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Razumikhin Theorems on Polynomial Stability of Neutral Stochastic Pantograph Differential Equations with Markovian Switching

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Abstract: This paper investigates the polynomial stability of neutral stochastic pantograph differential equations with Markovian switching (NSPDEsMS). Firstly, under the local Lipschitz condition and a more general nonlinear growth condition, the existence and uniqueness of the global solution to the addressed NSPDEsMS is considered. Secondly, by adopting the Razumikhin approach, one new criterion on the $q$th moment polynomial stability of NSPDEsMS is established. Moreover, combining with the Chebyshev inequality and the Borel–Cantelli lemma, the almost sure polynomial stability of NSPDEsMS is examined. The results derived in this paper generalize the previous relevant ones. Finally, two examples are provided to illustrate the effectiveness of the theoretical work.

Keywords: nonlinear growth condition; Itô formula; global solution; Borel–Cantelli lemma

MSC: 60H10; 34K40; 37H30; 93E15

1. Introduction

Due to the existence of random disturbances, neutral stochastic differential equations (NSDEs) can be utilized to characterize those complicated systems such as population system, chemical reaction process, heating control systems, complex networks and other systems [1–7]. The structures and parameters of some systems may encounter unpredictable variations, so Markovian jump systems are introduced to depict these phenomena. During the past several decades, many scholars have been absorbed in the neutral stochastic differential equations with Markovian switching (NSDEsMS), and large amounts of interesting results have been acquired [8–10].

The pantograph system was presented by Ockendon and Tayler in 1971 [11], which could be seen as one important class of systems with unbounded delays. Recently, network systems with pantograph delays as one class of pantograph systems have received extensive attention. Particularly, in [12], the exponential stability of switching neural networks with pantograph delays was discussed by adopting the average dwell-time (ADT) technique and Lyapunov stability approach. In [13], global h-stability criteria for pantograph delay high-order inertial neural networks were examined by utilizing the non-reduced order method. In [14,15], periodic solutions and anti-periodic solutions of neural networks with pantograph delays were analyzed by means of differential inequality techniques. In [16,17], based on the comparison principle and some analysis techniques, control issues, such as the synchronization and passivity of neural networks with pantograph delays were investigated. On the other hand, by employing the stochastic Lyapunov method, the stability of linear or highly nonlinear stochastic pantograph equations were extensively investigated [18–20]. Moreover, the referent results were generalized to the stochastic pantograph differential equations (SPDEs) or neutral stochastic pantograph differential equations with Markovian switching (NSPDEsMS) [21–23].
The Razumikhin approach is one effective tool to deal with the stability issue of the time delayed system. This approach was initiated in [24, 25] and it was developed in various different systems, including discrete systems, impulsive systems and stochastic systems, and many publications have been reported [26–35]. In particular, the Razumikhin technique was also extensively applied to NSDEs. For instance, Mao [28] adopted the Razumikhin technique to investigate the mean-square moment exponential stability of NSDEs in [29]. Huang and Deng [30] used the Razumikhin technique to examine the asymptotic stability of NSDEs. By incorporating the stability with general decay rate, Pavlović and Janković [31] established new Razumikhin theorems, which may be specialized on the different types of stability. Moreover, Razumikhin techniques were generalized to NSDEs with Markovian switching [32, 33] and NSDEs with unbounded delays [34]. For NSPDEs, Yu [35] constructed the criterion on Razumikhin-type $p$th moment asymptotic stability and discussed the stability of the numerical solutions in virtue of the backward Euler method.

In addition, different from exponential stability, polynomial stability is also one class important stability. In [36], Mao considered the almost sure polynomial stability of the stochastic systems by using the semimartingale theory. Inspired by several practical examples, Liu [37] investigated moment stability with general decay speeds. Lan et al. [38] proposed one modified truncated Euler–Maruyama (MTEM) approach and explored the almost sure and mean square polynomial stability of the numerical technique. For SPDEs, many scholars [39–43] analyzed the polynomial stability by using the stochastic Lyapunov function method and some numerical algorithms. More recently, Mao et al. [44] constructed the novel Razumikhin theorems on the $p$th moment polynomial stability of the SPDEs. For NSPDEs, it can be observed that the references listed above focus on two aspects. One is the $p$th moment exponential stability [20–22], the other one is the $p$th moment stability with general decay rate [23]. Meanwhile, all the results in Refs. [20–23] required that the coefficients of delayed terms keep time varying. Therefore, it is necessary to develop other stabilities, such as the polynomial stability of NSPDEs with constant coefficients and generalize the theory in [35, 44] to NSPDEsMS.

Inspired by the aforementioned discussions, this paper will investigate the polynomial stability of NSPDEsMS by virtue of the Razumikhin method and several stochastic analysis techniques. The contributions of our article are listed below. Firstly, the existence and uniqueness of the solutions to NSPDEsMS are analyzed, where the condition on upper bound of the operator $L$ is relaxed. Secondly, the Razumikhin theorem on the $q$th polynomial stability of NSPDEsMS is established, and the drift term does not need to meet the linear growth condition. Moreover, based on some stochastic theories, the criterion on almost sure polynomial stability of NSFDEsMS is provided. Thirdly, all the existing stability results [20–23] require that coefficients of the delay term be time-varying, but the restriction in this paper is removed and the coefficients may keep constant. This paper also generalizes the theory in [35, 44] to NSFDEsMS. The structure of this article is arranged appropriately. In Section 2, standard notations are introduced, and several importance assumptions are proposed. In Section 3, the existence and uniqueness of the global solutions to NSFDEsMS are considered. Furthermore, some criteria on polynomial stability are constructed by utilizing the Razumikhin approach and stochastic analysis techniques. Section 4 illustrates the validity of the theoretical work through two concrete examples, and a full summarization is made in the last part.

2. Preliminaries

Throughout this paper, the following standard notations are adopted. Set $t_0 > 0$, $0 < q < 1$. Let $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t \geq t_0}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq t_0}$ satisfying the usual conditions. Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be one $m$-dimensional Brownian motion defined on the probability space. $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. $C([\delta t_0, t_0]; \mathbb{R}^n)$ denotes the family of continuous functions $\zeta: [\delta t_0, t_0] \to \mathbb{R}^n$ with the norm.
\[ |\mathbf{\zeta}| = \sup_{t_0 \leq s \leq t} |\mathbf{\zeta}(s)|, \quad L^2_{F_t}([\delta t, t]; \mathbb{R}^n) \]

denotes the set of all \( \mathcal{F}_t \)-measurable, \( C([\delta t, t]; \mathbb{R}^n) \)-valued stochastic variables \( \mathbf{\psi} = \{ \mathbf{\psi}(s) : \delta t \leq s \leq t \} \) such that \( E[|\mathbf{\psi}(s)|^q] < +\infty \). \( Y(t), t \geq t_0 \) denotes a Markov chain on the probability space taking values in a finite state space \( S = \{1, 2, \cdots, N\} \) with generator \( \Gamma = (\rho_{ij})_{N \times N} \) given by

\[
P\{Y(t + \Delta) = j \mid Y(t) = i\} = \begin{cases} \rho_{ij} + o(\Delta), & \text{if } i \neq j, \\ 1 + \rho_{ii} + o(\Delta), & \text{if } i = j, \end{cases}
\]

in which \( \Delta > 0 \), and \( \rho_{ij} \) satisfies \( \rho_{ij} \geq 0, i \neq j \) and \( \rho_{ii} = -\sum_{j \neq i} \rho_{i j} \). Moreover, the Markov chain \( Y(\cdot) \) is supposed to be independent of the Brownian motion \( w(\cdot) \).

