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ON THE HECKE EIGENVALUES OF MAASS FORMS

By WENZHI LUO and FAN ZHOU

Abstract. Let \( \phi \) denote a primitive Hecke-Maass cusp form for \( \Gamma_0(N) \) with the Laplacian eigenvalue \( \lambda_\phi = \frac{1}{4} + t_\phi^2 \). In this work we show that there exists a prime \( p \) such that \( p \nmid N, |\alpha_p| = |\beta_p| = 1 \), and \( p \ll (N(1 + |t_\phi|))^c \), where \( \{\alpha_p, \beta_p\} \) is the Satake parameter of \( \phi \) at \( p \), and \( c \) is an absolute constant with \( 0 < c < 1 \). In fact, \( c \) can be taken as \( 8/11 + \epsilon \) (or even 0.27331 by a more elaborate numerical calculation). In addition, we prove that the natural density of such primes \( p \) \( (p \nmid N \text{ and } |\alpha_p| = |\beta_p| = 1) \) is at least \( 34/35 \).

1. Introduction. The celebrated Ramanujan-Petersson conjecture for an elliptic cuspidal Hecke eigenform \( f \) of weight \( k \geq 2 \) and level \( N \) asserts that for any prime \( p \nmid N \),

\[
|\lambda_f(p)| \leq 2p^{k-\frac{3}{2}},
\]

where \( \lambda_f(p) \) denotes the \( p \)-th Hecke eigenvalue of \( f \). This conjecture has been solved affirmatively by Deligne in [De1, De2] as a consequence of his proof of the Weil conjectures.

Now let \( \phi \) denote a primitive Hecke-Maass cusp form for \( \Gamma_0(N) \) and Dirichlet character \( \chi_\phi \) of conductor \( N \) with the Laplacian eigenvalue \( \lambda_\phi = \frac{1}{4} + t_\phi^2 \). Denote the \( n \)-th Hecke eigenvalue of \( \phi \) by \( \lambda_\phi(n) \) for \( n \in \mathbb{N} \). The generalized Ramanujan-Petersson conjecture predicts that for \( p \nmid N \),

\[
|\lambda_\phi(p)| \leq 2,
\]

which is equivalent to (see the Lemma 2.1 below) \( |\alpha_p| = |\beta_p| = 1 \), where \( \{\alpha_p, \beta_p\} \) is the Satake parameter of \( \phi \) at \( p \), i.e., the local component of \( \phi \) at \( p \) is tempered. This is an outstanding unsolved problem in number theory, which would follow from the Langlands functoriality conjectures. Currently the record of individual bounds towards this conjecture is due to Kim-Sarnak [KS]

\[
|\lambda_\phi(p)| \leq p^{\frac{2}{3}} + p^{-\frac{1}{3}}.
\]

We refer the readers to the recent survey paper by Blomer and Brumley [BB].
The goal of this paper is to prove the following theorem in which we show that the least (unramified) prime at which the local component of $\phi$ is tempered is bounded by $(N(1 + |t\phi|))^c$ for an explicit constant $c > 0$.

**Theorem 1.1.** Let $\phi$ be a Hecke-Maass cusp form for $\Gamma_0(N)$ as above, with $\lambda_\phi(n)$ its Hecke eigenvalue and $(\frac{1}{4} + t_\phi^2)$ its Laplace eigenvalue. Then for any $\epsilon > 0$, there exists a prime $p \nmid N$ such that $|\lambda_\phi(p)| \leq 2$ and $p \ll \epsilon (N(1 + |t\phi|))^{8/11 + \epsilon}$, where the implied constant depends only on $\epsilon$.

Our approach is based upon the following simple yet crucial observation that if the local component of $\phi$ at an unramified prime $p$ is not tempered, then (see Lemma 2.1) $\lambda_\phi(p^{2i})\chi_\phi(p^i) > 2i + 1$ for all $i \geq 1$, where $\chi_\phi$ is the Dirichlet character of $\phi$. Thus the following adjoint (square) $L$-function associated to $\phi$ comes into play (see [GJ]),

$$L^{(N)}(s, \text{Ad}(\phi)) = \zeta^{(N)}(2s) \sum_{\substack{n \equiv 1 \\ (n,N) = 1}} \chi_\phi(n)\lambda_\phi(n^2) n^{-s},$$

where $\zeta^{(N)}(s)$ stands for the partial zeta function of $\zeta(s)$ with local factors at $p|N$ removed. Then we relate our goal of bounding the least unramified prime at which the local component is tempered to the sieving idea in the work [IKS] of Iwaniec, Kohnen and Sengupta, which studies the sign changes of Hecke eigenvalues of holomorphic modular forms based on Deligne’s solution of the Ramanujan-Petersson conjecture. In the appendix, we improve the constant $8/11$ to 0.27331 by using a refined method, which incorporates numerical computation and some recent development in the theory of multiplicative functions.

Ramakrishnan in [Ram] proved that for a Maass cusp form $\phi$ as above, the Ramanujan-Petersson conjecture is true for (unramified) primes with the lower Dirichlet density (or analytic density) at least $9/10$. This lower Dirichlet density is later improved to $34/35$ by Kim and Shahidi in [KSh2] via the symmetric cube and fourth lifts of $\text{GL}_2(AQ)$ of Shahidi and Kim in [KSh1] and [Kim]. In Section 3, we refine the density results of [KSh2] (and [Ram]) from the Dirichlet density to the natural density. Our key ingredients is the nonvanishing theorem of Rankin-Selberg $L$-functions on $\Re(s) = 1$ in the paper [Sha] of Shahidi.

