Abstract

The discrete Schrödinger equation with potential belonging to $\mathbb{F}_2$ is solved explicitly. On this base the associated (1 + 1)-dimensional cellular automaton is examined and corresponding set of integrals of motions is constructed.

1 Introduction

Cellular automata are dynamical systems in the discrete space–time with values belonging to some finite field. For example, in the simplest situation the dependent variable $q^t_n$ takes values 0 and 1, i.e., belongs to $\mathbb{F}_2$ for any $n, t \in \mathbb{Z}$, where $n$ and $t$ are considered as space and time variables correspondingly. Initial data are given as a set of zeroes and ones numbered by $n \in \mathbb{Z}$ and time evolution is a law that enables us to construct $q^{t+1}_n$ also as a set of zeroes and ones if $q^t_n$ is known. Cellular automata attract essential interest in the literature because of the wide range of their applications in different sciences, from physics to biology, from chemistry to social sciences. Detailed references for these and many other applications can be found in [1–5]. These automata are also subject to intensive mathematical study, see for example [6–18], but nevertheless the mathematical theory of cellular automata is far from completeness. In some cases (see [4]) these dynamical system demonstrate stochastic behavior, in other cases they behave as solitonic systems [7–18]. The situation becomes even more intriguing if to take into account that, at least, two families of the cellular automata have Lax pairs. Nevertheless, the problem of integrability of such cellular automata is still opened as these Lax representation are satisfied only on corresponding finite fields as well.

Above mentioned families of the cellular automata that have associated Lax representations were considered in [12–15]. All of them are of so called filter type [6]. In works [12,15] these were automata with Lax operator $L$ to be the known Ablowitz–Ladik system [19]. In work [13] $L$ was chosen to be the discrete (1-dimensional) Schrödinger operator, that is also known in the literature [20,21]. As was mentioned in [12,13,15] (and also discussed below) known results on either Ablowitz–Ladik, or discrete Schrödinger equation cannot be applied to the case where potential belongs to a finite field as in this case.
case some of the Jost solutions do not exist. Correspondingly, this unable us to introduce
the spectral data, so the standard scheme [22–24] of the inverse spectral transform fails
for this kind of problems requires essential modification.

In this note we demonstrate that, on the other side, in consideration of the dis-
crete problems some essential simplifications occur. In particular we show that the Jost
solutions of the discrete Schrödinger equation (the existing ones!) can be constructed
explicitly. We apply result of this construction for the simplest example of the cellular
automata introduced in [13],

\[
q^{t+1}_n = q^t_n + q^{t+1}_{n-2}q^t_{n+1} + q^{t+1}_{n-1}q^t_{m+2}, \quad \text{mod 2},
\]

(1.1)

Here dependent variable \(q^t_n \in \{0, 1\}\) for any values of two independent variables \(n, t \in \mathbb{Z}\).

Lax representation for this equation is given in [13] as

\[
L^{t+1}A = AL^t, \quad \text{mod 2},
\]

(1.2)

where matrices \(L^t_{m,n}\) and \(A^t_{m,n}\) are equal to

\[
L^t_{m,n}(w) = \delta_{m,n+1} + (1 + q^t_m)\delta_{m,n-1} - \left( w + \frac{1}{w} \right) \delta_{m,n},
\]

(1.3)

\[
A^t_{m,n} = \delta_{m,n} - \delta_{m+2,n}q^{t+1}_{m-1}q^t_{m+2}.
\]

(1.4)

It was also shown in [13] that if we consider \(q^t_n\) at some moment of \(t \) to be finitely
supported then \(q^{t+1}_n\) and so on also have finite support. More exactly, let us introduce
support of sequence \(q^t_n\) at moment \(t\),

\[
K^t \equiv \{k^t_1 \ldots < k^t_N\} = \{k \in \mathbb{Z} | q^t_k = 1\},
\]

(1.5)

so that \(k^t_1\) and \(k^t_N\) are bottom and upper borders of the support and \(N^t < \infty\). Then
borders of the support are preserved under time evolution,

\[
k^{t+1}_1 = k^t_1, \quad k^{t+1}_{N+1} = k^t_N,
\]

(1.6)

and, moreover, support \(K^t\) is decomposed into “islands”—subsets of support separated
by three or more consequent sites occupied by zeros. These islands evolve independently,
in particular, their borders are also integrals of motion. Simple example of time evolution
of cellular automata (1.3) is presented in Fig. 1, where empty boxes correspond to zero
sites and black boxes correspond to the sites occupied by units. We presented here the
case of two islands. It is clear that time evolution is reduced to the motion of single and
double defects inside islands, and that it is enough to consider the case where \(K^t\) is a
single island. Finally, let us notice that equation (1.3) is invariant with respect to the
substitution \(q^t_n \rightarrow q^{t+1}_{-n}\), so this cellular automaton is time reversible.

