Quantum and classical chaos of a two-spinless-fermion system in a quantum wire

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Abstract
We study the classical and quantum dynamics of two spinless fermions confined in a quantum wire with a repulsive or an attractive Coulomb interaction. The interaction induces irregular dynamics in the classical mechanics, which reflects on the quantum properties of the system in the energy level statistics (the signatures of quantum chaos). We especially investigate the close correspondence between classical and quantum chaos. The classical dynamics studied has a scaling property, which the quantum counterpart does not have. However, we demonstrate that the energy level statistics implies the existence of a corresponding scaling property, even in the quantum system. Instead of an ordinary maximum Lyapunov exponent (MLE), we introduce a modified MLE, which is shown to be a suitable measure of the chaotic irregularity in the classical system studied. We show that the tendency of the energy dependence of the Brody parameter, which characterizes the energy level statistics in the quantum system, is consistent with that of the modified MLE.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Recent developments in high technology have led to the fabrication of nano-scale quantum dots containing a finite number of interacting electrons and optically trapped atoms, where a finite number of interacting macroscopic fermions are trapped in a small area. The quantum mechanics of these systems constitutes a topical subject. In such systems, the underlying classical motion is expected to play an important role. The nature of the classical motion, i.e. of a regular, mixed or chaotic character, reflects some of the quantum properties of the
systems, particularly in the energy level statistics. In this context, a number of studies on the quantum chaos of systems containing a few electrons in quantum dots have been reported [1–9]. However, the closer correspondence between the classical chaos and the quantum chaos in those systems has not been investigated enough.

The simplest system among them would be the one-dimensional (1D) system [5, 6]. In this paper we are concerned with the behavior of two fermions interacting with each other via the repulsive or attractive Coulomb potential in a 1D system and study the correspondence between the classical and the quantum chaos in detail. According to Fendrik et al [6], we use an effective Hamiltonian for a quantum wire, which reduces the original 3D system to the quasi-1D system. While its classical dynamics has a scaling property, the quantum counterpart has no such scaling property. We, however, show that the energy level statistics implies the existence of a corresponding scaling property, even in the quantum system. This is demonstrated by calculations of the Brody parameter for distributions of the nearest neighbor level spacing (NNLS). This subject, scaling in quantum chaos, has been examined for other systems: coupled harmonic or quartic oscillators [10–12] and the hydrogen atom in a magnetic field [13].

In order to clarify the closer correspondence between classical and quantum chaos, we introduce a modified maximum Lyapunov exponent (MLE) instead of an ordinary MLE. The ordinary MLE is a measure of the rate per unit of the time for the separation between two adjacent orbits, while the new MLE is the one per unit of the separation distance between them. The modified MLE is a suitable measure to compare the chaotic irregularity among classical orbits with different energies. We show that the tendency of the energy dependence of the Brody parameter is consistent with that of the modified MLE. We further show that the area of the chaotic region in Poincaré maps are not a suitable measure of the chaotic irregularity of this system, while several authors showed that it is a suitable measure of the irregularity of other systems [13–15].

This paper is organized as follows. In section 2, we construct a quasi-1D model of two fermions confined in a quantum wire. We introduce the modified MLE. In section 3, we explore the distribution of the NNLS in a wide range of energy and interaction strengths. Then we examine the chaotic irregularity of the corresponding classical system with the use of the MLE and Poincaré maps. We clarify the correspondence between the energy dependence of the distribution of the NNLS and the chaotic irregularity in the classical counterpart. The summary and conclusion are given in section 4.

2. Model and method

2.1. Quantum dynamics

We consider two spinless fermions confined in a quantum wire. We assume a narrow parabolic confinement in the transversal directions (x- and y-directions), which are much narrower than the confinement in the longitudinal direction (z-direction). We consider a hard wall potential in the z-direction. The fermions are interacting with each other via the repulsive or attractive Coulomb potential. The Hamiltonian of the system is written as

$$H = \sum_{i=1,2} \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2} \right) + \frac{1}{2m} \omega^2 (x_i^2 + y_i^2) \right]$$

$$\pm \frac{e^2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}. \quad (1)$$
Figure 1. The numerically calculated potential $V_{1D}$ (solid curve) and the analytical potential $V_m$ (broken curve). The lengths and potentials are scaled by $a$ and $e^2/a$, respectively.

