Minimum Supersymmetric Standard Model on the Noncommutative Geometry

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Abstract
We have obtained the supersymmetric extension of spectral triple which specify a noncommutative geometry(NCG). We assume that the functional space $\mathcal{H}$ constitutes of wave functions of matter fields and their superpartners included in the minimum supersymmetric standard model(MSSM). We introduce the internal fluctuations to the Dirac operator on the manifold as well as on the finite space by elements of the algebra $A$ in the triple. So, we obtain not only the vector supermultiplets which meditate $SU(3) \otimes SU(2) \otimes U(1)_Y$ gauge degrees of freedom but also Higgs supermultiplets which appear in MSSM on the same standpoint. According to the supersymmetric version of the spectral action principle, we calculate the square of the fluctuated total Dirac operator and verify that the Seeley-DeWitt coefficients give the correct action of MSSM. We also verify that the relation between coupling constants of $SU(3), SU(2)$ and $U(1)_Y$ is same as that of SU(5) unification theory.

1 INTRODUCTION

Yang-Mills gauge theory which provides the basis of the standard model of high energy physics and general relativity theory greatly have succeeded in describing the basic interactions in our Universe. But in the quantum level, they are difficult to be unified due to the unrenormalizability of gravity.

The standard model coupled to gravity was derived on the basis of noncommutative geometry(NCG) by Connes and his co-workers[1 2 3]. The framework of NCG is specified by a set called spectral triple $(\mathcal{H}, A, D)$[4]. Here, $A$ is a noncommutative complex algebra, acting on the Hilbert space $\mathcal{H}$, whose elements correspond to spinorial wave functions of physical matter fields, while the Dirac operator $D$ is a self adjoint operator with compact resolvent which play the role of metric of the geometry. Higgs, gauge and gravity fields which meditate all basic interactions that we know are introduced in the same standpoint, the fluctuations of Dirac operator.

The internal fluctuation of the Dirac operator is given as follows:

$$\tilde{D} = D + A + JAJ^{-1}, \quad A = \sum a_i [D, b_i], \quad a_i, b_i \in A. \quad (1.1)$$

For the Dirac operator on the manifold $D_M = i\gamma^\mu \nabla_\mu \otimes 1$, the fluctuation $A + JAJ^{-1}$ gives the gauge vector fields, while for the Dirac operator in the finite space, $D_F$, it gives the Higgs fields[5 6].

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The action for the NCG model is obtained by the spectral action principle and expressed by
\[ (\bar{\psi} \hat{D} \psi) + \text{Tr}(f(P)), \] (1.2)
where \( \psi \) is a fermionic field which is an element in \( \mathcal{H} \). \( f(x) \) is an auxiliary smooth function on a compact Riemann Manifold without boundary of dimension 4. The second term of the action is \( \text{Tr}U \) is the bosonic part which depends only on the spectrum of the squared Dirac operator \( P = \hat{D}^2 \) and represents the non-abelian gauge fields, Higgs fields and gravity.

In our previous papers\cite{8, 9}, we extended the spectral triple of NCG to a counterpart in the supersymmetric theory which may overcome various shortcomings of the standard model\cite{10} such as hierarchy problem, many free parameters to be determined by experiments. We derived the internal fluctuation not only to \( \mathcal{H} \) but also to the Dirac operator \( \mathcal{D}_F \) which acts on the finite space. The modified total Dirac operator is expressed by \( i\mathcal{D}_{\text{tot}} = i\mathcal{D}_M \otimes 1_F + \gamma_M \otimes \mathcal{D}_F \), where \( \gamma_M \) is a grading operator on the manifold and \( \mathcal{D}_F \) is the modified Dirac operator on the finite space. From the fluctuation in \( i\mathcal{D}_M \), we will obtain the vector supermultiplet which mediate \( U(3) \otimes U(2) \otimes U(1) \) gauge degrees of freedom. We take out \( U(1)'s \) from \( U(3) \) and \( U(2) \) and combine them with the other \( U(1) \) to produce \( U(1) \) of weak hypercharge. From the fluctuation to \( \gamma_M \otimes \mathcal{D}_F \), we also obtain supermultiplets which transform as Higgs fields of MSSM.

We will calculate the square of the modified total Dirac operator \( P = (i\mathcal{D}_{\text{tot}})^2 \), and Seeley-DeWitt coefficients of squared total Dirac operator which include the vector supermultiplets as well as the Higgs supermultiplets. In section 2, we will review the supersymmetric extension of the spectral triple which specifies NCG. In section 3, we will choose elements of \( \mathcal{A} \) as fluctuations of the total Dirac operator to produce vector supermultiplets and Higgs supermultiplets which appear in MSSM. In section 4, we will calculate the Seeley-DeWitt coefficients due to \( P = (i\mathcal{D}_{\text{tot}})^2 \). and verify that we obtain the correct action of MSSM. In the process, we will also obtain the relation between coupling constants of \( SU(3), SU(2), U(1)_Y \)-gauge degrees of freedom which is same as that of \( SU(5) \) unification theory.

## 2 THE TRIPLE FOR MSSM

Let us start with reviewing the supersymmetric counterpart \( (\mathcal{H}, \mathcal{A}, \mathcal{D}) \)\cite{10} extended from the spectral triple which specifies NCG. The functional space \( \mathcal{H} \) is the product denoted by
\[ \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F. \] (2.1)

The functional space \( \mathcal{H}_M \) on the the Minkowskian space-time manifold is the direct sum of two subsets, \( \mathcal{H}_+ \) and \( \mathcal{H}_- \):
\[ \mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_-, \] (2.2)
where \( \mathcal{H}_+ \) is the space of chiral supermultiplets which constitutes of Weyl spinors which transform as the \( (\frac{3}{2}, 0) \) of the Lorentz group \( SL(2, C) \) and their superpartners and \( \mathcal{H}_- \) is the space of antichiral supermultiplets. The elements are expressed by
\[ \Phi_+ = ((\Psi_+), 0) \in \mathcal{H}_+, \] (2.3)
\[ \Phi_- = (0, \Psi_-) \in \mathcal{H}_-. \] (2.4)

\( Z_2 \) grading \( \gamma_M \) on the space \( \mathcal{H}_M \) is given by
\[ \gamma_M(\Phi_+) = -i, \quad \gamma_M(\Phi_-) = i. \] (2.5)

A supersymmetric invariant product of wave functions \( \Psi, \Psi' \) is defined by
\[ \langle \Psi', \Psi \rangle = \int \bar{\Psi}' \Gamma_0 \Psi d^4x, \] (2.6)
where $\Psi$ is expressed on the base by

$$
\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \Phi_+ + \Phi_-,
$$

(2.7)

$\Gamma_0$ is given by

$$
\Gamma_0 = \begin{pmatrix} 0 & \Gamma_0 \\ \Gamma_0 & 0 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

(2.8)

The algebra $A_M = A_+ \oplus A_-$ which acts on $H_M$ constitutes of a space of elements expressed by

$$
u_a = \begin{pmatrix} (u_a)_{ij} \\ 0 \end{pmatrix} \in A_+, \quad (u_a)_{ij} = \frac{1}{m_0} \begin{pmatrix} \varphi_a & 0 & 0 \\ \psi_{aa} & \varphi_a & 0 \\ F_a & \psi_a^* & \varphi_a \end{pmatrix},
$$

(2.9)

$$
\bar{\nu}_a = \begin{pmatrix} 0 \\ 0 \\ (\bar{u}_a)_{ij} \end{pmatrix} \in A_-, \quad (\bar{u}_a)_{ij} = \frac{1}{m_0} \begin{pmatrix} \varphi_a^* & 0 & 0 \\ \psi_{aa}^* & \varphi_a^* & 0 \\ F_a^* & \psi_a & \varphi_a^* \end{pmatrix},
$$

(2.10)

where $\{\varphi_a, \psi_{aa}(\psi_a^*), F_a(F_a^*)\}$ is a chiral(antichiral) supermultiplet. We note that $A_\pm$ includes the constant functions expressed by

$$
1_+ = \begin{pmatrix} \delta_{ij} \\ 0 \\ 0 \end{pmatrix} \in A_+, \quad 1_- = \begin{pmatrix} 0 & 0 \delta_{ij} \end{pmatrix} \in A_-
$$

(2.11)

The Dirac operator which acts on $H_M$ is given by

$$
D_M = -i \begin{pmatrix} 0 & \tilde{D}_{ij} \\ \tilde{D}_{ij} & 0 \end{pmatrix},
$$

(2.12)

where

$$
D_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\sigma^\mu \partial_\mu & 0 \\ \square & 0 & 0 \end{pmatrix}, \quad \tilde{D}_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\sigma^\mu \partial_\mu & 0 \\ \square & 0 & 0 \end{pmatrix}.
$$

(2.13)

The algebra $A_F$ on the finite space $H_F$ is given by

$$
A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
$$

(2.14)

where $\mathbb{H}$ is the space of quaternions and $M_3(\mathbb{C})$ is the space of $3 \times 3$ complex matrices. The representation space of $A_F$, $H_F$ is the space of labels which denote quantum numbers of quarks and leptons as follows:

$$
Q_{AI} = \begin{pmatrix} (u_L)_A \\ (d_L)_A \end{pmatrix}, \quad (u_R)_A, (d_R)_A, \quad \ell_I = \begin{pmatrix} \nu_I \\ e_L \end{pmatrix}, \quad e_R,
$$

(2.15)

and labels of their antiparticles are also elements of $H_F$ which are expressed by

$$
Q^c = \begin{pmatrix} (u^c)_R \\ (d^c)_R \end{pmatrix} = \begin{pmatrix} (u^c)_L^* \\ (d^c)_L^* \end{pmatrix} = Q^*,
$$

(2.16)

$$
\ell^c = \begin{pmatrix} (\nu^c)_R \\ (e^c)_R \end{pmatrix} = \begin{pmatrix} (\nu^c)_L^* \\ (e^c)_L^* \end{pmatrix} = \ell^*, \quad (e^c)_L = (e_R)^*,
$$

where $A$ denotes indices on which $3 \times 3$ complex matrices act and $I$ denotes those on which quaternions act. $L$ and $R$ denote that the eigenvalue of $\mathbb{Z}/2$ grading $\gamma_F$ is $-1$ and $1$, respectively.

We define the action of $a \in A_F$ for the quark sector as follows:

$$
a \begin{pmatrix} u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \lambda^{m_1} & 0 \\ 0 & \lambda^{m_2} \end{pmatrix} \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \quad a \begin{pmatrix} u_L \\ d_L \end{pmatrix} = q_I \begin{pmatrix} u_L \\ d_L \end{pmatrix},
$$

(2.17)

$$
a \begin{pmatrix} Q^{cA} \\ u^{cA} \\ a^{cA} \end{pmatrix} = \begin{pmatrix} (m^*)^{A_B} & 0 & 0 \\ 0 & (m^*)^{A_B} & 0 \\ 0 & 0 & (m^*)^{A_B} \end{pmatrix} \begin{pmatrix} Q^{cB} \\ u^{cB} \\ a^{cB} \end{pmatrix},
$$

(2.18)
where $\lambda \in \mathbb{C}$, $q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$. When we denote the base of quark sector as

$$
\begin{pmatrix}
\Psi_q \\
\Psi^c_q
\end{pmatrix}, \quad
\Psi_q = (Q, u_R, d_R)^T, \quad
\Psi^c_q = (Q^c, (u_R)^c, (d_R)^c)^T,
$$

(2.19)
a is represented by a matrix given by

$$
a = \begin{pmatrix}
q \otimes 1_3 & \lambda^{m_1} \otimes 1_3 & 0 \\
1_3 \otimes \lambda^{m_2} & 0 & m^* \otimes 1_2 \\
0 & m^* & m^*
\end{pmatrix},
$$

(2.20)

where $1_3$ denotes the unit matrix of $M_3(\mathbb{C})$, $1_2$ is that of $\mathbb{H}$ and $m_i$ are integers.

