YORKE AND WRIGHT 3/2-STABILITY THEOREMS FROM A
UNIFIED POINT OF VIEW

EDUARDO LIZ
Departamento de Matemática Aplicada II, E.T.S.I. Telecomunicación
Universidad de Vigo, Campus Marcosende, 36280 Vigo, Spain

VÍCTOR TKACHENKO
Institute of Mathematics, National Academy of Sciences of Ukraine
Tereshchenkivs’ka str. 3, Kiev, Ukraine

SERGEI TROFIMCHUK
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Chile, Casilla 653, Santiago, Chile

Abstract. We consider a family of scalar delay differential equations
\[ x'(t) = f(t, x_t), \] with a nonlinearity \( f \) satisfying a negative feedback condition combined
with a boundedness condition. We present a global stability criterion for this
family, which in particular unifies the celebrated 3/2-conditions given for the
Yorke and the Wright type equations. We illustrate our results with some
applications.

1. Introduction and main result

In this paper we present a global stability criterion for a family of scalar functional
differential equations
\[ x'(t) = f(t, x_t), \quad (x_t(s) \overset{\text{def}}{=} x(t+s), \, s \in [-1, 0]), \]
where \( f : \mathbb{R} \times \mathcal{C} \to \mathbb{R} \) is a measurable functional, \( \mathcal{C} = \mathcal{C}([-1, 0], \mathbb{R}) \).
As we will show in Section 3, our setting allows us to prove global stability results
for a large family of functional delay differential equations, including a very general
form of the delayed logistic equation, and differential equations with maxima among
others.

Next we introduce the hypotheses that will be required in Eq. (1). In order
to understand the motivation for the choice of these conditions, we recall some classical
results. We refer to the famous 3/2 stability results due to Myshkis \[9\], Wright \[13\]
and Yorke \[14\].

In particular, the Wright equation can be written in the form
\[ x'(t) = f(x(t-1)), \]
with \( f(x) = p(e^{-x} - 1), \, p > 0 \). The famous 3/2 stability result by Wright says that
all solutions of this equation converge to zero if \( p \leq 3/2 \). The closeness between
this condition and the (local) asymptotic stability condition, \( p < \pi/2 \), suggests the

Date: March 29, 2022.
1991 Mathematics Subject Classification. 34K20.
Key words and phrases. 3/2 stability condition, global stability, delay differential equations.
This research was supported by FONDECYT (Chile), project 8990013. E. Liz was supported in
part by M. C. T. (Spain) and FEDER under project BFM2001-3884. V. Tkachenko was supported
in part by F.F.D. of Ukraine, project 01.07/00109.
Then the following "generalized Yorke condition": 

\[ x'(t) = -px(t - h(t)), \quad p > 0, \quad 0 < h(t) \leq h. \]

Myshkis proved that Eq. (3) is exponentially stable if \( ph < 3/2 \), and it is possible to find examples such that \( ph = 3/2 \) and (3) has a nontrivial periodic solution (see [7, p. 191]).

In 1970, Yorke extended the Myshkis criterion to a family of scalar functional differential equations (11) where \( f : \mathbb{R} \times C([-h,0]) \rightarrow \mathbb{R} \), \( h > 0 \), is continuous and satisfies the following conditions:

(Y1) There exists \( a < 0 \) such that

\[ aM(\phi) \leq f(t, \phi) \leq -aM(-\phi) \]

for all \( \phi \in C \), where \( M(\phi) = \max\{0, \max_{x \in [-h,0]} \phi(s)\} \).

(Y2) For all sequences \( t_n \rightarrow \infty \) and \( \phi_n \) converging to a constant nonzero function in \( C \), \( f(t_n, \phi_n) \) does not converge to 0.

Under these conditions, if \( 0 < |a|h < 3/2 \) then all solutions of (11) converge to zero as \( t \rightarrow \infty \).

We notice that condition (Y2) is only required to guarantee that the solutions of (11) that monotonically converge to a constant in fact should converge to zero.

One can check that condition (Y1) is not satisfied by the Wright equation, and therefore Wright’s theorem cannot be deduced from the Yorke result.

Trying to generalize the Wright theorem, in [7] we prove the following result:

**Theorem 1.** Assume that \( f \in C^3(\mathbb{R}, \mathbb{R}) \) and it satisfies the following conditions:

(A1) \( xf(x) < 0 \) for \( x \neq 0 \) and \( f'(0) < 0 \).

(A2) \( f \) is bounded below and has at most one critical point \( x^* \in \mathbb{R} \) which is a local extremum.

(A3) \( (Sf)(x) < 0 \) for all \( x \neq x^* \), where \( Sf = f'''(f')^{-1} - (3/2)(f'^2)(f'^{-2}) \) is the Schwarz derivative of \( f \).

If \( |f'(0)| < 3/2 \), then the steady state solution \( x(t) = 0 \) of Eq. (3) is globally attracting.

**Remark 1.** Conditions (A1)-(A3) are satisfied by the Wright equation and other more complicated equations arising in population dynamics (see [7] for details).

