Abstract. We show that for any monoid $M$, the family of languages accepted by $M$-automata (or equivalently, generated by regular valence grammars over $M$) is completely determined by that part of $M$ which lies outside the maximal ideal. In particular, every such family arises as a family of languages accepted by $N$-automata where $N$ is a simple or 0-simple monoid. A consequence is that every such family is either exactly the class of regular languages, contains all the blind one-counter languages, or is the family of languages accepted by $G$-automata where $G$ is a non-locally-finite torsion group.

We then consider a natural extension of the usual definition which permits the automaton to utilise more of the structure of each monoid, and additionally allows us to define $S$-automata for $S$ an arbitrary semigroup. In the monoid case, the resulting automata are equivalent to the valence automata with rational target sets which arise in the theory of regulated rewriting systems. We study these automata in the case that the register semigroup is completely simple or completely 0-simple, obtaining a complete characterisation of the classes of languages corresponding to such semigroups, in terms of their maximal subgroups. In the process, we obtain a number of results about rational subsets of Rees matrix semigroups which are likely to be of independent interest.

1. Introduction

Recently there has been increasing interest in finite automata augmented with a memory register which stores at any moment in time an element of a given monoid $M$ (or group $G$). The register is initialised with the identity element of the monoid, modified by right multiplication by monoid elements, and a word is accepted if and only if some computation reading the word returns the register to the identity with the finite state control in an accepting state. These automata are of considerable interest both to algebraists (who know them as blind monoid automata or $M$-automata) and to computer scientists (to whom they are often extended finite automata or valence automata). On the one hand, they provide algebraic characterizations of important language classes such as the context free languages, the recursively enumerable languages and the blind and partially blind counter languages [7, 13, 16, 19, 21]. On the other, they give insights into computational problems in group and monoid theory [4, 8, 13]. They are also closely related to regulated rewriting systems, and in particular to the valence grammars
introduced by Paun [18]; the languages accepted by $M$-automata are exactly the languages generated by regular valence grammars over $M$ [6].

While $M$-automata with $M$ a monoid appear at first sight to provide much more flexibility than their group counterparts, the extent to which such an automaton can fully utilise the structure of the register monoid is somewhat limited. Indeed, if the register ever contains an element of a proper ideal, then no sequence of actions of the automaton can cause it to contain the identity again; thus, the automaton has entered a “fail” state from which it can never accept a word. It follows that the automaton can make effective use of only that part of the monoid which does not lie in a proper ideal. This observation will be formalised below in Section 5 where we show that every $M$-automaton is equivalent to an $N$-automaton where $N$ is a simple or 0-simple monoid.

A natural way to circumvent this is to weaken the requirement that the identity element be the sole initial and accepting configuration of the register, and instead permit more general sets of initial and terminal configurations. Permitting more general terminal sets was suggested by Gilman [7] and the idea has also recently appeared in the study of regulated rewriting systems, where the introduction of valence grammars with target sets leads naturally to a corresponding notion of valence automata with target sets [5] [6]. An additional advantage of allowing more general initial and accepting sets is that we remove entirely the special role played by the identity, and hence the very need for identity. The resulting automata are thus not constrained to have register monoids, but instead can use arbitrary semigroups.

If we are to retain the advantages of monoid automata as an elegant and easily manipulated way of describing important language classes, it is clearly necessary to place some kind of restriction on the class of subsets permitted for initial and terminal configurations. Possible choices include the finite subsets or the finitely generated subsemigroups, but from a computational perspective, the most natural choice seems to be the more general rational subsets of the semigroup. These sets, which have been the subject of intensive study by both mathematicians and computer scientists (see for example [1] [15] [19] [22] [23]), are general enough to significantly increase the descriptive power of the automata, while remaining sufficiently well-behaved to permit the development of a meaningful theory.

The objective of this paper, then, is to begin the systematic study of finite automata augmented with a semigroup register with rational initial and accepting subsets. In Section 2, we briefly recall some elementary definitions from the theories of formal languages, monoid automata and rational subsets. In Section 3 we motivate the more general constructions which follow, by exhibiting a number of limitations on the capability of conventional $M$-automata to describe language classes.

In Section 4, we formally define rational semigroup automata, and obtain some foundational results about these automata and the classes of languages which they define. In Section 5, we turn our attention to Rees matrix constructions. We first study the relationship between rational subsets and Rees
matrix constructions, obtaining a number of results which may be of independent interest. Some of these results are then combined with a classical theorem of Rees [20] to yield a complete description of the classes of languages accepted by rational $S$-automata whenever $S$ is a completely simple or completely 0-simple semigroup.

2. Preliminaries

Firstly, we recall some basic ideas from formal language theory. Let $\Sigma$ be a finite alphabet. We denote by $\Sigma^*$ the set of all words over $\Sigma$ and by $\epsilon$ the empty word. Under the operation of concatenation and with the neutral element $\epsilon$, $\Sigma^*$ forms the free monoid on $\Sigma$. The set $\Sigma^* \setminus \{\epsilon\}$ of non-empty words forms a subsemigroup of $\Sigma^*$, called the free semigroup $\Sigma^+$ on $\Sigma$. A finite automaton over $\Sigma^*$ is a finite directed graph with each edge labelled with an element of $\Sigma$, and with a distinguished initial vertex and a set of distinguished terminal vertices. A word $w \in \Sigma^*$ is accepted by the automaton if there exists a directed path connecting the initial vertex with some terminal vertex labelled cumulatively with $w$. The set of all words accepted by the automaton is denoted $L$ or for an automaton $A$ sometimes $L(A)$, and is called the language accepted by $A$. A language accepted by a finite automaton is called rational or regular.

More generally, if $S$ is a semigroup then a finite automaton over $S$ is a finite directed graph with each edge labelled with an element of $S$, and with a distinguished initial vertex and a set of distinguished terminal vertices. An element $s \in S$ is accepted by the automaton if there exists some directed path connecting the initial vertex to some terminal vertex, the product of whose edge labels is $s$. If $S$ is a monoid then we admit a unique empty path at each vertex with label the identity element; otherwise we consider only non-empty paths. The subset accepted is the set of all elements accepted; a subset of $S$ which is accepted by some finite automaton is called a rational subset. The rational subsets of $S$ are exactly the homomorphic images in $S$ of regular languages.

We shall require the following result about rational subsets of groups, which is well-known and easy to prove.

**Proposition 2.1.** Let $G$ be a group. If $X \subseteq G$ is rational then the subset $X^{-1} = \{x^{-1} \mid x \in X\}$ is also rational.

