ON THE DUFLO FORMULA FOR $L_\infty$-ALGEBRAS AND $Q$-MANIFOLDS

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ABSTRACT. We prove a direct analogue of the classical Duflo formula in the case of $L_\infty$-algebras. We conjecture an analogous formula in the case of an arbitrary $Q$-manifold. When $G$ is a compact connected Lie group, the Duflo theorem for the $Q$-manifold $(\Pi^*T^*G, d_{DR})$ is exactly the Duflo theorem for the Lie algebra $\mathfrak{g} = \text{Lie} G$. The corresponding theorem for the $Q$-manifold $(\Pi^*T^*M, d_{DR})$, where $M$ is an arbitrary smooth manifold, is a generalization of the Duflo theorem for the case of smooth manifolds. On the other hand, the Duflo theorem for the $Q$-manifold $(\Pi^*T^*\text{hol} M, \partial)$, where $M$ is a complex manifold, is a generalization of the M. Kontsevich’s “theorem on complex manifold” [K1], Sect. 8.4.

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1. THE CLASSICAL DUFLO FORMULA [D], [K1]

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $S^\bullet(\mathfrak{g})$ be the symmetric algebra of the vector space $\mathfrak{g}$, and $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. Both spaces $S^\bullet(\mathfrak{g})$ and $U(\mathfrak{g})$ are $\mathfrak{g}$-modules with respect to the adjoint action; it follows from the Poincaré–Birkhoff–Witt theorem that these modules are isomorphic. Therefore, the vector spaces of invariants $[S^\bullet(\mathfrak{g})]^\mathfrak{g}$ and $[U(\mathfrak{g})]^\mathfrak{g}$ are isomorphic. The Duflo theorem states that $[S^\bullet(\mathfrak{g})]^\mathfrak{g}$ and $[U(\mathfrak{g})]^\mathfrak{g}$ are canonically isomorphic as algebras. Moreover ([K1], Sect. 8.3) the algebras $H^\bullet_{\text{Lie}}(\mathfrak{g}; S^\bullet(\mathfrak{g}))$ and $H^\bullet_{\text{Lie}}(\mathfrak{g}; U(\mathfrak{g}))$ are canonically isomorphic.

Let us recall the construction of this isomorphism. First of all, the map $\varphi_{PBW} : S^\bullet(\mathfrak{g}) \to U(\mathfrak{g})$, defined as follows

\begin{equation}
\varphi_{PBW}(g_1 \cdot g_2 \cdot \ldots \cdot g_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} g_{\sigma(1)} * g_{\sigma(2)} * \ldots * g_{\sigma(k)}
\end{equation}

is an isomorphism of the $\mathfrak{g}$-modules. The corresponding map $\varphi_{PBW} : [S^\bullet(\mathfrak{g})]^\mathfrak{g} \xrightarrow{\sim} [U(\mathfrak{g})]^\mathfrak{g}$ is not an isomorphism of the algebras.

There exists an isomorphism of $\mathfrak{g}$-modules $\varphi_{\text{strange}} : S^\bullet(\mathfrak{g}) \xrightarrow{\sim} S^\bullet(\mathfrak{g})$ such that the composition

$[S^\bullet(\mathfrak{g})]^\mathfrak{g} \xrightarrow{\varphi_{\text{strange}}} [S^\bullet(\mathfrak{g})]^\mathfrak{g} \xrightarrow{\varphi_{PBW}} [U(\mathfrak{g})]^\mathfrak{g}$
is an isomorphism of the algebras.

The map \( \varphi_{\text{strange}} : S^*(\mathfrak{g}) \to S^*(\mathfrak{g}) \) is defined as follows. Let us consider elements of \( \mathfrak{g}^* \) as derivations of the symmetric algebra \( S^*(\mathfrak{g}) \), then elements of \( S^k(\mathfrak{g}^*) \) are differential operators with constant coefficients acting on \( S^*(\mathfrak{g}) \). If the Lie algebra \( \mathfrak{g} \) is finite-dimensional, there exists a canonical element \( c_k \in S^k(\mathfrak{g}^*) \) for every \( k \geq 1 \), namely, it is the trace of the \( k \)-th power of the adjoint action, \( c_k = \{ g \mapsto \text{Tr} \text{Ad}^k g \} \). The elements \( c_k \) are invariant elements in \( S^k(\mathfrak{g}^*) \). We set:

\[
\varphi_{\text{strange}} = \exp \left( \sum_{k \geq 1} \alpha_{2k} \cdot c_{2k} \right)
\]

where the rational numbers \( \alpha_{2k} \) are defined by the formula:

\[
\sum_{k \geq 1} \alpha_{2k} \cdot x^{2k} = \log \sqrt{\frac{e^x - e^{-x}}{x}}.
\]

It is clear, that \( \varphi_{\text{strange}} : S^*(\mathfrak{g}) \to S^*(\mathfrak{g}) \) is an isomorphism of the \( \mathfrak{g} \)-modules.

**Theorem ([D]).** For any finite-dimensional Lie algebra \( \mathfrak{g} \) the composition

\[
\varphi_{\text{PBW}} \circ \varphi_{\text{strange}} : [S^*(\mathfrak{g})]^\mathfrak{g} \to [U(\mathfrak{g})]^\mathfrak{g}
\]

is an isomorphism of the algebras.

