Derivation of the Pauli exchange principle

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When the time-independent wave function of a multicomponent system is computed, it is necessary to require that it be consistent with the Pauli exchange principle which states that, for systems involving identical elements, wave functions must be antisymmetric to pair exchange if twice the spin \( s \) for each one is an odd integer and symmetric to pair exchange if the spin is an integer. For identical spin sets, each set containing spin \( s \), this can be expressed mathematically, for a single exchange, in the form,

\[
\Psi(x_1 \bar{s} \cdots x_n \bar{s} \cdots x_b \bar{s} \cdots x_n \bar{s}) = (-1)^{2s} \Psi(x_1 \bar{s} \cdots x_b \bar{s} \cdots x_n \bar{s})
\]  \( (1) \)

where \( x \) is a set of space coordinates, and \( \bar{s} \) represents a set of spin quantum numbers.

Pauli [1] showed that this relation connecting fermion and boson statistics with spin can be derived with the aid of relativistic quantum field theory. As will be seen below, it can also be derived directly from the properties of wave functions. Papers have been published by Feynman, Schwinger, and others in an effort to find a simpler and more understandable proof. Their work is discussed in a recent review article [2] in which we find this statement, “Finally we are forced to conclude that although the Spin-Statistics Theorem is simply stated, it is by no means simply understood or simply proved.”

Feynman’s position on the lack of a simple derivation in 1963 was expressed in his physics course [3] where he said, “This probably means we do not have a complete understanding of the fundamental principle involved.” Broyles [4] published an elementary proof of this theorem in 1976 in which he used an exchange operator defined in terms of rotation operators. Apparently unaware of this, Feynman presented a lecture [5] giving essentially the same proof in 1986. However, Duck and Sudarshan [2] objected to these proofs by expressing the need for evidence that the wave functions before and after the exchange are identical. This paper is written to answer this objection and to summarize the proof.

Feynman [6] introduced the term, “probability amplitude”, into quantum mechanics. The square of the magnitude of this complex number gives the probability of obtaining a specified collection of quantum numbers as the result of measurements. In order to identify this amplitude uniquely (aside from an overall constant phase), a complete set of quantum numbers must be specified. As an example, Kemble [7] relates how he and Feenberg proved that the probability amplitude over all space for position coordinates can be determined by repeated measurements. These measurements can be used to construct the probability distribution and its time derivative. They show that a phase factor, constant in space and time, remains undetermined. Phases of probability amplitudes can be measured relative to a standard by means of diffraction experiments [8], but no method is available for measuring the phase of the standard. We shall assume here that a complete set of quantum numbers can be found to determine any probability amplitude to within a constant phase.

These quantum numbers are associated in sets as though each set belonged to a particle. Each set can include the configuration space coordinates \( x \) as well as the spin coordinates in the subset \( \bar{s} \). We shall include the spin \( s \) and a spin component \( s_z \) in \( \bar{s} \). In addition, any other quantum numbers such as mass, charge, flavor, etc. that may be needed to identify the set, can be included in \( s \). In association with each set \( x_{\bar{s}} \), we can imagine a point in a representative spin configuration space whose coordinates are the components of \( x_{\bar{s}} \). In Fig. 1a, points representing three sets, that we shall take to have identical spin quantum numbers \( \bar{s} \), are pictured. We shall be interested in systems where all the sets \( \bar{s} \) are identical except possibly for the \( s_z \) quantum numbers. Let us first consider cases where all the \( s_z \)'s are the same and where each one has its maximum value \( s \).

The wave function \( \Psi \) in Eq. (1) is a convenient way to write down this probability amplitude and its associated sets of quantum numbers. However it is clear that additional information has been incorporated in this wave function, additional information beyond that contained in the quantum numbers that determine the probability amplitude. This information is provided by the arrangement of the sets in the argument of the wave function. The number represented by the function \( \Psi \) will, in general, depend upon the order in which the sets of quantum numbers are written. The probability amplitude, on the other hand, depends only on what quantum number sets are present, not on the order in which they appear. Thus if the wave function is to represent the probability amplitude, we must impose a restriction on it to remove the dependence on...
the order. If the spins are zero, all we need to do is make the wave function symmetric to the pair exchange of identical sets. For nonvanishing spins, however, the restriction is more complicated.

