Some algebraic aspects of the Turaev cobracket

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Abstract

The Turaev cobracket, a loop operation introduced by V. Turaev [50], which measures self-intersection of a loop on a surface, is a modification of a path operation introduced earlier by Turaev himself [49], as well as a counterpart of the Goldman bracket [21]. In this survey based on the author’s joint works with A. Alekseev, Y. Kuno and F. Naef, we review some algebraic aspects of the cobracket and its framed variants including their formal description, an application to the mapping class group of the surface and a relation to the (higher genus) Kashiwara-Vergne problem. In addition, we review a homological description of the cobracket after R. Hain [23].

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1 Introduction

Let $S$ be a connected oriented surface. By a surface we mean a 2-dimensional $C^\infty$ manifold possibly with boundary. The free homotopy set of free loops on $S$, $[S^1, S]$, which we denote by $\hat{\pi} = \hat{\pi}(S)$, is naturally identified with the set of conjugacy classes of the fundamental group $\pi_1(S)$, $\pi_1(S)/\text{conj}$. Goldman [21] introduced a binary loop operation on $\hat{\pi}$, which we call the Goldman bracket, on the free $\mathbb{Z}$ module over the set $\hat{\pi}$, $\mathbb{Z}\hat{\pi}$, as a topological extract of the Poisson structure on the moduli space of flat bundles over the surface $S$. Earlier than the Goldman bracket, Turaev [49] introduced a loop operation on the fundamental group $\pi_1(S)$. The Turaev cobracket [50] is its free loop version as well as a counterpart of the Goldman bracket. The cobracket together with the Goldman bracket makes the quotient of $\mathbb{Z}\hat{\pi}$ by the linear span of the trivial loop a Lie bialgebra [50]. We consider framed versions of the Turaev cobracket which make $\mathbb{Z}\hat{\pi}$ itself a Lie bialgebra.

Yusuke Kuno and the author have studied these loop operations and their applications to the mapping class group of a surface [31, 32, 33, 34]. There the formal description of the Goldman bracket through a special expansion plays a fundamental role. The formal description of the framed Turaev cobracket was desirable but in vain. In the genus 0 case it was carried out by Massuyeau [36]. To obtain the complete realization of the formal description, it was essential for us to collaborate with Anton Alekseev and Florian Naef. Here the key concept is the Kashiwara-Vergne problem [28, 8]. Alekseev-Kawazumi-Kuno-Naef proved that the set of special expansions which induce the formal description of the framed Turaev cobracket has a one-to-one correspondence with the set of solutions to the Kashiwara-Vergne problem. More precisely, the Kashiwara-Vergne problem involves genus 0 surfaces. We first proved the correspondence for the genus 0 case [3]. This leads us to formulate a positive genus version of the Kashiwara-Vergne problem [23, 8]. In [4] we proved the correspondence for all positive genus surface. Moreover we proved the existence of solutions to the positive genus version except some genus 1 cases, where there is no solution.

The present paper sketches this story, and is a continuation of the author’s joint survey with Kuno about the Goldman-Turaev Lie bialgebra and its applications to the mapping class group [34]. Independently, Hain is writing a survey on this topic [24]. His approach is ‘motivic’, i.e. based on his mixed Hodge theory, while our approach comes from low-dimensional topology. If the readers read both of Hain’s and the present surveys, their understanding of this topic must be deepened.

In [2] we recall the definition of the Turaev cobracket and its framed variants. The Turaev cobracket is defined for a generic immersed loop on the surface. It is invariant under regular homotopy, but not under homotopy. In fact, the birth-death move ($\omega 1$) of a monogon changes the value of the Turaev cobracket. This is the reason why we have to take the quotient by the linear span of the trivial loop. To avoid to take the quotient, we consider a framed version of the cobracket. We need the coefficient of the trivial loop in the framed Turaev cobracket to get the Enomoto-Satoh trace [19] and the Alekseev-Torossian divergence cocycle [8].
A straight-forward computation can show the regular homotopy invariance of the Turaev cobracket. But it is proved also by a homological description of the cobracket. Following Hain 23 we will give it in §3. Hain used his own description to prove that the framed cobracket is a morphism of mixed Hodge structures. Combining this result and our theorems (Theorems 5.2 and 5.6), he deduces the existence of solutions to the Kashiwara-Vergne problem except the genus 1 exception 23.

§4 has some overlaps with our previous survey paper 34. We formulate our formality problems in §4.1. §4.2 is devoted to a review on the formality of the Goldman bracket. An application to the mapping class group is mentioned in §4.3. A geometric version τ of the Johnson homomorphisms is introduced by the logarithm of mapping classes. We will give an alternative proof of the vanishing of the composite of the framed Turaev cobracket and the homomorphism τ on the Johnson group Kg,1.

We survey the results in 2, 3, 4 in §5. In §5.1 we sketch the proof of the correspondence between the formality of the Turaev cobracket and the Kashiwara-Vergne problem in genus 0. The formulation of our theorems in positive genus is stated in §5.2.

In this paper we confine ourselves to a compact connected oriented surface. Let $S$ be a compact connected oriented surface with non-empty boundary. It is classically known that $S$ is classified as an oriented $C^\infty$ manifold by its genus and the number of its boundary components. So we denote by $\Sigma_{g,n+1}$ a compact connected oriented surface of genus $g$ with $n+1$ boundary components. Such an $S$ is diffeomorphic to $\Sigma_{g,n+1}$ for some $g, n \geq 0$. The fundamental group $\pi_1(\Sigma_{g,n+1})$ is a free group of rank $2g+n$.

We conclude this section by introducing a symbol associated with an associative algebra. Let $A$ be an associative (topological) algebra over a commutative ring $R$ with unit. Then we denote by $|A|$ the quotient of $A$ by the commutator $[A, A]$, (the closure of) the $R$ submodule generated by the set $\{ab - ba; \ a, b \in A\}$, and by $|\cdot| : A \to |A|$, $a \mapsto |a|$, the quotient map. If we regard $A$ as a Lie algebra by the commutator $[a, b] := ab - ba$, $a, b \in A$, then $|A|$ equals the abelianization of the Lie algebra $A$. A priori, the quotient $|A|$ is just an $R$ module. For example, if $A = \widehat{T}(V) = \prod_{m=0}^{\infty} V^{\otimes m}$, the completed tensor algebra generated by a vector space $V$ over a field of characteristic 0, then the quotient $|A|$ equals the cyclic coinvariants in the completed tensor algebra: $|A| = \prod_{m=0}^{\infty} (V^{\otimes m})_{\mathbb{Z}/m}$, where the cyclic group $\mathbb{Z}/m$ acts on the space $V^{\otimes m}$ by cyclic permutation of the slots. Moreover, if $G$ is a group, we have $|RG| = R(G/\text{conj})$, the free $R$ module generated by the set of conjugacy classes in $G$, $G/\text{conj}$. For example, we have $|R\pi_1(S)| = R\tilde{\pi}(S)$.

First of all, the author thanks Athanase Papadopoulos for giving him a chance to take part in a tribute to Vladimir Turaev, whose works have great influences in mathematics in the twentieth and twenty-first centuries. Next he thanks Florian Naef for correction and simplification of arguments in 33. In particular, the proof of homotopy invariance of the framed Turaev cobracket in 3.2 is due to him. Finally he thanks Yusuke Kuno, who kindly read the whole manuscript carefully, and let the author know some typos and helpful suggestions for improvement in this survey.

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2 The Turaev cobracket and its framed variants

2.1 The original definition of the Turaev cobracket

Let $S$ be a connected oriented surface, i.e., a connected oriented 2-dimensional $C^\infty$ manifold possibly with boundary. We denote by $\bar{\pi}$ the free homotopy set of free loops on $S$, $\bar{\pi} = \bar{\pi}(S) := \{\pi_1, S\}$. Since $S$ is connected, it equals the set of conjugacy classes in the fundamental group $\pi_1(S)$, $\bar{\pi} = \pi_1(S)/\text{conj}$. Moreover, if we denote by $\text{Int}\, S$ the interior of $S$, then we have $\bar{\pi}(S) = \bar{\pi}(\text{Int}\, S)$. For any commutative ring $R$ with unit, we have $|R\pi_1(S)| = R\bar{\pi}(S)$, where the latter means the free $R$ module generated by the free homotopy set $\bar{\pi}$.

In general, any continuous map from a 1-dimensional manifold $T$ to the surface $S$ is homotopic to a generic immersion, a $C^\infty$ immersion at worst with transverse double points. Two generic immersions are homotopic if and only if they are deformed to each other by iteration of the following 3 kinds of moves $(\omega_1), (\omega_2), (\omega_3)$ and isotopy: The move $(\omega_1)$ is a birth-death of a monogon coming from a cusp, $(\omega_2)$ a birth-death of a bigon coming from a tangential double point, and $(\omega_3)$ a jumping over a double point coming from a transverse triple point. For details, see [21] 5.6 Lemma.

The moves $(\omega_2)$ and $(\omega_3)$ are regular homotopy, but $(\omega_1)$ is not so. One can slide a monogon along the given generic immersed loop by using the moves $(\omega_2), (\omega_3)$ and isotopy. Hence two generic immersed loops are regular homotopic if and only if they are deformed to each other by iteration of the moves $(\omega_2), (\omega_3)$ and isotopy.

Now we recall the definition of the Turaev cobracket for a generic immersed loop $\alpha : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to S$. We denote by $\partial_\alpha$ the set of pairs of parameters mapped to double points: $D_\alpha := \{(t_1, t_2) \in S^1 \times S^1; \alpha(t_1) = \alpha(t_2), t_1 \neq t_2\}$. Then the Turaev cobracket $\delta\alpha \in \mathbb{Z}\bar{\pi} \otimes \mathbb{Z}\bar{\pi} = [\mathbb{Z}\bar{\pi}_1(S)] \otimes [\mathbb{Z}\bar{\pi}_1(S)]$ is defined by

$$\delta\alpha := \sum_{(t_1, t_2) \in D_\alpha} \varepsilon(\hat{\alpha}(t_1), \hat{\alpha}(t_2))(\alpha|_{[t_1, t_2]} \otimes (\alpha|_{[t_2, t_1]}) \in \mathbb{Z}\bar{\pi} \otimes \mathbb{Z}\bar{\pi}.$$  

Here $\varepsilon(\hat{\alpha}(t_1), \hat{\alpha}(t_2)) \in \{\pm 1\}$ is the local intersection number of the velocity vectors with respect to the orientation of the surface $S$, and $\alpha|_{[t_1, t_2]} \in \bar{\pi}$ is the free homotopy class of the segment of $\alpha$ restricted the interval on $S^1$ running from $t_1$ to $t_2$ in the positive direction. One can prove that $\delta\alpha$ is invariant under the moves $(\omega_2), (\omega_3)$ and isotopy. This follows from straight-forward arguments, and will be proved later in §3.2 by using twisted homology. Hence the operation $\delta$ is regular homotopy invariant. The following is a typical example of the Turaev cobracket.

**Lemma 2.1.** For any simple closed curve $C$ in the interior $\text{Int}\, S$ and $m \in \mathbb{Z}$, we have $\delta(C^m) = 0 \in [\mathbb{Z}\pi] \otimes [\mathbb{Z}\pi]$.