We assume that the particles occupy the lowest energy state associated with the transverse motion, which is energetically well separated from the excited states. Then the two-particle wave function can be approximated as

$$\Psi(r_1, r_2) = \phi_0(x_1)\phi_0(y_1)\phi_0(x_2)\phi_0(y_2)\Phi(z_1, z_2),$$

(2)

where $\phi_0(x)$ is the lowest energy eigenstate of the harmonic oscillator. The wave function $\Phi(z_1, z_2)$ satisfies the equation

$$H_{1D}\Phi(z_1, z_2) = E\Phi(z_1, z_2),$$

(3)

where the effective Hamiltonian $H_{1D}$ is defined by

$$H_{1D} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} \right) + V_{1D}(z_1 - z_2).$$

(4)

$V_{1D}(z)$ is the effective potential given by

$$V_{1D}(z) = \pm e^2 \int |\phi_0(x_1)|^2|\phi_0(y_1)|^2|\phi_0(x_2)|^2|\phi_0(y_2)|^2 \frac{dx_1 dy_1 dx_2 dy_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + z^2}}.$$

$$= \pm \frac{e^2}{a} \int_0^\infty \frac{s \exp[-s^2/2]}{\sqrt{s^2 + (z/a)^2}} ds,$$

(5)

where $a = \sqrt{\hbar/m\omega}$ and $s = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + z^2}/a$. In this way, our system exhibits a quasi-1D property. Now we introduce a model potential [6] defined by

$$V_m(z) = \pm \frac{e^2}{\sqrt{a^2 + z^2}}.$$  

(6)

In figure 1, the solid and broken curves indicate the numerically calculated potential $V_{1D}$ and the analytical potential $V_m$, respectively. We see that $V_{1D}$ can be well approximated by $V_m$. We adopt $V_m$ instead of $V_{1D}$ as the interaction potential between particles since the analytical potential can be dealt with more easily. Thus our effective Hamiltonian is written as

$$H_{\text{eff}} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} \right) \pm \frac{e^2}{\sqrt{a^2 + (z_1 - z_2)^2}}.$$  

(7)

We scale lengths, angular momentums and masses by $L$, $\hbar$ and $2m$, respectively, where $L$ is the width of the system in the longitudinal direction. Then the effective Hamiltonian is...
distribution function of the even parity in this paper, when we examine the NNLS. The NNLS is fitted to the Brody feature of the system as varying $\lambda$ the latter system is also given by equation (8), in which the first and second terms represent the kinetic energy of the particle and the third term represents an external potential. We can choose energy eigenfunctions of the 2D system as being symmetric or antisymmetric against the exchange between $z_1$ and $z_2$, i.e., $\Phi(z_1, z_2) = \Phi(z_2, z_1)$ or $\Phi(z_1, z_2) = -\Phi(z_2, z_1)$, since the Hamiltonian does not change under this exchange. The symmetric and antisymmetric cases correspond to the boson and fermion cases, respectively, in the 1D two-particle system. In this paper we are concerned only with the cases of fermions.

Now we turn to the dynamics of the classical counterpart of the two-particle system, described by the Hamiltonian (8). Similarly to the quantum case, the lengths, angular momenta and

$$H_{\text{eff}} = -\left(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}\right) + \frac{\lambda}{\sqrt{(z_1 - z_2)^2 + \delta^2}},$$

where $\lambda$ is the effective interaction strength parameter given by $\lambda = \pm 2e^2 mL/\hbar^2$ and $\delta = a/L$. Note that the parameter $\lambda$ can be expressed as $\lambda = \pm 2L/a_B$, where $a_B$ is the Bohr radius defined by $a_B = h^2/e^2m$. The particles are confined by hard walls within $0 \leq z_1 \leq 1$ and $0 \leq z_2 \leq 1$. These hard walls describe the boundary of the quasi-1D wire. We examine the feature of the system as varying $\lambda$ while keeping $\delta$, which implies that we vary the system size, keeping the ratio between the longitudinal and transversal lengths.

