CLT for random quadratic forms based on sample means and sample covariance matrices

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Abstract: In this paper, we use dimensional reduction technique to study the central limit theory (CLT) random quadratic forms based on sample means and sample covariance matrices. Specifically, we use a matrix denoted by $U_{p \times q}$, to map $q$-dimensional sample vectors to a $p$ dimensional subspace, where $q \geq p$ or $q \gg p$. Under the condition of $p/n \to 0$ as $(p, n) \to \infty$, we obtain the CLT of random quadratic forms for the sample means and sample covariance matrices.

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1. Introduction

Consider the multivariate model

$$y_j = \mu + \Gamma x_j, 1 \leq j \leq n,$$

where $\mu$ is a mean vector in $\mathbb{R}^q$, $\Gamma$ is a $q$ by $m$ matrix, $q \leq m$, $\Sigma_q = \Gamma \Gamma^\top$ is a positive definite covariance matrix (denoted by $\Sigma_q \succ 0$) and $x_1, \ldots, x_n$ are independent and identically distributed (i.i.d.) $m$-dimensional real random vectors with the mean vector $\text{E}x_1 = 0_m$ and $\text{Cov}(x_1) = I_m$. So $y_1, \ldots, y_n$ are $q$-dimensional real random vectors and sample mean statistic $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ and sample covariance matrix $S_n = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})(y_i - \bar{y})^\top$ are very important in the mean vector test and covariance matrix test (see Anderson [1]). Here, $\top$ represents the transpose of a vector. However, as the dimension increases, there are problems in the mean vector test and covariance matrix test. For example, when $q > n - 1$, the inverse of $S_n$ does not exist so the Hotelling’s $T^2$ test obtained by Hotelling\textsuperscript{[13]}

$$T^2 = n(\bar{y} - \mu)^\top S_n^{-1}(\bar{y} - \mu),$$

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fails to test the high dimensional mean. There are many papers to study the high dimensional means and covariance matrix. For example, Bai et al.\cite{2}, Bai and Saranadasa\cite{3}, Bai and Silverstein\cite{5, 6}, Chen et al.\cite{11}, Chen and Qin\cite{12}, Pan and Zhou\cite{14}, Srivastava \cite{15}, Srivastava and Du\cite{16}, Srivastava and Li\cite{17}, etc.

Inspired by dimension reduction techniques such as principle component analysis, we aim to project the observations into a low dimensional subspace through a reduction matrix $U_{p \times q}$ ($p \ll q$). The projected observations can be written in a vector form as

$$z_j = U y_j = U \mu + U \Gamma x_j, 1 \leq j \leq n,$$

where $U = U_{p \times q}$ and $\Gamma = \Gamma_{q \times m}$ are nonrandom matrices and $x_1, \ldots, x_n$ are i.i.d. $m$-dimensional real random vectors with the mean vector $\mathbf{0}_m$ and covariance matrix $\mathbf{I}_m$. In view of (1.3), the centered sample covariance matrix is defined by

$$S_n = \frac{1}{n} \sum_{j=1}^{n} (z_j - \bar{z})(z_j - \bar{z})^\top = \frac{1}{n} \sum_{j=1}^{n} U \Gamma (x_j - \bar{x})(x_j - \bar{x})^\top \Gamma^\top U^\top$$

where $\bar{z} = n^{-1} \sum_{j=1}^{n} z_j = U \mu + U \Gamma \bar{x}$ and $\bar{x} = n^{-1} \sum_{j=1}^{n} x_j$.

This is the first paper in a series of two papers. In this paper, we will study the central limit theory (CLT) of random quadratic forms involving sample means and sample covariance matrices when $p/n \to 0$ as $(p, n) \to \infty$ but $q$ can be arbitrarily large. For the details, please see Theorem 1. In our second paper, we use Theorem 1 to derive the CLT Hotelling’s $T^2$ test when $p/n \to 0$.

To investigate this limit theory, we recall some basic definitions from random matrix theory.

Let $A = A_{p \times p}$ be any $p \times p$ square matrix with real eigenvalues denoted by $\lambda_1 \geq \cdots \geq \lambda_p$. The empirical spectral distribution (ESD) of $A$ is defined by

$$F^A(x) = \frac{1}{p} \sum_{j=1}^{p} I(\lambda_j \leq x), \ x \in \mathbb{R},$$

where $I(\cdot)$ is the indicator function. The Stieltjes transform of $F^A$ is given by

$$m_{F^A}(z) = \int \frac{1}{x - z} dF^A(x),$$

where $z = u + iv \in \mathbb{C}^+ \equiv \{ z \in \mathbb{C}, \Im(z) > 0 \}$. Let $X_n = (x_1, \ldots, x_n)$ and $\Sigma_m = \Gamma \Gamma^\top$. The famous Marcenko Pastur (M-P) law states that the ESD of $S_n = \frac{1}{n} \Gamma X_n X_n^\top \Gamma^\top$, i.e., $F^{S_n}(x)$, weakly converges to a nonrandom probability distribution function $F^{c,H}(x)$ whose Stieltjes transform is determined by the following equation

$$m(z) = \int \frac{1}{\lambda(1-c-czm(z)) - z} dH(\lambda)$$
for each $z \in \mathbb{C}^+$ as $m/n \to c \in (0, \infty)$ and $n \to \infty$, where $H$ is the limiting spectral distribution of $\Gamma \Gamma^\top$. One can refer to Bai and Silverstein [6] for more details.

The rest of this paper is organized as follows. Section 2 presents the CLT for random quadratic forms with dimensionality reduction, and its proof is presented in Section 3. Some auxiliary proofs are presented in Appendix A.1-A.2.

2. CLT for random quadratic forms

Assumption:

(A.1) Let $X_n = (x_{11}, \ldots, x_{nn}) = (X_{ij})$ be an $m \times n$ matrix whose entries are i.i.d. real random variables with $EX_{11} = 0$, $\text{Var}(X_{11}) = 1$ and $EX_{11}^4 < \infty$.

(A.2) For $p \leq q \leq m$, let $U = U_{p \times q}$, $\Gamma = \Gamma_{q \times m}$ and $\Sigma_p$ be nonrandom matrices satisfying $U \Gamma \Gamma^\top U^\top = \Sigma_p > 0$.

(A.3) Let $c_n = p/n$ and $c_n = O(n^{-\eta})$ for some $\eta \in (0, 1)$.

In the following, we will study the limiting distributions of random quadratic forms involving sample means and sample covariance matrices. Since $\Sigma_p > 0$, $\Sigma_p^{-1/2}$ exists. So we take the transform

$$
\tilde{\mu} = \Sigma_p^{-\frac{1}{2}} U \mu, \quad B = \Sigma_p^{-\frac{1}{2}} U \Gamma,
$$

$$
\tilde{z}_j = \Sigma_p^{-\frac{1}{2}} z_j = \Sigma_p^{-\frac{1}{2}} U y_j = \tilde{\mu} + B x_j, \quad 1 \leq j \leq n,
$$

$$
\tilde{S}_n = \frac{1}{n} \sum_{j=1}^n (\tilde{z}_j - \bar{\tilde{z}})(\tilde{z}_j - \bar{\tilde{z}})^\top = \frac{1}{n} \sum_{j=1}^n B (x_j - \bar{x})(x_j - \bar{x})^\top B^\top,
$$

where $\bar{x} = \frac{1}{n} \sum_{j=1}^n x_j$ and

$$
\bar{\tilde{z}} = \frac{1}{n} \sum_{j=1}^n \tilde{z}_j = \Sigma_p^{-\frac{1}{2}} U \mu + \frac{1}{n} \sum_{j=1}^n \Sigma_p^{-\frac{1}{2}} U \Gamma x_j = \tilde{\mu} + B \bar{x}.
$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ denote eigenvalues of $\tilde{S}_n$ defined by (2.3). For any analytic function $f(\cdot)$, define

$$
f(\tilde{S}_n) = V^\top \text{diag}(f(\lambda_1), \ldots, f(\lambda_p)) V,
$$

where $V^\top \text{diag}(\lambda_1, \ldots, \lambda_p) V$ denotes the spectral decomposition of $\tilde{S}_n$.

**Theorem 1.** Let assumptions (A.1)-(A.3) be satisfied. Assume that $g(x)$ is a function with a continuous first derivative in a neighborhood of 0 such that $g'(0) \neq 0$, $f(x)$ is analytic on an open region containing the interval

$$
[1 - \delta, 1 + \delta], \quad \exists \delta \in (0, 1)
$$

where
and satisfies $f(1) \neq 0$. Denote $\tilde{\mu} = \Phi_p^{-1/2} \Phi \mu$ and
\[
X_n = \frac{n}{\sqrt{p}} c_n \left[ \frac{(\tilde{z} - \tilde{\mu})^T f(\tilde{S}_n)(\tilde{z} - \tilde{\mu})}{\|\tilde{z} - \tilde{\mu}\|^2} - f(1) \right], \quad Y_n = \frac{n}{\sqrt{p}} \left[ g((\tilde{z} - \tilde{\mu})^T (\tilde{z} - \tilde{\mu}) - g(c_n) \right],
\]
where $\tilde{S}_n$ and $\tilde{z}$ are defined by (2.3) and (2.4), respectively. As $\min(p, n) \to \infty$,
\[
(X_n, Y_n) \overset{d}{\to} (X, Y),
\]
where $(X, Y) \sim N(0, \Gamma_1)$ with
\[
\Gamma_1 = \begin{bmatrix}
2f^2(1) & 2g'(0)f(1) \\
2g'(0)f(1) & 2(g'(0))^2
\end{bmatrix}.
\]

Remark 1. The conditions in (A.1) are usually used to study the random matrix (see [5, 6]). Condition (A.2) requests the reduced dimensional covariance matrix to be a positive definite matrix. When $q/n \to c \in (0, 1)$, Pan and Zhou [14] obtained the random quadratic forms based on sample means and sample covariance matrices and gave its application to the Hotelling’s $T^2$ test. In this paper, we use $\Phi_{p \times q}$ matrix to map $q$-dimensional sample vectors to a $p$ dimensional subspace, where $q \geq p$ or $q \gg p$. Under the condition of $p/n \to 0$ as $(p, n) \to \infty$, we obtain the CLT of random quadratic forms for the sample means and sample covariance matrices. In our second paper, we will use Theorem 1 to derive the CLT Hotelling’s $T^2$ test when $p/n \to 0$.

3. Outline of the proofs

The proof of Theorem 1 relies on Lemma 1 below that deals with the asymptotic joint distribution of
\[
X_n(z) = \frac{n}{\sqrt{p}} c_n \left[ \frac{\tilde{x}^T \Phi^{-1} (\tilde{S}_n - z\Phi) \Phi \tilde{x} - m(z)}{\|\Phi \tilde{x}\|^2} \right], \quad Y_n = \frac{n}{\sqrt{p}} \left[ g(\tilde{x}^T \Phi^{-1} \Phi \tilde{x}) - g(c_n) \right],
\]
where $\Phi = \Sigma^p \Phi \Gamma$, $c_n = p/n$, $m(z) = \int \frac{1}{x - z} dH(x)$ and $H(x) = I(1 \leq x)$ for $x \in \mathbb{R}$. The stochastic process $X_n(z)$ is defined on a contour $C$ as follows: Let $v_0 > 0$ be arbitrary and set $C_u = \{u + iv_0, u \in [u_l, u_r]\}$, where $u_l = 1 - \delta$ and $u_r = 1 + \delta$ for some $\delta \in (0, 1)$. Then, we define
\[
C^+ = \{u_l + iv : v \in [0, v_0]\} \cup C_u \cup \{u_r + iv : v \in [0, v_0]\}
\]
and denote $C^-$ to be the symmetric part of $C^+$ about the real axis. Then, let
\[
C = C^+ \cup C^-.
\]
Let
\[
\tilde{S}_n = \frac{1}{n} \sum_{j=1}^{n} \Phi \tilde{x}_j \tilde{x}_j^T \Phi.
\]
By random matrix theory, $\hat{S}_n$ defined by (2.3) can be replaced by $\tilde{S}_n$ defined by (3.1). It is difficult to control the spectral norm of $(\tilde{S}_n - zI_p)^{-1}$ or $(\hat{S}_n - zI_p)^{-1}$ on the whole contour $C$ (for example $v = 0$), we thus define a truncated version $\hat{X}_n(z)$ of $X_n(z)$ (see in Bai and Silverstein [5]). For some $\vartheta \in (0, 1)$, we choose a positive number sequence $\{\rho_n\}$ satisfying
\[
\rho_n \downarrow 0 \quad \text{and} \quad \rho_n \geq n^{-\vartheta}. \tag{3.2}
\]

Let $C_l = \{u_l + iv : v \in [n^{-1}\rho_n, v_0]\}$ and $C_r = \{u_r + iv : v \in [n^{-1}\rho_n, v_0]\}$. Write $C_l^+ = C_l \cup C_u \cup C_r$. Consequently, for $z = u + iv \in C$, a truncated process $\hat{X}_n(z)$ is defined as follows
\[
\hat{X}_n(z) = \begin{cases} 
X_n(z), & \text{if } z \in C_l^+ \cup C_r^-
\frac{nv + \rho_n}{2\rho_n}X_n(z_{r1}) + \frac{\rho_n - nv}{2\rho_n}X_n(z_{r2}), & \text{if } u = u_r, v \in [-\frac{\rho_n}{n}, \frac{\rho_n}{n}]
\frac{nv + \rho_n}{2\rho_n}X_n(z_{l1}) + \frac{\rho_n - nv}{2\rho_n}X_n(z_{l2}), & \text{if } u = u_l > 0, v \in [-\frac{\rho_n}{n}, \frac{\rho_n}{n}],
\end{cases} \tag{3.3}
\]

Here, $z_{r1} = u_r + \frac{i\rho_n}{n}$, $z_{r2} = u_r - \frac{i\rho_n}{n}$, $z_{l1} = u_l + \frac{i\rho_n}{n}$, $z_{l2} = u_l - \frac{i\rho_n}{n}$ and $C_l^-$ is the symmetric part of $C_l^+$ about the real axis.

We now give the asymptotic joint distribution of $(\hat{X}_n(z), Y_n)$ in Lemma 1.

**Lemma 1.** Under the conditions of Theorem 1, for $z \in C$, we have
\[
(\hat{X}_n(z), Y_n) \xrightarrow{d} (X(z), Y), \tag{3.4}
\]

where $(X(z), Y) \sim N(0, \Gamma_2)$,
\[
\Gamma_2 = \begin{bmatrix} \frac{2}{(1-z)^2} & \frac{2g'(0)}{1-z} \\ \frac{2g'(0)}{1-z} & \frac{1}{g'(0)^2} \end{bmatrix}.
\]

To transfer Lemma 1 to Theorem 1, we introduce a new ESD function
\[
F_{2n}^\delta(x) = \sum_{j=1}^{p} t_j^2 I(\lambda_j \leq x), \ x \in \mathbb{R},
\]

where $t = (t_1, \ldots, t_p)^\top = VBx/\|Bx\|$, $B = \Sigma^{1/2}_p U$ and $V$ is the eigenvector matrix of $\hat{S}_n$ defined by (2.3) (see Bai et al.[2]). Following Theorem 1 in Bai et al.[2] and Remark 3 in Pan and Zhou [14], one can similarly obtain that as $p/n \to 0$,
\[
F_{2n}^\delta(x) \to H(x), \ a.s.,
\]

where $H(x) = I(1 \leq x)$ for $x \in \mathbb{R}$. Then, by analyticity of $f(x)$, $\frac{x^\top B^\top f(\hat{S}_n)Bx}{\|Bx\|^2}$ in Theorem 1 is transferred to $\frac{x^\top B^\top (\hat{S}_n - zI_p)^{-1}Bx}{\|Bx\|^2}$ and Stieltjes transform of $F_{2n}^\delta(x)$. Let $A_n^{-1}(z) = (\hat{S}_n + zI_p)^{-1}$. Note that
\[
\frac{x^\top B^\top A_n^{-1}(z)Bx}{1 - x^\top B^\top A_n^{-1}(z)Bx} = \frac{x^\top B^\top (\hat{S}_n - zI_p)^{-1}Bx}{1 - x^\top B^\top (\hat{S}_n - zI_p)^{-1}Bx}, \tag{3.5}
\]
where we use $\tilde{S}_n = \tilde{S}_n - \bar{x}\bar{x}^\top$ and the identity
\[
\mathbf{r}^\top (\mathbf{C} + a\mathbf{r}\mathbf{r}^\top)^{-1} = \frac{\mathbf{r}^\top \mathbf{C}^{-1}}{1 + a\mathbf{r}^\top \mathbf{C}^{-1}\mathbf{r}}. 
\] (3.6)

Here, $\mathbf{C}$ and $\mathbf{C} + a\mathbf{r}\mathbf{r}^\top$ are both invertible, $\mathbf{r} \in \mathbb{R}^p$ and $a \in \mathbb{R}$ (see Bai and Silverstein [4]). In addition, by (B.85) in the Appendix,

\[
\|\mathbf{B}\bar{x}\|^2 - c_n = O_P(\sqrt{\frac{p}{n}}). 
\] (3.7)

So the stochastic process $X_n(z)$ in Lemma 1 can be presented as
\[
\frac{n}{\sqrt{p}} (\bar{x}^\top \mathbf{B}^\top (\tilde{S}_n - z\mathbf{I}_p)^{-1}\mathbf{B}\bar{x} - c_n m(z)) = \frac{n}{\sqrt{p}} \left( \frac{\bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x}}{1 - \bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x}} - c_n m(z) \right). 
\] (3.8)

But by (B.63) and $c_n = o(1)$, $\bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x}$ can be replaced by $c_n m(z)$, which implies that $1 - \bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x} = 1 + o_P(1)$. Consequently, $X_n(z)$ in Lemma 1 is then reduced to the stochastic process $M_n(z)$ such that

\[
M_n(z) = \frac{n}{\sqrt{p}} (\bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x} - c_n m(z))
\]

for large $p$ and $n$. For $z \in \mathbb{C}_n^+$, we decompose $M_n(z)$ into two parts as

\[
M_n(z) = M_n^{(1)}(z) + M_n^{(2)}(z),
\]

where

\[
M_n^{(1)}(z) = \frac{n}{\sqrt{p}} (\bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x} - \mathbf{E}\bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x}). \] (3.9)

and

\[
M_n^{(2)}(z) = \frac{n}{\sqrt{p}} (\mathbf{E}\bar{x}^\top \mathbf{B}^\top \mathbf{A}^{-1}_n(z)\mathbf{B}\bar{x} - c_n m(z)). \] (3.10)

Let

\[
\mathbf{B} = \Sigma_p^{-\frac{1}{2}}\mathbf{U}\mathbf{\Gamma} = (b_{ij})_{p \times m} = (\mathbf{b}_1, \ldots, \mathbf{b}_m),
\]

\[
\|\mathbf{b}_j\| = \left( \sum_{i=1}^{p} b_{ij}^2 \right)^{1/2}, \quad \mathbf{b}_j = (b_{1j}, \ldots, b_{pj})^\top, \quad 1 \leq j \leq m.
\]

In the following proof, we assume without loss of generally that $\mathbf{\mu} = 0$. As a consequence of Lemma A.7 in Appendix A.5, for $1 \leq i \leq m$, $1 \leq j \leq n$, the random variables $X_{ij}$ in the proof are truncated, i.e.

\[
\mathbf{E}X_{11} = 0, \quad EX_{11}^2 = 1, \quad EX_{11}^4 < \infty, \quad |X_{ij}| \leq (np)^{1/4}/\|\mathbf{b}_i\|, \quad 1 \leq i \leq m, 1 \leq j \leq n.
\] (3.11)
3.1. Convergence of finite dimensional distribution of $M_{n_1}^{(1)}(z)$

In this section we prove that for any positive integer $r$ and complex numbers $a_1, \ldots, a_r$,

$$
\sum_{i=1}^{r} a_i M_{n_1}^{(1)}(z_i), \quad \mathcal{I}(z_i) \neq 0,
$$

converges to a Gaussian random variable. We also derive the asymptotic covariance functions. Under the truncated assumption (3.11), for $1 \leq i, j \leq n$, write

$$x_j = (X_{1j}, X_{2j}, \ldots, X_{nj})^\top, \quad \bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j, \quad \bar{x}_j = \bar{x} - \frac{1}{n} x_j,$$

$$\tilde{S}_n = \frac{1}{n} \sum_{j=1}^{n} Bx_j x_j^\top B, \quad \tilde{S}_{nj} = \tilde{S}_n - \frac{1}{n} Bx_j x_j^\top B^\top,$$

$$\mathbf{A}_n^{-1}(z) = (\tilde{S}_n - zI_p)^{-1}, \quad \mathbf{A}_{n_1}^{-1}(z) = (\tilde{S}_{n_1} - zI_p)^{-1},$$

$$\mathbf{D}_{nj}(z) = \mathbf{A}_{n_1}^{-1}(z)Bx_j x_j^\top B^\top \mathbf{A}_{n_1}^{-1}(z),$$

$$\beta_j(z) = \frac{1}{1 + n^{-1}x_j^\top B^\top \mathbf{A}_{n_1}^{-1}(z)Bx_j} \beta_{tr}^j(z) = \frac{1}{1 + n^{-1} \text{tr}(\mathbf{A}_{n_1}^{-1}(z))},$$

$$\beta_{ij}(z) = \frac{1}{1 + n^{-1}x_i^\top B^\top \mathbf{A}_{n_1}^{-1}(z)Bx_i} \beta_{tr}^{ij}(z) = \frac{1}{1 + n^{-1} \text{tr}(\mathbf{A}_{n_1}^{-1}(z))},$$

$$b_1(z) = \frac{1}{1 + n^{-1} \text{Etr}(\mathbf{A}_{n_1}^{-1}(z))}, \quad b_{12}(z) = \frac{1}{1 + n^{-1} \text{Etr}(\mathbf{A}_{n_2}^{-1}(z))}.$$

