INITIAL DEGENERATIONS OF SPINOR VARIETIES

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Abstract. We construct closed immersions from initial degenerations of the spinor variety $S_n$ to inverse limits of strata associated to even $\Delta$-matroids. As an application, we prove that these initial degenerations are smooth and irreducible for $n \leq 5$ and identify the log canonical model of the Chow quotient of $S_5$ by the action of the diagonal torus of $\text{GL}(5)$.

Keywords: Spinor varieties, initial degenerations, Delta-matroids, Chow quotient.

Mathematics Subject Classification: 14T05 (primary), 15A66, 52B40 (secondary).

1. Introduction

The pure spinor variety, denoted $S^\pm_n$, parameterizes totally isotropic subspaces of a $2n$-dimensional vector space $V$ equipped with a symmetric, nondegenerate quadratic form of Witt index $n$. From the representation-theoretic viewpoint, $S^\pm_n$ is the Lie-type D analog of the Grassmannian. The Wick embedding, similar to the Plücker embedding, realizes $S^\pm_n$ as a closed subvariety of $\mathbb{P}(\wedge^* E)$, where $E$ is some fixed totally isotropic subspace of $V$. The exterior algebra of $E$ splits as a direct sum of its even and odd parts, i.e., $\wedge^* E = \wedge^\text{ev} E \oplus \wedge^\text{odd} E$, and $S^\pm_n$ has two (projectively equivalent) irreducible components $S^+_n$ and $S^-_n$, one contained in $\mathbb{P}(\wedge^\text{ev} E)$ and the other in $\mathbb{P}(\wedge^\text{odd} E)$. Our convention is that $S^+_n \subset \mathbb{P}(\wedge^\epsilon(n) E)$ where $\epsilon(n)$ is the parity of $n$.

We investigate the initial degenerations of $S^+_n$—the open dense cell of $S_n$ given by the nonvanishing of all Wick coordinates with $\epsilon(n)$ parity—via their relation to isotropical linear spaces in the sense of Rincón [23]. This work is an extension to the Lie-type D setting of [7] where the author investigates the initial degenerations of the Grassmannian via their relation to tropical linear spaces. Specifically, a vector in the tropical Grassmannian records an initial degeneration and a tropical linear space, and the latter records a diagram of matroid strata parameterized by a regular matroidal subdivision of the hypersimplex. The main result in loc. cit. produces a closed immersion from the initial degeneration into the inverse limit of this diagram. This result has a number of applications to $\text{Gr}_0(d,n)$—the open dense cell of $\text{Gr}(d,n)$ given by the nonvanishing of all Plücker coordinates—especially $\text{Gr}_0(3,7)$. Specifically, the initial degenerations of $\text{Gr}_0(3,7)$ are smooth and irreducible, the tropicalization of $\text{Gr}_0(3,7)$ with respect to the Plücker embedding is faithful, and the Chow quotient of $\text{Gr}(3,7)$ by the diagonal torus of $\text{PGL}(7)$ is the log canonical compactification of the moduli space of 7 lines in $\mathbb{P}^2$ in linear general position, partially resolving a conjecture of Hacking, Keel, and Tevelev [17, Conjecture 1.6]. As Grassmannians and matroids are Lie-type A partial flag varieties and Coxeter matroids, respectively, we expect that many of the results in [7] extend to a broader class of partial flag varieties. In this paper, we confirm this expectation in the Lie-type D setting.
There are numerous formulations of Lie-type D Coxeter matroids \([3,4,5]\); we use the language of \(\Delta\)-matroids. Given \(w\) in \(TS_n^\circ := \text{Trop}(S_n^\circ)\), we may form an initial degeneration \(\text{in}_w S_n^\circ\) and an isotropically linear space; the latter encodes a regular subdivision \(\Delta_w\) of the \(\Delta\)-hypersimplex \(\Delta(n)\) into \(\Delta\)-matroid polytopes \([23]\). The collection of totally isotropic subspaces realizing a \(\Delta\)-matroid \(M\) forms a locally-closed subscheme \(S_M \subset S_n^\circ\). If \(Q\) is a \(\Delta\)-matroid polytope and \(Q'\) is a face, then \(Q'\) is also a \(\Delta\)-matroid polytope (by the Gelfand-Serganova theorem \([3, \text{Theorem 6.3.1}]\)), and there is a morphism \(S_{MQ} \rightarrow S_{MQ'}\), where, e.g., \(M_Q\) denotes the \(\Delta\)-matroid of \(Q\). These morphisms determine a diagram of type \(\Delta_w\), hence we may form the inverse limit \(\lim \leftarrow_{Q \in \Delta_w} S_{MQ}\).

**Theorem 1.1.** There is a closed immersion \(\lim \leftarrow_{Q \in \Delta_w} S_{MQ} \leftarrow \lim \leftarrow_{Q \in \Delta_w} S_{MQ'}\).

We give a geometric interpretation of this closed immersion in §5; here is a summary. To define a morphism \(\lim \leftarrow_{Q \in \Delta_w} S_{MQ} \rightarrow \lim \leftarrow_{Q \in \Delta_w} S_{MQ'}\) for each \(Q \in \Delta_w\) (and these morphisms must be compatible with the face morphisms). Let \(\mathcal{K} = k((t^R))\), equip it with its usual \(t\)-adic valuation, and let \(x\) be a \(k\)-point of \(\text{in}_w S_n^\circ\). There is a \(\mathcal{K}\)-point \(q\) of \(S_n^\circ\) whose exploded tropicalization is \(x\) \([22]\). Let \(F_q \subset \mathcal{K}^{2n}\) be the totally isotropic subspace with Wick vector \(q\), and \(F_q^\circ = F_q \cap (\mathcal{K}^{*})^{2n}\). If \(v\) is in the cell of \(\text{Trop}(F_q^\circ)\) dual to \(Q \in \Delta_w\), then \(\text{in}_v F_q^\circ\) is a totally isotropic subspace of \(k^{n}\) that realizes \(M_Q\).

The map \(\text{in}_w S_n^\circ \rightarrow S_{MQ}\) takes \(x\) to \((\text{the Wick vector of})\; \text{in}_v F_q\).

Working with the full poset \(\Delta_w\) is impractical due to its size. Nevertheless, the inverse limit \(\lim \leftarrow_{Q \in \Delta_w} S_{MQ}\) depends only on the adjacency graph \(\Gamma_w\) of \(\Delta_w\). The vertices of \(\Gamma_w\) correspond to the codimension 0 cells, and two vertices are connected by an edge whenever their corresponding cells intersect along a facet. In §6, we show how this encodes a diagram of \(\Delta\)-matroid strata, form the inverse limit \(\lim \leftarrow_{\Gamma_w} S_{MQ}\), and deduce

\[
\lim \leftarrow_{Q \in \Delta_w} S_{MQ} \cong \lim \leftarrow_{\Gamma_w} S_{MQ},
\]

The analogous statement in the Grassmannian setting is due to Cueto \([7, \text{Appendix C}]\). We develop various techniques for evaluating these inverse limits, which we use, together with Theorem 1.1 to prove the following theorem.

**Theorem 1.2.** The initial degenerations of \(S_n^\circ\) are smooth and irreducible for \(n \leq 5\).

We conclude by studying the birational geometry of \(S_n/H\), the Chow quotient of \(S_n\) by the maximal torus \(H\) of the orthogonal group, for \(n \leq 5\). There is an ordered basis of \(V\) so that the matrix of the quadratic form is

\[
Q = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}
\]

The orthogonal group \(O(2n)\) (with respect to the symmetric form \(Q\)) and its maximal torus \(H\) are \(O(2n) = \{ X \in \text{GL}(2n) : X^T Q X = Q \}\) and \(H = \left\{ \begin{bmatrix} h^{-1} & 0 \\ 0 & h \end{bmatrix} : h \in \text{GL}(n) \text{ is diagonal} \right\}\).

Any totally isotropic subspace is the row-span of a full-rank \(n \times 2n\) matrix of the form \([A|B]\) where \(AB^T\) is skew-symmetric \([3, \text{Lemma 3.4.1}]\). The action \(H \sim S_n\) is given by
right multiplication:

\[(A|B) \begin{bmatrix} h^{-1} & 0 \\ 0 & h \end{bmatrix} = [Ah^{-1}|Bh]. \]

Let us describe how to realize $S_n/H$ as a compactification of $S_n^\circ/H$ inside of a toric variety. We may realize $H$ as a subtorus of the dense torus $T$ of $\mathbb{P}(\Lambda^e(n)E)$, and the scaling action of $T$ on $\mathbb{P}(\Lambda^e(n)E)$ restricts to the action of $H$ on $S_n$ from above. The Wick embedding induces an embedding of Chow quotients $S_n/H \hookrightarrow \mathbb{P}(\Lambda^e(n)E)/H$. By [15], $\mathbb{P}(\Lambda^e(n)E)/H$ is the toric variety of the secondary fan of $\Delta(n)/L_\mathbb{R}$, where $L$ is the cocharacter lattice of $H$ and $L_\mathbb{R} = L \otimes \mathbb{Z} \mathbb{R}$. The locus of matroidal subdivisions is the $\Delta$-Dressian $\text{Dr}(n)$, and we denote by $\Sigma_n$ the secondary fan of $\text{Dr}(n)$. The Chow quotient $S_n/H$ is the closure of $S_n^\circ/H$ in the toric variety $X(\Sigma_n/L_\mathbb{R})$.

As a consequence of Theorem [1.2] for $n \leq 5$, $S_n^\circ/H$ is schön in the sense of Tevelev [25], and $S_n/H$ is a schon compactification of $S_n^\circ/H$. By general results of Hacking, Keel, and Tevelev in [11], determining the log canonical model of $S_n/H$ amounts to finding a suitable fan structure of $\text{Trop}(S_n^\circ)$ (which equals $\text{Dr}(n)$ by [23, Theorem 4.5]). We show that $S_n/H$ is log canonical for $n = 4$, see Proposition [7.6]. For $n = 5$, there is a coarser fan $\Sigma'_5$ supported on $\text{Dr}(5)$; denote by $Y$ the closure of $S_5^\circ/H$ in $X(\Sigma'_5/L_\mathbb{R})$.

**Theorem 1.3.** The Chow quotient $S_5/H$ is smooth and has a simple normal crossings boundary. The log canonical model of $S_5/H$ is $Y$ and the refinement $\Sigma_5 \rightarrow \Sigma'_5$ induces a log crepant resolution $S_5/H \rightarrow Y$.

**Computations and data availability.** The software package gfan [14] was used to compute the tropicalization of $S_5^\circ$ in [7] and the $\Delta$-matroid subdivisions in Appendix [A] were computed using polymake [9]. Also, sage is used to verify Lemma [7.10]. No computation takes more than a few seconds on a standard desktop computer. The code may be found at the following website.

https://github.com/dcorey2814/tropicalSpinorVarieties

**Conventions.** Let $k$ be an algebraically closed field of characteristic 0. Let $\{n\} = \{0, 1, \ldots, n - 1\}$, $[n]^* = \{0^*, 1^*, \ldots, (n - 1)^*\}$, and $J = [n] \cup [n]^*$. Given a set $X$, let $\binom{X}{d}$ the set of all $d$-element subsets of $X$. We use juxtaposition of elements to denote small subsets of $J$, e.g., $ij = \{i, j\}$, $ijk = \{i, j, k\}$, etc. So, given $\lambda \subset [n]$, we write $\lambda \cup i = \lambda \cup \{i\}$, $\lambda \setminus i = \lambda \setminus \{i\}$, and $\lambda \Delta i = \lambda \Delta \{i\}$, etc.

## 2. The pure spinor variety

In this section, we recall the construction of the spinor variety following the conventions in [21, 23]. Let $V = k^{2n}$ and $Q$ the quadratic form from (1.1). An $n$-dimensional vector subspace $F \subset V$ is totally isotropic if $x^TQy = 0$ for all $x, y \in F$. The pure spinor variety is

$$S_n^\pm = \{F \subset V : F \text{ is an } n \text{ dimensional totally isotropic subspace of } V\}.$$ 

This space has two irreducible components $S_n$ and $S_n^\circ$, which may be distinguished in the following way. Given $\mu \subset [n]$, set

$$\bar{\mu} = \mu \cup \{i^* \in [n]^* : i \notin \mu\}.$$
Note that $|\bar{\mu}| = n$ for any $\mu$. Given any $d$-dimensional subspace $F$ of a $m$-dimensional vector space and $\lambda \in \binom{[n]}{d}$, let $p_\lambda(F)$ be the $\lambda$-th Plücker coordinate of $F$. The varieties $S_n$ and $S_n^-$ are

$$S_n = \{F \in S_n^\pm : p_\mu = 0 \text{ if } n - |\mu| \text{ is odd} \}, \quad S_n^- = \{F \in S_n^\pm : p_\mu = 0 \text{ if } n - |\mu| \text{ is even} \}.$$ 

Moreover, this characterization describes $S_n$ as a closed subvariety of the Grassmannian $\text{Gr}(n, 2n) \subset \mathbb{P}(\wedge^d V)$. Nonetheless, $S_n$ embeds into a smaller projective space, as we now describe.

Denote by $E(n)$ the set of subsets $\lambda$ of $[n]$ such that $n - |\lambda|$ is even. The Wick embedding realizes $S_n$ as a closed subvariety of $\mathbb{P}(k^{E(n)})$; we recall the definition of this map on the open dense affine chart

$$(2.1) \quad \mathcal{U} = \{ F \in S_n : p_{[\mu]}(F) \neq 0 \} .$$

the Wick embedding looks similar on other affine charts obtained by replacing $[n]$ by $\bar{\mu}$ for any $\mu \subset [n]$ (such a subset is called a transversal, see §5).

In terms of matrices, if $F \in \mathcal{U}$, then $F$ is the row span of the $n \times 2n$ matrix $[I_n | X]$ where $X$ is skew-symmetric. Given $\lambda \subset [n]$, let $X[\lambda]$ denote the skew-symmetric matrix obtained from the rows and columns of $X$ indexed by $\lambda$. Let $q_\lambda(F) = \text{Pf}(X[[n] \setminus \lambda])$ where Pf denotes the Pfaffian of a skew-symmetric matrix. Note that $q_\lambda(F) = 0$ if $n - |\lambda|$ is odd. The Wick embedding restricted to $\mathcal{U}$ is

$$\mathcal{U} \to \mathbb{P}(k^{E(n)}) \quad F \mapsto [q_\lambda(F) | \lambda \in E(n)].$$

Thus, $S_n$ is the closure of $\mathcal{U}$ in $\mathbb{P}(k^{E(n)})$.

Denote by $k[q_\lambda]$ the polynomial ring $k[q_\lambda | \lambda \in E(n)]$. The homogeneous ideal $I_n \subset k[q_\lambda]$ of $S_n$ is generated by the quadrics

$$(2.2) \quad P(\mu, \nu) = \sum_{i \in \nu \setminus \mu} (-1)^{|\mu| sgn(i; \mu, \nu) q_{\mu \cup i; \nu \setminus i} + \sum_{j \in \nu \setminus \mu} (-1)^{|\mu| sgn(j; \nu, \mu) q_{\mu \cup j; \nu \setminus j}}.$$ 

where $n - |\mu|$ and $n - |\nu|$ are odd and $|\mu \Delta \nu| \geq 4$; otherwise $P(\mu, \nu) = 0$. In particular, a monomial $q_\mu q_\lambda$ cannot appear in both sums in the above expression (this is be important in the proofs of Propositions 3.1 and 4.2). Here, $\text{sgn}(i; \mu, \nu)$ equals $(-1)^\ell$ where $\ell$ is the number of elements $j \in \nu$ with $i < j$ plus the number of elements $j' \in \mu$ such that $i > j'$.

