Tunable spin-orbit coupling and magnetic superstripe phase in a BEC

Xi-Wang Luo\textsuperscript{1} and Chuanwei Zhang\textsuperscript{1,}\textsuperscript{*}

\textsuperscript{1}Department of Physics, The University of Texas at Dallas, Richardson, Texas 75080-3021, USA

Superstripe phases in Bose-Einstein condensates (BECs), possessing both crystalline structure and superfluidity, opens a new avenue for exploring exotic quantum matters—supersolids. However, conclusive detection and further exploration of a superstripe is still challenging in experiments because of its short period, low visibility, fragility against magnetic field fluctuation or short lifetime. Here we propose a scheme in a spin-orbit coupled BEC which overcomes these obstacles and generates a robust magnetic superstripe phase, with only spin (no total) density modulation due to the magnetic translational symmetry, ready for direct real-space observation. In the scheme, two hyperfine spin states are individually Raman coupled with a largely-detuned third state, which induce a momentum-space separation between two lower band dispersions, yielding an effective spin-1/2 system with tunable spin-orbit coupling and Zeeman fields. Without effective Zeeman fields, spin-dependent interaction dominates, yielding a magnetic superstripe phase with a long tunable period and high visibility. Our scheme provides a platform for observing and exploring exotic properties of superstripe phases as well as novel physics with tunable spin-orbit coupling.

Introduction.— In supersolids, crystalline and superfluidity orders are formed through spontaneously breaking continuous translational and U(1) gauge symmetries [1]. The concept of supersolidity was originally discussed in solid $^4$He [2, 3], and later generalized to other superfluid systems that spontaneously form spatial periodicity. In particular, ultracold atomic gases provide a powerful platform for exploring quantum phases with supersolid-like properties [4–10]. For instance, a superstripe phase with spontaneously formed periodic density modulation has been theoretically proposed for a spin-orbit (SO) coupled Bose-Einstein condensate (BEC) with anisotropic spin interactions [11–15]. In this context, the recent experimental realization of SO coupling in ultracold atoms [16–27] paves a promising path for the observation and exploration of the long-sought supersolid phases. Here the pseudospin states could be formed by either two atomic hyperfine ground states [28–34] or two sites of a double well optical lattice [35]. For the later case, the crystalline structures of a BEC have been indirectly observed recently using Bragg reflection [36].

There are a few major obstacles [37–39] for conclusive observation and further exploration of superstripe phases in a SO coupled BEC: i) A superstripe is formed by the superposition of two plane waves separated by a large momentum, leading to a short period at the order of optical wavelength for the density modulation [16, 17]; ii) A superstripe phase is energetically unfavorable by density interaction $g_0$ due to its total density modulation, therefore could only exhibit a low visibility and exist in a small parameter region favored by weak spin interaction [12, 14]; iii) The superstripe phase for hyperfine state pseudospins is fragile against magnetic field fluctuation because the relative energy between two spin states is sensitive to the magnetic field [13, 18, 19]; iv) The superstripe phases for double-well lattice pseudospins (where SO coupling is realized by additional moving lattices) or dipole gases have a short lifetime [8–10, 36].

In this paper, we propose that all these obstacles can be completely overcome by engineering an effective spin-1/2 subsystem with tunable SO coupling in a spin-1 BEC (we use atomic hyperfine-state pseudospins to avoid heatings) [40–43], leading to a promising scheme for in-depth investigation of supersolidity. Our main results are:

i) We propose a generic and experimentally feasible scheme for generating an effective spin-1/2 system with tunable SO coupling through two individual Raman couplings of two spin states ($|\uparrow\rangle, |\downarrow\rangle$) with a third higher energy state ($|0\rangle$), which induce a momentum separation between two lower band dispersions, yielding SO coupling. The SO coupling strength can be widely tuned by varying laser and microwave intensities, in contrast to fixed SO coupling strength determined by the laser geometry in previous experiments [16].

ii) Because the SO coupling is induced by the Raman coupling with the third state, it can exist without an effective transverse field, where the total density modulation vanishes (due to magnetic translational symmetry) even when both band minima are occupied by the BEC. In this case, the spin interaction $g_2$, instead of density interaction $g_0$, dominates the phase diagram, leading to novel high-visibility ($\sim100\%$) magnetic superstripe phase with only spin density modulation. Depending on the SO coupling strength, the superstripe period is tunable up to $\sim5\mu m$, which can be directly imaged in the real space. Finally, the relative energy between two band minima is insensitive to magnetic field fluctuations, making the superstripe phase robust in experiments.

iii) Beside superstripe phases, we find a rich phase diagram with other novel phases in different parameter regions.