It is instructive to recognize a difference between the spatial exchange of two apparently identical classical objects and the exchange of two electron coordinate sets determining a probability amplitude. For example if we exchange two billiard balls, we produce a new state, a new configuration. Of course, if the two balls had exactly the same number of atoms of each kind, it might be possible to make them truly indistinguishable so that the state after the exchange is the same as the one before. Since, however, it is essentially impossible to equalize the number of atoms in the two balls, the probabilities are overwhelming that the initial and final states will be different. If however, two electron coordinate sets, \( \mathbf{x}_a \) and \( \mathbf{x}_b \), with identical \( \mathbf{s} \) subsets, are exchanged, the quantum numbers present are the same after the exchange as before. Thus the probability amplitude, since it depends only on the sets that are present and not on their order, will be unaltered by the exchange. This is illustrated in Fig. 1 where the exchange in Figs 1b and 1c leaves Fig 1d identical to 1a. This is equivalent to saying that the \( \mathbf{x}_a \)’s that specify a probability amplitude are generic sets rather than specific [9].

The argument of a wave function, is normally written down with the quantum number sets in specific positions. Thus there is one set in the first position to which we can assign a number one and label the symbols with subscripts so that we have \( \mathbf{x}_1 \mathbf{s}_1 \). Similarly, the particular set in the second position can be written as \( \mathbf{x}_2 \mathbf{s}_2 \), etc. Then an application of the exchange operator to sets \( a \) and \( b \) changes their positions in the argument of the wave function as we see in Eq. (1). In order to make the wave function equal the probability amplitude, it must not be allowed to change value as a result of this exchange.

To generate the \( N! \) specific positions that can appear in the argument of the wave function from the values of the wave function at the \( N \) generic positions that are available from the probability amplitudes, we must first identify each generic set. This can be done by assigning an order number to it. One way of accomplishing this is to select a point in the representation space to act as the center. The generic positions can then be numbered according to their distances from this center. The point nearest to the center will be point number one, the next nearest will be point number two, etc. The distance from the center to the nearest point can then be represented by \( r_1 \), to the next nearest one by \( r_2 \), etc. Then we have \( r_1 < r_2 < r_3 < \cdots < r_n \). The probability amplitudes for these \( N \) points then compose the wave function for the region where the \( r \)'s satisfy these inequalities. We shall call this, Region A.

Since the above procedure gives \( \Psi \) in a sizeable fraction of space, Region A, the assumption is made here that the \( \Psi \) in the remainder of space (where the \( r \)'s satisfy different inequalities) can be computed by analytic continuation. Since any order of \( r \)'s can be altered to a new order by exchanging two points in the representation space, it is convenient to obtain the collection of Taylor’s series for this continuation by expanding the rotation operators that move two points through angles. Such an exchange is illustrated in Figs. 1b and 1c. If it exchanges the points labeled 1 and 2 in the last paragraph, the distances from the center to the nearest point can then be represented by \( r_1 \), to the next nearest one by \( r_2 \), etc. Then we have \( r_1 < r_2 < r_3 < \cdots < r_n \). These points are in a region other than Region A.

If the sets to be exchanged are \( \mathbf{x}_a \mathbf{s}_a \) and \( \mathbf{x}_b \mathbf{s}_b \), the rotations required can be described by first placing the representation space coordinate system so that its \( x \) axis coincides with the line connecting the two points at \( \mathbf{x}_a \) and \( \mathbf{x}_b \) and with the origin (marked O in Figs. 1b and 1c) half way between these points. Then the operator [10] (with the subscript \( a, b \) indicating “a or b”),

\[
R^\phi_{a,b} = e^{i\phi_{a,b} J_{za,b}}
\]  

(2)

where

\[
J_{za,b} = L_{za,b} + S_{za,b}
\]  

(3)

and

\[
L_{za,b} = i\hbar \frac{\partial}{\partial \phi_{a,b}},
\]  

(4)

\( (S_{za,b} \) is the generator of the spin component in the \( z \) direction,) can be applied to \( \Psi \) to produce the \( \Psi \) after the required rotations. This rotation operator or its series can be used to compute the wave function with \( \mathbf{x}_a \) or \( \mathbf{x}_b \) moved by a rotation through an angle \( \phi_{a,b} \). If the spins in the argument of \( \Psi \) are zero, the spin
generator in Eq. (3) can be omitted. However, if the spins do not vanish, the spin generator will be required since the direction of spin in the argument of $\Psi$ after the rotation may be different from the initial one.