*Proof.* Since $C^{-1}$ is also a simple closed curve and $C^0$ is trivial, it suffices to show the lemma for $m > 0$. As a representative for $C^m$, we can take a spiral with $m - 1$ self-intersection points. Hence $\delta(C^m)$ equals $\sum_{k=1}^{m-1} C^k \otimes C^{m-k} - C^{m-k} \otimes C^k = 0$, which proves the lemma. \qed
It is easy to see that $\delta$ is not invariant under the move $(\omega_1)$. For example, if we insert a monogon in the positive direction to $\alpha$, then we have to add an extra term $\alpha \otimes 1 - 1 \otimes \alpha$ to the original $\delta \alpha$. To avoid this ambiguity, Turaev’s original definition [50] takes the quotient $\hat{\mathbb{Z}}/\mathbb{Z}$. Here $1 = |1| \in \hat{\pi}$ is the free homotopy class of the constant loop $1$. Then $\delta \alpha \mod \mathbb{Z}$ is homotopy invariant, that is, the cobracket

$$\delta: \hat{\mathbb{Z}}/\mathbb{Z} \to (\hat{\mathbb{Z}}/\mathbb{Z})^\otimes 2$$

is a well-defined operation. Moreover Turaev [50] observed that the cobracket together with the Goldman bracket $[-,-]$ makes the quotient $\hat{\mathbb{Z}}/\mathbb{Z}$ a Lie bialgebra in the sense of Drinfel’d [17]. Since the constant loop $1$ is in the center of the Goldman bracket, the Goldman bracket descends to the quotient $\hat{\mathbb{Z}}/\mathbb{Z}$. In particular, the cobracket satisfies the compatibility condition with the bracket, i.e.,

$$\delta[\alpha, \beta] = \text{ad}(\alpha)(\delta \beta) - \text{ad}(\beta)(\delta \alpha) \in (\hat{\mathbb{Z}}/\mathbb{Z})^\otimes 2 \quad (1)$$

for any $\alpha$ and $\beta \in \hat{\pi}$. Here $\text{ad}(\alpha) \in \text{End}((\hat{\mathbb{Z}}/\mathbb{Z})^\otimes 2)$ is the adjoint action defined by $\text{ad}(\alpha)(\beta_1 \otimes \beta_2) := [\alpha, \beta_1] \otimes \beta_2 + \beta_1 \otimes [\alpha, \beta_2]$ for $\beta_1, \beta_2 \in \hat{\pi}$. This condition can be regarded as the cocycle condition of $\delta$. Thus the Lie bialgebra $(\hat{\mathbb{Z}}/\mathbb{Z}, [-,-], \delta)$ is called the Goldman-Turaev Lie bialgebra of the surface $S$. It is naturally isomorphic to that of the interior $\text{Int} S$.

Later Chas [10] proved the Lie bialgebra is involutive. This survey covers only some algebraic aspects of the Turaev cobracket. But we should remark that geometric approach to the cobracket and the Goldman bracket by Chas and her coworkers yields fruitful results on surface topology [10, 11, 12, 13, 14, 15].

2.2 Framed Turaev cobrackets

As will be reviewed in [15], the Turaev cobracket vanishes on the logarithms of mapping classes of the surface $S$. So we can derive a constraint of the Johnson image from the cobracket. It includes the Morita trace [29], but not the Enomoto-Satoh trace. In order to recover the Enomoto-Satoh trace [19] we need to define a variant of the Turaev cobracket with values in $|K\pi|$ itself [29]. To do this, we consider a regular homotopy variant of the Turaev cobracket. In [50, §18] Turaev already studied such a variant. But we adopt a slightly different formulation using a framing of the surface $S$.

From now on, we assume $S$ is compact with non-empty boundary. In particular, the tangent bundle $TS$ is trivial. We mean by a framing the homotopy class of an orientation-preserving isomorphism $TS \cong S \times \mathbb{R}^2$ of vector bundles over $S$. The set $F(S)$ of all framings is an affine set modeled on the first cohomology group $H^1(S; \mathbb{Z})$.

For any framing $f \in F(S)$ one define the rotation number $\text{rot}_f \alpha \in \mathbb{Z}$ of an immersed loop $\alpha: S^1 \to S$ with respect to $f$. It is defined by the mapping degree

$$\text{rot}_f \alpha := \deg \left( S^1 \xrightarrow{\alpha} TS \setminus O(S) \cong S \times (\mathbb{R}^2 \setminus \{0\}) \xrightarrow{\text{proj}} \mathbb{R}^2 \setminus \{0\} \right) \in \mathbb{Z}.$$
The move \((\omega 1)\) means an insert of a monogon. If we insert a monogon in the positive direction to \(\alpha\), then \(\text{rot}_f \alpha\) increases by +1, while we have to add an extra term \(\alpha \otimes 1 - 1 \otimes \alpha\) to the original \(\delta \alpha\). Hence \(\delta \alpha + (\text{rot}_f \alpha)(1 \otimes \alpha - \alpha \otimes 1)\) is invariant under \((\omega 1)\), so is homotopy invariant, by which we define the \textit{framed Turaev cobracket} \(\delta^f\) with respect to a framing \(f\)

\[
\delta^f : |\mathbb{Z}\pi| \to |\mathbb{Z}\pi| 
\]

In other words, \(\delta^f \alpha\) is defined to be the \(\delta\) of a representative of \(\alpha\) whose rotation number \(\text{rot}_f\) vanishes. Similarly \((|\mathbb{Z}\pi|, [-,-], \delta^f)\) is an involutive Lie bialgebra. In particular, we have

**Lemma 2.2.** \(\delta^f\) is a cocycle with respect to the Goldman bracket, i.e.,

\[
\delta^f([\alpha, \beta]) = \text{ad}(\alpha)(\delta^f \beta) - \text{ad}(\beta)(\delta^f \alpha)
\]

for any \(\alpha, \beta \in \hat{\pi}\).

Thus we obtain the \textit{framed Goldman-Turaev Lie bialgebra} \((|\mathbb{Z}\pi|, [-,-], \delta^f)\) of the framed surface \((S, f)\). Since \(\delta^f(1) = 0\), \(\delta^f\) can be regarded as a map \(|\mathbb{Z}\pi|/\mathbb{Z}1 \to |\mathbb{Z}\pi| 
\|2\).

**Theorem 2.3** \((\mathbb{25} 30)\). Two framings \(f\) and \(f'\) of the surface \(S\) are in the same \(\mathcal{M}(S)\)-orbit if and only if

- All the rotation numbers along boundary loops coincide.
- The induced spin structures \(\mathfrak{s}_f\) and \(\mathfrak{s}_{f'}\) are in the same \(\mathcal{M}(S)\)-orbit.
- \(\tilde{A}(f) = \tilde{A}(f')\).

If \(g \geq 2\), the theorem is essentially due to Johnson \(\mathbb{25}\), where he proved that a spin structure is classified by the Arf invariant modulo the mapping class group \(\mathcal{M}(S)\). The orbit space \(F(S)/\mathcal{M}(S)\) with prescribed boundary data is an infinite set if and only if \(g = 1\) and the rotation number of each boundary loop equals –1. Only in this case, it is possible that there is no solution of the Kashiwara-Vergne problem associated with the framed surface (Theorem \(\mathbb{5.5}\)).
We can consider the relative version for Theorem 2.3 i.e., where our homotopy fixes a prescribed framing \(TS|_{\partial S} \cong \partial S \times \mathbb{R}^2\) on the boundary. This version was already established by Randal-Williams [42] for \(g \geq 2\). See §2.4 in [30] for \(g = 1\).

3 Homological descriptions of the cobracket

In this section we give a homological description of the Turaev cobracket after R. Hain. Our approach is slightly different from Hain’s original one [23].

3.1 A relative twisted chain \(s_a\)

We denote \(\mathbb{R}_{>0} := [0, +\infty[\) and \(\mathbb{R}_{\geq 0} := [0, +\infty[ \subset \mathbb{R}\). Let \(\pi : E \to B\) be a \(C^\infty\) \(\mathbb{R}\) vector bundle of rank 2 with Riemannian metric \(\| \cdot \|\). We identify the base space \(B\) with the zero section in the total space \(E\). The multiplicative group \(\mathbb{R}_{>0}\) acts on the complement \(E_0 := E \setminus B = \{e \in E; \|e\| \neq 0\}\) by scalar multiplication. The orbit space of the action \(S(E) := E_0/\mathbb{R}_{\geq 0} \cong \{e \in E; \|e\| = 1\}\) is a \(S^1\) bundle over \(B\). For any \(x \in E_0\), we denote \([x] := x \mod \mathbb{R}_{>0} \in S(E)\) and \([x]_0 := \mathbb{R}_{\geq 0} \cdot x \in E\). Then the space

\[
Q(E) := \{([x], e) \in S(E) \times E; e \in \{x\}_0\}
\]

is naturally diffeomorphic to the product \(S(E) \times \mathbb{R}_{\geq 0}\) by the map \(([x], \lambda) \in S(E) \times \mathbb{R}_{\geq 0} \mapsto ([x], \lambda ||x||^{-1}x) \in Q(E)\), and admits a natural projection \(\varpi : Q(E) \to E, ([x], e) \mapsto e\), which is the real oriented blow-up of \(E\) along \(B\). If \(\partial B = \emptyset\), we have \(\partial Q(E) = S(E) \times \{0\}\).

In this section we write simply \(M = \text{Int} S\), the interior of the surface \(S\). We denote by \(\Delta_M \subset M \times M\) the diagonal set, whose normal bundle is given by

\[
E := N_{M \times M/\Delta_M} = (TM \times_M TM)/\Delta(TM) \cong TM, \quad (u, v) \mapsto v - u.
\]

Choose a tubular neighborhood of the diagonal \(h : E \to M \times M\). Then we glue the space \(Q(E)\) with the configuration space \(C_2(M) := M \times M \setminus \Delta_M\) by the map \(h\):

\[
\hat{C}_2(M) := Q(E) \cup_h C_2(M),
\]

which admits a natural projection \(\varpi : \hat{C}_2(M) \to M \times M\) induced by the map \(Q(E) \xrightarrow{\cong} E \xrightarrow{h} h(E) \subset M \times M\).

We identify \(\varpi^{-1}(\Delta_M)\) with \(S(TM)\) as above. Let \(f\) be a framing of \(S\). We denote by \(f_f := f^*1_{S^1} \in H^1(S(TM); \mathbb{Z})\) the pullback of the positive generator \(1_{S^1} \in H^1(S^1; \mathbb{Z})\) by the framing \(f\). Hain uses the symbol \(\xi\) for a framing, and writes \(f_f\) for the cohomology class \(f_f\) in [23, §3]. On the other hand, we introduce a local system \(S\) on \(M \times M\) whose stalk is defined by

\[
S_{(p_1, p_2)} := \mathbb{Z}\Pi M(p_1, p_2) \otimes \mathbb{Z}\Pi M(p_2, p_1)
\]

for any \((p_1, p_2) \in M \times M\). Here we denote by \(\Pi M(p_1, p_2)\) the homotopy set of paths \(((0, 1], 0, 1), (M, p_1, p_2)]\). As in the previous section, we denote \(\hat{\pi} = [S^1, M] = [S^1, S]\). Then one can define a natural map \(\nu : H_0(S(TM); \varpi^*S) \to \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}\), and the composite

\[
H_2(\hat{C}_2(M), S(TM); \varpi^*S) \xrightarrow{\partial} H_1(S(TM); \varpi^*S) \xrightarrow{\nu} H_0(S(TM); \varpi^*S) \xrightarrow{\nu} \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}. \quad (2)
\]
Then a lift $\hat{\alpha}$ hand,

$$(0 \text{ does not lift to } \hat{\alpha})$$

The first line makes sense near the diagonal $\Delta$, $\alpha, \alpha \in \Delta$, we consider the map $\alpha \alpha$ and the map $\hat{\alpha}$.