Experimental scheme and Hamiltonian:— We consider an experimental setup shown in Fig. 1(a), which is similar as that in a recent experiment [26] but with different laser configuration and additional microwave fields. Three Raman lasers are employed to couple hyperfine
to a quasi-momentum basis, the resulting single-particle interaction between zero momentum transfer. After a unitary transformation, the corresponding two-photon Raman transition and microwave transition (MT) between three hyperfine spin states. (c) Mapping to an effective spin-1/2 system, with spin states \(|\uparrow\rangle\) and \(|\downarrow\rangle\). \(|\uparrow\rangle\) and \(|\downarrow\rangle\) are coupled by a two-photon microwave transition via an intermediate virtual state \(|\downarrow\rangle\to|\uparrow\rangle\to|\downarrow\rangle\) with zero momentum transfer. After a unitary transformation \(U = \exp(\frac{i2k_B x}{\hbar})|0\rangle\langle0|\) that only transforms state \(|0\rangle\) to a quasi-momentum basis, the resulting single-particle Hamiltonian becomes

\[
H_0 = \frac{k^2}{2m} - 4(k + 4) - 1 + \Delta F_z^2 + \sqrt{2m_F x + m_F^2} + \frac{1}{2}m_z + \Delta \left(\frac{F_F^1}{F_F^2} - \frac{F_F^2}{F_F^1}\right).
\]

Here we set \(\hbar = 1\) and use the energy and momentum units \(k_B^2 / 2m\) and \(k_B\). \(F_i (i = x, y, z)\) are spin vectors and \(4\hat{k}_F F_z\) describes the spin-tensor-momentum coupling [43]. \(m_z (\delta)\) is the Raman (microwave) coupling strength between \(|\downarrow\rangle\) and \(|\uparrow\rangle\), which can be tuned with high precision. The phase difference between two Raman lasers with frequencies \(\omega_x\) and \(\omega_y\) is locked to the same value as that between two microwave fields such that \(m_z\) and \(\delta\) become real and positive by gauging out irrelevant phases. \(m_z\) and \(\Delta\) are linear and quadratic Zeeman fields that can be tuned by laser detunings.

Tunable SO coupling strength.—We consider a large \(\Delta \ll 0\) such that low energy dynamics are mainly characterized by spin states \(|\uparrow\rangle\) and \(|\downarrow\rangle\) with two band minima near \(k = 0\) [see Figs. 1(c) and (d)]. By hybridizing \(|\uparrow\rangle\) \((\downarrow) = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle\)] with \(|0\rangle\) [see Fig. 1(c)] for state \(|\uparrow\rangle\), the Raman coupling \(m_z\) induces a momentum shift for \(|\uparrow\rangle\) band with band minimum \(k_m < 0\) [see Fig. 1(d)]. The band for \(|\downarrow\rangle\) \(\equiv |\downarrow\rangle\) is unaffected by \(m_z\). To restore the degeneracy between two band minima, a two-photon microwave transition with \(\delta > 0\) is used to tune their relative energy, forming an effective spin-1/2 system [see Fig. 1(d)]. Here \(\delta\) is crucial because \(|\uparrow\rangle\) band would be always higher than \(|\downarrow\rangle\) band without \(\delta\) [43].

The low energy effective Hamiltonian in the basis \(|\uparrow\rangle, |\downarrow\rangle\) can be written as

\[
H_{\text{eff}} = \begin{bmatrix}
\eta (k - k_m)^2 & 0 \\
0 & k^2
\end{bmatrix}
+ B_{\text{z}} \sigma_z + B_{x} \sigma_x,
\]

leading to a SO coupling \(\eta k_m k \sigma_z\). The effective “detrapping” \(B_z\) and “Raman coupling” \(B_x\) between \(|\uparrow\rangle\) and \(|\downarrow\rangle\) bands can be tuned by \(\delta\) and \(m_z\) respectively (see Appendix). \(\eta\) is the mass ratio between \(|\uparrow\rangle\) and \(|\downarrow\rangle\) and \(k_m\) characterizes the SO coupling strength, which can be tuned by varying Raman laser intensities (i.e., \(m_x\)). In contrast, the SO strength is preset by Raman laser geometry [34] in previous experiments and its modulation through periodic fast modulation of laser intensities [47–49] may lead to significant heating issues and complex interaction effects.

Our scheme for tunable 1D SO coupling only relies on the existence of three hyperfine ground states that can be coupled with each other, therefore it can be applied to other alkali (e.g., potassium) and alkaline-earth-like atoms (e.g., strontium, ytterbium). The corresponding laser configurations could be slightly different (see Appendix).

