Impurity Effects on the $A_1$-$A_2$ Splitting of Superfluid $^3$He in Aerogel

J. A. Sauls$^{1,2}$ and Priya Sharma$^1$

$^1$Department of Physics and Astronomy, Northwestern University, Evanston, IL, USA
$^2$Centre de Recherches sur les Très Basses Températures, Centre National de la Recherche Scientifique, Laboratoire Associé à l’Université J. Fourier, BP 166, 38042 Grenoble Cedex 9, France

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When liquid $^3$He is impregnated into silica aerogel a solid-like layer of $^3$He atoms coats the silica structure. The surface $^3$He is in fast exchange with the liquid on NMR timescales. The exchange coupling of liquid $^3$He quasiparticles with the localized $^3$He spins modifies the scattering of $^3$He quasiparticles by the aerogel structure. In a magnetic field the polarization of the solid spins gives rise to a splitting of the scattering cross-section of for ‘up’ vs. ‘down’ spin quasiparticles, relative to the polarization of the solid $^3$He. We discuss this effect, as well as the effects of non-magnetic scattering, in the context of a possible splitting of the superfluid transition for $\uparrow\uparrow$ vs. $\downarrow\downarrow$ Cooper pairs for superfluid $^3$He in aerogel, analogous to the $A_1-A_2$ splitting in bulk $^3$He. Comparison with the existing measurements of $T_c$ for $B < 5$ kG, which show no evidence of an $A_1-A_2$ splitting, suggests a liquid-solid exchange coupling of order $J \simeq 0.1$ mK. Measurements at higher fields, $B \simeq 20$ kG, should saturate the polarization of the solid $^3$He and reveal the $A_1-A_2$ splitting.

One focus of experimental investigations of $^3$He in aerogel has been the determination of the phase diagram. Torsional oscillator, NMR, vibrating wire and sound attenuation experiments on $^3$He in $\approx 98\%$ porosity aerogels, suggest that there is just one superfluid phase in zero magnetic field over the pressure range, $34$ atm $< p \lesssim 6$ atm. At lower pressures there appears to be only normal $^3$He down to zero temperature. In short, a B-like phase, with reduced susceptibility, is the only stable superfluid phase observed in zero field. An A-like phase, if it exists in zero field, may be stable only in a very small region $\Delta T \lesssim 20$ mK near $T_c$. As in bulk $^3$He an A-like phase can be stabilized in a magnetic field; the region of stability, $T_{AB} < T < T_A$, increases quadratically with field, $T_c - T_{AB} = -g_{AB}B^2$. However, a splitting of the transition for $\uparrow\uparrow$ and $\downarrow\downarrow$ pairs, analogous to the $A_1-A_2$ splitting in bulk $^3$He, has so far not been observed for fields up to $B \approx 5$ kG.

In pure $^3$He, on application of a magnetic field, the $A_1$ phase, characterized by spin-polarized Cooper pairs composed of only $\uparrow\uparrow$ spins nucleates at a temperature slightly higher than the zero-field transition, $T^{A_1}_c = T_c + \lambda A_1 B^{3/2}$. This transition is followed by a second transition, shifted below the zero-field transition, at $T^{A_2}_c = T_c - \lambda A_2 B$, in which the $\downarrow\downarrow$ pairs nucleate. The region of stability of pure $\uparrow\uparrow$ pairs linearly increases with field, $\Delta T_{A_1-A_2} \equiv (T^{A_1}_c - T^{A_2}_c) = (\lambda A_1 + \lambda A_2) B$, at a rate of $\lambda A_1 + \lambda A_2 \simeq 6.1$ mK/kG at $p = 33.4$ bar.

This splitting of the zero-field transition originates from the combined effect of the nuclear Zeeman coupling to the $^3$He spin, and particle-hole asymmetry in the normal-state density of states and pairing interaction. The original estimate of $\lambda A_1$ by Ambegaokar and Mermin was based on the asymmetry of the density of states for $\uparrow$ vs. $\downarrow$ quasiparticles at the Fermi level. More involved calculations include the effects of spin-polarization of the Fermi liquid on the pairing interaction. Estimates of the splitting of the $A_1$ transition are of order

$$\lambda A_1 = \Lambda \left| \frac{\gamma \hbar}{2} \right| \left( \frac{k_B T_c}{E_f} \right),$$

where $\gamma$ is the gyromagnetic ratio for $^3$He, $k_B T_c/E_f \sim 10^{-3}$ and $\Lambda \sim O(1-10)$ is a “high-energy” vertex. A first principles calculation of $\Lambda$ requires a solution of the many-body problem for the pairing interaction in $^3$He. Alternatively, we treat $\Lambda$ on the same level as other high-energy vertices in Fermi liquid theory; $\Lambda$ is a Fermi-liquid parameter, which can be determined by comparing physical predictions of the Fermi liquid theory with experiment - in this case the $A_1-A_2$ splitting. Once determined, the effects of $\Lambda$ on other low-energy properties of $^3$He can be calculated. Thus, the $A_2$ transition can be calculated in terms of $\Lambda$, and corrections to $T^{A_2}_c$ from the $\uparrow\uparrow$ pair condensate $^{11}$.

