Degenerations of Leibniz and anticommutative algebras

Nurlan Ismailova,b, Ivan Kaygorodovc, Yury Volkovd

a Universidade de São Paulo, IME, São Paulo, Brazil.
b Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.
c Universidade Federal do ABC, CMCC, Santo André, Brazil.
d Saint Petersburg State University, Saint Petersburg, Russia.

E-mail addresses:
Nurlan Ismailov (nurlan.ismail@gmail.com),
Ivan Kaygorodov (kaygorodov.ivan@gmail.com),
Yury Volkov (wolf86,666@list.ru).

Abstract. We describe all degenerations of three dimensional anticommutative algebras $\mathfrak{acom}_3$ and of three dimensional Leibniz algebras $\mathfrak{leib}_3$ over $\mathbb{C}$. In particular, we describe all irreducible components and rigid algebras in the corresponding varieties.

Keywords: degeneration, rigid algebra, orbits closure, anticommutative algebra, Leibniz algebra, Lie algebra

2010 Mathematics Subject Classification: 17A32, 14D06, 14L30

1. INTRODUCTION

Degenerations of algebras is an interesting subject, which was studied in various papers (see, for example, [1–3, 6–9, 12, 15–17, 20–23, 25–27, 29, 30]). In particular, there are many results concerning degenerations of algebras of low dimensions in a variety defined by a set of identities. One of important problems in this direction is the description of so-called rigid algebras. These algebras are of big interest, since the closures of their orbits under the action of the generalized linear group form irreducible components of a variety under consideration (with respect to the Zariski topology). For example, rigid algebras were classified in the varieties of low dimensional unital associative, Lie, Jordan and Leibniz algebras [19]. There are fewer works in which the full information about degenerations was found for some variety of algebras. This problem was solved for two dimensional pre-Lie algebras in [6], for three dimensional Novikov algebras in [7], for four dimensional Lie algebras in [9], for four dimensional Zinbiel algebras and nilpotent four-dimensional Leibniz algebras in [21], for nilpotent five and six dimensional Lie algebras in [15, 30], for nilpotent five and six dimensional Malcev algebras in [20], and for all two dimensional algebras in [22].

The most well known generalizations of Lie algebras are Leibniz, Malcev and binary Lie algebras. The Leibniz algebras were introduced as a non-anticommutative generalization of Lie algebras. The study of the structure theory and other properties of Leibniz algebras was initiated by Loday in [28]. An algebra $A$ is called a $Leibniz$ algebra if it satisfies the identity

$$(xy)z = (xz)y + x(yz).$$

The classification of all three dimensional Leibniz algebras can be found in [29]. Malcev algebras and binary Lie algebras are anticommutative. Gainov proved that there are no Malcev and binary Lie three dimensional algebras except Lie algebras [14]. The description of all three dimensional anticommutative algebras was given in [24] and the central extensions of three dimensional anticommutative algebras were described in [10]. In this paper we consider anticommutative algebras as a generalization of Lie algebras. Note that some steps towards a classification of all three dimensional algebras have been done in [5].

In this paper we give the full information about degenerations of three dimensional anticommutative and Leibniz algebras over $\mathbb{C}$. The vertices of this graph are the isomorphism classes of algebras in the variety under consideration. An algebra $A$ degenerates to an algebra $B$ if and only if there is a path from the
vertex corresponding to $A$ to the vertex corresponding to $B$. We also describe rigid algebras and irreducible components in the corresponding varieties.

2. Definitions and notation

All spaces in this paper are considered over $\mathbb{C}$, and we write simply $\dim$, $\text{Hom}$ and $\otimes$ instead of $\dim_{\mathbb{C}}$, $\text{Hom}_{\mathbb{C}}$ and $\otimes_{\mathbb{C}}$. An algebra $A$ is a set with a structure of a vector space and a binary operation that induces a bilinear map from $A \times A$ to $A$.

Given an $n$ dimensional vector space $V$, the set $\text{Hom}(V \otimes V, V) \cong V^* \otimes V^* \otimes V$ is a vector space of dimension $n^3$. This space has a structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1, \ldots, e_n$ of $V$. Then any $\mu \in \text{Hom}(V \otimes V, V)$ is determined by $n^3$ structure constants $c_{i,j,k} \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c_{i,j,k} e_k$. A subset of $\text{Hom}(V \otimes V, V)$ is Zariski-closed if it can be defined by a set of polynomial equations in the variables $c_{i,j,k}$ ($1 \leq i, j, k \leq n$).

