Convergence and error estimates of a penalization finite volume method for the compressible Navier–Stokes system

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Abstract

In numerical simulations a smooth domain occupied by a fluid has to be approximated by a computational domain that typically does not coincide with a physical domain. Consequently, in order to study convergence and error estimates of a numerical method domain-related discretization errors, the so-called variational crimes, need to be taken into account. In this paper we apply the penalty approach to control domain-related discretization errors. We embed the physical domain into a large enough cubed domain and study the convergence of a finite volume method for the corresponding domain-penalized problem. We show that numerical solutions of the penalized problem converge to a generalized, the so-called dissipative weak, solution of the original problem. If a strong solution exists, the dissipative weak solution emanating from the same initial data coincides with the strong solution. In this case, we apply a novel tool of the relative energy and derive the error estimates between the numerical solution and the strong solution. Extensive numerical experiments that confirm theoretical results are presented.

Keywords: compressible Navier–Stokes system, convergence, error estimates, finite volume method, penalization method, dissipative weak solution, relative energy

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1 Introduction

We study compressible fluid flow in a smooth domain $\Omega^f$, $\Omega^f \subset \mathbb{R}^d$, $d = 2, 3$, modelled by the Navier–Stokes system

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \\
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) &= \nabla \cdot S(\nabla u)
\end{align*}$$

subject to the following Dirichlet boundary condition and initial data, respectively,

$$\begin{align*}
\left. u \right|_{\partial \Omega^f} &= 0, \\
\rho(0, \cdot) &= \rho_0, \quad \rho u(0, \cdot) = m_0.
\end{align*}$$

Here, $\rho$ and $u$ are the fluid density and velocity, respectively. The viscous stress tensor $S$ reads

$$S(\nabla u) = \mu \left( \nabla u + \nabla^t u - \frac{2}{d} \nabla u \mathbb{I} \right) + \lambda \nabla u \mathbb{I},$$

where $\mu > 0$ and $\lambda \geq 0$ are the viscosity coefficients. For the sake of simplicity, we consider the isentropic state equation:

$$p(\rho) = a \rho^\gamma, \quad a > 0, \quad \gamma > 1 \quad \text{with the associated pressure potential} \quad \mathcal{P}(\rho) = \frac{a}{\gamma - 1} \rho^\gamma. \quad (1.2)$$

Throughout the paper we always assume that the initial data satisfy

$$\rho_0 \geq \rho > 0, \quad \rho_0 \in L^\infty(\Omega^f), \quad m_0 \in L^\infty(\Omega^f; \mathbb{R}^d). \quad (1.3)$$

On the one hand, mathematical analysis of the Navier–Stokes system (1.1) is currently available either for periodic boundary conditions or for no-slip boundary conditions applied in a smooth physical domain occupied by a fluid. On the other hand, numerical methods, e.g., finite volume or finite element methods, typically require a polygonal computational domain.

Hence, a smooth physical domain has to be approximated by a polygonal computational domain and additional approximation errors arise. Let us note that in the case of complicated geometry the generation of a suitable polygonal approximation may be computationally very costly.

To overcome these difficulties we apply a penalty method originally used in the context of incompressible Navier–Stokes equations by Angot et al. [2]. Thus, the physical domain $\Omega^f$ is embedded into a large cube on which the periodic boundary conditions are imposed, see Figure 1. The original boundary conditions are enforced through a penalty term, that is a singular friction term in the momentum equation. The resulting penalized problem is solved on a flat torus $\mathbb{T}^d$ by an upwind-type finite volume (FV) method. Specifically, the penalized Navier–Stokes system on $\mathbb{T}^d$ reads

$$\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) &= \nabla \cdot S(\nabla u) - \frac{1}{\epsilon} \mathbb{I} \rho u,
\end{align*}$$

where $\epsilon > 0$ is the penalty parameter.
where $\epsilon > 0$ is the penalty parameter and $\mathbf{1}_{\Omega^s}$ is the characteristic function

$$
\mathbf{1}_{\Omega^s}(x) = \begin{cases} 
1 & \text{if } x \in \Omega^s := \mathbb{T}^d \setminus \Omega^f, \\
0 & \text{if } x \in \Omega^f. 
\end{cases}
$$

The initial data of the penalized problem (1.4a)-(1.4b) are set to be

$$
\tilde{\varrho}_0 := \tilde{\varrho}(0, \cdot) = \begin{cases} 
some \varrho_0^0 & \text{if } x \in \Omega^s, \\
\varrho_0 & \text{if } x \in \Omega^f, 
\end{cases}
\quad \tilde{\vec{\mathbf{m}}}_0 := \tilde{\vec{\mathbf{m}}}(0, \cdot) = \begin{cases} 
0 & \text{if } x \in \Omega^s, \\
\varrho_0 u_0 & \text{if } x \in \Omega^f, 
\end{cases}
$$

with $\tilde{\varrho}_0 > 0$ satisfying the periodic boundary condition.

![Figure 1: A fluid domain $\Omega^f$ embedded into a torus $\mathbb{T}^d$.](image)

The main aim of the present paper is a rigorous convergence and error analysis of the Dirichlet boundary problem for the compressible Navier–Stokes equations. For a finite volume method using piecewise constant approximations a direct approach of approximating a smooth boundary by a sequence of polygonal domains leads to problems in the control of consistency errors arising from the convective terms. Alternatively, if a combined finite volume-finite element method is applied, the convergence and error analysis on a polygonal approximation of the fluid domain $\Omega^f$ has been done, see our recent works [7], [20]. The main reason of this discrepancy is a finite element approximation of the velocity that allows more degree of freedom inside a mesh cell.

The idea to penalize a complicated physical domain and solve numerically the corresponding problem on a simple domain is quite often used in the literature. We refer a reader to [23, 24] for the immersed boundary method and to [13, 14, 17] for the fictitious domain method developed in the context of incompressible Navier–Stokes equations. In [4] a penalty method has been applied to approximate a moving domain in the fluid-structure interaction problem. Further, in [15, 16] penalization of boundary conditions for the compressible Navier–Stokes-Fourier system was applied for the spectral method. Error estimates between exact and penalized numerical solutions were presented in [2] for the incompressible Navier–Stokes equations and in [22, 25, 27, 28] for elliptic boundary problems. We mention also Basarić et al. [3] and Feireisl et al. [11], where the penalty method was used to prove the existence of weak solutions. In [11] the penalization method has been used to show the existence of a weak solution to the compressible Navier–Stokes equations on a moving domain and in [3] the existence of a weak solution...
to the Navier–Stokes–Fourier system with the Dirichlet conditions on a rough (Lipschitz) domain was proved.

The present paper is organized in the following way. In Section 2 we introduce the concept of generalized, the so-called dissipative weak solution for the Dirichlet problem (1.1) and the corresponding penalized problem (1.4). A finite volume method for the approximation of the penalized problem (1.4) is introduced in Section 3. Section 4 and Section 5 are devoted to the main results of the paper: convergence analysis of the finite volume method as well as error estimates between the finite volume solutions of the penalized problem and the exact strong solution of the Navier–Stokes system with the Dirichlet boundary conditions. Here we consider the errors with respect to the mesh (discretization) parameter as well as the penalty parameter. The paper is closed with Section 6, where several numerical experiments illustrate our theoretical results.

2 Dissipative weak solution

Following [9] we define dissipative weak (DW) solutions to the Navier–Stokes system. We consider both, the penalized problem on $\mathbb{T}^d$ (1.4) and the original Dirichlet problem on $\Omega_f$ (1.1). We will show in Section 4 that a DW solution arises as a natural limit of numerical approximations and builds a suitable tool for the convergence analysis.

Definition 2.1 (DW solution of the penalized problem). We say that $(\rho, u)$ is a DW solution of the Navier–Stokes system (1.4) if the following hold:

- **Integrability.**
  \[
  \rho \geq 0, \quad \rho \in L^\infty(0, T; L^7(\mathbb{T}^d)), \quad u \in L^2(0, T; W^{1,2}(\mathbb{T}^d; \mathbb{R}^d)), \\
  \rho u \in L^\infty(0, T; L^{2\gamma}(\mathbb{T}^d; \mathbb{R}^d)), \quad S \in L^2((0, T) \times \mathbb{T}^d; \mathbb{R}^{d \times d}_{\text{sym}}). 
  \]  

- **Energy inequality.**
  \[
  \left[ \int_{\mathbb{T}^d} \left( \frac{1}{2} |u|^2 + \mathcal{P}(\rho) \right) \, dx \right](\tau, \cdot) + \int_0^\tau \int_{\mathbb{T}^d} S(\nabla u) : \nabla u \, dx \, dt \\
  + \frac{1}{\epsilon} \int_0^\tau \int_{\Omega_f} |u|^2 \, dx \, dt + \int_0^\tau \int_{\mathbb{T}^d} d\mathcal{E}(\tau) + \int_0^\tau \int_{\mathbb{T}^d} d\mathcal{D}(\tau) \, dt \\
  \leq \int_{\mathbb{T}^d} \left( \frac{|\mathbf{m}_0|^2}{2\epsilon\rho_0} + \mathcal{P}(\rho_0) \right) \, dx 
  \] (2.2)

  for any $\tau \in [0, T]$, where the energy and dissipation defect measures satisfy
  \[
  \mathcal{E} \in L^\infty(0, T; \mathcal{M}^+(\mathbb{T}^d)), \quad \mathcal{D} \in \mathcal{M}^+(\mathbb{T}^d). 
  \]

- **Equation of continuity.**
  \[
  -\int_{\mathbb{T}^d} \tilde{\rho}_0 \phi(0, \cdot) \, dx = \int_0^T \int_{\mathbb{T}^d} (\rho \partial_t \phi + \rho u \cdot \nabla \phi) \, dx \, dt 
  \] (2.3)

  for any test function $\phi \in C^1_c([0, T) \times \mathbb{T}^d)$. 

4
• Momentum equation.
\[
- \int_{T^d} m_0 \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_{T^d} (\rho u \cdot \partial_t \varphi + \rho u \otimes u : \nabla x \varphi + p(\varphi) \text{div} \varphi) \, dxdt
\]
\[
- \frac{1}{\epsilon} \int_0^T \int_{\Omega^f} u \cdot \varphi \, dxdt - \int_0^T \int_{T^d} S(\nabla x u) : \nabla x \varphi \, dxdt + \int_0^T \int_{T^d} \nabla x \varphi : d\mathcal{R}(t) \, dt \tag{2.4}
\]
for any test function \( \varphi \in C^1_c([0, T) \times T^d; \mathbb{R}^d) \) with the Reynolds defect
\[
\mathcal{R} \in L^\infty(0, T; \mathcal{M}^+(T^d; \mathbb{R}^{d \times d}_{sym}))
\]
satisfying\(^1\)
\[
d\mathcal{E} \leq \text{tr}[\mathcal{R}] \leq \overline{d}\mathcal{E} \text{ for some constants } 0 < d \leq \overline{d}. \tag{2.5}\]

Definition 2.2 (DW solution of the Dirichlet problem). We say that \((\rho, u)\) is a DW solution of the Navier–Stokes system (1.1) if the following hold:

• Integrability.
\[
\rho \geq 0, \quad \rho \in L^\infty(0, T; L^\gamma(\Omega^f)), \quad u \in L^2(0, T; W^{1,2}(\Omega^f; \mathbb{R}^d)),
\]
\[
\rho u \in L^\infty(0, T; L^{\frac{2\gamma}{\gamma + 1}}(\Omega^f; \mathbb{R}^d)), \quad S \in L^2((0, T) \times \Omega^f; \mathbb{R}^{d \times d}_{sym}). \tag{2.6}
\]

• Energy inequality.
\[
\left[ \int_{\Omega^f} \left( \frac{1}{2} |u|^2 + P(\varphi) \right) \, dx \right] (\tau, \cdot) + \int_0^T \int_{\Omega^f} S(\nabla x u) : \nabla x u \, dxdt
\]
\[
+ \int_{\Omega^f} d\mathcal{E}(\tau) + \int_0^T \int_{\Omega^f} d\mathcal{D}(\tau) \, dt \leq \int_{\Omega^f} \left( \frac{1}{2} |m_0|^2 + P(\varphi_0) \right) \, dx \tag{2.7}
\]
for any \( \tau \in [0, T] \), where the energy and dissipation defect measures satisfy
\[
\mathcal{E} \in L^\infty(0, T; \mathcal{M}^+(\Omega^f)), \quad \mathcal{D} \in \mathcal{M}^+([0, T] \times \overline{\Omega^f}).
\]

• Equation of continuity.
\[
- \int_{\Omega^f} \varphi_0 \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_{\Omega^f} (\rho \partial_t \varphi + \rho u \cdot \nabla x \varphi) \, dxdt \tag{2.8}
\]
for any test function \( \varphi \in C^1_c([0, T) \times \overline{\Omega^f}).\)

\(^1\mathcal{M}^+(T^d; \mathbb{R}^{d \times d}_{sym})\) denotes the set of symmetric square matrices of order \( d \), where each component is a positive Radon measure on \( T^d \). \text{tr}[\mathcal{R}]\) denotes the trace of the matrix \( \mathcal{R} \).
• Momentum equation.

\[- \int_{\Omega^f} \mathbf{m}_0 \cdot \varphi(0, \cdot) \, dx = \int_0^T \int_{\Omega^f} (\rho \mathbf{u} \cdot \partial_t \varphi + \varphi \otimes \mathbf{u} : \nabla \mathbf{u} + p(\varphi) \text{div} \varphi) \, dx \, dt \]

\[- \int_0^T \int_{\Omega^f} \mathbf{S}(\nabla \mathbf{u}) : \nabla \varphi \, dx \, dt + \int_0^T \int_{\Omega^f} \nabla \varphi : d\mathbf{R}(t) \, dt \]  

(2.9)

for any test function \( \varphi \in C^1_c([0, T) \times \Omega^f; \mathbb{R}^d) \) with the Reynolds defect

\[ \mathbf{R} \in L^\infty(0, T; \mathcal{M}^+(\Omega^f; \mathbb{R}^{d \times d})) \]

satisfies

\[ d\mathcal{E} \leq \text{tr}[\mathbf{R}] \leq \overline{d}\mathcal{E} \]

for some constants \( 0 < d \leq \overline{d} \).  

(2.10)

**Remark 2.3.** The existences of dissipative solutions to the penalized problem (1.4) as well as the original Dirichlet problem (1.1) will be proved in Theorems 4.6 and 4.7, respectively.

### 3 Numerical scheme

In this section we generalize the finite volume method proposed in [8] to approximate the penalized problem (1.4).

#### 3.1 Space discretization

**Mesh.** Let \( T_h \) be a uniform structured (square for \( d = 2 \) or cuboid for \( d = 3 \)) mesh of \( \mathbb{T}^d \) with \( h \) being the mesh parameter. We denote by \( \mathcal{E} \) the set of all faces of \( T_h \) and by \( \mathcal{E}_i, i = 1, \ldots, d \), the set of all faces that are orthogonal to \( e_i \) – the basis vector of the canonical system. Moreover, we denote by \( \mathcal{E}(K) \) the set of all faces of a generic element \( K \in T_h \). Then, we write \( \sigma = K|L \) if \( \sigma \in \mathcal{E} \) is the common face of neighbouring elements \( K \) and \( L \). Further, we denote by \( x_K \) and \( |x_K| = h^d \) (resp. \( x_\sigma \) and \( |x_\sigma| = h^{d-1} \)) the center and the Lebesgue measure of an element \( K \in T_h \) (resp. a face \( \sigma \in \mathcal{E} \)), respectively.

With the above notations, we further define \( \Omega^f_h \) as the set of all elements inside the physical domain \( \Omega^f \), i.e.

\[ \Omega^f_h = \left\{ K \in T_h \mid K \subset \Omega^f \right\} \quad \text{and} \quad \Omega^s_h = T_h \setminus \Omega^f_h, \]

cf. Figure 12. In what follows we write that \( a \lesssim b \) for \( a, b \in \mathbb{R} \), if there exists a generic constant \( C > 0 \) independent on the mesh and penalty parameters \( h, \epsilon \), such that \( a \leq Cb \). Further, we write \( a \approx b \) if \( a \lesssim b, b \lesssim a \). It is easy to verify that

\[ \Omega^f_h \subset \Omega^f, \quad \Omega^s \subset \Omega^s_h \quad \text{and} \quad \text{dist}(\partial \Omega^f, \Omega^f_h) \approx h, \quad |\Omega^f \setminus \Omega^f_h| \approx h, \quad |\Omega^s_h \setminus \Omega^s| \approx h, \]

(3.1)

where generic constants depend on the geometry of the fluid domain \( \Omega^f \).
**Dual mesh.** For any $\sigma = K|L \in \mathcal{E}_i$, we define a dual cell $D_\sigma := D_{\sigma,K} \cup D_{\sigma,L}$, where $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) is defined as

$$D_{\sigma,K} = \{ x \in K \mid x_i \in \text{co}\{ (x_K)_i, (x_\sigma)_i \} \}$$

for any $\sigma \in \mathcal{E}_i$, $i = 1, \ldots, d$ with $\text{co}\{A, B\} \equiv [\min\{A, B\}, \max\{A, B\}]$. We refer to Figure 2 for a two-dimensional illustration of a dual cell.

![Figure 2: Dual mesh $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$](image)

**Function space.** We introduce a space $Q_h$ consisting of piecewise constant functions on $T_h$. The notation $Q^n_h$ stands for the corresponding $n$-dimensional function space. Further, we define the standard projection operator $\Pi_T$ associated with $Q_h$

$$\Pi_T : L^1(T^d) \rightarrow Q_h, \quad \Pi_T \phi(x) = \sum_{K \in T_h} \frac{1}{|K|} \int_K \phi \, dx.$$

**Discrete differential operators.** To begin, we introduce the average and jump operators for $v \in Q_h$

$$\{\{v\}\} (x) = \frac{v^\text{in}(x) + v^\text{out}(x)}{2}, \quad \|v\| (x) = v^\text{out}(x) - v^\text{in}(x)$$

with

$$v^\text{out}(x) = \lim_{\delta \to 0^+} v(x + \delta n), \quad v^\text{in}(x) = \lim_{\delta \to 0^+} v(x - \delta n),$$

whenever $x \in \sigma \in \mathcal{E}$ and $n$ is the outer normal vector to $\sigma$. In addition, if $\sigma \in \mathcal{E}_i$, we also write $\{\{v\}\}$ and $\|v\|$ as $\{\{v\}\}^{(i)}$ and $\|v\|^{(i)}$, respectively.

