Null Lagrangians in Cosserat elasticity

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Abstract

In the framework of nonlinear theory of Cosserat elasticity, also called micropolar elasticity, we provide the complete characterization of null Lagrangians for three dimensional bodies as well as for shells. Using the Gibb’s rotation vector for description of the microrotation, this task is possible by an application of a theorem stated by Olver and Sivaloganathan in ‘the structure of null Lagrangians’ (Nonlinearity, 1, 1988, pp. 389-398). A set of necessary and sufficient conditions is also provided for the elasticity tensors to correspond to a null Lagrangian in linearized micropolar theory.

Introduction

According to Theorem 7 of Olver and Sivaloganathan in [21], for a star shaped $\Omega \subset \mathbb{R}^m$, with

$$x \in \Omega, \quad u : \overline{\Omega} \to \mathbb{R}^n, \quad F : \Omega \to M^{n \times m},$$

a function $L(x, u, F)$ (with $F = \nabla u = [\partial u_i/\partial x_j] \in M^{n \times m}$) is a null Lagrangian if and only if there exist an $m$-tuple of $C^1$ functions

$$P : \overline{\Omega} \times \mathbb{R}^n \times M^{n \times m} \to \mathbb{R}^m$$

such that

$$\mathcal{L}(x, u, \nabla u) = \nabla \cdot P(x, u, \nabla u) \quad \forall u \in C^1(\Omega),$$

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where the arbitrary scalar potential functions on $\Omega \times \mathbb{R}^n$ participating in the divergence above via the presence of $P$ can be also specified while their total number is given by the binomial coefficient $\binom{n+m}{m-1}$.

In the familiar case of three dimensional theory of elasticity, the number of arbitrary scalar potentials is known to be $\binom{3+3}{3-1}$ (with $m = 3, n = 3$), i.e., 15. For our purpose in this note, as another example, in the case of three dimensional Cosserat (micropolar) theory [1], the number of arbitrary scalar potentials in the sum appearing in Theorem 8 of Olver and Sivaloganathan [21] is anticipated to be $\binom{3+3+3}{3-1}$ (with $m = 3, n = 3 + 3 = 6$), i.e., 36, whereas for two dimensional shell theory (embedded in three dimensional space), this number is $\binom{2+3+3}{2-1}$, i.e., 8 (with $m = 2, n = 3 + 3 = 6$).

For the benefit of some readers, we recall that a null Lagrangian (see [16], [13], [21], [17]; [5], [9], [19], [4], [6]) is defined by the condition that its Euler–Lagrange equation is trivially satisfied; in other words, the so called functional $L$ given by the expression

$$ L[u] = \int_{\Omega} \mathcal{L}(x, u(x), \nabla u(x))dx $$

satisfies

$$ L[u + \varphi] = L[u] \quad \forall \varphi \in C_0^\infty(\Omega). $$

In the nonlinear theory of elasticity, the null Lagrangians have been found to have special importance in the questions of existence of solutions [11, 16, 34, 22] as well as in the surface potentials and handling certain boundary data [29, 15, 19]. The connection with the construction of polyconvex functions has a practical value as it is helpful in developing the rigorous framework for some very useful elastic models [23, 30] and also in the presence of various additional physical effects [40, 41]. Besides this the role of null Lagrangians in Noether symmetries is also well known [2, 3, 25]. Last but not the least, in the classical framework of calculus of variations [31], the null Lagrangians occupy a distinguished role in the field theory as any researcher can easily find out during an expedition on the ‘royal road of Caratheodory’.

In this short note, we apply the above mentioned Theorem of [21] to a Cosserat [1], or so called, micropolar elastic body [7, 8, 27, 20].

### 1 Nonlinear Cosserat media

We consider a body of Cosserat type with the reference configuration assumed to be a bounded domain denoted by $\Omega \subset \mathbb{R}^3$ (with a Lipschitz boundary $\partial \Omega$). However, it suffices to consider any smooth portion of the body as we are interested only in the characterization of the null Lagrangians on the lines of that in the nonlinear
theory of elasticity [24]. Following the standard notation for vectors and tensors in continuum mechanics [14], we denote the microdeformation (vector field), or placement, of a micropolar body by

$$\chi : \Omega \rightarrow \mathbb{R}^3$$

and the microrotation (describing the rotation of each particle in the micropolar body) with

$$R : \Omega \rightarrow SO(3).$$

An schematic is provided in Fig. 1 which illustrates the manner in which the microrotation field captures the rotation of an orthonormal triad of directors from reference configuration $\Omega$ to the current configuration $\chi(\Omega)$.

Here, we denote the standard basis vectors for $\mathbb{R}^3$ by the triplet $e_1, e_2, e_3$. The physical space $\mathbb{R}^3$ is assumed to be equipped with the cross product $\times$ corresponding to an orientation such that $e_1 \times e_2 \cdot e_3 = +1$. In (1.2), we use the symbol $SO(3)$ to denote the set of all rotation tensors in three dimensions, i.e., for $Q \in SO(3), Q^\top Q = \mathbb{I}, \det Q = +1$, where $\mathbb{I}$ stands for the identity tensor and $^\top$ denotes the transpose. For a skew tensor $W$ (i.e., $W^\top = -W$) the axial vector $w = axlW$, is defined by $Wa = w \times a, \forall a \in \mathbb{R}^3$. The relation $W = -\epsilon w$ provides the skew tensor corresponding to a given vector, where $\epsilon$ is the three dimensional alternating tensor which plays the role of a linear map from vectors $\mathbb{R}^3$ to tensors $\mathbb{M}^{3 \times 3}$ here; in components, the skew tensor $W$ corresponding to a vector $w$ is given by
\[ W_{ij} = -\epsilon_{ijk} w_k \] with \( \epsilon_{ijk} = e_i \cdot e_j \times e_k \). In other words, \( \text{skew}(w) := -\epsilon w \). We also employ a very convenient notation \([24]\) for a related entity, also called vector invariant (or Gibbsian Cross), \( A^\times \) with the components

\[(A^\times)_i := \epsilon_{ijk} A_{jk} \quad (1.3)\]

for any second order tensor \( A \). Thus, \( \text{axl} W = -\frac{1}{2} W^\times \) for skew tensor \( W \). The differentiation of a function \( f \) (which depends on position vector \( x \)) with respect to the \( x_j \) coordinate is written as \( f,j \). Note that \( Q^\top Q,j \) is a skew tensor for a rotation tensor field \( Q \) on \( \Omega \).

With above notation in place, the deformation gradient corresponding to (1.1) is expressed as

\[ F := \nabla \chi = \chi,j \otimes e_i, \quad \forall x \in \Omega, \quad (1.4) \]

while the nonsymmetric right stretch tensor is defined as

\[ U := R^\top F. \quad (1.5) \]

We define the relative Lagrangian \textit{stretch tensor} as strain measure by \([33]\)

\[ E := U - I. \quad (1.6) \]

In micropolar media, an additional dependent field is the axial vector of \( R^\top R,j \). The second order tensor

\[ K := \text{axl}(R^\top R,j) \otimes e_j, \quad (1.7) \]

is a Lagrangian measure for curvature \([33]\), called the \textit{wryness tensor}. In the nonlinear theory of Cosserat, i.e., the micropolar elasticity, the strain energy density function (in terms of the tensors of stretch \( E \) and wryness \( K \)) is

\[ W(E, K). \quad (1.8) \]

In order to proceed further for the characterization of the null Lagrangians, it is useful to employ the local coordinates for the microrotation \( R \) \((1.2)\). We utilize the Gibb’s rotation vector (or coordinates) to express the rotation \( R \),

\[ R = R(\theta) := \frac{1}{4 + \theta^2}((4 - \theta^2)I + 2\theta \otimes \theta - 4 \text{skew}(\theta)), \quad \theta^2 = \theta \cdot \theta. \quad (1.9) \]

It is easy to show that

\[ \text{axial}(R^\top R,i) = -\frac{1}{4 + \theta^2}(4\theta,i + 2\theta \times \theta,i), \]

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so that
\[ K = -\frac{1}{4 + \theta^2}(4\nabla\theta + 2\theta \times \theta, \otimes e_i). \]

In the context of the energy functionals (based on (1.8), for example) for a micropolar elastic body, we consider the null Lagrangian for the corresponding class of functionals
\[ L[\chi, \theta] = \int_\Omega \mathcal{L}(x, \chi, \theta, \nabla \chi, \nabla \theta) dx. \quad (1.10) \]

**Remark 1.** It is possible to combine \( \chi, \theta \) together as a single vector field \( u \) in \( \mathbb{R}^6 \) however we refrain from doing this in the first and second section. We insist on retaining the original fields so that the analysis yields a decomposition of the terms which can be utilized directly by the reader in various applications of interest.

