The nonlinear evolution of de Sitter space instabilities

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We investigate the quantum evolution of large black holes that nucleate spontaneously in de Sitter space. By numerical computation in the s-wave and one-loop approximations, we verify claims that such black holes can initially “anti-evaporate” instead of shrink. We show, however, that this is a transitory effect. It is followed by an evaporating phase, which we are able to trace until the black holes are small enough to be treated as Schwarzschild. Under generic perturbations, the nucleated geometry is shown to decay into a ring of de Sitter regions connected by evaporating black holes. This confirms that de Sitter space is globally unstable and fragments into disconnected daughter universes.

I. INTRODUCTION AND SUMMARY

A. Anti-evaporation and proliferation

Schwarzschild-de Sitter black holes have unusual quantum properties and instabilities\(^1\). They are of cosmological interest because they can be produced during inflation\(^2\). They are also theoretically significant because they change the global structure of de Sitter space\(^3,4\) fundamentally.

Unlike their asymptotically flat cousins, Schwarzschild-de Sitter black holes are surrounded by a cosmological horizon. This limits their size and puts them in a thermal bath. Their temperature is always larger than that of the cosmological horizon\(^5,6\). Nevertheless, Hawking and one of the authors argued in Ref.\(^4\) that some black holes accrete so much quantum radiation that they will grow, or ‘anti-evaporate,’ instead of evaporating (see also\(^7,8\)).

In the maximal solution, the black hole and cosmological horizon are of equal size, and the spatial geometry will be \(S^1 \times S^2\), with constant two-sphere radius. This geometry can nucleate semiclassically in de Sitter space through a gravitational tunneling process. Its evolution is unstable to the formation of \(n\) de Sitter regions, distributed around the \(S^1\) and connected by \(n\) black hole throats\(^7\). This configuration may be visualized as a ‘doughnut’ with \(n\) ‘wobbles,’ whose minima and maxima represent the black hole and cosmological horizons, respectively. For \(n = 1\), this reduces to the ordinary, submaximal Schwarzschild-de Sitter solution. For higher \(n\), it resembles a ring-like sequence of \(n\) Schwarzschild-de Sitter solutions. As the de Sitter regions form between the cosmological horizons, the black holes are expected to evaporate and shrink. When they finally disappear, the doughnut is pinched in \(n\) places, leaving \(n\) disconnected pieces. This corresponds to the fragmentation of space into \(n\) large daughter universes\(^7\).

Thus de Sitter space is globally unstable. Locally, however, the daughter universes are indistinguishable from de Sitter. Therefore, they will harbour more maximal black holes, whose evaporation can lead to further fragmentation. Iteratively, an unbounded number of disconnected de Sitter universes is produced: de Sitter space proliferates. This drastic conclusion depends crucially on the nonlinear evolution of the instabilities of the maximal Schwarzschild-de Sitter solution (also referred to as the ‘Nariai’ solution below).

B. Analytical shortcomings

In Refs.\(^1\) and\(^2,3\), the anti-evaporation and proliferation effects were derived analytically, using a number of approximations. The stability of the Nariai geometry was studied using linear perturbation theory. The evolution of the horizons was found only for very early times, as a power series in the time variable. This led to the discovery of anti-evaporation for a class of perturbations. But a power series can prove only initial anti-evaporation. It cannot answer an important question: Will such black holes continue to grow, asymptotically approaching the maximal
(Nariai) solution? Or will their growth eventually stall, and evaporation set in after all? This problem is highlighted by recent claims that anti-evaporation may permanently stabilize black holes nucleated during inflation.

At least for Euclidean boundary conditions, it was shown that the black holes do evaporate at late times. In fact, arguments might be made that the evaporating mode is attractive at late times for all initial conditions. In any case, however, the intermediate period containing the putative transition between anti-evaporation and evaporation is beyond the reach of the analytical methods employed. Can there be a smooth transition, and how does it proceed? To understand this crucial phase of the black hole evolution, it should be explored numerically.