Spectral problem for the operator (1.3),

\[
\sum_n L^t_{m,n}(w)\psi^t_n(w) = 0,
\]

(1.7)

here also has to be understood modulo 2. This means that for any \(n \) and \(t \) solution
\(\psi^t_n(w)\) is formal Laurent series in \(w \in \mathbb{C}\) with coefficients belonging to \(\mathbb{F}_2\). Omitting for
simplicity subscript \(t\) we rewrite (1.3) by means of (1.3) as

\[
\psi_{m-1} + (1 + q_m)\psi_{m+1} = \left( w + \frac{1}{w} \right) \psi_m, \quad \text{mod 2}.
\]

(1.8)
This is known discretization of the famous Schrödinger equation and for generic \( q_m \) it can be studied in analogy with the standard Schrödinger equation, see [20,21]. We already mentioned that it is just modulo 2 condition that prevents from the use of the corresponding results of [20,21] here. Indeed, for all \( m \not\in K \) (i.e., \( q_m = 0 \)) we can determine \( \psi_{m+1} \) if \( \psi_m \) and \( \psi_{m-1} \) are known. On the other side, if \( m \in K \) (i.e., \( q_m = 1 \)), then \( \psi_{m+1} \) drops out from (1.8) and \( \psi_m \) and \( \psi_{m-1} \) cannot be chosen independently. This means that in all cases where \( K \) is not empty equation (1.8) can be solved by means of the sweeping from the left, but cannot be swept from the right.

2 Solution of the discrete Schrödinger equation

In order to construct solutions of the discrete Schrödinger equation let us mention first that in the case where \( q_m \in \mathbb{F}_2 \) we have

\[
(1 + q_m) \mod 2 = 1 - q_m, \tag{2.1}
\]

so instead of (1.8) we start with construction of solutions of the infinite system

\[
\psi_{m-1} + (1 - q_m) \psi_{m+1} = \left( w + \frac{1}{w} \right) \psi_m, \tag{2.2}
\]

where \( q_m \in \{0, 1\} \) for any \( m \in \mathbb{Z} \), \( w \in \mathbb{C} \) and condition that (2.2) is satisfied by mod 2 is omitted. It is clear that such solutions of (2.2) resolves (1.8) in the sense mentioned in Introduction.

If \( m \leq k_1 \) or \( m \geq k_N + 1 \) (see (1.3)) we have two explicit solutions of (2.2): \( w^m \) and \( w^{-m} \). Thus for such \( m \) general solution of (2.2) is given as linear combination of these two solutions. Let

\[
x_m(w) = w^m \psi_m(w), \tag{2.3}
\]

that due to (2.2) obeys equation

\[
x_{m-1} - (1 + z)x_m + z(1 - q_m)x_{m+1} = 0, \tag{2.4}
\]

where for simplicity we denoted

\[
z = w^{-2}. \tag{2.5}
\]

In what follows we use the same notations, say \( x_m(z) \) or \( x_m(w) \), for functions of either \( z \), or \( w \), related by means of this substitution. It is clear that if \( \psi_m(w) \) solves (2.2) then

\[
\tilde{\psi}_m(w) = \psi_m(w^{-1}) \tag{2.6}
\]

gives another solution of (2.2). Let us introduce \( \tilde{x}_m \) in analogy with (2.3), then

\[
\tilde{x}_m(z) = z^{-m} x_m(z^{-1}) \tag{2.7}
\]

gives another solution of (2.4).

Mentioned in Introduction problem of the study of the spectrum (2.4) is reflected in the existence of the trivial solution

\[
x_{0,m}(z) = \begin{cases} 
1 - z^{kN - m} & , \quad m \geq k_N, \\
1 - 1/z & , \quad m \leq k_N.
\end{cases} \tag{2.8}
\]
This solution is invariant (up to \( m \)-independent factor) with respect to transformation (2.7).