Consider the following NSPDESMS

\[
\begin{align*}
\left\{ d[y(t) - G(y(\delta t), t, Y(t))] = F(y(t), y(\delta t), t, Y(t)) dt \\
y(s) = \zeta(s) \right\} \in [\delta t_0, t_0], \quad t_0 > 0, 0 < \delta < 1, \\
\end{align*}
\]

where \( G : \mathbb{R}^n \times [\delta t_0, +\infty) \times \mathbb{S} \to \mathbb{R}^n, F : \mathbb{R}^n \times \mathbb{R}^n \times [\delta t_0, +\infty) \times \mathbb{S} \to \mathbb{R}^n \) and \( H : \mathbb{R}^n \times \mathbb{R}^n \times [\delta t_0, +\infty) \times \mathbb{S} \to \mathbb{R}^{n \times n} \) are Borel-measurable functions. In order to discuss the polynomial stability of Equation (1), we suppose that the initial value \( \zeta \in L^2_{F_{t_0}} \) and \( F(0, 0, t, Y(t)) = 0, H(0, 0, t, Y(t)) = 0, G(0, 0, Y(t)) = 0 \). Obviously, this means that Equation (1) has one trivial solution.

To acquire our main results, the following definitions and assumptions on the addressed system are imposed.

**Definition 1.** The solution of Equation (1) is called to be polynomially stable in the \( p \)-th moment if there is one constant \( \bar{\eta} \) satisfying

\[
\lim_{t \to +\infty} \sup \frac{\ln E[|y(t)|^p]}{\ln(1 + t)} \leq -\bar{\eta}.
\]

**Definition 2.** The solution of Equation (1) is called to be almost surely polynomially stable if there is one constant \( \bar{\eta} \) satisfying

\[
\lim_{t \to +\infty} \sup \frac{\ln |y(t)|}{\ln(1 + t)} \leq -\bar{\eta} \quad \text{a.s.}
\]

**Assumption 1.** For arbitrary integer \( m > 1 \), there is one constant \( K_m > 0 \) satisfying that

\[
|F(v_1, v_2, t, i) - F(\bar{v}_1, \bar{v}_2, t, i)| \vee |H(v_1, v_2, t, i) - H(\bar{v}_1, \bar{v}_2, t, i)| \leq K_m (|v_1 - \bar{v}_1| + |v_2 - \bar{v}_2|),
\]

where all \( v_1, v_2, \bar{v}_1, \bar{v}_2 \in \mathbb{R}^n, |v_1| \vee |v_2| \vee |\bar{v}_1| \vee |\bar{v}_2| \leq m \) and \((t, i) \in [\delta t_0, +\infty) \times \mathbb{S} \).

**Assumption 2.** For all \( v_1, v_2 \in \mathbb{R}^n \) and \((t, i) \in [t_0, +\infty) \), there exists a real number \( \kappa \in (0, 1) \) satisfying

\[
|G(v_1, t, i) - G(v_2, t, i)| \leq \kappa |v_1 - v_2|,
\]

where \( G(0, t, i) = 0 \).

**Assumption 3.** Suppose that there exists one function \( U \in C^{2,1}([\delta t_0, +\infty) \times \mathbb{S}; \mathbb{R}_+) \) and several constants \( d_1 > 0, d_2 > 0, a > q \geq 1, \beta_i \geq 0 \) \((i = 0, 1, 2, 3, 4)\) such that

(i) \( d_1 |y|^q \leq U(y, t, i) \leq d_2 |y|^q \),

(ii) \( LU(x, t, i) \leq \beta_0 + \beta_1 |y|^q + \tilde{\beta}_2 |z|^q - \beta_3 |y|^a + \tilde{\beta}_4 |z|^a \).

**Assumption 4.** Suppose that there exists one function \( U \in C^{2,1}([\delta t_0, +\infty) \times \mathbb{S}; \mathbb{R}_+) \) and several constants \( d_1 > 0, d_2 > 0, a > q \geq 1, \beta_i \geq 0 \) \((i = 1, 2)\) such that

(i) \( d_1 |y|^q \leq U(y, t, i) \leq d_2 |y|^q \),
(ii) \[ \mathcal{L}u(y, z, t, i) \leq \beta_1 |y|^q + \delta \beta_2 |z|^q. \]

3. Main Results

In this section, the existence and uniqueness of the global solutions to NSPDEsMS are considered. Furthermore, some criteria on polynomial stability are constructed by utilizing the Razumikhin approach and stochastic analysis techniques.

**Theorem 1.** Under Assumptions 1, 2 and 3, for all \( y(s) = z(s), s \in [\delta t_0, t_0], t_0 > 0, \) there exists one unique global solution \( y(t) \) to Equation (1) on \( t \in [\delta t_0, +\infty). \)

**Proof.** According to Assumption 3, we can obtain that

\[ \mathcal{L}u(y, z, t, i) \leq \beta_1 |y|^q + \delta \beta_2 |z|^q, \]

where \( \beta_1 = 2^q \beta_1 \geq 0 \) and \( \beta_2 = 2^q \beta_1 \kappa^q \delta^{-1} + \beta_2 \geq 0. \) Moreover, since functions \( F, H \) and \( G \) satisfy Assumptions 1 and 2, by adopting the standing truncation technique, for all \( \forall \zeta(s) \in \mathcal{L}([\delta t_0, t_0]; R^n), \) there exists a unique maximal local solution \( y(t) \) on \( [\delta t_0, \sigma_\infty). \)

Let \( \theta_0 \) be large enough for \( ||y|| \leq \theta_0. \) For arbitrary integer \( b \geq \theta_0, \) define the stopping time sequence

\[ \theta_b = \inf\{t \in [t_0, \sigma_\infty]: |y(t)| > b\}. \]

Clearly, the sequence \( \{\theta_b\} \) keeps growing as \( b \to \infty. \) Set \( \theta_\infty = \lim_{b \to +\infty} \theta_b, \) whence \( \theta_\infty \leq \sigma_\infty \) a.s. Noting that \( \theta_\infty = +\infty \) a.s. means \( \sigma_\infty = +\infty \) a.s., we only need to prove \( \theta_\infty = +\infty \) a.s. Firstly, we will claim that \( \theta_\infty > \frac{\theta_0}{\beta} \) a.s. Let \( u(t) = y(t) - G(y(\delta t), t, Y(t)). \) By the Itô formula, for all \( t \in [t_0, t_1], \) we have that

\[ d_1 E|u(\theta_b \land t_1)|q \leq Q_1 + E \int_{t_0}^{t_1} \beta_1 |u(s)|^q ds, \]

where

\[ Q_1 = E\{u(t_0, Y(t_0))\} + E \int_{t_0}^{t_0} |\beta_0 + \beta_1|u(s)|^q + \delta \beta_2 |y(\delta s)|^q + \beta_4 |y(\delta s)|^4| ds \]

\[ \leq d_2 2^{t_1 - t_0 - 1} E|\zeta|^q + E \int_{\delta t_0}^{t_0} |\beta_0 + \beta_2| |y(s)|^q + \beta_4 |y(s)|^4| ds < +\infty. \]

It implies that

\[ E|u(\theta_b \land t_1)|q \leq Q_1 d_1 + \frac{1}{d_1} E \int_{t_0}^{t_1} \beta_1 |u(s)|^q ds, t_1 \in [t_0, t_0/\delta]. \]

