ON THE ANNIHILATOR GRAPH OF GROUP RINGS

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Abstract. Let $R$ be a commutative ring with nonzero identity and $G$ be a nontrivial finite group. Also, let $Z(R)$ be the set of zero-divisors of $R$ and, for $a \in Z(R)$, let $\text{ann}(a) = \{r \in R \mid ra = 0\}$. The annihilator graph of the group ring $RG$ is defined as the graph $AG(RG)$, whose vertex set consists of the set of nonzero zero-divisors, and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$. In this paper, we study the annihilator graph associated to a group ring $RG$.

1. Introduction

Let $R$ be a commutative ring with nonzero identity, and let $Z(R)$ be the set of zero-divisors of $R$. If $X$ is a subset of $R$, then the annihilator of $X$ is the ideal $\text{ann}(X) = \{r \in R \mid rX = 0\}$. The Jacobson radical of $R$ is denoted by $J(R)$. For any subset $Y$ of $R$, the cardinality of $Y$ is denoted by $|Y|$. Put $Y^* = Y \setminus \{0\}$. Let $G$ be a finite group that is defined multiplicatively. Also we denote the cyclic group of order $n$ by $C_n$, and a finite field with $q$ elements by $\mathbb{F}_q$.

The concept of the zero-divisor graph of a commutative ring $R$, denoted by $\Gamma(R)$, was introduced by Beck in [12], who let all elements of $R$ be vertices and was mainly interested in colorings. The work of Beck is further continued by Anderson and Naseer in [6] and, for other graph theoretical aspects, by Anderson and Livingston in [5]. While they focus just on the zero-divisors of the rings, there are many other kinds of graphs associated to ring, some of which are extensively studied, see for example [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 20].

In [9], Badawi introduced the concept of the annihilator graph for a commutative ring $R$, which is denoted by $AG(R)$. The annihilator graph $AG(R)$ is an undirected graph whose vertex set is the set of all nonzero zero-divisors of $R$, and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$. Also, the annihilator graph of a commutative semigroup is studied in [1].
Let $RG$ be a commutative group ring and $Z(RG)$ be its set of zero-divisors. In this paper, we study the annihilator graph of the group ring $RG$, which is denoted by $AG(RG)$. Also, we examine the planarity, outerplanarity of $AG(RG)$ and some properties of the line graph of $AG(RG)$.

Let $G$ be a graph with vertex set $V(G)$. For distinct vertices $x, y \in V(G)$, we use the notation $x \sim y$ to say that $x$ and $y$ are adjacent. The distance between two distinct vertices $x$ and $y$ in $G$ is the number of edges in a shortest path connecting them and it is denoted by $d(x, y)$. The diameter of a connected graph $G$, denoted by $\text{diam}(G)$, is the maximum distance between any pair of the vertices of $G$. The degree of a vertex $v$ of $G$, denoted by $\text{deg}(v)$, is the number of edges of $G$ incident with $v$ such that the maximum degree of a graph $G$, denoted by $\Delta(G)$. The girth of $G$, denoted by $\text{gr}(G)$, is the length of a shortest cycle in $G$. If $G$ does not contain a cycle, then $\text{gr}(G)$ is defined to be infinity. The complete graph is a graph in which any two distinct vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_n$. A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$. A complete bipartite graph is a bipartite graph in which every vertex of one part is adjacent to every vertex of the other part. If the size of one of the parts in a complete bipartite graph is 1, then the complete bipartite graph is said to be a star graph. The line graph $L(G)$ of $G$ is the graph whose vertices correspond to the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent.

2. Preliminaries

Throughout the paper, $R$ is a nontrivial commutative ring and $G$ is a nontrivial Abelian group. A group ring $RG$ is a construction which involves a group $G$ and a ring $R$. The group ring is a ring and the underlying set consists of formal sums

$$\sum_{g \in G} a_g g \ (a_g \in R, g \in G)$$

for which all but finitely many coefficients $a_g$ are zero. The addition of two elements of $RG$ is defined point-wise

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g)g,$$

and the multiplication is defined by

$$(\sum_{g \in G} a_g g)(\sum_{g \in G} b_g g) = (\sum_{g \in G} c_g g),$$

where

$$c_g = \sum_{x \in G} a_{g,b}.$$
Clearly, if $R$ and $G$ are commutative, then $RG$ is commutative. We can define an action of the ring $R$ on $RG$ by
\[ r \sum_{g \in G} a_g g = \sum_{g \in G} (ra_g)g. \]
This definition makes $RG$ into a left $R$-module. The group ring is then a free $R$-module with basis consisting (of copies) of elements of $G$, and it is of rank $|G|$. Indeed, \{1_{RG} : g \in G\} is a basis for $RG$. So if $R$ and $G$ are finite, then $|RG| = |R|^{|G|}$.