If anti-evaporation is only temporary, then how can we be sure that the evaporating phase, once entered, continues until the black hole disappears? This question is particularly important in the context of multiple black hole formation and proliferation. The fragmentation of space is possible only if all black holes become small and disappear like Schwarzschild black holes.

To show that black holes really evaporate completely, it is necessary to trace the backreaction at least until they are significantly smaller (perhaps by a factor of 10) than the cosmological horizon. Then they can be assumed to behave like Schwarzschild black holes, which presumably radiate all of their mass away. But linear perturbation theory requires that the small two-spheres inside the black hole and the large two-spheres between the cosmological horizons still be of nearly equal size. Thus, the analytic ‘late-time’ solution, showing black hole evaporation, is not really valid for arbitrarily late times. A full, non-linear numerical calculation is needed to interpolate to the Schwarzschild regime.

C. Numerical analysis

In Sec. we consider perturbations of the Nariai geometry, which can be parametrized by their initial amplitude, and phase, . Depending on , the black holes undergo initial periods of anti-evaporation or evaporation. We demonstrate the transitory nature of anti-evaporation for neutral black holes by showing that evaporation sets in at sufficiently late times, independently of . The evaporation rate is compared to an approximate solution of the linear equations in order to verify that this solution is an attractor. Beyond the linear regime, the black holes are found to shrink to a size much smaller than the cosmological horizon before our computation breaks down. Then they live in approximately flat space, and should simply continue to evaporate until they disappear.

In Sec. we show that maximal black holes become stable if their charge exceeds a critical value, in agreement with thermodynamic expectations. This confirms analytic arguments given in Ref. . In particular, we show that supercritical black holes, independently of their initial behavior, always end up anti-evaporating. This is the condition for a novel type of daughter-universe production proposed in Ref. .

In Sec. we show that a higher mode perturbation behaves as predicted by analytical arguments: First it oscillates, then it freezes out, forming black hole interiors and growing de Sitter regions. Depending on initial conditions, a period of anti-evaporation may follow. Finally, the black holes evaporate and become much smaller than the cosmological horizons.

II. ACTION AND PERTURBATIONS

A. Including back-reaction

The four-dimensional Lorentzian Einstein-Hilbert action with a cosmological constant, , and a Maxwell field, , is given by:

$$S = \frac{1}{16\pi} \int d^4x (-g^{IV})^{1/2} \left[ R^{IV} - 2\Lambda - F_{\mu\nu}F^{\mu\nu} \right],$$  \hspace{1cm} (1)

where and are the four-dimensional Ricci scalar and metric determinant.

Restricting to spherically symmetric fields and quantum fluctuations, the metric may be written as

$$ds^2 = e^{2\rho} (-dt^2 + dx^2) + e^{-2\rho} d\Omega^2, \hspace{1cm} (2)$$

\footnote{We point out, however, that even stable black holes nucleated semiclassically during inflation would be diluted by the cosmological expansion. Typically, they will lie outside the current horizon and will not be observable.}
where $x$ is the coordinate on the $S^1$, with period $2\pi$. Using this ansatz, and the on-shell condition for magnetic fields,

$$F_{\mu\nu} F^{\mu\nu} = 2Q^2 e^{4\phi},$$

the angular coordinates and the Maxwell field can be integrated out in Eq. (3), which reduces the action to

$$S = \frac{1}{16\pi} \int d^2x (-g)^{1/2} \left[ e^{-2\phi} \left[ R + 2(\nabla \phi)^2 + 2e^{2\phi} - 2\Lambda - 2Q^2 e^{2\phi} \right] \right],$$

(4)

In order to include back-reaction effects, a Polyakov term, which arises in the one-loop effective action of a two-dimensional scalar field, will be included. (It would be preferable to work with dilaton-coupled scalars [11–22], or even better, with a four-dimensional effective action reduced to two-dimensions. This would complicate the numerical computation enormously and will not be attempted here. For small perturbations, it has been shown that extra terms from dilaton-coupling do not affect results [4,8]. Thus one would not expect qualitative changes even for black holes noticeably smaller than the cosmological horizon. In order to corroborate our results for nearly-Schwarzschild black holes, however, a full four-dimensional treatment would be desirable in future work.)

