On \( p \)-form vortex-lines equations on extended phase space

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In differential-geometric language, vortex-lines equations on extended phase space of a system may be written as \( i_{\dot{\gamma}}d\sigma = 0 \), where \( \sigma \) is a differential 1-form. This is the structure, to give a paradigmatic example, of the Hamilton equations. Here, we study equations of the same structure, where \( \sigma \) is a differential \( p \)-form.

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I. INTRODUCTION

In hydrodynamics, vortex lines are streamlines of vorticity vector field \( \omega \), which is curl of velocity field \( v \). In the language of differential forms in \( E^3 \), we can incorporate \( v \) into 1-form

\[
\theta = v \cdot dx
\]  

(Ref. 1). Then, for a steady flow, the 2-form \( d\theta \) just encodes \( \omega \)

\[
d\theta = \text{curl} v \cdot dS = \omega \cdot dS
\]  

Now, for any vector field \( W \), the interior product with \( d\theta \) results in

\[
i_{\omega}(\omega \cdot dS) = (\omega \times W) \cdot dx
\]  

Consequently, if we consider a curve \( \gamma(t) \leftrightarrow r(t) \) in \( E^3 \), equation

\[
i_{\dot{\gamma}}d\theta = 0
\]  

is equivalent to

\[
\omega \times \dot{r} = 0
\]  

i.e., in each point, the curve is forced to proceed along the vorticity vector field \( \omega \) (its parametrization is irrelevant).

So, equation (4) represents the vortex-lines equation in hydrodynamics.

It turns out that we encounter the same type of equation in Hamiltonian mechanics (see a wonderful exposition in Ref. 1). Consider, first, the extended phase space of a system. Let local coordinates be \( (q^a, p_a, t) \). Introduce a distinguished 1-form, Poincaré-Cartan integral invariant

\[
\sigma = p_a dq^a - H dt
\]  

Then, Hamilton equations

\[
\dot{q}^a = \frac{\partial H}{\partial p_a} \quad \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}
\]  

may be succinctly written as

\[
i_{\dot{\gamma}}d\sigma = 0 \quad \quad \dot{\gamma} = \dot{q}^a \partial q^a + \dot{p}_a \partial p_a + \partial_t
\]  

Therefore, solutions of Hamilton equations (8) are said to represent, in analogy with (4), vortex lines of the form \( \sigma \). (The curve \( \gamma \) lives in the extended phase space and its parametrization is, according to (8), arbitrary. However, its projection onto the phase space itself, \( \gamma \leftrightarrow (q^a(t), p_a(t)) \), becomes uniquely parametrized in terms of the time variable \( t \).

Finally, as Takhtajan showed in Ref. 3 (see also Eq. (10) in Ref. 4), we also encounter very similar equation in Nambu mechanics, Ref. 3. Consider first Nambu extended phase space, a space with coordinates \( (x, y, z, t) \). Introduce, following Ref. 3, a 2-form ("generalized Poincaré-Cartan integral invariant")

\[
\hat{\sigma} := xdy \wedge dz - H_1 dH_2 \wedge dt
\]  

Then one easily checks that the "vortex-lines type" equation

\[
i_{\dot{\gamma}}d\hat{\sigma} = 0 \quad \quad \dot{\gamma} = \dot{x} \partial_x + \dot{y} \partial_y + \dot{z} \partial_z + \partial_t
\]  

reproduces Nambu equations

\[
\dot{x}_i = \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}
\]  

or, in vector notation,

\[
\dot{r} = \nabla H_1 \times \nabla H_2
\]  

Note, however, that \( \hat{\sigma} \) is a two-form rather than a one-form, now.

