FREEDOM OF EQUIVARIANT COHOMOLOGY AND MUTANTS OF COMPACTIFIED REPRESENTATIONS

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Abstract. We survey generalisations of the Chang–Skjelbred Lemma for integral coefficients. Moreover, we construct examples of compact manifolds with actions of tori of rank $> 2$ whose equivariant cohomology is torsion-free, but not free. This answers a question of Allday’s. The “mutants” we construct are obtained from compactified representations and involve Hopf bundles in a crucial way.

1. Introduction

Let $T = (S^1)^r$ be a torus and $X$ a “sufficiently nice” $T$-space, for example, a compact (differentiable) $T$-manifold. The equivariant cohomology $H^*_T(X; \mathbb{Q})$ is defined as the cohomology of the Borel construction $X_T = ET \times_T X$. It captures quite a lot of information about the $T$-action and sometimes also provides a link to other subjects such as combinatorics, cf. [BP]. For that reason, one is interested in efficient methods to compute $H^*_T(X; \mathbb{Q})$.

The projection $X_T \to ET/T = BT$ onto the classifying space of $T$ gives $H^*_T(X; \mathbb{Q})$ an algebra structure over the polynomial ring $R = H^*(BT; \mathbb{Q})$. A very important special case is when $H^*_T(X; \mathbb{Q})$ is a free $R$-module; in this situation, one says that “$X$ is (cohomologically) equivariantly formal”, “$X_T$ has a cohomology extension of the fibre” or just “$X$ is CEF”. Then, by a result of Chang–Skjelbred [CS (2.3)], the sequence

\[ 0 \to H^*_T(X; \mathbb{Q}) \to H^*_T(X_T; \mathbb{Q}) \xrightarrow{\partial} H^*_{T+1}(X_1, X_T; \mathbb{Q}) \]

is exact, where $X_T \subset X$ denotes the fixed point set, $X_1$ the union of all orbits of dimension at most 1, and $\partial$ the differential of the long exact cohomology sequence for the pair $(X_1, X_T)$. In other words, $H^*_T(X; \mathbb{Q})$ coincides, as subalgebra of $H^*_T(X_T; \mathbb{Q}) = H^*(X_T; \mathbb{Q}) \otimes R$, with the image of $H^*_T(X_1; \mathbb{Q}) \to H^*_T(X_T; \mathbb{Q})$. Usually $X_T$ and $X_1$ are much simpler than $X$, so that [L] gives an easy method to compute $H^*_T(X; \mathbb{Q})$. This can be used for instance for a short proof of (a rational version of) Jurkiewicz’s description of the cohomology of smooth projective toric varieties [J] because they are known, like all Hamiltonian $T$-manifolds, to be CEF.

The purpose of this paper is twofold. First we will survey generalisations of the Chang–Skjelbred sequence for integer coefficients instead of the rationals, as well as of a more general result due to Atiyah and Bredon. This is motivated by the strong interest that several participants of the Toric Topology Conference in Osaka expressed for that topic. Secondly, we will answer the following question raised by Chris Allday during his lecture at the conference:

**Question 1.1** (Allday). Let $T = (S^1)^r$ be a torus of rank $r > 2$. Does there exist a compact $T$-manifold $X$ such that the $R$-module $H^*_T(X; \mathbb{Q})$ is torsion-free, but not free?

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For rank $r = 1$ this is clearly false, even without assuming Poincaré duality: since $H^*_T(X; \mathbb{Q}) \subset H^*_T(X^T; \mathbb{Q}) = H^*(X^T; \mathbb{Q}) \otimes R$ is torsion-free and finitely generated, it is free over the principal ideal domain $R \cong \mathbb{Q}[t_1]$. For $r > 1$ it is easy to find $T$-spaces $X$ with $H^*_T(X; \mathbb{Q})$ as required, for example, the suspension $\Sigma T$ of $T$, cf. [PP2, Ex. 5.5]. But $\Sigma T$ is not smooth for $r > 1$, and when Allday posed the above question (for rational Poincaré duality spaces), he already proved that for $r = 2$ such spaces cannot exist [Al, Prop.].

In this paper we will answer Allday’s question in the affirmative by exhibiting an example for $r = 3$. Note that any example automatically gives further ones for higher rank tori by adding circle factors which act trivially on the space because in this case

$$H^*_{T \times S^1}(X; \mathbb{Q}) = H^*_T(X; \mathbb{Q}) \otimes H^*(BS^1; \mathbb{Q}).$$

Of course, this way one gets rather special modules over polynomial rings. Since our example turns out to be part of a (small) family of spaces which seems to be of independent interest, we present the general construction.

Allday’s question can be posed for “2-tori” $(\mathbb{Z}_2)^r$ as well, and, in a spirit similar to [DJ], we will treat this case as well.

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2. Exact sequences for equivariantly formal $T$-spaces

Cohomology is taken with coefficients in $\mathbb{Z}$ unless otherwise indicated. For any coefficients ring $k$ (including $\mathbb{Q}$), we write $R = H^*(BT; k) = k[t_1, \ldots, t_{rkT}]$ with $|t_i| = 2$, and $m \subset H^*(BT; k)$ for the ideal of elements of positive degree. All $R$-modules will be $\mathbb{N}$-graded. Note that $R/m = k$ is canonically an $R$-module (concentrated in degree 0). By the rank $rkM$ of an $R$-module $M$ we mean the dimension of the localisation of $M$ over the quotient field of $R$. Tensor products without additional specification are taken over $k$.

