Theorem 0.2 in the author’s paper [14] asserts that a Lagrangian Klein bottle in a projective complex surface must have non-zero mod 2 homology class. A gap in the topological part of the proof of this result was pointed out by Leonid Polterovich. (It is erroneously claimed in §3.6 that a diffeomorphism of an oriented real surface acts in some natural way on the spinor bundle of the surface.)

Recently, Vsevolod Shevchishin corrected both the statement and the proof of that theorem. On the one hand, he showed that the result is false as it stands by producing an example of a nullhomologous Lagrangian Klein bottle in a bi-elliptic surface. On the other hand, he proved that the conclusion holds true under an additional assumption which, in retrospect, appears to be rather natural.

**Theorem A** (Shevchishin [18]). Let \( K \subset X \) be an embedded Lagrangian Klein bottle in a compact symplectic four-manifold \( (X, \omega) \). Assume that \( c_1(X, \omega) \cdot [\omega] > 0 \). Then the homology class \( [K] \in H_2(X; \mathbb{Z}/2) \) is non-zero.

It was shown by Liu and Ohta–Ono that a symplectic four-manifold satisfies \( c_1(X, \omega) \cdot [\omega] > 0 \) if and only if it is symplectomorphic either to \( \mathbb{C}P^2 \) with its standard symplectic structure or to (a blow-up of) a ruled complex surface equipped with a suitable Kähler form (see [10], Theorem B, [16], Theorem 1.2, and also [12], Corollary 1.4).

Theorem A implies that the Klein bottle does not admit a Lagrangian embedding into the standard symplectic \( \mathbb{R}^4 \). Indeed, if such an embedding existed, one could produce a nullhomologous Lagrangian Klein bottle in a Darboux chart on any \( (X, \omega) \). (See also Remark 2.4 below.) The other results proved in [14] using Theorem 0.2 follow as well from Theorem A.

Shevchishin’s proof of Theorem A uses the Lefschetz pencil approach proposed in [14], the combinatorial structure of mapping class groups, and the above description of symplectic manifolds with \( c_1(X, \omega) \cdot [\omega] > 0 \). The purpose of the present paper is to give an alternative proof in which the first two ingredients are replaced by somewhat more traditional four-manifold topology. It should be noted that though closer in spirit to the work of Polterovich [17] and Eliashberg–Polterovich [4], this argument has been found by interpreting the results obtained in [18] in other geometric terms.

The contents of the paper should be clear from its section titles. For a more comprehensive discussion of Givental’s Lagrangian embedding problem for the Klein bottle [6], see the introductions to [14] and [18].

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1. **Self-linking indices for totally real surfaces**

1.1. **Characteristic circles.** Let \( S \) be a closed real surface. The characteristic homology class \( u \in H_1(S; \mathbb{Z}/2) \) is uniquely defined by the condition

\[
u \cdot x = x \cdot x = (w_1(S), x) \quad \text{for all } x \in H_1(S; \mathbb{Z}/2),
\]

where \( w_1(S) \) is the first Stiefel–Whitney class of \( S \). Note that \( u = 0 \) if and only if \( S \) is orientable.

**Lemma 1.1.** Let \( \xi \in H^1(S; \mathbb{Z}) \) be an integral cohomology class. Then \((\xi \mod 2, u) = 0 \).

**Proof.** Any integral cohomology class on a surface can be represented by the intersection index with a two-sided curve. The intersection of \( u \) with such a curve is zero by the definition of \( u \).

**Definition 1.2.** A **characteristic circle** is a simple closed curve \( \ell \subset S \) in the characteristic homology class.
Example 1.3. Represent the Klein bottle $K$ as the non-trivial circle bundle over the circle. A fiber $m \subset K$ of this bundle is a meridian of $K$. It is easy to check that $m$ is a characteristic circle on $K$.

Lemma 1.4. If $S \subset X$ is an embedded surface in a smooth manifold such that $[S] = 0 \in H_2(X;\mathbb{Z}/2)$, then $[\ell] = 0 \in H_1(X;\mathbb{Z}/2)$ for any characteristic circle $\ell \subset S$.