By applying the Gronwall inequality, we have that

\[ E|u(\theta_b \land t_1)|q \leq \frac{Q_1 d_1}{d_1} \left( \frac{1}{d_1} \right)^{t_1 - t_0} \leq \frac{Q_1 d_1}{d_1} \left( \frac{1}{d_1} \right)^{t_1 - t_0}, t_1 \in [t_0, t_0/\delta]. \]
According to the elementary inequality $|l_1 + l_2|^q \leq \frac{|l_1|^q}{(1-v)^v} + \frac{|l_2|^q}{v^q}, 0 < v = \kappa < 1$, we infer that

$$
\sup_{t_0 \leq t_1 \leq \frac{l_0}{\delta}} E|y(\theta_b \wedge t_1)|^q \leq \frac{E||\xi||^q}{1 - \kappa} + \frac{E||\xi||^q}{1 - \kappa} \leq \frac{Q_1 e^{l_0^q(1 - v) - 1}}{d_1 (1 - \kappa)^q} + \frac{E||\xi||^q}{1 - \kappa} = Q'_1, t_1 \in \left[t_0, \frac{l_0}{\delta}\right).
$$

In particular, when $t_1 = \frac{l_0}{\delta}$, we have that $E|y(\theta_b \wedge \frac{l_0}{\delta})|^q \leq Q'_1$. It means that $b^q P\{\theta_b \leq \frac{l_0}{\delta}\} \leq Q'_1$. Letting $b \to \infty$, we hence acquire that $P\{\theta_\infty \leq \frac{l_0}{\delta}\} = 0$, equivalently, $P\{\theta_\infty > \frac{l_0}{\delta}\} = 1$. Let us proceed to prove $\theta_\infty > \frac{l_0}{\delta}$ a.s. For $\forall t_1 \in \left[t_0, \frac{l_0}{\delta}\right]$, according to Assumption 3, we have that

$$
d_1 E|u(\theta_b \wedge t_1)|^q \leq E(U(u(\theta_b \wedge t_1), \theta_b \wedge t_1, Y(\theta_b \wedge t_1))) \leq Q_2 + E \int_{t_0}^{\theta_b \wedge t_1} \beta_1 |u(s)|^q ds,
$$

where

$$
Q_2 = d_2 E|u(t_0)|^q + E \int_{t_0}^{\frac{l_0}{\delta}} [\beta_0 + \delta \beta_2 |y(\delta s)|^q + \delta \beta_4 |y(\delta s)|^q] ds
$$

$$
\leq Q_1 + E \int_{t_0}^{\frac{l_0}{\delta}} [\beta_0 \delta^{-1} + \beta_2 |y(s)|^q + \beta_4 |y(s)|^q] ds < +\infty.
$$

Applying the Gronwall inequality yields that

$$
E|u(\theta_b \wedge t_1)|^q \leq \frac{Q_2 e^{l_0^q(1 - v) - 1}}{d_1}, t_1 \in \left[t_0, \frac{l_0}{\delta}\right].
$$

Similarly, we also have that

$$
\sup_{t_0 \leq t_1 \leq \frac{l_0}{\delta}} E|y(\theta_b \wedge t_1)|^q \leq \frac{E||\xi||^q}{1 - \kappa} + \frac{E||\xi||^q}{1 - \kappa} \leq \frac{Q_2 e^{l_0^q(1 - v) - 1}}{d_1 (1 - \kappa)^q} + \frac{E||\xi||^q}{1 - \kappa} = Q'_2, t_1 \in \left[t_0, \frac{l_0}{\delta}\right).
$$

In particular, when $t_1 = \frac{l_0}{\delta}$, we have that $E|y(\theta_b \wedge \frac{l_0}{\delta})|^q \leq Q'_2$. It means that $b^q P\{\theta_b \leq \frac{l_0}{\delta}\} \leq Q'_2$. Letting $b \to \infty$, we hence acquire that $P\{\theta_\infty \leq \frac{l_0}{\delta}\} = 0$, equivalently, $P\{\theta_\infty > \frac{l_0}{\delta}\} = 1$. Repeating this procedure, we can show that $P\{\theta_\infty > \frac{l_0}{\delta}\} = 1$ for any integer $j \geq 1$. Letting $j \to \infty$ yields that $\theta_\infty = +\infty$ a.s. It means that the above conclusion holds.

**Remark 1.** In Theorem 1, the existence and uniqueness of the global solutions to NSPDEsMS are investigated by combining stochastic analysis techniques and the Gronwall inequality. Compared with the results in [14–17], the assumption condition is more general since all the parameters only need to satisfy $\beta_1 \geq 0, i = \{0, 1, 2, 3, 4\}$.

**Lemma 1.** Let Assumption 2 be satisfied. Then, for $q \geq 1$,

$$
E|y(t) - G(y(\delta t), t, i)|^q \leq (1 + \kappa)^q - 1 \sup_{\delta t \leq s \leq t} E|y(s)|^q, t \geq t_0.
$$

**Proof.** By utilizing the inequality $|l_1 + l_2|^q \leq (1 + v)^{q - 1} (|l_1|^q + \frac{|l_2|^q}{v^{q - 1}})$, we derive that

$$
E|y(t) - G(y(\delta t), t, i)|^q \leq (1 + v)^q - 1 \sup_{\delta t \leq s \leq t} E|y(s)|^q.
$$

Noting Assumption 1, let $v = \kappa$, we can obtain that

$$
E|y(t) - G(y(\delta t), t, i)|^q \leq (1 + \kappa)^q \sup_{\delta t \leq s \leq t} E|y(s)|^q.
$$
Lemma 2. Let Assumption 2 hold. If \( y(t) \) satisfies that

\[
(1 + t)^{\eta} \mathbb{E}|y(t) - G(y(\delta t), t, i)|^{\eta} \leq \frac{d_2}{d_1} (1 + \kappa)^{\eta}(1 + t_0)^{\eta} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}, t_0 \leq s \leq c.
\]

where \( c \geq t_0 > 0 \) and \( 0 < \eta < \frac{2\ln \kappa}{\ln \sigma} \). Then,

\[
(1 + t)^{\eta} \mathbb{E}|y(t)|^{\eta} \leq \frac{d_2(1 + \kappa)^{\eta}(1 + t_0)^{\eta}}{d_1[1 - \kappa(\frac{1}{\eta})^{\eta}]} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}, \delta t_0 \leq t \leq c.
\]

Proof. Based on the inequality \( |l_1 + l_2|^{\eta} \leq (1 + \varepsilon)^{\eta-1}(|l_1|^{\eta} + |l_2|^{\eta}), \varepsilon \in (0, 1) \), we can derive that

\[
(1 + t)^{\eta} \mathbb{E}|y(t)|^{\eta} \leq (1 + t)^{\eta} \left[ \frac{\mathbb{E}|y(t) - G(y(\delta t), t, i)|^{\eta}}{(1 - \varepsilon)^{\eta-1}} + \frac{\kappa^{\eta}\mathbb{E}|y(\delta t)|^{\eta}}{\varepsilon^{\eta-1}} \right]
\]

\[
\leq \frac{d_2}{d_1} \frac{(1 + \kappa)^{\eta}(1 + t_0)^{\eta}}{(1 - \varepsilon)^{\eta-1}} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}
\]

\[
+ \frac{\kappa^{\eta}(1 + t)^{\eta}(1 + \delta t)^{\eta} \mathbb{E}|y(\delta t)|^{\eta}}{\varepsilon^{\eta-1}} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}
\]

\[
\leq \frac{d_2}{d_1} \frac{(1 + \kappa)^{\eta}(1 + t_0)^{\eta}}{(1 - \varepsilon)^{\eta-1}} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}
\]

\[
+ \frac{\kappa^{\eta}(1 + t)^{\eta}(1 + \delta t)^{\eta}}{\varepsilon^{\eta-1} \frac{1}{\delta}} \sup_{\delta t_0 \leq s \leq t} (1 + s)^{\eta} \mathbb{E}|y(t)|^{\eta}, t \in [t_0, c].
\]

