ON THE GEVREY ULTRADIFFERENTIABILITY
OF WEAK SOLUTIONS
OF AN ABSTRACT EVOLUTION EQUATION
WITH A SCALAR TYPE SPECTRAL OPERATOR
OF ORDERS LESS THAN ONE

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Abstract. It is shown that, if all weak solutions of the evolution equation
\[ y'(t) = Ay(t), \quad t \geq 0, \]
with a scalar type spectral operator \( A \) in a complex Banach space are Gevrey ultradifferentiable of orders less than one, then the operator \( A \) is necessarily bounded.

Too much of a good thing can be wonderful.

Mae West

1. Introduction

In [28,31,32], found are characterizations of the strong differentiability and Gevrey ultradifferentiability of order \( \beta > 1 \), in particular analyticity and entireness, on \([0, \infty)\) and \((0, \infty)\) of all weak solutions of the evolution equation
\[(1.1) \quad y'(t) = Ay(t), \quad t \geq 0,\]
with a scalar type spectral operator \( A \) in a complex Banach space.

As is shown by [31, Theorem 4.1] (see also [31, Corollary 4.1]), all weak solutions of equation (1.1) can be entire vector functions, i.e., belong to the first-order Beurling type Gevrey class \( \mathcal{E}^{(1)}([0, \infty), X) \) (see Preliminaries), while the operator \( A \) is unbounded, e.g., when \( A \) is a semibounded below self-adjoint operator in a complex Hilbert space (see [21, Corollary 4.1]). This remarkable fact contrasts the situation when, in (1.1), a closed densely defined linear operator \( A \) generates a \( C_0 \)-semigroup, in which case the strong differentiability of all weak solutions of (1.1) at 0 alone immediately implies boundedness for \( A \) (cf. [7], see also [24]).

It remains to examine whether all weak solutions of equation (1.1) with a scalar type spectral operator \( A \) in a complex Banach space can belong to the Gevrey classes of orders less than one (not necessarily to the same one) with \( A \) remaining unbounded.

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In this paper, developing the results of [28, 31, 32], we show that an unbounded scalar type spectral operator $A$ in a complex Banach space cannot sustain the strong Gevrey ultradifferentiability of all weak solutions of equation (1.1) for orders less than one, i.e., that imposing on all the weak solutions along with the entireness requirement, certain growth at infinity conditions (see Preliminaries) necessarily makes the operator $A$ bounded. Thus, we generalize the corresponding results for equation (1.1) with a normal operator $A$ in a complex Hilbert space found in [22].

**Definition 1.1 (Weak Solution).**

Let $A$ be a densely defined closed linear operator in a Banach space $(X, \| \cdot \|)$. A strongly continuous vector function $y : [0, \infty) \to X$ is called a weak solution of equation (1.1) if, for any $g^* \in D(A^*)$,

$$
\frac{d}{dt} \langle y(t), g^* \rangle = \langle y(t), A^* g^* \rangle, \quad t \geq 0,
$$

where $D(\cdot)$ is the domain of an operator, $A^*$ is the operator adjoint to $A$, and $\langle \cdot, \cdot \rangle$ is the pairing between the space $X$ and its dual $X^*$ (cf. [1]).

**Remarks 1.1.**

- Due to the closedness of $A$, the weak solution of (1.1) can be equivalently defined to be a strongly continuous vector function $y : [0, \infty) \mapsto X$ such that, for all $t \geq 0$,

$$
\int_0^t y(s) \, ds \in D(A) \text{ and } y(t) = y(0) + A \int_0^t y(s) \, ds
$$

and is also called a mild solution (cf. [7, Ch. II, Definition 6.3], [30, Preliminaries]).

- Such a notion of weak solution, which need not be differentiable in the strong sense, generalizes that of classical one, strongly differentiable on $[0, \infty)$ and satisfying the equation in the traditional plug-in sense, the classical solutions being precisely the weak ones strongly differentiable on $[0, \infty)$.

- When a closed densely defined linear operator $A$ in a complex Banach space $X$ generates a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators (see, e.g., [7, 14]), i.e., the associated abstract Cauchy problem (ACP)

$$
\begin{cases}
  y'(t) = Ay(t), & t \geq 0, \\
  y(0) = f
\end{cases}
$$

is well-posed (cf. [7, Ch. II, Definition 6.8]), the weak solutions of equation (1.1) are the orbits

$$
y(t) = T(t)f, \quad t \geq 0,
$$

with $f \in X$ [7, Ch. II, Proposition 6.4] (see also [1, Theorem]), whereas the classical ones are those with $f \in D(A)$ (see, e.g., [7, Ch. II, Proposition 6.3]).

- In our discourse, the associated $ACP$ need not be well-posed, i.e., the scalar type spectral operator $A$ need not generate a $C_0$-semigroup (cf. [24]).
2. Preliminaries

Here, for the reader’s convenience, we outline certain essential preliminaries.

2.1. Scalar Type Spectral Operators.

Henceforth, unless specified otherwise, \( A \) is supposed to be a scalar type spectral operator in a complex Banach space \((X, \| \cdot \|)\) with strongly \(\sigma\)-additive spectral measure (the resolution of the identity) \( E_A(\cdot) \) assigning to each Borel set \( \delta \) of the complex plane \( \mathbb{C} \) a projection operator \( E_A(\delta) \) on \( X \) and having the operator’s spectrum \( \sigma(A) \) as its support [2,3,6].

