Abstract. This paper aims at generalizing some geometric properties of Grassmannians of finite dimensional vector spaces to the case of Grassmannians of infinite dimensional ones, in particular for $\Gr(k((z)))$. It is shown that the Determinant Line Bundle generates its Picard Group and that the Plücker equations define it as closed subscheme of an infinite projective space. Finally, a characterization of finite dimensional projective spaces in Grassmannians allows us to offer an approach to the study of the automorphism group.

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1. Introduction

Infinite Grassmannians have recently emerged as a powerful tool to study certain moduli spaces as well as being fundamental objects in some algebraic approaches to Conformal Field Theories and String Theory. It seems reasonable that a better understanding of such spaces could throw some light on all these problems and the pursuit of this motivates the present work.

Before explaining how the paper is organized, we wish to remark that the method followed to reach the above-mentioned results is essentially

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to relate the infinite dimensional Grassmannian and a suitable increasing chain of finite dimensional Grassmannians contained in it (basically given in [14]).

Section §2 is introductory and should help us to fix notations and become familiarized with the main object of our study: infinite Grassmannian. Certain known facts ([1, 3, 12]) on infinite Grassmannians are addressed such as their scheme structure and the construction of global sections of (the dual of) the determinant bundle, $\text{Det}_V^*$. The rest of the section contains a deep study of the relationships of finite and infinite Grassmannians, in particular the restriction of sections of $\text{Det}_V^*$, as well as some auxiliary schemes such as the Grassmannian of the “dual” space and of a metric space.

In section §3 it is proved that the Picard group of (the index 0 connected component of) the infinite Grassmannian is $\mathbb{Z}$ and that the determinant bundle generates it. These results follow from study of the restriction homomorphisms (from the infinite Grassmannian to a finite one) induced on the group of Cartier divisors.

The first part of section §4 proves that the Plücker morphism is a closed immersion. In the second part, equations defining $\text{Gr}(k((z)))$ as a subscheme of $\mathbb{P}O^*$ are computed.

The last section, §5, begins with a characterization of finite dimensional projective spaces in (finite or infinite) Grassmannians (Theorem 5.8). This result is the cornerstone of Corollary 5.12 which recovers all the known results about the automorphism group of Grassmannians. In order to deal with the infinite dimensional case, some extra structure is added to $V$. It is now shown that the group of automorphisms preserving this structure is a group deeply related to the linear group.

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2. Backgrounds on Grassmannians

2.A. Infinite Grassmannian. In [13, 14] the infinite Grassmannian of $V$, a Hilbert space over a field $k$, and a certain decomposition of it, $V \simeq V^- \oplus V^+$, is defined as the set of subspaces $L \subseteq V$ such that the projection $L \to V^-$ has finite dimensional kernel and cokernel (our $+/-$ notation and that of Segal-Wilson have been exchanged).

This definition of the above infinite Grassmannian, however, can be slightly weakened ([13, 3]). Consider the following set:

$$\left\{ \text{subspaces } L \subset V \text{ such that } L \cap V^+ \text{ and } V/(L + V^+) \text{ are finite dimensional} \right\}$$

(2.1)
It then depends only on the choice of a subspace $V^+$ (instead of on a decomposition). But let us give an algebraic definition of it.

We shall fix a pair $(V, \mathcal{B})$ of a vector space and a family of neighbourhoods of $(0)$ such that:

1. $A, B \in \mathcal{B} \implies A + B, A \cap B \in \mathcal{B}$,
2. $A, B \in \mathcal{B} \implies \dim(A + B)/A \cap B < \infty$ (that is, $A$ is commensurable with $B$ and will be denoted by $A \sim B$),
3. the topology is separated: $\cap_{A \in \mathcal{B}} A = (0)$,
4. $V$ is complete: $\lim_{\to} A \in \mathcal{B} V/A$
5. every finite dimensional subspace of $V/B$ is a neighbourhood of $(0)$: $\lim_{\to} A \to V/B$ is surjective (for $B \in \mathcal{B}$).

The completion with respect to the topology induced by $\mathcal{B}$ will be denoted with $\hat{\cdot}$. If $T \to S$ is a morphism of schemes over $k$ and $U \subseteq V_S$ is a sub-$\mathcal{O}_S$-module, then $U_T$ will denote the $\mathcal{O}_T$-module $U \otimes_{\mathcal{O}_S} \mathcal{O}_T$ (completion with respect to the topology induced in $U$ by the restriction of the natural topology of $V_S$).

The important fact, is that the set (2.1) is actually the set of rational points of a reduced and separated scheme, $\text{Gr}^\bullet(V)$, over $\text{Spec}(k)$ (as was shown in [1] and in [3] §4.3). There it is proved that the affine schemes $F_A := \text{Hom}_k(L_A, A)$ (where $A \in \mathcal{B}$ and $L_A \oplus A \simeq V$) are an open covering of $\text{Gr}^\bullet(V)$. Note that $F_A$ represents the functor:

\[
S \mapsto \left\{ \begin{array}{l}
L \subseteq \hat{V}_S \text{ quasi-coherent} \\
\text{such that } L \oplus \hat{A}_S \simeq \hat{V}_S
\end{array} \right\}
\]

This idea is essentially the same as in the case of Grassmannians of finite dimensional vector spaces (see [4] I.9.7). But giving the functor of points of $\text{Gr}^\bullet(V)$ is by no means trivial, since one should define the set of $S$-valued points of $\text{Gr}^\bullet(V)$ for an arbitrary scheme $S$ ([4]).

From now on we will fix a subspace $V^+ \in \mathcal{B}$. This choice allows to introduce the index function as well as the determinant line bundle. Let $\mathcal{B}_0$ be the set of subspaces $A \in \mathcal{B}$ such that $\dim A/(A \cap V^+) = \dim V^+/V^+/(A \cap V^+)$. Note that the index function:

\[
\text{Gr}^\bullet(V) \xrightarrow{i} \mathbb{Z} \\
L \mapsto \dim_k(L \cap V^+) - \dim_k(V/L + V^+)
\]

is locally constant, and moreover $\text{Gr}^\bullet(V) = \prod \text{Gr}^n(V)$, where $\text{Gr}^n(V) := i^{-1}(n)$ is connected. For simplicity’s sake, $\text{Gr}^0(V)$ will be denoted by $\text{Gr}(V)$. 

Over the scheme $\text{Gr}(V)$ there is a canonical line bundle: the determinant bundle. In order to define it, one considers $L$, the universal object of $\text{Gr}(V)$. It is now not difficult to show that the complex $L \to \pi^*(V/V^+) \to O_{\text{Gr}(V)}$ (where $\pi: \text{Gr}(V) \to \text{Spec}(k)$) of $O_{\text{Gr}(V)}$-modules is a perfect complex ($\text{[1]}$). We know from ($\text{[1]}$) that its determinant, $\text{Det}_V$, exists.

Moreover, the complexes $C^\bullet_A \equiv L \to \pi^*(V/A)$ (for $A \in B_0$) are all canonically quasi-isomorphic, and their determinants are therefore isomorphic in a canonical way (since $\text{End}(\text{Det}_V) = H^0(\text{Gr}(V), O_{\text{Gr}(V)}) = k$, see ($\text{[1]}$)). Nevertheless, this allows us to construct global sections of $\text{Det}_V^\ast$, since the morphism $\pi_A$ induces a section $\text{det}(\pi_A) \in H^0(\text{Gr}(V), \text{Det}_V^\ast)$ (see ($\text{[1]}$)). Assume that it is possible to give isomorphisms $\phi_{AB} : \text{Det}(C^\bullet_A) \to \text{Det}(C^\bullet_B)$ in a compatible way; that is:

$$\phi_{AC} = \phi_{BC} \circ \phi_{AB} \quad (2.2)$$

We then have many sections:

$$\{ \phi_{AV}^*(\text{det}(\pi_A)) \mid A \in B_0 \} \subseteq H^0(\text{Gr}(V), \text{Det}_V^\ast)$$

that will enable us to introduce the Plücker morphism, since $\{ F_A \mid A \in B_0 \}$ is an open covering. If $\Omega \subseteq H^0(\text{Gr}(V), \text{Det}_V^\ast)$ is a subspace containing $\Omega_A$ for a subcovering of $\{ F_A \mid A \in B_0 \}$, there is a canonical surjection:

$$\Omega \otimes_k O_{\text{Gr}(V)} \to \text{Det}_V^\ast$$

that by the universal property of the projective space induces a morphism:

$$\text{Gr}(V) \to \mathbb{P}\Omega^*$$

which will be called the Plücker morphism. (Here and in the sequel $\mathbb{P} E^*$ will denote the scheme $\text{Proj}(S^*E)$, for a $k$-vector space $E$).