And, as is also mentioned in Ref. 3, the procedure even works for the \( n \)-dimensional version of the Nambu mechanics (Refs. 3 and 5). One should simply write a straightforward generalization of (9),

\[
\hat{\sigma} := x^1 dx^2 \wedge \cdots \wedge dx^n - H_1 dH_2 \wedge \cdots \wedge dH_{n-1} \wedge dt
\]  

and then

\[
i_{\dot{\gamma}}d\hat{\sigma} = 0 \quad \quad \dot{\gamma} = \dot{x}^1 \partial_{x^1} + \cdots + \dot{x}^n \partial_{x^n} + \partial_t
\]  

instead of (9) and (10), in order to obtain the corresponding Nambu equations

\[
\dot{x}_i = \epsilon_{ijk...l} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} \cdots \frac{\partial H_{n-1}}{\partial x_l}
\]  

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(\(\dot{\sigma}\) is an \((n-1)\)-form, now).

So, vortex-lines type of equation on extended phase space turns out to play central role in both Hamiltonian mechanics (where \(\sigma\) is 1-form) as well as Nambu mechanics (where \(\sigma\) is \((n-1)\)-form). Therefore, it might be interesting to have a look at the equation in general. Are there any interesting "forgotten" (or, possibly, well-known) equations for \(p\) (the degree of the form \(\sigma\)) somewhere between \(p = 1\) and \(p = n - 1\)?

II. DECOMPOSED FORM OF THE EQUATION

Consider an extended phase space \(\mathbb{R} \times M\) where \(M\), the phase space itself, is an \(n\)-dimensional manifold with local coordinates \(x^i, i = 1, \ldots, n\) and \(\mathbb{R}\) represents the time axis (with coordinate \(t\)). Vectors tangent to the factor \(M\) are called spatial. Then, if the trajectory of a system is given by a curve \(\gamma(t)\) on \(\mathbb{R} \times M\), its velocity (tangent) vector may be written as

\[
\dot{\gamma} = \partial_t v
\]

where \(v = \dot{x}^i \partial_i\) is spatial, see examples [8], [10] and [14]. The time component is standardly normalized by \(\langle \dot{t}, \dot{\gamma} \rangle = 1\). Let \(\sigma\) be a \(p\)-form on \(\mathbb{R} \times M\). Then, the main object of our interest will be the vortex-lines equation

\[
i_v d\sigma = 0 \quad \sigma \text{ is } p\text{-form}
\]

There is a simple machinery for decomposing differential forms on product manifold \(\mathbb{R} \times M\) (see Appendix A). In particular, the \(p\)-form \(\sigma\) may be decomposed as

\[
\sigma = dt \wedge \hat{s} + \hat{r}
\]

and for \(d\sigma\) we get

\[
d\sigma = dt \wedge (\hat{d}s + \partial_i \hat{r} + \hat{d}\hat{r})
\]

\[
\equiv dt \wedge \hat{S} + \hat{R}
\]

Then, taking into account (16), we get from (17) the following two equations

\[
i_v \hat{R} = -\hat{S} \quad i_v \hat{S} = 0
\]

Note, however, that the second one is a simple consequence of the first one (since \(i_v i_v = 0\)). Therefore, the following (single) equation

\[
i_v \hat{R} = -\hat{S}
\]

or, in more detail

\[
i_v (\hat{d}\hat{r}) = \hat{d}s - \partial_i \hat{r}
\]

is equivalent to (17). The two versions, (17) and (22), express the same content in two different languages. Namely, equation (17) uses the language of “complete” extended phase space \(\mathbb{R} \times M\), whereas (22) says everything in terms of spatial objects (and operations) \(\hat{r}, \hat{s}, \hat{R}, \hat{S}, \hat{d}, v\), i.e. via “time-dependent objects on \(M\”).

III. INTEGRAL INVARIANTS, LIOUVILLE THEOREM

There are features shared by solutions of (17) regardless of concrete \(p\), resulting from the structure of the equations alone. In particular, the structure alone guarantees existence of a series of integral invariants.

First, integral of \(\sigma\) over a \(p\)-cycle (closed \(p\)-dimensional surface) in \(\mathbb{R} \times M\) is, according to Cartan (see Ref. 6), relative integral invariant. This means that

\[
\oint_{c_0} \sigma = \oint_{c_1} \sigma
\]

holds for any two \(p\)-cycles \(c_0\) and \(c_1\), encircling the same tube of solutions. (For the proof, see Appendix B.)

