Static black holes in scalar tensor gravity

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Abstract

We study static black hole solutions in scalar tensor gravity. We present exact solutions in hiperextended models with a quadratic scalar potential.

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1 Introduction

The fact that the ratio between the electrostatic and gravitational forces between one proton and one electron in the vacuum is of the same order as the ratio between the atomic time for the electron in its classical description and the Hubble time, impressed Dirac in such a way that he postulated those relations as fundamental constants in Nature [1]. To keep them independent of the cosmological time and to not reformulate the Atomic Physics, Dirac postulated [1] a time variation for the gravitational constant, but his theory is in conflict with the observational results [2]. Partially motivated by the Dirac’s idea and the possible existence of extra dimensions of the space-time purposed by Kaluza and Klein [3], Jordan [4] formulated a gravity theory introducing the variations of the gravitational constant as an extra degree of freedom but, again, this theory is in conflict with the observational results [2]. These ideas culminated with the work of Brans and Dicke [5] who formulated a scalar tensor theory for gravity, the so called Jordan Brans Dicke theory.

Meanwhile Penrose and Hawking [6] proved a set of theorems that showed the existence of an initial singularity of the Universe and of the final state of a collapsing star, named by Wheeler as black hole [7]. Using a theorem due to Israel [8] and some results from Doroshkevich, Zel’dovich and Novikov [7], Wheeler conjectured that a black hole had no hair meaning that it would rapidly reach a stationary state uniquely determined by three parameters: its mass, angular momentum and electric charge, independently of the details of the body that had collapsed. This conjecture was rigorously proved later by the works of Israel, Carter, Hawking and Robinson [9].

Scalar tensor gravity seems to be the most promising alternative to Einstein’s theory of general relativity, at least at sufficiently high energy scales. Even this one is extremely successful at describing the dynamics of our solar system, and indeed the observable universe. These theories are indistinguishable by the observational tests in the Solar system [10] and therefore one has to look for their implications in other regimes such as in cosmological contexts, gravitational waves, neutron stars or black holes. In particular Hawking studied in reference [11] static black holes in the Jordan Brans Dicke theory and comparing them with those in Einstein gravity, he showed that they are equivalent. This result is stated as the Hawking theorem.
In this letter we review the Hawking’s theorem and generalize it to hiperextended Jordan Brans Dicke theories.

Therefore the layout of this letter is as follows: We first present some generalities for the hiperextended Jordan Brans Dicke theories. In Sec.III we review the Hawking’s theorem for the Jordan Brans Dicke theory and in Sec.IV we generalize it for some hiperextended theories.

2 The hiperextended Jordan Brans Dicke theories

In scalar tensor theories of gravity, the gravitational coupling is proportional to the inverse of a dynamical scalar field $\phi$, or in general to a function of a scalar field $\omega(\phi)$ which couples to the geometry by a generic coupling function, $\omega(\phi)$, and that can self interact in a scalar potential, $V(\phi)$.

These theories can be classified as extended or hiperextended depending on whether the coupling function is or is not a constant respectively.

In the canonical representation, the hiperextended theories, also known as the generalized Jordan Brans Dicke theories, are described by the action:

$$ S[g_{\mu\nu}, \phi, \chi] = \int d^4x \sqrt{-g} [\phi R - \frac{\omega(\phi)}{\phi} (\nabla \phi)^2 + V(\phi) + 16\pi L_{\text{matter}}[g_{\mu\nu}, \chi]] $$  \hspace{1cm} (1)

where $\chi$ denotes generically nongravitational fields described by the Lagrangian density $L_{\text{matter}}[g_{\mu\nu}, \chi]$.

Varying the action $S$ with respect to the Jordan Brans Dicke field and to the metric one obtains the equations of motion:

$$ R - \frac{\omega(\phi)}{\phi^2} (\nabla \phi)^2 + 2 \frac{\omega(\phi)}{\phi} \Box \phi + \frac{dV(\phi)}{d\phi} + \frac{d\omega(\phi)}{d\phi} \frac{1}{\phi} (\nabla \phi)^2 = 0 $$ \hspace{1cm} (2a)

$$ G_{\mu\nu} = \frac{\omega(\phi)}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi) + \frac{V(\phi)}{2\phi} g_{\mu\nu} + \frac{8\pi}{3} T_{\mu\nu}^{\text{matter}} $$ \hspace{1cm} (2b)

where $T_{\mu\nu}^{\text{matter}}$ is the energy-momentum tensor of the matter.

