Particle-dependent deformations of Lorentz symmetry

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I here investigate what is arguably the most significant residual challenge for the proposal of phenomenologically viable “DSR deformations” of relativistic kinematics, which concerns the description of composite particles, such as atoms. In some approaches to the formalization of possible scenarios for DSR-deformation of Lorentz symmetry it emerges that composite particles should have relativistic properties different from the ones of their constituent “fundamental particles”, but these previous results provided no clue as to how the mismatch of relativistic properties could be consistently implemented. I show that it is possible to implement a fully consistent DSR-relativistic description of kinematics endowing different types of particles with suitably different deformed-Lorentz-symmetry properties. I also contemplate the possibility that some types of particles (or macroscopic bodies) behave according to completely undeformed special relativity, which in particular might apply to the DSR description of the macroscopic bodies that constitute measuring devices (“observers”). The formalization is also applicable to cases where different fundamental particles have different relativistic properties, leading to a type of phenomenology which I illustrate by considering possible applications to the ongoing analyses of the “Lorentz-symmetry anomaly” that was recently tentatively reported by the OPERA collaboration. Some of the new elements here introduced in the formulation of relativistic kinematics appear to also provide the starting point for the development of a correspondingly novel mathematical formulation of spacetime-symmetry algebras.

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I. INTRODUCTION

One of the most active areas of quantum-gravity research over the last decade concerns the fate of Lorentz symmetry in the quantum-gravity realm\(^1\). It is turning out to be particularly useful to divide all such studies into three categories: (i) cases where Lorentz/Poincaré symmetry remains unaffected; (ii) cases where there are departures from classical Lorentz/Poincaré symmetry but the relativity of inertial frames is preserved. This third option, which was proposed in Refs. \([1]\), is the one that challenges us more significantly for what concerns formalization. The case of “broken Lorentz symmetry” (with a preferred frame) is technically not much more challenging that a standard special-relativistic case, since it allows formalizations that are rather familiar, already relevant for example in the analysis of conventional propagation of light in certain material media (which indeed provide a preferred frame for the analysis of those physical contexts). Instead formalizing “deformations of Lorentz symmetry”, in the sense of the “DSR” (“doubly-special”, or “deformed-special”, relativity) proposal of Refs. \([1]\), requires us to find ways of introducing departures from Lorentz/Poincaré symmetry while preserving the delicate balance that can assure the relativity of inertial frames. This DSR proposal focuses on the possibility of relativistic theories with two characterizing invariant scales, introducing a length/inverse-momentum scale \(\ell\) with relativistic properties analogous to the familiar ones of the speed-of-light scale \(c\). From a quantum-gravity perspective it would then be natural \([1]\) to assume that the new relativistic-invariant scale \(\ell\) be roughly of the order of the inverse of the Planck scale (the “Planck length”).

At this point there is a rich literature on DSR-deformations of Lorentz symmetry, with several encouraging results (see, e.g., Refs. \([1,2]\) and references therein). Most of these results concern DSR-relativistic formulations of the possibility of introducing relativistically some deformed on-shell relations and some associated deformations of the laws of composition of momenta. The DSR proposal was put forward \([1]\) as a conceptual path for pursuing a broader class of scenarios of interest for fundamental physics, and in particular for quantum-gravity research, including the possibility of introducing the second observer-independent scale primitively in spacetime structure or primitively at the level of the (deformed) de Broglie relation between wavelength and momentum. However, the bulk of the relevant preliminary results from quantum-gravity research concern departures from the special-relativistic on-shell relation, and this in turn became the main focus of DSR research.

I here investigate issues relevant for one of the most significant residual open issue for such studies of DSR-relativistic deformations of on-shell (and momentum-conservation) relations, which concerns the description of composite particles, such as atoms. In some of the most studied attempts of formulating DSR-relativistic theories it emerges that “DSR-composite particles” should have relativistic properties different from those of their constituents. The simplest way to see that this might be the case is to consider a bunch of \(N\) ultrarelativistic particles all propagating along the \(1\) direction and each governed, say, by

\[
p_0 \simeq p_1 + \frac{m^2}{2p_1} - \ell p_1^2
\]

We can then introduce some candidates for “total spatial momentum” and “total energy”, given by \(p_0 = Np_0\) and \(p_1 = Np_1\), and observe that the validity of (1) for each of the \(N\) particles implies

\[
p_0 \simeq p_1 + \frac{\mu^2}{2p_1} - \ell p_1^2 \frac{N}{N},
\]

where \(\mu = Nm\) is the rest energy of the \(N\)-particle system. The suppression by \(1/N\) of the last term of Eq. (2) illustrates the issue: the nonlinearity of the DSR laws has nontrivial consequences for particle composites.

Much more than this simple-minded argument supports the concern that composite particles should have relativistic properties

\(^1\) The relevant aspect of the quantum-gravity problem is the “quasi-Minkowski limit” of quantum gravity, the limit where quantum gravity should reproduce in first approximation (describing small modifications of) particle physics and its special-relativistic properties.
which are different from those of their constituents, and in particular the effects of the deformation should be more weakly felt by composites. I shall not review here these more sophisticated arguments, for which I refer my readers to Refs. [3, 8] and, most notably, Ref. [9]. Let me stress however that these technical arguments, based on the nonlinearity of the laws and the way it can affect the description of composites, also makes sense physically: while an on-shell relation of type (1) is certainly plausible, at least if \( \ell \) is indeed of the order of the inverse of the Planck scale, for microscopic particles, the same on-shell relation, even taking \( \ell \) as the inverse of the Planck scale, is unacceptable for macroscopic bodies composed of very many micro-particles. The Planck-scale is huge by the standards of elementary particles but is actually a small scale (\( \sim 10^{-5} \) grams) for macroscopic bodies, and as a result, unless there is a suppression of the type shown in Eq. (2), the DSR description of macroscopic bodies could be disastrous.

So there is technical evidence of the fact that composite particles should have DSR-relativistic properties different from those of their constituents, with weaker deformation effects, and this is much welcome from the point of view of reproducing the observed properties of macroscopic bodies. But this encouraging correspondence between features for composite (particles and) bodies found on the theory side and our desiderata for the phenomenology of macroscopic bodies has also provided a formidable challenge for DSR research: if composite particles (and macroscopic bodies) have relativistic properties which are different from the ones of their constituents then these DSR-relativistic theories should be theories that do not prescribe "universal" laws of kinematics but rather particle-dependent ones!! Is that even possible?

Can a theory be fully relativistic and yet attribute different laws of kinematics to different particles?

These questions have remained so far unanswered.

I shall here show that the correct answer is yes: there are logically consistent DSR-relativistic theories in which different particles (possibly "elementary" and "composite" particles) are governed by different laws of kinematics.

After a brief reminder, in the next Sec. [II] of the basic logical structure of DSR-relativistic theories, I set the stage for my analysis, in Sec. [III] by reviewing in some detail the relativistic kinematics of a much studied DSR framework. This is of course still a standard "universal" DSR framework, but a rather sophisticated possibility. My attitude here is not one of establishing the validity of general theorems, so the strength of my results is primarily exhibited in terms of the fact that I can take as starting point a rather sophisticated "universal" DSR framework. The fact that I am able to generalize such a framework to a "non-universal" version (with particle-type-dependent effects) suggests that such generalizations might be even easier when taking as starting point simpler "universal" DSR scenarios.

A first group of new results is reported in Sec. [IV] where I show that one can combine in the same relativistic theory particles with standard special-relativistic properties and particles with DSR-deformed relativistic properties. The key ingredient of this result is a "mixing composition law" suitable for writing a DSR-covariant law of conservation of momentum for processes involving different particles with different relativistic properties, and such that the covariance is assured by a suitably adapted action of boost generators on multiparticle systems.

I then show, in Sec. [V] that one can combine in the same relativistic theory two species of particles; one with some given DSR-deformed relativistic properties and the other with some other DSR-deformed relativistic properties. I verify that this is possible at least in cases such that the DSR deformation has the same formalization for the two types of particles, but with different magnitude. I introduce for this purpose a further generalization of the "mixing composition law" and of the laws of action of boost generators on multiparticle systems.

The new formulation of relativistic kinematics introduced in Secs. [IV] and [V] is most simply viewed from the perspective of applications to different types of particles (some "elementary" and some "composite"), but I also explore, in Sec. [VI] the possibility that the laws characterized by weaker DSR deformation apply to a macroscopic body. I find preliminary encouragement for such a possible application.

Then, in Sec. [VII] I consider the possibility of applications of the formulation of relativistic kinematics here introduced to the case of different types of elementary particles, establishing a few first points relevant for the phenomenology. As a way to illustrate more vividly the content of this possible application I use as "conceptual laboratory" the neutrino-superluminality-anomaly recently tentatively reported by the OPERA collaboration [10].

In the brief Sec. [VIII] I comment on the type of spacetime-symmetry algebra which could provide the formal/mathematical counterpart for the version of relativistic kinematics I here introduce. In this respect perhaps most notably I argue that a
suitable generalization of the Hopf-algebra notion of co-product, something of the sort of a “mixing co-product”, could be inspired by the “mixing composition laws” I here introduced.

I work throughout at leading order in the deformation scale $\ell$ (with $\ell$ that can be both positive and negative, in the sense than both scenarios with $\ell/|\ell| = 1$ and scenarios with $\ell/|\ell| = -1$ are admissible). This keeps formulas at reasonably manageable level, sufficiently characterizes the new concepts, and would be fully sufficient for phenomenology if indeed the deformation scale is roughly of the order of the huge Planck scale (in which case a leading-order analysis should be all we need for comparison to data we could realistically imagine to gather over the next few decades).

However, in Sec. [IX], I do offer a small aside contemplating possible generalizations of my results to “all-order analyses”.

Some speculations about possible future developments are offered in the brief closing Sec. [X].

I mostly focus on 1+1-dimensional cases, where all conceptual issues here relevant are already present and can be exposed more simply. Therefore my momenta will often have two components, $\{p_0, p_1\}$, and when I briefly switch to consider cases with more dimensions I will use the notation $\{p_0, p_j\}$.

II. DSR-DEFORMATIONS OF LORENTZ SYMMETRY

Before proceeding with the main part of the analysis, let me pause briefly, in this section, for summarizing the main points originally made in Ref. [1] concerning the consistency requirements that the relativity of inertial frames imposes on the relationship between the form of the on-shell/displacement relation and the form of laws of energy-momentum conservation. This is one of the most used DSR concepts, and plays a pivotal role in the analysis I report in the following sections.

This consistency between on-shell relation and laws of momentum conservation that follows from insisting on the relativity of inertial frames is also rather significant from the perspective of studies of the quantum-gravity problem, where in some cases one finds “preliminary theoretical evidence” of modifications of the on-shell relation but usually not accompanied so far by any information on whether or not there should also be modifications of the law of conservation of momentum. Indeed the idea of DSR-deformed Lorentz transformations was put forward [1] as a possible description of certain preliminary theory results suggesting that there might be violations of some special-relativistic laws in certain approaches to the quantum-gravity problem, most notably the ones based on spacetime noncommutativity and loop quantum gravity. The part of the quantum-gravity community interested in those results was interpreting them as a manifestation of a full breakdown of Lorentz symmetry, with the emergence of a preferred class of observers (an “ether”). But it was argued in Ref. [1] that departures from Special Relativity governed by a high-energy/short-distance scale may well be compatible with the Relativity Principle, the principle of relativity of inertial observers, at the cost of allowing some consistent modifications of the Poincaré transformations, and particularly of the Lorentz-boost transformations. And it was already observed in Ref. [1] that this in turn would require corresponding modifications of the laws of momentum conservation.

As mentioned above, the DSR proposal could provide [1] a conceptual path for pursuing a broader class of scenarios of interest for fundamental physics, and in particular for quantum-gravity research, including the possibility of introducing the second observer-independent scale primitively in spacetime structure or primitively at the level of the (deformed) de Broglie relation between wavelength and momentum. However, the bulk of the preliminary results from quantum-gravity research concern departures from the special-relativistic on-shell relation, and this in turn became the main focus of DSR research.

So let me consider a generic on-shell relation of the type

$$m^2 = p_0^2 - p^2 + \Delta(E, p; \ell)$$

(3)

where $\Delta$ is the deformation and $\ell$ is the deformation scale.

