INCLUSION OF FADING MEMORY TO BANISTER MODEL OF CHANGES IN PHYSICAL CONDITION

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Abstract. We introduced the fading memory effect to the model portraying the prediction in physical condition. The classical model is known as the Banister model. We presented the existence and uniqueness conditions of the exact solutions of this model using three different memory including the bad memory induced by the power law and the good memories induced by exponential decay law and the Mittag-Leffler law. We derived the exact solutions using the Laplace transform for the non-delay version.

1. Introduction. In the recent decades, the arena of sport industry has captured the minds of almost all mankind. We shall recall that, sports includes all types of competitive involving physical activity, games during which casual or organized participation, with aim to employ, maintain or enhance physical capabilities and skills while providing entertainment to fans and fulfilment to participants [8]-[10]. In more general sense, sport is acknowledged as a dynamic system of activities, which are underpinned, in physical athleticism, also physical dexterity; sometime this involves a largest class of competitions such as the Olympic games and other. It is nevertheless important to mention that, a careful analysis and in particular the understanding of training processes are off great importance to training science and also more importantly to practice of any sports. It is believed in the sport science that, training-effect analysis has commonly oriented itself, in a similar way like other fields of science for instance, geo-hydrology, medicine, biology, chemistry and psychology, with the principle of reduction [9]. In this science, an individual participant is considered as variable and they are rejected from the network of interactions under some conditions and rule as far as possible. Within these conditions, each participant or in mathematical terms variables of a training process can

2010 Mathematics Subject Classification. 92B05, 92C60.
Key words and phrases. Power law, exponential decay law, Mittag-Leffler law, Banister model.

The authors would like to extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No (RG-1438-086).

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be monitored and also some component phenomena scientifically grounded. Some mathematical models were constructed for instance that of Banister [7]-[15] and the dynamic model of football recently suggested in [12]. In another work, the Banister model was modified and the delays components were included. For the dynamic of football, the model was constructed using the concept of non-local operators that help not only to introduce the delay but also to introduce the non-locality of the system. In the Banister model and the work in [7], no sign of non-locality is presented, also a kind of delay was introduced in [7], worryingly this model does not take into account a fading memory that can be introduced using the non-local operators of differentiations. The aim of this work is to introduce the fading memory and the non-locality to the model suggested by Banister [3] -[16].

2. Banister model with power law memory. The classical model suggested by Banister to describe this dynamical system us given below as:

\[ v'(t) = \frac{1}{\theta_1} v(t) + w(t), \]  

(1)

The above equation does not include the non-locality and also the memory, in order to include the power memory in the above equation, we replace the left-hand sides by the Caputo-power law derivative to obtain:

\[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d}{dy} v(y)(t-y)^{-\alpha} dy + \frac{1}{\theta_1} v(t) = w(t) \]  

(2)

To establish the exact solution of the above equation, we make us of the Laplace transform operator on both sides of equation (2) to obtain:

\[
\begin{align*}
p^\alpha v(p) - p^{\alpha - 1} v(0) + \frac{1}{\theta_1} v(p) &= w(p), \\
v(p) \left[ p^\alpha + \frac{1}{\theta_1} \right] &= p^{\alpha - 1} v(0) + w(p), \\
v(p) &= \frac{p^{\alpha - 1} v(0)}{[p^\alpha + \frac{1}{\theta_1}]}.
\end{align*}
\]

(3)

The application of the inverse Laplace transform and the use of convolution theorem yield:

\[ v(t) = v(0) E_\alpha \left[ -\frac{1}{\theta_1} t^\alpha \right] + \int_0^t w(y) E_\alpha \left[ -\frac{1}{\theta_1} (t - y) \right] dy. \]  

(4)

The discrete version is obtained by a direct approximation of the integral component as follow:

\[ v(t_i) = v(0) E_\alpha \left[ -\frac{1}{\theta_1} t_i^\alpha \right] + \sum_{k=0}^{i-1} w(t_k) \int_{t_k}^{t_{k+1}} E_\alpha \left[ -\frac{1}{\theta_1} (t_i - y) \right] dy. \]  

(5)