For the lepton sector, we take the base expressed by

$$
\begin{pmatrix}
\Psi_l \\
\Psi^c_l
\end{pmatrix}, \quad
\Psi_l = (l, e_R)^T, \quad
\Psi^c_l = (\ell^c, (e^c)_L)^T
$$

(2.21)

and define $a$ as follows:

$$
a = \begin{pmatrix}
q & 0 \\
\lambda^{m_3} & 1 \\
0 & 1
\end{pmatrix}
$$

(2.22)

For an element $a \in A_F$, we define the real structure as an antilinear isometry of $\mathcal{H}_F$ as follows:

$$a \mapsto -J_F a^* J_F^{-1}.
$$

(2.23)

$J_F$ satisfies the following conditions:

$$[a, J_F b^* J_F^{-1}] = 0, \quad \forall a, b \in A_F
$$

(2.24)

$$[[D_F, a], J_F b^* J_F^{-1}] = 0.
$$

(2.25)

$J_F$ defines the right action of $A_F$ in $\mathcal{H}_F$ as follows:

$$\Psi a^T = J_F a^* J_F^{-1} \Psi, \quad \Psi \in \mathcal{H}_F.
$$

(2.26)

From (2.20), we see that the left action of $a$ for the antiparticle determine the right action for the particle and can be moved into the left action through $J_F$. For an example, let us consider $Q_{AI}$ in (2.15). From (2.20), the left and right actions of $A_F$ on $Q$ are summarized by

$$Q'_{AI} = q^I_J Q_{BJ} ((J_F m^* J_F^{-1})^B)_A = q^I_J m^B_A Q_{BJ}.
$$

(2.27)

The Dirac operator $D_F$ on $\mathcal{H}_F$ is given for each label of $s = u, d, e$ by

$$D_F = \begin{pmatrix}
0 & m^s \\
m^s & 0
\end{pmatrix},
$$

(2.28)

where $m_s$ is the mass matrix with respect to the family index.

In order to evade fermion doubling, we impose the following condition

$$\gamma M \gamma_F = i,
$$

(2.29)

and extract the physical wave functions as follows:

for quarks

$$
(\Phi_+ \oplus \Phi_-) \otimes u_L \begin{pmatrix}
\Psi_+ \\
0
\end{pmatrix} \rightarrow (\Psi^+_L \otimes u_L)^{\gamma_{M \gamma_F = 1}} / (\Psi_- \otimes u_R(d_R)),
$$

(2.30)
particle $s$ element $u_{a_0}^{(s)}$ $u_{a_1}^{(s)} = J u_{a}^{(s^c)} J^{-1}$

quark sector

$Q_{a}(x)$ $u_{a}^{[2]} = (q(x))_{J}^{J} \in \mathbb{H}$ $u_{a}^{[3]} = (m(x))_{A}^{B} \in M_{3}(\mathbb{C})$

$u_{R}^{a}(x)$ $u_{a}^{[1]} = \lambda_{m}^{1}(x) \in \mathbb{C}$ $u_{a}^{[3]}$

$d_{R}^{a}(x)$ $u_{a}^{[12]} = \lambda_{m}^{2}(x) \in \mathbb{C}$ $u_{a}^{[3]}$

lepton sector

$l_{a}(x)$ $u_{a}^{[2]} = 1$

$e_{R}^{a}(x)$ $u_{a}^{[3]} = \lambda_{m}^{3}(x) \in \mathbb{C}$ $1$

Table 1: The list of matter fields and the action of the algebra for them: The third column denotes the actions of elements of the algebra moved from the antiparticle part through the real structure $J$.

and for leptons

$$(\Phi_{+} \oplus \Phi_{-}) \otimes \begin{pmatrix} \nu_{L} \\ e_{L} \\ e_{R} \end{pmatrix} \gamma_{M}^{\gamma F=i} \rightarrow \begin{pmatrix} \Psi_{+} \\ 0 \\ (\Psi_{-}) \otimes e_{R} \end{pmatrix} , \quad (2.31)$$

From $(2.9), (2.10), (2.20), (2.22)$, we summarize in Table 1 the action of the elements of the algebra $A_{M} \otimes A_{F}$ on the physical states. The matrix form of these elements on the the basis $(2.30)$ is expressed by

$$U_{a} = \begin{pmatrix} u_{a}^{(Q)} \\ 0 \\ u_{a}^{(u_{R}(d_{R}))} \end{pmatrix} \in A_{+} \otimes A_{F} , \quad (2.32)$$

$$\bar{U}_{a} = \begin{pmatrix} \bar{u}_{a}^{(Q)} \\ 0 \\ \bar{u}_{a}^{(u_{R}(d_{R}))} \end{pmatrix} \in A_{-} \otimes A_{F} , \quad (2.33)$$

where $u_{a}^{(s)}$, $s = Q, u_{R}(d_{R})$ are elements of $A_{+}$ in the form of $(2.4)$ and they are linear combinations of $u_{a_0}^{(s)}$ and $u_{a_1}^{(s)}$ in the Table 1, while $\bar{u}_{a}^{(s)}$ are elements of $A_{-}$ in the form of $(2.10)$ and they are related to $u_{a}^{(s)}$ as follows:

$$(I_{0} u_{a})^{\dagger} = I_{0} \bar{u}_{a} \quad (2.34)$$

Constant elements of $A_{+} \otimes A_{F}$ are also given by

$$1_{+} = \begin{pmatrix} 1^{(Q)}_{+} \\ 0 \\ 1^{(u_{R}(d_{R}))}_{+} \end{pmatrix} , \quad 1_{-} = \begin{pmatrix} 1^{(Q)}_{-} \\ 0 \\ 1^{(u_{R}(d_{R}))}_{+} \end{pmatrix} , \quad (2.35)$$

where $1^{(s)}_{+}$ denotes the matrix form of $(2.11)$.

If we replace $Q \rightarrow l$ and $u_{R}(d_{R}) \rightarrow e_{R}$ in $(2.32), (2.33), (2.35)$, we obtain formulas for the lepton sector. Hereafter, we discuss the quark sector principally and obtain formulas for the lepton sector through this replacement.

On the basis, the total Dirac operators for the quark sector is given as follows:

$$iD_{tot} = i D_{M} \otimes 1_{F} + \gamma_{M} \otimes D_{F}$$

$$= \begin{pmatrix} 0 & \bar{D} & 0 & 0 \\ \bar{D} & 0 & 0 & i M^{(u_{R}(d_{R}))}^{\dagger} \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix} , \quad (2.36)$$

where $M^{(u_{R}(d_{R}))}$ represents the doublet term which includes mass of quarks expressed by

$$M^{(u)} = (m_{u}, 0) = y_{u}(\mu_{0}, 0) , \quad M^{(d)} = (0, m_{d}) = y_{d}(0, \mu_{0}) , \quad (2.37)$$

where $y_{u}$ and $y_{d}$ are Yukawa matrices with regard to generation.

For the antiparticle sector, the total Dirac operator is given by

$$J i D_{tot} J^{-1} , \quad (2.38)$$
where $J$ is the real structure of $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F$.

We note that under the definition of supersymmetric invariant product (2.30), $iD_{tot}$ has the following hermiticity:

$$ (\Gamma_0 iD_{tot})^\dagger = \Gamma_0 iD_{tot}. \quad (2.39) $$

## 3 INTERNAL FLUCTUATION, VECTOR AND HIGGS SUPERMULTIPLICIT

In the theory of noncommutative geometry without supersymmetry, gauge fields and Higgs fields are derived through internal fluctuation of Dirac operator in the form expressed by

$$ \sum_j a_j [D_j, b_j] + \sum_j J a_j [D_j, b_j] J^{-1}, \quad a_j, b_j \in \mathcal{A}. \quad (3.1) $$

The second term of (3.1) is the fluctuation given by the elements of algebra which act on the antiparticle part and transferred to the particle part through the real structure $J$. And gauge fields are derived from the fluctuations for the Dirac operator which acts on the Riemann manifold, while the Higgs fields are given by those for the Dirac operator on the finite space.

We will consider to extend these construction to supersymmetric version. The major differences from the non-supersymmetric case is due to the fact that the algebra which is the source of the fluctuation is the direct sum of the spaces $A_+ \otimes A_F$.

Using (2.32), (2.33), (2.35), we obtain the following expressions:

$$ \overline{U}_a [iD_{tot}, U_a] = \begin{pmatrix} 0 & \bar{u}^{(Q)}_{aij} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2) $$

$$ U_a [iD_{tot}, \overline{U}_a] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.3) $$

$$ U_a [iD_{tot}, U_a] - U_a U_a [iD_{tot}, 1_+] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.4) $$

$$ [iD_{tot}, \overline{U}_a] U_a - [iD_{tot}, 1_-] \overline{U}_a U_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.5) $$

where $s = u_F(d_F)$. Eq. (3.2) and (3.3) give fluctuations only to $iD_M \otimes 1_F$, so we will see later that this type of fluctuation produce vector superfields, while Eq. (3.4) and (3.5) give contribution only to $\gamma_M \otimes D_F$ so that this type of fluctuation will give Higgs superfields.

### 3.1 VECTOR SUPERMULTIPLICIT

We introduce a set of elements of $\mathcal{A}$ in (2.32), (2.33) expressed by

$$ \Pi_+ = \{ u_a^{[r]} ; a = 1, 2, \cdots, n_r \} \subset \mathcal{A}_+ \otimes \mathcal{A}_F, \quad (3.6) $$

$$ \Pi_- = \{ \bar{u}_a^{[r]} ; a = 1, 2, \cdots, n_r \} \subset \mathcal{A}_- \otimes \mathcal{A}_F, \quad (3.7) $$

where $r = 1, 2, 3$ and $u_a^{[r]}$, $\bar{u}_a^{[r]}$ are in the matrix form of (2.20), (2.21) and given in the Table 1. The elements of $\Pi_+$ and $\Pi_-$ are chosen such that the products of $u_a^{[r]} s$ and $\bar{u}_a^{[r]} s$ do not belong to $\Pi_+, \Pi_-$ any more.
We can define elements \((A^{[r]}_{a}, \lambda^{[r]}_{a}, D^{[r]})\) of a vector supermultiplet as follows:

\[
m_{0}^{2} A_{a}^{[r]} = i \sum_{a} c_{a}^{[r]} \left[ (\varphi_{a}^{[r]} + \partial_{\mu} \varphi_{a}^{[r]} - \partial_{\mu} \varphi_{a}^{[r]} \varphi_{a}^{[r]} ) - i \sqrt{2} \lambda_{a}^{[r]} \varphi_{a}^{[r]} \right], \\
m_{0}^{2} \lambda_{a}^{[r]} = -\sqrt{2} i \sum_{a} c_{a}^{[r]} \left( F_{a}^{[r]} \varphi_{a}^{[r]} - \sqrt{2} \lambda_{a}^{[r]} \varphi_{a}^{[r]} \partial_{\mu} \varphi_{a}^{[r]} \right), \\
m_{0}^{2} D^{[r]} = -\sum_{a} c_{a}^{[r]} \left( -2(\partial^{\rho} \varphi_{a}^{[r]} \partial_{\rho} \varphi_{a}^{[r]}) + i \left\{ \partial_{\mu} \varphi_{a}^{[r]} \sigma^{\mu\alpha} \varphi_{a}^{[r]} - \varphi_{a}^{[r]} \sigma^{\mu\alpha} \partial_{\mu} \varphi_{a}^{[r]} \right\} + 2 F_{a}^{[r]} F_{a}^{[r]} \right),
\]

where \(c_{a}^{[r]}\) are the real coefficients and elements \((C^{[r]}, \lambda^{[r]}, M^{[r]}, N^{[r]})\) can be defined as follows:

\[
m_{0}^{2} C^{[r]} = -\sum_{a} c_{a}^{[r]} \varphi^{[r]} \varphi_{a}^{[r]}, \\
m_{0}^{2} \lambda^{[r]} = i \sqrt{2} \sum_{a} c_{a}^{[r]} \varphi^{[r]} \varphi_{a}^{[r]}, \\
m_{0}^{2} \lambda^{[r]} \lambda^{[r]} = -i \sqrt{2} \sum_{a} c_{a}^{[r]} \varphi^{[r]} \varphi_{a}^{[r]}, \\
m_{0}^{2} (M + i N)^{[r]} = 2i \sum_{a} c_{a}^{[r]} \varphi^{[r]} F_{a}^{[r]}, \\
m_{0}^{2} (M - i N)^{[r]} = -2i \sum_{a} c_{a}^{[r]} F_{a}^{[r]} \varphi^{[r]}.
\]