## 2 Null Lagrangian in three dimensional Cosserat theory

Due to the presence of three different vector entities namely, \( x, \chi, \) and \( \theta, \) it is convenient to employ a more delicate indicial notation. Henceforth, let the local coordinates be denoted by \( x_A \) for \( x \) and \( y_i \) for \( \chi. \) As explained above, the local coordinates for \( \theta, \) essentially for \( \mathbb{R}, \) are \( \theta_\alpha. \) In the assumed framework for three dimensional Cosserat body, we have the following identification of the local coordinates with components
\[ x = x_A e_A, \quad y = \chi(x) = y_i e_i, \quad \theta = \theta_\alpha e_\alpha, \]
where except for the indices the orthonormal triad of vectors \( \{e_1, e_2, e_3\} \) can be chosen to be the same.

**Remark 2.** In indicial notation, according to (1.9),
\[ R_{\alpha\beta} = \frac{1}{4 + \theta_\eta \theta_\eta}((4 - \theta_\eta \theta_\eta)\delta_{\alpha\beta} + 2\theta_\alpha \otimes \theta_\beta + 4\epsilon_{\alpha\beta\gamma}\theta_\gamma), \]
while the inverse relation can be easily found to be
\[ \theta_\alpha = \frac{2}{1 + R_{\eta\eta}}\epsilon_{\alpha\beta\gamma}R_{\beta\gamma}, \]
provided \( R_{\eta\eta} \neq -1. \) With \( \epsilon \) in the role of a linear map from second order tensors to vectors, this can be also expressed as \( \theta = (2/(1 + \text{tr} R))\epsilon R. \)
The following is based on the result of [21, 26] for null Lagrangians (also termed as variationally trivial Lagrangians). Let

\[ \omega = \mathcal{A} dx_1 \wedge dx_2 \wedge dx_3 + \nabla dy_1 \wedge dy_2 \wedge dy_3 \]

\[ + \frac{1}{2} e_{ABC} B_{Ai} dy_i \wedge dx_B \wedge dx_C + \frac{1}{2} e_{ijk} C_{iA} dy_j \wedge dy_k \wedge dx_A \]

\[ + \frac{1}{2} e_{ABC} \tilde{B}_{Aa} d\theta_a \wedge dx_B \wedge dx_C + \frac{1}{2} e_{i\beta\gamma} \tilde{C}_{iA} d\theta_\beta \wedge d\theta_\gamma \wedge dx_A + \nabla d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \]

\[ + \frac{1}{2} e_{a\beta\gamma} \tilde{\beta}_{ai} dy_i \wedge d\theta_\beta \wedge d\theta_\gamma + \frac{1}{2} e_{ijk} \tilde{\epsilon}_{ia} dy_j \wedge dy_k \wedge d\theta_a + J_{a\beta\gamma} d\theta_a \wedge dy_\beta \wedge dx_\gamma, \]

which involves a total 84 arbitrary functions (as expected this number equals \((3^3 + 3)\)) of \(x, \chi, \) and \(\theta\). With details provided in Appendix A, the null Lagrangians are described by the general expression:

\[ \mathcal{A} + \mathbf{B}^\top \cdot \mathbf{F} + C \cdot \text{cof} \, \mathbf{F} + \nabla \det \mathbf{F} + \tilde{\mathbf{B}}^\top \cdot \mathbf{G} + \tilde{C} \cdot \text{cof} \, \mathbf{G} + \tilde{\nabla} \det \mathbf{G} \]

\[ + \tilde{\mathbf{B}} \cdot (\text{cof} \, \mathbf{G}) \mathbf{F}^\top + \tilde{\mathbf{C}} \cdot (\text{cof} \, \mathbf{G}) \mathbf{G}^\top + J \cdot e_a \otimes e_j \otimes (\mathbf{G}^\top e_a \otimes \mathbf{F}^\top e_j), \]

where

\[ \mathbf{F} = F_{iA} e_i \otimes e_A, \quad \mathbf{G} = G_{\alpha A} e_\alpha \otimes e_A, \]

\[ \mathbf{B} = B_{iA} e_i \otimes e_A, \quad \tilde{\mathbf{B}} = \tilde{B}_{\alpha A} e_\alpha \otimes e_A, \quad \tilde{\mathbf{B}} = \tilde{B}_{ia} e_i \otimes e_\alpha, \]

\[ \mathbf{C} = C_{iA} e_i \otimes e_A, \quad \tilde{\mathbf{C}} = \tilde{C}_{\alpha A} e_\alpha \otimes e_A, \quad \tilde{\mathbf{C}} = \tilde{C}_{ia} e_i \otimes e_\alpha, \]

\[ J = J_{a\beta\gamma} e_a \otimes e_j \otimes e_c. \]

We seek to obtain necessary and sufficient conditions on the coefficients in (2.2) so that it prescribes any arbitrary null Lagrangian (given the hypothesis on \(\Omega\) for the applicability of Poincaré Lemma [21, 26]). Indeed, the general form of the null Lagrangian of the form (1.10) is obtained by the exterior derivative of the 2-form

\[ \zeta = \frac{1}{2} e_{ABC} L_A dx_B \wedge dx_C + K_{iA} dy_i \wedge dx_A + \frac{1}{2} e_{ijk} M_i dy_j \wedge dy_k \]

\[ + \tilde{K}_{aA} d\theta_a \wedge dx_A + \frac{1}{2} e_{a\beta\gamma} \tilde{M}_a d\theta_\beta \wedge d\theta_\gamma + H_{aj} d\theta_a \wedge dy_j, \]

where the coefficients are functions of \(x_A, y_i, \theta_\alpha\), which form a total number of 36 functions of \(x_A, y_i, \theta_\alpha\). With details provided in Appendix B, we find that the characterizing condition \(\omega = d\zeta\) (and the Poincaré Lemma [21]) implies

\[ \mathcal{A} = L_{A,A}, \quad \nabla = M_{i,i}, \quad \tilde{\nabla} = \tilde{M}_{a,a}, \quad J_{aj\beta} = (K_{jC,a} - \tilde{K}_{aC,j} + H_{aj,C}), \]

\[ \frac{1}{2} e_{ABC} B_{Ai} = \left( \frac{1}{2} e_{ABC} L_{A,i} - K_{iC,B} \right), \quad \frac{1}{2} e_{ijk} C_{iA} = (K_{kA,j} + \frac{1}{2} e_{ijk} M_{i,A}), \]

\[ \frac{1}{2} e_{ABC} \tilde{B}_{Aa} = \left( \frac{1}{2} e_{ABC} L_{A,a} - \tilde{K}_{aC,B} \right), \quad \frac{1}{2} e_{i\beta\gamma} \tilde{C}_{iA} = (\tilde{K}_{jA,\beta} + \frac{1}{2} e_{i\beta\gamma} \tilde{M}_{a,A}), \]

\[ \frac{1}{2} e_{a\beta\gamma} \tilde{\beta}_{ai} = \left( \frac{1}{2} e_{a\beta\gamma} \tilde{M}_{a,i} + H_{i,a}, \beta \right), \quad \frac{1}{2} e_{ijk} \tilde{\epsilon}_{ia} = \left( 2 e_{ijk} M_{i,a} + H_{aj,k} \right). \]
Using the properties of the alternative tensor, moreover, starting from the second line above, the relations can be simplified as

\[ B_{Ai} = \epsilon_{ABC} \left( \frac{1}{2} \epsilon_{PBC} L_{Pi} - K_{iC,B} \right) = L_{A,i} - \epsilon_{ABC} K_{iC,B}, \]  
(2.5a)

\[ C_{iA} = \epsilon_{ijk} (K_{kA,j} + \frac{1}{2} \epsilon_{pjk} M_{p,i}) = \epsilon_{ijk} K_{kA,j} + M_{i,A}, \]  
(2.5b)

\[ \tilde{B}_{A\alpha} = \epsilon_{ABC} \left( \frac{1}{2} \epsilon_{PBC} L_{P,\alpha} - \tilde{K}_{\alpha C,B} \right) = L_{A,\alpha} - \epsilon_{ABC} \tilde{K}_{\alpha C,B}, \]  
(2.5c)

\[ \tilde{C}_{iA} = \epsilon_{i\beta\gamma} (\tilde{K}_{\gamma A,\beta} + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{M}_{\alpha,A}) = \epsilon_{i\beta\gamma} \tilde{K}_{\gamma A,\beta} + \tilde{M}_{i,A}, \]  
(2.5d)

\[ \hat{B}_{\alpha i} = \epsilon_{\alpha\beta\gamma} \left( \frac{1}{2} \epsilon_{\delta\beta\gamma} \tilde{M}_{\delta,i} + H_{\gamma i,\beta} \right) = \tilde{M}_{A,i} + \epsilon_{\alpha\beta\gamma} H_{\gamma i,\beta}, \]  
(2.5e)

\[ \hat{C}_{i\alpha} = M_{i,\alpha} + \epsilon_{ijk} H_{\alpha j,k}. \]  
(2.5f)

