Finite Rigid Sets in Arc Complexes

Emily Shinkle

Abstract

For any compact, connected, orientable, finite-type surface with marked point other than the sphere with three marked points, we construct a finite rigid set of its arc complex: a finite simplicial subcomplex of its arc complex such that any locally injective map of this set into the arc complex of another surface with arc complex of the same dimension is induced by a homeomorphism of the surfaces, unique up to homotopy in most cases. It follows that if the arc complexes of two surfaces are isomorphic, the surfaces are homeomorphic. We also give an exhaustion of the arc complex by finite rigid sets. This extends the results of Irmak–McCarthy.

1 Introduction

Let $S, S'$ be compact, connected, orientable, finite-type surfaces with marked points. By $S = S_{g,n}$ we mean $S$ has genus $g$ and $n \geq 1$ marked points. The arc complex $\mathcal{A}(S)$ of $S$, first defined by Harer in [7], is a simplicial complex whose vertices correspond to the isotopy classes of arcs on $S$ and whose $k$-simplices ($k > 0$) correspond with collections of $k + 1$ distinct isotopy classes of arcs which are pairwise disjoint. Homeomorphisms $H : S \to S$ induce simplicial isomorphisms $H_* : \mathcal{A}(S) \to \mathcal{A}(S)$. Conversely, in [19], Irmak–McCarthy prove that every simplicial automorphism (in fact, every injective simplicial self-map) of $\mathcal{A}(S)$ arises from a homeomorphism $H$ of $S$. Let $\text{Mod}^+ S$ denote the extended mapping class group of $S$, the group of homotopy classes of homeomorphisms of $S$. Irmak–McCarthy also show that if $S$ is not $S_{0,1}$, $S_{0,2}$, $S_{0,3}$ or $S_{1,1}$, the homeomorphism $H$ is unique up to homotopy, and in these exceptional cases, $H$ is unique up to the (finite) center $Z(\text{Mod}^+ S)$. Recall that a simplicial map is locally injective if the restriction to the star of every vertex is injective. In this paper, we adapt the arguments of Irmak–McCarthy to show the following.

**Theorem 1.1.** Let $S = S_{g,n}$ be a compact, connected, orientable surface of genus $g$ with $n \geq 1$ marked points. If $S \neq S_{0,3}$, then there exists a finite simplicial subcomplex $\mathcal{X}$ of $\mathcal{A}(S)$ such that for any surface $S'$ with $\dim(\mathcal{A}(S)) = \dim(\mathcal{A}(S'))$ and any locally injective simplicial map $\lambda : \mathcal{X} \to \mathcal{A}(S')$, there is a homeomorphism $H : S \to S'$ which induces $\lambda$. Moreover, if $S \neq S_{0,1}$, $S_{0,2}$, or $S_{1,1}$, $H$ is unique up to homotopy, and in these exceptional cases, $H$ is unique up to $Z(\text{Mod}^+ S)$.

We refer to any simplicial subcomplex of $\mathcal{A}(S)$ with this property as rigid. We provide a counterexample to the theorem for $S = S_{0,3}$; however we use a carnality argument to show the following corollary even for $S_{0,3}$.

**Corollary 1.2.** Let $S$ and $S'$ be compact, connected, orientable, finite-type surfaces with marked points. If $\mathcal{A}(S)$ and $\mathcal{A}(S')$ are isomorphic, then $S$ and $S'$ are homeomorphic.
We also give an exhaustion of \( \mathcal{A}(S) \) by finite rigid sets.

**Theorem 1.3.** For any compact, connected, orientable, finite-type surface \( S \) with marked points such that \( S \neq S_{0,3} \) there exists a sequence \( (X_i)_{i \in \mathbb{N}} \) of finite rigid sets of \( \mathcal{A}(S) \) such that \( X_0 \subseteq X_1 \subseteq \ldots \subseteq \mathcal{A}(S) \) and \( \bigcup_{i \in \mathbb{N}} X_i = \mathcal{A}(S) \).

**Remark.** Irmak–McCarthy prove their results for surfaces with \( n \) boundary components rather than \( n \) marked points; however, they consider arcs up to isotopy not necessarily fixing the endpoints of the arcs. This implies that Dehn twists around boundary components act trivially on \( \mathcal{A}(S) \). Hence we can think of these boundary components as marked points instead. In \( [6] \), Disarlo considers surfaces with nonempty boundary, with at least one marked point on each boundary component and with a finite number of punctures in the interior. In contrast to Irmak–McCarthy, she considers arcs up to isotopy fixing the endpoints; hence, in this setting Dehn twists around boundary components act nontrivially on \( \mathcal{A}(S) \). Disarlo proves that in this context, isomorphisms between arc complexes are induced by homeomorphisms of the associated surfaces.

Ivanov was the first to prove this type of result, demonstrating in \( [21] \) that any automorphism of the curve complex of a surface with genus \( g \geq 2 \) is induced by a homeomorphism of the surface. Korkmaz \( [23] \) and Luo \( [20] \) extend Ivanov’s result for \( g < 2 \). Similar results for other complexes associated to surfaces are often referred to as Ivanov-style rigidity or simply rigidity. Rigidity results were proved for the pants complex (Margalit \( [27] \), Aramayona \( [1] \)), the arc and curve complex (Korkmaz–Papadopoulos \( [24] \)), the ideal triangulation graph or flip graph (Korkmaz–Papadopoulos \( [25] \)), the Schumtz graph of nonseparating curves (Schaller \( [33] \)), the complex of nonseparating curves (Irmak \( [11] \)), the Hatcher-Thurston complex (Irmak–Korkmaz \( [18] \)), and the polygonalisation complex (Bell–Disarlo–Tang \( [4] \)), among others. See \( [30] \) for a survey. Rigidity results exist for complexes of non-orientable surfaces as well, such as the curve complex of non-orientable surfaces (Irmak \( [15], [16], [17] \)), the arc graph (Irmak \( [14] \)), and the two-sided curve complex (Irmak–Paris \( [20] \)). In \( [22] \), Ivanov conjectured that every object naturally associated to a surface with sufficiently rich structure has the extended mapping class group as its group of automorphisms. Brendle–Margalit \( [5] \) prove general rigidity results about subcomplexes of the complex of domains (introduced by McCarthy–Papadopoulos in \( [30] \)) for closed surfaces and McLeay \( [31] \) proves general rigidity results for subcomplexes of the complex of domains of punctured surfaces. These imply rigidity for a large class of complexes towards Ivanov’s conjecture.

