Cyclic Hypergraph Degree Sequences

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Abstract

The problem of efficiently characterizing degree sequences of simple hypergraphs is a fundamental long-standing open problem in Graph Theory. Several results are known for restricted versions of this problem. This paper adds to the list of sufficient conditions for a degree sequence to be hypergraphic. This paper proves a combinatorial lemma about cyclically permuting the columns of a binary table with length \( n \) binary sequences as rows. We prove that for any set of cyclic permutations acting on its columns, the resulting table has all of its \( 2^n \) rows distinct. Using this property, we first define a subset cyclic hyper degrees of hypergraphic sequences and show that they admit a polynomial time recognition algorithm. Next, we prove that there are at least \( 2^{(n-1)(n-2)/2} \) cyclic hyper degrees, which also serves as a lower bound on the number of hypergraphic sequences. The cyclic hyper degrees also enjoy a structural characterization, they are the integral points contained in the union of some \( n \)-dimensional rectangles.

1998 ACM Subject Classification G.2.1 Combinatorics, G.2.2 Hypergraphs

Keywords and phrases Hypergraph, Realizable Degree Sequence

Introduction

For a given list of positive integers \( D = (d_1, \ldots, d_n) \in \mathbb{Z}_+^n \), the graph realization problem asks if there exists a simple graph \( G_D \) on \( n \) vertices whose vertex degrees are given by the list \( D \). If such a graph exists then the degree sequence \( D \) is said to be realizable. A hypergraph \( H \) is said to be a simple \( k \)-hypergraph if every edge has \( k \) vertices. A \( k \)-hypergraph is called as a simple hypergraph if none of its edges are repeated and no edge contains a repeated vertex. The case of a simple graph can be seen to correspond to the case of 2-hypergraphs. Generally, given a degree sequence and a positive integer \( k \), one wants to find a \( k \)-hypergraph realizing it. When \( k = 2 \), Erdös-Gallai Theorem [10] gives necessary and sufficient conditions that must be satisfied by \( D \) to be realizable. Other characterizations for the realizability of graphs have been given by Havel [13] and Hakimi [11]. Sierksma and Hoogeveen [17] collected seven different characterizations for graph realizability and proved their equivalence.

An input degree sequence is assumed to be given in binary encoding, and an efficient characterization requires conditions which can be checked in time which is a polynomial in input size. For \( k \geq 3 \), the problem becomes difficult in the sense that there is no known efficient characterization of realizable degree sequences. A characterization was given by Dewdney [8] for all \( k \geq 3 \), which does not yield an efficient algorithm. Recently, some sharp sufficient conditions for realizability of a degree sequence based on a sequence’s length and degree sum were given [2]. For a fixed value of \( k \geq 3 \), the problem is easily seen to be in NP, but neither a polynomial time algorithm nor a hardness proof is known for this class.

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1 A loopless graph without repeated edges.
Cyclic Hypergraph Degree Sequences

Table 1: $n$-Bit-Table for $n = 3$

| $X$ | $c_{3,3}$ | $c_{2,3}$ | $c_{1,3}$ |
|-----|-----------|-----------|-----------|
| 0   | 0         | 0         | 0         |
| 1   | 0         | 0         | 1         |
| 2   | 0         | 1         | 0         |
| 3   | 0         | 1         | 1         |
| 4   | 1         | 0         | 0         |
| 5   | 1         | 0         | 1         |
| 6   | 1         | 1         | 0         |
| 7   | 1         | 1         | 1         |

of problems. However, Colbourn, Kocay and Stinson [7] proved that several other problems related to 3-graphic sequences are NP-Complete. Achuthan et al [1], Billington [5] and Choudum [3] gave several necessary conditions for 3-hypergraphs, however Achuthan et al [1] also showed that none of these conditions are sufficient. There are many surveys available for this problem [12, 16, 18, 20, 19], for a recent survey on related problems see [9].

If we give up the restriction on sizes of edges to be $k$, we get the Hypergraph Degree Sequence problem which was first stated in [3] for a restricted class of hypergraphs. In particular, the Hypergraph Degree Sequence problem, given a list of degrees $D$, asks if there exists a simple hypergraph which is a realization of $D$. This problem appears to be harder than the class of $k$-hypergraph problems in the sense that it not even known to be in NP. It can be easily seen to be in PSPACE. Several restricted versions of this problem have been studied in the past. Bhave, Bam, Deshpande [4] gave an Erdös-Gallai-type characterization of degree sequences of loopless linear hypergraphs, where a linear hypergraph is one in which any two edges have at most one common vertex. In another direction, characterization for a partial Steiner triple system (PSTS), which is a linear 3-hypergraph, were given by Keranen et al [14]. The results in [4] and [14] were recently generalized by Khan [15] using partial $(n,k,\lambda)$-systems.

