Tracer-particle dynamics in MHD fluids

Massimo Tessarotto, Claudio Asci, Alessandro Soranzo, and Gino Tirone
Department of Mathematics and Informatics, Trieste University, Trieste, Italy

Claudio Cremaschini
International School for Advanced Studies (SISSA) and INFN, Trieste, Italy

Marco Tessarotto
Civil Protection Agency, Regione Friuli Venezia-Giulia, Palmanova (Udine), Italy

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A key issue in fluid dynamics is the unique definition of the phase-space Lagrangian dynamics characterizing prescribed ideal fluids (i.e., continua), which is related to the dynamics of so-called ideal tracer particles (ITP) moving in the same fluids. These are by definition particles of infinitesimal size which do not produce significant perturbations of the fluid fields and do not interact among themselves. For Navier-Stokes (NS) fluids, the discovery by Tessarotto et al. (2005-2009) of the phase-space dynamical system advancing in time the state of the fluid, has made possible, in the case NS fluids, the actual definition of these trajectories. In this paper we intend to pose the problem in the case of compressible/incompressible magnetofluids based on the inverse kinetic theory which can be developed for their phase-space statistical description (see also accompanying paper). We propose the conjecture of the existence of a subset of ITP’s (i.e., particular solutions of the phase-space dynamical system), denoted as thermal ideal tracer particles (TITP). These particles are characterized by a relative velocity with respect to the fluid, whose magnitude is determined, by the kinetic pressure (in turn, related to the fluid pressure).

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I. INTRODUCTION

In this paper we report the discovery of a subset of so-called ideal tracer particles (ITP) belonging to Navier-Stokes (NS) fluids which are denoted as thermal tracer particles (TTP). Their states are found to be uniquely dependent on the local state of the fluid. The result applies to NS fluids described as mesoscopic, i.e., continuous fluids, which can be either viscous or inviscid, compressible or incompressible, thermal or isothermal, isentropic or non-isentropic. We shall assume that the state of these fluids is represented by an ensemble of observables \( \{Z(r, t)\} \) \( i = 1, \ldots, n \) (with \( n \) and integer \( \geq 1 \)), i.e., fluid fields, which can be unambiguously prescribed as continuous and suitably smooth functions, respectively, in \( \Omega \times I \) and in the open set \( \Omega \times I \). We intend to show that, as a remarkable consequence, the phase-space dynamical system which advances in time (the states of) these particles can be uniquely prescribed in such a way to determines self-consistently the time evolution of the complete set of fluid equations characterizing the fluid. This implies that TTPs must reproduce exactly the dynamics of the fluid. In other words, by means of an appropriate statistical averages on the ensemble of the TTPs, it is possible to determine the time-evolution of the fluid state, in such a way that it satisfies identically the required set of fluid equations.

A. Lagrangian dynamics of ideal tracer particles

A key aspect of fluid dynamics is the proper definition of the phase-space Lagrangian dynamics for continuous fluid systems, whereby possibly all the fluid fields characterizing the actual fluid state \( \{Z(r, t)\} \) can be identified with suitable statistical averages on appropriate ensembles of (fictitious) particles. This refers, in particular, to the phase-space dynamics of so-called ideal tracer particles (ITPs), namely rigid extended classical particles immersed in the fluid, all having the same support and infinitesimal size such that during their motion they do not
mutually interact and do not perturb the state of the fluid. Depending on their inertial mass $m_P$, ITPs can belong to different species of particles; thus, in general, their mass can differ from that of the corresponding displaced fluid element $m_F$. On the other hand, ITPs carrying the mass $m_P \equiv m_F$ will be denoted as the NS ideal tracer particles (NS-ITPs). In the following, in order to characterize the Lagrangian dynamics of NS fluids, ITPs will be identified only with NS-ITPs. In this framework, it follows that ITPs can undergo, by assumption, solely “unary” interactions with external force-fields and with the continuum fluid. Namely, in both cases they are subject only to the action of a continuum mean-field acceleration, which can depend solely on the states of the particle and of the fluid, i.e., is of the form $F = F(x, t)$. As a consequence, ITPs can be treated as Newtonian point-like particles characterized by a Newtonian state $x = (r, v)$ spanning the phase-space $\Gamma \equiv \Omega \times U$, with the position $r$ and the kinetic velocity $v$ belonging respectively to the configuration space of the fluid $\Omega$ (in the following to be identified with a bounded subset of $\mathbb{R}^3$) and the velocity space $U \equiv \mathbb{R}^3$.