Interacting phase diagram.—In the presence of atomic interaction, the effective spin-1/2 system with tunable SO coupling provides a path for realizing superstripe phases with long period and high visibility. For the simplicity of the presentation and accurate description of the results, we, however, still use the original spin-1 Hamiltonian (1) for our calculation.

The interaction energy density can be expressed as (see Appendix)

\[
\varepsilon_{\text{int}} = \frac{1}{V} \int dx \left[ \frac{g_0^2}{2} n_{\text{tot}}^2 + g_2 n_0 (n_{\downarrow} + n_{\uparrow}) + \frac{g_z^2}{2} x^2 \right],
\]

where \(V\) is the system volume and \(n_{\text{tot}}\), \(n_i (i = 0, \uparrow, \downarrow)\) are the total and spin densities, with \(F_z = n_{\uparrow} - n_{\downarrow}\) the polarization and \(g_0, g_z\) the density- and spin-interaction strengths. Under the Gross-Pitaevskii (GP) approximation, we adopt a variational ansatz as the general superposition of two plane waves around two band minima

\[
\Psi = \sqrt{n} (|c_1| \chi_1 e^{ik_1 x} + |c_2| \chi_2 e^{ik_2 x + i\alpha}),
\]

which is normalized by the average particle number density \(\bar{n} = V^{-1} \int d\Psi\dagger H_0 \Psi\), with three-component spinors \(\chi_j (\cos \theta_j, \cos \phi_j, -\sin \theta_j, \cos \phi_j \sin \theta_j)^T\) and \(|c_1|^2 + |c_2|^2 = 1\). The ground state is determined by minimizing the total energy density

\[
\varepsilon_{\text{tot}} = \varepsilon_{\text{int}} + \frac{1}{V} \int dx \Psi\dagger H_0 \Psi
\]
with respect to eight variational parameters $|c_1|$, $k_1$, $k_2$, $\theta_1$, $\theta_2$, $\phi_1$, $\phi_2$, and $\alpha$ (see Appendix). The phase diagram can be characterized by the atomic total density $n_{\text{tot}}$, spin density $n_i$ and polarization $\langle F_z \rangle$ which can be measured directly in experiments. We also obtain the ground states by directly simulating GP equation numerically, which are in good agreement with the variational results.

We first consider $m_z = 0$, where the spin states of two lower bands are orthogonal (i.e., $\langle \chi_1 | \chi_2 \rangle = 0$ for $B_z = 0$). Therefore, the total density is always a constant, and the density interaction $g_\delta$ plays no role for the phase diagram. The spin interaction $g_\delta$ tends to lower the energy by occupying both band minima, leading to a superstripe ground state. The phase diagram obtained from the variational method for ferromagnetic spin interaction (e.g., $^{87}$Rb with $g_\delta < 0$) is shown in Fig. 2(a) as a function of Raman couplings $m_x$ and $\delta$. There are four phases: the plane-wave phase PW1 (PW2) with zero spin polarization (i.e., $F_z = 0$) and single momentum occupation at the left (right) band minimum; the polarized plane-wave phase PW1-PPW with uniform spin polarization (i.e., $F_z \neq 0$) and single momentum occupation at the barrier between two band minima; the magnetic superstripe phase SS with striped spin polarizations $F_z$ (total density is uniform) and momentum occupations at both band minima [see the inset in Fig. 2(a)].

The plane-wave phases preserves the continuous translational symmetry with $T_d |\Psi\rangle = e^{i k_d d} |\Psi\rangle$, where $T_d$ is the translation operator. For the SS phase, we have $T_d |\Psi\rangle = \Lambda_d |\Psi\rangle$ with $\Lambda_d$ a spatially-independent unitary matrix because of $\langle \chi_1 | \chi_2 \rangle = 0$. In particular, we have $\Lambda_d = e^{i k_d d} |\chi_1\rangle \langle\chi_1| + e^{i k_d d} |\chi_2\rangle \langle\chi_2|$. This means that the SS phase breaks the translational symmetry but preserves a magnetic translational symmetry $T_m |\Psi\rangle = |\Psi\rangle$ with $T_m = \Lambda_d^\dagger T_d$. This magnetic translational symmetry is responsible to the uniform total density [since $n_{\text{tot}}(x + d) = |T_m |\Psi\rangle|^2 = |\Psi|^2 = n_{\text{tot}}(x)$].