When $\phi_a$ equals $\pi$, $\Psi$ will be calculated at the points resulting from the rotation of $x_{a,S_a}$ as shown in Fig. 1b. Next, setting $\phi_b$ to $\pi$ provides the $\Psi$ after the rotation of $x_{b,S_b}$ through $\pi$ as shown in Fig 1c. The two rotations together $R_aR_b$ then produce the $\Psi$ with the two points exchanged. See References [4] and [5] for additional discussions. By applying the proper choice of these rotation operators to $\Psi$, the wave function with any exchange of sets can be computed. By computing the proper choice of exchanges, the wave function for any specific configuration of sets can be determined from a generic one.

In the case under consideration where the spin components are identical, that is, $s_{za} = s_{zb} = s$, the $L_z$ operators will exchange the $x$'s while the $S_z$'s can be replaced by their eigenvalues. As a result, the two spin rotation operators applied to $\Psi$ to produce an exchange will give a factor, $e^{i2s_z\pi} = (-1)^{2s}$. Thus

$$R_aR_b\Psi(\cdots x_{a,S} \cdots x_{b,S} \cdots) = (-1)^{2s}\Psi(\cdots x_{b,S} \cdots x_{a,S} \cdots).$$

(5)

where the $\phi_{a,b}$ in Eq. (2) equals $\pi$ in $R_{a,b}$.

But, as we have seen, exchanging these coordinate sets does not introduce new sets or remove old ones and, therefore, does not change the probability amplitude. Thus $R_aR_b$ must have no effect on $\Psi$ if $\Psi$ is to equal the probability amplitude. As a result, the left hand side is equal to $\Psi(\cdots x_{a,S} \cdots x_{b,S} \cdots)$. This makes the last equation equivalent to Eq. (1). The effect of exchanging two points lying on a node can be computed by a limiting procedure.

If the quantum numbers in the various spin sets in the argument of $\Psi$ are identical except for the values of $s_z$, the Pauli principle again applies. To prove this, it is convenient to make use of the complete set of spin wave functions (defined with the aid of an integer $l$) with the property,

$$\chi(s, \theta_l) = e^{i\theta_l S_z} \chi(s, 0), \quad 0 \leq l \leq 2s, \quad \theta_l = l\pi/2s,$$

(6)

introduced in the appendix of Reference [4]. The $\theta_l$’s replace the $s_z$’s in the argument of the wave function. A wave function like that in Eq.(1) can then be written in the form,

$$\Psi(\cdots x_{a,s\theta_{l_a}} \cdots x_{b,s\theta_{l_b}} \cdots) = \cdots e^{i\theta_{l_a}S_{za}} \cdots e^{i\theta_{l_b}S_{zb}} \cdots \Psi(\cdots x_{a,s0} \cdots x_{b,s0} \cdots)$$

(7)

where the zero’s indicate values of the $\theta_l$’s. It is understood that the subscript $j$ on $S_{xj}$ is the same as the subscript on the $x_j$ in the set on which it operates. Since, as we have seen in the last section, $\Psi(\cdots x_{a,s0} \cdots x_{b,s0} \cdots)$ satisfies Eq.(1), and the operators, $e^{i\theta_{l_a}S_{za}}, e^{i\theta_{l_b}S_{zb}}, \cdots$ commute, it follows that $\Psi(\cdots x_{a,s\theta_{l_a}} \cdots x_{b,s\theta_{l_b}} \cdots)$ will also satisfy Eq.(1).

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Figure 1. The exchange, by means of rotations, of two points representing two sets of quantum numbers. Labels are unnecessary because we take the spin quantum numbers to be identical. The coordinate axes are located so that the origin is half way between the two points to be exchanged. 1a. The projection into the $xy$ plane of three points representing three sets of quantum numbers. 1b. The rotation of one of the points through an angle $\pi$ around the $z$ axis with the origin at $O$. 1c. The rotation of the other point through an angle $\pi$. 1d. The representative points after the two rotations that result in the exchange. Fig. 1d is identical to 1a.