No-deleting and no-cloning principles as consequences of conservation of quantum information

Michał Horodecki, Ryszard Horodecki, Aditi Sen(De) and Ujjwal Sen
Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland

We show that the no-deleting and no-cloning principles are implications of information conservation principle. This is unlike in classical physics, where cloning and deleting are possible, independently of information conservation. Connections with the second law of thermodynamics are also discussed.

I. INTRODUCTION

According to the generic information paradigm [1, 2], all physical processes in nature should be formulated in terms of processing of information, the latter being a fundamental quantity that can be defined independently of probability itself [3, 4]. This program got an excellent “tool” - the quantum information paradigm (see e.g. [5]). Then it seems reasonable to reformulate the quantum formalism in terms of quantum information. Such a task naturally requires some axiomatic approach to the latter.

In this spirit, the conservation of quantum information was proposed [2, 10] as a basic principle in processing of quantum information. It can be formulated as follows:

“In a closed system, one cannot change entanglement by local operations.”

Note that this formulation is “quantum”, as it does not say about probabilities but merely about entanglement - a purely quantum property of compound systems. In this context it is natural to ask: What are the fundamental consequences of the conservation of information principle? In this paper we show that both no-deleting and no-cloning principles, being consequences [11, 12, 13] of the quantum formalism, are implications of conservation of quantum information. It is also suggested that the converse implication holds. We also discuss the basic relations between the no-cloning and no-deleting principles and second law of thermodynamics, the latter being treated as part of the information conservation principle.

II. THE NO-DELETING PRINCIPLE

The no-deleting principle states that in a closed system, one cannot destroy quantum information. In closed systems, quantum information can only be moved from one place (subspace) to another. One formulation of this principle is the following:

Given two qubits (two-dimensional quantum systems, say an electron or a photon) in arbitrary but equal states, one cannot take one of them to a fixed state, while keeping the other in the original state. More formally, one cannot have the evolution

$$|\psi\rangle |\psi\rangle \rightarrow |\psi\rangle |0\rangle ,$$

where $|\psi\rangle$ is an arbitrary state of a qubit and $|0\rangle$ is a fixed state.

A sharper formulation is that it is not possible to delete a copy against another when given from among two nonorthogonal states. More formally, the evolution

$$|\psi_1\rangle |\psi_1\rangle \rightarrow |\psi_1\rangle |0\rangle ,$$

$$|\psi_2\rangle |\psi_2\rangle \rightarrow |\psi_2\rangle |0\rangle ,$$

is not possible, where $(\langle\psi_1| \langle\psi_2|) \neq 0$.

The above statements of the no-deleting principle can be proven as theorems, by assuming unitary evolution. This was done in [11], and was called the no-deleting theorem.

Note the importance of deleting one copy against another copy. In a closed system, even orthogonal states cannot be deleted alone, if an unitary evolution is assumed. One cannot take the orthogonal states $|0\rangle$ and $|1\rangle$ to a standard state $|0\rangle$, even with ancillas. If one assumes that

$$|0\rangle |A\rangle \rightarrow |0\rangle |A_0\rangle ,$$

$$|1\rangle |A\rangle \rightarrow |0\rangle |A_1\rangle ,$$

is true under an unitary evolution, the only option is to have orthogonal $|A_0\rangle$ and $|A_1\rangle$. That is the information is merely transfered from the system to the ancilla, and not deleted. Therefore even classical information cannot be deleted alone in unitary evolution. In contrast, classical information can of course be deleted against a copy. Therefore the transformations

$$|0\rangle |0\rangle \rightarrow |0\rangle |0\rangle ,$$

$$|1\rangle |1\rangle \rightarrow |1\rangle |0\rangle ,$$

is possible by a single unitary evolution, for example by the so-called CNOT gate.