**Theorem 1.2.** Let $\phi$ be a Hecke-Maass cusp form for $\Gamma_0(N)$ as above, with $\lambda_\phi(n)$ its Hecke eigenvalue. We have

$$\liminf_{X \to \infty} \frac{\#\{p \text{ prime} \mid p \nmid N, p \leq X, |\lambda_\phi(p)| \leq 2\}}{\#\{p \text{ prime} \mid p \leq X\}} \geq \frac{34}{35}.$$ 

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2. Effective Bound on the Least Tempered Component. Let $\phi$ be a primitive Hecke-Maass cusp form for $\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})$ with Dirichlet character $\chi_\phi : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}$. It has Laplacian eigenvalue $\lambda_\phi = \frac{1}{4} + t_\phi^2$ with the parameter $t_\phi$ lying in $\mathbb{R} \cup [-7i/64, 7i/64]$ by the Kim-Sarnak bound of [KS] at the archimedean place. We assume that $\phi$ is not of dihedral type, otherwise the full Ramanujan conjecture is known. Let $\{\alpha_p, \beta_p\}$ denote the Satake parameter at $p \nmid N$. The standard $L$-function of $\phi$ is given by

$$L(N)(s, \phi) = \sum_{n=1}^{\infty} \lambda_\phi(n) \frac{n^{s}}{(n, N)=1} \zeta(N)(2s) \prod_{p|N} \left(1 - \frac{\alpha_p}{p^s}\right) \left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

with $\alpha_p \beta_p = \chi_\phi(p)$, where $\lambda_\phi(n)$’s are normalized Hecke eigenvalues with $\lambda_\phi(1) = 1$ and $T_n\phi = \lambda_\phi(n)\phi$ for $(n, N) = 1$.

Our main tool is the adjoint $L$-function of $\phi$ mentioned in the Introduction

$$L(N)(s, \text{Ad}(\phi)) = \zeta(N)(2s) \sum_{n=1}^{\infty} \lambda_\phi(n^2) \frac{\chi_\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{A_\phi(n)}{n^s},$$

where we have

$$A_\phi(n) = \sum_{k^2|n} \lambda_\phi(n^2/k^4) \chi_\phi(n/k^2)$$

for $(n, N) = 1$. We denote

$$Q := N^2(1 + |t_\phi|)^2,$$

which bounds the analytic conductor defined in [IS] for $\text{Ad}(\phi)$.

The following lemma will be used in the proofs of the later sections, and a part of it is also an ingredient in [Ram]. It gives a criterion for the Ramanujan-Petersson conjecture by using the Fourier coefficients at even prime powers.

**Lemma 2.1.** Let $\{\alpha_p, \beta_p\}$ denote the Satake parameter at $p \nmid N$ of a primitive Hecke-Maass cusp form $\phi$ for $\Gamma_0(N)$ with Dirichlet character $\chi_\phi$. Then the Satake parameter at $p$ for $L(s, \text{Ad}(\phi))$ is given by $\{\alpha_p/\beta_p, 1, \beta_p/\alpha_p\}$. For any prime $p \nmid N$, we have

$$|\lambda_\phi(p)|^2 = \lambda_\phi^2(p) \chi_\phi(p) = \lambda_\phi(p^2) \chi_\phi(p) + 1.$$ 

In particular $\lambda_\phi(p^2) \chi_\phi(p)$ is real and $\lambda_\phi(p^2) \chi_\phi(p) \geq -1$. If the local component at $p$ is not tempered, i.e., $|\alpha_p| \neq 1$, $|\beta_p| \neq 1$, then we have $|\lambda_\phi(p)| > 2$ and $\alpha_p/\beta_p$
is real and \( > 0 \) and for \( n \geq 1 \)
\[
\lambda_\phi(p^{2n})\overline{\chi_\phi(p^n)} = \sum_{i=-n}^{n} \left( \frac{\alpha_p}{\beta_p} \right)^i > d(p^{2n}) = 2n + 1,
\]
where \( d \) is the divisor function.

**Proof.** The first assertion follows from the definition of \( L(s, \text{Ad}(\phi)) \) and the fact that the Satake parameter at \( p \) for the contragredient form \( \overline{\phi} \) is \( \{ \alpha_p^{-1}, \beta_p^{-1} \} \). For \( p \nmid N \), we have
\[
\lambda_\phi(p) = \chi_\phi(p)\overline{\lambda_\phi(p)}.
\]

By Hecke relation, we have \( \lambda_\phi(p^2) = \lambda_\phi(p)^2 - \chi_\phi(p) \). Then we have \( \lambda_\phi(p^2)\overline{\chi_\phi(p)} = \lambda_\phi(p)\overline{\lambda_\phi(p)} - 1 \) and obviously \( \lambda_\phi(p^2)\overline{\chi_\phi(p)} \) is real and \( \geq -1 \).