When \( t \in [\delta t_0, t_0] \), the above inequality still holds. Furthermore, we have that

\[
\sup_{\delta t_0 \leq s \leq t} (1 + s)^{\eta} \mathbb{E}|y(s)|^{\eta} \leq \frac{d_2}{d_1} \frac{(1 + \kappa)^{\eta}(1 + t_0)^{\eta}}{(1 - \varepsilon)^{\eta-1}} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}
\]

\[
+ \frac{\kappa^{\eta}(1 + t)^{\eta}(1 + \delta t)^{\eta}}{\varepsilon^{\eta-1} \frac{1}{\delta}} \sup_{\delta t_0 \leq s \leq t} (1 + s)^{\eta} \mathbb{E}|y(t)|^{\eta}.
\]

Letting \( \varepsilon = \kappa(\frac{1}{\eta})^{\eta} \), we see that

\[
\sup_{\delta t_0 \leq s \leq t} (1 + s)^{\eta} \mathbb{E}|y(s)|^{\eta} \leq \frac{d_2}{d_1} \frac{(1 + \kappa)^{\eta}(1 + t_0)^{\eta}}{[1 - \kappa(\frac{1}{\eta})^{\eta}]} \sup_{\delta t_0 \leq s \leq t_0} \mathbb{E}|y(s)|^{\eta}.
\]

\[
\square
\]

Theorem 2. Let Assumptions 1, 2 and 4 hold. If there exist two constants \( \mu > 0, \lambda > 0 \) such that

\[
\mathbb{E}[\mathcal{L}U(y(t), y(\delta t), t, i)] \leq -\mu \mathbb{E}[U(y(t) - G(y(\delta t), t, i), t, i)], i \in S, \tag{4}
\]

for all \( t \geq t_0 \) and function \( U \) satisfying

\[
\mathbb{E} \left[ \min_{1 \leq i \leq N} U(y(\delta t), \delta t, i) \right] \leq \lambda \mathbb{E} \left[ \max_{1 \leq i \leq N} U(y(t) - G(y(\delta t), t, i), t, i) \right],
\]
then, for $\forall \zeta(s) \in L^q_{F^*}$, the solution $y(t; \zeta)$ to Equation (1) has the property

$$\mathbb{E}[|y(t)|^q] \leq \left[ \lambda(1 + \kappa)^q (1 + t_0)^q \right] \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(s)|^q] (1 + t)^{-\bar{q}},$$

i.e.,

$$\lim_{t \to +\infty} \sup_{\delta_0 \leq s \leq t} \frac{\ln \mathbb{E}[|y(t)|^q]}{\ln(1 + t)} \leq -\bar{q},$$

where $\bar{q} = \min \left\{ \mu, \frac{1}{\ln 2} \ln \frac{\lambda_1}{(1 + x\lambda_1 q^q)} \right\}$ and $\lambda_1 = \frac{d_1}{d_2} \lambda > \frac{1}{(1 - \kappa)^q}.$

**Proof.** Let $\eta \in (0, \bar{q})$. Firstly, we will claim that

$$(1 + t)^{\bar{q}} \mathbb{E}[y(t) - G(y(\delta t), t, Y(t)), t, Y(t)] \leq d_2(1 + \kappa)^q (1 + t_0)^q \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(t)|^q], t \geq t_0. \quad (5)$$

When $t = t_0$, by Lemma 1, we have that

$$(1 + t_0)^{\bar{q}} \mathbb{E}[y(t_0) - G(y(\delta t_0), t_0, Y(t_0))] \leq d_2(1 + \kappa)^q (1 + t_0)^q \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(s)|^q]. \quad (6)$$

For $\forall t > t_0$, if assertion (5) does not hold, then we can find a constant $\chi_0$ satisfying

$$(1 + t)^{\bar{q}} \mathbb{E}[y(t) - G(y(\delta t), t, Y(t)), t, Y(t)] \leq d_2(1 + \kappa)^q (1 + t_0)^q \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(s)|^q], t < \chi_0. \quad (7)$$

and

$$(1 + \delta_0)^{\bar{q}} \mathbb{E}[y(\delta_0) - G(y(\delta \chi_0), \chi_0, Y(\chi_0)), \chi_0, Y(\chi_0)] = d_2(1 + \kappa)^q (1 + t_0)^q \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(s)|^q]. \quad (8)$$

There exists a time sequence $\{t_m\}_{m \geq 1}, t_m > \chi_0, t_m \to \chi_0$ such that

$$(1 + t_m)^{\bar{q}} \mathbb{E}[y(t_m) - G(y(t_m), t_m, Y(t_m)), t_m, Y(t_m)] > d_2(1 + \kappa)^q (1 + t_0)^q \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(s)|^q]. \quad (9)$$

Therefore, when $\delta \chi_0 \in [\delta t_0, \chi_0]$, combining (7), (8) and Lemma 2 result in that

