The Bimetric Theory Of Quantized Fields

M.W. AlMasri *

May 28, 2024

Abstract

We introduce a special class of bimetric theories of quantized fields with preserved classical energy conditions. More precisely, we describe the missing anti-particles in our visible universe as being trapped in a spacetime patch with different metric such as the de-Sitter spacetime and negative energies. In order to stabilize these anti-states, time must go in the opposite direction to our visible universe, in agreement with the Feynman-Stückelberg interpretation of antiparticles. Since the energy scale of our observed universe is low, we do not need to worry about these highly energetic anti-particles except for compatibility issues regarding the restoration of classical energy conditions. However, at the quantum gravity scale, these anti-states play a crucial role in the cancellation of gravitational anomalies, along with other possible quantum anomalies that may emerge.

1 Introduction

Despite the success of quantum theory, gravity can be considered classical for most of the observed phenomena in nature, except in some extreme cases, such as black holes and early universe immediately after the big bang, where gravity is quantized. The main problem with Quantum theory derives from the probabilistic nature associated with it. Observed reality, in contrast, seems to be so deterministic despite the fact that, at the atomic level, particles are described by wavefunctions which have probabilistic meanings. These facts were under serious investigation during the past century, and many approaches were developed to solve this issue. Probably the most famous approaches are the Copenhagen and many-world interpretations[1][2][3]. Before we can give a final answer, to determine which interpretation is correct, we have to answer the following important question: "What makes the quantum world so different from our intuitive daily life thinking?" The uncertainty principle of Heisenberg, and consequently, the vacuum fluctuations play an important role behind the scenes in making the quantum world different from classical world. The Heisenberg uncertainty principle is a collection of mathematical inequalities that bound the
accuracy of the measurement of two physical quantities, such as the momentum (energy) and the position (time) of a quantum particle.

In quantum theory, local energy densities can be negative for a short period of time [4, 5]. This fact violates the classical energy conditions in General Relativity [6, 7]. In [8], we proposed a mechanism for restoring classical energy conditions at a quantized level. The cost was to introduce antiparticles with negative energies located at different spacetime patches with metric $g$, trapped in the true vacuum configuration and with a reversed arrow of time with respect to us. These particles are the missing anti-particles in quantum field theory, but with a larger masses due to localization in a curved spacetime. Using quantum interest conjecture, we consider the trapped anti-particles with negative energies as the loan amount and the evolving positive energy particles (our observed universe) as an attempt to pay that loan with interest [9].

It is worth noting that Time is reversed for particles with negative energies in order to stabilize their structure. This is due to the fact that negative energies with time direction identical to us are deemed to collapse shortly. Any negative energy density in our reference frame will be overcompensated with a positive energy pulse. This restriction on negative energy densities has been studied by Ford and his collaborators[9, 10, 11, 12].

In this work, we formulate the mechanism presented in [8] using field theory techniques. The main idea is to construct a bi-metric field theory, which can be written as $L(\phi) - L(\varphi)$ where $\varphi$ represents the anti-particles of $\phi$ trapped in a curved spacetime with different metric tensor. The overall minus sign before $L(\varphi)$ is justified because time should be in the opposite direction with respect to the time for particles localized in spacetime patch with metric $g_1$. In this context, It is worth remembering the mass-shell relation $E^2 = p^2 + m^2$ where the square root of $E^2$ has two solutions: positive and negative. Since the time flow for an observer located near the trapped negative energy states are opposite to us; this leads to stability of these states by means of quantum interest conjecture. The only way to include unstable fields in our theory is by considering the Tachyons (fields with imaginary masses).

The idea of constructing theories with two metrics goes back to Rosen [13, 14, 15] through his construction of a bimetric theory of gravity. In [16], a bimetric theory were introduced with varying speed of light. The bimetric MOND gravity were introduced in [17]. In [18] a bimetric gravity was obtained from ghost-free massive gravity. In [19], massive gravity was obtained from bimetric gravity. The canonical structure of Tetrad bimetric gravity was studied in [20]. Recently, bimetric-affine quadratic gravity was introduced in [21]. Our theory developed here is different from previous ones since it has the property of preserving classical energy conditions at a quantized level.
Convention

- We use units $c = \hbar = G = 1$ in field theory computations;
- Our metric signature is $(+,-,-,\cdots)$;
- Spacetime dimension is $n$ in general; often with $n = 4$;
- The d’Alembertian (wave) operator: $\Box g = \nabla^\mu \nabla^\nu g$.
- Christoffel connection (no torsion): $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} - \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$;
- Riemann tensor: $R^\rho_{\tau\mu\nu} = \partial_\nu \Gamma^\rho_{\tau\mu} - \partial_\mu \Gamma^\rho_{\tau\nu} + \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\tau\mu} - \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\tau\nu}$;
- Ricci tensor: $R_{\mu\nu} = R^\rho_{\rho\mu\nu}$.

