Gravitational waves in the generalized Chaplygin gas model

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Abstract

The consequences of taking the generalized Chaplygin gas as the dark energy constituent of the Universe on the gravitational waves are studied and the spectrum obtained from this model, for the flat case, is analyzed. Besides its importance for the study of the primordial Universe, the gravitational waves represent an additional perspective (besides the CMB temperature and polarization anisotropies) to evaluate the consistence of the different dark energy models and establish better constraints to their parameters. The analysis presented here takes this fact into consideration to open one more perspective of verification of the generalized Chaplygin gas model applicability. Nine particular cases are compared: one where no dark energy is present; two that simulate the Λ-CDM model; two where the gas acts like the traditional Chaplygin gas; and four where the dark energy is the generalized Chaplygin gas. The different spectra permit to distinguish the Λ-CDM and the Chaplygin gas scenarios.

KEYWORDS: dark energy, dark matter, gravitational waves

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1 Introduction

The most recent results from type Ia supernovae observations [1, 2, 3] and the cosmic microwave background anisotropies detection [4] have led cosmologists and astrophysicists to the conclusion that most of the matter of the Universe interacts in a repulsive manner (as an example through a negative pressure). There are many candidates to describe this exotic fluid. The most natural candidate is a cosmological constant, but it presents a discrepancy of 120 order of magnitude between the theoretical predictions and the observational data [5]. A self-interacting scalar field, known as quintessence, is another proposal to explain dark energy [6]. However, it asks for fine tuning of microphysical parameters in order to have a suitable potential term [7]. Among many other possibilities, the Chaplygin gas model has receiveid recently special attention [8–10]. The Chaplygin gas model is based on a perfect fluid whose pressure, besides to be negative, varies inversely with density. One of the interesting aspects of this fluid is a connection with branes in the context of string theories [11–13]. Some phenomenological generalizations of this fluid have been proposed, leading to the so-called generalized Chaplygin gas model [8].

The aim of this paper is to investigate the particular signatures of the generalized Chaplygin gas, mainly in comparison to the standard Λ-CDM, in what concerns gravitational waves. Special attention will be given to the spectral distribution of energy density.
The gravitational waves are very important to Cosmology. They must have special signatures in the polarization \[14\] of the CMB anisotropies \[15\]. Moreover, since gravitational waves has decoupled from matter already in the deep early Universe, they can be a window to the primordial phase. Even if gravitational waves (and consequently the polarization of the CMB photons \(^1\)) have not been detected directly until now, great efforts are been done in this sense and there is a hope that the next generation of experiment in space (LISA, Planck, etc.), may allow this detection in perhaps ten years. The GW spectrum frequency range of observational interest extends from \(10^{-18} \text{ Hz}\) to \(10^{10} \text{ Hz}\) and the energy density spectrum is constrained by the CMB, in units of the critical density \(\Omega_{GW}\), as \[18\]

\[
\frac{d\Omega_{GW}}{d \ln \nu}\big|_{10^{-18}} \leq 10^{-10} .
\]  

This constraint will be used later in the evaluation of the spectra. Many works have been made in order to identify specific signatures of cosmological models in the spectra of gravitational waves, for example, in the case of quintessence model \[18\] and string cosmology \[19, 20, 21\].

The generalized Chaplygin gas is characterized by the equation of state

\[
p = -\frac{A}{\rho^\alpha} ,
\]  

where \(A\) and \(\alpha\) are constants such that \(A \geq 0\) and \(0 \leq \alpha \leq 1\). If the energy-momentum tensor conservation is taken into account, the relation between the generalized Chaplygin gas density and the scale factor \(a\) becomes

\[
\rho = \left( A + \frac{B}{a^{3(\alpha+1)}} \right)^{\frac{1}{\alpha+1}} ,
\]  

where \(B\) is an arbitrary integration constant. The main properties of the relation \[3\] are: (i) it interpolates the cosmological constant phase (when the scale factor is large) and a pressureless fluid (for small values of \(a\)) phase; (ii) the sound velocity associated with it goes from zero, for \(A = 0\), to the velocity of light, being always positive.

We will analyze the gravitational waves spectrum obtained from the generalized Chaplygin gas model, which is described in section \[2\]. In section \[3\] we discuss general properties of the gravitational waves differential equation in the context of two important dark matter models: cosmological constant and Chaplygin gas. The spectrum obtained from numerical calculations on these cases are presented in section \[4\] and analyzed in section \[5\] where we also present our conclusions.

