FACTORIZATION OF OPERATORS ON $C^*$-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a $C^*$-algebra. We prove that every absolutely summing operator from $\mathcal{A}$ into $\ell_2$ factors through a Hilbert space operator that belongs to the 4-Schatten-von Neumann class. We also provide finite dimensional examples that show that one can not replace the 4-Schatten-von Neumann class by $p$-Schatten-von Neumann class for any $p < 4$. As an application, we show that there exists a modulus of capacity $\varepsilon \to N(\varepsilon)$ so that if $\mathcal{A}$ is a $C^*$-algebra and $T \in \Pi_1(\mathcal{A}, \ell_2)$ with $\pi_1(T) \leq 1$, then for every $\varepsilon > 0$, the $\varepsilon$-capacity of the image of the unit ball of $\mathcal{A}$ under $T$ does not exceed $N(\varepsilon)$. This answers positively a question raised by Pełczyński.

1. Introduction

It is a well known consequence of a classical result of Grothendieck that if $X$ is a Banach space and $X^{**}$ is isomorphic to a quotient of a $C(K)$-space then every absolutely summing operator from $X$ into $\ell_2$ factors through a Hilbert-Schmidt operator. The present paper is an attempt to get a generalization of this fact for the setting of arbitrary $C^*$-algebras. Different structures of operators defined on arbitrary $C^*$-algebras was considered by Pisier in [11] and [12]; for instance he proved that every $(p, q)$-summing operators on an arbitrary $C^*$-algebra admit a factorization similar to that of operators on $C(K)$-spaces, every operator from any $C^*$-algebra into any Banach space of cotype 2 factors through Hilbert space. Using the notion of $C^*$-summing operators introduced by Pisier in [11], the author proved in [13], that absolutely summing operators from $C^*$-algebras into reflexive spaces are compact. The main result of this paper states that for the case of $C^*$-algebras and the range space being a Hilbert space, one can factor every absolutely summing operator through a Hilbert space operator that belongs to the 4-Schatten-von Neumann class (see definition below). The basic idea of the proof of this result is the factorization of $C^*$-summing operators used in [13] and some well known coincidence of different classes of Hilbert space operators. This result allows to
prove a quantitative result on the compactness of absolutely summing operators from $C^*$-algebras into Hilbert spaces, answering a question raised by Pełczyński in [3] (Problem 3') for the space of compact operators on Hilbert space. A finite dimensional approach shows that unlike the commutative case of $C(K)$-spaces, one cannot expect to factor every absolutely summing operators from general non-commutative $C^*$-algebras into Hilbert spaces through Hilbert-Schmidt operators. In fact our examples show that the result stated above cannot be improved to the case of $p$-Schatten-von Neumann class for any $p < 4$.

Our terminology and notation are standard. We refer to [2] and [17] for definitions from Banach space theory and [6] and [14] for basic properties from $C^*$-algebra and operator algebra theory.

2. Preliminaries

In this section we recall some definitions and facts which we use in the sequel. Throughout, the word operator will always mean linear bounded operator and $\mathcal{L}(E, F)$ will stand for the space of all operators from $E$ into $F$.

**Definition 1.** Let $E$ and $F$ be Banach spaces and $1 \leq p < \infty$. An operator $T \in \mathcal{L}(E, F)$ is said to be absolutely $p$-summing (or simply $p$-summing) if there exists a constant $C$ such that for any finite sequence $(e_1, e_2, \ldots, e_n)$ of $E$, one has

$$\left( \sum_{i=1}^{n} \|Te_i\|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left( \sum_{i=1}^{n} |\langle e_i, e^* \rangle|^p \right)^{\frac{1}{p}} ; e^* \in E^*, \|e^*\| \leq 1 \right\}.$$

The least constant $C$ for the inequality above to hold will be denoted by $\pi_p(T)$. It is well known that the class of all absolutely $p$-summing operators from $E$ to $F$ is a Banach space under the norm $\pi_p(.)$. This Banach space will be denoted by $\Pi_p(E, F)$.

**Definition 2.** Let $1 \leq q \leq p < \infty$. An operator $T \in \mathcal{L}(E, F)$ is said to be $(p, q)$-summing if there is a constant $K \geq 0$ for which

$$\left( \sum_{k=1}^{n} \|Te_i\|^p \right)^{\frac{1}{p}} \leq K \sup \left\{ \left( \sum_{i=1}^{n} |\langle e_i, e^* \rangle|^q \right)^{\frac{1}{q}} ; e^* \in E^*, \|e^*\| \leq 1 \right\}$$

for every finite sequence $(e_1, e_2, \ldots, e_n)$ in $E$.