Next, we list some important results of quadratic forms. In the following, to facilitate the analysis, we assume $v = \mathcal{I}(z) > 0$. It can be found that $\beta_j(z)$, $\beta_{ij}(z)$, $\beta_{tr}^j(z)$, $b_1(z)$, $b_{12}(z)$ are bounded by $|z|/v$ (see (3.4) of Bai and Silverstein [4]). For any matrix $C$, by Lemma 2.10 of Bai and Silverstein [4],

$$|\text{tr}[(\mathbf{A}_n^{-1}(z) - \mathbf{A}_{n_1}^{-1}(z))C]| \leq \|C\|/v, \quad (3.12)$$

where $\|\cdot\|$ denotes the spectral norm of a matrix. In addition, by (3.6),

$$\mathbf{A}_n^{-1}(z) - \mathbf{A}_{n_1}^{-1}(z) = \mathbf{A}_n^{-1}(z)[\mathbf{A}_{n_1}(z) - \mathbf{A}_n(z)]\mathbf{A}_{n_1}^{-1}(z)$$

$$= -\frac{1}{n} \beta_j(z) \mathbf{A}_{n_1}(z), \quad (3.13)$$
where \( \tilde{A}_{nij}(z) = A^{-1}_{nij}(z)Bx_i x_j^\top B^{-1} A^{-1}_{nij}(z) \). Then, by Lemma A.1 in Appendix A.1, it is easy to obtain that

\[
E|\gamma_1 - \xi_1|^k = n^{-k}E|\text{tr}(A^{-1}_{n1}(z)) - E\text{tr}(A^{-1}_{n1}(z))|^k = O(n^{-k/2}), \quad k \geq 2, \quad (3.14)
\]

(or see Bai and Silverstein [4]). This also holds when \( A^{-1}_{nij}(z) \) is replaced by \( A^{-1}_{nij}(z) \). Next, we list some useful results of (3.15)-(3.29). For simplicity, we assume that the spectral norms of nonrandom \( C, D, H, C_i, D_i \) involved in (3.16)-(3.26) below are all bounded by some positive constants. The proofs of (3.15)-(3.26) will be given in Appendix A.6.

\[
\begin{align*}
\frac{1}{n^k}E|x_1^\top B^\top CBx_1 - \text{tr}(C)|^k &= O\left(\frac{p^{k/2}}{n^{k/2+1}}\right), \quad 1 < k \leq 2, \quad (3.15) \\
\frac{1}{n^k}E|x_1^\top B^\top CBx_1 - \text{tr}(C)|^k &= O\left(\frac{p^{k/2}}{n^{k/2+1}}\right) + O\left(\frac{p^{k/2}}{n^{k/2+1}}\right), \quad k > 2, \quad (3.16) \\
E|\xi_1(z)|^2 &= O\left(\frac{p}{n}\right) + O(n^{-1}) = O(n^{-1}), \quad (3.17) \\
E|\xi_1(z)|^k &= O\left(\frac{p^{k+2}}{n^k}\right) + O\left(\frac{p^{k+2}}{n^k}\right), \quad k > 2, \quad (3.18) \\
E|x_1^\top B^\top Ce_je_j^\top DBx_1|^k &= O\left(\frac{p^{k/2-1}}{n^{k/2+1}}\right), \quad k \geq 2, \quad (3.19) \\
E|x_1^\top B^\top CBx_1|^2 &= O\left(\frac{p}{n}\right), \quad (3.20) \\
E|x_1^\top B^\top CBx_2|^k &= O\left((np)^{k/2-1}\right), \quad k \geq 4, \quad (3.21) \\
E|x_1^\top B^\top CBx_1|^k &= O\left(\frac{p^{k/2}}{n^{k/2}}\right), \quad k \geq 4, \quad (3.22) \\
E|x_1^\top B^\top CBx_1|^2 &= O\left(\frac{p^2}{n^2}\right), \quad (3.23) \\
E|x_1^\top B^\top CBx_1|^k &= O\left(\frac{p^{k/4}}{n^{k/4}}\right), \quad k \geq 4, \quad (3.24) \\
E|\alpha_1(z)|^2 &= O\left(\frac{p^2}{n}\right), \quad (3.25) \\
E\left|\prod_{i=1}^m \frac{1}{n}x_1^\top B^\top Cix_i Bx_i \prod_{j=1}^q \frac{1}{n}|x_1^\top B^\top D_j x_i Bx_1 - \text{tr}(D_j)|(x_1^\top B^\top HBx_1)|^\gamma\right| &= O(a_n), \quad (3.26)
\end{align*}
\]

where \( a_n = \frac{\sqrt{p}}{n} \), \( m \geq 0 \), \( q \geq 1 \), \( 0 \leq r \leq 2 \) and \( x \vee y \) means \( \max(x, y) \). Moreover, by \( p/n \to 0 \) as \( (p, n) \to \infty \), one has

\[
b_1(z) - 1 = -\frac{n^{-1}E\text{tr}(A^{-1}_{n1}(z))}{1 + n^{-1}E\text{tr}(A^{-1}_{n1}(z))} = O\left(\frac{p}{n}\right). \quad (3.27)
\]

In view of the fact that \( \beta_1(z), \beta_1^{tr}(z), b_1(z) \) are bounded by \( |z|/v \), by (3.15), we have

\[
\beta_1(z) - \beta_1^{tr}(z) = O\left(\frac{\sqrt{p}}{n}\right), \quad (3.28)
\]
and by (3.14), we have
\[ \beta_1^r(z) - b_1(z) = O_P(n^{-1/2}). \]  

(3.29)

Now, we consider the term of \( M_n^{(1)}(z) \). In the following, let \( F_0 = \sigma(\emptyset, \Omega) \) and \( F_j = \sigma(\mathbf{x}_1, \ldots, \mathbf{x}_j), \, j \geq 1 \). So \( E_j(\cdot) = E(\cdot | F_j) \) denotes the conditional expectation with respect to \( F_j, \, j \geq 0 \). It can be checked that
\[
M_n^{(1)}(z) = \frac{n}{\sqrt{p}} \left( \tilde{x}^\top B^\top A_n^{-1}(z)B\tilde{x} - E\tilde{x}^\top B^\top A_n^{-1}(z)B\tilde{x} \right)
\]
\[ = \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})[\tilde{x}^\top B^\top A_n^{-1}(z)B\tilde{x} - \tilde{x}_j^\top B^\top A_n^{-1}(z)B\tilde{x}_j]
\]
\[ = \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1} + a_{n2} + a_{n3}), \]  

(3.30)

where
\[
a_{n1} = (\tilde{x} - \tilde{x}_j)^\top B^\top A_n^{-1}(z)B\tilde{x}, \quad a_{n2} = \tilde{x}_j^\top B^\top (A_n^{-1}(z) - A_{n_j}^{-1}(z))B\tilde{x},
\]
\[
a_{n3} = \tilde{x}_j^\top B^\top A_{n_j}^{-1}(z)B(\tilde{x} - \tilde{x}_j).
\]

We estimate the first term \( a_{n1} \). In view of
\[
A_n^{-1}(z) = |A_n^{-1}(z) - A_{n_j}^{-1}(z)| + A_{n_j}^{-1}(z) \quad \text{and} \quad \tilde{x} = \tilde{x}_j + \mathbf{x}_j/n,
\]
we have by (3.13) that
\[
a_{n1} = a_{n1}^{(1)} + a_{n1}^{(2)} + a_{n1}^{(3)} + a_{n1}^{(4)}, \]  

(3.31)

where
\[
a_{n1}^{(1)} = -\frac{1}{n^3}(\mathbf{x}_j^\top B^\top A_{n_j}^{-1}(z)B\mathbf{x}_j)^2 \beta_j(z), \quad a_{n1}^{(2)} = -\frac{1}{n^2}(\mathbf{x}_j^\top B^\top A_{n_j}^{-1}(z)B\mathbf{x}_j) \beta_j(z)
\]

and
\[
a_{n1}^{(3)} = \frac{1}{n^2} \mathbf{x}_j^\top B^\top A_{n_j}^{-1}(z)B\mathbf{x}_j, \quad a_{n1}^{(4)} = \frac{1}{n} \mathbf{x}_j^\top B^\top A_{n_j}^{-1}(z)B\mathbf{x}_j.
\]

Since
\[
\beta_j(z) = \beta_j^r(z) - \beta_j(z)\beta_j^{tr}(z)\gamma_j(z), \]  

(3.32)

it follows that
\[
(E_j - E_{j-1})a_{n1}^{(1)} = (E_j - E_{j-1})[-\frac{1}{n^3}(\mathbf{x}_j^\top B^\top A_{n_j}^{-1}(z)B\mathbf{x}_j)^2 \beta_j^{tr}(z)] + \zeta_n
\]
\[ = (E_j - E_{j-1})[-\frac{2}{n} \gamma_j(z) \beta_j^{tr}(z)] + (E_j - E_{j-1})[-\frac{2}{n} \gamma_j(z) \beta_j^{tr}(z)\frac{1}{n} \text{tr}(A_{n_j}^{-1}(z))] + \zeta_n ,
\]

where
\[
\zeta_n = (E_j - E_{j-1})[-\frac{1}{n^3}(\mathbf{x}_j^\top B^\top A_{n_j}^{-1}(z)B\mathbf{x}_j)^2 \beta_j(z)\beta_j^{tr}(z)\gamma_j(z)].
\]
Thus, by (3.26),
\[
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})a_{n1}^{(1)} \right|^2 \leq C_1 \frac{n^2}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})a_{n1}^{(1)}]^2
\]
\[
\leq C_2 \frac{n^3}{p} \left\{ \frac{1}{n^2} E|\gamma_1(z)|^4 + \frac{p^2}{n^4} E|\gamma_1(z)|^2 + \frac{1}{n^2} E|\gamma_1(z)(n^{-\frac{1}{2}} x_1^T B^{-1} A_{n1}^{-1}(z) B x_1)|^2 \right\}
\]
\[
= O\left( \frac{n^3}{p n^3} \right) = O(p^{-1/2}),
\]
which implies
\[
\frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})a_{n1}^{(1)} = o_p(1).
\]
Similarly, by (3.15), $BB^\top = I_p$ and $p/n = o(1)$,
\[
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})a_{n1}^{(3)} \right|^2 \leq C_1 \frac{n^2}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})a_{n1}^{(3)}]^2
\]
\[
\leq C_2 \frac{n^3}{p} \left\{ \frac{1}{n^2} E\left|\beta_j(z)\right|^4 + \frac{p^2}{n^4} E\left|\beta_j(z)\right|^2 + \frac{1}{n^2} E\left|\beta_j(z)(n^{-\frac{1}{2}} x_1^T B^{-1} A_{n1}^{-1}(z) B x_1)\right|^2 \right\}
\]
\[
\leq C_2 \frac{n^3}{p} \left( \frac{p}{n^2} + \frac{p^2}{n^4} \right) = O\left( \frac{1}{n} + \frac{p}{n} \right).
\]
Then, we obtain that
\[
\frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})a_{n1}^{(3)} = o_p(1).
\]
In addition, it follows from (3.26) that
\[
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})\gamma_j(z) \frac{1}{n} x_j^T B^{-1} A_{n1}^{-1}(z) B x_j \beta_j(z) \right|^2
\]
\[
\leq C_1 \frac{n}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\gamma_j(z) x_j^T B^{-1} A_{n1}^{-1}(z) B x_j \beta_j(z)]^2 = O\left( \frac{n \sqrt{p}}{p n} \right) = O(p^{-1/2})
\]
and
\[
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2} x_j^T B^{-1} A_{nj}^{-1}(z) B x_j \beta_j(z) \gamma_j(z) \beta_j(z) \right|^2
\]
\[
\leq C_1 \frac{n}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\gamma_j^2(z) x_j^T B^{-1} A_{nj}^{-1}(z) B x_j \beta_j(z) \beta_j(z)]^2
\]
\[
+ C_2 \frac{n}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\frac{1}{n} \text{tr}(A_{nj}^{-1}(z)) \gamma_j(z) x_j^T B^{-1} A_{n1}^{-1}(z) B x_j \beta_j(z) \beta_j(z)]^2
\]
\[
= O\left( \frac{n}{p} \left( \frac{\sqrt{p}}{n} + \frac{p^2 \sqrt{p}}{n^2} \right) \right) = O(p^{-1/2}).
\]
Combining (3.32) with $BB^T = I_p$, we obtain that

$$
\frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) a_{n1}^{(2)}
$$

$$
= - \frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2} x_j^T B^T \hat{A}_{n1}(z) B x_j (\beta_j^{tr}(z) - \beta_j(z) \beta_j^{tr}(z) \gamma_j(z))
$$

$$
= - \frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2} x_j^T B^T \hat{A}_{n1}(z) B x_j \beta_j^{tr}(z) + o_P(1)
$$

$$
= - \frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2} x_j^T B^T \hat{A}_{n1}(z) B x_j \beta_j^{tr}(z) + o_P(1)
$$

Obviously, by (3.20) and Lemma A.1 in Appendix A.1,

$$
E \left| \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n} x_j^T B^T \hat{A}_{n1}(z) B x_j \right|^2
$$

$$
= \sum_{j=1}^{n} E \left| (E_j - E_{j-1}) \frac{1}{n} x_j^T B^T \hat{A}_{n1}(z) B x_j \right|^2 
\leq C_1 \frac{p^3}{n^4}.
$$

Moreover, by (3.27) and (3.29), $\beta_j^{tr}(z) = 1 + O_P(n^{-1/2}) + O(p/n)$. Thus,

$$
\frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) a_{n1}^{(2)} = \frac{n}{\sqrt{P}} \frac{p^3/2}{n^2} = O_P(\frac{p}{n}) = o_P(1). \quad (3.35)
$$

Consequently, by (3.31), (3.33), (3.34) and (3.35), we obtain that

$$
\frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) a_{n1} = \frac{n}{\sqrt{P}} \sum_{j=1}^{n} (E_j - E_{j-1}) a_{n1}^{(4)} + o_P(1)
$$

$$
= \sum_{j=1}^{n} E \left[ \frac{1}{\sqrt{P}} x_j^T B^T \hat{A}_{n1}(z) B x_j \right] + o_P(1). \quad (3.36)
$$

Now, we consider the second term $a_{n2}$. By $\mathbf{\bar{x}} = \mathbf{x}_j + \frac{1}{n} \mathbf{x}_j$, we have

$$
a_{n2} = \frac{1}{n^2} x_j^T B^T \hat{A}_{n1}(z) B x_j \beta_j(z) - \frac{1}{n} x_j^T B^T \hat{A}_{n1}(z) B \mathbf{\bar{x}}_j \beta_j(z). \quad (3.37)
$$
By $BB^T = I_p$, 

$$\frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})[\vec{x}_j^T B^T \tilde{A}_{nj}(z)B\bar{x}_j]$$

$$= \frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})[\vec{x}_j^T B^T A_{nj}^{-1}(z)B\bar{x}_j]$$

$$= \frac{n}{\sqrt{p}} \sum_{j=1}^{n} E_j \frac{1}{n} [\vec{x}_j^T B^T A_{nj}^{-1}(z)B\bar{x}_j] = \frac{n}{\sqrt{p}} \sum_{j=1}^{n} E_j \alpha_j(z).$$

By (3.25),

$$E\left[\frac{n}{\sqrt{p}} \sum_{j=1}^{n} E_j \alpha_j(z)\right]^2 = E\left[\frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \alpha_j(z)\right]^2$$

$$\leq C_1 n^2 \frac{p}{n} \sum_{j=1}^{n} E|(E_j - E_{j-1}) \alpha_j(z)|^2 = O\left(\frac{p}{n}\right).$$

Combining it with the fact that $p/n = o(1)$, we obtain that

$$\frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})[\vec{x}_j^T B^T \tilde{A}_{nj}(z)B\bar{x}_j] = \frac{n}{\sqrt{p}} \sum_{j=1}^{n} E_j \alpha_j(z) = o_P(1). \quad (3.38)$$

Obviously, it follows from (3.27), (3.28) and (3.29) that

$$\beta_j(z) = 1 + O_P(n^{-1/2}) + O\left(\frac{p}{n}\right) \quad \text{and} \quad \beta_j^s(z) = 1 + O_P(n^{-1/2}) + O\left(\frac{p}{n}\right).$$

Thus, similarly to $a_{n1}^{(2)}$, we establish by (3.35), (3.37) and (3.38) that

$$\frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})a_{n2}$$

$$= -\sum_{j=1}^{n} (E_j - E_{j-1})[(1 - \beta_j^s(z))\frac{1}{\sqrt{p}} \vec{x}_j^T B^T A_{nj}^{-1}(z)B\bar{x}_j]$$

$$-\frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})[\vec{x}_j^T B^T \tilde{A}_{nj}(z)B\bar{x}_j \beta_j(z)] + o_P(1)$$

$$= (O_P(n^{-1/2}) + O\left(\frac{p}{n}\right)) \sum_{j=1}^{n} E_j\left[\frac{1}{\sqrt{p}} \vec{x}_j^T B^T A_{nj}^{-1}(z)B\bar{x}_j\right]$$

$$- (1 + O_P(n^{-1/2}) + O\left(\frac{p}{n}\right)) \frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})[\vec{x}_j^T B^T \tilde{A}_{nj}(z)B\bar{x}_j]$$

$$= o_P(1). \quad (3.39)$$
Obviously, the term $a_{n3}$ has a similar result to that of $a_{n1}$. Then, it follows from (3.30), (3.36) and (3.39) that

$$M_n^{(1)}(z) = \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})(a_{n1} + a_{n3}) + o_P(1) = \sum_{j=1}^{n} Y_j(z) + o_P(1),$$

where

$$Y_j(z) = 2E_j\left(\frac{1}{\sqrt{p}}x_j^\top B^{-1}A_{nj}^{-1}(z)Bx_j\right), \quad 1 \leq j \leq n.$$ 

Consequently, for the finite dimensional convergence of $M_n^{(1)}(z)$, we only need to consider the sum

$$\sum_{i=1}^{r} a_i \sum_{j=1}^{n} Y_j(z_i) = \sum_{j=1}^{n} r \sum_{i=1}^{r} a_i Y_j(z_i), \quad (3.40)$$

where $r$ is any positive integer and $a_1, \ldots, a_r$ are any complex numbers. For any $\varepsilon > 0$, by (3.24),

$$\sum_{j=1}^{n} E\left[ a_i Y_i(z) \right]^2 I\left( \sum_{j=1}^{r} a_i Y_j(z_i) \geq \varepsilon \right) \leq \frac{C_1}{\varepsilon^2} \sum_{j=1}^{n} \sum_{i=1}^{r} \frac{|a_i|^4}{p^2} E|X_j^\top B^{-1}A_{nj}^{-1}(z)Bx_j|^4 = O\left(\frac{1}{p}\right) = o(1),$$

as $p \to \infty$. Therefore, the condition (ii) of Lemma A.2 in Appendix A.1 is satisfied. The next task is to verify the condition (i) of Lemma A.2, i.e., to calculate the limit of

$$\sum_{j=1}^{n} E_{j-1}(Y_j(z_1)Y_j(z_2))$$

$$= \frac{4}{p} \sum_{j=1}^{n} E_{j-1}\left[ E_j(x_j^\top B^{-1}A_{nj}^{-1}(z_1)Bx_j)E_j(x_j^\top B^{-1}A_{nj}^{-1}(z_2)Bx_j)\right] \quad (3.41)$$

in probability for $z_1, z_2 \in C \setminus \mathbb{R}$. By the Appendix A.2, the limit of (3.41) is obtained as

$$\text{RHS of (3.41)} \xrightarrow{P} \frac{1}{(1-z_1)(1-z_2)}, \quad (3.42)$$

as $p/n \to 0$ and $(p, n) \to \infty$.

### 3.2. Tightness of $\hat{M}_n^{(1)}(z)$

In this subsection we prove the tightness of $\hat{M}_n^{(1)}(z)$ for $z \in C$, which is a truncated version of $M_n(z)$ in (3.3). For $Y_j(z)$ defined in (3.40), by (3.20), we have

$$E\left[ \sum_{i=1}^{r} a_i \sum_{j=1}^{n} Y_j(z_i) \right]^2 = \sum_{j=1}^{n} E\left[ \sum_{i=1}^{r} a_i Y_j(z_i) \right]^2 \leq K_1, \quad v_0 = \mathcal{J}(z),$$
which ensures that the condition (i) of Theorem 12.3 in Billingsley [7] is satisfied. By the Appendix A.3, we obtain that

\[
E \left| \frac{M_n^{(1)}(z_1) - M_n^{(1)}(z_2)}{|z_1 - z_2|^2} \right| \leq K_2 \quad \text{for all} \quad z_1, z_2 \in C_n^+ \cup C_n^-, \quad (3.43)
\]

which completes the proof of tightness.

### 3.3. Convergence of \( M_n^{(2)}(z) \)

By the Appendix A.4, for all \( z \in C_n \), as \( (p, n) \to \infty \), we obtain that

\[
\sup_{z \in C_n} M_n^{(2)}(z) = \sup_{z \in C_n} \frac{n}{\sqrt{p}} \left( E(\overline{x}^\top B^\top A_n^{-1}(z)B\overline{x}) - c_n m(z) \right) \to 0, \quad (3.44)
\]

where \( p/n = c_n \) and \( m(z) = \int x I(x < z) dH(x) \) and \( H(x) = I(1 \leq x) \) for \( x \in \mathbb{R} \).