If we include the delay term and also the non-locality into the Banister model, the following model:

\[ C_0^\alpha D_0^\alpha v(t) = -\frac{1}{\theta_1} v(t) - \frac{1}{\theta_2 \Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} v(y) dy + w(t), \]  

(6)

We first present the conditions of existence and uniqueness of the exact solution of the above equation by using the fixed-point theorem. To do this we, apply on both sides of equation (6) the fractional integral to obtain:

\[ v(t) - v(0) = -\frac{1}{\theta_1 \Gamma(\alpha)} \int_0^t v(y)(t - y)^{\alpha - 1} dy - \frac{1}{\theta_2 \Gamma(\alpha + \beta)} \int_0^t (t - y)^{\alpha + \beta - 1} v(y) dy + \frac{1}{\theta_2 \Gamma(\alpha \gamma)} \int_0^t w(y)(t - y)^{\alpha - 1} dy, \]  

(7)
We consider the following compact cylinder

\[ \Xi_{a,b} = I_a(t_0) \times B_b(v_0), \]

\[ I_a(t_0) = [a - t_0, t_0 + a], \quad B_b(v_0) = [b - v_0, v_0 + b] \]

We equip the above compact cylinder with the uniform norm defined as:

\[ \|v(t)\|_\infty = \sup |v(t)| \]

The construct the following functional operator

\[
\Delta : \Xi_{a,b} \to \Xi_{a,b}
\]

\[ \Delta \Phi(t) = v(0) - \frac{1}{\alpha_1(\alpha)} \int_0^t \phi(y)(t - y)^{\alpha - 1}dy - \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t \phi(y)(t - y)^{\alpha + \beta - 1}dy + \frac{1}{\alpha_1(\alpha)} \int_0^t w(y)(t - y)^{\alpha - 1}dy, \]

We next show that, the above operator is well defined, or we construct the condition under which the above operator is well defined. To achieve our goal we evaluate the following:

\[
\|\Delta \Phi(t) - v_0\|_\infty = \left\| -\frac{1}{\alpha_1(\alpha)} \int_0^t \phi(y)(t - y)^{\alpha - 1}dy - \left[ -\frac{1}{\alpha_1(\alpha)} \int_0^t \phi(y)(t - y)^{\alpha - 1}dy \right] - \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t \phi(y)(t - y)^{\alpha + \beta - 1}dy + \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t \phi(y)(t - y)^{\alpha + \beta - 1}dy \right\|
\]

Employing the triangular inequality, we obtain:

\[
\|\Delta \Phi(t) - v_0\|_\infty \leq \left\| \frac{1}{\alpha_1(\alpha)} \int_0^t \phi(y)(t - y)^{\alpha - 1}dy \right\| + \left\| \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t \phi(y)(t - y)^{\alpha + \beta - 1}dy \right\| + \left\| \frac{1}{\alpha_1(\alpha)} \int_0^t w(y)(t - y)^{\alpha - 1}dy \right\| + \left\| \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t w(y)(t - y)^{\alpha + \beta - 1}dy \right\|
\]

\[
< (M+N)\alpha^\alpha + \frac{M\alpha^{\alpha + \beta}}{\alpha_1(1+\alpha)}, \quad N = \max |\phi(t)|, \quad M = \max |w(t)|
\]

For the operator to be well defined, we will need to have

\[
a^\alpha < \frac{b}{\alpha_1(\alpha + 1) + \alpha_2(\alpha + \beta + 1)}
\]

We next check on the Lipschitz condition of the defined operator. Let consider two different functions from the compact cylinder says, \( \chi(t) \) and \( \varphi(t) \), then

\[
\|\Delta \chi(t) - \Delta \varphi(t)\|_\infty = \left\| -\frac{1}{\alpha_1(\alpha)} \int_0^t \{\Delta \chi(y) - \Delta \varphi(y)\}(t - y)^{\alpha - 1}dy - \left[ -\frac{1}{\alpha_1(\alpha)} \int_0^t \{\Delta \chi(y) - \Delta \varphi(y)\}(t - y)^{\alpha - 1}dy \right] - \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t \{\Delta \chi(y) - \Delta \varphi(y)\}(t - y)^{\alpha + \beta - 1}dy + \frac{1}{\alpha_2(\alpha + \beta)} \int_0^t \{\Delta \chi(y) - \Delta \varphi(y)\}(t - y)^{\alpha + \beta - 1}dy \right\|
\]

\[
< a^\alpha \left( \frac{1}{\alpha_1(\alpha + 1)} + \frac{\alpha^{\alpha + \beta}}{\alpha_2(\alpha + \beta + 1)} \right) \|\Delta \chi(y) - \Delta \varphi(y)\|_\infty
\]

