Reduction of the optimal control problem for a magnetoelectric power drive

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Abstract. A linear-quadratic optimal control problem is considered for a singularly perturbed differential system, which describes the dynamics of a magnetoelectric. The optimal control law is constructed as feedback. Using the method of integral manifolds allows us to reduce the order of the matrix differential Riccati equation.

1. Introduction

The dynamics of a linear magnetoelectric actuator is described by the system of differential equations containing a small multiplier with some derivatives\textsuperscript{[1,2]}.

Using the geometric approach, integral manifolds theory, in particular, we can reduce the dimension of models and eliminate computational rigidity in some cases\textsuperscript{[2–4]}.

The method of integral manifolds was used to solve the optimal control problems for dynamic systems, the distinctive feature of which is the presence of small parameters and variables with significantly different rates of change\textsuperscript{[4–10]}.

We consider a mathematical model of the linear magnetoelectric power drive. Drive dynamics is described by a singularly perturbed system of differential equations. The presence of slow and fast variables makes the problems of reduction and decomposition topical.

One of the approaches to the problem of separating fast and slow variables is based on the geometric and asymptotic methods of analysis, the theory of integral manifolds in particular\textsuperscript{[1–3]}. The method of asymptotic decomposition was used to solve the problems of optimal control for multi-scale dynamic systems of various nature\textsuperscript{[4–10]}.

The problem of optimal speed for a singularly perturbed linear magnetoelectric drive model was considered in \textsuperscript{[2]}. Asymptotic expansions for switching points are constructed.

A linear-quadratic optimal control problem for the magnetoelectric drive was considered in \textsuperscript{[1]}. An approach based on the decision of the maximum principle boundary problem was used to construct the optimal control law. The decomposition of a singularly perturbed linear boundary value problem to a boundary value problem for slow variables and two initial problems for fast variables was performed.

In the present work, the approach based on the decision of matrix differential Riccati equation is used for the synthesis of the optimal control law. Using of integral manifolds method allows us to reduce the order of considered equation and to simplify the problem of building an optimal control law.
2. Statement of the problem
We consider a mathematical model of a system which consists of a linear magnetoelectric actuator (moving in the air gap of a permanent magnet) and a moving mass [11–13] (Figure 1).

\[ U(t) = I(t)R + L\dot{I}(t) + K_e v(t), \]
\[ m\ddot{v}(t) + Bv(t) = K_F I(t) \]

\[ \text{(1)} \]

where \( U = U(t) \) states for voltage of the motor, \( I = I(t) \) is the electric current, \( L \) is the coil inductivity, \( R \) is for the coil resistance, \( E = K_e v(t) \) is the back-EMF, \( m \) is the mass of moving part (including coil), \( B \) is the damping constant, \( F = K_F I \) is the actuating force (Lorenz force), \( x = x(t) \) is the load mass position, \( v = v(t) = \dot{x}(t) \) is the load mass linear speed. The coil inductivity is assumed to be small.

We introduce the state variables
\[ x_1 = x, \quad x_2 = v, \quad x_3 = I, \]
and small parameter \( \varepsilon = L \) and rewrite the system (1) as
\[ \dot{x}_1 = x_2, \]
\[ \dot{x}_2 = -\mu x_2 + \alpha_1 x_3, \]
\[ \varepsilon \dot{x}_3 = -\alpha_2 x_2 - Rx_3 + u, \]

\[ \text{(2)} \]

where \( \mu = B/m, \ \alpha_1 = K_F/m, \ \alpha_2 = K_e. \)

We consider the minimization problem for a quadratic cost functional
\[ J = \int_0^1 (\beta_1 x_1^2(t) + \beta_2 x_2^2(t) + \beta_3 x_3^2(t) + \gamma u^2(t)) dt, \quad \beta_j \geq 0, \ j = 1, 3, \ \gamma > 0, \]

\[ \text{(3)} \]
on the trajectories of the system (2) with initial conditions
\[ x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30}. \] 
(4)
The control parameter in this problem is the input voltage \( u = U = U(t) \).