More important is to realize that the no-deleting principle is true only in closed systems (see [14] in this regard). One can of course delete a state of a system, even alone, by simply throwing it out of the system. A more “sophisticated” way of saying it is the following: Given a qubit in a state $|\psi\rangle$, an observer (call him “Demon”) performs a measurement in the computational basis $\{|0\rangle, |1\rangle\}$. If the result is $|0\rangle$, Demon keeps the output as $|0\rangle$. However if the result is $|1\rangle$, Demon flips (by applying, for example, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, written in the computational basis) the output $|1\rangle$ to $|0\rangle$. Thus Demon is able...
to delete an arbitrary qubit to a standard state. And actually he is able to do it for a single copy alone, which we have seen to be not possible even for classical information if the system is closed. But the system here is open. The information about the result of measurement is now in Demon’s “head”. Formally, the superoperator (trace-preserving completely positive map) that effects the above transformation from \( |\psi\rangle \) to \( |0\rangle \) is the following:

\[
\rho \rightarrow \Lambda(\rho) = I |0\rangle \langle 0| \rho |0\rangle \langle 0| + \sigma_x |1\rangle \langle 1| \rho |1\rangle \langle 1| \sigma_x \]

\[
= |0\rangle \langle 0| \rho |0\rangle \langle 0| + |0\rangle \langle 1| \rho |1\rangle \langle 0| \]

(Here \( I \) denotes the identity operator on the qubit Hilbert space.) Every superoperator can be seen as an unitary evolution in an extended Hilbert space. Specifically, any superoperator evolution \( \rho \rightarrow \sum_i A_i \rho A_i^\dagger \), with \( \sum_i A_i^\dagger A_i \) being the identity operator on the Hilbert space on which \( \rho \) is defined, can be seen as the unitary evolution \( |\phi\rangle |E\rangle \rightarrow \sum_i A_i |\phi\rangle |i\rangle |E\rangle \), for orthogonal states \( |i\rangle \) of the “environment” \( E \). (Here \( |\phi\rangle \) is an arbitrary vector from the Hilbert space on which \( \rho \) is defined.) In our case, the superoperator \( \Lambda \) can be seen as the following unitary evolution:

\[
|\psi\rangle |0\rangle |E\rangle \rightarrow |0\rangle \langle 0| \psi \rangle \langle 0| + |0\rangle \langle 1| \psi \rangle \langle 1| |E\rangle.
\]

However the right hand side is simply

\[
|0\rangle \langle 0| \psi \rangle \langle 0| + |0\rangle \langle 1| \psi \rangle \langle 1| |E\rangle = \rho \text{ with no}\langle 0| \psi \rangle \langle 0| + \langle 1| \psi \rangle \langle 1| |E\rangle. \]

So the system is simply thrown out into the environment to effect the deletion. That is in doing the deletion, Demon simply transferred the whole information (about the state to be deleted) into his “head”.

Thus we see that the very notion of the no-deleting principle, naturally leads us to consider it in a scenario of a closed system. This (and more particularly, the action of Demon above) reminds us of the second law of thermodynamics. The whole point in exorcising the Maxwell’s demon (and save the second law of thermodynamics) was to include the measuring device into the system, that is to consider everything in a closed system (see [16] for a recent review).

III. THE NO-CLONING PRINCIPLE

The no-cloning principle states that quantum information cannot be “leaked out” while keeping the original information content intact. And this is true even in open systems.

More formally, given a qubit in an unknown state \( |\psi\rangle \), one cannot obtain

\[
|\psi\rangle |0\rangle \rightarrow |\psi\rangle |\psi\rangle,
\]

even for open systems. Considering the environment inside the dynamics, this states that one cannot obtain

\[
|\psi\rangle |0\rangle |0\rangle \rightarrow |\psi\rangle |\psi\rangle |\psi\rangle |E\rangle.
\]

Again a sharper formulation is possible, viz. that one cannot have

\[
|\psi_1\rangle |0\rangle \rightarrow |\psi_1\rangle |\psi_1\rangle,
\]

\[
|\psi_2\rangle |0\rangle \rightarrow |\psi_2\rangle |\psi_2\rangle,
\]

for nonorthogonal \( |\psi_1\rangle \) and \( |\psi_2\rangle \). Including the environment, this implies that

\[
|\psi_1\rangle |0\rangle |0\rangle \rightarrow |\psi_1\rangle |\psi_1\rangle e_{\psi_1} |E\rangle,
\]

\[
|\psi_2\rangle |0\rangle |0\rangle \rightarrow |\psi_2\rangle |\psi_2\rangle e_{\psi_2} |E\rangle,
\]

is not possible by a single evolution.

Again assuming an unitary dynamics, these statements can be proven. This was done in Refs. [12, 13], and is called the no-cloning theorem.

