Field Line Solutions
to the Einstein-Maxwell Equations

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Abstract
In this paper we are going to review the gravitating electromagnetic field in the 1+3 formalism on a general hyperbolic space-time manifold. Next, we are going to discuss in details the recent results on the existence of the local field line solutions to the Einstein-Maxwell equations that generalize the Rañada solutions from the flat space-time. The global field line solutions do not always exist since the space-time manifold could impose obstructions to the global extension of various geometric objects necessary to build the fields. One example of gravitating field line solution is the Kopiński-Natário field which it is discussed in some detail.

Keywords: Einstein-Maxwell's equations. Local field line solutions. Generalization of Rañada solutions.
1 Introduction

Recently, there has been an increasing interest in the topological aspects of the electromagnetism. This research line was pioneered by Trautman and Rañada in the seminal papers [1, 2, 3]. Since then, many developments and applications of the topological electromagnetic fields have been advanced ranging from the atomic particles in plasma to colloidal matter and to the realization of the mathematical Hopf mappings in classical and quantum electrodynamics. For a recent list of references and an updated review of these concepts we refer to the paper [4] and to the pedagogical introductory chapter from the present volume.

Most of the research on the topological electromagnetic fields has been done in the context of the classical electrodynamics applied to low energy phenomena or in vacuum. However, the electromagnetic fields have a larger range of applications which include cosmological and astrophysical medium energy processes. Generalizing the topological electromagnetism to these systems poses new challenges from the physical and mathematical point of view since in these cases one has to deal with the gravitating electromagnetic fields. Very recently, important steps in analysing the topological properties of gravitons as well as of higher spin particles have been taken in [5, 6, 7, 8, 9]. For the topological electromagnetism, these works are important since not only they generalize the topological properties of the classical photons but also contain new information on spin-2 field viewed as a particular case of higher spin fields. More directly related are the results from [10] where it was proved the existence of local field line solutions to the Einstein-Maxwell equations that are a direct generalization of the Rañada’s solutions to hyperbolic space-time manifolds. Also, a particular topological solution to the equations of motion of the electromagnetic field in the static Einstein universe was given in [11]. It is worthwhile mentioning the work [12] in which the electromagnetic knots on hyperbolic manifolds are treated formally from the point of the foliation theory.

In the present work we are going to give a rather pedagogical introduction to the problem of the topological gravitating electromagnetic fields. Since in the presence of gravity the space-time assumes the properties of a curved manifold, the difficulties in these systems are inherent from the general formulation of the field theories on curved manifolds. While in the flat space-time the topological electromagnetic solutions to the linear Maxwell’s equations can be calculated in vacuum, in the presence of the gravitational field the vacuum contains components of the metric that do not decouple, in general, from the electromagnetic field. In this case, topological fields like knots and tori are better described in terms of electric and magnetic field...
lines which make more transparent their geometrical and topological properties. However, the decomposition of the electromagnetic field in electric and magnetic components is not possible on general curved manifolds. Nevertheless, the decomposition is natural if the manifold is hyperbolic. One useful formulation of the Einstein-Maxwell equations is given by the $1 + 3$ - formalism that reproduces the Maxwell's equations from the Minkowski space-time in the no gravity limit [13]. By adopting this formalism, it has been shown in [10] that the Rañada’s solutions have a precise local generalization on hyperbolic space-time. However, one cannot expect global solutions to exist on general hyperbolic manifolds. Indeed, global field line solutions depend on the global properties of the space-time manifold that could contain obstructions to the extension of mathematical objects such as differential forms. Therefore, the global solutions are remarkable and they are associated to particular manifolds. One interesting solution that is known in the literature in the static Einstein universe that has the $\mathbb{R} \times S^3$ topology was given in [11].

The Kopiński-Natário field is only time-dependent and describes a radiation field, when viewed from the point of view of the Minkowski space-time.

This work is organized as follows. In Section 2 we are going to briefly review the basic features of the Maxwell’s equations in flat space-time in the formulation in terms of differential forms that is the most adequate one to the generalization to curved space-times. Since our main focus is the generalization of the Rañada’s solutions to the gravitating electromagnetic field, we will introduce in Section 3 the $1 + 3$-formalism which makes the correspondence between the equations of motion of the electromagnetic field in flat and in curved space-time, respectively, more transparent. This formalism will be used further to formulate the Einstein-Maxwell equations on hyperbolic manifolds. In Section 4 we will revisit the result from [10] that demonstrate that local field line solutions to the Einstein-Maxwell equations exist on hyperbolic manifolds in the sense of the Rañada’s fields. Lastly, we present in Section 5 the Kopiński-Natário solution in the Einstein space. We collect some useful mathematical definitions and properties in the Appendix.

The units used throughout this paper are natural with $c = G = 1$.

2 Field line solutions in flat space-time

In this section we are going to review the Maxwell’s equations in the Minkowski space-time and establish our notations. The material presented here is well-known and can be found in standard texts on classical electrodynamics, e. g. [15]. Then we will briefly recall the Rañada’s field line solutions given in [2, 3]. A more complete review was presented elsewhere in this volume.
2.1 The Maxwell’s equations

Consider the Minkowski space-time $\mathbb{R}^{1,3} = \{\mathbb{R}^4, \eta\}$, where $\eta$ is the Minkowski pseudo-metric tensor of signature $(-, +, +, +)$. The events from $\mathbb{R}^{(1,3)}$ are identified as usual by the coordinate four-vectors $x^\mu = (t, \mathbf{x}) = (t, x^i)$. The electromagnetic field on $\mathbb{R}^{1,3}$ can be described in terms of the four-vector potential $A^\mu = (\phi, \mathbf{A})$, where $\phi$ is the scalar potential and $\mathbf{A} = (A^1, A^2, A^3)$ is the (space-like) three-dimensional vector potential. Then the electromagnetic field is given by the rank-2 antisymmetric tensor defined by the following relation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