We rewrite the system (2), (4) as
\[ \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^3, \quad x(0) = x_0. \] 
(5)
Here
\[ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = A(\varepsilon) = \begin{pmatrix} A_1 & A_2 \\ A_3/\varepsilon & A_4/\varepsilon \end{pmatrix}, \quad B = B(\varepsilon) = \begin{pmatrix} B_1 \\ B_2/\varepsilon \end{pmatrix}, \]
\[ A_1 = \begin{pmatrix} 0 & 1 \\ 0 & -\mu \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ \alpha_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \end{pmatrix}, \quad A_3 = (0, -\alpha_2), \quad A_4 = -R, \quad B_2 = 1. \]
The first two components \( x_1, x_2 \) of \( x \) are slow variables, and \( x_3 \) is a fast variable.
The cost functional (3) takes the form
\[ J = \int_0^1 x^T(t)Qx(t) + \gamma u^2(t)dt, \] 
(6)
where
\[ Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_3 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad Q_3 = \beta_3. \]
The optimal control law for this problem takes the feedback form
\[ u = \frac{1}{\gamma} B^T K x, \] 
(7)
where the matrix function \( K = K(t, \varepsilon) \) is the solution of the matrix differential Riccati equation
\[ \dot{K} + A^T K + KA - KSK + Q = 0, \] 
(8)
with the boundary condition
\[ K(1, \varepsilon) = 0. \] 
(9)
Here
\[ S = \frac{1}{\gamma} BB^T = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad S_3 = \frac{1}{\gamma}. \]
Let matrix function \( K = K(t, \varepsilon) \) has a block form:
\[ K = \begin{pmatrix} K_1 & \varepsilon K_2 \\ \varepsilon K_1 & \varepsilon K_3 \end{pmatrix}. \] 
(10)
Then from (8), (9) we obtain the system of matrix equations for the blocks $K_1$, $K_2$, and $K_3$:

$$
\dot{K}_1 = -K_1 A_1 - A_1^T K_1 - K_2 A_3 - A_3^T K_2^T + K_2 S_3 K_2^T - Q_1,
$$

$$
\varepsilon \dot{K}_2 = -K_1 A_2 - K_2 A_4 - \varepsilon A_1^T K_2 - A_3^T K_3 + K_2 S_3 K_3,
$$

$$
\varepsilon \dot{K}_3 = -2 K_3 A_4 + S_3 K_3^2 + \varepsilon (-K_2^T A_2 - A_2^T K_2) - Q_3
$$

with the boundary conditions

$$
K_1(1, \varepsilon) = 0, \quad K_2(1, \varepsilon) = 0, \quad K_3(1, \varepsilon) = 0.
$$

The system (11) has a slow integral manifold

$$
K_2 = h_2(K_1, \varepsilon), \quad K_3 = h_3(K_1, \varepsilon).
$$

The matrix and the scalar functions $h_2$, $h_3$ can be found with any degree of accuracy in the form of asymptotic expansions in powers of a small parameter $\varepsilon$

$$
h_2(K_1, \varepsilon) = h_{20}(K_1) + \varepsilon h_{21}(K_1) + \ldots,
$$

$$
h_3(K_1, \varepsilon) = h_{30}(K_1) + \varepsilon h_{31}(K_1) + \ldots,
$$

from the equations

$$
\varepsilon \frac{\partial h_2}{\partial K_1} (-K_1 A_1 - A_1^T K_1 - h_2 A_3 - A_3^T h_2^T + h_2 S_3 h_2^T - Q_1) = -K_1 A_2 - h_2 A_4 - \varepsilon A_1^T h_2 - A_3^T h_3 + h_2 S_3 h_3,
$$

$$
\varepsilon \frac{\partial h_3}{\partial K_1} (-K_1 A_1 - A_1^T K_1 - h_2 A_3 - A_3^T h_2^T + h_2 S_3 h_2^T - Q_1) = -2 h_3 A_4 + S_3 h_3^2 + \varepsilon (-h_2^T A_2 - A_2^T h_2) - Q_3.
$$

