Cobordism classes of maps and covers for spheres

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Abstract

In this paper we show that for \( m > n \) the set of cobordism classes of maps from \( m \)-sphere to \( n \)-sphere is trivial. The determination of the cobordism homotopy groups of spheres admits applications to the covers for spheres.

Keywords: cobordism, homotopy group, covers

1 Introduction

Let \( M_1 \) and \( M_2 \) be compact oriented manifolds of dimension \( m \). Two continuous maps \( f_1 : M_1 \to X \) and \( f_2 : M_2 \to X \) are called cobordant if there are a compact oriented manifold \( W \) with \( \partial W = M_1 \sqcup M_2 \) and a continuous map \( F : W \to X \) such that \( F|_{M_i} = f_i \) for \( i = 1, 2 \).

Note that the set of cobordism classes \( f : S^m \to X \) form a group \( \pi^C_m(X) \) that is a quotient of \( \pi_m(X) \).

In Section 2 we consider assumptions for \( X \) such that \( \pi^C_m(X) = 0 \) (Theorem 2.1). In particular, Corollary 2.2 states that \( \pi^C_n(S^n) = \pi_n(S^n) = \mathbb{Z} \) and if \( m > n \) then

\[
\pi^C_m(S^n) = 0.
\]

In Section 3 we show that for manifolds the homotopy and cobordism classes of covers are equivalent to the homotopy and cobordism classes of their associated maps. Then we can apply results of Sections 2 for covers, in particular, see Corollary 3.6.

2 Cobordism classes of maps for spheres

Consider a group of oriented cobordism classes of maps \( \Omega_{*}^{SO}(X) \) [3, Chapter 1]. Let \( M_i, i = 1, 2 \), be compact oriented manifolds without boundary of dimension \( m \). Let \( f_i : M_i \to X, i = 1, 2 \), be continuous maps to a space \( X \). Then \( [f_1]_C = [f_2]_C \) in \( \Omega_{m}^{SO}(X) \), i.e. maps \( f_i \) are

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**cobordant** if there are a compact oriented manifold \(W\) with \(\partial W = M_1 \sqcup M_2\) and a continuous map \(F: W \to X\) such that \(F|_{M_i} = f_i\) for \(i = 1, 2\).

If \(M_2 = \emptyset\), then \([f_1]_C = 0\). In this case \(f_1\) is called **null-cobordant**.

Let \(M\) be a compact oriented manifold without boundary. We denote the set of cobordism classes of \(f: M \to X\) by \([M, X]_C\).

**Theorem 2.1.** Let \(X\) be a finite CW-complex whose integral homology \(H_*(X, \mathbb{Z})\) has only 2-torsion. Let \(f: S^m \to X\) be a map that induces zero homomorphism of \(m\)-dimensional cohomology with coefficients in \(\mathbb{Z}\) and \(\mathbb{Z}_2\). Then \(f\) is null-cobordant. in \(\Omega_{SO}^m(X)\) the image of \(f\) is 0. In particular, \([S^m, X]_C = 0\) if \(\dim X < m\).

**Proof.** By [3, Theorem 17.6], the cobordism class of \(f: S^m \to X\) is determined by the Pontrjagin numbers and the Stiefel-Whitney numbers of the map \(f\). From the definition, the Pontrjagin numbers and the Stiefel-Whitney numbers of the map \(f\) are determined by its induced homomorphisms on cohomology with coefficients in \(\mathbb{Z}\) and \(\mathbb{Z}_2\), respectively. The hypothesis in the statement guarantees that \(f\) and the constant map induce the same homomorphism on cohomology with coefficients in \(\mathbb{Z}\) and \(\mathbb{Z}_2\), and hence the result. 