Lemma 3.1. The lift $\hat{\alpha}$ in $\tilde{T}$ is diffeomorphic to the circle $S^1 = \mathbb{R}/2\pi \mathbb{Z}$. We choose the map $\Delta V \times V = V \times V \rightarrow V \times V$.

Moreover we assume the vectors $\hat{\alpha}(t) = (t, t)$ and $\hat{\alpha}(t) = (t, t)$ near $(t, t)$. We prove this proves the map $\hat{\alpha}$ is well-defined.

Next we consider two curves intersecting transversely at a point. Let $I \subset \mathbb{R}$ be an interval including the origin $0 \in \mathbb{R}$, and $\alpha_1(0) = \alpha_2(0) = p_0 \in V$. Moreover we assume the vectors $\hat{\alpha}(0)$ and $\hat{\alpha}(0)$ are linearly independent in $T_{p_0} V$ with local intersection number $\varepsilon_{p_0} = \varepsilon(\alpha_1(0), \alpha_2(0)) \in \{\pm 1\}$. Then we consider the map $\alpha := (\alpha_1, \alpha_2) : I \times I \rightarrow V \times V$, $(t_1, t_2) \mapsto (\alpha_1(t_1), \alpha_2(t_2))$. The map $\alpha$ does not lift to $\hat{\alpha}(V)$. We need to take a (real oriented) blow-up of $I \times I$ at the point $(0, 0)$.

Then a lift $\hat{\alpha} : I \times I \rightarrow \hat{\alpha}(V)$ is defined by

$$\hat{\alpha}(t_1, t_2) := \begin{cases} ([\alpha_2(t_2) - \alpha_1(t_1)], \alpha_1(t_1), \alpha_2(t_2)), & \text{if } (t_1, t_2) \neq (0, 0), \\ ([-(\cos \theta)\hat{\alpha}_1(t_1) + (\sin \theta)\hat{\alpha}_2(t_1), p_0, p_0), & \text{if } (t_1, t_2) = (0, 0). \end{cases}$$
The second line makes sense since the vectors $\dot{\alpha}_1(0)$ and $\dot{\alpha}_2(0)$ are linearly independent. In order to check the smoothness near $S^1 \times \{(0,0)\}$, we use a local coordinate $[0,\delta[\times S^1 \to \hat{I} \times I$, $(\rho, \theta) \mapsto (\theta, \rho \cos \theta, \rho \sin \theta)$ for a sufficient small $\delta > 0$, and recall the formula $\alpha_i(t) - p_0 = t \int_0^1 \dot{\alpha}_i(st) ds, i = 1, 2$. Then $[\alpha_2(\rho \sin \theta) - \alpha_1(\rho \cos \theta)]$ equals

\[
[(\sin \theta) \int_0^1 \dot{\alpha}_2(s \rho \sin \theta) ds - (\cos \theta) \int_0^1 \dot{\alpha}_1(s \rho \cos \theta) ds] \in S(V)
\]

which is $C^\infty$ in $(\rho, \theta)$ near $S^1 \times \{(0,0)\}$. This proves the smoothness of the map $\hat{\alpha}$. It should be remarked that the map $\hat{\alpha}$ is uniquely determined by $\alpha$. In fact, it must coincide with $(\alpha_1, \alpha_2)$ on the complement $\hat{I} \times I \setminus (S^1 \times \{(0,0)\})$ which is dense in $\hat{I} \times I$.

Now we go back to the situation where we defined the framed Turaev cobracket. Let $M$ and $f$ be as above, and $\alpha : S^1 \to M$ a generic $C^\infty$ immersion. The set parametrizing double points of $\alpha$

\[
D_\alpha = \{(t_1, t_2) \in S^1 \times S^1 \setminus \Delta_{S^1}; \alpha(t_1) = \alpha(t_2)\}
\]

is a finite set. Moreover, using the identification $S^1 = \mathbb{R}/2\pi \mathbb{Z}$, we introduce the map

$q : S^1 \times [0,2\pi] \to S^1 \times S^1, (\theta, t) \mapsto (\theta, \theta + t)$, and we take the real oriented blow-up of $S^1 \times [0,2\pi]$ at all points in the set $q^{-1}(D_\alpha), Q_\alpha(S^1 \times [0,2\pi])$. Then we have a canonical lift $\tilde{\alpha} : Q_\alpha(S^1 \times [0,2\pi]) \to \hat{C}_2(M)$ of the map $(\alpha, \alpha) \circ q : S^1 \times [0,2\pi] \to M \times M, (t_1, t_2) \mapsto (\alpha(t_1), \alpha(t_2))$. Here the lift near $S^1 \times \{0\}$ (resp. $S^1 \times \{2\pi\}$) comes from $\hat{\alpha}_+$ (resp. $\hat{\alpha}_-$) in Lemma 3.1.

A canonical horizontal lift $\tilde{s}_\alpha : S^1 \times [0,2\pi] \to S$ of the map $(\alpha, \alpha) \circ q : S^1 \times [0,2\pi] \to M \times M$ is defined by

\[
\tilde{s}_\alpha(\theta, t) := \begin{cases} 
\alpha|_{[\theta, \theta+t]} \otimes \alpha|_{[\theta+t, \theta]}, & \text{if } t \neq 0, 2\pi, \\
1 \otimes \alpha, & \text{if } t = 0, \\
\alpha \otimes 1, & \text{if } t = 2\pi.
\end{cases}
\]

Here the subset $[\theta, \theta+t] \subset S^1$ is defined to be the closed interval running from $\theta$ to $\theta + t$ in the positive direction, and $\alpha|_{[\theta, \theta+t]}$ is the based homotopy class of the segment of $\alpha$ restricted to the interval $[\theta, \theta + t]$. We denote by $s_\alpha$ the lift of $\tilde{s}_\alpha$ along the map $\hat{\alpha}$, which is a twisted cycle and defines a twisted homology class

\[
[s_\alpha] \in H_2(\hat{C}_2(M), S(TM); \mathbb{Z}_*)
\]

Proposition 3.2.

\[
\nu(f \cap \partial_\nu[s_\alpha]) = \delta \alpha + (\text{rot}_f \alpha)(1 \otimes \alpha - \alpha \otimes 1) \in \mathbb{Z}\pi \otimes \mathbb{Z}_{\pi}.
\]

The RHS exactly equals the framed Turaev cobracket $\delta^f \alpha$ with respect to $f$.

**Proof.** The contribution of the boundary $\partial(S^1 \times [0,2\pi])$ equals $(\text{rot}_f \alpha)(1 \otimes \alpha - \alpha \otimes 1)$. In order to compute the contribution of each $(t_1, t_2) \in D_\alpha$, we consider the construction $\theta_1$ for $\alpha_1(t) := \alpha(t_1 + t)$ and $\alpha_2(t) := \alpha(t_2 + t)$. Then the mapping degree of the map

\[
\theta \in S^1 \mapsto [-\cos(\theta)\dot{\alpha}_1(0) + \sin(\theta)\dot{\alpha}_2(0)] \in S^1
\]
equals $-\varepsilon(\omega_1(0), \omega_2(0)) = -\varepsilon(\omega(t_1), \omega(t_2))$. The orientation of $S^1$ induced from $S^1 \times S^1$ is the negative one. Hence the contribution equals $\varepsilon(\omega(t_1), \omega(t_2))(\alpha|_{t_1, t_2}) \otimes (\alpha|_{t_2, t_1})$. The sum of all these contributions is nothing but the Turaev cobracket. This proves the proposition.

**Remark 3.3.** The difference of this computation from Hain’s original one in [23, §5] comes from the blow-ups on $q^{-1}(D_\alpha)$.

### 3.2 Homotopy invariance

In this subsection we prove the homotopy invariance of the framed Turaev cobracket. The present proof is due to Florian Naef. The key to the proof is the following lemma.

**Lemma 3.4.** The map $\varpi : (\hat{C}_2(M), S(TM)) \to (M \times M, \Delta_M)$ induces an isomorphism of twisted homology groups

$$\varpi_* : H_*(\hat{C}_2(M), S(TM); \varpi^* S) \cong H_*(M \times M, \Delta_M; S).$$

**Proof.** Let $N \subset M \times M$ be a closed tubular neighborhood of the diagonal $\Delta_M$, and $N^\circ$ its interior. Then the pair $(M \times M \setminus N^\circ, \partial N)$ is homotopy equivalent to $(\hat{C}_2(M), S(TM))$ by a deformation supported near $N$. By excision, we have $H_*(M \times M \setminus N^\circ, \partial N; S) \cong H_*(M \times M, N; S) \cong H_*(M \times M, \Delta_M; S)$. Hence we obtain an isomorphism $H_*(\hat{C}_2(M), S(TM); \varpi^* S) \cong H_*(M \times M, \Delta_M; S)$, which can be realized by the induced map $\varpi_*$. \qed

By this lemma we can consider the map

$$\nu \circ (f \cap) \circ \partial_* \circ \varpi_*^{-1} : H_2(M \times M, \Delta_M; S) \to \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}. \hspace{1cm} (4)$$

The isomorphism $\varpi_*$ maps the relative homology class $[s_\alpha] \in H_2(\hat{C}_2(M), S(TM); \varpi^* S)$ to $[\tilde{s}_\alpha] \in H_2(M \times M, \Delta_M; S)$. Since the homology class $[\tilde{s}_\alpha]$ is homotopy invariant,

$$\nu \circ (f \cap) \circ \partial_* \circ \varpi_*^{-1}([\tilde{s}_\alpha]) = \delta \alpha + (\text{rot}_f \alpha)(1 \otimes \alpha - \alpha \otimes 1) \in \mathbb{Z}\hat{\pi} \otimes \mathbb{Z}\hat{\pi}$$

is also homotopy invariant, as was to be shown.

Here we remark that this proof establishes the invariance of the framed Turaev cobracket also under the move $(\omega 1)$.

### 4 Formality of the Goldman bracket

In this section we review the formality of the Goldman bracket for $\Sigma_{g,n+1}, g, n \geq 0$ over $K$, a field of characteristic 0. Some topics in this section overlap with those in our joint previous survey [34] with Y. Kuno. But the filtration we consider here is different from that in [34]. So we need to re-formulate the problem from the beginning.
4.1 Group-like expansions.

For any free group $\pi$ of finite rank, the formality problem of the group ring $\mathbb{K}\pi$ is classically solved by group-like expansions. See, for example, [8] Ch.2 and also [21]. The group ring $\mathbb{K}\pi$ admits the augmentation map $\varepsilon : \mathbb{K}\pi \to \mathbb{K}$ which maps a formal finite sum $\sum_{\gamma \in \pi} a_\gamma \gamma$, $a_\gamma \in \mathbb{K}$, to the finite sum $\sum_{\gamma \in \pi} a_\gamma$. The augmentation ideal $I_\pi := \ker \varepsilon$ defines a natural decreasing filtration $\{I_\pi^m\}_{m=0}^\infty$ on $\mathbb{K}\pi$, which induces the (completed) associated graded quotient $\text{gr}(\mathbb{K}\pi) := \prod_{m=0}^\infty I_\pi^m/(I_\pi^{m+1})$ and the completed group ring $\hat{\mathbb{K}}\pi := \lim_{\to} \mathbb{K}\pi/(I_\pi^m)$. The group ring $\mathbb{K}\pi$ is a Hopf algebra with coproduct $\Delta$ satisfying $\Delta_\gamma = \gamma \otimes \gamma$ for any $\gamma \in \pi$. The coproduct $\Delta$ descends to both of the algebras $\text{gr}(\mathbb{K}\pi)$ and $\hat{\mathbb{K}}\pi$, and make them complete Hopf algebras. The formality problem for $\mathbb{K}\pi$ asks if the complete Hopf algebras $\text{gr}(\mathbb{K}\pi)$ and $\hat{\mathbb{K}}\pi$ are isomorphic to each other.