The Jost solution is defined by condition \( \lim_{m \to +\infty} x_m = 1 \). Then by means of (2.4) we have

\[
x_m = 1, \quad m \geq k_N, \tag{2.9}
\]

and

\[
x_{m-1} - x_m = z q_m + o(z), \quad |z| \to 0, \tag{2.10}
\]

For \( m \leq k_1 - 2 \) this solution is linear combination of 1 and \( z^m \). It can be equivalently determined by means of the “integral” equation

\[
x_m = 1 - \sum_{k \geq m} q_k \frac{1 - z^{k-m}}{1 - 1/z} x_{k+1}. \tag{2.11}
\]

In order to resolve this equation we mention first that due to (1.5) it can be rewritten as

\[
x_m = 1 - \sum_{j: k_j \geq m} \frac{z^{k_j-m+1} - z}{1 - z} x_{k_j+1}. \tag{2.12}
\]

Thus we see that all \( x_m \) for \( m-1 \notin K \) are determined by those values for which \( m-1 \in K \). Substituting in (2.13) \( m \) by \( k_l + 1, k_l \in calK \) we get for these values the following linear algebraic system

\[
\sum_{j=l}^{N} \frac{z^{k_j-k_l} - z}{1 - z} x_{k_j+1} = 1, \quad l = 1, \ldots, N. \tag{2.13}
\]

Let us denote

\[
a_l(z) = 1 + \frac{z}{1 - z} \sum_{j=l}^{N} x_{k_j+1}, \tag{2.14}
\]

\[
b_l(z) = -\frac{1}{1 - z} \sum_{j=l}^{N} z^{k_j+1} x_{k_j+1}, \tag{2.15}
\]

where \( l = 1, \ldots, N \). These functions are not independent as because of (2.13) we have

\[
a_l + z^{-k_l-1}b_l = 0, \quad l = 1, \ldots, N, \tag{2.16}
\]

so that (2.12) takes the form

\[
x_m = \begin{cases} 
1, & m \geq k_N + 1 \\
a_{l+1}(1 - z^{k_{l+1} - m+1}), & k_{l+1} \geq m \geq k_l + 1
\end{cases}, \tag{2.17}
\]

where \( l = 0, \ldots, N - 1 \), and we introduced \( k_0 = -\infty \).

On the other side, considering difference \( a_{l+1} - a_l \) by (2.14) we get

\[
x_{k_{l+1}} = (1 - 1/z)(a_{l+1} - a_l), \quad l = 1, \ldots, N - 1. \tag{2.18}
\]

From (2.14) at \( l = N \) and the upper line of (2.17) we get that

\[
a_N = \frac{1}{1 - z} \tag{2.19}
\]
and by the second line and (2.18) we derive that

\[ a_l = a_{l+1} \frac{1 - z^{k_{l+1} - k_l + 1}}{1 - z}, \quad l = 1, \ldots, N - 1. \]  

(2.20)

Thus we get explicitly

\[ a_l(z) = \frac{1}{1 - z} \prod_{j=l}^{N-1} \frac{1 - z^{k_{j+1} - k_j + 1}}{1 - z}, \quad l = 1, \ldots, N, \]  

(2.21)

and then by (2.17)

\[ x_m = \begin{cases} 1, \\ 1 - z^{k_{l+1} - m + 1} \prod_{j=l+1}^{N-1} \frac{1 - z^{k_{j+1} - k_j + 1}}{1 - z}, & m \geq k_N \\ k_{l+1} \geq m \geq k_l \end{cases} \]  

(2.22)

where \( l = 0, \ldots, N - 1, \) and \( k_0 = -\infty. \) In fact we got the first line for \( m \geq k_N + 1. \) Value \( m = k_N \) was added as from the second line we have that \( x_{k_N} = 1. \) In the same way the interval of validity of the second equality was extended: we added points \( m = k_l \) as values at \( m = k_l \) and \( m = k_{l+1} \) differ by the shift \( l \rightarrow l + 1 \) only.