Finite rigidity results also exist for some of these complexes. Aramayona–Leininger construct finite rigid sets in the curve complex in \( [3] \) and provide an exhaustion of the curve complex by finite rigid sets in \( [2] \). Ilbira–Korkmaz construct finite rigid sets of the curve complex of non-orientable surfaces in \( [10] \) and Irmak gives an exhaustion of the curve complex of non-orientable surfaces by finite rigid sets in \( [12] \), strengthening her exhaustion by finite superrigid sets in \( [13] \). Maungchang \( [29] \) proves finite rigidity for the pants graph of a punctured sphere and Hernández-Hernández–Leininger–Maungchang extend the result for any surface \( S_{g,n} \) in \( [9] \). Maungchang also gives an exhaustion of the pants graph of punctured spheres by finite rigid sets in \( [28] \).

**Acknowledgements.** The author would like to thank Christopher Leininger for suggesting this project and for providing guidance and support throughout it. She would also like to thank Marissa Miller and Dan Margalit for their helpful comments on a draft of the paper.
2 The Arc Complex

Let \( S = S_{g,n} \) denote a compact, connected, orientable surface (without boundary) of genus \( g \) with \( n \geq 1 \) marked points. Let \( \mathcal{P}_S \) be the set of marked points of \( S \). Maps between surfaces will be assumed to map marked points to marked points.

Definition. An arc on \( S \) is a map \( \gamma : [0,1] \to S \) such that

- \( \gamma(0), \gamma(1) \in \mathcal{P}_S \),
- \( \gamma((0,1)) \cap \mathcal{P}_S = \emptyset \), and
- \( \gamma|_{(0,1)} \) is injective.

We will identify an arc \( \gamma \) with its image \( \gamma([0,1]) \) on \( S \) and we call \( \gamma((0,1)) \) the interior of the arc. Any isotopy of arcs will be through arcs. In particular, isotopies of arcs will be relative to the endpoints and are not permitted to pass through marked points. We will assume that all arcs are essential, i.e. cannot be isotoped into an arbitrary neighborhood of a marked point.

Definition. If \( a \) and \( b \) are isotopy classes of arcs on \( S \), then the geometric intersection number \( i(a,b) \), or intersection number, is the minimum number of intersection points of the interiors of representatives of \( a \) and \( b \).

Definition. The arc complex \( \mathcal{A}(S) \) is a simplicial complex whose vertices correspond to the isotopy classes of arcs on \( S \) and whose \( k \)-simplices \((k > 0)\) correspond with collections of \( k + 1 \) distinct isotopy classes of arcs which have pairwise disjoint representatives.

Unless necessary, we will not distinguish between a isotopy class of arcs, a representative of the class, and the corresponding vertex of the arc complex. By distinct arcs, we mean distinct isotopy classes of arcs. By disjoint arcs we mean \( i(a,b) = 0 \).

On \( S_{0,1} \), there are no essential arcs, hence \( \mathcal{A}(S_{0,1}) = \emptyset \). On \( S_{0,2} \), there is only one essential arc, hence \( \mathcal{A}(S_{0,2}) \) is a single point and has dimension 0. Any surface \( S = S_{g,n} \) is not \( S_{0,1} \) or \( S_{0,2} \), has at least three distinct, disjoint arcs, and we call a maximal collection of distinct pairwise disjoint arcs on \( S \) a triangulation of \( S \). There is a natural \( \Delta \)-complex associated to a triangulation these with the arcs as its 1-skeleton, hence the name “triangulation” (see e.g. Hatcher [8, Section 2.1] for more on \( \Delta \)-complexes). We will refer to the 2-cells as triangles of \( T \). We say that an arc \( a \) is a side of a triangle \( \Delta \) in \( T \) if \( a \) is contained in \( \partial \Delta \). We say a triangle is embedded if its sides are distinct arcs on \( S \) and that it is non-embedded otherwise. Note that an embedded triangle is not required to have distinct vertices. All non-embedded triangles are of the form pictured in Figure 1. Call the side of a non-embedded triangle which joins two distinct punctures the inner arc (e.g. arc \( a \) in Figure 1). Call the other arc the outer arc (e.g. arc \( b \) in Figure 1).

We can use the Euler characteristic to see that the number of arcs in a triangulation of \( S \) is \( 6g + 3n - 6 \), hence \( \dim(\mathcal{A}(S)) = 6g + 3n - 7 \). If \( \dim(\mathcal{A}(S)) = \dim(\mathcal{A}(S')) \), a locally injective simplicial map from a subcomplex \( \mathcal{A}' \) of \( \mathcal{A}(S) \) into \( \mathcal{A}(S') \) sends any triangulation \( T \) contained in \( \mathcal{A}' \) to a triangulation \( T' \) of \( S' \). The number of triangles in a triangulation is \( 4g + 2n - 4 \). Hence, if \( \dim(\mathcal{A}(S)) = \dim(\mathcal{A}(S')) \), then a triangulation of \( S \) has the same number of triangles as a triangulation of \( S' \).

We say that two triangulations are obtained from each other by a flip if they differ by exactly one arc. In this case, the distinct arcs have intersection number one. Conversely, if
two distinct arcs $a$ and $b$ have intersection number one, there exist triangulations $T_a$ and $T_b$ containing $a$ and $b$, respectively, such that $T_a \setminus \{a\} = T_b \setminus \{b\}$.

We will need the following result of Mosher regarding triangulations later.

**Proposition 2.1.** (Mosher, [32]). Let $S$ be a compact, connected, orientable, finite-type surface with marked points. Any two triangulations of $S$ differ by a finite number of flips.

