Non-existence of a short algorithm for multiplication
of $3 \times 3$ matrices with group $S_4 \times S_3$

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Аннотация

One of prospective ways to find new fast algorithms of matrix multiplication is to study algorithms admitting nontrivial symmetries. In the work possible algorithms for multiplication of $3 \times 3$ matrices, admitting a certain group $G$ isomorphic to $S_4 \times S_3$, are investigated. It is shown that there exist no such algorithms of length $\leq 23$. In the first part of the work, which is the content of the present article, we describe all orbits of length $\leq 23$ of $G$ on the set of decomposable tensors in the space $M \otimes M \otimes M$, where $M = M_3(\mathbb{C})$ is the space of complex $3 \times 3$ matrices. In the second part of the work this description will be used to prove that a short algorithm with the above-mentioned group does not exist.

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1. Introduction.

The fast matrix multiplication is one of the main questions about computational complexity, see e.g. [1,2]. It was proposed in works [3,4] and independently in [5] to consider algorithms admitting nontrivial symmetries. This may be a prospective way to find new fast algorithms. The exact definition of what is the automorphism group of an algorithm can be found in [4](also in [5], [6]).

In [3,4] the automorphism groups of well-known algorithms of Strassen, Hopcroft and Laderman were found. These groups are isomorphic to $S_3 \times S_3$, $S_3 \times Z_2$, and $S_4$, respectively. This suggests the following idea: take a group, which is more or less similar to these three, and investigate algorithms for multiplication of $3 \times 3$ matrices that are invariant under this group (with an intention to find an algorithm shorter than Laderman’s).

Recently several works [7–9, 9a] were published where algorithms invariant under a prescribed groups were studied. However, no algorithms better than the known ones (say, an algorithm for $3 \times 3$ matrices of length $\leq 22$; remind that Laderman’s algorithm is of length 23. On the other hand, a lower estimate $\geq 19$ for the length of such an algorithm is known, see [10]) were found.

In the present article we consider certain group isomorphic to $S_4 \times S_3$ as a candidate for the symmetry group. Unfortunately, the result is negative again: an algorithm of length $\leq 23$ with this group does not exist. The details of computations are nontrivial and may be useful for future research.

For the convenience of the reader who is not especially experienced in algorithms we formulate our result in purely representation-theoretic terms. Let

$$M = M_3(\mathbb{C}) = \langle e_{ij} \mid 1 \leq i, j \leq 3 \rangle_{\mathbb{C}}$$

be the space of complex $3 \times 3$ matrices. Here $e_{ij}$ are the usual matrix units. Consider the tensor

$$T = \sum_{1 \leq i,j,k \leq 3} e_{ij} \otimes e_{jk} \otimes e_{ki} \in M \otimes M \otimes M$$
(in complexity theory this tensor is usually denoted by \( \langle 3,3,3 \rangle \)).

Let \( A \leq GL(3, \mathbb{C}) \) be the group of all monomial \( 3 \times 3 \) matrices whose nonzero elements are \( \pm 1 \) and the determinant is \( \det = 1 \). It is easy to see that \( A \cong S_4 \) and \( A \) is irreducible.

For any \( a, b, c \in GL(3, \mathbb{C}) \) consider the transformation of \( \mathcal{M} \times \mathcal{M} \times \mathcal{M} \),

\[
T(a, b, c) : x \otimes y \otimes z \mapsto axb^{-1} \otimes byc^{-1} \otimes cza^{-1}.
\]

It is easy to see that always \( T(a, b, c) \mathcal{T} = \mathcal{T} \). In particular, \( T(a, a, a) \mathcal{T} = \mathcal{T} \) for \( a \in A \). Thus we can consider that \( A \) acts on \( \mathcal{M} \otimes \mathcal{M} \) and preserves \( \mathcal{T} \).

Also, consider the two transformations

\[
\rho(x \otimes y \otimes z) = y^t \otimes x^t \otimes z^t, \quad \sigma(x \otimes y \otimes z) = z \otimes x \otimes y
\]

(where \( t \) means transpose). It is easy to see that both \( \rho \) and \( \sigma \) preserve \( \mathcal{T} \), and that \( B := \langle \rho, \sigma \rangle \cong S_3 \). Finally, it is not hard to see that \( A \) and \( B \) commute elementwise (for the details of these (and more general) computations the reader is referred to [4] or [6]). Thus, the group \( G = A \times B \cong S_4 \times S_3 \) acts on \( \mathcal{M} \otimes \mathcal{M} \) and preserves \( \mathcal{T} \).

A decomposition of length \( l \) of \( \mathcal{T} \) is an (unordered) set of \( l \) decomposable tensors \( \{ x_i \otimes y_i \otimes z_i \mid i = 1, \ldots, l \} \) such that

\[
\sum_{i=1}^{l} x_i \otimes y_i \otimes z_i = \mathcal{T}.
\]

(Note that in the previous sentence the word “decomposition” was used twice, first in “additive” sense, then in the multiplicative one !).

Clearly, any element of \( G \) takes a length \( l \) decomposition into a length \( l \) decomposition also. So we can consider \( G \)-invariant decompositions. Now we can state the main result of the work.

**Theorem 1.** Let \( \mathcal{T} = \langle 3,3,3 \rangle \) and \( G = A \times B \) be as described above. Then there exists no \( G \)-invariant decomposition of \( \mathcal{T} \) of length \( \leq 23 \).

We divide the proof of this theorem into two parts. In the first part, which is the content of the present article, we describe the orbits of \( G \) on decomposable tensors in \( \mathcal{M} \otimes \mathcal{M} \), of length \( \leq 23 \) (this description turns out to be rather long). In the second part, which will be published soon, we make use of this description to prove Theorem 1.

The reader should be warned that this proof contains extensive calculations. To write all these calculations in full would be tiresome. So, we usually only explain the main idea of a calculation and give an example or two, and leave the rest of calculations to an interested reader.

**Remark.** Observe that the standard decomposition of \( \mathcal{T} \), that is, the set of all \( e_{ij} \otimes e_{jk} \otimes e_{ki} \), is \( G \)-invariant (but its full automorphism group is much larger, namely of the form \( (\mathbb{C}^*)^6 \times S_3 \)). It is interesting to find out whether there are \( G \)-invariant decompositions of smaller length (say, 26).

**2. The subgroups of** \( S_4 \times S_3 \). Thus, the main aim of the present article is to classify the orbits of \( G \) on the decomposable tensors \( x \otimes y \otimes z \), of length \( \leq 23 \). In fact, the length of such an orbit is \( \leq 18 \), because \( S_4 \times S_3 \) has no subgroups of index 19, 20, 21, 22, or 23. In this section we describe all subgroups of index \( \leq 18 \) of \( S_4 \times S_3 \). We write elements of \( S_4 \times S_3 \) as pairs of permutations. (The explicit correspondence between permutation notation and action of \( G \)
on $M \otimes M \otimes M$ will be described in the next section. We consider the permutations as acting on symbols $1, 2, 3, 4$ on the left (and so multiplied from right to left, like $(12)(13) = (132)$).

The subgroups of $S_3$ are, up to isomorphism, $1, Z_2, Z_3, S_3$ itself. And two isomorphic subgroups are always conjugated. So we denote the conjugacy class of a subgroup by the same symbol as its isomorphism class. Also, the subgroups of $S_4$ are, up to conjugacy, $1, Z_2^{(1)}, Z_2^{(2)}, Z_3, V^{(1)}, V^{(2)}, Z_4, S_3, D_8, A_4$, and $S_4$. Here $V = Z_2 \times Z_2$, and the superscript is used to distinguish between non-conjugate isomorphic subgroups:

\[
Z_2^{(1)} \sim \langle (12) \rangle, \quad Z_2^{(2)} \sim \langle (12)(34) \rangle, \\
V^{(1)} \sim \langle (12), (34) \rangle, \quad V^{(2)} \sim \langle (12)(34), (13)(24) \rangle.
\]

Recall that a subdirect product of groups $X$ and $Y$ is any subgroup $Z \leq X \times Y$ such that $\pi_X(Z) = X$ and $\pi_Y(Z) = Y$, where $\pi_X$ and $\pi_Y$ are the projections of $X \times Y$ onto factors. The subdirect products can be characterized by the following property (see [11], §5.5): there exists a group $W$, which is a quotient group for both $X$ and $Y$, and two epimorphism $\varphi_X : X \to W$ and $\varphi_Y : Y \to W$ such that

\[
Z = \{ (x, y) \in X \times Y \mid \varphi_X(x) = \varphi_Y(y) \}.
\]

Conversely, any subgroup of this form is a subdirect product of $X$ and $Y$.

A (nontrivial) subdirect product of $X$ and $Y$ will be denoted by $X \circ Y$ (or $X \circ_i Y$, if there are several such nontrivial products).

If $X$ and $Y$ are arbitrary groups and $Z$ any subgroup of $X \times Y$, then clearly $Z$ is a subdirect product of the projections $X_1 = \pi_X(Z)$ and $Y_1 = \pi_Y(Z)$. Moreover, if $X_1, X_2 \leq X$ are conjugate in $X$, and $Y_1, Y_2 \leq Y$ are conjugate also, and $Z_1$ is a subdirect product of $X_1$ and $Y_1$, then $Z_1$ is conjugate in $X \times Y$ with a subgroup $Z_2$ that is a subdirect product of $X_2$ and $Y_2$.

Note also that if $C$ and $D$ are subgroups of $X$ and $Y$ respectively, determined up to conjugacy, then $C \times D$ can be considered as a subgroup of $X \times Y$, determined also up to conjugacy.

For each subgroup of $S_4$ or $S_3$ to find all its quotients is trivial. Now, keeping in mind the previous discussion, it is easy to prove the following statement.

**Proposition 2.** All the subgroups of $S_4 \times S_3$ of index $\leq 18$ (that is, of order $\geq 8$) are the following, up to conjugacy:

\[
Z_2^{(1)} \times S_3, \quad Z_2^{(2)} \times S_3, \quad Z_3 \times Z_3, \quad Z_3 \times S_3, \quad V^{(1)} \times Z_2, \quad V^{(1)} \times Z_3, \\
V^{(1)} \times S_3, \quad V^{(1)} \circ_i S_3 \ (i = 1, 2), \quad V^{(2)} \times Z_2, \quad V^{(2)} \times Z_3, \quad V^{(2)} \times S_3, \\
V^{(2)} \circ S_3, \quad Z_4 \times Z_2, \quad Z_4 \times Z_3, \quad Z_4 \times S_3, \quad Z_4 \circ S_3, \quad S_3 \times Z_2, \quad S_3 \times Z_3, \\
S_3 \times S_3, \quad S_3 \circ S_3, \quad D_8 \times 1, \quad D_8 \times Z_2, \quad D_8 \times Z_3, \quad D_8 \circ S_3, \\
D_8 \circ_i Z_2 \ (i = 1, 2, 3), \quad D_8 \circ_i S_3, \quad A_4 \times 1, \quad A_4 \times Z_2, \quad A_4 \times Z_3, \quad A_4 \circ S_3, \\
A_4 \circ Z_3, \quad S_4 \times 1, \quad S_4 \times Z_2, \quad S_4 \times Z_3, S_4 \times S_3, \quad S_4 \circ Z_2, \quad S_4 \circ_i S_3 \ (i = 1, 2).
\]

**Sketch of proof.** We know from the previous discussion that any subgroup $Z \leq S_4 \times S_3$ is a subdirect product of $X \leq S_4$ and $Y \leq S_3$. And we can take $X$ and $Y$ up to conjugacy, if we are interested in the conjugacy class of $Z$ only.
Consider, for example, the case $X = Z_4$, $Y = S_3$. Notice that the only nontrivial quotient of both $Z_4$ and $S_3$ is $Z_2$, and the epimorphisms $Z_4 \twoheadrightarrow Z_2$ and $S_3 \twoheadrightarrow Z_2$ are unique. So there exists a unique nontrivial subdirect product of $Z_4$ and $S_3$. Thus, in the case under consideration we obtain two possibilities for $Z$: $Z = Z_4 \times S_3$ or $Z = Z_4 \circ S_3$.