The condition for the fields in \((3.11)\) to vanish is Wess-Zumino condition and given by

\[
\begin{aligned}
\sum_{a} c_{a}^{[r]} \varphi^{[r]} \varphi_{a}^{[r]} = 0, \\
\sum_{a} c_{a}^{[r]} \varphi^{[r]} \varphi_{a}^{[r]} \varphi^{[r]} = 0, \\
\sum_{a} c_{a}^{[r]} \varphi^{[r]} \varphi_{a}^{[r]} F_{a}^{[r]} = 0.
\end{aligned}
\]

From \((2.9), (2.10), (2.13)\) and \((3.8)\) with Wess Zumino condition \((3.16)\), we obtain the following expressions:

\[
\begin{aligned}
\sum_{a} c_{a}^{[r]} u_{a}^{[r]} D_{ik} \left( \eta_{a}^{[r]} \right) = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & \sqrt{2} \lambda_{a}^{[r]} & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\sum_{a} c_{a}^{[r]} (u_{a}^{[r]}) D_{ik} \left( (u_{a}^{[r]} \right) = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & \sqrt{2} \lambda_{a}^{[r]} & 0 \\
0 & 0 & 0
\end{pmatrix}, \\
\sum_{a} c_{a}^{[r]} c_{a}^{[r]} \left( (u_{a}^{[r]} \right) = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & \sqrt{2} \lambda_{a}^{[r]} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\end{aligned}
\]

where \(u_{a}^{[r]} = \eta_{a}^{[r]} u_{a}^{[r]}\) and \(u_{a}^{[r]} = \bar{u}_{a}^{[r]} u_{a}^{[r]}\).

In the Table 1, we see that the algebra which act on \(Q_{a}\) field is the space of functions in \(\mathbb{H} \oplus M_{3}(\mathbb{C})\). Let us assume that we choose elements of \(A\) in \((3.32)\) in the quark sector as follows:

\[
U_{a}^{[r]} = \begin{pmatrix}
\eta_{a}^{[r]} & 0 \\
0 & 0
\end{pmatrix}, \quad U_{a}^{[r]} = \begin{pmatrix}
\eta_{a}^{[r]} & \eta_{a}^{[r]} \\
0 & 0
\end{pmatrix}, \quad r, r' = 2, 3.
\]

\]
The fluctuation for \( D, \bar{D} \) which produces vector supermultiplet is given by

\[
V = 2 \sum_{r=2}^{3} \sum_{a_r} c_{a_r}^r \bar{U}_{a_r}^r [iD_{tot}, U_{a_r}^r] + 2 \sum_{r=2}^{3} \sum_{a_r} \sum_{a_r'} c_{a_r}^r c_{a_r'}^r \bar{U}_{a_r}^r [iD_{tot}, U_{a_r'}^r] \\
- 2 \sum_{r=2}^{3} \sum_{a_r} c_{a_r}^r \bar{U}_{a_r}^r [iD_{tot}, \bar{U}_{a_r}^r] + 2 \sum_{r=2}^{3} \sum_{a_r} c_{a_r}^r c_{a_r'}^r \bar{U}_{a_r}^r [iD_{tot}, \bar{U}_{a_r'}^r] \\
= \left( \frac{0}{(V_{WZ})_{ij}} \right) \left( \frac{0}{0} \right) .
\]

(3.22)

Hereafter, for simplicity, we abbreviate the upper index \( [r] \) of \( c_{a_r}^r \) in (3.22) as far as there is not confusion. Using (3.17)~(3.20), the modified \( D, \bar{D} \) by these fluctuations amount to

\[
\hat{D}_Q = D + V_{WZ}^{(Q)} = \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2} \lambda^{(Q) \alpha} & i\sigma D_{\mu}^{(Q)} & 0 \\ D_{\mu}^{(Q)} D_{\nu}^{(Q) \mu} - D^{(Q)} & -i\sqrt{2} \lambda^{(Q) \alpha} & 0 \end{pmatrix},
\]

(3.23)

\[
\bar{D}_Q = \bar{D} + \bar{V}_{WZ}^{(Q)} = \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2} \lambda^{(Q) \alpha} & i\sigma D_{\mu}^{(Q)} & 0 \\ D_{\mu}^{(Q)} D_{\nu}^{(Q) \mu} + D^{(Q)} & -i\sqrt{2} \lambda^{(Q) \alpha} & 0 \end{pmatrix},
\]

(3.24)

where \( D_{\mu}^{(Q)} \) is the covariant derivatives

\[
D_{\mu}^{(Q)} = \partial_{\mu} - iA_{\mu}^{(Q)}, \quad A_{\mu}^{(Q)} = A_{\mu}^{[2]} + A_{\mu}^{[3]},
\]

(3.25)

and

\[
\lambda^{(Q)}_\alpha = \lambda^{[3]}_\alpha + \lambda^{[2]}_\alpha, \quad D^{(Q)} = D^{[3]} + D^{[2]}.
\]

(3.26)

The algebra which acts on \( u_R (d_R) \) field is the space of functions in \( \mathbb{C} \oplus M_3 (\mathbb{C}) \). If we choose the following elements of \( \mathcal{A} \) in (3.21):

\[
U_{a_r}^{[r]} = \begin{pmatrix} 0 & 0 \\ 0 & u_{a_r}^{[r]} \end{pmatrix}, \quad U_{a_r a_r'}^{[r,r']} = \begin{pmatrix} 0 & 0 \\ 0 & u_{a_r}^{[r]} u_{a_r'}^{[r']} \end{pmatrix}, \quad r, r' = 1(12), 3,
\]

(3.27)

instead of (3.23), (3.24), we obtain the following expressions:

\[
\hat{D}_s = \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2} \lambda^{(s) \alpha} & i\sigma D_{\mu}^{(s)} & 0 \\ D_{\mu}^{(s)} D_{\nu}^{(s) \mu} - D^{(s)} & -i\sqrt{2} \lambda^{(s) \alpha} & 0 \end{pmatrix}, \quad \bar{D}_s = \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2} \lambda^{(s) \alpha} & i\sigma D_{\mu}^{(s)} & 0 \\ D_{\mu}^{(s)} D_{\nu}^{(s) \mu} + D^{(s)} & -i\sqrt{2} \lambda^{(s) \alpha} & 0 \end{pmatrix},
\]

(3.28)

where \( s = u_R (d_R) \) and

\[
A_{\mu}^{(s)} = A_{\mu}^{[3]} + A_{\mu}^{[1(12)]}, \quad \lambda^{(s)} = \lambda^{[3]} + \lambda^{[1(12)]}, \quad D^{(s)} = D^{[3]} + D^{[1(12)]}.
\]

(3.29)

We show the detail of the above operations in Appendix A. We also show in Appendix A that \( (A_{\mu}^{[r]}, \lambda_\alpha^{[r]}, D^{[r]}) \) is a vector supermultiplet which becomes to the adjoint representation of \( U (r) \) gauge symmetry.

As for \( \hat{D}_s, \bar{D}_s \) in the lepton sector, we again the same form as (3.28), but the vector supermultiplet meditates \( U (2), U (1) \) internal degrees of freedom for \( s = l, s = e_R \), respectively.

The internal symmetry corresponding to elements of \( (M_3 (\mathbb{C}), \mathbb{H}, \mathbb{C}) \) in the Table 1 amounts to \( U (3) \times U (2) \times U (1) \). In order to construct unified theory, we relate \( U (1) \)'s included in the \( U (2), U (3) \) to the \( U (1) \)'s induced by the fluctuations which correspond to \( u_a^{[1]}, j = 1, 2, 3 \) in Table 1. We choose appropriate \( u_{a,j}^{[k]}, j = 0, \ldots, 8 \) in (A.1) and \( u_{2a}^{[k]}, k = 0, \ldots, 3 \) in (A.35) such that we obtain the following expressions:

\[
\sum_a c_a (\tilde{u}_a^{[1]})_{ik} D_{kl} (u_a^{[1]})_{lj} \\
= m_1^{-1} \sum_a c_a (\tilde{u}_a^{[1]})_{ik} D_{kl} (u_a^{[1]})_{lj} = m_2^{-1} \sum_a c_a (\tilde{u}_a^{[1]}_{12})_{ik} D_{kl} (u_a^{[1]}_{12})_{lj} \\
= m_3^{-1} \sum_a c_a (\tilde{u}_a^{[13]}_{1})_{ik} D_{kl} (u_a^{[13]})_{lj}
\]

\[8\]
The matrix form of weak hypercharge is given by

$$U$$

The derived internal symmetry

$$U$$

describes the strength of coupling between

$$\{B_\mu, \lambda_{1\alpha}, D_1\} \otimes 1_3$$  \hspace{1cm} (3.31)

$$\{A_\mu, \lambda_{0\alpha}, D_2\} \otimes 1_2$$  \hspace{1cm} (3.32)

The derived internal symmetry $$U(2)$$ becomes $$SU(2) \otimes U(1)$$ and $$U(3)$$ becomes $$SU(3) \otimes U(1)$$. These $$U(1)'s$$ combined with the other $$U(1)'s$$ shall become to that of weak hypercharge.

