Fredholm’s Minors of Arbitrary Order: Their Representations as a Determinant of Resolvents and in Terms of Free Fermions and an Explicit Formula for Their Functional Derivative

Joshua Feinberg*

Department of Physics, Oramim-University of Haifa, Tivon 36006, Israel**

and

Department of Physics, Technion - Israel Institute of Technology, Haifa 32000 Israel

Abstract

We study the Fredholm minors associated with a Fredholm equation of the second type. We present a couple of new linear recursion relations involving the $n$th and $n-1$th minors, whose solution is a representation of the $n$th minor as an $n \times n$ determinant of resolvents. The latter is given a simple interpretation in terms of a path integral over non-interacting fermions. We also provide an explicit formula for the functional derivative of a Fredholm minor of order $n$ with respect to the kernel. Our formula is a linear combination of the $n$th and the $n \pm 1$th minors.

Key Words: Fredholm theory, Fredholm determinant, linear integral equations, non-interacting fermions.

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*e-mail: joshua@physics.technion.ac.il

**permanent address
1 Introduction

The ubiquity of linear integral equations, and in particular of Fredholm equations (FE) [1, 2, 3], in mathematical physics, and more broadly in analysis, cannot be overstated. Thus, new results in this classical field, such as the results presented in this paper (Eqs.(2.11) and (4.9)), should be of some interest.

To get oriented, let us briefly recall the basic definitions and facts of Fredholm theory, relevant to our discussion. (We have adopted throughout this paper the conventions and notations of Chapter II of [3].) Thus, consider a Fredholm integral equation of the second type in the unknown function $\phi(x)$,

$$\phi(x) = f(x) + \lambda \int_{\Omega} N(x,y) \phi(y) \, dy,$$  
(1.1)

with kernel $N(x,y)$ and given function $f(x)$. For simplicity, we shall take $N(x,y)$ and $f(x)$ as real functions. The generalization to the complex case is straightforward. The complex variable $\lambda$ is the spectral parameter of the equation, and $\Omega$ is the domain on which the equation is defined. To be concrete, we shall take $\Omega$ as a compact domain of $N$-dimensional Euclidean space of volume $V$

$$\int_{\Omega} dx = V.$$  
(1.2)

We further assume that $N(x,y)$ is bounded on $\Omega$

$$|N(x,y)| \leq M,$$  
(1.3)

and that it is integrable in $\Omega$ with respect to both $x$ and $y$. The given function $f(x)$ is assumed integrable as well.

It is also useful to introduce the operator $\hat{N}$ and the vectors $|\phi\rangle$ and $|f\rangle$, which correspond to the kernel $N(x,y)$ and functions $\phi(x)$ and $f(x)$. Thus, in obvious notations,

$$N(x,y) = \langle x|\hat{N}|y\rangle, \quad f(x) = \langle x|f\rangle, \quad \text{and} \quad \phi(x) = \langle x|\phi\rangle.$$  
(1.4)
In terms of (1.4), we can write Fredholm’s equation (1.1) as

\[(1 - \lambda \hat{N}) |\phi\rangle = |f\rangle.\]  

(1.5)

Next, we define the $n \times n$ determinant

\[N \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} = \det_{i,j} N(x_i, y_j),\]  

(1.6)

where $x_1, \ldots, y_n$ is a set of $2n$ points in $\Omega$. We shall refer to the $x_i$ as the row indices, and to the $y_j$ as the column indices of the symbol on the left-hand side of (1.6). In some of the mathematical literature (1.6) is known as the Fredholm determinant, however, we shall reserve this name, as is customary in vast portions of the physics and mathematics literature, to Fredholm’s first series $D(\lambda)$ defined below in (1.9).

Given (1.3), it follows from a theorem due to Hadamard that (1.6) is bounded according to

\[|N \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}| \leq n^{\frac{n^2}{2}} M^n.\]  

(1.7)

Fredholm’s $n$th minor is defined by the series

\[D_n \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} = N \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_{\Omega} N \begin{pmatrix} x_1 & x_2 & \ldots & x_n & s_1 & s_2 & \ldots & s_p \\ y_1 & y_2 & \ldots & y_n & s_1 & s_2 & \ldots & s_p \end{pmatrix} ds_1 \ldots ds_p \]  

(1.8)

By definition, $D_n$ is completely antisymmetric in the $x_i$, and also in the $y_i$. In view of (1.2) and (1.7), it is easy to see that the series (1.8) converges absolutely to an entire function of $\lambda$.

Fredholm’s first series

\[D(\lambda) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_{\Omega} N \begin{pmatrix} s_1 & s_2 & \ldots & s_p \\ s_1 & s_2 & \ldots & s_p \end{pmatrix} ds_1 \ldots ds_p.\]  

(1.9)
and second series
\[
D(x, y; \lambda) = N(x, y) + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_\Omega N \left( \begin{array}{cccc} x & s_1 & s_2 & \ldots & s_p \\ y & s_1 & s_2 & \ldots & s_p \end{array} \right) ds_1 \ldots ds_p
\]

(1.10)
correspond to setting \(n = 0\) and \(n = 1\) in (1.8), respectively.

