GENERIC COHEN-MACaulay MONOMIAL IDEALS

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Abstract. Given a simplicial complex, it is easy to construct a generic
deformation of its Stanley-Reisner ideal. The main question under in-
vestigation in this paper is how to characterize the simplicial complexes
such that their Stanley-Reisner ideals have Cohen-Macaulay generic de-
formations. Algorithms are presented to construct such deformations
for matroid complexes, shifted complexes, and tree complexes.

1. Introduction

Let \(S := k[x_1, \ldots, x_n]\) be a polynomial ring in \(n\) variables over a field
\(k\). A monomial \(x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\) is called square-free if \(a_i = 0\) or \(1\)
for all \(1 \leq i \leq n\). For monomials \(x^a\) and \(x^b\), we say \(x^a\) strictly divides \(x^b\) if, for \(1 \leq i \leq n\) and \(a_i > b_i\). A monomial ideal \(M\) is an ideal minimally generated by monomials, say \(m_1, \ldots, m_t\), (written as \(M = \langle m_1, \ldots, m_t \rangle\)). The monomial ideal \(M\) is called square-free if all \(m_i\) are square free monomials.

Definition 1.1. A monomial ideal \(M = \langle m_1, \ldots, m_t \rangle\) is called generic if for any two distinct minimal generators \(m_i\) and \(m_j\) with the same positive degree in some variable \(x_h\), there exists a third minimal generator \(m_l\) such that \(m_l\) strictly divides \(\text{lcm}(m_i, m_j)\).

Example 1.2. The monomial ideal \(\langle x^2y^2, x^2z^2, xz \rangle \subset k[x, y, z]\) is generic but \(\langle xy^2, xz^2, xz \rangle \subset k[x, y, z]\) is not generic.

This definition of generic monomial ideals has appeared in \([5]\), generalizing a definition in \([1]\). See \([10]\) for more information about generic monomial ideals.

Definition 1.3. A simplicial complex \(\Gamma\) on the vertex set \([n] := \{1, \ldots, n\}\) is a non-empty collection of subsets of \([n]\) such that \(\{i\} \in \Gamma\) for all \(i \in [n]\), and whenever \(\tau \subset \sigma\) and \(\sigma \in \Gamma\), then \(\tau \in \Gamma\). The elements of \(\Gamma\) are called faces, and the dimension of the face \(\sigma \in \Gamma\) is \(\dim(\sigma) := |\sigma| - 1\). The dimension of \(\Gamma\) is \(\dim(\Gamma) := \max\{\dim(\sigma) : \sigma \in \Gamma\}\). Faces of dimension zero are called vertices and faces that are maximal under inclusion are called facets. The
set of minimal non-faces of $\Gamma$ is $\Sigma_\Gamma := \{\sigma \subseteq [n] : \sigma \notin \Gamma$ but $\sigma \setminus \{i\} \in \Gamma$ for each $i \in \sigma\}$.

There is a one-to-one correspondence between square-free monomial ideals in $S$ and simplicial complexes on $[n]$. Given a simplicial complex $\Gamma$ on the vertex set $[n]$, the Stanley-Reisner ideal $I_\Gamma$ of $\Gamma$ is the square-free monomial ideal $\langle x_{i_1} \cdots x_{i_s} : \{i_1, \ldots, i_s\} \in [n] \setminus \Gamma \rangle \subset S$. Conversely, for a given nonzero square-free monomial ideal $M \subset S$, there exists a simplicial complex $\Gamma$ on $[n]$ such that $I_\Gamma = M$.

**Definition 1.4.** Let $I = \langle m_1, \ldots, m_t \rangle$ be a square-free monomial ideal. A deformation of $I$ is a monomial ideal $M = \langle m^*_1, \ldots, m^*_t \rangle$ such that, for all $1 \leq i \leq t$, $m^*_i = x_{i_1}^{j_1} \cdots x_{i_s}^{j_s}$, where $m_i = x_{i_1} \cdots x_{i_s}$ and $(j_1, \ldots, j_s) \in (\mathbb{N} \setminus \{0\})^s$.

Note that our definition of a deformation is different from the ones given at [1] [5]. In particular, our deformations are monomial ideals.

For a given simplicial complex $\Gamma$, one can easily find a deformation $M$ of the Stanley-Reisner ideal $I_\Gamma$ such that $M$ is a generic monomial ideal. In this paper, we characterize simplicial complexes such that their corresponding Stanley-Reisner ideals have deformations that are generic and Cohen-Macaulay. To be precise, we provide algorithms to construct generic Cohen-Macaulay deformations from the Stanley-Reisner ideals of matroids, shifted complexes, and tree complexes. A version of this question appeared in [3] where generic Cohen-Macaulay monomial ideals have been studied extensively.

**Definition 1.5.** Let $M = \langle m_1, \ldots, m_t \rangle$ be a monomial ideal in $S$. For $\sigma \subseteq [t]$, let $m_\sigma := \text{lcm}(m_i : i \in \sigma)$. The Scarf complex of $M$ is

\[ \Delta_M := \{\sigma \subseteq [t] : m_\sigma \neq m_\tau \text{ for all } \tau \subseteq [t] \text{ and } \tau \neq \sigma\}. \]

For $D$ large enough (larger than any exponent of any variable in any minimal generator of $M$), let

\[ M^* := M + \langle x_1^D, \ldots, x_n^D \rangle. \]

The extended Scarf complex of $M$ is the Scarf complex $\Delta_{M^*}$ of $M^*$.

It is easy to see that the induced subcomplex of $\Delta_{M^*}$ on the set of generators of $M$ is $\Delta_M$. On the other hand, Miller et. al. [5] proved that if $M$ is generic, then the induced subcomplex of $\Delta_{M^*}$ on the set $\{x_1^D, \ldots, x_n^D\}$ is the simplicial complex $V(M)$, where $I_{V(M)} = \text{rad}(M)$.

