Abstract – This paper introduces a modeling framework for distributed regression with agents/experts observing attribute-distributed data (heterogeneous data). Under this model, a new algorithm, the iterative covariance optimization algorithm (ICOA), is designed to reshape the covariance matrix of the training residuals of individual agents so that the linear combination of the individual estimators minimizes the ensemble training error. Moreover, a scheme (Minimax Protection) is designed to provide a trade-off between the number of data instances transmitted among the agents and the performance of the ensemble estimator without undermining the convergence of the algorithm. This scheme also provides an upper bound (with high probability) on the test error of the ensemble estimator. The efficacy of ICOA combined with Minimax Protection and the comparison between the upper bound and actual performance are both demonstrated by simulations.

Keywords: Distributed learning, heterogeneous data, cooperative training

1 Introduction

Distributed learning is a field that generalizes classical machine learning algorithms to a distributed framework. Unlike the classical learning framework, where one has full access to all the data and has unlimited central computation capability, in the framework of distributed learning, the data are distributed among a number of agents, who have limited access to the data. These agents are capable of exchanging certain types of information, which, due to limited computational power and communication restrictions (limited bandwidth or limited power), is usually restricted in terms of content and amount. Research in distributed learning seeks effective learning algorithms and theoretical limits within such constraints on computation, communication, and confidentiality.

Distributed learning can be categorized into many subareas. In terms of the way that the data are distributed, it can be categorized into homogeneous data (instance/horizontally distributed data) and heterogeneous data (attribute/vertically distributed data). In terms of the structure of the entire system, it can be categorized into systems with a fusion center and systems without a fusion center. Each category has its unique applications and challenges. We focus, in this paper, on the case in which data are attribute-distributed. The algorithm that we develop can be adapted to systems either with or without a fusion center.

The homogeneous-data problems have been widely studied. Two important types of models are established in [1] and [2] respectively: instance distributed learning with and without a fusion center. The relationship between the information transmitted among individual agents and the fusion center and the ensemble learning capability are discussed in these papers. Classical learning algorithms are more easily adapted to the homogeneous cases because for each agent, the form of the classifier/estimator is exactly the same as that of the centralized learning algorithm. The homogeneity in the individual classifiers/estimators is a great advantage for designing distributed learning algorithms that compare and combine them.

However, these advantages disappear in the heterogeneous data case, where different agents observe different attributes, and thus have many different forms of classifiers/estimators. This makes it harder to evaluate, compare and combine the estimators. Nevertheless, there are some research results in this area (e.g., [3], [4], [5]). Some basic ideas include voting/averaging, meta-learning, collective data mining, and residual refitting. The voting/averaging algorithm simply combines (linearly) the predictions of the individual agents. The training process is purely non-cooperative. In the meta-learning case (see [3] and [9]), the fusion center seeks a
more sophisticated way to integrate predictions of individual estimators by taking their predictions as a new training set (learning of learning results), i.e., the fusion center treats the output of individual estimators as the input covariates. Although this hierarchical training scheme looks more delicate, it is still non-cooperative and hence fails to learn hidden rules in which covariates of different agents intertwine in a complicated way.

In contrast, the collective data mining algorithms (see [5], [7] and [8]) are cooperative. They seek the information required to be shared among the agents so that the optimal estimator can be decomposed into an additive form without compromising the performance of the ensemble estimator (compared to the estimator trained by the centralized algorithm). Yet this requirement is rather strong and hence this technique relies on specific types of transformations, which require much prior knowledge of the problem, and thus is hard to generalize to other problems. The residual refitting algorithm, another cooperative training algorithm described in [4], has the advantage of not being dependent on individual learning algorithms. The only way that the agents communicate with each other is through their residuals. However, these algorithms are based on an additive model, and are susceptible to overtraining and pitfalls of local optima, though under some assumptions, optimality can be guaranteed.

In this paper, we develop another cooperative training scheme, using a modeling framework similar to that of the residual-refitting algorithms. However, instead of refitting the residuals directly, our new algorithm seeks to reshape the covariance matrix of the residuals generated by all the agents so that the linear combination of the estimators maintained by the agents can achieve a low ensemble test error. Again, residuals are the only information that the agents communicate to each other, yet they are used more intelligently than in the case of residual-refitting. In the case when residual-refitting is guaranteed to achieve global optimality, our new algorithm, iterative covariance optimization algorithm (ICOA) also achieves similar results - and due to its insusceptibility to overtraining, ICOA usually outperforms.