B. The Navier-Stokes dynamical system

It is possible to prove that the state $x$ of a generic ITP advances in time by means of a Newtonian classical dynamical system (DS). Contrary to a widespread misconception such a DS is finite-dimensional, i.e., it is characterized by a finite degree of freedom. In fact, the DS can be prescribed in terms of a, generally non-unique, vector field of the form $X(x, t) \equiv \left\{ v, \frac{1}{m_P}K = F \right\}$, with $\frac{1}{m_P}K \equiv F$ a suitable mean-field acceleration. This is identified with the flow $(T_{t,o})$ generated by the initial value problem associated to the deterministic equations of motion (Newton’s equations)

$$\begin{cases}
\frac{dx}{dt} = X(x, t), \\
x(t_o) = x_o.
\end{cases}$$

(1)

Such flow is referred to as a Navier-Stokes dynamical system (NS–DS) and is a homeomorphism in $\Gamma$ with existence domain $\Gamma \times I$, of the type

$$T_{t,o,t} : x_o \rightarrow x(t) = T_{t,o}x_o,$$

(2)

with $t \in I \subseteq \mathbb{R}$, $T_{t,o}$ being a measure-preserving evolution operator associated to $X(x, t)$. Thus, by definition, the NS-DS is uniquely prescribed by the couple $\{x, X(x, t)\}$, with $x(t) \equiv x = (r, v)$ to be identified with the instantaneous state of a generic ITP.

II. GEDANKEN EXPERIMENT

For a prescribed continuous fluid system, such as a compressible/incompressible NS thermonfluid, the problem arises whether there might exist a subset of the ensemble of ideal tracer particles, i.e., of the dynamical system (2), such that their Newtonian state and corresponding time evolution depend only on the state of the fluid $\{Z\}$.

We argue that it should be possible to prove the existence of the TTPs by performing a conceptual experiment (Gedanken experiment) on the fluid, i.e., looking at the properties of the IKT-statistical models $\{f, \Gamma\}$. The conjecture is suggested by the following arguments:

- The state of the fluid is solely dependent of the fluid fields, which in the case of a compressible NS thermonfluid can be identified with the set $\{Z_1\}$.
- The time-evolution of $\{Z\}$ is necessarily independent of the KDF $f(x, t)$ and of the NS-DS (2).
- On the other hand, the time evolution of the fluid fields is also generated by the Lagrangian IKE in terms of the NS-DS (2).

Here we conjecture that the ITPs belonging to such a subset should fulfill the following properties:

1. GDE-requirement $\#1$: their time evolution should be, at all times $t \in I$, independent of the particular form of the KDF $f(x, t)$. As a consequence, for them the form of the mean-field force $F$ should be independent of the KDF $f(x, t)$ [introduced in the IKT-statistical model $\{f, \Gamma\}$];

2. GDE-requirement $\#2$: for prescribed initial conditions, their Newtonian states $x(t) \equiv x = (r, v)$, and equivalently also $y(t) \equiv y = (r, u)$, should depend solely on the fluid fields $\{Z_1\}$.

In addition, one should expect that for all TTPs:
3. **GDE-requirement #3 - Local magnitude of** \( \mathbf{u}(t) \): the magnitude of their instantaneous relative velocity \( |\mathbf{u}(t)| \equiv |\mathbf{u}(\mathbf{r}, t)| \) remains at all times \( t \in I \) proportional to the local thermal velocity \( v_{th}(\mathbf{r}, t) \), i.e., of the form

\[
|\mathbf{u}(t)| = \beta v_{th}(\mathbf{r}, t),
\]

with \( p_1(\mathbf{r}, t) > 0, \mathbf{r} \equiv \mathbf{r}(t) \) and \( \beta \) denoting respectively the kinetic pressure (see accompanying paper [14]), the instantaneous position of the same particle and an appropriate non-vanishing constant, i.e., a function independent of \( (\mathbf{r}, t) \);

4. **GDE-requirement #4 - Kinetic constraint on the local direction of** \( \mathbf{u}(t) \): Let us introduce for \( \mathbf{u}(t) \) the representation

\[
\mathbf{u}(t) = \beta v_{th}(\mathbf{r}, t) \mathbf{n}(\mathbf{r}, t),
\]

with \( \mathbf{n}(\mathbf{r}, t) \) the unit vector proscribing the local direction of \( \mathbf{u}(t) \). Then \( \mathbf{n}(\mathbf{r}, t) \) should satisfy the kinetic constraint:

\[
\mathbf{n}(\mathbf{r}, t) \cdot \nabla \hat{\mathbf{p}}_1(\mathbf{r}, t) = 0.
\]

In fact, for a non-uniform kinetic pressure satisfying locally

\[
\nabla \hat{\mathbf{p}}_1(\mathbf{r}, t) \neq 0,
\]

the requirement [3] cannot generally be met unless the relative velocity \( \mathbf{u}(t) \) remains tangent to the local isobaric surface \( \hat{\mathbf{p}}_1(\mathbf{r}, t) = \text{const.} \). As a consequence, the direction of \( \mathbf{u}(t) \) is necessarily uniquely determined, once the initial conditions [1] and consequently its initial direction

\[
\mathbf{n}(\mathbf{r}, t_o) \equiv \mathbf{n}_o(\mathbf{r})
\]

have been set. Hence, in validity of [5] the unit vector \( \mathbf{n}(\mathbf{r}, t) \) must be orthogonal to the unit vector

\[
b(\mathbf{r}, t) = \frac{\nabla \hat{\mathbf{p}}_1(\mathbf{r}, t)}{|\nabla \hat{\mathbf{p}}_1(\mathbf{r}, t)|},
\]

i.e., the kinetic constraint

\[
\mathbf{n}(\mathbf{r}, t) \cdot b(\mathbf{r}, t) = 0
\]

must hold identically for all \((\mathbf{r}, t) \in \Omega \times I\).

