On one application of the Zigmund-Marczinkevich theorem

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Abstract

In this paper we aim to generalize results obtained in the framework of fractional calculus by the way of reformulating them in terms of operator theory. In its own turn, the achieved generalization allows us to spread the obtained technique on practical problems that connected with various physical - chemical processes.

Keywords: Positive operator; fractional power of an operator; semigroup generator; strictly accretive property.

MSC 26A33; 47A10; 47B10; 47B25.

1 Introduction

To write this paper, we were firstly motivated by the boundary value problems connected with various physical - chemical processes: filtration of liquid and gas in highly porous fractal medium; heat exchange processes in medium with fractal structure and memory; casual walks of a point particle that starts moving from the origin by self-similar fractal set; oscillator motion under the action of elastic forces which is characteristic for viscoelastic media, etc. It is worth noticing that the attention of many engineers are being attracted to the following processes: dielectric polarizations [27], electrochemical processes [5,18,26], colored noises [14], chaos [4]. The special interest is devoted to the viscoelastic materials [3,8,9,25]. As it is well-known, to describe these processes we have to involve the theory of differential equations of fractional order and such notions as the Riemann-Liouville, Marchaud, Weyl, Kipriyanov fractional derivatives, which have applications to the physical - chemical processes listed above. For instance, the model of the so-called intensification of scattering processes is based on the fractional telegraph equation which was studied in the papers [12,13]. In its own turn, the foundation of models describing the processes listed above can be obtained by virtue of fractional calculus methods, the central point of which is a concept of the Riemann-Liouville operator acting in the weighted Lebesgue space. In particular, we would like to point out the reader attention to the boundary value problems for the second order differential operator with the Riemann-Liouville fractional derivative in final terms.
Many papers were devoted to this question, for instance [1], [2], [10], [16]. It is quite reasonable that the variety of fractional derivative senses mentioned above create a motivation to consider abstract methods in order to solve concrete problems without inventing an additional technique in each case. In its own turn, the operator theory methods play an important role in applications and need not any of special advertising. Having forced by these reasons, we deal with mapping theorems for operators acting on Banach spaces in order to obtain afterwards the desired results applicable to integral operators. We also note that our interest was inspired by lots of previously known results related to mapping theorems for fractional integral operators obtained by mathematicians such as Rubin B.S. [19], [20], [21], Vakulov B.G. [28], Samko S.G. [23], [24], Karapetyants N.K. [6], [7].

2 Preliminaries

We consider a pair of spaces with a finite measure \((\Omega, \mathcal{F}, \mu_i), (i = 0, 1)\), the corresponding Banach spaces of real functions defined on a set \(\Omega\) are denoted by \(L^p(\Omega, \mu_i), 1 < p < \infty\). As usual, we assume that \(L^p(\Omega, \mu_0)\) are reflexive and the Riesz representation theorem is true. The dual normed spaces are denoted by \(L^p'(\Omega, \mu_1)\) respectively, where \(\frac{1}{p} + \frac{1}{p'} = 1\). Suppose \(L^p(\Omega, \mu_0) \cap L^p(\Omega, \mu_1) \neq \emptyset\), there exists a basis \(\{e_n\}_0^\infty\) in \(L^p(\Omega, \mu_1)\), \(\{e_n\}_0^\infty \subset L^p(\Omega, \mu_0) \cap L^p(\Omega, \mu_1)\), which is also a basis in \(L^p'(\Omega, \mu_1)\). Under this assumption, it is quite reasonable to involve the auxiliary construction

\[
S_k f := \sum_{n=0}^k f_n e_n, \quad f_n := \int f e_n d\mu_1, \quad k = 0, 1, \ldots, f \in L^p(\Omega, \mu_1).
\]