In some cases the subdirect product is not unique. For example, let $X = D_8$ and $Y = S_3$. The unique quotient of both $X$ and $Y$ is $Z_2$. The epimorphism $S_3 \twoheadrightarrow Z_2$ is unique, but there are three distinct epimorphisms $D_8 \twoheadrightarrow Z_2$. So there are three distinct subdirect products $D_8 \circ_i S_3$, $i = 1, 2, 3$. A similar argument applies for subgroups of the form $D_8 \circ_i Z_2$.

We mention also the case of groups $S_4 \circ S_3$. A common quotient of $S_4$ and $S_3$ is either $Z_2$ or $S_3$. In the case $Z_2$ the corresponding subgroup is unique, because the epimorphisms $S_4 \twoheadrightarrow Z_2$ and $S_3 \twoheadrightarrow Z_2$ are unique. In the case $S_3$ the subgroup is not unique, since there are distinct epimorphisms $S_4 \twoheadrightarrow S_3$ and $S_3 \twoheadrightarrow S_3$. However, since any automorphism of $S_3$ is an inner one, all these subgroups are conjugate in $S_4 \times S_3$ (by an element of the form $(1, x)$, where $x \in S_3$). The subdirect product with common quotient $Z_2$ will be denoted by $S_4 \circ_1 S_3$, and that with quotient $S_3$ by $S_4 \circ_2 S_3$. Similarly, the subgroups of the form $A_4 \circ Z_3$ are conjugate also (there exist two epimorphisms $A_4 \twoheadrightarrow Z_3$, but the corresponding subdirect products $A_4 \circ_1 Z_3$ and $A_4 \circ_2 Z_3$ are conjugate by an element of the form $(1, y)$, where $y \in S_3$ is a transposition).

Finally, consider subdirect products of the form $V^{(j)} \circ S_3$, $j = 1, 2$. Since there are three distinct epimorphisms $V^{(j)} \twoheadrightarrow Z_2$, there exists three products in each of two cases $j = 1, 2$. However, all three order 2 subgroups of $V^{(2)}$ are conjugate under normalizer $N_{S_4}(V^{(2)})$ ($= S_4$), and in the case $V^{(1)}$ there are two conjugacy classes of such subgroups. Hence three products of the form $V^{(2)} \circ S_3$ are conjugate in $S_4 \times S_3$, and there are two non-conjugate products of the form $V^{(1)} \circ S_3$.

Now, we choose some representatives of conjugacy classes of subgroups of $S_3$ and $S_4$, and, next, of subgroups of $S_4 \times S_3$.

As representatives for $Z_2$ and $Z_3$ in $S_3$ we take $\langle (12) \rangle_2$ and $\langle (123) \rangle_3$ respectively.

When choosing representatives for conjugacy classes of subgroups of $S_4$ we take care to take them in such a way that there will be inclusions among them, in natural situations. The representatives for $Z_2^{(1)}$, $Z_2^{(2)}$, $V^{(1)}$, and $V^{(2)}$ are chosen already. Then it is natural to take $\langle V^{(2)}, Z_2^{(1)} \rangle = \langle (12)(34), (13)(24), (12) \rangle$ as a representative of $D_8$; note that it contains the previous four groups. Now, it is natural to take the only subgroup isomorphic to $Z_4$ in the latter group, namely $\langle (1324) \rangle_4$, as a representative for $Z_4$. Next, as a representative of $Z_3$ take $\langle (123) \rangle$. Then $S_3 = \langle Z_3, Z_2^{(1)} \rangle$ is the subgroup of all permutations preserving $\{1, 2, 3\}$ and fixing 4. Finally, there is no need to choose representatives for $A_4$ and $S_4$, since such subgroups are unique. (In fact, there was no need to choose a representative for a class of subgroups conjugate to $V^{(2)}$ also, because $V^{(2)}$ is normal in $S_4_1$.)

In the sequel, when mentioning a subgroup $X \times Y \leq S_4 \times S_3$, we mean that $X$ and $Y$ are the standard representatives of their conjugacy classes, and similarly for $X \circ Y$.

Now we can describe more precisely the structure of nontrivial subdirect products in Proposition 2. Let $H = X \circ Y$ be such a product and $C = H \cap A$, $D = H \cap B$ (or, more exactly, $C$ is a subgroup of $A$, corresponding (with respect to the isomorphism $A \cong A \times 1$) to the subgroup $H \cap (A \times 1)$; and similarly for $D$). Then $H \geq C \times D$, and $X/C \cong Y/D \cong H/C \times D$ is a common quotient for $X$ and $Y$. If there exist $R_1 \leq X$ and $R_2 \leq Y$ such that $X = C \circ R_1$ and $Y = D \circ R_2$ (then necessarily $R_1 \cong R_2 \cong X/C$), then $H = (C \times D) \circ R$, where $R$ is a
diagonal subgroup of $R_1 \times R_2$.

In fact, the only case of all the cases listed in Proposition 2 when $R_1$ or $R_2$ do not exist is the group $Z_4 \circ S_3$. Also, in most cases $R \cong Z_2$. More exactly, the reader can easily verify the following proposition.

**Proposition 3.** The following equalities hold:

$$
V^{(1)} \circ_1 S_3 = (Z_2^{(1)} \times Z_3) \times \langle g_1 \rangle_2, \quad V^{(1)} \circ_2 S_3 = (Z_2^{(2)} \times Z_3) \times \langle g_2 \rangle_2,
$$

$$
V^{(2)} \circ S_3 = (Z_2^{(2)} \times Z_3) \times \langle g_3 \rangle_2, \quad Z_4 \circ S_3 = (Z_2^{(2)} \times Z_3) \langle g_4 \rangle_4,
$$

(Note nontrivial intersection in the latter case!)

$$
S_3 \circ S_3 = (Z_3 \times Z_3) \times \langle g_2 \rangle_2, \quad D_8 \circ_1 Z_2 = (V^{(1)} \times 1) \times \langle g_3 \rangle_2,
$$

$$
D_8 \circ_2 Z_2 = (V^{(2)} \times 1) \times \langle g_2 \rangle_2, \quad D_8 \circ_3 Z_2 = (Z_4 \times Z_3) \times \langle g_2 \rangle_2,
$$

$$
D_8 \circ_1 S_3 = (V^{(1)} \times Z_3) \times \langle g_3 \rangle_2, \quad D_8 \circ_2 S_3 = (V^{(2)} \times Z_3) \times \langle g_2 \rangle_2,
$$

$$
D_8 \circ_3 S_3 = (Z_4 \times Z_3) \times \langle g_2 \rangle_2, \quad A_4 \circ Z_3 = (V^{(2)} \times 1) \times \langle g_3 \rangle_3,
$$

$$
S_4 \circ Z_2 = (A_4 \times 1) \times \langle g_2 \rangle_2, \quad S_4 \circ_1 S_3 = (A_4 \times Z_3) \times \langle g_2 \rangle_2,
$$

$$
S_4 \circ_2 S_3 = (V^{(2)} \times 1) \times \langle g_2, g_5 \rangle_2,
$$

where

$$g_1 = ((12)(34), (12)), \quad g_2 = ((12), (12)), \quad g_3 = ((13)(24), (12)),
$$

$$g_4 = ((1324), (12)), \quad g_5 = ((123), (123)).$$

$\square$

(Note that in this proposition we implicitly introduced (and shall use in the sequel) a numbering for the products $X \circ_i Y$ in the cases where there exist several products of the form $X \circ Y$).

3. **Correspondences** $A \leftrightarrow S_4$, $B \leftrightarrow S_3$. For further considerations we need to choose isomorphisms $A \leftrightarrow S_4$ and $B \leftrightarrow S_3$ in an explicit form.

The isomorphism $B \leftrightarrow S_3$ is taken in such a way that the tensor factors of $M \otimes M \otimes M$ are permuted in the same way as the symbols 1, 2, 3, that is

$$
e \leftrightarrow (x \otimes y \otimes z \mapsto x \otimes y \otimes z), \quad (12) \leftrightarrow (x \otimes y \otimes z \mapsto y^t \otimes x^t \otimes z^t),
$$

$$
(13) \leftrightarrow (x \otimes y \otimes z \mapsto z^t \otimes y^t \otimes x^t), \quad (23) \leftrightarrow (x \otimes y \otimes z \mapsto x^t \otimes z^t \otimes y^t),
$$

$$
(123) \leftrightarrow (x \otimes y \otimes z \mapsto z \otimes x \otimes y), \quad (132) \leftrightarrow (x \otimes y \otimes z \mapsto y \otimes z \otimes x).
$$

This formulae mean the following: the element $(12) \in S_3$ corresponds to the transformation $x \otimes y \otimes z \mapsto y^t \otimes x^t \otimes z^t$ of $B$, etc. The fact that this is an isomorphism indeed easily follows from the observation that the permutation of factors in the tensor product commutes with taking componentwise transpose, i.e. with transformation $x \otimes y \otimes z \mapsto x^t \otimes y^t \otimes z^t$.

Next we describe an isomorphism between $A$ and $S_4$. First let $g \in S_3 \leq S_4$, $\tilde{g} \in GL(3, \mathbb{C})$ be the corresponding permutation matrix (i.e. $\tilde{g}e_i = e_{g_i}$, for all $i = 1, 2, 3$, whence $\tilde{g} = e_{g_11} + e_{g_22} + e_{g_33}$; here $e_1$, $e_2$, $e_3$ are unit column vectors (one of whose entries is 1, the others 0)), and $\tilde{g} = \text{sgn}(g) \cdot \tilde{g} = \pm \tilde{g}$. It is clear that $\tilde{g} \in A$, and the correspondence $\alpha : g \mapsto \tilde{g}$ is an injective homomorphism from $S_3$ to $A$. Its image will be denoted by $\hat{S}_3$. 
It is not hard to extend $\alpha$ to an isomorphism of $S_4$ onto $A$. Observe that

$$C = \{\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_{1,2,3} = \pm 1, \varepsilon_1\varepsilon_2\varepsilon_3 = 1\}$$

is a normal subgroup of $A$ isomorphic to $Z_2^3$ and $A = C \ltimes \hat{S}_3$. On the other hand, $V^{(2)}$ is normal in $S_4$ and isomorphic to $Z_2^3$ also, and $S_4 = V^{(2)} \ltimes S_3$. The unique element of $V^{(2)}$ commuting with (12) is (12)(34); the unique element of $C$ commuting with (12) = $-e_{12} - e_{21} - e_{33}$ is diag(−1, −1, 1). This suggests an idea to consider the correspondence $\beta : V^{(2)} \to C$ defined by

$$\beta : (12)(34) \mapsto \text{diag}(-1, -1, 1) = (12), \quad (13)(24) \mapsto \text{diag}(1, 1, 1), \quad e \mapsto E = \text{diag}(1, 1, 1).$$

Now it is possible to show that the map $\gamma : S_4 \to A$ defined by $\gamma(xy) = \beta(x)\alpha(y)$, where $x \in V^{(2)}$ and $y \in S_3$, is an isomorphism (the details are left to the reader. For example, we may check that $\gamma$ preserves defining relations for a certain system of generators of $S_4$).

