GLOBAL EXISTENCE OF ALMOST ENERGY SOLUTION TO THE TWO-DIMENSIONAL CHEMOTAXIS-NAVIER-STOKES EQUATIONS WITH PARTIAL DIFFUSION

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(Communicated by Michael Winkler)

Abstract. In this paper, we study Cauchy problem of the two-dimensional chemotaxis-Navier-Stokes equations with partial diffusion. Taking advantage of a coupling structure of the equations and using the damping effect of the growth term \( g(n) \), we obtain the necessary estimates of solution \((n, c, u)\) without the diffusion term \( \Delta n \). These uniform estimates enable us to establish the global-in-time existence of almost weak solutions for the system.

1. Introduction. The Cauchy problem of the two-dimensional incompressible chemotaxis-Navier-Stokes system with partial diffusion reads

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n &= -\nabla \cdot (n \nabla c) + g(n), & (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\
\partial_t c + u \cdot \nabla c - \Delta c &= -cn, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P &= -n \nabla \Phi, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

Here \( n = n(x, t) \), \( c = c(x, t) \) and \( P = P(x, t) \) stand for the cell density, certain chemoattractant concentration (for example, oxygen), the pressure of the fluid, respectively. The unknown vector-valued function \( u(x, t) = (u^1(x, t), u^2(x, t)) \) represents the velocity field of the surrounding water. In addition, the growth term \( g(n) \) is given by

\[ g(n) = n(1-n)(n-a), \quad \text{for} \quad 0 < a < \frac{1}{2}, \]

which \( g(n) \) is also a sufficiently smooth function which related to the cell’s growth rate including cooperation and competition effects and the degradation rate due to exterior forces such as predation or intoxication. And the gravitational potential \( \Phi \) is also a given smooth function.

System (1.1) describes a biological process in which cells (e.g. bacteria) move towards higher concentration of chemically more favorable environment. Some experiments have shown that the mechanism is a chemotactic movement of bacteria.

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Global existence, weak solutions, growth term, Chemotaxis-Navier-Stokes equations.

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often towards higher concentration of oxygen which they consume. At the same
time, a gravitational effect on the motion of the fluid is produced by the heavier
bacteria, and a convective transport of both cells and oxygen is happened through
the water. To know more about its physical background, please see [13, 16] for more
details.

Due to the significance of the biological background, many mathematicians have
studied the similar system (1.2) and made some progress in the past years.

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n - \Delta n &= -\nabla \cdot (\chi(c)n \nabla c), \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad d = 2, 3, \\
\partial_t c + u \cdot \nabla c - \Delta c &= -f(c)n, \\
\partial_t u + \kappa(u \cdot \nabla u) - \Delta u + \nabla P &= -n\nabla \Phi, \\
\nabla \cdot u &= 0, \\
(n, c, u)|_{t=0} &= (n_0, c_0, u_0).
\end{aligned}
\]

Lorz [13] obtained the local existence of the weak solutions for problem (1.2) in a
bounded domain in $\mathbb{R}^d, d = 2, 3$, with no-flux boundary conditions and the case of
inhomogeneous Dirichlet conditions for the oxygen in $\mathbb{R}^2$. Chae, Kang and Lee [2]
proved the local well-posedness and blow-up criterion of smooth solutions of (1.2)
in the framework $H^m$ with $m \geq 3$ in $\mathbb{R}^d, d = 2, 3$. Then the result was extended by
Zhang to Besov spaces [33].

As for the global-in-time existence of solutions. When $\kappa = 0$, a global existence
result of weak solutions obtained by Duan, Lorz and Markowich in [3] for problem
(1.2) under smallness assumptions on either $\nabla \Phi$ or the initial data $c_0$. The key
ingredient of their proof was to establish a priori estimates involving energy type
functionals. Subsequently, Liu and Lorz [11] removed this smallness conditions,
and obtained the global-in-time existence of weak solutions to the two-dimensional
Navier-Stokes version of system (1.2) with $\kappa = 1$ for arbitrarily large initial data,
der under the basically same assumptions on $\chi$ and $f$ made in [3]. Recently, Winkler [25]
proved that system (1.2) admitted a unique global classical solution in a bounded
convex domain $\Omega$ with smooth boundary in $\mathbb{R}^2$ under the weaker assumption on $\chi$,
$\Phi$ and initial data than [3, 11]. More recently, Zhang and Zheng [35] established
some new estimates and proved the global well-posedness of energy solution for
the two-dimensional chemotaxis-Navier-Stokes equations in $\mathbb{R}^2$ for the rough initial
data.

For the three-dimensional chemotaxis-Navier-Stokes equations, the global classical
solutions near constant steady states were constructed in [3] for the system
(1.2) with $\kappa = 1$. When $\kappa = 0$, Winkler [25] showed that problem (1.2), which was
considered in a bounded domain and supplemented with the Neumann boundary
condition, possessed at least one global weak solution. In [30, 31], Winkler fur-
ther studied existence, eventual regularity and asymptotic stabilization of solutions
even for the full chemotaxis-Navier-Stokes equations. However, whether solutions
of problem (1.2) with large initial data exist globally or may blow up appears to
remain an open problem.

Another case is that $\Delta n$ is replaced by the porous medium type expression $\Delta n^m$
with $m > 1$ in the first equation of (1.2). We chose $m$ large which should enhance
the balance effect of the nonlinear diffusion term, so solutions are likely to remain
bounded and global existence. To know more results, we can refer to [3, 11, 20] for
more details.
In 1970, Keller and Segel [8, 9] introduced one of the first mathematical models of chemotaxis to describe the aggregation of certain type of bacteria. A simplified version of their model is as follows

\[
\begin{align*}
\frac{\partial n}{\partial t} &= \Delta n - \chi \nabla \cdot (n \nabla c) + g(n), \\
\frac{\partial c}{\partial t} &= \alpha \Delta c - \beta c - \gamma c + f(n, c).
\end{align*}
\] (1.3)

Many mathematicians have analyzed several different mathematical models of partial differential equations arising in chemotaxis. When \( g \equiv 0 \), system (1.3) corresponds to the so-called minimal model

\[
\begin{align*}
\frac{\partial n}{\partial t} &= \Delta n - \chi \nabla \cdot (n \nabla c), \\
\frac{\partial c}{\partial t} &= \alpha \Delta c - \beta c - \gamma c + f(n, c).
\end{align*}
\] (1.4)

Tao and Winkler [19] shown this problem admits at least one global weak solution which approach spatially constant equilibria in the large time limit, under homogeneous Neumann boundary conditions in bounded convex domains. For the 2D homogeneous Neumann problem, Herrero and Velázquez [6] proved that there exist radial solutions that develop a Dirac-delta type singularity in finite time, a feature known in the literature chemotactic collapse. Also they studied the asymptotic of such solutions near the formation of the singularity and the structure of the inner layer around the unfolding singularity. Compared to (1.4), under the assume that \( f(n, c) = n - c \), problem (1.3) comprises a possible proliferation of cells, a growth restriction of logistic type being included by the assumption \( g(n) \leq a - \mu n^2 \) for all \( n \leq 0 \); accordingly, one might expect that the assumption prevents an unlimited increase of the cell density. This conjecture was supported by numerical experiments, which indicated that (1.3) possessed quite a large variety of dynamical properties, especially in respect of the spontaneous emergence of patterns, though apparently simple as a two-component parabolic system. Further evidence, both numerically and analytically, on the self-organizing features of (1.3) can be found in [16], where shock-type movements of interfaces are detected as

\[ g(n) = n(1 - n)(n - a), \quad 0 < a < \frac{1}{2} \]

for cell kinetics take place much faster than cell movement [5]. For the other interesting results, one can refer [10, 22, 24, 26, 27] for more detail.

The present paper is mainly devoted to study the global existence of weak solution to system (1.1) for initial data. Now we state our main result. For clarity, we set

\[ X_0 \triangleq \{ n_0 \in L^1 \cap L^2(\mathbb{R}^2), n_0 > 0; \quad c_0 \in L^2 \cap L^\infty(\mathbb{R}^2), c_0 > 0; \quad u_0 \in H^1(\mathbb{R}^2) \}. \]

**Theorem 1.1.** Let the triple \((n_0, c_0, u_0) \in X_0, \nabla \Phi \in L^\infty(\mathbb{R}^2)\) and \( g(n) = n(1 - a)(n - a)(0 < a < \frac{1}{2}) \). Then, system (1.1) admits at least one global-in-time weak solution \((n, c, u) \in \tilde{C}_{\text{loc}}(\mathbb{R}^+; L^1 \cap L^2, H^1, H^1)\) such that

\[ n \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) \cap L^4_{\text{loc}}(\mathbb{R}^+, L^3(\mathbb{R}^2)) \cap L^4_{\text{loc}}(\mathbb{R}^+, L^4(\mathbb{R}^2)), \]

\[ c \in L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^2(\mathbb{R}^2)), \]

\[ u \in L^\infty_{\text{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^2(\mathbb{R}^2)). \]

Here the weak solution defined as follows:

**Definition 1.2** (Weak solution). We call \((n, c, u)\) is a global-in-time weak solution of system (1.1) if for any \( T > 0, \)
The triple \((n, c, u) \in C_0(0, T; L^1 \cap L^2, H^1, H^1)\) satisfies
\[
\begin{align*}
n &\in L^\infty(0, T; L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) \cap L^3(0, T; L^3(\mathbb{R}^2)) \cap L^4(0, T; L^4(\mathbb{R}^2)), \\
c &\in L^\infty(0, T; L^\infty(\mathbb{R}^2)) \cap L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)), \\
u &\in L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)) \cap L^3(0, T; H^3(\mathbb{R}^2)).
\end{align*}
\]
\[(ii)\] The triple \((n, c, u)\) satisfies equations (1.1) in the sense of distribution.

**Remark 1.** Compared with (1.2), we obtain the global existence of weak solution to the two-dimensional chemotaxis-Navier-Stokes equations with partial diffusion. Until now, there are few mathematicians study the model without diffusion term \(\Delta n\). In our paper, the growth term \(g(n)\) ensures us to obtain the necessary estimates we need, even without the term \(\Delta n\). Taking advantage of a coupling structure of the equations, using the technology of double regularization and energy estimate, we explore the related estimates of approximate solutions. After that by establishing a priori estimates and using the Arzela-Ascoli theorem, we prove the global existence of weak solutions for the system we studied.

**Remark 2.** It seems, however, that our methods can readily be adapted to similar model by slightly modifying the proof of uniform estimates in section 4. In fact, one could conclude the so-called “energy-type inequality” which is the key ingredient of our proof, by combining our approach and the double regularization technology in [14].

The paper is organized as follows. In Section 2, we review the theory of Littlewood and Paley operator, the norm of two kinds of mixed space-time Besov space, and the related lemma which will be used in the following sections. In Section 3, we study the global well-posedness for the regularized problem. We prove the existence of global classical solution \((n^\varepsilon, c^\varepsilon, u^\varepsilon)\) for such a problem, which relying on some results presented in the recent literature. Section 4 is devoted to showing uniform estimates for the regularized problem which is the heart of our analysis, and we will provide \(\varepsilon\)-independent estimates for this classical solutions, which are attained by means of functional and algebra inequalities. The proof of the main result will be given in the last section.

**Notation.** Throughout the paper, \(\mathbb{R}^+ = (0, \infty)\) and \(C\) stands for a “generic” positive constant which may changes from line to line. For \(p, q \in [0, \infty)\), the usual Lebesgue space is denoted by \(L^p(\mathbb{R}^2)\) and \(\|\cdot\|_{L^q_t L^p}\) denotes the norm of \(\left(\int_0^t \|\cdot\|_{L^q_t L^p}^q \, dt\right)^{\frac{1}{q}}\). Finally, \(\mathcal{D}(\mathbb{R}^2)\) is a space of smooth compactly supported functions on \(\mathbb{R}^2\), \(\mathcal{C}^k(\mathbb{R}^2)\) is made up of all continuous function \(f(x)\) on \(\mathbb{R}^2\) with \(|\alpha| \leq k\) order continuous partial derivative \(\partial^\alpha f(x)\), and space \(\mathcal{S}(\mathbb{R}^2)\) is the Schwartz class of rapidly decreasing functions.

2. Preliminaries. In this preparatory section, we provide the definition of some function spaces and review some important lemmas. Here, we recall the theory of Littlewood-Paley which is useful to measure smoothing efforts of the linear heat equation. To define Besov Spaces, we start with the dyadic decomposition.

Let \(\psi \in C_0^\infty(\mathbb{R}^2)\) be supported in the ring \(\mathcal{E} := \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\) and such that
\[
\sum_{q \in \mathbb{Z}} \psi(2^{-q} \xi) = 1 \quad \text{for} \quad \xi \neq 0.
\]
We define also the function \( \chi(\xi) = 1 - \sum_{q \in \mathbb{N}} \psi(2^{-q}\xi) \). For \( u \in S'(\mathbb{R}^2) \), the inhomogeneous dyadic blocks are defined by:

\[
\Delta_{-1}u = \chi(D)u, \quad \Delta_q u = \psi(2^{-q}D)u, \quad \forall q \in \mathbb{N}.
\]

We also introduce the following low-frequency cutoff

\[
S_q u = \sum_{-1 \leq j \leq q-1} \Delta_j u.
\]

In terms of Littlewood-Paley operator \( \Delta_j \), we can define inhomogeneous Besov spaces as follows. Let \((p, r) \in [1, +\infty]^2 \) and \( s \in \mathbb{R} \), then the inhomogeneous space \( B^s_{p,r}(\mathbb{R}^2) \) is the set of tempered distributions \( u \) such that:

\[
\|u\|_{B^s_{p,r}(\mathbb{R}^2)} \triangleq \left( \sum_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^p(\mathbb{R}^2)}^r \right)^{\frac{1}{r}} < \infty.
\]

It is worthwhile to remark that \( B^s_{2,2} \) and \( B^s_{\infty,\infty} \) coincide with the usual Sobolev spaces \( H^s \) and the usual Hölder space \( C^s \) for \( s \in \mathbb{R} + \mathbb{Z} \) respectively. For more details, we can refer to [1, 15].