The tensor $F_{\mu\nu} = -F_{\nu\mu}$ has 6 independent degrees of freedom which are identified with the components of the electric and magnetic vector fields $\mathbf{E}$ and $\mathbf{B}$, respectively, as follows

$$E_i = \partial_i A_0 - \partial_0 A_i, \quad B_i = \varepsilon_{ijk} \partial_j A_k, \quad (2)$$

where the indices $i, j, k = 1, 2, 3$ denote the space-like components. The Hodge dual (pseudo)-tensor associated to $F_{\mu\nu}$ is defined by the following relation

$$\star F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} F_{\rho\sigma}. \quad (3)$$

One can write the electromagnetic field in terms of differential forms on $M$. This formulation will prove to be useful later when we will discuss the Einstein-Maxwell equations. Let us introduce the following electromagnetic field 2-form

$$F = B + E \wedge dx^0, \quad (4)$$
$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (5)$$

where $B$ is the magnetic 2-form, $E$ is the electric 1-form and $\wedge$ is the wedge product. Then the Hodge star operator is defined as

$$\star : \Omega^k(\mathbb{R}^{1,3}) \to \Omega^{4-k}(\mathbb{R}^{1,3}), \quad (6)$$
$$\omega \wedge (\star \sigma) = \langle \omega, \sigma \rangle n, \quad (7)$$

where $n$ is an unitary vector and $\langle \cdot, \cdot \rangle$ is the scalar product. Here, we have denoted by $\Omega^k(M)$ the linear space of $k$-forms on the manifold $M$. Then the Maxwell’s equations take the following form

$$dF = 0, \quad (8)$$
$$\star d \star F = J, \quad (9)$$
where \( J = J_\mu dx^\mu \) is the 1-form current density [16]. The equations (8) and (9) are covariant and so their symmetries are given by the Poincaré group of the Minkowski space-time.

In what follows, we are going to discuss a particular type of solutions to the Maxwell’s equations in the vacuum defined by the absence of currents \( J = 0 \), namely,

\[
\begin{align*}
  dF &= 0, \\
  \star d \star F &= 0,
\end{align*}
\]

(10) (11)

In this case, the Maxwell’s equations have an additional symmetry which is the electromagnetic duality given by the interchange of the forms \( F \) and \( \star F \), respectively,

\[
F \leftrightarrow \star F.
\]

(12)

It is important to recall here one property of the Hodge star operator. If \( \omega \) is an arbitrary \( k \)-dimensional form on a manifold \( M \) of dimension \( n \), then

\[
\star^2 \omega = (-1)^{k(n-k)} \text{sign} (\det \eta) \omega.
\]

(13)

From the (14) one can deduce that the operator \( \star^2 \) has the eigenvalues \( \pm i \) when acting on the 2-forms. Therefore, the fields \( F \) belong to one of the two classes of self-dual or anti-self-dual 2-forms. Due to the linearity of the Maxwell’s equations (10) and (11), respectively, a general solution in vacuum is a superposition of solutions from each class

\[
F = F_+ + F_-, \quad \star F_\pm = \pm i F.
\]

(15)

The above equations shows that \( F \) is a complex 2-form. It is a simple exercise in electrodynamics to show that the equations (10) and (11) reproduce the three-dimensional Maxwell’s equations. This can be seen by decomposing the differential forms defined on the Minkowski space-time with respect to its global spatial foliation of leaves isomorphic to \( \mathbb{R}^3 \). The derived electromagnetic forms \( dF \) and \( \star d \star F \) have the following decomposition

\[
\begin{align*}
  dF &= dB + dE \wedge dx^0, \\
  \star d \star F &= -\partial_0 E - \star d \star E \wedge dx^0 + \star d \star E \wedge dx^0,
\end{align*}
\]

(16) (17)
where $d$ is the three-dimensional differential derivative and $\star$ is the three-dimensional Hodge star operator. Then the Maxwell’s equations take the following form

\begin{align*}
dE + \partial_0 B &= 0, \\
dB &= 0, \\
\star d \star E &= \rho, \\
\star d \star B - \partial_0 E &= J,
\end{align*}

where $J = J_i dx^i$. In the vacuum the equations (21) take the form

\begin{align*}
dE + \partial_0 B &= 0, \\
dB &= 0, \\
\star d \star E &= 0, \\
\star d \star B - \partial_0 E &= 0.
\end{align*}

The formalism of the differential forms allows one to calculate the properties of the electromagnetic field in a coordinate independent manner. Also, the equations obtained in this formalism are more compact. For more details we refer to the chapter from this volume or to the excellent book [16].

### 2.2 Field line solutions

In order to describe the dynamics of the electromagnetic field in vacuum, the equations (22) - (25) must be solved with appropriate boundary conditions. As it is well known, the most general solution to the above set of equations is given by a superposition of monocromatic waves that satisfy the dispersion relation $k^2 = \omega^2(k)$ where $k$ is the wave vector and $\omega(k)$ is the correspondig wave frequency in the infinite empty space $\mathbb{R}^3$ [15]. Since these solutions are known for a long time, it came as a surprise when Trautman and, independently, Rañada, published in their seminal works [1, 2, 3] a new type of solution to the Maxwell’s equations in vacuum that displays non-trivial topological properties. This kind of solutions has a new non-zero topological charge which is the link number between the electric and magnetic field lines, respectively. Since these solutions that can come in to a number of types according to their topologies, like Hopf knots and tori, can be all expressed in terms of field lines we call them field line solutions. A more extended review is presented in another chapter from this book. Here, we are going only to recall the general form of the Rañada’s solutions which is relevant for the rest of our discussion.
The Rañada’s solution is described in terms of two complex scalar fields $\phi$ and $\theta$ as follows [4]

$$\phi : M \rightarrow \mathbb{C}, \quad \theta : M \rightarrow \mathbb{C}. \quad (26)$$

The main role of these fields is to serve as a backbone for the topology of the field lines in $\mathbb{R}^3$ in the following sense: the electric and magnetic field lines are the level curves of $\phi$ and $\theta$, respectively. Then one can show that the solutions to the Maxwell’s equations with the above property have the following form

$$F_{\mu\nu} = g(\bar{\phi}, \phi) \left( \partial_\mu \bar{\phi} \partial_\nu \phi - \partial_\nu \bar{\phi} \partial_\mu \phi \right), \quad (27)$$

$$\star F_{\mu\nu} = f(\bar{\theta}, \theta) \left( \partial_\mu \bar{\theta} \partial_\nu \theta - \partial_\nu \bar{\theta} \partial_\mu \theta \right). \quad (28)$$