**Remark 3 (Notation).** Let \( \nabla_x, \text{Div}_x, \text{Curl}_x \) denote the gradient, divergence, and rotation with respect to \( x \) keeping \( y, \theta \) fixed. Similarly, we suppose that \( \nabla_y, \text{Div}_y, \text{Curl}_y \) are the gradient, divergence, and rotation with respect to \( y \) keeping \( x, \theta \) fixed and \( \nabla_\theta, \text{Div}_\theta, \text{Curl}_\theta \) are the gradient, divergence, and rotation with respect to \( \theta \) keeping \( x, y \) fixed. In the case of indicial notation the same, we adopt the notation such that the comma followed by a subscript \( A \) (resp. \( i \) and \( \alpha \)) denotes the derivative with respect to \( x_A \) (resp. \( y_i \) and \( \theta_\alpha \)). Thus the useful definitions of curl are given by

\[ (\text{Curl}_x C)_{iA} = \epsilon_{ABC} C_{iB,C}, \quad (\text{Curl}_y D)_{Ai} = \epsilon_{ijk} D_{Aj,k}, \quad (\text{Curl}_\theta E)_{A\alpha} = \epsilon_{\alpha\beta\gamma} E_{A\beta,\gamma}. \]  
(2.6)

Based on the arguments provide so far, which fulfil the main ingredients of its proof following Olver and Sivaloganathan [21], we state the characterization theorem for null Lagrangians.

**Theorem 1.** The Lagrangian \( \mathcal{L} \) for the functional of the form (1.10) is a null Lagrangian if and only if there exist \( \mathcal{A}, \mathcal{D}, \tilde{\mathcal{D}} \) as scalar functions of \( (x, \chi, \theta) \), \( B, B, \hat{B}, C, \hat{C} \) as \( 3 \times 3 \) matrix functions of \( (x, \chi, \theta) \), and \( J \) as a \( 3 \times 3 \times 3 \) matrix function of \( (x, \chi, \theta) \) such that

\[
\mathcal{L} = \mathcal{A} + B^\top \cdot F + C \cdot \text{cof} \ F + \mathcal{D} \det F
+ \tilde{B}^\top \cdot G + \hat{C} \cdot \text{cof} \ G + \tilde{\mathcal{D}} \det G
+ \hat{B} \cdot (\text{cof} \ G) F^\top + \hat{C} \cdot (\text{cof} \ F) G^\top + J \cdot e_\alpha \otimes e_j \otimes (G^\top e_\alpha \wedge F^\top e_j),
\]  
(2.7)

where the 84 scalar functions appearing as coefficients (or its components) depend only on 36 scalar functions (as components of \( L, M, \tilde{M}, K, \tilde{K}, H \)), in terms of the
notation described in Remark 3, in the following way:

\[ A = \text{Div}_x L, \quad D = \text{Div}_y M, \quad \tilde{D} = \text{Div}_y \tilde{M}, \]  

\[ J = e_\alpha \otimes K_{\alpha} - e_\alpha \otimes (\nabla_y \tilde{K}^T e_\alpha)^T + \nabla_z H \]  

\[ B^\top = \text{Curl}_x K + (\nabla_y L)^\top, \quad C = \nabla_x M - (\text{Curl}_y K)^\top, \]  

\[ \tilde{B}^\top = \text{Curl}_x \tilde{K} + (\nabla_y L)^\top, \quad \tilde{C} = \nabla_x \tilde{M} - (\text{Curl}_y \tilde{K})^\top, \]  

\[ \hat{B}^\top = \nabla_y \tilde{M} - (\text{Curl}_y H)^\top, \quad \hat{C} = \nabla_y M + (\text{Curl}_y H)^\top. \]  

Here \( F = \nabla \chi, G = \nabla \theta \), i.e., \( F_{iA} = \chi_{i,A}, G_{\alpha A} = \theta_{\alpha,A} \).

**Remark 4.** In (2.8), \( L, M, \tilde{M} \) are 3 component vector functions of \( x, \chi, \theta \); \( K, \tilde{K}, H \) are \( 3 \times 3 \) matrix functions of \( x, \chi, \theta \). In indicial notation, (2.7) is alternately expressed as

\[ \mathcal{L} = \mathcal{A} + B_{iA} F_{iA} + C_{iA} (\text{cof} F)_{iA} + \mathcal{D} (\text{det} F) \]  

\[ + \tilde{B}_{iA} G_{\alpha A} + \tilde{C}_{iA} (\text{cof} G)_{iA} + \tilde{\mathcal{D}} (\text{det} G) \]  

\[ + \hat{B}_{iA} (\text{cof} G)_{\alpha A} F_{iA} + \hat{C}_{iA} (\text{cof} F)_{iA} G_{\alpha A} + J_{\alpha j C} G_{\alpha A} F_{jB} e_{CAB}. \]  

while, (2.8) is equivalent to the conditions

\[ \mathcal{A} = L_{i,A}, \quad \mathcal{D} = M_{i,A}, \quad \tilde{\mathcal{D}} = \tilde{M}_{i,A}, \]  

\[ J_{\alpha j C} = (K_{jC,\alpha} - \tilde{K}_{\alpha C,j} + H_{\alpha j C}), \]  

\[ B_{iA} = L_{i,A} - \epsilon_{ABC} K_{iC,B}, \quad C_{iA} = \epsilon_{ijk} K_{kA,j} + M_{i,A}, \]  

\[ \tilde{B}_{iA} = L_{i,A} - \epsilon_{ABC} \tilde{K}_{iC,B}, \quad \tilde{C}_{iA} = \epsilon_{ijk} \tilde{K}_{\gamma A,j} + \tilde{M}_{i,A}, \]  

\[ \hat{B}_{iA} = \tilde{M}_{i,A} + \epsilon_{A\beta\gamma} H_{\gamma i,\beta}, \quad \hat{C}_{iA} = M_{i,A} + \epsilon_{ijk} H_{iA,j}. \]  

**Remark 5.** Using the characterization of the null Lagrangians via the 2-form (2.2), it is natural to define the class of polyconvex functions relevant for a Cosserat elastic media as follows: A polyconvex Lagrangian function for a Cosserat elastic media is given by

\[ \Psi \left( F, \text{cof} F, \text{det} F, G, \text{cof} G, \text{det} G, \right. \]  

\[ \left. (\text{cof} G) F^T, (\text{cof} F) G^T, e_\alpha \otimes e_j \otimes (G^T e_\alpha \wedge F^T e_j) \right), \]  

where \( \Psi : \mathbb{R}^{83} \rightarrow \mathbb{R} \) is a convex function in each of its argument [11, 34]. Thus, a strain energy density function (1.8) for a Cosserat elastic media is polyconvex if
and only if there exists $\Psi$ of above form. At this point, it is also useful to list a special sub-class of above via additive decomposition, i.e.,

$$\Phi_1(F) + \Phi_2(\text{cof } F) + \Phi_3(\det F) + \Phi_4(G) + \Phi_5(\text{cof } G) + \Phi_6(\det G) + \Phi_7((\text{cof } G)F^\top) + \Phi_8((\text{cof } F)G^\top) + \Phi_9(e_\alpha \otimes e_j \otimes (G^\top e_\alpha \wedge F^\top e_j)),$$

(2.12)

where the nine functions $\{\Phi_i\}_{i=1}^9$ are convex functions of their arguments.

2.1 Divergence representation

The characterization theorem, stated above as Theorem 1, can be further applied to obtain a divergence representation of the null Lagrangians akin to Theorem 7 of Olver and Sivaloganathan in [21] (see also [18]). In fact, we find that

$$P = L + (F^\top K)^\times + (\text{cof } F^\top M)^\times + (\text{cof } G^\top \tilde{K})^\times + \frac{1}{2} (F^\top H^\top G - G^\top HF)^\times,$$

(2.13)

where we used the symbolic notation $\times$ (1.3). In indicial notation, (2.13) can be expressed as

$$P_A = L_A + \epsilon_{ABC} F_{iB} K_{iC} + (\text{cof } F)_{iA} M_i + \epsilon_{ABC} G_{aB} \tilde{K}_{aC} + (\text{cof } G)_{\alpha A} \tilde{M}_\alpha + \epsilon_{ABC} G_{aB} F_{iC} H_{ai}.$$

(2.14)

By a direct calculation, it is easy to verify that

$$\mathcal{L} = \nabla \cdot P$$

in (2.7) of Theorem above. The detailed steps justifying this identity are provided in Appendix C.