In the following we raise this theorem to a principle and look for its connections with thermodynamics.

IV. THE NO-DELETING AND NO-CLONING PRINCIPLES AND THE SECOND LAW

In classical physics, both cloning and deleting are possible. Independently of these, the second law of thermodynamics holds, which says that entropy cannot be decreased in a closed system. A different formulation of second law would be that the probabilities do not change, under Hamiltonian evolution in phase space. The latter could be called conservation of classical information, and it includes both no-increasing of entropy as well as no-decreasing of entropy.

In quantum mechanics, one cannot either delete or clone. The second law holds too: entropy cannot be decreased in a closed system.

Again the second law can be stated in a stronger way: that eigenvalues of density matrix do not change under quantum evolution. And this is actually the information conservation principle.

Since the second law holds for classical systems independently of deleting and cloning, one could expect that no-cloning and no-deleting principles will not be connected with this law, and with law of conservation of information.

However, as we have stressed above, the no-deleting principle has connections with the Maxwell’s demon approach to thermodynamics. Namely, it needs a closed system: One can always delete, by throwing away the copy. However the copy will reside somewhere in the world. This is similar to the second law, which says that you cannot decrease entropy in a closed system. In open systems, one can do that: By bringing in a pure state, and removing the initial state.

Below we will argue that both no-cloning and no-deleting principles are consequences of conservation of information.
A. The second law contains the no-deleting principle

In the following, we raise the statement of the no-deleting theorem to the level of a principle, and do not assume an unitary dynamics. We will then show that the no-deleting principle is contained in the second law of thermodynamics.

Suppose that deletion is possible. Then one can effect the following evolution (call it the “deleting evolution”) in a closed system:

$$|\psi\rangle |\psi\rangle \rightarrow |\psi\rangle |0\rangle,$$

for an arbitrary qubit $|\psi\rangle$ and a standard state $|0\rangle$. For arbitrary $|\psi\rangle$, the states $|\psi\rangle |\psi\rangle$ span a three-dimensional subspace (the so-called symmetric subspace) of the four-dimensional Hilbert space of two qubits. However the states $|\psi\rangle |0\rangle$ span only a two-dimensional subspace, as $|0\rangle$ is a fixed state. Thus the deleting evolution takes a system with von Neumann entropy $\log_2 3$ (the identity operator on the symmetric subspace of two qubits has von Neumann entropy $\log_2 3$) to one with von Neumann entropy $\log_2 2$. Since the system is closed, this decrease of entropy is therefore a violation of the second law of thermodynamics. In other words, the second law of thermodynamics implies the no-deletion principle.

One knows that the statement of the second law of thermodynamics can be that “entropy does not change in a closed system”. In this paper, unless stated otherwise, we will use only the version stating that “entropy does not decrease in a closed system”. This is the version that we have used above in showing that the second law implies the no-deleting principle.

The sharper form of the deleting, viz.

$$|\psi_1\rangle |\psi_1\rangle \rightarrow |\psi_1\rangle |0\rangle,$$
$$|\psi_2\rangle |\psi_2\rangle \rightarrow |\psi_2\rangle |0\rangle,$$

for $\langle \psi_1 | \psi_2 \rangle \neq 0$, also violates the second law. This is because $|\psi_1\rangle |\psi_1\rangle$ and $|\psi_2\rangle |\psi_2\rangle$ are more farther apart than $|\psi_1\rangle |0\rangle$ and $|\psi_2\rangle |0\rangle$ and consequently the average input state $\frac{1}{2}(|\psi_1\rangle |\psi_1\rangle \langle \psi_1 | \psi_1 \rangle + |\psi_2\rangle |\psi_2\rangle \langle \psi_1 | \psi_2 \rangle)$ has less von Neumann entropy than the average output state $\frac{1}{2}(|\psi_1\rangle |0\rangle \langle \psi_1 | 0 \rangle + |\psi_2\rangle |0\rangle \langle \psi_2 | 0 \rangle)$.

