CONTROL PROBLEM ON SPACE OF RANDOM VARIABLES AND
MASTER EQUATION

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Abstract

We study in this paper a control problem in a space of random variables. We show that its Hamilton
Jacobi Bellman equation is related to the Master equation in Mean field theory. P.L. Lions in ([14]),([15]
introduced the Hilbert space of square integrable random variables as a natural space for writing the
Master equation which appears in the mean field theory. W. Gangbo and A. Święch [10] considered this
type of equation in the space of probability measures equipped with the Wasserstein metric and use the
concept of Wasserstein gradient. We compare the two approaches and provide some extension of the
results of Gangbo and Święch.

1 INTRODUCTION

We study first an abstract control problem where the state is in a Hilbert space. We then show how this
model applies when the Hilbert space is the space of square integrable random variables, and for certain
forms of the cost functions. We see that it applies directly to the solution of the Master equation in Mean
Field games theory. We compare our results with those of W. Gangbo and A. Święch [10] and show that
the approach of the Hilbert space of square integrable random variables simplifies greatly the development.

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2 AN ABSTRACT CONTROL PROBLEM

2.1 SETTING OF THE PROBLEM

We begin by defining an abstract control problem, without describing the application. We consider a Hilbert space $\mathcal{H}$, whose elements are denoted by $X$. We identify $\mathcal{H}$ with its dual. The scalar product is denoted by $(\cdot, \cdot)$ and the norm by $\|\cdot\|$. We then consider functionals $F(X)$ and $F_T(X)$ which are continuously differentiable on $\mathcal{H}$. The gradients $D_X F(X)$ and $D_X F_T(X)$ are Lipschitz continuous

\begin{align}
\|D_X F(X_1) - D_X F(X_2)\| &\leq c\|X_1 - X_2\| \\
\|D_X F_T(X_1) - D_X F_T(X_2)\| &\leq c\|X_1 - X_2\|
\end{align}

(2.1)

To simplify notation, we shall also assume that

\begin{equation}
\|D_X F(0)\|, \|D_X F_T(0)\| \leq c
\end{equation}

(2.2)

So we have

\begin{equation}
\|D_X F(X)\| \leq c(1 + \|X\|)
\end{equation}

(2.3)

and

\begin{equation}
|F(X)| \leq C(1 + \|X\|^2),
\end{equation}

(2.4)

where we denote by $C$ a generic constant. The same estimates hold also for $F_T(X).

A control is a function $v(s)$ which belongs to $L^2(0, T; \mathcal{H})$. We associate to a control $v(.)$ the state $X(s)$ satisfying

\begin{equation}
\frac{dX}{ds} = v(s)
\end{equation}

(2.5)

\begin{align*}
X(t) &= X
\end{align*}

We may write it as $X_{X_t}(s)$ to emphasize the initial conditions and even $X_{X_t}(s; v(\cdot))$ to emphasize the dependence in the control. The function $X(s)$ belongs to the Sobolev space $H^1(t, T; \mathcal{H})$. We then define the cost functional
\[ J_{X_t}(v(.)) = \frac{\lambda}{2} \int_t^T \|v(s)\|^2 ds + \int_t^T F(X(s))ds + \mathcal{F}_T(X(T)) \] (2.6)

and the value function

\[ V(X, t) = \inf_{v(.)} J_{X_t}(v(.)) \] (2.7)

### 2.2 BELLMAN EQUATION

We want to show the following

**Theorem 1.** We assume (2.1), (2.2) and

\[ \lambda > cT(1 + T) \] (2.8)

The value function (2.7) is \( C^1 \) and satisfies the growth conditions

\[ |V(X, t)| \leq C(1 + \|X\|^2) \] (2.9)

\[ ||D_XV(X, t)|| \leq C(1 + ||X||), \quad \left| \frac{\partial V(X, t)}{\partial t} \right| \leq C(1 + ||X||^2) \]

where \( C \) is a generic constant. Moreover \( D_XV(X, t) \) and \( \frac{\partial V(X, t)}{\partial t} \) are Lipschitz continuous, more precisely

\[ ||D_XV(X^1, t^1) - D_XV(X^2, t^2)|| \leq C||X^1 - X^2|| + C|t^1 - t^2|(1 + ||X^1|| + ||X^2||) \] (2.10)

\[ \left| \frac{\partial V(X^1, t^1)}{\partial t} - \frac{\partial V(X^2, t^2)}{\partial t} \right| \leq C||X^1 - X^2||(1 + ||X^1|| + ||X^2||) + C|t^1 - t^2|(1 + ||X^1||^2 + ||X^2||^2) \]

It is the unique solution, satisfying conditions (2.7) and (2.10) of Bellman equation

\[ \frac{\partial V}{\partial t} \left( - \frac{1}{2\lambda} ||D_XV||^2 + F(X) \right) = 0 \]

\[ V(X, T) = \mathcal{F}_T(X) \] (2.11)

The control problem (2.5), (2.6) has a unique solution.

**Proof.** We begin by studying the properties of the cost functional \( J_{X_t}(v(.)) \). We first claim that \( J_{X_t}(v(.)) \) is
Gâteaux differentiable in the space $L^2(t, T; \mathcal{H})$, for $X, t$ fixed. Define $X_v(s)$ by

$$\frac{dX_v(s)}{ds} = v(s), \quad X_v(t) = X$$

and $Z_v(s)$ by

$$-\frac{dZ_v(s)}{ds} = D_X \mathcal{F}(X_v(s)), \quad Z_v(t) = D_X \mathcal{F}_T(X_v(T))$$

then we can prove easily that

$$\frac{d}{d\mu} J_{X_t}(v(\cdot) + \mu \tilde{v}(\cdot))|_{\mu=0} = \int_t^T ((\lambda v(s) + Z_v(s), \tilde{v}(s))) ds$$

(2.12)

Let us prove that the functional $J_{X_t}(v(\cdot))$ is strictly convex. Let $v_1(\cdot)$ and $v_2(\cdot)$ in $L^2(t, T; \mathcal{H})$. We write

$$J_{X_t}(\theta v_1(\cdot) + (1 - \theta) v_2(\cdot)) = J_{X_t}(v_1(\cdot) + (1 - \theta)(v_2(\cdot) - v_1(\cdot)))$$

$$= J_{X_t}(v_1(\cdot)) + \int_0^1 \frac{d}{d\mu} J_{X_t}(v_1(\cdot) + \mu(1 - \theta)(v_2(\cdot) - v_1(\cdot))) d\mu$$

From formula (2.12) we have also

$$\frac{d}{d\mu} J_{X_t}(v(\cdot) + \mu \tilde{v}(\cdot)) = \theta \int_t^T ((\lambda v(s) + \mu \tilde{v}(s)) + Z_{v+\mu \tilde{v}}(s), \tilde{v}(s))) ds$$

Therefore

$$\int_0^1 \frac{d}{d\mu} J_{X_t}(v_1(\cdot) + \mu(1 - \theta)(v_2(\cdot) - v_1(\cdot))) d\mu = (1 - \theta) \int_0^1 \frac{d}{d\mu} \left\{ \int_t^T ((\lambda v_1(s) + \mu(1 - \theta)(v_2(s) - v_1(s))) + Z_{v_1+\mu(1-\theta)(v_2-v_1)}(s), v_2(s) - v_1(s)) ds \right\}$$

Similarly we write

$$J_{X_t}(\theta v_1(\cdot) + (1 - \theta)v_2(\cdot)) = J_{X_t}(v_2(\cdot) + \theta(v_1(\cdot) - v_2(\cdot)))$$

$$= J_{X_t}(v_2(\cdot)) + \int_0^1 \frac{d}{d\mu} J_{X_t}(v_2(\cdot) + \mu \theta(v_1(\cdot) - v_2(\cdot))) d\mu$$

and
Going back to (2.13) we obtain easily

\[
\int_0^1 \frac{d}{d\mu}J_{X_t}(v_2(.)) + \mu\theta(v_1(.) - v_2(.)) d\mu = \theta \int_0^1 d\mu \left\{ \int_t^T ([\lambda(v_2(s) + \mu\theta(v_1(s) - v_2(s)) + \\
+ Z_{v_2 + \mu\theta(v_1 - v_2)}(s), v_1(s) - v_2(s))]ds \right\}
\]

We shall set \(Z_1(s) = Z_{v_1 + \mu(1-\theta)(v_2 - v_1)}(s)\) and \(Z_2(s) = Z_{v_2 + \mu\theta(v_1 - v_2)}(s)\). Combining formulas, we can write

\[
J_{X_t}(\theta v_1(.) + (1-\theta)v_2(.) = \theta J_{X_t}(v_1(.)) + (1-\theta)J_{X_t}(v_2(.)) + \\
+ \theta(1-\theta) \left[ -\frac{\lambda}{2} \int_t^T ||v_1(s) - v_2(s)||^2 ds + \int_0^1 d\mu \int_t^T ((Z_1(s) - Z_2(s), v_2(s) - v_1(s))ds \right]
\]

Let \(X_1(s)\) and \(X_2(s)\) denote the states corresponding to the controls \(v_1(.) + \mu(1-\theta)(v_2(.) - v_1(.)\) and \(v_2(.) + \mu\theta(v_1(.) - v_2(.)\). One checks easily that

\[
X_1(s) - X_2(s) = (1 - \mu) \int_t^s (v_1(\sigma) - v_2(\sigma)) d\sigma
\]

and from the definition of \(Z_1(.)\), \(Z_2(.)\) we obtain

\[
||Z_1(s) - Z_2(s)|| \leq c||X_1(T) - X_2(T)|| + \int_s^T ||X_1(\sigma) - X_2(\sigma)||d\sigma
\]

and combining formulas, we can assert

\[
||Z_1(s) - Z_2(s)|| \leq c(1 - \mu)(1 + T) \int_t^T ||v_1(\sigma) - v_2(\sigma)||d\sigma
\]

Going back to (2.13) we obtain easily

\[
J_{X_t}(\theta v_1(.) + (1-\theta)v_2(.) \leq \theta J_{X_t}(v_1(.)) + (1-\theta)J_{X_t}(v_2(.)) + \\
- \frac{\theta(1-\theta)}{2} (\lambda - cT(1 + T)) \int_t^T ||v_1(s) - v_2(s)||^2 ds
\]

and from the assumption (2.8) we obtain immediately that \(J_{X_t}(v(.))\) is strictly convex. Next we write

\[
\mathcal{F}(X(s)) - \mathcal{F}(X) = \int_0^1 ((D_X\mathcal{F}(X + \theta \int_t^s v(\sigma)d\sigma), \int_t^s v(\sigma)d\sigma))
\]

so, using (2.3) we obtain
\[ |\mathcal{F}(X(s)) - \mathcal{F}(X)| \leq c(1 + ||X||) ||\int_t^s v(\sigma) d\sigma|| + \frac{c}{2} ||\int_t^s v(\sigma) d\sigma||^2 \]

\leq \frac{c^2(1 + ||X||)^2}{2\delta} + \frac{c + \delta}{2} ||\int_t^s v(\sigma) d\sigma||^2

for any \( \delta > 0 \). Using (2.4) we can assert that

\[ |\mathcal{F}(X(s))| \leq C_\delta (1 + ||X||^2) + \frac{c + \delta}{2T} \int_t^T ||v(\sigma)||^2 d\sigma \]

A similar estimate holds for \( \mathcal{F}_T(X(T)) \). Therefore, collecting results, we obtain

\[ |\int_t^T \mathcal{F}(X(s)) ds + \mathcal{F}_T(X(T))| \leq C_\delta (1 + ||X||^2)(1 + T) + \frac{c + \delta}{2T} (1 + T) \int_t^T ||v(\sigma)||^2 ds \]

It follows that

\[ J_{X_t}(v(.)) \geq \frac{\lambda - (c + \delta)T(1 + T)}{2} \int_t^T ||v(\sigma)||^2 ds - C_\delta (1 + ||X||^2)(1 + T) \]

(2.15)

Since \( \lambda - cT(1 + T) > 0 \), we can find \( \delta > 0 \) sufficiently small so that \( \lambda - (c + \delta)T(1 + T) > 0 \). This implies that \( J_{X_t}(v(.)) \rightarrow +\infty \) as \( \int_t^T ||v(\sigma)||^2 ds \rightarrow +\infty \). This property and the strict convexity imply that the functional \( J_{X_t}(v(.)) \) has a minimum which is unique. The Gâteaux derivative must vanish at this minimum denoted by \( u(.) \). The corresponding state is denoted by \( Y(.) \). From formula (2.12) we obtain also the existence of a solution of the two-point boundary value problem

\[ \frac{dY}{ds} = -\frac{Z(s)}{\lambda}, \quad -\frac{dZ}{ds} = D_X \mathcal{F}(Y(s)) \]

\[ Y(t) = X, \quad Z(T) = D_X \mathcal{F}_T(Y(T)) \]

and the optimal control \( u(.) \) is given by the formula

\[ u(s) = -\frac{Z(s)}{\lambda} \]

(2.17)

In fact, the system (2.16) can be studied directly, and we can show directly that it has one and only one solution. We notice that it is a 2nd order differential equation, since
\[
\frac{d^2 Y}{ds^2} = \frac{1}{\lambda} D_X \mathcal{F}(Y(s))
\]

\[
Y(t) = X - \frac{s-t}{\lambda} D_X F_T(Y(T)) - \frac{1}{\lambda} \int_t^T D_X \mathcal{F}(Y(\sigma))(s \wedge \sigma - t) d\sigma
\] (2.19)

We can write also (2.18) as an integral equation

\[
Y(s) = X - \frac{s-t}{\lambda} D_X F_T(Y(T)) - \frac{1}{\lambda} \int_t^T D_X \mathcal{F}(Y(\sigma))(s \wedge \sigma - t) d\sigma
\] (2.19)

and we can view this equation as a fixed point equation in the space \(C^0([t, T]; \mathcal{H})\), namely \(Y(.) = \mathcal{K}(Y(.))\), where \(\mathcal{K}\) is defined by the right hand side of (2.19). One can show that \(\mathcal{K}\) is a contraction, hence \(Y(.)\) is uniquely defined. Note also, that if we have a solution of (2.16) and if \(u(.)\) is defined by (2.17) the control \(u(.)\) satisfies the necessary condition of optimality for the functional \(J_{X_t}(\nu(.))\). Since this functional is convex, the necessary condition of optimality is also sufficient and thus \(u(.)\) is optimal. The value function is thus defined by the formula

\[
V(X, t) = \frac{1}{2\lambda} \int_t^T ||Z(\sigma)||^2 d\sigma + \int_t^T \mathcal{F}(Y(\sigma)) d\sigma + \mathcal{F}_T(Y(T))
\] (2.20)

We now study the properties of the value function. We begin with the first property (2.9). Using (2.15) we obtain

\[
V(X, t) \geq -C(1 + ||X||^2)
\]

On the other hand, we have

\[
V(X, t) \leq J_{X_t}(0) = (T-t)\mathcal{F}(X) + \mathcal{F}_T(X) \leq C(1 + ||X||^2)
\]

and the first estimate (2.9) is obtained.