If $\varepsilon = 0$ in (14), we have

$$
-K_1 A_2 - h_{20} A_4 - A_3^T h_{30} + h_{20} S_3 h_{30} = 0,
$$

$$
-2 h_{30} A_4 + S_3 h_{30}^2 - Q_3 = 0.
$$

We choose the positive solution of the equation (16) as $h_{30}$

$$
h_{30} = M = -R\gamma + \sqrt{R^2\gamma^2 + \beta_3}.
$$

Then the inequality

$$
D_4 = A_4 - S_3 M = -\frac{1}{\gamma} \sqrt{R^2\gamma^2 + \beta_3} < 0
$$

takes place.

For $h_{20}$ we have

$$
h_{20} = K_1 M_1 + M_2,
$$
where
\[ M_1 = -A_2 D_4^{-1} = \frac{\gamma}{\sqrt{R^2 \gamma^2 + \beta_3}} \left( \begin{array}{c} 0 \\ \alpha_1 \end{array} \right), \quad M_2 = -A_3^T M D_4^{-1} = \frac{\gamma}{\sqrt{R^2 \gamma^2 + \beta_3}} \left( \begin{array}{c} 0 \\ -\alpha_2 \end{array} \right) M, \]
i.e.,
\[ h_{20} = \frac{\gamma}{\sqrt{R^2 \gamma^2 + \beta_3}} \left( \begin{array}{c} \alpha_1 K_1[1, 2] \\ \alpha_1 K_1[1, 2] + \alpha_2 (R \gamma - \sqrt{R^2 \gamma^2 + \beta_3}) \end{array} \right), \]
where \( K_1[i, j](i, j = 1, 2) \) are the elements of matrix \( K_1 \).

From (14), by equating the coefficients with the first degree of small parameter \( \varepsilon \), we get the system for \( h_{21} \) and \( h_{31} \):
\[ \{ \dot{K}_1 \}_0 (-A_2 D_4^{-1}) = -h_{21} A_4 - A_2^T h_{20} - A_3^T h_{31} + h_{20} S_3 h_{31} + h_{21} S_3 M, \]
(17)
\[ 0 = -h_{20}^T A_2 - A_2^T h_{20} - h_{31} A_4 - A_4^T h_{31} + h_{31} S_3 M + M S_3 h_{31}, \]
(18)
where
\[ \{ \dot{K}_1 \}_0 = -K_1 A_4 - A_4^T K_1 - h_{20} A_3 - A_3^T h_{20} + h_{20} S_3 h_{20} - Q_1. \]

The equation (18) is a linear algebraic equation for \( h_{31} \), therefore
\[ h_{31} = -\frac{1}{2 D_4} (h_{20}^T A_2 + A_2^T h_{20}) = \frac{\gamma^2}{R^2 \gamma^2 + \beta_3} \left( \begin{array}{c} 0 \\ \alpha_1 \end{array} \right) K_1 \left( \begin{array}{c} 0 \\ \alpha_1 \end{array} \right) - M \alpha_1 \alpha_2. \]

Then from (17) we have
\[ h_{21} = -[\{ \dot{K}_1 \}_0 (-A_2 D_4^{-1}) + A_2^T h_{20} + A_3 h_{31}] D_4^{-1}. \]
By similar way one can calculate \( h_{22}, h_{32}, \text{ etc.} \).

The system
\[ \dot{K}_1 = -A_1^T K_1 - K_1 A_1 - h_{20} A_3 - A_3^T h_{20} + h_{20} S_3 h_{20} - Q_1 \]
(19)
describes the motion on the integral manifold (13). Note that (19) is the differential matrix equation for \( K_1 \).