In our next study, we require two kinds of mixed space-time Besov space. The first one is defined in the following manner: for \( T > 0 \) and \( \rho \geq 1 \), we denote by \( L_T^p B^s_{p,r} \), the set of all tempered distributions \( u \) satisfying

\[
\|u\|_{L_T^p B^s_{p,r}(\mathbb{R}^2)} \triangleq \left( \int_0^T \left( \sum_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^p(\mathbb{R}^2)}^r \right)^{\frac{1}{r}} \, \tau \right)^{\frac{1}{p}} < \infty.
\]

The second mixed space is \( \tilde{L}_T^p B^s_{p,r} \), which consists of tempered distributions \( u \) satisfying

\[
\|u\|_{\tilde{L}_T^p B^s_{p,r}(\mathbb{R}^2)} \triangleq \left( \int_0^T \left( \sum_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^p(\mathbb{R}^2)}^r \right)^{\frac{1}{r}} \, \tau \right)^{\frac{1}{p}} < \infty.
\]

And the norm \( \| \cdot \|_{L_T^p} \) is defined as following

\[
\|f\|_{L_T^p} \triangleq \left( \int_0^T |f(\tau)|^p \, d\tau \right)^{\frac{1}{p}}.
\]

Minkowski’s inequality entails that if \( s \in \mathbb{R} \), \( \rho \geq 1 \), and \((p, r) \in [1, +\infty]^2 \), we have

\[
L_T^p B^s_{p,r} \hookrightarrow \tilde{L}_T^p B^s_{p,r} \quad \text{if} \quad r \geq \rho \quad \text{and} \quad \tilde{L}_T^p B^s_{p,r} \hookrightarrow L_T^p B^s_{p,r} \quad \text{if} \quad \rho \geq r.
\]

Finally, we would recall some useful lemmas which are powerful tools in the proof of our main result:

**Lemma 2.1** ([15]). Let \( 1 \leq p \leq q \leq \infty \). Assume that \( f \in L^p \), then there exists a constant \( C \) independent of \( f, j \) such that

\[
\sup \hat{f} \subset \{ |\xi| \leq C2^j \} \Rightarrow \|\partial^\alpha f\|_{L^q} \leq C2^{j(|\alpha|+2j)(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.
\]

\[
\sup \hat{f} \subset \{ C^{-1}2^j \leq |\xi| \leq C2^j \} \Rightarrow \|f\|_{L^p} \leq C2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_{L^p}.
\]

**Lemma 2.2** ([15]). There exists a constant \( C \) depending only on \( \varphi \) and such that for all \( q \in \mathbb{Z}, \lambda \geq 0, p \in [1, +\infty], \) and \( u \in S' \), we have

\[
\|\Delta_q e^{\lambda^2} u\|_{L^p} \leq C^{-1} e^{-C\lambda^2q^2}\|\Delta_q u\|_{L^p}.
\]
Lemma 2.3 ([14], Sobolev inequality). The space $H^{s+k}(\mathbb{R}^2)$, $s > 1$, $k \in \mathbb{Z}^+ \cup \{0\}$, is continuously embedded in the space $C^k(\mathbb{R}^2)$. That is, there exists $C > 0$ such that
\[
\|f\|_{C^k} \leq C\|f\|_{H^{s+k}} \quad \forall f \in H^{s+k}(\mathbb{R}^2).
\]

Lemma 2.4 ([14], Leibniz estimate). For all $m \in \mathbb{Z}^+ \cup \{0\}$, there exists a constant $C > 0$ such that for all $f, g \in L^\infty \cap H^m(\mathbb{R}^2)$,
\[
\|fg\|_{H^m} \leq C(\|f\|_{L^\infty}\|g\|_{H^m} + \|f\|_{H^m}\|g\|_{L^\infty}). \quad (2.1)
\]

3. Solutions to the regularized problem. This section is to modify equations in order to produce a family of global smooth solutions. We begin by a regularizing operator called a mollifier. Given any radial function
\[\phi(|x|) \in C_0^\infty(\mathbb{R}^2), \quad \phi \geq 0, \quad \int_{\mathbb{R}^2} \phi \, dx = 1,\]
define the standard mollifier
\[\phi^\varepsilon = \frac{1}{\varepsilon^2} \phi \left( \frac{x}{\varepsilon} \right).\]

Now let us consider the regularized system governed by
\[
\begin{aligned}
\partial_t n^\varepsilon + (v^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n^\varepsilon - \varepsilon \Delta n^\varepsilon &= -\nabla \cdot (n^\varepsilon \nabla (v^\varepsilon \ast \rho^\varepsilon)) + g(n^\varepsilon), \\
\partial_t c^\varepsilon + u^\varepsilon \cdot \nabla c^\varepsilon - \Delta c^\varepsilon &= -c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon), \\
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \Delta u^\varepsilon + \nabla P^\varepsilon &= -(n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon, \\
\nabla \cdot u^\varepsilon &= 0,
\end{aligned} \quad (3.1)
\]
where $g(n^\varepsilon) = (1 + a)(n^\varepsilon)^2 - an^\varepsilon - (n^\varepsilon)^3$.

We first establish the global existence of smooth solutions of (3.1), by using energy estimates and classical compactness argument.

Proposition 3.1. Let $\nabla \Phi \in L^\infty$ and the initial data $(n_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) \in (H^s(\mathbb{R}^2))^3$ with $s > 1$. Assume, in addition, that the initial data $n_0^\varepsilon$ and $c_0^\varepsilon$ are positive. Then the regularized system (3.1) admits a unique global solution
\[(n^\varepsilon, c^\varepsilon, u^\varepsilon) \in (C(\{0, \infty\}; H^s(\mathbb{R}^2)) \cap L^2_{loc}(\{0, \infty\}; H^{s+1}(\mathbb{R}^2)))^3 \cap (C^1(\{0, \infty\}; H^{s-2}))^3.
\]
Moreover, $n^\varepsilon(x, t) \geq 0$ and $c^\varepsilon(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^2 \times [0, \infty)$.

Let $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$ satisfy $\varphi(x) \geq 0$ and $\int_{\mathbb{R}^2} \varphi(x) \, dx = 1$. And we define the convolution operator $J_k$ as follows:
\[J_k u = \varphi_k \ast u = \int_{\mathbb{R}^2} \varphi_k(x-y)u(y) \, dy,\]
where $\varphi_k(x) = k^{-2}\varphi(\frac{x}{k})$.

We are going to construct the following approximate system:
\[
\begin{align*}
\partial_t n^{k,\varepsilon} + J_k \left( (u^{k,\varepsilon} \ast \rho^\varepsilon) \cdot \nabla J_k n^{k,\varepsilon} \right) - \varepsilon \Delta J_k n^{k,\varepsilon} &= -\nabla \cdot J_k \left( (n^{k,\varepsilon} \nabla (c^{k,\varepsilon} \ast \rho^\varepsilon)) \right) + J_k G(n^{k,\varepsilon}), \\
\partial_t c^{k,\varepsilon} + J_k (J_k u^{k,\varepsilon} \cdot \nabla J_k c^{k,\varepsilon}) - \Delta J_k^2 c^{k,\varepsilon} &= -J_k \left( (c^{k,\varepsilon} \ast \rho^\varepsilon) \right), \\
\partial_t u^{k,\varepsilon} + J_k (J_k u^{k,\varepsilon} \cdot \nabla J_k u^{k,\varepsilon}) - \Delta J_k^2 u^{k,\varepsilon} + \nabla J_k P^{k,\varepsilon} &= -J_k \left( (n^{k,\varepsilon} \nabla \Phi) \ast \rho^\varepsilon \right), \\
\nabla \cdot u^{k,\varepsilon} &= 0, \\
(n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon}) \big|_{t=0} &= (n_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon),
\end{align*}
\]
\tag{3.2}

where \(G(n^{k,\varepsilon}) = (1 + a)(J_k n^{k,\varepsilon})^2 - a J_k n^{k,\varepsilon} - (J_k n^{k,\varepsilon})^3\).

Following Leray operator, we eliminate the pressure \(P^{k,\varepsilon}\) and the incompressibility condition \(\text{div} u^{k,\varepsilon} = 0\) by protecting the third equation of the above system onto the space of divergence-free functions

\[H^{s,\sigma}(\mathbb{R}^2) \triangleq \{(n, c, u) \in (H^s(\mathbb{R}^2))^3 \mid \text{div } u = 0\}.
\]

Then, problem (3.2) reduces to an ODE in the Banach Space \(H^{s,\sigma}(\mathbb{R}^2)\):

\[
\frac{d}{dt} E^{k,\varepsilon}(x, 0) = E^{k,\varepsilon}(x, 0) = E_0^{k,\varepsilon}(x) = \begin{pmatrix} n_0^\varepsilon \\ c_0^\varepsilon \\ u_0^\varepsilon \end{pmatrix}, \text{ where } E^{k,\varepsilon} \triangleq \begin{pmatrix} n^{k,\varepsilon} \\ c^{k,\varepsilon} \\ u^{k,\varepsilon} \end{pmatrix}
\tag{3.3}
\]

and

\[
F_{k,\varepsilon}(E^{k,\varepsilon}) \triangleq \begin{pmatrix} F_{k,\varepsilon}^1 (E^{k,\varepsilon}) \\ F_{k,\varepsilon}^2 (E^{k,\varepsilon}) \\ F_{k,\varepsilon}^3 (E^{k,\varepsilon}) \end{pmatrix}
\]

\[= \begin{pmatrix} \varepsilon \Delta J_k n^{k,\varepsilon} - J_k \left( (u^{k,\varepsilon} \ast \rho^\varepsilon) \cdot \nabla J_k n^{k,\varepsilon} \right) - \nabla \cdot J_k \left( (n^{k,\varepsilon} \nabla (c^{k,\varepsilon} \ast \rho^\varepsilon)) \right) + J_k G(n^{k,\varepsilon}) \\
J_k (J_k u^{k,\varepsilon} \cdot \nabla J_k c^{k,\varepsilon}) + \Delta J_k^2 c^{k,\varepsilon} - J_k \left( (c^{k,\varepsilon} \ast \rho^\varepsilon) \right) \\
-\nabla J_k \left( (n^{k,\varepsilon} \nabla \Phi) \ast \rho^\varepsilon \right)
\end{pmatrix}.
\]

Let us imitate its relation proof process, we could know that problem (3.2) and (3.3) are equivalent. We can refer to [15] for the detailed proof.

**Step 1.** We first show the existence and uniqueness of solution of (3.2) for each \(k \in \mathbb{N}^+\). Specially, we have the following proposition:

**Proposition 3.2.** Let \(\Phi\) and the initial data \((n_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) \in (H^s(\mathbb{R}^2))^3\) be as in Proposition 3.1. Then, for each \(k \in \mathbb{N}^+\), system (3.2) has a unique global smooth solution \((n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon}) \in (C(\mathbb{R}^+; H^{s,\sigma}(\mathbb{R}^2)))^3\).

**Proof of Proposition 3.2.** Firstly, we are going to apply the Picard theorem to get a local-in-time solution of problem (3.3). It suffices to show that for each \(k \geq 1\) and all \(E_1^{k,\varepsilon}, E_2^{k,\varepsilon} \in H^{s,\sigma}(\mathbb{R}^2)\), we have

\[
\| F_{k,\varepsilon}^1 (E_1^{k,\varepsilon}) - F_{k,\varepsilon}^1 (E_2^{k,\varepsilon}) \|_{H^s} \leq C \left( k, \varepsilon, \| E_1^{k,\varepsilon} \|_{L^2}, \| E_2^{k,\varepsilon} \|_{L^2} \right) \| E_1^{k,\varepsilon} - E_2^{k,\varepsilon} \|_{H^s},
\]

\[
(3.4)
\]

\[
\| F_{k,\varepsilon}^2 (E_1^{k,\varepsilon}) - F_{k,\varepsilon}^2 (E_2^{k,\varepsilon}) \|_{H^s} \leq C \left( k, \varepsilon, \| E_1^{k,\varepsilon} \|_{L^2}, \| E_2^{k,\varepsilon} \|_{L^2} \right) \| E_1^{k,\varepsilon} - E_2^{k,\varepsilon} \|_{H^s},
\]

\[
(3.5)
\]
Because the proofs of (3.4), (3.5) and (3.6) are analogous, we just need to show the first inequality (3.4). Based on the Morse estimate (2.1), simple calculations yield

$$F_{k,\varepsilon}^1(E_{1}^{k,\varepsilon}) - F_{k,\varepsilon}^1(E_{2}^{k,\varepsilon}) \triangleq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$I_1 = \varepsilon \Delta J_k^2(n_1^{k,\varepsilon} - n_2^{k,\varepsilon}),$$

$$I_2 = -J_k \left( \left( (u_1^{k,\varepsilon} - u_2^{k,\varepsilon}) \ast \rho^\varepsilon \right) \cdot \nabla J_k n_1^{k,\varepsilon} \right) - J_k \left( (u_2^{k,\varepsilon} \ast \rho^\varepsilon) \cdot \nabla J_k (n_1^{k,\varepsilon} - n_2^{k,\varepsilon}) \right),$$

$$I_3 = -\nabla \cdot J_k \left( J_k (n_1^{k,\varepsilon} - n_2^{k,\varepsilon}) \nabla (c_1^{k,\varepsilon} \ast \rho^\varepsilon) \right) - \nabla \cdot J_k \left( J_k (n_2^{k,\varepsilon} \nabla ((c_1^{k,\varepsilon} - c_2^{k,\varepsilon}) \ast \rho^\varepsilon) \right),$$

$$I_4 = (1 + a) J_k \left( J_k n_1^{k,\varepsilon} \cdot (J_k n_2^{k,\varepsilon})^2 - (J_k n_2^{k,\varepsilon})^2 \right),$$

$$I_5 = -a J_k (J_k n_1^{k,\varepsilon} - J_k n_2^{k,\varepsilon}),$$

$$I_6 = -J_k \left( (J_k n_1^{k,\varepsilon})^3 - (J_k n_2^{k,\varepsilon})^3 \right).$$

By Lemma 2.1, the estimate (2.1) and the properties of mollifier, we see that

$$\|I_1\|_{H^s} = \varepsilon \|\Delta J_k^2(n_1^{k,\varepsilon} - n_2^{k,\varepsilon})\|_{H^s} \leq C(k,\varepsilon) \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s}.$$  

Similarly, we get that

$$\|I_2\|_{H^s} \leq \left| J_k \left( \left( (u_1^{k,\varepsilon} - u_2^{k,\varepsilon}) \ast \rho^\varepsilon \right) \cdot \nabla J_k n_1^{k,\varepsilon} \right) \right|_{H^s}$$

$$+ \left| J_k \left( (u_2^{k,\varepsilon} \ast \rho^\varepsilon) \cdot \nabla J_k (n_1^{k,\varepsilon} - n_2^{k,\varepsilon}) \right) \right|_{H^s} \triangleq J_1 + J_2.$$  

On one hand,

$$J_1 = \left| J_k \left( \left( (u_1^{k,\varepsilon} - u_2^{k,\varepsilon}) \ast \rho^\varepsilon \right) \cdot \nabla J_k n_1^{k,\varepsilon} \right) \right|_{H^s}$$

$$\leq C \left( \left( \left( (u_1^{k,\varepsilon} - u_2^{k,\varepsilon}) \ast \rho^\varepsilon \right) \right) \|\nabla J_k n_1^{k,\varepsilon}\|_{H^s} + \|\nabla J_k n_1^{k,\varepsilon}\|_{H^s} \right)$$

$$\leq C(k,\varepsilon) \left( \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{L^2} \|n_1^{k,\varepsilon}\|_{L^2} + \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s} \right)$$

$$\leq C(k,\varepsilon) \left( \|n_1^{k,\varepsilon}\|_{L^2} \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s} \right).$$

On the other hand, we see that

$$J_2 = \left| J_k \left( (u_2^{k,\varepsilon} \ast \rho^\varepsilon) \cdot \nabla J_k (n_1^{k,\varepsilon} - n_2^{k,\varepsilon}) \right) \right|_{H^s}$$

$$\leq C \left( \left( u_2^{k,\varepsilon} \ast \rho^\varepsilon \right) \|\nabla J_k (n_1^{k,\varepsilon} - n_2^{k,\varepsilon})\|_{H^s} + \|\nabla J_k (n_1^{k,\varepsilon} - n_2^{k,\varepsilon})\|_{H^s} \right)$$

$$\leq C(k,\varepsilon) \left( \|u_2^{k,\varepsilon}\|_{L^2} \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s} + \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{L^2} \|u_2^{k,\varepsilon}\|_{L^2} \right)$$

$$\leq C(k,\varepsilon) \|u_2^{k,\varepsilon}\|_{L^2} \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s}.$$

Those estimates together with (3.8) yield

$$\|I_2\|_{H^s} \leq C(k,\varepsilon) \left( \|n_1^{k,\varepsilon}\|_{L^2} + \|u_2^{k,\varepsilon}\|_{L^2} \right) \left( \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s} + \|n_1^{k,\varepsilon} - n_2^{k,\varepsilon}\|_{H^s} \right).$$
A similar estimate holds for $I_3$-$I_6$, and finally, we show that
\begin{align*}
\|I_3\|_{H^s} & \leq C(k, \varepsilon) \left( \|n_1^{k, \varepsilon}\|_{L^2} + \|c_1^{k, \varepsilon}\|_{L^2} \right) \left( \|n_1^{k, \varepsilon} - n_2^{k, \varepsilon}\|_{H^s} + \|c_2^{k, \varepsilon}\|_{H^s} \right), \\
\|I_4\|_{H^s} & \leq C(k, \varepsilon) \left( \|n_1^{k, \varepsilon}\|_{L^2} + \|n_2^{k, \varepsilon}\|_{L^2} \right) \|n_1^{k, \varepsilon} - n_2^{k, \varepsilon}\|_{H^s}, \\
\|I_5\|_{H^s} & \leq C(k) \|n_1^{k, \varepsilon} - n_2^{k, \varepsilon}\|_{H^s}, \\
\|I_6\|_{H^s} & \leq C(k) \left( \|n_1^{k, \varepsilon}\|_{L^2}^2 + \|n_2^{k, \varepsilon}\|_{L^2}^2 \right) \|n_1^{k, \varepsilon} - n_2^{k, \varepsilon}\|_{H^s}.
\end{align*}

Collecting these estimates leads to
\begin{align*}
\|F_{k, \varepsilon}^1 (E_{1}^{k, \varepsilon}) - F_{k, \varepsilon}^1 (E_{2}^{k, \varepsilon})\|_{H^s} 
& \leq C(k, \varepsilon) \left( \|n_1^{k, \varepsilon}\|_{L^2} + \|n_2^{k, \varepsilon}\|_{L^2} + \|u_2^{k, \varepsilon}\|_{L^2} + \|u_2^{k, \varepsilon}\|_{L^2} + \|u_2^{k, \varepsilon}\|_{L^2}^2 + \|u_2^{k, \varepsilon}\|_{L^2}^2 + \|u_2^{k, \varepsilon}\|_{L^2}^2 + \|u_2^{k, \varepsilon}\|_{L^2}^2 + 1 \right) \\
& \quad \times \left( \|n_1^{k, \varepsilon} - n_2^{k, \varepsilon}\|_{H^s} + \|c_1^{k, \varepsilon} - c_2^{k, \varepsilon}\|_{H^s} + \|u_1^{k, \varepsilon} - u_2^{k, \varepsilon}\|_{H^s} \right) \\
& \leq C(k, \varepsilon, \|E_{1}^{k, \varepsilon}\|_{L^2}, \|E_{2}^{k, \varepsilon}\|_{L^2}) \left( E_{1}^{k, \varepsilon} - E_{2}^{k, \varepsilon}\right) \|E_{1}^{k, \varepsilon}\|_{H^s}.
\end{align*}

(3.9)

Thanks to estimates (3.4) (3.5) and (3.6), we get that $F_{k, \varepsilon}$ maps $H^s(\mathbb{R}^2)$ into $H^s(\mathbb{R}^2)$ and $F_{k, \varepsilon}$ is locally Lipschitz continuous on any open set
\[ O^M \triangleq \{ E \in H^{s, \sigma} : \|E\|_{H^s} < M \} . \]

Hence, by Picard Theorem [14, Theorem 3.1], we have that for every $(n_0, c_0, u_0) \in H^s(\mathbb{R}^2))^3$, there exists a unique solution
\[ (n^{k, \varepsilon}, c^{k, \varepsilon}, u^{k, \varepsilon}) \in \left( C^1([0, T_k), H^s(\mathbb{R}^2) \cap O^M) \right)^3 \]

We finish the proof of the local existence. \quad \square

**Step 2.** In this step, we are devoted to proving that there exists $T > 0$ such that $T_k \geq T$ for all $k \in \mathbb{N}^+$. Thanks to blow up criterion, it is suffice to show the following proposition.