$$\mathbb{E}\left[ \min_{1 \leq i \leq N} U(y(\delta \chi_0), \delta \chi_0, i) \right] \leq \mathbb{E}(y(\delta \chi_0), \delta \chi_0, Y(\delta \chi_0)) \leq (1 + \delta \chi_0)^{\bar{q}} \mathbb{E}[y(\delta \chi_0), \delta \chi_0, Y(\delta \chi_0)] \leq d_2(1 + \delta \chi_0)^{-\bar{q}} (1 + \chi_0)^q \mathbb{E}[y(\delta \chi_0)]^q \leq d_2(1 + \delta \chi_0)^{-\bar{q}} \frac{d_2 (1 + \kappa)^q (1 + t_0)^q}{d_1} \sup_{\delta_0 \leq s \leq t_0} \mathbb{E}[|y(s)|^q] \leq \frac{d_2 (1 + \chi_0)^q}{d_1 (1 + \delta \chi_0)^q} \mathbb{E}[y(\chi_0) - G(y(\delta \chi_0), \chi_0, Y(\chi_0)), \chi_0, Y(\chi_0)] \leq \frac{d_2 (1 + \chi_0)^q}{d_1 (1 + \delta \chi_0)^q} \mathbb{E}[y(\chi_0) - G(y(\delta \chi_0), \chi_0, Y(\chi_0)), \chi_0, Y(\chi_0)]. \quad (10)$$
which implies that \( \frac{1}{\delta} \eta < \frac{\lambda_1}{(1 + \kappa \lambda_1^2)^\eta} \), i.e., \( \frac{(\frac{1}{\delta})^\eta}{(1 + \kappa \lambda_1^2)^\eta} \). Furthermore, it can be inferred that \( 1 - \kappa(\frac{1}{\delta})^\eta \geq 1 - \kappa \lambda_1^2 = \frac{1}{1 + \kappa \lambda_1^2} \). Therefore,
\[
\frac{(1/\delta)^\eta}{\lambda_1(1 - \kappa(1/\delta)^\eta)^\eta} < \lambda_1 \frac{1 - \kappa(1/\delta)^\eta)^\eta}{(1 + \kappa \lambda_1^2)^\eta} = \frac{d_1}{d_2} \lambda [1 - \kappa(1/\delta)^\eta]^\eta,
\]
which implies that \( \frac{d_2}{d_1} \frac{(\frac{1}{\delta})^\eta}{[1 - \kappa(\frac{1}{\delta})^\eta]^\eta} < \lambda \). It follows from Equation (10) that
\[
E \left[ \min_{1 \leq i \leq N} U(y(\delta \chi_0), \delta \chi_0, i) \right] \leq \lambda E U(y(\chi) - G(y(\delta \chi_0), \chi, Y(\chi_0)), \chi, Y(\chi_0)) \leq \lambda E \left[ \max_{1 \leq i \leq N} U(y(\chi_0) - G(y(\delta \chi_0), \chi, i), i, \chi_0) \right].
\]
Accordingly, we have
\[
E[LU(y(\chi_0), y(\delta \chi_0), \chi_0, i)] \leq -\mu E[U(y(\chi_0) - G(y(\delta \chi_0), \chi, i), i, \chi_0)], i \in S, \quad (11)
\]
Since \( \eta < \mu \) and functions \( F, H, G \) keep continuous, one sufficiently small \( h > 0 \) can be found such that
\[
E[LU(y(t), \eta(\delta t), t, i)] \leq -\eta E[U(y(t) - G(y(\delta t), t, i), t, i)], i \in S, t \in [\chi_0, \chi_0 + h]. \quad (12)
\]
Applying the Itô formula yields that
\[
(1 + \chi_0 + h)^\eta E U(y(\chi_0 + h) - G(y(\chi_0 + h)), \chi_0 + h, \chi_0 + h) - d_2(1 + \kappa)^\eta \sup_{\delta t_0 \leq s \leq t_0} E |y(s)|^\eta
\]
\[
= \int_{\chi_0}^{\chi_0 + h} (1 + s)^\eta \left[ \eta 1 + s E U(y(s) - G(y(\delta s), s, Y(s)), s, Y(s)) + E U(y(s), y(\delta s), s, Y(s)) \right] ds
\]
\[
\leq 0,
\]
which contradicts Equation (9). Therefore, we obtain that
\[
(1 + t)^\eta E U(y(t) - G(y(\delta t), t, Y(t)), t, Y(t)) \leq d_2(1 + \kappa)^\eta (1 + t)^\eta \sup_{\delta t_0 \leq s \leq t_0} E |y(s)|^\eta. \quad (13)
\]
By Lemma 2, we have that
\[
(1 + t)^\eta E |y(t)|^\eta \leq \lambda(1 + \kappa)^\eta (1 + t_0)^\eta \sup_{\delta t_0 \leq s \leq t_0} E |y(s)|^\eta.
\]
Letting \( \eta \to \tilde{\eta} \), we obtain that \( (1 + t)^\tilde{\eta} E |y(t)|^\tilde{\eta} \leq \lambda(1 + \kappa)^\tilde{\eta} (1 + t_0)^\tilde{\eta} \sup_{\delta t_0 \leq s \leq t_0} E |y(s)|^\tilde{\eta}, \)
which implies
\[
\lim_{t \to +\infty} \sup_{\delta t_0 \leq s \leq t_0} \frac{\ln E |y(t)|^\tilde{\eta}}{\ln(1 + t)} \leq -\tilde{\eta}. \quad (14)
\]
Remark 2. In Theorem 2, if all conditions are satisfied except condition (4) which is replaced by the stronger condition
\[
E \left[ \max_{1 \leq i \leq N} \mathcal{L}(y(t), y(\delta t), t, i) \right] \leq -\mu E \left[ \max_{1 \leq i \leq N} U(y(t) - G(y(\delta t), t, i), t, i) \right],
\]
then the assertion still holds. In fact, the above stronger condition is also difficult to be verified.

Remark 3. It is noted that all the existing stability results [20–23] require that the coefficients of the delay term be time varying, but the restriction in this paper is removed and the coefficients may keep constant. Theorem 2 also generalizes the theory in [35,44] to NSFDEsMS. In [45], an efficient method based on the generalized hat functions for solving nonlinear stochastic differential equations driven by the multi-fractional Gaussian noise was proposed, and the theory was applied to some stochastic population models. Moreover, dynamic properties of stochastic pantograph systems with multi-fractional Gaussian noise are worthy of exploration.

Lemma 3. Suppose that Assumption 2 is satisfied. Let \( q > 1 \). If there are two positive constants \( \eta \) and \( M_1 \) satisfying
\[
0 < \eta < \frac{1}{\ln \frac{1}{\delta}} \ln \frac{1}{\kappa^q},
\]
and
\[
|y(t) - G(y(\delta t), t, Y(t))| \leq M_1(1 + t)^{-\eta}, t \geq t_0,
\]
then we have that
\[
\lim_{t \to \infty} \sup_{t_0 \leq t \leq T} \frac{\sup_{\delta t_0 \leq t \leq T} |y(t)|^q}{\ln(1 + t)} \leq -\frac{\eta}{q} \text{ a.s.} \quad (15)
\]

Proof. For \( \forall T > 0 \), when \( t \in [t_0, T] \), we can find one constant \( v \) (0 < \( v < 1 \)) such that
\[
(1 + t)^\eta |y(t)|^q \leq (1 + t)^\eta \left[ \frac{|y(t) - G(y(\delta t), t, Y(t))|^q}{(1 - v)^q} + \frac{|G(y(\delta t), t, Y(t))|^q}{v^q-1} \right] \leq \frac{M_1}{(1 - v)^q} + \frac{\kappa^q}{v^q-1} \left( 1 + t \right)^\eta (1 + \delta t)^\eta |y(\delta t)|^q.
\]
We have that
\[
\sup_{\delta t_0 \leq t \leq T} (1 + t)^\eta |y(t)|^q \leq \frac{M_1}{(1 - v)^q} + \sup_{\delta t_0 \leq t \leq T} |y(t)|^q + \frac{\kappa^q}{v^q-1} \sup_{t_0 \leq t \leq T} (1 + \delta t)^\eta |y(\delta t)|^q
\]
\[
\leq \frac{M_1}{(1 - v)^q} + \sup_{\delta t_0 \leq t \leq T} |y(t)|^q + \frac{\kappa^q}{v^q-1} \sup_{\delta t_0 \leq t \leq T} (1 + t)^\eta |y(t)|^q. \quad (16)
\]
Since \( \eta < \frac{1}{\ln \frac{1}{\delta}}, \) i.e., \( \frac{1}{\delta}^q \kappa^q < 1 \), by choosing \( v = \left( \frac{1}{\delta} \right)^q \kappa \), we obtain that
\[
\sup_{\delta t_0 \leq t \leq T} (1 + t)^\eta |y(t)|^q \leq \frac{M_2}{1 - \left( \frac{1}{\delta} \right)^q \kappa}, \quad (17)
\]
where \( M_2 = \frac{M_1}{(1 - v)^q} + \sup_{\delta t_0 \leq t \leq T} |y(t)|^q \). It means that
\[
\lim_{t \to \infty} \sup_{t_0 \leq t \leq T} \frac{|y(t)|^q}{\ln(1 + t)} \leq -\frac{\eta}{q} \text{ a.s.} \quad (18)
\]
Theorem 3. Let $U(y, t, i) = y^2$. Let $q \geq 2$ and $\sigma_0 \in (1, \tilde{y})$. Suppose that there is one constant $L > 0$ satisfying $|H(y, z, t, i)| \leq L(|y| + |z|)$. If all the conditions of Theorem 2 hold, then
\[
\lim_{t \to +\infty} \sup_{t \in [t_0, \infty]} \frac{\ln |y(t)|}{\ln (1 + t)} \leq -\frac{\tilde{y} - \sigma_0}{q} \quad \text{a.s.}
\]  
(19)

