Non-divisibility of LCM Matrices by GCD Matrices on GCD-closed Sets

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Abstract

In this paper, we consider the divisibility problem of LCM matrices by GCD matrices in the ring $M_n(\mathbb{Z})$ proposed by Hong in 2002 and in particular a conjecture concerning the divisibility problem raised by Zhao in 2014. We present some certain gcd-closed sets on which the LCM matrix is not divisible by the GCD matrix in the ring $M_n(\mathbb{Z})$. This could be the first theoretical evidence that Zhao’s conjecture might be true. Furthermore, we give the necessary and sufficient conditions on the gcd-closed set $S$ with $|S| \leq 8$ such that the GCD matrix divides the LCM matrix in the ring $M_n(\mathbb{Z})$ and hence we partially solve Hong’s problem. Finally, we conclude with a new conjecture that can be thought as a generalization of Zhao’s conjecture.

Keywords: GCD matrix, LCM matrix, divisibility, greatest-type divisor, divisor chain, Möbius function

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $f$ be an arithmetical function. We denote by $(f(S))$ and $(f[S])$ the $n \times n$ matrices on $S$ having $f$ evaluated at the greatest common divisor $(x_i, x_j)$ and the least common multiple $[x_i, x_j]$ of $x_i$ and $x_j$ as their $ij$-entries, respectively. If $f = I$, the identity function, the matrix $(I(S))$ is called the GCD matrix on $S$ and denoted by $(S)$. The LCM matrix $[S]$ is defined similarly. Given any positive real number $e$, let $\xi_e$ be the $e$-th power function. If $f = \xi_e$, then the matrices $(\xi_e(S))$ and $(\xi_e[S])$ are called the power GCD matrix and the power LCM matrix and we simply denote them by $(S^e)$ and $[S^e]$, respectively. In 1876, Smith [26] proved that if $S = \{1, 2, \ldots, n\}$, then $\det(S) = \prod_{k=1}^{n}(f*\mu)(k)$, where $f*\mu$ is the Dirichlet convolution of $f$ and the Möbius function $\mu$. Since then, many results

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on these matrices have been published in the literature. For general accounts see e.g. [1, 2, 3, 4, 10, 18, 20, 24, 25].

Let $A$ and $B$ be in $M_n(\mathbb{Z})$. We say that $A$ divides $B$ or $B$ is divisible by $A$ in the ring $M_n(\mathbb{Z})$ if there exists a matrix $C$ in $M_n(\mathbb{Z})$ such that $B = AC$ or $B = CA$, equivalently, $A^{-1}B \in M_n(\mathbb{Z})$ or $BA^{-1} \in M_n(\mathbb{Z})$. We simply write $A \mid B$ if $A$ divides $B$ in the ring $M_n(\mathbb{Z})$ and $A \nmid B$ otherwise. Divisibility is an interesting topic in the study of GCD and LCM matrices and the first result on the subject belongs to Bourque and Ligh. In 1992, they [4] showed that if $S = \{x_1, x_2, \ldots, x_n\}$ is factor closed then $(S) \mid |S|$. A set $S$ is factor closed if it contains all divisors of $x$ for any $x \in S$. Then, in [6], they also proved that if $S$ is factor closed, $f$ is multiplicative and $(f \ast \mu)(x_i) \neq 0$ for all $x_i \in S$ then $(f(S)) \mid (f|S)|$. A set $S$ is said to be gcd-closed if $(x_i, x_j)$ is in $S$ for all $1 \leq i, j \leq n$. Hong [13] showed that for any gcd-closed set $S$ with $|S| \geq 3$, $(S) \mid |S|$; however, for any integer $n \geq 4$, there is a gcd-closed set $S$ with $|S| = n$ such that $(S) \nmid |S|$. Along with the aforementioned results, Hong raised the following open problem in the same paper.

**Problem 1.1** [13]. Let $n \geq 4$. Find necessary and sufficient conditions on the gcd-closed set $S$ with $|S| = n$ such that $(S) \mid |S|$.

Problem [13] was solved in particular cases $n = 4$ and $n = 5$ by Zhao [34] and Zhao-Zhao [35], respectively. Providing a complete solution of Problem 1.1 is a hard task because there is no general method to construct all possible gcd-closed sets with $n$-elements. In [12], Hong used greatest-type divisors of the elements in $S$ to overcome this difficulty. Actually, the concept of greatest-type divisor was introduced by Hong in [12] to prove the Bourque-Ligh conjecture [4]. For $x, y \in S$ and $x < y$, if $x \mid y$ and the conditions $x \mid z \mid y$ and $z \in S$ imply that $z \in \{x, y\}$, then we say that $x$ is a greatest-type divisor of $y$ in $S$. For $x \in S$, we denote by $G_S(x)$ the set of all greatest-type divisors of $x$ in $S$. In this frame, in [10], Hong conjectured that if $S$ is a gcd-closed set with $\max_{x \in S}\{|G_S(x)|\} = 1$, then $(S) \mid |S|$. Hong, Zhao and Yin [10] proved Hong’s conjecture and they solved Problem 1.1 for the particular case $\max_{x \in S}\{|G_S(x)|\} = 1$. Then, in [7], Feng, Hong and Zhao introduced a new method to investigate Problem 1.1 for the case $\max_{x \in S}\{|G_S(x)|\} \leq 2$. They gave a new and elegant proof of Hong’s conjecture. Let $e$ be a positive integer. Indeed, they proved that if $S$ is a gcd-closed set satisfying $\max_{x \in S}\{|G_S(x)|\} \leq 2$, then $(S^e) \mid |S^e|$ if and only if $\max_{x \in S}\{|G_S(x)|\} = 1$ or $\max_{x \in S}\{|G_S(x)|\} = 2$ with $S$ satisfying the condition $\mathcal{C}$. We say that an element $x \in S$ with $|G_S(x)| = 2$ satisfies the condition $\mathcal{C}$ if $(y_1, y_2) = x$ and $(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$, where $G_S(x) = \{y_1, y_2\}$. We say that the set $S$ satisfies the condition $\mathcal{C}$ if each element $x \in S$ with $|G_S(x)| = 2$ satisfies the condition $\mathcal{C}$.

In addition to the aforementioned results, in [8], Haukkanen and Korkee investigated the divisibility of unitary analogues of GCD and LCM matrices in the ring $M_n(\mathbb{Z})$ and also, in [21], they considered Problem 1.1 for meet and join matrices when $n \leq 5$. On the other hand, Hong [14] proved that $(f(S)) \mid (f|S)|$ if $f$ is completely multiplicative and $S$ is a divisor chain or a multiple closed set, namely we have $y \in S$ if $x \mid y \mid \text{lcm}(S)$ for any $x \in S$, where
lcm(S) denotes the least common multiple of all the elements in S. Moreover, in a different point of view, many results on the divisibility of GCD and LCM matrices defined on particular sets have been published in the literature, see e.g. [11, 17, 22, 23, 27, 28, 31, 32, 33, 37].

Recently, in [36], Zhao solved Problem 1.1 when $5 \leq |S| \leq 7$. Indeed, he proved that $(S^e) | [S^e]$ if and only if $\max_{x \in S}\{|G_S(x)|\} = 1$, or $\max_{x \in S}\{|G_S(x)|\} = 2$ and $S$ satisfies the condition $C$. Thus, Problem 1.1 was solved for the case $|S| \leq 7$. In the same paper, Zhao raised the following conjecture.