Proof. If $S$ is orientable, there is nothing to prove. Otherwise, let $\xi \in H^1(X;\mathbb{Z}/2)$ be any mod 2 cohomology class. The obstruction to lifting it to an integral cohomology class lives in $H^2(X;\mathbb{Z})$. Consider the commutative diagram

$$
\begin{array}{ccc}
H^2(X;\mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{Z}/2) \\
\downarrow & & \downarrow 0 \\
H^2(S;\mathbb{Z}) & \overset{\cong}{\longrightarrow} & H^2(S;\mathbb{Z}/2)
\end{array}
$$

where the vertical arrows are restrictions to $S$ and horizontal arrows reductions modulo 2. It follows that the map $H^2(X;\mathbb{Z}) \to H^2(S;\mathbb{Z})$ is trivial. Hence, the restriction of $\xi$ to $S$ lifts to an integral cohomology class on $S$ and has zero pairing with $u = [\ell]$ by the previous lemma. Thus, $[\ell] = 0 \in H_1(X;\mathbb{Z}/2)$ by Poincaré duality over $\mathbb{Z}/2$. \qed

1.2. Viro index (cf. [19], §4). An embedded surface $S \subset X$ in an almost complex four-manifold $(X,J)$ is called totally real if $J(T_pS)$ is transverse to $T_pS$ at every point $p \in S$.

Definition 1.5. Let $\ell \subset S$ be a two-sided simple closed curve on a totally real surface $S \subset X$. Its $\mathbb{C}$-normal pushoff $\ell^2$ is the isotopy class of curves in $X \setminus S$ containing the pushoff of $\ell$ in the direction of the vector field $J\nu_{\ell,S}$, where $\nu_{\ell,S}$ is a non-vanishing normal vector field to $\ell$ in $S$.

The $\mathbb{C}$-normal pushoff is well-defined because any two non-vanishing normal vector fields on $\ell \subset S$ are homotopic through non-vanishing sections of $TS|_\ell$.

Definition 1.6. Let $S \subset X$ be a totally real surface with $[S] = 0 \in H_2(X;\mathbb{Z}/2)$ in an almost complex four-manifold $(X,J)$, and let $\ell \subset S$ be a two-sided simple closed curve such that $[\ell] = 0 \in H_1(X;\mathbb{Z}/2)$. The Viro index of $\ell$ is the modulo 2 linking number

$$V(\ell) = \text{lk}(\ell^2, S),$$

where $\ell^2$ is the $\mathbb{C}$-normal pushoff of $\ell$.

Remark. It is known that $V(\ell)$ depends only on the homology class of $\ell$, and the map $[\ell] \mapsto 1 + V(\ell)$ is a quadratic function called the Viro form of $S \subset X$.

Lemma 1.7. Let $S \subset X$ be a totally real surface such that $(w_2(X),[S]) = 0$. Then every characteristic circle $\ell \subset S$ is two-sided.

Proof. Note that $TX|_S \cong TS \oplus TS$ and therefore $w_2(TX|_S) = w_1(S) \cup w_1(S)$ by the Whitney formula. Hence, $(w_1(S),[\ell]) = w_1(S)^2 = (w_2(TX),[S]) = 0$, which proves the lemma. \qed

Lemmas [14] and [17] show that the Viro index is defined for any characteristic circle on a nullhomologous totally real surface. If the ambient manifold is compact, then we have the following formula which is closely related to the results of Netsvetaev [15], Fiedler [5], and Polterovich [17].

Theorem 1.8. Let $\ell$ be a characteristic circle on a totally real surface $S$ with $[S] = 0 \in H_2(X;\mathbb{Z}/2)$ in a compact almost complex four-manifold $X$. Then

$$V(\ell) = 1 + \frac{\chi(S)}{4} \mod 2,$$

where $\chi(S)$ is the Euler characteristic of $S$.

Example 1.9 (Klein bottle case). If $\ell = m$ is a meridian of a totally real homologically trivial Klein bottle $K \subset X$, then the theorem shows that $V(m) = 1 \mod 2$. In other words, the $\mathbb{C}$-normal pushoff of the meridian is non-trivially linked with $K$, which is the result that will be used in the proof of Theorem A.
1.3. Proof of Theorem 1.8. Let us first construct a non-vanishing section of the restriction of the complex determinant bundle $\Lambda^2 TS$ to the surface $S$ which has standard form on $\ell$.

**Lemma 1.10.** Let $\tau_\ell$ be a non-vanishing vector field tangent to the curve $\ell$. Then the complex wedge product $\sigma_\ell := \tau_\ell \wedge \nu_\ell, S$ extends to a non-vanishing section of $\Lambda^2 TS|_S$.