Using (3.31), we can rewrite (A.13) which is the fluctuation of

$$\lambda_{0\alpha} = \frac{1}{2} \left( G_{\mu_0}^{(0)}, \lambda_{3\alpha}^{(0)}, D_3^{(0)} \right) = z_3 \{B_\mu, \lambda_{1\alpha}, D_1\} \otimes 1_3$$

$$\tau_{0\alpha} = \frac{1}{2} \left( A_\mu^{(0)}, \lambda_{0\alpha}^{(0)}, D_2^{(0)} \right) = z_2 \{B_\mu, \lambda_{1\alpha}, D_1\} \otimes 1_2$$

In the same way, using (3.32), we can rewrite the fluctuation (A.12) into

$$V_1^{[3]} = z_3 \left( \begin{array}{ccc} 0 & 0 & 0 \\ -i\sqrt{2}\lambda_1^i & \sigma^{i\alpha\delta} B_\mu & -i\sqrt{2}\lambda_1^i \\ -D_1 - i\partial^\mu B_\mu - 2iB_\mu \partial^\mu & 0 & 0 \end{array} \right) 1_3$$

$$+ \sum_{p=1, \ldots, 8} \left( \begin{array}{ccc} 0 & 0 & 0 \\ -i\sqrt{2}\lambda_3^{(p)i\alpha} & \sigma^{i\alpha\delta} G_\mu^{(p)} & -i\sqrt{2}\lambda_3^{(p)i\alpha} \\ -D_3^{(p)} - i\partial^\mu G_\mu^{(p)} - 2iG_\mu^{(p)} \partial^\mu & 0 & 0 \end{array} \right) \frac{\lambda_{0\alpha}}{2}.$$  \hspace{1cm} (3.33)

Through the above operations, the vector supermultiplet of the $$Q$$-sector is described as follows:

$$\sum_{p=2,3} A^{[p]}_\mu = \frac{Y}{2} B_\mu + \sum_n \frac{\tau_n}{2} A^{(n)}_\mu + \sum_u \frac{\lambda_u}{2} G^{(u)}_\mu$$  \hspace{1cm} (3.35)

$$\sum_{p=2,3} \lambda^{[p]}_{\alpha} = \frac{Y}{2} \lambda_{1\alpha} + \sum_n \frac{\tau_n}{2} \lambda^{(n)}_{\alpha} + \sum_u \frac{\lambda_u}{2} \lambda^{(u)}_{\alpha}$$  \hspace{1cm} (3.36)

$$\sum_{p=2,3} D^{[p]} = \frac{Y}{2} D_1 + \sum_n \frac{\tau_n}{2} D^{(n)} + \sum_u \frac{\lambda_u}{2} D^{(u)}_3$$  \hspace{1cm} (3.37)

The vector supermultiplets in the other sector can be written as well. The quantum number $$Y$$ which describes the strength of coupling between $$\{B_\mu, \lambda_{1\alpha}, D_1\}$$ and quark, lepton is just weak hypercharge. The matrix form of weak hypercharge is given by

$$Y = \begin{pmatrix} z_2 + z_3 & m_1 + z_3 \\ m_2 + z_3 & z_2 \\ m_3 & m_3 \end{pmatrix} \begin{pmatrix} Q \\ u_R \\ : \ell \\ e_R \end{pmatrix}$$  \hspace{1cm} (3.38)

The matrix elements shall be decided such that the electric charge of the particle is given by

$$Q = \frac{\tau_3}{2} + \frac{Y}{2}.$$  \hspace{1cm} (3.39)
The concrete form of Higgs supermultiplet is given by

\[
\begin{align*}
\begin{cases}
z_2 + z_3 = \frac{1}{2} \\
m_1 + z_3 = \frac{2}{3} \\
m_2 + z_3 = -\frac{1}{3} \\
z_2 = -\frac{1}{2} \\
m_3 = -1
\end{cases}
\end{align*}
\]  
(3.40)

The solution of (3.39) is given by

\[
\begin{align*}
m_1 &= 0, \\
m_2 &= -1, \\
m_3 &= -1, \\
z_2 &= -\frac{1}{2}, \\
z_3 &= \frac{2}{3}
\end{align*}
\]  
(3.41)

Here we can verify Tr \( Y = 0 \).

### 3.2 Higgs Supermultiplet

The fluctuation for \( \gamma_\mu \otimes D_F \) is given by (3.4) and (3.5). Taking the commutativity of \( M^{(u(d))} \) and \( u^3 \) into account, non-vanishing elements for the quark sector are given by

\[
\begin{align*}
\frac{1}{2} \overline{\pi}_{u(d)} &= \sum_a c_a (|\mu_1|^2 M^{(u(d))}|_{\pi_a}^{[1\,1]} - M^{\ast (u(d))}|_{\pi_a}^{[1\,1]} |_{\pi_a}^{[1\,1]}), \\
\frac{1}{2} \overline{\pi}'_{u(d)} &= \sum_a c_a (|u_1|^2 M^{(u(d))} u_a^{[2]} - u_a^{[1\,1]} u_a^{[1\,2]} M^{(u(d))}).
\end{align*}
\]  
(3.42, 3.43)

The total Dirac operator modified by the fluctuations which have appeared up to the present in the quark sector is given by

\[
i\overline{D}_{tot} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ \overline{D}_Q & 0 & 0 \\ -iy_s \gamma_s H'_d & 0 & 0 \end{array}\right),
\]  
(3.44)

where \( s = u_R(d_R), s' = u(d) \) and

\[
\begin{align*}
H'_u &= (\mu_0, 0) + \mathcal{H}'_u, \\
\overline{H}_d &= (0, \mu_0) + \mathcal{H}'_d, \\
\overline{H}'_{\mu} &= (\mu_0, 0) + \overline{\mathcal{H}}_u, \\
\overline{H}'_{\mu} &= (0, \mu_0) + \overline{\mathcal{H}}_d.
\end{align*}
\]  
(3.45-3.46)

The concrete form of Higgs supermultiplet is given by

\[
\begin{align*}
\overline{H}_u &= (\mu_0, 0) + 2\mu_0 \sum_a \epsilon_a \left[ \left( \frac{\overline{u}_a^{(0)}}{\overline{u}_a^{(2)}} - \frac{i\overline{u}_a^{(1)}}{\overline{u}_a^{(2)}} \right) \overline{u}_a^{[1]} - 1 \right] \overline{u}_a^{[1]} \overline{u}_a^{[1]} \right].
\end{align*}
\]  
(3.47)

Here, putting that

\[
1 + 2 \sum_a \epsilon_a \left( \frac{\overline{u}_a^{(0)}}{\overline{u}_a^{(2)}} - \frac{i\overline{u}_a^{(1)}}{\overline{u}_a^{(2)}} \right) = \overline{v}_a^{(0)}, \text{etc.},
\]  
(3.48)

(3.47) can be expressed by

\[
\begin{align*}
\overline{H}'_u &= \mu_u \left( \overline{v}_u^{(0)} - i\overline{v}_u^{(3)} \right) = \left( \overline{H}_u^0 \right)^*, \\
\overline{H}'_{\mu} &= \mu_u \left( \overline{v}_u^{(2)} - i\overline{v}_u^{(1)} \right) = -\left( \overline{H}_u^0 \right)^*.
\end{align*}
\]  
(3.49)

so that

\[
H_u = \overline{H}_u^0 \overline{H}_u^0, \quad \overline{H}_u = (i\tau_2) H_u^0
\]  
(3.50)

which constructs a isospin doublet. In the same way, we obtain that

\[
H_d = \overline{H}_d^0 \overline{H}_d^0.
\]  
(3.51)
\[ \mathcal{H}_d^\text{kinetic} = \sum_{s=Q,l} (\varphi_s^* D^{(s)}(s)_{\mu} \varphi_s - i\bar{\varphi}_s \sigma^\mu D^{(s)}(\mu)_{s} \varphi_s + F^s \bar{F} + i\sqrt{2} (\varphi_s^* \lambda(\mu) \varphi_s - \bar{\varphi}_s \lambda(\mu) \varphi_s) - \varphi_s^* D(\mu) \varphi_s + \varphi_s^* D(\mu) \varphi_s) + \sum_{s=u,R,d,R_c,R_c} (\varphi_s D^{(s)}(s)_{\mu} \varphi_s - i\varphi_s \sigma^\mu D^{(s)}(\mu)_{s} \varphi_s + F^T \bar{F} + i\sqrt{2} (\varphi_s^* \lambda(\mu) \varphi_s - \bar{\varphi}_s \lambda(\mu) \varphi_s) + \varphi_s D(\mu) \varphi_s^* ), \]

\[ \mathcal{L}_\text{mass} = \sum_{s=u,a,c} (-i\bar{y}_s)(\Psi_{s-})^* H' \Psi_{s+}. \]

On the other hand, in the noncommutative geometric approach, the action of the vector and Higgs supermultiplets are given by the supersymmetric version of the Seeley-DeWitt coefficients of heat kernel expansion of the elliptic operator \( P \):

\[ Tr_{L^2} f(P) \simeq \sum_{n \geq 0} c_n a_n(P), \]

where \( f(x) \) is an auxiliary smooth function on a smooth compact Riemannian manifold without boundary of dimension 4 similar to the non-supersymmetric case. Since the contribution to \( P \) from the antiparticles is the same as that of the particles, we consider only the contribution from the particles. Then the elliptic operator \( P \) in our case is given by the square of the Wick rotated Euclidean Dirac operator \( \bar{D}_{\text{tot}} \). Non-vanishing \( a_n \)'s for \( n \) in the flat space are given by

\[ a_0(P) = \frac{1}{16\pi^2} \int_M d^4x \text{tr} V(1), \]

\[ a_2(P) = \frac{1}{16\pi^2} \int_M d^4x \text{tr} V(3), \]

\[ a_4(P) = \frac{1}{32\pi^2} \int_M d^4x \text{tr} V(3) + \frac{1}{3} \Omega_{\mu\nu} \Omega^{\mu\nu}. \]
where $\mathbb{E}$ and the bundle curvature $\Omega^{\mu\nu}$ in the flat space are defined as follows:

$$\mathbb{E} = \mathbb{B} - (\partial_{\mu}\omega^{\mu} + \omega_{\mu}\omega^{\mu}),$$

$$\Omega^{\mu\nu} = \partial^{\mu}\omega^{\nu} - \partial^{\nu}\omega^{\mu} + [\omega^{\mu}, \omega^{\nu}],$$

$$\omega^{\mu} = \frac{1}{2} \tilde{A}^{\mu}. \quad (4.10)$$

We note that the trace for the spin degrees of freedom is the supertrace $[9]$ defined by

$$\text{Str} O = \sum_i \langle i|(-1)^{2s}|i\rangle$$

$$= \sum_b \langle b|O|b\rangle - \sum_f \langle f|O|f\rangle, \quad (4.11)$$

The square of total Dirac operator for the quark sector (3.41) is given by

$$(i\tilde{D}_{\text{tot}})^2 = \begin{pmatrix}
D_Q \bar{D}_Q & 0 & 0 & D_{14} \\
0 & \tilde{D}_Q \bar{D}_Q & D_{23} & 0 \\
0 & D_{32} & D_s \bar{D}_s & 0 \\
D_{41} & 0 & 0 & \tilde{D}_s \bar{D}_s
\end{pmatrix}, \quad (4.12)$$

where

$$D_{14} = ig^{\dagger}_s \begin{pmatrix}
F_{s}\tilde{\sigma}^{s}\left(\frac{1}{2}\lambda_1^{(Q)}h_{s}^{\dagger} + i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}\tilde{h}_{s}^{\dagger}\right)h_{s}^{\dagger} & -\tilde{h}_{s}^{\dagger} & h_{s}^{\dagger} \\
D_{\mu}^{(Q)}D_{\lambda}^{(Q)}\tilde{h}_{s}^{\dagger} & i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}\tilde{h}_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}\tilde{h}_{s}^{\dagger} & 0 \\
0 & -i\sqrt{2}\lambda_{\alpha}^{(Q)}\tilde{h}_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}\tilde{h}_{s}^{\dagger} & 0
\end{pmatrix},$$