\( \square \)

**Theorem ([K1], Sect. 8.3).** For any finite-dimensional Lie super-algebra \( \mathfrak{g} \) the composition

\[
\varphi_{\text{PBW}} \circ \varphi_{\text{strange}} : H^*_\text{Lie}(\mathfrak{g}; S^*(\mathfrak{g})) \to H^*(\mathfrak{g}; U(\mathfrak{g})).
\]

is an isomorphism of the algebras.

\( \square \)

In the present paper we prove the last Theorem also for differential graded Lie algebras. Moreover, after minor modifications the analogous statement is true also for strong homotopy Lie algebras (\( L_\infty \)-algebras). In fact, this result is a direct consequence of [KSh].

2. **STRONG HOMOTOPY LIE ALGEBRAS AND Q-MANIFOLDS**

2.1. A strong homotopy Lie algebra \( \mathfrak{g} \) (\( L_\infty \)-algebra) is, by definition, a \( \mathbb{Z} \)-graded vector space \( \mathfrak{g} \) and an odd vector field \( Q \) of degree +1 on the space \( \mathfrak{g}[1] \) such that \([Q, Q] = 0\).

On the other hand, it is an odd derivation \( Q \) of degree +1 on the algebra \( \bigwedge^\bullet(\mathfrak{g}^*) \), such that \( Q^2 = 0 \) (here \( \bigwedge^\bullet \) stands for the super-exterior algebra).

In the simplest case when \( \mathfrak{g} \) is a Lie algebra such a differential \( Q \) on \( \bigwedge^\bullet(\mathfrak{g}^*) \) is the cochain differential, it contains only a quadratic part. When \( \mathfrak{g} \) is a DG Lie algebra, the cochain differential on \( \bigwedge^\bullet(\mathfrak{g}^*) \) contain a linear and a quadratic parts. In other words, differential graded Lie algebras give us examples of \( L_\infty \)-algebras.

In the general case, the odd vector field \( Q \) contains also parts of 3-rd, 4-th, \ldots degree.

A Q-manifold is a smooth \( \mathbb{Z} \)-graded manifold \( X \) and an odd vector field \( Q \) on \( X \) of degree +1 such that \([Q, Q] = 0\). In other words, it is an odd derivation \( Q \) of the algebra of smooth functions \( C^\infty(X) \) such that \( Q^2 = 0 \). It is clear, that the case when \( X \) is an \( \mathbb{Z} \)-graded vector space is exactly the case of \( L_\infty \)-algebras.
Examples. (1) Let $Y$ be a smooth (purely even) manifold, $TY$ be its tangent bundle, and $X = T[1]Y$ be the tangent bundle with fibers $T_x[1]$. The algebra of functions on $T[1]Y$ is the algebra of differential forms on $Y$, the de Rham differential $d_{DR}$ acts on this algebra. Therefore $(T[1]Y, d_{DR})$ is an example of $Q$-manifold.

(2) Let $Y$ be a complex manifold, $X = \overline{T}_{hol}[1]Y$, where $T_{hol}Y$ is the holomorphic tangent bundle. Then $(\overline{T}_{hol}[1]Y, \overline{\partial})$ is an example of $Q$-manifold. The corresponding complex is the Dolbeault complex of the manifold $Y$.

2.2. Definition (algebra of polyvector fields on a $Q$-manifold). Let $(A, Q)$ be a differential graded commutative algebra, $\text{Der} A$ be the Lie algebra of derivations of the algebra $A$. Then $\text{Der} A$ is a complex, the differential $D : \text{Der} A \to (\text{Der} A)[1]$ is defined as the bracket with the derivation $Q$:

$$D(\xi) = [Q, \xi], \quad \xi \in \text{Der} A.$$ 

It is clear that $D(f \cdot \xi) = Q(f) \cdot \xi + f \cdot D\xi$, $f \in A$, $\xi \in \text{Der} A$. The differential $D$ acts by the Leibniz rule on the (super) exterior algebra over the algebra $A$, $\wedge^\bullet(\text{Der} A)$, we denote the last DG algebra by $T^\bullet_{poly}(A, Q)$, the algebra of polyvector fields. When $X$ is a $Q$-manifold, the algebra $T^\bullet_{poly}(X, Q)$ is, by definition, the DG algebra $T^\bullet_{poly}(\mathbb{C}\infty(X), Q)$.

Basic example. Let $g$ be a Lie algebra, $(\mathbb{g}[1], d_{\text{Lie}})$ be the corresponding $Q$-manifold. The DG algebra of functions on this $Q$-manifold is $C^\bullet_{\text{Lie}}(g; \mathbb{C})$, the cochain complex of the Lie algebra $g$. The DG algebra of polyvector fields $T^\bullet_{poly}(\mathbb{g}[1], d_{\text{Lie}})$ is the cochain complex $C^\bullet_{\text{Lie}}(g, S^\bullet(g))$ with the coefficient in the symmetric algebra of $g$.

