Quantum Field Theory of Vortices in Superfluid Films

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November 29, 1997

Abstract

A quantum field theory, consisting of the effective action of sound waves linearly coupled to a Chern-Simons term, is proposed to describe the dynamics of vortices in a superfluid film at the absolute zero of temperature.

PACS: 03.70.+k, 47.32.Cc, 67.70.+n
Keywords: Quantum vortices, Chern-Simons-theory, Phonon-vortex-scattering
In a superfluid film at the absolute zero of temperature, vortices are induced by quantum fluctuations. These vortices are to be distinguished from the thermally induced ones at finite temperature. Whereas the latter are classical objects, the zero-temperature vortices are quantum objects. Besides in superfluids, quantum vortices play an important role in various other two-dimensional quantum systems such as the fractional quantized Hall effect, superconducting films and Josephson junction arrays at the absolute zero of temperature.

Since zero-temperature vortices in a superfluid film are point-like quantum objects, one may wonder whether their dynamics can be described by a quantum field theory. We shall argue in this Letter that this is indeed the case. Because at zero temperature the normal fluid can be ignored, an important result of Helmholtz applies, stating that a vortex moves with the fluid, i.e., at the local velocity of the fluid. This theorem, implying that the dynamics of vortices is determined by the superfluid component, provides a stringent constraint on the quantum field theory we are seeking. An other constraint follows from the observation due to Haldane and Wu [1] that when a vortex encircles a boson, it accumulates a geometric phase of $2\pi$. It will be shown that in two dimensions these requirements can be fulfilled by a Chern-Simons theory, one of whose main characteristics is that it involves a vector field which has no independent dynamics.

Our approach is to be contrasted with other modern treatments of the dynamics of vortices in superfluid films [2, 3, 4, 5]. These theories give a quantum-mechanical description of vortices with delta functions representing their world-lines. In our quantum field approach, on the other hand, vortices are described by a nonsingular quantum field.

The theory we propose to describe sound waves as well as vortices in a superfluid film at zero temperature is given by the Lagrangian

$$L = L_{\text{eff}} - e n a_0 + \frac{e}{c} j \cdot a + L_{\text{CS}},$$

(1)

It consists of the effective Lagrangian $L_{\text{eff}}$ describing the superfluid without vortices, linearly coupled via the particle number current $(n, j)$ to a vector field $(a_0, a)$ governed by a Chern-Simons term

$$L_{\text{CS}} = \frac{1}{2c} a \times \partial_0 a - a_0 \nabla \times a,$$

(2)

with $c$ the sound velocity. The vector field, which has no independent dynamics, accounts for the vortices, with the Chern-Simons term encoding the geometric phase acquired by a vortex when it winds around a boson. The quantum-mechanical analog of such a term, representing the linking number of a closed boson and vortex trajectory [8], was introduced in the problem by Arovas and Freire [9]. The charge $e$ appearing in (1) will be determined shortly. The effective Lagrangian $L_{\text{eff}}$ is given by

$$L_{\text{eff}} = -\bar{n} \left[ \hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right] + \bar{n} \left[ \hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right],$$

(3)
where the dimensionless field $\varphi$ is the gapless Goldstone mode of the spontaneously broken global U(1) symmetry, $m$ is the mass of the Bose condensed atoms, and $\bar{n}$ is the particle number density of the fluid at rest. Physically, this nonlinear theory governs the sound mode, with the dispersion $\omega^{2} = c^{2}k^{2}$, where $\omega$ is the (angular) frequency and $k$ the wave vector. The effective theory can either be inferred from general symmetry arguments [7], or explicitly derived from the microscopic Bogoliubov theory by integrating out quantum fluctuations [8]. It gives a complete description of the superfluid valid at low energies and small momenta. The same effective theory appears in the context of (neutral) superconductors and of classical hydrodynamics [9].

The particle number current that follows from (3) reads

\[ n = \bar{n} - \frac{\bar{n}}{mc^{2}} \left[ \hbar \partial_{0} \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^{2} \right] \]

\[ j = n \mathbf{v}, \quad (4) \]

where $\mathbf{v} = (\hbar/m) \nabla \varphi$ is the superfluid velocity field. Physically, the first equation in (4) reflects Bernoulli’s principle which states that in regions of rapid flow, the density and therefore the pressure is low.