In any complete Hopf algebra $A$, the Lie-like elements $\text{Lie}(A) := \{a \in A ; \Delta(a) = a \otimes 1 + 1 \otimes a\}$ and the group-like element $\text{Gr}(A) := \{g \in A \setminus \{0\}; \Delta(g) = g \otimes g\}$ are in one-to-one correspondence by the exponential $\exp : \text{Lie}(A) \to \text{Gr}(A)$, $a \mapsto \exp(a) := \sum_{m=0}^\infty \frac{a^m}{m!}$, and the logarithm $\log : \text{Gr}(A) \to \text{Lie}(A)$, $g \mapsto \log(g) := \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m}(g - 1)^m$. The subset $\text{Gr}(A)$ is a subgroup of the multiplicative group of the algebra $A$.

Let $H$ be the first homology group of the group $\pi$, $H := H_1(\pi; \mathbb{K}) = [\pi/\pi, \pi]$ $\otimes \mathbb{K}$, and we denote by $[\gamma] \in H$ the homology class of $\gamma \in \pi$. The completed tensor algebra $\hat{T}(H) := \prod_{m=0}^\infty H^\otimes m$ has a natural filtration $\hat{T}(H)_{\geq p} := \prod_{m=p}^\infty H^\otimes m$, $p \geq 0$, and a natural coproduct $\Delta$ satisfying $\Delta_\gamma = [\gamma] \otimes 1 + 1 \otimes [\gamma]$ for any $\gamma \in \pi$, which makes $\hat{T}(H)$ a complete Hopf algebra. The Lie-like element $\text{Lie}(\hat{T}(H))$ equals the completed free Lie algebra over the vector space $H$. We write simply $\text{Lie}(\hat{T}(H))_{\geq p} := \text{Lie}(\hat{T}(H)) \cap \hat{T}(H)_{\geq p}$ for any $p \geq 1$.

A group-like expansion $\theta : \pi \to \hat{T}(H)$ is a group homomorphism $\theta : \pi \to \text{Gr}(\hat{T}(H))$ satisfying $\log(\theta(\gamma)) \equiv [\gamma] \pmod{\text{Lie}(\hat{T}(H))_{\geq 2}}$ for any $\gamma \in \pi$. Its linear extension induces isomorphisms of complete Hopf algebras $\theta : \text{gr}(\mathbb{K}\pi) \to \hat{T}(H)$ and $\theta : \hat{\mathbb{K}}\pi \to \hat{T}(H)$. The former one is independent of the choice of group-like expansions, so that we identify $\text{gr}(\mathbb{K}\pi) = \hat{T}(H)$. Thus any group-like expansion $\theta$ induces an isomorphism of complete Hopf algebras $\theta : \text{gr}(\mathbb{K}\pi) \to \hat{T}(H)$ whose associated graded equals the identity on $\text{gr}(\mathbb{K}\pi)$. In other words, the formality problem for the Hopf algebra $\mathbb{K}\pi$ is solved. This formulation is enough for two extreme cases $\Sigma_{0,n+1}$ and $\Sigma_{g,1}$, while we replace the filtration $\{I_\pi^m\}_{m=0}^\infty$ by a new filtration $\{\mathbb{K}\pi(m)\}_{m=0}^\infty$ defined below in order to obtain a formal description of the Goldman-Turaev Lie bialgebra.

We number the boundary components from 0 to $n$: $\partial S = \bigcup_{j=0}^n \partial_j S$, and choose a basepoint $*$ on the 0-th boundary component $\partial_0 S$. Denote by $\pi$ the fundamental group $\pi_1(S,*)$, which is a free group of rank $2g + n$. It admits free generators $\alpha_i$, $\beta_i$, $\gamma_j$, $1 \leq i \leq g$, $1 \leq j \leq n$, such that $\gamma_j$ is freely homotopic to the $j$-th boundary $\partial_j S$ with the positive direction and the product

$$\gamma_0 := \prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_{j=1}^n \gamma_j$$

is the based loop around the 0-th boundary in the negative direction. Choose a point
for each boundary component $\partial_j S$, $1 \leq j \leq n$. Then we can take $\gamma_j \in \pi$ as a concatenation of a path $\ell_j$ from $*$ to $*_j$, the boundary loop based at $*_j$ and the inverse $\ell_j^{-1}$. Moreover we write $*_0 := * \in \partial_0 S$.

The filtration $\{\mathbb{K}\pi(m)\}_{m=1}^\infty$, which we call the weight filtration, is defined to be a unique minimal multiplicative filtration of two-sided ideals of the algebra $\mathbb{K}\pi$ satisfying the conditions $\mathbb{K}\pi(0) = \mathbb{K}\pi$, $\alpha_i - 1, \beta_i - 1 \in \mathbb{K}\pi(1)$, $1 \leq i \leq g$, and $\gamma_j - 1 \in \mathbb{K}\pi(2)$, $1 \leq j \leq n$. See [4, §§3.1-2]. Here we remark this filtration coincides with (twice) the filtration $\{(I\pi)^m\}_{m=0}^\infty$ if $n = 0$ or $g = 0$. Even for any $\Sigma_{(g,n)}$, the topology on $\mathbb{K}\pi$ induced by the weight filtration coincides with that by the filtration $\{(I\pi)^m\}$. Hence we have a natural identification $\mathbb{K}\pi = \lim_{m \to \infty} \mathbb{K}\pi/\mathbb{K}\pi(m)$. The (completed) associated graded quotient

$$gr^{\text{wt}}(\mathbb{K}\pi) := \prod_{m=0}^\infty \mathbb{K}\pi(m)/\mathbb{K}\pi(m + 1)$$

also has a natural complete Hopf algebra structure, and is described as follows.

We denote

$$x_i := [\alpha_i], \ y_i := [\beta_i] \in H, \ 1 \leq i \leq g, \quad z_j := [\gamma_j] \in H, \ 1 \leq j \leq n,$$

and by $H^{(2)}$ the linear span of $z_j$’s, which is the annihilator of the intersection number $H \times H \to \mathbb{K}$. On the other hand, we cap closed disks on the boundary components from $1$ to $n$ to obtain a surface $\bar{S}$ diffeomorphic to $\Sigma_g$. The inclusion homomorphism $i_* : H = H_1(\bar{S}) \to H_1(\bar{S})$ is a surjection, and its kernel equals $H^{(2)}$. In particular, one can identify $H/H^{(2)} = H_1(\bar{S})$. Denote by $\bar{x}_i$ and $\bar{y}_i \in H_1(\bar{S}) = H/H^{(2)}$ the homology classes of $\alpha_i$ and $\beta_i$, respectively. Thus we obtain a two-step filtration on the homology group $H = H^{(1)} \cup H^{(2)} \cup 0$. One can consider the graded quotient $gr H = (H/H^{(2)}) \oplus H^{(2)}$ with weight $\text{wt}(H/H^{(2)}) = 1$ and $\text{wt}(H^{(2)}) = 2$. Any isomorphism $H \to gr H$ whose associated graded equals the identity on $gr H$ corresponds to a section of the surjection $i_* : H \to H_1(\bar{S})$, and vice versa. In particular, we often fix a unique isomorphism $H \cong gr H$ mapping $x_i$ and $y_i$ to $\bar{x}_i$ and $\bar{y}_i$, $1 \leq i \leq g$, respectively, and $z_j$ to itself, $1 \leq j \leq n$. As was proved in [4, Proposition 3.12], we have a natural isomorphism of graded Hopf algebras

$$gr^{\text{wt}}(\mathbb{K}\pi) = \hat{T}(gr H). \quad (5)$$

Here we change the definition of a group-like expansion: A group-like expansion is an isomorphism $\theta : \mathbb{K}\pi \to gr^{\text{wt}}(\mathbb{K}\pi)$ of complete filtered Hopf algebras whose associated graded equals the identity on $gr^{\text{wt}}(\mathbb{K}\pi)$. A unique group-like expansion $\theta_{\text{exp}}$ which maps $\alpha_i$ to exp$(\bar{x}_i)$, $\beta_i$ to exp$(\bar{y}_i), \ 1 \leq i \leq g$, and $\gamma_j$ to exp$(z_j), \ 1 \leq j \leq n$, is a typical example, which we call the exponential expansion associated with $\{\alpha_i, \beta_i, \gamma_j\}$.

A group-like expansion $\theta$ is tangential if we have $\theta(\gamma_j) = g_j \exp(z_j)g_j^{-1}$ for some $g_j \in \text{Gr}(gr^{\text{wt}}(\mathbb{K}\pi)), 1 \leq j \leq n$. The exponential expansion $\theta_{\text{exp}}$ is tangential. The group-like element $g_j$ is uniquely determined up to right multiplication by exp$(\lambda_j z_j), \lambda_j \in \mathbb{K}$. If we fix the elements $g_j$’s, then one can extend the expansion to paths connecting two different boundary components of the surface uniquely as in [34, Definition 7.2]. Similarly we fix the auxiliary data $g_j$’s in the definition of the group TAut in [34].
The most important notion in this section is that of a special expansion. A tangential expansion is special if \( \log \theta(\gamma_0) = \omega \), where we define

\[
\omega := \sum_i [\bar{e}_i, \bar{y}_i] + \sum_j z_j \in \text{gr}^\omega(\mathbb{K}\pi).
\]

Special expansions do exist for any \( g, n \geq 0 \), while the exponential one is not special. If \( n = 0 \), a special expansion is called a symplectic expansion, which was introduced by Massuyeau [33]. The discovery of the notion of a symplectic expansion by Massuyeau has been one of the important motives of the author’s studies in this decade. A special expansion often appears in genus 0 [1, 22, 36]. We should call it a special/symplectic expansion, but here we call just a special expansion for simplicity.

**Remark 4.1.** This definition is equivalent to that in [34] Definition 7.2. Let \( \theta : \mathbb{K}\pi \cong \text{gr}^\omega(\mathbb{K}\pi) \) be a special expansion. Then the composite of \( \log \theta|_\pi : \pi \to \text{Lie}(\text{gr}^\omega(\mathbb{K}\pi)) \) and the natural projection \( \text{Lie}(\text{gr}^\omega(\mathbb{K}\pi)) \to \text{gr}H \to H^{(2)} \) defines a splitting \( \pi \to H \to H^{(2)} \), from which we obtain a section \( s \) of the inclusion homomorphism \( i_* : H \to H_1(S) \). The composite of \( \theta \) and the isomorphism \( \text{gr}^\omega(\mathbb{K}\pi) \cong \hat{T}(H) \) induced by the section \( s \) satisfies the condition \((\ast)\) in [34] (7.3)]. Conversely, if we are given a section \( s \) of \( i_* \) and a Magnus expansion with condition \((\ast)\), then the composite of the Magnus expansion and the isomorphism \( \text{gr}^\omega(\mathbb{K}\pi) \cong \hat{T}(H) \) induced by the section \( s \) gives a special expansion in the sense of the present survey paper.