Next, we define the following discrete gradient, divergence and Laplace operators for piecewise constant functions $r_h \in Q_h$, $v_h = (v_1, \ldots, v_d) \in Q^d_h$

$$\nabla \mathcal{E} r_h(x) = \sum_{\sigma \in \mathcal{E}} (\nabla \mathcal{E} r_h)_D \mathbb{I}_{D_{\sigma}}(x), \quad (\nabla \mathcal{E} r_h)_D = \frac{1}{h} \| r_h \| \mathbf{n}, \quad \nabla \mathcal{E} v_h = (\nabla \mathcal{E} v_1, \ldots, \nabla \mathcal{E} v_d)^T,$$

$$\text{div}_h v_h(x) = \sum_{K \in T_h} (\text{div}_h v_h)_K \mathbb{1}_K(x), \quad (\text{div}_h v_h)_K = \frac{1}{h} \sum_{\sigma \in \mathcal{E}(K)} \{ v_h \} \cdot n,$$

$$\Delta_h r_h(x) = \sum_{K \in T_h} (\Delta_h r_h)_K \mathbb{1}_K(x), \quad (\Delta_h r_h)_K = \frac{1}{h^2} \sum_{\sigma \in \mathcal{E}(K)} \| r_h \|.$$
It is easy to verify the interpolation errors estimates
\[ ||\Pi_T \phi|| \lesssim h ||\nabla_x \phi||_{L^\infty(\Omega)} \text{ and } ||\Pi_T \phi - \phi||_{L^\infty(\Omega)} \lesssim h ||\nabla_x \phi||_{L^\infty(\Omega)} \text{ and } ||\nabla_x \Pi_T \phi||_{L^\infty(\Omega)} \lesssim ||\nabla_x \phi||_{L^\infty(\Omega)} \]  \hspace{1cm} (3.2)
for any \( \phi \in W^{1,\infty}(\Omega) \).

**Time discretization.** Given a time step \( \Delta t > 0 \) we divide the time interval \([0, T]\) into \( N_T = T/\Delta t \) uniform parts, and denote \( t_k = k\Delta t \). Consider a pointwise function \( v \), which is only known at time level \( t_k \), \( k = 0, \ldots, N_T \). Let \( L_{\Delta t}(0, T) \) denote a space of all piecewise constant functions \( v \), such that
\[ v(t) = v(t_0) \text{ for } t < \Delta t, \quad v(t) = v(t_k) \text{ for } t \in [k\Delta t, (k+1)\Delta t), \quad k = 1, \ldots, N_T. \]
Further, for \( v \in L_{\Delta t}(0, T) \) we define the backward Euler time discretization operator \( D_t \) as follows
\[ D_t v(t) = \frac{v(t) - v^{\Delta t}}{\Delta t} \quad \text{with} \quad v^{\Delta t} := v(t - \Delta t). \]
If \( v \) is a vector function then \( D_t v \) acts componentwisely.

### 3.2 Finite volume method for the penalized problem

We are now ready to propose a finite volume method for the penalized problem (1.4) that will be presented in the weak form.

**Definition 3.1 (Finite volume method).** We say that \((\tilde{\varrho}_h^t, \tilde{u}_h^t) \in L_{\Delta t}(0, T; Q_{\Delta t}^{d+1})\) is a finite volume approximation of the penalized problem (1.4) if the following algebraic equations hold
\[ \int_{T^d} D_t \tilde{\varrho}_h^t \phi_h \, dx - \int_{\mathcal{E}} F_h^\alpha (\tilde{\varrho}_h^t, \tilde{u}_h^t) \cdot [\phi_h] \, dS_x = 0, \quad \text{for all } \phi_h \in Q_h, \]  \hspace{1cm} (3.3a)
\[ \int_{T^d} D_t (\tilde{\varrho}_h^{t+1}, \tilde{u}_h^{t+1}) - \int_{\mathcal{E}} F_h^\alpha (\tilde{\varrho}_h^t, \tilde{u}_h^t) \cdot [\varphi_h] \, dS_x - \int_{T^d} p_h^t \div_h \varphi_h \, dx + \frac{1}{\epsilon} \int_{\Omega_h^t} \tilde{u}_h^t \cdot \varphi_h \, dx + \mu \int_{T^d} \nabla_{\mathcal{E}} \tilde{u}_h^t \cdot \nabla_{\mathcal{E}} \varphi_h \, dx + \nu \int_{T^d} \div_h \tilde{u}_h^t \div_h \varphi_h \, dx = 0, \quad \text{for all } \varphi_h \in Q_h^t, \]  \hspace{1cm} (3.3b)
where
\[ \tilde{\varrho}_h^0(0, \cdot) = \Pi_T \tilde{\varrho}_0, \quad (\tilde{\varrho}_h^t, \tilde{u}_h^t)(0, \cdot) = \Pi_T \tilde{m}_0, \quad \nu = \lambda + \frac{d - 2}{d - \mu}. \]

The numerical flux function \( F_h^\alpha(r_h, u_h) \) reads
\[ F_h^\alpha(r_h, u_h) = Up[r_h, u_h] - h^\alpha [r_h], \]
\[ Up[r_h, u_h] = r_h^{up} \{ u_h \} \cdot n, \quad r_h^{up} = \begin{cases} (r_h)^{in} & \text{if } \{ u \} : n \geq 0, \\ (r_h)^{out} & \text{if } \{ u \} : n < 0. \end{cases} \]

Here \( \alpha > -1 \) is the artificial viscosity parameter.

**Remark 3.2.** In what follows we shall write \((\varrho_h, u_h)\) and \((\varrho_h^0, u_h^0)\) instead of more precise notation \((\tilde{\varrho}_h^t, \tilde{u}_h^t)\) and \((\tilde{\varrho}_h^0(0, \cdot), \tilde{u}_h^0(0, \cdot))\) for simplicity, if there is no confusion. Consequently, we shall work with the couple \((\varrho_h, u_h)\) that represents the (piecewise constant in space and time) discrete density and velocity, respectively. Moreover, we set \( m_h = \varrho_h u_h \) and \( p_h = p(\varrho_h) \).
4 Convergence

In this section we study the convergence of the finite volume method (3.3). To this goal we first discuss its stability and consistency.

4.1 Stability

We begin with the following lemma reported by Feireisl et al. [9, Lemmas 11.2 and 11.3].

**Lemma 4.1** (Properties [9, Lemmas 11.2 and 11.3]). Let $\tilde{\rho}_0 > 0$. Then there exists at least one solution to the FV method (3.3). Moreover, any solution $(\rho_h, u_h)$ to (3.3a) satisfies for all $t \in (0, T)$ that

- Positivity of the density. \[ \rho_h(t) > 0. \]
- Mass conservation. \[ \int_{\Omega} \rho_h(t) \, dx = \int_{\Omega} \tilde{\rho}_0 \, dx. \]
- Internal energy balance. \[
\int_{\Omega} D_t P(\rho_h) \, dx + \int_{\Omega} p(\rho_h) \text{div}_h u_h \, dx \\
= - \int_{\Omega} \frac{\Delta t}{2} P''(\rho_h^*) |D_t \rho_h|^2 \, dx \\
- \int_{\mathcal{E}} \left( h^\alpha + \frac{1}{2} |\{u_h\} \cdot n| \right) P''(\rho_{h,\sigma}) \|\rho_h\|^2 \, dS_x, \tag{4.1}
\]

where $\rho_h^* \in \text{co}\{\rho_h^\text{in}, \rho_h\}$ and $\rho_{h,\sigma} \in \text{co}\{\rho_h^\text{in}, \rho_h^\text{out}\}$ for any $\sigma \in \mathcal{E}$.

**Lemma 4.2** (Energy stability). Let $(\rho_h, u_h)$ be a numerical solution of the FV method (3.3). Then it holds

\[
D_t \int_{\Omega} \left( \frac{1}{2} \rho_h |u_h|^2 + P(\rho_h) \right) \, dx + \int_{\Omega} \left( \mu |\nabla \varepsilon u_h|^2 + \nu |\text{div}_h u_h|^2 \right) \, dx \\
= - \frac{1}{\epsilon} \int_{\Omega}^* |u_h|^2 \, dx - D_{\text{num}} - \frac{1}{\epsilon} \int_{\Omega}^* |u_h|^2 \, dx - D_{\text{new}}^{\text{num}}, \tag{4.2}
\]

where $D_{\text{num}}^{\text{new}} \geq D_{\text{num}} \geq 0$ represent the numerical dissipations, which read

\[
D_{\text{new}}^{\text{num}} = D_{\text{num}} + \frac{1}{\epsilon} \int_{\Omega_h^\text{ex} \setminus \Omega_h^*} |u_h|^2 \, dx, \\
D_{\text{num}} = h^\alpha \int_{\mathcal{E}} \|\rho_h\| \|u_h\|^2 \, dS_x + \frac{\Delta t}{2} \int_{\Omega} \rho_h^\text{ex} |D_t u_h|^2 \, dx + \frac{1}{2} \int_{\mathcal{E}} \rho_h^{\text{ex}^2} |\{u_h\} \cdot n| \|u_h\|^2 \, dS_x \\
+ \int_{\Omega} \frac{\Delta t}{2} P''(\rho_h^*) |D_t \rho_h|^2 \, dx + \int_{\mathcal{E}} \left( h^\alpha + \frac{1}{2} |\{u_h\} \cdot n| \right) P''(\rho_{h,\sigma}) \|\rho_h\|^2 \, dS_x.
\]
Proof. Following the calculation in [8, equation (3.4)] we obtain the kinetic energy balance
\[
D_t \int_{\mathbb{T}^d} \frac{1}{2} \rho_h |u_h|^2 \, dx + \mu \int_{\mathbb{T}^d} |\nabla_x u_h|^2 \, dx + \nu \int_{\mathbb{T}^d} \text{div}_h u_h |u_h|^2 \, dx + \frac{1}{\epsilon} \int_{\Omega_h} u_h^2 \, dx - \int_{\mathbb{T}^d} p_h \text{div}_h u_h \, dx
+ h^\alpha \int_{\mathcal{E}} \|\psi_h\| \|u_h\|^2 \, dS_x + \frac{\Delta t}{2} \int_{\mathbb{T}^d} \rho_h^0 |D_t u_h|^2 \, dx + \frac{1}{2} \int_{\mathcal{E}} \rho_h^{up} \|u_h\| \cdot n \|u_h\|^2 \, dS_x = 0.
\]

Combining this with the internal energy balance (4.1) finishes the proof.

Next, thanks to the energy balance (4.2) and the Sobolev-Poincaré inequality, see Lemma A.1, we obtain the following a priori bounds for the finite volume solutions \(\{\theta_h, u_h\}_{h>0}^\star\).

**Lemma 4.3** (Uniform bounds). Let \((\theta_h, u_h)\) be a numerical solution of the FV method (3.3). Then the following hold
\[
\begin{align*}
\|\theta_h\|_{L^\infty(0,T;L^\infty(\Omega^d))} + \|\theta_h u_h\|_{L^\infty(0,T;L^{2+\gamma}(\Omega^d;\mathbb{R}^d))} + \|p_h\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \\
+ \|u_h\|_{L^2(0,T;L^p(\Omega^d;\mathbb{R}^d))} + \|\nabla_x u_h\|_{L^2(0,T \times \mathbb{T}^d;\mathbb{R}^d \times \mathbb{R}^d)} + \|\text{div}_h u_h\|_{L^2(0,T \times \mathbb{T}^d)} \leq C, \tag{4.3a}
\end{align*}
\]
\[
\int_0^T \int_{\mathcal{E}} \|\psi_h\| \|u_h\|^2 \, dS_x dt + \int_0^T \int_{\mathcal{E}} \left(h^\alpha + \frac{1}{\epsilon} \|\psi_h\| \cdot n\right) \mathcal{P}^{\psi}(\phi_h) \|\phi_h\|^2 \, dS_x dt \leq C, \tag{4.3b}
\]
\[
\frac{1}{\epsilon} \|u_h\|^2_{L^2(0,T;\Omega^d;\mathbb{R}^d)} \leq \frac{1}{\epsilon} \|u_h\|^2_{L^2(0,T \times \mathbb{T}^d;\mathbb{R}^d)} \leq C, \tag{4.3c}
\]
where \(\phi_h \in \text{co}\{\phi_{h_0}^{\psi}, \phi_{h}^{\text{out}}\}\). The parameter \(p \in [1, \infty)\) for \(d = 2\) and \(p = 6\) for \(d = 3\). The generic constant \(C\) depends on the initial mass \(M_0 := \int_{\mathbb{T}^d} \tilde{\phi}_0 \, dx > 0\) and the initial energy \(E_0 := \int_{\mathbb{T}^d} \left(\frac{1}{2} \tilde{\phi}_0 |\tilde{u}_0|^2 + \mathcal{P}(\tilde{\phi}_0)\right) \, dx > 0\), but it is independent of the computational parameters \((h, \Delta t)\) as well as the penalty parameter \(\epsilon\).

### 4.2 Consistency formulation

We proceed with the consistency analysis of the finite volume method (3.3). To begin, let us define two consistency errors:

- the consistency error \(e_\theta\)
  \[
e_\theta(\tau, \Delta t, h, \phi) := \left[ \int_{\mathbb{T}^d} \theta_h \phi \, dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{T}^d} \left(\theta_h \partial_t \phi + \theta_h \phi \cdot \nabla_x \phi\right) \, dx dt \tag{4.4}
\]
  for all \(\phi \in W^{1,\infty}((0, T) \times \mathbb{T}^d)\);

- the consistency error \(e_m\)
  \[
e_m(\tau, \Delta t, h, \varphi) := \left[ \int_{\mathbb{T}^d} \theta_h u_h \cdot \varphi \, dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{T}^d} \left(\theta_h u_h \cdot \partial_t \varphi + \theta_h u_h \otimes u_h : \nabla_x \varphi + p_h \text{div}_x \varphi\right) \, dx dt + \int_0^\tau \int_{\mathbb{T}^d} (\mu \nabla_x u_h : \nabla_x \varphi + \nu \text{div}_x u_h \text{div}_x \varphi) \, dx dt + \frac{1}{\epsilon} \int_0^\tau \int_{\Omega^d} u_h \cdot \varphi \, dx dt \tag{4.5}
\]
  for all \(\varphi \in W^{1,\infty}((0, T) \times \mathbb{T}^d; \mathbb{R}^d)\).
In order to have the consistency formulation, we estimate the consistency errors $e_\varepsilon, e_m$ by using more regular test functions. Since the consistency proof is quite technical and the ideas are analogous to [10, Section 2.7] and [9, Section 11.3], we postpone it to Appendix B. We point out that the result presented in Lemma 4.4 is new. Indeed, the consistency errors are improved since they are more precise and the test function in the continuity equation (4.4) is now more general.

**Lemma 4.4 (Consistency formulation).** Let $(\phi_h, u_h)$ be a solution of the FV scheme (3.3) with $(\Delta t, h, \epsilon) \in (0,1)^3$ and $\alpha > -1$. Then for any $\tau \in [0, T]$ it holds

\begin{equation}
|e_\varepsilon(\tau, \Delta t, h, \phi)| \leq C_\varepsilon(\Delta t + h^{(1+\alpha)/2} + h^{(1+\beta_3)/2} + h^{1+\beta_D});
\end{equation}

for all $\phi \in W^{1,\infty}((0, T) \times \mathbb{T}^d)$, $\partial^2_t \phi \in L^\infty((0, T) \times \mathbb{T}^d)$;

\begin{equation}
|e_m(\tau, \Delta t, h, \varphi)| \leq C_m\left(\sqrt{\Delta t} + h + h^{1+\alpha} + h^{1+\beta_M} + (h/\epsilon)^{1/2} + (\Delta t/\epsilon)^{1/2}\right)
\end{equation}

for all $\varphi \in W^{1,\infty}((0, T) \times \mathbb{T}^d; \mathbb{R}^d)$ \cap $L^\infty(0, T; W^{2,\infty}(\mathbb{T}^d; \mathbb{R}^d))$, $\partial^2_t \varphi \in L^\infty((0, T) \times \mathbb{T}^d; \mathbb{R}^d)$.

Here the constant $C_\varepsilon$ depends on $E_0, T$, $\|\phi\|_{W^{1,\infty}((0, T) \times \mathbb{T}^d)}$, $\|\partial^2_t \phi\|_{L^\infty((0, T) \times \mathbb{T}^d)}$ and $C_m$ depends on $E_0, T$, $\|\varphi\|_{W^{1,\infty}((0, T) \times \mathbb{T}^d; \mathbb{R}^d)}$, $\|\varphi\|_{L^\infty(0, T; W^{2,\infty}(\mathbb{T}^d; \mathbb{R}^d))}$, $\|\partial^2_t \varphi\|_{L^\infty((0, T) \times \mathbb{T}^d; \mathbb{R}^d)}$. Further, $\beta_D, \beta_R, \beta_M$ are defined as

\begin{equation}
\beta_D = \begin{cases}
\min_{\gamma \in [1, \infty)} \left\{ \frac{\gamma(\alpha+4)}{2p} + \frac{1}{2p} \right\} \cdot \frac{\gamma-2}{\gamma} & \text{if } d = 2, \gamma \in (1, 2), \\
\min \left\{ \frac{\alpha+2}{3} + 1, \frac{3(\gamma-2)}{2\gamma} \right\} & \text{if } d = 3, \gamma \in (1, 2), \\
0 & \text{if } \gamma \geq 2,
\end{cases}
\end{equation}

\begin{equation}
\beta_R = \begin{cases}
0 & \text{if } d = 2,
\min \left\{ \frac{1+\alpha}{2} + 1, \frac{5\gamma-6}{2\gamma} \right\} & \text{if } d = 3, \gamma \in \left(1, \frac{6}{5}\right), \\
0 & \text{if } d = 3, \gamma \geq \frac{6}{5},
\end{cases}
\end{equation}

\begin{equation}
\beta_M = \begin{cases}
\max_{\rho \in \left[\frac{2}{\gamma-3}, \infty\right)} \left\{ \frac{-\rho(\alpha+4)}{2p\gamma} + \frac{p(\gamma-2)-2\gamma}{p\gamma} \right\} & \text{if } d = 2, \gamma \leq 2,
0 & \text{if } d = 2, \gamma > 2,
\max \left\{ \frac{-\alpha+2}{2\gamma}, \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma} \right\} & \text{if } d = 3, \gamma \leq 2,
0 & \text{if } d = 3, \gamma \in (2, 3),
\end{cases}
\end{equation}

\begin{equation}
\max \left\{ \frac{-\alpha+2}{2\gamma}, \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma} \right\} & \text{if } d = 3, \gamma \geq 3.
\end{cases}
\end{equation}

**Remark 4.5 (Observations on the parameters $\beta_R, \beta_D, \beta_M$ and $\alpha$).** Our consistency errors (involving the terms $\beta_D, \beta_R, \beta_M$) are better than the results obtained in [9]. Further,

- It is easy to verify that $0 \geq \beta_R \geq \beta_D \geq \beta_M$ and $\beta_D > -1$.

Moreover, $\beta_M > -1$ if one of the following conditions holds

- $d = 2$;
\( d = 3 \) and \( \gamma > \frac{3}{2} \); \\
\( d = 3 \) and \( \gamma \leq \frac{3}{2} \) with \( \alpha < 2(\gamma - 1) \).

Consequently, we obtain a weaker constrain on \( \alpha \) than the one obtained in [9], where \( \alpha < 2\gamma - 1 - d/3 \) was needed for all \( \gamma \in (1, 2) \).