One may also see deletion in the following way [17]:

$$|\psi_1\rangle |\psi_1\rangle \rightarrow |\psi_1\rangle |a_{\psi_1}\rangle,$$
$$|\psi_2\rangle |\psi_2\rangle \rightarrow |\psi_2\rangle |a_{\psi_2}\rangle,$$

where $\langle \psi_1 | \psi_2 \rangle \neq 0$, and $|a_{\psi_1}\rangle$ and $|a_{\psi_2}\rangle$ are more nearer than $|\psi_1\rangle$ and $|\psi_2\rangle$. That is, $|\langle \psi_1 | \psi_2 \rangle| < |\langle a_{\psi_1} | a_{\psi_2} \rangle|$. Again we see that the average input state $\frac{1}{2}(|\psi_1\rangle |\psi_1\rangle \langle \psi_1 | \psi_1 \rangle + |\psi_2\rangle |\psi_2\rangle \langle \psi_1 | \psi_2 \rangle)$ has less von Neumann entropy than the average output state $\frac{1}{2}(|\psi_1\rangle |a_{\psi_1}\rangle \langle \psi_1 | a_{\psi_1} \rangle + |\psi_2\rangle |a_{\psi_2}\rangle \langle \psi_2 | a_{\psi_2} \rangle)$.

Therefore whatever is the form of deletion, it violates the second law of thermodynamics.

B. The second law for open systems contains the no-cloning principle

Can the no-cloning principle be also implied by the second law? To answer this question, we have first of all to formulate the second law of thermodynamics for open systems (see for example [18]). For open systems, the second law can be stated as follows:

The relative entropy distance between any two states $\rho$ and $\sigma$ does not increase.

The relative entropy distance of $\rho$ from $\sigma$ is denoted as $S(\rho|\sigma)$ as defined as $\text{tr}(\rho \log_2 \rho - \rho \log_2 \sigma)$ (which is in fact the Lindblad-Uhlmann monotonicity, see e.g. [19]).

Suppose now that the transformation in eq. (8) is possible. Then the average relative entropy distance of the input states from the average output state $\rho_{in} = \frac{1}{2}(|\psi_1\rangle |0\rangle \langle 0 | \psi_1 \rangle + |\psi_2\rangle |0\rangle \langle 0 | \psi_2 \rangle)$ is

$$\frac{1}{2}S(|\psi_1\rangle |0\rangle \langle 0 | \psi_1 \rangle |0\rangle |\rho_{in}) + S(|\psi_2\rangle |0\rangle \langle 0 | \psi_2 \rangle |0\rangle |\rho_{in})) = S(\rho_{in}).$$

However this quantity (which is actually the Holevo quantity for the input ensemble [20]) is less than the average relative entropy distance of the output states from the average output state $\rho_{out} = \frac{1}{2}(|\psi_1\rangle |e_{\psi_1}\rangle \langle e_{\psi_1} | \psi_1 \rangle + |\psi_2\rangle |e_{\psi_2}\rangle \langle e_{\psi_2} | \psi_2 \rangle)$ is

$$\frac{1}{2}S(|\psi_1\rangle |e_{\psi_1}\rangle \langle e_{\psi_1} | \psi_1 \rangle |e_{\psi_1} |\rho_{out}) + S(|\psi_2\rangle |e_{\psi_2}\rangle \langle e_{\psi_2} | \psi_2 \rangle |e_{\psi_2} |\rho_{out})) = S(\rho_{out}).$$

Therefore the second law of thermodynamics for open systems contains the no-cloning principle.

Again one can also considering in the following way:

$$|\psi_1\rangle |0\rangle_E \rightarrow |\psi_1\rangle |e_{\psi_1}\rangle_E,$$
$$|\psi_2\rangle |0\rangle_E \rightarrow |\psi_2\rangle |e_{\psi_2}\rangle_E,$$
1. The dynamics of the system is linear on the level of pure states (this can of course be easily generalised to mixed states),
2. The system is always in a pure state,
3. Addition of ancillas is allowed.

We do not assume unitary dynamics. And we will explicitly state when we allow tracing out as a valid operation.

We will show that in such physical systems,
1. Conservation of information (actually, no-decrease of entanglement under local operations) implies the no-deleting principle
2. Conservation of information (actually, no-increase of entanglement under local operations) implies the no-cloning principle

In the second implication, we allow tracing out as a valid operation.