**Proof of Theorem 1.** First, by the Cauchy integral formula, we have

\[
\int f(x) dG(x) = -\frac{1}{2\pi i} \oint f(z) m_G(z) dz,
\]

where the contour contains the support of \( G(x) \) on which \( f(x) \) is analytic and \( m_G(z) = \int x I(x < z) dH(x) \) for \( I(z) > 0 \) (see Bai and Silverstein [5]). Then, for all \( n \) large enough, with convergence in probability, we obtain

\[
\int f(x) dG_n(x) = -\frac{1}{2\pi i} \oint f(z) m_G(z) dz,
\]

where the complex integral is over \( C \), \( c_n = p/n \),

\[
G_n(x) = \frac{n}{\sqrt{p}} c_n \left[ F_n^\top (S_n - zI)^{-1} B\overline{x} \right] \frac{\overline{x}^\top B^\top (S_n - zI)^{-1} B\overline{x}}{||B\overline{x}||^2} - m(z),
\]

\[
X_n(z) = \frac{n}{\sqrt{p}} c_n \left[ F_n^\top (S_n - zI)^{-1} B\overline{x} \right] \frac{\overline{x}^\top B^\top (S_n - zI)^{-1} B\overline{x}}{||B\overline{x}||^2} - m(z),
\]
\[ m(z) = \int \frac{1}{x-z} dH(x), \quad F_{S_n}^2(x) = \sum_{j=1}^{p} t_j^2 I(\lambda_j \leq x), \quad H(x) = I(1 \leq x), \]

\[ t = (t_1, \ldots, t_p)^T = \frac{\text{VBl}}{\|x\|}, \text{V is the eigenvector matrix of } \tilde{S}_n \text{ and } \lambda = (\lambda_1, \ldots, \lambda_p)^T \text{ is the eigenvalue vector of } \tilde{S}_n. \]

Combining Lemma A.5 in Appendix A.1 with the rank inequality (see Bai and Silverstein [6]), \( \lambda_{\max}(\tilde{S}) \overset{p}{\to} \|I_p\| = 1 \) and \( \lambda_{\min}(\tilde{S}) \overset{p}{\to} \lambda_{\min}(I_p) = 1 \). Thus,

\[ \left| \int f(z)(X_n(z) - \hat{X}_n(z))dz \right| \leq \frac{C_1 \rho_n}{\sqrt{p}(ur - 1)} + \frac{C_2 \rho_n}{\sqrt{p}(1 - mu)} \overset{p}{\to} 0. \]

Secondly, for any real constants \( a_1 \) and \( a_2 \),

\((\hat{X}_n(z), Y_n) \to a_1 \int f(z)\hat{X}_n(z)dz + a_2 Y_n\)

is a continuous mapping. Furthermore,

\[
\begin{align*}
\text{Var}\left(-\frac{1}{2\pi i} \int f(z)\hat{X}(z)dz\right) &= -\frac{1}{4\pi^2} \int \int 2 \frac{f(z_1)f(z_2)}{(z_1 - 1)(z_2 - 1)} dz_1 dz_2 = 2f^2(1), \\
\text{Cov}\left(-\frac{1}{2\pi i} \int f(z)\hat{X}(z)dz, Y\right) &= -\frac{1}{2\pi i} \int 2g'(0) \frac{f(z)}{1 - z} dz \\
&= \frac{1}{\pi i} \int g'(0) \frac{f(z)}{z - 1} dz = 2g'(0)f(1), \\
\text{Var}(Y) &= 2(g'(0))^2.
\end{align*}
\]

Consequently, the proof of Theorem 1 is completed.

\[ \square \]

Appendices

In the Appendix A.1, Lemmas A.1-A.5 are listed. The proofs of (3.41), tightness of \( \tilde{M}_n^{(1)}(z) \) and convergence of \( M_n^{(2)}(z) \) are presented in the Appendices A.2-A.4, respectively. In the Appendix A.5, some results on truncated random variables are discussed for (3.11). The proofs of (3.15)-(3.26) are presented in the Appendix A.6. Lastly, the proof of Lemma 1 is listed in the Appendix A.7.

A.1. Some lemmas.

Lemma A.1 (Burkholder [9]). Let \( \{Y_n, n \geq 1\} \) be a sequence of complex martingale differences with respect to the increasing \( \sigma \)-field \( F_n \). Then, for any \( p \geq 2 \) and \( n \geq 1 \),

\[ E \left| \sum_{i=1}^{n} Y_i \right|^p \leq C_1 E \left( \sum_{i=1}^{n} E(Y_i^2 | F_{i-1}) \right)^{p/2} + C_2 \sum_{i=1}^{n} E|Y_i|^p, \]
where \( C_1 \) and \( C_2 \) are positive constants depending only on \( p \).

**Lemma A.2** ([Billingsley][8, Theorem 35.12]). For each \( n \), let \( \{Y_{n,1}, Y_{n,2}, \ldots, Y_{n,r_n}\} \) be a sequence of real martingale difference with respect to the increasing \( \sigma \)-field \( \mathcal{F}_{n,i} \) having finite second moments. If as \( n \to \infty \), (i) \( \sum_{i=1}^{r_n} E(Y_{n,i}^2|F_{i-1}) \xrightarrow{p} \sigma^2; \) (ii) \( \sum_{i=1}^{r_n} E(Y_{n,i}^2 I((Y_{n,i} \geq \varepsilon)) \to 0 \), where \( \sigma^2 \) is a positive constant and \( \varepsilon \) is an arbitrary positive number, then

\[
\sum_{i=1}^{r_n} Y_{n,i} \xrightarrow{d} N(0, \sigma^2).
\]

**Lemma A.3** Let \( Y = (Y_1, \ldots, Y_m)^\top \), where \( Y_1, \ldots, Y_m \) are i.i.d. real random variables with \( EY_1 = 0, EY_1^2 = 1, EY_1^4 = \gamma_1 < \infty \). Let \( p \leq m, A = (a_{ij})_{p \times p} \) and \( B = (b_{ij})_{p \times m} \) be a nonrandom matrix satisfying \( \|A\| = O(1), BB^\top = \Sigma_p \) and \( \|\Sigma_p\| = O(1) \), where \( \|A\| \) denote the spectral normal of matrix \( A \). Denote \( B = (b_{ij})_{p \times m} = (b_1, \ldots, b_m) \), where \( \|b_j\| = (\sum_{i=1}^{p} b_{ij}^2)^{1/2} \), \( b_j = (b_{1j}, \ldots, b_{pj})^\top, 1 \leq j \leq m \). Assume that \( |Y_j| \leq (pn)^{1/4}/\|b_j\| \) for all \( 1 \leq j \leq m \). Then,

\[
E|Y^\top B^\top A B Y - tr(A\Sigma_p)|^k \leq C_1 p^{k/2}, 1 < k \leq 2, \tag{B.1}
\]

\[
E|Y^\top B^\top A B Y - tr(A\Sigma_p)|^k \leq C_2 (p^{k/2+1} + n^{k/2-1}p^{k/2}), k > 2, \tag{B.2}
\]

where \( C_1 \) and \( C_2 \) denote some positive constant depending only on \( k \).

In addition, let \( Z = (Z_1, \ldots, Z_m)^\top \), where \( Z_1, \ldots, Z_m \) are i.i.d. real random variables with \( EZ_1 = 0, EZ_1^2 = 1, EZ_1^4 = \gamma_2 < \infty \). Assume that \( Z_1, \ldots, Z_m \) are independent of \( Y_1, \ldots, Y_m \) and \( |Z_j| \leq (pn)^{1/4}/\|b_j\| \) for all \( 1 \leq j \leq m \). Then

\[
E|Y^\top B^\top A B Z|^k \leq C_3 p^{k/2}, 1 < k \leq 2, \tag{B.3}
\]

\[
E|Y^\top B^\top A B Z|^k \leq C_4 (p^{k/2+1} + n^{k/2-2}p^{k/2}), k > 2, \tag{B.4}
\]

where \( C_3 \) and \( C_4 \) denote some positive constant depending only on \( k \).

**Proof of Lemma A.3.** Denote \( H = B^\top A B = (h_{ij})_{1 \leq i,j \leq m} \). It is used the partition

\[
Y^\top B^\top A B Y - tr(A\Sigma_p) = Y^\top H Y - tr(H)
= \sum_{i=1}^{m} h_{ii}(Y_i^2 - 1) + 2 \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Y_j. \tag{B.5}
\]

Firstly, it is easy to see

\[
|h_{ij}| = |e_i^\top B^\top A B e_j| \leq \|A\| \sqrt{e_i^\top B e_i} \sqrt{e_j^\top B e_j} = \|A\| \|b_i\| \|b_j\|, \tag{B.6}
\]

where \( e_i \) is a vector with \( i \)-th element 1 and remaining elements 0. In addition,

\[
\sum_{j=1}^{m} \|b_j\|^2 = \sum_{j=1}^{m} \sum_{i=1}^{p} b_{ij}^2 = \|B\|_F^2 = tr(B^\top B) = tr(BB^\top) \leq p \|\Sigma_p\|. \tag{B.7}
\]
and
\[ \|b_j\|^2 = b_j^\top b_j = \|b_j b_j^\top\| \leq \sum_{j=1}^m \|b_j b_j^\top\| = \|\Sigma_p\|, 1 \leq j \leq m. \quad (B.8) \]

By Hölder’s inequality and Lemma A.1, \( \|A\| = \|\Sigma_p\| = O(1) \) and (B.6)-(B.8), we have for \( 1 < k \leq 2 \) that
\[
E \left| \sum_{i=1}^m h_{ii} (Y_i^2 - 1) \right|^k \leq \left( E \left| \sum_{i=1}^m h_{ii} (Y_i^2 - 1) \right|^2 \right)^{k/2}
\leq C_1 \left( \sum_{i=1}^m |h_{ii}|^2 E |Y_i^2 - 1|^2 \right)^{k/2} \leq C_2 \left( \sum_{i=1}^m EY_i^4 \|b_i\|^4 \right)^{k/2} \leq C_3 p^{k/2}. \quad (B.9)
\]

Meanwhile,
\[
E \left| \sum_{i=1}^m \sum_{j=1}^{i-1} h_{ij} Y_i Y_j \right|^k \leq \left( E \left| \sum_{i=1}^m \sum_{j=1}^{i-1} h_{ij} Y_i Y_j \right|^2 \right)^{k/2}
\leq C_1 \left( \sum_{i=1}^m \sum_{j=1}^{i-1} h_{ij}^2 EY_i^2 EY_j^2 \right)^{k/2} \leq C_2 (\text{tr}(HH^\top))^k \leq C_3 p^{k/2}. \quad (B.10)
\]

Combining \( C_r \) inequality with (B.5), (B.9) and (B.10), we obtain that
\[
E|Y^\top H Y - \text{tr}(H)|^k \leq C_1 p^{k/2}, \quad 1 < k \leq 2.
\]

i.e. (B.1) is proved.

Next, we prove (B.2) for the case \( 2 < k \leq 4 \). By Lemma A.1 with \( 2 < k \leq 4 \),
\[
E \left| \sum_{i=1}^m h_{ii} (Y_i^2 - 1) \right|^k \leq C_1 \left\{ \left( \sum_{i=1}^m |h_{ii}|^2 E |Y_i^2 - 1|^2 \right)^{k/2} + \sum_{i=1}^m |h_{ii}|^k E |Y_i^2 - 1|^k \right\}
\leq C_2 \left\{ (\text{tr}(HH^\top))^{k/2} + \sum_{i=1}^m EY_i^{2k} \|b_i\|^{2k} \right\}
\leq C_3 (p^{k/2} + (np)^{k/2-1} \sum_{i=1}^m \|b_i\|^4)
\leq C_4 (p^{k/2} + n^{k/2-1} p^{k/2}). \quad (B.11)
\]

Let \( E_i(\cdot) \) be the conditional expectation with respect to \( \mathcal{F}_i = \sigma(Y_1, \ldots, Y_i) \) and
\[ \mathcal{F}_0 = \sigma(\emptyset, \Omega). \] Then, by Lemma A.1 with \( 2 < k \leq 4 \),

\[
E\left|\sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Y_j \right|^k \leq C_1 \left\{ E\left( \sum_{i=1}^{m} \left| \sum_{j=1}^{i-1} h_{ij} Y_i Y_j \right|^2 \right)^{k/2} + \sum_{i=1}^{m} E \left| \sum_{j=1}^{i-1} h_{ij} Y_i Y_j \right|^k \right\} 
\]

\[
= C_1 \left\{ E\left( \sum_{i=1}^{m} E\left| \sum_{j=1}^{i-1} h_{ij} Y_j \right|^2 \right)^{k/2} + \sum_{i=1}^{m} E \left| \sum_{j=1}^{i-1} h_{ij} Y_j \right|^k \right\} 
\]

\[
\leq C_1 E\left( \sum_{i=1}^{m} \left| E_{i-1} \sum_{j=1}^{i-1} h_{ij} Y_j \right|^2 \right)^{k/2} + C_2 \sum_{i=1}^{m} \left( \sum_{j=1}^{i-1} |h_{ij}|^2 E_{Y_i}^2 E_{Y_j}^2 \right)^{k/2} 
\]

\[
+ C_3 \sum_{i=1}^{m} \sum_{j=1}^{i-1} |h_{ij}|^k E |Y_i|^k E |Y_j|^k 
\]

\[
\leq C_4 \left( E|Y^\top HH^\top Y - \text{tr}(HH^\top)|^{k/2} + \|\text{tr}(HH^\top)\|^{k/2} + p^{k/2+1} + p^{k/2} \right) 
\]

\[
\leq C_5 p^{k/2+1}, \quad (B.12) 
\]

where we use inequality (B.1) with \( H \) replaced by \( HH^\top \) to obtain

\[
E|Y^\top HH^\top Y - \text{tr}(HH^\top)|^{k/2} \leq C p^{k/4}. 
\]

Consequently, by (B.11) and (B.12),

\[
E|Y^\top HY - \text{tr}(H)|^k \leq C_1 (p^{k/2+1} + n^{k/2-1} p^{k/2}), \quad (B.13) 
\]

i.e. (B.2) holds for \( 2 < k \leq 4 \).

Now, we proceed the proof of (B.2) by induction on \( k > 4 \). Assume that (B.2) is true for \( 2^t < k \leq 2^{t+1} \) with \( t \geq 2 \). Then, we consider the case \( 2^{t+1} < k \leq 2^{t+2} \) with \( t \geq 2 \). By the proof of (B.11),

\[
E\left| \sum_{i=1}^{m} h_{ii}(Y_i^2 - 1) \right|^k \leq C_1 \left\{ (\text{tr}(HH^\top))^{k/2} + \sum_{i=1}^{m} E Y_i^{2k} \|b_i\|^{2k} \right\} 
\]

\[
\leq C_2 (p^{k/2} + n^{k/2-1} p^{k/2}) \quad (B.14) 
\]
and by the proof of (B.12),
\[
E \left| \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Y_j \right|^k \\
\leq C_1 E \left( \sum_{i=1}^{m} \sum_{j=1}^{m} h_{ij} |Y_j|^2 \right)^{k/2} + C_2 \sum_{i=1}^{m} \| b_i \|^k \left( \sum_{j=1}^{m} \| b_j \|^2 \right)^{k/2} \\
+ C_3 \sum_{i=1}^{m} \| b_i \|^k E |Y_i|^k \sum_{j=1}^{m} \| b_j \|^k E |Y_j|^k \\
\leq C_4 \left( E|Y^\top HH^\top Y - \text{tr}(HH^\top)|^{k/2} + (\text{tr}(HH^\top))^{k/2} \right) \\
+ C_5 (p^{k/2+1} + (np)^{k/2-2} p^2).
\] (B.15)

Using the induction hypothesis with $H$ replaced $HH^\top$, we have for $2^{t+1} < k \leq 2^{t+2}$ that
\[
E|Y^\top HH^\top Y - \text{tr}(HH^\top)|^{k/2} \leq C_5 (n^{k/4-1} p^{k/4} + p^{k/4+1}).
\] (B.16)

From (B.14) to (B.16), for $2^{t+1} < k \leq 2^{t+2}$ and $t \geq 2$,
\[
E|Y^\top HY - \text{tr}(H)|^k \leq C_1 (p^{k/2+1} + n^{k/2-1} p^{k/2}).
\]

Thus, the proof of (B.2) is completed for $k > 2$.

To prove (B.3) and (B.4), we also use the same notion above. It is easy to see that
\[
Y^\top B^\top ABZ = Y^\top HZ = \sum_{i=1}^{m} h_{ii}Y_i Z_i + 2 \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Z_j.
\] (B.17)

Similarly to (B.9), for $1 < k \leq 2$,
\[
E \left| \sum_{i=1}^{m} h_{ii} Y_i Z_i \right|^k \leq \left( E \left| \sum_{i=1}^{m} h_{ii} Y_i Z_i \right|^2 \right)^{k/2} \\
\leq C_1 \left( \sum_{i=1}^{m} |h_{ii}|^2 E |Y_i|^2 E |Z_i|^2 \right)^{k/2} \leq C_2 \left( \sum_{i=1}^{m} \| b_i \|^4 \right)^{k/2} \leq C_3 p^{k/2}
\] (B.18)

and
\[
E \left| \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Z_j \right|^k \leq \left( E \left| \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Z_j \right|^2 \right)^{k/2} \\
\leq C_1 \left( \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij}^2 E |Y_i|^2 E |Z_j|^2 \right)^{k/2} \leq C_2 (\text{tr}(HH^\top))^{k/2} \leq C_3 p^{k/2}.
\] (B.19)

Combining with (B.17), we have
\[
E|Y^\top HZ|^k \leq C_1 p^{k/2}, \quad 1 < k \leq 2.
\]
i.e. (B.3) is proved.

Now, we prove (B.4) for the case $2 < k \leq 4$. Similarly to (B.11),

$$
E\left| \sum_{i=1}^{m} h_{ii}Y_{i}Z_{i}\right|^{k} \leq C_{1}\left\{ \left( \sum_{i=1}^{m} |h_{ii}|^{2} EY_{i}^{2}EZ_{i}^{2}\right)^{k/2} + \sum_{i=1}^{m} |h_{ii}|^{k} E|Y_{i}|^{k}E|Z_{i}|^{k}\right\} 
$$

$$
\leq C_{2}\left\{ (\text{tr}(HH^{\top}))^{k/2} + \sum_{i=1}^{m} \|b_{i}\|^{2k}\right\} 
$$

$$
\leq C_{3}(p^{k/2} + p) \leq C_{4}p^{k/2}.
$$

(B.20)

Let $E_{i}(\cdot)$ be the conditional expectation with respect to $\mathcal{F}_{i} = \sigma(Y_{1}, Z_{1}, \ldots, Y_{i}, Z_{i})$ and $\mathcal{F}_{0} = \sigma(\emptyset, \Omega)$. Then, similar to (B.12),

$$
E\left| \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij}Y_{i}Z_{j}\right|^{k} 
$$

$$
\leq C_{1}\left\{ E\left( \sum_{i=1}^{m} E_{i-1}\left| \sum_{j=1}^{i-1} h_{ij}Y_{i}Z_{j}\right|^{2}\right)^{k/2} + \sum_{i=1}^{m} E\left| \sum_{j=1}^{i-1} h_{ij}Y_{i}Z_{j}\right|^{k}\right\} 
$$

$$
= C_{1}\left\{ E\left( \sum_{i=1}^{m} EY_{i}^{2}\left| \sum_{j=1}^{i-1} h_{ij}Z_{j}\right|^{2}\right)^{k/2} + \sum_{i=1}^{m} E\left| \sum_{j=1}^{i-1} h_{ij}Y_{i}Z_{j}\right|^{k}\right\} 
$$

$$
\leq C_{1}E\left( \sum_{i=1}^{m} \left| E_{i-1}\sum_{j=1}^{i-1} h_{ij}Z_{j}\right|^{2}\right)^{k/2} + C_{2}\sum_{i=1}^{m} \left( \sum_{j=1}^{i-1} |h_{ij}|^{2} EY_{i}^{2}EZ_{j}^{2}\right)^{k/2} 
$$

$$
+ C_{3}\sum_{i=1}^{m} \sum_{j=1}^{i-1} |h_{ij}|^{k} E|Y_{i}|^{k}E|Z_{j}|^{k} 
$$

$$
\leq C_{1}E\left( \sum_{i=1}^{m} \left| \sum_{j=1}^{i-1} h_{ij}Z_{j}\right|^{2}\right)^{k/2} + C_{2}\sum_{i=1}^{m} \|b_{i}\|^{k} \left( \sum_{j=1}^{m} \|b_{j}\|^{2}\right)^{k/2} 
$$

$$
+ C_{3}\sum_{i=1}^{m} \|b_{i}\|^{k} E|Y_{i}|^{k} \sum_{j=1}^{m} \|b_{j}\|^{k} E|Z_{j}|^{k} 
$$

$$
\leq C_{4}\left( E|Z^{\top}HH^{\top}Z - \text{tr}(HH^{\top})|^{k/2} + (\text{tr}(HH^{\top}))^{k/2} + p^{k/2+1} + p^{2}\right) 
$$

$$
\leq C_{5}p^{k/2+1},
$$

(B.21)

where we use inequality (B.1) with $Y$ replaced by $Z$ and $H$ replaced by $HH^{\top}$ to obtain

$$
E|Z^{\top}HH^{\top}Z - \text{tr}(HH^{\top})|^{k/2} \leq C_{4}p^{k/4}.
$$

Consequently, by (B.20) and (B.21),

$$
E|Y^{\top}HZ|^{k} \leq C_{1}p^{k/2+1},
$$

(B.22)

i.e. (B.4) holds for $2 < k \leq 4$. 
In the following, we prove (B.4) for \( k > 4 \). By the proof of (B.20), for \( k > 4 \),
\[
E \left| \sum_{i=1}^{m} h_{ii} Y_i Z_i \right|^k \leq C_1 \left\{ \left( \sum_{i=1}^{m} |h_{ii}|^2 E Y_i^2 E Z_i^2 \right)^{k/2} + \sum_{i=1}^{m} |h_{ii}|^k E|Y_i|^k E|Z_i|^k \right\} \\
\leq C_1 \left\{ \left( \text{tr}(HH^T) \right)^{k/2} + \sum_{i=1}^{m} E|Y_i|^k E|Z_i|^k \|b_i\|^2k \right\} \\
\leq C_1 \left\{ \left( \text{tr}(HH^T) \right)^{k/2} + (np)^{k/2-2} \sum_{i=1}^{m} \|b_i\|^8 \right\} \\
\leq C_2 (p^{k/2} + n^{k/2-2} p^{k/2-1}) \tag{B.23}
\]
and by the proof of (B.21),
\[
E \left| \sum_{i=1}^{m} \sum_{j=1}^{i-1} h_{ij} Y_i Z_j \right|^k \leq C_1 E \left( \sum_{i=1}^{m} \left| \sum_{j=1}^{m} h_{ij} Z_j \right|^2 \right)^{k/2} + C_2 \sum_{i=1}^{m} \|b_i\|^k \left( \sum_{j=1}^{m} \|b_j\|^2 \right)^{k/2} \\
+ C_3 \sum_{i=1}^{m} \|b_i\|^k E|Y_i|^k \sum_{j=1}^{m} \|b_j\|^k E|Z_j|^k \leq C_4 \left( E|Z^T HH^T Z - \text{tr}(HH^T)\right)^{k/2} + (\text{tr}(HH^T))^{k/2} \right) \\
+ C_5 (p^{k/2+1} + (np)^{k/2-2} p^2). \tag{B.24}
\]
By using (B.2) with \( k > 4 \), we have
\[
E|Z^T HH^T Z - \text{tr}(HH^T)|^{k/2} \leq C_3 (p^{k/4+1} + n^{k/4-1} p^{k/4}). \tag{B.25}
\]
From (B.23) to (B.25), for \( k > 4 \),
\[
E|Y^T HZ|^k \leq C_1 (p^{k/2+1} + n^{k/2-2} p^{k/2}). \tag{B.26}
\]
Combining (B.22) and (B.26), we obtain (B.4) \( k > 2 \). \( \square \)

**Lemma A.4 (Vershynin\cite[Theorem 5.44]{vershynin})** Let \( A \) be an \( N \times N \) matrix whose row \( A_i \) are independent random vectors in \( \mathbb{R}^n \) with \( \Sigma = EA_i^T A_i \), \( 1 \leq i \leq N \). Let \( m \) be a number such that \( \|A_i\| = \sqrt{A_i A_i^T} \leq \sqrt{m} \) almost surely for all \( i \). Then for some positive constant \( c > 0 \) and every \( t > 0 \), the following inequality holds with probability at least \( 1 - n \exp(-ct^2) \):
\[
\lambda_{\max}(\frac{1}{N}A^T A - \Sigma) = \|\frac{1}{N}A^T A - \Sigma\| \leq \max(\|\Sigma\|^{1/2}, \delta^2),
\]
where \( \delta = t\sqrt{\frac{m}{N}} \).

