SOLUTIONS TO CERTAIN LINEAR EQUATIONS IN PIATETSKY-SHAPIRO SEQUENCES

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Abstract. Denote by $\text{PS}(\alpha)$ the image of the Piatetski-Shapiro sequence $n \mapsto \lfloor n^\alpha \rfloor$, where $\alpha > 1$ is non-integral and $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. We partially answer the question of which bivariate linear equations have infinitely many solutions in $\text{PS}(\alpha)$: if $a, b \in \mathbb{R}$ are such that the equation $y = ax + b$ has infinitely many solutions in the positive integers, then for Lebesgue-a.e. $\alpha > 1$, it has infinitely many or at most finitely many solutions in $\text{PS}(\alpha)$ according as $\alpha < 2$ (and $0 \leq b < a$) or $\alpha > 2$ (and $(a, b) \neq (1, 0)$). We collect a number of interesting open questions related to further results along these lines.

1. Introduction

A Piatetski-Shapiro sequence is a sequence of the form $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ for non-integral $\alpha > 1$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ and $\mathbb{N}$ is the set of positive integers. Denote by $\text{PS}(\alpha)$ the image of $n \mapsto \lfloor n^\alpha \rfloor$. We will say that the linear equation

$$y = ax + b, \quad a, b \in \mathbb{R} \quad (1)$$

is solvable in $\text{PS}(\alpha)$ if there are infinitely many distinct pairs $(x, y) \in \text{PS}(\alpha) \times \text{PS}(\alpha)$ satisfying $(1)$, and unsolvable otherwise. This terminology extends as expected to solving equations and systems of equations in other subsets of $\mathbb{N}$.

Theorem 1. Suppose that $(1)$ is solvable in $\mathbb{N}$. For Lebesgue-a.e. $\alpha > 1$,

i. if $\alpha < 2$ and $0 \leq b < a$, then $(1)$ is solvable in $\text{PS}(\alpha)$;
ii. if $\alpha > 2$ and $(a, b) \neq (1, 0)$, then $(1)$ is unsolvable in $\text{PS}(\alpha)$.

Piatetski-Shapiro sequences get their name from Ilya Piatetski-Shapiro, who proved a Prime Number Theorem for $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ for all $1 < \alpha < 12/11$; see [13]. Similar results regarding the distribution of $(\lfloor n^\alpha \rfloor)_{n \in \mathbb{N}}$ in arithmetic progressions and the square-free numbers hold for various ranges of $\alpha$ in both metrical and complete versions; see [1] for recent results in this direction and further references.

The motivation for this work comes from another line of thought. Since $\text{PS}(\alpha)$ is the (rounded) image of $\mathbb{N}$ under the Hardy field function $x \mapsto x^\alpha$, it is known to be a so-called set of multiple recurrence in ergodic theory (see [8]); thus, for example, every $E \subseteq \mathbb{N}$ with $\limsup_{N \to \infty} |E \cap \{1, \ldots, N\}|/N > 0$ contains arbitrarily long arithmetic progressions with step size in $\text{PS}(\alpha)$. That $\text{PS}(\alpha)$ is a set of multiple recurrence follows from it containing “many divisible polynomial patterns” (see [5],

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1Any subfield of the ring of germs at $+\infty$ of continuous real-valued functions on $\mathbb{R}$ which is closed under differentiation is a Hardy field, and its members are Hardy field functions.
Section 5); in particular, when $1 < \alpha < 2$, the set $\text{PS}(\alpha)$ contains arbitrarily long arithmetic progressions and arithmetic progressions of every sufficiently large step.

Another class of sets which enjoy strong recurrence properties are those which possess IP-structure. A finite sums set in $\mathbb{N}$ is a set of the form

$$\text{FS}\left((x_i)_{i=1}^n\right) = \left\{ \sum_{i \in I} x_i \mid I \subseteq \{1, \ldots, n\}, \ I \neq \emptyset \right\},$$

where $(x_i)_{i=1}^n \subseteq \mathbb{N}$, and a set $A \subseteq \mathbb{N}$ is called IP$_0$ if it contains arbitrarily large finite sums sets. Finite sums sets define "linear" IP-structure; we define a higher order analogue, VIP-structure, in Section 5. Sets with VIP-structure are known to be sets of multiple recurrence in both topological and measure-theoretic dynamical systems; see, for example, [3, 5, 6, 9].

It would be interesting, therefore, to identify IP-structure in sequences arising from Hardy field functions, and the Piatetski-Shapiro sequences provide an ideal first candidate in this search: they are easily described and already known to form sets of multiple recurrence. We elaborate on this further in Section 5.

In attempting to find IP-structure, a more basic question arises that does not seem to have been addressed in the literature: which linear equations are solvable in $\text{PS}(\alpha)$? In some cases, this question can already be easily answered. For example, for all $1 < \alpha < 2$, because $\text{PS}(\alpha)$ contains arbitrarily long arithmetic progressions, it contains infinitely many solutions to balanced, homogeneous linear equations: $a \cdot x = 0$ where $\sum a_i = 0$. Because $\text{PS}(\alpha)$ contains progressions of every sufficiently large step, it contains solutions to linear equations such as $x + y = z$, as well.