We conclude this section with a very useful remark.

**Remark.** Let $\pi \in S_3 \leq S_4$. Then action of an element $\hat{\pi} = \alpha(\pi) \in A$ on $M$ coincides with “the action of $\pi$ on indices”, that is $\pi e_{ij} = \hat{\pi} e_{ij} \hat{\pi}^{-1} = e_{\pi i, \pi j}$.

4. The theorem on classification of $G$-orbits. Now we can state the main result of the present article.

Below in the article $\text{St}_G(w)$ denotes the stabilizer of a tensor $w$ under the action of $G$; “subvariety” means an algebraic subset (i.e., closed in Zariski topology); $\zeta$ is a primitive cubic root of 1, $i = \sqrt{-1}$ (also, we use the same letter $i$ for indices, but hope this will not cause a confusion even in the formulae like $e_{ij} = i e_{ki}$). Next,

$$\delta = e_{11} + e_{22} + e_{33}, \quad \varkappa = \sum_{i \neq j} e_{ij} = e_{12} + e_{21} + e_{13} + e_{31} + e_{23} + e_{32},$$

$$\eta = e_{11} + \zeta e_{22} + \overline{\zeta} e_{33}, \quad \overline{\eta} = e_{11} + \overline{\zeta} e_{22} + \zeta e_{33},$$

$$\tau = e_{12} + e_{23} + e_{31} - e_{21} - e_{32} - e_{13}.$$

**Theorem 4.** The orbits of the group $G = A \times B$ on the set of nonzero decomposable tensors in $M^{\otimes 3}$ of length $\leq 18$ are described by the data written in the table below, in the following sense. If $\mathcal{O}$ is an orbit of length $\leq 18$, then there exists a point (i.e., a tensor) $w \in \mathcal{O}$, a row $(i, H_i, l_i, w_i(a, b, \ldots))$ of a table, and (in general, not uniquely determined) parameters $a, b, \ldots \in \mathbb{C}$ such that $w = w_i(a, b, \ldots)$, $\text{St}_G(w) = H_i$, and $|\mathcal{O}| = l_i$. Conversely, for each row $(i, H_i, l_i, w_i(a, b, \ldots))$ there exists a proper subvariety $Q_i \subset \mathbb{C}^{s_i}$ (where $s_i$ is the number of parameters $a, b, \ldots$, for given $i$) such that for $x = (a, b, \ldots) \in \mathbb{C}^{s_i} \setminus Q_i$ the orbit $\mathcal{O}$ of $w = w_i(x)$ has length $l_i$ and $\text{St}_G(w) = H_i$. If $x \in Q_i$, then $\text{St}_G(w)$ strictly contains $H_i$, and the length of its orbit is $< l_i$.

| $i$ | $H_i$ | $l_i$ | $w_i(a, b, \ldots)$ |
|-----|-------|------|---------------------|
| 1   | $Z_2^{(1)} \times S_3$ | 12   | $(a(e_{11} + e_{22}) + b(e_{12} + e_{21}) + c e_{33} + d(e_{13} + e_{23} + e_{31} + e_{32}))^{\otimes 3}$ |
| 2   | $Z_2^{(2)} \times S_3$ | 12   | $(a e_{11} + b e_{22} + c e_{33} + d(e_{12} + e_{21}))^{\otimes 3}$ |
| $i$ | $H_i$         | $l_i$ | $w_i(a, b, \ldots)$                                                                 |
|-----|--------------|------|--------------------------------------------------------------------------------------|
| 3   | $V^{(1)} \times S_3$ | 6    | $(a(e_{11} + e_{22}) + be_{33} + c(e_{12} + e_{21}))^{\otimes 3}$                   |
| 4   | $V^{(2)} \times S_3$ | 6    | $(ae_{11} + be_{22} + ce_{33})^{\otimes 3}$                                         |
| 5   | $D_8 \times S_3$   | 3    | $(a(e_{11} + e_{22}) + be_{33})^{\otimes 3}$                                         |
| 6   | $A_4 \times S_3$   | 2    | $an^{\otimes 3}$                                                                   |
| 7   | $S_4 \times S_3$   | 1    | $a\delta^{\otimes 3}$                                                               |
| 8   | $Z_3 \times Z_3$   | 16   | $(an + b(e_{12} + \zeta e_{23} + \bar{\zeta}e_{31}) + c(e_{21} + \zeta e_{32} + \bar{\zeta}e_{13}))^{\otimes 3}$ |
| 9   | $S_3 \times S_3$   | 4    | $(a\delta + b\lambda)^{\otimes 3}$                                                 |
| 10  | $Z_3 \times S_3$   | 8    | $(an + b(e_{12} + e_{21} + \zeta(e_{23} + e_{32}) + \bar{\zeta}(e_{31} + e_{13})))^{\otimes 3}$ |
| 11  | $S_3 \circ S_3$    | 8    | $(a\delta + b(e_{12} + e_{23} + e_{31}) + c(e_{21} + e_{32} + e_{13}))^{\otimes 3}$ |
| 12  | $D_8 \circ S_3$    | 6    | $(a(e_{11} + e_{22}) + be_{12} - e_{21} + ce_{33})^{\otimes 3}$                     |
| 13  | $V^{(1)} \circ_1 S_3$ | 12  | $(a(e_{11} + e_{22}) + be_{12} + e_{21} + ce_{33} + d(e_{13} + e_{23} - e_{31} - e_{32}))^{\otimes 3}$ |
| 14  | $V^{(1)} \circ_2 S_3$ | 12  | $(a(e_{11} + e_{22}) + be_{12} + ce_{21} + de_{33})^{\otimes 3}$                     |
| 15  | $V^{(2)} \circ S_3$ | 12  | $(ae_{11} + be_{22} + c(e_{12} - e_{21}) + de_{33})^{\otimes 3}$                     |
| 16  | $D_8 \times 1$     | 18   | $(a(e_{11} + e_{22}) + be_{33}) \otimes (c(e_{11} + e_{22}) + de_{33}) \otimes (f(e_{11} + e_{22}) + ge_{33})$ |
| 17  | $D_8 \times 1$     | 18   | $a(e_{11} - e_{22}) \otimes (e_{12} + e_{21}) \otimes (e_{12} - e_{21})$              |
| 18  | $D_8 \times Z_2$   | 9    | $(e_{11} - e_{22})^{\otimes 2} \otimes (a(e_{11} + e_{22}) + be_{33})$                |
| 19  | $D_8 \times Z_2$   | 9    | $(e_{12} + e_{21})^{\otimes 2} \otimes (a(e_{11} + e_{22}) + be_{33})$                |
| 20  | $D_8 \times Z_2$   | 9    | $(e_{12} - e_{21})^{\otimes 2} \otimes (a(e_{11} + e_{22}) + be_{33})$                |
| 21  | $D_8 \times Z_2$   | 9    | $(a(e_{11} + e_{22}) + be_{33})^{\otimes 2} \otimes (c(e_{11} + e_{22}) + de_{33})$    |
| 22  | $V^{(1)} \times Z_2$ | 18 | $(a(e_{11} + e_{22}) + b(e_{12} + e_{21}) + ce_{33})^{\otimes 2} \otimes (d(e_{11} + e_{22}) + f(e_{12} + e_{21}) + ge_{33})$ |
| 23  | $V^{(1)} \times Z_2$ | 18 | $(a(e_{11} - e_{22}) + b(e_{12} - e_{21})) \otimes (a(e_{11} - e_{22}) - b(e_{12} - e_{21})) \otimes (c(e_{11} + e_{22}) + d(e_{12} + e_{21}) + fe_{33})$ |
| 24  | $V^{(1)} \times Z_2$ | 18 | $(a(e_{13} + e_{23}) + b(e_{31} + e_{32})) \otimes (b(e_{13} + e_{23}) + a(e_{31} + e_{32})) \otimes (c(e_{11} + e_{22}) + d(e_{12} + e_{21}) + fe_{33})$ |
| 25  | $V^{(2)} \times Z_2$ | 18 | $(ae_{11} + be_{22} + ce_{33})^{\otimes 2} \otimes (de_{11} + fe_{22} + ge_{33})$     |
| $i$ | $H_i$ | $l_i$ | $w_i(a, b, \ldots)$ |
|-----|-------|------|---------------------|
| 26  | $V^{(2)} \times Z_2$ | 18   | $(ae_{12} + be_{21}) \otimes (be_{12} + ae_{21}) \otimes (ce_{11} + de_{22} + fe_{33})$ |
| 27  | $Z_4 \times Z_2$     | 18   | $(a(e_{11} + e_{22}) + b(e_{12} - e_{21}) + ce_{33}) \otimes (a(e_{11} + e_{22}) - b(e_{12} - e_{21}) + ce_{33}) \otimes (d(e_{11} + e_{22}) + fe_{33})$ |
| 28  | $Z_4 \times Z_2$     | 18   | $(a(e_{11} - e_{22}) + b(e_{12} + e_{21})) \otimes (c(e_{11} + e_{22}) + de_{33})$ |
| 29  | $Z_4 \times Z_2$     | 18   | $(a(e_{13} + ie_{23}) + b(e_{31} + i e_{32})) \otimes (b(e_{13} + i e_{23}) + a(e_{31} + i e_{32})) \otimes (c(e_{11} - e_{22}) + d(e_{12} + e_{21}))$ |
| 30  | $D_8 \circ_1 Z_2$    | 18   | $(a(e_{11} + e_{22}) + b(e_{12} + e_{21}) + ce_{33}) \otimes (a(e_{11} + e_{22}) - b(e_{12} + e_{21}) + ce_{33}) \otimes (d(e_{11} + e_{22}) + fe_{33})$ |
| 31  | $D_8 \circ_1 Z_2$    | 18   | $(a(e_{11} - e_{22}) + b(e_{12} - e_{21})) \otimes (c(e_{11} + e_{22}) + de_{33})$ |
| 32  | $D_8 \circ_2 Z_2$    | 18   | $(a(e_{13} + e_{23}) + b(e_{31} + e_{32})) \otimes (b(e_{13} - e_{23}) + a(e_{31} - e_{32})) \otimes (c(e_{11} - e_{22}) + d(e_{12} - e_{21}))$ |
| 33  | $D_8 \circ_2 Z_2$    | 18   | $(ae_{11} + be_{22} + ce_{33}) \otimes (be_{11} + ae_{22} + ce_{33}) \otimes (d(e_{11} + e_{22}) + fe_{33})$ |
| 34  | $D_8 \circ_2 Z_2$    | 18   | $(ae_{12} + be_{21}) \otimes (c(e_{11} + e_{22}) + de_{33})$ |
| 35  | $D_8 \circ_2 Z_2$    | 18   | $(ae_{13} + be_{31}) \otimes (be_{23} + ae_{32}) \otimes (ce_{12} + de_{21})$ |
| 36  | $D_8 \circ_3 Z_2$    | 18   | $(a(e_{11} + e_{22}) + b(e_{12} - e_{21}) + ce_{33}) \otimes (d(e_{11} + e_{22}) + f(e_{12} - e_{21}) + ge_{33})$ |
| 37  | $D_8 \circ_3 Z_2$    | 18   | $(a(e_{11} - e_{22}) + b(e_{12} + e_{21})) \otimes (a(e_{11} - e_{22}) - b(e_{12} + e_{21})) \otimes (c(e_{11} + e_{22}) + d(e_{12} - e_{21}) + fe_{33})$ |
| 38  | $D_8 \circ_3 Z_2$    | 18   | $(a(e_{13} + ie_{23}) + b(e_{31} + i e_{32})) \otimes (b(e_{13} - i e_{23}) + a(e_{31} - i e_{32})) \otimes (c(e_{11} + e_{22}) + d(e_{12} - e_{21}) + fe_{33})$ |
| 39  | $S_4 \circ Z_2$      | 6    | $\eta \otimes \overline{\eta} \otimes \delta$ |
| 40  | $S_3 \times Z_2$     | 12   | $(a\delta + b\kappa) \otimes (c\delta + d\kappa)$ |
| 41  | $S_3 \times Z_2$     | 12   | $\tau \otimes (a\delta + b\kappa)$ |
| 42  | $A_4 \circ Z_3$      | 12   | $(ae_{11} + be_{22} + ce_{33}) \otimes (ce_{11} + ae_{22} + be_{33}) \otimes (be_{11} + ce_{22} + ae_{33})$ |
| 43  | $S_4 \circ_2 S_3$    | 6    | $(ae_{11} + b(e_{22} + e_{33})) \otimes (ae_{22} + b(e_{11} + e_{33})) \otimes (ae_{33} + b(e_{11} + e_{22}))$ |
| 44  | $S_4 \circ_2 S_3$    | 6    | $(ae_{23} + be_{32}) \otimes (be_{13} + ae_{31}) \otimes (ae_{12} + be_{21})$ |