If $RG$ is a group ring and $X$ is a finite subset of $G$, then $\hat{X} := \Sigma_{x \in X} x$. In particular, if $X = G$ such that $|G| < \infty$, then $\hat{G} = \Sigma_{g \in G} g$. $\hat{G} \in Z(RG)$, since $\hat{G}(1 - g) = 0$.

The following lemmas are needed for the rest of the paper.

**Lemma 2.1.** The following statements hold.

(i) $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$;

(ii) $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Proof.** If $RG$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $|RG| = 4$. So $R \cong \mathbb{Z}_2$ and $G \cong C_2$ such that $G = \{1, g\}$. It is easy to see that $Z(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and $Z(\mathbb{Z}_2 C_2) = \{0, \hat{G}\}$. Hence $\mathbb{Z}_2 C_2 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, if $RG$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2$ and $G \cong C_3$ such that $G = \{1, g, g^2\}$. $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, since $Z(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(1, 1, 1)\}$ and $Z(\mathbb{Z}_2 C_3) = \{0, \hat{g}, 1 - g, 1 - g^2, g - g^2\}$. So the proof is completed.

**Lemma 2.2.** If $|RG| = 9$, then $RG \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

**Proof.** Suppose that $|RG| = 9$. Then $R \cong \mathbb{Z}_3$ and $G = \{1, g\} \cong C_2$. So $Z(RG) = \{0, \hat{G}, 2\hat{G}, \bar{1}, 2\bar{1}, \bar{2} + g, g + \bar{2}, \bar{g} + \bar{2}\}$. Thus $RG$ is a nonlocal ring. We know that $RG$ is a finite commutative ring. So $RG$ is a direct product of at least two local rings. On the other hand, $|RG| = 9$. So $RG \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

**Lemma 2.3.** If $AG(RG)$ is a complete graph, then $R$ is a local ring, $G$ is a $p$-group, and $p \in J(R)$.

**Proof.** Since $RG$ is a finite ring, $RG \cong R_1 \times \cdots \times R_n$ such that $R_i$ is a local ring for $1 \leq i \leq n$. If $n \geq 3$, then, by [9, Theorem 2.2], $d((0, 1, 0, \ldots, 0), (1, 1, 0, \ldots, 0)) = 2$ in $AG(RG)$. So $n \leq 2$. Suppose that $RG \cong R_1 \times R_2$ with $|R_2| \geq 3$. Then $d((0, 1), (0, r)) = 2$ in $AG(RG)$, where $1 \neq r \in R_2^*$. If $|R_1| = |R_2| = 2$, then $RG \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is impossible, by Lemma 2.1. Hence $RG$ is a local ring. So, by [19], the proof is completed.

3. Some properties of $AG(RG)$

We begin this section with the following proposition which is obtained from [9, Theorem 2.2] and [9, Corollary 2.11].

**Proposition 3.1.** The following statements hold.

(i) $AG(RG)$ is connected and $\text{diam}(AG(RG)) \leq 2$;
Theorem 3.2. \( \text{gr}(AG(RG)) = \text{gr}(\Gamma(RG)) \).

Proof. Clearly, \( \Gamma(RG) \) is a spanning subgraph of \( AG(RG) \). By [2, Proposition 2.8], \( \text{gr}(\Gamma(RG)) = 3 \) if and only if \( RG \) is neither \( Z_2 C_2 \) nor \( F_p C_q \) such that \( F_p C_q \cong F_{q_1} \times F_{q_2} \). First, we show that \( \text{gr}(\Gamma(RG)) = 3 \) if and only if \( \text{gr}(AG(RG)) = 3 \). Suppose that \( \text{gr}(\Gamma(RG)) = 3 \), which implies that \( \text{gr}(AG(RG)) = 3 \). If \( \text{gr}(AG(RG)) = 3 \), then, by [9, Corollary 2.11], \( \text{gr}(\Gamma(RG)) \in \{3, 4, \infty\} \). Let \( \text{gr}(\Gamma(RG)) = \infty \). Then, by [2, Proposition 2.8.1], \( RG \cong Z_2 C_2 \). Hence \( AG(RG) \cong K_1 \). Thus \( \text{gr}(AG(RG)) = \infty \), which is impossible. Now, suppose that \( \text{gr}(\Gamma(RG)) = 4 \). Then, by [2, Proposition 2.8.2] and Lemmas 2.1 and 2.2, \( RG \cong F_1 \times F_2 \) such that \( F_1 \) and \( F_2 \) are fields with at least three elements. In this situation, \( AG(F_1 \times F_2) \) is a complete bipartite graph. Hence \( \text{gr}(AG(RG)) = 4 \), which is impossible. So \( \text{gr}(\Gamma(RG)) = 3 \).