For \( p \nmid N \), we have
\[
\alpha_p + \beta_p = \lambda_\phi(p) \quad \text{and} \quad \alpha_p\beta_p = \chi_\phi(p).
\]

Then we get
\[
\frac{\alpha_p}{\beta_p} + \frac{\beta_p}{\alpha_p} = |\lambda_\phi(p)|^2 - 2 \geq -2 \quad \text{and} \quad \frac{\alpha_p}{\beta_p} \cdot \frac{\beta_p}{\alpha_p} = 1.
\]

The pair \( \{ \alpha_p/\beta_p, \beta_p/\alpha_p \} \) are the roots of the quadratic equation
\[
X^2 - (|\lambda_\phi(p)|^2 - 2) X + 1 = 0.
\]

If the local component of \( \phi \) at \( p \nmid N \) is not tempered, i.e., \( |\alpha_p/\beta_p| \neq 1 \), this implies that \( \{ \alpha_p/\beta_p, \beta_p/\alpha_p \} \) are two real positive distinct roots. Because their product is 1, one of them is \( > 1 \) and the other is \( < 1 \). Also, we have \( |\lambda_\phi(p)| > 2 \). From
\[
\lambda_\phi(p^n) = \frac{\alpha_p^{n+1} - \beta_p^{n+1}}{\alpha_p - \beta_p} \quad \text{and} \quad \alpha_p\beta_p = \chi_\phi(p),
\]
we get the last assertion. \( \square \)

**Proof of Theorem 1.1.** Assume the local component of \( \phi \) at \( p \) is not tempered for all primes \( p \leq y \) and \( p \nmid N \). Then by the Lemma 2.1 we have \( A_\phi(d) > 3 \) for \( 1 < d \leq y \) and \( (d, N) = 1 \). Take \( x = yz \) and \( z = y^\delta \) with \( 0 < \delta < 1/2 \). Consider the sum
\[
S(x) = \sum_{\substack{d < x, \\ (d, N) = 1}} A_\phi(d) \log \frac{x}{d} = S^+(x) + S^-(x),
\]
where \( S^+(x) \) and \( S^-(x) \) denote the partial sums over the positive and negative coefficients \( A_\phi(d) \) respectively.
We have $A_\phi(d) < 0$ in $S^-(x)$, if and only if $d$ uniquely splits into the product of two numbers $m$ and $p$ with $A_\phi(m) > 0$, $A_\phi(p) < 0$, $y < p < x$, because $x = y^{1+\delta}$ with $0 < \delta < 1/2$. Moreover we have $m < x/p < z$. From $A_\phi(p) \geq -1$ in Lemma 2.1, we deduce that

\[
S^-(x) = \sum_{pm<x, p>y, \quad (pm,N)=1, \quad A_\phi(p)<0} A_\phi(pm) \log \left( \frac{x}{pm} \right)
\]

\[
\geq - \sum_{m<z, \quad (m,N)=1} A_\phi(m) \sum_{p<x/m} \log \left( \frac{x}{pm} \right)
\]

\[
\geq - \left( \sum_{m<z, \quad (m,N)=1} \frac{A_\phi(m)}{m} \right) \frac{x}{\log y} \left( 1 + O \left( \frac{1}{\log y} \right) \right),
\]

in view of the asymptotics

\[
\pi(x) \log x - \sum_{p \leq x} \log p = \frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right),
\]

where $\pi(x)$ is defined to be the number of primes less than $x$. The last equality is a version of the prime number theorem (see page 61 and 138 of [Pra]).

Next we bound $S^+(x)$. By positivity, we have

\[
S^+(x) \geq \sum_{m<z, \quad (m,N)=1} A_\phi(m) \sum_{\substack{1<l<y/m, \quad p|l \Rightarrow z<p \leq y, \quad (l,N)=1}} A_\phi(l) \log \left( \frac{x}{lm} \right)
\]

\[
\geq 3 \sum_{m<z, \quad (m,N)=1} A_\phi(m) \left( \Phi'(x/m,y,z) + O \left( \frac{y}{m \log N} \right) \right),
\]

where we denote

\[
\Phi'(X,Y,Z) = \sum_{\substack{1<l<X \quad p|l \Rightarrow Z<p \leq Y}} \log \left( \frac{X}{l} \right).
\]
The error term in (3) comes from
\[
\sum_{1<l<x/m, \atop p
mid l \Rightarrow z<p \leq y, \atop q \mid l} \log \left( \frac{x}{lm} \right) \leq \frac{x}{qm} \log y \leq \frac{y}{m} \log y
\]
for a prime $q \mid N$ with $q > z$. There are at most $O\left( \frac{\log N}{\log y} \right)$ such $q$.

**Lemma 2.2.** If $Z$ is large, $Z < Y$ and $Y < X \leq YZ$, then we have
\[
\Phi'(X,Y,Z) > \frac{X}{2 \log Z} - \frac{X}{\log Y} + O \left( \frac{Z \log Y}{\log Z} + \frac{X}{\log^2 Z} \right).
\]

**Proof.** Define
\[
\Phi(X,Y,Z) = \sum_{1<l<X \atop p \mid l \Rightarrow Z<p \leq Y} 1 \quad \text{and} \quad \Phi(X,Z) = \sum_{1<l<X \atop p \mid l \Rightarrow Z<p} 1.
\]

Then we have
\[
\Phi'(X,Y,Z) = \int_{Y}^{X} \Phi(t,Y,Z) \frac{dt}{t} + \int_{Z}^{Y} \Phi(t,Z) \frac{dt}{t}.
\]

For $Y < t \leq YZ$, it is easy to see that
\[
\Phi(t,Y,Z) = \Phi(t,Z) - \Phi(t,Y).
\]

