CMV MATRICES WITH ASYMPTOTICALLY CONSTANT COEFFICIENTS. SZEGŐ OVER BLASCHKE CLASS, SCATTERING THEORY

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Abstract. We develop a modern extended scattering theory for CMV matrices with asymptotically constant Verblunsky coefficients. First we represent CMV matrices with constant coefficients as a multiplication operator in $L^2$-space with respect to a specific basis. This basis substitutes the standard basis in $L^2$, which is used for the free Jacobi matrix. Then we demonstrate that a similar orthonormal system in a certain "weighted" Hilbert space, which we call the Fadeev-Marchenko (FM) space, behaves asymptotically as the system in the standard (free) case discussed just before. The duality between the two types of Hardy subspaces in it plays the key role in the proof of all asymptotics involved. We show that the traditional (Fadeev-Marchenko) condition is too restrictive to define the class of CMV matrices for which there exists a unique scattering representation. The main results are: 1) Szegő-Blaschke class: the class of twosided CMV matrices acting in $l^2$, whose spectral density satisfies the Szegő condition and whose point spectrum the Blaschke condition, corresponds precisely to the class where the scattering problem can be posed and solved. That is, to a given CMV matrix of this class, one can associate the scattering data and related to them the FM space. The CMV matrix corresponds to the multiplication operator in this space, and the orthonormal basis in it (corresponding to the standard basis in $l^2$) behaves asymptotically as the basis associated with the free system. 2) $A_2$-Carleson class: from the point of view of the scattering problem, the most natural class of CMV matrices is that one in which a) the scattering data determine the matrix uniquely and b) the associated Gelfand-Levitan Marchenko transformation operators are bounded. Necessary and sufficient conditions for this class can be given in terms of an $A_2$ kind condition for the density of the absolutely continuous spectrum and a Carleson kind condition for the discrete spectrum. Similar close to the optimal conditions are given directly in terms of the scattering data.

1. Introduction

1.1. Why almost periodic operators are so important? One of the possible reason is the following rough statement: all operators with the "strong" absolutely continuous spectrum asymptotically looks as almost periodic operators with the same a.c. spectral set. To formulate a theorem (in support of the above "philosophy") let us give some definitions.

Of course

$$a_n = \cos(\kappa n + \alpha)$$

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is the best known example of an almost periodic sequence. Generally, they are of the form
\[ a_n = f(\kappa n), \]
where \( f \) is a continuous function on a compact Abelian group \( G \) and \( \kappa \in G \).

A Jacobi matrix
\[
J = J(\{p_n\}, \{q_n\}) = \begin{bmatrix}
p_0 & q_0 & p_1 \\
p_1 & q_1 & p_2 \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]
with almost periodic coefficient sequences \( \{p_n\}_{n \in \mathbb{Z}} \) and \( \{q_n\}_{n \in \mathbb{Z}} \) is called almost periodic. In what follows we assume \( p_n > 0 \) and \( q_n \in \mathbb{R} \).

Let \( E \subset \mathbb{R} \) be a compact of positive Lebesgue measure. Moreover, we require a certain kind of regularity of the set with respect to the Lebesgue measure: there exists \( \eta > 0 \) such that
\[
|E \cap (x - \delta, x + \delta)| \geq \eta \delta \quad \text{for all } x \in E \text{ and } 0 < \delta < 1. \tag{1.1}
\]

Such set \( E \) is called homogeneous \([1]\).

Denote
\[
J(E) = \{J \text{ is almost periodic} : \sigma(J) = \sigma_{a.c.}(J) = E\}. \tag{1.2}
\]
That is \( J(E) \) is the collection of all almost periodic Jacobi matrices acting in \( l^2(\mathbb{Z}) \), such that the spectral set of \( J \) is \( E \), moreover, the support of the absolutely continuous component of the spectrum covers the whole \( E \). It is important that the above set has a parametric representation

**Theorem 1.1.** \([11]\) Let \( \pi_1(\Omega) \) be the fundamental group of the domain \( \Omega := \mathbb{C} \setminus E \) and \( \pi_1^*(\Omega) \) denote its group of characters. There exist the character \( \mu \in \pi_1^*(\Omega) \) and the continuous functions \( P(\alpha) \) and \( Q(\alpha) \), \( \alpha \in \pi_1^*(\Omega) \), such that
\[
J(E) = \{J(\alpha), \alpha \in \pi_1^*(\Omega) : p_n(\alpha) = P(\alpha \mu^{-n}), q_n(\alpha) = Q(\alpha \mu^{-n})\}. \tag{1.3}
\]

**Remark 1.2.** \( \pi_1^*(\Omega) \) is a compact (multiplicative) Abelian group. In \([11]\) precise formulae for \( P(\alpha), Q(\alpha) \) and \( \mu \) are given.

Recall that one-sided Jacobi matrices \( J_+ \) acting as bounded self-adjoint operators in \( l^2(\mathbb{Z}_+) \) are in one-to-one correspondence with compactly supported measures on \( \mathbb{R} \): \( J_+ \) is the matrix of the multiplication operator by independent variable with respect to the basis of orthonormal polynomials \( \{P_n(z; \sigma)\}_{n=0}^{\infty} \in L^2_{\sigma} \), the measure \( \sigma \) is called the spectral measure of \( J_+ \).

Let \( \omega(dx) = \omega(dx, \infty; \Omega) \) and \( G(z) = G(z, \infty; \Omega) \) be the harmonic measure on \( E \) and the Green function in the domain \( \Omega \) with respect to infinity.

**Theorem 1.3.** \([6]\) Let \( E \) be a homogenous set and \( X \) be a discreet subset of \( \mathbb{R} \setminus E \), consisting of points accumulating to \( E \) only, moreover
\[
\sum_{x_j \in X} G(x_j) < \infty. \tag{1.4}
\]

Assume that \( \sigma \) is a measure on \( E \cup X \) such that
\[
\int_E |\log \sigma'_{a.c.}(x)| \omega(dx) < \infty, \tag{1.5}
\]
and $J_+$ is the Jacobi matrix associated to the $\sigma$.

Then there exists $\alpha \in \pi_1^\ast(\Omega)$ such that

$$p_n - P(\alpha \mu^n) \to 0, \quad q_n - Q(\alpha \mu^n) \to 0.$$ (1.6)

**Remark 1.4.** In [6] asymptotic for $P_n(z; \sigma)$ is also given.

In this paper we present a similar to Theorem 1.3 result for CMV matrices in the basic case, when the resolvent set is a simply–connected domain. The general case of a multi–connected domain with a homogeneous boundary will be considered in a forthcoming paper (we would like to mention this here, in particular, to indicate the power of our method).

1.2. **CMV matrices with constant coefficients.** For a given sequence of numbers from the unit disk $\mathbb{D}$

$$..., a_{-1}, a_0, a_1, a_2, ...$$ (1.7)

define unitary $2 \times 2$ matrices

$$A_j = \begin{bmatrix} a_j & \rho_j \\ \rho_j & -a_j \end{bmatrix}, \quad \rho_j = \sqrt{1 - |a_j|^2},$$

and unitary operators in $l^2(\mathbb{Z}) = l^2(\mathbb{Z}_-) \oplus l^2(\mathbb{Z}_+)$ given by block–diagonal matrices

$$\mathfrak{A}_0 = \begin{bmatrix} \vdots & A_{-2} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathfrak{A}_1 = S \begin{bmatrix} \vdots & A_{-1} & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} S^{-1},$$

where $S|j\rangle = |j+1\rangle$. The CMV matrix $\mathfrak{A}$, generated by the sequence (1.7), is the product

$$\mathfrak{A} = \mathfrak{A}(\{a_j\}) := \mathfrak{A}_0 \mathfrak{A}_1,$$ (1.8)

CMV matrices have been studied by M.J. Cantero, L. Moral, and L. Velásquez [2], for historical details see [10].

Recall that a Schur function $\theta_+(v)$, $|\theta_+(v)| \leq 1$, $v \in \mathbb{D}$, (a finite Blaschke product is a special case) is in a one to one correspondence with the so called Schur parameters

$$\theta_+(v) \sim \{a_0, a_1, \ldots\},$$ (1.9)

where

$$\theta_+(v) = \frac{a_0 + v \theta_+^{(1)}(v)}{1 + v \overline{a_0} \theta_+^{(1)}(v)},$$

$a_0 = \theta_+(0)$, $a_1 = \theta_+^{(1)}(0)$, and so on...

It is evident that the matrix $\mathfrak{A}(\{a_k\})$ is well defined by the two Schur functions $\{\theta_+(v), \theta_-(v)\}$, given by (1.9) and

$$\theta_-(v) \sim \{-a_{-1}, -a_{-2}, \ldots\}.$$ (1.10)

The spectral set of a CMV matrix $\mathfrak{A}_a$ with the constant coefficients $a_n = a \neq 0$ is an arc

$$E = E_a = \{e^{i\xi}: \xi_0 \leq \xi \leq 2\pi - \xi_0\},$$ (1.11)

where $\rho = \sqrt{1 - |a|^2} = \cos \frac{\xi_0}{2}$. 

The following construction is a very special case of a functional realization of almost periodic operators [5, 7]. The domain $\Omega = \mathbb{C} \setminus E$ is conformally equivalent to the unit disk $\mathbb{D}$:

$$z = \frac{1 + v}{1 - v}, \; v \in \Omega,$$

$$2 \tan \frac{\xi_0}{2} z = \zeta + \frac{1}{\zeta}, \; \zeta \in \mathbb{D}. \quad (1.12)$$

Put $\varkappa := -i \tan \frac{\pi - \xi_0}{4} \in \mathbb{D}$, so that

$$v(\varkappa) = 0, \quad v(\bar{\varkappa}) = \infty. \quad (1.13)$$

The Green function $G(v, v_0) = G(v, v_0; \Omega)$ is of the form

$$G(v(\zeta), v(\zeta_0)) = \log \frac{1}{|b_{\zeta_0}(\zeta)|}, \quad (1.14)$$

where

$$b_{\zeta_0}(\zeta) = e^{ic \frac{\zeta - \zeta_0}{1 - \zeta_0}} \quad (1.15)$$

is the Blaschke factor in $\mathbb{D}$. For $\Im \zeta_0 \neq 0$ it is convenient to use the normalization $b_{\zeta_0}(\zeta_0) > 0$. In particular,

$$b_{\varkappa}(\zeta) = \frac{1}{i} \frac{\zeta - \varkappa}{1 - \zeta \bar{\varkappa}}$$

and therefore $v(\zeta) = \frac{b_{\varkappa}(\zeta)}{b_{\bar{\varkappa}}(\zeta)}$. \quad (1.16)

For the Lebesgue measure $dm = dm(\tau)$ on the unit circle $\mathbb{T}$ as usual we define the $L^2$–norm by

$$||f||^2 = \int_\mathbb{T} |f(\tau)|^2 \; dm(\tau), \quad (1.17)$$

so that the reproducing kernel of the $H^2$ subspace is of the form $k_{\zeta_0}(\zeta) = k(\zeta, \zeta_0) = \frac{1}{1 - \zeta \bar{\zeta}_0}$. By $K(\zeta, \zeta_0)$ we denote the normalized kernel

$$K(\zeta, \zeta_0) = \frac{k_{\zeta_0}(\zeta)}{||k_{\zeta_0}||} = \sqrt{1 - |\zeta_0|^2} \frac{1}{1 - \zeta \bar{\zeta}_0}. \quad (1.18)$$

Using the above notation, to the given $\beta_k = e^{2\pi i \tau_k} \in \mathbb{T}$, $k = 0, 1$, we associate the space $H^2(\beta_0, \beta_1)$ of analytic multivalued functions $f(\zeta)$, $\zeta \in \mathbb{D} \setminus \{\varkappa, \bar{\varkappa}\}$, such that $|f(\zeta)|^2$ is singlevalued and has a harmonic majorant and

$$f \circ \gamma_i = \beta_i f,$$

where $\gamma_i$ is a small circle around $\varkappa$ and $\bar{\varkappa}$ respectively. Such a space can be reduced to the standard Hardy space $H^2$, moreover

$$H^2(\beta_0, \beta_1) = b_{\varkappa}^{-1} b_{\bar{\varkappa}}^{-1} H^2. \quad (1.19)$$

**Lemma 1.5.** Let $b = \sqrt{b_{\varkappa} b_{\bar{\varkappa}}}$. The space $bH^2(1, -1)$ is a subspace of $H^2(-1, 1)$ having a one dimensional orthogonal compliment, moreover

$$H^2(-1, 1) = \{ \sqrt{b_{\varkappa} k_{\varkappa}} \} \oplus bH^2(1, -1). \quad (1.19)$$

Iterating, now, the decomposition (1.19)

$$H^2(-1, 1) = \{ \sqrt{b_{\varkappa} k_{\varkappa}} \} \oplus bH^2(1, -1)$$

$$= \{ \sqrt{b_{\varkappa} k_{\varkappa}} \} \oplus b\{ \sqrt{b_{\varkappa} k_{\varkappa}} \} \oplus b^2 H^2(-1, 1) = ..., \quad (1.19)$$
one gets an orthogonal basis in $H^2(-1,1)$ consisting of vectors of two sorts
\[ b^{2m}\{\sqrt[2]{b_{2m}}K_{2m}\} \quad \text{and} \quad b^{2m+1}\{\sqrt[2]{b_{2m+1}}K_{2m+1}\}. \tag{1.20} \]
Note that this orthogonal system can be extended to the negative integers $m$ so that we obtain a basis in the standard $L^2$.

**Theorem 1.6.** With respect to the orthonormal basis
\[ e_n = \begin{cases} b^{2m}\sqrt[2]{b_{2m}}K_{2m}e^{ic}, & n = 2m \\ b^{2m+1}\sqrt[2]{b_{2m+1}}K_{2m+1}, & n = 2m + 1 \end{cases} \tag{1.21} \]
the multiplication operator by $v$ is the CMV matrix $A_v$, $a = e^{ie\frac{b_{2m}}{b_{2m+1}}} = e^{ie\sin \delta_0}^2$.

As it was mentioned the above theorem is a part of a very general construction. In this particular case, we will break the symmetry of the shift operation. Paying this price we can use the standard $H^2$ space. Factoring out $\sqrt[2]{b_{2m}}$, we get

**Theorem 1.7.** The system of functions
\[ e_{n,c} = \begin{cases} b^{2m}\sqrt[2]{b_{2m}}K_{2m}e^{ic}, & n = 2m \\ b^{2m+1}\sqrt[2]{b_{2m+1}}K_{2m+1}, & n = 2m + 1 \end{cases} \tag{1.22} \]
forms orthonormal basis in $H^2$ if $n \in \mathbb{Z}_+$ and in $L^2$ if $n \in \mathbb{Z}$. With respect to this basis the multiplication operator by $v$ is the CMV matrix $A_v$.

1.3. Szegő over Blaschke classes and the direct scattering. It follows from
\[ \mathfrak{A}\{2n-1|\rho_{2n-1} = 2n+1|\rho_{2n}\} = 2n+1|\rho_{2n}+1a_{2n+1} = 2n+2|\rho_{2n+1} \tag{1.23} \]
that the subspace formed by the vectors $| -1 \rangle, |0 \rangle$ is cyclic for $\mathfrak{A}$. The resolvent matrix–function is defined by the relation
\[ \mathcal{R}(v) = \mathcal{E} \frac{\mathfrak{A} + v}{\mathfrak{A} - v}, \tag{1.24} \]
where $\mathcal{E} : \mathbb{C}^2 \to l^2(\mathbb{Z})$ in such a way that
\[ \mathcal{E} \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = | -1 \rangle c_{-1} + |0 \rangle c_0. \]

This matrix–function possesses the integral representation
\[ \mathcal{R}(v) = \int_\Sigma^+ \frac{t + v}{t - v} d\Sigma(t) \tag{1.25} \]
with $2 \times 2$ matrix–measure. $\mathfrak{A}$ is unitary equivalent to the multiplication operator by an independent variable on
\[ L^2_{d\Sigma} = \left\{ f = \begin{bmatrix} f_{-1}(t) \\ f_0(t) \end{bmatrix} : \int_\Sigma f^*(t)d\Sigma(t)f(t) < \infty \right\}. \]

Note that $\Sigma$ is not an arbitrary $2 \times 2$ matrix–measure, but has a specific structure. In terms of the Schur functions (1.9), (1.10)
\[ \mathcal{R}(v) = \frac{I + vA_{-1}}{I - vA_{-1}^*} \begin{bmatrix} \theta_{-1}(v) & 0 \\ 0 & \theta_{+}(v) \end{bmatrix} \tag{1.26} \]
In particular, this means that the range of the measure $\Sigma(t_0)$ at an isolated spectral point $t_0$ is one, and the spectral density has the form

$$
\frac{d\Sigma(t)}{dm(t)} = \frac{I + R^+(t)}{2} \begin{bmatrix} 1 - |\theta^{-1}_+(t)|^2 & 0 \\ 0 & 1 - |\theta^+_+(t)|^2 \end{bmatrix} \frac{I + R(t)}{2}.
$$

(1.27)

**Definition 1.8.** Let $E$ be an arc of the form (1.11) and $X$ be a discreet set in $T \setminus E$, which satisfies the Blaschke condition in $\Omega = \mathbb{C} \setminus E$:

$$X = \{ t_k = v(\zeta_k) : \zeta_k \in \mathcal{Z} \},$$

$$\mathcal{Z} = \{ \zeta_k \in \mathbb{R} \cap \mathbb{D} : \sum (1 - |\zeta_k|) < \infty \}. \quad (1.28)
$$

We say that $\mathfrak{A}$ is in $\mathfrak{A}_{SB}(E)$ if $\sigma(\mathfrak{A}) = E \cup X$, the spectral measure is absolutely continuous on $E$,

$$d\mathfrak{A}|E = W(t) \, dm(t) \quad (1.29)$$

and the density satisfies the Szegö condition

$$\log \det W(\nu(\tau)) \in L^1. \quad (1.30)$$

**Remark 1.9.** Due to condition (1.28), $R(v)$ is of bounded characteristic in $\Omega$ [11, Theorem D], see Sect 5. Therefore (1.30) is equivalent to

$$\log(1 - |\theta^{-1}_+(\nu(\tau))|^2)(1 - |\theta^+_+(\nu(\tau))|^2) \in L^1. \quad (1.31)$$

With the set $X$ (1.28) we associate the Blaschke product

$$B(\zeta) = \prod_{\mathcal{Z}} \frac{|\zeta_k|}{\zeta_k - \zeta} \quad \frac{\zeta_k - \zeta}{1 - \zeta_k} \quad (1.32)$$

(this product contains the factor $\zeta$ if $0 \in \mathcal{Z}$).

**Theorem 1.10.** Let $\mathfrak{A} \in \mathfrak{A}_{SB}(E)$. Then there exists a generalized eigen vector (see (1.23))

$$v(\tau)\{ e^+(2m - 1, \tau)\rho_{2m-1} - e^+(2m, \tau)\rho_{2m-1} \}$$

$$= e^+(2m, \tau)\rho_{2m} + e^+(2m + 1, \tau)\rho_{2m} \quad (1.33)$$

$$v(\tau)^{-1}\{ e^+(2m, \tau)\rho_{2m} - e^+(2m + 1, \tau)\rho_{2m} \}$$

$$= e^+(2m + 1, \tau)\rho_{2m+1} + e^+(2m + 2, \tau)\rho_{2m+1}$$

such that (see (1.22))

$$T_+(\tau) e^+(-n - 1, \tau) = \tau \epsilon_{n,-}(-\tau) + R_-(n, \tau) \epsilon_{n,-}(\tau) + o(1),$$

$$T_+(\tau) e^+(n, \tau) = T_+(\tau) \epsilon_{n,+}(\tau) + o(1) \quad (1.34)$$

in $L^2$ as $n \to \infty$. Moreover the functions

$$b_{\kappa}^{-m} b_{\kappa}^{-m}(BT_+(\tau)) e^+(2m, \tau) \quad \text{and} \quad b_{\kappa}^{-m} b_{\kappa}^{-m-1}(BT_+(\tau)) e^+(2m + 1, \tau)$$

belong to $H^2$ for all $m \in \mathbb{Z}$, that is, $e^+(n, \zeta)$ is well defined in $\mathbb{D}$ and

$$\frac{1}{\nu_+}(\zeta_k) := \sum_{n=-\infty}^\infty |e^+(n, \zeta_k)|^2 < \infty \quad (1.36)$$

for all $\zeta_k : v(\zeta_k) \in X$. 