Observe that, in a complex finite-dimensional space, the scalar type spectral operators are all linear operators on the space, for which there is an eigenbasis (see, e.g., [3,6]) and, in a complex Hilbert space, the scalar type spectral operators are precisely all those that are similar to the normal ones [36].

Associated with a scalar type spectral operator in a complex Banach space is the Borel operational calculus analogous to that for a normal operator in a complex Hilbert space [3,5,6,34], which assigns to any Borel measurable function \( F : \sigma(A) \to \mathbb{C} \) a scalar type spectral operator

\[
F(A) := \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda)
\]

defined as follows:

\[
F(A)f := \lim_{n \to \infty} F_n(A)f, \quad f \in D(F(A)), \quad D(F(A)) := \left\{ f \in X \mid \lim_{n \to \infty} F_n(A)f \text{ exists} \right\},
\]

where

\[
F_n(\cdot) := F(\cdot) \chi_{\{\lambda \in \sigma(A) \mid |F(\lambda)| \leq n\}}(\cdot), \quad n \in \mathbb{N},
\]

\((\chi_\delta(\cdot))\) is the characteristic function of a set \( \delta \subseteq \mathbb{C} \), \( \mathbb{N} := \{1,2,3,\ldots\} \) is the set of natural numbers) and

\[
F_n(A) := \int_{\sigma(A)} F_n(\lambda) \, dE_A(\lambda), \quad n \in \mathbb{N},
\]

are bounded scalar type spectral operators on \( X \) defined in the same manner as for a normal operator (see, e.g., [5,34]).

In particular,

\[
A^n = \int_{\sigma(A)} \lambda^n \, dE_A(\lambda), \quad n \in \mathbb{Z}_+,
\]

\((\mathbb{Z}_+ := \{0,1,2,\ldots\})\) is the set of nonnegative integers, \( A^0 := I, \) \( I \) is the identity operator on \( X \) and

\[
e^{zA} := \int_{\sigma(A)} e^{z\lambda} \, dE_A(\lambda), \quad z \in \mathbb{C}.
\]

The properties of the spectral measure and operational calculus, exhaustively delineated in [3,6], underlie the entire subsequent discourse. Here, we underline a few facts of particular importance.
Due to its strong countable additivity, the spectral measure $E_A(\cdot)$ is bounded [4,6], i.e., there is such an $M \geq 1$ that, for any Borel set $\delta \subseteq \mathbb{C}$,
\begin{equation}
\|E_A(\delta)\| \leq M. \tag{2.5}
\end{equation}

Observe that the notation $\| \cdot \|$ is used here to designate the norm in the space $L(X)$ of all bounded linear operators on $X$. We adhere to this rather conventional economy of symbols in what follows also adopting the same notation for the norm in the dual space $X^*$.

For any $f \in X$ and $g^* \in X^*$, the total variation measure $v(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^* \rangle$ is a finite positive Borel measure with
\begin{equation}
v(f, g^*, \mathbb{C}) = v(f, g^*, \sigma(A)) \leq 4M\|f\|\|g^*\| \tag{2.6}
\end{equation}
(see, e.g., [25,26]).

Also (Ibid.), for a Borel measurable function $F : \mathbb{C} \to \mathbb{C}$, $f \in D(F(A))$, $g^* \in X^*$, and a Borel set $\delta \subseteq \mathbb{C}$,
\begin{equation}
\int_{\delta} |F(\lambda)| \, dv(E_A(\cup_{n=1}^{\infty} \Delta_n)f, g^*, \lambda) \leq 4M\|F(A)f\|\|g^*\|. \tag{2.7}
\end{equation}
In particular, for $\delta = \sigma(A)$,
\begin{equation}
\int_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) \leq 4M\|F(A)f\|\|g^*\|. \tag{2.8}
\end{equation}

Observe that the constant $M \geq 1$ in (2.6)–(2.8) is from (2.5).

Further, for a Borel measurable function $F : \mathbb{C} \to [0, \infty)$, a Borel set $\delta \subseteq \mathbb{C}$, a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint Borel sets in $\mathbb{C}$, and $f \in X$, $g^* \in X^*$,
\begin{equation}
\int_{\delta} F(\lambda) \, dv(E_A(\cup_{n=1}^{\infty} \Delta_n)f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) \, dv(E_A(\Delta_n)f, g^*, \lambda). \tag{2.9}
\end{equation}

Indeed, since, for any Borel sets $\delta, \sigma \subseteq \mathbb{C}$,
\begin{equation}
E_A(\delta)E_A(\sigma) = E_A(\delta \cap \sigma) \tag{3,6},
\end{equation}
for the total variation,
\[v(E_A(\delta)f, g^*, \sigma) = v(f, g^*, \delta \cap \sigma).\]

