The Graphical Traveling Salesperson Problem has no Integer Programming Formulation in the Original Space

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Abstract

The Graphical Traveling Salesperson Problem (GTSP) is the problem of assigning, for a given weighted graph, a nonnegative number $x_e$ to each edge $e$ such that the induced multi-subgraph is of minimum weight among those that are spanning, connected and Eulerian. Naturally, known mixed-integer programming formulations use integer variables $x_e$ in addition to others. Denis Naddef posed the challenge of finding a (reasonably simple) mixed-integer programming formulation that has integrality constraints only on these edge variables. Recently, Carr and Simonetti (IPCO 2021) showed that such a formulation cannot consist of polynomial-time certifyable inequality classes unless $\text{NP} = \text{coNP}$. In this note we establish a more rigorous result, namely that no such MIP formulation exists at all.

1 Introduction

Let $G = (V, E)$ be a graph and let $c \in \mathbb{R}^E$. The Graphical Traveling Salesperson problem is about finding $c$-minimum cost tour in $G$ that visits each node at least once, where edges can be used multiple times. It can be formulated as the following constraint integer program due to Cornéjols, Fonlupt and Naddef [2].

$$\begin{align*}
\text{min} \quad & c^T x \\
\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq 2 \quad \forall \emptyset \neq S \subsetneq V \quad (1a) \\
& \sum_{e \in \delta(v)} x_e \text{ is even} \quad \forall v \in V \quad (1b) \\
& x_e \in \mathbb{Z}_{\geq 0} \quad \forall e \in E
\end{align*}$$

Here, $\delta(S) := \{ e \in E : |e \cap S| = 1 \}$ and $\delta(v) := \delta(\{v\})$ denote the cuts induced by node set $S \subseteq V$ and node $v \in V$, respectively. For each edge $e \in E$, the variable $x_e$ indicates how often $e$ is traversed in the tour.
The authors of [2] describe several classes of inequalities that are valid for the GTSP polyhedron $P_{g tsp}(G)$ defined as the convex hull of all feasible solutions, i.e.,

$$P_{g tsp}(G) := \text{conv}\{x \in \mathbb{Z}^E : x \text{ satisfies (1b), (1c) and (1d)}\}.$$ 

Among these were path, wheelbarrow and bicycle inequalities. In order to turn (1) into a mixed-integer programming model (MIP), constraint (1c) can be replaced by this pair of constraints:

$$\sum_{e \in \delta(v)} x_e = 2y_v \quad \forall v \in V \quad (2a)$$

$$y_v \in \mathbb{Z} \quad \forall v \in V \quad (2b)$$

These additional $y$-variables are artificial and their presence has no impact on the linear programming relaxation of (1). For this reason, Naddef posed the challenge of finding a simple mixed-integer programming formulation that involves, apart from the $x$-variables, only continuous variables [4]. According to [1], he had the formulation from [2] with path, wheelbarrow and bicycle inequalities in mind. There exist other (mixed-)integer programming formulations for the GTSP, see [1, 3], all of which requiring additional integral variables.

Recently, Carr and Simonetti showed that such a formulation cannot be nice in the sense that it cannot consist of inequality families for which one can certify membership in polynomial time, provided $\text{NP} \neq \text{coNP}$ (see Section 4.2 in [1]).

The purpose of this paper is to show that the reason for the non-existence of a simple formulation does not lie in complexity theory. In fact, we show that no such formulation exists at all:

**Theorem 1.** The GTSP has no mixed-integer programming formulation whose only integer variables are the $x$-variables from (1).

## 2 Nonexistence of the formulation

![Figure 1: Example instance $G^*$ with unit costs $c^*$](image)

(a) Instance $G^* = (V^*, E^*)$ with unit costs $c^* = (1, 1, 1, 1, 1, 1)\top$.

To minimum-cost solutions are depicted in (1b) and (1c).
Proof of Theorem 1. We consider the graph $G^* = (V^*, E^*)$ from Fig. 1 with unit edge costs $c^* \in \mathbb{R}^{E^*}$. It is easy to see that $G^*$ has no Hamiltonian cycle, and therefore there is no solution of value $|V^*| = 5$. Hence, the tours in Figure 1(b) and 1(c) denoted by $x^*_1, x^*_2 \in \mathbb{Z}^{E^*}$ are optimal. Their midpoint is the point $x^* = (1, 1, 1, 1, 1)^T$, which is integral but infeasible for 1 as it violates (1c). Moreover, since the tours are optimal, so is $x^*$.

Assume, for the sake contradiction, that there exists a mixed-integer programming formulation $Q = \{(x, y) \in \mathbb{Z}^E \times \mathbb{R}^q : Ax + By \leq d\}$ that has integrality constraints only for the $x$-variables. Hence, the projection of $Q$ onto the $x$-variables is the set of feasible solutions to (1), and hence

$$\min \{c^T x : (x, y) \in Q\}$$

is equivalent to (1). In particular, feasibility of $x^*_1$ and $x^*_2$ for (1) implies that there exist $y^*_1, y^*_2 \in \mathbb{R}^q$ such that $(x^*_i, y^*_i) \in Q$ for $i = 1, 2$. Now let $y^* \in \mathbb{R}^q$ be the midpoint of $y^*_1$ and $y^*_2$. By convexity of the linear relaxation of $Q$ and integrality of $x^*$, also $(x^*, y^*)$ is an optimal solution to (3). This contradicts the fact that $x^*$ is infeasible for (1). \hfill \Box

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References

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