In the large $N$ limit, the contribution from the quantum fluctuations of the scalars dominates over that from the metric fluctuations. In order for quantum corrections to be small, one should take $N \ll 1$. One can obtain a local form of this action by introducing an independent scalar field $Z$ which mimics the trace anomaly [4]. The on-shell equivalence of the equations of motion can be seen by choosing a conformal gauge for the two-dimensional metric, as in Eq. (2). Thus the action of the one-loop model will be given by:

$$S = \frac{1}{16\pi} \int d^2x (-g)^{1/2} \left[ (e^{-2\phi} + \frac{N}{3} Z) R - \frac{N}{6} (\nabla Z)^2 + 2 + 2e^{-2\phi} (\nabla \phi)^2 - 2e^{-2\phi} \Lambda - 2Q^2 e^{2\phi} \right].$$

(5)

**B. Equations of motion**

Differentiation with respect to $t$ ($x$) will be denoted by an overdot (a prime). For any functions $f$ and $g$, define:

$$\frac{\partial f}{\partial g} \delta g \equiv - f' \delta g + f \delta g', \quad \partial^2 g \equiv - \ddot{g} + g''.$$ 

(6)

$$\frac{\delta f}{\partial g} \delta g \equiv f' \delta g + f \delta g', \quad \delta^2 g \equiv \ddot{g} + g''.$$ 

(7)

Variation with respect to $\rho$, $\phi$ and $Z$ yields the following equations of motion:

$$- \partial^2 \phi + 2(\partial \phi)^2 + \frac{N}{6} e^{2\phi} \partial^2 Z + e^{2\phi + 2\phi} (\Lambda e^{-2\phi} + Q^2 e^{2\phi} - 1) = 0;$$

(8)

$$\partial^2 \rho - \partial^2 \phi + (\partial \phi)^2 + \frac{N}{6} e^{2\phi} - Q^2 e^{2\phi} = 0;$$

(9)

$$\partial^2 Z - 2\partial^2 \rho = 0.$$ 

(10)

The constraint equations are:

$$(\delta^2 \phi - 2\delta \phi \delta \rho) - (\partial \phi)^2 = \frac{N}{12} e^{2\phi} \left[ (\delta Z)^2 + 2\delta^2 Z - 4\delta Z \delta \rho \right];$$

(11)

$$(\delta' - \rho \delta' - \rho' \phi) - \delta \phi' = \frac{N}{12} e^{2\phi} \left[ \dot{Z} Z' + 2 \dot{Z}' - 2 \left( \rho Z' + \rho' \dot{Z} \right) \right].$$

(12)

From Eq. (14), it follows that $Z = 2\rho + \eta$, where $\eta$ satisfies $\partial^2 \eta = 0$. The remaining freedom in $\eta$ can be used to satisfy the constraint equations for any choice of $\rho$, $\dot{\rho}$, $\phi$ and $\dot{\phi}$ on an initial spacelike section [3].
Maximal (Nariai) black holes nucleate semiclassically in de Sitter space. This process is mediated by gravitational instantons and has been described in detail in Refs. \[2,9,3\]. Here we are interested not in the nucleation, but in the further evolution of the Nariai solution. Its metric is given by

\[
e^{2\rho} = \frac{1}{A \cos^2 t}, \quad e^{2\phi} = B,
\]

where \(B\) is given by the cubic equation

\[
B \left[ 1 - Q^2 B \left( 1 + \frac{N}{3} B \right) \right] = \left[ 1 - \frac{N}{3} B \right] \Lambda;
\]

the physical solution is the one that limits to \(B = \Lambda\) for \(N = Q = 0\). \(A\) is given by

\[
A = \Lambda - Q^2 B^2.
\]

Quantum fluctuations will perturb this solution, so that the two-sphere radius, \(e^{-\phi}\), will vary slightly along the one-sphere coordinate, \(x\). Decomposition into Fourier modes on the \(S^1\) yields the perturbation ansatz

\[
e^{2\phi} = \Lambda^2 \left[ 1 + 2\epsilon \sum_n (\sigma_n(t) \cos nx + \tilde{\sigma}_n(t) \sin nx) \right],
\]

where \(\epsilon\) is taken to be small. This will be referred to as the metric perturbation, characterized by the \(\sigma_n, \tilde{\sigma}_n\) at the time \(t = 0\).

