HELMHOLTZ CONDITIONS AND SYMMETRIES FOR THE TIME DEPENDENT CASE OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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Abstract. We present a reformulation of the inverse problem of the calculus of variations for time dependent systems of second order ordinary differential equations using the Frölicher-Nijenhuis theory on the first jet bundle, $J^1\pi$. We prove that a system of time dependent SODE, identified with a semispray $S$, is Lagrangian if and only if a special class, $\Lambda^1_S(J^1\pi)$, of semi-basic 1-forms is not empty. We provide global Helmholtz conditions to characterize the class $\Lambda^1_S(J^1\pi)$ of semi-basic 1-forms. Each such class contains the Poincaré-Cartan 1-form of some Lagrangian function. We prove that if there exists a semi-basic 1-form in $\Lambda^1_S(J^1\pi)$, which is not a Poincaré-Cartan 1-form, then it determines a dual symmetry and a first integral of the given system of SODE.

1. Introduction

In this work we present a reformulation of the inverse problem for time dependent systems of second order ordinary differential equations in terms of semi-basic 1-forms. In this approach we solely make use of the Frölicher-Nijenhuis theory on $\pi_{10}: J^1\pi \rightarrow M$, the first jet bundle of an $(n+1)$-dimensional, real, smooth manifold $M$, which is fibred over $\mathbb{R}$, $\pi: M \rightarrow \mathbb{R}$. We characterize when the time dependent system of SODE

$$\frac{d^2x^i}{dt^2} + 2G^i\left(t, x, \frac{dx}{dt}\right) = 0$$

is equivalent to the system of Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{dt},$$

for some smooth Lagrangian function $L$ on $J^1\pi$, in terms of a class $\Lambda^1_S(J^1\pi)$ of semi-basic 1-forms on $J^1\pi$. This work is a natural extension of the time independent case studied in [4]. However, the time dependent framework has particular aspects that result into some differences of this approach from the one studied in [4]. Lagrangian systems of time independent differential equations are always conservative and this is not the case in the time dependent framework. In this approach we use the formalism developed for the inverse problem of the calculus of variations to search for symmetries as well. In other words, this approach gives the possibility...
of searching for dual symmetries for the time dependent system (1.1) of SODE in the considered class $\Lambda^1_S(J^1\pi)$ of semi-basic 1-forms.

Necessary and sufficient conditions under which the two systems (1.1) and (1.2) can be identified using a multiplier matrix are usually known as the Helmholtz conditions. This inverse problem was entirely solved only for the case $n=1$ by Darboux, in 1894, and for $n=2$ by Douglas, in 1941 [13]. A geometric reformulation of Douglas approach, using linear connections arising from a system of SODE and its associated geometric structures, can be found in [11, 33]. The relation between the inverse problem of the calculus of variations and the condition of self-adjointness for the equations of variation of the system (1.1) was studied by Davis in 1929, [12]. In 1935, Kosambi [19], has obtained necessary and sufficient conditions, that were called latter Helmholtz conditions, for the equations of variations of the system (1.1) to be self-adjoint. For various approaches to derive the Helmholtz conditions in both autonomous and nonautonomous case, we refer to Santilli [30], Crampin [8], Henneaux [10], Sarlet [31], Marmo et al. [24], Morandi et al. [28], Anderson and Thompson [1], Krupková and Prince [21]. See also [4] for a reformulation of the Helmholtz conditions in terms of semi-basic 1-forms in the time independent case.

In this paper we study the inverse problem of the calculus of variations when the time dependent system (1.1) of SODE is identified with a semispray $S$ on the first jet bundle $J^1\pi$. We seek for a solution of the inverse problem of the calculus of variations in terms of semi-basic 1-forms on $J^1\pi$. We first show, in Theorem 4.5, that a semispray $S$ is a Lagrangian vector field if and only if there exists a class of semi-basic 1-forms $\theta$ on $J^1\pi$ such that their Lie derivatives $L_S\theta$ are closed 1-forms. We denote this class by $\Lambda^1_S(J^1\pi)$. These results reformulate, in terms of a semi-basic 1-form, the results expressed in terms of a 2-form, obtained in [1, 3, 8, 10, 24]. In Proposition 4.6 we strengthen the results of Theorem 4.5 and prove that, if nonempty, the set of semi-basic 1-forms $\Lambda^1_S(J^1\pi)$ always contains the Poincaré-Cartan 1-form of some Lagrangian function $L$. Moreover, we show that any semi-basic 1-form $\theta \in \Lambda^1_S(J^1\pi)$, which is not the Poincaré-Cartan 1-form of some Lagrangian function, determines a first integral and a dual symmetry of the Lagrangian system.

In Theorems 5.1 and 5.2 we characterize the semi-basic 1-forms of the set $\Lambda^1_S(J^1\pi)$, depending if they represent or not Poincaré-Cartan 1-forms. First, we pay attention to $d_J$-closed, semi-basic 1-forms $\theta$, where $J$ is the vertical endomorphism. This class of semi-basic 1-forms $\theta$ coincides with the class of Poincaré-Cartan 1-forms corresponding to the Lagrangian function $L = i_S\theta$, see Lemma 4.2. For this class we prove, in Theorem 5.1, that the inverse problem has a solution if and only if the Poincaré-Cartan 1-form is $d_h$-closed, where $h$ is the horizontal projector induced by the semispray.

In Theorem 5.2 we formulate a coordinate free version of the Helmholtz conditions in terms of semi-basic 1-forms on the first jet bundle $J^1\pi$. If there exists a semi-basic 1-form $\theta$ that satisfies the Helmholtz conditions, then $\theta$ is equivalent (modulo $dt$) with the Poincaré-Cartan 1-form of some Lagrangian function. Moreover, if such $\theta$ is not $d_J$-closed then $i_Sd\theta$ is a dual-symmetry and induces a first integral for the semispray $S$. This theorem gives a characterization for conservative, Lagrangian, time dependent vector fields. To derive the Helmholtz conditions in Theorem 5.2 we make use of the Frölicher-Nijenhuis theory on $J^1\pi$ developed in
Section 2.2 as well as of the geometric objects induced by a semispray that are presented in Section 3.

Since semi-basic 1-forms play a key role in our work, we pay a special attention to this topic in Section 2.3. In this section we prove a Poincaré-type Lemma for the differential operator $d_J$ restricted to the class of semi-basic 1-forms on $J^1\pi$. The Poincaré-type Lemma will be very useful to characterize those semi-basic 1 forms that are the Poicaré-Cartan 1-forms of some Lagrangian functions.

An important tool in this work is discussed in Section 3.2, it is a tensor derivation along the bundle projection $\pi$. See also [5, 9, 17, 26, 34]. It can be defined also as a derivation on the total space $TJ^1\pi$, see [25, 29].

2. Preliminaries

2.1. The first jet bundle $J^1\pi$. For a geometric study of time dependent systems of SODE the most suitable framework is the affine jet bundle $(J^1\pi, \pi_{10}, M)$, see [9, 20, 25, 33]. We consider an $(n+1)$-dimensional, real and smooth manifold $M$, which is fibred over $\mathbb{R}$, $\pi : M \to \mathbb{R}$. The first jet bundle of $\pi$ is denoted by $\pi_{10} : J^1\pi \to M$, $\pi_{10}(j^1\pi) = \gamma(t)$, for $\gamma$ a local section of $\pi$ and $j^1\pi$ the first jet of $\gamma$ at $t$. A local coordinate system $(t, x^i)_{i\in\overline{1,n}}$ on $M$, where $t$ represents the global coordinate on $\mathbb{R}$ and $(x^i)$ the $\pi$-fiber coordinates, induces a local coordinate system on $J^1\pi$, denoted by $(t, x^i, y^i)$. Submersion $\pi_{10}$ induces a natural foliation on $J^1\pi$. Coordinates $(t, x^i)$ are transverse coordinates for this foliation, while $(y^i)$ are coordinates for the leaves of the foliation.

Throughout the paper we assume that all objects are $C^\infty$-smooth where defined. The ring of smooth functions on $J^1\pi$, the $C^\infty$ module of vector fields on $J^1\pi$ and the $C^\infty$ module of $k$-forms are respectively denoted by $C^\infty(J^1\pi)$, $\mathfrak{X}(J^1\pi)$ and $\Lambda^k(J^1\pi)$. The $C^\infty$ module of $(r,s)$-type tensor fields on $J^1\pi$ is denoted by $\mathcal{T}_r^s(J^1\pi)$ and $T(J^1\pi)$ denotes the tensor algebra on $J^1\pi$. By a vector valued l-form ($l \geq 0$) on $J^1\pi$ we mean an $(1,l)$-type tensor field on $J^1\pi$ that is skew-symmetric in its $l$ arguments.

A parameterized curve on $M$ is a section of $\pi$: $\gamma : \mathbb{R} \to M$, $\gamma(t) = (t,x^i(t))$. Its first jet prolongation $J^1\gamma : t \in \mathbb{R} \to J^1\gamma(t) = (t,x^i(t),dx^i/dt) \in J^1\pi$ is a section of the fibration $\pi_1 := \pi \circ \pi_{10} : J^1\pi \to \mathbb{R}$.

Let $VJ^1\pi$ be the vertical subbundle of $TJ^1\pi$, $VJ^1\pi = \{\xi \in TJ^1\pi, D\pi_{10}(\xi) = 0\} \subset TJ^1\pi$. Its fibers, $V_uJ^1\pi = \ker D_u\pi_{10}$, $u \in J^1\pi$, determine a regular, $n$-dimensional vertical distribution. The vertical distribution is integrable, being tangent to the natural foliation. Moreover, $VJ^1\pi = \text{spann}\{\partial/\partial y^i\}$ and its annihilators are the contact 1-forms

$$\delta x^i = dx^i - y^i dt, \quad i \in \{1, ..., n\}. \quad (2.1)$$

One can also view the vertical subbundle as the image of the vertical endomorphism (or tangent structure)

$$J = \frac{\partial}{\partial y^i} \otimes \delta x^i. \quad (2.2)$$
The vertical endomorphism \( J \) is a \((1, 1)\)-type tensor field, and hence a vector valued 1-form on \( J^1 \pi \), with \( \text{Im} \ J = VJ^1 \pi, \ VJ^1 \pi \subset \text{Ker} \ J \) and \( J^2 = 0 \).