### 4.2 The Goldman bracket

Now we recall the definition of the Goldman bracket. Choose generic immersions representing \( \alpha \) and \( \beta \in \hat{\pi} \). In particular, the intersection points \( \alpha \cap \beta \) are finite and transverse. Then the Goldman bracket is defined to be

\[
[\alpha, \beta] := \sum_{p \in \sigma(\alpha, \beta)} \varepsilon_p(\alpha, \beta)|\alpha_p|\beta_p| \in \mathbb{Z}\hat{\pi}
\]

[21]. Here \( \varepsilon_p(\alpha, \beta) \in \{ \pm 1 \} \) is the local intersection number of \( \alpha \) and \( \beta \) at the intersection point \( p \), and \( \alpha_p \) and \( \beta_p \in \pi_1(S, p) \) are the based loop with basepoint \( p \) along \( \alpha \) and \( \beta \), respectively. \( \alpha_p\beta_p \) means the product of the based loops, whose free homotopy class is denoted by \( |\alpha_p\beta_p| \in \hat{\pi} \). Then the bracket is well-defined, and makes the module \( \mathbb{Z}\hat{\pi} \) a Lie algebra [21]. One can check

\[
[|\mathbb{K}\pi(l)|, |\mathbb{K}\pi(m)|] \subset |\mathbb{K}\pi(l + m - 2)|, \quad \text{and} \quad (6)
\]

\[
\delta^f(|\mathbb{K}\pi(m)|) \subset \sum_{a+b=m-2} |\mathbb{K}\pi(a)| \otimes |\mathbb{K}\pi(b)| \quad (7)
\]

[4] Proposition 3.13]. Hence the Goldman bracket and the Turaev cobracket descend to \( |\mathbb{K}\pi| \) and define their associated graded

\[
[-, -]_{\text{gr}^\omega} : \text{gr}^\omega(|\mathbb{K}\pi|) \otimes \text{gr}^\omega(|\mathbb{K}\pi|) \to \text{gr}^\omega(|\mathbb{K}\pi|)
\]

\[
\text{gr}^\omega(\delta^f) : \text{gr}^\omega(|\mathbb{K}\pi|) \to \text{gr}^\omega(|\mathbb{K}\pi|) \otimes \text{gr}^\omega(|\mathbb{K}\pi|)
\]

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of degree $-2$. It should be remarked that $\text{gr}^{\text{wt}}(|\mathbb{K}\pi|) = |\text{gr}^{\text{wt}}(\mathbb{K}\pi)|$.

To describe these operations, we prepare some notation. The intersection number on $H = H_1(S;\mathbb{K})$ induces a non-degenerating skew-symmetric pairing $\langle -,- \rangle : (H/H^{(2)}) \times (H/H^{(2)}) \to \mathbb{K}$, which we extend to $\text{gr}H$ by zero on $H^{(2)}$. A symmetric operation $\delta : H^{(2)} \times H^{(2)} \to H^{(2)}$ is defined by $\delta(z_j, z_k) = \delta_{jk} z_j$, $1 \leq j, k \leq n$. The sum $z_0 := \sum_{j=1}^n z_j$ is the unit of the operation $\delta$, which we extend to $\text{gr}H$ by zero on $H/H^{(2)}$. Then the associated graded of the Goldman bracket is given by

$$[u,v]_{\text{gr}^{\text{wt}}} = \sum_{i,j} \langle u_i, v_j \rangle |u_{i+1} \cdots u_i u_1 \cdots u_{i-1} v_{j+1} \cdots v_m v_1 \cdots v_{j-1}|$$

Here $u = u_1 \cdots u_i$, $v = v_1 \cdots v_m \in \widehat{T}(\text{gr}H)$ for $u_i, v_j \in \text{gr}H$. For the framed Turaev cobracket, we need one more symbol: For a framing $F(S)$ we define $c_f : \text{gr}H \to \mathbb{K}$ by $c_f(\bar{x_i}) = c_f(\bar{y}_i) = 0$, $1 \leq i \leq g$, and $c_f(z_j) = \text{rot}_f(\partial_j S) + 1$, $1 \leq j \leq n$. Here we remark $z_0 = \sum_{j=1}^n z_j$ satisfies $c_f(z_0) = 2 - 2g - n - 1 - \text{rot}_f(\partial_0 S) + n = 1 - 2g - \text{rot}_f(\partial_0 S)$ from the Poincaré-Hopf theorem. Then the associated graded of the Turaev cobracket for $u = u_1 \cdots u_i \in \widehat{T}(\text{gr}H)$ with $u_i \in \text{gr}H$ is given by

$$\text{gr}^{\text{wt}}(\delta^f)([u]) = \sum_{j<k} \langle u_j, u_k \rangle |u_{j+1} \cdots u_k u_{k-1}| \wedge |u_{k+1} \cdots u_k u_1 \cdots u_{k-1}|$$

Here $P \wedge Q \in \text{gr}^{\text{wt}}(|\mathbb{K}\pi|) \otimes \text{gr}^{\text{wt}}(|\mathbb{K}\pi|)$ means $P \otimes Q - Q \otimes P \in \text{gr}^{\text{wt}}(|\mathbb{K}\pi|) \otimes \text{gr}^{\text{wt}}(|\mathbb{K}\pi|)$ for $P, Q \in \text{gr}^{\text{wt}}(|\mathbb{K}\pi|)$, and $1 = |1| \in \text{gr}^{\text{wt}}(|\mathbb{K}\pi|) = |\text{gr}^{\text{wt}}(\mathbb{K}\pi)|$. If $c_f = 0$, then $\text{gr}^{\text{wt}}(\delta^f)([u])$ equals Schedler’s cobracket [33]. In particular, the $1 \wedge \text{gr}^{\text{wt}}(|\mathbb{K}\pi|)$-part of $\text{gr}^{\text{wt}}(\delta^f)([u])$ equals

$$1 \wedge \sum_{i=1}^l \langle u_i, u_{i+1} \rangle - \delta(u_i, u_{i+1}) + c_f(u_i)u_{i+1}u_{i+2} \cdots u_1 u_i \cdots u_{i-1},$$

where we agree $l + 1 := 1$ and $l + 2 := 2$.

Any special expansion gives the formal description of the Goldman bracket.

**Theorem 4.2** ([31] [34] [37] [38] [40]). Let $\theta$ be a special expansion for $\Sigma_{(g,n+1)}$. Then $\theta : (|\mathbb{K}\pi|, [\cdot, \cdot]) \to (\text{gr}^{\text{wt}}(|\mathbb{K}\pi|), [\cdot, \cdot]_{\text{gr}^{\text{wt}}})$ is an isomorphism of Lie algebras.
From this theorem, any conjugate of a special expansion by a group-like element in Gr(gr\(^{\text{wt}}\)(\(\mathbb{K}\pi\))) also gives the formal description of the Goldman bracket. Very recently Alekseev-Kawazumi-Kuno-Naef [5] proved the converse.

**Theorem 4.3** ([5]). Let \( \theta \) be a group-like expansion and assume that \( \theta \) induces a Lie algebra isomorphism \( \theta : ([\mathbb{K}\pi], [\cdot, \cdot]) \rightarrow (\text{gr}\^{\text{wt}}([\mathbb{K}\pi]), [\cdot, \cdot]_{\text{gr}\^{\text{wt}}}) \). Then \( \theta \) is conjugate to a special expansion by a group-like element in Gr(gr\(^{\text{wt}}\)(\(\mathbb{K}\pi\))).

Lemma 4.4 and Theorem 4.5 are the keys to the proof of this theorem.

**Lemma 4.4** ([3] Proposition A.2 [5] Proposition 3.9). 1. Let \( V \) be a finite dimensional \( \mathbb{K} \) vector space. Assume \( x \in V \setminus \{0\} \) and \( a \in \hat{\mathcal{T}}(V) \) satisfy \( |ax|^l = 0 \in |\hat{T}(V)| \) for any \( l \geq 1 \). Then we have \( a \in [x, \hat{\mathcal{T}}(V)] \).

2. Let \( V \) be a finite dimensional symplectic \( \mathbb{K} \) vector space with symplectic form \( \omega_0 \in \wedge^2 V \). Assume an element \( a \in \hat{\mathcal{T}}(V) \) satisfies \( |a\omega_0|^l = 0 \in |\hat{T}(V)| \) for any \( l \geq 1 \). Then we have \( a \in [\omega_0, \hat{\mathcal{T}}(V)] \).

**Theorem 4.5** ([16]). The center of the Lie algebra \((\text{gr}\^{\text{wt}}([\mathbb{K}\pi]), [\cdot, \cdot]_{\text{gr}\^{\text{wt}}})\) equals

\[
\mathcal{Z}(\text{gr}\^{\text{wt}}([\mathbb{K}\pi]), [\cdot, \cdot]_{\text{gr}\^{\text{wt}}}) = |\mathbb{K}[\omega]| + \sum_{j=0}^n |\mathbb{K}[z_j]|.
\]

This theorem can be proved in an elementary way using Lemma 4.4 [1] Theorem 5.4.

**Remark 4.6.** Let \( S \) be the surface \( \Sigma_\theta \) and \( \theta \) a group-like expansion for \( \pi_1(S) \). Then, for any based loop \( \gamma \), \( \theta(\log \gamma) \in \text{Lie}(\hat{T}(H)) \) is a Lie-like element. Since \( |\text{Lie}(\hat{T}(H))| = |H| \), we have \( \theta(\log \gamma) = |[\gamma]| \in |H| \), where \( |[\gamma]| \in H \) is the homology class represented by the loop \( \gamma \). Hence, for any free loop \( \alpha \), the isomorphism \( [\theta] : |\mathbb{K}\pi| \rightarrow |\hat{T}(H)| \) maps \( \log \alpha : = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}(\alpha - 1)^m \in |\mathbb{K}\pi| \) to \( |[\alpha]| \in |H| \subset |\hat{T}(H)| \).