It should be noted that this 1D two-particle system is equivalent to a 2D system of one particle having a coordinate ($z_1, z_2$) within a hard-walled square billiard. The Hamiltonian of the latter system is also given by equation (8), in which the first and second terms represent the kinetic energy of the particle and the third term represents an external potential. We can choose energy eigenfunctions of the 2D system as being symmetric or antisymmetric against the exchange between $z_1$ and $z_2$, i.e., $\Phi(z_1, z_2) = \Phi(z_2, z_1)$ or $\Phi(z_1, z_2) = -\Phi(z_2, z_1)$, since the Hamiltonian does not change under this exchange. The symmetric and antisymmetric cases correspond to the boson and fermion cases, respectively, in the 1D two-particle system. In this paper we are concerned only with the cases of fermions.

In order to look for signatures of quantum chaos in this system, we examine the distributions of the NNLS. The eigenenergies are obtained by diagonalizing the Hamiltonian matrices numerically, whose elements are evaluated using energy eigenstates without the Coulomb interaction (Slater determinants) as a basis set:

$$\phi_{m,n}(z_1, z_2) = \sqrt{2} (\sin(m\pi z_1) \sin(n\pi z_2) - \sin(n\pi z_1) \sin(m\pi z_2)),$$

where $m$ and $n$ are integers larger than zero. The components of the Hamiltonian matrices are represented with respect to $\phi_{m,n}$ in equation (9) as

$$\langle \phi_{m,n}|H|\phi_{m',n'}\rangle = \pi^2 (m^2 + n^2) \delta_{m,m'} \delta_{n,n'} + 4\lambda [I(m, n|m', n') - I(m, n|m', n')]$$

where $I(m, n|m', n')$ is defined by

$$I(m, n|m', n') = \int_0^1 \int_0^1 d z_1 d z_2 \frac{\sin(m\pi z_1) \sin(n\pi z_2) \sin(m'\pi z_1) \sin(n'\pi z_2)}{\sqrt{(z_1 - z_2)^2 + \delta^2}}.$$

We further take into account the parity of the system. This system is invariant under the inversion associated with the center ($z_1, z_2$) = (1/2, 1/2). Therefore, the eigenstates are classified into ones having even parity with $(m, n)$ = (even, even) or (odd, odd) and those having odd parity with $(m, n)$ = (even, odd) or (odd, even). We concentrate on the eigenstates of the even parity in this paper, when we examine the NNLS. The NNLS is fitted to the Brody distribution function

$$P_b(S) = (\alpha + 1)bS^\alpha \exp(-bS^{\alpha+1}),$$

$$b = \left\{ \Gamma \left( \frac{\alpha + 2}{\alpha + 1} \right) \right\}^{\alpha+1},$$

which interpolates the Poisson and Wigner distributions. It coincides with the Poisson distribution for $\alpha = 0$ and recovers the Wigner distribution for $\alpha = 1$. We use the Brody parameter $\alpha$ as a measure for the degree of chaotic irregularity of the system.

2.2. Classical dynamics

Now we turn to the dynamics of the classical counterpart of the two-particle system, described by the Hamiltonian (8). Similarly to the quantum case, the lengths, angular momenta and
masses are scaled by $L$, $h$ and $2m$, respectively. The equations of motion are then given as

\[
\begin{align*}
\frac{1}{2} \frac{d^2 z_1}{d\tau^2} &= \frac{\lambda (z_1 - z_2)}{[(z_1 - z_2)^2 + \delta^2]^{3/2}}, \\
\frac{1}{2} \frac{d^2 z_2}{d\tau^2} &= \frac{\lambda (z_2 - z_1)}{[(z_1 - z_2)^2 + \delta^2]^{3/2}}.
\end{align*}
\]

The $\lambda$ defined by $\lambda = \pm 2e^2 mL/h^2$ is dimensionless. The total energy is represented as

\[
E = \frac{1}{4} \left( \frac{dz_1}{d\tau} \right)^2 + \frac{1}{4} \left( \frac{dz_2}{d\tau} \right)^2 + \frac{\lambda}{\sqrt{(z_1 - z_2)^2 + \delta^2}}.
\]