Let $X$ be a compact differentiable $T$-manifold or, more generally, a finite $T$-CW complex, cf. [AP, Def. (1.1.1)]. Denote by $X_i$, $-1 \leq i \leq r$, the equivariant $i$-skeleton of $X$, i.e., the union of all orbits of dimension $\leq i$. In particular, $X_{-1} = \emptyset$, $X_0 = X^T$ and $X_r = X$. Each $X_i$ is closed in $X$.

The inclusion of pairs $(X_i, X_{i-1}) \hookrightarrow (X, X_{i-1})$ gives rise to a long exact sequence

$$\rightarrow H^*_T(X, X_{i-1}; k) \rightarrow H^*_T(X, X_{i-1}; k) \rightarrow H^*_T(X_i, X_{i-1}; k) \delta \rightarrow H^*_T(X, X_i; k) \rightarrow,$$

and likewise $(X_{i+1}, X_{i-1}) \hookrightarrow (X_{i+1}, X_i)$ induces a map

$$H^*_T(X_i, X_{i-1}; k) \rightarrow H^*_T(X_{i+1}, X_i; k).$$

Roughly at the same time as Chang and Skjelbred, Atiyah proved a much more general theorem in the context of equivariant $K$-theory [At, Ch. 7]. Bredon [B] then observed that it applies equally to cohomology. Bredon’s version of Atiyah’s result is the following:

**Theorem 2.1** (Atiyah–Bredon). If $H^*_T(X; \mathbb{Q})$ is free over $R$, then the sequence

$$0 \rightarrow H^*_T(X; \mathbb{Q}) \rightarrow H^*_T(X_0; \mathbb{Q}) \rightarrow H^*_{T+1}(X_1, X_0; \mathbb{Q}) \rightarrow \cdots$$

$$\cdots \rightarrow H^{*+r-1}_T(X_{r-1}, X_{r-2}; \mathbb{Q}) \rightarrow H^{*+r}_T(X_r, X_{r-1}; \mathbb{Q}) \rightarrow 0$$

is exact.
A version for toric varieties was proven by Barthel–Brasselet–Fieseler–Kaup for cohomology and intersection homology [BBFK, Thm. 4.3], this time including the (easier) converse.

In previous publications we obtained two variants of Theorem 2.1 for integer coefficients, which we are going to recall now. To state the first generalisation more succinctly, we say that a closed subgroup $K \subset T$ “has at most one cyclic factor” if the quotient $K/K^0$ by the identity component is cyclic.

**Theorem 2.2** ([FP1]). Assume that $H^*_T(X)$ is free over $R$ and that each isotropy group of $X$ has at most one cyclic factor. Then the Atiyah–Bredon sequence is exact with integer coefficients.

We also obtained version for other subrings of $Q$ and for prime fields. The proof uses essentially the same techniques as in [At]. Using the cohomological grading, which is absent in $K$-theory, we could simplify the proof and obtain a different version which weakens the assumption on $H^*_T(X)$ at the expense of assuming more about the isotropy groups. This time, we obtained an equivalence between various conditions. (In fact, conditions (i) and (ii) below are equivalent without any assumption on the isotropy groups.)

**Theorem 2.3** ([FP2]). If all isotropy group of $X$ are connected, then the following conditions are equivalent:

(i) The inclusion of the fibre $\iota: X \hookrightarrow X_T$ induces a surjection $\iota^*: H^*_T(X) \to H^*(X)$. Equivalently, the second map in the factorisation

$$H^*_T(X) \to H^*_T(X) \otimes_R \mathbb{Z} \to H^*(X)$$

is an isomorphism.

(ii) $\text{Tor}^1_R(H^*_T(X), \mathbb{Z}) = 0$.

(iii) The Atiyah–Bredon sequence is exact with integer coefficients.

In both cases, one can prove a variant of the Chang–Skjelbred Lemma as well.

**Theorem 2.4.** The Chang–Skjelbred sequence (1.1) is exact over $\mathbb{Z}$ if

(i) $H^*_T(X)$ is free over $R$ and the isotropy group of each $x \notin X_1$ is contained in a proper subtorus, or

(ii) $\text{Tor}^1_R(H^*_T(X), \mathbb{Z}) = 0$ and $T_x$ is connected for all $x \in X_1$ and contained in a subtorus of rank $r - 2$ for $x \notin X_1$.

This result is best possible as the examples in [TW2, Sec. 4] and [FP2, Sec. 5] show. In the setting of Hamiltonian group actions, versions of the Chang–Skjelbred Lemma with integer coefficients have been obtained by Tolman–Weitsman [TW1, Prop. 7.2], [TW2, Sec. 4] and Schmid [Sc] Thm. 3.2.1] for connected as well as disconnected isotropy groups. As mentioned in the introduction, the Chang–Skjelbred Lemma (also in the version of Theorem 2.4) can be used to compute the equivariant cohomology of toric varieties in cases where it is known to be free over $R$. This is explained in [BFR, Prop. 2.3], for instance.

