Short proof and generalization of a Menon-type identity by Li, Hu and Kim

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Abstract

We present a simple proof and a generalization of a Menon-type identity by Li, Hu and Kim, involving Dirichlet characters and additive characters.

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1 Motivation and main result

Menon’s classical identity states that for every $n \in \mathbb{N}$,

$$\sum_{\substack{a=1 \\ (a,n)=1}}^{n} (a-1,n) = \varphi(n)\tau(n), \quad (1.1)$$

where $(a-1,n)$ stands for the greatest common divisor of $a-1$ and $n$, $\varphi(n)$ is Euler’s totient function and $\tau(n) = \sum_{d|n} 1$ is the divisor function. Identity (1.1) was generalized by several authors in various directions. Zhao and Cao [7] proved that

$$\sum_{a=1}^{n} (a-1,n)\chi(a) = \varphi(n)\tau(n/d), \quad (1.2)$$

where $\chi$ is a Dirichlet character (mod $n$) with conductor $d \ (n \in \mathbb{N}, \ d \mid n)$. If $\chi$ is the principal character (mod $n$), that is $d = 1$, then (1.2) reduces to Menon’s identity (1.1). Generalizations of (1.2) involving even functions (mod $n$) were deduced by the author [6], using a different approach.

Li, Hu and Kim [4] proved the following generalization of identity (1.2):
Theorem 1.1 ([4, Th. 1.1]). Let \( n \in \mathbb{N} \) and let \( \chi \) be a Dirichlet character \((\text{mod } n)\) with conductor \( d \) \((d \mid n)\). Let \( b \mapsto \lambda_\ell(b) := \exp(2\pi i w_\ell b/n) \) be additive characters of the group \( \mathbb{Z}_n \), with \( w_\ell \in \mathbb{Z} \) \((1 \leq \ell \leq k)\). Then

\[
\sum_{a,b_1,\ldots,b_k=1}^n (a-1,b_1,\ldots,b_k,n)\chi(a)\lambda_1(b_1)\cdots\lambda_k(b_k) = \varphi(n)\sigma_k((n/d,w_1,\ldots,w_k)),
\]

where \( \sigma_k(n) = \sum_{d \mid n} d^k \).

Note that in (1.2) and (1.3) the sums are, in fact, over \( 1 \leq a \leq n \) with \((a,n) = 1\), since \( \chi(a) = 0 \) for \((a,n) > 1\). In the case \( w_1 = \cdots = w_k = 0 \), identity (1.3) was deduced by the same authors in paper [3]. For the proof, Li, Hu and Kim computed first the given sum in the case \( n = p^t \), a prime power, and then they showed that the sum is multiplicative in \( n \).

It is the goal of this paper to present a simple proof of Theorem 1.1. Our approach is similar to that given in [6], and leads to a direct evaluation of the corresponding sum for every \( n \in \mathbb{N} \). We obtain, in fact, the following generalization of the above result. Let \( \mu \) denote the M"obius function and let \(*\) be the convolution of arithmetic functions.

Theorem 1.2. Let \( F \) be an arbitrary arithmetic function, let \( s_j \in \mathbb{Z} \), \( \chi_j \) be Dirichlet characters \((\text{mod } n)\) with conductors \( d_j \) \((1 \leq j \leq m)\) and \( \lambda_\ell \) be additive characters as defined above, with \( w_\ell \in \mathbb{Z} \) \((1 \leq \ell \leq k)\). Then

\[
\sum_{a_1,\ldots,a_m,b_1,\ldots,b_k=1}^n F((a_1 - s_1,\ldots,a_m - s_m,b_1,\ldots,b_k,n))\chi_1(a_1)\cdots\chi_m(a_m)\lambda_1(b_1)\cdots\lambda_k(b_k)
\]

\[
= \varphi(n)^m\chi_1^*(s_1)\cdots\chi_m^*(s_m) \sum_{e \mid (n/d_1,\ldots,n/d_m,w_1,\ldots,w_k)} \frac{e^k(\mu * F)(n/e)}{\varphi(n/e)^m}.
\]

where \( \chi_j^* \) are the primitive characters \((\text{mod } d_j)\) that induce \( \chi_j \) \((1 \leq j \leq m)\).

