Instantons and SL(2,R) Symmetry in Type IIB Supergravity

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We discuss the relation between the dual formulations of Type IIB supergravity emphasizing the differences between Lorentz and Euclidean signature. We demonstrate how the SL(2,R) symmetry of the usual action is manifested in the solution of the equations of motion with Euclidean signature for the dual theory.

I. INSTANTONS AND SL(2,R) SYMMETRY IN TYPE IIB SUGRA

It is well-known that the Lagrangian for Type IIB SUGRA possesses an SL(2,R) invariance.[1, 2] This symmetry is however only evident as a local symmetry when the RR potentials are chosen to be counterparts of the NS fields, the dilation $\phi$ and the antisymmetric tensor $B_{\mu\nu}$. By this, we mean that the RR potentials must be chosen to be the scalar $a$ and the two-form $(C_{\mu\nu})$. If we denote an element of SL(2,R) as

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

(1)

then $(B_{\mu\nu}, C_{\mu\nu})$ transform linearly as an SL(2,R) doublet, while $\tau = a + i \exp(-\phi)$ transforms non-linearly via the modular transformation

$$\tau \rightarrow \frac{p\tau + q}{r\tau + s}.$$  

(2)

Type IIB SUGRA possesses a $D = -1$ brane, an instanton.[3, 4] This is unique among D-branes in several respects — all other D-branes are static solutions of the classical fields equations,[5] so they take precisely the same forms in a space-time with Lorentzian (or pseudo-Riemannian) signature or, after Wick rotation, in a space-time with Euclidean (or Riemannian) signature. In contrast, the instanton is a finite action solution of the Euclidean field equations that depends non-trivially on all coordinates, including the Euclidean time. Although interpreted as a tunneling amplitude between states in the Hilbert space that is built up in Lorentzian space-time, it only exists as a solution of the field equations with Euclidean signature.

Under these circumstances, it is not so clear that the duality characteristic of other D-branes applies to the $D = -1$ brane or, if it does apply, how it is to be formulated. In particular, how the SL(2,R) symmetry can be seen in the dual, magnetic picture is not at all apparent. In a previous paper discussing the instanton solution,[4] we have discussed some of the relations between Lorentz signature (LS) and Euclidean signature (ES). In this note, we will concentrate on the role of the SL(2,R) symmetry.

As mentioned at the outset, the SL(2,R) symmetry is manifested in the formulation of Type IIB SUGRA in which the RR potentials have the same tensor structure as their NS counterparts. Since we are exclusively interested in the instanton, we will suppress the two-form fields for simplicity. The Lagrangian density in the Einstein frame is then simply

$$L_0 = -R + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}e^{2\phi}(\nabla a)^2$$

(3)

The last two terms, the “matter” Lagrangian, may also be written as

$$L_{0m} = \frac{1}{2} \frac{\nabla \mu \tau \nabla \nu \tau^*}{t_2^2},$$

(4)
where $\tau \equiv \tau_1 + i\tau_2 \equiv a + i e^{-\phi}$. From the point of view of the instanton, this is the “electric” picture, in which the instanton would be seen as an “elementary” source coupled to the RR field $a$.\(^1\)

To realize the instanton in its usual form as an extended solution of the source-free field equations requires the dual, or “magnetic,” formulation in which the dynamical variable $a$ is exchanged for an eight-form potential $C_8$, with Lagrangian

$$\mathcal{L}_8 = -R + \frac{1}{8}(\nabla \phi)^2 + \frac{1}{336} e^{-2\phi} F_9^2 \quad (5)$$

The nonlocal relationship between the fields $a$ and $C_8$ is implicit in the relation between their associated field strengths

$$F_9 = e^{2\phi} * F_1 \quad (6)$$

where $F_1 \equiv da$, $F_9 \equiv dC_8$, and the * denotes the usual Hodge dual. This relation is metric dependent and assumes Lorentzian signature. In form notation, the “matter” action above may be expressed as

$$\mathcal{L}_{0m} = \frac{1}{2} d\phi \wedge * d\phi + \frac{1}{2} e^{2\phi} da \wedge * da \quad (7)$$

