Abstract A new form of the light front Feynman propagators is proposed. It contains no energy denominators. Instead the dependence on the longitudinal subinterval $x^L = x^+_L x^-_L$ is explicit and a new formalism for doing the perturbative calculations is invented. These novel propagators are implemented for the one-loop effective potential and various 1-loop 2-point functions for a massive scalar field. The consistency with results for the standard covariant Feynman diagrams is obtained and no spurious singularities are encountered at all. Some remarks on the calculations with fermion and gauge fields in QED and QCD are added.

1 Introduction

Wightman function for a free massive scalar field $\langle 0| \phi(x)\phi(0)|0 \rangle = W_2(x)$ has its LF momentum representation

$$W_2(x^+, x^-, x_\perp) = \int_{\mathbb{R}^2} \frac{d^2k_\perp}{(2\pi)^2} e^{-i k_\perp \cdot x_\perp} \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-ik^+ x^-} e^{-\frac{m^2 + k^2}{2x^+}}.$$  \hspace{1cm} (1)

The LF propagator $\Delta_{LF}(x)$ is defined by the chronological (in $x^+$) ordering

$$\Delta_{LF}(x) = \langle 0| T^+ \phi(x)\phi(0)|0 \rangle := \Theta(x^+)|0| \phi(x)\phi(0)|0 + \Theta(-x^+)|0| \phi(0)\phi(x)|0 \rangle = \Theta(x^+) W_2(x^+, x^-, x_\perp) + \Theta(-x^+) W_2(-x^+, -x^-, -x_\perp).$$  \hspace{1cm} (2)

Within the standard LF approach [1] (for review see [2]) one introduces the Fourier representation for the Heaviside step function

$$\Theta(x^+) = \int_{\mathbb{R}} \frac{dk^-}{2\pi} e^{-ik^- x^+} \frac{i}{k^- + i0},$$  \hspace{1cm} (3)
then changes the order of integrations and finally one shifts the integration variable which gives
\[
\int_{\mathbb{R}} \frac{dk^+}{2\pi} e^{-ik^+x^+} e^{-i\pi^+(m^2+k^2)}(2k^+) = \int_{\mathbb{R}} \frac{dk^+}{2\pi} e^{-ik^+x^+} \frac{i}{k^-} \frac{e^{-ik^+x^-} + i0}{k^- - \frac{m^2+k^2}{2k^+} + i0}.
\]

For \( k^+ \to 0 \), the pole in \( k^- \) moves to infinity, so the naive implementation of the residua theorem can lead to false results. This problem overlaps with the usual LF singularity due to \( 1/k^+ \) pole. The very trick, presented in (4), is analogous to the equal-time propagators, where one makes the ordering in \( x^0 \) temporal variable, so one introduces
\[
\Theta(x^0) = \int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{-ik_0x^0} \frac{i}{k_0 + i0},
\]
and then one proceeds as follows
\[
\int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{-ik_0x^0} \frac{i}{k_0 + i0} \frac{1}{2\omega_k} e^{-i\lambda\omega_k} = \int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{-ik_0x^0} \frac{i}{2\omega_k} \frac{1}{k_0 - \omega_k + i0}.
\]

In this calculation the shifting of \( k_0 \) is reliable since the limit \( \omega_k \to \infty \) is suppressed by the inverse power of \( \omega_k \) in the invariant measure factor, on contrary in the Eq. (4). Therefore it is not strange that the LF propagator with the pole structure as in (4) may lead to various artificial singularities of Feynman diagrams, which are absent in the analogous equal-time calculation.

### 2 Novel LF Representation of Propagator and Convolutions of Propagators

We observe that we may make the following changes of integration variables: for \( x^+ > 0 \) we take \( k^+ = 2\lambda x^+ \)
\[
W_2(x) \simeq \int_{\mathbb{R}} \frac{dk^+}{4\pi k^+} e^{-ik^+x^-} e^{-i\lambda x^+} = \int_{\mathbb{R}} \frac{d\lambda}{4\pi\lambda} e^{-i\lambda x^+} e^{-i(\lambda^2+k^2)/(\lambda^2)},
\]
while for \( x^+ < 0 \) we take \( k^+ = -2\lambda x^+ \)
\[
W_2(-x) \simeq \int_{\mathbb{R}} \frac{dk^+}{4\pi k^+} e^{ik^+x^-} e^{i\lambda x^+} = \int_{\mathbb{R}} \frac{d\lambda}{4\pi\lambda} e^{i\lambda x^+} e^{i(\lambda^2+k^2)/(\lambda^2)}.
\]