A system \([1.1]\) of second order ordinary differential equations, whose coefficients depend explicitly on time, can be identified with a special vector field on \( J^1 \pi \), called a semispray. A semispray is a globally defined vector field \( S \) on \( J^1 \pi \) such that

\[
J(S) = 0 \quad \text{and} \quad dt(S) = 1.
\]

Therefore, a semispray is a vector field \( S \) on \( J^1 \pi \) characterized by the property that its integral curves are the first jet prolongations of sections of \( \pi_1 : J^1 \pi \to \mathbb{R} \).

Locally, a semispray has the form

\[
S = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} - 2G^i(t, x, y) \frac{\partial}{\partial y^i},
\]

where parameterized curves \( \gamma: I \to M \) is called a geodesic of \( S \) if \( S \circ J^1 \gamma = \frac{d}{dt}(J^1 \gamma) \).

In local coordinates, \( \gamma(t) = (t, x^i(t)) \) is a geodesic for the semispray \( S \) given by \([2.4]\) if and only if it satisfies the time dependent system \([1.1]\) of SODE.

2.2. Frölicher-Nijenhuis theory on \( J^1 \pi \). In this section we give a short review of the Frölicher-Nijenhuis theory on \( J^1 \pi \), following [14], [15], [13], [18].

Suppose that \( A \) is a vector valued \( l \)-form on \( J^1 \pi \) and \( B \) a (vector valued) \( k \)-form on \( J^1 \pi \). We can define an algebraic derivation [13] denoted by \( i_A B \):

\[
i_A B(X_1, \ldots, X_{k+l-1}) = \frac{1}{l!(k-1)!} \sum_{\sigma \in S_{k+l-1}} \text{sign}(\sigma) B(A(X_{\sigma(1)}, \ldots, X_{\sigma(l)}), X_{\sigma(l+1)}, \ldots, X_{\sigma(k+l-1)}),
\]

where \( X_1, \ldots, X_{k+l-1} \in \mathfrak{X}(J^1 \pi) \) and \( S_p \) is the permutation group of elements \( 1, \ldots, p \). We observe that \( i_A B \) is a \((k+l-1)\)-form (vector valued form, respectively if \( B \) is vector valued) on \( J^1 \pi \). In [14], [15] this algebraic derivation is denoted by \( B \wedge A \) and it is called exterior inner product. For \( B \) a scalar form and \( l = 0 \), \( A \) is a vector field on \( J^1 \pi \) and \( i_A B \) is the usual inner product of \( k \)-form \( B \) with respect to vector field \( A \). When \( l = 1 \), then \( A \) is a \((1, 1)\)-type tensor field and \( i_A B \) is the \( k \)-form (or vector valued \( k \)-form, if \( B \) is vector valued)

\[
i_A B(X_1, \ldots, X_k) = \sum_{i=1}^k B(X_1, \ldots, AX_i, \ldots, X_k).
\]

For any vector valued \( l \)-form \( A \) on \( J^1 \pi \) we define \( i_A B = 0 \), if \( k = 0 \), which means that \( B \in C^\infty(J^1 \pi) \) or \( B \in \mathfrak{X}(J^1 \pi) \). If \( k = 1 \), we have \( i_A B = B \circ A \).

Let \( A \) be a vector valued \( l \)-form on \( J^1 \pi \), \( l \geq 0 \). The exterior derivative with respect to \( A \) is the map \( d_A: \Lambda^k(J^1 \pi) \to \Lambda^{k+1}(J^1 \pi) \), \( k \geq 0 \),

\[
d_A = i_A \circ d - (-1)^{l-1} d \circ i_A.
\]

In [13], the exterior derivative \( d_A \) is called the Lie derivative with respect to \( A \) and it is denoted by \( \mathcal{L}_A \). When \( A \in \mathfrak{X}(J^1 \pi) \) and \( k \geq 0 \), we obtain \( d_A = \mathcal{L}_A \), the usual Lie derivative. In this case equation \([2.0]\) is the well known Cartan’s formula. If \( A = \text{Id} \), the identity \((1,1)\)-type tensor field on \( J^1 \pi \), then \( d_{\text{Id}} = d \), since \( i_{\text{Id}} \alpha = k \alpha \) for \( \alpha \in \Lambda^k(J^1 \pi) \).
Suppose $A$ and $B$ are vector valued forms on $J^1\pi$ of degrees $l \geq 0$ and $k \geq 0$, respectively. Then, the Frölicher-Nijenhuis bracket of $A$ and $B$ is the unique vector valued $(k+l)$-form on $J^1\pi$ such that

\begin{equation}
(2.7) \quad d_{[A,B]} = d_A \circ d_B - (-1)^{kl} d_B \circ d_A.
\end{equation}

When $A$ and $B$ are vector fields, the Frölicher-Nijenhuis bracket coincides with the usual Lie bracket $[A,B] = \mathcal{L}_A B$.

For a vector field $X \in \mathfrak{X}(J^1\pi)$ and a $(1,1)$-type tensor field $A$ on $J^1\pi$, the Frölicher-Nijenhuis bracket $[X,A] = \mathcal{L}_X A$ is the $(1,1)$-type tensor field

\begin{equation}
(2.8) \quad \mathcal{L}_X A = \mathcal{L}_X \circ A - A \circ \mathcal{L}_X.
\end{equation}

The Frölicher-Nijenhuis bracket of two $(1,1)$-type tensor fields $A, B$ on $J^1\pi$ is the unique vector valued 2-form $[A,B]$ defined by

\begin{equation}
(2.9) \quad [A,B](X,Y) = [AX,BY] + [BX,AY] + (A \circ B + B \circ A)[X,Y]
\end{equation}

\begin{equation}
- A[X,BY] - A[BX,Y] - B[X,AY]
\end{equation}

\begin{equation}
- B[AX,Y], \forall X,Y \in \mathfrak{X}(J^1\pi).
\end{equation}

In particular,

\begin{equation}
(2.10) \quad \frac{1}{2}[A,A](X,Y) = [AX,AY] + A^2[X,Y] - A[X,AY] - A[AX,Y].
\end{equation}

For a $(1,1)$-type tensor field $A$, the vector valued 2-form $N_A = \frac{1}{2}[A,A]$ is called the Nijenhuis tensor of $A$.

For the next commutation formulae on $\Lambda^k(J^1\pi)$, $k \geq 0$, that will be used throughout the paper, we refer to [15, chapter 2].

\begin{equation}
(2.11) \quad i_A d_B - d_B i_A = d_{B \circ A} - i_{[A,B]},
\end{equation}

\begin{equation}
(2.12) \quad \mathcal{L}_X i_A - i_A \mathcal{L}_X = i_{[X,A]},
\end{equation}

\begin{equation}
(2.13) \quad i_X d_A + d_A i_X = \mathcal{L}_{AX} - i_{[X,A]},
\end{equation}

\begin{equation}
(2.14) \quad i_A i_B - i_B i_A = i_{B \circ A} - i_{A \circ B},
\end{equation}

for $X \in \mathfrak{X}(J^1\pi)$ and $A, B \in T^1(\pi)$.

In this work we will also use the algebraic operator $A^*$,

\begin{equation}
A^* B(X_1, \cdots, X_k) = B(AX_1, \cdots, AX_k),
\end{equation}

for $A$ a $(1,1)$-type tensor field and $B$ a (vector valued) $k$-form on $J^1\pi$.

2.3. Poincaré-type Lemma for semi-basic forms. For a vector valued $l$-form $A$, we say that a $k$-form $\omega$ on $J^1\pi$ is $d_A$-closed if $d_A \omega = 0$ and $d_A$-exact if there exists $\theta \in \Lambda^{k-1}(J^1\pi)$ such that $\omega = d_A \theta$. For a vector valued 1-form $A$, from formulae \[2.7\] and \[2.10\] we obtain that $d_A^2 = d_{N_A}$. Therefore, if $A$ is not integrable, which means that $N_A \neq 0$, $d_A$-exact forms may not be $d_A$-closed.

Two forms $\omega_1$ and $\omega_2$ on $J^1\pi$ are called equivalent (modulo dt) if $\omega_1 \wedge dt = \omega_2 \wedge dt$.

For a vector valued $l$-form $A$, we say that a $k$-form $\theta \in \Lambda^k(J^1\pi)$ is $d_A$-closed (modulo dt) if $d_A \theta \wedge dt = 0$ and $d_A$-exact (modulo dt) if there exists $\omega$ in $\Lambda^{k-1}(J^1\pi)$ such that $\theta \wedge dt = d_A \omega \wedge dt$.

For the vertical endomorphism $J$, its Frölicher-Nijenhuis tensor is given by

\begin{equation}
(2.15) \quad N_J = -\frac{\partial}{\partial y^i} \otimes \delta x^i \wedge dt = -J \wedge dt.
\end{equation}
Consequently, using formula (2.7), we obtain $d_J^2 = d_{N_J} = -d_{J\wedge dt} \neq 0$. Therefore, $d_J$-exact forms on $J^1\pi$ may not be $d_J$-closed. However, we will prove in the first part of Lemma 2.4 that $d_J$-exact (modulo $dt$) forms are $d_J$-closed (modulo $dt$) forms. In the second part of Lemma 2.4 we will show that the converse is true for semi-basic $k$-forms, only.

For a $k$-form $\omega$, the following identities can be obtained immediately

$$
\begin{align*}
i_{J\wedge dt}\omega &= (-1)^{k+1}i_J\omega \wedge dt, \\
d_{J\wedge dt}\omega &= (-1)^{k+1}d_J\omega \wedge dt.
\end{align*}
$$

**Definition 2.1.** i) A $k$-form $\omega$ on $J^1\pi$, $k \geq 1$, is called *semi-basic* if $\omega(X_1, \cdots, X_k) = 0$, when one of the vectors $X_i$, $i \in \{1, \ldots, k\}$, is vertical.

ii) A vector valued $k$-form $A$ on $J^1\pi$ is called *semi-basic* if it takes values in the vertical subbundle and $A(X_1, \cdots, X_k) = 0$, when one of the vectors $X_i$, $i \in \{1, \ldots, k\}$ is vertical.

A semi-basic $k$-form verifies the relation $i_J\theta = 0$. The converse is true only for $k = 1$. Semi-basic 1-forms are annihilators for the vertical distribution and hence locally can be expressed as $\theta = \theta_0(t, x, y) dt + \theta_i(t, x, y) dx^i$. Contact 1-forms $dx^i$ given by formula (2.11) are semi-basic 1-forms.