- If \( \alpha \geq 1 \), the parameters \( \beta_D, \beta_R, \beta_M \) are independent of \( \alpha \). Indeed, for \( \alpha \geq 1 \) we have simpler forms of \( \beta_D, \beta_R, \beta_M \), i.e.

\[
\beta_D = \begin{cases} 
\frac{d(\gamma-2)}{2\gamma} & \text{if } \gamma < 2, \\
0 & \text{otherwise},
\end{cases} \quad \beta_R = \begin{cases} 
\frac{2\gamma-6}{2\gamma} & \text{if } d = 3, \gamma < \frac{6}{5}, \\
0 & \text{otherwise},
\end{cases} \\
\beta_M = \begin{cases} 
\max_{p \in \left[ \frac{2\gamma}{\gamma-1}, \infty \right]} \frac{p(\gamma-2)-2\gamma}{p^2} & \text{if } d = 2, \gamma \leq 2, \\
0 & \text{if } d = 2, \gamma > 2, \\
\max \left\{ \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma} \right\} & \text{if } d = 3, \gamma < 3, \\
0 & \text{if } d = 3, \gamma \geq 3.
\end{cases}
\]

### 4.3 Weak convergence for the penalized problem

In this section we consider \( \epsilon \) fixed and pass to the limit with \( \Delta t \approx h \rightarrow 0 \). The corresponding (weak) limit of \((\varrho_h, u_h)\) will be denoted by \((\varrho, u)\). First, we deduce from a priori estimates (4.3a) that up to a subsequence

\[
\varrho_h \rightarrow \varrho, \text{ weakly-(*) in } L^\infty(0, T; L^\gamma(T^d)), \quad \varrho \geq 0, \\
u_h \rightarrow u, \text{ weakly in } L^2(0, T; L^6(T^d; \mathbb{R}^d)), \\
\nabla_\epsilon u_h \rightarrow \nabla x u, \text{ weakly in } L^2((0, T) \times T^d; \mathbb{R}^d \times d), \quad \text{where } u \in L^2(0, T; W^{1,2}(T^d; \mathbb{R}^d))
\]

and

\[
\varrho_h u_h \rightarrow m, \text{ weakly-(*) in } L^\infty(0, T; L^\frac{2\gamma}{\gamma+1}(T^d; \mathbb{R}^d)) \quad \text{for } h \rightarrow 0.
\]

Realizing that \((\varrho_h, u_h)\) satisfies the consistency formulation (4.4) for the mass conservation equation, applying [1, Lemma 3.7] (see similar result in [18, Lemma 7.1]) we obtain

\[
m = \varrho u.
\]

Further, due to the fact that the total energy \( E = \frac{1}{2} \varrho |u|^2 + \mathcal{P}(\varrho) \) is a convex function of \((\varrho, m)\) and \( |\nabla x u|^2 + |\text{div} x u|^2 = |\nabla x u|^2 + |\text{tr}(\nabla x u)|^2 \) is a convex function of \( \nabla x u \), we deduce that, cf. [5, Lemma 2.7]

\[
\frac{1}{2} \varrho_h |u_h|^2 + \mathcal{P}(\varrho_h) \rightarrow \frac{m^2}{2\varrho} + \mathcal{P}(\varrho) \text{ weakly-(*) in } L^\infty(0, T; \mathcal{M}^+(T^d)), \\
\varrho_h u_h \otimes u_h + p(\varrho_h) I \rightarrow \frac{m \otimes m}{\varrho} + p(\varrho) I \text{ weakly-(*) in } L^\infty(0, T; \mathcal{M}^+(\mathbb{R}^d; \mathbb{R}^d_{\text{sym}})),
\]
\[ \mu |\nabla \varphi|^2 + \nu \text{div}_h \varphi|^2 \rightarrow \mu |\nabla x \varphi|^2 + \nu \text{div}_x \varphi|^2 \]

weakly-(*) in \( \mathcal{M}^+([0,T] \times \mathbb{T}^d) \) for \( h \to 0 \)

with the defects

\[
\mathcal{E} = \frac{m^2}{2h} + \mathcal{P}(\varphi) - \left( \frac{1}{2} \varphi \left| \nabla \varphi \right|^2 + \mathcal{P}(\varphi) \right) \geq 0,
\]

\[
\mathcal{R} = \frac{m \otimes m}{\varphi} + p(\varphi I - (\varphi \otimes \varphi \otimes \varphi) + p(\varphi)I) \geq 0,
\]

\[
\mathcal{O} = \mu |\nabla \varphi|^2 + \nu |\text{div}_h \varphi|^2 - (\mu |\nabla x \varphi|^2 + \nu |\text{div}_x \varphi|^2) \geq 0
\]

satisfying

\[ d\mathcal{E} \leq \text{tr}[\mathcal{R}] \leq d\mathcal{E}, \quad d = \min (2, d(\gamma - 1)), \quad \overline{d} = \max (2, d(\gamma - 1)) . \]

Together with

\[
\lim_{h \to 0} \int_{\mathbb{T}^d} E(\varphi_h^0, m_h^0) \, dx = \int_{\mathbb{T}^d} E(\varphi_0, m_0) \, dx,
\]

the consistency formulations (4.4) and (4.5), and the energy balance (4.2), the limit \((\varphi_h, u_h)\) is a DW solution of the penalized problem (1.4) in the sense of Definition 2.1. We summarize the obtained result on the weak convergence of FV solutions in the following theorem.

**Theorem 4.6. (Weak convergence for the penalized problem).** Let \( p \) satisfy (1.2) with \( \gamma > 1 \) and \( \epsilon > 0 \) be a fixed penalty parameter. Let \( \{\varphi_h, u_h\}_{h>0} \) be a family of numerical solutions obtained by the FV method (3.3) with \( \Delta t \approx h \in (0,1), \alpha > -1 \) and initial data satisfying (1.3). If \( d = 3 \) and \( \gamma \leq \frac{3}{2} \), we assume in addition that \( \alpha < 2(\gamma - 1) \).

Then, up to a subsequence, the FV solutions \( \{\varphi_h, u_h\}_{h>0} \) converge in the following sense

\[
\begin{align*}
\varphi_h & \to \varphi \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\mathbb{T}^d)), \\
u_h & \to u \text{ weakly in } L^2((0,T) \times \mathbb{T}^d; \mathbb{R}^d), \\
\nabla \nu u_h & \to \nabla x u \text{ weakly in } L^2((0,T) \times \mathbb{T}^d; \mathbb{R}^{d \times d}), \\
\text{div}_h u_h & \to \text{div}_x u \text{ weakly in } L^2((0,T) \times \mathbb{T}^d) \text{ for } h \to 0,
\end{align*}
\]

(4.8)

where \((\varphi, u)\) is a DW solution of the penalized problem (1.4) in the sense of Definition 2.1.

### 4.4 Weak convergence for the Dirichlet problem

We proceed by considering the limit process for \( \Delta t \approx h \to 0 \) and \( \epsilon \to 0 \). The corresponding (weak) limit on \( \mathbb{T}^d \) will be denoted by \((\bar{\varphi}, \bar{u})\). Moreover, we set \((\varphi, u) := (\bar{\varphi}, \bar{u})|_{\Omega^f} \). Analogously to Section 4.3 we have the following convergence results:

\[
\begin{align*}
\varphi_h & \to \varphi \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\mathbb{T}^d)), \quad \varphi \geq 0, \\
u_h & \to u \text{ weakly in } L^2((0,T) \times \mathbb{T}^d; \mathbb{R}^d), \\
\nabla \nu u_h & \to \nabla x u \text{ weakly in } L^2((0,T) \times \mathbb{T}^d; \mathbb{R}^{d \times d}), \quad \text{where } \bar{u} \in L^2(0,T; W^{1,2}(\mathbb{T}^d; \mathbb{R}^d)), \\
\varphi_h u_h & \to \bar{m} = \bar{\varphi} \bar{u} \text{ weakly-(*) in } L^\infty(0,T; L^\frac{2d}{d+\gamma}(\mathbb{T}^d; \mathbb{R}^d)) \text{ for } h \to 0, \epsilon \to 0.
\end{align*}
\]
Further, according to a priori bound (4.3c) we obtain that
\[ u_h \to 0 \text{ strongly in } L^2((0,T) \times \Omega^s; \mathbb{R}^d) \text{ for } h \to 0, \epsilon \to 0 \]
yielding
\[ u \in L^2(0,T; W^{1,2}_0(\Omega^f; \mathbb{R}^d)). \]
Together with the fact that \( \tilde{\varrho} \) satisfies the equation of continuity in the weak sense, we have
\[ \partial_t \tilde{\varrho} = 0 \text{ in } D'((0,T) \times \Omega^s). \]
Since \( \tilde{\varrho} \in C_{weak}(0,T; L^7(T^d)) \), cf. [12, Remark 1 of Section 3.3], taking any test function \( \phi \in C(\Omega^s) \) directly gives
\[ \int_{\Omega^s} \tilde{\varrho}(t,\cdot) \phi \, dx = \int_{\Omega^s} \tilde{\varrho}_0 \phi \, dx \text{ for all } t \in [0,T], \]
which means
\[ \tilde{\varrho} = \tilde{\varrho}_0 \text{ in } \Omega^s \text{ for all } t \in [0,T]. \] (4.9)
Consequently, we have
\[
\lim_{(h,\epsilon) \to 0} \int_{\Omega^s} \left( \frac{1}{2} \frac{|m_0|^2}{\epsilon_h^2} + \mathcal{P}(\varrho_h^0) \right) \, dx = \int_{\Omega^f} \left( \frac{1}{2} \frac{|m_0|^2}{\varrho_0} + \mathcal{P}(\varrho_0) \right) \, dx + \int_{\Omega^s} \mathcal{P}(\varrho_0) \, dx \quad (4.10)
\]
and
\[
\lim_{(h,\epsilon) \to 0} \int_{\Omega^s} \left( \frac{1}{2} \varrho_h |u_h|^2 + \mathcal{P}(\varrho_h) \right) \, dx = \int_{\Omega^f} \mathfrak{d}(\tau) + \int_{\Omega^f} \left( \frac{1}{2} \varrho |u|^2 + \mathcal{P}(\varrho) \right) \, dx,
\]
\[
\lim_{(h,\epsilon) \to 0} \int_{\Omega^f} \frac{1}{2} \varrho_h |u_h|^2 \, dx \geq 0, \quad \lim_{(h,\epsilon) \to 0} \int_{\Omega^s} \mathcal{P}(\varrho_h) \, dx \geq \int_{\Omega^s} \mathcal{P}(\varrho_0) \, dx, \quad (4.11)
\]
which yields the energy inequality (2.7).

Together with the consistency formulations (4.4), (4.5) and the energy balance (4.2), the limit \((\varrho, u)\) is a DW solution of the Navier–Stokes system (1.1) with the Dirichlet boundary conditions in the sense of Definition 2.2.

Theorem 4.7. (Weak convergence for the Dirichlet problem). In addition to the assumption of Theorem 4.6, let \( h, \epsilon \) satisfy
\[ h^3/\epsilon \to 0, \quad \text{as } h, \epsilon \to 0. \] (4.12)
Then, up to a subsequence, the FV solutions \( \{\varrho_h, u_h\}_{h,\epsilon \to 0} \) converge in the following sense
\[ \varrho_h \rightharpoonup \varrho \text{ weakly-(*) in } L^\infty(0,T; L^7(\Omega^f)), \]
\[ u_h \rightharpoonup u \text{ weakly in } L^2(0,T; L^2(\Omega^f; \mathbb{R}^d)), \]
\[ \nabla \mathfrak{e} u_h \rightharpoonup \nabla \mathfrak{e} u \text{ weakly in } L^2(0,T \times \Omega^f; \mathbb{R}^{d \times d}), \quad \text{where } u \in L^2(0,T; W^{1,2}_0(\Omega^f; \mathbb{R}^d)), \]
\[ \text{div}_h u_h \rightharpoonup \text{div}_x u \text{ weakly in } L^2((0,T) \times \Omega^f) \quad \text{for } h \to 0, \epsilon \to 0 \] (4.13)
where \((\varrho, u)\) is a DW solution of the Dirichlet problem of Navier–Stokes system (1.1) in the sense of Definition 2.2.
Proof. Compared to Theorem 4.6 we additionally require (4.12) in order to get a better control of the consistency error term $E_\epsilon$ stated in (B.1). With the test function $\phi \in C^2_c([0, T) \times \Omega^f)$ the estimate of $E_\epsilon$ can be improved with
\[
|E_\epsilon| = \left| \frac{1}{\epsilon} \int_0^{t_{n+1}} \int_{\Omega_h^c \setminus \Omega^s} u_h : \phi \, dx \, dt \right| \lesssim h^{1/2} \epsilon^{-1/2} \|\phi\|_{L^\infty((0, T) \times \Omega^c \setminus \Omega^s)} \lesssim h^{3/2} \epsilon^{-1/2}
\]
resulting $\epsilon_m \to 0$. This concludes the proof. \hfill \Box

4.5 Strong convergence for the Dirichlet problem

In this section we study the strong convergence of the numerical solutions \{\(\varrho_h, u_h\}\}_{h, \epsilon \searrow 0}. To begin, let us introduce the definition of the strong solution of the Navier–Stokes system (1.1). For the local existence, we refer a reader to Valli and Zajaczkowski [26], and Kawashima and Shizuta [19].

Definition 4.8 (Strong solution). Let $\Omega^f \subset \mathbb{R}^d, d = 2, 3$, be a bounded domain with a smooth boundary $\partial \Omega^f$. We say that $(\varrho, u)$ is the strong solution of the Navier–Stokes problem (1.1) if
\[
\varrho \in C^1([0, T] \times \overline{\Omega^f}) \cap C(0, T; W^{4,2}(\Omega^f)),
\]
\[
u \in C^1([0, T] \times \overline{\Omega^f}; \mathbb{R}^d) \cap C(0, T; W^{4,2}(\Omega^f; \mathbb{R}^d))
\]
and equations (1.1) are satisfied pointwise.

Theorem 4.9. (Strong convergence for the Dirichlet problem). Let the initial data $(\varrho_0, u_0)$ satisfy
\[
\varrho_0 \in W^{4,2}(\Omega^f), \quad \varrho_0 > 0, \quad u_0 \in W^{4,2}(\Omega^f; \mathbb{R}^d)
\]
and $(\varrho, u)$ be the corresponding strong solution of the Navier–Stokes system (1.1) belonging to the class (4.14). Let \{\(\varrho_h, u_h\)\}_{h, \epsilon \searrow 0} be a family of numerical solutions obtained by the FV method (3.3). Moreover, we assume that the parameters $\alpha, \Delta t, h, \epsilon$ satisfy the same conditions as in Theorem 4.7.

Then the FV solutions \{\(\varrho_h, u_h\)\}_{h, \epsilon \searrow 0} converge strongly to the strong solution $(\varrho, u)$ in the following sense
\[
\varrho_h \to \varrho \text{ strongly in } L^r((0, T); L^7(\Omega^f)),
\]
\[
u_h \to \nu \text{ strongly in } L^2((0, T) \times \Omega^f; \mathbb{R}^d) \text{ for } h \to 0, \epsilon \to 0
\]
for any $1 \leq r < \infty$.

Proof. By virtue of the weak-strong uniqueness principle established in [5, Theorem 4.1], analogously as in [9, Theorem 7.12], we obtain that the numerical solutions \{\(\varrho_h, u_h\)\}_{h, \epsilon \searrow 0} converge strongly to the strong solution $(\varrho, u)$ on $\Omega^f$. \hfill \Box
5  Error estimates

Having proven the convergence of the FV method (3.3), we proceed with the error analysis between the FV approximation of the penalized problem (1.4) and the strong solution to the Dirichlet problem (1.1). For simplicity of the presentation of main ideas, here and hereafter we consider a semi-discrete version of the FV method. In other words, we only study the error with respect to the spatial discretization.

In order to measure the distance between a FV solution of the penalized problem (1.4) and the strong solution to the Navier–Stokes problem (1.1), we introduce the relative energy functional

\[ R_E(\varrho_h, u_h; \bar{\varrho}, \bar{u}) = \int_{\Omega} \left( \frac{1}{2} \varrho_h |u_h - \bar{u}|^2 + E(\varrho_h; \bar{\varrho}) \right) \, dx, \quad E(\varrho_h; \bar{\varrho}) = P(\varrho_h) - P'(\bar{\varrho})(\varrho_h - \bar{\varrho}) - \mathcal{P}(\bar{\varrho}). \]  

Remark 5.3. Further, there hold

Theorem 5.2. (Error estimates). Let \( \gamma > 0, (\varrho_0, u_0) \in W^{4,2}(\Omega) \times W^{4,2}(\Omega; \mathbb{R}^d), \varrho_0 \in C^2(\Omega^*) \) and \( \bar{\varrho} > 0 \). Let \( \{\varrho_h, u_h\}_{h, \epsilon} \) be a family of numerical solutions obtained by the semi-discrete version of FV method (3.3) with \( (h, \epsilon) \in (0,1)^2 \) and \( \alpha > 0 \). Suppose that the numerical density \( \varrho_h \) is uniformly bounded

\[ 0 < \varrho_h < \overline{\varrho} \quad \text{uniformly for} \quad h, \epsilon \to 0 \]  

and the Navier–Stokes system (1.1) admits a global-in-time strong solution \((\varrho, u)\) belonging to the class (4.14).

Then the following error estimate holds

\[ R_E(\varrho_h, u_h; \bar{\varrho}, \bar{u})(\tau) + \int_0^\tau \int_{\Omega} \left( \mu |\nabla \cdot u_h - \nabla \cdot \tilde{u}|^2 + \nu |\nabla u_h - \nabla \tilde{u}|^2 \right) \, dx \, dt + \frac{1}{\epsilon} \int_0^\tau \int_{\Omega^*} |u_h|^2 \, dx \, dt \]

\[ \lesssim h^{\beta_{RE}} + \frac{\epsilon^3}{\epsilon} + \frac{\epsilon}{h}, \quad \beta_{RE} = \min \{1, (1 + \alpha)/2, \alpha\}. \]  

Further, there hold

\[ \| (\bar{\varrho}, \bar{u}, \tilde{\varrho}, \tilde{u}) - (\varrho_h, u_h, \varrho_h u_h) \|_{L^2((0,T) \times \Omega^*; \mathbb{R}^{2d+1})} + \| \nabla \varrho_h - \nabla \varrho_h \bar{u} \|_{L^2((0,T) \times \Omega^*)} \lesssim h^{\beta_{RE}} + \frac{\epsilon^3}{\epsilon} + \frac{\epsilon}{h} \]  

and

\[ \| \varrho_h - \varrho \|_{L^2(\Omega^*)} \lesssim h^{\beta_{RE}} + \frac{\epsilon^3}{\epsilon} + \frac{\epsilon}{h} \quad \text{if} \quad \gamma > 2. \]  

Remark 5.3. The error estimates in Theorem 5.2 confirm the strong convergence of the penalty method (3.3). By a closer inspection we observe the optimal first order convergence rate for the relative energy, i.e. \( h^{\beta_{RE}} + \frac{\epsilon^3}{\epsilon} + \frac{\epsilon}{h} \approx h \), by choosing \( \epsilon = h^2 \) and \( \alpha = 1 \).
Remark 5.4. We point out that the above error estimates can be generalized to the fully discrete method \((3.3)\) by the same argument as in \([21, \text{Theorem 6.2 or Appendix D}]\).

