Chapter 1

The Measurement of Polarization in Radio Astronomy

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Modern dual-polarization receivers allow a radio telescope to characterize the full polarization state of incoming interstellar radio waves. Many astronomers incorrectly consider a polarimeter to be the “backend” of the telescope. We go to lengths to dissuade the reader of this concept: the backend is the least complicated component of the radio telescope when it comes to measuring polarization. The feed, telescope structure, dish surface, coaxial cables, optical fibers, and electronics can each alter the polarization state of the received astronomical signal. We begin with an overview of polarized radiation, introducing Jones and Stokes vectors, and then discuss construction of digitized pseudo-Stokes vectors from the outputs of modern correlators. We describe the measurement and calibration process for polarization observations and illustrate how instrumental polarization can affect a measurement. Finally, we draw attention to the confusion generated by various polarization conventions and highlight the need for observers to state all adopted conventions when reporting polarization results.

1. Introduction

Astronomy involves the reception of light from objects beyond the Earth. Light from these distant objects can arrive at a telescope with its electric field having some preferred orientation or rotation. This tendency is known as polarization. Most astronomers are happy to just measure the intensity of light from distant sources, but radio astronomers can easily measure the full polarization state of the radio waves they collect. Sadly, many astronomers consider polarimetry an esoteric specialty that’s not worth their effort. The aim of this review is to offer a clear description of the fundamentals of measuring polarization in radio astronomy.
At radio wavelengths, we find a number of processes that can produce polarized radiation:

1. Linearly polarized blackbody emission from the solid surfaces of planets and moons;
2. Linearly polarized thermal emission from dust grains aligned with a magnetic field; synchrotron/cyclotron radiation emitted (or absorbed) by relativistic/non-relativistic electrons gyrating around magnetic field lines and producing linearly polarized light; Zeeman splitting of spectral lines emitted or absorbed in a region threaded by a magnetic field, producing elliptically polarized light; the Goldreich-Kylafis effect, producing linear polarization via scattering of anisotropic spectral-line radiation by atoms or molecules in a magnetic field; Thomson scattering and gravitational waves producing linear polarization in the cosmic microwave background. Radio sources that show some signs of polarization include our Sun, planets and moons in the solar system, pulsars, gas clouds in the interstellar medium, circumstellar disks, masers, synchrotron emission from galaxies, quasars, jets, and the cosmic microwave background. Most of these sources have low fractional polarization (pulsars, solid surfaces, cyclotron/synchrotron emission, and masers being notable exceptions, with fractional polarizations up to 100%).

The polarization of a radio wave can be affected as it travels through interstellar space. Faraday rotation causes the polarization angle of a linearly polarized wave to rotate (by an amount $\propto \lambda^2$) when the wave traverses an ionized medium threaded by a magnetic field having a component aligned with the direction of propagation. The Earth’s ionosphere produces Faraday rotation that must be corrected for; this is a complicated task for interferometers with intercontinental baselines.

Radio waves then interact with the antenna—typically a dish of some sort—where they are reflected and brought to a focus. At the focus the radio waves in free space are coupled to an antenna, known as the feed. The feed probes the electric field in an orthogonal basis, typically orthogonal linear polarizations (which we call X and Y) or left-hand and right-hand circular polarizations (LCP and RCP). In this paper, we will always use the IEEE definition of RCP and LCP (more of this in Sec. 6.2), for which a receiver would see the electric vector of incoming radiation rotate counterclockwise and clockwise, respectively, with time.

From this point forward, the signals are amplified and encounter a large number of electronic components that change the voltage gain (a complex number; Sec. 4.3). In addition, differences in cable length (e.g., from the telescope to the backend system) produce a differential phase change that is proportional to frequency (Sec. 4.6.1), and bandpass filters incur phase delays (Sec. 4.6.2). Finally, the voltages are sampled, digitized, correlated, Fourier transformed, and stored (Sec. 3).

In this chapter we discuss how various components of a single-dish radio telescope system create instrumental polarization and how one corrects or copes with this.

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\[a\] We highly recommend Ref. 1 for a gentle and clear introduction to the general characteristics of polarized light and the physical processes that produce polarized astronomical radiation.

\[b\] If one is interested in the details of polarization in interferometers, we refer you to Refs. 3 & 4.
There are some very comprehensive reviews of radioastronomical polarimetry in the literature, many of them are highly mathematical, employing elegant representations of polarization and invoking such tricks as Lorentz boosts. The aficionado should take the time to understand these papers, and those with a theoretical bent will really appreciate them, but the polarization newcomer is likely to be scared away. It’s our opinion that spectropolarimetrists should be doing more to convince observers to use this tool rather than obfuscating the methods with complex mathematical representations.

We begin in Sec. 2 by discussing the basic mathematical framework of polarization and how polarization is described by electric fields and, alternatively, by Stokes parameters. In Sec. 3 we discuss how we digitally create the self- and cross-products that are necessary for polarization measurement. In Sec. 4 we discuss how to create calibrated Stokes parameters from the digitally created products, including a thorough accounting of all the processes and components that change the polarization state of an incoming astronomical radio wave between the feed and the backend. The off-axis polarization response of a telescope is then considered in Sec. 5. Finally, in Sec. 6 we emphasize the important and necessary role played by polarization conventions—and the unfortunate tendency of astronomers to ignore those conventions.

2. Polarization: The Basics

2.1. The Description of Polarization by Electric Fields

The polarization of a radio wave is defined by the motion of its electric field vector as a function of time within a plane perpendicular to the direction of propagation. That plane is known as the plane of polarization and the general shape that the electric field traces with time is an ellipse. We can quantify this polarization ellipse in terms of any orthonormal basis in the plane of polarization; in radio astronomy, we encounter two—the standard Cartesian linear basis and a basis of circularly rotating unit vectors of opposite handedness.

The electric field vector of a monochromatic light wave travelling along the $+\hat{z}$ direction can be written in terms of both a linear and circular set of orthonormal bases:

$$E(z,t) = E_0 e^{i(2\pi\nu t - kz)} = (\mathcal{E}_x\hat{x} + \mathcal{E}_y\hat{y}) e^{i(2\pi\nu t - kz)}$$

$$= (\mathcal{E}_R\hat{R} + \mathcal{E}_L\hat{L}) e^{i(2\pi\nu t - kz)},$$

where $\hat{R} = (\hat{x} - i\hat{y})/\sqrt{2}$ and $\hat{L} = (\hat{x} + i\hat{y})/\sqrt{2}$ are the unit vectors of IEEE RCP and LCP. As seen from an observer somewhere at $z > 0$ and looking back towards the origin, IEEE RCP is seen to rotate counterclockwise with time and IEEE LCP clockwise.

We can write $E_0$ as a Jones vector in either of the bases:

$$E_0 = \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{bmatrix} = \begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{bmatrix} \quad \text{or} \quad E_0 = \begin{bmatrix} \mathcal{E}_R \\ \mathcal{E}_L \end{bmatrix} = \begin{bmatrix} E_{0R}e^{i\phi_R} \\ E_{0L}e^{i\phi_L} \end{bmatrix}. \quad (2)$$
At a given position $z$ along the direction of propagation (let’s take $z = 0$ for simplicity), the tip of the electric field vector $\mathbf{E}$ will trace out an ellipse in time with orthogonal components given in the linear basis by:

$$
E_x(t) = E_{0x}e^{i(2\pi \nu t + \phi_x)}, \quad E_y(t) = E_{0y}e^{i(2\pi \nu t + \phi_y)},
$$

or in the circular basis by:

$$
E_R(t) = E_{0R}e^{i(2\pi \nu t + \phi_R)}, \quad E_L(t) = E_{0L}e^{i(2\pi \nu t + \phi_L)}.
$$

These components define the previously mentioned polarization ellipse. Many treatments of polarization ignore the absolute phase (which must not be ignored when using an interferometer!) and define the relative phase as $\Delta \phi \equiv \phi_y - \phi_x$.

The major axis of the polarization ellipse will be oriented at an angle $\chi$ with respect to the $x$ axis (see Fig. 1) where

$$
\tan 2\chi = \frac{2E_{0x}E_{0y} \cos (\phi_y - \phi_x)}{E_{0x}^2 - E_{0y}^2} = \tan (\phi_R - \phi_L); \quad 0\degree \leq \chi \leq 180\degree.
$$

### 2.2. The Description of Polarization by Stokes Parameters

Astronomical radio signals are, in general, partially polarized. The polarization ellipse and Jones matrices cannot help us quantify partially polarized radiation. For this, we use the Stokes parameters. The Stokes parameters are most often denoted as $I$, $Q$, $U$, and $V$ in astronomical measurements and, because they are conveniently manipulated by matrix algebra, are often written as the Stokes vector:

$$
S = \begin{bmatrix}
S_0 \\
S_1 \\
S_2 \\
S_3
\end{bmatrix} = \begin{bmatrix}
I \\
Q \\
U \\
V
\end{bmatrix},
$$

where the Stokes parameters are defined in terms of the intensities of orthogonal polarization forms ($I_{0\degree}$, $I_{90\degree}$), ($I_{45\degree}$, $I_{-45\degree}$), and ($I_{RCP}$, $I_{LCP}$):

1. Stokes $I$ is the total intensity. It is the sum of the intensities of any two orthogonal polarization components and does not store any polarization information.

$$
I \equiv I_{tot} \equiv I_{0\degree} + I_{90\degree} \equiv I_{45\degree} + I_{-45\degree} \equiv I_{RCP} + I_{LCP}.
$$

While matrices are often represented by a bold font, here we have introduced the notation $\mathbf{A}$ to represent a $1 \times 4$ column matrix—known as a vector in the parlance of linear algebra—to differentiate from a physical vector $\mathbf{A}$, e.g., the electric field. (The Stokes vector comprises the Stokes parameters, which do not represent an orthonormal basis: Stokes $I$ can be a linear combination of Stokes $Q$, $U$, and $V$.) We later use the notation $\mathbf{A}$ to represent a square $4 \times 4$ matrix.