We remark that the sum in the left hand side of identity (1.4) vanishes provided that there is an \( s_j \) such that \((s_j,d_j) > 1\). If \( F(n) = n \) \((n \in \mathbb{N})\), \( m = 1 \) and \( s_1 = 1 \), then identity (1.4) reduces to (1.3). We also remark that the special case \( F(n) = n \) \((n \in \mathbb{N})\), \( m \geq 1 \), \( s_1 = \cdots = s_m = 1 \), \( k \geq 1 \), \( w_1 = \cdots = w_k = 0 \) was considered in the quite recent preprint [2]. Several other special cases of formula (1.4) can be discussed.

See the papers [3, 4, 5, 6, 7] and the references therein for other generalizations and analogues of Menon’s identity.

2 Proof

We need the following lemmas.
Lemma 2.1. Let \( n, d, e \in \mathbb{N} \), \( d \mid n, e \mid n \) and let \( r, s \in \mathbb{Z} \). Then

\[
\sum_{a=1\atop (a,n)=1}^{n} 1 = \begin{cases} 
\varphi(n)/(d,e), & \text{if } (r,d) = (s,e) = 1 \text{ and } (d,e) \mid r - s, \\
0, & \text{otherwise.}
\end{cases}
\]

In the special case \( e = 1 \) this is known in the literature, usually proved by the inclusion-exclusion principle. See, e.g., [1, Th. 5.32]. Here we use a different approach, in the spirit of our paper.

Proof of Lemma 2.1. For each term of the sum, since \((a,n) = 1\), we have \((r,d) = (a,d) = 1\) and \((s,e) = (a,e) = 1\). Also, the given congruences imply \((d,e) \mid r - s\). We assume that these conditions are satisfied (otherwise the sum is empty and equals zero).

Using the property of the M"obius function, the given sum, say \( S \), can be written as

\[
S = \sum_{a=1\atop a \equiv r \pmod{d}}^{n} \sum_{\delta \mid (a,n)} \mu(\delta) = \sum_{\delta \mid n} \mu(\delta) \sum_{j=1\atop \delta j \equiv r \pmod{d}}^{n/\delta} 1. \tag{2.1}
\]

Let \( \delta \mid n \) be fixed. The linear congruence \( \delta j \equiv r \pmod{d} \) has solutions in \( j \) if and only if \( (\delta,d) \mid r \), equivalent to \( (\delta,d) = 1 \), since \( (r,d) = 1 \). Similarly, the congruence \( \delta j \equiv s \pmod{e} \) has solutions in \( j \) if and only if \( (\delta,e) \mid s \), equivalent to \( (\delta,e) = 1 \), since \( (s,e) = 1 \). These two congruences have common solutions in \( j \) due to the condition \( (d,e) \mid r - s \). Furthermore, if \( j_1 \) and \( j_2 \) are solutions of these simultaneous congruences, then \( \delta j_1 \equiv \delta j_2 \pmod{d} \) and \( \delta j_1 \equiv \delta j_2 \pmod{e} \). Since \( (\delta,d) = 1 \), this gives \( j_1 \equiv j_2 \pmod{[d,e]} \). We deduce that there are

\[
N = \frac{n}{\delta [d,e]}
\]
solutions \( \pmod{n/\delta} \) and the last sum in (2.1) is \( N \). This gives

\[
S = \frac{n}{[d,e]} \sum_{\delta \mid n \atop (\delta,de)=1} \frac{\mu(\delta)}{\delta} = \frac{n}{[d,e]} \frac{\varphi(n)/n}{\varphi(de)/(de)} = \frac{\varphi(n)}{\varphi(de)}(d,e).
\]

The next lemma is a known result. See, e.g., [6] for its (short) proof.

Lemma 2.2. Let \( n \in \mathbb{N} \) and \( \chi \) be a primitive character \( \pmod{n} \). Then for any \( e \mid n, e < n \) and any \( s \in \mathbb{Z} \),

\[
\sum_{a=1\atop a \equiv s \pmod{e}}^{n} \chi(a) = 0.
\]
Now we prove

**Lemma 2.3.** Let \( \chi \) be a Dirichlet character \( \mod n \) with conductor \( d \) \((n \in \mathbb{N}, d \mid n) \) and let \( e \mid n, s \in \mathbb{Z} \). Then

\[
\sum_{a=1}^{n} \chi(a) = \begin{cases} \varphi(n) \chi^*(s), & \text{if } d \mid e \text{ and } (s, e) = 1, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( \chi^* \) is the primitive character \( \mod d \) that induces \( \chi \).