Whence, due to the nonnegativity of $F(\cdot)$ (see, e.g., [13]),
\begin{align*}
\int_{\delta} F(\lambda) \, dv(E_A(\cup_{n=1}^{\infty} \Delta_n)f, g^*, \lambda) &= \int_{\delta \cap \cup_{n=1}^{\infty} \Delta_n} F(\lambda) \, dv(f, g^*, \lambda) \\
&= \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) \, dv(f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) \, dv(E_A(\Delta_n)f, g^*, \lambda).
\end{align*}

The following statement, allowing to characterize the domains of Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures, is fundamental for our discourse.
Proposition 2.1 ([23, Proposition 3.1]).
Let \( A \) be a scalar type spectral operator in a complex Banach space \((X, \| \cdot \|)\) with spectral measure \( E_A(\cdot) \) and \( F : \sigma(A) \to \mathbb{C} \) be a Borel measurable function. Then \( f \in D(F(A)) \) iff

\[
\text{(i) for each } g^* \in X^*, \int_{\sigma(A)} |F(\lambda)| \, dv(f, g^*, \lambda) < \infty \text{ and } \\
\text{(ii) } \sup_{\{g^* \in X^* : \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) : |F(\lambda)| > n\}} |F(\lambda)| \, dv(f, g^*, \lambda) \to 0, \ n \to \infty,
\]

where \( v(f, g^*, \cdot) \) is the total variation measure of \( \langle E_A(\cdot), g^* \rangle \).

The succeeding key theorem provides a description of the weak solutions of equation (1.1) with a scalar type spectral operator \( A \) in a complex Banach space.

Theorem 2.1 ([23, Theorem 4.2]).
Let \( A \) be a scalar type spectral operator in a complex Banach space \((X, \| \cdot \|)\). A vector function \( y : [0, \infty) \to X \) is a weak solution of equation (1.1) iff there is an \( f \in \bigcap_{t \geq 0} D(e^{tA}) \) such that

\[
y(t) = e^{tA}f, \ t \geq 0,
\]

the operator exponentials understood in the sense of the Borel operational calculus (see (2.4)).

Remarks 2.1.

- Theorem 2.1 generalizing [20, Theorem 3.1], its counterpart for a normal operator \( A \) in a complex Hilbert space, in particular, implies
  
  - that the subspace \( \bigcap_{t \geq 0} D(e^{tA}) \) of all possible initial values of the weak solutions of equation (1.1) is the largest permissible for the exponential form given by (2.10), which highlights the naturalness of the notion of weak solution, and

  - that associated \( ACP \) (1.2), whenever solvable, is solvable uniquely.

- Observe that the initial-value subspace \( \bigcap_{t \geq 0} D(e^{tA}) \) of equation (1.1), containing the dense in \( X \) subspace \( \bigcup_{\alpha > 0} E_A(\Delta_\alpha)X \), where

  \[
  \Delta_\alpha := \{\lambda \in \mathbb{C} : |\lambda| \leq \alpha\}, \ \alpha > 0,
  \]

  which coincides with the class \( \mathcal{E}^{(0)}(A) \) of the entire vectors of \( A \) of exponential type (see below), is dense in \( X \).

- When a scalar type spectral operator \( A \) in a complex Banach space generates a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \),

  \[
  T(t) = e^{tA} \text{ and } D(e^{tA}) = X, \ t \geq 0,
  \]

[24], and hence, Theorem 2.1 is consistent with the well-known description of the weak solutions for this setup (see (1.3)).
Subsequently, the frequent terms “spectral measure” and “operational calculus” are abbreviated to s.m. and o.c., respectively.

### 2.2. Gevrey Classes of Functions.

**Definition 2.1** (Gevrey Classes of Functions).

Let \((X, \| \cdot \|)\) be a (real or complex) Banach space, \(C^\infty(I, X)\) be the space of all \(X\)-valued functions strongly infinite differentiable on an interval \(I \subseteq (-\infty, \infty)\), and \(0 \leq \beta < \infty\).

The following subspaces of \(C^\infty(I, X)\)

\[
\mathcal{E}^{(\beta)}(I, X) := \{ g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \exists \alpha > 0 \exists c > 0 : \\
\max_{a \leq t \leq b} \| g^{(n)}(t) \| \leq c \alpha^n [n!]^\beta, \ n \in \mathbb{Z}^+ \},
\]

are called the \(\beta\)th-order Gevrey classes of strongly ultradifferentiable vector functions on \(I\) of Roumieu and Beurling type, respectively (see, e.g., \([8, 15-17]\))

**Remarks 2.2.**

- In view of Stirling’s formula, the sequence \(\{n!^\beta\}_n^\infty\) can be replaced with \(\{n^{\beta n}\}_n^\infty\).

- For \(0 \leq \beta < \beta' < \infty\), the inclusions

\[
\mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{(\beta')} (I, X) \subseteq \mathcal{E}^{(\beta)} (I, X) \subseteq \mathcal{E}^{(\beta')} (I, X) \subseteq C^\infty(I, X)
\]

hold.

- For \(1 < \beta < \infty\), the Gevrey classes are non-quasianalytic (see, e.g., \([16]\)).

- For \(\beta = 1\), \(\mathcal{E}^{(1)}(I, X)\) is the class of all analytic on \(I\), i.e., analytically continuous into complex neighborhoods of \(I\), vector functions and \(\mathcal{E}^{(1)}(I, X)\) is the class of all entire, i.e., allowing entire continuations, vector functions \([19]\).