Another important issue for a distributed learning system is the trade-off between the amount of information exchanged among the agents and the performance of the ensemble estimator/classifier. To study this relationship, the major challenge is how to quantify the information exchanged and the ensemble performance. In this paper, based on ICOA with Minimax Protection, the relationship between the amount of information exchanged (measured by the compression rate) and the optimal test error of the ensemble estimator is discussed. More interestingly, an upper bound on the test error of the ensemble estimator is also derived.

The rest of this paper is organized as follows. In Section 2, we describe the basic model and abstract the problem of finding an optimal additive ensemble estimator into a two-stage optimization. In Section 3, we analyze this optimization problem and introduce ICOA, with its efficacy demonstrated by simulation. In Section 4, we discuss the problem of how to keep ICOA functioning when the covariance is not accurately estimated, which leads to Minimax Protection, and we demonstrate the trade-off between data transmission and system performance with an upper bound on the test error with respect to the data compression rate. Section 5 contains our conclusions.

2 Model and problem

Our discussion is based on an estimation/regression problem with attribute-distributed data. The estimation problem is specified as follow:

There are $M$ covariates (or attributes) $X_1, \ldots, X_M$ and one outcome $Y$, so the entire data set of $N$ instances is comprised of

$$\{(x_{i1}, x_{i2}, \ldots, x_{iM}, y_i)\}_{i=1}^N$$

where $N$ is the number of instances, $x_{ij} \in \mathbb{R}$ is the $i$-th instance of $X_j$, and $y_i \in \mathbb{R}$ is the $i$-th instance of $Y$.

We assume that there exists a hidden deterministic function (rule/hypothesis) $\phi : \mathbb{R}^M \rightarrow \mathbb{R}$ such that

$$y_i = \phi(x_{i1}, x_{i2}, \ldots, x_{iM}) + w_i, \quad (1)$$

where $\{w_i\}_{i=1}^N$ is an independently drawn sample from a zero-mean random variable $W$ that is independent of $X_1, \ldots, X_M$ and $Y$.

Suppose there are $D$ agents, each of which has only limited access to certain attributes. Define $F_j$ (where $j = 1, \ldots, D$) to be the set of attributes accessible by agent $j$, and define $F = \bigcup_{j=1}^D F_j$, assuming that $|F| = M$. The outcome $Y$, with all its instances, is visible to all the agents. These assumptions specify the “attribute-distributed” properties of our problem.

To highlight the distributed nature of the system, we add an extra restriction: the only information that the agents can communicate with each other is their training residuals (or information that can be locally derived from the training residuals). This is a reasonable assumption considering that the data observable by one agent are usually confidential or incompatible with the learning algorithm run by another agent.

Therefore, for these $D$ agents, each agent $i$ maintains an estimator $f_i$ of the outcome, which is a function that takes covariates $X_F$ as its input. Given individual estimators $f_i$ fixed, the problem of finding an optimal ensemble estimator of additive form can be described as an optimization problem

$$\min_{a_1, \ldots, a_D} \mathbb{E} \left[ \left( Y - \sum_{i=1}^D a_i f_i(X_{F_i}) \right)^2 \right], \quad (2)$$
where \( a_i \) are the weighting coefficients. Moreover, if we assume that each estimator has no “bias” after training, or equivalently, if we assume that the residuals have zero mean, then it follows that \( \mathbb{E}[f_i(X_F)] = \mathbb{E}[Y] \). Therefore, it is obvious that the sum of all weighting coefficients is equal to 1, i.e. \( \sum_{i=1}^{D} a_i = 1 \).