**Proof of Lemma 2.3.** We can assume \((a, n) = 1\) in the sum. If \( a \equiv s \mod e \), then \((s, e) = (a, e) = 1\). Given the Dirichlet character \( \chi \mod n \), the primitive character \( \chi^* \mod d \) that induces \( \chi \) is defined by

\[
\chi(a) = \begin{cases} \chi^*(a), & \text{if } (a, n) = 1, \\ 0, & \text{if } (a, n) > 1. \end{cases}
\]

We deduce

\[
T := \sum_{a=1}^{n} \chi(a) = \sum_{a=1}^{n} \chi^*(a) = \sum_{r=1}^{d} \chi^*(r) \sum_{a \equiv s \mod (d, e)} 1,
\]

where the inner sum is evaluated in Lemma 2.1. Since \((s, e) = 1\), as mentioned above, we have

\[
T = \sum_{r=1}^{d} \chi^*(r) \frac{\varphi(n)}{\varphi(de)}(d, e) = \frac{\varphi(n)}{\varphi(de)}(d, e) \sum_{r=1}^{d} \chi^*(r) = \frac{\varphi(n)}{\varphi(de)}(d, e)\chi^*(s),
\]

by Lemma 2.2 in the case \((d, e) = d\), that is \(d \mid e\). We conclude that

\[
T = \frac{\varphi(n)}{\varphi(de)} d\chi^*(s) = \frac{\varphi(n)}{\varphi(e)} \chi^*(s).
\]

If \( d \nmid e \), then \( T = 0 \).

**Proof of Theorem 1.2.** Let \( V \) denote the given sum. By using the trivial identity \( F(n) = \sum_{e \mid n} (\mu \ast F)(e) \), we have

\[
V = \sum_{a_1, \ldots, a_m, b_1, \ldots, b_k} \chi_1(a_1) \cdots \chi_m(a_m) \lambda_1(b_1) \cdots \lambda_k(b_k) \sum_{e \mid (a_1 - s_1, \ldots, a_m - s_m, b_1, \ldots, b_k, n)} (\mu \ast F)(e)
\]

\[
= \sum_{e \mid n} (\mu \ast F)(e) \sum_{\substack{a_1=1 \\atop a_1 \equiv s_1 \mod e}}^{n} \chi_1(a_1) \cdots \sum_{\substack{a_m=1 \\atop a_m \equiv s_m \mod e}}^{n} \chi_m(a_m) \sum_{\substack{b_1=1 \\atop e \mid b_1}}^{n} \lambda_1(b_1) \cdots \sum_{\substack{b_k=1 \\atop e \mid b_k}}^{n} \lambda_k(b_k).
\]

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Here for every $1 \leq \ell \leq k$,
\[
\sum_{\substack{b_\ell=1 \\ e|b_\ell}}^{n} \lambda_\ell(b_\ell) = \sum_{c_\ell=1}^{n/e} \exp\left(2\pi iw_\ell c_\ell/(n/e)\right) = \begin{cases} \frac{n}{e}, & \text{if } \frac{n}{e} \mid w_\ell, \\ 0, & \text{otherwise,} \end{cases}
\]
and using Lemma 2.3 we deduce that
\[
V = \chi_1^*(s_1) \cdots \chi_m^*(s_m) \sum' (\mu * F)(e) \left(\frac{\varphi(n)}{\varphi(e)}\right)^m \left(\frac{n}{e}\right)^k,
\]
where the sum $\sum'$ is over $e \mid n$ such that $d_\ell \mid e$, $(e, s_\ell) = 1$ for all $1 \leq \ell \leq m$ and $n/e \mid w_\ell$ for all $1 \leq \ell \leq k$. Interchanging $e$ and $n/e$, the sum is over $e$ such that $e \mid n/d_\ell$, $(e, s_j) = 1$ for all $1 \leq j \leq m$ and $e \mid w_\ell$ for all $1 \leq \ell \leq k$. This completes the proof.

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