We need the following essential assumption, the measure \(\mu_1\) and the domain \(\Omega\) are such that the Zigmund-Marczinkevich theorem (see [15], Appendix) is applicable. Let \(A, B\) be a pair of linear operators boundary acting on \(L^p(\Omega, \mu_0), L^p(\Omega, \mu_1)\) respectively, moreover both operators have the same restriction on \(L^p(\Omega, \mu_0) \cap L^p(\Omega, \mu_1)\). Assume that the operator \(A^{-1}\) is defined on \(\{e_n\}\) and denote the following functionals by

\[
(e_m, A e_n)_{\mu_1} = A_{mn}, \quad (e_m, A^{-1} e_n)_{\mu_1} = A'_{mn},
\]

here and further we use the denotations

\[
(f, g)_{\mu_i} := \int f g d\mu_i, \quad (f, g)_{\mu_0, n} := \int f g d\mu_0.
\]

3 Main results

The following theorems are formulated in terms of the Zigmund-Marczinkevich theorem (see Appendix) and provide a description of mapping properties of the quite wide operators class including fractional integral operators.

**Theorem 1.** Suppose \(2 \leq p < \infty, \beta, \lambda \in [0, \infty)\),

\[
\psi \in L^p(\Omega, \mu_1), \quad \left| \sum_{n=0}^\infty \psi_n A_{mn} \right| \sim m^{-\lambda}, \quad M_m \sim m^\beta, \quad m \to \infty;
\]
then

\[ A\psi = f \in L_q(\Omega, \mu_1), \]

where \( q = p, \) if \( 0 \leq \lambda \leq 1/2; \) \( q \) is an arbitrary large number satisfying

\[ q < \frac{\nu(2\beta + 1)}{\nu(\beta + 1 - \lambda) + 2\lambda - 1}. \]  \hfill (2)

if \( 1/2 < \lambda < [\nu(\beta + 1) - 1]/(\nu - 2); \) and \( q \) is an arbitrary large number, if \( \lambda \geq [\nu(\beta + 1) - 1]/(\nu - 2). \)

Moreover the image is represented by a convergent in \( L_q(\Omega, \mu_1) \) series

\[ f(x) = \sum_{m=0}^{\infty} e_m(x) \sum_{n=0}^{\infty} \psi_n A_{mn}. \]

This theorem can be formulated in the matrix form

\[ A \times \psi = f, \quad \sim \begin{pmatrix} A_{00} & A_{01} & \cdots \\ A_{10} & A_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix}. \]

Proof. Note, that in accordance with the basis property of \( \{e_n\}, \) we have

\[ \sum_{n=0}^{l} \psi_n e_n \xrightarrow{L_p(\Omega, \mu_1)} \psi \in L_p(\Omega, \mu_1), \; l \to \infty. \]

Since the operator \( A \) is bounded, then

\[ \sum_{n=0}^{l} \psi_n Ae_n \xrightarrow{L_p(\Omega, \mu_1)} A \left( \sum_{n=0}^{\infty} \psi_n e_n \right) = A\psi, \; l \to \infty. \]

Hence

\[ \sum_{n=0}^{l} \psi_n (Ae_n, e_m)_{\mu_1} \to (A\psi, e_m)_{\mu_1}, \; l \to \infty. \]

Applying first formula (1), we obtain

\[ f_m = (A\psi, e_m)_{\mu_1} = \sum_{n=0}^{\infty} \psi_n A_{mn}. \]

Since \( 0 \leq \lambda \leq 1/2, \) then it is not hard to prove that

\[ \frac{\nu(2\beta + 1)}{\nu(\beta + 1 - \lambda) + 2\lambda - 1} \leq 2. \]

This implies that we cannot use condition (2) to obtain the additional information on the function \( f. \) However, note that in any case by virtue of the conditions imposed on the operator \( A, \) we have
Using condition (2), by simple calculation in the case \(1/2 < \lambda < [\nu(\beta + 1) - 1]/(\nu - 2)\), we obtain

\[
\beta \frac{\nu(q - 2)}{\nu - 2} + \frac{\nu - 1}{\nu - 2}(q - 2) - \lambda q < -1.
\]

(3)