Contracting equation (2b) and substituting the expression for $R$ into equation (2a) one obtains an equation for the scalar field:

$$ \Box \phi = T^{\text{matter}}_{\mu\nu} \frac{8\pi}{3 + 2\omega(\phi)} + \left( 2V(\phi) - \phi \frac{dV(\phi)}{d\phi} \right) \frac{1}{3 + 2\omega(\phi)} - \frac{d\omega(\phi)}{d\phi} \frac{(\nabla \phi)^2}{3 + 2\omega(\phi)} $$ \hspace{1cm} (3)

where $T^{\text{matter}}_{\mu\nu}$ is the trace of $T^{\text{matter}}_{\mu\nu}$ and $\omega(\phi) \neq -\frac{3}{2}$ (nonsingular model).
3  The Hawking’s theorem

Let $\omega(\phi)$ be a constant and $V(\phi) = 0$ in the action in (3). This is the Jordan Brans Dicke action or the action for the minimum model $[4]$.

Outside the horizon of a static black hole it is vacuum and therefore one gets

$$\nabla \phi = 0 . \tag{4}$$

Let $\varphi = \phi - \phi_0$ with $\phi_0$ the scalar field at far distances from the horizon. Its equation of motion is also:

$$\nabla \varphi = 0 \tag{5}$$

Multiply both members of (5) by $\varphi$ and integrate covariantly by parts between two Cauchy surfaces, one placed on the horizon and the other on a distant region far from that. One obtains:

$$\int d^4x \sqrt{-g} (\varphi,_{\alpha})^2 = 0 . \tag{6}$$

Because $\varphi$ is a static field then $\varphi,_{\alpha}$ is a space time four vector and therefore equation (6) implies that $\varphi,_{\alpha} = 0$, i.e., the scalar field is constant outside the horizon and by continuity equals to $\phi_0$, with $[4]$:

$$\phi_0 = \frac{4 + 2\omega(\phi)}{3 + 2\omega(\phi)} . \tag{7}$$

This is the Hawking theorem $[11]$.

4  Generalisation of the Hawking theorem

The Hawking theorem can be immediately generalized to extended models with $V(\phi) = V_0\phi^2$, where $V_0$ is a constant, because the equation of motion for the scalar field is the same as for the minimum model.

Let us now suppose $V(\phi) = 0$, and $\omega(\phi)$ a generic “well behaved” function. This is the Nordtvedt model $[13]$. Proceeding as in the previous section one concludes that the Hawking theorem is verified when $[14]$:

$$- 1 + (\phi - \phi_0) \frac{d\omega(\phi)}{d\phi} \frac{1}{3 + 2\omega(\phi)} \neq 0. \tag{8}$$
Combining these results it is immediate to generalize the Hawking theorem to hiperex-
tended models with \( V(\phi) = V_0 \phi^2 \) and \( \omega(\phi) \) a generic “well behaved” function, i.e., satis-
fying the constraint in (8).

When the Hawking theorem is applicable as \( \phi \) is constant (equals to \( \phi_0 \)) the “Einstein’s”
equations become:

\[
G_{\mu\nu} = \frac{V(\phi_0)}{2\phi_0} g_{\mu\nu}
\]  

(9)

and the solution for the metric is given by:

\[
d s^2 = - \left( 1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\psi^2 \right)
\]  

(10)

with \( \Lambda = \frac{1}{2} V_0 \phi_0 \). This is a Schwarzschild’s type metric with a cosmological constant \( \Lambda \).

Now let us check that this is a metric of a black hole. Calculating the invariant scalar
\( I = R_{\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} \), with \( R_{\kappa\lambda\mu\nu} \) the Riemann tensor curvature [15] one obtains:

\[
I(r) = 48 \frac{M^2}{r^6} + \frac{8}{3} \Lambda^2
\]  

(11)

and therefore there is a singularity placed at \( r=0 \) [15]. The horizon is given by

\[
r_0 = \frac{1 + \sqrt{(-3\sqrt{-\Lambda M} + \sqrt{-1 - 9\Lambda M^2})^2}}{\sqrt{-\Lambda \sqrt{-3\sqrt{-\Lambda M} + \sqrt{-1 - 9\Lambda M^2}}}}
\]  

(12)

with \( M \) the mass which is greater than the critical mass \( M_{\text{critical}} \):

\[
M_{\text{critical}} = \sqrt{-\frac{1}{9\Lambda}}
\]  

(13)

Assuming that the Cosmic Censor Conjecture [15] is valid one concludes that this is indeed
a black hole type singularity.

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