Fredholm’s first series (1.9) is, by construction, the functional determinant
\[
D(\lambda) = \text{Det} \left( 1 - \lambda \hat{N} \right)
\]

(1.11)
of the operator on the left-hand side of (1.5). It is usually known in the literature as the *Fredholm determinant* associated with (1.1), and we shall adhere to this convention here.

From the definitions (1.8) and (1.9) we can prove the important relation
\[
\frac{d^n D(\lambda)}{d\lambda^n} = (-1)^n \int_\Omega D_n \left( \begin{array}{cccc} x_1 & x_2 & \ldots & x_n \\ x_1 & x_2 & \ldots & x_n \end{array} \right) |\lambda| dx_1 \ldots dx_n
\]

(1.12)
in a straightforward manner.

The motivation for introducing the minors stems from their important roles in solving Fredholm’s equation (1.1) in the most general case. This is briefly reviewed in the appendix. In particular, for values of \(\lambda\) such that \(D(\lambda) \neq 0\), the solution of (1.1) is determined by the resolvent kernel \(R(x, y; \lambda)\) according to (A.1). \(R(x, y; \lambda)\), in turn, is given in (A.2) as \(\frac{D(x, y; \lambda)}{D(\lambda)}\). For values of \(\lambda\) such that \(D(\lambda) = 0\), the solution is given by (A.6) and (A.10), and involves the higher minors.

The rest of the paper is organized as follows. In the next section we derive our new recursion relations (Eqs.(2.7) and (2.8)) for the minors (1.8). We then solve it and obtain the representation (2.11) for (1.8) as an \(n \times n\) determinant over resolvents. In section 3 we provide an interpretation of this representation in terms of non-interacting fermions. Finally, in section 4 we provide an explicit formula (Eq.(4.9)) for the functional derivative of the \(n\)th minor with respect to the kernel.
2 The Minor $D_n$ as an $n \times n$ Determinant

We start by deriving a couple of integral equations satisfied by $D_n$ [3]. To obtain the first equation, expand each of the determinants in the series (1.8) with respect to the row $x_i$, and integrate with respect to all the $s$-variables which occur in that term. It is easy to see, by elementary permutations of rows and columns, that in the $p$th term, all columns $s_1, \ldots, s_p$ yield the same integrated contribution. Then, after resumming over $p$, one obtains

$$D_n \begin{pmatrix} x_1 & \cdots & x_n \mid \lambda \\ y_1 & \cdots & y_n \end{pmatrix} = \sum_{k=1}^{n} (-1)^{i+k} N(x_i, y_k) D_{n-1} \begin{pmatrix} x_1 & \cdots & x_i' & \cdots & x_n \mid \lambda \\ y_1 & \cdots & y_k' & \cdots & y_n \end{pmatrix} + \lambda \int_{\Omega} N(x_i, s) D_n \begin{pmatrix} x_1 & \cdots & (x_i')s & \cdots & x_n \mid \lambda \\ y_1 & \cdots & y_i & \cdots & y_n \end{pmatrix} ds \quad (2.1)$$

where the symbol $x_i'$ in the upper row of $D_{n-1}$ indicates that the row index $x_i$ is to be omitted from the string $x_1, \ldots, x_n$ (and similarly for $y_k'$ in the lower row there), and $(x_i')s$ indicates that $x_i$ in the upper row of $D_n$ under the integral should be replaced by the integration variable $s$.

The second integral equation satisfied by $D_n$ is obtained similarly, by expanding each of the determinants in the series (1.8) with respect to the column $y_i$. One obtains

$$D_n \begin{pmatrix} x_1 & \cdots & x_n \mid \lambda \\ y_1 & \cdots & y_n \end{pmatrix} = \sum_{k=1}^{n} (-1)^{i+k} N(x_k, y_i) D_{n-1} \begin{pmatrix} x_1 & \cdots & x_k' & \cdots & x_n \mid \lambda \\ y_1 & \cdots & y_i & \cdots & y_n \end{pmatrix} + \lambda \int_{\Omega} N(s, y_i) D_n \begin{pmatrix} x_1 & \cdots & \cdots & \cdots & x_n \mid \lambda \\ y_1 & \cdots & (y_i)s & \cdots & y_n \end{pmatrix} ds \quad (2.2)$$
We now proceed to derive our own results. We assume henceforth that \(D(\lambda) \neq 0\). In this case, according to (A.1), Fredholm’s equation (1.1) has a unique solution

\[
\phi(x) = f(x) + \lambda \int_\Omega R(x, y; \lambda) f(y) \, dy.
\]  

(2.3)

From this solution we construct the quantity

\[
\Xi_n \left( \begin{array}{c} x_1 \cdots \cdots x_n \\ y_1 \cdots y_i \cdots y_n \end{array} \right) = \lambda \int_\Omega D_n \left( \begin{array}{c} x_1 \cdots \cdots x_n \\ y_1 \cdots y_i \cdots y_n \end{array} \right) \phi(y_i) \, dy_i
\]  

