ON GENERALIZATION OF DIFFERENT TYPE INEQUALITIES 
FOR HARMONICALLY QUASI-CONVEX FUNCTIONS VIA 
FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we obtained some new estimates on generalization 
of Hadamard, Ostrowski and Simpson-like type inequalities for harmonically 
 quasi-convex functions via Riemann Liouville fractional integral.

1. INTRODUCTION

In this section we will present definitions and some results used in this paper.
A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be a convex function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

The notion of quasi-convex functions generalizes the notion of convex functions.
More precisely, a function $f : [u, v] \to \mathbb{R}$ is said quasi-convex on $[u, v]$ if

$$f(\lambda x + (1 - \lambda)y) \leq \sup \{f(x), f(y)\},$$

for any $x, y \in [u, v]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex
function. Furthermore, there exist quasi-convex functions which are not convex
(see [4]).

Following inequalities are well known in the literature as Hermite-Hadamard
inequality, Ostrowski inequality, Simpson type inequalities, harmonically quasi-convex function.

**Theorem 1.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of
real numbers and $u, v \in I$ with $u < v$. The following double inequality holds

$$f \left( \frac{u + v}{2} \right) \leq \frac{1}{v - u} \int_{u}^{v} f(x)dx \leq \frac{f(u) + f(v)}{2}. \tag{1.1}$$

**Theorem 2.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in $I^\circ$, the interior of
$I$, and let $u, v \in I^\circ$ with $u < v$. If $|f'(x)| \leq M$ for all $x \in [u, v]$, then the following
inequality holds

$$\left| f(x) - \frac{1}{v - u} \int_{u}^{v} f(t)dt \right| \leq \frac{M}{v - u} \left[ \frac{(x - u)^2 + (v - x)^2}{2} \right]$$

for all $x \in [u, v]$.

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Ostrowski inequality, Simpson type inequalities, harmonically quasi-convex function.
Theorem 3. Let $f : [u, v] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(u, v)$ and $\|f^{(4)}\|_{\infty} = \sup_{x \in (u,v)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[ \frac{f(u) + f(v)}{2} + 2f \left( \frac{u + v}{2} \right) \right] - \frac{1}{v - u} \int_{u}^{v} f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (v - u)^4.$$ 

In [5], İscan defined harmonically convex functions and gave some Hermite-Hadamard type inequalities for this class of functions.

Definition 1. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f \left( \frac{xy}{\lambda x + (1 - \lambda)y} \right) \leq \lambda f(y) + (1 - \lambda)f(x)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality in (1.2) is reversed, then $f$ is said to be harmonically concave.

In [17], Zhang et al. defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

Definition 2. A function $f : I \subseteq (0, \infty) \to [0, \infty)$ is said to be harmonically convex, if

$$f \left( \frac{xy}{\lambda x + (1 - \lambda)y} \right) \leq \sup \{f(x), f(y)\}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

We would like to point out that any harmonically convex function on $I \subseteq (0, \infty)$ is a harmonically quasi-convex function, but not conversely. For example, the function

$$f(x) = \begin{cases} 1, & x \in (0, 1]; \\ (x - 2)^2, & x \in [1, 4]. \end{cases}$$

is harmonically quasi-convex on $(0,4]$, but it is not harmonically convex on $(0,4]$.

Let us recall the following special functions:

(1) The Beta function:

$$\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} = \int_{0}^{1} t^{x-1} (1 - t)^{y-1} dt, \quad x, y > 0,$$

(2) The hypergeometric function:

$$2F_1(a, b; c; z) = \frac{1}{\beta(b, c - b)} \int_{0}^{1} t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-a} dt, \quad c > b > 0, \ |z| < 1 \ (\text{see [11]})$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. Let $f \in L[u, v]$. The Riemann-Liouville integrals $J_{u+}^{\alpha} f$ and $J_{v-}^{\alpha} f$ of order $\alpha > 0$ with $u \geq 0$ are defined by
\[ J_u^\alpha f(b) = \frac{1}{\Gamma(\alpha)} \int_u^b (b-t)^{\alpha-1} f(t) \, dt, \quad b > u \]

and

\[ J_v^\alpha f(a) = \frac{1}{\Gamma(\alpha)} \int_a^v (t-a)^{\alpha-1} f(t) \, dt, \quad a < v \]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by
\[
\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt
\]

and \( J_u^0 f(x) = J_v^0 f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral. In recent years, many authors have studied error estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [1, 2, 3, 7, 8, 9, 10, 12, 13, 14, 15, 16].

In [6], Iscan gave the following new identity for differentiable functions and obtained some new general integral inequalities for harmonically convex functions via fractional integrals.