Let \(M\) be an \(m\)-dimensional sphere \(S^m\). In this case denote \([M, S^n]_C\) by \(\pi^C_m(S^n)\). It is easy to prove that the cobordism classes \(\pi^C_m(S^n)\) form a group. Moreover, there is a subgroup \(N\) in \(\pi_m(S^n)\) such that

\[
\pi^C_m(S^n) = \pi_m(S^n)/N.
\]

**Corollary 2.2.** If \(m \neq n\), then \(\pi^C_m(S^n) = 0\), otherwise \(\pi^C_n(S^m) = \mathbb{Z}\).

**Proof.** We obviously have the case \(m < n\). Theorem 2.1 yields the most complicated case.

Let \(m = n\). The Hopf degree theorem (see [6, Sect. 7]) states that two continuous maps \(f_1, f_2: S^n \to S^n\) are homotopic, i.e. \([f_1] = [f_2]\) in \(\pi_n(S^n)\), if and only if \(\deg f_1 = \deg f_2\). It is clear that \([f_1] = [f_2]\) implies \([f_1]_C = [f_2]_C\). Now we show that from \([f_1]_C = [f_2]_C\) follows \(\deg f_1 = \deg f_2\). Indeed, then we have \(F: W \to S^n\) with \(F|_{M_i} = f_i\). Note that \(Z := F^{-1}(x)\) for a regular \(x \in S^n\), is a manifold of dimension one. It is easy to see that a cobordism \((Z, Z_1, Z_2)\), where \(Z_i := f_i^{-1}(x)\), implies \(\deg f_1 = \deg f_2\). Thus, \(\pi^C_n(S^n) = \pi_n(S^n) = \mathbb{Z}\).

Corollary 2.2 states that \(f: S^m \to S^n\) is null-cobordant for \(m > n\). Therefore, we have the following result.

**Corollary 2.3.** Let \(m > n\). Then for any continuous map \(f: S^m \to S^n\) there are a compact oriented manifold \(W\) with \(\partial W = S^m\) and a continuous map \(F: W \to S^n\) such that \(F\) on the boundary coincides with \(f\).

**Remark.** In the earlier version of this paper, we had a proof that \(\pi^C_m(S^n) = 0\), where \(m > n\) only for particular cases. We formulated this statement as a conjecture and sent the preprint to several topologists. Soon, Diarmuid Crowley sent us a sketch of the proof of this conjecture. Later, Alexey Volovikov pointed out to us that Theorem 2.1 follows easily from [3, Theorem 17.6].
3 Homotopy and cobordism classes of covers

For open (or closed) covers $\mathcal{U}$ of a normal space $T$ we considered certain homotopy classes $[f_\mathcal{U}]$ in $[T, S^n]$ defined in [8]. In this section we define a homotopy equivalence for covers and prove that two covers $\mathcal{U}_1$ and $\mathcal{U}_2$ are homotopy equivalent if and only if $[f_\mathcal{U}_1] = [f_\mathcal{U}_2]$ in $[T, S^n]$ (Theorem 3.2). We also prove that two covers on manifolds of the same dimension are cobordant if and only if the corresponding cobordism classes $[f_\mathcal{U}]_C$ in $\Omega_*(S^n)$ are equal (Theorem 3.4).

The homotopy invariants $[f_\mathcal{U}]$ can be considered as obstructions for extending covers of a subspace $A \subset X$ to a cover of all of $X$. (Note that the classical obstruction theory (see [4, 10]) considers homotopy invariants that equal zero if a map can be extended from the $k$–skeleton of $X$ to the $(k+1)$–skeleton and are non-zero otherwise.) In our papers [8, 9] using these obstructions we obtain generalizations of the classic KKM (Knaster–Kuratowski–Mazurkiewicz) and Sperner lemmas [5, 11].

Let $X$ be any compact oriented manifold of dimension $(m+1)$ and $A = \partial X$ be its boundary. Let $\mathcal{U} = \{U_0, \ldots, U_{n+1}\}$ be a cover of $A$ such that the intersection of all subsets $U_i$ is empty. Then $[\mathcal{U}] \in [A, S^n]$, where the homotopy class $[\mathcal{U}]$ is defined in [8]. In the case $m = n$ we have $[A, S^n] = \mathbb{Z}$ and, if $[\mathcal{U}] \neq 0$, then for any extension of this cover to a cover $\mathcal{V} = \{V_0, \ldots, V_{n+1}\}$ of $X$ the intersection

$$\bigcap_{i=0}^{n+1} V_i \neq \emptyset \quad (4.1)$$

This fact is a generalization of the Sperner–KKM lemma [8 Theorem 2.6].