Whether other simple linear equations, such as $y = 2x$, are solvable in $\text{PS}(\alpha)$ does not follow from the aforementioned results. Theorem 1 serves as a partial answer to the question of which bivariate linear equations are solvable in $\text{PS}(\alpha)$. As a corollary, we find sets of the form $\text{FS}\left((x_i)_{i=1}^3\right)$ in $\text{PS}(\alpha)$ for Lebesgue-a.e. $1 < \alpha < 2$; this is a famous open problem in the set of squares, $\text{PS}(2)$.

**Notation.** For $x \in \mathbb{R}$, denote the distance to the nearest integer by $\|x\|$, the fractional part by $\{x\}$, the integer part (or floor) by $\lfloor x \rfloor$, and the ceiling by $\lceil x \rceil := - \lfloor -x \rfloor$. Denote the Lebesgue measure on $\mathbb{R}$ by $\lambda$, and denote the set of those points belonging to infinitely many of the sets in the sequence $(E_n)_{n \in \mathbb{N}}$ by $\limsup_{n \to \infty} E_n$. Given two positive-valued functions $f$ and $g$, we write $f \ll a_1, \ldots, a_k$ or $g \gg a_1, \ldots, a_k$ if there exists a constant $K > 0$ depending only on the quantities $a_1, \ldots, a_k$ for which $f(x) \leq Kg(x)$ for all $x$ in the domain common to both $f$ and $g$.

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## 2. Reduction to Diophantine approximation

We prove Theorem 1 by reducing it to the following theorem in Diophantine approximation.

**Theorem 2.** Let $I \subseteq [0,1)$ be a set with non-empty interior, $a, c > 0$, $a \neq 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$. For Lebesgue-a.e. $\alpha > 1$, the system

$$\begin{cases} \left\| a^{1/\alpha} n \right\| \leq \frac{c}{n^{\alpha-1}} \\ \{\gamma n^\alpha \} \in I \end{cases}$$

(3)
is solvable or unsolvable in \( N \) according as \( \alpha < 2 \) or \( \alpha > 2 \).

This theorem can be seen as “twisted” Diophantine approximation. Indeed, when \( \alpha - 1 < 1 \), the first inequality in (3) is solvable in \( N \) by Dirichlet’s Theorem; more concretely, any sufficiently large denominator of a continued fraction convergent of \( a^{1/\alpha} \) will yield a solution. The second condition in (3) provides the twist.

Proof of Theorem 1 assuming Theorem 2. Note that (1) is solvable in \( N \) if and only if

\[
a, b \in \mathbb{Q}, \; a = \frac{a_1}{a_2}, \; a_1, a_2 \in \mathbb{N}, \; (a_1, a_2) = 1, \; \text{and} \; a_2 b \in \mathbb{Z}.
\]

Combined with the fact that consecutive differences in \( PS(\alpha) \) tend to infinity, the theorem holds when \( a = 1 \). Assume \( a \neq 1 \). Note that there exists an integer \( 0 \leq d \leq a_2 - 1 \) for which

\[
a \lfloor n^\alpha \rfloor + b \in \mathbb{Z} \iff \lfloor n^\alpha \rfloor = d \pmod{a_2} \iff \left\{ \frac{n^\alpha}{a_2} \right\} \in \left[ \frac{d}{a_2}, \frac{d + 1}{a_2} \right).
\]

It follows that

\[
a \lfloor n^\alpha \rfloor + b \in \text{PS}(\alpha) \iff \exists \; k \in \mathbb{N}, \; a \lfloor n^\alpha \rfloor + b = \lfloor k^\alpha \rfloor \iff \left\{ \frac{n^\alpha}{a_2} \right\} \in \left[ \frac{d}{a_2}, \frac{d + 1}{a_2} \right) \text{ and } J_n \cap \mathbb{N} \neq \emptyset,
\]

where, by the Mean Value Theorem,

\[
J_n = (a \lfloor n^\alpha \rfloor + b)^{1/\alpha}, \; (a \lfloor n^\alpha \rfloor + b + 1)^{1/\alpha} = a^{1/\alpha} n + [L_n, R_n],
\]

\[
L_n = -\frac{a}{\alpha} \left( \{n^\alpha\} - \frac{b}{a} \right) l_n^{-1+1/\alpha}, \; l_n \text{ between } an^\alpha \text{ and } a \lfloor n^\alpha \rfloor + b,
\]

\[
R_n = \frac{a}{\alpha} \left( \frac{b + 1}{a} - \{n^\alpha\} \right) r_n^{-1+1/\alpha}, \; r_n \text{ between } an^\alpha \text{ and } a \lfloor n^\alpha \rfloor + b + 1.
\]

Note that \( J_n, L_n, R_n, l_n, \) and \( r_n \) all depend on \( \alpha \). This shows so far that (1) is solvable in \( \text{PS}(\alpha) \) if and only if the system in (1) is solvable in \( N \).