In the succeeding sections, we need an action of the Goldman Lie algebra \(|\mathbb{Z}\pi|\) on the group ring \( \mathbb{Z}\pi \) of the fundamental group \( \pi = \pi_1(S, \ast) \). A free loop \( \alpha \in \hat{\pi} \) acts on a based loop \( \gamma \in \pi \) as follows. Take generic immersions as their representatives. Then the intersection points \( \alpha \cap \gamma \) are finite, and we define \( \sigma(\alpha)(\gamma) \in \mathbb{Z}\pi \) by

\[
\sigma(\alpha)(\gamma) := \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma) \gamma_{sp} \alpha_p \gamma_{p*} \in \mathbb{Z}\pi.
\]

Here \( \varepsilon_p(\alpha, \gamma) \in \{ \pm 1 \} \) is the local intersection number of \( \alpha \) and \( \gamma \) at the intersection point \( p \), \( \gamma_{sp} \) and \( \gamma_{p*} \) are the segments of \( \gamma \) from \( \ast \) to \( p \) and from \( p \) to \( \ast \), respectively, and \( \alpha_p \in \pi_1(S, p) \) the based loop along \( \alpha \) with basepoint \( p \). Then it is well-defined, and defines a Lie algebra homomorphism \( \mathbb{Z}\pi \rightarrow \text{Der}(\mathbb{Z}\pi), \text{Der}(\mathbb{Z}\pi) \) the Lie algebra of derivations of the group ring \( \mathbb{Z}\pi \) [31]. We remark that the Goldman bracket of \( \alpha \in \hat{\pi} \) and the free loop \( |\gamma| \in \hat{\pi} \subset |\mathbb{Z}\pi| \) is given by \( |\alpha, |\gamma|\| = |\sigma(\alpha)(\gamma)| \). The homomorphism \( \sigma \)
is continuous with respect to the filtration \( \{|(I\pi)^m|\} = \{|[K\pi(m)]|\} \), and so descends to the Lie algebra homomorphism
\[
\sigma : [K\pi]/K1 \to \text{Der}(K\pi).
\]  
(11)

The RHS is the Lie algebra of \textit{continuous} derivations of \( K\pi \). The associated graded of the homomorphism \( \text{gr}^\text{wt}(\sigma) \) is given by
\[
\text{gr}^\text{wt}(\sigma)(u)(v) = \sum_{i,j} \langle u_i, v_j \rangle v_1 \cdots v_{j-1}u_{i+1} \cdots u_iu_1 \cdots u_{i-1}v_{j+1} \cdots v_m \tag{12}
\]
\[
+ \sum_{i,j} v_1 \cdots v_{j-1} S(13)(v_j, u_i)u_{i+1} \cdots u_iu_1 \cdots u_{i-1}v_{j+1} \cdots v_m
\]
\[
- \sum_{i,j} v_1 \cdots v_{j-1}u_{i+1} \cdots u_iu_1 \cdots u_{i-1} S(13)(u_i, v_j)v_{j+1} \cdots v_m.
\]

Here \( u = u_1 \cdots u_l, v = v_1 \cdots v_m \in \hat{T}(\text{gr}H) \) for \( u_i, v_j \in \text{gr}H \) as above. As is proved in \([31, 34, 37, 38, 40]\), any special expansion induces the formal description of the homomorphism \( \sigma \).

A free loop \( \alpha \) also acts on the free \( \mathbb{Z} \) module over the homotopy set of paths \( \{(0,1], 0,1), (S, *_{j}, *_{k})\} \), \( 0 \leq j,k \leq n \), \( \sigma : [\mathbb{Z}\pi]/\mathbb{Z}1 \to \text{End}(\mathbb{Z}\pi) \)).

Using the path \( \ell_j \) in \([41]\) we define \( \sigma_j(\alpha) \in \mathbb{Z}\pi \) by \( \sigma_j(\alpha) = (\sigma(\alpha)(\ell_j))\ell_j^{-1} \in \mathbb{Z}\pi \). Then we have \( \sigma(\alpha)(\ell_j^{-1}) = -\ell_j^{-1}\sigma_j(\alpha) \) and so \( \sigma_j([\alpha,\beta]) = \sigma(\alpha)(\sigma_j(\beta)) - \sigma(\beta)(\sigma_j(\alpha)) + \sigma_j(\alpha)\sigma_j(\beta) - \sigma_j(\beta)\sigma_j(\alpha) \) for any \( \alpha, \beta \in \hat{\pi} \). Hence the homomorphism \( \sigma \) lifts to a homomorphism to the semi-direct product
\[
\hat{\sigma} := (\sigma, \sigma_1, \ldots, \sigma_n) : [\mathbb{Z}\pi]/\mathbb{Z}1 \to \text{Der}(\mathbb{Z}\pi) \ltimes (\mathbb{Z}\pi)^{\oplus n},
\]
which also has the formal description by any special expansion \([34]\). The associated graded of the homomorphism \( \hat{\sigma} \)
\[
\text{gr}^\text{wt}(\hat{\sigma}) = (\text{gr}^\text{wt}(\sigma), \text{gr}^\text{wt}(\sigma_1), \ldots, \text{gr}^\text{wt}(\sigma_n)) : 
\text{gr}^\text{wt}([K\pi]) \to \text{Der}((\text{gr}^\text{wt}(K\pi)) \ltimes (\text{gr}^\text{wt}(K\pi))^{\oplus n}
\]
has its image in a subalgebra \( \text{tDer} \) defined by
\[
\text{tDer} := \{(u, u_1, \ldots, u_n) \in \text{Der}(\text{gr}^\text{wt}(K\pi)) \ltimes (\text{gr}^\text{wt}(K\pi))^{\oplus n} : u(z_j) = [z_j, u_j], 1 \leq j \leq n\}.
\]

The homotopy intersection form introduced by Papakyriakopoulos \([41]\) and Turaev \([49]\) independently is an upgrade of the homomorphism \( \sigma \). Massuyeau and Turaev \([37, 38]\) gave an explicit tensorial description of the homotopy intersection form by any special expansion.

### 4.3 The logarithms of mapping classes

In this subsection we confine ourselves to the case \( S = \Sigma_{g,1} \). See \([33, 34]\) for general compact surfaces \( S \). Instead of \( \gamma_0 \), we denote by \( \zeta \in \pi = \pi_1(S, *) \) the negative boundary
loop with basepoint \( * \in \partial S \). Then the image \( \sigma([\mathbb{Z}\pi]/\mathbb{Z}1) \) is included in \( \text{Der}_\zeta(\mathbb{K}\pi) \), where \( \text{Der}_\zeta(\mathbb{Z}\pi) \) is the Lie algebra of derivations of the group ring \( \mathbb{Z}\pi \) annihilating the boundary loop \( \zeta \in \pi \). Hence we have the Lie algebra homomorphism

\[
\sigma : \frac{\mathbb{K}\pi}{\mathbb{K}1} \to \text{Der}_\zeta(\mathbb{K}\pi).
\]

(13)

The RHS is the Lie subalgebra of \( \text{Der}(\mathbb{K}\pi) \) annihilating the boundary loop \( \zeta \). Using the formal descriptions of the Goldman bracket and the homomorphism \( \sigma \), one can prove the homomorphism \( \sigma \) in (13) is a homomorphism of Lie algebras. For details, see [34]. We denote by \( |\mathbb{K}\pi|^{+} \) the closure of \( |(\mathbb{Z}\pi)^{3}| \) in \( |\mathbb{K}\pi| \). From (6), \( |\mathbb{K}\pi|^{+} \) is a pro-nilpotent Lie subalgebra of \( |\mathbb{K}\pi| \). In particular, the Baker-Campbell-Hausdorff series defines a group structure on the subalgebra \( |\mathbb{K}\pi|^{+} \).

The mapping class group \( \mathcal{M}(S) \), the path-component group of the group of orientation-preserving diffeomorphisms fixing the boundary \( \partial S \) pointwise, acts on \( \mathbb{K}\pi \) in an obvious way. We consider the subset

\[
\mathcal{M}^{\circ} := \{ \varphi \in \mathcal{M}(S); \text{ the logarithm } \log \varphi := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}(\varphi - 1)^{m} \text{ on } \mathbb{K}\pi \text{ exists} \}.
\]

It is a proper subset of \( \mathcal{M}(S) \), and contains all the Dehn twists and the Johnson group \( \mathcal{K}_{g,1} := \ker(\mathcal{M}(\Sigma_{g,1}) \to \text{Aut}(\pi/[[\pi, \pi], \pi])) \). Then we can define a map

\[
\tau : \mathcal{M}^{\circ} \overset{\log}{\to} \text{Der}_\zeta(\mathbb{K}\pi) \overset{\sigma^{-1}}{\to} \frac{\mathbb{K}\pi}{\mathbb{K}1}, \quad \varphi \mapsto \sigma^{-1}(\log \varphi),
\]

which is a geometric extension of the Johnson homomorphisms [26]. As was proved in [31], we have \( \tau(t_{C}) = L(C) := \frac{1}{2}(\log C)^{2} \in \frac{\mathbb{K}\pi}{\mathbb{K}1} \) for any simple closed curve \( C \subset \text{Int } S \). This formula is generalized to any compact oriented surface by [32] and [37]. In particular, \( \tau(t_{C}) \) is an infinite sequence of the powers of \( C - 1 \), \( (C - 1)^{m} \), \( m \geq 0 \).

On the other hand, we have \( \tau(\mathcal{K}_{g,1}) \subset |\mathbb{K}\pi|^{+}/\mathbb{K}1 \), and \( \tau : \mathcal{K}_{g,1} \to |\mathbb{K}\pi|^{+}/\mathbb{K}1 \) is a group homomorphism. For details, see [34].

**Remark 4.7.** Following Turaev’s quantization of the Goldman-Turaev Lie bialgebra [50], S. Tsuji ‘skein-ize’ the logarithms of mapping classes in [44, 45, 47]. His constructions yield invariants of integral homology 3-spheres [46, 47]. Very recently he proved that the invariants recover the universal quantum \( sl(n) \) invariants [48].

Now we review a relation between the map \( \tau \) and the Turaev cobracket. In [33] it is proved that \( \delta \circ \tau = 0 : \mathcal{M}^{\circ} \to (|\mathbb{K}\pi|/\mathbb{K}1)^{\otimes 2} \) by using a based variant of the Turaev cobracket \( \mu \) [49]. Similarly one can prove \( \delta^{f} \tau(\varphi) = 0 \) for any \( \varphi \in \mathcal{M}^{\circ} \) satisfying \( f \varphi = f \). We remark that any \( \varphi \in \mathcal{K}_{g,1} \) acts on the set \( F(S) \) trivially. Here we give an alternative proof for \( \delta^{f} \tau(\mathcal{K}_{g,1}) = 0 \).

**Lemma 4.8.** For any simple closed curve \( C \) in \( M = \text{Int } S \), we have

\[
\delta^{f} L(C) = (\text{rot}_{f}C)(1 \otimes \log C - \log C \otimes 1) \in |\mathbb{K}\pi|^{\otimes 2}.
\]

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Proof. From Lemma 2.1 we have
\[ \delta f(C^m) = \delta(C^m) + \rot f(C^m)(1 \otimes C^m - C^m \otimes 1) = m \cdot \rot f(C)(1 \otimes C^m - C^m \otimes 1) \]
for any \( m \geq 0 \). Hence, for any formal power series \( h(z) \) in a variable \( z - 1 \), we have
\[ \delta f(h(C)) = \rot f(C)(1 \otimes C f'(C) - C f'(C) \otimes 1). \]
The lemma follows from the fact \( z \frac{d}{dz} \left( \frac{1}{2} \log z \right)^2 = \log z \).

As was stated in Remark 4.6, the logarithm \( \log C = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (C - 1)^m \in \hat{K}_\pi \) depends only on the homology class \([C] \in H_1(M; \mathbb{K})\). In particular, we have \( \delta f L(C) = 0 \) if \([C] = 0 \in H_1(M; \mathbb{K})\).

Theorem 4.9 ([33]). \( (\delta f \circ \tau)(K_{g,1}) = 0 \).

Proof. As was proved by Johnson [27], the group \( K_{g,1} \) is generated by Dehn twists along simple closed curves whose homology classes vanish. The value of the composite \( \delta f \circ \tau \) vanishes at any of such Dehn twists. The corollary follows from Lemma 2.2 and the fact that \( \tau \) is a group homomorphism.