Conditions (A1)-(A2) are not sufficient for the global attractivity in [7] (see [7, 12]). Hence an additional condition is required. We note that condition (A3) is not the unique option, in fact we only need some geometric consequences of the inequality \( Sf < 0 \) for the graph of \( f \). In particular, the following key result ([7, Lemma 2.1]) is very important:

**Lemma 1.** Assume that \( f \) satisfies (A1)-(A3), and \( f''(0) > 0 \). Let \( a = f'(0) < 0 \), \( b = -f''(0)/(2f'(0)) > 0 \), and \( r(x) = ax/(1 + bx) \). Then \( r(x) > f(x) \) if \( x \in (-1/b, 0) \), and \( r(x) < f(x) \) if \( x > 0 \).

From Lemma 1 we can obtain immediately the following

**Corollary 1.** Assume that \( f \) satisfies (A1)-(A3), and let \( f(t, \phi) = f(\phi(-1)) \). Then the following “generalized Yorke condition”:
(GY) There exist $a < 0, b \geq 0$ such that
\[
r(M(\phi)) = \frac{aM(\phi)}{1 + bM(\phi)} \leq f(t, \phi) \leq \frac{-aM(-\phi)}{1 - bM(-\phi)} = r(-M(-\phi)),
\]
where the first inequality holds for all $\phi \in C$, and the second one for all $\phi$ such that
\[
\min_{s \in [-1,0]} \phi(s) > -b^{-1} \in [-\infty, 0).
\]
Here $M(\phi) = \max\{0, \max_{s \in [-1,0]} \phi(s)\}$ is the Yorke functional.

This corollary suggests the unification of Yorke’s and Wright’s $3/2$ results by using condition (GY). In fact, we introduce the following three hypotheses (H):

- (H1) $f : \mathbb{R} \times C \to \mathbb{R}$ satisfies the Carathéodory condition. Moreover, for every $q \in \mathbb{R}$ there exists $\vartheta(q) \geq 0$ such that $f(t, \phi) \leq \vartheta(q)$ almost everywhere on $\mathbb{R}$ for every $\phi \in C$ which satisfies the inequality $\phi(s) \geq q, s \in [-1,0]$.

- (H2) Condition (GY) holds.

- (H3) $\int_0^{+\infty} f(s, p_s) ds$ diverges for every continuous $p(s)$ having nonzero limit at infinity.

We recall that $f(t, \phi)$ is a Carathéodory function if it is measurable in $t$ for each fixed $\phi$, continuous in $\phi$ for each fixed $t$, and for any fixed $(t, \phi) \in \mathbb{R} \times C$ there is a neighbourhood $V(t, \phi)$ and a Lebesgue integrable function $m$ such that $|f(s, \psi)| \leq m(s)$ for all $(s, \psi) \in V(t, \phi)$ (see [3, p. 58]).

The following result improves the above mentioned theorems due to Wright and Yorke respectively (notice that (H3) implies that $x(t) \equiv 0$ is the unique equilibrium of the equation):

**Theorem 2.** Assume that $f$ satisfies (H) and either $b > 0$ and $|a| \leq 3/2$, or $b = 0$ and $|a| < 3/2$. Then all solutions of (1) converge to zero as $t \to +\infty$.

**Remark 2.**

1. If (H2) holds with $b = 0$ (Yorke condition), then (H1) is satisfied automatically with $\vartheta(q) = -aM(-q)$.

2. Conditions (H1)- (H3) are satisfied for Eq. (2) under our hypotheses (A1)- (A3), with $a = f'(0), b = -f''(0)/2f'(0))$.

3. The constant $3/2$ in Theorem 2 is the best possible. For $b = 0$, this sharpness was shown in [14] (see also [4, 8]). For $b > 0$, Theorem 2 can be applied to prove that all positive solutions of the logistic equation with variable delays converge to zero if $p h \leq 3/2$ (see Theorem 3 in Section 3). Since the linearized equation of (1) at $x = 1$ is (3), constant $3/2$ cannot be improved. It is a remarkable fact that when Yorke’s result cannot be extended to the value $a = -3/2$ for $b = 0$, Theorem 2 allows this for every $b > 0$.

If $b > 0$, since $M$ is a positively homogeneous functional $(M(k\phi) = kM(\phi)$ for every $k \geq 0, \phi \in C)$, and since the global attractivity property of the trivial solution of (1) is preserved under the simple scaling $x = b^{-1}y$, the exact value of $b$ does not have importance and we can assume that $b = 1$. Also, the change of variables $x = -y$ transforms (1) into $y'(t) = -f(t, -y(t))$ so that it suffices that at least one of the two functionals $f(t, \phi), -f(t, -\phi)$ satisfies (GY).

**2. Proof of Theorem 2.**

2.1. **Auxiliary results.** Throughout this subsection, we will assume that $b = 1$ (and hence $r(x) = ax/(1 + x)$).
Lemma 2. Let (H) hold and \( x : [\alpha - 1, \omega) \to \mathbb{R} \) be a solution of \((H)\) defined on the maximal interval of existence. Then \( \omega = +\infty \) and \( M = \limsup_{t \to \infty} x(t) \), \( m = \liminf_{t \to \infty} x(t) \) are finite. Moreover, if \( m \geq 0 \) or \( M \leq 0 \), then \( M = m = 0 \).