We end this section by considering the first few examples. When $n = 1, 2, 3$, $S_n = \mathbb{P}(k^{E(n)})$. The first interesting case is $n = 4$, where

$$I_4 = \langle q_{00} q_{012} - q_{01} q_{23} + q_{02} q_{13} - q_{03} q_{12} \rangle \subset k[q_0, q_{01}, q_{02}, q_{12}, q_{03}, q_{13}, q_{23}, q_{0123}].$$

When $n = 5$, $I_5$ is the ideal of

$$k[q_0, q_1, q_2, q_3, q_4, q_{012}, q_{013}, q_{023}, q_{123}, q_{014}, q_{024}, q_{124}, q_{034}, q_{134}, q_{234}, q_{01234}]$$

generated by the quadrics

$$q_{01} q_{12} - q_{1} q_{02} + q_{2} q_{13} - q_{0} q_{23}, \quad q_{02} q_{12} - q_{1} q_{02} + q_{2} q_{13} - q_{0} q_{23},$$

$$q_{01} q_{13} - q_{1} q_{03} + q_{3} q_{12} - q_{0} q_{23}, \quad q_{02} q_{13} - q_{1} q_{03} + q_{3} q_{12} - q_{0} q_{23},$$

$$q_{01} q_{14} - q_{1} q_{04} + q_{4} q_{12} - q_{0} q_{23}, \quad q_{02} q_{14} - q_{1} q_{04} + q_{4} q_{12} - q_{0} q_{23},$$

$$q_{01} q_{12} q_{14} - q_{1} q_{02} q_{14} + q_{03} q_{14} - q_{0} q_{23} q_{14}, \quad q_{02} q_{12} q_{14} - q_{1} q_{02} q_{14} + q_{03} q_{14} - q_{0} q_{23} q_{14}.$$
3. Polytopes and strata of $\Delta$-matroids

A $\Delta$-matroid on $[n]$ is characterized by a nonempty collection $B(M)$ of subsets of $[n]$ satisfying the symmetric exchange axiom, i.e., for all $\mu, \nu \in B(M)$ and $i \in \mu \Delta \nu$ there is a $j \in \mu \Delta \nu$, such that $\mu \Delta ij \in B(M)$. Elements of $B(M)$ are called bases of $M$. A $\Delta$-matroid is even if $|\mu \Delta \nu|$ is even for all $\mu, \nu \in B(M)$.

3.1. $\Delta$-matroid polytopes. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, and given $\lambda = \{\lambda_1, \ldots, \lambda_k\}$, let $e_\lambda = e_{\lambda_1} + \cdots + e_{\lambda_k}$. Write $\langle u, v \rangle$ for the standard inner product of the vectors $u, v \in \mathbb{R}^n$. The polytope of a $\Delta$-matroid $M$ is

$$Q_M = \text{conv} \{e_\lambda : \lambda \in B(M)\}.$$ 

An (even) $\Delta$-matroid polytope is any polytope of the form $Q_M$ for some (even) $\Delta$-matroid $M$.

The faces of a polytope $Q \subset \mathbb{R}^n$ are all of the form

$$\text{face}_u Q = \{x \in Q : \langle u, x \rangle \leq \langle u, y \rangle \text{ for all } y \in Q\} \quad \text{where } u \in \mathbb{R}^n.$$ 

Because the status of being an even $\Delta$-matroid polytope is determined by its edges (a consequence of the Gelfand-Serganova theorem, see [23, Theorem 3.5] for a version adapted to $\Delta$-matroids), a face of an even $\Delta$-matroid polytope is an even $\Delta$-matroid polytope. Denote by $M_u$ the even $\Delta$-matroid such that $Q_{M_u} = \text{face}_u Q_M$; this is the analog of the initial matroid defined in [11]. The bases of $M_u$ are

$$B(M_u) = \{\lambda \in B(M) : \langle u, e_\lambda \rangle \leq \langle u, e_{\lambda'} \rangle \text{ for all } \lambda' \in B(M)\}.$$ 

3.2. $\Delta$-matroid strata. Given a totally isotropic subspace $F$ of $V$, its $\Delta$-matroid, denoted by $M(F)$, is defined by

$$B(M(F)) = \{\lambda \subset [n] : p_\lambda(F) \neq 0\}.$$ 

This $\Delta$-matroid is necessarily even. A $\Delta$-matroid is realizable if it is the $\Delta$-matroid of a totally isotropic subspace of some $V$. Let $S_M \subset S_\Delta^+$ be the locus of isotropic subspaces that realize $M$. We describe $S_M$ as a scheme in the following way. Define

- $B_M = k[q_\lambda : \lambda \in B(M)];$
- $I_M = (q_\lambda : \lambda \in E(n) \setminus B(M)) + I_n \cap B_M$;
- $S_M$ the multiplicative semigroup generated by $q_\lambda$ for $\lambda \in B(M)$.

Then $S_M = T_M \cap \text{Proj} (B_M/I_M)$ where $T_M$ is the dense torus of $\text{Proj} (B_M)$. Frequently it is easier to work with $\text{Spec} (R_M) \cong S_M \times G_m$, where $R_M = S_M^{-1} B_M / I_M$. The ideal $I_M$ is generated by the quadrics

$$P_M(\mu, \nu) = \sum_{i \in \nu \setminus \mu} (-1)^{\text{sgn}(i, \mu, \nu)} q_{|\mu \cup i \cap \nu|} + \sum_{j \in \mu \setminus \nu} (-1)^{\text{sgn}(j, \nu, \mu)} q_{|\mu \cap j \setminus \nu|},$$

such that $\mu \cup i, \mu \setminus j, \nu \setminus i, \nu \cup j$ are all bases of $M$; compare this with Equation (3.2).

Similar to the Grassmannian case, the face inclusion $Q_{M_u} \subset Q_M$ induces a morphism of strata $S_M \to S_{M_u}$, as we see in the following proposition.

**Proposition 3.1.** Suppose $M$ is a $k$-realizable even $\Delta$-matroid and $u \in \mathbb{R}^n$. Then the inclusion $B_{M_u} \subset B_M$ induces a morphism of strata $\varphi_{M, M_u} : S_M \to S_{M_u}$. These morphisms satisfy $\varphi_{M, M} = \text{id}_{S_M}$ and $\varphi_{M, (M_u)_c} = \varphi_{M_u, (M_u)_c} \circ \varphi_{M, M_u}$. 
Proof. It suffices to show that the extension of $I_M \subset B_M$ to $B_M$ is contained in $I_M$. Using the quadric generators from (3.3), we must show that $P_M(\mu, v) = 0$ or $P_M(\mu, v) = P_M(\mu, v)$. Suppose $P_M(\mu, v) \neq 0$. Then there is an $i_0 \in v \setminus \mu$ such that $\mu \cup i_0$ and $v \setminus i_0$ are bases of $M$, or there is a $j_0 \in \mu \setminus v$ such that $\mu \cup j_0$ and $v \cup j_0$ are bases of $M$. The two situations are symmetric, so we only consider the first one. By (3.1), we have

$$\langle u, e_{\mu \cup i_0} \rangle = \langle u, e_{v \setminus i_0} \rangle \leq \langle u, e_\lambda \rangle \text{ for all } \lambda \in B(M).$$

We must show that, for all $i \in v \setminus \mu$, (resp. $j \in \mu \setminus v$), $\mu \cup i$ and $v \setminus i$ are bases of $M$ if and only if they are bases of $M$ (resp. $\mu \cup j$ and $v \cup j$ are bases of $M$ if and only if they are bases of $M$). Because $B(M) \subset B(M)$, we need only show the “only if” directions. Suppose $\mu \cup i$ and $v \setminus i$ are bases of $M$. Then

$$\langle u, e_\mu \rangle + \langle u, e_i \rangle \leq \langle u, e_\mu \rangle + \langle u, e_i \rangle, \quad \text{and} \quad \langle u, e_\nu \rangle - \langle u, e_i \rangle \leq \langle u, e_\nu \rangle - \langle u, e_i \rangle,$$

so $\langle u, e_i \rangle = \langle u, e_i \rangle$. Therefore $\langle u, e_{\mu \cup i_0} \rangle = \langle u, e_{\mu \cup i} \rangle$ and $\langle u, e_{v \setminus i_0} \rangle = \langle u, e_{v \setminus i} \rangle$, that is, $\mu \cup i$ and $v \setminus i$ are bases of $M$. Now suppose $\mu \setminus j$ and $v \cup j$ are bases of $M$. Then

$$\langle u, e_\mu \rangle + \langle u, e_i \rangle \leq \langle u, e_\mu \rangle - \langle u, e_j \rangle, \quad \text{and} \quad \langle u, e_\nu \rangle - \langle u, e_i \rangle \leq \langle u, e_\nu \rangle + \langle u, e_j \rangle,$$

so $\langle u, e_j \rangle = -\langle u, e_i \rangle$. Therefore $\langle u, e_{\mu \cup i_0} \rangle = \langle u, e_{\mu \cup j} \rangle$ and $\langle u, e_{v \setminus i_0} \rangle = \langle u, e_{v \setminus j} \rangle$, that is, $\mu \setminus j$ and $v \cup j$ are bases of $M$, as required. That these morphisms satisfy the requisite functorial properties is clear.

3.3. Symmetries of $\Delta$-matroids. Given a set $X$, denote by $\mathcal{S}_X$ the symmetric group on $X$; if $X = [n]$, we write $\mathcal{S}_n := \mathcal{S}_n$. For $\tau \in \mathcal{S}_n$, let $s_\tau \in \mathcal{S}_E(n)$ be the permutation

$$s_\tau : E(n) \to E(n) \quad \lambda \mapsto \{\tau(i) : i \in \lambda\}.$$

If $\mu \subset [n]$ has an even number of elements, we have a permutation of $E(n)$ given by

$$t_\mu : E(n) \to E(n) \quad \lambda \mapsto \mu \Delta \lambda.$$

Let $G_n = \{t_\mu : \mu \subset [n], |\mu| \text{ even} \} \subset \mathcal{S}_E(n)$; this is a group isomorphic to $(\mathbb{S}_2)^{n-1}$. Following [6, VI.4.8], the type-$D_n$ Weyl group, denoted by $W(D_n)$, is

$$W(D_n) = \langle s_\tau, t_\mu : \tau \in \mathcal{S}_n, \mu \subset [n], |\mu| \text{ even} \rangle \leq \mathcal{S}_E(n).$$

The subgroup $G_n$ of $W(D_n)$ is normal since

$$s_\tau t_\mu s_\tau^{-1} = t_{s_\tau(\mu)}.$$

From this, we see that $W(D_n) \cong \mathcal{S}_n \rtimes (\mathbb{S}_2)^{n-1}$ and $|W(D_n)| = n!2^{n-1}$.

The action of $W(D_n)$ on $E(n)$ induces an action on the set of subsets of $E(n)$. Thus $W(D_n)$ acts on the set of even $\Delta$-matroids via its action on the bases sets. For example, given an even $\Delta$-matroid $M$ and an even-sized subset $\lambda \subset [n]$, the twist of $M$ by $\lambda$ is the even $\Delta$-matroid with bases $B(M \Delta \lambda) = \{\mu \Delta \lambda : \mu \in B(M)\}$; this is just the action of $t_\lambda \in W(D_n)$ on $B(M) \in \mathcal{P}(E(n))$.

4. Limits of Spinor Strata

4.1. The tropical spinor variety. We recall tropicalization of embedded varieties from the initial degenerations viewpoint; for details see [20, §§2.4-5]. Let $K$ be a field with valuation $\text{val}$, uniformizing parameter $t$, and residue field $k$ (e.g., $K = k((t^R)))$. The valuation plays a significant role in [5], the reader interested only in $\mathbb{S}_n$ may assume that the valuation is trivial. Let $\mathbb{P}^d = \text{Proj}(K[x_0, \ldots, x_d])$, $T \subset \mathbb{P}^d$ the dense torus, and $N$ the cocharacter lattice of $T$. Via the coordinates on $\mathbb{P}^d$, we identify $N$ with $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$. Let
$X \subset \mathbb{P}^d$ be a closed irreducible subvariety not contained in any coordinate hyperplane, $I \subset \mathbb{K}[x_0, \ldots, x_a]$ its homogeneous ideal, and $X^\circ = X \cap T$. If $z = (z_0, \ldots, z_a) \in \mathbb{Z}^{a+1}$, we write $x^z = x_0^{z_0} \cdots x_a^{z_a}$. Given $w \in N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, the $w$-initial form of $f \in \mathbb{K}[x_0, \ldots, x_a]$ is

$$in_w f = \sum_{\langle w, u \rangle + \text{val}(c_u) = W} i^{-\text{val}(c_u)}c_u x^u \in \mathbb{k}[x_0, \ldots, x_a]$$

where $f = \sum c_u x^u$ and $W = \min\{\langle w, u \rangle + \text{val}(c_u) : c_u \neq 0\}$. The initial ideal of $I$ is the homogeneous ideal

$$in_w I = \{in_w f : f \in I\} \subset \mathbb{k}[x_0, \ldots, x_a].$$

The tropicalization of $X^\circ$ is

$$\text{Trop} X^\circ = \{w \in N_{\mathbb{R}} : in_w I_0 \neq \langle 1 \rangle\}.$$

This set is the support of a rational polyhedral complex $\Sigma_{\text{Gr}}$, called the Gröbner complex, where $w, w'$ belong to the same relatively open cell of $\Sigma_{\text{Gr}}$ if and only if $in_w I = in_w I'$. If $\mathbb{K} = \mathbb{k}$, then $\Sigma_{\text{Gr}}$ is a polyhedral fan. The initial degeneration of $X^\circ$ (with respect to $w \in \text{Trop} X^\circ$) is the scheme

$$\text{in}_w X^\circ = T \cap \text{Proj}(\mathbb{k}[x_0, \ldots, x_a]/\text{in}_w I).$$

Frequently, it is easier to work with

$$\text{Spec}(\mathbb{k}[x_0^\pm, \ldots, x_a^\pm]/\text{in}_w I \cdot \mathbb{k}[x_0^\pm, \ldots, x_a^\pm]),$$

which is isomorphic to $\text{in}_w X^\circ \times \mathbb{G}_m$. We remark that $\text{in}_w X^\circ$ depends on the cone of $\Sigma_{\text{Gr}}$ that contains $w$ in its relative interior, but there may exist $w, w'$ belonging to different locally closed cones such that $\text{in}_w X^\circ = \text{in}_{w'} X^\circ$. When this happens, $\text{in}_w I$ and $\text{in}_{w'} I$ differ by primary components contained in $(x_0, \ldots, x_a)$.

We now specialize to the spinor variety. Let $n \geq 3$ and $S_n^\circ$ denote the intersection of $S_n$ with the dense torus in $\mathbb{P}(\mathbb{k}^{E(n)})$. Note that $S_n^\circ = S_M$ where $M$ is the uniform even $\Delta$-matroid, i.e., $B(M) = E(n)$. Viewing $S_M$ as a closed subvariety of the algebraic torus $\mathbb{G}_m^{B(M)}/\mathbb{G}_m$, we may form

$$\text{TS}_M := \text{Trop}(S_M), \quad \text{and} \quad \text{TS}_{n}^\circ := \text{Trop}(S_n^\circ).$$

Label the coordinates of $N_{\mathbb{R}} \cong \mathbb{R}^{E(n)}/\mathbb{R} \cdot (1, \ldots, 1)$ by $f_\lambda$ for $\lambda \in E(n)$. The tropicalization $\text{TS}_{n}^\circ$ is the support of a $(\binom{n}{2})$-dimensional polyhedral fan.

Consider the $L \cong \mathbb{Z}^n$-grading $\text{deg}_L(q_\lambda) = \sum_{i \in \lambda} e_i$. The following proposition is clear from the quadric generators of $I_n$ (2.2).