The corresponding dual action is

$$\mathcal{L}_{8m} = \frac{1}{2} d\phi \wedge * d\phi + \frac{1}{2} e^{-2\phi} dC_8 \wedge * dC_8 \quad (8)$$

The formal substitution in eq. (6), $dC_8 \rightarrow e^{2\phi} da$ does not map the two actions into each other, owing to the fact that $**F_p = (-)^{p+s} F_p$, where $s$ is the signature of the metric.\(^2\) As a result

$$e^{-2\phi} F_9 \wedge * F_9 = (-)^s e^{2\phi} F_1 \wedge * F_1. \quad (9)$$

so that, for Lorentzian signature, the substitution results in a wrong sign kinetic energy. However, the equations of motion (EOM) determined from these two actions are mapped into each other (again, only for Lorentzian signature). Duality is a property of the EOM, not of the action.

To be precise, the EOM stemming from eq. (7) are

$$\nabla_\mu (e^{2\phi} \nabla^\mu a) = 0$$

$$\nabla^2 \phi - e^{2\phi} (\nabla a)^2 = 0 \quad (10)$$

The EOM in the $C_8$ formulation, eq. (8) are

$$\nabla_\mu (e^{-2\phi} F_9^\mu_{\mu_1...\mu_8}) = 0$$

$$\nabla^2 \phi + e^{-2\phi} F_9^2 / 9! = 0 \quad (11)$$

These EOM are formally valid for either LS or ES. For LS, under the substitution eq. (6), the latter equations become the former.\(^3\)

If, on the other hand, we consider these same two Lagrangians for Euclidean signature, then they are mapped into one another under the formal substitution eq. (6), but the EOM are not. To see this explicitly, consider the solution of eq. (8). In the absence of any seven-branes, there are no source terms, so the first equation implies

$$e^{-2\phi} F_9 = db \quad (12)$$

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\(^1\) The notion of an “elementary” instanton is rather radical, but in this case, the source is to be interpreted as a boundary condition.\(^4\)

\(^2\) Here, we have assumed even space-time dimension since we are primarily interested in $D = 10$.

\(^3\) Because the metric in the Einstein frame is not transformed by SL(2,R), we have suppressed Einstein’s equations here. The instanton solution of course satisfies all the EOM.\(^4\)
for some scalar $b$, and the second equation then becomes
\[ \nabla^2 \phi + e^{2\phi}(\nabla b)^2 = 0. \] (13)

The Bianchi identity $dF_0 = 0$ then implies
\[ \nabla_\mu (e^{2\phi} \nabla^\mu b) = 0. \] (14)

If we compare eqs. (13) and (14) with eq. (10), we see that the EOM for ES for the scalar $b$ is the same as if we replaced $a$ by $ib$. For this reason, it is sometimes stated that the pseudoscalar $a \to ia$ under a Wick rotation.[7] This mnemonic is misleading, because this rule may only be used to obtain the form of the Euclidean EOM and not for the value of the action nor for variations of the action other than first order (e.g., to examine whether the solution is a local minimum in field space.) These EOM for ES are as if the action were
\[ S = \frac{1}{2} \int d^{10}x \sqrt{g} \left[ (\nabla \phi)^2 - e^{2\phi}(\nabla b)^2 \right] \] (15)

However, the value of the matter action is correctly given by
\[ S_m = \frac{1}{2} \int d^{10}x \sqrt{g} \left[ (\nabla \phi)^2 + e^{2\phi}(\nabla b)^2 \right] \geq 0. \] (16)

Nor are variational derivatives beyond the first correctly generated by eq. (15). For these reasons, we will refer to eq. (15) as the pseudo-action.

Given the clash between the EOM for Euclidean Signature in the dual formulations, it is not so clear that the instanton can be represented as an “elementary” excitation for eqs. (3) and (4). In particular, the SL(2,R) symmetry of Eq. (2) is not obviously a Noether symmetry of the pseudoaction eq. (15). Of course, any given instanton, like any other D-brane, will break the symmetry owing to its collective coordinates such as its origin. Whether different instantons solutions are related by an SL(2,R) transformation is not at all clear.