Let $\iota|n\rangle := |1 - n\rangle$. The following involution acts on the isospectral set of CMV matrices

$$\iota \mathfrak{A}\{a_n\} = \mathfrak{A}\{-\bar{a}_{-n-\frac{1}{2}}\}. \quad (1.37)$$

Theorem 1.10 with respect to $\iota A \iota$ can be rewritten into the form

**Corollary 1.11.** Simultaneously with the eigen vector (1.33) there exists the vector

$$v(\tau)\{e^{-(-2m, \tau)}\rho_{2m-1} - e^{-(-2m-1, \tau)}\bar{a}_{2m-1}\}$$

$$= e^{-(-2m-1, \tau)}a_{2m} + e^{-(-2m-2, \tau)}\rho_{2m}$$

$$v(\tau)^{-1}\{e^{-(-2m-1, \tau)}\rho_{2m} - e^{-(-2m-2, \tau)}a_{2m}\}$$

$$= e^{-(-2m-2, \tau)}a_{2m+1} + e^{-(-2m-3, \tau)}\rho_{2m+1}, \quad (1.38)$$

possessing the asymptotics

$$T_- (\tau)e^{-(-n-1, \tau)} = \bar{\tau}e_{n,e_+} (\bar{\tau}) + R_e^+(\tau) e_{n,e_+} (\tau) + o(1),$$

$$T_- (\tau)e^{-(-n, \tau)} = T_- (\tau)e_{n,e_-} (\tau) + o(1) \quad (1.39)$$

in $L^2$ as $n \to \infty$. And, also,

$$\frac{1}{\nu_- (\zeta_k)} := \sum_{n=-\infty}^{\infty} |e^{-n, \zeta_k}|^2 < \infty \quad (1.40)$$

for all $\zeta_k : v(\zeta_k) \in X$.

**Remark 1.12.** 1. We will see that Theorem 1.10 belongs to the family of Szegö kind results. Actually it claims that an orthonormal system in a certain "weighted" Hilbert space behaves asymptotically as such a system in the "unweighted" space.

2. On the other hand it belongs to the family of "direct scattering theorems" having the following specific: the class of CMV matrices is given in terms of their spectral properties, but not in terms of a behavior of the coefficients sequences.

$R_\pm , T_\pm$ in asymptotics (1.34), (1.39) are called the reflection and transmission coefficients respectively. They form the, so called, scattering matrix

$$S(\tau) = \begin{bmatrix} R_- & T_- \\ T_+ & R_+ \end{bmatrix} (\tau), \quad \tau \in \mathbb{T}. \quad (1.41)$$

**Proposition 1.13.** The matrix function $S$ possesses two fundamental properties: $S^*(\bar{\tau}) = S(\tau)$ and it is unitary-valued. The third property is analyticity of the entries $T_\pm$, each of them has analytic continuation in $\mathbb{D}$ as a function of bounded characteristic of a specific nature, — it is the ratio of an outer function and a Blaschke product. That is,

- $R_+$ is a contractive symmetric Szegö function on $\mathbb{T}$:

  $$|R_+(\tau)| \leq 1, \quad R_+(\tau) = \overline{R_+(\tau)},$$

  $$\int_{\mathbb{T}} \log(1 - |R_+(\tau)|^2)dm(\tau) > -\infty; \quad (1.42)$$

- All other coefficients are of the form

  $$T_- := \frac{O}{B}, \quad T_+(\tau) = \overline{T_- (\bar{\tau})} \quad \text{and} \quad R_- := -\frac{T_-}{T_+}, \quad (1.43)$$
where $O$ is the outer function in the unit disk $\mathbb{D}$, such that
\[ |O|^2 + |R_+|^2 = 1 \text{ a.e. on } \mathbb{T} \] (1.44)
and
\[ T_\pm(x) = -ie^{i\varphi}|T_\pm(x)|. \] (1.45)

Also
\[ \frac{1}{\nu_+(\zeta_k)\nu_-(\zeta_k)} = \left(\frac{1}{T_\pm} \right)'(\zeta_k)^2. \] (1.46)

Now, we define the Faddeev–Marchenko Hilbert space.

**Definition 1.14.** Set
\[ \alpha_+ := \{ R_+, \nu_+ \}. \] (1.47)

An element $f$ of the space $L^2_{\alpha_+}$ is a function on $\mathbb{T} \cup \mathbb{Z}$ such that
\[
||f||^2_{L^2_{\alpha_+}} = \sum_{\zeta_k \in \mathbb{Z}} |f(\zeta_k)|^2 \nu_+(\zeta_k) \\
+ \frac{1}{2} \int_{\mathbb{T}} |f(\tau)|^2 \left[ \frac{1}{R_+(\tau)} - \frac{\bar{R}_+(\tau)}{1} \right] |f(\tau)| \, dm
\] (1.48)
is finite.

**Theorem 1.15.** For $\mathfrak{A} \in \mathfrak{A}_{SB}(E)$ the system
\[ \{ e^+(n, \zeta) \}_{n=-\infty}^{\infty} \] (1.49)
forms an orthonormal basis in the associated space $L^2_{\alpha_+}$. Therefore, the map
\[ \mathcal{F}^+: l^2(\mathbb{Z}) \to L^2_{\alpha_+} \text{ such that } \mathcal{F}^+|n := e^+(n, \zeta) \] (1.50)
is unitary. Moreover $\mathcal{F}^+\mathfrak{A}(\mathcal{F}^+)^*$ is the multiplication operator by $v$.

(1.50) is called the scattering representation of $\mathfrak{A}$. Note that simultaneously we have the representation
\[ \mathcal{F}^-: l^2(\mathbb{Z}) \to L^2_{\alpha_-} \text{ such that } \mathcal{F}^-| -n-1 := e^-(n, \zeta). \] (1.51)

**Theorem 1.16.** The scattering representations (1.50), (1.51) determine each other by
\[
T_\pm(\tau)(\mathcal{F}^\pm \hat{f})(\tau) = \bar{\tau}(\mathcal{F}^\mp \hat{f})(\bar{\tau}) + R_\mp(\tau)(\mathcal{F}^\pm \hat{f})(\tau), \quad \tau \in \mathbb{T},
\]
\[
(\mathcal{F}^\mp \hat{f})(\zeta_k) = -\left(\frac{1}{T_\pm} \right)'(\zeta_k)\nu_+(\zeta_k)(\mathcal{F}^\pm \hat{f})(\zeta_k), \quad \zeta_k \in \mathbb{Z},
\] (1.52)
for $\hat{f} \in l^2(\mathbb{Z})$, and have the following analytic properties
\[ (BT_{\pm})\mathcal{F}^\pm(l^2(\mathbb{Z})) \subset H^2. \] (1.53)

Finally, let us mention the important Wronskian identity. Put formally
\[ e^+(2m, \tau) = e^{i\varphi} b^m b^m \lambda(2m, \tau), \]
(1.54)
and
\[ e^+(2m+1, \tau) = b^m b^{m+1} \lambda(2m+1, \tau), \]
\[ e^+(2m+2, \tau) + a_{2m+1} e^+(2m+1, \tau) = e^{i\varphi} b^m b^{m+1} \lambda(2m+1, \tau). \] (1.55)
Then
\[
\begin{vmatrix}
\tau L_{\hat{\alpha}}(n, \tau) & L_{\hat{\alpha}}(n, \tau) \\
\tau L_{\hat{\alpha}}(n, \tau) & \bar{L}_{\hat{\alpha}}(n, \tau)
\end{vmatrix} = \frac{d \log v(\tau)}{d\tau}.
\] (1.56)

1.4. Inverse scattering: a brief discussion. The unimodular constant \(e^{ic_+}\) and the pair \(\alpha_+\) (1.47) are called the scattering data.

A fundamental question is how to recover the CMV matrix from the scattering data? When can this be done? Do we have a uniqueness theorem?

We say that the scattering data are in the Szegő over Blaschke class, \(\alpha_+ \in A_{SB}(E)\), if

- \(R_+\) is of the form (1.42),
- \(\nu_+\) is a discrete measure supported on \(Z\) (1.28).

Let us point out that we did not even assume that the measure \(\nu_+\) is finite.

In short: to every scattering data of this class we can associate the system of reflection/transmission coefficients by (1.43), (1.44), (1.45), the dual measure \(\nu_-\) (1.46) and the constant \(e^{ic_-}\) (1.45) in such a way that there exists a CMV matrix from \(A_{SB}(E)\), which satisfies Theorem 1.10 and Corollary 1.11 with these data.

To this end we associate with \(\alpha_+\) the Faddeev–Marchenko space \(L^2_{\alpha_+}\), define a Hardy type subspace \(\hat{H}^2_{\alpha_+}\), in it, and, similar to (1.22), construct the orthonormal basis (at this place the constant \(e^{ic_+}\) is required). Then, the multiplication operator (with respect to this basis) is the CMV matrix and the claim of Theorem 1.10 is a Szegő kind result on the asymptotics of this orthonormal system.

For a brief explanation of the uniqueness problem we would like to use the following analogy. For the measure \(d\mu = w(\tau)dm(\tau)\), with \(w \in L^1\), we can define the Hardy space \(\hat{H}^2_{\alpha_+}\) as the closer of \(H^\infty\) (or polynomials) in \(L^2_{\mu}\)-sense. On the other hand, let us define the outer function \(\phi\) such that \(|\phi|^2 = w\) and then define
\[
\hat{H}^2_\phi := \left\{ f = \frac{g}{\phi} : g \in H^2 \right\}.
\] (1.57)

According to the Beurling Theorem [3] these two Hardy spaces are the same. But in the Faddeev–Marchenko setting their counterparts \(\hat{H}^2_{\alpha_+}\) and \(\hat{H}^2_{\alpha_-}\) not necessarily coincide. For the data \(\alpha_+\) the uniqueness takes place if and only if \(\hat{H}^2_{\alpha_+} = \hat{H}^2_{\alpha_-}\).

1.5. Hardy subspaces in the Faddeev–Marchenko space. Duality. Let \(\alpha_+ \in A_{SB}(E)\). Define \(T_{\pm}\), the dual data: \(e^{ic_-}\), \(\alpha_- := \{R_-, \nu_-\}\), and set
\[
\begin{bmatrix}
T_+f^+ \\
T_-f^-
\end{bmatrix}(\tau) = \begin{bmatrix} T_+ & 0 \\
R_+ & 1 \end{bmatrix} \begin{bmatrix} f^+(\tau) \\
\bar{f}^+(\bar{\tau}) \end{bmatrix}
= \begin{bmatrix} 1 & R_- \\
0 & T_- \end{bmatrix} \begin{bmatrix} f^-(\tau) \\
\bar{f}^-(\bar{\tau}) \end{bmatrix}
\] (1.58)

for \(\tau \in \mathbb{T}\) and
\[f^-(\zeta_k) = -\left(\frac{1}{T_-}\right)'(\zeta_k)\nu_+(\zeta_k)f^+(\zeta_k)\] (1.59)

for \(\zeta_k \in Z\). It is evident that in this way we define a unitary map from \(L^2_{\alpha_-}\) to \(L^2_{\alpha_+}\), in fact, due to (1.58)
\[
\frac{1}{2} \int_\mathbb{T} \left| f^+(\tau) \right|^2 \left| \bar{f}^+(\bar{\tau}) \right|^2 \left| \frac{1}{R_+(\tau)} \right|^2 \left| \bar{f}^+(\bar{\tau}) \right|^2 dm = \frac{||T_+f^+||^2 + ||T_-f^-||^2}{2},
\] (1.60)
where in the RHS we have the standard $L^2$ norm on $\mathbb{T}$. The key point is duality not only between these two spaces but, what is more important, between corresponding Hardy subspaces.

Actually now we give two versions of definitions of Hardy subspaces (in general, they are not equivalent!). Due to the first one $\hat{H}^2_{\alpha_+}$ basically is the closer of $H^\infty$ with respect to the given norm. More precisely, let $B = \{B_N\}$, where $B_N$ is a divisor of $B$ such that $B/B_N$ is a finite Blaschke product. Then

$$f := B_N g, \quad g \in H^\infty, \quad B_N \in B,$$

(1.61)

belongs to $L^2_{\alpha_+}$. By $\hat{H}^2_{\alpha_+}$ we denote the closer in $L^2_{\alpha_+}$ of functions of the form (1.61). Let us point out that every element $f$ of $\hat{H}^2_{\alpha_+}$ is such that $Of$ belongs to the standard $H^2$, see (1.60). Therefore, in fact, $f(\zeta)$ has an analytic continuation from $\mathbb{T}$ in the disk $\mathbb{D}$. Moreover, the value of $f$ at $\zeta_k$, due to this continuation, and $f(\zeta_k)$ that should be defined for all $\zeta_k \in \mathbb{Z}$, since $f$ is a function from $L^2_{\alpha_+}$, still perfectly coincide.

The second space also consists of functions from $L^2_{\alpha_+}$ having an analytic continuation in $\mathbb{D}$.

**Definition 1.17.** A function $f \in L^2_{\alpha_+}$ belongs to $\hat{H}^2_{\alpha_-}$ if $g(\tau) := (BT_+ f)(\tau)$, $\tau \in \mathbb{D}$, belongs to the standard $H^2$ and

$$f(\zeta_k) = \left( \frac{g}{BT_+} \right)(\zeta_k), \quad \zeta_k \in \mathbb{Z},$$

where in the RHS $g$ and $BT_+$ are defined by their analytic continuation in $\mathbb{D}$.

The following theorem clarifies relations between two Hardy spaces.

**Theorem 1.18.** Let $f^+ \in L^2_{\alpha_+} \ominus \hat{H}^2_{\alpha_+}$ and let $f^- \in L^2_{\alpha_-}$ be defined by (1.58), (1.59). Then $f^- \in \hat{H}^2_{\alpha_-}$. In short, we write

$$(\hat{H}^2_{\alpha_-})^+ = L^2_{\alpha_-} \ominus \hat{H}^2_{\alpha_+}. \quad (1.62)$$

### 1.6. Main results.

Both $\hat{H}^2_{\alpha_+}$ and $\hat{H}^2_{\alpha_-}$ are spaces of analytic functions in $\mathbb{D}$ with the reproducing kernels, which we denote by $\hat{k}_{\alpha_+,\zeta_0} = \hat{k}_{\alpha_+}(\zeta, \zeta_0)$ and $\hat{k}_{\alpha_-,\zeta_0} = \hat{k}_{\alpha_-}(\zeta, \zeta_0)$ respectively. We put

$$\hat{K}_{\alpha_+} = \hat{k}_{\alpha_+} / \|\hat{k}_{\alpha_+}\|, \quad \hat{K}_{\alpha_-} = \hat{k}_{\alpha_-} / \|\hat{k}_{\alpha_-}\|. \quad (1.63)$$

Define the following shift operation on the scattering data

$$\alpha^+_n := \{ R_+^{(n)} , \nu_+^{(n)} \} := \{ b_{\mu, \nu} R_+, b_{\mu, \nu}^+ \nu_+ \}, \quad n \in \mathbb{Z}. \quad (1.64)$$

**Theorem 1.19.** Let $K_{\alpha_+,\zeta_0}$ denote one of the normalized kernel in (1.63). The system of functions

$$e^+(n, \tau) = \begin{cases} \left( \frac{1}{b_{\mu, \nu}^{+m} K_{\alpha_+}^{n}(\tau) e^{ic_+}} \right), & n = 2m \\ \left( \frac{1}{b_{\mu, \nu}^{+m+1} K_{\alpha_+}^{n}(\tau) e^{ic_+}} \right), & n = 2m + 1 \end{cases} \quad (1.65)$$

forms orthonormal basis in $\hat{H}^2_{\alpha_+}$ and $\hat{H}^2_{\alpha_-}$ respectively, if $n \in \mathbb{Z}_+$ and in the whole $L^2_{\alpha_+}$ if $n \in \mathbb{Z}$. With respect to this basis the multiplication operator by $v(\tau)$ is
the CMV matrix $\mathcal{A} \in \mathfrak{A}_{SB}(E)$ with the coefficients given by (1.33). Moreover, the scattering data given by Proposition 1.13 and the dual orthonormal system

$$T_-(\tau)e^-(1-n,\tau) := \tau e^+(n,\tau) + R_+(\tau)e^+(n,\tau)$$

(1.66)

correspond to $\mathcal{A}$ in the sense of Theorem 1.10 and Corollary 1.11.

An important observation is the following

**Proposition 1.20.** Let $\mathcal{A} \in \mathfrak{A}_{SB}(E)$, and let $\alpha_+$ and $F^+$ correspond to this matrix. Then

$$\hat{H}^2_{\alpha_+} \subset F^+(l^2(\mathbb{Z}^+)) \subset \hat{H}^2_{\alpha_+}.$$  

(1.67)

Due to this observation Theorem 1.10 is proved as a corollary of Theorem 1.19.

Concerning the uniqueness problem:

**Theorem 1.21.** The scattering data $\alpha_+$, $e^{i\theta^+}$ determine $\mathcal{A} \in \mathfrak{A}_{SB}(E)$ if and only if

$$\hat{k}_{\alpha_+}(\tau,\bar{\zeta})\hat{k}_{\alpha_+}^{-1}(\bar{\tau},\bar{\zeta}) = \frac{1}{|T_\pm(\tau)|^2} \frac{1}{|1-|\tau|^2|^2}.$$  

(1.68)

**Corollary 1.22.** Let $\mathcal{A} \in \mathfrak{A}_{SB}(E)$ and $W$ be its spectral density (1.29). If

$$\int_E W^{-1}(t) \, dm(t) < \infty,$$  

(1.69)

then there is no other CMV matrix of $\mathfrak{A}_{SB}(E)$ class corresponding to the same scattering data.

In fact, (1.69) means that $e^\pm(n,\tau) \in L^2$ for $n = -1, 0$ and, therefore, for all $n \in \mathbb{Z}$. In this case there exist the decompositions

$$e^\pm(n,\tau) = \sum_{t \geq n} M^\pm_{t,n} \xi_{t,c^\pm}(\tau).$$  

(1.70)

The following matrix

$$\mathcal{M}_+ = 
\begin{bmatrix}
M^0_{0,0} & 0 & 0 & \ldots \\
M^0_{1,0} & M^0_{1,1} & 0 & \ldots \\
M^0_{2,0} & M^0_{2,1} & M^0_{2,2} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

(1.71)

yields the transformation (Gelfand–Levitan–Marchenko) operator, acting in $l^2(\mathbb{Z}^+)$. Similarly we define $\mathcal{M}_- : l^2(\mathbb{Z}^-) \to l^2(\mathbb{Z}^-)$ (for details see Sect. 8). Note that under condition (1.69) they are not necessary bounded. We present necessary and sufficient conditions when the scattering data determine the CMV matrix and both transformation operators $\mathcal{M}_\pm$ are bounded.

For $\theta \in \Theta_{SB}(E)$ consider the following two conditions:

(i) for all arcs $I \subset E$

$$\sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty,$$  

(1.72)

where $w(t) := \frac{1-|\theta(t)|^2}{|1-\theta(t)|^2}$, and

$$\langle w \rangle_I := \frac{1}{|I|} \int_I w(t) \, dm(t).$$
(ii) for all arcs of the form \( I = (e^{i\xi}, e^{i\xi_0}) \) or \( I = (e^{-i\xi_0}, e^{-i\xi}) \), \( I \subset T \setminus E \),

\[
\sup_I \left\{ \sum_{e^{i\eta_k} \in Y \cap I} \frac{1}{|I|} \frac{d \log v}{d \log \theta} (e^{i\eta_k}) \right\} < \infty,
\]

where \( Y = \{ e^{i\eta_k} \in T \setminus E : \theta(e^{i\eta_k}) = 1 \} \), and

\[
I_k = \begin{cases} 
(e^{i\xi_0}, e^{i(2\xi_0-\eta_k)}), & \eta_k > 0, \\
(e^{-i(2\xi_0-\eta_k)}, e^{-i\xi_0}), & \eta_k < 0.
\end{cases}
\]

**Theorem 1.23.** Let \( \mathfrak{A} \in \mathfrak{A}_{SB}(E) \) with the associated Schur functions \( \theta_{\pm} \) and scattering data \( \alpha_+ = \{ R_+, \nu_+ \} \). Then the following statements are equivalent.

1. The Schur functions \( \theta_{\pm} \) satisfy (i), (ii).
2. The scattering data \( \alpha_+ \) determine a CMV matrix of \( \mathfrak{A}_{SB}(E) \) class uniquely and both related transformation operators are bounded.

In Sect. 9 we propose the following sufficient condition given directly in terms of the scattering data.