Proof. Note that \((GY)\) implies that \( f(t, \phi) \geq a \) for all \( t \in \mathbb{R} \) and \( \phi \in C \). We claim that every solution \( x(t, \gamma) \) with initial value \( \gamma \) such that \( q \leq \gamma(s) \leq Q \), \( s \in [-1, 0] \), satisfies the inequality

\[
x(t, \gamma) \geq \min\{q, 0\} + a = \kappa, \quad t \geq 0.
\]

Indeed, if there is \( \delta > 0 \) such that \( x(t, \gamma) = \min\{q, 0\} + (1 + \delta)a \) for the first time at some point \( t = u \geq 0 \), then, for every \( w \in (u - 1 - \delta/2, u) \),

\[
x(u, \gamma) - x(w, \gamma) = \int_w^u f(s, x_s(\gamma))ds \geq a(u - w) > a(1 + \delta/2).
\]

Hence \( x(w, \gamma) \leq a\delta/2 < 0 \) for all \( w \in (u - 1 - \delta/2, u) \), and therefore \( M(x_s) = 0 \) for all \( s \in (u - \delta/2, u) \). Thus, by \((H2)\), \( x'(s, \gamma) = f(s, x_s(\gamma)) \geq 0 \) within some left neighborhood of \( u \), contradicting the choice of this point.

Proceeding analogously and using \((H1)\), we obtain that

\[
x(t, \gamma) \leq \max\{Q, 0\} + \vartheta(\kappa), \quad t \geq 0.
\]

Hence \( x(t) \) is bounded on the maximal interval of existence that implies the boundedness of the right hand side of Eq. \((H)\) along \( x(t) \). Thus \( \omega = +\infty \) due to the corresponding continuation theorem (see \(3\)).

Now, suppose that \( M < 0 \). Then \( x(t) < M/2 < 0 \) beginning from some \( t' = d - 1 \) so that, in view of \( x'(t) \geq 0 \), the solution \( x(t) \) converges monotonically to the negative value \( x(+\infty) = M \). We get a contradiction since, by \((H3)\),

\[
x(t) = x(d) + \int_d^t f(s, x_s)ds \to +\infty.
\]

Next we consider the situation when \( M = 0 \) and \( m < 0 \). In this case, \( x(t) \) necessarily oscillates about zero. Indeed, otherwise \( x(t) \leq 0 \) and thus \( x'(t) = f(t, x_t) \geq 0 \), so that \( x(t) \) converges monotonically to the trivial steady state (implying \( m = 0 \)). Now, since \( x(t) \) is oscillating, we can find a sequence of intervals \( I_k = (l_k, r_k) \) containing \( e_k \) such that \( x(t) < 0, t \in I_k \) and \( \min_{I_k} x(t) = x(e_k) \to m \) as \( k \to +\infty \), while \( e_k \) is the minimal point from \( I_k \) having this property. We claim that \( e_k - l_k \leq 1 \). On the contrary, let us suppose that \( e_k - l_k > 1 \). Then \( x_t < 0 \) (and, consequently, \( x'(t) \geq 0 \) for all \( t \)) from a small neighborhood of \( e_k \), contradicting to the choice of \( e_k \) as the leftmost point of global minimum in \( I_k \). Finally, observing that

\[
x(e_k) = \int_{l_k}^{e_k} f(s, x_s)ds \geq r(\max_{u \in [l_k - 1, l_k]} x(u)) \to 0, \quad k \to +\infty,
\]

we get a contradiction with the relation \( x(e_k) \to m < 0 \).

The case \( m \geq 0 \) is similarly addressed. \( \square \)

Remark 3. The last part of the above proof can be repeated to analyze the structure of the set of extreme points for every solution \( x(t) \) satisfying \( m < 0 \) and \( M > 0 \). We see that in that case there exist sequences of intervals \( A_k = (a_k, b_k) \), \( A_k' = (a_k', b_k') \) and points \( e_k \in A_k, \ e'_k \in A'_k \) such that \( x(a'_k) = x(a_k) = 0, \ e_k - a_k \leq 1, \ e_k - a'_k \leq 1 \) while \( x(e_k) \to m, \ x(e'_k) \to M \) and \( x(t) \) does not change sign over \( A_k, A'_k \). Moreover, for each \( k \), \( e_k \) and \( e'_k \) could be chosen as the points of global maximum of \( |x(t)| \) on \( A_k \) and \( A'_k \) respectively.
Now, we define continuous functions $A, B : (-1, +\infty) \to \mathbb{R}$ and $D : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = x + r(x) + \frac{1}{r(x)} \int_x^0 r(t)dt, \quad B(x) = \frac{1}{r(x)} \int_{-r(x)}^0 r(s)ds \quad \text{for } x \neq 0,$$

$$A(0) = B(0) = 0, \quad D(x) = \begin{cases} A(x) & \text{if } r(x) < -x, \\ B(x) & \text{if } r(x) \geq -x. \end{cases}$$

For the case $a < -1$, we will also use the rational function

$$R(x) = \frac{(A'(0))^2 x}{A'(0) - (A''(0)/2)x}$$

defined on the interval $(2A'(0)/A''(0), \infty) = (\nu, \infty)$. Note that $A'(0) = a + 1/2 < 0$, $A''(0) = -(2a + 1/3) > 0$. It is easy to check that $a < -1/6$ implies $(\nu, +\infty) \subset (-1, +\infty)$, and that $A(x_2) = B(x_2)$, where $r(x_2) = -x_2 < 0$. Also $B'(0) = -(r'(0))^2/2 = -a^2/2$.