Both PPW and SS phases result from the ferromagnetic spin interaction, and the total energy is minimized by generating non-zero spin polarizations $F_z$ (uniform in PPW and striped in SS). We note that only state $|0\rangle$ is transformed to the quasi-momentum basis, therefore the spin density modulation $F_z$ in SS phase are unaffected after transforming back to the real mechanical momentum. The uniform polarization $F_z$ in PPW phase can be either positive or negative due to the spontaneously breaking of the discrete $Z_2$ symmetry between states $|\uparrow\rangle$ and $|\downarrow\rangle$. In the supersolid-ordered SS phase, $n_\uparrow$ and $n_\downarrow$ exhibit out-of-phase density modulations (therefore leading to a nonzero spin-polarization modulation $F_z$) that spontaneously break the continuous translational symmetry due to the arbitrariness of relative phase $\alpha$ between two $k$ states. We can always choose the relative strength between the Raman and microwave couplings such that two band minima are degenerate [see the dashed line in Fig. 2(a)]; therefore, the SS phase can exist in a long ribbon along the degenerate line in the $m_x-\delta$ plane.

For a strong Raman coupling $m_x$, where two band minima are well separated [the upper part in Fig. 2(a)], the ground state prefers a plane wave (PW1) at the left band minimum when the microwave transition $\delta$ is weak. As we increase $\delta$ [which would rise (lower) the left (right) band minimum, the BEC starts to partially occupy the right band minimum, undergoing a second-order phase transition to the magnetic superstripe phase (SS) where both minima are populated. By further increasing $\delta$, the population of the right (left) minimum increases (decreases) until another second-order phase transition occurs where the BEC is fully transferred to the low-energy right minimum (PW2).

For weak Raman coupling $m_x$ [the lower part of the diagram of Fig. 2(a)], the two band minima are too close in momentum space to form the magnetic superstripe phase. If the system starts at the PW1 phase, it undergoes a second-order phase transition to PPW phase as

FIG. 2: (a) Phase diagram in the $m_x-\delta$ plane with $g_\delta n = -0.05$, $\Delta = -1$, and $m_x = 0$. Color bar shows the property of the polarization density (average of its absolute value). (b) Phase diagram in the $m_x-g_\delta$ plane with $\delta = 0.35$, other parameters are the same as in (a). Black (white) solid lines correspond to first (second) order phase transitions.

FIG. 3: Spin density modulations in the SS phase. (a) Ground state obtained from the variational ansatz for a non-trapped BEC. (b) Ground state for a trapped BEC (with trapping frequency 50Hz) obtained by directly solving the GP equation. Common parameters: $g_\delta n = -0.01$, $g_\Omega = 200|g_\delta|$, $\delta = 0.18$, and $m_x = 0.938$ ($m_x$ is chosen to obtain two degenerate band minima), and other parameters are the same as in Fig. 2(a). Blue (black) solid, green (light gray) solid, black dotted and purple dashed lines correspond to $n_\uparrow$, $n_\downarrow$, $n_0$ and $n_{\text{tot}}$, respectively. Raman lasers with 790nm wavelength (typical for alkali atoms) are used.

Common parameters:

- $\mu_\Omega = 0$.
- $\kappa = 0.001$, $\delta = 0.18$.
- $g_\delta n = -0.01$.
- $g_\Omega = 200|g_\delta|$.
- $m_\Omega = 0.938$.
- $\Delta = -1$.
- $\Delta = 1$.
- $\mu_\delta = 0.001$.
- $\mu_\delta = 0.0001$.
- $\mu_\delta = 0.00001$.
- $\mu_\delta = 0.000001$.
\( \delta \) increases, where the BEC would not partially occupy the right band minimum, but instead, starts to occupy two lower bands at the same momentum, generating a uniform spin polarization. Therefore, the BEC stays in a plane-wave state and shifts towards the right band minimum as a whole, where a second-order phase transition to PW2 occurs. The transition between SS and PPW phase is of first order, with their phase boundary ending at two triple points \( C_{1,2} \), as shown in Fig. 2(a). Compared with the SS phase, the PPW phase has a higher single-particle energy, but the total energy is favorable due to lower spin-interaction energy from its uniform spin polarization. As a result, the system prefers the PPW phase for weak Raman coupling \( m_x \) where the SO coupling is weak and band barrier is low. For conventional SO coupled spin-1/2 systems, atoms may condense at the barrier maximum only for very strong Raman coupling or interaction [14].