**Definition 2.2.** A semi-basic 1-form $\theta$ on $J^1\pi$ is called *non-degenerate* if the 2-form $d\theta + i_Sd\theta \wedge dt$ has rank $2\nu$ on $J^1\pi$.

If a vector valued $k$-form $A$ is semi-basic, then $J \circ A = 0$, $i_J A = 0$ and $J^* A = 0$. The converse is true only for $k = 1$. It follows that the vertical endomorphism $J$ is a vector valued, semi-basic 1-form.

Next lemma presents a characterization of forms that are equivalent (modulo $dt$) to the null form and will be used to prove a Poincaré-type Lemma for semi-basic forms.

**Lemma 2.3.** A $k$-form $\omega$ on $J^1\pi$ satisfies the condition $\omega \wedge dt = 0$ if and only if it is of the form $\omega = i_{S\otimes dt}\omega = (-1)^{k+1}i_S\omega \wedge dt$, for an arbitrary semispray $S$.

**Proof.** A simple computation gives $i_{S\otimes dt}\omega = (-1)^{k+1}i_S\omega \wedge dt$. Therefore, if $\omega = i_{S\otimes dt}\omega = (-1)^{k+1}i_S\omega \wedge dt$ it follows that $\omega \wedge dt = 0$. Conversely, if $\omega \wedge dt = 0$, we apply $i_S$ to this identity and get $0 = i_S\omega \wedge dt + (-1)^k \omega$. Hence, $\omega = (-1)^{k+1}i_S\omega \wedge dt = i_{S\otimes dt}\omega$, which completes the proof. □

**Lemma 2.4.** (Poincaré-type Lemma) Consider $\theta$ a $k$-form on $J^1\pi$.

i) If $\theta$ is $d_J$-exact (modulo $dt$) then $\theta$ is $d_J$-closed (modulo $dt$).

ii) If $\theta$ is a semi-basic form and $d_J$-closed (modulo $dt$), then $\theta$ is locally $d_J$-exact (modulo $dt$).

**Proof.** i) Suppose that $\theta \in \Lambda^k(J^1\pi)$ is $d_J$-exact (modulo $dt$). Then, there exists $\omega \in \Lambda^{k-1}(J^1\pi)$ such that $\theta \wedge dt = d_J\omega \wedge dt$. If we apply $d_J$ to both sides of this identity, and use formula (2.10), we obtain

$$
d_J\theta \wedge dt = d_J^2\omega \wedge dt = -d_{J\wedge dt}\omega \wedge dt = 0.
$$

which means that the $k$-form $\theta$ is $d_J$-closed (modulo $dt$).

ii) We have to prove that for $\theta \in \Lambda^k(J^1\pi)$ semi-basic, with $d_J\theta \wedge dt = 0$, there exists $\omega \in \Lambda^{k-1}(J^1\pi)$ (locally defined), such that $\theta \wedge dt = d_J\omega \wedge dt$. Since $\theta$ is
semi-basic, it follows that locally it has the form
\[ \theta = \frac{1}{k!} \partial_{i_1 \ldots i_k} \delta x^{i_1} \wedge \ldots \wedge \delta x^{i_k} + \frac{1}{(k-1)!} \partial_{i_1 \ldots i_{k-1}} \delta x^{i_1} \wedge \ldots \wedge \delta x^{i_{k-1}} \wedge dt. \]

Using the identities \( i_Jdt = 0 \), \( i_J \delta x^i = 0 \), and \( i_Jdy^i = \delta x^i \), we obtain
\[ d_J \theta \wedge dt = i_J d\theta \wedge dt = \frac{1}{k!} \frac{1}{(k+1)!} \epsilon_{i_1 i_2 \ldots i_{k+1}} \partial \theta_{j_1 \ldots j_{k+1}} \wedge \delta x^{i_1} \wedge \ldots \wedge \delta x^{i_k} \wedge dt, \]

where \( \epsilon_{i_1 i_2 \ldots i_{k+1}} \) is Kronecker’s symbol. Hence
\[ \frac{1}{(k+1)!} \epsilon_{i_1 i_2 \ldots i_{k+1}} \partial \theta_{j_1 \ldots j_{k+1}} = 0. \]

If one considers the transverse coordinates \( t, x^i \) of the natural foliation as parameters, one can use Poincaré Lemma on the leafs of this foliation to obtain (locally defined) functions \( \omega_{j_1 \ldots j_{k-1}}(t, x^i, y^i) \) such that
\[ \theta_i = \frac{1}{(k-1)!} \delta x^{i_1} \wedge \ldots \wedge \delta x^{i_k}. \]

In view of identity (2.17) we have \( \theta \wedge dt = d_J \omega \wedge dt \), which means that \( \theta \) is \( d_J \)-exact (modulo \( dt \)). □

For semi-basic forms, the differential operator \( d_J \) is closely related to the exterior differential \( d^J \) along the leafs of the natural foliation, studied by Vaisman in [36]. This relation as well as the proof of Theorem 3.1 from [36] can be used to give a different proof for Lemma 2.4. This proof will require to fix a distribution supplementary to the vertical distribution, which is always possible, see Section 3.1.

For semi-basic 1 and 2-forms a Poincaré-type Lemma and its usefulness for the inverse problem of the calculus of variations is discussed in Marmo et al. [24]. For forms along the tangent bundle projection a similar result as in Lemma 2.4 has been shown in [34, Prop. 2.1].

In Section 4 we will use Poincaré-type Lemma 2.4 to find necessary and sufficient conditions for a semi-basic 1-form on \( J^1 \pi \) to coincide, or to be equivalent (modulo \( dt \)), with the Poincaré-Cartan 1-form of some Lagrangian function.

3. Geometric objects induced by a semispray

In this section we present some geometric structures that can be derived from a semispray using the Frölicher-Nijenhuis theory: nonlinear connection, Jacobi endomorphism, dynamical covariant derivative. See [5 9 22 23 27 34]. These structures will be used later to express necessary and sufficient conditions for a given semispray to be a Lagrangian vector field.
3.1. **Nonlinear connection.** A *nonlinear connection* on the first jet bundle $J^1\pi$ is an $(n+1)$-dimensional distribution $H : u \in J^1\pi \mapsto H_u \subset T_u J^1\pi$, supplementary to the vertical distribution $V J^1\pi$. This means that for each $u \in J^1\pi$, we have the direct decomposition $T_u J^1\pi = H_u \oplus V_u$.

A semispray $S$ induces a nonlinear connection on $J^1\pi$, determined by the *almost product structure*, or dynamical connection [22]

$$\Gamma = -\mathcal{L}_S J + S \otimes dt, \quad \Gamma^2 = \text{Id}.$$  

The *horizontal projector* that corresponds to this almost product structure is

$$h = \frac{1}{2} (\text{Id} - \mathcal{L}_S J + S \otimes dt)$$

and the *vertical projector* is $v = \text{Id} - h$. Note that in this paper we chose to work with the *weak horizontal projector* $h$ defined by formula (3.2). We can also consider $h_0 = S \otimes dt$ and $h_1 = h - h_0$, the *strong horizontal projector*. The horizontal subspace is the eigenspace corresponding to the eigenvalue $+1$ of $\Gamma$, while the vertical subspace is the eigenspace corresponding to the eigenvalue $-1$. The horizontal subspace is spanned by $S$ and by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \quad \text{where} \quad N^j_i = \frac{\partial G^i}{\partial y^j}.$$  

From now on, whenever a semispray will be given, we prefer to work with the following adapted basis and cobasis

$$(3.3) \quad \left\{ S, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}, \quad \left\{ dt, \delta x^i, \delta y^i \right\},$$

with $\delta x^i = dx^i - y^i dt$ the contact 1-forms and

$$(3.4) \quad \delta y^i = dy^i + N^j_i dx^j + N^j_0 dt, \quad N^j_0 = 2G^j - N^j_i y^i.$$  

Functions $N^j_i$ and $N^j_0$ are the coefficients of the nonlinear connection induced by the semispray $S$. The 1-forms $\delta y^i$ are annihilators for the horizontal distribution.

With respect to the adapted basis and cobasis (3.3) the almost product structure can be locally expressed as

$$\Gamma = S \otimes dt + \frac{\delta}{\delta x^i} \otimes \delta x^i - \frac{\partial}{\partial y^i} \otimes \delta y^i.$$  

Therefore, the horizontal and vertical projectors are locally expressed as

$$h = S \otimes dt + \frac{\delta}{\delta x^i} \otimes \delta x^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i.$$  

We consider the (1,1)-type tensor field, which corresponds to the almost complex structure in the autonomous case,

$$(3.5) \quad F = h \circ \mathcal{L}_S h - J.$$  

Tensor $F$ satisfies $F^3 + F = 0$, which means that it is an $f(3,1)$ structure. It can be expressed locally as

$$F = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes \delta x^i.$$
For some useful future calculus, we will also need the next formulae
\begin{align}
\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] &= R^k_{ij} \frac{\partial}{\partial y^k}, \quad R^i_{jkl} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}, \\
\left[ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right] &= \frac{\partial N^k_i}{\partial y^j} \frac{\partial}{\partial y^k} = \frac{\partial^2 G^k}{\partial y^i \partial y^j} \frac{\partial}{\partial y^k},
\end{align}
for the Lie brackets of the vector fields of the adapted basis \((3.3)\). We also have:
\begin{align}
L_S S &= 0, \quad L_S \frac{\delta}{\delta x^i} = N^j_i \frac{\delta}{\delta x^j} + R^i_{jkl} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k}, \\
L_S \frac{\partial}{\partial y^i} &= -\frac{\delta}{\delta x^i} + N^i_j \frac{\partial}{\partial y^j}, \\
L_S \delta x^i &= -N^i_j \delta x^j + \delta y^i, \quad L_S \delta y^i = -R^i_{jkl} \delta x^j - N^i_j \delta y^j,
\end{align}
where
\begin{align}
R^i_{jkl} &= 2 \left( \frac{\partial G^i}{\partial x^j} - \frac{\partial G^i}{\partial y^k} \frac{\partial G^k}{\partial y^j} \right),
\end{align}
\(R^i_{jkl}\) are the components of a \((1,1)\)-type tensor field on \(J^1 \pi\), known as the second invariant in KCC-theory [2], the Douglas tensor [13, 15] or as the Jacobi endomorphism [5, 10, 17, 34].

