On numerical bifurcation analysis of periodic motions of autonomous Hamiltonian systems with two degrees of freedom

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Abstract. In this work we consider bifurcation problem for natural families of periodic motions of autonomous Hamiltonian systems with two degrees of freedom. While there exists a well-developed analytical approach to this problem, it is limited to small neighborhoods of known equilibria and stationary solutions. To explore the bifurcations of periodic motions for all admissible values of the problem’s parameters it is necessary to employ numerical methods. We propose an approach combining analytical and numerical computation of the natural families with numerical bifurcation analysis. We obtain the so-called base solutions either analytically or numerically for particular values of the problem’s parameters and then employ a numerical method to continue the base solutions to the borders of their existence domains and to identify bifurcation points. Linear orbital stability domains are also obtained in course of the continuation. To illustrate the proposed approach we analyze bifurcation of periodic motions emanating from Cylindrical precession of a dynamically-symmetric satellite on a circular orbit.

Introduction
Investigating the dynamics of an autonomous Hamiltonian system often requires to compute its periodic motions and analyzing their orbital stability. In an autonomous Hamiltonian system periodic motions constitute the so-called natural families defined by one or several of the problem’s parameters. Parameter variations may result in the orbitally-stable periodic motions becoming unstable or vice-versa. This change in orbital stability is often associated with bifurcation of a natural family constituted by these periodic motions. The existence, bifurcation and orbital stability problems for periodic motions had been previously explored analytically in multiple works [1, 2, 3, 4, 5, 6, 7]. Notably, a rigorous approach to the bifurcation and orbital stability problems for 2-DOF Hamiltonian systems was developed in [5, 6, 7]. Said approach involves computing the periodic motions analytically by normalizing the initial Hamiltonian system in the neighborhood of a known equilibrium or stationary motion. The resulting analytical expressions are then used for bifurcation and nonlinear orbital stability analysis. However, the results obtained in this manner are only valid in the small neighborhood of the aforementioned equilibria and stationary motions. To compute the natural families and conduct the bifurcation and orbital stability analysis for all admissible values of the problem’s parameters one has to employ a numerical method. In this work we expand upon previous
results presented in [8, 9, 10, 11, 12, 13] and propose a numerical approach for obtaining and analyzing the natural families of periodic motions of an autonomous Hamiltonian system with two degrees of freedom.

1. Problem formulation

We consider an autonomous Hamiltonian system with two degrees of freedom

\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2) \]  

(1)

defined by a Hamiltonian \( H(q_1, q_2, p_1, p_2, \alpha) \) with an energy constant \( h \) where \( \alpha = (a_1, ..., a_k, h) \) is a parameter vector. Let us assume that for a fixed set of parameter values \( \vec{A} = (A_1, ..., A_k, h_0) \) the system (1) possesses a \( T \)-periodic motion

\[ q_i(t, \vec{A}) = q_i(t + T, \vec{A}), \quad p_i = p_i(t, \vec{A}) = p_i(t + T, \vec{A}) \quad (i = 1, 2) \]  

(2)

with initial conditions \( q_{i0} = q_i(0, \vec{A}) = q_i(T, \vec{A}), \quad p_{i0} = p_i(0, \vec{A}) = p_i(T, \vec{A}) \quad (i = 1, 2) \). We then assume that the system (1) also possesses \( T^* \)-periodic motions

\[ q_i^*(t, \vec{a}) = q_i^*(t + T^*, \vec{a}), \quad p_i^* = p_i(t, \vec{a}) = p_i^*(t + T^*, \vec{a}) \quad (i = 1, 2) \]  

(3)

with initial conditions \( q_{i0}^* = q_i^*(0, \vec{a}) = q_i^*(T^*, \vec{a}), \quad p_{i0}^* = p_i^*(0, \vec{a}) = p_i^*(T^*, \vec{a}) \quad (i = 1, 2) \) which correspond to parameter variations

\[ \vec{a} \rightarrow \vec{a} = (a_1 - A_1, ..., a_k - A_k, \Delta h), \quad \Delta h = h_0 - h. \]  

(4)

Periodic motion (2) together with motions (3) satisfying the adherence conditions

\[ \lim_{\vec{a} \rightarrow \vec{0}} q_i^*(0, \vec{a}) = q_i(0, \vec{A}), \quad \lim_{\vec{a} \rightarrow \vec{0}} p_i^*(0, \vec{a}) = p_i(0, \vec{A}), \quad \lim_{\vec{a} \rightarrow \vec{0}} T^*(\vec{a}) = T(\vec{A}) \quad (i = 1, 2) \]

constitute a so-called natural family [14, 15] emanating from a motion (2).

