LATTICE CALCULATION OF POINT-TO-POINT
HADRON CURRENT CORRELATION
FUNCTIONS IN THE QCD VACUUM*

M.-C. Chu(1), J. M. Grandy(2), S. Huang(3) and J. W. Negele(2)

(1) Kellogg Laboratory, California Institute of Technology, 106–38, Pasadena, California 91125 U.S.A.

(2) Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139 U.S.A.

(3) Department of Physics, FM–15, University of Washington, Seattle, Washington 98195 U.S.A.

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ABSTRACT

Point-to-point correlation functions of hadron currents in the QCD vacuum are calculated on a lattice and analyzed using dispersion relations, providing physical information down to small spatial separations. Qualitative agreement with phenomenological results is obtained in channels for which experimental data are available, and these correlation functions are shown to be useful in exploring approximations based on sum rules and interacting instantons.

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One of the major challenges of lattice QCD is to provide as much insight as possible and to address as wide a range of experimental observables as possible using the Euclidean observables which are amenable to lattice calculations. In the past, the primary emphasis has been on correlation functions at large Euclidean separations. For example, the asymptotic decay of correlation functions between widely separated hadronic currents is used to measure hadron masses,\(^1\) distantly separated hadronic sources and sinks are used to filter out hadronic ground states, from which ground state observables, wave functions, and form factors are obtained,\(^2\) and spatially well-separated hadronic sources are used at finite temperature to extract analogous screening masses and wave functions.\(^3\) In these large distance calculations, it has been useful to integrate one of the currents over space to project onto a specific momentum and non-local sources are usually employed to increase the overlap with the desired states.

This present work addresses complementary QCD physics at short and intermediate distances, as well as large separations, by calculating the point-to-point Euclidean correlation functions \( R(x) \equiv \langle 0 | T J(x) J(0) | 0 \rangle \) where \( J \) is one of the point hadron currents listed in Table I. Physically, these Euclidean correlation functions correspond to space-like-separated or equal-time correlation functions describing the virtual propagation of quarks or hadrons. They complement bound state hadron properties in the same way that nucleon-nucleon scattering phase shifts provide detailed information about the spatial distribution of spin-spin, spin-orbit and tensor forces complementary to that provided by properties of the deuteron. These correlation functions have been determined phenomenologically in some channels by using dispersion relations to analyze hadron production and \( \tau \)-decay experiments,\(^4\) and have been studied extensively using QCD sum rules and the interacting instanton approximation.\(^4\)–\(^7\) Hence, in this work, we report what we believe to be the first exploratory lattice QCD calculation of

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the behavior of point-to-point meson and baryon correlation functions, from short distances 
where the physics is perturbative to large distances where it is highly non-perturbative. We 
demonstrate promising agreement with experimentally measured results where available, show 
how to analyze the lattice measurements using dispersion relations, and indicate the potential 
more extensive calculations have to explore, test, and refine approximations.

The lattice calculations were performed on a $16^3 \times 24$ lattice in the quenched approxi-
mation at inverse coupling $6/g^2 = 5.7$, corresponding to a physical lattice spacing defined by 
the proton mass of approximately 0.168 fm. This inverse coupling constant is large enough 
to give a semi-quantitative approximation and has the significant advantage that light quark 
propagators are available for point sources. Hence, propagators with point sources calculated 
by Soni et al.\(^8\) for 16 configurations with five values of the quark mass were used to calculate 
the meson and baryon correlation functions listed in Table I. For the benefit of non-specialists, 
we convert the hopping parameters $\kappa = (0.154, 0.160, 0.164, 0.166, \text{ and } 0.168)$ and critical 
value $\kappa_c = 0.1692$ to the intuitively useful bare quark mass $m \equiv (1/2\kappa - 1/2\kappa_c) a^{-1}$ which 
takes on the values (351, 199, 110, 67, and 25) MeV. Extrapolation to the value of $\kappa$ at which 
the pion acquires its correct mass corresponds to extrapolation to $m = 8$ MeV. In order to 
calculate the ratio of interacting to free correlation functions, corresponding free lattice prop-
agators at a very small quark mass, $ma = 0.05$ were calculated exactly on a $(48)^4$ lattice, 
where the lattice volume was chosen large enough to eliminate finite volume effects for spatial 
separation less than 2 fm and the value of $ma$ was chosen small enough to produce deviations 
from the massless result less than one percent at 1 fm. Because the propagators had hard-
wall boundary conditions in the time direction, all correlation functions were calculated on 
the central time slice containing the source.
Several lattice artifacts had to be eliminated to obtain a physical approximation to the continuum correlation functions. Significant anisotropy is introduced in the rotationally invariant continuum correlation functions by the Cartesian lattice. Although at very short physical distances the anisotropies in the numerator and denominator cancel by asymptotic freedom and at a sufficiently large number of lattice spacings the granularity of the lattice becomes negligible, for the present lattice spacing there remains an intermediate region in which some correlation function ratios display substantial anisotropy. As discussed and justified in detail in a subsequent article, since the lattice points far away from the Cartesian directions agree with the continuum result in the non-interacting case and should be the most reliable in the interacting theory, the lattice correlation functions $R(x)$ in this work are taken from sites $\vec{x}$ within an angular cone surrounding the diagonal directions $\vec{d} = (n, n, n)$ such that $\hat{d} \cdot \hat{x} \geq 0.9$. Also, although in principle one would like to normalize the ratios at an infinitesimally small separation, we have had to normalize our results at the first non-zero diagonal separation $(1, 1, 1)$, corresponding to a physical separation of $\sim 0.29$ fm.