**Conjecture 1.1.** Let $S = \{x_1, x_2, \ldots, x_n\}$ be a gcd-closed set with $\max_{x \in S}\{|G_S(x)|\} = m \geq 4$. If $n < \left(\frac{n}{2}\right) + m + 2$ then $(S^e) \nmid [S^e]$.

Organization of the paper is as follows. In Section 2, we present some well-known lemmas such as Lemmas 2.1, 2.2, and 2.4 and some novel lemmas which concern the inverse of the GCD matrix on gcd-closed sets and are important tools in the proof of our main results. In Section 3, firstly we give some results, in which we find some certain gcd-closed sets on which $(S)$ does not divide $[S]$. Secondly, using these results, which support the truth of Conjecture 1.1, we give the necessary and sufficient conditions on the gcd-closed set $S$ with $|S| \leq 8$ such that $(S) \mid [S]$ in the ring $M_n(\mathbb{Z})$, and hence a particular solution to Problem 1.1 when $|S| \leq 8$. In the last section, we present a new conjecture that can be thought as a generalization of Conjecture 1.1.

### 2. Preliminaries

We begin with a result of Bourque and Ligh [4] providing a formula for the entries of the inverse of $(S^e)$ when $S$ is gcd-closed. Throughout this section, we always assume that $S = \{x_1, x_2, \ldots, x_n\}$ and $S$ is gcd-closed.

**Lemma 2.1 ([5]).** The inverse of the power GCD matrix $(S^e)$ on $S$ is the matrix $W = (w_{ij})$, where

$$w_{ij} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik}c_{jk}}{\alpha_{e,k}}$$

with

$$c_{ij} = \sum_{\substack{d \mid x_i \\ x_j < x_i}} \mu(d)$$  \hspace{1cm} (2.1)

and

$$\alpha_{e,k} = \sum_{\substack{d \mid x_k \\ d \nmid x_i, x_j < x_k}} (\xi_e * \mu)(d)$$  \hspace{1cm} (2.2)

and $\xi_e(x) = x^e$.

The following lemma, which was presented by Hong [15], provides a simple way to calculate $\alpha_{e,k}$, and the proof follows from the inclusion-exclusion principle.
Lemma 2.2 ([15]). Let $G_S(x_k) = \{y_{k,1}, \ldots, y_{k,m}\}$ be the set of the greatest type divisors of $x_k$ in $S$ ($1 \leq k \leq n$). Then

$$\alpha_{c,k} = x_k^e + \sum_{t=1}^{m} (-1)^t \sum_{1 \leq t_1 < \cdots < t_r \leq m} (x_k, y_{k,t_1}, \ldots, y_{k,t_r})^r$$

(2.3)

with $\alpha_{c,k}$ defined as in (2.2).

Similarly, using the inclusion-exclusion principle, we obtain the following lemma for the values of $c_{ij}$ and $\alpha_{1,j}$.

Lemma 2.3. Let $G_S(x_j) = \{y_{j,1}, \ldots, y_{j,m}\}$ be the set of the greatest type divisors of $x_j$ in $S$ ($1 \leq j \leq n$). Then

$$c_{ij} = \sum_{d \mid x_j} \mu(d) + \sum_{r=1}^{m} (-1)^r \sum_{1 \leq t_1 < \cdots < t_r \leq m} \sum_{d \mid (y_{j,t_1}, \ldots, y_{j,t_r})} \mu(d)$$

(2.4)

and

$$\alpha_j := \alpha_{1,j} = \sum_{x_i \mid x_j} x_i c_{ij}.$$  

(2.5)

The values of $c_{ij}$ play an important role to determine the divisibility of LCM matrices by GCD matrices on gcd-closed sets. Therefore, we calculate the value of $c_{ij}$ in some particular cases. The first lemma belongs to Zhao [30].

Lemma 2.4 ([30]). If $x_i \in G_S(x_j)$ then $c_{ij} = -1$ and $c_{jj} = 1$.

Now, we introduce a new type subset of $S$. Let $G_S(x_k) = \{y_{k,1}, \ldots, y_{k,m}\}$ for $x_k \in S$. We define $D_S(x_k)$ as follows:

$$D_S(x_k) := \{(y_{k,i_1}, \ldots, y_{k,i_r}) : 2 \leq r \leq m \text{ and } 1 \leq i_1 < \cdots < i_r \leq m\}.$$  

In other words, $D_S(x_k)$ is the set of all possible greatest common divisors of different greatest-type divisors of $x_k$. Moreover, we recall the set $D_x = \{x \in S : x \mid x \text{ and } x > x_r\}$ for $x_r$ in $S$ which was defined by Feng, Hong and Zhao in [7].

We give the second lemma for the value of $c_{ij}$, which is, in fact, a generalization of Lemma 2.7 in [30].

Lemma 2.5. If $x_i \in D_S(x_j)$ and $D_S(x_j) \cap D_i = \emptyset$, then $c_{ij} = l_i - 1$, where $l_i = |D_i \cap G_S(x_j)|$.

Proof. Let $G_S(x_j) = \{y_{j,1}, \ldots, y_{j,m}\}$. Since $x_i \in D_S(x_j)$ it is obvious that $D_i \cap G_S(x_j) \neq \emptyset$. Without loss of generality, we can assume that $D_i \cap G_S(x_j) = \{y_{j,1}, \ldots, y_{j,l_i}\}$. Now, suppose that $G_S(x_j) - D_i = \emptyset$. Then, clearly $G_S(x_j) \subset D_i$. Since $D_i \cap D_S(x_j) = \emptyset$, $D_S(x_j)$ must consist of only $x_i$. In this case, it is clear that $|D_i \cap G_S(x_j)| = m$, and hence $l_i = m$. Then, by (2.4), we have

$$c_{ij} = (-1)^2 \binom{m}{2} + (-1)^3 \binom{m}{3} + \cdots + (-1)^m \binom{m}{m} = m - 1.$$  

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Now, consider the case \( G_S(x_j) - D_i \neq \emptyset \). Let \( y_{j,k} \in G_S(x_j) - D_i \). If \( y_{j,k} \in \{y_{j,i_1}, \ldots, y_{j,i_r}\} \) \( (2 \leq r \leq m) \), then, by the definition of \( D_i \), we have \( x_i \nmid y_{j,k} \), and hence \( x_i \nmid (y_{j,i_1}, \ldots, y_{j,i_r}) \). So, we have \( \frac{(y_{j,i_1}, \ldots, y_{j,i_r})}{x_i} \notin \mathbb{Z} \). Then, we can write \( c_{ij} \) as follows:

\[
c_{ij} = \sum_{d \mid i} \mu(d) + \sum_{r=1}^{l_i} (-1)^r \sum_{1 \leq i_1 < \cdots < i_r \leq l_i} \sum_{d \mid i} \mu(d). \tag{2.6}
\]

Since \( x_i \in D_S(x_j) \), we have \( x_i \notin G_S(x_j) \). So, by a well-known property of the Möbius function, \( \sum_{d \mid y_{i,t}} \mu(d) = 0 \) for \( 1 \leq i \leq l_i \). Thus, we can rewrite (2.6) as follows:

\[
c_{ij} = \sum_{r=2}^{l_i} (-1)^r \sum_{1 \leq i_1 < \cdots < i_r \leq l_i} \sum_{d \mid i} \mu(d). \tag{2.7}
\]

Since \( D_i \cap D_S(x_j) = \emptyset \) and \( \{y_{j,i_1}, \ldots, y_{j,i_r}\} \subset D_i \), we have \( (y_{j,i_1}, \ldots, y_{j,i_r}) = x_i \) for every \( 2 \leq r \leq l_i \) \((1 \leq i_1 < \cdots < i_r \leq l_i)\). Then, by (2.7), we have \( c_{ij} = l_i - 1 \).

