Routes to chaos, universality and glass formation

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Abstract

We review recent results obtained for the dynamics of incipient chaos. These results suggest a common picture underlying the three universal routes to chaos displayed by the prototypical logistic and circle maps. Namely, the period doubling, intermittency, and quasiperiodicity routes. In these situations the dynamical behavior is exactly describable through infinite families of Tsallis’ $q$-exponential functions. Furthermore, the addition of a noise perturbation to the dynamics at the onset of chaos of the logistic map allows to establish parallels with the behavior of supercooled liquids close to glass formation. Specifically, the occurrence of two-step relaxation, aging with its characteristic scaling property, and subdiffusion and arrest is corroborated for such a system.

Key words: Nonlinear dynamics, Renormalization group, Weak chaos, Glass formation

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1 Introduction

In recent years there has been a renewed interest in the analysis of the dynamical behavior at critical attractors in low dimensional maps. In such situations, criticality is triggered by the vanishing of the Lyapunov coefficient (coefficients in dimension higher than one) so that ordinary chaos paradigms are not applicable. Part of this interest is because Tsallis’ functional forms (the so-called $q$-exponential and $q$-logarithm) were found to mimic the role that the ordinary exponential and logarithm have in the usual theory of chaos (although, as we will see below, in a more complex and sophisticated sense). In fact,

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exact renormalization group (RG) solutions for the sensitivity to initial conditions at these critical attractors are found to assume the structure of (infinite families of) q-exponential functions. This paper attempts to rationalize in a single review some of the most relevant new results that stemmed from the application of exact RG approaches to critical attractors of low dimensional maps [1,2,3,4,5,6,7,8]. This research work moved from the understanding that RG techniques developed in the '80s for describing the static properties of the maps could indeed be extended in order to solve in all details the dynamical behavior at critical attractors. Then, contact with Tsallis’ and Mori’s formalisms were established a posteriori. The final result is the identification of a universal, intricate, dynamical mechanism common to the three paradigmatic routes to chaos of period doubling, intermittency and quasiperiodicity.

Another important result conveyed by the RG approach at the chaos threshold of the logistic map is the determination of the exponent characterizing the decay of correlations in presence of an external noise source [7,8]. By realizing that such an iteration equation is a discrete version of Langevin’s equation and that the underlying scenario is that of an ergodic to non-ergodic transition, one can in fact produce a “translation dictionary” between the dynamical behavior of the logistic map with additive noise and the phenomenology observed for a glass former. Translated words include Adam-Gibbs formula, time translation invariance and $\alpha$-relaxation, aging, subdiffusion and arrest.

There is a further interesting mechanism through which aging and weak ergodicity breaking can be obtained in low dimensional maps. This mechanism is related to the dynamical behavior close to the origin of a Manneville-like map (see below) and can be statistically described in terms of the continuous time random walk formalism. Such a discussion is however out of the scope of the present paper. The interested Reader is referred to [9] for an appropriate account.

The paper is divided in two main sections. Section 2 is devoted to the dynamical picture underlying the above mentioned routes to chaos. In Section 3 we describe the analogies between the behavior of the logistic map with additive noise and that of a glass former. Finally, some remarks are added.

2 Routes to chaos: a common dynamical picture

In the following, we will be mainly considering unimodal maps, i.e., discrete-time iterated maps governed by the equation

$$x_{t+1} = f_{\mu,\zeta}(x_t) = 1 - \mu|\lambda|^{\zeta}, \quad t = 0, 1, \ldots, \quad x \in [-1, 1],$$

(1)
where $\mu \in [0, 2]$ is the control parameter and $\zeta > 1$ the non linearity order of the maximum at $x = 0$ (Fig. 1). $\zeta = 2$ corresponds to the celebrated logistic map [10,11]. We define the sensitivity to initial conditions as

$$\xi(x_0, t) \equiv \lim_{|\Delta x_0| \to 0} \frac{|\Delta x_t|}{|\Delta x_0|},$$

(2)

where we denote explicitly the dependence on the initial condition $x_0$ in order to address cases in which the Lyapunov coefficient is zero. Typically (for all $\mu$ except a set of zero Lebesgue measure), $\xi$ is independent from $x_0$ and exponential for $t \gg 1$, $\xi(t) \sim \exp(\lambda t)$, where $\lambda \in \mathbb{R}$ is the Lyapunov coefficient [10,11].

