $ℍ$.

2. Generalization of Bessel–Struve Kernel Function

To discuss the structure of Generalized Bessel–Struve kernel function along with various properties, let’s recall about the Generalized Bessel function $J_{ν,b,c}$ from the article [8] and Generalized Struve function $S_{ν,b,c}$ from [9].

The functions $J_{ν,b,c}$ and $S_{ν,b,c}$ are, respectively, solutions of the differential equation

$$z^2 F''(z) + bz F'(z) + \left(cz^2 - ν^2 + (1 - b)ν\right)F(z) = 0, \quad (11)$$

and

$$z^2 F''(z) + bz F'(z) + \left(cz^2 - ν^2 + (1 - b)ν\right)F(z) = \frac{4(z/2)^{ν+1}}{\sqrt{πΓ(ν+1)}}. \quad (12)$$

Both functions have the power series representation as follows

$$J_{ν,b,c}(z) = \sum_{n=0}^{∞} \frac{(-c)^n}{n!Γ(n+κ)\left(\frac{z}{2}\right)^{2n+ν}}, \quad (13)$$

$$S_{ν,b,c}(z) = \sum_{n=0}^{∞} \frac{(-c)^n}{Γ(n+\frac{3}{2})Γ(n+κ+\frac{1}{2})\left(\frac{z}{2}\right)^{2n+ν+1}}, \quad (14)$$

where $κ = ν + (b + 1)/2$. The next result is about the power series of the Generalized Bessel–Struve kernel functions.

**Proposition 1 (Power Series).** For $ν > −1/2$, the generalized Bessel–Struve functions have the power series of the form

$$B_{ν,b,c}(z) = \sum_{n=0}^{∞} \frac{(c)^{n/2}Γ\left(\frac{ν+1}{2}\right)}{\sqrt{πn!Γ\left(\frac{ν}{2} + κ\right)}z^n}. \quad (15)$$

**Proof.** From the definition (8) of $B_{ν,b,c}$, it follows that
\[ B_{\nu,b,c}(z) = \Gamma(\kappa) 2^i i^{-\nu} z^{-\nu} j_{\nu,b,c}(iz) - i^{-\nu+1} \sqrt{z} 2^i i^{-\nu} i \sqrt{c} \Gamma(\kappa) s_{\nu,b,c}(iz) \]
\[ = \Gamma(\kappa) \sum_{m=0}^{\infty} \frac{(-c)^m}{m! \Gamma(m+\kappa)} \left( \frac{iz}{2} \right)^{2m} - i \sqrt{z} \Gamma(\kappa) \sum_{m=0}^{\infty} \frac{(-c)^m}{\Gamma(m+\frac{1}{2}) \Gamma(m+\kappa+\frac{1}{2})} \left( \frac{iz}{2} \right)^{2m+1} \]
\[ = \Gamma(\kappa) \sum_{m=0}^{\infty} \frac{(-c)^m}{m! 2^m \Gamma(m+\kappa)} z^{2m} + \sqrt{z} \Gamma(\kappa) \sum_{m=0}^{\infty} \frac{(-c)^m}{2^{m+1} \Gamma(m+\frac{1}{2}) \Gamma(m+\kappa+\frac{1}{2})} z^{2m+1}. \]  

(16)

The Legendre duplication formula (see [1,2]) shows that
\[ \Gamma(z) \Gamma \left( z + \frac{1}{2} \right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z) \]
shows that
\[ \frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}(2m)!} = \frac{1}{2^{2m} m!} \quad \text{and} \quad \frac{\Gamma(m+1)}{\sqrt{\pi}(2m+1)!} = \frac{1}{2^{2m+1} \Gamma(m+\frac{1}{2})}. \]

Using these identities and the arrangement of odd and even terms, (16) can be rewritten as
\[ B_{\nu,b,c}(z) = \sum_{m=0}^{\infty} \frac{\Gamma(\kappa) (c)^m \Gamma(m+\frac{1}{2})}{\sqrt{\pi}(2m)! \Gamma(m+\kappa)} z^{2m} + \sqrt{c} \sum_{m=0}^{\infty} \frac{\Gamma(\kappa) (c)^m \Gamma(m+1)}{\sqrt{\pi}(2m+1)! \Gamma(m+\kappa+\frac{1}{2})} z^{2m+1} \]
\[ = \sum_{n=0}^{\infty} \frac{\Gamma(\kappa) (c)^{n/2} \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi n! \Gamma \left( \frac{n}{2} + \kappa \right)}} z^n. \]

This complete the proof. \( \square \)

**Proposition 2 (Differential Equations).** The generalized Bessel–Struve function \( B_{\nu,b,c} \) is the solution of the differential equation
\[ z^2 F''(z) + (2\kappa - 1)z F'(z) - cz^2 F(z) = \frac{2 \sqrt{\pi} \Gamma(\kappa)}{\sqrt{\Gamma}(k - \frac{1}{2})}. \]  

(17)

**Proof.** In search of the series solution of (17), consider \( F(z) = \sum_{n=0}^{\infty} A_n z^n \) the solution of (17). From the second differentiation and by arrangement of terms, it follows that
\[ \sum_{n=2}^{\infty} n(n-1) A_n z^n + \sum_{n=1}^{\infty} n(2\kappa - 1) A_n z^n - c \sum_{n=0}^{\infty} A_n z^{n+2} = \frac{2 \sqrt{\pi} \Gamma(\kappa)}{\sqrt{\Gamma}(k - \frac{1}{2})} \]
\[ \Rightarrow \sum_{n=2}^{\infty} \left( n(n-1) A_n + n(2\kappa - 1) A_n - c A_{n-2} \right) z^n + (2\kappa - 1) A_1 z = \frac{2 \sqrt{\pi} \Gamma(\kappa)}{\sqrt{\Gamma}(k - \frac{1}{2})}. \]
Comparing the coefficients, we have
\[ A_1 = \frac{\sqrt{\pi} \Gamma(\kappa)}{\sqrt{\Gamma}(k - \frac{1}{2}) \Gamma \left( \frac{k - 1}{2} \right)} \]
\[ A_n = \frac{c}{(n^2 - 2n + 2nk) A_{n-2}} \quad \text{for} \quad n \geq 2. \]
This gives the odd coefficients as follows:

\[ A_3 = \frac{c}{(15 + 10x)} A_1 = \frac{c^{3/2} \Gamma(\kappa)}{6\sqrt{\pi} (\frac{1}{2} + \kappa) \Gamma\left(\left(\kappa + \frac{1}{2}\right)\right)} = \frac{c^{3/2} \Gamma(\kappa)}{3\sqrt{\pi} \Gamma(\kappa + \frac{1}{2})} \]

\[ A_5 = \frac{c}{(35 + 14x)} A_3 = \frac{c}{10(\frac{1}{2} + \kappa)} \frac{c^{3/2} \Gamma(\kappa)}{3!\sqrt{\pi} \Gamma(\kappa + \frac{1}{2})} = \frac{c^{5/2} \Gamma(\kappa) 2}{5!\sqrt{\pi} \Gamma(\kappa + \frac{1}{2})} \]

\[ A_7 = \frac{c}{(35 + 14x)} A_5 = \frac{c^{5/2} \Gamma(\kappa) 2}{14(\frac{1}{2} + \kappa) 5!\sqrt{\pi} \Gamma(\kappa + \frac{1}{2})} = \frac{c^{7/2} \Gamma(\kappa) 6}{7!\sqrt{\pi} \Gamma(\kappa + \frac{1}{2})} \]

and continuing this way, the odd coefficients have the general form

\[ A_{2n+1} = \frac{c^{2n+1} \Gamma(\kappa) 3(2n+1)}{(2n+1)!\sqrt{\pi} \Gamma(\kappa + \frac{2n+1}{2})}, \quad n \geq 0. \]

Similarly, the odd coefficients can be determined as follows:

\[ A_2 = \frac{c}{4x} A_0 = \frac{c \Gamma(\kappa) \Gamma\left(\frac{1}{2}\right)}{2!\sqrt{\pi} \Gamma(\kappa + 1)} A_0 \quad \text{using the fact that} \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}, \]

\[ A_4 = \frac{c}{8(1 + \kappa)} A_2 = \frac{c}{8(1 + \kappa)} \frac{c \Gamma(\kappa) \Gamma\left(\frac{2}{2}\right)}{2!\sqrt{\pi} \Gamma(\kappa + 1)} A_0 = \frac{c^2 \Gamma(\kappa) \Gamma\left(\frac{2}{2}\right)}{4!\sqrt{\pi} \Gamma(\kappa + 2)} A_0, \]

and continuing like this, the general form of even terms are as follows:

\[ A_{2n} = \frac{c^{2n} \Gamma(\kappa) \Gamma\left(\frac{2n+1}{2}\right)}{(2n)!\sqrt{\pi} \Gamma(\frac{2n+1}{2} + \kappa)} A_0, \quad n \geq 1. \]

Finally, by considering \( A_0 = 1 \), the series solution is

\[ f(z) = \sum_{n=0}^{\infty} A_n z^n = A_0 + \sum_{n=1}^{\infty} A_{2n} z^{2n} + \sum_{n=0}^{\infty} A_{2n+1} z^{2n+1} = A_0 + \sum_{n=1}^{\infty} \frac{c^{2n} \Gamma(\kappa) \Gamma\left(\frac{2n+1}{2}\right)}{(2n)!\sqrt{\pi} \Gamma\left(\frac{2n+1}{2} + \kappa\right)} z^{2n} + \sum_{n=0}^{\infty} \frac{c^{(2n+1)/2} \Gamma(\kappa) \Gamma\left(\frac{(2n+1)+1}{2}\right)}{(2n+1)!\sqrt{\pi} \Gamma\left(\kappa + \frac{2n+1}{2}\right)} z^{2n+1} \]

\[ = \sum_{n=0}^{\infty} \frac{(\sqrt{c})^n \Gamma(\kappa) \Gamma\left(\frac{n+1}{2}\right)}{n!\sqrt{\pi} \Gamma\left(\frac{n}{2} + \kappa\right)} z^n = B_{\nu,b,c}(z). \]

This completes the proof. \( \square \)

3. Geometric Properties of Generalized Bessel–Struve Kernel Functions

3.1. Relation with Lemniscate Class

This section finds the conditions on the parameters of the generalized Bessel–Struve kernel functions \( B_{\nu,b,c}(z) \) for which it is Lemniscate Carathéodory and convex in the unit disc. The first result finds the condition on \( \nu, b, c \) for which \( B_{\nu,b,c}(z) \prec \sqrt{1+z} \), while the second result discusses \( 1 + (zB_{\nu,b,c}(z) / B_{\nu,b,c}(z)) \prec \sqrt{1+z} \).
Theorem 1. For \( \kappa, c \in \mathbb{C} \), the generalized Bessel–Struve kernel function \( B_{\nu,b,c}(z) \in \mathcal{P}_L \) provided \( \text{Re}(\kappa - 1) > -3/4 \) and

\[
\left( 8 \text{Re}(\kappa - 1) + 3 - 16|c| \right) \sqrt{\pi} \Gamma(\kappa - \frac{1}{2}) \geq 16\sqrt{2c} |\Gamma(\kappa)|. \tag{18}
\]

Proof. Suppose that \( p(z) = f_\nu(z) = B_{\nu,b,c}(z) \). Since, \( f_\nu \) is the solution of the differential Equation (17), \( p \) is the solution of

\[
z^2p''(z) + (2\kappa - 1)zp'(z) - z(czp(z) + M) = 0. \tag{19}
\]

Let \( \Omega = \{0\} \subset \mathbb{C} \) and define \( \psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C} \) as

\[
\psi(r,s,t;z) := t + (2\kappa - 1)s - z(czr + M). \tag{20}
\]

It is clear from (19) that \( \psi(p(z),zp'(z),z^2p''(z);z) \in \Omega \). We shall apply Lemma 2 to show \( \psi(r,s,t;z) \not\in \Omega \), which further implies \( p(z) \prec \sqrt{1 - z} \).