### 3 Exceptional Cases

In this section, we dispense with the cases where $A(S)$ is empty or has dimension $\leq 2$, i.e., if $S$ is $S_{0,1}$, $S_{0,2}$, $S_{0,3}$, or $S_{1,1}$. As discussed above, $A(S_{0,1}) = \emptyset$ and if $S \neq S_{0,1}$, then $A(S) \neq \emptyset$. Also recall that $A(S_{0,2})$ is a single vertex, hence has dimension 0, and if $S \neq S_{0,2}$, then $\dim(A(S)) \neq 0$. In both of these cases, Theorem 1.1 follows from setting $\mathcal{X} = A(S)$ and applying the results of Irmak–McCarthy [19, Theorems 1.1, 1.2].

Now suppose $\dim(A(S)) = 2$. Then $S$ is $S_{0,3}$ or $S_{1,1}$. It is well-known that $A(S_{0,3})$ is isomorphic to a regular tessellation of a triangle by four triangles (see eg. [19], [30]) and that $A(S_{1,1})$ is isomorphic to the flag complex of the Farey graph, a decomposition by ideal triangles of $H^2 \cup \mathbb{QP}^1$ (see eg. [19]). Figure 2 shows a simplicial embedding of $A(S_{0,3})$ into $A(S_{1,1})$ but $S_{0,3}$ and $S_{1,1}$ are not homeomorphic. Hence, Theorem 1.1 cannot be extended for $S_{0,3}$. However, we can find a finite rigid set in $A(S_{1,1})$.

![Figure 2: A simplicial embedding of $A(S_{0,3})$ into $A(S_{1,1})$](image)

**Proof of Theorem 1.1 for $S = S_{1,1}$**. Let $\mathcal{X}$ be the simplicial subcomplex of $A(S)$ indicated in grey in Figure 3. Let $S'$ be such that $\dim(A(S)) = \dim(A(S'))$ and let $\lambda : \mathcal{X} \to A(S')$ be a locally injective simplicial map. Observe that $\mathcal{X}$ is contained in the closure of the star of the vertex marked $a$, hence $\lambda$ is injective. Then since there are five vertices in $\mathcal{X}$ connected
to $a$ by an edge in $\mathcal{X}$ and since every vertex in $A(S_{0,3})$ has degree at most four, $S' \neq S_{0,3}$. Hence $S' = S_{1,1}$. Now $A(S)$ is connected and each edge borders exactly two triangles, so an induction argument shows that $\lambda$ can be extended uniquely to to an automorphism of $A(S)$. Then we again apply the results of Irmak–McCarthy.

Now we can give an exhaustion of $A(S)$ by finite rigid sets in the cases that $S$ is $S_{0,1}$, $S_{0,2}$, and $S_{1,1}$.

**Proof of Theorem 1.3 for $S_{0,1}$, $S_{0,2}$, and $S_{1,1}$.** Suppose $S$ is $S_{0,1}$ or $S_{0,2}$. Then as discussed above, $A(S)$ is finite, so we can take $\mathcal{X}_i = A(S)$ for all $i \in \mathbb{N}$. Now suppose $S$ is $S_{1,1}$. Let $\mathcal{X}_0$ be the finite rigid set of $A(S)$ from the proof above. For $i \in \mathbb{N}$, let $\mathcal{X}_{i+1}$ be the simplicial subcomplex of $A(S)$ containing $\mathcal{X}_i$ and any triangles which share a side with a triangle in $\mathcal{X}_i$. Then $(\mathcal{X}_i)_{i \in \mathbb{N}}$ is an exhaustion of $A(S)$ by finite rigid sets by the same argument as above.

### 4 The General Case

In this section, $S$ and $S'$ will be surfaces with $\dim(A(S)) = \dim(A(S')) > 2$. We will use the following definition.

**Definition.** Let $V = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a collection of simplices of a simplicial complex $C$. The span of $V$, $\text{Span}_C(V)$, is the set of all simplices $\tau$ of $C$ such that each vertex of $\tau$ is the vertex of a simplex in $V$.

**Remark.** Observe that $V$ finite implies that $\text{Span}_C(V)$ is finite as well.

Roughly speaking, the proof of Theorem 1.3 will proceed as follows: We will include in $\mathcal{X}$ (the span of) a triangulation $T$ of $S$ which maps to a triangulation $T'$ of $S'$ under any locally injective map. We first show that, by adding finitely many arcs to $\mathcal{X}$, we can guarantee that triangles of $T$ map to triangles of the same type (embedded or non-embedded) in $T'$. Then we show that, by adding finitely many more arcs to $\mathcal{X}$, we can guarantee that the orientations of adjacent triangles in $T'$ match so that the map $T \to T'$ can be extended to a homeomorphism $H : S \to S'$. We use Proposition 2.1 to show that by including finitely many more arcs in $\mathcal{X}$ we can guarantee that any locally injective simplicial map $\lambda : \mathcal{X} \to A(S')$ agrees with the induced map $H_*$ on all of $\mathcal{X}$. Finally, we show uniqueness by proving that any other such homeomorphism $H'$ has induced map equal to $H_*$ and then applying the results of Irmak–McCarthy.
First, we need the following (cf. [19] Prop. 3.1]).

**Lemma 4.1.** If \((a, b)\) is a pair of arcs with \(i(a, b) = 1\), then there exists a finite simplicial subcomplex \(\mathcal{B}\) of \(\mathcal{A}(S)\) containing \(a\) and \(b\) with the following property: If \(\mathcal{X}\) is a simplicial subcomplex of \(\mathcal{A}(S)\) which contains \(\mathcal{B}\) and \(\lambda : \mathcal{X} \to \mathcal{A}(S')\) a locally injective simplicial map, then \(i(\lambda(a), \lambda(b)) = 1\).