Remark 6. It is easy to recognize that as a special case of nonlinear elasticity, i.e., absence of effect of microrotation, the expression of $P$ stated in (2.13) reduces to the well known one (Eq. (13.6.3) of [24]), i.e., $L + (F^\top K)^\times + (\text{cof } F)^\top M$.

3 Null Lagrangian in micropolar shell theory

In this section, due to a natural presence of curvilinear coordinates (local coordinate chart for two dimensional manifold embedded in three dimensional space [12, 28, 35]), we employ upper and lower indices in this section for contravariant and covariant components [32, 37]. It is natural to utilize the parameter space for the shell in place of $\Omega$ in this section; this also enables us to avoid the covariant
derivative (but it can be easily incorporated by multiplying the Lagrangian by a factor [37]). It is emphasized that in this section the capital Latin indices $A, B, \ldots$, range over 1, 2. Here $(x^1, x^2)$ (in place of the symbols $(s^1, s^2)$ as shown in Fig. 2) are local coordinates on the reference configuration of the shell. In the assumed framework for micropolar shells, we have the following identification of the local coordinates with components

$$x = x^A e_A = s^A e_A, \quad y^i = y^i e_i, \quad \theta^\alpha = \theta^\alpha e_\alpha.$$ 

With $\mathcal{A}, \mathcal{B}, C, \tilde{B}, \tilde{C},$ and $\hat{B}$ as local functions of $x^B, y^i, \theta^\beta$, (so that the total number of the scalar functions is $(2^{2+1}+3)$, i.e., 28), the general expression of a 2-form on the shell (as the counterpart of (2.1)) is found to be

$$\omega = \mathcal{A} dx^1 \wedge dx^2 + \mathcal{B}_{A} dy^i \wedge dx^A + \tilde{B}_{\alpha\theta} d\theta^\alpha \wedge dx^A$$
$$+ \frac{1}{2} \epsilon_{ijk} C^i dy^j \wedge dy^k + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{C}^\alpha d\theta^\beta \wedge d\theta^\gamma + \hat{B}_{\alpha i} dy^i \wedge d\theta^\alpha. \tag{3.1}$$

which leads to a ‘horizontal’ form

$$(\mathcal{A} + \mathcal{B}_{AB} y^i \epsilon^{AB} + \frac{1}{2} \epsilon_{ijk} C^i y^j \epsilon^{AB} + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{C}^\alpha \theta^\beta \epsilon^{AB}$$
$$+ \hat{B}_{\alpha A} \theta^\alpha \epsilon^{AB} + \hat{B}_{\alpha i} y^i \epsilon^{AB} ) dx^1 \wedge dx^2, \tag{3.2}$$
where the two dimensional Levi-Civita symbol is denoted by $\epsilon^{AB}$, i.e.,

$$
\epsilon^{AB} = \begin{cases} 
+1 & \text{if } (A, B) = (1, 2), \\
-1 & \text{if } (A, B) = (2, 1), \\
0 & \text{if } A = B.
\end{cases}
$$

(3.3)

Similar to the previous section, we continue to use the notation

$$
F = y_1^A e_i \otimes e^A = y_1^A e_i \otimes e^A, \\
G = \theta_1^\alpha e_\alpha \otimes e^A = \theta_1^\alpha e_\alpha \otimes e^A,
$$

(3.4)

Note that

$$
y_i^A e_i = F_{iA} e_i = F e_A = F_{iB} (e_i \otimes e^B) e_A,
$$

(3.5)

and similar relation for $\theta_1^\alpha e_\alpha$. The discussion so far enables us to write the general expression capturing the form of null Lagrangians, as the counterpart of (2.2) for the micropolar shell,

$$
\mathcal{L} = \mathcal{A} + \epsilon^{[BF]} + \epsilon^{[\tilde{B}G]} + \frac{1}{2} \epsilon^{AB} C \cdot F e_A \wedge F e_B + \frac{1}{2} \epsilon^{AB} \tilde{C} \cdot G e_A \wedge G e_B + \epsilon^{AB} \tilde{B} \cdot (F e_A \otimes G e_B),
$$

(3.6)

which can be further simplified to

$$
\mathcal{L} = \mathcal{A} + \epsilon^{[BF]} + \epsilon^{[\tilde{B}G]} \\
+ C \cdot F e_1 \wedge F e_2 + \tilde{C} \cdot G e_1 \wedge G e_2 \\
+ \tilde{B} \cdot (F e_1 \otimes G e_2 - F e_2 \otimes G e_1),
$$

(3.7a)

where

$$
B = B_{iA} e^A \otimes e^i, \quad \tilde{B} = \tilde{B}_{\alpha A} e^A \otimes e^\alpha, \\
C = C^i e_i, \quad \tilde{C} = \tilde{C}^\alpha e_\alpha,
$$

(3.7b)

and

$$
\epsilon^{[A]} = Ae_2 \cdot e_1 - Ae_1 \cdot e_2.
$$

(3.7c)

The coordinate expression of the relevant 1-form $\zeta$ (as counterpart of (2.4)) is written as

$$
\zeta = \bar{P}_A dx^A + \tilde{P}_k dy^k + \bar{P}_\alpha d\theta^\alpha,
$$

(3.8)

where $\bar{P}_A, \tilde{P}_k, \text{ and } \bar{P}_\alpha$ are local functions of $x^B, y^h, \theta^\beta$. The total number of the scalar functions is 8, as expected \((2+3+3)^2_1\). Using this expression, the exterior
derivative of \( \zeta \) (3.8) can be written as
\[
d\zeta = \frac{\partial P_A}{\partial x^B} dx^B \wedge dx^A + \frac{\partial P_\alpha}{\partial \theta^\beta} d\theta^\beta \wedge d\theta^\alpha + \frac{\partial \tilde{P}_k}{\partial y^k} dy^k \wedge dy^k + \left( \frac{\partial P_A}{\partial \theta^\alpha} - \frac{\partial \tilde{P}_\alpha}{\partial x^A} \right) d\theta^\alpha \wedge dx^A
\]
\[
+ \left( \frac{\partial P_A}{\partial y^k} - \frac{\partial \hat{P}_k}{\partial x^A} \right) dy^k \wedge dx^A + \left( \frac{\partial \tilde{P}_\alpha}{\partial y^k} - \frac{\partial \hat{P}_k}{\partial \theta^\alpha} \right) dy^k \wedge d\theta^\alpha.
\]
(3.9)

Using above expression of \( d\zeta \) and the expression of \( \omega \) (3.1), we find that the conditions corresponding to a null Lagrangian are
\[
A = \frac{\partial P_A}{\partial x^B} e_{BA}, B_A = \left( \frac{\partial P_A}{\partial y^i} - \frac{\partial \hat{P}_1}{\partial x^A} \right), \tilde{B}_\alpha = \left( \frac{\partial P_\alpha}{\partial \theta^\beta} - \frac{\partial \tilde{P}_1}{\partial x^A} \right),
\]
\[
\frac{1}{2} \epsilon_{ijk} C^i = \frac{\partial \tilde{P}_k}{\partial y^j}, \frac{1}{2} \epsilon_{\alpha \beta \gamma} \tilde{C}^\alpha = \frac{\partial \hat{P}_2}{\partial \theta^\beta}, \tilde{B}_\alpha = \left( \frac{\partial \tilde{P}_\alpha}{\partial y^i} - \frac{\partial \hat{P}_1}{\partial \theta^\alpha} \right).
\]
(3.10)

In direct notation, the set of conditions (3.10) can be re-written as
\[
\mathcal{A} = \mathcal{P}_{1,2} - \mathcal{P}_{2,1} = \epsilon [\nabla_x \mathcal{P}],
\]
\[
B = \nabla_y \mathcal{P} - (\nabla_x \hat{\mathcal{P}})^\top, \tilde{B} = \nabla_\theta \mathcal{P} - (\nabla_x \hat{\mathcal{P}})^\top,
\]
\[
C = \text{Curl}_y \hat{\mathcal{P}}, \tilde{C} = \text{Curl}_\theta \hat{\mathcal{P}}.
\]
(3.11)