It implies that corresponding series is convergent. By virtue of this fact, having applied the Zigmund-Marczinkevich theorem, we obtain the desired result. Now assume that \([\nu(\beta + 1) - 1]/(\nu - 2) \leq \lambda\), then consider relation (3) and let us prove that it is fulfilled for \(2 \leq q < \infty\). By easy calculations, we can rewrite (3) in the following form

\[
q \left\{ \frac{\beta \nu}{\nu - 2} + \frac{\nu - 1}{\nu - 2} - \lambda \right\} < 2 \left\{ \frac{\beta \nu}{\nu - 2} + \frac{\nu - 1}{\nu - 2} \right\} - 1.
\]

(4)

Since the multiplier of \(q\) is non-positive and the right side of inequality is positive (the proof of this fact is left to the reader), then we have the fulfilment of inequality (4) for \(2 \leq q < \infty\). Hence (3) holds and using the same reasonings we obtain the desired result.

The following result is formulated in terms of the coefficients of the expansion on the basis \(\{e_n\}\) and devoted to the conditions under which the inverse operator \(B^{-1}\) exists. Consider the following operator equation under most general assumptions concerning to the right part

\[
B\varphi = f.
\]

(5)

Let us consider the Riemann-Liouville operator to demonstrate applicability of such a consideration. In this case relation (5) provides the generalized Abel equation (see [11]) that becomes the ordinary Abel equation, if we make the following assumptions concerning to the right part. Thus, we know if the next conditions hold \(I_{\alpha+}^1 f \in AC(\bar{I})\), \((I_{\alpha+}^1 f)(a) = 0\), \(I = (a, b) \subset \mathbb{R}\), then there exists a unique solution of the ordinary Abel equation in the class \(L_1(I)\) (see [22]).

The sufficient conditions for solvability of equation (5) in the abstract case are established in the following theorem.

**Theorem 2.** Suppose \(\lambda, \beta \in [0, \infty)\), \(M_n \sim n^\beta\), the right part of equation (5) such that

\[
\|A^{-1}S_k f\|_{L_p(\Omega, \mu_1)} \leq C, \ k \in \mathbb{N}_0, \ \sum_{n=0}^{\infty} |f_n A'_m| \sim m^{-\lambda}, \ m \to \infty;
\]

(6)

then there exists a solution of equation (5) in \(L_p(\Omega, \mu_1)\), the solution belongs to \(L_q(\Omega, \mu_1)\), where \(q = p\), if \(0 \leq \lambda \leq 1/2\); \(q\) is an arbitrary large number satisfying

\[
q < \frac{\nu(2\beta + 1)}{\nu(\beta + 1 - \lambda) + 2\lambda - 1};
\]

(7)

if \(1/2 < \lambda < [\nu(\beta + 1) - 1]/(\nu - 2)\); and \(q\) is an arbitrary large number, if \(\lambda \geq [\nu(\beta + 1) - 1]/(\nu - 2)\). Moreover the solution is represented by a convergent in \(L_q(\Omega, \mu_1)\) series

\[
\psi(x) = \sum_{m=0}^{\infty} e_m(x) \sum_{n=0}^{\infty} f_n A'_{mn}.
\]

(8)
Moreover, if the measures $\mu_i$ such that: $\mu_0(M) = 0 \Rightarrow \mu_1(M) = 0, \forall M \subset \Omega$, there exists a sequence of sets

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega, \Omega_n \subset \Omega_{n+1}, L_p(\Omega, \mu_1) \subset L_p(\Omega_n, \mu_0), \mu_0(\Omega \setminus \Omega_n) \to 0, n \to \infty,$$

the corresponding sets of functions $\Theta_n, n \in \mathbb{N}_0$ such that

$$\Theta_1 \subset \Theta_2 \subset ... \subset \Theta,$$

and $\Theta, \Theta$ are dense sets in the spaces $L_p'(\Omega_n, \mu_0), L_p'(\Omega, \mu_0)$ respectively,

$$(g, \eta)_{\mu_0,n} = (g, \eta)_{\mu_0}, \eta \in \Theta_n, g \in L_p(\Omega_n, \mu_0),$$

$$\forall \eta \in \Theta, \forall \xi \in L_p(\Omega, \mu_1), \exists g \in L_p'(\Omega, \mu_1) : (\xi, \eta)_{\mu_0} = (\xi, B^*g)_{\mu_1};$$

then the existing solution is unique.