Here we briefly review the Johnson homomorphisms [26]. Let \( \{ \Gamma_k \pi \}_{k=1}^{\infty} \) be the lower central series of the fundamental group \( \pi = \pi_1(S, \ast) \). It is defined by \( \Gamma_1 \pi = \pi \) and \( \Gamma_{k+1} \pi = [\Gamma_k \pi, \pi] \) for \( k \geq 1 \). The Andreadakis-Johnson filtration \( \{ M(k) \}_{k=0}^{\infty} \) of the mapping class group \( M(S) = M(S, \ast) \) is defined by \( M(k) := \ker(M(S) \rightarrow \text{Aut}(\pi/\Gamma_{k+1} \pi)) \) for \( k \geq 0 \). We have \( M(0) = M(S), M(2) = K_{g,1} \) and \( M(1) \) is called the Torelli group.

One can check that \( \tau(M(k)) \) is included in the closure of \( |(I \pi)^{k+2}| \) in \( K_\pi/\mathbb{K} \) for any \( k \geq 1 \). Hence we obtain the associated graded
\[ \text{gr}(\tau) : \prod_{k=1}^{\infty} M(k)/M(k+1) \rightarrow \text{gr}^w(|K\pi|), \]
which equals the Johnson homomorphisms [34], and is a Lie algebra homomorphism [39]. From the facts \( \delta \circ \tau = 0 \) and \( \delta f \circ \tau = 0 \), we can derive some constraints of the image of the Johnson homomorphism \( \text{gr}(\tau) \). As was computed in [33] §6.4, the original Turaev cobracket \( \delta \) includes the Morita trace [39]. But, as was proved by Enomoto-Kuno-Satoh [18], the constraint coming from the original gr(\delta) does not include the Enomoto-Satoh trace [19], a refinement of the Morita trace. The Enomoto-Satoh trace equals the \( 1 \wedge \text{gr}^w(|K\pi|) \)-part of \( \text{gr}^w(\delta f(u)) \) [10] for \( S = \Sigma_{g,1} \) [29]. Thus we need to consider the formality problem of the framed Turaev cobracket \( \delta f \). Moreover, if \( g = 0 \) and \( c_f = 0 \), then the \( 1 \wedge \text{gr}^w(|K\pi|) \)-part equals minus the Alekseev-Torossian divergence cocycle div in the Kashiwara-Vergne problem [27]. This leads us to study the Kashiwara-Vergne problem.
5 Kashiwara-Vergne problems and the formality of the Turaev cobracket

As will be shown in this section, the Kashiwara-Vergne problem can be regarded as a problem on a group-like expansion for a framed surface $(\Sigma_{g,n+1}, f)$. But, in the formulation in [3], it is a problem for a (continuous) automorphism of the Lie algebra $\text{Lie}(\text{gr}^\text{wt}(K\pi))$ with some auxiliary data.

We write simply $A := \text{gr}^\text{wt}(K\pi)$ and $L := \text{Lie}(\text{gr}^\text{wt}(K\pi))$. The Lie algebra $L$ has a natural filtration coming from the weight filtration and a group structure by the Baker-Campbell-Hausdorff series, and we denote by $\text{Aut}^+(L)$ the group of filtration-preserving automorphisms of $L$ whose associated graded is the identity. Then one can introduce a subgroup $\text{TAut} = \text{TAut}^{(g,n+1)}$ of the semi-direct product of $\text{Aut}^+(L)$ and $L^\oplus n$ defined by

$$\text{TAut} := \{(F, f_1, \ldots, f_n) \in \text{Aut}^+(L) \ltimes L^\oplus n; F(z_j) = \exp(-f_j)z_j \exp(f_j), 1 \leq j \leq n\}.$$  

We denote by $\text{TAut}$ the image of the $\text{TAut}$ under the first projection $\text{TAut} \twoheadrightarrow \text{Aut}^+(L) \ltimes L^\oplus n$.

The Lie algebra of the group $\text{Aut}^+(L)$ is the Lie algebra $\text{Der}^+(L)$ of derivations of $L$ of positive degree with respect to the weight filtration, and the Lie algebra of $\text{TAut}$ coincides with a Lie subalgebra $\text{tder}^+$ of the semi-direct product of $\text{Der}^+$ and $L^\oplus n$ given by

$$\text{tder}^+ = \{(u, u_1, \ldots, u_n) \in \text{Der}^+(L) \ltimes L^\oplus n; u(z_j) = [z_j, u_j], 1 \leq j \leq n\}.$$  

In the sequel we write simply $u$ for $(u, u_1, \ldots, u_n) \in \text{tder}^+$ for simplicity. Here we remark that the exponential expansion $\text{tder}^+ \to \text{TAut}$ is a bijection, and that any graded Lie algebra action of $\text{tder}^+$ on a graded module $M$ integrates to a group action of the group $\text{TAut}$ on $M$. Moreover any graded Lie algebra 1-cocycle $c : \text{tder}^+ \to M$ integrates to a group cocycle $C : \text{TAut} \to M$ satisfying the condition

$$\frac{\partial}{\partial t}C(\exp(tu))|_{t=0} = c(u)$$  

for any $u \in \text{tder}^+$. Here we remark $\text{tder}^+$ can be regarded as a Lie subalgebra of $\text{tDer}$ in an obvious way.

Now we recall the free generator system $\{\alpha_i, \beta_i, (1 \leq i \leq \gamma), \gamma_j(1 \leq j \leq n)\}$ of the group $\pi = \pi_1(S, *)$, and the exponential expansion $\text{Exp} : K\pi \to \text{gr}^\text{wt}(K\pi)$ associated with the generator system introduced in [14]. Then any tangential expansion $\theta$ equals $F^{-1} \circ \text{Exp}$ for some unique automorphism $F \in \text{TAut}$. In other words, the set of tangential expansions is a torsor of the group $\text{TAut}$. From the definition of a special expansion $F^{-1} \circ \text{Exp}$ is special if and only if $F$ satisfies the condition

$$\text{(KVI)} \quad F \left( \sum_{i=1}^{g} [\bar{F}, \bar{y}_i] + \sum_{j=1}^{n} z_j \right) = \log \left( \prod_{i=1}^{g} (e^{2i}e^{\bar{y}_i}e^{-2i}e^{-\bar{y}_i}) \prod_{j=1}^{n} e^{z_j} \right),$$  

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which is exactly the first condition of the Kashiwara-Vergne problem. Hence we may assume that an automorphism \( F \in \text{TAut} \) satisfies the condition (KVI). In this section, under the assumption, we will provide a necessary and sufficient condition for \( F \) to induce the formality of the Turaev cobracket. This is exactly the same as the other condition of the Kashiwara-Vergne problem, that is, (KVII).

5.1 Formality in genus 0

A formal description of the framed cobracket \( \delta^f \) in genus 0 was given first by Massuyeau [36] by using the Kontsevich integral. After the work [3] we review here, Alekseev and Naef [7] gave it by using the Knizhnik-Zamolodchikov connections. This seems the simplest proof of the formality of the Goldman-Turaev Lie bialgebra in genus 0.

Now we recall the Kashiwara-Vergne problem [28] in the formulation of Alekseev-Torossian [8]. This corresponds to the genus 0 surface \( \Sigma_{0,n+1} \) with a unique framing \( f \) satisfying \( \text{rot}_f(\partial_j S) = -1 \) for any \( j \geq 1 \), which is realized by a standard embedding of \( \Sigma_{0,n+1} \) into the unit disk in \( \mathbb{R}^2 \) mapping \( \partial_0 S \) to the unit circle. Then we have \( c_f = 0 \), \( H = H^{(2)} \), and so \( \text{gr}^{\text{wt}}(\mathbb{K}\pi) = \hat{T}(H) \). In particular, \( L = \text{Lie}(\text{gr}^{\text{wt}}(\mathbb{K}\pi)) \) equals the complete free Lie algebra generated by \( H = \oplus_{j=1}^n \mathbb{K}z_j \). Any element \( a \in A = \hat{T}(H) \) is uniquely written as \( a = a^0 + \sum_{i=1}^n a^i z_i \) with \( a^0 \in \mathbb{K} \) and \( a^i \in A \). Then the Alekseev-Torossian divergence cocycle \( \text{div} : \text{tder}^+ \to |A| = |\hat{T}(H)| \) is defined by

\[
\text{div}(u) := \sum_{i=1}^n |z_i(u_i)^i| \in |A|
\]

for \( u = (u, u_1, \ldots, u_n) \in \text{tder}^+ \). We remark the divergence \( \text{div} \) naturally extends to the Lie algebra \( \text{tDer} \). The Jacobian cocycle \( j : \text{TAut} \to |A| \) is the integration of the divergence cocycle \( \text{div} \).

**Definition 5.1** ([8]). An element \( F \in \text{TAut} \) is a solution to the Kashiwara-Vergne problem (of type \((0,n+1)\)) if it satisfies the conditions (KVI) and

\[
\text{(KVII)} \quad j(F^{-1}) = \left| \sum_{j=1}^n h_j(z_j) - h \left( \sum_{i=1}^n z_i \right) \right|
\]

for some \( h(s) \) and \( h_j(s) \in \mathbb{K}[[s]] \), \( 1 \leq j \leq n \).

From Theorem 4.5 the condition (KVII) is equivalent to the condition

\[
\text{(KVII')} \quad j(F^{-1}) \in Z(\text{gr}^{\text{wt}}(\mathbb{K}\pi), [-,-]_{\text{gr}^{\text{wt}}}).
\]

The functions \( h_j \) and \( h \) are the same modulo their linear parts [3] Theorem 8.7] and called the *Duflo function*. The existence of solutions to the problem is proved by Alekseev-Meinrenken [6] and Alekseev-Torossian [8].

The main theorem in this subsection is
Theorem 5.2 (3). Let $F \in \text{TAut}$ satisfy the condition (KVI). Then the special expansion $F^{-1} \circ \theta_{\text{exp}}$ induces the formal description of the framed Turaev cobracket, or equivalently, the isomorphism of the Lie bialgebra

$$F^{-1} \circ \theta_{\text{exp}} : ([\mathbb{K}\pi], [\cdot, \cdot], \delta f) \rightarrow (\text{gr}^{\text{wt}}([\mathbb{K}\pi]), [\cdot, \cdot], \text{gr}^{\text{wt}}(\delta f)),$$

if and only if $F$ is a solution to the Kashiwara-Vergne problem.