Now, we show that \( \text{gr}(AG(RG)) = 4 \) if and only if \( \text{gr}(\Gamma(RG)) = 4 \). Let \( \text{gr}(AG(RG)) = 4 \). Then, by [2, Proposition 2.8], \( \text{gr}(\Gamma(RG)) \in \{4, \infty\} \). If \( \text{gr}(\Gamma(RG)) = \infty \), then, by [2, Proposition 2.8], \( RG \cong Z_2 C_2 \). Thus \( AG(RG) \cong K_1 \), and so \( \text{gr}(AG(RG)) = \infty \), which is impossible. Thus \( \text{gr}(\Gamma(RG)) = 4 \).

Now, if \( \text{gr}(\Gamma(RG)) = 4 \), then, by Proposition 3.1, \( \text{gr}(AG(RG)) \in \{3, 4\} \). By the above argument, \( \text{gr}(AG(RG)) = 3 \) if and only if \( \text{gr}(\Gamma(RG)) = 3 \). So we conclude that \( \text{gr}(AG(RG)) = 4 \).

Finally, let \( \text{gr}(AG(RG)) = \infty \). Then clearly, \( \text{gr}(\Gamma(RG)) = \infty \). If \( \text{gr}(\Gamma(RG)) = \infty \), then, by [2, Proposition 2.8], \( RG \cong Z_2 C_2 \). Hence \( AG(RG) \cong K_1 \).

Therefore \( \text{gr}(AG(RG)) = \infty \).

\[ \Box \]

4. Planarity of \( AG(RG) \)

In this section, we investigate when \( AG(RG) \) is planar, outerplanar or ring graph whenever \( RG \) is a finite ring.

Recall that a graph is said to be planar if it can be drawn in the plane, so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem says that a graph is planar if and only if it contains no subdivision of \( K_5 \) or \( K_{3,3} \). Let \( G \) be a graph with \( n \) vertices and \( q \) edges. We recall that a chord is any edge of \( G \) joining two nonadjacent vertices in a cycle of \( G \). We say that \( C \) is a primitive cycle if it has no chord. Also, a graph \( G \) has a primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The number \( \text{frank}(G) \) is called the free rank of \( G \) and it is the number of primitive cycles of \( G \). Also, the number \( \text{rank}(G) = q - n + r \) is called the cycle rank of \( G \), where \( r \) is the number of connected components of \( G \). By [16, Proposition 2.2], we have \( \text{rank}(G) \leq \text{frank}(G) \). A graph \( G \) is called a ring graph if it satisfies in one of the following equivalent conditions (see [16]).
(i) \( \text{rank}(G) = \text{rank}(G) \);
(ii) \( G \) satisfies the PCP and \( G \) does not contain a subdivision of \( K_4 \) as a subgraph.

Also, an undirected graph is outerplanar if and only if it does not contain a subdivision of \( K_4 \) or \( K_{2,3} \). Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

We begin this section with the following theorem.

**Theorem 4.1.** \( AG(RG) \) is planar if and only if \( RG \) is isomorphic to one of the following group rings.

(i) \( \mathbb{Z}_2 \mathbb{C}_2 \);
(ii) \( \mathbb{Z}_2 \mathbb{C}_3 \);
(iii) \( \mathbb{Z}_3 \mathbb{C}_2 \);
(iv) \( \mathbb{F}_4 \mathbb{C}_2 \).

**Proof.** First, suppose that \( AG(RG) \) is planar. Then we have the following cases.

**Case 1.** \(| Z(R) | \geq 3 \). Then there exist distinct nonzero zero-divisors \( r \) and \( s \) such that \( rs = 0 \), since \( \Gamma(RG) \) is a connected spanning subgraph of \( AG(RG) \). If \( 1 \neq g \in G \), then \( AG(RG) \) contains a copy of \( K_{3,3} \) with vertex set \( \{ g, rg, r \hat{G} \} \cup \{ 1 - g, sg, s \hat{G} \} \).

**Case 2.** \(| Z(R) | = 2 \). Since \( | R | \leq | Z(R) |^2 \) and \( R \) is not a field, \( | R | = 4 \). Let \( Z(R) = \{ 0, a \} \). Then \( a^2 = 0 \). Now, suppose that \(| G | \geq 5 \). Then there exist distinct nonidentity elements \( g_1, g_2, g_3 \) and \( g_4 \) in \( G \). Hence \( AG(RG) \) has a copy of \( K_5 \) with vertex set \( \{ g, a(1 - g_1), a(1 - g_2), a(1 - g_3), a(1 - g_4) \} \).

So we conclude that \(| G | < 5 \).