As above, the least constant $K$ for which the inequality holds is the $(p, q)$-summing norm of $T$ and is denoted by $\pi_{p,q}(T)$. The class of $(p, q)$-summing operators from $E$ into $F$ is a Banach space under the norm $\pi_{p,q}(.)$. This class will be denoted by $\Pi_{p,q}(E, F)$. 
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Another class of operators relevant for our discussion is the Schatten-von Neumann class.

**Definition 3.** For $1 \leq p < \infty$, $H_1$ and $H_2$ Hilbert spaces, the $p$-th Schatten-von Neumann class consists of all compact operators $U : H_1 \rightarrow H_2$ that has a representation of the form

$$U = \sum_{n=1}^{\infty} \alpha_n(e_n, f_n),$$

where $(e_n)_n$ is an orthonormal sequence in $H_1$, $(f_n)_n$ is an orthonormal sequence in $H_2$, and $(\alpha_n)_n \in \ell_p$.

We will refer to $(\ast)$ as an orthonormal representation of $U$. It is well known that one can always choose the sequence $(\alpha_n)_n$ in the representation $(\ast)$ to satisfy $0 \leq \alpha_{n+1} \leq \alpha_n$ for all admissible indices. The $p$-th Schatten-von Neumann norm is defined by

$$\sigma_p(U) = \left( \sum_{n=1}^{\infty} |\alpha_n|^p \right)^{\frac{1}{p}}$$

and the $p$-th Schatten-von Neumann class is denoted by $S_p(H_1, H_2)$.

**Definition 4.** Let $E$ and $F$ be Banach spaces, $1 \leq p \leq \infty$. We say that an operator $T \in \mathcal{L}(E, F)$ is $L_p$-factorable if there exist a measure space $(\Omega, \Sigma, \mu)$ and operators $U_1 \in \mathcal{L}(E, L_p(\mu))$ and $U_2 \in \mathcal{L}(L_p(\mu), F^{**})$ such that $i_F \circ T = U_2 \circ U_1$ where $i_F : F \rightarrow F^{**}$ denotes the natural embedding.

The $L_p$-factorable norm of $T$ is defined by $\gamma_p(T) := \inf \{ \|U_1\|, \|U_2\| \}$ where the infimum is taken over all possible factorizations as above.

For detailed discussion of $p$-summing operators, $(q, p)$-summing operators, $p$-Schatten-von Neumann operators and $L_p$-factorable operators, we refer to [3], [10] and [15].

We will now recall some basic facts on $C^*$-algebras and von-Neumann algebras. Let $\mathcal{A}$ be a $C^*$ algebra, we denote by $\mathcal{A}_h$ the set of Hermitian (self adjoint) elements of $\mathcal{A}$. For $x \in \mathcal{A}$ and $f \in \mathcal{A}^*$, $xf$ (resp. $fx$) denotes the element of $\mathcal{A}^*$ defined by $xf(y) = f(yx)$ (resp. $fx(y) = f(xy)$) for every $y \in \mathcal{A}$.

**Definition 5.** A von-Neumann algebra is said to be $\sigma$-finite if it admits at most countably many orthogonal projections.

We refer to [3] and [14] for some characterizations and examples of $\sigma$-finite von-Neumann algebras.
3. Main Theorem

**Theorem 1.** Let $\mathcal{A}$ be a $C^*$-algebra and $T \in \Pi_1(\mathcal{A}, \ell_2)$. Then for every $\varepsilon > 0$, there exists a Hilbert space $H$ and operators $J : \mathcal{A} \to H$ and $K : H \to \ell_2$ such that:

1. $T = K \circ J$;
2. $\|J\| \leq 1$;
3. $K \in S_4(H, \ell_2)$ with $\sigma_4(K) \leq 2(1 + \varepsilon)\pi_1(T)$.

To prove this theorem, we will consider first the following particular case:

**Proposition 1.** Let $\mathcal{M}$ be a $\sigma$-finite von-Neumann algebra, $T : \mathcal{M} \to \ell_2$ be a weak* to weakly continuous absolutely summing operator and $\varepsilon > 0$. Then there exist a Hilbert space $H$, operators $J : \mathcal{M} \to H$ and $K : H \to \ell_2$ such that:

1. $T = K \circ J$;
2. $\|J\| \leq 1$;
3. $K \in S_4(H, \ell_2)$ with $\sigma_4(K) \leq 2(1 + \varepsilon)\pi_1(T)$.

**Proof.** The proof is based on the factorization technique used in [13]. We will repeat the argument for completeness.

Let $T \in \Pi_1(\mathcal{M}, \ell_2)$ and assume that $T$ is weak* to weakly continuous. Fix $\delta > 0$ such that $(1 + \delta)^{1/2} \leq (1 + \varepsilon)$.