In Fig. 2(b) we plot the phase diagram in the \( g_2-m_x \) plane with a fixed \( \delta \). We see that the areas of PPW and SS phases shrink as \( |g_2| \) decreases. and the PPW phase in the weak SO coupling region is replaced by the SS phase as \( g_2 \) decreases. Therefore, the SS phase can have even longer period for weaker spin interaction. In Fig. 2(b) with spin interaction \( g_2\bar{n} = -0.05 \), the superstripe period can be up to around 3.8 \( \mu m \) (see Appendix). For \( g_2\bar{n} = -0.01 \), the period can be greater than 5 \( \mu m \), as shown in Fig. 3(a). Due to the uniform total density \( \langle n_0 \rangle \) is also uniform), the density interaction \( g_0 \) is irrelevant, and the spin interaction \( g_2 \) can lead to high-visibility (\( \sim 100\% \)) spin modulations in the SS phase, where the spin densities \( n_{\uparrow,\downarrow} \) show out-of-phase modulation with a long period and high visibility. In Fig. 3(b) we show the density distributions in the presence of a realistic harmonic trap, which are obtained by numerical simulation of the GP equation directly. Such long-period (\( \sim 5 \mu m \)) and high-visibility (\( \sim 100\% \)) magnetic superstripes can be directly detected by real-space imaging [50–52]. We emphasize that, here the long-period, high-visibility superstripe phase is the ground state possessing true supersolidity, which is different from the dynamically generated excited superstripe state [43].

Zeeman field effects. — So far we have focused on the case with zero linear Zeeman field \( m_z = 0 \). In a realistic experiment, though the detunings of laser frequencies can be tuned with high accuracy, the magnetic field fluctuation would lead to a non-zero \( m_z \). Therefore, the robustness of the superstripe phase against Zeeman field fluctuation is very important. In a conventional SO coupled spin-1/2 system [16, 19], the spin states are represented directly by the hyperfine states, leading to two band minima whose energies are sensitive to magnetic fields [14]. The superstripe phase is stable only in a narrow width \( |m_z| \lesssim g_2\bar{n}/4 \), which requires extreme control of ambient magnetic field fluctuations that is very challenging [36]. In our system, \( m_z \) acts like an effective “Raman coupling” which opens a band gap at the crossing point between two lower bands. We find that the SS phase could be very robust against such effective “Raman coupling”.

In the presence of \( m_z \), the spin state at the two band minima are no longer orthogonal, and the SS phase now possesses both spin and total density modulations (see Appendix), where \( g_0 \) becomes important and favors the plane-wave phases at large \( |m_z| \). In Fig. 4(a), we plot the phase diagram in the \( m_z-g_2/g_0 \) plane with fixed \( g_2 \), and a small \( m_x \) is used to obtain a small SO coupling \( k_m \sim k_R/4 \) (corresponding to a long superstripe period \( \sim 3.2 \mu m \) enough for direct real-space observation [50–52]). We find that even for strong density interaction \( |g_2|/g_0 \sim 0.005 \) (typical for \( ^8\text{Rb} \) atoms), the long-period, high visibility superstripes can exist up to a large Zeeman field \( |m_z| \sim g_2\bar{n} \) without involving strong total-density modulations.

The SS phase becomes more robust against \( m_z \) in the strong SO coupling region. Fig. 4(b) shows the phase diagram in the \( m_z-\delta \) for \( k_m \sim 1.5k_R \) (corresponding to a short superstripe period which may be observed by Bragg reflection). The system may stay in the SS phase until it shrinks to the triple point \( C_3 \) at extremely strong Zeeman field \( m_z \sim 10g_2\bar{n} \). The transition order can be revealed by looking at the behavior of \( \langle k \rangle, \langle F_z \rangle \) or the visibility (i.e., a jump in \( \langle k \rangle, \langle F_z \rangle \) or visibility represents a first-order transition). It is worth to mention that for \( m_z \neq 0 \), the Hamiltonian no longer has the symmetry between \( |\uparrow\rangle \) and \( |\downarrow\rangle \), and all phases have nonzero \( \langle F_z \rangle \). As a result, the phase transitions between PPW and PW1 (PW2) become crossovers (see Appendix).

Due to the hybridization between \( |+\rangle \) and \( |0\rangle \) for the \( |\uparrow\rangle \)-band, \( m_z \) would lower the right minimum more significantly. Therefore the global minimum may change from left to right as we increase \( m_z \), which drives the phase transitions from SS phase first to PW1 then to PW2 phases [see the left part in Fig. 4(a)]. In addition,
the transition from PW2 to PW1 occurs when the global band minimum is still the right one, which means that the BEC prefers the high-energy local minimum at the left [as schematically shown in the inset of Fig. 4(b)], where the hybridization between $|+\rangle$ and $|0\rangle$ leads to a lower interaction energy from $g_2n_0(n_t + n_l)$ that compensates the higher single-particle energy.

