Renormalization Group Approach to Casimir Effect and the Attractive and Repulsive Forces in Substance *

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Abstract

Electromagnetism in substance is characterized by permittivity (dielectric constant) and permeability (magnetic permeability). They describe the substance property effectively. We present a geometric approach to it. Some models are presented, where the two quantities are geometrically defined. Fluctuation due to the micro dynamics (such as dipole-dipole interaction) is taken into account by the (generalized) path-integral. Free energy formula (Lifshitz 1954), for the material composed of three regions with different permittivities, is explained. Casimir energy is obtained by a new regularization using the path-integral. Attractive force or repulsive one is determined by the sign of the renormalization-group $\beta$-function.

Keywords: geometric view, permittivity and permeability, path-integral, Casimir energy, induced geometry, hyper-surface, regularization, fluctuation

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1 Introduction

The recent development of the fundamental physics of the space-time-matter (quantum field theory, string theory, D-brane theory) has revealed the interesting relation between the ‘ordinary’ physics (boundary field theory) and the ‘bulk space’ geometry dynamics. It is called AdS/CFT relation\[1, 2, 3\]. One important outcome is that the renormalization-group flow can be regarded as the ‘classical path’ in the bulk (higher dimensional) space, which is called holographic renormalization\[4, 5\]. We interpret this new result in the context of the tribology physics.

Casimir phenomena\[6\] occur in substance at the (absolute) temperature zero: \(T = 0\). It is the zero-point oscillation of the quantum vacuum. Only free(kinetic) part contributes. Hence Casimir force (or energy) does not depend on couplings (interactions between the micro objects). The force works between two materials separated in the macro distance. Besides, Casimir force sensitively depends on the topology of the material. To define Casimir energy rigorously we need highly-sophisticated regularizations because the quantity severely diverges both in the infrared(IR) and in the ultraviolet(UV) regions.

In the science of friction, Casimir force (or energy) is one of important physical quantities where the renormalization procedure is necessary\[7\]. Usually the force works attractively. It plays the role of friction. The force (e.g., between two parallelly-placed metallic plates) is experimentally observed and coincides with the theoretical result\[8\]. It is caused by the quantum fluctuation of the electromagnetic field. Another similar example is Van der Waals force. It is also caused by the fluctuation of the micro objects which compose the substance. The dipole-dipole interaction is an example of the fluctuation forces. Van der Waals force occurs at the general temperature \(T\). The force also attractively works between the neutral micro objects. The electromagnetism in substance describes this property, in terms of the dielectric constant (permittivity) \(\varepsilon\) and the magnetic permeability (permeability) \(\mu\), as the effective continuum theory. Van der Waals force reduces to Casimir one as \(T \to +0\) limit\[9\].

The electromagnetic field in substance is characterized by the relation between the electric field \(\mathbf{E}\) (the magnetic field \(\mathbf{H}\)) and the electric flux density field \(\mathbf{D}\) (the magnetic flux density field \(\mathbf{B}\)).

\[
\mathbf{D} = \varepsilon(\omega)\mathbf{E} , \quad \mathbf{B} = \mu(\omega)\mathbf{H} .
\] (1)
If the micro model of the material is given (such as the elastic force model for the electron in the atom), the forms of $\varepsilon(\omega)$ and $\mu(\omega)$ are concretely obtained. Instead we approach the problem from the geometrical viewpoint.

2 Maxwell Equation in Substance

We consider the general continuous substance which has no real charge and current (classical vacuum) but has induced ones caused by the micro fluctuation. Let us explain the electromagnetism(EM) in substance with care for the $\omega$(frequency), $t$(time) and $x$(space) dependence of the permittivity and the permeability. Electric and magnetic fields, $\hat{E}(t, \mathbf{x})$ and $\hat{H}(t, \mathbf{x})$, with their flux density fields, $\hat{D}(t, \mathbf{x})$ and $\hat{B}(t, \mathbf{x})$, are denoted as

Upper-index Fields : $\hat{D}(t, \mathbf{x}) = (\hat{D}^i(x))$, $\hat{B}(t, \mathbf{x}) = (\hat{B}^i(x))$,

Lower-index Fields : $\hat{E}(t, \mathbf{x}) = (\hat{E}_i(x))$, $\hat{H}(t, \mathbf{x}) = (\hat{H}_i(x))$,

$i = 1, 2, 3$, $\mathbf{x} = (x^i) = (x, y, z)$; $\mu = 0, 1, 2, 3$, $x = (x^\mu) = (t, \mathbf{x})$. (2)

The dielectric function and the magnetic permeability are defined by (general form)

$$\hat{D}^i(x) = \hat{\varepsilon}^{ij}(x)\hat{E}_j(x), \quad \hat{B}^i(x) = \hat{\mu}^{ij}(x)\hat{H}_j(x).$$

(3)

We are usually considering in the 1+3 Minkowski (flat) space-time. The upper and lower indices appearing in (2) indicate that some curved geometry is expected to describe the EM phenomena in substance effectively. We will fix the geometry later.

