QUANTITATIVE
LOWER BOUND FOR LIFESPAN FOR SOLUTION
OF NAIVER-STOKES EQUATIONS.

E.Ostrovsky\textsuperscript{a}, L.Sirota\textsuperscript{b}

\textsuperscript{a} Corresponding Author. Department of Mathematics and computer science,
Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: galo@list.ru eugostrovsky@list.ru

\textsuperscript{b} Department of Mathematics and computer science. Bar-Ilan University, 84105,
Ramat Gan, Israel.
E-mail: sirota3@bezeqint.net

Abstract.

We find a simple quantitative lower bound for lifespan of solution of the multidimensional initial value problem for the Navier-Stokes equations in whole space when the initial function belongs to the correspondent Lebesgue - Riesz space, and give some a priory estimations for solution in some rearrangement invariant spaces.

Keywords and phrases: Multivariate Navier-Stokes (NS) equations, Riesz integral transform, rearrangement invariant, Grand and ordinary Lebesgue - Riesz spaces, initial value problem, Helmholtz-Weyl projection, divergence, Laplace operator, Besov, Morrey, Sobolev and Sobolev weight norms and spaces, pseudo - differential operator, global and short-time well - posedness, Young inequality, lifespan of solution.

2000 AMS Subject Classification: Primary 37B30, 33K55, 35Q30, 35K45; Secondary 34A34, 65M20, 42B25.

1 Notations. Statement of problem.

Statement of problem.

We consider in this article the initial value problem for the multivariate Navier-Stokes (NS) equations

\begin{equation}
\partial u_t - \Delta u + (u \cdot \nabla) u = \nabla P, \quad x \in \mathbb{R}^d, \quad d \geq 3, \quad t > 0; \quad (1.1)
\end{equation}

\begin{equation}
\text{Div}(u) = 0, \quad x \in \mathbb{R}^d, \quad t > 0; \quad (1.2)
\end{equation}

\begin{equation}
u(x,0) = a(x), \quad x \in \mathbb{R}^d. \quad (1.3)
\end{equation}
Here as ordinary

\[ x = (x_1, x_2, \ldots, x_k, \ldots, x_d) \in R^d, \quad ||x|| := \sqrt{\sum_{j=1}^{d} x_j^2}. \]

and

\[ u = u(t) = u(t, \cdot) = u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_d(x, t)\} \]

denotes the (vector) velocity of fluid in the point \( x \) at the time \( t \), \( P \) is represents the pressure.

Equally:

\[ \partial u_i/\partial t = \sum_{j=1}^{d} \partial^2_{x_j} u_i - \sum_{j=1}^{d} u_j \partial_{x_j} u_i + \partial u_i P; \]

\[ \sum_{j=1}^{d} \partial_{x_j} u_j = 0, \quad u(x, 0) = a(x), \]

\[ \text{Div} u = \text{Div} \bar{u} = \text{Div}\{u_1, u_2, \ldots, u_d\} = \sum_{k=1}^{d} \frac{\partial u_k}{\partial x_k} = 0 \]
in the sense of distributional derivatives.

As long as

\[ P = \sum_{j,k=1}^{d} R_j R_k(u_j \cdot u_k), \]

where \( R_k = R_k^{(d)} \) is the \( k^{th} \) \( d \) dimensional Riesz transform:

\[ R_k^{(d)} [f](x) = c(d) \lim_{\epsilon \to 0^+} \int_{||y|| > \epsilon} ||y||^{-d} \Omega_k(y) f(x - y) \, dy, \]

\[ c(d) = -\frac{\pi^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}, \quad \Omega_k(x) = x_k/||x||, \]

the system (1.1) - (1.3) may be rewritten as follows:

\[ \partial u_t = \Delta u + (u \cdot \nabla) u + Q \cdot \nabla \cdot (u \otimes u), \quad x \in R^d, \quad t > 0; \quad (1.4) \]

\[ \text{Div}(u) = 0, \quad x \in R^d, \quad t > 0; \quad (1.5) \]

\[ u(x, 0) = a(x), \quad x \in R^d, \quad (1.6) \]

where \( Q \) is multidimensional Helmholtz-Weyl projection operator, i.e., the \( d \times d \) matrix pseudo-differential operator in \( R^d \) with the matrix symbol

\[ a_{i,j}(\xi) = \delta_{i,j} - \xi_i \xi_j / ||\xi||^2, \quad \delta_{i,j} = 1, \quad i = j; \quad \delta_{i,j} = 0, \quad i \neq j. \]
We will understand henceforth as a capacity of the solution (1.4) - (1.6) the vector-function \( u = \vec{u} = \{u_1(x,t), u_2(x,t), \ldots, u_d(x,t)\} \) the so-called mild solution, see [28].

Namely, the vector-function \( u = u(t) \) satisfies almost everywhere in the time \( t \) the following non-linear integral-differential equation:

\[
\begin{align*}
    u(t) &= e^{t\Delta}a + \int_0^t e^{(t-s)\Delta}[(u \cdot \nabla)u(s) + Q \cdot \nabla \cdot (u \otimes u)(s)]ds \\
    &\overset{\text{def}}{=} e^{t\Delta}a + G[u](t) \\
    &\overset{\text{def}}{=} u_0(x,t) + G[u](t),
\end{align*}
\]

(1.7) the operator \( \exp(t\Delta) \) is the classical convolution integral operator with heat kernel:

\[
    u_0(x,t) := e^{t\Delta}[a](x,t) = w_t(x) * a(x),
\]

where

\[
    G(u) \overset{\text{def}}{=} F(u, u) = F(u), \quad F(u, v) := \int_0^t [(u \cdot \nabla)v(s) + Q \cdot \nabla \cdot (u \otimes v)(s)] ds,
\]

\[
    w_t(x) \overset{\text{def}}{=} (4\pi t)^{-d/2} \exp \left( -\frac{||x||^2}{4t} \right) \quad (1.8)
\]

The convolution between two functions \( r = f(t), g(t) \) defined on the set \( R_+ = (0, \infty) \) is defined as usually

\[
    f \odot g(t) = \int_0^t f(t - s) g(s) \ ds = g \odot f(t)
\]

(”time-wise” convolution) and between two, of course, measurable vector-functions \( u(x), v(x) \) defined on the whole space \( x \in R^d \)

\[
    u \ast v(x) = \int_{R^d} u(x - y) \ v(y) \ dy,
\]

(”space wise” and coordinate-wise convolution). The authors hope that this notations does not follow the confusion.

More results about the existence, uniqueness, numerical methods, and a priory estimates in the different Banach function spaces: Lebesgue-Riesz \( L_p \), Morrey, Besov for this solutions see, e.g. in [1]- [41]. The first and besides famous result belong to J.Leray [25]; it is established there in particular the global in time solvability and uniqueness of NS system in the space \( L_2(R^d) \) and was obtained a very interest a priory estimate for solution.

The immediate predecessor for offered article is the article of T.Kato [18]; in this article was considered the case \( a(\cdot) \in L_d(R^d) \). See also celebrate works of H. Fujita and T.Kato [10], Y.Giga [12] - [15], T.Kato [19] etc.

T.Kato in [18] proved in particular that if the initial function \( a = a(x) \) belongs to the space \( L_d(R^d) \) (in our notations), then there exists a positive time value \( T > 0 \) (lifespan of solution) such that the solution of NS system \( u = u(x,t) \) there exists for \( t \in (0,T) \), is smooth and satisfy some a priory integral estimates.
Furthermore, if the norm $||a||_{L_d(R^d)}$ is sufficiently small, then $T = \infty$, i.e. the solution $u = u(x,t)$ is global.

The upper estimate for the value $T$, conditions for finite-time blow-up and asymptotical behavior of solution as $t \to T - 0$ see in the articles [53], [54], [2], [1], [55], [27], [31], [32] etc.

**Our purpose in this report is to obtain the quantitative simple lower estimates for the lifespan of solution** $T$.

For the chemotaxis equations under some additional conditions this estimate is obtained in [16].

In detail, we understand as a solution $u = u(x,t)$, $t \in (0,T)$ together with T.Kato [18] the mild solution of NS equations such that for all the values $q \geq d$

$$t^{(1-d)/2} \cdot u \in BC([0,T], L^0_q), \quad t^{1-d/2q} \cdot \nabla u \in BC([0,T], L^0_q).$$

(1.9)

The critical value of the variable $T$, more exactly, for its lower estimate does not depend on the variable $q$.

Here $T \in (0,\infty)$; the case $T = \infty$ implies the global (in time) solution.