Evidently when $\Delta \neq 0$ such an on-shell relation (3) is not Lorentz invariant. If we insist on this law and on the validity of classical (undeformed) Lorentz transformations between inertial observers we clearly end up with a preferred-frame picture, and the Principle of Relativity of inertial frames must be abandoned: the scale $\ell$ cannot be observer independent, and actually the whole form of (3) is subject to vary from one class of inertial observers to another.

From the alternative DSR perspective one would have to enforce the relativistic invariance of laws such as (3), preserving the
relativity of inertial frames, at the cost of modifying the action of boosts on momenta. Then in such theories both the velocity scale $c$ (here mute only because of the choice of dimensions) and the length/inverse-momentum scale $\ell$ play the same role as invariant scales of the relativistic theory which govern the form of boost transformations.

Several examples of boost deformations adapted in the DSR sense to modified on-shell relations have been analyzed in some detail (see e.g. Refs. [1–7] and references therein). Clearly these DSR-deformed boosts $\mathcal{N}_j$ must be such that

$$[\mathcal{N}_j, p_0^2 - \mathbf{p}^2 + \Delta(E, \mathbf{p}; M_*)] = 0.$$  \hspace{1cm} (4)

This requirement (4) of DSR-relativity is completely analogous to the corresponding ones of Galilean Relativity and Special Relativity: of course in all these cases the on-shell relation is boost invariant (but respectively under Galilean boosts, Lorentz boosts, and DSR-deformed Lorentz boosts); for Special Relativity the action of boosts evidently must depend on the speed scale $c$ and boosts must act non-linearly on velocities (since they must enforce observer-independence of $c$-dependent laws), and for DSR relativity the action of boosts evidently must depend on both the scale $c$ and the scale $\ell$, with boosts acting non-linearly both on velocities and momenta, since it must enforce observer-independence of $c$-dependent and $\ell$-dependent laws.

Actually much of the logical structure of the conjectured transition from Special Relativity to a DSR theory can be understood in analogy with the transition from Galilean Relativity to Special relativity. Famously, as the Maxwell formulation of electromagnetism, with an observer-independent speed scale “$c$”, gained more and more experimental support (among which one should count the Michelson-Morley results) it became clear that Galilean relativistic symmetries could no longer be upheld. From a modern perspective we should see the pre-Einsteinian attempts to address that crisis (such as the ones of Lorentz) as attempts to “break Galilean invariance”, i.e. preserve the validity of Galilean transformations as laws of transformation among inertial observers, but renouncing to the possibility that those transformations be a symmetry of the laws of physics. The “ether” would be a preferred frame for the description of the laws of physics, and the laws of physics that hold in other frames would be obtained from the ones of the preferred frame via Galilean transformations. Those attempts failed. What succeeded is completely complementary. Experimental evidence, and the analyses of Einstein (and Poincaré) led us to a “deformation of Galilean invariance”: in Special Relativity the laws of transformation among observers still are a symmetry of the laws of physics (Special Relativity is no less relativistic then Galilean Relativity), but the special-relativistic transformation laws are a $c$-deformation of the Galilean laws of transformation with the special property of achieving the observer-independence of the speed scale $c$.

This famous $c$-deformation in particular replaces the Galilean on-shell relation $E = \text{constant} + \mathbf{p}^2/(2m)$ with the special-relativistic version $E = \sqrt{c^4\mathbf{p}^2 + c^4m^2}$ and the Galilean composition of velocities $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$ with the much more complex special-relativistic law of composition of velocities.

This interplay between $c$-deformation of Galilean transformations and the associated deformations of the law of composition of velocities, is analogous to the interplay between the DSR-type $\ell$-deformation of Lorentz transformations and the associated deformations of the law of composition of momenta.

III. A KNOWN EXAMPLE OF DSR SETUP WITH UNIVERSALITY

I shall now give more tangibility to the brief review of DSR concepts contained in the previous section, by discussing a known DSR setup and highlighting the connection between deformation of the on-shell relation and deformation of the laws of momentum conservation.

The specific DSR setup reviewed in this section will also provide the starting point for the generalization introduced in this manuscript. The one that I review in this section, following mainly the results of Ref. [11], still is a standard DSR setup with “universal effects”, i.e. the deformation of Lorentz symmetry affects all particles in exactly the same way. Then in the next sections I will take the DSR setup of this section as starting point for adding the possibility of “nonuniversal effects”, i.e. cases where the deformation of Lorentz symmetry affects different types of particles in different ways.

The DSR setup on which I focus in this section was analyzed from the perspective here relevant in the recent Ref. [11], and more preliminarily in previous DSR studies (some comments on it were already in Ref. [1]). It is centered on a choice of
on-shell relation and law of composition of momenta which became recently of interest\[12\] also in the study of the new proposal of “relative-locality momentum spaces” \[14\] \[15\], and provides a set of rules for kinematics which can be naturally described from the viewpoint of the κ-Poincaré Hopf algebra \[16\] \[17\].

The on-shell (“dispersion”) relation is (for the 1+1-dimensional case)

\[ m^2 = p_0^2 - p_1^2 + \ell p_0 p_1^2 \tag{5} \]

and the law of composition of momenta is

\[ (p \oplus \epsilon p')_1 = p_1 + p'_1 + \ell p_0 p'_1, \quad (p \oplus \epsilon p')_0 = p_0 + p'_0. \tag{6} \]

The on-shell relation \(5\) is invariant under the following action of a boost on the momentum of a particle:

\[ [N, p_0] = p_1, \quad [N, p_1] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2, \tag{7} \]

which indeed ensures

\[ [N, p_0^2 - p_1^2 + \ell p_0 p_1^2] = 2p_0 p_1 - 2p_1(p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2) + \ell p_1^3 + 2\ell p_0 p_1 = 0 \tag{8} \]

In light of the points highlighted in the brief review of DSR concepts offered in the previous section it should be clear that this invariance of the on-shell relation would not in itself establish the DSR-compatibility of this setup: we must also insist that the laws of conservation of momentum, written using the composition law \(6\), are covariant under the action of the boosts that leave the on-shell relation invariant.

And this requirement of covariance of conservation laws is rather challenging when the law of composition of momenta is non-commutative, as in the case of \(6\). I am not advocating that the law of composition of momenta should be non-commutative; on the contrary one may well prefer \[11\] commutative laws of composition of momenta in a DSR setup. But part of the strength of the results I am here reporting resides in the fact that I am able to generalize to “nonuniversal DSR deformations” not merely a particularly simple DSR setup previously known with universality, but actually a rather virulent DSR setup, which in particular relies on a noncommutative law of composition of momenta. This should reassure my readers of the fact that the new structures I introduce in this manuscript should allow to produce non-universal versions of a rather large variety of DSR setups.

However, as stressed and addressed in Ref. \[11\], the non-commutativity of the composition law \(6\) evidently poses a challenge for formulating the action of boosts on momenta obtained composing two or more single-particle momenta. In Special Relativity (and in DSR setups with commutative law of composition of momenta \[11\]) it is possible to simply impose that the boost of a two-particle event \(e_{p \oplus \epsilon p'}\) be governed by

\[ [N|p| + N|p'|, p \oplus \epsilon p'] \]

where I decomposed the action of boosts on the composed momentum into two pieces, each given in terms of a boost acting exclusively on a certain momentum in the event.

This means that in Special Relativity one has a “total boost generator” obtained by combining trivially the boost generators acting on each individual particle. But with a noncommutative law of composition of momenta this simplicity is lost: the lack of symmetry under exchange of particles precludes, as one can easily verify, the possibility of adopting a “total boost generator” given by a trivial sum of single-particle boost generators. There is in particular no choice \[11\] of \(N|p|\) capable of ensuring that \([N|p| + N|p'| + N|p''|, (p \oplus \epsilon p' \oplus \epsilon p'')_0] \) vanishes whenever \((p \oplus \epsilon p' \oplus \epsilon p'')_0 = 0\).

What does work, as shown in Ref. \[11\], is adopting a corresponding deformation of the “total-boost law”

\[ N|p \oplus \epsilon p'| = N|p| + N|p'| + \ell p_0 N|p'| \tag{9} \]
and accordingly

\[ N_{[p + p' + p'']} = N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p]} + \ell p_0 N_{[p']} + \ell p_0' N_{[p'']} \]. \tag{10} \]

In Ref. \[11\] I verified in some detail that this prescription produces a fully consistent relativistic framework, with the needed compatibility between on-shell relation (5) and law of composition of momenta (6): the on-shell relation is invariant and the laws of conservation of momentum obtained from the composition law are covariant.

Let me here just review briefly the specific result of Ref. \[11\] concerning the covariance of the conservation law for a “trivalent process” with \( p \oplus \ell p' \oplus \ell p'' = 0 \). Checking that \( N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p]} + \ell p_0 N_{[p']} + \ell p_0' N_{[p'']} \) does indeed ensure the relativistic covariance of the conservation law \( p \oplus \ell p' \oplus \ell p'' = 0 \) is best done considering separately the 0 (“time”) component and the spatial 1 (“spatial”) component. For the 0 component one easily finds \[11\]

\[ [N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p]} + \ell p_0 N_{[p']} + \ell p_0' N_{[p'']}], (p \oplus \ell p' \oplus \ell p'')_0 = \]
\[ = [N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p]} + \ell p_0 N_{[p']} + \ell p_0' N_{[p'']}], p_0 + p_0' + p_0'' = \]
\[ = p_1 + p_1' + p_1'' + \ell p_0 p_1 + \ell p_0' p_1 + \ell p_0'' p_1' + \ell p_0 p_1' + \ell p_0' p_1'' + \ell p_0 p_1'' = (p \oplus \ell p' \oplus \ell p'')_1 = 0, \tag{11} \]

where on the right-hand side I of course used the conservation law itself.

Similarly for the 1 component one easily finds \[11\]

\[ [N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p]} + \ell p_0 N_{[p']} + \ell p_0' N_{[p'']}], (p \oplus \ell p' \oplus \ell p'')_1 = \]
\[ = [N_{[p]} + N_{[p']} + N_{[p'']} + \ell p_0 N_{[p]} + \ell p_0 N_{[p']} + \ell p_0' N_{[p'']}], p_1 + p_1' + p_1'' + \ell p_0 p_1 + \ell p_0' p_1 + \ell p_0'' p_1' + \ell p_0 p_1' + \ell p_0' p_1'' + \ell p_0 p_1'' = \]
\[ = p_0 + p_0' + p_0'' + \ell (p_0 + p_0' + p_0'') + \ell \frac{1}{2} (p_1 + p_1' + p_1'')^2 = 0, \tag{12} \]

where again on the right-hand side I used the conservation law \( p \oplus \ell p' \oplus \ell p'' = 0 \) itself, and I took again into account that I am working at leading order in \( \ell \).

The results \[11\] and \[12\] establish that indeed the boosts \( \{7, 9, 10\} \), besides admitting the on-shell relation (5) as invariant, also admit \( p \oplus \ell p' \oplus \ell p'' = 0 \) as a covariant law.

IV. A FIRST EXAMPLE OF DSR SETUP WITHOUT UNIVERSALITY

The previous sections only provided the preliminaries for the main analysis that I report in this manuscript, which is contained in this and the next section. It was a rather bulky effort on preliminaries, but I felt this might be beneficial since the results I am reporting are to a large extent surprising/unexpected and it might be helpful for my readers to be equipped with a nearly self-contained summary of the previous results which provide the starting point for the analysis I am here reporting. As announced, I am going to show that there are logically consistent DSR-relativistic theories in which different types of particles

\[2\] Throughout this manuscript I write conservation laws at a process with conventions such that all momenta intervening in the process are incoming into the process, so that indeed a trivalent process would be characterized by a conservation law of the type \( p \oplus \ell p' \oplus \ell p'' = 0 \). The case of one (or two) of the momenta that is outgoing from the process, say the momentum \( p_o \), is recovered by simply substituting for \( p \) the “antipode” of the momentum of that outgoing particle, with the antipode \( \ominus p \) defined so that \( \ominus (p) \ominus p = 0 \). For the composition law (6) the antipode is such that \( (\ominus p)_0 = -p_0, \) and \( (\ominus p)_i = -p_i \).
are governed by different laws of kinematics.

