Solving the curvature and Hubble parameter inconsistencies through structure formation-induced curvature

Asta Heinesen and Thomas Buchert

1Univ Lyon, Ens de Lyon, Univ Lyon1, CNRS, Centre de Recherche Astrophysique de Lyon UMR5574, F–69007, Lyon, France

Emails: asta.heinesen@ens-lyon.fr and buchert@ens-lyon.fr

Abstract. Recently it has been noted by Di Valentino, Melchiorri and Silk (2019) that the enhanced lensing signal relative to that expected in the spatially flat ΛCDM model poses a possible crisis for the Friedmann-Lemaître-Robertson-Walker (FLRW) class of models usually used to interpret cosmological data. The ‘crisis’ amounts to inconsistencies between cosmological datasets arising when the FLRW curvature parameter Ω_k is determined from the data rather than constrained to be zero a priori. Moreover, the already substantial discrepancy between the Hubble parameter as determined by Planck and local observations increases to the level of 5σ. While such inconsistencies might arise from systematic effects of astrophysical origin affecting the Planck Cosmic Microwave Background (CMB) power spectra at small angular scales, it is an option that the inconsistencies are due to the failure of the FLRW assumption. In this paper we recall how the FLRW curvature ansatz is expected to be violated for generic relativistic spacetimes. We explain how the FLRW conservation equation for volume-averaged spatial curvature is modified through structure formation, and we illustrate in a simple framework how the curvature tension in a FLRW spacetime can be resolved—and is even expected to occur—from the point of view of general relativity. Requiring early-time convergence towards a Friedmannian model with a spatial curvature parameter Ω_{k0} equal to that preferred from the Planck power spectra resolves the Hubble tension within our dark energy-free model.

Keywords: relativistic cosmology—scalar curvature—Hubble tension—backreaction

1. Introduction

Since the founding of relativistic cosmology, the FLRW class of models has been used to interpret cosmological data and to constrain the dynamical nature of our Universe. While the FLRW spacetimes offer a simple framework for interpreting cosmological data, the spacetimes which the FLRW models can reasonably approximate are limited. In particular, the FLRW curvature ansatz of a single constant parameter describing the curvature of space throughout the evolution of the Universe excludes the general-relativistic coupling of spatial curvature to the matter sources. Such dynamical coupling is in general expected to be non-cancelling even when averaged over the largest scales.
Solving the curvature and Hubble parameter inconsistencies

It is worth recalling these limitations of the FLRW ansatz given the inconsistencies in cosmological parameters inferred by various experiments when using the FLRW framework for data reduction [1–4]. While the inconsistencies highlighted in [1] might partly be due to unknown astrophysical phenomena affecting the high multipoles of the Planck power spectra [5, 6] and the precise significance levels are dependent on the likelihood function used [7], we believe that there is reason to consider the possibility that the inferred discrepancies in cosmological parameters between datasets could be the result of neglected physics in the FLRW class of models. Various phenomenological extensions of the ‘base’ ΛCDM model (Cold Dark Matter and a cosmological constant Λ) with six parameters have been investigated for their potential to solve the parameter discrepancies, including for instance non-minimal dark sector physics and running of the spectral index [8, 9]. However, the effect of changing parameters within the FLRW paradigm appears to fall short with respect to the significance of the Hubble tension [10]. We argue that the physics driving the tensions between datasets within the ΛCDM paradigm might simply be general-relativistic interaction between structure in the matter distribution and curvature which generically introduces extra terms on the largest scales to the Friedmann equations of a strictly structureless universe model. Spatially averaging the Einstein field equations introduces non-cancelling correction terms to the large-scale evolution equations of FLRW model spacetimes. For example, on a compact domain $D$, the volume-averaged variance of the expansion rate $\Theta$, $\langle (\Theta - \langle \Theta \rangle_D)^2 \rangle_D$, acts as source of an effective Hubble rate $H_D = 1/3 \langle \Theta \rangle_D$, that positively accumulates differences in expansion rates, say between that of clusters and voids, from the smallest up to the largest scales. This variance counteracts gravity and couples to the average scalar curvature, and thus modifies the average dynamics relative to that expected in FLRW model universes. Hence, the aforementioned tensions might be solved from first principles within general relativity without the need for introducing dark energy or for introducing phenomenological parameters or exotic physics.

In this paper we highlight the differences between curvature in FLRW cosmology and in generic relativistic spacetimes, focusing on spacetimes with a single irrotational dust source. We illustrate how dynamical curvature expected from general relativity might account for the tensions encountered in the FLRW framework. We invoke—as a show-case and proof of concept—a simple and physically motivated solution to an exact scalar averaging scheme that quantifies spacetime dynamics on the largest scales. We emphasize that an FLRW solution for interpreting cosmological data can only make physical sense if it describes the Universe on average.

We start by discussing properties of spatial curvature, relations to topology and conservation laws in section 2. Then, we present a class of models that respects generic dynamical properties of average spatial curvature in a general-relativistic spacetime, and we employ a dark energy-free model that solves the curvature and Hubble parameter inconsistencies in section 3. We present our conclusions in section 4.
2. Remarks on spatial curvature

In general relativity, information about the curvature of spacetime is fully contained in the Riemann tensor. In FLRW cosmology the existence of six killing vector fields—which represent translational and rotational invariance and are physically motivated by the large-scale statistical homogeneity and isotropy suggested by cosmological data—is assumed in order to reduce the space of metric solutions. In the FLRW class of spacetimes the spatial Riemann tensor of spatial hypersurfaces is completely determined by the three-dimensional Ricci scalar $\mathcal{R} = 6k/a^2(t)$, where $k$ is the constant-curvature parameter of dimensions $1/\text{length}^2$, and $a(t)$ is the dimensionless scale factor evaluated at the hypersurface labelled by the time parameter $t$. The curvature parameter $k$ can be understood as an integration constant in a Newtonian derivation of the Friedmann equations, representing the conserved energy of accelerated particles located on the edge of an isolated uniformly expanding sphere.

