INTEGRABLE MODELS ASSOCIATED TO CLASSICAL REPRESENTATIONS OF $U_q(\widehat{sl(n)})$

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ABSTRACT

We describe a representation for $U_q(\widehat{sl(n)})$, when $q$ is not a root of unity, based on the fundamental representation of $sl(n)$. As $U_q(\widehat{sl(n)})$ has a Hopf algebra structure with a non-commutative co-product, we look for an intertwine matrix $R$ that relates two possible definitions of that co-product. We solve cases for $n = 2$ and $n = 3$, and then we generalize for any $n$. We obtain the hamiltonian associated to such matrix $R$, corresponding to a multi-state chain. As the case for $n = 2$ corresponds to the XXZ model with spin $1/2$, for $n > 2$ we have the generalization of the XXZ model to $sl(n)$. We show the case for $n = 3$ and its solution by Bethe ansatz.
1. Introduction

In these last years, the search for integrable models and related problems has deserved great attention. Some of the oldest and still most interesting models are the isotropic and anisotropic spin 1/2 chains of Heisenberg (XXX and XXZ models). The mathematical structure arising in these relatively simple models is astonishingly rich. The key featuring for it is the existence of a complete set of mutually commuting integrals of motion. They are usually defined by means of transfer matrix $F(u)$, which for these models is a function of a single spectral parameter $u$. Sutherland [1] showed that $[H, F(u)] = 0$ and later Baxter [2], in a more general context, proved the key property of the transfer matrix $[F(u), F(v)] = 0$. This implies that the logarithmic derivatives of $F(u)$,

$$Q_n = \frac{d^n}{du^n} \log F(u)|_{u=u_0}$$

mutually commute. The hamiltonian is related usually to $Q_1$.

With these results, Faddeev and collaborators stated the main ideas of quantum inverse scattering method (QISM) [3,4], where it is introduced the monodromy matrix $T(u)$ acting on an auxiliary space and whose elements are operators in the tensorial product of the site spaces. The trace of that matrix is transfer matrix $F(u)$. The main property of the monodromy matrix is that it verifies the Yang-Baxter equation (YBE), that resume the commutation relations of before. Besides it is possible to solve the model by using the method known as algebraic Bethe ansatz.

The development of quantum groups has introduced the YBE in a consistent mathematical structure and has guided the search of new solvable spin chains. It is know that these are related very closely to the representations of the quantum groups [5]. In fact, the XXZ model is associated to the dimension two representation of $U_q(sl(2))$ when $q$ is not a root of the unity. Other solvable models have been found for representations of $U_q(sl(2))$ when $q$ is a root of unity [6,7,8] and recently also for $SU_{p,q}(2)$ [9].
The present paper is organized as follows. In the next section, following the line used for the representations of \( U_q(\hat{sl}(2)) \), we build the representations of \( U_q(\hat{sl}(n)) \) when \( q \) is not a root of the unity [5,10], and find the \( R \) intertwine matrix in the product of representations. By using the \( R \) matrix we obtain the associate hamiltonian that corresponds to a multi-state spin chain with no-isotropic interaction. In that sense, we say that the models are a generalization to \( sl(n) \) of the \( sl(2) \) XXZ model.

In the third section, the eigenvalues of the transfer matrix of such models are calculated with the YBE. The nested algebraic Bethe ansatz is used [11]. For this calculation, we follow the method exposed in ref. [12]. A detailed account of the commutation rules derived from the YBE in every step of the development is given. The results are obtained for the eigenvalues of the transfer matrix in the \( U_q(\hat{sl}(n)) \), \( n = 2, 3 \) cases and the spectrum of energy is calculated for these cases and generalized to \( n > 3 \).

2. Formulation

Let \( sl(n) \) be the semisimple Lie algebra with generators \( e_i, f_i, h_i, (i = 1, \ldots, n-1) \) and \( A_{n-1} = (a_{i,j}) \) its Cartan matrix. One fundamental representation with dimension \( n \) is given by

\[
\begin{align*}
e_i & \to e_{i,i+1}, \\
f_i & \to e_{i+1,i}, \\
h_i & \to e_{i,i} - e_{i+1,i+1},
\end{align*}
\]

(2.1a) (2.1b) (2.1c)

where

\[
(e_{p,q})_{i,j} = \delta_{p,i} \delta_{q,j}, \quad (p, q, i, j = 1, \cdots, n).
\]

(2.2)