Another generalization of the KKM lemma is the following (see [8 Corollary 3.1]): Let $X$ is an $(m+1)$–disc and $A = S^m$. If $[\mathcal{U}] \neq 0$ in $\pi_m(S^n)$, then we have property (4.1).

However, for $m > n$ not all pairs $(X, A)$ satisfy property (4.1). For instance, $X = \mathbb{C}P^2 \setminus \text{Int}(D^4)$ and $A := \partial X = S^3$. Then the Hopf map $f : S^3 \to S^2$ can be extended to a continuous map $F : X \to S^2$. It implies that a corresonding cover $\mathcal{U} = \{U_0, U_1, U_2, U_3\}$ can be extened to $X$ such that the intersection of all $U_i$ is empty.

Let $\mathcal{U} = \{U_0, \ldots, U_{n+1}\}$ be a collection of open sets whose union contains a normal space $T$. In other words, $\mathcal{U}$ is a cover of $T$. Let $\Phi = \{\varphi_0, \ldots, \varphi_{n+1}\}$ be a partition of unity subordinate to $\mathcal{U}$. Let

$$f_{\mathcal{U}, \Phi}(x) := \sum_{i=0}^{n+1} \varphi_i(x)v_i,$$

where $v_0, \ldots, v_{n+1}$ are vertices of an $(n+1)$–simplex $\Delta^{n+1}$ in $\mathbb{R}^{n+1}$.

Suppose the intersection of all $U_i$ is empty. Then $f_{\mathcal{U}, \Phi}$ is a continuous map from $T$ to $S^n$. In [8] Lemmas 2.1 and 2.2 we proved that a homotopy class $[f_{\mathcal{U}, \Phi}]$ in $[T, S^n]$ does not depend on $\Phi$. We denote it by $[f_\mathcal{U}]$.

In fact, see [8 Lemma 2.4], the homotopy classes $[f_\mathcal{U}]$ of covers are also well defined for closed sets. We call a family of sets $\mathcal{S} = \{S_0, \ldots, S_{n+1}\}$ a cover of a space $T$ if $\mathcal{S}$ is either an open or closed cover of $T$. 

Homotopy invariants of covers we defined through homotopy invariants of maps. Let us define them directly for covers.

**Definition 3.1.** Let $S_i = \{S_{i0}^i, \ldots, S_{in}^i\}$, $i = 1, 2$, be covers of a normal space $T$ such that for $i = 1, 2$ the intersection of all subsets in $S_i$ is empty. We say that $S_1$ is homotopic to $S_2$ and write $[S_1] = [S_2]$ if $T \times [0, 1]$ can be covered by $Q = \{Q_0, \ldots, Q_{n+1}\}$ such that $Q$ is an extension of $S_1 \cup S_2$ of $T \times \{0, 1\}$ and the intersection of all $Q_k$ is empty.

The following theorem extends Theorem 2.2 in [8].

**Theorem 3.2.** Let $S_i = \{S_{i0}^i, \ldots, S_{in}^i\}$, $i = 1, 2$, be covers of a normal space $T$. Suppose the intersection of all the $S_{ij}^i$ in $S_i$ is empty. Then $[S_1] = [S_2]$ if and only if $[f_{S_1}] = [f_{S_2}]$ in $[T, S^n]$.

**Proof.** From [8] Lemma 1.11 it suffices to prove the theorem for open covers. It is clear that if $[S_1] = [S_2]$ then $[f_{S_1}] = [f_{S_2}]$. Now we prove the converse statement.

Suppose $[f_{S_1}] = [f_{S_2}]$. Let $\Phi_i, i = 1, 2$, be any partitions of unity subordinate to $S_i$. Then there is a homotopy $F_\Phi : T \times [0, 1] \rightarrow S^n$ between $f_{S_1, \Phi_1}$ and $f_{S_2, \Phi_2}$, where $\Phi := (\Phi_1, \Phi_2)$.