Conclusions.—In summary, we propose a scheme to realize a novel magnetic superstripe phase through engineering a spin-1/2 subsystem with tunable SO coupling in a spin-1 BEC. The tunable SO coupling could be generalized to other Bose and Fermi cold atomic systems, including Alkali-earth(-like) atoms. The system does not suffer heating issues and are robust against magnetic field fluctuations, making it a promising platform to explore supersolid physics (e.g., the phase transition, non-trivial dynamics, roton spectrum). More importantly, the superstripe phase has magnetic crystalline structure with a high visibility and long tunable period that can be directly detected by real-space imaging. Our scheme not only opens the possibility for exploring novel physics with tunable SO coupling; but also paves the way for conclusive (real-space) observation and exploration of long-sought supersolid phases in experiments.

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Appendix

Other experimental configurations for generating tunable spin-orbit (SO) coupling.—Our scheme for tunable 1D SO coupling only relies on the existence of three hyperfine ground states that can be coupled with each other, therefore it can be applied to other alkali (e.g., potassium) and alkaline-earth(-like) atoms (e.g., strontium, ytterbium). The corresponding laser configurations could be slightly different.

For instance, for fermionic $^{40}$K, we can choose $|\downarrow\rangle = |F = \frac{7}{2}, m_F = \frac{7}{2}\rangle$, $|\uparrow\rangle = |F = \frac{9}{2}, m_F = \frac{9}{2}\rangle$ and $|0\rangle = |F = \frac{9}{2}, m_F = \frac{9}{2}\rangle$ as the spin states [24] [Fig. A1(a)] with the same laser configuration as that in Fig. 1(a) in the main text. The phases of these Raman lasers are irrelevant because they can be gauged out in the definition of spin states. The coupling $\delta$ between $|\downarrow\rangle$ and $|\uparrow\rangle$ is realized directly by the Raman lasers $\omega_\uparrow$ and $\omega_\downarrow$ [24], and a positive $\delta > 0$ can be obtained using Raman lasers between D1 and D2 lines (no need for microwave fields). Notice that for $^{87}$Rb in the $F = 1$ manifold in the main text, the two-photon microwave transition is used to couple $|F = 1, m_F = 1\rangle (|\downarrow\rangle)$ and $|F = 1, m_F = -1\rangle (|\uparrow\rangle)$ because the corresponding Raman coupling need be near-resonance, yielding significant heating.

For the fermionic alkaline-earth(-like) atoms (e.g., $^{87}$Sr, $^{171}$Yb, $^{173}$Yb) [53–56], we can use two nuclear spin states in the $^1S_0$ manifold and one nuclear spin state in the $^3P_0$ manifold to represent a spin-1 system [see Fig. A1(b)]. Instead of a Raman process, the coupling between $|0\rangle$ and $|\uparrow, \downarrow\rangle$ is realized by the one-photon Rabi transition (i.e., the clock transition). The laser setup is similar as that in Fig. 1 in the main text, except that only two laser beams $\omega_\uparrow$ and $\omega_\downarrow$ are needed, whose polarizations are rotated by $\pi/4$ with respect to $z$-direction. These lasers can generate both $\pi$- and $\sigma^-$-clock transitions between $|0\rangle$ and $|\uparrow, \downarrow\rangle$. A one-photon microwave transition is also needed to achieve the coupling between $|\uparrow\rangle$ and $|\downarrow\rangle$. To obtain a positive $\delta$, the phase of the microwave field is locked to the same value.

FIG. A1: (a) Energy levels and Raman transitions to generate tunable SO coupling for $^{40}$K. (b) Energy levels and clock (microwave) transitions to generate tunable SO coupling for $^{171}$Yb.
as the phase difference between two clock lasers. The couplings with other nuclear spin states are suppressed due to the different Zeeman splitting and dipole potential [53, 54]. Similar spin-orbit coupling schemes can also be applied to fermionic species that are not considered in this work.

**Variational energy functional.**—In the basis \( \Psi = (\psi_\uparrow, \psi_0, \psi_\downarrow)^T \), the interaction energy in the laboratory frame is

\[
\varepsilon_{\text{int}} = \frac{1}{V} \int dx \left[ \frac{g_0}{2} n^2 + \frac{g_0}{2} (\Psi^\dagger F \Psi)^2 \right] = \frac{1}{V} \int dx \left[ \frac{g_0}{2} n^2 + g_2 n_0 (n_\uparrow + n_\downarrow) + \frac{g_2}{2} (n_\uparrow - n_\downarrow)^2 \right] + \frac{1}{V} \int dx g_2 \Re[\psi_\uparrow \psi_0^* \psi_0 \psi_\downarrow^*],
\]