**Theorem 2.** The Lagrangian \( \mathcal{L} \) for the functional of the form (1.10) for a micropolar shell is a null Lagrangian if and only if there exist \( \mathcal{A} \) as a scalar functions of \((x, \chi, \theta)\), \( B, \tilde{B} \) as \(2 \times 3\) matrix functions of \((x, \chi, \theta)\), \( \hat{B} \) as \(3 \times 3\) matrix function of \((x, \chi, \theta)\), and \( C \) and \( \tilde{C} \) as a 3 component vector functions of \((x, \chi, \theta)\) such that (3.7) holds where the 28 scalar functions appearing as coefficients (or its components) depend only on 8 scalar functions (as components of \( \mathcal{P}, \hat{\mathcal{P}}, \tilde{\mathcal{P}} \)) in the following way:
\[
\mathcal{A} = \epsilon [\nabla_x \mathcal{P}], \quad B = \nabla_y \mathcal{P} - (\nabla_x \hat{\mathcal{P}})^\top, \quad \tilde{B} = \nabla_\theta \mathcal{P} - (\nabla_x \hat{\mathcal{P}})^\top,
\]
\[
C = \text{Curl}_y \hat{\mathcal{P}}, \quad \tilde{C} = \text{Curl}_\theta \hat{\mathcal{P}}.
\]
(3.12a)

**Remark 7.** Using the characterization of the null Lagrangians (3.7) via the 2-form (3.1), it is natural to define the class of polyconvex functions for micropolar
shells \([36, 39]\) as follows. A polyconvex lagrangian for a micropolar shell is given by
\[
\mathcal{L}(x, \chi, \theta, \nabla \chi, \nabla \theta) := \Psi \left( F, G, Fe_1 \wedge Fe_2, Ge_1 \wedge Fe_2, \right.
\]
\[
\left. Fe_1 \otimes Ge_2 - Fe_2 \otimes Ge_1 \right),
\]
where \(\Psi : \mathbb{R}^{27} \to \mathbb{R}\) is a convex function in each of its argument \([11, 34]\).

3.1 Divergence representation

We expect to reduce above expression of \(\mathcal{L}\) in Theorem 2 as \(P_{1,1} + P_{2,2}\) for a vector \(P \sim (P_1, P_2)\). The expression (3.9) leads to a ‘horizontal’ form
\[
\left( \frac{\partial \tilde{P}_B}{\partial x^A} + \left( \frac{\partial \tilde{P}_B}{\partial \theta^a} - \frac{\partial \tilde{P}_a}{\partial x^B} \right) \theta_{,A}^a + \left( \frac{\partial \tilde{P}_B}{\partial y^k} - \frac{\partial \hat{P}_k}{\partial x^B} \right) y_{,A}^k \right)
\]
\[
- \frac{\partial \tilde{P}_a}{\partial x^A} \theta_{,B}^a + \frac{\partial \tilde{P}_a}{\partial \theta^a} \theta_{,B}^a + \frac{\partial \hat{P}_k}{\partial y^k} y_{,A}^k y_{,B}^k \right) dx^A \wedge dx^B.
\]

Using (3.14), the local expression of a null Lagrangian for micropolar shells is found to be
\[
\mathcal{L} = \left( \frac{\partial \tilde{P}_B}{\partial x^A} + \left( \frac{\partial \tilde{P}_B}{\partial \theta^a} - \frac{\partial \tilde{P}_a}{\partial x^B} \right) \theta_{,A}^a + \left( \frac{\partial \tilde{P}_B}{\partial y^k} - \frac{\partial \hat{P}_k}{\partial x^B} \right) y_{,A}^k \right)
\]
\[
- \frac{\partial \tilde{P}_a}{\partial x^A} \theta_{,B}^a + \frac{\partial \tilde{P}_a}{\partial \theta^a} \theta_{,B}^a + \frac{\partial \hat{P}_k}{\partial y^k} y_{,A}^k y_{,B}^k \right) \epsilon^{AB}.
\]

Indeed, (with \(|\) as a decoration to denote the ‘total’ derivative) by a repeated application of the product rule and chain rule of differentiation,
\[
\mathcal{L} = \left( \frac{\partial}{\partial x^A} | \tilde{P}_B - \tilde{P}_a \theta_{,A}^a - \hat{P}_k y_{,A}^k \right)
\]
\[
- \frac{\partial \tilde{P}_a}{\partial x^A} \theta_{,B}^a + \frac{\partial \tilde{P}_a}{\partial \theta^a} \theta_{,B}^a + \left( - \frac{\partial \hat{P}_k}{\partial \theta^a} y_{,A}^k \theta_{,B}^a + \frac{\partial \hat{P}_k}{\partial y^k} y_{,A}^k y_{,B}^k \right) \epsilon^{AB},
\]
which can be further written as
\[
\mathcal{L} = \left( \frac{\partial}{\partial x^A} | \tilde{P}_B + \tilde{P}_a \theta_{,A}^a + \hat{P}_k y_{,B}^k \right) \epsilon^{AB}
\]
\[
= \frac{\partial}{\partial x^A} \left( \tilde{P}_B + \tilde{P}_a \theta_{,B}^a + \hat{P}_k y_{,B}^k \right) \epsilon^{AB}.
\]
Above expression (3.17) motivates the definition
\[ P_A := \epsilon^{AB}(\overline{P}_B + \tilde{P}_a \theta^a_B + \hat{P}_k y^k_B). \] (3.18)
Thus, in direct notation,
\[ \mathcal{L} = \nabla \cdot P, \quad P = \epsilon [P + F^\top \hat{P} + G^\top \tilde{P}], \] (3.19)
where \( P = P(x, y(x), \theta(x)) \), and \( \overline{P}, \hat{P}, \tilde{P} \) are functions of \( x, y, \theta \).

4 Nilpotent energies in linearized micropolar theory

In the special case of homogenous, linearized micropolar theory \([10, 27, 38]\), using
the standard notation for fourth order tensors \([14]\), we are looking for Lagrangians
of the form
\[ \frac{1}{2}(\nabla u + \epsilon \phi) \cdot A(\nabla u + \epsilon \phi) + \frac{1}{2} \nabla \phi \cdot B \nabla \phi \]
\[ + (\nabla u + \epsilon \phi) \cdot D \nabla \phi, \] (4.1)
with \( A, B, D \) constant tensors such that
\[ A_{ijkl} = A_{klij}, \quad B_{ijkl} = B_{klij}; \] (4.2)
in indicial notation,
\[ \frac{1}{2} A_{ijkl}(u_{i,j} + \epsilon_{ijs} \phi_s)(u_{k,l} + \epsilon_{klt} \phi_t) + \frac{1}{2} B_{ijkl} \phi_{ij} \phi_{kl} + D_{ijkl}(u_{ij} + \epsilon_{ij} \phi_s) \phi_{kl}. \]
Here \( u \) represents the (infinitesimal) displacement field while \( \phi \) represents the
(infinitesimal) rotation vector field. In the context of the Lagrangian \( \mathcal{L} \) in (1.10),
with \( \chi \) (resp. \( \theta \)) replaced by \( u \) (resp. \( \phi \)), according to the characterization theorem
for first order null Lagrangians in the micropolar theory, by comparison of (4.1)
and (2.7), thus, we find that only terms bilinear in \( F = \nabla u \) and \( \phi \) and quadratic in
\( F, G = \nabla \phi \) (and \( \phi \)) are needed as well as mixed type where terms are bilinear in
\( F, G, \phi \) as well. Indeed, we conclude that \( \mathcal{S}(x, u, \phi) \) is bilinear in \( \phi \) and
independent of \( u \) and \( x \), \( \mathcal{G}(x, u, \phi) = \tilde{G}(x, u, \phi) = 0 \), \( B(x, u, \phi), \tilde{B}(x, u, \phi) \) is
linear in \( \phi \) and independent of \( u \) and \( x \), \( C(x, u, \phi), \tilde{C}(x, u, \phi) \) is a constant tensor,
\( \tilde{B}(x, u, \phi) = \tilde{C}(x, u, \phi) = 0 \), \( J(x, u, \phi) \) is a constant tensor.
Resorting to the indicial notation prescribed in Remark 3, here \( F_{iA} = u_{i,A}, G_{aA} = \phi_{a,A}, \) and \( e_A, e_j, e_{\alpha} \) are used to denote the same basis vectors \( e_1, e_2, e_3 \). However,
in the following sometimes we follow ordinary indicial notation while other times
we stick to Remark 3.
Theorem 3. A function of the form (4.1) (with the conditions (4.2)) is a null Lagrangian if and only if $A$, $B$ and $D$ satisfy (no sum for last two conditions)

\begin{align*}
A & = 0 \\
B_{ijkl} & = -B_{ilkj}, \\
D_{ijkl} & = -D_{ilkj}, \\
D_{ijji} & = -D_{ikki} \text{ for } i \neq j \neq k \neq i, \\
D_{ijjk} & = D_{kikk} + D_{jijk} \text{ for } i \neq j \neq k \neq i.
\end{align*}

(4.3)

For the proof of above theorem, the sufficiency can be checked by direct substitution in the Euler–Lagrange equations; the details of the same are also provided in [42]. Therefore, the only non-trivial part is the necessity which we establish in the following.

