We thank Arnoud van der Leer for their contributions to these notes.

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The source code for this document can be obtained from

https://github.com/benediktahrens/CT4P
Abstract  In these lecture notes, we give a brief introduction to some elements of category theory. The choice of topics is guided by applications to functional programming. Firstly, we study initial algebras, which provide a mathematical characterization of datatypes and recursive functions on them. Secondly, we study monads, which give a mathematical framework for effects in functional languages. The notes include many problems and solutions.
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1. Introduction

Category theory is a framework which allows one to formally describe and relate mathematical structures. By a mathematical structure, we mean, informally, a collection of things (like types, sets, etc.) and something which transforms one thing to another thing (like a program, function, etc.).

This framework started as a mathematical theory, but has now proven itself useful also in the world of Computer Science (and beyond). In this course we will introduce the necessary concepts from category theory with the goal of understanding its applications in the realm of programming.

1.1. About These Notes

These notes are not meant to give an exhaustive introduction to category theory. Instead, the aim is to develop just as much category theory as is necessary to discuss some interesting applications of category theory to computing, specifically, to programming.

Throughout these notes, pointers to other sources, such as textbooks and research articles, are given; it is highly recommended to consult these sources.

1.2. About Category Theory

Category theory is a mathematical area of endeavour and language developed to reconcile and unify mathematical phenomena from different disciplines. It was developed from the 1940s on, in particular by Samuel Eilenberg and Saunders Mac Lane.

Computer science is... well, you know what it is.

In this course, we learn about some fundamental applications of category theory to computer science, specifically, to programming. The power of category theory arises from abstraction: by boiling down constructions to their essence, analogous situations can be formally identified using category theory. One crucial concept provided by category theory to this end is that of universal property; we study some universal properties in Section 3. An application of universal properties to the theory of datatypes and structural recursion is studied in Section 8.

Another categorical concept that has proved particularly useful in programming is that of a monad. We study monads and their use in programming in Section 7.

1.3. Learning Material on Category Theory

The scientific literature on category theory in computer science is vast. We list some learning material on category theory.

Pierce’s book [9] (available for free) gives a brief introduction to category theory with some applications to computing.

Leinster’s book [7] (available for free online, under a free license) gives a concise introduction to category theory. It is a good resource for the basic concepts, but does not feature many examples from computer science.
The rather substantial textbook by Barr and Wells [1] (available for free online) covers a lot more than we are going to discuss in these notes. The Catsters [2] provide a lecture series on category theory on YouTube.

1.4. Notations

A list of notations which we use throughout these lectures notes.

• If \( X \) is a set and \( x \) is an element of \( X \), we write \( x \in X \).

• If \( X \) is a set and \( P \) and \( Q \) are properties dependent over the elements of \( X \), we write \( P \implies Q \) to express that if \( P(x) \) holds for an element \( x \in X \), then also \( Q(x) \) should hold for the element \( x \). Moreover, we write \( P \iff Q \) if \( P \implies Q \) and \( Q \implies P \).

• If \( X \) is a set and \( P \) is a property dependent over the elements of \( X \), we write:

  (*) \( \forall x \in X : P(x) \) to express that for every element in \( X \), the property \( P \) holds.

  (*) \( \exists x \in X : P(x) \) to express that there exists at least one element in \( X \) for which the property holds.

  (*) \( \exists! x \in X : P(x) \) to express that there exists a unique element in \( X \) for which the property holds.

• Let \( X \) and \( Y \) be sets. We denote:

  (*) \( X \times Y \) for the (cartesian) product of \( X \) and \( Y \).

  (*) \( X \sqcup Y \) for the disjoint union of \( X \) and \( Y \).

2. Categories

Further Reading. *The definition of categories is also given in [2, §2.1]. Plenty of examples of categories are given in [2, §§2.3–2.5]. The definition of categories is also given in [4, §1.1], together with some examples. There, also isomorphisms are discussed, which we define in Section 2.1. The tutorial [9] features the definition of categories in [9, §2.1]. It also introduces the notion of “diagram”, which we do not use in the present notes.*

Definition 1. A category \( \mathcal{C} \) consists of the following data:

1. A collection of objects, denoted by \( \mathcal{C}_0 \).

2. For any given objects \( X, Y \in \mathcal{C}_0 \), a collection of morphisms from \( X \) to \( Y \), denoted by \( \text{hom}_\mathcal{C}(X,Y) \) (or \( \text{hom}(X,Y) \) when the category \( \mathcal{C} \) is clear, or \( \mathcal{C}(X,Y) \) or \( X \to Y \)) and which is called a *hom-set*.

3. For each object \( X \in \mathcal{C}_0 \), a morphism \( \text{id}_X \in \text{hom}_\mathcal{C}(X,X) \), called the *identity morphism* on \( X \).
4. A binary operation

\[(\circ)_{X,Y,Z} : \text{hom}(Y,Z) \to \text{hom}(X,Y) \to \text{hom}(X,Z),\]

called the composition operator, and written infix without the indices \(X,Y,Z\) as in \(g \circ f\).

Moreover, this data should satisfy the following properties:

1. **(Left unit law)** For any morphism \(f \in \text{hom}(X,Y)\), we have

\[f \circ \text{Id}_X = f.\]

2. **(Right unit law)** For any morphism \(f \in \text{hom}(X,Y)\), we have

\[\text{Id}_Y \circ f = f.\]

3. **(Associative law)** For any morphisms \(f \in \text{hom}(X,Y)\), \(g \in \text{hom}(Y,Z)\) and 
\(h \in \text{hom}(Z,W)\), we have

\[h \circ (g \circ f) = (h \circ g) \circ f.\]

**Intuition 2.** So what does a category represent? There are (at least) 3 possible ways how one can think about this definition:

1. A category represents a type system in the sense that the objects are the types and each hom-set is the type of functions. See Definition 10.

2. A category represents a bag of instances of a particular mathematical structure (e.g. sets with a notion of addition). The objects are then instances of such a mathematical theory (e.g. \((\mathbb{N},+)\)) and the morphisms are structure preserving functions (e.g. functions \(f\) which satisfy \(f(x+y) = f(x) + f(y)\)). See Examples 4 and 14 and Definition 19.

3. A category represents a directed graph in the sense that an object is a vertex and a morphism is an edge.

4. Anything (almost at least) can be seen as a category in some exotic way.

**Notation 3.** Let \(\mathcal{C}\) be a category.

- We write \(X \in \mathcal{C}\) instead of \(X \in \mathcal{C}_0\).
- Let \(X, Y \in \mathcal{C}\) be objects. A morphism \(f \in \mathcal{C}(X,Y)\) can be visualized as

\[X \xrightarrow{f} Y.\]

\[\text{In this case, each hom-set is a type, so isn't each hom-set an object again? Categories which satisfy such a property are called cartesian closed.}\]
• Let \( X, Y, Z \in C_0 \) objects in \( C \) and consider the following morphisms:

\[ f \in C(X,Y), \quad g \in C(Y,Z), \quad h \in C(X,Z). \]

These morphisms can be visualized as a triangle:

\[
\begin{array}{c}
X \\
\downarrow^f \\
Y \\
\quad \downarrow^h \\
\quad \quad \downarrow^g \\
Z
\end{array}
\]

We say that such a triangle \textbf{commutes} if \( h = g \circ f \).

• Let \( X, Y_1, Y_2, Z \in C_0 \) objects in \( C \) and consider the following morphisms:

\[ f_1 \in C(X,Y_1), \quad f_2 \in C(X,Y_2), \quad g_1 \in C(Y_1,Z), \quad g_2 \in C(Y_2,Z). \]

These morphisms can be visualized as a square:

\[
\begin{array}{c}
X \\
\downarrow^{f_2} \\
Y_2 \\
\quad \downarrow^{f_1} \\
Y_1 \\
\quad \downarrow^{g_1} \\
\quad \quad \downarrow^{g_2} \\
Z
\end{array}
\]

We say that such a square \textbf{commutes} if \( g_1 \circ f_1 = g_2 \circ f_2 \).

\textbf{Example 4.} The \textbf{Category of sets}, denoted by \( \text{Set} \), is the category specified by the following data:

• An object is a set.

• If \( X \) and \( Y \) are sets, then is \( \text{Set}(X,Y) \) the set of all functions from \( X \) to \( Y \).

• The identity morphism \( \text{Id}_X \) (on \( X \in \text{Set}_0 \)) is the identity function on \( X \), i.e.

\[ \text{Id}_X : X \to X : x \mapsto x. \]

• The composition of functions is given by the usual composition of functions, i.e. for \( f \in \text{Set}(X,Y) \) and \( g \in \text{Set}(Y,Z) \), the composition of \( f \) and \( g \) is:

\[ g \circ f : X \to Z : x \mapsto g(f(x)). \]

\textbf{Lemma 5.} The data of \( \text{Set} \) satisfies the properties of a category, hence \( \text{Set} \) is indeed a category.

\textit{Proof.} We first show that the left unit law holds. Let \( X, Y \in \text{Set} \) be sets and \( f \in \text{Set}(X,Y) \) a function. We have to show that \( \text{Id}_X \cdot f = f \), hence it suffices to show that they pointwise equal which holds by the following calculation:

\[ \forall x \in X : (f \circ \text{Id}_X)(x) = f(\text{Id}_X(x)) = f(x), \]
where the first (resp. second) equality holds by definition of the composition (resp. identity morphism).

That the right unit law holds is analogous. To show that the associator law holds, let $X,Y,Z,W \in \text{Set}$ and $f \in \text{Set}(X,Y)$, $g \in \text{Set}(Y,Z)$ and $h \in \text{Set}(Z,W)$. We have to show $h \circ (g \circ f) = (h \circ g) \circ f$, hence it suffices again to show that they are pointwise equal which holds by the following calculation:

$$
\forall x \in X : (h \circ (g \circ f))(x) = h((g \circ f)(x)),
= h(g(f(x))),
= (h \circ g)(f(x)),
= ((h \circ g) \circ f)(x),
$$

where the first (resp. second, third, fourth) equality holds by definition of the composition of $h$ and $g \circ f$ (resp. composition of $g$ and $f$, composition of $h$ and $g$, composition of $h \circ g$ and $f$).

We are now going to describe the category whose collection of objects is given by collection of Lean types:

**Example 6.** Consider the following data:

- An object is a Lean type (of some fixed universe).
- If $X$ and $Y$ are Lean types, then is $\text{LEAN}(X,Y)$ the function type $X \to Y$.
- The identity morphism $\text{Id}_X$ (on $X \in \text{LEAN}_0$) is the identity function on $X$, i.e.
  ```Lean
  def idfun (X : Type) : X \to X := \lambda x, x.
  ```
- The composition of functions is given by the composition of functions:
  ```Lean
  def compfun {X Y Z} (f : X \to Y) (g : Y \to Z) : X \to Z := \lambda x, g(f(x))
  ```

Try it out, e.g., on https://leanprover.github.io/live/latest/ #eval compfun (+1) (\^ 3) 5

(You can get a pre-filled Lean input field by clicking here: ClickMe)

**Exercise 7.** Prove (on paper) that the data defined in Example 6 defines a category. That is, show that it satisfies the axioms of a category. You might need to use the axiom of functional extensionality:

```Lean
axiom funext_nondep : \forall {A B : Type} (f g : A \to B),
(\forall x, f x = g x) \to f = g
```

**Example 8.** We repeat the definitions of Example 6 in Haskell instead of Lean. Does this data satisfies the axioms of a category?

Due to Haskell allowing for the `undefined` value in each type, the situation is slightly more complicated; consider the following two functions:
undeﬁ1 : : a → a
undeﬁ1 = undeﬁned

undeﬁ2 : : a → a
undeﬁ2 = \x → undeﬁned

These are not equal by deﬁnition, but we have 1d · undeﬁ1 = undeﬁ2. So by the right
unit law, we must have that undeﬁ1 = undeﬁ2 (as morphisms in our sought category).

Exercise 9. Read the Haskell wiki page on the category Hask [4].

However, when considering functions to equal when they are pointwise
equal, we can deﬁne a category of Haskell types:

Deﬁnition 10. The category of Haskell types, denoted by Hask, is the category
speciﬁed by the following data:

• An object is a Haskell type.

• If X and Y are Haskell types, then is Hask(X,Y) the collection of functions
modulo the equivalence relation ∼ deﬁned by identifying pointwise equal func-
tions:

f ∼ g : ⇐⇒ ∀x : X, f(x) = g(x).

i.e. a morphism in Hask is an equivalence class of (Haskell) functions.

• The identity morphism IdX (on X ∈ Hask) is the equivalence class of the identity
function on X.

• The composition of (Haskell) functions is given by the equivalence class of the
composition of functions, i.e., for f ∈ Hask(X,Y) and g ∈ Hask(Y,Z), the
composition of f and g is the equivalence class of:

g ◦ f : X → Z : λx.g(f(x)).

Example 11. Recall that a preordered set (X, ≤) consists of a set X together with a
binary relation (≤) on X which satisﬁes the following properties:

• RefleXivity: ∀x ∈ X : x ≤ x.

• Transitivity: ∀x, y, z ∈ X : (x ≤ y ∧ y ≤ z) ⇒ x ≤ z.

Let (X, ≤) be a preordered set. The following data induces a category Pre(X, ≤):

• The objects are the elements of X.

• Let x, y ∈ X be elements. The hom-set hom(x, y) consists of a unique element
if x ≤ y and is empty otherwise.

• We need an identity morphism for each x ∈ X. By reﬂexivity (i.e., x ≤ x),
we have that hom(x, x) consists of a unique element, which we take to be the
identity.
• We need to define for each \( x, y, z \in X \), a composition operator:

\[
\text{hom}(y, z) \to \text{hom}(x, y) \to \text{hom}(x, z).
\]

By definition of the hom-sets, we only have to define it in case \( x \leq y \) and \( y \leq z \). But then, by transitivity (i.e. if \( x \leq y \) and \( y \leq z \), then \( x \leq z \)), we have that \( \text{hom}(x, z) \) consists of a unique element which we take to be the composition.

We are now going to show that the axioms of a category holds. To show the right unit law, we have to show that for each \( x, y \in X \) and \( f \in \text{hom}(x, y) \), we have \( f \circ \text{Id}_x = f \). This indeed holds since every hom-set has a unique element, but both \( f \circ \text{Id}_x \) and \( f \) live in the same hom-set, hence they must be equal. The proof that left unit law and associator law hold are analogous.

**Exercise 12** (Solution 210). A partially ordered set (poset) is a preordered set \((X, \leq)\) satisfying the following additional axiom:

- **Antisymmetry**: \( \forall x, y \in X : (x \leq y \land y \leq x) \Rightarrow x = y. \)

What does this axiom say about \text{Pre}(X, \leq)?

**Remark 13.** To understand a definition in category theory, it is very helpful to think about what the definition means in a preordered set, viewed as a category.

**Example 14.** The category of posets, denoted by \textbf{Pos}, is the category specified by the following data:

- An object is a poset \((X, \leq)\).
- A morphism from a poset \((X, \leq_X)\) to \((Y, \leq_Y)\) consists of a function \( f : X \to Y \) such that the following property holds:

\[
\forall x_1, x_2 \in X : x_1 \leq_X x_2 \Rightarrow f(x_1) \leq_Y f(x_2).
\]
- The identity morphism on \((X, \leq_X)\) is the identity function on \(X\).
- The composition given by the composition of functions.

Before we can show that this data satisfies the axioms of a category, notice that the identity function is a morphism of posets and that the composition of poset morphisms is again a poset morphism, indeed: If \( x_1 \leq_X x_2 \), then we also have \( \text{Id}_X(x_1) \leq_X \text{Id}_X(x_2) \) because \( \text{Id}_X(x) = x \). If \( f \in \textbf{Pos}((X, \leq_X), (Y, \leq_Y)) \) and \( g \in \textbf{Pos}((Y, \leq_Y), (Z, \leq_Z)) \) are morphisms of posets, then we have

\[
\forall x_1, x_2 \in X : x_1 \leq_X x_2 \Rightarrow f(x_1) \leq_Y f(x_2) \Rightarrow g(f(x_1)) \leq_Z g(f(x_2)),
\]

where the first (resp. second) inequality holds by \( f \) (resp. \( g \)) being a morphism of posets. So our data is indeed well-defined.

That the axioms of a category are satisfied by this data, is exactly the same proof as showing that \textbf{Set} is a category because the identity and composition are defined in the same way.
Exercise 15 (Solution 211). Is Pos a preorder-category itself? That is, is there at most one morphism between any two objects?

Question 16. In Example 14 we have shown that Pos is a category. However, by Example 11 we know that any poset also is a category. So we have that Pos is a category whose objects are certain categories. Can we also have some category whose collection of objects is the collection of all categories, and if so, what are the morphisms of categories?

Solution. Section 4 is devoted completely to this answer.

Lemma 17. Let $C$ be a category. For any object $X \in C$, $\text{Id}_X$ is the unique morphism which satisfies the following property: For any $Y \in C$ and $f \in C(X, Y)$, we have $f \circ \text{Id}_X = f$.

Proof. Assume $\tilde{\text{Id}}_X$ also satisfies this property, in particular we have $\text{Id}_X \circ \tilde{\text{Id}}_X = \text{Id}_X$. However, by the right unit law, we also must have $\text{Id}_X \circ \tilde{\text{Id}}_X = \tilde{\text{Id}}_X$. Hence, $\text{Id}_X = \tilde{\text{Id}}_X$. $
$.

Example 18. In this example we are going to associate a category which captures the multiplication of the rational numbers. Let $C$ be the category defined by the following data:

- There is a unique object $\star$.
- The (only) hom-set is given by $\text{hom}(\star, \star) = \mathbb{Q}$, i.e. each morphism corresponds with a rational number.
- The composition is defined by the multiplication of rational numbers: $\mathbb{Q} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q} : (p, q) \mapsto p \cdot q$.
- The identity morphism (of $\star$) is given by 1.

That $C$ is indeed a category follows because for each $p \in \mathbb{Q}$, we have $p \cdot 1 = p = 1 \cdot p$ (which shows the unit laws) and by associativity of multiplication, i.e. $(p \cdot q) \cdot h = p \cdot (h \cdot q)$ (which shows the associativity of the composition).

The construction in Example 18 uses no specific properties of the rational numbers, only that it has a multiplication which is associative and such that there is a special element which does not change an element when it is multiplied with this special element. Hence, Example 18 can be generalized as follows:
**Definition 19.** Recall that a monoid is a set $M$ equipped with binary operation $m : M \to M \to M$ which is associative, i.e.

$$\forall x, y, z \in M : m(x, m(y, z)) = m(m(x, y), z),$$

and such that there is an identity element, i.e.

$$\exists e \in M : \forall x \in M : m(e, x) = x = m(x, e).$$

Let $(M, m, e)$ be a monoid. The category $\textbf{Monoid}(M, m, e)$ is defined by the following data:

- There is a unique object $\star$.
- The (only) hom-set is given by $\text{hom}(\star, \star) = M$.
- The identity morphism on $\star$ is the identity element $e$.
- The composition of morphisms $x$ and $y$ is given by $y \circ x := m(x, y)$.

That for each monoid $(M, m, e)$, $\textbf{Monoid}(M, m, e)$ is indeed a category, follows directly by the properties of being a monoid. Indeed, the axioms of a category become precisely:

1. $\forall x \in M : m(x, e) = x$,
2. $\forall x \in M : m(e, x) = x$,
3. $\forall x, y, z \in M : m(m(x, y), z) = m(x, m(y, z))$.

**Remark 20.** Notice that this category illustrates that there is no relation between the collection of objects and the hom-sets since there is now only one object and the collection of the hom-set can be as small or as large as possible.

In fact, we can associate a different number of categories to a single monoid. We can for example consider an arbitrary set of objects $I$ and the defining the hom-sets as follows:

$$\text{hom}(i, j) := \begin{cases} M, & \text{if } i = j, \\ \emptyset, & \text{if } i \neq j. \end{cases}$$

**Exercise 21** (Solution [212]). Let $C$ be a category. When is $C$ of the form $\textbf{Monoid}(M, m, e)$, i.e. does there exists a monoid $(M, m, e)$ such that $C = \textbf{Monoid}(M, m, e)$?

**Exercise 22** (Solution [213]). Define a category $\textbf{Monoid}$ whose objects are monoids, i.e. define a suitable notion of morphism between monoids and moreover show that this indeed defines a category.

**Exercise 23** (Solution [214]). Define a category $C$ whose objects are the natural numbers (i.e. $C_0 = \mathbb{N}$) and whose hom-sets $C(n, m)$ are given by the $(n \times m)$-matrices.
Exercise 24 (Solution 215). Let \( \mathcal{C} \) be a category. Define a category \( \mathcal{C}^{\text{op}} \) such that

- the objects of \( \mathcal{C}^{\text{op}} \) are the same as those of \( \mathcal{C} \); and
- the morphisms \( \mathcal{C}^{\text{op}}(X,Y) \) are morphisms \( \mathcal{C}(Y,X) \).

The category \( \mathcal{C}^{\text{op}} \) is called the **opposite (category)** of \( \mathcal{C} \).

Exercise 25. Let \( G \) be a directed graph. Then \( G \) induces a category \( \text{Graph}(G) \) as follows:

- The collection of objects \( \text{Graph}(G)_0 \) is the set of vertices of \( G \).
- The morphisms between object are the (directed) paths, that is, finite sequences of composable edges, between them.
- For each object \( x \) (i.e. vertex), the identity morphism on \( x \) is the *identity path*.
- The composition of morphisms is the composition of paths.

We call \( \text{Graph}(G) \) the **category generated by** \( G \). Show that \( \text{Graph}(G) \) is indeed a category.

Exercise 26. Argue why the morphisms are chosen to be paths and it is not sufficient to just take the vertices.

Example 27. Consider the following graph \( G \):

\[
\begin{array}{c}
\text{x} \\
\end{array}
\]

i.e. the graph with only object vertex and no edges. The category generated by \( G \) is the category generated is the so-called **terminal category**, that is, the category with a single object and a single morphism (the identity morphism of the unique object).

Example 28. Consider the following graph \( G \):

\[
\begin{array}{c}
\text{x} \\
\text{y} \\
\end{array}
\]

The category generated by \( G \) is the category generated is the so-called **interval category**, that is, the category with two objects and, besides the identity morphisms, a unique morphism (living in \( \text{hom}(x,y) \)).

In the following example we use the following notation:

- If \( f \) is a morphism in a category, we denote \( f^2 := f \circ f \), \( f^3 := f^2 \circ f \), etc.
- We also label the edges in order to refer to them.

Example 29. Consider the following graph \( G \):

\[
\begin{array}{c}
\text{x} \\
\text{y} \\
\end{array}
\]

The category generated by \( G \) consists of the following data:
• The collection of objects is \{x, y\}.

• The hom-sets are given as follows:
  - \(\text{hom}(x, x)\) contains
    \[\text{id}_x, g \circ f, (g \circ f)^2, (g \circ f)^3, \cdots,\]
    But these are not the only ones, we also have that each of these can be
    precomposed or postcomposed with \(\text{id}_x\), however, by the unit laws, we know
    that these don’t give us any new morphisms. The same remark holds for
    the associativity law. This comment also holds for the upcoming hom-sets.
  - \(\text{hom}(y, y)\) contains
    \[\text{id}_y, f \circ g, (f \circ g)^2, (f \circ g)^3, \cdots,\]
  - \(\text{hom}(x, y)\) contains
    \[f, (f \circ g) \circ f, (f \circ g)^2 \circ f, (f \circ g)^3 \circ f, \cdots\]
  - \(\text{hom}(y, x)\) contains
    \[g, (g \circ f) \circ g, (g \circ f)^2 \circ g, (g \circ f)^3 \circ g, \cdots\]

Example 30. Consider the following graph \(G\):

\[
\begin{array}{ccc}
  & w & \\
  x & \rightarrow & z \\
  y & \rightarrow & \\
\end{array}
\]

The category generated by \(G\) has four objects (namely \(x, y, z, w\)) and the hom-sets are:

• \(\text{hom}(y, x), \text{hom}(y, z)\) and \(\text{hom}(z, w)\) are singleton sets,

• \(\text{hom}(x, y), \text{hom}(z, y), \text{hom}(x, w), \text{hom}(w, x)\text{ and }\text{hom}(w, z)\) are all empty.

