EXPLORATION NOISE FOR LEARNING LINEAR-QUADRATIC MEAN FIELD GAMES

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ABSTRACT. The goal of this paper is to demonstrate that common noise may serve as an exploration noise for learning the solution of a mean field game. This concept is here exemplified through a toy linear-quadratic model, for which a suitable form of common noise has already been proven to restore existence and uniqueness. We here go one step further and prove that the same form of common noise may force the convergence of the learning algorithm called ‘fictitious play’, and this without any further potential or monotone structure. Several numerical examples are provided in order to support our theoretical analysis.

1. INTRODUCTION

1.1. General context. Since their inception fifteen years ago in the seminal works of Lasry and Lions [49, 50, 51, 52] and of Huang, Caines and Malhamé [45, 47, 46], Mean Field Games (MFG) have met a tremendous success, inspiring mathematical works from different areas like partial differential equations, control theory, stochastic analysis, calculus of variation, toward a consistent and expanded theory for games with many weakly interacting rational agents. Meanwhile, the increasing number of applications has stimulated a long series of works on discretization and numerical methods for approximating the underlying equilibria (which we also call solutions); see for instance [1, 2, 3, 4] for discretization with finite differences, [14, 15] for semi-Lagrangian schemes, [9, 12, 11] for variational discretization, and the lecture notes [5] for an overview. These contributions often include numerical methods for solving the discrete schemes: Picard method, Newton method, fictitious play; we review the latter one in detail in the sequel. Recently, other works have also demonstrated the possible efficiency of tools from machine learning within this complex framework: standard equations for characterizing the equilibria may be approximately solved by means of a neural network; see for instance [19, 20]. Last (but not least), motivated by the desire to develop methods for models with partially unknown data, several authors have integrated important concepts from reinforcement learning in their studies; we refer to [21, 22, 35, 41].

The aim of our work is to make one new step forward in the latter direction with a study at the frontier between the theoretical analysis of MFGs and the development of appropriate forms of learning. In particular, our main objective is to provide a proof of concept for the notion of exploration noise, which is certainly a key ingredient in machine learning. For the reader who is aware of the MFG theory, our basic idea is to prove that common noise may indeed serve for the exploration of the space of solutions and, in the end, for the improvement of the existing learning algorithms. For sure, this looks a very ambitious program since there has not been, so far, any complete understanding of the impact of the common noise onto the shape of the solutions. However, several recent works clearly indicate that noise might be helpful for numerical purposes. Indeed, in a series of works [7, 29, 38], it was shown that common noise could help to force uniqueness of equilibria in a certain number of MFGs. This is a striking fact because non-uniqueness is the typical rule for...
MFGs, except in some particular classes with some specific structure; see for instance the famous uniqueness result of Lasry-Lions for games with monotonous coefficients \cite{lasry2007mean} and \cite{block2010mean} Chapter 3]. Conceptually, the key condition for forcing uniqueness is that, under the action of the (hence common) noise, the equilibria can visit sufficiently well the state space in which they live. This makes the whole rather subtle since, in full theory, the state space is the space of probability measures, which is typically infinite-dimensional. In this respect, the main examples for which such a smoothing effect has been rigorously established so far are: (i) a linear quadratic model with possibly nonlinear mean field interaction terms in the cost functional \cite{lasry2007mean}; (ii) a general model with an infinite dimensional noise combined with a suitable form of local interaction in the dynamics \cite{delarue2011unique}; (iii) models defined on finite state spaces and forced by a variant of the Wright-Fisher noise used in population genetics literature \cite{book}

1.2. **Our model.** In order to prove the possible efficiency of our approach, we here restrict the whole discussion to the first of the three aforementioned instances. In words, the MFG that we consider below comprises the state equation

$$\text{d}X_t = \alpha_t \text{d}t + \sigma dB_t + \varepsilon dW_t, \quad t \in [0, T], \quad (1.1)$$

together with the cost functional

$$J(\alpha; m) = \frac{1}{2} \mathbb{E} \left[ |RX_T + g(m_T)|^2 + \int_0^T \left\{ |QX_t + f(m_t)|^2 + |\alpha_t|^2 \right\} \text{d}t \right]. \quad (1.2)$$

Above, $X_t$ is the state at time $t$ of a representative agent evolving within a continuum of other agents. It takes values in $\mathbb{R}^d$, for some integer $d \geq 1$, and evolves from the initial time 0 to the terminal time $T$ according to the equation (1.1). Therein, $B = (B_t)_{0 \leq t \leq T}$ and $W = (W_t)_{0 \leq t \leq T}$ are two independent $d$-dimensional Brownian motions on a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and $\sigma \geq 0$ and $\varepsilon \geq 0$ account for their respective intensity in the dynamics. While $B$ is thought as a private (or idiosyncratic) noise felt by the representative player only (and not by the others), the process $W$ is said to be a common or systemic noise as all the others in the continuum also feel it. The initial condition $X_0$ may be random and, in any case, $X_0$ is independent of the pair $(B, W)$. The process $\alpha = (\alpha_t)_{0 \leq t \leq T}$ is a so-called control process, that is usually assumed to be progressively-measurable with respect to the augmented filtration $\mathcal{F}$ generated by $(X_0, B, \varepsilon W)$ (when $\sigma$ or $\varepsilon$ are zero, the corresponding process is no longer used to generate the filtration). The key fact in MFG theory is that the representative player aims at choosing the best possible $\alpha$ in order to minimize the cost functional $J$ in (1.2). The leading symbol $\mathbb{E}$ in the definition of $J$ is understood as an expectation with respect to all the inputs $(X_0, B, W)$. If there were no dependence on the extra term $m = (m_t)_{0 \leq t \leq T}$ in $f$ and $g$, the minimization of $J$ would reduce to a mere linear-quadratic stochastic control problem driven by $Q$ and $R$, with the latter being two $d$-square matrices.\footnote{We could take $Q$ and $R$ as $e \times d$ matrices, for a general $e \geq 1$ and then $f$ and $g$ as being $e$-dimensional. For simplicity, we feel easier to work with $e = d$.}

The spice of MFGs is that $(m_t)_{0 \leq t \leq T}$ accounts for the flow of marginal statistical states of the continuum of players surrounding the representative agent. In full generality, each $m_t$ should be regarded as a probability measure hence describing the statistical distribution of the other agents at time $t$. For simplicity, we here just assume $m_t$ to be the $d$-dimensional mean of the other agents at time $t$; consistently, $f = (f^1, \ldots, f^d)$ and $g = (g^1, \ldots, g^d)$ are $\mathbb{R}^d$-valued functions defined on $\mathbb{R}^d$. However, the notion of mean should be clarified, because of the distinction between the two private and common noises. From a modelling point of view, this mean should result from a law of large numbers taken over players that would be subjected to independent and identically distributed initial and private noises (consistently with our former description of $B$) but to the same common noise $W$. Because of this, $(m_t)_{0 \leq t \leq T}$ is itself required to be a stochastic process, progressively-measurable with respect to the filtration generated by $\varepsilon W$. When $\varepsilon = 0$, the filtration
becomes trivial and, accordingly, \((m_t)_{0 \leq t \leq T}\) is assumed to be deterministic. The notion of Nash equilibrium or MFG solution then comes through a fixed point argument. In short, \((m_t)_{0 \leq t \leq T}\) is said to be an equilibrium if the minimizer \((X_t^\ast)_{0 \leq t \leq T}\) of (1.1)–(1.2) satisfies:

\[
m_t = \mathbb{E}[X_t^\ast | \varepsilon \mathcal{W}], \quad t \in [0, T],
\]

with the conditional expectation becoming an expectation if \(\varepsilon = 0\). Throughout, \(f\) and \(g\) are typically assumed to be bounded and Lipschitz continuous functions. When \(\varepsilon = 0\), this is enough for ensuring the existence of solutions to the fixed point (1.3), but uniqueness is known to fail in general, except under some additional conditions. For instance, the Lasry-Lions monotonicity condition, when adapted to this setting, says that uniqueness indeed holds true if \(Q^1 f\) and \(R^1 g\) (with \(^\dagger\) denoting the transpose) are non-decreasing in the sense that (see [30])

\[
\forall x, x' \in \mathbb{R}^d, \quad (x - x') \cdot (Q^1 f(x) - Q^1 f(x')) \geq 0, \quad (x - x') \cdot (R^1 g(x) - R^1 g(x')) \geq 0,
\]

with \(\cdot\) denoting the standard inner product in \(\mathbb{R}^d\). When \(\varepsilon > 0\), existence and uniqueness hold true, even though the coefficients are not monotone (see [38]). This is a very clear instance of the effective impact of the noise onto the search of equilibria.

The reason why the MFG (1.1)–(1.2)–(1.3) becomes uniquely solvable under the action of the common noise may be explained as follows. In short (and this is the rationale for working in this set-up), the linear-quadratic structure of (1.1) forces uniqueness of the minimizer to (1.2) when \(m\) is fixed. Even more the optimal control is given in the Markovian form

\[
\alpha_t^\ast = -\eta t X_t^\ast - h_t, \quad t \in [0, T],
\]

where \(\eta = (\eta_t)_{0 \leq t \leq T}\) is the \(d \times d\)-matrix valued solution of an autonomous Riccati equation that only depends on \(Q\) and \(R\) and that is in particular independent of the input \(m\). In other words, only the intercept \((h_t)_{0 \leq t \leq T}\) in the above formula depends on the inputs \(f, g, R, Q\) and \(m\). The characterization of \(h\) is usually obtained by the (stochastic if \(\varepsilon > 0\)) Pontryagin principle, namely \(h\) solves the Backward Stochastic Differential Equation (BSDE)

\[
h_t = R^1 g(m_T) + \int_t^T \{Q^1 f(m_s) - \eta_s h_s\} \, ds - \varepsilon \int_t^T k_s dW_s, \quad t \in [0, T].
\]  

(1.6)

When \(\varepsilon = 0\), the above stochastic integral disappears and the equation (1.6) becomes a mere Ordinary Differential Equation (ODE) but set backwards in time. When \(\varepsilon > 0\), the solution is the pair \((h, k)\), which is required to be progressively measurable with respect to the filtration generated by \(W\). Existence and uniqueness are well-known facts in BSDE theory. By taking the conditional mean given \(W\) in (1.1), we deduce that solving the MFG problem thus amounts to find a pair \((m, h)\) satisfying the Forward-Backward Stochastic Differential Equation (FBSDE):

\[
\begin{align*}
dm_t &= -(\eta t m_t + h_t) dt + \varepsilon dW_t, \quad m_0 = \mathbb{E}(X_0), \\
dh_t &= -(Q^1 f(m_t) - \eta t h_t) dt + \varepsilon k_t dW_t, \quad h_T = R^1 g(m_T).
\end{align*}
\]

(1.7)

Unique solvability was proven in [28, 53]. The smoothing effect of the noise manifests at the level of the related system of Partial Differential Equations (PDEs), which is sometimes called the master equation of the game:

\[
\partial_t \theta(t, x) + \frac{\varepsilon^2}{2} \Delta_{xx} \theta(t, x) - (\eta x + \theta(t, x)) \cdot \nabla_x \theta(t, x) + Q^1 f(x) - \eta \theta(t, x) = 0,
\]

with \(\theta(T, \cdot) = R^1 g(\cdot)\) as boundary condition at terminal time, \(\theta\) being a function from \([0, T] \times \mathbb{R}^d \to \mathbb{R}^d\). When \(\varepsilon > 0\), the PDE is uniformly parabolic and hence has a unique classical solution (with bounded derivatives). When \(\varepsilon = 0\), it becomes an hyperbolic equation and singularities may emerge, precisely when the solutions to (1.7) (which are nothing but the characteristics of (1.8)) cease to be unique.
1.3. **Learning procedures.** In our setting, numerical solutions to the MFG may hence be caught by solving either the FBSDE (1.7) or the nonlinear equation (1.8). For sure, independently of any applications to MFGs, there have been well-known numerical methods for the two objects, see for instance (and for a tiny example) \[13, 14, 27, 31, 34, 48, 51, 55\]. In \[31, 48, 51, 55\], the problem is solved by constructing an approximation of \( \theta \) by means of a backward induction; this is a typical strategy in the field, which is fully consistent with the dynamic programming principle that holds true for uniquely solvable mean field games (see \[17\] Chapter 4)). Although formulated differently, \[10\] also relies on a backward induction. In comparison with \[10, 31, 48, 51, 55\], \[8, 27, 34\] proceed in a completely different manner since the backward equation therein is reformulated into a forward equation with an unknown initial condition; the goal is then to tune both the initial condition and the martingale representation term of the backward equation in order to minimize, at terminal time, the distance to the prescribed boundary condition. Those methods have the following main limitation within a learning prospect for MFGs: They require the two coefficients \( f \), \( g \), \( Q \) and \( R \), entering the model, to be explicitly known. Even more, they make no real use of the control structure (1.2) that underpins the game.

In fact, \[31, 48, 51, 55\] suffer from another drawback since all these works involve a space discretization that consists in approximating the function \( \theta \) at the nodes of a spatial grid. Accordingly, the complexity increases with the physical dimension \( d \) of the state variable. In particular, similar strategies would become even more costly for a more general mean field dependence than the one addressed in (1.2). Indeed, in the general case, the spatial variable is no longer the mean but the whole statistical distribution (which is an infinite dimensional object). For sure, the latter raises challenging questions that go beyond the scope of this paper since we cannot guess of a numerical method that would directly allow to handle the infinite dimensional statistical distribution of the solution. However, this is an objective that should be kept in mind. Say for instance that particle or quantization methods would be suitable candidates to overcome such an issue, see for instance \[25, 26\].

Whatever the method, the true spice is to decouple (1.7), since the two forward and backward time directions are conflicting. One of our basic concern here is thus to take benefit of the noise in order to define an iterative scheme that decouples (1.7) efficiently. At the same time, since the very structure of a MFG corresponds very naturally to the precepts of reinforcement learning, we want to have a method that may work with unknown coefficients \( f \), \( g \), \( Q \) and \( R \), and that may benefit from the observation of the cost if it is available. In this regard, we ask our scheme to have a learning structure that should manifest in a sequel of steps of the form

\[
\text{computation of a best action / update of the state variable,}
\]

and hence that would be adapted to real data. Surprisingly, this is not an easy question, even though the equation (1.7) is well-posed. As demonstrated in the recent work \[24\], naive Picard iterations may indeed fail. They would consist in solving inductively the backward equation

\[
dh^{n+1}_t = - (Q^t f(\overline{m}^{n}_t) - \eta h^{n+1}_t) dt + \varepsilon k^{n+1}_t dW_t, \quad h^{n+1}_T = R^t g(\overline{m}^{n}_T). \tag{1.9}
\]

for a given proxy \( \overline{m}^n := (\overline{m}^n_t)_{0 \leq t \leq T} \) of the forward equation and then in plugging the solution \( h^{n+1} := (h^{n+1}_t)_{0 \leq t \leq T} \) into the forward equation

\[
d\overline{m}^{n+1}_t = - (\eta \overline{m}^{n+1}_t + h^{n+1}_t) dt + \varepsilon dW_t, \quad m^{n+1}_0 = \mathbb{E}(X_0). \tag{1.10}
\]

Intuitively, the reason why it may fail is that the updating rule \( \overline{m}^n \mapsto \overline{m}^{n+1} \) is too ambitious. In other words, the increment may be too high and smaller steps are needed to guarantee the convergence of the procedure.

A more successful strategy is known in MFG theory (and more generally in game theory) under the name of fictitious play, see \[13, 35, 39, 12, 13, 58, 59\]. It is a learning procedure with a decreasing step of size \( 1/n \) at rank \( n \) of the iteration. Having as before a proxy \( \overline{m}^n \) for the state of the population at rank \( n \) of the learning procedure, \( h^{n+1} \) is computed as
above. Next, the same forward equation as before is also solved, but the solution is denoted by \( m^{n+1} \), namely
\[
dm^{n+1}_t = -(\eta m^{n+1}_t + h^n_t)dt + \varepsilon dW_t, \quad m^{n+1}_0 = \mathbb{E}(X_0),
\] (1.11)
and then the updating rule is given by
\[
m^{n+1}_t = \frac{1}{n+1}m^{n+1}_t + \frac{n}{n+1}m^n_t, \quad t \in [0, T].
\] (1.12)

Very importantly, both the Picard iteration and the fictitious play have a learning interpretation. In both cases, the backward equation (1.9) provides the best response to the minimization of the functional \( \alpha \mapsto J(\alpha; \bar{m}^n) \), which has indeed the same form as in (1.5):
\[
\alpha^{n+1,*}_t = -\eta X^{n+1,*}_t - h^{n+1}_t, \quad t \in [0, T],
\]
where \( (X^{n+1,*}_t)_{0 \leq t \leq T} \) is obtained implicitly by solving the corresponding state equation (1.1).

Equivalently, the above formula gives the form of the optimal feedback function to the minimization of the functional \( \alpha \mapsto J(\alpha; \bar{m}^n) \) (bearing in mind that the feedback function may be here random because of the common noise).

In fact, the fictitious play has been addressed so far in the following two main cases: potential MFGs (13) and MFGs with monotone coefficients (here, \( Q^f \) and \( R^g \) are non-decreasing; see [35, 39, 42, 43, 59] within a general setting). Also, except in the recent work [54] (whose framework is a bit different since the fictitious play is in continuous time), the analysis has just been carried out in the case \( \varepsilon = 0 \), i.e., when there is no common noise. Here, it is certainly useful to recall that potential MFGs are a class of MFGs for which there exists a potential, namely a functional \( J(\alpha) \) associated with the same state dynamics as in (1.1)–(1.2), such that any minimizer of \( J \) is a solution of the MFG. In our setting, the shape of \( J(\alpha) \) is
\[
J(\alpha) = \mathbb{E}\left[ G(\mathcal{L}(X_T|\varepsilon W)) + \int_0^T \frac{1}{2} [\mathcal{F}(\mathcal{L}(X_t|\varepsilon W)) + |\alpha_t|^2] dt \right],
\] (1.13)
where \( \mathcal{L}(X_t|\varepsilon W) \) denotes the conditional law of \( X_t \) given the common noise and
\[
G(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} |Rx|^2 d\mu(x) + G(\bar{\mu}), \quad \mathcal{F}(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} |Qx|^2 d\mu(x) + F(\bar{\mu}),
\] (1.14)
with \( G \) and \( F \) denoting primitives of \( R^g \) and \( Q^f \) (if any). In particular, the model is always potential when \( d = 1 \), but there is no potential structure when \( d \geq 2 \), unless \( Q^f \) and \( R^g \) both derive from (Euclidean) potentials, meaning that \( (Q^f)^t = \partial F/\partial x_t \) and \( (R^g)^t = \partial G/\partial x_t \).

1.4. Tilted fictitious play. Consistently with our agenda, our goal is to prove that the fictitious play may converge thanks to the presence of the common noise (i.e., \( \varepsilon > 0 \)). Seemingly, the above discussion about the potential structure of our model in dimension \( d = 1 \) demonstrates that this question becomes especially relevant in dimension greater than or equal to 2 (the results from [13] could be easily adapted to this setting, even in the presence of the common noise). In fact, as shown in Subsection 3.4, the question is also interesting in dimension \( d = 1 \) in cases when equilibria are not unique.

However, this program requires a modicum a care since we are not able, even in the presence of the common noise, to prove the convergence of the fictitious play stated in (1.9) and (1.11). Instead, we take benefit of the noise in order to reformulate the two equations (1.9) and (1.11) into a new system obtained by a mere shift of the common noise. By Girsanov theorem, the new shifted common noise has the same law as the original one but under a tilted probability measure. Using the updating rule (1.12), this so-called tilted scheme is then shown to converge.

In order to state the new fictitious play properly, we thus need to allow for another form of common noise in (1.1). For a process \( h = (h_t)_{0 \leq t \leq T} \), progressively-measurable with respect to
to the filtration generated by $W$, we thus introduce the shifted Brownian motion
\[ W^h = \left( W^h_t := W_t + \frac{1}{\varepsilon} \int_0^t h_s \, ds \right)_{0 \leq t \leq T}, \]



\[ \mathcal{E}(h) := \exp \left( -\frac{1}{\varepsilon} \int_0^T h_t \cdot dW_t - \frac{1}{2\varepsilon^2} \int_0^T |h_t|^2 \, dt \right). \]

Accordingly, for any two frozen continuous paths $n := (n_t)_{0 \leq t \leq T}$ and $w := (w_t)_{0 \leq t \leq T}$, we define the non-averaged cost functional
\[ \mathcal{R}(\alpha; n; w) = \frac{1}{2} \left[ R_{xt} + g(n_T) \right] + \int_0^T \left[ |Qx_t + f(n_t)|^2 + |\alpha_t|^2 \right] \, dt, \]

where
\[ x_t = x_0 + \int_0^t \alpha_s \, ds + \sigma B_t + \varepsilon w_t, \quad t \in [0, T], \]

the latter being nothing but the integral version of (1.17), when $W$ is replaced by the frozen trajectory $w$. Now, when $m$ and $h$ are two progressively-measurable processes with respect to the filtration generated by $\varepsilon W$, we have a look at the new cost functional
\[ J(\alpha; m; h) = \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; W^h) \right]. \]

We now define our fictitious play according to a two step iterative learning procedure, whose description at rank $n$ goes as follows:

**Best action:** For a proxy $m^n := (m^n_t)_{0 \leq t \leq T}$ of the conditional mean $m = (m_t)_{0 \leq t \leq T}$ of the in-equilibrium population (as given by the forward component of (1.7)) and a proxy $h^n = (h^n_t)_{0 \leq t \leq T}$ of the opposite of the $\mathbb{F}^W$-adapted intercept of the equilibrium feedback in (1.5) (as given by the backward component of (1.7)), solve
\[ \alpha^{n+1} = \text{argmin}_{\alpha} \mathbb{E}^{h^n} \left[ \mathcal{R}(\alpha; m^n; W^{h^n}) \right]. \]

the infimum being taken over all $\mathbb{F}$-progressively measurable ($\mathbb{R}^d$-valued) controls $\alpha$.

The optimal feedback being of the same linear form as in (1.5) (the proof is given below), we may call $h^{n+1} = (h^{n+1}_t)_{0 \leq t \leq T}$ the opposite of the resulting intercept.

**Update:** Given $h^{n+1}$, the optimal trajectory of the above minimization problem is
\[ X^{n+1}_t = X_0 - \int_0^t (\eta_s X^{n+1}_s + h^{n+1}_s) \, ds + \sigma B_t + \varepsilon W^{h^n}_t, \quad t \in [0, T]. \]

We then let
\[ m^{n+1}_t = \mathbb{E} \left[ X^{n+1}_t \mid W \right], \quad t \in [0, T], \]

together with
\[ m^{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} m^k = \frac{1}{n+1} m^{n+1} + \frac{n}{n+1} m^n_t, \quad t \in [0, T]. \]

For sure the rationale behind this strategy relies on Girsanov’s transformation. Under the tilted probability measure $\mathbb{P}^h$, the process $W^h$ is a new Brownian motion, with the same law as the original (or historical) common noise under $\mathbb{P}$. However, the main trick here is to dynamically change the form of the common noise (dynamically with respect to the rank of the iteration in the fictitious play). Precisely, this permits to decouple the two forward and backward equations, as clearly shown if we rewrite the state equation (1.20) as an equation with respect to the historical common noise:
\[ X^{n+1}_t = X_0 - \int_0^t (\eta_s X^{n+1}_s + h^{n+1}_s - h^n_s) \, ds + \sigma B_t + \varepsilon W_t, \quad t \in [0, T]. \]

\[ ^2 \text{The opposite comes from the sign – in the formula (1.5) of the optimal feedback.} \]
The forward equation then becomes asymptotically autonomous provided that the increment 
\( h^{n+1} - h^n \) tends to 0, which we succeed to prove in Section 2 by using available bounds on 
the process \( k \) in (1.6). Our main statements in this regard are Theorem 2.6 and Proposition 
2.8. (Notice however that we are not able to prove the convergence of the standard non-tilted 
fictitious play outside any further potential or monotonicity assumption, even in the presence 
of the common noise.)