Next we recall the usual definition of a monoid automaton. Let $M$ be a monoid with identity $1$ and let $\Sigma$ be an alphabet. An $M$-automaton over $\Sigma$ is a finite automaton over the direct product $M \times \Sigma^*$. We say that it accepts a word $w \in \Sigma^*$ if it accepts $(1, w)$, that is if there exists a path connecting the initial vertex to some terminal vertex labelled $(1, w)$. Intuitively, we visualize an $M$-automaton as a finite automaton augmented with a memory register which can store an element of $M$; the register is initialized to the identity element, is modified by right multiplication by element of $M$, and for a word to be accepted the element present in the memory register on completion must be the identity element. We write $F_1(M)$ for the class of all languages accepted by $M$-automata, or equivalently for the class of languages generated by regular valence grammars over $M$ [6].
3. Monoid Automata and Simple/0-Simple Semigroups

The aim of this section is to show that the extent to which an $M$-automaton can make use of the structure of a general monoid $M$ is severely limited. In particular, we formally justify and consider the consequences of our intuitive observation, made in the introduction, that a monoid automaton uses only that part of the monoid which does not lie in a proper ideal.

Recall that an **ideal** $I$ of a semigroup (or monoid) $S$ is a subset $I$ of $S$ with the property that $S^1 IS^1 \subseteq I$, where $S^1$ denotes $S$ with an identity element adjoined. To each ideal $I$ is associated a congruence $\rho_I$ on $S$ such that $(s, t) \in \rho_I$ if and only if either $s,t \in I$ or $s = t$. The quotient semigroup $S/\rho_I$, usually just denoted $S/I$, is called a **Rees quotient**, and takes the form

$$S/I = \{I\} \cup \{\{x\} \mid x \in S \setminus I\}.$$ 

It is usual to identify $\{x\}$ with $x$ for each $x \in S \setminus I$; the element $I$, which is easily seen to be a zero element in $S/I$, is often denoted 0.

**Proposition 3.1.** Let $I$ be a proper ideal of a monoid $M$. Then $F_1(M) = F_1(M/I)$.

**Proof.** Suppose $L \in F_1(M)$, and let $A$ be an $M$-automaton accepting $L$. First notice that any path containing an edge of the form $(x, w)$ with $x \in I$ will itself have label with first component in $I$; in particular, since $I$ is a proper ideal, $1 \notin I$ and such a path cannot be an accepting path. It follows that we may remove any such edges without changing the language accepted, and so that we may assume without loss of generality that $A$ has no such edges. Now for any $x_1, \ldots, x_n \in M \setminus I$, it follows from the definition of $M/I$ that $x_1 \ldots x_n = 1$ in $M$ if and only if $\{x_1\} \ldots \{x_n\} = \{1\}$ in $M/I$. Now if we let $B$ be the $(M/I)$-automaton obtained from $A$ by replacing edge labels of the form $(x, w)$ with $(\{x\}, w)$, it follows easily that $A$ has a path from the initial vertex to a terminal vertex labelled $(1, w)$ if and only if $B$ has a path from the initial vertex to a terminal vertex labelled $(\{1\}, w)$. Thus, $B$ accepts the language $L$ and $L \in F_1(M/I)$.

Conversely, if $L \in F_1(M/I)$ then $L$ is accepted by some $(M/I)$-automaton; by an argument akin to that above, we may assume without loss of generality that $B$ has no edges labelled by the zero element $I$. We now obtain from $B$ a new $M$-automaton $A$ by replacing edges labels of the form $(\{x\}, w)$ with $(x, w)$, and argue as above to show that $A$ accepts exactly the language $L$, so that $L \in F_1(M)$. \hfill $\Box$

A semigroup or monoid is called **simple** if it does not contain any proper ideals; similarly a semigroup with a zero element 0 is called **0-simple** if its only proper ideal is $\{0\}$.

**Corollary 3.2.** For every monoid $M$ there is a simple or 0-simple monoid $N$ such that $F_1(M) = F_1(N)$.

**Proof.** If $M$ has no proper ideals then it is simple, so we are done. Otherwise, let $I$ be the union of all the proper ideals of $M$. Then $I$ is an ideal and, since the identity element $1$ does not lie in any proper ideal, $1 \notin I$ and $I$ is a proper ideal of $M$. Setting $N = M/I$, it is easily verified that $N$ is
0-simple, and by Proposition 3.1 we have $F_1(M) = F_1(M/I) = F_1(N)$ as required. □

Corollary 3.2 tells us that the usual theory of $M$-automata really only involves the very restricted classes of simple and 0-simple monoids.

For $S$ a semigroup, we denote by $S^0$ the semigroup with a zero element $0$ adjoined, that is, with elements $S \cup \{0\}$ where $0$ is a new symbol not in $S$, and multiplication given by

$$st = \begin{cases} 
\text{the } S\text{-product } st & \text{if } s, t \in S; \\
0 & \text{otherwise.}
\end{cases}$$

We recall the following easy result from [21] which says that adjoining a zero to a monoid $M$ makes no difference to the class of languages accepted by $M$-automata.

**Proposition 3.3.** Let $M$ be a monoid. Then $F_1(M^0) = F_1(M)$.

It follows from standard results in semigroup theory (see [2, Theorem 2.54]) that a simple [0-simple] monoid is either a group [respectively, a group with 0 adjoined] or contains an embedded copy of the bicyclic monoid $B$. In [21] we observed that the languages accepted by $B$-automata are exactly the partially blind one-counter languages, while those accepted by $\mathbb{Z}$-automata are exactly the blind one-counter languages; both of these classes were introduced and studied by Greibach [10]. Combining this with Propositions 3.1 and 3.3 we obtain now the following.

**Theorem 3.4.** Let $M$ be a monoid. Then either $F_1(M^0) = F_1(G)$ for some group $G$, or $F_1(M)$ contains the partially blind one-counter languages.

Recall that a group $G$ is called *torsion* if every element has finite order, and *locally finite* if every finitely generated subgroup is finite. A straightforward consequence of Theorem 3.4 is the following trichotomy.

**Theorem 3.5.** Let $M$ be a monoid. Then $F_1(M)$ either

1. is exactly the class of regular languages; or
2. contains the class of blind one-counter languages; or
3. is equal to $F_1(G)$ for $G$ a torsion group which is not locally finite.

**Proof.** By Theorem 3.4 either $F_1(M)$ contains the partially blind one-counter languages, or $F_1(M) = F_1(G)$ for some group $G$. In the former case, since the class of partially blind one-counter languages contains that of blind one-counter languages, it is immediate that (ii) holds. So suppose $F_1(M) = F_1(G)$ for some group $G$.