5. **GDE-requirement #5 - Time evolution of** \( \mathbf{n}(\mathbf{r}, t) \): the unit vector \( \mathbf{n}(\mathbf{r}, t) \) satisfies an initial-value problem of the form

\[
\begin{aligned}
\frac{dn(\mathbf{r}, t)}{dt} &= \Omega(\mathbf{r}, t) \times \mathbf{n}(\mathbf{r}, t), \\
\mathbf{n}(\mathbf{r}(t_o), t_o) &= \mathbf{n}(\mathbf{r}_o, t_o),
\end{aligned}
\]

with \( \Omega(\mathbf{r}, t) \) denoting a suitable pseudo-vector. Without loss of generality we shall require that: 1) \( \Omega(\mathbf{r}, t) \) is a smooth real vector function defined in \( \Omega \times I \); 2) \( \Omega(\mathbf{r}, t) \) is defined also in the limit \( p_1(\mathbf{r}, t) \rightarrow 0^+ \).

An interesting issue concerns the physical interpretation TTP dynamics, and in particular the evolution equation for the unit vector \( \mathbf{n}(\mathbf{r}, t) \) [i.e., the direction of the particle relative velocity; see figure 1 and Eqs. (10)] and related the pseudovector \( \Omega(\mathbf{r}, t) \). In fact, that Eqs. (10) are analogous to those of a rigid body rotating with angular velocity \( \Omega_r = -\Omega(\mathbf{r}, t) \). This suggests that \( \Omega(\mathbf{r}, t) \) should be related to the local fluid vorticity \( \xi \equiv \nabla \times \mathbf{V} \). Indeed denoting by \( \frac{Db(\mathbf{r}, t)}{Dt} \equiv \frac{\partial b(\mathbf{r}, t)}{\partial t} + \mathbf{V}(\mathbf{r}, t) \cdot \nabla \) the fluid convective derivative, it follows

\[
\Omega(\mathbf{r}, t) \equiv b(\mathbf{r}, t) \times \frac{db(\mathbf{r}, t)}{dt} = b(\mathbf{r}, t) \times \frac{Db(\mathbf{r}, t)}{Dt} + b(\mathbf{r}, t) \times (\mathbf{u} \cdot \nabla) b(\mathbf{r}, t)
\]

where

\[
b(\mathbf{r}, t) \times (\mathbf{u} \cdot \nabla) b(\mathbf{r}, t) =
\]

\[
= -\xi \cdot \left[ 1 - bb \right] + \frac{1}{|\nabla p_1|} \left[ b \times \nabla (\nabla p_1 \cdot V) - b \times (\nabla p_1 \cdot \nabla) V \right].
\]

In the last equation the first term on the r.h.s. of denotes the tangential component of the vorticity (i.e., the component belonging to the local tangential plane w.r. to a local isobaric surface \( \hat{p}_1 = \text{const.} \)). This means that near a vortex the motion of TTPs is qualitatively similar to that of a rotating rigid body. However, by inspection of the remaining terms in Eqs. (11) and (12), it is evident that more complex particle-acceleration effects may be present, which are driven by time-dependent pressure and velocity-gradients contributions.
III. CONCLUSIONS

A fundamental issue for Navier-Stokes fluids, is their characterization in terms of the dynamics of ideal tracer particles (ITPs). Based on the formulation of an inverse kinetic theory for compressible/incompressible NS thermofluids (THM.1), in this paper the existence of a subset of ITPs denoted as TTPs whose states depend solely on the state of the fluid has been conjectured. A detailed proof of the statement will be reported elsewhere \[15\].

Applications of the present theory are in principle several, and deal, in particular, with the dynamics of small particles (such as solid particles or droplets, commonly found in natural phenomena and industrial applications) in compressible/incompressible thermofluids. The accurate description of particle dynamics, as they are pushed along erratic trajectories by fluctuations of the fluid fields, is essential, for example, in combustion processes, in the industrial production of nanoparticles as well as in atmospheric pollutant transport, cloud formation and air-quality monitoring of the atmosphere.

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[18] This means that a suitable statistical ensemble of TTPs should reproduce exactly the dynamics of the fluid. In other words, it should be possible to determine the fluid fields characterizing the fluid state by means of suitable statistical averages on the ensemble of TTPs so that they satisfy identically the required set of fluid equations.
[19] Thus, for example, in the case of an incompressible NS fluid, this would require to represent both the fluid velocity $\mathbf{V}(\mathbf{r}, t)$ and the fluid pressure $p(\mathbf{r}, t)$ in terms of suitable statistical averages of an appropriate probability density. This goal can be realized by means of the inverse kinetic theory (IKT) developed in Refs. \[1, 3\].