The rest of the article is devoted to the proof of this theorem.

5. Semi-invariants for the subgroups of $A$. Suppose $w = x \otimes y \otimes z \in M^{\otimes 3}$ is a decomposable tensor whose $G$-orbit is of length $\leq 18$. Then its stabilizer $H = St_G(w)$ is of
order $\geq 8$, and so has nontrivial intersection with $A$. Therefore, there exists $a \in A$ such that $axa^{-1} \otimes aya^{-1} \otimesaza^{-1} = x \otimes y \otimes z$, whence $axa^{-1}$, $aya^{-1}$, and $aza^{-1}$ must be proportional to $x$, $y$, and $z$, respectively. So the following task is reasonable: for each nontrivial subgroup $K \leq A$ find all its semiinvariants in $M$, i.e. all $x \in M$ such that $axa^{-1}$ is proportional to $x$, for all $a \in K$. Clearly, then there exists a homomorphism $\lambda : K \to \mathbb{C}^*$ (character) of $K$ such that $axa^{-1} = \lambda(a)x$ for each $a \in A$. (In the present article the word “character” always means one-dimensional (“linear”) character, that is, a homomorphism to $\mathbb{C}^*$, and is used without an adjective.)

Let $K^* = \text{Hom}(K, \mathbb{C}^*)$ be the group of characters of $K$, and for $\lambda \in K^*$ let

$$M_\lambda = \{x \in M \mid axa^{-1} = \lambda(a)x, \ \forall a \in K\}$$

be the corresponding invariant subspace. Then the sum $\sum_{\lambda \in K^*} M_\lambda$ is always direct. If $K$ is abelian, then $\bigoplus_{\lambda \in K^*} M_\lambda = M$, whereas for nonabelian $K$ this sum is a proper subspace of $M$.

Thus, we come to the following question: for each subgroup $1 \neq K \leq A$ find the subspaces $M_\lambda$, $\lambda \in K^*$.

It will be convenient to use the same notation for subgroups of $S_4$ and the corresponding subgroups of $A$. Let $K_1 = Z_2^{(1)}, \ldots, K_{10} = S_4$ be the canonical representatives of the conjugacy classes of the subgroups of $S_4$. In the following proposition we list the nontrivial subspaces $M_\lambda$, for $\lambda \in (K_i)^*$. We denote them by $L_{i,j}$.

**Proposition 5.** 1) The nontrivial subspaces of semiinvariants for $K_i$ in $M$ are the following:

- $K_1 = Z_2^{(1)}$: $L_{1,1} = \langle e_{11} + e_{22}, e_{12} + e_{21}, e_{13} + e_{32}, e_{23}, e_{31}, e_{33} \rangle$, $L_{1,2} = \langle e_{11} - e_{22}, e_{12} - e_{21}, e_{13} - e_{32}, e_{31}, e_{33} \rangle$,
- $K_2 = Z_2^{(2)}$: $L_{2,1} = \langle e_{11}, e_{22}, e_{21}, e_{23}, e_{33} \rangle$, $L_{2,2} = \langle e_{13}, e_{23}, e_{31}, e_{32} \rangle$,
- $K_3 = Z_3$: $L_{3,1} = \langle \delta = e_{11} + e_{22} + e_{33}, e_{12} + e_{23} + e_{31} + e_{21} + e_{32} \rangle$, $L_{3,2} = \langle \eta, e_{12} + \zeta e_{23} + \zeta e_{31}, e_{13} + \zeta e_{21} + \zeta e_{32} \rangle$, $L_{3,3} = \langle \overline{\eta}, e_{12} + \zeta e_{23} + \zeta e_{31}, e_{13} + \zeta e_{21} + \zeta e_{32} \rangle$;
- $K_4 = V^{(1)}$: $L_{4,1} = \langle e_{11} + e_{22}, e_{12} + e_{21}, e_{33} \rangle$, $L_{4,2} = \langle e_{11} - e_{22}, e_{12} - e_{21} \rangle$, $L_{4,3} = \langle e_{13} + e_{23}, e_{31} + e_{32} \rangle$, $L_{4,4} = \langle e_{13} - e_{23}, e_{31} - e_{32} \rangle$;
- $K_5 = V^{(2)}$: $L_{5,1} = \langle e_{11}, e_{22}, e_{33} \rangle$, $L_{5,2} = \langle e_{12}, e_{21} \rangle$, $L_{5,3} = \langle e_{13}, e_{31} \rangle$, $L_{5,4} = \langle e_{23}, e_{32} \rangle$;
- $K_6 = Z_4$: $L_{6,1} = \langle e_{11} + e_{22}, e_{12} - e_{21}, e_{33} \rangle$, $L_{6,2} = \langle e_{12} + e_{21}, e_{11} - e_{22} \rangle$, $L_{6,3} = \langle e_{13} + ie_{23}, e_{31} + ie_{32} \rangle$, $L_{6,4} = \langle e_{13} - ie_{23}, e_{31} - ie_{32} \rangle$, ($i^2 = -1$);
- $K_7 = S_3$: $L_{7,1} = \langle \delta, \zeta \rangle$, $L_{7,2} = \langle \tau \rangle$;
- $K_8 = D_8$: $L_{8,1} = \langle e_{11} + e_{22}, e_{33} \rangle$, $L_{8,2} = \langle e_{11} - e_{22} \rangle$, $L_{8,3} = \langle e_{12} + e_{21} \rangle$, $L_{8,4} = \langle e_{12} - e_{21} \rangle$;
- $K_9 = A_4$: $L_{9,1} = \langle \delta \rangle$, $L_{9,2} = \langle \eta \rangle$, $L_{9,3} = \langle \overline{\eta} \rangle$;
- $K_{10} = S_4$: $L_{10,1} = \langle \delta \rangle$.  


2) Denote by $\chi_{ij}$ the character of $K_i$ corresponding to $L_{ij}$. Then always $\chi_{i,1} = 1$. The following relations hold:

$$\chi_{s,2}^2 = 1, \ s = 1, 2, 7$$

(in these cases the group of characters $K_s^* \cong Z_2$);

$$\chi_{s,2}^2 = \chi_{s,3}, \ \ \ \chi_{s,2}^3 = 1, \ \ \ K_s^* \cong Z_3, \ \ \ s = 3, 9;$$

$$\chi_{s,2}^2 = \chi_{s,3}^2 = \chi_{s,4} = \chi_{s,2}\chi_{s,3}\chi_{s,4} = 1, \ \ \ K_s^* \cong Z_2 \times Z_2, \ \ \ s = 4, 5, 8,$$

$$\chi_{6,3}^2 = \chi_{6,2}, \ \ \ \chi_{6,3}^3 = \chi_{6,4}, \ \ \chi_{6,3}^4 = 1, \ \ \ K_6^* \cong Z_4.$$ 

**Sketch of a proof.** 1) This is a rather direct computation. For example consider $K_6 = Z_4$ (the hardest case). It is easy to see that the subgroup of $A$, corresponding to standard $Z_4 \leq S_4$, is generated by the matrix

$$g = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e_{12} - e_{21} + e_{33}.$$ 

Obviously, $g^{-1} = -e_{12} + e_{21} + e_{33}$. Hence it is easy to calculate that the conjugation by $g$ acts on matrix units by

$$e_{11} \leftrightarrow e_{22}, \ \ e_{12} \leftrightarrow -e_{21}, \ \ e_{21} \leftrightarrow -e_{12}, \ \ e_{33} \leftrightarrow e_{33},$$

$$e_{13} \leftrightarrow -e_{23}, \ \ e_{23} \leftrightarrow e_{13}, \ \ e_{31} \leftrightarrow -e_{32}, \ \ e_{32} \leftrightarrow e_{31}.$$ 

Now it is easy to see that the subspaces of matrices such that the conjugation by $g$ multiplies the matrix by $1, -1, i, \text{ or } -i$, are $L_{6,1}, L_{6,2}, L_{6,3}, \text{ and } L_{6,4}$, respectively.

To treat the other cases and to prove statement 2) is left to the reader. (For nonabelian groups it is useful to note that the commutator subgroup must act on any semiinvariant identically.)

6. Some invariance conditions. In this section we collected several general statements concerning invariance properties of decomposable tensors under various subgroups of $S_4 \times S_3$. (Particular subgroups will be considered later).

We identify $G$ with $S_4 \times S_3$ and write an element of $G$ as a pair of permutations.

Below $w$ usually denotes a decomposable tensor $w = x \otimes y \otimes z \in M^{\otimes 3}$. The next statement is obvious.

**Proposition 6.** Let $K \leq S_4$. Then $w$ is invariant under $K \times 1 \leq G$ if and only if $x, y, \text{ and } z$ are semiinvariants for $K$ belonging to characters $\mu_{1,2,3} \in K^*$ such that $\mu_1\mu_2\mu_3 = 1$. 

In the rest of the article we often use the following simple statement.

