Generalized permutahedra: Minkowski linear functionals and Ehrhart positivity

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Abstract
We characterize all signed Minkowski sums that define generalized permutahedra, extending results of Ardila–Benedetti–Doker (Discrete Comput. Geom. 43 (2010), no. 4, 841–854). We use this characterization to give a complete classification of all positive, translation-invariant, symmetric Minkowski linear functionals on generalized permutahedra. We show that they form a simplicial cone and explicitly describe their generators. We apply our results to prove that the linear coefficients of Ehrhart polynomials of generalized permutahedra, which include matroid polytopes, are non-negative, verifying conjectures of De Loera–Haws–Köppe (Discrete Comput. Geom. 42 (2009), no. 4, 670–702) and Castillo–Liu (Discrete Comput. Geom. 60 (2018), no. 4, 885–908) in this case. We also apply this technique to give an example of a solid-angle polynomial of a generalized permutahedron that has negative linear term and obtain inequalities for beta invariants of contractions of matroids.

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1 | INTRODUCTION

Generalized permutahedra form a combinatorially rich class of polytopes that naturally appear in many areas of mathematics such as combinatorics, geometry, representation theory, optimization, game theory, and statistics (see, e.g., [3, 11, 16, 22, 24, 28, 37, 38, 40, 41]). They contain a variety of interesting and significant classes of polytopes, in particular, matroid polytopes. Generalized permutahedra are sufficiently special to admit a thorough combinatorial description of their geometry as witnessed, for instance, by the discovery of Aguiar–Ardila of a Hopf monoid structure on generalized permutahedra [1], but also general enough to be widely applicable and to serve as useful test cases for questions in polyhedral combinatorics. In recent years, different groups of authors have explored generalizations of this class, leading to generalized nested permutahedra [10] and generalized Coxeter permutahedra [4].

The name generalized permutahedra was introduced by Postnikov in his pioneering work on the combinatorial aspects of this interesting class of polytopes [40]. It should, however, be noted that generalized permutahedra are equivalent to polymatroids, a class of polyhedra that were introduced by Edmonds [18] in 1970 as polyhedral generalization of matroids. Since then polymatroids have been intensively studied in optimization, game theory, and statistics due to their correspondence to submodular and supermodular functions (see [23, 39, 45]). For example, in game theory, well-studied objects are cooperative games, to each of which a polytope called the core of the game is associated, see [44, 49]. Generalized permutahedra turn out to be exactly equal to cores of convex cooperative games [32]. In the theory of discrete convex analysis [39] $M$-convex sets play a central role and there is a one-to-one correspondence between lattice points of generalized permutahedra and $M$-convex sets. For a thorough discussion of the equivalence of these concepts as well as connections to further areas such a conditional independence structures, we refer the reader to [49].

Recall that the (standard) permutahedron $\Pi_d \subset \mathbb{R}^d$ is the $(d - 1)$-dimensional polytope

$$\Pi_d = \text{conv}\{(\sigma(1), \sigma(2), \ldots, \sigma(d)) : \sigma \in S_d\} \subset \mathbb{R}^d,$$

where $S_d$ denotes the group of permutations on $[d] = \{1, 2, \ldots, d\}$. There are many equivalent ways of defining generalized permutahedra, the most concise one being via Minkowski summands of the permutahedron. The Minkowski sum of two polytopes $P, Q \subset \mathbb{R}^d$ is the polytope defined as the vector sum

$$P + Q = \{p + q : p \in P, q \in Q\}.$$

A polytope $R \subset \mathbb{R}^d$ is called a Minkowski summand of another polytope $Q \subset \mathbb{R}^d$ if there is a polytope $P \subset \mathbb{R}^d$ such that $P + R = Q$. We also call $R$ the Minkowski difference of $Q$ and $P$ and use the notation $R = Q - P$. Further, the polytope $R$ is called a weak Minkowski summand of $Q$ if it is a Minkowski summand of a dilate $\lambda Q$ for some $\lambda > 0$.

Definition 1.1. A polytope $P \subset \mathbb{R}^d$ is called a generalized permutahedron if it is a weak Minkowski summand of the permutahedron $\Pi_d$.

In the following we denote the class of all generalized permutahedra in $\mathbb{R}^d$ by $P_d$. In particular, every generalized permutahedron $P \in P_d$ is a polytope of dimension at most $d - 1$ and is contained in a hyperplane $\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = \ell\}$ for some $\ell \in \mathbb{R}$.
In [40], Postnikov studied the subclass of generalized permutahedra consisting of Minkowski sums of dilated standard simplices. Let \( \Delta_0 = \{0\} \) and for \( \emptyset \neq I \subseteq [d] \) let

\[ \Delta_I = \text{conv}\{e_i : i \in I\} \]

be the **standard simplices** where \( e_1, \ldots, e_d \) are the standard basis vectors in \( \mathbb{R}^d \). We will also use the notation \( \Delta_i \) to denote the \((i - 1)\)-dimensional simplex \( \Delta_{[i]} \) for all \( 1 \leq i \leq d \). Extending [40, Proposition 6.3], Ardila, Benedetti, and Doker [3, Proposition 2.4] proved that every generalized permutahedron is a Minkowski difference of sums of dilated standard simplices and can be uniquely represented as a signed Minkowski sum \( \sum_{I \subseteq [d]} y_I \Delta_I \). This representation was also considered in earlier works by Danilov and Koshevoy [16] where it was used to describe cores of cooperative games. Here, a **signed Minkowski sum** is a formal linear combination with coefficients \( y_I \in \mathbb{R} \) that describes a Minkowski difference:

\[
\sum_{I \subseteq [d]} y_I \Delta_I = \sum_{I \subseteq [d], y_I \geq 0} y_I \Delta_I - \sum_{I \subseteq [d], y_I < 0} (-y_I) \Delta_I.
\]

Not every set of coefficients \( \{y_I\}_{I \subseteq [d]} \) defines a generalized permutahedron, though, as we will see, the set of all possible coefficients forms a polyhedral cone. In Theorem 2.4 we give an explicit inequality description of this cone, thereby characterizing all coefficients \( \{y_I\}_{I \subseteq [d]} \) that define generalized permutahedra. We moreover prove that this cone is equal to the cone of **supermodular functions**, up to a change of coordinates. Interestingly, Theorem 2.4 has appeared in a very different context and language within game theory: it can be seen as a reincarnation of a result by Kuipers, Vermeulen, and Voorneveld [32, Theorem 9] who characterized all convex games given as a linear combination in the so-called unanimity basis. We offer a geometric proof of this result.