$$D_{41} = -ig\nu^{\dagger}_s \begin{pmatrix}
F_{s}\tilde{\sigma}^{s}\left(-i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} + i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}h_{s}^{\dagger}\right)h_{s}^{\dagger} & -\tilde{h}_{s}^{\dagger} & h_{s}^{\dagger} \\
D_{\mu}^{(Q)}D_{\lambda}^{(Q)}h_{s}^{\dagger} & i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}h_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & 0 \\
0 & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & 0
\end{pmatrix},$$

$$D_{23} = ig\nu^{\dagger}_s \begin{pmatrix}
0 & F_{s}\tilde{\sigma}^{s}\left(-i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} + i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}h_{s}^{\dagger}\right)h_{s}^{\dagger} & -\tilde{h}_{s}^{\dagger} & h_{s}^{\dagger} \\
-D_{\mu}^{(Q)}D_{\lambda}^{(Q)}h_{s}^{\dagger} & i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}h_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & 0 \\
0 & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & 0
\end{pmatrix},$$

$$D_{32} = -ig\nu^{\dagger}_s \begin{pmatrix}
0 & F_{s}\tilde{\sigma}^{s}\left(-i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} + i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}h_{s}^{\dagger}\right)h_{s}^{\dagger} & -\tilde{h}_{s}^{\dagger} & h_{s}^{\dagger} \\
D_{\mu}^{(Q)}D_{\lambda}^{(Q)}h_{s}^{\dagger} & i\tilde{\sigma}^{\mu}\alpha_{\tilde{\sigma}}D_{\mu}^{(Q)}h_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & 0 \\
0 & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & -i\sqrt{2}\lambda_{\alpha}^{(Q)}h_{s}^{\dagger} & 0
\end{pmatrix},$$

where $s = u_R(d_R), s' = u(d)$. When we decompose

$$(i\tilde{D}_{\text{tot}})^2 = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}1 + \tilde{A}^{\mu}\partial_{\mu} + \mathbb{B}, \quad (4.17)$$

$\omega = \frac{1}{2} \tilde{A}^{\mu}$ describes gauge connection and does not have non-diagonal elements of (4.12), because it does not act on the finite space $\mathcal{H}_F$. So the non-diagonal elements which include differential operators belong to $\mathbb{B}$. So, we obtain the following expressions:

$$E_{41} = \mathbb{B} - \partial^{\mu}\omega_{\mu} - \omega_{\mu}\omega^{\mu} = E_{41} = D_{41},$$

$$E_{32} = E_{32} = D_{32}, \quad E_{23} = E_{23} = D_{23}, \quad E_{14} = E_{14} = D_{14}. \quad (4.18)$$

We note that the field strength $\Omega^{\mu\nu}$ also does not have non-diagonal element.

In the previous paper [9], we have already obtained the heat kernel expansion coefficients $a_n(P)$ due to $(i\mathcal{D}^2) \otimes 1_F$. Let $a^{(M)}_n$ denotes these coefficients, $a^{(M)}_0(P)$ and $a^{(M)}_2$ vanish. The action for the gauge fields and its superpartners in MSSM comes from $a^{(M)}_1$ and is given by

$$I_{\text{gauge}} = 16\pi^2 f_0 \int_M dx^4 a^{(M)}_s(P) = f_0 \int_M dx^4 \sum_s \text{Tr} \left[ 2D^{(s)2} - 4\lambda_\beta^{(s)}\tilde{\sigma}^{\mu\beta}(D^{(s)}\lambda^{(s)}) - F^{(s)}F^{(s)\mu\nu} \right], \quad (4.19)$$
where $s$ runs over the elements of $\mathcal{H}_F$, $s = Q, u_R, d_R, l, e_R$ and $F_{\mu\nu}^s$ is the field strength of the matter particle that $s$ indicates.

Here, we rescale the vector supermultiplet with $\text{SU}(3)$ and $\text{SU}(2)$ gauge degrees of freedom as follows:

\begin{align}
(A^\mu, \lambda_\alpha, D) &\rightarrow g_3(A^\mu_3, \lambda_\alpha_3, D_3) = \sum_{p=1}^{8} g_3(A^{(p)}_3, \lambda^{(p)}_3, D^{(p)}_3) \lambda_p \left( \frac{\lambda_p}{2} \right), \\
(A^\mu, \lambda_\alpha, D) &\rightarrow g_2(A^\mu_2, \lambda_\alpha_2, D_2) = \sum_{p=1}^{3} g_2(A^{(p)}_2, \lambda^{(p)}_2, D^{(p)}_2) \tau_p \left( \frac{\tau_p}{2} \right),
\end{align}

where $\lambda_p, \tau_p$ denotes Gell-Mann matrix and Pauli matrix, respectively. For the vector supermultiplet with reference to weak hypercharge, we rescale those as

\begin{align}
(B_\mu, \lambda_\alpha, D) &\rightarrow (g_1 \frac{Y_s}{2} A_1, g_1 \lambda_1, g_1 D).
\end{align}

The field strength is expressed by

\begin{align}
F_{\mu\nu} = \partial_\mu A_{\nu} - \partial_\nu A_\mu - ig[A_{\mu}, A_\nu].
\end{align}

Since $A^Q_\mu = g_3 A_{3\mu} + g_2 A_{2\mu} + g_1 \frac{1}{2} \times \frac{1}{3} A_{1\mu}$ and $\text{Tr}(\lambda_\rho \lambda_\sigma) = \text{Tr}(\tau_\rho \tau_\sigma) = 2 \delta_{\rho\sigma}$, the contribution of $F_{Q\mu\nu}$ to the action is given by

\begin{align}
\text{Tr}(F^Q_{\mu\nu} F^{(Q)\mu\nu}) = 2 \sum_{j=1}^{6} F_{3\mu\nu}^{(j)} F_{3\mu\nu}^{(j)} + 3 \sum_{j=1}^{6} F_{2\mu\nu}^{(j)} F_{2\mu\nu}^{(j)} + 6 \sum_{j=1}^{6} \left( \frac{1}{3} \right)^2 g_1^2 F_{1\mu\nu} F_{1\mu\nu}.
\end{align}

In the r.h.s. of (4.24), the factor 2, 3, 6 is from the fact that the left-handed quark $Q$ transforms as $(3,2)$ under the gauge group $\text{SU}(3) \times \text{SU}(2)$. In the same way, we obtain that

\begin{align}
\sum_{s=1}^{6} \text{Tr}(F_{\mu\nu}^{(s)} F^{(s)\mu\nu}) = 2 \sum_{j=1}^{6} F_{3\mu\nu}^{(j)} F_{3\mu\nu}^{(j)} + 3 \sum_{j=1}^{6} F_{2\mu\nu}^{(j)} F_{2\mu\nu}^{(j)},
\end{align}

The sum of (4.24), (4.25) and (4.26) amounts to

\begin{align}
f_0 \sum_{s} \text{Tr}(F_{\mu\nu}^{(s)} F^{(s)\mu\nu}) = 2f_0 \left( g_3^2 \sum_{j=1}^{6} F_{3\mu\nu}^{(j)} F_{3\mu\nu}^{(j)} + g_2^2 \sum_{j=1}^{6} F_{2\mu\nu}^{(j)} F_{2\mu\nu}^{(j)} + \frac{5}{3} g_1^2 F_{1\mu\nu} F_{1\mu\nu} \right).
\end{align}

Normalizing the Yang-Mills terms to $-\frac{1}{4} F_{\mu\nu}^{(j)} F^{(j)\mu\nu}$ gives:

\begin{align}
g_3^2 = g_2^2 = \frac{5}{3} g_1^2, \quad 2f_0 g_3^2 = \frac{1}{4}.
\end{align}

Known already in [7], this expression is same as that of $\text{SU}(5)$ grand unification theory.

The non-diagonal elements of (4.12) produce Higgs Lagrangian. Let us calculate $\text{tr}_V((\mathbb{E}^2)^{\text{higgs}}) = \text{Str}(\sum_{i,j} \mathbb{E}_{ij} \mathbb{E}_{ij})$. Using (4.13) ~ (4.16), we obtain the result as follows:

\begin{align}
\|y_s\|^{-2} \text{Str} (\mathbb{E}_{14} \mathbb{E}_{41}) = \|y_s\|^{-2} \text{Str} (\mathbb{E}_{23} \mathbb{E}_{32}) = \|y_s\|^{-2} \text{Str} (\mathbb{E}_{42} \mathbb{E}_{24}) = \|y_s\|^{-2} \text{Str} (\mathbb{E}_{41} \mathbb{E}_{14}) \\
= F_{\mu}^{\nu} F_{\mu}^{\nu} - (\partial_\mu h_+ + i h_+ A^{[H]}_\mu) (\partial_\mu h_+ - i A^{[H]}_\mu h_+) - i h_+^{\mu\nu} \sigma^{\mu\nu} (\partial_\mu h_{s} - i A^{[H]}_\mu h_{s}) \\
+ i \sqrt{2} (\bar{h}_+ h_+^{[H]} h_+^{[H]} - h_+ h_+^{[H]} h_{s}) + h_+^{[H]} h_{s} h_{s},
\end{align}

where $s = u(d)$,

\begin{align}
A^{[H]}_\mu = A^{[Q]}_\mu - A^{[u_R]}_\mu, \quad \lambda^{[H]}_\alpha = \lambda^{Q}_\alpha - \lambda^{u_R}_\alpha, \quad D^{[H]} = D^{[Q]} - D^{[u_R]},
\end{align}

\begin{align}
A^{[H]}_\mu = A^{[Q]}_\mu - A^{[d_R]}_\mu, \quad \lambda^{[H]}_\alpha = \lambda^{Q}_\alpha - \lambda^{d_R}_\alpha, \quad D^{[H]} = D^{[Q]} - D^{[d_R]},
\end{align}

where

\begin{align}
\mathbb{E} = \begin{bmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{bmatrix},
\end{align}

\begin{align}
E_{ij} = \begin{bmatrix}
\mathbb{E}_{ij} & 0 \\
0 & 0
\end{bmatrix},
\end{align}

\begin{align}
\mathbb{E}_{ij} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}.
\end{align}
and
\[
\text{diag } y_+ y_s = \text{diag } y_s y_+ = ((y_s)_{11}, (y_s)_{22}, (y_s)_{33})^2, 
\]
\[
\text{Tr } y_+ y_s = \text{Tr } y_s y_+ = \sum_i |(y_s)_{i i}|^2 = \|y_s\|^2. 
\]

Taking \(\text{tr}_V(\mathbb{E}^2_{\text{higgs}})\) due to the lepton sector into account, the supersymmetric action of Higgs fields which interact with vector superfields is given by

\[
I_{\text{Higgs}} = 4f_0 \sum_{s = u, d, e} \|y_s\|^2 \left( F_s^{\mu} F_s^{\mu*} - |D_{\mu}^\mathcal{H}_s| h_s^{\mu*} \right|^2 - i h_s^{\mu*} \sigma_{\alpha}^{\mu} D_{\mu}^\mathcal{H}_s \tilde{h}_s^{\mu*} \\
+ i \sqrt{2} \left( \tilde{h}_s^{\mu*} \lambda_{\alpha}^{\mathcal{H}_s} h_s^{\mu*} - h_s^{\mu*} \tilde{\lambda}_{\alpha}^{\mathcal{H}_s} \tilde{h}_s^{\mu*} \right) + h_s^\mathcal{H}_s h_s^{\mu*} \\
+ \frac{1}{8g_3^2} \left( y_u^2 + y_d^2 \right) \left( F_d^{\mu} F_d^{\mu*} - |D_{\mu}^\mathcal{H}_d| h_d^{\mu*} \right|^2 - i h_d^{\mu*} \sigma_{\alpha}^{\mu} D_{\mu}^\mathcal{H}_d \tilde{h}_d^{\mu*} \\
+ i \sqrt{2} \left( \tilde{h}_d^{\mu*} \lambda_{\alpha}^{\mathcal{H}_d} h_d^{\mu*} - h_d^{\mu*} \tilde{\lambda}_{\alpha}^{\mathcal{H}_d} \tilde{h}_d^{\mu*} \right) + h_d^\mathcal{H}_d h_d^{\mu*}, \right. 
\]

where we use (4.28). Then we have obtained all the terms of correct Lagrangian which give MSSM.