The absence of real charges (electric and magnetic) requires the following conditions.

$$\text{div} \hat{D} = \partial_i \hat{D}^i = 0 \quad \text{electric charge density} = 0,$$
$$\text{div} \hat{B} = \partial_i \hat{B}^i = 0 \quad \text{magnetic charge density} = 0,$$  

(4)

where $\partial_i \equiv \partial/\partial x^i$. Ampère’s and Faraday’s laws are given by

Ampère’s Law : $\partial_t \hat{D}^i - \hat{\varepsilon}^{ijk}\partial_j \hat{H}_k = 0$ or $\partial_t \hat{D} - \nabla \times \hat{H} = 0$, (electric current density = 0),

Faraday’s Law : $\partial_t \hat{B}^i + \hat{\varepsilon}^{ijk}\partial_j \hat{E}_k = 0$ or $\partial_t \hat{B} + \nabla \times \hat{E} = 0$,  

(5)

\[^1\text{The permittivity and the permeability will be definitely introduced later (4, 8, 12 or 21).}\]
where $\partial_t \equiv \partial/\partial t$ and $\epsilon^{ijk}$ is the totally anti-symmetric tensor (Levi-Civita symbol) with $\epsilon^{123} = 1$.

Faraday’s law is solved by the vector and scalar potentials, $\hat{A}(x)$ and $\hat{\phi}(x)$.

$$
\hat{E}_i(x) = -\partial_i \hat{A}_i(x) - \partial_i \hat{\phi}(x) \quad \text{or} \quad \hat{\mathcal{E}}(x) = -\partial_i \hat{A}_i(x) - \nabla \hat{\phi}(x) \\
\hat{B}^i(x) = \epsilon^{ijk} \partial_j \hat{A}_k(x) \quad \text{or} \quad \hat{\mathcal{B}}(x) = \nabla \times \hat{A}(x),
$$

(6)

where $\hat{A}(x) = \hat{A}(t, x) = (\hat{A}_i(x))$. Let us re-express above quantities in the Fourier-transformed form with respect to time $t$ (t-to-$\omega$ Fourier-transformation).

$$
\hat{D}(x) = \hat{D}(t, x) = \int_{-\infty}^{\infty} D(\omega, x)e^{i\omega t} d\omega, \\
\hat{\mathcal{E}}(x) = \hat{\mathcal{E}}(t, x) = \int_{-\infty}^{\infty} \mathcal{E}(\omega, x)e^{i\omega t} d\omega, \\
\hat{B}(x) = \hat{B}(t, x) = \int_{-\infty}^{\infty} B(\omega, x)e^{i\omega t} d\omega \\
\hat{\mathcal{H}}(x) = \hat{\mathcal{H}}(t, x) = \int_{-\infty}^{\infty} H(\omega, x)e^{i\omega t} d\omega.
$$

(7)

As for the permittivity and permeability, we consider the following form.

$$
D^i(\omega, x) = \varepsilon^{ij}(\omega) E_j(\omega, x), \quad B^i(\omega, x) = \mu^{ij}(\omega) H_j(\omega, x).
$$

(8)

Note that we have replaced the definition of the permittivity and the permeability (3) with the above one. Later we will consider some generalization of (8).

We also do t-to-$\omega$ Fourier-transformation to the potentials.

$$
\hat{A}(t, x) = \int_{-\infty}^{\infty} A(\omega, x)e^{i\omega t} d\omega, \quad \hat{\phi}(t, x) = \int_{-\infty}^{\infty} \phi(\omega, x)e^{i\omega t} d\omega.
$$

(9)

The relations (6) are re-expressed as

$$
E_i(\omega, x) = -i\omega A_i(\omega, x) - \partial_i \phi(\omega, x) \quad \text{or} \quad E(\omega, x) = -i\omega A(\omega, x) - \nabla \phi(\omega, x), \\
B^i(\omega, x) = \epsilon^{ijk} \partial_j A_k(\omega, x) \quad \text{or} \quad B(\omega, x) = \nabla \times A(\omega, x),
$$

(10)

where $\nabla = (\partial_i)$. $E(\omega, x)$ and $B(\omega, x)$ are unchanged under the gauge transformation.

$$
A \rightarrow A + \nabla \Lambda, \quad \phi \rightarrow \phi - i\omega \Lambda.
$$

(11)

Note that this form is not the most general one: $\varepsilon^{ij} = \varepsilon^{ij}(\omega, x)$ and $\mu^{ij} = \mu^{ij}(\omega, x).$
where $\Lambda = \Lambda(\omega, x)$ is the local gauge freedom.

