Classical integrability of the squashed three-sphere, warped AdS$_3$ and Schrödinger spacetime via T-duality

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Received 29 November 2010
Published 22 February 2011
Online at stacks.iop.org/JPhysA/44/115401

Abstract
We discuss the integrability of 2D nonlinear sigma models with target space being the squashed three-sphere, warped anti-de Sitter space and the Schrödinger spacetime. These models can be obtained via T-duality from integrable models. We construct an infinite family of non-local conserved charges from the T-dual Lax currents, enhancing the symmetry of warped anti-de Sitter space and the Schrödinger spacetime to $\hat{sl}_2(\mathbb{R}) \oplus \hat{sl}_2(\mathbb{R})$.

PACS numbers: 02.30.Ik, 04.60.Cf, 02.20.Sv, 01.40.eg

1. Introduction

Our understanding of superconformal gauge theories is far from complete. However, for specific cases, integrability has enabled us to expand our insights and provide useful ways for determining different properties of the underlying theory (e.g. anomalous dimensions of operators, determination of other observables, etc). Indeed, for the case of AdS$_5$/CFT$_4$, the integrable structure of planar $\mathcal{N} = 4$ super Yang–Mills theory has allowed the determination of the spectrum via tools such as the Bethe ansatz, finite gap methods, S-matrices, etc [1–7]. Furthermore, integrability was also shown to be present in other AdS/CFT systems such as AdS$_4$/CFT$_3$ [8–10] and AdS$_3$/CFT$_2$ [11–13], supporting the idea that more examples of integrable systems should exist.

More recently, new examples of holographic systems have been proposed after novel target space metrics have been shown to emerge naturally as solutions of various three-dimensional gravity theories. In particular, backgrounds containing squashed geometries (spheres and anti-de Sitter spaces) have become an interesting arena for further explorations, as such configurations have long been known in the context of deformed CFTs and black holes in string theory [14–19]. For specific setups, it has been possible to determine the central charge...
of the dual CFT by looking at its asymptotic symmetry algebras [20–24], but little more is
known about its precise properties and only recently, integrability has come into play [13, 25].
Another interesting and related example is that of the Schrödinger spacetime [26–28]. This
background is invariant under the Schrödinger group which includes translations, rotations,
Galilean boosts and non-relativistic scale transformations. Its holographic properties have
been widely studied owing to its inherent appeal to condensed matter physics, where systems
described by strongly coupled non-relativistic QFTs are common fare [29, 30].

Independently, two-dimensional nonlinear sigma models which are integrable are
interesting in their own right, as it has been a longstanding question how to identify integrable
systems. In [25, 31, 32] it was shown that the T-duals of integrable systems often turn out to be
new integrable models, by explicit determination of their Lax pairs and the construction of
the infinite set of conserved charges. This was shown in [31] for the PCM and SU(2) sigma
models. This was also discussed in [33, 34]. Based on these results, the authors of [25]
discussed how integrability translates from the original model to the T-dual one, focusing on
the discussion of the (bosonic) AdS5 × S5 case which was known to be integrable and whose
T-dual model is again AdS5 × S5. The emerging picture was later generalized to the full
superstring action in [35].

In [13] the integrability approach was extended to AdS3 × X7 backgrounds supported
by RR fluxes in which standard worldsheet methods cannot be applied. It was shown that
backgrounds with 16 supercharges of the form AdS3 × S3 × M4 with M4 = T4 or S3 × S1 can
be described using a Green–Schwarz action admitting a Z4 grading [36], which in turn implies
integrability. This is due to the fact that the equations of motion and the Maurer–Cartan (MC)
equations of any Z4 coset can be re-expressed as flatness conditions for a Lax connection [2].

Given that AdS3 × S3 × M4 is equipped with an integrable structure, it is natural to
ask whether new examples of integrable systems can be obtained from this background
via T-duality. Squashed three-spheres (SqS3), warped anti-de Sitter spaces (WAdS3) and
the Schrödinger spacetime Sch3 are string theory backgrounds obtained via T-duality from
AdS3 × S3 [37]. Building on this observation we show that T-duality relates nonlinear
sigma models with group manifold G target space to their squashed counterparts SqG. The
construction that we present here has two main advantages compared to the treatment in [37]:
it is easier because it is not based on dimensional reduction and it is more general since it can
be applied to obtain the squashing in the compact directions of any Lie group. In this sense,
WAdS3 and SqS3 can be seen as the simplest examples. The example of Sch3 is different in
the sense that the T-duality has to be performed in a non-compact
direction.

From our point of view, T-duality is a linear transformation of the components of the
conserved currents of the initial model. This means that the integrability properties of the
nonlinear sigma model on G are inherited by the T-dual model on SqG. The initial manifold
has isometry G × G which is promoted via a Lax construction to affine ˆg ⊕ ˆg [45, 46].
This affine symmetry remains after the T-duality, but now the zero modes are not anymore
isometries of the target space. This is a crucial point: the squashing preserves only G × T
isometry (where T ⊂ G is the maximal torus), but the full symmetry of the T-dual model
is ˆg ⊕ ˆg, whose zero modes are the isometries supplemented by a set of currents generating
non-local charges that cannot be found via a Noether construction. In the case of WAdS3
for example, even though the isometry is SL2(R) × U(1), we find the full sl2(R) ⊕ sl2(R)
symmetry algebra. The construction also works in the limit case, when the T-duality connects

3 It might be that these ‘central charges’ are really a result of a generalized version of Cardy’s formula, and the jury
is still out with respect to these dual theories being conformal.

4 Related works with T-duality in the context of integrable models and conformal sigma models can be found in
[33, 38–44].
the nonlinear sigma model on $G$ to the one on $G/U(1)$. Also in this case we find an affine $\hat{\mathbb{g}} \times \hat{\mathbb{g}}$ algebra of symmetries. Since our analysis is classical, it is not surprising to find that the affine algebra is actually a loop algebra, without central element; it will not be the case once we add quantum corrections. In this paper we consider backgrounds with only the metric turned on; the analysis for the full type II solution will be presented in a forthcoming publication [47]. The results that we present will nevertheless remain true in the more general setting with RR fields.

The plan of this paper is as follows. In section 2 we describe the general framework. We review how squashed backgrounds are obtained via T-duality in section 2.1. In section 2.2 we review the integrability of the principal chiral model (PCM) and in section 2.3 extend the results to the T-dual squashed groups. Section 3 is devoted to explicitly working out the flat currents (Lax connections) and conserved charges of our examples. The squashed three-sphere $\text{Sq}^3$ is discussed in section 3.1, three-dimensional warped anti-de Sitter space in section 3.2 and the Schrödinger spacetime $\text{Schr}_3$ in section 3.3. The procedure is very similar to the one discussed in [40, 48, 49]. We show that the original $G \times G$ isometry group ($\text{SU}(2) \times \text{SU}(2)$ for $\text{Sq}^3$ and $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ for WAdS$_3$ and Sch$_3$) is realized non-locally in the T-dual model via the appearance of non-local currents. This generalizes and extends the results of [50] where the $\text{SU}(2) \times U(1)$ isometry of $\text{Sq}^3$ was extended to a hidden Yangian algebra, but non-local currents were not considered. Finally, section 4 is devoted to a discussion and summary of our results, with some outlook on future work.

Appendix A reviews in detail the integrable hierarchy of variations for the PCM. Appendix B discusses the infinite hierarchy of the squashed groups after T-duality. Appendix C details the geodesics on squashed groups, providing further insights into their nature.

2. General framework

In this section, we review the general framework. In section 2.1, we discuss how to construct NLSM on squashed groups via T-duality from PCM with group manifold target space $G$, generalizing the treatment in [37]. In section 2.2, we review the classical integrability of the PCM. In section 2.3, we show the classical integrability of the NLSM on squashed groups obtained via T-duality by explicitly constructing a one-parameter family of non-local conserved currents from the T-dualized Lax currents. In order to do so, we first have to find a gauge transformation on the original model to bring its currents into a form which is readily T-dualizable. It is this gauge transformation which ultimately leads to the non-local nature of the dualized Lax currents, which cannot be obtained from a Noether procedure.

2.1. Squashed groups via T-duality

Consider a nonlinear sigma model on a target space $M$ with coordinates $(u^I, z)$. If $\partial_z$ is a space-like Killing vector, the metric on $M$ can be put into the form

$$
\begin{pmatrix}
G_{ij}(u) + f_i(u) f_j(u) \\
\frac{f_i(u)}{f_j(u)} \\
1
\end{pmatrix}.
$$

(2.1)

The corresponding action on a surface $\Sigma$ with signature $(-, +)$ is given by

$$
S[u^I, z] = \int_{\Sigma} G_{ij}(u) \, du^i \wedge * du^j + (dz + f_i(u) \, du^i) \wedge *(dz + f_j(u) \, du^j),
$$

(2.2)

where the $u^I$ and $z$ are maps

$$
u^I : \Sigma \to \mathbb{R},
$$

(2.3)

$$
(x, t) \mapsto \nu^I(x, t).
$$

(2.4)
We want to T-dualize on \( z \) using a first-order formalism (see [51, 52]). Introduce a gauge field \( A \) and the Lagrange multiplier \( \tilde{z} \):

\[
S[u', A, \tilde{z}] = \int_{\Sigma} G_{ij}(u) \, du^i \wedge *du^j + (A + f_i(u) \, du^i) \wedge *(A + f_j(u) \, du^j) - 2 \tilde{z} dA. \tag{2.5}
\]

The equations of motion for \( \tilde{z} \) give

\[
dA = 0 \implies A = dz, \tag{2.6}
\]

which leads back to the original action \( (2.2) \). On the other hand, the equation of motion for \( A \) leads to

\[
* d\tilde{z} = A + f_i(u) \, du^i = dz + f_i(u) \, du^i. \tag{2.7}
\]

Note that \( dz + f_i(u) \, du^i \) is the current associated with the Killing vector \( \partial z \).