**Lemma A.5** Let \( X_n = (x_1, \ldots, x_n) = (X_{ij}) \) be an \( m \times n \) matrix whose entries are i.i.d. real random variables. Let \( \tilde{S}_n = \frac{1}{n}B X_n X_n^T B^T \), where \( B = \)
Consequently, it follows from Borel-Cantelli’s Lemma that

\[ P(\lambda_{\max}(\mathbf{S}_n) > 1 + \varepsilon) = o(n^{-k}) \quad \text{and} \quad P(\lambda_{\min}(\mathbf{S}_n) < 1 - \varepsilon) = o(n^{-k}). \] (B.27)

**Proof of Lemma A.5.** Let \( A = X_i^\top B^\top \) and take \( N = n, n = p \) in Lemma A.4. It is easy to check that \( A_i = x_i^\top B^\top \) and \( E A_i^\top A_i = B E(x_i x_i^\top)B^\top = BB^\top = I_p \). In addition, for any \( s > 2 \), by Markov inequality and (B.2), one has

\[
P\left( \max_{1 \leq i \leq s} |x_i^\top B^\top B x_i - \operatorname{tr}(B^\top B)| \geq \frac{n}{\log^3 n} \right) \leq \frac{\log^3 n E|x_i^\top B^\top B x_i - \operatorname{tr}(B^\top B)|^s}{n^{s-1}} \leq C_1 \frac{p^{s/2+1}}{n^{s/2}} \log^3 n + C_2 \frac{p^{s/2}}{n^{s/2}} \log^3 n. \] (B.28)

Since \( p/n = O(n^{-\eta}) \) for some \( 0 < \eta < 1 \), we take some large \( s \) in (B.28) and obtain that

\[
P\left( \max_{1 \leq i \leq n} |x_i^\top B^\top B x_i - \operatorname{tr}(B^\top B)| \geq \frac{n}{\log^3 n} \right) = O(n^{-2}).
\]

Consequently, it follows from Borel-Cantelli’s Lemma that

\[
\max_{1 \leq i \leq n} (x_i^\top B^\top B x_i) = \max_{1 \leq i \leq n} (A_i A_i^\top) \leq \operatorname{tr}(B^\top B) + \frac{n}{\log^3 n} = \operatorname{tr}(I_p) + \frac{n}{\log^3 n} \leq p + \frac{n}{\log^3 n} \leq 2 \frac{n}{\log^3 n}, \ a.s.
\]

In other words,

\[
\max_{1 \leq i \leq n} \|A_i\| = \sqrt{\max_{1 \leq i \leq n} A_i A_i^\top} \leq \sqrt{\frac{2n}{\log^3 n}}, \ a.s.
\]

For all \( t > 0 \), we apply Lemma A.4 with \( m = \frac{2n}{\log^3 n} \) and obtain that

\[
\lambda_{\max} (\mathbf{S}_n - I_p) = \|\mathbf{S}_n - I_p\| \leq t^2 \frac{2n}{n \log^3 n} + \|I_p\|^{1/2} \sqrt{\frac{2n}{n \log^3 n}} \leq t^2 \frac{2}{\log^3 n} + t \sqrt{\frac{2}{\log^3 n}}. \] (B.29)
with probability at least $1 - p \exp(-ct^2)$, where $c$ is some positive constant. Since $\lambda_{\max}(\hat{S}_n) = \|\hat{S}_n\| = \|S_n\| - \|I_p\| + \|I_p\|$, $\|I_p\| - \|\hat{S}_n - I_p\|$, we take $t = \log n$ in (B.29) and obtain that for any $k > 0$,

$$\|\hat{S}_n\| = \|I_p\|$$

with probability at least $1 - o(n^{-k})$. In addition, combining

$$|\lambda_{\min}(\hat{S}_n) - \lambda_{\min}(I_p)| \leq \|\hat{S}_n - I_p\|,$$

with (B.29), we also obtain that for any $k > 0$,

$$\lambda_{\min}(\hat{S}_n) = \lambda_{\min}(I_p)$$

with probability at least $1 - o(n^{-k})$. So (B.27) immediately follows from (B.30) and (B.31).

\section*{A.2. The limit of \eqref{eq:3.41}}

In order to show the limit of

$$\sum_{j=1}^{n} E_{j-1}(Y_j(z_1)Y_j(z_2))$$

$$= \frac{4}{p} \sum_{j=1}^{n} E_{j-1}[E_j(x_j^\top B^\top A_{n_j}^{-1}(z_1)B\bar{x}_j)E_j(x_j^\top B^\top A_{n_j}^{-1}(z_2)B\bar{x}_j)]$$

in probability for $z_1, z_2 \in \mathbb{C} \setminus \mathbb{R}$, we proceed by two steps.

\textbf{First step}, we aim to show that

$$\text{RHS of \eqref{eq:3.32}} = \frac{4}{p} \sum_{j=1}^{n} \frac{1}{n^2} \text{tr}[E_j(A_{n_j}^{-1}(z_2))E_j(A_{n_j}^{-1}(z_1))]$$

$$+ O_p\left(\frac{1}{\sqrt{p}}\right) + O_p\left(\frac{p^{1/2}}{n^{1/2}}\right).$$

Similarly to $A_{n_j}^{-1}(z)$ and $\bar{x}_j$, we define $\underline{A}_{n_j}^{-1}(z)$ and $\underline{x}_j$ by $x_1, x_2, \ldots, x_{j-1}, \underline{x}_{j+1}, \ldots, \underline{x}_n$, respectively, where $\{\underline{x}_{j+1}, \ldots, \underline{x}_n\}$ are i.i.d. copies of $x_{j+1}, \ldots, x_n$ and independent of $\{x_j, 1 \leq j \leq n\}$. Since $BB^\top = I_p$ and

$$E_{j-1}[E_j(x_j^\top B^\top A_{n_j}^{-1}(z_1)B\bar{x}_j)E_j(x_j^\top B^\top A_{n_j}^{-1}(z_2)B\bar{x}_j)]$$

$$= E_j(x_j^\top B^\top \underline{A}_{n_j}^{-1}(z_2))E_j(A_{n_j}^{-1}(z_1)B\bar{x}_j), 1 \leq j \leq n,$$

$$\text{RHS of \eqref{eq:3.32}} = \frac{4}{p} \sum_{j=1}^{n} E_j(x_j^\top B^\top A_{n_j}^{-1}(z_2))E_j(A_{n_j}^{-1}(z_1)B\bar{x}_j)$$

$$= \frac{4}{p} \sum_{j=1}^{n} E_j(x_j^\top B^\top A_{n_j}^{-1}(z_2)A_{n_j}^{-1}(z_1)B\bar{x}_j).$$

In addition, combining
In view of $x_j = \frac{1}{n} \sum_{i \neq j} x_i,$

$$E_j[x_j^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j]$$

$$= \frac{1}{n} \sum_{i \neq j} E_j[\beta_{ij}(z_2) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j]. \quad (B.35)$$

We will show that $\beta_{ij}(z_2)$ in the equality above can be replaced by $\beta_{ij}^t(z_2).$ First, we consider the case of $i > j$. Applying (3.26), we have

$$E|E_j[(\beta_{ij}(z_2) - \beta_{ij}^t(z_2)) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j]| = O(\sqrt{\frac{p}{n}}). \quad (B.36)$$

When $i < j$, we break $A_{nij}^{-1}(z_1)$ into the sum of $A_{nij}^{-1}(z_1)$ and $A_{nij}^{-1}(z_1) - A_{nij}^{-1}(z_1)$, respectively, where $A_{nij}(z_1) = A_{nij}(z_1) - n^{-1} Bx_i x_i^T B$ and $\tilde{x}_j = \tilde{x}_j - x_j / n$. Denote

$$\bar{\beta}_{ij}(z) = \frac{1}{1 + n^{-1} x_i^T B^T A_{nij}^{-1}(z_1) B x_i}.$$

Under the case of $i < j$, write

$$E_j[(\beta_{ij}(z_2) - \beta_{ij}^t(z_2)) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j] = c_{nk}, \quad k = 1, 2, 3, 4, \quad (B.37)$$

where

$$c_{n1} = E_j[(\beta_{ij}(z_2) - \beta_{ij}^t(z_2)) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j],$$

$$c_{n2} = \frac{1}{n} E_j[(\beta_{ij}(z_2) - \beta_{ij}^t(z_2)) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B x_i],$$

$$c_{n3} = -\frac{1}{n} E_j[(\beta_{ij}(z_2) - \beta_{ij}^t(z_2)) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B x_i x_i^T B A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j],$$

$$c_{n4} = -\frac{1}{n^2} E_j[(\beta_{ij}(z_2) - \beta_{ij}^t(z_2)) x_i^T B^T A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B x_i x_i^T B A_{nij}^{-1}(z_2) \Delta_{nij}^{-1}(z_1) B \tilde{x}_j].$$

In view of $BB^T = I_p,$

$$\beta_{ij}(z_2) - \beta_{ij}^t(z_2) = -\beta_{ij}(z_2) \beta_{ij}^t(z_2) (n^{-1} x_i^T B^T A_{nij}^{-1}(z_2) B x_i - n^{-1} tr(A_{nij}^{-1}(z_2))),$$

and (3.26),

$$E|c_{nk}| \leq C_1 \frac{\sqrt{p}}{n}, \quad k = 1, 2, 3, 4,$$

which yields

$$c_{nk} = O_P(\sqrt{\frac{p}{n}}), \quad k = 1, 2, 3, 4.$$

So, $\beta_{ij}(z_2)$ can be replaced by $\beta_{ij}^t(z_2)$ in probability.
where $d$ and $d^W$ we next show that the terms of $(B.36)$ and $(B.37)$, we obtain that

$$E_d[x^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)B\bar{x}_{ij}] = 0 \quad \text{when } i \neq j.$$ Combining

$$E_d[x^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)B\bar{x}_{ij}] + O_P(\sqrt{p}/n)$$

and

$$E_d[x^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)B\bar{x}_{ij}] + O_P(\sqrt{p}/n)$$

$$= d_{n1} + d_{n2} + d_{n3} + d_{n4} + O_P(\sqrt{p}/n), \quad (B.38)$$

where

$$d_{n1} = \frac{1}{n^2} \sum_{i < j} E_d[\beta_{ij}^T(2z)x_i^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)Bx_i\beta_{oij}(z_1)],$$

$$d_{n2} = \frac{1}{n} \sum_{i < j} E_d[\beta_{ij}^T(2z)x_i^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)Bx_i\beta_{oij}(z_1)],$$

$$d_{n3} = -\frac{1}{n^2} \sum_{i < j} E_d[\beta_{ij}^T(2z)x_i^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)Bx_i\beta_{oij}(z_1)],$$

and

$$d_{n4} = -\frac{1}{n^3} \sum_{i < j} E_d[\beta_{ij}^T(2z)x_i^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)Bx_i\beta_{oij}(z_1)].$$

We next show that the terms of $d_{n2}$, $d_{n3}$ and $d_{n4}$ are negligible. For $d_{n2}$,

$$d_{n2} = \frac{1}{n} \sum_{i < j} (E_d - E_{d-1})[\beta_{ij}^T(2z)x_i^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)B\bar{x}_{ij}].$$

By (3.20),

$$Ed^2_{n2} = \frac{1}{n^2} \sum_{i < j} E[(E_d - E_{d-1})[\beta_{ij}^T(2z)x_i^T B^T A_{nij}^{-1}(2z)A_{oij}^{-1}(z_1)B\bar{x}_{ij}]]^2$$

$$\leq C_1 \frac{p}{n^2}.$$ Thus,

$$d_{n2} = O_P(\sqrt{p}/n). \quad (B.39)$$
Now, we consider $d_{n3}$ and $d_{n4}$. By (3.20) and (3.26), we have

\[
d_{n3} = -\frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)C_{nij}(z_1, z_2)x_i^T B^T A^{-1}_{nij}(z_1)Bx_i B_{nij} \beta_j(z_1)] \\
= -\frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)[C_{nij}(z_1, z_2) - D_{nij}(z_1, z_2)]x_i^T B^T A^{-1}_{nij}(z_1)Bx_i B_{nij} \beta_j(z_1)] \\
- \frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)D_{nij}(z_1, z_2)x_i^T B^T A^{-1}_{nij}(z_1)Bx_i B_{nij} \beta_j(z_1)] \\
= O(p^{3/2} \sqrt{n} + O(\frac{p^{3/2}}{n^{3/2}}),
\]

where

\[
C_{nij}(z_1, z_2) = \frac{1}{n} x_i^T B^T A^{-1}_{nij}(z_2)A^{-1}_{nij}(z_1)Bx_i, \\
D_{nij}(z_1, z_2) = \frac{1}{n} \text{tr}(A^{-1}_{nij}(z_2)A^{-1}_{nij}(z_1)).
\]

By (3.15) and (3.25), we have

\[
d_{n4} = -\frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)\beta_j(z_1)F_{nij}(z_1, z_2)J_{nij}(z_1)] \\
= -\frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)\beta_j(z_1)[F_{nij}(z_1, z_2) - G_{nij}(z_1, z_2)]J_{nij}(z_1)] \\
- \frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)\beta_j(z_1)G_{nij}(z_1, z_2)[J_{nij}(z_1) - K_{nij}(z_1)] \\
- \frac{1}{n} \sum_{i<j} E_j[\beta^{ij}_0(z_2)\beta_j(z_1)G_{nij}(z_1, z_2)K_{nij}(z_1)] \\
= O(p^{3/2} \sqrt{n}) + O(p(\frac{p^{1/2}}{n}) + O(p^2 \frac{p^{2}}{n^2}) = O(p^{3/2} \sqrt{n} + O(p^2 \frac{p^{2}}{n^2})
\]

where

\[
F_{nij}(z_1, z_2) = \frac{1}{n} x_i^T B^T A^{-1}_{nij}(z_2)A^{-1}_{nij}(z_1)Bx_i, \\
G_{nij}(z_1, z_2) = \frac{1}{n} \text{tr}(A^{-1}_{nij}(z_2)A^{-1}_{nij}(z_1)), \\
J_{nij}(z_1) = \frac{1}{n} x_i^T B^T A^{-1}_{nij}(z_1)Bx_i, \\
K_{nij}(z_1) = \frac{1}{n} \text{tr}(A^{-1}_{nij}(z_1)).
\]

Therefore, we obtain that

\[
d_{n2} + d_{n3} + d_{n4} = O(p^{3/2} \sqrt{n} + O(p^2 \frac{p^{2}}{n^2})
\]

By the condition of $p/n = o(1)$ and (3.27)-(3.29), we have

\[
\beta^{ij}_0(z_2) - 1 = O(p \frac{p}{n}) \text{ and } \beta^{ij}_0(z_1) - 1 = O(p \frac{p}{n}).
\]
Moreover, using (3.15), $E|x_i^T B^T A_{ni12}^{-1}(z_2)A_{ni12}^{-1}(z_1)Bx_i| = O(p)$. Hence, we have

$$d_{n1} = \frac{1}{n^2} \sum_{i<j} E_j[x_i^T B^T A_{ni12}^{-1}(z_2)A_{ni12}^{-1}(z_1)Bx_i] + O_P\left(\frac{p^2}{n^2}\right). \quad (B.41)$$

Applying $BB^T = I_p$, (3.12), (3.15), (B.38), (B.40) and (B.41), we obtain that

$$E_{j-1}[E_j(x_j^T B^T A_{nj1j}^{-1}(z_1)Bx_j)] = \text{tr}[E_j(A^{-1}_{nj1j}(z_1)Bx_j)] = E_j(x_j^T B^T A_{nj1j}^{-1}(z_2)Bx_j) = E_j[x_j^T B^T A_{nj1j}^{-1}(z_2)A_{nj1j}^{-1}(z_1)Bx_j]

= \frac{1}{n^2} \sum_{i<j} E_j[x_i^T B^T A_{nj1j}^{-1}(z_2)A_{nj1j}^{-1}(z_1)Bx_i] + O_P\left(\frac{\sqrt{p}}{n}\right) + O_P\left(\frac{p^{3/2}}{n^{3/2}}\right)

= \frac{1}{n^2} \sum_{i<j} E_j[x_i^T B^T A_{nj1j}^{-1}(z_2)A_{nj1j}^{-1}(z_1)Bx_i] - \text{tr}(A_{nj1j}^{-1}(z_2)A_{nj1j}^{-1}(z_1))

+ \frac{1}{n^2} \sum_{i<j} E_j[\text{tr}(A^{-1}_{nj1j}(z_2)A_{nj1j}^{-1}(z_1))] + O_P\left(\frac{\sqrt{p}}{n}\right) + O_P\left(\frac{p^{3/2}}{n^{3/2}}\right)

= \frac{1}{n^2} \sum_{i<j} E_j[\text{tr}(A^{-1}_{nj1j}(z_2)A_{nj1j}^{-1}(z_1))] + O_P\left(\frac{\sqrt{p}}{n}\right) + O_P\left(\frac{p^{3/2}}{n^{3/2}}\right)

= \frac{1}{n^2} \sum_{i<j} E_j[\text{tr}(A^{-1}_{nj1j}(z_2)A_{nj1j}^{-1}(z_1))] + O_P\left(\frac{1}{n}\right) + O_P\left(\frac{\sqrt{p}}{n}\right) + O_P\left(\frac{p^{3/2}}{n^{3/2}}\right)

= \frac{j-1}{n^2} \text{tr}[E_j(A^{-1}_{nj1j}(z_2))E_j(A^{-1}_{nj1j}(z_1))] + O_P\left(\frac{\sqrt{p}}{n}\right) + O_P\left(\frac{p^{3/2}}{n^{3/2}}\right). \quad (B.42)$$

Consequently, (B.33) follows from (B.42) immediately.