• \(\text{hom}(y, w)\) consists of the path \(y \rightarrow z \rightarrow w\).

• For each vertex \(v\), we have that \(\text{hom}(v, v)\) consists only of the identity path on \(v\).

Exercise 31 (Solution 216). Describe the connection between the categories generated
by graphs and the categories associated to preordered sets. What is the property of
anti-symmetry translated under this connection with graphs?

Exercise 32. Define a category \textbf{Aut} whose objects are (deterministic finite) aut-
omatica.
2.1. Isomorphisms

Further Reading. In this section, we study properties of arrows in a category. More information on this topic is given in \cite{1}, \S2.7.

Also, \cite{9}, \S2.2 briefly discusses isomorphisms.

Definition 33 (Isomorphism). Given a category \(\mathcal{C}\), objects \(a, b \in \mathcal{C}_0\) and a morphism \(f : a \to b\) in \(\mathcal{C}\), we say that \(f\) is an isomorphism when there is a morphism \(g : b \to a\) (in the other direction!) such that \(f \cdot g = \text{id}\) and \(g \cdot f = \text{id}\). We write \(f : a \cong b\) for a morphism \(f\) that is an isomorphism.

In this case, we call \(g\) the inverse of \(f\) and \(f\) the inverse of \(g\). (The latter is justified by Exercise 34.)

Exercise 34 (Solution 217). Show that if \(f : a \to b\) is an isomorphism with inverse \(g : b \to a\), then \(g\) is an isomorphism with inverse \(f\).

Exercise 35 (Solution 218). Show that a morphism \(f : a \to b\) in \(\mathcal{C}\) is an isomorphism in at most one way, that is, show that its inverse is unique if it exists.

Exercise 36 (Solution 219). Show that the composition of two isomorphisms is an isomorphism.

Remark 37. Since any identity morphism is an isomorphism (check this!), we conclude by Exercise 35 that given any category \(\mathcal{C}\), we always get a new category \(\text{isos}(\mathcal{C})\) by restricting the morphisms to be isomorphisms, i.e.

\[
isos(\mathcal{C})_0 = \mathcal{C}_0, \quad isos(\mathcal{C})(X,Y) = \{ f \in \mathcal{C}(X,Y) \mid f \text{ is an isomorphism} \}
\]

and where the identity and composition is the same as in \(\mathcal{C}\).

Exercise 38 (Solution 220). Consider the Haskell datatype

\[\text{data BW = Black | White}\]

Construct two (different!) isomorphisms between \(\text{BW}\) and the type \(\text{Bool}\) of booleans.

Exercise 39 (Solution 221). Characterize the isomorphisms in \(\text{Set}\).

Exercise 40 (Solution 222). Describe the isomorphisms in \(\text{Pos}\).

Exercise 41 (Solution 223). Let \((X, \leq)\) be a poset. Can you characterize the isomorphisms in \(\text{Pos}(X, \leq)\)?

Exercise 42. Can you characterize the isomorphisms in \(\text{Monoid}\)?

Exercise 43. Let \(\mathcal{G}\) be the category generated by the following graph:

\[
\begin{array}{ccccccc}
& & w & & \text{x} & \text{y} & \text{z} & \text{w} & \\
& & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \\
& & z & & y & & x & & w
\end{array}
\]

Show that the only isomorphisms in \(\mathcal{G}\) are the identity morphisms (i.e. the identity paths).
2.2. Sections and Retractions

Definition 44 (Section, Retraction). A pair \((s, r)\) of morphisms \(s : a \to b\) and \(r : b \to a\) in \(\mathcal{C}\) is called a **section-retraction pair** if \(r \circ s = \text{Id}_b\).

In such a case, we call \(s\) a section and \(r\) a retraction.

Remark 45. Note that a morphism can be a retraction in more than one way, that is, there can be more than one section \(s\) such that \(r \circ s = \text{Id}_b\).

Intuitively, a section-retraction pair \((s, r)\) of morphisms \(s : a \to b\) and \(r : b \to a\) in a category \(\mathcal{C}\) provides a way for \(a\) to “live inside” \(b\). Note that for a given \(a\) and \(b\) there can be many ways for \(a\) to live inside \(b\).

Exercise 46 (Solution 224). Construct two different section-retraction pairs between the type \(\text{Bool}\) of booleans and the type \(\text{Int}\) of integers (e.g., in Haskell).

Exercise 47. Show that the type \(\text{Maybe } a\) is a retract of the type \([a]\).

Hint: The idea is that \(\text{Nothing}\) corresponds to the empty list \([\ ]\) and that \(\text{Just } x\) corresponds to the one-element list \([x]\). Make this idea precise by writing back and forth functions between these types so that they exhibit \(\text{Maybe } a\) as a retract of \([a]\).

2.3. Monomorphisms and Epimorphisms

Further Reading. See also [2, p. 134] and [1, §§2.8–2.9]. Also, [2, §2.2] briefly discusses monomorphisms and epimorphisms.

From undergraduate mathematics courses you know what injective and surjective functions between sets are. The definitions of “injective” and “surjective” do not carry over to any category (though they do for categories that are, in some sense, “similar” to the category of sets). In this section, we study two properties of morphisms in a category that, in the category of sets, are equivalent to “injective” and “surjective”, respectively.

Definition 48 (Monomorphism). Let \(f : a \to b\) be a morphism in \(\mathcal{C}\). We say that \(f\) is a **monomorphism** if, for any two morphisms \(g_1, g_2 : z \to a\), like in the following diagram,

\[
\begin{array}{c}
z \\
g_2 \downarrow \\
a \\
\downarrow f \\
b
\end{array}
\]

we have

\(f \circ g_1 = f \circ g_2\) implies \(g_1 = g_2\).

Exercise 49 (Solution 225). In the category of sets, show that a morphism \(f : X \to Y\) is a monomorphism if and only if it is injective.

Definition 50 (Epi). Let \(f : a \to b\) be a morphism in \(\mathcal{C}\). We say that \(f\) is an **epimorphism** if, for any two morphisms \(g_1, g_2 : b \to z\), like in the following diagram,
we have
\[ g_1 \circ f = g_2 \circ f \text{ implies } g_1 = g_2. \]

**Exercise 51.** In the category of sets, show that a morphism \( f : X \to Y \) is an epimorphism if and only if it is surjective.

**Exercise 52** (Solution 226). In the category of sets, show that if \((s, r)\) is a section-retraction pair, then the section \(s\) is injective. Hint: you can use Exercise 49.

**Exercise 53.** In the category of sets, show that if \((s, r)\) is a section-retraction pair, then the retraction \(r\) is surjective. Hint: you can use Exercise 51.

**Exercise 54** (Solution 227). Show that any isomorphism \( f : a \cong b \) (in some arbitrary category \( C \)) is both a monomorphism and an epimorphism.

**Exercise 55** (Solution 228). Show that the converse of Exercise 54 does not hold in general, i.e. give an example of a category where there exists a morphism which is both an epi- and a monomorphism, but which is not an isomorphism.

Hint: Consider a preordered set.

**Exercise 56.** Let \( \mathcal{G}_1 \) (resp. \( \mathcal{G}_2 \) and \( \mathcal{G}_3 \)) be the category generated by the following graph:

\[
\begin{array}{c}
\text{w} \rightarrow \\
\downarrow \\
\text{x} \rightarrow \text{y} \rightarrow \text{z} \rightarrow \\
\text{w} \rightarrow \\
\end{array}
\]

resp.

\[
\begin{array}{c}
\text{w} \rightarrow \\
\downarrow \\
\text{x} \rightarrow \text{y} \rightarrow \text{z} \rightarrow \\
\text{w} \rightarrow \\
\end{array}
\]

resp.

\[
\begin{array}{c}
\text{w} \rightarrow \\
\downarrow \\
\text{x} \rightarrow \text{y} \rightarrow \text{z} \rightarrow \\
\text{w} \rightarrow \\
\end{array}
\]

Can you characterize the mono- and epimorphisms in these categories?
Exercise 57. Can you characterize the monomorphisms, epimorphisms and isomorphisms in the category generated by the following graph:

\[ x \rightarrow y \]

3. Universal Properties

In this section, we discuss special objects in a category.

Further Reading. On initial and terminal objects, see also [1, §2.7.16] and [2, p. 48ff].

Products and coproducts, other special limits and colimits, and the general definition of limits and colimits, are discussed in [2, §§5.1, 5.2].

Pierce’s tutorial discusses the (co)limits defined here in [3, §§2.3–2.4], and further (co)limits in [3, §§2.5–2.7].

3.1. Initial Objects

Definition 58. Let \( \mathcal{C} \) be a category. An object \( A \in \mathcal{C}_0 \) is initial if there is exactly one morphism from \( A \) to any object \( B \in \mathcal{C}_0 \).

Exercise 59. Does the category • have an initial object?

Exercise 60. Does the category

\[ A \quad B \]

have an initial object?

Exercise 61. Does the category \( A \rightarrow B \) have an initial object?

Exercise 62. Does the category \( A \leftrightarrow B \) have an initial object? (Here, the morphism \( A \rightarrow B \) is inverse to the morphism \( B \rightarrow A \).)

Exercise 63. Does the category \( A \Rightarrow B \) have an initial object?

Exercise 64. Does the category

\[ \begin{array}{c}
A \\
\downarrow^f \\
B
\end{array} \]

have an initial object? (Here, the morphism \( f \) is different from \( \text{Id}_A \)).

Exercise 65 (Solution 229). Identify an initial object in the category \( \text{Set} \) of sets. Prove that it is indeed initial.

Exercise 66. Identify an initial object in the category \( \text{LEAN} \) of Lean types. Prove that it is indeed initial.
Exercise 67 (Solution 230). Let \((X, \leq)\) be a poset. Describe what an initial object looks like in \(\text{Pos}(X, \leq)\).

Exercise 68 (Solution 231). Let \(A\) and \(A'\) be initial objects in \(C\). Construct an isomorphism \(i : A \cong A'\).

Exercise 69 (Solution 232). Let \(A\) be an initial object in \(C\), and let \(A'\) be isomorphic to \(A\) (via an isomorphism \(i : A \cong A'\)). Show that \(A'\) is an initial object of \(C\).

Remark 70. Exercises 68 shows that initial objects in a category \(C\) are essentially unique, that is, they are unique up to (unique) isomorphism.

This justifies using the determinate article: we will say that \(A\) is the initial object of \(C\).

This is more generally the case for any object with a universal property, see, e.g., Exercises 82 and 94.

Exercise 71 (Solution 233). Construct a category that does not have an initial object.

Exercise 72 (Solution 240). Let \(\text{PtSet}\) be the category of pointed sets, that is the category whose objects are pairs \((X, x)\) with \(X\) a set and \(x \in X\) and a morphism from \((X, x)\) to \((Y, y)\) is defined as a function \(f : X \to Y\) such that \(f(x) = y\). Identify an initial object in \(\text{PtSet}\).

Remark 73. The concept of initial object seems trivial and boring in the categories considered above. However, in complicated categories, initial objects can be complicated and exciting; we will see this in Section 8.

3.2. Terminal Objects

Definition 74. Let \(C\) be a category. An object \(B \in C_0\) is terminal (or final) if there is exactly one morphism to \(B\) from any object \(A \in C_0\).

Exercise 75. Does the category \(\bullet\) have a terminal object?

Exercise 76. Does the category \(A \to B\) have a terminal object?

Exercise 77. Does the category \(A \rightrightarrows B\) have a terminal object?

Exercise 78. Does the category \(A \Rightarrow B\) have a terminal object?

Exercise 79 (Solution 241). Identify a terminal object in the category \(\text{Set}\) of sets. Prove that it is indeed terminal.

Exercise 80. Identify a terminal object in the category \(\text{LEAN}\) of Lean types. Prove that it is indeed terminal.

Exercise 81 (Solution 242). Let \((X, \leq)\) be a poset. Describe what a terminal object looks like in \(\text{Pos}(X, \leq)\).

Exercise 82 (Solution 243). Let \(B\) and \(B'\) be terminal objects in \(C\). Construct an isomorphism \(i : B \cong B'\).
Exercise 83 (Solution 244). Let $B$ be a terminal object in $\mathcal{C}$, and let $B'$ be isomorphic to $B$ (via an isomorphism $i : B \cong B'$). Show that $B'$ is a terminal object of $\mathcal{C}$.

Exercise 84 (Solution 245). Show that $\mathcal{C}$ has a terminal object if and only if $\mathcal{C}^{\text{op}}$ has an initial object.

Exercise 85 (Solution 246). Construct a category that does not have an terminal object.

### 3.3. (Binary) Products

**Definition 86.** Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}_0$ be objects of $\mathcal{C}$.

A triple $(P, \pi_l : P \to A, \pi_r : P \to B)$ is called a **product of $A$ and $B$** if for any triple $(Q, q_1 : Q \to A, q_2 : Q \to B)$ there is exactly one morphism $f : Q \to P$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{f} & P \\
\downarrow^{q_1} & & \downarrow^{\pi_l} \\
A & \xleftarrow{\pi_r} & B
\end{array}
\]

If $A$ and $B$ have a specified product $(P, \pi_l : P \to A, \pi_r : P \to B)$, then the object $P$ is often called $A \times B$. The morphism $f : Q \to A \times B$ determined by $(Q, q_1, q_2)$ is denoted by $(q_1, q_2)$.

Exercise 87. Does the category $\bullet$ have products?

Exercise 88. Does the category $A \to B$ have products?

Exercise 89. Does the category $A \iff B$ have products?

Exercise 90. Does the category $A \Rightarrow B$ have products?

Exercise 91 (Solution 247). Identify a product of sets $X$ and $Y$ in the category $\text{Set}$ of sets. Prove that it is indeed a product.

Exercise 92. Identify a product of types $A$ and $B$ in the category $\text{LEAN}$ of Lean types. Prove that it is indeed a product.

Exercise 93 (Solution 248). Let $(X, \leq)$ be a poset. Describe what a product looks like in $\text{Pos}(X, \leq)$.

Exercise 94 (Solution 250). Given two products of $A$ and $B$ in a category $\mathcal{C}$, construct an isomorphism between them, that is, between their underlying objects.

Exercise 95 (Solution 251). Given a product $(P, \pi_l : P \to A, \pi_r : P \to B)$ of $A$ and $B$ in $\mathcal{C}$, and an object $P'$ that is isomorphic to $P$ via an isomorphism $i : P \cong P'$, construct a product with object $P'$ of $A$ and $B$.  

20
Exercise 96 (Solution 252). Let $C$ be a category and $T \in C_0$ a terminal object. For any object $A \in C_0$, construct a product of $A$ and $T$.

Hint: to form an idea what the object $A \times T$ should be, solve the exercise first in a specific category, e.g., in the category of sets or in a category coming from a preordered set.

Exercise 97 (Solution 253). Let $C$ be a category and $A, B \in C_0$ be objects. Show that the product of $A$ and $B$ exists if and only if the following category has a terminal object:

- The objects are triples $(P, p_l : P \to A, p_r : P \to B)$.
- A morphism from $(P, p_l, p_r)$ to $(Q, q_l, q_r)$ is a morphism $f \in C_0$ such that the following diagram commutes:

\[
\begin{array}{ccc}
P & \xleftarrow{f} & Q \\
| & \downarrow{p_l} & | \\
A & \xleftarrow{q_l} & B
\end{array}
\]

- The composition and identity is inherited from the structure of $C$.

Exercise 98 (Solution 254). Let $C$ be a category with a choice of product $(A \times B, \pi_l, \pi_r)$ for any two objects $A, B \in C_0$. Given morphisms $f : A \to C$ and $g : B \to D$ in $C$, construct a morphism $f \times g : A \times B \to C \times D$.

Exercise 99 (Solution 255). Let $C$ be a category with a choice of product $(A \times B, \pi_l, \pi_r)$ for any two objects $A, B \in C_0$. For any $A, B \in C_0$, construct an isomorphism $A \times B \cong B \times A$.

3.4. (Binary) Coproducts

Definition 100. Let $C$ be a category and let $A, B \in C_0$ be objects of $C$.

A triple $(C, i_l : A \to C, i_r : B \to C)$ is called a coproduct of $A$ and $B$ if for any triple $(D, i_l : A \to D, i_r : B \to D)$ there is exactly one morphism $f : C \to D$ such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{i_l} & C \\
| & \downarrow{i_l} & | \\
D & \xleftarrow{f} & B
\end{array}
\]

If $A$ and $B$ have a specified coproduct $(C, i_l : A \to C, i_r : B \to C)$, then the object $C$ is often called $A + B$. The morphism $f : A + B \to D$ determined by $(D, i_l, i_r)$ is denoted by $[i_l, i_r]$.
Exercise 101. Write a sequence of suitable exercises about coproducts.

Exercise 102. Consider the category with one object and the rational numbers \( \mathbb{Q} \) as morphisms. Can you construct an initial object in this category? A terminal object? Products? Coproducts?

Remark 103. Initial and terminal objects and products and coproducts are special cases of limits and colimits. We are not studying, in these notes, the general notion of (co)limit. However, the examples above should suffice for you to understand, in your own time, other (co)limits, such as

- pullbacks and pushouts;
- products and coproducts of families of objects (not just of pairs of objects); and
- equalizers and coequalizers.

Further Reading. We do not discuss here whether/when/how (co)limits can be transported along functors. You can find some information on this in \[2, \S 5.3\].

4. Functors

An important aspect in computer programming is the transformation of data. For example, if you have a data type \( X \), then one can consider also the data type \( \text{List}(X) \) of lists with values in \( X \). If one thinks of the objects in a category to be data types, then we can ask even more. If \( f : X \to Y \) is a function (between the data types), then this also induces a function from the \( X \)-valued lists to the \( Y \)-valued lists as follows:

\[
\text{List}(f) : \text{List}(X) \to \text{List}(Y) \quad (1)
\]

\[
[x_1, \ldots, x_n] \mapsto [f(x_1), \ldots, f(x_n)].
\]

Remark 104. The “…” above are informal — a formal definition would define \( \text{List}(f) \) by structural recursion on lists, of course.

A functor formalizes this phenomenon:

Definition 105. Let \( C \) and \( D \) be categories. A functor \( F \) from \( C \) to \( D \) consists of the following data:

- A function \( C_0 \to D_0 \), written as \( X \mapsto F(X) \).
- For each \( X, Y \in C_0 \), a function \( C(X, Y) \to D(F(X), F(Y)) \), written as \( f \mapsto F(f) \).
Moreover, this data should satisfy the following properties:

- **(Preserves composition)** For $f \in \text{hom}_C(X, Y)$ and $g \in \text{hom}_C(Y, Z)$, we have $F(g \circ f) = Fg \circ Ff$.

- **(Preserves identity)** For $X \in \mathcal{C}$, we have $F(\text{id}_X) = \text{id}_{F(X)}$.

**Example 106.** The list-functor (on sets), denoted by $\text{List}$, is the functor from $\text{Set}$ to $\text{Set}$ defined by the following data:

- The function on objects is given by:
  \[
  \text{Set}_0 \to \text{Set}_0 : X \mapsto \text{List}(X).
  \]

- For each $X, Y \in \text{Set}$, the function on morphisms is given by
  \[
  \text{Set}(X, Y) \to \text{Set}(\text{List}(X), \text{List}(Y)) : f \mapsto \text{List}(f),
  \]
  where $\text{List}(f)$ is given in Eq. (1).

**Exercise 107.** Show that $\text{List}$ is a functor, that is, show that it preserves identity and composition of functions. Hint: use structural induction on lists.

**Exercise 108.** Consider the function $\text{Maybe}_0 : \text{Set}_0 \to \text{Set}_0$ sending a set $X$ to $X + \{\ast\}$. For any two sets $X$ and $Y$ and $f : X \to Y$, define a function

\[
\text{Maybe}(f) : \text{Maybe}_0 X \to \text{Maybe}_0 Y
\]

and show that this assignment satisfies the functor laws.

**Exercise 109.** Let $A \in \text{Set}_0$. Construct a functor $(\times A) : \text{Set} \to \text{Set}$ that, on objects, is given by

\[(\times A)X := X \times A.
\]

**Exercise 110.** Let $\mathcal{C}$ be a category with chosen products, and let $A \in \mathcal{C}_0$. Construct a functor $(\times A) : \mathcal{C} \to \mathcal{C}$ that, on objects, is given by

\[(\times A)X := X \times A.
\]

**Exercise 111.** Let $A \in \text{Set}_0$. Construct a functor $(+ A) : \text{Set} \to \text{Set}$ that, on objects, is given by

\[(+ A)X := X + A.
\]

**Exercise 112.** Let $\mathcal{C}$ be a category with chosen coproducts, and let $A \in \mathcal{C}_0$. Construct a functor $(+ A) : \mathcal{C} \to \mathcal{C}$ that, on objects, is given by

\[(+ A)X := X + A.
\]

**Exercise 113.** Let $R \in \text{Set}_0$ be a set. Construct a functor $(R \to) : \text{Set} \to \text{Set}$ that, on objects, is given by

\[(R \to)X := R \to X.
\]
Exercise 114. Let \( C \) be a category and let \( R \in C_0 \). Construct a functor \( C(R, -) : C \to \textbf{Set} \) that, on objects, is given by

\[
(C(R, -))X := C(R, X).
\]

Exercise 115. Let \((X, \leq_X)\) and \((Y, \leq_Y)\) be posets. Can you characterize/describe the functors from \( \textbf{Pos}(X, \leq_X) \) to \( \textbf{Pos}(Y, \leq_Y) \)? Before writing out the definitions, what would you expect the answer to be?

Exercise 116. Let \( C \) be a category with chosen products \((A \times B, \pi_A, \pi_B)\) for any two objects \( A \) and \( B \). Construct a functor

\[
(\times) : C \times C \to C
\]

from the product category \( C \times C \) to \( C \). The objects of \( C \times C \) are pairs of objects in \( C \), and morphisms \((C \times C)((X, X'), (Y, Y'))\) are pairs \((f : X \to Y, f' : X' \to Y')\) of morphisms in \( C \).

Exercise 117. Let \( C \) be a category with chosen coproducts \((A + B, i_A, i_B)\) for any two objects \( A \) and \( B \). Construct a functor

\[
(+) : C \times C \to C
\]

from the product category \( C \times C \) to \( C \).

Exercise 118. Let \((M, m, e)\) be a monoid and let \( \textbf{Monoid}(M, m, e) \) be its corresponding category as defined in Definition 19. Can you characterize/describe the functors from \( \textbf{Monoid}(M, m, e) \) to \( \text{Set} \)?

4.1. Categories as objects of a category?

Notice that a functor is a function between categories which preserves the structure of a category. So by the philosophy of category theory, this would define a category whose objects are categories and whose morphisms are functors. In order to make this precise, we would also need a identity functor and we should have a composition of functors.