3. **An algebraic version of Allday’s question**

If $X$ is a finite $T$-CW complex such that $H^*(X; \mathbb{Q})$ is a Poincaré duality algebra, then there is a minimal Hirsch–Brown model

$$(3.1) \quad HB(X) = (H^*(X; \mathbb{Q}) \otimes_R \delta),$$

see [AP] Rem. 1.2.10, Rem. 3.5.9, Cor. B.2.4]. The cohomology of the differential $R$-module (3.1) is the equivariant cohomology of $X$. The minimal Hirsch–Brown has the following properties:

1. The image of the differential $\delta$ lies in $m \cdot HB(X)$. 
Example 3.3. that Proposition 3.1 cannot be generalised to higher rank. The product on
\[ H^*(X; \mathbb{Q}) = HB(X) \otimes_R \mathbb{Q} \]
as \( \mathbb{Q} \)-algebras.

(3) The composition of the product \( \hat{\cup} \) in \( HB(X) \) with the \( R \)-linear extension \( \hat{\delta} \) of the orientation \( \sigma : H^*(X; \mathbb{Q}) \to \mathbb{Q} \) gives a non-degenerate \( R \)-bilinear pairing
\[ HB(X) \times HB(X) \overset{\hat{\cup}}{\longrightarrow} HB(X) \overset{\hat{\delta}}{\longrightarrow} R, \]
which is compatible with \( \delta \) and induces the Poincaré duality pairing on \( H^*(X; \mathbb{Q}) \) upon tensoring with \( \mathbb{Q} \) over \( R \).

The above properties of the minimal Hirsch–Brown model are essentially algebraic. Thus, it is natural to consider free differential \( R \)-modules satisfying (1)–(3) and to ask whether in this context torsion-free cohomology implies freeness.

Analysing Allday’s proof for the case \( r = 2 \) shows that his result is purely algebraic in nature and can essentially be stated in the following way:

Proposition 3.1 (Allday). Let \( \hat{C} \) be a free differential module over \( R = \mathbb{Q}[t_1, t_2] \) with algebraic properties corresponding to (1)–(3) above. Then \( H^*(\hat{C}) \) is a free \( R \)-module if it is torsion-free.

On the other hand, as the following example shows, any finitely generated \( R \)-module \( \hat{M} \) can be realised as a direct summand of \( H^*(\hat{C}) \) for suitable \( \hat{C} \). Of course, it is not clear at all whether this complex can be realised geometrically as the minimal Hirsch–Brown model of some \( T \)-CW complex.

Example 3.2. Let \( \hat{M} \) be a finitely generated \( R \)-module, and choose a minimal free presentation
\[ F_1 \overset{B}{\longrightarrow} F_0 \longrightarrow \hat{M}. \]
Define
\[ \hat{C} = R \oplus F_0 \oplus F_1 \oplus F'_1 \oplus F'_0 \oplus R[n], \]
where “[n]” denotes a degree shift by \( n \), which is chosen large enough such that \( F'_i \otimes_R \mathbb{Q} \) is dual to \( F_i \otimes_R \mathbb{Q} \) for \( i = 0, 1 \). Choose a homogeneous basis of \( F_i \) and adjust the degrees of the elements in the dual basis of \( F'_1 \) so that the degrees of an element and its dual add up to \( n \). The differential is given by \( B : F_1 \to F_0 \) and its (graded) transpose \( B^T : F'_0 \to F'_1 \). The resulting cohomology is
\[ H^*(\hat{C}) = R \oplus \text{coker} B \oplus \ker B \oplus \text{coker} B^T \oplus \ker B^T \oplus R[n]. \]
In particular, \( \hat{M} = \text{coker} B \) occurs as a direct summand in the cohomology of \( \hat{C} \). The product on \( \hat{C} \) is just the \( R \)-bilinear extension of the duality pairing on \( \hat{C} \otimes_R \mathbb{Q} \). The differential defined above is compatible with this product.

The above result of Allday’s implies in particular that if for \( r = 2 \) a non-free summand occurs in \( H^*(\hat{C}) \), then also torsion must occur. The next example shows that Proposition 3.1 cannot be generalised to higher rank.

Example 3.3. Let \( r = 3 \) and set
\[ \hat{C} = R \oplus (R \oplus R \oplus R)[1] \oplus (R \oplus R \oplus R)[2] \oplus R[3]. \]
The differential $\delta$ is zero except for a component $\delta: (R \oplus R \oplus R)[2] \to (R \oplus R \oplus R)[1]$ which is given by the matrix

$$
(3.8) \quad B = \begin{bmatrix}
0 & -t_3 & t_2 \\
t_3 & 0 & -t_1 \\
-t_2 & t_1 & 0
\end{bmatrix}.
$$

The complex $C = \tilde{C} \otimes_R \mathbb{Q}$ then is isomorphic to

$$
(3.9) \quad \mathbb{Q} \oplus (\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[1] \oplus (\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[2] \oplus \mathbb{Q}[3]
$$

with trivial differential. The multiplication $\mu: \tilde{C} \times \tilde{C} \to \tilde{C}$ is the $R$-linear extension of the multiplication $\mu: C \times C \to C$ given by the standard dual pairings $\mathbb{Q} \times \mathbb{Q}[3] \to \mathbb{Q}[3]$ and $(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[1] \times (\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[2] \to \mathbb{Q}[3]$.