If $G$ is not a torsion group then it has an element of infinite order; this element generates a subgroup isomorphic to $\mathbb{Z}$, from which it follows that $F_1(G)$ contains the class $F_1(\mathbb{Z})$ of blind one-counter languages and (ii) again holds. Now by [14, Proposition 1], every language in $F_1(G)$ is in $F_1(H)$ for some finitely generated subgroup of $H$ of $G$. If $G$ is locally finite, then such an $H$ must be finite, and so every language in $F_1(G)$ is regular. Conversely, $F_1(G)$ certainly contains the regular languages, so (i) holds. There remains only the case in which $G$ is a torsion group which is not locally finite, in which case (iii) holds. □
Elder and Mintz have observed that for $G$ a finitely generated infinite torsion group, $F_1(G)$ cannot contain the blind one-counter languages, but does always contain non-regular languages (for example, the word problem of $G$). It follows that the three possible conditions in Theorem 3.5 are in fact mutually exclusive. The theorem is of particular interest because torsion groups which are not locally finite are rather rare and difficult to construct. It would be interesting to study the language classes $F_1(G)$ corresponding to particular known examples of such groups [9, 11, 17].

4. Rational Semigroup Automata

In Section 3, we saw that the extent to which traditional monoid automata can utilise the differences in structure between groups and monoids was limited. In this section, we consider a generalisation which allows us to utilise more of the structure of arbitrary monoids, and indeed semigroups.

Let $S$ be a semigroup and $\Sigma$ a finite alphabet. We define a rational $S$-automaton over $\Sigma$ to be a finite automaton over the direct product $S \times \Sigma^*$ together with two rational subsets $X_0, X_1 \subseteq S$ called the initial set and terminal set respectively. The automaton accepts a word $w \in \Sigma^*$ if there exists $x_0 \in X_0$ and $x \in S$ such that $x_0x \in X_1$, and $(x, w)$ labels a path from the initial vertex to a terminal vertex in the automaton. For $S$ a semigroup, we let $F_{Rat}(S)$ be the class of languages accepted by rational $S$-automata.

Intuitively, a rational $S$-automaton is a non-deterministic finite automaton augmented with a register which stores an element of $S$. The register is (non-deterministically) initialised with an element of $X_0$, and a word is accepted if there is a computation which reads it and leaves the finite state control in an accept state and the register containing a value from $X_1$.

Note that if $S = M$ is a monoid then a rational $M$-automaton with initial set $\{1\}$ is just a valence automaton over $M$ with rational target set of the kind studied by Fernau and Stiebe [5] and the present authors [21]. Indeed, the following proposition says that, for $M$ a monoid, the initial set can be taken to be $\{1\}$ without loss of generality.

**Proposition 4.1.** Let $M$ be a monoid with identity 1, and $L \subseteq \Sigma^*$ a language. If $L \in F_{Rat}(M)$ then $L$ is accepted by a rational $M$-automaton with initial set $\{1\}$, that is, by a valence automaton over $M$ with rational target set.

**Proof.** Let $A$ be a rational monoid automaton with initial set $X_0 \subseteq M$ and terminal set $X_1 \subseteq M$ which accepts the language $L$. Let $L' \subseteq M \times \Sigma^*$ be the full subset accepted by $A$ interpreted as an automaton over $M \times \Sigma^*$. Since $X_0 \subseteq M$ is rational, the set

$$X'_0 = \{(x, \epsilon) \mid x \in X_0\} \subseteq M \times \Sigma^*$$

is rational. Now let

$$K = X'_0L' = \{(x_0x, w) \mid x_0 \in X_0, (x, w) \in L'\}.$$

Then $w \in L$ if and only if there exists $x_0 \in X_0$ and $x \in X$ such that $(x, w) \in L'$ and $x_0x \in X_1$. But this is true exactly if $(x', w) \in K$ for some $x' \in X_1$. 


Now $K$ is a product of two rational subsets, and hence is a rational subset. Let $B$ be a finite automaton over $M \times \Sigma^*$ recognizing $K$. If we interpret $B$ as a rational $M$-automaton with initial set $\{1\}$ and terminal set $X_1$, it is immediate that $B$ accepts exactly the language $L$. □

Combining Corollary 4.1 with a result of Fernau and Stiebe [5] we obtain the following.

**Theorem 4.2.** If $G$ is a group then $F_{\text{Rat}}(G) = F_1(G)$.

**Proof.** If $L \in F_{\text{Rat}}(G)$ then by Corollary 4.1 $L$ is accepted by a rational $G$-automaton with initial set $\{1\}$ and some rational terminal set $X_1$, that is, by a valence automaton with rational target set. But now by [5, Theorem 3.5], $L$ is accepted by a finite valence automaton, that is, a $G$-automaton, so that $L$ is in $F_1(G)$ as required. The converse is immediate. □

We now turn our attention to the relationship between rational relations and rational semigroup automata. Let $\Omega$ and $\Sigma$ be finite alphabets, and consider a finite automaton over the direct product $\Omega^+ \times \Sigma^*$; it recognizes a rational relation $R \subseteq \Omega^+ \times \Sigma^*$. The image of a language $L \subseteq \Omega^+$ under the relation $R$ is the set of words $y \in \Sigma^*$ such that $(x, y) \in R$ for some $x \in L$.

If $X_0, X_1 \subseteq S$ then their difference is the set
\[ X_0^{-1}X_1 = \{ x \in S \mid x_0x = x_1 \text{ for some } x_0 \in X_0, x_1 \in X_1 \}. \]

We say that a subset $X \subseteq S$ is a rational set difference if there exist rational subsets $X_0, X_1 \subseteq S$ such that $X = X_0^{-1}X_1$. Note that in a group, the rational set differences are exactly the rational subsets, but in a general semigroup this does not hold. The following statement is a semigroup analogue of [21, Proposition 3.1], which in turn generalises a well-known observation concerning $M$-automata (see for example [14, Proposition 2]).

**Proposition 4.3.** Let $X_0$ and $X_1$ be subsets of a semigroup $S$, and let $L \subseteq \Sigma^*$ be a language. Then the following are equivalent:

(i) $L$ is accepted by a $S$-automaton with initial set $X_0$ and terminal set $X_1$;

(ii) there exists a finite alphabet $\Omega$, a morphism $\omega : \Omega^+ \to S$ and a rational relation $\rho \subseteq \Omega^+ \times \Sigma^*$ such that
\[ L = (X_0^{-1}X_1)\omega^{-1}\rho. \]

If $S$ is finitely generated then the following condition is also equivalent to those above.