- For \(0 \leq \beta < 1\), the Gevrey class \(\mathcal{E}^{(\beta)}(I, X)\) consists of all functions \(g(\cdot) \in \mathcal{E}^{(1)}(I, X)\) such that, for some (any) \(\gamma > 0\), there is an \(M > 0\) for which

\[
\|g(z)\| \leq Me^{\gamma|z|^{1/(1-\beta)}}, \ z \in \mathbb{C},
\]

\([22]\). In particular, for \(\beta = 0\), \(\mathcal{E}^{(0)}(I, X)\) and \(\mathcal{E}^{(0)}(I, X)\) are the classes of entire vector functions of exponential and minimal exponential type, respectively (see, e.g., \([18]\)).

### 2.3. Gevrey Classes of Vectors.

One can consider the Gevrey classes in a more general sense.
**Definition 2.2** (Gevrey Classes of Vectors).

Let \((A, D(A))\) be a densely defined, closed linear operator in a (real or complex) Banach space \((X, \| \cdot \|)\), \(0 \leq \beta < \infty\), and

\[
C^\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)
\]

be the subspace of infinite differentiable vectors of \(A\).

The following subspaces of \(C^\infty(A)\)

\[
\mathcal{E}^{(\beta)}(A) := \{ x \in C^\infty(A) \mid \exists \alpha > 0 \exists c > 0 : \| A^n x \| \leq c \alpha^n n!^\beta, \ n \in \mathbb{Z}_+ \},
\]

\[
\mathcal{E}^{(\beta')}((A) := \{ x \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \| A^n x \| \leq c \alpha^n n!^\beta, \ n \in \mathbb{Z}_+ \}
\]

are called the \(\beta\)-th order Gevrey classes of ultradifferentiable vectors of \(A\) of the Roumieu and Beurling type, respectively (see, e.g., \([10–12]\)).

**Remarks 2.3.**

- In view of Stirling’s formula, the sequence \(\{n!^\beta\}_{n=0}^{\infty}\) can be replaced with \(\{n^{\beta n}\}_{n=0}^{\infty}\).
- For \(0 \leq \beta < \beta' < \infty\), the inclusions

\[
\mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{(\beta')}(A) \subseteq \mathcal{E}^{(\beta')}(A) \subseteq C^\infty(A)
\]

hold.
- In particular, \(\mathcal{E}^{(1)}(A)\) and \(\mathcal{E}^{(1)}(A)\) are the classes of analytic and entire vectors of \(A\), respectively \([9,33]\) and \(\mathcal{E}^{(0)}(A)\) and \(\mathcal{E}^{(0)}(A)\) are the classes of entire vectors of \(A\) of exponential and minimal exponential type, respectively (see, e.g., \([12,35]\)).
- In view of the closedness of \(A\), it is easily seen that the class \(\mathcal{E}^{(1)}(A)\) forms the subspace of initial values in \(X\) generating the (classical) solutions of \((1.1)\), which are entire vector functions represented by the power series

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f, \ t \geq 0, f \in \mathcal{E}^{(1)}(A),
\]

the classes \(\mathcal{E}^{(\beta)}(A)\) and \(\mathcal{E}^{(\beta)}(A)\) with \(0 \leq \beta < 1\) being the subspaces of such initial values for which the solutions satisfy growth estimate \((2.11)\) with some (any) \(\gamma > 0\) and some \(M > 0\), respectively (cf. \([18]\)).

As is shown in \([10]\) (see also \([11,12]\)), if \(0 < \beta < \infty\), for a normal operator \(A\) in a complex Hilbert space,

\[
\mathcal{E}^{(\beta)}(A) = \bigcup_{t > 0} D(e^{t|A|^{1/\beta}}) \ \text{and} \ \mathcal{E}^{(\beta)}(A) = \bigcap_{t > 0} D(e^{t|A|^{1/\beta}}),
\]

the operator exponentials \(e^{t|A|^{1/\beta}}, \ t > 0\), understood in the sense of the Borel operational calculus (see, e.g., \([5,34]\)).

In \([26,29]\), descriptions \((2.14)\) are extended to scalar type spectral operators in a complex Banach space, in which form they are basic for our discourse. In \([29]\), similar nature descriptions of the classes \(\mathcal{E}^{(0)}(A)\) and \(\mathcal{E}^{(0)}(A)\) \((\beta = 0)\), known for a
normal operator $A$ in a complex Hilbert space (see, e.g., [12]), are also generalized to scalar type spectral operators in a complex Banach space. In particular [29, Theorem 5.1],

$$
\mathcal{E}^{(0)}(A) = \bigcup_{\alpha > 0} E_A(\Delta_{\alpha})X,
$$

where

$$
\Delta_{\alpha} := \{\lambda \in \mathbb{C} | |\lambda| \leq \alpha\}, \alpha > 0.
$$

We also need the following characterization of a particular weak solution’s of equation (1.1) with a scalar type spectral operator $A$ in a complex Banach space being strongly Gevrey ultradifferentiable on a subinterval $I$ of $[0, \infty)$ proved in [30].