With \( \nu_+ \) we associate the measure \( \tilde{\nu}_+ \) by

\[
\tilde{\nu}_+(\zeta_k) = \frac{1}{|B'(\zeta_k)|^2 \nu_+(\zeta_k)}
\]

and with the reflection coefficient \( R_+ \) the Szegő function

\[
\tilde{R}_+(\tau) = R_+(\tau) B(\tau)^2.
\]

**Theorem 1.24.** Let \( \tilde{\nu}_+ \) be a Carleson measure in \( \mathbb{D} \) and \( \tilde{R}_+ \) satisfy the following modification of the \( A_2 \) condition

\[
\sup_I \frac{1}{|I|} \int_I \frac{|\tilde{R}_+-\langle \tilde{R}_+ \rangle_I|^2 + (1-|\tilde{R}_+|^2)}}{1-|\tilde{R}_+|^2} \, dm < \infty
\]

Then the data \( \alpha_+ = \{ R_+, \nu_+ \} \) determine the CMV matrix uniquely for any \( e^{ic_+} \). Moreover, the both GLM transformation operators are bounded.

It shows that the class of data, comparably with the classical Faddeev–Marchenko one, is indeed widely extended (an infinite set of mass points and the reflection coefficient is very far necessary to be a continuous function).

**Remark 1.25.** As we clarified in a discussion with A. Khelif, in fact, our condition is optimal in the class of conditions on the scattering data with the following two properties: a) the condition is stable with respect to the involution \( R_+(\tau) \mapsto -R_+(\tau) \); b) the assumptions on \( R_+ \) and \( \nu_+ \) are independent.

2. **Proof of the Duality Theorem**

The main goal of the Lemma below is to clarify notations that are, probably, a bit confusing. We believe that the diagram, given in it, and the proof will help to avoid misunderstanding: \( \pm \)-mappings \( L^2_{\pm}, \rightarrow L^2_{\pm}, \) defined by (1.58), (1.59), actually depend of the scattering data \( \{ R_\pm, \nu_\pm \} \), although we do not indicate this dependence explicitly.
**Lemma 2.1.** Let \( w(\zeta) \) be an inner meromorphic function in \( \mathbb{D} \) such that \( w(\zeta_k) \neq 0, w(\zeta_k) \neq \infty \) for all \( \zeta_k \in \mathbb{Z} \). Put \( w_*(\zeta) := w(\zeta) \). The following diagram is commutative

\[
\begin{array}{ccc}
L^2_{\{w_*, R_+, ww, \nu_+\}} & \xrightarrow{w} & L^2_{\alpha_+} \\
\downarrow & & \downarrow \\
L^2_{\{w^{-1}_* R_-, R_+, w^{-1}_* \nu_+\}} & \xrightarrow{\overline{w}^{-1}} & L^2_{\alpha_-}
\end{array}
\] (2.1)

Here the horizontal arrows are related to the unitary multiplication operators and the vertical arrows are related to two different \( \pm \)-duality mappings.

**Proof.** Note that both \( w \) and \( w_*^{-1} \) are well defined on \( \mathbb{T} \cup \mathbb{Z} \). Evidently, \( w f \in L^2_{\alpha_+} \) means that \( f \in L^2_{\{w_*, R_+, ww, \nu_+\}} \). Also, since \( |w(\tau)| = 1, \tau \in \mathbb{T} \), we have that \( \{w^{-1}_* R_-, w^{-1}_* \nu_+\} \) are minus–scattering data for \( \{ww_*, R_+, ww, \nu_+\} \) if \( \alpha_- \) corresponds to \( \alpha_+ \). In other words \( T_{\pm} \)-functions remain the same for both sets of scattering data. Then we use definitions (1.58), (1.59).

**Proof of Theorem 1.18.** Let us mention that \( f^+ \in L^2_{\alpha_+} \) implies

\[
(T_- f^-)(\tau) = R_+(\tau)f^+(\tau) + \overline{\tau} f^+(\overline{\tau}) \in L^2, \tau \in \mathbb{T}.
\]

Since

\[
\langle f^+, Bh \rangle_{\alpha_+} = \langle R_+(\tau)f^+(\tau) + \overline{\tau} f^+(\overline{\tau}), \overline{\tau} B(\overline{\tau}) h(\overline{\tau}) \rangle, \quad h \in H^2,
\]

it follows from \( f^+ \in L^2_{\alpha_+} \) that

\[
\langle BT_- f^- \rangle(\tau) = g(\tau) := B(\tau)(R_+(\tau)f^+(\tau) + \overline{\tau} f^+(\overline{\tau})) \in H^2.
\]

Now we calculate the scalar product

\[
\langle f^+, \frac{B(\tau)}{\tau - \zeta_k} \rangle_{\alpha_+} = f^+(\zeta_k) B'(\zeta_k) \nu_+(\zeta_k) + \langle BT_- f^-, \frac{1}{1 - \tau \zeta_k} \rangle = f^+(\zeta_k) B'(\zeta_k) \nu_+(\zeta_k) + g(\zeta_k) = 0.
\]

Therefore, by (1.59) we get

\[
f^-(\zeta_k) = \left( \frac{g}{BT_-} \right) (\zeta_k), \quad \zeta_k \in \mathbb{Z}.
\]

For the converse direction we calculate the scalar product of \( f^+ \in \tilde{H}^2_{\alpha_+} \) with a function of the form \( B_N g, B_N \in \mathbb{B}, g \in H^2 \) and use the fact that \( BT_- f^- \in H^2 \).

### 3. Reproducing Kernels

We prove several propositions concerning specific properties of the reproducing kernels in \( \tilde{H}^2_{\alpha_+} \) and \( \tilde{H}^2_{\alpha_-} \). The multiplication operator by \( v \) is playing an essential role in these constructions.

**Lemma 3.1.** Let \( \tilde{k}_{\alpha_+}(\zeta, \nu) \) and \( \tilde{k}_{\alpha_-}(\zeta, \nu) \) denote the reproducing kernels of the spaces \( \tilde{H}^2_{\alpha_+} \) and \( \tilde{H}^2_{\alpha_-} \) respectively. Then

\[
(\tilde{k}_{\alpha_+}(\zeta, \nu))^- = \frac{1 - \zeta\nu}{(\zeta - \overline{\nu})(1 - |\nu|^2)} \frac{1}{T_-(\overline{\nu})} \tilde{k}_{\alpha_-}^{-1}(\zeta, \overline{\nu}),
\]

(3.1)
and, therefore,
\[ \hat{k}_{\alpha_+}(\zeta, \xi)\hat{k}_{\alpha_-}(\bar{\zeta}, \bar{\xi}) = \frac{1}{|T_-(\bar{\zeta})|^2} \frac{1}{(1 - |\zeta|^2)^2}. \] (3.2)

\textbf{Proof.} First we note that the following one-dimensional spaces coincide:
\[ \{(\hat{k}_{\alpha_+}(\zeta, \xi))\}^- = \{b_\zeta^{-1}\hat{k}_{\alpha_-}(\bar{\zeta}, \bar{\xi})\}. \]

It follows immediately from Theorem 1.18, but let us give a formal prove. Starting with the orthogonal decomposition
\[ \{\hat{k}_{\alpha_+}(\zeta, \xi)\} = \hat{H}_{\alpha_+}^2 \ominus b_\zeta \hat{H}_{\alpha_-}^2, \]
we have
\[ \{\hat{k}_{\alpha_+}(\zeta, \xi)\}^- = (\hat{H}_{\alpha_+}^2)^- \ominus (b_\zeta \hat{H}_{\alpha_-}^2)^-, \]
or, due to (2.1),
\[ \{\hat{k}_{\alpha_+}(\zeta, \xi)\}^- = (\hat{H}_{\alpha_+}^2)^- \ominus b_\zeta^{-1}(\hat{H}_{\alpha_-}^2)^-. \]

Now we use Theorem 1.18
\[ \{\hat{k}_{\alpha_+}(\zeta, \xi)\}^- = (L_{\alpha_-}^2 \ominus \hat{H}_{\alpha_-}^2) \ominus b_\zeta^{-1}(L_{\alpha_-}^2 \ominus \hat{H}_{\alpha_-}^2) \]
\[ = b_\zeta^{-1}(\hat{H}_{\alpha_-}^2 \ominus b_\zeta \hat{H}_{\alpha_-}^2). \]

Thus
\[ (\hat{k}_{\alpha_+}(\zeta, \xi))^- = Cb_\zeta^{-1}\hat{k}_{\alpha_-}(\bar{\zeta}, \bar{\xi}). \] (3.3)

The essential part of the lemma deals with the constant $C$. We calculate the scalar product
\[ \left\langle \hat{k}_{\alpha_+}(\tau, \xi), \frac{B}{1 - \tau \bar{\zeta}} \right\rangle_{\alpha_+}. \]

On the one hand, since $\frac{B}{1 - \tau \bar{\zeta}}$ belongs to the intersection of $L_{\alpha_+}^2$ with $H^2$, we can use the reproducing property of $\hat{k}_{\alpha_+}$:
\[ \left\langle \hat{k}_{\alpha_+}(\tau, \xi), \frac{B}{1 - \tau \bar{\zeta}} \right\rangle_{\alpha_+} = \frac{B(\zeta)}{1 - |\zeta|^2} = \frac{B(\bar{\zeta})}{1 - |\bar{\zeta}|^2}. \] (3.4)

On the other hand we can reduce the given scalar product to the scalar product in the standard $H^2$. Since $B(\zeta) = 0$, the $\nu$-component vanishes and we get
\[ \frac{1}{2} \left\langle \begin{bmatrix} 1 & \bar{R}_+ \\ R_+ & 1 \end{bmatrix} (\tau) \begin{bmatrix} \hat{k}_{\alpha_+}(\tau, \xi) \\ \bar{\hat{k}}_{\alpha_+}(\bar{\tau}, \bar{\xi}) \end{bmatrix}, \begin{bmatrix} B(\tau) \\ B(\bar{\tau}) \end{bmatrix} \right\rangle_{1 - \tau \bar{\zeta}} = \left\langle T_-(\bar{\tau})(\hat{k}_{\alpha_+}(\tau, \xi))^-, \frac{B}{\tau - \bar{\zeta}} \right\rangle_{\alpha_+}. \]

Substituting here (3.3) we get
\[ C \left\langle (BT_-)(\tau)\hat{k}_{\alpha_-}^-(\tau, \bar{\zeta}), b_\zeta(\bar{\tau}) \frac{1}{\tau - \bar{\zeta}} \right\rangle_{\alpha_-}. \]

Since $(BT_-)(\zeta)\hat{k}_{\alpha_-}^-(\zeta, \bar{\xi})$ belongs to $H^2$ and $b_\zeta(\zeta)\frac{1}{\zeta - \bar{\xi}} = e^{i\xi} \frac{1}{\zeta - \bar{\xi}}$ is collinear to the reproducing kernel here, we get recalling (3.4)
\[ e^{-i\zeta}C(BT_-)(\bar{\zeta})\hat{k}_{\alpha_-}^-(\bar{\xi}, \bar{\zeta}) = \frac{B(\bar{\zeta})}{1 - |\bar{\zeta}|^2}. \]
Thus (3.1) is proved. Comparing the norms of that vectors and taking into account that the −−map is an isometry we get (3.2).

Consider the multiplication operator by $\overline{v}$, acting in
\begin{equation}
L_{\alpha_+}^2 = (\tilde{H}_{\alpha_+}^2)^+ \oplus \tilde{H}_{\alpha_+}^2.
\end{equation}

**Lemma 3.2.** The multiplication operator by $\overline{v}$ acts as a unitary operator from
\begin{equation}
\{ \tilde{k}^+_{\alpha_-}(\zeta, \kappa) \} \oplus \tilde{H}_{\alpha_+}^2
\end{equation}

\begin{equation}
to
\{ \tilde{k}^+_{\alpha_-}(\zeta, \overline{\kappa}) \} \oplus \tilde{H}_{\alpha_+}^2.
\end{equation}

**Proof.** It is evident that the multiplication by $\overline{v} = \frac{b_{\kappa}}{\nu}$ acts from
\begin{equation}
\{ f \in \tilde{H}_{\alpha_-}^2 : f(\kappa) = 0 \} = b_{\kappa}\tilde{H}_{\alpha_-}^2
\end{equation}

to
\begin{equation}
\{ f \in \tilde{H}_{\alpha_-}^2 : f(\overline{\kappa}) = 0 \} = b_{\overline{\kappa}}\tilde{H}_{\alpha_-}^2.
\end{equation}

Therefore it acts in their orthogonal complements (3.6), (3.7).

Recall definition of the characteristic function of a unitary node and its functional model. Let $U$ be a unitary operator acting from $K \oplus E_1$ to $K \oplus E_2$. We assume that the Hilbert spaces $E_1$ and $E_2$ are finite–dimensional (actually, in this section we need $\dim E_1 = \dim E_2 = 1$). The characteristic function is defined by
\begin{equation}
\Theta(w) := P_{E_2}U(I_{K \oplus E_1} - wP_KU)^{-1}|E_1.
\end{equation}

It is a holomorphic in the unit disk contractive–valued operator function. We make a specific assumption that $\Theta(w)$ has an analytic continuation in the exterior of the unite disk through a certain arc $(a, b) \subset \mathbb{T}$ due to the symmetry principle:
\begin{equation}
\Theta(w) = \Theta^*(\overline{1/w})^{-1}.
\end{equation}

For $f \in K$ define
\begin{equation}
F(w) := P_{E_2}U(I - wP_KU)^{-1}f.
\end{equation}

This $E_2$–valued holomorphic vector function belongs to the functional space $K_\Theta$ with the following properties.

- $F(w) \in H^2(E_2)$, moreover it has analytic continuation through the arc $(a, b)$.
- $F_*(w) := \Theta^*(w)F(\frac{1}{w}) \in H^2(E_1)$.
- For almost every $w \in \mathbb{T}$ the vector $\begin{bmatrix} F \\ F_* \end{bmatrix}(w)$ belongs to the image of the operator $\begin{bmatrix} I & \Theta \\ \Theta & I \end{bmatrix}(w)$, and therefore the scalar product
\begin{equation}
\left\langle \begin{bmatrix} I & \Theta \\ \Theta & I \end{bmatrix}^{-1} \begin{bmatrix} F \\ F_* \end{bmatrix}, \begin{bmatrix} F \\ F_* \end{bmatrix} \right\rangle_{E_1 \oplus E_2}
\end{equation}
has sense and does not depend of the choice of a preimage (the first term in the above scalar product). Moreover

$$\int_{\mathbb{T}} \left( \begin{bmatrix} I & \Theta \end{bmatrix}^{[-1]} \begin{bmatrix} F \end{bmatrix} : \begin{bmatrix} F \end{bmatrix} \right)_{E_1 \oplus E_2} dm < \infty.$$  \hspace{1cm} (3.10)

The integral in (3.10) represents the square of the norm of $F$ in $K_{\Theta}$. Note that in the model space $P_K U|K$ became a certain "standard" operator

$$f \mapsto F(w) \implies P_K Uf \mapsto \frac{F(w) - F(0)}{w},$$  \hspace{1cm} (3.11)

see (3.9).

The following simple identity is a convenient tool in the forthcoming calculation.

**Lemma 3.3.** For a unitary $U : K \oplus E_1 \to K \oplus E_2$

$$U^* P_{E_2} U (I - wP_K U)^{-1} = I + (w - U^*)P_K U (I - wP_K U)^{-1}.$$  \hspace{1cm} (3.12)

**Proof.** Since $I_{K \oplus E_2} = P_K + P_{E_2}$ and $U$ is unitary we have

$$U^* P_{E_2} U = (I - wP_K U) + (w - U^*)P_K U.$$  

Then we multiply this identity by $(I - wP_K U)^{-1}$. \hspace{2cm} $\Box$

**Theorem 3.4.** Let $e_1, e_2$ be the normalized vectors of the one-dimensional spaces (3.6) and (3.7)

$$e_1(\zeta) = \frac{1}{b_{\zeta}} \frac{\tilde{k}_{\alpha_+}^{-1}(\zeta, \bar{\zeta})}{\sqrt{\tilde{k}_{\alpha_+}^{-1}(\bar{\zeta}, \bar{\zeta})}} = -i \frac{T_+ (\bar{\zeta})}{|T_+ (\bar{\zeta})|} \frac{\tilde{k}_{\alpha_-}^{-1}(\zeta, \bar{\zeta})}{\sqrt{\tilde{k}_{\alpha_-}^{-1}(\bar{\zeta}, \bar{\zeta})}},$$  \hspace{1cm} (3.13)

$$e_2(\zeta) = \frac{1}{b_{\zeta}} \frac{\tilde{k}_{\alpha_+}^{-1}(\zeta, \bar{\zeta})}{\sqrt{\tilde{k}_{\alpha_+}^{-1}(\bar{\zeta}, \bar{\zeta})}} = i \frac{T_+ (\bar{\zeta})}{|T_+ (\bar{\zeta})|} \frac{\tilde{k}_{\alpha_-}^{-1}(\zeta, \bar{\zeta})}{\sqrt{\tilde{k}_{\alpha_-}^{-1}(\bar{\zeta}, \bar{\zeta})}}.$$

Then the reproducing kernel of $\hat{H}_{\alpha_+}^2$ is of the form

$$\tilde{k}_{\alpha_+}(\zeta, \zeta_0) = \frac{(we_2)(\zeta)(we_2)(\zeta_0) - e_1(\zeta)e_1(\zeta_0)}{1 - v(\zeta)v(\zeta_0)}.$$  \hspace{1cm} (3.14)

**Proof.** First, we are going to find the characteristic function of the multiplication operator by $\tilde{v}$ with respect to decompositions (3.6) and (3.7) and the corresponding functional representation of this node.

By (3.13) we fixed 'bases' in the one-dimensional spaces. So, instead of the operator we get the matrix, in fact the scalar function $\theta(w)$:

$$\Theta(w) e_1 := P_{E_2} U (I - wP_K U)^{-1} e_1 = e_2 \theta(w).$$  \hspace{1cm} (3.15)

Let us substitute (3.15) into (3.12)

$$v(\zeta)e_2(\zeta) \theta(w) = e_1(\zeta) + (w - v(\zeta)) (P_K U (I - wP_K U)^{-1} e_1)(\zeta).$$  \hspace{1cm} (3.16)

Recall an important property of $\tilde{k}_{\alpha_+}^{-1}(\zeta, \bar{\zeta})$: it has analytic continuation in the $\mathbb{D}$ with the only pole at $\bar{\zeta}$ (see Lemma 3.1). Therefore all terms in (3.16) are analytic
in ζ and we can chose ζ such that v(ζ) = w. Then we obtain the characteristic function in terms of the reproducing kernels

\[ \theta(v(ζ)) = \frac{e_1(ζ)}{v(ζ)e_2(ζ)} = \frac{\hat{k}_{α⁺}^{-1}(ζ, \bar{ζ})}{\hat{k}_{α⁺}^{-1}(ζ, ζ)}. \] (3.17)

Similarly for \( f ∈ K = \hat{H}^2_{α⁺} \) we define the scalar function \( F(w) \) by

\[ P_{E_2}(I - wP_K)^{-1}f = e_2F(w). \] (3.18)

Using again (3.12) we get

\[ v(ζ)e_2(ζ)F(w) = f(ζ) + (w - v(ζ))(P_KU(I - wP_K)^{-1}f)(ζ). \]

Therefore,

\[ F(v(ζ)) = \frac{f(ζ)}{v(ζ)e_2(ζ)}. \] (3.19)

Now we are in a position to get (3.14). Indeed, by (3.18) and (3.19) we proved that the vector

\[ P_K(I - v(ζ)U^∗P_K)^{-1}U^∗e_2v(ζ)e_2(ζ) \]

is the reproducing kernel of \( K = \hat{H}^2_{α⁺} \) with respect to \( ζ_0, |v(ζ_0)| < 1 \). Using the Darboux identity

\[ P_{E_2}(I - wP_K)^{-1}P_K(I - v(ζ_0)U^∗P_K)^{-1}U^∗|E_2 = \frac{I - Θ(w)Θ^∗(w_0)}{1 - w\bar{w}_0} \]

(in the given setting it is a simple and pleasant exercise) we obtain

\[ \hat{k}_{α⁺}(ζ, ζ_0) = v(ζ)e_2(ζ)\frac{I - \theta(v(ζ))\theta(v(ζ_0))}{1 - v(ζ)v(ζ_0)}v(ζ_0)e_2(ζ_0) \] (3.20)

for \( |v(ζ)| < 1, |v(ζ_0)| < 1 \). By (3.17) we have (3.14) that, by analyticity, holds for all \( |ζ| < 1, |ζ_0| < 1 \). \( ∎ \)