Plugging the condition in equation \( (2.7) \) back into the action \( (2.5) \) we obtain the T-dual sigma model on the space \( \tilde{M} \) where the fibration has become trivial, i.e. the geometry is now a direct product, and a \( B \) field appears. The action is

\[
S[u', \tilde{z}] = \int_{\Sigma} G_{ij}(u) \, du^i \wedge *du^j + d\tilde{z} \wedge *d\tilde{z} - 2 d\tilde{z} \wedge f_i(u) \, du^i, \tag{2.8}
\]

which corresponds to a target space metric on \( \tilde{M} \) given by

\[
\begin{pmatrix}
G_{ij}(u) + f_i(u) f_j(u) & 0 \\
0 & 1
\end{pmatrix}. \tag{2.9}
\]

From a more abstract point of view one can think of the initial target space \( M \) as the total space of an \( S^1 \) fibration whose fiber is parametrized by the coordinate \( z \) and whose base space \( N \) has coordinates \( u^i \) and metric \( G_{ij} \):

\[
\begin{array}{c}
S^1 \rightarrow M \\
\downarrow \\
N
\end{array} \tag{2.10}
\]

After the T-duality transformation, we obtain a nonlinear sigma model with target space \( \tilde{M} \) where the fibration has been traded for a direct product:

\[
\tilde{M} = N \times S^1, \tag{2.11}
\]

and the circle is parametrized by the new coordinate \( \tilde{z} \).

PCM are natural examples for this construction, since a Lie group \( G \) is a principal bundle with fiber \( H \) and base \( G/H \) for any closed subgroup \( H \):

\[
\begin{array}{c}
H \rightarrow G \\
\downarrow \\
G/H
\end{array} \tag{2.12}
\]

In our main examples we will concentrate on geometries described by fibrations in which the total space is \( G \times U(1) \), and the fiber is a rational linear combination of one direction in the Cartan subgroup of \( G \) and the extra \( U(1) \):

\[
\begin{array}{c}
U(1) \rightarrow G \times U(1) \\
\downarrow \\
\text{SqG}
\end{array} \tag{2.13}
\]

where the base space \( \text{SqG} \) is the squashed group with metric

\[
ds^2[\text{SqG}] = ds^2[G] + \tanh^2 \Theta j \otimes jc. \tag{2.14}
\]
where $\Theta$ is a real parameter related to the radius of the $U(1)$, and $j_c$ is the current in the Cartan direction. The $T$-dual geometry is the Cartesian product
\[
\tilde{M} = \operatorname{Sq} G \times S^1.
\]
In the limit case $\Theta \to \infty$ the $T$-dual geometry is the product
\[
\tilde{M} = (G/U(1)) \times U(1).
\]

**Proof.** To show this, decompose $g \in G \times U(1)$ as
\[
g = l \exp \left[ i \varphi T_C + i \frac{y}{\operatorname{Tr}[T_C^2]} T_{r+i} \right],
\]
where $T_C$ is a generator of the Cartan subalgebra of $g = \operatorname{Lie}(G)$, $T_{r+i}$ generates the extra $u(1)$ and $l \in G/U(1)$. The metric on the manifold $G \times U(1)$ is written as
\[
ds^2[G \times U(1)] = \frac{\operatorname{Tr}[dg \, d g^{-1}]}{\operatorname{Tr}[T_C^2]} = \frac{\operatorname{Tr}[dl \, dl^{-1}]}{\operatorname{Tr}[T_C^2]} + d y^2 + d y^2 \left( \frac{\operatorname{Tr}[T_{r+i}^2]}{\operatorname{Tr}[T_C^2]} - \frac{2i \varphi \operatorname{Tr}[l^{-1} \, dl_{T_C}]}{\operatorname{Tr}[T_C^2]} \right).
\]
Now introduce the variable
\[
\alpha = \varphi - \frac{y}{\operatorname{Tr}[T_C^2]}.
\]
The group element takes the form
\[
g = l \exp \left[ i \alpha T_C + i \frac{y}{\operatorname{Tr}[T_C^2]} (T_C + T_{r+i}) \right] = k \exp \left[ i \frac{y}{\operatorname{Tr}[T_C^2]} (T_C + T_{r+i}) \right],
\]
where $k \in G$. The metric becomes
\[
ds^2[G \times U(1)] = \frac{\operatorname{Tr}[dk \, dk^{-1}]}{\operatorname{Tr}[T_C^2]} + \frac{d y^2}{\operatorname{Tr}[T_C^2]} \left( \frac{\operatorname{Tr}[T_{r+i}^2]}{\operatorname{Tr}[T_C^2]} + 1 \right) - \frac{2i \varphi \operatorname{Tr}[l^{-1} \, dl_{T_C}]}{\operatorname{Tr}[T_C^2]} \frac{d \alpha \, d T_{r+i}^2}{\operatorname{Tr}[T_C^2]}.
\]
Introduce the parameter $\Theta$,
\[
\sinh^2 \Theta = \frac{\operatorname{Tr}[T_C^2]}{\operatorname{Tr}[T_{r+i}^2]},
\]
and rescale $y$ as
\[
\frac{y}{\operatorname{Tr}[T_C^2]} = z \tanh \Theta;
\]
then the metric becomes
\[
ds^2[G \times U(1)] = ds^2[G] + dz^2 - 2i \tanh \Theta \varphi \, dz_j c
\]
\[
= ds^2[G] + \tanh^2 \Theta j_C \otimes j_C + (dz - i \tanh \Theta j_C)^2,
\]
\[\text{as discussed in [37], $\Theta$ can assume only a discrete set of values in a string theory embedding.}
\]
\[\text{In our normalizations the metric on $G$ is decomposed as}
\]
\[
ds^2[G] = - \sum_{a=1}^{\dim G} j_a \otimes j_a.
\]
where $j_a = \operatorname{Tr}[g^{-1} \, dg T_a] / \operatorname{Tr}[T_a^2]$, and $T_a$ are the generators of $g = \operatorname{Lie}(G)$. 5

\[\text{6 In our normalizations the metric on $G$ is decomposed as}
\]
\[
ds^2[G] = - \sum_{a=1}^{\dim G} j_a \otimes j_a,
\]
where
\[
ds^2[G] = \frac{\text{Tr}[dk \, dk^{-1}]}{\text{Tr}[k^{-1} \, dk \, C]} , \quad jC = \frac{\text{Tr}[k^{-1} \, dk \, T_C]}{\text{Tr}[T_C^2]} .
\] (2.26)

The structure is precisely the same as for the metric in equation (2.1); hence, by applying
the T-duality transformation on \(z\) we obtain the condition
\[
* d\tilde{z} = dz - i \tanh \Theta_1 jC .
\] (2.27)

and the corresponding T-dual metric
\[
ds^2 = ds^2[G] + \tanh^2 \Theta_1 jC \otimes jC + d\tilde{z}^2 ,
\] (2.28)

which is the metric of a squashed group times \(S^1\).

The T-duality breaks the initial \(G \times G \times U(1)\) isometry to \(G \times T \times U(1)\), where \(T \subset G\)
is the maximal torus. Further insights on the geometry of squashed group manifolds can be
obtained by studying their geodesics. This is done in appendix C.

2.2. Integrability of the principal chiral model

Consider the two-dimensional PCM for a Lie group \(G\). The action is given by
\[
S = -\frac{1}{2} \int \Sigma \text{Tr}[dg(x, t) \wedge * dg^{-1}(x, t)] ,
\] (2.29)
where \(g\) is a map from the worldsheet \(\Sigma\) with coordinates \((x, t)\) to the group \(G\). The equations
of motion take the form
\[
d*(g^{-1} \, dg) = d*(dgg^{-1}) = 0 .
\] (2.30)

From these equations we can read off the two conserved currents which result from the explicit
invariance of the action under the transformations \(\delta g = \epsilon g\) and \(\delta g = g\epsilon\), where \(\epsilon, \epsilon \in g\):
\[
j = g^{-1} \, dg , \quad \overline{j} = -dgg^{-1} .
\] (2.31)

Those currents are flat and thus fulfill the MC equations:
\[
dj + j \wedge j = 0 , \quad d\overline{j} + \overline{j} \wedge \overline{j} = 0 .
\] (2.32)

These flatness conditions are the underlying reason for the integrability of the model. Note
that this theory is different from the WZW model which has an extra Wess–Zumino term.
Nevertheless also in this case it is possible to find an affine algebra of symmetries, as we show
in the following.