Asymptotic behavior at \( w^-\)infinity, i.e., at \( z \rightarrow 0 \) enables us to reconstruct potential, as by (2.11) we have

\[ x_m = 1 + z \sum_{j=m+1}^{\infty} q_j + o(z), \quad z \rightarrow 0. \]  

(2.23)

On the other side by (2.22)

\[ x_m = \begin{cases} 1, \\ 1 - z^{k_{l+1} - m + 1} \prod_{j=l+1}^{N-1} \frac{1 - z^{k_{j+1} - k_j + 1}}{1 - z}, & m \geq k_N \\ k_{l+1} \geq m \geq k_l \end{cases} \]  

(2.24)

that coincides with (2.23).

It was already mentioned above, that only one nontrivial solution exists in this case, i.e., solution given in (2.22). Indeed, if we construct solution \( \tilde{x}(z) \) by means of (2.7) we get thanks to (2.8) and (2.22) relation

\[ x_m - z^{k_N} \tilde{x}_m = \theta(m \geq k_N) (1 - 1/z) x_{0,m}. \]  

(2.25)

It is also impossible to introduce the Jost solution normalized at \(-\infty, \) cf. (2.9), as solution (2.8) obeys homogeneous equation

\[ x_{0,l} = \sum_{k\leq l} q_k \frac{1 - z^{k-l}}{1 - 1/z} x_{0,k+1}. \]  

(2.26)

Jost solution (2.22) of the equation (2.2) is polynomial with respect to \( z \) (i.e., \( w^-2 \) because of (2.5)) with coefficients belonging to \( \mathbb{N}. \) It is convenient to introduce new parametrization of this solution that will be useful in the construction of the integrals of motion below. So we introduce discrete measure

\[ f_i(m) = \sum_{n \geq m+1} q_n \prod_{j=1}^{i-1} (1 - q_{n+j}) q_{n+i} + \prod_{j=1}^{i-1} (1 - q_{m+j}) q_{m+i}. \]  

(2.27)
where \( m \in \mathbb{Z} \) and \( i = 1, 2, \ldots \). Writing down values for the lowest \( i \)'s

\[
f_1(m) = \sum_{n \geq m+1} q_n q_{n+1} + q_{m+1},
\]

\[
f_2(m) = \sum_{n \geq m+1} q_n (1 - q_{n+1}) q_{n+2} + (1 - q_{m+1}) q_{m+2},
\]

\[
f_3(m) = \sum_{n \geq m+1} q_n (1 - q_{n+1})(1 - q_{n+2}) q_{n+3} + (1 - q_{m+1})(1 - q_{m+2}) q_{m+3},
\]

etc., we see that for every \( m \) and \( i \) first term in \( f_i(m) \) is equal to the number of higher \((i-1)\)-defects, i.e., for all \( n \geq m+1 \) number of combinations \( \{q_n, \ldots, q_{n+i}\} = \{1, 0, \ldots, 0, 1\} \) with exactly \( i - 1 \) zeroes inside, while the second term is equal to 1 iff \( k_l - m = i \) for the lowest \( k_l \in K, k_l \geq m \). In particular, we see that if \( k_{l+1} \geq m \geq k_l \) then by (2.27)

\[
f_i(m) = f_i(k_{l+1}) + \delta_{k_{l+1}-m, i}.
\]

Taking into account that by (1.5) \( q_{k_l} = 1 \) we get

\[
f_i(k_l) = \sum_{n \geq k_l} q_n \prod_{j=1}^{i-1} (1 - q_{n+j}) q_{n+i},
\]

in particular

\[
f_i(k_1) = \sum_{n \geq k_1} q_n \prod_{j=1}^{i-1} (1 - q_{n+j}) q_{n+i} = \sum_{j \geq l} \delta_{k_{j+1}-k_j, i},
\]

i.e., \( f_i(k_1) \) is the number of neighbor cites in the support \( K \) of \( q_n \) occupied by units, \( f_i(k_1) \) are numbers of \((i-1)\)-defects in the support. As support \( K \) is bounded (see (1.5)) we have that

\[
f_i(k_1) = 0, \quad i \geq k_N - k_1 + 2, \quad l = 1, \ldots N,
\]

so that by (2.31)

\[
f_i(m) = \delta_{k_i - m, i}, \quad i \geq k_N - k_1 + 2.
\]

Also by (2.27) it is easy to see that

\[
f_i(m) \equiv 0, \quad \text{for} \quad m \geq k_N,
\]

\[
f_i(k_N - 1) = 1, \quad f_i(k_N - 1) = 0, \quad i \geq 2
\]