**Second step**, we will prove that

$$\text{RHS of (B.33)} \xrightarrow{p} \frac{2}{(1 - z_1)(1 - z_2)}, \quad (B.43)$$

as $p/n \to 0$ and $(p, n) \to \infty$. It can be found that

$$A_{nj1j}(z_1) + z_1 I_p - \frac{n-1}{n} b_1(z_1) I_p = \frac{1}{n} \sum_{i \neq j} Bx_i x_i^T B - \frac{n-1}{n} b_1(z_1) I_p.$$

Multiplying $(z_1 I_p - \frac{n-1}{n} b_1(z_1) I_p)^{-1}$ from the left-hand side, $A_{nj1j}^{-1}(z_1)$ from the right-hand side above, and using the fact that

$$Bx_i x_i^T B A_{nj1j}^{-1}(z_1) = \beta_{ij}(z_1) Bx_i x_i^T B A_{nj1j}^{-1}(z_1),$$
we obtain that
\[ A_{nij}^{-1}(z_1) = -(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1} \]
\[ + \frac{1}{n} \sum_{i \neq j} \beta_{ij}(z_1)(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}Bx_ix_i^T B^T A_{nij}^{-1}(z_1) \]
\[ - b_1(z_1)\frac{n-1}{n}(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}A_{nij}^{-1}(z_1) \]
\[ = -(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1} + b_1(z_1)C(z_1) + D(z_1) + E(z_1), \quad (B.44) \]

where
\[ C(z_1) = \sum_{i \neq j}(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}(-\frac{1}{n}Bx_ix_i^TB^T - \frac{1}{n}I_p)A_{nij}^{-1}(z_1), \]
\[ D(z_1) = \frac{1}{n} \sum_{i \neq j}((\beta_{ij}(z_1) - b_1(z_1))(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}Bx_ix_i^TB^T A_{nij}^{-1}(z_1), \]
\[ E(z_1) = \frac{1}{n}b_1(z_1)(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1} \sum_{i \neq j}(A_{nij}^{-1}(z_1) - A_{nij}^{-1}(z_1)). \]

Let \( K_1, K_2, \ldots, \) be some positive constants, which are independent of \( n \) and may have different values from line to line. It can be seen that
\[ \| (z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1} \| \leq \frac{K_1}{\nu_0}. \quad (B.45) \]

In the following (B.46)-(B.48), let \( M \) be any \( p \times p \) nonrandom matrix with bounded spectral norm \( \| M \| \). By (3.14), (3.15), (B.45) and Hölder’s inequality, we have
\[ E|\text{tr}(D(z_1)M)| \leq \frac{1}{n} \sum_{i \neq j} (E(\beta_{ij}(z_1) - b_1(z_1))^2)^{1/2} \times (E(x_i^TB^TA_{ij}^{-1}(z_1)M(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}Bx_i)^2)^{1/2} \]
\[ \leq K_1\| M \|\| \frac{1}{n} + \frac{p}{n^2} \|^{1/2}[p + p^2]^{1/2} \leq K_2\| M \|, \quad (B.46) \]

By (3.14) and (B.45),
\[ |\text{tr}E(z_1)M| \leq \frac{K_1\| M \|}{\nu_0}. \quad (B.47) \]

Moreover, by (3.15), (B.45) and \( BB^T = I_p \),
\[ E|\text{tr}C(z_1)M| \]
\[ = E|\text{tr} \sum_{i \neq j}(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}(\frac{1}{n}Bx_ix_i^TB^T - \frac{1}{n}I_p)A_{nij}^{-1}(z_1)M| \]
\[ \leq K_1\| M \| \sum_{i \neq j}(E|x_i^TBx_i - \text{tr}(I_p)|^2)^{1/2} \leq \frac{K_2\| M \|\sqrt{p}}{\nu_0}. \quad (B.48) \]
By (3.13), write
\[ \text{tr}E_j(C(z_1)A_{n_j}^{-1}(z_2)) = C_1(z_1, z_2) + C_2(z_1, z_2) + C_3(z_1, z_2), \tag{B.49} \]
where
\[
C_1(z_1, z_2) = -\text{tr} \sum_{i<j} (z_1 I_p - \frac{n - 1}{n} b_1(z_1) I_p)^{-1} \frac{1}{n} Bx_i x_i^T B^T \times E_j[A_{n_{ij}}^{-1}(z_1) \beta_{ij}(z_2) A_{n_{ij}}^{-1}(z_2) - \frac{1}{n} Bx_i x_i^T B^T A_{n_{ij}}^{-1}(z_2)]
\]
\[
= -\frac{1}{n^2} \sum_{i<j} x_i^T B^T E_j[\beta_{ij}(z_2) A_{n_{ij}}^{-1}(z_1) A_{n_{ij}}^{-1}(z_2) Bx_i x_i^T B^T A_{n_{ij}}^{-1}(z_2)]
\]
\[
\times (z_1 I_p - \frac{n - 1}{n} b_1(z_1) I_p)^{-1} Bx_i,
\]
\[
C_2(z_1, z_2) = -\frac{1}{n} \text{tr} \sum_{i<j} (z_1 I_p - \frac{n - 1}{n} b_1(z_1) I_p)^{-1} E_j[A_{n_{ij}}^{-1}(z_1) (A_{n_{ij}}^{-1}(z_2) - A_{n_{ij}}^{-1}(z_2))],
\]
\[
C_3(z_1, z_2) = \text{tr} \sum_{i<j} (z_1 I_p - \frac{n - 1}{n} b_1(z_1) I_p)^{-1} \frac{1}{n} Bx_i x_i^T B^T - \frac{I_p}{n} E_j[A_{n_{ij}}^{-1}(z_1) A_{n_{ij}}^{-1}(z_2)].
\]

We can get by (3.12) and (B.45) that
\[ |C_2(z_1, z_2)| \leq \frac{K_1}{v_0^2}. \tag{B.50} \]

Following the proof of (B.48), one can obtain that
\[ E|C_3(z_1, z_2)| \leq \frac{K_2}{v_0^2} \sqrt{p}. \tag{B.51} \]
For $i < j$, it follows from (3.15) and (3.26) that

$$
E\left| \frac{1}{n^2} x_i^\top B^\top E_j [\beta_{ij}(z_2)A_{ij}^{-1}(z_2)A_{nij}^{-1}(z_2)Bx_i \right.
\times x_i^\top B^\top A_{nij}^{-1}(z_2)(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}]Bx_i
- b_1(z_1)\frac{1}{n^2}\text{tr}[E_j(A_{nij}^{-1}(z_1)A_{nij}^{-1}(z_2))]
\times \text{tr}[E_j(A_{nij}^{-1}(z_1))](z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}]\bigg| 
\leq E\left| \frac{1}{n^2} E_j \left\{ x_i^\top B^\top A_{nij}^{-1}(z_2)(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}Bx_i \right. 
\times x_i^\top B^\top A_{nij}^{-1}(z_2)(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}Bx_i 
- \text{tr}(A_{nij}^{-1}(z_2)(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}) \right\} \right| 
\leq K_1(\frac{\sqrt{p}}{n} + \frac{p^{1/2}}{n^2}).
$$

By (3.12), it is easy to see that

$$
\left| \text{tr}[E_j(A_{nij}^{-1}(z_1)A_{nij}^{-1}(z_2))\text{tr}[E_j(A_{nij}^{-1}(z_2))(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}] 
- \text{tr}[E_j(A_{nij}^{-1}(z_1)A_{nij}^{-1}(z_2))\text{tr}[E_j(A_{nij}^{-1}(z_2))(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}] \bigg| 
\leq K_1p.
$$

Thus, it follows that

$$
E\left[ C_1(z_1, z_2) + \frac{j-1}{n^2}b_1(z_1)\text{tr}[E_j(A_{nij}^{-1}(z_1)A_{nij}^{-1}(z_2)) \right. 
\times \text{tr}[E_j(A_{nij}^{-1}(z_2))(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}] 
\leq K_1(n\frac{\sqrt{p}}{n} + n\frac{p}{n^2} + \frac{p^{1/2}}{n^2}) = O(\sqrt{p}). \quad (B.52)
$$

Thus, from (B.44) to (B.52), we have

$$
\text{tr}\left[ E_j(A_{nij}^{-1}(z_1)A_{nij}^{-1}(z_2)) \left\{ 1 + \frac{j-1}{n^2}b_1(z_1)b_1(z_2) 
\times \text{tr}\left[ E_j(A_{nij}^{-1}(z_2))(z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1} \right] \right\} 
= -\text{tr}\left[ (z_1I_p - \frac{n-1}{n}b_1(z_1)I_p)^{-1}E_j(A_{nij}^{-1}(z_2)) \right] + C_4(z_1, z_2),
$$
Therefore, by (3.7), (3.26), (B.42) and (B.53),

\[
\frac{4}{p} \sum_{j=1}^{n} \frac{j-1}{n^2} \text{tr}[E_j(A_{n,j}^{-1}(z_1))E_j(A_{n,j}^{-1}(z_2))] = \frac{4}{p} \sum_{j=1}^{n} \frac{j-1}{n^2} \frac{\text{tr}[(z_2 I_p - \frac{n-1}{n} b_1(z_2) I_p)^{-1}(z_1 I_p - \frac{n-1}{n} b_1(z_1) I_p)^{-1}]}{1 - \frac{j-1}{n}} + O_P\left(\frac{n \log p}{p}\right)
\]

\[
= \frac{2}{(1-z_1)(1-z_2)} + o_P(1), \quad (B.54)
\]

as \(p/n \to 0\) and \((p,n) \to \infty\). Thus, the proof of (B.43) is completed.

**A.3. Tightness of \(\hat{M}^{(1)}(z)\)**

This subsection is to prove the tightness of \(\hat{M}^{(1)}(z)\) for \(z \in \mathcal{C}\), which is a truncated version of \(M_n(z)\) as in (3.3). Let \(v_0 > 0\) be arbitrary and denote \(\mathcal{C}_u = \{u + iv_0, \ u \in [u_l, u_r]\}\), where \(u_l = 1 - \delta\) and \(u_r = 1 + \delta\) and \(\delta \in (0,1)\). Let \(\mathcal{C}_l = \{u_l + iv : v \in [n^{-1} \rho_n, v_0]\}\) and \(\mathcal{C}_r = \{u_r + iv : v \in [n^{-1} \rho_n, v_0]\}\), where \(\rho_n \geq n^{-\vartheta}, \vartheta \in (0,1)\) and \(\rho_n \downarrow 0\). Then it has \(\mathcal{C}^+ = \mathcal{C}_l \cup \mathcal{C}_u \cup \mathcal{C}_r\).

For \(Y_j(z)\) defined in (3.40), by (3.9), we have

\[
E\left[\sum_{i=1}^{r} a_i \sum_{j=1}^{n} Y_j(z_i)\right]^2 = \sum_{j=1}^{n} E\left[\sum_{i=1}^{r} a_i Y_j(z_i)\right]^2 \leq K_1, \quad v_0 = \mathcal{I}(z_i),
\]
which ensures that the condition (i) of Theorem 12.3 in Billingsley [7] is satisfied (see Bai and Silverstein [5]). Therefore, to complete the proof of tightness, we have to show that

\[ E \left| \frac{M_n^{(1)}(z_1) - M_n^{(1)}(z_2)}{|z_1 - z_2|^2} \right|^2 \leq K_2 \quad \text{for all} \quad z_1, z_2 \in C_n^+ \cup C_n^- . \]  

(B.55)

Since \( C_n^+ \) and \( C_n^- \) are symmetric, we only prove the above inequality on \( C_n^+ \). Throughout this section, all bounds including \( O(\cdot) \) and \( o(\cdot) \) expressions hold uniformly on \( z \in C_n^+ \).

Note that when \( z \in C_u \), \( \|A_{nj}^{-1}(z)\| \) is bounded in \( n \). But this is not the case for \( z \in C_r \) or \( z \in C_l \). In general, for \( z \in C_n^+ \), we have

\[ \|A_{nj}^{-1}(z)\| \leq K_3 + v^{-1}I(\|\hat{S}_{nj}\| \geq h_r \text{ or } \lambda_{\min}(\hat{S}_{nj}) \leq h_l) , \]  

(B.56)

where \( \hat{S}_{nj} = S_n - n^{-1}Bx_jx_j^\top B^\top \), \( h_l \in (u_l, 1) \), \( h_r \in (1, u_r) \), \( u_l = 1 - \delta \) and \( u_r = 1 + \delta \) for some \( \delta \in (0, 1) \).

It follows from (B.27) in Lemma A.5 that for any positive number \( k > 0 \),

\[ P(\|\hat{S}_n\| \geq h_r) = o(n^{-k}) , \quad P(\lambda_{\min}(\hat{S}_n) \leq h_l) = o(n^{-k}) . \]  

(B.57)

Obviously, \( A_n^{-1}(z_1) - A_n^{-1}(z_2) = (z_2 - z_1)A_n^{-1}(z_1)A_n^{-1}(z_2) \). Then, by the martingale method in (3.30), write

\[ \frac{M_n^{(1)}(z_1) - M_n^{(1)}(z_2)}{z_1 - z_2} = -\frac{n}{\sqrt{p}} \sum_{j=1}^{n}(E_j - E_{j-1})[\hat{x}^\top B^\top A_n^{-1}(z_1)A_n^{-1}(z_2)B\hat{x} - \hat{x}_j^\top B^\top A_n^{-1}(z_1)A_n^{-1}(z_2)B\hat{x}_j] \]

\[ = -\frac{n}{\sqrt{p}} \sum_{j=1}^{n}(E_j - E_{j-1})(q_n1 + q_n2 + q_n3) , \]  

(B.58)

where

\[ q_n1 = (\hat{x} - \hat{x}_j)^\top B^\top A_n^{-1}(z_1)A_n^{-1}(z_2)B\hat{x} , \]

\[ q_n2 = \hat{x}_j^\top B^\top [A_n^{-1}(z_1)A_n^{-1}(z_2) - A_n^{-1}(z_1)A_n^{-1}(z_2)]B\hat{x} , \]

and

\[ q_n3 = \hat{x}_j^\top B^\top A_n^{-1}(z_1)A_{nj}^{-1}(z_2)B(\hat{x} - \hat{x}_j) . \]

For \( 1 \leq j \leq n \), denote

\[ Q_{nj} = \{||\hat{S}_{nj}|| < h_r \text{ and } \lambda_{\min}(\hat{S}_{nj}) > h_l\} , \quad Q_{\bar{n}j} = \{||\hat{S}_{nj}|| \geq h_r \text{ or } \lambda_{\min}(\hat{S}_{nj}) \leq h_l\} . \]  

(B.59)
Then, for \( z \in \mathbb{C}_n^+ \), by \( \rho_n \geq n^{-\vartheta} \), \( 0 < \vartheta < 1 \), (3.9), (3.24), (B.56), (B.57) and (B.59), we establish that

\[
E\left[ \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) q_{n3} \right]^2 \leq \frac{K_1}{p} \sum_{j=1}^{n} E[\tilde{x}_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) B x_j]^2
\]

\[
= \frac{K_1}{p} \sum_{j=1}^{n} E[\tilde{x}_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) B x_j]^2 I(Q_{n_j})
\]

\[
+ \frac{K_1}{p} \sum_{j=1}^{n} E[\tilde{x}_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) B x_j]^2 I(Q_{n_j}^c)
\]

\[
\leq \frac{K_1}{p} \sum_{j=1}^{n} E[\tilde{x}_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) B x_j]^2 I(Q_{n_j})
\]

\[
+ \frac{K_1}{p} \sum_{j=1}^{n} E[\tilde{x}_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) B x_j]^4 (P(Q_{n_j}^c))^{1/2}
\]

\[
\leq \frac{K_2 n_p p}{n} + K_3 \frac{p^{1/2}}{n^{1/2} \eta^2} (P(\|\tilde{S}_1\| \geq h_r \text{ or } \lambda_{\min}(\tilde{S}_1) \leq h_t))^{1/2}
\]

\[
\leq K_4 + K_5 \rho_n^{1/2} n^4 \rho_n^{-4} n^{-k/2} \leq K_4 + K_5 \frac{p^{1/2}}{n^{1/2}} n^{4+4\vartheta} n^{-k/2} \leq K_6 (B.60)
\]

for \( k \geq 8 + 8\vartheta \) and \( 0 < \vartheta < 1 \).

For \( q_{n2} \), write

\[
q_{n2} = \sum_{j=1}^{6} q_{n2}^{(j)},
\]

where

\[
q_{n2}^{(1)} = \frac{1}{n^2} \tilde{x}_j^\top B^\top \beta_j(z_1) \beta_j(z_2) \tilde{A}_{n_j}(z_1) \tilde{A}_{n_j}(z_2) B x_j,
\]

\[
q_{n2}^{(2)} = -\frac{1}{n^2} \tilde{x}_j^\top B^\top \beta_j(z_1) \tilde{A}_{n_j}(z_1) A_{n_j}^{-1}(z_2) B x_j,
\]

\[
q_{n2}^{(3)} = -\frac{1}{n^2} \tilde{x}_j^\top B^\top \beta_j(z_2) A_{n_j}^{-1}(z_1) \tilde{A}_{n_j}(z_2) B x_j,
\]

\[
q_{n2}^{(4)} = \frac{1}{n^2} \tilde{x}_j^\top B^\top \beta_j(z_1) \beta_j(z_2) \tilde{A}_{n_j}(z_1) \tilde{A}_{n_j}(z_2) B x_j,
\]

\[
q_{n2}^{(5)} = -\frac{1}{n^2} \tilde{x}_j^\top B^\top \beta_j(z_1) \tilde{A}_{n_j}(z_1) A_{n_j}^{-1}(z_2) B x_j,
\]

\[
q_{n2}^{(6)} = -\frac{1}{n^2} \tilde{x}_j^\top B^\top \beta_j(z_2) A_{n_j}^{-1}(z_1) \tilde{A}_{n_j}(z_2) B x_j.
\]
For \( z \in \mathcal{C}_u \), by \( \mathbf{B}\mathbf{B}^\top = \mathbf{I}_p \), (3.24), (3.26) and (B.59),

\[
\begin{align*}
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n2}^{(6)} \right|^2 \\
\leq \frac{K_1}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\beta_j(z_2) - \frac{1}{n} \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_1) \hat{\mathbf{A}}_{nj}(z_2) \mathbf{Bx}_j]^2 \\
= \frac{K_1}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\beta_j(z_2) - \frac{1}{n} \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_1) \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j]^2 \\
\leq \frac{K_2}{p} \sum_{j=1}^{n} E[\mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_1) \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j \frac{1}{n} \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j - \text{tr}(\mathbf{A}^{-1}_{nj}(z_2))]^2 \\
+ \frac{K_3}{p} \sum_{j=1}^{n} E[\mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_1) \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j \frac{1}{n} \text{tr}(\mathbf{A}^{-1}_{nj}(z_2))]^2 \\
\leq \frac{K_4}{p} \frac{p^{1/2}}{n} + \frac{K_5}{n} \sqrt{\frac{p}{n} \frac{p^2}{n^2}} = o(1).
\end{align*}
\]

Then, similarly to the proof of (B.60), for \( z \in \mathcal{C}_u^+ \), by \( \mathbf{B}\mathbf{B}^\top = \mathbf{I}_p \), (3.16), (B.56), (B.57), (B.59) and Hölder’s inequality, we have

\[
\begin{align*}
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n2}^{(6)} \right|^2 \\
\leq \frac{K_1}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\beta_j(z_2) \\
\times \frac{1}{n} \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_1) \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j]^2 I(Q_{nj})] \\
+ \frac{K_1}{p} \sum_{j=1}^{n} E[(E_j - E_{j-1})\beta_j(z_2) \\
\times \frac{1}{n} \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_1) \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j \mathbf{x}_j^\top \mathbf{B}^\top \mathbf{A}^{-1}_{nj}(z_2) \mathbf{Bx}_j]^2 I(Q_{nj}^*)]
\end{align*}
\]
\begin{align*}
&\leq K_1 + \frac{K_2}{n^2 p} \sum_{j=1}^{n} (E|\beta_j(z_2)|^8)^{1/4} (E(x_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j)^8)^{1/4} \\
&\times (E(x_j^\top B^\top A_{n_j}^{-1}(z_2) Bx_j)^8)^{1/4} (P(\|\tilde{S}_{n1}\| \geq h_r \text{ or } \lambda_{\min}(\tilde{S}_{n1}) \leq h_l))^{1/4} \\
&\leq K_2 + \frac{K_3}{pm^2} \sum_{j=1}^{n} \frac{1}{n^2 v^2} (E(x_j^\top B^\top Bx_j)^8)^{1/4} \frac{1}{v^4} (E(x_j^\top B^\top Bx_j)^8)^{1/4} \\
&\times \frac{1}{v^2} (E(x_j^\top B^\top Bx_j)^8)^{1/4} (P(\|\tilde{S}_{n1}\| \geq h_r \text{ or } \lambda_{\min}(\tilde{S}_{n1}) \leq h_l))^{1/4} \\
&\leq K_2 + \frac{K_4}{pn^2} \frac{1}{n^2 v^2} (p^5 + p^4n^3 + p^8)^{1/4} \frac{1}{v^4} \frac{1}{v^2} (p^5 + p^4n^3 + p^8)^{1/4} n^{-k/4} \\
&\leq K_2 + \frac{K_4}{pn} \frac{1}{n^2 v^2} (p^4n^3 + p^8)^{1/2} \frac{1}{n^{1/2}} n^{-k/4} \\
&\leq K_2 + K_5 n^{6+8\theta} p^{1/2} n^{-k/4} \leq K_6
\end{align*}

for $k \geq 4(7 + 8\theta)$ and $0 < \theta < 1$. Here one uses the fact that on the event $(\|S_n\| \geq h_r \text{ or } \lambda_{\min}(S_n) \leq h_l)$,

\[
|\beta_j(z)| = \left| \frac{1}{1 + n^{-1}x_j^\top B^\top A_{n_j}^{-1}(z) Bx_j} \right| \leq \left| 1 - \frac{n^{-1}x_j^\top B^\top A_{n_j}^{-1}(z) Bx_j}{1 + n^{-1}x_j^\top B^\top A_{n_j}^{-1}(z) Bx_j} \right| \\
= \left| 1 - n^{-1}x_j^\top B^\top A_{n_j}^{-1}(z) Bx_j \right| \leq 1 + n^{-1}v^{-1}x_j^\top B^\top Bx_j. \quad (B.61)
\]