Let $T$ be a set of polynomial identities. All algebra structures on $V$ satisfying polynomial identities from $T$ form a Zariski-closed subset of the variety $\text{Hom}(V \otimes V, V)$. We denote this subset by $\mathbb{L}(T)$. The general linear group $GL(V)$ acts on $\mathbb{L}(T)$ by conjugations:

$$(g \ast \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in V$, $\mu \in \mathbb{L}(T) \subset \text{Hom}(V \otimes V, V)$ and $g \in GL(V)$. Thus, $\mathbb{L}(T)$ is decomposed into $GL(V)$-orbits corresponding to the isomorphism classes of algebras. Let $O(\mu)$ denote the orbit of $\mu \in \mathbb{L}(T)$ under the action of $GL(V)$ and $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$.

Let $A$ and $B$ be two $n$ dimensional algebras satisfying identities from $T$ and $\mu, \lambda \in \mathbb{L}(T)$ represent $A$ and $B$ respectively. We say that $A$ degenerates to $B$ and write $A \to B$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of $\mu$ and $\lambda$.

If $A \not\cong B$, then the assertion $A \to B$ is called a proper degeneration. We write $A \not\to B$ if $\lambda \not\in \overline{O(\mu)}$.

Let $A$ be represented by $\mu \in \mathbb{L}(T)$. Then $A$ is rigid in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an irreducible component. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $A$ is rigid in $\mathbb{L}(T)$ if and only if $O(\mu)$ is an irreducible component of $\mathbb{L}(T)$.

We denote by $\mathfrak{A}\text{Com}_n$ the variety of $n$ dimensional anticommutative algebras and by $\mathfrak{Leib}_n$ the variety of $n$ dimensional Leibniz algebras.

We use the following notation:

1. $\text{Ann}_L(A) = \{a \in A \mid xa = 0 \text{ for all } x \in A\}$ is the left annihilator of $A$;
2. $A^{(+2)}$ is the space $\{xy + yx \mid x, y \in A\}$.

Given spaces $U$ and $W$, we write simply $U > W$ instead of $\dim U > \dim W$.

3. Methods

In the present work we use the methods applied to Lie algebras in [9, 15, 16, 30]. First of all, if $A \to B$ and $A \not\cong B$, then $\text{Der}(A) < \text{Der}(B)$, where $\text{Der}(A)$ is the Lie algebra of derivations of $A$. We will compute the dimensions of algebras of derivations and will check the assertion $A \to B$ only for such $A$ and $B$ that $\text{Der}(A) < \text{Der}(B)$. Secondly, if $A \to C$ and $C \to B$ then $A \to B$. If there is no $C$ such that $A \to C$ and $C \to B$ are proper degenerations, then the assertion $A \to B$ is called a primary degeneration. If $\text{Der}(A) < \text{Der}(B)$ and there are no $C$ and $D$ such that $C \to A$, $B \to D$, $C \not\to D$ and one of the assertions $C \to A$ and $B \to D$ is a proper degeneration, then the assertion $A \not\to B$ is called a primary non-degeneration. It suffices to prove only primary degenerations and non-degenerations describing degenerations in the variety under consideration. It is easy to see that any algebra degenerates to the algebra with zero multiplication. From now on we use this fact without mentioning it.
To prove primary degenerations, we will construct families of matrices parametrized by $t$. Namely, let $A$ and $B$ be two algebras represented by the structures $\mu$ and $\lambda$ from $L(T)$ respectively. Let $e_1, \ldots, e_n$ be a basis of $V$ and $c_{i,j}^k (1 \leq i, j, k \leq n)$ be the structure constants of $\lambda$ in this basis. If there exist $a_i^t(t) \in \mathbb{C}$ $(1 \leq i, j \leq n, t \in \mathbb{C}^*)$ such that $E_i^t = \sum_{j=1}^n a_i^t(t)e_j (1 \leq i \leq n)$ form a basis of $V$ for any $t \in \mathbb{C}^*$, and the structure constants of $\mu$ in the basis $E_i^t, E_j^t, E_k^t$ are such polynomials $c_{i,j}^k(t) \in \mathbb{C}[t]$ that $c_{i,j}^k(0) = c_{i,j}^k$, then $A \to B$. In this case $E_i^1, \ldots, E_n^t$ is called a parametrized basis for $A \to B$.