This paper provides a sufficient condition for a degree sequence to be realizable by a hypergraph. We do not place any restriction on hypergraph being a linear hypergraph or a have bounded edge intersections like the cases studied earlier.

Our results and Contribution

We define a notion of cyclic permutations of a binary table (for example see Table 1) and use it to find an efficiently computable sufficient condition for a given degree sequence to be realizable by a simple hypergraph.

1. Firstly, we prove that for any set of cyclic permutations acting on the columns of a binary table, the resulting table has all of its $2^n$ rows distinct.

2. Next, we define a notion of cyclic hyper degrees viz. the degree sequences which are the sum of contiguous rows in a binary table in which each column has been permuted by different cyclic permutations. As cyclic permutations of binary tables have every row distinct, a cyclic hyper degree sequence is realizable for some simple hypergraph. Then we give an efficient algorithm which checks if a given degree sequence is a cyclic hyper degree. An equivalent definition of cyclic hyper degrees in Theorem 13 shows their structure, the cyclic hyper degrees are contained in the union of some $n$-dimensional rectangles.
3. Finally, we provide a lower bound of $2^{\frac{(n-1)(n-2)}{2}}$ for the number of cyclic hyper degree sequences, by proving the existence of a large $n$-dimensional rectangle such that every integral tuple contained in it is a realizable degree sequence. This also gives us a lower bound on the number of realizable sequences. For comparison, there is an upper bound of $O(2^n)$ for the number of realizable hypergraph degree sequences.

2 Preliminaries

The set of non-negative integers is denoted using the symbol $\mathbb{Z}_+$. The set of integers $\{1, \ldots, n\}$ is denoted by $[n]$. For $m, M \in \mathbb{Z}_+$, we refer to an integral interval $\{i : m \leq i \leq M\}$ as a range and specify it by providing its minimum and maximum element. An $n$-tuple, also simply referred to as a list, is $L = (\ell_1, \ell_2, \ldots, \ell_n)$, of size $n$, is an ordered collection of elements. We refer to its $i^{th}$ element $\ell_i$ as $L(i)$. We index any list or tuple with natural numbers starting with 1. The notation $a \cdot m$ denotes the tuple $(a, \ldots, a)$ consisting of $m$ repeated $a$.

For ease of notation, the sum of the entries in a list $L$ will be denoted by $\sum L = \sum_{i \in [L]} L(i)$. The sum of $L = (\ell_1, \ldots, \ell_n)$ and $L' = (\ell'_1, \ldots, \ell'_n)$, denoted by $L + L'$, is the $n$-tuple $(\ell_1 + \ell'_1, \ldots, \ell_n + \ell'_n)$. If $S$ is a list of lists then the sum of its elements will be denoted using $\sum S = \sum_{x \in S} x$.

For $k \geq 0$, a permutation $\pi$ is called a cyclic permutation of order $k$, for each $i \in [n]$ it maps $i \mapsto 1 + ((i + k - 1) \mod n)$.

A hypergraph $H$ is a pair $([n], \mathcal{F})$, where $\mathcal{F}$ is a family of subsets of $[n]$. A hypergraph is simple if no set is repeated in $\mathcal{F}$. The degree of a vertex $v \in [n]$ is equal to $|\{F \in \mathcal{F} : v \in F\}|$. We will be working with an equivalent version, which can be stated in terms of co-ordinate wise sum of binary sequences of length $n$.

For a given positive integer $n$, consider the set $S_n = \{0, 1\}^n$ consisting of all binary tuples of length $n$. The elements of $S_n$ will also be referred to as binary sequences. We construct a set $H_n = \{\sum_{x \in S} x : \emptyset \subseteq S \subseteq S_n\}$. Note that for the empty set $\emptyset$ the corresponding sum $\sum_{x \in S} x =$ is equal to $0_{n \times n}$. By construction, each element of $H_n$ is realized by some simple hypergraph and the degree sequence of every simple hypergraph is contained in $H_n$. Each element of $H_n$ is said to be representable or is said to admit a representation. Given this setting the realizability problem for simple hypergraphs can be restated as follows.