The space $BC([0,T], L^0_q)$ consists by definition on all the functions $v = v(x,t)$, $x \in R^d$, $t \in [0,T]$ with zero divergence and finite norm

$$||v(\cdot,\cdot)||_{BC([0,T], L^0_q)} := \sup_{t \in [0,T]} ||v(\cdot, t)||_q.$$  

(1.10)

Will be also presumed for all the functions from the space $BC([0,T], L^0_q)$ the continuity on the time $t$ in the $L_q$, $q \geq d$ sense:

$$\lim_{s \to t^-} ||v(\cdot, t) - v(\cdot, s)||_q = 0, \quad t \in [0,T].$$

At the value $t = 0+$

$$\lim_{t \to 0^+} ||u(\cdot, t) - u(\cdot, 0)||_q = 0, \quad q \geq d.$$  

For instance, the initial condition $a(x) = u(x,0)$ will be understood as follows:

$$\lim_{t \to 0^+} ||u(\cdot, t) - a(\cdot)||_d = 0.$$  

(1.11)

This estimates allow us to establish some new properties of solution and develop numerical methods.

Note that this statement of problem appeared in [65].

## 2 Some Notations, with Clarification.

As ordinary, for the measurable function $x \to u(x)$, $x \in R^d$
\[
\|u\|_p = \left[ \int_{R^d} |u(x)|^p \, dx \right]^{1/p}.
\]  

**Multidimensional case.**

Let \( u = \vec{u} = \{u_1(x), u_2(x), \ldots, u_d(x)\} \) be measurable vector - function: \( u_k : R^d \to R \). We can define as ordinary the \( L^p \), \( p \geq 1 \) norm of the function \( u \) by the following way:

\[
\|u\|_p := \max_{k=1,2,\ldots,d} \|u_k\|_p, \quad p \geq 1.
\]

Define also

\[
K_S(d, p) := \frac{\pi^{-1/2} \, d^{-1/p}}{\Gamma(d/2)^{(p-1)/p}} \left\{ \frac{\Gamma(1 + d/2) \, \Gamma(d)}{\Gamma(d/p) \, \Gamma(1 + d - d/p)} \right\}^{1/d}.
\]  

The function \( K_S(d, p) \) is the optimal (i.e. minimal) value in the famous Sobolev’s inequality

\[
||\phi||_r \leq K_S(d, q) \, ||\nabla \phi||_q, \quad 1 \leq q < d, \quad \frac{1}{r} = \frac{1}{q} - \frac{1}{d}, \quad r \geq 1,
\]  

see Bliss [43], (1930); Talenti, [52], (1995).

\[
\tilde{\omega}(d) := \frac{4\pi^{d/2-1}}{\Gamma(d/2)}, \quad \omega(d) := \frac{2\pi^{d/2}}{\Gamma(d/2)};
\]

\[
c(d) = -\frac{\pi^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)}. \quad \Omega_k(x) = x_k / ||x||.
\]

\[
x = (x_1, x_2, \ldots, x_k, \ldots, x_d) \in R^d \Rightarrow ||x|| = \sqrt{\sum_{j=1}^{d} x_j^2}.
\]

The explicit view for Riesz’s transform has a view

\[
R_k[f](x) = R_k^{(d)}[f](x) = c(d) \lim_{\epsilon \to 0^+} \int_{||y|| > \epsilon} ||y||^{-d} \Omega_k(y) \, f(x - y) \, dy.
\]  

The ultimate result in this direction belongs to T.Iwaniec and G.Martin [50]: the upper estimate value \( ||R_k||(L_p \to L_p) \) does not dependent on the dimension \( d \) and coincides with the Pichorides constant:

\[
K_R(p) := ||R_k||(L_p \to L_p) = \cot \left( \frac{\pi}{2p^*} \right), \quad p^* = \max(p, p/(p-1)), \quad p > 1.
\]  

For instance, \( K_R(3) = \sqrt{3} \).
T. Iwaniec and G. Martin considered also the vectorial Riesz transform. See for additional information [42], [35], chapter 2, section 4; [36], chapter 3.

We will use the famous Young inequality for the (measurable) functions \( f, g : \mathbb{R}^d \to \mathbb{R} \):

\[
||f * g||_r \leq K_{BL}(p, q) ||f||_p ||g||_q, \quad 1/r + 1 = 1/p + 1/q, \quad p, q, r > 1,
\]

where \( d \) is a dimension of arguments of a functions \( f, g \) and the optimal value of ”constant” \( K_{BL}(p, q) \) was obtained by H.J. Brascamp and E.H. Lieb [44]:

\[
K_{BL}(d; p, q) = K_{BL}(p, q) = \left[ \frac{p^{1/p} s^{-1/s} q^{1/q} t^{-1/t} r^{1/r} z^{-1/z}}{s}\right]^{d/2},
\]

\( s = p/(p - 1), \quad t = q/(q - 1), \quad z = r/(r - 1) \).

Note that \( K_{BL}(d; p, q) \leq 1 \).

Let us denote

\[
M(d, r) = ||w_1(\cdot)||_r, \quad r \geq 1.
\]

We deduce by direct computation

\[
M(d, r) = (4\pi)^{-d/2} \left[ \int_{\mathbb{R}^d} e^{-r||x||^2/(4t)} \, dx \right]^{1/r} = 2^{d/r} \pi^{-d(1-1/r)/2} r^{-d/2r}.
\]

Therefore

\[
||w_t(\cdot)||_r = t^{-d(1-1/r)/2} M(d, r) = t^{-d(1-1/r)/2} 2^{d/r} \pi^{-d(1-1/r)/2} r^{-d/2r}.
\]

Note that the value \( M(d, r) \) allows very simple estimates: \( M(d, r) < 2^d \).

Let us denote \( K_0 = K_0(d, \delta) = K_0(a; d, \delta) := \)

\[
\sup_{t > 0} \left[ t^{(1-\delta)/2} ||u_0(t)||_{d/\delta} \right] = \sup_{t > 0} \left[ t^{(1-\delta)/2} ||e^{t \Delta} a||_{d/\delta} \right] =
\]

\[
\sup_{t > 0} \left[ t^{(1-\delta)/2} ||w_t * a||_{d/\delta} \right].
\]

We conclude as a consequence using Young inequality: if \( t > 0, \ a \in L_d(\mathbb{R}^d), \ \delta = \text{const} \in (0, 1) \), then

\[
K_0(d, \delta) \leq K_{BL}(d; d, d/(d - (1 + \delta))) \cdot M(d, d/(d - (1 + \delta))) \cdot ||a||_d. \quad (2.10)
\]

We get analogously denoting \( K'_0 = K'_0(d) = K'_0(a; d) := \)

\[
\sup_{t > 0} \left[ t^{1/2} \cdot ||\nabla u_0(t)||_d \right] = \sup_{t > 0} \left[ t^{1/2} ||\nabla e^{t \Delta} a||_d \right] = \sup_{t > 0} \left[ t^{1/2} ||(\nabla w_t) * a||_d \right].
\]
\[ K_0' \leq 0.5 \cdot K_{BL}(d; d, d) \cdot M\left(d, \frac{d^2}{d-1}\right) \cdot ||a||_d. \quad (2.11) \]

Notice [18] that for the operator (non-linear) \( G \) are true the following estimates:

\[ ||Gu||_{d/\gamma} \leq K_R(d/\alpha) \cdot K_R(d/\beta) \]

\[ \int_0^t (t-s)^{-(\alpha+\beta-\gamma)/2} ||u(s)||_{d/\alpha} ||\nabla u(s)||_{d/\beta} \, ds = \quad (2.12) \]

\[ K_R(d/\alpha) \cdot K_R(d/\beta) \cdot t^{-(\alpha+\beta-\gamma)/2} \odot \left[ ||u(t)||_{d/\alpha} ||\nabla u(t)||_{d/\beta} \right]; \]

\[ ||\nabla Gu||_{d/\gamma} \leq K_R(d/\alpha) \cdot K_R(d/\beta) \times \]

\[ \int_0^t (t-s)^{-(1+\alpha+\beta-\gamma)/2} ||u(s)||_{d/\alpha} ||\nabla u(s)||_{d/\beta} \, ds = \quad (2.13) \]

\[ K_R(d/\alpha) \cdot K_R(d/\beta) \cdot t^{-(1+\alpha+\beta-\gamma)/2} \odot \left[ ||u(t)||_{d/\alpha} ||\nabla u(t)||_{d/\beta} \right], \]

\[ \alpha, \beta, \gamma > 0, \gamma \leq \alpha + \beta < d. \]

Another useful inequalities: \( ||e^{t\Delta v}||_q \leq \]

\[ K_{BL}\left(d; \frac{pq}{pq+p-q}, p\right) \cdot M\left(d, \frac{pq}{pq+p-q}\right) \cdot ||v||_p \cdot t^{-(d/p-d/q)/2}, 1 < p \leq q < \infty; \quad (2.14) \]

\[ ||\nabla e^{t\Delta v}||_q \leq 0.5 \cdot K_{BL}\left(d; \frac{pq}{pq+p-q}, p\right) \cdot M\left(d; d + \frac{pq}{pq+p-q}\right) \cdot ||v||_p \cdot t^{-(1+d/p-d/q)/2}, 1 < p \leq q < \infty; \quad (2.15) \]

\[ ||F(u, v)||_p \leq ||u||_r \cdot ||v||_s, \quad p, r, s > 1, 1/p = 1/r + 1/s. \quad (2.16) \]

We denote for simplicity

\[ M'(d, r) = 0.5 M(d, d + r), \]

so that

\[ ||\nabla w||_r = M'(d, r) \cdot t^{-1/2-d(1-1/r)/2} \]
and
\[ \| \nabla e^{t\Delta} v \|_q \leq K_{BL} (d; r_0(p,q), p) \cdot M'(d, r_0(p,q)) \cdot \| v \|_p \cdot t^{-(1+d(1-1/r_0(p,q)))/2}, \] (2.17)
where
\[ 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r_0(p,q)}. \] (2.18)
Note that \( M'(d, r) \leq 2^{d-1} \).