2.3. Conjecture. Let $(A_1, Q_1)$ and $(A_2, Q_2)$ be two commutative smooth DG algebras, which are quasi-isomorphic. Then

$$H^\bullet(T^\bullet_{poly}(A_1, Q_1)) \simeq H^\bullet(T^\bullet_{poly}(A_2, Q_2))$$

as algebras.

(A DG algebra $(A, Q)$ is called smooth if it is the algebra of functions on a smooth $Q$-manifold.)

3. Relationship with the Formality Conjecture [K1], [KSh]

3.1. Let $(\mathbb{g}[1], Q)$ be an $L_\infty$-algebra, $(\wedge^\bullet g^*, Q)$ be the corresponding DG algebra of functions. In the case when $\mathbb{g}$ is a Lie algebra, $Q = d_{\text{Lie}}$, it is well-known result that

$$HH^\bullet(C^\bullet_{\text{Lie}}(g; \mathbb{C})) \simeq HH^\bullet(\mathbb{U}(g))$$

as algebras, where $HH^\bullet$ here stands for the Hochschild cohomology. On the other hand, $T^\bullet_{poly}(\mathbb{g}[1], d_{\text{Lie}}) = C^\bullet_{\text{Lie}}(g, S^\bullet(g))$.

Therefore, the following theorem can be considered as a generalization of the Duflo formula for the case of $L_\infty$-algebras.

Theorem. The algebras $H^\bullet(T^\bullet_{poly}(\mathbb{g}[1], Q))$ and $HH^\bullet(\wedge^\bullet g^*, Q)$ are canonically isomorphic.
We prove this theorem and construct an explicit isomorphism, analogous to the isomorphism $\varphi_{PBW} \circ \varphi_{\text{strange}}$ from Sect. 1, in the next Section. Here we explain, following [KSh], Sect. 4, how the Duflo theorem itself follows from the Formality conjecture of Maxim Kontsevich and his theorem on the cup-products on tangent cohomology.

**Theorem** (Formality conjecture; proved in [K1]). Let $T^\bullet_{\text{poly}}(\mathbb{R}^{m|n})$ be the DG Lie algebra of polyvector field on $\mathbb{R}^{m|n}$ with zero-differential and the Schouten–Nijenhuis bracket and let $D^\bullet_{\text{poly}}(\mathbb{R}^{m|n})$ be the DG Lie algebra of polydifferential operators on $\mathbb{R}^{m|n}$ with the Hochschild differential and the Gerstenhaber bracket. Then there exists an $L_\infty$-quasi-isomorphism $\mathcal{U}: T^\bullet_{\text{poly}} \to D^\bullet_{\text{poly}}$.

(Here $\mathbb{R}^{m|n}$ is a super-space $\mathbb{R}^m \oplus \mathbb{R}^n[1].$)

More generally, one can consider an arbitrary finite-dimensional $\mathbb{Z}$-graded vector space.

3.2. An $L_\infty$-morphism of two $L_\infty$-algebras is a $Q$-equivariant (may be nonlinear) map $\varphi: (g_1[1],0) \to (g_2[1],0)$. In particular, if $\gamma \in (g_1)^1$ is such that $Q_1|_\gamma = 0$ than $Q_2|_{\varphi(\gamma)} = 0$. In the case of DG Lie algebra $g$ the equation $Q|_\gamma = 0$ is exactly the Maurer–Cartan equation:

\[(4) \quad \gamma \in g^1, \quad d\gamma + \frac{1}{2}[\gamma, \gamma] = 0.\]

Also, $d + \text{ad } \gamma$ defines a new differential on $g$. Moreover, if $\varphi: (g_1[1],0) \to (g_2[1],0)$ is an $L_\infty$-morphism of DG Lie algebras, it defines a map of the complexes

\[(g_1, d_1 + \text{ad } \gamma) \to (g_2, d_2 + \text{ad } \varphi(\gamma))\]

for each solution $\gamma \in g^1_1$ of the Maurer–Cartan equation. We denote the *tangent complex* $(g, d + \text{ad } \gamma)$ by $T_\gamma(g)$ and we denote by $\varphi_\gamma: T_\gamma(g_1) \to T_{\varphi(\gamma)}(g_2)$ the corresponding tangent map. The map $\varphi_\gamma$ is a map of the complexes.

For any solution $\gamma \in T^1_{\text{poly}}(\mathbb{R}^{m|n})$ of the Maurer–Cartan equation there exists a product on the tangent complex $T_\gamma(T^\bullet_{\text{poly}}(\mathbb{R}^{m|n}))$ which coincides with the usual cup-product of the polyvector fields. On the other hand, for any solution $\gamma \in D^1_{\text{poly}}(\mathbb{R}^{m|n})$ of the Maurer–Cartan equation the usual product of Hochshild cochains:

\[(5) \quad (\varphi \circ \psi)(a_1 \otimes \ldots \otimes a_{k_1+k_2}) = \varphi(a_1 \otimes \ldots \otimes a_{k_1}) \cdot \psi(a_{k_1+1}, \ldots, a_{k_1+k_2})\]

(here $\varphi: A^{\otimes k_1} \to A$, $\psi: A^{\otimes k_2} \to A$ are Hochshild cochains on an algebra $A$) defines a product on the tangent complex $T_\gamma(D^\bullet_{\text{poly}}(\mathbb{R}^{m|n}))$.