Consequently, we can rewrite the objective function as

\[
\mathbb{E} \left[ \left( \sum_{i=1}^{D} a_i [Y - f_i(X_F)] \right)^2 \right].
\]  

(3)

Note that the \( i \)th term in the parentheses is the residual of the \( i \)th agent, defined as \( R_i = Y - f_i(X_F) \). Therefore, the objective function can be rewritten as

\[
\sum_{i=1}^{D} a_i R_i.
\]  

(4)

To simplify our derivation, define the covariance matrix of the residuals as \( \mathbf{A} \) where \( [\mathbf{A}]_{ij} = \text{cov}(R_i, R_j) \); then the problem can be further simplified into a more concise form:

\[
\begin{align*}
\min_{\mathbf{a}} & \quad \mathbf{a}^{T} \mathbf{A} \mathbf{a} \\
\text{s.t.} & \quad \mathbf{1}^{T} \mathbf{a} = 1
\end{align*}
\]  

(5)

(6)

where \( \mathbf{a} = [a_1 \ a_2 \ \cdots \ a_D]^T \). This is our starting point for the distributed regression problem. The optimization problem of finding the best ensemble estimator of the form of a linear combination of individual estimators is equivalent to finding the best individual estimators that generate the most desirable residuals that cancel each other out.

The problem described by (5) and (6) is readily solved if the covariance matrix \( \mathbf{A} \) is known and fixed. This is equivalent to the problem of finding the best linear combination of the estimators with these estimators given and fixed - a case that appears in the non-cooperative training algorithms. However, in a cooperative training algorithm, by communicating with each other, the agents have a chance to change their training residuals intelligently and repeatedly so as to reshape the covariance matrix \( \mathbf{A} \) and to minimize the ensemble error. It is this step that makes the problem interesting and difficult. Therefore, the entire problem can be summarized as a two-stage optimization problem:

\[
\begin{align*}
\min_{\mathbf{A}} & \quad \min_{\mathbf{a}} \mathbf{a}^{T} \mathbf{A} \mathbf{a} \\
\text{s.t.} & \quad \mathbf{1}^{T} \mathbf{a} = 1 \\
& \quad \mathbf{A} \text{ subject to training restrictions}
\end{align*}
\]  

(7)

(8)

(9)

Note that \( \mathbf{A} \) is subject to training restrictions because the residual generated by each agent is not arbitrary. For \( R_i \), it must be achievable in the form of \( Y - f_i(X_F) \), which is highly restrictive because of the space to which \( f_i \) belongs.

3 Solution to the two-stage optimization

The first (inner) step of the optimization has a closed form solution (solved by Lagrange multipliers). When

\[
\mathbf{a} = \frac{\mathbf{A}^{-1} \mathbf{1}}{\mathbf{1}^{T} \mathbf{A}^{-1} \mathbf{1}},
\]  

(10)

the minimum value \( \eta \) is achieved:

\[
\eta = \frac{1}{\mathbf{1}^{T} \mathbf{A}^{-1} \mathbf{1}},
\]  

(11)

i.e. the minimum value is the inverse of the sum of all the elements of the inverse of \( \mathbf{A} \). Moreover, since \( \mathbf{A} \) is a covariance matrix, and thus must be positive definite, the second stage optimization problem is equivalent to

\[
\max_{\mathbf{A}} \mathbf{1}^{T} \mathbf{A}^{-1} \mathbf{1}.
\]  

(12)

It is necessary to bear in mind that \( \mathbf{A} \) is subject to training restrictions.

The optimization problem described in (12) is the key step of our algorithm. The most difficult step is to quantify the “training constraints” of the covariance matrix of the residuals. To tackle this problem, it is necessary to examine the inner structure of \( \mathbf{A} \).

As previously assumed, we have \( D \) agents, and each agent \( i \) maintains an estimator specified by the function \( f_i(X_F) \). Then, obviously, the covariance matrix \( \mathbf{A} \) can be expressed as

\[
[\mathbf{A}]_{ij} = \mathbb{E} \left[ (Y - f_i(X_F)) (Y - f_j(X_F)) \right].
\]  

(13)

where \( [\mathbf{A}]_{ij} \) stands for the element of \( \mathbf{A} \) in the \( i \)th row and \( j \)th column. However, for numerical purposes, we need to write the matrix in the form of a statistic by describing everything in terms of actual data. So we characterize the function \( f_i \) by a vector \( \mathbf{f}_i \), which is the prediction of the function \( f_i \) on all the training data points of agent \( i \). Similarly, we define \( \mathbf{y} \) as the value of the outcome \( Y \) for all data instances. Then, the covariance matrix \( \mathbf{A} \) can be estimated by (with the assumption of the unbiasedness of all the estimators)

\[
[\mathbf{A}]_{ij} = \frac{1}{N} \left[ (\mathbf{y} - \mathbf{f}_i)^T (\mathbf{y} - \mathbf{f}_j) \right].
\]  