Example 119. Let \( C \) be a category. The identity functor on \( C \), denoted by \( \text{Id}_C \), is the functor specified by the following data:

- The function on objects is given by

\[
C_0 \to C_0 : X \mapsto X.
\]

- For each \( X, Y \in C \), the function on morphisms is given by

\[
C(X, Y) \to C(X, Y) : f \mapsto f.
\]
Exercise 120. Show that \( \text{Id}_C \) (defined in Example 119) satisfies the properties of a functor, i.e. \( \text{Id}_C \) is indeed a functor.

Example 121. Let \( \mathcal{C}, \mathcal{D} \) and \( \mathcal{E} \) be categories and \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{E} \) functors. The **composition functor of \( F \) with \( G \)**, denoted by \( F \cdot G \), is the functor specified by the following data:

- The function on objects is given by
  \[ \mathcal{C}_0 \to \mathcal{E}_0 : X \mapsto G(F(X)). \]
- For each \( X, Y \in \mathcal{C} \), the function on morphisms is given by
  \[ \mathcal{C}(X, Y) \to \mathcal{E}(G(F(X)), G(F(Y))) : f \mapsto G(F(f)). \]

Exercise 122. Show that \( F \cdot G \) (defined in Example 121) satisfies the properties of a functor, i.e. \( F \cdot G \) is indeed a functor.

Definition 123. The **Category of categories**, denoted by \( \text{Cat} \), is the category specified by the following data:

- An object is a category.
- If \( \mathcal{C}, \mathcal{D} \in \text{Cat} \) are categories, then is \( \text{Cat}(\mathcal{C}, \mathcal{D}) \) the collection of all functors from \( \mathcal{C} \) to \( \mathcal{D} \).
- The identity morphism on a category \( \mathcal{C} \) is the identity functor on \( \mathcal{C} \) defined in Example 119.
- The composition of morphisms, i.e. functors, is the composition of functors defined in Example 121.

Exercise 124. Show that \( \text{Cat} \) satisfies the property of a category, i.e. \( \text{Cat} \) is indeed a category.

Remark 125. When showing that \( \text{Cat} \) is a category, one is forced to consider equality of objects when showing that two functors are equal. This goes against the spirit of category theory, where we only ever consider equality of (parallel) morphisms. We want to consider two objects “the same” when they are isomorphic, not when they are equal. Of course, any two equal objects are isomorphic to each other, but not the other way round; for instance, in the category of sets, the cartesian product \( A \times B \) is isomorphic to \( B \times A \), but they are not equal.

To stay within the spirit of category theory, one can instead consider \( \text{Cat} \) as a **bicategory**. In a bicategory, one has one more layer of things: objects, morphisms, and 2-cells between parallel morphisms. One also calls objects “0-cells” and morphisms “1-cells”, for consistency. Importantly, in a bicategory, the laws concerning 1-cells (as

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2See, e.g., [https://ncatlab.org/nlab/show/bicategory#detailedDefn](https://ncatlab.org/nlab/show/bicategory#detailedDefn) for a definition of bicategories.
stated in Definition 1 do not hold up to equality, but only up to isomorphism of 2-cells.

An important example is the bicategory given by the following data, which we only list partially:

1. 0-cells are categories;
2. 1-cells are functors;
3. 2-cells are natural transformations (see Section 5);
4. composition and identity of 1-cells is composition and identity of functors.

We do not delve into bicategories in these notes; an introductory text is, for instance, Leinster’s [6].

5. Natural transformations

**Definition 126.** Let $F, G : C \to D$ be functors. A natural transformation $\alpha$ from $F$ to $G$ consists of the following data:

- For each $X \in C_0$, a morphism $\alpha_X \in D(F(X), G(X))$.

Moreover, this data should satisfy the following naturality condition:
For each $f \in C(X, Y)$, the following diagram should commute:

\[
\begin{array}{ccc}
F(X) & \xrightarrow{\alpha_X} & G(X) \\
\downarrow F(f) & & \downarrow G(f) \\
F(Y) & \xrightarrow{\alpha_Y} & G(Y)
\end{array}
\]

Moreover, we call $\alpha$ a natural isomorphism if for each $X \in C_0$, we have that $\alpha_X$ is an isomorphism in $D$.

**Notation 127.** If $F, G : C \to D$ are functors. A natural transformation $\alpha$ from $F$ to $G$, is denoted as $\alpha : F \Rightarrow G$ or

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & D \\
U \downarrow & & \downarrow F
\end{array}
\]

**Example 128.** (Currying) Let $X$ be a set. Let $F := \text{Set}(X, -) \times X : \text{Set} \to \text{Set}$ be the functor induced by the following data (on objects):

\[Y \mapsto \text{Set}(X, Y) \times X.\]

The evaluation defines a natural transformation $ev : F \Rightarrow \text{Id}_{\text{Set}}$ as follows:

\[ev_Y : \text{Set}(X, Y) \times X \to Y : (f, x) \mapsto f(x).\]

Show that this indeed satisfies the naturality condition.
5.1. Functor categories

**Definition 129.** Let \( F : C \to D \) be a functor. The **identity natural transformation** \( \text{Id}_F \) on \( F \) is given by the following data:

\[
\forall X \in C_0 : (\text{Id}_F)_X := \text{Id}_{F(x)}.
\]

**Exercise 130.** Show that for any functor \( F : C \to D \), the identity natural transformation \( \text{Id}_F \) satisfies the properties of a natural transformation.

**Definition 131.** Let \( F, G, H : C \to D \) be functors and \( \alpha : F \Rightarrow G, \beta : G \Rightarrow H \) be natural transformations. The **(vertical) composition** of \( \alpha \) and \( \beta \) is the natural transformation \( \cdot \alpha \beta \) is given by the following data:

\[
\forall X \in C_0 : (\beta \circ \alpha)_X := \beta_X \circ \alpha_X.
\]

**Exercise 132.** Show that for any functors \( F, G, H : C \to D \) and \( \alpha : F \Rightarrow G, \beta : G \Rightarrow H \) natural transformations, the (vertical) composition of \( \alpha \) and \( \beta \) satisfies the properties of a natural transformation.

**Definition 133.** Let \( C, D \) be categories. The **category of functors** or the **functor category** from \( C \to D \), denoted by \( \text{Fun}(C, D) \) or \([C, D]\), is given by the following data:

- An object is a functor \( F : C \to D \).
- A morphism from \( F \) to \( G \) is a natural transformation \( \alpha : F \Rightarrow G \).
- The identity morphism on \( F \) is given by the identity natural transformation \( \text{Id}_F \) defined in Definition 129.
- The composition of \( F \) and \( G \) is given by the composition \( \beta \circ \alpha \) defined in Definition 131.

**Exercise 134.** Show that for any two categories \( C \) and \( D \), the functor category from \( C \to D \) satisfies the properties of a category.

**Definition 135.** Let \( F, G : C \to D \) and \( \tilde{F}, \tilde{G} : D \to E \) be functors and \( \alpha : F \Rightarrow G, \beta : G \Rightarrow H \) be natural transformations. The **horizontal composition** (also called the **Godement product**) of \( \alpha \) and \( \beta \), denoted by \( \beta \bullet \alpha \), is defined as:

\[
\forall X \in C_0 : (\beta \bullet \alpha)_X := \beta_{G(x)} \circ \tilde{F} (\alpha_X).
\]

**Exercise 136.** Show that \( \alpha \bullet \beta \) (defined as in Definition 135), is indeed a natural transformation.

**Exercise 137.** Show the following property:

\[
\forall X \in C_0 : (\beta \bullet \alpha)_X = \tilde{G} (\alpha_X) \circ \beta_{F(x)}.
\]

Hint: Write the equality as a (not-known commutative) square.
5.2. Exercises

**Exercise 138.** Let \((M, m, e)\) be a monoid and let \(\text{Monoid}(M, m, e)\) be its corresponding category. Recall from Exercise 118 that a functor from \(M\) to \(\text{Set}\) is a set \(X\) together with an action of \(M\) on \(X\), i.e. a function \(\mu : M \times X \to X\) such that

\[
\forall x \in X : \mu(e, x) = x, \quad \forall n_1, n_2 \in M, x \in X : \mu(n_1, \mu(n_2, x)) = \mu(m(n_1, n_2), x).
\]

We will call a set \(X\) with an action of \(M\) on \(X\) an \(M\)-set. Characterize the natural transformations between \(M\)-sets.

**Exercise 139.** Let \((X, \leq_X)\) and \((Y, \leq_Y)\) be posets. Recall from Exercise 115 that a functor between posets corresponds with an order-preserving function, i.e. \(x_1 \leq_X x_2 \implies f(x_1) \leq_Y f(x_2)\). Characterize the natural transformations between order-preserving functions.

5.3. Equivalence of categories

Recall that objects \(X, Y \in \mathcal{C}_0\) are isomorphic if there exist morphisms \(f \in C(X, Y)\) and \(g \in C(Y, X)\) such that \(g \circ f = \text{Id}_X\) and \(f \circ g = \text{Id}_Y\). So in particular we have the notion of an isomorphism in the category \(\text{Cat}\) of categories. Spelled out, this means categories \(\mathcal{C}\) and \(\mathcal{D}\) are isomorphic if there exist functors \(F : \mathcal{C} \to \mathcal{D}\) and \(G : \mathcal{C} \to \mathcal{D}\) such that \(G \circ F = \text{Id}_\mathcal{C}\) and \(F \circ G = \text{Id}_\mathcal{D}\).

However, the following exercise shows that isomorphism of categories is not the correct notion of equivalence/sameness between categories:

Let \(\text{FinSet}\) be the category whose objects are given by finite sets and whose morphisms are given by functions. That this is a category follows since \(\text{Set}\) is a category.

Let \(\text{FinOrd}\) be the category whose objects are given by sets of the form \([n] := \{0, 1, \ldots, n-1\}\), and whose morphisms are given by functions between these sets.

Every finite set \(X\) is always in bijection with a set of the form \([n]\) (where \(n\) is the size \(|X|\) of \(X\)). For each set \(X\), we fix a bijection \(\phi^X : X \to [|X|]\). Consequently, we have a functor:

**Definition 140.** Let \(U : \text{FinSet} \to \text{FinOrd}\) be the functor specified by the following data:

- For \(X \in \text{FinSet}_0\), we define \(U(X) := [|X|]\).
- For \(f \in \text{FinSet}(X, Y)\), we define \(U(f) : [|X|] \to [|Y|]\) as the unique function such that the following diagram commutes:

\[\text{Notice that the objects of } \text{FinSet} \text{ form a subset of the objects of } \text{Set}, \text{ but given any two finite sets } X, Y \text{ in } \text{FinSet}_0, \text{ we have } \text{FINSET}(X, Y) = \text{SET}(X, Y). \text{ We say in this case that } \text{FinSet} \text{ is a (full) subcategory of } \text{Set}.\]
Exercise 141. Show that $U : \mathbf{FinSet} \to \mathbf{FinOrd}$ is indeed a functor. In particular, you have to show that $U$ is well-defined on the morphisms.

In order to show that $U$ is not an isomorphism, one can use the following lemma/exercise:

Exercise 142. Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism if and only if $F$ satisfies the following properties:

- $F$ is injective on objects, i.e.
  \[
  \forall X,Y \in \mathcal{C}_0 : F(X) = F(Y) \implies X = Y.
  \]

- $F$ is surjective on objects, i.e.
  \[
  \forall Y \in \mathcal{D}_0 : \exists X \in \mathcal{C}_0 : F(X) = Y.
  \]

- $F$ is faithful, i.e. the following functions are injective
  \[
  \forall X,Y \in \mathcal{C}_0 : \mathcal{C}(X,Y) \xrightarrow{F_{X,Y}} \mathcal{D}(F(X),F(Y)) : f \mapsto F(f)
  \]

- $F$ is full, i.e. for all $X,Y \in \mathcal{C}_0$, $F_{X,Y}$ is surjective.

Exercise 143. Show that $U$ is not an isomorphism, i.e. state which part of an isomorphism fails and give a concrete example that it fails.

Remark 144. So the problem with $U$ (in the sense that it is not an isomorphism) is that multiple (finite) sets are mapped to the same set. For this reason, a good notion of equivalence between categories should not be injective on objects. Also, which is not clear from this example, we should also weaken the condition of $F$ being surjective on objects. Instead, we need that $F$ is essentially surjective on objects:

\[
\forall Y \in \mathcal{D}_0 : \exists X \in \mathcal{C}_0 : F(X) \cong Y.
\]

So motivated by the remark, we define:

Definition 145. Categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists a pair of functors $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C})$ such that there exists natural isomorphisms

\[
G \circ F \cong \text{Id}_\mathcal{C}, \quad F \circ G \cong \text{Id}_\mathcal{D}
\]

So although $U$ is not an isomorphism, it does induce an equivalence of categories:
Exercise 146. Show that $U$ induces an equivalence of categories.

Exercise 147. Let $C$ be the category whose objects are categories with a unique object and whose morphisms are functors between these one-object categories, i.e. $C$ is the (full) subcategory of $\text{Cat}$ generated by the categories with a unique object. Show that $C$ is equivalent to the category $\text{Monoid}$ of monoids.

What happens if we do not consider $C$ to consist of those categories with a unique object, but with a unique object up to isomorphism, i.e. $C$ is the category whose objects are categories $D$ which satisfy the following property:

$$
\forall X, Y \in D_0 : X \cong Y.
$$

The following exercise gives a characterization of a functor being an equivalence. However, in order to show this, one has to use the axiom of choice which means (informally) that if the following property holds:

$$
\exists x : P(x),
$$

then we can fix some $x$ such that $P(x)$ holds.

Exercise 148. Show that a functor $F : C \to D$ induces an equivalences of categories if and only if it is essentially surjective on objects and fully faithful.

6. Adjunctions

Definition 149. A pair $(F, G)$ of functors $F : C \to D, G : D \to C$ is called an adjoint pair if for every objects $C \in C_0$ and $D \in D_0$, there exists a bijection

$$
\alpha_{C,D} : D(F(C), D) \to C(C, G(D)),
$$

which are moreover natural in both $C$ and $D$, i.e. for each $f \in C(C_1, C_2)$ and $g \in D(D_1, D_2)$, the following diagrams commute:

$$
\begin{align*}
\begin{array}{c}
D(F(C_2), D) \xrightarrow{\alpha_{C_2,D}} C(C_2, G(D)) \\
\downarrow \alpha_{F(f)} \\
D(F(C_1), D) \xrightarrow{\alpha_{C_1,D}} C(C_1, G(D))
\end{array}
\end{align*}
$$

(3)

$$
\begin{align*}
\begin{array}{c}
D(F(C), D_1) \xrightarrow{\alpha_{C,D_1}} C(C, G(D_1)) \\
\downarrow \alpha_{G(g)} \\
D(F(C), D_2) \xrightarrow{\alpha_{C,D_2}} C(C, G(D_2))
\end{array}
\end{align*}
$$

(4)

If $(F, G)$ is an adjoint pair, we call $F$ the left adjoint of $G$ and we call $G$ the right adjoint of $F$ and we denote $F \dashv G$. 

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Exercise 150. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Show that if $F$ has a left (resp. right) adjoint $G$, then $G$ must be unique up to isomorphism.

Exercise 151. Let $U : \text{Monoid} \to \text{Set}$ be the forgetful functor (defined in Example 263) which maps a monoid to its underlying set and let $F : \text{Set} \to \text{Monoid}$ be the functor which maps a set to the free monoid of this set (defined in Example 266). Then $(F, U)$ is an adjoint pair. (Hint: Use Proposition 268).

Exercise 152. Let $Y$ be a set. Show that the functor (induced by the following mapping on objects)
\[- \times Y : \text{Set} \to \text{Set} : X \mapsto X \times Y,
\]
has a right adjoint which is given by the functor (induced by the following mapping on objects)
\[\text{Set}(Y, -) : \text{Set} \to \text{Set} : X \mapsto \text{Set}(Y, X).
\]

Is there an analogous statement in the category $\text{LEAN}$ instead of $\text{Set}$?

Theorem 153. Let $(F, G)$ be a pair of functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$. The following statements are equivalent:

1. $(F, G)$ is an adjoint pair.

2. There exists natural transformations
\[\eta : \text{id}_\mathcal{C} \Rightarrow G \circ F, \quad \epsilon : F \circ G \Rightarrow \text{id}_\mathcal{D},\]

such that for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$ the following diagrams commute:
\[\begin{array}{ccc}
F(C) & \xrightarrow{F(g)} & F(G(F(C))) \\
\downarrow_{\text{ld}_{F(C)}} & \quad & \downarrow_{\text{ld}_{F(G(F(C))}}} \\
F(C) & \quad & \text{G}(G(D))
\end{array}\]
\[\begin{array}{ccc}
G(D) & \xrightarrow{G(\eta_D)} & G(F(G(D))) \\
\downarrow_{\text{ld}_{G(D)}} & \quad & \downarrow_{\text{ld}_{G(F(G(D))}}} \\
G(D) & \quad & \text{G}(\epsilon_C)
\end{array}\]

In case $(F, G)$ satisfies these (equivalent) conditions, we call $\eta$ the unit of the adjunction and $\epsilon$ the counit of the adjunction. The equalities in condition 2 are called the triangle identities.

Proof. First assume that $(F, G)$ is an adjoint pair. We have to define the unit and counit and show that the triangle identities hold:

- Unit: For each $X \in \mathcal{C}$, we should first define $\eta_X \in \mathcal{C}(X, G(F(X)))$. Since $F \dashv G$, we have a bijection
\[\alpha_{X, FX} : \mathcal{D}(F(X), FX) \to \mathcal{C}(X, G(F(X))),\]

\footnote{Isomorphism w.r.t the functor category.}
hence, we define $\eta_X := \alpha_{X,FX}(\text{Id}_{FX})$. We now show that $(\eta_X)_{X \in C_0}$ forms a natural transformation: Assume $f \in C(X,Y)$. We have to show that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha_{X,FX}(\text{Id}_{FX})} & G(F(X)) \\
\downarrow f & & \downarrow G(f) \\
Y & \xrightarrow{\alpha_{Y,FY}(\text{Id}_{FY})} & G(F(Y))
\end{array}
$$

That this square is indeed commutative follows from the following computation:

$$
\alpha(\text{Id}_{FY}) \circ f = \alpha(\text{Id}_{FY} \circ F(f)) = \alpha(F(f)) = \alpha(F(f) \circ \text{Id}_{FX}) = G(F(f)) \circ \alpha(\text{Id}_{FX}),
$$

where the first and last equality hold by naturality of $\alpha$.

- **Counit**: For each $Y \in D_0$, we the counit $\epsilon_Y \in D(F(G(Y)),Y)$ is defined as the image of $\text{Id}_{G(Y)}$ of the bijection

$$
\alpha_{G,Y}^{-1} : C(GY,GY) \rightarrow D(F(G(Y)),Y).
$$

That $\epsilon$ indeed forms a natural transformation is analogous to the computation in Eq. (5).

- **Triangle identities**: Both the triangle identities are proved analogously, hence we will only show the first triangle identity, i.e. $\text{Id}_{FX} = \epsilon_{FX} \circ F(\eta_X)$. Unfolding the definition of $\epsilon$, this is equivalent to showing:

$$
\text{Id}_{FX} = \alpha_{GFX,FX}^{-1}(\text{Id}_{GFX}) \circ F(\eta_X).
$$

Since the components of $\alpha$ are bijections, this is equivalent to showing

$$
\alpha(\text{Id}_{FX}) = \alpha(\alpha_{GFX,FX}^{-1}(\text{Id}_{GFX}) \circ F(\eta_X)).
$$

This indeed holds by the following computation:

$$
\alpha(\alpha_{GFX,FX}^{-1}(\text{Id}_{GFX}) \circ F(\eta_X)) = \alpha(\alpha_{GFX,FX}^{-1}(\text{Id}_{GFX}) \circ \eta_X) = \text{Id}_{GFX} \circ \eta_X = \eta_X,
$$

where the first (resp. second) equality holds by naturality (resp. bijectiveness) of $\alpha$.

This concludes the proof of (1) $\Rightarrow$ (2). Now assume that (2) holds. We have to construct bijections $\alpha_{X,Y} : D(F(X),Y) \rightarrow C(X,G(Y))$, which are natural in $X$ and $Y$. Let $g \in D(F(X),Y)$. Define $\alpha_{X,Y}(g)$ as the composite:

$$
X \xrightarrow{\eta_X} G(F(X)) \xrightarrow{G(g)} G(Y).
$$

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For the other direction, let \( f \in \mathcal{C}(X, G(Y)) \). Define \( \alpha_{X,Y}^{-1}(f) \) as the composite:

\[
FX \xrightarrow{F(f)} F(G(Y)) \xrightarrow{\epsilon_Y} Y.
\]

That \( \alpha \) and \( \alpha^{-1} \) are inverses of each other follows from the following computation:

\[
\alpha(\alpha^{-1}(f)) = \alpha(\epsilon_Y \circ F(f)) = (G\epsilon_Y \circ GF(f)) \circ \eta_X \quad \text{by definition},
\]

\[
= (G\epsilon_Y \circ GF(f)) \circ \eta_X \quad \text{by functoriality of } G,
\]

\[
= G\epsilon_Y \circ (GF(f) \circ \eta_X) \quad \text{by associativity},
\]

\[
= G\epsilon_Y \circ (\eta_{GY}) \circ f \quad \text{by naturality of } \eta
\]

\[
= (G\epsilon_Y \circ \eta_{GY}) \circ f \quad \text{by associativity}
\]

\[
= f \quad \text{by triangle identity}.
\]

The other equality is shown analogous by using functoriality of \( F \), naturality of \( \epsilon \) and the other triangle identity.

It remains to show the naturality of \( \alpha \) in both \( x \) and \( y \) which is left to the reader as a good exercise on diagram chasing. \( \square \)

7. Monads and Effects

Recall that a monad (as defined, e.g., in Haskell) is a function \( m :: \star \to \star \) together with the additional data of a function \( \text{pure} :: a \to m a \) (for each type \( a \)) and a function \( (\gg\gg) :: m a \to (a \to m b) \to m b \) (for types \( a \) and \( b \)).

The operations \( \text{pure} \) and \( (\gg\gg) \) are expected to satisfy the following laws: such that they satisfy the following properties:

1. \( t \gg\gg \text{pure} == t \)
2. \( \text{pure}(x) \gg\gg f == f \ x \)
3. \( (t \gg\gg f) \gg\gg g == t \gg\gg (\lambda x \to f \ x \gg\gg g) \)

However, these laws cannot be enforced in Haskell, since Haskell does not have any infrastructure for logic.

In a category, however, we can define monads including the monad laws. We will actually give two different definitions of monad; one called “Kleisli triple” (Definition 154), which corresponds to what is called “monad” in Haskell, and one called “monad” (Definition 162). The formulation of monads uses that the arguments to \( (\gg\gg) \) can be reordered.

**Definition 154.** A **Kleisli triple** over a category \( \mathcal{C} \) is consisting of the following data:

- A function \( T : C_0 \to C_0 \).
- For each \( X \in C_0 \), a morphism \( \eta_X \in \mathcal{C}(X, T(X)) \).
• For each \( f \in \mathcal{C}(X, T(Y)) \), a morphism \( f^* \in \mathcal{C}(T(X), T(Y)) \).

such that the following properties holds:

1. For each \( X \in \mathcal{C}_0 \), we have \( \eta^*_X = \text{Id}_{T(X)} \).

2. For each \( f \in \mathcal{C}(X, T(Y)) \), the following diagram commutes:

\[
\begin{array}{c}
X \xrightarrow{\eta_X} T(X) \\
\downarrow f \quad \downarrow f^* \\
T(Y) 
\end{array}
\]

3. For each \( f \in \mathcal{C}(X, T(Y)) \) and \( g \in \mathcal{C}(Y, T(Z)) \), the following diagram commutes:

\[
\begin{array}{c}
T(X) \xrightarrow{f^*} T(Y) \\
\downarrow (g \circ f)^* \\
T(Z) 
\end{array}
\]

We denote a Kleisli triple as \((T, \eta, (\_)^*)\).