Using formulae \((2.9), (2.10), (3.6), (3.7)\), we obtain
\begin{align}
L_S J &= -\frac{\delta}{\delta x^i} \otimes \delta x^i + \frac{\partial}{\partial y^i} \otimes \delta y^i, \\
L_S h &= \frac{\delta}{\delta x^i} \otimes \delta y^i + R^i_{jkl} \frac{\partial}{\partial y^j} \otimes \delta x^i.
\end{align}

According to formulae \((2.9), (2.10), (3.6), (3.7)\), and \([J, \Gamma] = 2[J, h] - [J, \text{Id}] = 2[J, h]\) one can prove the following proposition.

**Proposition 3.1.** i) The weak torsion tensor field of the nonlinear connection \(\Gamma\) vanishes: \([J, h] = 0\), which is equivalent also with \([J, \Gamma] = 0\).

ii) The curvature tensor \(R = N_h\) of the nonlinear connection \(\Gamma\) is a vector valued semi-basic 2-form, locally given by
\begin{align}
R^i_j &= \frac{\partial G^i}{\partial x^j} - \frac{\partial G^i}{\partial y^k} \frac{\partial G^k}{\partial y^j} - S \left( \frac{\partial G^i}{\partial y^j} \right),
\end{align}
where \(R^i_j\) and \(R^i_{jkl}\) are given in formulae \((3.6)\) and \((3.10)\).

The Jacobi endomorphism can be defined as follows.
\begin{align}
\Phi &= v \circ L_S h = L_S h - \mathbb{F} - J.
\end{align}
It is a semi-basic, vector valued 1-form and satisfies \(\Phi^2 = 0\). Locally, it can be expressed as
\begin{align}
\Phi &= R^i_j \frac{\partial}{\partial y^j} \otimes \delta x^i.
\end{align}
As we can see from formula \((3.13)\), the Jacobi endomorphism is part of the curvature tensor \(R\), which is the third invariant in KCC theory [2]. Moreover, as we will see in the next proposition, the curvature tensor can be obtained, using Frölicher-Nijenhuis theory, from the Jacobi endomorphism. The result is similar to the one from [34], where different techniques are used.
Proposition 3.2. The Jacobi endomorphism and the curvature of the nonlinear connection are related by the following formulae.

\[ \Phi = i_S R, \]
\[ [J, \Phi] = 3R + \Phi \wedge dt. \]

Proof. Since \( 2R = [h, h] \), for \( X \) in \( \mathfrak{X}(J^1 \pi) \), we have

\[ 2(i_S R)(X) = [h, h](S, X) = 2[S, hX] = 2e[S, hX] = 2\Phi(X), \]

which proves formula (3.16). To prove formula (3.17) we use the local expression (2.2) and (3.15) of \( J \) and \( \Phi \) as well as the identity

\[ \frac{\partial R^k_i}{\partial y^j} - \frac{\partial R^k_j}{\partial y^i} = 3R^k_{ij}, \]

which follows by a direct computation. The Frölicher-Nijenhuis bracket of \( J \) and \( \Phi \) is given by

\[ [J, \Phi] = 2R^l_i \frac{\partial}{\partial y^j} \otimes dt \wedge \delta x^i + \frac{1}{2} \left( \frac{\partial R^k_i}{\partial y^j} - \frac{\partial R^k_j}{\partial y^i} \right) \frac{\partial}{\partial y^k} \otimes \delta x^i \wedge \delta x^j \]

\[ = 2dt \wedge \Phi + \frac{3}{2} R^l_{ij} \frac{\partial}{\partial y^k} \otimes \delta x^i \wedge \delta x^j. \]

Using the above formula and formula (3.13) that defines the curvature \( R \) we obtain formula (3.17), which completes the proof. \( \square \)

The following \((1,1)\)-type tensor field \( \Psi \) will allow us to express in a simpler form the dynamical covariant derivative induced by \( S \), which will be discussed in the next section.

\[ \Psi = h \circ L_S h + v \circ L_S v = \Gamma \circ L_S h = F + J - \Phi; \]
\[ \Psi = \frac{\delta}{\delta x^i} \otimes \delta y^j - R^l_i \frac{\partial}{\partial y^j} \otimes \delta x^i. \]

3.2. Dynamical covariant derivative. For a semispray \( S \) there are various possibilities to define a tensor derivation on \( J^1 \pi \). Such derivation was considered first by Kosambi in [19], with the name of biderivative. This derivation has been rediscovered latter and was called the dynamical covariant derivative [7]. It can be defined either as a derivation along the bundle projection \( \pi_{10} \), as in [5, 9, 17, 26, 34], or as a derivation on the total space \( T J^1 \pi \), [25, 29].

In this section we use Frölicher-Nijenhuis theory to define the dynamical covariant derivative as a tensor derivation on \( J^1 \pi \), following the time independent case developed in [4]. In Proposition 3.3 we present some useful commutation rules of the dynamical covariant derivative with the geometric structures studied in the previous sections.

A map \( \nabla : \mathcal{T}(J^1 \pi) \to \mathcal{T}(J^1 \pi) \) is a tensor derivation on \( \mathcal{T}(J^1 \pi) \) if it satisfies the following conditions.

i) \( \nabla \) is \( \mathbb{R} \)-linear.

ii) \( \nabla \) preserves the type of tensor fields.

iii) \( \nabla \) obeys the Leibnitz rule: \( \nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S \), \( \forall T, S \in \mathcal{T}(J^1 \pi) \).

iv) \( \nabla \) commutes with any contractions.
For a semispray \( S \) on \( J^1\pi \), we define the \( \mathbb{R} \)-linear map \( \nabla_0 : \mathfrak{X}(J^1\pi) \to \mathfrak{X}(J^1\pi) \)
\begin{equation}
\nabla_0 X = h[S, hX] + v[v, vX].
\end{equation}
It follows that \( \nabla_0(fX) = S(f)X + f\nabla_0 X \), for all \( f \in C^\infty(J^1\pi) \) and \( X \in \mathfrak{X}(J^1\pi) \).

Since any tensor derivation on \( T(J^1\pi) \) is completely determined by its action over the smooth functions and vector fields on \( J^1\pi \), there exists a unique tensor derivation \( \nabla \) on \( J^1\pi \) such that \( \nabla|_{C^\infty(J^1\pi)} = S \) and \( \nabla|_{\mathfrak{X}(J^1\pi)} = \nabla_0 \). This tensor derivation is called the dynamical covariant derivative induced by the semispray \( S \).

Next, we will obtain some alternative expressions for the action of the dynamical covariant derivative \( \nabla \) on \( \mathfrak{X}(J^1\pi), \Lambda^k(J^1\pi) \) and \((1,1)\)-type tensor fields on \( J^1\pi \).

From formula (3.20) we obtain
\begin{equation}
\nabla|_{\mathfrak{X}(J^1\pi)} = h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v = \mathcal{L}_S + h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v.
\end{equation}
Using the \((1,1)\)-type tensor field \( \Psi \) defined in formula (3.18), we obtain the following expression of the dynamical covariant derivative.
\begin{equation}
\nabla|_{\mathfrak{X}(J^1\pi)} = \mathcal{L}_S + \Psi.
\end{equation}
Since \( \nabla \) satisfies the Leibnitz rule, we deduce that the action of \( \nabla \) on \( k \)-forms is given by
\begin{equation}
\nabla|_{\Lambda^k(J^1\pi)} = \mathcal{L}_S - i\Psi.
\end{equation}
Above formula (3.22) implies that \( \nabla \) is a degree zero derivation on \( \Lambda^k(J^1\pi) \). Therefore [13 p. 69], it can be uniquely written as a sum of a Lie derivation, which is \( \mathcal{L}_S \), and an algebraic derivation, given by \( i\Psi \).

Similarly, we deduce that the action of \( \nabla \) on a vector valued \( k \)-form \( A \) on \( J^1\pi \) is given by
\begin{equation}
\nabla A = \mathcal{L}_S A + \Psi \circ A - i\Phi A.
\end{equation}

**Proposition 3.3.** The dynamical covariant derivative induced by a semispray \( S \) has the following properties.

i) \( \nabla S = 0 \) and \( \nabla i_S = i_S \nabla \).

ii) \( \nabla h = \nabla v = 0 \), which means that \( \nabla \) preserves by parallelism the horizontal and vertical distributions.

iii) \( \nabla J = \nabla F = 0 \), which means that \( \nabla \) acts similarly on both horizontal and vertical distributions.

iv) The restriction of \( \nabla \) to \( \Lambda^k(J^1\pi) \) and the exterior differential operator \( d \) satisfy the commutation formula
\begin{equation}
d\nabla - \nabla d = d\Psi.
\end{equation}

v) The restriction of \( \nabla \) to \( \Lambda^k(J^1\pi) \) satisfies the commutation rule
\begin{equation}
\nabla i_A - i_A \nabla = i\nabla A
\end{equation}
for a \((1,1)\)-type tensor field \( A \) on \( J^1\pi \). Hence the algebraic derivations with respect to \( h, v, J, F \) commute with \( \nabla|_{\Lambda^k(J^1\pi)} \).

**Proof.** First item follows directly using definition formula (3.20) and formula (3.22).

From the definition formula (3.18) of tensor \( \Psi \) we obtain
\begin{equation}
A \circ \Psi - \Psi \circ A = \mathcal{L}_S A,
\end{equation}
for \( A \in \{ h, v, J, F \} \). Using formula (3.23), it follows immediately that \( \nabla h = \nabla v = \nabla J = \nabla F = 0 \) and hence we proved item ii) and iii) of the proposition.
Using formula (3.22) we obtain
\[ d\nabla = d\mathcal{L}_S - di_{\Psi}d = \mathcal{L}_Sd - di_{\Psi}d + d\Phi = \nabla d + d\Phi, \]
for the restriction of \( \nabla \) to \( \Lambda^k(J^1\pi) \). Therefore, formula (3.24) is true.

From formulae (3.23), (2.12), as well as (2.14) we obtain
\[ \nabla i_A - i_A\nabla = \mathcal{L}_{SiA} - i_A\mathcal{L}_S - i_{\Psi i_A} + i_{\Psi i_{\Psi}} = i_{\Theta S,A} - i_{A S_{\Psi}} + i_{\Theta S A} = i_{\nabla A}, \]
which proves the last item of the proposition.