Using (A.7), $C_{iA} (\text{cof } F)_{iA} = C_{ij} \frac{1}{2} \epsilon_{imn} \epsilon_{jpq} F_{mp} F_{nq} = C_{ij} \frac{1}{2} \epsilon_{imn} \epsilon_{jpq} u_{m,p} u_{n,q}$, i.e.,

\begin{equation}
A_{iA} B = C_{mn} \frac{1}{2} \epsilon_{mij} \epsilon_{nAB}.
\end{equation}

\begin{equation}
\implies A_{iA} + A_{iB} = 0, \forall i, j, A, B;
\end{equation}

in particular, $A_{iA} = 0, A_{iA} = 0$. (4.5)

Similarly, a similar argument leads to the result for $B$, i.e.,

\begin{equation}
B_{iA} + B_{iB} = 0, \forall i, j, A, B.
\end{equation}

(4.6)

Comparing (4.1) with (2.7) and using (2.10) we get

\begin{align*}
D_{ij\alpha} e_{ij} \phi_s \phi_{\alpha,A} &= (L_{A,\alpha} - e_{ABC} \tilde{K}_{\alpha AB}) \phi_{\alpha,A}, \\
D_{ij\alpha} e_{ij} \phi_s &= L_{A,\alpha} - e_{ABC} \tilde{K}_{\alpha AB} \\
\therefore L_{A,\alpha} &= e_{ABC} \tilde{K}_{\alpha AB} = e_{ABC} \tilde{K}_{\alpha AB} = -e_{BAC} \tilde{K}_{\alpha AB} = 0.
\end{align*}

Now, $L_{A,\alpha} = \frac{1}{2} A_{ijkl} e_{ijs} e_{klt} \phi_s \phi_t$,

\begin{equation}
\therefore A_{ijkl} e_{ijs} e_{klt} \phi_s = L_{A,\alpha} = L_{A,\alpha} = 0 \quad \forall \alpha.
\end{equation}

(4.7)

We fix $\beta \in \{1, 2, 3\}$, then we put

\begin{equation}
\phi_s = \begin{cases}
0 & \text{if } s \neq \beta \\
1 & \text{if } s = \beta
\end{cases}
\end{equation}

(4.8)
Therefore we get \((A_{ijkl} - A_{ijkl})e_{klα} = 0\) if \(i \neq j \neq β \neq i\). Putting \(α = i\), the only possibility for \(k, l\) is β, j, (no sum)

\[
A_{ijβj} - A_{jiβj} - A_{ijjβ} + A_{jijβ} = 0 \quad (4.9)
\]

\((4.5)_2, (4.5)_3 \implies A_{ijβj} + A_{jiβj} = 0 \quad (4.10)\)

\[\therefore A_{jβij} = -A_{jjiβ}.\]

Again comparing (4.1) with (2.7) we get

\[
A_{iAkl}e_{klα}u_{i,A} = (L_{A,i} - e_{ABC}K_{iC,B})u_{i,A}
\]

\[
\implies A_{iAkl}e_{klα} = L_{A,i} - e_{ABC}K_{iC,B} \quad (4.11)
\]

\[D_{ijaA}e_{ijβ} = L_{A,α} - e_{ABC}K_{aC,B} \quad (4.12)\]

\[J_{αjC}G_{αA}F_{jB}e_{CAB} = D_{jBaAu}_{j,B}φ_{α,A}.\]

\[D_{jEαF} = e_{CFE}(K_{jC,α} - \tilde{K}_{αC,j} + H_{αj,C}).\]

From (4.13), we have

\[D_{ijkl} = -D_{ikjl}. \quad (4.14)\]

From (4.13) we get

\[D_{JαA} = 0 \quad ∀α, j, A. \quad (4.15)\]

Also if \(E \neq F\) then \(C\) has exactly one value for which RHS of (4.13) is nonzero. So there is no sum on \(C\). Now differentiating (4.13) with respect to \(x_B\) we get

\[K_{jC,α} - \tilde{K}_{aC,j} + H_{αj,C} = 0 \quad (4.16)\]

But

\[e_{ABC}H_{αj,CB} = e_{ABC}H_{αj,BC} = -e_{ACB}H_{αj,BC} = -e_{ABC}H_{αj,CB} = 0.\]

Therefore,

\[e_{ABC}K_{iC,αB} = e_{ABC}\tilde{K}_{αC,iB}. \quad (4.17)\]

Now differentiating (4.11) with respect to \(φ_{α}\) and (4.12) with respect to \(u_{i}\) we get

\[L_{A,ia} - A_{iAkl}e_{kla} = e_{ABC}K_{iC,aB} = e_{ABC}\tilde{K}_{aC,iB} = L_{A,ai}\]

\[
\implies A_{iAkl}e_{kla} = 0 \quad \text{as } L_{A,ai} = L_{A,ia}
\]

\[
\implies A_{iAkl} = A_{iAlk} \quad ∀i, A, k, l
\]

\[(4.4) \implies A_{ikA} = A_{klA} \quad ∀i, A, k, l
\]

\[\therefore A_{ikl} = A_{ikli} = -A_{ikli} = 0 \quad \text{by } (4.10)
\]

\[
\implies A_{ijl} = A_{jili} = 0 \quad ∀i, j, l
\]

\[
\implies A_{lij} = 0 = A_{jll} \quad ∀i, j, l.
\]
Therefore we get $A_{ijkl} = 0$ if any two of its subscript are equal as (no sum)

$$A_{llij} = 0, A_{lilj} = 0, A_{lijl} = 0, A_{illj} = 0, A_{iljl} = 0,$$

(4.18)

where in view of the symmetry $A_{llij} = A_{ijll}$ and $A_{lilj} = A_{ljil}$. Hence $A = 0$.

We expand and rewrite (4.1) as

$$L = \frac{1}{2} A_{ijkl} u_{i,j} u_{k,l} + \frac{1}{2} A_{ijkl} e_{ijs} u_{k,l} + \frac{1}{2} B_{ijkl} \phi_{i,j} \phi_{k,l} + \frac{1}{2} B_{ijkl} \phi_{i,j} \phi_{k,l} + \frac{1}{2} B_{ijkl} e_{ijs} \phi_{s,t} u_{k,l} + \frac{1}{2} B_{ijkl} e_{ijs} \phi_{s,t} \phi_{k,l} + \frac{1}{2} B_{ijkl} e_{ijs} \phi_{s,t} \phi_{k,l} + \frac{1}{2} B_{ijkl} e_{ijs} \phi_{s,t} \phi_{k,l},$$

(4.19)

where the terms are assigned sequentially. Applying the Euler operator (the sum on index $t$ ranges over $1, 2, 3$)

$$E_r := \frac{\partial}{\partial z_r} - \frac{d}{dx_t} \left( \frac{\partial}{\partial p_{rt}} \right) \quad \text{(with } r = 1, 2, 3 \text{ for } u; r = 4, 5, 6 \text{ for } \phi)$$

on (4.19), where $z$ refers to components of $u$ and $\phi$ while $p$ refers to the components of their gradients. Then for $r = 1, \ldots, 6$, $E_r(L_1) = E_r(L_2) = E_r(L_3) = 0$ identically, as $A = 0$; and $E_r(L_4) \equiv 0$. So if (4.19) is a null Lagrangian then for $r = 1, \ldots, 6$, $E_r(L_5 + L_6) \equiv 0$. For $r = 1, 2, 3$,

$$E_r(L_5 + L_6) = E_r(L_5) \equiv 0 \quad \Rightarrow \quad D_{rtkl} \phi_{k,l} = 0. \quad (4.20)$$