Following the Drinfeld-Jimbo method [2,8], the q-deformation of \( U_q(sl(2)) \) is gen-
erated by
\[ e_i, f_i, k_i, k_i^{-1}, \quad (i = 1, \cdots, n - 1) \] (2.3)
where \( k_i = q^{h_i/2} \). These satisfy the following relations

\[ [k_i, k_j] = 0, \quad (2.4a) \]
\[ k_i e_j k_i^{-1} = q^{a_{i,j}/2} e_j, \quad (2.4b) \]
\[ k_i f_j k_i^{-1} = q^{-a_{i,j}/2} f_j, \quad (2.4c) \]
\[ [e_i, f_j] = \delta_{i,j} \frac{k_i^2 - k_i^{-2}}{q - q^{-1}}, \quad (2.4d) \]
\[ k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad (2.4e) \]
\[ \sum_{\nu=0}^{1-a_{i,j}} (-1)^\nu \left[ 1 - \frac{a_{i,j}}{\nu} \right] q^{1-a_{i,j}-\nu} e_i^\nu f_i^\nu = 0, \quad (i \neq j), \quad (2.4f) \]
\[ \sum_{\nu=0}^{1-a_{i,j}} (-1)^\nu \left[ 1 - \frac{a_{i,j}}{\nu} \right] q^{1-a_{i,j}-\nu} f_j^\nu e_i^\nu = 0, \quad (i \neq j), \quad (2.4g) \]

where
\[ \begin{aligned}
[n]_j &= \frac{[n]_q [n - 1]_q \cdots [n - j + 1]_q}{[j]_q [j - 1]_q \cdots [1]_q}, \\
[n] &= 1, \quad j \in \mathbb{N}, \\
[n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (2.5b)
\end{aligned} \]

When \( q \) is not a root of the unity, an irreducible representation of \( U_q(sl(n)) \) with dimension \( n \) is obtained by using the fundamental representation of \( sl(n) \) given in (2.1). It is defined with the substitutions

\[ e_p \to e_{p,p+1}, \quad (2.6a) \]
\[ f_p \to e_{p+1,p}, \quad (2.6b) \]
\[ k_p \to q^{h_p/2}. \quad (2.6c) \]
Besides \( U_q(sl(n)) \) is a Hopf algebra with a coproduct \( \triangle \) defined with the elements of the base

\[
\begin{align*}
\triangle(e_i) &= k_i \otimes e_i + e_i \otimes k_i^{-1}, \\
\triangle(f_i) &= k_i \otimes f_i + f_i \otimes k_i^{-1}, \\
\triangle(k_i^{\pm1}) &= k_i^{\pm1} \otimes k_i^{\pm1}.
\end{align*}
\] (2.7a, 2.7b, 2.7c)

As this co-product is not commutative, we can as well define a new co-product \( \triangle' \) given by

\[
\begin{align*}
\triangle'(e_i) &= k_i^{-1} \otimes e_i + e_i \otimes k_i, \\
\triangle'(f_i) &= k_i^{-1} \otimes f_i + f_i \otimes k_i, \\
\triangle'(k_i^{\pm1}) &= k_i^{\pm1} \otimes k_i^{\pm1}.
\end{align*}
\] (2.8a, 2.8b, 2.8c)

Both co-products are related by a transformation \( R \) such that

\[
R \cdot \triangle(a) = \triangle'(a) \cdot R, \quad \forall a \in U_q(\widehat{sl(2)}),
\] (2.9)

and \( R \) verifies the Yang-Baxter equation

\[
R_{1,2} \cdot R_{1,3} \cdot R_{2,3} = R_{2,3} \cdot R_{1,3} \cdot R_{1,2}.
\] (2.10)

The models that we are going to describe, will be derived from a representation of the affine algebra \( U_q(\widehat{sl(n)}) \), with Cartan matrix \( A_{n-1}^{(1)} \). The generators \( \{E_i, F_i, H_i\}_{i=0}^{n-1} \) in a representation of such algebra can be obtained from another representation of \( U_q(sl(n)) \) using an affine parameter \( x \) as follows

\[
\begin{align*}
E_i &= x e_i, \\
F_i &= x^{-1} f_i, \\
H_i &= h_i, \quad (i = 1, 2, \cdots, n-1),
\end{align*}
\] (2.11a, 2.11b, 2.11c)
for the ordinary generators, and

\[ E_0 = x[f_{n-1}, f_1], \quad (2.12a) \]
\[ F_0 = x^{-1}[e_1, e_{n-1}], \quad (2.12b) \]
\[ H_0 = -h_1 - h_2 - \cdots - h_{n-1}, \quad (2.12c) \]

for the affine generators.

The product of representations must obey

\[
R(x, y) \cdot \triangle_{(n,x) \otimes (n,y)}(a) = \triangle'_{(n,x) \otimes (n,y)}(a) \cdot R(x, y), \quad \forall a \in U_q(\widehat{sl(n)}), \quad (2.13)
\]

where \((n, x)\) means a representation with dimension \(n\) and affine parameter \(x\).