(2.4)

Then, expand the \(D_n\) under the integral on the right-hand side of (2.4) according to (2.2). By exploiting the fact that \(\lambda \int_\Omega N(x, y) \phi(y) \, dy = \phi(x) - f(x) = \lambda \int_\Omega R(x, y; \lambda) f(y) \, dy\) from (1.1) and (2.3), we note that there appears a \(\Xi_n\) on the right-hand side of the equation, which cancels the original one on the left, leaving us with the identity

\[
\int_\Omega D_n \left( \begin{array}{c} x_1 \cdots \cdots x_n \\ y_1 \cdots y_i \cdots y_n \end{array} \right) f(y_i) \, dy_i =
\]

\[
\sum_{k=1}^n (-1)^{i+k} \int_\Omega R(x_k, y_i; \lambda) D_{n-1} \left( \begin{array}{c} x_1 \cdots x_k \cdots \cdots x_n \\ y_1 \cdots y_i \cdots y_n \end{array} \right) \phi(y_i) \, dy_i.
\]  

(2.5)

Since this identity holds for all admissible given functions \(f(x)\), we conclude that \(D_n\) must satisfy the recursion relation

\[
D_n \left( \begin{array}{c} x_1 \cdots x_n \\ y_1 \cdots y_n \end{array} \right) = \sum_{k=1}^n (-1)^{i+k} R(x_k, y_i; \lambda) D_{n-1} \left( \begin{array}{c} x_1 \cdots x_k \cdots \cdots x_n \\ y_1 \cdots y_i \cdots y_n \end{array} \right)
\]  

(2.6)

or equivalently,

\[
D(\lambda)D_n \left( \begin{array}{c} x_1 \cdots x_n \\ y_1 \cdots y_n \end{array} \right) = \sum_{k=1}^n (-1)^{i+k} D(x_k, y_i; \lambda) D_{n-1} \left( \begin{array}{c} x_1 \cdots x_k \cdots \cdots x_n \\ y_1 \cdots y_i \cdots y_n \end{array} \right),
\]  

(2.7)
from (A.2). Similarly, from (2.1), and by exploiting the associated (or transposed) Fredholm equation \( \psi(x) = g(x) + \lambda \int_{\Omega} \psi(y) N(y, x) \, dy \), we obtain the transposed identity

\[
D(\lambda)D_n \left( \begin{array}{c|c}
  x_1 & \cdots & x_n \\
  y_1 & \cdots & y_n \\
\end{array} ; \lambda \right) = \sum_{k=1}^{n} (-1)^{i+k} D(x_i, y_k; \lambda) D_{n-1} \left( \begin{array}{c|c}
  x_1 & \cdots & x_i' & \cdots & x_n \\
  y_1 & \cdots & y_k' & \cdots & y_n \\
\end{array} ; \lambda \right). \tag{2.8}
\]

The two identities (2.7) and (2.8) strongly suggest that the quantity

\[
\Delta_n \left( \begin{array}{c|c}
  x_1 & \cdots & x_n \\
  y_1 & \cdots & y_n \\
\end{array} ; \lambda \right) = \frac{1}{D(\lambda)} D_n \left( \begin{array}{c|c}
  x_1 & \cdots & x_n \\
  y_1 & \cdots & y_n \\
\end{array} ; \lambda \right), \tag{2.9}
\]

is simply the \( n \times n \) determinant with entries \( R(x_i, y_j; \lambda) \) and corresponding minor

\[
\Delta_{n-1} \left( \begin{array}{c|c}
  x_1 & \cdots & x'_{k} & \cdots & x_n \\
  y_1 & \cdots & y'_i & \cdots & y_n \\
\end{array} ; \lambda \right) \right).
\]

The proof of this proposition by induction is almost trivial: This proposition is indeed the content of (2.7) and (2.8) for \( n = 2 \):

\[
D(\lambda)D_2 \left( \begin{array}{c|c}
  x_1 & x_2 \\
  y_1 & y_2 \\
\end{array} ; \lambda \right) = D(x_1, y_1; \lambda)D(x_2, y_2; \lambda) - D(x_1, y_2; \lambda)D(x_2, y_1; \lambda). \tag{2.10}
\]

(For \( n = 1 \) (2.7) and (2.8) yield a trivial identity.) Then, by assuming it holds for \( \Delta_{n-1} \), we apply it to the \( \Delta_{n-1} \)s which appear on the right-hand sides of (2.7) and (2.8), and thus observe that the latter are just the expansion of an \( n \times n \) determinant with entries \( R(x_i, y_j; \lambda) \) according to the \( i \) column and \( i \)th row, respectively. The proposition of the induction is thus verified for \( \Delta_n \) as well.