**Theorem** (Theorem on cup-products on tangent cohomology, [K1], Sect. 8). Let $\mathcal{U}: T^\bullet_{\text{poly}}(\mathbb{R}^{m|n}) \to D^\bullet_{\text{poly}}(\mathbb{R}^{m|n})$ be the Formality $L_\infty$-morphism, and let $\gamma \in T^1_{\text{poly}}(\mathbb{R}^{m|n})$ be a solution of the Maurer–Cartan equation. Then the map

\[ [\mathcal{U}_\gamma]: H^\bullet(T_\gamma(T^\bullet_{\text{poly}})) \to H^\bullet(T_{\mathcal{U}(\gamma)}(D^\bullet_{\text{poly}})), \]

defined by the tangent map of the complexes:

\[ \mathcal{U}_\gamma: T_\gamma(T^\bullet_{\text{poly}}) \to T_{\mathcal{U}(\gamma)}(D^\bullet_{\text{poly}}), \]

is a morphism of the algebras. □
3.3. The differential graded Lie algebra $T^\bullet_{\text{poly}}(\mathbb{R}^m)$ is graded as follows:

$$T^i_{\text{poly}}(\mathbb{R}^m) = \{ (i + 1)\text{-polyvector fields} \}.$$  

In particular,

$$T^0_{\text{poly}}(\mathbb{R}^m) = \{ \text{vector fields on } \mathbb{R}^m \}.$$  

However, every odd vector field $Q$ of degree $+1$ on $\mathbb{R}^m$ lies in $T^1_{\text{poly}}(\mathbb{R}^m)$, and the Maurer–Cartan equation is exactly the equation $[Q, Q] = 0$.

Let $g$ be purely even finite-dimensional Lie algebra, and let $\gamma = \sum_{i,j,k} c^k_{ij} \xi_i \xi_j \frac{\partial}{\partial \xi_k}$ be the corresponding odd vector field on $g[1]$; the identity $[\gamma, \gamma] = 0$ is exactly the Jacobi identity.

Let us summarize some simplest facts on the tangent complex in this case.

**Lemma.** (i) $T_\gamma(T^\bullet_{\text{poly}}(g[1])) = C^\bullet_{\text{Lie}}(g; S^\bullet(g));$

(ii) $U(\gamma) = \left\{ f \mapsto \sum_{i,j,k} c^k_{ij} \xi_i \xi_j \frac{\partial f}{\partial \xi_k} \right\} \in \mathcal{D}^1_{\text{poly}}(g[1]);$

(iii) the tangent complex $T_\gamma(\mathcal{D}^\bullet_{\text{poly}}(g[1])) = CH^\bullet(C^\bullet_{\text{Lie}}(g; \mathbb{C}))$, the Hochschild cohomological complex of the cochain complex of the Lie algebra $g$.

It follows from Theorem 3.2 that the Formality $L_\infty$-morphism produces a map of the algebras

$$[U_\gamma]: H^\bullet_{\text{Lie}}(g; S^\bullet(g)) \rightarrow HH^\bullet(C^\bullet_{\text{Lie}}(g; \mathbb{C})).$$

3.3.1. **Lemma.** The map $[U_\gamma]$ is an isomorphism (of the vector spaces).

**Proof.** The statement of the Lemma follows from the homotopy theory of $L_\infty$-algebras, see [K1], Sect. 4.5.1. If $g_1, g_2$ are two DG algebras, and

$$\varphi: (g_1[1], 0) \rightarrow (g_2[1], 0)$$

is an $L_\infty$-quasi-isomorphism between them, and a solution $\gamma \in g^1_1$ of the Maurer–Cartan equation is sufficiently small, i.e. lies in an open neighbourhood of $0$ in $g[1]$, than the tangent map

$$[\varphi]: H^\bullet T_\gamma(g_1) \rightarrow H^\bullet T_{\varphi(\gamma)}(g_2)$$

is an isomorphism of the vector spaces. In our case, we can consider the vector field $\gamma_t = \sum_{i,j,k} t \cdot c^k_{ij} \xi_i \xi_j \frac{\partial}{\partial \xi_k}, t \in \mathbb{C}$, instead of the vector field $\gamma$. For sufficiently small $t$ the vector field $\gamma_t$ lies in any open neighbourhood of $0$ in $T^\bullet_{\text{poly}}(g[1])$; on the other hand, if $[U_{\gamma_t}]$ is an isomorphism for some $t \neq 0$, than $[U_\gamma]$ is also an isomorphism.

$\square$
4. Duflo formula for $L_\infty$-algebras

We want to describe explicitly the tangent map

$$\mathcal{U}_\gamma: C^*_\text{Lie}(g; S^\bullet(g)) \to CH^\bullet(C^*_\text{Lie}(g; \mathbb{C})).$$

Let $\gamma \in T^1_{\text{poly}}(\mathbb{R}^{m|n})$ be an arbitrary solution of the Maurer–Cartan equation, which is a vector field. The following description of the tangent map $\mathcal{U}_\gamma$ was found in [KSh].