**Proof.** The real wedge product $\tau_\ell \wedge \nu_\ell, S$ is a non-vanishing section of the real determinant bundle $\Lambda^2 TS$ restricted to $\ell$. Note first that it can be extended to a global non-vanishing section of the complexification $\Lambda^2 TS \otimes \mathbb{C}$ which is zero because the complement to the characteristic circle $\ell$ is orientable.

It remains to note that, for the totally real surface $S \subset X$, the map

$$\Lambda^2 TS \otimes \mathbb{C} \longrightarrow \Lambda^2 TX|_S$$

defined by replacing the real wedge product on $TS$ by the complex one on $TX$ is an isomorphism. \hfill \Box

**Example 1.11.** Consider the Klein bottle $K$ represented as the quotient of the two-torus by the equivalence relation $(\varphi, \psi) \sim (\varphi + \pi, -\psi)$ (see Subsec. 2.1 below). Then $e^{i\varphi} \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial \psi}$ is a well-defined non-vanishing section of $\Lambda^2 TK \otimes \mathbb{C}$ extending the section $\tau_m \wedge \nu_{m, K} = \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial \psi}$ from the meridian $m = \{\varphi = 0\} \subset K$.

Let $\sigma \in \Gamma(X, \Lambda^2 TX)$ be a generic global extension of the section constructed in the lemma. Then the zero set

$$\Sigma = \{x \in X \mid \sigma(x) = 0\}$$

is an oriented two-dimensional submanifold of $X$ disjoint from $S$. Note that $\Sigma$ is mod 2 Poincaré dual to the cohomology class $c_1(\Lambda^2 TX) \bmod 2 = w_2(X)$. Thus, both $\Sigma$ and $\Sigma \cup S$ are characteristic submanifolds of $X$, i.e., their mod 2 homology classes are Poincaré dual to the second Stiefel–Whitney class of $X$.

Let further $M \subset X$ be an embedded surface with boundary $\partial M = \ell$ that is normal to $S$ along $\ell$ and whose interior intersects $S$ and $\Sigma$ transversally. Such surfaces exist (because $[\ell] = 0 \in H_1(X; \mathbb{Z}/2)$ by Lemma 1.3) and are called membranes for $\ell \subset S \cup \Sigma$.

The **Rokhlin index** of $M$ (with respect to $\Sigma \cup S$) is defined by the formula

$$R(M) = n(M, \nu_{\ell, S}) + \#(M \cap S) + \#(M \cap \Sigma),$$

where $n(M, \nu_{\ell, S}) \in \mathbb{Z}$ is the obstruction to extending $\nu_{\ell, S}$ to a non-vanishing normal vector field on $M$, and $\#(M \cap S)$ and $\#(M \cap \Sigma)$ are the numbers of interior intersection points of $M$ with $S$ and $\Sigma$, respectively.

**Lemma 1.12.** $R(M) = \frac{S \cdot S}{4} \bmod 2$, where $S \cdot S \in \mathbb{Z}$ is the normal Euler number of $S \subset X$.

**Proof.** To any characteristic two-dimensional submanifold $F \subset X$ there is associated a quadratic function

$$q_F : \ker \iota_u \longrightarrow \mathbb{Z}/4$$

on the kernel of the inclusion homomorphism $\iota_u : H_1(F; \mathbb{Z}/2) \rightarrow H_1(X; \mathbb{Z}/2)$ called the **Rokhlin–Guillou–Marin form** of $F$ (see [7], [1], and [2], §2.6). The value of this function on the characteristic homology class $u \in H_1(F; \mathbb{Z}/2)$ (assuming that $u \in \ker \iota_u$) satisfies the congruence

$$q_F(u) = \frac{\text{sign}(X) - F \cdot F}{2} \bmod 4,$$  \hfill (1.1)

where sign$(X)$ is the signature of the four-manifold $X$. This formula is due to Rokhlin (at least in the case when $\iota_u = 0$, see [8], no 4). It may be obtained from the generalised Rokhlin–Guillou–Marin congruence (see [2], Theorem 2.6.1, or [1], Théorème 3) by reducing it modulo 8 and plugging in an elementary algebraic property of the Brown invariant (see [2], 3.4.4(2)).