This leads to the LF propagator in the form, which we call \( \lambda \)-representation,
\[
\langle 0|T^+\phi(x)\phi(0)|0 \rangle = \int_{\mathbb{R}^2} \frac{d^2k_L}{(2\pi)^2} e^{-ik_L} x_L e^{-iM^2/(4\lambda)} = \Delta_{LF}(x),
\]
where \( x_L^2 = 2x^+x^- \) and \( M^2 = m^2 + k^2 \). If one wishes to compare this new representation with the covariant formula in the 4-momentum space, then one takes the Fourier transform in the 2-dimensional longitudinal subspace (31) and then integrates over \( \lambda \), as defined in the sense of distributions in (30). This gives the equivalence between \( \lambda \)-representation and covariant Feynman propagators
\[
\Delta_{LF}(x) = \int_{\mathbb{R}^2} \frac{d^2k_L}{(2\pi)^2} e^{-ik_L} x_L e^{-iM^2/(4\lambda)} = \Delta_F(x).
\]

Evidently \( \lambda \)-representation of \( \Delta_{LF}(x) \) is singular at \( x^+ = 0 \), but one may check its consistency with the help of its Volterra equation, which is quite similar to the Volterra equation for the Wightman function of a free massive
scalar field \([3]\). Next we consider the convolutions of LF propagators and we begin with the convolution of two LF propagators defined as follows

\[
[\Delta_{LF} \ast \Delta_{LF}](x - z) = \Delta_{LF}^2(x - z) := \int d^4x \Delta_{LF}(x - y) \Delta_{LF}(y - z).
\]  

(11)

By inserting \(\lambda\)-representation, with \((\lambda_1, p_{1\perp})\) and \((\lambda_2, p_{2\perp})\) for respective propagators, one may directly integrate over the transverse and longitudinal coordinates using \((32a)\) and \((32c)\)

\[
\Delta_{LF}^2(x - z) = \frac{1}{4} \int \frac{d^2p_{1\perp}}{(2\pi)^2} e^{-i\lambda_1(x_{\perp} - z_{\perp})} \int_0^\infty \frac{d\lambda_1}{4\pi \lambda_1} e^{-i\lambda_1(x_{\perp} - z_{\perp})^2} e^{-i\frac{m^2 + p_{1\perp}^2}{4\lambda_1}}.
\]  

(12)

where \(M_1^2 = m^2 + p_{1\perp}^2\) and \(M_2^2 = m^2 + p_{2\perp}^2\). While the integration over \(p_{2\perp}\) is immediate, then for evaluating the integrals over \(\lambda_1, \lambda_2\) one needs to parameterize them as \(\lambda_1 = \lambda/\xi, \lambda_2 = \lambda/(1 - \xi)\), with new parameters \(\lambda \in (0, \infty)\) and \(\xi \in (0, 1)\). This allows to evaluate the integration over \(\xi\) explicitly, and finally we obtain \(\lambda\)-representation for the convolution of two LF propagators, (where we put \(p_{1\perp} = p_{\perp}\))

\[
\Delta_{LF}^2(x - z) = \frac{1}{4^n} \int \frac{d^2p_{1\perp}}{(2\pi)^2} e^{-i\lambda_1(x_{\perp} - z_{\perp})} \int_0^\infty \frac{d\lambda_n}{4\pi \lambda_n} e^{-i\lambda_n(x_{\perp} - z_{\perp})^2} e^{-i\frac{m^2 + p_{1\perp}^2}{4\lambda_n}}.
\]  

(13)

By induction one finds the convolution of \(n\) propagators

\[
\Delta_{LF}^n(x - z) = \frac{1}{4^n - 1} \left(\frac{-i}{n}\right)^n \sum_{n=1}^{\infty} \frac{-i}{n} \left(\frac{\lambda_1}{2\lambda_1}\right)^n \Delta_{LF}^n(0).
\]  

(15)

The 1-loop effective potential (for the \(g/(4!)\phi^4\) theory) is given by \([4]\)

\[
V_{eff}^{(1)}[\phi_c] = \frac{i}{2} \sum_{n=1}^{\infty} \frac{-i}{n} \left(\frac{g}{2\phi_c}\right)^n \Delta_{LF}^n(0),
\]  

(16)

\[
\Delta_{LF}^n(0) = \frac{1}{4^n - 1} \left(\frac{-i}{n}\right)^n \sum_{n=1}^{\infty} \frac{-i}{n} \left(\frac{\lambda_1}{2\lambda_1}\right)^n \Delta_{LF}^n(0)\]  

(17)

This expression gives nontrivial contribution to the effective potential, which can be compared with the result obtained within the standard LF formulation \([4]\)

\[
\Delta_{LF}^n(0) = \int \frac{d^2p_{1\perp}}{(2\pi)^2} \int \frac{dp^+}{4\pi(p^+)^n} \int \frac{dp^-}{4\pi(p^-)^n} \left[\frac{1}{p^- - m^2 + p_{1\perp}^2 + i\operatorname{sgn}(p^+)0}\right].
\]  