Definition (2.27) can be rewritten in the form

\[
f_i(m) = \sum_{n \geq m} \prod_{j=1}^{i-1} (1 - q_{n+j}) q_{n+i} - \sum_{n \geq m} \prod_{j=1}^{i} (1 - q_{n+j}) q_{n+i+1}
\]

that gives the following relations:

\[
\sum_{i=1}^{\infty} f_i(m) = \sum_{n \geq m+1} q_n
\]

\[
\sum_{i=1}^{\infty} if_i(m) = \begin{cases} 0, & k_N - m, \quad k_N \geq m, \\ k_N - m, & k_N \leq m. \end{cases}
\]
To prove these equalities, notice first that sums are convergent due to (2.35). Then (2.39) follows directly from (2.38), as well as (2.40) for \( m \geq k_N \). Let now \( k_N - 1 \geq m \), then thanks to (2.38)\[ \sum_{i=1}^{\infty} i f_i(m) = \sum_{i=1}^{\infty} \prod_{j=1}^{i} (1-q_{m+j})q_{m+i} = \sum_{n \geq m} \left\{ \sum_{i=0}^{k_N - m} \prod_{j=1}^{i} (1-q_{n+j}) - \sum_{i=0}^{k_N - m} \prod_{j=1}^{i} (1-q_{n+j}) \right\} \]that proves (2.40).

Let us turn now to (2.22) for \( k_{l+1} \geq m \geq k_l \), \( l = 0, \ldots, N-1 \), \( k_0 = -\infty \). Because of (2.33) this equality can be written for \( k_{l+1} > m \geq k_l \) as\[ x_m = \prod_{k=1}^{\infty} \left( \frac{1 - z^{k+1}}{1 - z} \right) \delta_{k_{l+1} - m, k} \prod_{k=1}^{\infty} \left( \frac{1 - z^{k+1}}{1 - z} \right) f_k(k_{l+1}) \].

Taking (2.31) now into account we get finally\[ x_m = \prod_{k=1}^{\infty} \left( \frac{1 - z^{k+1}}{1 - z} \right) f_k(m) \]that thanks to (2.36) gives also valid result in the first line of (2.22) at \( m \geq k_N \).

3 Solutions by mod 2

It was already mentioned that solution (2.41) is polynomial in \( z \) with coefficients belonging to \( \mathbb{N} \), so it solves (2.4) by modulo 2, i.e., it obeys\[ x_{m-1} + (1 + z)x_m + z(1 + q_m)x_{m+1} = 0, \quad \text{mod } 2. \quad (3.1)\]

In [13] it was proved that islands, i.e., intervals separated by at least three zeroes, evolve independently. Thus it is natural to consider here the case where support \( K \) is just one island. This means that we impose condition that all \( k_j \in K \) obey\[ k_{j+1} - k_j \leq 3, \quad j = 1, \ldots, N-1, \quad \text{(3.2)} \]
or equivalently,\[ (1 - q_n)(1 - q_{n+1})(1 - q_{n+2}) = 0, \quad k_N \geq n \geq k_1 - 2. \quad (3.3) \]

Then by (2.27)\[ f_i(m) = 0, \quad i \geq 4, \quad m \geq k_1 - 2, \quad (3.4) \]
so in the case of only one island our solution (2.41) is given as product of powers of three polynomials:

\[ x_m = (1 + z)^{f_1(m)}(1 + z + z^2)^{f_2(m)}(1 + z + z^2 + z^3)^{f_3(m)}, \quad m \geq k_1 - 2. \quad (3.5) \]

Polynomials \( 1 + z \) and \( 1 + z + z^2 \) are independent on \( \mathbb{F}_2 \), but for the third polynomial we can write

\[ 1 + z + z^2 + z^3 = (1 + z)^3, \quad \text{mod } 2, \quad (3.6) \]
and because of this
\[ x_m = (1 + z)f_1(m) + 3f_3(m)(1 + z + z^2)f_2(m), \quad \text{mod 2}. \]  
(3.7)

Moreover, by (2.40) and (3.4)
\[ f_1(m) + 2f_2(m) + 3f_3(m) = k_N - m, \quad k_N \geq m \geq k_1 - 2, \]  
(3.8)
so finally we get
\[ x_m = (1 + z)^{k_N - m}(1 + z)^{-2f_2(m)(1 + z + z^2)f_2(m)}, \quad \text{mod 2}, \]  
(3.9)
where again \( k_N \geq m \geq k_1 - 2 \). Thus the Jost solution is parametrized by the only one dynamical variable \( f_2 \).