**Lemma 1.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \) and \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \) then for all \( x \in [a, b] \), \( \lambda \in [0, 1] \) and \( \alpha > 0 \) we have:

\[
I_{f,g}(x, \lambda, \alpha, a, b) = \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \int_0^1 t^\alpha - \lambda \frac{ax}{A_t(a, x)} f'(\frac{ax}{A_t(a, x)}) \, dt
\]

\[
- \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \int_0^1 t^\alpha - \lambda \frac{bx}{A_t(b, x)} f'(\frac{bx}{A_t(b, x)}) \, dt,
\]

where \( A_t(u, x) = tu + (1-t)x \) and

\[
(1.3) I_{f,g}(x, \lambda, \alpha, a, b) = (1 - \lambda) \left[ \left( \frac{x-a}{ax} \right)^\alpha + \left( \frac{b-x}{bx} \right)^\alpha \right] f(x) + \lambda \left[ \left( \frac{x-a}{ax} \right)^\alpha f(a) + \left( \frac{b-x}{bx} \right)^\alpha f(b) \right] - \Gamma(\alpha + 1) \left[ J_{1/x^+}^\alpha (f \circ g)(1/a) + J_{1/x^-}^\alpha (f \circ g)(1/b) \right], \ g(u) = 1/u.
\]

In this paper, by using of Lemma 1, we obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for harmonically quasi-convex functions via Riemann Liouville fractional integral.

**2. Main Results**

Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \), the interior of \( I \), throughout this section we will take the notation \( I_{f,g}(x, \lambda, \alpha, a, b) \) as in [1,3], where \( a, b \in I \) with \( a < b \), \( x \in [a, b] \), \( \lambda \in [0, 1] \), \( g(u) = 1/u \), \( \alpha > 0 \) and \( \Gamma \) is Euler Gamma function. In order to prove our main results we need the following identity.
Theorem 4. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for some fixed $q \geq 1$, then for $x \in [a, b]$, $\lambda \in [0, 1]$ and $\alpha > 0$ the following inequality holds

$$|I_{f,g}(x, \lambda, \alpha, a, b)| \leq C_{1, \alpha}^{1/q} \left( \frac{2\alpha \lambda^{1/q} + 1}{\alpha + 1} - \lambda \right)$$

where

$$C_{1, \alpha}(\alpha, \lambda) = \frac{2\alpha \lambda^{1/q} + 1}{\alpha + 1} - \lambda,$$

$$C_{2}(\alpha, \lambda, q, x)$$

$$= \frac{1}{\alpha + 1} \cdot 2F(2q; \alpha + 1; \alpha + 2, 1 - \frac{a}{x}) - \lambda \cdot 2F(2q; 1; 2, 1 - \frac{a}{x})$$

$$+ 2\lambda^{1/q} \cdot \left[ 2F(2q; 1; 2, \lambda^{1/q} \left( 1 - \frac{a}{x} \right)) - \frac{1}{\alpha + 1} \cdot 2F(2q; 1; 1 + \alpha + 2, \lambda^{1/q} \left( 1 - \frac{a}{x} \right)) \right],$$

$$C_{3}(\alpha, \lambda, q, x, \beta)$$

$$= \frac{1}{\alpha + 1} \cdot 2F(2q; 1; 2, 1 - \frac{a}{x}) - \lambda \cdot 2F(2q; 1; 2, 1 - \frac{a}{x})$$

$$+ 2\lambda^{1/q} \cdot \left[ 2F(2q; 1; 2, \lambda^{1/q} \left( 1 - \frac{a}{x} \right)) - \frac{1}{\alpha + 1} \cdot 2F(2q; 1; 1 + \alpha + 2, \lambda^{1/q} \left( 1 - \frac{a}{x} \right)) \right].$$

Proof. Since $|f'|^q$ is harmonically quasi-convex on $[a, b]$, from Lemma 3 property of the modulus and using the power-mean inequality we have

$$|I_{f,g}(x, \lambda, \alpha, a, b)| \leq \left( \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \right) \int_0^1 \left| t^\alpha - \lambda \right| \left| f' \left( \frac{ax}{A_t(a, x)} \right) \right| dt$$

$$+ \left( \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \right) \int_0^1 \left| t^\alpha - \lambda \right| \left| f' \left( \frac{bx}{A_t(b, x)} \right) \right| dt$$

$$\leq \left( \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \right) \left( \int_0^1 \left| t^\alpha - \lambda \right| dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left| t^\alpha - \lambda \right|^q \left| f' \left( \frac{ax}{A_t(a, x)} \right) \right|^q dt \right)^{\frac{1}{q}}$$

$$+ \left( \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \right) \left( \int_0^1 \left| t^\alpha - \lambda \right| dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left| t^\alpha - \lambda \right|^q \left| f' \left( \frac{bx}{A_t(b, x)} \right) \right|^q dt \right)^{\frac{1}{q}}.$$
\[ C_1^{1-1/q}(\alpha, \lambda) \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} \left( \int_0^1 \frac{|t^\alpha - \lambda|}{A_2^{2q}(a, x)} dt \right)^{1/q} \right\} \]

(2.2) \[ \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} \left( \int_0^1 \frac{|t^\alpha - \lambda|}{A_2^{2q}(b, x)} dt \right)^{1/q} \right\} \]

where by simple computation we obtain

\[ C_1(\alpha, \lambda) = \left( \frac{1}{\alpha + 1} \right) \left[ \frac{2\alpha \lambda^{1+\frac{1}{\alpha}} + 1}{\alpha + 1} - \lambda \right], \]