For $\mu$ smaller than a critical value $\mu_\infty(\zeta)$, $\lambda$ is negative except for an infinite but numerable number of pitchfork bifurcations points in which $\lambda = 0$ (see Fig. 2). Within this region, the attractor is periodic of order $2^r$ ($r = 0, 1, \ldots$) with $r$ that increases by one unit at each pitchfork bifurcation and that becomes infinite at $\mu_\infty$. Inside each $\mu$-interval in which the period is $2^r$, there is a special orbit that passes through $x = 0$ and that has $\lambda \to -\infty$ (super-stable cycle [10]). Important quantities for the dynamics at the onset of chaos are the diameters of the bifurcation forks $d_{r,s}$ ($s = 0, 1, \ldots, 2^r - 1$), which are defined as the distances of two nearest neighbors attractor points in a super-stable cycle. As we will see, in the limit $r \to \infty$ these diameters contain all the information about the multifractal attractor at the onset of chaos. The chaos threshold $\mu_\infty$ is the accumulation point of both pitchfork bifurcations and super-stable cycles.

For $\mu > \mu_\infty$, $\lambda$ is positive except in an numerable number of intervals which are again characterized by a periodic attractor. The left extreme of such intervals is determined by a tangent bifurcation where $\lambda$ vanishes. Inside each interval, a new pitchfork bifurcation cascades produces a (higher order) chaos threshold before reaching the right extreme of the interval, so that the final result is an intricate and fascinating self-similar structure.

The study of unimodal maps marked an era in chaos theory. For example, the pitchfork cascade has been identified to occur in chemical reactions, optically bistable systems, electrical RCL oscillators, sound waves in water, Bénard convection fluid experiments, \ldots [10]. Also, the tangent bifurcation with the addition of a reinjection mechanism is the paradigm for the phenomenon of (type I) intermittency and $1/f$-noise observed, e.g., in nonlinear RCL oscillators and in Bénard convection [10]. The elegance and richness of this self-similar structure stimulated a number of theoretical results starting from the late 70’s. Perhaps the milestone was the RG approach developed by Feigenbaum [12] and, independently, by Coullet and Tresser [13] for the pitchfork bifurcations cascade. This work was extended by the group of Politi [14] and that of Mori.
[15] that focused on the properties of the fluctuation spectrum of generalized algebraic Lyapunov coefficients. An exact solution for the tangent bifurcation dynamics was found by Hu and Rudnick [17]. As we show below, such solution can be extended also to the pitchfork bifurcations. Another important contribution by Mori and colleagues was the introduction of the idea of dynamical transitions [16], a formalism closely related to the probabilistic large deviation theory [18], that is analogous to the description of first-order thermal transitions. More recently, the group of Tsallis [19] pointed out that the $q$-exponentials derived within the so-called nonextensive statistical mechanics formalism appropriately describe the envelope of the fluctuating sensitivity to initial conditions in which the ordinary Lyapunov coefficient vanishes. The work by Robledo and colleagues [1,2,3,4,5,6] that we are reviewing in this section put together all previous approaches in a single perspective, by deriving exact results supported by the RG functional composition.

2.1 Pitchfork and tangent bifurcations

Let us start by understanding what happens at pitchfork and tangent bifurcations. Reading such a transition along the $\mu$-axis of Fig. 2a, the transition is said of order $n$ if in the left (pitchfork) or in the right (tangent) neighborhood of the transition point the attractor is periodic of order $n$. If we consider the $n$-composed map $f_{\mu,\zeta}^{(n)}$ and shift the coordinate $x$ to one of the of the $n$ bifurcation points $1$, we have that in the neighborhood of the chosen point $f_{\mu,\zeta}^{(n)}$ has

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Unimodal map.}
\end{figure}

\footnote{With an abuse of notations we still indicate the shifted coordinate $x'$ as $x' \equiv x$.}
the universal expansion

\[ f^{(n)}_{\mu,\zeta}(x) = x + (-1)^z u \text{sgn}(x^z)|x|^z + o(|x|^z), \]  

(3)

where \( u > 0 \) is the leading expansion coefficient and \text{sgn} is the sign function (here and below repeated use of the sign function is made in order to unify the treatment of pitchfork and tangent bifurcations). For a pitchfork bifurcation \( z = 3 \), whereas for a tangent bifurcation \( z = 2 \) (see Fig. 3). The expansion is independent of the non-linearity parameter \( \zeta \).

With \( z = 2 \), by neglecting the \( o(|x|^z) \) term, Eq. (3) recovers the Manneville map [10]. In such a case, at the left side of the bifurcation point trajectories are converging to the bifurcation point, whereas at the right side they are expelled from the bifurcation point (Fig. 3b). As we will see below, in the converging side the dynamics is regular, being characterized by a weak (power law in place of exponential) insensitivity to initial conditions. In contrast, the expelling side is very strongly divergent, with a sensitivity to initial conditions stronger than exponential. Thus, the addition of an external reinjection mechanism connecting the right hand side with the left hand side allows an intermittent dynamics in which chaotic and regular patterns alternate. Recently, a remarkable connection between this mechanism and the dynamics of a cluster of spins at a thermal critical point has been established [20]. This opened the possibility for the description of such temporary critical fluctuations in terms of Tsallis’ functional forms [21].