1.5. Exploration. For sure, changing the common noise as we have done in the cost functional (1.19) raises many practical questions. In order to understand this properly, we should 
follow the presentation given in [21, 22] and think of \( R \) as being a black-box representing a 
decentralized unit. For instance, the box may regulate the consumption/production/storage 
of energy of a single individual connected to a smart grid, see for instance [6] and the refer-
ences therein; the box may also be an autonomous car moving in a flock of vehicles, see for 
instance [44].

The black box operation is described in Figure 1. In this picture, the decentralized black 
box receives three inputs: (i) the control, as tuned by the individual operating the black 
box; (ii) the two idiosyncratic and independent noises; (iii) the state of the population. In 
this representation, the only input that can be tuned by the individual operating the black 
box is the control itself. Consistently, the individual operating the black box can tilt the 
common noise by herself/himself by modifying her/his control accordingly.

![Figure 1. Black-box operation for an MFG with a common noise. In the three input arrows, only the plain line can be tuned. Given the input state of the population and the realizations of the two noises, the decentralized black box adjusts the control. The output is in the form of a cost depending on the input state and on the realizations of the noises.](image)

However, this picture makes sense only if the original model itself is subjected to a common 
oise. In the absence of common noise, we thus need to restore a form of common noise in 
order to conciliate our tilted fictitious play with the above picture. This comes through the 
notion of exploration. Below, we thus regard the common noise as a way to explore the space 
of possible solutions. This amounts to say that the original mean field game is no longer the 
mean field game with common noise, but the mean field game without common noise, i.e. 
\( \varepsilon = 0 \). Formally, any control \( \alpha \), as chosen above by a tagged individual, is then subjected to 
an additional randomization of the form

\[
\alpha = (\alpha_t)_{0 \leq t \leq T} \mapsto \left( \alpha_t + \varepsilon W_t^h \right)_{0 \leq t \leq T},
\]

The reader may find it reminiscent of the weak formulation of MFGs introduced in [18], but this is 
substantially different since the Girsanov transformation is here applied to the common noise (and not to the 
idiosyncratic one).
where $h$ is given as an information on the whole state of the population, in addition to $m$. Figure 1 becomes the new Figure 2 below.

![Diagram](image)

**Figure 2.** Black-box operation for an MFG without a common noise, but subjected to a common randomization of the control. In the four input arrows, only the plain lines (corresponding to yellow boxes) can be tuned. Given the input state of the population and the realizations of the two noises, the decentralized black-box adjusts the control. The output is in the form of a (random) cost depending on the input state and on the realizations of the noises.

This concept faces obvious mathematical difficulties, since the new control after randomization is no longer of finite energy. We solve this issue by replacing $W = (W_t)_{0 \leq t \leq T}$ by its piecewise linear interpolation along a mesh of $[0, T]$, say of uniform step $T/p$, for a given integer $p \geq 1$. We denote this interpolation by $W_p = (W_{pt})_{0 \leq t \leq T}$. Accordingly, the return of the black-box should be renormalized, letting:

$$R_p(\alpha; m; W) = R(\alpha + \varepsilon \dot{w}; m; 0) - \frac{1}{2} \varepsilon^2 p,$$

for a piecewise-affine path $w$, affine on each $[\ell T/p, (\ell + 1)T/p)$ for $\ell \in \{0, \cdots, p - 1\}$.

Our second main statement, see Theorem 2.14 and Proposition 2.19, is to prove that the fictitious play that is hence obtained by replacing $W$ by $W_p$ and $R(\alpha; m^n; W_{h^n})$ by $R_p(\alpha; m^n; W_{h^n})$, with

$$W_{t, h^n} = W_t + \frac{1}{\varepsilon} \int_0^t h^n_s \, ds, \quad t \in [0, T],$$

converges to the solution of the mean field game with $\varepsilon > 0$ as common noise, when the number $n$ of iterations tends to $\infty$ and $p$ also tends to $\infty$. (This also requires to force $h^n$ to be constant on any subdivision of the mesh, but we feel more appropriate not to detail all the ingredients here. We refer the reader to Subsection 2.2 for a complete description.) Importantly, the state dynamics over which the return $R_p$ is computed write

$$X_t = X_0 + \int_0^t \alpha_s \, ds + \sigma B_t + \varepsilon W_{t, h^n},$$

In this approach, the control after randomization is thus given by $\alpha + \varepsilon \dot{W}_{p, h^n}$, which clarifies the meaning of the common randomization in Figure 2. In the end, this fits well the concept of exploration, as stated by Sutton and Barto [60, Chapter 1, p. 1]: ‘The learner is not told which actions to take, but instead must discover which actions yield the most reward by trying them’. Last but not least, the algorithm runs from the sole observations of the returns.
of the black-box, and in particular without any further detailed of the cost coefficients \( f, g, Q \) and \( R \), provided we use a reinforcement learning method to solve the black-box.

Importantly, it must be emphasized that in the two Figures 1 and 2 the decentralized black-box does not return the expected cost, but only the realization of the cost for the given realizations of the noises. At each step of the fictitious play, the optimization of the expected cost is formally performed over all the possible trajectories of the independent and common noises, bearing in mind that the control is adapted to both noises and that the state of the population is adapted to the common noise. This is stylized in the form of Figure 3. Therein, this our choice to represent the possible trajectories of the two noises in the form of an infinite sequence of realizations from an I.I.D. (independent and identically distributed) sample of the idiosyncratic and common noises, this formal representation being very convenient for introducing next the numerical implementation.

1.6. **Exploitation.** In reinforcement learning, this is a common practice to distinguish exploration from exploitation. While exploration is intended as a way to visit the space of actions, exploitation is related to the error that is achieved by the learning method. When the learning addresses a stochastic control problem, the analysis goes through the loss (or the regret), which is the (absolute value of the) difference between the best possible cost and the cost to the strategy returned by the algorithm. Since we are dealing with a game, we use here the concept of approximated Nash equilibrium to define the regret. In brief, the point is to prove that the output of the algorithm is a \( \rho \)-Nash equilibrium, for \( \rho > 0 \) and to define the regret as the infimum of those \( \rho \). In our case, the situation is a bit more subtle since the learning procedure returns a random approximated equilibrium; ideally, we should associate with it a random regret. However, the analysis would be too difficult. Instead, we follow (1.19) and average the cost with respect to the common noise. We then define the notion of approximated equilibrium with respect to this averaged cost. The resulting regret can then be decomposed as the sum of two terms:

- The ‘error’ resulting from the approximation (as returned by the fictitious play) of the mean field game with common noise. The implementation of the algorithm involves \( n \) iterations and a \( p \)-step piecewise linear approximation of the Brownian motion \( W \). For a fixed \( \varepsilon > 0 \), this error is shown to tend to 0 as \( n \) and \( p \) tend to \( \infty \), with an explicit rate \( g_\varepsilon(n, p) \).
- The ‘error’ resulting from the common noise approximation of the original mean field game without common noise. We prove that the (unique) solution of the mean field with common noise produces a random equilibrium that forms an \( O(\varepsilon) \)-Nash equilibrium of the original mean field game, for a constant \( C \) in the symbol \( O(\cdot) \) only depending on the coefficients \( f, g, Q \) and \( R \).

In fact, the second item can be put in the form of a more general stability result that permits to evaluate in the end the trade-off between exploration and exploitation. The regret has the form

\[
O\left( \varepsilon + \frac{1}{\varepsilon} \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right) \left[ 1 + (\ln(np\varepsilon))^+ \right] \right),
\]

see Theorem 2.23. For fixed values of \( n \) and \( p \) (the latter two parametrizing the complexity of the algorithm and the required memory\(^4\)), we are hence able to tune the intensity of the noise.

To our mind, this result demonstrates the interest of our concept, even though it says nothing about the equilibria that are hence selected in this way when \( \varepsilon \) tends to 0. In fact, the latter is a difficult question, which has been addressed for instance in [30] (for the same model as in (1.1)–(1.2), but with \( d = 1 \) and for some specific choices of \( f \) and \( g \)) and [23] (for finite state potential games); generally speaking, this problem raises many theoretical

---

\(^4\)We feel better not to give any order of complexity and memory in terms of \( n \) and \( p \). This would have no sense since the integration and optimization steps are not discretized here.
Figure 3. Black-box (in red) inserted at the core of a learning step. Expectations are formally written as means over a sequence of realizations from an i.i.d. sample of idiosyncratic and common noises. For any respective realizations \( #i \) and \( #j \) of the idiosyncratic and common noises plugged into the black-box, we compute the corresponding state of the population (that depends on the realization \( #j \) of the common noise) and we choose the control (that is adapted to the realizations \( #i, #j \) of the two noises). The inputs are thus in green, except the control which is subjected to optimization (hence in yellow, as a mixture of green and red). The return (in red) of the black-box depends the realizations \( #i, #j \). The expectation is formally obtained by averaging with respect to \( #i, #j \). Once the optimizer has been found, we compute (in blue) the optimal states, depending on the realizations \( #i, #j \). The output state of the population after one learning step is obtained by averaging over \( i \).

Questions that are out of the scope of this paper and we just address it here through numerical examples (see the subsection below). The diagram right below hence summarizes the balance between exploitation and exploration in our case.

For sure, our analysis of exploitation is carried out in the ideal case when the underlying expectations in (1.19) are understood in the theoretical sense and when the optimal control problem at each step can be computed perfectly. We do not address here the approximation
EXPLORATION FOR MFG

Figure 4. Exploration vs. exploitation.

of those theoretical expectations by empirical means nor the numerical approximation of the optimizers.

1.7. Numerical examples. We complete the paper with some numerical examples that demonstrate the relevance of our concept. The numerical implementation requires additional ingredients that are explained in detail in Section 3. Obviously, the main difficulty is the encoding of the decentralized black-box, as represented in Figures 2 and 3. As clearly suggested by the latter figure, expectations are then approximated by averaging the costs over realizations from a finite I.I.D. sample of idiosyncratic and common noises. In this regard, the principle highlighted by Figure 3 is the same, but the optimization step needs to be clarified. Although we do not provide any further theoretical justification of the accuracy of the numerical optimization that is hence performed, we feel useful to stress that controls are chosen in a semi-feedback form, namely of the same linear form as in (1.5). Numerically, the coefficient \( \eta_t \) is hence parameterized in the form of a coefficient that is only allowed to depend on time; and the intercept \( h_t \) is sought as a function of the current mean state of the population. This latter function is parameterized in the form of a finite expansion along an Hermite polynomial basis, our choice for Hermite polynomials being dictated by the Gaussian nature of the trajectories in (1.23). In our numerical experiments, both the linear coefficient and the coefficients in the regression of the intercept along the Hermite polynomial basis are found by ADAM optimization method. The results exposed in Section 3 demonstrate the following features:

(1) For a given value of the intensity of the common noise, we run examples in dimension \( d = 2 \). The (tilted) fictitious play converges well. Solutions are compared to numerical solutions of (1.7) found by a BSDE solver that uses explicitly the shape of the coefficients \( f, g, Q \) and \( R \) and that does not use the observations of the cost.

(2) The standard fictitious play, with idiosyncratic noise but without common noise, may fail to converge. This is here the crucial point of the paper as it demonstrates that the common noise helps the algorithm to converge. From a conceptual point of view, this a key observation. In dimension 1, the usual algorithm is known to converge since the model is potential, see [13].

(3) In order to study the behavior of our fictitious play when the intensity of the common noise becomes small, we focus on a one-dimensional MFG that has multiple equilibria when there is no common noise. We observe the convergence of our fictitious play (with a small intensity \( \varepsilon \)) towards the equilibrium that should be selected according to the predictions of [30, 23]. The standard fictitious play (without common noise) also converges but may not select the right equilibrium (for the same choice of parameters).

It should be stressed that our numerical experiments are run under Tensorflow, using a pre-implemented version of ADAM optimization method in order to compute an approximation of the best response in (1.19). Accordingly, the optimization algorithm itself relies explicitly on the linear-quadratic structure of the mean field game through the internal automatic differentiation procedure (used for computing gradients in descents). In this sense, our
numerical experiments use in fact more than the sole observations of the costs. Anyhow, this
does not change the conclusion: descent methods, based on accurate approximations of the
gradients, do benefit from the presence of the common noise. For instance, the construction
of accurate approximations of the gradient is addressed in \cite{21,22} (within a slightly different
setting), in which a model-free reinforcement learning method is fully implemented.

1.8. Comparison with existing works. Exploration and exploitation are important con-
cepts in reinforcement learning and related optimal control.

In comparison with the time discrete literature, there have been less papers on the analy-
sis of exploration/exploitation in the time continuous setting. In both \cite{57} and \cite{61}, the
randomization of the actions goes through a formulation of the corresponding control prob-
lem in terms of relaxed controls. In \cite{57}, the authors address questions that are seemingly
different from ours, as the objective is to allow for a model with some uncertainty of the state
dynamics. Accordingly, the cost functional is averaged out with respect to some prior proba-
bility measure on the vector field driving the dynamics. Under suitable assumptions on this
prior probability measure, a dynamic programming principle and a then a Hamilton-Jacobi-
Bellman are derived. In fact, the paper \cite{61}, which addresses stochastic optimal controls, is
closer to the spirit of our work. Therein, relaxed controls are combined with an additional
entropic regularization that forces exploration. In case when the control problem has a
linear-quadratic structure, quite similar to the one we use here (except that there is no mean
field interaction), the entropic regularization is shown to work as a Gaussian exploration.
Although our choice for working with a Gaussian exploration looks consistent with the result
of \cite{61}, there remain however some conceptual differences between the two approaches: In
the theory of relaxed controls, the drifts in the dynamics are averaged out with respect to the
distribution of the controls; In our paper, the dynamics are directly subjected to the
randomized action. In this respect, our work is closer to the earlier contribution \cite{33}.

Recently, the approach initiated in \cite{61} has been extended to mean field games. In \cite{62}
and \cite{36}, the authors study the impact of an entropic regularization onto the shape of the
equilibria. In both papers, the models under study are linear-quadratic and subjected to a
sole idiosyncratic noise (i.e., there is no common noise). However, they differ on the following
important point: In \cite{36}, the intensity of the idiosyncratic noise is constant, whilst it depends
on the standard deviation of the control in \cite{62}. In this sense, \cite{36} is closer to the set-up that
we investigate here. Accordingly, the presence of the entropic regularization leads to different
consequences: In \cite{62}, the effective intensity of the idiosyncratic noise grows up under the
action of the entropy and this is shown to help numerically in some learning method (of a
quite different spirit than ours); In \cite{36}, the entropy plays no role on the structure of the
equilibria, which demonstrates, if needed, that our approach here is substantially different.

Within the mean field framework, there have been several recent contributions on rein-
fforcement learning for models featuring a common noise. In \cite{21}, the authors investigate
the convergence of a policy gradient method for a discrete time linear quadratic mean field
control problem (and not an MFG) with a common noise. In comparison with \cite{12}, the

5The reader may find in \cite{56} a nice explanation about the distinction between model-free and model-based
reinforcement learning. In any case, our algorithm is not model-based: It would be model-based if we tried
to learn first $f$, $g$, $Q$ or $R$. We refer to \cite{43,43,43} for a discussion about the possible numerical interest to learn
$Q$ and $R$ first.
common noise is used in the convergence analysis. In another work ([22]), the same three authors have developed a model free Q-learning method for a mean field control problem in discrete time and finite space. The model may feature a common noise, but the latter has no explicit impact onto the convergence analysis carried out in the paper. Lastly, in [35, 59], the authors deal with discrete and continuous time learning for MFGs using fictitious play and introducing a form of common noise. Their analysis is supported by various applications and numerical examples (including a discussion on the tools from deep learning to compute the best responses at any step of the fictitious play). The analysis relies on the notion of exploitability, which is close to our own definition of the regret. As the common noise therein is not used for exploratory reasons, the coefficients are assumed to satisfy the monotonicity Lasry-Lions condition in order to guarantee the convergence of the fictitious play.

1.9. Main assumption, useful notation and organization.

1.9.1. Assumption. Throughout the analysis, $\sigma$ in (1.1) is a fixed non-negative real. The intensity $\varepsilon$ of the common noise is taken in $[0, 1]$. Most of the time, it is implicitly required to be strictly positive, but we sometimes refer to the case $\varepsilon = 0$ in order to compare with the situation without common noise. As for the coefficients $f$ and $g$, there are assumed to be bounded and Lipschitz continuous. We write

$$\|f\|_{1, \infty} = \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x, x' \in \mathbb{R}^d, x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|} < \infty,$$

and similarly for $g$.

1.9.2. Notation. The notation $I_d$ stands for the $d$-dimensional identity matrix. For a process $Z = (Z_t)_{0 \leq t \leq T}$ with values in a Polish space $\mathcal{S}$, we denote by $F^Z$ the augmented filtration generated by $Z$. In particular, the notation $F^W$ is frequently used to denote the augmented filtration generated by the common noise when $\varepsilon > 0$. We recall that $F = F(X_0, \sigma B, W)$.

Also, for a filtration $\mathcal{G}$, we write $S^d(\mathcal{G})$ for the space of continuous and $\mathcal{G}$-adapted processes $Z = (Z_t)_{0 \leq t \leq T}$ with values in $\mathbb{R}^d$ that satisfy

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Z_t|^2 \right] < \infty.$$

1.9.3. Organization of the paper. The mathematical analysis is carried out in Section 2. Subsection 2.1 addresses the error associated with the scheme (1.19)–(1.20)–(1.21)–(1.22) without any time discretization, see Theorem 2.6. Similar results, but with the additional time discretization that makes the exploration possible, are established in Subsection 2.2, see in particular Theorem 2.14. In Subsection 2.3, we make the connection with the original mean field game without common noise. In particular, we prove that our learning procedure permits to construct approximate equilibria to the original problem. The bound (1.25) for the regret is established in Theorem 2.23. Following our agenda, we provide the results of some numerical experiments in Section 3. The method is tested on some benchmark examples that are presented in Subsection 3.1. The implemented version of the algorithm is explained in Subsection 3.2. The results, for a fixed intensity of the common noise, are exposed in Subsection 3.3. In the final Subsection 3.4, we provide an example that illustrates the behavior of the algorithm for a decreasing intensity of the common noise.

2. Theoretical results

The theoretical results are presented in three main steps. The general philosophy, as exposed in Figure 1, is addressed in Subsection 2.1. The analysis of the algorithm under a time-discrete randomization of the actions is addressed in Subsection 2.2. Lastly, the dilemma between exploration and exploitation is investigated in Subsection 2.3.

2.1. Tilted fictitious play with common noise. Throughout the subsection, the intensity $\varepsilon > 0$ of the common noise is fixed.
2.1.1. Construction of the learning sequence. We here formalize the scheme introduced in (1.19)–(1.20)–(1.21)–(1.22). We hence construct a sequence of proxies \((m^n)_n\geq 0\) and \((h^n)_n\geq 0\) in the following way. The two initial processes \(m^0\) and \(h^0\) are two \(\mathbb{F}^W\)-adapted continuous processes with values in \(\mathbb{R}^d\). Typically, we choose \(m^0 = (m^0_t = \mathbb{E}(X_0))_{0\leq t\leq T}\) and \(h^0 = (h^0_t = 0)_{0\leq t\leq T}\). Assuming that, at some rank \(n\), we have already defined \((m^1, \ldots, m^n)\) and \((h^1, \ldots, h^n)\), each in \(\mathbb{S}^d(\mathbb{F}^W)\), we call
\[
\alpha^{n+1} = \arg\min_\alpha \mathbb{E}^{h^n}[\mathcal{R}(\alpha; m^n; W^{h^n})],
\tag{2.1}
\]
the infimum being taken over controlled processes \(\alpha\) that are \(\mathbb{F}\)-progressively and that satisfy
\[
\mathbb{E}^{h^n} \int_0^T |\alpha_t| dt < \infty.
\]
With all the proxies up to rank \(n\), we can associate the process \(\overline{m}^n = (\overline{m}^n_t)_{0\leq t\leq T}\), by letting:
\[
\overline{m}^n_t = \frac{1}{n} \sum_{k=1}^{n} m^k_t, \quad t \in [0,T],
\]
which definition is consistent with (1.11). We then have the following lemma, as a direct consequence of the stochastic Pontryagin principle (see [63, Chapter 3]):

**Lemma 2.1.** Under the above assumptions, the process \(\alpha^{n+1}\) writes
\[
\alpha^{n+1}_t = -\left(\eta_t X^{n+1}_t + h^{n+1}_t\right), \quad t \in [0,T],
\]
where \(\eta = (\eta_t)_{0\leq t\leq T}\) solves the Riccati equation
\[
\dot{\eta}_t - \eta_t^2 + Q^t Q = 0, \quad t \in [0,T]; \quad \eta_T = R^T R,
\tag{2.2}
\]
\(h^{n+1} = (h^{n+1}_t)_{0\leq t\leq T} \in \mathbb{S}^d(\mathbb{F}^W)\) solves the backward SDE:
\[
dh^{n+1}_t = (-Q^t f(\overline{m}^n_t) + \eta_t h^{n+1}_t) dt + k^{n+1}_t dW_t^{h^n}, \quad t \in [0,T],
\]
\(h^{n+1}_T = R^t g(\overline{m}^n_T)\).

And \(X^{n+1} = (X^{n+1}_t)_{0\leq t\leq T}\) solves the foward SDE:
\[
dX^{n+1}_t = -\left(\eta_t X^{n+1}_t + h^{n+1}_t\right) dt + \sigma dB_t + \varepsilon dW_t^{h^n}, \quad t \in [0,T].
\]

The statement makes it possible to let
\[
m^{n+1}_t = \mathbb{E}[X^{n+1}_t | \sigma(W)], \quad t \in [0,T],
\]
together with
\[
\overline{m}^{n+1}_t = \frac{1}{n+1} \sum_{k=1}^{n+1} m^k_t = \frac{1}{n+1} m^{n+1}_t + \frac{n}{n+1} \overline{m}^n_t.
\]

**Remark 2.2.** Notice that since the density \(\mathcal{E}(h^n)\) is \(\sigma(W)\)-measurable, we also have
\[
m^{n+1}_t = \mathbb{E}^{h^n}[X^{n+1}_t | \sigma(W)], \quad t \in [0,T].
\]

2.1.2. Convergence of the fictitious play. As noticed in the introduction, it is especially convenient to reformulate the dynamics of \(X^{n+1}\) under the historical probability. Quite clearly, we can write
\[
dX^{n+1}_t = -\left(\eta_t X^{n+1}_t + h^{n+1}_t - h^*_t\right) dt + \sigma dB_t + \varepsilon dW_t, \quad t \in [0,T].
\tag{2.3}
\]
The analysis of the fictitious play then goes through the additional processes \((\overline{F}^n)_n\geq 1\), defined by
\[
\overline{F}^n_t = \frac{1}{n} \sum_{k=1}^{n} h^*_t, \quad t \in [0,T].
\]
Indeed, by averaging over the index \(n\) in (2.3), we observe that
\[
d\overline{m}^{n+1}_t = -\left(\eta_t \overline{m}^{n+1}_t + \overline{F}^{n+1}_t - \frac{n}{n+1} \overline{F}^n_t\right) dt + \varepsilon dW_t, \quad t \in [0,T].
\tag{2.4}
By coupling with the backward equation in the statement of Lemma 2.1, we obtain

\[
\begin{align*}
dt m_t^{n+1} &= -(\eta_t m_t^{n+1} + h_t^{n+1} - \frac{m_t}{n+1}) dt + \varepsilon dW_t, \\
dt h_t^{n+1} &= \left(-Q^t f(m_t^n) + \eta_t h_t^{n+1}\right) dt + k_t^{n+1} h_t^n dt + k_t^{n+1} dW_t, \quad t \in [0,T], \\
h_T^{n+1} &= R^t g(m_T^{n}).
\end{align*}
\]

(2.5)

One very first, but non optimal, result in this regard is the following statement:

**Proposition 2.3.** The scheme (2.5) converges (in \(L^2\)) to the decoupled version of the FBSDE system

\[
\begin{align*}
dt m_t &= -\eta_t m_t dt + \varepsilon dW_t, \\
dt h_t &= (-Q^t f(m_t) + \eta_t h_t + k_t dt + k_t dW_t, \quad t \in [0,T], \\
h_T &= R^t g(m_T),
\end{align*}
\]

(2.6)

with an explicit bound on the rate of convergence, namely

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( |m_t - m_t^n|^2 + |h_t - h_t^n|^2 \right) \right] \leq C \exp(C/\varepsilon^2)/n^2,
\]

for a constant \(C\) that depends on \(d, T\) and the norms \(\|Q^t f\|_{1,\infty}\) and \(\|R^t g\|_{1,\infty}\).