For \( r,s,t \) as given (9), it follows from (20) that

\[
|\psi(r,s,t;z)| = \left| (t + s) + 2(\kappa - 1)s - z(czr + M) \right|
\]
\[
> (t + s)e^{-3\theta} + \frac{(\kappa - 1)m}{\sqrt{2 \cos(2\theta)}} - |c|r - M - M
\]
\[
> \frac{3m^2}{8\sqrt{2 \cos(2\theta)}} + \frac{\text{Re}(\kappa - 1)m}{\sqrt{2 \cos(2\theta)}} - |c|\sqrt{2 \cos(2\theta)} - M
\]
\[
= \frac{1}{8\sqrt{2 \cos(2\theta)}} \left[ 3m^2 + 8\text{Re}(\kappa - 1)m - 16|c| \cos(2\theta) - 8|M| \sqrt{2 \cos(2\theta)} \right]
\]
\[
> \frac{1}{8\sqrt{2 \cos(2\theta)}} \left[ 3m^2 + 8\text{Re}(\kappa - 1)m - 16|c| - 8\sqrt{2}|M| \right].
\]

A calculation implies that \( 3m^2 + 8\text{Re}(\kappa - 1)m - 16|c| - 8\sqrt{2}|M| \) is increasing for \( m \geq 1 \) and \( \text{Re}(\kappa - 1) > -3/4 \). Thus, \( |\psi(r,s,t;z)| > 0 \) provided \( (8 \text{Re}(\kappa - 1) + 3 - 16|c|) \geq 8\sqrt{2}|M| \), which is equivalent to

\[
(8 \text{Re}(\kappa - 1) + 3 - 16|c|) \sqrt{\pi} \Gamma\left(\kappa - \frac{1}{2}\right) \geq 16\sqrt{2c} \Gamma(\kappa). \]

Finally, the conclusion follows from Lemma 2. \( \square \)

Consider \( b = 1 \) and \( \nu = 1/2 \), and hence, \( \kappa = 3/2 \), then \( B_{1/2,1,c} = (e^{cz} - 1)/cz \). Further, for \( \kappa = 3/2 \), the inequality (18) is equivalent to \( 16c + 8\sqrt{2c} < 7 \), which holds for all \( c \in \left(0, \left(11 - 6\sqrt{2}\right)/16\right) \). By this fact, we have the following result from Theorem 1.

Corollary 1. For \( c \in \left(0, \left(11 - 6\sqrt{2}\right)/16\right) \), the function \( (e^{cz} - 1)/cz \prec \sqrt{1 + z} \).

Corollary 2. The Classical Bessel–Struve kernel function \( B_{\nu,1,1}(z) \prec \sqrt{1 + z} \) for \( \nu \geq 5.4299\ldots \)

Corollary 2 follows from Theorem 1 by considering \( b = 1 = c \), and replacing \( \kappa \) by \( \nu + 1 \) in the inequality (18).
Our next result is related to the convexity of $B_{v,b,c}(z)$ in the lemniscate domain. For this purpose, define

$$
\kappa_0 := \min_{\theta \in (-\pi/4, \pi/4), c \in \mathbb{C}_{\text{fixed}}} \left\{ \begin{array}{l}
10\sqrt{2}(2\theta - \cos(\theta) \cos(2\theta) + \cos(3\theta) - \frac{37}{4\sqrt{2}\cos(2\theta)} - |c| \left(2 + \sqrt{2}\right) \\
-6\sqrt{2}(2\theta + \cos(\theta) \cos(2\theta) + \cos(3\theta) + \frac{\sqrt{2}}{\cos(2\theta)})
\end{array} \right\},\tag{21}
$$

for all $\theta \in (-\pi/4, \pi/4)$, and for a fixed $c \in \mathbb{C}$. The value of $\kappa_0$ is determined by $c$. Table 1 shows the values of $\kappa_0$ for some fixed $c \in \mathbb{C}$. Here, $\theta_0 \in (-\pi/4, \pi/4)$ is the point where the minimum is attained to obtain the values of $\kappa_0$ for a fixed $c$. The selective data in Table 1 suggest that the increase in $|c|$ decreases the values of $\kappa_0$. It is to be noted here that

$$
\min_{\theta \in (-\pi/4, \pi/4)} \left\{ -6\sqrt{2}(2\theta + \cos(\theta) \cos(2\theta) + \cos(3\theta) + \frac{\sqrt{2}}{\cos(2\theta)}) \right\} = 0.731925 > 0
$$

at $\theta = \pm 0.518768$, and $\kappa_0$ can not be more than $-4.12198$ for any $c \in \mathbb{C}$. The next theorem is about the lemniscate convexity of the generalized Bessel–Struve kernel function by consideration of the aforementioned fact.

**Table 1.** The values of $\kappa_0$.

| $c$   | 0          | $\pm \frac{1}{2}, \pm \frac{i}{2}$ | $\pm 1, \pm i$ | $1 \pm i$ | $\pm \frac{3}{2}, \pm \frac{3i}{2}$ | $\pm 2, \pm 2i$ |
|-------|------------|----------------------------------|----------------|-----------|----------------------------------|----------------|
| $\kappa_0$ | $-4.12198$ | $-5.59734$                       | $-7.1572$      | $-8.47739$ | $-8.75273$                      | $-10.3663$      |
| $\theta_0$ | $\pm 0.632754$ | $\pm 0.601589$                   | $\pm 0.583525$ | $\pm 0.573565$ | $\pm 0.57187$                 | $\pm 0.563755$      |

**Theorem 2.** Suppose that $v, b \in \mathbb{R}$ and $c \in \mathbb{C}$. For $\kappa_0$ as defined in (21), if $\kappa > -\kappa_0$, then the generalized Bessel–Struve kernel function $B_{v,b,c}(z) \in \mathcal{K}_C$.