**Proposition 3.3.** Let $\Phi$ and the initial data $(n_0^0, c_0^0, u_0^0) \in \left( H^s(\mathbb{R}^2) \right)^3$ be as in Proposition 3.1. Then, for each $k \in \mathbb{N}^+$, there exists $T > 0$ such that $(n^{k, \varepsilon}, c^{k, \varepsilon}, u^{k, \varepsilon}) \in \left( C([0, T_k), H^s(\mathbb{R}^2))^3 \right.$ Moreover, we have
\begin{align*}
\sup_{t \in [0, T]} \left\| (n^{k, \varepsilon}, c^{k, \varepsilon}, u^{k, \varepsilon}) (t) \right\|_{H^s} & \leq \frac{\left\| (n_0, c_0, u_0) \right\|_{H^m}}{1 - C(\varepsilon) T \left( \|n_0, c_0, u_0\|_{H^s} \right)} . \tag{3.10}
\end{align*}

**Proof of Proposition 3.3.** Taking the $L^2$-inner product with the second equation of (3.2) with $c^{k, \varepsilon}$ yields that
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left\| c^{k, \varepsilon} (t) \right\|_{L^2}^2 & + \| \nabla J_k c^{k, \varepsilon} (t) \|_{L^2}^2 \\
& = - \int_{\mathbb{R}^2} (J_k u^{k, \varepsilon} \cdot \nabla J_k c^{k, \varepsilon}) (J_k u^{k, \varepsilon} - J_k c^{k, \varepsilon} \ast \rho^\gamma) J_k c^{k, \varepsilon} \, dx - \int_{\mathbb{R}^2} J_k c^{k, \varepsilon} (n^{k, \varepsilon} \ast \rho^\gamma) J_k c^{k, \varepsilon} \, dx . \tag{3.11}
\end{align*}

The properties of $J_k$ and $\rho^\gamma$ together with the fact $(n^{k, \varepsilon}, c^{k, \varepsilon}, u^{k, \varepsilon}) \in \left( C^1([0, T_k), H^s(\mathbb{R}^2))\right)^3$ enable us to conclude that $J_k n^{k, \varepsilon}$, $J_k c^{k, \varepsilon}$, $J_k u^{k, \varepsilon}$, $n^{k, \varepsilon} \ast \rho^\gamma \in H^1(\mathbb{R}^2)$ on $[0, T_k)$ for all $l > 1$. Thus, the density argument that $\mathcal{D}(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$
allows us to perform integrations by parts without a boundary term in what follows. With this in hand, integrations by parts lead to
\[
\int_{\mathbb{R}^2} (J_k u^{k,\varepsilon} \cdot \nabla J_k c^{k,\varepsilon}) J_k c^{k,\varepsilon} \, dx = \frac{1}{2} \int_{\mathbb{R}^2} J_k u^{k,\varepsilon} \cdot \nabla (J_k c^{k,\varepsilon})^2 \, dx = 0.
\]
Inserting the above equality into (3.11) yields
\[
\frac{1}{2} \frac{d}{dt} \|c^{k,\varepsilon}(t)\|^2_{L^2} + \|\nabla J_k c^{k,\varepsilon}(t)\|^2_{L^2} = - \int_{\mathbb{R}^2} J_k c^{k,\varepsilon} (n^{k,\varepsilon} + \rho^\varepsilon) J_k c^{k,\varepsilon} \, dx \leq C(\varepsilon) \|n^{k,\varepsilon}\|^2_{L^2} \|c^{k,\varepsilon}\|_{L^2}.
\]
Taking the \(L^2\)-inner product with the first equation of (3.2) with \(n^{k,\varepsilon}\), we have
\[
\frac{1}{2} \frac{d}{dt} \|n^{k,\varepsilon}(t)\|^2_{L^2} + \varepsilon \|\nabla J_k n^{k,\varepsilon}(t)\|^2_{L^2} + a \|J_k n^{k,\varepsilon}(t)\|^2_{L^2} + \|J_k n^{k,\varepsilon}(t)\|^4_{L^4} = - \int_{\mathbb{R}^2} (\nabla \cdot J_k (J_k n^{k,\varepsilon} \nabla (c^{k,\varepsilon} + \rho^\varepsilon))) n^{k,\varepsilon} \, dx + (1 + a) \|J_k n^{k,\varepsilon}\|^3_{L^3} = K_1 + K_2.
\]
Obviously, by the Hölder inequality and the Young inequality, we have
\[
K_1 = - \int_{\mathbb{R}^2} (\nabla \cdot J_k (J_k n^{k,\varepsilon} \nabla (c^{k,\varepsilon} + \rho^\varepsilon))) n^{k,\varepsilon} \, dx = \int_{\mathbb{R}^2} (J_k n^{k,\varepsilon} \nabla (c^{k,\varepsilon} + \rho^\varepsilon)) \cdot \nabla J_k n^{k,\varepsilon} \, dx \leq \|J_k n^{k,\varepsilon}\|_{L^4} \|\nabla (c^{k,\varepsilon} + \rho^\varepsilon)\|_{L^\infty} \|\nabla J_k n^{k,\varepsilon}\|_{L^2} \leq C(\varepsilon) \|J_k n^{k,\varepsilon}\|^2_{L^2} \|c^{k,\varepsilon}\|^2_{L^2} + \varepsilon \|\nabla J_k n^{k,\varepsilon}\|^2_{L^2}.
\]
For \(K_2\), by the Interpolation inequality that
\[
\|J_k n^{k,\varepsilon}\|_{L^3} \leq C \|J_k n^{k,\varepsilon}\|_{L^2}^{\frac{1}{2}} \|J_k n^{k,\varepsilon}\|_{L^4}^{\frac{1}{2}}
\]
and the \(\varepsilon\)-Young inequality, we obtain that
\[
K_2 = (1 + a) \|J_k n^{k,\varepsilon}\|^3_{L^3} \leq C \|J_k n^{k,\varepsilon}\|^2_{L^2} + \frac{1}{2} \|J_k n^{k,\varepsilon}\|^4_{L^4}.
\]
Plugging (3.14) and (3.15) into (3.13), we have
\[
\frac{d}{dt} \|n^{k,\varepsilon}(t)\|^2_{L^2} + \varepsilon \|\nabla J_k n^{k,\varepsilon}(t)\|^2_{L^2} + 2a \|J_k n^{k,\varepsilon}(t)\|^2_{L^2} + \|J_k n^{k,\varepsilon}(t)\|^4_{L^4} \leq C(\varepsilon) \|J_k n^{k,\varepsilon}(t)\|^2_{L^2} \left(\|c^{k,\varepsilon}(t)\|^2_{L^2} + 1\right) \leq C(\varepsilon) \|n^{k,\varepsilon}(t)\|^2_{L^2} \left(\|c^{k,\varepsilon}(t)\|^2_{L^2} + 1\right).
\]
Taking the \(L^2\)-inner product with the third equation of (3.2) with \(u^{k,\varepsilon}\) implies
\[
\frac{1}{2} \frac{d}{dt} \|u^{k,\varepsilon}(t)\|^2_{L^2} + \|\nabla J_k u^{k,\varepsilon}(t)\|^2_{L^2} = - \int_{\mathbb{R}^2} \nabla J_k ((n^{k,\varepsilon} \nabla \Phi) * \rho^\varepsilon) u^{k,\varepsilon} \, dx \leq \|(n^{k,\varepsilon} \nabla \Phi) * \rho^\varepsilon\|_{L^2} \|J_k u^{k,\varepsilon}\|_{L^2} \leq \|n^{k,\varepsilon}\|_{L^2} \|
abla \Phi\|_{L^\infty} \|u^{k,\varepsilon}\|_{L^2} \leq C \|n^{k,\varepsilon}\|^2_{L^2} + \|u^{k,\varepsilon}\|^2_{L^2}.
\]
By the standard energy estimate, we can conclude that for \(s > 1\)
\[
\frac{d}{dt} \|E^{k,\varepsilon}(t)\|_{H^s} \leq C(\varepsilon) \|E^{k,\varepsilon}(t)\|^2_{H^s}.
\]
Step 3. Our target now is to prove that system (3.1) admits a local-in-time solution.

**Proposition 3.4.** When $k \to 0$, the solution $(n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$ has a limit $(n^\varepsilon, c^\varepsilon, u^\varepsilon)$ which is a local weak solution of system (3.1). Moreover, $n^\varepsilon(x, t) \geq 0$ and $c^\varepsilon(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^2 \times [0, T]$.

**Proof of Proposition 3.4.** First of all, by the estimate of (3.36), we know that the family $(\partial_t n^{k,\varepsilon}, \partial_t c^{k,\varepsilon}, \partial_t u^{k,\varepsilon})$ belongs to $(L^2_{loc}([0, T]; H^{s-1}({\mathbb{R}}^2)))^3$. In Step 1, we obtained that $(n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$ is bounded in $(C([0, T]; H^s({\mathbb{R}}^2)))^3$. Next, we introduce a sequence $(\phi_p)_{p \in \mathbb{N}}$ of smooth functions with values in [0, 1], supported in the ball $B(0, p + 1)$ and equal to 1 on $B(0, p)$. For all $p \in \mathbb{N}$, we conclude that $(\phi_p n^{k,\varepsilon}, \phi_p c^{k,\varepsilon}, \phi_p u^{k,\varepsilon})$ is uniformly bounded in $(C([0, T]; H^s({\mathbb{R}}^2)))^3$. On the other hand, we take notice of the application

$$(n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon}) \mapsto (\phi_p n^{k,\varepsilon}, \phi_p c^{k,\varepsilon}, \phi_p u^{k,\varepsilon})$$

is compact from $H^s$ into $H^s$ as $s > s'$. Therefore, Arzelà-Ascoli theorem ensures that there exists some function $(n^\varepsilon_p, c^\varepsilon_p, u^\varepsilon_p)$ such that, up to extraction,

$$(\phi_p n^{k,\varepsilon}, \phi_p c^{k,\varepsilon}, \phi_p u^{k,\varepsilon}) \to (n^\varepsilon_p, c^\varepsilon_p, u^\varepsilon_p) \quad \text{in} \quad (C([0, T]; H^{s'}({\mathbb{R}}^2)))^3.$$

Using the Cantor diagonal process, we will find a subsequence (which we still denoted by $(n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$) such that for all $p \in \mathbb{N}$,

$$(\phi_p n^{k,\varepsilon}, \phi_p c^{k,\varepsilon}, \phi_p u^{k,\varepsilon}) \to (n^\varepsilon_p, c^\varepsilon_p, u^\varepsilon_p) \quad \text{in} \quad (C([0, T]; H^{s'}({\mathbb{R}}^2)))^3.$$

As $\phi_p \phi_{p+1} = \phi_p$, we have, in addition, $(n^\varepsilon_p, c^\varepsilon_p, u^\varepsilon_p) = (\phi_p n^{\varepsilon}_{p+1}, \phi_p c^{\varepsilon}_{p+1}, \phi_p u^{\varepsilon}_{p+1})$. From that, we can easily deduce that there exists some function $(n^\varepsilon, c^\varepsilon, u^\varepsilon)$ such that for all $\phi \in \mathcal{D}$,

$$(\phi n^{k,\varepsilon}, \phi c^{k,\varepsilon}, \phi u^{k,\varepsilon}) \to (\phi n^\varepsilon, \phi c^\varepsilon, \phi u^\varepsilon) \quad \text{in} \quad (C([0, T]; H^{s'}({\mathbb{R}}^2)))^3.$$

This obviously entails that $(n^{k,\varepsilon}, c^{k,\varepsilon}, u^{k,\varepsilon})$ tends to $(n^\varepsilon, c^\varepsilon, u^\varepsilon)$ in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^2)$. The Fatou’s lemma ensures that $(n^\varepsilon, c^\varepsilon, u^\varepsilon) \in (C([0, T]; H^s) \cap L^2([0, T]; H^{s+1}))^3$. $\square$

We still have to prove that $(n^\varepsilon, c^\varepsilon, u^\varepsilon)$ is continuous in $H^s \times H^s \times H^s$. Let us apply the operator $\Delta_q (q \geq 0)$ to the first equation of (3.1), we get

$$\partial_t \Delta_q n^\varepsilon + \Delta_q ((u^\varepsilon * \rho^\varepsilon) \cdot \nabla n^\varepsilon) - \varepsilon \Delta_q n^\varepsilon = -\nabla \cdot \Delta_q (n^\varepsilon \nabla (c^\varepsilon * \rho^\varepsilon)) + (1 + a) \Delta_q (n^\varepsilon)^2 - a \Delta_q n^\varepsilon - \Delta_q (n^\varepsilon)^3.$$