Note also the following fact. Let $B(\alpha)$ be a series of algebras parametrized by $\alpha \in \mathbb{C}$ and $e_1, \ldots, e_n$ be a basis of $V$. Suppose also that, for any $\alpha \in \mathbb{C}$, the algebra $B(\alpha)$ can be represented by a structure $\mu(\alpha) \in L(T)$ having structure constants $c_{i,j}^k(\alpha) \in \mathbb{C}$ in the basis $e_1, \ldots, e_n$, where $c_{i,j}^k(t) \in \mathbb{C}[t]$ for all $1 \leq i, j, k \leq n$. Let $A$ be an algebra such that $A \to B(\alpha)$ for $\alpha \in \mathbb{C}\setminus S$, where $S$ is a finite subset of $\mathbb{C}$. Then $A \to B(\alpha)$ for all $\alpha \in \mathbb{C}$. Indeed, if $\lambda \in L(T)$ represents $A$, then we have $\mu(\alpha) \in \{\mu(\beta)\}_{\beta \in \mathbb{C}\setminus S} \subset O(\lambda)$ for any $\alpha \in \mathbb{C}$. Thus, to prove that $A \to B(\alpha)$ for all $\alpha \in \mathbb{C}$ we will construct degenerations that are valid for all but finitely many $\alpha$.

Let us describe the methods for proving primary non-degenerations. The main tool for this is the following lemma.

**Lemma 1** ([15]). Let $B$ be a Borel subgroup of $GL(V)$ and $R \subset L(T)$ be a $B$-stable closed subset. If $A \to B$ and $A$ can be represented by $\mu \in R$ then there is $\lambda \in R$ that represents $B$.

In particular, it follows from Lemma 1 that $A \not\to B$ in the following cases:

1. $Ann_L(A) > Ann_L(B)$;
2. $A^{(+2)} < B^{(+2)}$.

In the cases where all of these criteria cannot be applied to prove $A \not\to B$, we will define $R$ by a set of polynomial equations and will give a basis of $V$, in which the structure constants of $\mu$ give a solution to all these equations. We will omit everywhere the verification of the fact that $R$ is stable under the action of the subgroup of upper triangular matrices and of the fact that $\lambda \not\in R$ for any choice of a basis of $V$. These verifications can be done by direct calculations.

If the number of orbits under the action of $GL(V)$ on $L(T)$ is finite, then the graph of primary degenerations gives the whole picture. In particular, the description of rigid algebras and irreducible components can be easily obtained. Since the variety $\text{Lie}\theta_t$ contains infinitely many non-isomorphic algebras, we have to fulfill some additional work. Let $A(\ast) := \{A(\alpha)\}_{\alpha \in I}$ be a set of algebras, and let $B$ be another algebra. Suppose that, for $\alpha \in I$, $A(\alpha)$ is represented by the structure $\mu(\alpha) \in L(T)$ and $B(\alpha) \in L(T)$ is represented by the structure $\lambda$. Then $A(\ast) \to B$ means $\lambda \in \{O(\mu(\alpha))\}_{\alpha \in I}$, and $A(\ast) \not\to B$ means $\lambda \not\in \{O(\mu(\alpha))\}_{\alpha \in I}$.

Let $A(\ast), B, \mu(\alpha) (\alpha \in I)$ and $\lambda$ be as above. To prove $A(\ast) \to B$ it is enough to construct a family of pairs $(f(t), g(t))$ parametrized by $t \in \mathbb{C}^*$, where $f(t) \in I$ and $g(t) \in GL(V)$. Namely, let $e_1, \ldots, e_n$ be a basis of $V$ and $c_{i,j}^k(1 \leq i, j, k \leq n)$ be the structure constants of $\lambda$ in this basis. If we construct $a_i^t : \mathbb{C}^* \to \mathbb{C}$ $(1 \leq i, j \leq n)$ and $f : \mathbb{C}^* \to I$ such that $E_i^t = \sum_{j=1}^n a_i^t(t)e_j$ $(1 \leq i \leq n)$ form a basis of $V$ for any $t \in \mathbb{C}^*$, and the structure constants of $f(t)$ in the basis $E_i^t, E_j^t, E_k^t$ are such polynomials $c_{i,j}^k(t) \in \mathbb{C}[t]$ that $c_{i,j}^k(0) = c_{i,j}^k$, then $A(\ast) \to B$. In this case $E_i^t, \ldots, E_n^t$ and $f(t)$ are called a parametrized basis and a parametrized index for $A(\ast) \to B$ respectively.

We now explain how to prove $A(\ast) \not\to B$. Note that if $\dim Der(A(\alpha)) > \dim Der(B)$ for all $\alpha \in I$ then $A(\ast) \not\to B$. One can use also the following generalization of Lemma 1, whose proof is the same as the proof of Lemma 1.