(2.3)

\[ \int_0^1 \frac{|t^\alpha - \lambda|}{A_2^{2q}(a, x)} dt \]

(2.4)

\[ b^{-2q} \left\{ \frac{1}{\alpha + 1} \cdot 2F_1(2q; 1; 1, \alpha + 2, 1 - \frac{x}{b}) - \lambda \cdot 2F_1(2q; 1; 2, 1 - \frac{x}{b}) \right\}
+ 2\lambda^{1+\frac{1}{\alpha}} \left[ 2F_1(2q; 1; 2, \lambda^{1/\alpha} \left( 1 - \frac{a}{x} \right)) - \frac{1}{\alpha + 1} \cdot 2F_1(2q; \alpha + 1; \alpha + 2, \lambda^{1/\alpha} \left( 1 - \frac{a}{x} \right)) \right], \]

(2.5)

Hence, if we use (2.3), (2.4) and (2.5) in (2.2), we obtain the desired result. This completes the proof. \[ \Box \]

Corollary 1. Under the assumptions of Theorem 4 with \( q = 1 \), the inequality (2.1) reduced to the following inequality

\[ \left| I_{f,g}(x, \lambda, \alpha, a, b) \right| \leq \left\{ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}x^2} \left[ \sup \left\{ |f'(x)|, |f'(a)| \right\} \right] C_2 \left( \alpha, \lambda, 1, \frac{a}{x} \right) \right. \]
\[ + \left. \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}b^2} \left[ \sup \left\{ |f'(x)|, |f'(a)| \right\} \right]^{1/q} C_3 \left( \alpha, \lambda, 1, \frac{b}{x} \right) \right\}, \]
Corollary 3. In Theorem 4, the inequality (2.1) reduced to the following inequality

\[
\left( \frac{ab}{b-a} \right) |I_{f,g}(x, \lambda, \alpha, a, b)|
\]

\[
= |(1 - \lambda) f(x) + \lambda \left[ \frac{b(x-a) f(a) + a(b-x) f(b)}{x(b-a)} \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du |
\]

\[
\leq \left( \frac{ab}{b-a} \right) \left( \frac{2 \lambda^2 - 2 \lambda + 1}{2} \right)^{\frac{1}{4}} \left\{ \frac{(x-a)^2}{x^2 q} \left[ \sup \{|f'(x)|^q, |f'(a)|^q\} \right]^{1/q} C_{2,1/q}^{1/q} \left( 1, \lambda, q, \frac{a}{x} \right) \right.
\]

\[
+ \frac{(b-x)^2}{b^2 q} \left[ \sup \{|f'(x)|^q, |f'(a)|^q\} \right]^{1/q} C_{3,1/q}^{1/q} \left( 1, \lambda, q, \frac{x}{b} \right) \right\},
\]

specially for \( x = H = 2ab/(a+b) \), we get

\[
\left( 1 - \lambda \right) f(H) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \leq \frac{b-a}{4ab} \left( \frac{2 \lambda^2 - 2 \lambda + 1}{2} \right)^{\frac{1}{4}}
\]

\[
\times \left\{ \frac{a^2 H^2}{H^2 q} C_{1,1/q}^{1/q} \left( 1, \lambda, q, \frac{a+b}{2b} \right) \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{\frac{1}{q}} \right.
\]

\[
+ \frac{b^2 H^2}{b^2 q} C_{3,1/q}^{1/q} \left( 1, \lambda, q, \frac{a+b}{H} \right) \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{\frac{1}{q}} \right\}.
\]

Corollary 3. In Theorem 4

(1) If we take \( x = H = 2ab/(a+b) \), \( \lambda = \frac{1}{4} \), then we get the following Simpson type inequality for fractional integrals

\[
\left\{ \frac{1}{6} \right\} \left[ f(a) + 4f(H) + f(b) \right] - \left( \frac{ab}{b-a} \right)^{\alpha} 2^{\alpha-1} \Gamma(\alpha+1) \left[ J_{f/H+} (f \circ g) (1/a) + J_{f/H-} (f \circ g) (1/b) \right]
\]

\[
\leq \frac{b-a}{4ab} C_{1,1/q}^{1/q} \left( \alpha, \frac{1}{3} \right) \left\{ \frac{a^2 H^2}{H^2 q} \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{1/q} C_{2,1/q}^{1/q} \left( \alpha, \frac{1}{3}, q, \frac{a}{H} \right) \right.
\]

\[
+ \frac{b^2 H^2}{b^2 q} \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{1/q} C_{3,1/q}^{1/q} \left( \alpha, \frac{1}{3}, q, \frac{H}{b} \right) \right\}.
\]

specially for \( \alpha = 1 \), we get

\[
\left\{ \frac{1}{6} \right\} \left[ f(a) + 4f(H) + f(b) \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \leq \frac{b-a}{4ab} \left( \frac{5}{18} \right)^{1/4}
\]