In order to compute $H^*(\tilde{C})$, we choose the canonical $R$-bases $(x_1, x_2, x_3)$ of $(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[1]$ and $(y_1, y_2, y_3)$ of $(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[2]$. Then the kernel of $B$ is generated by $t_1 y_1 + t_2 y_2 + t_3 y_3$, which is of degree 4, and the cokernel is

$$
(3.10) \quad (\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[1] / (\langle -t_3 x_2 + t_2 x_3, t_3 x_1 - t_1 x_3, -t_2 x_1 + t_1 x_2 \rangle).
$$

The assignment $(\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})[1] \to \mathbb{Q}[-1], x_i \mapsto t_i$ induces an isomorphism of the quotient (3.10) with $m[-1]$. Hence,

$$
(3.11) \quad H^*(\tilde{C}) = R \oplus m[-1] \oplus R[3] \oplus R[4]
$$

as $R$-modules. In particular, $H^*(\tilde{C})$ is torsion-free, but not free.

The above example cannot be realised as the minimal Hirsch–Brown model of a finite $T$-CW complex $X$: If this were the case, then, being torsion-free, $H_T^*(X; \mathbb{Q})$ would inject into $H_T^*(X^T; \mathbb{Q}) = H^*(X^T; \mathbb{Q}) \otimes R$ by the Localisation Theorem, and the ranks over $R$ would be equal. More precisely, this would hold for even and odd degrees separately. But since $\dim H_T^*(X; \mathbb{Q}) = 3$ is greater than

$$
(3.12) \quad \dim H_T^*(X^T; \mathbb{Q}) \leq \rk H_T^{odd}(X^T; \mathbb{Q}) = \rk H_T^{odd}(X; \mathbb{Q}) = 2,
$$

the restriction map cannot be injective in degree 1.

**Remark 3.4.** The differential in the above example can be viewed as a part of the Koszul resolution of $\mathbb{Q}$ over $R$. (A good introduction to Koszul complexes can be found in [E] Sec. 17; see in particular Ex. 17.21 therein.) One can take a part of the Koszul resolution which is symmetric around the middle degree to obtain similar examples for $r > 3$ variables. In the next section, we will construct geometric examples for certain values of $r$, namely $r = 3, 5, 9$, which realise these complexes (over $\mathbb{Z}$ instead of $\mathbb{Q}$) up to degree shift. Example 3.3 is then realised by the minimal Hirsch–Brown model of the manifold $Z_3$ for $T = (S^1)^3$, see Section 5.

4. Mutants of compactified representations

Consider the standard action of $T = (S^1)^{r+1}$ on $\mathbb{C}^{r+1}$, which is given by

$$
(4.1) \quad (g_1, \ldots, g_{r+1}) \cdot (x_1, \ldots, x_{r+1}) = (g_1 x_1, \ldots, g_{r+1} x_{r+1})
$$

for $g_i \in S^1$ and $x_i \in \mathbb{C}$. The action extends to the one-point compactification $S^{2r+2}$ of $\mathbb{C}^{r+1}$ (which is a “torus manifold” in the sense of [HM]). It has two fixed points, the origin $0 \in \mathbb{C}^{r+1}$ and the added point $\infty$. The union $P = \mathbb{R}^{r+1}_0 \cup \{\infty\}$ of the positive orthant and the added point is topologically a cell $D^{r+1}$ of dimension $r+1$ and also a fundamental domain for the action. In other words,

$$
(4.2) \quad S^{2r+2} \cong (D^{r+1} \times T) / \sim
$$

as topological spaces, where $(x, g) \sim (x', g')$ if $x = x'$ and, in case $x \neq \infty$, $g_i = g_i'$ for each $i$ with $x_i > 0$.

Now assume that $r \in \{1, 2, 4, 8\}$, and consider the Hopf bundle

$$
(4.3) \quad S^{r-1} \hookrightarrow S^{2r-1} \xrightarrow{p} S^r.
$$
Topologically, we define $Z_r$ to be the quotient
\begin{equation}
Z_r = (D^{2r} \times T)/\sim,
\end{equation}
where again identification only takes place at the boundary $S^{2r-1} \subset D^{2r}$ and is induced by $p$:
\begin{equation}
(y, g) \sim (y', g') \iff y, y' \in S^{2r-1} \text{ and } ((p(y), g) \sim (p(y'), g')).
\end{equation}

The sphere bundle
\begin{equation}
S^{2r-1} \hookrightarrow Y_r := (\hat{D}^{2r} \times T)/\sim \twoheadrightarrow X_r := (\hat{D}^{r+1} \times T)/\sim.
\end{equation}
is induced from the Hopf bundle $[4.3]$ by a $T$-invariant map $f: X_r \to X_r/T \cong \hat{D}^{r+1} \to S^r$. In particular, it is orientable with $T$-invariant Euler class $e$. Note that the induced map of bundles is a retract, so that we can consider $[S^7] \in H_6(X_r)$ and $[S^{2r-1}] \in H_{2r-1}(Y_r)$ in a canonical way.