2 Quantum gravity scale

In 1889, M. Planck noticed that one could construct units of time, length and mass from the three fundamental constants in nature, namely the universal gravitational constant $G$, the speed of light $c$ and the Planck constant $\hbar$. Later on, these constants were referred to as the Planck time $t_P$, Planck mass $m_P$ and Planck length $l_P$. Such unit system is known as the Planck scale and assumes the following relations and approximate values

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-33} \text{ cm} \quad (1)$$
$$t_P = \frac{l_P}{c} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.40 \times 10^{-44} \text{ s} \quad (2)$$
$$m_P = \frac{\hbar}{l_P c} = \sqrt{\frac{\hbar c}{G}} \approx 2.17 \times 10^{-5} \text{ g} \approx 1.22 \times 10^{19} \text{ Gev} \quad (3)$$

Furthermore, one could use other related units (often in cosmology) such as the Planck temperature $T_P$

$$T_P = \frac{m_P c^2}{k_B} \approx 1.41 \times 10^{32} \text{ K} \quad (4)$$

and the Planck density $\rho_P$

$$\rho_P = \frac{m_P}{l_P^3} \approx 5 \times 10^{93} \frac{\text{ g}}{\text{ cm}^3} \quad (5)$$

Planck mass, or Planck energy, is enormously large compared with the energy of elementary particles in the standard model (SM) of particle physics. This huge gap is known as the hierarchy problem. In order to probe quantum gravity, one needs, in principle, to concentrate this amount of energy (the Planck energy) inside a space with Planck length dimensions. Thus, the Planck scale is very relevant to quantum gravity since all forces of nature are expected to be unified at this scale, including gravity.
3 Formulation

The bimetric theory of quantized fields with preserved classical energy conditions (BTQF for short) assumes the following action

$$S[g_1, g_2, \phi_i, \varphi_i] = \int_{\mathcal{M}_1} \text{dvol}_{g_1} \mathcal{L}(\phi_i) - \int_{\mathcal{M}_2} \text{dvol}_{g_2} \mathcal{L}(\varphi_i),$$

(6)

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are n-dimensional manifolds. $\text{dvol}_{g_1}$ and $\text{dvol}_{g_2}$ are the metric volume forms for the manifolds $\mathcal{M}_2$ and $\mathcal{M}_2$, respectively. Here, $\phi_i$ and $\varphi_i$ represent a set of quantum fields characterized by a superindex $i$, which can represent scalar, vector, tensor, and spinor fields with their derivatives that act on the relevant spacetime point (DeWitt notation [23]). $\varphi_i$ is the antiparticle of $\phi_i$ located at another spacetime patch with a different metric tensor.

We define the path integral as

$$Z = \int D\phi_i D\varphi_i \text{e}^{iS[g_1, g_2, \phi_i, \varphi_i]},$$

(7)

and the expectation value of any polynomial bounded function $F[\Phi_i]$ as

$$\langle F \rangle = \frac{\int D\phi_i D\varphi_i F \text{e}^{iS[g_1, g_2, \phi_i, \varphi_i]}}{\int D\phi_i D\varphi_i \text{e}^{iS[g_1, g_2, \phi_i, \varphi_i]}}.$$  

(8)

The functional variation of action 6 gives the Euler-Lagrange equation

$$\frac{\delta S}{\delta \Phi_i} = 0 \Rightarrow \frac{\partial L}{\partial \Phi_i} - \frac{\partial}{\partial \mu} \left( \frac{\partial L}{\partial (\partial_\mu \Phi_i)} \right) = 0,$$

(9)

where $\Phi_i$ can be $\phi_i$ or $\varphi_i$. The stress-energy tensor is [24, 25]

$$T^{\mu\nu} = -2|g_1|^{-1/2} \delta S_1 \delta g_{1\mu\nu} + 2|g_2|^{-1/2} \delta S_2 \delta g_{2\mu\nu},$$

(10)

where $|g_{1,2}| = \text{det}(g_{\mu\nu}^1, 2)$.