We proceed in getting estimates for the solution \(Y(.)\) of (2.19). We write

\[
||Y(.)|| = \sup_{t \leq s \leq T} ||Y(s)||
\]

Using easy majorations, we obtain
\[
\begin{align*}
||Y(.)|| &\leq \frac{||X|| + cT(T+1)}{\lambda - cT(T+1)}\quad (2.21) \\
||Z(.)|| &\leq \frac{\lambda(1+T)c(1 + ||X||)}{\lambda - cT(T+1)} \\
||u(.)|| &\leq \frac{(1+T)c(1 + ||X||)}{\lambda - cT(T+1)}
\end{align*}
\]

We then study how these functions depend on the pair \(X, t\). We recall that \(Y(s) = Y_{X,t}(s)\). Let us consider two points \(X_1, t_1\) and \(X_2, t_2\) and denote \(Y_1(s) = Y_{X_1,t_1}(s)\), \(Y_2(s) = Y_{X_2,t_2}(s)\). To fix ideas we assume \(t_1 < t_2\). For \(s > t_2\) we have

\[
Y_1(s) - Y_2(s) = X_1 - X_2 - \frac{1}{\lambda}(D_X F_T(Y_1(T)) - D_X F_T(Y_2(T))) (s - t_2) - \frac{1}{\lambda} D_X F_T(Y_1(T)) (t_2 - t_1) - \frac{1}{\lambda} \int_{t_2}^{T} (D_X F(Y_1(\sigma)) - D_X F(Y_2(\sigma))) (s \wedge \sigma - t_2) d\sigma - \frac{1}{\lambda} \int_{t_1}^{t_2} (D_X F(Y_1(\sigma)) (s \wedge \sigma - t_1) d\sigma
\]

From which we obtain

\[
\sup_{t_2 \leq s \leq T} ||Y_1(s) - Y_2(s)|| \leq ||X_1 - X_2|| + \frac{cT(1+T)}{\lambda} \sup_{t_2 \leq s \leq T} ||Y_1(s) - Y_2(s)|| + \frac{t_2 - t_1}{\lambda} [||D_X F_T(Y_1(T))|| + \int_{t_1}^{T} ||D_X F(Y_1(s))|| ds]
\]

Using the properties of \(D_X F\) and \(D_X F_T\) and (2.21) we can assert that

\[
\sup_{t_2 \leq s \leq T} ||Y_1(s) - Y_2(s)|| \leq \frac{\lambda}{\lambda - cT(T+1)} \left(||X_1 - X_2|| + (t_2 - t_1)(1+T)c \frac{1 + ||X_1||}{\lambda - cT(T+1)}\right)
\]

More globally we can write

\[
\sup_{\max(t_1, t_2) \leq s \leq T} ||Y_{X_1,t_1}(s) - Y_{X_2,t_2}(s)|| \leq \frac{\lambda}{\lambda - cT(T+1)} \left(||X_1 - X_2|| + |t_2 - t_1|(1+T)c \frac{1 + \max(||X_1||, ||X_2||)}{\lambda - cT(T+1)}\right)
\]

(2.22)

In particular

\[
\sup_{t \leq s \leq T} ||Y_{X,t}(s) - Y_{X,s}(s)|| \leq \frac{\lambda||X_1 - X_2||}{\lambda - cT(T+1)}
\]

(2.23)

Recalling that from the system (2.16) we have
\[
Z(s) = \int_s^T D_X \mathcal{F}(Y(\sigma))d\sigma + D_X \mathcal{F}_T(Y(T))
\]

and noting \(Z(s) = Z_X(t)\) we deduce from (2.23) that

\[
\sup_{t \leq s \leq T} \|Z_{X_1}(s) - Z_{X_2}(s)\| \leq \frac{c(T + 1)\|X_1 - X_2\|}{\lambda - c(T + 1)}
\]  
\[
(2.24)
\]

We next write

\[
J_{X_1}(u_1(.)) - J_{X_2}(u_1(.)) \leq V(X_1, t) - V(X_2, t) \leq J_{X_1}(u_2(.)) - J_{X_2}(u_2(.))
\]

where \(u_1(.)\) and \(u_2(.)\) are the optimal controls for the problems with initial conditions \((X_1, t)\) and \((X_2, t)\), respectively. Denoting by \(Y_{X_1}(s)\) and \(Y_{X_2}(s)\) the optimal states and by \(Y_{X_1}(s; u_2(.)), Y_{X_2}(s; u_1(.))\) the trajectories (not optimal) when the control \(u_2(.)\) is used with the initial conditions \((X_1, t)\) and when the control \(u_1(.)\) is used with the initial conditions \((X_2, t)\), we have

\[
Y_{X_1}(s; u_2(.)) - Y_{X_2}(s) = Y_{X_1}(s) - Y_{X_2}(s; u_1(.)) = X_1 - X_2
\]

Therefore

\[
V(X_1, t) - V(X_2, t) \leq \int_t^T (\mathcal{F}(Y_{X_2}(s)) + X_1 - X_2) - \mathcal{F}(Y_{X_2}(s)))ds + \mathcal{F}_T(Y_{X_2}(T) + X_1 - X_2) - \mathcal{F}_T(Y_{X_2}(T))
\]

and by techniques already used it follows

\[
V(X_1, t) - V(X_2, t) \leq (\int_t^T D_X \mathcal{F}(Y_{X_2}(s))ds + D_X \mathcal{F}_T(Y_{X_2}(T), X_1 - X_2)) + \frac{c}{2}(1 + T)\|X_1 - X_2\|^2
\]

which is in fact

\[
V(X_1, t) - V(X_2, t) \leq ((Z_{X_2}(t), X_1 - X_2)) + \frac{c}{2}(1 + T)\|X_1 - X_2\|^2
\]

(2.25)

By interchanging the roles of \(X_1\) and \(X_2\) we also obtain

\[
V(X_1, t) - V(X_2, t) \geq ((Z_{X_1}(t), X_1 - X_2)) - \frac{c}{2}(1 + T)\|X_1 - X_2\|^2
\]

(2.26)

Using the estimate (2.24) we can also write
\[ V(X_1, t) - V(X_2, t) \geq ((Z_{X,t}(t), X_1 - X_2)) - c(T + 1)\left[ \frac{\lambda}{\lambda - cT(T + 1)} + \frac{1}{2}\|X_1 - X_2\|^2 \right] \quad (2.27) \]

Combining (2.26) and (2.27) we immediately get

\[ |V(X_1, t) - V(X_2, t) - ((Z_{X,t}(t), X_1 - X_2))| \leq c(T + 1)\left[ \frac{\lambda}{\lambda - cT(T + 1)} + \frac{1}{2}\|X_1 - X_2\|^2 \right] \quad (2.28) \]

This shows immediately that \( V(X, t) \) is differentiable in \( X \) and that

\[ D_X V(X, t) = Z(t) = -\lambda u(t) \quad (2.29) \]

From the 2nd estimate (2.21) we immediately obtain the 2nd estimate (2.9). We continue with the derivative in \( t \). We first write the optimality principle

\[ V(X, t) = \frac{\lambda}{2} \int_t^{t+\varepsilon} ||u(s)||^2 ds + \int_t^{t+\varepsilon} \mathcal{F}(Y(s)) ds + V(Y(t+\varepsilon), t+\varepsilon) \quad (2.30) \]

which is a simple consequence of the definition of the value function and of the existence of an optimal control. From (2.28) we can write

\[ V(X_2, t) - V(X_1, t) - ((Z_{X,t}(t), X_2 - X_1)) \leq C\|X_1 - X_2\|^2 \]

where \( C \) is the constant appearing in the right hand side of (2.28). We apply with \( X_2 = Y(t + \varepsilon) \), \( X_1 = X, t = t + \varepsilon \). We note that \( Z_{Y(t+\varepsilon),t+\varepsilon}(t+\varepsilon) = Z_{X,t}(t+\varepsilon) = -\lambda u(t + \varepsilon) \), since \( u(s) \) for \( t + \varepsilon < s < t \) is optimal for the problem starting with initial conditions \( Y(t+\varepsilon), t+\varepsilon \). Therefore

\[ V(Y(t+\varepsilon), t+\varepsilon) - V(X, t+\varepsilon) \leq -\lambda((u(t+\varepsilon), \int_t^{t+\varepsilon} u(s) ds)) + C\| \int_t^{t+\varepsilon} u(s) ds \|^2 \]

Using this inequality in (2.30) yields

\[ V(X, t) - V(X, t+\varepsilon) \leq \frac{\lambda}{2} \int_t^{t+\varepsilon} ||u(s)||^2 ds + \int_t^{t+\varepsilon} \mathcal{F}(Y(s)) ds - \lambda((u(t+\varepsilon), \int_t^{t+\varepsilon} u(s) ds)) + C\| \int_t^{t+\varepsilon} u(s) ds \|^2 \]

from which we obtain
\[
\liminf_{\epsilon \to 0} \frac{V(X, t + \epsilon) - V(X, t)}{\epsilon} \geq \frac{\lambda}{2} \|u(t)\|^2 - F(X) \tag{2.31}
\]

Next we have

\[
V(X, t + \epsilon) \leq \frac{\lambda}{2} \int_{t+\epsilon}^{T} \|u(s)\|^2 ds + \int_{t+\epsilon}^{T} F(Y(s)) ds + \int_{t}^{t+\epsilon} u(\sigma) d\sigma - \int_{t}^{t+\epsilon} F(Y(s)) ds
\]

therefore

\[
V(X, t + \epsilon) - V(X, t) \leq -\frac{\lambda}{2} \int_{t}^{t+\epsilon} \|u(s)\|^2 ds - \int_{t}^{t+\epsilon} F(Y(s)) ds + \int_{t+\epsilon}^{T} (F(Y(s)) - \int_{t}^{t+\epsilon} u(\sigma) d\sigma - F(Y(s))) ds
\]

and using assumptions on \(F, F_T\) it follows that

\[
V(X, t + \epsilon) - V(X, t) \leq -\frac{\lambda}{2} \int_{t}^{t+\epsilon} \|u(s)\|^2 ds - \int_{t}^{t+\epsilon} F(Y(s)) ds + \int_{t+\epsilon}^{T} (F(Y(s)) - \int_{t}^{t+\epsilon} u(\sigma) d\sigma - F(Y(s))) ds
\]

which means

\[
\limsup_{\epsilon \to 0} \frac{V(X, t + \epsilon) - V(X, t)}{\epsilon} \leq \frac{\lambda}{2} \|u(t)\|^2 - F(X) \tag{2.32}
\]

and comparing with (2.31) we obtain immediately that \(V(X, t)\) is differentiable in \(t\) and the derivative is given by

\[
\frac{\partial V}{\partial t}(X, t) = \frac{\lambda}{2} \|u(t)\|^2 - F(X) \tag{2.33}
\]

Recalling (2.29) we see immediately that \(V(X, t)\) is solution of the HJB equation (2.11). The 2nd estimate
is an immediate consequence of the equation and the estimate on $||D_X V(X,t)||$. We next turn to check the additional estimates (2.10). We have

$$D_X V(X_1,t_1) - D_X V(X_2,t_2) = Z_{X_1t_1}(t_1) - Z_{X_2t_2}(t_2)$$

We assume $t_1 < t_2$ then we can write

$$Z_{X_1t_1}(t_1) - Z_{X_2t_2}(t_2) = \int_{t_1}^{t_2} (D_X F(Y_{X_1t_1}(s)) - D_X F(Y_{X_2t_2}(s)))ds +$$

$$+ D_X F_T(Y_{X_1t_1}(T)) - D_X F_T(Y_{X_2t_2}(T))$$

(2.34)

Using previously used majorations, we can check

$$||Z_{X_1t_1}(t_1) - Z_{X_2t_2}(t_2)|| \leq \frac{\lambda c(T+1)}{\lambda - cT(T+1)} \left(||X_1 - X_2|| + |t_2 - t_1|(1 + T)c \frac{1 + \max(||X_1||, ||X_2||)}{\lambda - cT(T+1)}\right)$$

(2.35)

and the first estimate (2.10) follows immediately. The 2nd estimate (2.10) is a direct consequence of the HJB equation and of the first estimate (2.10). So the value function has the regularity indicated in the statement and satisfies the HJB equation. Let us show that such a solution is necessarily unique. This is a consequence of the verification property. Indeed consider any control $v(.) \in L^2(t,T; H)$ and the state $X(s)$ solution of (2.5). Let $V(x,t)$ be a solution of the HJB equation which is $C^1$ and satisfies (2.9), (2.10). Then the function $V(X(s), s)$ is differentiable and

$$\frac{d}{ds} V(X(s), s) = \frac{\partial V}{\partial s}(X(s), s) + ((D_X V(X(s), s), v(s))) =$$

$$= - F(X(s)) + \frac{1}{2\lambda} ||D_X V(X(s), s)||^2 + ((D_X V(X(s), s), v(s))) \geq - F(X(s)) - \frac{\lambda}{2} ||v(s)||^2$$

from which we get immediately by integration $V(X, t) \leq J_{Xt}(v(.))$. Now if we consider the equation

$$\frac{d\hat{X}(s)}{ds} = - \frac{1}{\lambda} D_X V(\hat{X}(s), s), \hat{X}(t) = X$$