**Exercise 155.** Convince yourself that the operations and laws of a Kleisli triple correspond, in the category \( \textsf{Hask} \), to the operations and properties of a monad in Haskell.

**Exercise 156 (Solution 234).** Show how the following assignment induces a Kleisli triple over the category \( \textsf{Set} \):

\[
X \mapsto \text{List}(X).
\]

The resulting monad is called the \textbf{List monad}.

**Exercise 157 (Solution 235).** Show how the following assignment induces a Kleisli triple over the category \( \textsf{Set} \):

\[
X \mapsto \text{BinTree}(X),
\]

where \( \text{BinTree}(X) \) is the set of binary trees labelled with elements from \( X \) at the leaves, that is the set inductively generated by the constructors \( \text{leaf} : X \rightarrow \text{BinTree}(X) \) and \( \text{branch} : \text{BinTree}(X) \rightarrow \text{BinTree}(X) \rightarrow \text{BinTree}(X) \). The resulting monad is called the \textbf{Tree monad}.

**Exercise 158 (Solution 236).** Let \( E \) be a set (considered as a set of exceptions). Show how the following assignment induces a Kleisli triple over the category \( \textsf{Set} \):

\[
X \mapsto (X + E),
\]

The resulting monad is called the \textbf{Exception monad}.

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Exercise 159. Show how the following assignment induces a Kleisli triple over the category $\text{Set}$:

$$X \mapsto \mathbb{P}_{\text{fin}}(X) := \{A \subseteq X \mid A \text{ is finite} \}.$$ 

The resulting monad is called the **Monad of nondeterminism**.

Exercise 160 (Solution 238). Let $R$ be a set (considered as a set of results). Show how the following assignment induces a Kleisli triple over the category $\text{Set}$:

$$X \mapsto \text{Cont}^R(X) := (X \to R) \to R.$$ 

The resulting monad is called the **Continuation monad**.

Exercise 161 (Solution 239). Let $R$ be a set. Show how the following assignment induces a Kleisli triple over the category $\text{Set}$:

$$X \mapsto R \rightarrow X.$$ 

The resulting monad is called the **Monad of families of elements**.

The notion of a Kleisli triple can equivalently be described as follows:

**Definition 162.** A monad over a category $\mathcal{C}$ consists of the following data:

- A (endo)functor $T : \mathcal{C} \to \mathcal{C}$.
- A natural transformation $\eta : \text{id}_\mathcal{C} \Rightarrow T$.
- A natural transformation ("multiplication") $\mu : T \circ T \Rightarrow T$.

such that for each $X \in \mathcal{C}_0$ the following diagrams commute:

$$
\begin{array}{ccc}
T^3(X) & \xrightarrow{\mu_{T(X)}} & T^2(X) \\
\downarrow T(\mu_X) & & \downarrow \mu_X \\
T^2(X) & \xrightarrow{\mu_X} & T(X)
\end{array}
\quad
\begin{array}{ccc}
T(X) & \xrightarrow{\eta_{T(X)}} & T^2(X) \\
\downarrow T(\mu_X) & & \downarrow \mu_X \\
T(X) & \xrightarrow{\mu_X} & T(X)
\end{array}
$$

where we denote $T^2 := T \circ T$ and $T^3 := T \circ T \circ T$.

Exercise 163. Given a monad, construct a Kleisli triple from it. Conversely, given a Kleisli triple, construct a monad from it.

Exercise 164. For each of the Kleisli triples above, describe the monad multiplication $\mu$ obtained from it.

Every Kleisli triple induces a category:

**Definition 165.** Let $(T, \eta, (-)^*)$ be a Kleisli triple over $\mathcal{C}$. The **Kleisli category**, denoted by $\mathcal{C}_T$, is the category defined by the following data:

- $(\mathcal{C}_T)_0 := \mathcal{C}_0$. 

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• For each \(X, Y \in (\mathcal{C}_T)_0\), \(\mathcal{C}_T(X, Y) := \mathcal{C}(X, TY)\).

• The identity on \(X \in (\mathcal{C}_T)_0\) is \(\eta_X\).

• The composition of \(f \in \mathcal{C}_T(X, Y)\) and \(g \in \mathcal{C}_T(Y, Z)\) is \(g \circ f\).

**Exercise 166.** Show that for every Kleisli triple, its Kleisli category satisfies the properties of a category.

## 8. Inductive Datatypes and Initial Algebras

### 8.1. Examples

**Exercise 167.** Consider the datatype

\[
data \mathbb{N} ::= \\
| \text{zero} : \mathbb{N} \\
| \text{succ} : \mathbb{N} \to \mathbb{N}
\]

and the following category:

- Objects are triples \((X, z \in X, s : X \to X)\) with \(X\) a set/type;
- Morphisms from \((X, z, s : X \to X)\) to \((X', z', s' : X' \to X')\) are functions \(f : X \to X'\) such that the following diagrams commute:

\[
\begin{array}{ccc}
1 & \xrightarrow{z} & X \\
\downarrow & & \downarrow f \\
X' & \xrightarrow{s'} & X'
\end{array}
\]

- Composition and identity are given by composition of functions in \(\textbf{Set}\). (Check that this is well-defined, that is, that the composition of two functions making the above diagrams commute makes the right diagrams commute again.)

Show that the triple \((\mathbb{N}, \text{zero}, \text{succ})\) is an initial object in this category.

**Exercise 168.** Consider the datatype

\[
data \text{Exp} ::= \\
| \text{Int} : \mathbb{Z} \to \text{Exp} \\
| \text{Plus} : \text{Exp} \times \text{Exp} \to \text{Exp} \\
| \text{Squared} : \text{Exp} \to \text{Exp}
\]

and consider the following category:

- Objects are quadruples \((X, I : \mathbb{Z} \to X, P : X \to X \to X, S : X \to X)\) with \(X\) a set/type;
Morphisms from \((X, I, P, S)\) to \((X', I', P', S')\) are functions \(f : X \to X'\) such that the following diagrams commute:

\[
\begin{align*}
X & \xrightarrow{I} X \\
X & \xrightarrow{I} X \\
X' & \xrightarrow{f \times f} X' \\
X' & \xrightarrow{f \times f} X' \\
X' & \xrightarrow{S'} X'
\end{align*}
\]

Composition and identity are given by composition of functions in \(\text{Set}\). (Check that this is well-defined.)

Show that the quadruple consisting of the type \(\text{Exp}\) together with the functions \(\text{Int}, \text{Plus},\) and \(\text{Squared}\), is an initial object in this category.

8.2. Datatypes as Initial Algebras

Further Reading. This chapter is strongly inspired by Varma Vene’s Ph.D. thesis [10, Chapter 2]. A good explanation of recursion on lists is given in Graham Hutton’s tutorial paper [5].

In this section we introduce (initial) algebras which allows us to define inductive data types.

Definition 169. Let \(F : C \to C\) be an endofunctor. An \(F\)-algebra consists of the following data:

1. An object \(C \in C_0\).
2. A morphism \(\phi \in C(F(C), C)\).

Intuition 170. An algebra is roughly a set equipped with some operations, such as multiplication. The arities, that is, the inputs, of the operations are determined by the functor \(F\). An important class of functors are polynomial functors built using the coproduct (+) and the product (×). Intuitively, the different summands of a polynomial functor each correspond to one datatype constructor, whereas the use of the product indicates that a constructor takes several inputs.

Example 171. Let \(\text{Maybe} : \text{Set} \to \text{Set}\) be the endofunctor given on objects by \(\text{Maybe}(X) := 1 + X\). A \text{Maybe}-algebra is a pair \((X, \phi)\) of a set \(X\) and a function \(\phi : 1 + X \to X\). By the universal property of the coproduct, \(\phi\) is given, equivalently, by two functions \(z\) and \(s\) as follows.

\[
\begin{align*}
z : 1 & \to X \\
s : X & \to X
\end{align*}
\]

Here, think of “\(z\)” standing for “zero”, and “\(s\)” standing for “successor”.

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Example 172. Let $F$ be the endofunctor induced by:

$$F : \text{Set} \to \text{Set}$$

$$X \mapsto 1 + (X \times X).$$

An $F$-algebra consists of a set $X \in \text{Set}$ together with a function $\phi : \text{Set}(1 + (X \times X), X)$. Since $\phi$ is function from the disjoint union, we have that $\phi$ corresponds uniquely to two functions:

$$e : 1 \to X$$

$$m : X \times X \to X.$$

This is precisely the data of a monoid.

Conversely, if $(M, m, e)$ is a monoid, then this induces a function

$$1 + (M \times M) \xrightarrow{\phi = [\phi_e, \phi_m]} M,$$

defined by pattern matching as follows:

$$\phi_e : 1 \to M$$

$$\star \mapsto e$$

$$\phi_m : M \times M \to M$$

$$(x, y) \mapsto m(x, y)$$

The pair $(M, \phi)$ is an $F$-algebra.

Remark 173. Note that a monoid $(M, m, e)$ also satisfies some laws. The laws are not expressed in Example 172. To incorporate the laws, one studies instead algebras of a monad.

Definition 174. Let $F : C \to C$ be an endofunctor and $(C, \phi)$ and $(D, \psi)$ be $F$-algebras. A $(F$-algebra) homomorphism from $(C, \phi)$ to $(D, \psi)$ is a morphism $f \in C(C, D)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
FC & \xrightarrow{\phi} & C \\
\downarrow{Ff} & & \downarrow{f} \\
FD & \xrightarrow{\psi} & D
\end{array}
$$

Exercise 175. Let $F$ be the endofunctor defined as in Example 172, i.e. the endofunctor whose algebras corresponds with monoids. Characterize the $F$-algebra homomorphisms.

Definition 176. Let $F : C \to C$ be an endofunctor. The category of $F$-algebras, denoted by $\text{Alg}(F)$, is defined by the following data:

- The objects are the $F$-algebras.
• The morphisms are the $F$-algebra homomorphisms.

• The identity on $(C, \phi)$ is given by the identity $\text{Id}_C$ in $C$.

• The composition is given by the composition of morphisms in $C$.

Exercise 177. 1. Draw two diagrams to illustrate Definition 176.

2. Show that $\text{Alg}(F)$ satisfies the properties of a category.

We are interested in initial objects of $\text{Alg}(F)$, if they exist. We call these “initial $F$-algebras”. For a general endofunctor $F$, an initial $F$-algebra does not exist; but for many interesting choices of $F$, such an initial object does exist. Before coming to the general definition (see Definition 179), we consider an example.

Exercise 178. Consider the functor $\text{Maybe} : \text{Set} \to \text{Set}$.

1. Show that the initial $\text{Maybe}$-algebra is given by the pair $(N, [\text{zero}, \text{succ}])$, where $N$ is the set of natural numbers, and $\text{zero} : 1 \to N$ and $\text{succ} : N \to N$ are the function picking out zero and the successor function, respectively.

2. Given any other $\text{Maybe}$-algebra $(X, [z, s])$, unpack what it means for the square of Definition 174 to commute.

3. Compare the data from the previous exercise to a definition of a function $f : N \to X$ by pattern matching (e.g., in Haskell).

Definition 179. Let $F : C \to C$ be an endofunctor. An initial $F$-algebra (if it exists) is an initial object in $\text{Alg}(F)$. Unfolding the definition, this means that it is an $F$-algebra $(\mu F, \text{in})$ such that for any $F$-algebra $(C, \phi)$, there exists a unique morphism $\langle \phi \rangle \in C(\mu F, C)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F\mu F & \xrightarrow{\text{in}} & \mu F \\
\downarrow \phi & & \downarrow \langle \phi \rangle \\
FC & \xrightarrow{\phi} & C
\end{array}
$$

A morphism of the form $\langle \phi \rangle$ is called a catamorphism.

Exercise 180 (Solution 256). Let $F : C \to C$ be an endofunctor and let $(\mu F, \text{in})$ be an initial algebra. Show that $\langle \text{in} \rangle = \text{Id}_{\mu F}$.

Exercise 181. Let $\text{Bool}$ be the inductive data type generated by the following two constructors:

- **True** : $\text{Bool}$
- **False** : $\text{Bool}$
Define an endofunctor $F : \text{Set} \to \text{Set}$ such that the $F$-algebras can be characterized as triples $(X, b_1, b_2)$ with $X$ a set and $b_1, b_2 \in X$.

Moreover, show that $(\text{bool}, \text{true}, \text{false})$ is the initial object in $\text{Alg}([]F]$.

**Exercise 182.** The disjoint union (i.e. the coproduct) of two sets $X$ and $Y$ can also be characterized as an inductive data type, indeed: It is generated by the following two constructors:

- $f : X \to X + Y$
- $g : Y \to X + Y$

Define an endofunctor $F : \text{Set} \to \text{Set}$ such that the $F$-algebras can be characterized as triples $(X, i_l, i_r)$ with $X$ a set and $i_l : X \to C, i_r : Y \to C$ be functions.

Moreover, show that $(X + Y, \text{inl}, \text{inr})$ is the initial object in $\text{Alg}(F)$, where $X + Y$ is the disjoint union of $X$ and $Y$ (i.e. the coproduct in $\text{Set}$) and $\text{inl} : X \to X + Y, \text{inr} : Y \to X + Y$ the canonical inclusions.

**Exercise 183.** Let $\mathbb{N}^c$ be the conatural numbers, i.e. $\mathbb{N} + \{\infty\}$. Consider the maybe endofunctor (defined in Example 171), i.e. the functor induced by $F : \text{Set} \to \text{Set}$:

$$F : \text{Set} \to \text{Set} : X \mapsto 1 + X.$$ 

The functions

- $\text{zero}^c : 1 \to \mathbb{N}^c : * \mapsto 0,$
- $\text{succ}^c : \mathbb{N}^c \to \mathbb{N}^c : x \mapsto \begin{cases} n + 1, & \text{if } x := n \in \mathbb{N}, \\ \infty, & \text{if } x := \infty. \end{cases}$

form an $F$-algebra. However, show that $(\mathbb{N}^c, \text{zero}^c, \text{succ}^c)$ is not a initial $F$-algebra.

**Exercise 184** (Fusion Property, Solution 257). Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor and let $(\mu^F, \text{in})$ be an initial algebra. Show that for $F$-algebras $(C, \phi)$ and $(D, \psi)$ and $f \in \mathcal{C}(C, D)$, we have

$$f \circ \phi = \psi \circ F(f) \Rightarrow f \circ \llbracket \phi \rrbracket = \llbracket \psi \rrbracket.$$ 

This is summarized in the following diagram:

$$\begin{array}{c}
F \mu^F \xrightarrow{\text{in}} \mu^F \\
F \phi \downarrow \phantom{F \phi} \\
F C \xrightarrow{\phi} C \\
\phi \downarrow \phantom{\phi} \\
F D \xrightarrow{\psi} D
\end{array}$$

**Exercise 185** (Lambek’s theorem). Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor and let $(\mu^F, \text{in})$ be an initial algebra. Then in is an isomorphism whose inverse is given by $\text{in}^{-1} = \llbracket F(\text{in}) \rrbracket$. 

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Exercise 186 (Solution 268). Let $\mathcal{C}$ be a category with an initial object $\bot$. Show that $(\bot, \text{Id}_\bot)$ is the initial algebra for the identity (endo)functor on $\mathcal{C}$.

Exercise 187. Let $A$ be a set and define $F$ to be the functor induced by

$$F_A : \text{Set} \to \text{Set}$$

$$X \mapsto 1 + (A \times X).$$

1. Show that an $F_A$-algebra consists of a triple $(X, n, c)$, where $X$ is a set, $n \in X$ is an element of $X$, and $c : A \times X \to X$ is a function.

2. Show that the initial $F_A$-algebra is given by the set $\text{List}(A)$ of $A$-valued lists, together with constants $\text{nil} \in \text{List}(A)$ and $\text{cons} : A \times \text{List}(A) \to \text{List}(A)$.

3. Given any other $F_A$-algebra $(X, n, c)$, unpack what it means for the square of Definition 174 to commute. Compare it to a definition of a function $f : \text{List}(A) \to X$ by pattern matching.

Remark 188. In the case of lists, the operator $\langle \_ \rangle$ is also known as $\text{fold}$, which in Haskell is defined as follows:

$$\text{fold} :: (a \to b \to b) \to b \to ([a] \to b)$$

$$\text{fold} f v ~ [] ~ = ~ v$$

$$\text{fold} f v ~ (x : xs) ~ = ~ f x ~ (\text{fold} f v ~ xs)$$

Compare the input of $\text{fold}$ to the data of an $F_A$-algebra given in Item 1 of Exercise 187.

Exercise 189. Define the following functions as a catamorphism, that is, using $\text{fold}$. In each case, draw the diagram corresponding to Diagram 6 of Definition 179.

1. $\text{sum} :: [\text{Int}] \to \text{Int}$
2. $\text{product} :: [\text{Int}] \to \text{Int}$
3. $\text{and} :: [\text{Bool}] \to \text{Bool}$
4. $\text{or} :: [\text{Bool}] \to \text{Bool}$
5. $(++) :: [a] \to [a] \to [a]$  
6. $\text{length} :: [a] \to \text{Int}$
7. $\text{reverse} :: [a] \to [a]$  
8. $\text{map} :: (a \to b) \to ([a] \to [b])$  
9. $\text{filter} :: (a \to \text{Bool}) \to ([a] \to [a])$

Solutions are given in [5, §2].

Hint: a systematic approach to reformulating functions on lists defined by explicit recursion in terms of $\text{fold}$ is described in [5, §3.3].
Exercise 190 (§3.1). Show that \((-1) \cdot \text{sum} = \text{fold} (+) 1\), by showing that \((-1) \cdot \text{sum} \) makes Diagram 6 of Definition 179 commute.

Exercise 191. An exercise about (the limits of) representing functions as catamorphisms (or rewriting functions using ‘fold’):

1. We can represent numbers as big-endian binary numbers, in the set $\text{List}({0,1})$: the lists with elements in the set $\{0,1\}$. For example, 13 becomes $[1,1,0,1]$, which we represent as $(\text{cons } 1 (\text{cons } 1 (\text{cons } 0 (\text{cons } 1 \text{ nil }))))$. We define the function $\text{bin2int} : \text{List}({0,1}) \to \mathbb{N}$, that converts big-endian binary representations to positive integers. For example, $(\text{bin2int} (\text{cons } 1 (\text{cons } 0 \text{ nil }))) = 2$ and $(\text{bin2int} (\text{cons } 1 (\text{cons } 1 (\text{cons } 0 \text{ nil })))) = 13$.

   Show that we can give $\text{bin2int}$ as a catamorphism. i.e. with $F$ the functor for which $\text{List}({0,1})$ is an initial algebra, show that there exists an $F$-algebra $(\mathbb{N}, f)$ such that $(f) = \text{bin2int}$.

2. We can also represent a number as a little-endian binary number. Then 13 becomes $[1,0,1,1]$, which we represent as $(\text{cons } 1 (\text{cons } 0 (\text{cons } 1 \text{ nil })))$.

   We define the function $\text{bin2int2} : \text{List}({0,1}) \to \mathbb{N}$, that converts little-endian binary representations back to positive integers.

   Why can’t we give $\text{bin2int2}$ as a catamorphism? (Hint: how would such a catamorphism calculate the values for $[1]$, $[0,1]$ and $[0,0,1]$?)

3. Show that there exists an $F$-algebra $(\mathbb{N} \times \mathbb{N}, f)$ such that $\pi_l \circ (f) = \text{bin2int2}$, with $\pi_l$ the projection on the first coordinate.

Exercise 192. Let $A$ be a set and define $F$ to be the functor induced by

$$F_A : \text{Set} \to \text{Set} : X \mapsto A \sqcup (X \times X).$$

Show that the initial $F_A$-algebra is given by the set $BTree(A)$ of $A$-valued binary trees.

Remark 193. Notice that in Exercise 187 and Exercise 192 we can consider the functor $F_A$ as a bifunctor where we vary $A$, i.e.

$$F : \text{Set} \to \text{Set} \to \text{Set} : (A, X) \mapsto F_A(X).$$

In particular, under the assumption that for every $A \in \text{Set}$ the initial $F_A$-algebra exists, we can wonder if the assignment

$$\text{Set}_0 \to \text{Set}_0 : A \mapsto \mu F_A,$$

can be extended to a functor. The following exercise answers this question positively for arbitrary categories.
Exercise 194 (Solution 259). Let \( F : C \to C \to C \) be a bifunctor such that for any object \( A \in C \), the initial algebra for the functor induced by
\[
F_A : C \to C : X \mapsto F(A, X),
\]
exists. Show how
\[
C_0 \to C_0 : A \mapsto \mu F_A,
\]
induces a functor.

9. Fusion Property

The fusion property of Exercise 184 can be used to “fuse” a composition of functions into one function, possibly leading to more efficient code.

We are going to exemplify this using the datatypes of lists and of natural numbers. Recall that \((\mathbb{N}, [\text{zero}, \text{succ}])\) is the initial \texttt{Maybe}-algebra. We also write \((+1)\) for \texttt{succ}.

Further Reading. \textit{The content of this section is very much inspired by \cite{5, \S 3.2}. You are strongly encouraged to read that section before reading the present section. In this section, we show how}

Exercise 195 (\cite{5, \S 3.2}). Show that \((+1) \cdot \text{sum} = \text{fold} (+) 1\), by using the fusion property of Exercise 184.

Solution 196. We also write \(\sum\) for \texttt{sum}. Note that \texttt{sum} is defined as the catamorphism \((\{\text{zero}, (+)\})\). The situation is summarized in the following diagram

\[
\begin{align*}
\{\ast\} + \mathbb{N} \times \text{List}(\mathbb{N}) & \quad \xrightarrow{[\text{nil, cons}]} \quad \text{List}(\mathbb{N}) \\
\{\ast\} + \mathbb{N} \times \mathbb{N} & \quad \xrightarrow{[\text{zero, (+)}]} \quad \mathbb{N} \\
\{\ast\} + \mathbb{N} \times \mathbb{N} & \quad \xrightarrow{[1, (+)]}\quad \mathbb{N}
\end{align*}
\]

In order to show that \((+1) \cdot \text{sum} = \text{fold} (+) 1\), in the diagram expressed as \((+1) \circ \sum = (\{1, (+)\})\), we need to show that the lower rectangle commutes—this is what Exercise 184 says.

There are two cases to consider: the case of \(\ast\), and the case of a pair \((m, n) \in \mathbb{N} \times \mathbb{N}\).

In the case of \(\ast\), we obtain
\[
(+1) 0 = 1
\]
and in the case of \((m, n)\), we obtain
\[
(+1) \ ((+) m n) = (+) m ((+1) n)
\]
both of which hold.
Exercise 197. Prove that \( \text{map } g \cdot \text{map } f = \text{map } (g \cdot f) \) using the fusion property.

Further Reading. In Sections 8 and 9, we have only looked at functions defined via iteration, that is, functions defined as catamorphisms of some \( F \)-algebra. While primitive recursive functions can always be expressed as a catamorphism via tupling (see, e.g., [3, §4] and [10, §3.1]), it is more natural specify them via a slightly more sophisticated universal property, explained in detail in Vene’s dissertation [10, Chapter 3].

The interested reader might also study the tutorial by Fokkinga [3] or the paper by Meijer et al. [8].