On the other hand, the error estimates may be proven without assuming the upper bound on density \((5.3)\). However, we would obtain worse convergence rate and need some constrain on \(\gamma\). We leave the details to interested readers.

Proof of Theorem 5.2. The estimates \((5.5)\) and \((5.6)\) directly follow from Lemmas A.3 and D.2 once the estimates \((5.4)\) is proven. Hence, the key of the proof is to show \((5.4)\). In what follows we show the main idea of the proof leaving the technical details to Appendix D.

In order to show \((5.4)\), we study the relative energy balance by combining the energy estimate and the consistency formulations with suitable test functions. More precisely, we collect the energy estimate \((4.2)\), the density consistency formulation \((D.1)\) with the test function \(\phi = 1/2 |\vec{u}|^2 - \mathcal{P}(\vec{v})\) and the momentum consistency formulation \((4.5)\) with the test function \(\varphi = -\vec{u}\). It yields

\[
[R_E(q_h, u_h|\vec{v}, \vec{u})]_{t=0}^{t=T} + \int_0^T \int_{\Omega_h} \left( \mu |\nabla_x u_h - \nabla_x \vec{u}|^2 + \nu |\text{div}_h u_h - \text{div}_x \vec{u}|^2 \right) \, dx \, dt \\
+ \frac{1}{\epsilon} \int_0^T \int_{\Omega_h} |u_h|^2 \, dx \, dt = - \int_0^T D_{\text{num}} \, dt + e_S + e_R.
\]

see Appendix D.1 for details. Here, \(D_{\text{num}} \geq 0\) given in Lemma 4.2 represents the numerical dissipations. The consistency errors \(e_S\) and residual errors \(e_R\) are stated in \((D.3)\). They satisfy the following estimate

\[
|h_S| + |e_R| \lesssim h^{3+RE} + \int_0^T R_E(q_h, u_h|\vec{v}, \vec{u}) \, dt + \frac{\delta}{\epsilon} \|u_h\|_{L^2(0,T) \times \Omega_h}^2 + \delta \nu \|\text{div}_h u_h - \text{div}_x \vec{u}\|_{L^2(0,T) \times \Omega_h}^2.
\]

This inequality is obtained by splitting the grid into interior, exterior, and close to boundary parts, since \(\vec{v}, \nabla_x \vec{u}\) may lose regularity across the boundary. Then delicate estimates of the consistency and residual errors by means of the terms on the left hand side of \((5.7)\) yield the desired result, see Appendix D.2 (Lemma D.1) and Appendix D.3 (Lemma D.3) for details.

With the above estimate in hand, we choose any fixed \(\delta \in (0,1)\) to obtain the following relative energy inequality

\[
[R_E(q_h, u_h|\vec{v}, \vec{u})]_{t=0}^{t=T} + \int_0^T \int_{\Omega_h} \left( \mu |\nabla_x u_h - \nabla_x \vec{u}|^2 + \nu |\text{div}_h u_h - \text{div}_x \vec{u}|^2 \right) \, dx \, dt \\
+ \frac{1}{\epsilon} \int_0^T \int_{\Omega_h} |u_h|^2 \, dx \, dt \lesssim \int_0^T R_E(q_h, u_h|\vec{v}, \vec{u}) \, dt + h^{3+RE} + \frac{\epsilon}{h}.
\]

Finally, by Gronwall’s lemma and the continuity of the initial data, we obtain from the above inequality that

\[
R_E(q_h, u_h|\vec{v}, \vec{u})(\tau) + \int_0^\tau \int_{\Omega_h} \left( \mu |\nabla_x u_h - \nabla_x \vec{u}|^2 + \nu |\text{div}_h u_h - \text{div}_x \vec{u}|^2 \right) \, dx \, dt + \frac{1}{\epsilon} \int_0^\tau \int_{\Omega_h} |u_h|^2 \, dx \, dt \\
\lesssim \frac{h^3}{\epsilon} + \frac{\epsilon}{h} + h^{3+RE} + R_E(q_h^0, u_h^0|\Pi_T \vec{v}_0, \Pi_T \vec{u}_0) \lesssim \frac{h^3}{\epsilon} + \frac{\epsilon}{h} + h^{3+RE}.
\]

This concludes the proof.
6 Numerical experiments

In this section our aim is to validate the obtained theoretical convergence results. To this end, we compute the following errors. Firstly, the errors with respect to the discretization parameter \( h \), i.e. we fix a penalty parameter \( \epsilon \) and use a reference solution with a small \( h_{ref} \):

\[
E'_v = \| \varrho_h - \varrho_{h,ref} \|_{L^2(T^2)}, \quad E'_u = \| u_h - u_{h,ref} \|_{L^2(T^2)}, \quad E_{\nabla u} = \| \nabla \varphi_h - \nabla \varphi_{h,ref} \|_{L^2(T^2)}, \quad R_E = R_E \left( \varrho_h, u_h \mid \varrho_{h,ref}, u_{h,ref} \right).
\]

Secondly, we consider the errors with respect to two parameters \( h, \epsilon \). Thus, the reference solution has a fixed parameter pair \((h_{ref}, \epsilon_{ref})\):

\[
E'_v = \| \varrho_h - \varrho_{h,ref} \|_{L^2(T^2)}, \quad E'_u = \| u_h - u_{h,ref} \|_{L^2(T^2)}, \quad E_{\nabla u} = \| \nabla \varphi_h - \nabla \varphi_{h,ref} \|_{L^2(T^2)}, \quad R_E = R_E \left( \varrho_h, u_h \mid \varrho_{h,ref}, u_{h,ref} \right).
\]

In the simulation we take the following parameters

\[ \alpha = 0.6, \ T = 0.1, \ \mu = 0.1, \ \nu = 0, \ a = 1, \ \gamma = 1.4. \]

We recall that \( E'_v, E'_u, E_{\nabla u}, R_E \) are used to verify the convergence rate with respect to mesh parameter \( h \), cf. Theorem 4.6. Errors \( E'_v, E'_u, E_{\nabla u}, R_E \) (with respect to the parameter pair \((h, \epsilon(h))\)) are used to illustrate our convergence results in Theorem 4.7, Theorem 4.9 and Theorem 5.2.

6.1 Experiment 1: Ring domain - continuous extension

In this experiment we take the physical fluid domain to be a ring, i.e. \( \Omega^f \equiv B_{0.7} \setminus B_{0.2} \), where \( B_r = \{ x \mid |x| < r \} \). The initial data (including the smooth extension) are given by

\[
(\varrho, u)(0, x) = \begin{cases} 
(1, 0, 0), & x \in B_{0.2}, \\
\left(1, \frac{\sin(4\pi(|x|-0.2)x_2}{|x|}, -\frac{\sin(4\pi(|x|-0.2)x_1)}{|x|}\right), & x \in \Omega^f \equiv B_{0.7} \setminus B_{0.2}, \\
(1, 0, 0), & x \in T^2 \setminus B_{0.7}.
\end{cases}
\]

Figure 3 shows the numerical solutions \( \varrho_h \) and \( u_h \) at time \( T = 0.1 \) with fixed mesh size \( h = 0.2 \cdot 2^{-4} \) and various penalty parameter \( \epsilon = 4^{-3}, \ldots, 4^{-6} \). We can observe that the velocity vanishes in the penalized region with decreasing \( \epsilon \). Further, in Figure 4 we present the errors \( E'_v, E'_u, E_{\nabla u}, R_E \) with respect to \( h = 0.2 \cdot 2^{-m}, m = 0, \ldots, 3 \) for fixed \( \epsilon \in \{4^{-2}, 4^{-3}, 4^{-4}, 4^{-5}, 4^{-6}\} \) and \( h_{ref} = 0.2 \cdot 2^{-4} \). Figure 5 depicts the errors \( E'_v, E'_u, E_{\nabla u}, R_E \) with respect to the three parameter pairs \((h, \epsilon(h)) = (h, O(h^{1/2})), (h, O(h^2))\) and \((h, O(h^4))\) given by

\[
(h, \epsilon(h)) = \begin{cases} 
(0.2 \cdot 2^{-m}, 2^{-(m+1)/2}), & m = 0, \ldots, 3 \text{ with } h_{ref} = 0.2 \cdot 2^{-4}, \ \epsilon_{ref} = 2^{-9}; \\
(0.2 \cdot 2^{-m}, 4^{-(m+2)}), & m = 0, \ldots, 3 \text{ with } h_{ref} = 0.2 \cdot 2^{-4}, \ \epsilon_{ref} = 4^{-6}; \\
(0.2 \cdot 2^{-m}, 16^{-m}), & m = 0, \ldots, 3 \text{ with } h_{ref} = 0.2 \cdot 2^{-4}, \ \epsilon_{ref} = 16^{-4}.
\end{cases}
\]

Our numerical results indicate first order convergence rate for \( \varrho, u, \nabla_x u \) and second order convergence rate for \( R_E \). Note that the experimental convergence rates are better than our theoretical result.
Figure 3: Experiment 1: Numerical solutions $\varrho_h$ (left) and $u_h$ (right) obtained with $h = 0.2 \cdot 2^{-4}$ for different $\epsilon = 4^{-m-2}, m = 1, \ldots, 4$ from top to bottom.
Figure 4: Experiment 1: The errors $E_{\epsilon}^\varnothing$, $E_{\epsilon}^u$, $E_{\epsilon}^{\nabla x u}$, $R_E^\epsilon$ with respect to $h$ for different but fixed $\epsilon$. The black and red solid lines without any marker denote the reference slope of $h$ and $h^2$, respectively.

Figure 5: Experiment 1: Errors $E_{\epsilon}^\varnothing$, $E_{\epsilon}^u$, $E_{\epsilon}^{\nabla x u}$ and relative energy $R_E$ with respect to the pairs $(h, \epsilon(h)) = (0.2 \cdot 2^{-m}, 2^{-(m+14)/2})$ (left), $(0.2 \cdot 2^{-m}, 4^{-(m+2)})$ (middle) and $(0.2 \cdot 2^{-m}, 16^{-m})$ (right), $m = 0, 1, 2, 3$. The black solid and red dashed lines without any marker denote the reference slope of $h$ and $h^2$, respectively.
6.2 Experiment 2: Ring domain - discontinuous extension

In the second experiment we consider the same physical fluid domain, but different initial extension of density, i.e.

\[(\varrho, \mathbf{u})(0, x) = \begin{cases} 
(0.01, 0, 0), \\
\left(1, \sin(4\pi(|x| - 0.2)) \frac{x_2}{|x|} - \sin(4\pi(|x| - 0.2)) \frac{x_1}{|x|} \right), \\
(2, 0, 0), \\
\end{cases} \quad x \in B_{0.2},
\]
\[\begin{cases} 
\Omega^f = B_{0.7} \setminus \overline{B_{0.2}}, \\
x \in \Omega^f \equiv B_{0.7} \setminus \overline{B_{0.2}},
\end{cases}
\]
\[x \in \mathbb{T}^2 \setminus B_{0.7}.\]

The effect of different penalty parameters, \(\epsilon = 4^{-3}, \ldots, 4^{-6}\), is present in Figure 6. The errors \(E^\epsilon, E_u, E_{\nabla x u}, R_E\) with respect to \(h\) for fixed penalty parameters are shown in Figure 7. Figure 8 presents the errors \(E^\epsilon, E_u, E_{\nabla x u}, R_E\) with respect to the pair \((h, \epsilon(h)) = (h, \mathcal{O}(h^{1/2})), (h, \mathcal{O}(h^2))\) and \((h, \mathcal{O}(h^4))\). Figures 7 and 8 indicate similar convergence behaviours as in Experiment 1.

6.3 Experiment 3: Complex domain - discontinuous extension

In the last experiment, we consider a more complicated geometry of the fluid domain, i.e.

\[\Omega^f = \hat{B}_{0.7} \setminus \overline{B_{0.2}}, \quad \hat{B}_{0.7} := \left\{ x \mid |x| < (0.7 + \delta) + \delta \cos(8\phi), \phi = \arctan \frac{x_1}{x_2} \right\}.\]

The initial data (including a discontinuous extension) are given by

\[(\varrho, \mathbf{u})(0, x) = \begin{cases} 
(0.01, 0, 0), \\
\left(1, \frac{1-\cos(8\pi(|x| - 0.2))}{|x|} \frac{x_2}{|x|} - \frac{1-\cos(8\pi(|x| - 0.2))}{|x|} \frac{x_1}{|x|} \right), \\
(1, 0, 0), \\
(0.01, 0, 0), \\
\end{cases} \quad x \in B_{0.2},
\]
\[\begin{cases} 
\hat{B}_{0.7} \setminus \overline{B_{0.45}}, \\
x \in B_{0.45} \setminus \overline{B_{0.2}},
\end{cases}
\]
\[x \in B_{0.7} \setminus \overline{B_{0.45}},
\]
\[x \in \hat{B}_{0.7} \setminus \overline{B_{0.7}}, \quad x \in \mathbb{T}^2 \setminus \overline{B_{0.7}}.\]

In the simulation we set \(\delta = 0.05\) and the final time to \(T = 0.1\). The numerical solutions \(\varrho_h\) and \(\mathbf{u}_h\) at time \(T = 0.1\) for a fixed mesh size \(h = 0.2 \cdot 2^{-4}\) and various penalty parameters \(\epsilon = 4^{-m-2}, m = 1, \ldots, 4\), are presented in Figure 9. Figure 10 shows the errors with respect to \(h\) for a fixed \(\epsilon\). The errors with respect to both parameters \((h, \epsilon(h))\) are displayed in Figure 11. Analogously as above, the numerical results indicate the first order convergence rate for the numerical solutions \(\varrho, \mathbf{u}, \nabla x u\) and the second order convergence rate for the relative energy \(R_E\).

Let us point out that the initial data (including the extension) in Experiments 2 and 3 belong to the class \(L^\infty(\mathbb{T}^d)\), which is consistent with Theorem 5.2. Our results obtained in Theorem 5.2 requires \(\epsilon \in (\mathcal{O}(h^3), \mathcal{O}(h))\). However, the numerical results presented in Figures 5, 8, 11 indicate that the convergence rates hold for a more general setting, e.g., \((h, \epsilon(h)) = (h, \mathcal{O}(h^{1/2})), (h, \mathcal{O}(h^4))\).

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Figure 6: Experiment 2: Numerical solutions $\varrho_h$ (left) and $u_h$ (right) obtained with $h = 0.2 \cdot 2^{-k}$ for different $\epsilon = 4^{-m-2}, m = 1, \ldots, 4$, from top to bottom.
Figure 7: Experiment 2: The errors $E_\epsilon$, $E_{\epsilon u}$, $E_{\nabla \times u}$, $R_E$ with respect to $h$ for different but fixed $\epsilon$. The black and red solid lines without any marker denote the reference slope of $h$ and $h^2$, respectively.

Figure 8: Experiment 2: Errors $E_{\rho}$, $E_{\mathbf{u}}$, $E_{\nabla \times \mathbf{u}}$ and relative energy $R_E$ with respect to the pairs $(h, \epsilon(h)) = \left((0.2 \cdot 2^{-m}, 2^{-(m+14)/2}\right)$ (left), $(0.2 \cdot 2^{-m}, 4^{-(m+2)})$ (middle) and $(0.2 \cdot 2^{-m}, 16^{-m})$ (right), $m = 0, 1, 2, 3$. The black solid and red dashed lines without any marker denote the reference slope of $h$ and $h^2$, respectively.
Figure 9: Experiment 3: Numerical solutions $\rho_h$ (left) and $u_h$ (right) obtained with $h = 0.2 \cdot 2^{-4}$ for different $\epsilon = 4^{-m-2}, m = 1, \ldots, 4$, from top to bottom.
Figure 10: Experiment 3: The errors $E_\psi^\epsilon$, $E_u^\epsilon$, $E_{\nabla \times u}^\epsilon$, $R_E^\epsilon$ with respect to $h$ for different but fixed $\epsilon$. The black and red solid lines without any marker denote the reference slope of $h$ and $h^2$, respectively.

Figure 11: Experiment 3: Errors $E_\psi^\epsilon, E_u^\epsilon, E_{\nabla \times u}^\epsilon$ and relative energy $R_E^\epsilon$ with respect to the pairs $(h, \epsilon(h)) = \left(0.2 \cdot 2^{-m}, 2^{-(m+14)/2}\right)$ (left), $(0.2 \cdot 2^{-m}, 4^{-(m+2)})$ (middle) and $(0.2 \cdot 2^{-m}, 16^{-m})$ (right), $m = 0, 1, 2, 3$. The black solid and red dashed lines without any marker denote the reference slope of $h$ and $h^2$, respectively.
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A Preliminaries

In this section we list three useful results that are important to derive our theoretical results. First, we recall the generalized Sobolev-Poincaré inequality, see [9, Theorem 17].

**Lemma A.1** ([9, Theorem 17]). Let $q_h > 0$ satisfy

$$0 < c_M \leq \int_{\Omega} q_h \, dx \quad \text{and} \quad \int_{\Omega} q_h^2 \, dx \leq c_E,$$

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where $\gamma > 1$, $c_M$ and $c_E$ are positive constants. Then there exists $c = c(c_M, c_E, \gamma)$ independent of $h$ such that

$$
\|f_h\|_{L^q(T^d)}^2 \leq c \left( \|\nabla E f_h\|_{L^2(T^d)}^2 + \int_{T^d} |\rho_h| |f_h|^2 \, dx \right),
$$

where $q = 6$ if $d = 3$, and $q \in [1, \infty)$ if $d = 2$.

Next we recall a slightly reformulated lemma from [10, Lemma B.4].

**Lemma A.2** ([10, Lemma B.4]). Let $\gamma > 1$ and $\rho_h > 0, \rho_h \in L^\gamma(T^d), u_h \in L^2(T^d)$. Let $0 < \rho < \rho \leq \bar{\rho}, \bar{\rho} \in L^2(T^d)$. Then for any constant $\delta \in (0, 1)$ it holds that

$$
\int_0^t \int_{\Omega} |(\rho_h - \rho)(u_h - \bar{u})| \, dx \, dt \lesssim \delta \|u_h - \bar{u}\|_{L^2(T^d)}^2 + RE(\rho_h, u_h|\rho, \bar{u}).
$$

Further, we recall [10, Lemma C.1] for the following estimates, which are used to derive $L^p$-error estimates, cf. Theorem 5.2.

**Lemma A.3** ([10, Lemma C.1]). Let $\gamma > 1, \rho \in [\rho, \bar{\rho}], |\bar{u}| \leq \bar{u}, \rho, \bar{\rho}, \bar{u} > 0$. Moreover, let $0 < \rho_h < \rho$ uniformly for $h \to 0$. Then the following estimates hold

$$
\|\rho_h - \rho\|_{L^2(T^d)}^2 + \|\rho_h u_h - \rho \bar{u}\|_{L^2(T^d)}^2 \lesssim RE(\rho_h, u_h|\rho, \bar{u})
$$

and

$$
\|\rho_h - \rho\|_{L^\gamma(T^d)}^\gamma \lesssim RE(\rho_h, u_h|\rho, \bar{u}) \quad \text{if} \ \gamma > 2.
$$

**B Proof of consistency**

In this section our aim is to estimate the consistency errors $e_\rho$ and $e_m$ stated in Lemma 4.4. To begin, let us reformulate $e_\rho$ and $e_m$.