3 Main result.

We suppose in this section that the initial function \( a = a(x) \) belong to the space \( L_d : \| a \|_d < \infty \), and such that \( \text{Div} \ a = 0 \).

1. Derivation of the basic inequalities.

Let \( \delta \) be arbitrary fixed number from the set \((0, 1)\). We denote
\[ u_0 = u_0(x,t) = e^{t\Delta} a = [w_t * a](x) \] (3.1)
and consider together with T.Kato [18] the following recursion:
\[ u_{n+1} = u_0 + Gu_n, \quad n = 0, 1, \ldots \] (3.2)
Recall that the functions \( u_n = u_n(x,t) \) are vector functions.

We denote also \( K_n = K_n(d, \delta) = K_n(a; d, \delta) := \)
\[ \sup_{t>0} \| t^{(1-\delta)/2} u_n \|_{BC([0, \infty); L_d / \delta)} = \sup_{t>0} \| t^{(1-\delta)/2} u_n \|_d / \delta, \] (3.3)
\( K'_n = K'_n(d) = K'_n(a; d, \delta) := \)
\[ \sup_{t>0} \| t^{1/2} \nabla u_n \|_{BC([0, \infty); \ L_d)} = \sup_{t>0} \| t^{1/2} \nabla u_n \|_d, \] (3.4)
\( K_n(T) = K_n(d, \delta; T) = K_n(a; d, \delta; T) := \)
\[ \| t^{(1-\delta)/2} u_n \|_{BC([0,T); L_d / \delta)} = \sup_{t \in (0,T)} \| t^{(1-\delta)/2} u_n \|_{d / \delta}, \] (3.3')
\( K'_n(T) = K'_n(d; T) = K'_n(a; d; T) := \)
\[ \| t^{1/2} \nabla u_n \|_{BC([0,T); \ L_d)} = \sup_{t \in (0,T)} \| t^{1/2} \nabla u_n \|_d, \] (3.4')
and correspondingly
\( K_0 = K_0(d, \delta) = K_0(a; d, \delta) := \)
\[ ||t^{(1-\delta)/2} u_0||_{BC([0, \infty); L_{d/\delta})} = \sup_{t \geq 0} ||t^{(1-\delta)/2} u_0||_{d/\delta}, \quad (3.5) \]

\[ K'_0 = K'_0(d, \delta) = K'_0(a; d, \delta) := \]

\[ K'_0 = K'_0(d) = ||t^{1/2} \nabla u_0||_{BC([0, \infty); L_d)} = \sup_{t \geq 0} ||t^{1/2} \nabla u_0||_d. \quad (3.6) \]

\[ K_0(T) = K_0(d, \delta; T) = K_0(a; d, \delta; T) := \]

\[ K_0(T) = K_0(d, \delta; T) = ||t^{(1-\delta)/2} u_0||_{BC([0, T); L_{d/\delta})} = \sup_{t \in (0,T)} ||t^{(1-\delta)/2} u_0||_{d/\delta}, \quad (3.5') \]

\[ K'_0(T) = K'_0(d, \delta; T) = K'_0(a; d, \delta; T) := \]

\[ K'_0(T) = K'_0(d; T) = ||t^{1/2} \nabla u_0||_{BC([0, T); L_d)} = \sup_{t \in (0,T)} ||t^{1/2} \nabla u_0||_d. \quad (3.6') \]

Obviously,

\[ K'_n(d; T) < K'_n(d), \quad K_n(d; T) < K_n(d). \]

Moreover,

\[ \lim_{T \to 0} K_0(T) = 0, \quad \lim_{T \to 0} K'_0(T) = 0, \]

see [18].

As we know, see (2.10), (2.11),

\[ K_0(d, \delta) \leq K_{BL}(d; d, d/(d - (1 + \delta))) \cdot M(d, d/(d - (1 + \delta))) \cdot ||a||_d, \quad (3.7) \]

\[ K'_0(d) \leq 0.5 \cdot K_{BL}(d; d, d) \cdot M(d, 1) \cdot ||a||_d. \quad (3.8) \]

Further, we find using (2.12) and (2.13):

\[ ||Gu_n||_{d/\delta} \leq K_R(d/\delta) \cdot K_R(d) \cdot \int_0^t ||\nabla u_n(t - s)||_d \cdot ||u_n(s)||_{d/\delta} ds \leq \]

\[ K_R(d/\delta) \cdot K_R(d) \cdot K_n \cdot K'_n \cdot \int_0^t (t - s)^{-1/2} \cdot s^{-(1-\delta)/2} ds = \]

\[ K_R(d/\delta) \cdot K_R(d) \cdot K_n \cdot K'_n \cdot t^{-(1-\delta)/2} \cdot \frac{\Gamma(1/2) \Gamma(\delta/2)}{\Gamma((1+\delta)/2)} = \]

\[ K_R(d/\delta) \cdot K_R(d) \cdot K_n \cdot K'_n \cdot t^{-(1-\delta)/2} \cdot \frac{\sqrt{\pi} \Gamma(\delta/2)}{\Gamma((1+\delta)/2)}. \]

Therefore,
\[ K_{n+1} \leq K_{BL}(d, d/(d - (1 + \delta))) \cdot M(d, d/(d - (1 + \delta))) \cdot ||a||_d + \]
\[ K_R(d/\delta) \cdot K_R(d) \cdot K_n \cdot K_n' \cdot \frac{\sqrt{\pi} \Gamma(\delta/2)}{\Gamma((1 + \delta)/2)}. \] (3.9)

We obtain analogously
\[ ||\nabla G u(t)||_d \leq K_R^2(d) \int_0^t (t - s)^{-(1+\delta)/2} ||u(s)||_d ||\nabla u(s)||_d ds \leq \]
\[ K_R^2(d) \cdot K_n \cdot K_n' \cdot \int_0^t (t - s)^{-(1+\delta)/2} s^{-1+\delta/2} ds = \]
\[ K_R^2(d) \cdot K_n \cdot K_n' \cdot t^{-(1-\delta)/2} \int_0^1 (1 - z)^{-(1+\delta)/2} z^{\delta/2-1} \, dz = \]
\[ K_R^2(d) \cdot K_n \cdot K_n' \cdot t^{-(1-\delta)/2} \cdot \frac{\Gamma((1 - \delta)/2) \Gamma(\delta/2)}{\sqrt{\pi}}. \]

Following,
\[ K_{n+1}' \leq 0.5 \cdot K_{BL}(d, d) \cdot M(d, 1) \cdot ||a||_d + \]
\[ K_R^2(d) \cdot K_n \cdot K_n' \cdot \frac{\Gamma((1 - \delta)/2) \Gamma(\delta/2)}{\sqrt{\pi}}. \] (3.10)

To sum up the local. Let us denote
\[ S_1 = S_1(d, \delta) = K_{BL}(d, d/(d - (1 + \delta))) \cdot M(d, d/(d - (1 + \delta))), \] (3.11)
\[ J_1 = J_1(d, \delta) = K_R(d/\delta) \cdot K_R(d) \cdot \frac{\sqrt{\pi} \Gamma(\delta/2)}{\Gamma((1 + \delta)/2)}, \] (3.12)
\[ S_2 = S_2(d) = 0.5 \cdot K_{BL}(d, d) \cdot M(d, 1), \] (3.13)
\[ J_2 = J_2(d, \delta) = K_R^2(d) \cdot \frac{\Gamma((1 - \delta)/2) \Gamma(\delta/2)}{\sqrt{\pi}}. \] (3.14)

We obtained the following system of recurrent inequalities for the vector sequence \((K_n, K_n')\):
\[ K_{n+1} \leq K_0 + J_1 K_n K_n', \] (3.15)
\[ K_{n+1}' \leq K_0' + J_2 K_n K_n'. \] (3.16)

with initial conditions \((K_0, K_0')\), where \(K_0 \leq S_1(d, \delta) \cdot ||a||_d, \ K_0' \leq S_2(d) \cdot ||a||_d.