**Example 1.6.** Let $\Gamma$ is the simplicial complex in Figure [1] and let $M = \langle x_1^2 x_2^2 x_3^2, x_1 x_4, x_2^3 x_4^3 \rangle$. It is clear that $M$ is a generic deformation of $I_\Gamma$ (in particular, $\text{rad}(M) = I_\Gamma$ and hence $V(M) = \Gamma$). On the other hand, let $M^* := M + \langle x_1^3, x_2^3, x_3^3, x_4^3 \rangle$. Figure [2] gives $\Delta_M$ and $\Delta_{M^*}$. The set of facets of $\Delta_{M^*}$ is

\[ \{\{x_1 x_4, x_2^2 x_3^2, x_1 x_2, x_3^2\}, \{x_1 x_4, x_2^2 x_2^2 x_3, x_1 x_3^2, x_3^2\}, \{x_1 x_4, x_2^2 x_2^2 x_3, x_3^2, x_3^2\}, \{x_1 x_4, x_2^2 x_3^3, x_2 x_3^3, x_3^3\}, \{x_1 x_4, x_2^2 x_2^2 x_3, x_3^2, x_2^3, x_3^3\}, \{x_1 x_4, x_2^2 x_4^3, x_2 x_3^3, x_3^3\}, \{x_1 x_4, x_2^2 x_4^3, x_3^3, x_2^3\}, \{x_1 x_4, x_2^2 x_4^3, x_3^3, x_2^3\}\}. \]
The following theorem is proved as part of [5, Theorem 1.7].

**Theorem 1.7.** Let $M$ be a generic monomial ideal. For each facet $\sigma$ of $\Delta_{M^*}$, let

$$M_{\sigma} := \langle x_{s}^{p_{s}} : p_{s} := \deg_{x_{s}}(m_{\sigma}) \text{ and } p_{s} < D \rangle.$$ 

Then

$$M = \bigcap_{\sigma \text{ is a facet of } \Delta_{M^*}} M_{\sigma}$$

is a minimal irreducible decomposition of $M$.

**Example 1.6 (cont.).** By Theorem 1.7, each facet of $\Delta_{M^*}$ corresponds to an irreducible component of $M$, and hence $\Delta_{M^*}$ gives a minimal irreducible
decomposition of $M$ which implies a minimal primary decomposition of $M$:
\[ M = \langle x_2^2, x_4 \rangle \cap \langle x_1^2, x_4 \rangle \cap \langle x_1, x_2 \rangle \cap \langle x_1, x_2^2 \rangle. \]

**Theorem 1.8** ([5] Theorem 2.5]). Let $M$ be a generic monomial ideal. Then $M$ has no embedded associated primes if and only if $M$ is Cohen-Macaulay. In this case, both $\Delta_M$ and $V(M)$ are shellable.

For a given simplicial complex $\Gamma$, the theorem above implies that if $\Gamma$ is not shellable, then there is no generic Cohen-Macaulay monomial ideal $M$ such that $V(M) = \Gamma$. In particular, there is no generic Cohen-Macaulay deformation of $I_\Gamma$. The following theorem will play a major role subsequently.

**Theorem 1.9.** Let $M \subset S$ be a generic monomial ideal. Then $M$ is Cohen-Macaulay if and only if $\dim(\Delta_M) + \dim(V(M)) = n - 2$.

**Proof.** Suppose $M$ is Cohen-Macaulay. By Theorem 1.7 each facet of $\Delta_{M^*}$ gives an irreducible component of $M$. In particular, for any facet $\sigma$ of $\Delta_{M^*}$, we have
\[ |\sigma \cap \{m_1, \ldots, m_t\}| = \text{codim}(M) \quad \text{and} \quad |\sigma \cap \{x_1^D, \ldots, x_n^D\}| = \dim(R/M). \]
Furthermore, both cardinalities are independent of the facet $\sigma$, since $M$ has no embedded primes. But $\sigma \cap \{m_1, \ldots, m_t\}$ is a face of $\Delta_M$ and any facet of $\Delta_M$ is a restriction of a facet of $\Delta_{M^*}$. Thus $\dim(\Delta_M) = \text{codim}(M) - 1$. Hence $\sigma \cap \{x_1^D, \ldots, x_n^D\}$ is a facet of $V(M)$ and therefore $\dim(V(M)) = \dim(R/M) - 1$. But $\text{codim}(M) + \dim(R/M) = n$. Thus, $\dim(\Delta_M) + \dim(V(M)) = n - 2$.

Conversely, suppose $\dim(\Delta_M) + \dim(V(M)) = n - 2$. Then it is enough to show that for any facet $\sigma$ of $\Delta_{M^*}$, we have $|\sigma \cap \{m_1, \ldots, m_t\}| = n - d$, where $\dim(V(M)) = d - 1$, i.e., $\dim(\Delta_M) = n - d - 1$. This proves that all irreducible components are of the same codimension, namely $n - d$, and hence there are no inclusions among them. Thus the irreducible decomposition is minimal and hence all the associated primes of $M$ are of the same codimension. Thus $M$ has no embedded primes. By Theorem 1.8, $M$ is then Cohen-Macaulay.

To this end, let $\sigma$ be a facet of $\Delta_{M^*}$. If $|\sigma \cap \{m_1, \ldots, m_t\}| > n - d$, then $\dim(\Delta_M) > n - d - 1$, a contradiction. Now if $|\sigma \cap \{m_1, \ldots, m_t\}| < n - d$, then $|\sigma \cap \{x_1^D, \ldots, x_n^D\}| > d$. Thus $\dim(V(M)) > d - 1$, a contradiction. Therefore, $M$ is Cohen-Macaulay. \qed

**Example 1.10.** Let $\Gamma$ be the 1-dimensional simplicial complex (graph) in Figure 5. It is clear that $\Gamma$ is shellable, since it is connected. The Stanley-Reisner ideal of $\Gamma$ is
\[ I_\Gamma = \langle x_1x_2x_3, x_1x_4, x_1x_5, x_2x_4, x_3x_5 \rangle. \]
Let $M_1 = \langle x_1^2x_2^2x_3^2, x_2^3x_3^3, x_2^3x_3^3 \rangle$, $M_2 = \langle x_1^2x_4^2, x_2^2x_5^2 \rangle$, $M_3 = \langle x_1x_5 \rangle$, and let $M = M_1 + M_2 + M_3$. It is clear that $M$ is a deformation of $I_\Gamma$. Moreover, $M$ is
generic. Namely, for $1 \leq i \leq 2$, given any two distinct minimal generators $m_1, m_2 \in M_i$, there exists a third generator $m_3$ in $M_j$ for some $j > i$ such that $m_3$ strictly divides $\text{lcm}(m_1, m_2)$. Thus $\dim(\Delta_M) \leq 2 = 5 - 2 - 1 = n - d - 1$. But $\{x_1x_5, x_1^2x_4, x_2^3x_3\}$ is clearly a face of $\Delta_M$. So $\dim(\Delta_M) = 2$.