A guide to further literature on recursion operators is given in [2, §6].

10. Terminal Coalgebras and Coinductive Datatypes

Further Reading. We only give a brief introduction to (terminal) coalgebras in this section. A more systematic exploration of the topic is given in [2].

In this section we introduce (terminal) coalgebras which allows us to define coinductive data types.

Definition 198. Let \( F : C \to C \) be an endofunctor. An \( F \)-coalgebra consists of the following data:

- An object \( C \in C \).
- A morphism \( \phi \in C(C, F(C)) \).

Notice that an \( F \)-algebra consists of a morphism \( F(C) \to C \), while an \( F \)-coalgebra consists of a morphism \( C \to F(C) \) in the other direction.

Definition 199. Let \( F : C \to C \) be an endofunctor and \((C, \phi)\) and \((D, \psi)\) be \( F \)-coalgebras. A (\( F \)-coalgebra) homomorphism from \((C, \phi)\) to \((D, \psi)\) is a morphism \( f \in C(C, D) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\phi} & F(C) \\
\downarrow f & & \downarrow F(f) \\
D & \xrightarrow{\psi} & F(D)
\end{array}
\]

Exercise 200. Define the category \( \text{CoAlg}(F) \) of \( F \)-coalgebras analogously to the category \( \text{Alg}(F) \) of \( F \)-algebras (as in Definition 176).

Definition 201. Let \( F \) be an endofunctor on \( C \). The terminal \( F \)-coalgebra is the terminal object in \( \text{CoAlg}(F) \) which we denote by \((\nu^F, \text{out})\).

For \((C, \phi)\) an arbitrary \( F \)-coalgebra, we denote the unique morphism \((C, \phi) \to (\nu^F, \text{out})\) by \([\phi]\), and we call a morphism of this form an anamorphism (instead of a catamorphism as in Definition 179).
Exercise 202. Spell out what it means for a coalgebra to be the terminal coalgebra.

Exercise 203. Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor and assume that the terminal coalgebra $(\nu^F, \text{out})$ exists. Show that the following properties holds:

1. $\text{Id} = [\text{out}]$.

2. For $F$-algebras $(C, \phi)$ and $(D, \psi)$ and $f \in \mathcal{C}(C, D)$, we have
   \[ \psi \circ f = F(f) \circ \phi \implies [\psi] \circ f = [\phi]. \]

This is summarized in the following diagram:

\[
\begin{array}{c}
\nu^F \downarrow & & \downarrow F[\phi] \\
\psi \downarrow & & \\
F[\phi] \circ \phi & & \\
& & \psi \circ f = [\phi] \circ f = [\phi].
\end{array}
\]

Exercise 204. (Dual Lambek’s theorem) Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor and let $(\nu^F, \text{out})$ be a terminal coalgebra. Then is $\text{out}$ an isomorphism whose inverse is given by $\text{out}^{-1} = [F(\text{out})]$.

Exercise 205. Let $\mathcal{C}$ be a category with a terminal object $\top$. Then $(\top, \text{Id}_\top)$ is a terminal coalgebra for the identity (endo)functor on $\mathcal{C}$.

Exercise 206 (Solution 260). Let $F$ be the functor induced by

\[ F : \text{Set} \to \text{Set} : X \mapsto 1 + X. \]

Show that the terminal $F$-coalgebra is given by the following data:

- The underlying object is given by the set $\mathbb{N} + \{\infty\}$ of natural numbers with infinity.

- The underlying function is given by the predecessor defined as follows:

\[
\begin{align*}
\mathbb{N} + \{\infty\} & \to 1 + \mathbb{N} + \{\infty\} \\
0 & \mapsto * \\
s(n) & \mapsto n \\
\infty & \mapsto \infty
\end{align*}
\]

where $*$ is the unique element of 1.

Example 207 (Streams). The codata type of streams over a given set $A$ is given by the terminal coalgebra $(\nu^{F_A}, \text{out})$ of the functor $F_A(X) := A \times X$. We write $\text{Stream}(A)$.
for \( \nu^{F_A} \). The functions \( \text{head} : \text{Stream}(A) \to A \) and \( \text{tail} : \text{Stream}(A) \to \text{Stream}(A) \) are given by

\[
\text{head} = \pi_l \circ \text{out} : \text{Stream}(A) \to A \\
\text{tail} = \pi_r \circ \text{out} : \text{Stream}(A) \to \text{Stream}(A)
\]

respectively. Put differently (recall the definition of the product in Definition 86), we have

\[
\text{out} = \langle \text{head}, \text{tail} \rangle : \text{Stream}(A) \to A \times \text{Stream}(A)
\]

Given any two functions

\[
h : C \to A \\
t : C \to C
\]

the anamorphism \( \langle h, t \rangle \) is the unique solution \( f : C \to \text{Stream}(A) \) of the equation system

\[
\text{head} \circ f = h \\
\text{tail} \circ f = f \circ t
\]

that is, the unique function \( f : C \to \text{Stream}(A) \) making the following square commute:

\[
\begin{array}{ccc}
C & \overset{\langle h, t \rangle}{\longrightarrow} & A \times C \\
\downarrow\langle (h, t) \rangle & & \downarrow\text{id} \times \langle (h, t) \rangle \\
\text{Stream}(A) & \overset{\langle \text{head}, \text{tail} \rangle}{\longrightarrow} & A \times \text{Stream}(A)
\end{array}
\]

**Exercise 208** (Solution 261). Define, as an anamorphism, the function \( \text{nats} : \mathbb{N} \to \text{Stream}(\mathbb{N}) \) which returns the stream of all natural numbers starting with the natural number given as the argument.

**Exercise 209** (Solution 262). Define, as an anamorphism, the function \( \text{zip} : \text{Stream}(A) \times \text{Stream}(B) \to \text{Stream}(A \times B) \) that zips the argument streams together.

### 11. Solutions

**Solution 210** (Exercise 12). The inequality \( x \leq y \) (resp. \( y \leq x \)) means that we have a (unique) morphism from \( f \in \text{hom}(x, y) \) (resp. \( g \in \text{hom}(y, x) \)). Consequently, we get a loop \( g \circ f \in \text{hom}(x, x) \). Since the hom-sets are either empty or a singleton, we have \( g \circ f = \text{id}_x \). Hence, antisymmetry means that if \( \text{id}_X = g \circ f \) for some \( f \in \text{hom}(x, y) \) and \( g \in \text{hom}(y, x) \), we must have

\[
x = y, \quad f = \text{id}_y = g.
\]

Rephrased a little bit different, we get: The category \( \text{Pre}(X, \leq) \) has no non-trivial loops if \( (X, \leq) \) is antisymmetric.
Solution 211 (Exercise 15). That Pos would be a preorder-category means that each of the hom-sets is either empty or a singleton. Hence, it is not a preorder-category if there exists \((X, \leq_X)\) and \((Y, \leq_Y)\) such that \(\text{Pos}((X, \leq_X), (Y, \leq_Y))\) has more than one element.

We choose \((X, \leq) := (\mathbb{N}, \leq) =: (Y, \leq)\). A morphism \(f \in \text{Pos}((\mathbb{N}, \leq), (\mathbb{N}, \leq))\) consists of a function \(f : \mathbb{N} \to \mathbb{N}\) such that the following property holds:

\[
\forall n, m \in \mathbb{N} : n \leq m \implies f(n) \leq f(m).
\]

But there are a lot of functions from \(\mathbb{N}\) to \(\mathbb{N}\) which satisfies this property, indeed: For any \(k \in \mathbb{N}\), we have that

\[
f_k : \mathbb{N} \to \mathbb{N} : n \mapsto n + k,
\]

is a morphism in Pos. Hence \(\text{Pos}((\mathbb{N}, \leq), (\mathbb{N}, \leq))\) consists of an infinite amount of distinct morphisms.

Solution 212 (Exercise 21). A category \(C\) is of the form \((M, m, e)\) if and only if \(C\) has a unique object. Indeed, if \(C\) has a unique object, let’s denote this by \(X\), then we can define a monoid \((M, m, e)\) as follows:

- The underlying set of the monoid is \(M := C(X, X)\).
- The multiplication \(m\) is given by \(m(f, g) := g \circ f\).
- The identity element \(e\) is given by \(e := \text{Id}_X\).

That \((M, m, e)\) is indeed a monoid, i.e. satisfies the monoid laws, is just a translation of the axioms of \(C\) being a category.

Solution 213 (Exercise 22). Since a monoid consists of a set \(M\) together with a binary operation \(m : M \to M \to M\) (called the multiplication) and a identity element \(e \in M\), a suitable morphism of monoids, from \((M_1, m_1, e_1)\) to \((M_2, m_2, e_2)\) should consists of a function \(f : M \to N\) which preserves the structure (i.e. the multiplication and the identity element). More precisely:

- Preservation of the multiplication:

  \[
  \forall x, y \in M_1 : f(m_1(x, y)) = m_2(f(x), f(y)).
  \]

- Preservation of the identity:

  \[
  f(e_1) = e_2.
  \]

We will now prove that the monoids (as objects) and morphisms of monoids (as the morphisms) carry the data of a category, i.e. we have to define identity morphisms and the composition of morphisms:

- Let \((M, m, e)\) be a monoid. The identity morphism is given by the identity function \(\text{Id}_M\) on the underlying set \(M\).
Let \((M_i, m_i, e_i)\) be a monoid for \(i = 1, 2, 3\) and let

\[f : (M_1, m_1, e_1) \to (M_2, m_2, e_2), \quad g : (M_2, m_2, e_2) \to (M_3, m_3, e_3)\]

be morphisms of monoids. The composition \(g \circ f\) of \(f\) and \(g\) is defined as the composition of the underlying functions.

Before we show that this data satisfies the properties of a category, we first have to show that everything is well-defined, i.e. that the identity is a morphism of monoids and that the composition of morphisms of monoids is again a morphism of monoids:

- That the identity is a morphism of monoids follows by the following calculations:
  \[\forall x, y \in M : \text{Id}_M(m(x, y)) = m(x, y) = m(\text{Id}_M(x), \text{Id}_M(y)),\]
  \[\text{Id}_M(e) = e.\]
  The equalities hold because \(\text{Id}_M\) is the identity function on \(M\).

- That the composition of morphism of monoids is again a morphism of monoids follows by the following calculations:
  \[
g \circ f(m_1(x, y)) = g(f(m_1(x, y)))
  = g(m_2(f(x), f(y)))(f \text{ preserves mult.})
  = m_3(g(f(x)), g(f(y)))(g \text{ preserves mult.})
  = m_3(g \circ f(x), g \circ f(y))
  \]
  Composition preserves identity element by:
  \[
g \circ f(e_1) = g(f(e_1)) = g(e_2) = e_3,
  \]
  where the second (resp. third) equality holds since \(f\) (resp. \(g\)) preserves the identity element.

So everything is indeed well-defined. So we are now ready to show that composition of some morphism of monoids \(f\) with the identity morphism is again \(f\) (both on the left and right) and that the composition of morphisms of monoids is associative. This follows immediate since everything is defined using functions and we know that functions satisfy these properties.

**Solution 214** (Exercise [23]). In order to make \(C\) into a category, we have to define the identity morphisms and the composition of morphisms.

Let \(n \in \mathbb{N} =: C_0\) be a natural number. The identity on \(n\) is an element in \(C(n, n)\), which is the set of \(n \times n\)-matrices. Hence, for the identity on \(n\), we can take \(\text{Id}_n\) to be the identity \(n \times n\)-matrix, i.e. all coefficients are zero except on the diagonal where everything is 1.
Let \( l, m, n \in \mathbb{N} \) be natural numbers and \( M \) (resp. \( N \)) be an \( l \times m \)-matrix (resp. \( m \times n \)-matrix). The composition \( N \circ M \) should be an \( l \times n \)-matrix, hence we define \( N \circ M \) as the matrix multiplication of \( M \) and \( N \), i.e. \( N \circ M = M \cdot N \).

That this data satisfies the properties of being a category follows immediate because matrix multiplication is associative and that the multiplication of any matrix \( M \) with the identity matrix (of the right size and both on the left or on the right) is again \( M \).

**Solution 215** (Exercise 23). In order to avoid confusion, we use the following notation:

For any \( f \in C(X,Y) \) morphism, we denote by \( f^\text{op} \) the corresponding morphism in \( C^\text{op}(Y,X) \).

Let \( X \in C^0 = C_0 \). The identity morphism is defined as the morphism corresponding to the identity, i.e. it is \((\text{Id}_X)^\text{op}\).

Let \( g^\text{op} \in C^\text{op}(Z,Y), f^\text{op} \in C^\text{op}(Y,X) \). The composition is defined as: \( f^\text{op} \circ g^\text{op} := (g \circ f)^\text{op} \).

That this data satisfies the properties of a category, follows because \( C \) is a category, indeed:

- That the left unit law holds follows by the right unit law of \( C \) as follows:
\[
f^\text{op} \circ \text{Id}^\text{op} = (\text{Id} \circ f)^\text{op} = f^\text{op},
\]
where the first equality holds by definition of the opposite composition and the second holds by the right unit law of \( C \).

The right unit law holds analoguously by the left unit law of \( C \).

- That the associativity holds follows by the associativity of \( C \) as follows:
\[
(f^\text{op} \circ g^\text{op}) \circ h^\text{op} = (g \circ f)^\text{op} \circ h^\text{op} \\
= (h \circ (g \circ f))^\text{op} \\
= ((h \circ g) \circ f)^\text{op} \text{ by associativity of } C, \\
= f^\text{op} \circ (h \circ g)^\text{op} \\
= f^\text{op} \circ (g^\text{op} \circ h^\text{op})
\]

**Solution 216** (Exercise 31). Any preordered set \((X, \leq)\) can be described by a graph where the vertices are given by the elements of \( X \) and there exists an edge from \( x \) to \( y \) if and only if \( x \leq y \). Hence, if we denote by \( G \) the corresponding graph of \((X, \leq)\), we have that \( \text{Pos}(X, \leq) = \text{Graph}(G) \).

In particular we have that the number of edges is either 0 or 1. Hence, a category generated by a graph comes from a preordered set if and only if the number of morphisms in any fixed hom-set is either 0 or 1.

If \((X, \leq)\) is a poset, i.e. we have antisymmetry, then we have that the corresponding graph (and consequently the corresponding category) have no (non-trivial) loops.

**Solution 217** (Exercise 34). That \( f : a \to b \) is an isomorphism with inverse \( g \) means precisely that \( g \circ f = \text{Id}_a \) and \( f \circ g = \text{Id}_b \). But stating that \( g \) is an isomorphism with inverse \( f \) means precisely those conditions. Hence, this hold by definition.
Solution 218 (Exercise 35). Let \( f : a \to b \) be an isomorphism. That \( f \) has a unique inverse means that if \( g, h : b \to a \) are morphisms such that

\[
g \circ f = \text{id}_a, \quad f \circ g = \text{id}_b, \quad h \circ f = \text{id}_a, \quad f \circ h = \text{id}_b
\]

then we must have \( g = h \).

So assume \( g \) and \( h \) satisfy the condition of being an inverse of \( f \). Then we have:

\[
g = \text{id}_b \circ g, \quad \text{by left unit law,}
\]

\[
= (h \circ f) \circ g, \quad \text{since } h \text{ is inverse of } f,
\]

\[
= h \circ (f \circ g), \quad \text{by associativity,}
\]

\[
= h \circ \text{id}_a, \quad \text{since } g \text{ is inverse of } f
\]

\[
= h, \quad \text{by right unit law}
\]

Solution 219 (Exercise 36). Let \( f : a \to b \) and \( g : b \to c \) be isomorphisms. Denote their (unique) inverses by \( f^{-1} \) and \( g^{-1} \). We have to show that there exists a morphism \( h : c \to a \) such that

\[
h \circ (g \circ f) = \text{id}_a, \quad (g \circ f) \circ h = \text{id}_c.
\]

We define \( h := f^{-1} \circ g^{-1} \). The left equality then holds by the following computation:

\[
h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f)
\]

\[
= f \circ (g^{-1} \circ g) \circ f^{-1} \quad \text{by associativity,}
\]

\[
= f \circ \text{id}_b \circ f^{-1} \quad \text{since } g^{-1} \text{ inverse of } g,
\]

\[
= f \circ f^{-1} \quad \text{by unit law,}
\]

\[
= \text{id}_a \quad \text{since } f^{-1} \text{ inverse of } f.
\]

The right equality holds analogously.

Solution 220 (Exercise 37). The Haskell datatype \( \text{Bool} \) is given by:

\[
\text{data} \ \text{Bool} = \text{True} \mid \text{False}
\]

In order to construct a (Haskell) function \( f \) from \( \text{BW} \) to \( \text{Bool} \), it suffices to define \( f(\text{Black}) \) and \( f(\text{White}) \).

The first isomorphism, denoted by \( f_1 \), is given by \( f_1(\text{Black}) = \text{True} \) and \( f_1(\text{White}) = \text{False} \). Its inverse (denoted by \( g_1 \)) is given by \( g_1(\text{True}) = \text{Black} \) and \( g_1(\text{False}) = \text{White} \). To show that these are inverse, we have to show

\[
g_1(f_1(\text{White})) = \text{White}, \quad g_1(f_1(\text{Black})) = \text{Black}.
\]

These equalities holds by definition of \( f_1 \) and \( g_1 \).

The second isomorphism, denoted by \( f_2 \), is given by \( f_2(\text{Black}) = \text{False} \) and \( f_2(\text{White}) = \text{True} \). Its inverse (denoted by \( g_2 \)) is given by \( g_2(\text{False}) = \text{Black} \) and \( g_1(\text{True}) = \text{White} \). That \( g_2 \) is the inverse of \( f_2 \) is also immediate.

Solution 221 (Exercise 38). The isomorphisms in \( \text{Set} \) are precisely the bijective functions, indeed:
• Assume \( f : X \rightarrow Y \) is a bijection, i.e.

\[
\forall y \in Y : \exists! x_y \in X : f(x) = y
\]

We show that the inverse of \( f \) is given by:

\[
g : Y \rightarrow X : y \mapsto x_y.
\]

So we have to show \( g \circ f = \text{id}_X \) and \( f \circ g = \text{id}_Y \). Let \( x \in X \), since that \( g(f(x)) \) is the unique element \( z \in X \) such that \( f(z) = f(x) \) (and since \( x \) satisfies this condition), we have \( g(f(x)) = x \). Since this holds for all \( x \in X \), we have \( g \circ f = \text{id}_X \).

Let \( y \in Y \) and let \( x := x_y \) be the unique element in \( X \) such that \( f(x) = y \). So by definition of \( g \), we have \( g(y) = x \), hence \( f(g(y)) = f(x) = y \).

• Assume \( f : X \rightarrow Y \) is an isomorphism with inverse \( f^{-1} \). Let \( y \in Y \), we have to show that there exists a unique \( x \in X \) such that \( f(x) = y \). Define \( x := g(y) \).

Since \( f \circ g = \text{id}_Y \), we have \( y = f(g(y)) = f(x) \), hence, this \( x \) indeed satisfies the condition. To show that \( x \) is unique, let \( z \in X \) satisfy \( f(z) = y \). That \( z = x \) now follows from \( g \circ f = \text{id}_X \), indeed: \( z = g(f(z)) = g(y) = x \).

**Solution 222** (Exercise 40). The isomorphisms in \( \text{Pos} \) are precisely the bijections \( f : (X, \leq_X) \rightarrow (Y, \leq_Y) \) such that

\[
x_1 \leq_X x_2 \iff f(x_1) \leq_Y f(x_2),
\]

Indeed:

• Assume \( f \) is a bijection which satisfies Eq. (3). Since it is a bijection, we know (by the solution to Exercise 39), that there exists a function \( g : (Y, \leq_Y) \rightarrow (X, \leq_X) \) such that \( g \circ f = \text{id}_{(X, \leq_X)} \) and \( f \circ g = \text{id}_{(Y, \leq_Y)} \). However, this does not conclude the proof of the first implication, because we do not know appriori, that \( g \) is a morphism of posets. So we have to show

\[
\forall y_1, y_2 \in Y : y_1 \leq_Y y_2 \implies g(y_1) \leq_X g(y_2).
\]

Let \( y_1, y_2 \in Y \). Since \( f \) is bijective, there exist \( x_1, x_2 \in X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). If \( f(x_1) = y_1 \leq_Y y_2 = f(x_2) \), then by Eq. (3), we also have that \( x_1 \leq x_2 \). But by definition of \( g \), we have \( g(y_1) = x_1 \) and \( g(y_2) = x_2 \), hence \( g(y_1) \leq_X g(y_2) \) which shows that \( g \) is an order-preserving morphism, i.e. \( g \in \text{Pos}(Y, \leq_Y, (X, \leq_X)) \).

• Assume \( f \) is an isomorphism in \( \text{Pos} \) with inverse \( g \). Since \( f \) is a function which satisfies \( g \circ f = \text{id}_{(X, \leq_X)} \) and \( f \circ g = \text{id}_{(Y, \leq_Y)} \), we have (by the same argument as in the solution to Exercise 39), that \( f \) is a bijection. Hence, it remains to show that Eq. (3) holds. Let \( x_1, x_2 \in X \).

If \( x_1 \leq_X x_2 \), then we have \( f(x_1) \leq_Y f(x_2) \) since \( f \in \text{Pos}((X, \leq_X), (Y, \leq_Y)) \).

Assume \( f(x_1) \leq_Y f(x_2) \). Since \( g \in \text{Pos}((Y, \leq_Y), (X, \leq_X)) \), we have \( g(f(x_1)) \leq_X g(f(x_2)) \). But \( g \circ f = \text{id}_X \), hence \( x_1 \leq_X x_2 \).
Solution 223 (Exercise 41). First, let \((X, \leq_Y)\) be a preorder. A morphism \(f : x \to y\) is an isomorphism if and only if there exists a morphism \(g : y \to x\) such that \(g \circ f = \text{Id}_x\) and \(f \circ g = \text{Id}_y\). But, in a preorder category, each hom-set has a unique element if it is non-empty. So, for any \(g \in \text{hom}(y, x)\) and \(f \in \text{hom}(x, y)\), we always have \(g \circ f = \text{Id}_x\) and \(f \circ g = \text{Id}_y\). Hence a morphism \(f : x \to y\) in a preorder-category is an isomorphism if and only if there exists a morphism \(g : y \to x\). The existence of a morphism \(f : x \to y\) means precisely that \(x \leq y\). Hence, isomorphisms in a preorder-category corresponds with a pair of elements \((x, y)\) in \(X\) such that \(x \leq y\) and \(y \leq x\).

If \((X, \leq_X)\) is a poset, i.e. satisfies antisymmetry, then if \(x \leq y\) and \(y \leq x\), we must have \(x = y\). Consequently, in a poset-category, the only isomorphisms are the identity morphisms (i.e. corresponding with \(x \leq x\)).

Solution 224 (Exercise 46). Consider

\[
\begin{align*}
\text{bool2Int} & :: \text{Bool} \to \text{Int} \\
\text{bool2Int} \ False &= 0 \\
\text{bool2Int} \ True &= 1
\end{align*}
\]

We can go back, so that we get \(\text{False}\) and \(\text{True}\) from 0 and 1:

\[
\begin{align*}
\text{int2Bool} & :: \text{Int} \to \text{Bool} \\
\text{int2Bool} \ n \mid n == 0 &= \text{False} \\
\text{int2Bool} \ otherwise &= \text{True}
\end{align*}
\]

However, notice that not only 1 is converted back to \(\text{True}\), but also everything other than 0 is converted to \(\text{True}\).