In this work we follow terminology proposed by Wintner [2] and refer a periodic motion (2) to as critical if for a fixed set of parameter values \( \vec{a} \) there exist two or more natural families which emanate from it or if the periodic motion corresponds to a boundary point of a natural family’s existence domain in the problem’s parameter space. The nature of critical periodic motions is further specified in Paragraph 3.

Thus a bifurcation problem for a natural family consists of identifying its critical motions and the other natural families which emanate from them.

2. Computation of natural families

Computing a natural family implies obtaining periodic motions (3) emanating from a known motion (2). Since it’s generally impossible [14] to analytically obtain the periodic motions (3) for all admissible variations (4) of the problem’s parameters, we use a combined approach. On the first step we use the methods developed in works [5, 6, 7] to obtain analytical expressions for the periodic motions in the neighborhood of the problem’s known stationary solutions or equilibria. These periodic motions are then used as a starting point for a numerical continuation procedure. In this work we use a numerical continuation algorithm developed by A. Sokolskiy and S. Karimov [15] based on a method proposed by A. Deprit and J. Henrard [14]. Below we describe this algorithm with modifications proposed by E. Sukhov and B. Bardin [9, 10, 12].

To obtain initial conditions of a new periodic motion (3) which belongs to a natural family emanating from a known ‘base’ periodic motion (2) we first introduce local Cartesian coordinates

\[ \xi_i = q_i^* - q_i, \eta_i = p_i^* - p_i \quad (i = 1, 2) \]

and then apply a canonical univalent transformation [15]

\[ (\xi_1, \xi_2, \eta_1, \eta_2) = S\vec{w} \]

(5)
where \( \vec{w} = (n_u, m_u, n_v, m_v)^T \) are normal and tangential coordinates and \( S \) is a symplectic orthogonal matrix [10, 12]

\[
S = \frac{1}{V} \begin{bmatrix}
\dot{p}_2 & \dot{q}_1 & \dot{q}_2 & \dot{p}_1 \\
-p_1 & \dot{q}_1 & -\dot{q}_2 & \dot{p}_2 \\
\dot{q}_2 & \dot{p}_1 & -\dot{p}_2 & -\dot{q}_1 \\
-p_2 & \dot{p}_1 & -\dot{p}_2 & \dot{q}_2
\end{bmatrix},\ S^T IS = I, S^T S = E, V = \sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{p}_1^2 + \dot{p}_2^2}. \tag{6}
\]

The derivatives \( \dot{q}_1, \dot{q}_2, \dot{p}_1, \dot{p}_2 \) in (6) are calculated on the base motion (2). Transformation (5) rotates the Cartesian axes \( \xi_i, \eta_i \) and aligns them with the orbit of the base periodic motion. Resulting local coordinates \( n_u, n_v \) and \( m_u, m_v \) describe normal and tangential displacements to the base motion’s orbit. On applying these transformations the initial canonical system (1) takes on the following form

\[
n_u = \frac{\partial H_n}{\partial n_v}, n_v = -\frac{\partial H_n}{\partial n_u}, m_u = \frac{\dot{V}}{V} m_u + h_{14} n_u + h_{34} n_v + \sum_{j=1}^{k} h_{2j} \dot{\alpha}_j, m_v = -\frac{1}{V} H_n^T \dot{\alpha}_j, \tag{7}
\]

with a Hamiltonian \( H_n = \frac{1}{2} (h_{11} n_u^2 + h_{33} n_v^2 + 2n_u n_v h_{13}) + n_u \sum_{j=1}^{k} h_{1j} \dot{\alpha}_j + n_v \sum_{j=1}^{k} h_{1j} \dot{\alpha}_j \) where \( h_{11}, h_{13}, h_{14}, h_{33}, h_{34}, H_n^T, h_{11}, h_{12}, h_{1j} (j = 1...k) \) are time-dependent coefficients calculated on the base motion (2). In the system (7) the equations of normal displacements \( n_u, n_v \) are independent from tangential displacements \( m_u, m_v \) which allows to reduce the initial boundary value problem of computing the periodic motion (3) to a simpler Cauchy problem.