In this paper, one of our goals is to find the \(R(x, y)\) matrix associated to the fundamental \(n\)-dimensional representations of \(U_q(\widehat{sl(n)})\) when \(q\) is not a root of the unity.

We begin with \(U_q(\widehat{sl(2)})\), where the two-dimensional representation, for affine parameter \(x\), is given by

\[
E_1 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 \\ x^{-1} & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad (2.14a)
\]
\[
E_0 = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}, \quad (2.14b)
\]

We take now

\[
R_{sl(2)} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad (2.15)
\]

where \(a, b\) and \(c\) depend on the parameters \(x\) and \(y\) that characterize each representation involved in product representation \((n, x) \otimes (n, y)\).
If we impose the fulfilment of (2.13), we find

\[ a = y^2 - x^2 q^2, \quad b = q(y^2 - x^2), \quad c = xy(1 - q^2), \quad \text{(2.16)} \]

with what, we can write the \( R \) matrix in the base \((e_{p,r})_{i,j} = \delta_{p,i} \delta_{r,j}\),

\[
R^{sl(2)}(x, y) = (y^2 - x^2 q^2) \sum_{i=1}^{2} e_{i,i} \otimes e_{i,i} + q(y^2 - x^2) \sum_{i,j=1 \atop i \neq j}^{2} e_{i,i} \otimes e_{j,j} \\
+ (1 - q^2) \sum_{i,j=1 \atop i < j}^{2} (x^{j-i} y^{j-i} y^2 e_{i,j} \otimes e_{j,i} + x^{j-i} y^{j-i} x^2 e_{j,i} \otimes e_{i,j}).
\]

(2.17)

In the \( U_q(sl(3)) \) case, we can follow the same steps as before. With (2.11), (2.12) and (2.6), we build the generators and impose a matrix \( R \) of the form

\[
R^{sl(3)} = \begin{pmatrix}
  a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & b & 0 & c & 0 & 0 & 0 & 0 \\
  0 & 0 & b & 0 & 0 & 0 & d & 0 \\
  0 & d & 0 & b & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & b & 0 & c & 0 \\
  0 & 0 & c & 0 & 0 & b & 0 & 0 \\
  0 & 0 & 0 & 0 & d & 0 & b & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{pmatrix}.
\]

(2.18)

The fulfilment of the equation (2.13) needs that

\[ a = y^3 - x^3 q^2, \quad b = q(y^3 - x^3), \quad c = xy^2(1 - q^2), \quad d = x^2 y(1 - q^2), \quad \text{(2.19)} \]
that allow to express

\[ R_{\text{sl}}^{(3)}(x, y) = (y^3 - x^3 q) \sum_{i=1}^{3} e_{i,i} \otimes e_{i,i} + q(y^3 - x^3) \sum_{i,j=1 \atop i \neq j}^{3} e_{i,i} \otimes e_{j,j} \]

\[ + (1 - q^2) \sum_{i,j=1 \atop i < j}^{3} (x^{j-i}y^{i-j}y^3 e_{i,j} \otimes e_{j,i} + x^{i-j}y^{j-i}x^3 e_{j,i} \otimes e_{i,j}), \]

(2.20)

The generalization to \( U_q(\widehat{\text{sl}(n)}) \) is now obvious

\[ R_{\text{sl}}^{(n)}(x, y) = (y^n - x^n q) \sum_{i=1}^{n} e_{i,i} \otimes e_{i,i} + q(y^n - x^n) \sum_{i,j=1 \atop i \neq j}^{n} e_{i,i} \otimes e_{j,j} \]

\[ + (1 - q^2) \sum_{i,j=1 \atop i < j}^{n} (x^{j-i}y^{i-j}y^n e_{i,j} \otimes e_{j,i} + x^{i-j}y^{j-i}x^n e_{j,i} \otimes e_{i,j}). \]

(2.21)

Taking into account that the YBE is homogeneous, we can substitute

\[ \frac{y}{x} = \exp(u), \quad q = \exp(-\gamma), \]

(2.22)

in eq. (2.21) and write the \( R \)-matrix in the form

\[ R_{\text{sl}}^{(n)}(u) = \sinh \left( \frac{n}{2} u + \gamma \right) \sum_{i=1}^{n} e_{i,i} \otimes e_{i,i} + \sinh \left( \frac{n}{2} u \right) \sum_{i,j=1 \atop i \neq j}^{n} e_{i,i} \otimes e_{j,j} \]

\[ + \sinh(\gamma) \sum_{i,j=1 \atop i \neq j}^{n} \exp \left[ (-j + i - \frac{n}{2} \text{sign}(i-j))u \right] e_{i,j} \otimes e_{j,i}, \]

(2.23)

where \( u \) is the so called spectral parameter.