When studying the Grassmannian of $k((z))$ we assume that:

$$B := \{ \text{subspaces } A \subseteq V \text{ containing } z^n \cdot k[[z]] \text{ as a subspace of finite codimension (for } n \in \mathbb{Z}) \}$$

and $V^+ := k[[z]]$.

Let us now compute sections explicitly. Let us denote with $S$ the set of Maya diagrams of virtual cardinal zero; that is, the set of strictly increasing sequences of integers $S = \{ s_i \}_{i \geq 0}$ such that:

- there exists an integer $i_0$ such that $\{ i_0, i_0 + 1, \ldots \} \subseteq S$,
- $\#(S \cap \mathbb{Z}_{<0}) = \#(\mathbb{Z}_{\geq0} - S)$ (condition of virtual cardinal zero).
Let $e_i$ be $z^i \in V$, and let $A_S$ be $z$-adic completion of the subspace $<\{e_{s_i}\}_{i>0}>$ for a given Maya diagram $S$. Observe that for $A \in \mathcal{B}$ there exists a Maya diagram such that $A + V^+ \subseteq A_S$. Moreover, if $L \in F_A(k) \subset \text{Gr}(V)$ then there exists an $S' \in \mathcal{S}$ (contained in $S$) such that $L \in F_{A_S}(k)$; that is, the open subschemes $\{F_{A_S}|S \in \mathcal{S}\}$ cover $\text{Gr}(V)$.

Fix a Maya diagram $S = \{s_i\} \in \mathcal{S}$. Following [1] one obtains a canonical isomorphism:

$$\text{Det}^* C^*_{A_S} \otimes \wedge(A_S/A_S \cap V^+) \otimes \wedge(V^+/A_S \cap V^+)^* \xrightarrow{\sim} \text{Det}^*_V$$

(2.3)

(where $\wedge$ denotes the highest exterior power).

Since we have fixed a “dense” basis $\{e_i\}$ of $V$, we can give a generator of $\wedge(A_S/A_S \cap V^+) \otimes \wedge(V^+/A_S \cap V^+)^*$. Let $J = \{j_1, \ldots, j_n\} = (\mathbb{Z}_{<0} \cap S)$ and $K = \{k_1, \ldots, k_n\} = (\mathbb{Z}_{\geq 0} - S)$. Then:

$$< e_J \otimes e_K^* > = \wedge(A_S/A_S \cap V^+) \otimes \wedge(V^+/A_S \cap V^+)^*$$

(2.4)

where $e_j := e_{j_1} \wedge \cdots \wedge e_{j_n}$ and $e_K^* := e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*$ (where $e_i^*(e_j) = \delta_{ij}$). Moreover, from the choice of a basis of $V$ in order to obtain the isomorphisms (2.3) it follows that the compatibility condition (2.2) is satisfied. Summing up, if $s$ is a global section of $\text{Det}^* C^*_{A_S}$ then (by (2.3) and (2.4)) $s \otimes e_J \otimes e_K^*$ is a global section of $\text{Det}^*_V$.

Let us denote with $\Omega_S$ the section associated with $A_S$. In the sequel $\Omega$ will denote the vector space $<\{\Omega_S\}_{S \in \mathcal{S}}>$, and observe that the Plücker morphism $\text{Gr}(V) \xrightarrow{\pi} \mathbb{P}^n$ is well defined.

2.B. Finite Grassmannians. Let us now assume that $V$ is a finite dimensional $k$-vector space and that $\mathcal{B}$ is the set of subspaces of $V$. Choose a basis $\{e_1, \ldots, e_d\}$ of $V$ such that $V^+ = \langle e_1, \ldots, e_{d-r} \rangle$, and let $\{e_1^*, \ldots, e_d^*\}$ be its dual basis. Let $\mathcal{S}$ now be the set of strictly increasing sequences of $d - r$ integers $S = 0 < s_1 < \cdots < s_{d-r} \leq d$; and for $S \in \mathcal{S}$ define $A_S = \langle e_{s_1}, \ldots, e_{s_{d-r}} \rangle$.

Observe that $\{F_{A_S}|S \in \mathcal{S}\}$ is again a covering of $\text{Gr}(V)$ and that the rational points of $\text{Gr}(V)$ are precisely the $r$-dimensional subspaces.

What is $\Omega_S(L)$ now? (L being a rational point of $\text{Gr}(V)$, and $S \in \mathcal{S}$). Note that by the basic properties of $\text{Det}^* (\text{Det}_V)$, one has:

$$\text{Det}^* C^*_{A_S}|_L \xrightarrow{\sim} \wedge L^* \otimes \wedge(V/A_S)$$

and hence:

$$\Omega_S(L) = \text{det}(\pi_{A_S}|_L) \otimes e_J \otimes e_K^* =$$

$$= \pi_{A_S}(l_1) \wedge \cdots \wedge \pi_{A_S}(l_r) \otimes l_1^* \wedge \cdots \wedge l_r^* \otimes e_J \otimes e_K^*$$

(2.5)

where:
\begin{itemize}
\item \(\{l_1, \ldots, l_r\}\) is a basis of \(L\) and \(\{l_1^*, \ldots, l_r^*\}\) its dual (note that \((2.5)\) does not depend on the choice of the basis),
\item \(e_j = e_{j_1} \wedge \cdots \wedge e_{j_n}\) where \(\{e_{j_1}, \ldots, e_{j_n}\}\) generates \(A_S/A_S \cap V^+\); that is, \(\{j_1, \ldots, j_n\} = \{s_1, \ldots, s_{d-r}\} - \{1, 2, \ldots, d-r\}\),
\item \(e^n_k = e^n_{k_1} \wedge \cdots \wedge e^n_{k_n}\) where \(\{e^n_{k_1}, \ldots, e^n_{k_n}\}\) generates \((V^+/A_S \cap V^+)^\ast\); that is, \(\{k_1, \ldots, k_n\} = \{1, 2, \ldots, d-r\} - \{s_1, \ldots, s_{d-r}\}\).
\end{itemize}

Computing \((2.3)\) one has that:
\[
\Omega_S(L) = (e^n_{s_1} \wedge \cdots \wedge e^n_{s_r})(l_1^* \wedge \cdots \wedge l_r^*) \cdot e_1^* \wedge \cdots \wedge e_{d-r}^* \otimes l_1^* \wedge \cdots \wedge l_r^* \in (\text{Det}_V^\ast)|_L
\]
where \(\{s_1, \ldots, s_r\} := \{1, \ldots, d\} - \{s_1, \ldots, s_{d-r}\}\); that is, \(\{e^n_{s_1}, \ldots, e^n_{s_r}\}\) generates \((V/A_S)^\ast\) and \(\{l_1^*, \ldots, l_r^*\}\) is a basis of \(L\).

Using this calculations, one deduces a natural isomorphism:
\[
H^0(\text{Gr}(V), \text{Det}_V^\ast) \to \wedge^r V^\ast
\]
\[
\Omega_S \mapsto e^n_{s_1} \wedge \cdots \wedge e^n_{s_r}
\]
where \(\{s_1, \ldots, s_r\} = \{1, \ldots, d\} - \{s_1, \ldots, s_{d-r}\}\).

\[\text{2.C. Morphisms between Grassmannians.}\] Let us now relate both Grassmannians, the infinite one and the finite one. Let \((V, B, V^+)\) be as usual, and let \(N\) be a rational point of \(\text{Gr}^\ast(V)\) and \(M \subseteq V\) a subspace such that \(N \subset M\). If \(\text{Grass}(M/N)\) is the standard Grassmannian of a vector space (see \([7, \text{I}]\) we know from \([\text{III}]\) that the morphism:
\[
j : \text{Grass}(M/N) \to \text{Gr}^\ast(V)
\]
\[
L \mapsto \pi^{-1}(L)
\]
(where \(\pi : M \to M/N\) is a closed immersion. Moreover, if \(M \in \text{Gr}^\ast(V)\), there exists a canonical isomorphism:
\[
j^* \text{Det}_V \cong \wedge^{L^\prime} \otimes (M/N)^\ast \otimes (V^+ \cap M) \otimes (V/M + V^+)\)
\]
where \(L^\prime\) is the universal submodule of \(\text{Grass}(M/N)\). A very important consequence of this fact, is that the morphism:
\[
\text{Grass}^k(M/N) \xrightarrow{j} \text{Gr}(V) \xrightarrow{\text{Det}_V} \mathbb{P}H^0(\text{Gr}(V), \text{Det}_V^\ast)\)
\]
factors through the Plücker morphism of \(\text{Grass}^k(M/N)\) (for a certain \(k \in \mathbb{Z}\)).