First, suppose that \(| G | = 2 \). Then \( \text{Char}(R) = 4 \) or \( \text{Char}(R) = 2 \), since \( | R | = 4 \). If \( \text{Char}(R) = 4 \), then \( R \cong \mathbb{Z}_4 \) and \( Z(R) = \{ 0, 2 \} \). Clearly, we have

\[
\begin{align*}
\overline{2} & \in \text{ann}(g(1 - g)), \\
\overline{1} - g & \in \text{ann}(g(1 - g)), \\
\hat{G} & \in \text{ann}(g(1 - g)), \\
\overline{2} & \in \text{ann}(g(1 - g)).
\end{align*}
\]

Hence \( AG(RG) \) contains a copy of \( K_{3,3} \) with vertex set \( \{ \hat{G}, 2, 1 - g, \hat{G} \} \cup \{ 1 - g, \overline{2}, \overline{1} - g \} \).

If \( \text{Char}(R) = 2 \), then \( R \cong \mathbb{F}_4[\overline{2}] = \{ 0, \overline{1}, \overline{2}, \overline{3} \} \) such that \( Z(R) = \{ 0, \overline{1} \} \). Now, it is easy to see that \( \overline{1} \in \text{ann}(g(1 - g)) \) such that \( Z(R) = \{ 0, \overline{1} \} \).
\[\begin{align*}
\mathfrak{r} \in \text{ann}(\mathfrak{T} + \mathfrak{g})(\mathfrak{T} + (\mathfrak{T} + \mathfrak{g})) \setminus (\text{ann}(\mathfrak{T} + \mathfrak{g}) \cup \text{ann}(\mathfrak{T} + (\mathfrak{T} + \mathfrak{g}))) \\
(\mathfrak{T} + \mathfrak{g}) + \mathfrak{r} \in \text{ann}(\mathfrak{g})(\mathfrak{T} + (\mathfrak{T} + \mathfrak{g})) \setminus (\text{ann}(\mathfrak{g}) \cup \\
\text{ann}(\mathfrak{T} + \mathfrak{g}) + (\mathfrak{T} + \mathfrak{g})) \\
\mathfrak{r} \in \text{ann}(\mathfrak{T} + (\mathfrak{T} + \mathfrak{g})) \setminus (\text{ann}(\mathfrak{T} + \mathfrak{g}) + (\mathfrak{T} + \mathfrak{g})) \\
(\mathfrak{T} + \mathfrak{g}) + \mathfrak{r} \in \text{ann}(\mathfrak{T} + (\mathfrak{T} + \mathfrak{g})) \setminus (\text{ann}(\mathfrak{T} + \mathfrak{g}) + (\mathfrak{T} + \mathfrak{g})).
\end{align*}\]

Thus \(AG(RG)\) contains a copy of \(K_{3,3}\) with vertex set \(\{\mathfrak{g}, 1 + (\mathfrak{T} + \mathfrak{g}), 2, 3, 4, 5\}\).

Now, suppose that \(|G| = 3\). Then \(G = \{1, g, g^2\}\). In this situation, if \(\text{Char}(R) = 4\), then \(R \cong \mathbb{Z}_4\) such that \(Z(R) = \{0, \mathfrak{T}\}\). It is easy to see that \((\mathfrak{T} + \mathfrak{g})(\mathfrak{T} + g - g^2) = 0\). Hence \(\mathfrak{T} + g - g^2 \in Z(RG)\). Also, we have
\[
\begin{align*}
\mathfrak{T} \in \text{ann}(\mathfrak{T} + g - g^2) \setminus (\text{ann}(\mathfrak{T} + g - g^2) \cup \text{ann}(\mathfrak{T} + g - g^2)); \\
\mathfrak{T} \in \text{ann}(\mathfrak{T} + g - g^2) \setminus (\text{ann}(\mathfrak{T} + g - g^2) \cup \text{ann}(\mathfrak{T} + g - g^2)),
\end{align*}\]

So \(AG(RG)\) contains a copy of \(K_{3,3}\) with vertex set \(\{\mathfrak{g}, \mathfrak{T}, \mathfrak{T} - g^2, \mathfrak{T} - g^2\}\).

If \(\text{Char}(R) = 2\), then \(R \cong \mathbb{Z}_2[\mathfrak{g}]\) such that \(Z(R) = \{0, \mathfrak{T}\}\). Hence \(AG(RG)\) contains a copy of \(K_{3,3}\) with vertex set
\[
\{\mathfrak{T} - g, \mathfrak{T} - g^2, \mathfrak{T} - g^2\} \cup \{\mathfrak{g}, \mathfrak{T}, \mathfrak{T} - g^2\}.
\]

Finally, in this case, suppose that \(|G| = 4\). Then there exist nonidentity distinct elements \(g_1, g_2\) and \(g_3\) such that \(G = \{1, g_1, g_2, g_3\}\). Let \(r, s\) be nonzero and nonidentity distinct elements in \(R\). Thus \(AG(RG)\) contains a copy of \(K_{3,3}\) with vertex set \(\{1 - g_1, 2 - g_2, 3 - g_3\}\).