By calculating the associated quantum inequality to (10) we find

$$\hat{\rho} = \int (T_{00})(t, x)|S(t)|^2 dt \geq QI(\phi_i) - QI(\varphi_i) \geq 0$$

(11)

where $|S(t)|^2$ is an arbitrary sampling function, and $QI$ is an abbreviation for quantum inequality [8, 26, 27]. It is not difficult to realize that when $g_1 \to g_2$ the quantity $QI(\phi_i) = QI(\varphi_i)$ goes to zero, and when $QI(\varphi_i) > QI(\phi_i)$, the right-hand side of (11) becomes greater than zero since $-QI(\varphi_i)$ is positive.

Quantum energy inequalities are an uncertainty principle type of inequalities imposed on the magnitude and duration of negative energy fluxes introduced
by Ford in [5]. The general shape of averaged quantum energy inequalities on complete geodesics is

\[ \int_{-\infty}^{\infty} S(t) \langle T_{\mu\nu} u^\mu u^\nu \rangle dt \geq -\frac{C}{t_0^n}, \]  

(12)

where \( T_{\mu\nu} u^\mu u^\nu \) is the normal-ordered energy density operator, which is classically non-negative, \( t \) is the observer’s time, and \( S(t) \) is the sampling function with characteristic width \( t_0 \). The quantity \( C \) in (12) is a numerical constant in the case of massless quantum fields and some scaling function in the case of massive quantum fields multiplied by some numerical constants. The scaling function approaches 1 as the mass of the quantum field approaches zero \( m \to 0 \) and becomes identically 1 in the massless case. For example, the massive Klein-Gordon field in 4-dimensional Minkowski spacetime obeys the following bound [26, 28]:

\[ \int (\langle T_{00} \rangle(t,x) g(t) )^2 dt \geq -\frac{1}{16\pi^3} \int_{m}^{\infty} |\hat{g}(u)|^2 u^4 Q_3(u/m) du \]

(13)

where \( \hat{g}(u) \) is the Fourier transform of \( g(u) \) and \( Q_3 : [1, \infty) \to \mathbb{R}^+ \) is defined by

\[ Q_3(x) = (1 - \frac{1}{x^2})^{1/2}(1 - \frac{1}{2x^2}) - \frac{1}{2x^4} \ln(x + \sqrt{x^2 - 1}) \]

(14)

where \( 0 \leq Q_3(x) \leq 1 \) with \( Q_3 \to 1 \) as \( x \to \infty \).

We denote particles in our visible universe by \( \phi_i \), where \( g_1 \) is the flat metric in general, and their antiparticles \( \bar{\phi}_i \) trapped in the true vacuum with metric \( g_2 \) (de Sitter spacetime in general), then \( |\text{QI}(\phi_i)| > |\text{QI}(\bar{\phi}_i)| \). To prove this, we refer to the general shape of QI in [12] the behavior of QI is mostly dominated by time in the denominators. For example, in four-dimensional spacetimes, it scales with \( t^{-4} \) times some numerical constants and possibly a scaling function in the massive case. Since the time scale of particles in \( g_2 \) is extremely slow (due to the localization on a very curved spacetime patch) compared with the time scale for sub-atomic processes in our visible universe, then \( |\text{QI}(\phi_i)| \gg |\text{QI}(\bar{\phi}_i)| \) and since \( -\text{QI}(\bar{\phi}_i) \) is positive, this proves [11].

Thus, BTQF describes two sets of fields at different scales connected via a wormhole-like connection due to the huge time difference, and this corresponds to a large energy gap between these two sets since particles in our visible universe are below 1 Tev according to the standard model and antistates are trapped in the true vacuum near the Planck scale [8]. Note that although the energy scale of the trapping true vacuum is relative to the Planck scale, the masses of antiparticles don’t equal the Planck mass in general; however, they are higher than their corresponding antiparticles on a flat spacetime patch. This is supposed to have a connection with dark matter and dark energy [8].