First three items of Proposition 3.3 can be locally expressed as follows.
\[ \nabla S = 0, \quad \nabla dt = 0, \]
\[ (3.27) \]
\[ \nabla \delta x^i = N^i_j \delta x^j, \quad \nabla \delta x^j = -N^i_j \delta x^i, \]
\[ \nabla \partial y^i = N^i_j \partial y^j, \quad \nabla \partial y^j = -N^i_j \partial y^i. \]

Tensor derivation \( \nabla \) coincides with the dynamical covariant derivative induced by the Berwald linear connection \( \mathcal{\nabla} \) on \( J^1\pi \), studied by Massa and Pagani in [25], in the following sense \( \nabla = \mathcal{\nabla}_S \). See also [20] for a detailed study of Berwald-type
covariant connection associated to time dependent systems of SODE. The relation between
the dynamical covariant derivative and the Berwald connection implies that tensor
\( \Psi \) is the shape map \( A_S \) on the manifold \( N = J^1\pi \), studied by Jerie and Prince in
[17] on an arbitrary manifold.

Another tensor derivation on \( X(J^1\pi) \), induced by a semispray, was proposed by
Morando and Pasquero [29]. In our notations, Morando and Pasquero’s derivation
can be expressed as \( \nabla + \Psi = \mathcal{L}_S + 2\Psi \).

Next result is a technical lemma that expresses the action of the dynamical co-
variant derivative on semi basic 1-forms, and will be useful for the proof of Theorem 3.2.

**Lemma 3.4.** For a semi-basic 1-form \( \theta \) on \( J^1\pi \), its dynamical covariant derivative
can be expressed as follows.
\[ (3.28) \]
\[ \nabla \theta = d_h i_S \theta + i_S d_h \theta. \]

Moreover, \( \theta \) satisfies the identity
\[ (3.29) \]
\[ i_S d_h \theta = i_h i_S d \theta. \]

**Proof.** Using the identity \( d_h i_S + i_S d_h = \mathcal{L}_S - i_{[S,h]} \), one gets
\[ (3.30) \]
\[ \mathcal{L}_S \theta = d_h i_S \theta + i_S d_h \theta + i_{[S,h]} \theta. \]

Since \( \theta \) is semi-basic we have \( i_{[S,h]} \theta = i_{\Psi + J + \Phi} \theta = i_{\Psi} \theta \). Moreover, \( i_{\Phi} \theta = i_{\Psi + J - \Phi} \theta = i_{\Psi} \theta \) and hence \( \nabla \theta = \mathcal{L}_S \theta - i_{\Psi + J - \Phi} \theta \). Using \( i_{\Psi} \theta = i_{\Psi} \theta = i_{\Psi} \theta \) and \( i_{\Psi} \theta = i_{\Psi} \theta \) one gets \( i_{\Psi} \theta = i_{\Psi} \theta \).

For the second identity, we notice that \( i_S d_h \theta = i_S d_h \theta - i_S d_h \theta \). Using \( i_S i_h \theta - i_h i_S \theta = i_h \theta = i_{\Psi} \theta \) one gets \( i_{\Psi} \theta = i_{\Psi} \theta \).

3.3. **Dual symmetries.** In Section 5 we will search for solutions of the inverse
problem of the calculus of variations in terms of semi-basic 1-forms. We prove
that in the Lagrangian case there is always a solution that is a \( d_J \)-closed semi-
basic 1-form, which is the Poincaré-Cartan 1-form of some Lagrangian function.
If the solution of the inverse problem contains a semi-basic 1-form that is not \( d_J \-
closed then it induces a dual symmetry and a first integral of the semispray. For
discussions regarding adjoint symmetries in the context of the inverse problem of the calculus of variations we refer to [6, 7, 32]. A detailed discussion of symmetries, dual symmetries, adjoint symmetries and the relations among them can be found in [29]. In this section we use expression (3.22) of the dynamical covariant derivative to characterize dual symmetries in terms of a Jacobi equation.

**Definition 3.5.** A 1-form $\omega$ on $J^1\pi$ is called a *dual symmetry* (or an invariant form) for a semispray $S$ if $L_S\omega = 0$.

In this work we will be interested in dual symmetries for which $i_S\omega = 0$. Since we will work with equivalence classes (modulo $dt$) of dual symmetries, in each class we can choose a representant of this form.

To express the condition $L_S\omega = 0$ for a 1-form, locally expressed as $\omega = \tilde{\omega}_i \delta x^i + \omega_i \delta y^i$, we will use formulae (3.22) and (3.27) for the dynamical covariant derivative. Therefore $L_S\omega = 0$, which is equivalent to $\nabla \omega = -i_\Psi \omega$ can be locally expressed as

$$\begin{align*}
\nabla \tilde{\omega}_i \delta x^i + \nabla \omega_i \delta y^i &= -\tilde{\omega}_i \delta y^i + R^i_j \omega_j \delta x^i \\
\nabla \tilde{\omega}_i - R^i_j \omega_j &= 0.
\end{align*}$$

Hence $\omega$ is a dual symmetry, if and only if $\omega = -\nabla \omega_i \delta x^i + \omega_i \delta y^i$ and it satisfies the Jacobi equation

$$\nabla^2 \omega_i + R^i_j \omega_j = 0.$$  \hspace{1cm} (3.31)

If $\omega$ is a dual symmetry then $-i_\Psi \omega$ is an adjoint symmetry. Locally, an adjoint symmetry $\alpha$ with $i_S\alpha = 0$ is locally expressed as $\alpha = \nabla \omega_i \delta x^i + \omega_i \delta y^i$ and verifies the same equation (3.31).

4. **Lagrangian Vector Fields**

An approach to the inverse problem of the calculus of variations seeks for the existence of a non-degenerate multiplier matrix $g_{ij}(t,x,y)$ which relates the geodesic equations (1.1) of a semispray with the Euler-Lagrange equations (1.2) for a Lagrangian function $L$. In this case, the Lagrangian function $L$ is determined from the condition that the multiplier matrix $g_{ij}$ is the Hessian of $L$. Necessary and sufficient conditions for the existence of such multiplier matrix are known as Helmholtz conditions and were obtained, using various techniques for both autonomous and nonautonomous case, in [1, 4, 8, 16, 19, 20, 21, 28, 30, 31].

In our approach we look for solutions of the inverse problem in terms of semi-basic 1-forms, see Theorem 4.5 and Proposition 4.6. In the Lagrangian case, the Lagrangian function $L$ is determined by the fact that its Poincaré-Cartan 1-form coincides or it is equivalent (modulo $dt$) to a semi-basic 1-form that satisfies certain conditions. We will call these conditions *Helmholtz conditions* and we will present how they lead to the classic formulation of Helmholtz conditions in terms of a multiplier matrix.

4.1. **Poincaré-Cartan 1-forms.**

**Definition 4.1.** 1) A smooth function $L \in C^\infty(J^1\pi)$ is called a *Lagrangian function*.

2) The **Poincaré-Cartan 1-form** of the Lagrangian $L$ is the semi basic 1-form $\theta_L = L dt + dJ L$.

3) If for a Lagrangian $L$, the **Poincaré-Cartan 2-form** $d\theta_L$ has maximal rank $2n$, then $L$ is called a *regular Lagrangian*. 

13
For a Lagrangian function \( L \), the Poincaré-Cartan 2-form \( \theta_L \) can be written as follows, see [10].

\[
\theta_L = \frac{\partial^2 L}{\partial y^i \partial y^j} \delta y^i \wedge \delta x^j.
\]

Therefore, the Lagrangian function is regular if and only if the \( n \times n \) symmetric matrix with local components

\[
g_{ij}(t, x, y) = \frac{\partial^2 L}{\partial y^i \partial y^j}
\]

has rank \( n \) on \( J^1 \). Functions \( g_{ij} \), given by formula (4.2), are the components of a \((0,2)\)-type symmetric tensor \( g = g_{ij} \delta x^i \otimes \delta x^j \), which is called the metric tensor of the Lagrangian function \( L \).

Next lemma gives characterizations of Poincaré-Cartan 1-forms, as well as equivalent (modulo \( dt \)) Poincaré-Cartan 1-forms, as subsets of semi-basic 1-forms, using the exterior differential operator \( dJ \). In the time independent case, such characterization appears in [24].

Lemma 4.2. Let \( \theta \) be a semi-basic 1-form on \( J^1 \).

i) \( \theta \) is the Poincaré-Cartan 1-form of a Lagrangian function if and only if \( \theta \) is \( dJ \)-closed. Moreover, the Lagrangian function is given by \( L = i_S \theta \).

ii) \( \theta \) is equivalent (modulo \( dt \)) to the Poincaré-Cartan 1-form of a Lagrangian function if and only if \( \theta \) is \( dJ \)-closed (modulo \( dt \)).

Proof. i) If \( \theta = Ldt + d_j L \) is the Poincaré-Cartan 1-form of a Lagrangian function \( L \), then it follows that \( d_j \theta = 0 \) and \( L = i_S \theta \).

Conversely, let us assume that \( \theta \in \Lambda^1(J^1) \) is semi-basic and \( d_j \theta = 0 \). Since \( d_j i_S + i_S d_j = L_S - i_{[S,j]} = i_{h - S \otimes dt - e} \), we have that \( d_j i_S \theta = \theta - i_S \theta dt \). Therefore \( \theta = i_S \theta dt + d_j i_S \theta \) and hence \( \theta \) is the Poincaré-Cartan 1-form of the Lagrangian \( L = i_S \theta \).

ii) We apply the Poincaré-type Lemma 2.4 for \( k = 1 \). Suppose that the 1-form \( \theta \) is equivalent (modulo \( dt \)) to the Poincaré-Cartan 1-form of the Lagrangian function \( L \). Therefore \( \theta \wedge dt = d_j L \wedge dt \), which means that \( \theta \) is \( d_j \)-exact (modulo \( dt \)) and hence it is \( d_j \)-closed (modulo \( dt \)).

Conversely, if the semi-basic 1-form \( \theta \) is \( d_j \)-closed (modulo \( dt \)), it follows that there exists a (locally defined) Lagrangian function \( L \) such that \( \theta \wedge dt = d_j L \wedge dt = \theta_L \wedge dt \). Therefore, \( \theta \) is equivalent (modulo \( dt \)) with the Poincaré-Cartan 1-form of the Lagrangian function \( L \). \qed

Next Lemma will be applied in the next section in order to formulate Helmholtz-type conditions for the semispray \( S \).

Lemma 4.3. Let \( \theta \) be a semi-basic, \( d_j \)-closed, 1-form on \( J^1 \). Then, the next equivalent conditions are satisfied.

i) \( L_S \theta = d_i \theta + i_S d_k \theta \);

ii) \( i_S d \theta = i_S d_k \theta \);

iii) \( i_S d_r \theta = 0 \).