A second lattice artifact arises from the contributions of periodically repeated images of the physical sources on the finite lattice having periodic boundary conditions. As discussed in detail in Ref. [10], it is straightforward to perform a self-consistent subtraction of the image contributions, and we have applied and verified this image correction to all the correlation functions presented in this work.

Finally, because it is impractical to calculate quark propagators at a quark mass light enough to produce a physical pion, it is necessary to perform a sequence of calculations at a series of heavier quark masses and extrapolate to the pion limit. Although we do not know the proper functional form for the extrapolation in the chiral limit, the extrapolation is in fact
innocuous because we only have to extrapolate data at \( m_q = 351, 199, 110, 67, \) and 25 MeV down to 8 MeV. A worst case example will be shown below in Fig. 1. Thus, we believe the technical aspects of relating the present lattice calculations to the physical continuum theory are under reasonable control, and that our results provide a meaningful first comparison of quenched QCD with phenomenological and model results.

The correlation functions we considered are listed in Table I. For convenience, we have contracted the vector indices in the vector and axial channels. In addition, we inserted a factor of \( x_\mu\gamma^\mu \) in the baryonic correlation functions before taking the Dirac trace in order to project out the component which is stable in the massless quark limit. For a local field theory such as QCD, a two-point function is uniquely characterized by its absorptive part in momentum space (up to a polynomial) through the dispersion relation

\[
\int d^4x\, e^{iqx} \langle 0 | T J(x) J(0) | 0 \rangle = \left\{ \begin{array}{c} \frac{1}{3q^2} \\ -i\eta^\mu \gamma_\mu \\
\end{array} \right\} \left( \int ds \, \frac{f(s)}{s - q^2} + P(q^2) \right) + \ldots
\]

where the upper, middle and lower terms in brackets refer to scalar and pseudoscalar mesons, vector and axial vector mesons and baryon channels respectively and terms which vanish for the correlation functions in Table I have been omitted. Phenomenologically, we expect \( f(s) \) has two major contributions, a resonance piece and a continuum piece, parameterized as \( f(s) = \lambda^2 \delta(s - M^2) + f_p(s) \theta(s - s_0), \) where \( M \) is the resonance mass, \( \lambda \) is the coupling of the current to the resonance state and \( f_p(s) \) is the perturbative contribution. Since QCD is asymptotically free, the \( f_p(s) \)'s are well-approximated by the corresponding free correlation functions with massless quarks, also listed in Table I. This type of parameterization is widely used in QCD sum rule calculations.
An inverse Fourier transform of Eq. (1) defines the following phenomenological correlation functions in coordinate space as a function of $M$, $\lambda$ and $s_0$, where the polynomial $P(q^2)$ only contributes at the point $x = 0$ and can be ignored for finite $x$:

$$R\{m,b\}(x) = \lambda^2 M \left\{ \frac{x^{-1}}{3M^2x^{-1}} \right\} K \left\{ \frac{x}{M} \right\} + \int_{s_0}^{\infty} ds f_p(s) \left\{ \frac{\sqrt{s}x^{-1}}{3s^{3/2}x^{-1}} \right\} K \left\{ \frac{1}{2} \right\} \left( \sqrt{s}x \right)$$

(2)

Here, $m$ refers to the upper terms in brackets for scalar and pseudoscalar mesons and the middle terms in brackets for vector and axial vector mesons, and $b$ refers to the lower terms in brackets for baryon channels. In practice, we always normalize the phenomenological $R(x)$ by the corresponding massless results $R^m_0(x) \propto x^{-6}$ and $R^b_0(x) \propto x^{-9}$. Note that although we have not included lattice renormalization constants for non-conserved currents, they will not affect the shape of the normalized ratio of correlation functions, but only the value of the fitted parameters. Also, since the results below are sensitive to the presence of the continuum term in the fitting, it is clear that the lattice calculation is simultaneously providing short and long distance physics information.