The parameter variations \( \dot{\alpha} \) are set depending upon the desired rate and precision of the computation. To calculate the normal and tangential displacements \( n_u, n_v, m_u, m_v \) we present them in form of linear combinations of parameter variations

\[
n_u = \sum_{j=1}^{k} n_u j \dot{\alpha}_j, n_v = \sum_{j=1}^{k} n_v j \dot{\alpha}_j, m_u = \sum_{j=1}^{k} m_u j \dot{\alpha}_j, \tau = \sum_{j=1}^{k} \tau j \dot{\alpha}_j. \tag{8}
\]

By substituting (8) into (7) we obtain the so-called Predictor equations

\[
n_{11} = h_{11} n_{11} + h_{33} n_{12}, \quad n_{12} = -h_{11} n_{11} + h_{13} n_{12},
\]

\[
n_{21} = h_{11} n_{21} + h_{33} n_{22}, \quad n_{22} = -h_{11} n_{21} - h_{13} n_{22},
\]

\[
\dot{n}_{p1}^j = n_{p1} j h_{13} + n_{p2} h_{33} + h_{12}, \quad \dot{n}_{p2}^j = -n_{p1} j h_{11} - n_{p2} h_{33} - h_{11}, \quad j = 1...k+1
\]

\[
\dot{m}_1 = \dot{V} m_1 / V + h_{14} n_{11} + h_{34} n_{12}, \quad \dot{m}_2 = \dot{V} m_2 / V + h_{14} n_{21} + h_{34} n_{22},
\]

\[
\dot{m}_2 = \dot{V} m_2 / V + h_{14} n_{p1} + h_{34} n_{p2} + h_{12}, \quad j = 1...k+1
\]

\[
n_{u}(0) = \frac{n_{p1} (T) - n_{22} (T) n_{p2} (T) + n_{12} (T) n_{p1} (T)}{n_{11} (T) n_{22} (T) - n_{12} (T) n_{21} (T) - n_{11} (T) - n_{22} (T) + 1}, \tag{9}
\]

\[
n_{v}(0) = \frac{n_{p2} (T) - n_{11} (T) n_{p2} (T) + n_{21} (T) n_{p1} (T)}{n_{11} (T) n_{22} (T) - n_{12} (T) n_{21} (T) - n_{11} (T) - n_{22} (T) + 1},
\]

\[
m_v = \frac{1}{V(0)} \left( dh - \sum_{j=1}^{k} \tau j \frac{\partial H}{\partial \pi_j (0)} \right).
\]

Solving (9), (10) with initial conditions \( n_{11} = 1, n_{12} = 0, n_{21} = 0, n_{22} = 1, m_1 = 0, m_2 = 0, n_{p1} = 0, n_{p2} = 0 \) and \( m_{p} = 0, j = 1...k+1 \) we obtain (8) and then determine the normal and
tangential displacements. Returning to the initial variables $q_i, p_i, i = 1, 2$ gives us approximate values of initial conditions of a new periodic motion (3) belonging to a natural family emanating from a base solution (2). The new periodic motion is then taken as base motion and the process is repeated. Approximate initial conditions can be also corrected to a required level of precision using the Corrector part of the algorithm. The Corrector equations are derived from the system (7) assuming that parameter variations are equal to zero. A detailed description of the Corrector procedure is given in [12, 15].

The rate of computation using the algorithm described above considerably depends upon the choice of parameter variations (4). Setting excessive values of parameter variations either leads to a steep rise in the number of necessary corrections or makes the correction completely impossible and halts the algorithm. To address this problem we compute the parameter variations using a simple method proposed in [9]. We assert the computational precision of a $T$-periodic solution $z(t, \vec{A} + \vec{\alpha}) = (q_1, q_2, p_1, p_2)$ with the following formula

$$\Delta z(\vec{a}, \vec{\alpha}) = \sup |z(nT) - z(0)|.$$  

(11)

which denotes the computational error over $n$ periods $T$ (by default we assume $n = 1$). By expanding (11) into power series and omitting terms of the order two and higher we obtain an expression $\Delta z(\vec{a}, \vec{\alpha}) = \Delta z_0 + \Delta z_{aj}\alpha_j + \Delta z_{aj}\alpha_j, j = 1..k + 1.$ where $\Delta z_{aj} = \frac{\partial \delta z}{\partial a_j}, \Delta z_{aj} = \frac{\partial \delta z}{\partial \alpha_j}, j = 1..k + 1$ We can simplify it by assuming that $\alpha = const$ which gives

$$\Delta z(\vec{a}) = \Delta z_0 + \Delta z_{aj}\alpha_j, j = 1..k + 1,$$  

(12)

where $z_0$ is the error value (11) of the previous step. This assumption is justified if we execute two or more iterations of the Predictor procedure in a succession between computing the parameter variations. If we denote the desired level of precision with a small value $\varepsilon$ and analyze the expression (12) we arrive the estimates $\Delta z_{aj}\alpha_j \sim \varepsilon, j = 1..k + 1.$ which we use to derive the following formulae for the parameter variations:

$$\Delta a_j = \frac{\varepsilon}{\Delta z_{aj}}, j = 1..k + 1$$  

(13)