We proceed by showing that solutions to (3) yield solutions to (4) and vice versa when \( I, c, \) and \( \gamma \) are chosen appropriately. To this end, for \( i = 1, 2 \), let

\[
A = \{ \alpha > 1 \mid (1) \text{ is solvable in } \mathbb{N} \},
\]

\[
B_i = \{ \alpha > 1 \mid (3) \text{ is solvable in } \mathbb{N} \text{ for } I_i, c_i, \gamma_i, a \}.
\]

To prove Theorem 1, it suffices by Theorem 2 to find \( I_i, c_i, \gamma_i, i = 1, 2 \), for which

\[
B_1 \cap (1, 2) \subseteq A \subseteq B_2.
\]

We begin with the first containment in (5), which we will show under the assumption that \( 0 \leq b < a \). Let \( I_0 \) be the middle third sub-interval of the interval \( (b/a, \min(1, (b + 1)/a)) \). Let \( I_1 = d/a_2 + I_0/a_2, \; \gamma_1 = 1/a_2, \) and \( c_1 \) be a constant depending only on \( a \) and \( b \) to be specified momentarily.
Suppose \( \alpha \in B_1 \cap (1, 2) \) and that \( n \) is a solution to (3); we will show that if \( n \) is sufficiently large, then it solves the system in (4). By (3),
\[
\left\{ \frac{n^\alpha}{a_2} \right\} = \left\{ \gamma_1 n^\alpha \right\} \in I_1 \subseteq \left[ \frac{d}{a_2}, \frac{d+1}{a_2} \right).
\]
This also implies that \( \left\{ n^\alpha \right\} \in I_0 \), so
\[
\left\{ n^\alpha \right\} - \frac{b}{a} \gg_{a,b} 1, \quad \frac{b+1}{a} - \left\{ n^\alpha \right\} \gg_{a,b} 1.
\]
Combining these estimates with the facts that \( \alpha \in (1, 2) \) and, for \( n \) sufficiently large, \( a \left\{ n^\alpha \right\} + b + 1 \leq 2an^\alpha \), we get
\[
-L_n = \frac{a}{\alpha} \left( \left\{ n^\alpha \right\} - \frac{b}{a} \right) n_{1+1/\alpha} >_{a,b} 1\text{,} \quad R_n = \frac{a}{\alpha} \left( \frac{b+1}{a} - \left\{ n^\alpha \right\} \right) n_{1+1/\alpha} >_{a,b} 1\text{.}
\]
Set \( c_1 \) to be half the minimum of the constants implicit in these two expressions. It follows that \( J_n \) contains an open interval centered at \( a^{1/\alpha} n \) of length \( 2c_1/n^{\alpha - 1} \). By (3), \( \| a^{1/\alpha} n \| \leq c_1/n^{\alpha - 1} \), so the interval \( J_n \) contains the nearest integer to \( a^{1/\alpha} n \); in particular, \( J_n \cap \mathbb{N} \neq \emptyset \), so \( n \) solves (4).

The second containment in (5) is handled similarly. Let \( I_2 = [0,1) \), \( \gamma_2 = 1 \), and \( c_2 \) be a constant depending only on \( a \) to be specified momentarily. Suppose that \( \alpha \in A \) and \( n \) solves (4); we will show that \( n \) satisfies (3). The second condition in (3) is satisfied automatically by our choice of \( I_2 \). For \( n \) sufficiently large, \( a \left\{ n^\alpha \right\} + b \geq an^\alpha/2 \), whereby \( |L_n|, |R_n| \leq c_2/2n^{\alpha - 1} \), where \( c_2 \) is chosen (depending only on \( a \)) to satisfy both inequalities. Since \( J_n \) contains an integer, it must be that \( \| a^{1/\alpha} n \| \leq c_2/n^{\alpha - 1} \), meaning \( n \) satisfies (3).

\[\square\]

3. PROOF OF THEOREM 2

To prove Theorem 2 we first change variables under \( t_\alpha(x) = (\log_a x)^{-1} \) to arrive at the equivalent Theorem 3. Proof of the equivalence of these two theorems is a routine exercise using the fact that \( t_\alpha \) is measure-theoretically non-singular. The lemmas used in the proof of the following theorem may be found in Section 4.

**Theorem 3.** Let \( I \subseteq [0,1) \) be a set with non-empty interior, \( a, c > 0 \), \( a \neq 1 \), and \( \gamma \in \mathbb{R} \setminus \{0\} \). If \( a < 1 \), then for Lebesgue-a.e. \( a < \theta < 1 \), the system
\[
\left\{ \begin{align*}
\| \theta n \| &\leq \frac{c}{n^{\theta(a)-1}} \\
\gamma n^{t_\alpha(\theta)} &\in I
\end{align*} \right.
\]
is solvable or unsolvable in \( \mathbb{N} \) according as \( \theta < \sqrt{a} \) or \( \theta > \sqrt{a} \). If \( a > 1 \), then for Lebesgue-a.e. \( 1 < \theta < a \), the system is solvable or unsolvable in \( \mathbb{N} \) according as \( \theta > \sqrt{a} \) or \( \theta < \sqrt{a} \).

**Proof.** Fix \( I, a, c, \) and \( \gamma \). Suppose \( a < 1 \); the case \( a > 1 \) follows from the proof below with the obvious modifications. Without loss of generality, we may assume that \( I \) is an interval with non-empty interior. For brevity, we will suppress dependence on \( I, a, c, \) and \( \gamma \) in the asymptotic notation appearing in the proof.
Let $\Theta \subseteq (a, 1)$ be the set of those $\theta$ satisfying the conclusion of the theorem. We will show that $\Theta$ is of full Lebesgue measure by showing that it has full measure in the intervals $(a, \sqrt{a})$ and $(\sqrt{a}, 1)$.