**Proposition 4.1.** The ideal $I_n$ is homogeneous with respect to the $L$-grading.

Therefore, $\text{TS}_{n}^\circ$ has a $n$-dimensional lineality space $L_{\mathbb{R}} \subset N_{\mathbb{R}}$ where

$$L = \left\langle \sum_{\lambda \ni i} f_\lambda, \sum_{\lambda \ni i} f_\lambda : 0 \leq i \leq n - 1 \right\rangle \subset N.$$

Note that $L$ is a saturated subgroup of $N$, so $N/L$ is torsion-free. Let us describe $\text{TS}_{n}^\circ$ for small values of $n$. To simplify our description of $\text{TS}_{n}^\circ$, we list the $W(D_n)$-orbits of the cones, where $W(D_n)$ acts on $\text{TS}_{n}^\circ$ by

$$s_\tau \cdot f_\lambda = f_{s_\tau \cdot \lambda}, \quad t_\mu \cdot f_\lambda = f_{t_\mu \cdot \lambda}.$$
When $n = 3$, $TS_3^o = L_R = N_R$. Next, consider the case $n = 4$. The saturated subgroup $L \subset N$ is spanned by

$$f_{01} + f_{02} + f_{03} + f_{0123}, \quad f_{01} + f_{02} + f_{013} + f_{023}, \quad f_{01} + f_{12} + f_{13} + f_{0123}, \quad f_{01} + f_{12} + f_{23} + f_{0123}, \quad f_{02} + f_{23} + f_{0123}, \quad f_{03} + f_{13} + f_{23} + f_{0123},$$

$$f_0 + f_{01} + f_{02}, \quad f_0 + f_{01} + f_{03} + f_{13}, \quad f_0 + f_{02} + f_{03} + f_{23}, \quad f_0 + f_{12} + f_{13} + f_{23}.$$ 

There are 4 rays, which have primitive vectors (modulo $L_R$)

$$r_0 = f_0 + f_{0123}, \quad r_1 = f_{01} + f_{23}, \quad r_2 = f_{02} + f_{13}, \quad r_3 = f_{03} + f_{12},$$

and 6 maximal cones, one corresponding to each pair of rays. Up to $W(D_4)$-symmetry, there is only one ray and one maximal cone. The space $TS_4^o$ is described in [7]. In summary, it has a 5 dimensional lineality space, and $f$-vector (resp. $W(D_5)$-symmetric $f$-vector):

$$f(TS_5^o) = (1, 36, 280, 960, 1540, 912), \quad \text{resp.} \quad f(TS_5^o \mod W(D_5)) = (1, 2, 3, 5, 5, 4).$$

### 4.2. Subdivisions of $\Delta$-matroid polytopes.

Given a polytope $Q \subset \mathbb{R}^n$ with vertices $v_0, \ldots, v_k$ and $w \in \mathbb{R}^{k+1}$, the lifted polytope is

$$Q^w = \text{conv} \{ (v_i, w_i) : 0 \leq i \leq k \} \subset \mathbb{R}^n \times \mathbb{R}.$$ 

Any lower face of $Q^w$ is of the form face$_u Q^w$ where $u = (u, 1) \in \mathbb{R}^n \times \mathbb{R}$. The lower faces of $Q^w$ project onto $Q$, forming a polyhedral complex whose support is $Q$. This is called the regular subdivision of $Q$ induced by $w$. The adjacency graph of this subdivision is the graph with vertex $v_Q$ for each maximal cell $Q_i$ and an edge between $v_{Q_i}$ and $v_{Q_j}$ whenever $Q_i$ and $Q_j$ share a common facet. The secondary fan $\Sigma_S(Q)$ of $Q$ is the complete fan in $\mathbb{R}^{k+1}$ where $w$ and $w'$ belong to the relative interior of the same cone if and only if they induce the same regular subdivision on $Q$ [10, §7.C].

Given an even $\Delta$-matroid $M$ and $w \in \mathbb{R}^{B(M)}$, we write $\Delta_{M,w}$ for the regular subdivision of $Q_M$ induced by $w$. This subdivision is matroidal, or $\Delta_{M,w}$ is a matroid subdivision, if each $Q \in \Delta_{M,w}$ is an even $\Delta$-matroid polytope. The $\Delta$-Dressian of $M$ is the subfan of $\Sigma_S(Q_M)$ defined by

$$\text{Dr}_M = \left\{ w \in \mathbb{R}^{B(M)} : \Delta_{M,w} \text{ is matroidal} \right\}.$$ 

When $B(M) = E(n)$, we write $Q_M = \Delta(n)$, $\text{Dr}_M = \text{Dr}(n)$, and $\Sigma_S(Q_M) = \Sigma_n$.

Suppose $\Delta_{M,w}$ is matroidal. Given $u \in \mathbb{R}^n$, let $M^w_u$ be the matroid of the polytope in $\Delta_{M,w}$ determined by face$_u(Q^w_M)$. The bases of $M^w_u$ are

$$B(M^w_u) = \left\{ \lambda \in B(M) : w_\lambda + \langle u, e_\lambda \rangle \leq w_{\lambda'} + \langle u, e_{\lambda'} \rangle \text{ for all } \lambda' \in B(M) \right\}.$$ 

If $w \in TS_M$, then $\Delta_{M,w}$ is matroidal [23, Theorem 5.4]. In fact, for $n \leq 5$, there is an equality of fans $TS_n^o = \text{Dr}(n)$, and for $n \geq 7$, $\text{Dr}(n)$ is strictly larger than $TS_n^o$ by [loc. cit. Theorem 4.5].

#### 4.3. Finite limits of strata.

We consider the following partial order of $\Delta$-matroids: $M' \leq M$ means that $Q_{M'}$ is a face of $Q_M$, and $M' \prec M$ means that $Q_{M'}$ is a facet of $Q_M$. We may view, for $w \in \text{Dr}_M$, the polyhedral complex $\Delta_{M,w}$ as a finite poset, and hence we may form the inverse limit

$$S_{M,w} := \lim_{Q \in \Delta_{M,w}} S_{MQ}.$$
If $B(M) = E(n)$, then we write $S_w$ for this limit. Finite inverse limits exist in the category of affine $k$-schemes because this category has a terminal object and pullbacks \cite[Proposition 5.21]{red}. In fact, since Spec is left-adjoint to the global sections functor, it takes direct limits to inverse limits, so the affine coordinate ring of $S_{M,w}$ is the direct limit of the affine coordinate rings of the $S_w$.

**Proposition 4.2.** For any $w \in TS_M$ and $u \in \mathbb{R}^n$, the inclusion $B_{M^w_u} \subset B_M$ induces a morphism $\psi_{M,M^w_u} : \text{in}_w S_M \to S_{M^w_u}$.

**Proof.** We must show that the extension of $I_{M^w_u}$ to $B_M$ is contained in $\text{in}_w I_M$. We use the quadric generators listed in Equation (3.3). We must show that $P_{M^w_u}(\mu, v) = 0$ or $P_{M^w_u}(\mu, v) = \text{in}_w P_M(\mu, v)$.

Suppose $P_{M^w_u}(\mu, v) \neq 0$. Then either there is an $i_0 \in v \setminus \mu$ such that $\mu \cup i_0$ and $\mu \setminus i_0$ are bases of $M^w_u$, or there is a $j_0 \in \mu \setminus v$ such that $\mu \setminus j_0$ and $\mu \cup j_0$ are bases of $M^w_u$. The two situations are symmetric, so it suffices to consider the first one. We must show that $q_{\mu \cup i}q_{v \setminus i}$ is a monomial of $\text{in}_w P_M(\mu, v)$ if and only if $\mu \cup i$ and $\mu \setminus i$ are bases of $M^w_u$ (resp. $q_{\mu \setminus j}q_{v \cup j}$ is a monomial of $\text{in}_w P_M(\mu, v)$ if and only if $\mu \setminus j$ and $\mu \cup j$ are bases of $M^w_u$).

Let $\psi_{\lambda} = w_{\lambda} + \langle u, e_{\lambda} \rangle$. Observe that, for any $i, j \in v \setminus \mu$,

\begin{equation}
\psi_{\mu \cup j} + \psi_{v \setminus i} - \psi_{\mu \cup i} - \psi_{v \setminus j} = w_{\mu \cup j} + w_{v \setminus i} - w_{\mu \cup i} - w_{v \setminus j}.
\end{equation}

Recall from the discussion after Equation (2.2) that, since $|\mu \Delta v| \geq 4$, a monomial $q_{\mu}q_{\lambda}$ cannot appear in both sums in Equation (3.3), i.e., we need not be concerned with cancellations between the two sums. Let $q_{\mu \cup i}q_{v \setminus i}$ be a monomial in $P_M(\mu, v)$. The term $q_{\mu \cup i}q_{v \setminus i}$ is a monomial of $\text{in}_w P_M(\mu, v)$ if and only if

\begin{align*}
w_{\mu \cup i} + w_{v \setminus i} &\leq w_{\mu \cup j} + w_{v \setminus j} \quad \text{and} \\
w_{\mu \cup i} + w_{v \setminus i} &\leq w_{\mu \cup j} + w_{v \cup j},
\end{align*}

for all $i', j'$, if and only if

\begin{align*}
v_{\mu \cup i} + v_{v \setminus i} &\leq v_{\mu \cup j} + v_{v \setminus j} \quad \text{and} \\
v_{\mu \cup i} + v_{v \setminus i} &\leq v_{\mu \cup j'} + v_{v \cup j'},
\end{align*}

for all $i', j'$, if and only if $v_{\mu \cup i} + v_{v \setminus i} = v_{\mu \cup i_0} + v_{v \setminus i_0}$, if and only if $\mu \cup i$ and $\mu \setminus i$ are bases of $M^w_u$. A similar argument shows that, if $q_{\mu \setminus j}q_{v \cup j}$ is a monomial of $P_M(\mu, v)$, then $q_{\mu \setminus j}q_{v \cup j}$ is a monomial of $\text{in}_w P_M(\mu, v)$ if and only if $\mu \setminus j$ and $v \cup j$ are bases of $M^w_u$. \hfill \Box

**Theorem 4.3.** For any $w \in TS_M$ the morphisms $\psi_{M,M^w_u}$ induce a closed immersion

$$\psi_{M,w} : \text{in}_w S_M \hookrightarrow S_{M,w}.$$  

**Proof.** Clearly $\varphi_{M^w_u,(M^w_u)_v} \circ \psi_{M,M^w_u} = \psi_{M,(M^w_u)_v}$, so $\psi_{M,w}$ is defined by the universal property of inverse limits. For each $q_{\lambda}$ for $\lambda \in B(M)$, there is a $Q \in \Delta_{M,w}$ such that $\lambda \in B(M_Q)$, in which case

$$\psi^{\#}_{M,w}(q_{\lambda}) = \psi^{\#}_{M,M_Q}(q_{\lambda}) = q_{\lambda}.$$  

So the morphism $\psi^{\#}_{M,w} : \lim_{Q \in \Delta_{M,w}} R_{M_Q} \to S^{-1}B_M / \text{in}_w I_M$ is surjective, and therefore $\psi_{M,w}$ is a closed immersion. \hfill \Box
5. SYMMETRIC MATROIDS AND INITIAL DEGENERATIONS OF TOTALLY ISOTROPIC SUBSPACES

In this section, we provide the geometric description of Theorem 4.3 as described in the introduction. This requires a notion of circuits for \( M \)-matroids, which are better understood in the language of symmetric matroids. The reader interested in the applications of Theorem 4.3 may skip to the next section.

Let \( J = [n] \cup [n]^* \). Define an involution \( J \to J \) by \( i \mapsto i^* \) and \( (i^*)^* = i \) for \( i \in [n] \). Given \( \lambda \subset J \), let \( \lambda^* = \{ i^* : i \in \lambda \} \). The subset \( \lambda \) is admissible if \( \lambda \cap \lambda^* = \emptyset \), and is a transversal if, additionally, \( |\lambda| = n \). A symmetric matroid \( M \) is determined by a nonempty set of transversals \( B(M) \) satisfying the symmetric exchange axiom: for every \( \mu, \nu \in B(M) \) and \( i \in \mu \Delta \nu \), there is a \( j \in \mu \Delta \nu \) such that \( \mu i i^* j j^* \in B(M) \) (compare this to the symmetric exchange axiom used to define \( \Delta \)-matroids). An element of \( B(M) \) is called a basis of \( M \).

Symmetric matroids go by many different names in the literature, see [3 §4] and the references therein. The data of a symmetric matroid on \( \Delta \) is determined by a nonempty set of transversals \( B(M) \) satisfying the symmetric exchange axiom: for every \( \mu, \nu \in B(M) \) and \( i \in \mu \Delta \nu \), there is a \( j \in \mu \Delta \nu \) such that \( \mu i i^* j j^* \in B(M) \) (compare this to the symmetric exchange axiom used to define \( \Delta \)-matroids). An element of \( B(M) \) is called a basis of \( M \).

Let \( M \) be a symmetric matroid on \( J \). A subset of \( J \) is independent if it is contained in a basis, and dependent otherwise. A circuit is an admissible minimal dependent subset, and the set of all circuits of \( M \) is denoted by \( C(M) \). We use a similar notation for ordinary matroids, i.e., if \( M \) is an ordinary matroid, then we denote by \( B(M) \) the bases of \( M \) and the circuits of \( M \) by \( C(M) \).

Let \( F \) be a totally isotropic subspace of \( V \). The symmetric matroid of \( F \) is \( \overline{M(F)} \) where \( M(F) \) is the \( \Delta \)-matroid of \( F \) defined in 3.2. Explicitly,

\[
B(\overline{M(F)}) = \{ \lambda \subset J : \lambda \text{ is a transversal and } p_\lambda(F) \neq 0 \}.
\]

The (ordinary) matroid of \( F \), denoted by \( M_\Delta(F) \), is given by

\[
B(M_\Delta(F)) = \{ \lambda \subset J : p_\lambda(F) \neq 0 \}.
\]

That is, the bases of \( \overline{M(F)} \) are just the bases of \( M_\Delta(F) \) that are admissible.

Given an \( m \times n \) matrix \( X \) and subsets \( \mu \subset [m], \nu \subset [n] \), denote by \( X[\mu, \nu] \) the submatrix of \( X \) whose rows are indexed by \( \mu \) and columns are indexed by \( \nu \). We abbreviate \( X[\mu, \mu] \) by \( X[\mu] \).

**Lemma 5.1.** Let \( X \) be a skew-symmetric \( n \times n \) matrix and let \( \lambda \) be a subset of \( [n] \) with \( k \) elements. Then, for distinct \( i, j \in [n] \), we have

\[
\det(X[\lambda \setminus i, \lambda \setminus j]) = \begin{cases} 
\pm \text{Pf}(X[\lambda \setminus i]) \text{Pf}(X[\lambda \setminus j]) & \text{if } k \text{ is odd,} \\
\pm \text{Pf}(X[\lambda]) \text{Pf}(X[\lambda \setminus ij]) & \text{if } k \text{ is even.}
\end{cases}
\]

Additionally, \( \det(X[\nu, \mu]) = \pm \det(X[\mu, \nu]) \).

**Proof.** This was originally proved by Cayley, see [18 p. 11] and the references therein. \( \square \)
Lemma 5.2. Let $X$ be a skew-symmetric $n \times n$ matrix and let $F$ be the totally isotropic subspace given by the row span of $W := [I]X$. Let $\tau \in B(M(F))$ such that $|[n] \setminus \tau|$ is even. Then, for $i \in \tau$ and $j \notin \tau$, we have
\[
\det(W[[n], \tau\Delta ij]) = \pm \Pf(X[[n] \setminus \tau]) \Pf(X[[n] \setminus (\tau\Delta ii^* jj^*)]).
\]

Remark 5.3. In terms of Plücker and Wick coordinates, this proposition says that
\[
p_{\tau\Delta ij}(F) = \pm q_{\tau \cap |n|} (F) q_{(\tau\Delta ii^* jj^*) \cap |n|}(F).
\]

Proof. First, observe that if $i = j^*$, then $\det(W[[n], \tau\Delta ij])$ and $\Pf(X[[n] \setminus (\tau\Delta ii^* jj^*)])$ equal 0. Therefore, we assume that $i \neq j^*$. For brevity, let $\nu = [n] \setminus \tau$. Before considering the four cases, depending on whether $i$ or $j$ lie in $|n|$ or $[n]^*$, we record some useful formulas:

1. $\det(W[[n], \sigma]) = \pm \det(X[[n] \setminus \sigma, [n] \cap \sigma^*])$ for any $\sigma \subset J$ of size $n$. 
2. $\tau \Delta ij = \nu \cup ( [n] \cap i i^* ) \setminus jj^*$ 
3. $\tau \Delta ij^* = \nu \cup ( [n] \cap jj^* ) \setminus ii^*$

If $i, j \in [n]$, then $i \notin \nu, j \in \nu$. The above formulas and Lemma 5.1 applied to $\lambda = \nu \cup i$ yield
\[
\det(W[[n], \tau\Delta ij]) = \pm \det(X[\nu \Delta ij, \nu]) = \pm \Pf(X[\nu]) \Pf(X[\nu \Delta ij]).
\]

If $i, j \in [n]^*$, then $i^* \in \nu, j^* \notin \nu$ and we have
\[
\det(W[[n], \tau\Delta ij]) = \pm \det(X[\nu \Delta i^* j^*]) = \pm \Pf(X[\nu]) \Pf(X[\nu \Delta i^* j^*]).
\]

If $i \in [n]$ and $j \in [n]^*$, then $i \notin \nu, j^* \notin \nu$ and we have
\[
\det(W[[n], \tau\Delta ij]) = \pm \det(X[\nu \cup i, \nu \cup j^*]) = \pm \Pf(X[\nu \cup j^*])
\]

If $i \in [n]^*$ and $j \in [n]$, then $i^* \in \nu, j \in \nu$ and we have
\[
\det(W[[n], \tau\Delta ij]) = \pm \det(X[\nu \setminus j, \nu \setminus i^*]) = \pm \Pf(X[\nu \setminus i^*])
\]

In each of these cases, we get the desired result.