In correspondence with the tangent bifurcation \( (z = 2) \), Hu and Rudnick [17] discovered that the RG iteration equation

\[ f^*(f^*(x)) = \tilde{\alpha}^{-1} f^*(\tilde{\alpha} x), \quad \tilde{\alpha} = 2^{1/(z-1)} \]

(4)
possesses a solution, \( f^*(x) = x[1 - (z - 1)u \text{ sgn}(x)|x|^{z-1}]^{-1/(z-1)} \), that coincides with the expansion (3) up to order \( o(|x|^z) \). Remarkably, this is one of the very few RG solution exactly known [11]. The original solution [17] can be extended also to the pitchfork bifurcations by simply including a change of sign [2]:

\[
f^*(x) = x[1 - (z - 1)u (-1)^z \text{ sgn}(x^z) \text{ sgn}(x)|x|^{z-1}]^{-1/(z-1)}.
\] (5)

Close to the bifurcation point, by using the property:

\[
f^{*m}(x) = \frac{1}{m^{1/(z-1)}} f^*(m^{1/(z-1)}x), \quad m = 1, 2 \ldots
\] (6)

we can recast the \( m \)-times iterate of the \( n \)-composed map \( f_{\mu,\zeta}^{(n)} \) as

\[
x_t \equiv [f^{(n)}]^{(m)}(x_0) =
= x_0[1 - (z - 1)a(-1)^z \text{ sgn}(x_0^z)\text{ sgn}(x_0)|x_0|^{z-1}t]^{-1/(z-1)},
\] (7)

where \( t \equiv mn \) and \( a \equiv u/n \). Notice that while the \( n \)-composed map describes a single orbit of this form in the shifted coordinate \( x_0 \ll 1 \), for the original map \( f_{\mu,\zeta} \) there are \( n \) specific time subsequences, each corresponding to one of these orbits originated in one of the \( n \) bifurcation points. Eq. (7) satisfies the identity

\[
\frac{dx_t}{dx_0} = \left( \frac{x_t}{x_0} \right)^z.
\] (8)
Due to this property the sensitivity to initial conditions \( \xi(x_0, t) \equiv \lim_{\Delta x_0 \to 0} \frac{\Delta x_t}{\Delta x_0} \) assumes the form
\[
\xi(x_0, t) = \left[ 1 + (1 - q)\lambda_q(x_0)t \right]^{1-q} \equiv \exp_q(\lambda_q t), \quad q \in \mathbb{R}.
\] (9)

The parameter \( q \) can be read from \( z \). For all pitchfork (tangent) bifurcations of all order \( n \) we have \( q = 5/3 \) (\( q = 3/2 \)). The functional form \( \exp_q \) is a one-parameter deformation of the ordinary exponential function called Tsallis’ \( q \)-exponential \[19\]. In the limit \( q \to 1 \) the ordinary exponential is recovered. The inverse of the \( q \)-exponential, named \( q \)-logarithm, is given by \( \ln_q(x) \equiv (x^{1-q} - 1)/(1-q) \). The coefficient \( \lambda_q \) replaces the ordinary Lyapunov coefficient and depends on the shifted coordinate of the initial point:
\[
\lambda_q(x_0) = za (-1)^z \operatorname{sgn}(x_0) \operatorname{sgn}(x_0)|x_0|^z^{-1}.
\] (10)

As for the ordinary Lyapunov coefficient, the sign of \( \lambda_q \) discriminates if the dynamics is regular (\( \lambda_q < 0 \), insensitivity to initial conditions, meaning that two different orbits tend to converge in time) or chaotic (\( \lambda_q > 0 \), sensitivity to initial conditions, i.e., two different orbits tend to separate in time). In addition, the value of \( q \) determines whether the dependence with time is stronger or weaker than exponential. Fig. 4 reports the behavior of the \( q \)-exponential for different \( q \)'s and sign of \( \lambda_q \).

In the next subsection we will see that a similar scenario characterizes the onset of chaos. Namely, the ordinary exponential form for the sensitivity to
For the sake of simplicity we address first the logistic map \( \zeta = 2 \). We also change slightly our notations indicating for a moment with \( \tau = 0, 1, \ldots \) the iteration time. Eq. (1) reads then

\[
x_{\tau+1} = 1 - \mu x_\tau^2.
\]  

(11)

The chaos threshold is located at \( \mu_\infty(2) = 1.40115 \ldots \). Let us start by choosing the initial condition \( x_0 = 0 \). The Feigenbaum attractor can then be characterized [3,4] by a series of monotonic subsequences

\[
\tau_k = (2k + 1)2^{n-k}, \quad k = 0, 1, \ldots, \ n \geq k.
\]  

(12)