**Proof.** To prove the Lemniscate convexity of the generalized Bessel–Struve kernel function $f_v := B_{v,b,c}(z)$, let

$$
p(z) = 1 + \frac{zf_v'''(z)}{f_v''(z)}.
$$

A logarithmic differentiation gives

$$
\frac{p'(z)}{p(z)-1} = \frac{1}{z} + \frac{zf_v'''(z)}{f_v''(z)} - \frac{f_v''(z)}{f_v'(z)},
$$

which further implies

$$
\frac{zf_v'''(z)}{f_v''(z)} = \frac{zp'(z)}{p(z)-1} + p(z) - 2.
$$

Thus,

$$
\frac{z^2f_v'''(z)}{f_v'(z)} = \left( \frac{zp'(z)}{p(z)-1} + p(z) - 2 \right)(p(z)-1) = zp'(z) + (p(z) - 2)(p(z) - 1).	ag{22}
$$

Further calculation leads to

$$
\frac{z^3f_v^{(4)}(z)}{f_v'(z)} = z^2p''(z) + zp'(z)p(z) + (2p(z) - 5)zp'(z) + (p(z) - 3)(p(z) - 2)(p(z) - 1).	ag{23}
$$
Since \( f_r \) is the solution of the differential Equation (17) and hence
\[
z^2 f''_r(z) + (2\kappa - 1)z f'_r(z) - z(f_r(z) + M) = 0.
\]
Further differentiation leads to
\[
z^3 f^{(4)}_r(z) + (2\kappa + 1)z^2 f''_r(z) - cz^3 f'_r(z) - 2cz^2 f'_r(z) = 0.
\] (24)
An arrangement of (24) and application of (22) gives
\[
z^2 p''(z) + zp'(z)p(z) + (2p(z) - 5)zp'(z) + (p(z) - 3)(p(z) - 2)(p(z) - 1) + (2\kappa + 1)(zp'(z) + (p(z) - 2)(p(z) - 1)) - cz^2(p(z) + 1) = 0.
\] (25)
Let \( \Omega = \{0\} \subset \mathbb{C} \) and define \( \psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C} \) as
\[
\psi(r, s, t; z) := t + rs + (2\kappa - 5)s + (r - 1)(r - 2)(r - 3) + (2\kappa + 1)(s + r - 2)(r - 1)) - cz^2(r + 1)
\]
\[
= t + 3rs + (2\kappa - 4)s + r^3 + (2\kappa - 5)r^2 + (8 - 6\kappa)r + 4(\kappa - 1) - cz^2(r + 1).
\] (26)
It is clear from (25) that \( \psi(p(z), zp'(z); z) \in \Omega \). We apply Lemma 2 for two dimensions and show that \( \psi(r, s; z) \notin \Omega \), which further implies \( p(z) < \sqrt{1 + \frac{1}{4}} \).

For \( r, s \) as given in (9), it follows that
\[
|\psi(r, s, t; z)| \geq |(t + s)e^{-3i\theta} + \frac{m(2\kappa - 5)}{2\sqrt{2\cos(2\theta)}} + \frac{3me^{i\theta}}{2} + (2\cos(2\theta))^{3/2} + 2(2\kappa - 5)\cos(2\theta)e^{-i\theta} + (8 - 6\kappa)\sqrt{2\cos(2\theta)}e^{-2i\theta} + 4(\kappa - 1)e^{-3i\theta}| - |c|\left(\sqrt{2} + 1\right)
\]
\[
> \text{Re}((t + s)e^{-3i\theta}) + \frac{m(2\kappa - 5)}{2\sqrt{2\cos(2\theta)}} + \frac{3m\cos(\theta)}{2} + (2\cos(2\theta))^{3/2} - |c|\left(\sqrt{2} + 1\right)
\]
\[
+ 2(2\kappa - 5)\cos(2\theta)\cos(\theta) + (8 - 6\kappa)\sqrt{2\cos(2\theta)}\cos(2\theta) + 4(\kappa - 1)\cos(3\theta)
\]
\[
\geq \frac{3m^2}{8\sqrt{2\cos(2\theta)}} + \frac{m(2\kappa - 5)}{2\sqrt{2\cos(2\theta)}} + \frac{3m\cos(\theta)}{2} + (2\cos(2\theta))^{3/2} \cos(\theta) - |c|\left(\sqrt{2} + 1\right)
\]
\[
+ 2(2\kappa - 5)\cos(2\theta) + (8 - 6\kappa)\sqrt{2\cos(2\theta)}\cos(2\theta) + 4(\kappa - 1)\cos(3\theta).
\] (27)
Now for \( \theta \in (-\pi/4, \pi/4), \quad \left(3m^2 + 4m(2\kappa - 5) + 12m\cos(\theta)\sqrt{2\cos(2\theta)}\right)\left(8\sqrt{2\cos(2\theta)}\right)^{-1} \) is increasing on \( m \geq 1 \) provided \( \kappa > 7/4 \). Since \( \kappa > -\kappa_0 \) where \( \kappa_0 \) as defined in (21) has maximum value \( -4.12198 \) for any \( c \in \mathbb{C}, \kappa > -\kappa_0 > 7/4 \) holds for any \( c \).