To see under which circumstances the higher order terms in the effective theory $\mathcal{L}_{\text{eff}}$ can be neglected, we note that additional derivatives are always accompanied by additional factors of $\hbar k/mc$, with $k$ the wave number $k = |k|$. This means that third and higher order in the field $\varphi$ can be ignored provided the wave number is smaller than the inverse coherence length $\xi = \hbar/mc$

\[ k < 1/\xi. \]  

(5)

For $^4$He the coherence length, or Compton wavelength, is about 10 nm which is of the order of the interatomic spacing. In the experimentally determined spectrum of superfluid $^4$He, the inverse of this value, 0.1 nm$^{-1}$, marks the point beyond which the spectrum ceases to be linear. That is, the region defined by (5) coincides with the region where the superfluid $^4$He spectrum is linear and the description in terms of a sound mode is applicable.

Before proceeding, we emphasize that the coupling of the Chern-Simons vector field to the sound mode is linear and not minimal—as is usually the case in gauge theories. As a result, the coupled theory (1) does not possess a gauge invariance involving a simultaneous local gauge transformation of the Chern-Simons vector field and the matter field. Due to the Higgs mechanism, a minimal coupling would inevitably result in the disappearance of the gapless sound mode. This would not be in agreement with the physics we wish to describe in this Letter, which is the dynamics of vortices in a compressible superfluid film.

We next consider the field equation for $a_0$. As is common in theories containing a Chern-Simons term, this component of the vector field merely plays the role of a Lagrange multiplier. We find

\[ \nabla \times \mathbf{a} = -en, \]

(6)

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or when integrated

\[ \Phi = -eN, \tag{7} \]

where \( N = \int d^2x n \) is the particle number and \( \Phi = \int d^2x \nabla \times a \) the "magnetic" flux associated with the Chern-Simons vector field. That is, particles carry besides a particle number charge also a flux. In the Coulomb gauge \( \nabla \cdot a = 0 \), Eq. (3) yields the solution

\[ a(t, x) = -e \nabla_x \times \int d^2x' G(x - x') n(t, x'), \tag{8} \]

where \( G(x) \) is the two-dimensional Green function of the Laplace operator

\[ G(x) = -\frac{1}{2\pi} \ln(|x|), \tag{9} \]

i.e., \( \nabla^2 G(x) = -\delta(x) \), with \( \delta(x) \) the two-dimensional delta function. The solution shows that the Chern-Simons vector field is entirely determined by the particle number density \( n \).

To identify the flux carried by the particles, we calculate the Berry phase \( \gamma(\Gamma) \) by evaluating the Wilson loop \( W(\Gamma) = \exp[i\gamma(\Gamma)] \). The latter is obtained by integrating the Chern-Simons vector field \( a \) around a closed loop \( \Gamma \) in the superfluid,

\[ \gamma(\Gamma) = \frac{e}{\hbar c} \oint_{\Gamma} dl \cdot a(t, x) = \frac{e^2}{\hbar c} \int d^2x' \oint_{\Gamma} dl \cdot [\nabla_x \times \ln(|x - x'|)] n(t, x'), \tag{10} \]

where we substituted the explicit form (8) for \( a \). Provided we choose \( e^2 = \hbar c \), this geometric phase coincides with the one obtained by Haldane and Wu [1] who transported a vortex adiabatically around the closed path \( \Gamma \). Their starting point was an Ansatz for the multivortex wavefunction of the interacting Bose condensate. They concluded that, apart from corrections due to residual vortex interactions that become small in the dilute-vortex limit, the right-hand side of (10) yields \( \pm 2\pi \) times the mean number of superfluid particles enclosed by the closed loop \( \Gamma \). In other words, the vortex which was taken around the closed path \( \Gamma \) sees the encircled bosons as sources of geometric phase. In the theory proposed here, this counting is provided by the flux imparted to a particle by the Chern-Simons term.

We next introduce external point vortices into the theory to determine their action. To this end, we consider only the following terms of the Lagrangian (1):

\[ \mathcal{L}_{\text{ext}} = -\bar{n} \left[ \hbar \partial_\gamma \varphi - \frac{1}{2m}(\hbar \nabla \varphi)^2 \right] + \frac{e}{c} j \cdot a_{\text{ext}}, \tag{11} \]

where the particle number current \( j \) reads \( j = \bar{n} v \) in this approximation, and where the Chern-Simons vector field \( a_{\text{ext}} \) is given by (8) with \( n \) replaced by the external vortex density

\[ n_{\text{ext}}(t, x) = \sum_\alpha w_\alpha \delta[\mathbf{x} - \mathbf{X}_\alpha(t)]. \tag{12} \]
Explicitly,
\[ a_{\text{ext}}(t, x) = \frac{e}{2\pi} \sum_{\alpha} w_{\alpha} \nabla \arctan \left( \frac{x_2 - X_2^\alpha(t)}{x_1 - X_1^\alpha(t)} \right), \tag{13} \]
where \( X^\alpha(t) \) denotes the location of the \( \alpha \)th vortex with winding number \( w_{\alpha} \). The omitted terms in (11) are all of higher order in derivatives which can be ignored at low energies and small momenta. The field equation for \( \varphi \) obtained from (11) can be easily solved to yield
\[ \varphi(t, x) = -\frac{e}{\hbar c} \int d^2 x' G(x - x') \nabla_{x'} \cdot a_{\text{ext}}(t, x'). \tag{14} \]