Associated to every solution of the YBE, we can find a solvable model. So we introduce an one-dimensional lattice with a \( V_r = \mathbb{C}^n \) in every site \( r \) and an auxiliary vector space \( A = \mathbb{C}^n \). Now, we define an operator per site \( L_r(u) \), acting
on $V_r$ and $A$ and depending on the spectral parameter $u$ and a new operator $R(u)$ acting on $A \otimes A$. Then, the YBE (2.10) can be written in the usual form

$$R(u-v) \cdot (L_r(u) \otimes L_r(v)) = (L_r(v) \otimes L_r(u)) \cdot R(u-v), \quad (2.24)$$

where the $\otimes$ product is in the site space and $\cdot$ product is in $A \otimes A$ tensorial space. With this new form for YBE, the expression for $L$ is the same as for $R$ in (2.23) and the new $R$ is obtained from (2.23) by interchanging the indices $j$ and $m$ in every product $e_{i,j} \otimes e_{i,m}$ and in his coefficient. Then we have

$$R(u) = \sinh \left( \frac{n}{2}u + \gamma \right) \sum_{i=1}^{n} e_{i,i} \otimes e_{i,i} + \sinh \left( \frac{n}{2}u \right) \sum_{i,j=1 \atop i \neq j}^{n} e_{i,i} \otimes e_{j,j}$$

$$+ \sinh (\gamma) \sum_{i,j=1 \atop i \neq j}^{n} \exp \left[ \left( j - i - \frac{n}{2} \text{sign} (j-i) \right) u \right] e_{i,i} \otimes e_{j,j}, \quad (2.25a)$$

$$L_r(u) = \sinh \left( \frac{n}{2}u + \gamma \right) \sum_{i=1}^{n} e_{i,i} \otimes e_{i,i}^{r} + \sinh \left( \frac{n}{2}u \right) \sum_{i,j=1 \atop i \neq j}^{n} e_{i,i} \otimes e_{j,j}^{r}$$

$$+ \sinh (\gamma) \sum_{i,j=1 \atop i \neq j}^{n} \exp \left[ \left( j - i - \frac{n}{2} \text{sign} (j-i) \right) u \right] e_{i,j} \otimes e_{j,i}^{r}. \quad (2.25b)$$

Since $L_r$ is defined in $A \otimes V_r \sim \mathbb{C}^n \otimes \mathbb{C}^n$, we say that it is a $n$-component solvable model, and it is generalization of the model XXZ, defined in $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the Lie algebra $sl(2)$, to $sl(n)$.

To obtain the associated hamiltonian, we should proceed in the same way that in the $sl(2)$ case. We build the monodromy operator

$$T(u) = \prod_{i=N}^{1} L_i(u) = L_N(u) \cdot L_{(N-1)} \cdots L_1(u), \quad (2.26)$$

where the $\cdot$ product is understood as before in the auxiliary space and $N$ is the length of the chain.
Taking
\[ F(u) = \text{trace}_{aux.}(T(u)), \]  
the hamiltonian associated to the multi-state \( sl(n) \) chain is obtained when we evaluate in \( u = 0 \), the derivative with respect to \( u \) of the logarithm of \( F \). So, as in the \( sl(2) \) case, the hamiltonian is defined by
\[ H = \frac{2}{n} \sinh \gamma \frac{d}{du} \ln(F(u)) \bigg|_{u=0} - \frac{N}{n} \cosh \gamma, \]  
that can be expressed as
\[ H = \sum_{r=1}^{N-1} h_{r,r+1}, \]  
with
\[ h_{r,r+1} = \sum_{i,j=1 \atop i \neq j}^{n} e_{i,j}^{r} \otimes e_{j,i}^{r+1} + \frac{n-1}{n} \cosh (\gamma) \sum_{i=1}^{n} e_{i,i}^{r} \otimes e_{i,i}^{r+1} \]
\[ + \sum_{i,j=1 \atop i \neq j}^{n} \left( \frac{2(j-i)}{n} - \text{sign}(j-i) \right) \sinh (\gamma) - \frac{\cosh (\gamma)}{n} \right) e_{i,i}^{r} \otimes e_{j,j}^{r+1}. \]

If we specify for \( n = 2 \), we obtain the hamiltonian corresponding to the XXZ model
\[ H^{sl(2)} = \frac{1}{2} \sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh (\gamma) \sigma_n^z \sigma_{n+1}^z), \]  
where \( \sigma^x, \sigma^y \) and \( \sigma^z \) are the Pauli matrices. That model has a well known solution.