To show that $\Theta \cap (a, \sqrt{a})$ is of full measure, it suffices by Lemma 1 to show that there exists a $\delta > 0$ such that for all $a < \theta_1 < \theta_2 < \sqrt{a}$,

$$\lambda(\Theta \cap (\theta_1, \theta_2)) \geq \delta(\theta_2 - \theta_1).$$

(7)

To this end, fix $a < \theta_1 < \theta_2 < \sqrt{a}$. In what follows, the phrase “for all sufficiently large $n$” means “for all $n \geq n_0$,” where $n_0 \in \mathbb{N}$ may depend on any of the parameters introduced so far, including $\theta_1$ and $\theta_2$.

For $n \in \mathbb{N}$, define

$$E_n = \{ \theta \in (\theta_1, \theta_2) \mid \|\theta n\| \leq \psi(n) \}, \quad \psi(n) = \frac{1}{n},$$

$$F_n = \{ \theta \in (\theta_1, \theta_2) \mid \{ \gamma n^{t_a(\theta)} \} \in I \}, \quad G_n = E_n \cap F_n.$$ 

For $\theta \in (\theta_1, \theta_2)$, $t_a(\theta) < t_a(\theta_2) < 2$, so for sufficiently large $n$, we have $\psi(n) \leq c/n^{t_a(\theta)-1}$. It follows that $\limsup_{n \to \infty} G_n \subseteq \Theta \cap (\theta_1, \theta_2)$. Therefore, in order to show (7), it suffices to prove that there exists a $\delta > 0$, independent of $\theta_1, \theta_2$, for which

$$\lambda \left( \limsup_{p \to \infty} G_p \right) \geq \delta(\theta_2 - \theta_1).$$

(8)

Passing to primes here makes parts of the argument technically easier. To ease notation, any sum indexed over $p$ or $q$ will be understood to be a sum over prime numbers.

To prove (8), it suffices by Lemma 2 to prove that

$$\sum_{p=2}^{\infty} \lambda(G_p) = \infty$$

and that there exists a $\delta > 0$, independent of $\theta_1, \theta_2$ for which

$$\limsup_{N \to \infty} \left( \sum_{p=2}^{N} \lambda(G_p) \right)^2 \left( \sum_{p,q=2}^{N} \lambda(G_p \cap G_q) \right)^{-1} \geq \delta(\theta_2 - \theta_1).$$

(10)

First we show (10) using Lemma 3. Fix $0 < \eta < \min(\theta_1, 1 - \theta_2, (\theta_2 - \theta_1)/3)$. For $n \in \mathbb{N}$, let

$$S_n = \{ m \in \mathbb{Z} \mid \theta_1 + \eta < m/n < \theta_2 - \eta \},$$

$$T_n = \{ m \in \mathbb{Z} \mid \theta_1 - \eta < m/n < \theta_2 + \eta \},$$

and note that for $n$ sufficiently large,

$$(\theta_2 - \theta_1)n \ll |S_n| < |T_n| \ll (\theta_2 - \theta_1)n.$$ 

(12)

For $n$ sufficiently large, the set $E_n$ may be approximated by a disjoint union of intervals:

$$\bigcup_{m \in S_n} E_{n,m} \subseteq E_n \subseteq \bigcup_{m \in T_n} E_{n,m},$$

where

$$E_{n,m} = \left[ \frac{m}{n} - \frac{\psi(n)}{n}, \frac{m}{n} + \frac{\psi(n)}{n} \right].$$
Let $I_0 \subseteq I$ be the middle third sub-interval of $I$.

**Claim 1.** For $n$ sufficiently large and $m \in S_n$, if $\{\gamma n^{t_a(m/n)}\} \in I_0$, then $E_{n,m} \subseteq F_n$.

**Proof.** Note first that $|t_a'(x)| \ll 1$ for $x \in (a, \sqrt{a})$. For $\theta \in E_{n,m}$, by applying the Mean Value Theorem twice, we see that for some $\xi_1$ between $t_a(\theta)$ and $t_a(m/n)$ and some $\xi_2$ between $\theta$ and $m/n$,

$$\left| \gamma n^{t_a(\theta)} - \gamma n^{t_a(m/n)} \right| = |\gamma| |t_a(\theta) - t_a(m/n)| \log n n^{\xi_1} \leq |\gamma| \left| \theta - \frac{m}{n} \right| t_a'(\xi_2) \log n n^{t_a(\theta_2)} \ll \psi(n) \log n n^{t_a(\theta_2)} \leq \frac{\log n}{n^{2-t_a(\theta_2)}}.$$

Since $2 - t_a(\theta_2) > 0$, for $n$ sufficiently large,

$$\left| \gamma n^{t_a(\theta)} - \gamma n^{t_a(m/n)} \right| < \frac{\lambda(I)}{3}.$$

Therefore, if $\{\gamma n^{t_a(m/n)}\} \in I_0$, then for all $\theta \in E_{n,m}$, $\{\gamma n^{t_a(\theta)}\} \in I$, i.e. $E_{n,m} \subseteq F_n$. □

By the equidistribution result in Lemma 4 for $n$ sufficiently large,

$$\left| \left\{ m \in S_n \mid \{\gamma n^{t_a(m/n)}\} \in I_0 \right\} \right| \geq \frac{\lambda(I_0)}{2}.$$

Combining this with Claim 1 and the bounds on $\eta$, there are $\gg (\theta_2 - \theta_1)n$ integers $m \in S_n$ for which $E_{n,m} \subseteq F_n$. It follows that for $n$ sufficiently large,

$$\lambda(G_n) \gg (\theta_2 - \theta_1)n \frac{\psi(n)}{n} = (\theta_2 - \theta_1) \frac{1}{n}. \quad (13)$$

This proves (10).