The field line solutions given by the equations (27) and (28) are parametrized by two smooth functions $g$ and $f$ on $\theta$ and $\phi$. The solutions given in terms of the electromagnetic tensor and its dual can also be written in terms of differential forms. Their explicit expressions are given by the following equations

$$F = -\varepsilon_{jkl} B_j \, dx^k \wedge dx^l + E_j \, dx^j \wedge dx^0, \quad (29)$$

$$\star F = \varepsilon_{jkl} E_j \, dx^k \wedge dx^l + B_j \, dx^j \wedge dx^0. \quad (30)$$

The Rañada’s solution is an electromagnetic field of the form (27) and (28) with the following components

$$E_j = \frac{\sqrt{a}}{2\pi i} \left( 1 + |\theta|^2 \right)^{-2} \varepsilon_{jkl} \partial_k \bar{\theta} \partial_l \theta, \quad (31)$$

$$B_j = \frac{\sqrt{a}}{2\pi i} \left( 1 + |\phi|^2 \right)^{-2} \varepsilon_{jkl} \partial_k \bar{\phi} \partial_l \phi. \quad (32)$$

Here, $a$ is a parameter that fixes the physical dimension of the solution. The fields $E$ and $B$ defined as above correspond to particular choices of the functions $f$ and $g$, respectively. Since in the vacuum the electromagnetic field is subjected to the duality symmetry, the self-duality condition in three-dimensions imposes the following constraints on the parameters [4]

$$\left( 1 + |\phi|^2 \right)^{-2} \varepsilon_{jmn} \partial_m \phi \partial_n \bar{\phi} = \left( 1 + |\theta|^2 \right)^{-2} \left( \partial_0 \bar{\theta} \partial_j \theta - \partial_j \theta \partial_0 \bar{\theta} \right), \quad (33)$$

$$\left( 1 + |\theta|^2 \right)^{-2} \varepsilon_{jmn} \partial_m \bar{\theta} \partial_n \theta = \left( 1 + |\phi|^2 \right)^{-2} \left( \partial_0 \bar{\phi} \partial_j \phi - \partial_j \phi \partial_0 \bar{\phi} \right). \quad (34)$$

Some comments are in order here. Firstly, note that the Rañada’s solutions satisfy the orthogonality property

$$E_j B_k \delta_{jk} = 0. \quad (35)$$
Secondly, since the fields given by the relations (27) and (28) satisfy the Maxwell’s equations, they conserve the charges associated to the corresponding symmetries that form the Poincaré’s group and the $U(1)$ group. However, there are new charges associated to the topology of the field lines. The relevant topological observables associated with these charges are the helicities of the electromagnetic field. The helicities of observables are defined by the following relations

$$H_{ee} = \int d^3x \delta_{ij} E_i C_j = \int d^3x \varepsilon_{jkl} C_j \partial_k C_l,$$  \hspace{1cm} (36)

$$H_{mm} = \int d^3x \delta_{ij} B_i A_j = \int d^3x \varepsilon_{jkl} A_j \partial_k A_l,$$  \hspace{1cm} (37)

where $\mathbf{A}$ and $\mathbf{C}$ are the corresponding magnetic and electric potential vectors

$$E_j = \varepsilon_{jkl} \partial_k C_l, \quad B_j = \varepsilon_{jkl} \partial_k A_l.$$  \hspace{1cm} (38)

The fact that the helicities correspond to some topological objects is a consequence of the fact that the field line solutions are defined in terms of the complex scalars $\phi$ and $\theta$. If the electromagnetic solutions carry finite energy, then they should take zero value for $|\mathbf{x}| \to \infty$. In order for that to happen, the fields $\phi(t, \mathbf{x})$ and $\theta(t, \mathbf{x})$ should have the same asymptotic behaviour as the energy. This implies that the fields $\phi$ and $\theta$ can be interpreted as complex functions on $\mathbb{R} \times S^3$ or, equivalently, as two one-parameter families of maps $S^3 \to \mathbb{C}$. On the other hand, there is a natural identification of $\mathbb{C} \simeq \mathbb{R}^2$. This identification can be further refined if the inverse maps $\phi^{-1}$ and $\theta^{-1}$ do not depend on the complex phases of their arguments. The functions that satisfy this property can be interpreted as maps $S^3 \to S^2$. However, these are the Hopf’s maps that are characterized by the topological number called the Hopf’s index. It turns out that when this index is expressed as a Chern-Simons integral it takes the same form as the helicities defined above; For a detailed discussion of these aspects see [4].

### 3 The 1+3 - formalism of General Relativity

In this section we review very briefly those basic concepts of the 1+3 - formalism of the General Relativity necessary to discuss the Einstein-Maxwell equations on hyperbolic manifolds in the same way as we did with the Maxwell’s equations in the flat space-time. In our presentation we will follow mainly [13] with minor modifications of the notations.
3.1 Space-time foliation

According to the General Relativity, the gravity is a consequence of the non-trivial geometry of the space-time viewed as a four-dimensional differential manifold $M$ endowed with a metric tensor field $g = g_{\mu\nu}$ that locally can be approximated by the Minkowski space-time [14]. From the first principles, the metric $g_{\mu\nu}$ has to be a solution of the Einstein’s equations either in the presence of matter or with no matter at all. In this context, the gravitating electromagnetic field should be represented in terms of covariant equations on a differential manifold whose metric satisfies the Einstein’s equations. This problem is more general than our scope which is to investigate the existence of field line solutions of the gravitation electromagnetic field, therefore we will not dwell into the most genera aspects of the Einstein-Maxwell theory. Our main goal is to see whether there are locally any Rañada type solutions which reduce to the fields discussed in the previous section in the flat space-time limit. The existence problem is not well posed globally without further assumptions since an arbitrary manifold $M$ may have inner obstructions to defining globally the mathematical objects necessary for the analysis, such as the electromagnetic 2-form field.