\[
\times \left\{ \frac{a^2 H^2}{H^2 q} C_{1,1/q}^{1/q} \left( \frac{1}{3}, \frac{1}{3}, q, \frac{a+b}{2b} \right) \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{\frac{1}{q}} \right.
\]

\[
+ \frac{b^2 H^2}{b^2 q} C_{3,1/q}^{1/q} \left( \frac{1}{3}, \frac{1}{3}, q, \frac{2a}{a+b} \right) \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{\frac{1}{q}} \right\}.
\]
(2) If we take \( x = H = 2ab/(a+b) \), \( \lambda = 0 \), then we get the following midpoint type inequality for fractional integrals

\[
|f(H) - \left(\frac{ab}{b-a}\right)^\alpha 2^{\alpha-1} \Gamma(\alpha + 1) \left[J^{\alpha}_{1/H^+} (f \circ g) (1/a) + J^{\alpha}_{1/H^-} (f \circ g) (1/b)\right]|
\leq \frac{b-a}{4ab} \left(\frac{1}{\alpha + 1}\right)^{1-\frac{\alpha}{2}} \left\{ \frac{a^2 H^2}{H^{2q} a^{1/q}} \left(\alpha, 1, q, \frac{a + b}{2b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} + \frac{b^2 H^2}{b^{2q}} C_3^{1/q} \left(\alpha, 1, q, \frac{a + b}{b}ight) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} \right\}.
\]

specially for \( \alpha = 1 \), we get

\[
|f(H) - \frac{ab}{b-a} \int_a^b f(u) \frac{du}{u^2}| \leq \frac{b-a}{4ab} \left(\frac{1}{2}\right)^{1-\frac{\alpha}{2}} \left\{ \frac{a^2 H^2}{H^{2q} a^{1/q}} \left(\alpha, 1, q, \frac{a + b}{2b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} + \frac{b^2 H^2}{b^{2q}} C_3^{1/q} \left(\alpha, 1, q, \frac{a + b}{a+b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} \right\}.
\]

(3) If we take \( x = H = 2ab/(a+b) \), \( \lambda = 1 \), then we get the following trapezoid type inequality for fractional integrals

\[
|f(a) + f(b) - \left(\frac{ab}{b-a}\right)^\alpha \frac{a^{\alpha-1} \Gamma(\alpha + 1)}{2} \left[J^{\alpha}_{1/H^+} (f \circ g) (1/a) + J^{\alpha}_{1/H^-} (f \circ g) (1/b)\right]|
\leq \frac{b-a}{4ab} \left(\frac{\alpha}{\alpha + 1}\right)^{1-\frac{\alpha}{2}} \left\{ \frac{a^2 H^2}{H^{2q} a^{1/q}} \left(\alpha, 1, q, \frac{a + b}{2b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} + \frac{b^2 H^2}{b^{2q}} C_3^{1/q} \left(\alpha, 1, q, \frac{a + b}{b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} \right\}.
\]

specially for \( \alpha = 1 \), we get

\[
|f(a) + f(b) - \frac{ab}{b-a} \int_a^b f(u) \frac{du}{u^2}| \leq \frac{b-a}{4ab} \left(\frac{1}{2}\right)^{1-\frac{\alpha}{2}} \left\{ \frac{a^2 H^2}{H^{2q} a^{1/q}} \left(\alpha, 1, q, \frac{a + b}{2b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} + \frac{b^2 H^2}{b^{2q}} C_3^{1/q} \left(\alpha, 1, q, \frac{a + b}{a+b}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}} \right\}.
\]

Corollary 4. Let the assumptions of Theorem 4 hold. If \( |f'(x)| \leq M \) for all \( x \in [a, b] \) and \( \lambda = 0 \), then we get the following Ostrowski type inequality for fractional integrals

\[
\left\{ \left(\frac{x-a}{ax}\right)^\alpha + \left(\frac{b-x}{bx}\right)^\alpha \right\} f(x) - \left(\frac{ab}{b-a}\right)^\alpha 2^{\alpha-1} \Gamma(\alpha + 1) \left[J^{\alpha}_{1/x^+} (f \circ g) (1/a) + J^{\alpha}_{1/x^-} (f \circ g) (1/b)\right]
\leq \frac{M}{(\alpha + 1)^{1-\frac{\alpha}{2}}} \left\{ \left(\frac{x-a}{ax}\right)^{\alpha+1} + \left(\frac{b-x}{bax}\right)^{\alpha+1} \right\} C_3^{1/q} \left(\alpha, 0, q, \frac{a}{x}\right) \left[\sup \{ |f'(H)(a)|^q, |f'(a)|^q \} \right]^{\frac{q}{2}}.
\]
Theorem 5. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically quasi-convex on \( [a, b] \) for some fixed \( q \geq 1 \), then for \( x \in [a, b] \), \( \lambda \in [0, 1] \) and \( \alpha > 0 \) the following inequality holds

\[
(\mathcal{A}_q)_g (x, \lambda, \alpha, a, b) \leq C_2 \left( \alpha, \lambda, 1, \frac{a}{x} \right) \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}x^2} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} + C_3 \left( \alpha, \lambda, 1, \frac{x}{b} \right) \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}b^2} \left[ \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right]^{1/q}
\]

where \( C_2 \) and \( C_3 \) are defined as in Theorem 4.