A single iteration of the numerical continuation process as implemented in this work consists of two consequent runs of the Predictor procedure followed by one or several runs of the Corrector. After executing the two Predictor runs we check for an error criterion $\Delta z < \varepsilon$. If the criterion is fulfilled we set the parameter variations according to (13) and proceed to the Corrector phase otherwise we divide the variations in half and repeat the last two Predictor runs.

3. Bifurcation and linear orbital stability analysis

The numerical continuation process described in the previous paragraph terminates at a point $\vec{a}$ in the problem’s parameter space if it is not possible to find a solution to the equations of normal displacements (9) which are a part of the Predictor system. This condition is called termination of a natural family according to [2, 15] and can be also expressed by a criterion

$$\Delta = \det(N(T) - E) = 0$$  

(14)

where $N(t)$ is a fundamental matrix of and $E$ is a unit matrix. However, termination does not always imply principal impossibility to continue a natural family. Condition (14) can be either associated with reaching the border of natural family’s existence domain or with bifurcation of natural families. The latter case does not necessary result in halting of the computational
process as well since it is possible to ‘overstep’ a bifurcation point by proper choice of parameter variations while continuing a natural family numerically across the problem’s parameter space. While we do not give an analytical solution to the bifurcation problem we propose the following numerical approach for identifying and analyzing bifurcations of natural families.

To identify critical motions corresponding to bifurcation of natural families we calculate the value \( \Delta_n = \text{det}(N(T) - E) \) on each iteration \( n \) of the numerical continuation process and check for the following conditions

\[
\Delta_n \leq \varepsilon, \quad (15)
\]

\[
\text{sign}(\Delta_n) = -\text{sign}(\delta_{n+1}), \quad \text{sign}\left( \frac{\partial \Delta_n}{\partial a_j} \right) = -\text{sign}\left( \frac{\partial \Delta_{n+1}}{\partial a_j} \right). \quad (16)
\]

A periodic motion is marked as critical if condition (15) is met. Conditions (16) serve to identify possible ‘overstepping’ of the critical motion. In case one of these conditions is met we use the bisection method to choose the parameter variations and approach the critical motion until the condition (15) is fulfilled. While applying bisection method we also check for convergence of \( \Delta \) to zero to avoid indefinite looping of the computation. Since the system (9) is solved on each iteration of numerical continuation process we also analyze it’s characteristic equation \( \rho^2 + 2A\rho + 1 = 0 \) to investigate the linear orbital stability of newly-computed motions. If its roots \( \rho_i, i = 1, 2 \) satisfy the condition \(|A| < 1\) where \( A = Sp[N(t)]/2 \) and \( Sp[N(t)] \) is trace of the fundamental matrix then a periodic motion is linear orbitally stable.

After computing the existence domain of a natural family and identifying its critical motions we use Poincare maps to identify new natural families emanating from the critical solutions. To illustrate the bifurcations we plot the periods of the resulting motions against the parameters.

4. Periodic motions of a dynamically-symmetric satellite about its center of mass

As an illustration for the approach proposed herein we consider periodic motions of a rigid-body satellite about its center of mass on a circular orbit in central Newtonian gravitational field. To study these motions we introduce a mobile orbital reference frame \( OXYZ \) and a fixed reference frame \( Oxyz \) associated with the satellite. The axes \( OX \) and \( OY \) of the orbital frame are aligned with the transversal and normal vectors to the satellite’s orbit and the axis \( OZ \) is directed along the radius-vector \( \vec{R} \) of the satellite’s center of mass \( O \). The axes of the fixed frame are directed along the satellite’s principal axes with corresponding moments of inertia being \( J_1, J_2 \) and \( J_3 \). Relative position of the frames \( OXYZ \) and \( Oxyz \) is described by Euler’s angles \( \psi, \theta, \phi \) as shown on Figure 1-a.