(A1)

where \( F = (F_x, F_y, F_z) \). After the unitary transformation \( U = \exp(i 2 k_R x) |0\rangle \langle 0| \) to the quasi-momentum frame, the above equation is unchanged except that the last term becomes

\[
\frac{1}{V} \int dx g_2 \Re[\psi_\uparrow \psi_0^* \psi_0 \times \exp(4ik_R x)],
\]

(A2)

which is nonzero only when the state is a superposition of two plane waves with momentum separation \( 2k_R \). Here we focus on the case where the momentum separation is much smaller than \( 2k_R \), therefore this term becomes zero and we obtain the interaction energy Eq. (2) in the main text.

Using the variational ansatz

\[
\Psi = \sqrt{n} |c_1| \begin{pmatrix} \cos(\theta_1) \cos(\phi_1) \\ -\sin(\theta_1) \\ \cos(\theta_1) \sin(\phi_1) \end{pmatrix} e^{i k_1 x} + \sqrt{n} |c_2| \begin{pmatrix} \cos(\theta_2) \cos(\phi_2) \\ -\sin(\theta_2) \\ \cos(\theta_2) \sin(\phi_2) \end{pmatrix} e^{i k_2 x + i \alpha},
\]

(A3)

we obtain the single particle energy density

\[
\varepsilon_0 = \frac{1}{V} \int dx \Psi^\dagger H_0 \Psi \\
= \tilde{n} \sum_i |c_i|^2 \left\{ k_i^2 + \sin(2\theta_i) \sin(\phi_i + \pi/4) + \Delta + 4 - 4k_i + \delta \sin(2\phi_i) + m_z \cos(2\phi_i) \cos^2(\theta_i) \right\},
\]

(A4)

and the interaction energy density

\[
\varepsilon_{\text{int}} = \tilde{n} \frac{g_0}{2} \left\{ 1 + 2|c_1|^2 |c_2|^2 \sin(\theta_1) \sin(\theta_2) + \cos(\theta_1) \cos(\theta_2) \cos(\phi_1 - \phi_2) \right\} \\
+ \tilde{n} \frac{g_2}{2} \left\{ 2|c_1 c_2|^2 \left[ \cos^2(\theta_1) \cos^2(\theta_2) \cos^2(\phi_1 + \phi_2) + \sin(2\theta_1) \sin(2\theta_2) \cos(\phi_1 - \phi_2) \right] \\
+ \left[ \sum_i |c_i|^2 \cos^2(\theta_i) \cos(2\phi_i) \right]^2 + 2 \left[ \sum_i |c_i|^2 \sin^2(\theta_i) \right] \left[ \sum_i |c_i|^2 \cos^2(\theta_i) \right] \right\},
\]

(A5)

with the total energy density given by \( \varepsilon_{\text{tot}} = \varepsilon_0 + \varepsilon_{\text{int}} \).

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**FIG. A2:** (a) and (b) The dependence of \( B_z \) and \( k_m \) on the Raman and microwave couplings \( m_x, \delta \). The parameters are \( \Delta = -2, m_x = 1, \delta = 0.18, m_z = 0. \)
The transverse field $B$ Hamiltonian (in the basis low energy dynamics are characterized by an effective spin-1/2 system with tunable SO coupling, with an effective $\nu$ in Fig. 1(a) in the main text, with effects of density interaction can be suppressed by a weak or vanishing $B$. Maximum period is the SS phase is about 3.8 $\mu$m.

**Some details about tunable SO coupling and superstripe phase.** — As we discussed in the main text, the low energy dynamics are characterized by an effective spin-1/2 system with tunable SO coupling, with an effective Hamiltonian (in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$)

$$H_{\text{eff}} = \begin{bmatrix} \eta(k - k_m)^2 & 0 \\ 0 & k^2 \end{bmatrix} + B_z \sigma_z + B_x \sigma_x.$$  

(A6)

The transverse field $B_x$ is approximately given by the Zeeman field $m_z$, while the longitudinal field $B_z$ and the SO coupling strength $k_m$ can be tuned by varying Raman and microwave coupling strengths $m_x$ and $\delta$. In Fig. A2, we plot the dependence of $B_z$ and $k_m$ on $m_x$ and $\delta$.

Due to the tunability of the SO coupling, we can obtain superstripe phases (SS) with a tunable and long period, and the effects of density interaction can be suppressed by a weak or vanishing $B_x$, which leads to a high-visibility spin superstripe phase favored by the spin interaction. In Fig. A3, we plot the period and visibility of the superstripe phase in Fig. 1(a) in the main text.