**Lemma B.1.** Let $(\rho_h, u_h)$ be a solution of the FV scheme (3.3) with $(\Delta t, h, \epsilon) \in (0, 1)^3$ and $\alpha > -1$. Then it holds

$$
e_\rho(\phi) = E_t(\rho_h, \phi) + E_F(\rho_h, \phi),
$$

$$
e_m(\phi) = E_t(m_h, \phi) + E_F(m_h, \phi) + E_{\nabla u}(\phi) - E_p(\phi) + E_c(\phi)
$$

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with

\[
E_t(r_h, \phi) = \left[ \int_{T^d} r_h \phi \, dx \right]_{t=0}^{t_{n+1}} - \int_0^T \int_{T^d} D_t r_h(t) \Pi_T \phi(t) \, dx \, dt - \int_0^T \int_{T^d} r_h(t) \partial_t \phi(t) \, dx \, dt,
\]

\[
E_F(r_h, \phi) = \int_0^{t_{n+1}} \int_{T^d} F_h^{\sup}[r_h, u_h] \Pi_T \phi \, dS_x \, dt - \int_0^T \int_{T^d} r_h u_h \cdot \nabla_x \phi \, dx \, dt
\]

\[
= -4 \sum_{i=1}^4 E_i(r_h, \phi) + \int_0^{t_{n+1}} \int_{T^d} r_h u_h \cdot \nabla_x \phi \, dx \, dt,
\]

\[
\begin{align*}
E_{\nabla_x u}(\phi) &= \int_0^T \int_{T^d} (\mu \nabla_x u_h : (\nabla_x \phi - \nabla_x (\Pi_T \phi)) + \nu \text{div}_h u_h (\text{div}_x \phi - \text{div}_h (\Pi_T \phi))) \, dx \, dt \\
&\quad - \int_0^{t_{n+1}} \int_{T^d} (\mu \nabla_x u_h : \nabla_x \phi + \nu \text{div}_h u_h \text{div}_x \phi) \, dx \, dt, \\
E_p(\phi) &= \int_0^T \int_{T^d} p_h (\text{div}_x \phi - \text{div}_h (\Pi_T \phi)) \, dx \, dt - \int_0^{t_{n+1}} \int_{T^d} (p_h \text{div}_x \phi) \, dx \, dt, \\
E_c(\phi) &= \frac{1}{\epsilon} \int_0^T \int_{\Omega^\tau} u_h \cdot \phi \, dx \, dt - \frac{1}{\epsilon} \int_0^{t_{n+1}} \int_{\Omega_h^\tau} u_h \cdot \Pi_T \phi \, dx \, dt \\
&\quad - \frac{1}{\epsilon} \int_{t_{n+1}}^T \int_{\Omega^\tau} u_h \cdot \phi \, dx \, dt - \frac{1}{\epsilon} \int_0^{t_{n+1}} \int_{\Omega_h^\tau} u_h \cdot \phi \, dx \, dt,
\end{align*}
\]

and

\[
E_1(r_h, \phi) = \frac{1}{2} \int_0^{t_{n+1}} \int_{T^d} [u_h] \cdot n \, \Pi_T \phi \, dS_x \, dt, \quad E_2(r_h, \phi) = \frac{1}{4} \int_0^{t_{n+1}} \int_{T^d} [u_h] \cdot n \, \Pi_T \phi \, dS_x \, dt,
\]

\[
E_3(r_h, \phi) = h^\alpha \int_0^{t_{n+1}} \int_{T^d} [r_h] \, \Pi_T \phi \, dS_x \, dt, \quad E_4(r_h, \phi) = \int_0^{t_{n+1}} \int_{T^d} r_h u_h \cdot (\nabla_x \phi - \nabla_h (\Pi_T \phi)) \, dx \, dt.
\]

Proof. With the definition of \( e_\rho \), see (4.4), we have

\[
e_\rho(\tau, \Delta t, h, \phi) = \left[ \int_{T^d} \theta_h \phi \, dx \right]_{t=0}^{t_{n+1}} - \int_0^T \int_{T^d} (\theta_h \partial_t \phi + \theta_h u_h \cdot \nabla_x \phi) \, dx \, dt
\]

\[
= \left[ \int_{T^d} \theta_h \phi \, dx \right]_{t=0}^{t_{n+1}} - \int_0^T \int_{T^d} D_t \theta_h(t) \Pi_T \phi(t) \, dx \, dt - \int_0^T \int_{T^d} \theta_h(t) \partial_t \phi(t) \, dx \, dt
\]

\[
+ \int_0^{t_{n+1}} \int_{T^d} D_t \theta_h(t) \Pi_T \phi(t) \, dx \, dt - \int_0^T \int_{T^d} \theta_h \phi u_h \cdot \nabla_x \phi \, dx \, dt
\]

\[
= E_t(\theta_h, \phi) + \int_0^{t_{n+1}} \int_{T^d} F_h^{\sup}[r_h, u_h] \Pi_T \phi \, dS_x \, dt - \int_0^T \int_{T^d} \theta_h u_h \cdot \nabla_x \phi \, dx \, dt
\]

\[
= E_t(\theta_h, \phi) + E_F(\theta_h, \phi) = E_t(\theta_h, \phi) - \sum_{i=1}^4 E_i(r_h, \phi) + \int_0^{t_{n+1}} \int_{T^d} \theta_h u_h \cdot \nabla_x \phi \, dx \, dt,
\]

where we have used [8, Lemma 2.5] for the last equality. Analogous analysis applies to \( e_m \) and completes the proof.
The rest is to estimate $E_t, E_F, E_{\nabla u}, E_p, E_c$. For simplicity, hereafter we shorten $L^p(0, T; L^q(T^d))$ as $L^p L^q$. 

B.1 Negative estimates of density and momentum

Firstly we introduce two negative density estimates. For completeness we present the proof of Lemma B.2. Lemma B.3 states new improved results.

Lemma B.2 ([9, Lemma 11.4]). Let $(\theta_h, u_h)$ be a solution of the FV scheme (3.3) with $(\Delta t, h, \epsilon) \in (0, 1)^3$ and $\alpha > -1$. Then for $\gamma \in (1, 2]$ it holds that

$$\|\theta_h\|_{L^\gamma L^{p\gamma/2}} = \left\|\theta_h^{\gamma/2}\right\|_{L^2 L^p}^2 = \int_0^T \left\|\theta_h^{\gamma/2}\right\|_{L^p}^2 \, dt \lesssim h^{-\alpha - 1}, \tag{B.2}$$

$$\|\theta_h\|_{L^\gamma L^\infty}^2 = \int_0^T \|\theta_h\|_{L^\infty}^\gamma \, dt \lesssim h^{-s(2d + p(\alpha + 1))/(\gamma p)} \tag{B.3}$$

with $s \in (0, \gamma]$, $p = 6$ if $d = 3$ and $p \geq 1$ if $d = 2$.

Proof. First, it is easy to check the equivalence of the norms in (B.2), i.e.,

$$\|\theta_h\|_{L^\gamma L^{p\gamma/2}}^2 = \int_0^T \left(\int_{T^d} \theta_h^{\gamma/2} \, dx\right)^{2\gamma} \, dt = \int_0^T \left(\int_{T^d} (\theta_h^{\gamma/2})^p \, dx\right)^{\frac{1}{p}} \, dt = \int_0^T \|\theta_h^{\gamma/2}\|_{L^p}^2 \, dt = \|\theta_h^{\gamma/2}\|_{L^2 L^p}^2.$$

Further, we deduce from the estimate (4.3b) that

$$\left\|\nabla \epsilon \theta_h^{\gamma/2}\right\|_{L^2 L^2}^2 \lesssim h^{-1} \int_0^T \int_\mathcal{E} P''(\theta_h, \epsilon) \|\theta_h\|^2 \, dS_x \, dt \lesssim h^{-\alpha - 1} \text{ for } \gamma \in (1, 2].$$

Next, applying the Sobolev-Poincaré inequality, cf. Lemma A.1 we have from the density estimate (4.3a) that

$$\int_0^T \|\theta_h^{\gamma/2}\|_{L^p}^2 \, dt \lesssim \int_0^T \left(\left\|\nabla \epsilon \theta_h^{\gamma/2}\right\|_{L^2}^2 + \|\theta_h^{\gamma/2}\|_{L^2}^2\right) \, dt = \left\|\nabla \epsilon \theta_h^{\gamma/2}\right\|_{L^2 L^2}^2 + \|\theta_h\|_{L^\infty L^\gamma}^\gamma \lesssim h^{-\alpha - 1},$$

where $p > 1$ in the case of $d = 2$ and $p = 6$ for the case of $d = 3$. Together with the inverse estimates we obtain

$$\int_0^T \|\theta_h\|_{L^\infty} \, dt = \int_0^T \|\theta_h^{\gamma/2}\|_{L^\infty}^{2s/\gamma} \, dt \lesssim \int_0^T \left(h^{-d/p} \|\theta_h^{\gamma/2}\|_{L^p}^{2s/\gamma}\right) \, dt = h^{-2sd/(\gamma p)} \int_0^T \left(\|\theta_h^{\gamma/2}\|_{L^p}^{2s/\gamma}\right) \, dt \lesssim h^{-2sd/(\gamma p)} \int_0^T \left(\left\|\nabla \epsilon \theta_h^{\gamma/2}\right\|_{L^2 L^2}^{2s/\gamma} + \|\theta_h\|_{L^\infty L^\gamma}^{2s/\gamma}\right) \, dt = h^{-2sd/(\gamma p)} \left(\left\|\nabla \epsilon \theta_h^{\gamma/2}\right\|_{L^2 L^2}^{2s/\gamma} + \|\theta_h\|_{L^\infty L^\gamma}^{2s/\gamma}\right) \lesssim h^{-2sd/(\gamma p)} \left((h^{-\alpha - 1} s/\gamma + 1) \lesssim h^{-s(2d + p(\alpha + 1))/(\gamma p)}\right),$$

which completes the proof. 

Further, we derive the following negative estimates of density and momentum.
Lemma B.3 (Negative estimates of density and momentum). Let \((\varrho_h, \mathbf{u}_h)\) be a solution of the FV scheme (3.3) with \((h, \epsilon) \in (0,1)^2\), \(\alpha > -1\) and \(\gamma > 1\). Then the following hold:

\[
\|\varrho_h\|_{L^2((0,T)\times\mathbb{T}^d)} \lesssim h^{\beta_D}, \quad \beta_D = \begin{cases} 
\min_{p \in [1,\infty)} \left\{ \frac{p(\alpha+1)+4}{2p}, 1 \right\} \cdot \frac{3-2\gamma}{\gamma} & \text{if } d = 2, \gamma \in (1,2), \\
\min \left\{ \frac{\alpha+2}{3}, 1 \right\} \cdot \frac{3(\gamma-2)}{2\gamma} & \text{if } d = 3, \gamma \in (1,2), \\
0 & \text{if } \gamma \geq 2,
\end{cases} \tag{B.4}
\]

\[
\|\varrho_h\|_{L^2(0,T;L^{6/5}(\mathbb{T}^d))} \lesssim h^{\beta_R}, \quad \beta_R = \begin{cases} 
\min_{p \in [\frac{3}{2},\infty)} \left\{ \frac{1+\alpha p}{2(\gamma p-2)}, 1 \right\} \cdot \frac{5\gamma-6}{3\gamma} & \text{if } d = 2, \gamma \in (1,\frac{6}{5}), \\
\min \left\{ \frac{1+\alpha}{2}, 1 \right\} \cdot \frac{5\gamma-6}{3\gamma} & \text{if } d = 3, \gamma \in (1,\frac{6}{5}), \\
0 & \text{if } \gamma \geq \frac{6}{5},
\end{cases} \tag{B.5}
\]

\[
\|\varrho_h \mathbf{u}_h\|_{L^2((0,T)\times\mathbb{T}^d)} \lesssim h^{\beta_M}, \quad \beta_M = \begin{cases} 
\max_{p \in [\frac{3}{2},\infty)} \left\{ \frac{-\alpha p+2}{2p}, \frac{p(\gamma-2)-2}{p\gamma} \right\} & \text{if } d = 2, \gamma \leq 2, \\
0 & \text{if } d = 2, \gamma > 2, \\
\max \left\{ \frac{-\alpha+2}{2\gamma}, \frac{\gamma-3}{\gamma}, -\frac{3}{2\gamma} \right\} & \text{if } d = 3, \gamma \leq 2, \\
\frac{\gamma-3}{\gamma} & \text{if } d = 3, \gamma \in (2,3), \\
0 & \text{if } d = 3, \gamma \geq 3.
\end{cases} \tag{B.6}
\]

Proof. We start with estimating the density in the \(L^2L^2\)-norm, i.e. (B.4). For \(\gamma \geq 2\) we easily check

\[
\|\varrho_h\|_{L^2L^2} \lesssim \|\varrho_h\|_{L^\infty L^2} \lesssim 1 \quad \text{meaning } \beta_D = 0.
\]

Now, let us focus on the case \(\gamma < 2\). On the one hand, thanks to the inverse estimate we have

\[
\|\varrho_h\|_{L^2L^2} \lesssim h^{\frac{\gamma-2}{\gamma}} \|\varrho_h\|_{L^\infty L^\infty} \lesssim h^{\frac{\gamma-2}{\gamma}}.
\]

On the other hand, in view of Lemma B.2 we have

\[
\|\varrho_h\|_{L^2L^2}^{2-\gamma} \lesssim h^{-(2-\gamma)(2d+p(\alpha+1))/(\gamma p)},
\]

which yields

\[
\|\varrho_h\|_{L^2L^2} = \left( \int_0^T \int_{\mathbb{T}^d} \varrho_h^\gamma \varrho_h^{2-\gamma} \, dx \, dt \right)^{1/2} \leq \left( \int_0^T \|\varrho_h\|_{L^\infty L^2}^{2-\gamma} \int_{\mathbb{T}^d} \varrho_h^2 \, dx \, dt \right)^{1/2} \leq \|\varrho_h\|_{L^\infty L^\infty} \|\varrho_h\|_{L^{2-\gamma}L^\infty} \lesssim h^{\frac{\gamma(2-p)\gamma-2}{\gamma p}}.
\]

This completes the proof of (B.4).

Next, we prove (B.5) for the estimate of the \(L^2L^{6/5}\)-norm of the density. Considering \(\gamma \geq 6/5\) it is obvious that

\[
\|\varrho_h\|_{L^2L^{6/5}} \lesssim \|\varrho_h\|_{L^\infty L^6} \lesssim 1.
\]
For $\gamma < \frac{6}{5}$ the proof can be done in the following two ways. In the first approach we apply the inverse estimates to get

$$\| q_h \|_{L^2 L^p/\gamma} \lesssim h^{-\frac{p\gamma - 6d}{p\gamma - 2}} \| q_h \|_{L^\infty L^\gamma} \lesssim h^{-\frac{p\gamma - 6d}{p\gamma - 2}}.$$ 

In the second approach, recalling estimate (B.2) and applying the interpolation inequality we obtain

$$\| q_h \|_{L^2 L^6/\gamma} \lesssim \| q_h \|_{L^\infty L^\gamma} \| q_h \|_{L^{2\gamma/3} L^p/2} \lesssim h^{-\frac{1}{\gamma}} \times (1-q) \quad \text{for} \quad p > \frac{12}{5} \gamma > 2.$$ 

Here $q$ satisfies

$$\frac{1}{2} \geq \frac{q}{\gamma} + 1 - q \quad \text{and} \quad \frac{6}{5} \geq \frac{q}{\gamma} + \frac{1 - q}{\gamma p/2} \iff \frac{2 - \gamma}{2} \leq q \leq \frac{5\gamma p - 12}{6p - 12}.$$ 

Hence, the optimal bound is achieved by choosing $q = \frac{5\gamma p - 12}{6(p - 2)}$, i.e.

$$\| q_h \|_{L^2 L^6/\gamma} \lesssim h^{-\frac{1}{\gamma}} \times (1-q) = h^{-\frac{1}{\gamma} + \frac{(6 - 5\gamma)p}{6(p - 2)}}.$$ 

Collecting the above estimates we obtain (B.5).

Finally, we are left with the estimate of $\| q_h u_h \|_{L^2 L^2}$. It is easy to check

$$\| q_h u_h \|_{L^2 L^2} = \| q_h u_h \|_{L^2 L^2}^{1/2} \lesssim \left( \| q_h u_h \|_{L^\infty L^{p\gamma/(\gamma - 2)}} \| q_h \|_{L^{\gamma} L^{p\gamma/2}} \right)^{1/2} \lesssim \left( h^{-\frac{2d}{p\gamma}} h^{-\frac{\alpha + 1}{\gamma}} \right)^{1/2} = h^{-\frac{p(\alpha + 1) + 2d}{2p\gamma}} \quad \text{for} \quad \gamma \in (1, 2]$$ 

(B.7)

and

$$\| q_h u_h \|_{L^2 L^2} \lesssim h^{-\frac{d}{p\gamma}} \| q_h u_h \|_{L^\infty L^{2\gamma/(\gamma + 1)}} \lesssim h^{-\frac{d}{2\gamma}}.$$ 

(B.8)

Moreover, by Hölder’s inequality we have

$$\| q_h u_h \|_{L^2 L^{\gamma p/(\gamma + p)}} \lesssim \| q_h \|_{L^\infty L^\gamma} \| u_h \|_{L^2 L^p} \quad \text{for} \quad p > 1,$$

from which we obtain

for $d = 2$ : if $\gamma > 2$, $\| q_h u_h \|_{L^2 L^2} \lesssim \| q_h u_h \|_{L^2 L^{\gamma p/(\gamma + p)}} \lesssim 1$ with $p \geq 2 + \frac{4}{\gamma - 2},$

if $\gamma \leq 2$, $\| q_h u_h \|_{L^2 L^2} \lesssim h^{\frac{1}{\gamma - 2}} \| q_h u_h \|_{L^2 L^{\gamma p/(\gamma + p)}} \lesssim h^{\frac{p(\gamma - 2) - 2}{p\gamma}} \quad \text{for any} \quad p > 1$,

for $d = 3$ : if $\gamma \geq 3$, $\| q_h u_h \|_{L^2 L^2} \lesssim \| q_h u_h \|_{L^2 L^{\gamma p/(\gamma + 6)}} \lesssim 1,$

if $\gamma < 3$, $\| q_h u_h \|_{L^2 L^2} \lesssim h^{\frac{1}{\gamma - 6\gamma}} \| q_h u_h \|_{L^2 L^{\gamma p/(\gamma + 6)}} \lesssim h^{\frac{(\gamma - 3)d}{\gamma}} \lesssim h^{\frac{(\gamma - 3)\gamma}{\gamma}}.$

Consequently, collecting (B.7), (B.8) and the above estimates, we obtain

for $d = 2$ : if $\gamma > 2$, $\beta_M = 0,$

if $\gamma \leq 2$, $\beta_M = \max_{p \in [1, \infty)} \left\{ \frac{p(\alpha + 1) + 4}{2p\gamma}, \frac{p(\gamma - 2) - 2\gamma}{p\gamma}, \frac{1}{\gamma} \right\}$

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\[ p \in \left( \frac{2}{\gamma - 1}, \infty \right) \]

for \( d = 3 \) : if \( \gamma \geq 3 \), \( \beta_M = 0 \),
- if \( \gamma \in (2, 3) \), \( \beta_M = \max \left\{ \frac{\gamma - 3}{\gamma}, -\frac{3}{2\gamma} \right\} = \frac{\gamma - 3}{\gamma} \),
- if \( \gamma \leq 2 \), \( \beta_M = \max \left\{ -\frac{\alpha + 2}{2\gamma}, \frac{\gamma - 3}{\gamma}, -\frac{3}{2\gamma} \right\} \),

which concludes the proof. \( \square \)

### B.2 Negative variational estimates

Having shown the negative estimates of density and momentum we can present some useful negative variational estimates that shall be used later for the consistency formulation. These proofs are analogous to Lemma 11.5 and Lemma 11.6 of [9].