Recall the asymptotic formula of $\Phi(X,Z)$, $X \geq Z \geq 2$ (see Theorem 3, pp. 400, [Ten])
\[
\Phi(X,Z) = \omega \left( \frac{\log X}{\log Z} \right) \frac{X}{\log Z} - \frac{Z}{\log Z} + O \left( \frac{X}{\log^2 Z} \right), \tag{4}
\]
where $\omega(u)$ is the Buchstab function, that is the continuous solution to the difference-differential equation
\[
u \omega(u) = 1 \quad (1 \leq u \leq 2),
\]
\[
(u \omega(u))' = \omega(u-1) \quad (u > 2).
\]
Moreover the range of the Buchstab function is $1/2 \leq \omega(u) \leq 1$ [Ten, (22), p. 400]. We infer that

$$\Phi'(X, Y, Z) = \int_{Z}^{X} \Phi(t, Z) \frac{dt}{t} - \int_{Y}^{X} \Phi(t, Y) \frac{dt}{t} \geq \int_{Z}^{X} \left( \frac{1}{2} \frac{t}{\log Z} - \frac{Z}{\log Z} \right) \frac{dt}{t} - \int_{Y}^{X} \left( \frac{t}{\log Y} - \frac{Y}{\log Y} \right) \frac{dt}{t} + O \left( \frac{X}{\log^2 Z} \right)
$$

$$\geq \frac{X}{2 \log Z} - \frac{X}{\log Y} + O \left( \frac{Z \log Y}{\log Z} + \frac{X}{\log^2 Z} \right).$$

This completes the proof of Lemma 2.2. 

By Lemma 2.2, we have

$$\Phi'(x/m, y, z) > \left( \frac{1}{2\delta} - 1 + O \left( \frac{1}{\log y} \right) \right) \frac{x}{m \log y},$$

and

$$S^+(x) > \left( \frac{3}{2\delta} - 3 + O \left( \frac{1}{\log y} + \frac{\log y \log N}{z} \right) \right) \left( \sum_{\substack{m < z, \ \ (m, N) = 1 \ \ \ \ m \to \phi}} \frac{A_\phi(m)}{m} \right) \frac{x}{\log y}$$

from (3). Consequently, after combining with the lower bound of $S^-(x)$ in (2), we deduce that

$$S(x) > \left( \frac{3}{2\delta} - 4 + O \left( \frac{1}{\log y} + \frac{\log y \log N}{z} \right) \right) \left( \sum_{\substack{m < z, \ \ (m, N) = 1 \ \ \ m \to \phi}} \frac{A_\phi(m)}{m} \right) \frac{x}{\log y}.$$ 

Therefore we have

$$S(x) \gg \frac{x}{\log x}$$

on choosing $\delta = 3/8 - \epsilon$ for $0 < \epsilon < 1/8$, provided $y \gg N^{1/100}$.

If $\phi$ is of dihedral type, the Ramanujan conjecture is true for $\phi$. Hence we can assume that $\phi$ is not of dihedral type and the adjoint representation $\text{Ad}(\phi)$ is a cuspidal automorphic representation of $\text{GL}_3(A_Q)$. Its $L$-function $L(s, \text{Ad}(\phi))$ is
holomorphic on the whole complex plane. Now for \(\sigma > 1\), we have

\[
S(x) = \sum_{d \leq x, (d,N)=1} A_\phi(d) \log \left( \frac{x}{d} \right)
\]

(6)

\[
= \frac{1}{2\pi i} \int_{(\sigma)} L^{(N)}(s, \text{Ad}(\phi)) \frac{x^s}{s^2} ds
\]

\[
= \frac{1}{2\pi i} \int_{(1/2)} L^{(N)}(s, \text{Ad}(\phi)) \frac{x^s}{s^2} ds.
\]

The Phragmén-Lindelöf principle gives \(L(s, \text{Ad}(\phi))\) the convexity bound on the critical line (see (5.21) of [IK] on pp. 101). But since the ramified places in \(L(s, \text{Ad}(\phi))\) are well controlled (see Proposition 3.3 of [MS]), we have the same convexity bound

\[
L^{(N)}\left(\frac{1}{2} + it, \text{Ad}(\phi)\right) \ll Q^{1/4+\eta} (1 + |t|)^{3/4+\eta}
\]

for \(\eta > 0\). By applying the convexity bound for \(L^{(N)}(s, \text{Ad}(\phi))\) on the critical line, we obtain

(7)

\[
S(x) \ll \eta Q^{1/4+\eta} x^{1/2}.
\]

Comparing (5), (7) and \(x = y^{1+\delta}\), we obtain

\[
y \ll \epsilon \left(N (1 + |t_\phi|) \right)^{8/11+\epsilon},
\]

for any \(\epsilon > 0\). This completes the proof of Theorem 1.1.

\[
\Box
\]

3. **Natural density of tempered components.** Let \(\phi\) be a primitive Hecke-Maass cusp form for \(\Gamma_0(N)\) with Dirichlet character \(\chi_\phi \mod N\), with \(\lambda_\phi(n)\) its Hecke eigenvalue, following the same notations of the previous section. We assume that \(\phi\) is not of dihedral, tetrahedral, or octahedral type, since otherwise the Ramanujan-Petersson conjecture is obviously true. Therefore, the adjoint, symmetric cube and symmetric fourth power lifts of \(\phi\) are automorphic and cuspidal by [GJ, KSh1, Kim, KSh2] (see Theorem 3.3.7 of [KSh2], and Theorem B of [KSh1]).