**Remark 2.4.** The solution of the system (2.6) should be regarded as a solution for the MFG with common noise, whenever the latter is formulated in the weak form. Indeed, for \(m = (m_t)_{0 \leq t \leq T}\) as in (2.6), the optimal path \((X_t)_{0 \leq t \leq T}\) associated with the minimization problem

\[
\inf_{\alpha} \mathbb{E}^h [R(\alpha; m; W^h)],
\]

has exactly \(m = (m_t)_{0 \leq t \leq T}\) as conditional expectation given the common noise, which follows from an obvious adaptation of Lemma 2.7. Namely,

\[
m_t = \mathbb{E} [X_t | \sigma(W)], \quad t \in [0,T],
\]

which can be rewritten as

\[
m_t = \mathbb{E}^h [X_t | \sigma(W)], \quad t \in [0,T].
\]

Noticeably, under \(\mathbb{P}^h\), the process \((m, h)\) satisfies the forward-backward system

\[
\begin{align*}
dt m_t &= -\eta_t m_t dt - h_t dt + \varepsilon dW_t^h, \\
dt h_t &= (-Q^t f(m_t) + \eta_t h_t + k_t dt + k_t dW_t^h, \quad t \in [0,T], \\
h_T &= R^t g(m_T),
\end{align*}
\]

(2.7)

which is the standard representation of the solution \(\theta\) to the PDE (1.8) in the form of a forward-backward SDE. The connection between the two is given by the identities

\[
h_t = \theta(t, m_t), \quad t \in [0,T]; \quad k_t = \varepsilon \nabla_x \theta(t, m_t), \quad t \in [0,T),
\]

(2.8)

which holds true under both \(\mathbb{P}\) and \(\mathbb{P}^h\). By a standard application of the maximum principle (for PDEs), \(\theta\) is bounded in terms of \(d, T, \|Q^t f\|_{1,\infty}\) and \(\|R^t g\|_{1,\infty}\), and, by Lemma 2.5 right below, \(\nabla_x \theta\) is also bounded in terms of \(d, \varepsilon, T, \|Q^t f\|_{1,\infty}\) and \(\|R^t g\|_{1,\infty}\). The relationships stated in (2.8) show in fact how to construct easily a solution to (2.6) by solving first for \((m_t)_{0 \leq t \leq T}\) in the forward equation and then by expanding \((\theta(t, m_t))_{0 \leq t \leq T}\). The hence constructed solution \((h_t, k_t)_{0 \leq t \leq T}\) to the backward equation in (2.6) is bounded. In turn, uniqueness to the backward equation in (2.6) is easily shown to hold true in the class of bounded processes \((h_t, k_t)_{0 \leq t \leq T}\).
In fact, Proposition 2.3 is not our main result. It is just given as a warm-up in order to highlight the underlying difficulties of the analysis. In brief, the comparison between the two forward equations in (2.5) and (2.6) is trivial, see (2.10) for the key (but obvious) bound. What is more subtle is the comparison between the two backward equations, because of the two products \( k_t h_t \) and \( k_t h_t \) therein. A very elementary approach in order to handle these products is to use the bound on \( k_t \) given by:

**Lemma 2.5.** There exists a constant \( C_2 \), only depending on \( d, T, \|Q^1 f\|_1,\infty \) and \( \|R^1 g\|_1,\infty \), such that

\[
|\nabla_x \theta(t, x)| \leq \frac{C_2}{\varepsilon^2}, \quad t \in [0, T), \quad x \in \mathbb{R}^d,
\]

with \( \theta \) the solution of the PDE (1.8).

The drawback of this approach is obvious: the resulting rate of convergence in Proposition 2.3 blows-up exponentially fast with \( \varepsilon^2 \). Intuitively, the above \( L^\infty \) bound for the gradient is known to be sharp, but it remains rather poor since the estimate is precisely given in \( L^\infty \). In other words, \( \theta \) may indeed become very steep, but maybe only on some localized parts of the space. This is the point where things become highly subtle: as \( \varepsilon \) tends to 0, the process \( \overline{m}^n \) becomes localized itself since the diffusion coefficient \( \varepsilon \) tends to 0. So, the challenging question is to decide whether it may stay or not in parts of the space where the gradient is high. As exemplified in the analysis performed in Delarue and Foguen [30], this may be a challenging question, even in dimension 1.

We here succeed to bypass those difficulties. Surprisingly, we obtain a quite strong result with a short proof. Here is our main result:

**Theorem 2.6.** The weak error of the scheme for the Fortet-Mourier distance is upper bounded as follows:

\[
\sup_{F} \left| \mathbb{E}^h \left[ F(\overline{m}^n, h^n) \right] - \mathbb{E}^h \left[ F(m, h) \right] \right| \leq \frac{C}{n\varepsilon},
\]  

(2.9)

the supremum being taken over all the functions \( F \) on \( C([0, T]; \mathbb{R}^d \times \mathbb{R}^d) \) that are bounded by 1 and \( 1 \)-Lipschitz continuous, and for a constant \( C \) that only depends on \( d, T \) and the norms \( \|Q^1 f\|_1,\infty \) and \( \|R^1 g\|_1,\infty \).

The improvement from Proposition 2.3 to Theorem 2.6 is substantial. Firstly, the blow-up in \( \varepsilon \) is much less singular, which is an important feature when analysing the trade-off between exploration and exploitation, see Subsection 2.3. Secondly, we clearly compare the law of the equilibrium. The resulting error is called the weak error and is fully relevant from the practical point of view. In our context, the strong error under the measure \( \mathbb{P} \), as addressed in Proposition 2.3 (regardless of the rate), does not provide the same information. Indeed, since the two densities \( \mathcal{E}(h) \) and \( \mathcal{E}(h^n) \) become singular when \( \varepsilon \) tends to 0, the passage from the strong to the weak error is quite subtle.

2.1.3. Proof of Theorem 2.6. The proof relies on two preliminary results, whose statements are given right below. The proof of the following lemma is deferred to the end of the subsection.

**Lemma 2.7.** There exists a constant \( C_1 \), only depending on \( d, T, \|Q^1 f\|_1,\infty \) and \( \|R^1 g\|_1,\infty \), such that, \( \mathbb{P} \) almost surely,

\[
|h_t| \leq C_1; \quad |h^n_t| \leq C_1, \quad t \in [0, T], \quad n \geq 1.
\]

Moreover, there exists \( \delta \in (0, 1] \) such that

\[
\mathbb{E}^h \left[ \exp \left( \delta \int_0^T |k_t|^2 \, dt \right) \right] < \infty.
\]

The next step is then to prove the following:
Proposition 2.8. There is a constant $C$, only depending on $d$, $T$, $\|Q^1 f\|_{1, \infty}$ and $\|R^1 g\|_{1, \infty}$, such that

$$\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \left( |\mathbf{m}_t^n - m_t|^2 + |h_t^n - h_t|^2 \right) \right] \leq \frac{C}{n^2}.$$  

Remark 2.9. The result is somewhat amazing since the rate in the right-hand side does not depend on $\varepsilon$. This makes it very strong (much stronger than the original estimate given in Proposition 2.3) and it is even surprising at first sight. The key point in the proof is to use the second part in the statement of Lemma 2.7, which follows from a BMO (for bounded mean oscillation) inequality taken from the theory of continuous martingales. As we already explained, this turns out to be very much more powerful than using the bound for the gradient mean oscillation inequality taken from the theory of continuous martingales. As we already explained, this turns out to be very much more powerful than using the bound for the gradient in Lemma 2.7.

The subtlety in this result is that the process $(\mathbf{m}^{n+1}, h^{n+1})$ may have very different distributions under $\mathbb{P}^h$ (which is the law under which the estimate is given) and under $\mathbb{P}^h^n$ (which is the law under which the optimization problem (2.1) is solved). For sure, the two distributions are equivalent, but the bounds for the densities deteriorate when $\varepsilon$ tends to 0 and, in the limit $\varepsilon \downarrow 0$, the two measures become singular. In other words, the above result does NOT mean that the law of the approximation, as given by the fictitious play, converges with a rate that is uniform with respect to the intensity $\varepsilon$ of the common noise.

Proof of Proposition 2.8 Throughout, $C$ is a generic constant that is allowed to vary from line to line, as long as it only depends on $d$, $T$, $\|Q^1 f\|_{1, \infty}$ and $\|R^1 g\|_{1, \infty}$. By Lemma 2.7 we then have

$$|\bar{h}_t^{n+1} - \frac{n}{n+1} \bar{h}_t^n| = \frac{1}{n+1} |h_t^{n+1}| \leq \frac{C}{n}, \quad t \in [0, T],$$  

from which we deduce (consider the difference between (2.4) and (2.6)) that

$$\sup_{0 \leq t \leq T} |\mathbf{m}_t^n - m_t| \leq \frac{C}{n}.$$  

We now make the difference between the backward equations in (2.5) and (2.6). We obtain

$$\begin{align*}
d(h_t^{n+1} - h_t) &= -\left( Q^1 f(\mathbf{m}_t^n) - Q^1 f(m_t) \right) dt + \eta_t (h_t^{n+1} - h_t) dt + k_t (h_t^n - h_t) dt \\
&\quad + h_t^n (k_t^{n+1} - k_t) dt + (k_t^{n+1} - k_t) dW_t,
\end{align*}$$  

$$h_T^{n+1} - h_T = R^1 g(\mathbf{m}_T^n) - R^1 g(m_T).$$  

In particular, rewriting the above equation under $W^h$, we get

$$\begin{align*}
d(h_t^{n+1} - h_t) &= -\left( Q^1 f(\mathbf{m}_t^n) - Q^1 f(m_t) \right) dt + \eta_t (h_t^{n+1} - h_t) dt + k_t (h_t^n - h_t) dt \\
&\quad + (k_t^{n+1} - k_t) (h_t^n - h_t) dt + (k_t^{n+1} - k_t) dW_t^h, \quad t \in [0, T].
\end{align*}$$  

Then, taking the square, using (2.11) and Lemma 2.7, expanding by Itô’s formula and applying Young’s inequality, we get for $\delta$ as in the statement of 2.5

$$\begin{align*}
d|h_t^{n+1} - h_t|^2 &\geq -\frac{C}{n^2} |h_t^n - h_t|^2 dt - \frac{\delta}{2} |k_t^{n+1} - k_t|^2 dt - \left( C + \frac{\delta}{2} |k_t|^2 \right) |h_t^{n+1} - h_t|^2 dt \\
&\quad + \frac{1}{2} |k_t^{n+1} - k_t|^2 dt + 2 (h_t^{n+1} - h_t) (k_t^{n+1} - k_t) dW_t^h, \quad t \in [0, T].
\end{align*}$$  

Now, we let $K_t = \delta \int_0^t k_s^2 ds$, $t \in [0, T]$. For a constant $\lambda > 0$, we obtain

$$\begin{align*}
d \left[ \exp(\lambda t + K_t) |h_t^{n+1} - h_t|^2 \right] &\geq - \exp(\lambda t + K_t) \left( \frac{C}{n^2} + \frac{\delta}{2} \right) |h_t^n - h_t|^2 dt \\
&\quad + \left( \lambda - C + \frac{\delta}{2} |k_t|^2 \right) \exp(\lambda t + K_t) |h_t^{n+1} - h_t|^2 dt \\
&\quad + \frac{1}{2} \exp(\lambda t + K_t) |k_t^{n+1} - k_t|^2 dt \\
&\quad + 2 \exp(\lambda t + K_t) (h_t^{n+1} - h_t) \cdot [(k_t^{n+1} - k_t) dW_t^h].
\end{align*}$$
And then, integrating from $t$ to $T$ and taking expectation under $\mathbb{P}^h$,

$$
\mathbb{E}^h \left[ \exp(\lambda t + K_t) | h_t^{n+1} - h_t |^2 \right] + \frac{1}{2} \mathbb{E}^h \int_t^T \exp(\lambda s + K_s) | k_s^{n+1} - k_s |^2 ds \\
+ \mathbb{E}^h \int_t^T \left( \lambda - C + \frac{\delta}{2} | k_s |^2 \right) \exp(\lambda s + K_s) | h_s^{n+1} - h_s |^2 ds \\
\leq \frac{C}{n^2} \mathbb{E}^h \left[ \exp(\lambda T + K_T) \right] + \mathbb{E}^h \int_t^T \exp(\lambda s + K_s) \left( \frac{C}{n^2} + \frac{C}{\delta} | h_s^n - h_s |^2 \right) ds.
$$

(2.15)

Recall that $\delta$ is given by Lemma 2.7. We deduce that

$$
\mathbb{E}^h \left[ \exp(\lambda s + K_s) \right] \leq C_1 \exp(\lambda T), \quad s \in [0, T].
$$

Next, we can choose $\lambda = 2C/\delta + C$. This yields

$$
\mathbb{E}^h \int_0^T \exp(\lambda s + K_s) | h_s^{n+1} - h_s |^2 ds \leq \frac{C'}{n^2} + \frac{1}{2} \mathbb{E}^h \int_0^T \exp(\lambda s + K_s) | h_s^n - h_s |^2 ds,
$$

(2.16)

for a new constant $C'$, depending on the same parameters as $C$ and the value of which is allowed to vary from line to line. This suffices (by a standard induction argument) to show that

$$
\mathbb{E}^h \int_0^T \exp(\lambda s + K_s) | h_s^{n+1} - h_s |^2 ds \leq C' \sum_{k=0}^{n-1} 2^{-k} (n-k)^{-2}.
$$

The right-hand is less than $C'n^{-2}$. And then, by (2.15), we obtain

$$
\sup_{0 \leq t \leq T} \mathbb{E}^h \left[ \exp(\lambda t + K_t) | h_t^{n+1} - h_t |^2 \right] \leq \frac{C'}{n^2}.
$$

(2.17)

In order to get the result with the supremum inside and without the exponential, it suffices to return back to (2.14) and then apply Burkholder-Davis-Gundy inequality. The terms in $k_t$ and in $k_t^{n+1} - k_t$ are handled by means of (2.15). In brief, we get (for a new value of $C'$)

$$
\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \exp(\lambda t + K_t) | h_t^{n+1} - h_t |^2 \right] \\
\leq \frac{C'}{n^2} + C' \mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \int_0^t \exp(\lambda s + K_s) | h_s^{n+1} - h_s | \cdot | (k_s^{n+1} - k_s) dW_s | \right] \\
\leq \frac{C'}{n^2} + C' \mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \exp(\lambda t + K_t) | h_t^{n+1} - h_t | \left( \int_0^T \exp(\lambda t + K_t) | h_t^{n+1} - h_t |^2 dt \right)^{1/2} \right] \\
\leq \frac{C'}{n^2} + \frac{1}{2} \mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \exp(\lambda t + K_t) | h_t^{n+1} - h_t |^2 \right],
$$

where we used Cauchy-Schwarz and Young inequalities together with (2.15) in order to get the last line. The proof is easily completed.

We now complete our analysis:

**Proof of Theorem 2.8.** The conclusion follows from Pinsker’s inequality, which says that

$$
d_{TV}(\mathbb{P}^h, \mathbb{P}^{h_n}) \leq \mathbb{E}^h \left[ \ln \left( \frac{\mathcal{E}(h)}{\mathcal{E}(h_n)} \right) \right]^{1/2},
$$

where $d_{TV}$ in the left-hand side is the total variation distance. Now,

$$
\ln \left( \frac{\mathcal{E}(h)}{\mathcal{E}(h_n)} \right) = -\frac{1}{\varepsilon} \int_0^T (h_s - h_s^n) \cdot dW_s - \frac{1}{2\varepsilon^2} \int_0^T |h_s|^2 - |h_s^n|^2 | ds \\
= -\frac{1}{\varepsilon} \int_0^T (h_s - h_s^n) \cdot dW_s - \frac{1}{\varepsilon} \int_0^T (h_s - h_s^n) \cdot h_s ds - \frac{1}{2\varepsilon^2} \int_0^T |h_s|^2 - |h_s^n|^2 | ds \\
= -\frac{1}{\varepsilon} \int_0^T (h_s - h_s^n) \cdot dW_s + \frac{1}{2\varepsilon^2} \int_0^T |h_s - h_s^n|^2 ds.
$$
And then, by Proposition 2.8

\[ d_{TV}(\mathbb{P}^h, \mathbb{P}^{h_n}) \leq \frac{C}{\varepsilon n}. \] (2.18)

By Proposition 2.8 again, for any function \( F \) that is 1-bounded and 1-Lipschitz on the space \( C([0, T]; \mathbb{R}^d \times \mathbb{R}^d) \),

\[ \mathbb{E}^h \left[ F(m^n, h^n) \right] \leq C, \]

and then, by (2.18),

\[ \mathbb{E}^{h_n} \left[ F(m^n, h^n) \right] \leq C. \]

\[ \square \]

2.1.4. Proof of the auxiliary statements. It now remains to prove Lemma 2.7 and also to explain (if needed) how to adapt the proof of Proposition 2.8 in order to prove Proposition 2.3.

Proof of Lemma 2.7. Back to (2.5), we write the backward SDE therein as an equation driven by \( W^h \).

\[ dh^{n+1}_t = (-Q^1 f(m^n_t) + \eta_{h} h^{n+1}_t)dt + k^{n+1}_t dW^h_t, \quad t \in [0, T]. \]

We call \( (P_t)_{0 \leq t \leq T} \) the solution of the linear differential equation \( \dot{P}_t = -\eta_{h} P_t \), for \( t \in [0, T] \) with \( P_0 = I_d \) as initial condition, where \( I_d \) is the \( d \)-dimensional identity matrix. Then,

\[ d \left[ P_t h^{n+1}_t \right] = -P_t Q^1 f(m^n_t) dt + P_t k^{n+1}_t dW^h_t, \]

or equivalently,

\[ P_t h^{n+1}_t = P_T R^1 g(m^n_T) + \int_t^T P_s Q^1 f(m^n_s) ds - \int_t^T P_s k^{n+1}_s dW^h_s, \quad t \in [0, T]. \]

Taking conditional expectation given \( \mathcal{F}^W_t \) and next using the fact that \( Q^1 f \) and \( R^1 g \) are bounded, we deduce that the left hand side is bounded by a constant \( C_1 \) as in the statement. Moreover, squaring the identity and then taking the conditional expectation given \( \mathcal{F}^W_\tau \), for any stopping \( \tau \), we obtain (say for the same constant \( C_1 \)) that

\[ \mathbb{E}^{h_n} \left[ \int_\tau^T |P_s k^{n+1}_s|^2 ds \mid \mathcal{F}^W_\tau \right] \leq C_1. \]

Obviously, we can easily remove \( P_s \) in the integrand by modifying the constant \( C_1 \) accordingly. This shows that the \( \mathbb{E}^{h_n} \)-martingale

\[ \left( \int_0^t k^{n+1}_s dW^h_s \right)_{0 \leq t \leq T} \]

has a finite BMO norm, see for instance Kazamaki, (2.1) (with \( p = 2 \)). By Theorem 2.2 in the same reference, the proof is complete.

\[ \square \]

Proof of Proposition 2.3. It suffices to return back to (2.13) and to choose \( \delta = \varepsilon^2 \) therein and to use the bound \( |k_t| \leq C_2/\varepsilon^2 \) given by Lemma 2.5. We then choose \( \lambda = C/\varepsilon^2 \) in (2.15), for a suitable value of \( C \). The rest of the proof is similar to the proof of Proposition 2.8. \[ \square \]
2.1.5. Discussion about the rate of convergence. Beside any specific application to learning, this is another of our contributions to provide an explicit bound for the rate of convergence of our variant of the fictitious play for mean field games with common noise. We feel worth to point out that, to the best of our knowledge, there are very few results on the rate of convergence for the fictitious play in the absence of common noise, whether the mean field game be potential or monotone (as we already explained, no result is available without common noise outside the potential or monotone cases, except [59] which is for a time-continuous version of the fictitious play). In most of the existing references, the analysis indeed involves an additional compactness argument which complicates the computation of the rate. Still, the reader can find in [39] an explicit rate for the exploitability (which we called the regret in the introduction) for a monotone and potential game set in discrete time; the bound is of order $O(1/\sqrt{n})$ (hence weaker than ours). In [59], a bound is shown, also for the exploitability, but for the time-continuous version of the fictitious play, when the game is monotone; it is of order $O(1/n)$, and is hence comparable to ours.

Importantly, our analysis can be in fact pushed further and our scheme can be adapted to get an even finer rate of convergence. In (2.20), we can indeed replace $h^n$ (which is the proxy returned by the previous step of the scheme) by a more general proxy $\tilde{h}_n$. We hence get

$$dX_t^{n+1} = -\left(\tilde{h}_n X_t^{n+1} + \tilde{h}_t^{n+1} - \tilde{h}_t^n\right) dt + \sigma dB_t + \varepsilon dW_t, \quad t \in [0, T].$$

Now, we propose the following rule for computing $\tilde{h}_n$:

$$\tilde{h}_t^{n+1} = (1 + \frac{1}{n+1}) h_t^{n+1} - \frac{1}{n+1} \tilde{h}_t^n; \quad \tilde{h}_t^n = \frac{1}{n} \sum_{k=1}^n h_t^k, \quad t \in [0, T],$$

with the same initialization $\tilde{h}_0^n = \tilde{h}_0^n \equiv 0$ as before. Notice that, in the above scheme, $h_t^{n+1}$ is implicitly understood as the intercept of the (affine) optimal feedback function in the optimization problem (2.1) when the environment is $\overline{m}_t$ (as it is in (2.1)) and the tilted measure is no longer $\mathbb{P}^{h^n}$ but $\mathbb{P}^{\tilde{h}_n}$. Here, the updating rule for $m_t^n$ remains unchanged:

$$m_t^{n+1} = \frac{1}{n+1} m_t^{n+1} + \frac{n}{n+1} \overline{m}_t^n, \quad t \in [0, T].$$

In a way, the scheme (2.20) relies on a form of averaging over the past optimal controls. We refer to [58] for another form of averaging over past optimal controls in a fictitious play for mean field games.