We then use the characterization obtained in Theorem 2.4 to investigate **Minkowski linear functionals** on generalized permutahedra. In Theorem 3.1 and Proposition 3.2 we explicitly describe the rays of the cone of positive Minkowski linear functionals and provide an explicit geometric construction of the ray functionals. We then consider Minkowski linear functionals that are **symmetric**, that is, invariant under permutations of the coordinates. Minkowski linear functionals are valuations and structural results on valuations under the action of a group have been a focal point of research in classical convex geometry ever since Hadwiger’s seminal classification of continuous, rigid-motion invariant valuations on convex bodies [26]. In Theorem 3.3 we provide a complete classification of all positive, translation-invariant, symmetric Minkowski linear functionals: they form a simplicial cone and we explicitly determine the rays of this cone. We then apply our results to Ehrhart polynomials of generalized permutahedra that are also lattice polytopes.

The **Ehrhart polynomial** of a lattice polytope counts the number of lattice points in integer dilates of the polytope [19]. It is appealing to view Ehrhart polynomials as discrete analogs of the classical Minkowski volume polynomials of convex bodies [7, 30, 36], but unlike volume polynomials, the coefficients of Ehrhart polynomials need not be nonnegative. Understanding when we do have positivity is a fundamental question in Ehrhart theory (see, e.g., [5, 27]) and the study of **Ehrhart positive** [11] polytopes, namely, those that have only nonnegative coefficients is of current particular interest.

Known examples of Ehrhart positive polytopes include zonotopes [46] and integral cyclic polytopes [33]. However, there are elementary examples of non-Ehrhart-positive polytopes, the most classical being the Reeve tetrahedron [42]. In recent work, it has been shown that order
polytopes [2] and smooth polytopes [13] need not be Ehrhart positive. For a comprehensive survey on Ehrhart positivity, see [34].

In [11] Castillo and Liu conjectured Ehrhart positivity for generalized permutahedra expanding on a conjecture of De Loera, Haws, and Koepppe on matroid polytopes [17]. The conjecture was known to hold for all sums of standard simplices by an explicit combinatorial formula given in [40]. Ferroni [20] showed that hypersimplices, that is, matroid polytopes of uniform matroids, are Ehrhart positive. Using a valuation theoretic approach Castillo and Liu [11] proved that generalized permutahedra are Ehrhart positive in up to six dimensions, and moreover, showed that the third and the fourth highest coefficients are nonnegative for generalized permutahedra of any dimension. However, despite this evidence, both of the aforementioned conjectures have very recently, while this article was under review, simultaneously been disproved by Ferroni [21] who was able to construct examples of matroid polytopes with negative quadratic coefficients for all ranks greater or equal to three.

On the other hand, in [9, 11] strong computational evidence was given that the linear coefficient is always nonnegative by explicit calculations for $d \leq 500$. Using the classification of positive, symmetric, translation-invariant Minkowski linear functionals obtained in Theorem 3.3, we are able to prove in Theorem 4.5 that the linear coefficient is indeed always nonnegative. This has independently also been shown by Castillo and Liu [12] using different techniques from those developed in the present article. As an application, we then obtain an inequality among beta invariants of contractions of any given matroid in Corollary 4.10 using a result of Ardila, Benedetti, and Doker [3]. Further, we prove that the aforementioned formula for the number of lattice points in sums of standard simplices provided in [40] extends to arbitrary generalized permutahedra (Corollary 4.8). We conclude by applying our results to solid-angle polynomials and show the existence of a three-dimensional generalized permutahedron whose solid-angle polynomial has negative linear term.

### 2 | SIGNED MINKOWSKI SUMS

In the following we assume familiarity with the basics of polyhedral geometry and lattice polytopes. For further reading we recommend [5, 25, 50].

Let $P_1, \ldots, P_m$ be polytopes. A **signed Minkowski sum** is a formal sum $\sum_i y_i P_i$ with real coefficients $y_1, \ldots, y_m$. We say that $\sum_i y_i P_i$ defines a polytope if $P = \sum_{i: y_i \geq 0} y_i P_i$ is a Minkowski summand of $Q = \sum_{i: y_i \geq 0} y_i P_i$, in which case $\sum_i y_i P_i$ represents the Minkowski difference $Q - P$. In [3], Ardila, Benedetti, and Doker showed that every generalized permutahedron has a unique expression as a signed Minkowski sum of standard simplices. This decomposition was also considered in earlier works by Danilov and Koshevoy [16] where it was used to describe cores of cooperative games.

**Proposition 2.1** [3, Proposition 2.4]. For every generalized permutahedron $P \in \mathcal{P}_d$, there are uniquely determined real numbers $y_I$ for all $\emptyset \neq I \subseteq [d]$ and $y_{\emptyset} = 0$ such that

$$P = \sum_{\emptyset \neq I \subseteq [d]} y_I \Delta_I.$$ 

Equivalently, $\sum_{I: y_I < 0} (-y_I) \Delta_I$ is a Minkowski summand of $\sum_{I: y_I \geq 0} y_I \Delta_I$ and

$$P + \sum_{I: y_I < 0} (-y_I) \Delta_I = \sum_{I: y_I \geq 0} y_I \Delta_I. \quad (1)$$
Not every choice of coefficients \( \{ y_I \}_{I \subseteq [d]} \) yields a generalized permutahedron. The goal of this section is to complete the picture and to give a complete characterization of all coefficients \( \{ y_I \}_{I \subseteq [d]} \) for which \( \sum_{I \subseteq [d]} y_I \Delta_I \) defines a generalized permutahedron.

By a result of Shephard, Minkowski summands of polytopes can be characterized in terms of their edge directions and edge lengths (see [25, p. 318]). For any polytope \( P \subset \mathbb{R}^d \) and any direction \( u \in \mathbb{R}^d \setminus \{0\} \), let

\[
P^u = \{ x \in P \mid u^T x = \max_{y \in P} u^T y \}
\]

be the face of \( P \) in direction of \( u \).

**Theorem 2.2** [25, p. 318]. Let \( P, Q \subset \mathbb{R}^d \) be polytopes. Then \( P \) is a Minkowski summand of \( Q \) if and only if the following two conditions hold for all \( u \in \mathbb{R}^d \setminus \{0\} \).

(i) If \( Q^u \) is a vertex then so is \( P^u \).

(ii) If \( Q^u = [p, q] \) is an edge with endpoints \( p \) and \( q \) then up to translation, \( P^u = \lambda [p, q] \) for some \( 0 \leq \lambda \leq 1 \).

From Theorem 2.2 it follows that the possible edge directions of a Minkowski summand \( P \) of \( Q \) are given by the edge directions of \( Q \). Since the permutahedron \( \Pi_d \) equals, up to translation, the Minkowski sum over all line segments \([e_i, e_j], i \neq j \) (see, e.g., [47, Exercises 4.63 and 4.64]), all edge directions of \( \Pi_d \) are of the form \( e_i - e_j \) for \( i \neq j \). This property characterizes generalized permutahedra as shown by [4, Proposition 2.6], specialized to the permutahedron.

**Theorem 2.3** [4, Proposition 2.6]. A polytope is a generalized permutahedron if and only if all edge directions are of the form \( e_i - e_j \) for \( i \neq j \).