(18)

which for \(n > 1\), vanishes by residua, unless the contribution from the arc is properly taken into account.
3 One Loop 2-Point Diagrams

Now we will consider one loop 2-point diagrams with a flow of non-zero external 4-momentum $q^\mu$, starting with the simplest scalar self-energy diagram

$$\Sigma(q) = \int_{\mathbb{R}^4} d^4x e^{i q \cdot x} \Delta_F(x) \Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} \frac{i}{(k + q)^2 - m^2 + i0}, \quad (18)$$

where the standard covariant Feynman propagators are inserted and further after the Wick rotation one may evaluate the Euclidean momentum integrals. Our aim is to calculate $\Sigma(q)$ with $\lambda$-representation for $\Delta_{LF}(x)$

$$\Sigma(q) = \int_{\mathbb{R}^4} d^4x e^{i q \cdot x} \Delta_{LF}(x) \Delta_{LF}(x). \quad (19)$$

We denote $(\lambda_1, p_{1\perp})$ and $(\lambda_2, p_{2\perp})$ for $\Delta_{LF}(x)$ respectively and start with the integration over the space-time coordinates, using $(32b)$,

$$\Sigma(q) = \int_{\mathbb{R}^2} \frac{d^2p_{1\perp}}{(2\pi)^2} \int_{\mathbb{R}^2} d^2k_{\perp} \delta^2(p_{1\perp} + p_{2\perp} - q_{\perp}) \int_0^\infty \frac{d\lambda_1}{4\pi \lambda_1} \int_0^\infty \frac{d\lambda_2}{4\pi \lambda_2} e^{-i(\lambda_1 + \lambda_2)(x_{\perp})^2/2\lambda_1} e^{-iM_1^2/(4\lambda_1)} e^{i\frac{q_{\perp}^2}{4\lambda_1}} e^{-iM_2^2/(4\lambda_2)} e^{-iM_2^2/(4(1-\xi)\lambda_2)}. \quad (20)$$

The integration over $p_{\perp}$ is simple and due to the property $(33)$, one finds

$$\Sigma(q) = \int_{\mathbb{R}^2} \frac{d^2k_{\perp}}{(2\pi)^2} \int_0^\infty \frac{d\lambda_1}{16\pi \lambda_1^2} \int_0^1 \frac{d\xi}{\xi(1-\xi)} e^{i(q_{\perp}^2 - q_{\perp}^2)/(4\lambda_1)} e^{-i(m^2 + k_{\perp}^2)/(4\lambda_1 (1-\xi)\lambda_1)}. \quad (21)$$

The integral over $\lambda$ can be performed explicitly according to $(30)$ for $n = 1$, so

$$\Sigma(q) = \frac{i}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2k_{\perp}}{(2\pi)^2} \frac{1}{q_{\perp}^2 \xi(1-\xi) - m^2 - k_{\perp}^2 + i0}, \quad (23)$$

which coincides with the result of calculation with the covariant Feynman propagators. Then we wish to consider another 2-point function

$$\Sigma_\mu(q) = \int_{\mathbb{R}^4} d^4x e^{i q \cdot x} \partial_\mu \Delta_F(x) \Delta_F(x) = \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{k^2 - m^2 + i0} \frac{i^2}{(k + q)^2 - m^2 + i0} \quad (24)$$

We now insert the propagators in $\lambda$-representation and take steps analogous to those in the calculation of $\Sigma(q)$. There is a slight difference, because $i \partial_\mu \Delta_{LF}(x)$ appears instead of $\Delta_{LF}(x)$. First, for the longitudinal partial derivatives $(32d)$ one obtains the extra factor $\lambda_1 q_{\perp}/(\lambda_1 + \lambda_2)$, which after the re-parametrization of $\lambda_{1,2}$ boils down to the extra factor $q_{\mu} \xi$. Second, for the transverse partial derivatives one obtains another extra factor
which is local in time term 1/\((2\pi)^2 q^2\xi(1 - \xi) - m^2 - k^2_\perp + i0)\). Finally we may consider the 2-point function, which has been used by Melikhov and Simula in [5] for their discussion of spurious end-point singularities

\[
\Sigma_{\mu}(q) = i \frac{q_\mu}{4\pi} \int_0^1 d\xi \int_{\mathbb{R}^2} \frac{d^2 k_\perp}{(2\pi)^2} \frac{\xi}{q^2\xi(1 - \xi) - m^2 - k^2_\perp + i0}.
\]  

For the fermion field propagator, an analogous situation occurs for the fermion field propagator, where one obtains an additional local term for the “bad components” \(\psi_+, \psi_\perp\) of the fermion field. However in the LF Hamiltonian there are local interaction terms which exactly cancel the contribution from local terms in the LF propagators, thus effectively one obtains Feynman diagrams for the perturbative QED and QCD without local term contributions.