The log-log plot of Fig. 5 shows that for each fixed \( k, n \geq k \) defines a subsequence such that the absolute values of the iterates \( |x_{\tau_k}| \) lie along a diagonal
The Feigenbaum-Coullet-Tresser RG transformation defines the universality class of the onset of chaos for the logistic map. Such transformation is given by \[10,11\]

\[
g(x) = \alpha^n g^{(2^n)}(x/\alpha^n), \quad \alpha = 2.50290\ldots,
\]

where \(\alpha\) is one of the Feigenbaum’s universal constant and \(g\) is the universal Feigenbaum’s function (not known in closed form). This RG transformation allows to calculate explicitly the iterates for each subsequence \(\tau_k\). In fact, one obtains \[4\] the expansion

\[
x_{\tau_k} \equiv |g^{(\tau_k)}(x_0)| \approx \frac{g^{(2k+1)}(0)}{\alpha^{(n-k)}} + \frac{g^{(2k+1)''}(0)}{2\alpha^{(k-n)}} x_0^2,
\]

which is valid for initial conditions \(|x_0| < \alpha^{-(n-k)}\). We now shift to zero the initial time of each subsequence by redefining the time as

\[
t_k \equiv \tau_k - 2k - 1 = (2k + 1)2^{n-k} - 2k - 1, \quad n \geq k.
\]

Implementing this time shift in Eq. (14) and using the identity \(\alpha^{(n-k)} = [1 + t_k/(2k + 1)]^{\ln \alpha/\ln 2}\), the sensitivity to initial conditions for each time subsequence turns out to be \[4\]

\[
\xi_{t_k} = \exp_q \left[ \lambda_q^{(k)} t_k \right], \quad q = 1 - \frac{\ln 2}{\ln \alpha}, \quad \lambda_q^{(k)} = \frac{\ln \alpha}{(2k + 1)\ln 2}
\]

(see Fig. 6). The fact that the generalized Lyapunov coefficient \(\lambda_q^{(k)}\) varies with the subsequence \(k\) can be seen as a dependence of this coefficient on the initial condition. In fact, for each subsequence, the initial position at zero time \(t_k\), \(x_{\tau=2k+1}\), changes with \(k\).

Along these lines, it is also possible to show \[4\] that within each subsequence, Tsallis’ \(q\)-entropy grows linearly with the time \(t_k\) and the slope of the linear growth is equal to \(\lambda_q^{(k)}\). This, parallels the Pesin identity that relates the (sum of the) positive Lyapunov coefficient(s) to the Kolmogorov-Sinai entropy rate \[11\], and what happens for the ordinary entropy growth \((q = 1)\) when strong chaos is present.

\(^2\) The reader may be interested by the fact that in a different context, namely for a conservative bidimensional map with vanishing Lyapunov coefficients, similar numerical evidence has been found, although for the somewhat trivial (linear) case \(q = 0\) \[22\].
Fig. 6. Sensitivity to initial condition at the onset of chaos of the logistic map. For different subsequences, the $q$-logarithm/linear plot with $q = 1 - \frac{\ln 2}{\ln \alpha}$ exhibits straight lines whose slope corresponds to the generalized Lyapunov coefficient $\lambda_q$.

Although Eq. (16) gives the sensitivity to initial conditions for all the subsequences $k$, it is not yet completely general since such subsequences are those generated starting at $x_{\tau=0} = 0$. At this point, we have to say what happens if we choose another point of the attractor as the starting point. The answer is given by considering the Feigenbaum’s $\sigma$ function. This function is defined by the scaling properties of the diameters of the bifurcation forks $d_{r,s}$:

$$\sigma_r(s) \equiv \frac{d_{r+1,s}}{d_{r,s}}.$$  \hfill (17)

In the limit $r \to \infty$, $\sigma_r$ has an infinite number of jumps, each associated to a different attractor point which is here identified by the variable $s$. The sensitivity to initial conditions for the subsequence $k$ generated by the initial point $x_{\tau=0}$ defined by $s$ can be deduced from $\sigma$ through [5]

$$\xi_t(x_0) = \lim_{n \to \infty} \left| \frac{d_{n-k+1,s+t}}{d_{n-k,s}} \right| \sim \left| \frac{\sigma_{n-k}(s - 1)}{\sigma_{n-k}(s)} \right|^{n-k} = A^{n-k},$$ \hfill (18)

where $A$ is a number that depends on $s$ (and therefore on $x_{\tau=0}$). The quantity $A$ has been identified as the ratio of the values of $\sigma$ at each of its discontinuities [5]. Hence, again the use of the identity $A^n = [1 + \frac{\ln 2}{2k+1}]^\frac{n-1}{2}$ lead to the more
general result

\[ q = 1 - \frac{\ln 2}{\ln A}, \quad \lambda_q = \frac{\ln A}{(2k + 1) \ln 2}. \]  

(19)

We see then that the structure valid for \( x_{r=0} = 0 \) is maintained, but changing the starting points implies a change of the power law exponent \( q \).