The disorder introduced by the aerogel structure into liquid $^3$He is, on average, weak on the high-energy scale, $\hbar/\tau E_f \ll 1$. Thus, the Fermi-liquid interactions are essentially unaffected by the aerogel, and we can calculate the effects of aerogel on the low-energy excitations and superfluid properties within Fermi liquid theory. The main effect of the aerogel structure is to scatter $^3$He quasiparticles moving with the Fermi velocity. At temperatures below $T^* \approx 10$ mK elastic scattering by the aerogel dominates inelastic quasiparticle-quasiparticle collisions. This limits the mean free path of normal $^3$He quasiparticles to $\ell \simeq 130 - 180$ nm. In p-wave superfluids quasiparticle scattering is intrinsically pairbreaking and leads to renormalization of nearly all properties of the superfluid phases, including the $A_1$ and $A_2$ transition temperatures. The suppression of $T_c$, as well as pair-breaking effects on observable properties of the superfluid phases, have been analyzed theoretically for non-magnetic scattering. Here we analyze the effects of scattering by the aerogel on the $A_1$ and $A_2$ transitions.

The aerogel has a strong effect on the short-distance,
high-energy properties of the liquid locally near the silica strands. The first few atomic layers of \(^3\)He are adsorbed on the silica structure and form a highly polarizable solid-like phase, observable as a Curie-like component of the magnetization of \(^3\)He-aerogel. \(^3\)He is in fast exchange with the liquid on typical NMR timescales, implying a liquid-solid exchange interaction, \(|J|/\hbar \gtrsim 0.66\) MHz (\(|J| \gtrsim 0.03\) mK). The exchange coupling of liquid \(^3\)He quasiparticles with the localized \(^3\)He spins, \(J\), may modify the scattering of \(^3\)He quasiparticles by the aerogel structure. Here we include the effect of magnetic scattering of \(^3\)He quasiparticles by polarizable \(^3\)He spins coating the aerogel strands. The differential scattering of \(\uparrow\) vs. \(\downarrow\) spin quasiparticles by the polarized surface leads to an additional contribution to the splitting of the \(\uparrow\uparrow\) vs. \(\downarrow\downarrow\) transitions, \(\lambda_3 \propto J\), which is determined by the non-magnetic, \(u_0\), and magnetic, \(J\), interactions and the density of \(^3\)He coating the aerogel.

Below we extend the analysis of Ref. \(^{22}\) and examine the role of the exchange coupling on the possible \(A_1 - A_2\) splitting of the superfluid phases of \(^3\)He in aerogel.

The suppression of the AB transition in both pure and disordered \(^3\)He is quadratic in field on a scale set by \(g_{\text{AB}} \sim \text{mK}/k_B^2\). Thus, for fields, \(B \gg B^* = \lambda^{A_1}/g_{\text{AB}} \sim 1\) G, and temperatures, \(T \approx T_c\), the \(\uparrow\downarrow\) pairs are suppressed. In the field and temperature range of interest we can restrict the full p-wave, spin-triplet order parameter to two components, \(d_{\pm}\) for \(\uparrow\uparrow\) pairs and \(d_{-}\) for \(\downarrow\downarrow\) pairs. We assume that the orbital state is the same for both spin-components and of the axial (ABM) form, \(\chi(p) = p \cdot (\mathbf{m} + i\mathbf{n})/\sqrt{2}\). Thus, in pure \(^3\)He the full Ginzburg-Landau (GL) free energy functional (c.f. Ref. \(^{22}\)) reduces to:

\[
\Omega[d_{+},d_{-}] = \alpha (|d_{+}|^2 + |d_{-}|^2) - \eta B (|d_{+}|^2 - |d_{-}|^2) + \beta_{24} (|d_{+}|^2 + |d_{-}|^2)^2 + 4\beta_{4} |d_{+}|^2 |d_{-}|^2, \tag{2}
\]

where \(\alpha, \eta\) and \(\beta\) are the known material parameters for superfluid \(^3\)He; \(\alpha(T) = 4N_f \ln(T/T_c)\) determines the zero-field transition and \(\eta = \frac{1}{2}N_f \lambda^{A_1}/T_c\) determines the \(A_1\) (\(\uparrow\uparrow\)) transition, where \(N_f\) is the single-spin density of states at the Fermi level. The fourth-order coefficients determine the relative stability of the possible phases. In particular, \(\beta_{24} > 0\), and \(\beta_{4} < 0\) favors an equal-spin-pairing (ESP) phase with \(|d_{+}| = |d_{-}|\). The linear field term is symmetry breaking and competes with the fourth-order terms. The latter wins at lower temperatures and gives rise to the \(A_2\) transition where \(\downarrow\downarrow\) spins condense, with \(T_{cA_2} = T_c - \lambda^{A_2} B\) and

\[
\lambda^{A_2} \approx \left(\frac{\beta_{24} + \beta_{4}}{-\beta_{4}}\right) \lambda^{A_1}. \tag{3}
\]