If $\tau \in B(M_\Lambda(F))$ and $j \in J \setminus \tau$, then $\tau \cup j$ contains a unique circuit $\gamma(\tau, j)$ of $M_\Lambda(F)$ given by
\[
\gamma(\tau, j) = \{ i \in \tau : \tau\Delta ij \in B(M_\Lambda(F)) \} \cup \{ j \}
\]
Similarly, if $\tau \in B(M(F))$ and $j \in J \setminus \tau$, then $\tau \cup j$ contains a unique circuit $\overline{\gamma}(\tau, j)$ of $M(F)$ given by
\[
\overline{\gamma}(\tau, j) = \left\{ i \in \tau : \tau\Delta ii^* jj^* \in B(M(F)) \right\} \cup \{ j \}.
\]

Every circuit of $M_\Lambda(F)$, resp. $M(F)$, is of the form $\gamma(\tau, j)$, resp. $\overline{\gamma}(\tau, j)$.

Lemma 5.4. If $\tau \in B(M(F))$ and $j \in J \setminus \tau$, then
\[
\gamma(\tau, j) = \overline{\gamma}(\tau, j)
\]

Proof. Without loss of generality, assume that $|n| \in B(M(F))$. Note that $n - |\tau \cap |n||$ is even and $q_{|\tau \cap |n|}(F) \neq 0$. By Lemma 5.2, we have that $i \in \gamma(\tau, j) \setminus j$ if and only if $p_{\tau\Delta ij}(F) \neq 0$, if and only if $q_{(\tau\Delta ii^* jj^*) \cap |n|}(F) \neq 0$, if and only if $\tau\Delta ii^* jj^* \in B(M(F))$, if and only if $i \in \overline{\gamma}(\tau, j) \setminus j$. 

\[\square\]
The set $\gamma(\tau, j) = \mathcal{T}(\tau, j)$, is called the fundamental circuit of the pair $(\tau, j)$.

Fix a Plücker vector $p(F)$ and a Wick vector $q(F)$ for $F$; assume that these are compatible in the sense that $p_{\lambda}(F) = q_{\lambda \cap [n]}(F) = 1$ for some $\lambda \in B(M(F))$. View $F$ as a subscheme of $\text{Spec}(k[y_{i_j}, y_{j_i}])$ where $k[y_{i_j}, y_{j_i}] = k[y_{i_j}, y_{j_i} : i = 1, \ldots, n]$, and let $I_F$ be the ideal of $F$. For $\tau \in B(M_{\Lambda}(F))$ and $j \in J \setminus \tau$, there is, up to scaling, a unique linear form in $I_F$ with support $\gamma = \gamma(\tau, j)$. It is given by

$$
\ell_{\gamma}(F) = (-1)^{\sg(i_i, j)}p_{\tau}(F)y_j + \sum_{i_{\tau}}(-1)^{\sg(i_i, j)}p_{\tau \Delta i_j}(F) \cdot y_i.
$$

where $\sg$ is some sign function that is not important for us. Now suppose $\tau \in B(M(F))$ and $j \in J \setminus \tau$. Then $\ell_{\gamma}(F)$ is a scalar multiple of the form

$$
m_{\gamma}(F) = (-1)^{\sg'(i_i, j)}q_{\tau \cap [n]}(F)y_j + \sum_{i_{\tau}}(-1)^{\sg'(i_i, j)}q_{\tau \Delta i_j}(F) \cdot y_i.
$$

where $\sg'$ is some other sign function that is not important for us. Indeed, $\ell_{\gamma} = q_{\tau \cap [n]}m_{\gamma}$ by Lemma 5.2. Also note that $\text{supp}(\ell_{\gamma}(F)) = \text{supp}(m_{\gamma}(F)) = \gamma$ by (5.1), (5.2) and Lemma 5.4 where $\text{supp}$ denotes the support of a linear form.

**Proposition 5.5.** The ideal of $F$ is generated by these linear forms, i.e.,

$$I_F = \langle \ell_{\gamma}(F) : \gamma \in C(M_{\Lambda}(F)) \rangle = \langle m_{\gamma}(F) : \gamma \in C(M(F)) \rangle.$$

Moreover, \{ $\ell_{\gamma}(F) : \gamma \in C(M_{\Lambda}(F)) \}$ is a universal Gröbner basis for $I_F$.

**Proof.** The first equality and the last statement follows from [20, Lemma 4.1.4]. Now consider the last equality. Each $m_{\gamma}(F)$ lies in $I_F$ because $m_{\gamma}(F)$ is a multiple of $\ell_{\gamma}(F)$. Fix $\tau \in B(M(F))$. The set $\{ m_{\gamma(\tau, j)}(F) : j \notin \tau \}$ is a basis for the null space of $F$, and hence we get the second equality.

We end this section by making precise the geometric characterization of Theorem 4.3 presented in the introduction. Let $M$ be an even $\Delta$-matroid, $x$ be a $k$-point of $\text{in}_w S_M$, and $\mathbb{K} = k(t^R)$. By surjectivity of exploded tropicalization [22], there is a $\mathbb{K}$-point $q$ of $S_M$ such that $\text{Trop}(q) = x$. Let $F_q \subset \text{Spec}(\mathbb{K}[y_{i_j}, y_{j_i}])$ be the totally isotropic subspace with Wick vector $q$, and $F_q^\circ$ the intersection of $F_q$ with the dense torus $\text{Spec}(\mathbb{K}[y_{i_j}^\pm, y_{j_i}^\pm])$. By Proposition 5.6 below, if $(-u, u) \in \text{Trop}(F_q^\circ)$ then $\text{in}_{(-u, u)}F_q^\circ$ is totally isotropic, and a realization of $M_u^w$. Therefore, the map $\text{in}_w S_M \to S_{M_u^w}$ takes $x$ to (the Wick vector of) $\text{in}_{(-u, u)}F_q^\circ$.

**Proposition 5.6.** Suppose $(-u, u) \in \text{Trop}(F_q^\circ)$. The closure of $\text{in}_{(-u, u)}F_q^\circ$ in $k^{2n}$ is totally isotropic, and realizes the $\Delta$-matroid $M_u^w$.

**Proof.** By Proposition 4.2 the vector $\mathbf{x} = (x_{\mu} : \mu \in B(M_u^w)) \times 0 \in k^B(M_u^w) \times k^E\setminus B(M_u^w)$ lies in $S_{M_u^w}$. That is, $\mathbf{x}$ is the Wick vector of a totally isotropic subspace $F_{\mathbf{x}}$ realizing $S_{M_u^w}$. We claim that $\text{in}_{(-u, u)}F_q^\circ = F_{\mathbf{x}}$. Let $I_{\mathbf{x}}$ be the ideal of $F_{\mathbf{x}}$ in $k[y_{i_j}, y_{j_i}]$ and $I_{\mathbf{x}}$ the ideal of $F_{\mathbf{x}}$ in $k[y_{i_j}, y_{j_i}]$. The linear forms $\ell_{\gamma}(F_{\mathbf{x}})$ form a universal Gröbner basis for $I_{\mathbf{x}}$, so

$$\text{in}_{(-u, u)}I_{F_q} = \langle \text{in}_{(-u, u)}\ell_{\gamma}(F_q) : \gamma \in C(M_{\Lambda}(F_q)) \rangle \subset k[y_{i_j}, y_{j_i}].$$
Because $I_{\tau}$ and $\text{in}_{(-u,u)} I_{F_q}$ define $n$-dimensional linear subschemes of $\text{Spec}(k[y_i, y_j])$, it suffices to show $I_{\tau} \subset \text{in}_{(-u,u)} I_{F_q}$. Given $\tau \in B(M^w_u)$ and $j \in J \setminus \tau$, let $\gamma$ be the fundamental circuit of $(\tau, j)$ in $M$, and $\gamma'$ the fundamental circuit of $(\tau, j)$ in $M^w_u$. That is,

$$\gamma = \{i \in \tau : \tau \Delta i i^* j j^* \in B(M)\} \cup \{j\},$$

$$\gamma' = \{i \in \tau : \tau \Delta i i^* j j^* \in B(M) \text{ and } \langle u, e_{(\tau \Delta i i^* j j^*) \cap [n]} \rangle + w_{(\tau \Delta i i^* j j^*) \cap [n]} \text{ is minimal} \} \cup \{j\}.$$

We have

$$\text{in}_{(-u,u)} m_{\gamma}(F_q) = (-1)^{sg(i,j,\tau)} x_{\tau \cap [n]} y_j(F_q) + \sum_{i \in \gamma''} (-1)^{sg'(i,j,\tau)} x_{(\tau \Delta i i^* j j^*) \cap [n]}(F_q) \cdot y_i,$$

where

$$\gamma'' = \{i \in \tau \setminus j : \tau \Delta i i^* j j^* \in B(M) \text{ and } \langle (-u, u), e_i \rangle + w_{(\tau \Delta i i^* j j^*) \cap [n]} \text{ is minimal} \}.$$

By the equality

$$\langle u, e_{(\tau \Delta i i^* j j^*) \cap [n]} \rangle - \langle (-u, u), e_i \rangle = \langle u, e_{(\tau \Delta i i^* j j^*) \cap [n]} \rangle,$$

we see that $\text{supp}(\text{in}_{(-u,u)} m_{\gamma}(F_q)) = \gamma'' \cup j = \gamma'$, and therefore $\text{in}_{(-u,u)} m_{\gamma}(F_q) = m_{\gamma'}(F_q)$.

By Proposition 5.5 and because every circuit $\gamma'$ of $M^w_u$ is a fundamental circuit of some pair $(\tau, j)$, we see that $I_{\tau} \subset \text{in}_{(-u,u)} I_{F_q}$, as required. \qed

6. affine coordinates for strata

6.1. Affine coordinates. Throughout this subsection, we fix an even $\Delta$-matroid $M$ on $[n]$ that has $[n]$ as a basis. This means that $S_M$ is contained in open cell $U$ from (2.1), so every $F \in S_M(k)$ is the row span of a $k$-valued $n \times 2n$ matrix of the form $[I_n | X_F]$ where $X_F$ is skew-symmetric. Let $X$ be the skew-symmetric matrix of variables

$$X = \begin{bmatrix} 0 & x_{01} & x_{02} & \cdots & x_{0,n-1} \\ -x_{01} & 0 & x_{12} & \cdots & x_{1,n-1} \\ -x_{02} & -x_{12} & 0 & \cdots & x_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{0,n-1} & -x_{1,n-1} & -x_{2,n-1} & \cdots & 0 \end{bmatrix}.$$

As before, let $X[\lambda]$ denote the submatrix of $X$ whose rows and columns are indexed by $\lambda$, and $\text{Pf}(X[\lambda])$ the Pfaffian of $X[\lambda]$. Define

- $B_M^x = k[x_{ij} : [n] \setminus ij \in B(M)];$
- $I_M^x = \langle \text{Pf}(X[[n] \setminus \lambda]) : \lambda \in E(n) \setminus B(M) \rangle \cap B_M^x;$
- $S_M^x$ the multiplicative semigroup generated by $\{\pi(\text{Pf}(X[[n] \setminus \lambda])) : \lambda \in B(M)\}$ where $\pi : k[x_{ij}] \to k[x_{ij}]/(x_{ij} : [n] \setminus \{ij\} \in E(n) \setminus B(M)) \cong B_M^x$ is the quotient map.

The coordinate ring of $S_M$ is isomorphic to

$$R_M^x = (S_M^x)^{-1} B_M^x / I_M^x.$$

**Proposition 6.1.** Suppose $[n]$ is a basis of both $M$ and $M_u$. The induced morphism $\varphi_{M,M_u}^x : R_M^x \to R_M^x$ is given by $\varphi_{M,M_u}^x(x_{ij}) = x_{ij}$. 
Proof. Set
\[ \tilde{R}_M = S_M^{-1}k[q_\lambda/q_{[n]} \mid \lambda \in \mathcal{B}(M)]/I_M \]
The map \( q_\lambda/q_{[n]} \rightarrow Pf_\lambda(X) \) determines an isomorphism \( \theta_M : \tilde{R}_M \rightarrow R^*_M \). By Proposition 3.1, the morphism \( \varphi_{M,M_u} : S_M \rightarrow S_{M_u} \) induces the ring homomorphism \( \psi : \tilde{R}_{M_u} \rightarrow \tilde{R}_M \) given by \( \psi(q_\lambda/q_{[n]}) = q_\lambda/q_{[n]} \). The induced map \( \varphi^*_M \) is equal to \( \theta_{M_u} \circ \psi \circ \theta_M^{-1} \), which takes \( x_{ij} \) to \( x_{ij} \), as required.

We illustrate the affine coordinates construction with the following example, which we use in the proof of Theorem 6.12.

Example 6.2. Suppose \( K \) is the even \( \Delta \)-matroid on \([5]\) with
\[ \mathcal{B}(K) = \{0, 1, 012, 013, 014, 034, 134, 01234\} \]
Then
\[ B^x_K = k[x_{02}, x_{12}, x_{23}, x_{24}, x_{34}], \quad I^x_K = \langle 0 \rangle, \quad \text{and} \quad S^x_K = \langle x_{02}, x_{12}, x_{23}, x_{24}, x_{34} \rangle \text{semigp.} \]
and hence \( R^x_K = k[x_{02}^+, x_{12}^+, x_{23}^+, x_{24}^+, x_{34}^+] \). This shows that \( S^x_K \cong G^3_{10} \). The polytope \( Q_K \) has two facets not contained in \( \partial \mathcal{D}(S) \) which are defined by the vectors \( u = (1, 1, -1, 1, -1) \) and \( v = (1, 1, -1, -1, 1) \). The initial \( \Delta \)-matroids \( K_u \) and \( K_v \) have bases
\[ \mathcal{B}(K_u) = \{0, 1, 012, 013, 034, 134, 01234\} \quad \mathcal{B}(K_v) = \{0, 1, 012, 013, 034, 134, 01234\} \]
The coordinate rings of \( S_{K_u} \) and \( S_{K_v} \) are
\[ R^x_{K_u} = k[x_{02}^+, x_{12}^+, x_{23}^+, x_{24}^+, x_{34}^+], \quad R^x_{K_v} = k[x_{02}^+, x_{12}^+, x_{24}^+, x_{34}^+] \]
Therefore, \( S_{K_u} \) and \( S_{K_v} \) are isomorphic to \( G^4_{10} \). The morphisms
\[ \varphi_{K,K_u} : S_K \rightarrow S_{K_u} \quad \varphi_{K,K_v} : S_K \rightarrow S_{K_v} \]
may be identified with coordinate projections of tori; in particular, they are smooth and surjective with connected fibers.