Here we follow Ref. 10 in using each subscripted $I$ to represent intensities of a given polarization form. It might appear recursive to then also define the first Stokes parameter as $I$, but this is just a notational convention and the reader might wish to think of Stokes $I$ as always having an implicit “tot” subscript to clarify that it represents the total of intensities in any one pair of orthogonal polarization states.
(2) Stokes $Q$ is the difference in intensities between horizontal and vertical linearly polarized components and is a measure of the tendency of the radio wave to prefer the horizontal direction. If $Q > 0$ there is an excess of polarized radiation along the horizontal, while for $Q < 0$, there is a vertical excess (Fig. 1b). $Q \equiv I_{0^\circ} - I_{90^\circ}$.

(3) Stokes $U$ is the difference in intensities between linearly polarized components at $+45^\circ$ and $-45^\circ$ and represents the preference of the light to be aligned at $+45^\circ$, with $U < 0$ meaning an excess in linear polarization at an angle $-45^\circ$ to the horizontal (Fig. 1c). $U \equiv I_{+45^\circ} - I_{-45^\circ}$.

(4) Stokes $V$ is the difference between the intensities of the RCP and LCP components and describes the preference for the light to be RCP. For positive Stokes $V$, there is an excess of RCP over LCP when using the IEEE and IAU conventions (see Sec. 6.2; Fig. 1d). $V \equiv I_{\text{RCP}} - I_{\text{LCP}}$.

It’s important to note that these are definitions. Stokes himself used the notation $\{A, B, C, D\}$ a century before Chandrasekhar settled on $\{I, Q, U, V\}$, the latter three letters of which were assigned with no motivation. Given Chandrasekhar’s convention, there still remains room for ambiguity and confusion: for example, $Q$ could have been defined as $I_{90^\circ} - I_{0^\circ}$, and $V$ could have been defined as $I_{\text{LCP}} - I_{\text{RCP}}$ (and often is! See Sec. 6).

The degree of polarization, or fractional polarization, is the ratio of the intensity of the polarized emission to the total intensity:

$$p = \frac{I_{\text{pol}}}{I_{\text{tot}}} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}; \quad 0 \leq p \leq 1.$$  

We can also form fractional linear polarization

$$p_{\text{lin}} = \frac{\sqrt{Q^2 + U^2}}{I}, \quad 0 \leq p_{\text{lin}} \leq 1,$$

and fractional circular polarization

$$p_{\text{cir}} = \frac{V}{I}; \quad -1 \leq p_{\text{cir}} \leq 1.$$  

When combining (or spatially smoothing) polarized signals, one must combine (or smooth) Stokes parameters, not fractional polarizations, linearly polarized intensities, or polarization angles.

### 2.3. Stokes Parameters Expressed in Terms of Electric Fields

We can also write the Stokes parameters in terms of the time-averaged self- and cross-products of the electric field components as

$$I \equiv \langle E_x E_x \rangle + \langle E_y E_y \rangle \equiv \langle E_R E_R \rangle + \langle E_L E_L \rangle,$$

$$Q \equiv \langle E_x E_x \rangle - \langle E_y E_y \rangle \equiv \langle E_R E_L \rangle + \langle E_L E_R \rangle,$$

$$U \equiv \langle E_x E_y \rangle + \langle E_y E_x \rangle \equiv -i \left( \langle E_R E_L \rangle - \langle E_L E_R \rangle \right),$$

$$V \equiv -i \left( \langle E_x E_y \rangle - \langle E_y E_x \rangle \right) \equiv \langle E_R E_R \rangle - \langle E_L E_L \rangle.$$
where the angle brackets denote a time average of the electric field, and the overbar denotes complex conjugation. By substituting Eq. (3) and Eq. (4) into Eq. (10), we derive the more commonly found representation of the Stokes parameters:

\begin{align}
I &= \langle E_{0x}^2 \rangle + \langle E_{0y}^2 \rangle = \langle E_{0R}^2 \rangle + \langle E_{0L}^2 \rangle, \\
Q &= \langle E_{0x}^2 \rangle - \langle E_{0y}^2 \rangle = 2\langle E_{0R}E_{0L} \rangle \cos(\phi_R - \phi_L), \\
U &= 2\langle E_{0x}E_{0y} \rangle \cos(\phi_y - \phi_x) = 2\langle E_{0R}E_{0L} \rangle \sin(\phi_R - \phi_L), \\
V &= -2\langle E_{0x}E_{0y} \rangle \sin(\phi_y - \phi_x) = \langle E_{0R}^2 \rangle - \langle E_{0L}^2 \rangle.
\end{align}

From Eq. (11) and Eq. (5), it can be seen that the angle that the polarization ellipse makes with the horizontal (i.e., \(x\) axis) can be expressed by

\[ \chi = \frac{1}{2} \tan^{-1} \left( \frac{U}{Q} \right); \quad 0^\circ \leq \chi \leq 180^\circ, \]

where \(\chi\) is known as the position angle of linear polarization (or, more succinctly, the polarization angle) and has a total range of 180, not 360, degrees. Therefore, \(\chi\)

\footnote{This is necessary because the signal being received is being treated as quasi-monochromatic. Such light will not trace out an ellipse with time, but the ellipse can be recovered if the products are averaged over a time long relative to the period of the radio wave. Even for a very fast correlator that could accumulate only 100 ms of data, there will be millions of wave periods per integration at radio frequencies, which is plenty long to uncover the polarization properties of the astronomical radiation.}

\footnote{Textbooks covering polarization tend to denote complex conjugation as \(A^*\). Many authors reverse terms in some of the difference equations because they’ve either used the physics convention for Stokes \(V\) as IEEE LCP – RCP or they’ve defined the exponential propagation argument of the electric field as the negative of the IEEE convention that we’ve adopted in Eq. (1). Finally, there is an understood constant on the RHS of each equation accounting for the conversion of the square of the electric field to a temperature or flux density.}

\footnote{Optics, radiation, and astronomy texts usually provide this set of Stokes parameters, and will often include their representation as a function of the polarization ellipse parameters. The correlation representation of Eq. (11) is not widely presented.}
has an orientation, not a direction. Line segments are commonly used to represent the amplitude and orientation of linear polarization on the plane of the sky. The astronomical community regularly refers to such a line segment as a polarization vector even though a vector has a direction. We propose the adoption of the term sector.

3. Measuring Self- and Cross-Products with Digital Methods

Our dual-polarized receiver system has two orthogonal polarizations, which we denote by $A$ and $B$ because the discussion applies, unchanged, whether our feed system is native linear, native circular, or something in between. Having both polarizations allows us to synthesize all the Stokes parameters from self- and cross-products of the two polarizations using the digital equivalent of Eq. (10).

The time-averaged voltage products are derived from digital samples in one of two ways. Historically, the XF correlation technique\footnote{The "X" represents correlation and the "F" represents a Fourier transform.} prevailed because of its simpler hardware requirements. With XF, one uses a correlation spectrometer, which produces time-averaged auto- and cross-correlation functions (ACFs and CCFs, respectively). These are Fourier transformed, usually in a general-purpose computer, to produce power spectra. Each ACF is computed for $N$ positive lags; negative lags are unnecessary because autocorrelations are symmetric with respect to lag. The ACFs are averaged over time and the Fourier transform (FT) of the resulting average ACF gives the self-power spectrum. Because the ACF is symmetric with respect to lag, its Fourier transform is real and symmetric with frequency, so the self-product power spectrum has $N$ independent channels. Symbolically, for polarization $A$ we write

$$AA = \text{FT} \langle \text{ACF}(V_A) \rangle.$$  (13)

The cross-correlation of the two polarizations is not symmetric with lag, so it must be computed both for $N$ positive and $N$ negative lags. Its FT is complex with Hermitian symmetry, so the cross-power spectrum can be regarded as consisting of a real and imaginary part, each with $N$ independent channels. Symbolically, for polarizations $A$ and $B$ we write

$$AB = \text{Re} \{\text{FT} \langle \text{CCF}(V_A V_B) \rangle \},$$
$$BA = \text{Im} \{\text{FT} \langle \text{CCF}(V_A V_B) \rangle \}. (14)$$

Thus, for a native-linear feed connected to the inputs of a digital spectrometer in such a way that $(A, B) = (X, Y)$, the spectrometer will produce the four spectra $[XX, YY, XY, YX]$. Similarly, for a native-circular feed with $(A, B) = (R, L)$, the spectrometer will output $[RR, LL, RL, LR]$.