Proof. Consider \( \theta \) a semi-basic 1-form on \( J^1 \) such that \( d_j \theta = 0 \). Using Lemma 4.2, we deduce \( i_{[S,k]} \theta = i_F \theta = i_F (i_S \theta dt + d_j i_S \theta) = i_S d_j i_S \theta \).
Since \( i_\mathcal{E} d_J - d_\mathcal{E} i_J = d_I \mathcal{E} - i_{[\mathcal{E}, J]} = d_\mathcal{E} - i_{[\mathcal{E}, J]} \), it results that \( i_\mathcal{E} d_J i_S \theta = d J i_S \theta + d_\mathcal{E} i_J i_S \theta = d_\mathcal{E} i_S \theta \). Hence, \( i_{[\mathcal{E}, \mathcal{H}]} \theta = d_\mathcal{E} i_S \theta \). If we substitute this in formula (3.30) we obtain
\[
\mathcal{L}_S \theta = d_\mathcal{E} i_S \theta + i_S d_\mathcal{E} \theta + d_\mathcal{E} i_S \theta
\]
which proves condition i).

Evidently conditions i) and ii) are equivalent due to Cartan’s formula \( \mathcal{L}_S = d \circ i_S + i_S \circ d \).

From ii) and (3.29), we obtain \( i_\mathcal{E} d i_S \theta = i_S d \theta = i_\mathcal{H} i_S d \theta \) and hence \( i_S i_\mathcal{E} d \theta = 0 \). Using now the commutation rule \( i_S i_v = i_v i_S = i_v S = 0 \), we deduce that last two conditions are equivalent.

4.2. Lagrangian semisprays and dual symmetries. For a semispray \( S \), its geodesics, given by the system (1.1) of SODE coincide with the solutions of the Euler-Lagrange equations (1.2) of a regular Lagrangian \( L \) if and only if the two sets of equations are related by
\[
\text{(4.3)} \quad g_{ij} \left( t, x, \frac{dx}{dt} \right) \left( \frac{d^2 x^j}{dt^2} + 2G^j \left( t, x, \frac{dx}{dt} \right) \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i},
\]
with \( g_{ij} \) given by formula (1.2). Therefore, for a semispray \( S \), there exists a regular Lagrangian function \( L \) that satisfies equation (4.3) if and only if
\[
\text{(4.4)} \quad S \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0.
\]
Equations (4.3) can be globally expressed as
\[
\text{(4.5)} \quad \mathcal{L}_S \theta_L = dL.
\]

In view of Cartan’s formula, above equation (4.4) is equivalent to \( i_\mathcal{E} d_\mathcal{E} \theta_L = 0 \) and hence the regularity of the Lagrangian function \( L \) from Definition (1.1) is equivalent to the non-degeneracy of the Poincaré-Cartan 1-form \( \theta_L \) from Definition (2.2). For details regarding regularity aspects of Lagrangian systems see [20, chapter 6].

Definition 4.4. A semispray \( S \) is called a Lagrangian vector field (or a Lagrangian semispray) if and only if there exists a (locally defined) regular Lagrangian \( L \) that satisfies equation (4.5).

Next theorem provides necessary and sufficient conditions, in terms of a semi-basic 1-form, for a semispray to be a Lagrangian vector field. It corresponds the characterizations in terms of 2-forms of Lagrangian vector field in [11, 3, 8, 11, 24]. In [34], Proposition 8.3 gives a characterization of a Lagrangian vector field in terms of Poincaré-Cartan 1-forms.

Theorem 4.5. A semispray \( S \) is a Lagrangian vector field if and only if there exists a non-degenerate, semi-basic 1-form \( \theta \) on \( J^1 \pi \) such that \( \mathcal{L}_S \theta \) is closed.

Proof. We assume that the semispray \( S \) is a Lagrangian vector field for some regular Lagrangian function \( L \). Since \( L \) is regular it follows that its Poincaré-Cartan 1-form \( \theta_L = L dt + d_j L \) is non-degenerate. Moreover, \( L \) satisfies equation (4.5), which means that \( \mathcal{L}_S \theta_L \) is exact and hence it is a closed 1-form.
For the converse, consider a non-degenerate, semi-basic 1-form $\theta$ on $J^1\pi$, such that $L_S\theta$ is closed. It follows that there exists a (locally defined) Lagrangian function $L$ on $J^1\pi$ such that

$$L_S\theta = dL. \tag{4.6}$$

If we apply $i_S$ to both sides of formula (4.6) we obtain

$$S(i_S\theta) = S(L). \tag{4.7}$$

Now, we apply $i_J$ to both sides of formula (4.6) and obtain

$$i_JL_S\theta = d_JL. \tag{4.8}$$

Using $i_JL_S - L_Si_J = -i_{[S,J]} = i_h - S\otimes dt - v$, $i_{S\otimes dt}\theta = i_S\theta dt$ and the fact that $\theta$ is a semi-basic 1-form, which means that $i_v\theta = i_J\theta = 0$ and $i_h\theta = \theta$, from formula (4.8) we obtain

$$\theta = i_S\theta dt + d_JL. \tag{4.9}$$

From formula (4.9) it follows that $d\theta = d(i_S\theta) \wedge dt + dd_JL$, which can be written as $d\theta + i_Sd\theta \wedge dt = L_S\theta \wedge dt + dd_JL = d\theta_L$. Hence the non-degeneracy of the semi-basic 1-form $\theta$ implies the regularity of the Lagrangian function $L$. We will prove now that $L_S\theta_L = dL$. Using formula (4.9), we have $\theta_L = \theta + (L - i_S\theta)dt$ and hence, according to formulae (4.6) and (4.7), we get

$$L_S\theta_L = L_S\theta + L_S(L - i_S\theta)dt = dL,$$

which completes the proof of the theorem. $\square$

For a semispray $S$ let us introduce the set

$$\Lambda^1_S(J^1\pi) = \{ \theta \in \Lambda^1(J^1\pi), \theta \text{ semi-basic, non-degenerate, } L_S d\theta = 0 \}.$$

Theorem 4.5 states that a semispray $S$ is a Lagrangian vector field if and only if $\Lambda^1_S(J^1\pi) \neq \emptyset$. For a Lagrangian semispray $S$, on the set $\Lambda^1_S(J^1\pi)$ we introduce the following equivalence relation $\theta_1 \equiv \theta_2$ if $\theta_1 \wedge dt = \theta_2 \wedge dt$ and $S(i_S(\theta_1 - \theta_2)) = 0$. It follows immediately that $\theta_1 \equiv \theta_2$ if and only if there exists a first integral $f$ for $S$ such that $\theta_1 = \theta_2 + f dt$.

From formulae (4.9) it follows that each equivalence class $[\theta]$ in $\Lambda^1_S(J^1\pi)$ contains exactly one Poincaré-Cartan 1-form $\theta_L$ of some regular Lagrangian $L$, where

$$\theta = \theta_L + (i_S\theta - L) dt. \tag{4.10}$$

Therefore for each equivalence class $[\theta]$ in $\Lambda^1_S(J^1\pi)$ there exists a unique Lagrangian $L$ such that $[\theta] = [\theta_L]$. One can furthermore reformulate this using Lemma 4.2: each equivalence class in $\Lambda^1_S(J^1\pi)$ contains exactly one $d_J$-closed semi-basic 1-form.

If an equivalence class $[\theta_L]$ contains a semi-basic 1-form $\theta$, which is not $d_J$-closed, then $i_S\theta - L$ is a first integral and $i_Sd\theta$ is a dual symmetry for $S$. In this case the Lagrangian vector field $S$ is a conservative vector field.

According to the above discussion, we can strengthen the conclusion of Theorem 4.5 as follows.

**Proposition 4.6.** Consider $S$ a semispray.

1) $S$ is a Lagrangian vector field if and only if there exists $\theta \in \Lambda^1_S(J^1\pi)$ such that $d_J\theta = 0$. In this case $\theta = \theta_L$, for some locally defined Lagrangian function $L$, and $L_S\theta = dL$. 

16
ii) \( S \) is a conservative Lagrangian vector field if and only if there exists \( \theta \in \Lambda^1_S(J^1\pi) \) such that \( d_j\theta \neq 0 \). In this case \( \theta = \theta_k + f dt \), for some locally defined Lagrangian function \( L \) and \( f \) a conservation law for \( S \).

Necessary and sufficient conditions for the existence of semi-basic 1-forms that satisfy either condition i) or condition ii) of Proposition \ref{prop:necessary_sufficient} will be discussed in the next section in Theorems \ref{thm:semibasic_1_forms} and \ref{thm:semibasic_1_forms2}. However, we want to emphasize that a semi-basic 1-form \( \theta \in \Lambda^1_S(J^1\pi) \) which is not \( d_j \)-closed gives rise to a Lagrangian function, a first integral and a dual symmetry of the semispray, see Theorem \ref{thm:semibasic_1_forms2}.

5. **Helmholtz-type conditions**

In this section we use Frölicher-Nijenhuis theory on \( J^1\pi \) and geometric objects associated to a semispray \( S \), to obtain invariant conditions for a semi-basic 1-form \( \theta \) on \( J^1\pi \) that are equivalent with the condition that \( \mathcal{L}_S \theta \) is a closed 1-form. Therefore, in view of Theorem \ref{thm:frölicher_nijenhuis}, we obtain necessary and sufficient conditions, in terms of a semi-basic 1-form, for a semispray \( S \) to be a Lagrangian vector field. We will relate these conditions with the classic formulation of Helmholtz conditions in terms of a multiplier matrix in the next section.

5.1. **Semi-basic 1-forms, symmetries, and Helmholtz-type conditions.** In this section we present two theorems that give characterizations for those semi-basic 1-forms in the set \( \Lambda^1_S(J^1\pi) \). First theorem seeks for a solution of the inverse problem on a restricted class of semi-basic 1-forms: those forms that are \( d_j \)-closed, and hence represent the Poincaré-Cartan forms for some Lagrangian functions on \( J^1\pi \). Second theorem seeks for a solution of the inverse problem on a larger class of semi-basic 1-forms, which are not \( d_j \)-closed. It is important to note that in this case, if there is a solution then it induces a dual symmetry and a first integral of the given semispray.