**Case 3.** If \(|Z(R)| = 1\). Since \(R\) is finite and \(Z(R) = \{0\}\), we conclude that \(R\) is a field. So we have the following subcases.

**Subcase 1.** \(\text{Char}(R) | |G|\) and \(|R| \geq 6\). Then \(G^2 = 0\) and there exist distinct nonzero elements \(r_1, r_2, r_3, r_4\) and \(r_5\) in \(R\). Thus \(AG(RG)\) contains a copy of \(K_5\) with vertex set \(\{r_1G, r_2G, r_3G, r_4G, r_5G\}\).
Subcase 2. \( \text{Char}(R) \mid |G| \) and \( |R| = 5 \). So \( \hat{G}^2 = 0 \) and \( R \cong \mathbb{Z}_5 \). Let \( 1 \neq g \in G \). Then \( AG(RG) \) contains a copy of \( K_5 \) with vertex set \\
\{\hat{G}, 2\hat{G}, 3\hat{G}, 4\hat{G}, 1 - g\}.

Subcase 3. \( \text{Char}(R) \mid |G| \) and \( |R| = 4 \) such that \( R = \{0, 1, r, s\} \). Then \( \hat{G}^2 = 0 \). We know that \( R \) is a field. Hence \( \text{Char}(R) = 2 \). If \( |G| \geq 4 \), then there exists proper subgroup \( H \) of \( G \) such that \( |H| = 2 \). Hence there exists \( g_1 \in G \setminus H \). We have \( \text{Char}(R) \mid |H| \). Hence \( \hat{H}^2 = 0 \). Also \( H\hat{G} = 2\hat{G} = 0 \) and \( g_1\hat{H} \neq \hat{H} \). Thus \( AG(RG) \) contains a copy of \( K_5 \) with vertex set \\
\{\hat{G}, \hat{H}, r\hat{G}, s\hat{G}, g_1\hat{H}\}.

If \( \text{Char}(R) = 2 \) and \( |G| < 4 \), then \( |G| = 2 \). In this subcase, \( R \) is a local ring, \( G \) is a 2-group and \( 2 \in J(R) \). So, by [19], \( RG \) is a local ring. By [2, Definition 2.3], \( Z(RG) = \langle 1 - g; g \in G \rangle \). Since \( \text{Char}(R) \mid |G|, \ AG(RG) \cong K_3 \). Thus \( RG \cong F_4 C_2 \).

Subcase 4. \( \text{Char}(R) \mid |G| \) and \( |R| = 3 \). Then \( \text{Char}(R) = 3 \), \( R \cong \mathbb{Z}_3 \) and \( |G| = 3k \), for some positive integer \( k \).

Suppose that \( |G| > 3 \). Then \( G \) has a proper subgroup \( H \) such that \( |H| = 3 \). In this situation, \( \text{Char}(R) = 3 \). Hence \( \hat{H}^2 = 0 \). So \( H\hat{G} = 0 \). Let \( 1 \neq g \in G \setminus H \) and \( 1 \neq h \in H \). Then \( AG(RG) \) contains a copy of \( K_5 \) with vertex set \\
\{\hat{G}, \hat{H}, \hat{T} - h, \hat{G} - g\hat{H}, gh\}.

Let \( |G| = 3 \). Then \( G = \{1, g, g^2\} \). Then we have \\
\( \hat{T} - g \in \text{ann}(\hat{T} - g^2)(g - g^2)\setminus(\text{ann}(\hat{T} - g^2) \cup \text{ann}(g - g^2)) \);
\( \hat{T} - g \in \text{ann}(\hat{T} - g)(\hat{T} - g^2)\setminus(\text{ann}(\hat{T} - g) \cup \text{ann}(g - g^2)) \), and
\( \hat{T} - g \in \text{ann}(\hat{T} - g)(g - g^2)\setminus(\text{ann}(\hat{T} - g) \cup \text{ann}(g - g^2)) \).

Hence \( AG(RG) \) contains a copy of \( K_5 \) with vertex set \\
\{\hat{G}, 2\hat{G}, g - g^2, \hat{T} - g, \hat{T} - g^2\}.

Subcase 5. \( \text{Char}(R) \mid |G| \) and \( |R| = 2 \). Then \( R \cong \mathbb{Z}_2 \) and \( G \cong C_2 \). It is easy to see that \( Z(RG) = \{0, G\} \). So \( AG(RG) \cong K_1 \), which is planar.