**Lemma B.4.** Let \((\varrho_h, u_h)\) be a solution of the FV method (3.3) with \((h, \epsilon) \in (0, 1)^2\) and \(\gamma > 1\). Then the following hold:

\[
\begin{align*}
\int_0^T \int_{\Omega} \frac{[\varrho_h]^2}{\max\{\rho_{hi}, \rho_{hout}\}} \cdot (h^\alpha + [\{u_h\} \cdot n]) \, dS_x \, dt &\lesssim h^{\beta_D}, \\
\int_0^T \int_{\Omega} \|\varrho_h\| \, dS_x \, dt &\lesssim h^{-(\alpha + 1)/2}, \\
\int_0^T \int_{\Omega} \|\varrho_h\| \cdot |\{u_h\} \cdot n| \, dS_x \, dt &\lesssim h^{(\beta_R - 1)/2}, \\
\int_0^T \int_{\Omega} (|\varrho_h| + |\{\varrho_h\}|) |\{u_h\} \cdot n| \, dS_x \, dt &\lesssim h^{\beta_D}, \\
\int_0^T \int_{\Omega} [\varrho_h u_h] \cdot n \, dS_x \, dt &\lesssim h^{(\beta_R - 1)/2} + h^{\beta_D},
\end{align*}
\]

where \(\beta_D\) is given in (B.4) and

\[
\beta_R = \begin{cases} 0, & \text{if } d = 2, \\ \beta_R, & \text{if } d = 3. \end{cases}
\]

**Proof.** We start by showing (B.9a). For any \(\varrho > 0\), taking \(B(\varrho)\) and \(\phi_h\) in the renormalized continuity equation [9, Lemma 8.2] as \(\varrho \ln \varrho - \varrho\) and 1, respectively, we obtain

\[
B'(\varrho) = \ln(\varrho), \quad B''(\varrho) = \frac{1}{\varrho} > 0,
\]

\[
\int_{T^d} D_t(\varrho_h \ln \varrho_h) \, dx + \int_{T^d} \varrho_h \text{div}_h u_h \, dx \\
\leq - \sum_{K \in T_h} |K| \sum_{\sigma \in \partial(K)} |\sigma| \left( h^\alpha - (\{u_h\} \cdot n)^- \right) \left( \|B(\varrho_h)\| - B'(\varrho_h) \|\varrho_h\| \right). 
\]

(B.10)
Due to the convexity of $B(\varrho)$, i.e.

$$\|B(\varrho_h)\| - B'(\varrho_h)\|\varrho_h\|^2 = \frac{1}{2} B''(\xi)\|\varrho_h\|^2 \geq \frac{\|\varrho_h\|^2}{2 \max\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\}}, \quad \xi \in \text{co}\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\},$$

we obtain

$$h^\alpha \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (\|B(\varrho_h)\| - B'(\varrho_h)\|\varrho_h\|) \geq h^\alpha \int_{\mathcal{E}} \frac{\|\varrho_h\|^2}{\max\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\}} dS_x.$$

Moreover, we have

$$- \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (\|\varrho_h\| - B'(\varrho_h)\|\varrho_h\|) \geq \int_{\mathcal{E}} \|\varrho\| \cdot \|n\| \|B''(\xi)\|\varrho_h\|^2$$

with $\xi \in \text{co}\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\}$, which implies

$$- \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (\|\varrho_h\| - B'(\varrho_h)\|\varrho_h\|) \geq \int_{\mathcal{E}} \|\varrho\| \cdot \|n\| \|B''(\xi)\|\varrho_h\|^2$$

Collecting the above estimates we derive from (B.10) that

$$\int_{0}^{\tau} \int_{\mathcal{E}} \frac{\|\varrho_h\|^2}{\max\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\}} \left( h^\alpha + \frac{\|\varrho_h\| \cdot \|n\|}{2} \right) dS_x dt \leq \int_{0}^{\tau} \sum_{K \in \mathcal{T}_h} |K| \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (h^\alpha - (\|u\| \cdot |n|)) (\|B(\varrho_h)\| - B'(\varrho_h)\|\varrho_h\|) dt$$

$$\leq \int_{\mathcal{T}_h} \varrho_0^0 \ln(\varrho_0^0) dx - \int_{\mathcal{T}_h} \varrho_h \ln(\varrho_h) dx - \int_{0}^{\tau} \int_{\mathcal{T}_h} (\varrho_h \text{div}_h u_h) dx$$

$$\lesssim 1 + \int_{0}^{\tau} \int_{\mathcal{T}_h} (\varrho_h \text{div}_h u_h) dx \leq 1 + \|\varrho_h\|_{L^2 L^2} \|\text{div}_h u_h\|_{L^2 L^2}.$$

Note that we have used here the inequality $|\varrho \ln(\varrho)| \lesssim 1 + \varrho^\gamma$. Consequently, applying Lemma B.3 concludes the proof of (B.9a).

Secondly, thanks to Hölder’s inequality and trace inequality, together with the density dissipation (B.9a) and uniform bounds (4.3) we obtain (B.9b) in the following way:

$$\int_{0}^{\tau} \int_{\mathcal{E}} \|\varrho_h\| dS_x dt \lesssim \left( \int_{0}^{\tau} \int_{\mathcal{E}} \frac{\|\varrho_h\|^2}{\max\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\}} dS_x dt \right)^{1/2} \left( \int_{0}^{\tau} \int_{\mathcal{E}} \max\{\rho_{h}^{\text{in}}, \rho_{h}^{\text{out}}\} dS_x dt \right)^{1/2}$$

$$\lesssim h^{-\alpha/2} h^{-1/2} = h^{-(1+\alpha)/2} \quad \text{for } \gamma \geq 2,$$

$$\int_{0}^{\tau} \int_{\mathcal{E}} \|\varrho_h\| dS_x dt \lesssim \int_{0}^{\tau} \int_{\mathcal{E}} \|\varrho_h\| \sqrt{\mathcal{P}(\varrho_h)} (\varrho_{h+1} + 1) dS_x dt$$

$$\lesssim \left( \int_{0}^{\tau} \int_{\mathcal{E}} \|\varrho_h\|^2 \mathcal{P}(\varrho_h) dS_x dt \right)^{1/2} \left( \int_{0}^{\tau} \int_{\mathcal{E}} (\varrho_{h+1} + 1) dS_x dt \right)^{1/2}$$

$$\lesssim h^{-\alpha/2} h^{-1/2} = h^{-(1+\alpha)/2} \quad \text{for } \gamma \in (1, 2),$$

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where we have used the inequality
\[ 1_{(0,\infty)}(q) p''(q)(\omega + 1) \gtrsim 1 \text{ for } \omega \in (1, 2) \]
and \( \varrho \) is given in (4.3b).

Thirdly, we can derive (B.9c) in an analogous way. On the one hand, we have that for \( \gamma \geq 2 \)
\[
\int_0^\tau \int_E \left\| \varrho \right\| \left\| \{ u_h \} \cdot n \right\| dS_x dt
\lesssim \left( \int_0^\tau \int_E \max\{ \rho_{h}^{m}, \rho_{h}^{out} \} \left\| \{ u_h \} \cdot n \right\| dS_x dt \right)^{1/2} \left( \int_0^\tau \int_E \max\{ \rho_{h}^{m}, \rho_{h}^{out} \} \left\| \{ u_h \} \cdot n \right\| dS_x dt \right)^{1/2}
\lesssim h^{-1/2} \| \varrho \|_{L^2_L}^{1/2} \| u_h \|_{L^2_L}^{1/2} \lesssim h^{-1/2} \| \varrho \|_{L^2_L}^{1/2},
\]
where \( p' = \frac{p}{p+1} \), for any \( p > 1 \) in the case of \( d = 2 \) and \( p = 6 \) for the case of \( d = 3 \). On the other hand, for \( \gamma < 2 \) we have
\[
\int_0^\tau \int_E \left\| \varrho \right\| \left\| \{ u_h \} \cdot n \right\| dS_x dt
\lesssim \left( \int_0^\tau \int_E \varrho^2 \left\| \{ u_h \} \cdot n \right\| dS_x dt \right)^{1/2} \left( \int_0^\tau \int_E \varrho \left\| \{ u_h \} \cdot n \right\| dS_x dt \right)^{1/2}
\lesssim h^{-1/2} (\| \varrho \|_{L^2_L}^{1/2} + 1) \| u_h \|_{L^2_L}^{1/2} \lesssim h^{-1/2} \| \varrho \|_{L^2_L}^{1/2}.
\]
In view of the above two estimates, the proof of (B.9c) reduces to show \( \| \varrho \|_{L^2_{L'}} \lesssim h^{\beta \rho} \). If \( d = 2 \), we take \( p' \in (1, \gamma) \) (i.e. \( p \geq \frac{\gamma}{\gamma - 1} \)) and obtain \( | \| \varrho \|_{L^2_{L'}} \lesssim 1 \). If \( d = 3 \), we choose \( p = 6 \) and \( p' = \frac{6}{5} \). Then we apply (B.5) to get
\[
\| \varrho \|_{L^2_{L'}} = \| \varrho \|_{L^2_{L6/5}} \lesssim h^{\beta \rho},
\]
which completes the proof of (B.9c).

The fourth estimate (B.9d) is straightforward:
\[
\int_0^\tau \int_E (| \| \varrho \| | \left\{ u_h \right\} | \left\{ u_h \right\} \cdot n | dS_x dt \lesssim \left( \int_0^\tau \int_E h \left\{ | \varrho |^2 \right\} \left\{ u_h \right\} \cdot n | dS_x dt \right)^{1/2} \left( \int_0^\tau \int_E \frac{\left\{ u_h \right\}^2}{h} dS_x dt \right)^{1/2}
\lesssim \| \varrho \|_{L^2_{L2}} \| \nabla_E u_h \|_{L^2_{L2}} \lesssim h^{\beta \rho}.
\]
Finally, recalling the product rule for the equality
\[
| \| \varrho \| u_h \| = | \| \varrho \| | \left\{ u_h \right\} | + | \| \varrho \| | u_h \|\]
we may employ (B.9b) and (B.9d) to derive (B.9e)
\[
\int_0^\tau \int_E \left\| \varrho \right\| \left\{ u_h \right\} \cdot n \left\| dS_x dt \leq \int_0^\tau \int_E \left\| \varrho \right\| \left\{ u_h \right\} \cdot n \left\| dS_x dt + \int_0^\tau \int_E \left\{ \varrho \right\} \left\{ u_h \right\} \cdot n \left\| dS_x dt \right) \lesssim h^{(\beta \rho - 1)/2} + h^{\beta \rho},
\]
which completes the proof. \( \square \)
B.3 Consistency proof

Equipped with Lemmas B.1, B.3 and B.4, we are now ready to prove the consistency formulation, which is similar to the one in [9, Section 11.3], [8, Theorem 4.1] or [10, Section 2.7].

Proof of Lemma 4.4. Let $\tau \in [t_n, t_{n+1})$.

Step 1 – time derivative terms: Let $r_h$ stand for $\varrho_h$ or $\varrho_h u_h$. Recalling [10, equation(2.17)] we have
\[
E_t(r_h, \phi) \lesssim (\|\partial_t^2 \phi\|_{L^\infty L^\infty} + \|\partial_t \phi\|_{L^\infty L^\infty}) \Delta t \lesssim \Delta t. \tag{B.11}
\]

Step 2 – convective terms: Applying Hölder’s inequality with (3.2) and (B.9) we obtain for $r_h = \varrho_h$ that
\[
\sum_{i=1}^{4} |E_i(\varrho_h, \phi)| \lesssim h^{(1+\beta_R)/2} + h^{(1+\alpha)/2} + h^{1+\beta_D} \quad \text{for } \phi \in L^\infty(0, T; W^{1,1}(\mathbb{T}^d)).
\]

Directly following [8, Theorem 4.1] we obtain for $r_h = \varrho_h u_h$ that
\[
\sum_{i=1}^{4} |E_i(\varrho_h u_h, \varphi)| \lesssim h + h^{1+\alpha} + h^{1+\beta_M} \quad \text{for } \varphi \in L^\infty(0, T; W^{2,1}(\mathbb{T}^d; \mathbb{R}^d)).
\]

Moreover, it is obvious that
\[
\left| \int_{t_n}^{t_{n+1}} \int_{\mathbb{T}^d} r_h \cdot \nabla_x \phi \, dx \, dt \right| \lesssim \Delta t \|\nabla_x \phi\|_{L^\infty L^\infty} \|r_h\|_{L^1 L^1} \lesssim \Delta t, \quad r_h = \varrho_h \text{ or } \varrho_h u_h.
\]

Consequently, we obtain
\[
E_F(\varrho_h, \phi) \lesssim h^{(1+\beta_R)/2} + h^{(1+\alpha)/2} + h^{1+\beta_D} + \Delta t; \quad E_F(m_h, \varphi) \lesssim h + h^{1+\alpha} + h^{1+\beta_M} + \Delta t. \tag{B.12}
\]

Step 3 – viscosity terms: Applying the interpolation error estimates we have
\[
|E_{\nabla_x u}(\varphi)| \lesssim h \left( \|\nabla_x u_h\|_{L^2 L^2} + \|\text{div}_h u_h\|_{L^2 L^2} \right) \|\nabla_x^2 \varphi\|_{L^\infty L^\infty} + (\Delta t)^{1/2} \left( \|\nabla_x u_h\|_{L^2 L^2} + \|\text{div}_h u_h\|_{L^2 L^2} \right) \|\nabla_x \varphi\|_{L^\infty L^\infty}. \tag{B.13}
\]

Step 4 – pressure term: Applying the interpolation error estimates we have
\[
|E_p(\varphi)| \lesssim h \|p_h\|_{L^\infty L^1} \|\nabla_x^2 \varphi\|_{L^\infty L^\infty} + \Delta t \|p_h\|_{L^\infty L^1} \|\nabla_x \varphi\|_{L^\infty L^\infty}. \tag{B.14}
\]
Step 5 – penalization term: With
\[ \Delta t \left\| u_h(t_n) \right\|^2_{L^2(\Omega^s)} \leq \Delta t \sum_{i=0}^{N_T-1} \left\| u_h(t_i) \right\|^2_{L^2(\Omega^s)} = \left\| u_h \right\|^2_{L^2((0,T) \times \Omega^s)} \lesssim \epsilon \]
we have
\[ \frac{1}{\epsilon} \int_{\Omega^s} u_h \cdot \varphi \, dx \, dt \lesssim \frac{\| \varphi \|_{L^\infty L^\infty} \Delta t \| u_h(t_n) \|_{L^1(\Omega^s)}}{\epsilon} \lesssim \frac{\Delta t}{\epsilon} \| u_h(t_n) \|_{L^2(\Omega^s)} \lesssim (\Delta t / \epsilon)^{1/2}, \]
resulting the following estimate of \( E_\epsilon \)
\[ |E_\epsilon(\varphi)| \lesssim (\Delta t / \epsilon)^{1/2} + \frac{\| u_h \|_{L^2((0,T) \times \Omega^s_\epsilon)}}{\epsilon} \| \varphi \|_{L^\infty L^\infty} (|\Omega^s_\epsilon \setminus \Omega^s|)^{1/2} \lesssim (\Delta t / \epsilon)^{1/2} + (h/\epsilon)^{1/2}. \quad (B.15) \]
In summary, combining (B.11) – (B.15) we have
\[ |e_\varphi| \lesssim \Delta t + h^{(1+\alpha)/2} + h^{(1+\beta_R)/2} + h^{1+\beta_D}, \]
\[ |e_m| \lesssim (\Delta t)^{1/2} + h + h^{1+\alpha} + h^{1+\beta_M} + (\Delta t / \epsilon)^{1/2} + (h / \epsilon)^{1/2}, \]
which concludes the proof. \( \square \)

C Some useful estimates

In this section we derive some useful inequalities that help us to obtain the optimal error estimates in Theorem 5.2. To begin, let us recall the regularity of the extended strong solution \((\tilde{\varphi}, \tilde{u})\), cf. (5.2):
\[ \tilde{\varphi} \in \Omega^s \in C^2(\Omega^s; \mathbb{R}^{d+1}), \quad (\tilde{\varphi}, \tilde{u}) \in C^1((0,T) \times \Omega^f; \mathbb{R}^{d+1}) \cap C(0,T; W^{4,\infty}(\Omega^f; \mathbb{R}^{d+1})), \]
\[ \tilde{\varphi} \in L^\infty((0,T) \times \mathbb{T}^d), \quad \tilde{u} \in W^{1,\infty}((0,T) \times \mathbb{T}^d; \mathbb{R}^d). \]

Note that \((\tilde{\varphi}, \tilde{u})\) is piecewise smooth with possible discontinuities of \( \tilde{\varphi} \) and \( \nabla_x \tilde{u} \) on the boundary. To this end, we introduce an artificial splitting of the domain and derive suitable estimates related to the boundary part.