**Proof.** Applying formula (11) and using the theorem conditions, we obtain the following relation

$$(A^{-1}S_k f, e_m) \to \sum_{n=0}^{\infty} f_n A'_{mn}, k \to \infty, m \in \mathbb{N}_0. \quad (9)$$

Since relation (9) holds and the sequence $\{A^{-1}S_k f\}_0^\infty$ is bounded with respect to the norm $L_p(\Omega, \mu_1)$, then due to the well-known theorem, we have that the sequence $\{A^{-1}S_k f\}_0^\infty$ weakly converges to some function $\psi \in L_p(\Omega, \mu_1)$. Using the ordinary properties of inverse and adjoint operators, taking into account that $A^{-1}S_k f \in L_p(\Omega, \mu_1)$, we obtain the representation

$$(S_k f, e_m)_{\mu_1} = (AA^{-1}S_k f, e_m)_{\mu_1} = (B_0A^{-1}S_k f, e_m)_{\mu_1} =$$

$$= (A^{-1}S_k f, B_0^*e_m)_{\mu_1} = (A^{-1}S_k f, B^*e_m)_{\mu_1},$$

where $B_0$ is a restriction of $B$ to the set $A^{-1}S_k f, k \in \mathbb{N}_0$, thus we have $B_0^* \supset B^*$. Due to the week convergent of the sequence $\{A^{-1}S_k f\}_0^\infty$, we have

$$(A^{-1}S_k f, B^*e_m)_{\mu_1} \to (\psi, B^*e_m)_{\mu_1} = (B\psi, e_m)_{\mu_1}, k \to \infty.$$

It follows that

$$(S_k f, e_m)_{\mu_1} \to (B\psi, e_m)_{\mu_1}, k \to \infty. \quad (10)$$

Taking into account that

$$(S_k f, e_m)_{\mu_1} = \begin{cases} f_m, \ k \geq m, \\
0, \ k < m \end{cases},$$

we obtain

$$(B\psi, e_m)_{\mu_1} = f_m, \ m \in \mathbb{N}_0.$$

Using the uniqueness property of basis expansion, we obtain that the equality $B\psi = f$ holds on the set $\Omega \setminus M, \mu_1(M) = 0$. Hence there exists a solution of equation (10). Now let us proceed to the following part of the proof. Note that we have previously proved the fact $\psi \in L_p(\Omega, \mu_1)$, if
0 \leq \lambda < \infty. \text{ Let us show that } \psi \in L_q(\Omega, \mu_1), \text{ where } q \text{ is defined by condition } (7), \text{ if } 1/2 < \lambda < [\nu(\beta + 1) - 1]/(\nu - 2). \text{ In accordance with the above reasonings, we have }

\left( A^{-1}S_k f, e_m \right)_{\mu_1} \rightarrow (\psi, e_m)_{\mu_1}, \text{ } k \rightarrow \infty, \text{ } m \in \mathbb{N}_0.

Combining this fact with (9), we get

\psi_m = (\psi, e_m)_{\mu_1} = \sum_{n=0}^{\infty} f_n A'_{mn}, \text{ } m \in \mathbb{N}_0. \tag{11}

Using the theorem conditions, we have

\|\psi_m\| \sim m^{-\lambda}, \text{ } m \rightarrow \infty.