We fix a basis $x_1, \ldots, x_{m+n}$ on $\mathbb{R}^{m|n}$.

(i) The map $\varphi_{\text{HKR}}: T^\bullet\text{poly}(\mathbb{R}^{m|n}) \to D^\bullet\text{poly}(\mathbb{R}^{m|n})$

This map is defined as follows:

if $\eta = \xi \wedge \cdots \wedge \xi_k$, $\xi \in T^\bullet\text{poly}(\mathbb{R}^{m|n})$, $\xi_1, \ldots, \xi_k$ are vector fields, then

$$\varphi_{\text{HKR}}(\eta)(f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \text{supersign}(\sigma) \cdot \xi_{\sigma(1)}(f_1) \cdots \xi_{\sigma(k)}(f_k).$$

(ii) the map $c_T: \text{Vect}(\mathbb{R}^{m|n}) \to \text{Hom}_{\text{Fun}}(\mathbb{R}^{m|n}) (\text{Vect}(\mathbb{R}^{m|n}) \to \text{Vect}(\mathbb{R}^{m|n}))$

Let $\eta \in \text{Vect}(\mathbb{R}^{m|n})$. We set

$$c_T(\eta) = \left\{ \partial_{I(2)} \mapsto \sum_{I(1), I(3)} \eta(dx^{I(1)}) \cdot \partial_{I(1)} \partial_{I(2)} \langle \gamma, dx^{I(3)} \rangle \partial_{I(3)} \right\}.$$

Here $\partial_i = \frac{\partial}{\partial x_i}$ and $\langle dx^i, \partial_j \rangle = \delta_{ij}$, and $I$ runs through all possible maps $I: \{1, 2, 3\} \mapsto \{1, 2, \ldots, m+n\}$.

The $k$-th power of the map $c_T$ is the map

$$c^k_T: \text{Vect}^\otimes k(\mathbb{R}^{m|n}) \to \text{Hom}_{\text{Fun}}^\otimes k(\text{Vect}, \text{Vect})$$

There exists the trace map

$$\text{Tr}: \text{Hom}_{\text{Fun}}^\otimes (\text{Vect}, \text{Vect}) \to \text{Fun},$$

and the composition $\text{Tr} \circ c^k_T$ is a map

$$\text{Tr} \circ c^k_T = c_k: \text{Vect}^\otimes k(\mathbb{R}^{m|n}) \to \text{Fun}(\mathbb{R}^{m|n}).$$

After the (super-) symmetrization we consider $c_k$ as an operator

$$c_k: T^\bullet_{\text{poly}}(\mathbb{R}^{m|n}) \to T^\bullet_{\text{poly}}(\mathbb{R}^{m|n}).$$
Example. In the case of the odd field \( \gamma = \sum_{i,j,k} c_{ij}^k \xi_i \xi_j \frac{\partial}{\partial_k} \) on the space \( g[1] \), where \( g \) is a finite-dimensional Lie algebra, we have

\[
\langle \gamma, dx^{I(3)} \rangle = \sum_{i,j} c_{i,j}^{I(3)} \xi_i \xi_j.
\]

Let \( \eta = \partial_l \) for some \( l = 1, \ldots, \dim g \). Then, by formula (7), the map

\[
c_T(\partial_l) = \begin{cases} 
\partial_{I(2)} & \rightarrow \sum_{I(3)} c_{l,I(2)}^{I(3)} \partial_{I(3)} \\
= -\text{ad}(\partial_l)
\end{cases}
\]

coincides, up to a sign, with the adjoint action.

**Theorem (KSh).** Let \( \gamma \in T^1_{\text{pol}}(\mathbb{R}^m|n) \) be an odd vector field such that \([\gamma, \gamma] = 0\). Then the tangent map

\[
U_\gamma: T_\gamma(T^\bullet_{\text{pol}}(\mathbb{R}^m|n)) \rightarrow T_{U(\gamma)}(D^\bullet_{\text{pol}}(\mathbb{R}^m|n))
\]

coincides with the composition

\[
U_\gamma = \varphi_{\text{HKR}} \circ \varphi_{\text{strange}},
\]

where

\[
\varphi_{\text{strange}} = \exp \left( \sum_{k \geq 1} \alpha_{2k} c_{2k} \right).
\]

(the rational numbers \( \alpha_{2k} \) were defined in Sect. 1)

**Theorem.** Let \((g[1], Q)\) be a finite-dimensional \( L_\infty \)-algebra, and let \( Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \ldots \), where \( Q^{(i)} \) is the part of the vector field \( Q \) of \( i \)-th degree. Then the map

\[
\varphi_{\text{HKR}} \circ \varphi_{\text{strange}}: H^\bullet(T^\bullet_{\text{pol}}(g[1], Q)) \rightarrow HH^\bullet(\wedge \bullet g^*, Q)
\]

is an isomorphism of the algebras, where the operators

\[
c_{2k}: T^\bullet_{\text{pol}}(g[1], Q) \rightarrow T^\bullet_{\text{pol}}(g[1], Q)
\]

are defined by formulas (7) and (8) when \( \gamma = Q \).