Similarly, for $q_{n2}^{(1)}$ and $z \in C_{l}$, by $BB^\top = I_p$, (3.24) and (3.26),

\[
E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) q_{n2}^{(1)} \right|^2 \\
= E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n} x_j^\top B^\top \beta_j(z_1) \beta_j(z_2) \tilde{A}_{n_j}(z_1) \tilde{A}_{n_j}(z_2) Bx_j \right|^2 \\
= E \left| \frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n} x_j^\top B^\top \beta_j(z_1) \beta_j(z_2) A_{n_j}^{-1}(z_1) Bx_j \times x_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j \right|^2 \\
\leq \frac{K_1}{p} \sum_{j=1}^{n} E \left| x_j^\top B^\top A_{n_j}^{-1}(z_1) Bx_j \frac{1}{n} |x_j^\top B^\top A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j| \right|^2 \\
- \text{tr}(A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2)) |x_j^\top B^\top A_{n_j}^{-1}(z_2) Bx_j|^2 \\
+ \frac{K_2}{p} \sum_{j=1}^{n} (E(x_j^\top B^\top A_{n_j}^{-1}(z_1) Bx_j)^6)^{1/3} (E(\frac{1}{n} \text{tr}[A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2)])^6)^{1/3} \\
\times (E(x_j^\top B^\top A_{n_j}^{-1}(z_2) Bx_j)^6)^{1/3} \\
\leq \frac{K_3 n^{p/2}}{n} + \frac{K_4 n^{p^3}}{n^3} = O(1).
\]
For $q^{(2)}_{n^2}$, by (3.24),

$$E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) q^{(2)}_{n^2} \right|^2$$

$$= E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n} \tilde{x}_j^T B^T \beta_j(z_1) \tilde{A}_{nj}(z_1) A^{-1}_{nj}(z_2) B \tilde{x}_j \right|^2$$

$$= E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n} \tilde{x}_j^T B^T \beta_j(z_1) A^{-1}_{nj}(z_1) B x_j \tilde{x}_j^T B^T A^{-1}_{nj}(z_1) A^{-1}_{nj}(z_2) B \tilde{x}_j \right|^2$$

$$\leq \frac{K_1}{p} \sum_{j=1}^{n} E |\tilde{x}_j^T B^T A^{-1}_{nj}(z_1) B x_j \tilde{x}_j^T B^T A^{-1}_{nj}(z_1) A^{-1}_{nj}(z_2) B \tilde{x}_j |^2$$

$$\leq \frac{K_1}{p} \sum_{j=1}^{n} (E |\tilde{x}_j^T B^T A^{-1}_{nj}(z_1) B x_j |^4)^{1/2} (E |\tilde{x}_j^T B^T A^{-1}_{nj}(z_1) A^{-1}_{nj}(z_2) B \tilde{x}_j |^4)^{1/2}$$

$$\leq \frac{K_2n p}{p} n = O(1).$$

For $q^{(3)}_{n^2}$, by (3.24),

$$E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) q^{(3)}_{n^2} \right|^2$$

$$= E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n} \tilde{x}_j^T B^T \beta_j(z_2) \tilde{A}_{nj}(z_1) A^{-1}_{nj}(z_2) B x_j \tilde{x}_j^T B^T A^{-1}_{nj}(z_2) B \tilde{x}_j \right|^2$$

$$\leq \frac{K_1}{p} \sum_{j=1}^{n} E |\tilde{x}_j^T B^T A^{-1}_{nj}(z_1) A^{-1}_{nj}(z_2) B x_j \tilde{x}_j^T B^T A^{-1}_{nj}(z_2) B \tilde{x}_j |^2$$

$$\leq \frac{K_1}{p} \sum_{j=1}^{n} (E |\tilde{x}_j^T B^T A^{-1}_{nj}(z_1) A^{-1}_{nj}(z_2) B x_j |^4)^{1/2} (E |\tilde{x}_j^T B^T A^{-1}_{nj}(z_2) B \tilde{x}_j |^4)^{1/2}$$

$$\leq \frac{K_2n p}{p} n = O(1).$$
For $d_{n2}^{(4)}$, by $BB^T = I_p$, (3.16), (3.24), (3.26) and Hölder’s inequality,

$$E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n2}^{(4)} \right|^2 = E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2} x_j^T \beta_j(z_1) \beta_j(z_2) A_{n_j}^{-1}(z_1) Bx_j \times x_j^T B^T A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j \right|^2 \leq \frac{K_1}{p} \sum_{j=1}^{n} E|x_j^T B^T A_{n_j}^{-1}(z_1) Bx_j \frac{1}{n} x_j^T B^T A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j \times \frac{1}{n} (x_j^T B^T A_{n_j}^{-1}(z_2) Bx_j - \text{tr}(A_{n_j}^{-1}(z_2)))^2 \leq \frac{K_3 n p^{1/2}}{p} n + K_4 \frac{n p^{1/2}}{n^{1/2}} \left( \frac{p^3}{n^2} + \frac{p^6}{n^5} \right) \frac{p^6}{n^6} = O(1).$$

For $d_{n2}^{(5)}$, by $BB^T = I_p$, (3.24) and (3.26),

$$E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n2}^{(5)} \right|^2 = E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \frac{1}{n^2} x_j^T \beta_j(z_1) A_{n_j}^{-1}(z_1) Bx_j x_j^T B^T A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j \right|^2 \leq \frac{K_1}{p} \sum_{j=1}^{n} E|x_j^T B^T A_{n_j}^{-1}(z_1) Bx_j \frac{1}{n} (x_j^T B^T A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2) Bx_j - \text{tr}(A_{n_j}^{-1}(z_1) A_{n_j}^{-1}(z_2)))^2 \leq \frac{K_3 n p^{1/2}}{p} n + K_4 \frac{n p^{17/8}}{p n^{17/8}} = O(1).$$

Hence, we obtain that

$$E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n2}^{(j)} \right|^2 = O(1), \quad j = 1, \ldots, 6,$$

for $z \in C_n^+$. 

Now, we consider $q_{n1}$. Write
\[ q_{n1} = (x - x_j) \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j \]
\[ = \frac{1}{n^2} x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j + \frac{1}{n} x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j \]
\[ + \frac{1}{n} x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1)(A_{n,j}^{-1}(z_1) - A_{n,j}^{-1}(z_2)) \mathbf{B} x_j \]
\[ = q_{n1}^{(1)} + q_{n1}^{(2)} + q_{n1}^{(3)}, \]
where
\[ q_{n1}^{(1)} = \frac{1}{n^2} \beta_j(z_1) \beta_j(z_2) x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j, \]
\[ q_{n1}^{(2)} = \frac{1}{n} \beta_j(z_1) x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j \]
and
\[ q_{n1}^{(3)} = -\frac{1}{n^2} \beta_j(z_1) \beta_j(z_2) x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) \tilde{A}_{n,j}(z_2) \mathbf{B} x_j. \]

For $q_{n1}^{(1)}$ and $z \in C_u$, by (3.15),
\[ E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) q_{n1}^{(1)} \right|^2 \leq K_1 \frac{n}{n^2 p} \sum_{j=1}^{n} E \left| x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j \right|^2 \leq \frac{K_2 n}{n^2} \frac{p + p^2}{n} = O(1). \]

For $q_{n1}^{(2)}$, by (3.20),
\[ E \left| \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) q_{n1}^{(2)} \right|^2 \leq E \left| \frac{1}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1}) \beta_j(z_1) x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j \right|^2 \leq \frac{K_1 n}{n^2 p} \sum_{j=1}^{n} E \left| x_j^\top \mathbf{B}^\top \mathbf{A}_{n,j}^{-1}(z_1) A_{n,j}^{-1}(z_2) \mathbf{B} x_j \right|^2 \leq \frac{K_2 n}{n^2} \frac{p + p^2}{n} = O(1). \]
For \( q_{n1}^{(3)} \), by (3.16) and (3.24),

\[
E \left\{ \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n1}^{(3)} \right\}^2 = \sum_{j=1}^{n} \left( \frac{1}{\sqrt{p}} \sum_{i=1}^{n} E \left[ \beta_i(z_1)\beta_i(z_2)\beta_i(z_3)B_{i,j}^\top A_{n,-1}^{-1}(z_1)A_{n,-1}^{-1}(z_2)B_{i,j} x_i \right] \right)^2 \\
\leq \left( \frac{K_1}{n^2p} \sum_{j=1}^{n} (E|x_j^\top B^\top A_{n,-1}^{-1}(z_1)A_{n,-1}^{-1}(z_2)B x_j |^4)^{1/2} (E|x_j^\top B^\top A_{n,-1}^{-1}(z_2)B x_j |^4)^{1/2} \right) \\
\leq \left( \frac{K_2}{np} (p^3 + np^2 + p^4)^{1/2} \frac{p^3}{n^2} \right) = O(1).
\]

Consequently, similarly to the proof of \( q_{n2}^{(6)} \), for \( z \in C_n^+ \), we obtain that

\[
E \left\{ \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n1}^{(i)} \right\}^2 \leq K_3, \quad i = 1, 2, 3,
\]

which implies

\[
E \left\{ \frac{n}{\sqrt{p}} \sum_{j=1}^{n} (E_j - E_{j-1})q_{n1} \right\}^2 = O(1).
\]

So the proof of (B.55) is completed.

**A.4. Convergence of \( M_n^{(2)}(z) \)**

In this subsection, we will prove the convergence of \( M_n^{(2)}(z) \) defined by (3.10). In view of \( \beta_1(z) = b_1(z) - \beta_1(z)b_1(z)\xi_1(z) \) and \( BB^\top = I_p \), it can be seen that

\[
\frac{n}{\sqrt{p}} E(x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i) = \frac{1}{\sqrt{p}} \sum_{i=1}^{n} E[\beta_i(z)x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i] \\
= \frac{1}{\sqrt{p}} \sum_{i=1}^{n} E[\beta_i(z)x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i] + \frac{1}{n\sqrt{p}} \sum_{i=1}^{n} E[\beta_i(z)x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i] \\
= \frac{1}{n\sqrt{p}} \sum_{i=1}^{n} \sum_{i=1}^{n} b_1(z)E[x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i] - \frac{1}{n\sqrt{p}} \sum_{i=1}^{n} \sum_{i=1}^{n} b_1(z)E[\beta_i(z)\xi_i(z)x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i] \\
- \frac{1}{n\sqrt{p}} \sum_{i=1}^{n} \sum_{i=1}^{n} b_1(z)E[\beta_i(z)\xi_i(z)x_i^\top B^\top A_{n,-1}^{-1}(z)B x_i] \\
= \frac{b_1(z)}{\sqrt{p}} E(\text{tr}(A_{n,-1}^{-1}(z))) + b_1(z)t_{n1}(z) + b_1(z)t_{n2}(z),
\]
where
\[ t_{n1}(z) = - \frac{n}{\sqrt{p}} E(\beta_1(z) \xi_1(z) x_1^T B^T A_{n1}^{-1}(z) B x_1), \]
\[ t_{n2}(z) = - \frac{1}{\sqrt{p}} E(\beta_1(z) \xi_1(z) x_1^T B^T A_{n1}^{-1}(z) B x_1). \]

Write
\[ t_{n1}(z) = t_{n1}^{(1)}(z) + t_{n2}^{(2)}(z), \quad t_{n2}(z) = t_{n2}^{(1)}(z) + t_{n2}^{(2)}(z), \]
where
\[ t_{n1}^{(1)}(z) = - \frac{n}{\sqrt{p}} b_1(z) E(\beta_1(z) \xi_1^2(z) x_1^T B^T A_{n1}^{-1}(z) B x_1), \]
\[ t_{n2}^{(1)}(z) = - \frac{1}{\sqrt{p}} b_1(z) E(\beta_1(z) \xi_1^2(z) x_1^T B^T A_{n1}^{-1}(z) B x_1), \]
\[ t_{n1}^{(2)}(z) = \frac{n}{\sqrt{p}} b_1(z) E(\beta_1(z) \xi_1^2(z) x_1^T B^T A_{n1}^{-1}(z) B x_1), \]
\[ t_{n2}^{(2)}(z) = \frac{1}{\sqrt{p}} b_1(z) E(\beta_1(z) \xi_1^2(z) x_1^T B^T A_{n1}^{-1}(z) B x_1). \]

Note that \(|b_1(z)| \leq K_1\) for \(z \in C_n\). Then, by (3.18), (3.20) and Hölder’s inequality, for \(z \in C_n\), we have
\[ |t_{n1}^{(2)}(z)| \leq \frac{K_2 n^\frac{p^3}{n^3} + \frac{p^2}{n} + \frac{1}{n^2}}{n^{1/2}} = O\left(\frac{p}{n} + O(n^{-1/2})\right). \]

And by (3.18), (3.24), (B.56), (B.57), (B.59), (B.61) and Hölder’s inequality, for \(z \in C_n\),
\[ |t_{n1}^{(2)}(z)| \leq K_3 \frac{p}{n} + K_2 \frac{n}{\sqrt{p}} (E(\beta_1^2))^\frac{1}{2} (E(\xi_1^4))^\frac{1}{4} (E(x_1^T B A_{n1}^{-1}(z) B x_1))^\frac{1}{4} \times (P(\|S_{n1}\| \geq h_r) \lambda_{\min}(S_{n1}) \|S_{n1}\| \geq h_r) \leq h_1) ^\frac{1}{4} \]
\[ \leq K_3 \frac{p}{n} + K_2 \frac{n}{\sqrt{p}} \frac{1}{2}(p^3 + p^2 + p^2 + p^1)^\frac{1}{4} \frac{1}{n^2} (\frac{p^5}{n^5} + \frac{p^4}{n^3} + \frac{1}{n^2})^\frac{1}{4} (\frac{p}{n})^\frac{1}{4} n^{-\frac{1}{4}} \]
\[ \leq K_3 \frac{p}{n} + K_3 n^\frac{p}{2 + 5\theta} (p^\frac{p}{2} + p^\frac{p}{2} + p^\frac{p}{2}) n^{-\frac{1}{2}} = o(1) \] (B.62)

providing \(k \geq 20 + 20\vartheta\) and \(0 < \vartheta < 1\).
For \( z \in \mathcal{C}_u \), it follows from (3.16), (3.17) and Hölder’s inequality that
\[
|t_{n_2}^{(2)}(z)| \leq \frac{1}{\sqrt{p}} |b_1(z)| |E(\beta_1(z)\zeta_1^2(z)x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1)|
\]
\[
\leq \frac{K_1}{\sqrt{p}} (p^3 + p^2 + \frac{1}{n^2})^{1/2} (p + p^2) = O\left(\frac{p^{3/2}}{n^{3/2}}\right) + O\left(\frac{p^{1/2}}{n}\right).
\]

And similarly to the proof of (B.62), \( |t_{n_2}^{(2)}(z)| = o(1) \) for all \( z \in \mathcal{C}_n \). Note that
\[
t_{n_1}^{(1)}(z) = -\frac{n}{\sqrt{p}} b_1(z) E\left\{\frac{1}{n} x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1 - \text{tr}(A_{n_1}^{-1}(z)) x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1\right\}
\]
\[
-\frac{n}{\sqrt{p}} b_1(z) E[\text{tr}(A_{n_1}^{-1}(z)) - E\text{tr}(A_{n_1}^{-1}(z))] E[x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1]
\]
\[
= -\frac{n}{\sqrt{p}} b_1(z) E\left\{\frac{1}{n} x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1 - \text{tr}(A_{n_1}^{-1}(z))\right\}^2
\]
\[
-\frac{n}{\sqrt{p}} b_1(z) E[\text{tr}(A_{n_1}^{-1}(z)) - E\text{tr}(A_{n_1}^{-1}(z))]^2.
\]

Then, by (3.14) and (3.20), for \( z \in \mathcal{C}_u \),
\[
|t_{n_1}^{(1)}(z)| \leq \frac{n}{\sqrt{p}} |b_1(z)| \sqrt{\frac{1}{n} E(x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1)^2} \sqrt{E(x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1)^2}
\]
\[
= O\left(\frac{\sqrt{p}}{\sqrt{n}}\right).
\]

For \( t_{n_2}^{(1)}(z) \), it is easy to see that
\[
t_{n_2}^{(1)}(z)
\]
\[
= -\frac{1}{\sqrt{p}} b_1(z) E\left\{\frac{1}{n} x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1 - \text{tr}(A_{n_1}^{-1}(z)) + \text{tr}(A_{n_1}^{-1}(z)) - E\text{tr}(A_{n_1}^{-1}(z))\right\}
\]
\[
\times [x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1 - \text{tr}(A_{n_1}^{-1}(z)) + \text{tr}(A_{n_1}^{-1}(z))]
\]
\[
= -\frac{1}{\sqrt{p}} b_1(z) \frac{1}{n} E[|x_1^T B^{-1} A_{n_1}^{-1}(z) B x_1 - \text{tr}(A_{n_1}^{-1}(z))|^2
\]
\[
-\frac{1}{\sqrt{p}} b_1(z) \frac{1}{n} E[\text{tr}(A_{n_1}^{-1}(z)) - E\text{tr}(A_{n_1}^{-1}(z))]^2.
\]

and hence, by (3.14) and (3.15),
\[
|t_{n_2}^{(1)}(z)| = O\left(\frac{1}{\sqrt{p}}\right)
\]

for \( z \in \mathcal{C}_u \). So similarly to the proof of (B.62), \( |t_{n_2}^{(1)}(z)| = o(1) \) for all \( z \in \mathcal{C}_n \). Thus, we have \( |t_{n_1}^{(1)}(z)| = o(1) \) and \( |t_{n_2}^{(1)}(z)| = o(1) \) for all \( z \in \mathcal{C}_n \). Consequently, for all \( z \in \mathcal{C}_n \),
\[
\frac{n}{\sqrt{p}} E(x^T B^{-1} A_{n}^{-1}(z) B x) = \frac{b_1(z)}{\sqrt{p}} E\text{tr}(A_{n_1}^{-1}(z)) + o(1).
\]
Let \( m(z) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dH(x) \) and \( H(x) = I(1 \leq x) \) for \( x \in \mathbb{R} \). We will show that for all \( z \in \mathcal{C}_n \),
\[
\sqrt{p} \left( \frac{\text{Etr}(A_{n1}^{-1}(z))}{p} - m(z) \right) = \sqrt{p} \left( \frac{\text{Etr}(A_{n1}^{-1}(z))}{p} - \frac{1}{1-z} \right) \to 0.
\]
Similarly to the proof of (B.44),
\[
A_{n1}^{-1}(z) = -(zI_p - \frac{n-1}{n} I_p)^{-1} \\
+ \frac{1}{n} \sum_{i \neq 1} \beta_i(z)(zI_p - \frac{n-1}{n} I_p)^{-1} B x_i x_i^\top B^\top A_{n1}^{-1}(z) \\
- \frac{1}{n} (zI_p - \frac{n-1}{n} I_p)^{-1} A_{n1}^{-1}(z)
\]
\[
= -(zI_p - \frac{n-1}{n} I_p)^{-1} + C(z) + D(z) + E(z),
\]
where
\[
C(z) = \sum_{i \neq 1} (zI_p - \frac{n-1}{n} I_p)^{-1} \left( \frac{1}{n} B x_i x_i^\top B^\top - \frac{1}{n} I_p \right) A_{n1}^{-1}(z),
\]
\[
D(z) = \frac{1}{n} \sum_{i \neq 1} (\beta_i(z) - b_1(z))(zI_p - \frac{n-1}{n} I_p)^{-1} B x_i x_i^\top B^\top A_{n1}^{-1}(z),
\]
\[
E(z) = \frac{1}{n} b_1(z)(zI_p - \frac{n-1}{n} I_p)^{-1} \sum_{i \neq 1} (A_{n1}^{-1}(z) - A_{n1}^{-1}(z)).
\]
On the one hand,
\[
\text{Etr}(C(z)) = 0.
\]
On the other hand, from the proofs of (B.46) and (B.47),
\[
\frac{E|\text{tr}(D(z))|}{\sqrt{p}} = O(\sqrt{p/n}) = o(1), \quad \frac{E|\text{tr}(E(z))|}{\sqrt{p}} = O\left( \frac{1}{\sqrt{p}} \right) = o(1).
\]
Thus,
\[
\sqrt{p} \left( \frac{\text{Etr}(A_{n1}^{-1}(z))}{p} - m(z) \right) = \sqrt{p} \left( \frac{\text{Etr}(A_{n1}^{-1}(z))}{p} - \frac{1}{1-z} \right) \\
\quad = \sqrt{p} (-\text{tr}(zI_p - \frac{n-1}{n} I_p)^{-1}) - \frac{1}{1-z} + o(1) \\
\quad = \frac{\sqrt{p}}{n} \frac{1}{(\frac{n-1}{n} - z)(1-z)} + o(1) = o(1).
\]
Combining \( b_1(z) = 1 + O(p/n) \), we obtain that
\[
\frac{n}{\sqrt{p}} E (\bar{x}^\top B^\top A_{n1}^{-1}(z) B \bar{x}) = \frac{b_1(z)}{\sqrt{p}} \text{Etr}(A_{n1}^{-1}(z)) + o(1) \\
= \frac{n}{\sqrt{p}} p \text{Etr}(A_{n1}^{-1}(z)) + o(1) = \frac{n}{\sqrt{p}} m(z) + o(1).
\]
Consequently, for all \( z \in \mathcal{C}_n \),
\[
\sup_{z \in \mathcal{C}_n} M_n^{(2)}(z) = \sup_{z \in \mathcal{C}_n} \frac{n}{\sqrt{p}} \left( E(\bar{x}^T B A_n^{-1}(z) B \bar{x}) - c_n m(z) \right) \to 0 \tag{B.63}
\]
as \((p,n) \to \infty\).