(14)

With this notation, the optimization problem can now be converted into a more specific and implementable one:

\[
\begin{align*}
\max_{f_1, \ldots, f_D} & \quad \mathbf{1}^{T} \mathbf{A}^{-1} \mathbf{1} \\
\text{s.t.} & \quad \mathbf{f}_i \in \mathcal{H}_i, i = 1, \ldots, D,
\end{align*}
\]  

(15)

(16)

where \( \mathcal{H}_i \) denotes the space to which \( \mathbf{f}_i \) belongs, which depends on the class of functions in which \( f_i \) resides. Thus, the constraints on \( \mathbf{A} \) are implicitly included in the constraints of the vectors \( \mathbf{f}_1, \ldots, \mathbf{f}_m \).
The next step, ordinarily, is to massage the optimization problem and prove the convexity of the objective function and the domain so that we can apply gradient descent algorithms and guarantee global optimality. Yet for our problem, since the objective and constraints are both rather intricate and to some extent only implicitly specified, it is not very feasible to prove convexity without additional assumptions. Therefore, we will directly develop an algorithm based on gradient descent and test the algorithm empirically before we delve deeply into the problem of global optimality.

3.1 Iterative covariance optimization algorithm

The first thing required for a gradient-based algorithm is to find an expression for the gradient of the objective $\eta = 1^T A^{-1} 1$ with respect to $f_i$. By rather lengthy and intricate computation, a closed form expression for the gradient is given by

$$ \frac{\partial \eta}{\partial f_i} = \frac{2}{|A|^2} (1^T A^* 1) \left( \sum_{j=1}^{D} (y - f_j) [A^*]_{ij} \right) - \frac{2}{|A|} \left( \sum_{j \neq i} (f_k - f_j) [B^*(k)]_{ij} \right), $$

where $k \neq i$, $A^*$ denotes the adjoint of $A$, and $B(k)$ is a $(D - 1) \times (D - 1)$ matrix given by

$$ [B(k)]_{ij} = (f_k - f_{i+\zeta_ik})^T (f_k - f_{j+\zeta_jk}) $$

where $\zeta_{ik} = 0$ if $i < k$, and $\zeta_{ik} = 1$ if $i \geq k$.

This provides us with a feasible yet complex algorithm for estimating the gradient. It is worth noting that the gradient depends only on the residuals of the agents (through easy conversion of $f_i$ as $y - r_i$). In practice, we can also use numerical methods to estimate the gradient, i.e., we can perturb the components of $f_i$, compute the change of the objective and use the ratio between the change and the perturbation as an approximation of that component of the gradient.

There is another important issue before we develop the algorithm: we can search, by gradient descent, for a desirable $\hat{f}_i$ to replace $f_i$ so that we can increase the value of the objective function, yet $\hat{f}_i$ might not be achievable because agent $i$ may not be able to find a new estimator $\hat{f}_i$ such that $\hat{f}_i$ is realizable by $\hat{f}_i(X_{F_i})$. Therefore, what is reasonable to do is to use $\hat{f}_i$ as the new outcome for agent $i$ (instead of $y$) to train and find a new estimator $f_i$, i.e., we find the best projection of $\hat{f}_i$ onto the space $H_i$.

Based on the description above, the basic idea of ICOA is summarized as follows. First, cooperatively, all the agents determine the present covariance matrix of their residuals $A$. Then, one by one, each agent finds its estimate of the gradient $\frac{\partial 1^T A^{-1} 1}{\partial f_i}$, after which the selected agent $i$ updates its vector $f_i$ to $\hat{f}_i$ using gradient descent. After that, agent $i$ projects $\hat{f}_i$ onto $H_i$ by training with $f_i$ as the outcome and thus obtains the new version of $f_i$. Then, after agent $i$ updates its residual, all the agents update their estimates of covariance matrix $A$.