The dynamics of this system is equivalent to that of a particle confined in a 2D square box in $0 \leq z_1 \leq 1$ and $0 \leq z_2 \leq 1$ with the potential $\lambda/\sqrt{(z_1 - z_2)^2 + \delta^2}$ similar to the quantum case. The behavior of the system apparently depends on the interaction strength parameter $\lambda$. However, if we introduce a rescaled time $\tau$ defined by $\tau = \sqrt{\lambda} t$, the equations of motion are reduced to

\[
\begin{align*}
\frac{1}{2} \frac{d^2 z_1}{d\tau^2} &= \pm \frac{(z_1 - z_2)}{[(z_1 - z_2)^2 + \delta^2]^{3/2}}, \\
\frac{1}{2} \frac{d^2 z_2}{d\tau^2} &= \pm \frac{(z_2 - z_1)}{[(z_1 - z_2)^2 + \delta^2]^{3/2}}.
\end{align*}
\]

The rescaled energy $\varepsilon = E/|\lambda|$ is given by

\[
\varepsilon = \frac{1}{4} \left( \frac{dz_1}{d\tau} \right)^2 + \frac{1}{4} \left( \frac{dz_2}{d\tau} \right)^2 \pm \frac{1}{\sqrt{(z_1 - z_2)^2 + \delta^2}}.
\]

The plus and minus signs in equations (15) and (16) correspond to positive and negative $\lambda$, respectively. Consequently, the classical behavior of the system is independent of the value of $\lambda$ itself. This is because in the classical system, there is no such characteristic length as the Bohr radius due to the finite Plank constant in the quantum system. Even if we enlarge the system size, we can find the equivalent trajectory by increasing the total energy. On the other hand, the quantum behavior of the system depends on the value of $\lambda$ directly. Now the following question arises. Even in the quantum case, does the system with the same value of $E/|\lambda|$ but with different $\lambda$ exhibit a similar behavior to the classical system, especially concerning the degree of chaotic irregularity of the system? Otherwise there is no closer correspondence between the classical chaos and the quantum chaos. We investigate this point in this study.

We use two kinds of Poincaré map to see the behavior of the classical system. The first kind of Poincaré map, called Poincaré map 1, is defined in the section $v_2$ versus $z_2$ for the second particle taken at times when the first particle bounces off the left boundary of the well ($z_1 = 0$). The second kind of Poincaré map, called Poincaré map 2, is defined as follows. We take coordinate $l$ along the two sides of the square and a diagonal line connecting ($z_1$, $z_2$) = (0, 0) and (1, 1) where the ridge of the potential lies (see figure 2). $l$ is normalized so that the range is $0 \leq l \leq 1$. The trajectory can be recorded by two values. One is $l$ at the point where the particle is reflected on the hardwalls or intersects the line $z_1 = z_2$. The other is $p = \cos \theta$, where $\theta$ is the angle between the velocity vector after reflection and the normal to the solid line. The Poincaré map which records $l$ and $\theta$ of the orbits reflects the property of the classical system.

We also use the MLE as a measure for the degree of chaotic irregularity of the classical system. The ordinary Lyapunov exponent is defined as follows. We consider an orbit ($p(\tau)$, $q(\tau)$) (denoted as the reference orbit) and a slightly displaced orbit from the reference orbit in the phase space. The starting point of the displaced orbit is spaced apart by
Figure 2. The coordinates of the billiard system.

a small vector \((\Delta p(0), \Delta q(0))\) from \((p(0), q(0))\) at initial time \(\tau = 0\). The distance between the reference and displaced orbits is

\[
d_0 = |(\Delta p(0), \Delta q(0))|.
\]  

(17)

We follow these orbits for a time interval \(\Delta \tau\). The distance between the two orbits at \(\tau = \Delta \tau\) is represented as

\[
d_1 = |(\Delta p(\Delta \tau), \Delta q(\Delta \tau))|.
\]  

(18)

Then we choose a new starting point for the displaced trajectory at time \(\tau = \Delta \tau\) as

\[
(p(\Delta \tau), q(\Delta \tau)) + \frac{d_0}{d_1}(\Delta p(\Delta \tau), \Delta q(\Delta \tau))
\]  

(19)

so that the distance between the new starting points equals \(d_0\). The trajectory is followed up to time \(\tau = 2\Delta \tau\). The new deviation of the displaced orbit from the reference orbit

\[
d_2 = |(\Delta p(2\Delta \tau), \Delta q(2\Delta \tau))|
\]  