**Theorem 5.1.** Let \( \theta \) be a semi-basic, \( d_j \)-closed, 1-form on \( J^1\pi \). Then, the following conditions are equivalent

i) \( \mathcal{L}_S \theta \) is closed;
ii) \( \mathcal{L}_S \theta \) is exact;
iii) \( \mathcal{L}_S \theta = d_i S \theta \);
iv) \( d_i \theta = 0 \).

**Proof.** Implications iii) \( \Rightarrow \) ii) \( \Rightarrow \) i) are immediate. Therefore it remains to prove implications iv) \( \Rightarrow \) iii) and i) \( \Rightarrow \) iv).

We prove first that condition iv) implies condition iii). Consider \( \theta \) a semi-basic 1-form on \( J^1\pi \) such that \( d_j \theta = d_i \theta = 0 \). According to Lemma \ref{lem:commutation} we have that \( d_j \theta = 0 \) implies \( \mathcal{L}_S \theta = d_i S \theta + i_S d_i \theta \). Since \( d_i \theta = 0 \) it results \( \mathcal{L}_S \theta = d_i S \theta \), which is condition iii).

Finally, we have to prove implication i) \( \Rightarrow \) iv). If \( \mathcal{L}_S \theta \) is closed, we have that \( \mathcal{L}_S d \theta = 0 \) and hence \( i_j \mathcal{L}_S d \theta = 0 \). Using the commutation rule \( i_j \mathcal{L}_S - \mathcal{L}_S i_j = i_{\Gamma - S} d \theta \) and formula \( i_j d \theta = d_j \theta = 0 \) it results

\[
(5.1) \quad i_{\Gamma - S} d \theta = 0.
\]

Now, we have \( i_{\Gamma - S} d \theta = i_{2h - i_4} d \theta = 2i_h d \theta - i_4 d \theta = 2(i_h d \theta - i_4 d \theta) = 2(i_h d \theta - d_i i_4 \theta) = 2d_h \theta \). If we substitute this in formula \ref{eq:closed}, we obtain

\[ (5.2) \quad i_S d \theta = 2d_h \theta. \]
If we apply $i_\omega$ to both sides of equation (5.1) and use the identity $i_\omega \Gamma - \iota_{\omega \otimes dt}i_\omega = 0$, we obtain

$$i_\omega \Gamma - \iota_{\omega \otimes dt}i_\omega \theta = 0. \tag{5.3}$$

If we apply $i_\omega$ to both sides of the equation (5.2), use condition ii) from Lemma 4.3 and commutation formula $i_\omega \Gamma - \iota_{\omega \otimes dt}i_\omega = i_\omega$ we obtain $2i_\omega \theta = i_{\omega \otimes dt}i_\omega \theta$, which implies $i_\theta (i_\omega \theta) = i_{\omega \otimes dt}i_\omega \theta$. Last formula is equivalent to

$$i_\theta (i_\omega \theta) = i_{\omega \otimes dt}i_\omega \theta = 0. \tag{5.4}$$

Equations (5.3) and (5.4) lead to

$$i_{\omega \otimes dt}i_\omega \theta = 0. \tag{5.5}$$

Condition iii) from Lemma 4.3 can be written as

$$i_{\omega \otimes dt}i_\omega \theta = 0. \tag{5.6}$$

Using equations (5.5) and (5.6) we obtain $i_\omega \theta = 0$. Since $i_{\omega \otimes dt}i_\omega \theta = (i_\omega \theta) \wedge dt = 0$, from (5.2) we obtain that $d\theta = 0$, which is condition iv).

Condition iii) is equivalent with condition $i_\omega \theta = 0$ required in all previous works that deals with the inverse problem of the calculus of variations for the time dependent case [1, 10, 21, 34]. For example condition $i_\omega \theta = 0$ is reflected in the expression of the 2-form $\omega$ in formula (3.2) in [1]. The next theorem deals with a larger class of semi-basic 1-forms, where condition $i_\omega \theta = 0$ is not required. In this case, in view of Proposition 4.6 we obtain a characterization for conservative Lagrangian vector fields.

**Theorem 5.2.** Let $\theta$ be a semi-basic 1-form on $J^1\pi$, which is not $d_1$-closed. Then $\mathcal{L}_\omega \theta$ is a closed 1-form if and only if the following conditions hold true.

1. $d_\omega \theta \wedge dt = 0$ ($\theta$ is $d_1$-closed modulo $dt$);
2. $d_\omega \theta \wedge dt = 0$ ($\theta$ is $d_\omega$-closed modulo $dt$);
3. $d_\omega \theta \wedge dt = 0$ ($\theta$ is $d_\omega$-closed modulo $dt$);
4. $\nabla \theta \wedge dt = 0$;
5. The 1-form $i_\omega \theta$ is a dual symmetry for $S$.

**Proof.** Suppose that $\mathcal{L}_\omega \theta$ is a closed 1-form. If we apply $i_J$ to $\mathcal{L}_\omega \theta = 0$ and use commutation rule (2.12) we obtain

$$\mathcal{L}_\omega i_J \theta + i_{\omega \otimes dt}i_J \theta = 0. \tag{5.7}$$

If we apply again $i_J$ to above formula and use commutation rule (2.12) we obtain

$$\mathcal{L}_\omega i_J d_\omega \theta + i_{\omega \otimes dt}i_J d_\omega \theta + i_J i_{\omega \otimes dt}d_\omega \theta = 0.$$  

Since $\theta$ is semi-basic it follows that $i_J d_\omega \theta = i_J^2 \theta = 2d_\theta \circ J^* = 0$. Using commutation rule (2.12), $J \circ (\Gamma - S \otimes dt) = J$, and $(\Gamma - S \otimes dt) \circ J = -J$, we get

$$0 = 2i_J i_{\omega \otimes dt}d_\theta + i_J (i_{\omega \otimes dt})^* d_\theta - i_{\omega \otimes dt}i_J d_\theta = i_J i_{\omega \otimes dt}d_\theta + i_J d_\theta.$$  

Therefore, we have

$$d_\omega \theta + i_J i_{\omega \otimes dt}d_\theta - i_J i_{\omega \otimes dt}d_\theta = 0. \tag{5.8}$$

Since $i_\omega \theta = 2d_\theta$ is a semi-basic 2-form, it follows that $i_J i_\omega \theta = 0$. Also, $i_J i_{\omega \otimes dt}d_\theta = i_{\omega \otimes dt}i_J d_\theta = i_{\omega \otimes dt}d_\theta$.

Formula (5.8) becomes

$$d_\omega \theta = i_{\omega \otimes dt}d_\theta. \tag{5.9}$$
Applying Lemma 2.3 we obtain $d_{j}\theta \wedge dt = 0$, which is condition $(H_1)$.

If we use formulae (5.9), (5.7) and commutation rule $\mathcal{L}_{S}i_{S}\otimes dt = i_{S}\otimes d\mathcal{L}_{S}$, we obtain $i_{\Gamma}d\theta = i_{S}\otimes d\theta$ and therefore $i_{\Gamma}d\theta \wedge dt = 0$ and hence $d_{h}\theta \wedge dt = 0$, which is condition $(H_2)$.

From the action of the dynamical covariant derivative $\nabla$, expressed by formula (3.22), on 2-forms we obtain

\begin{equation}
\mathcal{L}_{S}d\theta = \nabla d\theta + i_{\Psi}d\theta.
\end{equation}

Applying $i_{\Gamma}$ to both sides of this identity and using $\nabla i_{\Gamma} = i_{\Gamma}\nabla$, we obtain $\nabla i_{\Gamma}d\theta + i_{\Gamma}i_{\Psi}d\theta = 0$, which in view of commutation formula (2.14) is equivalent to $\nabla i_{\Gamma}d\theta + i_{\Psi}i_{\Gamma}d\theta + i_{\Psi}\circ D_{\Gamma}d\theta = 0$. From Proposition 5.3 we obtain $\nabla D = 0$, which implies $\Psi \circ D \circ \Psi = -\mathcal{L}_{S}D = -2\mathcal{L}_{S}h = -2(\mathcal{F} + J + \Phi)$ and hence

\begin{equation}
\nabla i_{\Gamma}d\theta + i_{\Psi}i_{\Gamma}d\theta - 2i_{\Psi + J + \Phi}d\theta = 0.
\end{equation}

Using formula (5.11) and $\nabla i_{\Gamma}d\theta \wedge dt = i_{\Psi}i_{\Gamma}d\theta \wedge dt = 0$ we obtain $(i_{\Psi + J + \Phi}d\theta) \wedge dt = 0$.

From second formula (5.10) it results that $(\nabla d\theta + i_{\Psi + J + \Phi}d\theta) \wedge dt = 0$. Therefore

\begin{equation}
\begin{cases}
(\nabla d\theta + 2i_{\Psi + J + \Phi}d\theta) \wedge dt = 0, \\
(\nabla d\theta - 2i_{\Psi}d\theta) \wedge dt = 0.
\end{cases}
\end{equation}

From formula (5.12), it follows that there exists a 1-form $\omega$ on $J^1\pi$ such that $\nabla d\theta - 2i_{\Psi}d\theta = \omega \wedge dt$. Applying $i_{\Psi}$ to this identity, we get

\begin{equation}
i_{\Psi}i_{\Gamma}d\theta = i_{\Psi}i_{\Gamma}d\theta - i_{\Psi}\omega \wedge dt + \omega \wedge dt.
\end{equation}

We also have $i_{\Psi}i_{\Gamma}d\theta = i_{\Psi}i_{\Gamma}d\theta + i_{\Psi}\circ D_{\Psi}d\theta = i_{\Psi}i_{\Gamma}d\theta + i_{\Psi}d\theta$.

Since $i_{\Psi}d\theta \wedge dt = d\theta \wedge dt$ (H2), it exists a 1-form $\tilde{\omega}$ on $J^1\pi$ such that $i_{\Psi}d\theta = d\theta + \tilde{\omega} \wedge dt$. It results that

\begin{equation}
i_{\Psi}i_{\Gamma}d\theta = 2i_{\Psi}d\theta + i_{\Psi}\tilde{\omega} \wedge dt.
\end{equation}

Formulae (5.13) and (5.14) lead to

\begin{equation}
(\nabla d\theta - 4i_{\Psi}d\theta) \wedge dt = 0.
\end{equation}

Relations (5.12) and (5.15) imply $i_{\Phi}d\theta \wedge dt = 0$, which is condition (H3) as well as $\nabla d\theta \wedge dt = 0$, which is condition (H4).