**Hypergraph Degree Sequence**

**Input:** A tuple $w \in \mathbb{Z}_+^n$ which is provided as a binary input.

**Question:** Is $w \in H_n$?

If $w = (w_1, \ldots, w_n) \in H_n$ and $w_n = 0$, then $(w_1, \ldots, w_{n-1}) \in H_{n-1}$. As $\sum S_n = 2^{n-1}$, the maximum possible value of any entry in $w$ is $2^{n-1}$. Thus, if any entry of $w$ is outside the range $\{0, \ldots, 2^{n-1}\}$, then it is not a member of $H_n$. Even though the number of subsets of $S_n$ is $2^{2^n}$, we get that the cardinality of $H_n$ is at most $2^n$. 

CVIT 2016
3 Cyclic permutations and Binary Tables

In this section, we will be working with binary tables of size \(2^n \times n\) and study the action of cyclic permutations on the columns of the table. For a given number \(n \in \mathbb{Z}_+\), list out the binary expansion of numbers in increasing order from \(\{0, \ldots, 2^n - 1\}\) as rows in a table. We pad the binary expansion with sufficient numbers of zeros on the left to make the length of each row exactly equal to \(n\). For example, when \(n = 3\), the table is as given in Table 1.

To state it formally we need the following definitions. Given a number \(m\) we denote the \(i^{th}\) bit in its binary representation by \(\text{bin}(m, i)\). If the most significant bit in the binary expansion occurs at the \(s^{th}\)-position in the binary expansion of \(m\), then for all values of \(i > s\) the value of \(\text{bin}(m, i)\) is zero. For example, \(\text{bin}(4, 2) = 1\) and \(\text{bin}(4, i) = 0\) for every \(i \geq 3\).

For a given \(n\), we construct \(n\) lists \(c_{1, n}, \ldots, c_{n, n}\), with each list \(c_{i, n}\) having length equal to \(2^n\). For \(n = 2\), we have \(c_{1,2} = (0, 1, 0, 1)\) and \(c_{2,2} = (0, 0, 1, 1)\). For \(n = 3\), the lists \(c_{1,3}, c_{2,3}, c_{3,3}\) correspond to the columns of the Table 1. Formally, the lists are defined as follows.

**Definition 1** \((n\text{-Bit-Lists})\). For a given \(n\), we define \(n\)-Bit-List to consist of \(n\) lists \(c_{1, n}, \ldots, c_{n, n}\). For \(j \in [2^n]\), the value of \(c_{i, n}(j)\) is equal to \(\text{bin}(j, i - 1)\).

**Definition 2** \((n\text{-Bit-Table})\). For a positive integer \(n\), the \(n\)-Bit-Table \(T_n\) is defined to be a size \(2^n \times n\) table with \(T_n = [c_{n, n}, c_{n-1, n}, \ldots, c_{1, n}]\).

The lists \(c_{1, n}, \ldots, c_{n, n}\) have a nice recursive structure and can be generated in an alternative way by concatenation. Given a positive integer \(n\), the base case of \(n = 1\) is one list \(c_{1,1} = (0, 1)\). For \(n \geq 2\), the tuple \(c_{n, n} = 0_{2^{n-1}} \cdot 1 \cdot 0_{2^{n-1}}\) and for \(j \in [n-1]\), the list \(c_{j, n}\) is equal to the concatenated list \(c_{j, n-1} \cdot c_{j, n-1}\). Thus, we get the following observation about the lists.

**Observation 1.** For \(n \in \mathbb{Z}_+\) and \(i \in [n]\), we have \(c_{i, n} = (0_{2^{i-1}} \cdot 1 \cdot 0_{2^{i-1}})_{2^{n-1}}\).

Let \(C_n\) be the set of all cyclic permutations on an \(2^n\) length list. Let \(\Pi \subseteq C_n\) be a multi-set consisting of \(n\) arbitrary cyclic permutations \(\pi_1, \ldots, \pi_n\). Let \(\Pi(T_n) = [\pi_n(c_{n, n}), \ldots, \pi_1(c_{1, n})]\) be the new table obtained from \(T_n\). For clarity we will change the notation slightly, let \(\Pi(T_n, i)\) denote row \(i\) of \(\Pi(T_n)\), it is a length \(n\) binary tuple \((c_{n, n}(\pi_1(i)), c_{n-1, n}(\pi_{n-1}(i)), \ldots, c_{1, n}(\pi_1(i)))\), for \(i \in [2^n]\).

We next state a general lemma whose proof follows by induction over the order of a cyclic permutation.