(20)

is computed and a second rescaled trajectory is started. This process is continued, yielding a sequence of distances \(d_0, d_1, d_2, \ldots\). Using these values, the MLE is defined as

\[
\Gamma_L = \lim_{n \to \infty} \frac{1}{n\Delta \tau} \sum_{i=1}^{n} \ln \frac{d_i}{d_0},
\]  

(21)

where \(n\) is the number of the time segment. This quantity \(\Gamma_L\) is, however, not suitable as a measure for the degree of chaotic irregularity in this system. \(\Gamma_L\) only increases with \(\varepsilon\) evenly because the motion of particles becomes faster with the increase of \(\varepsilon\), while the classical dynamics becomes regular in a high energy regime, as shown by the numerical results in the next section.

We introduce a modified MLE, defined as

\[
\Gamma'_L = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n \xi} \ln \frac{d_i}{d_0}
\]  

(22)

The definition of \(\Gamma'_L\) is almost the same as \(\Gamma_L\), except that \(\Delta \tau\) in equation (21) is replaced by a small distance \(\xi\). The definition of \(\Gamma'_L\) in equation (22) is similar to equation (21). For \(\Gamma'_L\), we follow the two adjacent orbits while the reference orbit travels the small distance \(\xi\) and then evaluate \(d_i\). \(\Gamma'_L\) represents the degree of exponential divergence of adjacent orbits, similarly to \(\Gamma_L\). However, \(\Gamma'_L\) depends only on the geometry of the orbits but not on the quickness of the development of the orbits. We employ \(\Gamma'_L\) in equation (22) as a measure of the degree of chaotic irregularity in classical mechanics.
3. Numerical results

3.1. Repulsive interaction

We obtain energy eigenvalues by diagonalizing the effective Hamiltonian (8) and evaluate the smoothed counting function $N_{av}(E)$, which gives the cumulative number of states below an energy $E$. In order to analyze the structure of the level-fluctuation properties, we unfold the spectrum by applying the well-known transformation $x_n = N_{av}(E_n)$ to obtain a constant mean spacing, where $n$ denotes the number of the energy level. From the unfolded spectrum, we obtain the histogram of the NNLS distribution $P(S)$, where $S_n = x_{n+1} - x_n$. The histogram is fitted to the Brody distribution function $P_B(S)$ in equation (12). The integral of the Brody distribution function, $I_B(S) \equiv \int_0^S P_B(S') \, dS'$ satisfies

$$\ln \ln \left[ 1 / (1 - I_B(S)) \right] = (1 + \alpha) \ln S + \ln b, \quad (24)$$

where $b$ is given in equation (12). Using the above relation and the least-squares fitting method, we evaluate the Brody parameter $\alpha$ for the distribution $P(S)$ of the NNLS. Hereafter we take $\delta = 0.01$. As an example, the result of fitting for $\lambda = 200$ is shown in figure 3.

The total energy region is divided into several regions. In figure 4 we show the obtained NNLS distribution in each region for $\lambda = 200$. About 1000 eigenvalues are used in each region to compute each histogram. The range of the used energy levels and the Brody parameter $\alpha$ are shown below each panel. We see that the Brody parameter decreases with the increase of the average of the energy eigenvalues, which are used to obtain the histogram. Especially for the histograms in panels (e) and (f) in figure 4, with the Brody parameter less than 0.015, the histograms also fit well to the Poisson distribution. The histogram for $\lambda = 500$ is shown in figure 5.

We see that the Brody parameter $\alpha$ decreases with the increase of the average of the energy eigenvalues, similarly to the case of $\lambda = 200$; moreover, that the values of $\alpha$ are greater than those for $\lambda = 200$ in each energy level region.

We show the $\varepsilon$-dependence of the Brody parameter for $\lambda = 200, 500, 750, 1000$ in figure 6. For each data point we use energy eigenvalues in the energy interval $\Delta E = 50000,$
which includes about 1000 energy eigenvalues. The horizontal axis denotes the averaged value of $\varepsilon$ of the used eigenstates, $\langle \varepsilon \rangle$. We see that the Brody parameter $\alpha$ decreases almost monotonously with the increase of $\langle \varepsilon \rangle$. Moreover, the $\varepsilon$-dependences of $\alpha$ are quite similar for

Figure 4. Histograms of NNLS for $\lambda = 200$. Solid lines are the best fit Brody distributions. Each value of the Brody parameter $\alpha$ is shown below each panel.