For \( n = 3 \) the hamiltonian obtained is
\[ H^{sl(3)} = \frac{1}{2} \sum_{n=1}^{N} \left( J_9 (\lambda_n^8 \lambda_{n+1}^3 + \lambda_n^3 \lambda_{n+1}^8) + \sum_{\alpha=1}^{8} (J_\alpha \lambda_n^\alpha \lambda_{n+1}^\alpha) \right), \]  
with
\[ J_1 = J_2 = J_4 = J_5 = J_6 = J_7 = 1, \quad J_3 = J_8 = \cosh (\gamma), \quad J_9 = \frac{\sinh (\gamma)}{\sqrt{3}}, \]  
where \( \lambda^\alpha (\alpha = 1, \cdots, 8) \) are the Gell-Mann matrices.
3. Solutions in the \( sl(2) \) and \( sl(3) \) cases

The method that we are going to use is known as nested algebraic Bethe ansatz. This method is a generalization to \( n \) components of the normal algebraic Bethe ansatz in two dimensions proposed by Fadeev and Takhtajan [13]. We will follow in our calculation the steps such as they are described in ref. [12].

We start with \( sl(2) \) case or XXZ model. For this model the matrices \( R \) and \( L \) are

\[
R(u, \gamma) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad L(u, \gamma) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad (3.1)
\]

with

\[
a = \sinh (u + \gamma), \quad (3.2a) \\
b = \sinh (u), \quad (3.2b) \\
c = \sinh(\gamma). \quad (3.2c)
\]

The solution to this model can be found in many places, in particular in ref. [12]. We look for solutions of the eigenvalue equation for the operator \( F(u) \) defined in (2.27). They are

\[
\Lambda(u, u_1, \cdots, u_r) = \sinh (u + \gamma)^N \prod_{j}^{r} \frac{\sinh (u_j - u + \gamma)}{\sinh (u_j - u)} + \sinh (u)^N \prod_{j}^{r} \frac{\sinh (u - u_j + \gamma)}{\sinh (u - u_j)}. \quad (3.3)
\]

\( r \) being the number of sites with spin down in a lattice with length \( N \). The parameters \( u_j, j = 1, \cdots, r \), are the solutions of Bethe equations

\[
\frac{\sinh (u_k + \gamma)}{\sinh (u_k)} = \prod_{l \neq k \atop l=1}^{r} \frac{\sinh (u_l - u_k - \gamma)}{\sinh (u_l - u_k + \gamma)}, \quad k = 1, \cdots, r. \quad (3.4)
\]

The eigenvalues of the hamiltonian are obtained with the substitution in (2.28) of \( F(u) \) by \( \Lambda(u) \).
In the $sl(3)$ case we have the $R$ and $L$ matrices

\[
R(u) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & b & 0 \\
0 & b & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & 0 & b & 0 \\
0 & 0 & b & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & 0 & b & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad L(u) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & d & 0 \\
0 & d & 0 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & c & 0 \\
0 & 0 & c & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & d & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]  

(3.5)

where we have taken into account that the YBE is homogeneous, then we have redefined

\[
b(u) = \frac{\sinh \left( \frac{3}{2}u \right)}{\sinh \left( \frac{3}{2}u + \gamma \right)}, \quad c(u) = \frac{\sinh \gamma}{\sinh \left( \frac{3}{2}u + \gamma \right)} e^{\frac{u}{2}}, \quad d(u) = \frac{\sinh \gamma}{\sinh \left( \frac{3}{2}u + \gamma \right)} e^{-\frac{u}{2}}.
\]

(3.6a, 3.6b, 3.6c)

The monodromy matrix $T$ in (2.26) is specified as

\[
T(u) = \begin{pmatrix}
A(u) & B_2(u) & B_3(u) \\
C_2(u) & D_{22}(u) & D_{23}(u) \\
C_3(u) & D_{32}(u) & D_{33}(u) \\
\end{pmatrix}
\]

(3.7)

that is a matrix in the auxiliary space whose elements are operators on the sites of the chain. In order to solve the model we must apply the Bethe ansatz twice; in every step we must introduce a set of parameters on which the eigenvectors and the eigenvalues depend.
In the first step, inserting (3.7) in the YBE we find, with the usual notation,

\[ B(u) \otimes B(v) = R^{(2)}(u - v) \cdot (B(v) \otimes B(u)) = (B(v) \otimes B(u)) \cdot R^{(2)}(u - v), \]  
(3.8a)

\[ A(u)B(v) = g(v - u)B(v)A(u) - B(u)A(v) \cdot r^{(2)}(v - u), \]  
(3.8b)

\[ D(u) \otimes B(v) = g(u - v)B(v) \otimes (D(u) \cdot R^{(2)}(u - v)) - B(u) \otimes (r^{(2)}(u - v) \cdot D(v)), \]  
(3.8c)

where

\[ R^{(2)}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r^{(2)}(u) = \begin{pmatrix} h_- & 0 \\ 0 & h_+ \end{pmatrix}, \quad \tilde{r}^{(2)}(u) = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}, \]  
(3.9)

and \( g(u) = 1/b(u), h_+(u) = c(u)/b(u) \) and \( h_-(u) = d(u)/b(u) \).