For each event \( p \) in the spacetime patch with metric \( g_1 \) we define the ordinary light-cone as the collection of all light rays through \( p \). We have shown in a
previous work \cite{8} that in order to restore classical energy conditions at quantized level one must severely restrict the existence of closed timelike curves (CTC) \cite{6}. This implies that our spacetime manifold has a causal structure, so the causality is preserved in our theory. The occurrence of event $p$ means that another event called $q$ must happen in the spacetime patch with metric $g_2$. We can also define a light-cone associated with this event $q$ in the same way of $p$, however, the time direction of the future light cone of $q$ is opposite to the time direction of the future light-cone of $p$. The same applies for the past light-cones.

In order to have a universe that can survive for a relatively long time (like ours), we must choose one of the metrics $g_1$ or $g_2$ to be approximately flat, while the metric of the second spacetime patch be extremely curved. It is a well-known fact from general relativity, that states in curved spacetime evolve slowly in comparison with states in flat spacetime. This explains the fact that time near a black hole’s horizon moves slowly compared to the time on Earth, for example. Probably, the best choice is to describe the negative energy states that are trapped in a de-sitter spacetime. De-sitter space is a unique, maximally symmetric curved space that shares the same degree of symmetry as Minkowski spacetime (the same number of Killing vectors)\cite{24}.

4 Applications

Scalar fields. The scalar field theory, under the light of BTQF can be written as

$$S = \int d^n x_1 \mathcal{L}_1 - \int d^n x_2 \mathcal{L}_2,$$

(15)

Where the Lagrangian densities are\cite{1}

$$\mathcal{L}_1 = \frac{1}{2} |g_1|^{1/2} (g_1^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - \xi_1 R_1 \phi^2),$$

(16)

$$\mathcal{L}_2 = \frac{1}{2} |g_2|^{1/2} (g_2^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi - \mu^2 \varphi^2 - \xi_2 R_2 \varphi^2),$$

(17)

Here, $\xi_{1,2}$ and $R_{1,2}$ are dimensionless constants and the Ricci scalars respectively \cite{25}.

The Euler-Lagrange equations of scalar fields $\phi$ and $\varphi$ are

$$(\Box_{g_1} + m^2 + \xi_1 R_1) \phi = 0,$$

(18)

$$(\Box_{g_2} + \mu^2 + \xi_2 R_2) \varphi = 0.$$
and the corresponding stress energy tensor is

\[ T^{\mu\nu} = \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{2} g_1^{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi + \frac{1}{2} g_1^{\mu\nu} m^2 \phi^2 \]

\[ -\xi_1 (R_1^{\mu\nu} - \frac{1}{2} g_1^{\mu\nu} R_1) \phi^2 + \xi_1 [g_1^{\mu\nu} \Box (\phi^2) - \nabla^\mu \nabla^\nu (\phi^2)] \]

\[ \nabla^\mu \varphi \nabla^\nu \varphi + \frac{1}{2} g_2^{\mu\nu} \nabla_\rho \varphi \nabla_\sigma \varphi - \frac{1}{2} g_2^{\mu\nu} \mu^2 \varphi^2 \]

\[ + \xi_2 (R_2^{\mu\nu} - \frac{1}{2} g_2^{\mu\nu} R_2) \varphi^2 - \xi_2 [g_2^{\mu\nu} \Box (\varphi^2) - \nabla^\mu \nabla^\nu (\varphi^2)] \]

The special case \( R_1 = 0 \) would give remarkable results if one considers \( R_2 \) as the Ricci scalar of de-sitter spacetime with high energetic quantum states\[8\]. However, since this construction is new, one can speculate on many interesting other cases that need more investigations, which we leave for future studies.

**Spinor Fields.** Since our visible universe is flat, we shall take the Lagrangian to be composed of spinor fields localized in a flat spacetime, and anti-states trapped in a curved spacetime. However, we can assume the case where all spinors are localized in arbitrary curved spacetimes and in any representation, not only the Dirac representation that we shall follow here regarding the choice of \( \gamma_\mu \) matrices.