For simplicity, we consider the diagonal permittivity and permeability.

$$
\varepsilon^{ij} = \varepsilon(\omega)\delta^{ij}, \quad (\mu^{-1})_{ij} = \mu^{-1}(\omega)\delta_{ij},
$$

where $(\delta^{ij}) = (\delta_{ij}) = \text{diag}(1, 1, 1)$.

\textbf{Gauge 1}

First we take the following gauge.

$$
\partial_i \{i\omega \phi + (\varepsilon \mu)^{-1} \text{div} A \} = 0.
$$

Ampère’s law gives the field eq. of $A$

$$(\Delta + \omega^2 \varepsilon \mu) A(\omega, x) = 0, \quad E = -i\omega A - \frac{i}{\omega \varepsilon \mu} \nabla(\text{div} A).$$

When $\varepsilon$ and $\mu$ are constants (do not depend on $\omega$), $\varepsilon = \varepsilon_1, \mu = \mu_1$, $\hat{A}(x)$ satisfies the free wave equation with the velocity $v = 1/\sqrt{\varepsilon_1 \mu_1}$.

$$(\Delta - \frac{1}{v^2} \frac{\partial^2}{\partial t^2}) \hat{A}(t, x) = 0,$$

$$v = 1/\sqrt{\varepsilon_1 \mu_1}, \quad \hat{A}(t, x) = \int A(\omega, x) e^{i\omega t} d\omega.$$  

We keep the case: $\varepsilon = \varepsilon(\omega), \mu = \mu(\omega)$. From the condition $\text{div} D = \varepsilon \text{div} E = 0$, which is derived from (4), we obtain

$$(\Delta + \omega^2 \varepsilon \mu) \text{div} A(\omega, x) = 0.$$

Using this equation, we can show the energy density $E$ is given by

$$E = \frac{1}{2}(E \cdot D + H \cdot B) = \frac{1}{2}(\varepsilon^{ij} E_i E_j + \mu^{-1}_{ij} B^i B^j)$$

$$= \frac{1}{2} \mu^{-1} A \cdot (\Delta + \omega^2 \varepsilon \mu) A + \text{total derivative}.$$

\textbf{Gauge 2}\footnote{This gauge is used in the Landau-Lifshitz textbook\cite{LL}.}

We can take another gauge.

$$\partial_i \phi = 0.$$  

\footnote{This gauge is used in the Landau-Lifshitz textbook\cite{LL}.}
Ampère's law \((5)\) gives the field eq. of \(A\)
\[
\Delta A - \nabla(\text{div}A) + \omega^2 \varepsilon \mu A = 0, \quad \mathbf{E} = -i\omega \mathbf{A}.
\]
From the condition \(\text{div} \mathbf{D} = \varepsilon \text{div} \mathbf{E} = 0\), we obtain
\[
\text{div} \mathbf{A} = -\frac{1}{i\omega} \text{div} \mathbf{E} = 0.
\]
Hence the field eq. \((19)\) reduces to \((14)\) in the present case of no charge and no currents (classical vacuum). In this gauge too, we can confirm the relation \((17)\).

### 3  Geometry in 4D space \(((K^\mu) = (\omega, K^i))\) and Induced Geometry in 3D space \(((k^i))\)

From the previous description, we know the energy can be expressed by either \(A(\omega, x), E(\omega, x)\) and \(B(\omega, x)\),
\[
H' = \int d^3x \int d\omega \mathcal{E} = \int d^3x \int d\omega \frac{1}{2} (\varepsilon^{ij} \mathbf{E}_i \mathbf{E}_j + \mu^{-1} \varepsilon^{ij} \mathbf{B}_i \mathbf{B}_j)
\]
\[
= \int d^3x \int d\omega \frac{1}{2} \mu^{-1} \mathbf{A} \cdot (\Delta + \omega^2 \varepsilon \mu) \mathbf{A},
\]
\[
(21)
\]
or their \(x\)-to-\(k\) Fourier-transformed ones \(\tilde{A}(\omega, k), \tilde{E}(\omega, k)\) and \(\tilde{B}(\omega, k)\).
\[
H' = \int d^3k \int d\omega \frac{1}{2} (\varepsilon^{ij}(\omega, k) \tilde{E}_i(\omega, k) \tilde{E}_j(\omega, k) + \mu^{-1}(\omega, k)_{ij} \tilde{B}_i(\omega, k) \tilde{B}_j(\omega, k))
\]
\[
= \int d^3k \int d\omega \frac{1}{2} \mu^{-1}(\omega, k) \tilde{A}(\omega, k) \cdot (-k^2 + \omega^2 \varepsilon(\omega, k) \mu(\omega, k)) \tilde{A}(\omega, k)
\]
\[
(22)
\]
We stress again that the forms of \(\varepsilon^{ij}\) and \(\mu^{ij}\) represent the substance property.

