SURE shrinkage of Gaussian paths and signal identification

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Abstract

Using integration by parts on Gaussian space we construct a Stein Unbiased Risk Estimator (SURE) for the drift of Gaussian processes using their local and occupation times. By almost-sure minimization of the SURE risk of shrinkage estimators we derive an estimation and de-noising procedure for an input signal perturbed by a continuous-time Gaussian noise.

Key words: Estimation, SURE shrinkage, thresholding, denoising, Gaussian processes, Malliavin calculus.

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1 Introduction

Let $X$ be a Gaussian random vector on $\mathbb{R}^d$ with unknown mean $m$ and known covariance matrix $\sigma^2 I_d$ under a probability measure $\mathbb{P}_m$.

It is well-known [13] that given $g : \mathbb{R}^d \to \mathbb{R}^d$ a sufficiently smooth function, the mean square risk $\|X + g(X) - m\|^2_{\mathbb{R}^d}$ of $X + g(X)$ to $m$ can be estimated unbiasedly by

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from the identity
\[
E_m \left[ \| X + g(X) - m \|_R^2 \right] = \sigma^2 d + E_m \left[ \sum_{i=1}^d g_i(X)^2 + 2 \sum_{i=1}^d \nabla g(X) \right]
\] (1.2)
which is obtained by Gaussian integration by parts under \( P_m \). The estimator (1.1), which is independent of \( m \), is called the Stein Unbiased Risk Estimate (SURE).

When \((g^\lambda)_{\lambda \in \Lambda}\) is a family of functions it makes sense to almost surely minimize the Stein Unbiased Risk Estimate (1.1) of \( g^\lambda \) with respect to the parameter \( \lambda \). This point of view has been developed by Donoho and Johnstone [4] for the design of spatially adaptive estimators by shrinkage of wavelet coefficients of noisy data via
\[
X + g^\lambda(X) = \lambda \eta(X/\lambda),
\]
where \( \eta(x) \) is a threshold function.

In this paper we construct a Stein type Unbiased Risk Estimator for the deterministic drift \((u_t)_{t \in \mathbb{R}^+}\) of a one dimensional Gaussian processes \((X_t)_{t \in [0,T]}\) via an extension of the identity (1.2) introduced in [10], [9] on the Wiener space. For example, given \( \alpha(t) \) and \( \lambda(t) \) two functions given in parametric form, the SURE risk of the estimator
\[
X_t + \xi^{\alpha,\lambda}(X_t) = \alpha(t) + \lambda(t) \eta_S \left( \frac{X_t - \alpha(t)}{\lambda(t)} \right), \quad t \in [0,T],
\]
where \( \eta_H \) is the hard threshold function (5.1) below, is given by
\[
\text{SURE} (X + \xi^{\alpha,\lambda}(X)) = T + \int_0^T \frac{(X_t - \alpha(t))^2}{\gamma(t,t)} dt + 2\lambda \tilde{\ell}_T - 2\hat{L}_T,
\]
where \( \gamma(s,t) = \text{Cov}(X_s, X_t), \) \( 0 \leq s, t \leq T, \) denotes the covariance of \((X_t)_{t \in [0,T]}\) and \( \tilde{\ell}_T, \hat{L}_T \) respectively denote the local and occupation time of
\[
(|X_t - \alpha(t)|/\sqrt{\gamma(t,t)}))_{t \in [0,T]},
\]
cf. Proposition 5.1. We apply this technique to de-noising and identification of the input signal in a Gaussian channel via the minimization of \( \text{SURE} (X + \xi^{\alpha,\lambda}(X)) \). This
yields in particular an estimator of the drift of $X_t$ from the estimation of $\alpha(t)$, and an optimal noise removal threshold from the estimation of $\lambda$. This approach differs from classical signal detection techniques which usually rely on likelihood ratio tests, cf e.g. [8], Chapter VI. It also requires an a priori hypothesis on the parametric form of $\alpha(t)$.

We proceed as follows. In Section 2 we recall our framework of functional estimation of drift trajectories. In Section 3 we derive Stein’s unbiased risk estimate for the estimation of the drift of Gaussian processes. In Section 4 we discuss its application to soft thresholding for Gaussian processes using the local time and obtain an upper bound for the risk of such estimators. We also show the existence of an optimal parameter and the smoothness of the risk function. In Section 5 we consider the case of hard thresholding. In Section 6 we consider several numerical examples where $\alpha(t)$ is given in parametric form. In Section 7 we recall some elements of stochastic analysis of Gaussian processes.

## 2 Functional drift estimation

In this section we recall the setting of functional drift estimation to be used in this paper. Given $T > 0$ we consider a real-valued centered Gaussian process $X = (X_t)_{t \in [0,T]}$ with non-vanishing covariance function

$$\gamma(s, t) = \mathbb{E}[X_s X_t], \quad s, t \in [0, T],$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F})_{t \in [0,T]}$ is the filtration generated by $(X_t)_{t \in [0,T]}$. Assume that under a probability measure $\mathbb{P}_u$ we observe the paths of $(X_t)_{t \in [0,T]}$ decomposed as

$$X_t = u_t + X_t^u, \quad t \in [0, T],$$

where $u = (u_t)_{t \in [0,T]}$ is a square integrable $\mathcal{F}$-adapted process and $(X_t^u)_{t \in [0,T]}$ is a centered Gaussian process with covariance

$$\gamma(s, t) = \mathbb{E}_u[X_s^u X_t^u], \quad 0 \leq s, t \leq T,$$
where $\mathbb{E}_u$ denotes the expectation under $\mathbb{P}_u$. Given a continuous time observation of the process $(X_t)_{t\in[0,T]}$ we will propose estimators of the unknown drift function $u$.