2. Auxiliary facts.

Let us consider the following non-linear recurrent inequality: \(x_n \geq 0,\)
\[ x_{n+1} \leq \alpha + \beta x_n + \gamma x_n^2, \quad n = 0, 1, 2, \ldots \] (3.17)

with initial condition \( x_0 = x(0) > 0 \). Denote

\[ D(\alpha, \beta, \gamma) = (\beta - 1)^2 - 4\alpha\gamma, \quad Z(\alpha, \beta, \gamma) = \frac{1 - \beta + \sqrt{D(\alpha, \beta, \gamma)}}{2\gamma}. \] (3.18)

**Lemma 1.** Let

\[ \alpha, \gamma > 0, \beta \geq 0, \quad D(\alpha, \beta, \gamma) > 0, \quad Z(\alpha, \beta, \gamma) > 0, \quad x(0) < Z(\alpha, \beta, \gamma). \]

Then

\[ \sup_n x_n \leq Z(\alpha, \beta, \gamma). \] (3.19)

This assertion may be proved easily by means of induction over \( n \).

**Lemma 2.** Let us consider the following system of recurrent relations: \( x_n, y_n \geq 0 \),

\[ x_{n+1} \leq \alpha_1 + \beta_1 x_n y_n, \] (3.20)

\[ y_{n+1} \leq \alpha_2 + \beta_2 x_n y_n, \quad n = 0, 1, 2, \ldots \] (3.21)

with positive initial conditions \( x_0 = x(0), \quad y_0 = y(0) \).

We retain last notations and add some news:

\[ Det_1 = Det_1(\alpha_1, \alpha_2, \beta_1, \beta_2) \overset{\text{def}}{=} \alpha_2 \beta_1 - \alpha_1 \beta_2, \quad Det_2 = -Det_1, \] (3.22)

\[ D_1 = D_1(\alpha_1, \alpha_2, \beta_1, \beta_2) \overset{\text{def}}{=} (Det_1 + 1)^2 - 4\alpha_1 \beta_2, \] (3.23)

\[ D_2 = D_2(\alpha_1, \alpha_2, \beta_1, \beta_2) \overset{\text{def}}{=} (Det_2 + 1)^2 - 4\alpha_2 \beta_1. \] (3.24)

Suppose \( \alpha_1, \alpha_2, \beta_1, \beta_2, D_1, D_2 > 0, \)

\[ Z(\alpha_1, Det_1, \beta_2) > 0, \quad Z(\alpha_2, Det_2, \beta_1) > 0, \]

\[ 0 < x(0) < Z(\alpha_1, Det_1, \beta_2), \quad 0 < y(0) < Z(\alpha_2, Det_2, \beta_1). \]

We assert by induction based on the lemma 1:

\[ \sup_n x_n \leq Z(\alpha_1, Det_1, \beta_2), \quad \sup_n y_n \leq Z(\alpha_2, Det_2, \beta_1). \] (3.25)

3. Main result.

We restrict ourselves in relations (3.15) - (3.16) only the time interval \( t \in (0, T), \quad T < \infty \). We obtain then the following system of non-linear inequalities:
\[ K_{n+1}(T) \leq K_0(T) + J_1 K_n(T) \cdot K'_n(T), \quad (3.26) \]
\[ K'_{n+1}(T) \leq K'_0(T) + J_2 K_n(T) \cdot K'_n(T). \quad (3.27) \]

We have taken into account the notations (3.5') - (3.6'). It remains to use the estimates (3.25). Namely, we denote using last notations

\[ s_1 = s_1(d, \delta) = J_1(d, \delta)K'_0(T) - J_2(d, \delta)K_0(T), \quad s_2(d, \delta) = -s_1(d, \delta); \quad (3.28) \]

\[ V_1(a(\cdot); d, \delta; T) = Z(K_0(T); s_1(d, \delta), J_2(d, \delta)), \quad (3.29) \]
\[ V_2(a(\cdot); d, \delta; T) = Z(K'_0(T); s_2(d, \delta), J_1(d, \delta)). \quad (3.30) \]

We have proved in fact the following assertion:

**Theorem 3.1.** Let as before \( a \in L_d(R^d) \). Define the set \( L = L(a; d, \delta) \) as a set \( \{T_0\} \) of all the values \( T_0 \) as an arbitrary positive solutions of inequalities

\[ K_0(T_0) < V_1(a(\cdot); d, \delta, T_0), \quad (3.31) \]
\[ K'_0(T_0) < V_2(a(\cdot), d, \delta; T_0). \quad (3.32) \]

Then the lifespan of solution \( T^* \) of NS equations is greatest than \( T_0 \):

\[ T^* \geq \sup\{T_0(a; d, \delta), \ t_0 \in L\} \defeq \hat{T} = \hat{T}(a; d, \delta). \quad (3.33) \]

**Remark 3.1.** As long as \( \lim_{T \to 0^+} [K_0(T) + K'_0(T)] = 0 \) and

\[ |s_1|, \ |s_2| \leq J_1(d, \delta)K'_0(T) + J_2(d, \delta)K_0(T) \leq J_1(d, \delta)K'_0 + J_2(d, \delta)K_0, \]

the system of inequalities (3.31), (3.32) has at last one positive solution.

**Remark 3.2.** Evidently,

\[ T \geq \sup_{\delta \in (0,1)} \hat{T}(a; d, \delta) \defeq \hat{T}(a; d). \quad (3.33) \]

**4 Simplification.**

The system of inequalities (3.31) - (3.32) is very complicate. We aim to in this section simplification of this relations in order to obtain more convenient explicit view for the lower bound for lifespan \( T \).

1. Note first of all that
\[ J_1 \leq J^{(1)} = J^{(1)}(d, \delta) \overset{\text{def}}{=} \frac{9d^2}{2\delta^2}, \quad (4.1) \]
\[ J_2 \leq J^{(2)} = J^{(2)}(d, \delta) \overset{\text{def}}{=} \frac{81d^2}{4\sqrt{\pi}\delta(1-\delta)}, \quad (4.2) \]
as long as \( \delta \in (0, 1) \).

2. We denote

\[ K^{(0)}(T) = K^{(0)}(a(\cdot); d, \delta; T) = \max(K_0(a(\cdot); d, \delta; T), \ K'_0(a(\cdot); d, \delta; T)), \]
\[ X_n = X_n(T) = X_n(d, \delta; T) = \max(K_n(d, \delta; T), \ K_n(d, \delta; T)), \]
\[ J = J(d, \delta) = \max(J^{(1)}(d, \delta), \ J^{(2)}(d, \delta)), \quad (4.3) \]
\[ \delta_0 := \frac{2\sqrt{\pi}}{9 + 2\sqrt{\pi}} \approx 0.282577, \quad (4.4) \]
\[ \mathcal{J} = \mathcal{J}(d) = \min_{\delta \in (0,1)} J(d, \delta) = \frac{9d^2}{2\delta_0^2} = C_1 \cdot d^2, \quad (4.5) \]
then
\[ J(d, \delta) = J^{(1)}(d, \delta), \ \delta \in (0, \delta_0); \quad J(d, \delta) = J^{(2)}(d, \delta), \ \delta_0 \leq \delta < 1. \]

It is a reason to name the pair of values \((\delta_0, \mathcal{J})\) as a critical value for considered problem.

3. For the variables \(X_n = X_n(T)\) we can write one inequality:
\[ X_{n+1} \leq K^{(0)}(T) + \mathcal{J} X_n^2 \quad (4.6) \]
with initial condition \(X_0 = K^{(0)}(T)\). We conclude applying lemma 1:

**Theorem 4.1.** Define the value \(T_0\) as follows:
\[ \max(K_0(T_0), K'_0(T_0)) \leq \frac{3}{16 \mathcal{J}} = C_2/d^2, \quad (4.7) \]
\[ C_2 \approx 0.0033270; \quad (4.7a) \]
or equally
\[ \max\left( \max_{t \in (0, T)} ||t^{(1-\delta)/2} w_t * a||_{d/\delta}, \ \max_{t \in (0, T)} ||t^{1/2} \nabla w_t * a||_{d} \right) \leq \frac{3}{4 \mathcal{J}} = C_3/d^2. \quad (4.7b) \]
Then the lifespan $T$ of solution NS system is greater than $T_0$: $T \geq T_0$.