We have

\[
\text{Int2Bool} \ (\text{bool2Int} \ y) = y
\]

for every \(y :: \text{Bool}\), but we don’t have \(\text{bool2Int} \ (\text{int2Bool} \ x) = x\) for all \(x :: \text{Int}\).

We can say that there is enough room in the type integers for it to host a copy of the type of booleans, but there isn’t enough room in the type of booleans for it to host a copy of the type of integers.

But notice that there are other ways in which the type \text{Bool} lives inside the type \text{Int} as a retract: for example, we can send \(\text{False}\) to 13 and \(\text{True}\) to 17, and then send back everything bigger than 15 to \(\text{True}\) and everything else to \(\text{False}\).

Solution 225 (Exercise 49). The monomorphisms in \(\text{Set}\) correspond precisely with the injective functions, i.e. the functions \(f : X \to Y\) which satisfy

\[
\forall x_1, x_2 \in X : f(x_1) = f(x_2) \implies x_1 = x_2.
\]

Indeed:

\begin{itemize}
  \item Assume \(f\) is injective. Let \(g, h : Z \to X\) be functions such that \(f \circ g = f \circ h\). We have to show \(g = h\), i.e.

  \[
  \forall z \in Z : g(z) = h(z).
  \]

  Since \(f\) is injective, it suffices to show

  \[
  \forall z \in Z : f(g(z)) = f(h(z)).
  \]

  But this holds by the condition of \(g\) and \(h\). Hence, \(f\) is indeed a monomorphism.
\end{itemize}
• Assume \( f \) is a monomorphism. We have to show that for each \( x_1, x_2 \in X \), we have \( f(x_1) = f(x_2) \). Let \( 1 = \{ * \} \) be a singleton set and define

\[
g_1 : 1 \to X : * \mapsto x_1, \quad g_2 : 1 \to X : * \mapsto x_2,
\]

Since \( f(x_1) = f(x_2) \), we have \( f \circ g_1 = f \circ g_2 \). But \( f \) is a monomorphism, hence \( g_1 = g_2 \) which means \( x_1 = g_1(*) = g_2(*) = x_2 \). Thus, \( f \) is indeed injective.

Solution 226 (Exercise 52). Let \( f : X \to Y \) be a section with a retraction \( h : Y \to X \), i.e. \( h \circ f = \text{Id}_Y \). By Exercise 49, it suffices to show that \( f \) is a monomorphism. Let \( g_1, g_2 : Z \to X \) be morphisms in \( \text{Set} \) such that \( f \circ g_1 = f \circ g_2 \). We have to show \( g_1 = g_2 \), this follows from the following computation:

\[
g_1 = \text{Id}_Y \circ g_1 \quad \text{by unit law},
\]

\[
= h \circ f \circ g_1 \quad \text{since } f \text{ section},
\]

\[
= h \circ f \circ g_1 \quad \text{by associativity},
\]

\[
= h \circ f \circ g_2 \quad \text{by assumption},
\]

\[
= h \circ f \circ g_2 \quad \text{by associativity},
\]

\[
= \text{Id}_Y \circ g_2 \quad \text{since } f \text{ section},
\]

\[
= g_2 \quad \text{by unit law}.
\]

Notice that this proof shows that in an arbitrary category, a section is always a monomorphism.

Solution 227 (Exercise 54). Let \( f : a \cong b \) be an isomorphism with inverse \( f^{-1} \). There are multiple proofs which one can give, an abstract one (which is indirect in the sense that we use another exercise/lemma) and a more direct one.

• Indirect proof: By the solution of Exercise 52, we know that any section (from a section-retraction pair) is a monomorphism. An analogous argument shows that any retraction (section-retraction pair) is an epimorphism. Hence it suffices to show that \( f \) is both a section and a retraction, but this is immediate because \( f^{-1} \circ f = \text{Id}_a \) and \( f \circ f^{-1} = \text{Id}_b \).

• Direct proof: We first show that \( f \) is a monomorphism. Assume \( g_1, g_2 : c \to a \) are morphisms such that \( f \circ g_1 = f \circ g_2 \). We then have that \( g_1 = g_2 \) because

\[
g_1 = \text{Id}_a \circ g_1 = (f^{-1} \circ f) \circ g_1 \quad \text{since } f : a \cong b,
\]

\[
= f^{-1} \circ (f \circ g_1) \quad \text{by associativity},
\]

\[
= f^{-1} \circ (f \circ g_2) \quad \text{by assumption},
\]

\[
= (f^{-1} \circ f) \circ g_2 \quad \text{by associativity}
\]

\[
= \text{Id}_a \circ g_2 \quad \text{since } f : a \cong b
\]

\[
= g_2.
\]
That $f$ is also an epimorphism is analogous, indeed: Assume $g_1, g_2 : b \to c$ are morphisms such that $g_1 \circ f = g_2 \circ f$. We then have that $g_1 = g_2$ because

$$g_1 = g_1 \circ \text{Id}_b = g_1 \circ (f \circ f^{-1}) \quad \text{since } f : a \cong b,$$

$$= (g_1 \circ f) \circ f^{-1} \quad \text{by associativity},$$

$$= (g_2 \circ f) \circ f^{-1} \quad \text{by assumption},$$

$$= g_2 \circ (f \circ f^{-1}) \quad \text{by associativity}$$

$$= g_2 \circ \text{Id}_b \quad \text{since } f : a \cong b$$

$$= g_2.$$

**Solution 228** (Exercise 55). Let $(X, \leq_X)$ be a preordered set. Any morphism $f \in \text{Pos}(X, \leq_X)(x, y)$ is always both a monomorphism and an epimorphism because hom-sets have at most one element. But, by Exercise 41, we know that in a poset (not a preordered set!), the only isomorphisms are the identity morphisms. Hence, if $x \leq y$ but $x \neq y$ (living in a poset), then the corresponding morphism in $\text{hom}(x, y)$ is both an epimorphism and monomorphisms but not an isomorphism.

A concrete example is given by e.g. the poset of truth values $\{0, 1\}$. We have $0 \leq 1$ and those are not equal.

**Solution 229** (Exercise 65). An initial object (and the only one), is the emptyset $\emptyset$, indeed: Let $X$ be a set. Then there is clearly a unique function $\emptyset \to X$.

**Solution 230** (Exercise 67). A initial object in $\text{Pos}(X, \leq)$ is the minimal object, that is an element $\perp \in X$ such that

$$\forall y \in X : \perp \leq y. \quad (9)$$

Indeed: Assume $x$ is an initial object in $(X, \leq)$, i.e. for any other element $y \in X$, there exists a (unique) morphism $x \to y$, i.e. hence, by definition of the hom-sets, we have $x \leq y$. So $x$ is indeed the minimal object.

Conversely, assume $\perp$ is a minimal element, hence, for each $y \in X$, we have $\perp \leq y$. Hence $\text{hom}(\perp, y)$ is non-empty. So it must contain exactly one element. This means precisely that it is initial.

A somewhat more compact solution is as follows: By definition of $\text{Pos}(X, \leq)$, for each $x \in X$, we have:

$$\forall y \in X : (x \leq y \iff \exists ! f \in \text{hom}(x, y)).$$

Hence an object $x$ is initial if and only if, $x \leq y$ for all $y \in X$, if and only if it is minimal.

**Solution 231** (Exercise 68). Let $A$ and $B$ be initial objects in $\mathcal{C}$. By initiality of $A$ (resp. $B$), there exists a unique morphism $f \in \mathcal{C}(A, B)$ (resp. $g \in \mathcal{C}(B, A)$). That $f$ and $g$ are inverses follows because $g \circ f \in \mathcal{C}(A, A)$ (resp. $f \circ g \in \mathcal{C}(B, B)$). But by initiality of $A$ (resp. $B$), $\mathcal{C}(A, A)$ (resp. $\mathcal{C}(B, B)$) has a unique element, but both $g \circ f, \text{Id}_A \in \mathcal{C}(A, A)$ (resp. $f \circ g, \text{Id}_B \in \mathcal{C}(B, B)$), hence they must be equal.
Solution 232 (Exercise 69). Assume $A \in C_0$ is initial, $B \in C_0$ an arbitrary object and $i : A \cong B$ an isomorphism. We have to show that $B$ is initial, i.e. for each $X \in C_0$, there should exists a unique morphism $B \to X$.

Fix such an $X$. By initiality of $A$, there exists a (unique) morphism $f \in C(A, X)$. If we denote the inverse of $i$ by $j$, we have $f \circ j \in C(B, X)$. To show that $f \circ j$ is the unique morphism in this hom-set, let $g \in C(B, X)$. So we have $g \circ i \in C(A, X)$. By initiality of $A$, we have $g \circ i = f$. The claim now follows by the following computation:

$$f \circ j = (g \circ i) \circ j = g \circ (i \circ j) = g \circ \text{Id}_B = g.$$  

Solution 233 (Exercise 71). We give three solutions to this exercise.

- Consider the category generated by the graph:

$$\begin{array}{ccc}
  x & \xrightarrow{f} & y
\end{array}$$

This category can not have an initial object since there is no morphism from $x$ to $y$ or vice versa.

- Consider the category generated by the graph:

$$\begin{array}{ccc}
  x & \xrightarrow{g} & y
\end{array}$$

This category also can not have an initial object, indeed: There is no morphism from $y$ to $x$, hence $y$ can not be initial. But also $x$ can not be initial since $f$ and $g$ are different morphisms.

- Consider the category $\text{Pos}(\mathbb{Z}, \leq)$, i.e., the category

$$\begin{array}{ccccccc}
  \ldots & \longrightarrow & -2 & \longrightarrow & -1 & \longrightarrow & 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & \ldots
\end{array}$$

This category can not have an initial object: Suppose it has an initial object $x \in \mathbb{Z}$. Then we have another object $x - 1 \in \mathbb{Z}$. Since $x$ is initial, we have a morphism $x \to x - 1$, which means that $x \leq x - 1$, which is absurd. Therefore, this category does not have an initial object.

Solution 234 (Exercise 156). For any set $X$, we write

$$(+) : \text{List}(X) \times \text{List}(X) \to \text{List}(X)$$

for list concatenation.

For each set $X \in \text{Set}_0$, we define:

$$\eta_X : X \to \text{List}(X) : x \mapsto [x] := \text{cons}(x, \text{nil}).$$

For each function $f \in \text{Set}_0(X, \text{List}(Y))$, we define, by list recursion, the following function:

$$f^* : \text{List}(X) \to \text{List}(Y)$$

$$\begin{align*}
  \text{nil} & \mapsto \text{nil} \\
  \text{cons}(x, xs) & \mapsto f x + f^* xs
\end{align*}$$

We now show that the properties of a Kleisli triple hold:
1. For each set $X$, we have to show $\eta_X^\star = \text{id}_{\mathcal{P}(X)}$, that is, for a list $\ell \in \text{List}(X)$, we show $\eta_X^\star(\ell) = \ell$. We prove this by structural induction on the list $\ell$.

   In case $\ell = \text{nil}$, we have, by Eq. (11), that $\eta_X^\star(\text{nil}) = \text{nil}$.

   In case $\ell = \text{cons}(x, xs)$, we compute
   
   $$
   \eta_X^\star(\text{cons}(x, xs)) = \eta_X(x) + \eta_X^\star(xs)
   = \eta_X(x) + \text{id}_{\text{List}}(xs)
   = [x] + \eta_X(x)
   = [x] + \text{cons}(x, xs).
   $$

2. For each function $f : X \to Y$, we have to show $f^\star(\eta_X(x)) = f(x)$, this indeed holds by the following computation:

   $$
   f^\star(\eta_X(x)) = f^\star(\text{cons}(x, \text{nil})) = \text{cons}(f(x), f^\star(\text{nil})) = \text{cons}(f(x, \text{nil}),
   $$

   where the first equality holds by definition of $\eta_X$ and the second equality holds by definition of $f^\star$.

3. Let $f : X \to \text{List}Y$ and $g : Y \to \text{List}Z$ be functions, we have to show

   $$
   g^\star(f^\star(\ell)) = (g^\star \circ f)^\star(\ell),
   $$

   for any $\ell \in \text{List}X$. We prove this by structural induction on the list $\ell$.

   • In case $\ell = \text{nil}$, the equality holds trivially by Eq. (11).

   • In case $\ell := \text{cons}(x, s)$. By definition of $(f)^\star$, the left hand side of is:

   $$
   g^\star(f^\star(\text{cons}(x, s))) = g^\star(f(x) + f^\star(xs)),
   $$

   and the right hand side is:

   $$
   (g^\star \circ f)^\star(\text{cons}(x, s)) = g^\star(f(x)) + (g^\star \circ f)^\star(xs) = g^\star(f(x)) + g^\star(f^\star(xs)),
   $$

   where the first equality holds by definition of $(g^\star \circ f)^\star$ and the second holds by the induction hypothesis. Hence it remains to show the following equality:

   $$
   g^\star(f(x) + f^\star(xs)) = g^\star(f(x)) + g^\star(f^\star(xs)).
   $$

   We do a pattern matching on $f(x)$ to show Eq. (13):

   • In case $f(x) = \text{nil}$, we have

   $$
   g^\star(\text{nil} + f^\star(xs)) = g^\star(f^\star(xs)) = \text{nil} + g^\star(f^\star(xs)) = g^\star(\text{nil}) + g^\star(f^\star(xs)),
   $$

   where the third equality holds by Eq. (11).
In case \( f(x) = \text{cons}(y, u) \), with \( y \in Y \) and \( u \in \text{List} Y \), we have:

\[
g^* (\text{cons}(f(x), f^*(s))) = g^* (\text{cons}(\text{cons}(y, u), f^*(x))) = g^* (\text{cons}(y, u + f^*(x))) = g(y) + g^* (u + f^*(x)) = g(y) + g^* (\text{cons}(y, u)) + g^* (f^*(x)) = g^*(\text{cons}(y, u)) + g^* (f^*(x)).
\]

where the second equality holds by definition of \( \text{cons} \), the third and fifth by definition of \( g^* \) and the fourth by the induction hypothesis.

**Solution 235** (Exercise 157). For each set \( X \in \text{Set}_0 \), we define:

\[
\eta_X : X \to \text{BinTree}(X) : x \mapsto \text{leaf}(x).
\]

For each function \( f : X \to Y \), we define:

\[
f^* : \text{BinTree}(X) \to \text{BinTree}(Y) : t \mapsto \begin{cases} f(a) & \text{if } t = \text{leaf}(a), \\
\text{branch}(f^*(t_1), f^*(t_2)) & \text{if } t = \text{branch}(t_1, t_2).\end{cases}
\]

We now show that the properties of a Kleisli triple hold:

1. For each set \( X \), we have to show \( \eta_X^* = \text{id}_{\text{BinTree}(X)} \). We show this by pattern matching on \( t \):
   - If \( t = \text{leaf}(a) \), then
     \[
     \eta_X^*(t) = \eta_X^*(\text{leaf}(a)) = \eta_X(a) = \text{leaf}(a) = t.
     \]
   - If \( t = \text{branch}(t_1, t_2) \), then
     \[
     \eta_X^*(t) = \eta_X^*(\text{branch}(t_1, t_2)) = \text{branch}(\eta_X^*(t_1), \eta_X^*(t_2)) = \text{branch}((t_1, t_2) = t.
     \]

2. For each function \( f : X \to Y \), we have to show \( f^*(\eta_X(a)) = f(a) \), this indeed holds by the following computation:

\[
f^*(\eta_X(a)) = f^*(\text{leaf}(a)) = f(a).
\]

3. Let \( f : X \to \text{BinTree} Y \) and \( g : Y \to \text{BinTree} Z \) be functions, we have to show

\[
g^*(f^*(t)) = (g^* \circ f)^*(t).
\]

That this equality holds follows by pattern matching:
   - If \( t = \text{leaf}(a) \), then
     \[
g^*(f^*(t)) = g^*(f^*(\text{leaf}(a))) = g^*(f(a)) = (g^* \circ f)(a) = (g^* \circ f)^*(\text{leaf}(a)) = (g^* \circ f)^*(t).
     \]

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If \( t = \text{branch}(t_1, t_2) \), then is the left-hand-side given by

\[
g^*(f^*(t)) = g^*(f^*(\text{branch}(t_1, t_2))) = g^*(\text{branch}(f^*(t_1), f^*(t_2))) = \text{branch}(g^*(f^*(t_1)), g^*(f^*(t_2))).
\]

The right-hand-side is given by

\[
(g^* \circ f)^*(t) = (g^* \circ f)^*(\text{branch}(t_1, t_2)) = \text{branch}((g^* \circ f)(t_1), (g^* \circ f)(t_2)).
\]

Hence, by the induction hypothesis, the both sides are equal.

Solution 236 (Exercise 158). Before we continue with this exercise, we first fix some notation. Since \( X + E \) is the disjoint union of \( X \) and \( E \), we have the canonical inclusions which we denote by

\[
i^X_l : X \to X + E, \quad i^X_r : E \to X + E.
\]

Hence, a function whose domain is \( X + E \) is completely determined by specifying where each \( i^X_l(x) \) and each \( i^X_r(e) \) are mapped to. Notice that this is precisely the notation and the universal property of the coproduct (in \( \text{Set} \)).

For each set \( X \in \text{Set}_0 \), we define:

\[
\eta_X : X \to X + E : x \mapsto i^X_l(x).
\]

For each function \( f \in \text{Set}_0(X, Y + E) \), we define:

\[
f^* : X + E \to Y + E : z \mapsto \begin{cases} f(x) & \text{if } z = i^X_l(x), \\ i^Y_r(e) & \text{if } z = i^X_r(e). \end{cases}
\]

We now show that the properties of a Kleisli triple hold:

1. For each set \( X \), we have to show \( \eta^*_X = \text{Id}_{X + E} \):
   - If \( z = i^X_l(x) \), then
     \[
     \eta^*_X(z) = \eta^*_X(i^X_l(x)) = \eta_X(x) = i^X_l(x) = z = \text{ld}_{X + E}(z).
     \]
   - If \( z = i^X_r(e) \), then
     \[
     \eta^*_X(e) = \eta^*_X(i^X_r(e)) = i^X_r(e) = z = \text{ld}_{X + E}(z).
     \]

2. For each function \( f : X \to Y \), we have to show \( f^*(\eta_X(x)) = f(x) \) but this holds directly by the definition of \((-)^*\) since \( \eta_X(x) = i^X_l(x) \).

3. Let \( f : X \to Y + E \) and \( g : Y \to Z + E \) be functions, we have to show

\[
g^*(f^*(z)) = (g^* \circ f)^*(z).
\]

To show this, we do pattern matching on \( z \in X + E \):
First notice that \( \eta \). We now show that the properties of a Kleisli triple hold:

1. For each function \( f \):
   \[ g^*(f^*(z)) = g^*(f^*(i^*_X(x))) = g^*(f(x)) = (g^* \circ f)^*(i^*_X(x)) = (g^* \circ f)^*(z). \]

2. For each function \( f^* \):
   \[ g^*(f^*(z)) = g^*(f^*(i^*_Y(e))) = g^*(i^*_Y(e)) = i^*_Y(e) = (g^* \circ f)^*(i^*_Y(e)) = (g^* \circ f)^*(z). \]

**Solution 237** (Exercise 159). For each set \( X \in \text{Set}_0 \), we define:

\[ \eta_X : X \to \mathbb{P}_{\text{fin}}(X) : x \mapsto \{x\}. \]

For each function \( f \in \text{Set}_0(X, \mathbb{P}_{\text{fin}}(Y)) \), we define:

\[ f^* : \mathbb{P}_{\text{fin}}(X) \to \mathbb{P}_{\text{fin}}(Y) : A \mapsto \bigcup_{a \in A} f(a). \]

First notice that \( \eta_X \) and \( f^* \) are well-defined, indeed:

- \( \eta_X(x) = \{x\} \) is clearly finite since it only contains one element.
- Let \( A \in \mathbb{P}_{\text{fin}}(X) \). By definition of \( f \), for each \( a \in A \), \( f(a) \) is finite. But there are only a finite number of elements in \( A \), so \( \bigcup_{a \in A} f(a) \) is a finite union of finite sets, hence, it is again finite.

We now show that the properties of a Kleisli triple hold:

1. For each set \( X \), we have to show \( \eta^*_X = \text{id}_{\mathbb{P}_{\text{fin}}(X)} \). Let \( A \in \mathbb{P}_{\text{fin}}(X) \), the claim then follows by the following computation:
   \[ \eta^*_X(A) = \bigcup_{a \in A} \eta_X(a) = \bigcup_{a \in A} \{a\} = A = \text{id}_{\mathbb{P}_{\text{fin}}(X)}(A). \]

2. For each function \( f : X \to Y \), we have to show \( f^*(\eta_X(x)) = f(x) \) but this holds directly by the definition of \((-)^* \) since
   \[ f^*(\eta_X(x)) = f^*(\{x\}) = \bigcup_{a \in \{x\}} f(a) = f(x). \]

3. Let \( f : X \to \mathbb{P}_{\text{fin}}(Y) \) and \( g : Y \to \mathbb{P}_{\text{fin}}(Z) \) be functions, we have to show
   \[ g^*(f^*(A)) = (g^* \circ f)^*(A). \]

Let \( A \in \mathbb{P}_{\text{fin}}(X) \), the left-hand-side is given as:

\[ g^*(f^*(A)) = g^* \left( \bigcup_{a \in A} f(a) \right) = \bigcup_{b \in \bigcup_{a \in A} f(a)} g(f(a)) = \bigcup_{a \in A} \bigcup_{b \in f(a)} g(f(a)). \]

The right-hand-side is given as:

\[ (g^* \circ f)^*(A) = \bigcup_{a \in A} (g^* \circ f)(a) = \bigcup_{a \in A} g^*(f(a)) = \bigcup_{a \in A} \bigcup_{b \in f(a)} g(f(a)). \]

Hence, both sides are equal.
Solution 238 (Exercise 160). For each set \( X \in \text{Set}_0 \), we define:

\[
\eta_X : X \to (X \to R) \to R : x \mapsto (\lambda f.f(x)).
\]

For each function \( f \in \text{Set}_0(X, \text{Cont}^R(Y)) \), we define:

\[
f^* : \text{Cont}^R(X) \to \text{Cont}^R(Y) : i \mapsto \lambda (j : Y \to R). i(f(\cdot)(j)).
\]

Notice that this is indeed well-defined: Let \( i \in \text{Cont}^R(X) \), i.e. \( i : (X \to R) \to R \). Then \( f^*(i) : (Y \to R) \to R \). Let \( j : Y \to R \). Then \( f(\cdot)(j) : X \to R \), hence we can apply it to \( i \) and we have \( i(f(\cdot)(j)) \in R \).