More generally, one should also consider perturbations of the time derivative of the two-sphere radius, expressed by \(\dot{\sigma}_n\). As in Ref. \[8\], we parameterize the initial conditions for the perturbation as

\[
\sigma_n(0) = \sin \vartheta, \quad \dot{\sigma}_n(0) = \cos \vartheta,
\]

where \(\vartheta\) represents the phase of the initial perturbation.

Based on linear perturbation theory, one would expect such perturbations to lead to a classical, as well as a quantum instability. Classically, the regions on the \(S^1\) where the two-spheres are smaller than the Nariai value should collapse to form black hole interiors. The larger two-spheres, on the other hand, should grow exponentially, developing into asymptotic de Sitter regions. If the first mode dominates, this simply leads to a nearly-maximal Reissner-Nordström-de Sitter solution. But if higher modes are strongly excited, a whole sequence of Reissner-Nordström-de Sitter solutions can develop around the same one-sphere. This may be thought of as a necklace of de Sitter regions, strung together by black hole throats.

The expected quantum instability is the evaporation of these black holes. Actually, we will show below that black holes of sufficient charge are stable. But all other black holes would be expected to evaporate and get smaller, since their temperature is higher than that of the surrounding cosmological horizon.

The black hole evolution can be calculated numerically using Eqs. \(8\)–\(10\). It is important to stress that the black hole and cosmological horizons do not in general correspond to the minimal and maximal two-spheres along the \(S^1\), although such a slicing can always be found. In general, one must first find the positions of the horizons on the one-sphere; this can be done by finding the points where the gradient of the two-sphere size is null \(\frac{d}{dx} = 0\). The two-sphere sizes at those locations give the size of the black hole. The black hole evolution can be monitored by following the horizon location and plotting the horizon size vs. time. It will be convenient to define the horizon perturbation \(\delta\), for a black hole located at \(x_b\), by

\[
r_h(t)^{-2} = e^{2\phi(t,x_b(t))} = B \left[ 1 + 2\delta(t) \right].
\]

Thus \(\delta\) corresponds to the fractional difference between the current black hole size and the size of a maximal black hole of equal charge. Evaporation corresponds to increasing values of \(\delta\).

Because the \(S^1\) expands exponentially, the black hole and cosmological horizons of a Reissner-Nordström-de Sitter solution appear to move ever more closely together in comoving \(S^1\) coordinates (see Fig. \[3\]).
D. Numerical technique

The equations of motion (9) – (10) were solved numerically using the standard method of characteristics for second order quasi-linear hyperbolic equations \[23\]. For this purpose, the auxiliary field \(Z\) can be eliminated by inserting Eq. (10) into Eq. (9) and the remaining equations can be rewritten in manifestly hyperbolic form:

\[
\frac{\partial^2 \phi}{\partial t^2} = C \left[ e^{2\rho} \left( 3\Lambda - e^{2\phi}(3 + N\Lambda) + 3e^{4\phi}Q^2 + e^{6\phi}NQ^2 \right) - (e^{2\phi}N - 6)(\partial \phi)^2 \right], \tag{19}
\]

\[
\frac{\partial^2 \rho}{\partial t^2} = 3C \left[ e^{2\rho+\phi} \left( 2e^{2\phi}Q^2 - 1 \right) + (\partial \phi)^2 \right], \tag{20}
\]

where

\[
C = \frac{1}{3 - e^{2\phi}N}. \tag{21}
\]

The characteristic curves, along which the equations reduce to ordinary differential equations, are \(x_{\pm} = x \pm t\). Given the solution at two neighboring points on non-identical characteristics and approximating \(dx_{\pm}\) by \(\Delta x_{\pm}\), the solution at the intersection of the characteristic curves going through these points can be found by iteration. Assigning the initial conditions Eqs. (13) – (16) on a \(t = 0\) hypersurface thus allows the advancement of the solution along the \(x_{\pm}\)-grid, employing periodic boundary conditions at \(x = 0\) and \(x = 2\pi\).