We then have the following variant of Proposition 2.8

**Proposition 2.10.** There is a constant $C$, only depending on $d$, $T$, $\|Q^f\|_{1,\infty}$ and $\|R^g\|_{1,\infty}$, such that, for the new version (2.20) of the scheme,

$$\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \left( |\overline{m}_t^n - m_t| + |h_t^n - h_t| + |\tilde{h}_t^n - h_t| \right) \right] \leq \frac{C}{n^2}.$$

Before we explain the differences between the proofs of Propositions 2.8 and 2.10 we stress that Proposition 2.10 can be used to get a relevant version of Theorem 2.6 with $h^n$ replaced by $\tilde{h}_n$ and with $C/(n^2 \varepsilon)$ instead of $C/(n \varepsilon)$ as error. This improvement is mostly given to highlight the possible extensions of our method. For simplicity (and also because the variant (2.20) is heavier from a computational point of view), we feel better, in the rest of the paper, to stick to the scheme introduced in Subsection 2.1.1

**Sketch of the proof of Proposition 2.10.**

**First Step.** We first observe that, for $n \geq 1$,

$$h_t^{n+1} - \tilde{h}_t^n = h_t^{n+1} - \left(1 + \frac{1}{n} h_t^n - \frac{1}{n} \tilde{h}_t^{n-1}\right) = h_t^{n+1} - h_t^n - \frac{1}{n} (h_t^n - \tilde{h}_t^{n-1}).$$
Therefore,
\[
\frac{1}{n+1} \sum_{k=1}^{n} (h_{t}^{k+1} - \tilde{h}_{t}^{k}) = \frac{1}{n+1} \sum_{k=1}^{n} (h_{t}^{k+1} - h_{t}^{1}) - \frac{1}{n+1} \sum_{k=1}^{n} \frac{1}{k} (h_{t}^{k} - \tilde{h}_{t}^{k-1})
\]
\[
= \frac{1}{n+1} (h_{t}^{n+1} - h_{t}^{1}) - \frac{1}{n+1} \sum_{k=1}^{n} (\tilde{h}_{t}^{k} - \tilde{h}_{t}^{k-1})
\]
\[
= \frac{1}{n+1} (h_{t}^{n+1} - \tilde{h}_{t}^{n}).
\]

Recalling that $\tilde{h}_{t}^{n} = 0$, we deduce that
\[
\frac{1}{n+1} \sum_{k=1}^{n+1} (h_{t}^{k} - \tilde{h}_{t}^{k-1}) = \frac{1}{n+1} (h_{t}^{n+1} - \tilde{h}_{t}^{n}).
\]

In turn, the bound (2.11) becomes
\[
|m_{t}^{n} - m_{t}| \leq \frac{C}{n} \int_{0}^{t} |h_{s}^{n} - \tilde{h}_{s}^{n-1}| \, ds, \quad t \in [0, T],
\]
which yields
\[
\mathbb{E} \left[ \exp(\lambda t + K_{t}) |m_{t}^{n} - m_{t}|^{2} \right] \leq \frac{C}{n^{2}} \int_{0}^{t} \mathbb{E} \left[ \exp(\lambda s + K_{s}) |h_{s}^{n} - \tilde{h}_{s}^{n-1}|^{2} \right] \, ds, \quad t \in [0, T].
\]

The key fact is then to notice that the second inequality in the statement of Lemma 2.7 remains true in a conditional form, namely
\[
\mathbb{E}^{h} \left[ \exp(\delta \int_{t}^{T} |k_{s}|^{2} \, ds) \mid \mathcal{F}_{t}^{W} \right] \leq C',
\]
for a deterministic constant $C'$ depending on the same parameters as $C$. The proof of the latter is the same. We end-up with
\[
\mathbb{E} \left[ \exp(\lambda t + K_{t}) |m_{t}^{n} - m_{t}|^{2} \right] \leq \frac{C}{n^{2}} \int_{0}^{t} \mathbb{E} \left[ \exp(\lambda s + K_{s}) |h_{s}^{n} - \tilde{h}_{s}^{n-1}|^{2} \right] \, ds, \quad t \in [0, T].
\]

Second Step. Now, in the right-hand side of (2.12), the difference $h_{t}^{n} - h_{t}$ must be replaced by $\tilde{h}_{t}^{n} - h_{t}$. Observing that the first bound in Lemma 2.7 remains true, we are thus left with the following variant of (2.16):
\[
\mathbb{E}^{h} \int_{0}^{T} \exp(\lambda s + K_{s}) |h_{s}^{n+1} - h_{s}|^{2} \, ds \leq \frac{C'}{n^{2}} \mathbb{E}^{h} \int_{0}^{T} \exp(\lambda s + K_{s}) |h_{s}^{n} - \tilde{h}_{s}^{n-1}|^{2} \, ds
\]
\[
+ \frac{1}{2} \mathbb{E}^{h} \int_{0}^{T} \exp(\lambda s + K_{s}) |\tilde{h}_{s}^{n} - h_{s}|^{2} \, ds.
\]

By (2.20), we deduce (for a new value of $C'$ and for $n$ greater than some fixed rank $n_0$):
\[
\mathbb{E}^{h} \int_{0}^{T} \exp(\lambda s + K_{s}) |h_{s}^{n+1} - h_{s}|^{2} \, ds \leq \frac{C'}{n^{2}} \mathbb{E}^{h} \int_{0}^{T} \exp(\lambda s + K_{s}) |h_{s}^{n} - \tilde{h}_{s}^{n-1}|^{2} \, ds
\]
\[
+ \frac{3}{4} \mathbb{E}^{h} \int_{0}^{T} \exp(\lambda s + K_{s}) |\tilde{h}_{s}^{n} - h_{s}|^{2} \, ds.
\]  

(2.21)

Obviously, the first term in the right-hand side is less than $C'/n^2$ and we can proceed as in the proof of Proposition 2.8 to get that the left-hand side is also less than $C'/n^2$. In brief, the bound is as good as the bound obtained in Proposition 2.8. With this intermediary result, we get
\[
\mathbb{E} \int_{0}^{T} \exp(\lambda s + K_{s}) |\tilde{h}_{s}^{n} - h_{s}|^{2} \, ds \leq \frac{1}{n^{2}} \sum_{i,j=1}^{n} \mathbb{E} \int_{0}^{T} \exp(\lambda s + K_{s}) |h_{s}^{i} - h_{s}| |\tilde{h}_{s}^{j} - h_{s}| \, ds
\]
\[
\leq \frac{C'}{n^{2}} \sum_{i,j=1}^{n} \frac{1}{i j} \leq \frac{C'(1 + \ln(n))^{2}}{n^{2}},
\]  

(2.22)
where the second line follows from Cauchy-Schwarz inequality. Inserting the above bound in the right-hand side of (2.21) and repeating the argument used in the proof of Proposition 2.8, we get that the left-hand side in (2.21) is now bounded by $C'(1 + \ln(n))^2/n^4$. And then, (2.22) becomes

$$\mathbb{E} \int_0^T \exp(\lambda s + K_s) \left| \tilde{h}_n - h_s \right|^2 ds \leq \frac{C'}{n^2} \sum_{i,j=1}^n \frac{(1 + \ln(i))^2}{i^2} \frac{(1 + \ln(j))^2}{j^2} \leq \frac{C'}{n^2}.$$  

Back to (2.21), we deduce that the left-hand side therein is less than $C'/n^4$. Combined with the above inequality, this gives

$$\mathbb{E} \int_0^T \exp(\lambda s + K_s) \left| \tilde{h}_n - h_s \right|^2 ds \leq \frac{C'}{n^4}.$$  

We then get the analogue of (2.17), but with $C'/n^4$ in the right-hand side, which yields

$$\sup_{0 \leq t \leq T} \mathbb{E}^h \left| h^n_t - h_t \right|^2 \leq \frac{C'}{n^4}.$$  

As in the proof of Proposition 2.8, we can let the supremum in time enter inside the expectation. Following (2.22), we have the same bound with $h^n$ replaced by $\hat{h}^n$. The bound for $m^n - m$ then follows from the first step. \qed

2.2. The common noise as an exploration noise. Following the agenda explained in the introduction, we now regard the common noise as an exploration noise for an MFG without common noise. In words, $\varepsilon$ is set equal to 0 in the original MFG [1.1]-[1.3] and this is only in the choice of the controls that we restore the presence of the common noise, in the form of a randomization.

2.2.1. Presentation of the model. In the absence of common noise, the state dynamics merely write

$$dX_t = \beta_t dt + \sigma dB_t, \quad t \in [0,T].$$

However, we want $\beta = (\beta_t)_{0 \leq t \leq T}$ to be subjected to a random exploration of the form

$$\beta_t = \alpha_t + \varepsilon \tilde{W}_t, \quad t \in [0,T],$$

where $\alpha = (\alpha_t)_{0 \leq t \leq T}$ is the control effectively chosen by the agent (or by the ‘controller’) and $\varepsilon$ is (strictly) positive (and is kept fixed throughout the subsection). In this expansion, $(\tilde{W}_t)_{0 \leq t \leq T}$ is formally understood as the time-derivative of $(W_t)_{0 \leq t \leq T}$. Obviously, the latter does not exist as a function, which makes the above decomposition non tractable. However, it prompts us to introduce a variant based upon a mollification of the common noise. To make it clear, we introduce a family of regular processes $((W^p_t)_{0 \leq t \leq T})_{p \geq 1}$ such that, almost surely (under $\mathbb{P}$),

$$\lim_{p \to \infty} \sup_{0 \leq t \leq T} |W_t - W^p_t| = 0.$$  

and, for any $p \geq 1$, the paths of $(W^p_t)_{0 \leq t \leq T}$ are continuous and piecewise continuously differentiable. Below, we choose the following piecewise affine interpolation:

$$W^p_t = W_{\tau_p(t)} + \frac{p(t-\tau_p(t))}{T} (W_{\tau_p(t)+T/p} - W_{\tau_p(t)}),$$

where, by definition, $\tau_p(t) = \lfloor pt/T \rfloor (T/p)$. In words $\tau_p(t)$ is the unique element of $(T/p) \cdot \mathbb{N}$ such that $\tau_p(t) \leq t < \tau_p(t)+T/p = \tau_p(t+T/p)$, namely $\tau_p(t) = \ell T/p$, for $t \in [(\ell+1)T/p, \ell T/p)$ and $\ell \in \{0, \ldots, p-1\}$.

For an $\mathbb{F}^W$-adapted and continuous environment $m = (m_t)_{0 \leq t \leq T}$, the cost functional, as originally defined in [1.2], is turned into the following discrete time version

$$\bar{J}^p(\alpha; m) = \frac{1}{2} \mathbb{E} \left[ \left| R^1 X_T + g(m_T) \right|^2 + \int_0^T \left\{ \left| Q^1 X_{\tau_p(t)} + f(m_{\tau_p(t)}) \right|^2 + |\alpha_t + \varepsilon \tilde{W}_t|^2 \right\} dt \right],$$

where $\mathbb{F}^W$ is the filtration generated by the common noise $W$ and the control process $\alpha$.
where the expectation is taken over both the idiosyncratic and exploration (common) noises. Above, we require \( \alpha \) to be progressively-measurable with respect to the filtration \( \mathbb{F}_{T/p}.B,W = (\sigma(X_0,(B_{\tau_p(s)},W_{\tau_p(s)})_{s \leq T} \mid 0 \leq \tau_p \leq T) \) and to be constant on any interval \([\ell T/p,(\ell + 1)T/p)\), for \( \ell \in \{0,\ldots,p - 1\}\). The above minimization problem can be regarded as a discrete time control problem. The need for a time discretization is twofold: (i) On the one hand, it permits to avoid any anticipativity problem, since the linear interpolation at a time \( t \neq \tau_p(t) \) anticipates on the future realization of the exploration noise; (ii) On the other hand, it is more adapted to numerical purposes.

In the presence of the randomization, the cost functional might become very large as \( p \) tends to \( \infty \), even for a control \( \alpha \) of finite energy. This prompts us to renormalize \( \tilde{J}^p \). As a result of the adaptability constraint, we indeed have

\[
\mathbb{E} \int_0^T \alpha_t W_t^p dt = 0.
\]

Also, since \( W^p \) is piecewise linear, we have

\[
\mathbb{E} \int_0^T |\tilde{W}_t^p|^2 dt = \sum_{\ell=0}^{p-1} \int_{\ell T/p}^{(\ell + 1)T/p} \mathbb{E} \left[ \left( \frac{p}{T} (W_{(\ell + 1)T/p} - W_{\ell T/p}) \right)^2 \right] dt = \frac{T}{p} \frac{p^2}{T^2} = p.
\]

So, we must subtract to the cost a diverging term to recover the original cost functional. To make it clear, the effective cost must be

\[
J^p(\alpha;m) = \tilde{J}^p(\alpha;m) - \frac{1}{2} \varepsilon^2 p
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \left| \tilde{R}^1 X_T + g(m_T) \right|^2 + \int_0^T \left\{ |Q^1 X_{\tau_p(t)} + f(m_{\tau_p(t)})|^2 + |\alpha_{\tau_p(t)}|^2 \right\} dt \right].
\]

Noticeably, we recover (up to the time discretization) the same cost functional \( J \) as in (1.2), which explains why we have removed the tilde over the symbol \( \tilde{J}^p \). Anyway, \( J^p(\cdot;m) \) and \( \tilde{J}^p(\cdot;m) \) have the same minimizers.

Before we provide the form of the corresponding fictitious play, we need to clarify the notion of tilted measure. We assume that we are given a process \( h = (h_t)_{0 \leq t \leq T} \) that is piecewise constant and \( \mathbb{F}_{T/p}^W \)-adapted, with \( \mathbb{F}_{T/p}^W = \mathbb{F}^{W^p} \). For the same \( p \) as before, the process \( h \) is constant on each interval \( ((\ell T/p,(\ell + 1)T/p)_{\ell=0,\ldots,p-1} \) and each \( h_{\ell T/p} \) is \( \mathcal{F}_{\ell T/p}^W \)-measurable, with \( \mathcal{F}_{\ell T/p}^W = \sigma(W_s^p,s \leq \ell T/p) \). Then, as before, we can let \( W^h \):

\[
W_t^h = W_t + \frac{1}{\varepsilon} \int_0^t h_s ds.
\]

We compute the \( p \)-piecewise linear interpolation \( W^{p,h} \) of \( W^h \):

\[
W_t^{p,h} = W_{\tau_p(t)} + \frac{p(t - \tau_p(t))}{T} (W_{\tau_p(t) + T/p} - W_{\tau_p(t)})
\]

\[
= W_{\tau_p(t)} + \frac{p(t - \tau_p(t))}{T} (W_{\tau_p(t) + T/p} - W_{\tau_p(t)}) + \frac{1}{\varepsilon} \frac{p(t - \tau_p(t))}{p} h_{\tau_p(t)}
\]

\[
= W_{\tau_p(t)} + \frac{p(t - \tau_p(t))}{T} (W_{\tau_p(t) + T/p} - W_{\tau_p(t)}) + \frac{1}{\varepsilon} (t - \tau_p(t)) h_{\tau_p(t)}
\]

\[
= W_{\tau_p(t)} + \int_{\tau_p(t)}^t dW_s^p + \frac{1}{\varepsilon} \int_{\tau_p(t)}^t h_s ds,
\]

from which we deduce that

\[
W_t^{p,h} = W_t^p + \frac{1}{\varepsilon} \int_0^t h_s ds, \quad t \in [0,T].
\]
For sure, under the probability \( \mathcal{E}(h) \cdot \mathbb{P} \), the process \( W^{p,h} \) is the piecewise linear interpolation of \( W^h \) and the latter is a Brownian motion. Also,

\[
\dot{W}^{p,h}_t = \dot{W}^p_t + \frac{1}{\varepsilon} h_t,
\]

which permits to say that

\[
dX_t = (\alpha_t - h_t)dt + \sigma dB_t + \varepsilon dW^{p,h}_t,
\]

which is exactly the same as before. Importantly, since \( h \) is piecewise constant, \( \mathcal{E}(h) \) can be expressed in terms of the sole \( h \) and \( W^p \).

2.2.2. Fictitious play. This leads to the following new scheme, which should be regarded as the time discrete analogue of the scheme presented in Subsection 2.1.1. Fix an integer \( p \geq 1 \) and, for the same initialization \( m^0 = (m^0_t = \mathbb{E}(X_0))_{0 \leq t \leq T} \) and \( h^0 = (h^0_t = 0)_{0 \leq t \leq T} \) as in the time continuous setting, assume that we have defined two families of proxies \( (m^1, \ldots, m^n) \) and \( (h^1, \ldots, h^n) \) with the additional assumption that each process \( h^k \) is constant on each interval \( [T/p, (\ell + 1)/p) \), for \( \ell \in \{0, \ldots, p - 1\} \) and \( k \in \{1, \ldots, n\} \). And we assume each \( (m^k_{\ell T/p}, h^k_{\ell T/p}) \) to be measurable with respect to \( \mathcal{F}^{p,W}_{\ell T/p} \) if \( \ell \in \{0, \ldots, p\} \). Then, we solve for

\[
\alpha^{n+1} = \arg\min_\alpha \left( \mathbb{E}^{h^n}[\mathcal{R}^p(\alpha + \varepsilon \dot{W}^{p,h^n}; m^n; 0)] - \frac{1}{2} \varepsilon^2 p \right)
\]

with the same measurability rules as those explained above and where \( \mathcal{R}^p \) is the obvious time discrete version of \( \mathcal{R} \), namely

\[
\mathcal{R}^p(\alpha; n, w) = \frac{1}{2} \left( |Rx_T + g(n_T)|^2 + \sum_{\ell=0}^{p-1} \left( |Qx_{\ell T/p} + f(n_{\ell T/p})|^2 + |\alpha_{\ell T/p}|^2 \right) \right),
\]

where \( x_{\ell+1}T/p = x_{\ell T/p} + (T/p)\alpha_{\ell T/p} + \sigma(B_{\ell+1}T/p - B_{\ell T/p}) + \varepsilon(w_{\ell+1}T/p - w_{\ell T/p}) \). As before, \( m^\alpha \) in \( 2.23 \) is \( m^\alpha = (m^1 + \cdots + m^n)/n \).

The first step here is to solve the optimization problem \( 2.23 \). Very similar to Lemma 2.1 we can provide an explicit form for the optimal feedback through the solution of the following time-discrete Riccati equation:

\[
P_{\ell+1}T/p (\eta^{(p)}_{\ell+1}T/p - \eta^{(p)}_{\ell T/p}) - \eta^{(p)}_{\ell+1}T/p \eta^{(p)}_{\ell T/p} + Q^\dagger Q \left( I_d - \frac{T}{p} \eta^{(p)}_{\ell T/p} \right) = 0,
\]

\( \ell \in \{0, \ldots, p - 2\} \); \( \eta^{(p)}_{p T/p} = 0 \); \( \eta^{(p)}_{T/p} = R^\dagger R \).

The analysis of the latter (see the earlier reference [32]) goes through the auxiliary Riccati equation

\[
P_{\ell+1}T/p (P_{\ell+1}T/p - P_{\ell T/p}) - P_{\ell+1}T/p \left( I_d + \frac{T}{p} P_{\ell+1}T/p \right)^{-1} P_{\ell+1}T/p + Q^\dagger Q = 0,
\]

for \( \ell \in \{0, \ldots, p - 1\} \), with \( P_{T/p} = R^\dagger R \) as boundary condition. The Riccati equation \( 2.25 \) can be solved inductively: the solution is symmetric and non-negative, which guarantees that the inverse right above is well-defined. Then,

\[
\eta^{(p)}_{\ell T/p} = \left( I_d + \frac{T}{p} P_{\ell+1}T/p \right)^{-1} P_{\ell+1}T/p, \quad \ell \in \{0, \ldots, p - 1\},
\]

solves \( 2.24 \). Notice that the above left-hand side is symmetric and non-negative since the two matrices in the right-hand side commute.
Lemma 2.11. Under the above assumptions, the minimization problem \( \text{[2.23]} \) has a unique solution \( \alpha^{n+1} \), which writes

\[
\alpha^{n+1}_t = - \left( \eta^{(p)}_{\tau_p(t)} X^{n+1}_{\tau_p(t)} + \tilde{h}^{n+1}_{\tau_p(t)} \right), \quad t \in [0, T],
\]

where \( \tilde{h}^{n+1} \) solves the backward SDE:

\[
d\tilde{h}^{n+1}_t = \left\{ -\mathbb{E}^{h^n}[Q f(\overline{m}^{n}_{\tau_p(t)} + T/p)] \mathcal{F}^{p,W}_{\tau_p(t)} \mathbf{1}_{\{t \leq (p-1)/T\}} + \left( \eta^{(p)}_{\tau_p(t) + T/p} + \frac{T}{p} Q^{1} \mathbf{1}_{\{t \leq (p-1)/T\}} \right) \tilde{h}^{n+1}_{\tau_p(t)} \right\} dt + k^{n+1}_t \tilde{h}^{n+1}_t dt + k^{n+1}_t dW_t, \quad t \in [0, T),
\]

Accordingly, the optimal path \( X^{n+1} \) solves the forward SDE:

\[
dX^{n+1}_t = \alpha^{n+1}_t dt + \sigma dB_t + \varepsilon dW^{n}_t, \quad t \in [0, T].
\]

Remark 2.12. The convergence of \( (F^{(p)}_{\tau_p(t)})_{0 \leq t \leq T} \) to \( (\eta_{k})_{0 \leq t \leq T} \) in Lemma \( \text{[2.1]} \) is obvious.

By the symmetric and non-negative structure of the matrices \( (F^{(p)}_{\ell})_{\ell=0, \ldots, p-1} \), we can easily get uniform bounds on the latter. In turn, we can regard the equation as an Euler scheme for a matricial differential equation driven by a Lipschitz vector field. The rate of convergence is linear in \( p \). By \( \text{[2.26]} \), we deduce that

\[
\sup_{0 \leq t \leq T} |\eta^{(p)}_{t} - \eta_{t}| \leq \frac{C}{p},
\]

for a constant \( C \) independent of \( p \).

Remark 2.13. The backward SDE \( \text{[2.28]} \) is in fact a mere time discrete equation. Indeed,

\[
\tilde{h}^{n+1}_{\ell T/p} = \tilde{h}^{n+1}_{\ell T/p - T/p} + \frac{T}{p} Q^{1} f(\overline{m}^{n}_{\ell T/p - T/p}) \mathbf{1}_{\{t \leq (\ell+1)/T\}} - \int_{\ell T/p}^{(\ell+1)/T} \tilde{h}^{n+1}_t dW^{n}_t,
\]

for \( \ell \in \{0, \ldots, p-1\} \). Taking conditional expectation given \( \mathcal{F}^{W}_{\ell T/p} \), we can solve for

\[
\left( I_d + \frac{T}{p} (\eta^{(p)}_{(\ell+1)/T} + \frac{T^2}{p^2} Q^{1} \mathbf{1}_{\{t \leq (\ell+1)/T\}}) \right) \tilde{h}^{n+1}_{\ell T/p} = \mathbb{E}^{h^n}[\tilde{h}^{n+1}_{(\ell+1)/T} \mathbf{1}_{\{t \leq (\ell+2)/T\}} | \mathcal{F}^{W}_{\ell T/p}],
\]

which proves, by induction, that \( \tilde{h}^{n+1}_{\ell T/p} \) is \( \mathcal{F}^{W}_{\ell T/p} \)-measurable. It suffices to observe that, for \( Z \) an \( \mathcal{F}^{p,W}_{(\ell+1)T/p} \)-measurable random variable, the conditional expectation of \( Z \) given \( \mathcal{F}^{W}_{\ell T/p} \) is in fact \( \mathcal{F}^{p,W}_{\ell T/p} \)-measurable.

Taking for granted Lemma 2.11 (the proof is given at the end of the subsection), we put

\[
h^{n+1}_t = \tilde{h}^{n+1}_{\tau_p(t)},
\]

which satisfies the required measurability constraints thanks to \( \text{[2.13]} \). Then, we let

\[
m^{n+1}_t = \mathbb{E}[X^{n+1}_t | \sigma(W)] = \mathbb{E}^{h}[X^{n+1}_t | \sigma(W)] = \mathbb{E}^{h^n}[X^{n+1}_t | \sigma(W)], \quad t \in [0, T],
\]
together with
\[
\bar{m}_n^{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} m_n^k = \frac{1}{n+1} m_n^{n+1} + \frac{n}{n+1} \bar{m}_n^n, \quad t \in [0, T]. \tag{2.32}
\]

Importantly, notice that
\[
m_{nT/p}^{n+1} = \mathbb{E} \left[ X_{nT/p}^{n+1} \right] \mathbb{E} \left[ \sigma(W) \right] = \mathbb{E} \left[ X_{nT/p}^{n+1} \right] \mathbb{E} \left[ \mathcal{F}_{nT/p}^W \right], \quad \ell \in \{0, \ldots, p\},
\]
which proves in particular that the left-hand side is \( \mathcal{F}_{nT/p}^W \)-measurable.