(2.36)

it has a unique solution, since $D_X V(X, s)$ is uniformly Lipschitz in $X$. If we set $\hat{v}(s) = - \frac{1}{\lambda} D_X V(\hat{X}(s), s)$, we see easily that $V(X, t) = J_{Xt}(\hat{v}(.))$. So $V(X, t)$ coincides with the value function, and thus we have only
one possible solution. This completes the proof of the theorem. ■

3 THE MASTER EQUATION

3.1 FURTHER REGULARITY ASSUMPTIONS.

We now assume that

\[ \mathcal{F}, \mathcal{F}_T \text{ are } C^2 \] (3.1)

The operators \( D_X^2 \mathcal{F}(X), D_X^2 \mathcal{F}_T(X) \) belong to \( \mathcal{L}(\mathcal{H}; \mathcal{H}) \). According to the assumptions (2.1) we can assert that

\[ ||D_X^2 \mathcal{F}(X)||, ||D_X^2 \mathcal{F}_T(X)|| \leq c \] (3.2)

where the norm of the operators is the norm of \( \mathcal{L}(\mathcal{H}; \mathcal{H}) \). Recalling the equation (2.19) for \( Y(s) \), we differentiate formally with respect to \( X \) to obtain

\[ D_X Y(s) = I - \frac{s - t}{\lambda} D_X^2 \mathcal{F}_T(Y(T)) D_X Y(T) \]

\[ - \frac{1}{\lambda} \int_t^T D_X^2 \mathcal{F}(Y(\sigma)) D_X Y(\sigma)(s \wedge \sigma - t) d\sigma \] (3.3)

so, \( D_X Y(.) \) appears as the solution of a linear equation, and we see easily that it has one and only one solution verifying

\[ \sup_{t \leq s \leq T} ||D_X Y(s)|| \leq \frac{\lambda}{\lambda - cT(T + 1)} \] (3.4)

It is then easy to check that \( D_X Y(s) \) is indeed the gradient of \( Y_{Xt}(s) \) with respect to \( X \), and the estimate (3.4) is coherent with (2.10). Since \( D_X V(X, t) = Z(t) = Z_{Xt}(t) \) with

\[ Z(t) = \int_t^T D_X \mathcal{F}(Y(s)) ds + D_X \mathcal{F}_T(Y(T)) \]

we can differentiate to obtain

\[ D_X^2 V(X, t) = \int_t^T D_X^2 \mathcal{F}(Y(s)) D_X Y(s) ds + D_X^2 \mathcal{F}_T(Y(T)) D_X Y(T) \] (3.5)
\[ \| D_X^2 V(X, t) \| \leq \frac{\lambda c(T + 1)}{\lambda - cT(T + 1)} \] (3.6)

which is coherent with (2.21).

### 3.2 MASTER EQUATION

We obtain the Master equation, by simply differentiating the HJB equation (2.11) with respect to \( X \). We set \( U(X, t) = D_X V(X, t) \). We know from (2.21) that

\[ \| U(X, t) \| \leq \frac{\lambda(1 + T)c(1 + \| X \|)}{\lambda - cT(T + 1)} \] (3.7)

The function \( U(X, t) \) maps \( H \times (0, T) \) into \( H \). From (3.6) we see that it is differentiable in \( X \), with \( D_X U(X, t) : H \times (0, T) \rightarrow L(H; H) \) and

\[ \| D_X U(X, t) \| \leq \frac{\lambda c(1 + T)}{\lambda - cT(T + 1)} \] (3.8)

From the HJB equation we see that \( U(X, t) \) is differentiable in \( t \) and satisfies the equation

\[
\frac{\partial U}{\partial t} - \frac{1}{\lambda} D_X U(X, t) U(X, t) + D_X F(X) = 0
\] (3.9)

\[ U(X, T) = D_X F_T(X) \]

We have the

**Proposition 2.** We make the assumptions of Theorem 1 and (3.1). Then equation (3.9) has one and only one solution satisfying the estimates (3.7), (3.8).

**Proof.** We have only to prove uniqueness. Noting that

\[ D_X U(X, t) U(X, t) = \frac{1}{2} D_X \| U(X, t) \|^2 \]

we see immediately from the equation that \( U(X, t) \) is a gradient. So \( U(X, t) = D_X \tilde{V}(X, t) \). Therefore (3.9) reads
\[ D_X \left( \frac{\partial \tilde{V}}{\partial t} - \frac{1}{2\lambda} ||D_X \tilde{V}||^2 + F(X) \right) = 0 \]

\[ D_X \tilde{V}(X, T) = D_X F_T(X) \]

We thus can write

\[ \frac{\partial \tilde{V}}{\partial t} - \frac{1}{2\lambda} ||D_X \tilde{V}||^2 + F(X) = f(t) \]

\[ \tilde{V}(X, T) = D_X F_T(X) + h \]

where \( f(t) \) is purely function of \( t \) and \( h \) is a constant. If we introduce the function \( \varphi(t) \) solution of

\[ \frac{\partial \varphi}{\partial t} = f(t), \quad \varphi(T) = h \]

the function \( \tilde{V}(X, t) - \varphi(t) \) is solution of the HJB equation (2.11) and satisfies the regularity properties of Theorem [1]. From the uniqueness of the solution of the HJB equation we have \( \tilde{V}(X, t) - \varphi(t) = V(X, t) \) the value function, hence \( U(X, t) = D_X V(X, t) \), which proves the uniqueness. \( \square \)

4 FUNCTIONALS ON PROBABILITY MEASURES

4.1 GENERAL COMMENTS

If we have a functional on probability measures, the idea, introduced by P.L. Lions [14], [15] is to consider it as a functional on random variables, whose probability laws are the probability measures. Nevertheless, it is possible to work with the space of probability measures directly, which is a metric space. The key issue is to define the concept of gradient. For the space of probability measures, it is the Wasserstein gradient. We shall see that, in fact, it is equivalent to the gradient in the sense of the Hilbert space of random variables.

4.2 WASSERSTEIN GRADIENT

We consider the space \( \mathcal{P}_2(\mathbb{R}^n) \) of probability measures on \( \mathbb{R}^n \), with second order moments, equipped with the Wasserstein metric \( W_2^2(\mu, \nu) \), defined by

\[ W_2^2(\mu, \nu) = \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} ||\xi - \eta||^2 \gamma(d\xi, d\eta) \]  (4.1)
where $\Gamma(\mu, \nu)$ denotes the set of joint probability measures on $R^n \times R^n$ such that the marginals are $\mu$ and $\nu$ respectively. It is useful to consider a probability space $\Omega, \mathcal{A}, P$ and random variables in $\mathcal{H} = L^2(\Omega, \mathcal{A}, P; R^n)$. We then can write $\mu = \mathcal{L}_X$ and

$$W_2^2(\mu, \nu) = \inf_{X, Y \in \mathcal{H}} E|X - Y|^2$$

$\mathcal{L}_X = \mu$

$\mathcal{L}_Y = \nu$

When the probability law has a density with respect to Lebesgue measure, say $m(x)$ belonging to $L^1(R^n)$ and positive, we replace the law by its density. Note that $\int |x|^2 m(x) dx < +\infty$. We call $L^2_m(R^n; R^n)$ the space of functions $f : R^n \to R^n$ such that $\int_{R^n} |f(x)|^2 m(x) dx < +\infty$. We consider functionals $F(\mu)$ on $\mathcal{P}_2(R^n)$. If $\mu$ has a density $m$ we write $F(m)$. If $m \in L^2(R^n)$, we say that $F(m)$ has a Gâteaux differential at $m$, denoted by $\frac{\partial F(m)}{\partial m}(x)$ if we have

$$\lim_{\theta \to 0} \frac{F(m + \theta \mu) - F(m)}{\theta} = \int_{R^n} \frac{\partial F(m)}{\partial m}(x) \mu(x) dx, \forall \mu \in L^2(R^n) \quad (4.2)$$

and $\frac{\partial F(m)}{\partial m}(x) \in L^2(R^n)$. For probability densities, we shall extend this concept as follows. We say that $\frac{\partial F(m)}{\partial m}(x) \in L^1_m(R^n)$ is the functional derivative of $F$ at $m$ if for any sequence of probability densities $m_\epsilon$ in $\mathcal{P}_2(R^n)$ such that $W_2(m_\epsilon, m) \to 0$ then $\frac{\partial F(m)}{\partial m}(\cdot) \in L^1_{m_\epsilon}(R^n)$ and

$$\frac{F(m_\epsilon) - F(m) - \int_{R^n} \frac{\partial F(m)}{\partial m}(x)(m_\epsilon(x) - m(x)) dx}{W_2(m_\epsilon, m)} \to 0, \text{ as } \epsilon \to 0 \quad (4.3)$$

The function $\frac{\partial F(m)}{\partial m}(\cdot)$ is called the functional derivative of $F(m)$ at point $m$. Let us see the connection with the concept of Wasserstein gradient on the metric space $\mathcal{P}_2(R^n)$. We shall simply give the definition and the expression of the gradient. For a detailed theory, we refer to Otto [16], Ambrosio- Gigli- Savaré [1], Benamou-Brenier [2], Brenier [6], Jordan-Kinderlehrer-Otto [11], Otto [16], Villani [17].

The first concept is that of optimal transport map, also called Brenier’s map. Given a probability $\nu \in \mathcal{P}_2(R^n)$, the Monge problem

$$\inf_{\{T(.)|T(.)m=\nu\}} \int_{R^n} |x - T(x)|^2 m(x) dx$$

has a unique solution which is a gradient $T(x) = D\Phi(x)$. The notation $T(.) m = \nu$ means that $\nu$ is the image of the probability whose density is $m$. The optimal solution is the Brenier’s map. It is noted $T^*_m$. We do not
necessarily assume that \( \nu \) has a density. The following property holds

\[
W^2_2(m, \nu) = \int_{\mathbb{R}^n} |x - D\Phi(x)|^2 m(x) dx
\]

This motivates the definition of tangent space \( \mathcal{T}(m) \) of the metric space \( \mathcal{P}_2(\mathbb{R}^n) \) at point \( m \) as

\[
\mathcal{T}(m) = \{ D\Phi | \Phi \in C_\infty^\infty(\mathbb{R}^n) \}
\]

We next consider curves on \( \mathcal{P}_2(\mathbb{R}^n) \), defined by densities \( m(t) \equiv m(t)(x) = m(x, t) \). The evolution of \( m(t) \) is defined by a velocity vector field \( v(t) \equiv v(t)(x) = v(x, t) \) if \( m(x, t) \) is the solution of the continuity equation

\[
\frac{\partial m}{\partial t} + \text{div} (v(x, t)m(x, t)) = 0
\]

\[
m(x, 0) = m(x)
\]

We can interpret this equation in the sense of distributions, and it is sufficient to assume that \( \int_{0}^{T} \int_{\mathbb{R}^n} |v(x, t)|^2 m(x, t) dx dt < +\infty \) for all \( T < +\infty \). This evolution model has a broad spectrum and turns out to be equivalent to the property that \( m(t) \) is absolutely continuous in the sense

\[
W_2(m(s), m(t)) \leq \int_{s}^{t} \rho(\sigma) d\sigma, \forall s < t
\]

with \( \rho(\cdot) \) locally \( L^2 \). Now, for a given absolutely continuous curve \( m(t) \), the corresponding velocity field is not necessarily unique. We can define the velocity field with minimum norm, i.e. \( \hat{v}(x, t) \) solution of

\[
\inf \left\{ \int_{0}^{T} \int_{\mathbb{R}^n} |v(x, t)|^2 m(x, t) dx dt \left| \frac{\partial m}{\partial t} + \text{div} (v(x, t)m(x, t)) = 0 \right. \right\}
\]

The Euler equation for this minimization problem is

\[
\int_{0}^{T} \int_{\mathbb{R}^n} \hat{v}(x, t).v(x, t)m(x, t) dt = 0, \forall v(x, t) | \text{div} (v(x, t)m(x, t)) = 0 \text{ a.e.}
\]

which implies immediately that \( \hat{v}(t) \in \mathcal{T}(m(t)) \) a.e. \( t \). Consequently, to a given absolutely continuous curve \( m(t) \) we can associate a unique velocity field \( \hat{v}(t) \) in the tangent space \( \mathcal{T}(m(t)) \) a.e. \( t \). It is called the tangent vector field to the curve \( m(t) \). It can be expressed by the following formula

\[
\hat{v}(x, t) = \lim_{\epsilon \to 0} \frac{T_{m(t)}^{m(t+\epsilon)}(x) - x}{\epsilon}
\]
the limit being understood in \( L^2_{m(t)}(\mathbb{R}^n; \mathbb{R}^n) \). The function \( T^{m(t+\epsilon)}_m(x) \) is uniquely defined. Since by (4.3), 
\[ ||T^{m(t+\epsilon)}_m(x) - x||_{L^2_{m(t)}} = W_2(m(t), m(t + \epsilon)), \]
we see that, for any absolutely continuous curve 
\[ W_2(m(t), m(t + \epsilon)) \leq C(t)\epsilon \] (4.8)

In the definition of the functional derivative, see (4.3) we can write
\[
\frac{F(m_\epsilon) - F(m) - \int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x)(m_\epsilon(x) - m(x))dx}{\epsilon} \to 0, \text{ as } \epsilon \to 0
\] (4.9)
provided the sequence \( m_\epsilon \) is absolutely continuous.