Lemma 6.3. The morphism \( (\varphi_{K,K_u}, \varphi_{K_u,K_v}) : S_K \rightarrow S_{K_u} \times S_{K_v} \) is a closed immersion.

Proof. The induced map on coordinate rings is
\[ \varphi^*_K \otimes \varphi^*_K : R^x_{K_u} \otimes_k R^x_{K_v} \rightarrow R^x_K \]
which is surjective; see the explicit description of these maps above.

6.2. Inverse limits over a graph. In this subsection we show that, to compute the inverse limit \( S_{M,w} \) in (4.4), we do not need the full poset \( \Delta_{M,w} \) just those cells of codimension 0 and 1. In other words, we show that \( S_{M,w} \) may be computed as a limit over a diagram recorded by the adjacency graph \( \Gamma_{M,w} \). We begin by recalling this general construction, see [2] Appendix A for details.

Let \( C \) be a category that has finite limits; by [2] Proposition 5.21, it is necessary and sufficient that \( C \) has fiber products and a terminal object. Let \( G \) be a connected graph, possibly with loops or multiple edges. We view each edge as a pair of half-edges. Define a quiver \( Q(G) \) in the following way. The set of vertices of \( Q(G) \) is \( V(G) \cup E(G) \); write \( q_v \) (resp. \( q_e \)) for the vertex of \( Q(V) \) corresponding to the vertex \( v \) (resp. edge \( e \)). For each half-edge \( h \in e \) adjacent to \( v \), there is an arrow \( q_v \rightarrow q_e \). Viewing \( Q(G) \) as a category in the usual way, a diagram of type \( Q(G) \) in \( C \) is a functor \( X : Q(G) \rightarrow C \).
Let $\Gamma_{M,w}$ be the adjacency graph of a matroid subdivision $\Delta_{M,w}$. Let $M_v$, resp. $M_e$, denote the $\Delta$-matroid corresponding to the vertex $v$, resp. edge $e$, of $\Gamma_{M,w}$, and $\varphi_{M_v,M_e}$ : $S_{M_v} \to S_{M_e}$ whenever $e$ is incident to $v$. The data of $S_{M_v}, S_{M_e}$, and $\varphi_{M_v,M_e}$ defines a diagram of type $Q(\Gamma_{M,w})$ in $k$-sch.

Let $(\Delta_{M,w})^\text{top}$ be the set of top-dimensional cells of $\Delta_{M,w}$, and $A$ be a nonempty subset of $(\Delta_{M,w})^\text{top}$. We isolate some properties of $A$ that allows for different ways to study limits of spinor strata over full subgraphs $\Gamma_{M,w}[A] := \Gamma_{M,w}[\{v_Q : Q \in A\}]$ of the adjacency graph $\Gamma_{M,w}$ and their relation to initial degenerations of spinor varieties.

- The subset $A$ is basis-covering if
  \[
  \bigcup_{Q \in A} B(M_Q) = B(M).
  \]

- The subset $A$ is basis-intersecting
  \[
  \bigcap_{Q \in A} B(M_Q) \neq \emptyset.
  \]

- The subset $A$ is basis-connecting if, for each $\beta \in \bigcup_{Q \in A} B(M_Q)$, the induced subgraph $\Gamma_{M,w}[\{v_Q : Q \in A \text{ and } \beta \in B(M_Q)\}]$ is connected.

**Proposition 6.4.** For any even $\Delta$-matroid $M$ and $w \in \text{Dr}_M$, the set $(\Delta_{M,w})^\text{top}$ is basis-covering and basis-connecting. Moreover,

\[
S_{M,w} \cong \varprojlim \Gamma_{M,w}.
\]

**Proof.** The analog of this proposition for limits of thin Schubert cells in the Grassmannian follows from [7, Propositions C.11-12]. The proof in the spinor strata case is analogous. \hfill \Box

Next, we show how to compute the coordinate ring of inverse limits over the graphs $\Gamma_{M,w}$. First, in Wick coordinates, define

- $B = k[q_\lambda : \lambda \in \bigcup_{Q \in A} B(M_Q)]$;
- $I = \sum_{Q \in A} I_{M_Q} \cdot B$;
- $S$ is the multiplicative semigroup generated by $\{q_\lambda : Q \in A \text{ and } \lambda \in B(M_Q)\}$.

Set $R(A) = S^{-1}B/I$.

For affine coordinates, we must assume $A$ is basis-intersecting, say $\lambda$ lies in the intersection of all bases sets. After twisting each $M_Q$ by $[n] \setminus \lambda$, we may assume that $\lambda = [n]$. Define

- $B^x = k[x_{ij} : [n] \setminus ij \in \bigcup_{Q \in A} B(M_Q)]$;
- $I^x = \sum_{Q \in A} I_{M_Q}^x \cdot B^x$;
- $S^x$ is the multiplicative semigroup generated by

  \[
  \{\pi_Q(\text{Pf}(X[\lambda])) : Q \in A \text{ and } [n] \setminus \lambda \in B(M_Q)\}
  \]

where $\pi_Q$ is the composition

\[
k[x_{ij}] \to k[x_{ij}] / \langle x_{ij} : [n] \setminus ij \notin B(M_Q) \rangle \cong B^x_{M_Q} \subset B^x.
\]
Set $R^x(A) = (S^x)^{-1}B^x/I^x$.

**Proposition 6.5.** If $A$ is basis-connecting, then

$$\lim_{\Gamma_{M,w}[A]} R_{MQ} \cong R(A).$$

If, in addition, $[n] \in B(M_Q)$ for all $Q \in A$ (so $A$ is basis-intersecting), then

$$\lim_{\Gamma_{M,w}[A]} R^x_{MQ} \cong R^x(A).$$

**Proof.** We prove the second statement, the first one is similar, see also [7, Proposition 3.7]. Let $\hat{A}$ denote the set of all $Q \in \Delta_{M,w}$ such that either $v_Q \in V(\Gamma_{M,w}[A])$ or $e_Q \in E(\Gamma_{M,w}[A])$. For each $Q \in \hat{A}$, we have a ring homomorphism $R_{MQ}^x \rightarrow R^x(A)$ defined by $x_{ij} \mapsto x_{ij}$. These piece together to give a ring homomorphism $\Psi : \lim_{\Gamma_{M,w}[A]} R_{MQ}^x \rightarrow R^x(A)$. We define a ring homomorphism $\Theta$ that is an inverse to $\Psi$.

If $[n] \backslash ij \in B(M_Q)$ for some $Q \in A$ then set $\Theta(x_{ij}) = \phi_{MQ}^*(x_{ij})$. Suppose $Q' \in A$ is another cell such that $[n] \backslash ij \in B(M_{Q'})$. If $Q'' := Q \cap Q'$ lies in $\hat{A}$, then $\phi_{MQ}^*(x_{ij}) = \phi_{M_{Q''}}^*(x_{ij}) = \phi_{M_{Q'}}^*(x_{ij})$. Because $A$ is basis-connecting, there is a sequence $Q = Q_0, Q_1, \ldots, Q_k = Q'$ such that, for each $\ell$, we have $[n] \backslash ij \in B(M_{Q_\ell})$ and $Q_\ell \cap Q_{\ell+1}$ is a facet of $Q_\ell$ and $Q_{\ell+1}$. Then

$$\phi_{MQ}^*(x_{ij}) = \phi_{MQ_0}^* \cdots Q_1^* (x_{ij}) = \phi_{MQ_1}^* (x_{ij}) = \cdots = \phi_{MQ_k}^* (x_{ij})$$

so $\Theta : B^x \rightarrow \lim_{\Gamma_{M,w}[A]} R_{MQ}^x$ is well defined. Also, $\Theta(z)$ is invertible for any $z \in S^x$. Finally, we need to show that $I^x \subset \ker(\Theta)$. It suffices to show that $\Theta(zf) = 0$ for $z \in (S^x)^{-1}B^x$ and $f \in I_{MQ}^x$ for some $Q \in A$, which follows from the fact that $\Theta(af) = \Theta(a)\phi_{MQ}^*(f) = 0$. Therefore, $\Theta$ induces a ring homomorphism $\theta : R^x(A) \rightarrow \lim_{\Gamma_{M,w}[A]} R_{MQ}^x$, which is clearly an inverse to $\Psi$. \hfill \Box

Being basis-covering implies that the morphism $\lim_{\Gamma_{M,w}[A]} S_M \rightarrow \lim_{\Gamma_{M,w}[A]} S_{MQ}$ is a closed immersion.

**Proposition 6.6.** If $A \subset (\Delta_{M,w})_{\top}$ is basis-covering, then the morphisms $\lim_{\Gamma_{M,w}[A]} S_M \rightarrow S_{MQ}$ induce a closed immersion $\lim_{\Gamma_{M,w}[A]} S_M \rightarrow \lim_{\Gamma_{M,w}[A]} S_{MQ}$.

**Proof.** The morphism $\lim_{\Gamma_{M,w}[A]} S_M \rightarrow \lim_{\Gamma_{M,w}[A]} S_{MQ}$ is defined by the universal property of inverse limits, and the induced morphism on coordinate rings is surjective because every basis of $M$ is a basis of some $B(M_Q)$ for $Q \in A$. \hfill \Box

**Proposition 6.7.** If $A \subset (\Delta_{w})_{\top}$ is basis-covering, basis-intersecting, and basis-connecting, then $\lim_{\Gamma_{w}[A]} S_{MQ}$ is isomorphic to an open dense subvariety of $\mathbb{A}^{(2)}$. In particular, $\lim_{\Gamma_{w}[A]} S_{MQ}$ is isomorphic to a locally-closed subvariety of $\mathbb{A}^{(2)}$ by Proposition 6.5; in particular, the dimension of $\lim_{\Gamma_{w}[A]} S_{MQ}$ is at most $(n)_2$. Because $A$ is basis-covering, there is a closed immersion
in\(_w S_n^\circ \leftarrow \lim_{\Gamma_w[A]} S_{MQ}\) by Proposition 6.6, and the dimension of in\(_w S_n^\circ\) is \((\frac{n}{2})\) since this is a flat degeneration of \(S_n^\circ\). Therefore, the dimension of \(\lim_{\Gamma_w[A]} S_{MQ}\) is exactly \((\frac{n}{2})\), so \(\lim_{\Gamma_w[A]} S_{MQ}\) is an open subvariety of \(A(\frac{n}{2})\). The last statement follows from the fact that a closed immersion between integral affine schemes of the same dimension is an isomorphism, see [7, Proposition A.8]. □

Corollary 6.8. If \((\Delta_w)^{\text{top}}\) is basis-intersecting, then in\(_w S_n^\circ \cong S_w\). Furthermore, they are smooth and irreducible varieties of dimension \((\frac{n}{2})\).

Proof. By Proposition 6.4, \((\Delta_w)^{\text{top}}\) is already basis-covering and basis-connecting. The corollary now follows from Proposition 6.7. □

Example 6.9. Up to \(W(D_4)\)-symmetry, there are only 3 matroid subdivisions of \(\Delta(4)\). One is the trivial subdivision, and the adjacency graphs for the other two are recorded in Table 6.1. From this data, and Corollary 6.8, we conclude that in\(_w S_n^\circ\) is smooth and irreducible for all \(w \in TS_n^\circ\).

| \(w\) | Adjacency graph | Matroids |
|-------|----------------|---------|
| \(r_3\) | ![Adjacency graph](image) | \(M_0 : \{\emptyset, 01, 02, 03, 13, 23, 0123\}\), \(M_1 : \{\emptyset, 01, 02, 12, 13, 23, 0123\}\) |
| \(r_2 + r_3\) | ![Adjacency graph](image) | \(M_0 : \{\emptyset, 01, 12, 13, 23, 0123\}\), \(M_1 : \{\emptyset, 01, 02, 03, 23, 0123\}\), \(M_2 : \{\emptyset, 01, 02, 12, 23, 0123\}\), \(M_3 : \{\emptyset, 01, 03, 13, 23, 0123\}\) |

Table 6.1. Matroidal subdivisions of \(\Delta(4)\); the rays \(r_i\) are listed in (4.2)

Proposition 6.10. Suppose that there is a \(Q \in (\Delta_w)^{\text{top}}\) such that \(v_Q\) is adjacent to exactly 2 vertices \(v_{Q_1}, v_{Q_2}\) which are themselves adjacent, and \(A = (\Delta_w)^{\text{top}} \setminus \{Q\}\) is basis-covering. Then \(A\) is basis-connecting. If \(A\) is also basis-intersecting, then in\(_w S_n^\circ\) is smooth and irreducible.

Proof. First, we show that \(A\) is basis-connecting. Let \(\beta \in E(n)\) and

\[
H_\beta = \Gamma_w[\{v_{Q'} : Q' \in (\Delta_w)^{\text{top}} \text{ and } \beta \in B(M_{Q'})\}], \\
H'_\beta = \Gamma_w[\{v_{Q'} : Q' \in A \text{ and } \beta \in B(M_{Q'})\}].
\]

By Proposition 6.4, \(H_\beta\) is connected. If \(\beta \notin B(M_Q)\), then \(H'_\beta = H_\beta\) and hence connected. Suppose \(\beta \in B(M_Q)\). By hypothesis, there is an edge between \(v_{Q_1}\) and \(v_{Q_2}\). If \(v_{Q_1}, v_{Q_2} \in V(H'_\beta)\), then there is still an edge of \(H'_\beta\) between \(v_{Q_1}, v_{Q_2}\), and hence \(H'_\beta\) is connected. Otherwise, either \(v_{Q_1}\) or \(v_{Q_2}\) is in \(V(H'_\beta)\) since \(H_\beta\) is connected. This means that \(v_Q\) is a leaf-vertex of \(H_\beta\), so \(H'_\beta\) is connected. Therefore, \(A\) is basis-connected. The last statement now follows from Proposition 6.7. □
Lemma 6.11. Suppose we have a pullback diagram of finite-dimensional affine schemes

\[ W \times_Z X \longrightarrow X \]

\[ f' \downarrow \quad \downarrow f \]

\[ W \longrightarrow Z \]

where \( f : X \to Z \) is a closed immersion, \( W \) is irreducible, and \( W \times_Z X \) has the same dimension as \( W \). Then \( W \times_Z X \cong W \).

**Proof.** The morphism \( f' : W \times_Z X \to W \) is a closed immersion because closed immersions are preserved by arbitrary base change [12, Exercise II.3.11(a)], so \( f' \) is an isomorphism by [7] Proposition A.8].

**Theorem 6.12.** For \( n \leq 5 \), the initial degenerations of \( S_n^w \) are smooth and irreducible.

**Proof.** For \( n = 1, 2, 3 \), \( S_n = \mathbb{P}(k^{E(n)}) \), so the initial degenerations of \( S_n^w \) are clearly smooth and irreducible. The \( n = 4 \) case is handled in Example 6.9, but we can also see this directly because \( I_4 \) is a principal ideal. Up to \( W(D_4) \)-symmetry, there are only 3 cones of \( TS_4^w \). Representative weight vectors are

\[ w_0 = 0, \quad w_1 = f_{03} + f_{12}, \quad w_2 = f_{02} + f_{12} + f_{03} + f_{13}. \]

The initial ideals are

\[ \text{in}_{w_0} I_4 = \langle q_0 q_{0123} - q_0 q_{123} + q_0 q_{13} - q_0 q_{12} \rangle, \]

\[ \text{in}_{w_1} I_4 = \langle q_0 q_{0123} - q_0 q_{123} + q_0 q_{13} \rangle, \]

\[ \text{in}_{w_2} I_4 = \langle q_0 q_{0123} - q_0 q_{13} \rangle. \]

These define smooth and irreducible varieties in the dense torus of \( \mathbb{P}(k^{E(4)}) \cong \mathbb{P}^7 \).