Moreover:

$$\max(K_n(T), K'_n(T)) \leq \frac{3}{4J} = C_3/d^2, \quad (4.8)$$

$$C_3 \approx 0.0133308333. \quad (4.8a)$$

We have for instance in the ordinary three-dimensional case $d = 3$

$$C_2/d^2 \approx 0.00036967, \quad C_3/d^2 \approx 0.0014767.$$

4. In order to use the theorem 4.1 we need to derive a simple estimate for the values $K_0(T), K'_0(T)$ as $T \to 0 +$.

\textbf{α. Estimation of $K_0(T)$}.

Suppose in addition to the condition $||a||_d < \infty$ that for some $\theta \in (0, \min(1, (d - 1)/\delta))$

$$||a||_{d+\theta} < \infty. \quad (4.9)$$

Repeating the consideration for the inequality (2.10) and taking into account the inequalities $K_{BL}(\cdot) \leq 1$, $M(d, r) \leq 2^d$ we deduce:

$$||u_0||_{d/\delta} \leq t^{-(1-\delta)/2} \cdot t^{(\theta\delta)/(2d)} \cdot 2^{d+\theta} \cdot ||a||_{d+\theta};$$

therefore

$$K_0(T) \leq T^{\frac{\theta\delta}{2d}} \cdot 2^{d+\theta} \cdot ||a||_{d+\theta}. \quad (4.10)$$

\textbf{β. Estimation of $K'_0(T)$}.

We demonstrate in this pilcrow a different method. Namely, assume in addition that the initial condition $a(\cdot)$ belongs to the Sobolev space $W^d_1(R^d)$, which consists on all the (measurable) functions $a : R^d \to R^d$ which finite semi-norm

$$||a||_{W^d_1(R^d)} = ||\nabla a||_d < \infty. \quad (4.11)$$

We get using again Young’s inequality:

$$K'_0(T) = \sup_{t \in (0,T)} [t^{1/2} \nabla (w_t) \ast a]_d = \sup_{t \in (0,T)} [t^{1/2} w_t \ast \nabla a]_d \leq$$

$$\sup_{t \in (0,T)} [t^{1/2} ||w_t||_1 \cdot ||\nabla a||_d] = \sup_{t \in (0,T)} [t^{1/2} ||\nabla a||_d] = \sqrt{T} \cdot ||a||_{W^d_1(R^d)}, \quad (4.12)$$

since $||w_t||_1 = 1$.

\textbf{Remark 4.1.} The variables $K_0(T), K'_0(T)$ dependent in particular on the initial condition $a(\cdot)$:
\[ K_0(T) = K_0(T; a(\cdot)); \quad K'_0(T) = K'_0(T; a(\cdot)). \]

If it is so little that
\[
\max(K_0(a(\cdot)), K'_0(a(\cdot))) \leq \frac{3}{16 J} = C_2/d^2;
\]
then we can choose \( T = \infty \), i.e. this solution \( u = u(x, t) \) is global.

Recall that
\[ K_0(a(\cdot)) = K_0(a(\cdot); \infty); \quad K'_0(a(\cdot)) = K'_0(a(\cdot); \infty). \]

5 Mixed norm estimates for solution.

We suppose during this section \( ||a||_d < \infty \).

It is known, see [1], [18]-[19] that the global in time solution \( u(x, t) = u(t) \) obeys the property
\[
\lim_{t \to \infty} ||u(t)||_q = 0, \quad q > d.
\]
(”Energy” decay).

The case \( q = 2 \) was investigated in [29]; see also reference therein.

We want clarify in this section this fact; i.e. give the quantitative estimates one of main result if the article [18].

Recall that the so-called mixed, or equally anisotropic \((p_1, p_2)\) norm \( ||u||_{p_1, p_2}^* \) for the function of ”two” variables \( u = u(x, t), \ x \in R^d, \ t \in R^1_+ \) is defined as follows:
\[
||u||_{p_1, p_2}^* = \left( \int_{R^d} \left[ \int_0^\infty |u(x, t)|^{p_1} \ dx \right]^{p_2/p_1} \ dt \right)^{1/p_2}
\]
with evident modification in the case when \( p_2 = \infty \):
\[
||u||_{p_1, \infty}^* = \sup_{t \in (0, T)} \left[ \int_0^\infty |u(x, t)|^{p_1} \ dx \right]^{1/p_1}.
\]

We introduce here the following weight mixed norm, more precisely, the family of norms as follows:
\[
|||u|||_{q; T}^* = \sup_{t \in (0, T)} \left[ t^{(1-d/q)/2} ||u(\cdot, t)||_q \right], \quad q \geq d.
\] (5.1)

Let us introduce some new notations.
\[
\theta_1 = \frac{dq}{d(q + 1) - q(\delta + 1)}, \quad \theta_2 = \frac{q}{\delta + 1}, \quad \theta_3 = \frac{d}{\delta}, \quad \theta_4 = d;
\]
\[
\psi(d, q, \delta) = \psi(a; d, q, \delta) = K_{BL}(d; \theta_1, \theta_2) K^0(a; d, q, \delta; T) K^0(a; d, q, \delta; T)^\times
\]
\[ K_R(d/\delta) K_R(d) M(d, \theta_1) B \left( \frac{1-\delta}{2} + \frac{d}{2q}, \frac{\delta}{2} \right) + \\
0.5 \cdot K_{BL}(d; d,d) \cdot M \left( d, \frac{d^2}{d-1} \right) \cdot \|a\|_d, \]  
(5.2)

where \( B(\cdot, \cdot) \) is the classical Beta-function;

\[ \psi(q) = \psi(a; q) = \inf_{\delta \in (0,1)} \psi(a; d, q, \delta). \]  
(5.3)

Note that \( 1 < \theta_j < \infty, \ j = 1, 2, 3, 4. \)

**Theorem 5.1.** Let the lifespan of solution of NS system \( T \) be positive; may be infinite. Then

\[ \|\|u\||^*_q T \leq \psi(q), \ q \geq d. \]  
(5.4)

**Proof.** We follow T.Kato [18]. Indeed,

\[ -Gu(t) = \int_0^t w_{t-s}(\cdot) * F(u(\cdot), s) \, ds. \]

We use the triangle inequality for the \( L_q(R^d) \) norm:

\[ ||Gu(t)||_q \leq \int_0^t ||w_{t-s}(\cdot) * F(u(\cdot), s)||_q \, ds. \]  
(5.5)

The Young inequality gives us

\[ ||w_{t-s}(\cdot) * F(u(\cdot), s)||_q \leq K_{BL}(d; \theta_1, \theta_2) \cdot K_R(\theta_3) \cdot ||w_{t-s}(\cdot)||_{\theta_1} \cdot ||F||_{\theta_2} \leq \]

\[ K_{BL}(d; \theta_1, \theta_2) \cdot M(d, \theta_1) \cdot (t-s)^{-d(1-1/\theta_1)/2} \cdot K_R(\theta_3) \cdot K_R(\theta_4) \times \]

\[ ||u(\cdot, s)||_{\theta_3} \cdot ||\nabla u(\cdot, s)||_{\theta_4}, \]  
(5.6)

we have taken into account the norm of Riesz transform. Further, if \( t \in (0, T) \) then

\[ ||u(\cdot, s)||_{d/\delta} \leq K^0(a; d, q, \delta; T) \cdot s^{-(1-\delta)/2}, \]

\[ ||\nabla u(\cdot, s)||_d \leq K^0(a; d, q, \delta; T)' \cdot s^{-1/2}. \]

We conclude substituting into the inequality (5.5):

\[ ||Gu(t)||_q \leq C(a; d, q, \delta, T) \cdot \int_0^t (t-s)^{-(1+\delta-d/q)/2} \cdot s^{1+\delta/2} \, ds = \]

\[ C(a; d, q, \delta, T) \cdot t^{-(1-d/q)/2} \cdot B \left( \frac{1-\delta}{2} + \frac{d}{2q}, \frac{\delta}{2} \right). \]  
(5.7)
As long as

\[ ||u(t)||_q \leq ||a||_q + ||Gu(t)||_q \]

and the value \( ||a||_q \) was estimated in (2.11), we deduce after simple calculations

\[ ||u(t)||_q \leq t^{-(1-d/q)/2} \psi(a; d, q, \delta, T) \]

or equally

\[ ||u||_q^{*; T} \leq \psi(a; d, q, \delta, T). \] (5.8)

It remains to take the minimum over \( \delta; \delta \in (0, 1) \).