Now, by Theorem 1.9, $M$ is Cohen-Macaulay.

In the next sections, we show that the Stanley-Reisner ideals of matroid, shifted, or tree complexes have generic Cohen-Macaulay deformations.

2. Matroid Complexes

In this section, we prove that the Stanley-Reisner ideals of matroid complexes have generic Cohen-Macaulay deformations. There are many equivalent definitions of matroids, see [9], or [8, §III.3]. The following definition uses the so-called circuits axiom.

**Definition 2.1.** A simplicial complex $\Delta$ is a *matroid* if for any two minimal non-faces (circuits) $\alpha$ and $\beta$ with an $i \in \alpha \cap \beta$, there exists a minimal non-face $\gamma$ such that $\gamma \subseteq (\alpha \cup \beta) \setminus \{i\}$.

The following corollary follows directly from [8, Proposition 3.1].

**Corollary 2.2.** Let $\Gamma$ be a matroid complex on the vertex set $[n]$. For every subset $W$ of $[n]$, the induced subcomplex $\Gamma_W := \{\sigma \in \Gamma : \sigma \subseteq W\}$ of $\Gamma$ is a matroid complex. In particular, $\Gamma_W$ is pure and shellable.

**Example 2.3.** The simplicial complex $\Gamma$ in Figure 3 is a matroid. The Stanley-Reisner ideal of $\Gamma$ is $I_\Gamma = \langle x_1x_3, x_1x_2x_4, x_2x_3x_4 \rangle$.

The following theorem is the main result of this section.

**Theorem 2.4.** Let $\Gamma$ be a matroid complex on the vertex set $[n]$ and suppose $\dim(\Gamma) = d - 1$. Then there exists a generic Cohen-Macaulay deformation of $I_\Gamma$.

**Proof.** Pick a facet $\sigma$ of $\Gamma$. Without loss of generality, $\sigma = \{1, \ldots, d\} = [d]$. There exists a unique minimal non-face $\alpha \subset [d + 1]$: By Corollary 2.2, the induced subcomplex $\Gamma_{[d+1]}$ is pure and $\sigma \in \Gamma_{[d+1]}$, in particular,
dim(Γ_{d+1}) = d + 1. But Γ_{d+1} is pure, so there exists a unique minimal non-face α ⊂ [d+1]. Let m_α = \prod_{j \in \alpha} x_j. Let M_1 := \langle m_\alpha \rangle, a monomial ideal in S. By induction, for i > 1, let

\[ \Sigma_i := \{ \alpha \subset [d+i] : \alpha \text{ is a minimal non-face of } \Gamma \text{ and } d+i \in \alpha \}, \]

be the set of minimal non-faces in [d+i] but not in [d+i-1]. Let

\[ M_i := \langle \prod_{j \in \alpha} x_j^i : \alpha \in \Sigma_i \rangle. \]

Since we have n vertices, we will have all the minimal non-faces at the inductive step n-d.

Let M = M_1 + \cdots + M_{n-d}. Then M is generic and V(M) = Γ. It remains to show that M is Cohen-Macaulay. By Theorem 1.9, it is enough to show that dim(∆_M) = n - d - 1.

First, we show that, for all i > 0, M_i ≠ 0: at each inductive step i, 2 ≤ i ≤ n - d, Γ_{d+i} is a matroid complex on d+i vertices and dim(Γ_{d+i}) = d - 1. For if M_i = 0, there is no missing face with the vertex d+i, in particular σ ∪ \{d+i\} ∈ Γ. This is a contradiction, since σ is a facet of Γ. Furthermore, dim(∆_M) = n - d - 1: For each 1 ≤ i ≤ n - d, let

\[ F_i := \sigma \cup \{d+i\}. \]

Then there exists a unique minimal non-face (circuit) c_i of Γ such that d+i ∈ c_i ⊂ F_i. Let

\[ m_i := \prod_{j \in c_i} x_j^i. \]

Let G = \{m_1, \ldots, m_{n-d}\}. We will show that G is a face of ∆_M and hence dim(∆_M) = n - d - 1. Notice that

\[ \text{lcm}(G) := \text{lcm}(m_1, \ldots, m_{n-d}) = f \cdot x_{d+1}x_{d+2} \cdots x_{n-d}, \]

where f is a monomial in the variables x_1, ..., x_d. Moreover, for any proper subset H ⊂ G, it is easy to see that lcm(H) ≠ lcm(G).

Suppose that, for some 1 ≤ t ≤ n - d, there exists a generator m ∈ M_t such that m divides lcm(G), say m = x_{i_1}^t x_{i_2}^t \cdots x_{i_s}^t x_{d+t}, where 1 ≤ i_1 ≤
\[ \cdots \leq i_s < d + t. \] Thus \( i_s \leq d. \) This implies that \( m \) corresponds to the unique minimal non-face \( \alpha = \{i_1, \ldots, i_s, d + t\} \) which is \( m_i \). Hence \( G \) is a face of \( \Delta_M \) and therefore \( \dim(\Delta_M) = n - d - 1. \)

**Example 2.5.** For the simplicial complexes \( \Gamma \) in Figure 3, \( \sigma = \{1, 2\} \) and \( \Gamma[3] \) has the facets \( \sigma \) and \( \{2, 3\} \). Thus \( M_1 = \langle x_1x_3 \rangle \). It is clear that \( \Gamma[4] = \Gamma \). Thus we need to compute only \( M_2 \). But \( \Sigma_2 = \{\{1, 2, 4\}, \{2, 3, 4\}\} \). Thus \( M_2 = \langle x_1^2x_2^3x_3^4, x_2^2x_3^2x_4^2 \rangle \). Let \( M = M_1 + M_2 = \langle x_1x_3, x_1^2x_2^3x_3^4, x_2^2x_3^2x_4^2 \rangle \). It is clear that \( M \) is generic. Moreover, \( \dim(\Delta_M) = 1 \), see Figure 5. Thus \( M \) is Cohen-Macaulay, by Theorem 1.8.

![Figure 5. The Scarf complex \( \Delta_M \) of \( M \) in Example 2.5](image)

**3. Shifted Complexes**

In this section, we prove that shellable shifted complexes have generic Cohen-Macaulay deformations.