Within the homogeneous, isotropic scattering model (HSM) the rotational symmetry of \(^3\)He in aerogel is preserved on the coherence length scale, and the GL free energy has the same form as in pure \(^3\)He, but with material parameters, \(\bar{\alpha}, \bar{\eta}\), etc., that are modified by the effects of scattering by the aerogel (we use a ‘bar’ to denote the material parameters in the presence of aerogel scattering). These effects were calculated within the quasiclassical theory to leading order in \(T_c/E_f\) (weak-coupling), and one finds:

\[
\bar{\alpha} = \frac{1}{3}N_f [\ln(T/T_{c0}) - 2S_1(x)], \tag{4}
\]

where \(x = v_f/2\pi T\ell\), and \(\ell\) is the mean free path of quasiparticles scattering off the aerogel. In Eq. (4), and hereafter, we denote the transition temperature for pure \(^3\)He by \(T_{c0}\). The superfluid transition in aerogel is determined by the condition \(\bar{\alpha}(T_c) = 0\), and the function, \(S_1\), is

\[
S_1(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2n + 1 + x} - \frac{1}{2n + 1}\right). \tag{5}
\]

The parameter \(\eta\) is directly proportional to the high-energy vertex, \(A\), and so is un-renormalized by impurities to leading order in \(h/p\ell\). Thus, \(\bar{\eta} = \eta = \frac{1}{2N_f \lambda^{A_1}/T_{c0}}\), where \(\lambda^{A_1}\) is the rate for the splitting of the \(A_1\) transition in pure \(^3\)He. Although \(\eta\) is un-renormalized, the splitting parameters for the \(A_1\) and \(A_2\) transitions are renormalized by the impurity corrections to transition temperature and, in general, the \(\beta\) parameters.

The weak-coupling results for the fourth order coefficients are:

\[
\begin{align*}
\bar{\beta}_{24} &= -2\bar{\beta}_2 = 4(\bar{\beta}_{24c} + \bar{\beta}_{3c}) \tag{6} \\
\bar{\beta}_{24c} &= \frac{N_f}{30\pi^2 T^2} S_3(x) \\
\bar{\beta}_{3c} &= \frac{N_f}{9\pi^2 T^2} (\bar{\sigma}_0 - \frac{1}{2}) x S_4(x), \tag{7}
\end{align*}
\]

where \(\bar{\sigma}_0\) is the dimensionless, non-magnetic, s-wave scattering cross-section, \(0 < \bar{\sigma}_0 < 1\) (see below), and

\[
S_p(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2n + 1 + x}\right)^p, \quad p > 1. \tag{8}
\]

Note that the ratio, \(\bar{r}_\beta = \bar{\beta}_{24c}/(-\bar{\beta}_2) \approx 1\) in the weak-coupling limit, even with (isotropic) impurity scattering. However, this ratio deviates substantially from 1 in pure \(^3\)He, particularly at high pressures, e.g. \(r_\beta \approx 0.47\) at \(p = 33.4\) bar. Thus, the asymmetry of the \(A_1 - A_2\) transitions is a measure of the strong-coupling corrections to the \(\beta\) parameters, \(\delta \bar{\beta}_{24c} = \beta - \beta_{24c}\), which are of order

\[
\frac{\delta \bar{\beta}_{24c}}{\bar{\beta}_{24c}} \sim \frac{T_c}{E_f} \langle|\Gamma_n|^2\rangle_{FS}, \tag{9}
\]

compared to the weak-coupling values, where \(\langle\ldots\rangle_{FS}\) is a Fermi-surface average of the normal-state quasiparticle-quasiparticle collision rate, \(\propto |\Gamma_n|^2\).

Corrections to the weak-coupling \(\beta\) parameters from quasiparticle scattering off the aerogel strands are of order \(\delta \bar{\beta}_{24c}/\bar{\beta}_{24c} \sim x_c = v_f/2\pi T_c\), which is small for
high-porosity aerogels, but comparable to the strong-coupling corrections from quasiparticle-quasiparticle collisions. Based on the suppression of $T_c$ and the aerogel mean free path we estimate $x_c \simeq 0.12$ at high pressures.

If we neglect aerogel scattering corrections to the intermediate quasiparticle states that enter the strong-coupling self-energies, then the relative strong-coupling corrections for $^3$He in aerogel are scaled relative to their bulk ratios by the ratio of transition temperatures,

$$\frac{\delta \beta_{sc}}{\beta_{sc}} = \frac{\delta \beta_{sc}}{\beta_{sc}} \left( \frac{T_c}{T_{c0}} \right).$$

This approximation gives a good qualitative description of the suppression of strong-coupling parameters for $^3$He in aerogel as measured by the field-dependence of the AB transition.

A theoretical calculation of impurity scattering corrections to the strong-coupling $\beta$ parameters within the spin-fluctuation feedback theory of Brinkman and Anderson was carried out by Baramidze and Kharadze. Their theory predicts a suppression of strong-coupling effects with increased disorder, but at a rate that is slower than that predicted just on the basis of the suppression of $T_c$. We can use the results of Ref. 16 to estimate the strong-coupling correction to the predicted $A_2$ transition for $^3$He in aerogel. The results of Ref. 16 depend on a high-energy vertex, which we determine by corrections to the strong-coupling $\beta$-parameters calculated in Ref. 16, give $r_\beta = (1 - \frac{1}{2} \delta \beta_{sc}/\beta_{sc})/(1 + \frac{1}{2} \delta \beta_{sc}/\beta_{sc})$ where

$$\frac{\delta \beta_{sc}}{\beta_{sc}} = \frac{\delta \beta_{sc}}{\beta_{sc}} \left( \frac{T_c}{T_{c0}} \right) \left( \frac{S_2(x_c)/S_2(0)}{S_4(x_c)/S_4(0)} \right).$$