**Corollary 3.5.** The following Wronskian–kind identity is satisfied for the reproducing kernels

\[ \begin{vmatrix} T_+(e_2^-)(ζ) & T_+(e_1^-)(ζ) \\ e_2(ζ) & e_1(ζ) \end{vmatrix} = -(\log v(ζ))', \ |ζ| < 1. \] (3.21)

**Proof.** We multiply \( \hat{k}_{α⁺}^-(ζ, ζ_0) \) by \( b_{ζ_0}(ζ) \) and calculate the resulting function of ζ at \( ζ = ζ_0 \). By (3.1) we get

\[ \{b_{ζ_0}(ζ)\hat{k}_{α⁺}^-(ζ, ζ_0)\}_{ζ = ζ_0} = e^{ic} \frac{1}{T_-(ζ_0)(1 - |ζ_0|^2)}. \] (3.22)

Now we make the same calculation but using representation (3.14). We have

\[ \hat{k}_{α⁺}^-(ζ, ζ_0) = \frac{-v(ζ_0)}{v(ζ) - v(ζ_0)} \begin{vmatrix} v(ζ)e_2^- (ζ) & e_1^- (ζ) \\ e_2^- (ζ) & e_2^- (ζ) \end{vmatrix}, \]

or, after multiplication by \( b_{ζ_0}(ζ) \),

\[ \{b_{ζ_0}(ζ)\hat{k}_{α⁺}^-(ζ, ζ_0)\}_{ζ = ζ_0} = e^{ic} \frac{-v(ζ_0)}{v(ζ_0)(1 - |ζ_0|^2)} \begin{vmatrix} v(ζ)e_2^- (ζ) & e_1^- (ζ) \\ e_2^- (ζ) & e_2^- (ζ) \end{vmatrix}. \]
In combination with (3.22), we get
\[
- \frac{v'(\zeta_0)}{v(\zeta_0)T_-(\zeta_0)} = \begin{vmatrix} v(\zeta_0)e_2^-(\zeta_0) & e_1^-(\zeta_0) \\ e_2^-(\zeta_0) & v^{-1}(\zeta_0)e_2(\zeta_0) \end{vmatrix}.
\]

Due to the symmetry \( k_{\alpha-}(\zeta, \zeta_0) = \hat{k}_{\alpha-}(\zeta, \zeta_0) \), we have \( e_2(\zeta_0) = e_1(\zeta_0) \). Thus (3.21) is proved. \( \square \)

**Corollary 3.6.** Let \( \tau \in \mathbb{T} \), then
\[
|e_2(\tau)|^2 - |e_1(\tau)|^2 = \frac{d \log v(\tau)}{d \log \tau}. \tag{3.23}
\]

**Proof.** All terms of (3.21) have boundary values. Recall that
\[
(T_+ e_1)(\tau) = (R_+ e_{1,2})(\tau) + \bar{\tau} e_{1,2}(\bar{\tau}), \quad \tau \in \mathbb{T}.
\]
Then use again the symmetry of the reproducing kernel. \( \square \)

4. **A recurrence relation for reproducing kernels and the Schur parameters**

Let
\[
K_\alpha(\zeta, \zeta_0) := \frac{k_\alpha(\zeta, \zeta_0)}{\sqrt{k_\alpha(\zeta_0, \zeta_0)}}, \tag{4.1}
\]
where \( k_\alpha(\zeta, \zeta_0) \) denotes one of reproducing kernels \( \hat{k}_{\alpha-}(\zeta, \zeta_0) \) or \( \tilde{k}_{\alpha-}(\zeta, \zeta_0) \).

**Theorem 4.1.** Both systems
\[
\{K_\alpha(\zeta, \zeta_0), b_\alpha(\zeta)K_\alpha(\zeta, \zeta_0)\}
\]
and
\[
\{K_\alpha(\zeta, \zeta_0), b_\alpha(\zeta)K_\alpha(\zeta, \zeta_0)\}
\]
form an orthonormal basis in the two dimensional space spanned by \( K_\alpha(\zeta, \zeta_0) \) and \( K_\alpha(\zeta, \zeta_0) \). Moreover
\[
K_\alpha(\zeta, \zeta_0) = a(\alpha)K_\alpha(\zeta, \zeta_0) + \rho(\alpha)b_\alpha(\zeta)K_\alpha(\zeta, \zeta_0), \tag{4.2}
K_\alpha(\zeta, \zeta_0) = a(\alpha)K_\alpha(\zeta, \zeta_0) + \rho(\alpha)b_\alpha(\zeta)K_\alpha(\zeta, \zeta_0),
\]
where
\[
a(\alpha) = a = \frac{K_\alpha(\zeta_0, \zeta_0)}{K_\alpha(\zeta_0, \zeta_0)}, \quad \rho(\alpha) = \rho = \sqrt{1 - |a|^2}. \tag{4.3}
\]

**Proof.** The first claim is evident, therefore
\[
K_\alpha(\zeta, \zeta_0) = c_1 K_\alpha(\zeta, \zeta_0) + c_2 b_\alpha(\zeta)K_\alpha(\zeta, \zeta_0).
\]
Putting \( \zeta = \bar{\zeta} \) we get \( c_1 = a \). Due to orthogonality we have
\[
1 = |a|^2 + |c_2|^2.
\]
Now, put \( \zeta = \bar{\zeta} \). Taking into account that \( K_\alpha(\zeta, \zeta_0) = \overline{K_\alpha(\zeta, \zeta_0)} \) and that by normalization \( b_\alpha(\zeta_0) > 0 \) we prove that \( c_2 \) being positive is equal to \( \sqrt{1 - |a|^2} \).

Note that simultaneously we proved that
\[
\rho(\alpha) = b_\alpha(\zeta_0)\frac{K_\alpha(\zeta_0, \zeta_0)}{K_\alpha(\zeta_0, \zeta_0)}.
\] \( \square \)
Corollary 4.2. A recurrence relation for reproducing kernels generated by the shift of the scattering data is of the form
\[
\begin{align*}
  b_\alpha(\zeta) & \left[ K_{\alpha^1}(\zeta, \bar{x}), \ -K_{\alpha^1}(\zeta, \bar{x}) \right] \\
  = [K_\alpha(\zeta, \bar{x}) , \ -K_\alpha(\zeta, \bar{x})] \frac{1}{\rho} \begin{bmatrix} 1 & a \\ \bar{a} & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\end{align*}
\] (4.4)

Proof. Recalling \( v = b_\alpha / b_\bar{\alpha} \), we write
\[
\begin{align*}
  b_\alpha(\zeta) & \left[ K_{\alpha^1}(\zeta, \bar{x}), \ -K_{\alpha^1}(\zeta, \bar{x}) \right] \\
  = [b_\alpha(\zeta) K_{\alpha^1}(\zeta, \bar{x}) , \ -b_\alpha(\zeta) K_{\alpha^1}(\zeta, \bar{x})] \begin{bmatrix} v \\ 0 \end{bmatrix}.
\end{align*}
\]
Then, use (4.2). \( \square \)

Corollary 4.3. Let
\[
\theta_\alpha(v) := \frac{K_\alpha(\zeta, \bar{x})}{K_\alpha(\zeta, \bar{x})}.
\] (4.5)

Then the Schur parameters of the function \( e^{\imath \alpha} \theta_\alpha(v) \), are
\[
\{ e^{\imath \alpha} a(\alpha^n) \}_{n=0}^\infty.
\]

Proof. Let us note that (4.4) implies
\[
\theta_\alpha(v) = \frac{a(\alpha) + v \theta_\alpha(v)}{1 + a(\alpha)v \theta_\alpha(v)}
\]
and that \( |a(\alpha)| < 1 \). Then we iterate this relation. Also, multiplication by \( e^{\imath \alpha} \in \mathbb{T} \) of a Schur class function evidently leads to multiplication by \( e^{\imath \alpha} \) of all Schur parameters. \( \square \)

Theorem 4.4. The multiplication operator with respect to the basis (1.65) is CMV.

Proof. Recall (1.16), from which we can see that the decomposition of the vector \( v(\zeta) K_\alpha(\zeta, \bar{x}) \) is of the form
\[
v(\zeta) K_\alpha(\zeta, \bar{x}) = c_0 \frac{K_{\alpha^{-2}}(\zeta, \bar{x})}{b_\alpha(\zeta) b_\bar{\alpha}(\zeta)} + c_1 K_{\alpha^{-1}}(\zeta, \bar{x}) b_\bar{\alpha}(\zeta) + c_2 K_\alpha(\zeta, \bar{x}) + c_3 b_\bar{\alpha}(\zeta) K_{\alpha^1}(\zeta, \bar{x}).
\]

Multiplying by the denominator \( b_\alpha(\zeta) b_\bar{\alpha}(\zeta) \) we get
\[
b_\alpha^2(\zeta) K_\alpha(\zeta, \bar{x}) = c_0 K_{\alpha^{-2}}(\zeta, \bar{x}) + c_1 K_{\alpha^{-1}}(\zeta, \bar{x}) b_\bar{\alpha}(\zeta) + c_2 K_\alpha(\zeta, \bar{x}) + c_3 b_\bar{\alpha}(\zeta) K_{\alpha^1}(\zeta, \bar{x}) b_\alpha^2(\zeta).
\] (4.6)

First we put \( \zeta = \bar{x} \). By the definition of \( \rho(\alpha) \) we have
\[
c_0 = b_\alpha^2(\bar{x}) \frac{K_\alpha(\bar{x}, \bar{x})}{K_{\alpha^{-2}}(\bar{x}, \bar{x})} = \rho(\alpha^{-1}) \rho(\alpha^{-2}).
\]

Putting \( \zeta = \bar{x} \) in (4.6) and using the definition of \( a(\alpha) \), we have
\[
c_1 = -c_0 \frac{K_{\alpha^{-1}}(\bar{x}, \bar{x})}{b_\alpha(\bar{x})} = -\rho(\alpha \mu) \rho(\alpha^{-2}) \frac{a(\alpha^{-2})}{\rho(\alpha^{-2})} = -\rho(\alpha^{-1}) a(\alpha^{-2}).
\]

Doing in the same way we can find a representation for \( c_2 \) that would involve derivatives of the reproducing kernels. However, we can find \( c_2 \) in terms of \( a \) and \( \rho \) calculating the scalar product
\[
c_2 = \langle b_\alpha^2(\zeta) K_\alpha(\zeta, \bar{x}), b_\alpha(\zeta) b_\bar{\alpha}(\zeta) K_{\alpha^1}(\zeta, \bar{x}) \rangle.
\]
Since \( b_\omega(\zeta) \) is unimodular, using (4.2), we get
\[
c_2 = \left\langle \frac{K_{\alpha^{-1}}(\zeta, \bar{x}) - a(\alpha^{-1})K_{\alpha^{-1}}(\zeta, x)}{\rho(\alpha^{-1})}, b_\omega(\zeta)K_\alpha(\zeta, \bar{x}) \right\rangle.
\]
Recall that \( k_\alpha(\zeta, x) = K_\alpha(\zeta, x)K_\alpha(x, x) \) is the reproducing kernel. Thus
\[
c_2 = -\frac{a(\alpha^{-1})}{\rho(\alpha^{-1})} \frac{\bar{b}_\omega(x)K_\alpha(x, x)}{K_{\alpha^{-1}}(x, \bar{x})} = -\frac{a(\alpha^{-1})}{\rho(\alpha^{-1})} \bar{b}_\omega(x)K_\alpha(\zeta, x).
\]
And, similar,
\[
c_3 = \left\langle \frac{K_{\alpha^{-1}}(\zeta, \bar{x}) - a(\alpha^{-1})K_{\alpha^{-1}}(\zeta, x)}{\rho(\alpha^{-1})}, b_\omega(\zeta)K_\alpha(\zeta, \bar{x}) \right\rangle.
\]
Thus
\[
c_3 = \frac{-a(\alpha^{-1})}{\rho(\alpha^{-1})} \frac{\bar{b}_\omega(x)K_\alpha(x, x)}{K_{\alpha^{-1}}(x, \bar{x})} = -\frac{a(\alpha^{-1})}{\rho(\alpha^{-1})} \rho(\alpha) = -a(\alpha^{-1})\rho(\alpha).
\]
To find the decomposition of the vector \( v(\zeta) \frac{K_{\alpha^{-1}}(\zeta, x)}{b_\omega(\zeta)} \) is even simpler. Note that all other columns of the CMV matrix, starting from these two, can be obtain by the two step shift of the scattering data.

\[\square\]

5. From the spectral data to the scattering data: a special representation of the Schur function

In this section we use Theorem D [11], see also [9]. For readers convenience we formulate it here.

**Theorem 5.1.** Let \( r(v) \) be a meromorphic function in \( \Omega \) with the property
\[
\frac{r(v(\zeta)) + r(v(\zeta))}{i(\zeta - \bar{\zeta})} \geq 0.
\]
(5.1)
If poles \( \{t_j\} \) of \( r(v) \) (they should lie on \( \mathbb{T} \setminus E \) due to (5.1)) satisfy the Blaschke condition (1.28), then \( r(v(\zeta)) \) is of bounded characteristic in \( \mathbb{D} \), and in addition the inner (in the Beurling sense) factor of \( r(v(\zeta)) \) is a quotient of Blaschke products, i.e., it does not have a singular inner factor.

Note the evident fact: if \( r(v) \) is of bounded characteristic in \( \Omega \) then for the poles \( \{t_j\} \) the Blaschke condition (1.28) holds.

**Proposition 5.2.** If \( \mathfrak{A} \) belongs to \( \mathfrak{A}_{SB}(E) \) then the associated Schur functions \( \theta_\pm \) are of bounded characteristic in \( \Omega \) and
\[
\log |1 - |\theta_\pm(v(\tau))|^2| \in L^1.
\]
(5.2)

**Proof.** We use the formula (see (1.26))
\[
r_{\mathfrak{A}}(v) := \langle 0 | \frac{\mathfrak{A} + v}{\mathfrak{A} - v} | 0 \rangle = \frac{1 + v\theta_+(v)\theta_-(v)}{1 - v\theta_+(v)\theta_-(v)}.
\]
Since \( \mathfrak{A} \in \mathfrak{A}_{SB}(E) \) and \( r_{\mathfrak{A}}(v) \) is a resolvent function, its poles satisfy the Blaschke condition.

Now we note that
\[
r_+(v) + r_-(v) = \frac{1 + v\theta_+(v)}{1 - v\theta_+(v)} + \frac{1 + \theta_-(v)}{1 - \theta_-(v)} = \frac{2(1 - v\theta_+(v)\theta_-)(v)}{(1 - v\theta_+(v))(1 - \theta_-(v))}.
\]
Since zeros and poles of the last function interlace we get that poles of \( r_{\pm} \) also satisfy the Blaschke condition. By Theorem 5.1 they are of bounded characteristic in \( \Omega \). Hence \( \theta_{\pm} \) are also in this class.

By (1.27) we get (5.2).

\[ \square \]

**Definition 5.3.** A function \( \theta(v) \) belongs to the class \( \Theta_{SB}(E) \) if it is a function of bounded characteristic in \( \Omega \) with the following properties

\[
1 - \frac{\theta(v(\zeta))\overline{\theta(v(\zeta))}}{i(\zeta - \overline{\zeta})} \geq 0 \quad (5.3)
\]

and

\[
\log|1 - |\theta(v(\tau))|^2| \in L^1. \quad (5.4)
\]

Denote

\[ T_\pm = \{ \tau \in T : \text{Im} \tau < 0 \}, \quad D_\pm = \{ \zeta \in D : \text{Im} \zeta < 0 \}. \]

**Proposition 5.4.** Functions of the class \( \Theta_{SB}(E) \) possess the following parametric representation

\[
\theta(v(\zeta)) = e^{ic} \prod_{\lambda_k} \frac{\lambda_k - \zeta - 1 - \overline{\lambda_k}\zeta}{\lambda_k \overline{\lambda_k} - 1 - \overline{\lambda_k}\zeta} e^{-\int_{T_\pm}(\frac{\pm \zeta + \overline{\zeta}}{\pm \zeta - \overline{\zeta}}(d\mu(\tau) - \log \rho(\tau)dm(\tau)))}, \quad (5.5)
\]

where

- \( \Lambda = \{ \lambda_k \} \subset D_- \) is a Blaschke sequence,
- \( \mu \) is a singular measure on the (open) set \( T_- \),
- \( \rho, 0 \leq \rho \leq 1, \) is such that

\[
\int_{T_-} \log((1 - \rho(\tau))\rho(\tau))dm(\tau) > -\infty. \quad (5.6)
\]

**Proof.** First we note the symmetry

\[
\overline{\theta(v(\zeta))} = \frac{1}{\theta(v(\zeta))} \quad (5.7)
\]

and then use the parametric representation of functions of bounded characteristic and (5.4).

In the opposite direction to prove (5.3) we can use directly representation (5.5) or note that \( \theta(v(\zeta)) \) is of the Smirnov class in the domain \( D_- \) and then use the maximum principle.

\[ \square \]

**Remark 5.5.** \( \theta_{\pm} \in \Theta_{SB}(E) \) implies (1.28) and (1.30), but the spectral measure \( d\Sigma \) is not necessarily absolutely continuous on \( E \).

**Example 5.6.** On the other hand for every \( \theta_+ \in \Theta_{SB}(E) \) there exists \( \theta_- \in \Theta_{SB}(E) \) such that the associated to them \( \mathfrak{A} \) belongs to \( \mathfrak{A}_{SB}(E) \). Put, for instance,

\[
\theta_-(v(\zeta)) = e^{ic} \frac{1 - \overline{\zeta}}{1 - \zeta}, \quad (5.8)
\]

that corresponds to the constant Schur parameters (see Theorem 1.6). Since for every \( \epsilon > 0 \)

\[
\sup_{\{\zeta \in D_- : \text{Im} \zeta < -\epsilon\}} |\theta_-(v(\zeta))| < 1,
\]
we have that both resolvent functions
\[ \frac{1 + v\theta_-(v)\theta_+(v)}{1 - v\theta_-(v)\theta_+(v)} = \frac{1 + v\theta_0^{-1}(v)\theta_0^{(1)}(v)}{1 - v\theta_0^{-1}(v)\theta_0^{(1)}(v)} \]
are uniformly bounded in the such domain. Therefore the open arc \( E \setminus \{e^{i\xi_0}, e^{-i\xi_0}\} \) is free of the singular spectrum.

Consider the endpoints. Existence of a mass point here means that at least one of the following fore limits
\[ \lim_{v \to e^{i\xi_0}} \frac{1 + v\theta_+(v)\theta_-(v)}{1 - v\theta_+(v)\theta_-(v)} = \lim_{v \to e^{i\xi_0}} \frac{1 + v\theta_0^{(-1)}(v)\theta_0^{(1)}(v)}{1 - v\theta_0^{(-1)}(v)\theta_0^{(1)}(v)}, \quad v \in T \setminus E, \]
is infinite. In other words at least one of the following relations hold
\[ \lim_{\zeta \to \pm 1} v(\zeta)\theta_+(v(\zeta))\theta_-(v(\zeta)) = 1, \quad \lim_{\zeta \to \pm 1} v(\zeta)\theta_0^{(-1)}(v(\zeta))\theta_0^{(1)}(v(\zeta)) = 1, \quad (5.9) \]
for \( \zeta \in [-1, 1] \). Due to
\[ 1 - v\theta_0^{(-1)}(v)\theta_0^{(1)}(v) = \frac{\rho_0^2(1 - v\theta_-(v)\theta_+(v))}{(1 + \rho_0^-v_0\theta_-(v))(1 + \rho_0^-v_0\theta_+(v))} \quad (5.10) \]
the first and the second conditions in (5.9) are equivalent. Thus, up to two possible exceptional values
\[ e^{-ic-} = \frac{1 \pm \bar{\rho}}{1 \pm \rho} \lim_{\zeta \to \pm 1} \theta_+(v(\zeta)), \]
the endpoints also free of the mass of the measure \( d\Sigma \).

Now we prove a theorem on a representation of a Schur function of the above class in the form similar to (4.5).

**Theorem 5.7.** Let \( \theta(v) \in \Theta_{SB}(E) \). Then there exists and unique the representation
\[ \theta(v(\zeta)) = e^{ic-} \frac{L_{\bar{\nu}}(\zeta)}{L_{\nu}(\zeta)}, \quad L_{\bar{\nu}}(\bar{\zeta}) > 0, \quad L_{\nu}(\zeta) > 0, \quad (5.11) \]
such that \( L_{\bar{\nu}}(\zeta) \) and \( L_{\nu}(\zeta) \) are of Smirnov class in \( D \) with the mutually simple inner parts, and the (Wronskian) identity
\[ \frac{L_{\bar{\nu}}(\tau)}{L_{\nu}(\tau)} \frac{L_{\bar{\nu}}(\tau)}{L_{\nu}(\tau)} = \frac{d \log v(\tau)}{d \log \tau}, \quad \tau \in T, \quad (5.12) \]
holds.