**Definition 2.1.** The risk of an estimator $\xi := (\xi_t)_{t\in[0,T]}$ to $u$ is defined as

$$R(\gamma, \mu, \xi) := \mathbb{E}_u \left[ \int_0^T |\xi_t - u_t|^2 \mu(dt) \right]$$

where $\mu$ is a positive measure on $[0,T]$.

Examples of risk measures $\mu$ include the Lebesgue measure and

$$\mu(dt) = \sum_{i=1}^n a_i \delta_{t_i}(dt), \quad a_1, \ldots, a_n > 0,$$

in which case the risk of the estimator is computed from discrete values of the sample path observed at times $t_1, \ldots, t_n, n \geq 1$.

**Definition 2.2.** A drift estimator $(\xi_t)_{t\in[0,T]}$ is called unbiased if

$$\mathbb{E}_u[\xi_t] = \mathbb{E}_u[u_t], \quad t \in [0,T],$$

for all square-integrable $\mathcal{F}_t$-adapted process $(u_t)_{t\in[0,T]}$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by $(X_t)_{t\in[0,T]}$.

In the sequel we will consider the canonical process $(X_t)_{t\in[0,T]}$ as an unbiased estimator $\hat{u} := (X_t)_{t\in[0,T]}$ of its own drift $(u_t)_{t\in[0,T]}$ under $\mathbb{P}_u$, with risk

$$R(\gamma, \mu, \hat{u}) := \mathbb{E}_u \left[ \int_0^T |X_t - u_t|^2 \mu(dt) \right] = \int_0^T \gamma(t,t) \mu(dt)$$

Recall that the estimator $\hat{u} = (X_t)_{t\in[0,T]}$ is minimax i.e.

$$R(\gamma, \mu, \hat{u}) = \inf_{\xi} \sup_{v} \mathbb{E}_v \left[ \int_0^T |\xi_t - v_t|^2 \mu(dt) \right],$$

cf. Proposition 3.2 of [10]. In addition, when $(X_t)_{t\in[0,T]}$ has independent increments and $(u_t)_{t\in[0,T]} \in L^2(\Omega \times [0,T], \mathbb{P}_u \otimes \mu)$ is square-integrable and adapted, then for any adapted and unbiased estimator $\xi$ the Cramer-Rao bound

$$\mathbb{E}_u \left[ \int_0^T |\xi_t - u_t|^2 \mu(dt) \right] \geq R(\gamma, \mu, \hat{u}),$$

holds for any unbiased and adapted estimator $(\xi_t)_{t\in[0,T]}$ of $(u(t))_{t\in[0,T]} \in L^2(\Omega \times [0,T], \mathbb{P}_u \otimes \mu)$ and is attained by $\hat{u}$, cf. Proposition 4.3 of [10], hence $\hat{u} = (X_t)_{t\in[0,T]}$ is an efficient estimator of its own drift $u$. 


3 Stein’s unbiased risk estimate

Instead of using the minimax estimator \( \hat{u} \) we will estimate the drift of \( (X_t)_{t \in [0,T]} \) by the almost sure minimization of a Stein Unbiased Risk Estimator for Gaussian processes, constructed in the next proposition by analogy with (1.1). In the next proposition we use the gradient operator \( D_t \) whose definition and properties are recalled in the appendix, cf. Definition 7.2 and Lemma 7.3.

**Proposition 3.1.** For any \( (\xi_t)_{t \in [0,T]} \in L^2(\Omega \times [0,T], \mathbb{P}_u \otimes \mu) \) such that \( \xi_t \in \text{Dom} (\nabla) \), \( t \in [0,T] \), and \( (D_t\xi_t)_{t \in [0,T]} \in L^1(\Omega \times [0,T], \mathbb{P}_u \otimes \mu) \), the quantity

\[
\text{SURE}_\mu(X + \xi) := R(\gamma, \mu, \hat{u}) + \|\xi\|_{L^2([0,T],d\mu)}^2 + 2 \int_0^T D_t\xi_t d\mu(dt) \tag{3.1}
\]

is an unbiased estimator of the mean square risk \( \|X + \xi - u\|_{L^2([0,T],d\mu)}^2 \).

**Proof.** From Lemma 7.3 we have

\[
\mathbb{E}_u \left[ \|X + \xi - u\|_{L^2([0,T],d\mu)}^2 \right] = \mathbb{E}_u \left[ \int_0^T \left| X^u_t + \xi_t \right|^2 \mu(dt) \right] = \mathbb{E}_u \left[ \int_0^T |X^u_t|^2 \mu(dt) \right] + \mathbb{E}_u \left[ \|\xi\|_{L^2([0,T],d\mu)}^2 \right] + 2 \mathbb{E}_u \left[ \int_0^T X^u_t \xi_t d\mu(dt) \right] = R(\gamma, \mu, \hat{u}) + \mathbb{E}_u \left[ \|\xi\|_{L^2([0,T],d\mu)}^2 \right] + 2 \mathbb{E}_u \left[ \int_0^T D_t\xi_t d\mu(dt) \right] = \mathbb{E}_u [\text{SURE}_\mu(X + \xi)].
\]

Unlike the pointwise mean square risk \( \|X + \xi - u\|_{L^2([0,T],d\mu)}^2 \), the SURE risk estimator does not depend on the estimated parameter \( u \).