Looking at restrictions, we note first that if \(A \in B\) satisfies \(V/M + A = (0)\) and \(N \cap A = (0)\), then \(j^{-1}(F_A) = F_{M \cap A}\). It follows that the restriction of the section \(\Omega_A\) is \(\Omega_{M \cap A}\). One now concludes that the restriction homomorphism of global sections:
\[
\Omega \to H^0(\text{Grass}^k(M/N), \text{Det}_{M/N}^\ast)
\]
is surjective.
In the special case of $\text{Gr}(k((z))))$, there exist Maya diagrams $S(M)$ and $S(N)$ (but not of virtual cardinal zero) such that $M \in F_{A_S(M)}$ and $N \in F_{A_S(N)}$, where we can assume that $S(M) \subseteq S(N)$, since $N \subseteq M$. The restriction of $\Omega_S$ for $S \in S$ is now:

\[
S^{*} \Omega_{S} = \begin{cases} 
\Omega_{\tilde{S}} & \text{if } S(M) \subseteq S \subseteq S(N) \\
0 & \text{otherwise}
\end{cases}
\]

The case of a morphism between two finite Grassmannians is quite similar. Let $\pi : V \to V'$ be a surjective morphism between two $k$-vector spaces of dimensions $d$ and $d'$ respectively, and let $\bar{d}$ be $\dim_k(\ker \pi) = d - d'$. There is a natural morphism:

\[
\text{Grass}^r(V') \to \text{Grass}^{r+\bar{d}}(V) \\
L \mapsto \pi^{-1}(L)
\]

which is known to preserve the determinant sheaf, and it therefore induces a restriction homomorphism between global sections:

\[
\wedge^{r+\bar{d}} V^* \to \wedge^r V'^*
\]

given by the inner contraction with $\wedge \ker \pi$. Choose a basis $\{e_1, \ldots, e_d\}$ of $V$ and $\{e'_{d+1}, \ldots, e_{d'}\}$ of $V'$, such that: $\pi(e_i) = 0$ for $i \leq \bar{d}$, and $\pi(e_i) = e_i'$ for $i > \bar{d}$. The homomorphism (2.7) is now written as:

\[
e_{j_1}^* \wedge \cdots \wedge e_{j_{r+\bar{d}}}^* \mapsto \begin{cases} 
0 & \text{if } j_i = i \text{ for } i \leq \bar{d}, \\
e_{j_{\bar{d}+1}}^* \wedge \cdots \wedge e_{j_{r+\bar{d}}}^* & \text{otherwise}
\end{cases}
\]

(where we assume that $1 \leq j_1 < j_2 < \cdots < j_{r+\bar{d}} \leq d$).

2.D. Related Grassmannians.

2.D.1. The Grassmannian of the dual space. Let $(V, B, V^+)$ be as usual. For a given submodule $U \subseteq \hat{V}_S$ ($S$ a $k$-scheme), we introduce the following notation:

\[
U^* := \text{Hom}_{\hat{S}}(U, \mathcal{O}_S) \\
U^c := \{ f \in U^* \text{ continuous } \}
\]

where the topology in $U$ is given by $\{ \hat{A}_S \cap U | A \in B \}$, and $\mathcal{O}_S$ has the discrete topology. And define:

\[
U^o := \{ f \in \hat{V}_S^* | f|_U \equiv 0 \} \\
U^o := \{ f \in \hat{V}_S^c | f|_U \equiv 0 \}
\]
In order to make explicit the meaning of the expression “Grassmannian of the dual (continuous) space” we consider in $V^c$ the family:

$$\mathcal{B}^c := \{ A^c \text{ where } A \in \mathcal{B} \}$$

An easy consequence of linear algebra is the following:

**Lemma 2.8.**

1. $V^c = \lim_{\to} A^c$ ($A \in \mathcal{B}$),
2. $V = \lim_{\to} A$ ($A \in \mathcal{B}$),
3. the topology induced by $\mathcal{B}^c$ is separated,
4. $V^c$ is complete.

It is now easy to check that the family $\mathcal{B}^c$ satisfies the same conditions as $\mathcal{B}$ (given at the beginning of 2.3), and hence one obtains the following:

**Theorem 2.9.** The Grassmannian of $(V^c, \mathcal{B}^c)$ exists if and only if that of $(V, \mathcal{B})$ does.

Here, we shall construct a canonical isomorphism between the Grassmannian of $(V, \mathcal{B})$ and that of $(V^c, \mathcal{B}^c)$. The expression of this isomorphism for the rational points will be that given by the incidence:

$$I : \text{Gr}^\bullet(V) \longrightarrow \text{Gr}^\bullet(V^c)$$

$$L \longrightarrow L^c \quad (2.10)$$

However, the existence of such a morphism is equivalent to show that $L^c$ is in fact a point of $\text{Gr}^\bullet(V^c)$. Note that $L^c$ is a quasi-coherent sheaf since it is a direct limit of quasi-coherent sheaves; namely:

$$L^c = \lim_{\to} A (V/A + L)^*$$

Since $\{ F_A \}_{A \in \mathcal{B}}$ is a covering of $\text{Gr}^\bullet(V)$, it is thus enough to prove that $L^c$ and $\dot{A}_{F_A}$ are in direct sum (over $F_A$), but this follows from the canonical isomorphism $\dot{V}_{F_A} \simeq \mathcal{L}_{F_A} \oplus \dot{A}_{F_A}$.

An easy calculation shows now the relationships between the determinant bundles and index functions of both Grassmannians. The following formulae hold:

$$I^* \text{Det}_{V^c} \simeq \text{Det}^*_V \quad I^*(i_{V^c}) = -i_V$$

($i$ being the index function).

**2.D.2. The case of a metric space.** Assume now that there is a given irreducible and hemisymmetric metric on $V$:

$$T_2 : V \times V \to k$$
A subspace \( L \subseteq V \) is called totally isotropic when \( L \subseteq L^\perp \). Observe that since \( T_2 \) is hemisymmetric the condition of maximal totally isotropic (m.t.i.) on a subspace \( L \) implies \( L = L^\perp \). The subspace \( V^+ \) will be assumed to be m.t.i.

It is not difficult to prove that under these hypothesis the polarity:

\[
iT_2 : V \rightarrow V^c
\]

is a bicontinuous isomorphism of vector spaces (with respect to the topologies induced by \( \mathcal{B} \) in \( V \) and by \( \mathcal{B}^c \) in \( V^c \)) and therefore it induces an isomorphism:

\[
\text{Gr}^\bullet (V^c) \xrightarrow{\sim} \text{Gr}^\bullet (V)
\]

The composition of the latter isomorphism and (2.10) is a natural involution of the Grassmannian, whose expression at the rational points is given by:

\[
R : \text{Gr}^\bullet (V) \rightarrow \text{Gr}^\bullet (V)
L \mapsto L^\perp
\]  

(2.11)

Trivial calculation shows that \( R^* \text{Det}_V \simeq \text{Det}_V \) and that the index of a point \( L \in \text{Gr}^\bullet (V) \) is exactly the opposite of the index of \( R(L) = L^\perp \).

3. Picard Group of \( \text{Gr}(V) \)

**Theorem 3.1.** Let \( k \) be an algebraically closed field. Assume that \( \dim(\text{Gr}(V)) \geq 1 \). Then, the Picard group of \( \text{Gr}(V) \) is isomorphic to \( \mathbb{Z} \) and the line bundle \( \text{Det}_V \) is a generator.

**Proof.** Let us first deal with the \( \dim(\text{Gr}(V)) = 1 \) case. Recall that \( F_A \rightarrow \text{Hom}(L, A) \) (where \( V \simeq L \oplus A \) and \( A \in \mathcal{B} \)) is an open subscheme of \( \text{Gr}(V) \) and that its dimension equals \( \dim_k L \cdot \dim_k A \). It follows that \( \dim_k L = \dim_k A = 1 \) and that \( \text{Gr}(V) \) is the projectivization of a 2-dimensional vector space; i.e. \( \text{Gr}(V) \) is the projective line, whose Picard group is \( \mathbb{Z} \), as is well known.