To leading order in the pair-breaking parameter, $\delta \beta_{sc}/\beta_{sc} \approx (\delta \beta_{sc}/\beta_{sc})(1 - a x_c)$. Based on Eq. 16, $a \simeq 2.47$. The rate of suppression is reduced to $a \simeq 1.28$ based on Eq. 17. In what follows we use Eq. 17 to estimate the suppression of the strong-coupling correction for $T_{cA}$. This correction turns out to be small, and relatively unimportant on the scale of corrections that are required to explain the lack of an $A_1$-$A_2$ splitting for $B \leq 5kG$.

Finally, before discussing the effects of the liquid-solid exchange coupling, we consider a simplified version of an inhomogeneous scattering model discussed by Ref. 18 that incorporates correlations of the aerogel. The length scale at which aerogel reveals inhomogeneity, $\xi_a \sim 30 - 100$ nm, is typically comparable to the pair correlation length, $\xi$, and has a substantial effect on the transition temperature of $^3$He in aerogel, particularly at high pressures. The inhomogeneity of the aerogel on scales $\xi_a \sim \xi$ leads to higher superfluid transition temperatures than predicted by the HSM with the same quasiparticle mean free path. Regions of lower aerogel density, of size of order $\xi_a$, are available for formation of the condensate. Thus, the qualitative picture is that of a random distribution of low density regions, ‘voids’, with a typical length scale, $\xi_a$, in an aerogel with a quasiparticle mean-free path, $\ell$. When $\xi \sim \xi_a \ll \ell$, the superfluid transition is determined by the pairbreaking effects of dense regions surrounding the ‘voids’, and scales as $\delta T_c/T_{c0} \propto - (\xi/\xi_a)^2$. However, when the pair size is much larger than $\xi_a$ the aerogel is effectively homogeneous on the scale of the pairs and pairbreaking results from homogeneous scattering defined by the transport mean free path, which scales as $\delta T_c/T_{c0} \propto - (\xi/\ell)^2$. This latter limit is achieved at low pressures. We incorporate the correlation effect by introducing an effective pairbreaking parameter in Eq. 19 that interpolates between these two limits, $x \rightarrow \tilde{x} = x/(1 + \xi_a/\xi)$, where $\xi_a \equiv \xi_a/\ell$. This heuristic treatment of aerogel correlations provides a good description of the pressure dependence of $T_c$ in zero field for $^3$He in aerogel over the whole pressure range, as shown in Fig. 1 for $\ell = 1400$ Å and $\xi_a = 502$ Å. Alternatively, we can adjust the mean-free path, $\ell$, with pressure in order to simulate the correlation effect on $T_c$. However, we prefer to identify $\ell$ with the pressure-independent geometric mean-free path and introduce aerogel correlation effects via the effective pair-breaking parameter, $\tilde{x}$. In either scenario, the GL theory for $^3$He in aerogel predicts transitions for the $A_1$ and $A_2$ phases, which correspond to the condensation of $\uparrow\uparrow$ and $\downarrow\downarrow$ Cooper pairs as in pure $^3$He; the transition temperatures are of the same form,

$$T_{cA} = T_c + \lambda^{A_1} B, \quad T_{cA} = T_c - \lambda^{A_2} B,$$
but with renormalized parameters,
\[ \tilde{\lambda}^{A_1} = \lambda^{A_1} \left( \frac{T_{c}}{T_{c0}} \right) \left( 1 + 2a'_{s}S_{2}(\tilde{x}_{c}) \right), \quad \tilde{\lambda}^{A_2} = \tilde{\rho}_{s} \tilde{\lambda}^{A_1}, \quad (13) \]
where \( \tilde{x}_{c} = \frac{d}{dT_{c}} \) and \( \tilde{\rho}_{s} = \beta_{s} u_{s} / (-\tilde{\beta}_{s}) \) is calculated including both impurity scattering and strong-coupling corrections as described above. These results predict an \( A_1-A_2 \) splitting of \( \Delta T_{c}^{A_1-A_2}/B = (\tilde{\lambda}^{A_1} + \lambda^{A_2}) \approx 6.3 \mu K/\text{Kg} \) at \( p = 33.4 \text{bar} \), comparable to that of pure \(^3\text{He}\). There is currently no experimental evidence of an \( A_1-A_2 \) splitting in \(^3\text{He}\)-aerogel. Since the width of the transition is less than 20 \( \mu K \), inhomogeneities within the aerogel cannot account for the absence of the \( A_1-A_2 \) splitting.

For \(^3\text{He}\) in aerogel an additional mechanism contributing to the splitting of the \( \uparrow \uparrow \) and \( \downarrow \downarrow \) transitions is possible. It originates from an exchange coupling between liquid \(^3\text{He}\) quasiparticles and the surface \(^3\text{He}\) spins adsorbed on the silica structure. Such a surface solid of \(^3\text{He}\) has been observed for \(^3\text{He}\) impregnated into silica aerogel. The signature is a Curie-like susceptibility, \( \chi_{s} = C / (T - \Theta_{s}) \), with a Curie temperature, \( \Theta_{s} \approx 0.4 \text{mK} \).