Suppose that we consider the curve corresponding to a gradient \( D\Phi(x) \) where \( \Phi(x) \) is smooth with compact support, i.e. the curve \( m(t) \) is defined by
\[
\frac{\partial m}{\partial t} + \text{div } (D\Phi(x)m(x, t)) = 0
\]
\[
m(x, 0) = m(x)
\] (4.10)

Since it is a gradient, \( D\Phi(x) \) has minimal norm and we can claim from (4.7) that
\[
D\Phi(x) = \lim_{\epsilon \to 0} \frac{T^\epsilon_{m(t)}(x) - x}{\epsilon} \text{ in } L^2_{m(t)}(\mathbb{R}^n; \mathbb{R}^n)
\] (4.11)

We consider now a functional \( F(m) \) on \( \mathcal{P}_2(\mathbb{R}^n) \), and limit ourselves to densities. We say that \( F(m) \) is differentiable at \( m \) if there exists a function \( \Gamma(x, m) \) belonging to the tangent space \( T(m) \) with the property
\[
\frac{F(m_\epsilon) - F(m) - \int_{\mathbb{R}^n} \Gamma(x, m)(T^\epsilon_{m(t)}(x) - x)m(x)dx}{W_2(m, m(\epsilon))} \to 0, \text{ as } \epsilon \to 0
\] (4.12)

We recall that, see (4.4) \( W_2(m, m(\epsilon)) = ||T^\epsilon_{m(t)}(x) - x||_{L^2_{m(t)}} \). The function \( \Gamma(x, m) \) is called the Wasserstein gradient and denoted \( \nabla F_m(m)(x) \). If we apply this property to the map \( m(t) \) defined by (4.10), this is equivalent to
\[
\frac{F(m_\epsilon) - F(m)}{\epsilon} \to \int_{\mathbb{R}^n} \Gamma(x, m).D\Phi(x)m(x)dx
\]

From the continuity equation (4.10), using the regularity of \( \Phi \), we can state that
\[
\frac{m(x, \epsilon) - m(x)}{\epsilon} \to -\text{div } (D\Phi(x)m(x)), \text{ as } \epsilon \to 0, \text{ in the sense of distributions}
\]
If $F(m)$ has a functional derivative we obtain

$$\frac{F(m(\epsilon)) - F(m)}{\epsilon} \rightarrow - \int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x) \text{div} (D\Phi(x)m(x)) dx$$

Therefore we obtain

$$\int_{\mathbb{R}^n} \Gamma(x,m).D\Phi(x)m(x) dx = - \int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x) \text{div} (D\Phi(x)m(x)) dx$$

$$= \int_{\mathbb{R}^n} D\frac{\partial F(m)}{\partial m}(x).D\Phi(x)m(x) dx$$

If we assume that $D\frac{\partial F(m)}{\partial m}(x) \in L^2_m(R^n;R^n)$, we can replace $D\Phi(x)$ by any element of $T(m)$. Since $\Gamma(x,m)$ and $D\frac{\partial F(m)}{\partial m}(x)$ belong to $T(m)$, it follows that

$$\nabla F_m(m)(x) = D\frac{\partial F(m)}{\partial m}(x)$$

So the Wasserstein gradient is simply the gradient of the functional derivative.

**Remark 3.** The concept of functional derivative, defined in [4.3] uses a sequence of probability densities $m_\epsilon \rightarrow m$, so it is not equivalent to the concept of Gâteaux differential in the space $L^2(R^n)$, which requires to remove the assumptions of positivity and $\int_{\mathbb{R}^n} m(x) dx = 1$. We will develop the differences in examples in which explicit formulas are available, see section 8.

### 4.3 GRADIENT IN THE HILBERT SPACE $\mathcal{H}$.

The functional $F(m)$ can now be written as a functional $\mathcal{F}(X)$ on $\mathcal{H}$, with $m = L_X$. We assume that random variables with densities form a dense subspace of $\mathcal{H}$. Consider a random variable $Y \in \mathcal{H}$ and let $\pi(x,y)$ be the joint probability density on $R^n \times R^n$ of the pair $(X,Y)$. So $m(x) = \int_{\mathbb{R}^n} \pi(x,y) dy$. Consider then the random variable $X + tY$. Its probability density is given by

$$m(x,t) = \int_{\mathbb{R}^n} \pi(x-ty,y) dy$$

and it satisfies the continuity equation

$$\frac{\partial m}{\partial t} = -\text{div} (\int_{\mathbb{R}^n} \pi(x-ty,y) ydy)$$

We have $\mathcal{F}(X + tY) = F(m(t))$. Next
\[ \lim_{t \to 0} \frac{F(X + tY) - F(X)}{t} = ((D_X F, Y)) \]

and

\[ \lim_{t \to 0} \frac{F(m(t)) - F(m)}{t} = -\int_{\mathbb{R}^n} \frac{\partial F(m)}{\partial m}(x) \text{div} \left( \int_{\mathbb{R}^n} \pi(x, y) dy \right) dx \]

\[ = \int_{\mathbb{R}^n} D \frac{\partial F(m)}{\partial m}(x). \left( \int_{\mathbb{R}^n} \pi(x, y) dy \right) dx \]

\[ = ((D \frac{\partial F(m)}{\partial m}(X), Y)) \]

Thus necessarily

\[ D \frac{\partial F(m)}{\partial m}(X) = \nabla F_m(m)(X) = D_X F(X) \] (4.14)

So, the gradient in \( \mathcal{H} \) reduces to the Wasserstein gradient, in which the argument is replaced with the random variable. In the sequel, we will use the gradient in \( \mathcal{H} \).

5 MEAN FIELD TYPE CONTROL PROBLEM

5.1 PRELIMINARIES

Consider a function \( f(x, m) \) defined on \( \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \). As usual we consider only \( m \) which are densities of probability measures, and use also the notation \( f(x, \mathcal{L}_X) \). We then define \( \mathcal{F}(X) = E f(X, \mathcal{L}_X) \). This implies

\[ \mathcal{F}(X) = \Phi(m) = \int_{\mathbb{R}^n} f(x, m)m(x) dx \] (5.1)

We next consider the functional derivative

\[ \frac{\partial \Phi(m)}{\partial m}(x) = F(x, m) = f(x, m) + \int_{\mathbb{R}^n} \frac{\partial f}{\partial m}(\xi, m)(x)m(\xi) d\xi \] (5.2)

and we have

\[ D_X \mathcal{F}(X) = D_\mathcal{L} F(X, \mathcal{L}_X) \] (5.3)

We make the assumptions
\[ |D_x F(x, m)| \leq \frac{c}{2} (1 + |x| + (\int |\xi|^2 m(\xi) d\xi)^{\frac{1}{2}}) \] (5.4)

\[ |D_x F(x_1, m_1) - D_x F(x_2, m_2)| \leq \frac{c}{2} (|x_1 - x_2| + W_2(m_1, m_2)) \] (5.5)

which implies immediately the properties (2.1), (2.2).

5.2 EXAMPLES

We consider first quadratic functionals. We use the notation \( \bar{x} = \int_{R^n} x m(x) dx \). We then consider

\[ f(x, m) = \frac{1}{2} (x - S\bar{x})^* \bar{Q}(x - S\bar{x}) + \frac{1}{2} x^* Qx \] (5.6)

then assuming that \( \int_{R^n} m(x) dx = 1 \), i.e. \( m \) is a probability density we have

\[ F(x, m) = \frac{1}{2} x^* (Q + \bar{Q})x + \frac{1}{2} \bar{x}^* S^* \bar{Q}S \bar{x} - \bar{x}^* (\bar{Q}S + S^* \bar{Q} - S^* \bar{Q}S)x \] (5.7)

\[ D_x F(x, m) = (Q + \bar{Q})x - (\bar{Q}S + S^* \bar{Q} - S^* \bar{Q}S)\bar{x} \] (5.8)

We see easily that assumptions (5.4), (5.5) are satisfied.

We can give an additional example

\[ f(x, m) = \frac{1}{2} \int_{R^n} K(x, \xi) m(\xi) d\xi \] (5.9)

with \( K(x, \xi) = K(\xi, x) \) and

\[ |K(x_1, \xi_1) - K(x_2, \xi_2)| \leq C (1 + |x_1| + |x_2| + |\xi_1| + |\xi_2|)(|x_1 - x_2| + |\xi_1 - \xi_2|) \] (5.10)

\[ |D_x K(x_1, \xi_1) - D_x K(x_2, \xi_2)| \leq \frac{c}{2} (|x_1 - x_2| + |\xi_1 - \xi_2|) \] (5.11)

We have

\[ |D_x K(0, 0)| \leq \frac{c}{2} \]
\[
\frac{\partial f}{\partial m}(\xi, m)(x) = \frac{1}{2} K(\xi, x) = \frac{1}{2} K(x, \xi)
\]

hence \( \int_{\mathbb{R}^n} \frac{\partial f}{\partial m}(\xi, m)(x)m(\xi)d\xi = f(x, m) \) which implies

\[
F(x, m) = 2f(x, m) = \int_{\mathbb{R}^n} K(x, \xi)m(\xi)d\xi
\]  \hspace{0.5cm} (5.12)

We thus have

\[
|D_x F(x, m)| \leq \int_{\mathbb{R}^n} |D_x K(x, \xi)|m(\xi)d\xi \leq \\
\leq \frac{c}{2}(1 + |x| + \int |\xi|m(\xi)d\xi) \leq \\
\leq \frac{c}{2}(1 + |x| + (\int |\xi|^2m(\xi)d\xi)^{\frac{1}{2}})
\]

If we take 2 densities \( m_1, m_2 \), we may consider 2 random variables \( \Xi_1, \Xi_2 \) with the probabilities \( m_1, m_2 \). Therefore

\[
|D_x F(x_1, m_1) - D_x F(x_2, m_2)| \leq |\int D_x(K(x_1, \xi) - K(x_2, \xi))m_1(\xi)d\xi| + \\
+|E D_x(K(x_2, \Xi_1) - K(x_2, \Xi_2))| \leq \\
\leq \frac{c}{2}|x_1 - x_2| + \frac{c}{2} \sqrt{E|\Xi_1 - \Xi_2|^2}
\]

and since \( \Xi_1, \Xi_2 \) are arbitrary, with marginals \( m_1, m_2 \) we can write \( (5.5) \). In the sequel we also consider a functional \( h(x, m) \) with exactly the same properties as \( f \) and write

\[
F_T(x, m) = h(x, m) + \int_{\mathbb{R}^n} \frac{\partial h}{\partial m}(\xi, m)(x)m(\xi)d\xi
\]  \hspace{0.5cm} (5.13)

\[
\mathcal{F}_T(X) = \int_{\mathbb{R}^n} h(x, m)m(x)dx, \quad D_X \mathcal{F}_T(X) = D_x F_T(X, \mathcal{L}_X)
\]

5.3 MEAN FIELD TYPE CONTROL PROBLEM

We can formulate the following mean field type control problem. Let us consider a dynamical system in \( \mathbb{R}^n \)
\[
\frac{dx}{ds} = v(x(s), s) \tag{5.14}
\]

\[x(t) = \xi\]

where \(v(x, s)\) is a feedback to be optimized. The initial condition is a random variable with probability density \(m(x)\). The Fokker-Planck equation of the evolution of the density is

\[
\frac{\partial m}{\partial s} + \text{div}(v(x)m) = 0 \tag{5.15}
\]

\[m(x, t) = m(x)\]

We denote the solution by \(m_{v(.)}(x, s)\). Similarly we call the solution of \(x(s; v(.)\)) \(5.14\). We then consider the cost functional

\[
J_{m,t}(v(.) = \frac{\lambda}{2} \int_t^T \int_{R^n} |v(x(s), s)|^2 m_{v(.)}(x(s), s) dx ds + \int_t^T \int_{R^n} m_{v(.)}(x(s), s) f(x, m_{v(.)}(s)) dx ds + \\
\int_{R^n} m_{v(.)}(x(T)) h(x, m_{v(.)}(T)) dx
\]

which is equivalent to the expression

\[
J_{m,t}(v(.) = \frac{\lambda}{2} \int_t^T E|v(x(s; v(.)|)^2 ds + \int_t^T Ef(x(s; v(.)), \mathcal{L}_{x(s)}) ds + Eh(x(T; v(.)), \mathcal{L}_{x(T)}) \tag{5.17}
\]

This is a standard mean field type control problem, not a mean field game. In [3] we have associated to it a coupled system of HJB and FP equations, see p. 18, which reads here

\[
-\frac{\partial u}{\partial s} + \frac{1}{2\lambda} |Du|^2 = F(x, m(s)) \tag{5.18}
\]

\[u(x, T) = F_T(x, m(s))\]

\[
\frac{\partial m}{\partial s} - \frac{1}{\lambda} \text{div}(Du m) = 0
\]

\[m(x, t) = m(x)\]
This system expresses a necessary condition of optimality. The function \( u(x, t) \) is not a value function, but an adjoint variable to the optimal state, which is \( m(x, s) \). The optimal feedback is given by

\[
\hat{v}(x, s) = -\frac{1}{\lambda} Du(x, s) \tag{5.19}
\]

We proceed formally, although we shall be able to give an explicit solution of this system. If \( \hat{v}(x, s) \) is the optimal feedback, then the value function \( V(m, t) = J_{m,t}(\hat{v}(\cdot)) \) is given by

\[
V(m, t) = \frac{1}{2\lambda} \int_t^T \int_{\mathbb{R}^n} m(x, s)|Du(x, s)|^2 dx ds + \int_t^T \int_{\mathbb{R}^n} m(x, s)f(x, m(s)) dx ds + \int_{\mathbb{R}^n} m(x, T)h(x, m(T)) dx \tag{5.20}
\]