Assuming simply-connected three-manifolds, the sign of $k$ determines the topology of the spatial sections, such that $k > 0$ implies the topology of a hypersphere, $k = 0$ implies Euclidean topology, and $k < 0$ implies hyperbolic space. For generic spacetimes there are no ‘quantized’ scalar curvature states describing the topology of space. These topological implications hold because the scalar curvature coincides with the sectional curvatures of the manifold in this highly symmetric case. Contrary to what is the case for the FLRW class of spacetimes, the Universe may be described by spherical topology on spatial hypersurfaces while being equipped with a metric that everywhere has negative (and even constant) three-dimensional Ricci scalar curvature over the same hypersurfaces [11]. It does therefore not in general make physical sense to draw conclusions on the topology of the Universe based on the three-dimensional Ricci scalar. Moreover, several studies of inhomogeneous cosmological models point towards average negative three-dimensional Ricci scalar as an attractor in the late Universe irrespective of the exact initial conditions given at the CMB epoch. (See, e.g., [12–14, 16].) This generic feature is physically explained by (almost) empty void regions gaining volume dominance in the late Universe.

The FLRW three-dimensional Ricci scalar is associated with the conservation law $\mathcal{R}a^2(t) = \text{const.}$ In general, there does not exist such an integral constraint for the volume-averaged three-dimensional Ricci scalar as in the FLRW class of models. That is, the average scalar curvature does not obey a conservation law like the average rest-mass density [12]. It turns out, however, that there exists an integral constraint that couples the volume-averaged scalar curvature to the structure inhomogeneities, generalizing the FLRW conservation equation [17–19]. For a spacetime with a single irrotational dust source with four-velocity $u$ the integral constraint as formulated within the scalar averaging scheme reads [17]:

$$\frac{1}{a_D^2} \left( \frac{\mathcal{Q}_D}{a_D^6} \right)^\cdot + \frac{1}{a_D^2} \left( \langle \mathcal{R} \rangle_D a_D^2 \right)^\cdot = 0 \, ,$$

(1)

† See the following section for more introduction.
where $\mathcal{R}$ is the three-dimensional Ricci scalar defined on the spatial hypersurfaces normal to the fluid four-velocity. The averaging operation $\langle \cdot \rangle_D$ is the Riemannian average over a subdomain $\mathcal{D}$ of the same set of hypersurfaces, and $\dot{\cdot} \equiv d/d\tau$ denotes the derivative with respect to the proper time function $\tau$ of the fluid.\(\S\) The spatial domain $\mathcal{D}$ is defined to follow the fluid flow (no net flow of fluid elements into and out of the averaging domain) but might otherwise be chosen for the physical problem at hand. The volume of the domain, normalized by the initial volume, defines an effective dimensionless scale factor on the domain: $|\mathcal{D}|/|\mathcal{D}_i| =: a^3_D$.

The function $Q_D$ is the ‘kinematical backreaction’,\(\parallel\) and is defined from the variance of the rate of expansion and the averaged shear scalar of the fluid congruence over the domain $\mathcal{D}$, $Q_D \equiv \frac{2}{3} \langle (\Theta - \langle \Theta \rangle_D)^2 \rangle - 2 \langle \sigma^2 \rangle_D$. The average spatial curvature is generically not separately preserved but couples to the spacetime structure through $Q_D$. This in turn gives rise to an effective dark energy-like effect [17]. A coupling of this type is expected from first principles, since curvature generically couples to structure in the matter distribution. The dependence on domain of the conservation equation reflects the regional departure from homogeneity and isotropy which is in general present in a Universe with structure. For a structureless and isotropically expanding fluid $Q_D = 0$ and (1) reduces to the FLRW conservation equation for $\mathcal{R}$ for all domains $\mathcal{D}$. The no-backreaction conservation equation $\langle \mathcal{R} \rangle_D a^2_D \dot{=} 0$ remains true to linear order in perturbation theory around a FLRW background.\(\¶\) The integral constraint (1) suggests that the FLRW class of solutions forms a measure zero set (no interaction with structure) and that fine-tuning or restricting assumptions leading to exact cancelation are in general required to maintain a notion of conservation of curvature during a time interval and at some given spatial scale. In particular, it is expected for inhomogeneous model universes that the FLRW curvature conservation equation is violated at the onset of structure formation [14,15]. Furthermore, the exactly zero-curvature FLRW model forms a measure zero set within the FLRW class of models. The FLRW models have been shown to be globally gravitationally unstable in the directions of the dark energy and dark matter sectors, i.e. the average model is driven away from the FLRW solution, which forms a repeller within a dynamical systems analysis in the cases where $Q_D$ mimics the dark components [13]. We also note that it is a generic feature of relativistic spacetimes that average spatial curvature $\langle \mathcal{R} \rangle_D$ can change sign over cosmic epochs which is impossible in the FLRW class of models.

Table 1 shows a summary of important properties related to spatial curvature in the FLRW class of spacetimes and how these properties are modified in full general relativity.

\(\S\) The proper time function $\tau$ is defined uniquely from the family of possible proper time functions by requiring $u = \nabla \tau$.

\(\parallel\) For a detailed discussion on kinematical backreaction see [17,20].

\(\¶\) $Q_D$ also vanishes if deviations from homogeneity are evaluated on flat space sections with periodic boundary conditions [21,22]. This result carries over to relativistic perturbative settings, where the background of the perturbations is assumed to be spatially flat.
Table 1: Comparison of curvature properties within the FLRW class of cosmological models and for generic averaged globally hyperbolic spacetime models.

|                | FLRW                                                                 | General relativity                          |
|----------------|----------------------------------------------------------------------|---------------------------------------------|
| **Topology**   | \( \text{sign}(\mathcal{R}) \) determines the spatial topology for simply-connected domains | \( \langle \mathcal{R} \rangle_D \) does not in general allow conclusions on topological properties |
| **Integral constraint** | local ‘Newtonian’ energy conservation: \( (\mathcal{R} a^2) \cdot = 0 \) | general-relativistic coupling of \( \langle \mathcal{R} \rangle_D \) to structure: \( \frac{1}{a^2_D} (\mathcal{Q}_D a^2_D) + \frac{1}{a^2_D} (\langle \mathcal{R} \rangle_D a^2_D) \cdot = 0 \) |
| **Sign of curvature** | \( \text{sign}(\mathcal{R}) \) is preserved throughout the evolution of the Universe and on all scales | \( \text{sign}(\langle \mathcal{R} \rangle_D) \) can change in response to structure in the spacetime and may vary on different scales |
| **Copernican principle** | satisfied in its most strict interpretation. All fundamental observers are subject to the same local curvature | can be satisfied in a weaker sense than for FLRW. ‘Distributional equivalence’ between observers |