Thus, the model for scattering of quasiparticles by aerogel that we adopt is a modified version of the scattering model described above which includes an exchange coupling between \(^3\text{He}\) quasiparticles in the liquid and localized \(^3\text{He}\) atoms bound to the silica aerogel structure. This coupling is described by an exchange term in the quasiparticle-impurity potential,
\[ u = u_{0} + J S \cdot \sigma, \quad (14) \]
where \( J \) is the liquid-solid exchange coupling, \( S \) is the localized \(^3\text{He}\) spin operator and \( \sigma \) is the Pauli spin operator for the \(^3\text{He}\) quasiparticles. There are no direct measurements of \( J \) for \(^3\text{He}\) on aerogel, and theoretical calculations for \(^3\text{He}\) on planar substrates give indirect exchange interactions that vary over a wide range of values, \( J_{\text{mik}} \approx 0.1 \mu K - 1.0 \mu K \), and may be either ferromagnetic or anti-ferromagnetic depending on the specific mechanism and details of the theoretical model (c.f. Ref. 27).

In a magnetic field, \( B = -Bz \), the solid \(^3\text{He}\) spins are polarized, \( S = S(T, B)z \), with \( S(T, B) = \mathcal{P}(B, T)s \), where \( 0 \leq \mathcal{P} \leq 1 \) is the fractional polarization and \( s = \frac{1}{2} \). For sufficiently low fields, and temperatures well above the ordered temperature for the solid \(^3\text{He}\) spins, the polarization is expected to be linear in field with \( \mathcal{P}(B, T) \approx |B|/kB T \). In this limit the \( A_1-A_2 \) splitting is given by Eqs. (12) - (13), but with \( \tilde{\lambda}^{A_1} \rightarrow \lambda^{A_1} + \lambda_{t} \), where \( \lambda_{t} \) represents the effect of the surface polarization and exchange coupling on the transition temperatures for \( \uparrow \uparrow \) vs. \( \downarrow \downarrow \) pairs. The polarization-induced splitting, \( \lambda_{t} \propto J \), depends on the sign of the exchange coupling. Thus, this term may either enhance or reduce the intrinsic splitting, \( \tilde{\lambda}^{A_1} \). In what follows we calculate the exchange splitting, \( \lambda_{t} \), and discuss the result in relation to the existing data for \( T_{c} \).

To calculate the liquid-solid exchange contribution to the \( A_1-A_2 \) splitting we use the quasiclassical theory of superfluid \(^3\text{He}\) with effects of scattering by the aerogel described by the HSM modified to include the exchange coupling in Eq. (12). The instability temperatures for \( \uparrow \uparrow \) and \( \downarrow \downarrow \) Cooper pairs are obtained by solving the weak-coupling gap equation for the spin-triplet components of the order parameter,
\[
\frac{1}{3} \ln (T/T_{c0}) \Delta(p) = T \sum_{\varepsilon_{n}} \int \frac{d\Omega}{4\pi} (\hat{p} \cdot \hat{p'}) x (f(\hat{p'}; \varepsilon_{n}) - \frac{\Delta(\hat{p'})}{|\varepsilon_{n}|}), \quad (15)
\]
where \( f(\hat{p'}; \varepsilon_{n}) = (f_{1}, f_{2}, f_{3}) \) are the “pair” amplitudes for the three spin-triplet states: \( f_{\uparrow \uparrow} = f_{3}, f_{\downarrow \downarrow} = (-f_{1} + if_{2}) \) and \( f_{\uparrow \downarrow} = (f_{1} + if_{2}) \). The pairing interaction and density of states at the Fermi level, as well as the cutoff, have already been adsorbed into \( T_{c0} \).

The scattering of quasiparticles off the aerogel structure is described by a random distribution of scattering centers (“impurities”). The impurity self-energy, leading order in \( h/\tau E_{F} \), is determined by a t-matrix for multiple scattering by a single impurity and the mean density of impurities,
\[ \Sigma_{\text{imp}}(\hat{p}; \varepsilon_{n}) = n_{c}(\hat{p}, \hat{p'}; \varepsilon_{n}). \quad (16) \]

The model for scattering of quasiparticles by aerogel that we adopt is described by an isotropic, non-magnetic scattering amplitude, \( u_{0} \), and an exchange term in the quasiparticle-impurity scattering potential; in \( 4 \times 4 \) Nambu representation \( \hat{u} = u_{0} \hat{1} + J S \Sigma_{\varepsilon/2}, \) where \( \Sigma = (\hat{1} + \hat{r}_{3})\sigma/2 + (\hat{1} - \hat{r}_{3})\sigma' \) is the Nambu representation for the quasiparticle spin. For simplicity we also assume \( J \) to be isotropic. The t-matrix for repeated scattering of quasiparticles off a random distribution of these polarized scattering centers is
\[ \hat{i} = \hat{u} + N_{f} \hat{u}(\hat{g}) \hat{i}, \quad (17) \]
where \( (\hat{g}) \) is the Fermi-surface-averaged propagator. For normal \(^3\text{He}\) in aerogel and even in the presence of magnetic fields and magnetic scattering, the propagator reduces to \( \hat{g}_{N} = -i\pi \text{sgn}(\varepsilon_{n}) \hat{r}_{3} \). Thus, the solution to the scattering t-matrix is given by
\[ \hat{i} = \frac{1}{\pi N_{f}} (\hat{1} + is_{s} \hat{u}(\hat{g}) \hat{r}_{3})^{-1} \hat{u}, \quad (18) \]
where \( s_{s} = \text{sgn}(\varepsilon_{n}) \), and the dimensionless scattering potential is \( \hat{u} = u_{1} + v_{S} \hat{r}_{3} \), with \( u_{1} = \pi N_{f} u_{0}, v = \pi N_{f} J S \).