In this section I start by establishing that the type of particle discussed in the previous section, governed by a specific DSR $\ell$-deformation of Lorentz symmetry, can coexist with a second type of particle, governed by ordinary Special Relativity. The key point will be to show that there are laws of conservation of momentum allowing momentum to be exchanged between the two types of particles\(^3\) in a fully relativistic manner.

I shall consistently denote with \(p\) (or \(p'\) or \(p'' \ldots\)) the momenta of the type of particles affected by the DSR $\ell$-deformation of Lorentz symmetry discussed in the previous section, so that in particular

\[
m^2 = p_0^2 - p_j^2 + \ell p_0 p_j^2 .
\]  

(13)

And I shall consistently denote with \(k\) (or \(k'\) or \(k'' \ldots\)) the momenta of particles of the second type, the type governed by undeformed Special Relativity, so that in particular

\[
\mu^2 = k_0^2 - k_j^2
\]  

(14)

For the first type (“\(p\)-type”) of particles I shall insist again on

\[
\begin{align*}
[N_p, p_0] &= p_1, & [N_p, p_1] &= p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 , \\
[N_p, k_0] &= k_1, & [N_p, k_1] &= k_0 .
\end{align*}
\]  

(15)

while naturally, for the second type (“\(k\)-type”) of particles, I take

\[
\begin{align*}
[N_k, k_0] &= k_1, & [N_k, k_1] &= k_0 .
\end{align*}
\]  

(16)

And naturally the composition law for the special-relativistic momenta (“\(k\)-type”) is standard,

\[
(k \odot_\ast k')_j = k_j + k'_j , & \quad (k \odot_\ast k')_0 = k_0 + k'_0 ,
\]  

(17)

whereas the composition law for the $\ell$-deformed (“\(p\)-type”) particles was already introduced in the previous section:

\[
(p \oplus_\ell p')_j = p_j + p'_j + \ell p_0 p'_j , & \quad (p \oplus_\ell p')_0 = p_0 + p'_0 .
\]  

(18)

It is easy to see that the main challenge resides in finding a consistent way to compose momenta of different types, i.e. finding some “mixing composition law”, of the type \(p \oplus_\ell k\), while insisting that conservation laws written in terms of such a composition law would be covariant under a consistent prescription for the action of boosts. Equipped with no theorem, but rather the findings of a lengthy “trial and error exercise” I can simply exhibit an example of such a “mixing composition law” which does work, and it is remarkably simple:

\[
(p \oplus_\ell k')_j = p_j + k_j + \ell p_0 k_j , & \quad (p \oplus_\ell k')_0 = p_0 + k_0 .
\]  

(19)

In order to convince my readers that this does work let me start slowly, focusing at first on “bi-valent processes” in this theory with two types of particles. Some of these bi-valent processes, the ones without “mixing”, will pose no challenge: processes with conservation law \(k \odot_\ast k' = 0\) will have covariance ensured by standard total-boost actions

\[
N_{[k \odot_\ast k']} = N_k + N_{k'}
\]  

(20)

\(^3\) If the two types of particles could not interact then they would actually not “coexist”: the “Universe” of $\ell$-deformed particles would remain decoupled from (undetectable and irrelevant for) the “Universe” of $\ell$-deformed particles, and vice versa.
while processes with conservation law $p \oplus \ell p' = 0$ will have covariance ensured by the total-boost actions discussed in the previous section

$$N_{|p\oplus p'|} = N_{|p|} + N_{|p'|} + \ell p_0 N_{|p'|}.$$  \hspace{1cm} (21)

The only class of bi-valent processes which involves “mixing” is the one of processes with conservation law of the type $p \oplus \ell k, k = 0$. In light of the discussion offered in the previous section it should not be too surprising that a relativistically consistent description of such “mixing bi-valent processes” is obtained in terms of a total-boost action which itself involves a sort of “mixing”:

$$N_{|p\oplus \ell k|} = N_{|p|} + N_{|k|} + \frac{\ell}{2} p_0 N_{|k|}.$$  \hspace{1cm} (22)

Let us verify that indeed this total-boost action ensure the covariance of the conservation law $p \oplus \ell k = 0$. I start again from the 0 component of $p \oplus \ell k, k = 0$ for which I find

$$[N_{|p|} + N_{|k|} + \frac{\ell}{2} p_0 N_{|k|}, k_0 + p_0] = p_1 + k_1 + \frac{\ell}{2} p_0 k_1 = (p \oplus \ell k)_1 = 0$$  \hspace{1cm} (23)

where on the right-hand side I of course used again the conservation law itself.

Similarly for the 1 component of $p \oplus \ell k, k = 0$ I find

$$[N_{|p|} + N_{|k|} + \frac{\ell}{2} p_0 N_{|k|}, p_1 + k_1 + \frac{\ell}{2} p_0 k_1] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + k_0 + \frac{\ell}{2} p_1 k_1 + \ell p_0 k_0 =$$

$$= p_0 + k_0 + \ell p_0 (p_0 + k_0) + \frac{\ell}{2} p_1 (p_1 + k_1) = 0$$  \hspace{1cm} (24)

where again on the right-hand side I used the conservation law $p \oplus \ell k = 0$ and took again into account that I am working at leading order in $\ell$. [Working at leading order in $\ell$ one finds, e.g., $\ell p_1 [(p \oplus \ell k)_1] \simeq \ell p_1 [p_1 + k_1].$]

So I did find that my description of boosts and of the composition laws $\oplus_{\ell}, \oplus_{\ell}, \oplus_{\ast}$ ensures the covariance of conservation laws for all “bi-valent processes”.

I can now move on to the case of tri-valent processes. The covariance of the kinematics of processes with $k \oplus_{\ell} k' \oplus_{\ell} k'' = 0$ and $p \oplus \ell p' \oplus \ell p'' = 0$ is assured respectively by Special Relativity and by the results reviewed in the previous section. Of course the only potentially troublesome tri-valent processes are the ones whose conservation laws involve “mixing”, which leads me to focus on the cases $p \oplus \ell p' \oplus \ell k = 0$ and $p \oplus \ell k, k \oplus_{\ast} k' = 0$.

In order to show that these “tri-valent mixing conservation laws” are covariant under the action of the boosts

$$N_{|p|} + N_{|p'|} + N_{|k|} + \ell p_0 N_{|p'|} + \frac{\ell}{2} (p_0 + p_0'), N_{|k|}$$

let me start from the 0 component of $p \oplus \ell p' \oplus_{\ell k}, k = 0$, for which I easily find the expected result

$$[N_{|p|} + N_{|p'|} + N_{|k|} + \ell p_0 N_{|p'|} + \frac{\ell}{2} (p_0 + p_0'), N_{|k|}, p_0 + p_0' + k_0] = p_1 + p_1' + k_1 + \ell p_0 p_1' + \frac{\ell}{2} (p_0 + p_0'), k_1$$

$$= (p \oplus \ell p' \oplus_{\ell k}, k)_1 = 0$$  \hspace{1cm} (25)

Also successful is the verification for the 1 component of $p \oplus \ell p' \oplus_{\ell k}, k = 0$, which progresses as follows:

$$[N_{|p|} + N_{|p'|} + N_{|k|} + \ell p_0 N_{|p'|} + \frac{\ell}{2} (p_0 + p_0'), N_{|k|}, p_1 + p_1' + k_1 + \ell p_0 p_1' + \frac{\ell}{2} (p_0 + p_0'), k_1] =$$

$$= p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + p_0' + \ell p_0^2 + \frac{\ell}{2} p_1^2 + k_0 +$$

$$+ \ell p_1 p_1' + 2 \ell p_0 p_0' + \frac{\ell}{2} (p_1 + p_1') k_1 + \ell (p_0 + p_0') k_0 =$$

$$= p_0 + p_0' + k_0 + \ell (p_0 + p_0') (p_0 + p_0' + k_0) + \frac{\ell}{2} (p_1 + p_1') (p_1 + p_1' + k_1) = 0$$  \hspace{1cm} (26)
Having done this it is not hard to adapt the results I obtained for \( p \oplus_{\ell} p' \oplus_{k} k = 0 \) to the only slightly different case of \( p \oplus_{\ell} k \oplus_{\lambda} k' = 0 \). In order to establish also the covariance of \( p \oplus_{\ell} k \oplus_{\lambda} k' = 0 \) let me start again with its 0 component, for which I easily find a satisfactory result:

\[
[N_{[p]} + N_{[k]} + N_{[k']}, \frac{\ell}{2} p_0 (N_{[k]} + N_{[k']}), p_0 + k_0 + k_0'] = p_1 + k_1 + k_1' + \frac{\ell}{2} p_0 (k_1 + k_1') = (p \oplus_{\ell} k \oplus_{\lambda} k')_1 = 0
\]

(27)

And equally satisfactory is the corresponding verification for the 1-component of \( p \oplus_{\ell} k \oplus_{\lambda} k' = 0 \) which progresses as follows:

\[
[N_{[p]} + N_{[k]} + N_{[k']}, \frac{\ell}{2} p_0 (N_{[k]} + N_{[k']}), p_1 + k_1 + k_1' + \frac{\ell}{2} p_0 (k_1 + k_1')] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2 + k_0 + k_0' + \ell p_1 (k_1 + k_1') + \ell p_0 (k_0 + k_0') = p_0 + k_0 + k_0' + \ell p_1 (p_1 + k_1 + k_1') + \ell p_0 (p_0 + k_0 + k_0') = 0
\]

(28)

So I did show that there is at least one (and surely many more) example of “mixing interaction” which satisfies the demands of the relativity of inertial frames, producing rules of relativistic kinematics with a consistent description of interactions among particles with different relativistic properties: the results (25), (26), (27), (28) confirm that the boosts I introduced in (15), (16), (20), (21), (22), besides admitting the on-shell relations (13), (14) as invariants, also admit the conservation laws \( p \oplus_{\ell} p' \oplus_{\ell} p'' = 0 \) and \( p \oplus_{\ell} k \oplus_{\lambda} k' = 0 \) as covariant laws.

This establishes that it is possible to have meaningful (interacting) theories that are fully relativistic in spite of allowing for the coexistence of a type of particle whose relativistic properties are governed by ordinary Special Relativity and of a type of particle whose relativistic properties are governed by the type of DSR \( \ell \)-deformation of Lorentz symmetry reviewed in the previous section.

V. FROM HOPF-LIE TO HOPF-HOPF

The new results reported in the previous section establish a possibility for coexistence of DSR-relativistic and ordinarily special-relativistic particles. As I shall stress in later parts of this manuscript, this should also invite (in addition to possibly other schemes of mathematical implementation) the development of mathematical structures suitable for “mixing” the structure of a non-trivial Hopf algebra and the structure of a (Hopf algebra with primitive coproducts, i.e. a) Lie algebra. I am now going to report results that further generalize the realm of possibilities: the case of two types of particles, with both types governed by the specific DSR deformation of Lorentz symmetry reviewed in Sec. III but one type of particle has DSR deformation scale \( \ell \) while the other type of particle has DSR deformation scale \( \lambda \) (so that one could describe the framework as mixing the “Hopf-algebra properties” of one type of particles with the somewhat different “Hopf-algebra properties” of another type of particles). This will allow me, in later parts of this manuscript, to speculate about the possibility that \( \lambda \) for a “composite particle” might be obtained from \( \ell \) via some rescaling law possibly based on the number and nature of the constituents (an illustrative example of which would be \( \lambda = \ell/N \), with \( N \) the number of constituents).