**Proposition 2.2** ([30, Proposition 3.1]). Let $A$ be a scalar type spectral operator in a complex Banach space $(X, \| \cdot \|)$, $0 \leq \beta < \infty$, and $I$ be a subinterval of $[0, \infty)$. Then the restriction of a weak solution $y(\cdot)$ of equation (1.1) to $I$ belongs to the Gevrey class $\mathcal{E}^{(\beta)}(I, X)$ (or $\mathcal{E}^{(\beta)}(I, X)$) iff, for each $t \in I$,

$$
y(t) \in \mathcal{E}^{(\beta)}(A) (\mathcal{E}^{(\beta)}(A), \text{ respectively}),
$$

in which case

$$
y^{(n)}(t) = A^n y(t), \ n \in \mathbb{N}, t \in I.
$$

3. One Lemma

The following lemma generalizes [22, Lemma 4.1], its counterpart for a normal operator in a complex Hilbert space and, besides being an interesting result by itself, is necessary for proving our main statement.

**Lemma 3.1.** Let $A$ be a scalar type spectral operator in a complex Banach space $(X, \| \cdot \|)$ with spectral measure $E_A(\cdot)$ and $0 < \beta' < \infty$. If

$$
\bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{(\beta)}(A) = \mathcal{E}^{(\beta')}(A),
$$

then the operator $A$ is bounded.

**Proof.** First, observe that, in view of inclusions (2.12), for any $\beta' > 0$,

$$
\bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{(\beta')}(A),
$$

Let us prove the statement by contrapositive assuming $A$ to be unbounded.

The operator $A$ being scalar type spectral, this assumption implies that the spectrum $\sigma(A)$ of $A$ is an unbounded set in the complex plane $\mathbb{C}$ [3, 6]. Hence, the points of the spectrum can be found in infinitely many semi-open annuli of the form

$$
\delta_n := \{\lambda \in \mathbb{C} | n \leq |\lambda| < n + 1\}, \ n \in \mathbb{N},
$$

i.e., there is a sequence of natural numbers $\{n(k)\}_{k=1}^{\infty}$ such that

$$
k \leq n(k) < n(k + 1), \ k \in \mathbb{N},
$$

and, for each $k \in \mathbb{N}$, there is a

$$
\lambda_k \in \delta_{n(k)} \cap \sigma(A) \neq \emptyset.
$$
Setting $\varepsilon_0 := 1$, for each $k \in \mathbb{N}$, one can choose an
\begin{equation}
0 < \varepsilon_k < \min \left( \frac{1}{k}, \varepsilon_k-1 \right)
\end{equation}
such that
\begin{equation}
\lambda_k \in \Delta_k := \{ \lambda \in \mathbb{C} \mid n(k) - \varepsilon_k < |\lambda| < n(k) + 1 - \varepsilon_k \}.
\end{equation}
Since
\[ n(k) < n(k+1) \quad \text{and} \quad \varepsilon_{k+1} < \varepsilon_k, \quad k \in \mathbb{N}, \]
we have:
\[ n(k) + 1 - \varepsilon_k < n(k+1) - \varepsilon_{k+1}, \quad k \in \mathbb{N}, \]
and hence, the open annuli $\Delta_k$, $k \in \mathbb{N}$, are pairwise disjoint.

Whence, by the properties of the s.m.,
\[ E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j, \]
where 0 stands for the zero operator on $X$.

Observe also, that the subspaces $E_A(\Delta_k)X$, $k \in \mathbb{N}$, are nontrivial since
\[ \Delta_k \cap \sigma(A) \neq 0, \quad k \in \mathbb{N}, \]
with $\Delta_k$ being an open set in $\mathbb{C}$.

By choosing a unit vector $e_k \in E_A(\Delta_k)X$ for each $k \in \mathbb{N}$, we obtain a sequence $\{e_n\}_{n=1}^{\infty}$ in $X$ such that
\begin{equation}
\|e_k\| = 1, \quad k \in \mathbb{N}, \quad \text{and} \quad E_A(\Delta_i)e_j = \delta_{ij}e_j, \quad i, j \in \mathbb{N},
\end{equation}
where $\delta_{ij}$ is the Kronecker delta.

As is easily seen, (3.19) implies that the vectors $e_k$, $k \in \mathbb{N}$, are linearly independent.

Furthermore, there is an $\varepsilon > 0$ such that
\begin{equation}
d_k := \text{dist} (e_k, \text{span} (\{e_i \mid i \in \mathbb{N}, \ i \neq k\})) \geq \varepsilon, \quad k \in \mathbb{N}.
\end{equation}

Indeed, the opposite implies the existence of a subsequence $\{d_{k(m)}\}_{m=1}^{\infty}$ such that
\[ d_{k(m)} \to 0, \quad m \to \infty. \]

Then, by selecting a vector
\[ f_{k(m)} \in \text{span} (\{e_i \mid i \in \mathbb{N}, \ i \neq k(m)\}), \quad m \in \mathbb{N}, \]
such that
\[ \|e_{k(m)} - f_{k(m)}\| < d_{k(m)} + 1/m, \quad m \in \mathbb{N}, \]
we arrive at
\begin{align*}
1 &= \|e_{k(m)}\| \\
&= \|E_A(\Delta_{k(m)})e_{k(m)} - f_{k(m)}\| \leq \|E_A(\Delta_{k(m)})\|\|e_{k(m)} - f_{k(m)}\| \quad \text{by (2.5)}; \\
&\leq M\|e_{k(m)} - f_{k(m)}\| \leq M \left[ d_{k(m)} + 1/m \right] \to 0, \quad m \to \infty,
\end{align*}
which is a contradiction proving (3.20).