For non-magnetic scattering (\( S = 0 \)) the t-matrix is parameterized by the s-wave scattering phase shift, \( \delta_{0} = \tan^{-1}(u) \),
\[ \hat{i} = \frac{1}{\pi N_{f}} \sin \delta_{0} e^{-is_{s} \delta_{0} \hat{r}_{3}}, \quad (19) \]
In this minimal model for aerogel scattering, the mean density of impurities and scattering rate for normal quasi-particles are fixed by the mean free path, \( \ell \), and scattering cross-section, \( \sigma \),

\[
n_s = \frac{1}{\sigma \ell}, \quad \text{with} \quad \sigma = \frac{4 \pi h^2}{p_f^2} \bar{\sigma}_0 ,
\]  

where the normalized cross-section is related to the scattering potential by,

\[
\bar{\sigma}_0 = \frac{u^2}{1 + u^2} .
\]  

Note that \( \bar{\sigma}_0 \rightarrow 0 \) is the Born scattering limit, while \( \bar{\sigma}_0 \rightarrow 1 \) is the unitary limit.

When \( S \neq 0 \) there are different phase shifts for the scattering of \( \uparrow (\to) \) and \( \downarrow (\downarrow) \) spin quasiparticles, which we parameterize as

\[
\delta^\pm = \bar{\delta}_0 \pm \Delta \delta .
\]  

The t-matrix can now be expressed as,

\[
\hat{t} = \frac{1}{\pi N_f} \left\{ \sin \delta_0 \cos(\Delta \delta) \hat{1} + \cos \delta_0 \sin(\Delta \delta) \hat{\Sigma}_z \right\} \times e^{-i s_z \hat{\tau}_3} e^{-i s_\delta \Delta \Sigma_z \hat{\tau}_3} .
\]  

The quasiparticle-impurity scattering rates for \( \uparrow \) and \( \downarrow \) quasiparticles are calculated from the retarded self-energy, \( \Sigma^{\text{imp}}_n = n_s \hat{t}^2 \), obtained from Eq. (28) by setting \( s_x = + \). Thus, for quasi-particles the self-energy for spin \( \sigma_z = \uparrow \) and \( \sigma_z = \downarrow \) becomes,

\[
\Sigma^{\text{imp}}_{\uparrow \downarrow} = \Gamma_n \sin \delta_{\uparrow \downarrow} \left( \cos \delta_{\uparrow \downarrow} - i \sin \delta_{\uparrow \downarrow} \right) ,
\]  

where \( \Gamma_n = n_s/\pi N_f \). The scattering rates for \( \uparrow \) and \( \downarrow \) spin quasiparticles are then,

\[
\frac{1}{2 \tau_{\pm}} = \Gamma_n \sin^2 \delta^\pm = \Gamma_n \bar{\sigma}_{\pm} \equiv \Gamma_n \left( \frac{u \pm v}{1 + (u \pm v)} \right)^2 ,
\]  

where \( \bar{\sigma}_{\pm} \) is the dimensionless cross-section for scattering of \( \uparrow \) vs. \( \downarrow \) spin quasiparticles. In both the unitary (\( \delta_0 \rightarrow \pi/2 \)) and Born (\( \delta_0 \rightarrow 0 \)) limits, the \( \uparrow \) and \( \downarrow \) spin scattering rates are equivalent,

\[
\frac{1}{2 \tau_{\pm}} \rightarrow \begin{cases} \Gamma_n \cos^2(\Delta \delta), & \delta_0 = \pi/2 \\ \Gamma_n \sin^2(\Delta \delta), & \delta_0 = 0 . \end{cases}
\]  

Only when \( \delta_0 \neq 0, \pi \) is the scattering rate for \( \uparrow \) and \( \downarrow \) spin quasiparticles different. In general, we can parameterize the scattering rates as

\[
\frac{1}{\tau_\pm} = \frac{1}{\bar{\tau}} \frac{\bar{\sigma}_{\pm} - \bar{\sigma}_{-}}{\bar{\sigma}_{\pm} + \bar{\sigma}_{-}} ,
\]  

or equivalently,

\[
\frac{1}{\tau_\pm} = \frac{1}{\bar{\tau}} \left( \frac{\bar{\sigma}_{\pm} - \bar{\sigma}_{-}}{\bar{\sigma}_{\pm} + \bar{\sigma}_{-}} \right) ,
\]  

where \( 1/\bar{\tau} \) is the polarization-independent scattering rate. It is convenient to express the normal-state self-energy in terms of base particle-hole matrices,