Given a family \( (\xi^\lambda)_{\lambda \in \Lambda} \) of estimators indexed by a parameter space \( \Lambda \), we consider the estimator \( X + \xi^{\lambda^*} \) that almost-surely minimizes the SURE risk, with

\[ \lambda^* = \arg\min_{\lambda \in \Lambda} \text{SURE}_\mu(X + \xi^\lambda). \]

For all values of \( \lambda \) the SURE risk of the estimator \( X + \xi^{\lambda^*} \) improves on the mean square risk of \( X + \xi^\lambda \).
Precisely for all $\nu \in \Lambda$ we have
\[
\mathbb{E}_u[\text{SURE}_\mu (X + \xi^{\nu})] \leq \mathbb{E}_u[\text{SURE}_\mu (X + \xi^\nu)] = \mathbb{E}_u \left[ \|\xi^\nu - u\|_{L^2([0,T],\mu)}^2 \right] = \inf_\lambda \mathbb{E}_u \left[ \|\xi^\lambda - u\|_{L^2([0,T],\mu)}^2 \right].
\]

In the sequel we will apply the above to a process $\xi_t \in [0,T]$ given as a function $\xi_t = \xi_t(X_t)$ of $X_t$, $t \in [0,T]$. In particular we will discuss estimation and thresholding for estimators of the form
\[
X_t + \xi_t^{\alpha,\lambda}(X_t) = \alpha(t) + \lambda(t) \eta \left( \frac{X_t - \alpha(t)}{\lambda(t)} \right),
\]
where $\eta : \mathbb{R} \to \mathbb{R}$ is a threshold function with support in $(-\infty,-1] \cup [1,\infty)$.

In particular we will apply our method to the joint estimation of parameters $\alpha, \lambda$, successively in case $\alpha(t) = \alpha$, $\alpha(t) = \alpha t$, and $\lambda(t) = \lambda \sqrt{\gamma(t,t)}$.

## 4 Soft threshold

In this section we construct an example of SURE shrinkage by soft thresholding in the framework of Proposition 3.1, with application to identification and de-noising in a Gaussian signal. In case $\eta$ is the soft threshold function
\[
\eta_S(y) = \text{sign}(y)(|y| - 1)^+, \quad y \in \mathbb{R},
\]
the function $\xi_t^{\alpha,\lambda}$ in (3.2) becomes
\[
\xi_t^{\alpha,\lambda}(x) = -\text{sign}(x - \alpha(t)) \min(\lambda(t), |x - \alpha(t)|), \quad x \in \mathbb{R},
\]
where $\lambda(t) \geq 0$ is a given level function.

**Proposition 4.1.** We have $\mathbb{P}$-a.s
\[
\text{SURE}_\mu (X + \xi^{\alpha,\lambda}(X)) = R(\gamma, \mu, \hat{u}) + \int_0^T |X_t - \alpha(t)|^2 \wedge \lambda^2(t) \mu(dt) - 2 \int_0^T \int_{|X_t - \alpha(t)| \leq \lambda(t)} \gamma(t,t) \mu(dt).
\]
Proof. Since $\frac{d}{dt} \xi_t^{\alpha,\lambda}(x) = -\mathbf{1}_{\{|x-\alpha(t)|\leq \lambda(t)|}$, we have
\[
\int_0^T D_t \xi_t^{\alpha,\lambda}(X_t) \mu(dt) = -\int_0^T \mathbf{1}_{\{|X_t-\alpha(t)|\leq \lambda(t)|} D_t X_t \mu(dt) \\
= -\int_0^T \mathbf{1}_{\{|X_t-\alpha(t)|\leq \lambda(t)|} \gamma(t) \mu(dt),
\]
hence the conclusion from Proposition 3.1.

The risk associated to discrete observations $(X_{t_1}, \ldots, X_{t_n})$ can be computed via Proposition 4.1 by choosing the risk measure (2.1), in which case Relation (4.2) becomes
\[
\text{SURE}(X + \xi^{\alpha,\lambda}(X)) = \text{SURE}(X + \xi^{\alpha,\lambda}(X)) \]
\[
= \text{SURE}(X + \xi^{\alpha,\lambda}(X)) = d + \sum_{i=1}^d (|x_i| \wedge \lambda)^2 - 2 \sum_{i=1}^n \gamma(t_i) \mathbf{1}_{\{|X_{t_i}-\alpha(t_i)|\leq \lambda(t_i)|}.
\]
which is analog to the finite dimensional SURE risk
\[
\text{SURE}(X + \xi^{\alpha,\lambda}(X)) = d + \sum_{i=1}^d (|x_i| \wedge \lambda)^2 - 2 \# \{i; |x_i| \leq \lambda\} \quad (4.3)
\]
of [3]. In the simulations of Section 6 we effectively use such risk measures when discretizing the signal. More precisely, when $\mu(dt) = f(t)dt$ has a density $f(t)$ with respect to the Lebesgue measure and
\[
\mu_n(dt) = \sum_{i=1}^{n-1} f(t_i)(t_{i+1} - t_i) \delta_{t_i}(dt),
\]
Relation (4.2) shows that $\text{SURE}_{\mu_n}(X + \xi^{\alpha,\lambda}(X))$ becomes a consistent estimator of the risk $\text{SURE}_{\mu}(X + \xi^{\alpha,\lambda}(X))$ as $n$ goes to infinity.