The value function is solution of Bellman equation, see [4], [13], written formally (it will be justified later)

\[
\frac{\partial V}{\partial t} - \frac{1}{2\lambda} \int_{\mathbb{R}^n} |D_\xi \frac{\partial V(m, t)}{\partial m}(\xi)|^2 m(\xi) d\xi + \int_{\mathbb{R}^n} f(\xi, m) m(\xi) d\xi = 0 \tag{5.21}\]

\[
V(m, T) = \int_{\mathbb{R}^n} h(\xi, m) m(\xi) d\xi
\]

5.4 SCALAR MASTER EQUATION

We derive the master equation, by considering the function

\[
U(x, m, t) = \frac{\partial V(m, t)}{\partial m}(x)
\]

and we note that

\[
\frac{\partial U}{\partial m}(x, m, t)(\xi) = \frac{\partial^2 V(m, t)}{\partial m^2}(x, \xi)
\]

therefore the function is symmetric in \( x, \xi \) which means

\[
\frac{\partial U}{\partial m}(x, m, t)(\xi) = \frac{\partial U}{\partial m}(\xi, m, t)(x)
\]

By differentiating (5.21) in \( m \), and using the symmetry property, we obtain the equation
\[
\frac{\partial U}{\partial t} - \frac{1}{\lambda} \int_{\mathbb{R}^n} D_\xi \frac{\partial U}{\partial m}(x, m, t)(\xi) \cdot D_\xi U(\xi, m, t)m(\xi)d\xi - \frac{1}{2\lambda}|D_x U(x, m, t)|^2 + F(x, m) = 0
\]

\[U(x, m, T) = F_T(x, m)\]

This function allows to uncouple the system of HJB-FP equations, given in (5.18). Indeed, we first solve the FP equation, replacing \(u\) by \(U\), i.e.

\[
\frac{\partial m}{\partial s} - \frac{1}{\lambda} \text{div}(DU m) = 0
\]

\[m(x, t) = m(x)\]

then \(u(x, s) = U(x, m(s), s)\) is solution of the HJB equation (5.18), as easily checked. In particular, we have

\[u(x, t) = U(x, m, t)\]

5.5 VECTOR MASTER EQUATION

We next consider \(U(x, m, t) = D_x U(x, m, t)\). Differentiating (5.22) we can write

\[
\frac{\partial U}{\partial t} - \frac{1}{\lambda} \int_{\mathbb{R}^n} D_\xi \frac{\partial U}{\partial m}(x, m, t)(\xi) \cdot D_\xi U(\xi, m, t)m(\xi)d\xi - \frac{1}{\lambda}D_x U(x, m, t)U(x, m, t) + D_x F(x, m) = 0
\]

\[U(x, m, T) = D_x F_T(x, m)\]

6 CONTROL PROBLEM IN THE SPACE \(\mathcal{H}\).

6.1 FORMULATION

If we set
\[ F(X) = Ef(X, \mathcal{L}_X) = \int f(x, m) m(x) dx \]  
\[ F_T(X) = Eh(X, \mathcal{L}_X) = \int h(x, m) m(x) dx \]

\[ F(x, m) = f(x, m) + \int \frac{\partial f(\xi, m)}{\partial m}(x) m(\xi) d\xi \]  
\[ F_T(x, m) = h(x, m) + \int \frac{\partial h(\xi, m)}{\partial m}(x) m(\xi) d\xi \]

We have

\[ D_X F(X) = D_x F(X, \mathcal{L}_X) \]  
\[ D_X F_T(X) = D_x F_T(X, \mathcal{L}_X) \]

We assume that

\[ |D_x F(x_1, m_1) - D_x F(x_2, m_2)| \leq \frac{C}{2} (|x_1 - x_2| + W_2(m_1, m_2)) \]  
\[ |D_x F_T(x_1, m_1) - D_x F_T(x_2, m_2)| \leq \frac{C}{2} (|x_1 - x_2| + W_2(m_1, m_2)) \]

\[ |D_x F(x, m)| \leq \frac{C}{2} (1 + |x| + \sqrt{\int_{\mathbb{R}^n} |\xi|^2 m(\xi) d\xi}) \]  
\[ |D_x F_T(x, m)| \leq \frac{C}{2} (1 + |x| + \sqrt{\int_{\mathbb{R}^n} |\xi|^2 m(\xi) d\xi}) \]

It follows that

\[ ||D_X F(X_1) - D_X F(X_2)|| \leq ||D_x F(X_1, \mathcal{L}_{X_1}) - D_x F(X_2, \mathcal{L}_{X_1})|| + 
+ ||D_x F(X_2, \mathcal{L}_{X_1}) - D_x F(X_2, \mathcal{L}_{X_2})|| \]
\[ \leq c||X_1 - X_2|| \]

and similar estimate for \( F_T \). Therefore the set up of section 2.1 is satisfied. We can reinterpret the problem (5.14), (5.17) or (5.15), (5.16) as (2.15), (2.16) which has been completely solved in Theorem 1. We shall study the solution of the abstract setting. Of course, the initial state \( X \) has probability law \( \mathcal{L}_X = m \).

6.2 INTERPRETATION OF THE SOLUTION

The key point of the proof of Theorem 1 is the study of the system (2.16) which has one and only one solution. We proceed formally. Consider the HJB-FP system (5.18). The initial conditions are the pair \((m, t)\), so we can write the solution as \( u_{m,t}(x, s) \), \( m_{m,t}(x, s) \). We introduce the differential equation

\[
\begin{align*}
\frac{dy}{ds} &= -\frac{1}{\lambda} Du(y(s), s) \\
y(t) &= x
\end{align*}
\]

The solution (if it exists) can be written \( y_{xmt}(s) \). Now let us set \( z_{xmt}(s) = Du_{mt}(y_{xmt}(s), s) \). Differentiating the HJB equation (5.18) and computing the derivative \( \frac{dz}{ds} \) we obtain

\[
\begin{align*}
-\frac{dz}{ds} &= D_x F(y(s), m(s)) \\
z(T) &= D_x F_T(y(T), m(T))
\end{align*}
\]

Now, from the definition of \( m(s) \) solution of the FP equation, we can write

\[ m(s) = y_{mt}(s)(m) \]

in which we have used the notation \( y_{mt}(s)(x) = y_{mt}(x, s) = y_{xmt}(s) \) and \( y_{mt}(s)(.) (m) \) means the image measure of \( m \) by the map \( y_{mt}(.) \). So we can write the system (6.6), (6.7) as

\[
\begin{align*}
\frac{d^2 y}{ds^2} &= \frac{1}{\lambda} D_x F(y(s), y(s)(.)(m)) \\
y(t) &= x \frac{dy}{ds}(T) = -\frac{1}{\lambda} D_x F_T(y(T), y(T)(.)(m))
\end{align*}
\]
This is also written in integral form

\[
y(s) = x - \frac{s-t}{\lambda} D_x F_T(y(T), y(T)(.)(m)) \tag{6.10}
\]

\[
-\frac{1}{\lambda} \int_t^T D_x F(y(\sigma), y(\sigma)(.)(m))(s \land \sigma - t)d\sigma
\]

Now if we take \(y, \mathcal{L}_X, t(s)\), then \(y(s)(.)(\mathcal{L}_X) = \mathcal{L}_{y(s)}\). Writing \(y, \mathcal{L}_X, t(s) = Y(s)\) to emphasize that we are dealing with a random variable, we can write (6.10) as

\[
Y(s) = X - \frac{s-t}{\lambda} D_x F_T(Y(T), \mathcal{L}_Y(T)) - \frac{1}{\lambda} \int_t^T D_x F(Y(\sigma), \mathcal{L}_Y(\sigma))(s \land \sigma - t)d\sigma \tag{6.11}
\]

which is nothing else than (2.6) recalling the values of \(D_X F(X), D_X F_T(X)\), cf (6.3). We know from Theorem 1 that (6.11) has one and only one solution in \(C^0([t,T]; \mathcal{H})\) and in fact in \(C^2([t,T]; \mathcal{H})\). This result, of course, does not allow to go from (6.11) to (6.10), but it easy to mimic the proof. We state the result in the following

**Proposition 4.** We assume (6.4), (6.5) and condition (2.8). For given \(m, t\) there exists one and only one solution \(y_{mt}(x, s)\) of (6.10) in the space \(C(t, T; L^2_m(R^n; R^n))\).

**Proof.** We use a fixed point argument. We define a map from \(C(t, T; L^2_m(R^n; R^n))\) to itself. Let \(z(x, s)\) a function in \(C(t, T; L^2_m(R^n; R^n))\). We define

\[
\zeta(x, s) = x - \frac{s-t}{\lambda} D_x F_T(z(x, T), z(T)(.)(m)) - \frac{1}{\lambda} \int_t^T D_x F(z(x, \sigma), z(\sigma)(.)(m))(s \land \sigma - t)d\sigma
\]

We have

\[
|\zeta(x, s)| \leq |x| + \frac{Tc}{\lambda} (1 + |z(x, T)| + \int_{R^n} |z(\xi, T)|^2 m(\xi)d\xi)^{1/2} +
\]

\[+
\frac{cT}{2\lambda} \int_t^T (1 + |z(x, \sigma)| + \int_{R^n} |z(\xi, \sigma)|^2 m(\xi)d\xi)^{1/2}d\sigma
\]

hence, from norm properties
\[
\left(\int_{R^n} |\zeta(x,s)|^2 m(x) \, dx\right)^{1/2} \leq \sqrt{\int_{R^n} |x|^2 m(x) \, dx} + \frac{TC}{\lambda^2} \left(1 + 2\left(\int_{R^n} |z(\xi,T)|^2 m(\xi) \, d\xi\right)^{1/2}\right) + \frac{CT}{2\lambda} \left(1 + 2\left(\int_{R^n} |z(\xi,\sigma)|^2 m(\xi) \, d\xi\right)^{1/2}\right) \, d\sigma
\]

and we conclude easily that \(\zeta\) belongs to \(C(t,T;L^2_m(R^n;R^n))\). We set \(\zeta = T(z)\). Using the assumptions and similar estimates, one checks that \(T\) is a contraction. We prove indeed that

\[
\|T(z_1) - T(z_2)\|_{C(t,T;L^2_m)} \leq \left(1 - \frac{c(T+1)}{\lambda}\right)\|z_1 - z_2\|_{C(t,T;L^2_m)} \tag{6.12}
\]

It follows immediately that the solution \(y_{xmt}(s) = y_{mt}(x, s)\) satisfies the estimate

\[
\|y_{mt}(\cdot)\|_{C(t,T;L^2_m)} \leq \frac{\lambda \sqrt{\int_{R^n} |x|^2 m(x) \, dx} + cT(T+1)}{\lambda - cT(T+1)} \tag{6.13}
\]

Since \(Y_{Xt}(s) = y_{X,t}(s)\) we deduce the first estimate \((2.21)\). We consider next

\[
z_{xmt}(s) = z_{mt}(x, s) = D_x F_T(y(x,T), y(T)(\cdot)(m)) + \int_t^T D_x F(y(x,\sigma), y(\sigma)(\cdot)(m)) \, d\sigma \tag{6.14}
\]

and from the assumption \((6.5)\) we obtain easily

\[
\|z_{mt}(\cdot)\|_{C(t,T;L^2_m)} \leq c(1 + T)(1 + \|y_{mt}(\cdot)\|_{C(t,T;L^2_m)})
\]

hence

\[
\|z_{mt}(\cdot)\|_{C(t,T;L^2_m)} \leq \lambda c(1 + T) \frac{\sqrt{\int_{R^n} |x|^2 m(x) \, dx} + 1}{\lambda - cT(T+1)} \tag{6.15}
\]

Clearly \(Z(s) = Z_{Xt}(s) = z_{X,t}(s)\), see \((2.10)\), and we recover the 2nd estimate \((2.21)\).

We can give more properties on \(y_{xmt}(s)\). We write first
\[\begin{align*}
y_{mt}(x_1, s) - y_{mt}(x_2, s) &= x_1 - x_2 - \frac{s - t}{\lambda} (D_x F_T(y_{mt}(x_1, T), y_{mt}(T)(m)) - D_x F_T(y_{mt}(x_2, T), y_{mt}(T)(m))) - \\
&\quad - \frac{1}{\lambda} \int_t^T (D_x F(y_{mt}(x_1, \sigma), y_{mt}(\sigma)(m)) - D_x F(y_{mt}(x_2, \sigma), y_{mt}(\sigma)(m))) (s \land \sigma - t) d\sigma
\end{align*}\]

From (6.4) we obtain easily

\[\sup_{t<s<T} |y_{mt}(x_1, s) - y_{mt}(x_2, s)| \leq \frac{\lambda |x_1 - x_2|}{\lambda - cT(T + 1)} \quad (6.16)\]

Also

\[\sup_{t<s<T} |y_{mt}(x, s)| \leq \lambda \frac{|x| + \frac{Tc(1 + T)(1 + \sqrt{\int_{R^n} \|\xi\|^2 m(\xi) d\xi})}{\lambda - cT(T + 1)}}{\lambda - cT(T + 1)} \quad (6.17)\]

A similar estimate holds for \(\sup_{t<s<T} |z_{mt}(x, s)|\).