For the general case, recall that the Picard group of \( \text{Gr}(V) \) is canonically isomorphic to the class group of Cartier divisors, since \( \text{Gr}(V) \) is integral.

Fix \( A \in \mathcal{B}_0 \) and assume \( Z_A := \text{Gr}(V) - F_A \) (the locus where \( \Omega_A \) vanishes) to be irreducible. It then implies the exactness of the sequence:

\[
\mathbb{Z} \rightarrow \text{Pic}(\text{Gr}(V)) \rightarrow \text{Pic}(F_A)
1 \rightarrow \mathcal{O}_{\text{Gr}(V)}(-Z_A) = \text{Det}_V
\]

Observe that \( \text{Pic}(F_A) = 0 \), since \( F_A \) is the spectrum of a ring of polynomials (in finitely or infinitely many variables) which is factorial (see [2] cap. VII §3.5.). Finally, the line bundle \( \text{Det}_V \otimes n \) cannot be trivial
for \( n > 0 \), because its space of global sections has dimension greater than 1 and \( H^0(\text{Gr}(V), \mathcal{O}_{\text{Gr}(V)}) \) has dimension 1 ([1]).

The proof is therefore reduced to the following:

\[ \text{Lemma 3.2. Let } k \text{ be an algebraically closed field, and } \dim(\text{Gr}(V)) \geq 2. \text{ There exists a subspace } A \in B_0 \text{ of } V \text{ such that } Z_A \text{ is irreducible.} \]

\[ \text{Proof. If } \text{Gr}(V) \text{ is of finite type (that is, } V \text{ is finite dimensional) the proof is an easy consequence of the Bertini Theorem (see [8] II.8.18 and III.7.9.1), since } Z_A \text{ is precisely a hyperplane section of the Plücker morphism. Moreover, it implies that } Z_A \text{ is irreducible for } A \text{ generic.} \]

Assume now that \( V \) is not finite dimensional. We shall prove that if \( Z_A \) is reducible then its natural restriction to a (suitable) finite dimensional Grassmannian is also reducible. Bearing this in mind and since the restriction homomorphism of global sections is surjective ([2,3]), one concludes that this is not possible for generic \( A \), and the result follows.

Let \( Z_1, Z_2 \subset Z_A \) be two (different) irreducible components of \( Z_A \). If \( Z_1 \cap Z_2 \neq \emptyset \), take a point \( L_0 \in Z_1 \cap Z_2 \) and a subspace \( B \in B_0 \) such that \( L_0 \in F_B \). Let \( L_1, L_2 \) be points of \( Z_1 \cap F_B, Z_2 \cap F_B \) respectively, such that \( \dim_k(L_i/L_0 \cap L_1 \cap L_2) < \infty \) (for \( i = 1, 2 \)). Note that these three points can be assumed to be rational. From Subsection 2.C we know that a certain connected component, \( G \), of \( \text{Grass}((L_0 + L_1 + L_2)/(L_0 \cap L_1 \cap L_2)) \) is naturally mapped into \( \text{Gr}(V) \). Obviously, the points \( L_0, L_1 \) and \( L_2 \) lie in \( G \), and \( \emptyset \neq G \cap Z_i \neq G \) (\( i = 1, 2 \)) and \( G \cap Z_1 \neq G \cap Z_2 \). It follows that the divisor \( G \cap Z_A \subset G \) is a reducible hyperplane section of \( G \).

Let us finally prove the other case: \( Z_A \) is reducible and \( Z_1, Z_2 \subset Z_A \) are two irreducible components such that \( Z_1 \cap Z_2 = \emptyset \). Consider two rational points \( L_i \in Z_i \) (\( i = 1, 2 \)) such that \( \dim_k(L_i/L_1 \cap L_2) < \infty \), and let \( G \) be the connected component of \( \text{Grass}((L_1 + L_2)/(L_1 \cap L_2)) \) whose image lies in \( \text{Gr}(V) \). One then concludes using similar ideas as before.

\[ \square \]

4. Plücker equations

The Plücker morphism. We shall show that the Plücker morphism is a closed immersion. The proof is based on that of [7] I.9.8.4 for the finite dimensional case.

From [8] one canonically obtains a 1-dimensional subspace, \( < \Omega_A > (A \in B_0) \), of \( H^0(\text{Gr}(V), \text{Det}^*_V) \) by using the canonical isomorphism:

\[ \text{Det}^* C_A^* \otimes (\wedge A/A \cap V^+) \otimes (\wedge V^+/A \cap V^+)^* \xrightarrow{\sim} \text{Det}^*_V \quad (4.1) \]

and the canonical section of \( \text{Det}^* C_A^* \). Let \( \Omega \) be the subspace:

\[ \sum_{A \in B_0} < \Omega_A > \subseteq H^0(\text{Gr}(V), \text{Det}^*_V) \]
But let us offer another description of this. Observe that $\Omega$ is the image of the morphism of vector spaces:

$$\Psi : \bigoplus_{A \in B_0} (\Lambda A/A \cap V^+) \otimes (\Lambda V^+/A \cap V^+)^* \to H^0(\text{Gr}(V), \text{Det}_V^*)$$

defined by tensoring the $A$-component with $\text{det}(\pi_A)$.

**Lemma 4.2.** There exists a natural surjective morphism:

$$\lim_{\longrightarrow} \left( \bigoplus_{B \subseteq V^+} \Lambda^k V/V^+ \otimes \left( \Lambda^k V^+/B \right)^* \right)$$

such that $\Psi$ factors through $\Phi$ and an injection:

$$\lim_{\longrightarrow} \left( \bigoplus_{B \subseteq V^+} \Lambda^k V/V^+ \otimes \left( \Lambda^k V^+/B \right)^* \right) \hookrightarrow H^0(\text{Gr}(V), \text{Det}_V^*)$$

**Proof.** Let us fix $B \in B$ such that $B \subseteq V^+$. Note that the morphism:

$$\bigoplus_{A \in B_0 \atop A \cap V^+ = B} (\Lambda A/A \cap V^+) \otimes (\Lambda V^+/A \cap V^+)^* \to H^0(\text{Gr}(V), \text{Det}_V^*)$$

factors through a surjection onto $\Lambda^k V/V^+ \otimes (\Lambda^k V^+/B)^*$, where $k = \dim_k(V^+/B)$. Therefore, the linear map $\Psi$ factors through a surjection onto:

$$\bigoplus_{B \subseteq V^+} \left( \Lambda^k V/V^+ \otimes \left( \Lambda^k V^+/B \right)^* \right)$$

Fixing $B$ again, observe further that the induced morphism:

$$\bigoplus_{B \subseteq B', B' \subseteq V^+} \Lambda^k V/V^+ \otimes (\Lambda^k V^+/B')^* \to H^0(\text{Gr}(V), \text{Det}_V^*)$$

$(k = \dim_k(V^+/B))$ factors through a surjection onto:

$$\bigoplus_{k=0}^{\dim(V^+/B)} \Lambda^k V/V^+ \otimes \left( \Lambda^k V^+/B \right)^* \quad (4.3)$$

Note that the set $(4.3)$ is a direct system as $B$ varies. The statement now follows easily. \hfill \Box
\textbf{Theorem 4.4.} The Plücker morphism:
\[ \text{Gr}(V) \longrightarrow \mathbb{P} \Omega^* \]
is a closed immersion.

\textit{Proof.} Let \( U_A \) the affine open subscheme of \( \mathbb{P} \Omega^* \) where the \( A \)-coordinate has no zeroes. It is clear that \( p(F_A) \subseteq U_A \), and that it is enough to see that \( p|_{F_A} \) is a closed immersion.

For the sake of clarity, it will be assumed that \( A = V^+ \). Nevertheless, the general case presents no extra difficulty. By fixing sections of \( V \rightarrow V/V^+ \) and \( \Omega \rightarrow \Omega/\langle \Omega_+ \rangle \) one has identifications \( F\!V^+ \simeq \text{Hom}(V/V^+, V^+) \) and \( U\!V^+ \simeq \text{Hom}(\Omega/\langle \Omega_+ \rangle, \langle \Omega_+ \rangle) \). The restriction of the Plücker morphism to \( F\!V^+ \) is now a morphism:
\[ \text{Hom}(V/V^+, V^+) \rightarrow \text{Hom}(\Omega/\langle \Omega_+ \rangle, \langle \Omega_+ \rangle) \quad (4.5) \]

By Lemma 4.2 one has that:
\[ \Omega \simeq \lim_{\longrightarrow} \left( \bigoplus_{k=0}^{\dim(V^+/B)} \wedge^k V/V^+ \otimes \left( \wedge^k V^+/B \right)^* \right) \]
and that \( \langle \Omega_+ \rangle \) corresponds to \( B = V^+ \). Recalling that \( V^+ = \lim_{\longleftarrow} V^+/B \), it follows that (4.5) is the inverse limit of:
\[ \text{Hom}(V/V^+, V^+/B) \rightarrow \text{Hom} \left( \bigoplus_{k=1}^{\dim(V^+/B)} \wedge^k V/V^+ \otimes \left( \wedge^k V^+/B \right)^*, \langle \Omega_+ \rangle \right) \]
where \( B \in \mathcal{B} \) is such that \( B \subseteq V^+ \).