**Lemma 7.** Let $a \in A$ (or, more generally, $a$ is any orthogonal $3 \times 3$ matrix). Then the transformations $x \mapsto axa^{-1}$ and $x \mapsto x^t$ of $M$ commute, so that $(axa^{-1})^t = ax^ta^{-1}$.

**Proof.** We have $(axa^{-1})^t = (a^{-1})^tx^ta^{-1} = ax^ta^{-1}$, since $a$ is orthogonal. 

**Corollary 8.** If $K \leq S_4$, and if $x \in M$ is a semiinvariant for $K$ belonging to a character $\mu \in K^*$, then $x^t$ is also a semiinvariant belonging to the same character. Thus, all the
subspaces \( L_{i,j} \) of Proposition 5 are invariant under taking transpose (which, anyway, is visible immediately from the explicit form of the spaces \( L_{i,j} \)).

For an element \( h \in S_4 \), the statement of the lemma may be written as \((hx)^t = hx^t\), if we mean by \( hx \) the image of \( x \) under \( h \), i.e. \( hx = \hat{h}x\hat{h}^{-1} \), where \( \hat{h} = \gamma(h) \).

**Proposition 9.** 1) A decomposable tensor \( w \) is invariant under \( 1 \times Z_3 \subseteq G \) if and only if \( w = x^{\otimes 3} \), for some \( x \in M \).

2) \( w \) is invariant under \( 1 \times Z_2 \) if and only if \( w = x \otimes x^t \otimes y \), and \( y = y^t \).

3) \( w \) is invariant under \( 1 \times S_3 \) if and only if \( w = x^{\otimes 3}, \ x^t = x \).

4) \( w \) is invariant under \( K \times Z_3 \) if and only if \( w = x^{\otimes 3}, \ \) where \( x \) is a semiinvariant for \( K \) belonging to a character \( \mu \in K^* \) such that \( \mu^3 = 1 \).

5) \( w \) is invariant under \( K \times Z_2 \) if and only if \( w = x \otimes x^t \otimes y, \ y = y^t, \) and \( x \in M_\lambda, \ y \in M_\mu, \ \lambda^2 \mu = 1 \).

6) \( w \) is invariant under \( K \times S_3 \) if and only if \( w = x^{\otimes 3}, \ x = x^t, \ x \in M_\mu, \ \mu^3 = 1 \).

**Proof.** 1) We have \( 1 \times Z_3 = \langle \sigma \rangle \), where the element \( \sigma = (1, (123)) \in G \) acts by \( \sigma: x \otimes y \otimes z \mapsto z \otimes x \otimes y \). Obviously, any tensor of the form \( x \otimes x \otimes x \) is \( \sigma \)-invariant. Conversely, suppose that \( \sigma w = w \). Then \( z = \lambda_1 x, \ x = \lambda_2 y, \) and \( y = \lambda_3 z \) for some \( \lambda_1, \lambda_2, \lambda_3 \in C, \) and \( \lambda_1\lambda_2\lambda_3 = 1 \). Hence \( y \) and \( z \) are proportional to \( x \) with nonzero coefficients, and therefore \( w = mx^{\otimes 3}, \ m \neq 0, \) whence \( w = (x^t)^{\otimes 3}, \ x^t = m^{1/3}x \).

2) We have similarly \( 1 \times Z_2 = \langle \rho \rangle \), where \( \rho: x \otimes y \otimes z \mapsto y^t \otimes x^t \otimes z^t \). So a tensor of the form \( x \otimes x^t \otimes z \) is \( \rho \)-invariant under \( \rho \). Conversely, the equality \( \rho w = w \) implies \( y = \lambda x \), so \( w = x \otimes x^t \otimes z' \) for appropriate \( z' \). Hence \( \rho w = (x^t)^t \otimes x^t \otimes (z')^t \), and equality \( \rho w = w \) implies \( (z')^t = z' \). It remains to denote \( z' \) by \( y \).

3) Obviously, \( \rho(x^{\otimes 3}) = (x^t)^{\otimes 3} \) for any \( x \in M \). So the tensor \( x^{\otimes 3} \) with symmetric \( x = x^t \) is invariant under \( 1 \times S_3 = B = \langle \sigma, \rho \rangle \). Conversely, suppose \( w \) is invariant under \( B \). The invariance under \( \sigma \) implies \( w = x^{\otimes 3} \). Now the invariance under \( \rho \) gives \( (x^t)^{\otimes 3} = x^{\otimes 3} \), whence \( x^t = \lambda x, \ \lambda^3 = 1 \). But taking transpose can not multiply a matrix by a nontrivial cubic root of 1.

The statements 4), 5), and 6) easily follow from Proposition 6 and statements 1), 2), and 3), respectively.

In order to consider tensors invariant under groups of the form \( K \circ Z_2 \) or \( K \circ S_3 \) with common quotient group \( \cong Z_2 \), we need to know a condition of invariance under elements of the form \( (h, (12)) \in S_4 \times S_3 \), where \( h \in S_4 \) is of order 2 or 4.

**Proposition 10.** 1) Let \( h \in S_4 \) be an element of order 2, and \( g = (h, (12)) \in G \). Then \( w = x \otimes y \otimes z \) is invariant under \( g \) if and only if \( w \) is of the form \( x \otimes Rx \otimes z \), where \( R \) is the transformation of \( M \) defined by \( Ry = (hy)^t \), and \( z \) satisfies \( Rz = z \).

2) Suppose \( h \) is of order 1, 2, or 4. Then \( x^{\otimes 3} \) is invariant under \( g \) if and only if \( Rx = x \).

**Proof.** 1) For an arbitrary decomposable tensor we have

\[
g(x \otimes y \otimes z) = (1, (12))((h, 1)(x \otimes y \otimes z)) = (1, (12))(hx \otimes hy \otimes hz) = (hy)^t \otimes (hx^t) \otimes (hz^t) = Ry \otimes Rx \otimes Rz.
\]

So, if \( w \) is \( g \)-invariant, then \( y \) is proportional to \( Rx \), that is, \( w \) is of the form \( x \otimes Rx \otimes z' \).

Note that \( R \) is an involutive transformation, since the transformations \( v \mapsto v^t \) and \( v \mapsto hv \) commute and both are of order 2. Therefore, \( g(x \otimes Rx \otimes z') = R^2 x \otimes Rx \otimes Rz' = x \otimes Rx \otimes Rz' \). So \( gw = w \) implies \( Rz' = z' \). (And vice versa, if \( Rz = z \), then \( x \otimes Rx \otimes z \) is clearly \( g \)-invariant.)
2) Obviously, \( g(x^{\otimes 3}) = (Rx)^{\otimes 3} \). So the \( g \)-invariance of \( x^{\otimes 3} \) implies \( Rx = \mu x \), where \( \mu^3 = 1 \). Since \( h \) is of order 1, 2, or 4, the transformation \( R \) is of order 1, 2, or 4 also, and so can not multiply \( x \) by a nontrivial cubic root of 1. \( \square \)

Let \( X \) and \( Y \) be the subgroups consisting of all elements of the form \( (\pi, \pi) \), where \( \pi \) runs over \( Z_3 \) for \( X \) and over \( S_3 \) for \( Y \). That is \( X = \langle g_5 \rangle_3 \) and \( Y = \langle g_2, g_5 \rangle \) in the notation of Proposition 3. Let us find the general form of the decomposable tensors invariant under \( X \) or \( Y \).

It is clear that \( (\pi, \pi)(x \otimes y \otimes z) \) is obtained from \( x \otimes y \otimes z \) first by the componentwise action of \( \pi \), and then by the permutation of the factors according to \( \pi \). And in addition, if \( \pi \) is odd, then we must transpose each factor. In particular,

\[
((12), (12))(x \otimes y \otimes z) = ((12)y^t, (12)x^t, (12)z^t),
\]

\[
((123), (123))(x \otimes y \otimes z) = ((123)z, (123)x, (123)y).
\]

**Proposition 11.** The decomposable tensors invariant under \( X = \langle g_5 \rangle_3 \) are precisely all the tensors of the form \( x \otimes (123)x \otimes (132)x \), \( x \in M \), or, equivalently, all the tensors of the form

\[
(123)x \otimes (132)x \otimes x.
\]

The tensors invariant under \( Y \) are the tensors of the form (1) with \( x \) satisfying condition \( (12)x^t = x \).

**Proof.** The condition that \( x \otimes y \otimes z \) is invariant under \( ((123), (123)) \) is clearly equivalent to relations \( (123)z = \lambda_1 x, \ (123)x = \lambda_2 y, \ (123)y = \lambda_3 z \), with \( \lambda_1 \lambda_2 \lambda_3 = 1 \). Therefore, \( y \) is proportional to \( (123)x \) and \( z \) to \( (123)^{-1}x = (123)x \). Whence \( x \otimes y \otimes z \) is proportional to \( x \otimes (123)x \otimes (132)x \).

Note that any tensor proportional to a tensor of the latter form is of this form too (it is sufficient to divide \( x \) by the cubic root of the proportion coefficient). Conversely, it is clear that any tensor of the form \( x \otimes (123)x \otimes (132)x \) (or, equivalently, of the form \( (123)x \otimes (132)x \otimes x \)) is invariant under \( ((123), (123)) \).

The tensors invariant under \( Y \) also, clearly, are of the form (1). Moreover, it follows from Proposition 10, 1, that they satisfy condition \( (12)x^t = x \).

It remains to show the converse, that is the tensors of the form (1) with additional restriction \( (12)x^t = x \) are invariant under \( Y \). To do this it suffices to show that they are invariant under \( ((12), (12)) \), and for this, in turn, to check that \( x_1 = (123)x \) and \( x_2 = (132)x \) differ by an action of \( R \): \( v \mapsto (12)v^t \). Indeed,

\[
Rx_1 = (12)((123)x)^t = (12)(123)x^t = (12)(123)((12)x) = ((12)(123)(12))x = (132)x = x_2
\]

(we use here the relation \( x^t = (12)x \)). As \( R \) is involutive, we have also \( Rx_2 = x_1 \). \( \square \)

7. Beginning of the proof of Theorem 4. Now we start to prove Theorem 4. In this section we discuss the easier part of the theorem ("the orbit of the tensor \( w_i(a, b, \ldots) \) has length \( l_i \), for almost all \((a, b, \ldots)\)", and the more hard (but, in fact, easy enough also!) part ("any orbit is generated by a tensor \( w_i(a, b, \ldots) \)") consider in the next sections.

First of all, for each \( i = 1, \ldots, 44 \) we need to check that \( w_i(a, b, \ldots) \) is invariant under \( H_i \). By Proposition 3, any subgroup \( H = H_i \) can be represented as \( (C \times D)R \), where \( R \) is one of the groups \( 1, \langle g_i \rangle, \ i = 1, \ldots, 5, \) or \( \langle g_2, g_5 \rangle \). The invariance of \( w = x \otimes y \otimes z \) under \( C \) can
be easily checked by means of Proposition 6, under $D$ with Proposition 9, and under $R$ with Propositions 10 and 11.