**Remark 5.1.** If we define the so-called mixed Grand Lebesgue norm \( ||u||^{*G}(\psi; T) \) as follows:

\[ ||u||^{*G}(\psi; T) := \sup_{q \geq d} \left[ \frac{||u||^{*}_q; T}{\psi(a; q)} \right], \] (5.9)

then the assertion of the theorem 5.1 may be rewritten as follows: under the conditions of theorem 5.1

\[ ||u||^{*G}(\psi(a; T)) \leq 1. \] (5.10)

The detail investigation with applications of these norm and correspondent spaces see, e.g. in [45] - [51], [58], [60], [63], [64].

Let us investigate here the following weight mixed norm with derivative, (Sobolev’s weight norm) for solution \( u = u(x, t) \), more precisely, the family of semi-norms as follows:

\[ |||u|||^{**}_q; T = \sup_{t \in (0, T)} \left[ t^{(1-d/2q)} ||\nabla u(\cdot, t)||_q \right], \ q \geq d. \] (5.11)

New notations and restrictions:

\[ 1 + \frac{1}{q} = \frac{1}{\theta_5} + \frac{1}{d}, \ 1 + \frac{1}{q} = \frac{1}{\theta_6} + \frac{1}{d}, \]

\[ \frac{1}{\theta_7} = \frac{\delta}{d} + \frac{1}{d}, \]

\[ \nu(a; d, \delta, q, T) = K_{BL}(d; q, \theta_5) \cdot M'(d, \theta_5)||a||_d + K_{BL}(d; \theta_6, \theta_7) K_R(\theta_6) K_R(\theta_7) K(a; d, \delta, T) K'(a; d, \delta, T) M'(d, \theta_6), \]

\[ \nu(q) = \nu(a; q) := \inf_{\delta \in (0, 1)} \nu(a; d, \delta, q, T). \]

**Theorem 5.2.** Let the lifespan of solution of NS system \( T \) be positive; may be infinite. Then
\[||u||^*_q \leq \nu(q), \quad q \geq d. \quad (5.12)\]

**Proof.** First of all we estimate the influence of the initial condition \(a = a(x)\). Namely,

\[
||\nabla w_0||_q = ||\nabla w_t * a||_q \leq K_{BL}(d; q, \theta_5) \cdot ||\nabla w_t||_{\theta_5} \cdot ||a||_d =
\]
\[t^{-(1-d/(2q))} \cdot K_{BL}(d; q, \theta_5) \cdot M'(d, \theta_5) ||a||_d. \quad (5.13)\]

Further,

\[
||\nabla Gu(t)||_q = ||\int_0^t \nabla w_{t-s} * F(u(s)) \, ds||_q \leq \int_0^t ||\nabla w_{t-s} * F(u(s))||_q \, ds \leq 
\]
\[K_{BL}(d; \theta_6, \theta_7) \cdot \int_0^t ||\nabla w_{t-s}||_{\theta_6} \cdot ||F(u(s))||_{\theta_7} \, ds \leq 
\]
\[K_{BL}(d; \theta_6, \theta_7) \cdot \int_0^t ||\nabla w_{t-s}||_{\theta_6} \cdot ||u(s)||_{d/\delta} \cdot ||\nabla u(s)||_d \, ds. \quad (5.14)\]

As long as

\[
||u(s)||_{d/\delta} \leq s^{-(1-\delta)/2} K(a; d, \delta, T), \quad ||\nabla u||_d \leq K'(a; d, \delta, T) s^{-1/2}, \quad (5.15)\]

\[
||\nabla w_{t-s}||_{\theta_6} = M'(d, \theta_6) \cdot t^{-1/2-d(1-1/\theta_6)/2}, \quad (5.16)\]

we obtain substituting into (5.14): \(||\nabla Gu(t)||_q \leq \)

\[
K_{BL}(d; \theta_6, \theta_7) \cdot K_R(\theta_6) \cdot K_R(\theta_7) \cdot K(a; d, \delta, T) \cdot K'(a; d, \delta, T) \cdot M'(d, \theta_6) \times
\]
\[\int_0^t (t-s)^{-1/2-d(1-1/\theta_6)/2} s^{(-1+\delta)/2} \, ds = t^{-(1-d/(2q))} \cdot B(0.5 - d(1-\theta_6)/2, \delta/2) \times
\]
\[K_{BL}(d; \theta_6, \theta_7) \cdot K_R(\theta_6) \cdot K_R(\theta_7) \cdot K(a; d, \delta, T) \cdot K'(a; d, \delta, T) \cdot M'(d, \theta_6). \quad (5.17)\]

We get summing (5.13) and (5.17):

\[
||u(t)||_q \leq t^{-(1-d/(2q))} \cdot \nu(a; d, \delta, T)
\]

and after minimization over \(\delta\)

\[
\sup_{t \in (0, T)} \left[ t^{(1-d/(2q))} \cdot ||u(t)||_q \right] \leq \nu(q), \quad (5.18)\]

Q.E.D.
6 Concluding remarks.

1. It is known [12], [13], [18], [19] etc. that in general case, i.e. when the value $\epsilon = ||a||_a$ is not sufficiently small, then the lifespan of solution of NS equation $T$ may be finite (short-time solution). Perhaps, it is self-contained interest to find a quantitative computation of the exact value $T$.

For the non-linear Schrödinger’s equation the estimate

$$ T \geq \exp(C/\epsilon) $$

was obtained in the recent article [57].

2. At the same considerations may be provided for the NS equations with external force $f = f(x,t)$:

$$ \partial u_t = \Delta u + (u \cdot \nabla)u + Q \cdot \nabla \cdot (u \otimes u) + f(x,t), \quad x \in R^d, \ t > 0; \quad (6.1) $$

$$ u(x,0) = a(x), \ x \in R^d. $$

see [12] - [15], [21], [26], [29], [39].

More detail, the considered here problem may be rewritten as follows:

$$ u(x,t) = e^{t\Delta} a(x) + G[u](t) \overset{\text{def}}{=} u_0(x,t) + G[u](t) + v[f](x,t), \quad (6.2) $$

where

$$ v[f](x,t) = v(x,t) = v = \int_0^t ds \int_{R^d} w_{t-s}(x-y) f(y,s) \ dy = \int_0^t w_{t-s}(\cdot) \ast f(\cdot, s) \ ds. \quad (6.3) $$

In order to formulate a new result we introduce new Banach spaces on the (measurable) functions of two variables $f(x,t), \ x \in R^d, \ t > 0$:

$$ \|\|f\|\|_{\theta,\lambda} := \sup_{s > 0} \left[ \frac{||f(\cdot,s)||_\theta}{s^\lambda} \right], \quad (6.4) $$

$\theta = \text{const} \geq 1, \ \lambda = \text{const} \in (-1,0)$. The space of all the functions $\{f(\cdot,\cdot)\}$ with finite such a norm will be denoted $A(\theta, \lambda)$:

$$ A(\theta, \lambda) = \{f : \|\|f\|\|_{\theta,\lambda} < \infty\}. \quad (6.5) $$

We assume in this subsection

$$ f \in A(\theta_1, \lambda_1) \cap A(\theta_2, \lambda_2). \quad (6.6) $$

Define for arbitrary $\delta \in (0,1)$ the values $r_1, r_2$ as follows:

$$ 1 + \frac{\delta}{d} = \frac{1}{r_1} + \frac{1}{\theta_1}, \quad 1 + \frac{1}{d} = \frac{1}{r_2} + \frac{1}{\theta_2}. $$
with the following restrictions:

\[ r_{1,2} > 1, \theta_{1,2} \geq 1, \quad -1 < \lambda_{1,2} < 0, \]

\[ d \left( 1 - \frac{1}{r_1} \right) < 2, \quad d \left( 1 - \frac{1}{r_2} \right) < 1, \quad \frac{d}{2} \left( 1 - \frac{1}{r_1} \right) - 1 - \lambda_1 = \frac{1 - \delta}{2}, \quad \frac{d}{2} \left( 1 - \frac{1}{r_2} \right) - 1 - \lambda_2 = \frac{1}{2}. \]

We estimate using once more Young’s inequality:

\[
\|v\|_{d/\delta} \leq \int_0^t \|w_{t-s} \ast f(s)\|_{d/\delta} \, ds \leq K_{BL}(r_1, \theta_1) \int_0^t \|w_{t-s}\|_{r_1} \cdot \|f(s)\|_{\theta_1} \, ds \leq \\
K_{BL}(r_1, \theta_1) \cdot M(d, r_1) \cdot \|f\|_{\theta_1, \lambda_1} \int_0^t (t-s)^{-d(1-1/r_1)} s^\lambda \, ds = \\
t^{-1-\delta/2} K_{BL}(r_1, \theta_1) \cdot M(d, r_1) \cdot \|f\|_{\theta_1, \lambda_1} \cdot B(1-d(1-1/r_1), 1+\lambda_1). \tag{6.8}
\]