As follows from the Hahn-Banach Theorem, for any $k \in \mathbb{N}$, there is an $e^*_k \in X^*$ such that
\begin{equation}
\|e^*_k\| = 1, \quad k \in \mathbb{N}, \quad \text{and} \quad \langle e_i, e^*_j \rangle = \delta_{ij}d_i, \quad i, j \in \mathbb{N}.
\end{equation}
Let
\[ f := \sum_{k=1}^{\infty} n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k \in X \quad \text{and} \quad h := \sum_{k=1}^{\infty} n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k \in X, \]
the elements being well defined since \( \|e_k\| = 1, k \in \mathbb{N} \) (see (3.19)) and
\[ \left\{ n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} \right\}_{k=1}^{\infty}, \left\{ n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} \right\}_{k=1}^{\infty} \in l_1 \]
(\( l_1 \) is the space of absolutely summable sequences).
Indeed, in view of (3.16) and (3.17), for all \( k \in \mathbb{N} \) sufficiently large so that
\[ n(k) \geq 4^\varepsilon, \]
we have:
\[ n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} \leq n(k)^{-4} \leq 4^{-4}. \]
In view of (3.19), by the properties of the s.m.,
\[ E_A(\bigcup_{k=1}^{\infty} \Delta_k) f = f \quad \text{and} \quad E_A(\Delta_k) f = n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k, \quad k \in \mathbb{N}. \]
and
\[ E_A(\bigcup_{k=1}^{\infty} \Delta_k) h = h \quad \text{and} \quad E_A(\Delta_k) h = n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} e_k, \quad k \in \mathbb{N}. \]
For an arbitrary \( t > 0 \) and any \( g^* \in X^* \),
\[ \int_{\sigma(A)} e^{t|\lambda|^{1/\beta'}} \, dv(f, g^*, \lambda) \]
by (3.23);
\[ = \int_{\sigma(A)} e^{t|\lambda|^{1/\beta'}} \, dv(E_A(\bigcup_{k=1}^{\infty} \Delta_k) f, g^*, \lambda) \]
by (2.9);
\[ = \sum_{k=1}^{\infty} \int_{\sigma(A) \cap \Delta_k} e^{t|\lambda|^{1/\beta'}} \, dv(E_A(\Delta_k) f, g^*, \lambda) \]
by (3.23);
\[ = \sum_{k=1}^{\infty} n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\sigma(A) \cap \Delta_k} e^{t|\lambda|^{1/\beta'}} \, dv(e_k, g^*, \lambda) \]
since, by (3.18), for \( \lambda \in \Delta_k, |\lambda| < n(k) + 1 - \varepsilon_k; \)
\[ \leq \sum_{k=1}^{\infty} n(k)^{-\frac{1}{\beta}(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{t(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\sigma(A) \cap \Delta_k} 1 \, dv(e_k, g^*, \lambda) \]
\[ = \sum_{k=1}^{\infty} e^{-\ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{t(n(k)+1-\varepsilon_k)^{1/\beta'}} v(e_k, g^*, \Delta_k) \]
\[ \leq \sum_{k=1}^{\infty} e^{-\ln n(k)-t(n(k)+1-\varepsilon_k)^{1/\beta'}} v(e_k, g^*, \Delta_k) \]
by (2.6);
\[ \leq \sum_{k=1}^{\infty} e^{-\ln n(k)-t(n(k)+1-\varepsilon_k)^{1/\beta'}} 4M\|e_k\||g^*| \]
Similarly, for any $t > (3.26)$ we have:

Indeed, in view of (3.16) and (3.17), for all $k \in \mathbb{N}$ sufficiently large so that

\[ \ln n(k) - t \geq 1, \]

we have:

\[ e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} \leq e^{-k^{1/\beta'}}. \]

Similarly, for any $t > 0$ and $n \in \mathbb{N},$

\[ (3.26) \quad \sup_{\|g^*\|=1} \int_{\{\lambda \in \sigma(A) \mid e^{\|\lambda\|^{1/\beta'}} > n\}} e^{\|\lambda\|^{1/\beta'}} \, dv(f, g^*, \lambda) \]

\[ \leq \sup_{\|g^*\|=1} \sum_{k=1}^{\infty} e^{-(\ln n(k)-t)(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\{\lambda \in \sigma(A) \mid e^{\|\lambda\|^{1/\beta'}} > n\} \cap \Delta_k} 1 \, dv(e_k, g^*, \lambda) \]

\[ = \sup_{\|g^*\|=1} \sum_{k=1}^{\infty} e^{-1/2 \ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\{\lambda \in \sigma(A) \mid e^{\|\lambda\|^{1/\beta'}} > n\} \cap \Delta_k} 1 \, dv(e_k, g^*, \lambda) \]