In this section, we improve the density results of the tempered local components in [Ram, KSh2] from Dirichlet density to natural density. Our result appears to be new.

Let \(\{\alpha_p, \beta_p\}\) be the Satake parameter associated with \(\phi\) at an unramified prime \(p \nmid N\). The adjoint lift \(\text{Ad}(\phi)\) of Gelbart and Jacquet (see [GJ]), with its \(L\)-function
defined by

\[
L^{(N)}(s, \text{Ad}(\phi)) = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{A_{\phi}(n)}{n^s} = \prod_{p|N} \left( \left( 1 - \frac{\alpha_p/\beta_p}{p^s} \right) \left( 1 - \frac{1}{p^s} \right) \left( 1 - \frac{\beta_p/\alpha_p}{p^s} \right) \right)^{-1}, \quad \Re(s) > 1,
\]

is a cuspidal automorphic representation of $\text{GL}_3(\mathbb{A}_\mathbb{Q})$, with $\{\alpha_p/\beta_p, 1, \beta_p/\alpha_p\}$ its Satake parameter at $p \nmid N$. The symmetric cube lift $\text{Sym}^3 \phi$ and the twisted symmetric fourth power lift $\text{Sym}^4 \phi \times \chi_\phi^2$ are cuspidal automorphic representations of $\text{GL}_4(\mathbb{A}_\mathbb{Q})$ and $\text{GL}_5(\mathbb{A}_\mathbb{Q})$ respectively (see [KSh1, Kim, KSh2]). The Satake parameter of $\text{Sym}^3 \phi$ is given by $\{\alpha_3^3, \alpha_2^2, \beta_p, \alpha_p^2 \beta_p, \beta_3^3\}$, while that of $\text{Sym}^4 \phi \times \chi_\phi^2$ is given by $\{\alpha_2^2/\beta_p^2, \alpha_p/\beta_p, 1, \beta_p/\alpha_p, \beta_2^2/\alpha_2^2\}$ at $p \nmid N$. Let

\[
L^{(N)}(s, \text{Sym}^3 \phi) = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \frac{A_{\phi}^{[3]}(n)}{n^s} = \prod_{p|N} \left( \left( 1 - \frac{\alpha_3^3}{p^s} \right) \left( 1 - \frac{\alpha_2^2 \beta_p}{p^s} \right) \left( 1 - \frac{\alpha_p^2 \beta_p}{p^s} \right) \left( 1 - \frac{\beta_3^3}{p^s} \right) \right)^{-1}, \quad \Re(s) > 1,
\]

and

\[
L^{(N)}(s, \text{Sym}^4 \phi \times \chi_\phi^2) = \sum_{n=1}^{\infty} \frac{A_{\phi}^{[4]}(n)}{n^s} = \prod_{p|N} \left( \left( 1 - \frac{\alpha_p^2/\beta_p^2}{p^s} \right) \left( 1 - \frac{\alpha_p/\beta_p}{p^s} \right) \left( 1 - \frac{1}{p^s} \right) \right) \times \left( 1 - \frac{\beta_p/\alpha_p}{p^s} \right) \left( 1 - \frac{\beta_p^2/\alpha_p^2}{p^s} \right)^{-1}, \quad \Re(s) > 1,
\]

be their $L$-functions. The $L$-functions $L^{(N)}(s, \text{Ad}(\phi))$, $L^{(N)}(s, \text{Sym}^3 \phi)$ and $L^{(N)}(s, \text{Sym}^4 \phi \times \chi_\phi^2)$ are holomorphic on the whole complex plane.

Let $\pi(X)$ be the number of primes no greater than $X$. By the classical prime number theorem, we have $\pi(X) \sim X/\log X$ as $X \to \infty$. 

LEMMA 3.1. For a Hecke-Maass cusp form $\phi$ of level $N$ on $GL_2(\mathbb{A}_\mathbb{Q})$ that is not of dihedral, tetrahedral or octahedral type, we have the Prime Number Theorem for $Ad(\phi)$ and $\text{Sym}^4 \phi \times \overline{\chi}_\phi^2$

$$\sum_{p \nmid N, p \leq X} A_\phi(p) = o(\pi(X)) \quad \text{and} \quad \sum_{p \nmid N, p \leq X} A^4_\phi(p) = o(\pi(X)),$$

as $X \to \infty$.

Proof. Because $Ad(\phi)$ and $\text{Sym}^4 \phi \times \overline{\chi}_\phi^2$ are automorphic and cuspidal, their $L$-functions $L^{(N)}(s, Ad(\phi))$ and $L^{(N)}(s, \text{Sym}^4 \phi \times \overline{\chi}_\phi^2)$ have a standard zero-free region. The Prime Number Theorem for $L$-functions follows from the standard zero-free region. (see Theorem 5.13 of [IK]). □

We have $A_\phi(p) = \lambda_\phi(p^2)\overline{\chi}_\phi(p) \in \mathbb{R}$ for $p \nmid N$ by Lemma 2.1. We also have $A^4_\phi(p) = \lambda_\phi(p^4)\overline{\chi}_\phi(p^2) \in \mathbb{R}$ because of Lemma 2.1 and the Hecke relation $A_\phi(p^2) = A^4_\phi(p) + A_\phi(p) + 1$.