Subcase 6. \( \text{Char}(R) \mid |G| \). Then, by Perlis-Walker Theorem [18, Theorem 3.5.4], \( RG \) is a direct product of copies of at least two fields. First, if \( RG \) is a direct product of copies of two fields \( F_1 \) and \( F_2 \), then \( Z^*(F_1 \times F_2) = \{(r, 0) | r \in F_1^* \} \cup \{(0, r) | r \in F_2^* \} \). Hence \( AG(RG) \) is a complete bipartite graph with parts \( \{(r, 0) | r \in F_1^* \} \) and \( \{(0, r) | r \in F_2^* \} \). If \( |F_1^*|, |F_2^*| \geq 3 \), then \( AG(RG) \) contains a copy of \( K_{3,3} \). So, without loss of generality, we may assume that \( |F_1^*| \leq 2 \) and \( |F_2^*| \leq 3 \). Hence \( F_1 \times F_2 \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_4 \) such that \( F_4 \) is a field with four elements. By the definition of \( RG \), \( |F_1 \times F_2| \) can not be 6 and 12, so \( F_1 \times F_2 \) is not isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4 \). By Lemma 2.1, \( RG \) is not isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). If
$|RG| = 9$, then, by Lemma 2.2, $RG \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. In this situation, $AG(RG)$ is a cycle with length four such that $\hat{G} \sim \hat{T} + g \sim \mathbb{T}G \sim \mathbf{T} - g \sim \hat{G}$, where $1 \neq g \in G$. Hence $RG \cong \mathbb{Z}_2 C_2$. Suppose that $|RG| = 8$. Then we show that $RG \cong \mathbb{Z}_2 \times \mathbb{F}_4$. In this situation, $R \cong \mathbb{Z}_2$ and $G \cong C_3$. Let $G = \{1, g, g^2\}$. Then $Z(RG) = \{1 + g, 1 + g^2, g + g^3, \hat{G}\}$. $Z(RG) \not\cong RG$, since $|Z(RG)| = 5$. Thus $RG$ is nonlocal. On the other hand, $RG$ is finite. Hence $RG$ is a direct product of at least two local rings. We show that $RG$ is not isomorphic to three local rings. By the way of contradiction, assume that $RG$ is isomorphic to three local rings. Since $|RG| = 8$, $RG \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which is impossible, by Lemma 2.1. Thus $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $RG$ is a direct product of at least three local rings. Say $RG \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $RG$ is a direct product of at least three local rings. Since $|RG| = 8$, $RG \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which is impossible, by Lemma 2.1. Thus $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $RG$ is a direct product of at least three local rings. So $RG$ is isomorphic to one of the following rings.

$$Z_2 \times Z_4, Z_2 \times \frac{Z_2[x]}{(x^2)}, Z_2 \times F_4, Z_4 \times Z_2, \frac{Z_2[x]}{(x^2)} \times Z_2 \text{ and } F_4 \times Z_2.$$ 

We consider the rings $Z_2 \times Z_4, Z_2 \times \frac{Z_2[x]}{(x^2)}$ and $Z_2 \times F_4$. $\text{char}(RG) = 2$, since $R \cong \mathbb{Z}_2$. On the other hand, $\text{char}(Z_2 \times Z_4) = 4$. Thus $RG \not\cong Z_2 \times Z_4$. Also, we know that the nonzero element $(\overline{0}, \overline{1})^2 = (\overline{0}, \overline{0})$ in $Z_2 \times \frac{Z_2[x]}{(x^2)}$. Hence $RG \not\cong Z_2 \times \frac{Z_2[x]}{(x^2)}$. Now, we show that $RG$ is isomorphic to $Z_2 \times F_4$ which is planar. If $\varphi$ is a homomorphism from $RG$ to $Z_2 \times F_4$ given by

$$\varphi(0) = (0, 0), \varphi(1) = (1, 1), \varphi(\hat{G}) = (1, 0), \varphi(g + g^2) = (0, 1), \varphi(g) = (1, a), \varphi(g^2) = (1, a^2), \varphi(1 + g) = (0, a), \varphi(1 + g^2) = (0, a),$$

where $F_4 = \{0, 1, a, a^2 : a^3 = 1\}$, then $\varphi$ is a ring isomorphism. So $R \cong Z_2$ and $G \cong C_3$. In this situation, $\mathbb{Z}^*(RG) = \{\hat{G}, \mathbf{T} + g, \mathbf{T} + g^2, g + g^3\}$ such that $AG(Z_2 C_3) \cong K_{1,3}$, which is planar. Now, suppose that $RG$ is a direct product of three fields, say $F_1, F_2$ and $F_3$. Let $|F_1|, |F_2|$ and $|F_3| \geq 3$ such that $\{0, 1, r_i\} \subseteq F_i$ for $1 \leq i \leq 3$. Then since

$$\{1, 1, 0\} \in \text{ann}(0, r_2, 1)(1, 0, r_3) \setminus (\text{ann}(0, r_2, 1) \cup \text{ann}(1, 0, r_3)),$$