\[ = L \sup_{\|g^*\|=1} \sum_{k=1}^{\infty} e^{-1/2 \ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} \int_{\{\lambda \in \sigma(A) \mid e^{\|\lambda\|^{1/\beta'}} > n\} \cap \Delta_k} 1 \, dv(e_k, g^*, \lambda) \]

by (3.24);

\[ = L \sup_{\|g^*\|=1} \sum_{k=1}^{\infty} 1 \, dv(E_A(\Delta_k)h, g^*, \lambda) \]

by (2.9);

\[ = L \sup_{\|g^*\|=1} \int_{\{\lambda \in \sigma(A) \mid e^{\|\lambda\|^{1/\beta'}} > n\}} 1 \, dv(E_A(\bigcup_{k=1}^{\infty} \Delta_k)h, g^*, \lambda) \]

by (3.24);

\[ = L \sup_{\|g^*\|=1} \int_{\{\lambda \in \sigma(A) \mid e^{\|\lambda\|^{1/\beta'}} > n\}} 1 \, dv(h, g^*, \lambda) \]

by (2.7);
By Proposition 2.1 and (2.14), (3.25) and (3.26) jointly imply that
\begin{equation}
  f \in \bigcap_{t > 0} D(e^{t|A|^{1/\beta'}}) = \mathcal{E}(\beta')(A).
\end{equation}

Let
\begin{equation}
  h^* := \sum_{k=1}^{\infty} n(k)^{-2}e_k^* \in X^*,
\end{equation}
the functional being well defined since, by (3.16), \( \{n(k)^{-2}\}_{k=1}^{\infty} \in l_1 \) and \( \|e_k^*\| = 1 \), \( k \in \mathbb{N} \) (see (3.21)).

In view of (3.21) and (3.20), we have:
\begin{equation}
  \langle e_k, h^* \rangle = \langle e_k, n(k)^{-2}e_k^* \rangle = d_k n(k)^{-2} \geq \varepsilon n(k)^{-2}, \ n \in \mathbb{N}.
\end{equation}

Fixing an arbitrary \( 0 < \beta < \beta' \), for any \( t > 0 \), we have:
\begin{equation}
  \int_{\sigma(A)} e^{t|\lambda|^{1/\beta'}} \, dv(f, h^*, \lambda) = \sum_{k=1}^{\infty} n(k)^{-1}(n(k)+1-\varepsilon_k)^{1/\beta'} \int_{\sigma(A) \cap \Delta_k} e^{t|\lambda|^{1/\beta'}} \, dv(e_k, h^*, \lambda)
\end{equation}
\begin{equation}
  \geq \sum_{k=1}^{\infty} e^{-\ln n(k)(n(k)+1-\varepsilon_k)^{1/\beta'}} e^{t(n(k)-\varepsilon_k)^{1/\beta'}} v(e_k, h^*, \Delta_k)
\end{equation}
\begin{equation}
  \geq \sum_{k=1}^{\infty} e^{t(n(k)-1)^{1/\beta'}-\ln n(k)(n(k)+1)^{1/\beta'}} \varepsilon n(k)^{-2}
\end{equation}
\begin{equation}
  = \sum_{k=1}^{\infty} \varepsilon e^{t(n(k)-1)^{1/\beta'}-\ln n(k)(n(k)+1)^{1/\beta'}-2\ln n(k)} = \infty.
\end{equation}

Indeed, for \( k \geq 2 \),
\begin{equation}
  t(n(k)-1)^{1/\beta'}-\ln n(k)(n(k)+1)^{1/\beta'}-2\ln n(k)
  = (n(k)-1)^{1/\beta'} \left[ t - \left( \frac{n(k)+1}{n(k)-1} \right)^{1/\beta'} \frac{\ln n(k)}{(n(k)-1)^{1/\beta'-1/\beta}} + \frac{2\ln n(k)}{(n(k)-1)^{1/\beta}} \right]
\end{equation}
\begin{equation}
  \to \infty, \ k \to \infty,
\end{equation}
By Proposition 2.1 and (2.14), (3.30) implies that, for each $0 < \beta < \beta'$,

$$f \notin \bigcup_{t > 0} D(e^{t|A|^{1/\beta}}) = \mathcal{E}^{(\beta)}(A).$$

Whence, in view of inclusions (2.12), we infer that

$$f \notin \bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{(\beta)}(A). \tag{3.31}$$

Comparing (3.27) and (3.31), we conclude that

$$\bigcup_{0 \leq \beta < \beta'} \mathcal{E}^{(\beta)}(A) \neq \mathcal{E}^{(\beta')}(A),$$

which completes the proof by contrapositive. \[\square\]

4. Main Result

Lemma 3.1 affords a rather short proof for the following

**Theorem 4.1.** Let $A$ be a scalar type spectral operator in a complex Banach space $(X, \| \cdot \|)$. If every weak solution of equation (1.1) belongs to the $\beta$th-order Roumieu type Gevrey class $\mathcal{E}^{(\beta)}((0, +\infty), X)$ with $0 \leq \beta < 1$ (each one to its own), then the operator $A$ is bounded, and hence, all weak solutions of (1.1) are necessarily entire vector functions of exponential type.