(iii) for every finite choice of generators $\omega : \Omega^+ \to S$ for $S$, there exists a rational relation $\rho \subseteq \Omega^+ \times \Sigma^*$ such that
\[ L = (X_0^{-1}X_1)\omega^{-1}\rho. \]

**Proof.** To show that (i) implies (ii), suppose that $L$ is accepted by an $S$-automaton $A$ with initial set $X_0$ and terminal set $X_1$. Choose a finite alphabet $\Omega$ and a map $\omega : \Omega^+ \to S$ such that the image $\Omega^+\omega$ contains every element of $S$ which forms the first component of an edge label in the automaton. We now obtain from $A$ a finite automaton $B$ over $\Omega^+ \times \Sigma^*$ by replacing each edge label $(s, x)$ with $(w, x)$ for some $w \in \Omega^+$ is such that
It is a routine exercise to verify that $L$ is the image of $(X^{-1}_0X_1)\omega^{-1}$ under the relation accepted by $B$.

Conversely, suppose we are given a map $\omega : \Omega^+ \rightarrow S$ and an automaton $B$ over $\Omega^+ \times \Sigma^*$ such that $L$ is the image under the relation accepted by $B$ of the language $(X^{-1}_0X_1)\omega^{-1}$. We construct from $B$ a new automaton $A$ over $S \times \Sigma^*$ by applying the map $\omega$ to the first component of each edge label. Now interpreting $A$ as an $S$-automaton with initial set $X_0$ and terminal rational set $X_1$, it is easily seen that $A$ accepts the language $L$.

Suppose now that $S$ is finitely generated. Clearly (iii) implies (ii). Conversely, if (i) holds then we can extend $\omega$ arbitrarily to a finite choice of generators $\omega' : (\Omega')^+ \rightarrow S$ for $M$, and check that we still have the desired property, so that (iii) holds. □

As a corollary, we immediately obtain the following characterisation for language classes of the form $F_{\text{Rat}}(S)$.

**Proposition 4.4.** Let $S$ be a semigroup and $L \subseteq \Sigma^*$ a language. Then the following are equivalent.

(i) $L \in F_{\text{Rat}}(S)$;
(ii) there exists an alphabet $\Omega$, a morphism $\omega : \Omega^+ \rightarrow S$, a rational set difference $X \subseteq S$ and a rational relation $\rho \subseteq \Omega^+ \times \Sigma^*$ such that $L = X\omega^{-1}\rho$.

If $S$ is finitely generated then the following condition is also equivalent to those above.

(iii) there exists a rational set difference $X \subseteq S$ such that for every finite choice of generators $\omega : \Omega^+ \rightarrow S$ for $S$, there exists a rational relation $\rho \subseteq \Omega^+ \times \Sigma^*$ such that $L = X\omega^{-1}\rho$.

Note that, unlike in the monoid case [21], we cannot conclude that $F_{\text{Rat}}(S)$ is a rational cone. This is because the composition of a rational relation in $\Omega^+ \times \Sigma^*$ with a rational transduction from $\Sigma^*$ to another free monoid $\Gamma^*$ need not be a rational relation in $\Omega^+ \times \Gamma^*$ (although it will be rational in $\Omega^* \times \Gamma^*$).

5. **REES MATRIX CONSTRUCTIONS, COMPLETELY SIMPLE AND COMPLETELY 0-SIMPLE SEMIGROUPS**

In this section we apply the results of the previous sections to obtain a description of language classes $F_{\text{Rat}}(S)$ for semigroups $S$ belonging to the important classes of completely simple and completely 0-simple semigroups.

Recall that an idempotent $e$ in a semigroup is called *primitive* if for every non-zero idempotent $f$ such that $ef = fe = f$ we have $e = f$. A semigroup is *completely simple* [respectively, completely 0-simple] if it is simple [0-simple] and has a primitive idempotent. For more information about completely simple and completely 0-simple semigroups, see [12].

Now let $T$ be a semigroup, $0$ be a new symbol not in $T$ and let $I, J$ be non-empty sets. Let $P = (P_{ij})$ be a $J \times I$ matrix with entries in $T \cup \{0\}$. We define a new semigroup with set of elements

$$(I \times T \times J) \cup \{0\}$$
and multiplication defined by
\[
(i, t, j)(i', t', j') = \begin{cases} 
(i, tP_{ji}i't', j') & \text{if } P_{ji} \neq 0 \\
0 & \text{otherwise,}
\end{cases}
\]
and
\[
(i, t, j)0 = 0(i, t, j) = 00 = 0.
\]

It is simple to verify that this binary operation is associative; we call the semigroup constructed in this way a Rees matrix semigroup with zero over \(T\), and denote it \(M^0(T; I, J; P)\). The semigroup \(T\) is called the base semigroup and the matrix \(P\) the sandwich matrix of the construction. If \(P\) contains no zero entries then \(I \times T \times J\) forms a subsemigroup of \(M^0(T; I, J; P)\), called a Rees matrix semigroup (without zero) over \(T\) and denoted \(M(T; I, J; P)\).

Rees matrix semigroups play a crucial role in much of the structural theory of semigroups. Of particular importance is the case that the base semigroup \(T\) is a group \(G\). A Rees matrix semigroup with zero over a group is called regular if every row and every column of the sandwich matrix contains a non-zero entry. The importance of this construction can be seen from the following seminal result of Rees [20].

**Theorem 5.1 (The Rees Theorem).** Let \(S = M(G; I, J; P)\) [respectively, \(S = M^0(G; I, J; P)\)] be a [regular] Rees matrix semigroup over a group. Then \(S\) is a completely simple [respectively, completely 0-simple] semigroup. Conversely, every completely simple [completely 0-simple] semigroup is isomorphic to one constructed in this way.

We shall need the following proposition.

**Proposition 5.2.** Let \(S = M(T; I, J; P)\) or \(S = M^0(T; I, J; P)\) be a Rees matrix semigroup with or without zero over a semigroup \(T\). Let \(X \subseteq S\) be a rational subset and suppose \(i \in I\) and \(j \in J\). Then the set
\[
X_{ij} = \{ g \in T \mid (i, g, j) \in X \} \subseteq T
\]
is a rational subset of \(T\).