Taking
\[
\mu(dt) = \gamma^{-1}(t)dt \quad \text{and} \quad \lambda(t) = \lambda \sqrt{\gamma(t)}, \quad \lambda > 0, \quad t \in [0, T],
\]
and letting
\[
\bar{L}_T^\lambda := \int_0^T \mathbf{1}_{\{|X_t-\alpha(t)|\leq \lambda \sqrt{\gamma(t)}|} dt \quad (4.4)
\]
denote the occupation time of the process

\[ Z_t^{\alpha,\gamma} := \frac{X_t - \alpha(t)}{\sqrt{\gamma(t,t)}} \quad , \quad t \in [0, T], \]

up to time \( T \) in the set \([-\lambda, \lambda]\), Proposition 4.1 yields the identity

\[
\text{SURE}_{\mu}(X + \xi^{\alpha,\lambda}(X)) = T + \int_0^T (|Z_t^{\alpha,\gamma}| \wedge \lambda)^2 dt - 2\bar{L}_T^\lambda.
\]

(4.5)

As a consequence we obtain the following bound for the risk of the thresholding estimator \( X + \xi^{\alpha,\lambda}(X) \).

**Proposition 4.2.** Assume that \( u \in L^2([0, T], d\mu) \) is a deterministic function and let \( \mu(dt) := \gamma(t,t)^{-1} dt \). Then for all fixed \( \lambda \geq 0 \) we have

\[
\mathbb{E}_u[\|X + \xi^{\alpha,\lambda}(X) - u\|_{L^2([0, T], d\mu)}^2] \leq (1 + \lambda^2) \left( T \wedge \int_0^T |u(t) - \alpha(t)|^2 \mu(dt) \right) + T(1 + \lambda)e^{-\lambda^2/2}.
\]

Proof. We have

\[
\text{SURE}_{\mu}(X + \xi^{\alpha,\lambda}(X)) = T + \int_0^T (|Z_t^{\alpha,\gamma}| \wedge \lambda)^2 dt - 2\int_0^T 1_{\{|X_t - \alpha(t)| \leq \lambda \sqrt{\gamma(t,t)}\}} dt
\]

hence

\[
\mathbb{E}_u[\text{SURE}_{\mu}(X + \xi^{\alpha,\lambda}(X))] \leq T(1 + \lambda^2),
\]

and

\[
\mathbb{E}_u[\text{SURE}_{\mu}(X + \xi^{\alpha,\lambda}(X))] \leq \int_0^T 1 + \mathbb{E}_u[|Z_t^{\alpha,\gamma}|^2] \wedge \lambda^2 - 2\mathbb{P}_u(|Z_t^{\alpha,\gamma}| \leq \lambda) dt
\]

\[
\leq \int_0^T (1 + \lambda^2) \left( e^{-\lambda^2/2} + \frac{|u(t) - \alpha(t)|^2}{\gamma(t,t)} \right) dt
\]

\[
\leq (1 + \lambda^2) \int_0^T |u(t) - \alpha(t)|^2 \mu(dt) + T(1 + \lambda^2)e^{-\lambda^2/2},
\]

where we recall that from [3], Appendix 1, we have for every \( t \) in \([0, T]\) that

\[
1 + \mathbb{E}_u[|Z_t^{\alpha,\gamma}|^2] \wedge \lambda^2 - 2\mathbb{P}_u(|Z_t^{\alpha,\gamma}| \leq \lambda) \leq (1 + \lambda^2) \left( e^{-\lambda^2/2} + \frac{|u(t) - \alpha(t)|^2}{\gamma(t,t)} \right)
\]

and we conclude from Proposition 3.1. \( \square \)
From this proposition it follows that $\text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$ is independent of large values $\|u - \alpha\|_{L^2([0,T])}$, while its growth at most as $1 + \lambda^2$ in $\lambda \geq 0$.

Since $\lambda \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$ in (4.5) is lower bounded by $-T$ and equal to 0 when $\lambda = 0$, the optimal threshold

$$\lambda^* := \arg\min_\lambda \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$$

exists almost surely in $[0, \infty)$.

In addition we have the following proposition which important for the numerical search of an optimal parameter value.

**Proposition 4.3.** The function $\lambda \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$ is continuously differentiable.