The following theorem characterizes all signed Minkowski sums that define generalized permutahedra. It was brought to the authors’ attention by the anonymous referee that this theorem has appeared before in a different language in the game theory literature in an article by Kuipers–Vermuelen–Voorneveld [32]. There it yields a characterization of the class of convex games in terms of the unanimity basis introduced by Shapley in [44]. We offer two proofs: the second one, via supermodular functions, is similar in nature to the one given in [32]. Nevertheless, for reasons of completeness and to highlight the connection to supermodular functions, we have chosen to include it. Our first proof, in contrast, is, up to our knowledge, new and rather different in spirit, and offers a geometric perspective on this result.

In the following let \( \binom{[d]}{2} \) denote the set of all subsets of \([d] \) with 2 elements.

**Theorem 2.4.** Let \( \{ y_I \}_{I \subseteq [d]} \) be a vector of real numbers. Then the following are equivalent.

(i) The signed Minkowski sum \( \sum_{I \subseteq [d]} y_I \Delta_I \) defines a generalized permutahedron in \( \mathcal{P}_d \).

(ii) For all 2-element subset \( E \in \binom{[d]}{2} \) and all \( T \subseteq [d] \) such that \( E \subseteq T \)

\[
\sum_{E \subseteq I \subseteq T} y_I \geq 0.
\]

In particular, the collection of all coefficients \( \{ y_I \}_{I \subseteq [d]} \) such that \( \sum_{I \subseteq [d]} y_I \Delta_I \) defines a generalized permutahedron is a polyhedral cone. The inequalities (2) are facet-defining.
Proof. Let \( \alpha_I = -\min\{y_I, 0\} \) and \( \beta_I = \max\{y_I, 0\} \) and let \( P = \sum_I \alpha_I \Delta_I \) and \( Q = \sum_I \beta_I \Delta_I \). Then, by (1), we need to show that \( P \) is a Minkowski summand of \( Q \) if and only if

\[
\sum_{E \subseteq I \subseteq T} \alpha_I \leq \sum_{E \subseteq I \subseteq T} \beta_I \tag{3}
\]

for all 2-element subsets \( E \) of \([d]\) and all \( T \subseteq [d] \) such that \( E \subseteq T \).

We first prove the necessity of the inequality. Let \( E = \{i, j\} \) and let \( T \supseteq E \). Let \( u \in \mathbb{R}^d \setminus \{0\} \) be a vector such that

- \( u_i = u_j \) and \( u_k \neq u_l \) for \( k \neq l \) with \( \{k, l\} \neq \{i, j\} \), and
- further,

\[
\min_{k \notin T} u_k > u_i = u_j > \max_{k \in T \setminus E} u_k.
\]

A calculation shows that for such a vector \( u \), the face \( \Delta^u_I \) is either a point or an edge,

\[
\Delta^u_I = \begin{cases} [e_i, e_j], & \text{if } E \subseteq I \subseteq T, \\ e_k, & \text{if otherwise, where } k = \arg\max_{k \in I} u^T e_k. \end{cases}
\]

Therefore up to translation,

\[
P^u = \sum_I \alpha_I \Delta^u_I = \sum_{E \subseteq I \subseteq T} \alpha_I [e_i, e_j],
\]

and

\[
Q^u = \sum_I \beta_I \Delta^u_I = \sum_{E \subseteq I \subseteq T} \beta_I [e_i, e_j].
\]

Thus the desired inequality follows from Theorem 2.4.

For the converse direction, assume that \( u \in \mathbb{R}^d \setminus \{0\} \) is a vector such that \( Q^u \) is either a vertex or an edge. Let us first assume that \( Q^u \) is a vertex. We claim that \( P^u \) must also be a vertex. To see this, assume otherwise that there is an \( I \) with \( \alpha_I > 0 \) and \( \dim \Delta^u_I > 0 \). Then \( [e_i, e_j] \subseteq \Delta^u_I \) for some \( i, j \in I, i \neq j \). This further implies that \( [e_i, e_j] \subseteq \Delta^u_J \) for all \( \{i, j\} \subseteq J \subseteq I \). By (3),

\[
0 < \alpha_I \leq \sum_{\{i,j\} \subseteq I} \alpha_I \leq \sum_{\{i,j\} \subseteq I} \beta_I.
\]

Thus there must be a \( \{i, j\} \subseteq J \subseteq I \) with \( \beta_J > 0 \) and therefore \( \dim Q^u \geq \dim \Delta^u_J > 0 \), a contradiction.

If \( Q^u \) is an edge, by Theorem 2.3, we may assume that \( Q^u = \lambda [e_i, e_j] \) for some \( \lambda > 0 \), up to translation. Then necessarily, \( u_i = u_j \). Let \( M \) be the subset of all 2-element subsets \( \{k, l\} \) for which \( u_k = u_l \). For all \( F = \{k, l\} \in M \) let \( T_F = \{i \in [d]: u_i \leq u_k = u_l\} \). We observe that \([e_k, e_l] \subseteq \Delta^u_I \) if and only if \( F \subseteq I \subseteq T_F \). Therefore, for all \( F \neq E \) in \( M \) and all \( I \) with \( F \subseteq I \subseteq T_F \), we must have
\[ \beta_i = 0 \text{ since } Q^u = \lambda[e_i, e_j]. \] Thus we also obtain

\[ \sum_{F \subseteq I \subseteq T_F} \beta_I = 0, \]

and by (3) this equality remains true if we replace all \( \beta_I \) by \( \alpha_I \). This, in turn, implies that \( P^u \) equals \( \mu[e_i, e_j] \) with \( \mu = \sum_{E \subseteq I \subseteq T_E} \alpha_I \) which by (3) is smaller than \( \lambda = \sum_{E \subseteq I \subseteq T_E} \beta_I \). Thus \( P \) is a Minkowski summand of \( Q \) by Theorem 2.2.

Theorem 3.1 below together with its proof via cone duality imply that the inequalities (2) are facet-defining.

The previous proof of Theorem 2.4 made use of the characterization of the edge directions of generalized permutahedra given in Theorem 2.3. We now give a second proof that will display that the inequalities (2) given in Theorem 2.4 are exactly the defining inequalities of the cone of supermodular functions after a change of variables.

In what follows, we use the notation \( 2^d \) to denote the set of all subsets of \([d]\). A function \( 2^d \to \mathbb{R}, I \mapsto z_I \) is called supermodular if

\[ z_I + z_J \leq z_{I \cup J} + z_{I \cap J} \quad (4) \]

for all subsets \( I, J \subseteq [d] \). In particular, the set of all supermodular functions forms a polyhedral cone. This cone has been in the focus of research in game theory, statistics, and optimization. In optimization, typically the equivalent perspective of submodular functions is taken: a function \( f \) is submodular if and only if \( -f \) is supermodular. The facets of the pointed cone of supermodular functions, normalized such that \( z_\emptyset = 0 \), are well understood and are given by all inequalities of the form

\[ z_{K \cup \{i\}} + z_{K \cup \{j\}} \leq z_{K \cup \{i, j\}} + z_K \quad (5) \]

for all \( K \subseteq [d] \) and all \( i, j \in [d] \setminus K, i \neq j \) (see, e.g., [43, Theorem 44.1])). In contrast, the rays of the cone of supermodular functions are far less understood. In [45] Shapley gave an explicit description of the rays in the case \( d = 4 \). Also Edmonds [18] raised the question of determining the extreme submodular functions. In [48] operations preserving the rays are studied, and in [49] necessary and sufficient conditions for extremality of a supermodular function are given. For further references on extreme supermodular/submodular functions as well as their significance in the pertaining areas, we refer to [49].