Figure 5. The same as figure 4 except for $\lambda = 500$. 
Figure 6. $\varepsilon$-dependence of the Brody parameter for the repulsive interaction. For each data point, eigenvalues in the energy interval $\Delta E = 50000$ were used (their number is about 1000). The horizontal axis denotes the average of $\varepsilon$.

Figure 7. Poincaré 1 map for repulsive interaction with (a) $\varepsilon = 50$, (b) $\varepsilon = 200$ and (c) $\varepsilon = 1000$.

Figure 8. Poincaré 2 map for repulsive interaction with (a) $\varepsilon = 50$, (b) $\varepsilon = 200$ and (c) $\varepsilon = 1000$.

different $\lambda$, especially in $\langle \varepsilon \rangle < 600$. As mentioned in the previous section, the classical system has a scaling property characterized by the parameter $\varepsilon = E/|\lambda|$. The above results indicate that the distribution of the NNLS in quantum mechanics has the same scaling property on $\varepsilon$.

Now we see the behavior of the classical system with the equations of motion, equation (15). In figures 7(a)–(c) we show the Poincaré 1 maps defined in the previous section for $\varepsilon = 50, 200, 1000$, respectively, with $\delta = 0.01$. In figures 8(a)–(c), the Poincaré 2 maps are shown. We have taken about 20 different initial points in the phase space for
each map. These Poincaré maps show that this classical system exhibits mixed dynamics with coexisting Kolmogorov–Arnold–Moser (KAM) tori and chaotic regions. This is consistent with the fact that NNLS distributions in the quantum system are intermediate between the Poisson and Wigner distribution.

In figure 9 typical trajectories \((z_1(t), z_2(t))\) are shown for (a) \(\epsilon = 50\), (b) \(\epsilon = 200\) and (c) \(\epsilon = 1000\), respectively. The Poincaré maps for those orbits show that they are chaotic. It is seen that the potential bends the trajectories, especially near \(z_2 = z_1\) line for \(\epsilon = 50\) and \(\epsilon = 200\). It causes irregularity on the orbits. In contrast, the effect of the potential is much less for \(\epsilon = 1000\). The trajectory is composed of nearly straight lines. The orbits become more regular for larger \(\epsilon\) if \(\epsilon\) is large enough.

In order to confirm this point quantitatively, we evaluate \(\Gamma'_L\) in equation (22) with \(\xi = 0.01\), which reflects the irregularity of the trajectories. The numerical calculation of each trajectory is performed for times longer than \(1.5 \times 10^5\). We take the average of \(\Gamma'_L\) over about 20 orbits corresponding to the largest irregular region in the Poincaré maps for each \(\epsilon\). We show the \(\epsilon\)-dependence of \(\Gamma'_L\) in figure 10(a). The decrease of \(\Gamma'_L\) indicates the fact that the dynamics becomes more regular with the increase of \(\epsilon\), which is consistent with the above intuitive view from figure 9. It is also consistent with the \(\epsilon\)-dependence of \(\alpha\) in the quantum system.

We also calculate the ordinary MLE \(\Gamma_L\) in equation (21) with \(\Delta \tau = 0.01\). Numerical calculations for trajectories are performed for times longer than \(1.5 \times 10^5\). We take the average of \(\Gamma_L\) over about 20 orbits corresponding to the largest irregular region in the Poincaré maps, as well as for \(\Gamma'_L\). The \(\epsilon\)-dependence of \(\Gamma_L\) is shown in figure 10(b). We see that \(\Gamma_L\) increases with \(\epsilon\). This is because the motion of the particles become faster with the increase of \(\epsilon\), which does not necessarily imply an increase of chaotic irregularity. Therefore, \(\epsilon\)-dependence of \(\Gamma_L\) does not directly reflect the degree of chaotic irregularity.
We also point out that an area of the largest irregular region in the Poincaré map, which is adopted by several authors as a measure of chaotic irregularity [13–15], is irrelevant for this system. We calculate the ratio $R_0$ between two areas in the Poincaré maps: the area of the largest irregular region and the area of the total region reachable for a particle with $\varepsilon$. For the calculations we take meshes on the Poincaré map. The total number of meshes is $300 \times 300$. Then we count the number of meshes that an irregular trajectory visits and compare it to the total number of meshes energetically allowed. Numerical calculations of the trajectories are performed for times longer than $2.0 \times 10^5$. The $\varepsilon$-dependence of $R_0$ for Poincaré 1 and 2 maps are shown in figures 11(a) and (b), respectively. It is seen that $R_0$ increases with $\varepsilon$ for $\varepsilon > 100$, in which the trajectory in a 2D square can cross the potential hill (the diagonal line). On the other hand, the orbits become more regular, as can be clearly seen in the figure 9, when $\varepsilon$ increases for $\varepsilon > 100$. Therefore, $R_0$ is not a proper measure of the irregularity in this system in contrast to the other systems, in which $R_0$ can be adopted as a measure of the irregularity [13–15].