Applying (1.1) to the characteristic submanifolds $\Sigma \cup S$ and $\Sigma$, respectively, and using the orientability of $\Sigma$, we obtain that

$$q_{\Sigma \cup S}([\ell]) = \frac{\text{sign}(X) - \Sigma \cdot \Sigma - S \cdot S}{2} \bmod 4$$

and

$$0 = \frac{\text{sign}(X) - \Sigma \cdot \Sigma}{2} \bmod 4.$$  \hfill (The second congruence follows also from van der Blij’s lemma, see [13], §II.5.) Thus,

$$q_{\Sigma \cup S}([\ell]) = -\frac{S \cdot S}{2} \bmod 4.$$
whence

\[ R(M) = \frac{S \cdot S}{4} \mod 2 \]

because \( q_{\Sigma^2}(f) = 2R(M) \mod 4 \) by the definition of the Rokhlin–Guillou–Marin form.

Let us now choose the membrane \( M \) more carefully. Namely, assume henceforth that it has the following additional properties:

1) The tangent (half-)space to \( M \) at each point \( p \in \ell \) is spanned by \( \tau(\ell)(p) \) and \( J\nu_{\ell,S}(p) \). Note that these two vectors in \( T_pX \) are linearly independent over \( \mathbb{C} \) so that \( M \) is totally real near its boundary.

2) The complex points of \( M \) (i.e., the points \( p \in M \) such that \( J(T_pM) = T_pM \)) are generic.

The first property is achieved by spinning \( M \) around \( \ell \), and the second one by a small perturbation of the result. The Rokhlin index of this special membrane \( M \) can be calculated from a modulo 2 version of Lai’s formulas for the number of complex points of \( M \) (see \([9]\) and \([3]\), §4.3).

**Lemma 1.13.** \( R(M) = 1 + V(\ell) \mod 2 \).

**Proof.** Note first that the pushoff of \( \ell \) inside \( M \) is the \( \mathbb{C} \)-normal pushoff of \( \ell \) by property (1) of \( M \). Hence, \( V(\ell) = \#(M \cap S) \mod 2 \) by the definition of the Viro index. Thus, we need to show that

\[ n(M, \nu_{\ell,S}) + \#(M \cap \Sigma) = 1 \mod 2. \]

The obstruction \( n(M, \nu_{\ell,S}) \) can be computed as follows. Let \( \tau \) be a generic tangent vector field on \( M \) such that \( \tau = -J\nu_{\ell,S} \) on \( \ell = \partial M \). (Note that \( \tau \) is transverse to \( \partial M \).) Then \( J\tau \) fails to give a non-vanishing normal extension of \( \nu_{\ell,S} \) at the points \( p \in M \) such that \( J\tau(p) \in T_pM \). These points are, firstly, the zeroes of \( \tau \) and, secondly, the complex points of \( M \). Neglecting the signs involved, we get the modulo 2 formula

\[ n(M, \nu_{\ell,S}) = \chi(M) + c(M) \mod 2, \quad (1.2) \]

where \( \chi(M) \) is the Euler characteristic of \( M \) and \( c(M) \) is the number of complex points on \( M \).

Similarly, let us consider the modulo 2 obstruction to extending the section \( \sigma_\ell \) (from Lemma \[1.10\]) to a non-vanishing section of \( \Lambda^2 T^*X|_M \). On the one hand, it is equal to \( \#(M \cap \Sigma) \mod 2 \) because \( M \cap \Sigma \) is the transverse zero set of such an extension \( \sigma_\ell \). On the other hand, observe that

\[ \sigma_\ell = \tau_\ell \wedge_C \nu_{\ell,S} = -\sqrt{-1} \tau_\ell \wedge_C J\nu_{\ell,S} = -\sqrt{-1} \tau_\ell \wedge_C \nu_{\ell,M}. \]

As we have already seen in the proof of Lemma \[1.10\], the obstruction to extending \( \tau_\ell \wedge_C \nu_{\ell,M} \) from \( \partial M \) to a non-vanishing section of \( \Lambda^2 T^*X|_M \) is the sum of two obstructions. Firstly, \( \Lambda^2 T^*M \otimes \mathbb{C} \) can be non-trivial. Secondly, the map \( \Lambda^2 T^*M \otimes \mathbb{C} \to \Lambda^2 T^*X|_M \) degenerates at the complex points of \( M \). Altogether, we see that

\[ \#(M \cap \Sigma) = w^2_1(M, \partial M) + c(M) \mod 2. \quad (1.3) \]

Combining formulas (1.2) and (1.3) gives the congruence

\[ n(M, \nu_{\ell,S}) + \#(M \cap \Sigma) = \chi(M) + w^2_1(M, \partial M) \mod 2. \]

The right hand side is equal to 1 mod 2 for any surface \( M \) with a single boundary component, and the lemma follows.