I shall consistently denote with \( p \) (or \( p' \) or \( p'' \) ...) the momenta of the type of particles affected by the DSR \( \ell \)-deformation of Lorentz symmetry discussed Sec. III so that in particular

\[
m^2 = p_0^2 - p_j^2 + \ell p_0 p_j^2
\]

(29)

And I shall consistently denote with \( k \) (or \( k' \) or \( k'' \) ...) the momenta of particles of the second type, the type with DSR \( \lambda \)-deformation of Lorentz symmetry, so that in particular

\[
\mu^2 = k_0^2 - k_j^2 + \lambda k_0 k_j^2
\]

(30)
Of course, for boosts acting on a single-particle momentum I will rely again on

\[ [N[p], p_0] = p_1, \quad [N[p], p_1] = p_0 + \ell p_0^2 + \frac{\ell}{2} p_1^2, \]

and accordingly

\[ [N[q], k_0] = k_1, \quad [N[q], k_1] = k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2. \]

And also for the “laws of composition without mixing”, \( p \oplus \ell p' \) and \( k \oplus_\lambda k' \), everything is clear from the onset of the analysis:

\[ (p \oplus \ell p')_j = p_j + p'_j + \ell p_0 p'_j, \quad (p \oplus \ell p')_0 = p_0 + p'_0. \]

\[ (k \oplus_\lambda k')_j = k_j + k'_j + \lambda k_0 k'_j, \quad (k \oplus_\lambda k')_0 = k_0 + k'_0. \]

In particular, this ensures again that processes involving only “laws of composition without mixing” will be described in consistent relativistic manner by enforcing the “laws of action of boosts without mixing” which I already used above:

\[ N[p \oplus \ell p'] = N[p] + N[p'] + \ell p_0 N[p'] \]

\[ N[k \oplus_\lambda k'] = N[k] + N[k'] + \lambda k_0 N[k'] \]

So once again the main challenge resides in finding a consistent way to compose momenta of different types of particles, i.e. finding some “mixing composition law” \( p \oplus_\ell \lambda k \), while insisting that conservation laws written in terms of such a composition law would be covariant under a consistent prescription for the action of boosts. Relying on the findings of a lengthy “trial and error exercise” I can simply exhibit an example of such a “mixing composition law” which does work, and it is once again remarkably simple:

\[ (p \oplus_\ell \lambda k)_j = p_j + k_j + \frac{\ell + \lambda}{2} p_0 k_j, \quad (p \oplus_\ell \lambda k)_0 = p_0 + k_0. \]

And I shall also show that the needed counterpart taking the shape of a “mixing composition of boosts” that leads to a consistently relativistic description is given by

\[ N[p \oplus_\ell \lambda k] = N[p] + N[k] + \frac{\ell + \lambda}{2} p_0 N[k]. \]

With these characterizations I have completed the specifications of kinematics needed for the results being reported in this section. I shall now proceed to deriving these results. Readers are invited to notice that all specifications and results given in this section evidently reduce to the ones of the previous section in the limit \( \lambda \to 0 \).

Let us verify that indeed these boost actions ensure compatibility with the conservation laws obtained from the composition laws \( \oplus_\ell, \oplus_\lambda, \) and \( \oplus_\lambda \). Actually for processes involving either exclusively \( \oplus_\ell \), or exclusively \( \oplus_\lambda \) the compatibility with my deformed boosts was already verified in Sec. \textbf{III}. So I can focus on cases which involve at least one \( \oplus_\lambda \). Looking first at “bi-valent processes” the only case of interest then evidently is \( p \oplus_\ell \lambda k = 0 \). For the 0-th component of \( p \oplus_\ell \lambda k = 0 \) one finds

\[ [N[p] + N[k] + \frac{\ell + \lambda}{2} p_0 N[k], k_0 + p_0] = p_1 + k_1 + \frac{\ell + \lambda}{2} p_0 k_1 = (p \oplus_\ell \lambda k)_1 = 0 \]
where on the right-hand side I of course used again the conservation law itself.

Similarly for the 1-component of $p \oplus_{\ell, \lambda} k = 0$ one finds

$$
[N_p] + [N_k] + \frac{\ell + \lambda}{2} p_0 N_{[p]}; p_1 + k_1 + \frac{\ell + \lambda}{2} p_0 k_1 = p_0 + \ell \hat{p}_0 + \frac{\ell}{2} p_1^2 + k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2 + \frac{\ell + \lambda}{2} p_0 k_1 + (\ell + \lambda) p_0 k_0 =
$$

$$
p_0 + k_0 + \frac{\ell - \lambda}{2} (p_0^2 - k_0^2) + \frac{\ell - \lambda}{2} (p_1^2 - k_1^2) + \frac{\ell + \lambda}{2} (p_0^2 + k_0^2 + 2k_0 p_0) + \frac{\ell + \lambda}{4} (p_1^2 + k_1^2 + 2k_1 p_1) =
$$

$$
p_0 + k_0 + \frac{\ell - \lambda}{2} (p_0 - k_0)(p_0 + k_0) + \frac{\ell - \lambda}{4} (p_1 - k_1)(p_1 + k_1) + \frac{\ell + \lambda}{2} (p_0 + k_0)^2 + \frac{\ell + \lambda}{4} (p_1 + k_1)^2 = 0
$$

where again on the right-hand side I used the conservation law $p \oplus_{\ell, \lambda} k = 0$ and took again into account that I am working at leading order in $\ell, \lambda$. [Working at leading order in $\ell, \lambda$ one finds $(\ell + \lambda)(p_0 + k_1)^2 \approx (\ell + \lambda)(p_1 + k_1)^2$ and $(\ell - \lambda)(p_1 - k_1)(p_0 + k_1) \approx (\ell - \lambda)(p_1 - k_1)(p_0 + k_1).]

So I did find that my description of boosts (and of the composition laws $\oplus_{\ell, \lambda}, \oplus_{\ell, \ell}, \oplus_{\lambda, \lambda}$) ensures the covariance of conservation laws for “bi-valent processes”.

I can now move on to the case of tri-valent processes. The covariance of the kinematics of processes with $p \oplus_{\ell} p' \oplus_{\ell} p'' = 0$ and $k \oplus_{\lambda} k' \oplus_{\lambda} k'' = 0$ is already assured by the results here reviewed in Sec. III. I shall now verify that the formulation of boost transformations here introduced affords me the covariance also of the cases with $p \oplus_{\ell} p' \oplus_{\ell} k = 0$ and $p \oplus_{\ell} k \oplus_{\lambda} k' = 0$.

Let me start from the 0-th component of $p \oplus_{\ell} p' \oplus_{\ell} k = 0$ for which I easily find the expected result

$$
[N_p] + [N_p'] + [N_k] + \ell p_0 N_{[p']} + \frac{\ell + \lambda}{2} (p_0 + p_0') N_{[p], p_0 + p_0' + k_0} = p_1 + p_1' + k_1 + \ell p_0 p_1' + \frac{\ell + \lambda}{2} (p_0 + p_0') k_1 =
$$

$$
(p \oplus_{\ell} p' \oplus_{\ell} k)_1 = 0
$$

(38)

Also successful, but slightly more tedious, is the verification of the covariance for the 1-component of $p \oplus_{\ell} p' \oplus_{\ell} k = 0$, which progresses as follows:

$$
[N_p] + [N_p'] + [N_k] + \ell p_0 N_{[p']} + \frac{\ell + \lambda}{2} (p_0 + p_0') N_{[p], p_1 + p_1' + k_1 + \ell p_0 p_1' + \frac{\ell + \lambda}{2} (p_0 + p_0') k_1] =
$$

$$
p_0 + \ell \hat{p}_0 + \frac{\ell}{2} p_1^2 + p_0' + \ell \hat{p}_0' + \frac{\ell}{2} p_1'^2 + k_0 + \lambda k_0^2 + \frac{\lambda}{2} k_1^2 +
$$

$$+ \ell p_1 p_1' + 2 \ell p_0 p_1' + \frac{\ell + \lambda}{2} (p_1 + p_1') k_1 + (\ell + \lambda)(p_0 + p_0') k_0 =
$$

$$
p_0 + p_0' + k_0 + \frac{\ell - \lambda}{2} (p_0^2 + p_0'^2 - k_0^2) + \frac{\ell - \lambda}{2} (p_1^2 + p_1'^2 - k_1^2) + (\ell - \lambda) p_0 p_0' + \frac{\ell - \lambda}{2} p_1 p_1' +
$$

$$+ \frac{\ell + \lambda}{2} (p_0^2 + p_0'^2 + k_0^2) + \frac{\ell + \lambda}{4} (p_2^2 + p_2'^2 + k_2^2) + (\ell + \lambda)(p_0 p_0' + p_0 k_0 + p_0' k_0) + \frac{\ell + \lambda}{2} (p_1 p_1' + p_1 k_1 + p_1' k_1) =
$$

$$
p_0 + p_0' + k_0 + \frac{\ell - \lambda}{2} (p_0 - k_0)(p_0 + k_0) + \frac{\ell - \lambda}{4} (p_1 - k_1)(p_1 + k_1) + \frac{\ell - \lambda}{2} (p_1 - k_1)(p_1 + k_1) +
$$

$$+ \frac{\ell + \lambda}{2} (p_0 + p_0' + k_0)^2 + \frac{\ell + \lambda}{4} (p_1 + p_1' + k_1)^2 = 0
$$

(39)

Having done this it is not hard to adapt the results I obtained for $p \oplus_{\ell} p' \oplus_{\ell} k = 0$ to the only slightly different case of $p \oplus_{\ell, \lambda} k \oplus_{\lambda} k' = 0$. In order to establish also the covariance of $p \oplus_{\ell, \lambda} k \oplus_{\lambda} k' = 0$ let me start again with its 0-th component, for which I easily find a satisfactory result:

$$
[N_p] + [N_{[p]}] + \frac{\ell + \lambda}{2} p_0 N_{[p]} + \lambda k_0 N_{[\nu]}, p_0 + k_0 + k_0' = p_1 + k_1' + \ell + \lambda p_0 (k_1 + k_1') + \lambda k_0 k_1' =
$$

$$
(p \oplus_{\ell, \lambda} k \oplus_{\lambda} k')_1 = 0
$$

(40)
And equally satisfactory (but again slightly more tedious) is the corresponding verification for the 1-component of \( p \oplus_{\ell, \lambda} k \oplus_{\lambda} k' = 0 \) which progresses as follows:

\[
\begin{align*}
[N_p] + N_{[k]} + N_{[k']} + & \frac{\ell + \lambda}{2} p_0 (N_{[k]} + N_{[k']}) + \lambda k_0 N_{[k']} + p_1 + k_1 + k'_1 + & \frac{\ell + \lambda}{2} p_0 (k_1 + k'_1) + \lambda k_0 k'_1 = \\
= p_0 + \ell p_0^2 + & \frac{\ell}{2} p_1^2 + k_0 + \lambda k_0^2 + & \frac{\lambda}{2} k_1^2 + k'_0 + & \lambda k_0^2 + & \frac{\lambda}{2} k'_1^2 + \\
+ & \frac{\ell + \lambda}{2} p_1 (k_1 + k'_1) + (\ell + \lambda) p_0 (k_0 + k'_0) + & \lambda k_1 k'_1 + & 2 \lambda k_0 k'_0 = \\
= p_0 + k_0 + p'_0 + & \frac{\lambda - \ell}{2} (k_0^2 + k'_0^2 - p_0^2) + & \lambda - \ell}{4} (k_1^2 + k'_1^2 - p_1^2) + (\lambda - \ell) k_0 k'_0 + & \frac{\lambda - \ell}{2} k_1 k'_1 + \\
+ & \frac{\ell + \lambda}{2} (p_0^2 + k_0^2 + k'_0^2) + & \frac{\ell + \lambda}{4} (p_1^2 + k_1^2 + k'_1^2) + & (\ell + \lambda) (p_0 k_0 + p_0 k'_0 + k_0 k_0) + & \frac{\ell + \lambda}{2} (p_1 k_1 + p_1 k'_1 + k'_1 k_1) = \\
= p_0 + k_0 + p'_0 + & \frac{\lambda - \ell}{2} (k_0 + k'_0 - p_0) (k_0 + k'_0 + p_0) + & \lambda - \ell}{4} (k_1 + k'_1 - p_1) (k_1 + k'_1 + p_1) + \\
+ & \frac{\ell + \lambda}{2} (p_0 + k_0 + k'_0)^2 + & \frac{\ell + \lambda}{4} (p_1 + k_1 + k'_1)^2 = 0 \quad (41)
\end{align*}
\]

So I did manage to establish that it is possible to have meaningful (interacting) theories that are fully relativistic in spite of allowing for the coexistence of a type of particle whose relativistic properties are governed by a DSR \( \ell \)-deformation of Lorentz symmetry and of a type of particle whose relativistic properties are governed by a DSR \( \lambda \)-deformation of Lorentz symmetry.