Observe that all these spaces are affine schemes, and it then suffices prove that given \( B \in \mathcal{B} \) such that \( B \subseteq V^+ \) the morphism:
\[ \text{Hom}(V/V^+, V^+/B) \rightarrow \prod_{k=1}^{\dim(V^+/B)} \text{Hom}(\wedge^k V/V^+ \otimes \left( \wedge^k V^+/B \right)^*, \langle \Omega_+ \rangle) \]
is a closed immersion. This, however, is trivial since it is the graph of a morphism. \( \Box \)

\textit{Restriction of sections. (Case of } V = k((z))\).

Take \( L_i := \{ e_j \}_{j < i} \) for an integer \( i \). Note that: \( L_i \in \text{Gr}^*(V) \). Denote by \( j_i \) the induced morphism \( \text{Grass}(L_i/L_{-i}) \rightarrow \text{Gr}^*(V) \) and let \( G_i \) be \( \text{Grass}(L_i/L_{-i}) \cap \text{Gr}(V) \). Observe that the index corresponding to the connected component \( G_i \) is \( i \). The \( \text{Gr}_0 \) of the Segal-Wilson paper ([14]) is precisely the union of all \( G_i \).
Recall that the following diagram is commutative:

\[
\begin{array}{ccc}
G_i & \xrightarrow{p_i} & \mathbb{P}(\Lambda^i L_i/L_{-i}) \\
\downarrow j_i & & \downarrow i_i \\
\text{Gr}(V) & \xrightarrow{p} & \mathbb{P}\Omega^* 
\end{array}
\] (4.6)

Denote by \(\iota_i^*\) the restriction morphism \(\Omega \to \Lambda^i(L_i/L_{-i})^*\) which is known to be surjective by (2.6). Let us denote by \(<X>\) the free \(k\)-module generated by a set \(X\). Let \(S_i\) be the set of strictly increasing sequences \(-i \leq s_0 < s_1 < \cdots < s_{i-1} \leq i - 1\). Note that \(\Lambda^i(L_i/L_{-i})^* \simeq <S_i>\) (see 2.8), \(\Omega \simeq <S>\), and that \(\iota_i^*\) is the morphism induced by the map:

\[
\{s_i\}_{i \geq 0} \mapsto \begin{cases} 
\{s_0, \ldots, s_{i-1}\} & \text{if } -i \leq s_1 \text{ and } s_j = j \text{ for all } j \geq i \\
0 & \text{otherwise}
\end{cases}
\]

This morphism has a natural section; namely:

\[
S_i \rightarrow S \\
\{s_0, \ldots, s_{i-1}\} \mapsto \{s_0, \ldots, s_{i-1}, i, i+1, \ldots\}
\]

Let \(\sigma_i\) be the induced morphism \(S^* <S_i> \hookrightarrow S^* <S>\) (where \(S^*E\) denotes the symmetric algebra generated by a vector space \(E\)).

In a similar way, and bearing in mind the restriction morphism of global sections of determinant bundles on finite Grassmannians, one constructs surjections \(S_j \rightarrow S_i\) (for \(j \geq i\)) and sections \(S_i \rightarrow S_j\). It follows easily that the family \(\{S_i\}_{i \geq 1}\) is an inverse system (with respect to the surjections) and a direct system (with respect to the sections) in a compatible way. Moreover, one has:

\[
S \simeq \varinjlim_i S_i \sim \varprojlim_i S_i
\]

from which one has that \(S^* <S> \sim \varinjlim\varprojlim S^* <S_i>\); and hence:

\[
I = \varprojlim \left( I \cap \sigma_i(S^* <S_i>) \right) \sim \varinjlim \left( \iota_i^* I \cap S^* <S_i> \right)
\] (4.7)

for every submodule \(I \subset S^* <S>\).

**Plücker equations.** (Case of \(V = k((z))\)).

We shall now give explicit equations for the infinite Grassmannian, \(\text{Gr}(k((z)))\), in an infinite dimensional projective space. Such equations will in fact be the infinite set of Plücker relations, and in this sense we prove that our definition of infinite Grassmannian is the same as that of
Further, in this setting the Segal-Wilson Grassmannian $Gr_0$ (see §2 of [14]) is to be interpreted as the set of points of the Sato Grassmannian with finitely many non-zero coordinates. In a certain sense, the scheme $Gr(k(\langle z \rangle))$ unifies both Grassmannians.

By the compatibility of the Plücker morphisms of (4.6), one has also the following commutative diagram of sheaves:

\[
\begin{array}{ccc}
(S^*\Omega) & \xrightarrow{p^*} & \mathcal{O}_{Gr(V)} \\
\downarrow i_i^* & & \downarrow j_i^* \\
(S^*(\Lambda^i L_i/L_{i-1}^*)) & \xrightarrow{p_i^*} & \left( \bigoplus_{d \geq 0} H^0(G_i, \text{Det}^* \otimes d) \right)
\end{array}
\]

where $\sim$ denotes the homogeneous localization sheaf. Let $I$ be the kernel of $p^*$ and $I_i$ that of $p_i^*$. Theorem 4.4 implies that these ideals are the equations defining $Gr(k(\langle z \rangle))$ and $G_i$, respectively.

From (4.7), and since $S^*(\Lambda^i L_i/L_{i-1}^*) \sim S^* < S_i >$, one has that $I = \varprojlim(i_i^* I_i)$. The same argument also implies that $I$ is generated by its degree 2 homogeneous component, since $I_i$ does (this is a classical result). That is, the ideal $I$ is generated by the union of the generators of $I_i$ for all $i$.

Recall that a family of generators of $I_i$ is given by:

\[ X_S \cdot X_{S'} = \sum_{l \geq 0} X_{S_l} \cdot X_{S'_l} \quad \text{for } S, S' \in S_i, k \geq 0 \]

where:

\[ S_l := \{s_0, \ldots, s_{k-1}, s_i, s_{k+1}, \ldots, s_{i-1}\} \]

\[ S'_l := \{s'_0, \ldots, s'_{l-1}, s_k, s'_{l+1}, \ldots, s'_{i-1}\} \]

for sequences:

\[ S \equiv -i \leq s_0 < s_1 < \cdots < s_{i-1} \leq i-1 \]

\[ S' \equiv -i \leq s'_0 < s'_1 < \cdots < s'_{i-1} \leq i-1 \]

and $X_S$ is defined as follows:

1. for a sequence $S \in S_i$, $X_S$ denotes the degree 1 element of $S^* < S_i >$ canonically associated to $S$;
2. for an arbitrary set of $i$ distinct numbers $\{s_0, \ldots, s_{i-1}\}$ (such that $-i \leq s_j \leq i-1$ for all $j$), let $\sigma$ be the permutation of $i$-letters such that it takes $S$ in increasing order, and then $X_S := \text{sig}(\sigma) \cdot X_{S^*}$;
3. $X_S := 0$ otherwise.

We have thus proved the following:
Theorem 4.8. The ideal $I$ is generated by its degree 2 homogeneous component. More precisely, it is generated by the following equations:

$$X_S \cdot X_{S'} - \sum_{l \geq 0} X_{S'_l} \cdot X_{S''_l} \quad \text{for } S, S' \in S, k \geq 0$$

where $X_{S}$ is defined as above.

Note that these sums are finite. These equations will be called the set of all Plücker relations.

Corollary 4.9. The closed subscheme of $\mathbb{P}\Omega^*$ defined by the Plücker relations coincide with $\text{Gr}(k((z)))$ (via the Plücker morphism).

5. Automorphisms of Grassmannians

Our goal now is to describe the relation between the linear group of a vector space and the automorphism group of its Grassmannian (in the infinite dimensional case). It is a classical result that both groups coincide in the finite dimensional case.