The following relations were established in Lemma 2.4, Corollary 2.7:

**Lemma 3.** If $a < -1$, then $(A(x) - R(x))x > 0$ for $x \in (\nu, x_1) \setminus \{0\}$. Moreover, if $a \in [-1.5, -1.25]$, then $D(x) > R(x)$ for $x > 0$.

In the next stage of the proof, we establish various relations between $m$ and $M$, all of them being expressed in terms of the recently introduced functions. Notice that, by Lemma 2.4 the only case of interest is when $m < 0 < M$; thus we can suppose the existence of points $t_j, s_j$ of local maxima and local minima respectively such that $x(t_j) = M_j \to M, x(s_j) = m_j \to m$ and $s_j, t_j \to +\infty$ as $j \to \infty$.

**Lemma 4.** We have $m \geq D(M)$ and $m \geq r(-r(M)/2)$.

**Proof.** First we assume that $r(M) < -M$. Then $M/r(M) \in (-1, 0]$. Take now $\varepsilon > 0$ such that $t_1 = (M + \varepsilon)/r(M + \varepsilon) \in (-1, 0]$. Obviously, $M_j > m - \varepsilon$ and $M_j < M + \varepsilon$ for sufficiently large $j$. By Remark 4 there exists $\hat{s}_j \in [s_j - 1, s_j]$ such that $x(\hat{s}_j) = 0$ and $x(t) < 0$ for $t \in [\hat{s}_j, s_j]$.

Next, $z(t) = r(M + \varepsilon)(t - \hat{s}_j), \ t \in [\hat{s}_j + t_1, \hat{s}_j + t_1 + 1]$ solves the initial value problem $z(s) = M + \varepsilon, \ s \in [\hat{s}_j + t_1 - 1, \hat{s}_j + t_1]$ for

$$z'(t) = r(z(t) - 1)).$$

Clearly $M + \varepsilon = z(t) > x(t)$ for all $t \in [t_1 + \hat{s}_j - 1, \hat{s}_j + t_1]$. Moreover, we will prove that $z(t) > x(t)$ for all $t \in [\hat{s}_j + t_1, \hat{s}_j]$.

Indeed, if this is not the case we can find $t_\ast \in [\hat{s}_j + t_1, \hat{s}_j)$ such that $z(t_\ast) = x(t_\ast)$ and $z(t) > x(t)$ for all $t \in [\hat{s}_j + t_1, t_\ast]$. We claim that

$$x'(t) > z'(t) \quad \text{for all } t \in [t_\ast, \hat{s}_j].$$

We have

$$z'(t) = r(z(t) - 1)) < r(M(x_t)) \leq f(t, x_t) = x'(t).$$

After integration over $(t_\ast, \hat{s}_j)$, it follows from (5) that $x(t_\ast) < z(t_\ast)$, which is a contradiction.
Thus \( z(t) > x(t) \) for \( t \in [t_1 + \hat{s}_j, \hat{s}_j) \) and, arguing as above, we obtain
\[
m_j = \int_{\hat{s}_j}^{s_j} x'(s)ds = \int_{\hat{s}_j}^{s_j} f(s,x_s)ds \geq \int_{\hat{s}_j}^{s_j} r(M(x_s))ds > \int_{\hat{s}_j}^{s_j} r(z(s - 1))ds
\]
\[
= \int_{\hat{s}_j}^{s_j+1} r(M + \varepsilon)ds + \int_{\hat{s}_j+1}^{s_j} r(r(M + \varepsilon)(s - \hat{s}_j))ds
\]
\[
= r(M + \varepsilon)(t_1 + 1) + \int_{t_1}^{s_j-\hat{s}_j-1} r(r(M + \varepsilon)u)du
\]
\[
\geq M + \varepsilon + r(M + \varepsilon) + \int_{t_1}^{0} r(M + \varepsilon)u)du = A(M + \varepsilon).
\]
As a limit form of this inequality, we get \( m \geq A(M) \).

In the general case (i.e. we do not assume that \( r(M) < -M \)), we use the inequality \( f(t,x_t) \geq r(M(x_t)) > r(M + \varepsilon) \) to see that, for \( t \in (\hat{s}_j - 1, \hat{s}_j) \),
\[
x(t) = -\int_{t}^{\hat{s}_j} x'(s)ds < -\int_{t}^{\hat{s}_j} r(M + \varepsilon)ds = r(M + \varepsilon)(t - \hat{s}_j).
\]
In consequence,
\[
m_j = x(s_j) = \int_{\hat{s}_j}^{s_j} f(s,x_s)ds \geq \int_{\hat{s}_j}^{s_j} r(r(M + \varepsilon)(s - \hat{s}_j - 1))ds
\]
\[
\geq \int_{0}^{s_j-\hat{s}_j} r(r(M + \varepsilon)(s - 1))ds \geq \int_{0}^{1} r(r(M + \varepsilon)(u - 1))du = B(M + \varepsilon).
\]
Therefore, we obtain that \( m \geq B(M) \). Finally, applying Jensen’s integral inequality (see [10] p. 110) , we have
\[
m \geq B(M) = \frac{1}{r(M)} \int_{-r(M)}^{0} r(s)ds \geq r(-r(M)/2).
\]
This completes the proof. \( \square \)