The analogue of Theorem 2.6 becomes:

**Theorem 2.14.** With \((\mathbf{m}, h)\) denoting the solution of \((2.6)\) (or, equivalently, the unique equilibrium of the game with common noise), the weak error of the scheme for the Fortet-Mourier distance is upper bounded as follows:
\[
\sup_F \mathbb{E}^h \left[ F(\mathbf{m}^n, h^n) - F(\mathbf{m}, h) \right] \leq C \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right),
\]
where the supremum is taken over all the functions \( F \) on \( \mathcal{C}([0, T]; \mathbb{R}^d \times \mathbb{R}^d) \) that are bounded by 1 and 1-Lipschitz continuous, and for a constant \( C \) that only depends on \( d, T, \) the norms \( \|f\|_{1,\infty} \) and \( \|g\|_{1,\infty} \) and the norms \( |Q| \) and \( |R| \) of the matrices \( Q \) and \( R \).

**Remark 2.15.** Similar to the proof of Theorem 2.6 the proof of Theorem 2.14 also shows that
\[
d_{TV}(\mathbb{P}^h, \mathbb{P}^{h^n}) \leq C \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right).
\]

2.2.3. Analysis of the convergence. In the current discrete setting, the identity (2.4) becomes
\[
d\bar{m}_n^{n+1} = -(\eta_n^p(t) - \bar{h}_n^{n+1} - \frac{n}{n+1} \bar{h}_n^n) dt + \varepsilon dW^p_n, \quad t \in [0, T]. \tag{2.34}
\]
As in the analysis (2.5), the second term in the middle disappears asymptotically. This follows from the next result, which is the analogue of Lemma 2.7 (using (2.30) in order to control inductively the norm of \( h^n \) and, for reasons that will become clear in the proof, yielding the estimate on \( k^n \) in a conditional form):

**Lemma 2.16.** With \((\mathbf{m}, h)\) denoting the solution of \((2.6)\), there exists a constant \( C_1 \), only depending on \( d, T, \) \( \|Q\|_1, \|f\|_1, \) and \( \|R\|_1 \), such that, \( \mathbb{P}^h \) almost surely,
\[
|h|_1 \leq C_1; \quad |h^n|_1 \leq C_1, \quad t \in [0, T], \quad n \geq 1.
\]
Moreover, for the same constant \( C_1 \), there exists \( \delta \in (0, 1] \) such that, with probability 1 under \( \mathbb{P}^h \),
\[
\forall \tau \in [0, T], \quad \mathbb{E}^h \left[ \exp \left( \delta \int_\tau^T |k_s|^2 ds \right) \left| \mathcal{F}_t^W \right] \leq C_1.
\]

We will also need the following estimate on \( k^n \), the proof of which is proper to the discrete time setting that we use here.

**Lemma 2.17.** With \((\mathbf{m}, h)\) denoting the solution of \((2.6)\), there exists a constant \( C_2 \), only depending on \( d, T, \) \( \|f\|_1, \) \( \|g\|_1, \) \( |Q| \) and \( |R| \), such that, for any \( n \geq 1 \), \( \mathbb{P}^h \) almost surely,
\[
\forall \tau \in [0, T], \quad \int_{\tau(t)}^{\tau(t)+T/p} |k_s^n| ds \leq \frac{C_2}{p^{1/2}}.
\]
As a corollary, for a possibly new value of \( C_2 \) (provided it only depends on the same parameters as above),
\[
\left| \frac{p}{T} \int_{\tau(t)}^{\tau(t)+T/p} \mathbb{E}^h \left[ h_s^{n+1} \left| \mathcal{F}_{\tau(t)}^W \right] ds - \frac{p}{T} \int_{\tau(t)}^{\tau(t)+T/p} \mathbb{E}^h \left[ h_s^{n+1} \left| \mathcal{F}_{\tau(t)}^W \right] ds \right| \leq \frac{C_2}{p^{1/2}},
\]
with probability 1 under \( \mathbb{P}^h \).
The challenge in the first inequality is to prove that the bound holds almost everywhere. This makes it substantially different from the bound for \( k \) that is given in Lemma 2.16. In this sense, it is much closer to Lemma 2.5. As for the second line, we observe that the two conditional expectation make sense under the same probability \( P \), since \( P^h \) and \( P^{h^n} \) are equivalent. 

Proof. In order to simplify, we let \( \ell = p \tau_p(t)/T \), for \( t \in [0, T) \). Equivalently, \( \ell T/p \leq t < (\ell + 1)T/p \). Back to (2.29), we notice that the whole term in the right-hand side, the stochastic integral being excepted, is \( F^p_{(\ell + 1)/T/p} \)-measurable. Since all the variables \( (h^n_{s})_{s < (\ell + 1)/T/p} \) are \( F^p_{\ell T/p} \)-adapted, we can rewrite the equation in the form 

\[
\hat{h}^{n+1}_t = \Psi^{p,n}(W^n_s, W^{h^n}_{(\ell + 1)/T/p} - W^{h^n}_{\ell T/p}) - \int_t^{(\ell + 1)/T/p} k^{n+1}_s dW^h_s,
\]

for a (measurable) function \( \Psi^{p,n} : (\mathbb{R}^d)^{\ell+1} \to \mathbb{R}^d \), which is bounded by a constant \( C \) depending on the same parameters as \( C_1 \) in the statement. The latter bound is an obvious consequence of Lemma 2.16. Conditional on \( F^p_{\ell T/p} \), the above may be regarded as a backward SDE on the interval \( [\ell T/p, (\ell + 1)T/p] \), with a terminal condition of the form \( \psi^{p,n}(W^{h^n}_{(\ell + 1)/T/p} - W^{h^n}_{\ell T/p}) \). We know that, under \( P^{h^n}(\cdot | F^p_{\ell T/p}) \), the solution has the form \( \varphi(t - \ell T/p, W^n_t - W^{h^n}_{\ell T/p}) \) for \( \varphi \) being an \( F^p_{\ell T/p} \)-adapted random field solving the (backward) heat equation:

\[
\partial_t \varphi(r, x) + \frac{1}{2} \Delta_x \varphi(r, x) = 0, \quad r \in [0, T/p], \quad x \in \mathbb{R}^d.
\]

Moreover,

\[
k_t = \nabla_x \varphi(t - \ell T/p, W^n_t - W^{h^n}_{\ell T/p})
\]

We then recall the well-known bound \( |\nabla_x \varphi(r, x)| \leq C_2 \sqrt{T/p - r} \), for \( c \) a universal constant and for \( C \) as in the beginning of the proof. The first inequality easily follows. 

As for the second one, it suffices to observe that, for \( s \in [\ell T/p, (\ell + 1)p/T) \),

\[
\hat{h}^{n+1}_s = \hat{h}^{n+1}_{\ell T/p} + (s - \ell T/p) \hat{h}^{n+1}_{\ell T/p} + \int_s^{(\ell + 1)p/T} \hat{h}^{n+1}_{\ell T/p} dW^h_r
\]

Taking conditional expectation with respect to \( F^W_{\ell T/p} \) under \( P^{h^n} \) on the first and second lines and then taking conditional expectation with respect to \( F^W_{\ell T/p} \) under \( P^h \) on the first and third lines, we deduce that

\[
\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} |(W^p_t - W^h_t)|^q \right]^{1/q} \leq C_q \left( 1 + \ln(p) \right)^{1/2}.
\]

Lemma 2.18. For any exponent \( q \geq 1 \), there exists a constant \( C \), only depending on \( q \), \( d \), \( T \), \( \|f\|_{1,\infty} \), \( \|g\|_{1,\infty} \), \( |Q| \) and \( |R| \), such that

\[
\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} |(W^p_t - W^h_t)|^q \right]^{1/q} \leq C_q \left( 1 + \ln(p) \right)^{1/2}.
\]
Proof of Lemma 2.18 We write
\[
\varepsilon (W^p_t - W_t) = \varepsilon \left( W_{\tau_p(t)} + \frac{p(t - \tau_p(t))}{T} \left[ W_{\tau_p(t) + T/p} - W_{\tau_p(t)} \right] \right) - \varepsilon W_t \\
= \varepsilon \left( W^h_{\tau_p(t)} + \frac{p(t - \tau_p(t))}{T} \left[ W^h_{\tau_p(t) + T/p} - W^h_{\tau_p(t)} \right] \right) \\
- \left( \int_{\tau_p(t)}^{T/p} \frac{p(t - \tau_p(t))}{T} \int_{\tau_p(t)}^{T/p} h_s ds \right) \\
- \left( \int_{\tau_p(t)}^{T/p} \frac{p(t - \tau_p(t))}{T} \int_{\tau_p(t)}^{T/p} h_s ds \right) \\
= \varepsilon \left( W^h_{\tau_p(t)} + \frac{p(t - \tau_p(t))}{T} \left[ W^h_{\tau_p(t) + T/p} - W^h_{\tau_p(t)} \right] \right) \\
+ \int_{\tau_p(t)}^{T/p} \frac{p(t - \tau_p(t))}{T} \int_{\tau_p(t)}^{T/p} h_s ds.
\]

By Lemma 2.16 and from the fact that \( \tau_p(t) \leq t \leq \tau_p(t) + T/p \), it suffices to focus on
\[
\sup_{t=0, \ldots, p-1, \ell \leq (t+1)/p} \left| \int_{\tau_p(t)}^{T/p} \frac{p(t - \tau_p(t))}{T} \right| \int_{\tau_p(t)}^{T/p} h_s ds.
\]

The bound for the above term then follows from 37. □

We now have the analogue of Proposition 2.8

Proposition 2.19. With the same notation as in the statement of Theorem 2.14, there exists a constant \( C \), only depending on \( d, T, \|f\|_{1,\infty}, \|g\|_{1,\infty}, |Q| \) and \( |R| \), such that
\[
\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \left( |m^n_t - m_t|^2 + |h^n_t - h_t|^2 \right) \right] \leq C \left( \frac{1}{n^2} + \frac{1 + \ln(p)}{p} \right).
\]

Proof of Proposition 2.19 Throughout the proof, the constant \( C \) may vary from line to line as long as it only depends on the same parameters as those quoted in the statement.

First Step. We start with the analysis of \( m^n_t - m_t \). By making the difference between (2.34) and the forward equation in (2.6), we obtain
\[
\sup_{0 \leq t \leq T} |m^n_t - m_t| \leq \frac{C}{n} + C \sup_{0 \leq t \leq T} |\eta^n_{\tau_p(t)} - \eta_{\tau_p(t)}| + C \sup_{0 \leq t \leq T} |\eta_{\tau_p(t)} m_{\tau_p(t)} - \eta m_t| \\
+ C \varepsilon \sup_{0 \leq t \leq T} |W^p_t - W_t|.
\]

By Remark 2.12 the second term is less than \( C/p \). As for the third term, notice that \( \mathbb{E}^h [\sup_{|\tau_p - \tau_p(t)| \leq T/p} |\eta_{\tau_p(t)} m_{\tau_p(t)} - \eta m_{\tau_p(t)}|^q]^{1/q} \leq C q/p^{1/2} \). The last term is handled by means of Lemma 2.18. We easily get the bound on the \( L^2 \) moment of \( \sup_{0 \leq t \leq T} |m^n_t - m_t| \).

Second Step. We focus on the \( L^2 \) moment of \( \sup_{0 \leq t \leq T} |h^n_t - h_t| \). Making the difference between (2.6) and (2.28) (using Remark 2.13), we obtain
\[
d\tilde{h}^{n+1}_t - h_t \\
= -\left( Q^f (m^n_{\tau_p(t)}) - Q^f (m_t) \right) dt + \eta^{(p)}_{\tau_p(t) + T/p} \left( \tilde{h}^{n+1}_{\tau_p(t)} - h_t \right) dt \\
- \mathbb{E}^h \left[ Q^f (m^n_{\tau_p(t) + T/p}) - Q^f (m^n_{\tau_p(t)}) \right] \mathbb{1}_{t \leq (p-1)/p} dt \\
+ Q^f (m^n_{\tau_p(t) + T/p}) \mathbb{1}_{t > (p-1)/p} dt + \frac{T}{p} Q^f (\tilde{h}^{n+1}_{\tau_p(t + T/p)} \mathbb{1}_{t \leq (p-1)/p} dt \\
+ (\eta^{(p)}_{\tau_p(t) + T/p} - \eta_t) dt + (k_t^{n+1} - k_t)(h_t^n - h_t) dt + k_t (h_t^n - h_t) dt + (k_t^{n+1} - k_t) dW^h_t.
\]
Taking the square, using Remark 2.12 and then expanding as in (2.13), we get (with $\delta$ as in the statement of Lemma 2.16)

\[
\begin{align*}
&d|\tilde{h}^{n+1}_t - h_t|^2 
\leq -C \left( \frac{1}{p} + \sup_{0 \leq r \leq T} |m^n_{r,p(r)} - m_r|^2 + 1_{\{t > (p-1)T/p\}} \right) dt \\
&\quad - \left( C + \frac{\delta}{4} |k_t|^2 \right) |\tilde{h}^{n+1}_t - h_t|^2 dt - C|\tilde{h}^{n+1}_{r,p(t)} - h_t|^2 dt \\
&\quad - \frac{C}{\delta} |k^n_t - h_t|^2 dt + \frac{1}{2} |k^{n+1}_t - k_t|^2 dt + \langle \tilde{h}^{n+1}_t - h_t \rangle \cdot (k^{n+1}_t - k_t) dW_t^H.
\end{align*}
\]

Elaborating on the first step, notice that, on the first line, we can find a constant $C$. Taking the square, using Remark 2.12, and then expanding as in (2.13), we get (with $\ln(2)$ and we observe from Lemma 2.16 that $|\tilde{h}^{n+1}_t - h_T| = R \cdot (g(m^n_T) - g(m_T))$. Then, for $\lambda \geq C$, we can find a constant $C_\lambda$, possibly depending on $\lambda$, such that

\[
\begin{align*}
&\mathbb{E}^h \left[ \exp(\lambda t + K_t) |\tilde{h}^{n+1}_t - h_t|^2 \right] + \frac{1}{2} \mathbb{E}^h \left[ \int_T^t \exp(\lambda s + K_s) |k^{n+1}_s - k_s|^2 ds \right] \\
&\quad + \mathbb{E}^h \left[ \int_T^t \left( (\lambda - C) + \frac{\delta}{4} |k_s|^2 \right) \exp(\lambda s + K_s) |\tilde{h}^{n+1}_s - h_s|^2 ds \right] \\
&\leq C_\lambda \left( \frac{1}{n^2} + \frac{1 + \ln(p)}{p} \right) + C \mathbb{E}^h \left[ \int_T^t \exp(\lambda s + K_s) |h^n_s - h_s|^2 ds \right] \\
&\quad + 2C \mathbb{E}^h \left[ \int_T^t \exp(\lambda s + K_s) |\tilde{h}^{n+1}_{r,p(s)} - h_{r,p(s)}|^2 ds \right] + 2C \mathbb{E}^h \left[ \int_T^t \exp(\lambda s + K_s) |h_s - h_{r,p(s)}|^2 ds \right].
\end{align*}
\]

**Third Step.** We first address the term on the second line of the right-hand side of (2.35). By the second part in the statement of Lemma 2.16, it satisfies

\[
\begin{align*}
&\mathbb{E}^h \left[ \int_T^t \exp(\lambda s + K_s) |\tilde{h}^{n+1}_{r,p(s)} - h_{r,p(s)}|^2 ds \right] \\
&\quad = \mathbb{E}^h \left[ \int_T^t \mathbb{E}^h \left[ \exp(\lambda s + K_s) \left| F^{W}_{r,p(s)} \right| \right] |\tilde{h}^{n+1}_{r,p(s)} - h_{r,p(s)}|^2 ds \right] \\
&\quad \leq C_1 \exp \left( \frac{T \lambda}{p} \right) \mathbb{E}^h \left[ \int_T^t \exp(\lambda r + K_r(s)) |\tilde{h}^{n+1}_{r,p(s)} - h_{r,p(s)}|^2 ds \right].
\end{align*}
\]

Thanks to the forms of the two BSDEs (2.7) and (2.28), for $s \in [0, T)$,

\[
|\tilde{h}^{n+1}_{r,p(s)} - h_{r,p(s)}| \leq \frac{C'}{p} + \frac{p}{T} \int_{r,p(s)}^{r,p(s)+T/p} \mathbb{E}^h \left[ |\tilde{h}^{n+1}_r - F^{W}_{r,p(s)}| \right] dr - \frac{p}{T} \int_{r,p(s)}^{r,p(s)+T/p} \mathbb{E}^h \left[ |h_r - F^{W}_{r,p(s)}| \right] dr,
\]

for a constant $C'$ depending on the same parameters as $C$. By Lemma 2.17,

\[
|\tilde{h}^{n+1}_{r,p(s)} - h_{r,p(s)}| \leq \frac{C'}{pT^{3/2}} + \frac{p}{T} \int_{r,p(s)}^{r,p(s)+T/p} \mathbb{E}^h \left[ |\tilde{h}^{n+1}_r - F^{W}_{r,p(s)}| \right] dr - \frac{p}{T} \int_{r,p(s)}^{r,p(s)+T/p} \mathbb{E}^h \left[ |h_r - F^{W}_{r,p(s)}| \right] dr.
\]
Hence, squaring the above inequality and using Jensen’s inequality to ‘pass’ the square inside the integral and the expectation, we get
\[
\mathbb{E}^h \left[ \int_t^T \exp(\lambda s + K_s) |\tilde{r}^{n+1}_{\tau_p(s)} - h_{\tau_p(s)}|^2 \, ds \right]
\leq C_\lambda + 2C_1 \exp\left(\frac{\lambda T}{p} \right) \mathbb{E}^h \left[ \int_t^T \int_{\tau_p(s)}^{\tau_p(s)+T/p} \exp(\lambda r + K_r) |\tilde{r}^{n+1}_r - h_r|^2 \, ds \right]
\leq \frac{C_\lambda}{p^2} + 2C_1 \exp\left(\frac{\lambda T}{p} \right) \mathbb{E}^h \left[ \int_{\tau_p(t)}^T \exp(\lambda s + K_s) |\tilde{r}^{n+1}_{s} - h_s|^2 \, ds \right].
\] (2.36)

In the end, this term can be compared with the left-hand side in (2.35). We clarify this point in the fourth step.

**Fourth Step.** The other challenging term in (2.35) that we need to handle is
\[
\mathbb{E}^h \left[ \int_t^T \exp(\lambda s + K_s) |h_s - h_{\tau_p(s)}|^2 \, ds \right].
\] (2.37)

By the equation (2.7) satisfied by \( h \) under \( \mathbb{E}^h \), Then,
\[
|h_s - h_{\tau_p(s)}| \leq \int_{\tau_p(s)}^s k_r dW^h_r + \frac{C}{p}.
\]

In turn, by combining Cauchy-Schwarz inequality and Lemma 2.16 we obtain
\[
\mathbb{E}^h \left[ \int_t^T \exp(\lambda s + K_s) |h_s - h_{\tau_p(s)}|^2 \, ds \right]
\leq \mathbb{E}^h \left[ \exp(2\lambda T + 2K_T) \right]^{1/2} \mathbb{E}^h \left[ \left( \int_t^T |h_s - h_{\tau_p(s)}|^2 \, ds \right)^2 \right]^{1/2}
\leq \frac{C_\lambda}{p^2} \mathbb{E}^h \left[ \left( \int_{\tau_p(t)}^T \sup_{\tau_p(s) \leq r \leq (i+1)T/p} \left| \int_{\tau_p(s)}^r k_u dW^h_u \right|^2 \, ds \right)^2 \right] + \frac{C_\lambda}{p^2}.
\] (2.38)

We now write
\[
\mathbb{E}^h \left[ \left( \int_{\tau_p(t)}^T \sup_{\tau_p(s) \leq r \leq \tau_p(s)+T/p} \left| \int_{\tau_p(s)}^r k_u dW^h_u \right|^2 \, ds \right)^2 \right]
= \frac{T^2}{p^2} \sum_{i,j=0}^{n-1} \mathbb{E}^h \left[ \sup_{T/p \leq r \leq (i+1)T/p} \left| \int_{iT/p}^{(j+1)T/p} k_u dW^h_u \right|^2 \right] \sup_{T/p \leq r \leq (j+1)T/p} \left| \int_{iT/p}^{(j+1)T/p} k_u dW^h_u \right|^2
= \frac{T^2}{p^2} \sum_{i=0}^{T-1} \mathbb{E}^h \left[ \sup_{T/p \leq r \leq (i+1)T/p} \left| \int_{iT/p}^{(i+1)T/p} k_u dW^h_u \right|^4 \right].
\]

By Burkholder-Davis-Gundy inequality applied in a conditional form, we deduce that
\[
\mathbb{E}^h \left[ \left( \int_{\tau_p(t)}^T \sup_{\tau_p(s) \leq r \leq \tau_p(s)+T/p} \left| \int_{\tau_p(s)}^r k_u dW^h_u \right|^2 \, ds \right)^2 \right]
\leq \frac{C}{p^2} \sum_{i=0}^{T-1} \mathbb{E}^h \left[ \sup_{T/p \leq r \leq (i+1)T/p} \left| \int_{iT/p}^{(i+1)T/p} k_u dW^h_u \right|^2 \right] \mathbb{E}^h \left[ \int_{iT/p}^{(i+1)T/p} |k_u|^2 \, du \left| \mathcal{F}^{W}_{(i+1)T/p} \right| \right]
+ \frac{C}{p^2} \sum_{i=0}^{T-1} \mathbb{E}^h \left[ \left( \int_{iT/p}^{(i+1)T/p} |k_u|^2 \, du \right)^2 \right].
\]
By Lemma 2.16 we get
\[ \mathbb{E}^h \left[ \left( \int_0^T \sup_{\tau_p(t) \leq \tau_p(t) + T/p} \left| \int_{\tau_p(t)}^T k_u dW_u^h \right|^2 dt \right)^2 \right] \]
\[ \leq \frac{C}{p^2} \sum_{0 \leq m \leq T} \mathbb{E}^h \left[ \sup_{\tau_p(t) \leq \tau_p(t) + T/p} \left| \int_{\tau_p(t)}^T k_u dW_u^h \right|^2 \right] \]
\[ + \frac{C}{p^2} \mathbb{E}^h \left( \int_{\tau_p(t)}^T \left| k_u^2 du \right|^2 \right) \]
\[ \leq \frac{C}{p^2} \sum_{0 \leq m \leq T} \mathbb{E}^h \left[ \sup_{\tau_p(t) \leq \tau_p(t) + T/p} \left| \int_{\tau_p(t)}^T k_u dW_u^h \right|^2 \right] + \frac{C}{p^2} \]
\[ \leq \frac{C}{p^2} \sum_{0 \leq m \leq T} \mathbb{E}^h \left[ \int_{\tau_p(t)}^{(i+1)T/p} \left| k_u^2 du \right|^2 \right] + \frac{C}{p^2} \]
Back to (2.37) and (2.38) and by Lemma 2.16 again, we deduce that
\[ \mathbb{E}^h \left[ \int_0^T \exp(\lambda s + K_s) |h_s - h_{\tau_p(s)}|^2 ds \right] \leq \frac{C_\lambda}{p}. \]  

**Fifth Step.** In (2.35), we choose \( \lambda = C + (8CC_1/\delta) \exp(1) + 4CC_1 \exp(1) \), with \( C \) as therein and \( C_1 \) as in Lemma 2.16. Without any loss of generality, we can assume that \( p \geq \Lambda T \). Indeed, we want a bound on the \( L^2 \) moment of \( \sup_{0 \leq t \leq T} \left| h^n_t - h_t \right| \) in the statement of the proposition; using the fact that \( h^n \) and \( h \) are bounded (independently of \( n \) for the former), we observe that the bound claimed in the statement is always true when \( p \) is small.