Taking the $L^2$-inner product with the above equation with $\Delta_q n^\varepsilon$, yields

$$\frac{1}{2} \frac{d}{dt} \| \Delta_q n^\varepsilon(t) \|^2_{L^2} + \varepsilon \| \nabla \Delta_q n^\varepsilon(t) \|^2_{L^2} + a \| \Delta_q n^\varepsilon(t) \|^2_{L^2} = - \langle \Delta_q ((u^\varepsilon * \rho^\varepsilon) \cdot \nabla n^\varepsilon), \Delta_q n^\varepsilon \rangle - \langle \nabla \cdot \Delta_q (n^\varepsilon \nabla (c^\varepsilon * \rho^\varepsilon)), \Delta_q n^\varepsilon \rangle + (1 + a) \langle \Delta_q (n^\varepsilon)^2, \Delta_q n^\varepsilon \rangle - \langle \Delta_q (n^\varepsilon)^3, \Delta_q n^\varepsilon \rangle.$$
So, the Hölder inequality and the Young inequality ensure us to have
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q n^\varepsilon (t) \|^2_{L^2} + \varepsilon C 2^{2q} \| \Delta_q n^\varepsilon (t) \|^2_{L^2} + a \| \Delta_q n^\varepsilon (t) \|^2_{L^2} \\
\leq \| \Delta_q (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \|_{L^2} \| \nabla \Delta_q n^\varepsilon \|_{L^2} + \| \Delta_q (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon)) \|_{L^2} \| \nabla \Delta_q n^\varepsilon \|_{L^2} \\
+ (1 + a) \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2} + \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2} \| \Delta_q n^\varepsilon \|_{L^2} \\
\leq C(\varepsilon) \| \Delta_q (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \|^2_{L^2} + C(\varepsilon) \| \Delta_q (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon)) \|^2_{L^2} + \varepsilon C 2^{2q} \| \Delta_q n^\varepsilon \|^2_{L^2} \\
+ C \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2} + C \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2} + \frac{a}{2} \| \Delta_q n^\varepsilon \|^2_{L^2}.
\]
It implies that
\[
\frac{d}{dt} \| \Delta_q n^\varepsilon (t) \|^2_{L^2} + \varepsilon C 2^{2q} \| \Delta_q n^\varepsilon (t) \|^2_{L^2} + a \| \Delta_q n^\varepsilon (t) \|^2_{L^2} \\
\leq C(\varepsilon) \| \Delta_q (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \|^2_{L^2} + C(\varepsilon) \| \Delta_q (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon)) \|^2_{L^2} + C \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2} \\
+ C \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2}.
\]
Integrating with respect to time \( t \) leads to
\[
\| \Delta_q n^\varepsilon (t) \|^2_{L^2} + \varepsilon C \int_0^t 2^{2q} \| \Delta_q n^\varepsilon (t) \|^2_{L^2} \, dt + a \int_0^t \| \Delta_q n^\varepsilon (t) \|^2_{L^2} \, dt \\
\leq \| \Delta_q n^\varepsilon (0) \|^2_{L^2} + C(\varepsilon) \int_0^t \| \Delta_q (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \|^2_{L^2} \, dt + C(\varepsilon) \| \Delta_q (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon)) \|^2_{L^2} \\
+ C \int_0^t \| \Delta_q (n^\varepsilon)^3 \|^2_{L^2} \, dt.
\]
Multiplying the above inequality by \( 2^{2qs} \) and taking the \( L^2 \)-norm imply
\[
\left( \sum_{q \geq 0} 2^{2qs} \| \Delta_q n^\varepsilon \|^2_{L^2} \right)^{\frac{1}{2}} + \varepsilon C \left( \sum_{q \geq 0} 2^{2q(s+1)} \| \Delta_q n^\varepsilon \|^2_{L^2} \right)^{\frac{1}{2}} \\
+ a \left( \sum_{q \geq 0} 2^{2qs} \| \Delta_q n^\varepsilon \|^2_{L^2} \right)^{\frac{1}{2}} \\
\leq \| n^\varepsilon \|^2_{H^s} + C(\varepsilon) \| u^\varepsilon \ast \rho^\varepsilon \|_{L^1 H^s} + C(\varepsilon) \| n^\varepsilon \^2 \|_{L^2 H^s} + C \| (n^\varepsilon)^3 \|_{L^1 H^s} \\
+ C \| (n^\varepsilon)^3 \|_{L^1 H^s} \\
\leq \| n^\varepsilon \|^2_{H^s} + C(\varepsilon) \| u^\varepsilon \|^2_{L^2 H^s} + \| n^\varepsilon \|^2_{L^2 H^s} + C(\varepsilon) \| n^\varepsilon \|^2_{L^2 H^s} + C \| n^\varepsilon \|^2_{L^2 H^s} \\
+ C \| n^\varepsilon \|^2_{L^2 H^s} < \infty.
\]
Together with \( n^\varepsilon (t) \in L^\infty_t H^s \) gives
\[
\left( \sum_{q \geq -1} 2^{2qs} \| \Delta_q n^\varepsilon \|^2_{L^2} \right)^{\frac{1}{2}} < \infty.
\]
This is to say, the sequence \( \{ S_N n^\varepsilon \}_{N \in \mathbb{Z}^+} \) converges uniformly to \( n^\varepsilon \) in \( L^\infty_t H^s \). On the other hand, the fact \( \partial_t n^\varepsilon \in L^2_{loc} (\mathbb{R}^+; H^{s-1}(\mathbb{R}^2)) \) allows us to conclude that \( n^\varepsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \) with \( s' < s \). This means that \( S_N n^\varepsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \) for a fixed \( N \in \mathbb{Z}^+ \). As a result, we obtain \( n^\varepsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \). By the same argument with \( n^\varepsilon \), we can conclude that \( c^\varepsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \) and \( u^\varepsilon \in C(\mathbb{R}^+; H^s(\mathbb{R}^2)) \).
Now we need to show the positivity of \( n^\varepsilon > 0, c^\varepsilon > 0 \) for all \((x, t) \in \mathbb{R}^2 \times [0, T]\).

Let us set
\[
(n^\varepsilon)^- \triangleq \min\{n^\varepsilon, 0\}.
\]

Multiplying the first equation of (3.1) by \((n^\varepsilon)^-\) and then integrating in space variable \(x\), we readily have
\[
\frac{1}{2} \frac{d}{dt} \|(n^\varepsilon)^-(t)\|_{L^2}^2 + \varepsilon \|
abla(n^\varepsilon)^-(t)\|_{L^2}^2
= - \int_{\mathbb{R}^2} \nabla \cdot (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon))(n^\varepsilon)^- \, dx + \int_{\mathbb{R}^2} g(n^\varepsilon)(n^\varepsilon)^- \, dx.
\]

Integrating by parts and using the Hölder inequality, one has
\[
- \int_{\mathbb{R}^2} \nabla \cdot (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon))(n^\varepsilon)^- \, dx = \int_{\mathbb{R}^2} n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \nabla (n^\varepsilon)^- \, dx
\leq C(\varepsilon)\|c^\varepsilon\|_{L^2}^2 \|(n^\varepsilon)^-\|_{L^2}^2 + \frac{\varepsilon}{4} \|
abla(n^\varepsilon)^-(t)\|_{L^2}^2.
\]

On the other hand, we see that
\[
\int_{\mathbb{R}^2} g(n^\varepsilon)(n^\varepsilon)^- \, dx = (1 + a) \int_{\mathbb{R}^2} (n^\varepsilon)^-(n^\varepsilon) \, dx
- a \int_{\mathbb{R}^2} (n^\varepsilon)^-(n^\varepsilon) \, dx - \int_{\mathbb{R}^2} (n^\varepsilon)^-(n^\varepsilon) \, dx \leq 0.
\]

Therefore, we have
\[
\frac{d}{dt} \|(n^\varepsilon)^-(t)\|_{L^2}^2 + \varepsilon \|
abla(n^\varepsilon)^-(t)\|_{L^2}^2 \leq C(\varepsilon)\|c^\varepsilon\|_{L^2}^2 \|(n^\varepsilon)^-\|_{L^2}^2.
\]

The Grönwall inequality implies that for any \( t \leq T \)
\[
\|(n^\varepsilon)^-(t)\|_{L^2} \leq \|(n_0^\varepsilon)^-\|_{L^2} C(t) = 0,
\]
which means \( n^\varepsilon \geq 0 \) for almost everywhere \((x, t) \in \mathbb{R}^2 \times [0, T]\). Since \( s > 1 \), we have
\( H^s(\mathbb{R}^2) \hookrightarrow C_b(\mathbb{R}^2) \). Hence we have that \( n^{k, \varepsilon} \geq 0 \) for all \((x, t) \in \mathbb{R}^2 \times [0, T]\). By the same argument, we can get the positivity of \( c^{k, \varepsilon} \geq 0 \).

**Step 4.** In this step, we shall establish the global-in-time \( H^s \)-estimates for \((n^\varepsilon, c^\varepsilon, u^\varepsilon)\). Taking the \( L^2 \)-inner product with the second equation of (3.1) with \( c^\varepsilon \) yields that
\[
\frac{1}{2} \frac{d}{dt} \|c^\varepsilon(t)\|_{L^2}^2 + \|
abla c^\varepsilon(t)\|_{L^2}^2 = - \int_{\mathbb{R}^2} (u^\varepsilon \cdot \nabla c^\varepsilon)c^\varepsilon \, dx - \int_{\mathbb{R}^2} c^\varepsilon(n^\varepsilon \ast \rho^\varepsilon)c^\varepsilon \, dx. \tag{3.16}
\]

Integrations by parts lead to
\[
\int_{\mathbb{R}^2} (u^\varepsilon \cdot \nabla c^\varepsilon)c^\varepsilon \, dx = \frac{1}{2} \int_{\mathbb{R}^2} u^\varepsilon \cdot \nabla (c^\varepsilon)^2 \, dx = 0.
\]

Inserting the above equality into (3.16) and using \( n^\varepsilon \geq 0 \), one has
\[
\frac{1}{2} \frac{d}{dt} \|c^\varepsilon(t)\|_{L^2}^2 + \|
abla c^\varepsilon(t)\|_{L^2}^2 = - \int_{\mathbb{R}^2} c^\varepsilon(n^\varepsilon \ast \rho^\varepsilon)c^\varepsilon \, dx \leq 0,
\]
which implies that
\[
\|c^\varepsilon(t)\|_{L^2}^2 + 2 \int_0^t \|
abla c^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq \|c^\varepsilon_0\|_{L^2}^2. \tag{3.17}
\]
Taking the $L^2$-inner product with the first equation of (3.1) with $n^\varepsilon$, we have
\[
\frac{1}{2} \frac{d}{dt} \|n^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\nabla n^\varepsilon(t)\|_{L^2}^2 + a \|n^\varepsilon(t)\|_{L^4}^2 + \|n^\varepsilon(t)\|_{L^4}^4 = -\int_{\mathbb{R}^2} (\nabla \cdot (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon))) n^\varepsilon \, dx + (1 + a) \|n^\varepsilon\|_{L^3}^3 \tag{3.18}
\]
\[\triangleq K_1 + K_2.
\]
Obviously, by the Hölder inequality and the Young inequality, we have
\[
K_1 = -\int_{\mathbb{R}^2} (\nabla \cdot (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon))) n^\varepsilon \, dx
\]
\[
= \int_{\mathbb{R}^2} (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon)) \cdot \nabla n^\varepsilon \, dx \tag{3.19}
\]
\[\leq \|n^\varepsilon\|_{L^2} \|\nabla (c^\varepsilon \ast \rho^\varepsilon)\|_{L^\infty} \|\nabla n^\varepsilon\|_{L^2}
\]
\[\leq C(\varepsilon) \|n^\varepsilon\|_{L^2}^2 \|c^\varepsilon\|_{L^2}^2, + \frac{\varepsilon}{2} \|\nabla n^\varepsilon\|_{L^2}^2.
\]
For $K_2$, by the interpolation inequality that
\[
\|n^\varepsilon\|_{L^3} \leq C \|n^\varepsilon\|_{L^2}^{\frac{1}{2}} \|n^\varepsilon\|_{L^4}^{\frac{1}{4}}
\]
and the $\varepsilon$-Young inequality, we obtain that
\[
K_2 = (1 + a) \|n^\varepsilon\|_{L^3}^3 \leq C \|n^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|n^\varepsilon\|_{L^4}^4. \tag{3.20}
\]
Plugging (3.19) and (3.20) into (3.18), we have
\[
\frac{d}{dt} \|n^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\nabla n^\varepsilon(t)\|_{L^2}^2 + 2a \|n^\varepsilon(t)\|_{L^2}^2 + \|n^\varepsilon(t)\|_{L^4}^4
\leq C(\varepsilon) \|n^\varepsilon(t)\|_{L^2}^2 \|c^\varepsilon(t)\|_{L^2}^2 + 1) \leq C(\varepsilon) \|n^\varepsilon(t)\|_{L^2}^2 \|c^\varepsilon(t)\|_{L^2}^2 + 1).
\]
With the Grönwall inequality, we have
\[
\|n^\varepsilon(t)\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla n^\varepsilon(\tau)\|_{L^2}^2 d\tau + 2a \int_0^t \|n^\varepsilon(\tau)\|_{L^2}^2 d\tau + \int_0^t \|n^\varepsilon(\tau)\|_{L^4}^4 d\tau
\leq \|n_0^\varepsilon\|_{L^2}^2 e^{C(\varepsilon) \int_0^t \|c^\varepsilon(\tau)\|_{L^2}^2 + 1) d\tau} \leq \|n_0^\varepsilon\|_{L^2}^2 e^{C(\varepsilon) t},
\tag{3.21}
\]
we have used inequality (3.17) in the last line.

Taking the $L^2$-inner product with the third equation of (3.1) with $u^\varepsilon$ implies
\[
\frac{1}{2} \frac{d}{dt} \|u^\varepsilon(t)\|_{L^2}^2 + \|\nabla u^\varepsilon(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} \psi \left( (n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon \right) u^\varepsilon \, dx
\leq \|(n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon\|_{L^2} \|u^\varepsilon\|_{L^2}
\leq \|n^\varepsilon\|_{L^2} \|\nabla \Phi\|_{L^\infty} \|u^\varepsilon\|_{L^2}
\leq C \|n^\varepsilon\|_{L^2}^2 + \|u^\varepsilon\|_{L^2}^2,
\]
so we have
\[
\|u^\varepsilon(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 d\tau \leq \left( \|u_0^\varepsilon\|_{L^2}^2 + C \int_0^t \|n^\varepsilon(\tau)\|_{L^2}^2 d\tau \right) e^{Ct}
\leq \left( \|u_0^\varepsilon\|_{L^2}^2 + Ct \|n_0^\varepsilon\|_{L^2}^2 e^{C(\varepsilon) t} \right) e^{Ct}. \tag{3.22}
\]
Collecting estimates (3.17), (3.21) and (3.22), we easily conclude that
\[
\|E^\varepsilon(t)\|_{L^2}^2 + 2a \int_0^t \|n^\varepsilon(\tau)\|_{L^4}^4 d\tau + \int_0^t \|n^\varepsilon(\tau)\|_{L^4}^4 d\tau \\
\quad + \varepsilon \int_0^t \|\nabla n^\varepsilon(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\nabla c^\varepsilon(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 d\tau \\
\leq C(1 + t)\|E_0^\varepsilon\|_{L^2}^2 e^{Ct} \triangleq C(t, \varepsilon, E_0^\varepsilon, \|\nabla \Phi\|_{L^\infty}) e^{Ct}.
\]

Next, we will show the $H^1$-estimate of solution $(n^\varepsilon, c^\varepsilon, u^\varepsilon)$. Based on the equation
\[
\partial_t n^\varepsilon + ((u^\varepsilon \ast \rho^\varepsilon) \cdot \nabla (n^\varepsilon)) - \varepsilon \Delta n^\varepsilon \\
= -\nabla \cdot (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon)) + (1 + a)(n^\varepsilon)^2 - an^\varepsilon - (n^\varepsilon)^3,
\]
multiplying the above equation by $-\Delta n^\varepsilon$ and integrating with respect to space variable $x$ yields that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla n^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\Delta n^\varepsilon(t)\|_{L^2}^2 + a \|\nabla n^\varepsilon(t)\|_{L^2}^3 \leq 3\|n^\varepsilon \nabla n^\varepsilon(t)\|_{L^2}^2 + 3\|n^\varepsilon \nabla n^\varepsilon(t)\|_{L^2}^2.
\]

Clearly, the term $L_1$ can be bounded by using the Hölder inequality and the Young inequality, as follows
\[
L_1 = \int_{\mathbb{R}^2} ((u^\varepsilon \ast \rho^\varepsilon) \cdot \nabla (n^\varepsilon)) (\Delta n^\varepsilon) \, dx \\
\leq \|u^\varepsilon \ast \rho^\varepsilon\|_{L^\infty} \|\nabla n^\varepsilon\|_{L^2} \|\Delta n^\varepsilon\|_{L^2} \\
\leq C(\varepsilon) \|u^\varepsilon\|_{L^2}^2 \|\nabla n^\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{4} \|\Delta n^\varepsilon\|_{L^2}^2.
\]

For the term $L_2$, the Hölder inequality and the Young inequality allow us to infer that
\[
L_2 = \int_{\mathbb{R}^2} (\nabla \cdot (n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon))) (\Delta n^\varepsilon) \, dx \\
= \int_{\mathbb{R}^2} n^\varepsilon \Delta (c^\varepsilon \ast \rho^\varepsilon) (\Delta n^\varepsilon) \, dx + \int_{\mathbb{R}^2} (\nabla n^\varepsilon \cdot \nabla (c^\varepsilon \ast \rho^\varepsilon)) (\Delta n^\varepsilon) \, dx \\
\leq \|n^\varepsilon\|_{L^2} \|\Delta (c^\varepsilon \ast \rho^\varepsilon)\|_{L^\infty} \|\Delta n^\varepsilon\|_{L^2} + \|\nabla n^\varepsilon\|_{L^2} \|\nabla (c^\varepsilon \ast \rho^\varepsilon)\|_{L^\infty} \|\Delta n^\varepsilon\|_{L^2} \\
\leq C(\varepsilon) \|n^\varepsilon\|_{L^2}^2 \|c^\varepsilon\|_{L^2}^2 + C(\varepsilon) \|\nabla n^\varepsilon\|_{L^2}^2 \|c^\varepsilon\|_{L^2}^2 + \frac{\varepsilon}{4} \|\Delta n^\varepsilon\|_{L^2}^2.
\]