Today, the FX technique is favored because of the heavy computing ability of FPGAs and GPUs. With FX, each polarization is sampled at rate $t_s$ over time interval $2T$, providing $2N = \frac{2T}{t_s}$ samples. This block of data is Fourier transformed,
producing a complex transform of $2N$ channels with Hermitian symmetry having $N$ positive-frequency and $N$ negative-frequency channels. The self-product power spectrum is this FT times its complex conjugate, and because of the Hermitian symmetry, it is real with the $N$ negative- and positive-frequency portions identical. Thus, it is a power spectrum with $N$ independent channels. Similarly, one calculates cross-product power spectra by multiplying the Fourier transforms of the two polarizations with both possibilities of complex conjugate (Eq. (13)). This produces a complex cross-power spectrum having $2N$ independent channels, split between negative and positive frequencies. This cross-power spectrum does not have Hermitian symmetry, so has a real part and an imaginary part, each with $N$ independent channels. Thus, we have four spectra of length $N$. Symbolically, for the $V_A$ and $V_B$ self-product spectra we write

$$AA = \left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_A) \right\}, \quad BB = \left\{ \mathcal{FT}(V_B) \mathcal{FT}(V_B) \right\}. \tag{15}$$

The FX spectrometer will return either the complex cross-product spectrum

$$\left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_B) \right\} \quad \text{or} \quad \left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_B) \right\}, \tag{16}$$

but not both. Since these are a complex conjugate pair, we can symbolically represent the real and imaginary parts of these cross-product spectra as:

$$AB = \text{Re} \left\{ \left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_B) \right\} \right\} = \text{Re} \left\{ \left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_B) \right\} \right\} \tag{17}$$

$$BA = -i \text{Im} \left\{ \left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_B) \right\} \right\} = \text{Im} \left\{ \left\{ \mathcal{FT}(V_A) \mathcal{FT}(V_B) \right\} \right\} \tag{18}$$

(Note that ambiguity exists in the sign of the BA term because it won’t be known a priori which of the cross-product spectra an FX spectrometer will output; this is determined via calibration.) The real-valued Stokes parameter spectra can then be assembled from the self- and cross-product spectra following Eq. (10) as:

$$\begin{bmatrix} AA + BB \\ AA - BB \\ 2AB \\ 2BA \end{bmatrix}.$$
4. The Measurement and Calibration Process

We’ve treated everything in our system—from the source’s radiation incident on the Earth to the digital backend output—as a black box. To convert the resulting pseudo-Stokes vector into a true Stokes vector for the astronomical source being observed, we need to undo the effects of this black box.

4.1. Amplitude Calibration

The digitally produced pseudo-Stokes vector is generated in terms of arbitrarily scaled numbers derived from the correlator input voltages ($V_A, V_B$), which are instrumentally generated from the incoming electric fields ($E_A, E_B$). We must convert these arbitrary units to physically meaningful units (kelvins or janskys), which is done by inserting noise of known intensity using standard radioastronomical techniques. This is a standard process that is covered in other chapters of this book, so in the interest of brevity we will omit further discussion of this topic. Rather, we assume at this point that all the pseudo-Stokes vector $S_{cor}$ elements are properly calibrated with respect to amplitude and are brightness temperatures in units of kelvins.

4.2. For Illustrative Purposes: A Linearly Polarized Source

Measuring the polarization of a source means obtaining its four calibrated Stokes parameters. Here we focus the discussion for illustrative purposes by considering linearly polarized sources. Concentrating on linearly polarized sources is natural, because many polarized radioastronomical sources have only very small circular polarization; pulsars and masers can be exceptions.

Purely linearly polarized sources have $V = 0$ and are characterized by the fractional linear polarization $p_{src,lin}$ and the position angle $\chi_{src}$; these, in turn, are derived from Stokes $(Q, U)$:

$$S_{src} = I_{src} \cdot \begin{pmatrix} 1 & p_{src,lin} \cos 2\chi_{src} \\ p_{src,lin} \cos 2\chi_{src} & p_{src,lin} \sin 2\chi_{src} \\ 0 & 0 \end{pmatrix}. \tag{20}$$

We will consider both astronomical sources, which normally have $p_{src,lin} \ll 1$, and special-purpose test signals, which normally have $p_{cal,lin} = 1$.

For the purpose of measuring polarization, the receiver system needs a noise diode output that is injected into both polarizations as a correlated calibration signal (a.k.a. “cal”). This can be accomplished either by injecting it externally—e.g., by a linearly polarized vertex radiator—or by splitting the noise diode output and using two cables to inject the signal into both polarization paths, each with a directional coupler located just in front of its first amplifier. The position angle of

\[^{2}\text{See Chapter 1 of this volume.}\]
Fig. 2. Block diagram of dual-polarized (A and B) single-dish system (adapted from Ref. [13]). The noise diode (a.k.a. cal) output is injected through short cables and directional couplers with combined phase delays $\phi_A$ and $\phi_B$. The total voltage gain of polarization channel $A$ is $g_A = g_{A1}g_{A2}$; the voltage gains are complex, with an amplitude and a phase. The total cable length for channel $A$ is length $L_A$, which includes the run from the dish to the correlator input so it can be very long (more than 1 km at the Green Bank Telescope). The thick lines represent mechanical structures or passive electronics that do not change with time; the thin lines represent active electronics and other circuitry that do change with time and need calibration.

The vertex radiator should be $45^\circ$ away from the feed probes with the ideal that the injected noise has small, or ideally zero, $Q$. The cable-and-splitter option is the usual case, and it is depicted in the system block diagram in Fig. 2.

For the case of the cable-and-splitter injected cal, the powers injected into the two polarization channels are almost equal, so Stokes $Q$, which is their difference, is small and, ideally, zero; similarly, the total polarized power fraction (Eq. (7)) is unity. So for this ideal case, the Stokes cal vector is

$$S_{\text{cal}} = \begin{bmatrix} I_{\text{cal}} \\ Q_{\text{cal}} \\ I_{\text{cal}} \cos \Delta \phi_{\text{cal}} \\ I_{\text{cal}} \sin \Delta \phi_{\text{cal}} \end{bmatrix}.$$  

(21)

The angle $\Delta \phi_{\text{cal}} = \phi_{\text{cal},A} - \phi_{\text{cal},B}$ represents the phase difference between the injected noise diode signals. A primary contributor to this difference is the different
 lengths of the two noise diode cables, which makes $\Delta \phi_{\text{cal}}$ a linear function of frequency. If the cable length difference is exactly zero, and if the directional couplers and other devices in the circuit are perfectly matched for the two polarizations, then this injected cal signal is 100% polarized with $\Delta \phi_{\text{cal}} = 0$ and Stokes $U_{\text{cal}} = I_{\text{cal}}$.

Consider the two quantities $Q_{\text{cal}}$ and $\Delta \phi_{\text{cal}}$. The cal is injected through a power splitter, cables, and directional couplers. These are all mechanical devices and should be stable over long periods of time; indeed, we have observationally found that to be the case. This is fortunate, because we rely on the cal as the secondary standard for system calibration. Therefore, it is essential to determine $Q_{\text{cal}}$ and $\Delta \phi_{\text{cal}}$, and the process of determining them we call Mueller matrix calibration. We discuss this process below in Sec. 4.3.

4.3. Specifying the Stokes Vector Transfer Functions by Mueller Matrices

Figure 2 shows a block diagram of a typical dual-polarized radioastronomical receiver. The signal from the source first encounters the feed. The feed rotates with respect to the source: for an alt/az-mounted telescope observing an astronomical source, it rotates by the parallactic angle, while for an equatorially mounted telescope it doesn’t rotate at all. If it’s an injected test signal, e.g. from a vertex radiator, one intentionally rotates the feed for calibration purposes. The rotated feed converts the incoming electromagnetic radiation to voltages. Finally, these voltages are amplified to levels appropriate for the input to a digital spectrometer.

Each of these processes modifies the Stokes parameters. We can regard each process as having a transfer function for the four Stokes parameters. This transfer function is a $4 \times 4$ matrix, known as the Mueller matrix. We need the Mueller matrices for the above three processes, discussed here in the order in which the source radiation encounters them:

(1) **FEED ROTATION.** For the rotation of the feed by angle $\chi$ with respect to the source, the Mueller matrix is

$$\mathbf{M}_\chi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\chi & \sin 2\chi & 0 \\ 0 & -\sin 2\chi & \cos 2\chi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

The central $2 \times 2$ submatrix is, of course, nothing but a rotation matrix. When the telescope rotates with respect to the source, which is the operation described

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1You might think that rotating the vertex radiator is equivalent to rotating the feed. That is not the case! When you rotate the radiator, the transmitted signal changes, and along with it the reflections from portions of the telescope, such as feed legs, change. However, when you instead rotate the feed, the reflections of the transmitted signal remain unchanged.