For the elements of matrix \( K_1 \) we have the equations
\[ \dot{K}_1[1, 1] = \beta_1 - \frac{\gamma \alpha_1^2 K_1^2[1, 2]}{R^2 \gamma^2 + \beta_3} + O(\varepsilon), \]
\[ \dot{K}_1[1, 2] = -K_1[1, 1] + K_1[1, 2] \mu + \frac{\alpha_1 \gamma K_1[1, 2] (\alpha_1 K_1[2, 2] + \gamma \alpha_2 R)}{R^2 \gamma^2 + \beta_3} + O(\varepsilon), \]
\[ \dot{K}_1[2, 2] = -2 K_1[1, 2] + 2 K_1[2, 2] \mu - \beta_2 - \frac{\gamma (\alpha_2^2 \beta_3 - K_1^2[2, 2] \alpha_1^2 - 2 K_1[2, 2] \alpha_1 \alpha_2 \gamma)}{R^2 \gamma^2 + \beta_3} + O(\varepsilon). \]

Follow [2], we take
\[ K_1(1, \varepsilon) = \xi \]
(20)
as the boundary condition for the equation (19). Here

\[ \xi = \varepsilon(M_2N_0^{-1}M_2^T - L^TN^{-1}N_0^{-1}M_2^T - M_2N_0^{-1}N^{-1}L - L^TMN_0^{-1}N^{-1}L) + O(\varepsilon^2), \]

\[ N = \frac{1}{2}S_3D_4^{-1} = \frac{-1}{2\sqrt{R^2\gamma^2 + \beta_3}}, \quad N_0 = M + N^{-1} = -R\gamma - \sqrt{R^2\gamma^2 + \beta_3}, \]

\[ L = -D_4^{-1}(A_3 - S_3M_2^T) = \frac{R\gamma^2}{R^2\gamma^2 + \beta_3}(0 - \alpha_2). \]

The quasi-optimal control law takes the form

\[ u = -\frac{1}{\gamma}(0 0 1/\varepsilon) \left( \begin{array}{ccc} K_1 \varepsilon(h_{20}(K_1) + \varepsilon h_{21}(K_1)) \varepsilon(h_{20}(K_1) + \varepsilon h_{21}(K_1)) \varepsilon(h_{30}(K_1) + \varepsilon h_{31}(K_1)) \varepsilon(h_{30}(K_1) + \varepsilon h_{31}(K_1)) \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right), \]

where \( K_1 \) is a solution of the equation (19) with the initial condition (20).

The choice of such control law will give an error of order \( O(\varepsilon^2) \) in the cost functional [2].

As an illustration, we consider the problem for the following characteristic values of the parameters [1]:

\[ R = 2, \quad L = 0.001, \quad \alpha_1 = 2, \quad \alpha_2 = 2, \quad \beta_1 = 10, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \gamma = 0.1. \]

Figure 2 shows the dynamics of the elements of matrix \( K_1(t, \varepsilon) \) which are the slow components of the gain factors.

![Figure 2. The gain factors \( K_1 \).](image)

Figures 3, 4 show the dynamics of the elements of the matrices \( K_2(t, \varepsilon), K_3(t, \varepsilon) \) on the slow manifold (13). Figures 5, 6 demonstrate the coordinates of the optimal trajectory.

The results of numerical simulation are consistent well with the results obtained when the optimal control law was constructed in the form of programmed control using a boundary value problem for a singularly perturbed linear Hamiltonian system [1].

The use of an approach based on solving the matrix differential Riccati equation is preferred for practical implementation since the control law is constructed in the form of feedback.
3. Conclusions
We considered a linear-quadratic optimal control problem for the singularly perturbed differential system which describes the complex dynamics of the magnetoelectric actuator. The dynamics of the considered model of actuator combines fast and slow movements. This complicates the problem of construction and implementation of the optimal control law.

The use of the geometric theory of integral manifolds and asymptotic methods allows us to replace the singularly perturbed matrix differential Riccati equation with the nonlinear system without singular perturbations. The last system describes only slow motions on the attracting integral manifold. The order of this system is half of the order of the original system. However, it adequately reflects the behaviour of the original system and can be used as a simplified model. We used slow subsystem to construct the quasi-optimal feedback control law in the control problem for the magnetoelectric power drive.

4. References
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