Now consider \( n = 5 \). The subdivisions are listed in Appendix A. If \( \Delta_w \) is the subdivision 1, 2, 3, 5, 8, 11, 13, or 16, then \( \text{in}_{w} S_5 \) is smooth and irreducible by Proposition 6.4. Next, suppose \( \Delta_w \) is one of the subdivisions 0, 4, 6, 7, 9, 12, 14, 17, or 18. Observe that, for each of these subdivisions, \( \{0\} \) is a basis of \( M_Q \) for all but one \( Q \in (\Delta_w)^{\text{top}} \); the matroids missing \( \{0\} \) are \( M_0, M_4, M_5, M_4, M_3, M_9, M_4, M_{11}, M_4 \), respectively. For each of these subdivisions, \( v_{Q_0} \) has exactly two adjacent vertices, which are themselves adjacent. So \( \text{in}_{w} S_5 \) is smooth and irreducible by Proposition 6.10. The only subdivisions left are 10 and 15.

Consider subdivision 10 in Table A.4. Denote the \( i \)-th vertex by \( v_i \), its polytope by \( Q_i \), and its \( \Delta \)-matroid by \( M_i \). For the edge between \( v_i \) and \( v_j \), denote by \( Q_{ij} \) the corresponding polytope, and \( M_{ij} \) its \( \Delta \)-matroid. The set \( A = \{Q_0, Q_1, Q_2, Q_3, Q_4\} \) is basis-connecting. Also, we see that \( I_{M_i}^x = \langle 0 \rangle \) for \( i = 1, 3, 4 \) and \( I_{M_i}^x = \langle x_{02} x_{13} - x_{03} x_{12} \rangle \) for \( i = 0, 2 \). Therefore, the coordinate ring of \( \lim_{\leftarrow H} S_M \), where \( H = \Gamma_w[v_0, v_1, v_2, v_3, v_4] \), is isomorphic to

\[ R^x(A) = (S^x)^{-1}k[x_{ij}^\pm | 0 \leq i < j \leq 4] / (x_{02} x_{13} - x_{03} x_{12}) \]

where \( S^x \) is some finitely-generated multiplicative semigroup. Because we can solve for, e.g., \( x_{02} \), we see that \( \lim_{\leftarrow H} S_M \) is isomorphic to an open subvariety of \( G^*_n \). Therefore, \( \lim_{\leftarrow H} S_M \) is smooth and irreducible of dimension 9. Next, consider the pair \( (Q_5, Q_{25}) \).
Twist $M_5$ and $M_{25}$ by 1234 to get the $\Delta$-matroids $N$ and $N'$ with bases

$$B(N) = \{1, 2, 3, 012, 013, 123, 124, 134, 234, 01234\},$$

$$B(N') = \{1, 2, 3, 012, 013, 124, 134, 234, 01234\}.$$ 

The rings of $R_N^x$ and $R_{N'}^x$ are

$$R_N^x = (S_N^x)^{-1}k[x_{01}^\pm, x_{02}^\pm, x_{03}^\pm, x_{04}^\pm, x_{24}^\pm, x_{34}^\pm], \quad R_{N'}^x = (S_{N'}^x)^{-1}k[x_{01}^\pm, x_{02}^\pm, x_{03}^\pm, x_{24}^\pm, x_{34}^\pm]$$

and the map $\varphi^\#_{N,N'}: R_{N'}^x \to R_N^x$ is given by $x_{ij} \mapsto x_{ij}$, so $\varphi_{N,N'}: S_N \to S_{N'}$ is smooth and dominant with connected fibers. Set $H' = \Gamma_w[v_0, \ldots, v_5]$. By [7] Proposition A.5, we have an isomorphism

$$\mathrm{lim}_{\leftarrow H'} S_M \cong \mathrm{lim}_{\leftarrow H} S_M \times_{S_{N'}} S_N$$

which is smooth and irreducible [7] Proposition A.2.3 of dimension 10 [12] Proposition III.9.5. Next, observe that $M_6$ is equivalent to the $\Delta$-matroid $K$ from Example 5.2. Thus $\mathrm{lim}_{\leftarrow \Gamma_w} S_M$ fits into the pullback diagram

$$\begin{array}{ccc}
\mathrm{lim}_{\leftarrow \Gamma_w} S_M & \longrightarrow & S_M \\
\downarrow f & & \downarrow (\varphi_{M_6,M_06}, \varphi_{M_6,M_{56}}) \\
\mathrm{lim}_{\leftarrow H'} S_M & \longrightarrow & S_{M_06} \times S_{M_{56}}
\end{array}$$

The dimension of $\mathrm{lim}_{\leftarrow \Gamma_w} S_M$ is at least 10 by Theorem 4.3 and the fact that $\mathrm{in}_w S_5^0$ a flat degeneration of $S_5^x$, which is 10-dimensional. We conclude that $\mathrm{lim}_{\leftarrow \Gamma_w} S_M$ is smooth and irreducible of dimension 10 by Lemma 6.11 and therefore so is $\mathrm{in}_w S_5^0$.

Finally, we consider subdivision 15. We retain the notation convention for the vertices, polytopes, and $\Delta$-matroids. The set $A = \{Q_0, \ldots, Q_8\}$ is basis-connected, so we may compute its coordinate ring using Proposition 6.5. We see that $I^x_{M_i} = \langle 0 \rangle$ for $i = 0, 1, 2, 3, 5, 7, 8$ and $I^x_{M_i} = \langle x_{02}x_{13} - x_{03}x_{12} \rangle$ for $i = 4, 6$. By Proposition 6.5 the coordinate ring of $\mathrm{lim}_{\leftarrow H} S_{M_i}$, where $H = \Gamma_w[v_0, \ldots, v_8]$, is

$$R^x(A) \cong S^{-1}k[x_{ij}^\pm : 0 \leq i < j \leq 4]/(x_{02}x_{13} - x_{03}x_{12})$$

where $S$ is some finitely-generated multiplicative semigroup. As in the previous case, we deduce that $\mathrm{lim}_{\leftarrow H} S_{M_i}$ is isomorphic to an open subvariety of $G_{m_i}^\delta$, and therefore smooth and irreducible of dimension 9.

Next, consider the pair $(Q_9, Q_{69})$. Twist $M_9$ and $M_{69}$ by 1234 to get the $\Delta$-matroids $N$ and $N'$ with bases

$$B(N) = \{1, 2, 3, 012, 013, 123, 124, 134, 234, 01234\},$$

$$B(N') = \{1, 2, 3, 012, 013, 124, 134, 234, 01234\}.$$ 

The rings of $R_N^x$ and $R_{N'}^x$ are

$$R_N^x = k[x_{01}^\pm, x_{02}^\pm, x_{03}^\pm, x_{24}^\pm, x_{34}^\pm], \quad R_{N'}^x = k[x_{01}^\pm, x_{02}^\pm, x_{03}^\pm, x_{24}^\pm, x_{34}^\pm]$$

and the map $\varphi^\#_{N,N'}: R_{N'}^x \to R_N^x$ is given by $x_{ij} \mapsto x_{ij}$, so $\varphi_{N,N'}: S_N \to S_{N'}$ is smooth and dominant with connected fibers. Set $H' = \Gamma_w[v_0, \ldots, v_9]$. By [7] Proposition A.5, we have
an isomorphism

$$\lim_{H'} S_M \cong \lim_{H} S_M \times S_{N'} S_N$$

which is smooth and irreducible of dimension 10 (similar to the previous case). Finally, \(\Delta\)-matroids \(M_{10}\) and \(M_{11}\) are equivalent to the matroid \(K\) from Example 6.2 and by an argument similar to the previous case, we conclude that \(\text{in}_w S_5^o\) is smooth and irreducible. 

\[ \square \]

### 7. Log canonical model of \(S_5/H\)

#### 7.1. Schön compactifications

Throughout this section, we use the following notation for fans and toric varieties that is consistent with [13]. Given a finite rank lattice \(N\), let \(T_N\) be its torus, and given a torus \(T\) let \(N_T\) be its cocharacter lattice. If \(\tau\) is a rational polyhedral cone, let \(N_\tau\) be the saturated sublattice of \(N\) generated by \(\tau \cap N\), let \(N(\tau) = N/N_\tau\), and let \(\text{Star}(\tau)\) be the star of \(\tau\), viewed as a fan in \(N(\tau)_R\). If \(\Sigma\) is a rational polyhedral fan in \(N_R\), denote by \(|\Sigma|\) its support. If \(\Sigma\) is also pointed, denote by \(X(\Sigma)\) its toric variety.

Suppose \(Y^o\) is a closed subvariety of an algebraic torus \(T\). The closure \(Y\) of \(Y^o\) in a \(T\)-toric variety \(X(\Sigma)\) is a tropical, resp. schön, compactification if the multiplication map \(Y \times T \to X(\Sigma)\) is flat, resp. smooth, and surjective; in either case \(|\Sigma| = \text{Trop}(Y^o)\) [25]. The variety \(Y^o\) is schön if it admits a schön compactification, equivalently, if \(\text{in}_w Y^o\) is smooth for each \(w \in \text{Trop}(Y^o)\) [13, Proposition 3.9]. If \(Y^o\) is schön, then the closure of \(Y^o\) inside any toric variety \(X(\Sigma)\) with \(|\Sigma| = \text{Trop}(Y^o)\) is a schön compactification [19, Theorem 1.5].

A rational pointed polyhedral fan \(\Sigma\) in \(N_R\) is strictly simplicial if, for each cone \(\tau\) of \(\Sigma\), its rays can be extended to an integral basis of \(\Sigma\). The following proposition is well known to the experts, but for convenience we sketch a proof here.

**Proposition 7.1.** Suppose \(Y^o\) is schön and \(\Sigma\) is a strictly simplicial fan with support \(\text{Trop}(Y^o)\). Then the closure \(Y\) of \(Y^o\) in \(X(\Sigma)\) is smooth and its boundary \(B = Y \setminus Y^o\) is a simple normal crossings divisor.

**Proof.** As stated earlier, \(Y\) is a schön compactification of \(Y^o\), so the multiplication map \(Y \times T \to X(\Sigma)\) is smooth and surjective. Because \(\Sigma\) is strictly simplicial, the toric variety \(X(\Sigma)\) is smooth and its toric boundary is a simple normal crossings divisor. So \(Y \times T\) is smooth, and therefore so is \(Y\). Moreover, since the toric boundary of \(X(\Sigma)\) pulls back to \(B \times T\) under the smooth and surjective multiplication map, we have that \(B \times T\), and therefore \(B\), is a simple normal crossings divisor by [24, Lemma 0CBQ]. \[ \square \]

Let \(Y\) be a schön compactification of \(Y^o\) with ambient toric variety \(X(\Sigma)\), and set \(B = Y \setminus Y^o\). Then \(B\) is divisorial and \(Y\) has toroidal singularities [25, Theorem 1.4]. Given a cone \(\tau\) of \(\Sigma\), denote by \(Y_\tau\) the intersection of \(Y\) with the stratum of \(X(\Sigma)\) corresponding to \(\tau\). The following proposition gives a sufficient condition for \(K_Y + B\) to be ample, compare to [19, Theorem 4.9] and [7, Theorem 1.3].

**Proposition 7.2.** If \(\text{in}_w Y^o\) is smooth and irreducible for all \(w \in \text{Trop}(Y^o)\), and \(|\text{Star}(\tau)|\) is not preserved by a rational subspace of \(N(\tau)_R\) for all \(\tau \in \Sigma\), then \(K_Y + B\) is ample (where \(B = Y \setminus Y^o\)).
Lemma 7.5. Results in GIT. Note that $S$ is such, compare to [7, Lemma 7.1]. Let $\Sigma$ be the variation theory since, for each $S$ of this fan to Dr $[15]$, $P \Sigma$ set of the fan $\Sigma$. The Chow quotient of $n$ $T$ of $\Delta$. This is the cocharacter lattice of an $n$ $T$ and the scaling action of $T$ then $T$ and the scaling action of $T$ is not preserved under translation by a rational subspace of $N(\tau)_R$ [16, Lemma 5.2]. The proposition now follows from the fact that $\text{Trop}(Y_\tau) = |\text{Star}(\tau)|$ [20, Lemma 3.3.6].

We use the following lemma, which is essentially [7, Lemma 7.2], to determine whether $|\text{Star}(\tau)|$ is preserved under translation by a subspace.

Lemma 7.3. Suppose $\Sigma$ is a fan with lineality space $L_R$, and let $\tau \neq L_R$ be a nonmaximal cone of $\Sigma$. If there is a collection $A_\tau$ of maximal cones such that

\[(7.1) \bigcap_{\sigma \in A_\tau} (N_\tau)_R = (N_\tau)_R \]

then $|\text{Star}(\tau)|$ is not preserved under translation by any subspace of $N(\tau)_R$.

7.2. The Chow quotient of $S_n$. As before, let $N = Z^{E(n)} / Z \cdot (1, \ldots, 1)$ be the cocharacter lattice of the dense torus $T_N$ of $\text{Pic}(k^{E(n)})$, and let $L \leq N$ be the saturated subgroup from Equation (4.1). This is the cocharacter lattice of an $n$-dimensional subtorus $T_L \cong H$ of $T_N$, and the scaling action of $T_N \sim \text{Pic}(k^{E(n)})$ restricted to $S_n$ is the action $H \curvearrowright S_n$ from Equation (4.2). Therefore, there is an embedding of Chow quotients $S_n \sslash H \hookrightarrow \text{Pic}(k^{E(n)}) \sslash H$. By [15], $\text{Pic}(k^{E(n)}) \sslash H$ is the toric variety of the secondary fan of $\Delta(n)$. Denote by $\Sigma_n$ restriction of this fan to Dr $(n)$, which has $L_R$ in its lineality space. Thus Dr $(n) / L_R$ is the underlying set of the fan $\Sigma_n / L_R$, and because $T_{S_n} \subset \text{Dr}(n)$, the Chow quotient $S_n \sslash H$ is the closure of $S_n^o / H$ in $X(\Sigma_n / L_R)$.

Remark 7.4. Each point of the affine variety $S_n^o$ is $H$-stable in the sense of geometric invariant theory since, for each $x \in S_n^o$, the stabilizer is $\{\pm I_n\}$, which is finite, and the orbit $H \cdot x = (T_L \cdot x) \cap S_n^o$ is closed in $S_n^o$. Therefore, $S_n^o / H$ is a geometric quotient by standard results in GIT.

Lemma 7.5. For $n \leq 5$, the initial degenerations of $S_n^o / H$ are smooth and irreducible; in particular, $S_n^o / H$ is schön.

Note that $S_n^o$ is schön by Theorem 6.12

Proof. Compare to [7] Lemma 7.1. Let $w \in T \cap S$ and $\overline{w}$ the projection of $w$ to $N_R / L_R$. Then $\text{in}_w S_n^o \cong H \times \text{in}_{\overline{w}}(S_n^o / H)$. Therefore, for $n \leq 5$, the initial degenerations of $S_n^o / H$ are smooth and irreducible by Theorem 6.12. □

In particular, for $n \leq 5$, the Chow quotient $S_n \sslash H$ is a schön compactification of $S_n^o / H$. When $n = 1, 2, 3$, the spinor variety is $S_n = \text{Pic}(k^{E(n)})$ and $S_n \sslash H$ is a projective toric variety. The first interesting case is $n = 4$.

Proposition 7.6. The log canonical divisor $K_{S_4} \sslash H + B_4$ is ample, and therefore $S_4 \sslash H$ is log canonical.
The secondary fan of $T$ is described in §4.1. By Proposition 7.2 and Lemma 7.3, we need to show that $|\text{Star}(\tau)|$ is not invariant under translation by any rational subspace of $N(\tau)_{\mathbb{R}}$. It suffices to consider the nonmaximal cones $\tau \in \Sigma_4$. Up to $W(D_4)$-symmetry, there is only one cone, e.g., $\tau = \mathbb{R}_{\geq 0}(f_0 + f_{12}) + L_{\mathbb{R}}$. There are 3 maximal cones containing $\tau$:

$$\sigma_0 = \mathbb{R}_{\geq 0}(f_0 + f_{0123}, f_0 + f_{12}) + L_{\mathbb{R}}, \quad \sigma_1 = \mathbb{R}_{\geq 0}(f_0' + f_{23}, f_0 + f_{12}) + L_{\mathbb{R}},$$

$$\sigma_2 = \mathbb{R}_{\geq 0}(f_0 + f_{13}, f_0 + f_{12}) + L_{\mathbb{R}}.$$

Given $v \in N_{\mathbb{R}}$, denote by $v^*$ the linear functional given by $u \mapsto \langle u, v \rangle$. The subspace $(N_{\tau})_{\mathbb{R}}$ is the kernel of $v_i^*$ where

$$v_0 = f_0 - f_0 - f_1 + f_2, \quad v_1 = f_0 - f_0 - f_1 + f_0, \quad v_2 = f_0 - f_0 - f_2 + f_0.$$

Because $\text{span}\{v_0, v_1, v_2\}$ is 2-dimensional, the intersection in Equation (7.1) holds for $A_{\tau} = \{v_0, v_1, v_2\}$, and hence $|\text{Star}(\tau)|$ is not invariant under translation by any rational subspace of $N(\tau)_{\mathbb{R}}$ by Lemma 7.3.