When \( x_i \in D_S(x_j) \) and \( D_i \cap D_S(x_j) \neq \emptyset \), it is really a hard task to calculate the values of \( c_{ij} \) on all possible gcd-closed sets; however, by making some restrictions on the set \( S \), we can obtain a formula for the values of \( c_{ij} \). In order to do this, we denote by \( \text{Min}(D_i \cap D_S(x_j)) \) the set of all the minimal elements in \( D_i \cap D_S(x_j) \) with respect to the divisibility relation on \( S \).

**Lemma 2.6.** Let \( x_i \in D_S(x_j) \). \( D_i \cap D_S(x_j) \neq \emptyset \) and \( \text{Min}(D_i \cap D_S(x_j)) = \{x_{i_1}, \ldots, x_{i_k}\} \). Let \( |D_{i,t} \cap G_S(x_j)| \leq 1 \) for all \( 1 \leq r < t \leq k \) when \( k \geq 2 \). Then \( c_{ij} = l_i - \sum_{t=1}^{k} l_{i,t} + (k - 1) \), where \( l_{i,t} = |D_{i,t} \cap G_S(x_j)| \).

**Proof.** Let \( D_i \cap G_S(x_j) = \{y_{j,i_1}, \ldots, y_{j,i_k}\} \) without loss of generality. Since \( x_i \in D_S(x_j) \), we can calculate \( c_{ij} \) by (2.7). Now, we consider the summand for \( r = 2 \) in (2.7). We want to find the number of terms such that \( (y_{j,i_1}, y_{j,i_2}) = x_i \) or equivalently \( (y_{j,i_1}, y_{j,i_2})/x_i = 1 \) for \( 1 \leq i_1 < i_2 \leq l_i \). Since \( |D_{i,t} \cap G_S(x_j)| \leq 1 \), \( (y_{j,i_1}, y_{j,i_2}) \) is equal to \( x_i \) or a multiple of only one element in \( \text{Min}(D_{i,t} \cap G_S(x_j)) \). So, there exist \( \binom{l_{i,t}}{2} \) terms such that \( (y_{j,i_1}, y_{j,i_2}) \) is a multiple of \( x_{i,t} \). By the same argument, the number of 2-tuples of \( y_{j,i_1} \) and \( y_{j,i_2} \) \((i_1 < i_2)\) such that \( (y_{j,i_1}, y_{j,i_2}) \notin x_i \) is \( \sum_{t=1}^{k} \binom{l_{i,t}}{2} \). Here, it should be noted that there is no common subsets of \( D_{i,t} \cap G_S(x_j) \) and \( D_{i,t} \cap G_S(x_j) \) with two or more elements for \( 1 \leq r < t \leq k \) by the hypothesis of the theorem. If we continue in this manner for \( r = 3, \ldots, l_i \) we obtain that

\[
c_{ij} = (-1)^2 \left[ \binom{l_{i}}{2} - \sum_{t=1}^{k} \binom{l_{i,t}}{2} \right] + \cdots + (-1)^l_i \left[ \binom{l_{i}}{l_i} - \sum_{t=1}^{k} \binom{l_{i,t}}{l_i} \right]. \tag{2.8}
\]
Here, for convenience, we can assume \( \binom{n}{m} = 0 \) whenever \( n < m \). Thus, we obtain
\[
c_{ij} = \sum_{r=2}^{l_i} \left( -1 \right)^r \left( \frac{l_i}{r} - \sum_{t=1}^{k} \frac{l_i}{r} \right)
\]
\[
= (l_i - 1) - \sum_{t=1}^{k} (l_i,t - 1)
\]
\[
= l_i - \sum_{t=1}^{k} l_i,t + (k - 1),
\]
which concludes the proof.

Let \((L, \leq)\) be a finite meet semilattice. Haukkanen, Mattila and Mäntysalo determined the zeros of the Möbius function of \(L\), see [9, Lemma 3.1]. If we take \((L, \leq) = (S, |)\), where \(S\) is a gcd-closed set of distinct positive integers and | is the divisibility relation on \(\mathbb{Z}\), we can restate their claim as follows:
\[
\mu_S(x) = 0 \text{ unless } \gcd(G_S(x)) \mid z \mid x.
\]

The following lemma is a generalization of the above result in the number theoretical setting and by using it, we can determine the zeros of \(c_{ij}\) on a gcd-closed set.

**Lemma 2.7.** Let \(1 \leq i < j \leq n\). If \(x_i \not\mid x_j\) and \(x_i \not\in D_S(x_j)\), then \(c_{ij} = 0\).

**Proof.** If \(x_i \not\mid x_j\), then it is clear that \(c_{ij} = 0\). Now, let \(x_i \mid x_j\) and \(G_S(x_j) = \{y_{j,1}, \ldots, y_{j,m}\}\). Since \(\sum_{d \mid x_j} \mu(d) = 0\) whenever \(x_i \neq x_j\), by (2.4), we have
\[
c_{ij} = \sum_{r=1}^{m} (-1)^{r} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \sum_{d \mid y_{j,i_1, \ldots, y_{j,i_r}}} \mu(d).
\]

Now, consider the sum \(\sum_{d \mid y_{j,i_1, \ldots, y_{j,i_r}}} \mu(d)\) for \(1 \leq r \leq m\). Since \(x_i \not\in G_S(x_j)\) and \(x_i \not\in D_S(x_j)\), we always have \((y_{j,i_1}, \ldots, y_{j,i_r}) \neq x_i\). Therefore, by a well-known property of the Möbius function, \(\sum_{d \mid y_{j,i_1, \ldots, y_{j,i_r}}} \mu(d) = 0\) for all \(1 \leq r \leq m\). This completes the proof.

**Lemma 2.8.** For \(j > 1\), we have \(\sum_{i=1}^{n} c_{ij} = 0\) or equivalently \(\sum_{x_i \mid x_j} c_{ij} = 0\).

**Proof.** Since \(c_{jj} = 1\) we have to prove that \(\sum_{x_i \mid x_j} c_{ij} = -1\). Let \(G_S(x_j) = \{y_{j,1}, \ldots, y_{j,m}\}\). Then, by (2.4), we have
\[
\sum_{x_i \mid x_j} c_{ij} = \sum_{r=1}^{m} (-1)^{r} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \sum_{d \mid y_{j,i_1, \ldots, y_{j,i_r}}} \mu(d). \quad (2.9)
\]
Then, we have denote the largest integer such that $p^n \leq m$. For convenience, we assume our claim, we have

**Proof.** Firstly we claim that for positive integers $m$, $T_2 T_4 \ldots T_m = T_1 T_3 \ldots T_{m-1}$ if $m$ is even

$$[a_1, \ldots, a_m] = \frac{T_1 T_3 \ldots T_{m-1}}{T_2 T_4 \ldots T_m}$$

otherwise. Here $T_1 = \prod_{i=1}^m a_i$ and $T_k = \prod_{i<j} a_i a_j$ for $2 \leq k \leq m$. We will prove the claim when $m$ is even. It is sufficient to prove that $[a_1, \ldots, a_m] T_2 \ldots T_m = T_1 T_3 \ldots T_{m-1}$.