**Lemma 3.** Let \(L\) be a list and \(\pi\) be any cyclic permutation, then \(\pi(L \cdot L) = \pi(L) \cdot \pi(L)\).

Recall that for \(j \in [n-1]\), the tuple \(c_{j, n}\) consists of two copies of \(c_{j, n-1}\) concatenated together. Combining this fact with Lemma 3 we obtain the following as a special case.

**Corollary 4.** Given a positive integer \(n\) and a cyclic permutation \(\pi \in \mathbb{C}_n\). For \(j \in [n-1]\), the tuple \(\pi(c_{j, n})\) is equal to \(\pi(c_{j, n-1}) \cdot \pi(c_{j, n-1})\).

We next prove that for any set of cyclic permutations \(\Pi\), any two rows in \(\Pi(T_n)\) will never become equal.

**Theorem 5.** For any list of \(n\) cyclic permutations \(\Pi\), the rows of \(\Pi(T_n)\) are pair-wise distinct.
Proof. We prove this by induction on $n$. For the base case $n = 1$, the statement is trivially true. For the rest of the proof, assume that the list of cyclic permutations is $\Pi = (\pi_1, \ldots, \pi_n)$ and $\Pi_{\pi}$ is used to denote the list $(\pi_1, \ldots, \pi_{n-1})$.

The table $T_n$ can be constructed recursively by taking two copies of $T_{n-1}$ and appending the rows of $T_{n-1}$ below the other, after that we add $0_{n \times 2^{n-1}} \cdot 1_{n \times 2^{n-1}}$ as the first column. Consider the table

\[
T_{\pi} = [c_{n-1,n}, \ldots, c_{1,n}]
\]

\[
= [c_{n-1,n-1}, c_{n-1,n-1}, \ldots, c_{1,n-1} \cdot c_{1,n-1}].
\]

Apply the list of permutations $\Pi_{\pi}$ on $T_{\pi}$ to get

\[
\Pi_{\pi}(T_{\pi}) = [\pi_{n-1}(c_{n-1,n}), \ldots, \pi_1(c_{1,n})]
\]

\[
= [\pi_{n-1}(c_{n-1,n-1}) \cdot \pi_{n-1}(c_{n-1,n-1}), \ldots, \pi_1(c_{1,n-1}) \cdot \pi_1(c_{1,n-1})],
\]

where the second equality follows from Corollary 4. By the induction hypothesis, the first row-wise half of $\Pi_{\pi}(T_{\pi})$, which is the same as $\Pi_{\pi}(T_{n-1})$, consists of distinct rows. Therefore, the table $\Pi_{\pi}(T_{\pi})$ consists of rows which are repeated exactly twice. For $i < j$, the rows $\Pi_{\pi}(T_{\pi}, i)$ and $\Pi_{\pi}(T_{\pi}, j)$ are equal when $j = i + 2^{n-1}$. Therefore, it suffices to prove that the rows $\Pi_{\pi}(T_{n}, i)$ and $\Pi_{\pi}(T_{n}, i + 2^{n-1})$ are distinct. Observe that to obtain the table $\Pi(T_n)$, we need to append $\pi_n(c_{n,n})$ as the first column in $\Pi_{\pi}(T_{\pi})$. As $c_{n,n}$ is equal to $0_{n \times 2^{n-1}} \cdot 1_{n \times 2^{n-1}}$, we have $c_{n,n}(i) \neq c_{n,n}(i + 2^{n-1})$, this implies that $c_{n,n}(\pi_n(i)) \neq c_{n,n}(\pi_n(i + 2^{n-1}))$. \hfill □

Next, we define a notion of cyclic hyper degrees as follows.

Definition 6 (Cyclic Hyper Degree). Given $\Pi$, a list of $n$ cyclic permutations, a tuple $d \in \mathbb{Z}_n^+$ is said to be a cyclic hyper degree if there exist $i, N \in [2^n]$ such that $d = \sum_{k=i}^N \Pi(T_{n,k})$.

As the rows of $\Pi(T_n)$ are distinct, their contiguous sum is in $H_n$ by definition. This gives us the main theorem of this section.

Theorem 7. If $w \in \mathbb{Z}_n^+$ is a cyclic hyper degree, then $w \in H_n$.

We note that $(4, 1, 1, 1)$ is a realizable hypergraph degree sequence but it is not a cyclic hyper degree sequence. In the next section, we will show how to efficiently check if a given sequence $d$ is a cyclic hyper degree.