### 7 BELLMAN EQUATION AND MASTER EQUATION

#### 7.1 THE VALUE FUNCTION

The value function of the control problem in \(\mathcal{H}\) is given by

\[V(X, t) = \frac{1}{2\lambda} \int_t^T \|Z(s)\|^2 ds + \int_t^T F(Y(s))ds + F_T(Y(T))\]

in which \(Y(s) = y_{X,\mathcal{L}_X,t}(s)\) and \(Z(s) = z_{X,\mathcal{L}_X,t}(s)\). From this representation and the definition of \(F\) and \(F_T\), we can assert that \(V(X, t)\) depends only on \(\mathcal{L}_X\) and thus can be written \(V(m, t)\) with

\[V(m, t) = \frac{1}{2\lambda} \int_t^T \int_{R^n} |z_{mt}(s)|^2 m(x) dx ds + \int_t^T \int_{R^n} f(y_{mt}(s), y_{mt}(\cdot)(m)) m(x) dx ds + \]

\[+ \int_{R^n} h(y_{mt}(T), y_{mt}(\cdot)(m)) m(x) dx\quad (7.1)\]

From (6.15) we have

\[\int_t^T \int_{R^n} |z_{mt}(s)|^2 m(x) dx ds \leq \frac{T\lambda^2 c^2 (1 + T)^2 (1 + \int_{R^n} |x|^2 m(x) dx)}{\lambda - cT(T + 1)^2}\]

and \(|F(Y(s))| \leq C(1 + \|Y(s)\|^2)\), therefore
\[ \left| \int_{R^n} f(y_x(s), y_m(s))(\cdot)(m) m(x) dx \right| \leq C(1 + \int |y_{x}(s)|^2 m(x) dx) \]

and from the estimate (5.13) we obtain

\[ \left| \int_{t}^{T} \int_{R^n} f(y_x(s), y_m(s))(\cdot)(m) m(x) dx ds \right| \leq C T \left[ 1 + \frac{\lambda^2}{\lambda - cT(T + 1)} \right] \]

and the third term in the right hand side of (7.1) satisfies a similar estimate. We thus have obtained

\[ |V(m, t)| \leq C(1 + \int |x|^2 m(x) dx) \quad (7.2) \]

which is, of course, equivalent to the 1st estimate (2.9).

We turn now to \( U(x, m, t) = \partial V(m, t) / \partial m(x) \). We have seen formally in (5.24) that \( U(x, m, t) = u(x, t) = u_m(x, t) \). We need to prove it. We begin by giving a solution to the system HJB-FP equations (5.18). We have the

Lemma 5. We make the assumptions of Proposition 4. We can give an explicit formula to the system (5.18). We have

\[ u_m(x, t) = \frac{1}{2\lambda} \int_{t}^{T} |z_{x}(s)|^2 ds + \int_{t}^{T} F(y_x(s), y_m(s))(\cdot)(m) ds + \]

\[ + F_T(y_x(T), y_m(T))(\cdot)(m) \quad (7.3) \]

and \( m_m(s) = y_m(s)(\cdot)(m) \).

Proof. Indeed, if we look at \( F(x, m(s)) \) and \( F_T(x, m(T)) \) in which \( m(\cdot) \) is frozen, the HJB equation appears as a standard one for a deterministic control problem. This problem is simply

\[ \frac{dx}{ds} = v(s) \]
\[ x(t) = x \]

\[ J_{xt}(v(\cdot)) = \frac{\lambda}{2} \int_{t}^{T} |v(s)|^2 ds + \int_{t}^{T} F(x(s), m(s)) ds + F_T(x(T), m(T)) \]

in which the function \( m(s) \) is frozen, but not arbitrary. It is the function solution of the FP equation, in the system (5.18). If we write the necessary conditions of optimality, one checks easily that in view of the specific value of \( m(s) \), the optimal state is \( y_{x}(s) \) and the optimal control is \( -\frac{1}{\lambda} z_{x}(s) \). In plugging these values in the cost function, we obtain formula (7.3). ■
We may assume that

\[ |F(x, m)|, |F_T(x, m)| \leq C(1 + |x|^2 + \int |\xi|^2 m(\xi) d\xi) \quad (7.4) \]

We shall also assume that

\[ \left| \frac{\partial F(x, m)}{\partial m}(\xi) \right|, \left| \frac{\partial F_T(x, m)}{\partial m}(\xi) \right| \leq C(1 + |x|^2 + |\xi|^2 + \int |\eta|^2 m(\eta) d\eta) \quad (7.5) \]

\[ \left| D_x D_\xi \frac{\partial F(x, m)}{\partial m}(\xi) \right| \leq C \quad (7.6) \]

We also make an assumption which simplifies proofs, but which can be overcome, with technical difficulties.

\[ \int \mathbb{R}^n (F(x, m_1) - F(x, m_2))(m_1(x) - m_2(x)) dx \geq 0 \quad (7.7) \]
\[ \int \mathbb{R}^n (F_T(x, m_1) - F_T(x, m_2))(m_1(x) - m_2(x)) dx \geq 0 \]

This assumption allows to obtain the following interesting in itself result.

**Proposition 6.** We assume (7.7). Then considering the system of HJB-FP equations (5.18) with initial conditions \( m_1(x) \) and \( m_2(x) \) and calling \( u_1(x, s), m_1(x, s) \), respectively \( u_2(x, s), m_2(x, s) \) the solutions, we have the property

\[ \int \mathbb{R}^n (u_1(x, t) - u_2(x, t))(m_1(x) - m_2(x)) dx \geq 0 \quad (7.8) \]

**Proof.** From the system HJB-FP we can write

\[ - \frac{\partial}{\partial s} (u_1 - u_2) + \frac{1}{2\lambda} |Du_1|^2 - \frac{1}{2\lambda} |Du_2|^2 = F(x, m_1(s)) - F(x, m_2(s)) \]
\[ u_1(x, T) - u_2(x, T) = F_T(x, m_1(T)) - F_T(x, m_2(T)) \]
\[ \frac{\partial}{\partial s} (m_1 - m_2) = \frac{1}{\lambda} \text{div}(Du_1 m_1 - Du_2 m_2) \]
\[ m_1(x, t) - m_2(x, t) = m_1(x) - m_2(x) \]
then a simple calculation shows that

\[
\frac{d}{ds} \int_{\mathbb{R}^n} (u_1(x,s) - u_2(x,s))(m_1(x,s) - m_2(x,s))dx = -\int_{\mathbb{R}^n} (F(x,m_1(s)) - F(x,m_2(s)))(m_1(x,s) - m_2(x,s))dx \\
- \frac{1}{2\lambda} \int_{\mathbb{R}^n} (m_1(x,s) + m_2(x,s))|Du_1(x,s) - Du_2(x,s)|^2 dx
\]

and the result follows immediately, recalling that \(m_1, m_2\) are positive and using the assumption (7.7). ☐

We now state the

**Proposition 7.** We make the assumptions of Proposition 4 and (7.4), (7.5), (7.6), (7.7). We then have

\[
U(x,m,t) = \frac{\partial V}{\partial m}(m,t)(x) = u_{mt}(x,t) \tag{7.9}
\]

Moreover, we have the estimate

\[
|U(x,m,t)| \leq C(1 + |x|^2 + \int_{\mathbb{R}^n} |\xi|^2 m(\xi)d\xi) \tag{7.10}
\]

**Proof.** We recall the definition of the value function \(V(m,t)\), see section 5.3 and formulas (5.20) and (7.3). Let \(m_1(x)\) be some probability density and the functions \(u_1(x,s) = u_{m_1}(x,s), m_1(x,s) = m_{m_1}(x,s)\) solutions of the system HJB-FP (5.18). The feedback \(\hat{v}_1(x,s) = -\frac{1}{\lambda}Du_1(x,s)\) is optimal for the control problem (5.14), (5.15), (5.16). The corresponding optimal trajectory, starting from a deterministic value \(x\) is \(y_{xm_1}(s)\). The probability density \(m_{\hat{v}_1}(x,s)\) corresponding to the feedback \(\hat{v}_1(x,s)\) is the image of \(m_1\) by the map \(x \rightarrow y_{xm_1}(s)\), so we can write

\[m_1(s) = m_{\hat{v}_1}(s) = y_{xm_1}(s)(m_1)\]

We now consider another initial probability density \(m_2(x)\) and the same feedback \(\hat{v}_1\). Namely we compute \(J_{m_2}(\hat{v}_1(\cdot))\). The probability density at time \(s\), with initial condition at time \(t\) equal to \(m_2\) and feedback \(\hat{v}_1\) is \(y_{m_1}(s)(\cdot)(m_2)\) denoted \(m_{12}(s) = m_{12}(x,s)\). It is solution of the FP equation

\[
\frac{\partial m_{12}}{\partial s} - \frac{1}{\lambda} \text{div}(Du_1(x)m_{12}) = 0
\]

\[m_{12}(x,t) = m_2(x)\]

We can then write
\[ J_{m_2t}(\hat{v}_1(.) = \frac{1}{2\lambda} \int_t^T \int_{\mathbb{R}^n} |Du_1(x, s)|^2 m_{12}(x, s)dxds + \int_t^T \int_{\mathbb{R}^n} m_{12}(x, s) f(x, m_{12}(s))dxds + \int_{\mathbb{R}^n} m_{12}(x, t) h(x, m_{12}(T))dx \]

Therefore we have the inequality

\[ V(m_2, t) - V(m_1, t) \leq J_{m_2t}(\hat{v}_1(.)) - J_{m_1t}(\hat{v}_1(.)) \tag{7.11} \]

\[ = \frac{1}{2\lambda} \int_t^T \int_{\mathbb{R}^n} |Du_1(x, s)|^2 (m_{12}(x, s) - m_1(x, s))dxds + \int_t^T \int_{\mathbb{R}^n} (m_{12}(x, s)f(x, m_{12}(s)) - m_1(x, s)f(x, m_1(s)))dxds + \int_{\mathbb{R}^n} (m_{12}(x, T)h(x, m_{12}(T)) - m_1(x, T)h(x, m_1(T)))dx \]

We note that

\[ \frac{\partial(m_{12} - m_1)}{\partial s} - \frac{1}{\lambda} \text{div}(Du_1(x)(m_{12} - m_1)) = 0 \]

\[ m_{12}(x, t) - m_1(x, t) = m_2(x) - m_1(x) \]

\[-\frac{\partial}{\partial s} u_1 + \frac{1}{2\lambda} |Du_1|^2 = F(x, m_1(s)) \]

\[ u_1(x, T) = F_T(x, m_1(T)) \]

hence, as easily seen

\[ \int_{\mathbb{R}^n} u_1(x, t)(m_2(x) - m_1(x))dx = \frac{1}{2\lambda} \int_t^T \int_{\mathbb{R}^n} |Du_1(x, s)|^2 (m_{12}(x, s) - m_1(x, s))dxds + \]

\[ + \int_t^T \int_{\mathbb{R}^n} F(x, m_1(s))(m_{12}(x, s) - m_1(x, s))dxds + \int_{\mathbb{R}^n} F_T(x, m_1(T))(m_{12}(x, T) - m_1(x, T))dx \]
Combining with (7.11) we can write

$$V(m_2, t) - V(m_1, t) \leq \int_{R^n} u_1(x, t)(m_2(x) - m_1(x))dx +$$

$$\int_t^T \int_{R^n} [m_{12}(x, s)f(x, m_{12}(s)) - m_1(x, s)f(x, m_1(s)) - F(x, m_1(s))(m_{12}(x, s) - m_1(x, s))]dxds +$$

$$\int_{R^n} [m_{12}(x, T)h(x, m_{12}(T)) - m_1(x, T)f(x, m_1(s)) - F_T(x, m_1(T))(m_{12}(x, T) - m_1(x, T))]dx$$

Recalling that $F(x, m)$ is the functional derivative of $\int_{R^n} f(x, m)m(x)dx$, we can write the above inequality as follows

$$V(m_2, t) - V(m_1, t) \leq \int_{R^n} u_1(x, t)(m_2(x) - m_1(x))dx + (7.12)$$

$$\int_t^T \int_{R^n} \int_0^1 \int_0^1 \theta \frac{\partial F}{\partial m}(x, m_1(s) + \theta \mu(m_{12}(s) - m_1(s))(\xi)(m_{12}(x, s) - m_1(x, s))(m_{12}(\xi, s) - m_1(\xi, s)))dxdsd\theta d\mu +$$

$$\int_{R^n} \int_0^1 \int_0^1 \theta \frac{\partial F}{\partial m}(x, m_1(T) + \theta \mu(m_{12}(T) - m_1(T))(\xi)(m_{12}(x, T) - m_1(x, T))(m_{12}(\xi, T) - m_1(\xi, T)))dxdsd\theta d\mu$$

Recalling that $m_{12}(s) = y_{m_1t}(s)(.)(m_2)$ and $m_1(s) = y_{m_1t}(s)(.)(m_1)$, we can write for a test function $\varphi(x, \xi)$

$$\int_{R^n} \int_{R^n} \varphi(x, \xi)(m_{12}(x, s) - m_1(x, s))(m_{12}(\xi, s) - m_1(\xi, s))dxds =$$

$$\int_{R^n} \int_{R^n} \varphi(y_{x1m_1t}(s), y_{\xi m_1t}(s))(m_2(x) - m_1(x))(m_2(\xi) - m_1(\xi))dxds$$

We introduce a pair of random variables $X_1, X_2$ whose marginals are $m_1, m_2$. We then introduce an independent copy $Y_1, Y_2$. It is easy to convince oneself that we have the relation

$$\int_{R^n} \int_{R^n} \varphi(y_{x1m_1t}(s), y_{\xi m_1t}(s))(m_2(x) - m_1(x))(m_2(\xi) - m_1(\xi))dxds =$$

$$\int_0^1 \int_0^1 ED_xD_\xi \varphi(y_{X1m_1t}(s) + \theta(y_{X2m_1t}(s) - y_{X1m_1t}(s)), y_{Y1m_1t}(s) + \mu(y_{Y2m_1t}(s) - y_{Y1m_1t}(s)))$$

$$(y_{X2m_1t}(s) - y_{X1m_1t}(s))(y_{Y2m_1t}(s) - y_{Y1m_1t}(s))d\theta d\mu$$

If we have $||D_xD_\xi \varphi(x, \xi)|| \leq C$, then we get
or and from Proposition 6 and assumption (7.7) the 2nd integral is negative, which implies

\[ CE|y_{X_2 m_1 t}(s) - y_{X_1 m_1 t}(s)||y_{Y_2 m_1 t}(s) - y_{Y_1 m_1 t}(s) \]

and from the independence property

\[ \leq C(E|y_{X_2 m_1 t}(s) - y_{X_1 m_1 t}(s)|^2 \leq CE|y_{X_2 m_1 t}(s) - y_{X_1 m_1 t}(s)|^2 \]