The inequality (27) reduces to
\[
|\psi(r, s, t; z)| \geq \left(-6\sqrt{2}\cos^2(2\theta) + 4\cos(\theta)\cos(2\theta) + 4\cos(3\theta) + \frac{\sqrt{2}}{\sqrt{\cos(2\theta)}}\right)(k + \kappa_0).
\]
Finally, the fact
\[
\min_{\theta \in (-\pi/4, \pi/4)} \left\{-6\sqrt{2}\cos^2(2\theta) + 4\cos(\theta)\cos(2\theta) + 4\cos(3\theta) + \frac{\sqrt{2}}{\sqrt{\cos(2\theta)}}\right\} = 0.731925 > 0,
\]
along with the condition \( \kappa > -\kappa_0 \), implies \( |\psi(r, s, t; z)| > 0 \), and the conclusion follows from Lemma 2. \( \square \)

### 3.2. Relation with Exponential Class

In this part, we derive sufficient conditions on \( L \) and \( \eta \) for which \( f_v(z) \sim e^z \).

**Theorem 3.** The generalized Bessel–Struve kernel function \( B_{\nu, \beta, c}(z) \in \mathcal{P}_c \) for

\[
\text{Re}(2\kappa - 1) > |c|e^2 + |M|e.
\]

**Proof.** To prove the theorem, it is enough to consider the function \( \Psi(r, s, t; z) \) as defined in (20) and then apply Lemma 3 to show that \( \Psi(r, s, t; z) \notin \Omega \) for \( r, s \) and \( t \) as given in (10). For \( m \geq 1 \) and \( \text{Re}(2\kappa - 1) > |c|e^2 + |M|e \), it follows that

\[
|\psi(r, s, t; z)| = \left| (t + s) + (2\kappa - 1)s - z(czr + M) \right|
\]

\[
\geq e^{\cos(\theta)} \left| (t + s)e^{-it}e^{-i\theta} + (2\kappa - 1)m \right| - |c|r - |M|
\]

\[
> e^{\cos(\theta)} \left( \text{Re}(t + s)e^{-it}e^{-i\theta} + \text{Re}(2\kappa - 1)m \right) - |c|e^{-i\theta} - |M|
\]

\[
> e^{\cos(\theta)} \text{Re}(2\kappa - 1)m - |c|e^{\cos(\theta)} - |M|
\]

\[
> e^{\cos(\theta)} |c|(e^2 - 1) + |M|((e^{\cos(\theta) + 1} - 1) > 0.
\]

This together with Lemma 3 implies \( \Psi(r, s, t; z) \notin \Omega \), and hence, \( f_v(z) = B_{\nu, \beta, c}(z) \sim e^z \). This completes the proof. \( \square \)

### 3.3. Relation with Janowski Class

In this section, we shall discuss the relation of generalized Bessel–Struve kernel functions with the Janowski class \( \mathcal{P}[A, B] \).

**Theorem 4.** Let \(-1 \leq B < A \leq 1 \). Suppose \( c, \nu, b \in \mathbb{C} \) such that \( \kappa = \nu + (b + 1)/2 \neq 0, -1, -2, -3 \cdots \), and

\[
M := \frac{2\sqrt{\Gamma(\kappa)}}{\sqrt{\pi} \Gamma\left(\kappa - \frac{1}{2}\right)}.
\]

Consider any one of the following

(i) For \( B = -1, A > 3 - 2\sqrt{2} \) such that \( 2(1 - A) \text{Re}(\kappa - 1) \leq |c|(1 + A) \) and

\[
\text{Re}(\kappa - 1) \geq \max \left\{ \frac{|M|(1 + A)}{4A} + \frac{M^2(1 + A)^2}{16A^2} + \frac{|c|^2(1 + A)^2}{16A^2} + \frac{|c|(1 - A)}{4A} + \frac{2|M|}{17A} \right\}. \tag{28}
\]

(ii) For \( B = -1, B < A \leq 3 - 2\sqrt{2} \)

\[
\text{Re}(\kappa - 1) \geq \frac{|c|(1 + A)}{2(1 + A)} + \frac{|M|}{17A} + \sqrt{\left( \frac{|c|(1 - A)}{2(1 + A)} + \frac{|M|}{17A} \right)^2 + \frac{|c|^2}{16}}. \tag{29}
\]

(iii) For \(-1 < B < 0 \),

\[
\text{Re}(\kappa - 1) \geq \left( \frac{|c|(1 + A)}{1 + B} + |M| \right) \frac{(1 - B)^2}{2(A - B)} - \frac{1 + B}{2(1 - B)}. \tag{30}
\]

(iv) For \( B > 0 \),

\[
\text{Re}(\kappa - 1) \geq \left( \frac{|c|(1 + A)}{1 + B} + |M| \right) \frac{(1 + B)^2}{2(A + B)} - \frac{1 - B}{2(1 + B)}. \tag{31}
\]
If \((1 + B)\mathcal{B}_{\nu,b,c}(z) \neq (1 + A)\), then \(\mathcal{B}_{\nu,b,c}(z) \in \mathcal{P}[A, B] \).

**Proof.** Define the analytic function \(p : \mathbb{D} \to \mathbb{C}\) by

\[
p(z) = \frac{(1-A) - (1-B)f_\nu(z)}{(1-A) - (1-B)f_\nu(z)},
\]

where \(f_\nu(z) = \mathcal{B}_{\nu,b,c}(z)\). Then, a computation yields

\[
f_\nu(z) = \frac{(1-A) + (1+A)p(z)}{1-B} + (1+B)p(z), \tag{32}
\]

\[
f_\nu'(z) = \frac{2(A-B)p'(z)}{(1-B)(1+B)p(z))^{1/2}}, \tag{33}
\]

and

\[
f_\nu''(z) = \frac{2(A-B)((1-B)+(1+B)p(z)p'(z) - 4(1+B)(A-B)p^2(z))}{(1-B)(1+B)p(z))^{3/2}}. \tag{34}
\]