The Lagrangian is

\[ L = \bar{\psi} (i \gamma^\mu \nabla_\mu - m) \psi - |g|^{1/2} \bar{\chi} (i \gamma^\mu e^\nu_\mu D_\nu - M) \chi \]

where \( \bar{\psi} = \psi^\dagger \gamma^0 \) is the Dirac adjoint of \( \psi \) (the same applies for \( \bar{\chi} \)), \( \nabla_\mu \) is the covariant derivative associated with the spinor \( \psi \) and \( D_\mu \) is the covariant derivative associated with the spinor \( \chi \) that lives in a curved spacetime patch with metric \( g \) and \( e^\nu_\mu \) is the vierbein which defines a local rest frame. It is very important to consider the spinor \( \chi \) as the anti-spinor of \( \psi \). Let us consider the four-dimensional case, we define the 4D spinors as \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \) and \( \chi = \begin{pmatrix} \chi_2 \\ \chi_1 \end{pmatrix} \) where \( \psi_1, \psi_2, \chi_1 \) and \( \chi_2 \) are two-dimensional spinors. When the two metrics of both \( \psi \) and \( \chi \) are the same, \( \psi_1 = \chi_2 \) and \( \psi_2 = \chi_1 \). Under the quantum interest conjecture, matter in our visible universe will evolve during its attempt to pay back the loan with interest so that the flat metric will have the same shape as the curved spacetime patch that contains the spinors \( \chi \)\[8\]. In the case of pure spinors without applied electromagnetic fields, the covariant derivatives assume the following forms:

\[ \nabla_\mu = \partial_\mu \]

\[ D_\nu = \partial_\nu - \frac{i}{4} \omega^a_\nu \sigma_{ab} \]

where \( \sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b] \) is the commutator of Dirac matrices and \( \omega^a_\nu \) are the spin connection components. The calculation of stress tensor and conserved currents
are straightforward since we only need to subtract the stress energy tensor of $\psi$ from $\chi$. By applying relation 11 to spinor Lagrangian we find the connection between the components of spinors $\psi$ and $\chi$.

When $g_1 = g_2$, quantum anomalies vanish in BTQF. To see this, we compute the quantum action which gives all connected Green functions as

$$Z = e^{iW[g_1, g_2, \phi_i, \varphi_i]}$$

(24)

where $W[g_1, g_2, \phi_i, \varphi_i] = W[g_1, \phi_i] - W[g_2, \varphi_i]$ is the effective action. Quantum anomalies appear in the loop expansion of the quantum action, and obviously, when $g_1$ becomes identical to $g_2$ the loop corrections cancel each other. This happens to be the case for Chiral anomalies in ordinary and non-commutative spacetimes [30]. The same holds for other types of anomalies, such as conformal anomalies. Practically, the standard model of particle physics is anomaly-free. Furthermore, we do not need to worry about gravitational anomalies since, at the standard model scale, quantum gravitational effects are negligible, and when the energies of particles become relative to the Planck scale, then $g_1$ approaches $g_2$ which means cancellation of gravitational anomalies. Thus, nature always finds a way to correct itself.

**Electromagnetic, massive vector, and higher-tensor fields.** It is a straightforward task to build the BTQF for photons, massive vector (Proca), non-Abelian gauge (Yang-Mills) and any higher-tensor fields based on equation 6 and the analogy with scalar and spinor fields.

## 5 Outlook

In the present work, we introduce new types of theories that preserve classical energy conditions at quantized level. In the last decades, physicists followed a quantization program in building the fundamental theory of physics. In contrast, we started with General Relativity and the laws of thermodynamics (since quantum interest conjecture follows from thermodynamics) to construct a theory which has quantum ingredients inside, but the whole picture is close to classical world. We realized that nature has imposed extremely severe constraints on causality and chronological order in the universe. The most important question to answer: “How the very probabilistic and fuzzy nature at the microscopic level can lead to a deterministic and ordered picture at macroscopic level?” The present work and other previous work [8] give us the answer to this question by introducing BTQF. Since the metric of our flat spacetime $g_1$ will evolve so that it becomes curved and identical to $g_2$ at the end. We can construct a *dynamical bimetric theory of quantized fields* where the metric $g_1$ enters as a function of time. The differential equation for the metric $g_1(t)$ is the Ricci flow equation introduced by Hamilton in [31] and used by Perelman in proving the Poincaré conjecture [32]. However, this interesting theory will be the topic of
future work. It is important to note that the topic of BTQF is very broad and has many ramifications in different branches of physics and other sciences.

References

[1] P.A.M. Dirac: Principles of quantum mechanics. 4th edition, Clarendon Press, Oxford 1982.

[2] J. von Neumann: Mathematical Foundations of Quantum Mechanics. Princeton University Press, New Jersey 1955.

[3] H. Everett III: Relative state formulation of quantum mechanics. Reviews of Modern Physics 29 (3): 454–462 (1957).