The map $f$ can be used to give $Z_r$ a smooth structure in the following way: The map $\mathbb{R}^{r+1} \to \mathbb{R}^{r+1} \setminus \mathbb{R}_{>0}^{r+1}$ extending $x = (x_1, \ldots, x_{r+1}) \mapsto (||x_1||^2, \ldots, ||x_{r+1}||^2)$ is the quotient by $T$ and extends to $S^{2r+2} \to P \subset S^{2r+1}$. Now choose a stereographic chart of $S^{r+1}$ containing $P$ and take the radial projection with respect to some inner point of $P$. The composition of all these maps can be used as $f$, and pulling back the smooth bundle $[4.3]$ along it gives the smooth manifold $Y_r$. Because $Z_r$ is covered by $Y_r$ and $((\hat{D}^{2r} \setminus \mathfrak{S}^{2r-1}) \times T)/\sim = (\hat{D}^{2r} \setminus S^{2r-1}) \times T$, it is smooth, too. Since $Z_r$ is in addition compact, it satisfies Poincaré duality.

The $T$-action on $Z_r$ is smooth with fixed point set $S^0 \times S^{r-1}$ and quotient $D^{2r}$. Also note that for $r \neq 8$ the action of $S^{r-1}$ on $Y_r$ can be extended to $Z_r$ by defining the complement $Z_r \setminus Y_r \cong T$ to be fixed. The quotient of $Z_r$ by this action is $S^{2r+2}$.

We now calculate the integral homology of $Z_r$. In order to simplify this computation as well as that for equivariant cohomology in Section 5 we will consider the action of $\mathfrak{A} = H_6(T)$ along the way.

Recall that $\mathfrak{A}$ is an exterior algebra with the Pontryagin product induced by the group multiplication. It is generated by the classes $x_i$ of loops around the different circle factors of $T = (S^1)^{r+1}$. Moreover, the action of $T$ on a space $X$ induces an action of $\mathfrak{A}$ on $H_*(X)$ and also on $H^*(X)$. We will also need the quotient of $\mathfrak{A}$ by its top degree, $\mathfrak{A}^\vee = \mathfrak{A}/\mathfrak{A}^{r+1} \cong \Lambda^{<r+1}$.

Since $X_r$ is obtained from $S^{2r+2}$ by removing one $T$-orbit from the dense free stratum, we have an exact sequence of $\mathfrak{A}$-modules
\begin{equation}
\cdots \to H_{r+1}(S^{2r+2}) \to \mathfrak{A}[r+1] \to H_*(X_r) \to H_*(S^{2r+2}) \to \cdots,
\end{equation}
hence an isomorphism of $\mathfrak{A}$-modules
\begin{equation}
H_*(X_r) = \mathbb{Z} \oplus \mathfrak{A}^\vee[r],
\end{equation}
where the element $1 \in \mathfrak{A}^\vee[r]$ is mapped to $[S^7]$.

Consider the Gysin homology sequence (cf. [Sp] Sec. 7)) of the sphere bundle $[4.6]$,
\begin{equation}
\cdots \to H_{r+1}(X_r) \to H_*(Y_r) \to H_*(X_r) \xrightarrow{ce} H_{*-r}(X_r) \to \cdots,
\end{equation}
where $e$ is the Euler class. We claim that the map $H_k(X_r) \to H_{k-r}(X_r)$ is an isomorphism for $k = r$ and zero otherwise. The first part follows by naturality from the corresponding map for the Hopf bundle $[4.3]$. The second part uses that capping with the $\mathfrak{A}$-invariant class $e$ is a $\mathfrak{A}$-equivariant map which sends the generator of $\mathfrak{A}^\vee[r]$ to a $\mathfrak{A}$-invariant.

Hence, we get a short exact sequence of $\mathfrak{A}$-modules
\begin{equation}
0 \to \mathfrak{A}^\vee[2r-1] \to H_*(Y_r) \to \mathbb{Z} \oplus \mathfrak{A}^\vee[r] \to 0.
\end{equation}
Here $\mathfrak{A}^\circ$ denotes the kernel of the map $\mathfrak{A}^\vee \to \mathbb{Z}$ induced by the projection $T \to 1$. ($\mathfrak{A}^\circ$ is the direct sum of $\mathfrak{A}^k$ for $0 < k < r+1$.) The element $1 \in \mathfrak{A}^\vee[2r-1]$ is
mapped to \([S^{2r-1}]\). The sequence (4.10) actually splits, but it requires some work to see this, cf. Remark 5.4 below.