Evidently, the assumption $\mathcal{L}_{S}d\theta$ closed implies $\mathcal{L}_{S}i_{S}d\theta = i_{S}\mathcal{L}_{S}d\theta = 0$, and hence condition (DS) is also satisfied.

For the converse, we assume that there exists a semi-basic 1-form $\theta$ on $J^1\pi$ such that conditions $(H_1)$ - $(H_4)$ and (DS) are verified. We will prove that $\mathcal{L}_{S}d\theta$ is closed.

Using condition (DS) it follows that $i_{S}(\mathcal{L}_{S}d\theta \wedge dt) = \mathcal{L}_{S}(i_{S}d\theta \wedge dt) + \mathcal{L}_{S}d\theta = \mathcal{L}_{S}d\theta$. Therefore, it suffices to show that $\mathcal{L}_{S}d\theta \wedge dt = 0$. Using formula (3.22) and condition (H4) we obtain

\begin{equation}
\mathcal{L}_{S}d\theta \wedge dt = (\nabla d\theta + i_{\Psi}d\theta) \wedge dt = i_{\Psi}d\theta \wedge dt.
\end{equation}

We will show that the 2-form $i_{\Phi}d\theta \wedge dt$ vanishes by evaluating it on pairs of strong horizontal and/or vertical vector fields.

From conditions $(H_1)$ - $(H_3)$ we obtain that the 2-forms $d_{j}\theta$, $d_{h}\theta$ and $d_{\Phi}d\theta$ vanish on any pair of strong horizontal vector fields $h_{1}X, h_{1}Y$. From condition $(H_1)$ we have $i_{j}d\theta(h_{1}X, h_{1}Y) = 0$ and hence

\begin{equation}
d\theta(JX, h_{1}Y) + d\theta(h_{1}X, JY) = 0.
\end{equation}
From condition \((H_2)\) it follows that \((i_h d\theta - d\theta)(h_1X, h_1Y) = 0\), which implies
\begin{equation}
(5.17) \quad d\theta(h_1X, h_1Y) = 0.
\end{equation}
From condition \((H_3)\) we obtain that \(i_\Phi d\theta(h_1X, h_1Y) = 0\), which implies
\begin{equation}
(5.18) \quad d\theta(\Phi X, h_1Y) + d\theta(h_1X, \Phi Y) = 0.
\end{equation}

Using formulae \((5.16)\), \((5.17)\) and \((5.18)\) we obtain
\begin{align*}
(i_\Phi d\theta)(JX, JY) &= d(\Phi JX, JY) + d\theta(JX, JY) = 0, \\
(i_\Phi d\theta)(h_1X, h_1Y) &= d\theta(-\Phi X, h_1Y) + d\theta(h_1X, -\Phi Y) = 0, \\
(i_\Phi d\theta)(JX, h_1Y) &= d\theta(h_1X, h_1Y) + d\theta(JX, -\Phi Y) = 0,
\end{align*}
for any arbitrary pair of vector fields \(X, Y\) on \(J^1\pi\). Last three formulae imply that
\(i_\Phi d\theta \wedge dt = 0\) and hence \(L_\theta d\theta = 0\). \(\square\)

In view of Proposition \(1.6\) one can reformulate Theorem \(5.2\) as follows. A semispray \(S\) is a Lagrangian vector field if and only there exists a semi-basic 1-form \(\theta\) that satisfies the Helmholtz conditions \((H_1) - (H_4)\). Note that in this case we might have \(d_\theta \neq 0\) and hence \(\theta = \theta_L\) for some Lagrangian function \(L\). If \(d_\theta \neq 0\) then \(\theta\) is equivalent (modulo \(dt\)) with the Poincaré-Cartan 1-form of some Lagrangian function. In this case, \(S\) is a conservative Lagrangian vector field and \(i_\theta d\theta\) is a dual symmetry.

5.2. Semi-basic 1-forms and multiplier matrices. In this section we present a proof of the Theorem \(5.2\) using local coordinates. First, this allows to show the usefulness of the covariant derivative studied in Section \(3.2\) as well as the use of the adapted basis and cobasis \(\mathcal{B}_\theta\). Secondly, it will relate the Helmholtz-type conditions \((H_1) - (H_4)\) presented in Theorem \(5.2\) with their classic formulation for a multiplier matrix.

Consider that \(\theta = \theta_0 dt + \theta_j \delta x^j\) is a semi-basic 1-form on \(J^1\pi\) that is not \(d_\theta\)-closed and such that the 1-form \(L_\theta\) is closed. For the 1-form \(\theta\) we will use the following notations.
\begin{align*}
(5.19) \quad a_i &= \frac{\partial \theta_0}{\partial y^i} - \theta_i, \quad b_i = \frac{\delta \theta_0}{\delta x^i} - \nabla \theta_i, \\
b_{ij} &= \frac{\delta \theta_i}{\delta x^j} - \frac{\delta \theta_j}{\delta x^i}, \quad g_{ij} = \frac{\partial \theta_i}{\partial y^j}.
\end{align*}
A direct calculus using adapted cobasis \(\mathcal{B}_\theta\) leads to
\begin{equation}
(5.20) \quad d\theta = b_i \delta x^i \wedge dt + a_i \delta y^i \wedge dt + \frac{1}{2} b_{ij} \delta x^i \wedge \delta x^j + g_{ij} \delta y^i \wedge \delta x^j.
\end{equation}
We emphasize the presence of the two terms \(a_i\) and \(b_i\) in formula \((5.20)\) due to the fact that \(i_\phi d\theta \neq 0\), terms which do not appear in previous work for the time dependent case of the calculus of variations, see \([1, 10, 34]\).

Using formula \((5.22)\) for the dynamical covariant derivative, the 2-form \(L_\theta d\theta = \nabla d\theta + i_\phi d\theta\) can be expressed in terms of the adapted cobasis \(\mathcal{B}_\theta\) as follows.
\begin{align*}
L_\theta d\theta &= (\nabla b_i - a_j R^j_i \delta x^i \wedge dt + (b_i + \nabla a_i) \delta y^i \wedge dt \\
& \quad + \frac{1}{2} (\nabla b_{ij} - g_{ik} R^k_j + g_{jk} R^k_i) \delta x^j \wedge \delta x^i \\
& \quad + (\nabla g_{ij} + b_{ij} - b_{ji}) \delta y^j \wedge \delta x^i + \frac{1}{2} (g_{ij} - g_{ji}) \delta y^j \wedge \delta y^i.
\end{align*}
From above formula (5.21) it follows that the condition $\mathcal{L}_S \theta$ is closed is equivalent with the following two sets of conditions that correspond to Helmholtz conditions $(H_1)-(H_4)$ in Theorem 5.2.

\begin{align*}
(5.22) & \quad g_{ij} = g_{ji}, \\
(5.23) & \quad g_{ik} R_j^k = g_{jk} R_i^k, \\
(5.24) & \quad \nabla g_{ij} = 0, \\
(5.25) & \quad b_{ij} = 0,
\end{align*}

and respectively to condition (DS) in Theorem 5.2.

\begin{align*}
(5.26) & \quad b_i + \nabla a_i = 0, \\
(5.27) & \quad \nabla b_i - a_j R_i^j = 0.
\end{align*}

If the above two equations (5.25) and (5.26) hold good we deduce

\begin{equation}
(5.28) \quad \nabla^2 a_i + a_j R_j^i = 0
\end{equation}

which is equivalent with the fact that the 1-form

$$i_S d\theta = \nabla a_i \delta x^i - a_i \delta y^i = -b_i \delta x^i - a_i \delta y^i$$

is a dual symmetry of the semispray $S$, since it satisfies equation (3.31).

Next, we present the local expressions of the 2-forms $d_j \theta$, $d_h \theta$, $d_\phi \theta$ and $\nabla d \theta$.

\begin{align*}
d_j \theta &= a_i \delta x^i \wedge dt + \frac{1}{2} (g_{ij} - g_{ji}) \delta x^j \wedge \delta x^i, \\
d_h \theta &= b_i \delta x^i \wedge dt + \frac{1}{2} b_{ij} \delta x^j \wedge \delta x^i, \\
d_\phi \theta &= R_i^j a_j \delta x^j \wedge dt + \frac{1}{2} (g_{ik} R_j^k - g_{jk} R_i^k) \delta x^j \wedge \delta x^i, \\
\nabla d \theta &= \nabla b_i \delta x^i \wedge dt + \nabla a_i \delta y^i \wedge dt + \frac{1}{2} \nabla b_{ij} \delta x^j \wedge \delta x^i + \nabla g_{ij} \delta y^j \wedge \delta x^i.
\end{align*}

In view of above formulae, one can immediately see that Helmholtz-type conditions $(H_1) - (H_4)$ are equivalent with conditions (5.22)-(5.25). First three conditions (5.22)-(5.24) are usually known as the classic Helmholtz conditions of the multiplier matrix $g_{ij} = \partial \theta_i / \partial y^j$. Fourth classic Helmholtz condition

$$\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}$$

is identically satisfied in view of last notation (5.19).

The requirement that semi-basic 1-form $\theta$ is not $d_j$-closed implies that $i_S d_j \theta \neq 0$. Therefore $i_j i_S d\theta \neq 0$ and hence $i_S d\theta \neq 0$, which assures that a solution of equation (5.28) is not trivial. Locally, $d_j \theta \neq 0$ implies $a_i$ from formula (5.19) is not identically zero, and hence $i_S d\theta = \nabla a_i \delta x^i - a_i \delta y^i$ is a non trivial, dual symmetry of the semispray $S$.

Theorem 5.1 corresponds to the time independent case studied in Theorem 4.3, \[4\], where it is shown that for 0-homogeneous semi-basic 1-forms only two of the Helmholtz conditions are independent. These two conditions are $d_j \theta = 0$ and $d_h \theta = 0$ and appear in both Theorem 5.1 as well as Theorem 4.3 from \[4\].

Theorem 5.2 corresponds to the time independent case studied in Theorem 4.1, \[4\], where conditions $(H_1) - (H_4)$ are equivalent with the fact that a semi-basic
1-form is $d_J$, $d_h$ and $d_\Phi$-closed and satisfies $\nabla d\theta = 0$. It is important to emphasize that in general, for the time dependent case, Lagrangian semisprays are not conservative. Theorem 5.2 refers to the class of Lagrangian semisprays (conditions $(H_1) - (H_4)$ are satisfied) that have symmetries as well (condition $(DS)$ is satisfied).

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