Introduce the one-parameter families of currents
\[
J(x, t; \zeta) = -\frac{\zeta}{1 - \zeta^2} (\zeta j(x, t) + *j(x, t)) ,
\] (2.33)
\[
\overline{J}(x, t; \zeta) = -\frac{\zeta}{1 - \zeta^2} (\zeta \overline{j}(x, t) + *\overline{j}(x, t)) ,
\]
where \(\zeta \in C\) is the spectral parameter. Imposing the flatness of \(J\) and \(\overline{J}\) produces two
equations for the components \((J_x, J_t)\) and \((\overline{J}_x, \overline{J}_t)\), the so-called Lax equations:
\[
\partial_t J_x - \partial_x J_t + [J_x, J_t] = 0 , \quad \partial_t \overline{J}_x - \partial_x \overline{J}_t + [\overline{J}_x, \overline{J}_t] = 0 .
\] (2.34)

Each couple \((J_x, J_t)\) and \((\overline{J}_x, \overline{J}_t)\) forms a Lax pair [53]. In order to simplify the notation, we
introduce the operator
\[
\Lambda(\zeta) = \frac{\zeta}{1 - \zeta^2} (\zeta + *) ,
\] (2.35)
and we concentrate on the left current

$$J(x, t; \xi) = -\Lambda(\xi) j(x, t).$$

(2.36)

We can expand $\Lambda(\xi)$ in a power series as follows:

$$\Lambda(\xi) = \xi + \xi^2 + \xi^3 + \cdots + \sum_{n=1}^{\infty} \xi^{2n-1} + \xi^{2n}.$$  

(2.37)

Observe in particular that $J(\xi)$ has no zero mode in the expansion in powers of $\xi$. This is reflected in the fact that the model admits Noether charges. It is also useful to remark that

$$\star \Lambda(\xi) j = \frac{\xi}{\Lambda(\xi) + 1} j.$$  

(2.38)

The flatness of $J$ and $J$ implies both the EOM and the MC equations. Conversely, imposing the EOM and MC equations results in the flatness of the currents. This can be easily verified by observing that

$$dJ(\xi) + J(\xi) \wedge J(\xi) = \frac{\xi}{\xi^2 - 1} (d\star j + (d j + j \wedge j)).$$  

(2.39)

The flatness of $J$ and $J$ can be used to construct two infinite sets of conserved charges. Introduce a Wilson line $W(x, t; \xi)$ such that

$$J(x, t; \xi) = W^{-1}(x, t; \xi) dW(x, t; \xi).$$  

(2.40)

More precisely, $W$ depends on two points $(x, t)$ and $(x_0, t_0)$ on $\Sigma$ and can be written as

$$W(x, t|x_0, t_0; \xi) = P \left\{ \exp \left[ \int_{C(x_0, t_0) \rightarrow (x, t)} J(\xi, \tau; \xi) \right] \right\},$$  

(2.41)

where $P$ denotes the path-ordering and $C$ is a path from $(x_0, t_0)$ to $(x, t)$. We can now define a one-parameter family of conserved charges:

$$Q(t; \xi) = W(\infty, t | -\infty, t; \xi) = P \left\{ \exp \left[ \int_{-\infty}^{\infty} J_t(x, t; \xi) dx \right] \right\}.$$  

(2.42)

Using the Lax equations one can show that if the current $J$ vanishes at spatial infinity ($J(\pm \infty, t; \xi) = 0$), the one-parameter charge $Q(t; \xi)$ is conserved:

$$\frac{d}{dt} Q(t; \xi) = 0.$$  

(2.43)

Expanding in a power series in $\xi$,

$$Q(t; \xi) = 1 + \sum_{n=0}^{\infty} \xi^n Q^{(n)}(t).$$  

(2.44)

The condition in equation (2.43) is equivalent to the conservation of the infinite set of charges (see [54]):

$$\frac{d}{dt} Q^{(n)}(t) = 0, \quad \forall n = 0, 1, \ldots$$  

(2.45)

In other words, the model is classically integrable. The charge $Q^{(0)}$ is written explicitly as

$$Q^{(0)} = \int_{-\infty}^{\infty} j_t(x, t) dx.$$  

(2.46)

and it is the Noether charge corresponding to the manifest symmetry $g \mapsto g + \epsilon g$, for $\epsilon \in g$. All the other charges are non-local (i.e. they cannot be written as integrals of densities), but can always be understood in terms of transformations $g \mapsto g + \delta g$ that leave the EOM (2.30) invariant (but are not invariants of the action). The Poisson brackets of the set of $Q^{(n)}$ form a Kac–Moody algebra $\hat{g}$. This is explained in detail in appendix A. An alternative description of the charges can be obtained in terms of Yangian symmetry; this is discussed in detail in [55].
2.3. Integrability for squashed groups from T-duality

As remarked above, PCM are natural examples for the T-duality construction of section 2.1. Since we wish to study their classical integrability properties, we can think of the duality as of a linear transformation of the current components \((J(\zeta), \tilde{J}(\zeta)) \mapsto (\tilde{J}(\zeta), \tilde{J}(\zeta))\) that leaves the (on-shell) flatness conditions invariant:

\[
d\tilde{J} + \tilde{J} \wedge \tilde{J} = 0, \quad d\tilde{J} + \tilde{J} \wedge \tilde{J} = 0,
\]

where the T-dual Lax currents \(\tilde{J}(\zeta)\) and \(\tilde{J}(\zeta)\) are obtained by imposing the condition in equation (2.7):

\[
\ast d\tilde{z} = dz + f_i(u) du^i.
\]

In this case, in equation (2.25) we have found that the metric on \(G \times U(1)\) can be written as

\[
d\tilde{s}^2[G \times U(1)] = \tilde{\tilde{s}}^2[G] + \tanh^2 \Theta j_c \otimes j_c + (dz - \tau \tanh \Theta j_c)^2,
\]

and we want to perform a T-duality on \(z\) imposing the condition

\[
\ast d\tilde{z} = dz - \tau \tanh \Theta j_c.
\]

An important point is that in general the current \(dz - \tau \tanh \Theta j_c\) does not commute with all the currents \(J\). This is reflected by the fact that some of the components of \(J\) depend explicitly on \(z\) and not only on the differential \(dz\). For this reason it is necessary to introduce a suitable group-valued function \(h : \Sigma \rightarrow G\) and perform a gauge transformation:

\[
J' = h^{-1} Jh + h^{-1} dh,
\]

so that the new family of flat currents \(J'(\zeta)\) does not depend explicitly on \(z\) and can be dualized to \(\tilde{J}^0(\zeta)\).

\[
\tilde{J}^0(\zeta) = h^{-1} dh - \Lambda(\zeta) h^{-1} j h|_{dz=\ast d\xi-f_i(u)du^i} = \tilde{J}(0) - \Lambda(\zeta) \tilde{j}.
\]

Moreover, the zero mode depends explicitly on \(h\) which is a non-local object when expressed in terms of \(J\).

**Proof.** The explicit form of \(h\) can be found using the fact that \(g \in G \times U(1)\) can be written as in equation (2.21):

\[
g = k \exp[iz \tanh \Theta (T_c + T_{r+1})],
\]

where \(k \in G\) does not depend on \(z\). Then left and right currents read

\[
j = e^{iz \tanh \Theta (T_c + T_{r+1})} (k^{-1} \, dk + \tau \tanh \Theta \, dz (T_c + T_{r+1})) \, e^{z \tanh \Theta (T_c + T_{r+1})},
\]

\[
j = -dkk^{-1} - \tau \tanh \Theta \, dz (k T_c k^{-1} + T_{r+1}).
\]

The right-moving currents \(\tilde{j}\) do not depend on \(z\); this is not the case for \(j\). If we choose

\[
h = \exp[-iz \tanh \Theta (T_c + T_{r+1})],
\]

it is immediate to verify that

\[
J' = -\Lambda(\zeta) (k^{-1} \, dk + \tau \tanh \Theta \, dz (T_c + T_{r+1})) - \tau \, dz \tanh \Theta (T_c + T_{r+1})
\]

only depends on the differential \(dz\) and can be T-dualized, resulting in

\[
\tilde{J}' = -\tau (\Lambda(\zeta) + 1)(\ast d\xi + \tau \tanh \Theta j_c) \tanh \Theta (T_c + T_{r+1}) - \Lambda(\zeta) k^{-1} \, dk.
\]

The presence of the zero mode \(\tilde{J}(0)\) is related to the fact that \(\tilde{J}'\) cannot be understood in terms of a Noether current resulting from an isometry of the T-dual metric. Note that in the
T-dual model, we thus constructed an additional, non-local conserved current $\tilde{J}(\zeta)$ which does not stem from an isometry of the metric, but from a symmetry of the EOM. The integrability of the T-dual model can thus be ascertained via the existence of an infinite family of conserved charges constructed from the conserved current which is T-dual to the Lax current of the original model. $\tilde{J}(\zeta)$ on the other hand can be interpreted geometrically as a Noether current.

