\( L^p \) estimates for the Caffarelli-Silvestre extension operators

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\textbf{Introduction}. In this talk I present some new results, obtained in \cite{4}, on solvability and regularity of elliptic and parabolic problems associated to the degenerate operators

\[ \mathcal{L} = \Delta_x + D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2} \quad \text{and} \quad D_t - \mathcal{L} \]

in the half-space \( \mathbb{R}^{N+1}_+ = \{(x, y) : x \in \mathbb{R}^N, y > 0\} \) or \((0, \infty) \times \mathbb{R}^{N+1}_+\) under Dirichlet and Neumann boundary conditions at \( y = 0 \).

Here \( b, c \) are constant real coefficients and we use

\[ L_y = D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2} \]

Note that singularities in the lower order terms appear when either \( b \) or \( c \) is different from 0.

When \( b = 0 \), then \( L_y \) is a Bessel operator (I shall denote it by \( B_y \)) and both \( \mathcal{L} = \Delta_x + B_y \) and \( D_t - \mathcal{L} \) play a major role in the investigation of the fractional powers \((-\Delta_x)^s\) and \((D_t - \Delta_x)^s\), \( s = (1 - c)/2 \), through the “extension procedure” of Caffarelli and Silvestre, see \cite{2}.

The main results concern with the classical parabolic \( L^p \) estimates

\[ \| D_t v \|_p + \| \mathcal{L} v \|_p \leq C \|(D_t - \mathcal{L})v\|_p \]

(\( L^p \) norms on \((0, \infty) \times \mathbb{R}^{N+1}_+\)) and their elliptic counterpart

\[ \| \Delta_x u \|_p + \| L_y u \|_p \leq C \|\mathcal{L}u\|_p \]

(\( L^p \) norms on \( \mathbb{R}^{N+1}_+ \)) which we prove.

We make an extensive use of semigroup theory and vector-valued harmonic analysis relying upon explicit bounds on the heat kernels and weighted estimates, see \cite{1} and \cite{3}.

\textbf{References}

\[ \begin{align*} \\
[1]\text{P. Auscher, J.M. Martell. Weighted norm inequalities, off-diagonal estimates and elliptic operators. // Part I: General operator theory and weights, Adv. Math., 212 (2007), P. 225–276.} \\
[2]\text{L. Caffarelli, L. Silvestre. An extension problem related to the fractional Laplacian. // Comm. Partial Differential Equations. 2007. V. 32. 2007. No. 7–9. P. 1245–1260.} \\
[3]\text{P.C. Kunstmann, L. Weis. Maximal \( L^p \)-regularity for Parabolic Equations, Fourier Multiplier Theorems and } \mathcal{H}^{\infty}\text{-functional Calculus, in: Functional Analytic Methods for Evolution Equations, M. Iannelli, R. Nagel, S. Piazzera eds. // Lecture Notes in Mathematics. Springer. 2004. V. 1855.} \\
[4]\text{G. Metafune, L. Negro, C. Spina. } L^p \text{ estimates for the Caffarelli Silvestre extension operators. // arXiv: (2021). http://arxiv.org/abs/2103.10314} \\
\end{align*} \]

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