Using the Identities (32)–(34), the Bessel differential Equation (17) can be rewritten as

\[
z^2p''(z) - \frac{2(1+B)}{(1-B)+(1+B)p(z)}(zp'(z))^2 + (2\kappa - 1)zp'(z)
- (\frac{(1-B)+(1+B)p(z)((1-A)+(1+A)p(z))}{2(A-B)})cz^2 - zM\left(\frac{1-B+(1+B)p(z)}{2(A-B)}\right) = 0. \tag{35}
\]

Assume \(\Omega = \{0\}\), and define \(\Psi(r,s,t;z)\) by

\[
\Psi(r,s,t;z) := r - \frac{2(1+B)((1-B)+(1+B)p(z))}{(1-B)(1+B)r}s^2 + (2\kappa - 1)s
- (\frac{(1-B)+(1+B)r((1-A)+(1+A)r)}{2(A-B)})cz^2 - zM\left(\frac{1-B+(1+B)r}{2(A-B)}\right). \tag{36}
\]

The Equation (35) yields that \(\Psi(p(z),zp'(z),z^2p''(z)z)\in\Omega\). To ensure \(Re\ p(z) > 0\) for \(z \in \mathbb{D}\), we will use the Lemma 1. Hence, it suffices to establish \(Re\ \Psi(ip,\sigma,\mu + iv;z) \leq 0\) in \(\mathbb{D}\) for real \(\rho, \sigma\) such that \(\sigma \leq -(1 + \rho^2)/2\), and \(\sigma + \mu \leq 0\). Applying those inequalities we obtain

\[
Re\ \Psi(ip,\sigma,\mu + iv;z) \leq -Re(k-1)(1 + \rho^2) - \frac{2(1-B)^2\rho^2}{(1-B)^2 + (1+B)p^2}z^2
- Re\ \left(\frac{(1-B)+(1+B)p}{2(A-B)}\right)\cdot \frac{(1-A)+(1+A)p}{2(A-B)}\cdot \frac{2M}{1+A}z^2
\leq -Re(k-1)(1 + \rho^2) - \frac{(1-B)^2(1+\rho^2)^2}{2(1-B)^2 + (1+B)p^2}
+ \frac{|(1-B)+(1+B)p|\{1-A\} + (1+A)p|c|}{2(A-B)} + \frac{|(1-B)+(1+B)p|2|c|}{2(A-B)}. \tag{37}
\]

The proof will be divided into four cases. Consider first \(B = -1, A > 3 - 2\sqrt{2}\). According to (37), we have

\[
Re\ \Psi(ip,\sigma,\mu + iv;z) \leq -Re(k-1)(1 + \rho^2) + \frac{|(1-A)+(1+A)p|c|}{1+A} + \frac{2|M|}{1+A}
= -Re(k-1)(1 + \rho^2) + \frac{|c|}{1+A} \sqrt{(1-A)^2 + (1+A)^2\rho^2} + \frac{2|M|}{1+A}
= H(\rho).
\]

We note that the function \(H\) is even with respect to \(\rho\), and

\[
H(0) = \frac{|c|}{1+A} + \frac{2|M|}{1+A} - Re(k-1), \tag{38}
\]

that satisfies \(H(0) \leq 0\), by virtue of an inequality in (28) along with the fact that

\[
\frac{|M|(1+A)}{2A} + \sqrt{\frac{M^2(1+A)^2}{16A^2} + \frac{|c|^2(1+A)^2}{16A^2} - \frac{|c|(1-A)}{1+A} - \frac{2|M|}{1+A}} \geq \frac{|M|(1+A)^2}{2A(1+A)} + \frac{|c|(1+A)^2 - 4(1-A)\sqrt{A}(1+A)}{4\sqrt{A}(1+A)} > 0
\]
holds for $3 - 2\sqrt{2} \leq A \leq 1$. Moreover, $\lim_{\rho \to \infty} H(\rho) = -\infty$, and

$$H'(\rho) = -2 \Re((k - 1)\rho + \frac{|c|(1 + A)\rho}{\sqrt{(1 - A)^2 + (1 + A)^2}}),$$

with $H'(\rho) = 0$ if and only if $\rho = 0$ or

$$\rho_0^2 = \frac{|c|^2}{4 \Re((k - 1)^2) - \frac{(1 - A)^2}{(1 + A)^2}}.
$$

We observe that $\rho_0^2 > 0$ trivially for $A = 1$, and for $A < 1$, it holds by the inequality

$$\frac{|c|^2}{4 \Re((k - 1)^2) \geq \frac{(1 - A)^2}{(1 + A)^2}},$$

equivalently $2(1 - A) \Re(k - 1) \leq |c|(1 + A)$, (39)

which is true due to the right side inequality given in (28).

Further,

$$H''(\rho_0) = -2 \Re(k - 1) + \frac{8 \Re((k - 1)^2 (1 - A)^2)}{|c|^2 (1 + A)^2} = -\frac{8 \Re((k - 1)^2)}{|c|^2 (1 + A)^2} \left( \frac{|c|^2}{4 \Re((k - 1)^2) - \frac{(1 - A)^2}{(1 + A)^2}} - \frac{(1 - A)^2}{(1 + A)^2} \right) \leq 0,$$

in view of (39). Hence, $H(\rho_0) = H_{\text{max}}(\rho)$, and

$$H(\rho_0) = \frac{|c|^2}{4 \Re(k - 1)^2} - \frac{4A \Re((k - 1)^2)}{(1 + A)^2} + \frac{2|M|}{1 + A}
$$

$$= \frac{16A}{4 \Re((k - 1)^2)(1 + A)^2} \left( \Re(k - 1) + \frac{|M|(1 + A)}{4A} \right)^2 - \frac{M^2(1 + A)^2}{16A^2} - \frac{|c|^2(1 + A)^2}{16A^2} \leq 0,$$

due to (28).