As a consequence of Lemmas[3][4] we obtain that \( R(m), r(m) \) and \( r(-r(M)/2) \) are well-defined and that \( R(\nu, +\infty) \subset (\nu, +\infty) \) for suitable values of \( \alpha \):

**Corollary 2.** We have \( m > r(-r(M)/2) > -1 \) if \( a \in [-1,0) \) and \( m > \nu \) if \( a \in [-1.5, -1.25] \).

**Proof.** Indeed, for \( a \in (-2,0) \) we have
\[
m \geq r(-r(M)/2) > r(-r(+\infty)/2) = -\frac{a^2}{2-a} \geq -1.
\]
Next, \( -A'(0) = -(a+0.5) \leq 1 \) for \( a \geq -1.5 \), and Lemmas[3][4] lead to the estimate
\[
m \geq D(+\infty) = B(+\infty) \geq R(+\infty) = -A'(0)\nu \geq \nu.
\]
This proves the corollary. \( \square \)

**Lemma 5.** Let \( a \in [-1,0] \). We have \( M \leq r(m) \). Moreover, if \( a \in [-1.5, -1.25] \) then \( M < R(m) \).

**Proof.** We have that \( r(m) \) is well defined and \( [m,+\infty) \subset [-1,+\infty) \) since \( a \in [-1.5,0] \) (see Corollary[2]). Let us consider a solution \( x(t) \) of Eq. (1) and take \( s_j, t_j, m_j, M_j \), as in the paragraph below Lemma[3] By Remark[3] there exists \( \hat{t}_j \in [t_j - 1, t_j) \) such that \( x(\hat{t}_j) = 0 \).
First we note that, for \( \varepsilon > 0 \) and \( j \) sufficiently large, \( f(s, x_s) \leq r(M(x_s)) < r(m - \varepsilon) \) for \( s \in [\tilde{t}_j - 1, t_j] \) and \( a \in [-1.5, 0) \). Thus

\[
M_j = x(t_j) = \int_{\tilde{t}_j}^{t_j} f(s, x_s)ds < \int_0^1 r(m - \varepsilon)ds = r(m - \varepsilon).
\]

By taking the limits as \( \varepsilon \to 0 \) and \( j \to +\infty \), we obtain the inequality \( M \leq r(m) \).

Now, if \( a < -1 \) then \( r(m) > -m \), from which for all sufficiently small \( \varepsilon > 0 \) we obtain that \( t_2 = (m - \varepsilon)r(m - \varepsilon)^{-1} \in (-1, 0) \). Next, \( z(t) = r(m - \varepsilon)(t - \tilde{t}_j) \), with \( t \in [\tilde{t}_j + t_2, \tilde{t}_j + t_2 + 1] \), solves the initial value problem \( z(s) = m - \varepsilon, \ s \in [\tilde{t}_j + t_2 - 1, \tilde{t}_j + t_2] \) for Eq. (5). Now we only have to argue as in the proof of Lemma 3 to find out that

\[
M_j = \int_{\tilde{t}_j}^{t_j} f(t, x_t)dt \leq \int_{\tilde{t}_j}^{t_j} r(-M(x_t))dt \leq \int_{\tilde{t}_j}^{t_j} r(z(t - 1))dt \leq A(m - \varepsilon).
\]

Thus \( M \leq A(m) \). Finally, by Lemma 3 and Corollary 2 we obtain \( M \leq A(m) < R(m) \) when \( a \in [-1.5, -1.25] \).

2.2. Proof of Theorem 2. Let \( x : [\alpha - h, +\infty) \to \mathbb{R} \) be a solution of Eq. (1) and set \( M = \limsup_{t \to -\infty} x(t), \ m = \liminf_{t \to -\infty} x(t) \). We will reach a contradiction if we assume that \( m < 0 < M \) (note that the cases \( M \leq 0 \) and \( m \geq 0 \) were already considered in Lemma 2).

Assume first that \( b = 1 \). If \( a \in (-1.5, 0) \), in view of Lemmas 3, 5 and Corollary 2 we obtain that \( M \leq r(m) \leq r(r(-M)/2) = \lambda(M) \) with the rational function \( y = \lambda(x) \). Now, \( \lambda(M) < M \) for \( M > 0 \) if \( \lambda(0) = (1/2)a^3 < 1 \). Therefore, if \( a \in [-1.25, 0) \) we obtain the desired contradiction under the assumption \( M > 0 \).

Now let \( a \in [-1.5, -1.25] \) and, consequently, \( R'(0) = a + 0.5 \in [-1, -0.75] \). In this case Lemmas 3, 4 and 5 imply that \( M < R(R(M)) \). As \( R \circ R(x) \leq x \) for all \( x > 0 \) whenever \( (R \circ R')(0) = (R'(0))^2 \leq 1 \), we obtain a contradiction again. Therefore \( x(t) \equiv 0 \) is the global attractor of Eq. (1) if \( a \in [-1.5, 0) \).

If \( b = 0 \), we employ the linear function \( r(x) = ax \). Arguing as in the proofs of Lemmas 4 and 5, we obtain that \( m \geq (a + 1/2)M, \ M \leq (a + 1/2)m \) for \( a \in (-3/2, -1] \), and \( m \geq (-1/2)a^2M, \ M \leq am \) for \( a < 0 \) (see 7 for more details).