3.2. Attractive interaction

Now we turn to the results for the case of the attractive interaction with $\lambda < 0$ and $\delta = 0.01$. The $\varepsilon$-dependence of the Brody parameter is shown in figure 12. $\langle \varepsilon \rangle$ is the average of $\varepsilon$ of the eigenstates used. The decrease of the Brody parameter is seen with the increase of the scaled energy $\varepsilon$. We see that the Brody parameter depends almost only on $\varepsilon$ and not on $E$ and $\lambda$ separately, which is a situation similar to the case of repulsive interaction. Next, we consider the corresponding classical dynamics described by the equations of motion in equation (15) for the attractive interaction. The Poincaré 1 maps are shown for $\varepsilon = 5, 200, 1000$ in figures 13(a)–(c), respectively. The Poincaré 2 maps are also shown in figures 14(a)–(c). We have taken about 20 different initial points in the phase space for each map. These Poincaré maps show that this classical system exhibits mixed dynamics with coexisting KAM tori and chaotic regions, as well as in the case of repulsive interaction.

We calculate $\Gamma'_L$ in equation (22) with $\xi = 0.01$, which reflects the degree of chaotic irregularity. Numerical calculations of the trajectories are performed for times longer than $1.5 \times 10^5$. The results are shown in figure 15(a), where we see that $\Gamma'_L$ decreases with the increase of $\varepsilon$ for $\varepsilon > 0$. This is consistent with the fact that the dynamics of the particle becomes regular when $\varepsilon$ increases for $\varepsilon > 0$. On the other hand, $\Gamma_L$ in equation (21) increases with $\varepsilon$, as seen in figure 15(b), where we take $\Delta t = 0.01$. The increase of $\Gamma_L$ is due to the fact that the dynamics of the particle becomes faster with the increase of $\varepsilon$. $\Gamma_L$ does not reflect
the degree of chaotic irregularity of the classical system, which is similar to the case of the repulsive interaction.

We also calculate the ratio $R_a$ between two areas in the Poincaré maps, the area of the largest irregular region and the area of the total region reachable, in the same manner used for the repulsive interaction. The $\varepsilon$-dependence of $R_a$ for Poincaré 1 and 2 maps are shown in figures 16(a) and (b), respectively. $R_a$ increases with $\varepsilon$ for $\varepsilon > 500$, while the orbit becomes
Figure 15. $\varepsilon$-dependence of $\Gamma'_L$ and $\Gamma_L$ for the attractive interaction are shown in the left and right panels. The broken line is a guide for the eyes.

Figure 16. $\varepsilon$-dependence of the ratio $R_a$ between two areas in Poincaré maps are shown in the left and right panels, respectively. The broken line is a guide for the eyes.

more regular when $\varepsilon$ increases, as well as in the case of the repulsive interaction. Therefore, $R_a$ is not a proper measure of the irregularity, as is the case of the attractive interaction in this system.

4. Discussions and conclusion

We studied the dynamics of two spinless fermions confined in a quantum wire with a repulsive or attractive Coulomb interaction. The system is reduced to a quasi-one-dimensional system with an effective potential, under the assumption that the transverse confinement is much stronger than the longitudinal one.