Now we need to apply the Zigmund-Marczinkevich theorem. For this purpose, let us show that under the assumption 1/2 < \lambda < [\nu(\beta + 1) - 1]/(\nu - 2) the relation (3) is fulfilled in terms of this theorem conditions. Having done the reasonings analogous to the reasonings of Theorem 1, we obtain the desired result. Thus we have \psi \in L_q(\Omega, \mu_1), \text{ where } q \text{ is defined by condition } (7). \text{ The proof corresponding to the case } \lambda \geq [\nu(\beta + 1) - 1]/(\nu - 2) \text{ is also analogous to the proof given in Theorem 1. In a similar way, we obtain in this case that } \psi \in L_q(\Omega, \mu_1), \text{ where } q \text{ is arbitrary large. Taking into account the facts given above, by virtue of the Zigmund-Marczinkevich theorem, we also have the fulfillment of relation (3), if } \lambda > 1/2. \text{ Let us prove that under the second part of the theorem assumptions the existing solution is unique. Assume that there exists another solution } \phi \in L_p(\Omega, \mu_1), \text{ and let us denote } \xi := \psi - \phi. \text{ Due to the theorem conditions, we have }

(\xi, \eta)_{\mu_0} = (\xi, B^* g)_{\mu_1} = (B\xi, g)_{\mu_1} = 0, \text{ } \eta \in \Theta, \text{ } g \in L_{p'}(\Omega, \mu_1).

Hence

(\xi, \eta)_{\mu_0, n} = 0, \forall \eta \in \Theta_n.

We claim that \xi \neq 0. \text{ Therefore in accordance with the consequence of the Hahn-Banach theorem there exists the element } \vartheta \in L_{p'}(\Omega_n, \mu_0), \text{ such that }

(\xi, \vartheta)_{\mu_0, n} = \|\psi - \phi\|_{L_p(\Omega_n, \mu_0)} > 0.

On the other hand, there exists the sequence \{\eta_k\}_1^\infty \subset \Theta_n, \text{ such that } \eta_n \rightarrow \vartheta \text{ with respect to the norm } L_{p'}(\Omega_n, \mu_0). \text{ Hence }

0 = (\xi, \eta_n)_{\mu_0, n} \rightarrow (\xi, \vartheta)_{\mu_0, n}.

Thus, we have come to the contradiction. Due to this fact, it is not hard to prove that \psi = \phi on the set } \Omega \setminus M, \mu_0(M) = 0. \text{ It implies that } \psi = \phi \text{ on the set } \Omega \setminus M, \mu_1(M) = 0. \text{ The uniqueness has been proved. }

\[\square\]

4 Applications

In this section our aim is to justify the application of the obtained abstract results to the fractional integral operator. Throughout this section we use the following notation for the weighted complex Lebesgue spaces } L_p(I, \mu), \text{ } 1 \leq p < \infty, \text{ where } I = (a,b) \text{ is an interval of the real axis and } d\mu = \omega(x)dx, \text{ } x \in I, \text{ where the weighted function } \omega(x) = (x-a)^\beta(b-x)^\gamma, \text{ } \beta, \gamma \geq -1/2. \text{ If } \omega = 1,
then we use the notation $L_p(I)$. Using the denotations of the paper \cite{22}, let us define the left-side fractional integral and derivative of real order $\alpha \in (0, 1)$ respectively

$$
(I^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1+\alpha}} dt,
$$

$f \in L_1(I), (D^\alpha_{a+} f)(x) = \frac{d}{dx} (I^\alpha_{a+} f)(x), f \in L^\alpha_{a+}(L_1),$.

where $L^\alpha_{a+}(L_1)$ is the class of functions that can be represented by the fractional integral defined on $L_1(I)$ (see\cite{22}). The orthonormal system of the Jacobi polynomials, with the parameters $\beta, \gamma$ corresponding to the weighted function, is denoted by

$$
p_n^{(\beta, \gamma)}(x) = \delta_n (x-a)^{-\beta} (b-x)^{-\gamma} \frac{d^n}{dx^n} [(x-a)^{\beta+n} (b-x)^{\gamma+n}], n \in \mathbb{N}_0,
$$