By the integration by parts and simple calculations, we know
\[
L_3 = (1 + a) \int_{\mathbb{R}^2} (n^\varepsilon)^2 (\Delta n^\varepsilon) \, dx \\
= 2(1 + a) \int_{\mathbb{R}^2} n^\varepsilon \nabla n^\varepsilon \cdot \nabla n^\varepsilon \, dx \\
\leq C \|n^\varepsilon \nabla n^\varepsilon\|_{L^2} \|\nabla n^\varepsilon\|_{L^2} \leq C \|\nabla n^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|n^\varepsilon \nabla n^\varepsilon\|_{L^2}^2.
\]
Plugging the estimates of \(L_1-L_3\) into (3.24) leads to
\[
\frac{1}{2} \frac{d}{dt} \|\nabla n^\varepsilon(t)\|_{L^2}^2 + \frac{\varepsilon}{4} \|\Delta n^\varepsilon(t)\|_{L^2}^2 + a \|\nabla n^\varepsilon(t)\|_{L^2}^2 + \frac{5}{2} \| n^\varepsilon \nabla n^\varepsilon(t)\|_{L^2}^2 \leq C(\varepsilon) \|\nabla n^\varepsilon\|_{L^2}^2 (\|u^\varepsilon\|_{L^2}^2 + \|c^\varepsilon\|_{L^2}^2 + 1) + C(\varepsilon) \|n^\varepsilon\|_{L^2}^2 \|c^\varepsilon\|_{L^2}^2.
\]

(3.25)

Performing the same argument as above, we get that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla c^\varepsilon(t)\|_{L^2}^2 + \frac{1}{2} \|\Delta c^\varepsilon(t)\|_{L^2}^2 \leq C(\varepsilon) (\|\nabla u^\varepsilon\|_{L^2}^2 \|\nabla c^\varepsilon\|_{L^2}^2 + \|c^\varepsilon\|_{L^2}^2 \|n^\varepsilon\|_{L^2}^2).
\]

(3.26)

Taking the curl to the third equation of (3.2), multiplying the resulting equation by \(\omega^\varepsilon\) and integrating with respect to the space variable mean that
\[
\frac{1}{2} \frac{d}{dt} \|\omega^\varepsilon(t)\|_{L^2}^2 + \|\nabla \omega^\varepsilon(t)\|_{L^2}^2 = - \int_{\mathbb{R}^2} \text{curl} (\nabla \Phi) \omega^\varepsilon \, dx \leq \|\text{curl}(\nabla \Phi)\|_{L^2} \|\omega^\varepsilon\|_{L^2} \leq C(\varepsilon) \|n^\varepsilon\|_{L^2}^2 \|\omega^\varepsilon\|_{L^2}^2 \leq C(\varepsilon) \|n^\varepsilon\|_{L^2}^2 + \|\omega^\varepsilon\|_{L^2}^2.
\]

(3.27)

Summing up the estimates (3.25), (3.26) and (3.27), then by the Grönwall inequality, we obtain the \(H^1\) norm of \(n^\varepsilon, c^\varepsilon, \omega^\varepsilon\) are all closed. The expression is as follows
\[
\|\nabla E^\varepsilon(t)\|_{L^2}^2 + 2a \int_0^t \|\nabla n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau + 5 \int_0^t \|n^\varepsilon \nabla n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau + \frac{\varepsilon}{2} \int_0^t \|\Delta n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|\Delta c^\varepsilon(\tau)\|_{L^2}^2 \, d\tau + 2 \int_0^t \|\nabla \omega^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq \left( \|\nabla E_0^\varepsilon\|_{L^2}^2 + C(\varepsilon) \int_0^t (\|c^\varepsilon\|_{L^2}^2 + \|n^\varepsilon\|_{L^2}^2 + \|n^\varepsilon\|_{L^2}^2) (\tau) \, d\tau \right) \times \exp \left( C(\varepsilon) \int_0^t (\|\nabla u^\varepsilon\|_{L^2} + \|u^\varepsilon\|_{L^2} + \|c^\varepsilon\|_{L^2} + 1) (\tau) \, d\tau \right).
\]

Combining this with (3.23) yields
\[
\|E^\varepsilon(t)\|_{H^1}^2 + 2a \int_0^t \|n^\varepsilon(\tau)\|_{H^1}^2 \, d\tau + \frac{\varepsilon}{2} \int_0^t \|n^\varepsilon(\tau)\|_{H^2}^2 \, d\tau + \int_0^t \|c^\varepsilon(\tau)\|_{H^2}^2 \, d\tau + \int_0^t \|u^\varepsilon(\tau)\|_{H^2}^2 \, d\tau + 5 \int_0^t \|n^\varepsilon \nabla n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq (1 + \sqrt{t}) \|E_0^\varepsilon\|_{H^1}^2 \exp \left( \exp(\exp(C(\varepsilon, t, \|E_0^\varepsilon\|_{L^2}^2)) \right).
\]

Last, we shall give the \(H^s\)-norm for solutions by using the Fourier localization technique. Apply \(\Delta_\theta\) to the first equation of (3.2) and then multiplying the resulting equality by \(\Delta_\theta n^\varepsilon\), we have
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_\theta n^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\nabla \Delta_\theta n^\varepsilon(t)\|_{L^2}^2 + a \|\Delta_\theta n^\varepsilon(t)\|_{L^2}^2 \triangleq M_1 + M_2 + M_3 + M_4,
\]

(3.28)
Similarly, for the term $M_1$ and $M_2$, by integration by parts, the Hölder inequality and the Young inequality, we directly get that

$$
M_1 = -\int_{\mathbb{R}^2} \Delta_q \left( (u^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n^\varepsilon \right) \Delta_q n^\varepsilon \, dx
$$

$$
= \int_{\mathbb{R}^2} \Delta_q \left( (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \right) \cdot \nabla \Delta_q n^\varepsilon \, dx
$$

$$
\leq \\| \Delta_q \left( (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \right) \\|_{L^2} \| \nabla \Delta_q n^\varepsilon \|_{L^2}
$$

$$
\leq C(\varepsilon) \| \Delta_q \left( (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \right) \|_{L^2}^2 + \frac{\varepsilon}{4} \| \nabla \Delta_q n^\varepsilon \|_{L^2}^2.
$$

(3.29)

and

$$
M_2 = -\int_{\mathbb{R}^2} \nabla \cdot \Delta_q \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) \Delta_q n^\varepsilon \, dx
$$

$$
= \int_{\mathbb{R}^2} \Delta_q \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) \cdot \nabla \Delta_q n^\varepsilon \, dx
$$

$$
\leq \| \Delta_q \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) \|_{L^2} \| \nabla \Delta_q n^\varepsilon \|_{L^2}
$$

$$
\leq C(\varepsilon) \| \Delta_q \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) \|_{L^2}^2 + \frac{\varepsilon}{4} \| \nabla \Delta_q n^\varepsilon \|_{L^2}^2.
$$

(3.30)

Similarly, for the term $M_3$ and $M_4$, we obtain

$$
M_3 = (1 + a) \int_{\mathbb{R}^2} \Delta_q (n^\varepsilon)^2 \Delta_q n^\varepsilon \, dx \leq C \| \Delta_q (n^\varepsilon)^2 \|_{L^2}^2 + \frac{a}{4} \| \Delta_q n^\varepsilon \|_{L^2}^2
$$

(3.31)

and

$$
M_4 = -\int_{\mathbb{R}^2} \Delta_q (n^\varepsilon)^3 \Delta_q n^\varepsilon \, dx \leq C \| \Delta_q (n^\varepsilon)^3 \|_{L^2}^2 + \frac{a}{4} \| \Delta_q n^\varepsilon \|_{L^2}^2.
$$

(3.32)

Plugging (3.29)-(3.32) into (3.28), then we have the following estimate

$$
\frac{1}{2} \frac{d}{dt} \| \Delta_q n^\varepsilon(t) \|_{L^2}^2 + \frac{\varepsilon}{2} \| \nabla \Delta_q n^\varepsilon(t) \|_{L^2}^2 + \frac{a}{2} \| \Delta_q n^\varepsilon(t) \|_{L^2}^2
$$

$$
\leq C(\varepsilon) \| \Delta_q \left( (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \right) \|_{L^2}^2 + C(\varepsilon) \| \Delta_q \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) \|_{L^2}^2
$$

$$
+ C \| \Delta_q (n^\varepsilon)^2 \|_{L^2}^2 + C \| \Delta_q (n^\varepsilon)^3 \|_{L^2}^2.
$$

Now, multiplying both sides of the above inequality by $2^{2q\varepsilon}$, and then computing the $\ell^1$-norm, we have

$$
\frac{1}{2} \frac{d}{dt} \| n^\varepsilon(t) \|_{H^s}^2 + \frac{\varepsilon}{2} \| \nabla n^\varepsilon(t) \|_{H^s}^2 + \frac{a}{2} \| n^\varepsilon(t) \|_{H^s}^2
$$

$$
\leq C(\varepsilon) \| (u^\varepsilon \ast \rho^\varepsilon) n^\varepsilon \|_{H^s}^2 + C(\varepsilon) \| n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \|_{H^s}^2 + C(\varepsilon) \| (n^\varepsilon)^2 \|_{H^s}^2 + C(\varepsilon) \| (n^\varepsilon)^3 \|_{H^s}^2.
$$
By Leibniz estimate (2.1), we know that
\[ \|(n^\varepsilon)^2\|_{H^s}^2 \leq C\|n^\varepsilon\|^2_{L^\infty} \|n^\varepsilon\|^2_{H^s}, \]
and
\[ \|(n^\varepsilon)^3\|_{H^s}^3 \leq C\left(\|n^\varepsilon\|^2_{L^\infty} \|n^\varepsilon\|^2_{H^s} + \|(n^\varepsilon)^2\|_{L^\infty} \|n^\varepsilon\|^2_{H^s}\right). \]
Hence, according to \(|n^\varepsilon|_{L^\infty} \leq |n^\varepsilon|_{H^s}^2\) and \(n^\varepsilon \in L_t^\infty H^1 \cap L_t^2 H^2\), we get that
\[ \int_0^t \|n^\varepsilon(t)\|^4_{H^s} \, dt < \infty. \]
So,
\[ \frac{1}{2} \frac{d}{dt} \|n^\varepsilon(t)\|_{H^s}^2 + \frac{\varepsilon}{2} \|\nabla n^\varepsilon(t)\|_{H^s}^2 + \frac{\alpha}{2} \|n^\varepsilon(t)\|_{H^s}^2 \]
\[ \leq C(\varepsilon) \left(\|u^\varepsilon\|^2_{L^2} + \|c^\varepsilon\|^2_{L^2} + \|n^\varepsilon\|^2_{H^s} + \|n^\varepsilon\|^4_{H^s} \right) \|n^\varepsilon\|^2_{H^s} \]
\[ + C(\varepsilon) \left(\|u^\varepsilon\|^2_{L^2} + \|c^\varepsilon\|^2_{L^2} \right) |n^\varepsilon|^2_{H^s} + C(\varepsilon)|n^\varepsilon|^2_{H^s} |c^\varepsilon|^2_{L^2}. \]
\[ (3.33) \]
Similarly, applying \(\Delta_q\) to the second equation of (3.2) and then multiplying the resulting equality by \(\Delta_q c^\varepsilon\) yields that
\[ \frac{1}{2} \frac{d}{dt} \|\Delta_q c^\varepsilon(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} \Delta_q \left(u^\varepsilon \cdot \nabla (c^\varepsilon)\right) \Delta_q c^\varepsilon \, dx - \int_{\mathbb{R}^2} \Delta_q (\Delta c^\varepsilon) \Delta_q c^\varepsilon \, dx \]
\[ = - \int_{\mathbb{R}^2} \Delta_q \left(c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon)\right) \Delta_q c^\varepsilon \, dx. \]
By the Hölder inequality and the Young inequality, we have
\[ \frac{1}{2} \frac{d}{dt} \|\Delta_q c^\varepsilon(t)\|_{L^2}^2 + \|\nabla \Delta_q c^\varepsilon(t)\|_{L^2}^2 \]
\[ = \int_{\mathbb{R}^2} \Delta_q (u^\varepsilon \cdot \nabla c^\varepsilon) \, dx - \int_{\mathbb{R}^2} \Delta_q \left(c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon)\right) \Delta_q c^\varepsilon \, dx \]
\[ \leq \|\Delta_q (u^\varepsilon \cdot \nabla c^\varepsilon)\|_{L^2} \|\nabla \Delta_q c^\varepsilon\|_{L^2} + \|\Delta_q \left(c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon)\right)\|_{L^2} \|\Delta_q c^\varepsilon\|_{L^2} \]
\[ \leq C \|\Delta_q (u^\varepsilon \cdot \nabla c^\varepsilon)\|_{L^2}^2 + \frac{1}{4} \|\nabla \Delta_q c^\varepsilon\|_{L^2}^2 + C \|\Delta_q \left(c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon)\right)\|_{L^2}^2 + \|\Delta_q c^\varepsilon\|_{L^2}^2. \]
Multiplying \(2^{2q}\) on both sides of the above inequality, then taking the \(l^1\)-norm, we obtain
\[ \frac{d}{dt} \|c^\varepsilon(t)\|^2_{B_{2,q}^s} + \|\nabla c^\varepsilon(t)\|^2_{B_{2,q}^s} \leq C \|u^\varepsilon c^\varepsilon\|^2_{B_{2,q}^s} + C \|c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon)\|^2_{B_{2,q}^s} + \|c^\varepsilon\|^2_{B_{2,q}^s}. \]
From the above equation, we obtain
\[ \frac{d}{dt} \|c^\varepsilon(t)\|_{H^s}^2 + \|\nabla c^\varepsilon(t)\|_{H^s}^2 \]
\[ \leq C(\varepsilon) \left(\|u^\varepsilon\|^2_{H^1} |c^\varepsilon|^2_{H^1} + \|u^\varepsilon\|^2_{H^2} + \|n^\varepsilon\|^2_{H^s} + 1\right) \|c^\varepsilon\|_{H^s}^2 + \|c^\varepsilon\|^2_{L^2} |n^\varepsilon|^2_{L^2}. \]
\[ (3.34) \]
Apply \(\Delta_q\) to the third equation of (3.2) and then multiplying the resulting equality by \(\Delta_q u^\varepsilon\) implies
\[ \frac{1}{2} \frac{d}{dt} \|\Delta_q u^\varepsilon(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} \Delta_q \left(u^\varepsilon \cdot \nabla (u^\varepsilon)\right) \Delta_q u^\varepsilon \, dx + \int_{\mathbb{R}^2} (\nabla \Delta_q u^\varepsilon)^2 \, dx \]
\[ = - \int_{\mathbb{R}^2} \Delta_q \left((n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon\right) \Delta_q u^\varepsilon \, dx. \]
Similarly, by the Hölder inequality and the Young inequality, we get
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q u^\varepsilon(t) \|_{L^2}^2 + \| \nabla \Delta_q u^\varepsilon(t) \|_{L^2}^2 \\
= \int_{\mathbb{R}^2} \Delta_q (u^\varepsilon u^\varepsilon) \cdot \nabla (\Delta_q u^\varepsilon) \, dx - \int_{\mathbb{R}^2} \Delta_q ((n^\varepsilon \nabla \Phi) * \rho^\varepsilon) \Delta_q u^\varepsilon \, dx \\
\leq \| \Delta_q (u^\varepsilon u^\varepsilon) \|_{L^2} \| \nabla \Delta_q u^\varepsilon \|_{L^2} + \| \Delta_q (n^\varepsilon \nabla \Phi) * \rho^\varepsilon \|_{L^2} \| \Delta_q u^\varepsilon \|_{L^2} \\
\leq C \| \Delta_q (u^\varepsilon u^\varepsilon) \|_{L^2}^2 + \frac{1}{2} \| \nabla \Delta_q u^\varepsilon \|_{L^2}^2 + C \| \Delta_q ((n^\varepsilon \nabla \Phi) * \rho^\varepsilon) \|_{L^2}^2 + \frac{1}{2} \| \Delta_q u^\varepsilon \|_{L^2}^2.
\]

The above equation implies that
\[
\frac{d}{dt} \| \Delta_q u^\varepsilon(t) \|_{L^2}^2 + \| \nabla \Delta_q u^\varepsilon(t) \|_{L^2}^2 \\
\leq C \| \Delta_q (u^\varepsilon u^\varepsilon) \|_{L^2}^2 + C \| \Delta_q ((n^\varepsilon \nabla \Phi) * \rho^\varepsilon) \|_{L^2}^2 + \| \Delta_q u^\varepsilon \|_{L^2}^2.
\]