4 Efficiently Recognizing Cyclic Hyper Degrees

This section consists of two parts. In the first part will culminate with Theorem 12 which provides an efficiently computable closed form formula for the range of values taken by contiguous sum of $N$ elements in a list $c_{i,n}$, for any $i$. In the second part we will show how to use Theorem 12 to decide if a given degree sequence is a cyclic hyper degree.

The elements in the columns of $T_n$ do not change their relative position after application of a cyclic permutation when seen as a cyclic list. We shall use this property to efficiently search for possible bit subsets which may sum up to a given input degree sequence.

4.1 Contiguous Sum of Bit Lists

Definition 8 (Contiguous Sum). Given a list $L$ of length $m$, the contiguous sum of $N$ elements in $L$ starting at the index $i \in [m]$ is defined to be

\[
S(L, i, N) := \sum_{j=0}^{N-1} L(1 + ((i + j - 1) \mod m)).
\]
The summation above treats the list \( L \) as a cyclic list. Next, we prove that the contiguous sum function is a ‘continuous’ function, this property will allow us to specify the range of sum by stating the minimum and the maximum value taken by it. Note that if \( L \) is a 0-1 list, for any index \( \ell \in \{m\} \), we have \( |S(L, \ell, N) - S(L, \ell + 1, N)| \in \{0, 1\} \). This fact gives us the following property.

\[ \text{Lemma 9.} \quad \text{Let } j \in \{0, \ldots, n\}, \ i \in [n] \text{ and } N \in [2^n]. \text{ The minimum of the sum of } N \text{ contiguous bits in a bit list } c_{i,n} \text{ is } m \text{ if and only if its maximum is } N - m. \]

\[ \text{Proof.} \quad \text{Let } \bar{c}_{i,n} \text{ be the bit list obtained from the list } c_{i,n} \text{ by flipping each zero to one and vice versa. Let } \sigma_2^{-1} \text{ be an order } 2^{j-1} \text{ cyclic permutation, observe that } \bar{c}_{i,n} = \sigma_2^{-1}(c_{i,n}). \text{ If the minimum value is obtained at the contiguous segment which starts at the index } j \text{ in } c_{i,n}, \text{ then the value } N - m \text{ can be obtained by the contiguous sum starting at index } \sigma_2^{-1}(j). \text{ Finally, note that } m \text{ is the minimum value if and only if } N - m \text{ is the maximum value. } \]

Combining Observation 2 and Lemma 9 we get the following.

\[ \text{Lemma 10.} \quad \text{Let } N \in [2^n] \text{ and } m = \min_{j \in [2^n]} S(c_{i,n}, j, N). \text{ For every value } v \text{ in the range } \{m, \ldots, N - m\} \text{ there exists a } j \in [2^n] \text{ such that } S(c_{i,n}, j, N) = v. \]

The lemma above allows us to find the range of values taken by the contiguous sum by just finding the minimum value taken by it. Next we prove a simpler lemma about the range of values taken. Using that, in Theorem 12 we will find the range of values taken by the contiguous sum of \( N \) elements in any list \( c_{i,n} \).

\[ \text{Lemma 11.} \quad \text{For } j \in \{0, \ldots, n\} \text{ and } i \in [n], \text{ the sum of } 2^j \text{ contiguous bits in a bit list } c_{i,n} \text{ takes the following values.} \]

1. If \( j \leq (i - 1) \), then the range is \( \{0, \ldots, 2^j\} \), and
2. If \( j \geq i \), then the sum is exactly \( 2^i - 1 \).

\[ \text{Proof.} \quad \text{By Lemma 10 it suffices to find the minimum value of contiguous sum function. Notice that we have, } c_{i,n} = (0, 1, x_2^{-1})_{x_2^{-1}} \text{, by Observation 1.} \]

1. When \( j \leq (i - 1) \), we can pick a block of \( 2^j \) zeros giving a total of zero, which is the minimum possible value.
2. When \( j \geq i \), let \( L_k \) be a list of \( 2^j \) contiguous bits of \( c_{i,n} \) starting at the index \( k \) in \( c_{i,n} \). To prove that \( \sum L_k = \sum L_{k+1} \), it suffices to show that \( c_{i,n}(k) = c_{i,n}(k + 2^j) \). Rewriting \( c_{i,n} = ((0, 1, x_2^{-1})_{x_2^{-1}})_{x_2^{-1}} \) shows that any two indices with difference equal to \( 2^j \) store the same value. As the choice of \( k \) was arbitrary, the contiguous sum is equal to \( 2^j - 1 \).