**Proof.** Let \(A\) be a finite automaton over \(S\) accepting the rational subset \(X\), with vertex set \(Q\). Let \(J'\) be the set of all \(j \in J\) such that \(A\) has an edge label with third component \(j\); note that \(J'\) is necessarily finite. We construct from \(A\) a new finite automaton \(B\) over \(T\) with

- vertex set \((Q \times J') \cup \{q_0'\}\) where \(q_0'\) is a new symbol;
- start vertex \(q_0'\);
- terminal vertices \((q, j)\) such that \(q\) is a terminal vertex of \(A\);
- an edge from \(q_0'\) to \((q_1, j_1)\) labelled \(t_1\) whenever \(A\) has an edge from the initial vertex to \(q_1\) labelled \((i, t_1, j_1)\);
- for every \(j_1 \in J'\), an edge from \((q_1, j_1)\) to \((q_2, j_2)\) labelled \(P_{j_1j_2}t_2\) whenever \(A\) has an edge from \(q_1\) to \(q_2\) labelled \((i_2, t_2, j_2)\) with \(P_{j_1j_2} \neq 0\).

Since \(J'\) is finite and \(A\) has finitely many vertices and edges, we deduce that \(B\) has finitely many vertices and edges. Now we show that the subset accepted by \(B\) is exactly \(X_{ij}\). Let \(t \in X_{ij}\). Then \((i, t, j) \in X\) labels a path in
A from the initial vertex to some terminal vertex. Clearly this path cannot contain edges labelled 0, so it must have the form

$$\begin{align*}
\mathbf{P}_0 & \xrightarrow{(i_1,t_1,j_1)} \mathbf{P}_1 \xrightarrow{(i_2,t_2,j_2)} \mathbf{P}_2 \xrightarrow{(i_3,t_3,j_3)} \cdots \xrightarrow{(i_{m-1},t_{m-1},j_{m-1})} \mathbf{P}_{m-1} \xrightarrow{(i_{m},t_{m},j_{m})} \mathbf{P}_m
\end{align*}$$

where \( p_0 \) is the initial vertex of \( A \) and \( p_m \) is a terminal vertex. Since the path is labelled \((i,t,j)\) we must have

$$\begin{align*}
(i,t,j) &= (i_1,t_1,j_1)(i_2,t_2,j_2) \cdots (i_m,t_m,j_m)
\end{align*}$$

so that \( i_1 = i, j_m = j \). Now it follows easily from the construction of \( B \) that it has a path

$$\begin{align*}
q_0' & \xrightarrow{t_1} (p_{1},j_1) \xrightarrow{P_{1}j_1t_2} (p_{2},j_2) \cdots (p_{m-1},j_{m-1}) \xrightarrow{P_{m-1}j_{m-1}t_m} (p_{m},j),
\end{align*}$$

where \((p_m,j)\) is a terminal vertex of \( B \), so that \( B \) accepts

$$t = t_1P_{1}j_1t_2P_{2}j_2 \cdots P_{m-1}j_{m-1}t_m.$$ 

Thus \( X_{ij} \subseteq L(B) \).

Conversely, assume that \( t \in T \) is accepted by \( B \). Then there exists a path through \( B \) from the initial vertex to some terminal vertex labelled with \( t \). It follows from the definition of \( B \) that this path must have the form

$$\begin{align*}
q_0' & \xrightarrow{t_1} (p_{1},j_1) \xrightarrow{P_{1}j_1t_2} (p_{2},j_2) \cdots (p_{m-1},j_{m-1}) \xrightarrow{P_{m-1}j_{m-1}t_m} (p_{m},j),
\end{align*}$$

where \( p_m \) is a terminal vertex in \( A \),

$$t = t_1P_{1}j_1t_2P_{2}j_2 \cdots P_{m-1}j_{m-1}t_m$$

and \( A \) has a path

$$\begin{align*}
\mathbf{P}_0 & \xrightarrow{(i_1,t_1,j_1)} \mathbf{P}_1 \xrightarrow{(i_2,t_2,j_2)} \mathbf{P}_2 \xrightarrow{(i_3,t_3,j_3)} \cdots \xrightarrow{(i_{m-1},t_{m-1},j_{m-1})} \mathbf{P}_{m-1} \xrightarrow{(i_{m},t_{m},j_{m})} \mathbf{P}_m
\end{align*}$$

where \( p_0 \) is the initial vertex of \( A \). Hence, \( A \) accepts the element

$$\begin{align*}
(i,t_1,j_1)(i_2,t_2,j_2) \cdots (i_m,t_m,j) &= (i,t_1P_{1}j_1t_2P_{2}j_2 \cdots P_{m-1}j_{m-1}t_m,j) \\
&= (i,t,j).
\end{align*}$$

So \((i,t,j) \in X \) and hence \( t \in X_{ij} \).

So the automaton \( B \) accepts exactly the set \( X_{ij} \), and hence \( X_{ij} \) is a rational subset of \( T \).

As a corollary, we obtain a result about the intersections of rational subgroups with maximal subgroups in completely simple semigroups.

**Corollary 5.3.** Let \( H \) be a maximal subgroup of a completely simple or completely 0-simple semigroup \( S \). Let \( X \) be a rational subset of \( S \). Then \( X \cap H \) is a rational subset of \( H \).

**Proof.** By the Rees theorem, we may assume that \( S \) is a Rees matrix semigroup without zero \( M(G;I,J;P) \) or a regular Rees matrix semigroup with zero \( S = M^0(G;I,J;P) \) over a group \( G \). It follows easily from the definition of the Rees matrix construction that either \( H = \{0\} \) or

$$H = \{(i,g,j) \mid g \in G\}$$

for some \( i \in I \) and \( j \in J \) with \( P_{ji} \neq 0 \). In the former case the result is trivial, so we assume the latter. By Proposition 5.2 the set

$$X_{ij} = \{ g \in G \mid (i,g,j) \in X \} = \{ g \in G \mid (i,g,j) \in H \cap X \}$$
is a rational subset of $G$. It follows that

$$P_{ji}X_{ij} = \{P_{ji}g \mid g \in X_{ij}\} = \{P_{ji}g \mid (i, g, j) \in X\}$$

is also a rational subset of $G$. Now define a map

$$\phi : G \rightarrow H, \ g \mapsto (i, P_{ji}^{-1}g, j)$$

where $P_{ji}^{-1}$ is the inverse of $P_{ji}$ in the group $G$. It is readily verified that $\phi$ is an isomorphism from $G$ to $H$, and so the image

$$(P_{ji}X_{ij})\phi = \{(i, P_{ji}^{-1}g, j) \mid g \in P_{ji}X_{ij}\}$$

$$(i, P_{ji}^{-1}P_{ji}g, j) \mid (i, g, j) \in X\}$$

$$(i, g, j) \mid (i, g, j) \in X\}$$

is a rational subset of $G$, as required.

