Methodology of robust inverted pendulum controllers on a vehicle

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Abstract. In the theory of controllers, the simple and inverted pendulum play an important role due to the equations that result from them, which imply non-linearities and perturbations, thus, in this article, a brief classification of inverted pendulums is presented: inverted pendulum, inverted double pendulum, inverted rotary pendulum (Furuta pendulum). Subsequently, a mathematical model of the inverted pendulum is described through the deduction of the equations of motion that represent the dynamics of the system. Robust control is presented that allows expanding the richness of the mathematical equations, for this case, a control with output feedback is presented and applied to the inverted pendulum to control the unstable dynamics of this model. The results are compared with a post placement control and a robust control using a norm that analyses the characteristics of the system.

1. Introduction
The inverted pendulum is a classic example of a nonlinear system; the inherent dynamics in the model are commonly used to represent problems associated with the balance, such as biped’s walkers exposed in [1-3], and in personal transport systems "Segway" [4]. In addition, due to the own system instability, the inverted pendulum system is often considered a sub-acted mechanic system, where the control variables are fewer than the system's liberty grades [5,6]. Hence, the systems' natural complexity makes them perfect for new control techniques [7].

Therefore, like other systems, the models are not perfect. Because of this, the uncertainties addition, noisy sensors, and unknown extern perturbations can convert a system previously stable into an unstable system [8]. Xin Y [9] presents a technique to absorb the variability of the system through a neural network controller. Although the approximation is accurate, there is no consideration for possible external perturbations.

Within the family of robust controllers, we must highlight the control for non-linear systems called infinite H control (H∞), which is useful when not only the system is non-linear, but also when there are disturbances. Said control technique consists of minimizing the associated cost function that describes the dynamics of the system; therefore, the purpose of this article is to design a controller H∞ considering external perturbations and noisy sensors.

2. Inverted pendulum systems
An inverted pendulum is a mechanical system consisting of a mobile device that holds a rigid extended object in one extreme and in the other an axis that can be moved bidimensional [10]. Other variants are double pendulum, triple, rotational base pendulum or Furuta pendulum, and inertial wheel pendulum.
Figure 1 shows a simple inverted pendulum. In this inverted pendulum, the center of mass is above a cart that can move horizontally over a linear axis. The double or triple inverted pendulum corresponds to an inverted pendulum with one or two additional articulations, respectively, introducing new variables to control [11]. For example, Figure 2 shows a double inverted pendulum.

Furuta pendulum or inverted rotational pendulum consists of a controlled arm that can rotate horizontally while join with a pendulum with a free rotation in the vertical plane, as shown in Figure 3. Figure 1 shows a simple inverted pendulum. In this inverted pendulum, the center of mass is above a cart that can move horizontally over a linear axis. The double or triple inverted pendulum corresponds to an inverted pendulum with one or two additional articulations, respectively, introducing new variables to control [11].

The mathematical model of the inverted pendulum on a cart is developed with the forces described in Figure 4 and the abbreviations used are reported in Table 1.

| Parameter | Description |
|-----------|-------------|
| M         | cart mass (kg) |
| m         | pendulum mass (kg) |
| g         | gravitational acceleration (\(\frac{m}{s^2}\)) |
| N         | normal force of the rail on the cart (N) |
| B         | friction coefficient (\(\frac{N}{m}\)) |
| l         | pendulum length (m) |
| I         | moment of inertia of the pendulum mass (kg \* m^2) |
| F         | cart applied force (N) |
| P         | vertical reaction force (N) |
| N         | horizontal reaction force (N) |
| X         | cart coordinate position (m) |
| V         | cart horizontal speed (m/s) |
| a         | horizontal cart acceleration (m/s^2) |
| \(\Phi\)  | pendulum angle relative to vertical (rad) |
| w         | pendulum angular speed (rad/s) |
| \(\propto\) | pendulum angular acceleration (\(\frac{rad}{s^2}\)) |
The analysis of the forces acting on the cart in the horizontal axis allows us to obtain the system of equations represented in Equation (1) [12].

\[
\begin{align*}
\sum F_{x_{\text{cart}}} &= M\ddot{x}, \\
M\ddot{x} &= F - b\dot{x} - N, \\
N &= F - b\dot{x} - M\ddot{x},
\end{align*}
\]  

(1)

Similarly, the system of equations of acting forces in the horizontal axis for the pendulum in describe Equation (2) [12].

\[
\begin{align*}
\sum F_{x_{\text{pendulum}}} &= m\ddot{x}, \\
m\ddot{x} &= N - ml\dot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta, \\
N &= m\ddot{x} + ml\dot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta.
\end{align*}
\]  

(2)

Where \( \dot{x} = v, \ddot{x} = a, \theta = \Phi + \pi, \dot{\theta} = w, \ddot{\theta} = \alpha \). Matching the equations of the cart and pendulum is obtained by Equation (3), that represents the first system dynamic equation; likewise, the sum of perpendicular forces to the pendulum is used to find an additional dynamic equation given by the Equation (4).