It is now easy to complete the proof of the theorem. Lemmas \[1.13\] and \[1.12\] show that

\[ V(\ell) = 1 + \frac{S \cdot S}{4} \mod 2, \]

and it remains to observe that \( S \cdot S = -\chi(S) \) for any totally real embedded surface \( S \).

\[ \square \]

2. Application of Luttinger’s surgery

2.1. Explicit model (cf. \([11]\) and \([4]\), §2). Let us first consider the (trivial) cotangent bundle \( T^*T \) of the two-torus with the coordinates

\[ (\varphi, \psi, r, \theta) \in \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z}, \]

where \( (\varphi, \psi) \) are the standard coordinates on the torus and \( (r, \theta) \) are the polar coordinates on the fibre.

The Luttinger twist \( f_{n,k} \) is the diffeomorphism of the hypersurface \( \{ r = 1 \} \subset T^*T \) given by

\[ (\varphi, \psi, \theta) \mapsto (\varphi + n\theta, \psi + k\theta, \theta), \]
for a pair of integers \((n,k) \in \mathbb{Z}^2\). The crucial property of this map is that it preserves the restriction of the canonical symplectic form to the hypersurface \(\{r = 1\}\).

The Klein bottle \(K\) is the quotient of the torus by the equivalence relation
\[
(\varphi, \psi) \sim (\varphi + \pi, -\psi).
\]
Hence, the cotangent bundle of \(K\) is the quotient of the cotangent bundle of the torus by the relation
\[
(\varphi, \psi, r, \theta) \sim (\varphi + \pi, -\psi, r, -\theta).
\]
It follows that the Luttinger twists \(f_{0,k}\) with \(n = 0\) descend to the hypersurface \(\{r = 1\} \subset T^*K\).

On \(T^*K\) we consider the Kähler structure \((\gamma_0, \omega_0, J_0)\) defined by the flat metric and the canonical symplectic form. This structure lifts to the similarly defined structure on \(T^*T\) and is given by the same formulas in the coordinates \((\varphi, \psi, r, \theta)\).

**Lemma 2.1.** The \(\mathbb{C}\)-normal pushoff of the meridian \(m = \{\varphi = 0\} \subset K\) in \((T^*K, J_0)\) is given by the curve \(\{\varphi = 0, r = 1, \theta = 0\}\).

**2.2. Reconstructive surgery.** Let \(K \subset X\) be a totally real Klein bottle embedded in an almost complex four-manifold \((X, J)\). Then there exists a closed tubular neighbourhood \(N \supset K\) orientation preserving diffeomorphic to the unit disc bundle \(DT^*K = \{r \leq 1\} \subset T^*K\). Furthermore, the diffeomorphism can be chosen so that the almost complex structure \(J\) on \(TX|_K\) corresponds to the standard complex structure \(J_0\).

**Theorem 2.2.** Let \(K \subset X\) be a nullhomologous totally real Klein bottle in a compact almost complex four-manifold. Then the Luttinger surgery

\[
X' = N \cup_{f_{0,1}} (X \setminus N)
\]
makes the Klein bottle \(K\) homologically non-trivial.

**Proof.** By Lemma 1.3 the meridian \(m \subset K\) is nullhomologous in \(X\) (modulo 2). The Mayer–Vietoris sequence for \(X = N \cup (X \setminus N)\) shows that there exists a homology class \(\zeta \in H_1(\partial N; \mathbb{Z}/2)\) such that

1) \(i_* \zeta = [m] \in H_1(N; \mathbb{Z}/2),\)

2) \(j_* \zeta = 0 \in H_1(X \setminus N; \mathbb{Z}/2),\)

where \(i : \partial N \to N\) and \(j : \partial N \to X \setminus N\) are inclusion maps. It follows from the second property that

\[
\text{lk}(\zeta, K) = 0.
\]

Therefore, Theorem 1.8 shows that \(\zeta\) differs from the homology class of the \(\mathbb{C}\)-normal pushoff \(m^2\) (see Example 1.9).