Observe that the flatness of $\tilde{J}^{(0)}$ implies the flatness of $\tilde{J}^{(0)}(0)$:

$$d\tilde{J}^{(0)} + \tilde{J}^{(0)} \wedge \tilde{J}^{(0)} = 0.$$  \tag{2.59}

This means that both $\tilde{J}(\zeta)$ and $\tilde{J}^{(0)}$ can be written in terms of Wilson lines $\tilde{W}$ and $\tilde{W}^{(0)}$:

$$\tilde{J}(\zeta) = \tilde{W}(\zeta)^{-1} d\tilde{W}(\zeta), \quad \tilde{J}^{(0)} = (\tilde{W}^{(0)})^{-1} d\tilde{W}^{(0)},$$  \tag{2.60}

where $\tilde{W}(\zeta) = P \left\{ \exp \left[ \int_{\mathcal{C}} \tilde{J}(\zeta) \right] \right\}$, $\tilde{W}^{(0)} = P \left\{ \exp \left[ \int_{\mathcal{C}} \tilde{J}^{(0)} \right] \right\}$, \tag{2.61}

and $\mathcal{C}$ is a path in $\Sigma$. Note that $\tilde{W}^{(0)}$ is the first term in the development in $\zeta$ of $\tilde{W}(\zeta)$:

$$\tilde{W}^{(0)} = \tilde{W}(\zeta) |_{\zeta = 0}.$$  \tag{2.62}

The flatness of $\tilde{J}$ is all we need to define a one-parameter family of conserved charges:

$$\tilde{Q}(t; \zeta) = \tilde{W}(\infty, t - \infty, t; \zeta) = P \left\{ \exp \left[ \int_{-\infty}^{\infty} \tilde{J}(\xi, t; \zeta) d\xi \right] \right\},$$  \tag{2.63}

which are in general non-local. It is the existence of this infinite family of conserved charges which makes the model integrable.

These charges and the transformations they generate via Poisson brackets (which form a $\hat{g}$ algebra) can be understood in terms of symmetries of the EOM and a hierarchy of first-order equations (see appendix B).

3. Examples

In this section, we put the general formalism developed in section 2 into practice by explicitly working out three examples. In section 3.1, we discuss the example of the squashed three-sphere, a background appearing frequently in the context of conformal field theory applications, black holes in string theory and the AdS/CFT correspondence (squashed giant gravitons). In an analogous manner, section 3.2 treats the warped anti-de Sitter space which has been featured prominently in the discussion of topologically massive gravity. Differing slightly from the other examples because of the T-duality being performed in a non-compact direction, section 3.3 spells out the Lax currents of the Schrödinger spacetime $\text{Sch}_3$, a background which has met with sustained interest in the context of strongly coupled non-relativistic quantum field theories.

3.1. Example one: the squashed three-sphere

As a first application of this general construction let us consider the case of the squashed three-sphere. In the simplest case, the geometry of $\text{SU}(2) \times U(1)$ can be understood as the fibration

$$S^1 \times S^1 \rightarrow S^3 \times S^1 \quad \downarrow \quad S^2$$  \tag{3.1}

$$9$$
where one of the directions in the torus fibration is the Hopf fiber in $S^3$. If we instead consider an $S^1$ sub-bundle $A$ of the torus, obtained as a rational linear combination of the two $S^1$ directions above, we obtain the fibration

$$A \to S^3 \times S^1 \downarrow \text{SqS}^3$$

where $\text{SqS}^3$ is the squashed three-sphere.

It is convenient to choose a coordinate system in which the group element $g \in SU(2) \times U(1)$ is written as

$$g(\phi, \theta, \psi, y) = e^{i\phi T_3} e^{i\theta T_2} e^{i\psi T_3} e^{2iyT_4},$$

where $\Theta_1$ is a real parameter and

$$T_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_4 = \frac{1}{2\sinh\Theta_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are the generators of the algebra $su(2) \oplus u(1)$:

$$[T_1, T_2] = iT_3, \quad [T_2, T_3] = iT_1, \quad [T_3, T_1] = iT_2, \quad [T_4, T_a] = 0,$$

with scalar product

$$2\text{Tr}[T_a T_b] = \begin{cases} 1 & \text{if } a = b = 1, 2, 3 \\ \sinh^{-2}\Theta_1 & \text{if } a = b = 4 \\ 0 & \text{if } a \neq b. \end{cases}$$

Following [37] we introduce the variables $\alpha$ and $z$ as in equation (2.20):

$$\alpha = \psi - 2y, \quad z = \frac{2y}{\tanh\Theta}.$$  

The resulting metric on $SU(2) \times U(1)$ ($ds^2 = 2\text{Tr}[dg\,dg^{-1}]$) takes the form

$$ds^2 = \left[ d\theta^2 + \sin^2\theta \, d\phi^2 + \frac{1}{\cosh^2\Theta} (d\alpha + \cos\theta \, d\phi)^2 \right] + (dz + \tanh\Theta \, (d\alpha + \cos\theta \, d\phi))^2,$$

which describes a fibration with fiber $z$, as in equation (2.1) where $u' = \theta, \phi, \alpha$ and $G_{ij}$ is the metric of a squashed three-sphere. Performing the T-duality on $z$ leads to the condition

$$*d\tilde{z} = dz + \tanh\Theta \, (d\alpha + \cos\theta \, d\phi),$$

and substituting this into the action we obtain the metric in equation (2.14):

$$\tilde{ds}^2 = \left[ d\theta^2 + \sin^2\theta \, d\phi^2 + \frac{1}{\cosh^2\Theta} (d\alpha + \cos\theta \, d\phi)^2 \right] + d\tilde{z}^2,$$

where $\tilde{z}$ is the dual variable. This is precisely the metric on $\text{SqS}^3 \times S^1$. Observe that by construction, the initial $SU(2) \times SU(2) \times U(1)$ isometry group has been broken to $SU(2) \times U(1)^\ast$. The corresponding Killing vectors are

$$k^3 = \partial_\alpha, \quad k^4 = \partial_{\tilde{z}}.$$
\[ k^1 = \sin \phi \partial_\phi + \cos \phi \frac{\cos \theta}{\sin \theta} \partial_\theta - \frac{\cos \phi}{\sin \theta} \partial_t, \]  
\[ k^2 = \cos \phi \partial_\phi - \sin \phi \frac{\cos \theta}{\sin \theta} \partial_\theta + \frac{\sin \phi}{\sin \theta} \partial_t, \]  
\[ k^3 = \partial_\phi, \]  
\tag{3.11c} \tag{3.11d} \tag{3.11e}

The Killing vectors do not depend on the deformation parameter \( \Theta \) and are the same as for the initial three-sphere.

Our main result is that the initial symmetry can be restored and promoted to affine\( \hat{\mathfrak{su}}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \) thanks to the presence of non-local charges that cannot be found via a standard Noether construction.

In order to realize this symmetry explicitly, we start with the conserved currents of the initial \( \mathfrak{su}(2) \times U(1) \) PCM. In terms of variables \( \theta, \phi, \alpha, z \), the conserved currents \( j \) and \( \bar{j} \) are

\[ j^1 = t (\cos(\alpha + \tanh \Theta z) \sin \phi \partial_\phi - \sin(\alpha + \tanh \Theta z) \partial_t), \]  
\[ j^2 = t (\sin(\alpha + \tanh \Theta z) \sin \phi \partial_\phi + \cos(\alpha + \tanh \Theta z) \partial_t), \]  
\[ j^3 = t (\partial_\alpha + \cos \phi \partial_\phi + \tanh \Theta \partial_z), \]  
\[ j^4 = t \tanh \Theta \partial_z, \]  
\tag{3.12a} \tag{3.12b} \tag{3.12c} \tag{3.12d}

and

\[ \bar{j}^1 = t (\cos \phi \sin \theta \partial_\phi - \sin \phi \partial_t + \tanh \Theta \cos \phi \sin \theta \partial_z), \]  
\[ \bar{j}^2 = -t (\sin \phi \sin \theta \partial_\phi + \cos \phi \partial_t + \tanh \Theta \sin \phi \sin \theta \partial_z), \]  
\[ \bar{j}^3 = -t (\cos \phi \partial_\phi + \partial_\theta + \tanh \Theta \cos \theta \partial_z), \]  
\[ \bar{j}^4 = -t \tanh \Theta \partial_z, \]  
\tag{3.12e} \tag{3.12f} \tag{3.12g} \tag{3.12h}

where \( j = g^{-1} dg \) and \( \bar{j} = -dgg^{-1} \) have been decomposed on the generators \( T_a \) of the algebra introduced in equation (3.4)\(^7\).