By [13] (Proposition 1.1) and [11] (Lemma 4.1), there exists a normal positive functional $g$ on $\mathcal{M}$ such that $\|g\| \leq 1$ and

$$\|Tx\| \leq \pi_1(T)g(|x|) \text{ for every } x \in \mathcal{M}_h.$$  

Since the von-Neumann algebra $\mathcal{M}$ is $\sigma$-finite, there exists a faithful normal functional $f_0$ in $\mathcal{M}_*$ (see [14] Proposition II-3.19). We can choose $f_0$ such that $\|f_0\| \leq \delta$. Let $f = (g + f_0)/(1 + \delta)$; clearly $\|f\| \leq 1$ and

$$\|Tx\| \leq (1 + \delta)\pi_1(T)f(|x|) \text{ for every } x \in \mathcal{M}_h.$$  

From Lemma 2 of [13], we deduce that

$$\|Tx\| \leq 2(1 + \delta)\pi_1(T)xf + fx\|_{\mathcal{M}_*} \text{ for every } x \in \mathcal{M}.$$  

As in [13], we equip $\mathcal{M}$ with the scalar product

$$\langle x, y \rangle = f\left(\frac{xy^* + y^*x}{2}\right).$$
Since \( f \) is faithful, \( \mathcal{M} \) with \( \langle \cdot, \cdot \rangle \) is pre-Hilbertian. We denote the completion of this space by \( L_2(\mathcal{M}, f) \) (or simply \( L_2(f) \)). From [13], we have the following factorization:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{T} & \ell_2 \\
\downarrow J & & \downarrow L \\
L_2(f) & \xrightarrow{\theta} & L_2(f)^\ast \\
\end{array}
\]

where \( \theta(Jx) = \langle \cdot, J(x^\ast) \rangle \) for every \( x \in \mathcal{M} \); \( L(\frac{xf + fx}{2}) = Tx \) for every \( x \in \mathcal{M} \) and \( J \) is the inclusion map (one can easily check as in [13] that \( J^* \circ \theta \circ J(x) = (xf + fx)/2 \)).

Set \( H := L_2(f) \) and \( K := L \circ J^* \circ \theta \). Clearly (1) and (2) are satisfied.

To prove (3), let us consider the adjoint maps:

\[
\begin{array}{ccc}
\ell_2 & \xrightarrow{L^*} & \mathcal{M}^\ast \\
\downarrow J & & \downarrow J^* \\
\mathcal{M} & \xrightarrow{L_2(f)} & L_2(f)^\ast \\
\end{array}
\]

The proposition will be deduced from the following lemma:

**Lemma 1.** For every \( p \geq 1 \), \( K^* \in \Pi_{2p, p}(\ell_2, H^\ast) \) with "\( \pi_{2p, p}(K^*) \leq \pi_p(T)^{\frac{1}{2}} \|L\|^{\frac{1}{2}} \)."

To see the lemma, let \((z_n)_n\) be a sequence in \( \ell_2 \) such that

\[
\sup \left\{ \left( \sum_{n=1}^{\infty} |\langle z_n, z^\ast \rangle|^p \right)^{\frac{1}{p}} ; \|z^\ast\| \leq 1 \right\} = C < \infty.
\]

Then

\[
\sup \left\{ \left( \sum_{n=1}^{\infty} |\langle L^*(z_n), \xi \rangle|^p \right)^{\frac{1}{p}} ; \xi \in \mathcal{M}^\ast, \|\xi\| \leq 1 \right\} \leq \|L\|C.
\]

Similarly,

\[
\sup \left\{ \left( \sum_{n=1}^{\infty} |\langle (L^*(z_n))^\ast, \xi \rangle|^p \right)^{\frac{1}{p}} ; \xi \in \mathcal{M}^\ast, \|\xi\| \leq 1 \right\} \leq \|L\|C.
\]
where \((L^*(z_n))^*\) is the adjoint of the operator \(L^*(z_n)\) in \(\mathcal{M}\) for every \(n \in \mathbb{N}\). Since \(((L^*(z_n))^*)_n\) is a sequence in \(\mathcal{M}\), one can apply \(T\). The fact that \(T\) is \(p\)-summing implies that

\[
\left(\sum_{n=1}^{\infty} \|T(L^*(z_n))^*\|^p\right)^{\frac{1}{p}} \leq \pi_p(T) \sup \left\{ \left(\sum_{n=1}^{\infty} |\langle L^*(z_n)^*, \xi \rangle|^p\right)^{\frac{1}{p}} : \|\xi\| \leq 1 \right\}
\]

\[
\leq \pi_p(T) \|L\|C.
\]

But since \((z_n)_n\) is bounded (in fact it is bounded by \(C\)) we get that

\[
\left(\sum_{n=1}^{\infty} |\langle T((L^*(z_n))^*), z_n \rangle|^p\right)^{\frac{1}{p}} \leq \pi_p(T) \|L\|C^2.
\]