Here recall the Lie algebra homomorphism $\text{gr}^{\text{wt}}(\hat{\delta}) = (\text{gr}^{\text{wt}}(\delta), \text{gr}^{\text{wt}}(\delta_1), \ldots, \text{gr}^{\text{wt}}(\delta_n)) : \text{gr}^{\text{wt}}([\mathbb{K}\pi]) \rightarrow \text{tDer}$ introduced in \cite{12}. In the present case $S = \Sigma_{0,n+1}$, for any element $u = z_{k_1} \cdots z_{k_m} \in A$, $1 \leq k_i \leq n$, we have $\text{gr}^{\text{wt}}(\delta_j)(|u|) = \sum \delta_{j \cdot z_{k_{i+1}}} \cdots \cdot z_{k_m}$, and so the $1 \wedge \text{gr}^{\text{wt}}([\mathbb{K}\pi])$-part of $\text{gr}^{\text{wt}}(\delta f)(|u|)$ equals $-1 \wedge \text{div}^\cdot(\text{gr}^{\text{wt}}(\delta)(|u|))$. A ‘double version’ of this observation is one of the main ingredients in our approach to the formality of the Turaev cobracket. So we introduce the double divergence $t\text{Div} : \text{tDer} \to |A| \otimes |A|$ which maps $(u, u_1, \ldots, u_n) \in \text{tDer}$ to

$$t\text{Div}(u) := \sum_{j=1}^n |(z_j \otimes 1)(\frac{\partial}{\partial z_j} u_j) - (\frac{\partial}{\partial z_j} u_j)(1 \otimes z_j)|.$$

Here $| \cdot : A \otimes A \to |A| \otimes |A|$, $a_1 \otimes a_2 \mapsto |a_1| \otimes |a_2|$, is the quotient map, and $\frac{\partial}{\partial z_j} : A \to A \otimes A$ is defined by $\frac{\partial}{\partial z_j}(z_{k_1} \cdots z_{k_m}) = \sum l \delta_{j \cdot z_{k_{i+1}}} \cdots \cdot z_{k_m} \otimes z_{k_{i+1}}, 1 \leq k_i \leq n$. The divergences div and tDiv satisfy the commutative diagram

$$\begin{array}{ccc}
t\text{der}^+ & \xrightarrow{\text{div}} & |A| \\
\text{incl} & & \downarrow \hat{\Delta} \\
t\text{Der} & \xrightarrow{t\text{Div}} & |A| \otimes |A|,
\end{array}$$

where $\hat{\Delta} = (1 \otimes \text{antipode}) \circ \Delta : |A| \to |A| \otimes |A|$. We remark $t\text{der}^+$ in the diagram (15) cannot be replaced by $t\text{Der}$. The cocycle $t\text{der}^+ : |A| \otimes |A|$ is integrated to a group cocycle $J : \text{TAut} \to |A| \otimes |A|$. The diagram (15) implies $J = \Delta \circ j : \text{TAut} \to |A| \otimes |A|$.

Now we take the exponential expansion $\theta_{\text{exp}} : \mathbb{K}\pi \xrightarrow{\exp} \text{gr}^{\text{wt}}[\mathbb{K}\pi] = A$, and the conjugates $\delta_{\text{exp}} : |A| \to |A| \otimes |A|$ and $\hat{\delta}_{\text{exp}} : |A| \to \text{tDer}$ by the isomorphism $\theta_{\text{exp}}$. Then we have

**Theorem 5.3.**

$$\delta f_{\text{exp}} = t\text{Div} \circ \hat{\delta}_{\text{exp}} : |A| \to |A| \otimes |A|.$$

The theorem is the key to the proof of the formality of the framed Turaev cobracket. Its proof is straightforward and long, and involves the homotopy intersection form \cite{11, 49, 37} in the framework of non-commutative geometry in the sense of van den Bergh \cite{52}. As a corollary we have $\text{gr}^{\text{wt}}(\delta f) = t\text{Div} \circ \text{gr}^{\text{wt}}(\delta)$.

Let $F$ be an element of $\text{TAut}$ such that $\theta := F^{-1} \circ \theta_{\text{exp}}$ is a special expansion. The conjugate $F^* \delta_{\text{exp}}$ by $F$ equals the induced cobracket $\delta f_{\text{exp}}$ on $|A|$ by the isomorphism $\theta$. 21
Then we have $F^*\delta^f_{\exp} = (F^*\text{tDiv}) \circ (F^*\delta_{\exp})$. Any special expansion induces the formal description of $\delta$, i.e., $F^*\delta_{\exp} = \text{gr}^{\text{wt}}(\partial)$. On the other hand, if we denote the Lie algebra coboundary operator by $\partial$, i.e., $(\partial w)(u) = u \cdot w$ for $u \in \text{tDer}$ and $w \in |A| \otimes |A|$, one can prove $F^*\text{tDiv}(u) = \text{tDiv}(u) + u \cdot J(F^{-1}) = (\text{tDiv} + \partial J(F^{-1}))(u) = (\text{tDiv} + \partial \Delta j(F^{-1}))(u)$ [4 Proposition 4.9]. Hence we have

$$\delta^f_\theta = F^*\delta^f_{\exp} = \text{gr}^{\text{wt}} \delta^f + (\partial \Delta j(F^{-1})) \circ \text{gr}^{\text{wt}}(\partial).$$

Hence, if $j(F^{-1})$ is in the center of the bracket $[-, -]_{\text{gr}^{\text{wt}}}$, we obtain $\delta^f_\theta = \text{gr}^{\text{wt}} \delta^f$. Conversely assume $\delta^f_\theta = \text{gr}^{\text{wt}} \delta^f$. Looking at the $1 \otimes |A|$-part, we find out that $|\text{gr}^{\text{wt}}(\sigma)(j(F^{-1}))(w)| = |j(F^{-1}), |w||_{\text{gr}^{\text{wt}}} = 0$ for any $w \in A$ of degree $\geq 2$. From Lemma 4.4 the derivation $\text{gr}^{\text{wt}}(\sigma)(j(F^{-1}))$ is inner, and so $j(F^{-1}) \in |A|$ is in the center of the bracket $[-, -]_{\text{gr}^{\text{wt}}}$. This proves Theorem 5.2.

5.2 Higher genus Kashiwara-Vergne problems

Theorem 5.2 provides a necessary and sufficient condition for a special expansion to induce the formal description of the Turaev cobracket in genus 0. This leads us to generalize the Kashiwara-Vergne problem to positive genus surfaces: If an element $F \in \text{TAut}^{(g,n+1)}$ satisfies the condition (KVI), and the expansion $F^{-1} \circ \theta_{\exp}$ induces the formal description of the Turaev cobracket, then it should be a solution to the higher genus Kashiwara-Vergne problem.

For this purpose, we need to modify the divergence. In fact, the topological nature of $\alpha_i$’s and $\beta_i$’s are quite different from that of $\gamma_j$’s, which brings us a function

$$r(s) := \log((e^s - 1)/s).$$

Moreover we consider all possible framings on the surface $S = \Sigma_{g,n+1}$. Recall we fix the generating system $\{\alpha_i, \beta_i, \gamma_j\}$ of the fundamental group $\pi = \pi_1(S, \ast)$. There is a unique framing $f_{\text{adp}}$ on $S$ such that $\text{rot}_{\text{adp}}(\langle \alpha_i \rangle) = \text{rot}_{\text{adp}}(\langle \beta_i \rangle) = 0$ for any $1 \leq i \leq g$ and $\text{rot}_{\text{adp}}(\partial_j S) = -1$ for any $1 \leq j \leq n$. For any framing $f$ on $S$, there exists a unique $\chi \in H^1(S; \mathbb{Z})$ such that $\text{rot}_{\text{adp}}(\langle \alpha \rangle) = \text{rot}_{\text{adp}}(\langle \alpha \rangle) + \langle \chi, [\alpha] \rangle$ for any immersed loop $\alpha$. We introduce $q \in \text{Hom}(H^{(2)}, \mathbb{Z})$ and $p \in H/H^{(2)} \subset \text{gr}H$ by $q := c_{f/H^{(2)}} = \chi|_{H^{(2)}}$ and $p(q, \cdot) = \chi - c_f : H/H^{(2)} \to \mathbb{Z}$. Here we remark the intersection number on $H/H^{(2)}$ is non-degenerate. The map $c_q : \text{tDer} \to |A|, (u, u_1, \ldots, u_n) \mapsto \sum_{j=1}^n q(z_j)|u_j|$ restricts to a cocycle on $\text{tDer}^+$, and integrates to a group cocycle $C_q : \text{TAut} \to |A|$. In this situation the double divergence $\text{tDiv} : \text{tDer} \to |A| \otimes |A|$ maps $(u, u_1, \ldots, u_n)$ to

$$\left| \sum_i \frac{\partial}{\partial x_i} u(x_i) + \frac{\partial}{\partial y_i} u(y_i) \right| + \sum_{j=1}^n ((z_j \otimes 1)(\frac{\partial}{\partial z_j} u_j) - (\frac{\partial}{\partial z_j} u_j)(1 \otimes z_j)).$$

It is a cocycle and integrates to a group cocycle $J : \text{TAut} \to |A| \otimes |A|$. In a similar way to the genus 0 case, there exists a unique group cocycle $j : \text{TAut} \to |A|$ such that $J = \Delta \circ j$. We denote $j_q := j - C_q : \text{TAut} \to |A|$, and $r := \sum_i |r(x_i) + r(y_i)|$ for $r(s) = \log((e^s - 1)/s)$.  

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Definition 5.4 ([2]), An element $F \in \text{TAut}$ is a solution to the Kashiwara-Vergne for the framed surface $(\Sigma_{g,n+1}, f)$ if it satisfies the condition (KVI) and the second condition

$$j_q(F) = r + |p| + \sum_{j=1}^{n} |h_j(z_j)| - |h(\xi)|$$

for some $h(s)$ and $h_j(s) \in \mathbb{K}[[s]]$, $1 \leq j \leq n$. Here $\xi := \log \left( \prod_{i=1}^{g} (e^{x_i} e^{y_i} e^{-x_i} e^{-y_i}) \prod_{j=1}^{n} e^{z_j} \right) \in L \subset A$ is the RHS of the condition (KVI).

In the case $g > 0$ we don’t know if there is a non-trivial relation among the functions $h$ and $h_j$’s or not. By Theorem 4.5, the second condition is equivalent to the condition

$$(\text{KVII'}) \quad j_q(F^{-1}) + F^{-1}(r + |p|) \in Z(\text{gr wt}(|K\pi|), [-, -]_{\text{gr wt}}).$$

With few exceptions there exist solutions to the problem.

Theorem 5.5 ([4, Theorem 6.1]).
1. If $g \geq 2$, then the Kashiwara-Vergne problem has a solution for any framing.
2. Assume $g = 1$. Then the Kashiwara-Vergne problem has a solution if and only if $q \neq 0$ or $q = p = 0$.

We construct solutions by gluing solutions in low genera. In particular, our construction in the case $g = 1$ follows Enriquez’ construction of elliptic associators [20].

Recently Hain [23] proved that the framed cobracket is a morphism of mixed Hodge structures. Combining his result and Theorems 5.2 and 5.6, one can deduce the existence of solutions to the Kashiwara-Vergne problem except the genus 1 case with $q = 0$ and $p \neq 0$ [23].

The problem is related to the formality of the Turaev cobracket, as desired.

Theorem 5.6 ([4]). Let $F \in \text{TAut}$ satisfy the condition (KVI). Then the special expansion $F^{-1} \circ \theta_{\text{exp}}$ induces the formal description of the framed Turaev cobracket $\delta^f$, or equivalently, the isomorphism of the Lie bialgebra

$$F^{-1} \circ \theta_{\text{exp}} : (|K\pi|, [-, -], \delta^f) \to (\text{gr wt}(|K\pi|), [-, -]_{\text{gr wt}}, \text{gr wt}(\delta^f)),$$

if and only if $F$ is a solution to the Kashiwara-Vergne problem for the framed surface.

An analog of Theorem 5.3 is the key to the proof of the theorem. We define $\text{gDiv}^f : \text{tDer} \to |A| \otimes |A|$ by $\text{gDiv}^f := \text{tDiv} + \partial(\Delta r + |p| \wedge 1) - c_q \wedge 1$. Here $\partial$ means the Lie algebra coboundary operator.

Theorem 5.7 ([4, Corollary 4.8]).

$$\delta^f_{\text{exp}} = \text{gDiv}^f \circ \delta_{\text{exp}} : |A| \to |A| \otimes |A|.$$
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