5 CONCLUSIONS

We defined the "triple" extended from the spectral triple which was to specify NCG. As the functional space, we take chiral and antichiral supermultiplets which correspond to matter fields and their superpartners in the supersymmetric standard model. We introduced algebra \(\mathcal{A}\) and total Dirac operators \(i\mathcal{D}_{\text{tot}} = i\mathcal{D}_M \otimes 1 + \gamma_M \otimes \mathcal{D}_F\) in the flat space-time which acted on the functional space \(\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F\). We considered the internal fluctuations induced by the elements of \(\mathcal{A}\). The fluctuation for \(i\mathcal{D}_M \otimes 1\) generated vector supermultiplets. The vector supermultiplets mediate \(U(3) \otimes U(2) \otimes U(1)\) gauge degrees of freedom. We took out \(U(1)\)’s from \(U(3), U(2)\) and combined the other \(U(1)\)’s to obtain \(U(1)\) of weak hypercharge so that the gauge degrees of freedom amounted to those of MSSM, \(SU(3) \times SU(2) \times U(1)_Y\) and each matter particle was distributed to adequate quantum numbers. On the other hand, the fluctuation for \(\gamma_M \otimes \mathcal{D}_F\) generated supermultiplets which transformed as Higgs fields of MSSM.

Following the supersymmetric version of spectral action principle, we calculated the square of the total Dirac operator and Seeley-Dewitt coefficients of heat kernel expansion. From the coefficient \(a_4(P)^{(3)}\), we obtained the action of vector supermultiplets of MSSM. Normalizing the coefficients of the squared field strength of each of \(SU(3), SU(2), U(1)_Y\) gauge field to the same value, we found the relations between coupling constants which was same as that of SU(5) grand unification theory. We also verified that the coefficient due to non-diagonal elements of total Dirac operator gave the action for the Higgs supermultiplets and that we arrived at the correct whole action of MSSM.

All formulae in this paper were established in the Minkowskian space. We are now preparing the theory in which the total Dirac operator in the curved Riemannian space is assumed in order to take gravity into account. It will give the supersymmetric version of NC geometric view of unifying the gravity and gauge, Higgs fields.
Appendix

A Vector supermultiplet with U(r) gauge symmetry

In this appendix, we will see that choosing suitable components of \([2,32,2,33]\), we can construct the vector supermultiplet in \([(3,8)\sim(3,10)]\) which meditates the \(U(r)\) gauge degrees of freedom.

A.1 U(1)

Choosing as the algebra the space of complex functions, \(u_a^{[1]}(x), j = 1, 2, 3\) in \([2,32,2,33]\), the vector supermultiplet defined in \([(3,8)\sim(3,10)]\) meditate U(1) gauge symmetry of which the phase \(m_j(j = 1, 2, 3)\) in the Table 1 represents the quantum number of \(u_R, d_R, c_R\) field, respectively.

A.2 U(3)

In this case, the elements of \(A_F \otimes A_M\) are \(3 \times 3\) matrix-valued functions \((u_a^{[3]}(x))_A^B, A, B = 1, 2, 3\) which act on the internal degrees of freedom of quarks \(A = 1, 2, 3\). The \(3 \times 3\) matrix-valued functions \((u_a^{[3]}(x))_A^B \in A_F \otimes A_+\) and \((\bar{u}_a^{[3]}(x))_A^B \in A_F \otimes A_-\) can be expressed in the matrix form as follows:

\[
(u_a^{[3]}(x))_A^B = \sum_{k=0,1,\ldots,8} u_a^{(k)}(x) \left( \frac{\lambda_k}{2} \right)_A^B \quad (A.1)
\]

\[
(\bar{u}_a^{[3]}(x))_A^B = \sum_{k=0,1,\ldots,8} \bar{u}_a^{(k)}(x) \left( \frac{\lambda_k}{2} \right)_A^B \quad (A.2)
\]

where \(\lambda_k\) are Gell-Mann matrices.

The internal fluctuation due to \((u_a^{[3]}(x))_A^B\) and \((u_a^{[3]}(x))_A^B\) is given by

\[
(V_{ij})_{ij}^C = 2 \sum_{a} c_a (\bar{u}_a^{[3]}(x))_{ik} A' D_{kl} [(u_a^{[3]}(x))_{lj}]_{A'}^C + \sum_{a, b} c_0 c_b (\bar{u}_b^{[3]}(x))_{ik} A' D_{kl} [(u_a^{[3]}(x))_{lj}]_{A'}^B C
\]

Substituting \((A.1), (A.2)\), the fluctuation due to the part of \((u_a^{[3]}(x))_A^B\) is obtained by

\[
(V_{ij})_{ij}^C = 2 \sum_{a} c_a \left\{ \sum_{p=0,1,\ldots,8} (\bar{u}_a^{(p)}(x))_{ik} D_{kl} [(u_a^{(p)}(x))_{lj}]_{A'}^C \right. + \sum_{u} \left[ (\bar{u}_a^{(0)}(x))_{ik} D_{kl} [(u_a^{(0)}(x))_{lj}]_{A'}^B \left( \frac{\lambda_0}{2} \right)_A^C \right) \left\}
\]

where we use the following formulae:

\[
\lambda_0 = \sqrt{2/3} \text{ diag}(1, 1, 1) \quad (A.5)
\]

\[
[\lambda_p, \lambda_t] = 2i \delta_{pt} \lambda_u, \quad \{\lambda_p, \lambda_t\} = 2d_{ptu} \lambda_u + \frac{4}{3} \delta_{pt} (p, t, u = 1, \cdots, 8) \quad (A.6)
\]

From \((2.9), (2.10), (2.13)\), the product of \(\bar{u}_a, D_{itu}\) is given by

\[
\bar{u}_a \psi_{a}^{\alpha} F_\alpha = \frac{1}{m_0^2} \psi_{a}^{\alpha} F_\alpha - \psi_{a}^{\alpha} \varphi_{a} \quad (A.7)
\]
In the way same as (A.7), we obtain the following formula:

\[
\begin{align*}
\sum_a c_a (\bar{u}_a(3)^{(p)})_{ik} D_{k\ell} (u_3^{(t)})_{\ell j} &= \frac{1}{2} \begin{pmatrix}
0 & 0 \\
-\sqrt{2}\lambda_3^{(p)} & -i\gamma^\mu G_{\mu}^{(p)} - 2iG_{\mu}^{(p)} \gamma^\mu - i\sqrt{2} \lambda_3^{(p)}
\end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix}
0 & 0 \\
-\sqrt{2} \lambda_3 & -i\gamma^\mu G_{\mu} - 2iG_{\mu} \gamma^\mu - i\sqrt{2} \lambda_3
\end{pmatrix} (A.8)
\end{align*}
\]

where, we use the Wess-Zumino condition

\[
\sum_a c_a (\bar{\varphi}_a^{(3)} + \varphi_a^{(3)}) A = \sum_a c_a (\bar{\varphi}_a^{(3)} + \varphi_a^{(3)}) A = \sum_a c_a (\varphi_a^{(3)} + \bar{\varphi}_a^{(3)}) A = 0. (A.12)
\]

Substituting (A.8) into (A.4) we obtain the following formula:

\[
\begin{align*}
(V_{i3})^C &= 2 \sum_a c_a ([\bar{u}_a^{(3)}])_{ik} B D_{k\ell} ([u_3^{(3)})_{\ell j}] B \\
&= \sum_{\mu=0,\ldots,8} \begin{pmatrix}
0 & 0 \\
-\sqrt{2} \lambda_3 & -i\gamma^\mu G_{\mu} - 2iG_{\mu} \gamma^\mu - i\sqrt{2} \lambda_3
\end{pmatrix} \left( \frac{\lambda_{\mu}}{2} \right) A, (A.13)
\end{align*}
\]

where

\[
G_{\mu}^{(0)} = \frac{1}{\sqrt{6}} \sum_{\mu=0,\ldots,8} G_{\mu}^{(p,p)} (A.14)
\]

\[
\lambda_3^{(0)} = \frac{1}{\sqrt{6}} \sum_{\mu=0,\ldots,8} \lambda_3^{(p,p)} (A.15)
\]

\[
D_3^{(0)} = \frac{1}{\sqrt{6}} \sum_{\mu=0,\ldots,8} D_3^{(p,p)} (A.16)
\]

and

\[
G_{\mu}^{(u)} = \frac{1}{\sqrt{6}} G_{\mu}^{(0,u)} + \frac{1}{\sqrt{6}} G_{\mu}^{(u,0)} + \frac{1}{2} \sum_{\mu,t=1,\ldots,8} (i f_{ptu} + d_{ptu}) G_{\mu}^{(p,t)} (A.17)
\]

\[
\lambda_3^{(u)} = \frac{1}{\sqrt{6}} \lambda_3^{(0,u)} + \frac{1}{\sqrt{6}} \lambda_3^{(u,0)} + \frac{1}{2} \sum_{\mu,t=1,\ldots,8} (i f_{ptu} + d_{ptu}) \lambda_3^{(p,t)} (A.18)
\]

\[
D_3^{(u)} = \frac{1}{\sqrt{6}} D_3^{(0,u)} + \frac{1}{\sqrt{6}} D_3^{(u,0)} + \frac{1}{2} \sum_{\mu,t=1,\ldots,8} (i f_{ptu} + d_{ptu}) D_3^{(p,t)}. (A.19)
\]

In (A.8) and (A.11),

\[
G_{\mu}^{(p,t)*} = G_{\mu}^{(p,t)} (A.17)
\]

\[
D_3^{(p,t)*} = D_3^{(p,t)} (A.18)
\]

so that \(G_{\mu}^{(u)}\) and \(D_3^{(u)}\) are real functions, while \(\lambda_3^{(u)}\) is a complex spinor field. The vector supermultiplet \((G_{\mu}^{(u)}, \lambda_3^{(u)}), D_3^{(u)})\), \(u = 0, 1, \ldots, 8\) obeys the transformation law of the adjoint representation of \(U(3)\). (A.14) is equivalent to decomposing the tensor product \(8 \otimes 8\) under \(SU(3) \subset U(3)\) into the direct sum of irreducible representations and extract the term of \(1 \otimes 8\).

Next we deal with the fluctuation due to the part of \(u^{[3]}_{ab} \Lambda C = (u_a^{[3]} B (u_a^{[3]} C B)\). As for the representation of \(SU(3)\) it is to extract \((1 \otimes 8) \times (1 \otimes 8)\) from irreducible representations into which the tensor
product \((8 \otimes 8) \otimes (8 \otimes 8)\) is decomposed, where \(\times\) denote the product of representation matrix. So we contract the tensor product as shown in the second term of \((A.3)\). We let \(V'_{ij}^{[33]}\) denote the second term of \((A.3)\). Under the Wess-Zumino condition, we calculate the matrix elements of \(V'_{ij}^{[33]}\):

\[
\frac{1}{2} (V'_{ij}^{[33]})_A^C = \frac{1}{m_0^2} \sum_{a,b} c_a c_b (\varphi_{ab}^{[33]})^A_{AB} (\varphi_{ab}^{[33]})^B_{A'B'} D_{km} \left( u_{ab}^{[33]} \right)_{A'B'}^{BC}.
\]

Continuing the similar calculations, we see that the matrix elements vanish except for \(V'_{31}^{[33]}\).