Definition C.1 (Splitting of the mesh). We split the mesh \( T_h \) into three pieces, see Figure 12. \( \Omega^C_h \) denotes the area containing the neighbourhood of the fluid boundary \( \partial \Omega^f \)
\[ \Omega^C_h = \left\{ K \mid \cup_{L \cap K \neq \emptyset} L \cap \partial \Omega^f \neq \emptyset \right\}. \]
The inner domain \( \Omega^I_h \) and the outer domain \( \Omega^O_h \) are given as
\[ \Omega^I_h : = \Omega^f \setminus \Omega^C_h \quad \text{and} \quad \Omega^O_h : = \Omega^s \setminus \Omega^C_h. \]
By the above definition, for any cell $K \in \Omega_h^C$, we know that either $K$ or one of its neighbours intersects with the fluid boundary $\partial\Omega_f$. Moreover, we have

$$\Omega_h^C \lesssim h, \quad \|\mathbf{u}\|_{W^{1,\infty}(\Omega_f)} \leq h$$ if $x \in \Omega_h^C$, \hfill (C.1a)

$$|\nabla_h(\Pi_T \mathbf{u})| + |\text{div}_h(\Pi_T \mathbf{u})| + |\nabla\mathbf{e}(\Pi_T \mathbf{u})| \lesssim \begin{cases} \|\mathbf{u}\|_{W^{1,\infty}(\Omega_f)} & \text{if } x \in \Omega_h^I \cup \Omega_h^C, \\ 0 & \text{if } x \in \Omega_h^O, \end{cases}$$ \hfill (C.1b)

$$|\Delta_h \Pi_T \mathbf{u}| \lesssim \begin{cases} \|\mathbf{u}\|_{W^{2,\infty}(\Omega_h)} & \text{if } x \in \Omega_h^I, \\ \|\mathbf{u}\|_{W^{1,\infty}(\Omega_f)} h^{-1} & \text{if } x \in \Omega_h^C, \\ 0 & \text{if } x \in \Omega_h^O. \end{cases}$$ \hfill (C.1c)

**Lemma C.2.** Let $\delta \in (0, 1)$ be an arbitrary constant, $(\varrho_h, \mathbf{u}_h)$ be a solution of the FV method (3.3) with $(h, \epsilon) \in (0, 1)^2$. Let $(\tilde{\varrho}, \tilde{\mathbf{u}})$ be the extended strong solution in the sense of Definition 5.1. Then

$$\int_0^\tau \int_{\Omega_h^C} \varrho_h |\mathbf{u}_h|^2 \, dx \, dt \lesssim h^4 + \int_0^\tau R_E(\varrho_h, \mathbf{u}_h | \tilde{\varrho}, \tilde{\mathbf{u}}) \, dt,$$ \hfill (C.2a)

$$\int_0^\tau \int_{\Omega_h^I \setminus \Omega_h^O} |\mathbf{u}_h| \, dx \, dt \lesssim \epsilon + \delta h \frac{\|\mathbf{u}_h\|_{L^2((0, \tau) \times \Omega_h^O)}^2}{\epsilon},$$ \hfill (C.2b)

$$\int_0^\tau \int_{\Omega_h^C} |\mathbf{u}_h| \, dx \, dt \leq \int_0^\tau \int_{\Omega_h^C} |\mathbf{u}_h| \, dx \, dt \lesssim \epsilon + \delta \frac{\|\mathbf{u}_h\|_{L^2((0, \tau) \times \Omega_h^O)}^2}{\epsilon},$$ \hfill (C.2c)

$$\int_0^\tau \int_{\Omega_h^C} |\nabla \mathbf{e} \mathbf{u}_h| \, dx \, dt \lesssim h + \delta \mu \|\nabla \mathbf{e} \mathbf{u}_h - \nabla x \tilde{\mathbf{u}}\|_{L^2((0, \tau) \times \mathbb{T}^d)}^2,$$ \hfill (C.2d)

$$\int_0^\tau \int_{\Omega_h^C} |\text{div}_h \mathbf{u}_h| \, dx \, dt \lesssim h + \delta \nu \|\text{div}_h \mathbf{u}_h - \text{div}_x \tilde{\mathbf{u}}\|_{L^2((0, \tau) \times \mathbb{T}^d)}^2,$$ \hfill (C.2e)
\[
\int_0^\tau \int_{\Omega_h^C_t} |u_h| \, dx \, dt \lesssim h^2 + \epsilon + h \left( \delta \mu \|\nabla_x u_h - \nabla_x \bar{u}\|_{L^2((0,\tau) \times \mathbb{T}^d)}^2 + \delta \frac{\|u_h\|_{L^2((0,\tau) \times \Omega_h^C_t)}^2}{\epsilon} \right). \quad \text{(C.2f)}
\]

**Remark C.3.** We point out that the above estimates (C.2d)–(C.2f) on \( \Omega_h^C \) also hold on any general domain \( \Omega_h^{C+} \) given by

\[
\Omega_h^{C+} := \{ x \in \mathbb{T}^d : \text{dist}(x, \partial \Omega') \leq h \},
\]

which can be seen as an extension of \( \Omega_h^C \).

**Proof.** Firstly, we use the triangular inequality, the boundedness of \( \varrho_h \), and (C.1a) to get (C.2a)

\[
\int_0^\tau \int_{\Omega_h^C} \varrho_h |u_h|^2 \, dx \, dt \lesssim \int_0^\tau \int_{\Omega_h^C} \varrho_h |u_h - \bar{u}|^2 \, dx \, dt + \int_0^\tau \int_{\Omega_h^C} \varrho_h |\bar{u}|^2 \, dx \, dt \lesssim \int_0^\tau \int_{\Omega_h^C} \varrho_h |\bar{u}|^2 \, dx \, dt + h^3.
\]

Secondly, by Young’s inequality we get (C.2b)

\[
\int_0^\tau \int_{\Omega_h^{C+}} |u_h| \, dx \, dt \leq \int_0^\tau \int_{\Omega_h^{C+}} \left( \frac{\delta h}{2e} |u_h|^2 + \frac{\epsilon}{2\delta h} \right) \, dx \, dt \lesssim \delta h \frac{\|u_h\|_{L^2((0,\tau) \times \Omega_h^{C+})}^2}{\epsilon} + \epsilon.
\]

Further, denoting \( \Omega_h^{C-} = \{ x \in \Omega_h^C : \text{dist}(x, \partial \Omega') \leq h \} \) we can exactly follow the above estimate to get

\[
\int_0^\tau \int_{\Omega_h^{C-}} |u_h| \, dx \, dt \lesssim \delta h \frac{\|u_h\|_{L^2((0,\tau) \times \Omega_h^{C-})}^2}{\epsilon} + \epsilon. \quad \text{(C.3)}
\]

Analogously, we have the third estimate (C.2c)

\[
\int_0^\tau \int_{\Omega_h^C} |u_h| \, dx \, dt \leq \int_0^\tau \int_{\Omega_h^C} |u_h| \, dx \, dt \lesssim \epsilon^{1/2} \frac{\|u_h\|_{L^2((0,\tau) \times \Omega_h^C)}^2}{\epsilon^{1/2}} \lesssim \epsilon + \delta \frac{\|u_h\|_{L^2((0,\tau) \times \Omega_h^C)}^2}{\epsilon}.
\]

Fourthly, we use triangular inequality, Young’s inequality, and (C.1a) to get (C.2d)

\[
\int_0^\tau \int_{\Omega_h^C} |\nabla_x u_h| \, dx \, dt \leq \int_0^\tau \int_{\Omega_h^C} |\nabla_x u_h - \nabla_x \bar{u}| \, dx \, dt + \int_0^\tau \int_{\Omega_h^C} |\nabla_x \bar{u}| \, dx \, dt \lesssim \int_0^\tau \int_{\Omega_h^C} \left( \delta \mu |\nabla_x u_h - \nabla_x \bar{u}|^2 + \frac{1}{\delta \mu} \right) \, dx \, dt + h \|\nabla_x \bar{u}\|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)}
\]

\[
\lesssim h + \delta \mu \frac{\|\nabla_x u_h - \nabla_x \bar{u}\|_{L^2((0,\tau) \times \mathbb{T}^d)}^2}{\epsilon}.
\]

The same process applies to \( \nabla_h u_h \) and yields (C.2e).

To show the sixth estimate (C.2f), we apply (C.2d) and (C.3) and find

\[
\int_0^\tau \int_{\Omega_h^C} |u_h| \, dx \, dt \leq h \int_0^\tau \int_{\Omega_h^C} |\nabla_x u_h| \, dx \, dt + \int_0^\tau \int_{\Omega_h^{C-}} |u_h| \, dx \, dt \lesssim h^2 + \epsilon + h \left( \delta \mu \|\nabla_x u_h - \nabla_x \bar{u}\|_{L^2((0,\tau) \times \mathbb{T}^d)}^2 + \delta \frac{\|u_h\|_{L^2((0,\tau) \times \Omega_h^C)}^2}{\epsilon} \right),
\]

\[\square\]
Lemma C.4. Let \( v \in L^1(\mathbb{T}^d) \), \( u \in W^{2,\infty}(\Omega^f;\mathbb{R}^d) \), \( \tilde{u} \) be given by Definition 5.1. Let \((q_h, u_h)\) be a solution of the FV method (3.3) with \((h, \epsilon) \in (0,1)^2\). Then

\[
\left| \int_{\mathbb{T}^d} v(\nabla_x \tilde{u} - D_1 \tilde{u}) \, dx \right| + \left| \int_{\Omega^e_h} v(\text{div}_x \tilde{u} - D_2 \tilde{u}) \, dx \right| \lesssim h + \int_{\Omega^e_h} |v| \, dx,
\]

where \( D_1 = \nabla \Pi T, \nabla \Pi T, \Pi \nabla_x \) and \( D_2 = \text{div}_h \Pi T, \Pi \text{div}_x \) with

\[
\Pi \phi = \left( \Pi_{\xi}^{(1)} \phi_1, \ldots, \Pi_{\xi}^{(d)} \phi_d \right), \quad \Pi_{\xi}^{(1)} \phi = \sum_{\sigma \in E_\xi} \frac{1}{|\sigma|} \int_{\sigma} \phi \, dS_x; \quad \Pi_{\xi} \phi|_K := \frac{1}{|\xi|} \int_{\xi} \phi \, dS_x, \quad \xi \in \partial D_\sigma \cap K.
\]

Further, it holds

\[
\| \nabla_x \tilde{u} - \nabla \Pi T \tilde{u} \|^2_{L^2(\mathbb{T}^d)} + \| \nabla_x \tilde{u} - \Pi \nabla_x \Pi T \tilde{u} \|^2_{L^2(\mathbb{T}^d)} + \| \text{div}_x \tilde{u} - \Pi \text{div}_x \Pi T \tilde{u} \|^2_{L^2(\mathbb{T}^d;\mathbb{R}^d)} \lesssim h. \tag{C.4b}
\]

Proof. Thanks to the regularity of \( \tilde{u} \), the estimate (C.4b) directly follows from (C.4a) by taking \( v = \nabla_x \tilde{u} - D_1 \tilde{u} \), \( \text{div}_x \tilde{u} - D_2 \tilde{u} \), respectively. To see (C.4a) it is enough to show the proof for one term, e.g., \( \int_{\mathbb{T}^d} v(\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}) \, dx \), as the rest can be done exactly in the same way. Splitting \( \mathbb{T}^d \) into three regions \( \Omega^I_h, \Omega^C_h, \Omega^O_h \) we obtain

\[
\left| \int_{\mathbb{T}^d} v(\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}) \, dx \right| \\
\leq \int_{\Omega^I_h} v(\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}) \, dx + \int_{\Omega^C_h} v(\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}) \, dx + \int_{\Omega^O_h} v(\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}) \, dx \\
\lesssim \|v\|_{L^1(\Omega^I_h)} \|\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}\|_{L^\infty(\Omega^I_h;\mathbb{R}^d)} + \|v\|_{L^1(\Omega^C_h)} \|\nabla_x \tilde{u} - \nabla \Pi T \tilde{u}\|_{L^\infty(\Omega^C_h;\mathbb{R}^d)} + 0 \\
\lesssim h \|u\|_{W^{2,\infty}(\Omega^I_h;\mathbb{R}^d)} \|v\|_{L^1(\Omega^I_h)} + \|\nabla_x \tilde{u}\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)} \|v\|_{L^1(\Omega^C_h)} \lesssim h + \|v\|_{L^1(\Omega^C_h)},
\]

which completes the proof. \( \square \)

D Proof of error estimates

In this section we show the details of the proof of the error estimates stated in Theorem 5.2.

D.1 Relative energy balance

We start with deriving the relative energy balance by means of the consistency formulation. Noticing the lower regularity of \( \tilde{\rho} \) on \( \mathbb{T}^d \), i.e. \( \tilde{\rho} \in L^\infty((0,T) \times \mathbb{T}^d) \), let us modify the density consistency formulation (4.4):

\[
e_{\rho}^{\text{new}}(\tau, \Delta t, h, \phi) = \left[ \int_{\mathbb{T}^d} \rho_{h} \phi \, dx \right]_{t=\tau}^{t} - \int_{\Omega^f} \left( \rho_{h} \partial_t \phi + \rho_{h} u_{h} \cdot \nabla \phi \right) \, dx \, dt, \tag{D.1}
\]
where $\phi_{|\Omega^*} \in L^\infty(\Omega^*)$, $\phi_{|\Omega^f} \in W^{1,\infty}((0,T) \times \Omega^f)$. Following the decomposition of $e_\varphi$ in Lemma B.1 we have

$$e_\varphi^{\text{new}} = - \sum_{i=1}^{3} E_i(\varphi_h, \varphi) - E_4^{\text{new}}(\varphi_h, \varphi) + \int_0^{T_{n+1}} \int_{\Omega_f^*} \varphi_h \nabla \cdot \nabla \varphi \, dx,$$

where

$$E_4^{\text{new}}(\varphi_h, \varphi) = \int_0^{T_{n+1}} \int_{\Omega_f^*} \varphi_h \nabla \cdot \nabla \varphi \, dx + \int_0^{T_{n+1}} \int_{\Omega_f} \varphi_h u_h \cdot \nabla \varphi \, dx dt - \int_0^{T_{n+1}} \int_{\Omega_f} \varphi_h u_h \cdot \nabla \varphi \cdot (\Pi \varphi) \, dx dt.$$

Hence, collecting the energy estimate (4.2), the density consistency formulation (D.1) with the test function $\varphi_1 = \frac{1}{2} |\tilde{u}|^2 - P'(\tilde{\varphi})$ and the momentum consistency formulation (4.5) with the test function $\varphi = -\tilde{u}$, we obtain

$$[R_E(\varphi_h, u_h|\tilde{\varphi}, \tilde{u})]_{t=0}^{t=T} + \frac{1}{\epsilon} \int_0^T \int_{\Omega_f^*} |\varphi_h|^2 \, dx dt + \int_0^T D_{\text{num}} \, dt$$

$$= \int_0^T \int_{\Omega_f^*} \left( \frac{|\tilde{u}|^2}{2} + \varphi_h \nabla \cdot \nabla \tilde{u} \right) \, dx dt$$

$$- \int_0^T \int_{\Omega_f^*} \left( \varphi_h \partial_t P'(\tilde{\varphi}) + \varphi_h u_h \cdot \nabla P'(\tilde{\varphi}) \right) \, dx dt + e_\varphi^{\text{new}}(\tau, h, \varphi_1)$$

$$- \int_0^T \int_{\Omega_f^*} \left( \varphi_h u_h \partial_t \tilde{u} + \varphi_h u_h \otimes u_h : \nabla \tilde{u} + p_h \text{div} \tilde{u} \right) \, dx dt$$

$$+ \int_0^T \int_{\Omega_f^*} (\nabla \cdot \tilde{u} + \nu \text{div} \nabla \tilde{u}) \, dx dt + \frac{1}{\epsilon} \int_0^T \int_{\Omega^*} u_h \cdot \tilde{u} \, dx dt + e_m(\tau, h, -\tilde{u})$$

Further, let us split the integrals in the right hand side into two parts: $A = \int_{\Omega^*} \cdot dx$ and $B = \int_{\Omega_f^*} \cdot dx$. Applying (5.2) we have $A = 0$. On the other hand, thanks to

$$\partial_t \tilde{\varphi} + \text{div} \tilde{\varphi} = 0 \quad \text{and} \quad \tilde{\varphi}(\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u}) + \nabla P(\tilde{\varphi}) - \mu \Delta \tilde{u} - \nu \nabla \text{div} \tilde{u} = 0 \quad \text{on} \quad \Omega^f,$$

we can follow calculations in [10, Section 3] and obtain $B = \sum_{i=1}^5 R_i$ with

$$R_1 = \int_0^T \int_{\Omega_f^*} \left( \varphi_h - \tilde{\varphi} \right) \left( \tilde{u} - u_h \right) \cdot \left( \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \cdot \nabla \tilde{u} \right) \, dx dt,$$

$$R_2 = -\int_0^T \int_{\Omega_f^*} \varphi_h(u_h - \tilde{u}) \otimes (u_h - \tilde{u}) : \nabla \tilde{u} \, dx dt,$$

$$R_3 = -\mu \int_0^T \int_{\Omega_f^*} (\nabla \cdot \tilde{u}) u_h : \nabla u_h \cdot \nabla \tilde{u} \, dx dt,$$

$$R_4 = -\nu \int_0^T \int_{\Omega_f^*} (\text{div} \nabla u_h \text{div} \tilde{u} + u_h \cdot \nabla \text{div} \tilde{u}) \, dx dt,$$

$$R_5 = -\nu \int_0^T \int_{\Omega_f^*} (p_h - P'(\tilde{\varphi})(\varphi_h - \tilde{\varphi}) - P(\tilde{\varphi})) \text{div} \tilde{u} \, dx dt.$$
Altogether, we obtain the following relative energy balance
\[ |R_E(\varrho_h, u_h|\tilde{\varrho}, \tilde{u})|_{t=0}^{T} + \int_0^T \int_{\mathbb{T}^d} \left( \mu|\nabla \varphi u_h - \nabla x \tilde{u}|^2 + \nu |\text{div} \, u_h - \text{div} \, \tilde{u}|^2 \right) \, dx \, dt + \frac{1}{\epsilon} \int_0^T \int_{\Omega_h^i} |u_h|^2 \, dx \, dt = - \int_0^T D_{num} \, dt + e_S + e_R, \]

where \( D_{num} \geq 0 \) is given in Lemma 4.2 and
\[ e_S = e_{\text{new}}^{\text{new}}(\tau, h, \phi_1) + e_m(\tau, h, -\tilde{u}), \quad \phi_1 = \frac{1}{2} |\tilde{u}|^2 - \mathcal{P}'(\tilde{\varrho}); \quad e_R = \sum_{i=1}^5 R_i. \]  

### D.2 Estimates on \( e_S \)

In this section we estimate new consistency error \( e_S \) given in (D.3) for the semi-discrete version of the FV scheme with bounded density. It means that time evolution is not approximated \( D_t = \partial_t \) and \( \varrho_h \lesssim 1 \).