Hence, if \( a \in (-3/2, -1] \), we get the contradiction \( M \leq (a + 1/2)a^2M < M \). Finally, if \( a \in (-1, 0) \), we obtain \( M \geq (-1/2)a^2M \geq (-1/2)a^3m \). Thus, \( -a^3/2 \geq 1 \), a contradiction.

3. An Application

Probably, the most interesting object to which we can apply our results is the following generalization of the logistic delayed equation:

(7) \( x'(t) = \lambda(t)x(t)f(t, L(t, x_t)), \ t \geq 0, \ x \geq 0 \).

Here \( \lambda : [-h, \infty) \to (0, \infty) \) is measurable and \( \int_0^\infty \lambda(s)ds = \infty, \sup_{t \geq 0} \int_{-h}^t \lambda(s)ds < \infty \). We suppose that \( f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) and \( L(t, \phi) : \mathbb{R}_+ \times \mathcal{C}([-h, 0]) \to \mathbb{R} \) are Carathéodory functions, and that

\[
\min \phi \leq L(t, \phi) \leq \max \phi
\]

for every \( \phi \in \mathcal{C}([-h, 0]) \), \( t \geq 0 \). We are interested in the case when, apart from \( x(t) \equiv 0 \), Eq. (7) has another equilibrium \( x(t) \equiv \kappa \). Without loss of generality we can assume that \( \kappa = 1 \) (and, consequently, that \( f(t, 1) \equiv 0 \)). Finally, we will require the divergence of \( \int_0^x \lambda(s)f(s, w(s))ds \) for every continuous function \( w \) converging
to some positive number different from 1, as well as the following negative feed-back
condition:
\[ x f(t, 1 + x) < 0, \forall x > -1, \ x \neq 0. \]
As a simple observation shows, every nontrivial solution of (7) is eventually positive,
so that we will only study the behavior of the positive solutions.

The next result is a consequence of Theorem 2.

**Theorem 3.** Assume that
\[ x (f(t, x + 1) - r(x)) \geq 0 \]
for some \( r(x) = ax/(1 + bx) \) with \( a < 0, \ b \geq 0 \) and for all \( x > \max\{-1, -b^{-1}\} \).
If \( b \neq 0.5 \) and, for some \( T > 0, |a| \Lambda \leq 3/2 \), then \( \lim_{t \to +\infty} x(t) = 1 \) for every
nontrivial solution of (7), where \( \Lambda = \sup_{t \geq r} \int_{t-h}^{t} \lambda(s)ds \). If \( b = 0.5 \), then the same
conclusion holds when \( |a| \Lambda \leq 3/2 \).

**Proof.** First, let \( b \geq 0.5 \). The change of variables \( y(s) = \ln x(t) \), where \( s = s(t) = \Lambda^{-1} \int_{0}^{t} \lambda(u)du \) (with the inverse \( t = t(s) \) to \( s = s(t) \)), reduces (7) to
\[ y'(s) = g(s, \mathcal{K}(s, y)), \ y \in \mathbb{R}, \ s \geq 0. \]
Here, \( g(s, x) = f(t(s), x) \) and \( \mathcal{K}(s, \phi) = \mathcal{L}(t(s), \Psi) \), where \( \Psi = \Psi(s, \phi) \) is defined by \( [\Psi(s, \phi)](u) = \exp[\phi(\sigma(s, u))] \), being \( \sigma(s, u) = \Lambda^{-1} \int_{t(s)}^{u} \lambda(v)dv \). Notice that
\( \mathcal{K} : \mathbb{R}_{+} \times C([-1, 0]) \to \mathbb{R} \) is a Carathéodory function, \( \exp(-\mathcal{M}(\phi)) \leq \mathcal{K}(s, \phi) \leq \exp(\mathcal{M}(\phi)) \) for every \( \phi \in C([-1, 0]) \), \( s \geq 0 \), and \( \mathcal{K}(s, 0) \equiv 1 \). Since \( g(s, \mathcal{K}(s, \phi)) = f[t(s), 1 + \mathcal{K}(s, \phi) - 1] \) we have, for \( \mathcal{K}(s, \phi) \leq 1 \), that
\[ r_{1}(\mathcal{M}(\phi)) \leq 0 \leq \Lambda g(s, \mathcal{K}(s, \phi)) \leq \Lambda r(\mathcal{K}(s, \phi) - 1) \leq \Lambda r(\exp[-\mathcal{M}(\phi)] - 1) \leq r_{1}(-\mathcal{M}(\phi)), \]
where \( r_{1}(x) = ax/(1 + (b - 0.5)x) \). Indeed, function \( v(x) = r(e^{x} - 1) \) satisfies conditions (A1)-(A3) in Theorem 1 and therefore, by Lemma 1 \( r(e^{x} - 1) \geq r_{1}(x) \) for \( x < 0 \), and \( r(e^{x} - 1) \geq r_{1}(x) \) for \( x > 0 \), since \( v'(0) = a, v''(0) = a(1 - 2b) \).