One might argue that the FLRW solutions are compatible with the high degree of isotropy of the CMB together with the Copernican principle, and that the FLRW constraints on spatial curvature are natural in a physical Universe with no preferred observers. While it is true that comoving observers in the FLRW spacetimes can be considered strictly identical, the FLRW models constitute an idealized limit of realistic statistically homogeneous and isotropic models. In 1968 Ehlers, Geren, and Sachs proved that for a solution of the Einstein equations with the only matter source being a radiative fluid with an isotropic distribution function, the spacetime is either stationary, or given by an FLRW solution, or a special solution with non-zero rotation and acceleration of the radiation fluid [23]. These results have been generalized to the case of a radiative fluid with an almost isotropic distribution function along with realistic matter content [24] as is relevant for our observations of the CMB. Here it was shown that if the CMB temperature and its derivatives are almost isotropic everywhere in a dust-dominated and expanding universe model, and the observers of the CMB are geodesic, the spacetime is almost described by an FLRW metric. However, as pointed out in [25], the results in [24] follow from the assumptions about smallness of derivatives in the temperature field of the CMB photons, which are not directly observed and which are directly related to local derivatives of the metric tensor which are expected to be large in the real Universe. However, as a conservative assumption, we may adopt the FLRW model in the radiation-dominated phase and an ample time thereafter. We expect any significant deviations from the FLRW class of models to emerge at the time of onset of structure formation.

3. Scaling solutions as a case study and a proof of concept

Here we provide a brief introduction to the scalar averaging scheme (the ‘Buchert equations’) and introduce a class of ‘scaling solutions’ which satisfy the global equations.
For an overview of this averaging scheme see, e.g., [14,17–20,26]. The Buchert scheme of inhomogeneous cosmology replaces local spacetime variables by volume-averaged variables which represent the ‘macro-state’ on a given domain of the spacetime. The global dynamics is constrained by the local spacetime variables which obey the Einstein equations. Consequently the macroscopic variables must obey a set of equations which are similar in form to the Friedmann equations, but with additional terms accounting for the large-scale integrated effect of local inhomogeneity and anisotropy, named ‘cosmological backreaction’. The volume-averaged energy constraint in an irrotational dust universe model reads:

$$\Omega^D_m + \Omega^D_\Lambda + \Omega^D_R + \Omega^D_Q = 1,$$

(2)

where the four domain-dependent cosmological ‘parameters’ $\Omega^D_m$, $\Omega^D_\Lambda$, $\Omega^D_R$, and $\Omega^D_Q$ constituting a ‘cosmic quartet’ are defined by:

$$\Omega^D_m \equiv \frac{8\pi G}{3H^2_D} \langle \rho \rangle_D; \quad \Omega^D_\Lambda \equiv \frac{\Lambda}{3H^2_D}; \quad \Omega^D_R \equiv -\frac{\langle R \rangle_D}{6H^2_D}; \quad \Omega^D_Q \equiv -\frac{Q_D}{6H^2_D},$$

(3)

where $\rho$ is the local rest mass density, $R$ is the local three-Ricci scalar, $\Lambda$ is the cosmological constant, $Q_D$ is the non-local kinematical backreaction term, $H_D \equiv \dot{a}_D/a_D$ is the ‘effective Hubble parameter’, $a_D \equiv (|D|/|D_0|)^{1/3}$ the ‘effective scale factor’ as introduced above, and $\langle \cdot \rangle_D$ is the Riemannian volume average over the three-dimensional hypersurfaces defined as being orthogonal to the fluid flow. + The averaged energy constraint (2), together with the averaged Raychaudhuri equation—which is analogue to the acceleration equation of FLRW—and the integral constraint (1), form a set of three independent equations with four unknown macroscopic variables $a_D$, $\langle \rho \rangle_D$, $\langle R \rangle_D$, and $Q_D$. A physically motivated closure condition is thus needed.

3.1. A large-scale exact scaling approximation

Following a similar approach as in [13,27–29] we consider the following simple, exact closure to the system of equations on the largest scales $D$:

$$\langle R \rangle_D = \mathcal{W}_D a_D^p + 6k a_D^2; \quad Q_D = Q^0_D a_D^p,$$

(4)

which we denote scaling solutions. The constant $\mathcal{W}_D$ is the backreaction-induced curvature $\mathcal{W}_D \equiv \langle R \rangle_D - 6k a_D^{-2}$ evaluated at the present epoch, and $p$ is a scaling index determining the power law dependence with $a_D$. The ‘integration constant’ term $6k a_D^{-2}$—where $k \equiv \Omega^0_k H^2_D$—is the Friedmannian component of the curvature that can be added to any solution satisfying (1) to obtain a new solution. Plugging (4) into the integral constraint (1) provides the linear relation

$$\langle R \rangle_D = 6k a_D^{-2} - \frac{n + 6}{n + 2} Q_D$$

(5)

+ The dependence on foliation is expected to be weak on large scales in physical applications of this covariant averaging scheme. This has been discussed in [30] and contrasted with coordinate-dependent statements in the literature. We note that the choice of comoving foliation applied here can be defined coordinate-independently and is distinct from the choice of gauge in standard model perturbation theory [31]. For the explicit demonstration of 4−covariance of the averaging formalism adapted in this paper, see [32].
Solving the curvature and Hubble parameter inconsistencies

between kinematical backreaction and the average spatial curvature, where the second term models the deviations from the Friedmannian behaviour that we below determine from perturbative considerations and observational data. We may rewrite the averaged energy constraint \( (2) \) as follows:

\[
\Omega^D_m + \Omega^D_\Lambda + \Omega^D_\chi + \Omega^\text{FLRW}_k = 1 ;
\]

\[
\Omega^D_\chi \equiv \frac{4}{n+6} \Omega^D_W ; \quad \Omega^\text{FLRW}_k \equiv -\frac{k a_D^{-2}}{H_D^2} ,
\]

where \( \Omega^D_W \equiv -W^D / 6 H^2 D \) is the backreaction-induced curvature parameter. The dimensionless cosmological parameter \( \Omega^D_\chi \) can be seen as incorporating the collection of effects due to inhomogeneous structure (both the backreaction term itself \( Q^D_D \), but also the backreaction-induced curvature \( W^D_D \)).