By combining (2.35), (2.36) and (2.39), we obtain
\[ \mathbb{E}^h \left[ \int_0^T \left( \frac{8CC_1}{\delta} \exp(1) + \frac{\delta}{4} |k_t|^2 \right) \exp(\lambda t + K_t) |h_t^{n+1} - h_t|^2 dt \right] \]
\[ \leq C' \left( \frac{1}{n^2} + \frac{1 + \ln(p)}{p} \right) + \frac{C}{\delta} \mathbb{E}^h \left[ \int_0^T \exp(\lambda t + K_t) |h_t^n - h_t|^2 dt \right], \]  
for a new value of the constant \( C' \) introduced in the third step. By (2.31), we know that \( h_t^n = h_{\tau_p(t)}^n \) and then, by (2.36) and (2.39),
\[ \mathbb{E}^h \left[ \int_0^T \exp(\lambda t + K_t) |h_t^n - h_t|^2 dt \right] \leq \frac{C'}{p} + 4C_1 \exp(1) \mathbb{E}^h \left[ \int_0^T \exp(\lambda s + K_s) \left| h_s^n - h_s \right|^2 ds \right], \]
for a possibly new value of \( C' \). Inserting the above inequality in (2.40) and proceeding as in the derivation of (2.17), we deduce that
\[ \mathbb{E}^h \left[ \int_0^T \exp(\lambda s + K_s) \left| h_s^n - h_s \right|^2 ds \right] \leq C' \left( \frac{1}{n^2} + \frac{1 + \ln(p)}{p} \right), \]
for a new value of \( C' \). Then, we get a similar bound for the left-hand side in (2.35).

The derivation of the inequality with a sup inside (as in the statement) is then similar to the end of the proof of Proposition 2.8.

**2.2.4. Proof of auxiliary results.**

**Proof of Lemma 2.17**

**First Step.** We first notice that, for a given choice of \( (\mathbf{m}_n, h^n) \), the problem (2.23) has at least one minimizer. Indeed, the cost functional \( \alpha \mapsto \mathbb{E}^{h^n} \left[ R^p(\alpha + \varepsilon W^{h^n}; \mathbf{m}_n; 0) \right] \) is lower semicontinuous with respect to \( \alpha \) when the latter is identified with a collection of random variables \( (\alpha_0, \ldots, \alpha_{(p-1)T/p}) \in \times_{n=0}^{p-1} L^2(\Omega, \mathcal{F}_{T/p}^n, \mathcal{B}, \mathbb{P}^{h^n}; \mathbb{R}^d) \), with the product space being equipped with the weak topology. The identification is made possible by the fact that the controls \( \alpha \) are chosen to be piecewise constant. Moreover, the cost functional blows up
when the $L^2$-norm of $\alpha$ tends to $\infty$. The minimization problem can hence be restricted to a bounded subset of $\mathcal{X}^{\delta}_{t=0:T} L^2(\Omega, \mathcal{F}_{T/p}^{p, n}, B_\mathcal{W}, \mathbb{P}^n; \mathbb{R}^d)$, the latter being obviously compact for the weak topology. Existence of a minimizer easily follows.

We check below that any critical point of the cost functional is a solution of the forward-backward system \[2.27\]–\[2.28\]. Since the latter is uniquely solvable, this indeed provides a characterization of the minimizer.

**Second Step.** In order to prove that critical points solve \[2.27\]–\[2.28\], we follow the usual lines of the Pontryagin principle. Although the result is certainly not new, we feel better to provide the complete proof, since the formulation we use is tailor-made to our needs. We start to notice that, under the probability $\mathbb{P}^n$, the path driven by $\alpha$ reads

$$dX_t = \alpha_t dt + \sigma dB_t + \varepsilon dW_t^{p, n}, \quad t \in [0, T].$$

We then introduce an additive perturbation of the cost and hence replace $\mathbf{c} = (\alpha_t)_{0 \leq t \leq T}$ by $\mathbf{c} + \mathbf{\delta c} = (\alpha_t + \delta \beta_t)_{0 \leq t \leq T}$ in the above expansion, with $\mathbf{c}$ and $\mathbf{\beta}$ being identified as before as elements of $\mathcal{X}^{\delta}_{t=0:T} L^2(\Omega, \mathcal{F}_{T/p}^{p, n}, B_\mathcal{W}, \mathbb{P}^n; \mathbb{R}^d)$. We then write $\mathbf{X}^\delta = (X_t^\delta)_{0 \leq t \leq T}$ in order to emphasize the dependence of $\mathbf{X}$ upon $\delta$. When $\delta = 0$, we merely write $\mathbf{X}$ instead of $\mathbf{X}^0$.

The formal derivative with respect to $\delta$ at $\delta = 0$ reads

$$d(\partial X_t) = \beta_t dt, \quad t \in [0, T]; \quad \partial X_0 = 0.$$

In turn, the derivative of the cost $\tilde{J}^p(\alpha + \delta \beta; \mathbf{m})$, with respect to $\delta$, is

$$\frac{d}{d\delta} \tilde{J}^p(\alpha + \delta \beta; \mathbf{m})|_{\delta=0} = \mathbb{E}^n \left[ (RX_T + g(\mathbf{m}^\alpha_T)) \cdot R \partial X_T 
\quad + \int_0^T \left\{ (QX_{\tau(t)} + f(\mathbf{m}^\alpha_{\tau(t)})) \cdot Q \partial X_{\tau(t)}(p) + \alpha_{\tau(t)}(p) \cdot \beta_{\tau(t)}(p) \right\} dt \right].$$

We now solve the BSDE (under $\mathbb{P}^n$)

$$dY_t = -(Q^1 QX_{\tau(t)} + Q^1 f(\mathbf{m}^\alpha_{\tau(t)})) dt + Z^B_t dB_t + Z^W_t dW_t^{h, n}, \quad Y_T = R^1 RX_T + R^1 g(\mathbf{m}^\alpha_T).$$

By discrete integration by parts, we have

$$\mathbb{E}^n [Y_T \cdot \partial X_T] = \sum_{\ell=0}^{N-1} \mathbb{E}^n \left[ (Y_{\ell+1} T/p - Y_{\ell T/p}) \cdot \partial X_{\ell T/p} + Y_{(\ell+1) T/p} \cdot (\partial X_{(\ell+1) T/p} - \partial X_{\ell T/p}) \right] = \frac{T}{p} \sum_{\ell=0}^{N-1} \mathbb{E}^n \left[ -(Q^1 QX_{\ell T/p} + Q^1 f(\mathbf{m}^\alpha_{\ell T/p})) \cdot \partial X_{\ell T/p} + Y_{(\ell+1) T/p} \cdot \beta_{\ell T/p} \right].$$

Rewriting the above sum as an integral, we obtain

$$\frac{d}{d\delta} \tilde{J}^p(\alpha + \delta \beta)|_{\delta=0} = \mathbb{E}^n \left[ \int_0^T (Y_{\tau(t)} + \alpha_{\tau(t)}) \cdot \beta_{\tau(t)}(p) dt \right],$$

which means that the optimizer is given by

$$\alpha_{\ell T/p}^{n+1} = -\mathbb{E}^n \left[ Y_{\ell+1} T/p + B_\mathcal{W} \right] = -Y_{\ell T/p} + \frac{T}{p} \left( Q^1 QX^{n+1}_{\ell T/p} + Q^1 f(\mathbf{m}^\alpha_{\ell T/p}) \right), \quad (2.41)$$

for $\ell \in \{0, \ldots, p-1\}$, where

$$dX_t^{n+1} = \alpha_t^{n+1} dt + \sigma dB_t + \varepsilon dW_t^{p, n},$$

$$dY_t^{n+1} = -(Q^1 QX_{\tau(t)}^{n+1} + Q^1 f(\mathbf{m}^\alpha_{\tau(t)})) dt + Z^{n+1, B}_t dB_t + Z^{n+1, W}_t dW_t^{h, n}, \quad t \in [0, T],$$

$$Y_T^{n+1} = R^1 RX_T^{n+1} + R^1 g(\mathbf{m}^\alpha_T).$$
Importantly, we stress that $X_{\tau_n}^{n+1}$ is $(\mathcal{F}_p^{p,X_0,B,W})_{0 \leq t \leq T}$ adapted. We then let, for the same solution $\eta^{(p)}$ as in \eqref{2.24}–\eqref{2.26},

$$
\tilde{h}^{n+1}_{\ell T/p} = E^{h^n} \left[ Y^{n+1}_{(\ell+1)T/p} \mid \mathcal{F}_{\ell T/p} \right] = \frac{T}{p} \left( Q^1 f \left( m^{n}_{\ell T/p} \right) \right) - \mathcal{F}_{\ell T/p} Y^{n+1}_{\ell T/p} \quad \text{for } \ell \in \{0, \cdots, p-1\},
$$

for $\tilde{h}^{n+1}_{(\ell+1)T/p} - \tilde{h}^{n+1}_{\ell T/p} = - \frac{T}{p} \left( Q^1 f \left( m^{n}_{(\ell+1)T/p} \right) \right) - \mathcal{F}_{\ell T/p} Y^{n+1}_{(\ell+1)T/p} + \int_{\ell T/p}^{(\ell+1)T/p} k_{s}^{n+1,B} dB_s + \int_{\ell T/p}^{(\ell+1)T/p} k_{s}^{n+1,W} dW_s,
$$

for some square integrable and $\mathcal{F}_p^{p,X_0,B,W}$-progressively measurable processes $k^{n+1,B}_s$ and $k^{n+1,W}_s$. Above, we used the fact that $W^{h^n}_{\tau_p(t)+T/p} - W^{h^n}_{\tau_p(t)} = W^{h^n}_{\tau_p(t)+T/p} - W^{h^n}_{\tau_p(t)}$. Now, using \eqref{2.41} and conditioning on $\mathcal{F}_{\tau_p(t)}^{p,X_0,B,W}$ in \eqref{2.42}, we obtain

$$
e^{n+1}_{\ell T/p} = - \mathcal{F}_{\ell T/p} X^{n+1}_{\ell T/p} - E^{h^n} \left[ \tilde{h}^{n+1}_{\ell T/p} \mid \mathcal{F}_{\ell T/p} \right], \quad \ell \in \{0, \cdots, p-1\},
$$

which permits to write

$$
X^{n+1}_{(\ell+1)T/p} = X^{n+1}_{\ell T/p} + \frac{T}{p} \mathcal{F}_{\ell T/p} X^{n+1}_{(\ell+1)T/p} + \mathcal{F}_{\ell T/p} \left[ W^{h^n}_{(\ell+1)T/p} \right] - W^{h^n}_{\ell T/p} \quad \text{for } \ell \in \{0, \cdots, p-2\},
$$

where

$$
\rho^{n+1}_{\ell T/p} = \frac{T}{p} Q^1 + \mathcal{F}_{\ell T/p} X^{n+1}_{\ell T/p} - \mathcal{F}_{\ell T/p} \left[ W^{h^n}_{\ell T/p} \right] + \int_{\ell T/p}^{(\ell+1)T/p} k_{s}^{n+1,B} dB_s + \int_{\ell T/p}^{(\ell+1)T/p} k_{s}^{n+1,W} dW_s,
$$

By recalling that $m^{n}_{(\ell+1)T/p}$ is $\mathcal{F}_{(\ell+1)T/p}^{p,X_0,B,W}$-measurable and by modifying the process $k^{n+1,W}_s$ accordingly, we can rewrite \eqref{2.43} in the form

$$
\tilde{h}^{n+1}_{(\ell+1)T/p} - \tilde{h}^{n+1}_{\ell T/p} = - \frac{T}{p} \mathcal{F}_{\ell T/p} X^{n+1}_{(\ell+1)T/p} + \mathcal{F}_{\ell T/p} \left[ W^{h^n}_{(\ell+1)T/p} \right] - W^{h^n}_{\ell T/p} + \int_{\ell T/p}^{(\ell+1)T/p} k_{s}^{n+1,B} dB_s + \int_{\ell T/p}^{(\ell+1)T/p} k_{s}^{n+1,W} dW_s,
$$

for $\ell \in \{0, \cdots, p-2\}$.
for $\ell \in \{0, \ldots, p-2\}$. When $\ell = p-1$, we deduce from (2.42) that
\[
\tilde{h}^{n+1}_{(p-1)/p} = R^1 R \mathbb{E}^{h^{n}}[X^{n+1}_T|F_{(p-1)/p}] + R^1 g(\mathbb{m}_{T}^{n}) - \eta^{(p)}_{(p-1)/p}X^{n+1}_{(p-1)/p} + R^1 g(\mathbb{m}_{T}^{n}) - \eta^{(p)}_{(p-1)/p}X^{n+1}_{(p-1)/p}
\]
which yields, as a boundary condition,
\[
\tilde{h}^{n+1}_{(p-1)/p} = R^1 g(\mathbb{m}_{T}^{n}) - \eta^{(p)}_{(p-1)/p}X^{n+1}_{(p-1)/p} + R^1 g(\mathbb{m}_{T}^{n}) - \eta^{(p)}_{(p-1)/p}X^{n+1}_{(p-1)/p}.
\]
(2.45)

We now recall that, by construction, $\tilde{h}^{n+1}_{(p-1)/p}$ is $\mathcal{F}_{(p-1)/p}^{X_0, \varepsilon W}$-measurable. Arguing as in Remark 2.13 we can prove inductively (over $\ell$) that $\tilde{h}^{n+1}_{(p-1)/p}$ is in fact $\mathcal{F}_{(p-1)/p}^{W}$-measurable. As a consequence, $k^{n+1,\varepsilon}$ has to be zero. By combining (2.44) and (2.45), we hence recover BSDE (2.28).

□

2.3. Exploitation versus exploration. We now address the regret, when we play the equilibrium strategy given by the fictitious play.

2.3.1. Approximate Nash equilibrium formed by the solution of the MFG with common noise. We first prove (which is easier) that the solution of the MFG with common noise forms an approximate Nash equilibrium of the game without common noise. In order to do so, we call $\alpha^*$ the equilibrium strategy of the MFG with an $\varepsilon$ common noise, as given by Remark 2.4.

Then
\[
\mathbb{E}^h \left[ \mathcal{R}(\alpha^*; m; \varepsilon W^h) \right],
\]
with $m$ and $h$ as in the statement of Remark 2.4 is the value of the game with the $\varepsilon$-common noise, the intensity $\varepsilon$ being kept fixed in $(0, 1)$ throughout the subsection. In particular, it must be clear that both $m$ and $h$ depend on $\varepsilon$. We claim:

Proposition 2.20. There exists a constant $C$, only depending on the parameters $d$, $T$, $\|f\|_{1,\infty}$, $\|g\|_{1,\infty}$, $|Q|$ and $|R|$, such that
\[
\inf_{\alpha} \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\alpha^*; m; 0) \right] - C\varepsilon,
\]
the infimum in the left-hand side being taken over $\mathbb{F}^{X_0, \varepsilon W}$-progressively measurable and square-integrable processes $(\alpha_t)_{0 \leq t \leq T}$.

We start with the following lemma:

Lemma 2.21. There exists a constant $C$, only depending on the parameters $d$, $T$, $\|f\|_{1,\infty}$, $\|g\|_{1,\infty}$, $|Q|$ and $|R|$, such that, for any $\mathbb{F}^{X_0, \varepsilon W}$-progressively measurable process $(\alpha_t)_{0 \leq t \leq T}$,
\[
\mathbb{E}^h \left[ \mathcal{R}(\alpha; m; \varepsilon W^h) \right] - \mathbb{E}^h \left[ \mathcal{R}(\alpha^*; m; 0) \right] \leq C \left( 1 + \mathbb{E}[|X_0|^2]^{1/2} + \mathbb{E} \left[ \int_0^T |\alpha_t|^2 dt \right]^{1/2} \right) \varepsilon.
\]

The symbol $\mathbb{E}$ instead of $\mathbb{E}^h$ for the $L^2$ moment of the initial condition is fully justified by the fact that $X_0$ is $\mathcal{F}_0$-measurable.

Proof. For the same initial condition $X_0$, we let
\[
dX_t^\varepsilon = \alpha_t dt + \sigma dB_t + \varepsilon dW_t^h, \quad t \in [0, T].
\]
In particular, we write $X^0 = (X^0_t)_{0 \leq t \leq T}$ when there is no common noise. Then, we obviously have
\[
\sup_{0 \leq t \leq T} |X_t - X^0_t| \leq C\varepsilon \sup_{0 \leq t \leq T} |W_t^h|.
\]
Meanwhile, we observe that
\[
\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \left( |X_t|^2 + |X_0|^2 \right) \right] \leq C \left( 1 + \mathbb{E} \left[ |X_0|^2 \right] + \mathbb{E}^h \left[ \int_0^T |\alpha_t|^2 dt \right] \right).
\]
The result easily follows. □

The next lemma says that we can easily restrict ourselves to controls with a finite energy.

**Lemma 2.22.** There exists a constant \( A \), only depending on the parameters \( d, Q, R, T, \| f \|_{1,\infty} \) and \( \| g \|_{1,\infty} \) such that, for any \( \varepsilon \in (0, 1) \) and any \( \mathbb{F}^W \)-progressively measurable process \( \alpha = (\alpha_t)_{0 \leq t \leq T} \),
\[
\mathbb{E}^h \left[ \int_0^T |\alpha_t|^2 dt \right] \geq A \Rightarrow \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(0; m; \varepsilon W) \right] + 1.
\]

**Proof.** It suffices to observe that
\[
\mathbb{E}^h \left[ \mathcal{R}(\alpha; m; 0) \right] \geq \frac{1}{2} \mathbb{E}^h \left[ \int_0^T |\alpha_t|^2 dt \right].
\]
Meanwhile,
\[
\mathbb{E}^h \left[ \mathcal{R}(0; m; \varepsilon W) \right] \leq C.
\]
The proof is easily completed. □

We now complete the

**Proof of Proposition 2.20.** By Lemma 2.22, we can reduce ourselves to the set \( \mathcal{E}_A \) of processes \( \alpha \) such that
\[
\mathbb{E}^h \left[ \int_0^T |\alpha_t|^2 dt \right] \leq A.
\]
We then apply Lemma 2.21 which says that
\[
\sup_{\alpha \in \mathcal{E}_A} \left| \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; \varepsilon W^h) \right] - \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; 0) \right] \right| \leq C\varepsilon.
\]
The proof is easily completed, using the fact that, for any \( \alpha \in \mathcal{E}_A \),
\[
\mathbb{E}^h \left[ \mathcal{R}(\alpha^*; m; \varepsilon W^h) \right] \leq \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; \varepsilon W^h) \right] \leq \mathbb{E}^h \left[ \mathcal{R}(\alpha; m; 0) \right] + C\varepsilon.
\]

\( \Box \)

### 2.3.2. Approximate Nash equilibrium formed by the return of the fictitious play.

We now state a similar result, but for the approximation returned by the fictitious play subjected to the \( \varepsilon \)-randomization.

The point is hence to consider, under the environment \( m^{n+1} \), the strategy \( \alpha^{n+1} \) (returned by the fictitious play, see (2.23)) as an approximate Nash equilibrium under the probability \( \mathbb{P}^n \). Notice that this is fully legitimated by the fact that
\[
dX_t^{n+1} = \alpha_t^{n+1} dt + \sigma dB_t + \varepsilon dW_t^p h^n, \quad t \in [0, T],
\]
with
\[
m_t^{n+1} = \mathbb{E}^n \left[ X_t^{n+1} | \sigma(W) \right], \quad t \in [0, T].
\]
In other words, the conditional state under the candidate strategy is the environment itself, which is a pre-requisite in mean field games. Notice that, here as well, \( \alpha^{n+1}, h^n \) and \( m^{n+1} \) do depend on \( \varepsilon \).
Following the proof of Lemma 2.18, And then, by Remark 2.12 and Lemma 2.18, Meanwhile, we write With the same notation as in the statement of Theorem 2.14, there exists a constant \(C\), only depending on the parameters \(d, T\), \(\|f\|_{1,\infty}, \|g\|_{1,\infty}\), \(|Q|\) and \(|R|\), such that

\[
\inf_{\alpha} \mathbb{E}^{\alpha} \left[ R(\alpha; m^{n+1}, 0) \right] \geq \mathbb{E}^{\alpha} \left[ R(\alpha^{n+1}, m^{n+1}, 0) \right] - C\varepsilon
\]

\[
- \frac{C}{\varepsilon} \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right) \left[ 1 + (\ln(np\varepsilon))_+ \right],
\]

the infimum in the left-hand side being taken over \(\mathbb{E}^{X_0,B,W}\)-progressively measurable and square-integrable processes \((\alpha_t)_{0 \leq t \leq T}\).

The reader must realize that, differently from what is done in (2.23), the environment used in the cost functional is the conditional state of the reference particle when using the strategy \(\alpha^{n+1}\). This looks subtle, but this makes a big difference in the analysis.

Notice that Proposition 2.19 just gives a bound on the difference between \(m^{n+1}\) and \(m\), while we here need a bound on the difference between \(m^{n+1}\) and \(\alpha\). This prompts us to start with the following refinement of Proposition 2.19.

**Proposition 2.24.** With the same notation as in the statement of Theorem 2.14, there exists a constant \(C\), only depending on \(d, T\), \(\|f\|_{1,\infty}\) and \(\|g\|_{1,\infty}\) such that

\[
\mathbb{E}^{\alpha} \left[ \sup_{0 \leq t \leq T} \left( |X^n_t - X_t|^2 + |m^n_t - m_t|^2 \right) \right] + \mathbb{E}^{\alpha} \left[ \int_0^T |\alpha^n_t - \alpha_t|^2 dt \right] \leq C \left( \frac{1}{n^2} + \frac{1 + \ln(p)}{p} \right).
\]

**Proof of Proposition 2.24.** We recall from Lemma 2.11 and (2.31) that

\[
dX^n_t = -\left( \eta_t^{R(t)} X^{n+1}_t + h_t^{n+1} \right) dt + \sigma dB_t + \varepsilon dW^{h^n}_t,
\]

which we rewrite

\[
dX^n_t = -\left( \eta_t^{R(t)} X^{n+1}_t + h_t^{n+1} \right) dt + (h^n_t - h_t) dt + \sigma dB_t + \varepsilon d\left( W^{h^n}_t - W^{h^n}_t \right) + \varepsilon dW_t,
\]

Meanwhile, we write

\[
dX_t = -\left( \eta_t X_t + h_t \right) dt + \sigma dB_t + \varepsilon dW_t,
\]

Following the proof of Lemma 2.18,

\[
\mathbb{E}^{\alpha} \left[ \sup_{|t-s| \leq 1/p} |X_t - X_s|^2 \right] \leq C \frac{1 + \ln(p)}{p}.
\]

We can rewrite the dynamics for \(X\) in the form

\[
dX_t = -\left( \eta_t^{R(t)} X^{R(t)}_t + h_t \right) dt + \left( \eta_t^{R(t)} X^{R(t)}_t - \eta_t X_t \right) dt + \sigma dB_t + \varepsilon dW_t,
\]

And then, by Remark 2.12 and Lemma 2.18,

\[
\mathbb{E}^{\alpha} \left[ \sup_{0 \leq r \leq t} |X^{n+1}_r - X_r|^2 \right] \leq C \int_0^t \mathbb{E}^{\alpha} \left[ \sup_{0 \leq r \leq s} |X^{n+1}_r - X_r|^2 \right] dr + C \frac{1 + \ln(p)}{p} + \mathbb{E}^{\alpha} \int_0^T |h^n_t - h_t|^2 dt + \mathbb{E}^{\alpha} \int_0^T |h^{n+1}_t - h_t|^2 dt.
\]

By Proposition 2.19, the two terms on the last line are less than \(C/n^2 + C(1 + \ln(p))/p\). The result for \(X^{n+1} - X\) then follows from Gronwall’s lemma. The result for \(m^{n+1} - m\) can be proved in the same way. As for the bound for \(\alpha^{n+1} - \alpha^*\), it follows from the feedback form of the control.