Then, integral of \(d\sigma\) over an arbitrary \((p + 1)\)-chain ((\(p + 1\))-dimensional surface, not necessarily a cycle) in \(\mathbb{R} \times M\) is absolute integral invariant (meaning there is no restriction to integrate over cycles).

In the same way one easily checks that relative invariants emerge from integrating

\[
\sigma, \quad \sigma \wedge d\sigma, \quad \sigma \wedge d\sigma \wedge d\sigma, \ldots
\]

over appropriate cycles and for corresponding absolute invariants we are to integrate

\[
d\sigma, \quad d\sigma \wedge d\sigma, \quad d\sigma \wedge d\sigma \wedge d\sigma, \ldots
\]

over appropriate chains.

We can choose the chains (in particular, cycles) to be spatial, i.e. lying in hypersurfaces of constant time. Then we get original Poincaré integral invariants. In their explicit expressions one can restrict the forms under integral sign to their spatial parts (since \(dt\) vanishes on the hypersurfaces). Therefore, we can write Poincaré invariants as integrals of the forms

\[
\hat{r}, \quad \hat{r} \wedge \hat{d}\hat{r}, \quad \hat{r} \wedge \hat{d}\hat{r} \wedge \hat{d}\hat{r}, \ldots
\]

(absolute) and

\[
\hat{d}\hat{r}, \quad \hat{d}\hat{r} \wedge \hat{d}\hat{r}, \quad \hat{d}\hat{r} \wedge \hat{d}\hat{r} \wedge \hat{d}\hat{r}, \ldots
\]

If the degree of some of the forms in (28) matches the dimension of the phase space \(M\), we can use it as the volume form on \(M\). The statement concerning the corresponding integral invariant is then nothing but the Liouville theorem, expressing invariance of volume of arbitrary domain on \(M\) under time development.

IV. EXAMPLES

Here we just summarize examples mentioned in Section II in terms of objects introduced in Section II and also provide an example, where the scheme does not work properly.
Example 1. Hamiltonian mechanics. Here
\[ \dot{s} = -H \quad \dot{\mathbf{r}} = p_a dq^a \]  
and \[ \dot{S} = \dot{\mathbf{r}} H \quad \dot{\mathbf{R}} = dp_a \wedge dq^a \]
Therefore, equation (22) says
\[ i_v(dp_a \wedge dq^a) = -\dot{d}H \]
which leads, taking into account
\[ v = \dot{q}^a \partial_{q^a} + \dot{p}_a \partial_{p_a} \]
to Hamilton equations (7). There is, in general, a series of relative integral invariants of type (27)
\[ p_a dq^a, \quad p_a dq^a \wedge dp_b \wedge dq^b, \ldots \]
as well as the corresponding number of absolute invariants of type (28)
\[ dp_a \wedge dq^a, \quad dp_a \wedge dq^a \wedge dp_b \wedge dq^b, \ldots \]
the last one providing the (standard) volume form
\[ d^m dq^m dp \text{ on } M, \quad n = 2m, \text{ used in Liouville theorem.} \]

Example 2. Basic Nambu mechanics. Here
\[ \dot{s} = H_1 \dot{H}_2 \quad \dot{\mathbf{r}} = xdy \wedge dz \]
\[ \dot{S} = -\dot{H}_1 \dot{H}_2 \quad \dot{\mathbf{R}} = dx \wedge dy \wedge dz \]
Therefore, equation (22) says
\[ i_v(dx \wedge dy \wedge dz) = \dot{d}H_1 \dot{\mathbf{H}}_2 \]
which leads, taking into account
\[ v = \dot{x} \partial_x + \dot{y} \partial_y + \dot{z} \partial_z \]
to Nambu equations (11) or (12).