The element \(V'_{31}^{[33]}\) is given as follows:

\[
\frac{1}{2} (V'_{31}^{[33]})_A^C = \frac{1}{m_0^2} \sum_{a,b} c_a c_b \left( (\varphi_{ab}^{[33]} A'B')_{AB} (\varphi_{ab}^{[33]} A'B')_{AB} - i (\psi_{ab}^{[33]} A'B') \right)
\]

so that

\[
\frac{1}{2} (V'_{31}^{[33]})_A^C = \frac{1}{m_0^2} \sum_{a,b} c_a c_b \left[ \frac{1}{2} (\varphi_{ab}^{[33]} A'B')_{AB} (\varphi_{ab}^{[33]} A'B')_{AB} - i (\psi_{ab}^{[33]} A'B') \right]
\]

Here, using \((A.13)\) and \((A.17)\), \(A^{[3]}_{\mu} \) is given by

\[
A^{[3]}_{\mu} = \frac{i}{m_0^2} \sum_{a} c_a \left[ \left( (\varphi_{a}^{[3]} A'B') \right)_{AB} - i (\psi_{a}^{[3]} A'B') \right]
\]

The internal fluctuation \((A.3)\) due to \((A.13)\) amounts to

\[
(V'_{ij}^{[33]})_{ij} = V^{[3]}_{ij} + V^{[33]}_{ij}
\]

where

\[
A^{[3]}_{\mu} = \frac{i}{m_0^2} \sum_{a} c_a \left[ \left( (\varphi_{a}^{[3]} A'B') \right)_{AB} - i (\psi_{a}^{[3]} A'B') \right]
\]

\[
A^{[3]}_{\mu} = \frac{i}{m_0^2} \sum_{a} c_a \left[ \left( (\varphi_{a}^{[3]} A'B') \right)_{AB} - i (\psi_{a}^{[3]} A'B') \right]
\]
where
\[
\lambda^{[3]}_\alpha = \sum_{p=0,\ldots,8} \lambda^{(p)}_{3\alpha} \frac{\lambda_2}{2}, \tag{A.30}
\]
\[
D^{[3]} = \sum_{p=0,\ldots,8} D^{(p)}_3 \frac{\lambda_2}{2}. \tag{A.31}
\]

Repeating the similar calculation, the fluctuation for the opposite chirality sector is given by
\[
(\nabla^{[3]}_{WZ})_{ij} = \begin{pmatrix} 0 & 0 & -i\sqrt{2}\lambda^{[3]}_\alpha & 0 \\ -i\sqrt{2}\lambda^{[3]}_\alpha & 0 & 0 & 0 \\ 0 & 0 & -i\sigma_{\alpha\alpha}^{\mu} A^{[3]}_\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.32}
\]

### A.3 U(2)

Here, we calculate internal fluctuation due to quaternion-valued function. Quaternion-valued chiral multiplet \((u^{[2]}_a)_{\mu} \in \mathcal{A}_F \otimes \mathcal{A}_+\) is expressed into matrix form as follows:
\[
(u^{[2]}_a)^{J} = \begin{pmatrix} u^{(0)}_a(x) + iu^{(2)}_a(x) & u^{(2)}_a(x) + iu^{(1)}_a(x) \\ -iu^{(3)}_a(x) + iu^{(1)}_a(x) & u^{(0)}_a(x) - iu^{(3)}_a(x) \end{pmatrix}^J \tag{A.33}
\]
where \(u^{(k)}_a(x), (k = 0, 1, 2, 3)\) is given by
\[
(u^{(k)}_a)_{ij} = \frac{1}{m_0} \begin{pmatrix} \varphi^{(k)}_a & 0 & 0 \\ \psi^{(k)}_a & \varphi^{(k)}_a & 0 \\ 0 & 0 & (\varphi^{(k)}_a) \end{pmatrix} \tag{A.34}
\]

The matrix form of \((\bar{\pi}^{[2]}_a)_{IJ} \in \mathcal{A}_F \otimes \mathcal{A}_-\) is also given by
\[
(\bar{\pi}^{[2]}_a)^{J} = \begin{pmatrix} \bar{u}^{(0)}_a(x) - i\bar{u}^{(2)}_a(x) & \bar{u}^{(2)}_a(x) - i\bar{u}^{(1)}_a(x) \\ -i\bar{u}^{(1)}_a(x) - i\bar{u}^{(3)}_a(x) & \bar{u}^{(0)}_a(x) + i\bar{u}^{(3)}_a(x) \end{pmatrix}^J \tag{A.35}
\]
where \(\bar{u}^{(k)}_a(x)\) is given by
\[
(\bar{u}^{(k)}_a)_{ij} = \frac{1}{m_0} \begin{pmatrix} \varphi^{(k)}_{a\dot{\alpha}} & 0 & 0 \\ \psi^{(k)}_{a\dot{\alpha}} & \varphi^{(k)}_{a\dot{\alpha}} & 0 \\ 0 & 0 & (\varphi^{(k)}_{a\dot{\alpha}}) \end{pmatrix} \tag{A.36}
\]

Developing \(\bar{u}^{[2]}_a D^{(2)} u^{[2]}_a\) with the aid of \((A.33), (A.35)\), we obtain
\[
\sum_a c_a (\bar{u}^{[2]}_a)_{i\dot{k}} D_{\dot{k}k} [(u^{[2]}_a)_{\mu}]_{iJ} = \sum_a c_a \left[ \left\{ (\bar{u}^{(0)}_a)_{i\dot{k}} D_{\dot{k}k}(u^{(2)}_a)_{\mu} \right\}_J + \sum_{m=1,2,3} (\bar{u}^{(m)}_a)_{i\dot{k}} D_{\dot{k}k}(u^{(m)}_a)_{\mu} \right] (\tau_0)_{iJ} \\
+ i \sum_{n=1,2,3} \left\{ (\bar{u}^{(0)}_a)_{i\dot{k}} D_{\dot{k}k}(u^{(n)}_a)_{\mu} - (\bar{u}^{(n)}_a)_{i\dot{k}} D_{\dot{k}k}(u^{(0)}_a)_{\mu} \right\} (\tau_n)_{iJ} \\
+ \frac{i}{2} \sum_{m,s,n=1,2,3} \left\{ (\bar{u}^{(m)}_a)_{i\dot{k}} D_{\dot{k}k}(u^{(s)}_a)_{\mu} - (\bar{u}^{(s)}_a)_{i\dot{k}} D_{\dot{k}k}(u^{(m)}_a)_{\mu} \right\} \varepsilon_{msn}(\tau_n)_{iJ} \tag{A.37}
\]

We also obtain the following expressions
\[
2 \sum_a c_a (\bar{u}^{(m)}_a)_{i\dot{k}} D_{\dot{k}k} (u^{(s)}_a)_{\mu} = \begin{pmatrix} 0 & 0 & -i\sqrt{2} \bar{\pi}^{(m,s)}_a & 0 \\ -i\sqrt{2} \bar{\pi}^{(m,s)}_a & 0 & 0 & 0 \\ 0 & 0 & -i\sigma_{\alpha\alpha}^{\mu} A^{(m,s)}_\mu & 0 \\ 0 & 0 & 0 & -i\sqrt{2} \lambda^{(m,s)}_\alpha \end{pmatrix} \tag{A.38}
\]
\[ A_{\mu}(m,s) = \frac{i}{m_0^2} \sum_a c_a \left( \bar{\varphi}_{2\alpha}^{(m)} \partial_\mu \varphi_{2\alpha}^{(s)} - \partial_\mu \varphi_{2\alpha}^{(m)} \bar{\varphi}_{2\alpha}^{(s)} - i \bar{\psi}_{2\alpha}^{(m)} \sigma_{\mu \alpha} \psi_{2\alpha}^{(s)} \right), \]  
\[ \lambda_{2\alpha}(m,s) = -\sqrt{2} \frac{i}{m_0^2} \sum_a c_a \left( \bar{F}_{2\alpha}^{(m)}(\varphi_{2\alpha}^{(s)}) - i \sigma_{a\alpha} \bar{\varphi}_{2\alpha}^{(m)} \partial_\mu \varphi_{2\alpha}^{(s)} \right), \]  
\[ D_{2}^{(m,s)} = -\frac{1}{m_0^2} \sum_a c_a \left[ 2F_{2\alpha}^{(m)} F_{2\alpha}^{(s)} - 2(\partial_\mu \varphi_{2\alpha}^{(m)} - \partial_\mu \varphi_{2\alpha}^{(s)}) \right] + i \left\{ \partial_\mu \bar{\psi}_{2\alpha}^{(m)} \sigma_{\mu \alpha} \psi_{2\alpha}^{(s)} - \bar{\psi}_{2\alpha}^{(m)} \sigma_{\mu \alpha} \partial_\mu \psi_{2\alpha}^{(s)} \right\}. \]

Substituting these into (A.37), we obtain

\[ V_1^{[2]} = 2 \sum_a c_a (\bar{u}_{\mu}^{[2]})_{ik} D_{k\ell} (u_{\alpha}^{[2]})_{\ell j} = \begin{pmatrix} 0 & 0 & 0 \\ -i \sqrt{2} \lambda^{[2] \alpha} & 0 & 0 \\ -D^{[2]} - i \partial_\mu A_{\mu}^{[2]} & 2 i A_{\mu}^{[2]} \partial_\mu & -i \sqrt{2} \lambda^{[2] \alpha} & 0 \end{pmatrix}, \]

where

\[ A_{\mu}^{[2]} = 2(A_{\mu}^{(0,0)} + \sum_m A_{\mu}(m,m)) \frac{\tau_0}{2} + \sum_n \left( 2 i (A_{\mu}^{(0,n)} - A_{\mu}^{(n,0)}) + i \sum_{m,s} \varepsilon_{nms} (A_{\mu}^{(m,s)} - A_{\mu}^{(s,m)}) \right) \frac{\tau_0}{2} = A_{\mu}^{(0)} + \sum_n \frac{\tau_0}{2} A_{\mu}^{(n)}, \]
\[ A_{\mu}^{(0)} = 2(A_{\mu}^{(0,0)} + \sum_m A_{\mu}(m,m)), \]
\[ A_{\mu}^{(n)} = \left( 2 i (A_{\mu}^{(0,n)} - A_{\mu}^{(n,0)}) + i \sum_{m,s} \varepsilon_{nms} (A_{\mu}^{(m,s)} - A_{\mu}^{(s,m)}) \right). \]

In the same way, we have

\[ \lambda_{\alpha}^{[2]} = 2(\lambda_{2\alpha}^{(0,0)} + \sum_n \lambda_{2\alpha}^{(n,n)} \frac{\tau_0}{2} + \sum_n \left( 2 i (\lambda_{2\alpha}^{(0,n)} - \lambda_{2\alpha}^{(n,0)}) + i \sum_{m,s} \varepsilon_{nms} (\lambda_{2\alpha}^{(m,s)} - \lambda_{2\alpha}^{(s,m)}) \right) \frac{\tau_0}{2} = \lambda_{2\alpha}^{(0)} + \sum_n \lambda_{2\alpha}^{(n)} \frac{\tau_0}{2}, \]
\[ D^{[2]} = 2(D_2^{(0,0)} + \sum_n D_2^{(n,n)} \frac{\tau_0}{2} + \sum_n \left( 2 i (D_2^{(0,n)} - D_2^{(n,0)}) + i \sum_{m,s} \varepsilon_{nms} (D_2^{(m,s)} - D_2^{(s,m)}) \right) \frac{\tau_0}{2} = D_2^{(0)} + \sum_n D_2^{(n)} \frac{\tau_0}{2}. \]