**Proof.** There exist triangulations \(T_a\) and \(T_b\) of \(S\) which share all arcs except for \(a \in T_a\) and \(b \in T_b\). Let \(T_0 = T_a \cap T_b\) and let

\[
\mathcal{B} = \text{Span}_{\mathcal{A}(S)}(T_0 \cup \{a, b\})
\]

Now suppose \(\mathcal{X}\) is a simplicial subcomplex of \(\mathcal{A}(S)\) which contains \(\mathcal{B}\) and \(\lambda : \mathcal{X} \to \mathcal{A}(S')\) a locally injective simplicial map. Note that \(\lambda|_\mathcal{B}\) is injective since \(\lambda\) is locally injective. The simplex spanned by \(T_0\) has codimension one in \(\mathcal{A}(S)\), so \(T_0' = \lambda(T_0)\) has codimension one in \(\mathcal{A}(S')\). Let \(a' = \lambda(a)\) and \(b' = \lambda(b)\). Then \(\lambda(T_a') = \{a'\} \cup T_0'\) and \(\lambda(T_b') = \{b'\} \cup T_0'\) are both triangulations of \(S'\). It follows that \(i(a', b') = 1\). \(\square\)

We will also use the following result of Irmak–McCarthy [19, Prop. 3.2]:

**Proposition 4.2.** Let \(\Delta\) be an embedded triangle on \(S\) with sides \(a, b,\) and \(c\). Then there exists a triangulation \(T\) on \(S\) containing \(a, b,\) and \(c\) such that the unique triangles \(\Delta_a, \Delta_b,\) and \(\Delta_c\) of \(T\) on \(S\) which are different from \(\Delta\) and have, respectively, \(a\) as a side, \(b\) as a side, and \(c\) as a side, are distinct triangles of \(T\) on \(S\).

The above proposition is proved by taking an arbitrary triangulation containing \(a, b,\) and \(c\) and using flips to obtain a new triangulation with the desired properties. Applying Proposition 4.2 we deduce the following (cf. [19] Prop. 3.3)).

**Lemma 4.3.** If \(a, b,\) and \(c\) are the edges of an embedded triangle on \(S\), then there exists a finite simplicial subcomplex \(\mathcal{C}\) of \(\mathcal{A}(S)\) containing \(a, b,\) and \(c\) with the following property: If \(\mathcal{X}\) is a simplicial subcomplex of \(\mathcal{A}(S)\) which contains \(\mathcal{C}\) and \(\lambda : \mathcal{X} \to \mathcal{A}(S')\) a locally injective simplicial map, then \(\lambda(a), \lambda(b),\) and \(\lambda(c)\) are the edges of an embedded triangle on \(S'\).

**Proof.** By Proposition 4.2 there exists a triangulation \(T\) of \(S\) containing \(\{a, b, c\}\) so that the unique triangles \(\Delta_a, \Delta_b,\) and \(\Delta_c\) of \(T\) on \(S\) which are different from \(\Delta\) and have, respectively, \(a\) as a side, \(b\) as a side, and \(c\) as a side, are distinct triangles of \(T\) on \(S\).

Let \(P_1\) be the vertex of \(\Delta_a\) opposite \(a,\) \(P_2\) the vertex of \(\Delta_b\) opposite \(b,\) and \(P_3\) the vertex of \(\Delta_c\) opposite \(c\) (see Figure 4). There exists an arc \(d\) connecting \(P_1\) to \(P_2\) which intersects \(a\) and \(b\) once and is disjoint from all other arcs in \(T\). Similarly, there exists an arc \(e\) connecting \(P_2\) to \(P_3\) which intersects \(b\) and \(c\) once and is disjoint from all other arcs in \(T\). And finally, there is an arc \(f\) connecting \(P_1\) to \(P_3\) which intersects \(a\) and \(c\) once and is disjoint from all other arcs in \(T\). Note that \(d, e,\) and \(f\) are pairwise disjoint.

Now, consider the following six pairs of arcs: \(\mathcal{A}_1 = (a, d), \mathcal{A}_2 = (a, f), \mathcal{A}_3 = (b, d), \mathcal{A}_4 = (b, e), \mathcal{A}_5 = (c, e),\) and \(\mathcal{A}_6 = (c, f).\) Each of these pairs has intersection number one.

Let \(B_i\) be the finite simplicial complex of \(\mathcal{A}(S)\) from Lemma 4.1 corresponding to \(\mathcal{A}_i\) for each \(1 \leq i \leq 6.\)

Let

\[
\mathcal{C} = \text{Span}_{\mathcal{A}(S)} \left( T \cup \{d, e, f\} \cup \bigcup_{1 \leq i \leq 6} B_i \right)
\]
Suppose \( \mathcal{X} \) is any simplicial subcomplex of \( \mathcal{A}(S) \) which contains \( C \) and \( \lambda : \mathcal{X} \to \mathcal{A}(S') \) a locally injective simplicial map. Let \( T' = \lambda(T) \), a triangulation of \( S' \). Additionally, let \( a' = \lambda(a), b' = \lambda(b), c' = \lambda(c), d' = \lambda(d), e' = \lambda(e) \), and \( f' = \lambda(f) \). Lemma 4.1 and the local injectivity of \( \lambda \) guarantee each pair in \( \{a', b', c', d', e', f'\} \) has the same intersection number as its preimage under \( \lambda \).

Since \( d' \) intersects \( a' \) and \( b' \) once each and is disjoint from all other arcs in the triangulation \( T' \), it must be the case that \( a' \) and \( b' \) border an embedded triangle in \( T' \) — call it \( \Delta_1 \). Similarly, since \( c' \) intersects \( b' \) and \( c' \) once each and is disjoint from all other arcs in the triangulation \( T' \), there is an embedded triangle \( \Delta_2 \) in \( T' \) with sides \( b' \) and \( c' \). Finally, a similar argument using \( f' \) demonstrates the existence of an embedded triangle \( \Delta_3 \) with sides \( a' \) and \( c' \). Suppose that the third side of \( \Delta_1 \) is \( r' \), the third side of \( \Delta_2 \) is \( s' \), and the third side of \( \Delta_3 \) is \( t' \). If \( r' = c', s' = a' \), or \( t' = b' \), then we are done.

Suppose \( r' \neq c', s' \neq a' \), and \( t' \neq b' \), hence \( \Delta_1 \neq \Delta_2 \), \( \Delta_2 \neq \Delta_3 \) and \( \Delta_3 \neq \Delta_1 \). Up to homeomorphism (and ignoring admissible identifications among arcs and among vertices of the triangles), there are two possible configurations of \( \Delta_1 \) and \( \Delta_2 \), depending on their relative orientations. For each configuration, there are two configurations (up to homeomorphism and ignoring admissible identifications among arcs and among vertices of the triangles) of \( \Delta_3 \) with \( \Delta_1 \) and \( \Delta_2 \), depending on the relative orientation of \( \Delta_3 \) with \( \Delta_1 \) and \( \Delta_2 \). See Figure 5. In each of these cases, \( d' \) and \( e' \) must intersect, a contradiction. \( \square \)

Now we can use Lemma 4.3 to prove the corresponding result for non-embedded triangles (cf. [19, Prop. 3.4]).