2A reminder that we adopt the notation $\mathbf{A}$ to represent a $1 \times 4$ column matrix and $\mathbf{A}$ to represent a square $4 \times 4$ matrix.
by Eq. (22), it is equivalent to keeping the feed stationary and having a purely linearly polarized source emitting with position angle \((\chi_{\text{src}} - \chi)\):

\[
S_{\text{src}, \chi} = M_{\chi} \cdot S_{\text{src}} = I_{\text{src}} \cdot \begin{bmatrix}
1 \\
p_{\text{src,lin}} \cos(2(\chi_{\text{src}} - \chi)) \\
p_{\text{src,lin}} \sin(2(\chi_{\text{src}} - \chi)) \\
0
\end{bmatrix}.
\] (23)

Note that we can regard both \(S_{\text{src}}\) and \(S_{\text{src}, \chi}\) as true Stokes vectors in the sense of Eq. (6) as long as we specify that each has its own reference coordinate system. In terms of the source’s reference system, \(S_{\text{src}, \chi}\) is a pseudo-Stokes vector because its elements are not \([I, Q, U, V]\).

(2) FEED COUPLING. Next comes the feed. Here we consider perfect dual-polarized feeds with native-linear or native-circular polarization, where the term ‘perfect’ means that the two polarizations are orthogonal, the two polarizations are either purely linear or purely circular, and there are no losses. We consider these extremes for several reasons: (1) many feeds are, in fact, close to perfection; (2) the discussion can focus on fundamentals without the fog of excess detail; (3) in practice, when you’re sitting at the telescope and want to know how well things are working, a quick and approximate assessment of the receiver system is often adequate.

The feed’s Mueller matrix must be obtained from its Jones matrix. The Jones and Mueller matrices for the general case of imperfect feeds are given by Eqs. (10)–(11) of Ref. 14. For perfect feeds of arbitrary polarization, the matrices depend on two angles, called \(\alpha_{\text{feed}}\) and \(\chi_{\text{feed}}\). Ref. 14 uses \(\tan \alpha_{\text{feed}}\) to specify the voltage coupling between the input \(E\)-field and output voltages and \(\chi_{\text{feed}}\) to represent the phase of that coupling (not to be confused with the position angle \(\chi\) used in the current chapter). Perfect native linear feeds have \(\alpha_{\text{feed}} = 0^\circ\) and perfect native circular feeds have \(\alpha_{\text{feed}} = \pm 45^\circ\) and \(\chi_{\text{feed}} = \pm 90^\circ\).

Our two feed types are:

(a) Native-Linear Feeds. The Mueller matrix for a perfect native-linear feed whose probes are aligned with the azimuth and elevation directions is just the unitary matrix, i.e.,

\[
M_{F, \text{lin}} = I. \quad \text{(24)}
\]

More generally, if the native-linear feed is mounted at angle \(\chi_F\) with respect to being aligned, the Mueller matrix is simply

\[
M_{F, \chi} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\chi_F & \sin 2\chi_F & 0 \\
0 & -\sin 2\chi_F & \cos 2\chi_F & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}. \quad \text{(25)}
\]
(b) **Native-Circular Feeds.** The Mueller matrix for a perfect native-circular feed is

\[
M_{\text{Fcir}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & \pm 1 \\
0 & 0 & 1 & 0 \\
0 & \mp 1 & 0 & 0 \\
\end{bmatrix},
\]

where the signs depend on the values of \(\alpha_{\text{feed}}\) and \(\chi_{\text{feed}}\); the case \(\alpha_{\text{feed}} = +45^\circ\) and \(\chi_{\text{feed}} = +90^\circ\) has the signs on top (i.e., +1 in the second row and -1 in the fourth row).

(3) **AMPLIFICATION AND ELECTRONICS.** The Mueller matrix for the electronics chains deals with amplitude, so we must define our intensity units. First, our uppercase \(G\) means power gain (which has no phase), while the lowercase \(g\) means voltage gain, which is complex; \(G = g_\text{eff}^\ast\). Following Ref. [14], we assume that good, but not perfect, intensity calibration has been previously applied to the two polarization channels so that the Stokes parameters have the correct units (e.g., temperature), and, in addition, that the total intensity, Stokes \(I\), has perfect intensity calibration (to simplify the following equations). Then we define \((G_A, G_B)\) to be the power gains for the two polarization channels. Because of our assumptions we write \(G_A = (1 + \delta G)\) and \(G_B = (1 - \delta G)\), where \(\delta G\) is unitless and \(|\delta G| \ll 1\). For consistency with Ref. [14], we define \(\Delta G_{AB} = 2\delta G\). Then, to first order in \(\Delta G_{AB}\), the Mueller matrix for the electronics chains (see Fig. 2) is

\[
M_{AB} = \begin{bmatrix}
1 & \Delta G_{AB} & 0 & 0 \\
\Delta G_{AB}^2 & 1 & 0 & 0 \\
0 & 0 & \cos \Delta \phi_{AB} - \sin \Delta \phi_{AB} & 0 \\
0 & 0 & \sin \Delta \phi_{AB} & \cos \Delta \phi_{AB} \\
\end{bmatrix}.
\]

The two parameters in \(M_{AB}\) are the relative power gain (\(\Delta G_{AB}\)) and phase delay between the two polarization channels (\(\Delta \phi_{AB}\)) and are both associated with the electronics and the circuitry, including cable lengths. These quantities can change with time because they are associated with active electronics, so they need to be measured often enough to keep up with the variability of system electronics—and at least once per observing session.

### 4.4. The Measured Pseudo-Stokes Vector \(S_{\text{cor}}\) for Several Cases

After being operated on by these three Mueller matrices, the original source Stokes vector becomes the previously defined pseudo-Stokes vector, producing voltages \(V_A\) and \(V_B\) at the input to the correlator. The correlator generates the auto- and cross-products as discussed above, producing the pseudo-Stokes vector output \(S_{\text{cor}}\).

When the system looks at 'blank sky', the input noise is mainly from the receiver, with a small contribution from the sky and ground pickup. For purposes of illustration, we include only the receiver contribution. In this case, the noise is
injected after the feed so the only Mueller matrix that operates is $M_{AB}$. Denote the associated pseudo-Stokes vector by $S_{rx}^{\text{cor}}$:

$$S_{rx}^{\text{cor}} = M_{AB} \cdot S_{rx}.$$  \hfill (28)

When on the source, with the cal off, we see

$$S_{src}^{\text{cor}} = M_{AB} \cdot M_{F} \cdot M_{\chi} \cdot S_{src} + S_{rx}^{\text{cor}}.$$  \hfill (29)

And when off the source with the cal on:

$$S_{cal}^{\text{cor}} = M_{AB} \cdot S_{cal} + S_{rx}^{\text{cor}}.$$  \hfill (30)

For an accurate measurement of the source or cal deflection, we must subtract the off-source contribution $S_{rx}^{\text{cor}}$, as is usual for all single-dish measurements. Denote these deflections with the prefix $\Delta$. Then for any type of feed, the cable-injected cal response is

$$\Delta S_{cal}^{\text{cor}} = M_{AB} \cdot S_{cal} = I_{cal} \cdot \begin{bmatrix} 1 \\ \frac{\Delta G_{AB}}{2} + \frac{Q_{cal}}{I_{cal}} \cos(\Delta \phi_{AB} + \Delta \phi_{cal}) \\ \frac{\Delta G_{AB}}{2} + p_{src,\text{lin}} \cos(2[\chi_{src} - \chi]) \\ p_{src,\text{lin}} \cos(\Delta \phi_{AB} \sin(2[\chi_{src} - \chi])) \end{bmatrix}. \hfill (31)$$

We have assumed $\frac{Q_{cal}}{I_{cal}} \ll 1$ and kept only first-order terms.

Similarly, for the source deflection, we get

$$\Delta S_{src}^{\text{cor}} = M_{AB} \cdot M_{F} \cdot M_{\chi} \cdot S_{src}.$$  \hfill (32)

For a perfect native-linear feed with probes aligned with the azimuth and elevation directions, $M_{F} = M_{F,\text{lin}} = I$ (Eq. (24)) and

$$\Delta S_{src,\text{lin}}^{\text{cor}} = I_{src} \cdot \begin{bmatrix} 1 + \frac{\Delta G_{AB}}{2} p_{src,\text{lin}} \cos(2[\chi_{src} - \chi]) \\ \frac{\Delta G_{AB}}{2} + p_{src,\text{lin}} \cos(2[\chi_{src} - \chi]) \\ p_{src,\text{lin}} \cos(\Delta \phi_{AB} \sin(2[\chi_{src} - \chi])) \end{bmatrix}. \hfill (33)$$

The $\Delta S_{src,\text{lin},0}^{\text{cor}}$ element is the pseudo-Stokes $I$ and is not equal to unity. This can be awkward for Mueller matrix calibration, when one almost always forces $\Delta S_{src,\text{lin},0}^{\text{cor}}$ to be unity to eliminate the influence of overall system gain changes. These can occur, for example, from pointing errors or position-dependent telescope surface distortions and other circumstances that reduce the overall system gain. So one must divide the other three pseudo-Stokes parameters by $\Delta S_{src,\text{lin},0}^{\text{cor}}$. Fortunately, for the common case when an astronomical source is used for the calibration, we almost always have $p_{src,\text{lin}} \ll 1$; this makes the contribution of the non-unity portion of $\Delta S_{src,\text{lin},0}^{\text{cor}}$ to the other three pseudo-Stokes parameters second-order, so it can be neglected. However, for a locally generated test signal, $p_{cal,\text{lin}}$ is likely to be unity. The easiest way to deal with this is to rescale the amplitudes so that $\Delta G_{AB}$ itself becomes second order.