The contraction will be obtained if and only if

\[
a^\alpha < \frac{1}{\alpha_1(\alpha + 1) + \alpha_2(\alpha + \beta + 1)},
\]
The Banach fixed point theorem will therefore be valid in this case of the following condition is satisfied:

$$a^\alpha < \min \left\{ \frac{1}{\sigma_1(\alpha+1)} + \frac{a^\beta}{\sigma_1(\alpha+\beta+1)}, \frac{(M+N)}{\sigma_1(\alpha+1)} + \frac{Ma^\beta}{\sigma_2(\alpha+\beta+1)} \right\}$$

Since the exact solution of this equation can be obtained via the use of Laplace transform, however the problem is to obtain the inverse Laplace transform, to avoid this difficulty, we chose to solve the equation using some numerical scheme. To discretize equation (6), we proceed as follow at the point \(t_i\)

$$C^D_t v(t_i) = \frac{(\Delta t)^{-\alpha}}{\Gamma(\alpha+1)} \sum_{k=0}^{i-1} (v(t_{k+1}) - v(t_k)) \Lambda_{\alpha,i,k},$$

$$\Lambda_{\alpha,i,k} = (i - k)^{1-\alpha} - (i - k - 1)^{1-\alpha}$$

(15)

Replacing the above in equation (6) yields

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{i-1} (v(t_{k+1}) - v(t_k)) \Lambda_{\alpha,i,k} + \frac{1}{\sigma_1} v(t_i)$$

$$+ \frac{1}{\sigma_2} \left\{ \frac{(\Delta t)^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=0}^{i-1} v(t_{k+1}) \{(i - k)^{\alpha} - (i - k - 1)^{\alpha}\} \right\} = w(t_i)$$

(17)

The convergence and stability analysis of the above scheme are obvious to investigate; therefore we will not pay attention on them.

3. Banister model with exponential decay memory. In this section, we introduce the fading memory uses in Cattaneo model, which also happens to be Caputo-Fabrizio fractional differentiation we power law memory, which is considered in the paper as good memory. In this case an introduction of Cattaneo fading memory yields:

$$\frac{M(\alpha)}{1-\alpha} \int_0^t \frac{d}{dy} v(y) \exp \left( -\frac{t - y}{1-\alpha} \right) dy + \frac{1}{\sigma_1} v(t) = w(t)$$

(18)

The exact solution of the above equation is obtain by simple application of Laplace transform as follows:

$$\frac{M(\alpha)}{1-\alpha} \left[ \frac{p v(p) - v(0)}{p^+, \frac{1}{M(\alpha)}} + \frac{1}{\sigma_1} v(p) \right] = w(p),$$

$$\frac{p v(p) - v(0)}{p^+, \frac{1}{M(\alpha)}} + \frac{1-\alpha}{\sigma_1 M(\alpha)} v(p) = \frac{1-\alpha}{M(\alpha)} w(p),$$

$$\left( \frac{p^+}{p^+, \frac{1}{M(\alpha)}} + \frac{1-\alpha}{\sigma_1 M(\alpha)} \right) v(p) = \frac{v(0)}{p^+, \frac{1}{M(\alpha)}} + \frac{1-\alpha}{\sigma_1 M(\alpha)} w(p),$$

(19)

$$\frac{v(p)}{1+\frac{1}{\sigma_1 M(\alpha)}} = \frac{v(0)}{p^+, \frac{1}{M(\alpha)}} + \frac{1-\alpha}{\sigma_1 M(\alpha)} w(p),$$