Proof. For convenience, let $u(t) = y(t) - G(y(\delta t), t, Y(t))$. For any positive integer $n \geq 0$, by using the Itô formula, we compute that
\[
\begin{align*}
E \left[ \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-n+1} t_0} |u(t)|^q \right] &
\leq E|u(\delta^{-n} t_0)|^q + \frac{1}{q} \int_{\delta^{-n} t_0}^t \left( L \left( y(s), y(\delta s), s, Y(s) \right) + \left( L \left( u(s), s, Y(s) \right) + H(y(s), y(\delta s), s, Y(s)) \right) ds \right)^{\frac{q}{2}} \\
&+ E \left[ \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-n+1} t_0} \int_{\delta^{-n} t_0}^t |L_x(u(s), s, Y(s))| ds \right] \\
&= E|u(\delta^{-n} t_0)|^q + I_1 + I_2. 
\end{align*}
\]
(20)
Noting that $L \left( y, z, t, Y(t) \right) \leq \beta_1 |y|^q + \beta_2 |\delta z|$, $\beta_1 \geq 0, \beta_2 \geq 0$, we acquire that
\[
I_1 \leq \int_{\delta^{-n} t_0}^{\delta^{-n+1} t_0} |\beta_1 |y(s)|^q + \beta_2 |y(\delta s)|^q| ds. 
\]  
(21)

On the other hand, by using BDG inequality, we have that
\[
\begin{align*}
I_2 &\leq C_q \left( \int_{\delta^{-n} t_0}^{\delta^{-n+1} t_0} \left| L_x(u(s), s, Y(s)) \right| \left| H(y(s), y(\delta s), s, Y(s)) \right|^{\frac{q}{2}} ds \right)^{\frac{2}{q}} \\
&\leq qC_q L \left( \int_{\delta^{-n} t_0}^{\delta^{-n+1} t_0} |u(s)|^{2q-2} \left( |y(s)| + |y(\delta s)| \right)^{\frac{q}{2}} ds \right)^{\frac{1}{q}} \\
&\leq qC_q L \left( \epsilon \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-n+1} t_0} |u(s)|^q \left( \frac{1}{1} \int_{\delta^{-n} t_0}^{\delta^{-n+1} t_0} |y(s)|^{q-2} \left( |y(s)| + |y(\delta s)| \right)^2 ds \right)^{\frac{1}{2}} \right). 
\end{align*}
\]
Applying the Young inequality to the above equation yields that
\[
I_2 \leq \frac{\epsilon qC_q L}{2} \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-n+1} t_0} |u(s)|^q \\
+ \frac{2^{(q-3)\nu_0} qC_q L}{2\epsilon} \int_{\delta^{-n} t_0}^{\delta^{-n+1} t_0} \left( |y(s)|^{q-2} + \kappa^{q-2} |y(\delta s)|^{q-2} \right) (2 |y(s)|^2 + 2 |y(\delta s)|^2) ds. 
\]
(22)
Substituting $I_1$ and $I_2$ into Equation (20), we have

$$
E \left[ \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-(n+1)} t_0} |u(s)|^q \right] \leq M_2 E[|u(\delta^{-n} t_0)|^q + M_2 \int_{\delta^{-n} t_0}^{\delta^{-(n+1)} t_0} |y(s)|^q ds
$$

$$
+ M_3 \int_{\delta^{-n} t_0}^{\delta^{-(n+1)} t_0} |y(\delta s)|^q ds,
$$

where $M_1 = 2$, $M_2 = 2(\theta-3)\sqrt{2q^2 C_4^2 L^2} + 2\beta_1$, $M_3 = 2(\theta-3)\sqrt{2q^2 C_4^2 L^2} + 2\beta_2 \delta$. When Theorem 2 holds, we obtain that

$$
E[|u(t)|^q] \leq \frac{d_2}{d_1} (1 + \kappa)^\theta (1 + t_0)^\theta \sup_{\delta t_0 \leq s \leq t_0} E[|y(s)|^q (1 + s)^{-\theta}],
$$

and

$$
E[|y(t)|^q] \leq \lambda (1 + \kappa)^\theta (1 + t_0)^\theta \sup_{\delta t_0 \leq s \leq t_0} E[|y(s)|^q (1 + s)^{-\theta}].
$$

Substituting Equations (24) and (25) into Equation (23) yields that

$$
E[|y(s)|^q] \leq M_4 (1 + \delta^{-n} t_0)^{-\theta} + M_5 \int_{\delta^{-n} t_0}^{\delta^{-(n+1)} t_0} (1 + s)^{-\theta} ds
$$

$$
+ M_6 \int_{\delta^{-n} t_0}^{\delta^{-(n+1)} t_0} (1 + s)^{-\theta} ds,
$$

where $M_4 = M_1 (1 + t_0)^\theta (1 + \kappa)^\theta \sup_{\delta t_0 \leq s \leq t_0} E[|y(s)|^q]$, $M_5 = \lambda (1 + \kappa)^\theta (1 + t_0)^\theta \sup_{\delta t_0 \leq s \leq t_0} E[|y(s)|^q] M_2$, and $M_6 = \lambda M_3 (1 + \kappa)^\theta (1 + t_0)^\theta \sup_{\delta t_0 \leq s \leq t_0} E[|y(s)|^q]$. According to Chebyshev’s inequality, we obtain that

$$
P\left\{ \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-(n+1)} t_0} |u(t)|^q > (1 + \delta^{-n} t_0)^{-(\theta-\epsilon_0)} \right\}
$$

$$
\leq (1 + \delta^{-n} t_0)^{-\theta+\epsilon_0} E\left( \sup_{\delta^{-n} t_0 \leq t \leq \delta^{-(n+1)} t_0} |u(t)|^q \right)
$$

$$
\leq (1 + \delta^{-n} t_0)^{-\theta+\epsilon_0} \left[ M_4 + M_5 (\delta^{-(n+1)} t_0 - \delta^{-n} t_0) + M_6 (\delta^{-n} t_0 - \delta^{-(n-1)} t_0) \right] = \Phi(n).
$$

Noting $\epsilon_0 \in (1, \tilde{\eta})$, we have that

$$
\sum_{n=0}^{\infty} \Phi(n) = [M_4 + M_5 (\delta^{-1} - 1) + M_6 (1 - \delta)] \sum_{n=0}^{\infty} \frac{(1 + \delta^{-n} t_0)^\epsilon_0}{\delta^n} < \infty.
$$

By utilizing the Borel–Cantelli lemma, there exists one set $\Omega_0$ with $P(\Omega_0) = 1$ and one integer $n_0(\omega) > 0$, for $n > n_0(\omega)$, $\omega \in \Omega_0$ and $\delta^{-n} t_0 \leq t \leq \delta^{-(n+1)} t_0$, satisfying that

$$
|u(t)|^q = |y(t) - G(y(\delta t), t, Y(t))|^q \leq (1 + \delta^{-n} t_0)^{-(\theta-\epsilon_0)} \leq \left( \frac{1}{\delta} \right)^{\tilde{\eta} - \epsilon_0} (1 + t)^{-(\tilde{\eta} - \epsilon_0)}.
$$

By Lemma 3, we can conclude that

$$
\lim_{t \to +\infty} \sup_{n(1+t)} \frac{\ln |y(t)|}{\ln (1+t)} \leq -\frac{\tilde{\eta} - \epsilon_0}{\theta} a.s.
$$

\( \square \)
4. Examples

In this section, two examples are exhibited to show the validity of the proposed theoretical results.