Proof. Since \( |f'|^q \) is harmonically quasi-convex on \( [a, b] \), from Lemma 1 using property of the modulus and the power-mean inequality we have

\[
|I_{f,g} (x, \lambda, \alpha, a, b)| \leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \left( \int_0^1 \left| t^\alpha - \lambda \right| A_t^\alpha(a,x) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| t^\alpha - \lambda \right| A_t^\alpha(a,x) f' \left( \frac{ax}{A_t(a,x)} \right)^q dt \right)^{\frac{1}{q}} + \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \left( \int_0^1 \left| t^\alpha - \lambda \right| A_t^\alpha(b,x) dt \right)^{1-\frac{1}{q}} \left( \int_0^1 \left| t^\alpha - \lambda \right| A_t^\alpha(b,x) f' \left( \frac{bx}{A_t(b,x)} \right)^q dt \right)^{\frac{1}{q}}
\]

\[
\leq \ C_2 \left( \alpha, \lambda, 1, \frac{a}{x} \right) \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}x^2} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} + C_3 \left( \alpha, \lambda, 1, \frac{x}{b} \right) \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}b^2} \left[ \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right]^{1/q}
\]

which completes the proof. \( \square \)

Corollary 5. Under the assumptions of Theorem 5 with \( q = 1 \), the inequality (2.6) reduced to the following inequality

\[
|I_{f,g} (x, \lambda, \alpha, a, b)| \leq \ C_2 \left( \alpha, \lambda, 1, \frac{a}{x} \right) \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}x^2} \left[ \sup \left\{ |f'(x)|, |f'(a)| \right\} \right] + C_3 \left( \alpha, \lambda, 1, \frac{x}{b} \right) \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}b^2} \left[ \sup \left\{ |f'(x)|, |f'(b)| \right\} \right]
\]
Corollary 6. Under the assumptions of Theorem 5 with \( \alpha = 1 \), the inequality (2.6) reduced to the following inequality

\[
\left( \frac{ab}{b-a} \right) |I_{f,g}(x, \lambda, \alpha, a,b) |
\]

\[
= \left| (1-\lambda) f(x) + \lambda \left[ \frac{b(x-a)f(a) + a(b-x)f(b)}{x(b-a)} \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right|
\]

\[
\leq \left( \frac{ab}{b-a} \right) \left\{ \frac{(x-a)^2}{x^2} \sup \{|f'(x)|^q, |f'(a)|^q\} \right\}^{1/q} C_2 \left( 1, \lambda, 1, \frac{a}{x} \right)
\]

\[
+ \frac{(b-x)^2}{b^2} \sup \{|f'(x)|^q, |f'(a)|^q\} \right\}^{1/q} C_3 \left( 1, \lambda, 1, \frac{b}{a} \right)
\]

specially for \( x = H = 2ab/(a+b) \), we get

\[
\left| (1-\lambda) f(H) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{b-a}{4ab}
\]

\[
\times \left\{ a^2 C_2 \left( 1, \lambda, 1, \frac{a+b}{2b} \right) \left( \sup \{|f'(H)|^q, |f'(a)|^q\} \right)^{1/q} \right\}^{1/2}
\]

\[
+ H^2 C_3 \left( 1, \lambda, 1, \frac{2a}{a+b} \right) \left( \sup \{|f'(H)|^q, |f'(b)|^q\} \right)^{1/2}
\]

Corollary 7. In Theorem 5

(1) If we take \( x = H = 2ab/(a+b) \), \( \lambda = \frac{1}{4} \), then we get the following Simpson type inequality for fractional integrals

\[
\frac{1}{6} \left[ f(a) + 4f(H) + f(b) \right] - \frac{ab}{b-a} \left[ J_{a/H+}^a \Gamma (\alpha + 1) J_{1/(a/b)}^b f \circ g (1/a) + J_{H-}^b \Gamma (\alpha + 1) J_{a/b}^a f \circ g (1/b) \right] \leq \frac{b-a}{4ab}
\]

\[
\times \left\{ a^2 \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{1/q} C_2 \left( \alpha, \frac{1}{3}, 1, \frac{a}{x} \right) \right\}^{1/2}
\]

\[
+ H^2 \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{1/q} C_3 \left( \alpha, \frac{1}{3}, 1, \frac{x}{b} \right)
\]

specially for \( \alpha = 1 \), we get

\[
\left| \frac{1}{6} \left[ f(a) + 4f(H) + f(b) \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{b-a}{4ab}
\]

\[
\times \left\{ a^2 C_2 \left( 1, \frac{1}{3}, 1, \frac{a+b}{2b} \right) \left[ \sup \{|f'(H)|^q, |f'(a)|^q\} \right]^{1/2} \right\}^{1/2}
\]