We now specify the forms in the geometrical way. First let us introduce the metric \(G_{\mu \nu}(K)\) in the 4 dim space \((\omega, K^i) \equiv (K^\mu)\), \(ds^2 = G_{\mu \nu}(K)dK^\mu dK^\nu\). As interesting metrics, we can consider the following ones.

1. Minkowski \(ds^2 = -d\omega^2 + \sum_{i=1}^3 dK_i^2\)
2. dS\(_4\) \(ds^2 = -d\omega^2 + e^{2H_0} \sum_{i=1}^3 dK_i^2\), \(H_0 > 0\)
3. AdS\(_4\) \(ds^2 = (dK^3)^2 + e^{-2H_0}(-d\omega^2 + (dK^1)^2 + (dK^2)^2)\)

\(^4\) The metric treatment of the permittivity and the permeability has been frequently suggested in the past \([10][11][12][13]\).
Figure 1: 3 dim hyper-surface \((24)\). \(0 \leq \omega \leq T\). Renormalization group flow.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{3 dim hyper-surface.}
\end{figure}

\(H_0\) is a model parameter (constant) which expresses the 4 dim curvature. With the aim of specifying (parametrizing) the 3 dim metric \(g_{ij}(\omega)\), we \textit{introduce} 3 dim hyper-surface in this 4 dim space \((\omega, K^i)\).

\[
\text{Dispersion relation :} \quad (k^i)^2 = p(\omega)^2, \quad (24)
\]

where the \textit{isotropy} of the 3 spacial directions \((K^1, K^2, K^3)\) is assumed, and \(p(\omega)\), which specify the hypersurface, is some function (of \(\omega\)) to be explained below. See Fig.1. In the following, we explain taking the case 1, Minkowski metric.

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5 Three types of metric, 1) Minkowski, 2) de Sitter (dS), 3) anti de Sitter (AdS), are all \textit{maximally symmetric}. \(H_0\) is a model parameter (constant) which expresses the 4 dim curvature.

6 Fig.1 shows the renormalization flow in the "holographic approach". We can regard the 3 dim hyper-surface as the "physical world" where the actual (not formally artificial) things occur. We are looking the physical world from the higher-dimensional (4 dim) space. On the point \(\omega\) of the \(\omega\)-axis (called "extra-axis"), there exists 3 dim \textit{ball} with the boundary of \(S^2\) sphere (radius \(p(\omega)\)). The center of the ball is \((\omega, K = 0)\). The ball is like a \textit{brane}. If we can regard the radius \(p(\omega)\) as a scale, the whole configuration surrounded by the hyper-surface describes the scale flow of the system. The "brane" moves along the \(\omega\)-axis changing its radius \(p(\omega)\).
Figure 2: Behavior of $p_1(\omega)$ in (26). Dispersion relation of the massless particle.

The *induced* metric $g_{ij}$, is given as

$$ds^2 = \{- (p')^2 k^i k^j + \delta_{ij}\} dk^i dk^j = g_{ij}(\omega, k) dk^i dk^j, \quad \dot{p} = \frac{dp}{d\omega}.$$ (25)

When $p(\omega)$ is specified, $g_{ij}(\omega, k)$ is explicitly given. We will soon show how to geometrically determine the form, but we here show some examples.

$$g_{ij}(\omega, k) = \begin{cases} 
\delta_{ij} - \frac{c^4}{\omega^2} k_{(1)}^i k_{(1)}^j, & p_1(\omega) = \frac{\omega}{c}, \quad \dot{p}_1 = \frac{1}{c}, \quad (k_{(1)}^i)^2 = \frac{\omega^2}{c^2}, \\
\delta_{ij} - \frac{c^4}{\omega^2} k_{(2)}^i k_{(2)}^j, & p_2(\omega) = \sqrt{\frac{\omega^2 - (mc^2)^2}{c^2}}, \quad \dot{p}_2 = \frac{\omega}{c^2} \frac{1}{p_2}, \quad (k_{(2)}^i)^2 = \frac{\omega^2 - (mc^2)^2}{c^2}.
\end{cases}$$ (26)

where $c$(light velocity) and $m$(mass) are some constants. See Fig.2 and Fig.3.

The system energy expression (22) is the summation over all frequency-momentum points $(\omega, k)$. The quantity, however, is divergent. Let us propose that the summation should be replaced by that of all hypersurfaces

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7 See eq.(32).
Figure 3: Behavior of $p_2(\omega)$ in (26). Dispersion relation of the massive particle.