3.2. Template metric, distances and structure-emergent curvature evolution

We employ the following template metric—as motivated by Ricci flow smoothing of Riemannian hypersurfaces [33]—to convert cosmological parameters into predictions for angular diameter distance for observations on the largest scales,

\[
^4 g^D \equiv -dt^2 + L^2_{D_0} a^2_D \left( \frac{dr^2_D}{1 - \kappa_D(t) r^2_D} + r^2_D d\Omega^2 \right) ,
\]

where \( t \) labels the fundamental hypersurfaces of averaging, and \( r_D \) is a dimensionless radial coordinate, which also has the interpretation as a comoving distance.

\[
L_{D_0} = \begin{cases} \sqrt{|\Omega^D_{R_0}| H_{D_0}^{-1}} , & \Omega^D_{R_0} \neq 0 , \\ H_{D_0}^{-1} , & \Omega^D_{R_0} = 0 , \end{cases}
\]

is the spatial curvature scale for curved models and the Hubble horizon in the spatially flat case. The dimensionless scale factor is set equal to unity at the present epoch \( a_{D_0} = a_{D}(t_0) = 1 \). \( d\Omega^2 \equiv (d\theta^2 + \sin(\theta)^2 d\phi^2) \) is the angular element on the unit sphere, and \( \kappa_D \) is a dimensionless spatial constant-curvature function

\[
\kappa_D(t) \equiv \begin{cases} \langle R \rangle_D(t) a^2_D(t) , & \Omega^D_{R_0} \neq 0 , \\ 0 , & \Omega^D_{R_0} = 0 . \end{cases}
\]

The metric \( (7) \) reduces to a spatial FLRW template metric on each spatial slice with scalar curvature equal to \( \langle R \rangle_D(t) / 6 \) on each spatial hypersurface of constant proper time \( t = \text{const.} \), but the union of hypersurfaces does not in general correspond to a single four-dimensional FLRW metric. This dynamical curvature feature for the template metric reflects the lack of a ‘Newtonian’ conservation law for the average three-Ricci scalar discussed in section 2. We note that there are other possible extrapolations of the FLRW metric yielding the same spatial FLRW metric on each of the \( t = \text{const} \) hypersurfaces, but which are associated with a different union of the surfaces into a four-metric. We employ the form of the metric \( (7) \) in this analysis, keeping in mind the limited space of FLRW extrapolations investigated, c.f. [34]. For investigations of the
Solving the curvature and Hubble parameter inconsistencies

application of a spatially flat template metric in an interesting statistically homogeneous
test case within the Buchert and Green & Wald schemes, see [35], c.f. [36].

Let us consider a Universe with a statistically homogeneous and isotropic matter
distribution which is slowly evolving compared to the time it takes for light to cross a
homogeneity scale. The redshift associated with typical observers and emitters comoving
with the slices of statistical homogeneity and isotropy and separated by distances
larger than an approximate homogeneity scale is then well-approximated as [37, 38]
\[ 1 + z = 1/a_D. \]
This identification of redshift is different from that used in [28, 29] where
the redshift function was calculated from the geodesic equation for null rays propagating
on the ‘template metric background’ (7). Such a phenomenological procedure is at
odds for long-time evolution with the more rigorous calculations from local spacetime
dynamics in [37, 38], and we thus employ the approximative result \( 1 + z = 1/a_D \) in the
following analysis.

∗ Note in this context that ‘Ricci-dominated metrics’ (like the FLRW
model that only features a Ricci curvature component) is at odds with the reality of
light propagation in the sense that light predominantly ‘sees’ the Weyl tensor (and it
does so exclusively in the case of propagation through voids).

We assume that the angular diameter distance is well-described by that of the
template metric,
\[ d_A(z_D) = L_{D_0} a_D(z_D) r_D(z_D), \]
with \( r_D(z_D) \) given by the radial null lines in (7),
\[ \frac{dr_D}{da_D} = -\frac{L_{D_0} H_{D_0}}{a_D^2} \sqrt{1 - \kappa_D(a_D) r_D^2(a_D)} \Omega_{D_0} m a_D^{-3} + \Omega_{k_0}^{FLRW} a_D^{-2} + \Omega_X a_D^n, \]
with \( r_D(a_D = 1) \equiv 0 \). With the scaling closure (4), the macroscopic variables \( a_D, \langle \rho \rangle_D, \langle R \rangle_D, \langle Q \rangle_D \) and the corresponding template metric are fully determined by the initial
conditions. Assuming \( \Lambda = 0 \), the four parameters \( \Omega_{m_0}, \Omega_{k_0}^{FLRW}, H_{D_0}, \) and \( n \) uniquely
determine the scaling solution.

It is useful to consider the following curvature function
\[ \frac{\langle R \rangle_D(t) a_D^2(t)}{6 H_{D_0}^2} = |\Omega_{n_0}^{D_0}| \kappa_D(t) = -\Omega_{m_0}^{D_0} a_D^{n+2}(t) - \Omega_{k_0}^{FLRW}, \]
where we have used the definitions given in (3), (6), and (9). The function (12) can be
seen as an effective FLRW ‘present-epoch curvature parameter’ for each hypersurface
\( t = const., \) and might be derived from the generic curvature statistic [39] for models
where an angular diameter distance and a Hubble parameter can be formulated as
functions of redshift,
\[ k_H \equiv \frac{1}{D^2} \left( 1 - \left( \frac{dD}{dz} \frac{H}{H_0} \right)^2 \right), \]