**Corollary.** Let \( g \) be a finite-dimensional differential graded Lie algebra, i.e. \( Q = Q^{(1)} + Q^{(2)} \). Then \( c_T, c_{2k} \) do not depend on \( Q^{(1)} \), i.e. on the differential in the DG Lie algebra \( g \). Therefore, the usual Duflo formula (Section 1) defines an isomorphism of the algebras

\[
\varphi_{\text{PBW}} \circ \varphi_{\text{strange}}: H^\bullet_{\text{Lie}}(S^\bullet(g)) \rightarrow HH^\bullet(C^\bullet_{\text{Lie}}(g; \mathbb{C})).
\]

**Proof.** The expression for the “Atiyah class” \( c_T \) depends only on the second derivatives of the vector field \( Q \) (see formula (7)), but \( Q^{(1)} \) is the linear term.
5. Duflo formula for $Q$-manifolds

Here we propose a conjecture giving an explicit formula for the isomorphism of the algebras:

$$H^\bullet(T^\bullet_{\text{poly}}(C^\infty(X), Q)) \to HH^\bullet(C^\infty(X), Q)$$

for any smooth $Q$-manifold $X$.

5.1. Let us explain why the Duflo isomorphism for a smooth $Q$-manifold $X$ should exist.

**Theorem** (Formality conjecture for smooth manifold; proved in [K1], Sect. 7). Let $X$ be a smooth super-manifold, $T^\bullet_{\text{poly}}(X)$ be the DG Lie algebra of smooth polyvector fields on $X$, and let $D^\bullet_{\text{poly}}(X)$ be the DG Lie algebra of smooth polydifferential operators on $X$. Then there exists an $L_\infty$-quasi-isomorphism $\mathcal{U}: T^\bullet_{\text{poly}}(X) \to D^\bullet_{\text{poly}}(X)$.

More generally, one can consider any smooth Zariski-graded manifold $X$.

There does not exist a canonical choice of this $L_\infty$-quasi-isomorphism. It was constructed in [K1], Sect. 7 canonically “up to a homotopy”.

**Conjecture** ([K2], Sect. 6.4). Let $X$ be a smooth super-manifold, $\mathcal{U}$ be an $L_\infty$-quasi-isomorphism $\mathcal{U}: T^\bullet_{\text{poly}}(X) \to D^\bullet_{\text{poly}}(X)$ constructed in [K1], Sect. 7, and let $\gamma \in T^1_{\text{poly}}(X)$ be such that $[\gamma, \gamma] = 0$ (the Maurer–Cartan equation). Then the map

$$[\mathcal{U}_\gamma]: H^\bullet T_\gamma(T_{\text{poly}}^\bullet(X)) \to H^\bullet T_\gamma(D_{\text{poly}}^\bullet(X))$$

induced by the tangent map $\mathcal{U}_\gamma$, is a morphism of the algebras.

One can also suppose, that $U_1 = \varphi_{\text{HKR}}$ and that $U(\gamma) = \varphi_{\text{HKR}}(\gamma)$.

Let $X$ be a smooth super-manifold, $Q$ be an odd vector field of degree $+1$ on $X$ such that $[Q, Q] = 0$ (i.e., $X$ is a $Q$-manifold). It follows from the above Conjecture that the tangent map

$$[\mathcal{U}_Q]: H^\bullet T^\bullet_{\text{poly}}(C^\infty(X), Q) \to HH^\bullet(C^\infty(X), Q)$$

is a morphism of the algebras. Then the arguments analogous to Lemma 3.3.1 shows that $[\mathcal{U}_Q]$ is in fact an *isomorphism* of the algebras.

It would be very interesting to find a description of the tangent map

$$\mathcal{U}_Q: T^\bullet_{\text{poly}}(C^\infty(X), Q) \to CH^\bullet(C^\infty(X), Q),$$

analogous to the description given in Theorem 6 in the local case. The problem is that the $L_\infty$-quasi-isomorphism $\mathcal{U}: T^\bullet_{\text{poly}}(X) \to D^\bullet_{\text{poly}}(X)$ is not defined canonically and therefore the tangent map $\mathcal{U}_Q$ also is not defined canonically. On the other hand, the map

$$[\mathcal{U}_Q]: H^\bullet T^\bullet_{\text{poly}}(C^\infty(X), Q) \to HH^\bullet(C^\infty(X), Q)$$

is defined canonically. The Conjecture 5.2 below (Duflo formula in the case of $Q$-manifolds) describes explicitly the map $[\mathcal{U}_Q]$.

5.1.1. One can apply Conjecture 2.3 instead of Conjecture 5.1. Indeed, the DG algebra $(C^\infty(X), Q)$ is quasi-isomorphic to a finite-dimensional $L_\infty$-algebra $(\mathfrak{g}[1], \tilde{Q})$. It is easy to see that the Hochschild cohomology of quasi-isomorphic DG algebras coincides. On the other hand, Conjecture 2.3 states that

$$H^\bullet(T^\bullet_{\text{poly}}(C^\infty(X), Q)) \simeq H^\bullet(T^\bullet_{\text{poly}}(\mathfrak{g}[1], \tilde{Q}))$$

as algebras. However, this approach does not lead us to the explicit formula.
5.2. In this section we formulate a Conjecture about the tangent map $[\mathcal{U}_Q]$.