\[
+ H^2 C_3 \left( 1, \frac{1}{3}, 1, \frac{2a}{a+b} \right) \left[ \sup \{|f'(H)|^q, |f'(b)|^q\} \right]^{1/2}
\]
(2) If we take $x = H = 2ab/(a+b)$, $\lambda = 0$, then we get the following midpoint type inequality for fractional integrals
\[
\left| f(H) - \left( \frac{ab}{b-a} \right)^{\alpha} 2^\alpha - 1 \Gamma (\alpha + 1) \left[ J_{1/H+}^\alpha (f \circ g) (1/a) + J_{1/H-}^\alpha (f \circ g) (1/b) \right] \right|
\leq \frac{b-a}{4ab} \left\{ a^2 C_2 \left( \alpha, 0, 1, \frac{a+b}{2b} \right) \left[ \sup \left\{ |f'(H)|^q, |f'(a)|^q \right\} \right]^\frac{1}{q} \right.
\right.
\left. + H^2 C_3 \left( \alpha, 0, 1, \frac{2a}{a+b} \right) \left[ \sup \left\{ |f'(H)|^q, |f'(b)|^q \right\} \right]^\frac{1}{q} \right\},
\]
specially for $\alpha = 1$, we get
\[
\left| f(H) - \left( \frac{ab}{b-a} \right)^{\alpha} \int_a^b \frac{f(u)}{u^2} du \right|
\leq \frac{b-a}{4ab} \left( \frac{1}{2} \right) \left\{ a^2 H^2 H^{-2q} C_{1/q} \left( 1, 0, q, \frac{a+b}{2b} \right) \left[ \sup \left\{ |f'(2ab/a+b)|^q, |f'(a)|^q \right\} \right]^\frac{1}{q} \right.
\right.
\left. + \frac{b^2 H^2}{b^2 q} C_{3/1/q} \left( 1, 0, q, \frac{2a}{a+b} \right) \left[ \sup \left\{ |f'(2ab/a+b)|^q, |f'(b)|^q \right\} \right]^\frac{1}{q} \right\}.\]

(3) If we take $x = H = 2ab/(a+b)$, $\lambda = 1$, then we get the following trapezoid type inequality for fractional integrals
\[
\left| \frac{f(a) + f(b)}{2} - \left( \frac{ab}{b-a} \right)^{\alpha} 2^\alpha - 1 \Gamma (\alpha + 1) \left[ J_{1/H+}^\alpha (f \circ g) (1/a) + J_{1/H-}^\alpha (f \circ g) (1/b) \right] \right|
\leq \frac{b-a}{4ab} \left\{ a^2 C_2 \left( \alpha, 1, 1, \frac{a+b}{2b} \right) \left[ \sup \left\{ |f'(H)|^q, |f'(a)|^q \right\} \right]^\frac{1}{q} \right.
\right.
\left. + H^2 C_3 \left( \alpha, 1, 1, \frac{2a}{a+b} \right) \left[ \sup \left\{ |f'(H)|^q, |f'(b)|^q \right\} \right]^\frac{1}{q} \right\},
\]
specially for $\alpha = 1$, we get
\[
\left| \frac{f(a) + f(b)}{2} - \left( \frac{ab}{b-a} \right)^{\alpha} \int_a^b \frac{f(u)}{u^2} du \right|
\leq \frac{b-a}{4ab} \left\{ a^2 C_{1/q} \left( 1, 1, 1, \frac{a+b}{2b} \right) \left[ \sup \left\{ |f'(H)|^q, |f'(a)|^q \right\} \right]^\frac{1}{q} \right.
\right.
\left. + H^2 C_{3/1/q} \left( 1, 1, 1, \frac{2a}{a+b} \right) \left[ \sup \left\{ |f'(H)|^q, |f'(b)|^q \right\} \right]^\frac{1}{q} \right\}.\]

Corollary 8. Let the assumptions of Theorem 5 hold. If $|f'(x)| \leq M$ for all $x \in [a,b]$ and $\lambda = 0$, then we get the following Ostrowsky type inequality for fractional from the inequality (2.6) integrals
\[
\left| \left[ \left( \frac{x-a}{ax} \right)^{\alpha} + \left( \frac{b-x}{bx} \right)^{\beta} \right] f(x) - \left( \frac{ab}{b-a} \right)^{\alpha} 2^\alpha - 1 \Gamma (\alpha + 1) \left[ J_{1/x+}^\alpha (f \circ g) (1/a) + J_{1/x-}^\alpha (f \circ g) (1/b) \right] \right|
\leq M \left[ \left( \frac{x-a}{ax} \right)^{\alpha+1} C_2 \left( \alpha, 0, 1, \frac{a}{x} \right) + \left( \frac{b-x}{bx} \right)^{\alpha+1} C_3 \left( \alpha, 0, 1, \frac{b}{x} \right) \right].
Theorem 6. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \(|f'|^q\) is harmonically quasi-convex on \([a, b]\) for some fixed \( q > 1 \), then for \( x \in [a, b], \lambda \in [0, 1] \) and \( \alpha > 0 \) the following inequality holds

\[
(2.7) \quad |I_{f,g}(x, \lambda, \alpha, a, b)| \leq C_1^{1/q}(\alpha, \lambda) \left\{ C_2^{1/p}(\alpha, \lambda, a, b) \left( \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} \right) \right. \\
\left. + C_3^{1/p}(\alpha, \lambda, \frac{x}{b}) \left( \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \left[ \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right]^{1/q} \right) \right\}
\]

where \( C_1, C_2 \) and \( C_3 \) are defined as in Theorem 3.