More precisely, the algorithm is as shown below:

```
while |\eta_n - \eta_{n-1}| > \epsilon do
  for i from 1 to D do
    1. Given current $A$, compute $\frac{\partial 1^T A^{-1} 1}{\partial f_i}$;
    2. Back-search for the optimal step size $\Delta$;
    3. $\hat{f}_i \leftarrow f_i + \Delta \times \frac{\partial 1^T A^{-1} 1}{\partial f_i}$;
    4. Train $f_i(X_{F_i})$ with $\hat{f}_i$ as the outcome;
    5. Use $\hat{f}_i$ to update the training residual of agent $i$ and update $A$;
  end
  $\eta_{n+1} \leftarrow 1^T A^{-1} 1$;
  $n \leftarrow n + 1$;
end
```

3.2 Simulation for regression problems

In order to compare distributed regression implemented by ICOA to other multi-dimensional regression algorithms (distributed or non-distributed), we use three functions used in [10] as the hidden rule to generate our simulation training data sets. The three functions and the corresponding joint distribution of the covariates are:

- **Friedman-1**:
  $$ \phi(x) = 10 \sin(\pi x_1 x_2) + 20(x_3 - 1/2)^2 + 10x_4 + 5x_5 + w, $$
  where $x_j \sim U[0, 1], \ j = 1, \ldots, 5$;

- **Friedman-2**:
  $$ \phi(x) = \left( x_1^2 + (x_2 x_3 - \frac{1}{x_2 x_4})^{2/3} \right) + w, $$
  where
  $$ x_1 \sim U[1, 100], \ x_2 \sim U[40\pi, 560\pi], \ x_3, x_5 \sim U[0, 1], \ x_4 \sim U[1, 11]. $$

- **Friedman-3**:
  $$ \phi(x) = \tan^{-1} \left( \frac{x_2 x_3 - \frac{1}{x_2 x_4}}{x_1} \right) + w, $$
  where the distributions of the covariates are the same as those of Friedman-2.
All the covariates are independent of one another, and before running the algorithm, the outcomes are normalized to the range [0, 1]. Also, to highlight the effects of the distributed nature of the system, the independent additive white noise \( w \) is set to a negligible level in our simulation. Furthermore, it is worth pointing out that in Friedman-2 and Friedman-3, attribute \( X_5 \) is irrelevant, serving purely as a nuisance variable.

The structure of the entire distributed system is as follows. There are 5 attributes, \( X_1, \ldots, X_5 \), and we assume that there are 5 agents, with agent \( i \) observing attribute \( X_i \) exclusively. Each agent uses a regression tree as its individual estimator.

With the setup above, the simulation results of the two algorithms are as shown in Table 1. As a comparison, we ran two other distributed regression algorithms: averaging and residual refitting (or ICEA, see [5] for details).

| Friedman Data set | 1   | 2   | 3   |
|-------------------|-----|-----|-----|
| ICOA              | .0047 | .0095 | .0086 |
| Residual Refit    | .0047 | .0101 | .0096 |
| Averaging         | .0277 | .0355 | .0312 |

Table 1: Test errors (mean squared) of ICOA, the residual refitting algorithm and the averaging algorithm on Friedman-1, -2 and -3.

Generally speaking, the performance, measured in terms of test error, of ICOA is slightly better than that of the residual refitting algorithm for these three cases, while being much better than the averaging algorithm. More importantly, ICOA has shown little sign of overtraining, yet this is not the case for residual refitting. This is demonstrated in Figure [1].

![Figure 1](image1.png)

Figure 1: Comparison between the convergence of ICOA and the residual refitting algorithm for Friedman-1. ICOA is less susceptible to overtraining. The training error of ICOA basically parallels the trends of the test error. Yet for the residual-refitting algorithm, although the training error converges to 0 rapidly, it does not correctly reflect the trend of the test error of the ensemble estimator.

Note that for the residual refitting algorithm, the test error curve turns up as the rounds of iteration increase, even if the training error is consistently decreasing. On the contrary, the test error curve and training error curve of ICOA are almost parallel and horizontal. This suggests that ICOA knows when to stop unnecessary overtraining and its training error is a good indicator of its test error. This property is highly desirable.