We can now apply the procedure described in section 2.3 to obtain the T-dual currents for the metric with the metric in equation (3.10). In order to impose the condition in equation (3.9)\((\ast d^2 = dz + \tanh \Theta (\partial \alpha + \cos \phi \partial \phi))\) on the \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \) Lax currents, we first have to perform a gauge transformation since \( j^3 \) (appearing in the T-duality transformation) does not commute with the currents \( j^1 \) and \( j^2 \) in equation (3.1), as one can see in this coordinate system since they depend explicitly on the variable \( z \). Instead of \( J(\xi) \), consider the flat current

\[ J'(\xi) = h^{-1} J(\xi) h + h^{-1} dh, \]  
\tag{3.14}

where\(^8\)

\[ h = \exp \left[ -t (\alpha + \tanh \Theta z) T_3 \right]. \]  
\tag{3.15}

\(^7\) Since the currents are maps \( j : \Sigma \to g \), they can be decomposed on a basis \( T_a \) of \( g \):

\[ j(x, t) = \sum_{a=1}^{\dim G} j^a(x, t) T_a, \quad \bar{j}(x, t) = \sum_{a=1}^{\dim G} \bar{j}^a(x, t) T_a. \]  
\tag{3.13}

\(^8\) Note that this transformation differs from the one in equation (2.56) and it has been chosen because it leads to simpler expressions for the currents in the new gauge.
Explicitly,
\[ J_1^1(\zeta) = -i \Lambda(\zeta) \sin \theta \, d\phi, \]  
\[ J_1^2(\zeta) = -i \Lambda(\zeta) \, d\theta, \]  
\[ J_1^3(\zeta) = -i \left[ (d\alpha + \tanh \Theta \, d\zeta) + \Lambda(\zeta) \left( d\alpha + \cos \theta \, d\phi + \tanh \Theta \, d\zeta \right) \right], \]  
\[ J_1^4(\zeta) = -i \tanh \Theta \Lambda(\zeta) \, d\zeta. \]  

We can now impose the T-duality condition in equation (3.9):
\[ \ast d\tilde{z} = d\zeta + \tanh \Theta (d\alpha + \cos \theta \, d\phi), \]  
and find
\[ \tilde{J}_1^1(\zeta) = -i \Lambda(\zeta) \sin \theta \, d\phi, \]  
\[ \tilde{J}_1^2(\zeta) = -i \Lambda(\zeta) \, d\theta, \]  
\[ \tilde{J}_1^3(\zeta) = -i \left[ (1 + \Lambda(\zeta)) \left( \frac{d\alpha + \cos \theta \, d\phi}{\cosh^2 \Theta} + \tanh \Theta \ast d\tilde{z} \right) - \cos \theta \, d\phi \right], \]  
\[ \tilde{J}_1^4(\zeta) = -i \tanh \Theta \Lambda(\zeta) (\ast d\tilde{z} - \tanh \Theta (d\alpha + \cos \theta \, d\phi)), \]  
and
\[ \tilde{J}_1^5(\zeta) = \left. \frac{1}{\cosh^2 \Theta} \cos \phi \sin \theta \, d\alpha - \sin \phi \, d\theta \right. \]  
\[ + \tanh^2 \Theta \cos \phi \sin \theta \cos \theta \, d\phi + \tan h \Theta \cos \phi \sin \theta \ast d\tilde{z} \right], \]  
\[ \tilde{J}_1^6(\zeta) = i \Lambda(\zeta) \left[ \frac{1}{\cosh^2 \Theta} \sin \phi \sin \theta \, d\alpha + \cos \phi \, d\theta - \tanh^2 \Theta \sin \phi \sin \theta \cos \theta \, d\phi \right. \]  
\[ + \tan h \Theta \sin \phi \sin \theta \ast d\tilde{z} \right], \]  
\[ \tilde{J}_1^7(\zeta) = i \Lambda(\zeta) \left[ \frac{1}{\cosh^2 \Theta} \cos \theta \, d\alpha + (1 - \tanh^2 \Theta \cos^2 \theta) \, d\phi + \tanh \Theta \cos \theta \ast d\tilde{z} \right], \]  
\[ \tilde{J}_1^8(\zeta) = i \tan h \Theta \Lambda(\zeta) (\ast d\tilde{z} - \tan h \Theta (d\alpha + \cos \theta \, d\phi)). \]  

Note that the power series expansion in \( \zeta \) of the current \( \tilde{J}(\zeta) \) has a zero-order component:
\[ \tilde{J}^{(0)} = h^{-1} dh = -i (d\alpha + \tanh \Theta \ast d\tilde{z} - \tanh^2 \Theta (d\alpha + \cos \theta \, d\phi)) T_3. \]

Note also that while \( \tilde{J}(\zeta) \) generates non-local charges which stem from a symmetry of the EOM which is not an isometry of the metric, \( \tilde{J}(\zeta) \) descends from an isometry of the metric and thus has a geometrical interpretation as a Noether current.

The \( S^2 \times S^1 \) geometry and currents can be found in the limit \( \Theta \rightarrow \infty \).
3.2. Example two: warped anti-de Sitter space

In a completely analogous fashion to the last example, we can instead start from the group $\text{SL}_2(\mathbb{R}) \times \text{U}(1)$. Also in this case we will see how the $\text{SL}_2(\mathbb{R})^2 \times \text{U}(1)$ isometry is promoted to affine $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{u}(1)$.

It is convenient to choose a coordinate system in which the group element $g \in \text{SL}_2(\mathbb{R}) \times \text{U}(1)$ is written as

$$g(\tau, \omega, \sigma, y) = e^{-\tau T_1} e^{\omega T_3} e^{i\sigma T_4} e^{2y T_4},$$

(3.20)

where $\Theta$ is a real parameter and

$$T_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$T_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad T_4 = \frac{1}{2 \sinh \Theta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(3.21)

are the generators of the algebra $\mathfrak{sl}_2$:

$$[T_1, T_2] = -i T_3, \quad [T_2, T_3] = i T_1, \quad [T_3, T_1] = i T_2, \quad [T_4, T_a] = 0,$$

(3.22)

with scalar product

$$-2 \text{Tr}[T_a T_b] = \begin{cases} 1 & \text{if } a = b = 1, 2 \\ -1 \sinh^{-2} \Theta & \text{if } a = b = 3 \\ 0 & \text{if } a \neq b. \end{cases}$$

(3.23)

Following [37] we introduce the variables $\beta$ and $z$:

$$\beta = \sigma - 2y, \quad z = \frac{2y}{\tanh \Theta},$$

(3.24)

The resulting metric on $\text{SL}_2(\mathbb{R}) \times \text{U}(1)$ takes the form

$$ds^2 = \left[ d\omega^2 - \cosh^2 \omega d\tau^2 + \frac{1}{\cosh^2 \Theta} (d\beta + \sinh \omega d\tau)^2 \right]$$

$$+ (dz + \tanh \Theta (d\beta + \sinh \omega d\tau))^2,$$

(3.25)

which describes a fibration with fiber $z$, as in equation (2.1) where $u^i = \{\omega, \tau, \beta\}$, and $G_{ij}$ is the metric of a warped anti-de Sitter space. Performing the T-duality on $z$ leads to the condition

$$*dz = dz + \tanh \Theta (d\beta + \sinh \omega d\tau).$$

(3.26)

Substituting this into the action we obtain the following metric:

$$\tilde{ds}^2 = \left[ d\omega^2 - \cosh^2 \omega d\tau^2 + \frac{1}{\cosh^2 \Theta} (d\beta + \sinh \omega d\tau)^2 \right] + d\tilde{z}^2,$$

(3.27)

where $\tilde{z}$ is the dual variable. This is precisely the metric on $W\text{AdS}_3 \times S^1$, where by construction the initial $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \times \text{U}(1)$ isometry group has been broken to $\text{SL}_2(\mathbb{R}) \times \text{U}(1)^2$. The corresponding Killing vectors (that do not depend on the deformation parameter $\Theta$) are

$$k^2 = \partial_{\beta},$$

(3.28a)

$$k^4 = \partial_{\tilde{z}},$$

(3.28b)

$$\overline{k}^1 = \partial_{\tau},$$

(3.28c)
\[ k^2 = \cos \tau \tanh \omega \partial_\tau + \frac{\cos \tau}{\cosh \omega} \partial_\beta + \sin \tau \partial_\omega, \]  
\[ k^3 = \sin \tau \tanh \omega \partial_\tau + \frac{\sin \tau}{\cosh \omega} \partial_\beta - \cos \tau \partial_\omega. \]  

Once more, the initial symmetry can be restored and promoted to affine \( \hat{sl}_2 \oplus \hat{sl}_2 \oplus \hat{u}(1) \) thanks to the presence of a non-local current that cannot be found via a standard Noether construction.

In the following, we will construct this symmetry explicitly. In terms of the variables \( \omega, \tau, \beta, z \), the conserved currents before the T-duality \( j \) and \( \bar{j} \) are given by

\[ j^1 = - \cosh(\beta + \tanh \Theta z) \cosh \omega \mathrm{d}\tau + \sinh(\beta + \tanh \Theta z) \mathrm{d}\omega, \]  
\[ j^2 = t (\mathrm{d}\beta + \sinh \omega \mathrm{d}\tau + \tanh \Theta \mathrm{d}z), \]  
\[ j^3 = \cosh(\beta + \tanh \Theta z) \mathrm{d}\omega - \sinh(\beta + \tanh \Theta z) \cosh \omega \mathrm{d}\tau, \]  
\[ j^4 = t \tanh \Theta \mathrm{d}z, \]

and

\[ \bar{j}^1 = \mathrm{d}\tau - \sinh \omega \mathrm{d}\beta - \tanh \Theta \sinh \omega \mathrm{d}z, \]  
\[ \bar{j}^2 = -t (\cos \tau \cosh \omega \mathrm{d}\beta + \sin \tau \cosh \omega \mathrm{d}\omega + \cos \tau \cosh \Theta \mathrm{d}z), \]  
\[ \bar{j}^3 = - \cos \tau \mathrm{d}\omega + \sin \tau \cosh \omega \mathrm{d}\beta + \sin \tau \cosh \Theta \mathrm{d}z, \]  
\[ \bar{j}^4 = -t \tanh \Theta \mathrm{d}z. \]

where \( j = g^{-1} \mathrm{d}g \) and \( \bar{j} = -
\mathrm{d}gg^{-1} \) have been decomposed on the generators \( T_\alpha \) of the algebra introduced in equation (3.21).