Now for each \(n \in \mathbb{N}\),

\[
\langle T((L^*(z_n))^*), z_n \rangle = \langle L \circ J^* \circ \theta \circ J (L^*(z_n)^*), z_n \rangle
\]

\[
= \langle \theta \circ J (L^*(z_n)^*), J \circ L^*(z_n) \rangle
\]

\[
= \langle J (L^*(z_n)), J (L^*(z_n)) \rangle
\]

\[
= \|J (L^*(z_n))\|_{L_2(f)}^2.
\]

So

\[
\left(\sum_{n=1}^{\infty} \|J \circ L^*(z_n)\|^{2p}\right)^{\frac{1}{p}} \leq \pi_p(T) \|L\|C^2.
\]

Hence

\[
\left(\sum_{n=1}^{\infty} \|K^*(z_n)\|^{2p}\right)^{\frac{1}{p^2}} \leq \pi_p(T) \|L\|^{1/2}C
\]

which shows that \(K^* \in \Pi_{2p,p}(\ell_2, H^*)\) with \(\pi_{2p,p}(K^*) \leq \pi_p(T) \|L\|^{1/2}C\). The lemma is proved.

To complete the proof of the proposition, we apply the above lemma for \(p = 2\); we get that \(\pi_{4,2}(K^*) \leq \pi_2(T)^{1/2}\|L\|^{1/2}\). We note also from the proof of Theorem 1 of \([13]\) that the set \(\{xf + fx; x \in \mathcal{M}\}\) is norm dense in \(\mathcal{M}_s\) so from the estimate \(\|L(xf + fx)\| = \|Tx\| \leq 2(1 + \delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}}\) for every \(x \in \mathcal{M}\), we get that

\[
\|L(xf + fx)\| \leq 4(1 + \delta)\pi_1(T)\|xf + fx\|_{\mathcal{M}}\) for every \(x \in \mathcal{M}\).
\]

We conclude that \(\|L\| \leq 4(1 + \delta)\pi_1(T)\) and therefore \(\pi_{4,2}(K^*) \leq \pi_2(T)^{1/2} 2(1 + \delta)^{1/2}\pi_1(T)^{1/2} \leq 2(1 + \epsilon)\pi_1(T)\).

From a result of Mitjagin (which appeared for the first time in a paper of Kwapien \([8]\); see also \([3]\) Theorem 10.3 or \([13]\) Proposition 11.8), the space \(\Pi_{4,2}(\ell_2, H^*)\) is isometrically
isomorphic to $S_4(\ell_2, H^*)$ so $\sigma_4(K^*) \leq 2(1 + \varepsilon)\pi_1(T)$ and from Proposition 4.5 of [3] (p. 80), $K \in S_4(H, \ell_2)$ with $\sigma_4(K) = \sigma_4(K^*) \leq 2(1 + \varepsilon)\pi_1(T)$. The proof of the proposition is complete.

**Proof of Theorem 1.** Assume first that $\mathcal{A}$ is separable and $T \in \Pi_1(\mathcal{A}, \ell_2)$. The space $\mathcal{A}^{**}$ is a von-Neumann algebra and $T^{**} \in \Pi_1(\mathcal{A}^{**}, \ell_2)$. Let $i_\mathcal{A} : \mathcal{A} \to \mathcal{A}^{**}$ be the natural embedding and choose $(a_n)_n$, a countable dense subset of $\mathcal{A}$. If $\mathcal{M}$ is the von-Neumann algebra generated by $\{i_\mathcal{A}(a_n); n \geq 1\}$, then $\mathcal{M}$ is $\sigma$-finite. Also if we denote by $I$ the inclusion of $\mathcal{M}$ into $\mathcal{A}^{**}$, then $I$ is weak* to weak* continuous. From Proposition 1, the operator $T^{**} \circ I$ factors through a Hilbert space operator $K$ that belongs to the class $S_4$ and so does $T = T^{**} \circ I \circ i_\mathcal{A}$.

One can easily verify that this factorization satisfies the conclusion of the theorem.

For the general case, we will use ultraproduct technique. Let $(\mathcal{A}_s)_{s \in S}$ be the collection of all separable $C^*$-subalgebras of $\mathcal{A}$. As a particular case of Theorem 3.3 of [3] (which is the $C^*$-version of Proposition 6.2 of [1]), there exists a subset $\Lambda$ of $S$ and an ultrafilter $\mathcal{U}$ on $\Lambda$ such that $\mathcal{A}$ is (completely) isometric to a subspace of $(\mathcal{A}_s)_{\mathcal{U}}$. Inspecting the proof of [3], one notices that in our case $\Lambda = S$.