\[
\begin{align*}
F - b\dot{x} - M\ddot{x} &= m\ddot{x} + ml\dot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta, \\
F &= \ddot{x}(M + m) + b\dot{x} + ml\dot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta, \\
P_{\text{sen}}(\theta) + N\cos(\theta) - mg\text{sen}(\theta) &= ml\dot{\theta} + m\ddot{x}\cos(\theta).
\end{align*}
\]  

(3)

The torque due to the forces on the pendulum bar generates a second dynamic equation of the system together with Equation (4); to do this, combining Equation (4) and Equation (5) we obtain the Equation (6).

\[
\begin{align*}
T &= \sum F_{r_1} = I\ddot{\theta}; I\ddot{\theta} = -P_{\text{sen}}(\theta) - N\cos(\theta), \\
m\ddot{x}\cos(\theta) + ml\dot{\theta} + mg\text{sen}(\theta) &= -\frac{I\ddot{\theta}}{I}; \ddot{\theta}(1 + ml^2) + mg\text{sen}(\theta) = -m\ddot{x}\cos(\theta).
\end{align*}
\]  

(5)

The dynamics of the inverted pendulum in the cart is represented in Equation (3) and Equation (6); however, this representation is acceptable for designing the infinite H control system as a linear system. The control of the inverted pendulum is made in an unstable balance point or higher. The equations must be linearized using the pendulum angle relative to the vertical, \( \Phi \rightarrow 0 \). If \( \theta \) is the pendulum angle relative to the vertical axis, \( \theta = \Phi + \pi \) the system Equation (7) obtained are.

\[
\begin{align*}
\cos(\theta) &= \cos(\Phi + \pi) \approx -1, \\
\sin(\theta) &= \sin(\Phi + \pi) \approx -\Phi, \\
\dot{\theta}^2 &= \Phi \dot{\theta}^2 \approx 0, \\
\ddot{\theta} &= \Phi, \\
\ddot{\theta} &= \Phi.
\end{align*}
\]  

(7)

When substituting these expressions in Equation (3) and Equation (6) and considering that force \( F \) corresponds to the control signal \( u \), the linearized system Equations (8) are obtained in the operation point \( \Phi \rightarrow 0 \).

\[
\begin{align*}
\ddot{x}(M + m) + b\dot{x} - ml\dot{\Phi} &= u, \\
\Phi(ml^2 + 1) - mg\Phi &= m\ddot{x}.
\end{align*}
\]  

(8)
From the dynamic equations of the system, the Laplace transform can be applied to obtain Equation (9) [12], corresponding to the transfer function of the system; where \( U(s) \) is the input in the Laplace domain and \( \Phi(s) \) is the output in the Laplace domain.

\[
\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{d}s^2}{s^4 + \frac{b(1+ml^2)}{d}s^3 - \frac{(M+m)mgl}{d}s^2 - \frac{bmgl}{d}s} = \frac{\frac{ml}{d}s^2}{s^4 + \frac{b(1+ml^2)}{d}s^3 - \frac{(M+m)mgl}{d}s^2 - \frac{bmgl}{d}s}, \tag{9}
\]

\( d = ((M+m)(1+ml^2) - (ml)^2). \)

From the transference function in Equation (9) can be seen a pole and a zero in origin, these can be simplified, and the transference function can be written as follows Equation (10).

\[
\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{d}s^2}{s^3 + \frac{b(1+ml^2)}{d}s^2 - \frac{(M+m)mgl}{d}s - \frac{bmgl}{d}} \tag{10}
\]

Making possible to know the system stability in an open loop, as shown in Figure 5.

![Figure 5. Impulse response of the state space model.](image)

The \( x \) vector determines the system state; this consists of four elements (cart position, cart's first derivate, angle position, and the angle position derivate). For the \( y \) vector has been considered that the pendulum has three sensors, one for the cart position \( r \), the other for the angle \( \phi \), and the last for the cart speed \( \dot{r} \). The \( u \) vector has a single element, the force applied to the cart. Once meet this information, matrixes \( C \) and \( D \) can be determined (where \( D = 0 \)) Equation (11) [12].