Finally, all previous results for the onset of chaos can be further generalized to general nonlinearity \( \zeta \) in unimodal maps [5], in which case the families of \( q \)-exponential functions characterizing the sensitivity to initial conditions are defined by

\[ q = 1 - \frac{\ln 2}{(\zeta - 1) \ln A}, \quad \lambda_q = \frac{(\zeta - 1) \ln A}{(2k + 1) \ln 2}, \]  

(20)

where \( A \) now also depends on \( \zeta \).

Summarizing, we have seen that the sensitivity to initial conditions at the chaos threshold can be understood in terms of a complex hierarchical structure that can be exactly deduced from the Feigenbaum-Coullet-Tresser RG approach. For a given point of the attractor, this structure corresponds to an infinite family of \( q \)-exponentials, each with the same \( q \) but with a different generalized Lyapunov coefficient \( \lambda_q^{(k)} \). Spanning the initial point along the attractor corresponds to an infinite number of such families, each of the families being now characterized by a different value of \( q \) which is given in terms of the discontinuities of Feigenbaum \( \sigma \) function. Since the stronger of such discontinuities happens for \( s = 0 \) \( (x_{r=0} = 0) \), the value of \( q \) corresponding to this discontinuity is in some sense the most important one. Such a value, for the logistic map, is \( q = 1 - \ln 2/ \ln \alpha = 0.2445 \ldots \).

Another fundamental result [5] for the period doubling onset of chaos (and actually also for the quasi periodic route to chaos [6]) is that the Tsallis’ \( q \)-exponential structure for the dynamics can be exactly linked with the formalism of Mori’s transitions [16], which we briefly sketch below. Mori’s formalism, when applied to the transition to chaos, starts by defining a different generalization of the Lyapunov coefficient, called generalized finite-time Lyapunov coefficient

\[ \lambda(x_0, t) = \frac{1}{\ln t} \sum_{i=0}^{t-1} \ln \left| \frac{df_{\mu^\infty}(x_i)}{dx_i} \right|. \]  

(21)
The probability for getting the value \( \lambda \) at the finite-time \( t \) is postulated in the form
\[
p(\lambda, t) \equiv t^{-\psi(\lambda)} p(0, t).
\]

(22)

This allows the introduction of a dynamic partition function \( Z(q, t) \) and of the associated free energy \( \phi(q) \)
\[
Z(q, t) \equiv \int d\lambda p(\lambda, t) t^{-(q-1)\lambda}, \quad \phi(q) \equiv -\lim_{t \to \infty} \frac{\ln Z(q, t)}{\ln t},
\]

(23)

where \( q \in \mathbb{R} \) is Mori’s \( q \)-parameter. The “coarse grained” function of generalized finite-time Lyapunov coefficients is
\[
\lambda(q) \equiv \frac{d\phi(q)}{dq},
\]

(24)

and \( \psi(\lambda) \) is the Legendre transform of \( \phi(q) \). In analogy with ordinary thermal phase transitions, Mori and colleagues detected a single first order phase transition for the onset of chaos [16].

The important identification [23]
\[
\lambda \equiv \frac{1}{\ln n} \ln A^{n-k} = \frac{1}{n} \ln_q A^{n-k},
\]

(25)

together with the previous results, leads to the conclusion [5] that the dynamics at the onset of chaos is actually characterized by an infinite family of Mori’s \( q \)-phase transitions each occurring at one of the allowed values of the Tsallis’ \( q \)-index in the sensitivity:
\[
q = q.
\]

(26)

2.3 Quasiperiodic onset of chaos

The prototype for the quasiperiodic route to chaos is the circle map (see, e.g., [24])
\[
\theta_{t+1} = f_{\Omega,K}(\theta_t) = \theta_t + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_t) \quad (\theta \text{ mod } 1),
\]

(27)

which depends on the real parameters \( \Omega \) (bare winding number) and \( K \) (amount of nonlinearity). In order to describe the dynamical behavior it is important to
consider the influence of the parameter $\Omega$ through the dressed winding number

$$w \equiv \lim_{t \to \infty} \frac{\theta_t - \theta_0}{t}, \quad (28)$$

that gives the average increment of $\theta_t$ per iteration. The orbits for $K < 1$ are periodic (locked motion) if $w$ is rational and quasiperiodic (unlocked motion) if $w$ is irrational. In fact, $w(\Omega)$ describes the so-called “devil staircase” making a step at each rational value of $\Omega$ [24]. $K = 1$ defines the critical circle map for which the periodic motion covers the whole $\Omega$ interval apart from a multifractal set of values for which the motion is quasiperiodic. For $K > 1$ regions of periodic motion (Arnold tongues) overlap in chaotic bands. The chaos threshold can be studied by fixing $K = 1$ and by approximating a target irrational value of $w$ through a sequence of rational values. Perhaps the most interesting case is the one in which the target value is the reciprocal of the golden mean $w_{gm} \equiv (\sqrt{5} - 1)/2$ and the sequence of rationals is given by Fibonacci numbers $w_n \equiv F_{n-1}/F_n$ (with $F_{n+1} = F_n + F_{n-1}$) [24]. The universal properties of the onset of chaos are then described by the fixed-point map $g_{gm}(\theta)$ of the following RG transformation