5.A. The Linear Group. Fix a pair $(V, B)$ as usual. Given a $k$-scheme $S$, $\text{Aut}_{O_S}(\hat{V}_S)$ will denote the automorphism group of $\hat{V}_S$ as an $O_S$-module.

Definition 5.1.

- A sub-$O_S$-module $A \subseteq \hat{V}_S$ belongs to $B$ if there exists $B \in B$ such that $\hat{B}_S \subseteq A$ and the quotient is free of finite type.
- An automorphism $g \in \text{Aut}_{O_S}(\hat{V}_S)$ is bicontinuous (w.r.t. $B$) if there exists $A \in B$ such that both $g(\hat{A}_S)$ and $g^{-1}(\hat{A}_S)$ belong to $B$.
- The linear group, $\text{Gl}(V)$, associated to $(V, B)$, is the contravariant functor over the category of $k$-schemes given by:

$$S \mapsto \text{Gl}(V)(S) = \{ g \in \text{Aut}_{O_S}(\hat{V}_S) \text{ such that } g \text{ is bicontinuous} \}$$

Π

Theorem 5.2. There exists a canonical action of $\text{Gl}(V)$ on (the functor of points of) $\text{Gr}^*(V)$:

$$\text{Gl}(V) \times \text{Gr}^*(V) \xrightarrow{\Delta} \text{Gr}^*(V) \quad (g, L) \mapsto g(L)$$

Moreover, this action preserves $\text{Det}^*_V$.

Proof. Fix $g \in \text{Gl}(V)(S)$. Note that it suffices check that $g(L)$ is a point of the Grassmannian for arbitrary $L \in F_A(S)$.

From $L \oplus \hat{A}_S \simeq \hat{V}_S$ it follows that $g(L) \oplus g(\hat{A}_S) \simeq \hat{V}_S$. Let $B \in B$ be such that $g(\hat{A}_S)/\hat{B}_S$ is free of finite type. It follows from [\[\]] that
$g(L) \cap \hat{B}_S = 0$ and $\hat{V}_S/(g(L) + \hat{B}_S)$ is locally free of finite type, and hence $g(L) \in \text{Gr}^*(V)(S)$, as desired.

Let us explain what “preserve” means. Let $f : S \to \text{Gr}(V)$ be an $S$-valued point of $\text{Gr}(V)$, $g$ an element of $\text{Gl}(V)(S)$, and $f_g : S \to \text{Gr}(V)$ the transform of $f$ under $g$. The claim is:

$$f^* \det_V \simeq f_g^* \det_V$$

(as line bundles over $S$). But this is a direct consequence of the properties of the determinant \([10]\) and the exactness of the sequence of complexes (written vertically):

$$
\begin{array}{c}
g(L) \oplus \hat{B}_S \longrightarrow g(L) \oplus g(\hat{V}_S^+) \longrightarrow g(\hat{V}_S^+)/\hat{B}_S \\
\downarrow \quad \downarrow \quad \downarrow \\
\hat{V}_S \quad \longrightarrow \quad \hat{V}_S \quad \longrightarrow \quad 0
\end{array}
$$

(where $B \in \mathcal{B}$ is such that $g(\hat{V}_S^+)/\hat{B}_S$ is free of finite type). □

5.B. Projective Spaces in $\text{Gr}(V)$. For the sake of notation, let us denote simply by $D$ the dual of the determinant sheaf $\det_V^*$ over $\text{Gr}(V)$, and by $D_L$ its stalk at a rational point $L \in \text{Gr}(V)$. Further, $\text{Gr}(V)$ will be thought of as a closed subscheme of $\mathbb{P} \Omega^*$ (Theorem \([14]\)) and we shall make no distinction between a point of $\text{Gr}(V)$ and the same point considered as a point of $\mathbb{P} \Omega^*$. Let us first study the structure of the projective lines contained in $\text{Gr}(V)$.

For a point $L \in \text{Gr}(V)$, $\mathcal{H}_L$ is defined by:

$$\mathcal{H}_L := \{ \text{hyperplanes } \mathbb{P} \Omega^* \text{ passing through } L \}$$

For a family $\{L_i\}$ of points of $\text{Gr}(V)$ (from now on all the points are assumed to be rational) we define:

$$\mathcal{H}_{\{L_i\}} := \bigcap_{i \in I} \mathcal{H}_{L_i}$$

Observe that (set-theoretically):

$$\mathcal{H}_{\{L_i\}} = \text{Projectivization of } \ker \left( \Omega \to \bigoplus_{i \in I} D_{L_i} \right)$$

Fix a family $\{L_i\}_{1 \leq i \leq n}$, and assume that there exists $A \in \mathcal{B}$ such that $(V/(A + L_i)) = (0)$. It then follows that dim$(A \cap L_i)$ is constant; call it $k$. Now, for $1 \leq j \leq n$ define:

$$\Lambda_j := \bigwedge^k (A \cap L_1 + \cdots + A \cap L_j)^*$$
Lemma 5.3. The restriction homomorphism:
\[ \Omega \to \bigoplus_{i=1}^{j} D_{L_i} \]
factors through \( \Lambda_j \). Further, \( \Omega \to \Lambda_j \) is surjective.

Proof. Recall that \( D \) is isomorphic to the determinant of the complex \( \mathcal{L} \to V/A \) for \( A \in \mathcal{B} \). Bearing in mind that there exists an isomorphism \( \wedge^k (A \cap L_i)^* \sim D_{L_i} \) (for \( A \) as above) and Lemma 4.2, the claim reduces to an exercise of linear algebra.

Theorem 5.4. Three rational points \( L_1, L_2, L_3 \) of \( \text{Gr}(V) \) lie in a line (as points of \( \mathbb{P} \Omega^* \)) if and only if \( L_1 \cap L_2 \subseteq L_3 \subseteq L_1 + L_2 \) and both inclusions have codimension 1.

Proof. Observe that \( \{ L_1, L_2, L_3 \} \) lie in a line if and only if:
\[ \mathcal{H}_{L_1, L_2, L_3} = \mathcal{H}_{L_1, L_2} \]  
\[ (5.5) \]
Consider the following commutative diagram:
\[ \begin{array}{ccc}
\Omega & \xrightarrow{p_3} & \Lambda_3 \\
\phantom{\Omega} \downarrow \cong & \phantom{\Omega} \downarrow \pi & \phantom{\Omega} \downarrow \pi_{12} \\
\Omega & \xrightarrow{p_2} & \Lambda_2 \\
& \phantom{\Omega} \downarrow \cong & \phantom{\Omega} \\
& \not\Omega & \not\Lambda_2 \\
\end{array} \]
\[ \bigoplus_{i=1}^{3} D_{L_i} \]
\[ \bigoplus_{i=1}^{2} D_{L_i} \]
and note that \( p_3, p_2 \) are surjective (by the above Lemma), \( p_2 \) is also surjective (since \( L_1 \) and \( L_2 \) are distinct), and finally \( \pi \) is surjective too. Now, condition \[ (5.5) \] is equivalent to:
\[ \ker(p_3) = \ker(p_2 \circ \pi) \]
But from the very definition of \( \Lambda_3 \) one easily sees that:
\[ \ker(p_3) = (\wedge^k (A \cap L_1) + \wedge^k (A \cap L_2) + \wedge^k (A \cap L_3))^o \subseteq \Lambda_3 \]
\[ \ker(p_2 \circ \pi) = (\wedge^k (A \cap L_1) + \wedge^k (A \cap L_2))^o \subseteq \Lambda_3 \]
and they are equal if and only if:
\[ \wedge^k (A \cap L_3) \subseteq \wedge^k (A \cap L_1) + \wedge^k (A \cap L_2) \]  
\[ (5.6) \]
It is easy to see that \[ (5.6) \] implies that \( A \cap L_3 \subseteq A \cap L_1 + A \cap L_2 \) and that it has codimension 1. Observe, however, that the whole argument also holds for every \( B \in \mathcal{B} \) such that \( A \subseteq B \), and hence:
\[ \lim_{A \subseteq B} B \cap L_3 \subseteq \lim_{A \subseteq B} B \cap L_1 + \lim_{A \subseteq B} B \cap L_2 \]
is of codimension 1. From this, one concludes that \( L_3 \subseteq L_1 + L_2 \) has codimension 1, as desired.
Let us now compute the codimension of the other inclusion. Observe again that the whole construction can be repeated with \( B \in \mathcal{B} \) such that \( B \cap L_i = (0) \) instead of \( A \), and:

\[
\bigwedge^k \left( V/(B \cap L_1 + \cdots + B \cap L_j) \right)
\]

instead of \( \Lambda_j \). In this case the analogous inclusion of (5.6) implies that:

\[
(B + L_1) \cap (B + L_2) \subseteq (B + L_3)
\]

is of codimension 1. However, we can assume that \( B \cap (L_1 + L_2) = (0) \), since \( L_3 \subseteq L_1 + L_2 \) has codimension 1. One then concludes that \( L_1 \cap L_2 \subseteq L_3 \) and that it is of codimension 1.