∗ The two methods give comparable results in the low-redshift Universe, with \( \sim 0.3\% \) differences at
the mean redshift of the Joint Lightcurve Analysis (JLA) supernova sample \( z \sim 0.3, \) but we here wish
to span the whole cosmic epoch since decoupling. We skip the index \( D \) at the redshift for notational ease.
where $D$ is related to the angular diameter distance $d_A$ by $D = H_0/c (1+z)d_A$, and where $H$ is the Hubble parameter of the model. We have omitted the label $D$ in the expression (13) for ease of notation, and any scale-dependence of $k_H$ remains implicit. From the expression for the FLRW comoving distance $D = 1/\sqrt{\Omega_{k_0}} \sinh(\sqrt{\Omega_{k_0}} \int_0^z dz' H(z'))$—where $\Omega_{k_0}$ is the FLRW cosmological curvature parameter evaluated at the present epoch—it follows that $k_H = -\Omega_{k_0}$ in the FLRW class of metrics by identity. In generic models the expression (13) need not coincide with a curvature parameter entering in an energy constraint equation, and it will in general fail to be a constant in redshift. Using the expressions for the angular diameter distance of the scaling solutions (10), (11), we find that

$$k_H = \left| \Omega_{D_0}^{D_0} \right| \kappa_{D}(t) = -\Omega_{D_0}^{D_0} a_D^{n+2}(t) - \Omega_{FLRW}^{FLRW},$$

(14)

which is equal to the curvature function in (12). Thus, if one were able to determine the right-hand side of equation (13) model-independently at different redshifts, we would expect the outcome (14) in the case of the scaling solution with scaling index $n$ being an accurate phenomenological model for describing the largest scales of the Universe. For $n > -2$ we expect convergence to constant FLRW-type curvature at high redshifts, whereas at low redshifts it is expected to be dominated by curvature induced by structure formation.

3.3. Asymptotic positive FLRW curvature at the last scattering epoch

We are interested in investigating whether we might be able to account for the apparent curvature discrepancy between Planck power spectra and low redshift datasets together with the FLRW Hubble parameter discrepancy within this model of backreaction. For the purpose of constraining the investigated class of scaling solutions, we fix the scaling index $n$ by its theoretical prediction obtained in [40] in a perturbative framework around an Einstein–de Sitter background, where the leading-order backreaction was found to obey the scaling law $Q_D \propto a_{EdS}^{-1}$ corresponding to $n = -1$. Interestingly, this theoretical prediction is supported by the fit of the scaling solutions with $\Lambda = 0$ to the Joint Lightcurve Analysis (JLA) dataset [41] of type Ia supernovae, keeping the scaling index as a free parameter [29]. Fixing the scaling index to its theoretical prediction $n = -1$, the ‘1σ’ confidence bounds on the matter cosmological parameter was constrained to

‡ This identity is purely geometrical, and robust to tuneable features within the FLRW class of metrics such as matter content, dark energy equation of state, and modifications of the Einstein field equations.

†† Model-dependent determinations of (13) should be treated with care as the assumptions might a priori impose specific model curvature behaviour which need not correspond to that of the underlying spacetime.

† The scaling solution $Q_D \propto a_D^{-1}$ corrects the background-dependent scaling by following the domain scale factor $a_D$ rather than $a_{EdS}$, accounting for the volume difference in a curved space compared to that of a flat space.

‡‡ In practice the luminosity distance–redshift relation of the scaling solutions is obtained by applying Etherington’s reciprocity theorem.
Solving the curvature and Hubble parameter inconsistencies

\[ \Omega^D_{m_0} = 0.25^{+0.04}_{-0.04}, \]

with a quality of fit comparable to that of the ΛCDM model. We note that this result was derived assuming \( \Omega^{\text{FLRW}}_{k_0} = 0 \).

However, for sufficiently small values of \( \Omega^{\text{FLRW}}_{k_0} \), the ‘energy budget’ (6) at low redshifts is dominated by \( \Omega^D_m \) and \( \Omega^D_X \). The model \( \Omega^D_{m_0} = 0.25 - \Omega^{\text{FLRW}}_{k_0} / 2; |\Omega^{\text{FLRW}}_{k_0}| \lesssim 0.05 \)§ produces angular diameter distances of \( \lesssim 0.5\% \) differences to the \( \Omega^D_{m_0} = 0.25, \Omega^{\text{FLRW}}_{k_0} = 0 \) model for the redshift range \( z \lesssim 1.3 \) of the JLA sample. We thus conclude that we might safely use the modified best-fit model,

\[ \Omega^D_{m_0} = 0.25 - \Omega^{\text{FLRW}}_{k_0} / 2; |\Omega^{\text{FLRW}}_{k_0}| \lesssim 0.05, \]

as an approximation for the purposes of this paper.

Let us now consider the scaling solution model with \( \Omega^{\text{FLRW}}_{k_0} = -0.04 \)—which converges (already at moderately high redshifts) toward an FLRW universe model with \( \Omega_{k_0} = -0.04 \) at early times—consistent with the best-fit value of the curvature parameter of the Planck power spectra [1,6]. Note that we might transform the Planck inferred FLRW curvature parameter \( \Omega_{k_0} \) to the corresponding value \( \Omega_k(z^*) \) evaluated at the last scattering surface of central redshift \( z^* \), and match the scaling solution to the FLRW model at this epoch by requiring \( \Omega^{\text{FLRW}}_k(z^*) = \Omega_k(z^*) \). This matching procedure gives slightly different results than a simple matching at the present epoch \( \Omega^{\text{FLRW}}_{k_0} = \Omega_{k_0} \) due to the different evolution of the Hubble parameter as a function of redshift in the models and the potential difference in redshift of the last scattering epoch. However, the difference in the evolution in Hubble parameter between the models and the model-independent constraint on redshift to the last scattering surface (see the below analysis) gives \( \lesssim 0.002 \) differences in these two determinations of \( \Omega^{\text{FLRW}}_{k_0} \), and our conclusions are robust to the exact choice of matching procedure.