Two important details of the mathematical structure of the Maxwell’s equations presented in the previous section are: i) the splitting between the electric and the magnetic fields, necessary to define the electric and magnetic field lines, and ii) the explicit time-evolution of the system during which the helicities are conserved. This structure can be reproduced in the presence of gravity if the manifold $M$ is globally hyperbolic, that is if it admits a foliation in terms of an one-parameter family of space-like manifolds $\Sigma \simeq \mathbb{R}^3$ parametrized by a global time $t \in \mathbb{R}$. That means that $M$ has the topology $M = \mathbb{R} \times \Sigma$. The method of splitting the covariant equations on $M$ with respect to the foliation is called the $1+3$ - formalism. In order to make the discussion more concrete, we need to introduce these concepts in a more formal way. For a short review of some definitions and properties see the Appendix and for more details on the 1+3 - formalism see [13].

Let $(M, g)$ be a four-dimensional space-time manifold $\mathcal{M}$ endowed with a smooth metric tensor field $g$ of signature $(-, +, +, +)$ that has two properties: 1) $M$ is time-orientable and 2) $M$ is hyperbolic. Then there is a globally defined scalar field $t$ such that $M$ is a foliation generated by $t$ with the leaves defined as follows

$$\Sigma_t = \{ p \in \mathcal{M} : t(p) = t = \text{constant} \}.$$  \hspace{1cm} (39)

Equivalently, one can write this property as $\mathcal{M} \simeq \mathbb{R} \times \Sigma$. There are two remarkable vector fields and one scalar function on $M$ that are induced by...
the above structure of \( M \), namely: the normal vector field \( n \), the normal evolution vector field \( m \) and the lapse function \( N \). These are defined as follows

\[
n := -N \nabla t, \quad m := \sigma n, \quad N := \left[ -g_{\mu \nu} \nabla^\mu t \nabla^\nu t \right]^{-\frac{1}{2}}. \tag{40}
\]

Since we want to identify the leaves \( \Sigma_t \) at every value of the parameter \( t \) and at every point \( p \) with the space-like submanifold of \( M \) and \( t \) with the time vector field throughout the entire \( M \), the gradient \( \nabla t \) must be time-like.

In order to be able to make physical measurements at a point \( p \in \Sigma_t \) one needs to define local coordinates with time- and space-like characteristics, respectively. Since the foliation of \( M \) was constructed with this thought in mind, we can choose the local coordinates \( x^\mu = (t, x^i) \) adapted to the foliation. Here, the Latin indices \( i, j = 1, 2, 3 \) correspond to the objects defined on the leaf. The time evolution of a physical system that contains the event \( p \) at \( t \) takes place along the field line of the vector field \( \partial_t \) defined by the following equation

\[
\partial_t := m + \beta, \quad g_{\mu \nu} \beta^\mu n^\nu = 0, \tag{41}
\]

where \( \partial_t \) is the derivative along the adapted time and \( \beta \in T_p(M) \) is the shift vector corresponding to the displacement of the origin of space-like coordinates on infinitesimally closed leaves.

The above decomposition of the manifold \( M \) which corresponds to its topological structure as a foliation is reflected into the geometrical properties by the decomposition of the metric tensor \( g \). Indeed, one can define an induced metric \( \gamma \) on each three-dimensional leaf \( \Sigma_t \) by the canonical reduction

\[
\gamma = g|\Sigma_t \iff \gamma_{ij} = g_{ij}. \tag{42}
\]

The equation (42) can be interpreted as the action of the projector \( P^{\mu \nu} \) on the tensor \( g \) where

\[
P^{\mu \nu} := g^{\mu \nu} + n^\mu n^\nu, \quad P_{\mu \nu} n^\nu = 0. \tag{43}
\]

This second interpretation is more operational as it allows one to project other mathematical objects onto \( \Sigma_t \).

If we denote the components of the covariant derivative on the leaf by \( D_i \), the three-dimensional connection associated to it is torsionless providing that the metric is compatible with the covariant derivative \( D^i \gamma_{ij} = 0 \). Since the leaf \( \Sigma_t \) is embedded into the four-dimensional manifold \( M \), one can define its exterior curvature (see the Appendix)

\[
K_{ij} := P^{\mu \nu} P^\mu_j \nabla_\mu n_\nu, \quad K := \gamma^{ij} K_{ij}. \tag{44}
\]
where \( \nabla_\mu \) are the components of the four-dimensional covariant derivative compatible with the metric \( g_{\mu\nu} \). The line element can be easily written as follows
\[
ds^2 = -(N dt)^2 + \gamma_{ij} \left( dx^i + \beta^i dt \right) \left( dx^j + \beta^j dt \right). \tag{45}\]
The above decomposition can be applied to the covariant gravitational and electromagnetic fields and field-derived objects, too.

### 3.2 Einstein-Maxwell equations

The equations that govern the dynamics of the gravitating electromagnetic field can be obtained from the following action functional
\[
S[g, A] = - \int d^4 x \sqrt{-g} \left( F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right), \tag{46}\]
where
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{47}\]
\[
 j^\mu = \frac{1}{\sqrt{-g}} \partial_\nu D^{\mu\nu}, \tag{48}\]
\[
 D^{\mu\nu} = \sqrt{-g} F^{\mu\nu}. \tag{49}\]