Consider a prime number $p$ such that $p \nmid [a_1, \ldots, a_m]$. For $a \in \mathbb{Z}^+$, let $\nu_p(a)$ denote the largest integer such that $p^{\nu_p(a)}$ divides $a$. Without loss of generality, we can assume that $\nu_p(a_1) \leq \cdots \leq \nu_p(a_m)$. Then, we have

$$\nu_p ([a_1, \ldots, a_m] T_2 T_4 \ldots T_m) = \nu_p (a_m) + \sum_{i=1}^{m-1} \left( \sum_{j=1}^{m/2} \binom{m-i}{2j-1} \right) \nu_p (a_i)$$

$$= \nu_p (a_m) + \sum_{i=1}^{m-1} \left( \sum_{j=1}^{m/2} \binom{m-i}{2j} \right) \nu_p (a_i)$$

$$= \nu_p (T_1 T_3 \ldots T_{m-1}).$$

Here, for convenience, we assume $\binom{i}{j} = 0$ whenever $j > i$. Thus,

$$[a_1, \ldots, a_m] T_2 T_4 \ldots T_m = T_1 T_3 \ldots T_{m-1}.$$

We can similarly prove the case that $m$ is odd. Assuming that $m$ is even, by our claim, we have

$$[y_{n,1}, \ldots, y_{n,m}] = \prod_{i=1}^m \frac{y_{n,i} T_3 T_5 \ldots T_{m-1}}{T_1 T_3 \ldots T_m}$$
where $T_k = \prod_{1 \leq i_1 < \cdots < i_k \leq m} (y_{n,i_1}, \ldots, y_{n,i_k})$. Since every $(y_{n,i_1}, \ldots, y_{n,i_k})$ is in $D_S(x_n)$ for $k \geq 2$, we can write $T_k = \beta_{n_1} \cdots \beta_{n_t}$, where each $\beta_{n_r}$ is a nonnegative integer for $1 \leq r \leq t$. Indeed,

$$\beta_{n_r} = |\{(y_{n,i_1}, \ldots, y_{n,i_k}) : (y_{n,i_1}, \ldots, y_{n,i_k}) = x_{n_r}, 1 \leq i_1 < \cdots < i_k \leq m\}|.$$

Thus, by (2.4), it is clear that the exponent of $x_{n_r}$ in the fraction $\frac{\prod_{1 \leq i_1 < \cdots < i_k \leq m} (y_{n,i_1}, \ldots, y_{n,i_k})}{T_k T_{k+1} \cdots T_{m-1}}$ is equal to $c_{n_r,n}$. Furthermore, by Lemma 2.8 we obtain that $\sum_{k=1}^{m} c_{n_k n} = m - 1$.

3. Main Results

In this section, we give main results of our paper. For the proof of the first three results, we use Zhao’s approach [36], that is, we will prove that an entry of the product $[S](S)^{-1}$ is in the interval $(0, 1)$. Throughout this section, we denote $[S](S)^{-1}$ by $U$, where $[S]$ is the LCM matrix and $(S)$ is the GCD matrix, and we assume that $S$ is gcd-closed.

**Theorem 3.1.** Let $S = \{x_1, x_2, \ldots, x_n\}$ with $m > 5$. Let $x_n \in S$ such that $n \geq 5$, $G_S(x_n) = \{x_2, \ldots, x_{n-1}\}$ and $\gcd(G_S(x_n)) = x_1$. If $x_i \mid x_n$ and $x_i \notin D_S(x_n)$ for all $n < i \leq m$, then $(S) \nmid [S]$.

**Proof.** We have to prove that $U \notin M_m(\mathbb{Z})$. To perform this, it is sufficient to show that $U_{2n} \notin \mathbb{Z}$. By Lemmas 2.1 and 2.4 we have

$$U_{2n} = \frac{x_n - \sum_{i=2}^{n-1} [x_2, x_i] + \sum_{i=n+1}^{m} [x_2, x_i] c_{in} + [x_2, x_1] c_{1n}}{\alpha_n}.$$

By Lemma 2.7 we have $c_{in} = 0$ for $n + 1 \leq i \leq m$ since $x_i$ is neither in $G_S(x_n)$ nor in $D_S(x_n)$ whenever $i > n$. In addition to this, $c_{in} = n - 3$ by Lemma 2.5. Then, we have $U_{2n} = \frac{x_n - \sum_{i=2}^{n-1} [x_2, x_i] + x_2(n-3)}{\alpha_n}$. Also, by Lemma 2.3 $\alpha_n = x_n - \sum_{i=2}^{n-1} x_i + x_1(n-3)$. Letting

$$\beta_n := x_n - \sum_{i=2}^{n-1} [x_2, x_i] + x_2(n-3),$$

we can write $U_{2n}$ as $U_{2n} = \frac{\beta_n}{\alpha_n}$. Here, one can show that $\beta_n > 0$ and $\alpha_n > \beta_n$ using Zhao’s approach as in the proof of Lemma 2.9 in [36]. So, we have $0 < U_{2n} < 1$ which means that $U_{2n} \notin \mathbb{Z}$.

**Theorem 3.2.** Let $S = \{x_1, x_2, \ldots, x_t\}$ with $t > 5$. Let $x_n \in S$ such that $n \geq 5$, $G_S(x_n) = \{x_2, \ldots, x_{n-1}\}$, $\gcd(G_S(x_n)) = x_1$ and $D_S(x_n) = \{x_1, x_{n+1}, \ldots, x_m\}$ ($m < t$). If $x_i \mid x_n$ for all $n < i \leq t$ and $D_S(x_n)$ is a divisor chain, then $(S) \nmid [S]$. 