**Lemma 2.** Let $B$ be a Borel subgroup of $GL(V)$ and $R \subset L(T)$ be a $B$-stable closed subset. If $A(\ast) \to B$ and for any $\alpha \in I$ the algebra $A(\alpha)$ can be represented by a structure $\mu(\alpha) \in R$, then there is $\lambda \in R$ representing $B$.
4. CLASSIFICATION AND DEGENERATIONS OF THREE DIMENSIONAL ANTIMUTATIVE ALGEBRAS

First we consider the variety \( \mathfrak{ACom}_3 \). Let us fix the basis \( e_1, e_2, e_3 \) of \( V \). Any structure \( \mu \in \mathfrak{ACom}_3 \) with structure constants \( c_{kij}^s \) \((1 \leq i, j, k \leq 3)\) is determined by the \( 3 \times 3 \) matrix \( A^\mu \) whose \((i,j)\)-entry is \((-1)^{i+j}c_{i,j,\mu}^s\), where \((u,v)\) is a unique pair of numbers such that \( u, v \in \{1, 2, 3\} \setminus \{i\} \) and \( u < v \). Since \( \mathbb{C} \) is an algebraically closed field the structure \( \lambda \) belongs to \( O(\mu) \) if and only if there is a nonsingular matrix \( X \) such that \( A^\lambda = X^T A^\mu X \) by [24, Corollary 2.4]. Then the classification of three dimensional anticommutative algebras modulo isomorphism can be obtained from the classification of bilinear forms modulo congruence given in [18].

We denote by \( W \) some four dimensional space that contains \( V \) as a subspace and by \( e_4 \) some fixed vector of \( W \) such that \( W = V \oplus \mathbb{C}e_4 \). Let us now consider four dimensional algebras \( A \) such that \( A(A^2) = (A^2)A = 0 \) and \( \dim A^2 \leq 1 \). It is easy to see that such an algebra can be represented by a structure \( \chi \) on \( W \) such that \( \chi(W, W) \subset \mathbb{C}e_4 \) and \( \chi(W, e_4) = \chi(e_4, W) = 0 \). Such a structure is defined by the \( 3 \times 3 \) matrix \( B^\chi \), whose \((i,j)\)-entry is \( d_{ij}^\chi \), where \( d_{ij}^\chi (1 \leq i, j, k \leq 4) \) are the structure constants of \( \chi \). It is clear that two such structures \( \chi \) and \( \eta \) lie in the same orbit if and only if there is a nonsingular matrix \( X \) such that \( B^\eta = X^T B^\chi X \). Now we put in correspondence to an anticommutative algebra structure \( \mu \) on \( V \) the structure \( \chi_\mu \) on \( W \) satisfying the properties above with \( B^{\chi_\mu} = A^\mu \). As it was explained above, we get a bijection between orbits of \( \mathfrak{ACom}_4 \) and isomorphism classes of four dimensional algebras \( A \) such that \( A(A^2) = (A^2)A = 0 \) and \( \dim A^2 \leq 1 \). Moreover, it is clear that if \( \lambda \in O(\mu) \), then \( \chi_\lambda \in O(\chi_\mu) \).

The converse assertion follows from [15, Proposition 1.7] and the fact that the set of structures \( \chi \) on \( W \) satisfying \( \chi(W, W) \subset \mathbb{C}e_4 \) and \( \chi(W, e_4) = \chi(e_4, W) = 0 \) is a closed subset stable under the action of lower triangular matrices.

Thus, isomorphism classes and degenerations of three dimensional anticommutative algebras can be transferred from the isomorphism classes and degenerations of four dimensional algebras \( A \) such that \( A(A^2) = (A^2)A = 0 \) and \( \dim A^2 \leq 1 \). The last mentioned problem is a part of the problems that were solved in [13, 21]. Unfortunately both of the mentioned works have problems. Some degenerations are missed in the paper [13]. All degenerations between algebras that we are interested in are described correctly in [21], but the classification used in this paper lost one algebra. In the current work we will use the results of [21].

Let us first deduce the classification of three dimensional anticommutative algebras using the last mentioned paper. We do this in Table A.1 below, where in the first column we put the names of anticommutative three dimensional algebras, in the second column we put the corresponding names of algebras from [21], in the third and fourth columns we put multiplication tables and dimensions of algebras of derivations of the corresponding anticommutative algebras. We omit products of basic elements whose values are zero or can be recovered from the anticommutativity and given products. Note that if \( \dim \text{Der}(\mu) = k \), then \( \dim \text{Der}(\chi_\mu) = k + 4 \).