**Lemma 4.4.** If \( a \) and \( b \) border a non-embedded triangle on \( S \) with \( a \) the inner arc, then there exists a finite simplicial subcomplex \( D \) of \( \mathcal{A}(S) \) containing \( a \) and \( b \) with the following property: If \( \mathcal{X} \) is a simplicial subcomplex of \( \mathcal{A}(S) \) which contains \( D \) and \( \lambda : \mathcal{X} \to \mathcal{A}(S') \) a locally injective simplicial map, then \( \lambda(a) \) and \( \lambda(b) \) border a non-embedded triangle on \( S' \) with \( \lambda(a) \) as the inner arc.

**Proof.** Since \( S \) is not \( S_{0,3} \), there exists an embedded triangle having \( b \) as a side. Call the other sides of this triangle \( c \) and \( d \). Let \( e \) be the arc pictured in Figure 6.

We can see that arcs \( a, c, e \) and \( e \) border an embedded triangle on \( S \). Let \( T' \) be the corresponding finite simplicial subcomplex of \( \mathcal{A}(S) \) from Lemma 4.3 respectively. Also, observe that \( i(b,e) = 1 \). Then let \( B \) be the corresponding finite simplicial subcomplex of \( \mathcal{A}(S) \) from Lemma 4.1. Let

\[
D = \text{Span}_{\mathcal{A}(S)}(\{a, b, c, d, e\} \cup B \cup C_1 \cup C_2).
\]

Let \( \mathcal{X} \) be any simplicial subcomplex of \( \mathcal{A}(S) \) which contains \( D \) and \( \lambda : \mathcal{X} \to \mathcal{A}(S') \) a locally injective simplicial map. Let \( a' = \lambda(a), b' = \lambda(b), c' = \lambda(c), d' = \lambda(d), \) and
Lemma 4.3 and the local injectivity of $\lambda$ guarantee each pair in $\{a', b', c', d', e'\}$ has the same intersection number as the preimage under $\lambda$. By Lemma 4.3, we know that $a'$, $c'$, and $e'$ border an embedded triangle on $S'$ and that $a'$, $d'$, and $e'$ also border an embedded triangle on $S'$. Since $b'$ is disjoint from $a'$, $c'$, and $d'$ and intersects $e'$ once, the only possible arrangement of arcs then guarantees that $a'$ and $b'$ border a non-embedded triangle on $S'$ with $a'$ as the inner arc.

The two lemmas above give conditions such that a locally injective simplicial map takes a triangulation $T$ of $S$ to a triangulation $T'$ of $S'$ in such a way that triangles in $T$ are sent to triangles of the same type (embedded or non-embedded) in $T'$. The following lemma provides conditions under which two triangles in $T$ which share an edge map to consistently oriented triangles in $T'$, so that the map from $T$ to $T'$ can be extended to a homeomorphism from $S$ to $S'$ (cf. [19, Prop. 3.5–3.7]). We make this precise in the proof of Proposition 4.6.

If $\Delta$ is a triangle, let $\bar{\Delta}$ denote $\Delta \setminus P_3$, the triangle minus its vertex set. The edges of $\bar{\Delta}$ are the interiors of arcs; however, for simplicity, we will not use a separate notation for them.

Lemma 4.5. Suppose $\Delta_1$ is an embedded triangle on $S$ with sides $a$, $b$, and $c$, and $\Delta_2$ an embedded triangle with sides $c$, $d$, and $e$, as shown in Figure 7. Then there exists a finite simplicial subcomplex $\mathcal{E}$ of $\mathcal{A}(S)$ containing $a$, $b$, $c$, $d$, and $e$ with the following property: Let $\mathcal{X}$ be any simplicial subcomplex of $\mathcal{A}(S)$ containing $\mathcal{E}$ and $\lambda: \mathcal{X} \to \mathcal{A}(S')$ a locally injective simplicial map. Let $a' = \lambda(a)$, $b' = \lambda(b)$, $c' = \lambda(c)$, $d' = \lambda(d)$, and $e' = \lambda(e)$. Then there exist an embedded triangle $\Delta'_1$ on $S'$ with sides $a'$, $b'$, and $c'$, an embedded triangle $\Delta'_2$ on $S'$ with sides $c'$, $d'$, and $e'$, and the natural homeomorphisms $F_1 : (\bar{\Delta}_1, a, b, c) \to (\bar{\Delta}'_1, a', b', c')$.
Figure 6: Arc configurations. Outer vertices may be identified.

Figure 7: Two embedded triangles sharing a side. We allow for the possibility that $a = d$, $a = e$, $b = d$ or $b = e$. We also allow identifications among vertices.

and $F_2 : (\Delta_2', c, d, e) \rightarrow (\Delta_2', c', d', e')$ can be made to agree along $c$.

Proof. Let $f$ be the arc shown in Figure 7. Then $f$ is an arc on $S$ which intersects $c$ exactly once and is disjoint from $a$, $b$, $d$, and $e$. Then $a$, $d$, and $f$ are the sides of a triangle on $S$. This triangle is non-embedded if $a = d$ and embedded otherwise. Let $K$ be the finite simplicial subcomplex of $A(S)$ from either Lemma 4.3 or Lemma 4.4 corresponding to this triangle. Further, let $B$ be the finite simplicial complex of $A(S)$ from Lemma 4.1 corresponding to the pair $(c, f)$ and let $C_1$ and $C_2$ be the finite simplicial complexes of $A(S)$ from Lemma 4.3 corresponding to $\Delta_1$ and $\Delta_2$, respectively. Let

$$E = \text{Span}_{A(S)}(\{a, b, c, d, e, f\} \cup B \cup C_1 \cup C_2 \cup K).$$

Let $\mathcal{X}$ be any simplicial subcomplex of $A(S)$ containing $E$ and $\lambda : \mathcal{X} \rightarrow A(S')$ a locally injective simplicial map. By Lemma 4.3 there is an embedded triangle $\Delta'_1$ on $S'$ with sides $a'$, $b'$ and $c'$ and an embedded triangle $\Delta'_2$ on $S'$ with sides $c'$, $d'$, and $e'$. Further, by Lemma 4.1 we know that $i(c', f') = 1$. Since $\lambda$ is simplicial, $a'$, $b'$, $d'$, and $e'$ are disjoint from $f'$.