The pre-image \(i^{-1}([m]) \in H_1(\partial N; \mathbb{Z}/2)\) consists of exactly two homology classes, represented by the curves \(m' = \{\varphi = 0, r = 1, \theta = 0\}\) and \(m'' = \{\varphi = 0, r = 1, \theta = \psi\}\). The first curve gives the \(\mathbb{C}\)-normal pushoff of the meridian by Lemma 2.1. Thus, \(\zeta = [m'']\) and therefore \(m''\) bounds in \(X \setminus N\).

Now consider the fibre \(\{\varphi = 0, \psi = 0, r = 1\}\) of the projection \(\partial N \to K\). The Luttinger twist

\[
f_{0,1}(\varphi, \psi, 1, \theta) := (\varphi, \psi + \theta, 1, \theta)
\]
maps this fibre to the curve \(m''\). It follows that the disc \(\Delta = \{\varphi = 0, \psi = 0, r \leq 1\}\) bounded by the fibre in \(N\) and the chain bounded by \(m''\) in \(X \setminus N\) are glued together into a 2-cycle in \(X'\). The intersection index of this cycle with \(K\) equals \(\#(K \cap \Delta) = 1\) mod 2, and therefore \(K\) is homologically non-trivial in \(X'\).

**Corollary 2.3.** \(\dim_{\mathbb{Z}/2} H_1(X'; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_1(X; \mathbb{Z}/2) + 1\).

**Proof.** Consider the following long exact sequences in cohomology with compact support:

\[
\cdots \to H^0_c(X; \mathbb{Z}/2) \to H^2_c(K; \mathbb{Z}/2) \to H^1_c(X \setminus K; \mathbb{Z}/2) \to H^0_c(X; \mathbb{Z}/2) \to H^2_c(K; \mathbb{Z}/2) \cong 0
\]
and

\[
\cdots \to H^1_c(X'; \mathbb{Z}/2) \to H^3_c(K; \mathbb{Z}/2) \to H^2_c(X' \setminus K; \mathbb{Z}/2) \to H^1_c(X'; \mathbb{Z}/2) \to H^3_c(K; \mathbb{Z}/2) \cong 0.
\]

The first map in the first sequence is trivial because \(K\) is nullhomologous in \(X\). Hence,

\[
\dim_{\mathbb{Z}/2} H^3_c(X; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^3_c(X \setminus K; \mathbb{Z}/2) - \dim_{\mathbb{Z}/2} H^3_c(K; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^3_c(X' \setminus K; \mathbb{Z}/2) - 1.
\]

On the other hand, the first map in the second sequence is onto because \([K] \neq 0\) in \(X'\), and therefore

\[
\dim_{\mathbb{Z}/2} H^3_c(X'; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^3_c(X' \setminus K; \mathbb{Z}/2).
\]
Since $X \setminus K$ and $X' \setminus K$ are diffeomorphic, we conclude that
\[\dim_{\mathbb{Z}/2} H^3_c(X'; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^3_c(X; \mathbb{Z}/2) + 1,\]
and the result follows because $H_1(Y; \mathbb{Z}/2) \cong H^3_c(Y; \mathbb{Z}/2)$ for any four-manifold $Y$ by Poincaré duality. \qed

2.3. Symplectic rigidity: Proof of Theorem \[A\]. Let now $K \subset X$ be a Lagrangian Klein bottle in a compact symplectic four-manifold $(X, \omega)$. Then $K$ is totally real with respect to any almost complex structure $J$ on $X$ compatible with $\omega$.

Let $X'$ be the manifold obtained from $X$ by the Luttinger surgery used in Theorem \[A\] since $K$ is Lagrangian, it follows that the Luttinger surgery can be performed symplectically, i.e., there exists a symplectic form $\omega'$ on $X'$ that coincides with $\omega$ on $X' \setminus N = X \setminus N$. This is proved in exactly the same way as the analogous statement for Lagrangian tori in \[A\]. A different proof using symplectic Lefschetz pencils is given in \[A\].