Proof. Since \(|f'|^q\) is harmonically quasi-convex on \([a, b]\), from Lemma 4 using property of the modulus and the Hölder inequality we have

\[
|S_f(x, \lambda, \alpha, a, b)| \leq \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \left( \int_0^1 \frac{|\alpha - \lambda|}{\Lambda' \alpha (a, x)} \, dt \right)^{\frac{1}{q}} \left( \int_0^1 \frac{t^\alpha - \lambda}{t^\alpha} \left| f' \left( \frac{ax}{\Lambda'(a, x)} \right) \right|^q \, dt \right)^{\frac{1}{q}} \\
\left. + \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \left( \int_0^1 \frac{|\alpha - \lambda|}{\Lambda' \alpha (b, x)} \, dt \right)^{\frac{1}{q}} \left( \int_0^1 \frac{t^\alpha - \lambda}{t^\alpha} \left| f' \left( \frac{bx}{\Lambda'(b, x)} \right) \right|^q \, dt \right)^{\frac{1}{q}} \right.
\]

\[
\leq C_1^{1/q}(\alpha, \lambda) \left\{ C_2^{1/p}(\alpha, \lambda, a, b) \left( \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha-1}} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} \right) \right. \\
\left. + C_3^{1/p}(\alpha, \lambda, \frac{x}{b}) \left( \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1}} \left[ \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right]^{1/q} \right) \right\}
\]

which completes the proof. \( \square \)

Corollary 9. Under the assumptions of Theorem 4 with \( \alpha = 1 \), the inequality (2.7) reduced to the following inequality

\[
\left( \frac{ab}{b-a} \right) |I_{f,g}(x, \lambda, \alpha, a, b)| \leq \left( 1 - \lambda \right) f(x) + \lambda \left[ \frac{b(x-a)f(a) + a(b-x)f(b)}{x(b-a)} \right] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \]

\[
\leq \left( \frac{ab}{b-a} \right) \left( \frac{2\lambda^2 - 2\lambda + 1}{2} \right)^{\frac{1}{2}} \left\{ \frac{(x-a)^2}{x^2} \left[ \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right]^{1/q} C_2^{1/p}(1, \lambda, a) \right. \\
\left. + \frac{(b-x)^2}{b^2} \left[ \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right]^{1/q} C_3^{1/p}(1, \lambda, \frac{x}{b}) \right\},
\]
specially for \( x = H = 2ab/(a+b) \), we get

\[
\left| (1 - \lambda) f(H) + \lambda \left( \frac{f(a) + f(b)}{2} \right) - \frac{ab}{b-a} \int_a^b \frac{f'(u)}{u^2} \, du \right| \leq \frac{b-a}{4ab} \left( \frac{\lambda^2 - 2\lambda + 1}{2} \right) \frac{1}{2}^{\frac{1}{2}}
\]

\[
\times \left\{ \frac{a^2 H^2}{H^2 p} C_2^{1/p} \left( 1, \frac{1}{3}, \frac{a+b}{2b} \right) (\sup \{ |f'(H)|^q, |f'(a)|^q \}) \right\}^{\frac{1}{2}}
\]

\[
+ \frac{b^2 H^2}{b^2 p} C_3^{1/p} \left( 1, \frac{1}{3}, \frac{2a}{a+b} \right) \left( \sup \{ |f'(H)|^q, |f'(b)|^q \} \right) \frac{1}{2}^{\frac{1}{2}} \right\}.
\]

**Corollary 10.** In Theorem 7

(1) If we take \( x = H = 2ab/(a+b) \), \( \lambda = \frac{1}{2} \), then we get the following Simpson type inequality for fractional integrals

\[
\left( \frac{ab}{b-a} \right)^\alpha |J_{f,g}(x, \lambda, \alpha, a, b)|
\leq \frac{b-a}{4ab} C_1^{1/\alpha} \left( \alpha, 1 \right) \left\{ \frac{a^2 H^2}{H^2 p} (\sup \{ |f'(H)|^q, |f'(a)|^q \}) \right\}^{1/\alpha} C_2^{1/p} \left( \alpha, 1, \frac{1}{3}, \frac{a+b}{2b} \right)
\]

\[
+ \frac{b^2 H^2}{b^2 p} C_3^{1/p} \left( 1, \frac{1}{3}, \frac{2a}{a+b} \right) \left( \sup \{ |f'(H)|^q, |f'(b)|^q \} \right) \frac{1}{2}^{\frac{1}{2}} \right\},
\]