We find analogously

\[
\nabla v = \int_0^t ds \int_{\mathbb{R}^d} \nabla w_{t-s}(x-y) \cdot f(y,s) \, ds;
\]

\[
\|\nabla v\|_d \leq K_{BL}(r_2, \theta_2) \cdot M'(d, r_2) \cdot \|f\|_{\theta_2, \lambda_2} \int_0^t (t-s)^{-1/2-0.5d(1-1/r_2)} s^{\lambda_2} \, ds = \\
K_{BL}(r_2, \theta_2) \cdot M'(d, r_2) \cdot \|f\|_{\theta_2, \lambda_2} \cdot B(1/2-0.5d(1-1/r_2), \lambda_2 + 1) \cdot t^{-1/2}. \tag{6.9}
\]

Summing with the estimations for \(\|u_0\|_{d/\delta}, \|\nabla u_0\|_d\) with corresponding estimations (6.8), (6.9) for \(\|v\|_{d/\delta}, \|\nabla v\|_d\) we conclude that if the norms \(\|a\|_d, \|f\|_{\theta_1, \lambda_1}, \|f\|_{\theta_2, \lambda_2}\) are finite, then \(T > 0\); if these norms are sufficiently small, then \(T = \infty\).

3. Analogously to the content of this report may be considered a more general case of abstract (linear or not linear) parabolic equation of a view

\[
\partial_t u = Au + F(u, \nabla u; x, t) + f(x, t), \quad u(x, 0) = a(x).
\]

The detailed investigation of this case when the initial condition and external force belong to some Sobolev’s space may be found, e.g. in [36] - [38], [17], [34].
4. Let us consider the general non-linear parabolic equation (may be multivariate, i.e. system of equations) of a view

\[ \frac{\partial u}{\partial t} = Au + F(u, \nabla u) \]

with initial condition \( u(x, 0^+) = a(x) \). Here \( A \) be negative definite linear operator, may be unbounded, for example

\[ Au = \sum_{i,j=1}^{d} b_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \]

i.e \( A \) be strictly elliptical differential operator of a second order with bounded coefficients:

\[ 0 < \inf_{\xi:||\xi||=1} \sum_{i,j=1}^{d} b_{i,j}(x) \xi_i \xi_j \leq \sup_{\xi:||\xi||=1} \sum_{i,j=1}^{d} b_{i,j}(x) \xi_i \xi_j < \infty, \]

satisfying the Hölder’s condition: for some positive \( \alpha = \text{const} \in (0, 1] \)

\[ \max_{i,j} |b_{i,j}(x) - a_{i,j}(y)| \leq C \cdot ||x - y||^\alpha. \]

We set ourselves a goal to obtain a positive lower estimate for lifespan \( T \) for solution likewise to the third section, following, e.g. [38], p. 272-275.

We understand as before in the capacity of solution the mild solution:

\[ u(t) = e^{At}a + \int_0^t e^{(t-s)A} F(u(s), \nabla u(s)) \, ds \overset{def}{=} e^{At}a + \int_0^t e^{(t-s)A} \Phi(u(s)) \, ds \overset{def}{=} \Psi[u](t), \]

where \( \Phi(u(s) = F(u(s), \nabla u(s)). \]

We enumerate the conditions.

**A.** There exist two Banach spaces \( X, Y \) such that \( a(\cdot) \in X \) and such that the semigroup \( \{S_t\} = \{\exp(At)\} \), \( X \rightarrow X \ t \geq 0 \) is strong continuous.

**B.** The operator (non linear, in general case) \( \Phi : X \rightarrow Y \) is locally Lipshitz map.

**C.**

\[ \forall t > 0 \ S_t = e^{At} : Y \rightarrow X \]

and moreover

\[ \exists \gamma \in (0, 1), \ \exists C = C(\gamma) \Rightarrow ||e^{At}||(Y \rightarrow X) \leq C(\gamma) \ t^{-\gamma}, \ t \in (0, 1). \]

Many examples of such a semigroups are given in [38], p. 274 - 286.

Let \( \alpha = \text{const} \in (0, 1) \); we define

\[ Z = Z(\alpha) = \{u \in C([0, T_1], X : u(0) = a, ||u(t) - f||X \leq \alpha\}. \quad (6.10) \]
There exists a value $T_2 = T_2(\alpha) > 0$ for which

$$\sup_{t \in (0, T_2)} \| e^{tA}a - a \| X \leq \frac{\alpha}{2}. \quad (6.11)$$

Denote also

$$K_1 = K_1(\alpha) = \sup_{u \in Z(\alpha)} \| \Phi(u)(\cdot) \| Y < \infty;$$

then

$$\| \int_0^t e^{A(t-s)}\Phi(u(s)) \, ds \| X \leq K_1(\alpha) \, C(\gamma) \frac{T_1^{1-\gamma}}{1-\gamma}. \quad (6.12)$$

We can choose $T_3$ such that

$$K_1(\alpha) \, C(\gamma) \frac{T_3^{1-\gamma}}{1-\gamma} < \frac{\alpha}{2}.$$  

As long as

$$\forall u, v \in Z(\alpha) \Rightarrow \| \Phi(u)(s) - \Phi(v(s)) \| Y \leq K_2(\alpha) \| u(s) - v(s) \| X, \quad K_2 = K_2(\alpha) < \infty,$$

we deduce

$$\| \Psi[u](t) - \Psi[v](t) \| \leq \int_0^t e^{A\xi} \| \Phi(u)(s) - \Phi(v(s)) \| Y \, ds \leq$$

$$K_2(\alpha)C(\gamma) \frac{T_1^{1-\gamma}}{1-\gamma} \sup_{s \in (0, T_4)} \| u(s) - v(s) \| X \leq$$

$$0.5 \sup_{s \in (0, T_4)} \| u(s) - v(s) \| X,$$

if

$$K_2(\alpha) C(\gamma) \frac{T_4^{1-\gamma}}{1-\gamma} \leq 0.5.$$

Thus, we can take $T = \min(T_1, T_2, T_3, T_4)$.

On the other hand, the examples of blow up in finite time solutions of non-linear parabolic equations see, e.g. in articles [53], [61]. Examples of non-uniqueness for this equations see in [59].

5. It may be investigated in addition analogously the boundary value problem, for instance, Dirichlet or Neuman, when the variable $x$ belongs to some domain $\Omega$ (bounded or not) with boundary of the class $C^{1,1}$.
References

[1] Shangbin Cui. Global well-posedness of the 3-dimensional Navier-Stokes initial value problem in $L(p) \cap L(2)$ with $3 < p < \infty$. arXiv:1204.5040v1 [math.AP] 23 Apr 2012

[2] Chae Dongho and Lee Jihoon. On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. arXiv:1305.4681v1 [math.AP] 21 May 2013

[3] Barraza O. Self-similar solutions in weak $L_p$– spaces of the Navier-Stokes equations. Revista Mat. Iberoamer., 12(1996), 411–439.

[4] Caffarelli L., Kohn R. and Nirenberg L. Partial regularity of suitable weak solutions of the Navier-Stokes equations. Comm. Pure Appl. Math., 35, (1082), 771–831.

[5] Calderon C. Existence of weak solutions for the Navier-Stokes equations with initial data in $L(p)$. Trans. A.M.S., 318(1990), 179–207.

[6] Cannone M. A generalization of a theorem by Kato on Navier-Stokes equations. Revista Matematica Iberoamericana. V. 13 3, (1997), 515–542.

[7] Escauriaza L., Seregin G., and Sverak V. $L(3,\infty)$ -Solutions to the Navier-Stokes Equations and Backward Uniqueness. Uspekhi Mat. Nauk, 58(2003), no.2, 3–44.

[8] Fabes E., Johns B. and Riviere N. The initial value problem for the Navier-Stokes equations with data in $L(p)$. Arch. Rat. Mech. Anal., 45(1972), 222–240.

[9] Foias C., Guillope C. and Temam R. New a priori estimates for Navier-Stokes equations in dimension 3. Comm. in Part. Dif. Eq., 6, (1981), 329–359.

[10] Fujita H. and Kato T. On the Navier-Stokes initial value problem I. Arch. Ration. Mech. Anal., 16(1964), 269–315.

[11] Germain P. Multipliers, paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations. J. Diff. Equations., 226(2006), 373–428.

[12] Giga Y. Solutions of semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system. J. Diff. Equations, 62(1986), 186–212.

[13] Giga Y. and Miyakawa T. Navier-Stokes flows in $R^3$ with measureas as initial vorticity and the Morrey spaces. Comm. P. D. E., 14(1989), 577–618.