$AG(RG)$ has a copy of $K_{3,3}$ with vertex set

$$\{(0, r_2, 1), (0, r_2, 0), (0, 1, 0)\} \cup \{(1, 0, r_3), (1, 0, 0), (r_1, 0, 0)\}.$$

Without loss of generality, we may assume that $|F_1| = 2, |F_2| \leq 3$ and $|F_3| \leq 3$. Then $RG$ can be isomorphic to $Z_2 \times Z_2 \times Z_2, Z_2 \times Z_4 \times Z_2, Z_2 \times Z_2 \times Z_3$ and $Z_2 \times Z_2 \times Z_2$. We know that $|RG|$ is neither $12$ nor $18$. Also, by Lemma 2.1, $RG$ is not isomorphic to $Z_2 \times Z_2 \times Z_2$. Thus $RG$ is not isomorphic to direct product of copies of three fields. Finally, suppose that $RG$ is isomorphic to direct product of copies of at least four fields. Then the element $(1, 1, 0, 1, \ldots, 1)$ belongs to

$$\text{ann}(1, 0, 1, 0, \ldots, 0)(0, 0, 1, 1, 1, 0, \ldots, 0) \setminus (\text{ann}(1, 0, 1, 0, \ldots, 0) \cup \text{ann}(0, 0, 1, 1, 0, \ldots, 0)).$$
(1, 1, 0, 1, . . . , 1) belongs to
\[ \text{ann}((1, 0, 1, 0, . . . , 0)(0, 1, 1, 0, . . . , 0)) \backslash \text{ann}(0, 1, 0, 1, . . . , 0) \cup \text{ann}(0, 1, 0, . . . , 0), \]
(1, 0, 1, . . . , 1) belongs to
\[ \text{ann}((1, 1, 0, . . . , 0)(0, 1, 0, . . . , 0)) \backslash \text{ann}(1, 0, 1, 0, . . . , 0) \cup \text{ann}(0, 1, 0, . . . , 0), \]
(1, 0, 1, . . . , 1) belongs to
\[ \text{ann}((1, 1, 0, . . . , 0)(0, 1, 0, . . . , 0)) \backslash \text{ann}(1, 0, 1, 0, . . . , 0) \cup \text{ann}(0, 1, 0, . . . , 0), \]
(1, 1, 1, . . . , 0) belongs to
\[ \text{ann}((1, 0, 0, 1, 0, . . . , 0)(0, 0, 1, 1, 0, . . . , 0)) \backslash \text{ann}(1, 0, 0, 1, 0, . . . , 0) \cup \text{ann}(0, 0, 1, 1, 0, . . . , 0), \]
and (1, 1, 1, . . . , 0) belongs to
\[ \text{ann}((1, 0, 0, 1, 0, . . . , 0)(0, 0, 1, 1, 0, . . . , 0)) \backslash \text{ann}(1, 0, 0, 1, 0, . . . , 0) \cup \text{ann}(0, 1, 0, 1, 0, . . . , 0), \]
\[ \text{AG}(\text{RG}) \] has a copy of \( K_{3,3} \) with vertex sets
\[ \{(0, 0, 1, 1, 0, . . . , 0), (0, 1, 0, 1, 0, . . . , 0), (0, 1, 1, 0, . . . , 0)\}, \]
and
\[ \{(1, 0, 0, 1, 0, . . . , 0), (1, 0, 1, 0, . . . , 0), (1, 1, 0, . . . , 0)\}. \]
Thus \( \text{RG} \) is not isomorphic to at least four fields.
The converse statement is clear. \( \square \)

Now, the following corollaries are obtained from Theorem 4.1.

**Corollary 4.2.** \( \text{AG}(\text{RG}) \) is a ring graph if and only if \( \text{AG}(\text{RG}) \) is planar.

**Corollary 4.3.** \( \text{AG}(\text{RG}) \) is outerplanar if and only if \( \text{AG}(\text{RG}) \) is planar.

### 5. Line graph of \( \text{AG}(\text{RG}) \)

We begin this section with the following lemma.

**Lemma 5.1.** ([15, Lemma 2.1]). If \( G \) is a graph, then \( \text{diam}(L(G)) = 1 \) if and only if \( G \) is isomorphic to \( K_3 \) or \( K_{1,n} \).

In the following lemma, which is from [21], the planarity of a line graph \( L(G) \) is characterized by using the planarity of \( G \) and its vertex degrees.

**Lemma 5.2.** A nonempty graph \( G \) has a planar line graph \( L(G) \) if and only if

(i) \( G \) is planar;

(ii) \( \triangle(G) \leq 4 \), and
such that \( G \) is impossible. So

**Proof.** Suppose that \( G \mid G \) does not contain any cycle. If \( |G| = 2 \) then \( Z(RG) \) is not a field. Hence the proof is completed.