**Proof.** By (5.12) and (5.11) we have
\[ |L_{\bar{\nu}}(\tau)|^2 (1 - |\theta(v(\tau))|^2) = \frac{d \log v(\tau)}{d \log \tau}. \quad (5.13) \]
Due to (5.4) we can define the outer function \( O_{\bar{\nu}} \) such that
\[ |O_{\bar{\nu}}(\tau)|^2 = (1 - |\theta(v(\tau))|^2)^{-1} \frac{d \log v(\tau)}{d \log \tau}, \quad O_{\bar{\nu}}(\bar{\zeta}) > 0, \]
and the outer function \( O_{\nu} \) such that
\[ |O_{\nu}(\tau)|^2 = |O_{\nu}(\tau)|^2 |\theta(v(\tau))|^2, \quad O_{\bar{\nu}}(\bar{\zeta}) > 0. \]
We represent the inner part of the function \( \theta(v(\zeta)) \) as the ration of the inner holomorphic functions

\[
\frac{I_{\zeta}(\zeta)}{I_{\zeta}(\zeta)} \quad I_{\zeta}(\tilde{x}) > 0, \quad I_{\zeta}(x) > 0.
\]

Finally we put

\[
L_{\zeta}(\zeta) := I_{\zeta}(\zeta)O_{\zeta}(\zeta), \quad L_{\zeta}(\zeta) := I_{\zeta}(\zeta)O_{\zeta}(\zeta).
\]

Then the left- and right-hand sides of (5.11) coincide up to a unimodular constant and this defines \( e^{ic} \). By (5.13) relation (5.12) also holds.

It is evident that \( L_{\zeta}(\zeta) \) and \( L_{\tilde{\zeta}}(\zeta) \) as functions of the Smirnov class are defined uniquely.

Note that due to the uniqueness and property (5.7) we have

\[
\text{Note that due to the uniqueness and property (5.7) we have}
\]

and let

\[
\theta(n) = e^{ic_n} = e^{ic_n} \quad (5.15)
\]

Then \( e^{ic_n} = e^{ic} \) and

\[
L_{\tilde{\zeta}}(n, \zeta) = (e^{-ic_n}a_n)L_{\tilde{\zeta}}(n, \zeta) + \rho_n b_{\tilde{\zeta}}(\zeta)L_{\tilde{\zeta}}(n + 1, \zeta),
\]

\[
L_{\tilde{\zeta}}(n, \zeta) = (e^{ic_n}a_n)L_{\tilde{\zeta}}(n, \zeta) + \rho_n b_{\tilde{\zeta}}(\zeta)L_{\tilde{\zeta}}(n + 1, \zeta).
\]

**Theorem 5.8.** Let \( \theta \in \Theta_{SB}(E) \) and let \( \{a_k\}_{k=0}^{\infty} \) be the sequence of its Schur parameters. Put

\[
\theta(n) = e^{ic_n}L_{\zeta}(n, \zeta)
\]

Then \( e^{ic_n} = e^{ic} \) and

\[
L_{\tilde{\zeta}}(n, \zeta) = (e^{-ic}a_n)L_{\tilde{\zeta}}(n, \zeta) + \rho_n b_{\tilde{\zeta}}(\zeta)L_{\tilde{\zeta}}(n + 1, \zeta),
\]

\[
L_{\tilde{\zeta}}(n, \zeta) = (e^{ic}a_n)L_{\tilde{\zeta}}(n, \zeta) + \rho_n b_{\tilde{\zeta}}(\zeta)L_{\tilde{\zeta}}(n + 1, \zeta).
\]

**Proof.** By definition

\[
\theta^{(1)} = \theta - a_0 b_{\zeta} = e^{icL_{\zeta} - (e^{-ic}a_0)L_{\zeta}b_{\zeta}} = e^{ic\left[\begin{array}{c}
L_{\zeta} \\
L_{\zeta}
\end{array}\right]}
\]

By the uniqueness of representation (5.11) we get

\[
\left[\rho_1 b_{\tilde{\zeta}} L^{(1)}_{\tilde{\zeta}} \quad \rho b_{\tilde{\zeta}} L^{(1)}_{\tilde{\zeta}}\right] = [L_{\tilde{\zeta}} L_{\tilde{\zeta}}] \left[\begin{array}{cc}
1 & -e^{-ic}a_0 \\
-e^{-ic}a_0 & 1
\end{array}\right],
\]

with \( \rho_1 = \rho e^{ic_1 - c} \). Using

\[
a_0 = e^{ic}L_{\tilde{\zeta}}(z)
\]

we have in particular

\[
\rho_1 b_{\tilde{\zeta}} L^{(1)}_{\tilde{\zeta}}(\tilde{z}) = L_{\tilde{\zeta}}(\tilde{z})(1 - |a_0|^2), \quad \rho b_{\tilde{\zeta}} L^{(1)}_{\tilde{\zeta}}(\tilde{z}) = L_{\tilde{\zeta}}(\tilde{z})(1 - |a_0|^2).
\]

That is, both \( \rho, \rho_1 \) are positive and therefore \( \rho_1 = \rho \) or \( e^{ic_1} = e^{ic} \).

From (5.17) we have the matrix identity

\[
\tilde{\rho}L^{(1)}_{\tilde{\zeta}}(\tilde{z}) = \left[\begin{array}{cc}
L_{\tilde{\zeta}}(\tilde{z}) & L_{\tilde{\zeta}}(\tilde{z}) \\
L_{\tilde{\zeta}}(\tilde{z}) & L_{\tilde{\zeta}}(\tilde{z})
\end{array}\right] \left[\begin{array}{cc}
1 & -e^{-ic}a_0 \\
-e^{-ic}a_0 & 1
\end{array}\right].
\]

Finally using (5.14) we have \( \tilde{\rho}^2 = 1 - |a_0|^2 \). Hence \( \tilde{\rho} = \rho_0 \). Thus (5.16) holds for \( n = 0 \) and we can iterate this procedure.
Lemma 5.9. For the spectral density $W$ the following factorization holds

$$W^{-1}(v(\tau)) \frac{dm(v(\tau))}{dm(\tau)} = \rho^{-1}_{-1} \Phi(\tau) \Phi^*(\tau),$$  \hspace{1cm} (5.19)$$

where

$$\Phi(\tau) = \left[ \begin{array}{cc} \frac{1}{\rho \Phi(\tau)} L_{-+,\kappa}(\tau) & -e^{ic} L_{+-,\kappa}(\tau) \\ e^{ic} L_{-+,\kappa}(\tau) & \frac{1}{\rho \Phi(\tau)} L_{++,\kappa}(\tau) \end{array} \right].$$  \hspace{1cm} (5.20)$$

Proof. Due to (5.11)

$$\theta_{+}(v) = e^{ic} L_{+-,\kappa}(\zeta), \quad \theta_{-}^{(1)}(v) = e^{ic} L_{-+,\kappa}^{(1)}(\zeta).$$  \hspace{1cm} (5.21)$$

Besides, due to (5.13)

$$1 - |\theta_{+}(v(\tau))|^2 = \frac{1}{|L_{-+,\kappa}(\tau)|^2} \frac{dm(v(\tau))}{dm(\tau)}$$  \hspace{1cm} (5.22)$$

and

$$1 - |\theta_{-}^{(1)}(v(\tau))|^2 = \frac{1}{|L_{-+,\kappa}^{(1)}(\tau)|^2} \frac{dm(v(\tau))}{dm(\tau)}.$$  \hspace{1cm} (5.23)$$

By definition (1.27) and (5.23), (5.22), we have

$$W^{-1}(v(\tau)) \frac{dm(v(\tau))}{dm(\tau)} = \frac{2}{I + \mathcal{R}(v)} \left[ \frac{|L_{-+,\kappa}^{(1)}(\tau)|^2}{0} \right] \frac{2}{I + \mathcal{R}(v)}.$$  \hspace{1cm} (5.24)$$

By definition (1.26)

$$\frac{2}{I + \mathcal{R}(v)} = I - v A_{-1}^* \left[ \begin{array}{cc} \theta_{-}^{(1)} & 0 \\ 0 & \theta_{+} \end{array} \right].$$

Therefore we get (5.19) with

$$\rho^{-1}_{-1} \Phi(\tau) = \frac{b_{\kappa}}{b_{\kappa}} \left[ \begin{array}{cc} L_{-+,\kappa}^{(1)} & 0 \\ 0 & L_{++,\kappa} \end{array} \right](\tau) - A_{-1}^* \left[ \begin{array}{cc} e^{ic} L_{-+,\kappa}^{(1)} & 0 \\ 0 & e^{ic} L_{++,\kappa} \end{array} \right](\tau).$$  \hspace{1cm} (5.25)$$

By (5.16)

$$\rho^{-1}_{-1} b_{\kappa}(\zeta) L_{++,\kappa}(\zeta) = L_{-+,\kappa}^{(-1)}(\zeta) - e^{ic} \bar{a}_{-1} L_{-+,\kappa}^{(-1)}(\zeta)$$

$$= e^{ic} a_{-1}(L_{++,\kappa}^{(-1)}(\zeta) + \rho^{-1}_{-1} b_{\kappa}(\zeta)L_{++,\kappa}(\zeta))$$

$$=(\rho^{-1})^2 L_{++,\kappa}^{(-1)}(\zeta) - \rho^{-1} e^{ic} \bar{a}_{-1} L_{++,\kappa}(\zeta),$$

that is

$$b_{\kappa}(\zeta)L_{-+,\kappa}(\zeta) + e^{ic} \bar{a}_{-1} b_{\kappa}(\zeta)L_{++,\kappa}(\zeta) = \rho^{-1} L_{-+,\kappa}^{(-1)}(\zeta).$$  \hspace{1cm} (5.26)$$

and similarly

$$b_{\kappa}(\zeta)L_{++,\kappa}^{(-1)}(\zeta) + e^{ic} \bar{a}_{-1,\kappa}^{(-1)} b_{\kappa}(\zeta)L_{-+,\kappa}^{(-1)}(\zeta) = \rho_{-1} L_{-+,\kappa}^{(-1)}(\zeta).$$  \hspace{1cm} (5.27)$$

Note that $a_{-1}^{(-1)} = -\bar{a}_{-1}$ (generally $a_{-k}^{(-1)} = -\bar{a}_{-k-2}$). Thus, using (5.26), (5.25), we get (5.20) from (5.24). The lemma is proved.
Lemma 5.10. Define

\[ S(\tau) = \begin{bmatrix} R_- & T_- \\ T_+ & R_+ \end{bmatrix} = -\Phi^{-1}(\tau)\tau\Phi(\tau). \]  

(5.27)

Then (1.42), (1.43), (1.44) and (1.45) hold true.

Proof. \( S(\tau) \) is unitary–valued since \( W(v(\tau)) = W(v(\tau)) \).

Due to \( L_{\pm,\zeta}(\zeta) = L_{\pm,\zeta}(\zeta) \) and \( A_n = A_n^* = A_n^{-1} \) we get directly from (5.24)

\[ \Phi(\bar{\tau}) = -vA_{-1}\Phi(\tau) \begin{bmatrix} e^{-ic_+} & 0 \\ 0 & e^{-ic_+} \end{bmatrix}. \]  

(5.28)

And, therefore, the following symmetry property of \( S \)

\[ \overline{S(\bar{\tau})} = -\begin{bmatrix} e^{ic_-} & 0 \\ 0 & e^{ic_-} \end{bmatrix} \Phi(\bar{\tau})^{-1}\bar{\tau}\Phi(\tau) \begin{bmatrix} e^{-ic_-} & 0 \\ 0 & e^{-ic_-} \end{bmatrix} = \begin{bmatrix} e^{ic_-} & 0 \\ 0 & e^{ic_-} \end{bmatrix} S(\tau) \begin{bmatrix} e^{-ic_-} & 0 \\ 0 & e^{-ic_-} \end{bmatrix}, \]  

(5.29)

is proved.

Let us show that \( T_+(\tau) = \bar{T}_-(\bar{\tau}) \). We have

\[ \begin{bmatrix} R_- & T_- \\ T_+ & R_+ \end{bmatrix} = -\frac{\bar{\tau}}{\Delta} \begin{bmatrix} \frac{1}{b_{-\zeta}} L_{-,-,\bar{\zeta}}^{(-1)} & e^{ic_+} L_{+,\bar{\zeta}} \\ e^{ic_-} L_{+,\zeta} & \frac{1}{b_{+\zeta}} L_{-,-,\zeta} \end{bmatrix}(\tau) \begin{bmatrix} \frac{1}{b_{-\zeta}} L_{-,-,\bar{\zeta}}^{(-1)} & -e^{ic_+} L_{+,\bar{\zeta}} \\ -e^{ic_-} L_{+,\zeta} & \frac{1}{b_{+\zeta}} L_{-,-,\zeta} \end{bmatrix}(\bar{\tau}), \]  

(5.30)

where \( \Delta = \det \Phi \). Therefore

\[ T_+ = -e^{ic_+} \frac{\bar{\tau}}{\Delta} \begin{bmatrix} \frac{1}{b_{-\zeta}(\bar{\tau})} L_{-,-,\bar{\zeta}}(\bar{\tau}) & L_{-,-,\bar{\zeta}}^{(-1)}(\bar{\tau)} \\ \frac{1}{b_{-\zeta}(\tau)} L_{-,-,\zeta}(\tau) & L_{-,-,\zeta}^{(-1)}(\tau) \end{bmatrix} \]  

(5.31)

\[ = -e^{ic_+} \frac{\bar{\tau}}{\rho_1\Delta} \begin{bmatrix} v^{-1} L_{-,-,\bar{\zeta}}^{(-1)}(\bar{\tau}) - e^{ic_-} a_{-1} L_{-,-,\bar{\zeta}}^{(-1)}(\bar{\tau}) \\ v^{-1} L_{-,-,\zeta}(\tau) - e^{ic_-} a_{-1} L_{-,-,\zeta}^{(-1)}(\tau) \end{bmatrix} = -e^{ic_+} \frac{(v^{-1})'}{\rho_1\Delta}. \]

Similarly \( T_- = -e^{ic_+} \frac{(v^{-1})'}{\rho_1\Delta} = e^{i(c_+ - c_-)}T_+ \). Due to (5.29) \( \overline{T_+(\bar{\tau})} = e^{i(c_+ - c_-)}T_+(\tau) \), therefore the symmetry \( S^\ast(\bar{\tau}) = S(\tau) \) is completely proved.

Note also that (5.31) implies the following normalization

\[ T_+(\bar{\zeta}) = e^{ic_+} \frac{b_{\bar{\zeta}}(\bar{\zeta}) b_{+\zeta}'(\zeta)}{\rho_1 L_{-,-,\bar{\zeta}}^{(-1)}(\zeta) L_{-,-,\zeta}(\bar{\zeta})} = e^{ic_+} \frac{b_{+\zeta}'(\zeta)}{L_{-,-,\bar{\zeta}}^{(-1)}(\zeta) L_{-,-,\zeta}(\bar{\zeta})}. \]  

(5.32)

That is, \( T_+(\bar{\zeta}) = -ie^{ic_+} L_{-,-,\zeta}(\bar{\zeta}) \).

Finally we have to prove that \( T_{-\pm} \) is a ratio of an outer function and a Blaschke product. In other words, by (5.31), we need to show that the inner part of the
Proof. By definition (1.25) and (1.26) we have

$$\nu$$

Lemma 5.11. For every

Proposition 5.4.

is a Blaschke product (actually related to the spectrum of the associated CMV matrix).

Since

by Theorem 5.1 the inner part of this fraction is a ratio of two Blaschke products. Thus, any other inner divisor of the inner part of \( \rho_1 b_2^2 \Delta \) should simultaneously divide the inner part of the numerator \( b_2 L_{-\infty} L_{+\infty} + e^{i(c+e-c)} b_{\mathcal{P}} L_{+\infty} L_{-\infty} \) and \( e^{i(c+e-c)} b_{\mathcal{P}} L_{+\infty} L_{-\infty} \) possess a nontrivial common inner factor in this case. But they are coprime since the inner part of the first function is supported in the upper half plane and of the second one in the lower part, see Proposition 5.4.

\[ \square \]

Lemma 5.11. For every \( \zeta_k \in \mathbb{Z} \) the following two vectors are collinear

\[
\begin{bmatrix}
L_{-\infty}^{(1)} & \frac{1}{b_2} L_{-\infty} \\
0 & 0
\end{bmatrix} (\zeta_k) = - \left( \frac{1}{L_2} \right)' (\zeta_k) \nu_+ (\zeta_k) \begin{bmatrix}
\theta_+^{(1)} (v) & 0 \\
0 & \theta_+ (v)
\end{bmatrix}^{-1}.
\]

Moreover \( \nu_+ (\zeta_k) > 0 \).

Proof. By definition (1.25) and (1.26) we have

\[
t_k \Sigma(t_k) = \left\{ (t_k - v) \begin{bmatrix}
I - v A_{-1}^{*} & \begin{bmatrix}
\theta_+^{(1)} (v) & 0 \\
0 & \theta_+ (v)
\end{bmatrix}
\end{bmatrix}^{-1} \right\}_{v = t_k}.
\]

Since

\[
\rho_1 v \Phi (\tau) = \left( I - v A_{-1}^{*} \begin{bmatrix}
\theta_+^{(1)} (v) & 0 \\
0 & \theta_+ (v)
\end{bmatrix} \right) \begin{bmatrix}
L_{-\infty}^{(1)} & 0 \\
0 & L_{+\infty}
\end{bmatrix} (\tau),
\]

we get

\[
t_k \Sigma(t_k) = \begin{bmatrix}
L_{-\infty}^{(1)} & 0 \\
0 & L_{+\infty}
\end{bmatrix} (\zeta_k) \begin{bmatrix}
(t_k - v) & 0 \\
0 & \rho_1 v \Phi (\tau)
\end{bmatrix}^{-1} \bigg|_{\tau = \zeta_k}.
\]

\[
= \begin{bmatrix}
L_{-\infty}^{(1)} & 0 \\
0 & L_{+\infty}
\end{bmatrix} (\zeta_k) \begin{bmatrix}
\frac{1}{b_2} L_{+\infty}^{(-1)} & e^{i c_+ + \bar{c} c_-} L_{+\infty} \\
e^{i c_-} L_{-\infty}^{(-1)} & \frac{1}{b_2} L_{-\infty}
\end{bmatrix} (\zeta_k) \begin{bmatrix}
(t_k - v) & 0 \\
0 & \rho_1 v \Delta
\end{bmatrix}^{-1} \bigg|_{\tau = \zeta_k}.
\]

\[
= \begin{bmatrix}
L_{-\infty}^{(1)} & 0 \\
0 & L_{+\infty}
\end{bmatrix} (\zeta_k) \begin{bmatrix}
\frac{1}{b_2} L_{+\infty}^{(-1)} & e^{i c_+ + \bar{c} c_-} L_{+\infty} \\
e^{i c_-} L_{-\infty}^{(-1)} & \frac{1}{b_2} L_{-\infty}
\end{bmatrix} (\zeta_k) \begin{bmatrix}
0 & 0 \\
0 & \rho_1 v \Delta
\end{bmatrix}^{-1} \bigg|_{\tau = \zeta_k}.
\]
or, using \( T_\pm = -e^{i\zeta_+ (\nu^{-1}_-)^t} \).

\[
\Sigma(t_k) = \begin{bmatrix} L_{\pm,\nu}^{(1)} & 0 \\ 0 & L_{\pm,\nu} \end{bmatrix} (\zeta_k) \begin{bmatrix} \frac{1}{b_{\nu}} L_{\pm,\nu}^{(-1)} & e^{i\zeta_+} L_{\pm,\nu} \\ e^{i\zeta_-} L_{\pm,\nu}^{(-1)} & \frac{1}{b_{\nu}} L_{\pm,\nu} \end{bmatrix} (\zeta_k) \left( - \left( \frac{e^{i\zeta_+}}{T_\pm} \right)^t (\zeta_k) \right)^{-1}.
\]

(5.36)

From this formula we conclude the vector in the RHS (5.34) does not vanish. Otherwise, by \( L_{\pm,\nu}(\zeta_k) = L_{\pm,\nu}(\zeta_k) \), we have \( \Sigma(t_k) = 0 \), which is impossible. On the other hand rank of the second matrix in (5.36) is one, therefore (5.34) is proved.