**Proof.** Letting

$$\Delta(s,t) = \text{Var}_u(Z_{t}^{\alpha,\gamma} - Z_{s}^{\alpha,\gamma}) = 2 - 2\frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}, \quad 0 \leq s, t \leq T,$$

under Condition (7.4), the local time

$$\bar{\ell}_T^\lambda := \frac{d}{d\lambda} \bar{L}_T^\lambda$$

of $(|Z_{t}^{\alpha,\gamma}|)_{t \in [0,T]}$ exists almost surely, cf. Section 7, and we have

$$\frac{\partial}{\partial \lambda} \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X)) = \frac{\partial}{\partial \lambda} \int_0^T (|Z_{t}^{\alpha,\gamma}| \wedge \lambda)^2 \, dt - 2\bar{\ell}_T^\lambda$$

$$= 2\lambda \int_0^T 1_{\{|X_{t} - \alpha(t)| \geq \lambda \sqrt{\gamma(t,t)}\}} \, dt - 2\bar{\ell}_T^\lambda$$

$$= 2\lambda(T - \bar{\ell}_T^\lambda) - 2\bar{\ell}_T^\lambda,$$

which is a continuous function of $\lambda$ since the covariance $\gamma(s,t)$ does not vanish, cf. e.g. Theorem 26.1 of [5]. \qed
Consequently we have
\[ \frac{\partial}{\partial \lambda} \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))|_{\lambda=0} = -2\ell^0_T, \]
hence \( \lambda^* > 0 \) a.s. when \( \ell^0_T \) is a.s. positive, which is the case for example when \( X_t \) is a Brownian motion, see Corollary 2.2 of page 240 of [12], Chapter VI.

In practice we will compute \( \lambda^* \) numerically by minimization of \( \lambda \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X)) \) over \( \lambda \) in a range \( \Lambda = [0, C(T)] \) where \( C(T) \) is such that
\[ \lim_{T \to \infty} \mathbb{P}_u \left( \sup_{t \in [0,T]} |Z^{\alpha,\gamma}_t| \leq C(T) \right) = 1. \]
This condition is analog to Condition (31) in [3] and allows us to restrict the range of \( \lambda \) when searching for an optimal threshold.

The function \( \alpha(t) \) can be given in parametric form, in which case the parameters will be used to minimize \( \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X)) \), cf. Section 6.

## 5 Hard threshold

Here we use the threshold function
\[ \eta_H(y) = y 1_{\{|y| > 1\}}, \quad y \in \mathbb{R}, \quad (5.1) \]
hence
\[ \xi^{\alpha,\lambda}_t(x) = -(x - \alpha(t)) 1_{\{|x - \alpha(t)| < \lambda \sqrt{\gamma(t,t)}\}}, \quad x \in \mathbb{R}, \]
where \( \lambda \geq 0 \) is a level parameter.

In finite dimensions [3] the SURE estimator (1.1) can not be computed due to the non-differentiability of \( \eta_H \), however a deterministic optimal threshold equal to \( \sqrt{2 \log d} \) can be obtained by other methods, cf. Theorem 4 of [3].

In continuous time the situation is different due to the smoothing effect of the integral over time. In the next proposition we compute the SURE risk using the local time of Gaussian processes when \( \mu(dt) = \gamma^{-1}(t,t)dt \).
**Proposition 5.1.** We have \( \mathbb{P}\text{-a.s} \)

\[
\text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X)) = T + \int_0^T \frac{(X_t - \alpha(t))^2}{\gamma(t,t)} \mathbf{1}_{|X_t - \alpha(t)| \leq \lambda \sqrt{\gamma(t,t)}} dt + 2\lambda \bar{T}^\lambda - 2\bar{L}^\lambda. 
\]

(5.2)

**Proof.** Let \( \phi \in C^\infty([-1,1]), \phi \geq 0 \) be symmetric around the origin, such that \( \int_{-1}^1 \phi(x)dx = 1 \), and let

\[
\phi_\varepsilon(x) = \varepsilon^{-1} \phi(\varepsilon^{-1} x), \quad x \in \mathbb{R}, \quad \varepsilon > 0.
\]

Let

\[
\xi^{\alpha,\lambda}_t, \varepsilon(x) = \phi_\varepsilon \sqrt{\gamma(t,t)} * \xi^{\alpha,\lambda}_t(x) = \int_{-\infty}^{\infty} \phi_\varepsilon \sqrt{\gamma(t,t)}(y) \xi^{\alpha,\lambda}_t(x - y)dy,
\]

denote the convolution of \( \phi_\varepsilon \sqrt{\gamma(t,t)} \) with \( \xi^{\alpha,\lambda}_t \), with

\[
\frac{d}{dx} \phi_\varepsilon \sqrt{\gamma(t,t)} * \xi^{\alpha,\lambda}_t(x) = \phi_\varepsilon \sqrt{\gamma(t,t)} * \frac{d}{dx} \xi^{\alpha,\lambda}_t(x)
\]

\[
= \lambda \sqrt{\gamma(t,t)} \phi_\varepsilon \sqrt{\gamma(t,t)} (-\lambda \sqrt{\gamma(t,t)} + x - \alpha(t)) + \lambda \sqrt{\gamma(t,t)} \phi_\varepsilon \sqrt{\gamma(t,t)} (\lambda \sqrt{\gamma(t,t)}) + x - \alpha(t)
\]

\[
- \int_{-\infty}^{\infty} \phi_\varepsilon \sqrt{\gamma(t,t)}(y) \mathbf{1}_{|x-y-\alpha(t)| < \lambda \sqrt{\gamma(t,t)}} dy.
\]