2 Outline of the model

We consider a flat, homogeneous and isotropic Universe described by the Friedman-Robertson-Walker metric, which may be written as

\[
ds^2 = c^2 dt^2 - a^2(t) \ g_{ij}^{(0)} \ dx^i dx^j , \quad g_{ij}^{(0)} = \delta_{ij} ,
\]  

and leads the Einstein’s equations to assume the form

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left( \rho_m + \rho_c \right) ,
\]  

\[
\frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 = -8\pi G \left( p_m + p_c \right) ,
\]  

where \(a\) is the scale factor of the Universe, while \(\rho_m\) and \(\rho_c\) are the pressureless fluid and the Chaplygin gas densities. The pressures \(p_m\) and \(p_c\) of the fluids are related to their densities by the equations of state \(p_m = 0\) and \(p_c = -A/\rho_0^\alpha\).

\(^1\)There are two polarizations modes, called \(E\) and \(B\). The mode \(E\) has already been identified, but only the detection of the mode \(B\) will allow to identify gravitational waves \[16, 17\].
The option of working in a flat Universe is justified by the recent data from the CMB measurements. The curvature of the Universe is characterized by the parameter $\Omega_k$ which is defined, in terms of the total observed density $\rho_T$ and the critical density $\rho_{cr}$, as $\Omega_k = 1 - \Omega_T$, $\Omega_T = \rho_T / \rho_{cr}$. The CMB data gives $\Omega_k = 0 \pm 0.06$ [22].

If the fluids interacts only through geometry, the energy-momentum tensor for each component is conserved and we get

$$\rho_m = \frac{\rho_{mo}}{a^3}, \quad \rho_c = \left( A + \frac{B}{a^{3(\alpha+1)}} \right)^{\bar{\alpha}},$$  \hspace{1cm} (7)

We take the scale factor today as the unity, $a_0 = 1$, and thus $\rho_{mo}$ and $\rho_{co} = (A + B)^{\bar{\alpha}}$ are the present densities of the fluids (The subscripts $o$, according to the current notation, indicate we are considering the present values of these quantities). From this last equation we can express the integration constant $B$ in terms of $A$, $B = \rho_c^{\alpha+1} - A$. The Chaplygin gas density is then rewritten as

$$\rho_c = \rho_{co} \left[ \bar{A} + \frac{(1 - \bar{A})}{a^{3(\alpha+1)}} \right]^{\frac{1}{\bar{\alpha}}}, \quad \bar{A} = \frac{A}{\rho_c^{\alpha+1}},$$  \hspace{1cm} (8)

and the parameter $\bar{A}$ is connected to the sound velocity in the gas, $v_s$, by the expression

$$v_{s0} = c \sqrt{\frac{\partial p_c}{\partial \rho_c}} |_{t_0} = c \sqrt{\alpha A}.$$  \hspace{1cm} (9)

Taking the first of the equations (7) (which refers to the pressureless fluid) and the equation (8), we use the relation $\Omega_{io} = \rho_{io} / \rho_{cr}$ (where $i = m, c$) in (8) to obtain

$$\frac{\dot{a}}{a} = H_0 \left[ \frac{\Omega_{mo}}{a^3} + \Omega_{co} \left( \bar{A} + \frac{1 - \bar{A}}{a^{3(\alpha+1)}} \right)^{\frac{1}{\bar{\alpha}}} \right]^{1/2}.$$  \hspace{1cm} (10)

$$\frac{\ddot{a}}{a} = H_0^2 \left[ \frac{-\Omega_{mo}}{2a^3} + \Omega_{co} \left( \bar{A} + \frac{1 - \bar{A}}{a^{3(\alpha+1)}} \right)^{\frac{1}{\bar{\alpha}}} \left( 1 - \frac{3}{2} \frac{1 - \bar{A}}{a^{3(\alpha+1)}} \left( \bar{A} + \frac{1 - \bar{A}}{a^{3(\alpha+1)}} \right) \right) \right],$$  \hspace{1cm} (11)

where the Hubble constant $H_0$ is defined by the expression $H_0 = \dot{a}_0 / a_0$. Since we are restricted to a flat Universe, the fractions of pressureless matter and Chaplygin gas today, $\Omega_{mo}$ and $\Omega_{co}$, obey to the relation $\Omega_{mo} + \Omega_{co} = 1$.