For example consider the case $i = 38$, where the tensor $w_i(a, b, \ldots)$ looks probably most complex. We have $w_{38}(a, b, c, d, f) = x \otimes y \otimes z$, where

$$x = a(e_{13} + ie_{23}) + b(e_{31} + ie_{32}), \qquad y = b(e_{13} - ie_{23}) + a(e_{31} - ie_{32}),$$

$$z = c(e_{11} + e_{22}) + d(e_{12} - e_{21}) + f e_{33};$$

$$H_{38} = D_8 \circ_3 Z_2 = (Z_4 \times 1) \times \langle g_2 \rangle_2.$$

By Proposition 5, $x \in L_{6,3}$, $y \in L_{6,4}$, $z \in L_{6,1}$, and $\chi_{6,3}\chi_{6,4}\chi_{6,1} = 1$, so $w$ is invariant under $Z_4 \times 1$. Further, the condition for invariance under $g_2 = ((12), (12))$ is that $y$ is proportional to $Rz$, and $z = Rz$, where $Rv = ((12)v)^t$.

Taking into account the remark in the end of Section 3 (that $\pi e_{ij} = \hat{\pi} e_{ij} \hat{\pi}^{-1} = e_{\pi_i \pi_j}$), we have

$$Rx = ((12)x)^t = ((12)[a(e_{13} + ie_{23}) + b(e_{31} + ie_{32})])^t = (a(e_{23} + ie_{13}) + b(e_{32} + ie_{31}))^t = a(e_{32} + ie_{31}) + b(e_{23} + ie_{13}) = iy,$$

and also

$$Rz = ((12)[c(e_{11} + e_{22}) + d(e_{12} - e_{21}) + f e_{33}])^t = (c(e_{22} + e_{11}) + d(e_{21} - e_{12}) + f e_{33})^t = c(e_{11} + e_{22}) + d(e_{12} - e_{21}) + f e_{33} = z,$$

which proves that $w$ is invariant. The detailed calculations for other cases are left to the reader.

Next, we should show that for almost all sets of parameters $x = (a, b, \ldots)$ the orbit of tensor $w_i(x)$ is of length $l_i$, or equivalently, the stabilizer coincides with $H_i$ (the reader can easily see that $l_i = |G : H_i|$ in all the cases listed in the table), an that the exceptional $x$ form a proper subvariety in $\mathbb{C}^{s_i}$ (where $s_i$ is the number of parameters $a, b, \ldots$, for given $i$).

It is clear that $\varphi_i : x \mapsto w_i(x)$ is a polynomial map from $\mathbb{C}^{s_i}$ to $M^{s_i}$. The set of all (not necessary decomposable) tensors invariant under given $g \in G$ is a subspace of $M^{s_i}$. So its $\varphi_i$-preimage is a Zariski closed subset of $\mathbb{C}^{s_i}$. That is, the set of all $x$ such that $w_i(x)$ is $g$-invariant, is closed. The set of all “exceptional” $x$ is the union of the latter sets over all $g \in G \setminus H_i$ and so is closed also. To show that this is a proper subvariety it suffices to produce, for each $1 \leq i \leq 44$, a set of parameters for which the length of the orbit is $\geq l_i$. This usually can be done orally, without many computations. As an example consider again $i = 38$.

Take $w$ general enough, an at the same time simple enough. Say, in the case under consideration we can take

$$w = (e_{13} + ie_{23}) \otimes (e_{31} - ie_{32}) \otimes e_{33}.$$

Let $O$ be its orbit. It is clear that for each tensor $x \otimes y \otimes z \in O$ one of the vectors $x, y, z$ is of the form $\pm e_{ik} \pm ie_{jk}$ (do not confuse here subscript $i$ with the imaginary unit $i = \sqrt{-1}$!), the other of the form $\pm e_{ki} \pm ie_{kj}$, and the remaining third of the form $e_{kk}$. This vector $e_{kk}$ may be in the place of $x, y, or z$, and this gives us a partition of $O$ into three disjoint subsets.
\( \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \), which are clearly of the same cardinality (it is sufficient to consider the action of \( 1 \times Z_3 \leq G \)). We can assume that \( \mathcal{O}_1 \) is the subset corresponding to \( z = e_{kk} \).

Next, \( \mathcal{O}_1 \) splits into three subsets \( \mathcal{O}_1 = \mathcal{O}_1' \cup \mathcal{O}_1'' \cup \mathcal{O}_1''' \), corresponding to the cases \( k = 1, 2, 3 \) (consider the action of \( Z_3 \times 1 \)). Finally, \( \mathcal{O}_1''' \) contains at least two points \( w \) and \( w' \neq w \),

\[
w' = (e_{13} - i e_{23}) \otimes (e_{31} + i e_{32}) \otimes e_{33}
\]

(\( w' = (h, 1)w \), where \( h = (13)(24) \), which corresponds to \( \hat{h} = \text{diag}(-1, 1, -1) \in A \)). Thus, we see that \( |\mathcal{O}| = 3|\mathcal{O}_1'| = 9|\mathcal{O}_1'''| \geq 18 \), as required.

**8. G-orbits. The case of \( 1 \times Z_3 \leq H \).** Now we begin to prove the more difficult part of Theorem 4. Observe first of all that for a given \( G \)-orbit \( \mathcal{O} \) of length \( \leq 18 \) on decomposable tensors the stabilizers of points of \( \mathcal{O} \) form a conjugacy class of subgroups of \( G \), so there exists a point \( w \in \mathcal{O} \) whose stabilizer is a standard representative of conjugacy class of subgroups, that is one of the subgroups listed in Proposition 2. We call this subgroup, denote it \( H \), the stabilizer of the orbit. So, we should find all \( H \)-invariant decomposable tensors, for each of these standard subgroups \( H \).

However, note that the set of all \( G \)-orbits whose stabilizer is \( H \) is not, in general, in a bijection with the set of all \( H \)-invariant decomposable tensors, for two reasons. First, it may be that for a given \( H \)-invariant decomposable tensor \( w \) its \( G \)-stabilizer \( St_G(w) \) strictly contains \( H \). Then the length of its orbit is actually \( < |G : H| \). In particular, it may happen that the orbits with a given stabilizer \( H \) do not exist at all. Second, it may be that the normalizer \( N_G(H) \) strictly contains \( H: N_G(H) > H \). Then the orbit \( Gw \) contains several tensors, whose stabilizers are equal to \( H \) (namely, all the tensors of the form \( gw \), with \( g \in N_G(H)/H \)).

Taking these remarks into account, the following terminology will be used. We say that Theorem 4 is true for a standard subgroup \( H \), if any orbit \( \mathcal{O} \), whose stabilizer is \( H \), is generated by a tensor of the form \( w = w_i(a, b, \ldots) \), where \( 1 \leq i \leq 44 \) is one of the indices such that \( H_i = H \). In particular, the theorem will be true for \( H \) if there exist no orbits with stabilizer \( H \) and in the table of Theorem 4 there are no rows with \( H_i = H \) at all.

Obviously, to prove Theorem 4 it is sufficient to prove it for all standard subgroups \( H \). It is clear that if each decomposable tensor whose stabilizer is \( H \), is of the form \( w_i(a, b, \ldots) \) with \( H_i = H \), then the theorem is true for \( H_i \); but the converse is not true, in general.

Note that the majority of the subgroups listed in Proposition 2 contains \( 1 \times Z_3 \) as a subgroup. In this section we prove the theorem for all such subgroups.

Consider \( H = Z_2^{(1)} \times S_3 \). By Proposition 9.6), any \( H \)-invariant tensor is of the form \( x^{\otimes 3} \), where \( x \) is a semiinvariant belonging to a character \( \mu \) of \( Z_2^{(1)} \) such that \( \mu^3 = 1 \). Moreover, \( x \) must be symmetric. But there are no such \( \mu \), except for \( \mu = 1 \). So \( x \) is an invariant for \( Z_2^{(1)} \), \( x \in L_{1,1} = \langle e_{11} + e_{22}, e_{12} + e_{21}, e_{13} + e_{23}, e_{31} + e_{32}, e_{33} \rangle \). In particular, \( x \) is necessary symmetric. Therefore any \( H \)-invariant tensor is of the form \( w_1(a, b, c, d, f) \), and the theorem is proved for this \( H \).

The same argument applies for \( H = Z_2^{(2)} \times S_3 \), \( V^{(1)} \times S_3 \), \( V^{(2)} \times S_3 \), \( S_3 \times S_3 \), \( D_8 \times S_3 \), \( S_4 \times S_3 \). On the other hand, there exists no orbits whose stabilizer is one of the groups \( K \times Z_3 \), \( K = V^{(1)}, V^{(2)}, S_3 \), \( D_8, S_4 \). For a \( K \times Z_3 \)-invariant tensor must be of the form \( x^{\otimes 3} \), where \( x \) is an invariant for \( K \). But in all these cases all the \( K \)-invariants in \( M \) are symmetric, so in fact \( x^{\otimes 3} \) is invariant under \( K \times S_3 \).
Next consider three groups $Z_3 \times Z_3$, $Z_3 \times S_3$, and $S_3 \circ S_3$ (note that the latter two contain $Z_3 \times Z_3$). A tensor invariant under $Z_3 \times Z_3$ is $x^\otimes_3$, where $x \in L_{3,1}$, $L_{3,2}$, or $L_{3,3}$. By Proposition 3, $S_3 \circ S_3 = (Z_3 \times Z_3) \rtimes \langle g_2 \rangle$, where $g_2 = ((12), (12))$. We have $g_2(x \otimes y \otimes z) = R_y \otimes R_x \otimes R_z$, where $R_v = ((12)v)^t$. It is easy to see that for $v \in L_{3,1}$ we have $R_v = v$. Moreover, $(12) \in S_4$ (keep in mind that here "$(12) \in S_4$" actually means the element of $A$, or $A \times 1$, corresponding to $(12) \in S_4$), and therefore $R_1$ interchanges $L_{3,2}$ with $L_{3,3}$. Hence the tensor $x^{\otimes 3}$ is $g_2$-invariant when $x \in L_{3,1}$ and is not $g_2$-invariant when $x \in L_{3,2}, L_{3,3}$. So, the $S_3 \circ S_3$-invariant tensors are precisely the tensors of the form $w_{11}(a, b, c)$, and the theorem is true for $S_3 \circ S_3$. And since $(12, 1)$ interchanges the tensors $x^{\otimes 3}$, where $x \in L_{3,2}$, with such tensors with $x \in L_{3,3}$, then any orbit whose stabilizer is $Z_3 \times Z_3$ has a representative $x^{\otimes 3}$ with $x \in L_{3,2}$, i.e., of the form $w_8(a, b, c)$, which proves the theorem for $Z_3 \times Z_3$.

Finally assume that $x^{\otimes 3}$ is invariant under $Z_3 \times S_3$. Then $x \in L_{3,1}$, $L_{3,2}$, or $L_{3,3}$, and $x$ is symmetric. If $x \in L_{3,1}$, then $x$ is invariant under $(12) \in S_4$ and so $x^{\otimes 3}$ is invariant under $S_3 \times S_3$, a contradiction. And since $(12) \in S_4$ normalizes $Z_3 \times S_3$ and interchanges $L_{3,2}$ with $L_{3,3}$, we see that any orbit whose stabilizer is $Z_3 \times S_3$ contains a tensor of the form $x^{\otimes 3}$, where $x \in L_{3,2}$ and is symmetric, that is a tensor of the form $w_{10}(a, b)$, which proves the theorem for $H = Z_3 \times S_3$.