In a completely simple semigroup, where every element lies in a maximal subgroup, Corollary 5.3 easily yields the following complete characterisation of rational subsets.

**Theorem 5.4.** The rational subsets of a completely simple semigroup are exactly the finite unions of rational subsets of maximal subgroups.

**Proof.** Let $S$ be a completely simple semigroup. If $X_1, \ldots, X_n$ are rational subsets of maximal subgroups of $S$ then certainly they are rational subsets of $S$, and so is their union. Conversely, suppose $X$ is a rational subset of $S$. It follows easily from the Rees theorem that $X$ lies inside a finitely generated completely simple subsemigroup $S'$ of $S$. Now $S'$ is the union of finitely many maximal subgroups, so $X$ is the union of its intersections with these subgroups. By Corollary 5.3 these intersections are rational, so $X$ is a finite union of rational subsets of maximal subgroups of $S'$. But maximal subgroups of $S'$ are subgroups of $S$, and hence lie in maximal subgroups of $S'$. It follows that $X$ is a finite union of rational subsets of maximal subgroups of $S$, as required. \hfill \Box

**Proposition 5.5.** Let $S = M(T; I, J; P)$ or $S = M^0(T; I, J; P)$ be a Rees matrix semigroup with or without zero over a semigroup $T$, and let $P' \subseteq T$ be the set of non-zero entries of the sandwich matrix $P$. Suppose $T = P'T$ or $T = TP'$. Then for any $i \in I$, $j \in J$ and rational subset $X$ of $T$, the set

$$\{(i, t, j) \mid t \in X\}$$

is a rational subset of $S$.

**Proof.** By symmetry of assumption, it suffices to consider the case in which $T = P'T$. Let $A$ be a finite automaton over $T$ accepting $X$, with vertex set $Q$. Let $Y \subseteq T$ be the set of edge labels in $A$, and for every $t \in Y$, let $j_t \in J$, $i_t \in I$ and $s_t \in T$ be such that $t = P_{ji_t}s_t$. Let $J' = \{j_t \mid t \in Y\} \cup \{j\}$. Then $J'$ is a finite subset of $J$. We define a new automaton $B$ over $S$ with

- vertex set $(Q \times J') \cup \{q_0\}$ where $q_0$ is a new symbol;
- initial vertex $q_0$;
- terminal vertices $(q, j)$ such that $q$ is a terminal vertex of $A$;
for every edge in $A$ from the start vertex to a vertex $q$ labelled $t$, and every $j' \in J'$, an edge from $q_0$ to $(q, j')$ labelled $(i, t, j')$;

- for every edge in $A$ from a vertex $p$ to a vertex $q$ labelled $t$, and every $j' \in J'$, an edge from $(p, j_i)$ to $(q, j')$ labelled $(i_i, s_i, j')$;

A routine argument, akin to that in the proof of Proposition 5.2, shows that $B$ accepts the required subset of $S$. \hfill $\Box$

Note in particular that the conditions on the sandwich matrix in the hypothesis of Proposition 5.5 are satisfied in the case of a regular Rees matrix construction over a group.

Recall that the rational subset problem for a finitely semigroup $S$ is the algorithmic problem of deciding, given a rational subset (described as a finite automaton over a fixed generating set for $S$) and an element (described as a word over the same generating set), deciding whether the latter belongs to the former. While the phrasing of the problem is independent on the precise choice of finite generating set, the decidability or undecidability of the problem is independent of this choice [15, Corollary 3.4], so one can meaningfully say that the abstract semigroup $S$ has decidable or undecidable rational subset problem.

**Corollary 5.6.** Let $S = M(T; I, J; P)$ or $S = M^{0}(T; I, J; P)$ be a finitely generated Rees matrix semigroup with or without zero over a semigroup $T$. If $T$ has decidable rational subset problem then $S$ has decidable rational subset problem.

**Proof.** We prove the statement for Rees matrix semigroups with zero. The result for Rees matrix constructions without zero can be obtained as an easy consequence, or proved directly using a similar method.

Let $\omega : \Omega^* \rightarrow T$ and $\sigma : \Sigma^* \rightarrow S$ be finite choices of generators for $T$ and $S$ respectively. For every $x \in \Sigma$ such that $x\sigma \neq 0$, suppose $x\sigma = (i_x, g_x, j_x)$ and let $w_x \in \Omega^*$ be a word with $w_x \omega = g_x$. For $j \in J$ and $i \in I$ such that $P_{ji} \neq 0$ let $w_{ji} \in \Omega^*$ be a word with $w_{ji} \omega = P_{ji}$.

Now suppose we are given a word $w = w_1 \ldots w_n \in \Sigma^*$, where each $w_i \in \Sigma$, and a rational subset $X$ of $S$. Clearly, we can test whether $w$ represents 0, and in the case that it does, whether $0 \in X$. Assume now that $w$ does not represent 0. Then

$$w \omega = (w_1 \omega) \ldots (w_n \omega) = (i_{w_1}, g_{w_1} P_{j_{w_1} i_{w_2} g_{w_2} \ldots g_{w_n} i_{w_n} j_{w_n}}).$$

Let $Y = \{ t \in T \mid (i_{w_1}, t, j_{w_n}) \in X \}$, so that $w \omega \in X$ if and only if

$$(w_{g_{w_1} P_{j_{w_1} i_{w_2} g_{w_2} \ldots g_{w_n}}} \sigma = g_{w_1} P_{j_{w_1} i_{w_2} g_{w_2} \ldots g_{w_n} i_{w_n} j_{w_n}} \in Y.$$  \hfill (1)

Now by Proposition 5.2, $Y$ is rational and it follows moreover from the proof that we can effectively compute an automaton for $Y$. By assumption, we can solve the rational subset problem for $Y$, so we can decide whether (1) holds, as required. \hfill $\Box$

We now turn our attention to languages accepted by rational $S$-automata, where $S$ is a Rees matrix semigroup. We begin with a lemma which simplifies the case of Rees matrix semigroups with zero, by allowing us to restrict attention to automata for which neither the initial set nor the terminal set contains the zero element.
Lemma 5.7. Let $S = M^0(T; I, J; P)$ be a finitely generated Rees matrix semigroup with zero over a semigroup $T$. If $L$ is accepted by a rational $S$-automaton, then $L$ is accepted by a rational $S$-automaton for which neither the initial set nor the terminal set contain $0$.