Multiplying $2^k \cdot 2^q$ on both sides of the above inequality, then taking the $l^1$-norm, we get
\[
\frac{d}{dt} \| u^\varepsilon(t) \|_{H^s}^2 + \| \nabla u^\varepsilon(t) \|_{H^s}^2 \leq C \| u^\varepsilon u^\varepsilon \|_{H^s}^2 + C \| (n^\varepsilon \nabla \Phi) * \rho^\varepsilon \|_{H^s}^2 + \| u^\varepsilon \|_{H^s}^2 \\
\leq C \| u^\varepsilon \|_{H^s}^2 + \| u^\varepsilon \|_{H^s}^2 + C(\varepsilon) \| n^\varepsilon \|_{L^2}^2 + \| u^\varepsilon \|_{H^s}^2 \hspace{0.6em} (3.35)
\]

Summing up (3.33), (3.34) and (3.35), and using the Grönwall inequality, we obtain
\[
\| E^\varepsilon(t) \|_{H^s}^2 + a \int_0^t \| n^\varepsilon(\tau) \|_{H^s}^2 \, d\tau + \varepsilon \int_0^t \| n^\varepsilon(\tau) \|_{H^{s+1}}^2 \, d\tau \\
+ \int_0^t \| \epsilon^\varepsilon(\tau) \|_{H^{s+1}}^2 \, d\tau + \int_0^t \| u^\varepsilon(\tau) \|_{H^{s+1}}^2 \, d\tau \\
\leq \left( \| E_0^\varepsilon \|_{H^s}^2 + C(\varepsilon) \int_0^t \| n^\varepsilon(\tau) \|_{L^2}^2 (1 + \| \epsilon^\varepsilon(\tau) \|_{L^2}^2) \, d\tau \\
+ C(\varepsilon) \int_0^t \left( \| u^\varepsilon(\tau) \|_{L^2}^2 + \| \epsilon^\varepsilon(\tau) \|_{L^2}^2 \right) \| n^\varepsilon(\tau) \|_{H^s}^2 \, d\tau \right) e^{C(\varepsilon) \int_0^t A \, d\tau},
\]

where
\[
A \triangleq \| (n^\varepsilon, \epsilon^\varepsilon, u^\varepsilon) \|_{L^2}^2 + \| n^\varepsilon \|_{H^s}^2 + \| n^\varepsilon \|_{H^{s+1}}^2 + \| (\epsilon^\varepsilon, u^\varepsilon) \|_{H^s}^2 + 1.
\]

Then we obtain
\[
\| E^\varepsilon(t) \|_{H^s}^2 + a \int_0^t \| n^\varepsilon(\tau) \|_{H^s}^2 \, d\tau + \varepsilon \int_0^t \| n^\varepsilon(\tau) \|_{H^{s+1}}^2 \, d\tau \\
+ \int_0^t \| \epsilon^\varepsilon(\tau) \|_{H^{s+1}}^2 \, d\tau + \int_0^t \| u^\varepsilon(\tau) \|_{H^{s+1}}^2 \, d\tau \hspace{0.6em} (3.36)
\]

Moreover, by continuation of an autonomous ODE on a Banach space [14, Theorem 3.3], we can conclude that the local weak solution can be continued for all time.

**Step 5.** This step is devoted to proving the uniqueness of solutions. Let $(n_i^\varepsilon, \epsilon_i^\varepsilon, u_i^\varepsilon)$ $i = 1, 2$ be two solutions of the system (3.1) with the same initial data $(n_0^\varepsilon, \epsilon_0^\varepsilon, u_0^\varepsilon)$. Denote $\delta n = n_1^\varepsilon - n_2^\varepsilon$, $\delta \epsilon = \epsilon_1^\varepsilon - \epsilon_2^\varepsilon$, $\delta u = u_1^\varepsilon - u_2^\varepsilon$. Then we have the difference
equations as follows:

\[
\begin{aligned}
\partial_t \delta n^\varepsilon + (u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n_1^\varepsilon + (u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla \delta n^\varepsilon - \varepsilon \Delta \delta n^\varepsilon &= H(x, t), \\
\partial_t \delta c^\varepsilon + \Delta u_1^\varepsilon - \Delta \delta c^\varepsilon &= -\delta c^\varepsilon (n_1^\varepsilon \ast \rho^\varepsilon) - c_2^\varepsilon (\delta n^\varepsilon \ast \rho^\varepsilon), \\
\partial_t \delta u_1^\varepsilon + \partial_t \delta u_2^\varepsilon \cdot \nabla u_1^\varepsilon + u_2^\varepsilon \cdot \nabla \delta u_1^\varepsilon - \Delta \delta u_1^\varepsilon + \nabla (P_1^\varepsilon - P_2^\varepsilon) &= -\delta n^\varepsilon \nabla \Phi \ast \rho^\varepsilon, \\
\nabla \cdot u^\varepsilon &= 0,
\end{aligned}
\]

(3.37)

\[
(n^\varepsilon, c^\varepsilon, u^\varepsilon) \big|_{t=0} = (n_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon),
\]

where \( H(x, t) \triangleq -\nabla \cdot (\delta n^\varepsilon \nabla (c_1^\varepsilon \ast \rho^\varepsilon)) \) - \( \nabla \cdot (n_2^\varepsilon \nabla (\delta c^\varepsilon \ast \rho^\varepsilon)) + g(n_1^\varepsilon) - g(n_2^\varepsilon) \)
and

\[
g(n_1^\varepsilon) - g(n_2^\varepsilon) = (1 + a)(n_1^\varepsilon + n_2^\varepsilon)\delta n^\varepsilon - a\delta n^\varepsilon - \delta n^\varepsilon ((n_1^\varepsilon)^2 + n_1^\varepsilon n_2^\varepsilon + (n_2^\varepsilon)^2).
\]

Taking the \( L^2 \)-inner product with the first equation with \( \delta n^\varepsilon \), we have

\[
\frac{1}{2} \frac{d}{dt} \| \delta n^\varepsilon(t) \|_{L^2}^2 + \varepsilon \| \nabla \delta n^\varepsilon(t) \|_{L^2}^2 + a \| \delta n^\varepsilon(t) \|_{L^2}^2 + \int_{\mathbb{R}^2} (\delta n^\varepsilon(t))^2 \left( (n_1^\varepsilon)^2 + n_1^\varepsilon n_2^\varepsilon + (n_2^\varepsilon)^2 \right) \, dx
\]

(3.38)

\[
\triangleq O_1 + O_2 + O_3 + O_4 + O_5.
\]

where

\[
O_1 = -\int_{\mathbb{R}^2} (\delta u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n_1^\varepsilon \delta n^\varepsilon \, dx,
\]

\[
O_2 = -\int_{\mathbb{R}^2} (u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla \delta n^\varepsilon \delta n^\varepsilon \, dx,
\]

\[
O_3 = -\int_{\mathbb{R}^2} \nabla \cdot (\delta n^\varepsilon \nabla (c_1^\varepsilon \ast \rho^\varepsilon)) \delta n^\varepsilon \, dx,
\]

\[
O_4 = -\int_{\mathbb{R}^2} \nabla \cdot (n_2^\varepsilon \nabla (\delta c^\varepsilon \ast \rho^\varepsilon)) \delta n^\varepsilon \, dx,
\]

\[
O_5 = (1 + a) \int_{\mathbb{R}^2} (n_1^\varepsilon + n_2^\varepsilon) \delta n^\varepsilon \delta n^\varepsilon \, dx.
\]

Now we need to estimate the above equations, respectively. For the first term \( O_1 \), we have

\[
O_1 = -\int_{\mathbb{R}^2} (\delta u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n_1^\varepsilon \delta n^\varepsilon \, dx = \int_{\mathbb{R}^2} n_1^\varepsilon (\delta u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla \delta n^\varepsilon \, dx
\]

\[
\leq \| \delta u_2^\varepsilon \ast \rho^\varepsilon \|_{L^\infty} \| n_1^\varepsilon \|_{L^2} \| \nabla \delta n^\varepsilon \|_{L^2} \leq C(\varepsilon) \| \delta u_2^\varepsilon \|_{L^2} \| n_1^\varepsilon \|_{L^2} \| \nabla \delta n^\varepsilon \|_{L^2}
\]

(3.39)

\[
\leq C(\varepsilon) \| \delta u_2^\varepsilon \|_{L^2}^2 \| n_1^\varepsilon \|_{L^2}^2 + \frac{\varepsilon}{8} \| \nabla \delta n^\varepsilon \|_{L^2}^2.
\]

Similarly, we have

\[
O_2 = -\int_{\mathbb{R}^2} (u_2^\varepsilon \ast \rho^\varepsilon) \cdot \nabla \delta n^\varepsilon \delta n^\varepsilon \, dx = 0.
\]

(3.40)

Applying the integration by parts, the Hölder inequality and the Young inequality, we get

\[
O_3 = -\int_{\mathbb{R}^2} \nabla \cdot (\delta n^\varepsilon \nabla (c_1^\varepsilon \ast \rho^\varepsilon)) \delta n^\varepsilon \, dx = \int_{\mathbb{R}^2} \nabla \delta n^\varepsilon \nabla (c_1^\varepsilon \ast \rho^\varepsilon) \cdot \nabla \delta n^\varepsilon \, dx
\]

\[
\leq \| \nabla \delta n^\varepsilon \|_{L^2} \| \nabla (c_1^\varepsilon \ast \rho^\varepsilon) \|_{L^\infty} \| \nabla \delta n^\varepsilon \|_{L^2}
\]

\[
\leq C(\varepsilon) \| \nabla \delta n^\varepsilon \|_{L^2}^2 \| c_1^\varepsilon \|_{L^2}^2 + \frac{\varepsilon}{8} \| \nabla \delta n^\varepsilon \|_{L^2}^2.
\]

(3.41)
Let's just assume that

\[
C(\|n_1^{\varepsilon}\|_{L^2}^2 + \|n_2^{\varepsilon}\|_{L^2}^2) + C\|\delta n^\varepsilon\|_{L^2}^2.
\]

as well as

\[
O_5 = (1 + a) \int_{\mathbb{R}^2} (n_1^{\varepsilon} + n_2^{\varepsilon}) \delta n^\varepsilon \delta n^\varepsilon \, dx
\leq C\left(\|n_1^{\varepsilon}\|_{L^2}^2 + \|n_2^{\varepsilon}\|_{L^2}^2\right) \|\delta n^\varepsilon\|_{L^2}^2
\leq \frac{1}{4}\left(\|n_1^{\varepsilon}\|_{L^2}^2 + \|n_2^{\varepsilon}\|_{L^2}^2\right) + C\|\delta n^\varepsilon\|_{L^2}^2.
\]

Let's just assume that \( n_1^{\varepsilon} < n_2^{\varepsilon} \), then

\[
LHS \geq \frac{1}{2} \frac{d}{dt}\|\delta u^\varepsilon(t)\|_{L^2}^2 + \varepsilon \|\nabla \delta n^\varepsilon(t)\|_{L^2}^2 + a\|\delta n^\varepsilon(t)\|_{L^2}^2 + 2\|n_1^{\varepsilon}\|_{L^2}^2 + \|n_2^{\varepsilon}\|_{L^2}^2.
\]

Plugging (3.39)-(3.44) into (3.38), we have

\[
1 \frac{d}{dt}\|\delta u^\varepsilon(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|\nabla \delta n^\varepsilon(t)\|_{L^2}^2 + a\|\delta n^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2}\|n_1^{\varepsilon}\|_{L^2}^2 + \frac{3}{4}\|n_2^{\varepsilon}\|_{L^2}^2 + \frac{3}{4}\|\delta n^\varepsilon(t)\|_{L^2}^2
\leq C(\varepsilon)\|\delta u^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon)\|\delta n^\varepsilon(t)\|_{L^2}^2 (c_1^2 + 1) + C(\varepsilon)\|\delta u^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon)\|\delta n^\varepsilon(t)\|_{L^2}^2.
\]

Similarly, we can infer that

\[
1 \frac{d}{dt}\|\delta c^\varepsilon(t)\|_{L^2}^2 + \|\nabla \delta c^\varepsilon(t)\|_{L^2}^2
\geq 1 \frac{d}{dt}\|\delta n^\varepsilon(t)\|_{L^2}^2 + \|\nabla \delta c^\varepsilon(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|\delta n^\varepsilon(t)\|_{L^2}^2 + a\|\delta n^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2}\|n_1^{\varepsilon}\|_{L^2}^2 + \frac{3}{4}\|n_2^{\varepsilon}\|_{L^2}^2 + \frac{3}{4}\|\delta n^\varepsilon(t)\|_{L^2}^2
\leq C(\varepsilon)\|\delta u^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon)\|\delta c^\varepsilon(t)\|_{L^2}^2 (c_1^2 + 1) + C(\varepsilon)\|\delta u^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon)\|\delta n^\varepsilon(t)\|_{L^2}^2.
\]

and

\[
1 \frac{d}{dt}\|\delta n^\varepsilon(t)\|_{L^2}^2 + \|\nabla \delta n^\varepsilon(t)\|_{L^2}^2
\geq 1 \frac{d}{dt}\|\delta n^\varepsilon(t)\|_{L^2}^2 + \|\nabla \delta n^\varepsilon(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|\delta n^\varepsilon(t)\|_{L^2}^2 + a\|\delta n^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2}\|n_1^{\varepsilon}\|_{L^2}^2 + \frac{3}{4}\|n_2^{\varepsilon}\|_{L^2}^2 + \frac{3}{4}\|\delta n^\varepsilon(t)\|_{L^2}^2
\leq C(\varepsilon)\|\delta n^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon)\|\delta n^\varepsilon(t)\|_{L^2}^2 (c_1^2 + 1) + C(\varepsilon)\|\delta u^\varepsilon(t)\|_{L^2}^2 + C(\varepsilon)\|\delta n^\varepsilon(t)\|_{L^2}^2.
Summing up the above three equations, we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} (\|\delta n^\varepsilon(t)\|_{L^2}^2 + \|\delta c^\varepsilon(t)\|_{L^2}^2 + \|\delta u^\varepsilon(t)\|_{L^2}^2) + a \|\delta n^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2} \|n^\varepsilon_1 \delta n^\varepsilon(t)\|_{L^2}^2 \\
+ \frac{3}{4} \|n^\varepsilon_2 \delta n^\varepsilon(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|\nabla \delta n^\varepsilon(t)\|_{L^2}^2 + \frac{3}{4} \|\nabla \delta c^\varepsilon(t)\|_{L^2}^2 + \frac{7}{8} \|\nabla \delta u^\varepsilon(t)\|_{L^2}^2 \\
\leq C(\varepsilon) \left( \|\delta n^\varepsilon\|_{L^2}^2 + \|\delta c^\varepsilon\|_{L^2}^2 + \|\delta u^\varepsilon\|_{L^2}^2 \right) \left( \|(n^\varepsilon_1, n^\varepsilon_2, c^\varepsilon_t)\|_{L^2}^2 + \|(c^\varepsilon_1, c^\varepsilon_2, u^\varepsilon_t)\|_{H^s}^2 + 1 \right).
\end{align*}
\]

Consequently, we get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|\delta n^\varepsilon(t)\|_{L^2}^2 + \|\delta c^\varepsilon(t)\|_{L^2}^2 + \|\delta u^\varepsilon(t)\|_{L^2}^2 \right) \\
\leq C(\varepsilon) \left( \|\delta n^\varepsilon\|_{L^2}^2 + \|\delta c^\varepsilon\|_{L^2}^2 + \|\delta u^\varepsilon\|_{L^2}^2 \right) \left( \|(n^\varepsilon_1, n^\varepsilon_2, c^\varepsilon_t)\|_{L^2}^2 + \|(c^\varepsilon_1, c^\varepsilon_2, u^\varepsilon_t)\|_{H^s}^2 + 1 \right).
\end{align*}
\]

Therefore, we obtain the uniqueness by using the Grönwall inequality on time \([0, T]\).

Lastly, we want to show that \((n^\varepsilon, c^\varepsilon, u^\varepsilon) \in (C^1(0, \infty); H^{s-2})^3\). Since
\[
\partial_t u^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon + \Delta u^\varepsilon + \nabla P^\varepsilon - (n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon,
\]
we can show by the Leibniz estimates that for \(s > 1\)
\[
\|\partial_t u^\varepsilon\| \leq C\|u^\varepsilon\|_{H^s} + \|u^\varepsilon\|_{H^s}^2 + \|n^\varepsilon\|_{L^2}.
\]

It follows from argument in [14, Theorem 3.5] that \(\partial_t u^\varepsilon \in C(0, T; H^{s-2})\). The proof of Proposition 3.1 is completed.