Here \( g_1, g_2, g_3 \) and \( g_4 \) are Lie algebras and \( A_2 \) corresponds to the algebra missed in [21] that is denoted by \( \mathfrak{N}_0 \) in this paper. We have \( g_3 \cong g_3^\alpha \) and \( A_1^\alpha \cong A_1^\alpha \) if \( \alpha \beta = 1 \) and there are no other nontrivial isomorphisms between the algebras in the table. All degenerations and non-degenerations between the algebras from the column \( \mathfrak{N} \) that do not involve \( \mathfrak{N}_0 \) are described in [21]. Thus, it remains to describe degenerations involving \( A_2 \).

Note that \( \dim \text{Ann}_L(\mathfrak{N}_0) > \dim \text{Ann}_L(\mathfrak{N}_3(0)) = \dim \text{Ann}_L(\mathfrak{N}_{30}) \) and \( \dim \text{Der}(\mathfrak{N}_0) < \dim \text{Der}(\mathfrak{N}_3(0)) = \dim \text{Der}(\mathfrak{N}_{30}) \), and hence there are no degenerations between \( \mathfrak{N}_0, \mathfrak{N}_3(0) \) and \( \mathfrak{N}_{30} \). Note now that, for any \( \alpha \in \mathbb{C} \) we have

- the degeneration \( A_2 \rightarrow g_3^\alpha \) given by the parametrized basis
  \[ E'_1 = te_3, E'_2 = te_1, E'_3 = e_1 + (\alpha + t)e_2 + e_3; \]

- the degeneration \( A_1^\alpha \rightarrow A_2 \) given by the parametrized basis
  \[ E'_1 = te_2, E'_2 = -e_1, E'_3 = \alpha e_1 - e_2 + e_3. \]

Thus, we get the following result.
Theorem 3. The graph of primary degenerations for $\mathfrak{ACom}_3$ can be obtained from the graph given in Figure 1 below by deleting all vertices named with $L$.

Since $\mathfrak{ACom}_3$ is isomorphic to $\mathbb{C}^9$ as an algebraic variety, it is irreducible and equals $O(A^0_\alpha)$

5. DEGENERATIONS OF THREE DIMENSIONAL LEIBNIZ ALGEBRAS

The classification of three dimensional non-Lie Leibniz algebras is presented in Table A.2 below.

Theorem 4. The graph of primary degenerations for $\mathfrak{Lieb}_3$ can be obtained from the graph given in Figure 1 below by deleting all vertices named with $A$.

Proof. We prove all the required primary degenerations in Table A.3 below. Let us consider the degeneration $\mathfrak{L}_t^0 \rightarrow \mathfrak{L}_t$ to clarify our formulas. Write nonzero products in $\mathfrak{L}_t^0$ in the basis $E_i^t$:

$$E_2^t E_2^t = \beta i^2 E_1^t, \quad E_3^t E_2^t = t E_1^t, \quad E_3^t E_3^t = E_1^t.$$  

It is easy to see that for $t = 0$ we obtain the multiplication table of $\mathfrak{L}_2$. The remaining degenerations can be interpreted in the same way.

A part of non-degenerations is given in Table A.4 below. Whenever an algebra named by $ACom_3$ since $\mathfrak{L}_t$ is formed by

Corollary 5. The irreducible components of $\mathfrak{Lieb}_3$ are

$$\mathfrak{C}_1 = \overline{O(\{g_1^0\} \alpha \in C)} = \{g_1, g_2, g_3^0, C^1\}_{\alpha \in C};$$

$$\mathfrak{C}_2 = \overline{O(g_4)} = \{g_1, g_2^{-1}, g_4, C^1\};$$

$$\mathfrak{C}_3 = \overline{O(\{g_2^0\} \alpha \in C)} = \{g_1, g_2^0, g_3^0, g_4, g_5, g_6^0, C^1\}_{\alpha, \beta \in C};$$

$$\mathfrak{C}_4 = \overline{O(\{g_5\} \alpha \in C)} = \{L_2, L_3, L_4, L_5, C^3\};$$

$$\mathfrak{C}_5 = \overline{O(\{L_6^0\} \alpha \in C)} = \{L_1^0, L_2, L_6^0, L_7, L_8, L_9, C^3\}_{\alpha \in C}.$$  

In particular, the set of rigid algebras in the variety $\mathfrak{Lieb}_3$ is formed by $g_4$ and $L_5$.