Fix $i, j, h \in \{1, 2, 3\}$. Put $\phi_{h,ij} = 1$ (so that $\phi_{h,ji} = 1$), while let other components of $\phi_{k,l} = 0$. Then from (4.20) we get, $D_{rijkl} + D_{rjki} = 0$ which implies

$$D_{rikj} = -D_{rjki} \quad (4.21)$$

$$D_{rik} = 0. \quad (4.22)$$

Now for $r = 4, 5, 6$, ($h=r-3$),

$$E_r(L_5 + L_6) \equiv 0, \quad \Rightarrow \quad D_{ijht} u_{i,j} + D_{ijht} e_{ijs} \phi_{st} + D_{ijht} e_{ijh} \phi_{k,l} = 0. \quad (4.23)$$

With proper choice of $u$ we again get (4.21). Therefore, as a result of (4.23), we get

$$(D_{ijht} e_{ijs} - D_{ijst} e_{ijh}) \phi_{s,t} \equiv 0, \quad (4.24)$$

i.e.,

$$D_{ijht} e_{ijs} = D_{ijst} e_{ijh} \quad \forall h, s, t. \quad (4.25)$$

With ($h, s, t$) = $\pi(1, 2, 3)$, with the notation that $\pi$ stands for the circular permutation map, then (4.25) gives (no sum)

$$D_{thht} - D_{htht} = D_{stst} - D_{tsst} \quad (4.26)$$
\[ \Rightarrow D_{thht} = -D_{tstt} \quad \text{by (4.22).} \quad (4.27) \]

Also if \((h, s, t) = -\pi(1, 2, 3)\) then by same calculation (4.27) holds. With \(s = t \neq h\), we assume that \((s, h, k) = \pi(1, 2, 3)\). Then (4.25) gives (no sum)

\[ D_{kkhs} - D_{khhs} = D_{ksss} - D_{skss} \quad \text{(4.28)} \]

\[ \Rightarrow D_{khhs} = D_{hkhs} + D_{skss} \quad \text{by (4.22)} \quad (4.29) \]

Also if \((s, h, k) = -\pi(1, 2, 3)\) then by same calculation (4.29) holds. Also for \(h = t \neq s\), by similar argument (4.29) holds.

**Remark 8.** Some other aspects pertaining to the sufficiency part of the Theorem 3 are discussed in [42] along with a few other generalizations of linearized theory of elasticity.

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**A Expansion of \(\omega\)**

For the purpose of convenience of derivation, we assume that \(\det G \neq 0\), let \(\nabla_{\theta} \chi = T = FG^{-1}\), then

\[ F_{iA} = y_{i,A}, G_{\alpha A} = \theta_{\alpha,A}. \quad (A.1) \]

Then

\[ \omega = \omega \Omega_v + \frac{1}{2} \epsilon_{ABC} B_{Aa} F_{IM} dx_M \wedge dx_B \wedge dx_C + \frac{1}{2} \epsilon_{ijk} C_{iA} dy_j \wedge dy_k \wedge (F^{-1})_{AI} dy_l + \partial dy_1 \wedge dy_2 \wedge dy_3 \]

\[ + \frac{1}{2} \epsilon_{ABC} \tilde{B}_{Aa} G_{\alpha M} dx_M \wedge dx_B \wedge dx_C + \frac{1}{2} \epsilon_{ij\gamma} \tilde{C}_{iA} d\theta_\beta \wedge d\theta_\gamma \wedge (G^{-1})_{AI} d\theta_\alpha + \partial d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \]

\[ + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{B}_{AI} T_{ia} d\theta_\alpha \wedge d\theta_\beta \wedge d\theta_\gamma + \frac{1}{2} \epsilon_{ijk} \tilde{C}_{ia} dy_j \wedge dy_k \wedge (T^{-1})_{AI} dy_l \]

\[ + J_{ajC} G_{Aa} F_{jB} dx_A \wedge dx_B \wedge dx_C, \quad (A.2) \]

where

\[ \Omega_v := dx_1 \wedge dx_2 \wedge dx_3. \quad (A.3) \]
Expanding further,
\[ \omega = \varepsilon ABC \epsilon_{MBC} B_{Ai} F_{iM} \Omega_v + \frac{1}{2} \epsilon_{ijk} C_{iA} (F^{-1})_{Ai} \epsilon_{jkl} (\det F) \Omega_v + \mathcal{D} (\det F) \Omega_v \]
\[ + \frac{1}{2} \epsilon_{ABC} \epsilon_{MBC} B_{Ai} G_{\alpha M} \Omega_v + \frac{1}{2} \epsilon_{\delta \beta \gamma} \epsilon_{\alpha \beta \gamma} \tilde{B}_{iA} T_{ia} (\det G) \Omega_v + \frac{1}{2} \epsilon_{ijk} \tilde{C}_{ia} (T^{-1})_{ai} \epsilon_{jkl} (\det F) \Omega_v \]
\[ + \epsilon_{ABC} J_{\alpha j} G_{\alpha A} F_{jB} \Omega_v. \]  
\[(A.4)\]

Simplifying above expression, we find that
\[ \omega = (\mathcal{A} + B_{Ai} F_{iA} + C_{iA} (F^{-1})_{Ai} (\det F) + \mathcal{D} (\det F) \]
\[ + \tilde{B}_{Ai} G_{\alpha A} + \tilde{C}_{iA} (G^{-1})_{Ai} (\det G) + \tilde{\mathcal{D}} (\det G) \]
\[ + \tilde{B}_{ai} T_{ia} \det G + \tilde{C}_{ia} (T^{-1})_{ai} (\det F) + J_{\alpha j} G_{\alpha A} F_{jB} \epsilon_{CAB} ) \Omega_v, \]
\[(A.5)\]

which can be also written as
\[ \omega = (\mathcal{A} + B^\top \cdot F + C \cdot \text{cof} \ F + \mathcal{D} \det F + \tilde{B}^\top \cdot G + \tilde{C} \cdot \text{cof} \ G + \tilde{\mathcal{D}} \det G \]
\[ + \tilde{B}^\top \cdot FG^{-1} (\det G) + \tilde{C} \cdot \text{cof} \ (FG^{-1}) (\det G) + J_{\alpha j} G_{\alpha A} F_{jB} \epsilon_{CAB} \) \Omega_v, \]
\[(A.6)\]

where \((.)_{\alpha j} = G_{\alpha A} F_{jB} \epsilon_{CAB} = G^\top e_{\alpha} \wedge e_{j} \cdot e_{C} ; (.) = e_{\alpha} \otimes e_{j} \otimes (G^\top e_{\alpha} \wedge F^\top e_{j}).\]

Replacing the inverse of \( G \) by the cofactor, we get the form \((2.2)\) which does not depend on the invertibility of \( G \). The components of the cofactor of \( A \) are given by
\[ (\text{cof} \ A)_{ij} = \frac{1}{2} \epsilon_{imn} \epsilon_{j pq} A_{mp} A_{nq}, \]  
\[(A.7)\]

\section{B Expansion of \( d\zeta \)}

The expression \((2.4)\) leads to its exterior derivative
\[ d\zeta = \frac{1}{2} \epsilon_{ABC} dL_{A} \wedge dx_{B} \wedge dx_{C} + dK_{iA} \wedge dy_{i} \wedge dx_{A} + \frac{1}{2} \epsilon_{ijk} dM_{i} \wedge dy_{j} \wedge dy_{k} \]
\[ + d\tilde{K}_{\alpha A} \wedge d\theta_{\alpha} \wedge dx_{A} + \frac{1}{2} \epsilon_{\alpha \beta \gamma} d\tilde{M}_{\alpha} \wedge d\theta_{\beta} \wedge d\theta_{\gamma} + dH_{\alpha j} \wedge d\theta_{\alpha} \wedge dy_{j}, \]  
\[(B.1)\]
which can be expanded further given that $L, M, \tilde{M}, K, \tilde{K}, H$ are functions of $(x, \chi, \theta)$ so that

$$d\zeta = \frac{1}{2} \epsilon_{ABC} L_{A,D} dx_D \wedge dx_B \wedge dx_C + \frac{1}{2} \epsilon_{ABC} L_{A,i} dy_i \wedge dx_B \wedge dx_C + \frac{1}{2} \epsilon_{ABC} L_{A,\alpha} d\theta_\alpha \wedge dx_B \wedge dx_C$$

$$+ K_{iA,B} dx_B \wedge dy_i \wedge dx_A + K_{iA,j} dy_j \wedge dy_i \wedge dx_A + K_{iA,\beta} d\theta_\beta \wedge dy_i \wedge dx_A$$

$$+ \frac{1}{2} \epsilon_{ijk} M_{i,A} dx_A \wedge dy_j \wedge dy_k + \frac{1}{2} \epsilon_{ijk} M_{i,A} dy_i \wedge dy_j \wedge dy_k + \frac{1}{2} \epsilon_{ijk} M_{i,A} d\theta_\alpha \wedge dy_j \wedge dy_k$$

$$+ \frac{1}{2} \epsilon_{ijk} M_{i,A} dx_A \wedge d\theta_\alpha \wedge dx_A + \tilde{K}_{\alpha A,i} dy_i \wedge d\theta_\alpha \wedge dx_A + \tilde{K}_{\alpha A,\beta} d\theta_\beta \wedge d\theta_\alpha \wedge dx_A$$

$$+ \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{M}_{\alpha,A} dx_A \wedge d\theta_\beta \wedge d\theta_\gamma + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{M}_{\alpha,A} dy_i \wedge d\theta_\beta \wedge d\theta_\gamma + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{M}_{\alpha,A} d\theta_\beta \wedge d\theta_\beta \wedge d\theta_\gamma$$

$$+ H_{\alpha,j,A} dx_A \wedge d\theta_\alpha \wedge dy_j + H_{\alpha,j,k} dy_k \wedge d\theta_\alpha \wedge dy_j + H_{\alpha,j,\beta} d\theta_\beta \wedge d\theta_\alpha \wedge dy_j.$$