Since \(Z_r\) is obtained from \(Y_r\) by gluing in an equivariant \(T\)-cell of dimension \(2r\) along \(D^{2r} \times T\), we get an exact sequence of \(\Lambda\)-modules

\[
\cdots \longrightarrow \Lambda[2r]_{i+1} \overset{\partial}{\longrightarrow} H_*(Y_r) \longrightarrow H_*(Z_r) \longrightarrow \Lambda[2r]_* \longrightarrow \cdots
\]

where the element \(1 \in \Lambda[2r]_1\) is also mapped to \([S^{2r-1}]\), which generates \(\Lambda^\psi[2r-1] \subset H_*(Y_r)\). By \(\Lambda\)-equivariance, the sequence therefore splits into the exact sequence

\[
0 \longrightarrow \Lambda^\psi[2r-1] \longrightarrow H_*(Y_r) \longrightarrow H_*(Z_r) \longrightarrow \mathbb{Z}[3r+1] \longrightarrow 0.
\]

Because the same submodule of \(H_*(Y_r)\) appears in both (4.10) and (4.12) and moreover \(\mathbb{Z}[3r+1]\) is the only contribution in degrees \(\geq 3r+1\), we finally obtain an isomorphism of \(\Lambda\)-modules

\[
(4.13) \quad H_*(Z_r) = \mathbb{Z} \oplus \Lambda^\psi[r] \oplus \mathbb{Z}[3r+1].
\]

Note that \(Z_1\) has the homology of \(S^2 \times S^2\), and for \(r \in \{2, 4, 8\}\) the homology of \(Z_r\) is (additively) that of the connected sum

\[
(4.14) \quad\left(\frac{r+1}{1}\right) \ast (S^{r+1} \times S^{2r}) \ast \cdots \ast \left(\frac{r+1}{r/2}\right) \ast (S^{3r/2} \times S^{3r/2+1}),
\]

where “\(n \ast X\)” means taking \(n\) copies of the space \(X\) in the sum. In Section 5 we will identify \(Z_1\) and \(Z_2\).

5. Computing the equivariant cohomology

By replacing each \(T\)-space \(X\) by its Borel construction \(ET \times_T X\), one could compute the equivariant cohomology of \(Z_r\) in a way analogous to the calculation of the homology in Section 4. Because this time the extension problems one is faced with are more intricate, we follow a different approach which makes use of the \(\Lambda\)-structure on \(H^*(Z_r)\).

Consider the Leray–Serre spectral sequence of principal \(T\)-bundle \(ET \times_T Z_r \to BT\). Its \(E_2\)-term is of the form

\[
E_2^{p,q} = R^p \otimes H^q(Z_r), \quad d_2(s \otimes \gamma) = \sum_{i=1}^{r+1} t_i s \otimes x_i \cdot \gamma,
\]

see for example [F, Sec. 5.1].

Given the form (4.13) of \(H^*(Z_r)\), (5.1) is essentially the Koszul resolution of \(Z\) over \(R\), with lowest and highest degree moved apart from the central piece as in Remark 5.4. We therefore obtain an isomorphism of \(R\)-modules

\[
E_3 = \begin{cases} 
R \oplus R[2] \oplus R[2] \oplus R[4] & \text{if } r = 1, \\
R \oplus m[r-1] \oplus R[2r+2] \oplus R[3r+1] & \text{if } r \in \{2, 4, 8\}. 
\end{cases}
\]

In all cases, the rank of the \(E_3\)-term over \(R\) is 4. Since \(E_3\) is torsion-free, any higher differential would lower the rank. By the Localisation Theorem, the rank of \(H^*_T(Z_r)\) is the same as that of

\[
(5.3) \quad H^*_T(Z_r^T) = H^*(S^0 \times S^{r-1}) \otimes R
\]

which is again 4. Hence, higher differentials cannot not occur.

Clearly, there is no extension problem for \(r = 1\). For \(r > 1\), there is none either because of the equality

\[
(5.4) \quad H^*_T(Z_r) = m[r-1]_k
\]
for \( k = r + 1 \) and \( r + 3 \): the generators \( m_i \) of \( m[r - 1] \) live in degree \( r + 1 \), and the relations \( t_i m_j \) for \( i \neq j \) in degree \( r + 3 \). We therefore get an isomorphism of \( R \)-modules

\[
H^*_T(Z_r) \cong \begin{cases} 
R \oplus R[2] \oplus R[3] & \text{if } r = 1, \\
R \oplus m[r - 1] \oplus R[2r + 2] \oplus R[3r + 1] & \text{if } r \in \{2, 4, 8\}.
\end{cases}
\]

In particular, \( H^*_T(Z_r) \) is free over \( R \) for \( r = 1 \) and torsion-free, but not free for larger \( r \).

**Remark 5.1.** Alternatively, one can prove that the isomorphism (4.13) is induced by a quasi-isomorphism of \( A \)-modules

\[
\mathbb{Z} \oplus A^0[r] \oplus \mathbb{Z}[3r + 1] \to C_*(Z_r)
\]

where \( C_*(\cdot) \) denotes the normalised singular chain functor. An element \( a \in A^0[r] \) is mapped to \( a \cdot c \), where \( c \in C_*(Y_r) \subset C_*(Z_r) \) is a suitable transgression chain of the fibre \( Y_r \to X_r \) and the action of \( A \) is lifted from \( H_*(Z_r) \) to \( C_*(Z_r) \). Using the “singular Cartan model” (see [F, Sec. 5.1]), this implies that the differential \( R \)-module (5.1) computes \( H^*_T(Z_r) \) without extension problem.

6. **The analogous construction for 2-tori**

By replacing \( S^1 \) by \( \mathbb{Z}_2 \) in the previous constructions, one arrives at a similar family of spaces, which we describe in more detail in this section. We assume again \( r \in \{1, 2, 4, 8\} \) and consider the canonical action of the “2-torus” \( G = (\mathbb{Z}_2)^{r+1} \) on the one-point compactification \( S^{r+1} \) of \( \mathbb{R}^{r+1} \) (which may be called a “2-torus manifold”). As before, a fundamental domain is the compactified positive orthant,

\[
S^{r+1} \cong (D^{r+1} \times G)/\sim,
\]

where identification only takes place along the boundary of \( D^{r+1} \).