VI. COMPOSITE PARTICLES AND POTENTIAL IMPLICATIONS FOR MACROSCOPIC BODIES

The main result I am here announcing is contained in the previous two sections. In the remainder of this manuscript my only objective is to show that the new class of DSR-relativistic theories introduced in the previous two sections may have application in quite a few different physical pictures:

(i) It could evidently be used to describe pictures in which different “elementary/fundamental” particles have different relativistic properties.

(ii) It could also be used to describe pictures in which all “elementary/fundamental” particles have the same DSR-deformed relativistic properties, but “composite microscopic particles”, such as atoms (because of the known mechanisms mentioned in Section I) have different relativistic properties, with weaker deformation of special-relativistic properties than the fundamental particles that compose them.

(iii) And perhaps it could also be used to describe pictures in which microscopic particles have DSR-deformed relativistic properties, but macroscopic bodies (again because of the known mechanisms mentioned in Section I) have ordinary special-relativistic properties.

For cases (i) and (ii), assuming indeed \( |\ell|^{-1} \) is of the order of the Planck scale the new class of DSR-relativistic theories introduced in the previous two sections certainly provides plausible physical pictures, since for microscopic particles (even for atoms) the Planck scale is a gigantic scale and all effects of DSR deformation amount to small corrections.

The physical picture of case (iii) instead does not look too promising: even for this case (iii) the fact that I have shown here how different relativistic properties can coexist is a significant step forward, but for macroscopic bodies the Planck scale is actually a small energy scale and there is therefore the risk of predicting hugely unrealistic effects. It is also for this reason that so far the most popular way to handle macroscopic bodies in DSR research has been (see, e.g., Refs. [3, 9]) the one of renouncing to the introduction of direct interactions between macroscopic and microscopic particles: the interaction between a macroscopic particle could be of course also described in terms of the microscopic interactions involving the constituents of the macroscopic body.
It may well be the case that in spite of the option I managed to produce in the previous two sections one should still proceed in this way. But it is no longer so obvious that this should be the case: in this section I am going to report a simple derivation which provides encouragement for the possibility of allowing direct interactions between microscopic particles and macroscopic bodies (without needing to decompose such interactions in terms of the microscopic interactions between the microscopic particle and the constituents of the macroscopic body).

The simple derivation I am reporting does not establish anything of general validity, but it does show that at least some of the derivations which could turn pathological with a DSR description of macroscopic bodies actually do not.

This simple derivation concerns an elastic collision between an elementary particle and a macroscopic body

\[ e^- + X \rightarrow e^- + X \]

where \( e^- \) stands for any elementary particle (e.g. an electron) governed by a DSR-relativistic description of the type in Sec. [III]

\[ p_0 = p + \frac{m^2}{2p} - \frac{\ell}{2}p^2 \]  

(42)

while \( X \) is a macroscopic body of large mass \( M (M > |\ell|^{-1}) \) in a nonrelativistic regime but possibly with large spatial momentum \( k (|\ell|^{-1} < k \ll M) \)

\[ k_0 = M + \frac{k^2}{2M} \]  

(43)

Since I am considering \( M > |\ell|^{-1} \) (and \( k > |\ell|^{-1} \)) one might fear that this elastic scattering might give pathological results, such as a huge transfer of momentum (of order, e.g. \( \ell k \)) from the macroscopic body to the micro particle. But this is not what the formalization I developed in the previous two sections predicts.

In seeing this the nontrivial point of the derivation is of course the “mixing composition law” which I introduced in Sec. [IV]. The process I am here considering has two incoming and two outgoing particles, so it must be written in terms of two antipodes (see Ref. [11] and references therein)

\[ (\ominus \ell p^i) \ominus \ell p \ominus \ell \phi \ominus \ell k^i \ominus \ell k = 0 . \]  

(44)

Since \( \ominus \ell \phi \) is the undeformed composition law one has that \( (\ominus \ell \phi)_{ij} = -k_{ij} \), whereas for the \( \ominus \ell \) composition law the antipode is such that still \( (\ominus \ell p)_0 = -p_0 \) but \( (\ominus \ell p)_j = -p_j + \ell p_0 p_j \) (in fact \( [(\ominus \ell p) \ominus \ell p]_j = -p_j + \ell p_0 p_j + p_j + \ell (-p_0) p_j = 0 \).

So for the process I am considering one has

\[ -p_1' + \ell p_0 p_1' + p_1 - \ell p_0 p_1 - k_1' + k_1 + \frac{\ell}{2} (p_0 - p_0') (k_1 - k_1') = 0 . \]  

(45)

and

\[ -p_0' + p_0 - k_0' + k_0 = 0 . \]  

(46)

From (45) one finds that

\[ p_1' - p_1 \simeq k_1 - k_1' + \frac{\ell}{2} (p_0 + p_0') (p_1' - p_1) \simeq k_1 - k_1' + \ell p_1 (p_1' - p_1) . \]  

(47)

where on the right-hand side, consistently with the fact that I am working throughout at leading order, I used zero-th order properties in a reexpression of the first-order term.

Then from (46) one has that

\[ p_1' + \frac{m^2}{2p_1} - \frac{\ell}{2} p_1'^2 + M + \frac{k_1'^2}{2M} = p_1 + \frac{m^2}{2p_1} - \frac{\ell}{2} p_1^2 + M + \frac{k_1^2}{2M} , \]  

(48)
which gives

\[ p'_1 - p_1 = \frac{1}{2M} (k^2_1 - k'^2_1) + \frac{m^2}{2p_1} - \frac{m^2}{2p'_1} - \frac{\ell}{2} (p^2_1 - p'^2_1) \simeq \frac{1}{2M} (k^2_1 - k'^2_1) + \frac{m^2}{2p_1} - \frac{m^2}{2p'_1}, \quad (49) \]

where on the right-hand side I used again zero-th order properties in a reexpression of the first-order term.

Combining (47) and (49) one sees that, at least in the specific context of an elastic collision between a DSR-deformed micro particle and special-relativistic macroscopic body no pathology arises: neither in (47) nor in (49) one finds pathological correction terms of the type \( \ell k \) or \( \ell M \).

If it turned out to be possible to extend this observation to a wider class of phenomena we might also have an exciting opportunity for the description of “observers” in DSR-relativistic theories of this sort. In any relativistic theory an observer is to be identified with a macroscopic device (or, idealizing, a network of such devices), and so far studies of this type of on-shell-relation-centered DSR scenarios have kept such observers in a sort of “limbo”, protected artificially (by hand) from the implications of the deformation of relativistic symmetries. If it turned out to be possible to generalize the observation I reported in this section those artificial abstractions could be eliminated in favor of a more satisfactory picture of DSR observers.

VII. POTENTIAL IMPLICATIONS FOR OPERA-ANOMALY-TYPE PHENOMENOLOGY

Having discussed a possible implication of the results I reported in Secs. IV and V for the inclusion of special-relativistic macroscopic bodies in an otherwise DSR-relativistic framework, in this section I go to the opposite extreme of the range of possible applications of the results I reported in Secs. IV and V, by considering the possibility that different microscopic (possibly “fundamental”) particles be governed by different DSR-relativistic properties.

As I already stressed above (this was characterized as case (i) in the previous section) this possible application is clearly viable if one assumes indeed that \( |\ell|^{-1} \) is of the order of the Planck scale (or some other ultralarge momentum scale), since for microscopic particles (even for atoms) the Planck scale is gigantic and all effects of DSR deformation amount to small corrections.

While this is evident, it is nonetheless valuable for me to stress that, in spite of the apparently “invasive” prescription of different DSR-relativistic properties for different types of particles, the deformation is still a very smooth deformation of special-relativistic symmetries. I shall do this by showing that some pathologies that are expected in cases where different particles have different relativistic properties (if this is the result of a full breakdown of relativistic symmetries, with emergence of a preferred frame), are not present when this feature is introduced in the DSR-compatible manner I proposed in this manuscript.

In order to give some tangibility to my discussion (well, some tentative tangibility) I shall take as illustrative example the type of phenomenology of departures from Lorentz symmetry that was recently motivated by the fact that the OPERA collaboration reported [10] tentative evidence of superluminality for neutrinos.

The possibility that superluminal particles might well be describable in DSR-compatible fashion (without the introduction of a preferred frame) has already been raised in a few OPERA-motivated papers (see, e.g., Refs. [18–23]).

The fact that particles with the DSR-relativistic properties here described in Sec. III and IV are “superluminal” for negative \( \ell \) (i.e. \( \ell/|\ell| = -1 \)) has been established in several previous studies [13, 24–27]: when these particles have spatial momentum \( p \) such that \( \ell p > m^2 / p^2 \) their speed is higher than the speed of low-energy photons, and in that sense they indeed are “superluminal”.

But, while previous quantum-gravity-inspired studies of DSR-relativistic theories usually assumed “universality”, it appears that any attempt to do OPERA-inspired phenomenology should require a non-universal picture: for photons of, say, about 20 GeV the agreement of the speed law with the special-relativistic prescription is confirmed at the level of 1 part in 10^{18} (see, e.g., Refs. [28–30]), whereas taking the OPERA result at face value one should assume that \( \sim 20 \text{-GeV} \) neutrinos experience departures from the special-relativistic speed law at the level of a few parts in 10^5.
Ref. [19] observed that one might well have a universal DSR deformation and yet have “effectively particle-dependent effects”: the illustrative example adopted in Ref. [19] is centered on an on-shell relation of the type

\[ p_0^2 = p^2 + m^2 + 2\ell \frac{p^2}{m}, \]  

(50)

universally for all particles\(^4\), but yet effectively attributing stronger departures from special relativity to lighter particles, because of the explicit dependence on the mass (a relativistic invariant) in the correction term.

This idea of a universal DSR deformation with “effectively particle-dependent effects” may well be valuable in OPERA-inspired phenomenology. However, from this perspective, I have provided in this manuscript an interesting alternative by showing that it is possible to have a consistent DSR-relativistic framework even endowing different types of particles with genuinely different DSR-relativistic properties. It is now possible to contemplate genuinely “non-universal” DSR-relativistic pictures.

In light of this it is an amusing exercise, to which I devote the remainder of this section, to take indeed negative \(\ell\) and assume for the purposes of the exercise that \(|\ell|^{-1}\) might be much smaller than the Planck scale, but still much higher than 20 GeV. This will allow me to contemplate some of the issues that have taken center stage in the OPERA-related literature, as possible pathologies of superluminal neutrinos, and show that because of the smoothness of the DSR deformations (even the ones I here introduced, with a dependence on the type of particle) these concerns do not apply to the DSR formulation I here introduced as first illustrative example of “non-universal” DSR-relativistic framework.

My OPERA-inspired exercise will be conducted (for the sake of the argument) assuming that only neutrinos are governed by the DSR picture of Sec. III with

\[ p_0 = p + \frac{\nu^2}{2p} - \frac{\ell}{2} p^2, \]  

(51)

while all other microscopic particles are also governed by the DSR picture of Sec. III but with a weaker deformation

\[ k_0 = k + \frac{m^2}{2k} - \frac{\lambda}{2} k^2 \]  

(52)

with \(\lambda\) also negative and \(0 \leq |\lambda| \leq |\ell|\) (importantly including the case \(\lambda = 0, \ell \neq 0\) and the case \(\lambda = \ell\) as limiting cases).

A. Absence of Cherenkov-like \(\nu \to \nu + X\) processes

Among the possible “pathologies" for superluminal neutrinos which have been of interest in OPERA-inspired phenomenology much attention has been devoted [31] (also see, e.g., Refs. [32, 33]) to in-vacuo Cherenkov-like processes \(\nu \to \nu + X\) (where \(X\) may also be, e.g. an electron-positron pair). In-vacuo Cherenkov-like processes are forbidden in special relativity, but in theories with superluminal neutrinos, if they violate/break relativistic invariance (with associated emergence of a preferred “ether” frame), they can be allowed if the energy of the incoming neutrino is above a certain threshold value.

Ref. [19] (on the basis of results already discussed, for other purposes, in Refs. [34, 35]) already observed that in standard “universal” DSR-relativistic descriptions of neutrino superluminality, which are fully relativistic and do not have a preferred frame, such in-vacuo Cherenkov-like processes remain forbidden, in spite of the superluminality of the neutrinos.