Let $T : \mathcal{A} \to \ell_2$ be a 1-summing operator and $i_s : \mathcal{A}_s \to \mathcal{A}$ be the inclusion map. It is clear that $T \circ i_s \in \Pi_1(\mathcal{A}_s, \ell_2)$ with $\pi_1(T \circ i_s) \leq \pi_1(T)$. From the proposition above, there exists a Hilbert space $H_s$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A}_s & \xrightarrow{T \circ i_s} & \ell_2 \\
\downarrow{J_s} & & \downarrow{K_s} \\
H_s & & \\
\end{array}
\]

with $\|J_s\| \leq 1$ and $\sigma_4(K_s) \leq 2(1 + \varepsilon)\pi_1(T)$.

From this, one can verify that the following diagram commutes:

\[
\begin{array}{ccc}
(\mathcal{A}_s)_{\mathcal{U}} & \xrightarrow{(T \circ i_s)_\mathcal{U}} & (\ell_2)_{\mathcal{U}} \\
\downarrow{(J_s)_\mathcal{U}} & & \downarrow{(K_s)_\mathcal{U}} \\
(H_s)_{\mathcal{U}} & & \\
\end{array}
\]

It is clear that $\|(J_s)_\mathcal{U}\| \leq 1$ and since $S_4$ is a maximal ideal operator, we get that $(K_s)_\mathcal{U} \in S_4(\ell_2)_\mathcal{U}$ with $\sigma_4((K_s)_\mathcal{U}) \leq \lim_{\mathcal{U}} \sigma_4(K_s) \leq 2(1 + \varepsilon)\pi_1(T)$ (see [3] Theorem 8.1).

Let $Q : (\ell_2)_\mathcal{U} \to \ell_2$ defined by $Q((y_s)_s) = \text{weak} - \lim_{s, \mathcal{U}} y_s$ and $I : \mathcal{A} \to (\mathcal{A}_s)_{\mathcal{U}}$ be the isometric embedding. We claim that $Q \circ (T \circ i_s)_\mathcal{U} \circ I = T$. 


To see this, notice that for every \( x \in A \), \( I(x) = 0 \) if \( x \notin A \), and \( I(x) = x \) if \( x \in A \).
So \((T \circ i_s)_U(Ix) = (y_s)_{s \in S}\) where \( y_s = 0 \) if \( x \notin A \)
and \( y_s = Tx \) if \( x \in A \), and by the definition of \( Q \) the claim follows.

We get the conclusion of the theorem by setting \( J = (J_s)_U \circ I \), \( K = Q \circ (K_s)_U \) and \( H = (H_s)_U \).

For the next simple extension of Theorem 1, we refer to [10] for definitions and examples of \( JB^* \)-triples and \( JBW^* \)-triples.

**Corollary 1.** If \( A \) is a \( JB^* \)-triple then every absolutely summing operator from \( A \) into \( \ell_2 \)
factors through an operator that belongs to the \( 4 \)-Schatten-von Neumann class.

**Proof.** Let \( T : A \to \ell_2 \) be absolutely summing operator. The space \( A^{**} \) is a \( JBW^* \)-triple. But every \( JBW^* \)-triple is (as Banach space) isometric to a complemented subspace of a von-Neumann algebra (see [1]). From Theorem 1, \( T^{**} \) (and consequently \( T \)) factors through an operator that belongs to the class \( S_4 \).

**Remark 1.** We remark that Lemma 1 is valid for any weak* to weakly continuous absolutely summing operator from a \( \sigma \)-finite von-Neumann algebra into a general Banach space; in particular, the adjoint of any such operator belongs to the class ideal \( \Pi_{2p,p} \) for every \( p \geq 1 \).

The following finite dimensional examples show that one can not improve Theorem 1 to the case of \( p \)-Schatten-von Neumann class for \( p < 4 \). The type of operators considered below were suggested to the author by Pelczyński.

For \( n \geq 1 \), \( B(\ell_2^n) \) (resp. \( HS(\ell_2^n) \)) denotes the space of \( n \times n \) matrices with the usual operator norm (resp. the Hilbert-Schmidt norm).

Let \( I_n : B(\ell_2^n) \to HS(\ell_2^n) \) be the identity operator and set \( \alpha_n = \pi_1(I_n) \).

**Theorem 2.** For every \( n \geq 1 \), let \( T_n = I_n/\alpha_n \). There exists an absolute constant \( \beta > 0 \) (independent of \( n \)) such that if \( H \) is a Hilbert space, \( J \in L(B(\ell_2^n), H) \) and \( K \in L(H, HS(\ell_2^n)) \) satisfying:

(i) \( \|J\| \leq 1 \);
(ii) \( T_n = K \circ J \).

Then for every \( p \geq 2 \), \( \sigma_p(K) \geq \beta n^{\frac{4-p}{2p}} \).
For the proof of this theorem, we will recall few well-known facts about the operator $I_n$.