Depending on the identifications of sides, either Lemma 4.3 or Lemma 4.4 implies that $a'$, $d'$ and $f'$ border a triangle. By inspection, we see that these conditional hold simultaneously only if the orientation of $\Delta'_1$ relative to $\Delta'_2$ is the same as the orientation of $\Delta_1$ relative to $\Delta_2$. Figure 8 for example, shows the case where no sides are identified and the relative orientations are reversed. As another example, Figure 9 shows the case where $b = e$ and the relative orientations are reversed. In both cases, there is no possible placement of $f'$ which satisfies all the conditions above. All possible identifications of the sides of $\Delta_1$ and $\Delta_2$ yield this result, hence the natural homeomorphisms $F_1 : (\Delta_1, a, b, c) \rightarrow (\Delta'_1, a', b', c')$ and $F_2 : (\Delta_2, c, d, e) \rightarrow (\Delta'_2, c', d', e')$ can be made to agree along $c$. \qed
Figure 8: Triangle configurations reversing relative orientation

Figure 9: Triangle configurations reversing relative orientation

Now we apply the above results to find a candidate homeomorphism (cf. [19, Prop. 3.8]).

**Proposition 4.6.** Let $T$ be a triangulation of $S$. Then there exists a finite simplicial subcomplex $F$ of $A(S)$ containing $T$ with the following property: If $X$ is any simplicial subcomplex of $A(S)$ containing $F$ and $\lambda : X \to A(S')$ a locally injective simplicial map, there exists a homeomorphism $H : S \to S'$ whose induced map on $A(S)$ agrees with $\lambda$ on $T$.

**Proof.** There are $N = 4g + 2n - 4$ triangles in $T$. Denote them by $\{\Delta_i : 1 \leq i \leq N\}$. Since $S$ is not $S_{0,3}$ or $S_{1,1}$, no two components $\Delta_i$, $\Delta_j$ share the same three sides. Suppose that $M$ is the number of embedded triangles in $T$. Then after reordering, we may assume that $\{\Delta_i : 1 \leq i \leq M\}$ are embedded triangles and $\{\Delta_j : M + 1 \leq j \leq N\}$ are non-embedded triangles. Then for each $1 \leq i \leq M$, let $C_i$ be the finite simplicial subcomplex of $A(S)$ from Lemma 4.3 corresponding to $\Delta_i$, and for each $M + 1 \leq i \leq N$, let $D_i$ be the finite simplicial subcomplex of $A(S)$ from Lemma 4.4 corresponding to $\Delta_i$.

Suppose there are $K$ arcs in $T$ which border two distinct triangles of $T$. (The inner arc of a non-embedded triangle does not have this property, so it may be that $K < N$.) Let $E_i$, $1 \leq i \leq K$ be the finite simplicial complexes of $A(S)$ from Lemma 4.5 corresponding to each pair. Then let

$$F = \text{Span}_{A(S)} \left( T \cup \bigcup_{1 \leq i \leq M} C_i \cup \bigcup_{M + 1 \leq i \leq N} D_i \cup \bigcup_{1 \leq i \leq K} E_i \right)$$

Let $X$ be any simplicial subcomplex of $A(S)$ containing $F$ and let $\lambda : X \to A(S')$ be a locally injective simplicial map. Let $T' = \lambda(T)$, a triangulation of $S'$. Since $\dim(A(S)) =$
dim(\(A(S')\)), there are also \(N\) triangles in \(T'\). Denote them by \(\{\Delta'_i : 1 \leq i \leq N\}\). Suppose \(\Delta_i\) is a triangle in \(S\) with sides \(a, b,\) and \(c\). Then Lemmas 4.3 and 4.4 ensure that \(\lambda(a), \lambda(b),\) and \(\lambda(c)\) border a triangle in \(T'\) of the same type (embedded or non-embedded). Then after reordering, we may assume that \(\Delta'_i\) corresponds to \(\Delta_i\) in this way.

Suppose \(\Delta_i\) is embedded with sides \(a_i, b_i,\) and \(c_i\) and that \(a'_i = \lambda(a_i), b'_i = \lambda(b_i),\) and \(c'_i = \lambda(c_i)\) border \(\Delta'_i\). Then there exists a homeomorphism \(F_i : (\Delta_i, a_i, b_i, c_i) \to (\Delta'_i, a'_i, b'_i, c'_i)\). This homeomorphism is well-defined up to relative isotopies and its orientation type is fixed.

Suppose \(\Delta_i\) is non-embedded with inner arc \(a_i\) and outer arc \(b_i\). Then \(a'_i = \lambda(a_i)\) is the inner arc of a triangle \(\Delta'_i\) with outer arc \(b'_i = \lambda(b_i)\). Further, there exists two homeomorphisms \(F_i, F^*_i : (\Delta_i, a_i, b_i) \to (\Delta'_i, a'_i, b'_i)\) with opposite orientation types.

Now suppose \(\Delta_i\) and \(\Delta_j\) are two distinct triangles in a triangulation \(T\) of \(S\) which share the side \(s\). Since \(S\) is not \(S_{0,3}\), it cannot be the case that \(\Delta_i\) and \(\Delta_j\) are both non-embedded triangles. Suppose \(\Delta_i\) is non-embedded and \(\Delta_j\) is embedded. The shared side \(s\) must be the outer arc of \(\Delta_j\). Then one of \(F_i\) or \(F^*_i\) can be made to agree with \(F_j\) along \(s\). If \(\Delta_i\) and \(\Delta_j\) are both embedded, Lemma 4.5 guarantees that \(F_i\) and \(F^*_i\) can be made to agree along \(s\).