Example 1. Consider the following NSPDEsMS

\[ d[y(t) - G(y(\delta t), t, Y(t))] = F(y(t), y(\delta t), t, Y(t))dt + H(y(t), y(\delta t), t, Y(t))dw(t), \]

where \( \delta = 0.2 \), \( Y(t) \) denotes one Markov chain taking values in \( S = \{1, 2\} \) with the generator

\[ \Gamma = \begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix}. \]

Here,

\[ G(y(0.2t), i, t) = \begin{cases} 0.4y(0.2t), & i = 1, \\ 0.3y(0.2t), & i = 2. \end{cases} \]

\[ F(y(t), y(0.2t), t, i) = \begin{cases} -12[y(t) - 0.4y(0.2t)] - [y(t) - 0.4y(0.2t)]^3 + 0.5y(0.2t), & i = 1, \\ -10[y(t) - 0.3y(0.2t)] + 0.6y(0.2t), & i = 2. \end{cases} \]

\[ H(y(t), y(0.2t), t, i) = \begin{cases} 0.6y(0.2t), & i = 1, \\ 0.5y(0.2t), & i = 2. \end{cases} \]

We choose

\[ U(y, i, t) = y^2. \]

By computing, one obtains that

\[ \mathcal{L}U(y(t), y(0.2t), t, 1) \leq -23.5[y(t) - 0.4x(0.2t)]^2 + 0.86y^2(0.2t), \]

\[ \mathcal{L}U(y(t), y(0.2t), t, 2) \leq -19.4[y(t) - 0.3x(0.2t)]^2 + 0.85y^2(0.2t). \]

Furthermore, when

\[ U(y(0.2t), t, i) \leq \lambda U[y(t) - G(y(0.2t), t, i), t, i], \]

by choosing \( \lambda = 20 \), we have that

\[ \mathbb{E} \mathcal{L}U(y, i, t) \leq [-19.4 + 0.86 \times 20]U(y(t) - G(y(0.2t), t, i), t, i) \]

\[ \leq -2.2U(y(t) - G(y(0.2t), t, i)). \]

Let \( \tilde{\eta} = \min\{\mu, \frac{1}{\ln 2} \ln \frac{\lambda_1}{(1 + \lambda_2^2)^y}\} = \min\{2.2, \frac{1}{\ln 2} \ln \frac{20}{(1 + 0.2 \times 20^y)}\} = 1.0674 > 1. \] According to Theorems 2 and 3, we can deduce that the above system is polynomially stable in mean square and almost surely polynomially stable.

Example 2. Consider the two-dimensional system

\[ d[y(t) - G(y(0.6t), t, Y(t)))] = [A(Y(t))y(t) + B(Y(t))y(0.3t)]dt + CY(t)y(0.3t)dw(t), \]

where \( Y(t) \) denotes one Markov chain with the generator

\[ \Gamma = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}. \]
Meanwhile,

$$G(y(0.6t), i, t) = \begin{bmatrix} 0.3y_1(0.6t) \\ 0.3y_2(0.6t) \end{bmatrix}.$$ 

$$A(1) = A_1 = \begin{bmatrix} -4 & 0.2 \\ 0.1 & -0.4 \end{bmatrix}, B(1) = B_1 = \begin{bmatrix} 1.5 & 0.34 \\ 0.17 & 1.7 \end{bmatrix}, C(1) = C_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{bmatrix},$$

$$A(2) = A_2 = \begin{bmatrix} 0.3 & 0.5 \\ 0.6 & -3 \end{bmatrix}, B(2) = B_2 = \begin{bmatrix} 1.1 & -0.05 \\ 0.12 & 1.1 \end{bmatrix}, C(2) = C_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}.$$ 

We choose the Lyapunov function

$$U(y, i, t) = \theta_i[(y_1 - G(y_1(0.6t), i, t))^2 + (y_2 - G(y_2(0.6t), i, t))^2], i = 1, 2.$$ 

Let $\theta_1 = 2, \theta_2 = 3, d_1 = 2, d_2 = 3, \lambda = 6, \lambda_1 = 4$. If

$$U(y, i, t) \leq \lambda U[y(t) - G(y(0.6t), i, t), i, t],$$

then we have

$$LU(y(t), y(0.6t), i, t, i) \leq \lambda_{\max}[A_i + A_i^T + I + \frac{1}{\theta_i} \sum_{l=1}^{2} r_{ij}\theta_l I] + \lambda_2 \max{[(0.3A_i + B_i)^T (0.3A_i + B_i)}$$

$$+ C_i^TC_i]U[y(t) - G(y(0.6t), t, i, t, i)].$$

We compute that

$$LU(y(t), y(0.6t), t, 1) \leq (-7.7 + 6.1626)U(y(t) - G(y(0.6t), t, 1), t, 1)$$

$$\leq -1.5374U(y(t) - G(y(0.6t), t, 1), t, 1),$$

and

$$LU(y(t), y(0.6t), t, 2) \leq -2.0404U(y(t) - G(y(0.6t), t, 2), t, 2),$$

which indicates that

$$LU(y(t), y(0.6t), t, i) \leq -1.5U(y(t) - G(y(0.6t), i, t), t, i).$$

Let $\mu = 1.5$. Noting that

$$\frac{1}{\ln \frac{\lambda_1}{(1 + \kappa\lambda_1^{\frac{1}{2}})^2}} = \frac{1}{\ln \frac{1}{0.6}} \ln \frac{4}{(1 + 0.6)^2} = 0.8737,$$

then $\eta = \min \left\{ \mu \frac{1}{\ln \frac{\lambda_1}{(1 + \kappa\lambda_1^{\frac{1}{2}})^2}} \right\} = \min \{1.5, 0.8737\} = 0.8737 < 1$. Hence, according to Theorems 2 and 3, we can conclude that the above system is polynomially stable in mean square rather than almost surely polynomially stable.

5. Conclusions

In this paper, the new Razumikhin theorem on the $q$th moment polynomial stability of NSPDEsMS is established. Furthermore, combining with several stochastic analysis techniques, the almost sure polynomial stability of NSPDEsMS is explored. In the end, the effectiveness of the main results is demonstrated through two concrete examples. In years to come, our theoretical work can be further generalized to the SPDEs with Lévy noise or neural network systems.
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References