A direct application of the inverse Laplace transform on equation (19) produces:

$$v(t) = \left( \frac{1}{1+\frac{1-\alpha}{\sigma_1 M(\alpha)}} \right) v(0) \exp \left( -\frac{\sigma_1 M(\alpha)}{1+\frac{1-\alpha}{\sigma_1 M(\alpha)}} t \right)$$

$$+ \int_0^t \exp \left( -\frac{\sigma_1 M(\alpha)}{1+\frac{1-\alpha}{\sigma_1 M(\alpha)}} (t - y) \right) w(y) dy,$$

(20)
We now consider the Banister model with fading memory with non-local power law delay.

\[
\frac{M(\alpha)}{1 - \alpha} \int_0^t \frac{dv(y)}{dy} \exp \left[ -\alpha \frac{t - y}{1 - \alpha} \right] dy = -\frac{1}{\theta_1} v(t) - \frac{1}{\theta_2 \Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} v(y) dy + w(t),
\]

(21)

Here, we next present the conditions under which the existence and uniqueness of exact solution is obtained. To do this, we let

\[
F(t, v(t)) = -\frac{1}{\theta_1} v(t) - \frac{1}{\theta_2 \Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} v(y) dy + w(t),
\]

(22)

since the function \(v(t)\) is bounded then,

\[
\|F(t, v(t))\|_\infty = \left\| -\frac{1}{\theta_1} v(t) - \frac{1}{\theta_2 \Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} v(y) dy + w(t) \right\|_\infty,
\]

\[
< \frac{1}{\theta_1} \|v(t)\|_\infty + \frac{a}{\theta_2 \Gamma(\alpha + 1)} + \|w(t)\|_\infty,
\]

(23)

In addition to above, we also check the conditions under which the function is Lipschitz:

\[
\|F(t, v(t)) - F(t, u(t))\|_\infty = \left\| -\frac{1}{\theta_1} (v(t) - u(t)) - \frac{1}{\theta_2 \Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} (v(y) - u(y)) dy \right\|_\infty,
\]

\[
\leq \frac{1}{\theta_1} \|(v(t) - u(t))\|_\infty + \left\| \frac{1}{\theta_2 \Gamma(\alpha + 1)} \int_0^t (t - y)^{\alpha - 1} (v(y) - u(y)) dy \right\|_\infty,
\]

\[
\leq \frac{1}{\theta_1} \|(v(t) - u(t))\|_\infty + \frac{1}{\theta_2 \Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} \|(v(y) - u(y))\|_\infty dy,
\]

(24)

Therefore the function has Lipschitz condition, far more if the following inequality is reached

\[
\theta_1 a + \theta_2 \Gamma(\alpha + 1) < \theta_1 \theta_2 \Gamma(\alpha + 1)
\]

(25)

Then the function is a contraction. To set the conditions under which the exact solution exists and is unique, we revert the fractional differential equation (21) to integral equation as

\[
v(t) - v(0) = \frac{1 - \alpha}{M(\alpha)} F(t, v(t)) + \frac{\alpha}{M(\alpha)} \int_0^t F(y, v(y)) dy
\]

(26)

We next consider the compact cylinder constructed earlier and also we consider the following functional operator

\[
\psi \omega(t) = v(0) + \frac{1 - \alpha}{M(\alpha)} F(t, \omega(t)) + \frac{\alpha}{M(\alpha)} \int_0^t F(y, \omega(y)) dy
\]

(27)

Thus,

\[
\|\psi \omega(t) - v(0)\|_\infty = \left\| \frac{1 - \alpha}{M(\alpha)} F(t, \omega(t)) + \frac{\alpha}{M(\alpha)} \int_0^t F(y, \omega(y)) dy \right\|_\infty,
\]

\[
< \frac{1 - \alpha}{M(\alpha)} \|F(t, \omega(t))\|_\infty + \frac{\alpha}{M(\alpha)} \int_0^t \|F(y, \omega(y))\|_\infty dy,
\]

\[
< \frac{1 - \alpha}{M(\alpha)} \left( \frac{1}{\theta_1} M + \frac{a}{\theta_2} + N \right) + \frac{\alpha}{M(\alpha)} \left( \frac{1}{\theta_1} M + \frac{a}{\theta_2} + N \right),
\]

