REALITY AND TRANSVERSALITY FOR SCHUBERT CALCULUS IN $\text{OG}(n, 2n+1)$

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Abstract. We prove an analogue of the Mukhin-Tarasov-Varchenko theorem (formerly the Shapiro-Shapiro conjecture) for the maximal type $B_n$ orthogonal Grassmannian $\text{OG}(n, 2n+1)$.

1. The Mukhin-Tarasov-Varchenko Theorem

For any non-negative integer $k$, let $\mathbb{C}_k[z]$ denote the $(k+1)$-dimensional complex vector space of polynomials of degree at most $k$:

$$\mathbb{C}_k[z] := \{ f(z) \in \mathbb{F}[z] \mid \deg f(z) \leq k \}.$$  

Fix integers $0 \leq d \leq m$, and consider the Grassmannian $X = \text{Gr}(d, \mathbb{C}_{m-1}[z])$, the variety of all $d$-dimensional linear subspaces of the $m$-dimensional vector space $\mathbb{C}_{m-1}[z]$. A point $x \in X$ is real if $x$ is is spanned by polynomials in $\mathbb{R}_{m-1}[z]$; a subset of $S \subset X$ is real if every point in $S$ is real.

The Mukhin-Tarasov-Varchenko theorem (formerly the Shapiro-Shapiro conjecture) asserts that any zero-dimensional intersection of Schubert varieties in $X$, relative a special family of flags in $\mathbb{C}_{m-1}[z]$, is transverse and real. This theorem is remarkable for two immediate reasons: first, it is a rare example of an algebraic geometry problem in which the solutions are always provably real; second, the usual arguments to prove transversality involve Kleiman’s transversality theorem [5], which requires that the Schubert varieties be defined relative to generic flags. We recall the most relevant statements here, and refer the reader to the survey article [14] for a discussion of the history, context, reformulations and applications of this theorem.

To begin, we define a full flag in $\mathbb{C}_{m-1}[z]$, for each $a \in \mathbb{C}^1$:

$$F \cdot (a) : \{ 0 \} \subset F_1(a) \subset \cdots \subset F_{m-1}(a) \subset \mathbb{C}_{m-1}[z].$$

If $a \in \mathbb{C}$,

$$F_i(a) := (z + a)^{m-i}\mathbb{C}[z] \cap \mathbb{C}_{m-1}[z]$$

is the set of polynomials in $\mathbb{C}_{m-1}[z]$ divisible by $(z + a)^{m-i}$. For $a = \infty$, we set $F_i(\infty) := \mathbb{C}_{i-1}[z] = \lim_{a \to \infty} F_i(a)$. The flag $F \cdot (a)$ is often described as the flag osculating the rational normal curve $\gamma : \mathbb{C}^1 \to \mathbb{P}(\mathbb{C}_{m-1}[z]), \gamma(t) = (z + t)^{m-1}$, which simply means that $F_i(a)$ is the span of $\{ \gamma(a), \gamma'(a), \ldots, \gamma^{(i-1)}(a) \}$.

Let $\Lambda = \Lambda_{d,m}$ be the set of all partitions $\lambda : (\lambda^1 \geq \cdots \geq \lambda^d)$, where $\lambda^1 \leq m-d$ and $\lambda^d \geq 0$. We say $\lambda$ is a partition of $k$ and write $\lambda \vdash k$ or $|\lambda| = k$ if $k = \lambda^1 + \cdots + \lambda^d$.  

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For every \( \lambda \in \Lambda \), the Schubert variety in \( X \) relative to the flag \( F_\bullet(a) \) is

\[
X_\lambda(a) := \{ x \in X \mid \dim (x \cap F_{n-d-\lambda_i+i}(a)) \geq i, \text{ for } i = 1, \ldots, d \}.
\]

The codimension of \( X_\lambda(a) \) in \( X \) is \( |\lambda| \).

**Theorem 1** (Mukhin-Tarasov-Varchenko [6, 7]). If \( a_1, \ldots, a_s \in \mathbb{R}P^1 \) are distinct real points, and \( \lambda_1, \ldots, \lambda_s \in \Lambda \) are partitions with \( |\lambda_1| + \cdots + |\lambda_s| = \dim X \), then the intersection

\[
X_{\lambda_1}(a_1) \cap \cdots \cap X_{\lambda_s}(a_s)
\]

is finite, transverse, and real.

In [13], Sottile conjectured an analogue of Theorem 1 for \( \text{OG}(n, 2n+1) \), the maximal orthogonal Grassmannian in type \( B_n \). In Section 2 of this note, we give a proof of this conjecture (our Theorem 3). We discuss some of its consequences in Section 3; in particular, we note that Theorem 3 should yield a geometric proof of the Littlewood-Richardson rule for \( \text{OG}(n, 2n+1) \).