In the second case, we consider $B = -1, B < A \leq 3 - 2\sqrt{2}$. The inequality (37) reduces then to the following

$$\Re\Psi(i\rho, \sigma, \mu + iv; z)$$

$$\leq -\Re(k - 1)(1 + \rho^2) + \Re\left( \frac{|1 - A|(1 + A)\rho}{2(1 + A)} \right) + \frac{2|M|}{1 + A}$$

$$\leq -\Re(k - 1)(1 + \rho^2) + \frac{|c|}{2(1 + A)} ((1 - A) + (1 + A) |\rho|) + \frac{2|M|}{1 + A}$$

$$= -\Re(k - 1)\rho^2 + \frac{|c|}{2} |\rho| + \frac{|c|(1 - A)}{2(1 + A)} - \Re(k - 1) + \frac{2|M|}{1 + A}$$

$$= -\Re(k - 1) \left( |\rho| - \frac{|c|}{4 \Re(k - 1)} \right)^2 + \frac{|c|^2}{16 \Re(k - 1)} + \frac{|c|(1 - A)}{2(1 + A)} - \Re(k - 1) + \frac{2|M|}{1 + A}$$

$$= G(\rho).$$

Clearly the quadratic function $G$ is nonpositive for any $\rho \in \mathbb{R}$, if

$$\frac{|c|^2}{16 \Re(k - 1)} + \frac{|c|(1 - A)}{2(1 + A)} - \Re(k - 1) + \frac{2|M|}{1 + A} \leq 0,$$

equivalently

$$-\left( \Re(k - 1) - \left( \frac{|c|(1 - A)}{2(1 + A)} + \frac{|M|}{1 + A} \right) \right)^2 + \left( \frac{|c|(1 - A)}{2(1 + A)} + \frac{|M|}{1 + A} \right)^2 + \frac{|c|^2}{16} \leq 0,$$

which holds if

$$\Re(k - 1) \geq \frac{|c|(1 - A)}{2(1 + A)} + \frac{|M|}{1 + A} + \sqrt{\left( \frac{|c|(1 - A)}{2(1 + A)} + \frac{|M|}{1 + A} \right)^2 + \frac{|c|^2}{16}},$$

which is the assumption (29). Therefore, the assertion $(ii)$ holds.
Now, let \(-1 < B \leq 0, A > B.\) By the fact \(\frac{1-A}{1+B} < \frac{1-B}{1+B},\) we obtain
\[
|(1-B) + (1+B)i\rho|||1-A) + (1+A)i\rho| = (1+A)(1+B)\sqrt{\left(\frac{1-B}{1+B}\right)^2 + \rho^2} \leq (1+A)(1+B)\left[\left(\frac{1-B}{1+B}\right)^2 + \rho^2\right].
\]
Furthermore, for \(B \leq 0\) we have \((1+B)/(1-B) \leq 1,\) therefore
\[
\frac{1+\rho^2}{(1-B)^2+(1+B)^2\rho^2} = \frac{1}{(1-B)^2} \frac{1+\rho^2}{1+(1-B)^2\rho^2} \geq \frac{1}{(1-B)^2},
\]
for any real \(\rho.\) Thus
\[
\text{Re} \Psi(ip, \sigma, \mu + iv; z) \leq -\left(\text{Re}(\kappa - 1) + \frac{(1+B)}{2(1-B)}\right)(1+\rho^2) + \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1-B)^2+(1+B)^2\rho^2}{2(A-B)}
\]
\[
= \rho^2\left(-\text{Re}(\kappa - 1) - \frac{1+B}{2(1-B)} + \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1+B)^2}{2(A-B)}\right)
\]
\[
- \text{Re}(\kappa - 1) - \frac{1+B}{2(1-B)} + \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1+B)^2}{2(A-B)}
\]
and the last expression is nonpositive in view of (30), and then the assertion follows.

Finally, consider \(0 \leq B < A \leq 1.\) In this case, \(\beta = (1-B)/(1+B) \leq 1.\) Hence, setting \(t = \beta^2 + \rho^2, t \geq \beta^2,\) using (40), we obtain from (37)
\[
\text{Re} \Psi(ip, \sigma, \mu + iv; z) \leq -\left(\text{Re}(\kappa - 1) + \frac{(1+B)}{2(1-B)}\right)(1+\rho^2) + \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1-B)^2+(1+B)^2\rho^2}{2(A-B)}
\]
\[
= -\left(\text{Re}(\kappa - 1) + \frac{\beta}{2}\right)(1-\beta^2 + t) + \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1+B)^2}{2(A-B)}t
\]
\[
= t\left(-\text{Re}(\kappa - 1) - \frac{\beta}{2} + \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1+B)^2}{2(A-B)}\right)
\]
\[
- \left(\text{Re}(\kappa - 1) + \frac{\beta}{2}\right)(1-\beta^2).
\]
That is nonpositive because of the inequality
\[
\text{Re}(\kappa - 1) \geq \left(\frac{|c(1+A)|}{(1+B)} + |M|\right)\frac{(1+B)^2}{2(A-B)} - \frac{1-B}{2(1+B)},
\]
which is equivalent to the assumption (31).

Taking into account the above reasoning, we see that \(\Psi\) satisfies the hypothesis of Lemma 1, and thus \(\text{Re} \ p(z) > 0,\) that is,
\[
\frac{(1-A)-(1-B)f_0(z)}{(1+A)-(1+B)f_0(z)} < \frac{1+z}{1-z}.
\]
Hence, there exists an analytic self-map \(w\) of \(\mathbb{D}\) with \(w(0) = 0\) such that
\[
\frac{(1-A)-(1-B)f_0(z)}{(1+A)-(1+B)f_0(z)} = \frac{1+w(z)}{1-w(z)} \implies f_0(z) = \frac{1+Aw(z)}{1+Bw(z)},
\]
which is equivalent to say \(f_0 \sim (1 + Az)/(1 + Bz).\) \(\square\)
Taking $A = 1$ and $B = -1$ in Theorem 4 gives the following result:

**Corollary 3.** $\text{Re}(B_{\nu, b, c}(z)) > 0$ for $\sqrt{\pi} \text{Re}(k - 1)\Gamma\left(k - \frac{1}{2}\right) \geq 2|\sqrt{c}|\Gamma(k)$. In particular, for $\nu \in \mathbb{R}$, $\text{Re}(B_{\nu, 1, 1}(z)) > 0$ when $\nu > 1.5$.