There is a one-to-one correspondence of supermodular functions and generalized permutahedra via their facet description: for every vector \( \{z_I\}_{I \subseteq [d]} \in \mathbb{R}^{2^d} \) with \( z_\emptyset = 0 \) let

\[ P(\{z_I\}) = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = z_{[d]}, \sum_{i \in I} x_i \geq z_I \text{ for all } \emptyset \subseteq I \subseteq [d] \right\}, \]

where we assume that all \( z_I \) are chosen maximally, that is, all defining inequalities of the polytope \( P(\{z_I\}) \) are tight. Every generalized permutahedra in \( P_d \) is a polytope of the form \( P(\{z_I\}) \), but not every such polytope is a generalized permutahedra. The following theorem characterizes all vectors \( \{z_I\} \) for which \( P(\{z_I\}) \) is a generalized permutahedron. This characterization appeared in [41, Proposition 3.2]. The equivalence to Definition 1.1 follows from [49, Lemma 9 and Corollary 11].
Theorem 2.5. Let \( \{z_I\}_{I \subseteq [d]} \) be a vector in \( \mathbb{R}^{2^d} \) with \( z_\emptyset = 0 \). Then the polytope \( P(\{z_I\}) \) is a generalized permutahedron if and only if the function \( 2^d \to \mathbb{R}, I \mapsto z_I \) is supermodular.

In [3], Ardila, Benedetti, and Doker explicitly described the representation of \( P(\{z_I\}) \) as signed Minkowski sum.

Proposition 2.6 [3, Proposition 2.4]. For every generalized permutahedron \( P(\{z_I\}) \in \mathcal{P}_d \) there are uniquely determined real numbers \( y_I \) for all \( \emptyset \neq I \subseteq [d] \) and \( y_\emptyset = 0 \) such that
\[
P(\{z_I\}) = \sum_{I \subseteq [d]} y_I \Delta_I,
\]
namely,
\[
y_I = \sum_{J \subseteq I} (-1)^{|I| - |J|} z_J.
\]

Second proof of Theorem 2.4. Let \( U \) be the linear transformation defined by
\[
U : \mathbb{R}^{2^d} \to \mathbb{R}^{2^d}, \quad z_I \mapsto y_I = \sum_{J \subseteq I} (-1)^{|I| - |J|} z_J.
\]

Then, by Möbius inversion, \( U \) is a bijection with \( z_I = U^{-1}(y_I) = \sum_{J \subseteq I} y_J \) for all \( I \). By Theorem 2.5, \( P(\{z_I\}) \) is a generalized permutahedron if and only if \( \{z_I\} \) satisfies the supermodularity condition \( (4) \). On the other hand, by Theorem 2.6, \( P(\{z_I\}) = \sum y_I \Delta_I \) where \( y_I = \sum_{J \subseteq I} (-1)^{|I| - |J|} z_J = U(z_I) \). In particular, a signed Minkowski sum \( \sum y_I \Delta_I \) defines a generalized permutahedron if and only if \( \{y_I\} = U(\{z_I\}) \) where \( \{z_I\} \) satisfies the supermodularity condition \( (4) \). In other words, the set of all vectors \( \{y_I\} \) such that \( \sum y_I \Delta_I \) defines a generalized permutahedron is a polyhedral cone, namely, the image of the cone of supermodular functions under the linear bijection \( U \). By \( (5) \) \( \{z_I\} \) defines a supermodular function if and only if for all \( K \subseteq [d] \) and all \( i, j \in [d] \setminus K, i \neq j \),
\[
z_{K \cup \{i\}} + z_{K \cup \{j\}} \leq z_{K \cup \{i, j\}} + z_K.
\]

These inequalities are facet-defining and equivalent to
\[
\sum_{J \subseteq K \cup \{i\}} y_J + \sum_{J \subseteq K \cup \{j\}} y_J \leq \sum_{J \subseteq K \cup \{i, j\}} y_J + \sum_{J \subseteq K} y_J \quad \Leftrightarrow \quad (6)
\]
\[
0 \leq \sum_{J \subseteq K} y_{J \cup \{i, j\}}. \quad (7)
\]

We conclude by observing that the inequality \( (7) \) is equivalent to condition \( (2) \) when interchanging \( K \) with \( T \setminus \{i, j\} \).

\[\Box\]

3 MINKOWSKI LINEAR FUNCTIONALS

We call a function \( \varphi : \mathcal{P}_d \to \mathbb{R} \) **Minkowski linear** if \( \varphi(\emptyset) = 0 \) and
\[
\varphi(\lambda P + \mu Q) = \lambda \varphi(P) + \mu \varphi(Q)
\]
for all \(P, Q \in P_d\) and all \(\lambda, \mu \geq 0\). The function \(\varphi\) is **positive** if \(\varphi(P) \geq 0\) for all \(P \in P\) and **translation-invariant** if \(\varphi(P + t) = \varphi(P)\) for all \(P \in P_d\) and all \(t \in \mathbb{R}^d\). If \(\varphi : P \to \mathbb{R}\) is a Minkowski linear functional, then by linearity we obtain

\[
\varphi \left( \sum_I y_I \Delta_I \right) = \sum_I y_I \varphi(\Delta_I)
\]

and \(\varphi(\Delta_\emptyset) = 0\). By Theorem 2.6, every generalized permutahedron has a unique representation as a signed Minkowski sum \(\sum_I y_I \Delta_I\) given \(y_\emptyset = 0\). Consequently, we may identify every Minkowski linear map \(\varphi : P_d \to \mathbb{R}\) with the vector \(\{\varphi(\Delta_I)\}_{\emptyset \neq I \subseteq [d]} \in \mathbb{R}^{2^d \setminus \emptyset}\).

For any 2-element subset \(E \in \binom{[d]}{2}\) and any \(T \subseteq [d]\) such that \(E \subseteq T\) let \(v^T_E\) be the Minkowski linear functional defined by

\[
v^T_E(\Delta_I) = \begin{cases} 1 & \text{if } E \subseteq I \subseteq T, \\ 0 & \text{otherwise}. \end{cases}
\]

Since \(\Delta_{\{i\}} = e_i\) and \(v^T_E(\Delta_{\{i\}}) = 0\) for all \(1 \leq i \leq d\), these functionals are translation-invariant.

The following theorem characterizes all positive, translation-invariant Minkowski linear functionals on \(P_d\).