$$g_{gm}(\theta) = \alpha_{gm} g_{gm} \left( \frac{\theta}{\alpha_{gm}} \right), \quad (29)$$

where $\alpha_{gm} = -1.288575...$ is a universal constant. The same techniques implemented for the pitchfork bifurcation onset of chaos lead to the previous universal picture, although this transition to chaos is much more involved and reach in detail [6]. Namely, the sensitivity to initial condition at the chaos threshold is given by an infinite number of Tsallis’ $q$-exponential families, each $q$-exponential being associated to a specific time subsequence\(^3\):

$$\xi_{tk} = \exp_q \left[ \lambda_{q}^{(k)} t_k \right], \quad q = 1 + \frac{\ln w_{gm}}{2 \ln |A|}, \quad \lambda_{q}^{(k)} = 2 \frac{\ln |A|}{k \ln w_{gm}}, \quad (30)$$

where $A$ depends on the initial condition that generates the time subsequence $k$ ($A = \alpha_{gm}$ if the initial condition is at $\theta_0 = 0$). Correspondingly, the dynamics is characterized by an infinite family of Mori’s $q$-transitions occurring at $q = q$ [6].

\(^3\) More precisely, the time subsequences are in this case identified by two different numbers $k = 1, 2, \ldots$ and $l = 0, 1, \ldots, k - 1$. Eq. (30) is valid if $l = 0$ (see [6] for details).
2.4 Tsallis’ \(q\)-exponential sensitivity to initial conditions

We have seen that if one is interested in extracting significant trends for the dynamics at critical attractors where the ordinary Lyapunov coefficient vanishes, some precise time subsequences that depend on the initial condition must be selected. Within these subsequences, Tsallis’ \(q\)-exponentials and \(q\)-generalized Lyapunov coefficients \(\lambda_q\) appear to be convenient tools for extracting these trends. Table 1 summarizes the different behaviors obtained from various combinations of \(q\) and \(\lambda_q\). The only case not yet observed is \(q < 1\) and \(\lambda_q < 0\), which would amount to super-strong (stronger than exponential) insensitivity to initial conditions. In private conversations with A. Robledo, we conjectured that this could be the case of the super-stable cycles of unimodal maps. Efforts to clarify this point are currently in course [25].

| \(\lambda_q\) | \(q < 1\)         | \(q = 1\)     | \(q > 1\)     |
|---------------|-------------------|---------------|---------------|
| \(\lambda_q < 0\) | super-strong insensitivity | strong insensitivity | weak insensitivity |
| super-stable cycles? | \(\mu < \mu_\infty\) | \(\mu = \mu_\infty\) | typically for \(\mu > \mu_\infty\) |
| \(\lambda_q > 0\) | weak sensitivity | strong sensitivity | super-strong sensitivity |
| \(\mu = \mu_\infty\) | | typically for \(\mu > \mu_\infty\) | tangent rhs |

Table 1
Sensitivity to initial conditions and Tsallis’ \(q\)-exponentials.

3 Logistic map with additive noise and glassy dynamics

The theoretical understanding of the dynamical and thermodynamical features of supercooled liquids is nowadays a very important research topic in condensed matter physics and in statistical mechanics. Theoretical efforts are motivated by the search for general common features of non-equilibrium systems and applications span from new material science to protein folding. In a glass former there are two different dynamical scenarios, separated by the glass transition temperature \(T_g\). Above \(T_g\) the system equilibrates while below \(T_g\) it is out-of-equilibrium. Phenomenology of glassy systems includes [26]:

- Time translation invariant relaxation processes. Above, but close to, the glass transition two-time correlations do not depend on the first (waiting) time and display power law decays (\(\alpha\)-decays). Some glass formers also exhibit a further initial power law decay called \(\beta\)-decay.
- Loss of time translation invariance (aging) at and below the glass transition.
• Intriguing connection between kinetics and thermodynamics – Adam-Gibbs formula:

\[ t_x = A \exp \left( \frac{B}{TS_c} \right) \]  

(31)

where \( t_x \) is a relaxation time (equivalently the viscosity), \( A \) and \( B \) are constants, \( T \) is temperature, and \( S_c \) configurational entropy

• Critical slowing down: transition from normal- to sub-diffusion to localization of molecular motion.