**4. Concluding Remarks and Future Problems**

By applying Lemma 2, we are able to drive the criteria for the convexity of generalized Bessel–Struve kernel functions $B_{\nu, b, c}(z)$ in the lemniscate domain. The exponential convexity and Janowski convexity, however, cannot be produced in the same way. Using (26) and applying Lemma 3, we attempt to derive conditions on $\kappa$. However, there is no feasible $\kappa$ for which the Lemma 3 assumptions are satisfied. Using Lemma 1, one can make a similar observation that the relationship with the Janowski Convex or convex with $B_{\nu, b, c}(z)$ is not possible. Thus, further theoretical concepts or different approaches require studying the exponential or Janowski convexity or convexity of $B_{\nu, b, c}(z)$.

**Author Contributions:** Formal analysis, N.M.A. and S.R.M.; Investigation, N.M.A. and S.R.M.; Methodology, N.M.A. and S.R.M.; Supervision, N.M.A. and S.R.M.; Writing—original draft, N.M.A. and S.R.M. All authors have read and agreed to the published version of the manuscript.

**Funding:** The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

**Institutional Review Board Statement:** Not Applicable.

**Informed Consent Statement:** Not Applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors have no relevant financial or non-financial interest to disclose.

**References**

1. Abramowitz, M., Stegun, I. A. *A Handbook of Mathematical Functions*; National Bureau of Standards Applied Mathematics Series No. 55; U. S. Government Printing Office: Washington, DC, USA, 1965.
2. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions*; Encyclopedia of Mathematics and its Applications, 71; Cambridge University Press: Cambridge, UK, 1999.
3. Baricz, A.; Mondal, S.R.; Swaminathan, A. Monotonicity properties of the Bessel-Struve kernel. *Bull. Korean Math. Soc.* 2016, 53, 1845–1856. [CrossRef]
4. Baricz, A.; Mondal, S.R.; Swaminathan, A. Geometric properties of generalized Bessel functions. *Integral Transforms Spec. Funct.* 2010, 21, 641–653. [CrossRef]
5. Gasmi, A.; Sifi, M. The Bessel-Struve intertwining operator on C and mean-periodic functions. *Int. J. Math. Math. Sci.* 2004, 57–60, 3171–3185. [CrossRef]
6. Gasmi, A.; Sifi, M. Fock spaces for the Bessel-Stuve kernel. *J. Anal. Appl.* 2005, 3, 91–106.
7. Hamem, S.; Kamoun, L.; Negzaoui, S. Cowling-Price type theorem related to Bessel-Struve transform. *Arab J. Math. Sci.* 2013, 19, 187–198. [CrossRef]
8. Kamoun, L.; Sifi, M. Bessel-Struve intertwining operator and generalized Taylor series on the real line. *Integral Transforms Spec. Funct.* 2005, 16, 39–55. [CrossRef]
9. Kanas, S.; Mondal, S.R.; Mohammed, A.D. Relations between the generalized Bessel functions and the Janowski class. *Math. Inequal. Appl.* 2018, 21, 165–178.
10. Kanas, S.; Mondal, S.R.; Mohammed, A.D. Relations between the generalized Bessel functions and the Janowski class. *Math. Inequal. Appl.* 2018, 21, 165–178.
11. Kanas, S.; Mondal, S.R.; Mohammed, A.D. Relations between the generalized Bessel functions and the Janowski class. *Math. Inequal. Appl.* 2018, 21, 165–178.
12. Kanas, S.; Mondal, S.R.; Mohammed, A.D. Relations between the generalized Bessel functions and the Janowski class. *Math. Inequal. Appl.* 2018, 21, 165–178.
17. Naz, A.; Nagpal, S.; Ravichandran, V. Exponential starlikeness and convexity of confluent hypergeometric, Lommel, and Struve functions. *Mediterr. J. Math.* 2020, 17, 204. [CrossRef]
18. Selinger, V. Geometric properties of normalized Bessel functions. *Pure Math. Appl.* 1995, 6, 273–277.
19. Szász, R.; Kupán, P.A. About the univalence of the Bessel functions. *Stud. Univ. Babeş-Bolyai Math.* 2009, 54, 127–132.
20. Janowski, W. Some extremal problems for certain families of analytic functions. *I Ann. Polon. Math.* 1973, 28, 297–326. [CrossRef]
21. Ali, R.M.; Ravichandran, V.; Seenivasagan, N. Sufficient conditions for Janowski starlikeness. *Int. J. Math. Math. Sci.* 2007, 2007, 62925. [CrossRef]
22. Ali, R.M.; Chandrashekar, R.; Ravichandran, V. Janowski starlikeness for a class of analytic functions. *Appl. Math. Lett.* 2011, 24, 501–505. [CrossRef]
23. Miller, S.S.; Mocanu, P.T. Differential subordinations and inequalities in the complex plane. *J. Differ. Equ.* 1987, 67, 199–211. [CrossRef]
24. Miller, S.S.; Mocanu, P.T. *Differential Subordinations*; Monographs and Textbooks in Pure and Applied Mathematics, 225; Dekker: New York, NY, USA, 2000.
25. Madaan, V.; Kumar, A.; Ravichandran, V. Starlikeness associated with lemniscate of Bernoulli. *Filomat* 2019, 33, 1937–1955. [CrossRef]
26. Naz, A.; Nagpal, S.; Ravichandran, V. Star-likeness associated with the exponential function. *Turkish J. Math.* 2019, 43, 1353–1371. [CrossRef]