**Proposition 2.**

1) There exists a universal constant $c > 0$ such that $\alpha_n = \pi_1(I_n) \leq cn$ for every $n \geq 1$;

2) There exists a universal constant $c' > 0$ such that $\gamma_1(I_n) \geq c'n^{\frac{3}{2}}$ for every $n \geq 1$.

To prove the theorem, let $H$ be a Hilbert space and $J$ and $K$ be operators as in the statement. Since $HS(\ell^2_2)$ is a finite dimensional Hilbert space, $K : H \to HS(\ell^2_2)$ is a Hilbert-Schmidt operator. Similarly, the adjoint $K^* : HS(\ell^2_2) \to H$ is also a Hilbert-Schmidt operator. One can choose a probability space $(\Omega, \Sigma, \lambda)$ such that:

\[
\begin{array}{c}
\begin{array}{ccc}
\text{HS}(\ell^2_2) & \xrightarrow{K} & H \\
V & \downarrow & U \\
L_\infty(\lambda) & \xrightarrow{i_2} & L_2(\lambda)
\end{array}
\end{array}
\]

with $\|V\| = 1$ and $\|U\| = \pi_2(K^*) = \pi_2(K)$. Taking the adjoints,

\[
\begin{array}{c}
\begin{array}{ccc}
H & \xrightarrow{K} & HS(\ell^2_2) \\
U^* & \downarrow & V^* \\
L_2(\lambda) & \xrightarrow{i_2^*} & L_1(\lambda)
\end{array}
\end{array}
\]

Hence the operator $T_n$ factors through $L_1(\lambda)$ as follows:

\[
\begin{array}{c}
\begin{array}{ccc}
B(\ell^m_2) & \xrightarrow{T_n} & HS(\ell^m_2) \\
U_1 & \downarrow & V^* \\
L_1(\lambda)
\end{array}
\end{array}
\]

where $U_1 = i_2^* \circ U^* \circ J$. From the definition of $\gamma_1(T_n)$, we get the following estimate:

\[
\begin{align*}
\gamma_1(T_n) & \leq \|U_1\| \cdot \|V^*\| \\
& \leq \|i_2^*\| \cdot \|U^*\| \cdot \|J\| \cdot \|V^*\| \\
& \leq \|U^*\| = \pi_2(K).
\end{align*}
\]

From the above proposition, $\frac{c'n^{\frac{3}{2}}}{c} \leq \frac{c'n^{\frac{3}{2}}}{\alpha_n} \leq \pi_2(K)$.

If we set $\beta := \frac{c'}{c}$, we get $\sigma_2(K) = \pi_2(K) \geq \beta n^{\frac{3}{2}}$ and the theorem is proved for the case $p = 2$. 

For \( p > 2 \), note that \( B(\ell^2_n) \) and \( HS(\ell^2_n) \) are of dimension \( n^2 \) so we can assume without loss of generality that \( \dim(H) = n^2 \). Let \( (s_i(K))_{1 \leq i \leq n^2} \) be the singular numbers of \( K \). It is well known that for every \( q > 0 \), \( \sigma_q(K) = \left( \sum_{i=1}^{n^2} s_i(K)^q \right)^{\frac{1}{q}} \). Using Holder’s inequality, we get for every \( p > 2 \),

\[
\sigma_2(K) = \left( \sum_{i=1}^{n^2} s_i(K)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{n^2} s_i(K)^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n^2} 1 \right)^{(1-\frac{2}{p})\frac{1}{2}} = \sigma_p(K)n^{1-\frac{2}{p}}.
\]

Hence \( \beta n^{\frac{1}{2}} \leq \sigma_2(K) \leq \sigma_p(K)n^{1-\frac{2}{p}} \) which implies that \( \sigma_p(K) \geq \beta n^{-\frac{1}{2} + \frac{2}{p}} = \beta n^{\frac{4-p}{2p}} \). The proof of the theorem is complete.

The operator \( T_n \) satisfies \( \pi_1(T_n) = 1 \) but any factorization through any Hilbert space operator has large \( p \)-Schatten-von Neumann norm for \( p < 4 \). This shows that the class \( S_4 \) in the statement of Theorem 1 cannot be improved.

The results above lead us to the question of characterizing operators from a \( C^* \)-algebra into \( \ell_2 \) that can be factored through Hilbert-Schmidt operators.

**Theorem 3.** Let \( \mathcal{A} \) be a \( C^* \)-algebra. An operator \( T : \mathcal{A} \to \ell_2 \) factors through a Hilbert-Schmidt operator if and only if it is \( L_1 \)-factorable.