The original potential can be reconstructed by means of \( f_2 \). Indeed, by (2.29)
\[ f_2(m - 1) - f_2(m) = (1 - q_m)(q_{m+1} - (1 - q_{m+1})q_{m+2}) \]
that thanks to (3.3) can be written in the form
\[ f_2(m - 1) - f_2(m) = (1 - q_m)(2q_{m+1} - 1), \quad k_N \geq m \geq k_1 - 2, \]  
(3.10)
and then
\[ q_m = 1 - |q_2(m) - q_2(m - 1)| \]  
(3.11)
as \( |2q_{m+1} - 1| = 1 \). Let us emphasize that equality (3.11) is exact, i.e., not the mod 2 one.

### 4 Time evolution and integrals of motion

In order to study time evolution by means of the above construction we need to consider two neighbor values of time only, say, \( t \) and \( t + 1 \). So let us simplify notations used in Introduction by denoting
\[ q_n = q_n^t, \quad x_n = x_n^t, \quad \text{etc.}, \]
\[ q_n = q_n^{t+1}, \quad x_n = x_n^{t+1}, \quad \text{etc.} \]
(4.1)
(4.2)
Then equation of motion (1.1) takes the form
\[ \dot{q}_m = q_m + \dot{q}_{m-2}q_{m+1} + \dot{q}_{m-1}q_{m+2}, \quad \text{mod 2}, \]  
(4.3)
and corresponding transformation of the Jost solution is given due to (1.1) by means of
\[ \dot{\psi}_m = \psi_m + \dot{q}_{m-1}q_{m+2}\psi_{m+2}, \quad \text{mod 2}, \]  
(4.4)
or using notations (2.3) and (2.5) we have
\[ \dot{x}_m = x_m + z\dot{q}_{m-1}q_{m+2}x_{m+2}, \quad \text{mod 2}. \]  
(4.5)
Time evolution of \( f_2(m) \) is given by substitution of \( x_m \) in (4.5) by (3.9):
\[ (1 + z)^{2(1-f_2(m))}(1 + z + z^2)f_2(m) = (1 + z)^{2(1-f_2(m))}(1 + z + z^2)f_2(m) + + z\dot{q}_{m-1}q_{m+2}(1 + z)^{-2f_2(m+2)(1 + z + z^2)f_2(m+2),} \quad \text{mod 2}. \]  
(4.6)
Considering this (mod 2)-equation separately for the cases $\hat{q}_{m-1}q_{m+2}$ equal to 0 and to 1, we arrive to the **exact** (not mod 2) equation

$$\hat{f}_2(m) = f_2(m) + \hat{q}_{m-1}q_{m+2}(2q_{m+1} - 1), \quad m \geq k_1 - 2$$

(\(k_N \geq m \geq k_1 - 2\) is interval of validity of \((3.9)\) and validity on the interval \(m \geq k_N + 1\) follows from \((2.36)\)). Using \((3.10)\) for \(f_2\) we get

$$\hat{f}_2(m - 1) - \hat{f}_2(m) =$$

$$= \left(1 - q_m - \hat{q}_{m-1}q_{m+2} + \hat{q}_{m-2}q_{m+1}(2q_m - 1)\right)(2q_{m+1} - 1),$$

and then it is easy to check that \(|\hat{f}_2(m) - \hat{f}_2(m)|\) is equal to 0 or 1, so we can use \((3.11)\) to define

$$\hat{q}_m = 1 - |\hat{f}_2(m) - \hat{f}_2(m)|.$$  

Thanks to this relation we derive the exact equation of time evolution of \(q_m\), that substitutes (mod 2)-equation \((4.3)\). Indeed, using \((4.8)\) we get

$$\dot{\hat{q}}_m = 1 - |1 - q_m - \hat{q}_{m-1}q_{m+2} + \hat{q}_{m-2}q_{m+1}(2q_m - 1)|.$$  