Now we show (11) by estimating the “overlaps” between the $G_p$’s. It suffices to prove that there exists a constant $K$ (which may depend on any of the parameters introduced so far) such that for all sufficiently large primes $p$ and for all $N \geq p$,

$$\sum_{q > p}^N \lambda(E_x \cap E_y) \ll (\theta_2 - \theta_1) \sum_{q > p}^N \psi(p)\psi(q) + K\psi(p). \quad (14)$$

Indeed, suppose (13) and (14) both hold for all primes $p$ greater than some sufficiently large $p_0 \in \mathbb{N}$. Using the trivial bound $\lambda(G_p \cap G_q) \leq \lambda(E_x \cap E_y)$, it follows
that
\[
\sum_{p, q=2}^{N} \lambda(G_p \cap G_q) \leq 2 \left( \sum_{p \geq p_0, q > p}^{N} \lambda(G_p \cap G_q) + \sum_{p < p_0, q > p}^{N} \lambda(G_q) \right) + \sum_{q=2}^{N} \lambda(G_q)
\]
\[
\ll (\theta_2 - \theta_1) \sum_{p \geq p_0, q > p}^{N} \psi(p) \psi(q) + K \sum_{p \geq p_0}^{N} \psi(p) + \sum_{p < p_0, q > p}^{N} \lambda(G_q)
\]
\[
\ll \frac{1}{\theta_2 - \theta_1} \sum_{p, q=2}^{N} \lambda(G_p) \lambda(G_q) + p_0 \left( \frac{K}{\theta_2 - \theta_1} + 1 \right) \sum_{q=2}^{N} \lambda(G_q),
\]
where the last line follows from (13). This combines with (9) to yield (10).

To show (14), note that the set \( E_p \) is covered by a union of intervals \( E_{p, r} \), each of length \( 2 \psi(p)/p \). If \( p < q \) and \( E_{p, r} \cap E_{q, s} \neq \emptyset \), then by estimating the distance between the midpoints of the intervals,

\[
|sp - rq| < pq 3 \max \left( \frac{\psi(p)}{p}, \frac{\psi(q)}{q} \right) = 3q \psi(p),
\]

\[\lambda(E_{p, r} \cap E_{q, s}) \leq \min \left( 2 \frac{\psi(p)}{p}, 2 \frac{\psi(q)}{q} \right) = 2 \frac{\psi(q)}{q}.
\]

The left hand side of (14) is then

\[
\sum_{q > p} \lambda(E_p \cap E_q) = \sum_{q > p} \sum_{r \in T_p}^{q} \lambda(E_{p, r} \cap E_{q, s}) \ll \sum_{q > p} \frac{\psi(q)}{q} \sum_{r \in T_p, s \in T_q}^{q} 1.
\]

Now (14) will follow by partitioning the range of the sum on \( q \) and applying Lemma 3. Indeed, the right hand side of the previous expression is equal to

\[
\sum_{\ell=0}^{\infty} \sum_{2^\ell p \leq q \leq 2^{\ell+1} p}^{q \leq N} \psi(q) \sum_{r \in T_p, s \in T_q} \frac{1}{|sp - rq| < 3q \psi(p)} \leq \sum_{\ell=0}^{\infty} \sum_{2^\ell p}^{Q_\ell} \frac{\psi(2^\ell p)}{2^\ell p} \sum_{r \in T_p, s \in T_q} \frac{1}{|sp - rq| < L_\ell}
\]

where \( L_\ell = 3 \cdot 2^{\ell+1} p \psi(p) \) and \( Q_\ell = 2^\ell p \). For each \( \ell \), we apply Lemma 3 with \( N \\nand \( p \) as they are, \( Q_\ell \) as \( Q \), \( L_\ell \) as \( L \), and \( (\theta_1 - \eta, \theta_2 + \eta) \) as \( (\eta_1, \eta_2) \) : there exists a \( K > 0 \) depending only on \( \theta_1, \theta_2 \) (since \( \eta \) depends only on \( \theta_1, \theta_2 \)) such that the
right hand side of the previous expression is

\[
\ll \sum_{\ell=0}^{\infty} \frac{\psi(2^\ell p)}{2^\ell p} \left( (\theta_2 - \theta_1 + 2\eta)L_{\ell} \sum_{q \leq N} 1 + K 2^\ell p \right) \\
\ll (\theta_2 - \theta_1) \sum_{\ell=0}^{\infty} \sum_{q < 2^{\ell+1} p} \psi(p)\psi(2^\ell p) + K \sum_{\ell=0}^{\infty} \psi(2^\ell p) \\
\leq (\theta_2 - \theta_1) \sum_{\ell=0}^{\infty} \sum_{q \leq N} \psi(p)\psi(1/2)\psi(q) + K \psi(p) \sum_{\ell=0}^{\infty} \psi(2^\ell) \\
\ll (\theta_2 - \theta_1) \sum_{q > p} N \psi(p)\psi(q) + K \psi(p),
\]

where the third line and fourth lines follow by noting that \(\psi(2^\ell p) \leq \psi(q/2)\), that \(\psi\) is multiplicative, and that \(\sum_{\ell=0}^{\infty} \psi(2^\ell)\) converges. This shows (14), completing the proof of (10).