(B.2)

Collecting the terms accompanying the same exterior product of differentials, we get

$$d\zeta = L_{A,A} \Omega_v + \left( \frac{1}{2} \epsilon_{ABC} L_{A,i} - K_{iC,B} \right) dy_i \wedge dx_B \wedge dx_C$$

$$+ (K_{kA,j} + \frac{1}{2} \epsilon_{ijk} M_{i,A}) dy_j \wedge dy_k \wedge dx_A + M_{i,A} \Omega_v + \tilde{M}_{\alpha,A} \Omega_\theta$$

$$+ \left( \frac{1}{2} \epsilon_{ijk} M_{i,A} + H_{\alpha,j,k} \right) dy_j \wedge dy_k \wedge d\theta_\alpha + \left( \frac{1}{2} \epsilon_{ABC} L_{A,\alpha} - \tilde{K}_{\alpha C,B} \right) d\theta_\alpha \wedge dx_B \wedge dx_C$$

$$+ (\tilde{K}_{\gamma A,\beta} + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{M}_{\alpha,A}) d\theta_\gamma \wedge d\theta_\beta \wedge dx_A + (K_{jC,\alpha} - \tilde{K}_{\alpha j,C} + H_{\alpha,j,C}) d\theta_\alpha \wedge dy_j \wedge dx_C$$

$$+ \left( \frac{1}{2} \epsilon_{\alpha\beta\gamma} \tilde{M}_{\alpha,i} + H_{\gamma i,\beta} \right) dy_i \wedge d\theta_\beta \wedge d\theta_\gamma.$$

(B.3)

where (A.3) is used.

### C Expansion of $\nabla \cdot P$

The expression (2.13) is equivalent to

$$P = L + (F^T K)^x + (\text{cof} \ F)^\top M + (G^T \tilde{K})^x$$

$$+ (\text{cof} \ G)^\top \tilde{M} + \epsilon_{ABC} G_{aB} F_{iC} H_{a} e_A,$$

(C.1)
as
\[
\begin{align*}
\epsilon_{ABC}G_{ab}F_{ic}H_{ai}e_A &= e_A(e_A \cdot (e_B \wedge e_C))G_{ab}F_{ic}H_{ai} \\
&= e_A \otimes e_A(e_B \wedge e_C)G_{ab}F_{ic}H_{ai} \\
&= (G_{ab}e_B \wedge F_{ic}e_C)H_{ai} \\
&= (G^\top e_a \wedge F^\top e_j)H_{ai} = (G^\top e_a \wedge F^\top e_j)H \cdot e_a \otimes e_j \\
&= (G^\top e_a \wedge F^\top e_j)H \cdot e_a (\text{Cof} A \cdot e_\alpha)(\text{Cof} e_\beta)G \cdot \epsilon(e_\alpha \wedge F^\top H \cdot e_a) \\
&= \text{axl}(F^\top H \cdot e_a \wedge G^\top e_a - G^\top e_a \wedge F^\top H \cdot e_a) \\
&= \frac{1}{2}(F^\top H \cdot G - G^\top HF) \\
\end{align*}
\]
(recall axl(b \otimes c - c \otimes b) = c \wedge b). Thus, using the conditions stated in §2,
\[
\nabla \cdot P = \mathcal{A} + (\nabla_y L)^\top \cdot F + (\nabla_y L)^\top \cdot G + (\text{Curl}_x K \cdot F - \text{cof} F \cdot (\text{Curl}_y K)^\top) + \epsilon_{ABC}F_{ib}K_{ic,a}G_{aa} \\
+ \text{cof} F \cdot (\nabla_x M + (\nabla_y M)F + (\nabla_y L)G) \\
+ (\text{Curl}_y \tilde{K} \cdot G - \text{cof} G \cdot (\text{Curl}_y \tilde{K})^\top) + \epsilon_{ABC}G_{ab}\tilde{K}_{ac,i}F_{iA} \\
+ (\text{cof} G) \cdot (\nabla_x \tilde{M} + (\nabla_y \tilde{M})F + (\nabla_y \tilde{M})G) \\
+ \epsilon_{ABC}G_{ab}F_{ic}H_{ai,A} + \epsilon_{ABC}G_{ab}F_{ic}H_{ai,B}G_{\beta A} \\
+ \epsilon_{ABC}G_{ab}F_{ic}H_{ai,j}F_{jA},
\]
i.e.,
\[
\nabla \cdot P = \mathcal{A} + B \cdot F + C \cdot \text{cof} F + \tilde{D} \text{det} F + \tilde{B} \cdot G + \tilde{C} \cdot \text{cof} G + \tilde{D} \text{det} G \\
+ (\text{cof} F)G^\top \cdot (\nabla_y M) + (\nabla_y \tilde{M})^\top \cdot F(\text{cof} G)^\top \\
+ \epsilon_{ABC}F_{ib}K_{ic,a}G_{aa} + \epsilon_{ABC}G_{ab}\tilde{K}_{ac,i}F_{iA} + \epsilon_{ABC}G_{ab}F_{ic}H_{ai,A} \\
+ \epsilon_{ABC}\beta A G_{ab}F_{ic}H_{ai,B} - \epsilon_{ABC}F_{jA}F_{ib}G_{ac}H_{ai,j},
\]
i.e.,
\[
\nabla \cdot P = \mathcal{A} + B \cdot F + C \cdot \text{cof} F + \tilde{D} \text{det} F + \tilde{B} \cdot G + \tilde{C} \cdot \text{cof} G + \tilde{D} \text{det} G \\
+ \tilde{B} \cdot (\text{cof} G)F^\top + \tilde{C} \cdot (\text{cof} F)G^\top \\
- (\text{cof} F)G^\top \cdot (\text{Curl}_y H)^\top + (\text{Curl}_y H^\top) \cdot F(\text{cof} G)^\top \\
+ J \cdot e_a \otimes e_j \otimes (G^\top e_a \wedge F^\top e_j) \\
+ \epsilon_{ABC}F_{ib}K_{ic,a}G_{aa} + \epsilon_{ABC}G_{ab}\tilde{K}_{ac,i}F_{iA} \\
- \epsilon_{ABC}(K_{ic,a} - \tilde{K}_{ac,i} + H_{ai,C})G_{aa}F_{ib} + \epsilon_{ABC}G_{ab}F_{iB}H_{ai,C} \\
+ \epsilon_{ABC}G_{aa}\beta B F_{iC}H_{\beta,a} - \epsilon_{ABC}F_{jA}F_{jB}G_{ac}H_{ai,j},
\]
so that finally,
\[
\nabla \cdot P = \mathcal{L} - (\text{cof } F)G^\top \cdot (\text{Curl } H)^\top + (\text{Curl } H^\top) \cdot F(\text{cof } G)^\top \\
+ \epsilon_{\alpha\beta\gamma}(\text{cof } G)_{\gamma\delta} F_{\delta C} H_{i\beta,\alpha} - \epsilon_{ijk}(\text{cof } F)_{kC} G_{\alpha C} H_{\alpha i} \\
= \mathcal{L}.
\]

(C.6)

Note that
\[
\epsilon_{ABC} A_{iA} A_{jB} B_{ijC} = \epsilon_{ABC} A_{iA} A_{jB} \delta_{CD} B_{ijD} \\
= \epsilon_{ABC} A_{iA} A_{jB} A_{Ck}^\top (A^{-\top})_{kD} B_{ijD} \\
= \epsilon_{ijk}(\text{cof } A)_{kD} B_{ijD}.
\]

(C.7)

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