**Lemma D.1.** Let \( \gamma > 1 \) and \((\varrho_h, u_h)\) be a solution of the FV method (3.3) with \((h, \epsilon) \in (0,1)^2 \) and \( \alpha > 0 \). Let \((\tilde{\varrho}, \tilde{u})\) be the extended strong solution in the sense of Definition 5.1. Suppose that \( \varrho_h \) is uniformly bounded from above by a positive constant, cf. (5.3). Then
\[ |e_S| \lesssim h + \frac{h^{(1+\alpha)/2}}{\epsilon} + \frac{\nu \delta}{\epsilon} + \int_0^T R_E(\varrho_h, u_h|\tilde{\varrho}, \tilde{u}) \, dt + \delta \| \text{div} \, u_h - \text{div} \, \tilde{u} \|^2_{L^2((0,\tau) \times \Omega_h^i)} \]
\[ + \delta \mu \| \nabla \varphi u_h - \nabla x \tilde{u} \|^2_{L^2((0,\tau) \times \mathbb{T}^d)}. \]

**Proof.** Let us recall the error terms in \( e_S \) given in (D.2) and (B.1). We study each term separately.

- \( E_1(r_h, \phi) \): To begin, we recall the integration by parts formula for any \( f_h \in Q_h \)
\[ \int_{\mathbb{T}^d} \partial \varphi f_h v_h \, dx = - \int_{\mathbb{T}^d} f_h \partial_T \varphi v_h \, dx, \]  

where \( v_h \) is piecewise constant on the dual grid \( \{D_\sigma|\sigma \in E_i\} \), \( \partial \varphi \) := \nabla \varphi \) on \( D_\sigma \) when \( \sigma \in E_i \) and \( \partial_T v_h|K := \frac{v_h|_{x<K^+} - v_h|_{x>K^-}}{h} \). Here, \( \sigma_{K^\pm} \in E_i \) are the left and right faces of \( K \in T_h \) in the \( i \)th direction. Denoting \( \Delta_h f_h = \partial \varphi h \partial_T f_h \) and noticing \( \Delta_h f_h = \sum_{i=1}^d \partial \varphi h \partial_T f_h \) we have
\[ |E_1(r_h, \phi)| = \int_0^T \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}^d} \left| \sum_{i=1}^d \partial \varphi h \partial_T f_h \right| \right| \, dx \, dt \]
\[ = \int_0^T \int_{\mathbb{T}^d} \left| \sum_{i=1}^d \partial \varphi h \partial_T f_h \right| \, dx \, dt \]
\[ \lesssim h \| r_h \|_{L^2((0,\tau) \times \mathbb{T}^d \setminus \Omega_h^i)} \| \phi \|_{L^\infty(0,\tau; W^1_\infty(\mathbb{T}^d \setminus \Omega_h^i))} \left( \| u_h \|_{L^2((0,\tau) \times \mathbb{T}^d \setminus \Omega_h^i)} + \| \nabla \varphi u_h \|_{L^2((0,\tau) \times \mathbb{T}^d \setminus \Omega_h^i)} \right) \]
\[ + h \| r_h \|_{L^2((0,\tau) \times \mathbb{T}^d \setminus \Omega_h^i)} \left( \| u_h \|_{L^2((0,\tau) \times \mathbb{T}^d \setminus \Omega_h^i)} + \| \nabla \varphi u_h \|_{L^2((0,\tau) \times \mathbb{T}^d \setminus \Omega_h^i)} \right) \]
\[ + \left( \int_0^T \int_{\mathbb{T}^d} \left| \sum_{i=1}^d \partial \varphi h \partial_T f_h \right| \, dx \, dt \right) \]
\[
\lesssim h \|\phi\|_{L^{\infty}(0, T; W^{2, \infty}(\Omega_h^C))} + h \int_0^T \int_{\Omega_h^C} |r_h u_h| \, dx \, dt \|\Delta_h \Pi_T \phi\|_{L^{\infty}(0, T; \Omega_h^C)} \\
+ h \int_0^T \int_{\Omega_h^C} |r_h \Phi_T(\{\{u_h, h\}\})| \, dx \, dt \|\nabla \delta \Pi_T \phi\|_{L^{\infty}(0, T; \Omega_h^C)}
\]
which implies that for \((r_h, \phi) = (\varrho_h, \phi_1)\) and \((\varrho_h u_h, \tilde{u})\) we have

\[
|E_1(\varrho_h, \phi_1)| \lesssim h + h^{-1} \int_0^T \int_{\Omega_h^C} |u_h| \, dx \, dt + \int_0^T \int_{\Omega_h^C} |\nabla \delta \phi| \, dx \, dt,
\]

\[
|E_1(\varrho_h u_h, \tilde{u})| \lesssim h + \int_0^T \int_{\Omega_h^C} \varrho_h |u_h|^2 \, dx \, dt.
\]

- **\(E_2(r_h, \phi)\):** Let us define by \(E_C\) the set of all faces inside \(\Omega_h^C\). Then applying triangular inequality, Hölder’s inequality and the boundedness of \(\varrho_h\) we have

\[
|E_2(\varrho_h, \phi_1)| \lesssim h \int_0^T \int_{E \setminus E_C} |\{\{\varrho_h u_h\}\}| \, dS_x \, dt + \int_0^T \int_{E_C} |\{\{\varrho_h u_h\}\}| \cdot n \, dS_x \, dt + \lesssim h + \int_0^T \int_{\Omega_h^C} |\nabla \delta \phi| \, dx \, dt
\]
and

\[
|E_2(\varrho_h u_h, \tilde{u})| \lesssim h \|\tilde{u}\|_{L^{\infty}(0, T; W^{1, \infty}(T^h))} \left(\int_0^T \int_{E} |\{\{\varrho_h u_h\}\}|^2 \, dS_x \, dt\right)^{1/2} \left(\int_0^T \int_{E} |\varrho_h u_h|^2 \, dS_x \, dt\right)^{1/2} = h^{3/2} \left(\int_0^T \int_{E} |\varrho_h u_h|^2 \, dS_x \, dt\right)^{1/2} \approx h.
\]

- **\(E_3(r_h, \phi)\):** Using triangular inequality and (B.9b) we have

\[
|E_3(\varrho_h, \phi_1)| = h^\alpha \int_0^T \int_{E} |\varrho_h| |\Pi_T \phi| \, dS_x \, dt \\
\lesssim h^\alpha \int_0^T \int_{E_C} 1 \, dS_x \, dt + h^{1+\alpha} \int_0^T \int_{E \setminus E_C} |\varrho_h| \, dS_x \, dt \lesssim h^\alpha + h^{(1+\alpha)/2}.
\]
Moreover, using (D.4), (C.1c), and (C.2f) we have

\[
|E_3(\varrho_h u_h, \tilde{u})| = h^{\alpha+1} \int_0^T \int_{T^h} \varrho_h u_h \cdot \Delta_h \Pi_T \tilde{u} \, dx \, dt \lesssim h^\alpha + 1 + h^\alpha \int_0^T \int_{\Omega_h} |u_h| \, dx \, dt.
\]

- **\(E_4(\varrho_h u_h, \tilde{u})\):** By Hölder’s inequality and (C.4a) we have

\[
|E_4(\varrho_h u_h, \tilde{u})| \lesssim h + \int_0^T \int_{\Omega_h} \varrho_h |u_h|^2 \, dx \, dt.
\]

- **\(E_4^{\text{new}}(\varrho_h, \phi_1)\):** Analogously, we have

\[
|E_4^{\text{new}}(\varrho_h, \phi_1)| = \left(\int_0^T \int_{\Omega_h} \varrho_h u_h \cdot (\nabla_h (\Pi_T \phi_1) - \nabla \phi_1) \, dx \, dt\right) + \int_0^T \int_{\Omega_h} \varrho_h u_h \cdot \nabla_h (\Pi_T \phi_1) \, dx \, dt.
\]

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Finally, applying Lemma C.2 finishes the proof.

In this section we estimate $e_R$ defined in (D.3). We begin with the following lemma.

**Lemma D.2.** Let $\gamma > 1$ and $(\varrho_h, u_h)$ be a solution obtained by the FV method (3.3) with $(h, \epsilon) \in (0,1)^2$. Let $(\tilde{\varrho}, \tilde{u})$ be the extended strong solution in the sense of Definition 5.1. Then the following holds

$$\|\mathbf{u}_h - \tilde{\mathbf{u}}\|_{L^2(T^d; \mathbb{R}^d)} \lesssim h + \|\nabla \varphi \mathbf{u}_h - \nabla \varphi \tilde{\mathbf{u}}\|_{L^2(T^d)} + R_E(\varrho_h, u_h; \tilde{\varrho}, \tilde{\mathbf{u}}).$$

**Proof.** Firstly, by setting $f_h = \mathbf{u}_h - \Pi_T \tilde{\mathbf{u}}$ in Lemma A.1 we know that

$$\|\mathbf{u}_h - \Pi_T \tilde{\mathbf{u}}\|_{L^2(T^d; \mathbb{R}^d)} \lesssim \left( \|\nabla \varphi (\mathbf{u}_h - \Pi_T \tilde{\mathbf{u}})\|_{L^2(T^d; \mathbb{R}^d)} + \int_{T^d} \varrho_h \|\mathbf{u}_h - \Pi_T \tilde{\mathbf{u}}\|^2 \, dx \right).$$

Then by the triangular inequality, projection error estimate and (C.4b) we derive

$$\|\mathbf{u}_h - \tilde{\mathbf{u}}\|_{L^2(T^d; \mathbb{R}^d)} \leq \|\mathbf{u}_h - \Pi_T \tilde{\mathbf{u}}\|_{L^2(T^d; \mathbb{R}^d)} + \|\Pi_T \mathbf{u} - \tilde{\mathbf{u}}\|_{L^2(T^d; \mathbb{R}^d)}$$

$$\lesssim \|\nabla \varphi (\mathbf{u}_h - \Pi_T \tilde{\mathbf{u}})\|_{L^2(T^d; \mathbb{R}^d)} + \int_{T^d} \varrho_h \|\mathbf{u}_h - \Pi_T \tilde{\mathbf{u}}\|^2 \, dx + \left( h \|\nabla \tilde{\mathbf{u}}\|_{L^\infty(T^d; \mathbb{R}^d)} \right)^2.$$
where div
\[
\lesssim \|\nabla \epsilon u_h - \nabla_x \tilde{u}\|_{L^2(T^d;\mathbb{R}^{d \times d})}^2 + \int_{T^d} \theta_h |u_h - \tilde{u}|^2 \, dx
\]

+ \|\nabla_x \tilde{u} - \nabla \epsilon \Pi_T \tilde{u}\|_{L^2(T^d;\mathbb{R}^{d \times d})}^2 + \int_{T^d} \theta_h \|\Pi_T \tilde{u} - \tilde{u}\|^2 \, dx + h^2
\]

\[
\lesssim \|\nabla \epsilon u_h - \nabla_x \tilde{u}\|_{L^2(T^d;\mathbb{R}^{d \times d})}^2 + R_E(\theta_h, u_h|\tilde{\phi}, \tilde{u}) + h,
\]

which completes the proof. \(\square\)

We are now ready to show the estimate of \(e_R\) given in (D.3).

**Lemma D.3.** Let \(\gamma > 1\) and \((\theta_h, u_h)\) be a solution of the FV method (3.3) with \((h, \epsilon) \in (0, 1)^2\). Let \((\tilde{\phi}, \tilde{u})\) be the extended strong solution in the sense of Definition 5.1. Suppose that \(\theta_h\) is uniformly bounded from above by a positive constant. Then

\[
|e_R| \lesssim h + \frac{\epsilon}{h} + \int_0^\tau R_E(\theta_h, u_h|\tilde{\phi}, \tilde{u}) \, dt + \delta \frac{\|u_h\|_{L^2((0, \tau) \times \Omega^*_h)}^2}{\epsilon}
\]

\[
+ \delta \mu \|\nabla \epsilon u_h - \nabla_x \tilde{u}\|_{L^2((0, \tau) \times T^d)}^2 + \delta \nu \|\text{div}_h u_h - \text{div}_x \tilde{u}\|_{L^2((0, \tau) \times T^d)}^2.
\]

**Proof.** Firstly, thanks to the regularity of the strong solution we can control \(R_2 + R_5\) by means of relative energy

\[
|R_2 + R_5| \lesssim \int_0^\tau R_E(\theta_h, u_h|\tilde{\phi}, \tilde{u}) \, dt.
\]

Secondly, thanks to Lemmas A.2 and D.2 we directly obtain the estimate of \(R_1\)

\[
|R_1| \lesssim h + \int_0^\tau R_E(\theta_h, u_h|\tilde{\phi}, \tilde{u}) \, dt + \delta \|\nabla \epsilon u_h - \nabla_x \tilde{u}\|_{L^2((0, \tau) \times T^d)}^2.
\]

Thirdly, by the Gauss theorem we know that

\[
\int_K \text{div}_x U \, dx = \int_K \text{div}_T \Pi_\epsilon U \, dx \text{ for any } K \in T_h,
\]

where \(\text{div}_T = \sum_{i=1}^d \tilde{\phi}_T^{(i)}\) and \(\tilde{\phi}_T^{(i)}\) is defined in the previous subsection. Thus we have

\[
\int_{\Omega_f} u_h \cdot \Delta_x \tilde{u} \, dx = \int_{\Omega_f} u_h \cdot \text{div}_T \Pi_\epsilon \nabla_x \tilde{u} \, dx.
\]

Using this identity and \(\tilde{u}|_{\Omega^*_T} = 0\) we observe

\[
R_3 = -\mu \int_0^\tau \int_{T^d} \nabla \epsilon u_h : \nabla_x \tilde{u} \, dx \, dt - \mu \int_0^\tau \int_{\Omega_f} u_h \cdot \Delta_x \tilde{u} \, dx \, dt
\]

\[
= -\mu \int_0^\tau \int_{T^d} \nabla \epsilon u_h : \nabla_x \tilde{u} \, dx \, dt - \mu \int_0^\tau \int_{\Omega_f} u_h \cdot \text{div}_T \Pi_\epsilon \nabla_x \tilde{u} \, dx \, dt - \mu \int_0^\tau \int_{\Omega_f \setminus \Omega_f^h} u_h \cdot \Delta_x \tilde{u} \, dx \, dt
\]

\[
= -\mu \int_0^\tau \int_{T^d} \left( \nabla \epsilon u_h : \nabla_x \tilde{u} + u_h \cdot \text{div}_T \Pi_\epsilon \nabla_x \tilde{u} \right) \, dx \, dt + I_1 = I_1 + I_2,
\]

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where
\[
I_1 = \mu \int_0^T \int_{\Omega_h^0} \mathbf{u}_h \cdot \nabla \Pi E \nabla_x \tilde{\mathbf{u}} \, dx \, dt - \mu \int_0^T \int_{\Omega^f \setminus \Omega_h^0} \mathbf{u}_h \cdot \Delta_x \tilde{\mathbf{u}} \, dx \, dt,
\]
\[
I_2 = \mu \int_0^T \int_{\mathcal{T}^d} \nabla \epsilon \mathbf{u}_h : (\Pi E \nabla_x \tilde{\mathbf{u}} - \nabla_x \tilde{\mathbf{u}}) \, dx \, dt = 0.
\]
Noticing that \( \nabla \Pi E \nabla_x \tilde{\mathbf{u}} \big|_{\Omega_h^0} = 0 \), \( |\nabla \Pi E \nabla_x \tilde{\mathbf{u}}|_{\Omega_h^0} \lesssim h^{-1} \) and \( |\Delta_x \tilde{\mathbf{u}}|_{\Omega^f \setminus \Omega_h^0} \lesssim 1 \), we can control \( R_3 = I_1 \) by (C.2f) as
\[
|R_3| \lesssim (1 + h^{-1}) \int_0^T \int_{\Omega_h^0} |\mathbf{u}_h| \, dx \, dt \lesssim h + \frac{\epsilon}{h} + \delta \mu \|\nabla E \mathbf{u}_h - \nabla_x \tilde{\mathbf{u}}\|^2_{L^2((0,\tau) \times \mathcal{T}^d)} + \delta \frac{\|\mathbf{u}_h\|^2_{L^2((0,\tau) \times \Omega_h^0)}}{\epsilon}.
\]
To estimate \( R_4 \) we use the Gauss theorem again for \( D_\sigma, \sigma \in \mathcal{E} \)
\[
\int_{D_\sigma} \partial_\sigma \nabla \cdot \tilde{\mathbf{u}} \, dx = \int_{D_\sigma} \partial_\sigma^{(i)} \Pi E \nabla \cdot \tilde{\mathbf{u}} \, dx,
\]
where \( \Pi E \phi|_K := \frac{1}{|E|} \int_E \phi \, dS_x \) for \( \epsilon \in \partial D_\sigma \cap K \). Thanks to this equality and the integration by parts (D.4) and the equality \( \nabla \cdot \mathbf{u}_h = \sum_{i=1}^d \partial_\sigma^{(i)} \{u_{i,h}\}^{(i)} \) we have
\[
\int_{\Omega_f} \mathbf{u}_h \cdot \nabla \cdot \tilde{\mathbf{u}} \, dx = \int_{\Omega_h^f} \mathbf{u}_h \cdot \nabla \cdot \tilde{\mathbf{u}} \, dx + \int_{\Omega^f \setminus \Omega_h^f} \mathbf{u}_h \cdot \nabla \cdot \tilde{\mathbf{u}} \\
= \int_{\Omega_h^f} \{u_h\} \cdot \nabla \cdot \tilde{\mathbf{u}} \, dx + \int_{\Omega^f \setminus \Omega_h^f} \mathbf{u}_h \cdot \nabla \cdot \tilde{\mathbf{u}} \, dx \\
= \sum_{i=1}^d \int_{\mathcal{T}^d} \{u_{i,h}\}^{(i)} \partial_\sigma^{(i)} \Pi E \nabla \cdot \tilde{\mathbf{u}} \, dx + J_1 - \sum_{i=1}^d \int_{\Omega_h^f} \{u_{i,h}\}^{(i)} \partial_\sigma^{(i)} \Pi E \nabla \cdot \tilde{\mathbf{u}} \, dx \\
= - \sum_{i=1}^d \int_{\mathcal{T}^d} \{u_{i,h}\}^{(i)} \partial_\sigma^{(i)} \Pi E \nabla \cdot \tilde{\mathbf{u}} \, dx + J_1 + J_2 = - \int_{\mathcal{T}^d} \nabla \cdot \mathbf{u}_h \Pi E \nabla \cdot \tilde{\mathbf{u}} \, dx + J_1 + J_2.
\]
Thus, together with (C.4a), (C.2b), (C.2e), and (C.2f) we have
\[
|R_4| = \left| \int_0^T \int_{\mathcal{T}^d} \nabla \cdot \mathbf{u}_h (\Pi E \nabla \cdot \tilde{\mathbf{u}} - \nabla \cdot \tilde{\mathbf{u}}) \, dx \, dt \right| - \int_0^T (J_1 + J_2) \, dt \\
\lesssim h + \int_{\Omega_h^f} |\nabla \cdot \mathbf{u}_h| \, dx + h \int_0^T \int_{\Omega_h^f} |\nabla \cdot \mathbf{u}_h| \, dx \, dt + \int_0^T \int_{\Omega^f \setminus \Omega_h^f} |\mathbf{u}_h| \, dx \, dt + h^{-1} \int_0^T \int_{\Omega_h^0} |\mathbf{u}_h| \, dx \, dt \\
\lesssim \epsilon + h + \frac{\epsilon}{h} + \delta \frac{\|\mathbf{u}_h\|^2_{L^2((0,\tau) \times \Omega_h^0)}}{\epsilon} + \delta \frac{\|\nabla \cdot \mathbf{u}_h - \nabla \cdot \tilde{\mathbf{u}}\|^2_{L^2((0,\tau) \times \mathcal{T}^d)}}{\epsilon} + \delta \nu \|\nabla \cdot \mathbf{u}_h - \nabla \cdot \tilde{\mathbf{u}}\|^2_{L^2((0,\tau) \times \mathcal{T}^d)}.
\]
Consequently, collecting the estimates of \( R_i \)-terms completes the proof. □