In order to elaborate the counterpart of such behaviors in low-dimensional maps, let us start by noticing that in the Langevin theory the effect of collisions of the diffusive particle with molecules in the fluid is represented by an additive noise term. In the same spirit, nonlinear low-dimensional maps with external noise can be used to model systems with many degrees of freedom. The discrete form for a Langevin equation is

\[ x_{\tau+1} = x_{\tau} + h_{\mu}(x_{\tau}) + \sigma \Gamma_{\tau} \quad \tau = 0, 1, \ldots, \]  

(32)

where \( \Gamma_{\tau} \) is a Gaussian white noise \((\langle \Gamma_{\tau} \Gamma_{\tau'} \rangle = \delta_{\tau,\tau'} \)) and \( \sigma \) measures the noise intensity. Taking \( h_{\mu}(x) = 1 - x - \mu x^2 \), Eq. (32) becomes the iteration equation of the Logistic map with additive noise

\[ x_{\tau+1} = 1 - \mu x_{\tau}^2 + \sigma \Gamma_{\tau}, \quad x \in [-1, 1], \quad \mu \in [0, 2]. \]  

(33)

For small noise amplitudes the onset of chaos \( \mu_\infty(\sigma) \) still separates an “equilibrium” ergodic phase characterized by a chaotic dynamics with positive Lyapunov coefficient from an “out of equilibrium” non-ergodic one where the Lyapunov coefficient is negative [10].

We will be studying the chaos threshold from \( \mu > \mu_\infty \). In our analogy with the glass former this means that we will be looking at the glass formation from the side of the liquid. In the absence of noise, the attractor, besides being the accumulation point of the bifurcations cascade, can also be viewed as the accumulation point of the band splittings occurring above the onset of chaos. For \( \mu = 2 \), the chaotic attractor is in fact characterized by a single band that spans the interval \([-1, 1]\) (see Fig. 7a). As \( \mu \) is reduced towards \( \mu_\infty \), this unique band reduces its width and then splits in two disjoint bands. This band splitting continues up to \( \mu_\infty \), where the attractor can be viewed as \( 2^\infty \) disjoint bands. The same occurs if \( \sigma \neq 0 \), the only difference being that the band splittings end (and the bifurcations end) at a finite value \( 2^{N(\sigma)} \) (Fig. 7b). This phenomenon is called bifurcation gap.

If the noise term is small enough, the RG perturbative approach can be applied in order to calculate the position of the iterates around \( x_0 = 0 \) [7]. The result
Fig. 7. Bifurcation gap phenomenon. (b) and (c) are magnification of the region enclosed by the box in (a). The infinite number of bands (points) at the noiseless chaos threshold in (b) become a finite number of chaotic bands in presence of the noise in (c).

is

\[ x_{\tau_k} = \tau_k^{-q} g \left( \tau_k^{-q} x_0 \right) + \sigma \Gamma \tau_k^{-r} + G_\Lambda \left( \tau_k^{-r} x_0 \right), \]  

(34)

where \(|x_0| \ll 1\), \(G_\Lambda(x)\) is the first order perturbation eigenfunction, \(r \simeq 0.6332\) is determined from the noisy power spectrum of the map [10], and we are using a notation consistent with the one in Section 2.2. For the leading subsequence \(k = 0\), by introducing the previous time shift \(t \equiv t_0 = \tau_0 - 1\), the iterate positions can be viewed in terms of \(q\)-exponentials:

\[ x_t = \exp_{2-q} (-\lambda_q t) \left[ 1+\sigma \Gamma \exp_r (-\lambda_r t) \right]. \]  

(35)

This implies the presence of a “crossover” or “relaxation” time in the noisy dynamical behavior:

\[ t_x = \sigma^{r-1}. \]  

(36)

For \(t < t_x = 2^{N(\sigma)} - 1\) the dynamics is restricted to non-overlapping bands attractor, similarly to the noiseless case; for \(t > t_x\) the dynamics becomes more and more chaotic as a consequence of the band merging (Fig. 8).

This crossover furnishes the key for identifying the following marks of glassy-like behavior for the noisy onset of chaos (see [7,8] for details).
Fig. 8. Crossover dynamics observed for the main subsequence \( k = 0 \) for two different values of the noise intensity. For \( t < t_x(\sigma) \) the iterates closely follow the noiseless attractor. For \( t > t_x(\sigma) \) the iterates become more chaotic and they jump among different bands of the noiseless attractor.

- **Analogue of the Adam-Gibbs formula.** The entropy associated to the distribution of the iterate positions within the \( 2^N \) bands has the form

\[
S_c = 2^N \sigma s,
\]

where \( \sigma s \equiv -\sigma \int d\Gamma p(\Gamma) \ln p(\Gamma) \) is the entropy associated to a single band. Since \( 2^N = 1 + t_x \) and \( t_x = \sigma^{(r-1)} \) we get [7]:

\[
t_x = \left( \frac{s}{S_c} \right)^{\frac{1-r}{r}}.
\]

Notice that in contrast with the Adam-Gibbs exponential law, in this case \( t_x \) has a power law behavior in \( 1/S_c \).