From the occupation time density formula (7.5) we have

\[
\int_0^T D_t \xi^{\alpha,\lambda,\varepsilon}_t(X_t) \mu(dt) = \lambda \int_0^T \sqrt{\gamma(t,t)} \phi(\varepsilon \sqrt{\gamma(t,t)})(-\lambda \sqrt{\gamma(t,t)} + X_t - \alpha(t)) dt 
\]

\[
+ \lambda \int_0^T \sqrt{\gamma(t,t)} \phi(\varepsilon \sqrt{\gamma(t,t)})(\lambda \sqrt{\gamma(t,t)}) + X_t - \alpha(t)) dt 
\]

\[
- \int_0^T \int_{-\infty}^{\infty} \phi(\varepsilon \sqrt{\gamma(t,t)}(y) \mathbf{1}_{|x-y-\alpha(t)| > \lambda \sqrt{\gamma(t,t)}} dydt 
\]

\[
= \lambda \int_{-\infty}^{\infty} (\phi(\varepsilon(-\lambda + Z^{\alpha,\gamma}_t)) + \phi(\varepsilon(-\lambda - Z^{\alpha,\gamma}_t)) dt 
\]

\[
- \int_0^T \int_{-\infty}^{\infty} \phi(\varepsilon \sqrt{\gamma(t,t)}(y) \mathbf{1}_{|x-y-\alpha(t)| > \lambda \sqrt{\gamma(t,t)}} dydt 
\]

\[
= \lambda \int_{-\infty}^{\infty} \phi(\varepsilon(\alpha - \lambda)) \bar{T}_T^\alpha da 
\]

\[
- \int_0^T \int_{-\infty}^{\infty} \phi(\varepsilon \sqrt{\gamma(t,t)}(y) \mathbf{1}_{|x-y-\alpha(t)| < \lambda \sqrt{\gamma(t,t)}} dydt,
\]

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which converges in $L^2(\Omega, \mathbb{P}_u)$ to

$$
\lambda_{\ell_T}^\lambda - \int_0^T 1_{\{|X_{\ell_T} - \alpha(t)| < \lambda \sqrt{\gamma(t)} \}} \, dt
$$
as $\varepsilon$ tends to zero.

6 Numerical examples

In this section we assume that $X^u$ is a centered stationary Ornstein-Uhlenbeck process solution of

$$
dX^u_t = -aX^u_t \, dt + \sigma dB_t, \quad t \in [0, T],
$$
with $X^u_0 \sim \mathcal{N}(0, \frac{\sigma^2}{2a})$ and covariance function $\gamma(s, t) = \frac{\sigma^2}{2a} e^{-a|t-s|}$, $s, t \in [0, T]$, for $\sigma, a > 0$. As a consequence of the following proposition we can take $\Lambda = [0, \sqrt{2\log T}]$ as parameter range when $T$ is large.

**Proposition 6.1.** Assume that $\|\alpha\|_{L^\infty([0,\infty))} < \infty$ and $\|u\|_{L^\infty([0,\infty))} < \infty$. Then for any $r > 1$ we have

$$
\lim_{T \to \infty} \mathbb{P}_u \left( \sup_{t \in [0,T]} |Z_t| \leq \sqrt{2r \log T} \right) = 1.
$$

**Proof.** From Theorem 1.1 of [14] (see also [7], Theorem 2.1 of [11], and [2], page 488) there exists a universal constants $c_1, c_2 > 0$ such that for all $\lambda, T > 0$,

$$
\mathbb{P}_u \left( \sup_{t \in [0,T]} |Z_t| > \lambda \right) \leq c_1 M(2aT, c_2/\lambda) \Psi(\lambda),
$$

where $\Psi(x) = \int_x^{\infty} e^{-y^2/2} \, dy / \sqrt{2\pi}$ and $M(2aT, c_2/\lambda)$ is the maximal cardinal of all sequences $S$ in $[0, 2aT]$ such that

$$
|Z_t - Z_s|_{L^2(\Omega)} = \sigma \sqrt{1 - e^{-a|t-s|}} > \frac{c_2}{\lambda}, \quad s, t \in S.
$$

Setting $\lambda = \sqrt{2r \log T}$, $r > 0$, $T > 1$, and using the bound $\Psi(\lambda) \leq e^{-\lambda^2/2}(\lambda\sqrt{2\pi})$ this yields, for all $T$ large enough:

$$
\mathbb{P} \left( \sup_{t \in [0,T]} |Z_t| \leq \sqrt{2r \log T} \right) \geq 1 - c r \frac{T^{1-r}}{\sqrt{a}},
$$

which tends to 1 as $T \to \infty$ provided $r > 1$. \qed
In the next figures we present some numerical simulations when the signal \((X_t)_{t \in [0,T]}\) is a deterministic function \((u(t))_{t \in [0,T]}\) perturbed by a centered Ornstein-Uhlenbeck process, with parameters \(a = 0.5, \sigma = 0.05, T = 1\).

We represent simulated samples path with the optimal thresholds obtained by soft thresholding, the de-noised signal after hard thresholding, and the corresponding risk function \((\alpha, \lambda) \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))\) whose minimum gives the optimal parameter value(s). The hard threshold function has not been used for estimation due to increased numerical instabilities linked to the simulation of the local time in \([0,2]\).

**Simple thresholding**

Here we take \(u_t = 0.2 \times \max(0, \sin(3\pi t)), \lambda(t) = \lambda\sqrt{\gamma},\) and we aim at de-noising the signal around the level \(\alpha(t) = 0, t \in [0, T]\).