Proof. All degenerations and non-degenerations that do not follow directly from Theorem 4 follow from Table A.5.

\[ \square \]
FIGURE 1: THE GRAPH OF PRIMARY DEGENERATIONS FOR
LIE, ANTICOMMUTATIVE AND LEIBNIZ THREE DIMENSIONAL ALGEBRAS.
### Appendix A. Tables

#### Table A.1. Classification of three dimensional anticommutative algebras.

| A | B | Multiplication tables | Det |
|---|---|------------------------|-----|
| $g_1$ | $\mathfrak{g}_1$ | $e_2e_2 = e_1$ | 6 |
| $g_2$ | $\mathfrak{g}_2$ | $e_1e_3 = e_1$, $e_2e_3 = e_2$ | 6 |
| $g_3^{\alpha}$ | $\mathfrak{g}_3^{\alpha}$ if $\alpha = -1$, $\mathfrak{g}_3^{\alpha} \left( \frac{\alpha}{1+\alpha^2} \right)$ otherwise | $e_1e_3 = e_1 + e_2$, $e_2e_3 = \alpha e_2$ | 4 |
| $g_4$ | $\mathfrak{g}_4(0)$ | $e_1e_2 = e_3$, $e_1e_3 = -e_2$, $e_2e_3 = e_1$ | 3 |
| $A_1^{\alpha}$ | $\mathfrak{a}_1^{\alpha}$ if $\alpha = -1$, $\mathfrak{a}_1^{\alpha}$ if $\alpha = 1$, $\mathfrak{a}_1^{\alpha} \left( \frac{\alpha}{1+\alpha^2} \right)$ otherwise | $e_1e_2 = e_3$, $e_1e_3 = e_1 + e_3$, $e_2e_3 = \alpha e_2$ | 1 |
| $A_2$ | $\mathfrak{a}_2$ | $e_1e_2 = e_1$, $e_2e_3 = e_2$ | 2 |
| $A_3$ | $\mathfrak{a}_3$ | $e_1e_2 = e_3$, $e_1e_3 = e_1$, $e_2e_3 = e_2$ | 3 |

#### Table A.2. Classification of three dimensional Leibniz (non-Lie) algebras.

| A | $S_\mu$ | Multiplication tables | Det |
|---|---|------------------------|-----|
| $g_1^{(1)}$ | 2(a) | $e_2e_2 = 3e_1$, $e_3e_2 = e_1$, $e_3e_3 = e_1$ | 4 |
| $g_2$ | 2(b) | $e_3e_3 = e_1$ | 5 |
| $\mathfrak{l}_1$ | 2(c) | $e_2e_2 = e_1$, $e_3e_3 = e_1$ | 4 |
| $g_3^{\alpha}$ | 2(e), 2(f) | $e_1e_3 = \alpha e_1$, $e_2e_3 = e_2$, $e_3e_2 = -e_2$, $e_2e_3 = e_1$ | 3 |
| $g_4$ | 2(g) | $e_1e_3 = 2e_1$, $e_2e_2 = e_1$, $e_2e_3 = e_2$, $e_3e_2 = -e_2$, $e_3e_3 = e_1$ | 2 |
| $g_5^{\alpha, \beta}$ | 2(d), 3(a) | $e_1e_3 = e_1$, $e_2e_3 = e_3$ | $2 + \delta_{\alpha,0} + 2\delta_{\alpha,1}$ |
| $g_6^{\beta}$ | 3(b) | $e_1e_3 = e_1$, $e_2e_3 = e_2$ | 2 |
| $g_7^{\alpha}$ | 3(c) | $e_1e_3 = e_2$, $e_3e_3 = e_1$ | 3 |
| $g_8^{\alpha}$ | 3(d) | $e_1e_3 = e_2$, $e_2e_3 = e_2$, $e_3e_3 = e_1$ | 2 |