\footnote{$\Delta S_{src,\text{lin},0}^{\text{cor}}$ is the zeroth element of the $\Delta S_{src,\text{lin}}^{\text{cor}}$ pseudo-Stokes vector; see Eq. (33).}
For a perfect native-circular feed, $M_F$ is given by Eq. (26), and

$$\Delta S_{\text{src,cir}}^{\text{cor}} = I_{\text{src}} \cdot \begin{bmatrix} 1 & \Delta G_{AB} \\ \frac{1}{2} & \frac{1}{2} \sin(\Delta \phi_{AB} + 2[\chi_{\text{src}} - \chi]) \\ -p_{\text{src,lin}} \sin(\Delta \phi_{AB} + 2[\chi_{\text{src}} - \chi]) & -p_{\text{src,lin}} \cos(\Delta \phi_{AB} + 2[\chi_{\text{src}} - \chi]) \end{bmatrix} .$$  \hspace{1cm} (34)

4.5. Discussion: The Process of ‘Mueller Matrix Calibration’

Suppose you make a single measurement $\Delta S_{\text{src}}^{\text{cor}}$ of the deflection of a linearly polarized source and wish to derive the source’s linear polarization fraction $p_{\text{src,lin}}$ and position angle $\chi_{\text{src}}$ from the measured $\Delta S_{\text{src}}^{\text{cor}}$. If all of the off-diagonal terms in the three Mueller matrices were zero, this would be easy. However, this is never the case. If you know the three Mueller matrices, then you can calculate the inverse of their matrix product and derive $S_{\text{src}}$ from the measured $\Delta S_{\text{src}}^{\text{cor}}$ using Eq. (32); alternatively, if you know $p_{\text{src,lin}}$ and $\chi_{\text{src}}$ (because it’s a polarization calibration source, for example), then you can analytically calculate $\Delta S_{\text{src}}^{\text{cor}}$ from $S_{\text{src}}$ using Eq. (32). Either way, the parameters $\Delta G_{AB}$ and $\Delta \phi_{AB}$ need to be known. To determine them we need to use the calibration noise diode, which produces the deflection given by Eq. (31). This deflection depends on four quantities: our two required amplifier-chain parameters $\Delta G_{AB}$ and $\Delta \phi_{AB}$ (which change with time), and the two cal-injection parameters $Q_{\text{cal}}$ and $\Delta \phi_{\text{cal}}$ (which do not change with time).

We cannot determine $\Delta G_{AB}$ and $\Delta \phi_{AB}$ without knowing $Q_{\text{cal}}$ and $\Delta \phi_{\text{cal}}$. We call the process of determining $Q_{\text{cal}}$ and $\Delta \phi_{\text{cal}}$ the Mueller matrix calibration. Mueller matrix calibration is done by observing a polarization calibrator with known intensity and polarization to obtain $\Delta S_{\text{src}}^{\text{cor}}$ over a range of parallactic angle $\chi$ and, in addition, obtaining the cal deflection $\Delta S_{\text{cal}}^{\text{cor}}$. One then plots the $\chi$-dependence of the four elements of $\Delta S_{\text{src}}^{\text{cor}}$. The first element, Stokes $I$, is constant by definition, because we always deal with fractional Stokes parameters. The remaining three elements vary periodically with $\chi$, and from the amplitudes and phases of their variation one can use least-squares fitting of Eq. (33) or Eq. (34) to derive all of the parameters.

Least-squares fitting is best for accuracy, but referring to that process does not aid our phenomenological understanding. We can develop our understanding by solving for the parameters using basic algebra. First, obtain $\Delta \phi_{AB}$ and $\Delta \phi_{\text{cal}}$ from Eq. (33) and Eq. (31) \text{m}\n
$$\Delta \phi_{AB} = \tan^{-1}\left(\frac{\Delta S_{\text{src,lin},1}^{\text{cor}}}{\Delta S_{\text{src,lin},2}^{\text{cor}}}\right) ,$$

$$\Delta \phi_{AB} + \Delta \phi_{\text{cal}} = \tan^{-1}\left(\frac{\Delta S_{\text{cal},1}^{\text{cor}}}{\Delta S_{\text{cal},2}^{\text{cor}}}\right) .$$

\text{m} \Delta S_{\text{src,lin},i}^{\text{cor}} is the $i$th element of the $\Delta S_{\text{src,lin}}^{\text{cor}}$ pseudo-Stokes vector; see Eq. (6).
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Next, plot $\Delta S_{\text{src,lin},1}^{\text{cor}}$ versus $\chi$. The part that varies with $\chi$ gives $p_{\text{src,lin}}$ and the offset of this cosine wave from zero gives $\Delta G_{AB}$. Combine this with $\Delta S_{\text{cal,1}}^{\text{cor}}$ to obtain $Q_{\text{cal}}$. One assumes, of course, that during the time interval for this calibration the parameters stay fixed—in particular, that the electronics parameters $\Delta G_{AB}$ and $\Delta \phi_{AB}$ stay fixed. Experience shows that with modern electronics at the 305-m Arecibo telescope and the 100-m Green Bank Telescope (GBT) this assumption is good.

Figure 3 shows a set of 1666 MHz Mueller matrix calibration data from the famous polarization calibrator 3C 286 for the native-linear polarization system at the GBT. The crosses (solid line) show $(\Delta S_{\text{src,lin},1}^{\text{cor}})/(\Delta S_{\text{src,lin},0}^{\text{cor}})$ and the diamonds (dashed line) show $(\Delta S_{\text{src,lin},2}^{\text{cor}})/(\Delta S_{\text{src,lin},0}^{\text{cor}})$. If the data were perfectly calibrated for polarization, these two outputs would equal $Q_{\text{src}}$ and $U_{\text{src}}$ and would vary sinusoidally with twice the parallactic angle, with the two sinusoids having equal amplitude and no offsets from zero. This is definitely not the case. The squares (dash-dot line) in Fig. 3 represent $(\Delta S_{\text{src,lin},3}^{\text{cor}})/(\Delta S_{\text{src,lin},0}^{\text{cor}})$ and reveal a major leakage of linear polarization into Stokes $V$. A nonlinear least-squares fit of these data yields the first seven parameters listed below the left plot in Fig. 3. The associated Mueller matrix is listed at the bottom of the left panel. The nonzero off-axis elements quantify the leakage of one uncalibrated Stokes parameter into another. If $S_{\text{src,lin}}^{\text{cor}}$ is corrected by this Mueller matrix, the proper $\chi$-dependencies of the elements of $S_{\text{src,lin}}^{\text{cor}}$ are recovered, as depicted in the right panel of Fig. 3.

4.6. Two Important Subtleties Regarding Relative Phase $\phi_{AB}$

4.6.1. System Cable Lengths

Various electronics components in the signal path between the feed and the correlator introduce complex voltage gains that can include amplification, attenuation, and phase changes (e.g., some amplifiers introduce a phase shift of 180°). Of particular importance: the combined lengths of the coaxial cables and optical fibers differ between the two signal paths ($L_A$ and $L_B$ in Fig. 2). Environmental factors can cause these lengths to change with time. A difference between the path lengths produces a phase difference in radians of

$$\delta \phi_{AB} = \frac{2\pi(L_A - L_B)}{\lambda},$$  \hspace{1cm} (36)

and this phase difference depends on frequency as

$$\frac{d\delta \phi_{AB}}{d\nu} = \frac{2\pi(L_A - L_B)}{c}. \hspace{1cm} (37)$$

This phase difference, $\delta \phi_{AB}$, adds to other contributions to produce the total phase difference $\Delta \phi_{AB}$. Measured values of the total phase gradient $\frac{d\Delta \phi_{AB}}{d\nu}$ at Arecibo

$^a$In Fig. 3, the first two parameters are labelled DELTAG and PSI and correspond to our $Q_{\text{cal}}$ and $\phi_{\text{cal}}$; the next three deal with feed imperfections; and the last four are the source polarization. For a detailed description of all the listed parameters, see Sec. 7.1 of Ref. 14.
and the GBT are about 0.3 rad MHz$^{-1}$, corresponding to a difference in cable/fiber length of $\sim$20 m. This is surprisingly large, even considering the extreme distances between the feed and correlator for these telescopes.

4.6.2. System Band-Limiting Filters and Their Induced Kramers-Kronig Phase Shifts

At some point in the receiver chain one always has a band-limiting filter. Frequency-dependent gains automatically introduce phase delays, which can be calculated from the Kramers-Kronig relations. If the filters in the two polarizations are not perfectly matched, a frequency-dependent phase difference between the two polarization channels ensues. This can be particularly serious when the filters have significant gain changes within the usable portion of the band.