![Figure 6.1: Risk function \(\lambda \mapsto \text{SURE}_\mu(X + \xi^{0,\lambda}(X))\)](image)

From Figure 6.1 we estimate the optimal threshold to \(\lambda^* \sqrt{\gamma} = 0.018\), after numerical minimization on a grid, which leads to the thresholding described in Figure 6.2 below.
Level detection and thresholding

We apply our method to the joint estimation of parameters $\alpha$, $\lambda$, in case $u_t = 0.3 + 0.2 \text{sign}(\sin(2\pi t)) \times \max(0, \sin(3\pi t))$, $\alpha(t) = \alpha$ and $\lambda(t) = \lambda \sqrt{\gamma}$, i.e. we aim at detecting simultaneously the level $\alpha = 0.3$ and the threshold $\lambda \sqrt{\gamma}$ at which the noise can be removed. For this we have the following proposition that completes Proposition 4.3.

**Proposition 6.2.** The function $(\alpha, \lambda) \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$ is continuously differentiable.

**Proof.** We have

$$\frac{\partial}{\partial \alpha} \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X)) = -2 \int_0^T \frac{X_t - \alpha}{\gamma(t, t)} 1_{\{|X_t - \alpha| \leq \lambda \sqrt{\gamma(t, t)}\}} dt + 2 \ell_T^{\alpha,\lambda} - 2 \ell_T^{\alpha,-\lambda},$$

where $\ell_T^{\alpha,\lambda}$ denotes the local time at level $\alpha$ of the process $(X_t + \lambda \sqrt{\gamma(t, t)})_{t \in [0, T]}$. □

---

**Figure 6.2:** Process trajectory Estimated trajectory

**Figure 6.3:** Risk function $(\alpha, \lambda) \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$
From Figure 6.3 we estimate the optimal threshold and shift parameters at $\lambda^* \sqrt{\gamma} = 0.017$ and $\alpha^* = 0.30$, which leads to the thresholding described in Figure 6.4 below.

Figure 6.4: Process trajectory  
Estimated trajectory

Figure 6.3 also shows that the values 0.5 and 0.1 are other candidates to an estimation of $\alpha$. These values correspond to the extrema in the sample trajectory of Figure 6.4.

Drift detection and thresholding

We apply our method to the joint estimation of parameters $\alpha$, $\lambda$, in case $u_t = 0.3t + 0.2 \text{sign}(\sin(2\pi t)) \times \max(0, \sin(3\pi t))$, $\alpha(t) = \alpha t$, and $\lambda(t) = \lambda \sqrt{\gamma}$, i.e. we aim at locating noise with threshold $\lambda \sqrt{\gamma}$ around a line of slope $\alpha = 0.3$. Analogously to Propositions 4.3 and 6.2 we have the following result.

**Proposition 6.3.** The function $(\alpha, \lambda) \mapsto \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X))$ is continuously differentiable.

**Proof.** We have

$$\frac{\partial}{\partial \alpha} \text{SURE}_\mu(X + \xi^{\alpha,\lambda}(X)) = -2 \int_0^T \frac{X_t - \alpha t}{\gamma(t,t)} 1_{|X_t - \alpha t| \leq \lambda \sqrt{\gamma(t,t)}} tdt + 2\ell_T^{\alpha,\lambda} - 2\ell_T^{\alpha,-\lambda},$$

where $\ell_T^{\alpha,\lambda}$ denotes the local time at level $\alpha$ of the process $((X_t + \lambda \sqrt{\gamma(t,t)})/t)_{t \in [0,T]}$. □
The optimal threshold and slope parameters are numerically estimated at $\lambda^* \sqrt{\gamma} = 0.0093$ and $\alpha^* = 0.294$.

The threshold and slope and actually slightly underestimated, as the larger noise at the right end of the slope line has been interpreted as being part of the signal.

7 Appendix

In this section we review three aspects of stochastic analysis for Gaussian processes, including local time and the Malliavin calculus calculus.
Malliavin calculus on Gaussian space

Here we recall some elements of the Malliavin calculus on Gaussian space for the centered Gaussian process \((X_t)_{t \in [0,T]}\), see e.g. [6]. Let \(\mu\) be a finite Borel measure on \([0,T]\) and let \(\Gamma\) the operator defined as

\[
(\Gamma g)(t) = \int_0^T g(s)\gamma(s,t)\mu(ds), \quad t \in [0,T],
\]
on the Hilbert space \(H\) of functions on \([0,T]\) with the inner product

\[
\langle h, g \rangle_H = \langle h, \Gamma g \rangle_{L^2([0,T],d\mu)}.
\]
The process \((X_t)_{t \in [0,T]}\) can be used to construct an isometry \(X : H \rightarrow L^2(\Omega, \mathcal{F}, P)\) as

\[
X(h) = \int_0^T X_u h(s)\mu(ds), \quad h \in H.
\]

Then \(\{X(h) : h \in H\}\) is an isonormal Gaussian process on \(H\), i.e. a family of centered Gaussian random variables satisfying

\[
\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_H, \quad h, g \in H.
\]

For any orthonormal basis \((h_k)_{k \in \mathbb{N}}\) of \(L^2([0,T], d\mu)\), we have the Karhunen-Loève expansion

\[
X_t = \sum_{k=0}^{\infty} h_k(t) X(h_k), \quad t \in [0,T]. \tag{7.1}
\]

Let now \(S\) denote the space of cylindrical functionals of the form

\[
F = f_n(X^u(h_1), \ldots, X^u(h_n)), \tag{7.2}
\]
where \(f_n\) is in the space of infinitely differentiable rapidly decreasing functions on \(\mathbb{R}^n\), \(n \geq 1\).