When substituting this back into the Lagrangian (11), which is tantamount to integrating out the phonons, we find for the action \( S_{\text{ext}} = \int d^3 x \mathcal{L}_{\text{ext}} \) describing the external charges:
\[ S_{\text{ext}} = \bar{n} m \int dt \left[ \frac{1}{2} \sum_{\alpha} \gamma_{\alpha} \mathbf{X}^\alpha \times \dot{\mathbf{X}}^\alpha + \frac{1}{2\pi} \sum_{\alpha < \beta} \gamma_{\alpha} \gamma_{\beta} \ln(|X^\alpha - X^\beta|) \right], \tag{15} \]
where \( \gamma_{\alpha} = (h/m) w_{\alpha} \) is the circulation of the \( \alpha \)th vortex. This action yields the well-known equations of motion for point vortices in an incompressible two-dimensional fluid [10, 11], which we thus have shown to be correctly reproduced by the field theory proposed here. The restriction to an incompressible fluid, where the particle density \( n \) is a constant \( \bar{n} \), stems from ignoring the higher-order derivative terms in (11).

Let us continue by calculating the scattering amplitude of two phonons interacting via the Chern-Simons vector field at the tree level. In the frame where the sum of the two incoming momenta is zero, this corresponds to the scattering of a phonon from a vortex. An analogous situation arises in the case of Aharonov-Bohm scattering, i.e., scattering of a massive nonrelativistic particle from an infinitely thin magnetic flux tube. It was pointed out by Bergman and Lozano [12] that such a scattering process can be described by a vacuum field theory consisting of a nonrelativistic \( |\psi|^4 \)-theory also coupled—albeit minimally—to a Chern-Simons term.

The scattering of sound waves from a vortex, formulated as a quantum mechanical problem, was first studied by Pitaevskii in the Born approximation [13]. (For a recent account and an extensive list of references see Ref. [14], where also the close connection with Aharonov-Bohm scattering is discussed.) It was found that in this approximation, the two-dimensional scattering amplitude \( f(\theta, k) \) is given by
\[ f(\theta, k) = \frac{1}{2} \sqrt{\frac{k}{2\pi \hbar c}} \frac{e^{i\theta}}{1 - \cos \theta}, \tag{16} \]
where \( \theta \) is the angle between the incoming and the scattered sound wave and \( k \) the wave number of both waves. For small scattering angles, the Born approximation breaks down.
At small energies and momenta, we can ignore the higher-order terms in the Lagrangian (1) and restrict ourselves to terms at most quadratic in the phonon field:

\[
\mathcal{L}^{(2)} = \frac{n}{mc^2} \left\{ \frac{1}{2} (\hbar \partial_0 \varphi)^2 - \frac{e^2}{2} (\hbar \nabla \varphi)^2 + e a_0 \left[ \hbar \partial_0 \varphi + \frac{1}{2m} (\hbar \nabla \varphi)^2 \right] \right\} + \mathcal{L}_{CS}.
\] (17)

If we again introduce external vortices by replacing the Chern-Simons vector field \( a \) with (13), the field equation for \( \varphi \) becomes

\[
\partial_0^2 \varphi - c^2 \nabla^2 \varphi = 2 \frac{e}{mc} \partial_0 \nabla \cdot a_{\text{ext}}
\] (18)

which is the equation found by Pitaevskii [13] with the contribution from the vortex motion ignored.

Let us return to the theory (17) without external vortices. In the gauge \( \nabla \cdot a = 0 \), the nonzero components of the Chern-Simons vector-field propagator are given by

\[
0 \quad p \quad i : \quad i \hbar G_{i0}(p_0, p) = -i \hbar G_{0i}(p_0, p) = -\hbar \epsilon_{ij} \frac{p_j}{p^2},
\] (19)

while the phonon propagator reads

\[
p \quad : \quad i \hbar G(p_0, p) = \frac{mc^2}{n} \frac{i \hbar}{p_0^2 - c^2 p^2 + i \eta},
\] (20)

where \( \eta \) is a small positive constant that has to be taken to zero after the loop integration over the energy \( p_0 \) has been carried out. The vertices of the theory are