With these two last equations we are able to write the GW amplitude differential equation as a function of the observable variables $H_0$, $a$, $\Omega_{mo}$ and $\Omega_{co}$, and of the Chaplygin gas parameters $\bar{A}$, $\alpha$. It’s also important to remark that if $\bar{A} = 0$, the gas behaves like the pressureless fluid (and the situation is the same as if we had set $\Omega_{mo} = 1$) while, on the other hand, it behaves like the cosmological constant fluid when $\bar{A} = 1$ (and therefore we can simulate the $\Lambda$-CDM scenario).

Among the many possible cases produced by the combinations of parameters we chose a few important ones and classify them according to the fractions of dark energy and matter and to the kind of dark energy, as shown in table 1 below.

In the first case no dark energy is present, while the other eight cases reflect the situation where the dark energy constitutes all the non-visible matter (that means, no dark matter exists and $\Omega_{mo} = 0.04$) or the one where the dark matter and the visible matter altogether represent 30% of the total energy density ($\Omega_{mo} = 0.3$).

After establishing the characteristics of the studied models, we start in section 3 the description of the behavior of gravitational waves due to different cosmic fluids contents, namely the cosmological constant and the generalized Chaplygin gas.
3 GW equation in Chaplygin gas and Λ-CDM models

Cosmological gravitational waves are obtained by means of a small correction \( h_{ij} \) on equation (4), which represents the metric. Hence, the tensor \( g_{ij}^{(0)} \), related to the unperturbed metric, is replaced by \( g_{ij} = g_{ij}^{(0)} + h_{ij} \) and the resulting expression is [23, 24]:

\[
\ddot{h} - \frac{\dot{a}}{a} \dot{h} + \left( \frac{k^2}{a^2} - 2 \frac{\ddot{a}}{a} \right) h = 0 ,
\]

where \( k \) is the wave number times the velocity of light \( (k = 2\pi c/\lambda) \), the dots indicate time derivatives and we have written, \( h_{ij}(t, \vec{x}) = h(t)Q_{ij} \), where \( Q_{ij} \) are the eigenmodes of the Laplacian operator such that \( Q_{ii} = Q_{ki,k} = 0 \).

Performing a variable transformation, from time to the scale factor \( a \), and representing the derivatives with respect to \( a \) by primes, equation (12) assumes the form

\[
\dddot{h} + \left( \frac{\ddot{a}}{a^2} - \frac{1}{a} \right) \dot{h} + \frac{1}{a^2} \left( \frac{k^2}{a^2} - 2 \frac{\ddot{a}}{a} \right) h = 0 .
\]

By using the background equations (10) and (11), or similar ones concerning to other fluids, into (13), one can easily express \( h \) in terms of the parameters of the Chaplygin gas or any other model.

3.1 Chaplygin Gas

Let us perform the last operation mentioned above and find \( h \) as a function of the redshift \( z \), with the following steps: (i) use the fairly known relations \( 1 + z = \frac{a_{0}}{a_{0}} \), \( a_{0} = 1 \); (ii) perform a second variable changing (from \( a \) to \( z \)); and (iii) take back the dots to indicate, from now on, the new integration variable. These operations result in

\[
\dddot{h} + \left[ \frac{2}{1+z} + \frac{3}{2}(1+z)^2f_1 \right] \dot{h} + \left[ \frac{k^2}{f_2} - \frac{2}{(1+z)^2} + 3(1+z)f_1 \right] h = 0 ,
\]

\[
f_1 = \Omega_{m0} + \Omega_{c0}(1 - \bar{A})(1 + z)^{3\alpha} \left[ \bar{A} + (1 - \bar{A})(1 + z)^{3(\alpha + 1)} \right]^{\alpha+1}^{-1} ,
\]
\[ f_2 = \Omega_m(1 + z)^3 + \Omega_c \left[ \bar{A} + (1 - \bar{A})(1 + z)^3(\alpha + 1) \right]^{\frac{1}{\alpha + 1}}, \tag{16} \]

where \( k \) has been redefined to absorb the Hubble constant, i.e., \( k = 2\pi cH_0/\lambda \). Therefore, \( h \) is a dimensionless quantity.

The solutions \( h(z) \) of the differential equation oscillates with greater amplitudes as \( z \to 0 \): these amplitudes are increasing from the decoupling era up to the present.

Equations (14-16) above may be used to perform all the cases of interest mentioned before. Setting \( \Omega_m = 1 \) and \( \Omega_c = 0 \), for example, we have the pressureless fluid equation and for the cosmological constant we set \( \bar{A} = 1 \).