**Definition 3.1.** A simplicial complex \( \Gamma \) on the vertex set \( [n] \) is **shifted** if, for all \( \sigma \in \Gamma \), whenever \( j \in \sigma, i \in [n], \) and \( i < j \), we have \( (\sigma \setminus \{j\}) \cup \{i\} \in \Gamma \).

**Example 3.2.** Let \( \Gamma \) be the 2-dimensional simplicial complex on the vertex set \( [6] \) with the set of facets \( \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 5\}, \{1, 2, 6\}\} \), see Figure 6. It is easy to see that \( \Gamma \) is a shifted complex.

![Figure 6. The shifted complex \( \Gamma \) from Example 3.2](image)
Notice that the labeling plays a major role in this definition. So relabeling (permuting the vertices) a shifted complex might yield a non-shifted one, although they are combinatorially equivalent (their face lattices are isomorphic).

The following theorem \[3\, \text{Theorem 3}\] gives simple geometrical and combinatorial characterizations for Cohen-Macaulay, and shellable shifted complexes.

**Theorem 3.3.** Let \( \Gamma \) be a \((d - 1)\)-dimensional shifted simplicial complex.

1. \( \Gamma \) is homotopically equivalent to a wedge of spheres,
2. \( \Gamma \) is shellable if and only if \( \Gamma \) is Cohen-Macaulay if and only if \( \Gamma \) is pure if and only if for every \( \sigma \in \Gamma \) and \( |\sigma| = t \), \( \sigma \cup [d - t] \in \Gamma \).

Let \( \Gamma \) be a shellable \((d - 1)\)-dimensional shifted simplicial complex on the vertex set \([n]\). We give a partition for the set of minimal non-faces \( \Sigma_\Gamma \).

Recall that
\[
\Sigma_\Gamma := \{ \sigma \subseteq [n] : \sigma \notin \Gamma \text{ but } \sigma \setminus \{i\} \in \Gamma \text{ for each } i \in \sigma \}.
\]

Now, for \( 1 \leq j \leq n - d \), let
\[
\Sigma_j := \{ \sigma \in \Sigma_\Gamma : \sigma = A \cup \{d - |A| + j\} \text{ and } A \subseteq [d - |A| + j] \}.
\]

In the next proposition, we present some facts that will be very useful in proving the main result of this section. But before that we give an example.

**Example** \[3,2\] (cont.). It is clear that
\[
\Sigma_\Gamma = \{\{2, 3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}.
\]

Moreover, it is easy to see that
\[
\Sigma_1 = \{\{2, 3, 4\}, \{3, 5\}, \{3, 6\}\},
\Sigma_2 = \{\{4, 5\}, \{4, 6\}\},
\Sigma_3 = \{\{5, 6\}\}.
\]

Hence, \( \Sigma_\Gamma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \).

**Proposition 3.4.** Let \( \Gamma, \Sigma_\Gamma \), and \( \Sigma_j \) be as above.

1. For \( 1 \leq j \leq n - d \), we have \( \Sigma_j \neq \emptyset \),
2. For \( 1 \leq i < j \leq n - d \), \( \Sigma_i \cap \Sigma_j = \emptyset \), and
3. \( \Sigma_\Gamma = \cup_{j=1}^{n-d} \Sigma_j \).

**Proof.**
1. Let \( 1 \leq j \leq n - d \) and let \( H := \{d + j, d + j - 1, \ldots \} \). Since \( |H| = d + 1 \) and \( \dim(\Gamma) = d - 1 \), for some \( j \leq t \leq d + j \), we have that \( \{d + j, d + j - 1, \ldots , t\} \) is a minimal non-face of \( \Gamma \). Let \( A := \{d + j, d + j - 1, \ldots , t - 1\} \). It is clear that \( |A| = d + j \) and hence \( t = d - |A| + j \). Therefore, \( A \cup \{t\} \in \Sigma_j \) which implies that \( \Sigma_j \neq \emptyset \).
2. Straightforward.
3. It is clear that, for any \( 1 \leq j \leq n - d \), we have \( \Sigma_j \subseteq \Sigma_\Gamma \). Let \( \sigma \in \Sigma_\Gamma \), say \( \sigma = \{i_1, \ldots , i_{t+1}\} \) where \( 1 \leq i_{t+1} < \cdots < i_1 \leq n \). Let \( A := \{i_1, \ldots , i_t\} \).
Since $\sigma$ is a minimal non-face, we have $A \in \Gamma$. Since $1 \leq i_{t+1} \leq n-t$, there exists $1 \leq j_0 \leq n - d$ such that $i_{t+1} = d - t + j_0$. Notice that $|A| = t$. Thus $A \subseteq [d - |A| + j_0]$ and $\sigma = A \cup \{d - |A| + j_0\}$. Therefore $\sigma \in \Sigma_{j_0}$. Hence $\Sigma_\Gamma = \bigcup_{j=1}^{n-d} \Sigma_j$.

**Proposition 3.5.** Let $\Gamma$, $\Sigma_j$ be as above. Let $\sigma, \gamma \in \Sigma_j$ and $\sigma \neq \gamma$. There exists $s > j$ and $\tau \in \Sigma_s$ such that $\tau \subseteq \tau \cup \gamma$.

**Proof.** Since $\sigma, \gamma \in \Sigma_j$, there exists $A, B \in \Gamma$ such that $\sigma = A \cup \{d - |A| + j\}$ and $\gamma = B \cup \{d - |B| + j\}$ where $A \subseteq [d - |A| + j]$ and $B \subseteq [d - |B| + j]$. Moreover, $A \neq B$, since $\sigma \neq \gamma$. Without loss of generality, assume $|A| \leq |B|$.

**Case 1.** Suppose $b \leq d - |A| + j$, for every $b \in B \setminus A$. Thus

$$B \setminus A \subseteq \{d - |A| + j, \ldots, d - |B| + j + 1\}.$$ 

Hence

$$|B \setminus A| \leq |d - |A| + j, \ldots, d - |B| + j + 1|$$

$$= (d - |A| + j) - (d - |B| + j)$$

$$= |B| - |A|,$$

which implies that $A \subseteq B$. But $d - |A| + j \notin B$, otherwise $\sigma \subseteq B$. Therefore,

$$B \subseteq A \cup \{d - |A| + j - 1, \ldots, d - |B| + j + 1\}.$$ 

Hence

$$|B| \leq |A \cup \{d - |A| + j, \ldots, d - |B| + j + 1\}|$$

$$= |A| + (d - |A| + j - 1) - (d - |B| + j)$$

$$= |B| - 1,$$

a contradiction.