2. The theorem for \( \text{OG}(n, 2n+1) \)

Fix a positive integer \( n \), and consider the non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on the \( (2n+1) \)-dimensional vector space \( \mathbb{C}_{2n}[z] \) given by

\[
\left\langle \sum_{k=0}^{2n} a_k \frac{z^k}{k!}, \sum_{\ell=0}^{2n} b_\ell \frac{z^\ell}{\ell!} \right\rangle = \sum_{m=0}^{2n} (-1)^m a_m b_{2n-m}.
\]

Let \( Y = \text{OG}(n, \mathbb{C}_{2n}[z]) \) be the orthogonal Grassmannian in \( \mathbb{C}_{2n}[z] \), which is the variety of all \( n \)-dimensional isotropic subspaces of \( \mathbb{C}_{2n}[z] \). The dimension of \( Y \) is \( \frac{n(n+1)}{2} \).

The definition of a Schubert variety in \( Y \) requires our reference flags to be orthogonal flags. As explained in the next proposition, the bilinear form on \( \mathbb{C}_{2n}[z] \) has been chosen so that this is true for the flags \( F_\bullet(a) \).

**Proposition 2.** For \( a \in \mathbb{C}P^1 \), then the flag \( F_\bullet(a) \) is an orthogonal flag; that is, \( F_i(a) = F_{2n+1-i}(a) \), for \( i = 0, \ldots, 2n+1 \).

**Proof.** For \( a = 0, \infty \), this is straightforward to verify. We deduce the result for all other \( a \) by showing that \( \langle f(z), g(z) \rangle = \langle f(z+a), g(z+a) \rangle \).

To see this, note that \( \langle \frac{df}{dz}(\frac{z}{z+1}), \frac{df}{dz}(\frac{z}{z+1}) \rangle = -\langle \frac{df}{dz}(\frac{z}{z+1}), \frac{df}{dz}(\frac{z}{z+1}) \rangle \), so \( \frac{df}{dz} \) is a skew-symmetric operator on \( \mathbb{C}_{2n}[z] \). It follows that \( \exp(a \frac{df}{dz}) \) is an orthogonal operator on \( \mathbb{C}_{2n}[z] \) and so \( \langle f(z+a), g(z+a) \rangle = \langle \exp(a \frac{df}{dz})f(z), \exp(a \frac{df}{dz})g(z) \rangle = \langle f(z), g(z) \rangle \). \( \square \)

The Schubert varieties in \( Y \) are indexed by the set \( \Sigma \) of all strict partitions \( \sigma : \sigma_1 > \sigma_2 > \cdots > \sigma_k \), with \( \sigma_1 \leq n, \sigma_k > 0, k \leq n \). For convenience, we put \( \sigma^j = 0 \) for \( j > k \). We associate to \( \sigma \) a decreasing sequence of integers, \( \sigma_1 > \cdots > \sigma_n \), such that \( \sigma_i = \sigma^i \) if \( \sigma^i > 0 \), and \( \{ |\sigma_1|, \ldots, |\sigma| \} = \{ 1, \ldots, n \} \). It is not hard to see that \( \sigma_i \) is given explicitly by the formula

\[
\sigma_i = \sigma^i - i + \# \{ j \in \mathbb{N} \mid j \leq i < j + \sigma^j \}.
\]

For \( \sigma \in \Sigma \), the Schubert variety in \( Y \) relative to the flag \( F_\bullet(a) \) is defined to be

\[
Y_\sigma(a) := \{ y \in Y \mid \dim (y \cap F_{1+n-\sigma_i}(a)) \geq i, \text{ for } i = 1, \ldots, n \}.
\]
The codimension of $Y_\sigma(a)$ in $Y$ is $|\sigma|$. We refer the reader to [2, 12] for further details.

**Theorem 3.** If $a_1, \ldots, a_s \in \mathbb{R}P^1$ are distinct real points, and $\sigma_1, \ldots, \sigma_s \in \Sigma$, with $|\sigma_1| + \cdots + |\sigma_s| = \dim Y$, then the intersection

$$\bigcap_{i=1}^{s} Y_{\sigma_i}(a_i)$$

is finite, transverse, and real.

**Proof.** Let $X = \text{Gr}(n, \mathbb{C}^{2n})$, and let $\Lambda = \Lambda_{n, 2n+1}$. We prove this result by viewing $Y$ as a subvariety of $X$, and the Schubert varieties $Y_\sigma$ as the intersections of Schubert varieties in $X$ with $Y$. Note that $\dim X = 2 \dim Y = n(n+1)$.