Note that it is natural that we do not allow tracing out as a valid operation in considerations of the no-deleting principle. When we are trying to show that reducing information from the system is not possible, we cannot afford to be “careless” and allow throwing out parts of the system. On the other hand, in considerations with no-cloning, when we are trying to show that production of information in a system is not possible, we must allow throwing out. After all, throwing out part of a system cannot produce information.

A. No-decrease of entanglement in a closed system implies the no-deleting principle

We show here that the no-deleting principle is implied by the principle that entanglement cannot decrease in a closed bipartite system.

By entanglement for a bipartite pure state $|\alpha_{AB}\rangle$, we mean the von Neumann entropy of the local density matrix $\rho_{AorB}|\alpha_{AB}\rangle \langle\alpha_{AB}|$ of $|\alpha_{AB}\rangle$.  

Again we prove by contradiction. Suppose therefore that an observer, Alice, prepares the state

$$ |\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |\psi_1\rangle |\psi_1\rangle + |1\rangle |\psi_2\rangle |\psi_2\rangle ) $$

and sends the second and third systems to a distant party, Bob, while keeping the first system to herself.

If deleting is possible, that is if there is a machine (available to Bob) that effects the transformation in eq. 1, then Bob can locally change the state $|\Psi_{AB}\rangle$ into the state

$$ |\Psi'_{AB}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |\psi_1\rangle |0\rangle + |1\rangle |\psi_2\rangle |0\rangle ).$$

Clearly, $|\Psi'_{AB}\rangle$ possesses less entanglement than $|\Psi_{AB}\rangle$, for nonorthogonal $|\psi_1\rangle$ and $|\psi_2\rangle$.

B. No-increase of entanglement implies the no-cloning principle

We now show that the no-cloning principle is implied by the principle that entanglement cannot increase in bipartite systems (see 2 in this regard).

Suppose again that Alice, prepares the state

$$ |\Phi_{AB}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |\psi_1\rangle |0\rangle |0\rangle + |1\rangle |\psi_2\rangle |0\rangle |0\rangle ) $$

and sends all but the first system to a distant party, Bob. The first system she keeps to herself.

If cloning were possible, that is if Bob had access to a machine that performs the operation in eq. 3, then application of that machine by Bob (locally) changes the state $|\Phi_{AB}\rangle$ into

$$ |\Phi'_{AB}\rangle = \frac{1}{\sqrt{2}} (|0\rangle |\psi_1\rangle |\psi_1\rangle |e_{\psi_1}\rangle + |1\rangle |\psi_2\rangle |\psi_2\rangle |e_{\psi_2}\rangle). $$

One can see that the state $|\Phi'_{AB}\rangle$ possesses more entanglement than $|\Phi_{AB}\rangle$, for nonorthogonal $|\psi_1\rangle$ and $|\psi_2\rangle$.

VI. DISCUSSION

That the no-deleting and no-cloning principles follows from a principle of conservation of information is a remarkable result, as it allows us to understand the physical reason for which perfect deleting or cloning is impossible. They are forbidden because they infringe a basic law of nature - conservation of quantum information. In particular it follows that somehow two clones means more (subjective) information than one clone, as opposed to arguments in 11. Deleting means increasing objective information (negentropy), but decreasing subjective information. In the case of no-cloning, we have the converse. This is compatible with the observation that cloning increases distinguishability, and deleting decreases it. Let us add here that it is conceivable that suitably extended forms of the no-deleting and no-cloning principles would imply the information conservation principle.

It is perhaps dangerous to say that the second law is a part of an information conservation principle. Usually the second law is associated with the arrow of time, and with irreversibility. Conservation of information is something completely converse. However in our approach, there is a place for these two contradicting laws together. Namely, if the universe is a closed system in a pure state, then we can think that due to dynamics, the subsystems may get irreversibly entangled with each other (as discussed e.g. in 23, and consistent with the decoherence point of view 24). Observers have access only to some of them, hence cannot change this entanglement. Thus the irreversibility is here put by hands (it should follow from features of the dynamics), and then the second law is expressed by two facts:

1) Observers can act only on part of the system
2) Information is conserved - implying, in particular, that the local disorder (equal to entanglement [23]) cannot decrease.