Using property (6.16) we obtain also

\[ \left| \int_{R^n} \int_{R^n} \varphi(y_{x m_1 t}(s), y_{\xi m_1 t}(s))(m_2(x) - m_1(x))(m_2(\xi) - m_1(\xi))dx d\xi \right| \leq CE|X_2 - X_1|^2 \]

and since \( X_1, X_2 \) have an arbitrary correlation, this implies also

\[ \left| \int_{R^n} \int_{R^n} \varphi(y_{x m_1 t}(s), y_{\xi m_1 t}(s))(m_2(x) - m_1(x))(m_2(\xi) - m_1(\xi))dx d\xi \right| \leq CW_2^2(m_1, m_2) \]

We may apply this result with \( \varphi(x, \xi) = \frac{\partial F}{\partial m}(x, m_1(s) + \theta \mu(m_12(s) - m_1(s))(\xi) \). Thanks to assumption (7.6) the same result carries over. Therefore we conclude easily the estimate

\[ V(m_2, t) - V(m_1, t) \leq \int_{R^n} u_1(x, t)(m_2(x) - m_1(x))dx + CW_2^2(m_1, m_2) \quad (7.13) \]

Interchanging the role of \( m_1, m_2 \), we have also

\[ V(m_1, t) - V(m_2, t) \leq \int_{R^n} u_2(x, t)(m_1(x) - m_2(x))dx + CW_2^2(m_1, m_2) \]

\[ \leq \int_{R^n} u_1(x, t)(m_1(x) - m_2(x))dx + \int_{R^n} (u_2(x, t) - u(x, t)(m_1(x) - m_2(x))dx + CW_2^2(m_1, m_2) \]

and from Proposition and assumption (7.7) the 2nd integral is negative, which implies

\[ V(m_1, t) - V(m_2, t) \leq \int_{R^n} u_1(x, t)(m_1(x) - m_2(x))dx + CW_2^2(m_1, m_2) \]

or

\[ V(m_2, t) - V(m_1, t) \geq \int_{R^n} u_1(x, t)(m_2(x) - m_1(x))dx - CW_2^2(m_1, m_2) \]
and comparing with (7.13) we can assert

\[
|V(m_2, t) - V(m_1, t) - \int_{\mathbb{R}^n} u_1(x, t)(m_2(x) - m_1(x))dx| \leq CW_2^2(m_1, m_2)
\] (7.14)

Now we have

\[
|u_{mt}(x, t)| \leq C(1 + |x|^2 + \int_{\mathbb{R}^n} |\xi|^2 m(\xi) d\xi)
\] (7.15)

So for any curve \( m_\epsilon \in \mathcal{P}_2, u_{mt}(x, t) \in L^1_m \). From the estimate (7.14) we get immediately the result (7.9). The proof has been completed. ■

### 7.2 OBTAINING BELLMAN EQUATION

We have seen in section 6.2 that

\[
z_{xmt}(s) = Du_{mt}(y_{xmt}(s), s)
\]

and thus

\[
D_x U(x, m, t) = z_{xmt}(t)
\] (7.16)

Therefore from the estimate (6.17) we can assert that

\[
|D_x U(x, m, t)| \leq C[1 + |x| + \sqrt{\int_{\mathbb{R}^n} |\xi|^2 m(\xi) d\xi}]
\] (7.17)

In particular, we can see that \( D_x U(x, m, t) \) belongs to \( L^2_m(\mathbb{R}^n; \mathbb{R}^n) \). But then, recalling the correspondance \( V(X, t) = V(m, t)|_{m=\mathcal{L}_X} \), we can write

\[
D_X V(X, t) = D_x U(X, \mathcal{L}_X, t)
\] (7.18)

and

\[
||D_X V(X, t)||^2 = \int_{\mathbb{R}^n} |D_x \frac{\partial V(m, t)}{\partial m}(x)|^2 m(x)dx
\]

If we look at the Bellman equation in the Hilbert space \( \mathcal{H} \), see (2.11) we obtain exactly (5.21). So we can state

**Proposition 8.** We make the assumptions of Proposition 7. The value function \( V(m, t) \) of the problem
or equivalently \((5.15),(5.16)\) satisfies the estimates \((7.2),(7.10)\) (with 
\[U(x, m, t) = \frac{\partial V(m, t)}{\partial m}(x)\]) and \((7.17)\). It is the unique solution, satisfying these estimates, of the Bellman equation \((5.21)\). Moreover, we have the explicit formula \((7.1)\) with 
\[y_{xmt}(s)\text{ being the unique solution of (6.10) and } z_{xmt}(s) = -\lambda \frac{dy_{xmt}(s)}{ds}\]

### 7.3 Obtaining the scalar master equation

We can derive the scalar master equation from the probabilistic master equation \((3.9)\), which we write as follows

\[
\frac{\partial U}{\partial t} - \frac{1}{2\lambda} D_X ||U(X,t)||^2 + D_X F(X) = 0
\]

\[U(X,T) = D_X F_T(X)\] \hspace{1cm} (7.19)

We know that \(U(X,t) = D_x U(X, L_X, t)\) and \(U(x, m, t) = u_{mt}(x, t)\) and \(D_x u_{mt}(x,t) = z_{xt}(t)\). Therefore, 
\[ ||U(X,t)||^2 = \int_{R^n} |D_\xi U(\xi, m, t)|^2 m(\xi) d\xi.\]

Since this functional of \(m\) only has a derivative in the Hilbert space \(H\) it can be written as follows

\[
D_X ||U(X,t)||^2 = D_x \frac{\partial}{\partial m} \left( \int_{R^n} |D_\xi U(\xi, m, t)|^2 m(\xi) d\xi \right)(X)
\]

Recalling that \(D_X F(X) = D_x F(X, m), D_X F_T(X) = D_x F_T(X, m)\), we see that \((7.19)\) can be written as follows

\[
D_x \left[ \frac{\partial U(X,m,t)}{\partial t} \right] - \frac{1}{2\lambda} \frac{\partial}{\partial m} \left( \int_{R^n} |D_\xi U(\xi, m, t)|^2 m(\xi) d\xi \right)(X) + F(X, m) = 0
\]

\[D_x U(X,m,T) = D_x F_T(X,m)\]

This leads to

\[
\frac{\partial U(x,m,t)}{\partial t} - \frac{1}{2\lambda} \frac{\partial}{\partial m} \left( \int_{R^n} |D_\xi U(\xi, m, t)|^2 m(\xi) d\xi \right)(x) + F(x, m) = 0
\]

\[U(x,m,T) = F_T(x,m)\]

which we can write as \((5.22)\), taking account of the symmetry property \(\frac{\partial}{\partial m} U(x,m,t)(\xi) = \frac{\partial}{\partial m} U(\xi, m, t)(x).\)
This proof is not fully rigorous. It assumes implicitly the existence of \( \frac{\partial}{\partial m} U(x, m, t)(\xi) \), which is the 2nd derivative of the function \( V(m, t) \). To study it rigorously and give an implicit formula for \( \frac{\partial}{\partial m} U(x, m, t)(\xi) \), one can use the system of HJB-FP equations (5.18) and write the solution as \( u_{xt}(x, s), m_{xt}(x, s) \) to emphasize the initial conditions \( m, t \). We then consider the functional derivatives \( \frac{\partial u_{xt}(x, s)}{\partial m}(\xi) \), \( \frac{\partial m_{xt}(x, s)}{\partial m}(\xi) \) and differentiate formally the system of HJB-FP equations. To simplify notation, we take a test function \( \tilde{m}(\xi) \) and consider

\[
\int_{\mathbb{R}^n} \frac{\partial u_{xt}(x, s)}{\partial m}(\xi) \tilde{m}(\xi) d\xi, \quad \int_{\mathbb{R}^n} \frac{\partial m_{xt}(x, s)}{\partial m}(\xi) \tilde{m}(\xi) d\xi
\]

which we note \( \tilde{u}_{xt}; \tilde{m}(x, s) \), and to simplify further \( \tilde{u}(x, s), \tilde{m}(x, s) \). In particular, \( \tilde{u}(x, t) = \int \frac{\partial}{\partial m} U(x, m, t)(\xi) \tilde{m}(\xi) d\xi \). The pair \( \tilde{u}(x, s), \tilde{m}(x, s) \) is solution of a system of linear P.D.E. as follows

\[
- \frac{\partial \tilde{u}}{\partial s} + \frac{1}{\lambda} D\tilde{u}.D\tilde{u} = \int \frac{\partial F}{\partial m}(x, m(s))(\xi) \tilde{m}(\xi, s) d\xi
\]

\[
\tilde{u}(x, T) = \int \frac{\partial F}{\partial m}(x, m(T))(\xi) \tilde{m}(\xi, T) d\xi
\]

\[
\frac{\partial \tilde{m}}{\partial s} - \frac{1}{\lambda} \text{div}(D\tilde{u} m + Du \tilde{m}) = 0
\]

\[
\tilde{m}(x, t) = \tilde{m}(x)
\]

This system is obtained by linearization of the system (5.18). The functions \( u(x, s), m(x, s) \) are solutions of the system (5.18). We can write also

\[
\tilde{u}(x, s) = \int_s^T \int_{\mathbb{R}^n} \frac{\partial F}{\partial m}(y_{xt}(\sigma), y_{mt}(\sigma)(\cdot)(m))(\xi) \tilde{m}(\xi, \sigma) d\xi d\sigma + \int_{\mathbb{R}^n} \frac{\partial F}{\partial m}(y_{xmt}(T), y_{mt}(\cdot)(m))(\xi) \tilde{m}(\xi, T) d\xi
\]

We can then study (7.21) as a fixed point equation in the function \( \tilde{u}(x, s) \).

8 QUADRATIC CASE

8.1 ASSUMPTIONS

We consider the quadratic case, (5.6). We also take
\[ h(x, m) = \frac{1}{2}(x - S_T\bar{x})^*Q_T(x - S_T\bar{x}) + \frac{1}{2}x^*Qx \] (8.1)

In the space \( \mathcal{H} \) we have

\[
\mathcal{F}(X) = \frac{1}{2}EX^*(Q + \bar{Q})X + \frac{1}{2}EX^*(S^*QS - \bar{Q}S - S^*\bar{Q})EX
\]

\[
\mathcal{F}_T(X) = \frac{1}{2}EX^*(Q_T + \bar{Q})X + \frac{1}{2}EX^*(S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})EX
\] (8.2)

We can write \( \mathcal{F}(X) = Ef(X, \mathcal{L}_X) \) with

\[
f(x, m) = \frac{1}{2}(x - S\bar{x})^*\bar{Q}(x - S\bar{x}) + \frac{1}{2}x^*Qx
\] (8.3)

in which \( \bar{x} = \int x m(x)dx \), assuming the probability law of \( X \) has a density, \( m \). So we can also write

\[
\mathcal{F}(X) = \Phi(m) = \frac{1}{2} \int_{R^n} x^*(Q + \bar{Q})xm(x)dx + \frac{1}{2} \int_{R^n} x^*m(x)dx \left( S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q} \right) \int_{R^n} xm(x)dx
\]

\[
= \int_{R^n} f(x, m)m(x)dx
\]

If we take \( m \in L^2(R^n) \cap L^1(R^n) \) not necessarily a probability density, then we have to introduce \( m_1 = \int_{R^n} m(x)dx \) and write

\[
\Phi(m) = \int_{R^n} f(x, m)m(x)dx = \frac{1}{2} \int_{R^n} x^*(Q + \bar{Q})xm(x)dx + \frac{1}{2} \bar{x}^*S^*\bar{Q}S\bar{x} m_1 - \frac{1}{2} \bar{x}^*(\bar{Q}S + S^*Q)\bar{x}
\] (8.4)

We have noted \( F(x, m) = \frac{\partial \Phi(m)}{\partial m} (x) \). Then as a Gâteaux differential we have

\[
F(x, m) = \frac{1}{2}x^*(Q + \bar{Q})x + \bar{x}^*(S^*\bar{Q}S m_1 - \bar{Q}S - S^*\bar{Q})x + \frac{1}{2} \bar{x}^*S^*\bar{Q}S\bar{x}
\] (8.5)

We note that

\[
D_X\mathcal{F}(X) = (Q + \bar{Q})X + (S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})EX
\] (8.6)

So the equality \( D_X\mathcal{F}(X) = D_x F(X, m) \) is true only when \( m_1 = 1 \). It is important to keep in mind that
when we work with Gâteaux differentials, we have to make calculations with the term \( m_1 \), even though that eventually, when applied to \( m = \) probability density, we shall have \( m_1 = 1 \). To understand further this point, let us compute the 2nd derivative. We have

\[
\frac{\partial F}{\partial m}(x, m)(\xi) = \bar{x}^* S^* QS(x + \xi) + \xi^*(S^* QS m_1 - QS - S^* \bar{Q})x
\]

(8.7)

We see that this formula is symmetric in \( x, \xi \) as needed. Without the term \( m_1 \) in (8.5) this will not be true.

We have

\[
D_x^2 F(x, m) = Q + \bar{Q}, \quad D_x D_\xi \frac{\partial F}{\partial m}(x, m)(\xi) = (S^* QS m_1 - QS - S^* \bar{Q})
\]

and, see [5]

\[
D_X^2 F(X)Z = D_x^2 F(X, L_X)Z + E_Y D_x D_y \frac{\partial F}{\partial m}(X, L_X)(Y)Z = (Q + \bar{Q})Z + (S^* QS - QS - S^* \bar{Q})EZ
\]

which is exactly what we obtain by differentiating (8.6) in the Hilbert space.