For most of our computations, we used a resolution of 10000 grid points on \(x\)-hypersurfaces. The accuracy of the results was verified by comparison with perturbative solutions \[\[\]\] and with results obtained from an independent second-order finite difference code for the same equations. With the exception of the very last time step, the closest to the conformal time coordinate singularity, excellent agreement of all solutions was found.

III. ANTI-EVAPORATION AND TURNAROUND

Anti-evaporation was first found in Ref. \[\[\]\] for the \(\vartheta = \pi/2, n = 1\) perturbation of the maximal (Nariai) neutral black hole solution in de Sitter space. A power series approximation showed that black holes formed by this perturbation will grow initially. A numerical calculation, however, allows us to probe the black hole evolution beyond the range of validity of the power series. Interestingly, this proves the anti-evaporation effect to be transitory. This is demonstrated by the long-dashed line in Fig. 2. The horizon perturbation first decreases quadratically, signaling anti-evaporation, in quantitative agreement with the result of Ref. \[\[\]\]. At a time \(t \approx 0.2\), however, it turns around and starts to
increase. This means that the black hole eventually stops growing. Instead, it starts to shrink in size, corresponding to evaporation at late times.

For other initial phases, the behavior of the black hole can be quite complicated, going through various periods of evaporation and anti-evaporation (Fig. 2). The important result is, however, that it always ends up evaporating at late times, as conjectured in Ref. [8]. The numerical result for the final evaporation rate agrees with the asymptotic solutions given there to within better than one percent:

$$\delta \propto (\cos t)^{1-c_+},$$  
(22)

where $c_+$ is the larger root of

$$c(c + 1) = 2\sqrt{1 + \frac{2}{3}N\Lambda + \frac{1}{9}N^2\Lambda^2}.$$  
(23)

An initial phase just under $\vartheta = \pi/2$ corresponds to a large time-derivative of the metric perturbation directed opposite to the perturbation. In this case, the metric overshoots the Nariai value and a cosmological horizon forms. The corresponding black hole is located where the cosmological horizon would usually sit. Its evaporation is mirrored by the growth of the cosmological horizon. In Fig. 2 this cosmological horizon shows up for $\vartheta = 5\pi/6$ as a negative value of $\delta$ which increases in magnitude.

![FIG. 2](image1)

**FIG. 2.** Evolution of the black hole horizon perturbation $\delta$ for different initial values of $\vartheta$: $\vartheta = 0$ (solid), $\vartheta = \pi/6$ (dotted), $\vartheta = \pi/3$ (short dashed), $\vartheta = \pi/2$ (dashed-single dotted), $\vartheta = 2\pi/3$ (dashed-triple dotted), $\vartheta = 5\pi/6$ (long dashed) for $\Lambda = 0.1$, $N = 5$, and $w = 0$. The behavior is independent of the initial amplitude $\epsilon$ as long as $\epsilon \ll 1$; all plots are for $\epsilon = .0005$ unless otherwise indicated.

![FIG. 3](image2)

**FIG. 3.** Numerically one can show that perturbed Nariai black holes evaporate until they are significantly smaller than the cosmological horizon. Then they can simply be viewed as living in asymptotically flat space, so evaporation will continue. Model parameters here are $\epsilon = 0.5$, $\vartheta = \pi/3$. For this run, an RST-type counterterm [24] was included in the effective action in order to ensure that the black hole singularity resides at $r = 0$. 

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How can we be sure that the evaporation seen in the later stages of Fig. 2 is permanent? By the nature of the conformal time $t$, an infinite amount of proper time is contained in the final step of the numerical calculation. No matter how refined the step size, this region cannot be resolved. It is sufficient, however, to trace the evolution until the black hole size differs significantly from the size of the cosmological horizon. This is precisely the condition for non-linearity, so it cannot be verified analytically using the power series for $\delta$. Our numerical calculation, however, can reach this regime (see Fig. 3). Once it is so small, the black hole will be much hotter than the cosmological radiation, and will no longer be influenced by it. It will behave like a Schwarzschild black hole, so we can trust that it will continue to evaporate.