**Case 2.** There exists $b \in B \setminus A$ such that $b > d - |A| + j$, say $b = d - |A| + j + h$ for some $h \geq 1$. Since $\sigma$ is a non-face and $b > d - |A| + j$, the set $A \cup \{b\}$ must contain a minimal non-face. Let

$$A^0 := \{a \in A : a > b\} = \{a_1 > \cdots > a_t\},$$

and

$$A^0 := \{a \in A : a < b\} = \{b_1 > \cdots > b_{|A| - 1}\},$$

if $A^0 = \emptyset$, let $t := 0$. If $A^0 \neq \emptyset$, then for $1 \leq i \leq |A| - t$,

$$b_i = d - |A| + j + s_i,$$

where $s_i \geq |A| - t - i + 1$. Thus

$$h > s_1 > \cdots > s_{|A| - t} \geq 1.$$ 

In the following, we will discuss all possible cases.

1. Suppose $A^0 \cup \{b\}$ is a minimal non-face of $\Gamma$. We need to show that $A^0 \cup \{b\} \in \Sigma_s$ for some $s > j$. So it is enough to show that $b = d - |A^0| + s =$
$d - t + s$, for some $s > j$. But

$$b > b_1 = d - |A| + j + s_1$$

$$\geq d - |A| + j + |A| - t$$

$$= d - |A^\geq| + j.$$

This implies that $b = d - |A^\geq| + s$ for some $s > j$. Therefore, $A^\geq \cup \{b\} \in \Sigma_s$ and $s > j$.

2. Suppose $A^\geq \cup \{b\} \cup \{b_1, \ldots, b_i\}$ is a minimal non-face of $\Gamma$. As above,

$$b_i = d - |A| + j + s_i$$

$$\geq d - |A| + j + |A| - t - i + 1$$

$$= d - (t + i) + j + 1.$$

Thus $b_i = d - |A^\geq \cup \{b\} \cup \{b_1, \ldots, b_{i-1}\}| + s$ for some $s > j$. Therefore,

$$(A^\geq \cup \{b\} \cup \{b_1, \ldots, b_{i-1}\}) \cup \{b_i\} \in \Sigma_s.$$

\[\square\]

In the following we construct a generic deformation of $I_{\Gamma}$. For $1 \leq j \leq n - d$, let

$$M_j := \langle \prod_{i \in \sigma} x_i^{n-d-j+1} : \sigma \in \Sigma_j \rangle,$$

a monomial ideal in $R := k[x_1, \ldots, x_n]$. Let

$$M := M_1 + \cdots + M_{n-d}.$$

The following corollary is straightforward from Proposition 3.5.

**Corollary 3.6.** For any $1 \leq j < n - d$, let $m_1, m_2 \in M_j$ be two distinct generators of $M_j$. Then there exists $s > j$ and a minimal generator $m_3 \in M_s$ such that $m_3$ divides $\text{lcm}(m_1, m_2)$. Thus $M$ is a generic deformation of $I_{\Gamma}$.

The following theorem shows that $M$ is Cohen-Macaulay. Before we prove that, we verify it for our Example.

**Example 3.2 (cont.).** The above construction yield the following monomial ideals:

$$M_1 = \langle x_2^3 x_3^3 x_4^3, x_3^3 x_5^3, x_3 x_6^3 \rangle,$$

$$M_2 = \langle x_2^2 x_5^2, x_4 x_6^2 \rangle,$$

$$M_3 = \langle x_5 x_6 \rangle.$$

It is easy to check that the monomial ideal

$$M = \langle x_2^3 x_3^3 x_4^3, x_3^3 x_5^3, x_3^2 x_6^2, x_3 x_5^2, x_4 x_6^2, x_4^2 x_6, x_5 x_6 \rangle$$

is generic and $\text{rad}(M) = I_{\Gamma}$. Moreover, by Corollary 3.6, $\dim(\Delta_M) \leq 2$. To show that $M$ is Cohen-Macaulay, it is enough to show that $\dim(\Delta_M) = 2$,
by Theorem [1.9] But it is clear that \( \text{lcm}\{x_5x_6, x_4^2x_5, x_3^3x_6\} \) is uniquely attained and hence \( \{x_5x_6, x_4^2x_5, x_3^3x_6\} \) corresponds to a facet of \( \Delta_M \). Therefore \( \dim(\Delta_M) = 2 \) and hence \( M \) is Cohen-Macaulay.

**Theorem 3.7.** The ideal \( M \), constructed above, is a generic Cohen-Macaulay integral deformation of \( I_\Gamma \).

**Proof.** By Corollary 3.6 \( M \) is a generic integral deformation of \( I_\Gamma \). So it remains to show that \( M \) is Cohen-Macaulay. By Theorem 1.9 it is enough to show that \( \dim(\Delta_M) = n - d - 1 \). By Corollary 3.6 \( \dim(\Delta_M) \leq n - d - 1 \). Thus to show that \( \dim(\Delta_M) = n - d - 1 \), it is enough to find a face \( F \) of \( \Delta_M \) such that \( \dim(F) = n - d - 1 \).

For \( 1 \leq j \leq n - d \), let \( \sigma_j := \{d + j, d + j - 1, \ldots, i_j\} \), where \( i_j \) is small enough such that \( \sigma_j \in \Sigma_\Gamma \), i.e., \( \sigma_j \) is a minimal non-face of \( \Gamma \). Such an \( i_j \) exists since \( \{d + j\} \) is a face of \( \Gamma \) for \( j \) and \( \dim(\Gamma) = d - 1 \). First we will show that \( \sigma_j \in \Sigma_j \). Let \( A := \sigma_j \setminus \{i_j\} \). It is clear that \( |A| = (d + j) - i_j \). Thus \( i_j = d - |A| + j \). Therefore \( A \subseteq [d - |A| + j] \) and hence \( \sigma = A \cup \{i_j\} \in \Sigma_j \).