Example 3. n-dimensional Nambu mechanics. Here
\[ \dot{s} = H_1 \dot{H}_2 \ldots \dot{H}_{n-1} \quad \dot{\mathbf{r}} = \mathbf{x}^i dx^2 \wedge \ldots \wedge dx^n \]
\[ \dot{S} = -\dot{H}_1 \dot{H}_2 \ldots \dot{H}_{n-1} \quad \dot{\mathbf{R}} = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \]
Therefore, equation (22) says
\[ i_v(dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n) = \dot{d}H_1 \dot{H}_2 \ldots \dot{H}_{n-1} \]
which leads, taking into account
\[ v = \dot{x} \partial_1 + \ldots + \dot{x}^n \partial_n \]
to Nambu equations (15). There is, for general \( n \), just a single relative invariant as well as a single absolute invariant of type (27) and (28), in both cases the first one in the series (the degrees of remaining forms are too high to accomodate the forms on \( M \)). The absolute invariant, integral of \( dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \), leads to Liouville theorem.

Example 4. The case \( \text{dim } M = 3, \text{ deg } \sigma = 1 \). This should serve as an elementary counterexample which demonstrates why one has to choose \( \sigma \) in (17) with due caution. So, consider phase space with coordinates \( (x, y, z, q, p, z) \), just like in Example 2, but now take
\[ \dot{s} = -H(q, p, z, t) \quad \dot{\mathbf{r}} = pdq \]
\[ \dot{S} = \dot{\mathbf{R}} = dp \wedge dq \]
so that
\[ \sigma = pdq - H dt \]
is one-form, now, rather than two-form. Then, equation (22) says
\[ i_v(dp \wedge dq) = -\dot{d}H \]
which leads, taking into account
\[ v = \dot{q} \partial_q + \dot{p} \partial_p + \dot{z} \partial_z \]
\[ \dot{d}H = (\partial_q H) dq + (\partial_p H) dp + (\partial_z H) dz \]
to the following system of equations
\[ \dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q} \quad 0 = -\frac{\partial H}{\partial z} \]
Clearly, we can not determine \( (q(t), p(t), z(t)) \) from (51), since there is no equation containing \( \dot{z} \).

[The last equation is a constraint on \( H \) rather than a differential equation. It excludes \( z \)-dependence of the Hamiltonian \( H \). If it is fulfilled, there is already no contradiction in (51). Now, we can determine \( (q(t), p(t)) \), but still we have no information about \( z(t) \).]

So, we cannot use forms (45), built into (18), to produce a reasonable vortex-lines equation (17). The problem lies in ranks of forms concerned. In particular, the form \( \dot{\mathbf{R}} \) has rank (only) 2 so that in \( v \mapsto i_v \dot{\mathbf{R}} \) one dimension is lost. That’s why equation (22) does not fix \( v \) uniquely. Therefore, one should take care of ranks, in general. This is done in more detail in the next section.

V. WHEN IT DOES WORK

Equation (22) may be regarded as a linear inhomogeneous system of the structure
\[ Av = a \]
for unknown vector \( v \), whereas the linear operator \( A \) and the vector \( a \) are given. If (17) is to represent a reasonable system of first-order differential equations (like (1), (11) or (15) and unlike (1)), the system (52) has to possess unique solution \( v \) for any given \( a \). In more detail, the map \( v \mapsto i_v \dot{\mathbf{R}} \) (see the l.h.s. of (22)) has to be injective, otherwise a part of \( v \) disappears, i.e. we miss some dotted coordinates in the resulting system of equations. But it
also has to be surjective, otherwise, for some choice of \( S \) on the r.h.s. of (22), we get contradictory equation. So, the map \( v \mapsto i_v \hat{R} \) should be a linear isomorphism of the \((n\text{-dimensional})\) linear space of all vectors \( v \) to the \((\binom{n}{p}\text{-dimensional})\) space of all \( p \)-forms \( \hat{S} \) in \( n\)-dimensional space. This needs, first, the equality of the dimensions

\[
n = \binom{n}{p}
\]  

(53)

Assuming \( n \) being given, we have just two solutions for \( p \), the degree of \( \sigma \):

\[
p = 1 \quad \text{or} \quad p = n - 1
\]  

(54)

Under these conditions, linear isomorphism is possible. In order the possibility be materialized, the rank of the map \( v \mapsto i_v \hat{R} \), which is, by definition, the rank of the form \( \hat{R} \), is to be maximal, i.e. \( n \).