IV. STABILITY OF CHARGED BLACK HOLES

Nariai solutions can be electrically or magnetically charged. In this case, their spacelike sections will still be given by a direct product of $S^1 \times S^2$ (with constant $S^2$ radius), but the $S^1$ and $S^2$ radii will not be equal. The flux runs around the $S^1$. The metric is classically unstable to small perturbations. Reissner-Nordström-de Sitter black holes form where the two-sphere size is smaller, and asymptotic de Sitter regions develop where the two-spheres are larger.

These black holes are of nearly maximal size. For a small charge, one would expect their initial behavior to be similar to the neutral Schwarzschild-de Sitter black holes. When they are highly charged, however, they become nearly extremal. The temperature of a black hole decreases as it approaches extremality. Thermodynamic arguments
were given in Ref. [8] which showed that there is a critical value of the charge \( Q_c^2 = 3/(16\Lambda) = 3/4Q_{\text{max}}^2 \). Roughly speaking, when a subcritical Nariai solution is perturbed, the temperature of the black hole will be larger than that of the cosmological horizon because it is smaller. When a supercritical Nariai solution is perturbed, the black hole will be colder than the cosmological horizon, even though it is smaller, simply because its mass is already very close to the mass of the extremal, zero-temperature solution.

Thus one would expect subcritically charged black holes to evaporate at late times. Supercritically charged black holes, on the other hand, should absorb quantum radiation from the cosmological horizon and grow, their size approaching the Nariai value asymptotically. This is confirmed numerically, as shown in Figs. [4] and [5]. The value of the critical charge is confirmed quantitatively. This is a non-trivial check that this simple, two-dimensional model reflects the thermodynamic properties of Reissner-Nordström-de Sitter black holes accurately.

The anti-evaporation of supercritically charged black holes has an important consequence for the quantum global structure of de Sitter space. It gives rise to a new type of proliferation effect, as pointed out in Ref. [8]. Because the two-sphere size is forever nearly constant in the Hubble-size region between the two horizons, a small quantum fluctuation can easily invert the role of the horizons. In other words, it can increase the two-sphere size on the black hole horizon, turning it into a cosmological horizon, and vice-versa. This amounts to the insertion of a new black hole ‘bead’ into the \( S^1 \) ‘necklace.’ Such processes repeat endlessly, so that an unbounded number of causally disconnected de Sitter regions develop on the same \( S^1 \).

V. HIGHER MODES AND PROLIFERATION

We now return to uncharged solutions, but consider modes with \( n > 1 \). Such perturbations lead to a spatial geometry that can be described as a ‘doughnut with \( n \) wobbles.’ Like for \( n = 1 \), the metric perturbation is classically unstable. The mode will oscillate until the \( S^1 \) expansion has stretched it enough to leave the horizon. Then it will freeze out, and grow exponentially. This was shown in a linear approximation in Ref. [7], and is demonstrated numerically in Fig. [6] for \( n = 3 \).

![Fig. 6](image)

**Fig. 6.** Metric perturbation evolution for \( n = 3 \) as function of \( x \) and \( t \). Model parameter values as in Fig. [2].

The evolution of the horizon perturbation for \( n = 3 \) is shown in Fig. [7]. While the metric perturbation oscillates, the \((\nabla \phi)^2 = 0\) surfaces move rapidly and ‘cross over’ (see Fig. [8]). They only represent black hole horizons after the metric perturbation freezes out. Fig. [7] shows that in the \( n = 3, \vartheta = \pi/2 \) case considered there, the metric perturbation freezes out while it of the opposite sign compared to its initial value. This means that cosmological horizons develop from the initial minimal two-spheres. This is reflected in the figure in the negative values for the ‘black hole horizon’ perturbation. Note that the absolute value is increasing at late times, signalling evaporation.
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