(28)

\[
< \left( \frac{\alpha}{M(\alpha)} + \frac{1 - \alpha}{M(\alpha)} \right) \left( \frac{1}{\theta_1} M + \frac{a}{\theta_2} + N \right)
\]
The function is then well defined if
\[
\left( \frac{aa}{M(\alpha)} + \frac{1 - \alpha}{M(\alpha)} \right) < \frac{b}{\left( \frac{M}{\theta_1} + \frac{a_{\alpha}}{\theta_2} + N \right)}.
\]
(29)

We now check the condition for contraction
\[
\|\psi(t) - \psi(u)\|_{\infty} = \left\| \frac{1 - \alpha}{M(\alpha)} \int_{t}^{\infty} \{ F(\omega, (t)) - F(\omega, (u)) \} \, d\omega \right\|_{\infty},
\]
\[
\leq \frac{1 - \alpha}{M(\alpha)} \left( \frac{1}{\theta_1} + \frac{a_{\alpha}}{\theta_2} \right) \int_{t}^{\infty} \|F(\omega, (y)) - F(\omega, (u))\|_{\infty} \, dy.
\]

The contraction is reached if and only if the following inequality holds
\[
\left( \frac{aa}{M(\alpha)} + \frac{1 - \alpha}{M(\alpha)} \right) < \frac{1}{\left( \frac{1}{\theta_1} + \frac{a_{\alpha}}{\theta_2} \right)}.
\]
(31)

The Banach fixed-point theorem will therefore be applied under the condition that:
\[
\left( \frac{aa}{M(\alpha)} + \frac{1 - \alpha}{M(\alpha)} \right) < \min \left\{ \frac{1}{\left( \frac{1}{\theta_1} + \frac{a_{\alpha}}{\theta_2} \right)}, \frac{b}{\left( \frac{1}{\theta_1} + \frac{a_{\alpha}}{\theta_2} + N \right)} \right\}
\]
(32)

Under the above condition our modified Banister has unique solution.

4. Banister model with Mittag-Leffler memory. In this section, we introduce the generalized and non-local fading memory, which also happens to be Atangana-Baleanu fractional differentiation we Mittag-Leffler law memory, which is considered in the paper as good memory. In this case an introduction of generalized and non-local fading memory yields:

\[
\frac{AB(\alpha)}{1 - \alpha} \int_{0}^{t} \frac{d\psi(y)}{dy} E_{\alpha} \left( \frac{-\alpha(t-y)^{\alpha}}{1 - \alpha} \right) \, dy + \frac{1}{\theta_1} \psi(t) = \psi(t)
\]
(33)

The exact solution of the above equation is obtain by simple application of Laplace transform as follows:

\[
\frac{AB(\alpha)}{1 - \alpha} \frac{p^{\alpha} \psi(p) - p^{\alpha-1} \psi(0)}{p^{\alpha} + \frac{1}{\theta_1} \psi(p)} + \frac{1}{\theta_1} \psi(p) = \psi(p),
\]
\[
\frac{p^{\alpha} \psi(p) - p^{\alpha-1} \psi(0)}{p^{\alpha} + \frac{1}{\theta_1} \psi(p)} + \frac{1 - \alpha}{1 - \alpha} \frac{AB(\alpha)}{1 - \alpha} \psi(p) = \frac{1 - \alpha}{1 - \alpha} \frac{AB(\alpha)}{1 - \alpha} \psi(p),
\]
\[
\frac{\psi(p)}{1 + \frac{1}{\theta_1} A_{\alpha}(\psi)} = \frac{p^{\alpha-1} \psi(0)}{p^{\alpha} + \frac{1}{\theta_1} A_{\alpha}(\psi)} + \frac{p^{\alpha} + \frac{1}{\theta_1} A_{\alpha}(\psi)}{1 + \frac{1}{\theta_1} A_{\alpha}(\psi)} \psi(p),
\]
(34)