Next we will show that \( F := \{m_{\sigma_1}, \ldots, m_{\sigma_{n-d}}\} \) is a face of \( \Delta_M \). Now
\[
\text{lcm}(F) = \text{lcm}(m_{\sigma_1}, \ldots, m_{\sigma_{n-d}}) = x_nx_{n-1}^2 \ldots x_{d+1}^x_d \ldots x_{n-d}^x_{i_1}.
\]
It is clear that, for \( 1 \leq j \leq n - d \), we have \( \text{lcm}(F) \neq \text{lcm}(F \setminus \{m_{\sigma_i}\}) \). So we only need to show that \( \text{lcm}(F) \neq \text{lcm}(F \cup \{m_\sigma\}) \) for any \( \sigma \in \Sigma_\Gamma \setminus F \). Let \( \sigma \in \Sigma_\Gamma \setminus F \), say \( \sigma \in \Sigma_i \) for some \( 1 \leq j \leq n - d \). Thus \( \sigma = A \cup \{d - |A| + j\} \), for some \( A \in \Gamma \) and \( A \subseteq [d - |A| + j] \). Since \( \sigma \notin F \), there exists \( a \in A \) such that \( a > d + j \), otherwise \( \sigma = \sigma_j \). Suppose \( a = d + i \) for some \( i > j \). Thus \( a \in \sigma_i \) and hence \( \deg_{\sigma_i}(\text{lcm}(F)) = n - (d + i) + 1 \). But \( \deg_{\sigma_i}(m_{\sigma_\sigma}) = n - (d + j) + 1 \). Therefore, \( \text{lcm}(F) \leq \text{lcm}(F \cup \{m_\sigma\}) \). Hence \( F \) is a face of \( \Delta_M \) of dimension \( n - d - 1 \). Therefore, \( \dim(\Delta_M) = n - d - 1 \). By Theorem 1.9 \( M \) is Cohen-Macaulay.

\( \square \)

4. **Tree Complexes: Shellable Clique Complexes of Chordal Graphs**

In this section, we study the class of shellable clique complexes of chordal graphs, which we call tree complexes because they generalize tree graphs. The main result of this section is that tree complexes have generic Cohen-Macaulay deformations. Moreover, we give an exact formula to compute the \( f \)-vector of tree complexes.

Throughout this section, let \( G \) be a simple graph (undirected graph with no loops and no multiple edges) on the vertex set \([n]\).

**Definition 4.1.** Let \( G \) be as above.

(1) A subset \( \sigma \subseteq [n] \) is a **clique** if the induced subgraph \( G_\sigma \) is complete.
(2) The **clique complex** of \(G\) is
\[
K(G) := \{\sigma \subseteq [n] : \sigma \text{ is a clique of } G\}.
\]

**Example 4.2.** A graph \(G\) and its clique complex \(K(G)\) are in Figure 7. Notice that \(G\) is the 1-skeleton of \(K(G)\).

![Figure 7. A graph G and its clique complex K(G).](image)

The proof of the following lemma is straightforward.

**Lemma 4.3.** Every minimal non-face of \(K(G)\) has precisely two elements. Moreover, if \(\Delta\) is a simplicial complex such that every minimal non-face has exactly two elements, then \(\Delta = K(G)\), where \(G\) is the 1-skeleton of \(\Delta\).

Clique complexes are also called **flag complexes**, see [9].

**Definition 4.4.** A simplicial complex \(\Delta\) is a **tree complex** if there exists a shelling \(F_1, \ldots, F_t\) of \(\Delta\) such that, for \(1 < i \leq t\), there exists \(j_i \in F_i\) such that \(j_i \notin F_1 \cup \cdots \cup F_{i-1}\). In this case, we say that \(F_i\) introduces the vertex \(j_i\).

It is easy to see that any tree graph is a 1-dimensional tree complex. Hence tree complexes are a natural generalization of tree graphs.

**Example 4.5.** The simplicial complex \(\Gamma\) in Figure 8 is a 2-dimensional tree complex.

One of the most interesting problems in combinatorics is characterizing the \(f\)-vectors of flag complexes [2, 9, 4]. For tree complexes, the next theorem gives an exact formula for the \(f\)-vector of these complexes.

**Theorem 4.6.** Let \(\Delta\) be a \((d-1)\)-dimensional tree complex on \([n]\). Then the number of facets of \(\Delta\) is 
\[
f_{d-1}(\Delta) = n - d + 1,
\]
and for \(1 < i \leq d - 1\),
\[
f_{i-1}(\Delta) := \binom{d}{i} + (n - d) \left\{ \binom{d}{i} - \binom{d-1}{i} \right\}
= (n - d + 1) \binom{d}{i} - (n - d) \binom{d-1}{i}.
\]

**Proof.** Let \(F_1, \ldots, F_t\) be a shelling of \(\Delta\). By Theorem 4.10, every vertex \(d \leq i \leq n\) introduces a facet. Thus \(f_{d-1}(\Delta) = n - d + 1\). Now for \(1 < i \leq d - 1\), it is clear that \(F_1\) has \(\binom{d}{i}\) \((i-1)\)-faces. Moreover, every facet
$F_j$, where $j > 1$ has $\binom{d}{i} (i - 1)$-faces minus the number of $(i - 1)$-faces that $F_j$ shares with $F_1, \ldots, F_{j-1}$. But $F_j$ shares only $d - 1$ vertices with $F_1, \ldots, F_{j-1}$. Hence the number of $(i - 1)$-faces it shares is $\binom{d-1}{i}$ which implies the equality above.

**Definition 4.7.** A graph $G$ is **chordal** if every cycle of length four or more has a chord (an edge linking two non-adjacent nodes in the cycle).

**Example 4.8.** The graph $G$ in Figure 7 is chordal.

The following theorem gives many useful characterizations of chordal graphs.

**Theorem 4.9** ([7, Theorem 7, p 112]). The following statements are equivalent.

1. $G$ is a chordal graph.
(2) All vertices of $G$ can be deleted by arranging them in separate piles, one for each maximal clique, and then repeatedly applying the following two operations:
- Delete a vertex that occurs in only one pile.
- Delete a pile if all its vertices appear in another pile.

(3) There is a spanning tree (called a clique tree) $T$ of the facet graph of $K(G)$ such that for every vertex $i$ of $G$, if we remove from $T$ all cliques not containing $i$, the remaining subtree stays connected. In other words, any two cliques containing $i$ are either adjacent in $T$ or connected by a path made entirely of cliques that contain $i$.

Suppose $\Delta$ is a $(d-1)$-dimensional tree complex on $[n]$. Without loss of generality (after relabeling the vertices), we may assume $F_1 := [d]$ and, for $1 < i \leq t$, $F_i$ introduces the vertex $d + i - 1$, i.e., $F_i := A \cup \{d + i - 1\}$ for some $A \subseteq [d + i - 2]$ and $|A| = d - 1$. The following theorem shows that the class of tree complexes is equal to the class of shellable clique complexes of chordal graphs.