where $\delta_n$ are constants depending of $\beta, \gamma$. For the case corresponding to $\beta = \gamma = 0$, we have the Legendre polynomials. It is worth seeing that the criterion of the basis property for the Jacobi polynomials was proved by Pollard H. in the work \cite{17}. In that paper Pollard H. formulated the theorem proposing that the Jacobi polynomials have the basic property in the space $L_p(I_0, \mu), I_0 := (-1, 1), \beta, \gamma \geq -1/2, M(\beta, \gamma) < p < m(\beta, \gamma)$ and do not have the basis property, if $p < M(\beta, \gamma)$ or $p > m(\beta, \gamma)$, where $M(\beta, \gamma), m(\beta, \gamma)$ are constants depending on $\beta, \gamma$. According to the denotations of the paper \cite{11}, we have

$$A_{mn}^{\alpha, \beta, \gamma} := \delta_m \sum_{k=0}^n (-1)^k \frac{C_n^{(k)}(\beta, \gamma) B(\alpha + \beta + k + 1, \gamma + m + 1)}{\Gamma(k + \alpha - m + 1)},
$$

where $\delta_m$ is a constant depending on $\alpha, \beta, \gamma$. Suppose $\beta, \gamma \in [-1/2, 1/2], M(\beta, \gamma) < p < m(\beta, \gamma)$. Further, we will use the short-hand notation $p_n := p_n^{(\beta, \gamma)}$. In accordance with the results of the paper \cite{11}, we have

$$
\int_I p_m I^\alpha_{a+} \mu_n d\mu = (-1)^n A_{mn}^{\alpha, \beta, \gamma},

\int_I p_m D^\alpha_{a+} p_n d\mu = (-1)^n A_{mn}^{\alpha, \beta, \gamma}. \quad (12)
$$

Let us assume that

$$
\Omega := I, \mu_0 := x, \mu_1 := \mu, L_p(\Omega, \mu_0) := L_p(I), L_p(\Omega, \mu_1) := L_p(I, \mu), e_n := p_n,

A := I^\alpha_{a+}, A^{-1} := D^\alpha_{a+}, A_{mn} := A_{mn}^{\alpha, \beta, \gamma}, A_{mn}' := A_{mn}^{\alpha, \beta, \gamma}, B := I^\alpha_{a+}.
$$

Taking into account the facts given above, it can be easily proved (see \cite{11}) that all assumptions of section 1 are fulfilled except the following ones. In terms of the given interpretation, the proof of uniqueness of the Abel equation solution does not seem obvious. Thus we need to produce it, for this purpose assume that

$$
\Omega_n := \left( a + \frac{1}{n}, b - \frac{1}{n} \right), \Theta_n := C^\infty_0(\Omega_n), \Theta := C^\infty(\Omega).
$$

Then the following assumptions of the uniqueness part of Theorem \cite{13} are fulfilled

$$
\bigcup_{n=1}^\infty \Omega_n = \Omega, \Omega_n \subset \Omega_{n+1}, L_p(\Omega, \mu_1) \subset L_p(\Omega_n, \mu_0), \mu_0(\Omega \setminus \Omega_n) \to 0, n \to \infty.
$$
The verification is left to the reader. It is also not hard to prove that 

\[(g, \eta)_{\mu_0, n} = (g, \eta)_{\mu_0}, \eta \in \Theta_n, \ g \in L_p(\Omega_n, \mu_0).\]

Let us show that 

\[\forall \eta \in \Theta, \ \forall \xi \in L_p(\Omega, \mu_1), \ \exists h \in L_p(\Omega, \mu_1) : (\xi, \eta)_{\mu_0} = (\xi, B^* h)_{\mu_1}.\]

Consider 

\[\omega^{-1}(x) D^a_{b-} \eta(x) = (x-a)^{-\alpha}(b-x)^{-\gamma} D^a_{b-} \eta(x).\]

If we note that \(D^a_{b-} \eta(b) = 0\) (see [22]) then it can be easily shown that 

\[(b-x)^{-1} D^a_{b-} \eta(x) \to D^{a+1}_{b-} \eta(b), \ x \to b.\]