The exact formula for the phase shift (in radians) induced by a power gain
change in an electrical circuit is given by Eq. (2) of Ref. 15:

$$\phi(\nu) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\bar{G}(u)}{du} \ln \left[ \coth \left( \frac{|u|}{2} \right) \right] du,$$

where $\bar{G}(u)$ is the filter power gain in nepers, $u = \ln \left( \frac{\nu}{\nu_c} \right)$, $\nu$ is frequency, and $\nu_c$ is an arbitrarily chosen frequency. The weighting function $\ln \left[ \coth \left( \frac{|u|}{2} \right) \right]$ is sharply peaked at $u = 0$ where $\nu = \nu_c$, so a good approximation eliminates the integral and uses only the local derivative (Eq. 22 of Ref. 16):

$$\phi(\nu_c) = -\frac{\pi}{2} \left. \frac{d\bar{G}(u)}{du} \right|_{u=0}.$$  (39)

Thus a non-flat filter produces phase shifts.

The left panel of Fig. 4 depicts the power gains and phase delays for Arecibo’s interim correlator, for which the baseband low-pass filters (cutoff frequency 6.25 MHz) are digitally defined and are remarkably flat. We show only the positive-frequency half. Phase shifts occur only at the high-frequency end, where the responses of the filters drop precipitously.

*You would miss a lot if you pass up the opportunity to read Bode’s paper [15], particularly the first six pages. Go to [http://www.alcatel-lucent.com/bstj/](http://www.alcatel-lucent.com/bstj/). N.B.: Bode’s derivation treats changes in logarithmic attenuation $A$; since we’re treating changes in logarithmic power gain $G$, we’ve set $\bar{G} = -A$ in his equations.*
In contrast, the GBT Radar Backend filters (Fig. 4, right-hand panel) fall to zero gradually, with no sharp cutoff frequency. Thus, the power gain varies rapidly within the observing band (top panel), with correspondingly large (huge!) phase delays (middle panel), peaking at ~600°! With such large phase delays, even small differences between the A and B filters lead to significant frequency-dependent relative phase delays $\delta\phi_{AB}$ (bottom panel). For native-linear polarization, these delays interchange power between U and V; for native-circular, they interchange Q and U. These phase differences must be corrected.

For the GBT, the bottom plot shows both the theoretical (the noisy curve, from Eq. (39)) and measured (the smoother curve, from correlated noise injection) phase differences between the two signal paths. The theory and the data do not agree at all. The reason is inaccuracy in the filter shape resulting from uncorrected 4-bit quantized voltage sampling. Specifically, the calculated filter responses do not fall to zero at high frequencies, as they actually do. If, as a numerical experiment, we displace the BB curve downwards by 0.03, the theory curve becomes equal to the smoother measured one above 0.35 MHz. Thus, the theory curve is inaccurate and noisy, because it is the difference between two large numbers, neither of which is itself very accurate.

5. Off-Axis Instrumental Polarization

Thus far we’ve considered the polarization properties of radiation entering the feed along the optical axis of a telescope’s main beam. However, radio telescopes pick up radiation off-axis via sidelobe response. The polarization state of incoming radiation can be altered in these polarized sidelobes (and even inside the main beam!) in such a way that unpolarized astronomical radiation can be converted to a polarized response affecting the on-axis signal.

Understanding the mechanisms that create this off-axis instrumental polarization is the domain of antenna engineers whose interests lie in building efficient dual-polarized communication systems that carry a pure polarized signal (what they call the co-polarized signal) in one channel without allowing that information to leak into the orthogonal polarization state (what they call a cross-polarized signal). In order to accomplish this, the two E-field polarization states (horizontal and vertical for a dual-linear feed, RCP and LCP for a dual-circular feed) need to be perfectly orthogonal across the aperture plane of the telescope. This is an impossible task: there is always some cross-polarization inherent in the system. We investigate below some of the most common causes of this cross-polarization from both the engineer’s viewpoint of transmitting from the focus and the astronomer’s reciprocal perspective of receiving at the focus.

Ref. 17 lists the various terms that engineers and astronomers use for the singular concept of instrumental polarization, among them: cross-polarization, feed or polarization leakage, $D$-terms, cross-coupling, mutual coupling, cross-talk, barrel distortion, and beam squish.
5.1. Cross-Polarization Induced by the Feed and Dish Surface: Beam Squash

Ref. [18] uses multiple methods to analytically derive the cross-polarization response of a circularly symmetric paraboloidal reflector with a feed located at the primary focus. The resulting cross-polar pattern depends on the analysis method, but two components are always present: a depolarization pattern caused by the curvature of the reflector surface and a pattern from the inherent cross-polarization of the feed. Both contributions will produce \( E \)-field aperture distributions with nulls along the principal plane \( \phi \) and field maxima in the \( \pm 45^\circ \) plane.\(^{18-21} \) The far-field \( E \)-field radiation pattern can then be produced from this \( E \)-field aperture distribution via 2-D Fourier transform integration.\(^{19} \)

Astronomers are interested in knowing how their telescope responds to an unpolarized source of radiation at any angle off of the optical axis. This can be measured in practice for a single-dish telescope by mapping out the Stokes parameter response of a strong unpolarized continuum source as the main beam is driven around an area centered on the source. Fractional Stokes parameter beam maps are then generated by dividing these Stokes beam maps by \( I_{\text{peak}} \), the peak Stokes \( I \) response of the main beam.\(^{2} \) For notational efficiency, we will refer to the fractional quantities \{\( I, Q, U, V \)\}/\( I_{\text{peak}} \) in the remainder of this section as simply \{\( I, Q, U, V \)\}.

Before inspecting a measured polarized beam pattern, we can investigate what one might expect from a perfect telescope. For the last few decades, the commercial software package GRASP has developed into a sophisticated tool allowing the far-field vector \( E \)-field response of reflector antennas to be precisely modelled using efficient algorithms for physical optics and the physical theory of diffraction. We follow the lead of Ref. [22] and use GRASP to model the transmitted far-field pattern of the circularly symmetric DRAO 25.9-m diameter paraboloidal telescope (\( f/D = 0.2941 \)) fed from the primary focus by a simulated feed pattern for a circular-waveguide feed (with inherent cross-polarization) with four \( \lambda/4 \) chokes and dual-linear probes.\(^{23} \) The Stokes parameters were constructed from the simulated far-field \( E \)-field distribution for a given orientation of the feed probes via Eq. (10). (To simplify the modelling even further, we exclude any feed-support legs and aperture blockage.) Then, invoking the principle of reciprocity, the feed was rotated through 180\(^\circ\) and each of the Stokes patterns averaged over these orientations to simulate the transmission of unpolarized light in the far field. Figure 5 shows these averaged fractional Stokes parameter beam patterns, which also represent the tele-

\(^{9}\)For a dual-linear feed, the two principal planes are those that contain the reflector axis and the orthogonal feed probes.
\(^{1}\)If one wanted to estimate the instrumental contribution to the on-axis Stokes \( Q \) response from an unpolarized source in the first sidelobe, one would multiply the source’s Stokes \( I \) brightness by the fractional polarization at the appropriate location in the \( Q/I_{\text{peak}} \) pattern. These fractional Stokes parameter beam maps should not be confused with point-for-point maps of fractional Stokes parameters, e.g., the \( Q \) pattern divided by the \( I \) pattern. While an interferometer might be able to measure such a pattern readily, a single-dish telescope does not have enough dynamic range or angular resolution to quickly measure point-for-point fractional polarization in far-out sidelobes.
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scope’s response to unpolarized radiation. The rightmost panels show the simulated beam patterns for a perfect linear dipole feed transmitting onto the same reflector geometry; this feed has absolutely no inherent cross-polarization, so that any Stokes $Q$ or $U$ response will be entirely brought about by the reflector surface. Some important properties are immediately evident:

1. The Stokes $Q$ and $U$ cross-polarization patterns resemble a four-lobed clover leaf with lobes on opposite sides of the beam center having identical signs; the signs of adjacent lobes alternate in beam azimuth. We call this pattern beam squash. For a Stokes $Q$ pattern with its positive-response lobes aligned along the vertical axis, this is equivalent to the beamwidth being larger in the vertical direction than in the horizontal direction (meaning the feed pattern illuminating the primary reflector is wider in the horizontal direction than in the vertical).

2. The lobes of the beam squash pattern are aligned with the feed probe orientation for Stokes $Q$ and are aligned at $45^\circ$ for Stokes $U$.

3. The sign of the beam squash response reverses between the main beam and the first sidelobe.

4. The beam squash produced by the dish is dwarfed (by a factor of 3000 in this instance) by the beam squash inherent in the feed response. This situation almost always obtains, even for corrugated conical horns whose cross-polarization response can be designed to be significantly smaller than other types of feed.

5.2. Polarization Induced by the Feed Location: Beam Squint

If a feed is tilted or displaced from the focus of a reflector such that the feed axis and the reflector axis are misaligned, an amplitude or phase slope is induced across the reflector’s aperture plane. In the far-field response, this translates to the RCP and LCP beams pointing in slightly different directions on either side of boresight; the displacement occurs in the plane that is orthogonal to the plane of symmetry of the reflector and is known as beam squint. So if a feed is tilted and/or displaced from the reflector axis in the azimuth direction, the beam squint lobes will lie along the elevation direction.