---

\(^4\) Massless particles should anyway be excluded in Eq. (50). A more sophisticated way to achieve similar goals would be to adopt a dispersion relation of the type

\[ p_0^2 = p^2 + m^2 + \ell \frac{p^2}{p_0^2 - p^2}. \]
The notion of “non-universal” DSR-relativistic framework, which I here introduced, also fully preserves the relativity of inertial frames, so in-vacuo Cherenkov-like processes must also be forbidden \cite{11, 34} in this new class of relativistic theories. But it is still valuable to see this in an explicit calculation. So let me analyze the relativistic kinematics of

\[ \mathbf{v} \rightarrow \mathbf{v} + X \]

in the novel setup of Sec.\[V\] assuming indeed, as I announced for this section, that neutrinos are affected by deformation with parameter \( \ell \) while all other particles are affected by deformation with parameter \( \lambda \), with \( 0 \leq |\lambda| \leq |\ell| \).

Within the framework here introduced in Sec.\[V\] such a process should be described in terms of the “mixing conservation law”

\[ (\oplus_{\ell} p') \oplus_{\ell} p \oplus_{\lambda} k = 0 . \] (53)

where \( p \) is the (four-)momentum of the outgoing neutrino, \( k \) is the momentum of the outgoing \( X \), and \( \oplus_{\ell} p' \) is the \( \oplus_{\ell} \)-antipode of the momentum of the incoming neutrino.

In light of the properties specified for \( \oplus_{\ell} \) in Secs.\[III, IV, V\] one finds that the 0-component of this conservation law is

\[ 0 = -p'_0 + p_0 + k_0 = -p'_0 - \frac{m^2}{2p'_1} + \frac{\ell}{2} p'_1^2 + p_L + \frac{m^2 + p_L^2}{2p_L} - \frac{\ell}{2} p_L^2 + k_L + \frac{m^2 + p_L^2}{2k_L} - \frac{\ell}{2} k_L^2 . \] (54)

where I also used \((\ref{eq:51})\) and \((\ref{eq:52})\) with \( m \) the mass of the incoming and outgoing neutrino, and \( m_L \) the rest energy of \( X \). Also notice that I am assuming that the incoming neutrino is ultrarelativistic (in particular \( p \gg m_L \)) and the momenta transverse to the direction of the incoming neutrino are small: along that transverse direction both the outgoing neutrino and the outgoing \( X \) have transverse momenta of roughly the same magnitude, here denoted with \( p_T \), which is very small compared to the momenta these outgoing particles have along the direction of the incoming neutrino.

For the momenta (with index \( L \)) along that direction of the incoming neutrino one obtains from \((\ref{eq:55})\)

\[ -p' + \ell p'_0 p' + p_L - \ell p'_0 p_L + k_L + \frac{\ell + \lambda}{2} (p_0 - p'_0) k_L = 0 . \] (55)

i.e.

\[ p' = p_L + k_L + \ell p'_0 k_L - \frac{\ell + \lambda}{2} k_0 k_L \] (56)

where I used again in the leading-order correction properties established at 0-th order.

Combining \((\ref{eq:54})\) with \((\ref{eq:56})\) one obtains

\[ \frac{p_T^2}{2p_L} + \frac{p_L^2}{2k_L} = \frac{m^2}{2p'_1} - \frac{m^2}{2p_L} + \ell p'_0 k_L - \frac{\ell + \lambda}{2} k_0 k_L - \frac{\ell}{2} p_L^2 + \frac{\ell}{2} p_L^2 + \frac{\ell}{2} k_L^2 \simeq \frac{m^2}{2p'_1} - \frac{m^2}{2p_L} - \frac{m^2}{2k_L} , \] (57)

where on the right-hand side I observed that

\[ \ell p'_0 k_L - \frac{\ell + \lambda}{2} k_0 k_L - \frac{\ell}{2} p_L^2 + \frac{\ell}{2} p_L^2 + \frac{\ell}{2} k_L^2 \simeq 0 \]

using properties valid at 0-th order in rearranging this leading-order correction.

Eq. \((\ref{eq:57})\) is the main result of this subsection. For scenarios with neutrino superluminality and breakdown of Lorentz symmetry, with emergence of a preferred frame, the key observation concerning in-vacuo Cherenkov-like processes is that the analog of Eq. \((\ref{eq:57})\) contains strong corrections \([31]\) from Lorentz-symmetry-breaking terms such that then real values of \( p_T \) are allowed. One way to see that in-vacuo Cherenkov-like processes are forbidden in special relativity is through the fact that it would require imaginary values of \( p_T \). And in my DSR-relativistic analysis I found, as codified in Eq. \((\ref{eq:57})\), that independently of the value of the momentum of the incoming neutrino the \( p_T \) is necessarily imaginary; in-vacuo Cherenkov-like processes are indeed still forbidden in my DSR-relativistic framework.
B. Absence of anomalies for $\pi \to \mu + \nu$

Another "pathology" for superluminal neutrinos which has been of interest in OPERA-inspired phenomenology concerns the pion-decay channel $\pi \to \mu + \nu$, for which, in theories with superluminal neutrinos, if they violate/break relativistic invariance (with associated emergence of a preferred "ether" frame), one finds that for high-energy pions the muon/muon-neutrino phase space available for the decay is severely reduced [36–38].

Also with respect to this other concern for OPERA-inspired phenomenology the notion of "non-universal" DSR-relativistic framework, which I here introduced, turns out to be immune, mainly as a result of the fact that it fully preserves the relativity of inertial frames. To see this I shall analyze the relativistic kinematics of the process

$$\pi \to \mu + \nu$$

in the novel setup of Sec. V assuming indeed, as I announced for this section, that neutrinos are affected by deformation with parameter $\ell$ while all other particles are affected by deformation with parameter $\lambda$, with $0 \leq |\lambda| \leq |\ell|$, so that in particular

$$p_{\nu 0} = p_{\nu L} + \frac{m_\nu^2 + p_T^2}{2p_{\nu L}} - \frac{\ell}{2} p_{\nu L}^2 \quad (58)$$

$$k_{\pi 0} = k_\pi + \frac{m_\pi^2}{2k_\pi} \frac{\lambda}{2} k_\pi^2 \quad (59)$$

$$k_{\mu 0} = k_{\mu L} + \frac{m_\mu^2 + p_T^2}{2k_{\mu L}} - \frac{\lambda}{2} k_{\mu L}^2 \quad (60)$$

I am evidently again focusing on the case that the incoming particle (this time the pion) is ultrarelativistic ($p_\pi \gg m_\pi$) and the momenta transverse to the direction of the incoming pion are small: along that transverse direction both the outgoing neutrino and the outgoing muon have transverse momenta of roughly the same magnitude, here denoted again with $p_T$, which is very small compared to the momenta these outgoing particles have along the direction of the incoming pion.

On the basis of the findings here reported in Sec. V the process $\pi \to \mu + \nu$ should be described in terms of the "mixing conservation law"

$$p_\nu \oplus_\lambda [k_\mu \oplus_\lambda (\ominus_\lambda k_\pi)] = 0 \quad (61)$$

where $\ominus_\lambda k_\pi$ is the $\oplus_\lambda$-antipode of the momentum of the incoming pion.

For the 0 component this gives

$$0 = -k_{\pi 0} + p_{\nu 0} + p_{\mu 0} = -k_\pi - \frac{m_\pi^2}{2k_\pi} + \frac{\lambda}{2} k_\pi^2 + p_{\nu L} + \frac{m_\nu^2 + p_T^2}{2p_{\nu L}} - \frac{\ell}{2} p_{\nu L}^2 + k_{\mu L} + \frac{m_\mu^2 + p_T^2}{2k_{\mu L}} - \frac{\lambda}{2} k_{\mu L}^2 \quad (62)$$

where I also used (58), (59) and (60).

For the momenta (with index $L$) along the direction of the incoming pion one obtains from (61):

$$p_{\nu L} + k_{\mu L} - k_\pi + \lambda k_{\pi 0} k_\pi - \lambda k_{\mu 0} k_\pi + \frac{\ell + \lambda}{2} p_{\nu 0} (k_{\mu L} - k_\pi) = 0 \quad (63)$$

i.e.

$$k_\pi = p_{\nu L} + k_{\mu L} + \frac{\ell + \lambda}{2} p_{\nu 0} k_\pi \quad (64)$$
where again on the right-hand side I used properties of the 0-th order result in rearranging the leading-order correction. Combining (62) with (64) one obtains

\[
\frac{p_T^2}{2p_{\nu L}} + \frac{\ell p_{\mu L}}{2k_{\mu L}} = \frac{m_\pi^2}{2k_{\pi}} - \frac{m_\mu^2}{2k_{p_L}} - \frac{\mu}{2} p_{\nu\nu p_{\nu L}} + \frac{\ell}{2} p_{\nu L} + \frac{\lambda}{2} k_{p_L}^2 - \frac{\nu}{2} k_{p_L}^2 = \frac{m_\pi^2}{2k_{\pi}} - \frac{m_\mu^2}{2p_{\nu L}} - \frac{\mu}{2} k_{p_L}^2,
\]

where on the right-hand side I observed that

\[
\ell p_{\nu\mu L} - \frac{\nu}{2} p_{\nu\nu L} + \frac{\mu}{2} k_{p_L}^2 - \frac{\nu}{2} k_{p_L}^2 \simeq 0
\]

using properties valid at 0-th order in rearranging this leading-order correction.

Eq. (65) is the main result of this subsection. For scenarios with neutrino superluminality and breakdown of Lorentz symmetry, with emergence of a preferred frame, the key observation concerning \(\pi \rightarrow \mu + \nu\) is that the analog of Eq. (65) contains strong corrections [36–38] from Lorentz-symmetry-breaking terms, such that the combinations of \(p_{\nu L}\) and \(k_{\mu L}\) which satisfy the requirement \(p_T^2 > 0\) only amount to a very small phase space. Instead in my DSR-relativistic analysis I found Eq. (65), in which all correction terms canceled each other out, so that the phase space available for the decay in the DSR case is identical to the phase space available for the decay in the standard special-relativistic case.

I should stress that this result holds in a leading-order analysis, and it is therefore reliable only for \(p_\pi \ll |\ell|^{-1}\) (which however is the case of interest for Refs. [36–38]). For \(p_\pi \sim |\ell|^{-1}\) one might have sizeable modifications of the phase space even in a DSR case.

C. Back to quantum gravity and the Planck scale

The observations reported in the previous two subsections may be used in attempts of interpreting the OPERA anomaly as an actual manifestation of physics beyond the reach of special relativity. But chances are the OPERA anomaly will be eventually found to be described by much less exotic physics. One should therefore expect that the most promising applications of the results in the previous two subsections will be in the context where DSR-deformations of Lorentz symmetry were first conceived, studies of the quantum-gravity problem, for which such deformations would not be surprising, in the case with deformation scale roughly of the order of the Planck scale.

There were some studies (see, e.g., Refs. [39–41]) proposing mechanisms by which quantum-gravity/quantum-spacetime effects could produce departures from Lorentz symmetry of different magnitude for different particles. The results I reported in this manuscript show that such scenarios do not necessarily have to "break" Lorentz invariance, producing a preferred "ether" frame. One could attempt to discuss such scenarios in terms of the particle-type-dependent DSR-deformations of Lorentz symmetry I here proposed. And then the results reported in the previous two subsections could be valuable assets for the corresponding "Planck-scale phenomenology".

From that perspective I should stress that, as the careful reader can easily verify, the point made in Subsec. [VII A] for the process \(\nu \rightarrow \nu + X\) can be generalized to all Cherenkov-like processes \(A \rightarrow A + X\). And similarly the point made in Subsec. [VII B] for the process \(\pi \rightarrow \mu + \nu\) can be generalized to all decay processes \(A \rightarrow B + C\).