For a strict partition $\sigma \in \Sigma$, let

$$\tilde{\sigma}^i := \sigma^i + i = \sigma^i + \# \{ j \in \mathbb{N} \mid j \leq i < j + \sigma^i \}.$$ 

Observe that $\tilde{\sigma}^i - \tilde{\sigma}^{i+1} = \tilde{\sigma}^i - \tilde{\sigma}^{i+1} - 1 \geq 0$, and $\tilde{\sigma}^1 \leq \sigma^1 + 1 \leq n + 1$; hence we see that

$$\tilde{\sigma} : (\tilde{\sigma}^1 \geq \tilde{\sigma}^2 \geq \cdots \geq \tilde{\sigma}^n)$$

is a partition in $\Lambda$.

It follows directly from the definitions of Schubert varieties in $X$ and $Y$ that

$$X_{\tilde{\sigma}}(a) \cap Y = Y_\sigma(a).$$

Moreover, we have,

$$|\tilde{\sigma}| = |\sigma| + \sum_{i \geq 1} \# \{ j \in \mathbb{N} \mid j \leq i < j + \sigma^i \}$$

$$= |\sigma| + \sum_{j \geq 1} \# \{ i \in \mathbb{N} \mid j \leq i < j + \sigma^j \}$$

$$= |\sigma| + \sum_{j \geq 1} \sigma^j = 2|\sigma|.$$ 

Thus, if $|\sigma_1| + \cdots + |\sigma_s| = \dim Y$, then $|\tilde{\sigma}_1| + \cdots + |\tilde{\sigma}_s| = 2 \dim Y = \dim X$, and so by Theorem 1 the intersection

$$X_{\tilde{\sigma}_1}(a_1) \cap \cdots \cap X_{\tilde{\sigma}_s}(a_s)$$

is finite, transverse, and real; in particular this intersection is a zero-dimensional reduced scheme. It follows immediately that

$$Y_{\sigma_1}(a_1) \cap \cdots \cap Y_{\sigma_s}(a_s) = Y \cap X_{\tilde{\sigma}_1}(a_1) \cap \cdots \cap X_{\tilde{\sigma}_s}(a_s)$$

is finite and real. To see that the intersection on the left hand side is also transverse, note that it is proper, so it suffices to show that it is scheme-theoretically reduced. But this is immediate from the fact that the right hand side is the intersection of $Y$ with a zero-dimensional reduced scheme. \qed
3. Consequences

Let $0 \leq d \leq m$, $X = \text{Gr}(d, \mathbb{C}_{m-1}[z])$, be as in Section 1. We can consider the Wronskian of $d$ polynomials $f_1(z), \ldots, f_d(z) \in \mathbb{C}_{m-1}[z]$:

$$W_{f_1, \ldots, f_d}(z) := \begin{vmatrix} f_1(z) & \cdots & f_d(z) \\ f_1'(z) & \cdots & f_d'(z) \\ \vdots & \cdots & \vdots \\ f_1^{(d-1)}(z) & \cdots & f_d^{(d-1)}(z) \end{vmatrix}.$$  

This is a polynomial of degree at most $\dim X = d(n - d)$. If $f_1, \ldots, f_d$ are linearly dependent, the Wronskian is zero; otherwise up to a constant multiple, $W_{f_1, \ldots, f_d}(z)$ depends only on the linear span of $f_1(z), \ldots, f_d(z)$ in $\mathbb{C}_{m-1}[z]$. Thus the Wronskian gives us a well defined morphism of schemes $W : X \rightarrow P(\mathbb{C}_{d(n-d)}[z])$, called the \textit{Wronski map}. This morphism is flat and finite [1]. For $x \in X$ we will write $W(x; z)$ for any representative of $W(x)$ in $\mathbb{C}_{d(n-d)}[z]$.

The Wronski map has a deep connection to the Schubert varieties on $X$ relative to the flags $F_a(a), a \in \mathbb{C}P^1$. A proof of the following classical result may be found in [1, 9, 14].

\textbf{Theorem 4.} The Wronskian $W(x; z)$ is divisible by $(z+a)^k$ if and only if $x \in X_\lambda(a)$ for some partition $\lambda \vdash k$. Also, $x \in X_\mu(\infty)$ for some $\mu \vdash (\dim X - \deg W(x; z))$.