We can again apply the procedure described above to the T-duality transformation that leads to the WAdS metric. In order to impose the condition in equation (3.26) \([\ast \mathrm{d}z = \mathrm{d}z + \tanh \Theta (\mathrm{d}\beta + \sinh \omega \mathrm{d}\tau)]\) to the Lax currents for \( \hat{sl}_2 \oplus \hat{sl}_2 \oplus \hat{u}(1) \), we first have to perform a gauge transformation

\[ J'(\xi) = h^{-1} J(\xi) h + h^{-1} \mathrm{d}h, \]

where

\[ h = \exp [-t (\beta + \tanh \Theta z) T_2]. \]  

Explicitly,

\[ J'^1(\xi) = \Lambda(\xi) \cosh \omega \mathrm{d}\tau, \]  
\[ J'^2(\xi) = -t [(\mathrm{d}\beta + \tanh \Theta \mathrm{d}z) + \Lambda(\xi) (\mathrm{d}\beta + \sinh \omega \mathrm{d}\tau + \tanh \Theta \mathrm{d}z)], \]  
\[ J'^3(\xi) = -\Lambda(\xi) \mathrm{d}\omega, \]  
\[ J'^4(\xi) = -t \tanh \Theta \Lambda(\xi) \mathrm{d}z. \]

We can now impose the T-duality condition in equation (3.26)

\[ \ast \mathrm{d}z = \mathrm{d}z + \tanh \Theta (\mathrm{d}\beta + \sinh \omega \mathrm{d}\tau) \]

and find

\[ \bar{J}'^1(\xi) = \Lambda(\xi) \cosh \omega \mathrm{d}\tau, \]  

\[ \bar{J}'^2(\xi) = -t \Lambda(\xi) (\mathrm{d}\beta + \tanh \Theta \mathrm{d}z), \]  
\[ \bar{J}'^3(\xi) = -\Lambda(\xi) \mathrm{d}\omega, \]  
\[ \bar{J}'^4(\xi) = -t \tanh \Theta \Lambda(\xi) \mathrm{d}z. \]
\[ \tilde{J}^2(\zeta) = -i \left[ 1 + \Lambda(\zeta) \right] \left( \frac{d\beta + \sinh \omega \, dr}{\cosh \Theta} + \tanh \Theta \ast \tilde{d} \right) - \sinh \omega \, dr \right], \quad (3.34b) \\
\tilde{J}^3(\zeta) = -\Lambda(\zeta) \, d\omega, \quad (3.34c) \\
\tilde{J}^4(\zeta) = -i \tanh \Theta(\ast \tilde{d} - \tanh (d\beta + \sinh \omega \, dr)), \quad (3.34d) \\
\text{and} \\
\tilde{J}^1(\zeta) = \Lambda(\zeta) \left[ -(1 + \tanh^2 \Theta \sinh^2 \omega) \, d\tau + \frac{1}{\cosh^2 \Theta} \sinh \omega \, d\beta + \tanh \Theta \sinh \omega \ast \tilde{d} \right], \quad (3.34e) \\
\tilde{J}^2(\zeta) = i \Lambda(\zeta) \left[ \frac{1}{\cosh^2 \Theta} \cos \tau \cosh \omega \, d\beta + \sin \tau \, d\omega \\
- \cos \tau \cosh \omega \sinh \tau \, \Theta \, dr + \cos \tau \cosh \omega \tanh \Theta \ast \tilde{d} \right], \quad (3.34f) \\
\tilde{J}^3(\zeta) = -\Lambda(\zeta) \left[ \frac{1}{\cosh^2 \Theta} \sin \tau \cosh \omega \, d\beta - \cos \tau \, d\omega \\
- \sin \tau \cosh \omega \sinh \tau \, \Theta \, dr + \sin \tau \cosh \omega \tanh \Theta \ast \tilde{d} \right], \quad (3.34g) \\
\tilde{J}^4(\zeta) = i \tanh \Theta \Lambda(\zeta) \left( \ast \tilde{d} - \tanh \Theta (d\beta + \sinh \omega \, dr) \right). \quad (3.34h) \\
\text{Note that also the power series expansion in } \zeta \text{ of the current } \tilde{J}(\zeta) \text{ has a zero-order component:} \\
\tilde{J}^{(0)} = h^{-1} \, dh = i(d\beta + \tanh \Theta \ast \tilde{d} - \tanh^2 \Theta (d\beta + \sinh \omega \, dr)) T_2. \quad (3.35) \\
\text{Just like in the previous example, the current } \tilde{J}(\zeta) \text{ can be regarded as a Noether current, while this is not the case for } \tilde{J}'(\zeta). \\
The limit } \Theta \to \infty \text{ describes the currents in the geometry AdS}_2 \times S^1. \\

3.3. Example three: Schrödinger spacetime \\
Cartan subgroups of non-compact groups are in general not related by inner automorphisms. This leaves more freedom in the choice of the direction in which to perform the T-duality. The Schrödinger spacetime is another example of geometry that can be obtained starting from AdS_3. The main difference with respect to the previous example is that the T–duality is performed in a non-compact direction. The general form of a (d + 1)-dimensional Schrödinger spacetime is given by \\
\[ ds^2 = -\frac{b^2 \, dx_{-2}^2}{r^4} + \frac{2 \, dx^- \, dx^+ + dx^i \, dx^i + dr^2}{r^2} \quad (3.36) \]
\text{with } i = 1, \ldots, d - 2 \text{. A list of all the isometries of this metric can be found in [29]. It was initially believed that this background could be holographically dual to a critical non-relativistic system in } (d - 1) \text{ spacetime dimensions having the same symmetries. It was later shown that one requires the coordinate } x^+ \text{ to be a compact null direction, which introduces a series of complications when looking at quantum corrections [28]. Nonetheless, an holographic dictionary has been established for the case in which } x^+ \text{ is non-compact and it has been shown}
that the space described by equation (3.36) is dual to a $d$-dimensional QFT which is non-local in the $x^+$ direction.

We will look at $d = 2$ which corresponds to Sch$_3$. We start by considering a choice of coordinates where the group element $g \in \text{GL}_2(\mathbb{R})$ is written as
\[
g(r, u^+, x^-, z) = e^{x^-(T_3 - T_4)} e^{i u^+ T_2} e^{-i u^+ (T_2 + T_1)} e^{2 i z T_4},
\]
where
\[
T_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]
\[
T_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad T_4 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
are the generators of the algebra $\text{gl}_2$, with scalar product
\[
-2 \text{Tr}[T_a T_b] = \begin{cases} 1 & \text{if } a = b, a = 1, 2, 4 \\ -1 & \text{if } a = b, a = 3 \\ 0 & \text{if } a \neq b. \end{cases}
\]
In this case we do not introduce the parameter $\Theta$, since we are interested in a non-compact direction and this parameter could be reabsorbed by a coordinate redefinition. Let us now introduce the coordinate $x^+$:
\[
x^+ = x^+ + z.
\]
The resulting metric on $\text{GL}_2(\mathbb{R})$ takes the form
\[
ds^2 = \left[ \frac{dr^2}{r^2} + \frac{dx^+ dx^-}{r^2} - \frac{(dx^-)^2}{4r^4} \right] + \left( dz + \frac{dx^-}{2r^2} \right)^2,
\]
which describes a fibration with fiber $z$, as in equation (2.1) where $u^i = \{r, x^+, x^-\}$, and $G_\theta$ is the metric of a three-dimensional Schrödinger spacetime. Performing the T-duality on $z$ leads to the condition
\[
\|dz\|^2 = dz + \frac{dx^-}{2r^2}.
\]
Substituting this into the action we obtain the metric
\[
\frac{r^2}{2} \partial \tilde{s} = \left[ \frac{dr^2}{r^2} + \frac{dx^+ dx^-}{r^2} - \frac{(dx^-)^2}{4r^4} \right] + dz^2,
\]
where $z^2$ is the dual variable. This is precisely the metric on Sch$_3 \times S^1$, where by construction the initial $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \times \mathbb{R}$ isometry group has been broken to $\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2$. The corresponding Killing vectors are
\[
\begin{align*}
k^- &= \partial_x, & k^4 &= \partial_z, \\
\tilde{k}^- &= r x^- \partial_r - r^2 \partial_{x^+} + (x^-)^2 \partial_{x^-}, & \tilde{k}^4 &= r \partial_r + 2x^- \partial_{x^-}.
\end{align*}
\]
Once more, the initial symmetry can be restored and promoted to affine $\tilde{\mathfrak{sl}}_2 \oplus \tilde{\mathfrak{sl}}_2 \oplus \tilde{\mathfrak{u}}(1)$ thanks to the presence of a non-local current that cannot be found via a standard Noether construction.