4. Uniform estimate for the regularized problem. In this section, we prove the uniform estimates for smooth solutions \((n^\varepsilon, c^\varepsilon, u^\varepsilon)\) to the regularized problem (3.1) which are independent neither of \(\varepsilon > 0\) nor the mollifier \(\rho^\varepsilon\). By Proposition 3.1, we know that the regularized system (3.1) admits a unique global solution \((n^\varepsilon, c^\varepsilon, u^\varepsilon) \in (C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2_{\text{loc}}([0, \infty); H^{s+1}(\mathbb{R}^2)))^3\). On the other hand, for \(l > 1\) space \(H^l(\mathbb{R}^2)\) is a Banach algebra embedded in the set of continuous functions going to 0 at infinity. It means that the solutions to the regularized problem (3.1) are smooth and decay sufficiently fast at infinity, so when we integrate by parts in our calculations below, there are no boundary terms. We distinguish especially two kinds: the first one deals with some easy estimates that one can obtained by energy estimates. The second one is concerned with some strong estimates which are the heart of the proof of our main result.

**Proposition 4.1.** Let the triple \((n^\varepsilon_0, c^\varepsilon_0, u^\varepsilon_0) \in X_0 \cap (H^s(\mathbb{R}^2))^3\) with \(s > 1\) and \(\nabla \Phi \in L^\infty\). Assume \((n^\varepsilon, c^\varepsilon, u^\varepsilon)\) is a smooth solution of system (3.1). Then there exists a constant \(C > 0\) independent of \(\varepsilon > 0\) such that
\[
\|n^\varepsilon(t)\|_{L^1} + a \int_0^t \|n^\varepsilon(\tau)\|_{L^1} \, d\tau + \frac{1}{2} \int_0^t \|n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq \|n^\varepsilon_0\|_{L^1} e^{Ct},
\]
\[
\|c^\varepsilon(t)\|_{L^p} \leq \|c^\varepsilon_0\|_{L^p}, \quad \text{for} \quad 2 \leq p \leq \infty,
\]
and
\[
\|u^\varepsilon(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq \|u^\varepsilon_0\|_{L^2}^2 + Cte^{Ct}.
\]
Lastly, multiplying both sides of the third equation of (3.1) by $c\varepsilon$ and $p\varepsilon$

Proof. Since $n_0^\varepsilon \geq 0$ and $c_0^\varepsilon \geq 0$, we know from Proposition 3.1 that $n^\varepsilon(x,t) > 0$ and $c^\varepsilon(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$. Firstly, we deal with the first equation of (3.1). Since $\nabla \cdot u^\varepsilon = 0$, we have

$$\frac{d}{dt}\|n^\varepsilon(t)\|_{L^1} + a\|n^\varepsilon(t)\|_{L^1} + \|n^\varepsilon(t)\|^3_{L^3} = (1 + a)\|n^\varepsilon(t)\|^2_{L^2}$$

$$\leq (1 + a)\|n^\varepsilon(t)\|^2_{L^2} + \|n^\varepsilon(t)\|^3_{L^3}$$

$$\leq \frac{1}{2}\|n^\varepsilon(t)\|^2_{L^3} + C\|n^\varepsilon(t)\|_{L^1}.$$

Then we have

$$\frac{d}{dt}\|n^\varepsilon(t)\|_{L^1} + a\|n^\varepsilon(t)\|_{L^1} + \frac{1}{2}\|n^\varepsilon(t)\|^3_{L^3} \leq C\|n^\varepsilon(t)\|_{L^1}.$$

Hence, by the Grönwall inequality, we get that

$$\|n^\varepsilon(t)\|_{L^1} + a\int_0^t \|n^\varepsilon(\tau)\|_{L^1} d\tau + \frac{1}{2} \int_0^t \|n^\varepsilon(\tau)\|^3_{L^3} d\tau \leq \|n_0^\varepsilon\|_{L^1} e^{Ct}. \quad (4.1)$$

Then we tackle the second inequality. Multiplying the second equation of (3.1) by $(c^\varepsilon)^{p-1}(2 \leq p < \infty)$ and integrating the resulting equation, yields

$$\frac{1}{p} \frac{d}{dt}\|c^\varepsilon(t)\|^p_{L^p} - \int_{\mathbb{R}^2} \Delta c^\varepsilon(p(c^\varepsilon)^{p-1} dx$$

$$= - \int_{\mathbb{R}^2} u^\varepsilon \cdot \nabla c^\varepsilon(p-1) dx - \int_{\mathbb{R}^2} (n^\varepsilon \ast \rho^\varepsilon)(c^\varepsilon)^p dx.$$

Then we get

$$\frac{1}{p} \frac{d}{dt}\|c^\varepsilon(t)\|^p_{L^p} + \frac{4(p-1)}{p} \int_{\mathbb{R}^2} \|\nabla(c^\varepsilon)^{\frac{p}{2}}(t)\|^2_{L^2} \leq 0, \quad \text{for} \quad 2 \leq p < \infty.$$

So, by the Grönwall inequality, we obtain

$$\|c^\varepsilon(t)\|^p_{L^p} + \frac{4(p-1)}{p} \int_0^t \|\nabla(c^\varepsilon)^{\frac{p}{2}}(\tau)\|^2_{L^2} d\tau \leq \|c_0^\varepsilon\|^p_{L^p}, \quad \text{for} \quad 2 \leq p < \infty.$$

Thus, we have

$$\|c^\varepsilon(t)\|_{L^p} \leq \|c_0^\varepsilon\|_{L^p}, \quad 2 \leq p < \infty.$$

This estimate implies

$$\|c_0^\varepsilon\|_{L^p} \leq \|c_0^\varepsilon\|_{L^2 \cap L^\infty},$$

when $p \rightarrow \infty$, we have

$$\|c^\varepsilon(t)\|_{L^\infty} \leq \|c_0^\varepsilon\|_{L^2 \cap L^\infty}.$$

Lastly, multiplying both sides of the third equation of (3.1) by $u^\varepsilon$, we have

$$\frac{1}{2} \frac{d}{dt}\|u^\varepsilon(t)\|^2_{L^2} + \|\nabla u^\varepsilon(t)\|^2_{L^2} = - \int_{\mathbb{R}^2} ((n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon) \cdot u^\varepsilon dx$$

$$\leq \|((n^\varepsilon \nabla \Phi) \ast \rho^\varepsilon\|_{L^2} \|u^\varepsilon\|_{L^2}$$

$$\leq \|n^\varepsilon\|_{L^2} \|\nabla \Phi\|_{L^\infty} \|u^\varepsilon\|_{L^2}. \quad (4.2)$$

The above inequality implies that

$$\frac{1}{2} \frac{d}{dt}\|u^\varepsilon(t)\|_{L^2} \leq \|n^\varepsilon(t)\|_{L^2} \|\nabla \Phi\|_{L^\infty}.$$
Let the triple

\[ \text{Proposition 4.2.} \]

By the Grönwall inequality and the Interpolation theorem, we directly know that

\[ \|u^\varepsilon(t)\|_{L^2} \leq \|u_0^\varepsilon\|_{L^2} + C \int_0^t \|n^\varepsilon(\tau)\|_{L^2} \, d\tau \]
\[ \leq \|u_0^\varepsilon\|_{L^2} + C\sqrt{t}\|n^\varepsilon(t)\|_{L^2}^{\frac{1}{2}} \|n^\varepsilon(\tau)\|_{L^2}^{\frac{3}{2}} \]
\[ \leq \|u_0^\varepsilon\|_{L^2} + C\sqrt{t}e^{Ct}, \]

where we have used estimate (4.1).

Plugging (4.3) into (4.2), we have

\[ \frac{1}{2} \frac{d}{dt} \|u^\varepsilon(t)\|_{L^2}^2 + \|\nabla u^\varepsilon(t)\|_{L^2}^2 \leq \|n^\varepsilon(t)\|_{L^2} \|\nabla \Phi\|_{L^\infty} (\|u_0^\varepsilon\|_{L^2} + C\sqrt{t}e^{Ct}) \]
\[ \leq C\|n^\varepsilon(t)\|_{L^2}(1 + \sqrt{t}e^{Ct}). \]

By the Grönwall inequality, the Hölder inequality and the Interpolation theorem, we obtain that

\[ \|u^\varepsilon(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \leq \|u_0^\varepsilon\|_{L^2}^2 + C\int_0^t \|n^\varepsilon(\tau)\|_{L^2}(1 + \sqrt{t}e^{C\tau}) \, d\tau \]
\[ \leq \|u_0^\varepsilon\|_{L^2}^2 + Cte^{Ct}. \]

This completes the proof of Proposition 4.1.

\[ \square \]

**Proposition 4.2.** Let the triple \((n_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) \in X_0 \cap (H^s(\mathbb{R}^2))^3\) with \(s > 1\) and \(\nabla \Phi \in L^\infty\). Assume \((n^\varepsilon, c^\varepsilon, u^\varepsilon)\) is a smooth solution of system (3.1). Then there exists a constant \(C > 0\) depending only on the initial data such that

\[ \|n^\varepsilon(t)\|_{L^2}^2 + \|c^\varepsilon(t)\|_{H^s}^2 + 2a \int_0^t \|n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \left( \|n^\varepsilon(\tau)\|_{L^4}^4 + \|\Delta c^\varepsilon(\tau)\|_{L^2}^2 \right) \, d\tau \leq C\exp(\exp(Ct)), \]

and

\[ \|u^\varepsilon(t)\|_{H^1}^2 + \frac{3}{2} \int_0^t \|u^\varepsilon(\tau)\|_{H^2}^2 \, d\tau \leq \|u_0^\varepsilon\|_{H^2}^2 + Ce^{Ct}. \]

**Proof.** Multiplying \(n^\varepsilon\) on both sides of the first equation of (3.1), and integrating the resulting equation yields

\[ \frac{1}{2} \frac{d}{dt} \|n^\varepsilon(t)\|_{L^2}^2 + \|\nabla n^\varepsilon(t)\|_{L^2}^2 + a \|n^\varepsilon(t)\|_{L^2}^2 + \|n^\varepsilon(t)\|_{L^4}^4 \]
\[ = \int_{\mathbb{R}^2} n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n^\varepsilon \, dx + (1 + a)\|n^\varepsilon(t)\|_{L^2}^3 \]
\[ = - \frac{1}{2} \int_{\mathbb{R}^2} (n^\varepsilon)^2 \Delta (c^\varepsilon \ast \rho^\varepsilon) \, dx + (1 + a)\|n^\varepsilon(t)\|_{L^2}^3. \]

By the Hölder inequality, the Interpolation theorem and the Young inequality, we find that the above equation can be bounded as following

\[ \frac{1}{2} \|n^\varepsilon\|_{L^4}^4 + \|\Delta (c^\varepsilon \ast \rho^\varepsilon)\|_{L^2}^2 + (1 + a)\|n^\varepsilon\|_{L^2}^2 \|n^\varepsilon\|_{L^4}^2 \]
\[ \leq \frac{1}{2} \|n^\varepsilon\|_{L^4}^4 + \|\Delta c^\varepsilon\|_{L^2}^2 + (1 + a)\|n^\varepsilon\|_{L^2}^2 \|n^\varepsilon\|_{L^4}^3 \]
\[ \leq \frac{1}{4} \|n^\varepsilon\|_{L^4}^4 + \frac{1}{4} \|\Delta c^\varepsilon\|_{L^2}^2 + C\|n^\varepsilon\|_{L^2}^2 + \frac{1}{8} \|n^\varepsilon\|_{L^4}^4, \]
from which we have
\[ \frac{1}{2} \frac{d}{dt} \| n^\varepsilon(t) \|_{L^2}^2 + a \| n^\varepsilon(t) \|_{L^2}^2 + \frac{1}{2} \| n^\varepsilon(t) \|_{L^2}^4 \leq \frac{1}{4} \| \Delta c^\varepsilon(t) \|_{L^2}^2 + C \| n^\varepsilon(t) \|_{L^2}^2. \] (4.4)

Now, we turn to the second equation of (3.1), multiplying it by \(-\Delta c^\varepsilon\) and integrating the resulting equation, we could get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla c^\varepsilon(t) \|_{L^2}^2 + \| \Delta c^\varepsilon(t) \|_{L^2}^2 = \int_{\mathbb{R}^2} (u^\varepsilon \cdot \nabla) \Delta c^\varepsilon \, dx + \int_{\mathbb{R}^2} c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon) \Delta c^\varepsilon \, dx
\]
\[
= -\int_{\mathbb{R}^2} (\nabla c^\varepsilon \cdot \nabla) u^\varepsilon \cdot \nabla c^\varepsilon \, dx + \int_{\mathbb{R}^2} c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon) \Delta c^\varepsilon \, dx.
\]

Similarly, by the Hölder inequality and the Young inequality, we obtain
\[
-\int_{\mathbb{R}^2} (\nabla c^\varepsilon \cdot \nabla) u^\varepsilon \cdot \nabla c^\varepsilon \, dx + \int_{\mathbb{R}^2} c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon) \Delta c^\varepsilon \, dx \leq \| \nabla c^\varepsilon \|_{L^4}^2 \| \nabla u^\varepsilon \|_{L^2} + \| c^\varepsilon \|_{L^\infty} \| n^\varepsilon \ast \rho^\varepsilon \|_{L^2} \| \Delta c^\varepsilon \|_{L^2}
\]
\[
\leq \| \nabla u^\varepsilon \|_{L^2} \| \nabla c^\varepsilon \|_{L^2} \| \Delta c^\varepsilon \|_{L^2} + \| c^\varepsilon \|_{L^\infty} \| n^\varepsilon \|_{L^2} \| \Delta c^\varepsilon \|_{L^2}
\]
\[
\leq C \| \nabla u^\varepsilon \|_{L^2}^2 \| \nabla c^\varepsilon \|_{L^2}^2 + C \| c^\varepsilon \|_{L^\infty}^2 \| n^\varepsilon \|_{L^2}^2 + \frac{1}{4} \| \Delta c^\varepsilon \|_{L^2}^2,
\]
from which we have
\[ \frac{1}{2} \frac{d}{dt} \| \nabla c^\varepsilon(t) \|_{L^2}^2 + \frac{3}{4} \| \Delta c^\varepsilon(t) \|_{L^2}^2 \leq C \| n^\varepsilon(t) \|_{L^2}^2 \| \nabla \Phi \|_{L^\infty}^2. \] (4.5)

Next, we shall consider the last one. According to the third equation of (3.1),
\[ \omega = \partial_1 u^\varepsilon - \partial_2 u^\varepsilon \] solves
\[ \partial_1 \omega^\varepsilon + (u^\varepsilon \cdot \nabla) \omega^\varepsilon - \Delta \omega^\varepsilon = -\partial_1 (n^\varepsilon \partial_2 \Phi) + \partial_2 (n^\varepsilon \partial_1 \Phi). \]

Then multiplying the above equality by \(\omega^\varepsilon\) and integrating with respect to space, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \omega^\varepsilon \|_{L^2}^2 + \| \nabla \omega^\varepsilon \|_{L^2}^2 = -\int_{\mathbb{R}^2} \partial_1 (n^\varepsilon \partial_2 \Phi) \omega^\varepsilon \, dx + \int_{\mathbb{R}^2} \partial_2 (n^\varepsilon \partial_1 \Phi) \omega^\varepsilon \, dx
\]
\[
= \int_{\mathbb{R}^2} n^\varepsilon \partial_2 \Phi \partial_1 \omega^\varepsilon \, dx - \int_{\mathbb{R}^2} n^\varepsilon \partial_1 \Phi \partial_2 \omega^\varepsilon \, dx.
\]