- **Time translation invariance and \( \alpha \)-relaxation.** Whereas a study of the relaxation processes of iterates starting with initial conditions \( x_0 \) outside the attractor could furnish some analogue to the \( \beta \) initial fast relaxation observed for some glass formers, the \( \alpha \) subsequent relaxation is mimicked by the bifurcation-gap crossover for \( t > t_x \). This becomes apparent if one studies the behavior of ensemble-averaged correlations \( c_e(t_1, t_2) \) [8]. There is a “thermalization” process occurring for small values of \( t_1 \). If \( t_1 \) is large enough inside the chosen subsequence \( k \), the correlation becomes time translation invariant and \( t_x \) characterizes the decay of correlations.

- **Aging.** At the onset of chaos when \( \sigma = 0 \) trajectories are non-ergodic and retain memory of the initial data. This property is equivalent to a loss of
time translation invariance and to a “built-in” aging. Since ensemble and
time averages are not equivalent, a time-averaged definition of correlations
allows the exhibition of such a property [8].

- **Subdiffusion and arrest.** The sharp slowing down of dynamics in supercooled
liquids implies a crossover from normal diffusion to sub-diffusion and finally
to arrest. This deceleration is caused by the confinement of any molecule
by a “cage” formed by its neighbors. Such a “cage” can be reproduced by
a periodic map obtained via repetition of a single cell map in such a way
that diffusion is due only to the noise term. The periodic map is given by

\[ x_{t+1} = F(x_t), \quad F(l + x) = l + F(x), \quad l = \ldots -1, 0, 1, \ldots, \]  

where

\[ F(x) = \begin{cases}  
-|1 - \mu_c x^2| + \sigma \xi, & -1 \leq x < 0, \\
|1 - \mu_c x^2| + \sigma \xi, & 0 \leq x < 1.
\end{cases} \]  

The diffusion process displayed by the map in Eq. (39) reproduces the
crossover normal/sub-diffusion/arrest as \( \sigma \) tends to zero [8].

4 **Final remarks**

Several nonlinear dissipative systems, like those mentioned in this review,
exhibit both regular and chaotic behavior depending on some external param-
eter. The transition from regularity to chaoticity implies a dramatic alteration
in the dynamical behavior, the situation changing from an exponentially fast
insensitivity to initial conditions to an exponentially fast sensitivity to initial
conditions. There are only few possible mechanisms through which such transi-
tions occur. These are the well known (universal) routes to chaos. Thus, when
the external parameters are tuned at the transition point, the ordinary Lya-
punov coefficients vanish and they do not allow to extract significant trends
characterizing the critical dynamics. Under these conditions, intricate oscillations
typically appear. Since these oscillations are reminiscent of the regular
behavior, the possibility arises to characterize the dynamics exactly if one
accepts to describe it via a set of specific temporal subsequences.

Here we have given an account of the extension of the static RG approaches,
discovered in the 80’s, to the dynamics at the critical attractors of period
doubling, intermittency and quasiperiodicity routes to chaos [1,2,3,4,5,6]. The
outcome of this extension, obtained basically from an \textit{a priori} approach, are
significant new results that enlarge considerably the previous knowledge. For
example, it is now clear that all the properties found for the sensitivity to ini-
tial conditions are associated to the discontinuities of Feigenbaum’s universal
$\sigma$ scaling function for the bifurcation cascade onset of chaos [5] (with an analogous extension for the quasiperiodic critical attractor [6]). In the language of Tsallis' functional forms these properties can be read as Tsallis' generalization of the Lyapunov coefficient that exhibits a new spectrum of behaviors ranging from weaker to stronger than exponential laws [2,3,4]. Also, precise contact with the formalism of Mori's dynamical transitions at the transition point and the fixed $q$-parameter in Tsallis' formalism has been established [5,6]. In view of this relation one can conclude that the dynamics for the period doubling and the quasiperiodic critical attractors is in fact characterized by an infinite number of Mori's dynamical transitions [5,6], in place of the single one known previously. In relation to a recent debate in the literature [27], let us then conclude that the novel exact knowledge on the critical dynamics obtained in [1,2,3,4,5,6] puts into a different fresh context the fundamental scaling laws discovered in the 80's for these nonlinear dynamical systems.

As an example of how to implement the detailed knowledge of the dynamics at the period doubling onset of chaos, we have reported [7,8] that specific parallels with the behavior of a glass former close to the glass transition can be established if one adds a noise term to the logistic map equation. Of course through this very simple one-dimensional system we can aim only at qualitative analogies with respect to the behavior of a supercooled liquid close to the glass transition, nevertheless these similarities have the appeal of allowing a precise numerical and analytical inspection.

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