**Theorem 3.1.** Let \(\varphi : P_d \to \mathbb{R}\) be a Minkowski linear functional. Then \(\varphi\) is positive and translation-invariant if and only if there are nonnegative real numbers \(c^T_E\) such that

\[
\varphi = \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c^T_E v^T_E.
\]

In particular, the family of positive, translation-invariant Minkowski linear functionals is a polyhedral cone with rays \(v^T_E\).

**Proof.** Let \(C \subseteq \mathbb{R}^{2^d \setminus \emptyset}\) be the set of all vectors \(\{y_I\}\) such that \(\sum y_I \Delta_I\) defines a generalized permutahedron. Then, by Theorem 2.4, \(C\) is a polyhedral cone with inequality description

\[
C = \bigcap_{E \in \binom{[d]}{2}} \bigcap_{T \supseteq E} \left\{ \{y_I\} : \sum_{E \subseteq I \subseteq T} y_I \geq 0 \right\}.
\]

Thus, by cone duality, a Minkowski functional \(\varphi\) is positive if and only if \(\varphi = \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c^T_E v^T_E\) for some nonnegative numbers \(c^T_E\). Since \(v^T_E(\Delta_I) = 0\) for all 1-element subsets \(I \subseteq [d]\), the functional \(\varphi\) is also translation-invariant in this case. To see that the functionals \(v^T_E\) are rays of the cone of positive, translation-invariant Minkowski functionals, we observe that none of them can be expressed as a positive linear combination of the others. For that assume that \(v^T_E = \sum \lambda^T_{E'} v^T_{E'}\) for some nonnegative \(\lambda^T_{E'}\). Then \(\lambda^T_{E'} = 0\) for all \(E \neq E'\) and all \(T' \nsubseteq T\). From evaluating \(v^T_E\) at \(\Delta_T\) it follows that \(\lambda^T_E = 1\). Then evaluating at \(\Delta_E\) yields \(\lambda^T_{E'} = 0\) for all \((E', T') \neq (E, T)\). This finishes the proof. \(\square\)

Next, we provide a geometric description of the ray generators \(v^T_E\). Let \(E = \{i, j\} \in \binom{[d]}{2}\) and \(T \subseteq [d]\) such that \(E \subseteq T\). We say that a vector \(u \neq 0\) is **compatible** with \((E, T)\) if \(u_i = u_j\), all other
coordinates of \( u \) are different and distinct from each other, and

\[
\min_{k \notin T} u_k > u_i = u_j > \max_{k \in T} u_k.
\]

**Proposition 3.2.** Let \( E \in \binom{[d]}{2} \) and \( T \subseteq [d] \) such that \( E \subseteq T \). Let \( u \neq 0 \) be compatible with \((E, T)\). Then for all \( P \in \mathcal{P}_d \), \( P^u \) is one dimensional and

\[
u^T_E(P) = \text{vol}_1(P^u),
\]

where \( \text{vol}_1 \) denotes the normalized volume where \( \text{vol}_1([e_i, e_j]) = 1 \).

**Proof.** Let \( E = \{i, j\} \). Since \( u \) is compatible, we have that, up to translation, \( \Pi_d^u = \sum_{e_i, e_j}^u = [e_i, e_j] \). Since every generalized permutahedron is a weak Minkowski summand of \( \Pi_d \), by Theorem 2.3, \( P^u = \lambda [e_i, e_j] \), and \( \text{vol}_1(P^u) \) is therefore well defined. Since \( (\lambda P + \mu Q)^u = \lambda P^u + \mu Q^u \) for all polytopes \( P, Q \) and all \( \lambda, \mu \geq 0 \), Equation (8) defines a Minkowsk linear functional on \( \mathcal{P}_d \). We observe that since \( u \) is compatible with \((E, T)\), we have \( \Delta_i^u = [e_i, e_j] \) if and only if \( E \subseteq I \subseteq T \). In this case \( \text{vol}_1(\Delta_i^u) = 1 \). Otherwise, \( \Delta_i^u \) is a vertex and \( \text{vol}_1(\Delta_i^u) = 0 \). Since every Minkowski linear function is uniquely defined by its values on \( \Delta_I \) for all \( I \subseteq [d] \), this finishes the proof. \( \square \)

### 3.1 Symmetric Minkowski linear functionals

We conclude this section by classifying all positive Minkowski linear functionals that are invariant under coordinate permutations. We call such functionals symmetric. The natural action of the symmetric group \( S_d \) on \( \mathbb{R}^d \) which acts by permuting the coordinates induces an action on the class of generalized permutahedra which, in turn, induces an action on Minkowski linear functionals on generalized permutahedra by \( (\sigma \cdot \varphi)(P) = \varphi(\sigma(P)) \) for all \( P \in \mathcal{P}_d \). Then every symmetric, translation-invariant Minkowski linear functional \( \varphi \) can be identified with the \((d - 1)\)-dimensional vector \( \{\varphi(\Delta_{i+1})\}_{1 \leq i \leq d-1} \in \mathbb{R}^{d-1} \). For all \( 1 \leq k \leq d - 1 \) let \( f_k : \mathcal{P}_d \to \mathbb{R} \) be the symmetric, translation-invariant Minkowski linear functional defined by

\[
(f_k)(\Delta_{i+1}) = \binom{i + 1}{2} \binom{d - i - 1}{k - i}
\]

for all \( 1 \leq i \leq d - 1 \).

**Theorem 3.3.** Let \( \varphi : \mathcal{P}_d \to \mathbb{R} \) be a Minkowski linear functional. Then \( \varphi \) is positive, translation, and symmetric if and only if there are real numbers \( c_1, \ldots, c_{d-1} \geq 0 \) such that

\[
\varphi = \sum_{k=1}^{d-1} c_k f_k.
\]

In particular, the family of all positive, symmetric, and translation-invariant Minkowski linear functionals forms a simplicial cone of dimension \( d - 1 \).
Proof. By Theorem 3.1, \( \varphi \) is a positive, Minkowski linear and translation invariant linear functional if and only if \( \varphi = \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c^T_E u^T_E \) for nonnegative numbers \( u^T_E \). If \( \varphi \) is moreover invariant under permutation of the coordinates, we obtain

\[
d! \cdot \varphi = \sum_{\sigma \in S_d} \sigma \cdot \varphi \tag{10}
\]
\[
= \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c^T_E \sum_{\sigma \in S_d} \sigma \cdot u^T_E \tag{11}
\]
\[
= \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c^T_E \cdot |\text{Stab}(u^T_E)| \sum_{\psi \in \mathcal{O}(v^T_E)} \psi, \tag{12}
\]

where \( \text{Stab}(u^T_E) = \{ \sigma \in S_d : \sigma u^T_E = u^T_E \} \) denotes the stabilizer and \( \mathcal{O}(u^T_E) = \{ \sigma \cdot u^T_E : \sigma \in S_d \} \) denotes the orbit of \( u^T_E \). We observe that if \( |T| = k \), then \( \mathcal{O}(u^T_E) = \{ u^T_E : |T| = k \} \). Clearly, \( \sum_{\psi \in \mathcal{O}(u^T_E)} \psi \) is symmetric. Therefore, since

\[
\sum_{\psi \in \mathcal{O}(u^T_E)} \psi(\Delta_{i+1}) = \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E \atop |T| = k} \sum_{\psi \in \mathcal{O}(u^T_E)} \psi(\Delta_{i+1})
\]
\[
= \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq \{i+1\} \atop |T| = k} 1
\]
\[
= \binom{i + 1}{2} \binom{d - i - 1}{k - i - 1},
\]

we see that \( f_{k-1} = \sum_{\psi \in \mathcal{O}(u^T_E)} \psi \) whenever \( |T| = k \). Thus, by (12), every symmetric translation-invariant valuation is a nonnegative linear combination of the functionals \( f_1, \ldots, f_{d-1} \) which are easily seen to be linearly independent and positive by Theorem 3.1. This finishes the proof. \( \square \)