Before we move on to the construction of the currents it is useful to comment on the norms of the explicit Killing vectors for the metrics obtained via the change of variables and T-duality. In the initial metric $ds^2 \propto \text{Tr}[dg dg^{-1}]$, we have
\[
\|\partial_r\|^2 = 0, \quad \|\partial_x\|^2 = 0, \quad \|\partial_z\|^2 = 1.
\]
After the change of coordinates $u^* = x^* + z$, the variable $x^*$ describes a null direction and $z$ remains space-like:

$$\|\partial_{x^*}\|^2 = 0, \quad \|\partial_z\|^2 = 1.$$  \hfill (3.46)

The T-duality changes the nature of $x^-$, which in the metric in equation (3.41) is time-like:

$$\|\partial_{x^-}\|^2 = 0, \quad \|\partial_z\|^2 = -\frac{1}{4r^4}, \quad \|\partial_{x^+}\|^2 = 1.$$  \hfill (3.47)

This can be changed to $\|\partial_{x^-}\|^2 = 1/(4r^4)$ via a double analytic continuation $(x^+, x^-) \mapsto (ix^+, -ix^-)$ which does not affect the $(-, +, +, +)$ signature of the metric.

Let us now move to the currents. In terms of the variables $\{r, x^+, x^-, z\}$, the conserved currents before the T-duality $j$ and $\bar{j}$ are given by

$$j^+ = -\frac{2\Lambda}{r^2} \frac{dz}{r}, \quad j^- = \frac{2\Lambda}{r^2} \frac{dx^-}{r}, \quad j^2 = 2\Lambda \left( \frac{dr}{r} + \frac{z}{r} \frac{dx^-}{r} + \frac{x^+}{r} \frac{dx^-}{r} \right), \quad j^4 = 2\Lambda \frac{dz}{r^2},$$  \hfill (3.48a)

$$\bar{j}^+ = 2\Lambda \left( \frac{dz}{r^2} + \frac{dx^+}{r^2} \right), \quad \bar{j}^- = -2\Lambda \left( \frac{dr}{r} + \frac{z}{r} \frac{dx^-}{r} + \frac{x^-}{r} \frac{dx^-}{r} \right), \quad \bar{j}^2 = -2\Lambda \left( \frac{dr}{r} + \frac{dz}{r} + \frac{dx^+}{r} \right), \quad \bar{j}^4 = -2\Lambda \frac{dz}{r^2},$$  \hfill (3.48f)

where $j^\pm = j^1 \pm j^3$.

Using a by now familiar procedure, in order to impose the condition in equation (3.42) $(\ast d\xi = dz + dx^-/(2r^2))$ on the Lax currents for $sl_2 \oplus sl_2 \oplus u(1)$, we first have to perform a gauge transformation

$$J'(\zeta) = h^{-1} J(\zeta) h + h^{-1} dh,$$  \hfill (3.49)

where

$$h = \exp[i(x^* + z)(T_1 + T_3)].$$  \hfill (3.50)

Explicitly,

$$J'^- = 2\Lambda \frac{dx^-}{r^2}, \quad J'^2 = -2\Lambda \frac{dr}{r}, \quad J'^+ = 2\Lambda (1 + \Lambda(\zeta)) \frac{dz + dx^+}{r},$$  \hfill (3.51a)

$$\bar{J}'^+ = 2\Lambda \frac{dz}{r^2} + \frac{dx^+}{r^2}, \quad \bar{J}'^- = -2\Lambda \frac{dr}{r}, \quad \bar{J}'^2 = 2\Lambda \left( \frac{dr}{r} + \frac{dz}{r^2} + \frac{dx^+}{r^2} \right).$$  \hfill (3.51c)
\( J'^4(\zeta) = -2t \Lambda(\zeta) \, d\zeta. \) (3.51d)

We can now impose the T-duality condition in equation (3.42),
\( *d\tilde{\zeta} = d\zeta + \frac{dx^-}{2r^2}, \) (3.52)
and find
\[
\begin{align*}
\tilde{J}^- &= 2t \Lambda(\zeta) \frac{dx^-}{r^2}, \\
\tilde{J}^0 &= -2t \Lambda(\zeta) \frac{dr}{r}, \\
\tilde{J}^+ &= 2t (1 + \Lambda(\zeta)) \left( *d\tilde{\zeta} - \frac{dx^-}{2r^2} + dx^+ \right), \\
\tilde{J}'^4 &= -2t \Lambda(\zeta) \left( *d\tilde{\zeta} - \frac{dx^-}{2r^2} \right),
\end{align*}
\]
and
\[
\begin{align*}
\tilde{J}^- &= 2t \Lambda(\zeta) \left( \frac{2x^-}{r^2} dr - dx^- + \frac{(x^-)^2}{r^2} \left( *d\tilde{\zeta} - \frac{dx^-}{2r^2} + dx^+ \right) \right), \\
\tilde{J}^0 &= 2t \Lambda(\zeta) \left( \frac{dr}{r} + \frac{x^-}{r^2} d\zeta + \frac{x^-}{r^2} dx^+ \right), \\
\tilde{J}^+ &= -2t \frac{r}{t^2} \Lambda(\zeta) \left( *d\tilde{\zeta} - \frac{dx^-}{2r^2} + dx^+ \right), \\
\tilde{J}'^4 &= 2t \Lambda(\zeta) \left( *d\tilde{\zeta} - \frac{dx^-}{2r^2} \right). 
\end{align*}
\] (3.53)

Just like in the previous example, the current \( \tilde{J}(\zeta) \) can be regarded as a Noether current, while this is not the case for \( \tilde{J}'(\zeta) \).

4. Conclusions

From a world-sheet point of view, T-duality is a linear transformation of the components of the currents. It preserves the integrable structure of the original model, at least at the classical level, and can thus be used to generate new integrable sigma models.

This mechanism is so powerful that it has allowed us to treat in a unified way three different examples, which emerge naturally as target space backgrounds in string theory and that can be used to further study certain types of black holes and through holography, non-relativistic quantum field theories.

Inspired by the fact that backgrounds containing squashed spheres can be obtained via T-duality from an \( \text{AdS}_3 \times S^3 \) background and that the corresponding sigma model is integrable, we showed by direct computation of the Lax pairs and the infinite set of conserved charges how the integrability emerges in the T-dual model for the cases of the squashed three-sphere, warped \( \text{AdS} \) and the Schrödinger spacetime.

It is very interesting that despite the fact that the isometry group of the T-dual model is only a subgroup of the original isometry group, the T-dual currents lead to the full \( \mathfrak{g} \oplus \mathfrak{g} \)
symmetry, realizing the ‘hidden’ symmetry just as in the cases discussed in [25]. In the T-dual model, the symmetry group can be promoted to an affine symmetry due to the emergence of non-local charges that cannot be obtained via a Noether construction. In this way we also generalize the hidden Yangian algebra found in [50].

It should be remarked that our analysis is purely classical, which is consistent with the fact that the algebra of the symmetries is a loop algebra (i.e. there is no central term). This changes once quantum corrections are taken into account. Nevertheless we expect the integrable structure to be preserved in the full supersymmetric sigma model.

The natural next step is to extend the analysis performed here to the case in which both RR fluxes and fermions are turned on. It would be important to understand whether the original symmetries can be realized in a non-local way in an analogous fashion and whether the promotion to the affine symmetry can still be realized. This will be the subject of a forthcoming publication [47].

Acknowledgments

We would like to thank Arkady Tseytlin for inspiring discussions and detailed comments on the manuscript. We furthermore would like to thank Konstadinos Sfetsos for correspondence. Moreover, DO and SR would like to thank the participants of the IPMU string theory group meetings for stimulating discussions. DO would like to thank Io Kawaguchi and Kentaroh Yoshiida for collaboration on a related topic. The research of DO and SR was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. LIU acknowledges the support of an STFC Postdoctoral Fellowship.

Appendix A. Hierarchy of variations for the principal chiral model

The PCM has an explicit $G \times G$ symmetry. The infinitesimal version of this symmetry implies that its action is invariant under

\[
\delta g = \epsilon g \quad \text{and} \quad \delta g = g \tau, \quad d\epsilon = 0, \quad d\tau = 0, \quad (A.1)
\]

where $\epsilon$ and $\tau$ are constants $\epsilon, \tau \in \mathfrak{g}$. Following [45] we can rewrite the left action $G_L$ $\delta g = \epsilon g$ in terms of a function $\eta : \Sigma \to \mathfrak{g}$:

\[
\delta g(x,t) = \epsilon g(x,t) = g(x,t)\eta(x,t) = \delta \eta g(x,t), \quad \eta(x,t) = g^{-1}(x,t)\epsilon g(x,t). \quad (A.2)
\]

The function $\eta$ is not constant on $\Sigma$ and this symmetry should not be confused with $G_R$.