Taking the Euler angles as generalized coordinates and assuming that the satellite is dynamically symmetric \((J_1 = J_2)\) we deduce that \( \phi \) is a cyclic coordinate with its respective impulse \( p_\phi \) retaining constant value \( p_\phi = \frac{J_1 r_0}{J_1 \sin \theta} = \gamma, \quad r_0 = \psi \cos \theta - \omega_0 \cos \psi \sin \theta \) where \( \omega_0 \) is the angular velocity of the radius-vector \( \vec{R} \). Taking \( \gamma \) and \( \delta = 3 \left( \frac{J_2}{J_1} - 1 \right), (-3 < \delta < 3) \) as parameters, we obtain canonical equations describing the satellite’s motion about its center of mass [16]

\[
\frac{dv_\psi}{d\nu} = \frac{\partial H}{\partial p_\psi}, \quad \frac{d\theta}{d\nu} = \frac{\partial H}{\partial p_\theta}, \quad \frac{dp_\psi}{d\nu} = -\frac{\partial H}{\partial \psi}, \quad \frac{dp_\theta}{d\nu} = -\frac{\partial H}{\partial \theta}, \quad (17)
\]

with Hamiltonian \( H = \frac{p_\psi^2}{2\sin \theta} + \frac{p_\theta^2}{2} - \left( \frac{\gamma \cos \theta}{\sin \theta} + \frac{\cos \psi}{\tan \theta} \right) p_\psi - p_\theta \sin \psi + \frac{\gamma^2}{2\tan \theta} + \frac{\gamma \cos \psi}{\sin \theta} + \frac{1}{2} \delta \cos \theta^2 \)

where \( \nu = \omega_0 t \). Equations (17) possess particular solutions known as Regular precessions [17]

\[
\theta_0 = \frac{\pi}{2}, \quad \cos \psi_0 = -\gamma, \quad p_{\theta 0} = \sin \psi_0, \quad p_{\psi 0} = 0, \quad (18)
\]

\[
\theta_0 = \frac{\pi}{2}, \quad \psi_0 = \pi, \quad p_{\theta 0} = 0, \quad p_{\psi 0} = 0, \quad (19)
\]
\[
\sin \theta_0 = \frac{\gamma}{\delta - 1}, \quad \psi_0 = 0, \quad p_{\theta 0} = 0, \quad p_{\psi 0} = \delta \sin \theta_0 \cos \theta_0.
\] (20)

In case of a Regular precession the satellite’s principal axis \( z \) describes either a hyperboloidal, a cylindrical or a conical surface in the absolute space as shown on Figure 1-b, c and d, respectively. Thus the stationary motions given by expressions (18), (19) and (20) are also known as Hyperboloidal, Cylindrical and Conical precessions.

Figure 1. Reference frames and coordinates (a) and Regular precessions (b-d) of a dynamically symmetric satellite on a circular orbit with radius \(| \vec{R} | = R_0 \) and gravitating center \( G_0 \).

If a Regular precession is Lyapunov-stable there exist two types of periodic motions in its neighborhood [3]: short-periodic motions with period close to \( \frac{2\pi}{\omega_2} \) and long-periodic motions with period close to \( \frac{2\pi}{\omega_1} \) where \( \omega_1 \) and \( \omega_2 \) are the frequencies of the linearized system. Said periodic motions describe oscillations of the satellite’s principal axis \( Oz \) about a Regular precession.

Periodic motions originating from Regular precessions were previously obtained analytically and numerically in works [8, 11, 13]. In this work we explore the bifurcation problem for the short-periodic motions originating from Cylindrical precession. To compute the natural family constituted by these motions and analyze it for bifurcations we must first obtain the base motions. Following Lyapunov’s method we normalize the initial Hamiltonian system (17) in the neighborhood of Cylindrical precession and obtain the following small parameter power series representing the short-periodic motions [11]

\[
\begin{align*}
\psi &= \pi + c \frac{\kappa_2 (\omega_2^2 - \gamma^2 - \delta + 1)}{\omega_2} \sin \Omega_2 (\nu - \nu_0) + O(c^2), \\
\theta &= \frac{\pi}{2} + c \kappa_2 (\gamma - 2) \cos \Omega_2 (\nu - \nu_0) + O(c^2), \\
p_{\psi} &= c \kappa_2 (\omega_2^2 - \gamma^2 + 2\gamma - \delta - 1) \cos \Omega_2 (\nu - \nu_0) + O(c^2), \\
p_{\theta} &= \pi + c \frac{\kappa_2 (\omega_2^2 - \omega_2^2 \gamma + \gamma + \delta - 1)}{\omega_2} \sin \Omega_2 (\nu - \nu_0) + O(c^2)
\end{align*}
\] (21)