Because $\text{Sym}^3 \phi$ and $\text{Sym}^4 \phi \times \overline{\chi}_\phi^2$ are cuspidal automorphic representations, we further look at their Rankin-Selberg $L$-functions. Unlike $L^{(N)}(s, Ad(\phi))$ and $L^{(N)}(s, \text{Sym}^4 \phi \times \overline{\chi}_\phi^2)$, we don’t have a standard zero-free region for the Rankin-Selberg $L$-functions. However, in this particular case, nonvanishing results are available from Shahidi’s earlier work [Sha].

LEMMA 3.2. For a Hecke-Maass cusp form $\phi$ of level $N$ on $GL_2(\mathbb{A}_\mathbb{Q})$ that is not of dihedral, tetrahedral or octahedral type with the Satake parameter $\{\alpha_p, \beta_p\}$ at an unramified prime $p \nmid N$, we have

$$\sum_{p^k \leq X, p \nmid N} \log(p) \left| \alpha_p^{3k} + \alpha_p^{2k}\beta_p^k + \alpha_p^k\beta_p^{2k} + \beta_p^{3k} \right|^2 \sim X$$

and

$$\sum_{p^k \leq X, p \nmid N} \log(p) \left| \alpha_p^{2k}/\beta_p^k + \alpha_p^k/\beta_p^k + 1 + \beta_p^k/\alpha_p^k + \beta_p^{2k}/\alpha_p^{2k} \right|^2 \sim X$$

as $X \to \infty$.

Proof. Shahidi proved in Theorem 5.2 of [Sha] that the Rankin-Selberg convolution $L$-functions are non-zero on the line $\Re(s) = 1$. Thus, $-\mathcal{L}'/\mathcal{L}(s)$, where $\mathcal{L}(s) = L(s, \Pi \times \overline{\Pi})$, and $\Pi = \text{Sym}^3 \phi$ or $\text{Sym}^4 \phi \times \overline{\chi}_\phi^2$, is regular on $\Re(s) = 1$ except a pole at $s = 1$. Moreover they are Dirichlet series with non-negative coefficients. By the Tauberian theorem of Wiener and Ikehara [MV, Corollary 8.8] we obtain the Prime Number Theorem for such $\mathcal{L}(s)$. □
Recall that
\[ A \beta \]
the proof. □

Thus we have
\[ \text{By Lemma 3.1 and Remark 3.3, we have} \]
\[ \limsup_{X \to \infty} \frac{\sum_{p \leq X, p \mid N} |A^{[3]}_\phi(p)|^2}{\pi(X)} \leq 1 \]
and
\[ \limsup_{X \to \infty} \frac{\sum_{p \leq X, p \mid N} A^{[4]}_\phi(p)^2}{\pi(X)} \leq 1, \]
because \( A^{[3]}_\phi(p) = \alpha_p^3 + \alpha_p^2 \beta_p + \alpha_p \beta_p^2 + \beta_p^3 \) and \( A^{[4]}_\phi(p) = \alpha_p^2 / \beta_p^2 + \alpha_p / \beta_p + 1 + \beta_p / \alpha_p + \beta_p^2 / \alpha_p^2 \) with \( A^{[4]}_\phi(p) \in \mathbb{R} \).

**Proof of Theorem 1.2.** For \( p \nmid N \), define \( U(p) := (1 + 3A_\phi(p) + 5A^{[4]}_\phi(p))^2 \).
Recall that \( A_\phi(p), A^{[4]}_\phi(p) \in \mathbb{R} \) for \( p \nmid N \). We have \( U(p) \geq 0 \) and if the local component at \( p \) is not tempered, we have by Lemma 2.1
\[ U(p) > 35^2. \]
Because of \( \{ \alpha_p, \beta_p \} = \{ \alpha_p^{-1}, \beta_p^{-1} \} \), we have the Hecke relations
\[ A_\phi(p)A^{[4]}_\phi(p) = |A^{[3]}_\phi(p)|^2 - 1 \quad \text{and} \quad A_\phi(p)^2 = A^{[4]}_\phi(p) + A_\phi(p) + 1. \]
Thus we have
\[ U(p) = 1 + 9A_\phi(p)^2 + 25A^{[4]}_\phi(p)^2 + 6A_\phi(p) + 10A^{[4]}_\phi(p) + 30A_\phi(p)A^{[4]}_\phi(p) \]
\[ = -20 + 15A_\phi(p) + 19A^{[4]}_\phi(p) + 30|A^{[3]}_\phi(p)|^2 + 25A^{[4]}_\phi(p)^2. \]
By Lemma 3.1 and Remark 3.3, we have
\[ \limsup_{X \to \infty} \frac{\sum_{p \leq X, p \mid N} U(p)}{\pi(X)} \leq 35. \]
We have from Lemma 2.1
\[ \frac{\sum_{p \leq X, p \mid N} U(p)}{\pi(X)} \geq 35^2 \left( \frac{\pi(X) - \# \{ \text{prime} \mid p \nmid N, p \leq X, |\lambda_\phi(p)| \leq 2 \} \} }{\pi(X)} \]
and then
\[ \frac{\# \{ \text{prime} \mid p \nmid N, p \leq X, |\lambda_\phi(p)| \leq 2 \} }{\pi(X)} \geq 1 - \frac{\sum_{p \leq X, p \mid N} U(p)}{35^2 \pi(X)}. \]
Hence by (8) we get \( \liminf_{X \to \infty} \frac{\# \{ \text{prime} \mid p \nmid N, p \leq X, |\lambda_\phi(p)| \leq 2 \} }{\pi(X)} \geq 34/35 \) and complete the proof. □
Remark 3.4. The construction of $U(p)$ originates from [KSh2] and [Ram], which can be proved to be optimal by the Cauchy-Schwarz inequality. The difference between [KSh2] and this article is that [KSh2] considered only nonvanishing of the aforementioned $L$-functions at $s = 1$, whereas we have nonvanishing results of standard automorphic $L$-functions (zero-free region) and the Rankin-Selberg $L$-functions on $\Re(s) = 1$ from [Sha]. It is not yet proved that the general Rankin-Selberg $L$-functions have a standard zero-free region.