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} =
\begin{bmatrix}
    \dot{r} \\
    \dot{\phi} \\
    \dot{r} \\
    \dot{\phi}
\end{bmatrix} ;
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} ;
\begin{bmatrix}
    u
\end{bmatrix} =
\begin{bmatrix}
    F
\end{bmatrix}. \tag{11}
\]

To obtain matrixes \( A \) and \( B \) from the space state system is obligatory to express the Equation (1) and Equation (6) in the way Equation (12).

\[
\dot{x} = f(x, u), \text{ hence,} \\
\dot{x}_1 = f_1(x, u) = x_3, \\
\dot{x}_2 = f_2(x, u) = x_4. \tag{12}
\]

Fully expressing the system in its state-space representation is obtained Equation (13).
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-l(1+m^2)b}{l(M+m)+Mml^2} & \frac{m^2gl^2}{l(M+m)+Mml^2} & 0 \\ 0 & 0 & \frac{-mb}{l(M+m)+Mml^2} & 0 \\ 0 & \frac{m^2gl^2}{l(M+m)+Mml^2} & 0 & 0 \end{bmatrix} ; B = \begin{bmatrix} 0 \\ \frac{l(1+m^2)}{l(M+m)+Mml^2} \\ 0 \\ \frac{ml}{l(M+m)+Mml^2} \end{bmatrix} ; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; D = 0. \quad (13)

### 3. Control system design

Figure 5 shows the system answer to an impulse entry and the assignation of the physical parameters presented in Table 2. Analyzing Figure 5, the need to use a control technique to enhance the system dynamic is clear. A robust control technique is needed to define the weights of weighting functions $W$, which must fulfill closed-loop criteria.

**Table 2 Physical parameters.**

| M (kg) | m (kg) | b (N/m/s) | l (m) | I (kg*m²) |
|--------|--------|-----------|-------|-----------|
| 0.7    | 0.7    | 0.1       | 0.3   | 0.006     |

The magnitude of the frequency response of the weight functions $W$ is shown in Figure 6. The value for the weight component $W_i$ is 5, given $|d_i| \leq 5$, to satisfy the robust control criteria [10].

![Figure 6. System representation.](image)

The weight signal $u$, $W_u$, is chosen, so that great importance is placed on control within the confines of the bandwidth of the closed-loop system, and it is not penalized at high frequencies. However, locating a "huge" weight out of the system bandwidth implies noise amplification and degrades performance [13]. Therefore, the system will turn challenging to control out of the bandwidth in a closed loop. Furthermore, the weight function $W_n$ models the noise introduced by the sensor in the system due to low frequencies. Finally, the weight function $W_e$ guarantees noise attenuation in low frequencies and avoids this amplification in relevant frequencies, as shown in Figure 7.

The nominal driver form to stabilize the pendulum in the vertical direction is the classic pole relocation system full state feedback (FSF), also known as pole replacement; however, it is transformed into a state-space system for ease. Using the described system of Equation (14), poles were chosen in closed loop to stabilize the system in approximately 10 seconds.

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}. \quad (14)$$

Corresponding to dominant poles located in $-0.3 \pm j0.52$. It is essential to underline that all chosen poles are in the left semi-plane to secure the closed-loop stability [14]. Limits were defined; the beyond
limit $\gamma = 0.001$ and an above limit of 50. Additionally, tolerance $\gamma$ of 0.01 is defined. Once the weights of the weighting functions $W$ are defined, robust control is performed in $H_\infty$ and in a robust control that analyzes the characteristics (Control $H_2$) of the inverted pendulum system on the cart.

4. Results
Figure 8 and Figure 9 show the pendulum angle and the required control quantity to stabilize the pendulum in a vertical direction. The perturbation was modeled as an impulse signal with a peak value of 0.1, presenting a relation between angle control and the effort of the control system; based on Figure 8, the control $H_\infty$ produces minor angle variation in the vertical axis; the controller $H_2$ can be considered the second most effective relative to angle variation. Nevertheless, the used system as a nominal controller presents the most huge-angle variation. This result was expected since the FSF control does not guarantee robustness when assuming uncertainties in the system.

Figure 9 shows that controller $H_\infty$ applies a higher control effort at instant $t = 2s$; however, the average effort made by controller $H_\infty$ is significantly minor than that of the other controllers. Among the techniques used, the advantage in terms of time and stability of the $H_\infty$ controller can be observed; but this advantage implies a more complex elaboration in the description of the plant, since it implies constructing a cost function that relates the minimizer and the penalty; for practical purposes in which the disturbances are small and the return time can be flexible, the $H_2$ type control is adequate.

5. Conclusions
The weights of weighting functions were made through heuristic techniques; thus, optimum techniques can improve the control design; for the case of an inverted pendulum above a cart, it is possible to evidence a better controller $H_\infty$ performance compared with control $H_2$ and a model of relocation poles FSF in terms.
The determination of the weights is much more complex through exact techniques, on the contrary, metaheuristic techniques are not only more efficient, but also allow to be more flexible. For this reason, the optimization techniques improve as in our case, since the cost function associated with the infinite $H$ control must minimize said function, thus, the implementation of metaheuristics improves the optimization theory in the same way as that of the control design; for the case of an inverted pendulum on a carriage, it is possible to demonstrate a better performance of the $H_{\infty}$ controller compared to the $H_2$ control and a relocation model through full state feedback.

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