[4] H. Epstein, V. Glaser, A. Jaffe: Nonpositivity of the energy density in quantized field theories. Nuovo Cim 36 1016–1022 (1965).

[5] L.H. Ford: Quantum Coherence Effects and the Second Law of Thermodynamics. Proc. R. Soc. Lond. A. 364, 227-236 (1978).

[6] S.W. Hawking, G. Ellis: The Large Scale Structure of Space-Time. Cambridge University Press, Cambridge 1973.

[7] R.M. Wald: General Relativity. The University of Chicago Press, Chicago 1984.

[8] M.W. AlMasri: Restoring Classical Energy Conditions At Microscopic Level. (2020), https://arxiv.org/abs/2005.09182.

[9] L. H. Ford, T.A. Roman: The Quantum Interest Conjecture. Phys. Rev. D 60, 104018 (1999).

[10] L.H. Ford, T.A. Roman: Averaged Energy Conditions and Quantum Inequalities. Phys. Rev. D 51, 4277-4286 (1995).

[11] L.H. Ford, T.A. Roman: Restrictions on Negative Energy Density in Flat Spacetime. Phys. Rev. D 55, 2082-2089 (1997).

[12] M.J. Pfennig, L.H. Ford: Quantum Inequalities on the Energy Density in Static Robertson-Walker Spacetimes. Phys. Rev. D 55 4813-4821 (1997).

[13] N. Rosen: A bi-metric theory of gravitation. General Relativity and Gravitation volume 4, pages 435–447 (1973).

[14] N. Rosen: Bimetric theory of gravitation, In: De Sabbata V., Weber J. (eds) Topics in Theoretical and Experimental Gravitation Physics. NATO Advanced Study Institutes Series (Series B: Physics), Vol 27. Springer, Boston, MA, (1977).
[15] N. Rosen: Bimetric General Relativity and Cosmology. General Relativity and Gravitation, Vol. 12, No. 7, (1980).

[16] J. W. Moffat: Bimetric Gravity Theory, Varying Speed of Light and the Dimming of Supernovae. Int.J.Mod.Phys.D 12 :281-298 (2003).

[17] M. Milgrom: Bimetric MOND gravity. Phys.Rev.D 80 :123536 (2009).

[18] S.F. Hassan, R.A. Rosen: Bimetric gravity from ghost-free massive gravity. J. High Energ. Phys. 126 (2012).

[19] V. Baccetti, P. Martin-Moruno, M. Visser: Massive gravity from bimetric gravity. Class. Quantum Grav. 30 (2013) 015004.

[20] S. Alexandrov: Canonical structure of Tetrad Bimetric Gravity. Gen.Rel.Grav. 46 (2014) 1639.

[21] I. D. Gialamas and K. Tamvakis: Bimetric-affine quadratic gravity. Phys. Rev. D 107, 104012 (2023).

[22] C. Kiefer: Quantum Gravity. Oxford University Press , Oxford 2012.

[23] B. DeWitt: The Global Approach to Quantum Field Theory. Clarendon Press, Oxford 2003.

[24] N. D. Birrell, P.C.W. Davies: Quantum Fields in Curved Space. Cambridge University Press, Cambridge 1982.

[25] L. Parker, D. Toms: Quantum Field Theory in Curved Spacetime . Cambridge University Press, Cambridge 2009.

[26] C.J. Fewster: Lectures on quantum energy inequalities. Lectures given at the Albert Einstein Institute, Golm (2012).

[27] E. Kontou, K. Sanders: Energy conditions in general relativity and quantum field theory. Classical and Quantum Gravity 37 (19), 193001 (2020).

[28] C.J. Fewster and S.P. Eveson: Bounds on negative energy densities in flat spacetime. Phys. Rev. D 58 084010 (1998).

[29] R.A. Bertlmann: Anomalies in Quantum Field Theory, Oxford University Press, Oxford (2000).

[30] M.W. AlMasri: Axial-anomaly in noncommutative QED and Pauli–Villars regularization. International Journal of Modern Physics A 34, No. 26, 1950150 (2019).

[31] R.S. Hamilton: Three manifolds with positive Ricci curvature. Jour. Diff. Geom. 17, 255-306 (1982).

[32] G. Perelman: The entropy formula for the Ricci flow and its geometric applications. arXiv:math.DG/0211159 (2002).