A direct application of the inverse Laplace transform on equation (34) produces:

\[
v(t) = \left( 1 + \frac{1 - \alpha}{1 + \frac{1}{\theta_1} A_{\alpha}(\psi)} \right) \psi(0) E_{\alpha} \left[ \frac{\alpha}{1 + \frac{1}{\theta_1} A_{\alpha}(\psi)} (t-y) \right]
\]
\[
+ \int_{0}^{t} E_{\alpha} \left[ \frac{1}{1 + \frac{1}{\theta_1} A_{\alpha}(\psi)} (t-y) \right] \psi(0) \, dy,
\]
(35)
We next consider the following Banister model with the more generalized fading memory and an addition power law memory:

$$AB(\alpha) \frac{d}{dy} v(y) E_\alpha \left( -\alpha \frac{(t-y)\gamma}{1-\alpha} \right) dy + \frac{1}{\theta_1} v(t) - \frac{1}{\theta_2} \int_0^t (t-y)^{\alpha-1} v(y) dy = w(t)$$

(36)

We next consider the following Banister model with the more generalized fading memory and an addition power law memory:

$$AB(\alpha) \frac{d}{dy} v(y) E_\alpha \left( -\alpha \frac{(t-y)\gamma}{1-\alpha} \right) dy + \frac{1}{\theta_1} v(t) - \frac{1}{\theta_2} \int_0^t (t-y)^{\alpha-1} v(y) dy = w(t)$$

(36)

We now check the condition for contraction

$$\| \psi(u) - \psi(t) \| \leq \frac{1-\alpha}{M(\alpha)} \| F(t, u(t)) - F(t, u(t)) \| + \frac{\alpha}{M(\alpha)} \int_0^t \| F(y, u(y)) - F(y, u(y)) \| (t-y)^{-\alpha} dy$$

(37)

Thus,

$$\| \psi(u) - \psi(t) \| \leq \frac{1-\alpha}{M(\alpha)} \| F(t, u(t)) - F(t, u(t)) \| + \frac{\alpha}{M(\alpha)} \int_0^t \| F(y, u(y)) - F(y, u(y)) \| (t-y)^{-\alpha} dy$$

(38)

The function is then well defined providing that the following holds

$$\left( \frac{\alpha}{AB(\alpha) \Gamma(\alpha)} + \frac{1-\alpha}{AB(\alpha)} \right) < \frac{1}{\frac{1}{\theta_1} M + \frac{a^\alpha}{\theta_2 (1+\alpha) + N}}$$

(39)

We now check the condition for contraction

$$\| \psi(u) - \psi(t) \| \leq \frac{1-\alpha}{M(\alpha)} \| F(t, u(t)) - F(t, u(t)) \| + \frac{\alpha}{M(\alpha)} \int_0^t \| F(y, u(y)) - F(y, u(y)) \| (t-y)^{-\alpha} dy$$

(40)

The contraction is reached if and only if the following inequality holds

$$\left( \frac{\alpha}{AB(\alpha) \Gamma(\alpha)} + \frac{1-\alpha}{AB(\alpha)} \right) < \frac{1}{\frac{1}{\theta_1} M + \frac{a^\alpha}{\theta_2 (1+\alpha) + N}}$$

(41)

The Banach fixed-point theorem will therefore be applied under the condition that:

$$\left( \frac{\alpha}{AB(\alpha) \Gamma(\alpha)} + \frac{1-\alpha}{AB(\alpha)} \right) < \min \left\{ \frac{1}{\frac{1}{\theta_1} + \frac{a^\alpha}{\theta_2 (1+\alpha) + N}}, \frac{1}{\frac{1}{\theta_1} M + \frac{a^\alpha}{\theta_2 (1+\alpha) + N}} \right\}$$

(42)

Under the above condition our modified Banister with generalized fading memory and addition power memory has unique solution.
5. Conclusion. We make use of three different concept of fractional differentiation to include into mathematical formulation of Banister model of prediction in physical condition the concept of fading memory including the exponential decay memory, the power law memory and the generalized Mittag-Leffler memory. Therefore in this paper four modified Banister models were introduced. We constructed using the Laplace transform the exact solutions of three generalized models. Then we constructed the conditions under which the more complex models have exact unique solution using the Banach fixed-point theorem.

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Received May 2018; revised June 2018.

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