The symmetry of the solutions of the magnetic EOM’s are the same as the symmetries of the pseudo-action eq. (15). Even though this is not the true action, it suffices to examine whether this action possesses an SL(2,R) symmetry. Moreover, since conserved currents only depend on the EOM, the symmetries of the pseudo-action may be used to determine them via the usual Noether procedure. In order to see that the pseudo-action does in fact possess an SL(2,R) symmetry, let us observe the following little lemma: The expression
\[ -\frac{2\nabla_\mu W \nabla^\mu Z}{(W - Z)^2} \] (17)

is SL(2,R) invariant if $W$ and $Z$ are any two (real or complex) scalar fields that transform by the modular transformation eq. (2). It can easily be shown by explicit calculation that $\nabla W \to \frac{\nabla W}{\tau W + \bar{s}}$ and similarly for $Z$, where $(W - Z) \to \frac{W - Z}{\tau W + \bar{s}}$. In the case of the Lagrangian eq. (3), SL(2,R) is realized for $W = \tau$, $Z = \tau^*$. For the pseudo-action eq. (15), SL(2,R) is realized by taking $W = i(b + e^{-\phi})$, $Z = i(b - e^{-\phi})$. (Requiring $W$ and $Z$ to transform by eq. (2), one may deduce the transformation properties of $b$ and $e^{-\phi}$.) In other words, SL(2,R) symmetry is in a sense preserved is by the replacement of $a \to ib$, provided one makes the replacement in $\tau$ and $\tau^*$ independently. This may be seen as an automorphism of the original SL(2,R) which may be obtained by the formal replacement $q \to iq, r \to -ir$, with $ps - qr = 1$ as before.

\[ \text{Because } p, q, r, s \text{ are real, } \tau \text{ and } \tau^* \text{ transform the same way.} \]
For completeness, let us record the three SL(2,R) currents for each form. In the original form eq. (3), we may choose these as

\[ J_\mu = e^{2\phi} \nabla_\mu a \] (18)

\[ K_\mu = \nabla_\mu \phi - a J_\mu \] (19)

\[ L_\mu = (a^2 + e^{-2\phi}) J_\mu + 2a K_\mu \] (20)

The motivation for the particular choices here is that \( J_\mu \) is associated with the shift \( a \to a + c \), \( K_\mu \) with the scaling \( e^\phi \to e^\nu e^\phi \), \( a \to e^{-\nu} a \). The third is inherently non-linear. We may express these in a more familiar language, using the isomorphism of the SL(2,R) algebra with the three-dimensional Lorentz group SO(2,1):

\[ [J, K_1] = iK_2, \ [J, K_2] = -iK_1, \ [K_1, K_2] = -iJ. \] (21)

Then the generators associated with each current transform as follows:

\[ J_\mu \sim \frac{1}{2}(K_2 + J) \] (22)

\[ K_\mu \sim K_1 \] (23)

\[ L_\mu \sim \frac{1}{2}(K_2 - J) \] (24)

Correspondingly, for the solution of the dual theory, we have the conserved currents

\[ J_\mu = e^{2\phi} \nabla_\mu b \] (25)

\[ K_\mu = \nabla_\mu \phi + b J_\mu \] (26)

\[ L_\mu = (e^{-2\phi} - b^2) J_\mu + 2b K_\mu. \] (27)

These of course are only conserved for the source-free EOM in either case.

All the instanton solutions obtained in Ref. [4] are related to each other by SL(2,R). Thus, any change in the three integration constants discussed there may be interpreted as an SL(2,R) transformation. These relations and currents are useful for understanding the role of the instantons and the properties of the ground state. However, inasmuch as variations higher than the first are not correctly obtained from the pseudoaction, it remains for future work to determine the implications of this symmetry for transition amplitudes mediated by instantons.

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