The constant $34/35$ could be improved if there was progress toward the functoriality of symmetric powers on $GL_2(\mathbb{A}_\mathbb{Q})$. If the symmetric $i$-th powers of $GL_2(\mathbb{A}_\mathbb{Q})$ for $i = 2, 3, \ldots, 2n$ are proved to be automorphic and cuspidal, we may replace the constant $34/35$ with $(1 - 3/(n+1)(4n^2 + 8n + 3))$.

Appendix: Refinement of Section 2. Technology in the theory of multiplicative functions as developed from [GS, KLSW, LLW, Mat], combined with numerical computation, would improve the exponent further. We consider a sum over squarefree numbers for $x > 1$

$$S^\phi(x) = \sum_{n \leq x}^{\sharp} A_\phi(n) \log \left( \frac{x}{n} \right),$$

where the summation $\sum^{\sharp}$ is taken over squarefree numbers. This is similar to but different from $S(x)$. A lower bound for (9) is a solution to some differential-difference equation. One can find its numerical solution with a computer and a computational software program, and improve the exponent in Theorem 1.1 to 0.27331.

Let us assume that the local component of $\phi$ at $p$ is not tempered for all $p \leq y$ and $p \nmid N$. Thus we have $A_\phi(p) > 3$ for all $p \leq y$ and $p \nmid N$ by Lemma 2.1.

Lemma A.1. We have the bound

$$S^\phi(x) \ll_\epsilon x^{3/4} Q^{1/8 + \epsilon}$$

for $\epsilon > 0$.

Proof. Define a Dirichlet series

$$G(s) = \prod_{p|N} \left( 1 - \frac{A_\phi(p)}{p^s} + \frac{A_\phi(p)}{p^{2s}} - \frac{1}{p^{3s}} \right) \left( 1 + \frac{A_\phi(p)}{p^s} \right).$$

The analytic function $G(s)$ is absolutely convergent in $\{ \Re(s) > 1/2 + \epsilon \}$, and uniformly bounded by $Q^\epsilon$ with any $\epsilon > 0$, in view of the Rankin-Selberg convolution of $Ad(\phi) \times Ad(\phi)$. The bound $Q^\epsilon$ is a consequence of [Bru, Corollary 2] and more
generally [Li, Theorem 2]. Now

\[
L^{(N)}(s, \text{Ad}(\phi))G(s) = \sum_{n=1}^{\infty} \frac{A_\phi(n)}{n^s}
\]

is absolutely convergent in \(\{\Re(s) > 1\}\). For \(c > 1\), we have

\[
S^\phi(x) = \frac{1}{2\pi i} \int_{(c)} L^{(N)}(s, \text{Ad}(\phi))G(s) \frac{x^s}{s^2} ds
\]

\[= \frac{1}{2\pi i} \int_{(3/4)} L^{(N)}(s, \text{Ad}(\phi))G(s) \frac{x^s}{s^2} ds.
\]

By using the convexity bound

\[
L^{(N)}(\frac{3}{4} + it, \text{Ad}(\phi)) \ll (Q(1 + |t|)^3)^{1/8+\epsilon},
\]

we obtain \(S^\phi(x) \ll x^{3/4}Q^{1/8+\epsilon}\).

\[\square\]

Define a multiplicative function supported on squarefree numbers with

\[
h(p) = \begin{cases} 3, & p \leq y, \\ -1, & p > y. \end{cases}
\]

It extends to all squarefree numbers. For convenience, we define \(h(n) = 0\) if \(n\) is not squarefree. Define the sum

\[S^\phi(x) = \sum_{n \leq x} A_\phi(n).
\]

**Lemma A.2.** If \(\sum_{n \leq t} h(n) \geq 0\) for all \(t \leq x\), we have \(S^\phi(x) \geq \sum_{n \leq x} h(n)\).

**Proof.** The proof follows from (2.4) of [KLSW]. Let us define a multiplicative function \(g\) supported on squarefree numbers defined by the Dirichlet convolution

\[A_\phi = h * g, \quad \text{or} \quad A_\phi(n) = \sum_{d | n} h(d)g \left(\frac{n}{d}\right)
\]
for \((n, N) = 1\). By Lemma 2.1, we have \(g(p) = A_\phi(p) - h(p) \geq 0\) for \(p \nmid N\) and \(g(1) = 1\). Then we have

\[
S^b(x) = \sum_{n \leq x} \sum_{d|n, (n, N) = 1} h(d)g\left(\frac{n}{d}\right)
\]

\[
= \sum_{d \leq x} g(d) \sum_{b \leq x/d, (b, N) = 1} h(b)
\]

\[
\geq \sum_{n \leq x} h(n)
\]

since both \(g(d)\) and \(\sum h(b)\) are non-negative.