**Proof.** If \( T \) factors through a Hilbert-Schmidt operator then it is \( L_1 \)-factorable since Hilbert-Schmidt operators are \( L_1 \)-factorable.

Conversely, assume that \( T \) is \( L_1 \)-factorable i.e. there exits a measure space \( (\Omega, \Sigma, \lambda) \), operators \( U_1 : \mathcal{A} \to L_1(\Omega, \Sigma, \lambda) \) and \( U_2 : L_1(\Omega, \Sigma, \lambda) \to \ell_2 \) such that \( T = U_2 \circ U_1 \). From Grothendieck’s theorem \( U_2 \) is 1-summing. Since \( L_1(\Omega, \Sigma, \lambda) \) is of cotype 2, \( U_1 \) factors through a Hilbert space (see [11]) which shows that \( T \) factors through a Hilbert-Schmidt operator. \( \square \)

4. **Measure of compactness.**

In this section, we will provide an application of the main theorem to measure compactness of any absolutely summing operator from \( C^* \)-algebras into Hilbert spaces.
Let $L$ be a normed linear space with norm $\| \cdot \|$ and $A$ be a totally bounded set in $L$.

For any given $\varepsilon > 0$, we set $N_\varepsilon(A) := \inf m$ such that there exists of subsets $E_1, E_2, \ldots, E_m$ of $L$ whose diameters do not exceed $2\varepsilon$ and whose union contains $A$ i.e.,

$$\bigcup_{k=1}^n E_k \supseteq A \text{ and } \text{diam}(E_k) \leq 2\varepsilon.$$  

**Definition 6.** $H_\varepsilon(A) := \log_2 N_\varepsilon(A)$ is called the $\varepsilon$-capacity of the set $A$.

This definition was introduced by Kolmogorov and Tihomirov (among other related notions) in [7].

Our main result in this section answers positively a question raised by Pełczyński and can be viewed as a quantitative version of Theorem 1 of [13].

**Theorem 4.** There exists an absolute constant $C$ such that if $A$ is a $C^*$-algebra and $T \in \Pi_1(A, \ell_2)$ with $\pi_1(T) \leq 1$, then for every $\varepsilon > 0$,

$$H_\varepsilon(T(B_A)) \leq \frac{C}{\varepsilon^4}.$$  

We will show that Theorem 4 is a consequence of the following result.

**Theorem 5.** Let $H$ be a separable Hilbert space and $S \in S_p(H, \ell_2)$, then for every $\varepsilon > 0$,

$$H_\varepsilon(S(B_H)) \leq \frac{\sigma_p(S)^p \cdot \rho(p)}{\varepsilon^p}$$  

where $\rho(p) = \left( \frac{p^p}{p} + \int_0^{\frac{1}{1}} \ln(\frac{1}{t}) dt + 1 \right)^p$.

The proof is based on a notion of entropy of operators introduced by Pietsch (see [10] p. 168).

**Definition 7.** Let $E$ and $F$ be Banach spaces and $S \in \mathcal{L}(E, F)$. The $n$-th (outer) entropy number $e_n(S)$ of the operator $S$ is the minimum of $\delta > 0$ such that there exists a finite sequence $y_1, y_2, \ldots, y_q \in F$ with $q \leq 2^{n-1}$ and $S(B_E) \subseteq \bigcup_{i=1}^q \{ y_i + \delta B_F \}$.

Clearly $e_{n+1}(S) \leq e_n(S)$ for every operator $S$ and every $n \in \mathbb{N}$.

For diagonal Hilbert space operators, the following proposition was proved by Pietsch.

**Proposition 3.** ([10] p. 174) Let $S \in \mathcal{L}(\ell_2)$ such that $S((\xi_n)_n) = (\alpha_n \xi_n)_{n \geq 1}$ and $(\alpha_n)_n \in c_0$. Then

$$\left( \sum_{n=1}^{\infty} e_n(S)^p \right)^{\frac{1}{p}} \leq K_p \left( \sum_{n=1}^{\infty} |\alpha_n|^p \right)^{\frac{1}{p}}.$$
Proof of Theorem 5.

Let $1 < p < \infty$ and $S \in S_p(H, \ell_2)$. The operator $S$ admits an orthonormal representation

\[(*) \quad S = \sum_{n=1}^{\infty} \alpha_n (\cdot, h_n)f_n,\]

where $(h_n)$ and $(f_n)$ are orthonormal sequences in $H$ and $\ell_2$ respectively and $(\alpha_n)_{n} \in \ell_p$. We can choose this representation so that $0 \leq \alpha_{n+1} \leq \alpha_n$ for all admissible indices. Let $(e_{n})_{n}$ be the unit vector basis of $\ell_2$. Let $Z = \text{span}\{h_{n}; n \in \mathbb{N}\}$ in $H$. Since $S(B_Z) = S(B_H)$, we can assume without loss of generality that $H = Z$.