The rest of this section is devoted to the $n = 5$ case. Recall that $T^0_S = Dr(5)$, and the Gröbner fan of $T^0_S$ equals $\Sigma_5$. Here is an explicit description of $\Sigma_5$, adapted from a gfan computation. By §4.1 the lineality space of $\Sigma_5$ is the 5-dimensional subspace $L_{\mathbb{R}}$ where $L \subset N$ is the saturated subgroup generated by

$$l_i = \sum_{\lambda \ni i} f_\lambda, \quad m_i = \sum_{\lambda \not\ni i} f_\lambda \quad \text{for} \quad 0 \leq i \leq 4.$$

This fan has 36 rays, which have primitive vectors (modulo $L_{\mathbb{R}}$):

$$r_0 = f_0, \quad r_1 = f_1, \quad r_2 = f_2, \quad r_3 = f_3, \quad r_4 = f_4, \quad r_5 = f_{012},$$

$$r_6 = f_0 + f_1 + f_2 + f_{012}, \quad r_7 = f_0 + f_1 + f_3 + f_{013}, \quad r_8 = f_0 + f_1 + f_4 + f_{014}, \quad r_9 = f_0 + f_1 + f_2 + f_{023}.$$

$$r_{10} = f_1 + f_2 + f_3 + f_{123}, \quad r_{11} = f_1 + f_2 + f_4 + f_{124}, \quad r_{12} = f_0 + f_3 + f_{034}, \quad r_{13} = f_0 + f_4 + f_{043}.$$

$$r_{14} = f_1 + f_3 + f_{134}, \quad r_{15} = f_2 + f_3 + f_{234}, \quad r_{16} = f_0 + f_3 + f_{013}.$$

Notice that, e.g., $r_{25}$ and $r_{35}$ belong to the same $W(D_5)$-orbit modulo $L$ because

$$t_{02} \cdot r_{25} = m_1.$$

The cones of $\Sigma_5$ partition into 20 $W(D_5)$-orbits, which we denote by $O_i$ for $-1 \leq i \leq 18$. Orbit representatives are listed in Table 7.1.

The space $T^0_S$ is supported on a slightly coarser fan $\Sigma'_5$, which we now describe.

**Lemma 7.7.** Each $\tau \in O_{11}$ has exactly two cones in its star, and these belong to the orbit $O_{18}$. Similarly, each $\sigma \in O_{18}$ has only one face in $O_{11}$. 


That is, the cones in $\mathcal{O}_{18}$ partition in pairs, one pair $(\sigma_\tau, \sigma'_\tau)$ for each $\tau \in \mathcal{O}_{11}$. For example, if $\tau = \mathbb{R}_+ (r_3, r_4, r_6, r_9) + L \mathbb{R}$, then
\[(7.2) \quad \sigma_\tau = \mathbb{R}_+ (r_3, r_4, r_6, r_9, r_{25}) + L \mathbb{R} \quad \sigma'_\tau = \mathbb{R}_+ (r_3, r_4, r_6, r_9, r_{26}) + L \mathbb{R}.
\]

**Proof.** For the first part, it suffices to show that the star of $\tau = \mathbb{R}_+ (r_3, r_4, r_6, r_9, r_{25}) + L \mathbb{R}$ consists of exactly $\sigma_\tau$ and $\sigma'_\tau$ from Equation (7.2). That Star$(\tau)$ contains no cones from the orbits $\mathcal{O}_{15}, \mathcal{O}_{16}, \mathcal{O}_{17}$ follows from the fact that, for
\[
S_{15} = \{ f_4, f_{2023}, f_{123}, f_{014} \}, \quad S_{16} = \{ f_3, f_4, f_{012}, f_{023}, f_{014} \}, \quad S_{17} = \{ f_3, f_4, f_{012}, f_{013} \},
\]
there is no even subset $\mu$ such that $t_\mu \cdot S_\mu$ contains some $\{ f_k, f_\ell, f_{ijk}, f_{ij\ell} \}$ for some distinct $i, j, k, \ell$, see §3.3. Now, the $W(D_5)$-stabilizer of $S_{18} = \{ f_3, f_4, f_{013}, f_{014} \}$ is
\[
\text{Stab}_{W(D_5)}(S_{18}) = \langle s_{01}, s_{34}, t_{01}, t_{34}, s_{03}(14)t_{03} \rangle.
\]
The $\text{Stab}_{W(D_5)}(S_{18})$-orbit of $r_{25} = f_2 + f_3 + f_4 + f_{234}$ modulo $L \mathbb{R}$ is $\{ r_{25}, r_{26} \}$, and therefore the only cones in the star of $\tau$ are $\sigma_\tau$ and $\sigma'_\tau$, as required.

For the second statement, it suffices to consider the cone $\sigma = \mathbb{R}_+ (r_3, r_4, r_6, r_9, r_{25}) + L \mathbb{R}$. Every facet of $\sigma$ other than $\mathbb{R}_+ (r_3, r_4, r_6, r_9) + L \mathbb{R}$ has one ray in the $W(D_5)$-orbit of $\mathbb{R}_+ r_{25} + L \mathbb{R}$, and therefore is not in $\mathcal{O}_{11}$. \hfill \Box

**Lemma 7.8.** For any $\tau \in \mathcal{O}_{11}$, we have that $\sigma_\tau \cup \sigma'_\tau$ is the convex polyhedral cone spanned by the rays of $\sigma_\tau$ and $\sigma'_\tau$. Furthermore, each face of this cone is contained in $\mathcal{O}_j$ for some $j \neq 11, 18$.

**Proof.** By symmetry, it suffices to show that
\[
\sigma_\tau \cup \sigma'_\tau = \mathbb{R}_+ (r_3, r_4, r_6, r_9, r_{25}, r_{26}) + L \mathbb{R}
\]
where $\tau = \mathbb{R}_+ (r_3, r_4, r_6, r_9) + L \mathbb{R}$. Because the cones $\sigma_\tau / L \mathbb{R}$ and $\sigma'_\tau / L \mathbb{R}$ are simplicial, we must show that
\[
(\mathbb{R}_+ (r_{25}, r_{26})) \cap \tau \neq L \mathbb{R}.
\]
Indeed, consider the vector $w = f_3 + f_4 + f_{013} + f_{014}$. We may express $w$ as
\[
w = r_3 + r_4 + r_6 + r_9 = r_{25} + r_{26} + \frac{1}{2}(\ell_0 + \ell_1 - \ell_2 - \ell_3 + m_4).
\]
This shows that $w$ lies in $(\mathbb{R}_+ (r_{25}, r_{26})) \cap \tau$ but not in $L \mathbb{R}$, as required. The last statement follows from Lemma [7.7]. \hfill \Box

The fan $\Sigma'_5$ is obtained by gluing $\sigma_\tau$ and $\sigma'_\tau$ along $\tau$, for each $\tau \in \mathcal{O}_{11}$. Explicitly, let $\mathcal{O}'_{18} = \{ \sigma_\tau \cup \sigma'_\tau : \tau \in \mathcal{O}_{11} \}$. Define $\Sigma'_5$ by
\[
\Sigma'_5 = \{ \tau : \tau \in \mathcal{O}_i \text{ for } i \neq 11, 18, \text{ or } \tau \in \mathcal{O}'_{18} \}.
\]

**Lemma 7.9.** The collection of cones $\Sigma'_5$ is a fan whose support is $T \Sigma^0_5$, and $\Sigma_5$ is a refinement of $\Sigma'_5$.

**Proof.** It is clear that $T \Sigma^0_5$ is the union of the cones in $\Sigma'_5$ and each cone of $\Sigma_5$ is contained in a cone of $\Sigma'_5$. Therefore, it suffices to show that $\Sigma'_5$ is a fan. That is, we must show that every face of a cone in $\Sigma'_5$ is in $\Sigma'_5$, and the intersection of any two cones in $\Sigma'_5$ is a face of both of them.

Suppose $\sigma \in \Sigma'_5$, and $\sigma'$ is a face of $\sigma$. If $\sigma \in \mathcal{O}_i$ for some $i \neq 11, 18$, then $\sigma'$ is in $\mathcal{O}_j$ for some $j \neq 11, 18$ by Lemma [7.7] and because $\Sigma_5$ is a fan; in particular $\sigma' \in \Sigma'_5$. If $\sigma \in \mathcal{O}'_{18}$ then $\sigma' \in \mathcal{O}_j$ for some $j \neq 11, 18$ by Lemma [7.8].
Suppose $\sigma_1, \sigma_2 \in \Sigma_5'$ and let $\sigma_3 = \sigma_1 \cap \sigma_2$. If $\sigma_1 \in \mathcal{O}_i$ and $\sigma_2 \in \mathcal{O}_j$ for some $i, j \neq 11, 18$, then $\sigma_3 \in \mathcal{O}_k$ for some $k \neq 11, 18$ because $\Sigma_5$ is a fan and Lemma 7.7. Now suppose $\sigma_2 \in \mathcal{O}_{18}'$, by symmetry it suffices to consider $\sigma_2 = \sigma_\tau \cup \sigma_\tau'$ from Equation (7.2). If $r_{25}, r_{26} \in \sigma_1$, then $\sigma_1$ meets the relative interior of $\sigma_\tau$ and $\sigma_\tau'$ by Lemma 7.8 and hence $\sigma_1 = \sigma_2 = \sigma_3$ because $\Sigma_5$ is a fan. If $r_{25} \in \sigma_1$ but $r_{26} \notin \sigma_1$, then $\sigma_3$ is a face of $\sigma_\tau$ not contained in $\tau$, and therefore is a face of $\sigma_1$ and $\sigma_2$. Finally, if that $r_{25}, r_{26} \notin \sigma_1$, then $\sigma_3$ is a face of $\sigma_1$ and $\tau$, but it cannot equal $\tau$ by Lemma 7.7. Therefore, $\sigma_3$ is also a face of $\sigma_2$, as required. 

Let $Y$ denote the closure of $S_5^\circ / H$ in the toric variety $X(\Sigma_5' / L_{\mathbb{R}})$.

**Lemma 7.10.** For each $\tau \in \Sigma_5' / L_{\mathbb{R}}$, the space $|\text{Star}(\tau)|$ is not preserved under translation by any rational subspace of $N(\tau)_{\mathbb{R}}$.

**Proof.** By Lemma 7.3 we must find a collection of maximal cones $A_\tau$ satisfying Equation (7.1). We need only check one cone, e.g., $\tau_\mathcal{O}$, from each non-maximal $W(D_5)$-orbit $\mathcal{O}_i$. Sets $A_\tau$ that satisfy (7.1) are listed in Table 7.1.

Equation (7.1) may be verified for $A_\tau$ in the following way. For each $\sigma \in A_\tau$, let $R_{\sigma\tau}$ be the matrix whose rows consist of the rays of $\sigma$ together with the vectors $\ell_0, \ldots, \ell_4, m_0, \ldots, m_4$ spanning the lineality space. Let $S_{\sigma\tau}$ be a matrix such that rowsp $R_{\sigma\tau} = \ker S_{\sigma\tau}$. Now let $S_{\tau}$ be the matrix whose (block) rows are the $S_{\sigma\tau}$. The kernel of $S_{\tau}$ is $\bigcap_{\sigma \in A_\tau} (N_\sigma)_{\mathbb{R}}$. Thus, to establish (7.1) for $A_\tau$, one must show that $\text{rank}(S_{\tau}) = \text{codim}_{\mathbb{R}^{E(5)}}(\tau)$, which is a routine verification.

| Rep. $\tau_i \in \mathcal{O}_i$ | Dim | $A_{\tau_i}$ |
|-------------------------------|-----|-------------|
| -1   | $\emptyset$   | $t_{23} \cdot \tau_{15}, s_{(13)} t_{03} \cdot \tau_{15}$ |
| 0    | $\{r_{26}\}$   | $s_{(12)} \cdot \tau_{15}, s_{(34)} \cdot \tau_{15}, s_{(12)(34)} \cdot \tau_{15}$ |
| 1    | $\{r_4\}$   | $s_{(142)} t_{14} \cdot \tau_{15}, s_{(024)} t_{03} \cdot \tau_{15}$ |
| 2    | $\{r_3, r_4\}$   | $s_{(021)} t_{34} \cdot \tau_{16}, s_{(02)(34)} \cdot \tau_{16}$ |
| 3    | $\{r_4, r_5\}$   | $s_{(032)(14)} t_{23} \cdot \tau_{15}, s_{(02)(13)} \cdot \tau_{15}$ |
| 4    | $\{r_4, r_{26}\}$   | $\tau_{15}, s_{(012)} \cdot \tau_{15}, s_{(12)} \cdot \tau_{15}$ |
| 5    | $\{r_4, r_5, r_9\}$   | $\tau_{16}, s_{(243)} t_{23} \cdot \tau_{17}$ |
| 6    | $\{r_4, r_9, r_{26}\}$   | $\tau_{15}, s_{(03)(142)} t_{02} \cdot \tau_{17}, s_{(03)(142)} t_{03} \cdot \tau_{17}$ |
| 7    | $\{r_4, r_6, r_{26}\}$   | $s_{(012)} \cdot \tau_{15}, s_{(12)} \cdot \tau_{15}$ |
| 8    | $\{r_3, r_4, r_5\}$   | $\tau_{16}, \tau_{17}$ |
| 9    | $\{r_3, r_4, r_{26}\}$   | $s_{(04132)} t_{03} \cdot \tau_{17}, s_{(032)(14)} t_{03} \cdot \tau_{17}$ |
| 10   | $\{r_4, r_7, r_8, r_9\}$   | $\tau_{15}, s_{(23)} \cdot \tau_{15}$ |
| 11   | $\{r_3, r_4, r_6, r_9\}$   | $\text{N/A}$ |
| 12   | $\{r_3, r_4, r_5, r_{25}\}$   | $\tau_{17}, s_{(34)} \cdot \tau_{17}$ |
| 13   | $\{r_3, r_4, r_5, r_6\}$   | $s_{(12)} \cdot \tau_{16}, s_{(01)(34)} \cdot \tau_{16}$ |
| 14   | $\{r_3, r_4, r_6, r_{26}\}$   | $s_{(03)(14)} t_{03} \cdot \tau_{17}, s_{(03)(14)} t_{03} \cdot \tau_{17}$ |
| 15   | $\{r_4, r_7, r_8, r_9, r_{26}\}$   | $\text{N/A}$ |
| 16   | $\{r_3, r_4, r_5, r_7, r_9\}$   | $\text{N/A}$ |
| 17   | $\{r_3, r_4, r_5, r_6, r_{25}\}$   | $\text{N/A}$ |
| 18   | $\{r_3, r_4, r_6, r_9, r_{26}\}$   | $\text{N/A}$ |

**Table 7.1.** Sets $A_{\tau_i}$ satisfying (7.1).
Proof of Theorem [1.3] By Lemma 7.5, the initial degenerations of $S_5^5/H$ are smooth and irreducible, so $S_5^5/H$ and $Y$ are schön compactifications of $S_5^5/H$. Each $Y_\tau$ for $\tau \in \Sigma_5^5$ is not preserved by a rational subspace of $N(\tau)_R$ by Lemma 7.10. Therefore, $K_Y + B$ is ample by Proposition 7.2, so $Y$ is the log canonical model of $S_5^5/H$.