**Theorem 4.10.** A shellable simplicial complex $\Delta$ is a tree complex if and only if $\Delta = K(G)$, the clique complex of a chordal graph $G$.

**Proof.** Let $F_1, \ldots, F_t$ be a shelling of $\Delta$. First suppose that $\Delta = K(G)$, for some chordal graph $G$, and assume that $\Delta$ is not a tree complex. Let $s$ be the smallest index such that $F_s$ does not introduce a new vertex. There exists a simple cycle in the facet graph of $K(G)$ that contains $F_s$, without loss of generality, $F_1, \ldots, F_s$ is a simple cycle of minimal length. Thus every vertex is contained in more than one clique. This is a contradiction to (2) of Theorem 4.9. Therefore, for $1 < i \leq t$, there exists $i_j \in F_i$ such that $i_j \notin F_1 \cup \cdots \cup F_i$, and hence $\Delta$ is a tree complex.

Now suppose that $\Delta$ is a tree complex. Let $G$ be the 1-skeleton of $\Delta$ ($G$ is the subcomplex of simplexes of dimension less than or equal to 1). It is clear that $G$ is a graph on the set of vertices $[n]$. Indeed $G$ is a chordal graph: For every facet $F_i$, let $P_i$ be the pile that contains the vertices of $F_i$. Since, for $1 < i \leq t$, $F_i$ introduces the vertex $j_i$, it is clear that one can empty all the piles by deleting each vertex that is in only one pile and then deleting any pile where its vertices are in another pile. Thus $G$ is chordal. So we only need to show that $\Delta = K(G)$. It is clear that $\Delta$ is a subcomplex of $K(G)$. Suppose $C$ is a clique of $G$ and $h \in C$ is the maximal element. Thus the vertex $h$ is introduced by the facet $F_{i_h}$, for some $1 \leq i_h \leq t$. Thus every vertex connected to $h$ and less than $h$ is in $F_{i_h}$ which implies that $C \subset F_{i_h}$. Hence $\Delta = K(G)$. \qed

Let $FG(\Delta)$ be the facet graph of $\Delta$. By Theorem 4.5, $FG(\Delta)$ has $n - d + 1$ facets. Let $T_\Delta$ be a clique tree of $FG(\Delta)$, it has $n - d$ edges and each edge in $T_\Delta$ corresponds to a missing edge in $\Delta$.

If $F_i$ and $F_j$ are vertices connected by an edge $e$ in $T_\Delta$, then $F_i$ and $F_j$ are facets of $\Delta$, i.e., $|F_i| = |F_j| = d$ and $|F_i \cap F_j| = d - 1$. Thus, since $\Delta$ is a
clique complex, there is a unique minimal non-face in $F_i \cup F_j$, say $\{p_e, q_e\}$. Let
\[ \Sigma_0 := \{\{p_e, q_e\} : e \in T_\Delta \} . \]
Now for the edge $e$, let
\[ F_{ij} := F_i \cup F_j . \]
Let $\Delta_1$ be the $d$-dimensional simplicial complex on $[n]$ with the set of facets \{ $F_{ij} : e \in T_\Delta$ \}. It is easy to see that $\Delta_1$ is a tree complex. If $\Delta_1$ is not a simplex, let $T_{\Delta_1}$ be a clique tree of $FG(\Delta_1)$. Repeat the steps above for this case $(\Delta_t$ is a simplex only when $t = n - d)$. Hence, in general, for $1 \leq t < n - d$, let
\[ \Sigma_t := \{\{p_e, q_e\} : e \in T_{\Delta_t}\} . \]
This process gives a partition of $\Sigma$. Before we give a proof of this fact, we will verify it for our example.

**Example 4.5 (cont.)**. It is easy to check that
\[ \Sigma = \{\{3, 5\}, \{4, 5\}, \{1, 6\}, \{2, 7\}, \{3, 4\}, \{4, 6\}, \{1, 7\}, \{3, 6\}, \{4, 7\}, \{3, 7\}\} . \]
Moreover, the graph in Figure (i) is $FG(\Gamma_i)$ and the subgraph of dashed edges is $T_{\Gamma_i}$. This yields the following partition of $\Sigma$:
\[
\begin{align*}
\Sigma_0 &= \{\{3, 5\}, \{4, 5\}, \{1, 6\}, \{2, 7\}\} \\
\Sigma_1 &= \{\{3, 4\}, \{4, 6\}, \{1, 7\}, \{3, 6\}, \{4, 7\}\} \\
\Sigma_2 &= \{\{3, 6\}, \{4, 7\}\} \\
\Sigma_3 &= \{\{3, 7\}\} .
\end{align*}
\]
The proof of the following proposition is straightforward.

**Proposition 4.11.** Let $\Delta$ be a tree complex and $\Sigma_t$ be as above.
1. For $0 \leq t < n - d$, we have $\Sigma_t \neq \emptyset$,
2. For $0 \leq t_1 < t_2 < n - d$, $\Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset$, and
3. $\Sigma_\Delta = \bigcup_{t=0}^{n-d-1} \Sigma_t$.

The following proposition will help us to construct a generic deformation from $I_\Delta$.

**Proposition 4.12.** Let $\Delta$, $\Sigma_t$ be as above. Let $e_1, e_2 \in \Sigma_t$ and $e_1 \neq e_2$. There exists $s > t$ and $e_3 \in \Sigma_s$ such that $e_3 \subset e_1 \cup e_2$.

**Proof.** Without loss of generality, let
\[ e_1 = \{p_1, q_1\}, \quad e_2 = \{p_2, q_2\} \]
be two edges of the clique tree $T_{\Delta_t}$ where the edge $e_1$ (resp. $e_2$) has as vertices the $(d + t - 1)$-simplexes (facets) $\tau_1, \tau_2$ (resp. $\gamma_1, \gamma_2$) of $\Delta_t$, say
\[
\begin{align*}
\tau_1 &= A \cup \{p_1\}, & \tau_2 &= A \cup \{q_1\}, \\
\gamma_1 &= B \cup \{p_2\}, & \gamma_2 &= B \cup \{q_2\} .
\end{align*}
\]
**Case 1.** $p := p_1 = p_2$ ($q_1 \neq q_2$). If $\tau_1 = \gamma_1$, then $A = B$. Thus the edge $\{q_1, q_2\}$ is in the facet graph $FG(\Delta_t)$ but not in $T_{\Delta_t}$, otherwise
τ₁, τ₂, γ₂ form a cycle in T₁. So assume that τ₁ ≠ γ₁. There exists a path τ₁ = G₁, . . . , Gᵣ = γ₁ such that p ∈ Gᵢ for 1 ≤ i ≤ r. This implies that τ₂, G₁, . . . , Gᵣ, γ₂ is the only path between τ₂ and γ₂ in T₁. Hence {q₁, q₂} is not an edge in T₁, otherwise we have a cycle in T₁. In this case, let e₃ = {q₁, q₂}.