To show that the set \(\Theta \cap (\sqrt{a}, 1)\) is of full measure, we will show that for all \(\theta_3 > \sqrt{a}\), the set \((\theta_3, 1) \setminus \Theta\) has zero measure. Let

\[
H_n = \left\{ \theta \in (\theta_3, 1) \mid \|\theta n\| \leq \frac{c}{n t_n(\theta_3) - 1} \right\}.
\]

If \(\theta \in (\theta_3, 1) \setminus \Theta\), then for infinitely many \(n \in \mathbb{N}\), \(\|\theta n\| \leq c/n t_n(\theta_3) - 1\). It follows that

\[
(\theta_3, 1) \setminus \Theta \subseteq \limsup_{n \to \infty} H_n.
\]

Since \(H_n\) is a union of \(\ll (1 - \theta_3)n\) intervals, each of length \(2c/n t_n(\theta_3)\), and since \(t_n(\theta_3) > 2\), \(\sum_{n=1}^{\infty} \mu(H_n) < \infty\). By the first Borel-Cantelli Lemma, \(\limsup_{n \to \infty} H_n\) has zero measure, so \((\theta_3, 1) \setminus \Theta\) has zero measure by (15).

4. SUPPORTING LEMMATA

Here we collect some supporting lemmata. Recall that \(\lambda\) denotes the Lebesgue measure on \(\mathbb{R}\) and that \(t_a(x) = (\log_a x)^{-1}\).

**Lemma 1** ([11], Lemma 1.6). Let \(I \subseteq \mathbb{R}\) be an interval and \(A \subseteq I\) be measurable. If there exists a \(\delta > 0\) such that for every sub-interval \(I_0 \subseteq I\), \(\lambda(A \cap I_0) \geq \delta \lambda(I_0)\), then \(A\) is of full measure in \(I\): \(\lambda(I \setminus A) = 0\).

**Lemma 2** ([11], Lemma 2.3). Let \((X, B, \mu)\) be a measure space with \(\mu(X) < \infty\). If \((G_n)_{n \in \mathbb{N}} \subseteq B\) is a sequence of subsets of \(X\) for which \(\sum_{n=1}^{\infty} \mu(G_n) = \infty\), then

\[
\mu \left( \limsup_{n \to \infty} G_n \right) \geq \limsup_{N \to \infty} \left( \sum_{n=1}^{N} \mu(G_n) \right)^2 \left( \sum_{n,m=1}^{N} \mu(G_n \cap G_m) \right)^{-1}.
\]

**Lemma 3.** Let \(N, Q, p \in \mathbb{N}\), \(p\) prime with \(p \leq Q\), \(L > 0\), and \(0 < \eta_1 < \eta_2 < 1\). There exists a constant \(K > 0\) depending only on \(\eta_1, \eta_2\) such that the number of
triples \((q, r, s) \in \mathbb{N}^3\) satisfying
\[ Q < q < \min(2Q, N + 1), \ q \text{ prime, } \frac{r}{p}, \frac{s}{q} \in (\eta_1, \eta_2), \ |sp - rq| < L, \]
is
\[ \ll (\eta_2 - \eta_1)L \sum_{Q < q < \min(2Q, N + 1)} 1 + KQ. \]

**Proof.** This lemma follows immediately from [11], Lemma 6.2, by putting the set of integers \(\mathcal{A}\) to be those primes strictly between \(Q\) and \(\min(2Q, N + 1)\) and taking the worst error \(KQ\).

**Lemma 4.** Let \(a > 0\), \(a \neq 1\), \(\gamma \in \mathbb{R} \setminus \{0\}\), and \(\min(\sqrt{a}, a) < \eta_1 < \eta_2 < \max(\sqrt{a}, a)\). Then
\[
\frac{1}{n(\eta_2 - \eta_1)} \sum_{m \in \mathbb{Z} : m/n \in (\eta_1, \eta_2)} \delta_{\{\gamma n^a(m/n)\}} \longrightarrow \lambda|_{[0,1)} \text{ weakly as } n \to \infty,
\]
where \(\delta_n\) denotes the point mass at \(x \in [0,1)\).