We hope that our novel representation for propagators may shed a new light on the equivalence problem between the LF and equal-time perturbative calculations [8]. At last, it will be quite interesting to apply \(\lambda\)-regularization for the LF Bethe-Salpeter equations as in [9].
Appendix: Definitions and Useful Formulas

The LF longitudinal coordinates are defined as $x^± = (x^0 ± x^3)/\sqrt{2}$ and the partial derivatives are denoted as $∂_± = ∂/∂x^±$. The Minkowski space-time metric tensor has non-vanishing components $g_+− = g_−+ = 1$, $g_12 = −δ_12$. We have the formula for $n ∈ \mathbb{N} − \{1\}$

$$\int_0^{∞} \frac{dλ}{λ^{n+1}} e^{iA/λ} = i^n(n−1)! (A+i0)^n. \quad (30)$$

The Fourier transform in the longitudinal coordinates

$$e^{-iλx^2_L} = e^{-i2x^±x^−λ} = \int_\mathbb{R}^2 \frac{dk^+dk^−}{4πλ} e^{−i(k^+x^−k^−x^−)} e^{iλk^−/2λk^+} = \int_\mathbb{R}^2 \frac{d^2k_L}{4πλ} e^{−iλk_L·x_L} e^{iλk_L^2/(4λ)}. \quad (31)$$

The integrations over the transverse and longitudinal coordinates give respectively

$$\int d^2y_L e^{−iλ_1(y_L−x_L)} = (2π)^2λ^2 (p_{1⊥}−p_{2⊥}) e^{−iλ_1y_L·p_{1⊥}+p_{2⊥}·z_L}, \quad (32a)$$

$$\int d^2x_L e^{−iλ_1(p_{1⊥}−x_L)} = (2π)^2λ^2 (p_{1⊥}+p_{2⊥}−q_{⊥}), \quad (32b)$$

$$\int d^2y_L e^{−iλ_2(y_L−x_L)} = \frac{π}{λ_1+λ_2} \exp{iε} \left( \frac{λ_1λ_2}{λ_1+λ_2} (x_L−z_L)^2 \right). \quad (32c)$$

$$\int d^2x_L \left[ e^{−i\lambda_1λ_2}, (i∂_±e^{−i\lambda_1λ_2}) \right] e^{−iλ_2λ_2q_L·x_L} = \frac{π}{λ_1+λ_2} \left[ 1, \frac{λ_1q_{±}}{λ_1+λ_2} \right] \exp{iq_L^2 / 4(λ_1+λ_2)}. \quad (32d)$$

The parameterization of transverse momenta leads to

$$\frac{M^2}{ξ} + \frac{M^2}{1−ξ} = \frac{m^2+(p_{⊥}ξ+k_{⊥})^2}{ξ} + \frac{m^2+(p_{⊥}(1−ξ)−k_{⊥})^2}{1−ξ} = \frac{m^2+k^2}{ξ(1−ξ)} + p_{⊥}^2. \quad (33)$$

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References

1. Chang, S.-J., Yan, T.-M.: Quantum field theories in the infinite-momentum frame. II. Scattering matrices of scalar and Dirac fields. Phys. Rev. D 7, 1147–1161 (1973)
2. Brodsky, S.J., Pauli, H.-C., Pinzsky, S.: Quantum chromodynamics and other field theories on the light cone. Phys. Lett. C (Phys. Rep.) 301, 299–486 (1998)
3. Przeszowski, J.A.: Lorentz symmetry for the light-front Wightman functions. Acta Phys. Pol. Proc. Suppl. B 6, 327–333 (2013)
4. Convery, M.E., Taylor, C.C., Jun, J.W.: Vacuum structure, zero modes, and the effective potential in light-cone quantization. Phys. Rev. D 51, 4445–4450 (1995)
5. Melikhov, D., Simula, S.: End-point singularities of Feynman graphs on the light cone. Phys. Lett. B 556, 135–141 (2003)
6. Heinzl, T.: Alternative approach to light-front perturbation theory. Phys. Rev. D 75, 025013 (2007)
7. Bassetto, A.: Free vector propagator in the light-cone gauge and the Mandelstam-Leibbrandt prescription. Phys. Rev. D 46, 3676–3677 (1992)
8. Bakker, B.L.G., DeWitt, M.A., Ji, C.-R., Mischenko, Y.: Restoring the equivalence between the light-front and manifestly covariant formalisms. Phys. Rev. D 72, 076005 (2005)
9. Sales, J.H.O., Frederico, T., Carlson, B.V., Sauer, P.U.: Light-front Bethe-Salpeter equation. Phys. Rev. C 61, 044003 (2000)