4 | APPLICATIONS

4.1 | Ehrhart positivity

A lattice polytope is a polytope in \( \mathbb{R}^d \) with vertices in the integer lattice \( \mathbb{Z}^d \). A famous result by Ehrhart states that the number of lattice points in integer dilates of a lattice polytope is given by a polynomial [19].

**Theorem 4.1** [19]. Let \( P \subseteq \mathbb{R}^d \) be a lattice polytope. Then there is a polynomial \( E_P \) of degree \( \dim P \) such that

\[
E_P(n) = |nP \cap \mathbb{Z}^d|
\]

for all integers \( n \geq 1 \).
The polynomial $E_P(n) = E_0(P) + E_1(P)n + \cdots + E_{\dim P}(P)n^{\dim P}$ is called the **Ehrhart polynomial** of $P$. In this section we show that the linear coefficient $E_1(P)$ of the Ehrhart polynomial of every generalized permutahedra $P$ with vertices in the integer lattice is nonnegative. This has independently been proved by Castillo and Liu [12]. In [8], the authors make the useful observation that the linear coefficient is additive under taking Minkowski sums of lattice polytopes.

**Lemma 4.2** [8, Corollary 23]. Let $P$ and $Q$ be lattice polytopes and $k, \ell \geq 0$ be integers. Then

$$E_1(kP + \ell Q) = kE_1(P) + \ell E_1(Q).$$

Let $\mathcal{E} : \mathcal{P}_d \to \mathbb{R}$ be the symmetric Minkowski linear functional defined by

$$\mathcal{E}(\Delta_{i+1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{i} =: h_i$$

for all $1 \leq i \leq d - 1$. Then $\mathcal{E}$ agrees with $E_1$ on all generalized permutahedra that are lattice polytopes.

**Proposition 4.3.** Let $P \in \mathcal{P}_d$ be a generalized permutahedron with vertices in the integer lattice. Then $\mathcal{E}(P) = E_1(P)$.

**Proof.** We recall that for all $1 \leq i \leq d - 1$

$$E_{\Delta_{i+1}}(n) = \left\{ x \in \mathbb{R}^{i+1} \times \{0\}^{d-i-1} : \sum_{k=1}^{d} x_k = n \right\} = \binom{n + i}{i}.$$

In particular, $E_1(\Delta_{i+1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{i} = \mathcal{E}(\Delta_{i+1})$. It follows from [3, Proposition 2.3] that every generalized permutahedron that is a lattice polytope is a signed Minkowski sum of standard simplices $\Delta_i$ with integer coefficients. Furthermore, $E_p(n)$ and therefore $E_1(n)$ is invariant under permutations of the coordinates. Thus, the claim follows from Lemma 4.2. \qed

Thus, to prove that $E_1(P)$ is always nonnegative for any generalized permutahedron $P$, by Theorem 3.3, we are left to prove that $\mathcal{E} = \sum_{k=1}^{d-1} c_k f_k$ for nonnegative real numbers $c_1, \ldots, c_{d-1}$. Let $A = (a_{ik}) = (f_1, \ldots, f_{d-1}) \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ be the matrix with column vectors $f_1, \ldots, f_{d-1}$. Then

$$a_{ik} = \binom{i + 1}{2} \binom{d - i - 1}{k - 1}$$

and

$$c = A^{-1}h,$$

where $h = (h_1, \ldots, h_{d-1})^T$. 
Lemma 4.4.

\[ A^{-1} = \frac{(-1)^{k+j}}{\binom{j+1}{2}} \binom{d-k-1}{j-k} = : (b_{k,j}) = B. \]

**Proof.** We calculate

\[
(AB)_{ij} = \sum_{k=1}^{d-1} \binom{i+1}{2} \binom{d-i-1}{k-i} \frac{(-1)^{k+j}}{\binom{j+1}{2}} \binom{d-k-1}{j-k}
\]

\[
= \binom{i+1}{2} \sum_{k=1}^{d-1} (-1)^{k+j} \binom{d-1-i}{j-i} \binom{j-i}{k-i}
\]

\[
= (-1)^{j-i} \binom{d-1-i}{j-i} (\binom{i+1}{2}) \sum_{k=1}^{d-1} (-1)^{k-i} \binom{j-i}{k-i}
\]

\[
= (-1)^{j-i} \binom{d-1-i}{j-i} (\binom{i+1}{2}) (1-1)^{j-i}
\]

\[
= \begin{cases} 
1 & \text{if } j = i, \\
0 & \text{otherwise.}
\end{cases}
\]

That is, \(AB = I_{d-1}\) and thus \(A\) is invertible with inverse equal to \(B\).

\(\square\)

**Theorem 4.5.** Let \(P \in \mathcal{P}_d\) be a generalized permutahedron. Then \(\mathcal{E}(P) \geq 0\).

**Proof.** We consider the polynomial

\[ p_k = \sum_{j=k}^{d-1} b_{k,j} t^j \]

and observe that

\[
\int_0^1 \frac{p_k(1) - p_k(t)}{1-t} dt = \int_0^1 \sum_{j=k}^{d-1} b_{k,j} (1 + t + \cdots + t^{j-1}) dt = (Bh)_k = c_k,
\]

which we need to show is nonnegative. It therefore suffices to show that

\[ p'_k(t) \geq 0 \]

for all \(t \in [0,1]\). Let

\[ q_k(t) = \frac{t^2 p'_k(t)}{2} = \sum_{j=k}^{d-1} (-1)^{k+j} \binom{d-k-1}{j-k} \frac{t^{j+1}}{j+1}. \]
Then
\[ q_k'(t) = \sum_{j=k}^{d-1} (-1)^{k+j} \binom{d-k-1}{j-k} t^j = \sum_{\ell=0}^{d-1-k} (-1)^\ell \binom{d-k-1}{\ell} t^{\ell+k}. \]

We conclude by observing that
\[ q_k(t) = \int_0^t q_k'(t) dt \]
and
\[ q_k'(t) = t^k(1-t)^{d-k-1}, \]
which is nonnegative for all \( t \in [0, 1] \).