[14] Giga Y. and Sohr H. Abstract $L^p$ – estimates for the Cauchy problem with Applications to the Navier-Stokes equations in exterior domains. Hokkaido University, Preprint, Series 60 on Mathematics, November 1989.
[15] GIGA Y. AND SOHR H. Abstract $L^p$ – estimates for the Cauchy problem with Applications to the Navier-Stokes equations in extero ir domains. J. Funk. Anal., 102 (1991), 72 - 94.

[16] FARHAD HATAMI, MOHAMMAD BAGHER GHAEMI. On the global existence solution for a chemotaxis model. Applied Mathematics, (2013), 78, 134 - 143.

[17] IWASHITA H. $L^p−L^r$ estimates for solution of non-stationary Stokes equations in exterior domain and the Navier-Skokes initial value problems in $L_q$ spaces. Math. Ann., 285, (1989), 265 - 288.

[18] KATO T. Strong $L_p$ solutions of the Navier-Stokes equations in $R^m$ with applications to weak solutions. Math. Z., 187(1984), 471 - 480.

[19] KATO T. AND PONCE G. Commutator estimates and the Euler and Navier-Stokes equations. Comm. P. D. E., 41(1988), 891 - 907.

[20] KENIG C.E. AND KOCH G.S. An alternative approach to regularity for the Navier-Stokes equations in a critical space. arXiv:0908.3349.

[21] KOCH H. AND TATARU D. Well-posedness for the Navier-Stokes equations. Adv. in Math., 157(2001), 22 – 35.

[22] KOZONO H. AND TANIUCHI Y. Bilinear estimates in BMO and the Navier-Stokes equations. Math. Z., 235(2000), 173 – 194.

[23] LEMARIE-RIEUSSET P.G. Weak infinite-energy solutions for the Navier-Stokes equations in $R^3$, Preprint, 1998.

[24] LEMARIE-RIEUSSET P.G. Recent developments in the Navier-Stokes problems. Research Notes in Mathematics, Chapman, Hall/CRC, 2002.

[25] LERAY J. Sur le mouvement dun liquide visqueux emplissant lespace. Acta Math., 63(1934), 193 – 248.

[26] MASUDA K. Weak solutions of Navier-Stokes equations. Tohoku Math. J., 36(1984), 623 – 646.

[27] MONTGOMERY-SMITH S. Finite-time blow up for a Navier-Stokes like equation. Proc. Amer. Math. Soc., 129, (2001,) pages 3025 - 3029.

[28] MIURA H. Remarks on uniqueness of mild solutions to the Navier-Stokes equations. J. Funct. Anal., 218(2005), 110129.

[29] OGAWA T., RAJOPADHYE SH. V. AND SCHONBEK M.E. Energy Decay for a Weak Solution of the Navier-Stokes Equation with Slowly Varying External Forces. Journal of Functional Analysis, 144, (1997), 325 – 358.

[30] PLANCHON F. Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $R^3$, Ann. Inst. H. Poincare Anal. Non Lineaire, 13(1996), 319 – 336.
[31] Seregin G. A certain necessary condition of potential blow up for Navier-Stokes equations. arXiv:1104.3615. 21 Aug 2010.

[32] Seregin, G. (2011) Necessary conditions of potential blowup for the Navier-Stokes equations. J. Math. Sci. (N.Y.) 178, (2011), 345 - 352.

[33] Serrin J. The initial value problem for the Navier-Stokes equations. In: R.E. Langer, (Ed.), Nonlinear Problems, 1963, University of Wisconsin Press, Madison, 1963, pp. 69 - 98.

[34] Solonnikov V.A. Estimates for Solutions of non-stationary Navier - Stokes equations. J. Soviet Math., 8, (1977), 467 - 523.

[35] Stein E. M. Singular Integrals and Differentiability Properties of Functions. Princeton, University Press, (1970), Princeton, New Jersey.

[36] Taylor M.E. Pseudodifferential Operators. Princeton, University Press; Princeton, New Jersey, (1981)

[37] Taylor M.E. Partial Differential Equations I. Linear Equations. Applied Math. Sciences, 117, Springer, (1996).

[38] Taylor M.E. Partial Differential Equations III. Non-linear Equations. Applied Math. Sciences, 117, Springer, (1996).

[39] Temam R. Navier - Stokes Equations. Theory and Numerical Analysis. North-Holland Publishing Company. Amsterdam, New York, Oxford, (1977).

[40] Vishik M.I. Hydrodynamics in Besov spaces. Arch. ration. Mech. Anal., 145, 197-214 (1998)

[41] Zeng Zhang, Zhaoyang Yin Global Well-posedness for the Generalized Navier-Stokes System. arXiv:1306.3735v1 [math.AP] 17 Jun 2013

[42] Bañuelos R. and Osekowski A. Sharp martingale inequalities and application to Riesz transforms on manifolds, Lie group and Gauss space. arXiv:1305.1492v1 [math.PR] 7 May 2013

[43] Bliss G. An integral inequality. J. London Math. Soc., (1930), vol. 5, 40 - 46.

[44] Brascamp H.J. and E.H. Lieb E.H. Best constants in Youngs inequality, its converse and its generalization to more than three functions. Journ. Funct. Anal., 20(1976), 151 - 173.

[45] Fiorenza A., and Karadzhov G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
[46] Fiorenza A. *Duality and reflexivity in grand Lebesgue spaces.* Collect. Math. **51**(2000), 131 - 148.

[47] Fiorenza A. Karadzhov G.E. *Grand and small Lebesgue spaces and their analogs.* Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine”, Sezione di Napoli, Rapporto tecnico 272/03(2005).

[48] Iwaniec T. and Sbordone C. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat. Mech. Anal., **119**(1992), 129 - 143.

[49] Iwaniec T., Koskela P. and Onninen J. *Mapping of Finite Distortion: Monotonicity and Continuity.* Invent. Math. **144**(2001), 507 - 531.

[50] Iwaniec T. and Martin G. *Riesz transforms and related singular integrals.* J. Reine Angew. Math. 473 (1996), 25 - 57.

[51] Jawerth B. and Milman M. *Extrapolation theory with applications.* Mem. Amer. Math. Soc. **440**, (1991).

[52] G.Talenti. *Inequalities in Rearrangement Invariant Function Spaces. Non-linear Analysis, Function Spaces and Applications.* Prometheus, Prague, **5**, (1995), 177 - 230.

[53] Ball J. *Remarks on blow-up and nonexistence theorems for nonlinear evolution equations.* Quarterly Journal of Mathematics 28 (1977) 473-486.

[54] Benamour, J. (2010) *On the blow-up criterion of 3D Navier-Stokes equations.* J. Math. Anal. Appl. 371, (2010) 719 - 727.

[55] Gallagher I. and Paicu M. *Remarks on the blow-up of solutions to a toy model for the Navier-Stokes equations.* Proceedings of the American Mathematical Society, 137, 2009, pages 2075 - 2083.

[56] German P. *The second iterate for the Navier-Stokes equation.* arXiv:0806.4525v1 [math.AP] 27 Jun 2008

[57] Masahiro Ikeda, Soichiro Katayama, and Hideaki Sunagawa. *Null structure in a system of quadratic derivative nonlinear Schrödinger equations.* arXiv:1305.3662v1 [math.AP] 16 May 2013

[58] Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian type.* Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, **32**, 43 - 57.

[59] Ladyzhenskaya, O. A. *Example of non-uniqueness in the Hopf class of weak solutions for the navier-Stokes equations.* Izv. Ahad. Nauk SSSR, Ser. Mat. Tom 33 (1969), No. 1, pp. 229236.

[60] Liflyand E., Ostrovsky E., Sirota L. *Structural Properties of Bilateral Grand Lebesgue Spaces.* Turk. J. Math.; **34** (2010), 207 - 219.
[61] Marino V., Pacella F., aan Scuinzi B. *Blow up of solution of semilinear heat equations in general domains*. arXiv:1306.1417v1 [math.AP] 6 Jun 2013

[62] Ostrovsky E.I. (1999). *Exponential estimations for random Fields and its applications (in Russian)*. Moscow - Obninsk, OINPE.

[63] Ostrovsky E. and Sirota L. *Moment Banach spaces: theory and applications*. HIAT Journal of Science and Engineering, C, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).

[64] Ostrovsky E., Sirota L. *Nikolskii-type inequalities for rearrangement invariant spaces*. arXiv:0804.2311v1 [math.FA] 15 Apr 2008

[65] Ostrovsky E., Sirota L. *Solvability of Navier-Stokes equations in some rearrangement invariant spaces*. arXiv:1305.5321v1 [math.AP] 23 May 2013

[66] Pichorides S.K. *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*. Studia Math. 44 (1972), 165 - 179.