5.2.1. The Atiyah class in Lie algebra cohomology. Let $X$ be a smooth super-manifold, $\text{Vect}(X)$ be the graded Lie algebra of the smooth vector fields on $X$. Let $TX$ be the tangent bundle of the manifold $X$. We denote by $J^1(TX)$ the bundle of 1-jets of the tangent bundle. The space of global sections $\Gamma_X(J^1(TX))$ has a natural structure of $\text{Vect}(X)$-module. There exists the canonical map of the $\text{Vect}(X)$-modules

$$\tilde{p} : \Gamma_X(J^1(TX)) \to \text{Vect}(X).$$

It is clear that the kernel of the map $\tilde{p}$ is the $\text{Vect}(X)$-module $\Omega^1_X \otimes C^\infty(X)$ $\text{Vect}(X)$. We obtain a short exact sequence

$$0 \to \Omega^1_X \otimes C^\infty(X) \to \Gamma_X(J^1(TX)) \to \text{Vect}(X) \to 0.$$  \hfill (9)

Let us note that both maps in (9) are maps of $C^\infty(X)$-modules. The short exact sequence (9) defines the “Atiyah class” $c_T \in C^1_{\text{Lie}}(\text{Vect}(X); \Omega^1_X \otimes C^\infty(X) \text{Hom}_{C^\infty(X)}(\text{Vect}(X), \text{Vect}(X))).$

5.2.2. Let $X$ be a smooth $Q$-manifold. The value $c_T(Q)$ of the Atiyah class (10) on the odd vector field $Q$ gives us an element

$$c_T(Q) \in \Omega^1_X \otimes C^\infty(X) \text{Hom}_{C^\infty(X)}(\text{Vect}(X), \text{Vect}(X)).$$

This is an explicit analogue of the Atiyah class in the case of $L_\infty$-algebras given by formula (7).

The $k$-th power of the element $c_T(Q)$ is a map

$$c^k_T(Q) : \text{Vect}^\otimes k(X) \to \text{Hom}_{C^\infty(X)}^\otimes k(\text{Vect}(X), \text{Vect}(X)).$$

Furthermore, there exists the trace map

$$\text{Tr} : \text{Hom}_{C^\infty(X)}^\otimes k(\text{Vect}(X), \text{Vect}(X)) \to C^\infty(X),$$

and we obtain an element

$$c_k = \text{Tr} \circ c^k_T(Q) : \text{Vect}^\otimes k(X) \to C^\infty(X).$$

After the (super-) symmetrization we can consider $c_k$ as operators

$$c_k : T^\bullet_{\text{poly}}(X) \to T^\bullet_{\text{poly}}(X).$$

Lemma. The map $c_k$ is a map of the complexes

$$c_k : T^\bullet_{\text{poly}}(X, Q) \to T^\bullet_{\text{poly}}(X, Q).$$

Proof. It is sufficient to prove that $[Q, c_T(Q)] = 0$. But $c_T$ is a 1-cocycle on $\text{Vect}(X)$, and therefore $(d_{\text{Lie}} c_T)(Q, Q) = 0$. By the definition of the cochain differential $d_{\text{Lie}}$, we have

$$(d_{\text{Lie}} c_T)(Q, Q) = c_T([Q, Q]) - 2[Q, c_T(Q)].$$

The desired result follows now from the identity $[Q, Q] = 0$. \hfill $\square$

We set:

$$\varphi_{\text{strange}} = \exp \left( \sum_{k \geq 1} \alpha_{2k} c_{2k} \right) : T^\bullet_{\text{poly}}(X, Q) \to T^\bullet_{\text{poly}}(X, Q),$$

where the numbers $\alpha_{2k}$ are defined by formula (3).
Conjecture (Duflo formula for $Q$-manifolds). Let $X$ be a smooth $Q$-manifold.

(i) the map

$$[\varphi_{\text{HKR}} \circ \varphi_{\text{strange}}]: H^\bullet(T^\bullet_{\text{poly}}(X,Q)) \to HH^\bullet(C^\infty(X),Q)$$

is an isomorphism of the algebras;

(ii) the map $[\mathcal{U}_Q]$, induced by the tangent map $\mathcal{U}_Q$, coincides with the map $[\varphi_{\text{HKR}} \circ \varphi_{\text{strange}}]$.

5.3. Examples.

5.3.1. (the de Rham complex). Let $Y$ be a (purely even) smooth manifold, and let $X = (T[1]Y, d_{\text{DR}})$. We consider $X$ as a $Q$-manifold, the DG algebra of functions $C^\infty(X,Q)$ coincides with the de Rham complex of the manifold $Y$.

In the case when $Y = G$ be a (connected compact) Lie group, the corresponding Duflo formula for $Q$-manifold $(T[1]G, d_{\text{DR}})$ can be considered as the classical Duflo formula for the Lie algebra $\mathfrak{g}$. We consider the Duflo formula for the $Q$-manifold $X = (T[1]Y, d_{\text{DR}})$ as an analogue of the classical Duflo formula for smooth manifolds.