\[ \text{Theorem 12.} \quad \text{For } i \in [n], N \in [2^n] \text{ and } p = 2^i, \text{ the sum of } N \text{ contiguous bits in a bit list } c_{i,n} \text{ takes values in the range, } range(i, N) \equiv \left\{ \left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2} + \max\left(\left(\frac{N \mod p}{p} - \frac{p}{2}, 0\right), \ldots, \left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2} + \min\left(\frac{N \mod p}{p}, \frac{p}{2}\right)\right\} \right\}. \]
Proof. For a fixed $i \in [n]$ consider the list $c_{i,n}$. Assuming that the minimum value of the range is as claimed, by Lemma 10, the maximum value is

$$\max_{j \in [2^n]} S(c_{i,n}, j, N) = N - \min_{j \in [2^n]} S(c_{i,n}, j, N)$$

$$= N - \left( \left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2} + \max \left( (N \mod p) - \frac{p}{2}, 0 \right) \right)$$

$$= \frac{N}{p} \left( p - (N \mod p) - \left( \left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2} + \max \left( (N \mod p) - \frac{p}{2}, 0 \right) \right) \right)$$

$$= \frac{N}{p} \left( p - (N \mod p) - \left( \left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2} + \max \left( (N \mod p) - \frac{p}{2}, 0 \right) \right) \right)$$

$$= \frac{N}{p} \left( p - (N \mod p) - \left( \left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2} + \max \left( (N \mod p) - \frac{p}{2}, 0 \right) \right) \right)$$

As proved in case 2 of Lemma 11, the sum of $\left\lfloor \frac{N}{p} \right\rfloor p$ contiguous bits is equal to $\left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2}$ irrespective of the starting index. Therefore, it suffices to find the minimum sum of $R = (N \mod 2^n)$ contiguous bits. Let $L_k$ be a list of $R$ bits occurring contiguously in $c_{i,n}$ starting at index $k$. If the first bit of $L_k$ is 1, then $\sum L_{k+1} \leq \sum L_k$. Therefore, we can keep on increasing the value of $k$ until the first bit is zero, without increasing the value of the contiguous sum. On the other hand, if $c_{i,n}(k-1) = 0$, then $\sum L_{k-1} \leq \sum L_k$. Therefore, we can keep on decreasing the value of $k$ one at a time until $c_{i,n}(k-1) = 1$, without increasing the value of the contiguous sum. Thus the minimum value of the contiguous sum is achieved when the index $k$ points to the start of any block $0_{2^{k-1}}$ contained in $c_{i,n}$. The value of minimum is $\max(R - \frac{p}{2}, 0)$ as the ones start appearing after $\frac{p}{2}$ indices from the start of a list $0_{2^{k-1}}1_{2^{k-1}}$. Adding it to $\left\lfloor \frac{N}{p} \right\rfloor \frac{p}{2}$ gives the required minimum value. ▶

### 4.2 Algorithm

We next state a theorem which gives an equivalent definition of cyclic hyper degrees.

**Theorem 13.** A list $w = \{w_1, \ldots, w_n\} \in \mathbb{Z}_+^n$ is a cyclic hyper degree if and only if there exist $N \in [2^n]$ and a permutation $\pi$, such that for each $i \in [n]$, $w_{\pi(i)} \in \text{range}(i, N)$.

**Proof.** Forward direction is a direct consequence of the definition.

Using Definitions 6 and 8 we get that there exist numbers $s_1, \ldots, s_n \in [2^n]$ such that for each $i \in [n]$, we have $w_{\pi(i)} = S(c_{i,n}, s_i, N)$. Let $\Pi^{-1} = (\sigma_{s_1}^{-1}, \ldots, \sigma_{s_n}^{-1})$ be the list of cyclic permutations, where for each $i \in [n]$, $\sigma_{s_i}^{-1}$ is the inverse of the cyclic permutation of order $s_i$. Consider the table $\Pi^{-1}(T_n)$, by Theorem 5 all its rows are distinct. In particular, the first $N$ rows are distinct and their sum is $\pi(w)$. Finally, $w \in H_n$ if and only if $\pi(w) \in H_n$. ▶

Theorem 12 gives us a way to efficiently find the number of bits in a contiguous sum of $N$ bits. If we know the number of distinct bit sequences that can sum up to a given vector $w \in \mathbb{Z}_+^n$, then using Theorem 12 we can generate all the possible ranges of values which can be taken by each coordinate of the sum. Finally, we need to check if each coordinate of $w$ is contained in different ranges, this corresponds to finding the permutation $\pi$ in Theorem 13. In the next lemma, we will find the number of possible distinct bit-sequences which can sum up to a given $w$ using cyclic shifts, this corresponds to finding $N$ in Theorem 13.
Lemma 14. If \( w = \{w_1, \ldots, w_n\} \in \mathbb{Z}_+^n \) is a cyclic hyper degree, then the number of bit sequences which sum up to \( w \) is an element of the set
\[
N_w = \{2w_i + j : i \in [n], j \in \{-1, 0, 1\}\}.
\]