Conversely, given \( L_1, L_2, L_3 \) in the hypothesis, we have that \( L_1 \cap L_2, L_1 + L_2 \in \text{Gr}(V) \). This implies that:

\[
X := \text{Grass} \left( L_1 + L_2/L_1 \cap L_2 \right) \hookrightarrow \text{Gr}^\ast(V)
\]

The condition on the codimensions implies that \( L_i \) lies in \( X^0 := X \cap \text{Gr}(V) \) (for \( i = 1, 2, 3 \)) and that \( X^0 \) is a projective line.

**Corollary 5.7.** Let \( X \subset \text{Gr}(V) \) be a projective line, and let \( L_1, L_2 \) be two distinct points of it. It then holds that:

\[
X = \text{Grass}^1 \left( (L_1 + L_2)/(L_1 \cap L_2) \right)
\]

Moreover, \( L_1 + L_2, L_1 \cap L_2 \) do not depend on the choice of \( L_1, L_2 \).

The same methods can easily be generalized to prove the main result of this subsection and it is the following characterization of finite dimensional projective spaces in (finite or infinite) Grassmannians. Recall that \( n + 2 \) points of a projective space define a reference in it if and only if there is no \( n + 1 \) of them lying in a \((n-1)\)-dimensional projective space.

**Theorem 5.8.**

- Let \( \{L_i\}_{1 \leq i \leq n+2} \) be points of \( \text{Gr}(V) \) defining a \( n \)-dimensional reference (as points in \( \mathbb{P}\Omega^\ast \)) and assume:

\[
\dim_k (L_1 + \cdots + L_{n+2})/(L_1 \cap \cdots \cap L_{n+2}) = n + 1
\]

It then holds that:

\[
\text{Grass}^k (L_1 + \cdots + L_{n+2})/(L_1 \cap \cdots \cap L_{n+2})
\]

\((k being \dim_k L_i/(L_1 \cap \cdots \cap L_{n+2})) \) is a \( n \)-dimensional projective space contained in \( \text{Gr}(V) \).
• If $X = \mathbb{P}_n \subseteq \text{Gr}(V)$, then there exists a reference $\{L_i\}_{1 \leq i \leq n+2}$ in $X$ such that $\dim_k (L_1 + \cdots + L_{n+2})/(L_1 \cap \cdots \cap L_{n+2}) = n+1$ and:

$$X = \text{Grass}^k(L_1 + \cdots + L_{n+2})/(L_1 \cap \cdots \cap L_{n+2}) \subseteq \text{Gr}(V)$$

where $k = 1$ or $k = n$. (Note that $k$ does not depend on $\{L_i\}$ but on $X$ only).

5.C. Automorphisms of $\text{Gr}(V)$. In the sequel, Aut$_k$-scheme$(\text{Gr}(V))$ will be simply denote by Aut$(\text{Gr}(V))$ and similarly for $\text{Gr}^\bullet(\mathbb{V})$.

**Lemma 5.9.** Let $X \subseteq \text{Gr}(V)$ be a finite dimensional projective space and $\phi$ be an automorphism of $\text{Gr}(V)$. Then, $\phi(X)$ is a finite dimensional projective space.

**Proof.** An automorphism $\phi$ of $\text{Gr}(V)$ induces automorphism of the universal submodule, $\phi^* \mathcal{L} \overset{\sim}{\rightarrow} \mathcal{L}$, from which one deduces $\phi^* \text{Det}_V^* \overset{\sim}{\rightarrow} \text{Det}_V^*$, and hence a projectivity $\phi^*$ of $\mathbb{P}^0(\text{Gr}(V), \text{Det}_V^*)$. Now, the result follows easily.

**Lemma 5.10.** Fix $\phi \in \text{Aut}(\text{Gr}(V))$. There then exists an unique $\bar{\phi} \in \text{Aut}(\text{Gr}^\bullet(\mathbb{V}))$ with the following properties:

1. it is an extension of $\phi$ ($\bar{\phi}|_{\text{Gr}(V)} = \phi$),
2. $\bar{\phi}$ is an inclusion-preserving or inclusion-reversing automorphism.

**Proof.** Let us first define $\bar{\phi}(L)$ for $L \in \text{Gr}^k(\mathbb{V})$ ($k > 0$). Choose $L' \in \text{Gr}^{-1}(\mathbb{V})$ such that $L' \subset L$, and hence $\mathbb{P}(L/L') \subseteq \text{Gr}(V)$ is a finite dimensional projective space. Theorem 5.8 implies that there exist a finite family of points $\{M_i\}$ of $\mathbb{P}(L/L')$, such that:

$$\mathbb{P}(L/L') = \mathbb{P}\left( \bigcup M_i / \bigcap M_i \right)$$

Using this theorem again and Lemma 5.9, it follows that:

$$\phi\left( \mathbb{P}(L'/L) \right) = \text{Grass}^r\left( \bigcup \phi(M_i) / \bigcap \phi(M_i) \right)$$

where $r = 1$ or $r = -1$ (codimension 1). Observe that $r$ does not depend on $L$ but on the connected component of $\text{Gr}^\bullet(\mathbb{V})$ where $L$ lies, that is, $r$ only depend on $k$. Now take $L'' \in \text{Gr}^k(\mathbb{V})$ such that $L \subset L''$ (and $k < k'$), then $\mathbb{P}(L/L') \subseteq \mathbb{P}(L''/L) \subseteq \text{Gr}(V)$, and it follows that $r$ does not depend on $k > 0$. An analogous argument shows that it does not depend on $k < 0$ neither, that is, it only depends on $\phi$.

Continuing with the above notations, we define:

$$\bar{\phi}(L') := \begin{cases} \bigcup \phi(M_i) & \text{if } r = 1 \\ \bigcap \phi(M_i) & \text{if } r = -1 \end{cases}$$
(observe that this definition does not depend on the choice of \( \{ M_i \} \)). It is now clear that \( \hat{\phi} \) is the desired extension which is inclusion-preserving when \( r = 1 \) and inclusion-reversing when \( r = -1 \).

**Corollary 5.11.** Let \( V^+ \) be finite dimensional. Then the following conditions are equivalent:

1. there exists \( \phi \in \text{Aut}(\text{Gr}(V)) \) with an inclusion-reversing extension,
2. \( V \) is finite dimensional and \( \dim_k V = 2 \dim_k V^+ \),
3. there is an irreducible hemisymmetric metric in \( V \), such that \( V^+ \) is m.t.i.

**Proof.** Conditions 2 and 3 are clearly equivalent. The third implies the first since \( R \), the automorphism of \( \text{Gr}(V) \) constructed in (2.11), extends naturally to an inclusion-reversing automorphism of \( \text{Gr}^*(V) \).

Let us prove that 1 implies 2. Let \( \phi \in \text{Aut}(\text{Gr}(V)) \) and let \( \tilde{\phi} \in \text{Aut}(\text{Gr}^*(V)) \) be its inclusion-reversing extension. Observe that:

\[
\text{Gr}(V) = \text{Grass}^{-k}(V)
\]

where \( k = \dim_k(V^+) \). Since \( \tilde{\phi} \) reverses inclusions and leaves \( \text{Gr}(V) \) invariant, it follows that:

\[
\tilde{\phi} : \text{Grass}^{-k-r}(V) \rightarrow \text{Grass}^{-k+r}(V) \quad \forall r \in \mathbb{Z}
\]

Observe that for \( r = -k - 1 \), the scheme on the left hand side is a projective space. By Theorem 5.8, one has that \( \dim_k V = 2k \).

We now arrive at the classical results on the automorphism group of Grassmannians:

**Corollary 5.12.** The group \( \text{Aut}(\text{Gr}(V)) \) is canonically isomorphic to:

- \( \mathbb{P}GL(V) \) if \( \dim_k V < \infty \) and \( \dim_k V \neq 2 \dim_k V^+ \);
- \( \mathbb{P}GL(V) \times \mathbb{Z}/2 \) if \( \dim_k V < \infty \) and \( \dim_k V = 2 \dim_k V^+ \);
- \( \mathbb{P}GL(V) \) if \( \dim_k V = \infty \) and \( \dim_k V^+ < \infty \);
- \( \mathbb{P}GL(V^c) \) if \( \dim_k V = \infty \) and \( \dim_k(V/V^+) < \infty \).