A.5. Some results on truncated random variables

Let
\[
X_n = (x_1, \ldots, x_n), \quad B = \Sigma_p^{-1/2} U^T = (b_{ij})_{p \times m} = (b_1, \ldots, b_m),
\]
\[
b_j = (b_{1j}, \ldots, b_{pj})^T, \quad \|b_j\| = \sqrt{\sum_{i=1}^p b_{ij}^2}, 1 \leq j \leq m.
\]

In view of \(BB^T = I_p\), it is easy to check that
\[
\sum_{j=1}^m \|b_j\|^2 = \sum_{j=1}^m \sum_{i=1}^p b_{ij}^2 = \|B\|_F^2 = \text{tr}(B^T B) = \text{tr}(BB^T) \leq p, \tag{B.64}
\]
and
\[
\|b_j\|^2 = b_j^T b_j = \|b_j b_j^T\| \leq \| \sum_{j=1}^m b_j b_j^T \| = \|I_p\| = 1, \quad 1 \leq j \leq m. \tag{B.65}
\]

Since \(EX_{11}^4 < \infty\), it has
\[
\lim_{(p,n) \to \infty} \sup_{1 \leq i \leq m} E[|X_{11}|^4 I(|X_{11}| > (np)^{1/4}/\|b_i\|)] = 0, \tag{B.66}
\]
as \((p,n) \to \infty\). In fact, by (B.65) and \(EX_{11}^4 < \infty\),
\[
0 \leq \lim_{(p,n) \to \infty} \sup_{1 \leq i \leq m} E[|X_{11}|^4 I(|X_{11}| > (np)^{1/4}/\|b_i\|)] \\
\leq \lim_{(p,n) \to \infty} E[|X_{11}|^4 I(|X_{11}| > (np)^{1/4})] = 0.
\]

Denote
\[
\hat{X}_{ij} = X_{ij} I(|X_{ij}| \leq \frac{(pn)^{1/4}}{\|b_i\|}) - EX_{ij} I(|X_{ij}| \leq \frac{(pn)^{1/4}}{\|b_i\|}), \quad \hat{X}_n = (\hat{X}_{ij}).
\]
Then, by (B.64), (B.65) and (B.66)

\[
P(X_n \neq \hat{X}_n) \leq P \left( \bigcup_{i,j} \{ |X_{ij}| > (np)^{1/4}/\|b_i\| \} \right)
\]

\[
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} P \left( |X_{ij}| > (np)^{1/4}/\sqrt{\|b_i\|} \right)
\]

\[
\leq \frac{1}{np} \sum_{i=1}^{m} \sum_{j=1}^{n} \|b_i\|^4 E(|X_{ij}|^4 I(|X_{ij}| > (np)^{1/4}/\|b_i\|))
\]

\[
\leq \frac{1}{np} \sum_{i=1}^{m} \sum_{j=1}^{n} \|b_i\|^2 \sup_{1 \leq i \leq m} E[|X_{11}|^4 I(|X_{11}| > (np)^{1/4}/\|b_i\|)]
\]

\[
\leq \sup_{1 \leq i \leq m} E[|X_{11}|^4 I(|X_{11}| > (np)^{1/4}/\|b_i\|)] \to 0 \quad \text{(B.67)}
\]

as \((p, n) \to \infty\). Denote

\[
\tilde{X}_{ij} = X_{ij} I \left( |X_{ij}| > \left( \frac{pm}{\|b_i\|} \right)^{1/4} \right) - EX_{ij} I \left( |X_{ij}| > \left( \frac{pm}{\|b_i\|} \right)^{1/4} \right),
\]

\[
\tilde{X}_n = X_n - \hat{X}_n = (\tilde{X}_{ij}).
\]

Let

\[
\sigma_n = \sqrt{E|\tilde{X}_{11}|^2}, \quad \tilde{S}_n = \frac{1}{n\sigma_n^2} B\tilde{X}_n \tilde{X}_n^\top B^\top, \quad \tilde{A}_n^{-1}(z) = (\tilde{S}_n - zI_p)^{-1}.
\]

In addition, denote

\[
\bar{x} = \frac{1}{n} \sum_{j=1}^{n} \tilde{x}_j,
\]

where \(\tilde{x}_j\) is the \(j\)th column of the matrix \(\frac{1}{n} \tilde{X}_n\).

**Lemma A.6** Assume that \(\{X_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}\) is an i.i.d. random array sequence with \(EX_{11} = 0, EX_{11}^2 = 1\) and \(EX_{11}^4 < \infty\). Let \(p/n = O(n^{-\eta})\) for some \(\eta \in (0, 1)\). Then for \(z \in \mathbb{C}_n^+\), we have that

\[
\frac{n}{\sqrt{p}} (\bar{x}^\top B A_n^{-1}(z) B\bar{x} - \bar{x}^\top B A_n^{-1}(z) B\bar{x}) \overset{P}{\to} 0. \quad \text{(B.68)}
\]

Moreover,

\[
\frac{n}{\sqrt{p}} (\bar{x}^\top B A_n^{-1}(z) B\bar{x} - \bar{x}^\top B B\bar{x}) \overset{P}{\to} 0. \quad \text{(B.69)}
\]
Proof of Lemma A.6. It can be seen that
\[
\frac{n}{\sqrt{p}} (\bar{x}^\top B A_n^{-1}(z) B \bar{x} - \tilde{x}^\top B A_n^{-1}(z) B \tilde{x})
\]
\[
= \frac{n}{\sqrt{p}} (\bar{x}^\top B A_n^{-1}(z) B \bar{x} - \tilde{x}^\top B A_n^{-1}(z) B \bar{x})
\]
\[
+ \frac{n}{\sqrt{p}} (\tilde{x}^\top B A_n^{-1}(z) B \bar{x} - \tilde{x}^\top B A_n^{-1}(z) B \tilde{x})
\]
\[
+ \frac{n}{\sqrt{p}} (\tilde{x}^\top B A_n^{-1}(z) B \tilde{x} - \tilde{x}^\top B A_n^{-1}(z) B \tilde{x})
\]
\[
:= u_{n1} + u_{n2} + u_{n3}.
\]
where
\[
u_{n1} = \frac{n}{\sqrt{p}} [(\bar{x} - \tilde{x})^\top B A_n^{-1}(z) B \bar{x}],
\]
\[
u_{n2} = \frac{n}{\sqrt{p}} [\tilde{x}^\top B (A_n^{-1}(z) - A_n^{-1}(z)) B \bar{x}]
\]
and
\[
u_{n3} = \frac{n}{\sqrt{p}} [\tilde{x}^\top B A_n^{-1}(z) B (\bar{x} - \tilde{x})].
\]
First, we consider \(u_{n1}\) on \(C_u\). Obviously,
\[
|u_{n1}| \leq \frac{n}{\sqrt{p}} |(\bar{x} - \tilde{x})^\top B A_n^{-1}(z) B \bar{x}|
\]
\[
\leq \frac{n}{\sqrt{p} \sigma} ||(\bar{x} - \tilde{x})^\top B A_n^{-1}(z) B \bar{x}||
\]
\[
\leq \frac{n}{\sqrt{p} \sigma} |(\bar{x} - \tilde{x})^\top B A_n^{-1}(z) B \bar{x}||+\frac{n}{\sqrt{p} \sigma} \sigma_b ||B \bar{x}||,\]
by the fact that \(\bar{x} - \tilde{x} = (1 - \frac{1}{\sigma}) \bar{x} + \frac{1}{\sigma} \tilde{x}\) with \(\tilde{x} = \frac{1}{n} \sum_{j=1}^n \bar{x}_j\) and \(\bar{x}_j\) being the \(j\)th column of \(\bar{X}_n\). It follows from \(EX_{11} = 0, EX_{11}^2 = 1\), (B.64), (B.66) and \(m \geq p\) that
\[
1 - \sigma_n^2 = \frac{1}{m} \sum_{i=1}^m (EX_{11}^2 - \sigma_n^2)
\]
\[
\leq \frac{1}{m} \sum_{i=1}^m E[|X_{11}|^2 I(|X_{11}| > (pn)^{1/4}/|b_i||)]
\]
\[
\leq \frac{2}{m(pn)^{1/2}} \sum_{i=1}^m ||b_i||^2 E[|X_{11}|^4 I(|X_{11}| > (pn)^{1/4}/|b_i||)]
\]
\[
\leq \frac{2}{m(pn)^{1/2}} \sup_{1 \leq i \leq m} E[|X_{11}|^4 I(|X_{11}| > (pn)^{1/4}/|b_i||)]
\]
\[
= O\left(\frac{1}{(np)^{1/2}}\right),
\]
which yields
\[
\frac{n}{\sqrt{p}} |1 - \sigma_n^2| = \frac{n}{\sqrt{p} \sigma_n (1 + \sigma_n)} = O\left(\frac{\sqrt{m}}{p}\right).
\]
(B.70)
In addition, by $E X_{ij} = 0$, Cov($\tilde{x}_1$) = $E X_{11}^2 I_m$, (B.64), (B.65) and (B.66),

$$E\|B\tilde{x}\|^2 = E\tilde{x}^\top B^\top B \tilde{x} = \frac{1}{n^2} \left[ \sum_{j=1}^{n} E\tilde{x}_j^\top B^\top B \tilde{x}_j + 2 \sum_{1 \leq i < j \leq n} E\tilde{x}_i^\top B^\top B \tilde{x}_j \right]$$

$$= \frac{1}{n^2} \sum_{j=1}^{n} E\tilde{x}_j^\top B^\top B \tilde{x}_j = \frac{1}{n} E\tilde{x}_1^\top B^\top B \tilde{x}_1$$

$$= \frac{1}{n} \text{tr}(B^\top B E\tilde{x}_1 \tilde{x}_1^\top) = \frac{1}{n} \sum_{i=1}^{m} \|b_i\|^2 E\tilde{x}_i^2$$

$$\leq \frac{2}{n \sqrt{mp}} \sum_{i=1}^{m} \|b_i\|^4 E X_{11}^4 I(|X_{11}| > (np)^{1/4} / \|b_i\|)$$

$$\leq \frac{2}{n \sqrt{np}} \sum_{i=1}^{m} \|b_i\|^2 E X_{11}^4 I(|X_{11}| > (np)^{1/4} / \|b_i\|)$$

$$= O(p^{1/2} / n^{3/2}),$$

which yields

$$\|B\tilde{x}\| = O_P(p^{1/4} / n^{3/4}). \quad \text{(B.71)}$$

In addition, by (3.23), one has $\|Bx\|^2 = O_P(\frac{p}{n})$ and $\|B\tilde{x}\| = O_P(\sqrt{n})$. So, by $p/n = o(1)$,

$$|u_n| = O(\sqrt{n}) O_P(\frac{p}{n}) + n \sqrt{p} O_P(p^{1/4}/n^{3/4}) O_P(\sqrt{n} / p) = O_P(n^{-1/2}) + O_P(p^{1/4} / n^{1/4}) = o_P(1)$$

uniformly on $C_n$.

Second, we analyze $u_{n2}$. In view of $E X_{ij} = 0$, $X_n - \hat{X}_n = \tilde{X}_n$,

$$X_n - \frac{1}{\sigma_n} \tilde{X}_n = X_n - \frac{1}{\sigma_n} (X_n - \tilde{X}_n) = (1 - \frac{1}{\sigma_n}) X_n + \frac{1}{\sigma_n} \tilde{X}_n,$$

and

$$A_n(z) - \hat{A}_n(z) = \tilde{S}_n - \hat{S}_n = \frac{1}{n} B X_n X_n^\top B^\top - \frac{1}{n} \frac{1}{\sigma_n} B \tilde{X}_n \tilde{X}_n^\top B^\top$$

$$= \frac{1}{n} B (X_n - \frac{1}{\sigma_n} \tilde{X}_n) X_n^\top B^\top + \frac{1}{n} \frac{1}{\sigma_n} B \tilde{X}_n (X_n - \tilde{X}_n)^\top B^\top$$

$$= \frac{1}{n} (1 - \frac{1}{\sigma_n}) B X_n X_n^\top B^\top + \frac{1}{n} \frac{1}{\sigma_n} B \tilde{X}_n X_n^\top B^\top + \frac{1}{n} \frac{1}{\sigma_n} B \tilde{X}_n \tilde{X}_n^\top B^\top.$$
we have that
\[ |u_{n2}| \leq \frac{n}{\sqrt{n}} \| B^\top \| \| A_n^{-1}(z) - \hat{A}_n^{-1}(z) \| \| Bx \| \]
\[ \leq \frac{n}{v_0 \sqrt{n}} \| B^\top \| \| A_n(z) - \hat{A}_n(z) \| \| Bx \| \]
\[ \leq \frac{\| \tilde{b}^\top B \| \| Bx \|}{v_0 \sqrt{n}} \left| 1 - \frac{1}{\sigma_n} \| BX_n \| \| X_n^\top B^\top \| \right| \]
\[ + \frac{\| \tilde{b}^\top B \| \| Bx \|}{v_0 \sqrt{n}} \frac{1}{\sigma_n} \| BX_n \| \| X_n^\top B^\top \| \]
\[ + \frac{\| \tilde{b}^\top B \| \| Bx \|}{v_0 \sqrt{n}} \frac{1}{\sigma_n} \| BX_n \| \| \tilde{X}_n^\top B^\top \|. \]

It follows from (3.23) that \( \| Bx \| = O_P(\sqrt{p}) \) and \( \| \tilde{B}x \| = O_P(\sqrt{p}). \) Meanwhile, by Lemma A.5 with \( p/n = O(n^{-\eta}) \) for some \( \eta \in (0, 1), \) we establish that \( \| Bx \| \overset{P}{\rightarrow} \sqrt{\Sigma_p} \) and \( \| \tilde{B}x \| \overset{P}{\rightarrow} \sqrt{\Sigma_p}. \) Similarly, \( \| B\tilde{X}_n \| / \sqrt{nEX_{11}^2 \overset{P}{\rightarrow} \sqrt{\Sigma_p}}. \) In addition, by (B.64), (B.66) and \( m \geq p, \)

\[ nEX_{11}^2 \leq \frac{2n}{m} \sum_{i=1}^{m} EX_{11}^2 I(|X_{11}| > \sqrt{np/\|b_i\|}) \]
\[ \leq \frac{2n}{m(np)^{1/2}} \sum_{i=1}^{m} \| b_i \|^2 EX_{11}^2 I(|X_{11}| > \sqrt{np/\|b_i\|}) \]
\[ \leq \frac{2p}{m(np)^{1/2}} \sup_{1 \leq i \leq m} EX_{11}^4 I(|X_{11}| > \sqrt{np/\|b_i\|}) \]
\[ = O(n^{1/2} p^{1/2}) = O(\sqrt{n/p}), \quad \text{(B.72)} \]

which yields \( \| B\tilde{X}_n \| = O_P(\sqrt{n/p}). \) Consequently, by \( p/n = o(1), \)

\[ |u_{n2}| = O_P(\frac{1}{\sqrt{p}}) O_P(\frac{p}{n}) \left( O_P(\frac{1}{\sqrt{np}}) O_P(n) + O_P(\sqrt{\frac{n}{p}}) O_P(\sqrt{n}) \right) \]
\[ = O_P(\frac{1}{n^{1/2}}) + O_P(\frac{p^{1/4}}{n^{1/4}}) = o_P(1). \]

Thus, \( u_{n2} \) converges in probability to zero uniformly on \( C_u. \) Similarly to \( u_{n1}, \)
\( u_{n3} \) also converges in probability to zero uniformly on \( C_u. \)

Moreover, for \( z \in C_l \) or \( z \in C_r, \) we apply Lemma A.5 and obtain that
\[ \lim_{n \to \infty} \min(u_r - \lambda_{\max}(\tilde{S}_n), \lambda_{\min}(\tilde{S}_n) - u_l) > 0, \quad \text{in probability,} \]
\[ \lim_{n \to \infty} \min(u_r - \lambda_{\max}(\tilde{S}_n), \lambda_{\min}(\tilde{S}_n) - u_l) > 0, \quad \text{in probability.} \]

Therefore, the above argument for \( u_{nj}, \ j = 1, 2, 3 \) for \( z \in C_u \) also holds for \( z \in C_l \) and \( C_r. \) Thus, (B.68) is completely proved. In addition, the above argument for (B.68) also works for (B.69). \( \square \)
A.6. The proofs of (3.15)-(3.26)

As a consequence of Lemma A.6, for $1 \leq i \leq m$ and $1 \leq j \leq n$, we assume that $EX_{ij} = 0$, $EX_{ij}^2 = 1$ and

$$|X_{ij}| \leq \frac{(np)^{1/4}}{\|b_i\|}, \quad \lim_{(p,n) \to \infty} \sup_{1 \leq i \leq m} E[|X_{11}|^4 I(|X_{11}| > \frac{(np)^{1/4}}{\|b_i\|})] = o(1). \quad (B.73)$$

We use the notion $A^{-1}_{ij}(z)$, $A^{-1}_{nij}(z)$, $D_{nj}(z)$, $\beta_j(z)$, $\beta_j^T(z)$, $b_1(z)$, $\gamma_j(z)$, $\xi_j(z)$, $\alpha_j(z)$, $\beta_{ij}(z)$, $\beta_{ij}^T(z)$, $b_{12}(z)$, $\gamma_{ij}(z)$, $\xi_{ij}(z)$ defined in Section 5. Let $\|A\|$ denote the spectral normal of matrix $A$. For $k \geq 2$ and $x_1 = (X_{11}, \ldots, X_{m1})^T$, by $BB^T = I_p$, $\|C\| = O(1)$, (B.64), (B.65), (B.73) and Lemma A.3, it has

$$E|\gamma_1(z)|^k = E|\frac{1}{n} x_1^T B^T CBx_1 - \frac{1}{n} \text{tr}(C)|^k = O(\frac{p^{k/2}}{nk}), \quad 1 < k \leq 2,$$

$$E|\gamma_1(z)|^k = O(\frac{p^{k/2+1}}{nk}) + O(\frac{p^{k/2}}{nk^{k/2+1}}), \quad k > 2$$

i.e. (3.15) and (3.16) hold.

By $C_\tau$’s inequality, (3.14) and (3.15),

$$E|\xi_1(z)|^2 = E|\frac{1}{n} x_1^T B^T A^{-1}_{n1}(z)Bx_1 - \frac{1}{n} E\text{tr}(A^{-1}_{n1}(z))|^2 = O(n^{-1}),$$

$$E|\xi_1(z)|^k = O(\frac{p^{k/2+1}}{nk^2}) + O(\frac{p^{k/2}}{nk^{k/2+1}}) + O(n^{-k/2}) \quad k > 2,$$

i.e. (3.17) and (3.18) hold.

Obviously, by $BB^T = I_p$, $\|C\| = O(1)$, $\|D\| = O(1)$, the proof of (B.1) and (B.2) in Lemma A.3,

$$n^{-2} E|x_1^T B^T C e_j^T D Bx_1|^2 \leq \frac{C_1}{n^2} E|x_1^T B^T C e_j^T D Bx_1 - \text{tr}(B^T C e_j^T D B)|^2 + E|e_j^T D B B^T C e_i|^2$$

$$= O(\frac{1}{n^2}), \quad (B.74)$$

and

$$n^{-k} E|x_1^T B^T C e_j^T D Bx_1|^k \leq \frac{C_1}{n^k} E|x_1^T B^T C e_j^T D Bx_1 - \text{tr}(B^T C e_j^T D B)|^k + E|e_j^T D B B^T C e_i|^k$$

$$= O(\frac{(np)^{k/2-1}}{n^k}) = O(\frac{p^{k/2-1}}{nk^{k/2+1}}), \quad k > 2. \quad (B.75)$$
So \((3.19)\) follows from \((B.74)\) and \((B.75)\). Then, by \(\|C\| = O(1)\),
\[
E|\mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_1|^2 \\
= E|\text{tr}(C \mathbf{B} \mathbf{x}_1 \mathbf{x}_1^\top \mathbf{B}^\top)|^2 = E(\mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_1)^2 \\
= \frac{1}{n^2} E\left( \sum_{j=2}^n \mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_j \right)^2 \\
= \frac{1}{n^2} \left[ \sum_{j=2}^n E \mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_j \mathbf{x}_j^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_1 + 2 \sum_{2 \leq j < k \leq n} E \mathbf{x}_j^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_j \mathbf{x}_k^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_1 \right] \\
= \frac{1}{n^2} \sum_{j=2}^n E \text{tr}(\mathbf{x}_j \mathbf{x}_j^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_j \mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B}) \\
+ \frac{2}{n^2} \sum_{2 \leq j < k \leq n} E \text{tr}(\mathbf{x}_j \mathbf{x}_k^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_j \mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B}) \\
= \frac{1}{n^2} \sum_{j=2}^n \text{tr}(E(\mathbf{x}_j \mathbf{x}_j^\top \mathbf{B}^\top C \mathbf{B})E(\mathbf{x}_1 \mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B})) \\
+ \frac{2}{n^2} \sum_{2 \leq j < k \leq n} \text{tr}(E \mathbf{x}_j \mathbf{x}_k^\top \mathbf{B}^\top C \mathbf{B} \mathbf{E}(\mathbf{x}_1 \mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B})) \\
= \frac{1}{n^2} \sum_{j=2}^n \text{tr}(\mathbf{B}^\top \mathbf{C} \mathbf{B} \mathbf{B} \mathbf{B}^\top \mathbf{C} \mathbf{B}) = \frac{1}{n^2} \sum_{j=2}^n \text{tr}(\mathbf{B} \mathbf{B}^\top \mathbf{C} \mathbf{B} \mathbf{B}^\top \mathbf{C}^\top) \\
= \frac{C_1}{n^2} \sum_{j=2}^n \text{tr}(\mathbf{C} \mathbf{C}^\top) = O\left(\frac{p}{n}\right),
\]

i.e. \((3.20)\) holds. Now we consider \((3.21)\). For \(\mathbf{x}_1 = (X_{11}, \ldots, X_{m_1})^\top\), \(\mathbf{x}_2 = (X_{12}, \ldots, X_{m_2})^\top\), by \((B.65)\), \((B.64)\), \((B.73)\), \(\|C\| = O(1)\) and \((B.4)\) in Lemma A.3, it is easy to obtain that
\[
E|\mathbf{x}_1^\top \mathbf{B}^\top C \mathbf{B} \mathbf{x}_2|^k = O(p^{k/2+1} + n^{k/2-2}p^{k/2}) = O((np)^{k/2-1}), \quad k \geq 4
\]
with \(p \leq n\). We turn to prove \((3.22)\). For \(k \geq 4\), it can be checked that
\[
E|\mathbf{x}_1^\top \mathbf{B}^\top \mathbf{B} \mathbf{x}_1|^k \leq \frac{C_1}{n^{2k}} \left[ E\left| \sum_{i=2}^n \mathbf{x}_i^\top \mathbf{B}^\top \mathbf{B} \mathbf{x}_i \right|^k + E\left| \sum_{i \neq j, |i-j| > 1} \mathbf{x}_i^\top \mathbf{B}^\top \mathbf{B} \mathbf{x}_j \right|^k \right] \\
= O\left(\frac{p^k}{n^{2k}} + \frac{p^{k/2}}{n^{k/2} p} \right) = O\left(\frac{p^{k/2}}{n^{k/2}}\right).
\]