VIII. ASIDE ON A "\(\kappa\)-MINKOWSKI SPACETIME" AND ALGEBRAS WITH "MIXING CO-PRODUCTS"

A. Contemplating algebras with "mixing co-products"

I have here introduced a new class of (DSR-)relativistic theories, focusing in this first study on kinematics. I expect that in order to achieve a full empowerment of this new class of theories it will be necessary to identify a symmetry-algebra counterpart to the novel type of relativistic kinematics I here proposed. This balance is rather visible in special relativity, whose full
understanding requires combining the Poincaré symmetry algebra and Einstein kinematics. And this "balance of powers" appears to be preserved also in the illustrative example of "universal" DSR-relativistic theory which I here took as starting point, in Sec. III for the specific example of DSR deformation of relativistic kinematics here reviewed in Sec. III one can find numerous points of contact with the structure of the κ-Poincaré Hopf algebra [16, 17]. In particular, the most crucial aspect of that construction, the composition law

\[
(p \otimes p')_1 = p_1 + p'_1 + \ell p_0 p'_1 , \quad (p \otimes p')_0 = p_0 + p'_0 ,
\]

(66)
can be placed in correspondence with the law of "co-product" that characterizes the κ-Poincaré Hopf algebra in the Majid-Ruegg basis [16], which in leading order reads

\[
\Delta(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \frac{1}{\kappa} \lambda P_0 \otimes P_1 , \quad \Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 .
\]

(67)

In order to establish a similar connection between relativistic kinematics and symmetry algebras for the novel class of relativistic theories I here proposed in Secs. IV and V it would seem necessary to introduce on the algebra side enough structure to accommodate at least 3 laws of coproduct: two different laws of the type (67)

\[
\Delta_\ell(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \ell P_0 \otimes P_1 , \quad \Delta_\ell(P_0) = P_0 \otimes 1 + 1 \otimes P_0 ,
\]

\[
\Delta_\lambda(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \lambda P_0 \otimes P_1 , \quad \Delta_\lambda(P_0) = P_0 \otimes 1 + 1 \otimes P_0 ,
\]

and a novel "mixing coproduct" of a type illustrated by

\[
\Delta_{\ell,\lambda}(P_1) \simeq P_1 \otimes 1 + 1 \otimes P_1 + \frac{\ell + \lambda}{2} P_0 \otimes P_1 , \quad \Delta_{\ell,\lambda}(P_0) = P_0 \otimes 1 + 1 \otimes P_0 .
\]

(68)

I am not aware of any work on symmetry algebras (some sort of "κκ-Poincaré algebra") already providing these structures, but it appears natural to expect that such a construction should be possible.

B. Toward a "κκ-Minkowski spacetime"

There is at least one more ingredient that would be desirable, in addition to looking for a "symmetry-algebra counterpart", as an enrichment of the non-universal DSR deformations I here introduced focusing temporarily on aspects pertaining kinematics from a momentum space perspective. This other desired ingredient is (one form or another of) a "spacetime picture". I shall not speculate much about this here. I expect it may require several striking steps of abstraction, which will take time to mature. Even for universal DSR deformation we are still in the "digestion process" for some of the striking new features that the associate spacetime pictures typically introduce, such as the relativity of spacetime locality [25, 26]. We must expect features possibly even more virulent (but again not necessarily in conflict with established experimental facts) to be typical for the novel non-universal DSR deformations I am here introducing.

Hoping to offer a useful contribution to the study of these issues I venture to formulate here only a preliminary speculation. This concerns the fact that for the illustrative example of "universal" DSR-relativistic theory which I here took as starting point, in Sec. III there is an established "formal link" to the so-called "κ-Minkowski spacetime" (see, e.g., Refs. [16, 17, 42]), in which the same scale κ of (67) appears in a form of spacetime noncommutativity

\[
[\hat{x}_j, \hat{\ell}] = i \frac{1}{\kappa} \hat{\ell} , \quad [\hat{x}_j, \hat{x}_k] = 0 .
\]

(69)
The core feature of this formal link is visible already in how the composition law (6)

\[
(p \otimes p')_1 = p_1 + p'_1 + \ell p_0 p'_1
\]

(70)
emerges in cases where certain ordering prescriptions are applied in analyses of $\kappa$-Minkowski noncommutativity, such as in

$$e^{ip_j \hat{\ell}} e^{ip_0 \hat{\lambda}} e^{i\ell \hat{\kappa} k_j} e^{ip_0 \hat{\lambda}} e^{i\ell \hat{\kappa} p_0} \sim e^{i[p_j + p_k + \ell p_0] \hat{\kappa}} e^{ip_0 \hat{\lambda}}$$  \hspace{1cm} (71)

where I used (69) for $\ell \equiv 1/\kappa$.

Inspired by this observation, one may consider attempting to provide a formal spacetime picture for a non-universal DSR-relativistic scenario of the type in Sec. [V] by seeking a generalization/deformation of $\kappa$-Minkowski spacetime, suitable for accommodating the “mixing composition law” introduced in Sec. [V] and particularly, from (35),

$$(p \oplus \ell \lambda k) = p_j + k_j + \frac{\ell + \lambda}{2} p_0 k_j$$  \hspace{1cm} (72)

or similarly$^5$

$$(p \oplus \ell \lambda k) = p_j + k_j + \frac{\lambda}{2} p_0 k_j - \frac{\ell}{2} k_0 p_j$$  \hspace{1cm} (73)

I feel that one tempting possibility would be the one of contemplating a notion of quantum spacetime in which the coordinates of different types of particles have different noncommutativity properties, which taking as starting point $\kappa$-Minkowski spacetime may lead one to consider a generalization suitable for being labeled as “$\kappa\kappa$-Minkowski spacetime”.

Postponing a more in depth analysis of possible alternatives to future work, I just want to notice, concerning this $\kappa\kappa$-Minkowski speculation, that structures such as the one found in (72) and (73) might be naturally encountered in cases where suitable ordering prescriptions are applied in analyses of a scenario such that my "$p$-particles" have coordinates with noncommutativity

$$[\hat{x}_j, \hat{t}] = i\ell \hat{x}_j, \quad [\hat{x}_j, \hat{x}_k] = 0$$  \hspace{1cm} (74)

while my "$k$-particles" have coordinates with noncommutativity

$$[\hat{x}_j', \hat{t}] = \hat{\alpha} \hat{x}_j', \quad [\hat{x}_j', \hat{x}_k'] = 0$$  \hspace{1cm} (75)

I conjecture that some of the findings of recent works on a possible relativity of spacetime locality [14, 25, 26] will lead to a conceptualization of spacetime for which it would not be cumbersome to introduce different types of particles “sharing the same (relative-locality) spacetime” and yet with coordinates governed by different laws of noncommutativity.

And on the basis of (74) and (75) one could produce terms of the form $\lambda p_0 k_j$ from applying suitable ordering prescriptions in such a $\kappa\kappa$-Minkowski spacetime, as shown by

$$e^{ip_j \hat{\ell}} e^{ip_0 \hat{\lambda}} e^{i\ell \hat{\kappa} k_j} e^{ip_0 \hat{\lambda}} e^{i\ell \hat{\kappa} p_0} \sim e^{i[p_j + (j + \lambda p_0)] \hat{\kappa}} e^{ip_0 \hat{\lambda}}$$  \hspace{1cm} (76)

and one could produce terms of the form $\ell k_0 p_j$ from other applications of suitable ordering prescriptions in such a $\kappa\kappa$-Minkowski spacetime, as shown by

$$e^{ik_j \hat{\ell}} e^{ik_0 \hat{\lambda}} e^{i\ell \hat{\kappa} k_j} e^{ik_0 \hat{\lambda}} e^{i\ell \hat{\kappa} p_0} \sim e^{i[p_j + (j + \lambda p_0)] \hat{\kappa}} e^{ik_0 \hat{\lambda}}$$  \hspace{1cm} (77)

$^5$ Note that a conservation law of the type $p_0 + k_0 = 0$, $p_j + k_j + (\ell + \lambda)p_0 k_j/2 = 0$ is equivalent to $p_0 + k_0 = 0$, $p_j + k_j + \lambda p_0 k_j - \ell k_0 p_j/2 = 0$ since (for $p_0 + k_0 = 0$ and working at leading order) one has that

$$p_j + k_j + \frac{\ell + \lambda}{2} p_0 k_j = 0 \quad \Rightarrow \quad (1 + \lambda k_0/2)[p_j + k_j + \frac{\ell + \lambda}{2} p_0 k_j] = 0 \quad \Rightarrow \quad p_j + k_j + \frac{\lambda}{2} p_0 k_j - \frac{\ell}{2} k_0 p_j = 0$$
IX. ASIDE BEYOND LEADING ORDER

I have worked throughout this manuscript at leading order in the deformation scale. As stressed above this is the only reasonable choice since in quantum-gravity/Planck-scale phenomenology it would be already very fortunate to ever uncover leading-order effects and at least presently going beyond leading order is unjustified. As I stressed already in Refs. [1, 3], enforcing some sort of requirement of mathematical consistency beyond leading order is not only inappropriate because of the limitations of the expected experimental sensitivities, but may also be inappropriate in light of the complexity of the quantum-gravity problem: even if DSR-deformations of Lorentz symmetry do end up being actually relevant for quantum gravity (and of course this is only a remote hypothesis) it may well be the case that only their “leading-order formulation” makes sense physically. This is because the availability of a description in terms of DSR deformations essentially still assumes a rather standard picture of spacetime, novel enough to include striking new features such as a relativity of spacetime locality[14, 25, 26], but still conventional enough to allow a description to a large extent still consistent with the standard role of spacetime in physics. However, the nature of the quantum-gravity problem is such that it would not be surprising if, as the characteristic energy scales get closer to the Planck scale (just as the leading-order analysis starts to be insufficient), at some point there would be the onset of a completely foreign regime of the laws of physics, not even affording us the luxury of the abstraction of an (however exotic) spacetime formulation.

So I do not view the development of DSR pictures beyond leading order as an important priority. It is nonetheless conceptually intriguing and provides amusing challenges. I shall not dwell much here on this issue, but let me nonetheless at least exhibit, in this section, partial results that provide some encouragement for the idea that such “all-order particle-type-dependent DSR-deformations” are indeed possible.

The first ingredient I introduce for this purpose is an “all-order generalization” of the type of particle described in Sec. III. This generalization replaces Eqs. (5) (6), (7), (9) with

\[
\cosh(\ell m) = \cosh(\ell p_0) - \frac{\ell^2}{2} e^{-\ell p_0} p_1^2
\]

\[
(p \oplus \ell p')_1 = p_1 + e^{\ell p_0} p_1', \quad (p \oplus \ell p')_0 = p_0 + p_0'.
\]

\[
[N, p_0] = p_1, \quad [N, p_1] = \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2} p_1,
\]

\[
N[p \oplus \ell p'] = N[p] + e^{\ell p_0} N[p']
\]

These formulas may be viewed as a kinematical counterpart from some of the structures, including the mentioned co-products, of one of the descriptions [16] of the the \( \kappa \)-Poincaré Hopf algebra.

In this section I refer to particles governed by Eqs. (78) (79), (80), (81) as “\( p \)-particles” and denote their momenta consistently with \( p \) (or \( p' \) or \( p'' \) ...).