### 3.2 Cosmological constant

In order to establish the GW equation for the cosmological constant fluid, we use the fact that the Chaplygin gas reproduces its behavior if \( \bar{A} = 1 \), for any value of \( \alpha \), as stated before. Hence, the equations (15) and (16) become

\[ f_1 = \Omega_m, \tag{17} \]
\[ f_2 = \Omega_m(1 + z)^3 + \Omega_\Lambda. \tag{18} \]

The resulting equation is easier to integrate than the one obtained before and presents the same qualitative behavior with respect to the redshift. Another important feature to be mentioned is the existence of quite simple analytical solutions, which permit us write \( h(z) \) in terms of a combination of sines and cosines:

\[ h(z) = \frac{\sqrt{2/\pi}}{k^{3/2}(1 + z)^3} \left[ (k(1 + z)C_1 + C_2) \cos[k(1 + z)] + (k(1 + z)C_2 - C_1) \sin[k(1 + z)] \right]. \tag{19} \]

The arbitrary constants \( C_1 \) and \( C_2 \) remain undetermined since the set of initial conditions is unspecified. Nevertheless, it is important to remark that, for a fixed \( k \), the argument \( \theta \) of the cosine and sine functions in (19) is \( \theta = k(1 + z) \) and so the oscillation frequency decreases when \( z \) tends to 0.

Figure (1) illustrates the case where \( \ddot{h}(z_i) = \dot{h}(z_i) = 10^{-6} \), \( z_i = 4000 \) are the initial conditions for two different values of \( k \).

![Figure 1](image)

Figure 1: Gravitational waves curves for Λ-CDM model. The dash line refers to \( k = 1/100 \) and the solid line to \( k = 1/50 \).

The behavior described is evident and one can also observe that the amplitude grows with time (i.e., as \( z \to 0 \)), as we have remarked for the Chaplygin gas. This result implies that initial small primordial fluctuations on the gravitational field are amplified.
4 GW spectra

The power spectrum of gravitational waves, defined as

\[
\frac{d\Omega_{GW}}{d\ln \nu} = |h_0(\nu)|\nu^{5/2},
\]

(where \(h_0(\nu) = h(0)\) and \(\nu = H_0 k/2\pi\)), is generally obtained directly from the solutions of \(14\). In the particular case of the \(\Lambda\)-CDM model, an analytical result is also possible, as shown in \(19\), and this fact may be used to verify the accuracy of the calculation and the applicability of the algorithm.

For each of the cases of interest found in table \(1\), we have assigned some common parameters, namely the initial conditions \(h(z_i) = \nu^{-3}10^{-5}, z_i = 4000\); the range of frequencies \(10^{-18}\text{Hz} \leq \nu \leq 10^{-15}\text{Hz}\), and the normalization constant imposed by the constraint equation \(11\). The factor \(h^{-3}\) in the initial conditions fix the primordial spectrum. The resulting spectra are presented in figures \(2-12\).

Figure 2: Graphic \((d\Omega_{GW}/dln\nu) \times (\nu \times 10^{-18}\text{Hz})\) for the model where \(\Omega_{c0} = 0\) and \(\Omega_{m0} = 1\)

Figure 3: Graphic \((d\Omega_{GW}/dln\nu) \times (\nu \times 10^{-18}\text{Hz})\) for the model where \(A = 1, \Omega_{c0} = 0.96\) and \(\Omega_{m0} = 0.04\)

Figure 4: Graphic \((d\Omega_{GW}/dln\nu) \times (\nu \times 10^{-18}\text{Hz})\) for the model where \(A = 1, \Omega_{c0} = 0.7\) and \(\Omega_{m0} = 0.3\)

Figure 2 refers to the pure pressureless fluid case and one can see that for \(\nu \to 0\) the amplitude reaches its maximum. It decreases very quickly and, for greater values of \(\nu\), it increases slowly.

On the other hand, the spectrum corresponding to a cosmological constant dominated Universe \(3\) has a different behavior. The amplitude grows faster as the frequency increases and reaches a value six times greater than the previous one. Figures \(5\) and \(6\) shows the behaviour of the spectrum for very low frequencies for the \(\Lambda\)-CDM case. For the other models, the spectrum is very similar. Still referring to the \(\Lambda\)-CDM model, figure \(4\) shows a behavior similar to the one from the pressureless fluid, but the amplitude growing is faster.
Figure 5: Graphic \( (d\Omega_{GW}/d\ln \nu) \times (\nu \times 10^{-18} \text{Hz}) \) for the model where \( \bar{A} = 1, \Omega_{c0} = 0.96 \) and \( \Omega_{m0} = 0.04 \) for small values of \( k \).