Using the Hopf bundle \( p \) (1.3), we define

\[
Z_r = (D^{2r} \times G)/\sim
\]

with the identification induced by \( p \). Also define \( Y_r \) and \( X_r \) analogously to the constructions in Section 5. \( Z_r \) is smooth by a reasoning similar to the previous one, and there is again a \( G \)-invariant retract from \( Y_r \to X_r \) to the Hopf bundle. The \( G \)-action on \( Z_r \) has a fixed point set \( S^0 \times S^{r-1} \) and quotient \( D^{2r} \). For \( r \in \{1, 2, 4\} \) the quotient of \( Z_r \) by the action of \( S^{r-1} \) is \( S^{r+1} \).

To describe the homology of \( Z_r \), we consider the group algebra \( A = \mathbb{H}_*(G) = \mathbb{Z}[G] \) and its quotient \( A^0 \) by the “top element”

\[
\omega = (1 - g_1) \cdots (1 - g_{r+1}),
\]

where the \( g_i \) are the canonical generators of \( G \). (Note that the line through \( \omega \) is \( A \)-stable.) Also let \( A^0 \) be the kernel of the augmentation \( A \to 1 \), divided by \( \mathbb{Z} \omega \).

Since \( \omega \cdot [S^r] = 0 \in \mathbb{H}_r(X_r) \), we get

\[
H_*(X_r) = \mathbb{Z} \oplus A^0[r],
\]

where \( 1 \in A^0[r] \) corresponds to \([S^r]\). The Gysin homology sequence splits again into a short exact sequence,

\[
0 \to A^0[2r - 1] \to H_*(Y_r) \to \mathbb{Z} \oplus A^0[r] \to 0
\]

because the Euler class \( e \) of the bundle is killed by all elements \( 1 - g_i \).

Arguing as before, we get an exact sequence of \( A \)-modules,

\[
0 \to A^0[2r - 1] \to H_*(Y_r) \to H_*(Z_r) \to \mathbb{Z} \omega[2r] \to 0,
\]

and finally

\[
H_*(Z_r) = \mathbb{Z} \oplus A^0[r] \oplus \mathbb{Z} \omega[2r].
\]
Note that \( Z_r \) has the homology of a connected sum of \( 2^r - 1 \) copies of \( S^r \times S^r \).

The group algebra \( H_*(G; \mathbb{F}_2) \) is a strictly exterior algebra on generators \( 1 - g_i \) of degree 0, and \( R = H^*(BG; \mathbb{F}_2) \) a polynomial algebra on generators \( t_i \) of degree 1. Also observe that all \( g_i \) act trivially on \( \mathbb{F}_2 \omega \).

We claim that the minimal Hirsch-Brown model of the \( G \)-space \( Z_r \) is given by
\[
H^*(Z_r; \mathbb{F}_2) \otimes R, \quad \delta(\gamma \otimes s) = \sum_{i=1}^{r+1} (1 - g_i) \cdot \gamma \otimes t_i s.
\]

Since \( Z^G \) is not empty, the differential \( \delta \) does not hit \( H^0(Z_r; \mathbb{F}_2) \otimes R \), and \( \delta \) vanishes on \( H^{2r}(Z_r; \mathbb{F}_2) \otimes R \) because of Poincaré duality, see [AP, Cor. (5.3.4)]. So, for degree reasons, \( \delta \) only contains terms linear in the \( t_i \)'s, and these are given by the induced action in cohomology, cf. [AP] p. 453. This proves the claim.

We therefore get
\[
H^*_G(Z_r; \mathbb{F}_2) \cong \begin{cases} R \oplus R[1] \oplus R[1] \oplus R[2] & \text{if } r = 1, \\
R \oplus m[r - 1] \oplus R[r + 1] \oplus R[2r] & \text{if } r \in \{2, 4, 8\}.
\end{cases}
\]

Again, \( H^*_G(Z_r; \mathbb{F}_2) \) is free over \( R \) for \( r = 1 \) and torsion-free, but not free for larger \( r \).

7. Identifying Some Mutants

Following a suggestion of M. Kreck, we use classification results for highly connected manifolds to identify some of the manifolds from Sections 4 and 5 up to homeomorphism. In all cases we have already computed the integral homology. To apply the classification results, we need to know the cup product structure in cohomology, as well as certain characteristic classes.

7.1. The torus case. The spaces \( X_r, Y_r \) and \( Z_r \) are \( r \)-connected and – except for \( r = 1 \) – it follows already from Poincaré duality and degree reasons that \( H^*(Z_r) \) is isomorphic as graded ring to the cohomology of a connected sum of products of spheres.

Case \( r = 1 \). According to the construction given in Section 4, \( Z_1 \) is the quotient of \( D^2 \times T \) by the following equivalence relation: For points \((x, g)\) with \( x \) contained in the interior of the disk no identification takes place. The boundary \( S^1 \subset D^2 \) is divided into 4 segments, and for \( x \) in two opposite segments one identifies one coordinate circle \( S^1 \subset T = S^1 \times S^1 \) and for the other two segments the other coordinate circle. For the 4 points \( x \) separating the segments one identifies all of \( T \).