### 8.2 Bellman Equation

Bellman equation (5.21) writes

\[
\frac{\partial V}{\partial t} - \frac{1}{2\lambda} \int_{\mathbb{R}^n} |D_\xi \frac{\partial V(m, t)}{\partial m}(\xi)|^2 m(\xi) d\xi + \frac{1}{2} \int_{\mathbb{R}^n} \xi^*(Q + \bar{Q})m(\xi)d\xi + \frac{1}{2} \bar{x}^* S^* QS \bar{x} m_1 - \frac{1}{2} \bar{x}^* (\bar{Q} S + S^* \bar{Q})\bar{x} = 0
\]

(8.8)

\[
V(m, T) = \frac{1}{2} \int_{\mathbb{R}^n} \xi^* (Q_T + \bar{Q}_T)m(\xi)d\xi + \frac{1}{2} \bar{x}^* S_T^* Q_T S_T \bar{x} m_1 - \frac{1}{2} \bar{x}^* (Q_T S_T + S_T^* \bar{Q}_T)\bar{x}
\]

The solution is

\[
V(m, t) = \frac{1}{2} \int_{\mathbb{R}^n} \xi^* P(t)m(\xi)d\xi + \frac{1}{2} \bar{x}^* \Sigma(t; m_1)\bar{x}
\]

(8.9)

with
\[
\frac{dP}{dt} - \frac{P^2}{\lambda} + Q + \bar{Q} = 0 \tag{8.10}
\]
\[
P(T) = Q_T + \bar{Q}_T
\]

\[
\frac{d\Sigma}{dt} - \frac{1}{\lambda}(\Sigma P + P\Sigma) - \frac{1}{\lambda}\Sigma^2 m_1 + S^* \bar{Q} S m_1 - (\bar{Q} S + S^* \bar{Q}) = 0 \tag{8.11}
\]
\[
\Sigma(T; m_1) = S^*_T \bar{Q}_T S_T m_1 - (\bar{Q}_T S_T + S^*_T \bar{Q}_T)
\]

### 8.3 MASTER EQUATION

The scalar Master equation (5.22) reads

\[
\frac{\partial U}{\partial t} - \frac{1}{\lambda} \int_{\mathbb{R}^n} D_{\xi} \frac{\partial U}{\partial m}(x, m, t)(\xi) D_\xi U(x, m, t)(\xi) d\xi - \frac{1}{2\lambda} |D_x U(x, m, t)|^2 + \frac{1}{2} x^*(Q + \bar{Q})x + \bar{x}^*(S^* \bar{Q} S m_1 - \bar{Q} S - S^* \bar{Q})x + \frac{1}{2} \bar{x}^* S^* \bar{Q} S \bar{x} = 0 \tag{8.12}
\]
\[
U(x, m, T) = \frac{1}{2} x^*(Q_T + \bar{Q}_T)x + \bar{x}^*(S^*_T \bar{Q}_T S_T m_1 - \bar{Q}_T S_T - S^*_T \bar{Q}_T)x + \frac{1}{2} \bar{x}^* S^*_T \bar{Q}_T S_T \bar{x}
\]

Its solution is

\[
U(x, m, t) = \frac{\partial V(m, t)}{\partial m}(x) = \frac{1}{2} x^* P(t)x + \bar{x}^* \Sigma(t; m_1)x + \frac{1}{2} \bar{x}^* \frac{\partial \Sigma(t; m_1)}{\partial m_1} \bar{x} \tag{8.13}
\]

We have

\[
D_x U(x, m, t) = P(t)x + \Sigma(t; m_1)\bar{x} \tag{8.14}
\]

\[
\frac{\partial U}{\partial m}(x, m, t)(\xi) = \bar{x}^* \frac{\partial \Sigma(t; m_1)}{\partial m_1}(x + \xi) + \xi^* \Sigma(t; m_1)x + \frac{1}{2} \xi^* \frac{\partial^2 \Sigma(t; m_1)}{\partial m_1^2} \bar{x}
\]
\[
D_\xi \frac{\partial U}{\partial m}(x, m, t)(\xi) = \frac{\partial \Sigma(t; m_1)}{\partial m_1} \bar{x} + \Sigma(t; m_1)x
\]

We note that \(\Gamma(t; m_1) = \frac{\partial \Sigma(t; m_1)}{\partial m_1}\) satisfies the equation
\[
\frac{d\Gamma}{dt} - \frac{1}{\lambda}(\Gamma(P + \Sigma) + (P + \Sigma)\Gamma) - \frac{1}{\lambda^2}\Sigma^2 + S^*\bar{Q}S = 0
\]  
\[\Gamma(T; m_1) = S_T^*\bar{Q}_TS_T\]  

and we check easily that the function \(U(x, m, t)\) defined by (8.13) is solution of the scalar master equation (8.12).

We turn to the vector master equation (5.25) which reads

\[
\frac{\partial U}{\partial t} - \frac{1}{\lambda} \int_{\mathbb{R}^n} D_\xi \frac{\partial U}{\partial m}(x, m, t)(\xi) U(\xi, m, t) m(\xi) d\xi - \frac{1}{\lambda} D_x U(x, m, t) U(x, m, t) + (Q + \bar{Q})x + (S^*\bar{Q}m_1 - \bar{Q}S - S^*\bar{Q})\bar{x} = 0
\]

whose solution is

\[
U(x, m, T) = \frac{1}{2} x^* (Q_T + \bar{Q}_T)x + (S_T^*\bar{Q}_TS_T m_1 - \bar{Q}_T S_T - S_T^*\bar{Q}_T)\bar{x}
\]

This statement is easily verified.

### 8.4 SYSTEM OF HJB-FP EQUATIONS

We now look at the system (5.18) which reads

\[
-\frac{\partial u}{\partial s} + \frac{1}{2\lambda}|Du|^2 = \frac{1}{2} x^* (Q + \bar{Q})x + \bar{x}^*(s)(S^*\bar{Q}S m_1(s) - \bar{Q}S - S^*\bar{Q})x + \frac{1}{2} \bar{x}^*(s)S^*\bar{Q}\bar{x}(s)
\]

\[
u(x, T) = \frac{1}{2} x^* (Q_T + \bar{Q}_T)x + \bar{x}^*(T)(S_T^*\bar{Q}_TS_T m_1(T) - \bar{Q}_T S_T - S_T^*\bar{Q}_T)\bar{x} + \frac{1}{2} \bar{x}^*(T)S_T^*\bar{Q}_TS_T\bar{x}(T)
\]

\[
\frac{\partial m}{\partial s} - \frac{1}{\lambda} \text{div}(Du m) = 0
\]

\[m(x, t) = m(x)
\]

It is immediate to see that \(m_1(s) = m_1\). The function \(\bar{x}(s)\) represents the mean \(\int_{\mathbb{R}^n} \xi m(\xi, s)d\xi\). We do not need to obtain the full probability \(m(x, s)\). The mean is sufficient. One can check the formula
\[ u(x, s) = \frac{1}{2} x^* P(s)x + x^* \Sigma(s; m_1) \bar{x}(s) + \frac{1}{2} \bar{x}(s)^* \frac{\partial \Sigma(s; m_1)}{\partial m_1} \bar{x}(s) \quad (8.19) \]

In particular \( u(x, t) = U(x, m, t) \) given by (8.13). Also \( u(x, s) = U(x, m(s), s) \). Note that \( \bar{x}(s) \) evolves as follows

\[ \frac{d \bar{x}}{ds} + \frac{1}{\lambda} (P(s) + m_1 \Sigma(s; m_1)) \bar{x}(s) = 0 \quad (8.20) \]

\[ \bar{x}(t) = \bar{x} \]

We have seen that \( U(x, m, t) \) is differentiable in \( m \) with

\[ \frac{\partial U}{\partial m}(x, m, t)(\xi) = \bar{x}^* \frac{\partial \Sigma(t; m_1)}{\partial m_1} (x + \xi) + \xi^* \Sigma(t; m_1)x + \frac{1}{2} \bar{x}^* \frac{\partial^2 \Sigma(t; m_1)}{\partial m_1^2} \bar{x} \]

If we consider a test function \( \tilde{m}(\xi) \) and define

\[ \tilde{u}(x, t) = \tilde{u}_{m; \tilde{m}}(x, t) = \lim_{\theta \to 0} \frac{u_{m + \theta \tilde{m}, t}(x, t) - u_{m, t}(x, t)}{\theta} \]

we get

\[ \tilde{u}(x, t) = x^* \Sigma(t; m_1) \tilde{x} + \bar{x}^* \frac{\partial \Sigma(t; m_1)}{\partial m_1} \tilde{x} + \tilde{m}_1(\bar{x}^* \frac{\partial \Sigma(t; m_1)}{\partial m_1} \bar{x} + \frac{1}{2} \bar{x}^* \frac{\partial^2 \Sigma(t; m_1)}{\partial m_1^2} \bar{x}) \quad (8.21) \]

We can also compute

\[ \tilde{u}(x, s) = \tilde{u}_{m; \tilde{m}}(x, s) = \lim_{\theta \to 0} \frac{u_{m + \theta \tilde{m}, t}(x, s) - u_{m, t}(x, s)}{\theta} \]

We have

\[ \tilde{u}(x, s) = x^*(\Sigma(s; m_1) \tilde{x}(s) + \frac{\partial \Sigma(s; m_1)}{\partial m_1} \bar{x}(s) \tilde{m}_1) + \tilde{x}(s)^* \frac{\partial \Sigma(s; m_1)}{\partial m_1} \bar{x}(s) + \tilde{m}_1 \frac{1}{2} \bar{x}(s)^* \frac{\partial^2 \Sigma(s; m_1)}{\partial m_1^2} \bar{x}(s) \quad (8.22) \]

in which

\[ \frac{d \tilde{x}(s)}{ds} + \frac{1}{\lambda} (P(s) + m_1 \Sigma(s; m_1)) \tilde{x}(s) + \frac{1}{\lambda} \tilde{m}_1(\Sigma(s; m_1) + m_1 \frac{\partial \Sigma(s; m_1)}{\partial m_1}) \bar{x}(s) = 0 \quad (8.23) \]

and where \( \tilde{m}_1 = \int_{R^n} \tilde{m}(\xi)d\xi \). The function \( \bar{u}(x, s) \) is the solution of the linearized equation (7.20), namely
\[
\frac{\partial \tilde{u}}{\partial s} + \frac{1}{\lambda} D\tilde{u} (P(s)x + \Sigma(s;m_1)\tilde{x}(s)) = x^* \left( S^* \tilde{Q} S \tilde{x}(s) \tilde{m}_1 + (S^* \tilde{Q} S m_1 - \tilde{Q} S - S^* \tilde{Q}) \tilde{x}(s) \right) + \lambda D \tilde{u} \tilde{x}(s)
\]

\[
\tilde{u}(x, T) = x^* \left( S^* \tilde{Q} S \tilde{x}(T) \tilde{m}_1 + (S^* \tilde{Q} S m_1 - \tilde{Q} S - S^* \tilde{Q}) \tilde{x}(T) \right) + \lambda D \tilde{u} \tilde{x}(T)
\]

which can be checked by direct calculation.

### 8.5 STATE EQUATION

We consider equation (6.10) in the quadratic case. Since we know the function \( u_{mt}(x, s) \) see (8.19) the best is to use the fact that \( y_{xmt}(s) \) is the solution of

\[
\frac{dy}{ds} = -\frac{1}{\lambda} D\tilde{u}(y(s), s)
\]

\[
y(t) = x
\]

We get the explicit solution

\[
y_{xmt}(s) = \exp \left\{ \frac{1}{\lambda} \int_t^s P(\sigma) d\sigma x - \frac{1}{\lambda} \int_t^s \left( \exp \left\{ -\frac{1}{\lambda} \int_\tau^s P(\tau) \Sigma(\sigma) \right\} \exp \left\{ -\frac{1}{\lambda} \int_t^\sigma (\Sigma(\tau) + P(\tau)) d\tau \right\} \right) d\sigma \right\} (8.25)
\]

### 8.6 FORMULATION IN THE HILBERT SPACE

We can formulate Bellman equation and the Master equation in the Hilbert space \( \mathcal{H} \). We have first Bellman equation

\[
\frac{\partial V}{\partial t} - \frac{1}{2\lambda} ||D_X V||^2 + \frac{1}{2} EX^*(Q + \tilde{Q}) X + \frac{1}{2} EX^*(S^* \tilde{Q} S - \tilde{Q} S - S^* \tilde{Q}) EX = 0
\]

\[
V(X, T) = \frac{1}{2} EX^*(Q_T + \tilde{Q}_T) X + \frac{1}{2} EX^*(S_T^* \tilde{Q}_T S_T - \tilde{Q}_T S_T - S_T^* \tilde{Q}_T) EX
\]

whose solution is
\[ V(X, t) = \frac{1}{2}EX^*P(t)X + \frac{1}{2}EX^*\Sigma(t)EX \]  \hspace{1cm} (8.27)

with \( \Sigma(t) = \Sigma(t; 1) \). The Master equation reads

\[ \frac{\partial U}{\partial t} - \frac{1}{\lambda}D_XU(X, t)U(X, t) + (Q + \bar{Q})X + (S^*\bar{Q}S - \bar{Q}S - S^*\bar{Q})EX = 0 \]  \hspace{1cm} (8.28)

\[ U(X, T) = (Q_T + \bar{Q}_T)X + (S^*_TQ_TS_T - \bar{Q}_TS_T - S^*_T\bar{Q}_T)EX \]

whose solution is \( U(X, t) = P(t)X + \Sigma(t)EX \). Note that \( D_XU(X, t)Z = P(t)Z + \Sigma(t)EZ \). The state equation is the solution of

\[ \frac{dY}{ds} = -\frac{1}{\lambda}(P(s)Y(s) + \Sigma(s)EY(s)) \]  \hspace{1cm} (8.29)

\[ Y(t) = X \]

hence the formula

\[ Y(s) = \exp -\frac{1}{\lambda} \int_t^s P(\sigma)d\sigma X - \int_t^s (\exp -\frac{1}{\lambda} \int_\sigma^s P(\tau)d\tau \Sigma(\sigma) \exp -\frac{1}{\lambda} \int_t^\sigma (P(\tau) + \Sigma(\tau))d\tau)d\sigma)EX \]  \hspace{1cm} (8.30)

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