Case 2. pᵢ ≠ qⱼ for 1 ≤ i, j ≤ 2. Suppose for 1 ≤ i, j ≤ 2, the edge {pᵢ, qⱼ} ∈ T₁. Thus p₁, q₁, p₂, q₂ form a 4-cycle in the chordal graph G, hence either the edge {p₁, q₁} or {p₂, q₂} is in G, a contradiction. Without loss of generality, suppose the 1-simplex {p₁, p₂} /∈ T₁. If {p₁, p₂} /∈ T₁, then we are done. Suppose {p₁, p₂} is an edge in T₁ with the vertices F and G such that p₁ ∈ F and p₂ ∈ G. Since p₁ ∈ F ∩ τ₁, there exists a path between τ₁ and F made entirely from facets containing p₁. Similarly, there exists a path between G and γ₁ made entirely from facets containing p₂. Thus there is a path between τ₂ and γ₂. It is clear that if {q₁, q₂} ∈ T₁ or {p₁, p₂} is an edge in T₁, then we have a cycle in T₁. Once again, in this case, let e₃ = {q₁, q₂}.

In the following we construct a generic integral deformation of I₁. For 0 ≤ t < n − d, and {pₑ, qₑ} ∈ Σ₁, let

\[ M₁ := \langle xⁿ⁻ᵈ⁻ᵗₑ⁻¹ xⁿ⁻ᵈ⁻ᵗₑ : \{pₑ, qₑ} ∈ Σ₁ \rangle. \]

It is clear that M₁ ≠ 0, otherwise there are no missing faces and hence we have a simplex which is not the case unless t = n − d.

Let M = M₀ + M₁ + · · · + Mₙ−₁. It is clear that \( \text{rad}(M) = I₁ \). Moreover, in the next Lemma, we show that M is generic. Before that we continue with our example.

Example 4.3 (cont.). For each i; 0 ≤ i ≤ 4, the corresponding ideals to the partition above are:

\[ M₀ = \langle x⁴ₓ₅, x⁴ₓ₅, x₁ₓ₆, x₂ₓ₇ \rangle, \]
\[ M₁ = \langle x₃ₓ₄, x₄ₓ₆, x₁ₓ₇ \rangle, \]
\[ M₂ = \langle x₂ₓ₆, x₁ˣ₇ \rangle, \]
\[ M₃ = \langle x₃x₇ \rangle. \]

Since Γ₄ is a simplex, the facet graph FG(Γ₄) is a vertex with no edges and hence M₄ = 0.

Let M = M₀ + · · · + M₃. It is easy to see that M is a generic monomial ideal and V(M) = Γ.

The following lemma follows from Corollary 4.12.

Lemma 4.13. For any 0 ≤ j < n − d, let m₁, m₂ ∈ M_j be two distinct generators of M_j. Then there exists s > j and a minimal generator m₃ ∈ M_s such that m₃ divides lcm(m₁, m₂). Thus M is a generic integral deformation of I₁.
The following theorem shows that $M$ is Cohen-Macaulay. Before we prove that, we verify it for our example.

**Example 4.5** (cont.). It is easy to see that $\{x_3x_7, x_3^2x_6, x_2^3x_4, x_3^4x_5\}$ is a face of $\Delta_M$ and hence $\dim(\Delta_M) = 3 = 7 - 3 - 1$. Thus, by Theorem 1.9 $M$ is Cohen-Macaulay.

**Theorem 4.14.** The ideal $M$ is a generic Cohen-Macaulay integral deformation of $I_\Delta$.

**Proof.** It remains to show that $M$ is Cohen-Macaulay. Since $V(M) = \Delta$, by Theorem 1.9 we need to show that $\dim(\Delta_M) = n - d - 1$. By Lemma 4.13, each face of $\Delta_M$ is of dimension $\leq n - d - 1$. To show that $\dim(\Delta_M) = n - d - 1$, it is enough to find a face of dimension $n - d - 1$. For $0 \leq i < n - d$, there exists a unique minimal non-face in $\Delta_i$ which is a subset of $[d + i + 1]$ and contains $d + i + 1$, say $\{\alpha_i, d + i + 1\} \in \Sigma_i$. Let $m_i = x_{d+i}^{n-d-i}x_{d+i+1}^{n-d-i}$ be the corresponding monomial in $M_i$.

**Claim.** $C := \{m_0, \ldots, m_{n-d-1}\}$ is a facet of $\Delta_M$ and hence $\dim(\Delta_M) = n - d - 1$.

\[
lcm(C) := \text{lcm}\{m_0, \ldots, m_{n-d-1}\} = f \cdot x_{d+1}^{n-d}\cdot x_{d+2}^{n-d-1}\cdots x_{d+i+1}^{n-d-i+1}\cdots x_{n-1}^2 x_n,
\]

where $f$ is a monomial on the variables $x_1, \ldots, x_d$. It is clear that $\text{lcm}(C) \neq \text{lcm}(H)$ for any proper subset $H \subset C$. Suppose there exists $i, j$ such that $j < d + i + 1$, $\{j, d + i + 1\} \in \Sigma$ and $x_j^a x_{d+i+1}^b$ divides $\text{lcm}(C)$, i.e., $a \leq n - d - i + 2$. This implies that $\{j, d + i + 1\}$ corresponds to an edge in $\Delta_s$ where $s \geq i$. Since $\{\alpha_i, d + i + 1\}$ is the only non-face in $\Delta_i$ that is a subset of $[d + i + 1]$, we must have $s = i$ and $j = \alpha_i$ and hence $m = m_i$. Therefore, $C \in \Delta_M$ and hence $\dim(\Delta_M) = n - d - 1$. By Theorem 1.9 $M$ is Cohen-Macaulay. \qed

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