**Remark 1.** These results have been already proved in three remarkable papers on the subject; namely [4], [5] and [9]. The two first statements were proved by Chow in [4], while the last two are algebraic versions of the results of Cowen ([5]) and Kaup ([9]), which are given for Hilbert spaces.

While Chow and Cowen’s techniques ([4], [5]) are based on the study of Schubert cycles and adjacency, our study deals with the structure of finite dimensional projective spaces contained in infinite Grassmannians. Nevertheless, such a study does imply “adjacency type” properties.
Recall that such a group isomorphism can not hold for arbitrary infinite Grassmannians (see §1 of [9] for the case of a Banach space).

**Remark 2.** Recall that a collineation of a vector space over a field \( k \) is a semi-linear transformation; that is, there is an automophism of \( k \) involved. If \( E \) is a finite dimensional vector space, then \( \text{Aut}(\mathbb{P}E^*) \) consists of semi-linear transformations. However, if we consider only \( \text{Spec}(k) \)-automorphisms then the automorphism of \( k \) has to be the identity, and hence the automorphism group of \( \mathbb{P}E^* \) as Spec\((k)\)-scheme is \( \mathbb{P}Gl(E) \). Bearing this in mind, Corollary 5.12 can easily be generalized to the case of Spec\((\mathbb{Z})\)-schemes.

Our goal now is to give a similar result for the infinite-dimensional case: \( V = k((z)) \). In view of the following Proposition we can restrict our study to automorphisms with an inclusion-preserving extension, with no loss of generality.

**Proposition 5.13.** The subset:

\[
\{\text{automorphisms with an inclusion-preserving extension}\}
\]

is a index 2 subgroup of \( \text{Aut}(\text{Gr}(V)) \). In the case of \( \text{Gr}(k((z))) \), the quotient is generated by the automorphism \( R \) of (2.11). (On \( k((z)) \) the metric: \( T_2(f, g) := \text{Res}_{z=0} f(z)g(-z)dz \) satisfies the hypothesis of [2.D.2]).

Recall that these two types of automorphisms correspond to collineations and correlations in the finite dimensional case. Nevertheless, study of the automorphism group is rather complicated, and we shall therefore add some extra structure to the pair \( (V, B) \). This consists of a separated linear topology on \( V \) with a basis \( \mathcal{C} \) such that:

- \( A \sim B \) for all \( A, B \in \mathcal{C} \),
- \( A \in \text{Gr}^\ast(V) \) for all \( A \in \mathcal{C} \).

In this setting \( \hat{\cdot} \) will denote the completion w.r.t. the topology induced by \( \mathcal{C} \).

Now, we define:

\[
\text{Gl}(\mathcal{C}) := \left\{ \phi \in \text{Gl}(V)(k) \text{ bicontinuous w.r.t. } \mathcal{C} \mid \phi(\text{Gr}(V)) = \text{Gr}(V) \right\}
\]

\[
\mathcal{A}(\mathcal{C}) := \left\{ \phi \in \text{Aut}(\text{Gr}(V)) \text{ bicontinuous w.r.t. } \mathcal{C}, \right\}
\]

where bicontinuous w.r.t. \( \mathcal{C} \) means that both \( \phi(A) \) and \( \phi^{-1}(A) \in \mathcal{C} \) contain an element of \( \mathcal{C} \) (for all \( A \in \mathcal{C} \)).
Since \( k^* \) is the kernel of the morphism \( \text{Gl}(V)(k) \to \text{Aut} (\text{Gr}^\bullet(V)) \), one has the exactness of the following sequence:

\[
0 \to k^* \to \text{Gl}(\mathfrak{C}) \to A(\mathfrak{C}) \quad (5.14)
\]

The following result is to be interpreted as the natural generalization of Corollary 5.12 to the infinite dimensional case.

**Theorem 5.15.** There exist injective morphisms of groups:

\[
P \text{Gl}(\mathfrak{C}) \hookrightarrow A(\mathfrak{C}) \hookrightarrow P \text{Gl}(\mathfrak{C})(\tilde{V})
\]

such that the composite maps a continuous automorphism of \( V \) to that canonically induced on \( \tilde{V} \).

**Proof.** Since \( \bar{\phi} \) is inclusion-preserving, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Gr}(V/L) & \hookrightarrow & \text{Gr}(V/L') \\
\downarrow & & \downarrow \\
\text{Gr}(V/\phi(L)) & \hookrightarrow & \text{Gr}(V/\phi(L'))
\end{array}
\quad (5.16)
\]

where \( L' \subseteq L \) and \( L, L' \in \mathfrak{C} \). Applying Corollary 5.12, one obtains a family of linear applications (up to a scalar):

\[
T_L : V/L \to V/\phi(L)
\]

Bearing in mind the inclusion \( L' \hookrightarrow L \) and (5.16), the commutativity of the following diagram becomes apparent:

\[
\begin{array}{ccc}
V/L & \xrightarrow{T_L} & V/\phi(L) \\
\downarrow & & \downarrow \\
V/L' & \xrightarrow{T_{L'}} & V/\phi(L')
\end{array}
\]

or, what amounts to the same, \( \{T_L\}_{L \in \mathfrak{C}} \) is a morphism of inverse systems. Taking inverse limits one obtains \( T_{\phi} : \tilde{V} \to \tilde{V} \), which is bicontinuous w.r.t. \( \mathfrak{C} \) by the very construction.

Let us show that if \( T_{\phi} = \lambda \cdot \text{Id} \ (\lambda \in k^*) \), then \( \phi = \text{Id} \). Note first that \( T_{\phi} = \text{Id} \) implies that \( \phi \) induces the identity on \( \text{Gr}(V/L) \) for all \( L \in \mathfrak{C} \), and hence the commutative diagram of affine schemes:

\[
\begin{array}{ccc}
F_A & \xrightarrow{\phi} & F_A \\
\downarrow j_L & & \downarrow j_L \\
\check{j}_L^{-1}(F_A) & \xrightarrow{\text{Id}} & \check{j}_L^{-1}(F_A)
\end{array}
\]

where \( j_L : \text{Gr}(V/L) \to \text{Gr}(V) \). Taking inverse limits in the inverse system of morphisms between the associated rings, it is easy to conclude
that $\phi|_{F_A} = Id|_{F_A}$ and hence $\phi = Id$, since there is no non-constant function on $F_A$ vanishing on $F_A \cap \text{Gr}(V/L)$ for all $L \in \mathcal{C}$.

**Example 3.** Assume that $\dim(V/V^+) = n$ and let $\mathcal{C}$ be the set of all rational points of $\text{Gr}(V)$. It then holds that $\mathbb{P}\text{Gl}(V) \xrightarrow{\sim} \mathbb{P}\text{Gl}(\mathcal{C})$ and $\mathcal{A}(\mathcal{C}) \xrightarrow{\sim} \text{Aut}(\text{Gr}(V))$. Summing up:

$$\mathbb{P}\text{Gl}(V) \xrightarrow{\sim} \text{Aut}(\text{Gr}(V))$$

**Example 4.** It is clear from Theorem 5.15 that, in general, it is not possible to represent elements of the linear groups above as matrices. Nevertheless, let us study the case of $V = k((z))$.

Let $\mathcal{C}$ be the set of subspaces $\{z^n \cdot k[z^{-1}]\}_{n \in \mathbb{Z}}$. One checks that $\tilde{V} = k[[z^{-1}, z]]$, and hence to $g \in \text{Gl}(\tilde{V})$ there is an associated $\mathbb{Z} \times \mathbb{Z}$-matrix $(g_{ij})_{i,j \in \mathbb{Z}}$ ($g_{ij}$ being the coefficient of $z^i$ in $g(z^j)$). But one recovers $g$ from the matrix if and only if $g$ is continuous w.r.t. the $z$-adic topology.

Moreover, the image of:

$$\text{Gl}(\mathcal{C}) \longrightarrow M_{\mathbb{Z} \times \mathbb{Z}}(k)$$

consists of matrices $(g_{ij})_{i,j \in \mathbb{Z}}$ such that there exists $n(j) : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying: $g_{ij} = 0$ for $i < n(j)$, and $g_{n(j)j} \in k^*$. Further, it is clear that the matrices of the group $\text{Gl}_{\text{res}}$ of Segal-Wilson belong to the image.

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