From the requirement of the general coordinate invariance, the energy expression $H'$ of (22) is replaced by

$$H'' = \int d[\text{hyper-surface}(24)] \times$$

$$\frac{1}{2} \sqrt{\det g_{ij}(\omega, k)} \varepsilon^{ij}(\omega, k) \bar{E}_i(\omega, k) \bar{E}_j(\omega, k)$$

$$+ \mu^{-1}(\omega, k)_{ij} \bar{B}^i(\omega, k) \bar{B}^j(\omega, k))$$

$$\equiv \int d[\text{hyper-surface}(24)] \sqrt{\det g_{ij}(\omega, k)} \mathcal{E}[\mathbf{A}(\omega, k)]$$, \hspace{1cm} (27)$$

where we assume

$$\varepsilon^{ij}(\omega, k) = e_1 g^{ij}(\omega, k), \hspace{0.5cm} \mu(\omega, k)^{ij} = m_1 g^{ij}(\omega, k),$$ \hspace{1cm} (28)$$

where $e_1$ and $m_1$ are constants. Note that the expression $H''$ can be written by the vector potential $\mathbf{A}(\omega, x)$ in the non-local form.

We notice that $H''$ depends on the hyper-surface (24). There are many hyper-surfaces by varying the form of $p(\omega)$. The present model of the electromagnetism in substance should describe the fluctuation of the micro dynamics. In order to take it into account, we propose here to promote the
expression $H''$, (27), to the following generalized path-integral expression (31).

Before presenting (31), we explain a geometrical quantity, area $A$, of the hyper-surface. On the hyper-surface (24), $\sum_{i=1}^3 (k^i)^2 = p(\omega)^2$, the induced metric $g_{ij}$ gives us the area as the functional of the path $\{p(\omega) : 0 \leq \omega \leq T\}$ where $T$ is introduced as the upper bound (a boundary parameter) for the frequency $\omega$.

$$A[p(\omega)] = \int \sqrt{\det g_{ij}} \, d^3 k = \int_0^T \sqrt{\dot{p}^2 + 1} \, p^2 d\omega. \quad (29)$$

In order to express the statistical ensemble due to the micro fluctuation we take the following distribution $\Omega[p(\omega)]$ for the energy expression (27).

$$\Omega[p(\omega)] = \frac{1}{N} \exp\left(-\frac{1}{2\alpha'} A[p(\omega)]\right) = \frac{1}{N} \exp\left\{-\frac{1}{2\alpha'} \int_0^T \sqrt{\dot{p}^2 + 1} \, p^2 d\omega\right\}, \quad (30)$$

where a new model parameter $\alpha'$ (string tension) is introduced. $N$ is the normalization factor. Taking the above distribution $\Omega$, the system energy $H = E(T)$ is finally given by

$$E(T) = \frac{1}{N} \int_0^\infty d\rho \int_{p(0) = \rho}^{p(T) = \rho} \prod_{\omega,i} Dk^i(\omega) \times \tilde{E}[\tilde{A}(\omega, k)] \exp \left[-\frac{1}{2\alpha'} \int_0^T \sqrt{\dot{p}^2 + 1} \, p^2 d\omega\right], \quad (31)$$

where the integral is over all hyper-surfaces (24) and $\tilde{E}[\tilde{A}(\omega, k)]$ is defined in (27).

Among all possible paths $\{p(\omega) : 0 \leq \omega \leq T\}$, the dominant one is given by the minimal area principle.

$$\delta A[p(\omega)] = \delta \int_0^T \sqrt{\dot{p}^2 + 1} \, p^2 d\omega = 0. \quad (32)$$

The solution is explained in ref. [14]. In the calculation of (31), infrared (IR) and ultraviolet (UV) divergences appear. To regularize them, we calculate

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8 The ordinary path-integral is the summation over all possible lines (under the given boundary condition), whereas the present one is over all possible hyper-surfaces.
4 Lifshitz Formula

Let us take the material composed of three regions with different permittivities $\varepsilon_1(\omega), \varepsilon_m(\omega), \varepsilon_2(\omega) (\mu = 1)$, (Fig. 4). We consider the free energy at temperature $T$. Instead of the action derived from (14), we take the following simplified model of the Maxwell theory [15].

$$S = \frac{1}{2} \int d^3 x \int \frac{d\omega}{2\pi} \phi_\omega^* (\Delta + \omega^2 \varepsilon(\omega)) \phi_\omega,$$

where $\varepsilon(x, \omega) = \{ \varepsilon_1(\omega) \text{ when } x \in R_1, \varepsilon_m(\omega) \text{ when } x \in R_m, \varepsilon_2(\omega) \text{ when } x \in R_2 \}$.
This is the action for each region. Compare the simplified model above with that of Maxwell theory \([21]\). When \(\varepsilon(\omega) = c_1(\text{const.})\), the above expression is the action of the \((3+1)\) dim massless complex free scalar.

\[
S_{\text{free}} = \frac{1}{2} \int d^3 x \int \frac{dt}{2\pi} \hat{\phi}^*(x, t)(\Delta - c_1 \frac{\partial^2}{\partial t^2})\hat{\phi}(x, t),
\]

\[
\hat{\phi}(x, t) = \int_{-\infty}^{\infty} \phi_{\omega}(x)e^{i\omega t}d\omega.
\] \(\text{ (35)}\)

We keep the general case \((34)\). The field equation is given by

\[
(\Delta + \omega^2 \varepsilon(\omega))\phi_{\omega} = 0, \quad \phi_{\omega}(x_\perp, z) = \tilde{\phi}_{\omega}(z)e^{i\mathbf{q} \cdot \mathbf{x}_\perp}.
\] \(\text{ (36)}\)

This system is in the thermal equilibrium at temperature \(T\). We can realize this situation by imposing the periodicity condition on the time variable \(t\). The period is \(1/T\).