Replacing the disk by a square, one arrives at the usual topological construction of the toric manifold \( S^2 \times S^2 \) where one looks at a 2-sphere as a quotient of \([0, 1] \times S^1 \).

Hence,
\[
Z_1 \cong S^2 \times S^2.
\]

Since we know \( H^*(Z_1) \) to be free over \( R \), we could alternatively use Theorem 2.4 to compute the cup product in \( H^*_G(Z_1) \) and \( H^*(Z_1) = H^*_G(Z_1) \otimes R \mathbb{Z} \) and then apply classification results for 4-manifolds (see [F2]) to conclude (7.1).

Case \( r = 2 \). We claim that the first Pontryagin class \( p_1(Z_2) \) vanishes. Since the map \( H^4(Z_2) \to H^4(Y_2) \) is injective, it suffices to show that \( p_1(Y_2) = 0 \). It follows from the fibration \( S^1 \to Y_2 \to X_2 \) that the tangent bundle of \( Y_2 \) is the Whitney sum of the pull-back of the tangent bundle of \( X_2 \) and the tangent bundle along the fibres, and the latter plus a one-dimensional trivial bundle is the pull-back of the vector bundle associated to the above fibration (cf. [Sz], for example). The first Pontryagin class of this vector bundle vanishes (since it is induced from the Hopf bundle over \( S^2 \)) and so does \( p_1(X_2) = 0 \) (since \( X_2 \) is an open subset of \( S^6 \)). So by
the product formula for Pontryagin classes (see [MS]) and the fact that \( H^*(Y_2) \) has no 2-torsion, one has \( p_1(Y_2) = 0 \). From [Wi] one gets
\[
Z_2 \cong 3 \ast (S^3 \times S^4).
\]

**Cases \( r = 4 \) and \( r = 8 \).** Here the cup product structure is clear as well (see above) and one could prove the vanishing of certain characteristic classes along the same line of arguments, but we do not know of a classification result which allows to identify \( Z_r \) in these cases up to homeomorphism.

### 7.2. The finite case

The spaces \( X_r, Y_r \) and \( Z_r \) are clearly \((r - 1)\)-connected in this case.

**Case \( r = 1 \).** It is elementary to see that
\[
Z_1 \cong S^1 \times S^1.
\]

For \( r \in \{2, 4\} \) we consider the restriction of the \( S^{r-1} \)-action on \( Z_r \) to \( K = S^1 \subset S^{r-1} \). This action has \( l = 2^{r+1} \) fixed points. Since \( H^*(Z_r) \) and \( H^*(Z^K) \) are free \( \mathbb{Z} \)-modules of the same rank, \( H^*_K(Z_r) \) is free over \( H^*(BK) \). As a consequence we get an injection
\[
H^*_K(Z_r) \to H^*_K(Z^K) \cong \mathbb{Z}[t]^l
\]
(which can be seen as a very special case of Theorem [ZA]).

We write \( b^{(a)} \) as a shorthand for \( (b, \ldots, b) \in \mathbb{Z}^a \). As shown in [P, Sec. 2], the cohomology \( H^*(Z_r) \) can be described as the graded algebra associated to the following filtration of \( \mathbb{Z}^l \): \( F_0 = \cdots = F_{r-1} = 0 \) is the \( \mathbb{Z} \)-module generated by \( 1^{(i)} \), \( F_r = \cdots = F_{2r-1} = \ker \sigma \) where \( \sigma : \mathbb{Z}^l \to \mathbb{Z} \) is given by \( \sigma(e_i) = (-1)^i \) for \( i = 1, \ldots, l \), and \( F_{2r} = \mathbb{Z}^l \).

The intersection form on \( H^r(Z_r) \cong F_r/F_{r-1} \) with respect to the basis represented by
\[
v_i = \begin{cases} (0^{(i)}, 1, 1, 0^{(l-i-2)}) & \text{for odd } i, \\ (1^{(i)}, 0^{(l-i)}) & \text{for even } i, \end{cases} \quad i = 1, \ldots, l - 2,
\]
is given as a direct sum of \((l-2)/2\) copies of the form \( \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \). Hence also for \( r \in \{2, 4\} \) we have an isomorphism of graded rings between the cohomology of \( Z_r \) and that of a connected sum of \( 2^r - 1 \) copies of \( S^r \times S^r \).

**Case \( r = 2 \).** It now follows from [F] that
\[
Z_2 \cong 3 \ast (S^2 \times S^2).
\]

**Case \( r = 4 \).** As before, one can show that \( p_1(Z_4) = 0 \) by using the fibration \( S^3 \to Y_4 \to X_4 \) and the facts that the pull-back of the first Pontryagin class to the total space of the Hopf bundle over \( S^4 \) vanishes (which can be seen e.g. from the Gysin cohomology sequence of the sphere bundle) and that \( p_1(X_4) = 0 \). By [Wa] and [St] this together with the cup product structure implies
\[
Z_4 \cong 15 \ast (S^4 \times S^4).
\]

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