Figure 6: Graphic \( (d\Omega_{GW}/d\ln \nu) \times (\nu \times 10^{-18} \text{Hz}) \) for the model where \( \bar{A} = 1, \Omega_{c0} = 0.7 \) and \( \Omega_{m0} = 0.3 \) for small values of \( k \).

Figure 7: Graphic \( (d\Omega_{GW}/d\ln \nu) \times (\nu \times 10^{-18} \text{Hz}) \) for the model where \( \bar{A} = 0.5, \alpha = 1, \Omega_{c0} = 0.96 \) and \( \Omega_{m0} = 0.04 \).

Figure 8: Graphic \( (d\Omega_{GW}/d\ln \nu) \times (\nu \times 10^{-18} \text{Hz}) \) for the model where \( \bar{A} = 0.5, \alpha = 1, \Omega_{c0} = 0.7 \) and \( \Omega_{m0} = 0.3 \).

Figure 9: Graphic \( (d\Omega_{GW}/d\ln \nu) \times (\nu \times 10^{-18} \text{Hz}) \) for the model where \( \bar{A} = 0.5, \alpha = 0.5, \Omega_{c0} = 0.96 \) and \( \Omega_{m0} = 0.04 \).

Figure 10: Graphic \( (d\Omega_{GW}/d\ln \nu) \times (\nu \times 10^{-18} \text{Hz}) \) for the model where \( \bar{A} = 0.5, \alpha = 0.5, \Omega_{c0} = 0.7 \) and \( \Omega_{m0} = 0.3 \).

The other graphics correspond to the Chaplygin gas and present a similar shape. However, in general the Chaplygin gas models lead to a smaller amplitude in the gravitational wave spectra. This amplitude is still smaller for the case of pure dark matter model. Its also important to remark that figures 7-8 refers to the Chaplygin gas in its traditional form (with the equation of state \( p = -A/\rho \), since \( \alpha = 1 \) while the others correspond to the generalized Chapligyn gas.
5 Conclusions

In this work we have studied the fate of gravitational waves in the context of cosmological models where dark energy is described by the generalized Chaplygin gas. We have exploited the cases where dark energy and dark matter are unified by the Chaplygin gas, as well as the cases where a pressureless dark component is present besides the Chaplygin gas. In all situations, however, we have taken into account the baryonic component as inferred from the primordial nucleosynthesis. We perform also an analysis, at same time analytical and numerical, of the Λ-CDM model and of a pure dark matter model (where pressureless matter is the only component of the Universe). In this sense, the goal of the present work was to try to identify special signature of each model in the energy spectrum as function of the frequency. The spectra for all models exhibit the same shape: initially the energy decreases with frequency, becomes almost constant, and then it increases with frequency. However, from the results it comes out that the Λ-CDM model presents a spectra with an amplitude greater then the pure dark matter model by a factor of 6. The Chaplygin gas model interpolates these two cases. In a typical situation, for example when $\alpha = 1$, $\bar{A} = 0.5$ and there are only Chaplygin gas and baryons, the amplitude of the spectra is about 4 times smaller than in the Λ-CDM case. As the Λ-CDM case is approached ($\bar{A} \to 1$), the amplitude decreases, but in a very low rate, exceptly at the neighborhood of $\bar{A} = 1$.

These results show that gravitational waves discriminate very poorly the different models. Somehow, this is natural since the gravitational waves equation is sensitive essentially to the background behavior, in contrast with density perturbations which depends strongly on the kind of matter content. This is more accentuated in the comparison of different generalized Chaplygin gas models, that is, models with different $\alpha$. In fact, the Chaplygin gas models are more sensitive to the parameter $\bar{A}$ than $\alpha$.

However, it must be stressed that even if the different models based on the generalized Chaplygin gas may be quite degenerate in what concerns the behaviour of gravitational waves, it is possible to distinguish them from the Λ-CDM model by analyzing the typical amplitude of the spectra. But, a complete study of the value of this amplitude asks for a complementation of the analysis here done by introducing the primordial spectrum of gravitational waves, for example, that one coming from the primordial inflationary phase, which allows to fix precisely the amplitude and the initial power spectrum \[25\].

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