**Theorem 5.5.** \( gr(L(AG(RG))) = 3 \) or \( RG \) is isomorphic to one of the following rings.

(i) \( \mathbb{Z}_2C_2 \);

(ii) \( \mathbb{Z}_3C_2 \).

**Proof.** By Theorem 3.2, we know that \( gr(AG(RG)) = 3 \) if and only if \( gr(\Gamma(RG)) = 3 \). So, by [2, Proposition 2.8], \( gr(AG(RG)) = 3 \) if and only if \( RG \cong \mathbb{Z}_2C_2 \) and \( RG \cong \mathbb{F}_pC_q \) such that \( p \) and \( q \) are distinct prime numbers, \( p \) is a generator for \( \left( \frac{\mathbb{F}_q}{q} \right) \) and \( \text{gcd}(r, q - 1) = 1 \) such that \( \mathbb{F}_pC_q \) is isomorphic to the direct product of two fields. Suppose that \( RG \cong \mathbb{Z}_2C_2 \). Then \( AG(RG) \cong K_1 \). So \( L(AG(RG)) \) is a null graph. So \( L(AG(RG)) \) does not contain any cycle. If \( RG \cong \mathbb{F}_1 \times \mathbb{F}_2 \) such that \( |\mathbb{F}_1|, |\mathbb{F}_2| \geq 4 \), then \( AG(RG) \cong K_{|\mathbb{F}_1|, |\mathbb{F}_2|} \) such that \( AG(RG) \) contains a copy of \( K_{3,3} \). So \( gr(L(AG(RG))) = 3 \). Without loss of generality, we may assume that \( |\mathbb{F}_1| \leq 3 \) and \( |\mathbb{F}_2| \leq 4 \). In this situation, by Lemmas 2.1 and 2.2, \( RG \cong \mathbb{Z}_2C_2 \). Hence \( AG(\mathbb{Z}_3 \times \mathbb{Z}_3) \) is a cycle of length four. Hence \( gr(L(AG(RG))) = 4 \). Thus the proof is completed. \( \square \)
Theorem 5.6. \( \text{diam}(L(AG(RG))) = 1 \) if and only if \( RG \) is isomorphic to \( F_4C_2 \) or \( Z_2C_3 \).

Proof. First, suppose that \( \text{diam}(L(AG(RG))) = 1 \). Then, by Theorem 4.1, if \( RG \notin \{Z_2C_2, Z_2C_3, Z_3C_2, F_4C_2\} \), then \( RG \) contains a copy of \( K_{3,3} \) or \( K_5 \), which is not a star graph. Now, suppose that \( RG \in \{Z_2C_2, Z_2C_3, Z_3C_2, F_4C_2\} \). Then \( AG(Z_2C_2) \cong K_1, AG(Z_2C_3) \cong K_{1,3}, AG(F_4C_2) \cong K_3 \) and \( AG(Z_3C_2) \) is a cycle. Also, by Lemma 5.1 and Proposition 5.3, \( RG \) is isomorphic to \( F_4C_2 \) or \( Z_2C_3 \).

The converse statement is clear. \( \square \)

Theorem 5.7. \( L(AG(RG)) \) is planar if and only if \( RG \) is isomorphic to one of the following rings.

(i) \( Z_2C_2 \);
(ii) \( Z_3C_2 \);
(iii) \( Z_2C_3 \);
(iv) \( F_4C_2 \).

Proof. First, suppose that \( L(AG(RG)) \) is planar. Then, by Theorem 4.1, \( RG \) is one of the rings \( Z_2C_2, Z_2C_3, Z_3C_2 \) or \( F_4C_2 \). We know that \( |V(AG(Z_2C_2))| = 1 \). Hence \( L(AG(RG)) \) is a null graph, which is planar. Now, suppose that \( RG \) is isomorphic to \( Z_3C_2 \). Then \( Z(Z_3C_2) = \{0, \hat{G}, \hat{2G}, \hat{T} + \hat{2g}, \hat{2} + g\} \). Hence \( L(AG(RG)) \) is isomorphic to the cycle of length four, which is planar. If \( RG \) is isomorphic to \( Z_2C_3 \), then \( Z(Z_2C_3) = \{0, g, 1 - g, 1 - g^2, g - g^3\} \). In this situation, \( L(AG(RG)) \) is isomorphic to \( K_3 \), which is planar. Finally, suppose that \( RG \) is isomorphic to \( F_4C_2 \). Then \( L(AG(RG)) \cong K_3 \), which is planar.

The converse statement is clear. \( \square \)

The following corollaries are obtained from Theorem 5.7.

Corollary 5.8. \( L(AG(RG)) \) is outerplanar if and only if \( L(AG(RG)) \) is planar.

Corollary 5.9. \( L(AG(RG)) \) is ring graph if and only if \( L(AG(RG)) \) is planar.

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