Let $I : \ell_2 \longrightarrow H$ defined by $Ie_n = h_n$ for every $n \in \mathbb{N}$ and $J : \ell_2 \longrightarrow \ell_2$ so that $J(f_n) = e_n$ for every $n \in \mathbb{N}$. Let $\tilde{S} := J \circ S \circ I$. Clearly $\tilde{S} \in S_p(\ell_2, \ell_2)$, $I$ and $J$ are isometries.

For every $x \in \ell_2$, we have

\[
\tilde{S}x = \sum_{n=1}^{\infty} \alpha_n (Ix, h_n)J(f_n)
= \sum_{n=1}^{\infty} \alpha_n (x, I^* h_n)e_n
= \sum_{n=1}^{\infty} \alpha_n (x, e_n)e_n.
\]

So for every $x = (x_n)_{n} \in \ell_2$, $Sx = (\alpha_n x_n)_{n \geq 1}$. Hence $\tilde{S}$ satisfies the assumption of the above proposition and therefore

\[
\left(\sum_{n=1}^{\infty} \left|e_n(\tilde{S})\right|^p\right)^{\frac{1}{p}} \leq K_p \left(\sum_{n=1}^{\infty} \left|\alpha_n\right|^p\right)^{\frac{1}{p}} \leq K_p \sigma_p(S).
\]

For $\varepsilon > 0$, define $k(\varepsilon) := \max\{k : e_k(\tilde{S}) \geq \varepsilon\}$. We have

\[
(K_p \sigma_p(S))^p \geq \sum_{n=1}^{\infty} \left|e_n(\tilde{S})\right|^p \geq \sum_{n=1}^{k(\varepsilon)} \left|e_n(\tilde{S})\right|^p \geq \varepsilon^p k(\varepsilon)
\]

so

\[
k(\varepsilon) \leq \left(\frac{K_p \sigma_p(S)}{\varepsilon}\right)^{\frac{1}{p}}.
\]

From the definition of $k(\varepsilon)$, $e_{k(\varepsilon)+1}(\tilde{S}) \leq \varepsilon$ and the definition of the $n$-th entropy of $\tilde{S}$ implies that there exists $\delta \leq \varepsilon$ and $\{y_1, y_2, \ldots, y_q\} \subseteq \ell_2$, with $q \leq 2^{k(\varepsilon)}$ so that $\tilde{S}(B_{\ell_2}) \subseteq \text{span}\{y_1, y_2, \ldots, y_q\}$.
\( \{y_1, y_2, \ldots, y_q\} + \delta B_{\ell_2} \) i.e., the set \( \tilde{S}(B_{\ell_2}) \) can be covered by \( 2^{k(\varepsilon)} \) balls of radius \( \delta \leq \varepsilon \) so \( N_{\varepsilon} (\tilde{S}(B_{\ell_2})) \leq 2^{k(\varepsilon)} \) and

\[
H_\varepsilon (\tilde{S}(B_{\ell_2})) \leq k(\varepsilon) \leq \frac{\sigma_p(S)K_p}{\varepsilon^p}.
\]

Now since \( J \) is an isometry, \( H_\varepsilon (\tilde{S}(B_{\ell_2})) = H_\varepsilon (S \circ I(B_{\ell_2})) \); also by the definition of \( I \), \( I(B_{\ell_2}) = B_H \) so

\[
H_\varepsilon (S(B_H)) = H_\varepsilon (\tilde{S}(B_{\ell_2})) \leq \frac{\sigma_p(S)^pK_p^p}{\varepsilon^p}
\]

and setting \( \rho(p) = K_p^p \), the theorem is proved.

The estimate on \( K_p \) can be found in Pietsch’s book \([10]\) (p. 174).

\[\square\]

**Proof of Theorem 4.**

If \( \mathcal{A} \) is a \( C^* \)-algebra and \( T \in \Pi_1(\mathcal{A}, \ell_2) \) with \( \pi_1(T) \leq 1 \), then one can deduce from Theorem 1 and Theorem 5 that \( H_\varepsilon (T(B_{\mathcal{A}})) \leq \frac{3^4\rho(4)}{\varepsilon^4} \). In fact, one can choose \( H \), \( J \) and \( K \) (as in Theorem 1) so that \( \sigma_4(K) \leq 3 \) so from Theorem 5, \( H_\varepsilon (K(B_H)) \leq \frac{3^4\rho(4)}{\varepsilon^4} \) and since \( \|J\| \leq 1 \), \( H_\varepsilon (T(B_{\mathcal{A}})) \leq \frac{3^4\rho(4)}{\varepsilon^4} \). Hence if we set \( C = 3^4\rho(4) \), the proof of the theorem is complete. \[\square\]

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