By applying the variational principle to the action (46) on obtains the Einstein-Maxwell equations. In the 1+3 - formalism presented above the field strength tensor \( F_{\mu\nu}(t, x) \) can be decomposed locally in to the electric \( E^\mu(t, x) \) and magnetic \( B^\mu(t, x) \) components, respectively, as follows
\[
E^\mu(t, x) = F^\mu(t, x)n^\nu(t, x), \quad B^\mu(t, x) = \frac{1}{2} \varepsilon_{\mu\nu\sigma}(t, x) F^{\nu\sigma}(t, x). \tag{50}\]
Here, \( \varepsilon_{\mu\nu\sigma}(t, x) \) is the contracted four-dimensional Levi-Civita tensor. Since the following relations hold
\[
E^\mu(t, x)n^\mu(t, x) = 0, \quad B^\mu(t, x)n^\mu(t, x) = 0, \tag{51}\]
one concludes that \( E^\mu(t, x), B^\mu(t, x) \in T_p(\Sigma_t) \), that is the electric and magnetic vectors are tangent to the leaf at \( p \). By using the above equations one can show that the field strength has the following local form
\[
F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \varepsilon_{\mu\nu\rho\sigma} n^\rho B^\sigma. \tag{52}\]
Note that in what follows we are going to discuss the local properties of the Einstein-Maxwell equations but in order to simplify the notations we...
are going to drop off the local space-time coordinates unless their explicit presence is strictly necessary. With these remarks, we can write down the Einstein-Maxwell equations without sources. They have the following form

\begin{align}
L_m E^i - N K E^i - \varepsilon^{ijk} D_j (N B_k) &= 0, \\
L_m B^i - N K B^i + \varepsilon^{ijk} D_j (N E_k) &= 0, \\
D_i E^i &= 0, \\
D_i B^i &= 0.
\end{align}

Note that the equations (53) and (54) are dynamical and correspond to the Faraday’s and Ampère’s laws, respectively, while the equations (55) and (56) are constraints on the fields $E^i$ and $B^i$ and generalize the Gauss’ laws.

### 4 Existence of local field line solutions

The 1+3 - formalism introduced in the previous section allows us to separate the covariant electromagnetic field into the electric and the magnetic fields, respectively. This sought for decomposition is useful if one wants to generalize the field line solutions from the Minkowski space-time reviewed in the Section 2. In the most general case of an arbitrary hyperbolic space-time manifold $(M, g)$, the field line solutions are local. However, for particular manifolds $M$ one might be able to find global field line electromagnetic fields. The proof of the existence of local field line solutions to the Einstein-Maxwell equations was given in [10] which we will follow closely.

#### 4.1 Field line solutions

In order to find magnetic field line solutions we consider the equations (54) and (56), respectively. Since the problem is local, one needs to work within an (arbitrary) neighbourhood $U_p \in M$ where $p \in \Sigma_t$. Also, it is necessary to define the adapted coordinates $(t, x)$ in $U_p$ as discussed in the previous section.

As we have seen in previously, the field lines from the Minkowski space-time are defined in terms of a scalar field. Therefore, we also introduce a scalar field $\phi : U_p \to \mathbb{C}$ on $U_p$. The equation of motion of $B^i(t, x)$ is given by the equation (54) and at all times the magnetic field should obey locally the constraint given by the equation (56). This constraint suggests the following ansatz

\begin{equation}
B^i(t, x) = f(t, x) \varepsilon^{ijk}(t, x) D_j \phi(t, x) D_k \tilde{\phi}(t, x).
\end{equation}
Here, \( f(t, x) \) is a smooth but otherwise arbitrary field on \( U_p \). By plugging \( B^i(t, x) \) into the equation (56) we can verify that the functions \( f(t, x) \)'s that depend on the space-time coordinates implicitly via their arguments \( f(\phi(t, x), \bar{\phi}(t, x)) \) are admissible in the ansatz (57).

Let us look at the electric component corresponding to the magnetic field (57). The Ampère's law (54) rules the evolution of \( B^i(t, x) \) in terms of metric and the components \( E^i(t, x) \) of the electric field. It turns out that the electric field should have the following form

\[
E^i(t, x) = \frac{f(t, x)}{N(t, x)} \left[ (\mathcal{L}_m \bar{\phi}(t, x)) D^i \phi(t, x) - (\mathcal{L}_m \phi(t, x)) D^i \bar{\phi}(t, x) \right]. \tag{58}
\]

The fields proposed in the equations (57) and (58) must verify the equation of motion (54). The verification can be done by plugging the fields into the Ampère's law. The most rapid way to prove that the equation (54) is satisfied is to show that both left and right hand sides of it take the same form, namely,

\[
\varepsilon^{ijk} f \left( \partial_i \partial_j \phi - \beta^r \partial_i \partial_j \beta^r - \partial_i \phi \partial_j \beta^r \right) \partial_k \bar{\phi}
+ \varepsilon^{ijk} f \left( \partial_i \partial_k \bar{\phi} - \beta^r \partial_i \partial_k \bar{\phi} - \partial_i \bar{\phi} \partial_k \beta^r \right) \partial_j \phi. \tag{59}
\]

This result can be proved by the direct calculation of each side of the equation. For more details we refer to [10].

We can conclude that the fields \( B^i(t, x) \) and \( E^i(t, x) \) given by the relations (57) and (58) are field line solutions of two of the Einstein-Maxwell equations (54) and (56), respectively. These fields satisfy the orthogonality property

\[
\gamma^{ij}(t, x) E_i(t, x) B_j(t, x) = 0, \quad \forall p \in U_p. \tag{60}
\]

The solution given above offers a clear geometric picture to the magnetic field whose field lines are the level lines of the complex function \( \phi \), as in the flat space-time. However, the interpretation of the electric field in terms of field lines is not transparent.

In order to remediate that, we recall that the electric field lines were obtained in flat space-time from the duality between the electric and magnetic fields. The same duality also holds in curved space-time as it has been shown in [17]. In the present case, this duality allows one to find solutions to the Faraday’s law and electric Gauss’s law given by the equations (53) and (55), respectively. To this end, we introduce the complex field \( \theta : U_p \to \mathbb{C} \) and note that the constraint on the electric field is the same as the one on the magnetic field. This allows us to make the same assumption about the electric field line, namely

\[
E^i(t, x) = g(t, x) \varepsilon^{ijk}(t, x) D_j \bar{\theta}(t, x) D_k \theta(t, x), \tag{61}
\]
where \( g(t, x) \) is an arbitrary real smooth field on \( U_p \) that depends on the adapted coordinates implicitly \( g(\theta(t, x), \bar{\theta}(t, x)) \). Then the magnetic field should have the following form

\[
B^i(t, x) = \frac{g(t, x)}{N(t, x)} \left[ (\mathcal{L}_m \bar{\theta}(t, x)) D^i \theta(t, x) - (\mathcal{L}_m \theta(t, x)) D^i \bar{\theta}(t, x) \right].
\]

(62)

Since the equation of motion and the constraints have very similar structures with the ones discussed in the case of the magnetic field lines, the proof that the fields given by the equations (61) and (62) are solutions to the second set of Einstein-Maxwell equations (55) and (53) can be done in the same manner.