**Periodicity:** \(t \to t + \frac{1}{T}, \quad \omega_n = \frac{2\pi T}{\hbar}n.\) \(\text{ (37)}\)

In the plane perpendicular to \(z\)-axis, we impose the periodicity with the length \(L\). This is for the IR regularization. \(L\) is considered to be sufficiently large.

**Periodicity:** \(x_\perp \equiv (x, y) \to (x + L, y + L),\)

\[
\mathbf{q}(n_x, n_y) = (\frac{2\pi}{L}n_x, \frac{2\pi}{L}n_y).
\] \(\text{ (38)}\)

The wave function \(\tilde{\phi}_{\omega}(z)\) in \((36)\) satisfies.

\[
(-\mathbf{q}^2 + \partial_z^2 + \omega^2 \varepsilon(\omega))\tilde{\phi}_{\omega}(z) = 0.
\] \(\text{ (39)}\)

Assuming the form \(\tilde{\phi}_{\omega}(z) \propto e^{\pm \rho z}\), we obtain

\[
\tilde{\phi}_{j}^z(z) = A_j(\omega)e^{\rho_j z} + B_j(\omega)e^{-\rho_j z},
\]

\[
-q^2 + \rho_j^2 + \omega^2 \varepsilon_j(\omega) = 0 \quad (j = 1, m, 2).
\] \(\text{ (40)}\)

The proper boundary condition (damping in the remote regions) finally determines the wave function for each region as

\[
\begin{align*}
\text{region 1} & \quad z < -l, \quad \tilde{\phi}_{\omega}(z) = A(\omega)e^{\rho_1 z}, \\
\text{region m} & \quad -l < z < l, \quad \tilde{\phi}_{\omega}(z) = C_1(\omega)e^{\rho_m z} + C_2(\omega)e^{-\rho_m z}, \\
\text{region 2} & \quad z > l, \quad \tilde{\phi}_{\omega}(z) = B(\omega)e^{-\rho_2 z}
\end{align*}
\] \(\text{ (41)}\)
Imposing the continuity and the smoothness at the two boundaries, the boundary between $R_1$ and $R_m$ and that between $R_m$ and $R_2$, we can get the solution under the condition:

$$\Delta = 1 - \frac{(\rho_1 - \rho_m)(\rho_2 - \rho_m)}{\rho_1 + \rho_m}(\rho_2 + \rho_m)e^{-4\rho_m l} = 0 . \quad (42)$$

This condition comes from avoiding the trivial solution: $A = B = C_1 = C_2 = 0$.

Now we can define the free energy $F$ using the path-integral.

$$e^{-F_\chi} = \int D\phi D\phi^* e^{iS[\phi^*, \phi; \chi_1, \chi_m, \chi_2]} = \text{det}(\Delta + \omega^2 \varepsilon_\alpha(\omega)) = \exp \text{Tr} \ln (\Delta + \omega^2 (1 + \chi_\alpha(\omega))) , \quad (43)$$

where $\varepsilon_\alpha = 1 + \chi_\alpha$ is given by

- In $R_1$\, $1 + \chi_1(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2 + n_y^2) - \frac{\rho_1^2}{\omega^2}$ ,

- In $R_m$\, $1 + \chi_m(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2 + n_y^2) - \frac{\rho_m^2}{\omega^2}$ ,

- In $R_2$\, $1 + \chi_2(\omega) = \frac{1}{\omega^2}(\frac{2\pi}{L})^2(n_x^2 + n_y^2) - \frac{\rho_2^2}{\omega^2} . \quad (44)$

If the three functions $\chi_1(\omega), \chi_m(\omega), \chi_2(\omega)$, which characterize the material, are given, the wave function is completely solved at the IR-regularized (xy-plane) level. The free energy defined in $(43)$ still has UV-divergences. To deal with it, we use the ambiguity of energy origin $F = 0$.

REGULARIZATION 1
First we may subtract the value at $\chi = 0$. This is because we suppose the region of no substance does not contribute to the system energy.

$$F_C \equiv F_\chi - F_{\chi=0} = -\text{Tr} \ln(1 + \frac{\omega^2 \chi_\alpha(\omega)}{\Delta + \omega^2}) . \quad (45)$$

Next we may subtract trivial constants. We have interest only in the interaction between different materials, not in the each material’s property.