We notice that the SO coupling is written in the basis $|\uparrow\rangle$ and $|\downarrow\rangle$ (which are approximately given by $|\uparrow\rangle \approx |+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\downarrow\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$), where the spin density modulation is formed in a different basis $|\uparrow\rangle$ and $|\downarrow\rangle$. A natural question to ask is whether high-visibility spin density modulation can be obtained in the basis $|+\rangle$ and $|-\rangle$ for conventional SO coupling scheme in the basis $|\uparrow\rangle$ and $|\downarrow\rangle$. The answer is no and the reason is illustrated below. In our scheme, only state $|0\rangle$ is transformed to the quasi-momentum frame, and states $|\uparrow\rangle$ and $|\downarrow\rangle$ are associated with atomic mechanical momentum, therefore the plane-wave superposition of $|+\rangle$ and $|-\rangle$ at different momenta gives rise to spin density modulation in the laboratory frame, with period directly determined by the momentum difference between two plane waves. While for conventional SO coupling scheme in the basis $|\uparrow\rangle$ and $|\downarrow\rangle$, both states $|\uparrow\rangle$ and $|\downarrow\rangle$ are transformed to the quasi-momentum frame, and the superstripe state in the quasi-momentum frame is [14]

$$(|\uparrow\rangle + \epsilon |\downarrow\rangle)e^{-ikx} + (|\downarrow\rangle + \epsilon |\uparrow\rangle)e^{ikx}.$$  

(A7)

![FIG. A3: (a) and (b) The corresponding visibility ($v$) and period ($P$) of the spin density modulations in the SS phase shown in Fig. 1(a) in the main text, with $v \equiv \frac{\max(n_{\uparrow,\downarrow}) - \min(n_{\uparrow,\downarrow})}{\max(n_{\uparrow,\downarrow}) + \min(n_{\uparrow,\downarrow})}$. We set $P = v = 0$ in the (polarized) plane-wave phases, and the maximum period is the SS phase is about 3.8 $\mu$m.](image)

![FIG. A4: (a) Spin density modulation in the SS phase with $\delta = 0.3$ and $m_z = 0.8g_0\bar{n}$. Both the total density and spin densities have slight periodic modulations. Other parameters are the same as Fig. 4(a) in the main text. (b) Change from second-order phase transitions to crossovers due to finite Zeeman field $m_z$. From the first order derivative of $F_x$ over $m_x$ (which equals to the second-order derivative of $\varepsilon_{\text{tot}}$ over $m_x$ due to the Hellmann-Feynman theorem, i.e., $F_{x}' = \varepsilon_{\text{tot}}''$), we see that The PPW-PW1 and PPW-PW2 boundaries change from second-order boundaries with $m_x = 0$ (blue solid line) to crossover boundaries with $m_x \neq 0$ (red dash-dotted line for $m_z = 10^{-4}$ and purple dashed line for $m_z = 10^{-3}$). Other parameters are $\Delta = -1$, $\delta = 0.1$.](image)
In the ideal case, we may have $k \approx k_R (1 - g_2 / 2g_0)$ and $\epsilon \approx \sqrt{g_2 / 2g_0}$. After transforming back to the laboratory frame, the above state in the basis $|\pm\rangle$ can be written as

$$[\cos(k x - k_R x) + \epsilon \cos(k x + k_R x)]|\pm\rangle + [\sin(k x - k_R x) + \epsilon \sin(k x + k_R x)]|\mp\rangle.$$  \hspace{1cm} (A8)

Without loss of generality, we consider the $|\pm\rangle$ state, where $\epsilon \cos(k x + k_R x)$ gives a short-period modulation ($\sim 0.4 \mu m$) with a low visibility $\sim \sqrt{g_2 / 2g_0}$ that is around 5% for typical parameters of $^{87}$Rb, while $\cos(k x - k_R x)$ gives an extremely long-period modulation around 300 $\mu m$ that is invisible for typical BEC cloud size (less than 100 $\mu m$)

In the presence of $m_z$ (i.e., $B_x \neq 0$), the spin states at two band minima are no longer orthogonal, and the SS phase now possesses both spin and total density modulations, as shown in Fig. A4(a). The Zeeman field $m_z$ breaks the $Z_2$ symmetry between $|\uparrow\rangle$ and $|\downarrow\rangle$, and all phases now have nonzero $\langle F_x \rangle$. The phase transitions between PPW and PW1 (PW2) become crossovers, as confirmed by our numerical results of the derivative of the ground-state energy [see Fig. A4(b)].

* Corresponding author. 
Email: chuanwei.zhang@utdallas.edu

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