**Lemma A.3.** If \(\sum_{n \leq t} h(n) \geq 0\) for all \(t \leq x\), we have \(S^b(x) \geq \sum_{n \leq x} h(n) \log \left(\frac{x}{n}\right)\).

**Proof.** It follows from the formula \(S^b(x) = \int_1^x S^b(t) \frac{dt}{t}\) and Lemma A.2.

The following lemma evaluates the mean of the multiplicative function \(h(n)\) over a long range \(1 \leq n \leq x\) where \(x\) equals \(y^u\) for some \(u > 1\). The special case of this lemma appears in [KLSW, LLW] and a more elaborate version is available in [Mat]. Its idea originates from [GS].

**Lemma A.4.** Let \(U \geq 1\) and let \(h(n)\) be as above. We have

\[
\sum_{n \leq y^u} h(n) = c(N)(\sigma(u) + o_U(1))(\log y)^2 y^u
\]

uniformly for \(u \in [1/U, U]\), where \(\lim_{y \to \infty} o_U(1) = 0\) and

\[
c(N) = \frac{1}{2} \left(\frac{\phi(N)}{N}\right)^3 \prod_{p|N} \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p}\right) \gg (\log \log N)^{-3}.
\]

The constant \(\sigma(u)\) is the continuous function of \(u \in (0, \infty)\) uniquely determined by the differential-difference equation

\[
\sigma(u) = u^2, \quad 0 < u \leq 1,
\]

\[
(u^{-2}\sigma(u))' = -\frac{4\sigma(u-1)}{u^3}, \quad u > 1.
\]

**Proof.** In Lemma 6 of [Mat], take \(K = 1, x_0 = 0, x_1 = 1, \chi_0 = 3, \chi_1 = -1, q = N\). The function \(\sigma(u)\) can be computed from Lemma 8 of [Mat].
Lemma A.5. Let $U > 1$ be such that $\sigma(u) > 0$ for $1 < u \leq U$. We have for $y \gg_U 1$,

$$\sum_{n \leq y^U \atop (n,N)=1} h(n) \log \left( \frac{y^U}{n} \right) \gg_U c(N)y^U.$$  

Proof. Define $H(x) = \sum_{n \leq x \atop (n,N)=1} h(n)$. We have

$$\sum_{n \leq y^U \atop (n,N)=1} h(n) \log \left( \frac{y^U}{n} \right) = \int_1^{y^U} H(t) \frac{dt}{t} = \int_0^U H(y^u) \log y \, du$$

$$\geq \int_{1/U}^U H(y^u) \log y \, du.$$  

By Lemma A.4, we have for $1/U \leq u \leq U$ uniformly

$$H(y^u) = c(N)(\sigma(u) + o_U(1))(\log y)^2 y^u.$$  

For $y \gg_U 1$, we hence have

$$\int_{1/U}^U H(y^u) \log y \, du \gg_U c(N)y^U$$

and this completes the proof. \qed

Let $U$ be the same as defined in Lemma A.5. We have $c(N) \gg Q^{-\epsilon}$ for $\epsilon > 0$. Comparing Lemma A.1, Lemma A.3 and Lemma A.5, we infer that

$$y^U Q^{-\epsilon} \ll_U \sum_{n \leq y^U \atop (n,N)=1} h(n) \log \left( \frac{y^U}{n} \right) \ll S^b(y^U) \ll (y^U)^{3/4} Q^{1/8+\epsilon}$$

and this in turn gives

$$y \ll_U Q^{\frac{1}{12}+\epsilon} = (N(1+|t_{\phi}|))^{1/12+2\epsilon}.$$  

By numerical computation of Mathematica, we find the smallest zero of $\sigma(u)$ is approximately 3.65887. Then taking $U$ to be microscopically less than 3.65887 we get:

Theorem A.6. Let $\phi$ be a Hecke-Maass cusp form for $\Gamma_0(N)$, with $\lambda_\phi(n)$ its Hecke eigenvalue and $(\frac{1}{4}+t_{\phi}^2)$ its Laplace eigenvalue. There exists a prime $p \nmid N$ such that $|\lambda_{\phi}(p)| \leq 2$ and $p \ll (N(1+|t_{\phi}|))^{0.27331}$. 


Remark A.7. To estimate the smallest zero of $\sigma(u)$ without numerical computation, we have from Lemma A.4

$$\sigma(u) = 7u^2 - 8u + 2 - 4u^2 \log u$$

for $1 \leq u \leq 2$. It is not hard to prove that $\sigma(u)$ is monotone for $1 \leq u \leq 2$ and this leads us to conclude $\sigma(u)$ is positive for $1 \leq u \leq 2$. Without numerical computation, we can have $1/2$ as the exponent in Theorem A.6.

For $2 \leq u \leq 3$, we have

$$\sigma(u) = 16u^2 Li_2(1-u) + (4\pi^2 u^2)/3 + 35u^2 - 24u^2 \log(u-1)$$

$$+ 16u^2 \log(u-1) \log(u) - 4u^2 \log(u) - 80u$$

$$+ 32u \log(u-1) - 8\log(u-1) + 34,$$

where $Li_2$ is the dilogarithm function (see [Zag]).
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