#### Table A.3. Degenerations of Leibniz algebras of dimension 3.

| Degenerations | Parametrized bases |
|---------------|--------------------|
| $g_1^{(1)} \rightarrow g_2$ | $E_1^2 = e_1$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_1$, $E_2^2 = \frac{t}{1+\alpha^2}e_1 + \frac{1}{1+\alpha^2}e_3$, $E_3^2 = \frac{1}{1+\alpha^2}e_2 + te_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = -t^{-1}e_1 + te_2$, $E_2^2 = t^{-1}e_1$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = te_1$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_2$, $E_2^2 = e_1$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_1$, $E_2^2 = \frac{t}{1+\alpha^2}e_1 + \frac{1}{1+\alpha^2}e_3$, $E_3^2 = \frac{1}{1+\alpha^2}e_2 + te_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = -t^{-1}e_1 + te_2$, $E_2^2 = t^{-1}e_1$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = te_1$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_2$, $E_2^2 = e_1$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_1$, $E_2^2 = e_1 + e_2$, $E_3^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_1$, $E_2^2 = e_1 + e_2$, $E_3^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = t^2e_1$, $E_2^2 = e_1 + e_2$, $E_3^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
| $g_1^{(1)} \rightarrow g_1^{(1)}$ | $E_1^2 = e_1 + e_2$, $E_2^2 = te_2$, $E_3^2 = e_3$ |
Table A.4. Non-degenerations of Leibniz algebras of dimension 3.

| Non-degenerations | Reasons |
|-------------------|---------|
| $L_4 \nRightarrow L_3 \cup \{L_4^{\alpha \neq 2}\}$ | $\mathcal{R} = \begin{cases} \mathcal{L} & A = (f_1, f_2, f_3), c_{11} = 0, c_{12} = -c_{12}' \\ \mathcal{L}_1 & A_1 + A_2 + A_3 \subseteq A_1, A_1 \subseteq A_2 \end{cases}$ |
| $L_5 \nRightarrow L_3 \cup \{L_5^\beta, L_5^{\alpha \neq 0}\}$ | $\mathcal{L}_5 \in \Omega(\mathcal{R})$ (take $f_1 = e_3, f_2 = e_2, f_3 = e_1$), but $\mathcal{B} \notin \Omega(\mathcal{R})$ |
| $L_6 \leftrightarrow L_4, L_2 \in \{L_4^\alpha, L_2^\beta\}$ | $Ann_L(\mathcal{B}) > Ann_L(\mathcal{B})$ |

Table A.5. Orbit closures for some families of three dimensional Leibniz algebras.

| Degenerations | Parametrized bases | Indices |
|---------------|-------------------|---------|
| $L_4^\gamma \leftrightarrow L_6^\alpha$ | $E_1 = e_2$, $E_3 = e_1$, $E_3 = e_3$ | $\epsilon = t^{-1}$ |
| $L_4^\gamma \leftrightarrow L_7$ | $E_1 = e_1 + e_2$, $E_3 = e_2$, $E_3 = e_3$ | $\epsilon = 1 - t$ |
| $L_4^\gamma \leftrightarrow L_9$ | $E_1 = e_1 + e_2$, $E_3 = (1 - t)e_1$, $E_3 = e_1 + e_2 + te_3$ | $\epsilon = t^{-1}$ |
| $L_4^\gamma \leftrightarrow L_3$ | $E_1 = t^2e_1$, $E_3 = t^2e_2$, $E_3 = t^3e_3$ | $\epsilon = t^{-2}$ |

| Non-degenerations | Reasons |
|-------------------|---------|
| $L_4^\gamma \leftrightarrow L_5^{\beta, \gamma} \in \{L_5^\beta, L_5^{\alpha \neq 0}\}$ | $(L_4^\gamma)^{t^{2}} < L_5^{(t^{2})}$ |
| $L_4^\gamma \leftrightarrow L_2, L_3 \in \{L_2^\beta, L_3^{\alpha \neq 0}\}$ | $Ann_L(L_4^\gamma) > Ann_L(\mathcal{B})$ |
| $L_6^\alpha \leftrightarrow L_2, L_3 \in \{L_2^\beta, L_3^{\alpha \neq 0}\}$ | $Ann_L(L_6^\alpha) > Ann_L(\mathcal{B})$ |