Let \( i \in \text{Cont}^R(X) \). We now show that this data satisfies the properties of a Kleisli triple:

1. For each set \( X \), we have to show \( \eta^*_X = \text{id}_{\text{Cont}^R(X)} \). Let \( x \in X \). The claim then follows by the following computation:

\[
\eta^*_X(i) = \lambda j, i(\eta_X(\cdot)(j)) = \lambda j, i(\lambda x, j(x)) = i
\]

2. For each function \( f : X \to Y \), we have to show \( f^*(\eta_X(x)) = f(x) \), this follows by the following computation:

\[
f^*(\eta_X(x)) = f^*(\lambda g, g(x)) = \lambda j, ((\lambda g, g(x))(f(\cdot)(j))) = \lambda j, (f(x)(j)) = f(x).
\]

3. Let \( f : X \to \text{Cont}^R(Y) \) and \( g : Y \to \text{Cont}^R(Z) \) be functions, we have to show

\[
g^*(f^*(i)) = (g^* \circ f)^*(i).
\]

The left-hand-side is given as:

\[
g^*(f^*(i)) = g^*(\lambda j, i(f(\cdot)(j))) = \tilde{\lambda} j, (\lambda j, i(f(\cdot)(j))) (g(\cdot)(\tilde{\lambda} j)) = \tilde{\lambda} j, i(f(\cdot)(g(\cdot)(\tilde{\lambda} j))).
\]

The right-hand-side is given as:

\[
(g^* \circ f)^*(i) = \lambda j, i((g^* \circ f)(\cdot)(j))
\]

So to show that both sides are equal, it suffices to show that for each \( j \), we have

\[
f(\cdot)(g(\cdot)(j))(x) = f(x)(g(\cdot)(j)).
\]

Notice that these are functions \( X \to R \). Hence we will show this pointwise for each \( x \in X \). The left-hand-side is given by:

\[
f(\cdot)(g(\cdot)(j))(x) = f(x)(g(\cdot)(j))
\]

The right-hand-side is given by:

\[
(g^* \circ f)(\cdot)(j)(x) = (g^* \circ f)(x)(j) = ((g^* \circ f)(x))(j) = (g^*(f(x)))(j) = (\lambda k, f(x)(g(\cdot)(k)))(j) = f(x)(g(\cdot)(j)).
\]

Hence, both sides are equal.
Solution 239 (Exercise 161). For each set $X \in \text{Set}_0$, we define:

$$\eta_X : X \to (R \to X) : x \mapsto (\lambda_r x).$$

For each function $f \in \text{Set}_0(X, R \to Y)$, we define:

$$f^* : (R \to X) \to (R \to Y) : g \mapsto \lambda r, f(g(r))(r).$$

Notice that this is indeed well-typed: Let $g : R \to X$ and $r \in R$. Then $g(r) \in X$, therefore, $f(g(r))(r) \in Y$.

We now show that this data satisfies the properties of a Kleisli triple:

1. For each set $X$, we have to show $\eta_X^* = \text{id}_{R \to X}$. Let $r \in R$ and $g : R \to X$. The claim then follows by the following computation (using functional extensionality):

$$\eta_X^*(g)(r) = \eta_X(g(r))(r) = (\lambda_r g(r))(r) = g(r).$$

2. For each $x \in X$ and $r \in R$, we have to show $f^* \circ \eta_X = f$. The claim then follows by the following computation (using functional extensionality):

$$f^*(\eta_X(x))(r) = f^*(\lambda_r x)(r) = (\lambda_r', f((\lambda_r x)(r'))(r'))(r) = f((\lambda_r x)(r))(r) = f(x)(r).$$

3. Let $f : X \to (R \to Y)$ and $g : Y \to (R \to Z)$. We have to show, by function extensionality, that for any $\phi \in R \to X$, we have

$$(g^* \circ f)^*(\phi) = g^*(f^*(\phi)).$$

This indeed follows, yet again by functional extensionality. Let $r \in R$. We calculate both the left and right-hand side:

$$\text{LHS} = (g^* \circ f)^*(\phi)(r)$$
$$= f \circ g^*(\phi(r))(r)$$
$$= g^*(f(\phi(r)))(r)$$
$$= g(f(\phi(r))(r))(r).$$

$$\text{RHS} = g^*(f^*(\phi))(r)$$
$$= g^*(\lambda s, f(\phi(s))(s))(r)$$
$$= g(f(\phi(r))(r))(r).$$

Thus, the left and right hand sides compute to the same term, hence they are equal.

Solution 240 (Exercise 72). An initial object is a one-element set ($\{\star\}$, $\star$). Let $(X, x)$ be a pointed set. Then we have a unique function $\star \to X$ that sends $\star$, the chosen (and only) point of $\{\star\}$, to $x$, the chosen point of $X$, namely $f : \star \mapsto x$. 
Solution 241 (Exercise 79). A terminal set in the category of sets is a one-element set \( \{\star\} \). Given any set \( X \), we have a unique function \( X \to \{\star\} \), since any element of \( X \) must be sent to \( \star \).

Solution 242 (Exercise 81). Given a poset \( (X, \leq) \), a terminal object in the category \( \text{Pos}(X, \leq) \) is exactly a maximal element in \( X \). Indeed, given such a maximal element \( x \in X = \text{Pos}(X, \leq)_0 \), we have for all \( y \in \text{Pos}(X, \leq)_0 \), since \( x \) is maximal, that \( y \leq x \). Therefore, we have a morphism \( f : x \to y \). By the definition of the hom-sets in \( \text{Pos}(X, \leq) \), \( f \) is unique and we conclude that \( x \) is a terminal object. Conversely, unfolding the definition of terminal object shows that a terminal object yields a maximal element.

Solution 243 (Exercise 82). Suppose that we have a category \( C \) and two terminal objects \( B, B' \in C_0 \). Since \( B' \) is terminal, we have a morphism \( f : B \to B' \) and since \( B \) is terminal, we have a morphism \( g : B' \to B \). Note that we have two morphisms from \( B \) to \( B' \), namely \( g \circ f \) and \( \text{Id}_B \). Also, because \( B \) is terminal, there exists a unique morphism \( B \to B \). Therefore, \( g \circ f = \text{Id}_B \). In the same way, we have \( f \circ g = \text{Id}_{B'} \). Therefore, \( f \) is the isomorphism (with inverse \( g \)) between \( B \) and \( B' \) that we are looking for.

Solution 244 (Exercise 83). Let \( C \) be a category and take \( B, B' \in C_0 \) objects in \( C \). Suppose that we have an isomorphism \( i : B \cong B' \) and that \( B \) is a terminal object in \( C \). We have to show that \( B' \) is terminal. In other words, for all \( A \in C_0 \), we have to show that there exists a unique morphism \( f : A \to B' \).

Now, given such an object \( A \in C_0 \), we have a morphism \( f : A \to B \) by terminality of \( B \). Therefore, we have a morphism \( g = i \circ f : A \to B' \). This proves existence. For uniqueness, suppose that we also have another morphism \( h : A \to B' \). Then we have morphisms \( i^{-1} \circ g \) and \( i^{-1} \circ h \) from \( A \) to \( B \). Since \( B \) is terminal, there exists only one morphism from \( A \) to \( B \), so \( i^{-1} \circ g = i^{-1} \circ h \). Therefore, we have
\[
g = i \circ (i^{-1} \circ g) = i \circ (i^{-1} \circ h) = h
\]
and this concludes the proof.

Solution 245 (Exercise 84). Let \( C \) be a category. Suppose that \( C \) has a terminal object \( B \in C_0 \). Note that \( C_0 = C^{\text{op}}_0 \). We will show that \( B \) is an initial object in \( C^{\text{op}} \). That is, for all \( A \in C^{\text{op}}_0 \), we will show that \( C^{\text{op}} \) has a unique morphism from \( B \) to \( A \). Let \( A \in C^{\text{op}}_0 \) be an arbitrary object. Since \( B \) is terminal in \( C \), we have that \( C(A, B) \) contains exactly one element. Then \( C^{\text{op}}(B, A) \) contains exactly one element as well, because \( C(A, B) = C^{\text{op}}(B, A) \). Therefore, \( B \) is an initial object in \( C^{\text{op}} \).

Conversely, suppose that \( C^{\text{op}} \) has an initial object \( B \in C^{\text{op}}_0 = C_0 \). Given any object \( A \in C_0 = C^{\text{op}}_0 \), since \( B \) is initial in \( C^{\text{op}} \), \( C^{\text{op}}(B, A) \) contains exactly one element \( f \). Then \( C(A, B) \) contains exactly one element (this is \( f \) again) as well. Therefore, \( B \) is a terminal object in \( C \).

Solution 246 (Exercise 85). We give three solutions to this exercise.
• Consider the category generated by the graph:

\[
\begin{array}{ccc}
x & \rightarrow & y
\end{array}
\]

This category can not have a terminal object since there is no morphism from \(x\) to \(y\) or vice versa.

• Consider the category generated by the graph:

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y
\end{array}
\]

This category also can not have a terminal object, indeed: There is no morphism from \(y\) to \(x\), hence \(x\) can not be terminal. But also \(y\) can not be terminal since \(f\) and \(g\) are different morphisms.

• Consider the category \(\text{Pos}(\mathbb{N}, \leq)\), i.e., the category

\[
\begin{array}{ccc}
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & \ldots
\end{array}
\]

This category can not have a terminal object: Suppose it has a terminal object \(n \in \mathbb{N}\). Then we have another object \(n + 1 \in \mathbb{N}\). Since \(n\) is terminal, we have a morphism \(n + 1 \rightarrow n\), which means that \(n + 1 \leq n\), which is absurd. Therefore, this category does not have a terminal object.

**Solution 247** (Exercise 91). Given \(A, B \in \text{Set}_0\), we claim that the cartesian product \(A \times B\) with the left projection \(\pi_l : (a, b) \mapsto a\) and right projection \(\pi_r : (a, b) \mapsto b\) is a product of \(A\) and \(B\).

Indeed, given an object \(Q \in \text{Set}_0\) with morphisms \(l : Q \rightarrow A\) and \(r : Q \rightarrow B\), we have a morphism \(f := (l, r) : Q \rightarrow A \times B\), given by \(q \mapsto (l(q), r(q))\). For all \(q \in Q\), we have \(\pi_l \circ f(q) = \pi_l(l(q), r(q)) = l(q)\) and \(\pi_r \circ f(q) = \pi_r(l(q), r(q)) = r(q)\), so the diagram commutes, which proves existence.

Now, for uniqueness, suppose that we also have another morphism \(g : Q \rightarrow A \times B\) that makes the diagram commute. Note that we can write all elements \(p \in A \times B\) as \(p = (\pi_l(p), \pi_r(p))\). Then we have, for all \(q \in Q\),

\[
g(q) = (\pi_l(g(q)), \pi_r(g(q))) = (\pi_l \circ g(q), \pi_r \circ g(q)) = (l(q), r(q)) = f(q),
\]

which completes the proof.

**Solution 248** (Exercise 93). For a poset \((X, \leq)\), note that morphisms between objects always are unique if they exist, so uniqueness conditions on morphisms (and whether diagrams commute or not) are not relevant here. Furthermore, any diagram commutes automatically.

For objects \(A, B \in \text{Pos}(X, \leq)_0\), a product of \(A\) and \(B\) is an object \(C \in \text{Pos}(X, \leq)_0\) with morphisms \(C \rightarrow A\) and \(C \rightarrow B\) such that for any object \(D \in \text{Pos}(X, \leq)_0\) with morphisms \(D \rightarrow A\) and \(D \rightarrow B\), we have a morphism \(D \rightarrow C\).
This means that we need an object $C$ such that $C \leq A$ and $C \leq B$, and for all objects $D$ such that $D \leq A$ and $D \leq B$, we also have $D \leq C$. Therefore, a product of $A$ and $B$ is the (unique up to isomorphism) greatest element that is less than or equal to $A$ and $B$, if it exists.

**Remark 249.** The uniqueness up to isomorphism is necessary. For example, consider the poset-category generated by the following diagram:

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \\
& & \\
\end{array}
\]

For the product of $A$ and $B$, the candidates are $X$ and $Y$, since we have both $X \leq A$ and $X \leq B$, and $Y \leq A$ and $Y \leq B$. However, $X$ cannot be the product, since if we are given $D = Y$, we need $Y \leq X$, which is not true. In the same way, $Y$ cannot be the product, because $X \leq Y$ does not hold.

**Solution 250** (Exercise 94). In a category $C$, given two objects $A, B \in C_0$, suppose that we have two products $C, C' \in C_0$, with projections $\pi_l : C \rightarrow A$, $\pi_r : C \rightarrow B$, $\pi'_l : C' \rightarrow A$ and $\pi'_r : C' \rightarrow B$.

Since $C'$ is a product, and we have the morphisms $\pi_l$ and $\pi_r$, the universal property gives a morphism $f : C \rightarrow C'$ such that $\pi_l = \pi'_l \circ f$ and $\pi_r = \pi'_r \circ f$. In the same way, we have a morphism $g : C' \rightarrow C$ such that $\pi'_l = \pi_l \circ g$ and $\pi'_r = \pi_r \circ g$.

Since $C$ is a product, the universal property gives that there is exactly one morphism $h : C \rightarrow C$ such that $\pi_l \circ h = \pi_l$ and $\pi_r \circ h = \pi_r$. The morphism $h = \text{Id}_C$ satisfies this property. However, we also have the morphism $g \circ f : C \rightarrow C$, such that $\pi_l \circ g \circ f = \pi'_l \circ f = \pi_l$ and $\pi_r \circ g \circ f = \pi'_r \circ f = \pi_r$. Therefore, $g \circ f = h = \text{Id}_C$. In the same way, $f \circ g = \text{Id}_{C'}$, so $f : C \cong C'$ is an isomorphism, with inverse $g$.

**Solution 251** (Exercise 95). Let $C$ be a category and take objects $A, B \in C_0$. Let $(P, \pi_l : P \rightarrow A, \pi_r : P \rightarrow B)$ be a product of $A$ and $B$ in $C$. Let $P' \in C_0$ be another object, and suppose that we have an isomorphism $f : P \cong P'$.

We claim that $(P', \pi'_l, \pi'_r)$, with $\pi'_l = \pi_l \circ f^{-1}$ and $\pi'_r = \pi_r \circ f^{-1}$ is also a product of $A$ and $B$. Now, given any object $Q \in C_0$ with morphisms $l : Q \rightarrow A$ and $r : Q \rightarrow B$, we have to show that there exists a unique morphism $g : Q \rightarrow P'$ such that $\pi'_l \circ g = l$ and $\pi'_r \circ g = r$. See also the following diagram:

\[
\begin{array}{ccc}
& & Q \\
& h \searrow & \downarrow r \\
A \quad \quad & \rightarrow & B \\
\downarrow & & \downarrow \\
& \pi_l \swarrow & \pi_r \\
P \quad & \rightarrow & P' \\
\end{array}
\]

By the universal property of the product, we have a morphism $h : Q \rightarrow P$ such that $\pi_l \circ h = l$ and $\pi_r \circ h = r$. We take $g = f \circ h$. We have

$$\pi'_l \circ g = (\pi_l \circ f^{-1}) \circ (f \circ h) = \pi_l \circ h = l$$
and
\[ \pi_r \circ g = (\pi_r \circ f^{-1}) \circ (f \circ h) = \pi_r \circ h = r. \]
This proves existence.

For uniqueness, suppose that we have two morphisms, \( g, g' : Q \to P' \) such that \( \pi'_l \circ g = l = \pi'_l \circ g' \) and \( \pi'_r \circ g = r = \pi'_r \circ g' \). By the universal property of the product, there exists exactly one morphism \( h : Q \to P \) such that \( \pi_l \circ h = l \) and \( \pi_r \circ h = r \). Since
\[ \pi_l \circ (f^{-1} \circ g) = (\pi_l \circ f^{-1}) \circ g = \pi'_l \circ g = l \]
and
\[ \pi_r \circ (f^{-1} \circ g) = (\pi_r \circ f^{-1}) \circ g = \pi'_r \circ g = r, \]
we have \( h = f^{-1} \circ g \). In the same way, we have \( h = f^{-1} \circ g' \). Therefore,
\[ g = f \circ f^{-1} \circ g = f \circ h = f \circ f^{-1} \circ g' = g', \]
which concludes the proof.

**Solution 252** (Exercise 99). Let \( \mathcal{C} \) be a category, let \( A \in \mathcal{C}_0 \) be an object and let \( T \in \mathcal{C} \) be the terminal object.

We claim that \((A, \pi_l, \pi_r)\) (with \( \pi_l = \text{id}_A \) and \( \pi_r \) the unique morphism \( A \to T \)) is a product of \( A \) and \( T \).

We have to prove that for all \( B \in \mathcal{C}_0 \) with morphisms \( l : B \to A \) and \( r : B \to T \), there exists a unique morphism \( f : B \to A \) such that \( \pi_l \circ f = l \) and \( \pi_r \circ f = r \). To that end, let \( B \in \mathcal{C}_0 \) be such an object with morphisms \( l \) and \( r \).

To prove existence, we take \( f = l \). Then we have \( \pi_l \circ f = \text{id}_A \circ l = l \). Since \( T \) is a terminal object, the morphism \( B \to T \) is unique, so \( l = \pi_r \circ f \).

To prove uniqueness of \( f \), suppose that there exists also another morphism, \( f' : B \to A \) such that \( \pi_l \circ f' = l \). Then we have
\[ f' = \text{id}_A \circ f' = \pi_l \circ f' = l = f, \]
which concludes the proof.

**Solution 253** (Exercise 97). Let \( \mathcal{C} \) be a category and \( A, B \in \mathcal{C}_0 \) objects. Let us call the category mentioned in the exercise \( \mathcal{C}_{A \times B} \).

Suppose that we have a terminal object \((C, \pi_l, \pi_r) \in \mathcal{C}_{A \times B_0}\). We claim that this is a product of \( A \) and \( B \) in \( \mathcal{C} \). Indeed, given any object \( D \in \mathcal{C}_0 \) with morphisms \( l : D \to A \), \( r : D \to B \), we have an object \((D, l, r) \in \mathcal{C}_{A \times B_0}\). Since \((C, \pi_l, \pi_r) \) is a terminal object, there exists a unique morphism \( f : (D, l, r) \to (C, \pi_l, \pi_r) \). By the definition of morphisms in \( \mathcal{C}_{A \times B} \), \( f \) is the unique morphism \( f : D \to C \) such that \( \pi_l \circ f = l \) and \( \pi_r \circ f = r \).

Conversely, suppose that we have a product \( C \in \mathcal{C}_0 \) of \( A \) and \( B \) with morphisms \( \pi_l : C \to A \) and \( \pi_r : C \to B \). This gives an object \((C, \pi_l, \pi_r) \in \mathcal{C}_{A \times B_0} \). We claim that this is a terminal object. Indeed, given any object \((D, l, r) \in \mathcal{C}_{A \times B_0}\), since \( C \) is a product in \( \mathcal{C} \), we have a unique morphism \( f : D \to C \) such that \( \pi_l \circ f = l \) and \( \pi_r \circ f = r \). Therefore, \( f \) is a unique morphism in \( \mathcal{C}_{A \times B}(D, l, r), (C, \pi_l, \pi_r) \). Since this holds for any \((D, l, r) \in \mathcal{C}_{A \times B_0}\), \((C, \pi_l, \pi_r) \) is a terminal object in \( \mathcal{C}_{A \times B} \), which concludes the proof.

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Solution 254 (Exercise 98). Let \( \mathcal{C} \) be a category with a choice of product \((A \times B, \pi_l, \pi_r)\) for any two objects \(A, B \in \mathcal{C}_0\). Take objects \(A, B, C, D \in \mathcal{C}_0\) and morphisms \(f : A \to C\) and \(g : B \to D\).

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_l} & A \times B \\
\downarrow f & & \downarrow \pi_r \\
C & \xrightarrow{\pi'_l} & C \times D
\end{array}
\begin{array}{ccc}
\downarrow g \circ \pi_r & & \downarrow g \\
D & \xleftarrow{\pi'_r} & B \times A
\end{array}
\]

We have the products \((A \times B, \pi_l, \pi_r)\) and \((C \times D, \pi'_l, \pi'_r)\). We have morphisms \(f \circ \pi_l : A \times B \to C\) and \(g \circ \pi_r : A \times B \to D\). By the universal property of the product \(C \times D\), there exists a (unique) morphism \(h : A \times B \to C \times D\) that makes the diagram commute. This is the morphism we are looking for.

Solution 255 (Exercise 99). Let \( \mathcal{C} \) be a category with a choice of product \((A \times B, \pi_l, \pi_r)\) for any two objects \(A, B \in \mathcal{C}_0\). Let \(A, B \in \mathcal{C}_0\) be objects.

\[
\begin{array}{ccc}
A \times B & \xrightarrow{\pi_l} & A \\
\downarrow f & & \downarrow \pi_r \\
B \times A & \xleftarrow{\pi'_l} & B
\end{array}
\begin{array}{ccc}
\downarrow g & & \downarrow \pi'_r \\
A & \xleftarrow{\pi'_r} & A \times B
\end{array}
\]

We have the products \((A \times B, \pi_l, \pi_r)\) and \((B \times A, \pi'_l, \pi'_r)\). By the universal properties of the products \(A \times B\) and \(B \times A\), we get morphisms \(f : A \times B \to B \times A\) and \(g : B \times A \to A \times B\), which make the diagram commute.

By the universal property of the product \(A \times B\), there exists a unique morphism \(h : A \times B \to A \times B\) such that \(\pi_l \circ h = \pi_l\) and \(\pi_r \circ h = \pi_r\). Since \(\text{Id}_{A \times B}\) satisfies this, we have \(h = A \times B\). We also have the morphism \(g \circ f : A \times B \to A \times B\). We have

\[\pi_r \circ g \circ f = \pi'_l \circ f = \pi_r\]

and in the same way, we have \(\pi_l \circ g \circ f = \pi_l\). Therefore, \(g \circ f = h = \text{Id}_{A \times B}\). In the same way, we have \(f \circ g = \text{Id}_{B \times A}\). Therefore, we conclude that \(f\) is the isomorphism we are looking for, with inverse \(g\).

Solution 256 (Exercise 100). Let \(F : \mathcal{C} \to \mathcal{C}\) be an endofunctor and let \((\mu^F, \text{in})\) be an initial algebra. By definition, \((\text{in})\) is the unique morphism \(\mu^F \to \mu^F\) that makes the following diagram commute:

\[
\begin{array}{ccc}
F(\mu^F) & \xrightarrow{\text{in}} & \mu^F \\
\downarrow F(\text{in}) & & \downarrow \text{in} \\
F(\mu^F) & \xrightarrow{\text{in}} & \mu^F
\end{array}
\]
Since $F(\text{Id}_F) = \text{Id}_{F(F)}$, the morphism $\text{Id}_F$ satisfies this property, and therefore $\langle \text{in} \rangle = \text{Id}_{\mu F}$.

**Solution 257** (Exercise 184). Let $F : C \to C$ be an endofunctor and let $(\mu_F, \text{in})$ be an initial algebra. Given $F$-algebras $(C, \phi)$ and $(D, \psi)$ and $f \in C(D, C)$, we have the following commutative diagrams:

$$
\begin{array}{ccc}
F\mu^F & \xrightarrow{\text{in}} & \mu^F \\
\downarrow F(\phi) & & \downarrow F(\phi) \\
C & \xrightarrow{\phi} & D \\
\end{array}
$$

Now, suppose that $f \circ \phi = \psi \circ F(f)$. By definition, $\langle \psi \rangle$ is the unique morphism from $\mu^F$ to $D$ such that $\langle \psi \rangle \circ \text{in} = \psi \circ F(\langle \psi \rangle)$. Therefore, to show that $\langle \psi \rangle = f \circ \langle \phi \rangle$, it suffices to show that $(f \circ \langle \phi \rangle) \circ \text{in} = \psi \circ F(f \circ \langle \phi \rangle)$.

And indeed, we have

\[
f \circ \langle \phi \rangle \circ \text{in} = f \circ \phi \circ F(\langle \phi \rangle) = \psi \circ F(f) \circ F(\langle \phi \rangle) = \psi \circ F(f \circ \langle \phi \rangle).
\]

Therefore, $\langle \psi \rangle = f \circ \langle \phi \rangle$.

**Solution 258** (Exercise 186). Let $C$ be a category with an initial object $\perp$. Let $F : C \to C$ be the identity endofunctor.