\[
\hat{\Sigma}_n = \Sigma_{11} \hat{1} + \Sigma_{13} \hat{\Sigma}_z + \Sigma_{31} \hat{\tau}_3 + \Sigma_{33} \hat{\tau}_3 \hat{\Sigma}_z ,
\]  

with components

\[
\Sigma_{11} = -\frac{1}{2} \Gamma_n \sin(2\delta_0) \cos(\Delta \delta) , \quad \Sigma_{13} = \frac{1}{2} \Gamma_n \cos(2\delta_0) \sin(\Delta \delta) , \quad \Sigma_{31} = -\frac{i}{2} \Gamma_n \bar{\epsilon}_\pm [1 - \cos(2\delta_0) \cos(2\Delta \delta)] , \quad \Sigma_{33} = -\frac{i}{2} \Gamma_n \bar{\epsilon}_\pm \sin(2\delta_0) \sin(2\Delta \delta) .
\]  

To calculate the instability temperatures for \( \uparrow \uparrow \) and \( \downarrow \downarrow \) pairs we need the off-diagonal propagator to linear order in the pairing self-energy. Thus, we expand the transport equation, self-energies and normalization condition in powers \( \Delta \). The zeroth-order terms are the normal-state propagator and self-energy (Eq. (22)). To first-order we obtain

\[
\begin{bmatrix} i \epsilon_n \hat{\tau}_3 - \hat{\Sigma}_n, \hat{\varrho}^{(1)} \end{bmatrix} = \begin{bmatrix} \hat{\Delta}, \hat{\varrho}_{\pm} \end{bmatrix} ,
\]  

and \( \hat{\varrho}_{\pm}^{(1)} + \hat{\varrho}_{\pm}^{(1)} \hat{\tau}_3 = 0 \) from the normalization condition. We reduce the equations to \( 2 \times 2 \) spin-space by writing \( \hat{H}_n = i \epsilon_n \hat{\tau}_3 - \hat{\Sigma}_n = \frac{i}{2} (1 + \hat{\tau}_3) \hat{H}_n + \frac{i}{2} (1 - \hat{\tau}_3) \hat{H}_n \) with \( \hat{H}_n = i \epsilon_n - \hat{\Sigma}_n \) and \( \hat{H}_n = -i \epsilon_n - \hat{\Sigma}_n \). Note that \( \hat{\varrho}_{\pm}^{(1)} \) is purely off-diagonal, with the upper-right pair amplitude satisfying the equation in spin-space,

\[
\hat{H}_n f^{(1)} - f^{(1)} \hat{H}_n = 2i \pi \text{sgn}(\epsilon_n) \Delta .
\]  

Projecting out the spin-triplet components, we obtain

\[
(i \epsilon_n - \Sigma_{13}^{\pm}) f^{\pm} = i \pi \text{sgn}(\epsilon_n) \Delta_{\pm} \quad \text{with} \quad \Sigma_{13}^{\pm} = \Sigma_{31} \pm \Sigma_{33} = -\text{sgn}(\epsilon_n)/2 \tau_{\pm} , \text{and} \quad f_{\uparrow \downarrow (\downarrow \uparrow)} \equiv f_{\pm}^{\uparrow \downarrow (\downarrow \uparrow)} = \mp f_1 + i f_2 ,
\]  

\[
f_{\pm} = \frac{\pi \Delta_{\pm}}{|\epsilon_n| + 1/2 \tau_{\pm}} .
\]  

The linearized gap equations for \( \Delta_{\pm} \) are given by Eq. (I9) with \( \mathbf{f} \rightarrow f^{\pm}, \Delta \rightarrow \Delta_{\pm} \) and \( T \rightarrow T_c^{\pm} \). For non-unitary, axial states,

\[
\Delta_{\pm} = d_{\pm} \left( \hat{\varrho}_x + i \hat{\varrho}_y \right) / \sqrt{2} ,
\]  

the eigenvalue equation for \( d_{\pm} \) yields the weak-coupling equation for the instability temperatures, \( T_c^{\pm} \). In the absence of the polarization, the aerogel transition temperature is given by

\[
\ln \left( T_c / T_{c0} \right) = 2 S_1(x_c) ,
\]  

where \( x_c = 1/2 \pi \tau T_c \), and the spin-independent rate for quasiparticles scattering off the aerogel is given by

\[
\frac{1}{2 \tau} = \frac{1}{\bar{\tau}} \Gamma_n \sin^2 \delta_0 \equiv \Gamma_n \bar{\sigma}_0 .
\]
In the presence of a liquid-solid exchange coupling, and polarization of the solid $^3$He, the instability temperatures for $\uparrow\uparrow$ and $\downarrow\downarrow$ condensates are given by
\begin{equation}
\ln \left( \frac{T_c^-}{T_c^0} \right) = 2 S_1(x^\pm),
\end{equation}
where $x^\pm = 1/2\pi r_\lambda T_c^\pm$. For $u_0 \neq 0$, the leading order polarization correction to the scattering cross-sections gives
\begin{equation}
\frac{1}{\bar{\tau}} = \frac{1}{\tau}, \quad \frac{1}{2\tau_{\bar{\tau}}} = 2\pi \lambda J T_c, \quad \text{and we obtain,}
\lambda_3 = g_1 \left( \frac{\mu J}{k_B} \right) \left( \frac{1 - \bar{\sigma}_0^{3/2}}{\sigma_0} \right) \left( -2x_c S_2(x_c) - 1 - 2x_c S_2(x_c) \right),
\end{equation}
where $x_c = (\bar{\sigma}_0/\epsilon)(T_c/T_c^0)$, and the dimensionless exchange coupling is
\begin{equation}
g_1 = 2\pi N F J s.
\end{equation}
Note the impurity-induced exchange splitting vanishes in the unitary limit.$^{31}$ Equation (42) is easily generalized to include aerogel correlations within the heuristic ‘random void’ model described above; the result for $\lambda_3$ has the same form as Eq. (42), but with $-x_c S_2(x_c) \rightarrow x_c^2 S_2(x_c), \quad x_c = x_c^2/(x_c + \zeta^2), \quad x_c = 1/2\pi r T_c, \quad \text{and} \quad x_c^2 \equiv T_c d\tilde{\epsilon} / dT_c$.