\[
\begin{align*}
0 & \quad p \quad i \quad : \quad -\frac{i}{\hbar} \frac{e \bar{n}}{m^2 c^2} (p_0 q_i + p_i q_0) \quad (21) \\
0 & \quad p \quad i \quad : \quad -\frac{i}{\hbar} \frac{e \bar{n}}{m^2 c^2} p \cdot q \quad (22) \\
p & \quad : \quad \frac{1}{\hbar} \frac{e \bar{n}}{mc^2} p_0. \quad (23)
\end{align*}
\]
The tree graphs for the phonon-phonon scattering are depicted in Fig. 1. Of these, only the first graph survives; the second graph is identical zero, while the third and fourth graph cancel each other. Before evaluating the first graph, let us pause for a moment and compare it with a similar graph where instead of a Chern-Simons vector-field quantum, a phonon is exchanged. The required vertices are contained in the higher-order terms of the theory which have been ignored in (17). A dimensional analysis of the type commonly used in the context of effective field theories [15], shows that the exchange of a phonon can be ignored in comparison to the exchange of a vector-field quantum provided

$$k < \xi \tilde{n}, \tag{24}$$

where $k$ is the wave number of the incoming and outgoing phonons. For superfluid $^4$He this implies that the wave number should be smaller than about 10 nm, which happens to coincide with the previous bound [5]. This conclusion also applies to graphs involving an exchange of multiple quanta.

Let us now evaluate the first graph in Fig. 1. To determine the normalization of the external lines we consider the energy density $\mathcal{E}$ of the free phonon theory,

$$\mathcal{E} = \frac{\tilde{n}}{2mc^2} \left[ (\hbar \partial_0 \varphi)^2 + c^2 (\hbar \nabla \varphi)^2 \right]. \tag{25}$$

The incoming and outgoing plane waves have the form

$$\varphi(t, x) = \zeta \left[ e^{i(\omega t - k \cdot x)} + e^{-i(\omega t - k \cdot x)} \right], \tag{26}$$

where $\zeta$ is the normalization constant which we fix by demanding that the integrated energy density yields $\hbar \omega = \hbar ck$, the energy of such a wave,

$$\hbar \omega = \int d^2 x \mathcal{E} = 2\zeta^2 \frac{\tilde{n} \hbar^2 \omega^2}{mc^2} V, \tag{27}$$

where $V$ is the volume of the system. In this way, we obtain for the normalization constant

$$\zeta = \sqrt{\frac{mc^2}{2\hbar \omega \tilde{n} V}} \tag{28}$$

The first graph in Fig. 1 is now easily evaluated, with the result for the scattering
amplitude $A(\theta, k)$

\[
\begin{align*}
A(\theta, k) &= - \frac{\hbar k}{2 m^2 c} \frac{\sin \theta \cos \theta}{1 - \cos \theta},
\end{align*}
\]

(29)

where in the frame chosen by us, the energy-momenta of the incoming and outgoing phonons satisfy the equations

\[
p_0 = q_0 = p'_0 = q'_0, \quad |p| = |q| = |p'| = |q'|, \quad p = -q, \quad p' = -q'.
\]

(30)

The scattering amplitude calculated here agrees with the result of Pitaevskii. The difference in kinematical factors in (16) and (29) is due to a different definition of the scattering amplitude in quantum mechanics and in quantum field theory.

For identical particles, we have to add to the tree graph we just calculated the graph with the two outgoing lines exchanged. This yields the same result as in (29) with $\theta \rightarrow \theta - \pi$. Adding the two contributions, we find

\[
A^{\text{tot}}(\theta, k) = i \frac{\hbar k}{m^2 c} \cot \theta,
\]

(31)

which diverges for both $\theta = 0$ and $\pi$.

A dimensional analysis of the loop corrections to this tree result reveals that the expansion parameter is given by $(\xi k)^2$. This is in agreement with a conclusion of Sonin, which was based on the analogy between the scattering of sound waves from a vortex and Aharonov-Bohm scattering. It shows that in the region where the field theory is applicable, the higher loop corrections are small.

In conclusion, we have shown that the quantum field theory proposed here to describe zero-temperature vortices in a superfluid film, correctly reproduces previously known results. In a future publication, we plan to report the one-loop correction to the scattering amplitude considered here and discuss the renormalizability of the theory.

Acknowledgments

The authors gratefully acknowledge discussions with D. Arovas and H. Kleinert. This work was performed as part of a scientific network supported by the European Science Foundation, an association of 62 European national funding agencies (see network’s URL, http://defect.unige.ch/defect.html).

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