Hence we can choose homeomorphisms \(G_i : \hat{\Delta}_i \to \hat{\Delta}_i'\) for each \(1 \leq i \leq N\) where \(G_i\) is isotopic to \(F_i\) if \(\Delta_i\) is embedded and to \(F^*_i\) if \(\Delta_i\) is non-embedded, so that if \(\Delta_i\) and \(\Delta_j\) share a common side \(s\), then \((G_i)_s = (G_j)_s\).

Then there is a homeomorphism of the punctured surfaces \(G : S\setminus\mathcal{P}_S \to S'\setminus\mathcal{P}_{S'}\), whose restriction to \(\hat{\Delta}_i\) is equal to \(G_i\) for \(1 \leq i \leq N\). This can be extended uniquely to a homeomorphism \(H : S \to S'\), and by construction the induced map by \(H\) on \(A(S)\) agrees with \(\lambda\) on \(T\).

We can now prove the general case of Theorem 1.1 (cf. [19, Prop. 3.11]).

**Proof of Theorem 1.1.** We dispensed with the cases where \(\dim(A(S)) \leq 2\) in Section 3. Suppose \(\dim(A(S)) > 2\).

Let \(T\) be a triangulation of \(S\) and let \(\mathcal{F}\) be as in Lemma 4.6. Suppose \(y_1, \ldots, y_k\) are the vertices of \(\mathcal{F}\). Then each \(y_i\) is contained in some triangulation \(T_i\) of \(S\). By Proposition 2.1, there exists a finite sequence of triangulations \(T = T_0, T_1, \ldots, T_{m-1}, T_m = T^*\) such that for each \(0 \leq j \leq m - 1\), \(T_j\) and \(T_{j+1}\) differ by a flip. Then let

\[
\mathcal{X} = \text{Span}_{A(S)} \left( \mathcal{F} \cup \bigcup_{1 \leq i \leq k} \bigcup_{0 \leq j \leq m_i} \{T^*_j\} \right).
\]

Then \(\mathcal{X}\) is a finite simplicial subcomplex of \(A(S)\) and has the property that any vertex \(y \in \mathcal{X}\) is contained in a triangulation \(T^*\) of \(S\) such that there exists a finite sequence of simplices \(T = T_0, T_1, \ldots, T_{m-1}, T_m = T^*\), all contained in \(\mathcal{X}\), where for each \(0 \leq j \leq m - 1\), \(T_j\) and \(T_{j+1}\) differ by a flip.

Let \(\lambda : \mathcal{X} \to A(S')\) be any locally injective simplicial map and \(H : S \to S'\) be as in Proposition 4.6. Let \(H_* : A(S) \to A(S')\) be the induced map of \(H\). Proposition 4.6 says that \(H_*(x) = \lambda(x)\) for \(x \in T\). We now show that this is true for any vertex \(y \in \mathcal{X}\) so that \(H_*|\mathcal{X} = \lambda\).

Suppose \(\phi = (H_*)^{-1} \circ \lambda : \mathcal{X} \to A(S)\). We already know that \(\phi\) is the identity on \(T\). Suppose \(y\) is a vertex in \(\mathcal{X}\). Let \(T = T_0, T_1, \ldots, T_{m-1}, T_m = T^*\) be the finite sequence of triangulations, all contained in \(\mathcal{X}\), such that \(y \in T^*\) and for each \(0 \leq j \leq m - 1\), \(T_j\) and \(T_{j+1}\) differ by a flip. We will show that if \(\phi\) is the identity on \(T^*\), this implies that it is the identity
Theorem 1.1 above shows that $X$ is a vertex set, so we can enumerate the vertices $T = \cup_{i} T_{i}$ by construction contains a triangulation $T$. For $T_{i} \cap T_{i+1}$, there are two arcs $a$ and $b$. We know that $\phi(a) = a$ and $\phi(T_{i} \cap T_{i+1}) = T_{i} \cap T_{i+1}$ by our hypothesis. Then since $\phi$ is a locally injective simplicial map, it must be the case that $\phi(b) = b$, hence $\phi$ is the identity on $T_{i+1}$. Then by induction, it follows that $\phi(y) = y$ and hence that $\phi$ is the identity on all of $X$. This implies that $H_{i}|_{X} = \lambda$. If $H'$ is another homeomorphism whose induced map agrees with $\lambda$ on $X$, then $H$ and $H'$ agree on the triangulation $T$, thus they are homotopic.

We can now prove Corollary 1.2.

Proof of Corollary 1.2. Suppose $\varphi: A(S) \to A(S')$ is an isomorphism. If $S \neq S_{0,3}$, we apply Theorem 1.1 to the injective simplicial map $\varphi|_{X}: X \to A(S')$, where $X$ is the finite rigid set given in the theorem. Suppose $S = S_{0,3}$. Then dim($A(S)$) = 2, so $S'$ is either $S_{0,3}$ or $S_{1,1}$. However $A(S_{0,3})$ is finite and $A(S_{1,1})$ is infinite, so it must be that $S'$ is $S_{0,3}$, hence $S$ and $S'$ are homeomorphic.

Finally, we extend the proof of Theorem 1.1 given above to construct an exhaustion of $A(S)$ by finite rigid sets.

Proof of Theorem 1.3. We dispensed with the cases where dim($A(S)$) $\leq$ 2 in Section 3 so suppose dim($A(S)$) $>$ 2. Let $X_{0}$ be the finite rigid set of $A(S)$ from Theorem 1.1 which by construction contains a triangulation $T$ of $S$. It is well-known that $A(S)$ has countable vertex set, so we can enumerate the vertices $A^{(0)}(S) = \{x_{1}, x_{2}, \ldots\}$. As above, for each $i \in \mathbb{N}$, there is a triangulation $T^{i}$ of $S$ containing $x_{i}$ and a finite sequence of simplices $T = T_{0}^{i}, T_{1}^{i}, \ldots, T_{m_{i}-1}^{i}, T_{m_{i}}^{i} = T^{i}$ where for each $0 \leq j \leq m_{i} - 1$, $T_{j}^{i}$ and $T_{j+1}^{i}$ differ by a flip. Then for $i \geq 1$, let

$$X_{i} = \text{Span}_{A(S)} \left( X_{i-1} \cup \bigcup_{0 \leq j \leq m_{i}} T_{j}^{i} \right)$$

Observe that $X_{i}$ is finite. An identical argument to the one employed in the proof of Theorem 1.1 above shows that $X_{i}$ is rigid. It is clear that $X_{0} \subseteq X_{1} \subseteq \ldots \subseteq A(S)$ and $\bigcup_{i \in \mathbb{N}} X_{i} = A(S)$.