For $X = \text{Gr}(n, \mathbb{C}_{2n}[z])$, and $Y = \text{OG}(n, \mathbb{C}_{2n}[z])$ we deduce the following analogue:

\textbf{Theorem 5.} If $y \in Y$ then $W(y; z) = P(y; z)^2$ for some polynomial $P(y; z) \in \mathbb{C}_{2n+1}[z]$. $P(y; z)$ is divisible by $(z+a)^k$ if and only if $y \in Y_\sigma(a)$ for some strict partition $\sigma \vdash k$ in $\Sigma$. Also, $y \in Y_{\tau}(\infty)$ for some strict partition $\tau \vdash (\dim Y - \deg P(y; z))$.

\textit{Proof.} Let $y \in Y$, and let $(z+a)^\ell$ be the largest power $(z+a)$ that divides $W(x; z)$. By Theorem 4, there exists a partition $\lambda \vdash \ell$ such that $y \in X_\lambda(a)$. Since $\ell$ is maximal, $y$ is in the Schubert cell

$$X_\lambda(a) := \{ x \in X \mid \dim (x \cap F_k(a)) \geq i, \, n+1-\lambda^i+i \leq k \leq n+1-\lambda^{i+1}+i, \, 0 \leq i \leq n \} = X_\lambda(a) \setminus \left( \bigcup_{|\mu|>|\lambda|} X_\mu(a) \right).$$

(Here, by convention, $\lambda^0 = n+1, \lambda^{n+1} = 0$.) The Schubert cells in $Y$ are of the form

$$Y_\sigma(a) := \{ y \in Y \mid \dim (y \cap F_k(a)) \geq i, \, n+1-\sigma^i \leq k \leq n-\sigma^{i+1}, \, 0 \leq i \leq n \} = X_\sigma(a) \cap Y$$

(Here, by convention, $\sigma^0 = n+1, \sigma^{n+1} = -n-1$.) Now, the intersection $X_\lambda(a) \cap Y$ is nonempty, since it contains $y$, and is therefore a Schubert cell in $Y$. It follows that $\lambda = \kappa$ for some strict partition $\kappa \in \Sigma$. Thus $\ell = |\lambda| = 2|\kappa|$ is even, which proves that $W(y; z) = P(y; z)^2$ is a square.

We have shown that $(z+a)^{|\kappa|}$ is the largest power of $(z+a)$ that divides $P(y; z)$, and $y \in Y_\sigma(a)$. If $y \in Y_\sigma(a)$ then we must have $Y_\sigma(a) \supset Y_\kappa(a)$, which implies that $|\sigma| \leq |\kappa|$, and hence $(z+a)^k$ divides $P(y; z)$. Conversely, for any $k \leq |\kappa|$ there exists $\sigma \vdash k$ such that $Y_\sigma(a) \supset Y_\kappa(a)$, and so $y \in Y_\sigma(a)$. This proves the second
assertion. The third is proved by the same argument, taking \( \ell = \dim Y - \deg P(y; z) \) and \( a = \infty \). □

If we write \( P(y) \) for the class of \( P(y; z) \) in projective space \( \mathbb{P}(\mathbb{C}_{n(n+1)/2}[z]) \), then \( y \mapsto P(y) \) defines a morphism of schemes \( P : Y \to \mathbb{P}(\mathbb{C}_{n(n+1)/2}[z]) \).

**Theorem 6.** \( P \) is a flat, finite morphism.

**Proof.** Let \( h(z) = (z + a_1)^{k_1} \cdots (z + a_s)^{k_s} \in \mathbb{C}_{n(n+1)/2}[z] \). By Theorem 5,

\[
P^{-1}(h(z)) = \bigcap_{i=1}^{s} \left( \bigcup_{\sigma_i \vdash k_i} \sigma_i(a_i) \right),
\]

which, by Theorem 3, is a finite set. Since \( P \) is a projective morphism, this implies that that \( P \) is flat and finite [4, Ch. III, Exer. 9.3(a)]. □

In [9] we showed that the properties of the Wronski map and Theorem 1 can be used to give geometric interpretations and proofs of several combinatorial theorems in the jeu de taquin theory, including the Littlewood-Richardson rule for Grassmannians in type \( A_n \). The map \( P \) and Theorem 3 are the appropriate analogues for \( \text{OG}(n, 2n+1) \).

With a few modifications, it should be possible to use the arguments in [9] to give geometric proofs of the analogous results in the theory of shifted tableaux, as developed in [3, 8, 10, 11, 15], including the Littlewood-Richardson rule for \( \text{OG}(n, 2n+1) \). The main ingredients required to adapt these proofs are Theorems 3, 5 and 6, and the Gel’fand-Tsetlin toric degeneration of \( \text{OG}(n, 2n+1) \), which can be also be computed by considering \( Y \subset X \). The complete details should be straightforward but somewhat lengthy, and we will not include them here.

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