Now, making of use (5.34) and the symmetry of \( T_- \), we get from (5.36)

\[
\Sigma(t_k) = \frac{1}{b_{\nu}(\zeta_k)} L_{\pm,\nu}^{(-1)}(\zeta_k) \left[ \frac{1}{b_{\nu}(\zeta_k)} L_{\pm,\nu}^{(-1)}(\zeta_k) e^{i\zeta_+} L_{\pm,\nu}(\zeta_k) \right] \nu_+(\zeta_k),
\]

(5.37)

here \( \Sigma(t_k) \geq 0 \) implies \( \nu_+(\zeta_k) > 0 \).

\[\Box\]

**Remark 5.12.** Similarly

\[- \left( \frac{1}{T_+} \right)^t (\zeta_k) \nu_-(\zeta_k) \left[ e^{i\zeta_-} L_{\pm,\nu}^{(1)} \frac{1}{b_{\nu}} L_{\pm,\nu} \right] (\zeta_k) \nu_+(\zeta_k) = \left[ \frac{1}{b_{\nu}} L_{\pm,\nu}^{(-1)} e^{i\zeta_+} L_{\pm,\nu} \right] (\zeta_k).
\]

(5.38)

Therefore (1.46) holds for \( \nu_\pm \) defined by (5.34) and (5.38).

6. FROM THE SPECTRAL REPRESENTATION TO THE SCATTERING REPRESENTATION

In this section an essential part of Theorem 1.10 will be proved.

**Theorem 6.1.** Let \( A \in \mathfrak{A}_{SB}(E) \). Define \( S \) by (5.27) and \( \nu_\pm \) by (5.34) and (5.38). Then

\[
e^\pm(n, \zeta) = \begin{cases} \frac{b_m}{b_{\nu}}(\zeta)b_{\nu}(\zeta)L_{\pm,\nu}(n, \zeta)e^{i\zeta_\pm}, & n = 2m \\ \frac{b_m}{b_{\nu}}(\zeta)b_{\nu}(\zeta)L_{\pm,\nu}(n, \zeta), & n = 2m + 1 \end{cases}
\]

(6.1)

is an orthonormal basis in \( L^2_{A_{\pm}}, \alpha_\pm = \{R_{\pm, \nu_\pm}\} \).

The proof is based on the following lemma.

**Lemma 6.2.** For \( f^+ \in L^2_{A_+} \)

\[
\left[ \begin{array}{c} \left\langle \frac{u(\tau)+w}{v(\tau)-w} f^+, e^+(-1, \tau) \right\rangle_{A_+} \\ \left\langle \frac{u(\tau)+w}{v(\tau)-w} f^+, e^+(0, \tau) \right\rangle_{A_+} \end{array} \right] = \int \frac{t+w}{t-w} d\Sigma(t) \hat{f}(t),
\]

(6.2)

where

\[
\hat{f}(t) := \frac{1}{A(\tau)} \Phi \left[ \int_{A(\tau)}^\tau \right] (\tau), \quad t = v(\tau), \quad \tau \in \mathcal{T}_-.
\]

(6.3)

\[
\hat{f}(t_k) := \left[ \frac{e^+(0, \zeta_k)}{e^+(0, \zeta_k)} \frac{f^+(\zeta_k)}{|e^+(0, \zeta_k)|^2 + |e^+(0, \zeta_k)|^2} \right] \tau = v(\zeta_k), \quad \zeta_k \in \mathcal{Z}.
\]

(6.4)

**Proof.** Note that in this notations (see (5.37))

\[
\Sigma(t_k) = \left[ \frac{e^+(0, \zeta_k)}{e^+(0, \zeta_k)} \left[ e^+(0, \zeta_k) \right] e^+(0, \zeta_k) \nu_+(\zeta_k),
\]

(6.5)
and (see (5.27))

\[
\begin{bmatrix}
e^-(1, \tau) & -e^+(0, \tau) \\
-e^-(0, \tau) & e^-(1, \tau)
\end{bmatrix}
\begin{bmatrix}R_- & T_-
\end{bmatrix}(\tau) = -\bar{\tau}
\begin{bmatrix}e^-(1, \bar{\tau}) & -e^+(0, \bar{\tau})
\end{bmatrix}.
\]

Therefore, by definition of the scalar product in \(L^2_{\alpha+}\), we have

\[
\begin{bmatrix}
\langle \frac{v(\tau) + w}{v(\tau) - w} f^+, e^+(1, \tau) \rangle \\
\langle \frac{v(\tau) + w}{v(\tau) - w} f^+, e^+(0, \tau) \rangle
\end{bmatrix} = \sum_{\zeta_k \in \mathbb{Z}} \frac{e^+(0, \zeta_k) f^+(\zeta_k)}{v(\zeta_k) - w} v(\zeta_k) + w
\]

\[
+ \int_{\tau, -} \left\{ \begin{bmatrix} T_+(\tau) e^+(1, \tau) & T_+(\tau) e^+(0, \tau) \\
T_-(\tau) e^+(0, \tau) & T_-(\tau) e^+(1, \tau)
\end{bmatrix} \right\} \frac{1}{\Delta} \left\{ \begin{bmatrix} \Phi^+ \langle f^+ \rangle & \Phi^+ \langle f^- \rangle
\end{bmatrix} \right\} \frac{v(\tau) + w}{v(\tau) - w} dm(\tau).
\]

Using (6.5), definition (6.4) and

\[
\Phi^{-1}(\tau) = \frac{1}{\Delta} \begin{bmatrix} e^+(1, \tau) & e^+(0, \tau) \\
e^-(0, \tau) & e^-(1, \tau)
\end{bmatrix},
\]

we get

\[
\begin{bmatrix}
\langle \frac{v(\tau) + w}{v(\tau) - w} f^+, e^+(1, \tau) \rangle \\
\langle \frac{v(\tau) + w}{v(\tau) - w} f^+, e^+(0, \tau) \rangle
\end{bmatrix} = \sum_{t_k \in \mathbb{C}} \frac{t_k + w}{t_k - w} \sum (t_k) \tilde{f}(t_k)
\]

\[
+ \int_{\tau, -} \frac{1}{\Delta(\tau)} \left\{ \Phi^{-1}(\tau) \right\} \left\{ \begin{bmatrix} T_+^2 f^+ & T_-^2 f^-
\end{bmatrix} \right\} \frac{v(\tau) + w}{v(\tau) - w} dm(\tau)
\]

\[
= \sum_{t_k \in \mathbb{C}} \frac{t_k + w}{t_k - w} \sum (t_k) \tilde{f}(t_k) + \int_{E} \frac{t + w}{t - w} W(t) \left\{ \begin{bmatrix} \Phi \langle f^+ \rangle & \Phi \langle f^- \rangle
\end{bmatrix} \right\} \frac{1}{\Delta(\tau)} \frac{v(\tau) + w}{v(\tau) - w} dm(\tau),
\]

since \( W = (\Phi^{-1})^* \Phi^{-1} \frac{|\nu^+|}{c+1} \) and \( |T_+|^2 = |T_+|^2 = \frac{|\nu^+|^2}{c+1} \).

**Proof of Theorem 6.1.** It was shown that \( e^+(1, \tau), e^+(0, \tau) \) form a cyclic subspace for the multiplication operator by \( v(\tau) \) in \( L^2_{\alpha+} \), moreover, the resolvent matrix function

\[
(\mathcal{E}^+) \frac{v(\tau) + w}{v(\tau) - w} \mathcal{E}^+, \quad \mathcal{E}^+ \begin{bmatrix} c-1 & c_0 \\
0 & c_0 \end{bmatrix} := e^+(1, \tau) c-1 + e^+(0, \tau) c_0,
\]

coincides with \( \mathcal{R}(w) \) (1.24). Therefore the operator \( \mathcal{F}^+: l^2(\mathbb{Z}) \rightarrow L^2_{\alpha+} \), defined by

\[
\mathcal{F}^+(w - n)^{-1} = (v(\tau) - w)^{-1} e^+(n, \tau), \quad n = -1, 0,
\]

is unitary.

Recurrences (5.16) implies (1.54), (1.55), and therefore, (1.33). Thus

\[
\mathcal{F}^+|n = e^+(n, \tau), \quad n \in \mathbb{Z},
\]

and the theorem is proved.

**Proof of Proposition 1.20.** Note that \( e^+(n, \tau) \)'s are in the Smirnov class for \( n \in \mathbb{Z} \). Therefore \( (BT_+)(\tau) e^+(n, \tau) \in L^2 \) implies \( (BT_+)(\tau) e^+(n, \tau) \in H^2 \). Thus

\[
\mathcal{F}^+(\mathbb{Z}_+) \subset \hat{H}^2_{\alpha+}.
\]

In the same way \( \mathcal{F}^-(\mathbb{Z}_-) \subset \hat{H}^2_{\alpha-} \). Therefore, due to the duality Theorem 1.18,

\[
\hat{H}^2_{\alpha+} \subset \mathcal{F}^+(\mathbb{Z}_+).
\]
Remark 6.3. Let us note the following fact
\[
\lim_{n \to -\infty} \mathcal{F}^+(\mathbb{Z}+, n) = L^2_{\mathfrak{a}+}, \quad \lim_{n \to -\infty} \mathcal{F}^+(\mathbb{Z}+, n) = \{0\},
\] (6.8)
where \(\mathbb{Z}+ := \{m \in \mathbb{Z}, m \geq n\}\).

Also, in the standard way,
\[
l^{n,+}(\zeta, \zeta_0) := \sum_{m=n}^{\infty} e^+(m, \zeta) e^+(m, \zeta_0), \quad \zeta, \zeta_0 \in \mathbb{D},
\] (6.9)
is the reproducing kernel in \(\mathcal{F}^+(\mathbb{Z}+, n)\). In particular,
\[
L_{+, \mathfrak{x}}(\zeta) = \frac{l^+(\zeta, \mathfrak{x})}{l^+((\mathfrak{x}, \mathfrak{x})}, \quad L_{+, \mathfrak{x}}(\zeta) = \frac{l^+(\zeta, \mathfrak{x})}{l^+((\mathfrak{x}, \mathfrak{x})},
\]
where \(l^+(\zeta, \zeta_0) := l^{0,+}(\zeta, \zeta_0)\).

Proof of (1.36). Let
\[
\delta_k(\tau) = \begin{cases} \nu_+(\zeta_k), & \tau = \zeta_k, \\ 0, & \tau \in (\mathbb{T} \cup \mathcal{Z}) \setminus \{\zeta_k\}. \end{cases}
\]
Then
\[
(f^+(\tau), \delta_k(\tau))_{\alpha_+} = f^+(\zeta_k)
\]
for every \(f^+ \in \mathcal{F}^+(\mathbb{Z}+, n)\). Therefore the projection of \(\delta_k(\tau)\) onto \(\mathcal{F}^+(\mathbb{Z}+, n)\) is the reproducing kernel \(l^{n,+}(\tau, \zeta_k)\). Since by (6.8)
\[
\|\delta_k(\tau)\| = \lim_{n \to -\infty} \|l^{n,+}(\tau, \zeta_k)\|,
\]
we get by (6.9)
\[
\frac{1}{\nu_+(\zeta_k)} = \lim_{n \to -\infty} \sum_{m=n}^{\infty} |e^+(m, \zeta_k)|^2 = \sum_{m=-\infty}^{\infty} |e^+(m, \zeta_k)|^2.
\]

Thus, to complete the proof of Theorem 1.10, we have to show asymptotics (1.34).

7. Asymptotics

In this section we prove the main claim of Theorem 1.10. Recall briefly notations. With \(\mathfrak{a} \in \mathfrak{A}_{SB}(E)\) we associate the Schur functions \(\theta_+, \theta_-^{(1)}\). They belong to \(\Theta_{SB}(E)\) and, therefore, possess the special representation (5.11). We put
\[
e^+(0, \tau) = e^{ic_+}L_{+, \mathfrak{x}}(\tau), \quad \rho_1 e^+(1, \tau) = \frac{1}{v(\tau)}L_{+, \mathfrak{x}}(\tau) + e^{ic_+} \tilde{a}_1 L_{+, \mathfrak{x}}(\tau),
\]
\[
e^-(0, \tau) = e^{ic_1}L_{-, \mathfrak{x}}^{(1)}(\tau), \quad \rho_1 e^-(1, \tau) = \frac{1}{v(\tau)}L_{-, \mathfrak{x}}^{(1)}(\tau) - e^{ic_1} \tilde{a}_1 L_{-, \mathfrak{x}}^{(1)}(\tau).
\] (7.1)
The scattering matrix \(S\) is defined by (6.6) and the measures \(\nu_\pm\) on \(\mathcal{Z}\) are defined by
\[
\nu_\pm(\zeta_k)(|e^+(1, \zeta_k)|^2 + |e^+(0, \zeta_k)|^2) = \text{tr} \Sigma(t_k).
\]
Our goal is to prove asymptotics (1.34), (1.39) for the systems defined by recurrence relations (1.33), (1.38) with the initial data \(e^\pm(n, \tau), n = -1, 0\).
This is a standard fact that such asymptotics can be obtained from a convergence of a certain system of analytic functions just in a one fixed point of their domain. More specifically, our first step is a reduction to the convergence of the reproducing kernels $L_{\pm,\zeta_0}(n,\zeta_0)$ to the standard one $K_{\zeta_0}(\zeta_0)$ in a fixed point of the unite disk.

**Lemma 7.1.** Let $\chi_T(\tau), \tau \in \mathbb{T} \cup \mathbb{Z}$, be the characteristic function of the set $\mathbb{T}$. Then (1.34), (1.39) can be deduced from

$$\lim_{n \to \infty} \| \chi_T \{ L_{\pm,\zeta_0}(n,\tau) - K_{\zeta_0}(\tau) \} \|_{L^2_{\alpha,\pm}} = 0, \quad \zeta_0 \in \mathbb{D}. \tag{7.2}$$

**Proof.** Using definition (6.6) and the recurrence relations for $e^\pm(n,\tau)$ we have

$$\tau e^\pm(n,\tau) + R^\pm(\tau)e^\pm(n,\tau) = T^\pm(\tau)e^\mp(-n-1,\tau).$$

Therefore conditions (1.34), (1.39) are equivalent to

$$T^\pm(\tau)e^\pm(n,\tau) = T^\pm(\tau)e^\mp(n,\tau) + o(1),$$

$$\tau e^\pm(n,\tau) + R^\pm(\tau)e^\pm(n,\tau) = \tau e^\mp(n,\tau) + R^\mp(\tau)e^\pm(n,\tau) + o(1)$$

in $L^2$ as $n \to \infty$. That is,

$$\lim_{n \to \infty} \| \chi_T \{ e^\pm(n,\tau) - e^\mp(n,\tau) \} \|_{L^2_{\alpha,\pm}} = 0. \tag{7.3}$$

Then we use (6.1), (1.22) and definition (1.64) to rewrite (7.3) into the form

$$\lim_{n \to \infty} \| \chi_T \{ L_{\pm,\zeta}(n,\tau) - K_{\zeta}(\tau) \} \|_{L^2_{\alpha,\pm}} = 0,$$

$$\lim_{n \to \infty} \| \chi_T \{ L_{\pm,\zeta}(n,\tau) - K_{\zeta}(\tau) \} \|_{L^2_{\alpha,\pm}} = 0.$$

\[
\square
\]

**Lemma 7.2.** Assume that

$$\lim_{n \to \infty} L_{\pm,\zeta_0}(n,\zeta_0) = K_{\zeta_0}(\zeta_0), \quad \zeta_0 \in \mathbb{D}. \tag{7.4}$$

Then

$$\lim_{n \to \infty} \| L_{\pm,\zeta_0}(n,\tau) - \chi_T K_{\zeta_0}(\tau) \|_{L^2_{\alpha,\pm}} = 0. \tag{7.5}$$

**Proof.** For $\epsilon > 0$ chose $N$ such that $\Re B_N(\zeta_0) \geq 1 - \epsilon$. Note that $B_N K_{\zeta_0} \in \hat{H}^2_{\alpha,\pm}$

and consider

$$\| L_{\pm,\zeta_0}(n,\tau) - (B_N K_{\zeta_0})(\tau) \|_{L^2_{\alpha,\pm}}^2 = 2 - 2\Re \left( \frac{(B_N K_{\zeta_0})(\zeta_0)}{L_{\pm,\zeta_0}(n,\zeta_0)} \right) - \Re \langle P_\beta b_\alpha^* b_\alpha^* R_\pm B_N K_{\zeta_0}, \tau (B_N K_{\zeta_0})(\tau) \rangle_{L^2}.$$

Note that for any two functions $f, g \in L^2$ the following limit exists

$$\lim_{n \to \infty} \langle P_\beta b_\alpha^* b_\alpha^* f, g \rangle_{L^2} = 0. \tag{7.6}$$

Therefore, if (7.4) is satisfied then

$$\limsup_{n \to \infty} \| L_{\pm,\zeta_0}(n,\tau) - (B_N K_{\zeta_0})(\tau) \|_{L^2_{\alpha,\pm}}^2 \leq 2\epsilon. \tag{7.7}$$
Similarly,
\[
\|(B_N K_{\zeta_0})(\tau) - \chi_T K_{\zeta_0}(\tau)\|_{\alpha_{\pm}^n}^2 = \sum \left| B_N K_{\zeta_0}(\zeta_0) \nu_\pm(\zeta_0) b_\pm(\zeta_0) \right|^2
\]
\[+ 2 - 2\Re B_N(\zeta_0) + \Re(\bar{P} b_\pm^* b_\pm R_\pm (B_N - 1)K_{\zeta_0} - \bar{\tau}(B_N - 1)K_{\zeta_0})(\tau) \rangle L^2.\]

Note that the sum over $\mathbb{Z}$ here contains just a fixed finite number of nonvanishing terms, and therefore it goes to zero as $n \to \infty$. Thus, taking also into account (7.6), we get
\[
\limsup_{n \to \infty} \| (B_N K_{\zeta_0})(\tau) - \chi_T K_{\zeta_0}(\tau) \|_{\alpha_{\pm}^n}^2 \leq 2\epsilon. \tag{7.8}
\]
Combining (7.7) and (7.8) we have
\[
\limsup_{n \to \infty} \| L_{\pm,\zeta_0}(n, \tau) - \chi_T K_{\zeta_0}(\tau) \|_{\alpha_{\pm}^n}^2 \leq 8\epsilon.
\]
Since $\epsilon > 0$ is arbitrary the lemma is proved. \hfill \Box

**Remark 7.3.** Note that, in addition to (7.2), (7.5) contains
\[
\lim_{n \to \infty} \sum \left| L_{\pm,\zeta_0}(n, \zeta_0) \right|^2 |b_\pm(\zeta_0)|^{2n} \nu_\pm(\zeta_0) = 0.
\]

In the proof of (7.4) we follow the line that was suggested in [6] and then improved in [12] and [4]. Actually, the general idea is very simple. There are two natural steps in approximation of the given spectral data by “regular” ones. First, to substitute the given measure $\nu_\pm$ by a finitely supported $\nu_{\alpha_{\pm}^n}$. Second, to substitute $R_\pm$ by $q R_\pm$, with $0 < q < 1$. Then the corresponding data produce the Hardy space which is topologically equivalent to the standard $H^2$. In particular
\[
\hat{K}_{\alpha_{\pm}^n}(\zeta_0, \zeta_0) = \hat{K}_{\alpha_{\pm}^n}(\zeta_0, \hat{0}).
\]

**Lemma 7.4.** Let $\mathbb{Z}$ contain a finite number of points and $\| R_+ \|_{L^\infty} < 1$. Then the limit (7.4) exists.

Basically, it follows from (7.6) and $|b_\pm(\zeta_0)|^n \to 0$. It is a fairly easy task and we omit a proof here.

Further, due to
\[
\hat{H}^2_{\alpha_{\pm}^n} \subset \hat{H}^2 \subset \mathcal{F}^+(\mathbb{Z}_+) \subset \hat{H}^2_{\alpha_{\pm}^n} \subset \hat{H}^2_{\alpha_{\pm}^n},
\]
we have the evident estimations
\[
\hat{K}_{\alpha_{\pm}^n}(\zeta_0, \zeta_0) \leq L_{\alpha_{\pm}^n}(\zeta_0, \zeta_0) \leq \hat{K}_{\alpha_{\pm}^n}(\zeta_0, \zeta_0).
\]
And the key point is that, due to the duality principle, (3.2) holds. It allow us to use the left or right side estimation whenever it is convenient for us.

**Theorem 7.5.** Let $\mathfrak{A} \in \mathfrak{B}(E)$. For $\zeta_0 \in \mathfrak{D}$ the limit (7.4) exists, and therefore (1.34), (1.39) hold true.

**Proof.** Recall Lemma 2.1 on the relation between $\pm$ mappings and the notations $|T_\pm(\zeta)| = |O(\zeta)| |O(\zeta)|$, where $B$ is a Blaschke product and $O$ is an outer function (1.43).