**Proof.** Let \(N_n = |\{m \in \mathbb{Z} \mid m/n \in (\eta_1, \eta_2)\}|\), and note that \(N_n/(n(\eta_2 - \eta_1)) \to 1\) as \(n \to \infty\). For \(n, h \in \mathbb{N}\), let
\[
g_n(x) = \gamma n^{\log_a(x + |\eta_n|)/n} \text{ and } g_n(x) = g_n(x + h) - g_n(x).
\]
In this notation, we must show
\[
(16) \quad \frac{1}{N_n} \sum_{i=1}^{N_n} \delta_{\{g_n(i)\}} \longrightarrow \lambda|_{[0,1)} \text{ weakly as } n \to \infty.
\]
Since \(t_a\) is either increasing \((a < 1)\) or decreasing \((a > 1)\) between \(\sqrt{a}\) and \(a\), we can fix \(\sigma_1, \sigma_2\) such that for all \(x \in (\eta_1, \eta_2)\),
\[
1 < \sigma_1 < t_a(x) < \sigma_2 < 2.
\]
To handle the exponential sum estimates that follow, we will show that there exist positive constants \(C_1\) and \(C_2\) (depending on \(a\) and \(\gamma\)) such that for all \(h \in \mathbb{N}\), all sufficiently large \(n \in \mathbb{N}\), and all \(x \in [1, N_n - h]\),
\[
C_1 h \frac{(\log n)^3}{n^{3-\sigma_2}} \leq |g_{n,h}''(x)| \leq C_2 h \frac{(\log n)^3}{n^{3-\sigma_2}}.
\]
By the Mean Value Theorem, \(g_{n,h}''(x) = h g_{n}''(\xi_x)\) for some \(\xi_x \in (x, x + h)\), so it suffices to show that for all \(h \in \mathbb{N}\), all sufficiently large \(n \in \mathbb{N}\), and all \(x \in [1, N_n]\),
\[
(18) \quad C_1 \frac{(\log n)^3}{n^{3-\sigma_1}} \leq |g_{n}''(x)| \leq C_2 \frac{(\log n)^3}{n^{3-\sigma_2}}.
\]
Writing \(g_{n}''(x)\) explicitly reveals that
\[
g_{n}''(x) = g_{n}(x) \left(\frac{\log n}{n}\right)^3 \left(\frac{x + |\eta_n|}{n}\right)^{-3} \left(t_a \left(\frac{x + |\eta_n|}{n}\right)^6 \frac{r(x)}{\log n}\right),
\]
where \(|r(x)| \ll_a 1\) for \(x \in [1, N_n]\) because \((x + |\eta_n|)/n \in (\eta_1, \eta_2)\). The inequality in (18) follows for \(n\) sufficiently large since \(|\gamma n^{\sigma_1} \leq |g_{n}(x)| \leq |\gamma n^{\sigma_2}|\).
To prove (16), it suffices by Weyl’s Criterion ([12], Chapter 1, Theorem 2.1) to show that for all \( b \in \mathbb{Z} \setminus \{0\} \),

\[
\frac{1}{N_n} \sum_{i=1}^{N_n} e\left(bg_n(i)\right) \longrightarrow 0 \quad \text{as} \quad n \to \infty,
\]

where \( e(x) = e^{2\pi i x} \). By the van der Corput Difference Theorem ([12], Chapter 1, Theorem 3.1) and another application of Weyl’s Criterion, it suffices to prove that for all \( h \in \mathbb{N} \) and for all \( b \in \mathbb{Z} \setminus \{0\} \),

\[
\frac{1}{N_n - h} \sum_{i=1}^{N_n - h} e\left(bg_{n,h}(i)\right) \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\]

An exponential sum estimate ([12], Chapter 1, Theorem 2.7) gives us that

\[
\left| \frac{1}{N_n - h} \sum_{i=1}^{N_n - h} e\left(bg_{n,h}(i)\right) \right| \leq \left( \frac{4}{\sqrt{|b\rho|} + 3} \right),
\]

where \( \rho = C_1 h (\log n)^3 / n^{3 - \sigma_1} \) from (17). By the Mean Value Theorem and the upper bound from (17), we see the right hand side is bounded from above for sufficiently large \( n \) by

\[
\left( |b| C_2 h \frac{(\log n)^3}{n^{3 - \sigma_2}} + \frac{2}{N_n - h} \right) \left( \frac{4n^{(3 - \sigma_1)/2}}{\sqrt{|b|C_1 h (\log n)^3}} + 3 \right) \ll \frac{n^{(3 - \sigma_1)/2}}{n \sqrt{|b|h(\log n)^3}},
\]

where the implicit constant depends on \( a, \gamma, \eta_1, \) and \( \eta_2 \). The limit in (19) follows since \((3 - \sigma_1)/2 < 1\).

\[
\square
\]

5. Remarks and conclusions

Here are a number of natural questions and further directions.

i. Equations of the form \( y = ax - b \) where \( 0 \leq b < 1 \) are handled by Theorem 1 by rearranging, but there are still simple linear equations which the theorem does not handle. For example, is the equation \( y = 2x + 2 \) solvable in \( \text{PS}(\alpha) \) for Lebesgue-a.e. \( 1 < \alpha < 2 \)?

**Question 1.** Does Theorem 1 hold with the assumption \( 0 \leq b < a \) in part i. replaced by \( a \notin \{0,1\} \)?

The answer is likely ‘yes.’ Using the technique above, we need infinitely many \( n \)’s for which the interval \( a^{1/\alpha}n + [L_n, R_n] \) contains an integer, where now \( L_n \) and \( R_n \) are both positive or both negative. Accounting for \( \{n^\alpha\} \) and changing variables, this requires control on the set of \( n \)’s for which \( \{\theta n\} \) falls within a shrinking annulus about 0. These shrinking annuli are not nested, making them difficult to handle with the established theory.

ii. Is there a non-metrical version of Theorem 1?

**Question 2.** Does Theorem 1 hold with “Lebesgue-a.e.” replaced by “all”?

The inequality in (3) is solvable in \( \mathbb{N} \) when \( a^{1/\alpha} \) is irrational by Dirichlet’s Theorem, and the whole system is solvable in \( \mathbb{N} \) when \( a^{1/\alpha} \) is rational since \( \{n^\alpha\} \) is uniformly distributed along arithmetic progressions containing 0. Therefore, exceptional \( \alpha \)’s would only arise because of the second condition, the “twist,” in (3) when \( a^{1/\alpha} \) is irrational.
Here are two thoughts for proving the result for all $1 < \alpha < 2$. The result would be immediate from Theorem 2 if $(n^\alpha)_{n \in \mathbb{N}}$ was known to be equidistributed (or even dense) modulo 1 along denominators of the continued fraction convergents of $a^{1/\alpha}$. Alternatively, perhaps the set $PS(\alpha)$ is sufficiently pseudorandom as to contain solutions to linear equations; see [7] for recent results regarding combinatorial structure in sparse random sets.

iii. Asymptotics are known for the distribution of Piatetski-Shapiro sequences in arithmetic progressions, the square-free numbers, and the primes; it is feasible that analogous asymptotics hold for the number of solutions to linear equations, as well.