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Proof. Since $D_S(x_n)$ is a divisor chain and $G_S(x_n) = \{x_2, \ldots, x_{n-1}\}$, we can assume that $x_1 \mid x_{n+1} \mid x_{n+2} \mid \cdots \mid x_m$ and $x_m \mid x_2$ without loss of generality. By Lemma 2.1, we have

$$U_{2n} = \frac{\sum_{i=1}^{t} [x_2, x_i]c_{sn}}{\alpha_n}.$$ 

By Lemmas 2.4-2.7, we have $c_{sn} = \begin{cases} 0 & \text{if } s > m, \\ l_s - l_{s+1} & \text{if } n + 1 \leq s \leq m - 1, \\ l_m - 1 & \text{if } s = m, \\ 1 & \text{if } s = n, \\ -(n - 2) - l_{n+1} & \text{if } s = 1, \end{cases}$

and hence

$$U_{2n} = \frac{x_n - \sum_{i=2}^{n-1} [x_2, x_i] + \sum_{i=n+1}^{m-1} (l_i - l_{i+1})x_2 + (l_m - 1)x_2 + [(n - 2) - l_{n+1}]x_2}{\alpha_n}.$$ 

Then, we have

$$U_{2n} = \frac{x_n - \sum_{i=2}^{n-1} [x_2, x_i] + x_2(n - 3)}{\alpha_n}.$$ 

In what follows we let $\gamma_n := x_n - \sum_{i=2}^{n-1} [x_2, x_i] + x_2(n - 3)$. Since $n \geq 5$, we have

$$\gamma_n > [x_2, \ldots, x_{n-1}] - \sum_{i=3}^{n-1} [x_2, x_i].$$

By Lemma 2.9 we know that

$$[x_2, \ldots, x_{n-1}] = \frac{\prod_{i=2}^{n-1} x_i}{x_1^{c_{1n}} x_{n+1}^{c_{n+1,n}} \cdots x_m^{c_{mn}}},$$

where $c_{1n} + c_{n+1,n} + \cdots + c_{mn} = n - 3$ and $c_{in} > 0$ for $i = 1$ and $n + 1 \leq i \leq m$.

Suppose that $\max([x_2, x_i] : 3 \leq i \leq n - 1) = [x_2, x_r]$. By the definition of $D_S(x_n)$, it is clear that $(x_2, x_r) \in D_S(x_n)$. Without loss of generality, we can assume that $(x_2, x_r) = x_1$. Then

$$\gamma_n \geq \frac{x_2x_r}{x_1} \left( \frac{\prod_{i=2}^{n-1} x_i}{x_1^{c_{1n}} x_{n+1}^{c_{n+1,n}} \cdots x_m^{c_{mn}}} - (n - 3) \right) \geq \frac{x_2x_r}{x_1} \left( 2^{n-4} - (n - 3) \right) \geq 0.$$ 

On the other hand, by Lemma 2.3 we have

$$\alpha_n = x_n - \sum_{i=2}^{n-1} x_i + \sum_{i=n+1}^{m-1} (l_i - l_{i+1})x_i + (l_m - 1)x_m + [(n - 2) - l_{n+1}]x_1.$$
Now, we show that $\alpha_n$ is greater than $\gamma_n$.

\[
\alpha_n - \gamma_n = \sum_{i=3}^{n-1} ([x_2, x_i] - x_i) + (x_1 - x_2)(n - 2 - l_{n+1}) + (x_m - x_2)(l_m - 1) \\
+ \sum_{i=n+1}^{m-1} (x_i - x_2)(l_i - l_{i+1}).
\]

We claim that $\{x_i \in G_S(x_n) : (x_2, x_i) = x_s\} = c_{sn}$ for $s = 1$ or $n + 1 \leq s \leq m$. For $s = 1$,

\[
(x_2, x_i) = x_1 \Leftrightarrow x_{n+1} \nmid x_i \\
\Leftrightarrow x_i \notin D_{n+1} \\
\Leftrightarrow x_i \in (G_S(x_n) \cap D_1) - (G_S(x_n) \cap D_{n+1})
\]

and for $n + 1 \leq s \leq m - 1$,

\[
(x_2, x_i) = x_s \Leftrightarrow x_{s+1} \nmid x_i \\
\Leftrightarrow x_i \notin D_{s+1} \\
\Leftrightarrow x_i \in (G_S(x_n) \cap D_s) - (G_S(x_n) \cap D_{s+1}).
\]

Also, our claim for $s = m$ is a direct consequence of Lemma 2.5. Now, we can rewrite $\alpha_n - \gamma_n$ according to $(x_2, x_i)$ for $3 \leq i \leq n - 1$.

\[
\alpha_n - \gamma_n = \sum_{(x_2, x_i) = x_1} ([x_2, x_i] - x_i + x_1 - x_2) + \sum_{k=n+1}^{m} \sum_{(x_2, x_i) = x_k} ([x_2, x_i] - x_i + x_1 - x_2).
\]

It is clear that in the first sum

\[
[x_2, x_i] - x_i + x_1 - x_2 = \frac{x_2}{x_1} - 1)(x_i - x_1) > 0
\]

and in the second sum

\[
[x_2, x_i] - x_i + x_k - x_2 = \frac{x_2}{x_k} - 1)(x_i - x_k) > 0.
\]

Thus, $\alpha_n - \gamma_n > 0$, and hence $U_{2n} = \frac{2n}{\alpha_n}$ is not an integer. \qed

**Theorem 3.3.** Let $S = \{x_1, x_2, \ldots, x_m\}$ with $m > 5$. Let $x_n \in S$ such that $G_S(x_n) = \{x_2, \ldots, x_{n-1}\}$, $gcd(G_S(x_n)) = x_1$ and $D_S(x_n) = \{x_1, x_{n+1}, x_{n+2}\}$. If $x_i \mid x_n$ for all $n < i \leq m$, then $(S) \not| [S]$.

**Proof.** If $D_S(x_n)$ is a divisor chain then the proof is a direct consequence of Theorem 3.2. Now, let $D_S(x_n)$ be a $x_1$-set, namely $(x_{n+1}, x_{n+2}) = x_1$. We will prove the claim of the theorem in two cases as the set $(G_S(x_n) \cap D_{n+1}) \cap (G_S(x_n) \cap D_{n+2})$ can be empty or a singleton subset of $G_S(x_n)$.

Now, let $G_S(x_n) \cap D_{n+1} \cap D_{n+2} \neq \emptyset$. The set $G_S(x_n) \cap D_{n+1} \cap D_{n+2}$ cannot have more than one element. Suppose the contrary, that is, $x_i, x_j \in
$G_S(x_n) \cap D_{n+1} \cap D_{n+2}$. Since $S$ is gcd-closed and $D_S(x_n) = \{x_1, x_{n+1}, x_{n+2}\}$, we have $(x_i, x_j) = x_{n+1}$ or $x_{n+2}$. Now, assume that $(x_i, x_j) = x_{n+1}$. On the other hand, $x_{n+2} \mid (x_i, x_j)$ since $x_i, x_j \in D_{n+2}$. Then, we have $x_{n+2} \mid x_{n+1}$, a contradiction. Thus, we can assume that $G_S(x_n) \cap D_{n+1} \cap D_{n+2} = \{x_2\}$ without loss of generality. We will show that $U_{2n} \notin \mathbb{Z}$. By Lemmas 2.4-2.7, it is clear that

$$c_{s,n} = \begin{cases} n - l_{n+1} - l_{n+2} - 1 & \text{if } s = 1, \\
-1 & \text{if } 2 \leq s \leq n - 1, \\
1 & \text{if } s = n, \\
l_{n+1} - 1 & \text{if } s = n + 1, \\
l_{n+2} - 1 & \text{if } s = n + 2, \\
0 & \text{if } s > n + 2. \end{cases}$$

Thus, by Lemma 2.1, we have

$$U_{2n} = \frac{1}{\alpha_n} \left( x_n - \sum_{i=2}^{n-1} [x_2, x_i] + [x_2, x_{n+1}](l_{n+1} - 1) + [x_2, x_{n+2}](l_{n+2} - 1) + [x_2, x_1](n - l_{n+1} - l_{n+2} - 1) \right)$$