Now consider the last statement. One readily verifies that the fan $\Sigma_5/L_R$ is strictly simplicial as defined in the beginning of this section. So $S_5^5/H$ is smooth with simple normal crossings boundary by Proposition 7.1. The morphism $S_5^5/H \to Y$ is log crepant by [25, Theorem 1.4]. □

Funding. This research was partially supported by NSF RTG Award DMS–1502553 and “Symbolic Tools in Mathematics and their Application” (TRR 195, project-ID 286237555).

Acknowledgments. The author would like to thank Michael Joswig and anonymous referees for their constructive feedback on earlier drafts.

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APPENDIX A. SUBDIVISIONS FOR \( n = 5 \)

In this appendix, we record the matroidal subdivisions of \( \Delta(5) \). Each table consists of the subdivisions corresponding to the cones of \( \text{Dr}(5) \) of the prescribed dimension. The leftmost column corresponds to the leftmost column of Table 7.1. The Adjacency graph column records the adjacency graph of the subdivision. Finally, in the Bases of the \( \Delta \)-matroids \( M_i \) column, we list the bases of the matroids \( M_i \), where \( M_i \) is the matroid corresponding to the vertex \( i \) in the adjacency graph.

### TABLE A.1. Dimension 6

| Index | Adjacency graph | Bases of the \( \Delta \)-matroids \( M_i \) |
|-------|------------------|------------------------------------------|
| 0     | ![Adjacency Graph](image) | \( M_0 : \{0, 1, 2, 3, 012, 123, 124, 134, 234, 101234\} \),  
\( M_1 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_2 : \{0, 1, 2, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_3 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_4 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_5 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \) |

### TABLE A.2. Dimension 7

| Index | Adjacency graph | Bases of the \( \Delta \)-matroids \( M_i \) |
|-------|------------------|------------------------------------------|
| 2     | ![Adjacency Graph](image) | \( M_0 : \{0, 1, 2, 3, 012, 123, 124, 134, 234, 101234\} \),  
\( M_1 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_2 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_3 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_4 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_5 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \) |
| 3     | ![Adjacency Graph](image) | \( M_0 : \{0, 1, 2, 3, 012, 123, 124, 134, 234, 101234\} \),  
\( M_1 : \{0, 1, 2, 3, 012, 123, 124, 134, 234, 101234\} \),  
\( M_2 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_3 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \) |
| 4     | ![Adjacency Graph](image) | \( M_0 : \{0, 1, 2, 3, 012, 123, 124, 134, 234, 101234\} \),  
\( M_1 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_2 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_3 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_4 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_5 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \),  
\( M_6 : \{0, 1, 2, 3, 4, 014, 024, 124, 134, 1234, 101234\} \) |
| Index | Adjacency graph | Bases of the $\Delta$-matroids $M_i$ |
|-------|----------------|------------------------------------|
| 5     | ![Adjacency graph](image) | $M_0: \{0, 1, 2, 3, 013, 023, 123, 024, 124, 134, 234, 01234\}$, $M_1: \{0, 1, 013, 014, 024, 124, 034, 134, 01234\}$, $M_2: \{0, 1, 012, 013, 014, 024, 124, 01234\}$, $M_3: \{0, 1, 2, 012, 013, 023, 123, 024, 124, 01234\}$, $M_4: \{0, 1, 2, 3, 4, 024, 124, 034, 134, 134, 01234\}$, $M_5: \{0, 1, 4, 014, 024, 124, 034, 134, 134\}$ |
| 6     | ![Adjacency graph](image) | $M_0: \{0, 1, 2, 3, 012, 012, 013, 023, 123, 023, 134, 234, 01234\}$, $M_1: \{0, 2, 012, 023, 024, 234, 01234\}$, $M_2: \{0, 1, 2, 012, 024, 124, 034, 134, 01234\}$, $M_3: \{0, 1, 012, 013, 014, 034, 134, 01234\}$, $M_4: \{0, 1, 012, 014, 024, 124, 034, 134, 01234\}$, $M_5: \{1, 2, 012, 123, 124, 134, 234, 01234\}$, $M_6: \{0, 1, 2, 3, 4, 034, 134, 234\}$, $M_7: \{0, 1, 2, 4, 024, 124, 034, 134, 234\}$, $M_8: \{0, 1, 4, 014, 024, 124, 034, 134, 134\}$ |
| 7     | ![Adjacency graph](image) | $M_0: \{0, 1, 2, 3, 012, 012, 013, 023, 123, 023, 134, 234, 01234\}$, $M_1: \{0, 1, 3, 012, 013, 023, 123, 034, 134, 01234\}$, $M_2: \{0, 1, 2, 012, 014, 024, 124, 034, 134, 01234\}$, $M_3: \{0, 1, 012, 013, 014, 034, 134, 01234\}$, $M_4: \{1, 2, 012, 123, 124, 134, 234, 01234\}$, $M_5: \{0, 2, 012, 023, 024, 234, 01234\}$, $M_6: \{0, 1, 2, 3, 4, 034, 134, 234\}$, $M_7: \{0, 1, 2, 4, 014, 024, 124, 034, 134, 234\}$, $M_8: \{0, 1, 4, 014, 024, 124, 034, 134, 134\}$ |
| 8     | ![Adjacency graph](image) | $M_0: \{0, 1, 2, 013, 023, 123, 014, 024, 124, 034, 134, 234, 01234\}$, $M_1: \{0, 1, 2, 012, 013, 023, 123, 014, 024, 124, 01234\}$, $M_2: \{0, 1, 2, 3, 4, 034, 134, 234\}$, $M_3: \{0, 1, 2, 4, 014, 024, 124, 034, 134, 234\}$, $M_4: \{0, 1, 2, 3, 013, 023, 123, 034, 134, 234\}$ |
| 9     | ![Adjacency graph](image) | $M_0: \{0, 1, 2, 012, 014, 024, 124, 034, 134, 234, 01234\}$, $M_1: \{0, 2, 012, 023, 024, 34, 01234\}$, $M_2: \{0, 1, 2, 012, 013, 023, 123, 034, 134, 234, 01234\}$, $M_3: \{1, 2, 012, 123, 124, 134, 234, 01234\}$, $M_4: \{0, 1, 012, 013, 014, 034, 134, 01234\}$, $M_5: \{0, 1, 2, 3, 4, 034, 134, 234\}$, $M_6: \{0, 1, 2, 4, 014, 024, 124, 034, 134, 234\}$, $M_7: \{0, 1, 2, 3, 013, 023, 123, 034, 134, 234\}$ |

**Table A.3.** Dimension 8
| Index | Adjacency graph | Bases of the Δ-matroids $M_i$ |
|-------|------------------|-------------------------------|
| 10    | ![Adjacency graph](image) | $M_0 : \{0, 1, 012, 013, 014, 024, 124, 034, 134, 01234\}$,  
      |      | $M_1 : \{2, 3, 012, 013, 023, 123, 234, 01234\}$,  
      |      | $M_2 : \{0, 1, 2, 3, 012, 013, 024, 124, 034, 134, 01234\}$,  
      |      | $M_3 : \{0, 2, 3, 012, 013, 023, 024, 034, 124, 134, 01234\}$,  
      |      | $M_4 : \{1, 2, 3, 012, 123, 124, 134, 134, 01234\}$,  
      |      | $M_5 : \{0, 1, 2, 3, 4, 024, 124, 034, 134, 01234\}$,  
      |      | $M_6 : \{0, 1, 2, 3, 4, 014, 024, 124, 034, 0134\}$ |
| 11    | ![Adjacency graph](image) | $M_0 : \{0, 1, 012, 023, 123, 024, 124, 034, 134, 1234, 01234\}$,  
      |      | $M_1 : \{0, 1, 012, 013, 023, 123, 034, 134, 01234\}$,  
      |      | $M_2 : \{0, 1, 012, 014, 024, 124, 034, 134, 01234\}$,  
      |      | $M_3 : \{0, 1, 012, 013, 014, 034, 134, 01234\}$,  
      |      | $M_4 : \{0, 1, 2, 3, 0, 043, 134, 234\}$,  
      |      | $M_5 : \{0, 1, 2, 044, 024, 124, 034, 134, 234\}$,  
      |      | $M_6 : \{0, 1, 2, 3, 023, 123, 034, 134, 234\}$,  
      |      | $M_7 : \{0, 1, 2, 014, 024, 124, 034, 134\}$,  
      |      | $M_8 : \{0, 1, 3, 013, 023, 123, 034, 134\}$ |
| 12    | ![Adjacency graph](image) | $M_0 : \{0, 1, 013, 023, 123, 014, 024, 124, 034, 134, 01234\}$,  
      |      | $M_1 : \{0, 1, 012, 013, 023, 123, 034, 134, 01234\}$,  
      |      | $M_2 : \{0, 1, 023, 123, 024, 124, 034, 134, 234\}$,  
      |      | $M_3 : \{0, 1, 2, 3, 043, 134, 234\}$,  
      |      | $M_4 : \{0, 1, 2, 4, 024, 124, 034, 134, 234\}$,  
      |      | $M_5 : \{0, 1, 2, 3, 023, 123, 034, 134, 234\}$,  
      |      | $M_6 : \{0, 1, 4, 014, 024, 124, 034, 134\}$,  
      |      | $M_7 : \{0, 1, 3, 013, 023, 123, 034, 134\}$,  
      |      | $M_8 : \{0, 1, 2, 012, 023, 123, 034, 124, 01234\}$,  
      |      | $M_9 : \{023, 123, 024, 124, 034, 134, 234, 01234\}$ |
| 13    | ![Adjacency graph](image) | $M_0 : \{0, 1, 2, 023, 123, 014, 024, 124, 034, 134, 234, 01234\}$,  
      |      | $M_1 : \{0, 1, 013, 023, 123, 014, 034, 134, 01234\}$,  
      |      | $M_2 : \{0, 1, 012, 013, 023, 123, 014, 024, 124, 01234\}$,  
      |      | $M_3 : \{0, 1, 2, 012, 023, 123, 014, 024, 124, 01234\}$,  
      |      | $M_4 : \{0, 1, 2, 3, 043, 134, 234\}$,  
      |      | $M_5 : \{0, 1, 2, 3, 023, 123, 034, 134, 234\}$,  
      |      | $M_6 : \{0, 1, 2, 014, 024, 124, 034, 134, 234\}$,  
      |      | $M_7 : \{0, 1, 3, 013, 023, 123, 034, 134\}$,  
      |      | $M_8 : \{0, 1, 2, 012, 023, 123, 034, 134, 01234\}$,  
      |      | $M_9 : \{0, 1, 3, 013, 023, 123, 034, 134\}$ |
| 14    | ![Adjacency graph](image) | $M_0 : \{0, 1, 012, 013, 023, 123, 034, 134, 01234\}$,  
      |      | $M_1 : \{0, 1, 2, 012, 014, 024, 124, 034, 134, 01234\}$,  
      |      | $M_2 : \{0, 1, 2, 012, 023, 123, 034, 134, 01234\}$,  
      |      | $M_3 : \{0, 1, 012, 013, 014, 034, 134, 01234\}$,  
      |      | $M_4 : \{1, 2, 012, 123, 124, 134, 234, 01234\}$,  
      |      | $M_5 : \{0, 2, 012, 023, 024, 034, 234, 01234\}$,  
      |      | $M_6 : \{0, 1, 2, 3, 043, 134, 234\}$,  
      |      | $M_7 : \{0, 1, 2, 3, 023, 123, 034, 134, 234\}$,  
      |      | $M_8 : \{0, 1, 2, 014, 024, 124, 034, 134, 234\}$,  
      |      | $M_9 : \{0, 1, 3, 013, 023, 123, 034, 134\}$ |

Table A.4. Dimension 9
| Index | Adjacency graph | Bases of the Δ-matroids $M_i$ |
|-------|----------------|-------------------------------|
| 15    | ![Adjacency graph](image) | $M_0 : \{2, 3, 012, 013, 023, 123, 234, 01234\}$,  
       |                       | $M_1 : \{0, 2, 3, 012, 013, 023, 034, 234, 01234\}$,  
       |                       | $M_2 : \{1, 2, 3, 012, 013, 123, 134, 01234\}$,  
       |                       | $M_3 : \{0, 1, 2, 013, 013, 034, 134, 01234\}$,  
       |                       | $M_4 : \{0, 1, 012, 014, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_5 : \{0, 2, 012, 023, 024, 234, 01234\}$,  
       |                       | $M_6 : \{0, 1, 012, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_7 : \{0, 1, 012, 013, 014, 034, 134, 01234\}$,  
       |                       | $M_8 : \{0, 1, 012, 123, 124, 034, 134, 01234\}$,  
       |                       | $M_9 : \{0, 1, 2, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_{10} : \{0, 1, 4, 014, 024, 124, 034, 134\}$,  
       |                       | $M_{11} : \{0, 1, 2, 3, 4, 034, 134, 234\}$ |
| 16    | ![Adjacency graph](image) | $M_0 : \{0, 1, 013, 023, 123, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_1 : \{0, 2, 013, 023, 123, 024, 034, 134, 01234\}$,  
       |                       | $M_2 : \{0, 1, 013, 014, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_3 : \{0, 1, 012, 013, 014, 024, 124, 01234\}$,  
       |                       | $M_4 : \{0, 1, 2, 013, 013, 034, 123, 024, 01234\}$,  
       |                       | $M_5 : \{0, 2, 012, 013, 023, 123, 024, 01234\}$,  
       |                       | $M_6 : \{0, 1, 2, 3, 4, 034, 134, 234\}$,  
       |                       | $M_7 : \{0, 1, 2, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_8 : \{0, 1, 4, 014, 024, 124, 034, 134\}$,  
       |                       | $M_9 : \{0, 2, 3, 013, 023, 123, 034, 234\}$,  
       |                       | $M_{10} : \{0, 1, 2, 3, 013, 123, 024, 134, 01234\}$ |
| 17    | ![Adjacency graph](image) | $M_0 : \{0, 1, 013, 023, 123, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_1 : \{0, 1, 023, 123, 014, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_2 : \{0, 1, 012, 023, 123, 014, 024, 124, 01234\}$,  
       |                       | $M_3 : \{0, 1, 012, 013, 023, 123, 014, 01234\}$,  
       |                       | $M_4 : \{0, 1, 012, 123, 023, 124, 034, 134, 01234\}$,  
       |                       | $M_5 : \{0, 1, 2, 023, 024, 124, 034, 134, 134, 01234\}$,  
       |                       | $M_6 : \{0, 1, 2, 3, 013, 023, 123, 034, 134, 01234\}$,  
       |                       | $M_7 : \{0, 1, 2, 3, 4, 034, 134, 134, 01234\}$,  
       |                       | $M_8 : \{0, 1, 4, 014, 024, 124, 034, 01234\}$,  
       |                       | $M_9 : \{0, 1, 2, 023, 123, 024, 124\}$,  
       |                       | $M_{10} : \{0, 1, 2, 3, 013, 123, 023, 134, 01234\}$ |
| 18    | ![Adjacency graph](image) | $M_0 : \{0, 2, 012, 023, 024, 034, 234, 01234\}$,  
       |                       | $M_1 : \{0, 1, 012, 014, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_2 : \{0, 1, 2, 012, 023, 034, 134, 01234\}$,  
       |                       | $M_3 : \{0, 1, 012, 013, 023, 123, 034, 134, 01234\}$,  
       |                       | $M_4 : \{1, 2, 012, 123, 124, 134, 01234\}$,  
       |                       | $M_5 : \{0, 1, 012, 013, 014, 034, 134, 01234\}$,  
       |                       | $M_6 : \{0, 1, 2, 012, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_7 : \{0, 1, 2, 4, 024, 124, 034, 134, 01234\}$,  
       |                       | $M_8 : \{0, 1, 2, 3, 023, 123, 034, 134, 01234\}$,  
       |                       | $M_9 : \{0, 1, 4, 014, 024, 124, 034, 134\}$,  
       |                       | $M_{10} : \{0, 1, 3, 013, 023, 123, 034, 134\}$,  
       |                       | $M_{11} : \{0, 1, 2, 3, 4, 034, 134, 234\}$ |

**Table A.5. Dimension 10**