The insusceptibility of ICOA to overtraining is a result of the fact that whenever an agent optimizes its estimator, it takes into consideration the predictions (represented by the residuals) of all the other estimators, instead of only one (as in the case of the residual refitting algorithm). Compared with residual refitting, ICOA is a less “greedy” algorithm in reducing its training error, and once it reaches the optimal ensemble estimator, the covariance matrix, known to all agents, would prevent the agents from changing their estimators any further, unlike in the residual refitting cases, where the agents are busy fitting the very last noise left in the residuals, the culprit for overtraining.

4 Optimization under inaccurate covariance

Although ICOA has an advantage in the performance of its ensemble estimator compared to other distributed algorithms, it requires more communication among the agents. In the voting/averaging algorithm, no residual transmission is required. In the residual refitting algorithm, for each iteration, residuals need to be transmitted \( D \) times in total, or once for each agent (as soon as an agent finishes training, it sends its residual to the next agent). However, for ICOA, residuals need to be transmitted \( D(D−1) \) times in total, or \( D−1 \) times for each agent (each agent needs to send its residual to other agents whenever an agent finishes training, because each agent needs all the latest training residuals to compute the new covariance matrix). If the total number of data instances is \( N \), then in terms of data transmission, the complexity is \( O(1) \) for voting/averaging, \( O(ND) \) for the residual refitting algorithm, and \( O(ND^2) \) for ICOA. The most communication-intensive algorithm among the three is ICOA. This is highly undesirable when the number of agents is large. This is illustrated in Figure [2].

To reduce the amount of data needed to be transmitted among the agents for ICOA, we can relax the accuracy for the estimation of the covariances. Yet this compromises the first (inner) step of our two-stage optimization specified by [7]. We need to develop an algorithm that still functions when \( A \) is not fixed, but can take values over a non-singleton domain.

4.1 A minimax problem

Given the covariance matrix \( A \), the standard optimization problem for ICOA is given by [5] and [6]. However, when we restrict the amount of data that can be transmitted, the estimation of the covariance matrix is
not accurate enough. We can model this by allowing $A$ to be of any value in a range, i.e.

$$A \in C \cap P,$$

where

$$C = \{ S \mid [S]_{ij} \in [ [A_0]_{ij} - \delta_{ij}, [A_0]_{ij} + \delta_{ij} ] \}$$

and $P$ stands for the class of semi-positive definite matrices of the same size as $A$. In other words, the element $[A]_{ij}$ has a range of length $2\delta_{ij}$ centered at $[A_0]_{ij}$, with the semi-positivity of $A$ guaranteed.

This assumption addresses the inaccuracy of the estimation of the covariance matrix $A$. Moreover, the choice of $a$ should take consideration of the worst case of $A$, which could be anywhere in $C$ (here we neglect the extra constraint of $P$, and this makes our adversary $A$ even worse for the minimization), i.e., we have a minimax optimization problem for the choice of $a$:

$$\min_{a} \max_{A \in C} a^T A a$$

s.t. $A^T 1 = 1$.  

The solution to the inner maximization is straightforward, because we can decompose this problem into $D \times D$ independent optimization problems:

$$\max_{[A]_{ij}} a_i a_j [A]_{ij}.$$  

Obviously, when $a_i a_j > 0$, $[A]_{ij} = [A_0]_{ij} + \delta_{ij}$, otherwise, $[A]_{ij} = [A_0]_{ij} - \delta_{ij}$. More concisely,

$$[A]_{ij} = [A_0]_{ij} + \delta_{ij} \text{sgn}(a_i a_j).$$

To simplify our next step, we now need to make a few more assumptions. First, since the diagonal elements of $A$ are estimated locally, i.e., no data need be transmitted to estimate residual variances of individual agents, it is reasonable to assume $\delta_{ii} = 0$, $i = 1, \ldots, D$. Another assumption is $\delta_{ij} = \delta > 0$, $i \neq j$; thus we can characterize the uncertainty of the estimation of covariance using a single number. This might sacrifice some accuracy of our model, yet it simplifies our problem at least for a preliminary exploration.