specially for \( \alpha = 1 \), we get

\[
\left| \frac{1}{6} [f(a) + 4f(H) + f(b)] - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \right| \leq \frac{b-a}{4ab} \left( \frac{5}{18} \right) \frac{1}{2}^{\frac{1}{2}}
\]

\[
\times \left\{ \frac{a^2 H^2}{H^2 p} C_2^{1/p} \left( 1, \frac{1}{3}, \frac{a+b}{2b} \right) [\sup \{ |f'(H)|, |f'(a)| \}] \right\}^{\frac{1}{2}}
\]

\[
+ \frac{b^2 H^2}{b^2 p} C_3^{1/p} \left( 1, \frac{1}{3}, \frac{2a}{a+b} \right) [\sup \{ |f'(H)|^q, |f'(b)|^q \}] \frac{1}{2}^{\frac{1}{2}} \right\}.
\]

(2) If we take \( x = H = 2ab/(a+b) \), \( \lambda = 0 \), then we get the following midpoint type inequality for fractional integrals

\[
|f(H) - \left( \frac{ab}{b-a} \right)^\alpha 2^{\alpha-1} \Gamma (\alpha + 1) \left[ J_H^\alpha (f \circ g) (1/a) + J_{H-}^\alpha (f \circ g) (1/b) \right] |
\leq \frac{b-a}{4ab} \left( \frac{1}{\alpha + 1} \right)^{\frac{1}{2}} \left\{ \frac{a^2 H^2}{H^2 p} C_2^{1/p} \left( \alpha, 0, \frac{a+b}{2b} \right) [\sup \{ |f'(H)|^q, |f'(a)|^q \}] \right\}^{\frac{1}{2}}
\]

\[
+ \frac{b^2 H^2}{b^2 p} C_3^{1/p} \left( \alpha, 0, \frac{2a}{a+b} \right) [\sup \{ |f'(H)|^q, |f'(b)|^q \}] \frac{1}{2}^{\frac{1}{2}} \right\},
\]

specially for \( \alpha = 1 \), we get

\[
\left| f(H) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} \, du \right|
\leq \frac{b-a}{4ab} \left( \frac{1}{2} \right)^{\frac{1}{2}} \left\{ \frac{a^2 H^2}{H^2 p} C_2^{1/p} \left( 1, 0, \frac{a+b}{2b} \right) [\sup \{ |f'(H)|^q, |f'(a)|^q \}] \right\}^{\frac{1}{2}}
\]

\[
+ \frac{b^2 H^2}{b^2 p} C_3^{1/p} \left( 1, 0, \frac{2a}{a+b} \right) [\sup \{ |f'(H)|^q, |f'(b)|^q \}] \frac{1}{2}^{\frac{1}{2}} \right\}.
(3) If we take $x = H = 2ab/(a+b)$, $\lambda = 1$, then we get the following trapezoid type inequality for fractional integrals

$$\left| \frac{f(a) + f(b)}{2} - \left( \frac{ab}{b-a} \right)^\alpha \Gamma (\alpha + 1) \frac{1}{2^\alpha} \left[ J_{f(x)}^\alpha (f \circ g) (1/a) + J_{f(x)}^\alpha (f \circ g) (1/b) \right] \right| \leq \frac{b-a}{4ab} \left( \frac{\alpha}{\alpha + 1} \right)^\frac{1}{p} \left\{ \frac{a^2H^2}{H^{2p}} C_2^{1/p} \left( \sup \left\{ |f''(H)|^q, |f''(a)|^q \right\} \right) \right\}^\frac{1}{q}$$

specially for $\alpha = 1$, we get

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b f(u) \, du \right| \leq \frac{b-a}{4ab} \left( \frac{1}{2} \right)^\frac{1}{p} \left\{ \frac{a^2H^2}{H^{2p}} C_2^{1/p} \left( \sup \left\{ |f''(H)|^q, |f''(a)|^q \right\} \right) \right\}^\frac{1}{q}$$

Corollary 11. Let the assumptions of Theorem 6 hold. If $|f'(x)| \leq M$ for all $x \in [a,b]$ and $\lambda = 0$, then we get the following Ostrowski type inequality for fractional integrals

$$\left| \left( \frac{x-a}{ax} \right)^\alpha + \left( \frac{b-x}{bx} \right)^\alpha \right| f(x) - \left( \frac{ab}{b-a} \right)^\alpha \Gamma (\alpha + 1) \frac{1}{2^\alpha} \left[ J_{f(x)}^\alpha (f \circ g) (1/a) + J_{f(x)}^\alpha (f \circ g) (1/b) \right] \right| \leq \frac{M}{(\alpha + 1)^\frac{1}{p}} \left[ \frac{(x-a)^{\alpha+1}}{(ax)^{\alpha+1-2p}} C_2^{1/p} \left( \alpha, 0, p, \frac{a}{x} \right) + \frac{(b-x)^{\alpha+1}}{(bx)^{\alpha-1+2p}} C_3^{1/p} \left( \alpha, 0, p, \frac{x}{b} \right) \right]$$

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