Proof. As one of the coordinates of \( w \), say \( w_k \), is the contiguous sum of \( c_{1,n} \), we need to find the number of bits which sum up to \( w_k \). From the structure of \( c_{1,n} \), it is easily seen that there are just three values viz. \( 2w_k - 1, 2w_k, 2w_k + 1 \) which contain \( w_k \) in their range of sums. Conversely, for any number \( x \) not contained in \( \{2w_i + j : i \in [n], j \in \{-1, 0, 1\}\} \), the sum of \( x \) contiguous bits \( c_{1,n} \) will not contain any of \( w_i \), for \( i \in [n] \).

Lemma 15. Given \( w \in \mathbb{Z}_+^n \) and a list of integer intervals \( R_1, \ldots, R_n \subset \mathbb{Z}_+^2 \). There exists an algorithm running in time polynomial in \( n \) which correctly answers if there exists a permutation \( \pi \) such that for each \( i \in [n] \), \( w_{\pi(i)} \in R_i \).

Proof. Construct a bipartite graph \( G = (A, B, E) \) on \( 2n \) vertices. Let \( A = B = [n] \) and \((i, j) \in E \) if and only if \( w_i \in R_j \). Using a polynomial time algorithm one can find if there exists a perfect matching in \( G \). If there is a perfect matching then the answer is Yes, otherwise it is No.

Theorem 16. There is a polynomial time algorithm in \( n \) which decides if a given \( w \in \mathbb{Z}_+^n \) is a cyclic hyper degree.

Proof. For each \( N \in N_w \), given by Lemma 14 and \( i \in [n] \) compute range\((i, N)\) as given by Theorem 12. Now, use Lemma 15 on these ranges of numbers and decide if \( w \) is a cyclic hyper degree, if it is not then try the next number from the set \( N_w \). If it succeeds for at least one element of \( N_w \), we answer Yes, otherwise we answer No. Finally, note that \( |N_w| \leq 3n \) and all the other steps can be performed in time which is a polynomial function of \( n \).

5 Lower Bound on the number of Cyclic Hyper Degrees

In this section we will lower bound the number of cyclic hyper degrees for a given value of \( n \). For a given \( N \) we will find the size of range of contiguous sum, denoted by \( R_{i,N} \), for each value of \( i \) as given by Theorem 12. The number of distinct cyclic degree sequences which are the sum of \( N \) bit sequences is then \( \prod_{i \in [n]} R_{i,N} \). This is so because we have \( R_{i,N} \) choices for the \( i^{th} \) coordinate. We proceed to find \( R_{i,N} \), the size of each range, as a corollary of Theorem 12 as follows.

Corollary 17. For \( i \in [n] \), \( N \in [2^n] \) and \( p = 2^i \), the sum of \( N \) contiguous bits in a bit list \( c_{i,n} \) takes \( 1 + \min(N \mod p, p - (N \mod p)) \) distinct values.

Proof. The number of values is
\[
1 + \max_{j \in [2^n]} S(c_{i,n}, j, N) - \min_{j \in [2^n]} S(c_{i,n}, j, N)
\]
\[
= 1 + N - 2 \min_{j \in [2^n]} S(c_{i,n}, j, N)
\]
\[
= 1 + N - \left\lfloor \frac{N}{p} \right\rfloor p - 2 \max (N \mod p - \frac{p}{2}, 0)
\]
\[
= 1 + N \mod p - 2 \max (N \mod p - \frac{p}{2}, 0)
\]
\[
= 1 + \min (N \mod p - 2(N \mod p) + p, N \mod p)
\]
\[
= 1 + \min (p - (N \mod p), N \mod p).
\]
Lemma 18. For $n \in \mathbb{Z}_+$, the number of cyclic hyper degrees is at least $2^{(n-1)(n-2)/2}$.