With these additional assumptions, the optimal value $\zeta$ of the maximization step in (19) is given by

$$\zeta = a^T A_0 a + 2\delta \sum_{i \neq j} |a_i||a_j|.  \quad (23)$$

Thus, we can rewrite the objective of the minimax problem as

$$\min_{a} a^T A_0 a + 2\delta \sum_{i \neq i} |a_i||a_j|.  \quad (24)$$

Unfortunately, the objective function of this problem is not always convex, and there is no closed form solution. To show the conditions for convexity, we rewrite the objective function as

$$a^T (A_0 - \delta I) a + \delta \sum_i |a_i|^2.  \quad (25)$$

It is easy to show that the second term, the “penalization term”, $\delta(\sum_i |a_i|^2)^2$, is a convex function. And the convexity of the first term is dependent on the value of $\delta$. Since $A_0$ is a covariance matrix, i.e. it is positive definite, the convexity of $a^T (A_0 - \delta I) a$ is hence equivalent to $\delta \leq \lambda_{\min}$, where $\lambda_{\min}$ is the smallest eigenvalue of $A_0$.

The second term serves as a penalization, restricting the magnitudes of the coefficients. It is similar to Lasso Regression, except for the square. This term can be crudely interpreted as follows: when the covariance matrix is not accurately known, it is not wise to fully minimize the ensemble training residual without paying attention to the complexity of the ensemble model (measured by the squared L-1 magnitude of the weighting coefficients).

Even if the problem is not convex, if the change in $A$ is not too large, the solution to (5) is a fairly good initial value and gradient descent can be applied to solve the problem specified by (24) and (6).

4.2 ICOA with Minimax Protection

The above derivation actually changes the inner step of our two-stage optimization, and we no longer have a closed form solution. Nonetheless, we can still run ICOA numerically, because we can still use perturbation to estimate the influence of the change of $\epsilon_i$ on the value of $\zeta$, given that we can numerically solve the inner minimization.

Obviously, if we know the covariance accurately, changing the inner step from minimization to minimax, i.e. changing (5) to (24) has no advantage. On the contrary, it compromises the performance of the ensemble estimator and slows down the convergence speed of ICOA. However, if we add restrictions on the number of data instances exchanged between two agents, this makes $A_0$, the estimate of $A$, less accurate, and then the minimax optimization is of utmost importance for...
the convergence of ICOA. We call this procedure Minimax Protection.

For instance, if we transmit only $1/\alpha$ of the total $N$ data instances (randomly sampled from all the data instances) for covariance estimation, say, $\alpha = 100$, then the estimate $A_0$ has a large variance. Thus if we directly substitute this estimate into the ICOA algorithm, it causes inaccurate and unstable estimation of the direction for gradient descent and prevents the algorithm from converging. This phenomenon is illustrated in Figure 3 where the compression rate is $\alpha = 100$ (only 1% of the data are transmitted for each iteration) and $\delta = 0$ (no Minimax Protection).

![Figure 3: ICOA without Minimax Protection for Friedman-1. The training/test errors oscillates wildly and fail to converge. There is no way to decide when to stop the iterations.](image)

However, if we foresee the inaccuracy and error of the estimation of the covariance matrix, and properly choose a value for $\delta$, then not only can we prevent the oscillation of the training/test error, but we also sacrifice little in the performance of the ensemble estimator. In Figure 4, Minimax Protection is applied to ICOA, with $\alpha = 100$ and $\delta = 0.8$.

![Figure 4: ICOA with Minimax Protection for Friedman-1. The training/test errors decrease almost monotonically and converge rather quickly and smoothly, with a reasonable compromise in performance.](image)

The results of a series of simulations are shown in Table 2 for different values of compression rate $\alpha$ and $\delta$. In this simulation, the data set is Friedman-1, and the system configuration is the same as the simulation in the previous section. The individual estimator is of the form of a 4th order polynomial.

| $\alpha$ | 1   | 10  | 50  | 200 | 800 |
|--------|-----|-----|-----|-----|-----|
| $\delta$ | 0.00 | .0037 | NaN | NaN | NaN |
| $\delta$ | .050 | .0044 | .0045 | NaN | NaN |
| $\delta$ | .500 | .0051 | .0056 | .0052 | NaN |
| $\delta$ | .750 | .0071 | .0071 | .0073 | .0077 | NaN |
| $\delta$ | 1.00 | .0086 | .0086 | .0086 | .0090 | .0098 |
| $\delta$ | 2.00 | .0112 | .0111 | .0112 | .0114 | .0113 |

Table 2: Test errors (mean squared) of ICOA with Minimax Protection for Friedman-1. For certain values of $\alpha$ and $\delta$, ICOA does not converge, and the test error exceeds machine limits and hence cannot be obtained.