References

[1] Alvarez M.A., On rigid 2-step nilpotent Lie algebras, Algebra Colloquium, 25 (2018), 2, 349–360.
[2] Alvarez M.A., The variety of 7-dimensional 2-step nilpotent Lie algebras, Symmetry, 10 (2018), 1, 26.
[3] Alvarez M.A., Hernández I., On degenerations of Lie superalgebras, Linear and Multilinear Algebra, DOI: 10.1080/03081087.2018.1498060
[4] Alvarez M.A., Hernández I., Kaygorodov I., Degenerations of Jordan superalgebras, Bulletin of the Malaysian Mathematical Sciences Society, 43 (2020), DOI: 10.1007/s40840-018-0664-3
[5] Bekbaev U., Complete classification of a class of 3-dimensional algebras, arXiv:1710.10734
[6] Benes T., Burde D., Degenerations of pre-Lie algebras, Journal of Mathematical Physics, 50 (2009), 11, 112102, 9 pp.
[7] Benes T., Burde D., Classification of orbit closures in the variety of three dimensional Novikov algebras, Journal of Algebra and Its Applications, 13 (2014), 2, 1350081, 33 pp.
[8] Burde D., Degenerations of nilpotent Lie algebras, Journal of Lie Theory, 9 (1999), 1, 193–202.
[9] Burde D., Steinhoff C., Classification of orbit closures of 4–dimensional complex Lie algebras, Journal of Algebra, 214 (1999), 2, 729–739.
[10] Calderón A., Fernández Ouaridi A., Kaygorodov I., The classification of $n$–dimensional anticommutative algebras with $(n - 3)$–dimensional annihilator, Communications in Algebra, 47 (2019), DOI:10.1080/00927872.2018.1468909
[11] Casas J., Insua M., Ladra M., Ladra S., An algorithm for the classification of 3-dimensional complex Leibniz algebras, Linear Algebra and Its Applications, 436 (2012), 9, 3747–3756.
[12] Casas J., Khudoyberdiyev A., Ladra M., Omirov B., On the degenerations of solvable Leibniz algebras, Linear Algebra and Its Applications, 439 (2013), 2, 472–487.
[13] Fialowski A., Penkava M., The moduli space of 4-dimensional nilpotent complex associative algebras, Linear Algebra and Its Applications, 457 (2014), 408–427.
9

[14] Gainov A., Binary Lie algebras of lower ranks [Russian], Algebra i Logika Sem., 2 (1963), 4, 21–40.
[15] Grunewald F., O’Halloran J., Varieties of nilpotent Lie algebras of dimension less than six, Journal of Algebra, 112 (1988), 315–325.
[16] Grunewald F., O’Halloran J., A Characterization of orbit closure and applications, Journal of Algebra, 116 (1988), 163–175.
[17] Grunewald F., O’Halloran J., Deformations of Lie algebras, Journal of Algebra, 162 (1993), 1, 210–224.
[18] Horn R., Sergeichuk V., Canonical matrices of bilinear and sesquilinear forms, Linear Algebra and its Applications, 428 (2008), 1, 193–223.
[19] Ismailov N., Kaygorodov I., Volkov Yu., The geometric classification of Leibniz algebras, International Journal of Mathematics, 29 (2018), 5, 1850035.
[20] Kaygorodov I., Popov Yu., Volkov Yu., Degenerations of binary Lie and nilpotent Malcev algebras, Communications in Algebra, 46 (2018), 11, 4929–4941.
[21] Kaygorodov I., Popov Yu., Pozhidaev A., Volkov Yu., Degenerations of Zinbiel and nilpotent Leibniz algebras, Linear and Multilinear Algebra, 66 (2018), 4, 704–716.
[22] Kaygorodov I., Volkov Yu., The variety of 2-dimensional algebras over an algebraically closed field, Canadian Journal of Mathematics, (2018), DOI: 10.4153/S0008414X18000056
[23] Kaygorodov I., Volkov Yu., Complete classification of algebras of level two, Moscow Mathematical Journal, (2018), to appear, arXiv:1710.08943
[24] Kobayashi Yu., Shirayanagi K., Takahasi S., Tsukada M., Classification of three dimensional zero-potent algebras over an algebraically closed field, Communications in Algebra, 45 (2017), 12, 5037–5052.
[25] Khudoyberdiyev A., Omirov B., The classification of algebras of level one, Linear Algebra and its Applications, 439 (2013), 11, 3460–3463.
[26] Khudoyberdiyev A., Ladra M., Masutova K., Omirov B., Some irreducible components of the variety of complex \((n + 1)\)-dimensional Leibniz algebras, Journal of Geometry and Physics, 121 (2017), 228–246.
[27] Lauret J., Degenerations of Lie algebras and geometry of Lie groups, Differential Geometry and its Applications, 18 (2003), 2, 177–194.
[28] Loday J.-L., Pirashvili T., Universal enveloping algebras of Leibniz algebras and (co)homology, Mathematische Annalen, 296 (1993), 1, 139–158.
[29] Rakhimov I., Mohd Atan K., On contractions and invariants of Leibniz algebras, Bulletin of the Malaysian Mathematical Sciences Society, 35 (2012), 557–565.
[30] Seeley C., Degenerations of 6-dimensional nilpotent Lie algebras over \(\mathbb{C}\), Communications in Algebra, 18 (1990), 3493–3505.