where \( c \) is the oscillation amplitude of the Satellite’s principal axis \( Oz \) about Cylindrical precession, \( \kappa_2 = \frac{1}{\omega_2} \gamma^2 + (\delta - 4) \gamma^2 + (6 - 2\omega_2^2 - 2\delta) \gamma - 3 + 2\delta + \omega^4 + \delta^2 + 2\omega_2^2 - 2\delta \omega_2^2 \), \( T = \frac{2\pi}{\Omega} \) is period of the motions, \( \Omega = \omega_2 + 4c^2 a + O(c^4) \) and \( a \) is a coefficient dependant upon the parameters \( \gamma \) and \( \delta \). Expressions (21) remain valid in the neighborhood of unstable Cylindrical precession as well. We will denote the natural family containing these short-periodic motions as \( Z_s \).

Using (21) as base solutions we numerically continue the natural family of short-periodic motions originating from Cylindrical precession to the borders of its existence domain. We will further refer this family to as \( Z_s \). To illustrate the individual periodic motions we use traces of
the satellite’s principal axis $Oz$ on a unit sphere. Figure 2 shows these traces for short-periodic motions originating from Hyperboloidal, Cylindrical and Conical precessions and corresponding to parameter values $\gamma = 0.5, \delta = 1.0, \Delta h = 0.01$ (black curve) and $\gamma = 0.5, \delta = 1.0, \Delta h = 0.1$ (white curve). Here $\Delta h$ is deviation of the energy constant from its value for a Regular precession. The middle image on Figure 2 represents short-periodic motions of the natural family $Z_s$.

![Figure 2. Short-periodic motions originating from Hyperboloidal (left), Cylindrical (center) and Conical (right) precessions of a dynamically-symmetric satellite shown as traces of the satellite’s principal axis $Oz$ on a unit sphere.](image)

Figure 3 shows existence and orbital stability domains of the natural family $Z_s$ computed for constant value $\gamma = 1$ as well as bifurcation curves $S_1, S_2, S_3$ and $S_4$. Short-periodic motions of the family $Z_s$ exist everywhere in the region showed on this figure. Stability analysis shows that for small deviations of the energy constant $h$ form the Cylindrical precession the short-periodic motions of the family $Z_s$ are orbitally unstable.

![Figure 3. Existence and orbital stability domains of short-periodic motions originating from Cylindrical precession of a dynamically-symmetric satellite.](image)
With the increase of the value of energy constant the motions belonging of the family $Z_s$ become orbitally stable upon crossing the curve $S_1$. On this curve a bifurcation takes place and a family of orbitally unstable short-periodic motions originating from Conical precession branches off from the family $Z_s$. Upon further increasing the value of $h$ the family $Z_s$ traverses the curve $S_2$ and becomes orbitally unstable. This is accompanied by a bifurcation as the unstable short-periodic motions of the Conical family coincide with the natural family $Z_s$. On crossing the curve $S_3$ the family $Z_s$ becomes linear orbitally stable again as it coincides with the family of linear orbitally stable short-periodic motions arising from Hyperboloidal precession.

The family $Z_s$ continues above the curve $S_3$ and experiences bifurcations with long-periodic motions. One of the curves representing this type of bifurcation is shown on the Figure as $S_4$.

The details of these bifurcations remain a matter of further study.

Figure 3 also shows the traces of the satellite’s figure axis $Oz$ on the unit spheres for the short-periodic motions corresponding to four regions separated by the bifurcation curves $S_1$, $S_2$, $S_3$ and $S_4$ in case of parameter values $\gamma = 1$, $\delta = 1.5$. The shape of these traces transforms continuously with the increase of energy constant $h$ from the shape shown on Figure 3-a to the shape shown on Figure 3-d.

5. Conclusion
In this work using established analytical methods and a numerical continuation algorithm we have proposed a new approach to investigating the bifurcation problem for the natural families of a 2-DOF Hamiltonian systems. This approach allows to identify and study the bifurcations for any admissible values of a problem’s parameters. To illustrate the results we have obtained bifurcation curves for the natural family of short-periodic motions originating from Cylindrical precession of a dynamically-symmetric satellite on a circular orbit.

Acknowledgements
This study was performed at the Moscow Aviation Institute (National Research University) and funded by RFBR, project Nr. 20-01-00637.

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