Next consider $A_4 \times Z_3, A_4 \times S_3$, and $S_4 \circ S_3$. The tensors, invariant under $A_4 \times Z_3$, are $x^{\otimes 3}$, where $x \in L_{0,1}, L_{0,2}$, or $L_{0,3}$. That is they are multiples of $\delta^{\otimes 3}$, $\eta^{\otimes 3}$, or $\bar{\eta}^{\otimes 3}$, respectively. The tensor $a^{\otimes 3}$ is invariant under $S_4 \times S_3$, that is it is a one-point orbit. The pair $\{a\eta^{\otimes 3}, a\bar{\eta}^{\otimes 3}\}$ is an orbit with stabilizer $A_4 \times S_3$. So an orbit whose stabilizer is $H = A_4 \times S_3$ contains $a\eta^{\otimes 3} = w_6(a)$, and there exist no orbits with stabilizer $H = A_4 \times Z_3$ or $S_4 \circ S_3$.

Next consider group $Z_4 \times Z_3$ and two groups containing it, namely $Z_4 \times S_3$ and $D_8 \circ S_3$. If $x^{\otimes 3}$ is invariant under $Z_4 \times Z_3$, then $x$ is invariant under $Z_4$, whence $x \in L_{6,1} = \langle e_{11} + e_{22}, e_{12} - e_{21}, e_{33} \rangle$. But any $x \in L_{6,1}$ is invariant under $R: x \mapsto ((12)x)^t$. So $x^{\otimes 3}$ is invariant under $g_2 = ((12), (12))$, and therefore under $(Z_4 \times Z_3) \rtimes \langle g_2 \rangle = D_8 \circ S_3$. Thus, in the case $H = D_8 \circ S_3$ we have $w = w_{12}(a, b, c)$, and the cases $H = Z_4 \times Z_3$ or $Z_4 \times S_3$ are impossible (in the last case $St_G(w)$ would contain $\langle Z_4 \times S_3, g_2 \rangle = D_8 \times S_3$).

The cases $D_8 \circ S_3$ and $D_8 \circ S_3$ are impossible. Indeed, this subgroups contain $V^{(1)} \times Z_3$ and $V^{(2)} \times Z_3$, respectively. But, as we have seen earlier, any $x^{\otimes 3}$ that is invariant under these subgroups is invariant also under $1 \times S_3$, so $St_G(w)$ is strictly larger than $D_8 \circ S_3$, $i = 1, 2$.

Next we consider, in a uniform way, three groups $Q_1 = V^{(1)} \circ S_3, Q_2 = V^{(1)} \circ S_3$, and $Q_3 = V^{(2)} \circ S_3$. These groups can be represented as $Q_i = (P_i \times Z_3) \rtimes \langle (h_i, (12)) \rangle$, where $P_{i,2,3} = Z_2^{(1)}, Z_2^{(2)}, Z_2^{(3)}$, respectively, and $h_{1,2,3} = (12)(34), (12), (13)(24)$. So the set of $Q_i$-invariant decomposable tensors coincides with the set of tensors $x^{\otimes 3}$, where $x$ is invariant under both $P_i$ and $R_i$, where $R_i: x \mapsto (h_i x)^t$ (see Proposition 10.2).

The spaces of $P_i$-invariants in $M$ are $N_1 = L_{1,1}$ and $N_2 = N_3 = L_{2,1}$. The spaces of $R_i$-invariants are

\[
N'_1 = \langle e_{11}, e_{22}, e_{12} + e_{21}, e_{33}, e_{13} - e_{31}, e_{23} - e_{32} \rangle, \\
N'_2 = \langle e_{11} + e_{22}, e_{12}, e_{21}, e_{33}, e_{13} + e_{32}, e_{23} + e_{31} \rangle, \\
N'_3 = \langle e_{11}, e_{22}, e_{33}, e_{12} - e_{21}, e_{13} + e_{31}, e_{23} - e_{32} \rangle.
\]

The intersections $U_i = N_i \cap N'_i$ are

\[
U_1 = \langle e_{11} + e_{22}, e_{12} + e_{21}, e_{33}, e_{13} - e_{31} + e_{23} - e_{32} \rangle,
\]
$$U_2 = \langle e_{11} + e_{22}, e_{12}, e_{21}, e_{33} \rangle, \quad U_3 = \langle e_{11}, e_{22}, e_{33}, e_{12} - e_{21} \rangle.$$ 

The tensors $x^{\otimes 3}$ with $x \in U_{1,2,3}$ are precisely the tensors of the form $w_i(a, b, \ldots)$ with $i = 13, 14, 15$, which proves the theorem for the three considered groups.

It remains to show that the case $H = Z_4 \circ S_3$ is impossible. We have $H = (Z_2^{(2)} \times Z_3) \langle ((1324), (12)) \rangle$. The tensors invariant under $Z_2^{(2)} \times Z_3$ are $x^{\otimes 3}, x \in N = \langle e_{11}, e_{22}, e_{33}, e_{12}, e_{21} \rangle$.

Find the invariants in $N$ of the transformation $R: x \mapsto ((1324)x)^t$. This $R$ can be written as $R(x) = (\hat{h}x\hat{h}^{-1})^t$, where $\hat{h} = e_{12} - e_{21} + e_{33}$ is the element of $A$ corresponding to $(1324) \in S_4$.

Trivially, $\hat{h}^{-1} = -e_{12} + e_{21} + e_{33}$. It is easy to calculate that $R$ transforms basis elements of $N$ as $e_{11} \leftrightarrow e_{22}, e_{33} \leftrightarrow e_{33}, e_{12} \leftrightarrow -e_{12}, e_{21} \leftrightarrow -e_{21}$. So the $R$-invariant elements of $N$ are in $\langle e_{11} + e_{22}, e_{33} \rangle$. But for these $x$ the tensor $x^{\otimes 3}$ is invariant under $D_8 \times S_3$, a contradiction.

Thus, the theorem is true in all the cases where $1 \times Z_3 \leq H$.

9. The orbits whose stabilizer is a 2-group. In this section we list the $G$-orbits (of length $\leq 18$) on decomposable tensors, whose stabilizer is a 2-group. Obviously, in such a case the length of the orbit is 9 or 18, and the order of the stabilizer is either 16 or 8, respectively. The group listed in Proposition 2 that are 2-groups are $V^{(1)} \times Z_2$, $V^{(2)} \times Z_2$, $Z_4 \times Z_2$, $D_8 \times 1$, $D_8 \times 2$, and the three groups $D_8 \circ_1 Z_2$, $i = 1, 2, 3$.

First consider $H = D_8 \times 1$. Suppose $w = x \otimes y \otimes z$ is invariant under $H$. Then $x \in L_{8,i}$, $y \in L_{8,j}$, and $z \in L_{8,k}$. We shall say in such a case that $w$ is of type $(i, j, k)$. The invariance condition (Proposition 6) implies $\chi_{8,i} \chi_{8,j} \chi_{8,k} = 1$. Taking into account that the group of characters $D_8^8 \cong Z_2^2$, we see that $(i, j, k)$ is one of $(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1),$ or $(2, 3, 4)$ as a multiset.

The subgroup $1 \times S_3$ normalizes $D_8 \times 1$ and so acts on the set of $D_8 \times 1$-invariant tensors. It is easy to see that if $w = x \otimes y \otimes z$ is of type $(i, j, k)$ and $w' = (1 \times \pi)w = x' \otimes y' \otimes z'$, where $\pi \in S_3$, then $w'$ is of type $(i', j', k')$, where $(i', j', k')$ is obtained from $(i, j, k)$ by permutation $\pi$. So an orbit whose stabilizer is $D_8 \times 1$ contains an element of one of types $(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1),$ or $(2, 3, 4)$.

The elements of type $(1, 1, 1)$ are precisely the tensors of the form $x \otimes y \otimes z$ with $x, y, z \in L_{8,1} = \langle e_{11} + e_{22}, e_{33} \rangle$. The elements of types $(2, 2, 1), (3, 3, 1), (4, 4, 1)$ are the tensors of the form $(e_{11} - e_{22}) \otimes e_{33}, (e_{12} + e_{21}) \otimes e_{33}$, and $(e_{12} - e_{21}) \otimes e_{33}$ (since the subspaces $L_{8,i}$ with $i = 2, 3, 4$ are of dimension one). Since $z' = z$, Proposition 9.2 implies that these elements are invariant under $(1, 12)$, so $w$ is invariant under $D_8 \times Z_2$, a contradiction.

Finally, the elements of type $(2, 3, 4)$ are proportional to $(e_{11} - e_{22}) \otimes (e_{12} + e_{21}) \otimes (e_{12} - e_{21})$.

Thus, either $w = x \otimes y \otimes z$ where $x, y, z \in L_{8,1}$ or $w = a(e_{11} - e_{22}) \otimes (e_{12} + e_{21}) \otimes (e_{12} - e_{21})$, $a \in \mathbb{C}^\times$.

An element of the former of these two forms is $w_{16}(a, \ldots, g)$, and of the latter $w_{17}(a)$.

This proves the theorem for $D_8 \times 1$.

Next consider $H = P \times Z_4$, where $P = V^{(1)}, V^{(2)}, Z_4$, or $D_8$. Similarly to the case $D_8 \times 1$, if $w = x \otimes y \otimes z$ is a $P \times 1$-invariant decomposable tensor, then define its type as $(l, m, n)$, where $l, m, n$ are such that $x \in L_{j,l}$, $y \in L_{j,m}$, $z \in L_{j,n}$, and $j$ is the number such that $K_j = P$ (in the notation of Proposition 5). The invariance condition gives $\chi_{j,l} \chi_{j,m} \chi_{j,n} = 1$.

If $w$ is $P \times Z_2$-invariant, then it is of the form $x \otimes x^t \otimes z$. As the spaces $L_{p,q}$ are all invariant under transpose, the type of the latter tensor is $(l, l, n)$. Hence $\chi_{j,l} \chi_{j,n} = 1$. Note that in the cases $P = V^{(1)}, V^{(2)}, D_8$ we have $P^* \cong Z_2^2$, so the relation $\chi_{j,l} \chi_{j,n} = 1$ implies $\chi_{j,n} = 1$, that is $n = 1$, and the type of $w$ is $(l, l, 1), l = 1, \ldots, 4$. In the case $P = Z_4$ the type is one of $(1, 1, 1), (2, 2, 1), (3, 3, 2), (4, 4, 2)$. 

In fact, these types are sometimes equivalent, in the following sense. We say that two types \((l, m, n)\) and \((l', m', n')\) are equivalent, if every \(G\)-orbit containing an element of one of these types necessarily contains an element of the other type also.

Let \(N = N_{S4}(P)\) be the normalizer of \(P\) in \(S4\). Then \(N\) permutes the subspaces \(L_{i;i}\), and so acts on \(\{1, 2, 3, 4\}\). The orbits of this action are \(\{1\}, \{2\}, \{3\}, \{4\}\) when \(P = D_8\), \(\{1\}, \{2\}, \{3, 4\}\) when \(P = V^{(1)}\) or \(P = Z_4\), and \(\{1\}, \{2, 3, 4\}\) when \(P = V^{(2)}\).