The effects of the liquid-solid exchange coupling, $g_1$, and the polarization of the solid $^3$He coating the aerogel strands on the $A_1$-$A_2$ splitting are shown in Fig. 2 and compared with measurements of the superfluid transition in 98% aerogel reported in Ref. 23; these authors found no evidence of an $A_1$-$A_2$ splitting for fields up to $B = 5$ kG. The data for the superfluid transition of $^3$He in 98% aerogel for fields up to $B = 5$ kG are shown in Fig. 2. The error bars are conservative estimates of the uncertainty in defining $T_c$; the experiment shows no evidence of a splitting to within the error of determining $T_c$, and consequently we can assume that the splitting to be less than the error bars for $T_c$.\cite{23}

The calculation of the $A_1$-$A_2$ splitting includes aerogel correlations, which are most important at high pressures. Both the mean-free path, $\ell$, and the aerogel correlation length, $\xi_a$, contribute. The values of $\ell = 1400$ A and $\xi_a = 502$ A correspond to $T_c = 2.12$ mK at $p = 33.4$ bar and yield close agreement with $T_c(p)$ over the full pressure range. The dimensionless cross-section, $\sigma_0$, is not known with any certainty; there is likely a distribution of cross-sections provided by the aerogel. In the absence of detailed knowledge we assume an average value of $\bar{\sigma}_0 = 1/2$. The values of $\lambda^{A1}$ and $\lambda^{A2}$ for pure $^3$He, and thus the strong-coupling parameter, $r_{3A}$, are taken from Ref. 6. The effects of non-magnetic scattering by aerogel lead to small corrections for $\lambda^{A1}$ and $\lambda^{A2}$; these terms alone (shown in Fig. 2 as $g_1 = 0$) generate an $A_1$-$A_2$ splitting that is substantially larger ($\approx 2$) than the error reported for the superfluid $^3$He transition in Ref. 6. An anti-ferromagnetic exchange coupling ($g_1 > 0$) decreases the $A_1$-$A_2$ splitting. The magnitude of the predicted splitting is reduced to lie within the error bars for $T_c$ for $g_1 = 1.8 \times 10^{-4}$, which corresponds to an exchange coupling of $J \approx 0.1$ mK per liquid $^3$He spin.

The existing data, while suggestive that the liquid-solid layer coupling may be playing an important role in suppressing the $A_1$-$A_2$ splitting, is not conclusive. If
scattering by polarized $^3$He is responsible for the suppressed $A_1$-$A_2$ splitting for pure $^3$He in aerogel, then heat capacity or acoustic attenuation measurements with $^4$He added to displace the solid $^3$He, should exhibit an $A_1$-$A_2$ splitting that is comparable to that of pure bulk $^3$He. Measurements of Sprague et al. at $p = 18.7$ bar and at $B = 1.47$ kG do show and increase in $T_c$ from 1.69 mK without $^4$He coverage, to 1.76 mK with the addition of one monolayer of $^3$He to remove the solid $^3$He; thus, $\Delta T_c \approx 70 \mu$K. By comparison, if we suppress the polarization component of the scattering rate in our theoretical calculation we obtain and increase in $\Delta$ polarization component of the scattering rate in our theoretical calculation we obtain and increase in $\Delta T_c$, a factor of almost an order of magnitude smaller than the change in $\Delta T_c$. 

$\mu_0 = 1$ and at $B \lesssim B_s$, the $A_1$-$A_2$ splitting should emerge for fields above $B \approx 20$ kG.

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29. The $A_1$ phase corresponds to pairs of $^3$He quasiparticles with their magnetic moments aligned along the field. We follow the notation of Ref. 22 and define the quantization axis for the spin to be $\hat{z} - |\vec{B}|$ to compensate for the negative gyromagnetic ratio of $^3$He.
30. This is the result for $\sigma_0 = 1/2$. There is a correction to the third term on the right-side of Eq. 11 for $\sigma_0 \neq 1/2$.
31. The splitting also vanishes in the Born limit; although Eq. 28 is not valid in the Born limit since it is based on an expansion of $J/u_0$. Nevertheless, in the limit $u_0 = 0$, i.e. with only pure exchange coupling the cross-sections for $\uparrow$ and $\downarrow$ spin scattering are equal and the splitting vanishes.
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