Analogously, if \( \mathcal{K}(s, \phi) \geq 1 \) then
\[ r_{1}(\mathcal{M}(\phi)) \geq 0 \geq \Lambda g(s, \mathcal{K}(s, \phi)) \geq \Lambda r(\mathcal{K}(s, \phi) - 1) \geq \Lambda r(\exp[\mathcal{M}(\phi)] - 1) \geq r_{1}(\mathcal{M}(\phi)). \]
Let now \( b \in [0, 0.5] \). In this case, the change of variables \( z(s) = -\ln x(t) \), where
\( s = s(t) \) reduces (7) to the form
\[ z'(s) = -\Lambda g(s, \mathcal{K}(s, -z)), \ z \in \mathbb{R}, \ s \geq 0. \]
We have, for \( \mathcal{K}(s, -\phi) \geq 1 \), that
\[ r_{2}(\mathcal{M}(\phi)) \leq 0 \leq -\Lambda g(s, \mathcal{K}(s, -\phi)) \leq -\Lambda r(\mathcal{K}(s, -\phi) - 1) \leq -\Lambda r(\exp[-\mathcal{M}(\phi)] - 1) \leq r_{2}(-\mathcal{M}(\phi)), \]
where \( r_{2}(x) = a\Delta x/(1 + (0.5 - b)x) \). (Here we use \( u(x) = -r(e^{x} - 1) \) and argue
as before). Next, if \( \mathcal{K}(s, -\phi) \leq 1 \), we obtain
\[ r_{2}(-\mathcal{M}(\phi)) \geq 0 \geq -\Lambda g(s, \mathcal{K}(s, -\phi)) \geq -\Lambda r(\mathcal{K}(s, -\phi) - 1) \geq -\Lambda r(\exp[-\mathcal{M}(\phi)] - 1) \geq r_{2}(\mathcal{M}(\phi)). \]

Hence, both equations (9) and (12) are of Carathéodory type and satisfy (GY).
Now, if \( \lim_{s \to +\infty} w(s) = w_{*} \neq 0 \), then \( \int_{\mathbb{R}_{+}} g(s, w(s))ds = \int_{\mathbb{R}_{+}} \lambda(t)f(t, w_{1}(t))dt \)
for some \( w_{1}(t) \) with \( \lim_{t \to +\infty} w_{1}(t) = \exp(w_{*}) \neq 1 \) so that (H3) is also fulfilled.
Finally, the boundedness requirement from (H1) follows easily from [10], [11], [12], [13], [14]. In this way, we can finish the proof of Theorem 3 by applying Theorem 2 to Eqns. 15 and 12.

**Remark 4.** Notice that Theorem 3 still remains true if, in its statement, we replace the rational function \( r(x) \) by any decreasing function \( \tilde{r} : (-1, \infty) \to \mathbb{R} \) having negative Schwarz derivative, satisfying \( 5 \), and such that \( \tilde{r}'(0) = a < 0 \) and \( \tilde{r}' \) is below bounded if \(-\tilde{r}''(0)/(2\tilde{r}'(0))\geq1/2\). Indeed, in this case we still can use Lemma 4 while evaluating \( \nu(x) = \tilde{r}(\exp(\pm x) - 1) \) (see [10], [13], [15], [16]). The sign of the second derivative of \( \tilde{r} \) (associated with the sign of \( b \) before) does not matter now since \( \tilde{r} \) is defined for all \( t > -1 \). It is clear that \(|a|\Lambda \leq 3/2 \) implies the global stability when \( \rho''(0) \neq 0 \), while for \( \rho''(0) = 0 \) we should assume that \(|a|\Lambda < 3/2 \).

**Example 1.** Let us apply Theorem 3 to study the food-limited population model [11, 2, 5, 6, 11] with variable, continuous and generally unbounded delay \( h(t) \geq 0 \) such that \( \inf_{t \geq 0} (t - h(t)) = h_* \) is finite (obviously, \( h_* \leq 0 \)):

\[
N'(t) = \lambda(t)N(t) \frac{k - N(t - h(t))}{k + \nu(t)N(t - h(t))}, \quad t \geq 0.
\]

Here \( k, l > 0, \lambda \in C([h_*^+, \infty), (0, \infty)), \nu \in C([0, \infty), [0, \infty)) \) and \( \int_{R_+} \lambda = +\infty \).

**Corollary 3.** Let \( \nu_0 = \inf_{t \geq 0} \nu(t) \geq 0 \), and assume that for some \( T \geq 0 \) either \( \nu_0 \neq 1 \) and \( \Lambda/1 + \nu_0 \leq \frac{3}{2} \), or \( \nu_0 = 1 \) and \( \Lambda/1 + \nu_0 \leq \frac{2}{3} \), where \( \Lambda = \sup_{t \geq T} \left\{ \int_{t-h(t)} \lambda(s)ds \right\} \). Then every positive solution of equation (14) converges to \( k^{1/l} \).

**Proof.** With \( N(t) = k^{1/l} x(s) \) and \( s = s(t) = \Lambda^{-1} \int_0^t \lambda(u)du \), Eq. 15 is transformed into \( x'(s) = \Lambda x(s) f(s, \mathcal{L}(s, x(s))) \), where \( \mathcal{L}(s, \phi) = \phi(-\Lambda^{-1} \int_{t(s)-h(t(s))}^{t(s)} \lambda(u)du) \), and \( f(s, x) = (1 - x^l)/(1 + \nu(t(s)))x^l \).