REGULARIZATION 2 (Entanglement)

$$F \equiv F_C(R_1 \cup R_m \cup R_2) - F_C(R_1) - F_C(R_m) - F_C(R_2) . \quad (46)$$
Finally we obtain the well-defined (finite) quantity\(^{15}\).

In Ref.\(^{15}\), it is shown that the Casimir force between two materials related by reflection is always attractive.

5 Ordinary Regularization for Casimir Energy

Let us consider 1+3 dim electromagnetism (free field theory, \((\varepsilon_0, \mu_0)\): constants of vacuum values) in Minkwski space \(^{19}\), \(^{20}\):

\[
S = \int d^4x \frac{1}{2} \mathbf{A} \cdot (\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \mathbf{A} = 0, \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \equiv 1,
\]

\[
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.
\]

2 perfectly-conducting plates parallel with the separation \(2l\) in the z-direction. See Fig.5. As for x- and y-directions, we impose the periodicity \(2L\) for the
The eigen frequencies and Casimir energy are
\[ \omega_{m_1, m_2, n} = \sqrt{\left(\frac{n\pi}{l}\right)^2 + \left(\frac{m_1\pi}{L}\right)^2 + \left(\frac{m_2\pi}{L}\right)^2}, \]
\[ E_{\text{Cas}} = 2 \cdot \sum_{m_1, m_2, n \in \mathbb{Z}} \frac{1}{2} \omega_{m_1, m_2, n} \geq 0, \tag{49} \]
where \( \mathbb{Z} \) is the set of all integers. \( \frac{1}{2} \omega_{m_1, m_2, n} \) is the zero-point oscillation energy.

Introducing the cut-off function \( g(x) \) (= 1 \( [0 < x < 1] \) or 0 [otherwise]),
\[ E_{\Lambda, \text{Cas}} = \sum_{m_1, m_2, n \in \mathbb{Z}} \omega_{m_1, m_2, n} g \left( \frac{\omega_{m_1, m_2, n}}{\Lambda} \right) \geq 0, \tag{50} \]
where \( \Lambda \) is the UV-CutOff. Taking the continuum limit \( L \rightarrow \infty, L \ll l \rightarrow \infty \), we obtain
\[ E_{\Lambda, \text{Cas}}^0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_x dk_y dk_z}{(\frac{\pi}{L})^2} \sqrt{k_x^2 + k_y^2 + k_z^2} g \left( \frac{k}{\Lambda} \right) \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k| \leq \Lambda \frac{dk_x dk_y dk_z}{(\frac{\pi}{L})^2} \sqrt{k_x^2 + k_y^2 + k_z^2} \geq 0. \tag{51} \]
Note that \( E_{\text{Cas}}, E_{\Lambda, \text{Cas}} \) and \( E_{\Lambda, \text{Cas}}^0 \) are all positive-definite. In a familiar way, regarding \( E_{\Lambda, \text{Cas}}^0 \) as the origin of the energy scale, we consider the quantity \( u = (E_{\Lambda, \text{Cas}} - E_{\Lambda, \text{Cas}}^0)/(2L)^2 \) as the physical Casimir energy and evaluate it with the help of the Euler-MacLaurin formula as \( u = \left( \pi^2 / (2L)^3 \right) \left( B_4 / 4! \right) = -\left( \pi^2 / 720 \right) \left( 1 / (2L)^3 \right) < 0. \]
\[ \] The final result is negative. In the present analysis we take a new regularization which keeps positive-definiteness.

## 6 New Regularization for Casimir Energy

First we re-express \( E_{\Lambda, \text{Cas}}^0 \) using a simple identity: \( l = \int_0^l dw \) (\( w \): a regularization axis).
\[ E_{\Lambda, \text{Cas}}^0 / (2L)^2 = \frac{1}{2^2 \pi^3} \int_0^1 dw \int_{k \leq \Lambda} P(k) 2\pi k^2 dk \]
\[ B_4 \] is the 4-th Bernoulli number.
where the integration variable changes from the momentum \((k)\) to the coordinate \((r \equiv 1/k)\). The integration region in \((R, w)\)-space is the infinite rectangular shown in Fig.6.