It is easy to see that Eqs. (78) (79), (80), (81) ensure relativistic consistency for the description of such particles. In particular,

\[
[N, \cosh(\ell p_0) - \frac{\ell^2}{2} e^{-\ell p_0} p_1^2] = 0
\]

and interactions among “\( p \)-particles”, with conservation law \( p \oplus \ell p' \oplus \ell p'' = 0 \), evidently admit consistent relativistic description, as shown by the following two equations:

\[
[N[p], + e^{\ell p_0} N[p]] + e^{\ell(p_0 + p_0')} N[p_1, p_0 + p_0' + p_0''] = p_1 + e^{\ell p_0} p_1' + e^{\ell(p_0 + p_0')} p_1'' = (p \oplus \ell p' \oplus \ell p'')_1 = 0
\]
[\mathcal{N}_p] + e^\ell p_0 \mathcal{N}_{p'} + e^{\ell(p_0 + p_0')} \mathcal{N}_{p'} + p_1 + e^\ell p_1' + e^{\ell(p_0 + p_0')} p_1' = \\
= \left( \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2 p_1} \right) + e^{2\ell p_0} \left( \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2 p_1} \right) + e^{2\ell(p_0 + p_0')} \left( \frac{e^{2\ell p_0} - 1}{2\ell} + \frac{\ell}{2 p_1} \right) + \\
+ \ell e^{\ell p_0} p_1' + e^{\ell(p_0 + p_0')} p_1 p_1'' + \ell e^{\ell(p_0 + p_0')} p_1' p_1'' = \\
= \frac{\ell}{2} \left( p_1 + e^{\ell p_0} p_1' + e^{\ell(p_0 + p_0')} p_1' \right)^2 + \left( \frac{e^{2\ell(p_0 + p_0')} - 1}{2\ell} \right) = 0 \quad (84)

where for both these results I of course enforced on the right-hand side the conservation law \( p \oplus \ell p' \oplus \ell p'' = 0 \) itself. Note that from (83) and (84) it also follows that, for the conservation law \( p \oplus \ell p' = 0 \), one has

\[ [\mathcal{N}_p] + e^\ell p_0 \mathcal{N}_{p'} + \mathcal{N}_0 + p_0' = 0 \]

and

\[ [\mathcal{N}_p] + e^\ell p_0 \mathcal{N}_{p'} + p_1 + e^{\ell p_1'} = 0 \]

So I have a fully consistent (and consistent to all orders in \( \ell \)) DSR-relativistic description of \( p \)-particles, propagating (with \( p \oplus \ell p' = 0 \) conservation) and interacting among themselves (with \( p \oplus \ell p' \oplus \ell p'' = 0 \) conservation).

My next task is to introduce a second type of particles, with different DSR-relativistic properties. For these \( \sim k \)-particles (whose momenta I shall consistently denote with \( k \) or \( k' \)) I take the following on-shell relation

\[ \cosh(\ell \mu) = \cosh(\ell k_0) - \frac{\ell^2}{2} k_1^2 \quad (85) \]

and boost such that

\[ [\mathcal{N}_k, k_0] = k_1 \quad [\mathcal{N}_k, k_1] = \frac{\sinh(\ell k_0)}{\ell} \quad (86) \]

which indeed is compatible with the on-shell relation

\[ [\mathcal{N}_k, \cosh(\ell k_0) - \frac{\ell^2}{2} k_1^2] = 0 \quad (87) \]

Notice that I am now considering a case where also the DSR-deformation of the second type of particle is characterized by the same deformation scale \( \ell \), but the form of the laws of transformation that apply to the two types of particles are very significantly different.

For this section of “aside beyond leading order” I do not go as far as introducing a proper composition law for \( k \)-particles and/or a proper “\( p \)-particle/\( k \)-particle mixing composition law” to be used as general rules applicable to all sorts of processes. I will instead directly show that some acceptable conservation laws, relevant for certain specific processes, do admit a fully relativistic description.

And I will not elaborate on the ways to construct such conservation laws, but the careful reader will notice that their structure can be seen as inspired by either one of two ways of characterizing equivalently the covariance of my choice of conservation law for “\( k \)-particle propagation processes”, which is the undeformed one:

\[ k_0 + k_0' = 0 \quad k_1 + k_1' = 0 \quad (88) \]

This evidently is compatible with a correspondingly undeformed law of “composition of boosts” \( \mathcal{N}_k' \): 

\[ [\mathcal{N}_k + \mathcal{N}_k', k_0 + k_0'] = k_1 + k_1' = 0 \quad [\mathcal{N}_k + \mathcal{N}_k', k_1 + k_1'] = \frac{\sinh(\ell k_0)}{\ell} + \frac{\sinh(\ell k_0')}{\ell} = 0 \quad (89) \]
where I of course also used the conservation law \( k_0 + k'_0 = 0, k_1 + k'_1 = 0 \) itself.

The anticipated interesting alternative way to characterize this conservation law for “\( k \)-particle propagation processes” is centered on noticing that \((88)\) can be rewritten equivalently as

\[
    k_0 + k'_0 = 0, \quad e^{-\ell k_0/2} k_1 + e^{\ell k_0/2} k'_1 = 0, \tag{90}
\]

whose compatibility with the \( N_{\llbracket k \rrbracket} \) boost could be described as follows

\[
    \left[ e^{-\ell k_0/2} N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket k' \rrbracket}, k_0 + k'_0 \right] = e^{-\ell k_0/2} k_1 + e^{\ell k_0/2} k'_1 = 0, \tag{91}
\]

\[
    \left[ e^{-\ell k_0/2} N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket k' \rrbracket}, e^{-\ell k_0/2} k_1 + e^{\ell k_0/2} k'_1 \right] = e^{-\ell k_0/2} \frac{\sinh(\ell k_0)}{\ell} + e^{\ell k_0/2} \frac{\sinh(\ell k'_0)}{\ell} = 0. \tag{92}
\]

I do not provide any other conservation law for processes involving exclusively \( k \)-particles (as such I provide a picture for \( k \)-particles that do not self-interact). But I do provide “mixing conservation laws”, for processes involving both \( k \)-particles and \( p \)-particles, with structure which may be viewed as inspired either by the form of \((88), (89)\) or by the form of \((90), (91), (92)\).

The first such “mixing composition law” which I exhibit is for simple “oscillation processes”

\[
    k_0 + p_0 = 0, \quad k_1 + e^{\ell k_0/2} p_1 = 0
\]

and is compatible with the following law of “mixing of boosts”

\[
    N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket p \rrbracket}
\]

This is easily verified from the following two observations:

\[
    \left[ N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket p \rrbracket}, k_0 + p_0 \right] = k_1 + e^{\ell k_0/2} p_1 = 0
\]

and

\[
    \left[ N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket p \rrbracket}, k_1 + e^{\ell k_0/2} p_1 \right] = \\
    \frac{\sinh(\ell k_0)}{\ell} + e^{\ell k_0} \left( \frac{e^{2k_0} - 1}{2\ell} + \frac{\ell}{2} p_1^2 \right) + \\
    + \frac{\ell}{2} e^{\ell k_0/2} k_1 p_1 = \\
    = \frac{\ell}{2} e^{\ell k_0/2} p_1 \left( k_1 + e^{\ell k_0/2} p_1 \right) + \left( \frac{e^{k_0 + 2k_0} - e^{-k_0}}{2\ell} \right) = 0
\]

A first example of consistently relativistic conservation law for interactions among \( k \)-particles and \( p \)-particles is the following:

\[
    k_0 + p_0 + p'_0 = 0, \quad k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1 = 0,
\]

which is for one \( k \)-particle interacting with two \( p \)-particles and is compatible with the following law of “\( kpp \) mixing of boosts”

\[
    N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket p \rrbracket} + e^{\ell k_0/2} e^{\ell p_0} N_{\llbracket p' \rrbracket}.
\]

This is easily verified as follows:

\[
    \left[ N_{\llbracket k \rrbracket} + e^{\ell k_0/2} N_{\llbracket p \rrbracket} + e^{\ell k_0/2} e^{\ell p_0} N_{\llbracket p' \rrbracket}, k_0 + p_0 + p'_0 \right] = k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p'_1 = 0
\]
as one can again easily verify:

\[ \sinh(\ell k_0) + e^{\ell k_0/2} N[p] + e^{\ell k_0/2} e^{\ell p_0} N[p'] k_1 + e^{\ell k_0/2} e^{\ell p_0} p_1' = \]

\[= \sinh(\ell k_0) + e^{\ell k_0} \left( \frac{e^{2\ell p_0 - 1}}{2\ell} + \frac{\ell}{2} p_1'^2 \right) + e^{\ell k_0} e^{2\ell p_0} \left( \frac{e^{2\ell p_0 - 1}}{2\ell} + \frac{\ell}{2} p_1'^2 \right) + \]

\[+ \frac{\ell}{2} e^{\ell k_0/2} k_1 p_1 + \frac{\ell}{2} e^{\ell k_0/2} e^{\ell p_0} k_1 p_1' + \ell e^{\ell k_0} e^{\ell p_0} p_1' = \]

\[= \frac{\ell}{2} \left( e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p_1' \right) \left( k_1 + e^{\ell k_0/2} p_1 + e^{\ell k_0/2} e^{\ell p_0} p_1' + \left( \frac{e^{\ell k_0+2\ell p_0+2\ell p_0} - e^{-\ell k_0}}{2\ell} \right) \right) = 0 \]

Finally I also exhibit a consistently relativistic conservation law for interactions involving two $k$-particles and one $p$-particle:

\[ k_0 + p_0 + p_0' = 0 , \quad e^{-\ell k_0} e^{-\ell k_0/2} k_1 + e^{-\ell k_0/2} k_1 + p_1 = 0 \]

which is compatible with the following law of “kkp mixing of boosts”

\[ e^{-\ell k_0} e^{-\ell k_0/2} N[k] + e^{-\ell k_0/2} N[k] + N[p] \]

as one can again easily verify:

\[ \left[ e^{-\ell k_0} e^{-\ell k_0/2} N[k] + e^{-\ell k_0/2} N[k] + N[p], k_0 + k_0' + p_0 = e^{-\ell k_0} e^{-\ell k_0/2} k_1 + e^{-\ell k_0/2} k_1 + p_1 = 0 \right] \]

\[ e^{-\ell k_0} e^{-\ell k_0/2} N[k] + e^{-\ell k_0/2} N[k] + N[p], e^{-\ell k_0} e^{-\ell k_0/2} k_1 + e^{-\ell k_0/2} k_1 + p_1 = \]

\[= e^{-2\ell k_0} e^{-\ell k_0} \sinh(\ell k_0) + \frac{\sinh(\ell k_0)}{\ell} + \left( \frac{e^{2\ell p_0 - 1}}{2\ell} + \frac{\ell}{2} p_1'^2 \right) - \frac{\ell}{2} e^{-2\ell k_0} e^{-\ell k_0} k_1^2 - \frac{\ell}{2} e^{-2\ell k_0} k_1^2 - \ell e^{-3\ell k_0} e^{-\ell k_0} k_1 k_1' = \]

\[= \frac{\ell}{2} \left( p_1 - e^{-\ell k_0} e^{-\ell k_0/2} k_1 - e^{-\ell k_0/2} k_1' \right) \left( p_1 + e^{-\ell k_0} e^{-\ell k_0/2} k_1 + e^{-\ell k_0/2} k_1' \right) + \left( \frac{e^{2\ell k_0+2\ell p_0+2\ell p_0} - e^{-2\ell k_0}}{2\ell} \right) = 0 \]

X. CLOSING REMARKS

The fact, here established, that there are consistent scenarios of “non-universal” deformation of Lorentz symmetry, with particle-type-dependent deformations of relativistic kinematics, can be used to address some long-standing issues in DSR research, mentioned in Sec. I, and can also be the starting point for several further developments and further generalizations.

Some of these future studies could be directed toward the understanding of the associated relativity of spacetime locality. It is established that already for some “universal” DSR-deformation schemes spacetime locality becomes relative \[25\,27\]. The case of “non-universal” DSR-deformation here introduced should have subtle implications also for the characterization of the relativity of spacetime locality.

And it is established that for some schemes of “universal” DSR deformation it is possible to provide a formulation within the “relative-locality framework” \[14\,15\], centered on the geometry of momentum space. It would therefore be interesting to attempt to generalize the relative-locality-framework formulation also to the case of the “non-universal” DSR deformations here introduced, but this raises some intriguing questions: could then one describe the different types of particles on the same geometry of momentum space? or should one rather seek a formulation based on different momentum-space geometries for different particles? and in that case which sort of geometric requirement could codify the feature here formulated in terms of a “mixing composition law”?

One more intriguing question I want to mention here concerns the types of different relativistic properties which can be found to be compatible. In the construction of the cases I here analyzed as illustrative examples a consistent particle-dependent
(“non-universal”) DSR deformation was achieved also exploiting in part the fact that I confined myself to considering only rather mild differences of relativistic properties: I managed to “mix” particles governed by quantitatively very different deformations of relativistic kinematics, but all members of a class of related such deformations. Having established that this can be done consistently, one may now ask whether it is possible to “mix” in a single consistent relativistic framework particles with more profoundly different relativistic properties. Ideally one would like to establish some sort of theorem characterizing the types of different relativistic properties which can be made compatible in the sense I here introduced. Such a theorem appears to be extremely challenging, but even gaining some expertise on the basis of a few “trial-and-error exercises” might be valuable.