The variation of the current $j = g^{-1} dg$ under $\delta \eta$ is

\[
\delta \eta \ j = d\eta + [j, \eta] = \nabla_j \eta. \quad (A.3)
\]

We thus conclude that $\delta \eta$ is a symmetry of the action if $\eta$ is covariantly constant with respect to $j$:

\[
\nabla_j \eta(x,t) = 0. \quad (A.4)
\]

In the spirit of integrability, instead of $\eta$, we introduce a one-parameter family of Lie algebra-valued functions $\eta_\zeta : \Sigma \to \mathfrak{g}$ which are covariantly constant with respect to the Lax current $J(\zeta)$:

\[
\nabla_j \eta_\zeta = d\eta_\zeta + [J(\zeta), \eta_\zeta] = 0. \quad (A.5)
\]

It follows that

\[
\nabla_j \eta_\zeta = \nabla_j \eta_\zeta + [j - J(\zeta), \eta_\zeta] = [(1 + \Lambda(\zeta)) j, \eta_\zeta] = \frac{1}{\zeta} [\Lambda(\zeta) j, \eta_\zeta] = -\frac{1}{\zeta} [\ast J(\zeta), \eta_\zeta] = \frac{1}{\zeta} * d\eta_\zeta, \quad (A.6)
\]

\[
\]
where we used the property in equation (2.38). This implies that the EOM \( \dd j = 0 \) are conserved under the variation \( \delta \eta \)

\[
\delta \eta (\dd j) = \dd (\delta \eta) = \dd (\nabla_j \eta) = \frac{1}{\xi} \dd (\ast \dd \eta) = 0. \tag{A.7}
\]

We thus have found a one-parameter family of symmetries of the EOM. The condition \( \nabla_j \eta = 0 \) can be rewritten in terms of a hierarchy of first-order equations. Expand \( \eta \) in powers of \( \xi \):

\[
\eta = \sum_{n=0}^{\infty} \xi^n \eta^{(n)}. \tag{A.8}
\]

Then \( \eta \) is covariantly constant with respect to \( J(\xi) = -\Lambda(\xi) j \) if and only if

\[
\dd \eta = [\Lambda(\xi) j, \eta] \tag{A.9}
\]

and, order by order

\[
\begin{align*}
\dd \eta^{(0)} &= 0, \\
\dd \eta^{(1)} &= [\ast j, \eta^{(0)}], \\
\dd \eta^{(2)} &= [\ast j, \eta^{(0)}] + [j, \eta^{(1)}], \\
&\vdots \\
\dd \eta^{(n)} &= \begin{cases} 
\sum_{k=0}^{n/2-1} [\ast j, \eta^{(2k+1)}] + [j, \eta^{(2k)}] & \text{if } n \text{ is even} \\
[\ast j, \eta^{(n-1)}] + \sum_{k=0}^{(n-1)/2-1} [\ast j, \eta^{(2k)}] + [j, \eta^{(2k+1)}] & \text{if } n \text{ is odd}
\end{cases} \tag{A.10d}
\end{align*}
\]

Note that \( \eta^{(0)} = \text{const} \) is not the \( \eta \) we started with, but reproduces the \( \tau \) from the symmetry of the right action.

Alternatively, the condition \( \nabla_j \eta = 0 \) is satisfied if we require

\[
\eta = W^{-1}(\xi) \tau W(\xi), \tag{A.11}
\]

where \( \epsilon \in \mathfrak{g} \) and \( W \) is defined by \( J = W^{-1} \dd W \) and is given explicitly in equation (2.41). Since \( \eta \) is a function of \( \epsilon \), we can write explicitly this dependence by introducing the variation operator

\[
\delta(\epsilon, \xi) = g \dd \eta = g W^{-1}(\xi) \tau W(\xi). \tag{A.12}
\]

Expanding \( \delta(\tau, \xi) \) in series of \( \xi \)

\[
\delta(\tau, \xi) = \sum_{n=0}^{\infty} \delta^{(n)}(\tau) \xi^n, \tag{A.13}
\]

one finds that the \( \delta^{(n)} \) form (half of) the loop algebra of the original algebra \( \mathfrak{g} \) (see [45]):

\[
[\delta^{(n)}(\tau_1), \delta^{(m)}(\tau_2)] = \delta^{(n+m)}([\tau_1, \tau_2]), \quad \forall n, m = 0, 1, \ldots \tag{A.14}
\]

The zero modes of the algebra describe the explicit symmetry \( G_R \) and not \( G_L \) as one could have expected. The algebra can be extended to a full \( \hat{\mathfrak{g}} \) as shown in [46]. The same procedure can be repeated for the right current \( J(\xi) \), thus leading to another infinite set of symmetries commuting with these ones. The full hidden symmetry algebra is then \( \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \).
Appendix B. Deformed hierarchy for the squashed groups

In section 2.3 we found that the Lax current \( \tilde{J}(\zeta) \) can be expanded as
\[
\tilde{J}(\zeta) = \tilde{J}^{(0)}(\zeta) - \Lambda(\zeta) \tilde{j}.
\] (B.1)

The presence of the zero mode changes the hierarchy of variations described in appendix A.

The equations of motion are invariant under the variation
\[
\delta \tilde{\eta}(\zeta) \equiv g \tilde{\eta}(\zeta),
\] (B.2)
where \( \tilde{\eta} \) is covariantly constant with respect to \( \tilde{J}'(\zeta) \):
\[
d\tilde{\eta}(\zeta) + [\tilde{J}'(\zeta), \tilde{\eta}(\zeta)] = 0.
\] (B.3)

Writing \( \tilde{J}'(\zeta) = \tilde{J}^{(0)}(\zeta) - \Lambda(\zeta) \tilde{j} \) we see that
\[
\nabla_{J^{\prime}(\zeta)} \tilde{\eta}(\zeta) = d\tilde{\eta}(\zeta) + [\tilde{J}'(\zeta), \tilde{\eta}(\zeta)] = \nabla\tilde{J}'(\zeta) \tilde{\eta}(\zeta) - \frac{1}{\Lambda(\zeta)} [\tilde{j}, \tilde{\eta}(\zeta)] = 0,
\] (B.4)
which can be expanded in powers of \( \zeta \) as follows:
\[
\nabla_{J^{\prime}(\zeta)} \tilde{\eta}(n)(\zeta) = \begin{cases} 
\frac{n}{2} - 1 \sum_{k=0}^{(n-1)/2-1} [\tilde{j}, \tilde{\eta}(2k+1)] + [\tilde{j}, \tilde{\eta}(2k)] & \text{if } n \text{ is even} \\
[\tilde{j}, \tilde{\eta}^{(n-1)}] + \sum_{k=0}^{(n-1)/2-1} [\tilde{j}, \tilde{\eta}(2k)] + [\tilde{j}, \tilde{\eta}^{(2k+1)}] & \text{if } n \text{ is odd}
\end{cases}
\] (B.5d)

Observe that the only difference with respect to the hierarchy in the PCM case obtained in equation (A.10) is that the differential has been traded for a covariant derivative with respect to \( \tilde{J}'(\zeta) \). In particular, \( \tilde{\eta}(0) \) is not a constant, but is covariantly constant. One can easily verify that the variation \( \tilde{\eta}(\zeta) \) can be written as
\[
\tilde{\eta}(\zeta) = \tilde{W}^{-1}(\zeta) \tau \tilde{\eta}(\zeta),
\] (B.6)
where \( \tau \in \mathfrak{g} \). At zero order in \( \zeta \) this becomes
\[
\tilde{\eta}(0)(x, t) = (\tilde{W}^{(0)}(x, t))^{-1} \tau \tilde{W}^{(0)}(x, t).
\] (B.7)

Appendix C. Geodesics on the squashed groups

The geodesics of a manifold can be identified with the trajectories of free point particles. In other words, they are the minima of the action
\[
S = \int_{\mathbb{R}} dt [G_{ij}(u(t)) \dot{u}^i(t) \dot{u}^j(t) + (\dot{z}(t) + f_i(u(t)) \dot{u}^i(t))^2],
\] (C.1)
which is the one-dimensional counterpart of the action we have considered in this note.

In the case of a squashed group, this reads
\[
S = \int_{\mathbb{R}} dt [-\text{Tr} [g^{-1} \dot{g}^2] + \tanh^2 \Theta \text{Tr} [g^{-1} \dot{g} T_C]^2].
\] (C.2)
where now $g$ is a map
\[ g : \mathbb{R} \to G. \]
(C.3)

The effect of the squashing is to break the left symmetry group $G$ to its Cartan subgroup $T \subset G$. From the variation
\[ \delta g = \eta g, \quad \eta : \mathbb{R} \to \mathfrak{g}, \]
(C.4)

we find the conserved right currents
\[
\frac{d}{dt} J(t) = \frac{d}{dt} \left[ gg^{-1} - \tanh^2 \Theta g T_C g^{-1} \text{Tr}[g^{-1} \dot{g} T_C] \right] = 0,
\]
while from the variation
\[ \delta g = g \eta, \quad \eta : \mathbb{R} \to \mathfrak{t}, \]
(C.6)

where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{g}$, we find the conservation of the left currents:
\[
\frac{d}{dt} J_\alpha = \frac{d}{dt} \text{Tr}[T_\alpha g^{-1} \dot{g}] = 0, \quad \alpha = 1, 2, \ldots, \text{rank}(G),
\]
(C.7)

where $T_\alpha$ are the generators of $\mathfrak{t}$.

These first-order equations can be integrated to give the trajectories $g = g(t)$ of a geodesic starting from a point $g(0)$. From the left current conservation we find
\[
g(t) = g(0) e^{L_t l_t} \quad l_t \in \mathbb{R}, \quad T_t \in \mathfrak{t},
\]
(C.8)

while from the right currents
\[
g(t) = g(0) e^{R t} e^{C \tanh^2 \Theta t R} \quad R \in \mathfrak{g}, l_C \in \mathbb{R}.
\]
(C.9)

These are to be compared with the usual geodesics on the group manifold $G$, which are obtained for $\Theta = 0$:
\[
g(t) = g(0) e^{L t} \quad L \in \mathfrak{g}.
\]
(C.10)

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