We have
\[
L_{\alpha_{\pm}^n}(\zeta_0, \zeta_0) \leq \hat{K}_{\alpha_{\pm}^n}(\zeta_0, \zeta_0) \leq \hat{K}_{\alpha_{\pm}^n}(\zeta_0, \zeta_0) = \frac{1}{|T_{\alpha_{\pm}^n}(\zeta_0)|} \frac{K_2(\zeta_0, \zeta_0)}{K_{\alpha_{\pm}^n}(\zeta_0, \zeta_0)}.
\]
\[
\leq \frac{1}{|T_{\alpha_{\pm}^n}(\zeta_0)|} \frac{K_2(\zeta_0, \zeta_0)}{K_{\alpha_{\pm}^n}(\zeta_0, \zeta_0)} = |O(\zeta)| |O(\zeta)| \hat{K}_{\alpha_{\pm}^n}(\zeta_0, \zeta_0).
\]
(7.9)
And from the other side
\[
L_{\alpha_+}^{(n)}(\zeta_0, \zeta_0) \geq \hat{K}_{\alpha_+}^{(n)}(\zeta_0, \zeta_0) \geq \tilde{K}_{\alpha_+}^{(n)}(\zeta_0, \zeta_0) = \frac{1}{|T_{q, \pm}(\zeta_0)|} \frac{K^2(\zeta_0, \zeta_0)}{|K_{\alpha_{q, \pm}}^{(-n-1)}(\zeta_0, \zeta_0)|}
\]
(7.10)
Passing to the limit in (7.9) and (7.10) we get
\[
|B_N(\zeta_0)| \leq \lim_{n \to \infty} L_{\alpha_+}^{(n)}(\zeta_0, \zeta_0)
\]
(7.11)
Since
\[
\lim_{N \to \infty} |B_N(\zeta_0)| = 1 \quad \text{and} \quad \lim_{q \to \infty} |O_q(\zeta_0)| = |O(\zeta_0)|,
\]
(7.11) implies (7.4) and thus asymptotics (1.34), (1.39) are proved.

\[\Box\]

8. HILBERT TRANSFORM

Recall definition (1.71) of the transformation operator. In terms of the decomposition (1.70) the operator \( \mathcal{M}_- : l^2(\mathbb{Z}_-) \to l^2(\mathbb{Z}_-) \) is defined by
\[
\mathcal{M}_- = \tau^* \begin{bmatrix}
M_{0,0} & 0 & 0 & \ldots \\
M_{1,0} & M_{1,1} & 0 & \ldots \\
M_{2,0} & M_{2,1} & M_{2,2} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
(8.1)
where \( \tau : l^2(\mathbb{Z}_-) \to l^2(\mathbb{Z}_+) \), \( \tau(m) = |1 - m| \). Also, the shifted transformation operator is of the form
\[
\mathcal{M}_+^{(n)} = \begin{bmatrix}
M_{n,0} & 0 & 0 & \ldots \\
M_{1+n,0} & M_{1+n,1+n} & 0 & \ldots \\
M_{2+n,0+n} & M_{2+n,1+n} & M_{2+n,2+n} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
(8.2)
for even \( n \) and
\[
\mathcal{M}_+^{(n)} = \begin{bmatrix}
A & A \\
A & A \\
\vdots & \vdots
\end{bmatrix} S_n^* \mathfrak{A}_+^* S_n
\]
(8.3)
for odd \( n \), where \( S_n : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+|n) \), \( S_n|m| = |n + m| \) and \( A = \begin{bmatrix} a & \rho \\ \rho & -a \end{bmatrix} \) is related to the matrix \( \mathfrak{A}_n \) with constant coefficients.

**Lemma 8.1.** \( \mathcal{M}_+ \) is bounded if and only if
\[
\int_{\mathbb{R}} |F(\tau)|^2 dm(\tau) \leq C \|F\|^2_{\alpha+}
\]
(8.4)
is satisfied for all \( F \in \mathcal{F}^+(\mathbb{Z}_+) \). If \( \mathcal{M}^{(n)}_+ \) is bounded for a certain \( n = n_0 \) then it is bounded for all \( n \in \mathbb{Z} \).

**Proof.** (8.4) follows directly from (1.70). \( \|\mathcal{M}^{(n+1)}_+\| \leq \|\mathcal{M}^{(n)}_+\| \). So the only thing required to be proved is that \( \|\mathcal{M}^{(n)}_+\| < \infty \) implies \( \|\mathcal{M}^{(n-1)}_+\| < \infty \). It follows from the recurrence (1.33).

Let

\[
\frac{1 + \theta(v)}{1 - \theta(v)} = \text{i} \text{Im} \frac{1 + \theta(0)}{1 - \theta(0)} + \int_T \frac{t + v}{t - v} d\sigma(t)
\]

\[
= \text{i} \text{Im} \frac{1 + \theta(0)}{1 - \theta(0)} + \sum_{t_k \in \mathbb{T} \setminus E} \frac{t_k + v}{t_k - v} \sigma_k + \int_E \frac{t + v}{t - v} \{ \mathbf{w}(t) d\alpha(t) + d\sigma_s(t) \},
\]

(8.5)

where \( \sigma_s \) is a singular measure on \( E \) and

\[
\mathbf{w}(t) = \frac{1 - |\theta(t)|^2}{|1 - \theta(t)|^2}.
\]

Then

\[
\frac{1 - \theta(v)\bar{\theta}(v_0)}{(1 - \theta(v))(1 - \theta(v_0))} = \int_T \frac{1 - v\bar{v}_0}{(t - v)(t - v_0)} d\sigma(t).
\]

(8.7)

**Lemma 8.2.** Let \( \theta \in \Theta_{SB}(E) \). Put \( \theta_+ = \theta \) and select \( \theta_- \) as in Example 5.6, so that the associate CMV matrix \( \mathfrak{A} \) belongs to \( \mathfrak{A}_{SB}(E) \). Then

\[
F(\xi) := (L(\zeta, \xi) - e^{ic}L(\bar{\zeta}, \bar{\xi})) \int_T \frac{t}{b_{\bar{\xi}}(\xi)t - b_{\bar{\xi}}(\xi)} d\sigma(t) f(t)
\]

(8.8)

is a unitary map from \( L^2_\sigma \) to \( \mathcal{F}^+(\mathbb{Z}_+, 1) \).

**Proof.** Put \( f(t) = \frac{L(\bar{\zeta}, \zeta)L(\zeta, \xi) - L(\bar{\zeta}, \xi)L(\zeta, \xi)}{b_{\bar{\xi}}(\xi)\bar{b}_{\bar{\xi}}(\xi) - b_{\bar{\xi}}(\xi)b_{\bar{\xi}}(\xi)} \). Note that

\[
l^{1,+}(\zeta, \xi_0) = \frac{L(\bar{\xi}, \xi)L(\xi, \xi_0) - L(\bar{\xi}, \xi)L(\xi, \xi_0)}{b_{\bar{\xi}}(\xi)b_{\bar{\xi}}(\xi_0) - b_{\bar{\xi}}(\xi)b_{\bar{\xi}}(\xi_0)}
\]

(8.9)

is the reproducing kernel in \( \mathcal{F}^+(\mathbb{Z}_+, 1) \). By (8.7) we have

\[
\|f\|_{L^2_\sigma}^2 = \|F\|_{\mathcal{A}_+}^2.
\]

Thus the map is an isometry. Since the set of such functions is dense, it is unitary. \( \square \)

**Proposition 8.3.** The transformation operator \( \mathcal{M}^{(1)}_+ \) is bounded if and only if

\[
\int_E |(\mathcal{S}f)(v)|^2 \frac{d\alpha(v)}{\mathbf{w}(v)} \leq C\|f\|_{L^2_\sigma}^2, \quad f \in L^2_{\alpha_+},
\]

(8.10)

where

\[
(\mathcal{S}f)(v) := \int_T \frac{t}{t - v} d\sigma(t) f(t).
\]

(8.11)

**Proof.** We use (8.4). Then, by (8.8) and (5.12), we have

\[
\int_E \frac{|1 - \theta(v)|^2}{|1 - \theta(v)|^2} |(\mathcal{S}f)(v)|^2 d\alpha(v) \leq C\|F\|_{\mathcal{A}_+}^2 = C\|f\|_{L^2_\sigma}^2.
\]

Thus (8.10) is proved. \( \square \)
We give necessary and sufficient conditions on measure $\sigma$ that guarantee (8.10).

Let us reformulate our problem and change the notations slightly. Obviously we can straighten up by fractional linear transformation the arc $E$ and point part of $\sigma$ in such a way that $E$ becomes the segment $[-2,2]$, points $\{\zeta_k\}$ are transformed to $\{x_k\}$ accumulating only to $-2$ and $2$, measure $\sigma$ goes to $\tilde{\sigma}$, and $d\tilde{\sigma} = \tilde{w}\,dx$ on $[-2,2]$, $\tilde{\sigma}(x_k) = \sigma_k$. It is easy to see that inequality (8.10) becomes equivalent to the following one

$$
\int_{-2}^{2} |(Hf)(y)|^2 \frac{dy}{\tilde{w}(y)} \leq C\|f\|_{L^2}^2, \quad \forall f \in L^2(d\tilde{\sigma}), \tag{8.12}
$$

where

$$(Hf)(\nu) := \int_{-2}^{2} \frac{f(x)}{x - y} \, d\tilde{\sigma}(x). \tag{8.13}$$

If we choose all $f$'s from $L^2([-2,2], \tilde{w}\,dx)$ we get that (8.12) is equivalent to $\tilde{w} \in A_2[-2,2]$. In fact, with such test functions $f$ (8.12) becomes

$$
\int_{-2}^{2} \left( \int_{-2}^{2} \frac{f(x)\tilde{w}(x)\,dx}{x - y} \right)^2 \frac{dy}{\tilde{w}(y)} \leq C \int_{-2}^{2} |f|^2 \tilde{w}(x)\,dx, \quad \forall f \in L^2(\tilde{w}\,dx). \tag{8.14}
$$

Put $F := f\tilde{w}$. Then the previous estimate becomes the boundedness of

$$
H : L^2([-2,2], \tilde{w}^{-1}\,dx) \to L^2([-2,2], \tilde{w}^{-1}\,dx).
$$

This is of course equivalent to $\tilde{w}^{-1} \in A_2[-2,2]$, namely, to

$$
\sup_{I, \tilde{I} \subset [-2,2]} \langle \tilde{w}\rangle_I \langle \tilde{w}^{-1}\rangle_{\tilde{I}} < \infty, \tag{8.15}
$$

where $\langle \tilde{w}\rangle_I := \frac{1}{|I|} \int_I \tilde{w}\,dx$. This is obviously the same as $\tilde{w} \in A_2[-2,2]$.

Notice that it is easy to proof that $\tilde{w} \in A_2[-2,2]$ if and only if $\tilde{w}$ is a restriction onto $[-2,2]$ of an $A_2$ weight on the whole real line.

**Lemma 8.4.** Condition (8.10) implies that the measure $\sigma$ is absolutely continuous on the arc $E$ and moreover $\tilde{w} \in A_2$.

Therefore, to prove Theorem 1.23 we have to answer the following question: what is the property of the singular part on $T \setminus E$?

To continue with (8.12) we write it down now for all $f \in L^2(X, d\tilde{\sigma})$:

$$
\int_{-2}^{2} \left( \int_{X} \frac{f(x)\tilde{\sigma}(x)}{x - y} \, dy \right)^2 \frac{dy}{\tilde{w}(y)} \leq C \int_{X} |f(x)|^2 \tilde{\sigma}(x), \quad \forall f \in L^2(X, d\tilde{\sigma}). \tag{8.16}
$$

Let us write down the dual inequality. Fix $g \in L^2([-2,2], \tilde{w}\,dx)$, we have that (8.16) is equivalent to

$$
\sup_{\|g\|_{L^2([-2,2])} \leq 1} \int_{-2}^{2} g(y) \int_{X} \frac{f(x)\tilde{\sigma}(x)}{x - y} \, dy \leq C \left( \int_{X} |f|^2 \tilde{\sigma} \right)^{1/2}.
$$

Thus, we can conclude that (8.16) is equivalent to the following inequality:

$$
\int_{X} \left( \int_{-2}^{2} \frac{g(y)}{y - x} \, dy \right)^2 \tilde{\sigma} \leq C \int_{-2}^{2} |g|^2 \tilde{w}\,dx, \quad \forall g \in L^2([-2,2], \tilde{w}\,dx). \tag{8.17}
$$
To understand necessary and sufficient conditions for (8.17) we introduce Smirnov class \( E^2(\Omega) \), where \( \Omega = \mathbb{C} \setminus [-2, 2] \). Recall that this is the class of analytic functions \( f \) on \( \Omega \) having the property that \( \int_{\partial \Omega} |f(z)|^2 |dz| \leq C \) for a sequence of smooth contours converging to \([-2, 2]\) (class does not depend on the sequence of contours). Let us denote by \( \phi(z) \) the outer function in \( \Omega \) such that \( \tilde{w} = |\phi|^2 \) on the boundary \([-2, 2]\) of \( \Omega \) (the same boundary value on both sides of \([-2, 2]\)), \( \phi(\infty) > 0 \). The fact that \( \tilde{w} \in A_2[-2, 2] \) is way sufficient for the fact that \( \phi \) exists (as \( \tilde{w} \in A_2[-2, 2] \) obviously ensures \( \int_{-2}^2 \frac{\log |\tilde{w}(z)|}{\sqrt{4-z^2}} \, dx < \infty \), and the latter condition means the existence of outer function in \( \Omega \) with absolute value \( \tilde{w} \) on the boundary).

**Lemma 8.5.** Let \( \tilde{w} \in A_2[-2, 2], \int_{-2}^2 |g|^2 |\tilde{w}| \, dx < \infty \), and let

\[
G(z) = \int_{-2}^2 \frac{g(t) \, dt}{t - z}.
\]

Then \( G(z)\phi(z) \in E^2(\Omega) \).

**Proof.** Consider

\[
G_+(x) := \lim_{y \to 0^+} \int_{-2}^2 \frac{g(t) \, dt}{t - x - iy}
\]

and

\[
G_-(x) := \lim_{y \to 0^-} \int_{-2}^2 \frac{g(t) \, dt}{t - x - iy}.
\]

Jump formula says that \( G_+(x) - G_-(x) = c \cdot g(x) \) for a.e. \( x \). On the other hand, \( G_+(x) + G_-(x) = c \cdot \mathcal{H}g(x) \) for a.e. \( x \). We conclude that both \( G_+ \), \( G_- \) are in \( L^2([-2, 2], \tilde{w} \, dx) \) if and only if both \( g, \mathcal{H}g \in L^2([-2, 2], \tilde{w} \, dx) \). The latter is the same as \( \mathcal{H}g \in L^2([-2, 2], \tilde{w} \, dx) \) (because of course \( g \in L^2([-2, 2], \tilde{w} \, dx) \) by assumption). We conclude that both boundary values are in \( L^2([-2, 2], \tilde{w} \, dx) \) if and only if \( \mathcal{H}g \) is. But the latter condition is equivalent to (we discussed this already) \( \tilde{w} \in A_2[-2, 2] \).

We are left to prove that \( G_+, G_- \in L^2([-2, 2], \tilde{w} \, dx) \) implies that \( G(z)\phi(z) \in E^2(\Omega) \). Actually these claims are equivalent, and this does not depend on \( A_2 \) anymore. Notice that our function \( G(z) \) is a Cauchy integral of \( L^1(-2, 2) \) function, and, as such, belongs to Smirnov class \( E^p(\Omega) \) for any \( p \in (0, 1) \).

For any outer function \( h \) in \( \Omega \) and for any analytic function \( G \), say, from \( E^{1/2}(\Omega) \) we have that \( Gh \in E^2(\Omega) \) if and only if \( (Gh)_+ \in L^2(-2, 2), (Gh)_- \in L^2(-2, 2) \). This is the corollary of the famous theorem of Smirnov (see [8]) that says that if in a domain \( \Omega \) one has a holomorphic function \( F \) which is the ratio of two bounded holomorphic functions such that the denominator does not have singular inner part (the class of such functions is denoted by \( \mathcal{N} \)), and if \( f \partial \Omega \in L^q(\partial \Omega) \) then \( f \in E^q(\Omega) \). In our case one should only see that any \( G \in E^{1/2}(\Omega) \) and any outer function \( h \) are functions from \( \mathcal{N} \). Then we apply this observation to \( G \) and to outer function \( h = \phi \), and we see that the requirement \( G(z)\phi(z) \in E^2(\Omega) \) is equivalent to \( G_+, G_- \in L^2([-2, 2], \tilde{w} \, dx) \). We finished the lemma’s proof. \( \square \)
Remark 8.6. A little bit more is proved. Namely, given a weight \( \tilde{w} \) on \([-2, 2]\), we can claim that for every \( g \) such that \( \int_{-2}^{2} |g|^2 \tilde{w} \, dx < \infty \) we have that function

\[
G(z) = \int_{-2}^{2} \frac{g(t) \, dt}{t - z}
\]

satisfies \( G(z) \phi(z) \in E^2(\Omega) \) if and only if \( \tilde{w} \in A_2[-2, 2] \). We need this claim only in “if” direction.

Lemma 8.5 is very helpful as it allows us to write yet another inequality equivalent to eqrefdual1:

\[
\sum_{x_k \in X} |G\phi(x_k)|^2 \frac{\sigma_k}{|\phi(x_k)|^2} \leq C \int_{-2}^{2} |G\phi(x)|^2 \, dx.
\] (8.18)

We want to see now that when \( g \) runs over the whole of \( L^2(\tilde{w}) \), function \( G\phi \) runs over the whole of \( E^2(\Omega) \) (recall that \( G(z) := \int_{-2}^{2} \frac{g(t) \, dt}{t - z} \)). Lemma 8.5 gives one direction: if \( g \in L^2(\tilde{w}) \) then \( G\phi \in E^2(\Omega) \).

Let us show the other inclusion. So suppose \( F \in E^2(\Omega) \). Consider \( G(z) = \frac{F(z)}{\phi(z)} \). We want to represent it as follows:

\[
F(z) \phi(z) = \int_{-2}^{2} f(t) \, dt \quad f \in L^2(\tilde{w}).
\] (8.19)

To do that notice that both boundary value functions \( \left( \frac{F(z)}{\phi(z)} \right)_+ \) \( \left( \frac{F(z)}{\phi(z)} \right)_- \) are in \( L^2(\tilde{w}) \). Here we use again the fact that \( \tilde{w} \in A_2[-2, 2] \). So these two boundary value functions are in \( L^1 \). And \( \frac{F(z)}{\phi(z)} \in \mathcal{N} \) of course. We use again Smirnov’s theorem (see [8]) to conclude that \( \frac{F(z)}{\phi(z)} \in E^1(\Omega) \). Then put

\[
f(t) := c \cdot \left( \left( \frac{F(z)}{\phi(z)} \right)_+ - \left( \frac{F(z)}{\phi(z)} \right)_- \right).
\]

It is in \( L^2(\tilde{w}) \) and so is in \( L^1 \). We apply Cauchy integral theorem to function \( \frac{F(z)}{\phi(z)} \) from \( E^1(\Omega) \). We get exactly (8.19) if constant \( c \) is chosen correctly.

All this reasoning shows that in (8.18) when \( g \) runs over the whole of \( L^2(\tilde{w}) \), function \( G\phi \) runs over the whole of \( E^2(\Omega) \). Therefore (8.18) can be rewritten as follows:

\[
\sum_{x_k \in X} |F(x_k)|^2 \frac{\sigma_k}{|\phi(x_k)|^2} \leq C \int_{-2}^{2} |F(x)|^2 \, dx, \quad \forall F \in E^2(\Omega).
\] (8.20)

This is very nice because (8.20) is a familiar Carleson measure condition, only not in Hardy class \( H^2 \) in the unit disc, but for its full analog \( E^2 \) in \( \Omega = \mathbb{C} \setminus [-2, 2] \).