As in the case of the field line solutions in flat space-time discussed somewhere else in this volume, the electromagnetic duality implies the existence of a relationship between the scalar fields defined by the following equations [10]

\[
f(\phi, \bar{\phi}) \varepsilon^{ijk} D_j \phi D_k \bar{\phi} = \frac{g(\theta, \bar{\theta})}{N} \left[ (\mathcal{L}_m \bar{\theta}) D^i \theta - (\mathcal{L}_m \theta) D^i \bar{\theta} \right],
\]

(63)

\[
g(\theta, \bar{\theta}) \varepsilon^{ijk} D_j \bar{\theta} D_k \theta = \frac{f(\phi, \bar{\phi})}{N} \left[ (\mathcal{L}_m \bar{\phi}) D^i \phi - (\mathcal{L}_m \phi) D^i \bar{\phi} \right].
\]

(64)

These equations represent non-linear local constraints on the functions \( f \) and \( g \) that should be obeyed at all times. There are no other constraints on the functions \( f \) and \( g \) which shows that there are actually families of field line solutions. One particular solution is interesting since in the flat space-time limit it reduces to the Rañada’s field.

### 4.2 Local generalization of Rañada’s solutions

As discussed in another place in this volume, one interesting field line solution in flat space-time was given by Rañada in [2, 3]. This solution is characterized by the following choice of functions

\[
f = \frac{1}{2\pi i} \frac{1}{(1 + |\phi|^2)^2},
\]

(65)

\[
g = \frac{1}{2\pi i} \frac{1}{(1 + |\theta|^2)^2}.
\]

(66)

It is interesting to see what is the generalization of the electromagnetic field constructed with the above functions to the gravitating electromagnetism. To this end, we use the equations (65) and (66) to calculate the corresponding electric and magnetic field line solutions to the Einstein-Maxwell equations.
discussed in the previous subsection. After some algebraic manipulation the result takes the following form

\[ B^i(t, x) = \varepsilon^{ijk} D_j \alpha_1(t, x) D_k \alpha_2(t, x), \]  
\[ E^i(t, x) = \varepsilon^{ijk} D_j \beta_1(t, x) D_k \beta_2(t, x), \]

(67)  

(68)

where \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are real scalar fields related to the complex fields \( \phi \) and \( \theta \) by the following equations

\[ \alpha_1 = \frac{1}{1 + |\phi|^2}, \quad \alpha_2 = \frac{\Phi}{2\pi}, \quad \phi = |\phi|e^{i\Phi}, \]  
\[ \beta_1 = \frac{1}{1 + |\theta|^2}, \quad \beta_2 = \frac{\Theta}{2\pi}, \quad \theta = |\theta|e^{i\Theta}. \]

(69)  

(70)

The fields given by the relations (67) and (68) represent the local generalization of the Rañada’s solution to the hyperbolic space-time \( M \). It is important to note that the boundary conditions on \( \phi \) and \( \theta \) that generate the electromagnetic knots in the Minkowski space-time cannot be imposed automatically on the fields from the equations (69) and (70) since the \( x \to \infty \) limit is not in general well defined in the neighbourhood \( U_p \). However, it is an interesting problem to see what kind of topologies can be obtained from the local boundary conditions.

Thus, we have seen that by using the 1 + 3-formalism, one can analyse the gravitating electromagnetic field in close analogy with the flat space-time. This is advantageous at several levels: firstly, it allows one a simple interpretation of the fields in terms of the local geometry and topology on the leaf which is three-dimensional and euclidean. Secondly, one can easily obtain the flat space-time limit with the choice of the Gauss normal coordinate system \( N = 1, \beta = 0 \) for which the solutions from [2, 3] are recovered. This fact allows us to conclude that the field line solutions of the gravitating electromagnetism are the natural local generalization of the corresponding solutions in flat space-time. However, global solution could also exist if the topology of \( M \) permits the global extension of the geometrical objects involved in the description of the field line solutions.

5 Kopiański-Natário field

In this section we are going to discuss a particular field line solution to the gravitating electromagnetic field given in [11] in the case of the Einstein universe. Let us start describing this solution by recalling the line element
of the cosmological FRW metric \[14\]

\[ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right]. \tag{71}\]

The parameter \(a\) and the variable \(r\) can be rescaled such that \(k\) take the integer values \(-1, 0\) or \(+1\). For \(k = 0\) the universe is flat, for \(k = +1, r = \sin(\chi)\) the universe is closed while for \(k = -1, r = \sin \psi\) the universe is open. In what follows we are interested in the case when the space-time has an \(\Sigma \simeq S^3\) foliation, that is in the close universe. Matter content can be added to the model of the universe such that the space-time properties of homogeneity and isotropy are preserved. In this case, the matter can be characterized by the following energy-momentum tensor

\[T_{\mu\nu} = (\rho + P) u_\mu u_\nu + Pg_{\mu\nu}, \tag{72}\]

where \(u^\mu\) is unitary time-like vector, \(\rho\) is the density of mass and \(P\) is the density of pressure. The tensor \(T_{\mu\nu}\) is characteristic to the perfect fluid, although many matter models can be put into this form including the electromagnetic field. In the co-moving frame one can choose \(u^\mu = (1, 0, 0, 0)\) and the energy-momentum tensor becomes diagonal \(T^\mu_\nu = \text{diag}(-\rho, P, P, P)\).