In fact, by Lemma A.1, \(C_r\)'s inequality, \(\mathbf{B} \mathbf{B}^\top = \mathbf{I}_p\), \((3.15)\) and \((3.16)\), we have
that
\[ E \left| \sum_{i=2}^{n} x_i^\top B^\top B x_i \right|^k \]
\[ \leq C_1 E \left| \sum_{i=2}^{n} (x_i^\top B^\top B x_i - E(x_i^\top B^\top B x_i)) \right|^k + C_2 \left| \sum_{i=2}^{n} E(x_i^\top B^\top B x_i) \right|^k \]
\[ \leq C_3 \left( \sum_{i=2}^{n} E(x_i^\top B^\top B x_i - E(x_i^\top B^\top B x_i))^2 \right)^{k/2} + C_4 \sum_{i=2}^{n} E|x_i^\top B^\top B x_i - E(x_i^\top B^\top B x_i)|^k + C_5 (np)^k \]
\[ \leq C_6 \left( \sum_{i=2}^{n} E|x_i^\top B^\top B x_i|^2 \right)^{k/2} + K_7 \sum_{i=2}^{n} E|x_i^\top B^\top B x_i|^k + K_5 (np)^k \]
\[ \leq C_6 \left( \sum_{i=2}^{n} [E(x_i^\top B^\top B x_i - \text{tr} I_p)^2 + (\text{tr} I_p)^2] \right)^{k/2} \]
\[ + C_7 \sum_{i=2}^{n} E|x_i^\top B^\top B x_i - \text{tr} I_p|^k + |\text{tr} I_p|^k \right) + K_5 (np)^k \]
\[ \leq C_6 [(n(p + p^2))^{k/2} + n(p^{k/2+1} + p^{k/2}n^{k/2-1} + p^k) + (np)^k] \]
\[ = O((np)^k). \]

We introduce the random variables \( \xi_j = \sum_{i=2}^{j-1} x_i^\top B^\top B x_j \) for \( j = 3, \ldots, n \). Then it is straightforward to check that
\[ E[\xi_j|\mathcal{G}_{j-1}] = 0, \]
where \( \mathcal{G}_j = \sigma(x_2, \ldots, x_j) \), \( j = 3, \ldots, n \). This means that \( \{\xi_j, \mathcal{G}_j, j \geq 3\} \) are martingale differences. Thus, we have by Lemma A.1, (3.16) and (3.21) that, for \( k \geq 4, \)
\[ E \left| \sum_{2 \leq i \neq j \leq n} x_i^\top B^\top B x_j \right|^k = E \left| 2 \sum_{j=3}^{n} \xi_j \right|^k \]
\[ \leq C_1 \left\{ E \left[ \sum_{j=3}^{n} E(\xi_j^2|\mathcal{G}_{j-1}) \right]^{k/2} + \sum_{j=3}^{n} E|\xi_j|^k \right\} \]
\[ \leq C_2 \left\{ E \left[ \sum_{j=3}^{n} \sum_{i=2}^{j-1} x_i^\top B^\top B x_j \right]^{k/2} + \sum_{j=3}^{n} E \left| \sum_{i=2}^{j-1} x_i^\top B^\top B x_j \right|^k \right\} \]
\[ \leq C_3 n^k E|x_2^\top B^\top B x_2|^k + C_4 n^{k+1} E|x_2^\top B^\top B x_3|^k \]
\[ \leq C_5 n^k [E|x_2^\top B^\top B x_2 - \text{tr} I_p|^k + |\text{tr} I_p|^k] + C_4 n^{k+1} E|x_2^\top B^\top B x_3|^k \]
\[ \leq C_6 n^k [p^{k/2+1} + p^{k/2}n^{k/2-1} + p^k] + C_7 n^{k+1} (np)^{k/2-1} \]
\[ = O(n^{3k/2}p^{k/2-1}). \]
So the proof of (B.76) is completed. On the other hand, for \( k \geq 4 \), it follows from (B.76) that
\[
E|\bar{x}_1^T B^T C B \bar{x}_1|^k = E||\bar{x}_1^T B^T C B \bar{x}_1||^k \leq E(||\bar{x}_1^T B^T||||C||||B \bar{x}_1||)^k \leq ||C||^k E|\bar{x}_1^T B^T B \bar{x}_1|^k = O\left(\frac{p^{k/2}}{n^{k/2}}\right).
\] (B.77)

Thus, (3.22) follows from (B.76) and (B.77). Similarly to (B.76), for \( k = 2 \), it can be seen that
\[
E|\bar{x}_1^T B^T B \bar{x}_1|^2 \leq \frac{C_1}{n^2} \left[ E\left| \sum_{i=2}^{n} x_i^T B^T B x_i \right|^2 + E\left| \sum_{i \neq j, j > 1} x_j^T B^T B x_i \right|^2 \right] = O\left(\frac{p^2}{n^2}\right).
\] (B.78)

In fact, by Lemma A.1, \( C_r \) inequality and (3.15),
\[
E\left| \sum_{i=2}^{n} x_i^T B^T B x_i \right|^2 \leq C_1 E \left| \sum_{i=2}^{n} (x_i^T B^T B x_i - E(x_i^T B^T B x_i))^2 \right| + C_2 \left| \sum_{i=2}^{n} E(x_i^T B^T B x_i) \right|^2 \leq C_3 \sum_{i=2}^{n} E(x_i^T B^T B x_i)^2 + C_4 (np)^2 \leq C_5 \sum_{i=2}^{n} \left[ E(x_i^T B^T B x_i - trI_p)^2 + |trI_p|^2 \right] + C_4 (np)^2 \leq C_6 n(p + p^2) + C_4 (np)^2 = O((np)^2).
\]

For \( j = 3, \ldots, n \), let \( \xi_j = \sum_{i=2}^{j-1} x_i^T B^T B x_j \). Then, we apply Lemma A.1 with
Thus, (B.78) holds. In addition, similarly to (B.77),

\[
E|x_i^\top B^\top CBx_i|^2 = E\|x_i^\top B^\top CBx_i\|^2 \leq E(\|x_i^\top B^\top\|C\|Bx_i\|)^2 \\
\leq \|C\|^2 E\|x_i^\top B^\top CBx_i\|^2 = O\left(\frac{p^2}{n^2}\right). \quad \text{(B.79)}
\]

Then, by (B.78) and (B.79), (3.23) is proved. Combining Lemma A.1 with (3.21), we establish that

\[
E|x_i^\top B^\top CBx_i|^k = E\|x_i^\top B^\top CBx_i\|^k \leq E(\|x_i^\top B^\top\|C\|Bx_i\|)^k \\
\leq \|C\|^k E\|x_i^\top B^\top CBx_i\|^k = \|C\|^k \frac{1}{n^k} E\left(\sum_{i=2}^{n} x_i^\top B^\top Bx_i\right)^k \\
\leq \frac{C_1}{n^k} \left(\sum_{i=2}^{n} E|x_i^\top B^\top Bx_i|^2\right)^{k/2} + \frac{1}{n^k} E\left(\sum_{i=2}^{n} x_i^\top B^\top Bx_i\right)^k \\
\leq \frac{C_2}{n^k} \left((n(np)^{1/2})^{k/2} + n(np)^{k/2-1}\right) \\
= O\left(\frac{p^{k/4}}{n^{k/4}}\right), \quad k \geq 4, \quad \text{(B.80)}
\]
which implies (3.24). By (B.78) and the proof of (B.1) in Lemma A.3 with $k = 2,$

\[
E|\alpha_1(z)|^2
= \frac{1}{n^2} E|x_1^T B A_{n1}^{-1}(z) B x_1 x_1^T B A_{n1}^{-1}(z) B x_1 - \text{tr}(B^T A_{n1}^{-1}(z) B x_1 x_1^T B A_{n1}^{-1}(z) B)|^2 
\leq \frac{C_1}{n^2} [EX_1^4 \|A_{n1}^{-1}(z)\|^2] E|x_1^T B^T B x_1|^2 = O\left(\frac{p^2}{n^2}\right),
\]

which implies (3.25).

Lastly, we consider (3.26). As for (3.26), if $m = 0$ and $r = 0$, then (3.26) directly follows from (3.15), (3.16) and Hölder’s inequality. If $m \geq 1$ and $r = 0$, then by induction on $m$, we have

\[
E\left|\prod_{i=1}^{m} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right|
\leq E\left|\prod_{i=1}^{m-1} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right|
+ C_1 \frac{p}{n} E\left|\prod_{i=1}^{m-1} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right|
\leq O\left(\sqrt{\frac{p}{n} \frac{1}{n^{1/2}}}\right).
\]

Repeating the argument, we have

\[
E\left|\prod_{i=1}^{m} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right|^2 = O\left(\frac{p}{n^2}\right)
\]

for $m = 0$ by (3.15), (3.16) and $m \geq 1$ by induction. Then, for $m \geq 1$ and $1 \leq r \leq 2$, we have by (3.24) that

\[
E\left|\prod_{i=1}^{m} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right| (x_1^T B^T H B x_1)^r
\leq \left(E\left|\prod_{i=1}^{m} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right|^2\right)^{1/2} \left(E(x_1^T B^T H B x_1)^{2r}\right)^{1/2}
\leq \left(E\left|\prod_{i=1}^{m} \frac{1}{n} x_i^T B^T C_i B x_1 \prod_{j=1}^{q} \frac{1}{n} [x_1^T B^T D_j B x_1 - \text{tr}(D_j)]\right|^2\right)^{1/2} \left(E(x_1^T B^T H B x_1)^{4r}\right)^{1/4}
= O\left(\frac{p^{1/2}}{n}\right).
\]

If $m = 0$ and $1 \leq r \leq 2$, (3.26) can be obtained similarly.
A.7. Proof of Lemma 1

Proof of Lemma 1. Since $BB^T = I_p$, we have

$$Ex^\top B^\top Bx = \text{tr}(x^\top B^\top Bx) = \frac{\text{tr}(BB^\top)}{n} = \frac{p}{n} = c_n.$$ 

Then

$$\frac{n}{\sqrt{p}}(x^\top B^\top Bx - c_n) = \frac{n}{\sqrt{p}}(x^\top B^\top Bx - Ex^\top B^\top Bx)$$

$$= \frac{n}{\sqrt{p}} \sum_{j=1}^n (E_j - E_{j-1})(x^\top_j B^\top Bx - x^\top_j B^\top Bx_j)$$

$$= \frac{n}{\sqrt{p}} \sum_{j=1}^n (E_j - E_{j-1})(2 \frac{x^\top_j B^\top Bx_j}{n} + \frac{x^\top_j B^\top Bx_j}{n^2}). \quad (B.81)$$

By Lemma A.1 in Appendix A.1, \((3.15)\) and $BB^\top = I_p$, we have

$$E\frac{n}{\sqrt{p}} \sum_{j=1}^n (E_j - E_{j-1})(\frac{x^\top_j B^\top Bx_j}{n^2})^2$$

$$= \frac{1}{n^2 p} E\sum_{j=1}^n (x^\top_j B^\top Bx_j - Ex^\top_j B^\top Bx_j)^2$$

$$= \frac{1}{n^2 p} E\sum_{j=1}^n (x^\top_j B^\top Bx_j - \text{tr}(B^\top B))^2 = \frac{1}{n^2 p} E\sum_{j=1}^n (x^\top_j B^\top Bx_j - \text{tr}(I_p))^2$$

$$\leq \frac{C_1}{n^2 p} \sum_{j=1}^n E|x^\top_j B^\top Bx_j - \text{tr}(I_p)|^2 \leq \frac{C_2}{n^2 p} np = O\left(\frac{1}{n}\right), \quad (B.82)$$

where $C_1, C_2$ are some positive constants. Next, we verify the condition \((i)\) of Lemma A.2 in Appendix A.1. Obviously, it follows that

$$E_{j-1}[E_j(x^\top_j B^\top Bx_j)^2] = E_{j-1}[E_j(x^\top_j B^\top Bx_j x^\top_j B^\top Bx_j)] = E_j(x^\top_j B^\top) E_j(Bx_j)$$

$$= \frac{1}{n^2} \sum_{k_1 < j, k_2 < j} x^\top_{k_1} B^\top Bx_{k_2}.$$ 

Note that for the above terms corresponding to the case of $k_1 = k_2$, we have

$$E\frac{1}{n^2} \sum_{k_1 < j} [x^\top_{k_1} B^\top Bx_{k_1} - E_{x^\top_{k_1} B^\top Bx_{k_1}}]^2$$

$$= \frac{1}{n^4} \sum_{k_1 < j} E|x^\top_{k_1} B^\top Bx_{k_1} - E(x^\top_{k_1} B^\top Bx_{k_1})|^2 = O\left(\frac{p}{n^3}\right).$$
On the other hand, when \( k_1 \neq k_2 \), by (3.21),

\[
E \left[ \frac{1}{n^2} \sum_{k_1 \neq k_2} x_{k_1} \top B x_{k_2} \right]^2 = \frac{1}{n^4} \sum_{k_1 \neq k_2, k_1 \neq k_2} E[ x_{k_1} \top B x_{k_2} x_{k_1} \top B x_{k_2} ] \\
\leq \frac{C_1}{n^4} \sum_{k_1 \neq k_2} E( x_{k_1} \top B x_{k_2} )^2 = O \left( \frac{p^{1/2}}{n^{3/2}} \right).
\]

Consequently,

\[
\frac{4}{p} \sum_{j=1}^n E_j [ E_j ( x_j \top B x_j ) ]^2 \\
= \frac{4}{p} \sum_{j=1}^n \sum_{k_1=1}^{j-1} E x_{k_1} \top B x_{k_1} \frac{1}{n^2} + \frac{n}{p} ( O_p \left( \frac{p}{n} \right) + O_p \left( \frac{p^2}{n^2} \right) ) \\
= \frac{4}{p} \sum_{j=1}^n \frac{(j-1) \text{tr}(BB \top)}{n^2} + o_p(1) = 4 \sum_{j=1}^n \frac{(j-1)}{n^2} + o_p(1) P \to 2, \tag{B.83}
\]

since \( BB \top = I_p \). Applying Lemma A.2 in Appendix A.1, one can establish that

\[
\frac{n}{\sqrt{p}} (x \top B x - c_n) \xrightarrow{d} N(0, 2), \tag{B.84}
\]

which implies

\[
\|Bx\|^2 - c_n = O_p \left( \frac{\sqrt{p}}{n} \right). \tag{B.85}
\]

Denote

\[
X_n(z) = \frac{n}{\sqrt{p}} \left[ c_n \frac{x \top \left( \tilde{S}_n / p - z \right)^{-1} B x}{\|Bx\|^2} - c_n m(z) \right], \\
Y_n = \frac{n}{\sqrt{p}} \left( g(x \top B x) - g(c_n) \right),
\]

where \( m(x) = \int \frac{1}{2} dH(x) \) and \( H(x) = I(1 \leq x) \) for \( x \in \mathbb{R} \). By (B.63) and \( c_n = o(1), \) \( E x \top B A_n^{-1}(z) B x \) can be estimated by \( c_n m(z) \), which implies \( 1 - x \top B A_n^{-1}(z) B x = 1 + o_p(1). \) Then, by (3.5), (3.8), (3.9), (3.10), (B.81), (B.82), (B.85), \( c_n \to 0 \) and delta method, we obtain that for any constants \( a_1 \) and \( a_2, \)

\[
a_1 X_n(z) + a_2 Y_n = a_1 \frac{n}{\sqrt{p}} \left( \frac{x \top B A_n^{-1}(z) B x}{1 - x \top B A_n^{-1}(z) B x} - c_n m(z) \right) + a_2 \frac{n}{\sqrt{p}} \left( g(x \top B B x) - g(c_n) \right) + o_p(1) \\
= a_1 \frac{n}{\sqrt{p}} \left( \frac{x \top B A_n^{-1}(z) B x - c_n m(z)}{1 - x \top B A_n^{-1}(z) B x} \right) + a_2 \frac{g'(0) n}{\sqrt{p}} \left( x \top B B x - c_n \right) + o_p(1) \\
:= \sum_{j=1}^n l_j(z) + o_p(1),
\]
where
\[
l_j(z) = 2a_1 \frac{1}{\sqrt{p}} E_j(x_j^T B^T A_{n_j}^{-1}(z)B x_j) + \frac{2a_2 g'(0)}{\sqrt{p}} E_j(\tilde{x}_j^T B^T B x_j).
\]

Next, we apply Lemma A.2 in Appendix A.1 to complete the proof of Lemma 1. Obviously, for all \( \varepsilon > 0 \), we obtain by (3.24) that
\[
\sum_{j=1}^n E[|l_j(z)|^2 I(|l_j(z)| > \varepsilon)] \\
\leq 2 \sum_{j=1}^n \frac{4a_1^2}{p} E[|x_j^T B^T A_{n_j}^{-1}(z)B x_j|^2 I(|\frac{2a_1}{\sqrt{p}} x_j^T B^T A_{n_j}^{-1}(z)B x_j| > \frac{\varepsilon}{2})] \\
+ 2 \sum_{j=1}^n \frac{4a_2^2 (g'(0))^2}{p} E[|\tilde{x}_j^T B^T B x_j|^2 I(|\frac{2a_2 g'(0)}{\sqrt{p}} \tilde{x}_j^T B^T B x_j| > \frac{\varepsilon}{2})] \\
\leq \frac{C_1}{p^2} \sum_{j=1}^n (E|x_j^T B^T A_{n_j}^{-1}(z)B x_j|^4 + E|\tilde{x}_j^T B^T B x_j|^4) = O(\frac{1}{p}).
\]

So the condition (ii) of Lemma A.2 is satisfied. Next, we aim to verify its condition (i). Suppose that we have
\[
\frac{4}{p} \sum_{j=1}^n E_{j-1}[E_j(x_j^T B^T A_{n_j}^{-1}(z)B x_j)] E_j(\tilde{x}_j^T B^T B x_j) \xrightarrow{p} \frac{2}{p(z)}.
\]  

Combining (B.43) with (B.83) and (B.86), one can easily obtain that
\[
\sum_{j=1}^n E_{j-1}[l_j(z_1)l_j(z_2)] = 2a_1^2 \frac{1}{(1-z_1)(1-z_2)} + 2a_2^2 (g'(0))^2 \\
+ 2a_1 a_2 g'(0) \frac{1}{1-z_1} + 2a_1 a_2 g'(0) \frac{1}{1-z_2} + o_P(1).
\]

Therefore, the condition (i) of Lemma A.2 is completely verified.

Lastly, we have to prove (B.86). Write
\[
E_{j-1}[E_j(x_j^T B^T A_{n_j}^{-1}(z)B x_j)] = E_{j-1}[E_j(x_j^T B^T B x_j)] \\
= E_{j-1}[E_j(\tilde{x}_j^T B^T B x_j)] \\
= E_{j-1}[E_j(\tilde{x}_j^T B^T A_{n_j}^{-1}(z)B x_j)] \\
= \frac{1}{n} \sum_{i<j} E_j(x_i^T B^T A_{n_{i,j}}^{-1}(z)B x_j B x_j) \\
= \frac{1}{n^2} \sum_{i<j} E_j(x_i^T B^T A_{n_{i,j}}^{-1}(z)B x_j B x_j) + \frac{1}{n} \sum_{i<j} E_j(x_i^T B^T A_{n_{i,j}}^{-1}(z)B x_j B x_j).
\]  

(B.87)
It follows from (3.27) and (3.29) that \( \beta_{ij}(z) = 1 + O\left( \frac{p}{n} \right) + O_p\left( \frac{\sqrt{p}}{n} \right) + O_P(n^{-1/2}). \)

Then by the proof of (B.39) in Appendix A.2, it follows that

\[
\frac{1}{n} \sum_{i<j} E_j (x_i^\top B^\top A_{nij}^{-1}(z) B x_{ij}) = O_P\left( \frac{\sqrt{p}}{n} \right).
\]  

(B.88)

In addition, by the proof of (B.42) in Appendix A.2,

\[
\frac{1}{n^2} \sum_{i<j} E_j (x_i^\top B^\top A_{nij}^{-1}(z) B x_{ij}) = \frac{j-1}{n^2} E_j \text{tr}[A_{nij}^{-1}(z)] + O_p\left( \frac{\sqrt{p}}{n} \right) + O_P\left( \frac{p^{3/2}}{n^{3/2}} \right),
\]

and by the proofs of (B.44), (B.46), (B.47) and (B.48),

\[
\frac{j-1}{n^2} E_j \text{tr}[A_{nij}^{-1}(z)] = \frac{j-1}{n^2} p \text{tr}[A_{nij}^{-1}(z)] + O_P\left( \frac{\sqrt{p}}{n} \right)
\]

which yields

\[
\frac{1}{n^2} \sum_{i<j} E_j (x_i^\top B^\top A_{nij}^{-1}(z) B x_{ij}) = \frac{j-1}{n^2} p \text{tr}[A_{nij}^{-1}(z)] + O_P\left( \frac{p^{1/2}}{n^{1/2}} \right) + O_P\left( \frac{p^{3/2}}{n^{3/2}} \right).
\]

(B.89)

Moreover, by (B.44), it follows that

\[
A_{nij}^{-1}(z) = -(zI_p - \frac{n-1}{n} b_1(z)I_p)^{-1} + b_1(z)C(z) + D(z) + E(z),
\]

where \( C(z), D(z) \) and \( E(z) \) are defined by (B.44). Consequently, by (B.44), (B.46), (B.47), (B.48), (B.87), (B.88) and (B.89), we establish that

\[
\text{LHS of (B.86)} = \frac{4}{p} \sum_{j=1}^{n} \frac{j-1}{n^2} \text{tr}[A_{nij}^{-1}(z)] + O_P\left( \frac{1}{\sqrt{p}} \right) + O_P\left( \frac{p^{1/2}}{n^{1/2}} \right)
\]

\[
= -4 \sum_{j=1}^{n} \frac{j-1}{n^2} \frac{\text{tr}[(zI_p - \frac{n-1}{n} b_1(z)I_p)^{-1}]}{p} + O_P\left( \frac{1}{\sqrt{p}} \right) + O_P\left( \frac{p^{1/2}}{n^{1/2}} \right)
\]

\[
= 4 \sum_{j=1}^{n} \frac{j-1}{n^2} \frac{1}{n} b_1(z) - \frac{1}{z} + O_P\left( \frac{1}{\sqrt{p}} \right) + O_P\left( \frac{p^{1/2}}{n^{1/2}} \right)
\]

\[
= \frac{2}{1-z} + O_P(1),
\]

So the proofs of Lemma 1 is completed. \( \square \)
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