Since $x_2$ is a multiple of lcm($D_S(x_n)$), by Lemma 2.8 we have

$$U_{2n} = \frac{x_n - \sum_{i=2}^{n-1} [x_2, x_i] + x_2(n - 3)}{\alpha_n},$$

where

$$\alpha_n = x_n - \sum_{i=2}^{n-1} x_i + x_{n+1}(l_{n+1} - 1) + x_{n+2}(l_{n+2} - 1) + x_1(n - l_{n+1} - l_{n+2} - 1).$$

Let

$$\gamma_n = x_n - \sum_{i=2}^{n-1} [x_2, x_i] + x_2(n - 3).$$

Using the same method as in the proof of Theorem 3.2, one can easily show that $\gamma_n$ is positive and $|\{x_k \in G_S(x_n) : x_k = (x_2, x_i)\}| = c_{k,n}$ for $k = 1, n + 1, n + 2$. So, it is sufficient to show that $\alpha_n - \gamma_n$ is positive. To do this, we write $\alpha_n - \gamma_n$
as follows:

\[\alpha_n - \gamma_n = \sum_{(x_2, x_i) = x_{n+1}, \ x_i \in G_S(x_n)} ([x_2, x_i] - x_i + (x_{n+1} - x_2)) + \sum_{(x_2, x_i) = x_{n+2}, \ x_i \in G_S(x_n)} ([x_2, x_i] - x_i + (x_{n+2} - x_2)) + \sum_{(x_2, x_i) = x_1, \ x_i \in G_S(x_n)} ([x_2, x_i] - x_i + (1 - x_2))\]

\[= \sum_{(x_2, x_i) = x_{n+1}, \ x_i \in G_S(x_n)} \left( \frac{x_2}{x_{n+1}} - 1 \right)(x_i - x_{n+1}) + \sum_{(x_2, x_i) = x_{n+2}, \ x_i \in G_S(x_n)} \left( \frac{x_2}{x_{n+2}} - 1 \right)(x_i - x_{n+2}) + \sum_{(x_2, x_i) = x_1, \ x_i \in G_S(x_n)} \left( \frac{x_2}{x_1} - 1 \right)(x_i - x_1).\]

Then, it is clear that \(\alpha_n - \gamma_n > 0\).

Now, we investigate the case \([D_{n+1} \cap D_{n+2}] \cap G_S(x_n) = \emptyset\). Without loss of generality, we can assume that \(D_{n+1} \cap G_S(x_n) = \{x_2, \ldots, x_k\}\) and \(D_{n+2} \cap G_S(x_n) = \{x_{k+2}, \ldots, x_{k+s+1}\}\). In this case, by Lemmas 2.1, 2.4 - 2.7, we have

\[U_{2n} = \frac{1}{\alpha_n} \left( x_n - \sum_{i=2}^{n-1} [x_2, x_i] + [x_2, x_{n+1}](k-1) + [x_2, x_{n+2}](s-1) + [x_2, x_1](n-k-s-1) \right).\]

Also, by Lemma 2.8

\[\alpha_n = x_n - \sum_{i=2}^{n-1} x_i + (k-1)(x_{n+1}) + (s-1)x_{n+2} + (n-k-s-1)x_1.\]

Let

\[\gamma_n := x_n - \sum_{i=2}^{n-1} [x_2, x_i] + [x_2, x_{n+1}](k-1) + [x_2, x_{n+2}](s-1) + [x_2, x_1](n-k-s-1).\]

Using a similar method as in the proof of Theorem 3.2, one can show that \(\gamma_n > 0\). Now, we will prove that \(\alpha_n - \gamma_n\) is positive.
\[ \alpha_n - \gamma_n = (k - 1)(x_{n+1} - x_2) + \sum_{i=3}^{k+1} ([x_2, x_i] - x_i) \]

\[ + (s - 1)(x_{n+2} - [x_2, x_{n+2}]) + \sum_{i=k+2}^{k+s+1} ([x_2, x_i] - x_i) \]

\[ + (n - k - s - 1)(x_1 - x_2) + \sum_{i=k+s+2}^{n-1} ([x_2, x_i] - x_i) \]

\[ = \sum_{i=3}^{k+1} ([x_2, x_i] - x_i + (x_{n+1} - x_2)) \]

\[ + \sum_{i=k+2}^{k+s+1} ([x_2, x_i] - x_i + (x_{n+2} - x_2)) \]

\[ + \sum_{i=k+s+2}^{n-1} ([x_2, x_i] - x_i + (x_1 - x_2)) \]

\[ + [x_2, x_{n+2}] - x_{n+2} + x_1 - x_2 \]

\[ = \sum_{i=3}^{k+1} \left( \frac{x_2}{x_{n+1}} - 1 \right)(x_i - x_{n+1}) + \sum_{i=k+2}^{k+s+1} \left( \frac{x_2}{x_1} - 1 \right)(x_i - x_{n+2}) \]

\[ + \sum_{i=k+s+2}^{n-1} \left( \frac{x_2}{x_1} - 1 \right)(x_i - x_1) + \left( \frac{x_2}{x_1} - 1 \right)(x_{n+2} - x_1) \]

\[ > 0. \]

This completes the proof. \( \square \)

After the proof of Theorems 3.1-3.3 we can say that Zhao’s approach works when \( x_n \) is a maximal element of \( S \) with respect to the divisibility relation. Does the same method work if \( S \) contains some multiples of \( x_n \)? It appears to be difficult to answer this question without the following lemma.

**Lemma 3.1.** Let \( S = \{x_1, x_2, \ldots, x_m\} \) such that \( i \leq j \) whenever \( x_i \mid x_j \). Also, let \( x_n \in S \) and \( D_n \cup \{x_n\} = \{x_n = x_{n_1}, \ldots, x_{n_t}\} \). Then, for each \( 1 \leq q \leq m \),

\[ \sum_{i=1}^{t} U_{qn_i} = \sum_{s=1}^{m} [x_q, x_s] C_{sn} \theta_n. \]
we can write (3.1) as follows

\[
\sum_{i=1}^{t} U_{qn_i} = \sum_{i=1}^{t} \left[ \sum_{s=1}^{m} [x_q, x_s] \sum_{x_i \in x_k} c_{nk} \frac{c_{nk}}{\alpha_k} \right]
\]

\[
= \sum_{s=1}^{m} [x_q, x_s] \sum_{k=1}^{m} k \sum_{i=1}^{t} c_{nk} \frac{c_{nk}}{\alpha_k}
\]

\[
= \sum_{k=1}^{m} \sum_{s=1}^{m} [x_q, x_s] \frac{c_{nk}}{\alpha_k} \sum_{i=1}^{t} c_{nk}
\]

Here \( \sum_{i=1}^{t} c_{nk} = c_{nn} = 1 \) and \( \sum_{i=1}^{t} c_{nk} = \sum_{x_i \in x_k} c_{nk} \). The last sum is over \( x_n \in D_n \cup \{x_n\} \) dividing the fixed \( x_k \in D_n \). Since \( S \) is gcd-closed, \( D_n \cup \{x_n\} \) is also gcd-closed. Now, let \( (c_{ij})_A \) denote \( c_{ij} \) for a gcd-closed set \( A \), as defined in Lemma 2.1. We want to show that \( (c_{nk})_S = (c_{nk})_{D_n \cup \{x_n\}} \). Let \( G_S(x_n) = \{y_j, \ldots, y_j, k\} \). By Lemma 2.3

\[
(c_{nk})_S = \sum_{d | x_n} \mu(d) + \sum_{r=1}^{k} (-1)^r \sum_{1 \leq i_1 < \ldots < i_r \leq k} \sum_{d | (y_j, i_1, \ldots, y_j, i_r)} \mu(d).
\]  