We now turn to
Proof of Theorem 2.23.

First Step. Recalling the shape of the optimizer $\alpha^{n+1}$ in Lemma 2.11 together with the fact that $\hat{h}^{n+1}$ is bounded, uniformly in $n \geq 1$, see Lemma 2.16, we deduce that
\[ |\mathbb{E}^h_n \left[ \mathcal{R}(\alpha^{n+1}; m^{n+1}; 0) \right] | \leq C. \]

Here, we must invoke the fact that the problem
\[ \inf_\alpha \mathbb{E}^h_n \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \]
is a linear quadratic-problem in a random environment. Similar to the minimization problem studied in Remark 2.4, its solution is given in feedback form. If we call $\hat{\alpha}^{n+1}$ the minimizer, it reads
\[ \hat{\alpha}^{n+1}_t = - (\eta_t \hat{X}^{n+1}_t + \hat{h}^{n+1}_t), \quad t \in [0, T], \]
with
\[ d\hat{X}^{n+1}_t = - (\eta_t \hat{X}^{n+1}_t + \hat{h}^{n+1}_t) dt + \sigma dB_t, \]
and where, $\mathbb{P}^h_n$ almost surely,
\[ \forall t \in [0, T], \quad |\hat{h}^{n+1}_t| \leq C. \]

As a result, with probability 1 under $\mathbb{P}^h_n$,
\[ \sup_{0 \leq t \leq T} |\hat{X}^{n+1}_t| \leq C \left( 1 + \sup_{0 \leq t \leq T} |B_t| \right), \]
\[ \sup_{0 \leq t \leq T} |\hat{\alpha}^{n+1}_t| \leq C \left( 1 + \sup_{0 \leq t \leq T} |B_t| \right). \tag{2.46} \]

In particular,
\[ \mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) \leq C \left( 1 + \sup_{0 \leq t \leq T} |B_t| \right)^2. \tag{2.47} \]

Second Step. As a result, for any $\rho > 0$,
\[ \inf_\alpha \mathbb{E}^h_n \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] = \mathbb{E}^h_n \left[ \mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) \right] \geq \mathbb{E}^h_n \left[ \mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) 1_{\{\mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) \leq \rho\}} \right]. \]

Using the upper bound on $d_{TV}(\mathbb{P}^h, \mathbb{P}^h_n)$ (see Remark 2.15), we deduce that
\[ \inf_\alpha \mathbb{E}^h_n \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) 1_{\{\mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) \leq \rho\}} \right] - C \frac{\rho}{\varepsilon} \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right). \]

And then, by (2.47), we obtain
\[ \inf_\alpha \mathbb{E}^h_n \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) \right] - C \exp(-\eta\rho) - C \frac{\rho}{\varepsilon} \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right), \tag{2.48} \]
for some $\eta > 0$. Now, we notice from (2.46) that
\[ \mathbb{E}^h \left[ \sup_{0 \leq t \leq T} |\hat{X}^{n+1}_t|^2 \right] \leq C. \]

Therefore, by Proposition 2.24, we get
\[ \mathbb{E}^h \left[ |\mathcal{R}(\hat{\alpha}^{n+1}; m^{n+1}; 0) - \mathcal{R}(\hat{\alpha}^{n+1}; m; 0)| \right] \leq C \left( \frac{1}{n} + \frac{(1 + \ln(p))^{1/2}}{p^{1/2}} \right). \]
Plugging the latter into (2.48), we obtain
\[
\inf_{\alpha} \mathbb{E}^h \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\hat{\alpha}^{n+1}; m; 0) \right] - C \exp(-\eta \varrho) - C \frac{\varrho^2}{\varepsilon} \left( \frac{1}{n} + \frac{1 + \ln(p)^{1/2}}{p^{1/2}} \right),
\]
and then, by Proposition 2.20
\[
\inf_{\alpha} \mathbb{E}^h \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\alpha^*; m; 0) \right] - C \varepsilon - C \exp(-\eta \varrho) - C \frac{\varrho^2}{\varepsilon} \left( \frac{1}{n} + \frac{1 + \ln(p)^{1/2}}{p^{1/2}} \right). \tag{2.49}
\]

**Third Step.** We now revert the computations achieved in the second step. To do so, we observe that the bounds (2.46) and (2.47) remain true if we replace \(\hat{X}^{n+1}\) by \(X^{n+1}\) and similarly \(\hat{\alpha}^{n+1}\) by \(\alpha^{n+1}\), with \(X^{n+1}\) and \(\alpha^{n+1}\) being defined as in Lemma 2.11. Then, by Proposition 2.24, (2.49) yields
\[
\inf_{\alpha} \mathbb{E}^h \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\alpha^*; m^{n+1}; 0) \right] - C \varepsilon - C \exp(-\eta \varrho) - C \frac{\varrho^2}{\varepsilon} \left( \frac{1}{n} + \frac{1 + \ln(p)^{1/2}}{p^{1/2}} \right),
\]
and then,
\[
\inf_{\alpha} \mathbb{E}^h \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\alpha^{n+1}; m^{n+1}; 0) \right] - C \varepsilon - C \exp(-\eta \varrho^2) - C \frac{\varrho^2}{\varepsilon} \left( \frac{1}{n} + \frac{1 + \ln(p)^{1/2}}{p^{1/2}} \right),
\]
and then,
\[
\inf_{\alpha} \mathbb{E}^h \left[ \mathcal{R}(\alpha; m^{n+1}; 0) \right] \geq \mathbb{E}^h \left[ \mathcal{R}(\alpha^{n+1}; m^{n+1}; 0) \right] - C \varepsilon - C \exp(-\eta \varrho^2) - C \frac{\varrho^2}{\varepsilon} \left( \frac{1}{n} + \frac{1 + \ln(p)^{1/2}}{p^{1/2}} \right). \tag{2.49}
\]

**Fourth Step.** Choosing
\[
\eta \varrho = \left( -\ln \left[ \frac{1}{\varepsilon} \left( \frac{1}{n} + \frac{1 + \ln(p)^{1/2}}{p^{1/2}} \right) \right] \right)_+,
\]
and noticing that
\[
\eta \varrho \leq \left( -\ln \left[ \frac{1}{\varepsilon} \left( \frac{1}{n} + \frac{1}{p^{1/2}} \right) \right] \right)_+ \leq \left( \ln(\varepsilon) + \ln(n) + \ln(p) \right)_+,
\]
we complete the proof. \(\square\)

3. Numerical experiments

This section is devoted to several numerical experiments based on our learning algorithm. We first provide in Subsection 3.1 a theoretical analysis of a benchmark example that we use throughout the section. In Subsection 3.2, we present several variants of the implemented version of the algorithm. We also explain how to compute a reference solution. Then, we study in Subsection 3.3 the numerical behavior of the algorithm for a fixed intensity \(\varepsilon > 0\) of the common noise. Finally, in Subsection 3.4, we discuss a numerical vanishing viscosity method based on the learning method.
3.1. A benchmark example. As a particular instance of the cost functional $J$ in (1.2), we focus on

$$J(\alpha; m) = \frac{1}{2} \mathbb{E} \left[ |X_T + g(m_T)|^2 + \int_0^T |\alpha_t|^2 dt \right]$$  \hspace{1cm} (3.1)

which implicitly means that $f = 0$. The examples that are addressed below are in dimension $d = 1$ or $d = 2$, with $T = 1$ in any cases. For simplicity, we also work with $X_0 = 0$.

In dimension $d = 1$, we choose

$$g(x) = \cos(\kappa x), \; x \in \mathbb{R},$$  \hspace{1cm} (3.2)

for a free positive parameter $\kappa$ that we tune from smaller to larger values. Obviously, the Lipschitz constant of $g$ becomes larger with $\kappa$. Accordingly, the coupling between the two forward and backward equations in (1.7) becomes stronger as $\kappa$ gets larger, which makes it more difficult to solve, especially when $\varepsilon = 0$. We illustrate this in Lemma 3.1 below: it says that, when $\varepsilon = 0$, (1.7) is uniquely solvable when $|\kappa| < 2$. Even more, the analysis performed in [24] shows that the more standard (and obviously simpler) Picard scheme (1.9)–(1.10) would converge when $\kappa$ is small, whether there is a common noise or not. Our result is thus especially relevant when $\kappa$ gets larger.

In dimension 2, we work with a similar terminal cost $g = (g_1, g_2)$:

$$g_1(x_1, x_2) = \cos(\kappa x_1) \cos(\kappa x_2), \quad g_2(x_1, x_2) = \sin(\kappa x_1) \sin(\kappa x_2), \quad x_1, x_2 \in \mathbb{R},$$  \hspace{1cm} (3.3)

where, as in dimension 1, $\kappa$ is a free positive parameter that we tune from smaller to larger values.

Regardless of the choice of $g$, the Riccati equation (2.2) associated with (3.1) writes

$$\dot{\eta}_t - \eta_t^2 = 0, \; t \in [0, 1]; \quad \eta_1 = 1,$$  \hspace{1cm} (3.4)

the solution of which is given by

$$\eta_t = \frac{1}{2 - t}, \; t \in [0, 1].$$  \hspace{1cm} (3.5)

3.1.1. Solutions without common noise. Even though this is not needed for all the examples we handle below, we feel useful to have a preliminary discussion about the shape of the solutions when there is no common noise, keeping in mind that the MFG without common noise is our original model. Recalling (1.7), we indeed have that, whenever there is no common noise (i.e., $\varepsilon = 0$), the equilibria are given as the solutions of the deterministic system

$$\dot{m}_t = - (\eta m_t + h_t), \quad \dot{h}_t = \eta h_t, \; t \in [0, 1]; \quad h_1 = g(m_1),$$  \hspace{1cm} (3.6)

with $m_0 = 0$, which prompts us to perform the changes of variable:

$$\tilde{m}_t = \frac{m_t}{2 - t}, \quad \tilde{h}_t = (2 - t) h_t, \; t \in [0, 1].$$  \hspace{1cm} (3.7)

We then have that $(m_t, h_t)_{0 \leq t \leq 1}$ solves (3.6) if and only if

$$\tilde{m}_t = - \frac{1}{2 - t} h_t = - \frac{1}{(2 - t)^2} \tilde{h}_t, \quad \tilde{h}_t = 0, \; t \in [0, 1]; \quad \tilde{h}_1 = g(\tilde{m}_1),$$

from which we deduce the following simpler characterization (recalling that $m_0 = 0$)

$$\tilde{m}_1 = - g(\tilde{m}_1) \int_0^1 \frac{dt}{(2 - t)^2} = - \frac{g(\tilde{m}_1)}{2}.$$  \hspace{1cm} (3.8)

There are as many equilibria as solutions to the equation $x = -g(x)/2$. We thus have the following obvious lemma:

**Lemma 3.1.** When $\varepsilon = 0$ and regardless of the dimension, the solutions of the MFG associated with the cost functional (3.1) are given by the roots $\tilde{m}_1$ of the equation $2x + g(x) = 0$ and then by the changes of variable (3.7). In particular, if the Lipschitz constant of the function $g$ is strictly less than 2, then the MFG has one and only one solution.
3.1.2. Potential structure when $\varepsilon = 0$. Following (1.13) and (1.14), we may associate a mean field control problem with the MFG (with $\varepsilon = 0$) if the function $g$ derives from a potential $G$. The cost functional is given by

$$J(\alpha) = \mathbb{E}\left[\frac{1}{2}|X_1|^2 + G(\mathbb{E}(X_1)) + \frac{1}{2}\int_0^1 |\alpha_t|^2 dt\right],$$

where, in the right-hand side,

$$X_1 = \int_0^1 \alpha_t dt.$$

The analysis of the minimization problem $\inf_{\alpha} J(\alpha)$ is straightforward. Indeed, by an obvious convexity argument, we observe that

$$J(\alpha) \geq J\left(\mathbb{E}\int_0^1 \alpha_t dt\right),$$

where, in the right-hand side, it is obviously understood that the argument of $J$ is a constant control. This shows that the minimizers are deterministic and constant in time. Using the generic notation $\beta$ for $\int_0^1 \alpha_t dt$, the problem is thus to minimize

$$\mathcal{J}(\beta) = \beta^2 + G(\beta), \quad \beta \in \mathbb{R}.$$ 

Of course, we observe that any minimizer of $\mathcal{J}$ is also a solution of the equation $\beta = -g(\beta)/2$: We recover the fact that any solution to the mean field control problem is an MFG solution. Obviously, the converse may not be true. Accordingly, the set of solutions to the mean field control problem may be strictly included in the set of equilibria of the corresponding mean field game. This principle is illustrated by Figure 5 below with $g$ as in (3.2): For all the values of $\kappa \in \{2, 3, \cdots, 10\}$, the potential $\mathcal{J}$ has a unique minimizer; but, for $\kappa \in \{7, 8, 9, 10\}$, the derivative of $\mathcal{J}$ has several zeros, hence proving that the MFG has several solutions.

![Figure 5](attachment:image.png)

(A) Evolution of $\mathcal{J}$ with $\kappa$. (B) Evolution of $\frac{d}{d\kappa} \mathcal{J}$ with $\kappa$.

Figure 5. Minimizers of the mean field control problem and equilibria of the mean field game for $\kappa \in \{2, \cdots, 10\}$ when $d = 1$ and $g(x) = \cos(\kappa x)$: Solutions of the mean field control problem are the minimizers of $\mathcal{J}$; Solutions of the mean field game are the roots of $\frac{d}{d\kappa} \mathcal{J}$.

Interestingly, the recent contribution [23] says that, when the MFG is potential, the equilibria that are not associated with a minimizer of the potential should be ruled out. Equivalently, only the MFG solutions that minimize (globally) the potential should be selected. Even more, the guess (which has been rigorously established in [23, 30] but in different settings that the one addressed here) is that the minimizers of the potential are precisely those that appear by forcing uniqueness with a common noise and then by tuning down the
intensity of the common noise. We check this prediction numerically in Subsection 3.4 by using our learning method.

3.2. Algorithms for a fixed intensity of the common noise. We here explain how to implement the fictitious play in the two benchmark examples (3.2) and (3.3) along the lines of Figure 3. Throughout the subsection, the intensity of the common noise is fixed. For simplicity, the intensity of the common noise is chosen as \( \varepsilon = 1 \) and the intensity of the independent noise is chosen as \( \sigma = 0 \) or 1. Numerical experiments are discussed in the next subsection.

3.2.1. Numerical reference solution. We first explain how to compute a reference solution by solving numerically the decoupled forward-backward equation (2.6). Observe that there is no contradiction in using a numerical method for testing our learning algorithm: the numerical method relies on the explicit knowledge of the coefficient \( g \), whilst the learning algorithm is intended to work on the sole observations of the costs.

The numerical method we use is tailor-made to our problem. Indeed, we observe that the solution to the backward equation in (2.6) writes

\[
h_t = \mathbb{E}^h \left[ \exp \left( - \int_t^1 \eta_t ds \right) g(m_1) \right | \mathcal{F}^W_t], \quad t \in [0, 1],
\]

where \((m_t)_{0 \leq t \leq 1}\) solves is an Ornstein-Uhlenbeck process (with independent coordinates)

\[
dm_t = -\eta m_t dt + dW_t, \quad t \in [0, 1]; \quad m_0 = 0.
\]

We then employ a Picard scheme for solving the fixed point equation (3.11), by iterating

\[
h_{t_{n+1}}^{ref} = \mathbb{E}^{h_{t_{n}}^{ref}} \left[ \exp \left( - \int_t^1 \eta_t ds \right) g(m_1) \right | \mathcal{F}^W_t], \quad t \in [0, 1],
\]

Numerically, we use a time grid \( \{t_k = k/p\}_k \) with \( p \) uniform steps (for the same \( p \) as the one used in the learning method, which makes the comparison easier). Following the seminal work of [40], we find, at each iteration \( n \) of the Picard sequence and at each node \( t_j > 0 \) of the time mesh, an approximation of the conditional expectation in the right-hand side of (3.11) in the form of a (deterministic) function of \( m_{t_j} \) (the forward component in (3.11)). This function is taken in the linear span of \( \{H^\ell_j(\cdot/\sigma_j)\}_{|\ell| \leq D} \), where \( \sigma_j \) is the common standard deviation of the coordinates of \( m_{t_j} \) (which is explicitly computable) and \( \{H^\ell_j(\cdot)\}_{|\ell| \leq D} \) is the collection of Hermite polynomials of dimension \( d \) (\( d = 1, 2 \) in our case) and of degree less than \( D \), for a given integer \( D \) (here \( \ell \) is a \( d \)-tuple of integers and \(|\ell| \) is the sum of its entries).

For completeness, we recall that, when \( d = 1 \),

\[
H^1_\ell(x) = \frac{(-1)^\ell}{\sqrt{2^\ell \ell!}} e^{x^2} \frac{d}{dx} \left( e^{-x^2} \right), \quad x \in \mathbb{R}.
\]

When \( d = 2 \),

\[
H^2_{(\ell_1, \ell_2)}(x_1, x_2) = H^1_{\ell_1}(x_1) H^1_{\ell_2}(x_2), \quad x_1, x_2 \in \mathbb{R}.
\]

The weights in the Hermite decomposition are computed by a regression method:

**Algorithm 1.** [Reference solution]

*Input:* Introduce \( \Delta_{t_k} w^{(j)} \) with \( j \in \{1, \ldots, N\} \) and \( k \in \{1, \ldots, p\} \), a collection of \( N \) realizations of independent Gaussian variables \( \mathcal{N}(0, I_d) \).

*Task:* For each \( i \), compute \( (m_{t_k}^{(j)})_{j=0, \ldots, p} \) the realizations of the Euler scheme associated with (3.10) and driven by the simulations

\[
\left( w_{t_k}^{(j)} = \frac{1}{\sqrt{p}} \left[ \Delta_{t_k} w^{(j)} + \cdots + \Delta_{t_k} w^{(j)} \right] \right)_{k=1, \ldots, p}.
\]
Loop: One iteration consists of one Picard iteration. The input at iteration number \( n \) is encoded in the form of an array \((h_{tk}^{n,j})_{j,k}\) with entries in \(\mathbb{R}^d\). The following procedure returns the next input for iteration number \( n + 1 \).

(a) Look for \((h_{tk}^{n+1,j})_{j,k}\) in terms of \((h_{tk}^{n,j})_{j,k}\) by solving, for each \( k = 0, \ldots, p - 1 \), the minimization problem

\[
\min_{c=(c_{\ell})_{|\ell|\leq D}} \frac{1}{N} \sum_{j=1}^{N} c_{\ell}^{n,j} \left| \exp\left(-\sum_{s=k}^{p-1} \eta_s \right) g(m_{t}^{(j)}) - \sum_{|\ell|\leq D} c_{\ell} H_{\ell}^{t} \left(m_{t}^{(j)}\right) \right|^2,
\]

where each \( c_{\ell} \) is in \( \mathbb{R}^d \) and with

\[
c_{\ell}^{n,j} = \exp\left(\sum_{s=k}^{p-1} h_{ts}^{n,j} \cdot \Delta_{t+k+1} w^{(j)} - \frac{1}{2p} \sum_{s=k}^{p-1} |h_{ts}^{n,j}|^2 \right).
\]

(When \( k = 0 \), only \( c_0 \), where \( |0| = 0 \), matters and it is given by a mere empirical mean.)

(b) To enforce some stability, update each coordinate of \((h_{tk}^{n+1,j})_{j,k}\) by projecting it onto \([-1,1]\).

**Remark 3.2.** Here, the coordinates in (3.10) have the same marginal variances because \((\eta_t)_{0\leq t\leq T}\) is scalar-valued (or equivalent is a d-diagonal matrix). This is no longer true when \( R \) and \( Q \) in (1.2) are general matrices. In such a case (and more generally when the components are not centered), the components of \((m_t)_{0\leq t\leq T}\) are no longer independent and the ‘normalized’ Hermite polynomials \(\{H_{\ell}^{t}(\cdot/\sqrt{\gamma_t})\}_{|\ell|\leq D}\) used above should be instead replaced by \(\{H_{\ell}^{t}(\mathbb{K}^{-1/2}(m_t)(\cdot - \mathbb{E}(m_t)))\}_{|\ell|\leq D}\), where \(\mathbb{K}(m_t)\) is the covariance matrix of \(m_t\) and \(\mathbb{K}^{1/2}(m_t)\) is any root of it (e.g., as given by the Cholesky decomposition).

3.2.2. Numerical approximation by fictitious play. We now address the implementation of the fictitious play according to the principle stipulated by Figure 3. The black-box therein is discretized in a suitable manner. It is used to solve numerically the optimization problem (2.23), and then to implement the updating rule (2.32) for the environment. Since Lemma 2.1 says that the corresponding optimal law has an affine Markov feedback form, we thus perform the optimization in (2.23) over closed loop controls that are affine in the state variable, with the linear coefficient being a scalar only depending on time and with the intercept possibly depending on the environment. From the practical point of view, this choice is fully justified if the model is expected to be linear-quadratic in the space/action variables (as it is in (1.1)–(1.2), with \( Q \) and \( R \) being scalars), but this does not require the coefficients \( Q, R, f \) and \( g \) to be known explicitly. Mathematically, this writes as follows. We restrict ourselves to controls \( \alpha = (\alpha_{tk})_{k=1,\ldots,p} \) of the form

\[
\alpha_{tk} = a_{tk-1} X_{tk-1} + C_{tk-1},
\]

where \( a_{tk-1} \) is a scalar and \( C_{tk-1} \) is a \( d \)-dimensional random vector. In comparison with the formula (2.27), \( a_{tk-1} \) should be understood as a proxy for \( -\eta_{tk-1} \) and \( C_{tk-1} \) as a proxy for the intercept therein. Part of the numerical difficulty is thus to have a tractable regression method for capturing the randomness of \( C_{tk-1} \). Recalling that the intercept \( h \) in the solution of the mean field game with common noise is a function of the environment \((m_t)_{0\leq t\leq T}\), see Remark 2.4, a key point in our numerical method is to look for \( C_{tk-1} \) as a \( \sigma(m_{tk-1}) \)-measurable random variable, where \( m_{tk-1} \) is the current proxy for the state of the environment at time \( t_{k-1} \). Numerically, we use a regression method based on a \( d \)-dimensional Hermite polynomials of degree less than \( D \), for a fixed value of \( D \).