Note that $(\perp, \text{Id}_\perp)$ is an $F$-algebra since $\text{Id}_\perp : \perp \to \perp$ and $\perp = F(\perp)$. Now, to show that it is initial, suppose we have another $F$-algebra $(X \in C_0, f : FX \to X)$. By initiality of $\perp$ in $C$, we have a unique morphism $g \in C(\perp, X)$. To show that $g$ is a $F$-algebra morphism, we have to show that the following diagram commutes:

$$
\begin{array}{ccc}
\perp = F(\perp) & \xrightarrow{\text{Id}_\perp} & \perp \\
\downarrow g = F(g) & & \downarrow g \\
X = F(X) & \xrightarrow{f} & X \\
\end{array}
$$

Because $\perp$ is initial, there exists exactly one morphism from $\perp$ to $X$. Therefore, $g = f \circ g$ and $g$ is an $F$-algebra morphism. It is also unique, because $\perp$ is initial in $C$. Therefore, $(\perp, \text{Id}_\perp)$ is an initial $F$-algebra.

**Solution 259** (Exercise 194). Let $F : C \to C \to C$ be a bifunctor such that for any object $A \in C$, the initial algebra $(\mu_{FA}, \text{in}_A)$ for the functor induced by $F_A : C \to C : X \mapsto F(A, X)$,

\[
F_A : C \to C : X \mapsto F(A, X),
\]
exists.

We define a functor $G : C \to C$ by $G(A) = \mu^{F_A}$. For $A, B \in C_0$, we have an initial $F_A$-algebra $(\mu^{F_A}, i_A : F_A(\mu^{F_A}) \to \mu^{F_A})$ and an initial $F_B$-algebra $(\mu^{F_B}, i_B : F_B(\mu^{F_B}) \to \mu^{F_B})$. Now, for all $f \in C(A, B)$, we have by the bifunctoriality of $F$, a morphism

$$F(\mu^{F_B})(f) : F_A(\mu^{F_A}) = F(A, \mu^{F_A}) \to F(B, \mu^{F_B}) = F_B(\mu^{F_B}).$$

Then $(\mu^{F_B}, i_B \circ F(\mu^{F_B})(f)) : F_A(\mu^{F_B}) \to \mu^{F_B}$ is an $F_A$-algebra, so by the initiality of $\mu^{F_A}$, we have a unique morphism $f' : \mu^{F_A} \to \mu^{F_B}$ such that

$$i_B \circ F(\mu^{F_B})(f) \circ F_A(f') = f' \circ i_A$$

and we set $G(f) = f'$.

We now check that $G$ is a functor. To show that, we need to show that for all $A \in C_0$, $G(\text{id}_A) = \text{id}_{\mu^{F_A}}$ and for all $f \in C(A, B)$ and $g \in C(B, C)$, $G(g \circ f) = G(g) \circ G(f)$.

For the first property, given an object $A \in C_0$, $G(\text{id}_A) : \mu^{F_A} \to \mu^{F_A}$ is the unique morphism such that

$$\text{id}_A \circ F(\mu^{F_A})(\text{id}_A) \circ F_A(\text{id}_{\mu^{F_A}}) = G(\text{id}_A) \circ \text{id}_A.$$

Hence, to show that $G(\text{id}_A) = \text{id}_{\mu^{F_A}}$ holds, it suffices to show

$$\text{id}_A \circ F(\mu^{F_A})(\text{id}_A) \circ F_A(\text{id}_{\mu^{F_A}}) = \text{id}_{\mu^{F_A}} \circ \text{id}_A.$$

This indeed follows by the following computation:

$$\text{id}_A \circ F(\mu^{F_A})(\text{id}_A) \circ F_A(\text{id}_{\mu^{F_A}}) = \text{id}_A \circ \text{id}_{\mu^{F_A}} \circ F_A(\mu^{F_A})$$

$$= \text{id}_A \circ \text{id}_{\mu^{F_A}} \circ \text{id}_A.$$

For the second property, given objects $A, B, C \in C_0$, and $f : C(A, B)$ and $g : C(B, C)$, we have that

$$\text{id}_B \circ F(\mu^{F_B})(f) \circ F_A(G(f)) = G(f) \circ \text{id}_A$$

and

$$\text{id}_C \circ F(\mu^{F_C})(g) \circ F_B(G(g)) = G(g) \circ \text{id}_B,$$

and we know that $G(g \circ f) : \mu^{F_A} \to \mu^{F_C}$ is the unique morphism such that

$$\text{id}_C \circ F(\mu^{F_C})(g \circ f) \circ F_A(G(g \circ f)) = G(g \circ f) \circ \text{id}_A.$$

Note that $F$ is a bifunctor, and therefore, the following diagram commutes:

$$\begin{array}{ccc}
F(A, \mu^{F_B}) & \xrightarrow{F(\mu^{F_B})(f)} & F(B, \mu^{F_B}) \\
\downarrow F(A, G(g)) & & \downarrow F(B, G(g)) \\
F(A, \mu^{F_C}) & \xrightarrow{F(\mu^{F_C})(f)} & F(B, \mu^{F_C})
\end{array}$$
Then we have the following equality:

\[
\begin{align*}
\text{in}_C \circ F(\_ \cdot \mu_{FC})(g \circ f) & \circ F_A(G(g) \circ G(f)) \\
= \text{in}_C \circ F(\_ \cdot \mu_{FC})(g) \circ F(\_ \cdot \mu_{FC})(f) \circ F_A(G(g)) \circ F_A(G(f)) \\
= \text{in}_C \circ F(\_ \cdot \mu_{FC})(g) \circ F_B(G(g)) \circ F(\_ \cdot \mu_{FB})(f) \circ F_A(G(f)) \\
= G(g \circ \text{in}_B \circ F(\_ \cdot \mu_{FB})(f)) \circ F_A(G(f)) \\
= G(g \circ \text{in}_B \circ F(\_ \cdot \mu_{FB})(f) \circ F_A(G(f))) \\
= G(g \circ \text{in}_B \circ F(g) \circ \text{in}_A).
\end{align*}
\]

Therefore, \( G(g \circ f) = G(g) \circ G(f) \).

**Solution 260** (Exercise 206). Let \( F : \text{Set} \to \text{Set} \) be the functor induced by \( X \mapsto 1 + X \) with \( 1 = \{ \star \} \). We claim that \( (\mathbb{N}^c, f) \) is a terminal coalgebra for this functor, with \( \mathbb{N}^c = \mathbb{N} + \{ \infty \} \) and \( f : \mathbb{N} + \{ \infty \} \to 1 + \mathbb{N} + \{ \infty \} \) given by

\[
\begin{align*}
0 & \mapsto \star \\
\begin{array}{r}
s(n) \mapsto n \\
\infty \mapsto \infty
\end{array}
\end{align*}
\]

We need to show that for all \( F \)-coalgebras \((X, g)\), there exists a unique \( F \)-coalgebra morphism \( \varphi : (X, g) \to (\mathbb{N}^c, f) \). To this end, take an arbitrary \( F \)-coalgebra \((X, g)\).

We first show existence. Borrowing some notation from monads, given the morphism \( g : X \to FX \), we write \( g^* \) for the induced morphism \( g^* : FX \to FX \) and we write \( (g^*)^n = g^* \circ g^* \circ \cdots \circ g^* \). We define \( \varphi : X \to \mathbb{N}^c \) as follows: For \( x \in X \), if for some \( n \in \mathbb{N} \), \( (g^*)^n(x) \in X \) and \( (g^*)^{s(n)}(x) = \star \), we set \( \varphi(x) = n \). Else, we set \( \varphi(x) = \infty \). We need to show that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & 1 + X \\
\downarrow{\varphi} & & \downarrow{\text{id}_1 + \varphi} \\
\mathbb{N} + \{ \infty \} & \xrightarrow{f} & 1 + \mathbb{N} + \{ \infty \}
\end{array}
\]

To that end, take \( x \in X \). Suppose that \( (\text{id}_1 + \varphi)(g(x)) = \star \). Then \( g^1(x) = \star \), so \( \varphi(x) = 0 \) and \( f(\varphi(x)) = \star \). Suppose that \( (\text{id}_1 + \varphi)(g(x)) = \infty \). Then (by the definition of \( \varphi \)) for all \( n \in \mathbb{N} \), \( (g^*)^{s(n)}(x) = (x)(g^*)^n(g(x)) \in X \), so \( \varphi(x) = \infty \) and \( f(\varphi(x)) = \infty \). Lastly, suppose that \( (\text{id}_1 + \varphi)(g(x)) = n \). Then (by the definition of \( \varphi \)), \( (g^*)^{s(n)}(x) = (g^*)^n(g(x)) \in X \) and \( (g^*)^{s(s(n))}(x) = (g^*)^{s(n)}(g(x)) = \star \). By the definition of \( \varphi \), we then have \( \varphi(x) = s(n) \). Then \( f(\varphi(x)) = n \) and we conclude that the diagram commutes and \( \varphi \) is a \( F \)-coalgebra morphism.

For uniqueness, suppose that we also have a morphism \( \psi : X \to \mathbb{N}^c \) that makes the diagram commute.

Suppose that for some \( x \in X \), \( g(x) = \star \). Then, to make the diagram commute, we must have \( f(\psi(x)) = (\text{id}_1 + \psi)(\star) = \star \), so by the definition of \( f \), we must have \( \psi(x) = 0 \).

By induction on \( n \), we will show that for all \( x \in X \), if \( (g^*)^n(x) \in X \) and \( (g^*)^{s(n)} = \star \), we have \( \psi(x) = n \) (which equals \( \varphi(x) \)). Since we just showed the case for \( n = 0 \), we
will now show the induction step: suppose that it holds for some \( n \). Take \( x \in X \). If \((g^*)^n(g(x)) = (g^*)^s(n)(g(x)) \in X\) and \((g^*)^{s(n)}(g(x)) = (g^*)^{s(n)}(g(x)) = \ast\), we have \( \psi(g(x)) = n \). Then we have \( f(\psi(x)) = (\text{Id}_1 + \psi)(g(x)) = n \). Therefore, \( f(\psi(x)) = s(n) \), which proves the induction step.

Now, take \( x \in X \) and suppose that for all \( n \in \mathbb{N} \), \((g^*)^n(x) \in X\). If \( \psi(x) = n \) for some \( n \), then (since the diagram commutes), we have \( \psi(g(x)) = f(\psi(x)) = f(n) = n - 1 \). Repeating this \( n \) times, we have \((\text{Id}_1 + \psi)((g^*)^n(x)) = 0\). We take \( x' = (g^*)^n(x) \in X\). We have \( \psi(x') = 0 \) and \( f(x') = \ast\). However, \( g(x') \in X \), so \((\text{Id}_1 + \varphi)(g(x')) \neq \ast \) and the diagram does not commute. Therefore, we cannot have \( \psi(x) \in \mathbb{N} \), so \( \psi(x) = \infty = \varphi(x) \).

Therefore, \( \psi = \varphi \) and we conclude that \( \varphi \) is unique, so \((\mathbb{N}, f)\) is a terminal \( F \)-coalgebra.

**Solution 261** (Exercise 208). We wish to define \( h, t : N \to N \) such that the following diagram commutes:

\[
\begin{array}{ccc}
N & \xrightarrow{(h,t)} & N \times N \\
\downarrow \text{nats} & & \downarrow \text{id} \times \text{nats} \\
\text{Stream}(N) & \xrightarrow{(\text{head},\text{tail})} & N \times \text{Stream}(N)
\end{array}
\]

This means:

\[
\begin{align}
\text{head} \circ \text{nats} &= h \quad (14) \\
\text{tail} \circ \text{nats} &= \text{nats} \circ t \quad (15)
\end{align}
\]

By definition of \( \text{nats} \), Eq. (14) means

\[
\forall n : N : n = \text{head}(\text{nats}(n)) = h(n),
\]

and Eq. (15) means

\[
\forall n : \text{nats}(n + 1) = \text{tail}(\text{nats}(n)) = \text{nats}(t(n)).
\]

Hence, we define (as candidates)

\[
\begin{align}
h &= \text{id}_N, \quad t &= \text{succ}.
\end{align}
\]

By construction, we have that Eq. (14) and Eq. (15) indeed holds for \( h := \text{id}_N \) and \( t := \text{succ} \), which shows that the sought function \( \text{nats} : N \to \text{Stream}(N) \) is the unique solution of the equation system

\[
\begin{align}
\text{head} \circ \text{nats} &= \text{id} \\
\text{tail} \circ \text{nats} &= \text{nats} \circ \text{succ}
\end{align}
\]

and thus can be defined as the anamorphism \( [\text{id}, \text{succ}] \).

**Solution 262** (Exercise 209). We wish to define \( h : \text{Stream}(A) \times \text{Stream}(B) \to A \times B \) and \( t : \text{Stream}(A) \times \text{Stream}(B) \to \text{Stream}(A) \times \text{Stream}(B) \) such that the following diagram commutes:

\[
\end{align}
\]
\[
\begin{array}{c}
\text{Stream}(A) \times \text{Stream}(B) \xrightarrow{(h,t)} A \times B \times \text{Stream}(A) \times \text{Stream}(B) \\
\text{zip} \downarrow & \downarrow \text{Id} \times \text{zip} \\
\text{Stream}(A \times B) \xrightarrow{(\text{head}, \text{tail})} \text{Stream}(A \times B)
\end{array}
\]

This means:
\[
\begin{align*}
\text{head} \circ \text{zip} &= h \\
\text{tail} \circ \text{zip} &= \text{zip} \circ t
\end{align*}
\]

By definition of \text{zip}, Eq. (16) means
\[
\forall as : \text{Stream}(A), bs : \text{Stream}(B) : (\text{head}(as), \text{head}(bs)) = \text{head}(\text{zip}(as, bs)) = h(as, bs),
\]
and Eq. (17) means
\[
\forall as : \text{Stream}(A), bs : \text{Stream}(B) : \text{zip}(\text{tail}(as), \text{tail}(bs)) = \text{tail}(\text{zip}(as, bs)) = \text{zip}(t(as, bs)).
\]
Hence, we define (as candidates)
\[
h = \text{head} \times \text{head}, \quad t = \text{tail} \times \text{tail}.
\]

By construction, we have that Eq. (16) and Eq. (17) indeed holds for \( h := \text{head} \times \text{head} \) and \( t := \text{tail} \times \text{tail} \),
which shows that the sought function \( \text{zip} : \text{Stream}(A) \times \text{Stream}(B) \to \text{Stream}(A \times B) \)
is the unique solution to the equation system
\[
\begin{align*}
\text{head} \circ \text{zip} &= \text{head} \times \text{head} \\
\text{tail} \circ \text{zip} &= \text{zip} \circ (\text{tail} \times \text{tail})
\end{align*}
\]
This function can, therefore, be defined as \( [[\text{head} \times \text{head}, \text{tail} \times \text{tail}]] \).

### A. Forgetful and free functors

A lot of (mathematical) structures are defined as some other kind of mathematical structure, but where extra structure is added. An example of this is the following:
Recall that a monoid is a set \( M \) together with a binary operation \( m : M \to M \to M \) which is associative and such that there is an identity element \( e \) (see Definition 19).
In particular, any monoid has an underlying set and any morphism of monoids has an underlying function (between those sets). So forgetting the binary operation and identity element defines a functor from \textbf{Monoid} to \textbf{Set} which is called a \textit{forgetful functor}:

**Example 263.** The \textit{forgetful functor} from \textbf{Monoid} to \textbf{Set} is the functor specified by the following data:
• The function on objects is given by
  \[ \text{Monoid}_0 \to \text{Set}_0 : (M, m, e) \mapsto M. \]

• The function on objects is given by
  \[ \text{Monoid}((M_1, m_1, e_1), (M_2, m_2, e_2)) \to \text{Set}(M_1, M_2) : f \mapsto f. \]

Notice that if one defines a category whose objects are sets \( M \) together with an associative binary operation \( m : M \to M \to M \), then one could analogously also define a forgetful functor from \( \text{Monoid} \) to this category by only forgetting the neutral element.

**Lemma 264.** The forgetful functor from \( \text{Monoid} \) to \( \text{Set} \) satisfies the properties of a functor.

**Proof.** We clearly have that everything is well-defined since the codomain is \( \text{Set} \).
That the identity morphism is preserved holds by definition because the identity morphism of \((M, m, e)\) (in \( \text{Monoid} \)) is given by the identity function and the identity morphism of \( M \) (in \( \text{Set} \)) is also given by the identity function.
That the composition of morphisms is preserved also holds by definition because the composition of morphisms (in \( \text{Monoid} \)) is given by the composition of the underlying functions which is also the composition in \( \text{Set} \).

The forgetful functor \( \text{Forget} \) from \( \text{Monoid} \) to \( \text{Set} \) forgets the algebraic structure of a monoid and since there are multiple monoid structures on the same set (given an example of this), hence we do not have that there exists some functor \( G : \text{Set} \to \text{Monoid} \) such that \( \text{Forget} \cdot G \) is the identity on \( \text{Monoid} \). However, to each set, one can define a monoid which satisfies an important property (this is Proposition 268). The associated monoid is called the free monoid:

**Example 265.** Let \( X \) be a set. The free monoid generated by \( X \), denoted by \( \text{Free}(X) \), is specified by the following data:

- The underlying set consists of all finite sequences/strings of elements in \( X \) (including the empty sequence).
- The multiplication is defined by concatenating the sequences, i.e.
  \[ m \left( (x_1, \cdots, x_n), (y_1, \cdots, y_m) \right) := (x_1, \cdots, x_n, y_1, \cdots, y_m). \]
- The identity element is given by the empty sequence.

**Example 266.** The free functor from \( \text{Set} \) to \( \text{Monoid} \) is specified by the following data:

- The function on objects is given by
  \[ \text{Set}_0 \to \text{Monoid}_0 : X \mapsto \text{Free}(X). \]
• The function on morphisms is given as follows: A morphism \( f \in \text{Set}(X,Y) \) (i.e. a function) is mapped to the monoidal morphism which is given by pointwise application of \( f \), i.e.

\[
\text{Free}(f)(x_1, \ldots, x_n) := (f(x_1), \ldots, f(x_n)).
\]

**Exercise 267.** Show that \( \text{Free} \) satisfies the properties of a functor.

For any set \( X \), we have the **canonical function**

\[
\text{Free}_X^X : X \to \text{Free}(X) : x \mapsto (x).
\]

This function satisfies the property that any function from \( X \) to an arbitrary monoid corresponds with a unique morphism (of monoids) from the free monoid generated by \( X \) to that monoid:

**Proposition 268.** Let \((M,m,e)\) be a monoid and \( X \) be a set. For any morphism \( f \in \text{Set}(X,M) \) (i.e. a function), there exists a unique morphism \( \phi^f \in \text{Monoid}(\text{Free}(X),(M,m,e)) \), such that \( f = \text{Free}_X^X \cdot \phi^f \).

**Proof.** For elements \( a,b \in M \), we denote their multiplication by \( a \times b := m(a,b) \) (note that by associativity we have that \( a_1 \times a_2 \times \cdots \times a_n \) is well-defined). Let \( f \in \text{Set}(X,M) \) be a function. Define

\[
\phi^f : \text{Free}(X) \to (M,m,e) : (x_1 \cdots, x_n) \mapsto f(x_1) \times \cdots \times f(x_n),
\]

We have to define separately what happens with the empty sequence. The empty sequence we map to the identity element \( e \), so in particular we have that the identity element is preserved under \( \phi^f \), so in order to conclude that \( \phi^f \) is a morphism of monoids, it remains to show that it preserves the binary operation but this is clear by definition.

That \( f = \text{Free}_X^X \cdot \phi^f \) holds follows immediately by the definition of \( \text{Free}_X^X \) and \( \phi^f \), indeed:

\[
(\text{Free}_X^X \cdot \phi^f)(x) = \phi^f((x)) = f(x).
\]

So it only remains to show uniqueness. Assume that \( \psi \) also satisfies \( \text{Free}_X^X \cdot \psi = f \). The claim now follows because \( \psi \) is a morphism of monoids, indeed: Since \( \psi \) is morphism of monoids, we have that it preserves the multiplication, but the multiplication is given by concatenation, hence, we have that \( \psi \) is uniquely determined by the images of the sequences of length 1 (and length 0, but this sequence of length 0 should be mapped under \( \psi \) to \( e \)). But a sequence of length 1 is of the form \((x) = \text{Free}_X^X(x)\). So the claim indeed follows by the following computation:

\[
\psi((x)) = \psi(\text{Free}_X^X(x)) = f(x) = \phi^f(\text{Free}_X^X(x)) = \phi^f((x)).
\]
The following exercise shows that there is a special connection between the forgetful functor \( \text{forget} : \text{Monoid} \to \text{Set} \) and the free functor \( \text{free} : \text{Set} \to \text{Monoid} \). This connection expresses that these form a so-called adjoint pair (see Section 6).

**Exercise 269.** Show that for any set \( X \) and monoid \((M, m, e)\), there exist bijections between the hom-sets:

\[
\alpha_X^{(M, m, e)} : \text{Monoid}(\text{Free}(X), (M, m, e)) \to \text{Set}(X, \text{forget}(M, m, e)).
\]

Hint: use Proposition 268.

**Exercise 270.** Define a forgetful functor from the category \( \text{Pos} \) of posets (defined in Example 14) to \( \text{Set} \) analogous to the the forgetful functor from \( \text{Monoid} \) to \( \text{Set} \) (defined in Example 263).

**Remark 271.** The story about free monoids can not be repeated for posets, i.e. there is no free poset structure on all sets. But in order to prove this one needs more machinery.

### A.1. Contravariant functors

A variation on functors are **contravariant functors**. A contravariant functor consists of a map on objects just like a functor. However, the map on morphisms turns the morphisms around. We give the formal definition:

**Definition 272.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. A **contravariant functor** \( F \) from \( \mathcal{C} \) to \( \mathcal{D} \) consists of the following data:

- A function \( C_0 \to D_0 \), written as \( X \mapsto F(X) \).
- For each \( X, Y \in C_0 \), a function \( C(X, Y) \to D(F(Y), F(X)) \), written as \( f \mapsto F(f) \).

Moreover, this data should satisfy the following properties:

- **(Preserves composition)** For \( f \in \text{hom}_C(X, Y) \) and \( g \in \text{hom}_C(Y, Z) \), we have \( F(g \circ f) = F(f) \circ F(g) \).
- **(Preserves identity)** For \( X \in \mathcal{C} \), we have \( F(\text{Id}_X) = \text{Id}_{F(X)} \).

**Exercise 273.** Notice that the preservation of composition has now changed, why is this the case?

An example of a contravariant functor is given by the powerset-functor:
Example 274. Recall that the powerset of a set $X$, denoted by $\mathbb{P}(X)$, is the set of all subsets of $X$, i.e.

$$\mathbb{P}(X) := \{A \mid A \subseteq X\}.$$ 

The contravariant powerset-functor (on sets), denoted by $\mathbb{P}$, is the functor from $\text{Set}$ to $\text{Set}$ defined by the following data:

- The function on objects is given by:
  $$\text{Set}_0 \to \text{Set}_0 : X \mapsto \mathbb{P}(X).$$

- For each $X, Y \in \text{Set}$, the function on morphisms is given by
  $$\text{Set}(X, Y) \to \text{Set}(\mathbb{P}(X), \mathbb{P}(Y)) : f \mapsto f^{-1},$$
  where $f^{-1}$ given by
  $$f^{-1} : \mathbb{P}(Y) \to \mathbb{P}(X) : B \mapsto f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

Exercise 275. Show that $\mathbb{P}$, defined in Example 274 satisfies the properties of a contravariant functor.

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\(^5\)Since a function is mapped to the inverse-image function, one also calls the powerset-functor, an inverse image-functor.
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