Now it is interesting to write the Einstein’s equations for the FRW metric given by the equation (71) and for the energy-momentum tensor from the equation (72). In this case, the Einstein’s equations in the general form

\[G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{73}\]

can be put into the following form

\[3 \frac{\dot{a}^2 + k}{a^2} = 8\pi \rho + \Lambda, \tag{74}\]

\[2a\ddot{a} + \dot{a}^2 + k \frac{a^2}{a^2} = 8\pi P + \Lambda, \tag{75}\]

\[\frac{\ddot{a}}{a^2} = -\frac{4\pi}{3} \left( \rho + 3P \right) + \frac{\Lambda}{3}. \tag{76}\]

where \(\Lambda\) is the cosmological constant. We note that the equations (74)-(76) are not all independent, but the first two of them are. These are the fundamental equations of the cosmology and they describe a large variety of cosmological models that are homogeneous and isotropic. In particular, one obtains the \(\text{Einstein universe}\) if the following constraints are imposed

\[\dot{a} = \ddot{a} = 0, \quad P = 0. \tag{77}\]
It is easy to see that by introducing the equation (77) into the cosmological equations (74)-(76) one obtains the following relations

\[ \frac{3k}{a^2} = \Lambda + 8\pi\rho, \quad \frac{k}{a^2} = \Lambda, \quad k = 4\pi a^2\rho. \]  

(78)

The Einstein universe is a homogeneous model in space and time, but it is unstable to perturbations \( a \to a + \varepsilon \), where \( \varepsilon \ll 1 \). Also, for baryonic matter it requires \( k = +1 \), so the universe is closed. One can determine its parameters up to a constant \( C \) from the above equations and the result is

\[ a = \frac{3C}{2}, \quad \Lambda = \frac{4}{9C^2}. \]  

(79)

Although the experimental data favours an expanding universe, the Einstein universe is an interesting model to explore from both mathematical and physical point of view due to its highly homogeneous structure.

One can give an alternative description of the above model that predicts a cylindrical space-time of topology \( M \simeq \mathbb{R} \times S^3 \) that exploits the spherical symmetry of the leaves. To this end, recall that one can introduce an adapted coordinate system \( (t, x^i) \) at any point \( p \in M \) and associate to it the canonical basis \( (e_0 = \partial_t, e_i = \partial_i) \) of the tangent space \( T_p(M) \). By Weyl’s postulate, \( e_0 \) is orthogonal to the leaf \( \Sigma_t \simeq S^3 \), that is

\[ g(e_0, e_i) = 0. \]  

(80)

Now in order to take into account the symmetry of the leaf \( S^3 \simeq SU(2) \), one chooses the basis \( (e_i) \in T_p(S^3) \) such that it satisfy the \( su(2) \) algebra

\[ e_0 = \partial_t \in T_p(\mathbb{R}), \]  

(81)

\[ [e_i, e_j] = 2\varepsilon_{ijk}e_k, \quad e_i \in T_p(S^3). \]  

(82)

In the original paper the authors used the notation \( e_\mu = X_\mu \) so we adopt that here in order to facilitate the reading of [11]. The dual basis to \( (e_\mu) \) is denoted by \( (\theta^\mu) \) and it follows the same decomposition of the foliated space-time manifold \( M \). Also, they agree with the symmetry of the leaf as they satisfy the following equation

\[ d\theta^i = -\varepsilon_{ijk}d\theta^j \wedge d\theta^k. \]  

(83)

One important feature of the Einstein universe is that it is locally conformally equivalent to the Minkowski space-time. By using this equivalence, the local Einstein-Maxwell equations can be treated as the Maxwell’s equations in some region of the flat space-time. This is possible since the Maxwell’s
equations are invariant under the conformal transformations \([11]\). Thus, the Maxwell’s equations without sources are written as
\[
dF = 0, \quad d \star F = 0.
\] (84)

The decomposition of the electromagnetic 2-form in to the electric and magnetic components is standard and takes the following form in the chosen basis
\[
F = E^i \theta^i \wedge \theta^0 + \frac{1}{2} B^i \varepsilon_{ijk} \theta^j \wedge \theta^k.
\] (85)

By plugging the equation (85) into the equations (84) one can easily obtain the following equations
\[
X_i(E^i) = X_i(B^i) = 0,
\] (86)
\[
\dot{B}^i - 2E^i + \varepsilon_{ijk} X_j(E^k) = 0,
\] (87)
\[
\dot{E}^i + 2B^i - \varepsilon_{ijk} X_j(B^k) = 0.
\] (88)

The Kopiński-Natário field is obtain by making the ansatz that the components of the electric and magnetic fields depend only on time as measured in a stationary frame \([11]\). This assumption is valid in the Einstein universe which is a stationary closed model. As such, the variations along the directions of the \(SU(2)\) manifold vanish and the equations (87) and (88) take the following much simpler form
\[
\dot{B}^i - 2E^i = 0,
\] (89)
\[
\dot{E}^i + 2B^i = 0.
\] (90)

The simplest solution found in \([11]\) is given by the following fields
\[
E(t) = E_0 \cos(2t) X_1,
\] (91)
\[
B(t) = B_0 \cos(2t) X_1.
\] (92)

However, from the equations (89) and (90) one can see that many other solutions are possible all of them generated by the second degree differential equations
\[
\ddot{f} \pm 4f = 0.
\] (93)

One important remark is that the above fields are not orthogonal to each other as they share the same direction of \(SU(3)\). Also, the field lines corresponding to them are unstable under small perturbations.

As the authors noted in their paper, the \([11]\) Einstein universe and the Minkowski space-time are conformally equivalent to each other in some open
regions, which allows one to conformally map the objects defined in the Einstein universe into similar objects from the Minkowski space-time. The conformal mapping is defined by the following transformation of the metric tensor
\[ g_{\mu\nu} = \Omega \eta_{\mu\nu}, \tag{94} \]
where \( \eta_{\mu\nu} \) is the metric of the Minkowski space-time and the conformal factor. In particular, one can interpret the energy of the field line solution in terms of the flat space-time energy and it was found in [11] that it has the following form
\[ \mathcal{E}(t) = \frac{E_0^2}{12} \left[ 9\pi \sin(t) + \pi \sin(3t) + 12\pi^2 \cos(t) - 12\pi t \cos(t) \right], \tag{95} \]
which indicates a radiation field.