5.3.2. (the Dolbeault complex). Let $Y$ be a complex manifold, $\overline{T}_{\text{hol}}Y$ be its holomorphic tangent bundle, $T_{\text{hol}}Y$ be its anti-holomorphic tangent bundle. We consider $X = (\overline{T}_{\text{hol}}[1]Y, \overline{\partial})$ as a $Q$-manifold, the DG algebra of functions $C^\infty(X,Q)$ coincides with the Dolbeault complex of the manifold $Y$.

There exist at least two different ways to define the notion of the “Hochschild cohomology of the structural sheaf $\mathcal{O}_Y$.”

5.3.2.1 (M. Kontsevich). One can define $HH^\bullet(\mathcal{O}_Y)$ as the algebra of $Ext$-s $Ext^\bullet_{\text{Coh}(Y \times Y)}(\mathcal{O}_{\text{diag}}, \mathcal{O}_{\text{diag}})$. The direct analogue of the Hochschild–Kostant–Rosenberg theorem states that

$$(12) \quad Ext^k_{\text{Coh}(Y \times Y)}(\mathcal{O}_{\text{diag}}, \mathcal{O}_{\text{diag}}) = \bigoplus_{i+j=k} H^i_{\text{sheaf}}(X, T^j)$$

(here $T^j$ be the sheaf of holomorphic $j$-polyvector fields on $Y$). There exist canonical products on both sides of (12): the Yoneda product on the Hochschild cohomology (see formula (5)) and the product induced by the usual cup-product of polyvector fields on the right-hand side.

The “theorem on complex manifold” of M. Kontsevich states that both algebras are canonically isomorphic. Let us recall the construction of this isomorphism.

Let $\alpha_T \in H^1_{\text{sheaf}}(Y, \Omega^1_{\text{hol}} \otimes \text{End} T_{\text{hol}})$ be the Atiyah class of the holomorphic tangent bundle (in the classical sense), it define the elements $c_k = \text{Tr} \circ \alpha_T^k \in H^k_{\text{sheaf}}(Y, \Omega^k_{\text{hol}})$, which can be considered as the Chern classes of the tangent bundle. One can consider $c_k$ as an operator

$$c_k: H^\bullet_{\text{sheaf}}(Y, T^\bullet_{\text{poly}}) \to H^\bullet_{\text{sheaf}}(Y, T^\bullet_{\text{poly}}).$$

Theorem (M. Kontsevich). The map

$$\varphi_{\text{HKR}} \circ \varphi_{\text{strange}}: H^\bullet_{\text{sheaf}}(Y, T^\bullet_{\text{poly}}) \to Ext^\bullet_{\text{Coh}(Y \times Y)}(\mathcal{O}_{\text{diag}}, \mathcal{O}_{\text{diag}})$$

is an isomorphism of the algebras.
5.3.2.2. We define the Hochschild cohomology \( HH^\bullet(\mathcal{O}_Y) \) of the structural sheaf \( \mathcal{O}_Y \) as the Hochschild cohomology of the corresponding Dolbeault complex, \( HH^\bullet(C^\infty(X), \overline{\partial}) \).

We claim, that this definition does not coincide with the definition given in Sect. 5.3.2.1. Conjecture 5.2 states, that the algebras \( HH^\bullet(C^\infty(X), \overline{\partial}) \) and \( H^\bullet T^\bullet_{\text{poly}}(C^\infty(X), \overline{\partial}) \) are isomorphic, and gives an explicit formula for the isomorphism.

We claim, that \( \bigoplus H^i(Y, T^j_{\text{poly}}) \) is a proper subalgebra of the algebra \( H^\bullet T^\bullet_{\text{poly}}(C^\infty(X), \overline{\partial}) \), and \( \text{Ext}^\bullet_{\text{Coh}(Y,Y)}(\mathcal{O}_\text{diag}, \mathcal{O}_\text{diag}) \) is a proper subalgebra of the algebra \( HH^\bullet(C^\infty(X), \overline{\partial}) \). Indeed, let us consider the derivations of the Dolbeault complex of the manifold \( Y \) of the form \( C^\infty(X) \otimes \mathcal{O} w \), where \( w \) is a holomorphic vector field on \( Y \); the differential \( \text{ad}(\overline{\partial}) \) on these derivations is the same as the differential in the Dolbeault complex \( D^\bullet(\text{Hol}) \) of the sheaf of holomorphic vector fields on \( Y \). It is clear that \( \bigwedge (C^\infty(X), \overline{\partial}) D^\bullet(\text{Hol}) \) is a proper DG subalgebra in \( T^\bullet_{\text{poly}}(C^\infty(X), \overline{\partial}) \); on the other hand,

\[
H^\bullet \left( \bigwedge (C^\infty(X), \overline{\partial}) D^\bullet(\text{Hol}) \right) \simeq \bigoplus H^\bullet_{\text{sheaf}}(Y, T^\bullet_{\text{poly}}).
\]

Therefore, Conjecture 5.2 in the case of the \( Q \)-manifold \( X = (T^\text{hol}[1] Y, \overline{\partial}) \) gives us a generalization of the M. Kontsevich’s theorem on complex manifold.

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