It is worth pointing out two phenomena. First, when ICOA with Minimax Protection converges, the performance is almost independent of the compression rate $\alpha$. Second, given $\alpha$, when the value of $\delta$ is above a certain level, ICOA almost always converges, yet below that level, ICOA does not converge. These two phenomena allow us to find an optimal $\delta$ for every given $\alpha$, so that we can optimize the performance of ICOA under a given compression rate.

In Table 2, another dramatic phenomenon is the case for $\alpha = 800$ and $\delta = 1.00$. Since we have only 4000 training data instances, this means in each iteration, we use only 5 pairs of numbers to estimate the covariance between two agents. And Minimax Protection with properly selected $\delta$, enables us to achieve a decent test mean square error of .0098, only about 2.5 times of the optimal value .0037. Yet only 1% of the data transmission is needed compared with the amount needed in the optimal case (after taking into consideration the longer convergence time). Thus ICOA provides us with a very useful tool to trade off between performance and data transmission.

### 4.3 Upper bound of the test error

From the simulation results shown in Table 2 it is of interest to investigate the relationship between the compression rate $\alpha$ and the optimal performance (measured by test error) of the system. As analyzed previously, the key is to select a proper $\delta$ so that we neither under-protect ICOA (leading to unstable convergence) nor over-protect (leading to worse performance). This requires us to investigate the statistical properties of the estimator of the correlation coefficient between two random variables. In [11], it is shown that the pivot statistic $T_N$ of the sample correlation coefficient has the Student’s t-distribution; that is,

$$T_N = \frac{\sqrt{N-2} \hat{\rho}}{\sqrt{1-\hat{\rho}^2}} \sim t_{N-2}, \quad (26)$$
where $N$ is the number of data instances. Therefore the 95% confidence interval of the correlation coefficient is given by $[\hat{\rho} - \xi, \hat{\rho} + \xi]$, where $\xi = 1.96(1 - \hat{\rho}^2)/\sqrt{N}$.

If we assume that the largest variance of all the residuals is $\sigma^2_{\text{max}}$, then an approximation to the optimal $\delta$ (as a function of $\alpha$) can be given by

$$\delta_{\text{opt}}(\alpha) = \min\{1.96\sigma^2_{\text{max}}/\sqrt{N/\alpha}, 2\sigma^2_{\text{max}}\}.$$  \hspace{1cm} (27)

The basic idea is to find the smallest $\delta$ that covers, with high probability, the possible domain of the covariance matrix, given a crude estimate $A_0$.

With this approximation, we are able to develop an upper bound on the test mean square error as a function of $\alpha$. Define $A_{\text{ini}}$ as the covariance matrix (accurate) of the residuals of all individual estimators before we run ICOA. For each step, ICOA with Minimax Protection improves the test error (not merely training error), because Minimax Protection guarantees, with high probability, that the true covariance matrix is in the range $C$ defined in (18). Therefore, the solution to

$$\min_{\mathbf{a}} \mathbf{a}^T (A_{\text{ini}} - \delta_{\text{opt}}(\alpha)I) \mathbf{a} + \delta_{\text{opt}}(\alpha)(\sum_i |a_i|)^2$$  \hspace{1cm} (28)

with constraint (6) provides us with an upper bound (with high probability) on the generalization error with respect to the compression rate $\alpha$. Figure 5 illustrates the comparison between this upper bound and the simulated optimal performance.

Figure 5: Comparison between the upper bound given by (28) and the simulated test error.

5 Conclusions

In this paper, we have shown that ICOA, as a cooperative training algorithm, demonstrates its efficacy for finding an optimal ensemble estimator of additive form, while demonstrating an insusceptibility to overtraining. Moreover, Minimax Protection provides us with a tool to run ICOA when covariances are not accurately estimated, and hence enable us to trade off between performance and data transmission. Minimax Protection, combined with ICOA, also helps us to develop an upper bound on the test error for the ensemble estimator.

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