Similar considerations were made for a general FRW closed universe described by the line element given by the equation (71). In this case, the vector fields corresponding to the metric is given by the following relations
\[ \bar{X}_0 = \partial_t, \quad \bar{X}_i = a^{-1}(t) \partial_i. \tag{96} \]
The dual basis associated to it contains the following tetrad fields
\[ \bar{\theta}^0 = dt, \quad \bar{\theta}^i = a(t) \partial_i, \tag{97} \]
as can be easily verified. Then the electromagnetic 2-form field has the following decomposition
\[ F = -E^1 \bar{\theta}^0 \wedge \bar{\theta}^1 + B^1 \bar{\theta}^2 \wedge \bar{\theta}^3. \tag{98} \]
The corresponding Einstein-Maxwell’s equations in vacuum take the following form
\[ \frac{d}{dt} [a^2(t)B^1] = 2aE^1, \tag{99} \]
\[ \frac{d}{dt} [a^2(t)E^1] = -2aB^1, \tag{100} \]
where the same assumption on the dependence of the fields on time only has been made. The solutions to the equations (101) and (102) can be easily found by integrating the above equations and has the following form
\[ E^1(t) = \frac{E_0}{a^2(t)} \cos \left[ \int dt \ a^{-1}(t) \right], \tag{101} \]
\[ B^1(t) = \frac{E_0}{a^2(t)} \sin \left[ \int dt \ a^{-1}(t) \right]. \tag{102} \]
The energy of this solution goes as \( \sim E_0^2 a^{-4}(t) \). The Einstein universe represents a particular case for which \( a \) is constant. For a more complete discussion of the properties of these solution we refer to the original work [11].
Appendix

In this appendix we collect some definitions and properties of the Lie derivative, the interior derivative and the extrinsic curvature that have been used in the text. This is standard material and can be found in any classic text on differential geometry. For applications in physics, see e.g. [16].

In all the definitions reviewed here we will consider a differentiable manifold \(M\) of dimension \(\dim(M) = n\).

**Definition 5.1.** If \(T \in T^p_q(M)\) is a tensor field of rank \((p,q)\) and \(X \in \mathcal{X}(M)\) is a differentiable vector field, then the **Lie derivative** of \(T\) along \(X\) is defined as follows

\[
(L_X T)_p = \frac{d}{dt} \bigg|_{t=0} \left[ (\mu - t)_* T_{\varphi_t(p)} \right] = \frac{d}{dt} \bigg|_{t=0} \left[ (\mu_t)_* T_p \right].
\]  

(103)

Here, \(\mu : I \times M \to M, I \subset \mathbb{R}\) is the one-parameter semigroup of diffeomorphisms on \(M\) generated by the flow of \(X\) with the action

\[
x \to \mu_t(x) = \mu(t, x), \quad \forall x \in M, \quad X(x) = \frac{d\mu(t, x)}{dt} \bigg|_{t=0}.
\]  

(104)

The Lie derivative obeys the following axioms

\[
\mathcal{L}_X f = X(f),
\]  

(105)

\[
[\mathcal{L}_X, d] = 0,
\]  

(106)

\[
\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S),
\]  

(107)

\[
\mathcal{L}_X(T(X_1, \ldots, X_n)) = (\mathcal{L}_X T)(X_1, \ldots, X_n)
+ T(\mathcal{L}_X X_1, \ldots, X_n) + \cdots + T(X_1, \ldots, \mathcal{L}_X X_n),
\]  

(108)

where \(T\) and \(S\) are arbitrary tensor fields and \(d\) is the exterior derivative. In the above relations, the objects considered there have natural properties. It is easy to show that the following properties of the Lie derivatives hold

\[
\mathcal{L}_X Y(f) = X(Y(f)) - Y(X(f)) = [X,Y](f),
\]  

(109)

\[
\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega.
\]  

(110)

In the equation (110) we have denoted by \(\omega\) a differential form on \(M\) and by \(\iota\) the interior product to be defined below.

**Definition 5.2.** Let \(X \in \mathcal{X}(M)\) be a smooth vector field on \(M\) and \(\omega \in \Omega^k(M)\) a \(k\)-form. Then the **interior product** of \(X\) and \(\omega\) is a \((k-1)\)-form \(\iota_X \omega\) defined by the following property

\[
\iota_X \omega(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}).
\]  

(111)

If \(k = 0\) we take by definition \(\iota_X \omega = 0\).
It is easy to show by using the above definition that the following properties of the interior derivative hold

\begin{align}
\iota_X (\omega \wedge \gamma) &= \iota_X \omega \wedge \gamma + (-1)^k \omega \wedge \iota_X \gamma, \quad \forall \omega \in \Omega^k(M), \forall \gamma \in \Omega^q(M), \quad (112) \\
\iota_{[X,Y]} &= [\mathcal{L}_X, \iota_Y], \quad \forall X,Y \in \mathcal{X}(M), \quad (113) \\
\iota_X \iota_Y \omega &= -\iota_Y \iota_X \omega, \quad \forall X,Y \in \mathcal{X}(M), \forall \omega \in \Omega^k(M). \quad (114)
\end{align}

Let us review the extrinsic curvature of an embedded surface in the intuitive three-dimensional context where the construction is more easily to visualize. The generalization to four-dimensions is natural. Let \( \Sigma \in \mathbb{R}^3 \) be an oriented surface on which the orientation is defined by the unit vector field \( \mathbf{n} \). The \textit{Gauss map} is defined as

\[ \nu : \Sigma \to S^2, \quad \nu(x) = \mathbf{n}(x). \quad (115) \]

Since \( \nu \) is smooth it induces the map

\[ D_x \nu : T_x(\Sigma) \to T_{\nu(x)}(S^2). \quad (116) \]

Since \( T_{\nu(x)}(S^2) \simeq T_x(\Sigma) \), the derivative \( D_x \nu \) defines the \textit{Weingarten map}

\[ W_x = -D_x \nu : T_x(\Sigma) \to T_x(\Sigma). \quad (117) \]

The map \( W_x \) defines the symmetric bilinear two-form curvature map

\[ K_x(X,Y) = \langle W_x X, Y \rangle \quad (118) \]

for any \( X,Y \in T_x(\Sigma) \).
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