(3.1)

Without loss of generality, let \( x_n \uparrow y_k \). Then \( x_n \uparrow (y_j, i_1, \ldots, y_j, i_r, y_k) \), and hence \( (y_j, i_1, \ldots, y_j, i_r, y_k) / x_n \notin \mathbb{Z} \). So, if \( y_j, k \in \{y_j, i_1, \ldots, y_j, i_r\} \), then the summation \( \sum_{d | (y_j, i_1, \ldots, y_j, i_r) / x_n} \mu(d) \) is empty, and hence it is equal to zero. Thus, letting \( G_S(x_n) \cap (D_n \cup \{x_n\}) = \{y_j, 1, \ldots, y_j, u\} \) without loss of generality, we can write (3.1) as follows

\[
(c_{nk})_S = \sum_{d | x_n} \mu(d) + \sum_{r=1}^{u} (-1)^r \sum_{1 \leq i_1 < \ldots < i_r \leq u} \sum_{d | (y_j, i_1, \ldots, y_j, i_r)} \mu(d).
\]

On the other hand, it is clear that \( G_S(x_n) \cap (D_n \cup \{x_n\}) \subset G_S(x_n) \cap (D_n \cup \{x_n\}) \) and \( G_{D_n \cup \{x_n\}}(x_n) = G_S(x_n) \cap (D_n \cup \{x_n\}) \). Thus, we obtain \( (c_{nk})_S = (c_{nk})_{D_n \cup \{x_n\}} \).

Now, since \( D_n \cup \{x_n\} \) is a gcd-closed set and \( (c_{nk})_{D_n \cup \{x_n\}} = (c_{nk})_S \), we have \( \sum_{i=1}^{t} c_{nk} = 0 \) by Lemma 2.8 for \( x_k \in D_n \). Thus,

\[
\sum_{i=1}^{t} U_{qn_i} = \sum_{s=1}^{m} [x_q, x_s] \frac{c_{nn}}{\alpha_n}.
\]
Putting Theorems 3.1-3.3 and Lemma 3.1 together, we have the following result.

**Theorem 3.4.** Let \( S = \{ x_1, x_2, \ldots, x_m \} \) and let \( S \) have an element \( x \) with \( |G_S(x)| \geq 3 \). If \( D_S(x) \) is a divisor chain or \( |D_S(x)| \leq 3 \), then \( (S) \mid [S] \).

**Proof.** Without loss of generality, we can assume that \( x_n \in S \) such that \( 5 \leq n \leq m \), \( G_S(x_n) = \{ x_2, \ldots, x_{n-1} \} \), and \( \gcd(G_S(x_n)) = x_1 \). Also, let \( D_n \cup \{ x_n \} = \{ x_n = x_n_1, \ldots, x_n_t \} \). By Lemma 3.1, we have

\[
\sum_{i=1}^{t} U_{2n_i} = \sum_{s=1}^{m} [x_2, x_s] \frac{c_{2n_i}}{\alpha_n}.
\]

We have two cases that \( D_S(x_n) \) could be a divisor chain or not. In both cases, one can show that \( \sum_{i=1}^{m} [x_2, x_s] \frac{c_{2n_i}}{\alpha_n} \notin \mathbb{Z} \) by similar methods to the proofs of Theorems 3.2 and 3.3, respectively. \( \square \)

So far, we have proven that if \( S \) has an element \( x \) such that \( |G_S(x)| \geq 3 \), and \( |D_S(x)| \leq 3 \) or \( D_S(x) \) is a divisor chain, then the divisibility does not hold. On the other hand, for the complete solution of Problem 1.1 for \( |S| \leq 8 \), whether the divisibility holds when \( S \) has an element \( x \) such that \( |G_S(x)| = 3 \) and \( |D_S(x)| = 4 \) remains unsolved. The following condition is a key to the divisibility for this case. For \( x \in S \), we say that \( x \) satisfies the condition \( \mathcal{M} \) if \( [x_i, x_j] = x \) for all different \( x_i, x_j \in G_S(x) \) when \( |G_S(x)| \geq 2 \). Also, we say that the set \( S \) satisfies the condition \( \mathcal{M} \) if each element \( x \in S \) with \( |G_S(x)| \geq 2 \) satisfies the condition \( \mathcal{M} \). Recall that the condition \( \mathcal{C} \) is defined for the elements with only two greatest-type divisors. If \( x \in S \) satisfies the condition \( \mathcal{C} \), then it clearly satisfies the condition \( \mathcal{M} \). On the other hand, an element satisfying the condition \( \mathcal{M} \) need not satisfy the condition \( \mathcal{C} \).

**Theorem 3.5.** Let \( S = \{ x_1, x_2, \ldots, x_8 \} \), \( |G_S(x_8)| = 3 \) and \( |D_S(x_8)| = 4 \). Then, \( (S) \mid [S] \) if and only if \( S \) satisfies the condition \( \mathcal{M} \).

**Proof.** Under the hypothesis of the theorem we can assume that the Hasse diagram of \( S \) with respect to the divisibility relation is as follows:

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If \([x_{k,i}, x_{k,j}] = x_k\) for all different \(x_{k,i}, x_{k,j} \in G_S(x_k)\) when \(|G_S(x_k)| \geq 2\), then by a direct computation, one can obtain that

\[
U_{ij} = \begin{cases} \frac{x_i}{x_1} & \text{if } [x_i, x_j] = x_8 \text{ and } (x_i, x_j) = x_1, \\ 0 & \text{otherwise.} \end{cases}
\]

We will show non-divisibility of the LCM matrix by the GCD matrix on \(S\) in two cases.

Case 1. Let \(S\) have an element \(x_k\) such that \(G_S(x_k) = \{x_{k,1}, x_{k,2}\}\) and \([x_{k,1}, x_{k,2}] < x_k\). Without loss of generality, we can take \(x_k = x_5\). Then, it is clear that \([x_2, x_3] < x_5\). By Lemmas 2.1, 2.4-2.6, and 3.1, we have

\[
U_{25} + U_{28} = \alpha_5 = \sum_{s=1}^{8} \frac{[x_2, x_s]}{c_{s,5}} = \frac{x_5 - [x_2, x_3]}{\alpha_5}.
\]

By Lemma 5.3, we have \(\alpha_5 = x_5 - x_2 - x_3 + x_1\). Since \([x_2, x_3] < x_5\), we have \(x_5 - [x_2, x_3] > 0\) and

\[
\alpha_5 - (x_5 - [x_2, x_3]) = \left(\frac{x_3}{x_1} - 1\right)(x_2 - x_1) > 0.
\]

Thus, \(0 < U_{25} + U_{28} < 1\). That is \(U \notin M_8(\mathbb{Z})\).