The expression (52) severely diverges as \(\Lambda \to \infty\). In order to regularize it, we first replace the summation over all coordinate-space points \((R, w)\) with the summation over all possible paths (path-integral). For each path we introduce the damping factor as in the same way in Sec.3. Hence the above expression is replaced (regularized) by the following path-integral expression:

\[
E_{\text{Cas}}^W / (2L)^2 = \frac{2\pi}{2^2\pi^3} \int_{\text{all paths}} dr(w) \prod_w D r(w) \left[ \int dw' P\left(\frac{1}{r(w')}\right)r(w')^{-4} \right] \exp \left\{ -W[r(w)] \right\},
\]

(53)

where the integral is over all paths \(r(w)\) which are defined between \(0 \leq w \leq l\) and whose value is above \(\Lambda^{-1}\), as shown in Fig.7. \(W[r(w)]\) is some damping functional. \(W[r(w)] = 0\) corresponds to (52). The slightly-more-restrictive regularization is

\[
E_{\text{Cas}}^W / (2L)^2 = \frac{2\pi}{2^2\pi^3} \int_{\Lambda^{-1}} d\rho \int_{r(0) = r(l) = \rho} dr(w) \prod_w D r(w) \left[ \int dw' P\left(\frac{1}{r(w')}\right)r(w')^{-4} \right] \exp \left\{ -W[r(w)] \right\} \geq 0,
\]

(54)

where the integral is over all periodic paths. Note that the above regularization keeps the positive-definite property. It is mainly defined by the choice of \(W[r(w)]\). In order to specify it, we introduce the following metric in \((R, w)\)-space [519].

\[
\text{Dirac Type : } ds^2 = dR^2 + V(R)dw^2, \quad V(R) = \Omega^2 R^2,
\]

(55)
or

\[
\text{Standard Type : } ds^2 = \frac{1}{dw^2} (dR^2 + V(R)dw^2)^2,
\]

(56)

\[
V(R) = \Omega^2 R^2,
\]
where $\Omega$ is the regularization constant. (When $V(R) = 1$, $w$ is the familiar Euclidean time.) On a path $R = r(w)$, the induced metric and the length $L$ is given as follows. As the damping functional $W[r(w)]$, we take the length $L$.

$$ds^2 = dw^2(r'^2 + \Omega^2 r^2)^2, \quad r' \equiv \frac{dr}{dw},$$

$$L = \int ds = \int (r'^2 + \Omega^2 r^2) dw,$$

$$W[r(w)] = \frac{1}{2\alpha'}L = \frac{1}{2\alpha'} \int (r'^2 + \Omega^2 r^2) dw,$$

(57)

where $\alpha'$ and $\Omega$ are the regularization parameters. They can be regarded as the model parameters for the statistical ensemble which is taken as the regularization. The limit $\alpha' \to \infty$ corresponds to (52).

Numerical calculation can evaluate $E^W_{\text{Cas}}$, and we expect the following form\cite{14,16,17,18}.

$$\frac{E^W_{\text{Cas}}}{(2L)^2} = \frac{a}{b^3}(1 - 3b \ln(l\Lambda)),$$

(58)

where $a$ and $b$ are some constants. $a$ should be positive because of the positive-definiteness of (54). The present regularization result has, like the ordinary renormalizable ones such as the coupling in QED, the log-divergence. The divergence can be renormalized into the boundary parameter $l$. This means $l$ flows according to the renormalization group.

$$l' = l(1 - 3b \ln(l\Lambda))^{-\frac{1}{4}},$$

$$\beta \equiv \frac{d\ln(l'/l)}{d\ln \Lambda} = b, \quad |b| \ll 1,$$

(59)

where $\beta$ is the renormalization group function, and we assume $|b| \ll 1$. The sign of $b$ determines whether the length separation increases ($b > 0$) or decreases ($b < 0$) as the measurement resolution becomes finer ($\Lambda$ increases). In terms of the usual terminology, attractive case corresponds to $b > 0$, and repulsive case to $b < 0$. Compare the above way of determining the force-direction with that in Sec.3 (ordinary case) where the relation $F = -\partial V/\partial x$ is necessarily used.
7 Conclusion

The electromagnetism in substance is formulated in the geometrical way. The dispersion relation is introduced by the 3 dim hyper-surface \( \omega, K^i \) in \( \omega, K^i \) space. The permittivity and the permeability are regarded as the metric defined on the hyper-surface. The micro fluctuation effect is taken into account by the generalized path-integral \( \alpha' \) (string tension) is introduced, which is necessary in the present formulation using the geometrical quantity, area \( A \). We point out the renormalization of the boundary parameters, \( T \) (boundary of the frequency \( \omega \)) and \( H_0 \) (4 dim curvature), takes place in the treatment of IR and UV divergences. In relation to the problem of the attractive or repulsive force, Lifshitz formula is explained in the context of the regularization of the quantum field theory. The regularization is basically the same as that in Sec[5], and there appears no renormalization of \( l \). The new regularization is applied to Casimir energy calculation and compared with the ordinary case. The advantageous points are 1) the positivity is preserved in the regularization, 2) attractive or repulsive is determined by the sign of the renormalization group \( \beta \)-function for the boundary parameter \( l \).

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Figure 6: The integral region of (52).

Figure 7: A general path \( r(w) \) of (53) and a periodic path \( r(w) \) of (54).