So, we can summarize the conditions under which the equation (17) represents a reasonable system of first-order differential equations as follows:

\[
\hat{R} \text{ is exact 2-form or } n\text{-form}
\]  

(55)

rank of \( \hat{R} \) is \( n \)

(56)

(exactness of \( \hat{R} \) follows from (19) and (20)).

For the second possibility in (55), the additional requirement (56) is void. Indeed, each (non-zero) \( n\)-form has automatically rank \( n \) (in \( n\)-dimensional space; more generally, a \( p\)-form has rank at least \( p \), see e.g. Ref. 2). This possibility is realized in Nambu mechanics (see (42)). So, in Nambu mechanics, there is no concern about the rank of \( \hat{R} \), since it is automatically correct. Similarly, there is no restriction on the dimension \( n \) of the phase space; it is arbitrary.

For the first possibility in (55), on the contrary, the additional requirement (56) is non-trivial. In general, a closed 2-form can only have, due to the classical result of Darboux, even rank: 2, 4, …, \( n \) or \( n - 1 \) depending on whether \( n \) is even or odd. So, necessarily, our \( n \) is to be even, \( n = 2m \), and the rank of \( \hat{R} \) is to be \( 2m \). In Darboux coordinates, then, \( \hat{R} \) takes the form (30). This possibility is realized in standard Hamiltonian mechanics (see (30)).

VI. CONCLUSIONS

Closer inspection of ranks of relevant forms (and corresponding mappings) reveals, that Hamiltonian and Nambu equations actually represent the only meaningful realizations of the vortex-lines equation (17) on extended phase space. There are, to answer the question addressed at the end of section 1, no “forgotten” equations of this type, already.

Appendix A: Decomposition of forms

On \( \mathbb{R} \times M \), a \( p \)-form \( \alpha \) may be uniquely decomposed as

\[
\alpha = dt \wedge \dot{s} + \dot{r}
\]  

(A1)

where both \( \dot{s} \) and \( \dot{r} \) are spatial, i.e. they do not contain the factor \( dt \) in its coordinate presentation (here, we assume adapted coordinates, \( t \) on \( \mathbb{R} \) and some \( x^i \) on \( M \)). Simply, after writing the form in coordinates, one groups together all terms which do contain \( dt \) once and, similarly, all terms which do not contain \( dt \) at all. Note, however, that \( t \) still can enter components of any (even spatial) form. Therefore, when performing exterior derivative \( d \) of a spatial form, say \( \dot{r} \), there is a part, \( d\dot{r} \), which does not take into account the \( t \)-dependence of the components (if any; as if it was performed just on \( M \)), plus a part which, on the contrary, only operates on the \( t \) variable. Putting both parts together, we have

\[
d\dot{r} = dt \wedge \partial_t \dot{r} + d\dot{r}
\]  

(A2)

Then, for a general form (A1), we get

\[
d\alpha = dt \wedge (-d\dot{s} + \partial_t \dot{r}) + d\dot{r}
\]  

(A3)

Appendix B: A proof of (24)

The proof is amazingly simple (see Ref. 1). Consider integral of \( d\alpha \) over the \((p+1)\)-chain \( \Sigma \) given by the family of trajectories (solutions) connecting \( c_0 \) and \( c_1 \) (so that \( \partial \Sigma = c_0 - c_1 \)). Then

\[
\int_\Sigma d\alpha \bigg|_1^2 = \int_{\partial\Sigma} \sigma = \int_{c_0} - \int_{c_1} \sigma
\]

The second line (zero) comes from observation, that \( \dot{\gamma} \) is tangent to \( \Sigma \), so that integral of \( d\sigma \) over \( \Sigma \) consists of infinitesimal contributions proportional to \( d\sigma(\dot{\gamma}, \ldots) \), vanishing because of (17).

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