Proof. Given a fixed number $n$, we are going to count the number of cyclic hyper degrees which are the sum of exactly $M$ bit-sequences, where $M = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} 2^j$. By Corollary 17, the range of values possible for $i^{th}$ bit is $B_i = 1 + \min(2^i - M \mod 2^i, M \mod 2^i) = 1 + \min(2^i - \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} 2^j, \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} 2^j)$. Depending on whether $i$ is even or odd, it can be broken into two cases, but in both the cases, for $i \geq 2$ we have $B_i \geq 2^{i-2}$. The number of representable bit sequences possible with these ranges of coordinates is

$$\prod_{k \in [n]} B_k \geq \prod_{k=2}^{n} 2^{k-2} = \prod_{k=0}^{n-2} 2^k = 2^{(n-1)(n-2)/2}.$$ 

6 Conclusion

We looked at the Hypergraph Degree Sequence problem and gave a sufficient condition for hypergraphic sequences. In particular, we proved that there is a subset of hypergraphic sequences having size at least $2^{(n-1)(n-2)/2}$ which is efficiently recognizable. It would be interesting to look at the hypergraphic sequences which are not cyclic hyper degrees, these sequences may be helpful in finding hardness reductions for it. Lastly, it would be interesting to further study the structure of cyclic hyper degrees.

References

1. Nirmala Achuthan, N Achuthan, and Mudin Simanihiruk. On 3-uniform hypergraphic sequences. J. Combin. Math. Combin. Comput., 14:3–13, 1993.
2. Sarah Behrens, Catherine Erbes, Michael Ferrara, Stephen G Hartke, Benjamin Reiniger, Hannah Spinoza, and Charles Tomlinson. New results on degree sequences of uniform hypergraphs. the electronic journal of combinatorics, 20(4):P14, 2013.
3. Claude Berge. Hypergraphs: combinatorics of finite sets, volume 45. Elsevier, 1984.
4. Nirupama S Bhave, BY Bam, and CM Deshpande. 13 a characterization of degree sequences of linear hypergraphs. Journal of the Indian Mathematical Society, 75(1):151, 2008.
5. David Billington. Conditions for degree sequences to be realisable by 3-uniform hypergraphs. The Journal of Combinatorial Mathematics and Combinatorial Computing, 3:71–91, 1988.
6. Sheshayya A Choudum. On graphic and 3-hypergraphic sequences. Discrete Mathematics, 87(1):91–95, 1991.
7. Charles J Colbourn, William L Kocay, and Douglas R Stinson. Some np-complete problems for hypergraph degree sequences. Discrete applied mathematics, 14(3):239–254, 1986.
8. Alexander K Dewdney. Degree sequences in complexes and hypergraphs. Proceedings of the American Mathematical Society, 53(2):535–540, 1975.
9. Michael Ferrara. Some problems on graphic sequences. Graph Theory Notes of New York, 64:19–25, 2013.
10. P Erdos-T Gallai and Paul Erdos. Graphs with prescribed degree of vertices (hungarian), mat. Lapok, 11:264–274, 1960.
11. S Hakami. On the realizability of a set of integers as degrees of the vertices of a graph. SIAM Journal Applied Mathematics, 1962.
12. S Louis Hakimi and Edward F Schmeichel. Graphs and their degree sequences: A survey. In Theory and applications of graphs, pages 225–235. Springer, 1978.
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13 Václav Havel. A remark on the existence of finite graphs. *Casopis Pest. Mat.*, 80(477-480):1253, 1955.

14 Melissa S Keranen, DL Kreher, W Kocay, and Pak Ching Li. Degree sequence conditions for partial steiner triple systems. *Bull. Inst. Combin. Appl.*, 57:71–73, 2009.

15 Marcin Mucha and Piotr Sankowski. Maximum matchings via gaussian elimination. In *Foundations of Computer Science, 2004. Proceedings. 45th Annual IEEE Symposium on*, pages 248–255. IEEE, 2004.

16 SB Rao. A survey of the theory of potentially p-graphic and forcibly p-graphic degree sequences. In *Combinatorics and graph theory*, pages 417–440. Springer, 1981.

17 Gerard Sierksma and Han Hoogeveen. Seven criteria for integer sequences being graphic. *Journal of Graph theory*, 15(2):223–231, 1991.

18 RI Tyshkevich, AA Chernyak, and Zh A Chernyak. Graphs and degree sequences. i. *Cybernetics*, 23(6):734–745, 1987.

19 RI Tyshkevich, AA Chernyak, and Zh A Chernyak. Graphs and degree sequences: A survey. iii. *Cybernetics*, 24(5):539–548, 1988.

20 RI Tyshkevich, AA Chernyak, and Zh A Chernyak. Graphs and degree sequences. ii. *Cybernetics*, 24(2):137–152, 1988.