Now, with the help of the relations (3.8), we look for solutions of the equation

\[ F(u)\Psi(\mu_1, \cdots, \mu_r) = \Lambda(u, \mu_1, \cdots, \mu_r)\Psi(\mu_1, \cdots, \mu_r), \]  
(3.10)

of the form

\[ \Psi(\bar{\mu}) = \Psi(\mu_1, \cdots, \mu_r) = X_{i_1, \cdots, i_r} B_{i_1}(\mu_1) \otimes \cdots \otimes B_{i_r}(\mu_r) \| 1 >, \]  
(3.11)

being

\[ \| 1 > = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]  
(3.12)

This condition introduces the first set of parameters \( \{ \mu_j \}_{j=1}^r \).

To begin with, since \( \| 1 > \) is eigenvector of \( A(u) \) and \( D_{i,i} \) with eigenvalues 1 and \( b(u)^N \delta_{i,j} \) respectively, we apply these operators on \( \Psi \) and, by using the
commutation relations (3.8), we push the operators $A$ or $D_{i,j}$ through the $B$ to the right. When either $A$ or $D$ reaches $\parallel 1 >$ they reproduce this vector again. Since the commutation relations have two terms, this procedure generates a lot of terms. Some of them have the same order of the arguments in the $B$ product; we call them wanted terms. The others have some $B(\mu_j)$ replaced by $B(u)$ and we call them unwanted terms.

When we apply $F = A + D_{2,2} + D_{3,3}$ to $\Psi(\mu_1, \cdots, \mu_r)$, we collect the unwanted terms and require them to have a vanishing sum. This condition gives us a set of equations for the parameters. The sum of the wanted terms will be required to be proportional to $\Psi$, providing us with the second part of equation (3.10).

So, the application of $A(u)$ to $\Psi$, gives the wanted term

$$\prod_{j=1}^r g(\mu_j - u)B_{j_1}(\mu_1) \cdots B_{j_r}(\mu_r)X_{j_1,\cdots,j_r} \parallel 1 >, \quad (3.13)$$

and the $k - th$ unwanted term

$$\prod_{j=1}^r g(\mu_j - \mu_k) \left( B(u)\tilde{\nu}(2)(\mu_k - u) \right) \otimes B(\mu_{k+1}) \otimes \cdots \otimes B(\mu_r) \otimes B(\mu_1) \otimes B(\mu_{k-1}) M^{(k-1)} X \parallel 1 >, \quad (3.14)$$

$M^{(k-1)}$ being the tensor obtained from the commutation relations (3.8a), which produces a cyclic permutation

$$B(\mu_1) \otimes \cdots \otimes B(\mu_r) = B(\mu_{k+1}) \otimes \cdots \otimes B(\mu_r) \otimes B(\mu_1) \cdots \otimes B(\mu_{k-1}) M^{(k-1)}, \quad (3.15)$$

In the same form, the application of $D_{2,2} + D_{3,3}$ to $\Psi$ produces the wanted term

$$\left( \frac{1}{g(\mu_1)} \right)^N \prod_{j=1}^r g(u - \mu_j)B(\mu_1) \otimes \cdots \otimes B(\mu_r) F_r^{(2)}(u, \vec{\mu}) X \parallel 1 >, \quad (3.16)$$
and the $k$-th unwanted term

$$
\left( \frac{1}{g(\mu_k)} \right)^{N} \prod_{j=1}^{r} g(\mu_j - \mu_k) \left( B(u) r^{(2)}(\mu_k - u) \right) \otimes B(\mu_{k+1}) \otimes \cdots \\
\cdots \otimes B(\mu_r) \otimes B(\mu_1) \otimes \cdots \otimes B(\mu_{k-1}) F_r^{(2)}(\mu_k, \vec{\mu}) M^{(k-1)} X \| 1 >, \tag{3.17}
$$

the operator

$$
F_r^{(2)} = \sum_{j=2,3} T_r^{(2)}(u, \vec{\mu})_{j,i_1,\cdots,i_r}^{i_i,\cdots,i_r}, \tag{3.18}
$$

being the trace of a tensor acting on the index of the $B$’s with the values $i = 2, 3$ and defined by

$$
T_r^{(2)}(u, \vec{\mu})_{j,i_1,\cdots,i_r}^{i_i,\cdots,i_r} = R^{(2)}_{l_1,m_1}(u-\mu_1) R^{(2)}_{l_2,m_2}(u-\mu_2) \cdots R^{(2)}_{l_{r-1},m_r}(u-\mu_r), \tag{3.19}
$$