The Hölder inequality and the Young inequality enable us to conclude that
\[
\int_{\mathbb{R}^2} n^\varepsilon \partial_2 \Phi \partial_1 \omega^\varepsilon \, dx - \int_{\mathbb{R}^2} n^\varepsilon \partial_1 \Phi \partial_2 \omega^\varepsilon \, dx \leq \| n^\varepsilon \|_{L^2} \| \partial_2 \Phi \|_{L^\infty} \| \partial_1 \omega^\varepsilon \|_{L^2} + \| n^\varepsilon \|_{L^2} \| \partial_1 \Phi \|_{L^\infty} \| \partial_2 \omega^\varepsilon \|_{L^2}
\]
\[
\leq \| n^\varepsilon \|_{L^2} \| \nabla \Phi \|_{L^\infty} \| \nabla \omega^\varepsilon \|_{L^2} + \| n^\varepsilon \|_{L^2} \| \nabla \Phi \|_{L^\infty} \| \nabla \omega^\varepsilon \|_{L^2}
\]
\[
\leq C \| n^\varepsilon \|_{L^2}^2 \| \nabla \Phi \|_{L^\infty}^2 + C \| n^\varepsilon \|_{L^2}^2 \| \nabla \Phi \|_{L^\infty}^2 + \frac{1}{4} \| \nabla \omega^\varepsilon \|_{L^2}^2,
\]
from which we get
\[ \frac{1}{2} \frac{d}{dt} \| \omega^\varepsilon \|_{L^2}^2 + \frac{3}{4} \| \nabla \omega^\varepsilon \|_{L^2}^2 \leq C \| n^\varepsilon \|_{L^2}^2 \| \nabla \Phi \|_{L^\infty}^2. \] (4.6)
By the Grönwall inequality, we get
\[ \|\omega^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2} \int_0^t \|\nabla\omega^\varepsilon(t)\|_{L^2}^2 \, d\tau \leq \|\omega^\varepsilon_0\|_{L^2}^2 + C \int_0^t \|\nabla \Phi\|_{L^\infty} \|n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \]
\[ \leq \|\omega^\varepsilon_0\|_{L^2}^2 + Ce^{Ct}. \]
According to \( \|\nabla u^\varepsilon\|_{L^2}^2 = \|\nabla \cdot u^\varepsilon\|_{L^2}^2 + \|\omega^\varepsilon\|_{L^2}^2 \) and \( \nabla \cdot u^\varepsilon = 0 \), we have
\[ \|\nabla u^\varepsilon\|_{L^2}^2 = \|\omega^\varepsilon\|_{L^2}^2. \]
Together with (4.4)-(4.6) and the thirs estimate in Proposition 4.1, we conclude that
\[ \|n^\varepsilon(t)\|_{L^2}^2 + \|\nabla c^\varepsilon(t)\|_{L^2}^2 + 8a \int_0^t \|n^\varepsilon(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \left( \|n^\varepsilon(\tau)\|_{L^4}^4 + \|\Delta c^\varepsilon(\tau)\|_{L^2}^2 \right) \, d\tau \]
\[ \leq C \|(a_0, \nabla C_0, \omega_0)\|_{L^2} \int_0^t (1 + \|\nabla u^\varepsilon\|_{L^2}^2) \, ds \leq C \exp(\exp(Ct)). \]
We end the proof of Proposition 4.2. \( \square \)

5. **Proof of the Theorem 1.1.** In this section, we are devoted to show the global-in-time existence of weak solution of system (1.1). Recall from the regularized problem (3.1) that:
\[
\begin{aligned}
\partial_t n^\varepsilon + (u^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n^\varepsilon - \varepsilon \Delta n^\varepsilon &= -\nabla \cdot \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) + g(n^\varepsilon), \\
\partial_t c^\varepsilon + u^\varepsilon \cdot \nabla c^\varepsilon - \Delta c^\varepsilon &= -c^\varepsilon (n^\varepsilon \ast \rho^\varepsilon), \\
\partial_t u^\varepsilon \cdot \nabla u^\varepsilon - \nabla P^\varepsilon &= -\left( n^\varepsilon \nabla \Phi \right) \ast \rho^\varepsilon, \\
\nabla \cdot u^\varepsilon &= 0,
\end{aligned}
\]
where \( g(n^\varepsilon) = (1 + a)(n^\varepsilon)^2 - an^\varepsilon - (n^\varepsilon)^3 \). According to the property of the mollifier \( \rho^\varepsilon \) and \((n_0, c_0, u_0) \in X_0 \), we know that \((n_0^\varepsilon, c_0^\varepsilon, u_0^\varepsilon) \in X_0 \cap (H^\infty)^3 \) with \( H^\infty \triangleq \bigcap_{s \geq 0} H^s \). Moreover, by Proposition 3.1 we know that the system (3.1) admits at least one global smooth solution. More importantly, Proposition 4.1 and Proposition 4.2 guarantee the following bounds uniform in \( \varepsilon \):
\[ n^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) \cap L^3_{\text{loc}}(\mathbb{R}^+; L^3(\mathbb{R}^2)) \cap L^4_{\text{loc}}(\mathbb{R}^+; L^4(\mathbb{R}^2)), \]
\[ c^\varepsilon \in L^\infty(\mathbb{R}^+; L^\infty_{\text{loc}}(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)), \]
\[ u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)). \]

In order to show that \((n^\varepsilon, c^\varepsilon, u^\varepsilon) \) converges (up to extraction), a boundedness information over \((\partial_t n^\varepsilon, \partial_t c^\varepsilon, \partial_t u^\varepsilon) \) is needed. As for \( \partial_t n^\varepsilon \), because
\[ \partial_t n^\varepsilon(t) = \varepsilon \Delta n^\varepsilon - (u^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n^\varepsilon - \nabla \cdot \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right) + (1 + a)(n^\varepsilon)^2 - an^\varepsilon - (n^\varepsilon)^3, \]
we claim that \( \partial_t n^\varepsilon \) is bounded in \( L^2_{\text{loc}}(\mathbb{R}^+; H^{-2}) \). Indeed,
\[ \|\partial_t n^\varepsilon(t)\|_{L^2_{H^{-2}}} \leq \varepsilon \|\Delta n^\varepsilon\|_{L^2_{H^{-2}}} + \|(u^\varepsilon \ast \rho^\varepsilon) \cdot \nabla n^\varepsilon\|_{L^2_{H^{-2}}} + \|\nabla \cdot \left( n^\varepsilon \nabla (c^\varepsilon \ast \rho^\varepsilon) \right)\|_{L^2_{H^{-2}}} \]
\[ + (1 + a)(n^\varepsilon)^2\|_{L^2_{H^{-2}}} + a\|n^\varepsilon\|_{L^2_{H^{-2}}} + \|(n^\varepsilon)^3\|_{L^2_{H^{-2}}}. \]
By the Gagliardo-Nirenberg-Sobolev inequality, we have
\[ \|\nabla c^\varepsilon\|_{L^4_{L^4}} \leq \|\nabla c^\varepsilon\|_{L^\infty L^2} \|\Delta c^\varepsilon\|_{L^2_{L^2}}, \]
moreover, we have
\[ \|\nabla c^\varepsilon\|_{L^4_{L^4}}^2 \leq \|c^\varepsilon\|_{L^\infty L^1} \|c^\varepsilon\|_{L^2_{L^2}}. \]
Some simple calculations ensure us to obtain that (5.1) can be bounded by
\[
\|\partial_t n^\varepsilon(t)\|_{L^2_t H^{-2}} \leq \varepsilon \|\Delta n^\varepsilon\|_{L^2_t L^2} + \|u^\varepsilon\|_{L^2_t L^4}^2 + 3\|n^\varepsilon\|_{L^4_t L^4}^2 + \|c^\varepsilon\|_{L^\infty_t H^1} \|c^\varepsilon\|_{L^2_t H^2}^2 + (1 + a)\|n^\varepsilon\|_{L^2_t L^4}^2 + a\|n^\varepsilon\|_{L^2_t L^4} + \|n^\varepsilon\|_{L^2_t L^4}^4 \leq C.
\]
Here we have used Proposition 4.1 and Proposition 4.2.

Similarly, we claim that \(\partial_t c^\varepsilon\) is bounded in \(L^2_{\text{loc}}(\mathbb{R}^+; H^{-1})\). Indeed
\[
\|\partial_t c^\varepsilon(t)\|_{L^2_t H^{-1}} \leq \|\Delta c^\varepsilon\|_{L^2_t H^{-1}} + \|u^\varepsilon \cdot \nabla c^\varepsilon\|_{L^2_t H^{-1}} + \|\nabla c^\varepsilon(n^\varepsilon * \rho^\varepsilon)\|_{L^2_t H^{-1}} \leq \|c^\varepsilon\|_{L^2_t H^1} + \|u^\varepsilon c^\varepsilon\|_{L^2_t L^2} + \|\nabla c^\varepsilon(n^\varepsilon * \rho^\varepsilon)\|_{L^2_t L^2} \leq \|c^\varepsilon\|_{L^2_t H^1} + \|u^\varepsilon\|_{L^2_t L^2} + \|\nabla c^\varepsilon\|_{L^2_t L^2} + \|\nabla c^\varepsilon(n^\varepsilon * \rho^\varepsilon)\|_{L^2_t L^2} \leq C.
\]

By the same argument with \(\partial_t c^\varepsilon\), we can infer that \(\partial_t u^\varepsilon\) is bounded in \(L^2_{\text{loc}}(\mathbb{R}^+; H^{-1})\).

We know that \(L^2\) local compactly embeds in \(H^s\) and \(H^s\) continuously embed in \(H^{-1}\) with \(s \in (-1, 0)\). Thanks to Proposition 4.1 and Proposition 4.2, we know that
\[
n^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) \cap L^3_{\text{loc}}(\mathbb{R}^+; L^3(\mathbb{R}^2)) \cap L^4_{\text{loc}}(\mathbb{R}^+; L^4(\mathbb{R}^2)),
\]
\[
c^\varepsilon \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)),
\]
\[
u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)).
\]

By Arzelà-Ascoli theorem, we know that there exists some function \((n_p^\varepsilon, c_p^\varepsilon, u_p^\varepsilon)\) such that, up to extraction,
\[(\phi_p n^\varepsilon, \phi_p c^\varepsilon, \phi_p u^\varepsilon) \to (n_p, c_p, u_p) \quad \text{in} \quad \mathcal{C}([0, T]; H^{-1}(\mathbb{R}^2)) \times (\mathcal{C}([0, T]; L^2(\mathbb{R}^2)))^2.
\]
Using the Cantor diagonal process, we will find a subsequence (which we still denoted by \(\{(n^\varepsilon, c^\varepsilon, u^\varepsilon)_p\}_{p \in \mathbb{N}}\) such that for all \(p \in \mathbb{N}\),
\[(\phi_p n^\varepsilon, \phi_p c^\varepsilon, \phi_p u^\varepsilon) \to (n_p, c_p, u_p) \quad \text{in} \quad \mathcal{C}([0, T]; H^{-1}(\mathbb{R}^2)) \times (\mathcal{C}([0, T]; L^2(\mathbb{R}^2)))^2.
\]
As \(\phi_p \phi_{p+1} = \phi_p\), we have, in addition, \((n_p^\varepsilon, c_p^\varepsilon, u_p^\varepsilon) = (\phi_p n_{p+1}^\varepsilon, \phi_p c_{p+1}^\varepsilon, \phi_p u_{p+1}^\varepsilon)\).
From that, we can easily deduce that there exists some function \((n, c, u)\) such that for all \(\phi \in \mathcal{D}\),
\[(\phi n^\varepsilon, \phi c^\varepsilon, \phi u^\varepsilon) \to (\phi n, \phi c, \phi u) \quad \text{in} \quad \mathcal{C}([0, T]; H^{-1}(\mathbb{R}^2)) \times (\mathcal{C}([0, T]; L^2(\mathbb{R}^2)))^2.
\]
This obviously entails that \((n^\varepsilon, c^\varepsilon, u^\varepsilon)\) tends to \((n, c, u)\) in \(\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^2)\). The Fatou’s lemma ensures that
\[
n \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)) \cap L^3_{\text{loc}}(\mathbb{R}^+; L^3(\mathbb{R}^2)) \cap L^4_{\text{loc}}(\mathbb{R}^+; L^4(\mathbb{R}^2)),
\]
\[
c \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)),
\]
\[
u \in L^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)).
\]
This completes the proof of Theorem 1.1.

**Acknowledgments.** This work was supported by grants 11771423, 11871087 and 11771423 from the National Natural Science Foundation of China.
REFERENCES

[1] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer Berlin Heidelberg, 2011.

[2] M. Chae, K. Kang and J. Lee, Existence of smooth solutions to coupled Chemotaxis-fluid equations, *Discrete Continous Dynam. Systems-A*, 33 (2013), 2271–2297.

[3] R. J. Duan, A. Lorz and P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, *Comm. Partial Differential Equations*, 35 (2010), 1635–1673.

[4] M. D. Francesco, A. Lorz and P. Markowich, hemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior, *Discrete Continous Dynam. Systems-A*, 28 (2010), 1437–1453.

[5] M. Henry, D. Hilhorst and R. Schatzle, Convergence to a viscosity solution for an advection-reaction-diffusion equation arising from a chemotaxis-growth model, *Hiroshima Math. J.*, 29 (1999), 591–630.

[6] M. A. Herrero, E. Medina and J. J. L. Velázquez, Finite-time aggregation into a single point in a reaction-diffusion system, *Nonlinearity*, 10 (1997), 1739–1754.

[7] M.A.Herrero, J.J. L.Velázquez, A blow-up mechanism for a chemotaxis model, *Ann. Scuola Normale Superiore*, 24 (1997), 633–683.

[8] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399–415.

[9] E. F. Keller and L. A. Segel, A model for chemotaxis, *J. Theoret. Biol.*, 30 (1971), 225–234.

[10] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, *J. Differential Equations*, 258 (2015), 1158–1191.

[11] J. G. Liu and A. Lorz, A coupled chemotaxis-fluid model: Global existence, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 643–652.

[12] A. Lorz, A coupled Keller-Segel-Stokes model: Global existence for small initial data and blow-up delay, *Commun. Math. Sci.*, 10 (2012), 555–574.

[13] A. Lorz, Coupled chemotaxis fluid model, *Math. Models Methods Appl. Sci.*, 20 (2010), 987–1004.

[14] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[15] C. Miao, J. Wu and Z.Zhang, *Littlewood-Paley Theory and Applications to Fluid Dynamics Equations*, Monogr. Modern Pure Math. Science Press, Beijing, 42, 2012.

[16] M. Mimura and T. Tsujikawa, Aggregating pattern dynamics in a chemotaxis model including growth. *Physica A*, 230 (1996), 499–543.

[17] K. Osaki and A. Yagi, Finite dimensional attractors for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.*, 44 (2001), 441–469.

[18] K. J. Painter and T. Hillen, Spatio-temporal chaos in a chemotaxis model, *Phy. D*, 240 (2011), 363–375.

[19] Y. Tao and M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, *J. Differential Equations*, 252 (2012), 2520–2543.

[20] Y. Tao and M. Winkler, Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion, *Discrete Continous Dynam. Systems-A*, 32 (2012), 1901–1914.

[21] Y. Tao and M. Winkler, Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30 (2013), 157–178.

[22] J. I. Tello and M. Winkler, A chemotaxis system with logistic source, *Comm. Part. Differ. Eq.*, 32 (2007), 849–877.

[23] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessker, R. E. Goldstein and H. L. Swinney, Bacterial swimming and oxygen transport near constant lines, *Proc. Natl. Acad. Sci. USA*, 102 (2005), 2277–2285.

[24] G. Viglialoro, Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source, *J. Math. Anal. Appl.*, 439 (2016), 197–212.

[25] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations*, 35 (2010), 1516–1537.

[26] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, *J. Math. Anal. Appl.*, 384 (2011), 261–272.
[27] M. Winkler, Chemotaxis with logistic source: Very weak global solutions and their boundedness properties, \textit{J. Math. Anal. Appl.}, 348 (2008), 708–729.

[28] M. Winkler, Does a ‘volume-filling effect’ always prevent chemotactic collapse?, \textit{Math. Methods Appl. Sci.}, 33 (2010), 12–24.

[29] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, \textit{J. Math. Pures Appl.}, 100 (2013), 748–767.

[30] M. Winkler, Global weak solutions in a three-dimensional chemotaxis–Navier–Stokes system, \textit{Ann. Inst. H. Poincaré Anal. Non Linéaire}, 33 (2016), 1329–1352.

[31] M. Winkler, How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system? \textit{Trans. Amer. Math. Soc.}, 369 (2017), 3067–3125.

[32] M. Winkler, Stabilization in a two-dimensional chemotaxis-Navier-Stokes system, \textit{Arch. Ration. Mech. Anal.}, 211 (2014), 455–487.

[33] Q. Zhang, Local well-posedness for the chemotaxis-Navier-Stokes equations in Besov spaces, \textit{Nonlinear Anal. Real World Appl.}, 17 (2014), 89–100.

[34] Q. Zhang, On the inviscid limit of the three dimensional incompressible chemotaxis-Navier-Stokes equations, \textit{Nonlinear Anal. Real World Appl.}, 27 (2016), 70–79.

[35] Q. Zhang and X. Zheng, Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations, \textit{SIAM J. Math. Anal.}, 46 (2014), 3078–3105.

Received August 2018; 1st revision December 2018; 2nd revision December 2018.

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