Consider $S^n$ as the boundary of $\Delta^{n+1}$. Let $B_i$ be the open star of a vertex $v_i$ of $\Delta^{n+1}$. Let

$$U_\ell(\Phi) := F_\Phi^{-1}(B_\ell), \quad U(\Phi) := \{U_0(\Phi), \ldots, U_{n+1}(\Phi)\}.$$ Then $U(\Phi)$ is a cover of $T \times \{0, 1\}$.

Denote by $\Pi$ the set of all pairs $\Phi := (\Phi_1, \Phi_2)$, where $\Phi_i$ is a partition of unity subordinate to $S_i$. Let

$$Q_\ell := \bigcup_{\Phi \in \Pi} U_\ell(\Phi), \quad Q := \{Q_0, \ldots, Q_{n+1}\}.$$ Then $Q$ is a cover of $T \times [0, 1]$ and

$$Q|_{T \times \{0\}} = S_1, \quad Q|_{T \times \{1\}} = S_2.$$ This yields $[S_1] = [S_2]$. □

**Definition 3.3.** Let $M_i, i = 1, 2$, be compact oriented manifolds without boundary with $\dim M_1 = \dim M_2$. Let $S_i = \{S_{i0}^i, \ldots, S_{in}^i\}$, $i = 1, 2$, be covers of $M_i$ such that for $i = 1, 2$ the intersection of all subsets in $S_i$ is empty. We say that $S_1$ is cobordant to $S_2$ and write $[S_1]_C = [S_2]_C$ if there are a compact oriented manifold $W$ with $\partial W = M_1 \sqcup M_2$ and its cover $Q = \{Q_1, \ldots, Q_n\}$ such that $Q|_{M_i} = S_i$, $i = 1, 2$, and the intersection of all $Q_k$ is empty. If $M_2 = \emptyset$, then we say that $S_1$ is null–cobordant and write $[S_1]_C = 0$.

Note that if $[S_1]_C = [S_2]_C$, then $[f_{S_1}]_C = [f_{S_2}]_C$ in $\Omega_S^E(S^n)$, where for a continuous $f : M \rightarrow S^n$ by $[f]_C$ we denote the correspondent cobordism class.

**Theorem 3.4.** Let $M_1$ and $M_2$ be compact oriented homotopy equivalent manifolds without boundary. Let $S_i, i = 1, 2$, be covers of $M_i$ such that the intersection of all covers in $S_i$ is empty. Then $[S_1]_C = [S_2]_C$ if and only if $[f_{S_1}]_C = [f_{S_2}]_C$. 

4
Proof. By definition if \([f_{S_i, \Phi_i}]_C = [f_{S_2, \Phi_2}]_C\), then there is a map \(F_\Phi : W \to S^n\) such that \(F|_M = f_{S_i, \Phi_i}\). Actually, the theorem can be proved by the same arguments as Theorem 3.2 if we substitute \(T \times [0, 1]\) by a cobordism \(W\).

From this theorem it is easy to prove the following corollary.

**Corollary 3.5.** Let \(S\) be a cover of a compact oriented manifold \(M\) such that the intersection of all subsets in \(S\) is empty. Suppose \([S]_C = 0\). Then there is a compact oriented manifold \(W\) with \(\partial W = M\) such that \(S\) can be extended to a cover \(Q\) of \(W\) (i.e. \(Q|_M = S\)) with the empty intersection of all subsets \(Q_k\).

Theorem 3.4 and Corollary 2.2 yield

**Corollary 3.6.** Let \(m > n\). Then for any cover \(U = \{U_0, \ldots, U_{n+1}\}\) of \(S^m\) with the empty intersection of all subsets in \(U\) there are a compact oriented manifold \(W\) with \(\partial W = S^m\) and a cover \(Q\) of \(W\) such that \(Q\) is an extension of \(U\) with the empty intersection of all subsets in \(Q\).

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