In order to the sum of the wanted terms solve (3.10), $X_{i_1,\cdots,i_r}$ must be eigenstate of $F_r^{(2)}$

$$
F_r^{(2)}(u, \vec{\mu}) X = \Lambda_r^{(2)}(u, \vec{\mu}) X, \tag{3.20}
$$

However, the cancelation of unwanted terms gives

$$
\Lambda_r^{(2)}(\mu_k, \vec{\mu}) = (g(\mu_k))^{N} \prod_{j=1}^{r} \frac{g(\mu_j - \mu_k)}{g(\mu_k - \mu_j)}, \tag{3.21}
$$

The second step then is to diagonalize the equation (3.20). We proceed analogously to the first step. The operator $T_r^{(2)}(u, \vec{\mu})_{j,i_1,\cdots,i_r}^{i_i,\cdots,i_r}$ is as a dimension two matrix in the index $i, j$ written as

$$
T_r^{(2)}(u, \vec{\mu}) = \begin{pmatrix}
A_r^{(2)}(u, \vec{\mu}) & B_r^{(2)}(u, \vec{\mu}) \\
C_r^{(2)}(u, \vec{\mu}) & D_r^{(2)}(u, \vec{\mu})
\end{pmatrix}. \tag{3.22}
$$

It satisfies a YBE with the matrix $R^{(2)}$ in (3.9),

$$
R^{(2)}(u - v) \cdot \left( T_r^{(2)}(u, \vec{\mu}) \otimes T_r^{(2)}(v, \vec{\mu}) \right) = \left( T_r^{(2)}(v, \vec{\mu}) \otimes T_r^{(2)}(u, \vec{\mu}) \right) \cdot R^{(2)}(u - v), \tag{3.23}
$$
that gives the relations

\[
A^{(2)}(u) \cdot B^{(2)}(v) = g(v-u)B^{(2)}(v) \cdot A^{(2)}(u) - h_+(v-u)B^{(2)}(u) \cdot A^{(2)}(v)
\]

\[
D^{(2)}(u) \cdot B^{(2)}(v) = g(u-v)B^{(2)}(v) \cdot D^{(2)}(u) - h_-(u-v)B^{(2)}(u) \cdot v^{(2)}
\]

\[[B^{(2)}(u), B^{(2)}(v)] = 0.\]  

(3.24c)

We take now

\[
\|1 >^{(2)} = \bigotimes_{i=1}^{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i
\]

(3.25)

that is a eigenvector of \(A^{(2)}\) and \(D^{(2)}\)

\[
A^{(2)}(u, \mu) \|1 >^{(2)} = \|1 >^{(2)},
\]

(3.26a)

\[
D^{(2)}(u, \mu) \|1 >^{(2)} = \left( \prod_{i=1}^{r} \frac{1}{g(u-\mu_i)} \right) \|1 >^{(2)},
\]

(3.26b)

\[
C^{(2)}(u, \mu) \|1 >^{(2)} = 0,
\]

(3.26c)

and, as in the first step, we look for eigenvectors of the form

\[
\Psi^{(2)} = B^{(2)}(\lambda_1, \mu) \cdots B^{(2)}(\lambda_s, \mu) \|1 >^{(2)},
\]

(3.27)

that introduce the dependence of the eigenvalues on a new set of parameters \(\{\lambda_i\}_{i=1}^{s}\).

We follow the same procedures as in the first step, but now in two dimensions, and we find that the wanted terms give the eigenvalues

\[
\Lambda^{(2)}(u, \mu, \bar{\lambda}) = \prod_{i=1}^{s} g(\lambda_i - u) + \prod_{i=1}^{s} g(u - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(u - \mu_j)},
\]

(3.28)

whereas the cancelation of the unwanted terms imposes the equations

\[
h_+(\lambda_k - u) \prod_{i=1}^{s} g(\lambda_i - \lambda_k) + h_-(u - \lambda_k) \prod_{i=1}^{s} g(\lambda_k - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\lambda_k - \mu_j)} = 0.
\]

(3.29)