Thus, we have derived our first main result:

\[
\frac{1}{D(\lambda)} D_n \left( \begin{array}{c|c}
  x_1 & \cdots & x_n \\
  y_1 & \cdots & y_n \\
\end{array} ; \lambda \right) = \det_{ij} R(x_i, y_j; \lambda). \tag{2.11}
\]

Note that for \( n = 1 \), (2.11) coincides with (A.2), as it should.
3 Interpretation of (2.11) in Terms of Non-interacting Fermions

The determinantal representation (2.11) suggests, due to Wick’s theorem for non-interacting fermions, an interpretation of $D_n$ as the correlation function of $n$ fermions and $n$ anti-fermions.

To this end, consider the non-interacting complex Grassmann valued field $\psi(x)$, living on $\Omega$, with action

$$S = \int_\Omega \bar{\psi}(x) \left[ \delta(x - y) - \lambda N(x, y) \right] dx dy.$$  \hfill (3.1)

Its partition function\cite{4} is given by the path integral

$$Z = \int D\psi D\bar{\psi} e^S = \text{Det} \left( 1 - \lambda \hat{N} \right) = D(\lambda).$$  \hfill (3.2)

As is well-known from the annals of quantum field theory, the non-vanishing correlation functions of (3.2) are those which contain equal numbers of $\psi$s and $\psi^\dagger$s, namely,

$$\langle \psi^\dagger(y_1)\psi(x_1) \cdots \psi^\dagger(y_n)\psi(x_n) \rangle = \frac{1}{Z} \int D\psi D\bar{\psi} \left[ \psi^\dagger(y_1)\psi(x_1) \cdots \psi^\dagger(y_n)\psi(x_n) \right].$$  \hfill (3.3)

The latter are determined according to Wick’s theorem as

$$\langle \psi^\dagger(y_1)\psi(x_1) \cdots \psi^\dagger(y_n)\psi(x_n) \rangle = \text{det}_{ij} \langle \psi^\dagger(y_i)\psi(x_j) \rangle,$$  \hfill (3.4)

where the two-point function is

$$\langle \psi^\dagger(y)\psi(x) \rangle = \langle x \left| \frac{1}{1 - \lambda N} \right| y \rangle.$$  \hfill (3.5)

Then, observe from (3.5) and (A.3) that

$$\int_\Omega N(x, z)\langle \psi^\dagger(y)\psi(z) \rangle dz = \langle x \left| \frac{\hat{N}}{1 - \lambda N} \right| y \rangle = R(x, y; \lambda).$$  \hfill (3.6)

Thus, by linearity, we obtain from (3.4) and (3.6) that

$$\langle \psi^\dagger(y_1)(\hat{N}\psi)(x_1) \cdots \psi^\dagger(y_n)(\hat{N}\psi)(x_n) \rangle = \text{det}_{ij} R(x_i, y_j; \lambda),$$  \hfill (3.7)
where \((\tilde{N}\psi)(x) = \int N(x, y)\psi(y) dy\). Thus, we interpret the \(n\)th minor \(D_n\), according to (2.11) and (3.7), as the multi-fermion correlator

\[
D_n \left( \begin{array}{cccc} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{array} \right) = D(\lambda) \langle \psi^\dagger(y_1)(\tilde{N}\psi)(x_1) \cdots \psi^\dagger(y_n)(\tilde{N}\psi)(x_n) \rangle
\]

that is, \(D_n\) is the \(2^n\)th moment of the Grassmann weight \(e^S\) (convoluted against \(n\) powers of \(\tilde{N}\)). As such, it might be thought of as some kind of a \(continuum\) (supplementary) compound matrix associated with \(1 - \lambda \tilde{N}\) [5]. The latter interpretation might be useful in studying minors of very large order, such that the \(2n\) points \(x_i\) and \(y_i\) become typically dense in \(\Omega\).

As a consistency check of (3.8), let us trace it over all coordinates, and see if we recover (1.12). Thus,

\[
\int_{\Omega} D_n \left( \begin{array}{cccc} x_1 & \cdots & x_n \\ x_1 & \cdots & x_n \end{array} \right) d\lambda \right) = D(\lambda) \langle \psi^\dagger(y_1)(\tilde{N}\psi)(x_1) \cdots \psi^\dagger(y_n)(\tilde{N}\psi)(x_n) \rangle
\]

From (3.1), (3.2) and (3.3), we see that

\[
\left\langle \left( \int_{\Omega} \psi^\dagger(x)(\tilde{N}\psi)(x) dx \right)^n \right\rangle = \frac{1}{D(\lambda)} \left( -\frac{d}{d\lambda} \right)^n D(\lambda).
\]

Thus,

\[
\int_{\Omega} D_n \left( \begin{array}{cccc} x_1 & \cdots & x_n \\ x_1 & \cdots & x_n \end{array} \right) dx_1 \cdots dx_n = \left( -\frac{d}{d\lambda} \right)^n D(\lambda),
\]

in accordance with (1.12).
4 The Functional Derivative of $D_n$ with Respect to the Kernel

As we have discussed above, the minors $D_n$ (1.8) determine the solution of the Fredholm equation (1.1). In some applications of (1.1), the kernel $N(x, y)$ may depend on a set of parameters or functions, and it may be important to determine how the solutions vary with these quantities. To this end we have first to determine the functional derivative of the minors $D_n$ with respect to the kernel.