Obviously, \((h, 1)\) normalizes \(P \times Z_2\) if \(h \in N\) and so acts on the set of \(P \times Z_2\)-invariant decomposable tensors. Also, if \(w\) is of type \((l, m, n)\), then \((h, 1)w\) is of type \((h(l), h(m), h(n))\). It follows that if two types are in the same orbit with respect to the componentwise action of \(N\), then they are equivalent. Therefore, for \(P = V^{(1)}\) any orbit, containing a \(P \times Z_2\)-invariant decomposable tensor, contains a tensor of one of types \((1, 1, 1), (2, 2, 1),\) or \((3, 3, 1)\). When \(P = V^{(2)}\), such an orbit contains a tensor of one of types \((1, 1, 1)\) or \((2, 2, 1)\), and when \(P = Z_4\) of types \((1, 1, 1)\), \((2, 2, 1)\), or \((3, 3, 2)\). It remains to write explicitly for each type \((l, l, m)\), appropriate for a given \(P\), the general form of a tensor \(x \otimes x^t \otimes z\) such that \(x \in L_{j,l}\), \(z \in L_{j,m}\), and \(z^t = z\), in the last column of the table. This proves the theorem for the groups of the form \(P \times Z_2\).

It remains to consider the groups \(H = Q_i = D_8 \circ Z_2\), \(i = 1, 2, 3\). We have \(Q_i = (P_i \times 1) \lhd \langle (h_i, (12)) \rangle\), by Proposition 3, where \(P_{1,2,3} = V^{(1)}, V^{(2)}, Z_4\), and \(h_{1,2,3} = (13)(24), (12), (12)\), respectively. Each \(Q_i\)-invariant decomposable tensor is \(P_i \times 1\)-invariant also, an thus we can define its type, with respect to the spaces \(L_{j,l}\), where \(j\) is the number such that \(K_j = P_i\), where \(K_j\) as in Proposition 5. By Proposition 10.1, a \(Q_i\)-invariant decomposable tensor is of the form \(x \otimes R_i x \otimes z\), where \(R_i: v \mapsto (h_i v)^t\) and \(R_i z = z\).

Note that the transformation \(h_i\), and therefore \(R_i\), permutes the spaces \(L_{j,l}\): \(R_i(L_{j,l}) = h_i(L_{j,l}) = L_{j,\lhat}\), for some transformation \((l \mapsto \lhat) \in S_4\). It is not hard to check that this transformation \(l \mapsto \lhat\) is the same in all three cases, namely it fixes 1 and 2 and interchanges 3 with 4.

It is clear that a tensor of type \((l, m, n)\) goes to a tensor of type \((\lhat, \mhat, \nhat)\) under action of \(g = (h_i, (12))\). So the type \((l, m, n)\) of an \(H\)-invariant tensor must satisfy two conditions:

\[
\chi_{j,l} \chi_{j,m} \chi_{j,n} = 1, \quad m = \lhat, \quad n = \nhat.
\]

The types, satisfying these conditions, are the following: \((1, 1, 1), (2, 2, 1), (3, 4, 2), (4, 3, 2)\) when \(P = V^{(1)}\) or \(V^{(2)}\); and \((1, 1, 1), (2, 2, 1), (3, 4, 1), (4, 3, 1)\) when \(P = Z_4\).

Finally, we can take into account the invariance under \(N_G(H)\), similarly to the way how it was done earlier. Namely, under action of \((h_i, 1)\) a tensor of type \((l, m, n)\) goes to a tensor of type \((\lhat, \mhat, \nhat)\). So there is no need to consider the last of the four types (i.e. \((4, 3, 2)\) or \((4, 3, 1)\)).

Thus, in all the cases the orbits have representatives of the form \(x \otimes R_i x \otimes z\), where one of the following three conditions holds: (a) \(x, z \in L_{j,1}\), \(R_i z = z\); (b) \(x \in L_{j,2}\), \(z \in L_{j,1}\), \(R_i z = z\), or (c) \(x \in L_{j,3}, z \in L_{j,2}\) when \(P_i = V^{(1)}, V^{(2)}, z \in L_{j,1}\) when \(P_i = Z_4\), and \(R_i z = z\). To check that in all the cases the explicit form of the tensor \(x \otimes R_i x \otimes z\) coincides with the corresponding table entry is left to the reader.

10. The other orbits. In this section we treat the remaining cases, namely \(S_3 \times Z_2, A_4 \times 1, A_4 \times Z_2, A_4 \circ Z_3, S_4 \times 1, S_4 \times Z_2, S_4 \circ Z_2,\) and \(S_4 \circ_2 S_3\).

First let \(H = S_4 \times 1\) or \(S_4 \times Z_2\). The unique (up to a scalar) semiinvariant for \(S_4\) in \(M\) is \(\delta = e_{11} + e_{22} + e_{33}\). So the unique invariant of \(S_4 \times 1\) or \(S_4 \times Z_2\) in \(M^\otimes 3\) is \(\delta^\otimes 3\). But this tensor is invariant under a larger group \(S_4 \times S_3\), a contradiction.
Next consider $H = A_1 \times 1$, $A_4 \times Z_2$, and $S_4 \circ Z_2$. All these three groups contain $A_4 \times 1$ as a normal subgroup. The semiinvariants of $A_4$ in $M$ are $\delta, \eta = \epsilon_{11} + \zeta \epsilon_{22} + \zeta \epsilon_{33}$, and $\overline{\eta} = \zeta \epsilon_{11} + \epsilon_{22} + \zeta \epsilon_{33}$. And they belong to distinct characters in $A_4^* \cong Z_3$. So the invariants of $A_4 \times 1$ in $M^{\otimes 3}$ are, up to proportionality, $\alpha_1 \otimes \alpha_2 \otimes \alpha_3$, where $\alpha_i \in \{\delta, \eta, \overline{\eta}\}$ and either $\alpha_1 = \alpha_2 = \alpha_3$ or $\{\alpha_1, \alpha_2, \alpha_3\} = \{\delta, \eta, \overline{\eta}\}$. In the case where $\alpha_1 = \alpha_2 = \alpha_3$ we have $w = \delta^{\otimes 3}$, $\eta^{\otimes 3}$, or $\overline{\eta}^{\otimes 3}$. These $w$ correspond to groups $S_4 \times S_3$ and $A_4 \times S_3$, respectively, so they are not suitable. On the other hand, all the tensors with $\{\alpha_1, \alpha_2, \alpha_3\} = \{\delta, \eta, \overline{\eta}\}$ form an orbit under $1 \times S_3$. (Note that $\delta$, $\eta$, and $\overline{\eta}$ are symmetric, so the action of $S_3$ is the permutations of factors without transposing.) As the action of $(12) \in S_4$ interchanges $\eta, \overline{\eta} \in M$, the latter orbit is invariant under $S_4 \times 1$ and so is a $G$-orbit.

It remains to find out to which of the groups $A_4 \times 1$, $A_4 \times Z_2$, or $S_4 \circ Z_2$ this orbit corresponds. The group $A_4 \times 1$ is not appropriate evidently, because its index 12 is not equal to 6, the length of the orbit. On the other hand, the element $(12) \in S_4$ interchanges $\eta$ with $\overline{\eta}$, so $w = \eta \otimes \overline{\eta} \otimes \delta$ is invariant under $g_2 = ((12), (12))$. Therefore this $w$ is invariant under $(A_4 \times 1, g_2) = S_4 \circ Z_2$ (and $A_4 \times 1$ and $A_4 \times Z_2$ are impossible as $H$).

Next consider $H = S_4 \times Z_2$. As before, for a decomposable $S_3 \times 1$-invariant tensor $w$ we can consider its type $(l, m, n)$. The condition of $S_3 \times 1$-invariance immediately implies that $\{l, m, n\} = \{1, 1, 1\}$ or $\{2, 2, 1\}$ (note that $S_3^* \cong Z_2$). It follows from the invariance under $1 \times Z_2$ that $l = m$. So $(l, m, n) = \{1, 1, 1\}$ or $\{2, 2, 1\}$. Each $1 \times Z_2$-invariant tensor of type $(1, 1, 1)$ is $x \otimes x' \otimes y$, where $x, y \in L_{7,1}$ and $y'^t = y$. However, as all the tensors in $L_{7,1}$ are symmetric, the latter condition can be rewritten as $w = x \otimes x \otimes y, x, y \in L_{7,1}$. Similarly, an $S_3 \times Z_2$-invariant tensor of type $(2, 2, 1)$ is of the form $\tau \otimes \tau \otimes x$, where $x \in L_{7,1}$ (note that the space $L_{7,2}$ is one-dimensional).

It remains to consider two groups $A_4 \circ Z_3$ and $S_4 \circ_2 S_3$. By Proposition 3,

$$A_4 \circ Z_3 = (V^{(2)} \times 1) \wedge \langle g_5 \rangle_3, \quad S_4 \circ_2 S_3 = (V^{(2)} \times 1) \wedge \langle g_2, g_5 \rangle,$$

where $g_2 = ((12), (12))$ and $g_5 = ((123), (123))$.

We consider first the larger group $S_4 \circ_2 S_3$. By Proposition 11, any $\langle g_2, g_5 \rangle$-invariant decomposable tensor is of the form

$$(123)x \otimes (132)x \otimes x \quad (1)$$

where $x$ satisfies condition $Rx = x$, $R: v \mapsto (12)v^t$. Next, the $V^{(2)} \times 1$-invariance of the latter tensor implies that $x \in L_{5,j}$ for some $j$. Note that $R$ leaves the spaces $L_{5,1}$ and $L_{5,2}$ invariant, and interchanges $L_{5,3}$ with $L_{5,4}$. So $x \in L_{5,1}$ or $L_{5,2}$. But, $R$ acts identically on $L_{5,2}$, and for such $x$ the tensor (1) is of the form $w_{44}(a, b)$. Also, the subspace of $R$-invariants in $L_{5,1}$ is $\langle e_{11} + e_{22}, e_{33} \rangle$, and for $x$ in this subspace the tensor (1) is $w_{43}(a, b)$. This proves the theorem for $S_4 \circ_2 S_3$.

Now consider $H = A_4 \circ Z_3$. Again, $g_5$-invariant decomposable tensor must be of the form (1), but not necessarily $Rx = x$. Note that (123) $\in S_4$ permutes the subspaces $L_{5,j}$ as $L_{5,1} \mapsto L_{5,1}$, $L_{5,2} \mapsto L_{5,4} \mapsto L_{5,3} \mapsto L_{5,2}$. So the type of $w$ with respect to $V^{(2)} \times 1$ is one of $(1, 1, 1)$, $(2, 4, 3)$, $(3, 2, 4)$, or $(4, 3, 2)$. Note that $1 \times Z_3$ normalizes $H$ (even commute with it elementwise), and cyclically permutes the three latter types. So we can assume that the type of $w$ is either $(1, 1, 1)$ or $(4, 3, 2)$. If the type is $(4, 3, 2)$, then $x \in L_{5,2}$, whence $w$ is invariant under $S_4 \circ_2 S_3$, a contradiction. The only type left is $(1, 1, 1)$. It corresponds to the tensors of the form $w_{42}(a, b, c)$. 


The proof of Theorem 4 is now complete.

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