The r.h.s. can be simplified if we notice that

$$|1 - q_m - \hat{q}_{m-1}q_{m+2} + \hat{q}_{m-2}q_{m+1}(2q_m - 1)| = 1 - q_m + |\hat{q}_{m-1}q_{m+2} - \hat{q}_{m-2}q_{m+1}(2q_m - 1)|,$$

so finally we get equation

$$\dot{\hat{q}}_m = q_m - |\hat{q}_{m-1}q_{m+2} - \hat{q}_{m-2}q_{m+1}|(2q_m - 1).$$

In order to consider integrals of motion we introduce elements of the monodromy matrix by means of the standard relation

$$x_m = a(z) + z^{-m}b(z), \quad m \leq k_1,$$

and analogously for \(\hat{x}_m\). Thanks to \((4.3)\) for \(m \leq k_1\)

$$\hat{x}_m = x_m,$$

thus

$$\hat{a}(z) = a(z), \quad \hat{b}(z) = b(z)$$

i.e., both \(a(z)\) and \(b(z)\) are integrals of motion as it was demonstrated in [13]. Here because of \((3.7)\) we single out functionally independent integrals explicitly. Indeed, by \((2.17)\)

$$b(z) = z^{1+k_1}a(z), \quad (1 - z)a(z) = x_{k_1}(z), \mod 2,$$

so \(x_{k_1}(z)\) is also integral of motion and by \((3.9)\) we have

$$x_{k_1}(z) = (1 + z)^{k_N-k_1}(1 + z)^{2f_2(k_1)}(1 + z + z^2)^{f_2(k_1)}, \mod 2.$$  

Now \((4.13)\) and \((4.14)\) are equivalent to

$$\hat{k}_1 = k_1, \quad \hat{k}_N = k_N,$$

$$\hat{f}_2(k_1) = f_2(k_1),$$

(4.16)
where the first line was already known (see (1.6)).

Thus in addition to two known integrals—borders of the island, $k_1$ and $k_N$,—we proved that $f_2(k_1)$, number of the single defects (isolated zeroes, see comments after (2.33)) inside the island, is preserved under time evolution. This new integral is explicitly demonstrated on Fig. 1. It is clear that these three integrals are mutually independent and thanks to (4.14) and (4.15) they uniquely determine $a(z)$ and $b(z)$. Thus all coefficients of the Taylor expansion of $a(z)$ are functions of these three integrals only. We emphasize that in spite of the fact that Eq. (4.15) is equation by mod 2 all these three integrals are exact ones.

An example of (mod 2)-integral is provided by $N$. Indeed, in analogy with (3.8) we have from (2.39) that

$$f_1(m) + f_2(m) + f_3(m) = \sum_{j \geq m+1} q_j.$$  

On the other side (3.8) gives

$$f_1(m) + f_3(m) = k_N - m, \mod 2,$$

so that

$$\sum_{j \geq m+1} q_j = k_N - m + f_2(m), \mod 2.$$  

Then

$$N = \sum_{m \geq k_1+1} q_m + 1 = k_N - k_1 + 1 + f_2(k_1), \mod 2,$$

and as any exact integral of motion is also (mod 2)-integral we obtain that

$$\hat{N} = N, \mod 2.$$  

Explicit formulas for $f_1(k_1)$, $f_2(k_1)$, and $f_3(k_1)$ are given in (2.33) and Fig. 1 shows that indeed neither $f_1(k_1)$, nor $f_3(k_1)$ are preserved under time evolution. Of course, this does not mean that there cannot be some other integrals of motion. Moreover, it is unclear if existence of three exact integrals (4.16) and one (mod 2)-integral (4.17) is enough for the system to be integrable in some sense.

One of results of the above construction is substitution of (mod 2)-equation (1.1) (or (1.3)) by (4.11). The latter is the exact one, as for $q_m$ with range equal $\{0, 1\}$ we get $\hat{q}_m$ with the same range $\{0, 1\}$ without using summation by mod 2. Thus the problem of integrability of this cellular automaton is reduced to the problem of existence of an (exact!) Lax pair for the equation (4.11). This problem, as well as problem of definition of the reasonable spectral data for the linear operator (1.1) on $\mathbb{F}_2$ are still opened. Moreover, we have to stress, that such notions as “integrability” and “degree of freedom” themselves must be modified to be applicable to the case of systems on finite fields.

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Fig. 1. Here the horizontal axis is the $n$-axis and the downward vertical axis is the $t$-axis.