As \(h_+(x) + h_-(x) = 0\), the last relations give the equations for the \(\{\lambda_i\}_{i=1}^{s}\)
parameters
\[
\prod_{j=1}^{r} g(\lambda_k - \mu_j) = \prod_{i=1, i \neq k}^{s} \frac{g(\lambda_k - \lambda_j)}{g(\lambda_j - \lambda_k)}, \quad k = 1, \ldots, s
\]  
(3.30)
and, due to \( \frac{1}{g(0)} = 0 \), the eigenvalues \( \Lambda(u, \vec{\mu}) \) in (3.10), for \( u = \mu_k \), are
\[
\Lambda(\mu_k, \vec{\mu}) = \prod_{i=1}^{s} g(\lambda_i - \mu_k).
\]  
(3.31)
Then our results are the eigenvalues of \( F(u) \), given by the equations
\[
\Lambda(u, \vec{\mu}, \vec{\lambda}) = \prod_{i=1}^{r} g(\mu_i - u) + \left( \frac{1}{g(u)} \right)^N \prod_{i=1}^{r} g(u - \mu_i) \Lambda(2)(u, \vec{\mu}, \vec{\lambda}), \quad (3.32a)
\]
\[
\Lambda(2)(u, \vec{\mu}, \vec{\lambda}) = \prod_{i=1}^{s} g(\lambda_i - u) + \prod_{i=1}^{s} g(u - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(u - \mu_j)}, \quad (3.32b)
\]
and the parameters \( \{\mu_i\}_{i=1}^{r} \) and \( \{\lambda_i\}_{i=1}^{s} \), solutions of the coupled equations
\[
\prod_{i=1}^{s} g(\lambda_i - \mu_k) = (g(\mu_k))^N \prod_{j=1, j \neq k}^{r} \frac{g(\mu_j - \mu_k)}{g(\mu_k - \mu_j)}, \quad (3.33a)
\]
\[
\prod_{j=1}^{r} g(\lambda_k - \mu_j) = \prod_{i=1, i \neq k}^{s} \frac{g(\lambda_k - \lambda_i)}{g(\lambda_i - \lambda_k)}, \quad (3.33b)
\]
Every set of solutions for \( 1 \leq s \leq r \leq N \) of these coupled equations determines an eigenvalue of \( F \).
We can now substitute the function \( g(x) \)
\[
g(x) = \frac{\sinh \left( \frac{3}{2} x + \gamma \right)}{\sinh \left( \frac{3}{2} x \right)},
\]  
(3.34)
and use it to calculate the energy spectrum of the hamiltonian with (2.28). Then,
as \( \frac{1}{g(0)} = 0 \), we have from (3.32a)

\[
\Lambda(0, \bar{\mu}, \bar{\lambda}) = \prod_{i=1}^{r} g(\mu_i),
\]

(3.35)

and

\[
\left. \frac{d}{du} \Lambda(u, \bar{\mu}, \bar{\lambda}) \right|_{u=0} = \sum_{j=1}^{r} \left. \frac{d}{du} [g(\mu_j - u)] \right|_{u=0} \prod_{i=1}^{r} g(\mu_i) = \sum_{j=1}^{r} f(\mu_j) \prod_{i=1}^{r} g(\mu_i),
\]

(3.36)

with

\[
f(x) = -\frac{3}{2} \frac{\sinh(\gamma)}{\sinh(\frac{3}{2}x)}.
\]

(3.37)

Then, the energy is

\[
E = \frac{3}{2} \sinh(\gamma) \sum_{j=1}^{r} \left[ \sinh \left( \frac{3}{2} \mu_j + \gamma \right) \sinh \left( \frac{3}{2} \mu_j \right) \right]^{-1}
\]

(3.38)

As we can see in the last expression, the energy only depends on the first set of parameters \( \{\mu_j\}_{j=1}^{r} \).

As conclusion, we have found a family of integrable models based in the \( U_q(\widehat{sl(n)}) \) quantum group. The \( n = 2 \) it is the same as the six vertex model that can be found in ref [12]. For \( n=3 \) we have a model described with a matrix whose elements are the functions \( b, c \) and \( d \) given in (3.2) and depending on \( \gamma \) and the spectral parameter \( u \). It is different of the models studied in ref. [12] since its hamiltonian can be proved to be different.

The generalization to \( n > 3 \) follows easily. The number of functions on which depend the matrix elements of the model is increased. Analogously to the \( c \) and \( d \) functions, now we will have other functions with different exponentials in \( \gamma \). The
function $b$, as can be see from (2.25), will be

$$b(u, \gamma) = \frac{\sinh \left( \frac{n}{2} u \right)}{\sinh \left( \frac{n}{2} u + \gamma \right)},$$

(3.39)

and the energy spectrum, for any value of $n$, depends only of the parameters $\{\mu_j\}_{j=1}^r$ introduced in the first step. It turns out

$$E = \frac{n}{2} \sinh (\gamma) \sum_{j=1}^r \left[ \sinh \left( \frac{n}{2} \mu_j + \gamma \right) \sinh \left( \frac{n}{2} \mu_j \right) \right]^{-1},$$

(3.40)

obtained from eq. (3.18) by the substitution of $\frac{3}{2}$ by $\frac{n}{2}$. Eq (3.40) is applicable also in the case $n = 2.$

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