From the best-fit supernovae result we thus construct the solution \( \Omega^D_{m_0} = 0.25 - \Omega^{\text{FLRW}}_{k_0} / 2 = 0.27 \), implying \( \Omega^D_{D_0} = 0.77 \) and \( \Omega^D_{X_0} = 5/4 \cdot 0.77 \) from (6). The value \( \Omega^D_{m_0} = 0.27 \) is in good agreement with the recent model-independently determined matter density parameter from gas mass fraction measurements in galaxy clusters, supernovae observations and cosmic baryon abundance measurements from absorption systems at high redshifts [42], \( \Omega_m = 0.285 \pm 0.013 \), and with the ΛCDM inferred value from the Dark Energy Survey galaxy clustering and weak lensing [43], \( \Omega_m = 0.267^{+0.030}_{-0.017} \). These smaller values of \( \Omega^D_{m_0} \) relative to that expected in ΛCDM are in line with what has been found in relativistic simulations done within a class of ‘silent universe models’ [44], where the emergence of spatial curvature was found to cause \( \Omega^D_{m_0} \) to be smaller, with \( \sim 0.05 \) relative to a ΛCDM model with the same initial conditions.

The function \( k_H \) for this model is shown in figure 1. At large redshifts we have asymptotic convergence towards \( k_H = -\Omega^{\text{FLRW}}_{k_0} \), whereas smaller redshifts are dominated by the negative induced average curvature due to structure formation.

§ Where we compensate for the introduction of a non-zero value of \( \Omega^{\text{FLRW}}_{k_0} \) by modifying \( \Omega^D_{m_0} \) and \( \Omega^D_{X_0} \) by the same amount.
Figure 1: The functions $k_H$ and $-\Omega^D_W$ as predicted by the scaling solution $n = -1$, $\Omega^D_m = 0.27$, $\Omega^{\text{FLRW}}_{k0} = -0.04$, and with zero cosmological constant. The horizontal line shows the $k_H$ line for an FLRW model with $\Omega_{k0} = -0.04$, which the scaling solution approaches asymptotically. For a FLRW model universe $k_H = -\Omega_{k0}$, where $\Omega_{k0}$ is the spatial curvature parameter evaluated at the present epoch.

Figure 1 also displays the backreaction-induced curvature parameter $\Omega^D_W$ of the scaling solutions. Backreaction-induced curvature reaches percent levels of the ‘energy budget’ with $\Omega^D_W \sim 0.01$ at $z = 15$. Thus, our model reflects the conservative assumption of an almost FLRW universe model at early stages. The onset of backreaction-induced curvature is early as compared to what was found in numerical simulations within the ‘silent universe model’ mentioned above [16], where $\Omega^D_W \sim 0.01$ was reached at $z = 4$, which agrees with the general idea that backreaction effects might become significant when nonlinear structures become volume-dominant, solving the coincidence problem [14]. The early onset of backreaction in the scaling solution investigated is directly linked to the closure condition (4), which only allows for backreaction and the associated average curvature to follow a simple power law. Such a solution might not be adequate for extrapolation over large cosmological time intervals. We consider the scaling solution as a case study and proof of concept, keeping in mind that the extrapolation between redshifts probed by supernovae $z \lesssim 1$ and redshifts around the epoch of decoupling $z \sim 1000$ is given by a large-scale leading-mode approximation.

The magnitude of the curvature statistic $k_H$ at low redshifts shown in figure 1 might seem to contradict the FLRW results in the literature, given the interpretation of $-k_H$ as an effective FLRW ‘present epoch’ spatial curvature parameter at each $t = \text{const.}$ hypersurface. However, the tight combined constraints excluding a negatively curved
universe model within the FLRW models, is not applicable to models which are not contained in the FLRW class of spacetimes.

Interestingly, constraints on the FLRW curvature parameter $\Omega_{k0}$—independent of the theory of gravity on cosmological scales and matter content, but assuming the FLRW class of metrics—from supernovae and strong lensing probing redshifts $\lesssim 1.8$ hints at moderate negative curvature [45] even though results are compatible with $\Omega_{k0} = 0$ at the level of one standard deviation. We also note that recent analysis [46] points to non-accelerating negatively curved FLRW universe models as good fits to data, especially when accounting for the dipole moment in the FLRW inferred acceleration [47, 48], suggesting that bulk flow caused by a regional under density can mimic the observed positive acceleration [49]. For the scaling solutions there is no local cosmological constant or energy component contributing to accelerating expansion—rather cosmological acceleration is an emergent large-scale effect caused by the rapidly expanding (almost) empty void regions gaining volume dominance in the late Universe. The transition from negative to positive global acceleration $\ddot{a}_D$ occurs at a redshift $z \sim 0.7$ for the model investigated in this section, which is comparable to the onset of acceleration as estimated in the $\Lambda$CDM model.

The curvature statistic $k_H$ has been analyzed model-independently using the JLA sample, SDSS-III BOSS BAO measurements, and differential age measurements of galaxies [50]. In this analysis, tendencies for preferred negative $k_H$—corresponding to negative effective FLRW spatial curvature—was inferred (see their Fig. 6). Despite these tendencies, the analysis [50] showed consistency of the flat FLRW expectation $k_H = 0$ within $2\sigma$ confidence bounds.

3.4. Solving the Hubble discrepancy

We now analyze the implications for the scaling solution Hubble parameter as inferred from the acoustic scale of the CMB. We use two different estimates of the redshift and angular diameter distance to the epoch decoupling. The results from Planck [6] quoted in the first column of their table 2 gives $\{z^*_\text{Planck} = 1090.3 \pm 0.4, d_A\text{Planck}(z^*_\text{Planck})/\text{Mpc} = 12.72 \pm 0.05\}$. More model-independent constraints based on the allowance for a rescaling of the angular diameter distance to the epoch of decoupling in the Einstein–de Sitter model [51] gives the constraints $\{z^*_\text{MI} = 1094 \pm 1, d_A\text{MI}(z^*_\text{MI})/\text{Mpc} = 12.7 \pm 0.2\}$. Requiring the angular diameter distance as parameterized by the scaling solution with $\Omega_{k0} = 0.27$ and $\Omega_{k0}^{\text{FLRW}} = -0.04$ to coincide with the best-fit values of these empirical determinations of $\{z^*, d_A(z^*)\}$, we obtain:

$$H_{D_0} = 74.5 \text{ km/s/Mpc}$$
(Planck: $z^* = 1090.3$ ; $d_A(z^*)/\text{Mpc} = 12.72$);

$$H_{D_0} = 74.2 \text{ km/s/Mpc}$$
(Model indep.: $z^* = 1094$ ; $d_A(z^*)/\text{Mpc} = 12.70$),

(15)
which are in agreement with the low-redshift measurements of the expansion rate from cepheids and type Ia supernovae [2, 3]. Even though the results [2, 3] are derived in a ΛCDM model-dependent manner, the low-redshift value of the cepheids used to calibrate the Hubble parameter should make these measurements relative insensitive to the model cosmology and valid for comparison between models. The negative curvature component $\Omega_D^{W}$, which falls off slowly with redshift due to the scaling index being $n = -1$, contributes to large distances to the CMB.