**Definition 7.1.** The \(H\)-valued Malliavin derivative is defined as

\[
\nabla_t F = \sum_{i=1}^n h_i(t) \partial_i f_n(X^u(h_1), \ldots, X^u(h_n)),
\]

for \(F \in S\) of the form \((7.2)\).
It is known that $\nabla$ is closable, cf. Proposition 1.2.1 of [6], and its closed domain will be denoted by $\text{Dom}(\nabla)$.

**Definition 7.2.** Let $D_t$ be defined on $F \in \text{Dom}(\nabla)$ as

$$D_tF := (\Gamma \nabla F)(t), \quad t \in [0,T].$$

Let $\delta : L^2_u(\Omega;H) \rightarrow L^2(\Omega,\mathbb{P}_u)$ denote the closable adjoint of $\nabla$, i.e. the divergence operator under $\mathbb{P}_u$, which satisfies the integration by parts formula

$$\mathbb{E}_u[F\delta(v)] = \mathbb{E}_u[(v, \nabla F)_H], \quad F \in \text{Dom}(\nabla), \quad v \in \text{Dom}(\delta), \quad (7.3)$$

where $\mathbb{E}_u$ denotes the expectation under $\mathbb{P}_u$, with the relation

$$\delta(hF) = FX(h) - \langle h, \nabla F \rangle_H,$$

cf. [6], for $F \in \text{Dom}(\nabla)$ and $h \in H$ such that $hF \in \text{Dom}(\delta)$. The next lemma will be needed in Proposition 3.1 below to establish Stein’s Unbiased Risk Estimate for Gaussian processes.

**Lemma 7.3.** For any $F \in \text{Dom}(\nabla)$ and $u \in H$ we have

$$\mathbb{E}_u[FX^u_t] = \mathbb{E}_u[D_tF], \quad t \in [0,T].$$

**Proof.** We have

$$\mathbb{E}_u[FX^u_t] = \sum_{k=0}^{\infty} h_k(t) \mathbb{E}_u[FX^u(h_k)]$$

$$= \sum_{k=0}^{\infty} h_k(t) \mathbb{E}_u[F\delta(h_k)]$$

$$= \sum_{k=0}^{\infty} h_k(t) \mathbb{E}_u[\langle h_k, \nabla F \rangle_H]$$

$$= \sum_{k=0}^{\infty} h_k(t) \mathbb{E}_u[\langle h_k, \Gamma \nabla F \rangle_{L^2([0,T],\mu)}]$$

$$= \mathbb{E}_u[(\Gamma \nabla F)(t)], \quad F \in \text{Dom}(\nabla), \quad t \in [0,T].$$

$\square$
Note that since \( u \in H \) we have \( \nabla_s X_t(h) = \nabla_s X^u_t(h) = h(s) \) and
\[
D_t X_t = (\Gamma \nabla X_t)(t) = \int_0^T \gamma(s,t) \nabla_s X_t \mu(ds) = \sum_{k=0}^{\infty} h_k(t) \int_0^T \gamma(s,t) \nabla X(h_k) \mu(ds) = \sum_{k=0}^{\infty} h_k(t) \langle \gamma(\cdot,t), h_k \rangle_{L^2([0,T],d\mu)} = \gamma(t,t), \quad t \in [0,T].
\]

**Local time of Gaussian processes**

Given \((Z_t)_{t \in [0,T]}\) a Gaussian process let
\[
\Delta(s,t) = \text{Var}(Z_t - Z_s), \quad 0 \leq s,t \leq T,
\]
and denote by
\[
L^\lambda_T := \int_0^T 1_{\{Z_t \leq \lambda\}} dt
\]
the occupation time of \((Z_t)_{t \in [0,T]}\) up to \(T\) in the set \((-\infty, \lambda]\).

Recall that a classical result of Berman [1], see Theorem 21.9 of [5], shows that if
\[
\int_0^T \int_0^T \Delta^{-1}(s,t) dsdt < \infty, \quad (7.4)
\]
then for any \( \lambda \in \mathbb{R} \) the local time
\[
\ell^\lambda_T := \frac{\partial}{\partial \lambda} L^\lambda_T
\]
of \((Z_t)_{t \in [0,T]}\) at the level \( \lambda \) exists and the occupation time density formula
\[
\int_0^T f(Z_t) dt = \int_{\mathbb{R}} f(\lambda) \ell^\lambda_T d\lambda \quad (7.5)
\]
holds for every positive measurable function \( f \) on \( \mathbb{R} \). The local time \( \ell_T^\lambda \) of \(|Z_t|\) is given by \( \ell_T^\lambda = \ell_T^\lambda a + \ell_T^\lambda a \) and the related occupation time formula can be obtained under the same condition from the relation
\[
\int_0^T f(|Z_t|) dt = \int_{-\infty}^{\infty} f(|a|) \ell_T^a da = \int_{0}^{\infty} f(a) \ell_T^a da.
\]
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