Note that $c_1(X, \omega) \cdot [\omega] = c_1(X', \omega') \cdot [\omega']$. (The proof follows easily from the fact that $H^2(N; \mathbb{R}) = 0$ and is left to the reader.) Thus, if $c_1(X, \omega) \cdot [\omega] > 0$, then $c_1(X', \omega') \cdot [\omega'] > 0$, and hence each of the manifolds $X$ and $X'$ is diffeomorphic to $\mathbb{C}P^2$ or to (a blow-up of) a ruled surface by the result of Liu and Ohta–Ono cited in the introduction. In particular, the $\mathbb{Z}/2$-dimensions of $H_1(X; \mathbb{Z}/2)$ and $H_2(X'; \mathbb{Z}/2)$ are even.

However, if $[K] = 0 \in H_2(X; \mathbb{Z}/2)$, then $\dim_{\mathbb{Z}/2} H_1(X'; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_1(X; \mathbb{Z}/2) + 1$ by Corollary \[B\] and at least one of these dimensions is odd. This contradiction proves Theorem \[A\]. \qed

Remark 2.4. It is not hard to specialise our proof of Theorem \[A\] to yield just the non-existence of embedded Lagrangian Klein bottles in $(\mathbb{R}^4, \omega_0)$. Several steps in the argument are then simplified or avoided. For instance, Gromov’s original theorem about symplectic four-manifolds symplectomorphic to $(\mathbb{R}^4, \omega_0)$ at infinity may be used instead of the more involved results of Liu and Ohta–Ono.

References

[1] C. Baily, A. Vdovina, *Sous-espaces déterminant l’invariant de Arf et un théorème de Rohlin sur la signature*, C. R. Acad. Sci. Paris Sér. I Math. 330-3 (2000), 221–223.

[2] A. Degtyarev, I. Itenberg, V. Kharlamov, *Real Enriques Surfaces*, Lecture Notes in Mathematics 1746, Springer-Verlag, Berlin, 2000.

[3] A. Degtyarev, V. Kharlamov, *Topological properties of real algebraic varieties: Rohlin’s way*, Russian Math. Surveys 55:4 (2000), 735–814.

[4] Y. Eliashberg, L. Polterovich, *New applications of Luttinger’s surgery*, Comment. Math. Helv. 69 (1994), 512–522.

[5] Th. Fiedler, *Totally real embeddings of the torus into $\mathbb{C}^3$*, Ann. Global Anal. Geom. 5:2 (1987), 117–121.

[6] A. B. Givental, *Lagrangian embeddings of surfaces and the open Whitney umbrella*, Functional Anal. Appl. 20:3 (1986), 197–203.

[7] L. Guillou, A. Marin, *Une extension d’un théorème de Rohlin sur la signature*, in *À la recherche de la topologie perdue*, pp. 97–118, Birhäuser, Boston, 1986.

[8] Appendice: *Rohlin et son théorème (d’après une lettre de O. Viro et V. Harlamov)*, in *À la recherche de la topologie perdue*, pp. 153–155, Birhäuser, Boston, 1986.

[9] H. F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. 172 (1972), 1–33.

[10] A.-K. Liu, *Some new applications of general wall crossing formula, Gompf’s conjecture and its applications*, Math. Res. Lett. 3 (1996), 569–585.

[11] K. M. Luttinger, *Lagrangian tori in $\mathbb{R}^4$*, J. Differential Geom. 42 (1995), 220–228.

[12] D. McDuff, D. Salamon, *A survey of symplectic 4-manifolds with $b^+ = 1$*, Turkish J. Math. 20 (1996), 47–60.

[13] J. Milnor, D. Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York-Heidelberg, 1973.

[14] S. Nemirovski, *Lefschetz pencils, Morse functions, and Lagrangian embeddings of the Klein bottle*, Izv. Math. 66:1 (2002), 151–164; arXiv:math/0106122v2.

[15] N. Netsvetaev, *On an analogue of the Maslov index*, J. Math. Sci. 81 (1996), no. 2, 2535–2537.

[16] H. Ohta, K. Ono, *Notes on symplectic 4-manifolds with $b^+ = 1$*, II. Internat. J. Math. 7 (1996), 755–770.

[17] L. Polterovich, *Strongly optical Lagrangian manifolds*, Math. Notes 45 (1989), 152–158.

[18] V. Shevchishin, *Lagrangian embeddings of the Klein bottle and combinatorial properties of mapping class groups*, Preprint, May 2006, 48 pp; arXiv:0707.2085v1.

[19] O. Viro, *Complex orientations of real algebraic surfaces*, in *Topology of manifolds and varieties*, pp. 261–284, Adv. Soviet Math. 18, AMS, Providence, RI, 1994.

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