While the modification of the best-fit scaling solution from the JLA sample $\{\Omega_m^D = 0.25, \Omega_{k0}^{\text{FLRW}} = 0\} \mapsto \{\Omega_m^D = 0.27, \Omega_{k0}^{\text{FLRW}} = -0.04\}$ does not change the angular diameter distance–redshift relation significantly at the low redshifts probed by the JLA sample, it does change the relation at high redshifts. For the model $\{\Omega_m^D = 0.25, \Omega_{k0}^{\text{FLRW}} = 0\}$ without a small positive FLRW curvature component the predictions (15) would have been $H_D^0 = 82.2$ km/s/Mpc and $H_D^0 = 81.9$ km/s/Mpc for the Planck and model-independent determination of $z^*, d_A(z^*)$, respectively. Thus, assuming the validity of the applied constraints for angular diameter distance and local Hubble parameter estimation in the context of the scaling solutions, a positive curvature component is not only allowed for in the scaling solutions but is necessary to fit the angular diameter distance to the recombination epoch.

4. Summary and Conclusion

We have discussed the assumptions about spatial curvature which are inherent in the FLRW ansatz usually imposed in cosmological analysis. We have discussed how the situation differs in a generic average model subject to the laws of general relativity where curvature and structure in the matter distribution are dynamically coupled. Models with dynamical curvature are not a priori excluded from any physical principle nor from any existing cosmological dataset, rather they are natural in a general-relativistic universe model with structure on a hierarchy of scales.

As a case study of models allowing for spatial curvature we have considered a class of scaling solutions which are obtained by imposing constraints on the generic solutions of the averaged Einstein equations. We have shown that the best-fit scaling solution from the JLA sample of type Ia supernovae accounts for the Hubble parameter anomaly in FLRW cosmology when an asymptotic FLRW curvature parameter coinciding with that preferred from the Planck power spectra [1, 6] is required in the model. The coincidence of the best-fit scaling solution obtained from type Ia supernovae data fitting the peak of the angular diameter distance with an asymptotic FLRW spatial curvature parameter of $\Omega_{k0} = -0.04$ indicates that general-relativistic dynamical spatial curvature models are natural candidates for accounting for the tensions between high- and low-redshift cosmological datasets in FLRW cosmology.

In FLRW cosmology the preferred positive curvature of the Planck power spectra arises from an enhanced lensing signal—resulting in smoothing of high multipoles—relative to what would be expected within a flat ΛCDM model [5, 6]. It is beyond the
solving the curvature and Hubble parameter inconsistencies

scope of this paper to aim for quantifying lensing within the scaling solutions which would require quantification in a yet undeveloped perturbative framework. Here we merely point out that large-scale dynamical curvature models—as exemplified by the scaling solutions—can account for positive curvature in the early universe while being consistent with local expansion rate measurements as a result of dynamical, structure-emergent average curvature [17].

The overall results of our analysis align with those found in [16] where general-relativistic simulations in the ‘silent universe model’—with relativistic ray-tracing based on the Sachs optical equations implemented—showed a backreaction-induced transition to negative curvature towards late cosmological epochs. Within the same simulation, initial data consistent with Planck was shown to generate a Hubble parameter consistent with low-redshift measurements [2,3] as a direct consequence of the emergent negative spatial curvature.

The reader may recall that our model simply assumed a large-scale leading-mode approximation for backreaction, which is a member of a generic realization of average properties of the $3 + 1$ Einstein equations, together with consistent and physically motivated metric and distance notions. As a proof of concept, this model respects (i) the generic coupling of geometry (curvature) to the sources, (ii) the generic non-conservation of the average scalar curvature, and (iii) it reflects the generic possibility of the change of sign of the averaged scalar curvature. The result is a natural and consistent explanation of (i) dark energy, (ii) the coincidence problem (here conceptually, not quantitatively), (iii) positive initial curvature, (iv) the small matter density cosmological parameter found in local probes of the matter density, (v) the large angular diameter distance to the CMB consistent with JLA supernova sample parameter constraints, and (vi) the local expansion rate measurements (removal of the ‘Hubble tension’).

We believe that this model architecture needs convincing arguments to be rejected as a physically viable show-case, on the basis of which the model ingredients can be improved in order to build a physical cosmology in the future.

Acknowledgements: This work is part of a project that has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement ERC advanced grant 740021–ARTHUS, PI: TB). We wish to thank Léo Brunswic for discussions on topology, Jan Ostrowski for many valuable comments during this work, and Krzysztof Bolejko, Syksy Räsänen and David Wiltshire for fruitful comments on the manuscript.

References

[1] Di Valentino E, Melchiorri A and Silk J 2020 Planck evidence for a closed Universe and a possible crisis for cosmology Nature Astron. 4 196–203 (2020) [arXiv:1911.02087]
[2] Riess A G et al. 2018 New Parallaxes of Galactic Cepheids from Spatially Scanning the Hubble Space Telescope: Implications for the Hubble Constant Astrophys. J. 855 136 [arXiv:1801.01120]
[3] Riess A G et al. 2019 Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for
Solving the curvature and Hubble parameter inconsistencies

the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM *Astrophys. J.* 876 85 [arXiv:1903.07603]

[4] Hildebrandt H et al. 2017 KiDS-450: Cosmological parameter constraints from tomographic weak gravitational lensing *Mon. Not. Roy. Astron. Soc.* 465 1454 [arXiv:1606.05338]

[5] Ade P A R, et al. [Planck collaboration] (2016) Planck 2015 results. XIII. Cosmological parameters *Astron. Astrophys.* 594 A13 [arXiv:1502.01589]

[6] Aghanim N, et al. [Planck collaboration] (2018) Planck 2018 results. VI. Cosmological parameters [arXiv:1910.09853]