For example, in a recent paper [6], we have calculated the variation of the solution of the Gelfand-Levitan-Marchenko equation with the reflection amplitude of scattering theory, and deduced from it the corresponding variation of the Schrödinger potential and wave-function.

The minor $D_n$ is expressed in (2.11) in terms of $D(\lambda)$ and $R(x, y; \lambda)$. It is straightforward to obtain the functional derivatives of these two objects with respect to the kernel $N(x, y)$ directly from (1.11) and (A.3). Thus, consider a perturbation $\hat{N} \to \hat{N} + \delta \hat{N}$. From (1.11) we see that under this variation $\delta D(\lambda) = D(\lambda) \delta \log D(\lambda) = -\lambda D(\lambda) \text{tr} \left( \frac{1}{1 - \lambda \hat{N}} \delta \hat{N} \right)$, from which we infer

$$\frac{\delta D(\lambda)}{\delta N(a, b)} = -\lambda D(\lambda) \left[ \delta (b - a) + \lambda R(b, a; \lambda) \right]$$

$$= -\lambda D(\lambda) \delta (b - a) - \lambda^2 D(b, a; \lambda).$$  \hspace{1cm} (4.1)

Under this variation we also have $\hat{R} \to \hat{R} + \frac{1}{1 - \lambda \hat{N}} \delta \hat{N} \frac{1}{1 - \lambda \hat{N}}$. Consequently

$$\frac{\delta R(x, y; \lambda)}{\delta N(a, b)} = \langle x | (1 + \lambda \hat{R}) | a \rangle \langle b | (1 + \lambda \hat{R}) | y \rangle$$

$$= (\delta (x - a) + \lambda R(x, a; \lambda)) (\delta (b - y) + \lambda R(b, y; \lambda)) .$$  \hspace{1cm} (4.2)

We can then calculate

$$\delta \left( D(\lambda) \det_{ij} R(x_i, y_j; \lambda) \right)$$

by applying (4.1) and (4.2) as necessary. The expression we obtain in this way is rather cumbersome, but the plethora of terms thus obtained can be organized into a linear combination of the minors $D_n$ and $D_{n\pm 1}$. 
Instead of pursuing this line of derivation, we shall now sketch the calculation of the functional derivative of \(D_n\) directly from (1.8), by taking the derivative of this series term by term. Thus, consider taking the derivative of the \(p\)th term. Let us split the matrix whose determinant

\[
\int \Omega \begin{pmatrix} x_1 & x_2 & \ldots & x_n & s_{n+1} & s_{n+2} & \ldots & s_{n+p} \\ y_1 & y_2 & \ldots & y_n & s_{n+1} & s_{n+2} & \ldots & s_{n+p} \end{pmatrix} ds_{n+1} \ldots ds_{n+p} \quad (4.3)
\]

is being integrated in that term into four blocks, according to NN, NI, IN and II, where ‘I’ stands for an integrated coordinate index, and ‘N’ for a non-integrated one. Let us now scan systematically through these blocks.

When the derivative \(\frac{\delta}{\delta N(a,b)}\) hits the term \(N(x_i, y_j)\) in the NN sector, it produces a factor \(\delta(x_i - a)\delta(y_j - b)\) which is multiplied by the minor of \(N(x_i, y_j)\), times a sign factor \((-1)^{i+j}\). The total contribution of the NN sector to the derivative is the sum of all these terms:

\[
\text{NN} = \sum_{i,j=1}^{n} (-1)^{i+j} \delta(x_i - a)\delta(y_j - b) \times \\
\int \Omega \begin{pmatrix} x_1 & \ldots & x_i' & \ldots & x_n & s_{n+1} & \ldots & s_{n+p} \\ y_1 & \ldots & y_j' & \ldots & y_n & s_{n+1} & \ldots & s_{n+p} \end{pmatrix} ds_{n+1} \ldots ds_{n+p}.
\]

(4.4)

Move now to the NI block. When the derivative \(\frac{\delta}{\delta N(a,b)}\) hits the term \(N(x_i, s_{n+l})\), it produces a factor \(\delta(x_i - a)\delta(s_{n+l} - b)\) which is multiplied by the minor of \(N(x_i, s_{n+l})\), which contains \(s_{n+l}\) as a row index (but not as a column index), times a sign factor \((-1)^{i+n+l}\). Integration over \(s_{n+l}\) thus replaces the row index \(s_{n+l}\) in that minor by \(b\). Now, permute the row which used to be that of \(s_{n+l}\) in that minor, and move it in between the rows corresponding to \(x_{i-1}\) and \(x_{i+1}\). This means permuting it across \(n+l-i-1\) rows and costs a sign factor \((-1)^{n+l-i-1}\), which combines with the previous sign factor to \((-1)\), independently of \(l\). Thus, all the \(p\) columns which intersect the original row \(x_i\) in the NI block make the same contribution to the functional derivative, after integration over the remaining \(p-1\) variables. Finally, summing over all the
\(x_i\) in the NI block we obtain the total contribution by that block to the functional derivative as

\[
\text{NI} = -p \sum_{i=1}^{n} \delta(x_i - a) \times 
\int_{\Omega} N \begin{pmatrix}
    x_1 & \ldots & (x_i)b & \ldots & x_n & s_{n+1} & \ldots & s_{n+p-1} \\
    y_1 & \ldots & \ldots & \ldots & y_n & s_{n+1} & \ldots & s_{n+p-1} 
\end{pmatrix} ds_{n+1} \ldots ds_{n+p-1}.
\]