An important subclass of generalized permutahedra consists of polytopes that can be written as Minkowski sums of standard simplices. Postnikov [40, Theorem 11.3] gave a combinatorial formula for the number of lattice points in generalized permutahedra contained in this subclass that shows Ehrhart positivity in this case (see Equation (13) below). In the remainder of this section we will see that this formula extends to signed Minkowski sums and thus to arbitrary generalized permutahedra.

A valuation on lattice polytopes is a function \( \varphi \) such that
\[ \varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q) \]
for all lattice polytopes \( P, Q \) such that \( P \cup Q \) (and thus also \( P \cap Q \)) are lattice polytopes. A valuation \( \varphi \) is called translation-invariant if \( \varphi(P + t) = \varphi(P) \) for all lattice polytopes \( P \) and all \( t \) in the integer lattice. The volume and the number of lattice points in a lattice polytope present examples of such valuations. A multivariate version of Theorem 4.1 was proved by Bernstein [7] for the number of lattice points in Minkowski sums of lattice polytopes, and, more generally, by McMullen [36] for arbitrary translation-invariant valuations on lattice polytopes. The following result is often referred to as the Bernstein–McMullen theorem.

**Theorem 4.6** [36, Theorem 6]. Let \( P_1, \ldots, P_k \) be lattice polytopes and let \( \varphi \) be a translation-invariant valuation. Then \( \varphi(n_1P_1 + n_2P_2 + \cdots + n_kP_k) \) agrees with a polynomial \( \varphi_{P_1,\ldots,P_k}(n_1,\ldots,n_k) \) of total degree at most \( \dim(P_1 + \cdots + P_k) \) for all integers \( n_1, \ldots, n_k \geq 0 \).

The following extension of Theorem 4.6 complements results in [29, 31] where a multivariate Ehrhart–Macdonald reciprocity was established and generalizes results by Ardila, Benedetti, and Doker [3, Proposition 3.2] from volumes to translation-invariant valuations using a similar argument.

**Proposition 4.7.** Let \( P_1, \ldots, P_k, Q_1, \ldots, Q_\ell \) be lattice polytopes and let \( n_1, \ldots, n_k, m_1, \ldots, m_\ell \geq 0 \) be integers such that \( Q = m_1Q_1 + \cdots + m_\ell Q_\ell \) is a Minkowski summand of \( P = n_1P_1 + \cdots + n_kP_k \). Let
\( \varphi \) be a translation-invariant valuation. Then
\[
\varphi(P - Q) = \varphi_{P_1, \ldots, P_k, Q_1, \ldots, Q_{\ell}}(n_1, \ldots, n_k, -m_1, \ldots, -m_{\ell}).
\]

**Proof.** By Theorem 4.6 the number of lattice points in \((P - Q) + tQ\) agrees with a polynomial for all integers \( t \geq 0 \). Let \( f(t) \) denote this polynomial. On the other hand, since \((P - Q) + Q = P\), we obtain
\[
\varphi((P - Q) + tQ) = \varphi(n_1 P_1 + \cdots + n_k P_k + (t - 1)m_1 Q_1 + \cdots + (t - 1)m_{\ell} Q_{\ell})
\]
\[
= \varphi_{P_1, \ldots, P_k, Q_1, \ldots, Q_{\ell}}(n_1, \ldots, n_k, (t - 1)m_1, \ldots, (t - 1)m_{\ell})
\]
for all \( t \geq 1 \), again by Theorem 4.6. Since two polynomials which agree infinitely many times must be equal, we conclude
\[
\varphi(P - Q) = f(0) = \varphi_{P_1, \ldots, P_k, Q_1, \ldots, Q_{\ell}}(n_1, \ldots, n_k, -m_1, \ldots, -m_{\ell}),
\]
as desired. \( \square \)

The following expression for the number of lattice points in generalized permutahedra \( \sum_{I \subseteq [d]} y_I \Delta_I \) has been proved by Postnikov \([40, \text{Theorem 11.3}]\) in the case when all coefficients \( y_I \) are nonnegative integers. Proposition 4.7 allows us to extend formula (13) to signed Minkowski sums and thus to all generalized permutahedra.

**Corollary 4.8.** For all integer vectors \( \{y_I\}_{I \subseteq [d]} \) that satisfy Equation (2) in Theorem 2.4
\[
\left| \sum_{I \subseteq [d]} y_I \Delta_I \cap \mathbb{Z}^d \right| = \sum_{a} \left( y_{[d]} + a_{[d]} \right) \prod_{I \subseteq [d]} \left( y_I + a_I - 1 \right),
\]
where the sum is over all nonnegative integer vectors \( \{a_I\}_{I \subseteq [d]} \) such that \( \sum_{I \subseteq [d]} a_I = d - 1 \) and for all \( M \subseteq 2^{[d]} \) we have
\[
\left| \bigcup_{J \in M} J \right| \geq 1 + \sum_{J \in M} a_J.
\]

**Proof.** By the Bernstein–McMullen Theorem 4.6, the number of lattice points in \( \sum_{I \subseteq [d]} y_I \Delta_I \) is given by a polynomial for all integers \( y_I \geq 0 \). Indeed, the right-hand side of Equation (13), which was proved in \([40, \text{Theorem 11.3}]\) in this case, is a polynomial in the coefficients \( y_I, I \subseteq [d] \), since
\[
\binom{x}{k} = 1/k! \cdot x \cdot (x - 1) \cdots (x - k + 1)
\]
for all nonnegative integers \( x \geq 0 \). By Theorem 2.4, an integer vector \( \{y_I\}_{I \subseteq [d]} \) satisfies Equation (2) if and only if \( \sum_{I \subseteq [d]} y_I \Delta_I \) defines a generalized permutahedra, and this holds if and only if \( Q := \sum_{I: y_I \leq 0} (-y_I) \Delta_I \) is a Minkowski summand of \( P := \sum_{I: y_I \geq 0} y_I \Delta_I \). Thus, by Proposition 4.7, the polynomial expression for the number of lattice points in \( \sum_{I \subseteq [d]} y_I \Delta_I \) given by Equation (13) extends to all vectors \( \{y_I\}_{I \subseteq [d]} \) satisfying Equation (2). \( \square \)
4.2 Matroid polytopes

In this section we apply our results to matroid polytopes and matroid independent set polytopes to obtain inequalities for the beta invariant of a matroid. Let $M$ be a matroid on a groundset $E$ with rank function $r$. The matroid polytope $P_M$ is a polytope that is defined as the convex hull of all indicator functions of bases of $M$. The beta invariant $\beta(M)$ of $M$ is defined as

$$\beta(M) = (-1)^r(M) \sum_{X \subseteq E} (-1)^{|X|} r(X).$$

In [3] a signed version, the signed beta invariant,

$$\tilde{\beta}(M) = (-1)^{r(M)+1} \beta(M)$$

was introduced in order to express the matroid polytope as a signed Minkowski sum of standard simplices.