The trasfer from the disc to \( \Omega \) is obvious:
Lemma 8.7. Let $D_I$ denote two discs centered at $-2$ and $2$ and of radius $I$. Measure $d\mu$ in $\Omega$ satisfies
\[
\int |F(z)|^2 d\mu(z) \leq C \int_{-2}^2 |F(x)|^2 \, dx
\]
for all $F \in E^2(\Omega)$ if and only if
\[
\int_{D_I} \frac{d\mu(z)}{|z^2 - 4|} \leq C' \sqrt{I}.
\] (8.21)

Proof. Let $\psi$ be conformal map from the disc $\mathbb{D}$ onto $\Omega$. If $F \in E^2$ then $F \circ \psi \cdot (\psi')^{1/2} \in H^2$. We apply Carleson measure theorem to the new measure $\tilde{\mu} := \psi^{-1} \ast \mu$ in the disc and see that $\tilde{\mu}/|\psi'|$ is a usual Carleson measure (see [3]). Coming back to $\Omega$ gives (8.21). □

Immediately we obtain the following necessary and sufficient condition for (8.16) (or (8.17)) to hold:
\[
\sum_{k:x_k \pm 2 \leq \tau} \frac{\sigma_k}{|\phi(x_k)|^2 \sqrt{x_k^2 - 4}} \leq C \sqrt{\tau}, \quad \forall \tau > 0.
\] (8.22)

The condition (8.22) plus $\tilde{\omega} \in A_2$ give the full necessary and sufficient condition for (8.12) to hold, and so for the $L^2$ boundedness of the operators of transformation.

However we want to simplify (8.22). The problem with this condition as it is shown now lies in the fact that we have to compute the outer function $\phi$ with given absolute value $\sqrt{\tilde{\omega}}$ on $[-2, 2]$. This might not be easy in general. We want to use the fact that $\tilde{\omega} \in A_2[-2, 2]$ once again to replace $\phi(x_k)$ by a simpler expression. We need one more lemma.

Lemma 8.8. Let $\tilde{\omega} \in A_2[-2, 2]$ and let $x > 2$. There are two constants $0 < c < C < \infty$ independent of $x$ such that
\[
c \frac{1}{x - 2} \int_{4-x}^2 \tilde{\omega} dt \leq |\phi^2(x)| \leq C \frac{1}{x - 2} \int_{4-x}^2 \tilde{\omega} dt.
\] (8.23)

Proof. Let $P_z(s)$ stands for the Poisson kernel for domain $\Omega$ with pole at $z \in \Omega$. It is easy to write its formula using conformal mapping onto the disc, but we prefer to write its asymptotic behavior when $z > 2$ and $z - 2$ is small:
\[
P_z(s) \asymp \frac{\sqrt{z - 2}}{\sqrt{2 - s} (z - s)}.
\] (8.24)

Notice that it is sufficient to prove only the right inequality in (8.23). In fact, the left one then follows from the right one applied to $\tilde{\omega}^{-1}$ if one uses $\tilde{\omega} \in A_2$. So let us have $\delta$ be a number very close to 1, but $\delta < 1$. There exists such a $\delta$ that $\tilde{\omega}^{\delta}$ is still in $A_2$.

Having this in mind we write
\[
\phi^2(x) = c \int_{-2}^2 \log \tilde{\omega} P_z(s) ds \leq \left( \int_{-2}^2 \tilde{\omega}^{\delta} \frac{1}{\sqrt{2 - s} \sqrt{x - s}} ds \right)^\frac{1}{\delta}.
\]
We can split the last integral into two:
\[ I := \int_{4-x}^{2} \tilde{w}^{\delta} \frac{1}{\sqrt{x-s}} \sqrt{x-2} \frac{1}{s} ds \leq C \frac{1}{\sqrt{x-2}} \int_{4-x}^{2} \tilde{w}^{\delta} \frac{1}{\sqrt{2-s}} ds \]

and
\[ II := \int_{-2}^{4-x} \tilde{w}^{\delta} \cdot ds \leq C \int_{-2}^{2} \tilde{w}^{\delta} \frac{\sqrt{x-2}}{(x-s)^{\frac{1}{2}}} ds. \]

It is easy to take care of \( II \). In fact, it is well known that for any \( A_{2}[-2,2] \) weight \( u \)
\[ \int_{-2}^{2} u(s) (x-2)^{a} (s)^{1+a} ds \leq \frac{1}{x-2} \int_{4-x}^{2} u(s) ds. \]

But this is false to claim that for any \( A_{2} \) weight \( u \) one has
\[ \frac{1}{\sqrt{x-2}} \int_{4-x}^{2} u(s) ds \leq C \frac{1}{\sqrt{x-2}} \int_{4-x}^{2} \tilde{w} ds. \]

As a result we get \( |\phi^{2}(x)| \leq C \frac{1}{x-2} \int_{4-x}^{2} \tilde{w} ds \), which is the right inequality of the lemma. We already noticed that the left inequality follows from the right one (using \( A_{2} \) property and applying what we proved to \( \frac{1}{\tilde{w}} \)). Lemma is completely proved. \( \square \)

Now we can rewrite (8.22) in an equivalent form.

**Proposition 8.9.** Let \( x_{k} \to 2 \) (we consider accumulation to point 2 only, accumulation to \(-2\) is symmetric). Consider condition
\[ \sum_{k : x_{k} - 2 \leq \tau} \frac{\sigma_{k}}{\tilde{w}(s)} \sqrt{x_{k} - 2} \leq C \sqrt{\tau}, \quad \forall \tau > 0. \] (8.25)

Then (if points accumulate only to 2) (8.25) plus \( \tilde{w} \in A_{2}[-2,2] \) are equivalent to (8.12).

If points accumulate to both \( \pm 2 \) we need to add an obvious symmetric condition near \(-2\). Thus Theorem 1.23 is completely proved.

**Remark 8.10.** (step backward—step forward). Let
\[ \hat{\theta}(v) := \frac{b_{0} + v \theta^{(1)}(v)}{1 + b_{0} v \theta^{(1)}(v)}, \quad b_{0} \in \mathbb{D}. \] (8.26)

Then
\[ \hat{\theta}(v) := \frac{1 + \epsilon_{0} \theta(0)}{1 + \epsilon_{0} \theta(0)} \frac{c_{0} + \theta(v)}{1 + \epsilon_{0} \theta(v)}, \] (8.27)
where \( c_0 = \frac{b_0 - a_0}{1 - a_0 b_0} \) is actually an arbitrary point in \( \mathbb{D} \). Obviously multiplication of \( \hat{\theta} \) by \( e^{ic} \) does not change the norm of the transformation operator. Thus arbitrary fraction-linear transformation

\[
\hat{\theta}(v) := e^{ic} \frac{c_0 + \theta(v)}{1 + c_0 \theta(v)},
\]

preserves \( A_2 \) (1.72) and ”Carleson” (1.73) conditions.

9. SUFFICIENT CONDITION IN TERMS OF SCATTERING DATA

Proof of Theorem 1.24. Let

\[
W = \begin{bmatrix} 1 & \bar{R}_+ \\ \bar{R}_+ & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \bar{B} & 0 \\ 0 & B \end{bmatrix}.
\]

Condition (1.76) means that the matrix weight \( BWB^* \) is in \( A_2 \).

First we prove that

\[
||f^-||^2 \leq Q||f^-||^2_{L^2_{\alpha_-}} \tag{9.1}
\]

for \( f^- (t) \in \hat{H}^2_{\alpha_-} \). In fact even \( ||f^-||^2 \leq Q||f^-||^2_{\hat{H}^2_{\alpha_-}} \).

Recall

\[
T_-(\tau) f^-(\tau) = \bar{\tau} f^+(\tau) + R_+ (\tau) f^+(\tau) \in \bar{B} H^2,
\]

where \( f^+ \in L^2_{\alpha_+} \otimes \hat{H}^2_{\alpha_+} \). Therefore

\[
P_+ B W \begin{bmatrix} f^+(\tau) \\ \bar{\tau} f^+(\bar{\tau}) \end{bmatrix} = \begin{bmatrix} 0 \\ B(\tau) T_-(\tau) f^{-} (\tau) \end{bmatrix},
\]

and

\[
\langle BW^{-1} B^* P_+ BW \begin{bmatrix} f^+(\tau) \\ \bar{\tau} f^+(\bar{\tau}) \end{bmatrix}, P_+ BW \begin{bmatrix} f^+(\tau) \\ \bar{\tau} f^+(\bar{\tau}) \end{bmatrix} \rangle = ||f^-||^2. \tag{9.2}
\]

Due to the \( A_2 \) condition we get

\[
||f^-||^2 \leq Q \langle W \begin{bmatrix} f^+(\tau) \\ \bar{\tau} f^+(\bar{\tau}) \end{bmatrix}, \begin{bmatrix} f^+(\tau) \\ \bar{\tau} f^+(\bar{\tau}) \end{bmatrix} \rangle = Q||f^-||^2_{L^2_{\alpha_-}}. \tag{9.3}
\]

Now we will prove the second part of the claim, that is,

\[
||f^+||^2 \leq Q||f^+||^2_{L^2_{\alpha_+}} \tag{9.4}
\]

for \( f^+ (t) \in \hat{H}^2_{\alpha_+} \).

Since (9.1) holds then \( \hat{H}^2_{\alpha_-} = \hat{H}^2_{\alpha_-} \) (moreover \( \hat{H}^2_{\alpha_-} \subset H^2 \)). Evidently, this implies \( \hat{H}^2_{\alpha_+} = H^2_{\alpha_+} \). Indeed,

\[
\hat{H}^2_{\alpha_+} = (L^2_{\alpha_+} \otimes \hat{H}^2_{\alpha_+})^+ = (L^2_{\alpha_+} \otimes \hat{H}^2_{\alpha_-})^+ = L^2_{\alpha_+} \otimes (\hat{H}^2_{\alpha_-})^+ = H^2_{\alpha_+}.
\]

Therefore (9.4) is guarantied by the inequality

\[
\langle f, f \rangle \leq Q \left\{ \langle W \begin{bmatrix} f(\tau) \\ \bar{\tau} f(\bar{\tau}) \end{bmatrix}, \begin{bmatrix} f(\tau) \\ \bar{\tau} f(\bar{\tau}) \end{bmatrix} \rangle + \sum |f(\zeta_k)|^2 \nu(\zeta_k) \right\} \tag{9.5}
\]

for functions of the form \( f = f_1 + B f_2 \), where

\[
f_1(\zeta) = \sum_{k=1}^{N} \frac{B(\zeta)}{(\zeta - \zeta_k) B'(\zeta_k)} f(\zeta_k), \quad f_2 \in H^2.
\]
Note that $f_1$ and $Bf_2$ orthogonal with respect to the standard metric in $H^2$, i.e.,

$$\|f\|^2 = \|f_1\|^2 + \|f_2\|^2.$$ 

Let us calculate the matrix of the matric in $\tilde{H}^2_{\alpha+}$ which is generated by this decomposition.

$$\langle Bf_2, Bf_2 \rangle_{L^2_{\alpha+}} = \frac{1}{2} \left< B\mathbb{W}B^* \left[ f_2(\tau) \right], \left[ f_2(\tau) \right] \right> = \langle (I + H_2)f_2, f_2 \rangle, \quad (9.6)$$

where $H_2$ is the Hankel operator generated by the symbol $\tilde{R}+\tilde{\nu}_+$. The matrix generated by this decomposition is

$$\mathbf{H}_2 f_2 = P_+(\tilde{R}+\tilde{\nu}_+)(\tilde{R}_+ f_2)(\tilde{\tau}).$$

Similarly

$$\langle f_1, f_1 \rangle_{L^2_{\alpha+}} = \langle (I + H_1)f_1, f_1 \rangle + \delta(f_1, f_1), \quad (9.7)$$

where $\delta$ is the quadratic form corresponding to the scalar product in $L^2_{\nu_+}$. Finally,

$$\langle f_2, f_1 \rangle_{L^2_{\alpha+}} = \langle T f_2, f_1 \rangle, \quad (9.8)$$

where $T$ is the truncated Toeplitz operator

$$T f_2 = P_+(B - \bar{\tau}(\tilde{R}+f_2)(\tilde{\tau})).$$

In these terms, according to (9.5) and the above (9.6)–(9.8), we have to show that there exists $\epsilon(= \frac{1}{Q}) > 0$ such that

$$\epsilon \left[ \begin{array}{ccc} I & 0 & \delta \\
0 & I & \mathbf{T} \\
\mathbf{T}^* & I + H_2 \end{array} \right] \leq \left[ \begin{array}{ccc} I + H_1 + \delta & \mathbf{T} \\
\mathbf{T}^* & I + H_2 \end{array} \right]. \quad (9.9)$$

By (1.76) we have $\|H_2\| < 1$. Therefore we can substitute (9.9) by

$$\epsilon \left[ \begin{array}{ccc} I & 0 & \delta \\
0 & I + H_2 \end{array} \right] \leq \left[ \begin{array}{ccc} I + H_1 + \delta & \mathbf{T} \\
\mathbf{T}^* & I + H_2 \end{array} \right]. \quad (9.10)$$

It is equivalent to

$$\left[ \begin{array}{ccc} I + H_1 + \delta - \epsilon & \mathbf{T} \\
\mathbf{T}^* & (1 - \epsilon)(I + H_2) \end{array} \right] \geq 0$$

or

$$\left[ \begin{array}{ccc} (1 - \epsilon)(I + H_1 + \delta - \epsilon) & \mathbf{T} \\
\mathbf{T}^* & (I + H_2) \end{array} \right] \left[ \begin{array}{ccc} I + H_1 & \mathbf{T} \\
\mathbf{T}^* & (I + H_2) \end{array} \right] + \left[ \begin{array}{ccc} (1 - \epsilon)(\delta - \epsilon) - \epsilon(I + H_1) & 0 \\
0 & 0 \end{array} \right] \geq 0. \quad (9.11)$$

Since the first term in the RHS (9.11) is nonnegative it is enough to find $\epsilon$ such that

$$\frac{\epsilon}{1 - \epsilon}(I + H_1) + \epsilon I \leq \frac{3 - \epsilon}{1 - \epsilon} I \leq \delta.$$ 

Note that the last inequality is the same as

$$\frac{3 - \epsilon}{1 - \epsilon}\|f_1\|^2 \leq \|f_1\|^2_{L^2_{\alpha+}}. \quad (9.12)$$

Due to the below (well known) lemma, the Carleson condition for $\tilde{\nu}_+$, given by (1.74), implies

$$\|f_1\|^2 \leq Q\|f_1\|^2_{L^2_{\alpha+}}. \quad (9.13)$$

Thus (9.12) and consequently the whole theorem is proved.
Lemma 9.1. The following condition
\[ \|f\|_{K_B}^2 \leq Q \|f\|_{L^2}^2, \quad \forall f \in K_B := H^2 \ominus BH^2, \] (9.14) is satisfied if and only if \( \tilde{\nu}, \tilde{\nu}(\zeta_k) = \frac{B(\zeta)}{(\zeta - \zeta_k)B'(\zeta_k)} \) is a Carleson measure.

Proof. A function \( f \in K_B \) we represent in the form
\[ f = \sum \frac{B(\zeta)}{(\zeta - \zeta_k)B'(\zeta_k)} f(\zeta_k) \]
so (9.14) is equivalent to the boundness of the operator
\[ A(\{x_k\}) = \sum \frac{B(\zeta)}{(\zeta - \zeta_k)B'(\zeta_k)} x_k \rightarrow K_B. \]

Note that
\[ (f, g) = \sum \frac{x_k}{\sqrt{\nu(\zeta_k)}} \bar{g}_k \quad \text{for} \quad g = \sum \frac{y_k}{1 - \zeta_k}. \]
That is,
\[ A^*(g) = \left\{ \frac{y_k}{\sqrt{\nu(\zeta_k)}} \right\} \]
and (9.14) can be rewritten into the form
\[ \sum |y_k|^2 \leq Q \|g\|_{K_B}^2. \] (9.15)
Note that
\[ g = \sum \frac{y_k}{1 - \zeta_k} \mapsto \tilde{g} = \sum \frac{B(\zeta)y_k}{(\zeta - \zeta_k)} \]
is a unitary mapping, and thus we get from (9.15)
\[ \sum |\tilde{g}(\zeta_k)|^2 \frac{1}{|B'(\zeta_k)|^2 \nu(\zeta_k)} \leq Q \|\tilde{g}\|_{K_B}^2 \] (9.16)
for all \( \tilde{g} \in K_B. \) □

10. ATTACHMENT

The space \( L_\theta \) is defined as the set of 2D vector functions with the scalar product
\[ \|f\|^2 = \int_T \left( \begin{bmatrix} 1 & \theta \end{bmatrix} f_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right) d\mu(t). \] (10.1)
By \( K_\theta \) we denote its subspace
\[ K_\theta = L_\theta \ominus \begin{bmatrix} H^2_\theta \\ H^2_\theta \end{bmatrix}. \]

Lemma 10.1. The vector
\[ \begin{bmatrix} 1 \\ -\theta(\mu) \end{bmatrix} \begin{bmatrix} 1 \\ 1 - \mu \end{bmatrix} \]
is the reproducing kernel in \( K_\theta. \)
Proposition 10.2. Let $E = \{t : |\theta| \neq 1\}$. For $f \in K_\theta$

\[
\int_E \frac{|f_1 + \theta f_2|^2}{1 - |\theta|^2} dm \leq C\|f\|^2
\]

(10.2)

(compare (8.10)) if and only if

\[
\int_E \left\langle \begin{bmatrix} 1 & \theta \\ \bar{\theta} & 1 \end{bmatrix}^{-1} Hf, Hf \right\rangle dm(t) \leq C \int_T \left\langle \begin{bmatrix} 1 & \theta \\ \bar{\theta} & 1 \end{bmatrix} f, f \right\rangle dm(t),
\]

(10.3)

where $(Hf)(z)$ is the Hilbert Transform

\[
(Hf)(z) = \int \begin{bmatrix} 1 & \theta \\ \bar{\theta} & 1 \end{bmatrix} \frac{f_1}{t - z} dm(t).
\]

Proof. By definition

\[
Hf(z) = \int \begin{bmatrix} 1 & \theta \\ \bar{\theta} & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \frac{t}{t - z} dm(t)
\]

\[
= P_+ \left\{ \begin{bmatrix} f_1 + \theta f_2 \\ \bar{\theta} f_1 + f_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \bar{\theta} \end{bmatrix} \begin{bmatrix} h_- \\ h_+ \end{bmatrix}, \begin{bmatrix} h_+ \end{bmatrix} \right\}(z) = \left[ \begin{bmatrix} (f_1 + \theta f_2)(z) \\ 0 \end{bmatrix} + \begin{bmatrix} \theta(z) \\ 1 \end{bmatrix} h_+(z) \right].
\]

Therefore

\[
\int_E \left\langle \begin{bmatrix} 1 & -\theta \\ -\bar{\theta} & 1 \end{bmatrix} \frac{1}{1 - |\theta|^2} Hf, Hf \right\rangle dm
\]

\[
= \int_E \left\langle \begin{bmatrix} 1 & -\theta \\ -\bar{\theta} & 1 \end{bmatrix} \left( f_1 + \theta f_2 \right) \right\rangle + \begin{bmatrix} 0 \\ 1 \end{bmatrix} h_+, \left[ \begin{bmatrix} f_1 + \theta f_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \theta \end{bmatrix} h_+ \right] dm
\]

\[
= \int_E \frac{|f_1 + \theta f_2|^2}{1 - |\theta|^2} dm + \int_E |h_+|^2 dm.
\]

That is (10.3) is equivalent to

\[
\int_E \frac{|f_1 + \theta f_2|^2}{1 - |\theta|^2} dm + \int_E |h_+|^2 dm \leq C(\|f\|^2 + \int_T |h_+|^2 dm).
\]

(10.6)

Lemma 10.3. Condition (10.3) implies

\[
\sup_I \frac{1}{|I|} \int_I |\theta - \langle \theta \rangle_I|^2 + (1 - |\langle \theta \rangle_I|^2) \frac{1}{1 - |\theta|^2} dm < \infty,
\]

(10.7)

where for an arc $I \subset E$ we put

\[
\langle \theta \rangle_I := \frac{1}{|I|} \int_I \theta dm.
\]

(10.8)
Proof. In particular (10.3) implies that the matrix weight \[
\begin{bmatrix}
1 & \theta \\
\bar{\theta} & 1
\end{bmatrix}
\] is in $A_2$ on $E$. And this means
\[
\int_I \begin{bmatrix}
1 & -\theta \\
-\bar{\theta} & 1
\end{bmatrix} \frac{1}{1-|\theta|^2} \, dm \leq C \begin{bmatrix}
1 & \langle \theta \rangle_I \\
\langle \bar{\theta} \rangle_I & 1
\end{bmatrix}^{-1},
\]
(10.9)
or
\[
\int_I \frac{1-|\theta|^2 + |\theta - \langle \theta \rangle_I|^2}{(\langle \theta \rangle_I - \theta)\sqrt{1-|\langle \theta \rangle_I|^2}} \frac{1-|\langle \theta \rangle_I|^2}{1-|\theta|^2} \, dm \leq C,
\]
(10.10)
which is equivalent to (10.7) □

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