Similarly, when we consider efficiency of Carnot cycle, we derive it from second law, even though the cycles are reversible. Thus while deriving efficiencies, we do not deal with irreversibility “in action”, but rather with the implications of irreversible changes, “conserved” by information conservation: Once the entropy is increased (which occurred in the past) we cannot decrease it, and can only transfer it somewhere else. We could do some bad cycles, increasing entropy, but if we are interested in optimal performance, one can keep the information constant (which is in practice impossible, but in principle can be done). Going back to our picture: After the information has been partially changed into entanglement with environment, one cannot regain it. The only thing one can do is to keep constant the information that is still accessible.

Acknowledgments

This work is supported by the University of Gda´ nsk, Grant No. BW/5400-5-0256-3 and EC grants RESQ (IST-2001-37559) and QUPRODIS (IST-2001-38877).

[1] R. Horodecki, Ann. Phys. (Leipzig), 48, 479 (1991).
[2] M. Horodecki and R. Horodecki, Phys. Lett. A 244, 473 (1998).
[3] A.N. Kolmogorov, Wiestnik AN SSRR 5, 9 (1955); Problems of Information Transmission 1, 1 (1965).
[4] R. Solomonoff, technical report ZTB-138, Zator Company, Cambridge, Mas. (1960).
[5] G. Chaitin, Journal of the ACM 13, 547 (1966).
[6] R.S. Ingarden and K. Urbanik, Colloq. Mat. 9, 131 (1962).
[7] R.S. Ingarden and A. Kossakowski, Bull. Acad. Polon. Sci. 16, 61 (1968).
[8] R. Horodecki, M. Horodecki, and P. Horodecki, Quantum information isomorphism: beyond the dilemma of Scylla of ontology and Charybdis of instrumentalism, quant-ph/0305024.
[9] R. Jozsa, Illustrating the concept of quantum information, quant-ph/0305114.
[10] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Rev. A 63, 022310 (2001).
[11] A.K. Pati and S.L. Braunstein, Nature 404, 164 (2000); A.K. Pati and S.L. Braunstein, Quantum no-deleting principle and some of its implications, quant-ph/0007121.
[12] W. K. Wootters and W.H. Žurek, Nature 299, 802 (1982).
[13] D. Dieks, Phys. Lett. 92A, 271 (1982).
[14] R. Jozsa, A stronger no-cloning theorem, quant-ph/0204153.
[15] M.B. Plenio and V. Vitelli, Contemporary Physics 42, 25 (2001).
[16] The von Neumann entropy of a state $\rho$, denoted as $S(\rho)$, is defined to be $-\text{tr}\rho \log_2 \rho$.
[17] This weaker form of deletion is not possible in unitary dynamics [11].
[18] R. Alicki and K. Lendi, Quantum Dynamical Semigroups and Applications, Springer, Berlin (1987).
[19] M. Ohya and D. Petz, Quantum Entropy and its use, Springer, Verlag (1993).
[20] J.P. Gordon, in Proc. Int. School Phys. “Enrico Fermi, Course XXXI”, ed. P.A. Miles, pp 156 (Academic Press, NY 1964); L.B. Levitin, in Proc. VI National Conf. Inf. Theory, Tashkent, pp 111 (1969); A.S. Holevo, Probl. Pereda. Inf. 9, 3 1973 [Probl. Inf. Transm. 9, 110 (1973)].
[21] Again this weaker formulation of cloning is not possible, once we assume an unitary dynamics [22].
[22] C.H. Bennett, G. Brassard, and N.D. Mermin, Phys. Rev. Lett. 68, 557 (1992).
[23] C.H. Bennett, H. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996); C.H. Bennett, D.P. DiVincenzo, J. Smolin, and W.K. Wootters, Phys. Rev. A 54, 3824 (1997).
[24] J.P. Paz and W.H. Zurek, Environment-Induced Decoherence and the Transition From Quantum to Classical, quant-ph/0010011.
[25] Let us note here that recently, in Ref. [26], it was shown that equilibration in thermodynamics can be explained by entanglement.
[26] J. Gemmer, A. Otte, and G. Mahler, Quantum Approach to a Derivation of the Second Law of Thermodynamics, quant-ph/0101140; J. Gemmer, and G. Mahler, Distribution of local entropy in the Hilbert space of bipartite quantum systems: Origin of Jaynes’ principle, quant-ph/0201136.