(4.5)

Similarly, the contribution of the entire IN block to the functional derivative is

\[
\text{IN} = -p \sum_{j=1}^{n} \delta(y_j - b) \times 
\int_{\Omega} N \begin{pmatrix}
    x_1 & \ldots & \ldots & \ldots & x_n & s_{n+1} & \ldots & s_{n+p-1} \\
    y_1 & \ldots & (y_j)a & \ldots & y_n & s_{n+1} & \ldots & s_{n+p-1} 
\end{pmatrix} ds_{n+1} \ldots ds_{n+p-1}.
\]

(4.6)

The last block is II. Clearly, we should discuss the diagonal terms and the non-diagonal terms separately. When \(\frac{\delta}{\delta N(a,b)}\) hits the diagonal term \(N(s_{n+l}, s_{n+l})\), it produces a factor \(\delta(s_{n+l} - a)\delta(s_{n+l} - b)\) times the corresponding diagonal minor, which comes with a positive sign and does not depend on \(s_{n+l}\). Thus, integration over \(s_{n+l}\) simply produces a factor \(\delta(a - b)\) which multiplies the remaining integral. The latter is the same for all diagonal terms. Thus, the overall contribution of diagonal terms from the II block to the functional derivative is

\[
\text{II}_{\text{diag}} = p \delta(a - b) \int_{\Omega} N \begin{pmatrix}
    x_1 & \ldots & x_n & s_{n+1} & \ldots & s_{n+p-1} \\
    y_1 & \ldots & y_n & s_{n+1} & \ldots & s_{n+p-1} 
\end{pmatrix} ds_{n+1} \ldots ds_{n+p-1}.
\]

(4.7)

Finally, when \(\frac{\delta}{\delta N(a,b)}\) hits the non-diagonal term \(N(s_{n+l}, s_{n+k})\), \(l \neq k\), it produces a factor \(\delta(s_{n+l} - a)\delta(s_{n+k} - b)\) times the minor of \(N(s_{n+l}, s_{n+k})\), which contains \(s_{n+k}\) as a row index (but not as a column index), and also contains \(s_{n+l}\) as a column index (but not as a row index), times a sign factor \((-1)^{n+k+n+l}\).

Integration over \(s_{n+l}\) thus replaces the column index \(s_{n+l}\) in that minor by \(a\). Similarly, integration over \(s_{n+k}\) replaces the row index \(s_{n+k}\) by \(b\). Now, move the row
which used to be that of $s_{n+k}$ right below the row $x_n$, and the column which used to be that of $s_{n+l}$ immediately to the right of the column $y_n$. These permutations produce a sign factor $(-1)^{k-1+l-1}$, which combines with the previous sign simply to $(-1)$. The remaining integral is independent of $s_k$ and $s_l$ and yields the same contribution for all the $p(p-1)$ non-diagonal terms. Thus, their total contribution to the functional derivative is

$$
\Pi_{\text{non-diag.}} = -p(p-1) \int_{\Omega} N \left( \begin{array}{cccc}
x_1 & \ldots & x_n & b \\
y_1 & \ldots & y_n & a
\end{array} \right) ds_{n+1} \ldots ds_{n+p-2}.
$$

(4.8)

Gathering all contributions (4.4)-(4.8) together, multiplying their sum by $\frac{(-\lambda)^p}{p!}$ and summing over $p$, we finally arrive, after some rearrangement of terms, at our second main result:

$$
\frac{\delta}{\delta N(a, b)} D_n \left( \begin{array}{c}
x_1 \ldots x_n \\
y_1 \ldots y_n
\end{array} \right| \lambda \right) =
$$

$$
\sum_{i,j=1}^n (-1)^{i+j} \delta(x_i - a) \delta(y_j - b) D_{n-1} \left( \begin{array}{cccc}
x_1 & \ldots & x_{i-1} & x_i & \ldots & x_n \\
y_1 & \ldots & y_{j-1} & y_j & \ldots & y_n
\end{array} \right| \lambda 
$$

$$
+ \lambda \sum_{i=1}^n \delta(x_i - a) D_n \left( \begin{array}{cccc}
x_1 & \ldots & (x_i) b & \ldots & x_n \\
y_1 & \ldots & \ldots & y_n
\end{array} \right| \lambda 
$$

$$
+ \lambda \sum_{j=1}^n \delta(y_j - b) D_n \left( \begin{array}{cccc}
x_1 & \ldots & \ldots & x_n \\
y_1 & \ldots & (y_j) a & \ldots & y_n
\end{array} \right| \lambda 
$$

$$
- \lambda \delta(a - b) D_n \left( \begin{array}{cccc}
x_1 & \ldots & x_n \\
y_1 & \ldots & y_n
\end{array} \right| \lambda \right) - \lambda^2 D_{n+1} \left( \begin{array}{cccc}
x_1 & \ldots & x_n & b \\
y_1 & \ldots & y_n & a
\end{array} \right| \lambda \right)
$$