In order to apply Theorem 3 to the transformed equation, we consider function

\[
f^*(s, x) = f(t(s), x + 1) = \frac{1 - (x + 1)^l}{1 + \nu(t(s))(x + 1)^l}, \quad x > -1.
\]

It is obvious that \( x f^*(s, x) < 0 \) for all \( x \neq 0 \). Next, \((1 - (x + 1)^l)/(1 + u(x + 1)^l)\) is increasing with respect to \( u \) for \( x > 0 \) fixed and decreasing with respect to \( u \) for fixed \( x < 0 \). Hence, \( f^*(s, x) \geq \tilde{r}(x) \) for \( x > 0 \), and \( f^*(s, x) \leq \tilde{r}(x) \) for \( x < 0 \), where

\[
\tilde{r}(x) = \frac{1 - (x + 1)^l}{1 + \nu_0(x + 1)^l}, \quad x > -1.
\]

Finally, it is easy to check that \( \tilde{r} \) satisfies all conditions indicated in Remark 4. Notice that \( a = \tilde{r}'(0) = -l/(1 + \nu_0) < 0 \) and \( \rho''(0) = 0 \) only if \( \nu_0 = 1 \). \( \square \)

**Remark 5.** In the particular case when the delay is constant, Corollary 3 was proved in [2] under the stronger assumptions \( l\Lambda/(1 + \nu_0) \leq 1 \) and

\[
\int_0^\infty \frac{\lambda(s)}{1 + \nu(s)} ds = \infty.
\]

Also, Liu in [5] considered

\[
N'(t) = \lambda(t)N(t) \left( \frac{k - N(t - h)}{k + \nu(t)N(t - h)} \right)^{\beta}, \quad t \geq 0,
\]
where $\beta = (2m + 1)/(2n + 1) \geq 1$, $m, n \in \mathbb{N}$, getting our $3/2$ condition only in the particular case when $\nu_0 = 1$ (see [8, Theorem 2]). We remark that, for $\beta > 1$, the global attractivity of (17) can be always proved once (16) is assumed. Notice also that the statement of Corollary 3 remains true if we replace all entries of $N^t(t-h(t))$ in (15) by $N^t(\mu t)$, with $\mu \in (0, 1)$, or by $\max_{t-h_{0}} N(u)$ for some $h_{0} > 0$ (thus we can include in our considerations equations with linearly transformed argument and equations with maxima).

**Remark 6.** Assume (16). In [11], it has been established that the steady state of (15) with $h(t) = h$ is (locally) uniformly and asymptotically stable if

$$
\int_{t-h}^{t} \frac{\lambda(s)}{1 + \nu(s)} ds \leq \alpha < 3/2, \ t \geq h.
$$

This inequality is less restrictive than the one we obtained in Corollary 3; thus, inspired by the Wright conjecture about the equivalence of global and local asymptotic stability, one can try to improve our result up to (18). However, it would be impossible: even with (18) satisfied, Eq. (15) can have nontrivial periodic solutions.

**References**

[1] K. Gopalsamy, “Stability and oscillations in delay differential equations of population dynamics,” Mathematics and its Applications, 74, Kluwer, Dordrecht, 1992.
[2] E. A. Grove, G. Ladas and C. Qian, Global attractivity in a ‘food limited’ population model, Dynamic Syst. Appl., 2 (1993), 243-250.
[3] J. K. Hale and S. M. Verduyn Lunel, “Introduction to functional differential equations”, Applied Mathematical Sciences, Springer-Verlag, 1993.
[4] A. Ivanov, E. Liz and S. Trofimchuk, Halanay inequality, Yorke $3/2$ stability criterion, and differential equations with maxima, Tohoku Math. J. 54 (2002), 277-295.
[5] Y. Kuang, “Delay differential equations with applications in population dynamics,” Academic Press, 1993.
[6] Y. Liu, Global attractivity for a differential-difference population model, Appl. Math. E-Notes 1 (2001), 56-64 (electronic).
[7] E. Liz, M. Pinto, G. Robledo, V. Tkachenko and S. Trofimchuk, Wright type delay differential equations with negative Schwarzian, Discrete Contin. Dynam. Systems 9 (2003), 399-321.
[8] E. Liz, V. Tkachenko and S. Trofimchuk, A global stability criterion for scalar functional differential equations, submitted (available at [arXiv.org/abs/math.DS/0112047](http://arXiv.org/abs/math.DS/0112047)).
[9] A. D. Myshkis, “Lineare Differentialgleichungen mit Nacheilendem Argument,” Deutscher Verlag Wiss., Berlin, 1955.
[10] H. L. Royden, “Real Analysis,” The Macmillan Company, 1969.
[11] J. W.-H. So and J. S. Yu, On the uniform stability for a ‘food-limited’ population model with time delay, Proc. Roy. Soc. Edinburgh Sect. A 125 (1995), 991–1002.
[12] H.-O. Walther, Contracting return maps for monotone delayed feedback, Discrete Contin. Dynam. Systems 7 (2001), 259–274.
[13] E. M. Wright, A nonlinear difference-differential equation, J. Reine Angew. Math. 194 (1955), 66–87.
[14] J. A. Yorke, Asymptotic stability for one dimensional differential-delay equations, J. Differential Equations 7 (1970), 189–202.

E-mail address: eliz@dma.uvigo.es
E-mail address: vitk@imath.kiev.ua
E-mail address: trofimch@uchile.cl