Hence the function \(\omega^{-1} D^a_{b-} \eta\) is bounded. It implies that we have a representation \(D^a_{b-} \eta = \omega h\), where \(h\) is a bounded function and due to this fact it belongs to \(L_p(\Omega, \mu_1)\). By virtue of the fact \(\eta \in C_0^\infty(\Omega)\) (see [22]), we have \(I^a_{b-} D^a_{b-} \eta = \eta\). Hence 

\[\eta = I^a_{b-} \omega h, \ h \in L_p(\Omega, \mu_1).\]

Taking into account the reasonings given above, we get 

\[\int_\Omega \xi \eta d\mu_0 = \int_\Omega \xi \eta d\mu_0 = \int_\Omega \xi \omega^{-1} I^a_{b-} \omega h d\mu_1.\]

On the other hand, since we have 

\[|l_\eta(\xi)| = \left| \int_\Omega \xi \eta d\mu_0 \right| = \left| \int_\Omega \xi \eta \omega^{-1} d\mu_1 \right| \leq \|\xi\|_{L_p(\Omega, \mu_1)} \|\eta \omega^{-1}\|_{L_p(\Omega, \mu_1)},\]

then using the Riesz representation theorem, we get 

\[\int_\Omega \xi \eta d\mu_0 = \int_\Omega \xi \eta^* d\mu_1, \ \eta^* \in L_p(\Omega, \mu_1).\]

Hence \(\omega^{-1} I^a_{b-} \omega h \in L_p(\Omega, \mu_1)\) and we obtain the desired result, where \(B^* h := \omega^{-1} I^a_{b-} \omega h\). Applying the analogous reasoning we prove that \(B^* e_m = \omega^{-1} I^a_{b-} \omega p_m, \ (m = 0, 1, \ldots)\). As a result we have come to conclusion the theoretical results of section 1 have a concrete application to the questions connected with existence and uniqueness of the Abel equation solution in the weighted case, which has an important role in the description of physical processes in porous medium.

5 Conclusions

In this paper, our first aim is to construct an operator model describing the fractional integral action in the weighted Lebesgue spaces. The approach used in the paper is the following: to generalize known results of fractional calculus and by this way achieve a novel method of studying operators action in Banach spaces. Besides the theoretical results of the paper, we produce the
relevance of such a consideration that provided by a plenty of applications in various engendering sciences. More precisely a description of such process as electrochemical processes, dielectric polarizations, colored noises are provided by the theoretical part of this paper. The foundation of models describing the processes listed above can be obtained by virtue of fractional calculus methods, the central point of which is a concept of the Riemann-Liouville operator acting in the weighted Lebesgue space. In its own turn this concept is covered by the theoretical part of this paper. Thus, the obtained results are harmoniously connected with the concrete models of physical-chemical process.

Appendix

In this section, we formulate the Zigmund-Marczinkevich theorem (see [15]).

Theorem. Suppose that

\[
\left( \int_{\Omega} |e_n|^\nu \, d\mu_1 \right)^{1/\nu} \leq M_n, \quad 2 \leq q < \nu, \quad M_n \leq M_{n+1}, \quad (n = 0, 1, \ldots),
\]

and the series

\[
\sum_{n=0}^{\infty} |c_n|^q M_n^{\frac{\nu(q-2)}{q-2}} n^{-\frac{q-1}{2(q-2)}} < \infty
\]

converges. Then there exists a function \( f \in L_q(\Omega, \mu_1) \) such that

\[
\left( \int_{\Omega} |f|^q \, d\mu_1 \right)^{1/q} \leq A_{q,\nu} \left( \sum_{n=0}^{\infty} |c_n|^q M_n^{\frac{\nu(q-2)}{q-2}} n^{-\frac{q-1}{2(q-2)}} \right)^{1/q}.
\]

(13)

Here \( A_{q,\nu} \) depends on \( q \) and \( \nu \) only. If by \( A_{q,\nu} \) we mean the least number satisfying (13) for all sequences \( \{c_n\} \), then

\[
A_{q,\nu} \leq G \frac{\nu - 2}{\nu - q},
\]

where \( G \) is an absolute constant.

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