(4.9)
It is gratifying that the functional derivative of $D_n$ is expressed as a relatively simple linear combination of $D_n$ and $D_{n\pm 1}$. In particular, we could think of (4.9) as recursively defining $D_{n+1}$ in terms of the lower minors. This is analogous to Jacobi’s recursive definition of supplementary compound matrices in the finite dimensional case [5]. The formula (4.9) for the functional derivative of $D_n$ should coincide, of course, with the expression we would obtain by taking the derivative of $D(\lambda) \det_{ij} R(x_i, y_j; \lambda)$.

As a simple application, let us check (4.9) for $n = 0$ and $n = 1$. For $n = 0$, it yields

$$\frac{\delta D(\lambda)}{\delta N(a, b)} = -\lambda \delta(a - b) D(\lambda) - \lambda^2 D(b, a; \lambda),$$

which coincides with (4.1). Similarly, for $n = 1$, we obtain

$$\frac{\delta D(x, y; \lambda)}{\delta N(a, b)} = \delta(x - a) \delta(y - b) D(\lambda) + \lambda \delta(x - a) D(b, y; \lambda) + \lambda \delta(y - b) D(x, a; \lambda)$$

$$- \lambda \delta(a - b) D(x, y; \lambda) - \lambda^2 D_2 \begin{pmatrix} x & b \\ y & a \end{pmatrix}.$$ (4.10)

Thus, from the last two equation we can derive that

$$\frac{\delta R(x, y; \lambda)}{\delta N(a, b)} = \frac{\delta}{\delta N(a, b)} \left( \frac{D(x, y; \lambda)}{D(\lambda)} \right) =$$

$$\delta(x - a) \delta(y - b) + \lambda \delta(x - a) R(b, y; \lambda) + \lambda \delta(y - b) R(x, a; \lambda)$$

$$+ \lambda^2 \begin{pmatrix} R(x, y; \lambda) R(b, a; \lambda) - \Delta_2 \begin{pmatrix} x & b \\ y & a \end{pmatrix} \end{pmatrix}.$$ (4.11)

which coincides with (4.2) due to our determinantal representation (2.11).

**Appendix: The Solution of Fredholm’s Equation**

The theory of the Fredholm’s equation (1.1) and its solution is summarized in Fredholm’s celebrated three theorems (sometimes referred to collectively as “Fredholm’s alternatives”). The exposition in this appendix will be very telegraphic. We will
describe the content of Fredholm’s theorems in a semi-quantitative way, sufficient for our purposes, and refer the interested reader to the cited literature for more details.

Broadly speaking, the solution of (1.1) depends on whether $D(\lambda) \neq 0$ or not:

Case (1): $D(\lambda) \neq 0$

In this case, the solution of (1.1) involves the $n = 1$ minor $D(x, y; \lambda)$. For values of $\lambda$ such that $D(\lambda) \neq 0$, the operator $1 - \lambda \hat{N}$ is invertible, and (1.1) (or, equivalently (1.5)) has a unique solution, given by

$$\phi(x) = f(x) + \lambda \int_{\Omega} R(x, y; \lambda) f(y) \, dy,$$

where $R(x, y; \lambda)$ is the resolvent kernel of (1.1), which is given by

$$R(x, y; \lambda) = \frac{D(x, y; \lambda)}{D(\lambda)}.$$

From (1.5) we can read off the operator $\hat{R}$, which corresponds to $R(x, y; \lambda) = \langle x | \hat{R} | y \rangle$ as

$$\hat{R} = \frac{1}{\lambda} \left( \frac{1}{1 - \lambda \hat{N}} - 1 \right) = \frac{\hat{N}}{1 - \lambda \hat{N}}.$$

Thus, from (A.2), (1.12) (at $n = 1$), (1.11) and (A.3) we conclude that

$$\text{tr} \hat{R} = \int_{\Omega} R(x, x; \lambda) \, dx = -\frac{d}{d\lambda} \log D(\lambda),$$

which shows that the poles of $\hat{R}$ as a function of $\lambda$ are the zeros of $D(\lambda)$.