As for \( V_2^{[22]} \), we employ the similar operations to \( V_2^{[33]} \) to \( V_2^{[22]} \), and obtain

\[ (V_2^{[22]} )_{31} = -A_{\mu}^{[2]} A_{\mu}^{[2]}, \text{ the other elements } = 0. \]

The \( V_{ij}^{[2]} \) and \( V_{ij}^{[22]} \) amount to the same form as (A.29) replacing the upper indices [3], [33] to [2], [22]:

\[ (V_{WZ}^{[22]})_{ij} = V_{ij}^{[2]} + V_{ij}^{[22]} = \begin{pmatrix} 0 & 0 & 0 \\ -i \sqrt{2} \lambda^{[2] i} & 0 & \sigma_{\mu \alpha} A_{\mu}^{[2]} \\ -D^{[2]} - i \partial_\mu A_{\mu}^{[2]} - 2 i A_{\mu}^{[2]} \partial_\mu - A_{\mu}^{[2]} A_{\mu}^{[2]} & -i \sqrt{2} \lambda^{[2] i} & 0 \end{pmatrix}, \]

As for the opposite chirality sector, \( V_{WZ}^{[2]} \), we also obtain the same form as (A.32) replacing the upper index [3], [33] to [2], [22].
A.4 \( U(3) \otimes U(2), U(3) \otimes U(1) \)

In Eq. (A.22), in addition to the elements in \( V_{WZ, WZ}', \tilde{V}_{WZ, WZ}', \tilde{V}_{WZ}' \), there exist elements which correspond to the second and forth term for \((r, r') = (2, 3), (3, 2)\). Under the Wess-Zumino gauge condition, we calculate these terms as follows:

\[
\frac{1}{Z_2} [(V_{2}^{[23]}_{31})_{11}]_{I_A}^{IB} = \sum_{a,b} c_a c_b \left( \mu_{[23]}_{ab} \right)_{I_A}^{I'_{A'}} D_{k \ell} \left[ (u_{[23]}^{[3]}_{ab})_{\ell_1} \right]_{I_{A'}}^{IB} \\
= \sum_{a,b} c_a c_b \left( (u_{[23]}^{[3]}_{ab})_{I_{A}} \right)_{I_{A}}^{I'_{A'}} D_{k \ell} \left[ (u_{[23]}^{[3]}_{ab})_{\ell_1} \right]_{I'_{A'}}^{IB} \\
= \frac{1}{m_0} \sum_{a,b} c_a c_b \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} \left[ (F_{a}^{[23]}_{B})_{I_{A}}^{I'_{A'}} - \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} \left( \gamma_{B}^{[3]} \right)_{I_{A}}^{I'_{A'}} + \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} \left( \gamma_{B}^{[3]} \right)_{I_{A}}^{I'_{A'}} \right] \\
= \frac{1}{m_0} \sum_{a,b} c_a c_b \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} \left( \gamma_{B}^{[3]} \right)_{I_{A}}^{I'_{A'}} = 0. \tag{A.50} \]

In \( (A.50) \), the contraction of indices of internal degrees of freedom corresponds to extracting \((3, 8)\) representation in \( SU(2) \otimes SU(3) \subset U(2) \otimes U(3) \) from the tensor product \((3, 8) \otimes (3, 8)\). Repeating the similar calculations, we have vanishing elements but \((V_{2}^{[23]}_{31})_{31}\). As for \((V_{2}^{[23]}_{31})_{31}\), we obtain the following expressions:

\[
\frac{1}{Z_2} [(V_{2}^{[23]}_{31})_{31}]_{I_A}^{IB} = \sum_{a,b} c_a c_b \left( \mu_{[23]}_{ab} \right)_{31}^{I_{A}} D_{k \ell} \left[ (u_{[23]}^{[3]}_{ab})_{\ell_1} \right]_{I_{A'}}^{IB} \\
= \frac{1}{m_0} \sum_{a,b} c_a c_b \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} \left( \gamma_{B}^{[3]} \right)_{I_{A}}^{I'_{A'}} + \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} \left( \gamma_{B}^{[3]} \right)_{I_{A}}^{I'_{A'}} = 0. \tag{A.51} \]

where we have

\[
(F_{ab}^{[23]}_{I_{A}})^{I'_{A'}} = m_0 \left( \mu_{[23]}_{ab} \right)_{31}^{I_{A}} = m_0 \sum_{\ell} \left( \mu_{[23]}_{ab} \right)_{\ell_1}^{I_{A}} \left( \mu_{[23]}_{ab} \right)_{\ell}^{I'_{A'}} \tag{A.52} \]

\[
(F_{ab}^{[23]}_{I_{A}})^{I'_{A'}} = m_0 \left( u_{[23]}^{[3]}_{ab} \right)_{\ell_1}^{I_{A}} = m_0 \sum_{\ell} \left( u_{[23]}^{[3]}_{ab} \right)_{\ell_1}^{I_{A}} \left( u_{[23]}^{[3]}_{ab} \right)_{\ell}^{I'_{A'}} \tag{A.53} \]

\[
m_0 \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} = m_0 \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} = m_0 \sum_{\ell} \left( \mu_{[23]}_{ab} \right)_{\ell_1}^{I_{A}} \left( \mu_{[23]}_{ab} \right)_{\ell}^{I'_{A'}} \tag{A.54} \]

\[
\left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} = -m_0 \left( \mu_{[23]}_{ab} \right)_{32}^{I_{A}} = -m_0 \sum_{\ell} \left( \mu_{[23]}_{ab} \right)_{\ell_1}^{I_{A}} \left( \mu_{[23]}_{ab} \right)_{\ell}^{I'_{A'}} \tag{A.55} \]

\[
\left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} = m_0 \left( u_{[23]}^{[3]}_{ab} \right)_{21}^{I_{A}} = m_0 \sum_{\ell} \left( u_{[23]}^{[3]}_{ab} \right)_{21}^{I_{A}} \left( u_{[23]}^{[3]}_{ab} \right)_{\ell}^{I'_{A'}} \tag{A.56} \]

\[
m_0 \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} = m_0 \left( \sigma_{[23]}^{[3]}_{ab} \right)_{I_{A}}^{I'_{A'}} = m_0 \sum_{\ell} \left( \mu_{[23]}_{ab} \right)_{\ell_1}^{I_{A}} \left( \mu_{[23]}_{ab} \right)_{\ell}^{I'_{A'}} \tag{A.57} \]
In result, we obtain

\( (\varphi_{ab}^{[23]})' I_A' = m_0[\bar{\varphi}_{ab}^{[23]}]_A I_A' = m_0 \sum_\ell \langle \bar{\varphi}_{ab}^{[23]} \rangle_\ell' \langle \bar{\varphi}_{ab}^{[3]} \rangle_\ell A' \)

\[ = \frac{1}{m_0} (\varphi_{ab}^{[23]})' I_A' = \frac{1}{m_0} (\varphi_{ab}^{[3]})' I_A' , \tag{A.58} \]

\( (\varphi_{ab}^{[23]})' I_B' = m_0[\bar{u}_{ab}^{[23]}]_A I_B' = m_0 \sum_\ell \langle u_{ab}^{[23]} \rangle_\ell' \langle u_{ab}^{[3]} \rangle_\ell A' \)

\[ = \frac{1}{m_0} (\varphi_{ab}^{[23]})' I_I' B_A' , \tag{A.59} \]

\( m_0 (\varphi_{ab}^{[23]})' I_A' = (\varphi_{ab}^{[23]})' I_A' \partial^\mu \partial_\mu (\varphi_{ab}^{[23]})' I_A' \)

\[ = 2 (\varphi_{ab}^{[2]} \partial^\mu \varphi_{ab}^{[2]}) I_A' (\varphi_{ab}^{[3]} \partial^\mu \varphi_{ab}^{[3]}) B_A' \]

+ (terms which vanish by W.Z. condition). \( \tag{A.60} \)

In result, we obtain

\[ [(V_2^{[23]} I_A')_{31 B}] = \frac{2}{m_0} \sum_{a,b} c_{a,b} \left[ -\frac{1}{2} \left( \bar{\sigma}_{\mu}^{[2]} \sigma_{\mu}^{[2]} \right) I_A' \left( \bar{\psi}_{b}^{[3]} \sigma_{\mu}^{[3]} \psi_{b}^{[3]} \right) B_A' - i \left( \bar{\psi}_{a}^{[2]} \sigma_{\mu}^{[2]} \psi_{a}^{[2]} \right) I_A' \left( \bar{\psi}_{b}^{[3]} \sigma_{\mu}^{[3]} \psi_{b}^{[3]} \right) B_A' \right. \]

\[ - i \left( \bar{\psi}_{a}^{[2]} \sigma_{\mu}^{[2]} \psi_{a}^{[2]} \right) I_A' \left( \bar{\psi}_{b}^{[3]} \sigma_{\mu}^{[3]} \psi_{b}^{[3]} \right) B_A' + 2 \left( \bar{\psi}_{a}^{[2]} \sigma_{\mu}^{[2]} \psi_{a}^{[2]} \right) I_A' \left( \bar{\psi}_{b}^{[3]} \sigma_{\mu}^{[3]} \psi_{b}^{[3]} \right) B_A' \]

\[ = - (A_{\mu}^{[2]})' I_A' (A_{\mu}^{[3]}) B_A' \]

\[ = [(V_2^{[32]} I_A')_{31 B}] = - (A_{\mu}^{[2]})' I_A' (A_{\mu}^{[3]}) B_A' , \tag{A.61} \]

where the last equation is due to the commutativity of \( (A_{\mu}^{[2]})' I_A' \) and \( (A_{\mu}^{[3]}) B_A' \).

Continuing the similar calculations for the opposite chirality sector, we obtain \( \bar{V}_2^{[23]} \) expressed by

\[ [(V_2^{[23]} I_A')_{31 B}] = [(V_2^{[32]} I_A')_{31 B}] = - (A_{\mu}^{[2]})' I_A' (A_{\mu}^{[3]}) B_A' , \text{ the other elements are } = 0. \tag{A.62} \]

Adding \( \mathcal{D}, \mathcal{D} \) and fluctuations due to the algebra which act on \( Q \) field, we obtain the fluctuated Dirac operator \( \mathcal{D}, \mathcal{D} \) in \( (3.23) \) and \( (3.24) \). The derived vector supermultiplet become to be adjoint representation of \( U(3) \times U(2) \).

To the fluctuation for \( \mathcal{D} \) in the \( u_R, d_R \) sector due to the elements expressed by Eq. \( (3.27) \), similar operations carry the same result as \( \text{[A.61], [A.62]} \) but replacing the index \( [32] \) to \( [31] \) so that we obtain \( (3.28) \) in which the vector supermultiplet \( (A_{\mu}^{[s]}, \lambda_{a}^{[s]}, D^{(s)}), s = u_R(d_R) \) becomes to be adjoint representation of \( U(3) \times U(1) \).

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