Case 2. Let \([x_5, x_6] < x_8\) without loss of generality. Now, we must have \([x_2, x_3] = x_5\), \([x_2, x_4] = x_6\) and \([x_3, x_4] = x_7\) otherwise the proof is obvious by Case 1. Under these assumptions, we have

\([x_5, x_6, x_7] = [x_5, x_6] < x_8\).

We will show that \(U_{58} \notin \mathbb{Z}\). By Lemmas 2.1, 2.4-2.6, we have

\[
U_{58} = \frac{x_8 - [x_5, x_6] - [x_5, x_7] + [x_4, x_5]}{\alpha_8}.
\]

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Let \( \gamma_8 = x_8 - [x_5, x_6] - [x_5, x_7] + [x_4, x_5] \). Since \([x_5, x_6, x_7] < x_8\) and clearly \([x_5, x_6, x_7] | x_8\), we have \( x_8 \geq 2 \cdot [x_5, x_6, x_7] \), and hence
\[
\gamma_8 > x_8 - [x_5, x_6] - [x_5, x_7] \geq 2 \cdot [x_5, x_6, x_7] - [x_5, x_6] - [x_5, x_7] \geq 0.
\]
By Lemma 2.3 we have \( \alpha_8 = x_8 - x_5 - x_6 - x_7 + x_2 + x_3 + x_4 - x_1 \). Then,
\[
\alpha_8 - \gamma_8 = ([x_5, x_6] - x_6 - x_5 + x_2) + ([x_5, x_7] - x_7 - x_5 + x_3)
\]
\[
+ (-[x_4, x_5] - x_1 + x_5 + x_4)
\]
\[
= \left( \frac{x_5}{x_1} - 1 \right) (x_6 - x_2) + \left( \frac{x_5}{x_3} - 1 \right) (x_7 - x_3) + \left( \frac{x_4}{x_1} - 1 \right) (x_1 - x_5)
\]
\[
= \left( \frac{x_4}{x_1} - 1 \right) (x_5 - x_2 + x_1 - x_3)
\]
\[
= \left( \frac{x_4}{x_1} - 1 \right) \frac{x_3}{x_1} - 1 \right) (x_2 - x_1) > 0.
\]
This completes the proof. \( \Box \)

**Corollary 3.1.** Let \( S \) be a gcd-closed set with \(|S| \leq 8\). \((S) \ | \ [S]\) if and only if
i) \( \max_{x \in S} \{|G_S(x)|\} = 1 \) or
ii) \( \max_{x \in S} \{|G_S(x)|\} = 2 \) and \( S \) satisfies the condition \( C \) or
iii) \( \max_{x \in S} \{|G_S(x)|\} = 3 \) and \( S \) satisfies the condition \( M \).

**Proof.** If (i) or (ii) holds then by Theorems 3.4 and 4.7 in [1] we know \((S) \ | \ [S]\).
Now, let (iii) hold. Let \(|S| = n \) with \( n \leq 8 \), \( G_S(x_n) = \{x_{n,1}, x_{n,2}, x_{n,3}\} \) and let \( S \) satisfy the condition \( M \). Then we claim that \( |D_S(x_n)| = 4 \) and \(|S| = 8\). Since \( S \) satisfies the condition \( M \), we must have
\[
[x_{n,1}, x_{n,2}] = [x_{n,1}, x_{n,3}] = [x_{n,2}, x_{n,3}]
\]
and hence \((x_{n,1}, x_{n,2}), (x_{n,1}, x_{n,3})\) and \((x_{n,2}, x_{n,3})\) must be different elements in \( S \). This means that \(|D_S(x_n)| = 4\), and hence \(|S| = 8\). So, we must investigate the case that \(|S| = 8\) and \(|D_S(x_8)| = 4\). Thus, by Theorem 3.5 we have \((S) \ | \ [S]\).

Now, we prove the necessary part of the theorem by contrapositive. If \( \max_{x \in S} \{|G_S(x)|\} = 2 \) and \( S \) does not satisfy the condition \( C \), then by Theorems 4.7 in [1] we know \((S) \notin [S]\). Consider the case that \( \max_{x \in S} \{|G_S(x)|\} = 3 \) and \( S \) does not satisfy the condition \( M \). If \(|D_S(x)| \leq 3\) for the element \( x \) with three greatest-type divisors, then we have \((S) \notin [S]\) by Theorem 3.4. If \(|D_S(x)| = 4\), then we have \((S) \notin [S]\) by Theorem 3.5. Since \(|S| \leq 8\), \( D_S(x) \leq 3 \) if \( \max_{x \in S} \{|G_S(x)|\} \geq 4 \). Thus, by Theorem 3.4 we have \((S) \notin [S]\). This completes the proof of the necessary part.

Let \( e \geq 1 \) be an integer. All the results that we have obtained in this section are valid for the \( e \)th power GCD matrix and the \( e \)th power LCM matrix. In this paper, we have only considered the original version of Problem 1.1 for the sake of brevity.
4. A new conjecture

Let $k$ and $i$ be arbitrary positive integers. Consider the set

$$S_i = \{p, p^2, \ldots, p^k, p^kq_1, p^kq_2, \ldots, p^kq_i, p^kq_1q_2, \ldots q_i\},$$

where $q_1, \ldots, q_i$ and $p$ are different prime numbers. It is clear that

$$\max_{x \in S_i} |G_{S_i}(x)| = i \quad \text{and} \quad |D_{S_i}(p^kq_1q_2 \cdots q_i)| = 1.$$ 

If $i \geq 3$, then, by a direct consequence of Theorem 3.4, we have $(S_i) \mid [S_i]$. Let $k = 10$ and $i = 4$. We have $\max_{x \in S_i} |G_{S_i}(x)| = 4$ and $|S_i| = 15$. Thus, we have a gcd-closed set, not satisfying the hypothesis of Conjecture 1.1, but the divisibility for this set cannot hold. Moreover, $S_4$ does not satisfy the condition $\mathfrak{M}$. Therefore, in the light of our results, we can say that the non-divisibility depends on not only the number $\max_{x \in S} |G_S(x)|$ but also the condition $\mathfrak{M}$. Indeed, a reason preventing the divisibility is that $S$ does not satisfy the condition $\mathfrak{M}$.

If a set $S$ satisfies the hypothesis of Zhao’s conjecture, then there must be at least three elements $x_{m,i_1}, x_{m,i_2},$ and $x_{m,i_3}$ such that $(x_{m,i_1}, x_{m,i_2}) = (x_{m,i_2}, x_{m,i_3}) = (x_{m,i_1}, x_{m,i_3})$ where $|G_{S}(x)| = m$ and $x_{m,i_k} \in G_{S}(x)$ for $1 \leq k \leq 3$. Then, we have $x_{m,i_1} < x_{m,i_3}$. This means that if the set $S$ with $|S| = n$ holds the hypothesis of Zhao’s conjecture, then $S$ does not satisfy the condition $\mathfrak{M}$.

Finally, after the above observations, we conclude our paper with a new conjecture, which is a generalization of Conjecture 1.1.

**Conjecture 4.1.** Let $S$ be a gcd-closed set with $\max_{x \in S} |G_{S}(x)| \geq 2$. If $S$ does not satisfy the condition $\mathfrak{M}$, then $(S) \mid [S]$.

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