Case (2): $D(\lambda) = 0$

If $\lambda = \lambda_0$ such that $D(\lambda_0) = 0$, the homogeneous equation

$$\phi(x) = \lambda_0 \int_{\Omega} N(x, y) \phi(y) \, dy$$

has one or more non-trivial, linearly independent solutions $\Phi_i(x), \quad i = 1, \ldots, \nu$, where $\nu \geq 1$. In this case, we say that $\lambda_0$ is an eigenvalue of $\hat{N}$ of rank $\nu$, and refer to the functions $\Phi_i(x)$ as the characteristic functions corresponding to the eigenvalue $\lambda_0$. (This nomenclature deviates from that of linear algebra, which would refer to $\frac{1}{\lambda_0}$ as the
eigenvalue.) Any solution of the homogeneous equation (A.5) is a linear combination of the characteristic functions, i.e., the characteristic functions span \( \text{Ker}(1 - \lambda_0 \hat{N}) \).

As it turns out, the \( \nu \) characteristic functions are proportional to the \( \nu \)th minor \( D_\nu \). More precisely, for a fixed set of \( 2\nu \) points \( x_1, \ldots, y_\nu \), such that

\[
D_\nu \begin{pmatrix} x_1 & \cdots & x_\nu \\ y_1 & \cdots & y_\nu \end{pmatrix} \neq 0,
\]

we have

\[
\Phi_i(x) = \frac{D_\nu \begin{pmatrix} \hat{x}_i & x_1 & \cdots & x_\nu \\ y_1 & \cdots & y_\nu \end{pmatrix} - D_\nu \begin{pmatrix} x_1 & \cdots & x_\nu \\ y_1 & \cdots & y_\nu \end{pmatrix} \lambda_0}{D_\nu \begin{pmatrix} x_1 & \cdots & x_\nu \\ y_1 & \cdots & y_\nu \end{pmatrix} \lambda_0},
\]

where the symbol \((\hat{x}_i)x\) indicates that the \( i \)th row index \( x_i \) in the upper row of the minor in the numerator is to be replaced by the coordinate \( x \). The functions (A.6) are normalized such that

\[
\Phi_i(x_k) = \delta_{i,k},
\]

as can be seen from (1.8), since \( D_n \) vanishes when any two of its row (or column) indices coincide.

Since the characteristic functions are expressed in terms of \( D_\nu \), it does not vanish identically. In fact, it is the minor of lowest order which does not vanish identically as a function of its \( 2\nu \) arguments at \( \lambda = \lambda_0 \). In fact, (A.6) is obtained by setting \( n = \nu \) (and \( \lambda = \lambda_0 \)) in (2.1). Furthermore, it follows from (1.12) that \( \lambda_0 \) must be a zero of \( D(\lambda) \) of multiplicity which is greater or equal to \( \nu \), since \( \frac{d^n D(\lambda)}{d\lambda^n} \) might still vanish at \( \lambda = \lambda_0 \).

As for solving the inhomogeneous equation (1.1) at \( \lambda = \lambda_0 \), one proceeds as follows. First, one has to consider the transposed, or associated homogeneous Fredholm equation

\[
\psi(x) = \lambda_0 \int_\Omega \psi(y)N(y, x) \, dy.
\]

(A.8)
Since its kernel is the transpose of the kernel of (1.1), it has $\lambda_0$ as an eigenvalue of the same rank $\nu$. Thus, there are $\nu$ independent characteristic solutions $\Psi_i(x)$, which are also expressed in terms of the $\nu$th minor $D_\nu$, similarly to (A.6), as

$$
\Psi_i(x) = \frac{D_\nu \left( \begin{array}{c|c} x_1 & \ldots & x_i & \ldots & x_\nu \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1 & \ldots & (y_i x) & \ldots & y_\nu \\ \hline \lambda_0 & & & & \end{array} \right)}{D_\nu \left( \begin{array}{c|c} x_1 & \ldots & x_\nu \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1 & \ldots & y_\nu \\ \hline \lambda_0 & & & & \end{array} \right)},
$$
(A.9)

which span $\text{Ker}(1 - \lambda_0 \hat{N}^T)$.

A necessary and sufficient condition for the existence of a solution of (1.1) at $\lambda = \lambda_0$ is then that the given function $f(x)$ be orthogonal to all the characteristic functions $\Psi_i(x)$, i.e., that $\int_{\Omega} \Psi_i(x) f(x) = 0$, $i = 1, \ldots, \nu$.

If this condition holds, the solution (which exists) is not unique, since given a particular solution, one can always add to it an arbitrary solution of the homogeneous equation (A.5). The piece in the general solution of (1.1) which is linear in $f(x)$ (i.e., a particular solution of (1.1)) is

$$
\phi_p(x) = f(x) + \lambda_0 \int_{\Omega} \frac{D_{\nu+1} \left( \begin{array}{c|c} x & x_1 & \ldots & x_\nu \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y & y_1 & \ldots & y_\nu \\ \hline \lambda_0 & & & & \end{array} \right)}{D_\nu \left( \begin{array}{c|c} x_1 & \ldots & x_\nu \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1 & \ldots & y_\nu \\ \hline \lambda_0 & & & & \end{array} \right)} f(y) \, dy.
$$
(A.10)
Acknowledgments

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