Offset paraboloidal reflectors are now commonly used in place of primary focus-fed circularly symmetric paraboloids in order to overcome the blockage and scattering brought about by the feed, receiver housing, and feed-support legs. In such a system, an elliptical section can be cut out of a circularly symmetric paraboloid in such a way that the primary focus is outside the main beam of the primary reflector. It is well known that such a system suffers a cross-polarization penalty in the form of beam squint. An off-axis secondary reflector can be added to the optical path and designed to minimize the squinting at a secondary focus. The GBT and the planned Square Kilometer Array dishes employ this design.

*The reflector cross-polarization decreases with increasing $f/D$. 

19,24
Fig. 5. GRASP-generated far-field beam patterns at 1420 MHz for a circularly symmetric paraboloidal reflector of diameter 25.9 m and \(f/D = 0.2941\) with no feed legs or aperture blockage. All beam patterns are normalized to the peak main-beam Stokes \(I\) response and each frame covers \(3^\circ \times 3^\circ\) on beam center. A simulated feed pattern for a circular-waveguide feed (with inherent cross-polarization) with four \(\lambda/4\) chokes and dual-linear probes was used to illuminate the primary, producing beam patterns for: (top left) Stokes \(I\) with grayscale covering \(0–100\%\) (white to black), solid white contours covering \((10\%, 30\%, 50\%, 70\%, 90\%)\), and dashed black contours covering \((1\%, 3\%, 5\%, 7\%, 9\%)\); Stokes \(Q\) (top middle), \(U\) (bottom middle), and \(V\) (bottom left) with grayscale covering (white to black) \(\pm 0.2\%\) of the peak Stokes \(I\) (thus white is \(-0.2\%\), black \(+0.2\%\), and gray \(0\%)\), dashed black contours covering \((-0.02\%, -0.10\%, -0.18\%, -0.26\%, -0.34\%, -4.2\%)\), dashed white contours covering \((0.02\%, 0.10\%, 0.18\%, 0.26\%, 0.34\%, 4.2\%)\), and \(0\%\) contour omitted. The orientation of the dual-linear feed probes is indicated; the lobes of the Stokes \(Q\) pattern align with the probes while the \(U\) pattern is oriented at 45\(^\circ\). There is no discernible \(V\) response. The Stokes \(Q\) (top right) and \(U\) (bottom right) patterns are also shown for the same primary reflector being fed by a perfect feed with no inherent cross-polarization. Grayscale covers \(\pm 0.00012\%\) (white to black). These patterns show that the cross-polarization induced by the reflector alone has the same character and orientation as that produced by the waveguide feed and reflector working in conjunction, but the reflector-only pattern is narrower and more than 1000 times weaker.

A circularly symmetric parabolic reflector in a Cassegrain or Gregorian configuration can also suffer beam squint when the feed is positioned at a secondary focus that is located off of the primary’s axis of symmetry. This arrangement obtains for multiple feeds at the Effelsberg 100-m telescope and at the NRAO Very Large Array (VLA), where significant beam squints have been measured\cite{2007}. If observing a large-scale region of emission for which the Stokes \(I\) brightness temperature varies with position, beam squint will respond to the first derivative of
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Fig. 6. Fractional Stokes $V$ sidelobe response of the now-collapsed 85-ft Hat Creek Telescope out to $24^\circ$ from beam center (adapted from Ref. 34). Only positive Stokes $V$ is shown in grayscale; negative values appear as blank areas. The four thick gray lines show the inner portions of the four feed-leg scatter rings, with each ring represented by a different line style for clarity. Each ring is a small circle in the sky centered on the direction the feed leg points, whose angular diameter is twice the angle the leg makes with the symmetry axis of the primary reflector. (Each ring—there are as many rings as there are feed legs—passes through and draws its energy from the main beam. Feed-leg scattering therefore reduces telescope gain.) For each ring, the sign of the Stokes $V$ response is reversed on either side of beam center.

Stokes $I$ with position. Measurements of 21-cm Zeeman splitting can be seriously affected by spatial gradients in the diffuse 21-cm emission interacting with the beam squint in such a way as to produce an artificial Stokes $V$ response that exactly mimics a Zeeman splitting signature.

5.3. Instrumental Polarization Induced by Aperture Blockage and Feed-Support Legs

Structures that block the primary aperture are also a source of polarized sidelobes; these include feed-support legs, cables, subreflectors, and receiver cabins. These can produce instrumental polarization in sidelobes both near-in to and far-out from the main beam. While receiver cabins and subreflectors are complex structures whose effect on the telescope’s polarized response cannot be easily modelled, the effect of feed-support legs is relatively easy to simulate. Figure 5.3 shows the measured Stokes $V$ response within $24^\circ$ of the main beam of the now-collapsed 85-ft Hat Creek Telescope. The dashed lines trace four circular features whose Stokes $V$ polarization
response reverses sign on either side of beam center. While Refs. 3, 4, and 21 have correctly pointed out that these arcs are related to the scatter cones generated by the quadrapod feed-support structure, they were at a loss to explain why the circular polarization should display the observed pattern. Modern full-polarization simulations of feed-leg scattering using GRASP easily reveal this exact pattern, including the observed sign reversals. Such simulations also reveal significant structure in the Stokes $Q$ and $U$ patterns, which can affect measurements of diffuse polarized Galactic continuum radiation. Because this radiation covers the entire sky, a polarized sidelobe sitting on the sky will pick up unpolarized radiation and alter the polarized component of the measured signal. Even polarized sidelobes sitting on the ground will affect the measured on-axis polarization via two possible mechanisms: (a) the ground’s thermal radio emission is linearly polarized and (b) unpolarized off-axis Galactic emission will reflect off the ground, becoming polarized in the process. Spectropolarimetric studies of the 21-cm line can also be affected since the diffuse Galactic 21-cm line emission covers the entire sky: this emission can reflect off the ground (becoming polarized in the process) and be picked up by sidelobes sitting on the ground.

It might seem obvious that offset reflector telescopes with unblocked apertures have no (or at least much reduced) distant sidelobes, and therefore remove the complications just described. However, spillover is unrelated to aperture blockage, and if an unblocked aperture is overilluminated, producing spillover (around the primary or subreflectors), complications remain.

5.4. Putting It All Together: The Full-Stokes Off-Axis Response of the Arecibo Telescope

The Arecibo telescope is a very complicated system: it has a 305-m spherical primary reflector with shaped secondary and tertiary Gregorian reflectors located in a focus cabin mounted on an azimuth arm. The cabin travels along a track on the arm allowing for the beam to be pointed in zenith angle (ZA), and the arm swings 360° in azimuth. The azimuth arm and focus cabin are suspended from a large multistory triangular platform that is itself suspended via cables from three towers positioned around the primary’s perimeter. The platform and azimuth arm block

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1. The authors haven’t yet gleaned the phenomenological reason for the sign flip through beam center, but they take great comfort in seeing this empirically measured feature borne out by electromagnetic simulations.

2. Another significant cause of polarized sidelobes involves the spillover of the feed response around the reflector or subreflector that it illuminates; depending on the geometry and orientation of the telescope, the spillover sidelobe can end up positioned on the ground or the sky.

3. Note that the GBT L-band feed was designed with too shallow a taper, such that a significant 20° diameter spillover sidelobe exists around the secondary with its center offset from the main beam by 40°. At certain local sidereal times, 21-cm emission from the plane of the Milky Way can align with this spillover lobe and cause the on-axis response to change. Ref. 38 showed that the instrumental polarization due to this spillover cannot be easily parametrized for the GBT, so that the advantages of the unblocked aperture are completely ruined for studies of 21-cm emission Zeeman splitting.
Fig. 7. Arecibo beam maps at 1175 MHz for all four Stokes parameters, normalized to the peak main-beam Stokes I response (adapted from Ref. 36). Azimuth direction is horizontal, zenith angle (ZA) direction is vertical. Each map is $19'' \times 19''$. For Stokes I (top left) the grayscale covers 0–100% of the peak Stokes I (white to black), solid white contours cover (40%, 50%, ..., 90%), solid black cover (10%, 20%, 30%), dashed black (1%, 2%, ..., 9%). For Stokes Q (top right) and U (bottom right) the grayscale covers (white to black) $\pm 2.8\%$ of the peak Stokes I; thus, black is $+2.8\%$, white is $-2.8\%$, and gray is 0%. For V (bottom right) the total range is $\pm 1.6\%$. Contours are spaced by 0.4% for Q and U, 0.2% for V (with the 0% contour omitted for all); white contours are negative, black positive. The feed is native linear with probes at 45° with respect to the azimuth and ZA directions.

$\sim 5$–15% of the aperture. Despite these complexities, a team set out to map and parametrize the polarized beam patterns of the telescope at 1175 MHz by driving the main beam across the unpolarized continuum source PKS B1749+096 [35,36]. Figure 7 shows the resulting fractional Stokes beam patterns; the azimuth direction is along the horizontal and the ZA direction is along the vertical.
The far-out polarization response of this telescope—especially given the incredibly complicated structure of the suspended platform and the shaped subreflectors—is likely beyond the reach of accurate modelling via software such as GRASP. However, remarkably, some of the fundamental instrumental polarization features that we described for a primary focus-fed circularly symmetric paraboloid in Sec. 5.1–Sec. 5.2 are clearly seen for Arecibo:

1. We saw in Sec. 5.2 that a displacement of the feed from the center of symmetry of the primary reflector will induce a beam squint in the Stokes $V$ pattern. At Arecibo, any feed at the tertiary focus will always be displaced along the azimuth arm, which points along the ZA direction. The beam squint lobes therefore ought to be aligned with the azimuth direction. This is exactly what is seen in Fig. 7.