In Fig. 1 we show the complete lattice data and the three-parameter fit in the pseudoscalar meson (pion) channel for each of five values of the bare quark mass. The result at the physical pion mass is obtained by binning the lattice data in two-lattice-unit bins and extrapolating the binned data with the results shown by the solid circles. The striking result in this channel is the extremely rapid rise in the correlation function ratio, necessitating a log plot, which arises from the strong attraction and corresponding light pion mass. The lattice result agrees qualitatively with Shuryak’s phenomenological estimate, denoted by the dot-dashed line, based on the value $\lambda_\pi = (480 \text{ MeV})^2$ and the fact that the peak is proportional to $\lambda_\pi/m_\pi^2 \sim f_\pi/m_q$ explains the particularly large quark mass dependence in this channel. Detailed treatment of the
extrapolation, error analysis of fitted parameters, and lattice renormalization corrections will be deferred to the longer paper, and we only show extrapolated results and phenomenological fits for all other channels, where the quark mass dependence is much weaker.

Figure 2a displays the vector meson (rho) channel result. As emphasized by Shuryak, the salient feature in this channel is the fact that although the free correlator falls four orders of magnitude between 0.3 and 1.5 fm, the ratio is nearly one over the whole range, and our lattice result is consistent with his phenomenological analysis of $e^+e^- \to \text{even number } \pi$’s, denoted by the dashed curve. The result in the axial meson ($A_1$) channel, shown in Fig. 2b, is qualitatively similar to the phenomenological analysis of $\tau \to 3\pi$ decay denoted by the dashed line, although finite lattice effects render it difficult to reproduce the rising tail due to mixing with the pion. The result for the scalar meson channel, for which the extrapolation was more problematic than any other, is shown in Fig. 2c. Note that in both the axial and scalar channels, Eq. (2) did not produce reasonable physical parameters when fit to the data, so the solid curves are smooth curves to guide the eye in these cases. For comparison, the predictions of the interacting instanton approximation for mesons using a Pauli–Villars cutoff $\Lambda_{PV} = 130$ MeV are shown by dotted lines, and in each case are in qualitative agreement with the lattice results. Comparison of calculations on the same lattice with “cooled” configurations which are in progress will be particularly instructive in this connection.

Since there are no phenomenological data in the baryon channels, in Fig. 3 we have compared nucleon and delta correlators with the sum rule calculations of Belyaev and Ioffe (dashed lines) and Farrar et al. (dot-dashed lines). Given the agreement with phenomenology in the meson channels and drastic differences in sum rule results for the delta, lattice correlation functions can play a useful role in testing and refining sum rule approximations.
In summary, we believe these exploratory calculations demonstrate the feasibility and utility of calculating vacuum correlation functions on a lattice and motivate definitive calculation on larger lattices with \( 6/g^2 \geq 6 \).

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FIGURE CAPTIONS

Fig. 1: Ratio of interacting to free correlation functions measured at five quark masses (open circles), binned data extrapolated to the physical pion mass (closed circles), three-parameter fits (solid lines) and the phenomenological results of Ref. [1] (dot-dashed line).

Fig. 2: Extrapolated ratio of meson correlation functions (closed circles) and fits (solid curves) as in Fig. 1. Dashed lines denote phenomenological results\textsuperscript{1} and dotted lines show the interacting instanton approximation.\textsuperscript{5}

Fig. 3: Extrapolated ratio of baryon correlation functions (closed circles) and fits (solid curves) as in Fig. 1. Dashed and dot-dashed lines denote sum rule results from Ref. [6] and [7], respectively.
| Channel | Current | Correlator | $f_\rho(s)$ |
|---------|---------|------------|------------|
| Vector  | $J_\mu = \bar{u}_\gamma \gamma_\mu d$ | $\langle 0 | T J_\mu(x) \bar{J}_\mu(0) | 0 \rangle$ | $\frac{1}{12\pi^2}$ |
| Axial   | $\bar{J}_\mu^5 = \bar{u}_\gamma \gamma_\mu \gamma_5 d$ | $\langle 0 | T \bar{J}_\mu^5(x) \bar{J}_\mu^5(0) | 0 \rangle$ | $\frac{1}{12\pi^2}$ |
| Pseudoscalar | $J^p = \bar{u}_\gamma d$ | $\langle 0 | T J^p(x) \bar{J}^p(0) | 0 \rangle$ | $\frac{3s}{8\pi^2}$ |
| Scalar  | $J^s = \bar{u}d$ | $\langle 0 | T J^s(x) \bar{J}^s(0) | 0 \rangle$ | $\frac{3s}{8\pi^2}$ |
| Nucleon | $J^N = \epsilon_{abc} [u^a C \gamma_\mu u^b] \gamma_\mu \gamma_5 d^c$ | $\frac{1}{4} \text{Tr} \left( \langle 0 | T J^N(x) \bar{J}^N(0) | 0 \rangle x_\nu \gamma_\nu \right)$ | $\frac{s^2}{64\pi^4}$ |
| Delta   | $J^\Delta_\mu = \epsilon_{abc} [u^a C \gamma_\mu u^b] u^c$ | $\frac{1}{4} \text{Tr} \left( \langle 0 | T J^\Delta_\mu(x) \bar{J}^\Delta_\mu(0) | 0 \rangle x_\nu \gamma_\nu \right)$ | $\frac{3s^2}{256\pi^4}$ |