**Question 3.** Is it true that for Lebesgue-a.e. or for all $1 < \alpha < 2$,

$$\left\{1 \leq m, n \leq N \mid \lfloor m^\alpha \rfloor = a \lfloor n^\alpha \rfloor + b\right\} \sim_{a,b,\alpha} N^{2-\alpha}?$$

It is not hard to verify that $N^{2-\alpha}$ is the correct asymptotic for $\alpha \leq 1$.

iv. Which systems of linear equations are solvable in $PS(\alpha)$? Consider, for example, the system $y = 2x$, $z = 3x$. Just as is done in Section 2, this can be reduced after a change of variables to the system

\[
\begin{cases}
\|\theta n\| \leq \frac{c}{n^{t_1(\theta)-1}} \\
\|\theta^{\log_3 2} n\| \leq \frac{c}{n^{t_1(\theta)-1}} \\
\{\gamma n^{t_1(\theta)}\} \in I
\end{cases}
\]

This is "twisted" Diophantine approximation on the curve $x \mapsto (x, x^{\log_3 2})$; see [2], Theorem 1. Assuming the twist does not interfere with the approximation, it is conceivable that this system is solvable for Lebesgue-a.e. $1 < \alpha < 3/2$.

v. Solving the system $y = 2x$, $z = 3x$ in $PS(\alpha)$ is the same as finding $FS((x, x, z))$ in $PS(\alpha)$; recall [2]. It is an open problem ([10], D18) to determine whether or not the set of squares contains a set of the form $FS((x_i)^3_{i=1})$. We can use Theorem 1 to solve this problem in almost all $PS(\alpha)$.

**Corollary 1.** For Lebesgue-a.e. $1 < \alpha < 2$, the set $PS(\alpha)$ contains infinitely many sets of the form $FS((x_i)^3_{i=1})$.

**Proof.** By Theorem 1, there are infinitely many $x \in PS(\alpha)$ for which $2x \in PS(\alpha)$. For $x$ sufficiently large (depending on $\alpha$), the set $PS(\alpha)$ contains an arithmetic progression of step $x$ and length 3 ([8], Proposition 5.1). If $z$ starts such a progression, then $FS((x, x, z)) \subseteq PS(\alpha)$.

**Question 4.** (V. Bergelson) Is $PS(\alpha)$ an $IP_0$ set for Lebesgue-a.e. or all $1 < \alpha < 2$?

V. Bergelson remarked that while $PS(\alpha)$ may not always possess "linear structure," it may contain higher order structure. Indeed, the set $PS(m/n)$ contains the set of $m$th powers, and this implies that $PS(m/n)$ is a set of multiple recurrence; see [4].
The set \( A \subseteq \mathbb{N} \) possesses VIP-structure if it contains arbitrarily large sub-
sets of the form
\[
\left\{ f \left( \sum_{i \in I} x_i^{(1)} , \ldots , \sum_{i \in I} x_i^{(k)} \right) \bigg| I \subseteq \{1, \ldots , n\}, I \neq \emptyset \right\},
\]
where \( f \in \mathbb{Z}[z_1, \ldots , z_k] \) has zero constant term and \( (x_i^{(1)})_{i=1}^n , \ldots , (x_i^{(k)})_{i=1}^n \subseteq \mathbb{N} \). Thus, the set of \( m^{\text{th}} \) powers possesses VIP-structure. Note that when \( \deg(f) = 1 \), the set above is a finite sums set. Recall from Section \[1\] that any set with
VIP-structure is a set of multiple recurrence; perhaps VIP-structure in \( \text{PS}(\alpha) \)
gives an alternate explanation of the set’s recurrence properties.

**Question 5.** (V. Bergelson) Does \( \text{PS}(\alpha) \) possess VIP-structure for all \( \alpha > 1 \)?

vi. Theorem \[1\] gives that for many linear equations, \( \alpha = 2 \) is a threshold value for
being solvable or unsolvable in \( \text{PS}(\alpha) \). Do other linear equations have such a
threshold, and can we compute it?

**Question 6.** Does there exist an \( \alpha_S > 1 \) with the property that for Lebesgue-
ea.e. or all \( \alpha > 1 \), the equation \( x + y = z \) is solvable or unsolvable in \( \text{PS}(\alpha) \)
according as \( \alpha < \alpha_S \) or \( \alpha > \alpha_S \)?

As mentioned in Section \[1\] the equation \( x + y = z \) is solvable in \( \text{PS}(\alpha) \)
for all \( \alpha < 2 \); when \( \alpha \geq 3 \) is an integer, the equation is unsolvable in \( \text{PS}(\alpha) \).
What happens for \( \alpha \) just larger than 2? The same question is meaningful and
interesting for more general (systems of) linear equations.

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\[2\] For a more general definition, see \[3, 5\].
SOLUTIONS TO LINEAR EQUATIONS IN PIATELSKI-SHAPIRO SEQUENCES

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