The Coulomb interaction induces irregular dynamics in classical mechanics. Examining Poincaré maps for this system, we have found that the classical system exhibits mixed dynamics with coexisting KAM tori and chaotic regions. To see the signatures of the quantum chaos in the corresponding quantum system, we analyzed the distributions of the nearest neighbor level spacing (NNLS), which are fitted to the Brody distribution function characterized by the Brody parameter $\alpha$. The results indicate that they are intermediate between the Poisson and Wigner distributions, which is consistent with the mixed character of the classical dynamics.

This classical system has a scaling property: its dynamics is characterized by the rescaled energy parameter $\varepsilon = E / |\lambda|$, where $\lambda$ is the interaction strength parameter. In contrast, the quantum system has no such scaling property. However, it has turned out that the distribution of the NNLS in the quantum system has a scaling property similar to the case of classical mechanics. The Brody parameter $\alpha$ depends almost only on the average value of $\varepsilon$ and is insensitive to the value of $\lambda$ itself.
In the classical system, we found that orbits are more regular for larger values of $\varepsilon$. The ordinary maximum Lyapunov exponent (MLE) $\Gamma_L$ is not a suitable measure of the chaotic irregularity for this system, because it increases with $\varepsilon$ whereas the irregularity of the system decreases. In order to improve this discrepancy, we introduced a modified MLE $\Gamma'_L$, which represents the rate of the exponential divergence of two adjacent orbits (reference and displaced orbits) with respect to the length of the reference orbit, while the ordinary Lyapunov exponent describes the one with respect to time. The dependence of $\Gamma'_L$ on $\varepsilon$ quantitatively shows the decrease of the chaotic irregularity with an increase of $\varepsilon$. Therefore, the $\Gamma'_L$ is a suitable measure of the chaotic irregularity of this classical system rather than $\Gamma_L$. On the other hand in the quantum system, the Brody parameter $\alpha$ decreases almost monotonically with the increase of the average value of $\varepsilon$ in both cases of repulsive and attractive interactions, which indicates the distribution function of the NNLS approaches the Poisson distribution with the increase of $\varepsilon$. Consequently, we have shown a closer correspondence between the classical chaos and quantum chaos in this system.

We also showed that the area of the irregular region in the Poincaré maps are not a suitable measure of the chaotic irregularity for this system, in contrast with other systems in which the area has been adopted as a measure of the irregularity by several authors [13–15]. We have shown numerically that the distribution of the NNLS has approximately a scaling property existing in the classical system, though this Hamiltonian itself has no such scaling property in the quantum system. Similar situations have been found as numerical evidence in other systems [10–12]. However, so far no explanations have been proposed: it is a very difficult problem. One of the reasons for the difficulty is a lack of any sufficient explanation for the transition of the distributions of the NNLS corresponding to a change of the chaotic degree in the classical counterpart. Fitting them to the Brody distribution is just a phenomenological method and there is no analytical justification for it. The analytical or semi-analytical explanation for the scaling in quantum chaos is left for a future issue.

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References

[1] Ulloa S E and Pfannkuche D 1997 Superlatt. Microstruct. 21 21
[2] Meza-Montes L, Ulloa S E and Pfannkuche D 1998 Physica E 1 274
[3] Ahn K-H and Richter K 1999 Ann. Phys., Lpz. 8 1
[4] Ahn K-H, Richter K and Lee I-H 1999 Phys. Rev. Lett. 83 4144
[5] Van Vessen M Jr, Santos M C, Cheng B K and da Luz M G E 2001 Phys. Rev. E 64 026201
[6] Fendrik A P, Schmelcher P and Diakonos F K 2004 Phys. Rev. B 69 035333
[7] Xavier E P S, Santos M C, Dias da Silva L G G V, da Luz M G E and Beimsaand M W 2004 Physica A 342 377
[8] Sawada S, Terai A and Nakamura K 2009 Chaos Solitons Fractals 40 862
[9] Haller E, Köppel H and Cederbaum L S 1984 Phys. Rev. Lett. 52 1665
[10] Zimmermann Th, Meyer H D, Köppel H and Cederbaum L S 1986 Phys. Rev. A 33 4334
[11] Zeng Y H and Serota R A 1994 Phys. Rev. B 50 2492
[12] Winter D and Friedrich H 1987 Phys. Rev. A 35 1464
[13] Terasaka T and Matsushita T 1985 Phys. Rev. A 32 538
[14] Harada A and Hasegawa H 1983 J. Phys. A Math. Gen. 16 L259