**Proof.** Let $y(\cdot)$ be an arbitrary weak solution of equation (1.1). Then, by the premise,

$$y(\cdot) \in \mathcal{E}^{(\beta)}([0, \infty), X)$$

with some $0 < \beta < 1$ ($\beta$ depends on $y(\cdot)$).

This, by Proposition 2.2, implies that

$$y(t) \in \mathcal{E}^{(\beta)}(A), \ t \geq 0.$$

In particular,

$$y(0) \in \mathcal{E}^{(\beta)}(A).$$

Since, by Theorem 2.1, the initial values of all weak solutions of equation (1.1) form the subspace

$$\bigcap_{t \geq 0} D(e^{tA}),$$

in view of $D(e^{0A}) = D(I) = X$, we have the inclusion

$$\bigcap_{t > 0} D(e^{tA}) = \bigcap_{t \geq 0} D(e^{tA}) \subseteq \bigcup_{0 \leq \beta < 1} \mathcal{E}^{(\beta)}(A). \tag{4.32}$$

Since, for an arbitrary $t > 0$ and any $f \in X, g^* \in X^*$,

$$\int_{\sigma(A)} |e^{t\lambda}| \ dv(f, h^*, \lambda) = \int_{\sigma(A)} e^{t\Re \lambda} \ dv(f, h^*, \lambda) \leq \int_{\sigma(A)} e^{t|\lambda|} \ dv(f, h^*, \lambda)$$
and, considering the inclusion
\[ \{ \lambda \in \sigma(A) \mid e^{t \Re \lambda} > n \} \subseteq \{ \lambda \in \sigma(A) \mid e^{t|\lambda|} > n \}, \quad t > 0, n \in \mathbb{N}, \]
for any \( n \in \mathbb{N} \),
\[
\int_{\{ \lambda \in \sigma(A) \mid |e^{t\lambda}| > n \}} |e^{t\lambda}| \, dv(f, h^*, \lambda) = \int_{\{ \lambda \in \sigma(A) \mid e^{t \Re \lambda} > n \}} e^{t \Re \lambda} \, dv(f, h^*, \lambda)
\]
\[
\leq \int_{\{ \lambda \in \sigma(A) \mid e^{t|\lambda|} > n \}} e^{t|\lambda|} \, dv(f, h^*, \lambda),
\]
by Proposition 2.1, we infer that, for each \( t > 0 \),
\[
D(e^{tA}) \supseteq D(e^{t|A|}),
\]
which, in view of (2.14), with \( \beta = 1 \), implies that
\[
\bigcap_{t>0} D(e^{tA}) \supseteq \bigcap_{t>0} D(e^{t|A|}) = \mathcal{E}^{(1)}(A),
\]
the equality here being necessary and sufficient for \(-A\) to be a generator of an analytic \( C_0 \)-semigroup [27] (see also [7, 14]).

Observe that inclusion (4.33) also directly follows from the fact that \( \mathcal{E}^{(1)}(A) \) is the subspace of initial values in \( X \) generating the (classical) solutions of (1.1), which are entire vector functions represented by power series (2.13) (see Preliminaries).

Inclusions (4.32) and (3.15) for \( \beta' = 1 \) jointly with (4.33), imply the following closed chain
\[
\bigcap_{t>0} D(e^{tA}) \subseteq \bigcup_{0 \leq \beta < 1} \mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{(1)}(A) \subseteq \bigcap_{t>0} D(e^{tA}),
\]
which proves that
\[
\bigcup_{0 \leq \beta < 1} \mathcal{E}^{(\beta)}(A) = \mathcal{E}^{(1)}(A).
\]

Whence, by Lemma 3.1 with \( \beta' = 1 \), we infer that the operator \( A \) is bounded, which completes the proof and implies that each weak solution \( y(\cdot) \) of equation (1.1) is an entire vector function of the form
\[
y(z) = e^{zA} f = \sum_{n=0}^{\infty} \frac{z^n}{n!} A^n f, \quad z \in \mathbb{C}, \text{ with some } f \in X,
\]
and hence, satisfying the growth condition
\[
\|y(z)\| \leq \|f\| e^{\|A\| |z|}, \quad z \in \mathbb{C},
\]
is of exponential type (see Preliminaries). \( \square \)

Theorem 4.1 generalizes [22, Theorem 5.1], its counterpart for a normal operator \( A \) in a complex Hilbert space and shows that, while a scalar type spectral operator \( A \) in a complex Banach space can be unbounded while all weak solutions of equation (1.1) are entire vector functions (see [31, Theorem 4.1] and [31, Corollary 4.1]), it cannot remain unbounded, if each weak solutions \( y(\cdot) \) of (1.1), in addition to being entire, is to satisfy the growth condition
\[
\|y(z)\| \leq Me^{\gamma |z|^{1/(1-\beta)}}, \quad z \in \mathbb{C},
\]
with some $0 \leq \beta < 1$, $\gamma > 0$, and $M > 0$ depending on $y(\cdot)$ (see Preliminaries, cf. (2.11)).

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