References

[1] Javier Aramayona. “Simplicial embeddings between pants graphs”. In: Geometriae Dedicata 144.1 (2010), pp. 115–128.

[2] Javier Aramayona and Christopher Leininger. “Exhausting curve complexes by finite rigid sets”. In: Pacific Journal of Mathematics 282.2 (2016), pp. 257–283.

[3] Javier Aramayona and Christopher J Leininger. “Finite rigid sets in curve complexes”. In: Journal of Topology and Analysis 5.02 (2013), pp. 183–203.
[4] Mark C Bell, Valentina Disarlo, and Robert Tang. “Cubical geometry in the polygonalisation complex”. In: Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 167. 1. Cambridge University Press. 2019, pp. 1–22.
[5] Tara Brendle and Dan Margalit. “Normal subgroups of mapping class groups and the metaconjecture of Ivanov”. In: arXiv preprint arXiv:1710.08929 (2017).
[6] Valentina Disarlo. “Combinatorial rigidity of arc complexes”. In: arXiv preprint arXiv:1505.08080 (2015).
[7] John L Harer. “The virtual cohomological dimension of the mapping class group of an orientable surface”. In: Inventiones mathematicae 84.1 (1986), pp. 157–176.
[8] Allen Hatcher. Algebraic topology. Cambridge University Press, 2001.
[9] Jesús Hernández Hernández, Christopher J. Leininger, and Rashimate Maungchang. “Finite rigid subgraphs of pants graphs”. In: arXiv preprint arXiv:1907.12734 (2019).
[10] Sabahattın İlbir and Mustafa Korkmaz. “Finite Rigid Sets in Curve Complexes of Non-Orientable Surfaces”. In: arXiv preprint arXiv:1810.07964 (2018).
[11] Elmas Irmak. “Complexes of nonseparating curves and mapping class groups”. In: arXiv preprint math/0407285 (2004).
[12] Elmas Irmak. “Exhausting Curve Complexes by Finite Rigid Sets on Nonorientable Surfaces”. In: arXiv preprint arXiv:1906.09913 (2019).
[13] Elmas Irmak. “Exhausting Curve Complexes by Finite Superrigid Sets on Nonorientable Surfaces”. In: arXiv preprint arXiv:1903.04926 (2019).
[14] Elmas Irmak. “Injective Simplicial Maps of the Arc Complex on Nonorientable Surfaces”. In: arXiv preprint arXiv:0803.0498 (2008).
[15] Elmas Irmak. “Injective Simplicial Maps of the Complexes of Curves of Nonorientable Surfaces”. In: arXiv preprint arXiv:1203.4271 (2012).
[16] Elmas Irmak. “On simplicial maps of the complexes of curves of nonorientable surfaces”. In: Algebraic & Geometric Topology 14.2 (2014), pp. 1153–1180.
[17] Elmas Irmak. “Superinjective simplicial maps of the complexes of curves on nonorientable surfaces”. In: Turkish Journal of Mathematics 36.3 (2012), pp. 407–421.
[18] Elmas Irmak and Mustafa Korkmaz. “Automorphisms of the Hatcher-Thurston complex”. In: Israel Journal of Mathematics 162.1 (2007), pp. 183–196.
[19] Elmas Irmak and John D McCarthy. “Injective simplicial maps of the arc complex”. In: Turkish Journal of Mathematics 34.3 (2010), pp. 339–354.
[20] Elmas Irmak and Luis Paris. “Superinjective simplicial maps of the two-sided curve complexes on nonorientable surfaces”. In: arXiv preprint arXiv:1707.09937 (2017).
[21] Nikolai V Ivanov. “Automorphisms of complexes of curves and of Teichmuller spaces”. In: International Mathematics Research Notices 1997.14 (1997), pp. 651–666.
[22] Nikolai V Ivanov. “Fifteen problems about the mapping class groups”. In: arXiv preprint math/0608325 (2006).
[23] Mustafa Korkmaz. “Automorphisms of complexes of curves on punctured spheres and on punctured tori”. In: Topology and its Applications 95.2 (1999), pp. 85–111.
[24] Mustafa Korkmaz and Athanase Papadopoulos. “On the arc and curve complex of a surface”. In: Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 148. 3. Cambridge University Press. 2010, pp. 473–483.

[25] Mustafa Korkmaz and Athanase Papadopoulos. “On the ideal triangulation graph of a punctured surface”. In: Annales de l’Institut Fourier. Vol. 62. 4. 2012, pp. 1367–1382.

[26] Feng Luo. “Automorphisms of the complex of curves”. In: arXiv preprint math/9904020 (1999).

[27] Dan Margalit. “Automorphisms of the pants complex”. In: Duke Mathematical Journal 121.3 (2004), pp. 457–479.

[28] Rasimate Maungchang. “Exhausting pants graphs of punctured spheres by finite rigid sets”. In: Journal of Knot Theory and Its Ramifications 26.14 (2017), p. 1750105.

[29] Rasimate Maungchang. “Finite rigid subgraphs of the pants graphs of punctured spheres”. In: Topology and its Applications 237 (2018), pp. 37–52.

[30] John D McCarthy and Athanase Papadopoulos. “Simplicial actions of mapping class groups”. In: Handbook of Teichmüller theory 3 (2012), pp. 297–423.

[31] Alan McLeay. “Geometric normal subgroups in mapping class groups of punctured surfaces”. In: arXiv preprint arXiv:1810.00742 (2018).

[32] Lee Mosher. “Tiling the projective foliation space of a punctured surface”. In: Transactions of the American Mathematical Society (1988), pp. 1–70.

[33] Paul Schmutz Schaller. “Mapping class groups of hyperbolic surfaces and automorphism groups of graphs”. In: Compositio Mathematica 122.3 (2000), pp. 243–260.