[7] Efstathiou G and Gratton S (2020) The evidence for a spatially flat Universe [arXiv:2002.06892]

[8] Di Valentino E, Melchiorri A, Mena O and Vagnozzi S (2019) Non-minimal dark sector physics and cosmological tensions [arXiv:1910.09853]

[9] Di Valentino E, Melchiorri A and Silk J (2020) Cosmological constraints in extended parameter space from the Planck 2018 Legacy release *J. Cosmol. Astropart. Phys.* JCAP 01 013 [arXiv:1908.01391]

[10] Dhawan S, Brout D, Scolnic D, Goobar A, Riess A G, Miranda V (2020) Cosmological model insensitivity of local $H_0$ from the Cepheid distance ladder [arXiv:2001.09260]

[11] Lohkamp J (1994) Metrics of negative Ricci curvature *Ann. of Math.* 140 655

[12] Buchert T and Carfora M (2008) On the curvature of the present-day Universe *Class. Quantum Grav.* 25 195001 [arXiv:0803.1401]

[13] Roy X, Buchert T, Carloni S and Obadia N (2011) Global gravitational instability of FLRW backgrounds—interpreting the dark sectors *Class. Quantum Grav.* 28 165004 [arXiv:1103.1146]

[14] Buchert T and Räsänen S (2012) Backreaction in Late–Time Cosmology *Annu. Rev. Nucl. Part. Sci.* 62 57 [arXiv:1112.5335]

[15] Räsänen S (2011) Backreaction: directions of progress *Class. Quant. Grav.* 28 164008 [arXiv:1102.0408]

[16] Bolejko K (2018) Emerging spatial curvature can resolve the tension between high-redshift CMB and low-redshift distance ladder measurements of the Hubble constant *Phys. Rev. D* 97 103529 [arXiv:1712.02967]

[17] Buchert T (2000) On average properties of inhomogeneous fluids in general relativity. I: Dust cosmologies *Gen. Relativ. Gravit.* 32 105 [arXiv:gr-qc/9906015]

[18] Buchert T (2001) On average properties of inhomogeneous fluids in general relativity. II: Perfect fluid cosmologies *Gen. Relativ. Gravit.* 33 1381 [arXiv:gr-qc/0102049]

[19] Buchert T, Mourier P and Roy X (2020) On average properties of inhomogeneous fluids in general relativity III: general fluid cosmologies *Gen. Relativ. Gravit.*, accepted [arXiv:1912.04213]

[20] Buchert T (2008) Dark Energy from Structure: A status report *Gen. Relativ. Gravit.* 40 467 [arXiv:0707.2153]

[21] Buchert T and Ehlers J (1997) Averaging inhomogeneous Newtonian cosmologies *Astron. Astrophys.* 320, 1 [arXiv:astro-ph/9510056]

[22] Buchert T (2018) On backreaction in Newtonian cosmology *Mon. Not. Roy. Astron. Soc.* 473 L46 [arXiv:1704.00703]

[23] Ehlers J., Geren P and Sachs R K (1968) Isotropic Solutions of the Einstein-Liouville Equations *J. Math. Phys.* 9 1344

[24] Stoeger W R, Maartens R and Ellis G F R (1995) Proving Almost-Homogeneity of the Universe: an Almost Ehlers-Geren-Sachs Theorem *Astroph. J.* 443 1

[25] Räsänen S (2009) On the relation between the isotropy of the CMB and the geometry of the universes *Phys. Rev. D* 79 123522 [arXiv:gr-qc/0903.3013]

[26] Ellis G F R and Buchert T (2005) The universe seen at different scales *Phys. Lett. A* 347 38 [arXiv:gr-qc/0506106]

[27] Buchert T, Larena J and Alimi J M (2006) Correspondence between kinematical backreaction and scalar field cosmologies—the ‘morphon field’ *Class. Quantum Grav.* 23 6379 [arXiv:gr-qc/0606020]
Solving the curvature and Hubble parameter inconsistencies

[28] Larena J, Alimi J M, Buchert T, Kunz M and Corasaniti P-S (2009) Testing backreaction effects with observations Phys. Rev. D. 79 083011 [arXiv:0808.1161]

[29] Desgrange C, Heinesen A and Buchert T (2019) Dynamical spatial curvature as a fit to type Ia supernovae Int. J. Mod. Phys. D 28 1950143 [arXiv:1902.07915]

[30] Buchert T, Mourier P and Roy X (2018) On cosmological backreaction and its dependence on spacetime foliation Class. Quantum Grav. 35 24LT02 [arXiv:1805.10455]

[31] Clifton T, Gallagher C S, Goldberg S and Malik A (2020) Viable Gauge Choices in Cosmologies [arXiv:2001.00394]

[32] Buchert T and Carfora M (2002) Regional averaging and scaling in relativistic cosmology Class. Quantum Grav. 19 6109 [arXiv:gr-qc/0210037]

[33] Koksbang S M (2019) Another look at redshift drift and the backreaction conjecture J. Cosmol. Astropart. Phys. JCAP 10 03 [arXiv:1909.13489]

[34] Sikora S and Glöd K (2017) Example of an inhomogeneous cosmological model in the context of backreaction Phys. Rev. D 95 063517 [arXiv:1612.03604]

[35] Räsänen S (2010) Light propagation in statistically homogeneous and isotropic universes with general matter content J. Cosmol. Astropart. Phys. JCAP 03 018 [arXiv:0912.3370]

[36] Bolejko K (2017) Relativistic numerical cosmology with Silent Universes Class. Quantum Grav. 35 024003 [arXiv:1708.09143]

[37] Betoule M et al. (2014) Improved cosmological constraints from a joint analysis of the SDSS–II and SNLS supernova samples Astron. Astrophys. 568 A22 [arXiv:1401.4064]

[38] Colin J, Mohayaee R, Rameez M and Sarkar S (2019) Evidence for anisotropy of cosmic acceleration Astron. Astrophys. 651 L13 [arXiv:1808.04597]

[39] Tsagas C G (2010) Large-scale peculiar motions and cosmic acceleration Mon. Not. Roy. Astron. Soc. 405 503 [arXiv:0902.3232]