Proof. Suppose $L$ is accepted by a rational $S$-automaton $A$ with initial set $X_0$ and terminal set $X_1$. Suppose first that $0 \in X_0$. If also $0 \in X_1$ then we have $0x \in X_1$ for all $x \in S$, so the language accepted is just the set of all words $w$ such that $(x, w)$ labels a path from the initial vertex to a terminal vertex of $A$ for some $x \in S$. It follows that $L$ is regular, and hence lies in $F_1(S)$. On the other hand, if $0 \notin X_1$ then there is no $x \in S$ such that $0x \in X_1$; hence we may replace the initial set $X_0$ with $X_0 \setminus \{0\}$ without changing the language accepted. Thus, we may assume that $0 \notin X_0$.

Clearly we can write $L = L_0 \cup L_1$ where $L_0$ is accepted by a rational $S$-automaton with $0$ not in the initial or terminal sets, and $L_0$ is accepted by a rational $S$-automaton with terminal set $0$. We claim that $L_0$ is regular; it will follow that $L$ is the union of $L_1$ with a regular language, and so can clearly be accepted by a rational $S$-automaton without $0$ in the terminal set.

Let $\omega : \Omega^* \to S$ be a finite choice of generators for $S$. For each $x \in \Omega$ such that $x\omega \neq 0$ suppose $x\omega = (i_x, g_x, j_x)$. Now let $K$ be the set of all words representing elements of the initial set of $A$, and let $K'$ be the (necessarily finite) set of all final letters of words in $K$. It is easily seen that the language

$$\{ v \in \Omega^* | (vw)\omega = 0 \text{ for some } w \in K \}$$

is regular. Indeed, it consists of all words which

- contain a generator representing zero; or
- contain consecutive generators $x$ and $y$ with $P_{j_xy} = 0$; or
- start with a generator $y$ with $P_{j_y} = 0$ for some $x \in K'$

and so can be easily described by a regular expression. It now follows from Proposition 4.3 that $L_0$ is a rational transduction of the above regular language and hence is itself regular. This completes the proof. □

We are now ready to prove the main theorem of this section, the essence of which is that rational $S$-automata where $S$ is a completely simple or completely 0-simple semigroup are no more powerful than $G$-automata where $G$ is the maximal subgroup of $S$.

Theorem 5.8. Let $S$ be a completely simple or completely 0-simple semigroup with maximal non-zero subgroup $G$. Then

$$F_\text{Rat}(S) = F_\text{Rat}(G) = F_1(G).$$

Proof. That $F_\text{Rat}(G) = F_1(G)$ is Theorem 4.2 while the inclusion $F_\text{Rat}(G) \subseteq F_\text{Rat}(S)$ is immediate. Hence, we need only prove that $F_\text{Rat}(S) \subseteq F_\text{Rat}(G)$. It follows easily from the Rees theorem that every completely simple semigroup $S$ embeds in a completely 0-simple semigroup $S'$ with the same maximal non-zero subgroup, so that $F_\text{Rat}(S) \subseteq F_\text{Rat}(S')$. Hence, it suffices to prove the result in the case that $S$ is completely 0-simple.

Suppose, then, that $S$ is completely 0-simple. By the Rees theorem, we may assume that $S$ is a regular Rees matrix semigroup $M^0(G^0; I, J; P)$ where $G$ is a group. Suppose now that a language $L \subseteq \Sigma^*$ lies in $F_\text{Rat}(S)$. Let
Let $A$ be a rational $S$-automaton accepting $L$, with initial rational set $X_0 \subseteq S$ and terminal rational set $X_1 \subseteq S$. By Lemma 5.7 we may assume that $0 \not\in X_0$ and $0 \not\in X_1$.

Let $C$ and $D$ be automata over $S$ accepting $X_0$ and $X_1$ respectively. Since $C$, $D$ and $A$ have only finitely many edges between them, we may choose finite subsets $I' \subseteq I$ and $J' \subseteq J$ such that the edge labels of $C$ and $D$ all lie in $I' \times G \times J'$, and the edge labels of $A$ all lie in $(I' \times G \times J') \times \Sigma^*$.

For each $i \in I'$ and $j \in J'$, we let $X_{ij} = \{g \in G \mid (i, g, j) \in X_0\}$. By Proposition 5.2 each $X_{ij}$ is a rational subset of $G$. It follows that

$$X_{ij}' = X_{ij} \times \{\epsilon\}$$

is a rational subset of $G \times \Sigma^*$; let $C_{ij}$ be an automaton accepting $X_{ij}'$.

Similarly, for each $i \in I'$ and $j \in J'$ we define $Y_{ij} = \{g^{-1} \in G \mid (i, g, j) \in X_1\}$. By Propositions 5.2 and 5.7 $Y_{ij}$ is a rational subset of $G$, and so

$$Y_{ij}' = Y_{ij} \times \{\epsilon\}$$

is a rational subset of $G \times \Sigma^*$; let $D_{ij}$ be an automaton accepting $Y_{ij}'$.

Assume without loss of generality that the automata $A$ and all the automata $C_{ij}$ and $D_{ij}$ have disjoint vertex sets. We construct from these automata a $G$-automaton $B$ with

- vertex set the union of the vertex sets of $C_{ij}$ and $D_{ij}$ (for $i \in I'$ and $j \in J'$) together with $I' \times Q \times J'$ where $Q$ is the vertex set of $A$, and a new vertex $q_0'$;
- initial vertex $q_0'$;
- terminal vertices the terminal vertices of the automata $D_{ij}$;
- all the edges of the automata $C_{ij}$ and $D_{ij}$;
- for each $i \in I'$ and $j \in J'$, an edge from $q_0'$ to the initial vertex of $C_{ij}$ labelled $(1, \epsilon)$;
- for each $i \in I'$ and $j \in J'$, an edge from each terminal vertex of $C_{ij}$ to $(i, q_0, j)$ labelled $(1, \epsilon)$, where $q_0$ is the initial vertex for $A$;
- for each edge in $A$ from $p$ to $q$ labelled $((i, g, j), w)$ and each $i' \in I'$ and $j' \in J'$, an edge from $(i', p, j')$ to $(i', q, j)$ labelled $(P_{ij}, g, w)$;
- for each $i \in I'$, $j \in J'$ and terminal vertex $p$ of $A$, an edge from $(i, p, j)$ to the initial vertex of $D_{ij}$ labelled $(1, \epsilon)$.