We now make this construction explicit when \( \varepsilon = 1 \), noticing that the algorithm must rely on simulations for the independent and common noises. Below, we call \( M \) the number of particles and \( N \) the number of simulations of the whole system, with \( i \) denoting the generic
Remark 3.3.\label{rem:representation} where \( c \) the intercept in the optimal law (2.27) is given in the form of a collection label for a particle and \( C \) Moreover, in the minimization problem, to the realization of the common noise. Following our introductory discussion, we solve, as an approximation of the stochastic control problem (2.23), the minimization problem: in independent of \( i \) is required to be of the form \( \Delta t_k b(i,j) \) \( \Delta t_k u(j) \), with \( h^n_{i,k} \), also with \( j \) running from 1 to \( N \) and \( k \) running from 0 to \( p - 1 \). The fact that the two proxies are independent of \( i \) is fully consistent with the fact that \( m^n \) and \( h^n \) in Remark 2.13 are adapted to the realization of the common noise. Following our introductory discussion, we solve, as an approximation of the stochastic control problem (2.23), the minimization problem:

\[
\min \left\{ \mathcal{E}^{n,(j)} \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \left[ \frac{1}{2p} \sum_{k=1}^{p} |a_t(i,j)|^2 + \frac{1}{2} \left| x^{(i,j)}_1 + g(m^{n,(j)}_1) \right|^2 \right] \right\}, \tag{3.13}\]

where

\[
\mathcal{E}^{n,(j)} = \exp \left( -\sqrt{\frac{1}{p} \sum_{k=0}^{p-1} h^{n,(j)}_{k} \cdot \Delta t_{k+1} w^{(j)} - \frac{1}{2p} \sum_{k=0}^{p-1} |h^{n,(j)}_k|^2} \right).
\]

Moreover, in the minimization problem, \( \alpha_t(i,j), x_t(i,j) \) are required to be of the form

\[
x^{(i,j)}_k = x^{(i,j)}_{k-1} + \frac{1}{p} \alpha_t(i,j) + \frac{1}{\sqrt{p}} \Delta t_k b(i,j), \quad \ell = 1, \ldots, p; \quad x^{(i,j)}_0 = x_0,
\]

\[
\alpha_t(i,j) = a_t(i,j) + C_t(i,j) + h^{n,(j)}_{k-1} + \sqrt{p} \Delta t_k w^{(j)}, \quad k = 1, \ldots, p. \tag{3.14}
\]

In particular, \( (x^{(i,j)}_k)_{k=0,\ldots,p} \) solves the following Euler scheme:

\[
x^{(i,j)}_k = x^{(i,j)}_{k-1} + \frac{1}{p} \left( a_t(i,j) + C_t(i,j) + h^{n,(j)}_{k-1} \right) + \frac{1}{\sqrt{p}} \Delta t_k b(i,j) + \frac{1}{\sqrt{p}} \Delta t_k w^{(j)}, \quad k = 1, \ldots, p.
\]

Moreover, \( C_t(i,j) \) is required to be of the form

\[
C_t(i,j) = \sum_{|\ell| \leq D} c_t(\ell) H(\Delta U^{n}_t)^{-1} \left( m_{t,k}^{n,(j)} - \frac{1}{N} \sum_{r=1}^{N} m_{t,k}^{n,(r)} \right), \tag{3.15}
\]

where \( c_t(\ell) \in \mathbb{R}^d \) and \( U^{n}_t \) is the diagonal matrix obtained by taking the roots of the diagonal of the empirical covariance matrix:

\[
\Sigma_t^n = \frac{1}{N} \sum_{j=1}^{N} \left( m_{t,k}^{n,(j)} - \frac{1}{N} \sum_{r=1}^{N} m_{t,k}^{n,(r)} \right) \otimes \left( m_{t,k}^{n,(j)} - \frac{1}{N} \sum_{r=1}^{N} m_{t,k}^{n,(r)} \right).
\]

Remark 3.3. Following Remark 3.3 we can define \( \Delta U^{n}_t \) (in a more general fashion) as the upper triangular matrix given by the Cholesky decomposition of \( \Sigma_t^n \) in the form \( \Sigma_t^n = \Delta U^{n}_t \Delta U^{n}_t \).

The minimization problem is thus defined over \( a = (a_t(i,j))_k \in \mathbb{R}^p \) and \( c = (c_t(\ell))_{k,\ell} \in (\mathbb{R}^d)^{p \times L} \), where \( L \) is the number of distinct \( d \)-tuples of (non-negative) integers whose sum is less than \( D \).

We can summarize the algorithm in the following form.

**Algorithm 2.** [Learning with two noises, \( \sigma = \varepsilon = 1 \)]

**Input:** Take as input the realizations (3.12), which are the same at any iteration of the algorithm.

**Loop:** At rank \( n + 1 \) of the fictitious play:
(a) Take as input the proxies \((m^{n,j}_{t,k})_{j,k}\) and \((h^{n,j}_{t,k})_{j,k}\) for (respectively) the environment and the intercept.

(b) Solve the minimization problem (3.13) over \(\mathbf{a} = (a_{t,k})_k\) in (3.14) and \(\mathbf{c} = (c_{t,k}(\ell))_{k,\ell}\) in (3.15). Call \(\mathbf{a}^{n+1} = (a^{n+1}_{t,k})_k\) and \(\mathbf{c}^{n+1} = (c^{n+1}_{t,k}(\ell))_{k,\ell}\) the optimal points (returned by any optimization method).

(c) With \(\mathbf{a}^{n+1} = (a^{n+1}_{t,k})_k\) and \(\mathbf{c}^{n+1} = (c^{n+1}_{t,k}(\ell))_{k,\ell}\), associate \((x^{n+1,(i,j)}_{t,k})_{i,j,k}\) as in (3.14) and \((C^{n+1,(j)}_{t,k})_{k,j}\) as in (3.15).

(d) Update the proxies by letting

\[
m^{n+1,(j)}_{t,k} = \frac{1}{M} \sum_{i=1}^{M} x^{n+1,(i,j)}_{t,k}, \quad m^{n+1,(j)}_{t,k} = \frac{1}{n+1} m^{n+1,(j)}_{t,k} + \frac{n}{n+1} m^{n,(j)}_{t,k},
\]

\[
h^{n+1,(j)}_{t,k} = -C^{n+1,(j)}_{t,k}.
\]

In some of the numerical examples below, we will use variants of Algorithm 2. We first focus on the shape of the algorithm when \(\sigma = 0\), recalling that our results do not require the presence of an idiosyncratic noise. Implicitly, we then have a single particle only, i.e. \(M = 1\). Moreover, \(\Delta_{t,k} b^{(i,j)}\) no longer appears in (3.14). We then have the following variant of Algorithm 2, which is of lower complexity:

**Algorithm 3.** [Learning with common noise only, \(\sigma = 0, \varepsilon = 1\)]

Input: Take as input the realizations \((\Delta_{t,k} w^{(j)})_{j,k}\) in (3.12), which are the same at any iteration of the algorithm.

Loop: At rank \(n+1\) of the fictitious play, apply the same loop as in Algorithm 2, but with \(M = 1\) and with \(\Delta_{t,k} b^{(1,j)} = 0\) in (3.14).

We end up our presentation with the case when there is no common noise, i.e. \(\sigma = 1\) and \(\varepsilon = 0\). In that case, our strategy no longer applies. The point is thus to implement the standard version of the fictitious play. Accordingly, there is no Girsanov density in (3.13) and \(\alpha^{(i,j)}_{t,k}\) in (3.14) merely writes

\[
\alpha^{(i,j)}_{t,k} = a_{t,k-1}x^{(i,j)}_{t,k-1} + C_{t,k-1}, \quad k = 1, \cdots, p.
\]

with \(C_{t,k}\) being a deterministic \(d\)-dimensional vector (which is independent of \(j\)). In particular, there is no need to use the proxy \((h^{n,j}_{t,k})_{j,k}\) for the intercept. Equivalently, we can assume that \(N = 1\) and \(D = 0\). The rest of the algorithm is similar. It may be written as follows.

**Algorithm 4.** [Learning with idiosyncratic noise only, \(\sigma = 1, \varepsilon = 0\)]

Input: Take as input the realizations \((\Delta_{t,k} b^{(1,j)})_{k}\) in (3.12), which are the same at any iteration of the algorithm.

Loop: At rank \(n+1\) of the fictitious play:

(a) Take as input the proxy \((m^{n}_{t,k})_{k}\) for the environment and the intercept.

(b) Solve the minimization problem (3.13) over \(\mathbf{a} = (a_{t,k})_k\) in (3.14) and \(\mathbf{c} = (c_{t,k})_k\) in (3.15), with \(\mathbf{c}^{n,(j)} = 1\) in (3.13) and with the last line in (3.14) being replaced by

\[
\alpha^{(i,1)}_{t,k} = a_{t,k-1}x^{(i,1)}_{t,k-1} + c_{t,k-1}(0), \quad k = 1, \cdots, p.
\]

Call \(\mathbf{a}^{n+1} = (a^{n+1}_{t,k})_k\) and \(\mathbf{c}^{n+1} = (c^{n+1}_{t,k}(0))_k\) the optimal points.

(c) With \(\mathbf{a}^{n+1} = (a^{n+1}_{t,k})_k\) and \(\mathbf{c}^{n+1} = (c^{n+1}_{t,k}(0))_k\), associate \((x^{n+1,(1,1)}_{t,k})_{i,k}\) as in (3.14) (with the same prescription as above).

(d) Update the proxy by letting

\[
m^{n+1,(1,1)}_{t,k} = \frac{1}{M} \sum_{i=1}^{M} x^{n+1,(i,1)}_{t,k}, \quad m^{n+1,(1,1)}_{t,k} = \frac{1}{n+1} m^{n+1,(1,1)}_{t,k} + \frac{n}{n+1} m^{n,(1,1)}_{t,k}.
\]
In all the numerical experiments, we use ADAM optimizer (as implemented in TensorFlow) in order to solve the optimization step in Algorithms 2, 3 and 4. We recall from the discussion in Subsection 1.7 that the code in TensorFlow relies on some automatic differentiation and thus makes an explicit use of the linear-quadratic form of the coefficients. In this sense, our numerical implementation requires part of the model to be known, but, as we already explained in Subsection 1.7, this does not really affect the scope of our conclusions: Provided the optimization method used in Algorithms 2, 3 and 4 is sufficiently accurate, the whole works well and demonstrates the interest for exploring the state space by means of the common noise.

3.3. Numerical experiments. The numerical examples that we provide below are here to illustrate several features of our form of fictitious play. In all the examples treated below, we choose $d = 2$, $f \equiv 0$ and $g$ as in (3.3) with $\kappa = 10$.

We emphasize that our aim is to demonstrate, numerically, the positive impact of the common noise, not to compare the numerical rates of convergence obtained in the numerical experiments with the bounds obtained in the theoretical analysis. The reason is that the algorithms we present below include additional features (like numerical optimization methods and regressions) that are not addressed in the theoretical analysis. This makes difficult any precise comparison. Also, it is fair to say that the time of execution of the full algorithm becomes quite long (several hours for the longest experiments), even for low values of $n$ and $p$ ($n$ less than 50 and $p$ less than 100).

3.3.1. Evolution of the learnt cost with one type of noise only. We first test the influence of each of the two types of noises onto the behavior of the algorithm. We thus compare the evolution of the cost learnt by the fictitious play in two scenarios: In the first scenario, there is an independent noise, but no common noise (Algorithm 4); in the second scenario, there is a common noise but no independent noise (Algorithm 3). We will see next what happens when there are the two noises.

![Figure 6](image)

Figure 6. Comparison of the learnt cost depending on the type of noise. In $x$-axis: number of iterations; In $y$-axis: learnt cost.

The experiments are computed with: $n = 20$ learning iterations, $p = 30$ time steps, $M = 4 \times 10^5$, $N = 1$, $\sigma = 1$ and $\varepsilon = 0$ in case (A) and $n = p = 20$, $M = 1$, $N = 4 \times 10^5$, $\sigma = 0$, $\varepsilon = 1$ and $D = 4$ in case (B). In ADAM method, the learning rate is 0.01, with 15 epochs.

In plot (B), the orange line is the theoretical equilibrium cost as computed by the BSDE method explained in Subsection 3.2.1. In plot (A), there is no computed reference cost. As we already explained, there might not be a unique equilibrium and the notion of reference cost no longer makes sense. Notice by the way that the equilibrium cost in case (B) is not an equilibrium cost in case (A) since the two problems are different (as being set over different
forms of dynamics). Anyway, the conclusion is clear: the learnt cost does not convergence in case (A), whilst it does in case (B). This is a clear evidence of the numerical impact of the common noise onto the behavior of the fictitious play.

3.3.2. Error in the optimal path/intercept with common noise and no independent noise. We now compute the error achieved by the learning algorithm. Using the same notation as in the statement of Proposition 2.19, we focus on the following $L^2$ error:

$$\left[ \mathbb{E} \int_0^1 \left( |\bar{m}^n_t - m_t|^2 + |\bar{h}^n_t - h_t|^2 \right) dt \right]^{1/2},$$

which slightly differs from the error appearing in Proposition 2.19. Focusing here on the time-averaged error is obviously more advantageous from the numerical point of view, but it is sufficient to demonstrate the efficiency of the algorithm and also to identify some of its limitations. Notice also that the expectation in the above error is taken under the non-tilted expectation. Since $\varepsilon$ is here equal to 1, the Girsanov density $\mathcal{E}(h)$ and its inverse can indeed be easily estimated. Also, since we have bounds for the moments of $\bar{m}^n$ and $m$, Proposition 2.19 shows that the above expectation tends to 0 as $n$ tends to $\infty$. This is sufficient for our purpose since we do not try to identify the numerical rate of convergence precisely.

Numerically, the error is approximated by

$$\left[ \frac{1}{Np} \sum_{j=1}^N \sum_{k=0}^{p-1} \left| \bar{m}^{n,(j)}_{tk} - m^{(j)}_{tk} \right|^2 + \left| \bar{h}^{n,(j)}_{tk} - h^{(j)}_{tk} \right|^2 \right]^{1/2},$$

(3.16)

where $(\bar{m}^{n,(j)}_{tk})_{j,k}$ and $(\bar{h}^{n,(j)}_{tk})_{j,k}$ are the returns of Algorithm 2 or 3 (depending on the value of $\sigma$) and $(m^{(j)}_{tk})_{j,k}$ and $(h^{(j)}_{tk})_{j,k}$ are the reference solutions as computed with Algorithm 1 (with a sufficiently high number of Picard iterations: 10 in practice).

The first experiment is performed by running Algorithm 3 (no idiosyncratic noise), but assuming that the solution $(\eta_t)_{0 \leq t \leq 1}$ to the Riccati equation (3.4) is known, see (3.5): In Algorithm 3, $a_{tk}$ is replaced by $\eta_{tk}$. This amounts to say that the coefficients $Q$ and $R$ in (1.2) are explicitly known. The evolution of the error with the number of iterations is plotted in (A), Figure 7. The second experiment is performed by running Algorithm 3, but without any further knowledge of the solution of the Riccati equation (3.4). The evolution of the error with the number of iterations is plotted in (B), Figure 7.

![Figure 7](image)

**Figure 7.** Comparison of the error returned by Algorithm 3 (common noise only), depending on whether the solution to the Riccati equation is known or not.

The experiments are computed with: $n = 20$ learning iterations, $p = 30$ time steps, $M = 1$, $N = 5 \times 10^4$, $\sigma = 0$, $\varepsilon = 1$ and $D = 4$. On Plot (A), the error is less than 0.01 after iteration 3.
The conclusion is quite clear: When randomness only comes from the common noise, it is very difficult for the optimization method to distinguish between \( x_{t_k}^{(j)} \) and \( C_{t_k}^{(j)} \) in (3.14).

### 3.3.3. Error in the optimal path/intercept with both noises

We now proceed with the same analysis but putting the two noises. As the total number of simulated paths is \( N \times M \), this may be however rather costly. We proceed below by freezing the quantity \( N \times M \).

#### (a) \( M = 4 \times 10^4, N = 1 \), time (cpu): 4867

#### (b) \( M = 2 \times 10^4, N = 20 \), time (cpu): 4096

#### (c) \( M = 10^4, N = 40 \), time (cpu): 8409

**Figure 8.** Comparison of the error returned by Algorithm 2, depending on the number of particles/realizations.

The experiments are computed with: \( n = 10 \) learning iterations, \( p = 30 \) time steps, \( \sigma = 1 \) and \( \varepsilon = 1 \); 4000 units in cpu time is around 1h. The conclusion is that idiosyncratic noises demonstrate to be useful from the numerical point of view, even though the results stated in Section 2 remain the same whatever the value of \( \sigma \). A careful inspection of the numerical results show that, in fact, idiosyncratic noises provide a better fit of the solution to the Riccati equation, which is fully consistent with the conclusion of the previous paragraph. In turn, we guess that a model-based method, in which \( Q \) and \( R \) first, and then the solution to the Riccati equation, would be learnt separately, would make sense in this specific setting.

### 3.3.4. Validation

The reader could worry about a possible overfitting in our numerical experiments. Actually, in order to validate our results, we can learn the coefficients \( a^n \) and \( c^n \) in Algorithm 2 for a first series of data in (3.12). In brief, this first series of data is used to learn the equilibrium feedback. Then, we can use a second series of data (say, of the same size) in (3.12) in order to compute the error. In clear, given this second series of data, we can implement Algorithm 1 and then compute the resulting error (3.16) when \((m_{t_k}^{n,(j)}, h_{t_k}^{n,(j)})_{j,k}\) therein are obtained from the formulas (3.14) and (3.15) by using the coefficients \( a^n \) and \( c^n \) returned by the first series of data and then by implementing the Euler scheme with respect to the second series of data. The resulting plots are very similar to those represented in Figure 8. For this reason, we feel useless to insert them here. Intuitively, what happens is that there are sufficiently many realizations of the common noise in our experiments to guarantee, in the regression (3.13), a convenient form of averaging with respect to all these realizations.

### 3.4. Vanishing viscosity

The next question in our numerical experiments is to address the behavior of the algorithm as the viscosity tends to 0. As made clear by the analysis performed in Section 2, the influence of the small viscosity may manifest in several manners, depending on the error that is addressed: recall indeed that the bound in the statement of Proposition 2.19 does not depend on \( \varepsilon \), whilst the bound provided by the main Theorem 2.14 blows up as \( \varepsilon \) tends to 0.

For sure, this is a main issue from the numerical point of view since we may expect the numerical method to become somewhat instable as \( \varepsilon \) tends to 0. Independently of the bounds provided by the results, the numerical difficulty is easy to understand: The Girsanov density
underlying our algorithm becomes highly singular and this may produce high variances in the computations. Our strategy to overcome this obvious drawback is inspired from simulated annealing method: The viscosity parameter is decreased step by step; At each new step, we use the return of the algorithm at the previous value of the viscosity as a new guess in our algorithm. As exemplified below (see Figure 10), this turns out to be an efficient method in the benchmark example introduced in Subsection 3.1.

The use of the previous return (at the previous value of the viscosity) may be formalized as follows. Assume indeed that we have a decreasing sequence \((\varepsilon_q)_{q \geq 1}\) that converges to 0, with each \(\varepsilon_q > 0\) being understood as the value of the viscosity at step \(q\). With each \(q\), we may associate a return \(h^{\varepsilon_q}\). The symbol \(\infty\) in superscript should be understood as an ‘idealization’ in the interpretation of \(h^{\infty,q}\): This is the return of our algorithm, when run with viscosity \(\varepsilon_q\), after sufficiently many iterations \(n\) of the fictitious play; Since we do not quantify here which value of \(n\) we use (this would be doable, but this would lead to additional cumbersome notations and computations), we just think of \(n\) as being large, ideally equal to \(\infty\), whence our notation. At the next step \(q + 1\), when using viscosity \(\varepsilon_{q+1}\), we then implement (1.19) (for simplicity, we also take the time discretization parameter \(1/p\) to be equal to 0, but the same idea can be obviously implemented when \(p\) is finite as in (2.23)) with \(h^n - h^{\infty,q}\) in lieu of \(h^n\). In words, (1.19) becomes

\[
\alpha^{n+1} = \arg\min_{\alpha} \mathbb{E}^{\infty,q} \left[ R(\alpha; m^n, W^{h^n - h^{\infty,q}}) \right]. \tag{3.17}
\]

Implicitly, the viscosity that is used in the above formula is \(\varepsilon_{q+1}\). Also, it may be useful to recall that \(n\) here denotes the iteration in the fictitious play: Whatever the value of \(n\), we use the same guess \(h^{\infty,q}\).

The reader must understand that this modification of the guess does not change our theoretical analysis. In short, the analysis performed in Section 2 is written with 0 as guess. The fact that our analysis still holds can be justified by changing the probability space. We may indeed equip \((\Omega, \mathcal{A})\) with the new probability measure \(\tilde{\mathbb{P}} = 1_{\infty,q}^{\infty,q}\) and with the new common noise \(\tilde{W} = W^{\infty,q}\). Under \(\tilde{\mathbb{P}}\), the common noise \(\tilde{W}\) is a Brownian motion. Since the latter is the same in (3.17) for any \(n\) (which is a prescription in our method), we can invoke the results of Section 2 in order to study (3.17). For instance, Proposition 2.8 gives

\[
\mathbb{E}^h \left[ \sup_{0 \leq t \leq T} \left( |m^n_t - m_t|^2 + |h^n_t - h_t|^2 \right) \right] \leq \frac{C}{n^2},
\]

where \(h\) is a sloppy notation for the backward component of the solution to the MFG with common noise of intensity \(\varepsilon_{q+1}\). If, as we said above, \(h^{\infty,q+1}\) is indeed understood as the limit of \(h^n\) as \(n\) tends to \(\infty\), then we can identify \(h^{\infty,q+1}\) itself, hence leading to

\[
\mathbb{E}^{h^{\infty,q+1}} \left[ \sup_{0 \leq t \leq T} \left( |m^n_t - m_t^{\infty,q+1}|^2 + |h^n_t - h_t^{\infty,q+1}|^2 \right) \right] \leq \frac{C}{n^2},
\]

where, for consistency, we wrote \(m^{\infty,q+1}\) for the forward component of the solution to the MFG with common noise of intensity \(\varepsilon_{q+1}\). Recalling the definition of \(\tilde{\mathbb{P}}\), the above may be rewritten as

\[
\mathbb{E}^{h^{\infty,q+1} - h^{\infty,q}} \left[ \sup_{0 \leq t \leq T} \left( |m^n_t - m_t^{\infty,q+1}|^2 + |h^n_t - h_t^{\infty,q+1}|^2 \right) \right] \leq \frac{C}{n^2}.
\]

Now, if we tune the increment \(\varepsilon_q - \varepsilon_{q+1}\) in such a manner that \(h^{\infty,q+1}\) and \(h^{\infty,q}\) are sufficiently close compared to \(\varepsilon_q\), it becomes reasonable to expect that the same expectation but under the historical probability measure (hence without the Girsanov transformation) is small, namely

\[
\lim_{n \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( |m^n_t - m_t^{\infty,q+1}|^2 \right) \right] = 0,
\]

with a precise decay that would depend on the bound for the Girsanov density. We feel better not to address this question mathematically here. This will be the subject of a
separate note. However, we prove the guess to be satisfied in a numerical experiment based on the aforementioned benchmark example. We choose a variant of $g$ in (3.2), with $d = 1$:

$$g(x) = \cos(\kappa(x - x_0)) - 2x_0,$$

with $\kappa = 10$ and where $x_0$ is a root of the equation

$$\cos(\kappa x_0) = 2x_0.$$

Numerically, we find that a choice is $x_0 \approx -0.384$. The motivation for such an $x_0$ is that $0$ is a solution of (3.8). In other words, $0$ is an equilibrium. Even more, we observe that, if the viscosity is zero, then the iterative sequence defined through the two updating rules (1.9) and (1.10) remains in $(m^n, h^n) = (0, -2x_0)$ for any $n \geq 1$ if $m^0$ is chosen as $0$. In other words, the standard fictitious play converges (as predicted by the theory since this 1-dimensional MFG is potential, see §3.1.2) and chooses the $0$-equilibrium. Now, in order to implement the same prediction method as in §3.1.2, we plot the corresponding potential (which is given by a primitive $G$ of $g$). The plots are given in Figure 9. We observe that 0 is just a local minimizer of the potential and that the global minimizer is around $-0.5$: Our selection method is hence to rule the 0-equilibrium and to retain the other one. In order to check this numerically, we apply the vanishing viscosity method (3.17), with the following values: We choose $\sigma = 0$ (no idiosyncratic noise), $p = 10$ time steps, $M = 10^6$ Monte-Carlo simulations, $n = 10$ iterations in the fictitious play. Our regression on the Hermite polynomials goes up to polynomials of degree 6; We start from $\varepsilon = 1$ as viscosity and we decrease by 0.1 its value up to viscosity 0.1; at each new step, we use the last return of the fictitious play as guess in (3.17). The plot below (Figure 10) shows that our method is able to capture the global minimizer even if initialized from the local minimizer.

![Figure 9](image1.png)  ![Figure 10](image2.png)

**Figure 9.** Equilibrium predicted by the potential rule: Potential $G$ on the left pane; Zeros of the function $g$ in the right pane.

**Figure 10.** Selection of a solution by vanishing viscosity: Plots of the histogram of the approximate mean state of the population at terminal time.
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