1. Mao, X. Stochastic Differential Equations and Applications; Horwood Publishing: Chichester, UK, 1997.
2. Liu, K.; Xia, X. On the exponential stability in mean square of neutral stochastic functional differential equations. Syst. Control Lett. 1999, 37, 207–215. [CrossRef]
3. Shaikhet, L. Some new aspects of Lyapunov-type theorems for stochastic differential equations of neutral type. Siam J. Control Optim. 2010, 48, 4481–4499. [CrossRef]
4. Jovanovic, M.; Jankovic, S. Neutral stochastic functional differential equations with additive perturbations. Appl. Math. Comput. 2009, 213, 370–379. [CrossRef]
5. Jiang, F.; Shen, Y.; Wu, F. A note on order of convergence of numerical method for neutral stochastic functional differential equations. Mathematics 2022, 10, 866. [CrossRef]
6. Wang, Q.; Chen, H.; Yuan, C. A note on exponential stability for numerical solution of neutral stochastic functional differential equations. Mathematics 2022, 10, 866. [CrossRef]
7. Chen, H.; Shi, P.; Lim, C. Exponential synchronization for markovian stochastic coupled neural networks of neutral-type via adaptive feedback control. IEEE Trans. Neural Netw. Learn. Syst. 2017, 28, 1618–1632. [CrossRef]
8. Mao, X.; Yuan, C. Stochastic Differential Equations with Markovian Switching; Imperial College Press: London, UK, 2006.
9. Ji, Y.; Chizeck, H.J. Controllability, stabilizability and continuous-time markovian jump linear quadratic control. IEEE Trans. Autom. Control 1990, 35, 777–788. [CrossRef]
10. Li, X.; Mao, X. A note on almost sure asymptotic stability of neutral stochastic delay differential equations with Markovian switching. Automatica 2012, 48, 2329–2334. [CrossRef]
11. Ockendon, J.R.; Taylor, A.B. The dynamics of a current collection system for an electric locomotive. Proc. R. Soc. Lond. A 1971, 322, 447–468.
12. Wang, X.; Park, J.H.; Yang, H.; Zhong, S. Delay-Dependent Stability Analysis for Switched Stochastic Networks with Proportional Delay. IEEE Trans. Cybern. 2020, 52, 6369–6378. [CrossRef]
13. Wang, J.; Wang, X.; Wang, Y.; Zhang, X. Non-reduced order method to global h-stability criteria for proportional delay high-order inertial neural networks. Appl. Math. Comput. 2021, 407, 126308. [CrossRef]
14. Zhou, L.; Zhao, Z. Global polynomial periodicity and polynomial stability of proportional delay Cohen–Grossberg neural networks. ISA Trans. 2021, 122, 205–217. [CrossRef] [PubMed]
15. Luo, D.; Jiang, Q.; Wang, Q. Anti-periodic solutions on Clifford-valued high-order Hopfield neural networks with multi-proportional delays. Neurocomputing 2022, 472, 1–11. [CrossRef]
16. Xiao, Q.; Huang, T.; Zeng, Z. Synchronization of Timescale-type nonautonomous neural networks with proportional delays. IEEE Trans. Syst. Man Cybern. A 2021, 52, 2168–2216. [CrossRef]
17. Padmaja, N.; Balasubramaniam, P. New delay and order-dependent passivity criteria for impulsive fractional-order neural networks with switching parameters and proportional delays. Neurocomputing 2021, 454, 113–123. [CrossRef]
18. Fan, Z.; Liu, M.; Cao, W. Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations. J. Math. Anal. Appl. 2007, 325, 1142–1159. [CrossRef]
19. Appleby, J.; Buckwar, E. Sufficient conditions for polynomial asymptotic behaviour of the stochastic pantograph equation. Electron. J. Qual. Theory 2016, 45, 1–32.
20. Liu, L.; Deng, F. nth moment exponential stability of highly nonlinear neutral pantograph stochastic differential equations driven by Lévy noise. Appl. Math. Lett. 2018, 86, 313–319. [CrossRef]
21. Shen, M.; Fei, W.; Mao, X.; Deng, S. Exponential stability of highly nonlinear neutral pantograph stochastic differential equation. Asian J. Control 2020, 22, 436–448. [CrossRef]
22. Caraballo, T.; Mchiri, L.; Mohsen, B.; Rhaima, M. nth moment exponential stability of neutral stochastic pantograph differential equations with Markovian switching. Commun. Nonlinear Sci. Numer. Simulat. 2021, 102, 105916. [CrossRef]
23. Mao, W.; Hu, L.; Mao, X. Almost sure stability with general decay rate of neutral stochastic pantograph equations with Markovian switching. Electron. J. Qual. Theory Differ. Equ. 2019, 52, 1–17. [CrossRef]
24. Razumikhin, B.S. On the stability of systems with a delay. Prikl. Mat. Mekh. 1956, 20, 500–512.
25. Razumikhin, B.S. Application of Lyapunov’s method to problems in the stability of systems with a delay. *Avtomat. I. Telemekh.* 1960, 21, 740–749.

26. Mao, X. Razumikhin-type theorems on exponential stability of stochastic functional differential equations. *Stoch. Proc. Appl.* 1996, 65, 233–250. [CrossRef]

27. Zong, X.; Lei, D.; Wu, F. Discrete Razumikhin-type stability theorems for stochastic discrete-time delay systems. *J. Frankl. Inst.* 2018, 355, 8245–8265. [CrossRef]

28. Mao, X. Razumikhin-type theorems on exponential stability of neutral stochastic functional differential equations. *SIAM J. Math. Anal.* 1997, 28, 389–401. [CrossRef]

29. Janković, S.; Randjelović, J.; Jovanović, J. Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations. *J. Math. Anal. Appl.* 2009, 355, 811–820. [CrossRef]

30. Huang, L.; Deng, F. Razumikhin-type theorems on stability of neutral stochastic functional differential equations. *IEEE Trans. Automat. Control* 2008, 53, 1718–1723. [CrossRef]

31. Pavlović, G.; Janković, S. The Razumikhin approach on general decay stability for neutral stochastic functional differential equations. *J. Frankl. Inst.* 2013, 350, 2124–2145. [CrossRef]

32. Zhou, S.; Hu, S. Razumikhin-type theorems of neutral stochastic functional differential equations. *Acta Math. Sci.* 2009, 29, 181–190.

33. Song, Y.; Zeng, Z. Razumikhin-type theorems on pth moment boundedness of neutral stochastic functional differential equations with Markovian switching. *J. Frankl. Inst.* 2018, 355, 8296–8312. [CrossRef]

34. Wu, F.; Hu, S. Razumikhin-type theorems for neutral stochastic functional differential equations with unbounded delay. *Acta Math. Sci.* 2011, 31, 1245–1258.

35. Yu, Z. Razumikhin-type theorem and mean square asymptotic behavior of the backward Euler method for neutral stochastic pantograph equations. *J. Inequalities Appl.* 2013, 229, 2–15. [CrossRef]

36. Mao, X. Polynomial stability for perturbed stochastic differential equations with respect to semimartingales. *Stoch. Proc. Appl.* 1992, 41, 101–116. [CrossRef]

37. Liu, K.; Chen, A. Moment decay rates of solutions of stochastic differential equations. *Tohoku Math. J.* 2001, 53, 81–93. [CrossRef]

38. Lan, G.; Xia, F.; Wang, Q. Polynomial stability of exact solution and a numerical method for stochastic differential equations with time-dependent delay. *J. Comput. Appl. Math.* 2019, 346, 340–356. [CrossRef]

39. Hu, L.; Mao, X.; Shen, Y. Stability and boundedness of nonlinear hybrid stochastic differential delay equations. *Syst. Control Lett.* 2013, 62, 178–187. [CrossRef]

40. Milošević, M. Existence, uniqueness, almost sure polynomial stability of solution to a class of highly nonlinear pantograph stochastic differential equations and the Euler–Maruyama approximation. *Appl. Math. Comput.* 2014, 237, 672–685.

41. Milošević, M. Convergence and almost sure polynomial stability of the backward and forward–backward Euler methods for highly nonlinear pantograph stochastic differential equations. *Math. Comput. Simulat.* 2018, 150, 25–48. [CrossRef]

42. Guo, P.; Liu, M.; He, Z.; Jia, H. Stability of numerical solutions for the stochastic pantograph differential equations with variable step size. *J. Comput. Appl. Math.* 2021, 388, 325–342. [CrossRef]

43. Mao, W.; Hu, L.; Mao, X. The asymptotic stability of hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise. *J. Frankl. Inst.* 2020, 357, 1174–1198. [CrossRef]

44. Mao, W.; Hu, L.; Mao, X. Razumikhin-type theorems on polynomial stability of hybrid stochastic systems with pantograph delay. *Discret. Contin. Dyn. Syst.* 2020, 25, 3217–3232. [CrossRef]

45. Eftekhar, T.; Rashidinia, J. A novel and efficient operational matrix for solving nonlinear stochastic differential equations driven by multi-fractional Gaussian noise. *Appl. Math. Comput.* 2022, 429, 127218. [CrossRef]