Since $I'$, $J'$ and all the automata $A$, $C_{ij}$ and $D_{ij}$ are finite, it follows that the $G$-automaton $B$ is finite. We now show that $B$ accepts the language $L$.

Let $w \in L$. Then there exists a path through the automaton $A$ labelled $((i, g, j), w)$ connecting the initial vertex with some terminal vertex ($p_t$ say), such that

$$(i_0, g_0, j_0)(i, g, j) = (i', g', j') \in X_1$$

for some $(i_0, g_0, j_0) \in X_0$. Suppose this path has the form

$q_0 \xrightarrow{((i_1, g_1, j_1), x_1)} q_1 \xrightarrow{((i_2, g_2, j_2), x_2)} q_2 \xrightarrow{((i_3, g_3, j_3), x_3)} \cdots \xrightarrow{((i_{m-1}, g_{m-1}, j_{m-1}), x_{m-1})} q_m$.

where $q_0$ is the initial vertex and $g_m = p_t$ is a terminal vertex of $A$ and $w = x_1 \ldots x_m$. Note that we must have $i' = i_0$, $j' = j_m$ and

$$g = g_1 P_{j_1 i_2} g_2 \cdots P_{j_{m-1} i_m} g_m.$$
Now by construction, $B$ has a path $\pi_2$ of the form

$$(i_0, q_0, j_0) \xrightarrow{(P_{j_0,1}g_1, x_1)} (i_0, q_1, j_1) \xrightarrow{(P_{j_1,2}g_2, x_2)} (i_0, q_2, j_2) \xrightarrow{(P_{j_2,3}g_3, x_3)} \ldots \xrightarrow{(P_{j_m-1,m}g_m, x_m)} (i_0, q_m, j_m)$$

Moreover, from the fact that $(i_0, q_0, j_0) \in X_0$ we see that $q_0 \in X'_{i_0,j_0}$, so that $(g_0, \epsilon) \in X'_{i_0,j_0}$. Hence, $(g_0, \epsilon)$ labels a path in $C_{i_0,j_0}$ from the initial vertex to a terminal vertex. It follows easily that $(g_0, \epsilon)$ labels a path $\pi_1$ in $B$ from the initial vertex $q'_0$ to $(i_0, q_0, j_0)$ where $q_0$. Similarly, since $(i', g', j') \in X_1$ we deduce that $((g')^{-1}, \epsilon) \in Y_{i'j'} = Y_{i_0j_m}$ so that $B$ has a path $\pi_3$ from $(i_0, q_m, j_m)$ to a terminal vertex labelled $((g')^{-1}, \epsilon)$.

Composing the paths $\pi_1, \pi_2$ and $\pi_3$, we see that $B$ has a path from the initial vertex to a terminal vertex with label

$$(g_0P_{j_0i_1}g_1P_{j_1i_2}g_2 \ldots P_{j_{m-1}i_m}g_m(g')^{-1}, x_1x_2 \ldots x_m)$$

But we know that $(i_0, q_0, j_0)(i, g, j) = (i', g', j')$, so we must have

$g_0P_{j_0i_1}g_1P_{j_1i_2}g_2 \ldots P_{j_{m-1}i_m}g_m = g'$

and hence

$g_0P_{j_0i_1}g_1P_{j_1i_2}g_2 \ldots P_{j_{m-1}i_m}g_m(g')^{-1} = 1.$

It follows that $w$ is accepted by the $G$-automaton $B$, as required.

Conversely, suppose $w$ is accepted by the $G$-automaton $B$. Then there is a path in $B$ from the initial vertex to a terminal vertex labelled $(1, w)$. We deduce easily from the construction of $B$ that this path must have the form $\pi_1\pi_2\pi_3$ where

- $\pi_1$ runs from the start vertex to some vertex $(i_0, q_0, j_0)$ with label of the form $(g_0, \epsilon)$ for some $g_0 \in X'_{i_0,j_0}$, so that $(i_0, q_0, j_0) \in X_0$;
- $\pi_2$ runs from $(i_0, q_0, j_0)$ to a vertex $(i_0, q_m, j_m)$ where $q_m$ is a terminal vertex of $A$; and
- $\pi_3$ runs from $(i_0, q_m, j_m)$ to a terminal vertex with label $((g')^{-1}, \epsilon)$ where $(g')^{-1} \in Y_{i_0j_m}$, so that $(i_0, g', j_m) \in X_1$.

Moreover, $\pi_2$ must have the form

$$(i_0, q_0, j_0) \xrightarrow{(P_{j_0i_1}g_1, x_1)} (i_0, q_1, j_1) \xrightarrow{(P_{j_1i_2}g_2, x_2)} (i_0, q_2, j_2) \xrightarrow{(P_{j_2i_3}g_3, x_3)} \ldots \xrightarrow{(P_{j_{m-1}i_m}g_m, x_m)} (i_0, q_m, j_m)$$

where, since the label of the entire path $\pi$ is $(1, w)$, we must have $w = x_1 \ldots x_m$ and $g_0P_{j_0i_1}g_1 \ldots P_{j_{m-1}i_m}g_m(g')^{-1} = 1$, that is,

$g_0P_{j_0i_1}g_1 \ldots P_{j_{m-1}i_m}g_m = g'$.

We deduce from the path above and the construction of $B$ that $A$ has a path

$q_0 \xrightarrow{((i_1, g_1, j_1), x_1)} q_1 \xrightarrow{((i_2, g_2, j_2), x_2)} q_2 \xrightarrow{((i_3, g_3, j_3), x_3)} \ldots q_{m-1} \xrightarrow{((i_m, g_m, j_m), x_m)} q_m$

Since $q_0$ and $q_m$ are initial and terminal vertices of $A$ respectively, it follows that $A$ accepts $(x, w)$ where

$x = (i_1, g_1, j_1)(i_2, g_2, j_2) \ldots (i_m, g_m, j_m)$. 


But \((i_0, g_0, j_0)\) lies in \(X_0\) and
\[
(i_0, g_0, j_0) x = (i_0, g_0, j_0)(i_1, g_1, j_1) \cdots (i_m, g_m, j_m)
= (i_0, g_0 P_{j_0} i_1 g_1 \cdots P_{j_{m-1}} i_m g_m, j_m)
= (i_0, g', j_m)
\]
lies in \(X_1\), from which we deduce that the rational \(S\)-automaton \(A\) accepts the word \(w\), and so \(w \in L\) as required. \(\square\)

**Acknowledgements**

The research of the second author was supported by an RCUK Academic Fellowship.

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