2. The expected beam squash cloverleaf pattern is seen in the Stokes $Q$ and $U$ response. The Stokes $Q$ pattern shows the expected reversal of sign in the first sidelobe. At the time these polarized beams were measured, the “old” $L$-band-wide feed was aligned with probes at $45^\circ$ to the azimuth and ZA directions (since then, the “new” feed has replaced the “old” one and is aligned at $0^\circ$). In a simple primary focus circularly symmetric paraboloidal reflector system, the Stokes $Q$ squash pattern for this feed orientation would have its lobes aligned at $45^\circ$ to the (Az, ZA) directions and the Stokes $U$ pattern would be aligned with (Az, ZA). Neither is quite the case, and the $Q$ and $U$ patterns are certainly not offset from one another by the expected $45^\circ$.

3. The Stokes $I$ beam is highly elliptical (by design) and shows a significant coma lobe to the left of the main beam. The first sidelobe response is extreme on the coma side of the main beam and the Stokes $U$ pattern shows significant instrumental linear polarization response in this coma-side sidelobe response.

6. Polarization Conventions

The history of polarization studies is fraught with confusion that arises because of conventions. As early as 1896, Pieter Zeeman, in discovering his eponymous effect, measured the charge of (what would turn out to be) the electron to be positive! Why? Because he had used a mislabelled quarter-wave plate and therefore swapped his sense of circulars.

We’ll say it now, and we’ll say it again: When presenting polarization results, you must state your conventions.

6.1. Linear Polarization

There are two linear polarization conventions defined by the IAU (1) the polarization angle $\chi$ is zero at north; and (2) $\chi$ is measured east of north. Thus, when

\[\chi\]See Ref. for a detailed discussion of these patterns.
represented on an image of the sky, a line segment representing polarization rotates counterclockwise as \( \chi \) increases, and \( \chi = 0^\circ \) corresponds to a vertical orientation.

In December 2015, the IAU sent an open letter to the astronomical community pointing out that researchers studying the polarization of the Cosmic Microwave Background (CMB) have been defining polarization angle to increase clockwise on the sky. This effectively swaps the sign of Stokes \( U \) and causes confusion for astronomers studying Galactic polarization using CMB satellite data.

6.2. Circular Polarization

If you’re interested in studying circular polarization, there are a few things you really need to worry about. The most important things to be aware of are:

(1) Radio astronomers use the IEEE convention for the sense of circular polarization\(^{15}\) (which has been around since 1942) and have been doing so at least since Pawsey & Bracewell’s 1955 seminal textbook\(^{16}\) on the subject. Stick both your thumbs along the direction of propagation: whichever hand has its fingers wrapped in the direction that the electric field is rotating with time defines the handedness of the polarization sense. To wit, if radiation is incoming, then stick both your thumbs towards you. If the electric field is rotating counterclockwise around the direction of propagation—your thumb—then your right hand describes the circular polarization state of IEEE RCP. The IEEE logo even has a drawing of the right-hand rule, in case you ever forget which sense is RCP. This is opposite to the definition used by physicists and optical astronomers.

(2) That last point leads to a serious problem: how should astronomers define Stokes \( V \) if optical and radio observers are using different definitions? A working group chaired by Gart Westerhout tried to tackle this problem at the 1973 IAU meeting in Sydney\(^{21}\) by establishing an IAU definition for Stokes \( V \) to be IEEE RCP minus IEEE LCP. Unfortunately, that definition just didn’t stick—not even among radio astronomers. This is likely because by 1974, the opposite convention was firmly established in many fundamental radio astronomy references. When Cohen introduced Stokes parameters to radio astronomers in 1958\(^{47}\) he had defined \( V \) as IEEE LCP − RCP. Kraus’s \textit{Radio Astronomy}\(^{48}\)—“the bible” for many generations of radio astronomers—had also defined \( V \) as IEEE LCP − RCP in 1966 (and again in the 1986 2nd edition).

Seemingly all pulsar observers (as well as Heiles and his Zeeman effect collaborators), unaware of the IAU definition, have used the Kraus LCP − RCP definition for decades. The pulsar crowd have further muddied the situation by ac-

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\(^{a}\)IAU Commissions 25 and 40 resolved to align the horizontal and vertical axes of the Stokes parameter reference frame along the Declination and Right Ascension axes, respectively. This might seem somewhat paradoxical as we tend to think of Declination as the vertical equatorial axis, but the choice sensibly retains a right-handed coordinate system for which \( \chi = 0^\circ \) and \( Q/I = +1 \) for completely linearly polarized radiation aligned with the Declination axis.

\(^{b}\)The immense confusion encountered in dealing with circular polarization and Stokes \( V \) definitions has been outlined at length over the last two decades\(^{15}\)\(^{16}\).
knowing the discrepancy and—rather than adopting the IAU conventions—
introducing a special pulsar Stokes $V$ convention that is defined oppositely from
the IAU definition; this is implemented in their software and data storage definitions.

We collected a sample of 53 radio Zeeman papers and found: 71% failed to
state whether they were using IEEE circular conventions, but we can give them
the benefit of the doubt; 57% failed to define their Stokes $V$ convention; in the
cases where the Stokes $V$ convention is defined or can be clearly inferred, 56%
used the IAU definition.

(3) The sense of circular polarization reverses upon reflection. For telescopes with
a feed at the primary or tertiary focus (e.g., Parkes, Arecibo, WSRT, GMRT),
the Stokes $V$ measured by the correlator will be the negative of the Stokes $V$
signal incident on the primary surface. For telescopes with a feed at the sec-
secondary focus (e.g., $L$ band at the GBT, Effelsberg, VLA), the sense of Stokes $V$
measured by the correlator will match that of the incoming radiation. This
subtlety was overlooked when Verschuur discovered 21-cm Zeeman splitting
in the Perseus Arm absorption feature towards Cas A using the NRAO 140-ft
(a prime-focus telescope). He plotted Stokes $V$ as IEEE RCP − LCP incident
on the feed and inferred a magnetic field pointing towards the observer; how-
ever, in a follow-up publication he shows the same exact Stokes $V$ spectrum
and labels it as RCP − LCP, but this time as incident on the dish, with a note
added in proof that he had previously assigned an incorrect sign for the derived
magnetic field vector. The clear lesson here is that, in addition to stating the
adopted definition of Stokes $V$, one must state what one’s Stokes $V$ spectrum
represents—the difference in circular polarization incident on the dish or inci-
dent on the feed. The authors suggest that presenting Stokes $V$ incident on
the primary dish is the sensible choice: this represents the circular polarization
state of the astronomical signal and removes the onus of tracking reflections
from the reader.

(4) The sense of circular polarization must be calibrated in order to tie the sign of
the pseudo-Stokes correlator output $S_{\text{src},3}^{\text{cor}}$ to IEEE RCP or LCP. The incoming
astronomical Stokes $V$ signal must be positive for IEEE RCP, so if an astro-
nomical source emits a signal with net RCP and produces $S_{\text{src},3}^{\text{cor}} < 0$, then the
sign of the correlator output must be corrected.

6.3. Magnetic Field Direction

There is a further conventional complication when comparing the direction of the
line-of-sight component of magnetic fields in interstellar space that have been mea-
sured by means of Zeeman splitting and Faraday rotation. Zeeman observers have
always taken positive $\mathbf{B}$ to point away from the observer, analogous to Doppler

$^a$A helical antenna of known circular polarization sense can also be broadcast directly into the
feed.
velocity, but Manchester changed the convention in 1972 for Faraday rotation enthusiasts, who take positive B to point towards the observer in order to match with the convention that rotation measures are positive when the field points towards the observer.

6.4. A Factor of Two in the Stokes Parameters

Some observatories (e.g., the VLA) define Stokes I as the straight average of the autocorrelations in orthogonal feed responses rather than their sum. So if one were observing a continuum source producing a flux density of 30 mJy in the AA output and 30 mJy in the BB output, the reported Stokes I value would also have a flux density of 30 mJy. This does not conform to the convention for the Stokes parameters. Stokes I is defined as the sum of the orthogonal outputs and should have a value of 60 mJy in the above example. The AIPS and CASA software packages divide all the Stokes parameters by 2. At least they’re consistent: the fractional polarization of a source should be the same whether using the AIPS/CASA convention or the proper Stokes convention. But the intensities of the Stokes parameters themselves will be half those of the proper convention, so if comparing fluxes between two telescopes, one needs to know what conventions were used to create Stokes I. The sheer momentum of this usage means that it will never be changed, so one must keep this in mind.

Given the muddled history of polarization and magnetic field conventions over the last 50 years, there appears little chance that any single set of conventions (even those resolved by the IAU) will be adopted by all radio observers. The only possible way that we can reconcile different polarimetric observations is for you, the observer, to state your conventions when presenting results!

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