**Proposition 4.9** [3]. Let $M$ be a matroid of rank $r$ on $E$ and let $P_M$ be its matroid polytope. Then

$$P_M = \sum_{A \subseteq E} \tilde{\beta}(M/A) \Delta_{E-A}.$$

As a consequence of Theorem 4.5 together with Proposition 4.9 and recalling that $E_1(\Delta_i) = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$, we obtain the following inequality for signed beta invariants of contractions.

**Corollary 4.10.** Let $M$ be a matroid with groundset $E$. Then

$$\sum_{A \subseteq E} h_{|E-A|-1} \tilde{\beta}(M/A) \geq 0,$$

where $h_i := 1 + \frac{1}{2} + \cdots + \frac{1}{i}$.

The independent set polytope $I_M$ of a matroid $M$ is defined as the convex hull of indicator functions of all independent sets of $M$. For $I \subseteq E$ let

$$D_I = \text{conv}([0] \cup \{e_i : i \in I\}).$$

In [3] these simplices were used to express the matroid independence polytope as a signed Minkowski sum.

**Proposition 4.11** [3]. Let $M$ be a matroid of rank $r$ on $E$ and let $I_M$ be its independent set polytope. Then

$$I_M = \sum_{A \subseteq E} \tilde{\beta}(M/A) D_{E-A}.$$
Corollary 4.12. Let $M$ be a matroid with groundset $E$. Then

$$\sum_{A \subseteq E} h_{|E-A|} \tilde{\beta}(M/A) \geq 0,$$

where $h_i := 1 + \frac{1}{2} + \cdots + \frac{1}{i}$.

Proof. After a lattice preserving affine transformation $\mathbb{R}^{|E|} \to \mathbb{R}^{|E|+1}$, $e_i \mapsto e_i$, $0 \mapsto e_{|E|+1}$, $I_M$ is a generalized permutahedron and $D_I$ are standard simplices. The proof follows then from Theorem 4.5. □

4.3 Solid angles

We conclude by applying our results of the previous chapters to a close relative of the Ehrhart polynomial, the solid-angle polynomial of a lattice polytope. Let $q \in \mathbb{R}^d$ be a point, $P \subseteq \mathbb{R}^d$ be a polytope, and let $B_{\varepsilon}(q)$ denote the ball with radius $\varepsilon$ centered at $q$. The solid angle of $q$ with respect to $P$ is defined by

$$\omega_q(P) = \lim_{\varepsilon \to 0} \frac{\text{vol}(P \cap B_{\varepsilon}(q))}{\text{vol} B_{\varepsilon}(q)}.$$ 

We note that the function $q \mapsto \omega_q(P)$ is constant on relative interiors of the faces of $P$. In particular, if $q \notin P$, then $\omega_q(P) = 0$; if $q$ is in the interior of $P$, then $\omega_q(P) = 1$ and if $q$ lies inside the relative interior of a facet, then $\omega_q(P) = \frac{1}{2}$. The solid-angle sum of $P$ is defined by

$$A(P) = \sum_{q \in \mathbb{Z}^d} \omega_q(P).$$

By an analog of Ehrhart’s theorem (Theorem 4.1) for solid-angle sums due to Macdonald [35] $A(P) = A_0(P) + A_1(P)n + \cdots + A_d(P)n^d$ is a polynomial for all lattice polytopes $P$. This follows also from the Bernstein-McMullen Theorem 4.6 since $A(P)$ is a translation-invariant valuation (see, e.g., [6]). Indeed, since $\omega_q(P)$ is constant on relative interiors of faces

$$A(nP) = \sum_{F \subseteq P} \sum_{q \in \text{relint } F \cap \mathbb{Z}^d} \omega_q(nP) = \sum_{F \subseteq P} \omega_F(P)E_{\text{relint } F}(n),$$

where the first sum is over all faces $F$ of $P$, $\omega_F(P)$ is the solid angle of a point in the relative interior of $F$ and $E_{\text{relint } F}(n) = |\text{relint } nF \cap \mathbb{Z}^d|$ is the Ehrhart polynomial of the relative interior of $F$ (see [5, Lemma 13.2]). For lattice polygons $P$ in $\mathbb{R}^2$, the solid-angle sum $A(P)$ agrees with the area, $\text{area}(P)$, of the polygon [5, Corollary 13.11]. In particular, $A(nP) = \text{area}(P)n^2$ has only non-negative coefficients. As in the case of Ehrhart polynomials, for polytopes $P$ of higher dimension the coefficients $A_i(P)$ can be negative in general [6, Proposition 1], even in dimension 3. We supplement this result by showing that for the class of generalized permutahedra, unlike the case of Ehrhart polynomials, the linear terms of solid-angle polynomials can be negative. Here, we view generalized permutahedra in $P_d$ as polytopes in $\{x \in \mathbb{R}^d : \sum x_i = \ell\}$ for some $\ell \in \mathbb{Z}$.
Proposition 4.13. Let \( Q \in P_4 \) be the 3-dimensional generalized permutahedron defined by

\[
Q = \sum_{|I| = 2}^{I \subseteq [4]} \Delta_I - \Delta_4.
\]

Then \( A_1(Q) < 0 \). In particular, there is a 3-dimensional generalized permutahedron in \( \mathbb{R}^4 \) such that the linear term of its solid-angle polynomial is negative.

Proof. It is easy to check that the coefficients in the signed Minkowski sum by which \( Q \) is given satisfy the inequalities (2), and therefore, by Theorem 2.4, \( Q \) is a generalized permutahedron. Since the solid-angle sum is a translation-invariant valuation and by observing that Lemma 4.2 and its proof in [8] via the Bernstein–McMullen Theorem 4.6 carry over verbatim to translation-invariant valuations, we see that the linear term \( A_1(P) \) is Minkowski additive. By definition, \( A(P) = 0 \) whenever \( \dim P < 3 \) and therefore \( A_1(\Delta_I) = 0 \) for all \( I \subseteq [4] \) with \(|I| < 4\). In particular, by Minkowski additivity, we have \( A_1(Q) = -A_1(\Delta_4) \). It thus suffices to prove \( A_1(\Delta_4) > 0 \). By (14),

\[
A(n\Delta_4) = \alpha E_{\text{relint} \Delta_4} + 4\beta E_{\text{relint} \Delta_3} + 6\gamma E_{\text{relint} \Delta_2} + 4\delta E_{\text{relint} \Delta_1}
\]

where \( \alpha, \beta, \gamma, \delta \) denote the solid angle of \( \Delta_4 \) at a lattice point in the interior, on a facet, on an edge and at a vertex, respectively. Inserting the values \( \alpha = 1, \beta = \frac{1}{2}, \) and \( \gamma = \frac{\cos^{-1}(\frac{1}{3})}{2\pi} \) (see, e.g., [14]) we obtain

\[
A_1(\Delta_4) = \frac{3}{\pi} \cos^{-1}\left(\frac{1}{3}\right) - \frac{7}{6} \approx 0.00881298 ... > 0
\]

as desired. This completes the proof. \( \square \)

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