RADON TRANSFORM ON REAL SYMMETRIC VARIETIES: KERNEL AND COKERNEL

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Date: February 1, 2008.
1. Introduction

Our concern is with

\[ Y = G/H \]

a semisimple irreducible real symmetric variety (space)\(^1\)

Our concern is with

\[ L^2(Y) \]

the space of square integrable function on \( Y \) with respect to a \( G \)-invariant measure. This Hilbert space has a natural splitting

\[ L^2(Y) = L^2_{mc}(Y) \oplus L^2_{mc}(Y)^\perp \]

into most continuous part and its orthocomplement.

Another function space is of need, namely:

\[ \mathcal{A} := L^1(Y)^\omega, \]

the space of analytic vectors for the left regular representation of \( G \) on \( L^1(Y) \). Further we set

\[ \mathcal{A}_{mc} := \mathcal{A} \cap L^2_{mc}(Y) \quad \text{and} \quad \mathcal{A}_\perp^{mc} := \mathcal{A} \cap L^2_{mc}(Y)^\perp. \]

We believe that \( \mathcal{A}_{mc} \) is dense in \( L^2(Y)_{mc} \) and that \( \mathcal{A}_\perp^{mc} \) is dense in the subspace of \( L^2(Y)_{mc}^{\perp} \) which corresponds to principal series which are induced from integrable representations of their Levi subgroups – a proof of this and similar facts for \( \mathcal{A} \) replaced by other function spaces is desirable.

Our concern is with an open domain in parameter space of generic real horospheres

\[ \Xi = G/(M \cap H)N \]

where \( MAN \) is a minimal \( \sigma\theta \)-stable\(^2\) parabolic subgroup of \( G \).

Write \( C_0^\omega(\Xi) \) for the space of analytic functions on \( \Xi \) which vanish at infinity. In this paper we verify the following facts:

- The map

\[ \mathcal{R} : \mathcal{A} \to C_0^\omega(\Xi), f \mapsto \left( gM_HN \mapsto \int_N f(gnH) \, dn \right) \]

is well defined. (We call \( \mathcal{R} \) the (minimal) Radon transform)

- \( \mathcal{R}|_{\mathcal{A}_{mc}} = 0 \).

- \( \mathcal{R}|_{\mathcal{A}_{mc} \cap \mathcal{S}(Y)} \) is injective\(^3\).

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\(^1\)This means \( G \) is a connected real semisimple Lie group, \( H \) is the fixed point group of an involutive automorphism \( \sigma \) of \( G \) such that there is no \( \sigma \)-stable normal subgroup \( H \subset L \subset G \) with \( \dim H < \dim L < \dim G \).

\(^2\)\( \theta \) is a Cartan involution commuting with \( \sigma \).

\(^3\)\( \mathcal{S}(Y) \) is the Schwartz space of rapidly decaying functions
Acknowledgement: The above results are motivated by discussions with Simon Gindikin during my stay at IAS in October 2006. I am happy to acknowledge his input and the hospitality of IAS.

I would like to thank Henrik Schlichtkrull for pointing out a mistake and several inaccuracies in an earlier version of the paper. Also I thank the anonymous referee for pointing out a mistake and his many very useful requests on more detail.

2. Real symmetric varieties

2.1. Notation

The objective of this section is to introduce notation and to recall some facts regarding real symmetric varieties.

Let $G_C$ be a simply connected linear algebraic group whose Lie algebra $g_C$ we assume to be semi-simple. We fix a real form $G$ of $G_C$: this means that $G$ is the fixed point set of an involutive automorphism $\sigma$ of $G_C$ and that $g$, the Lie algebra of $G$, yields $g_C$ after complexifying.

Let now $\tau$ be a second involutive automorphism of $G_C$ which we request to commute with $\sigma$. In particular, $\tau$ stabilizes $G_C$. We write $H_C := G^\tau_C$ and $H := G^\tau$ for the corresponding fixed point groups of $\tau$ in $G$, resp. $G_C$. We note that $H_C$ is always connected, but $H$ usually is not; the basic example of $(G_C, G) = (\text{Sl}(2, \mathbb{C}), \text{Sl}(2, \mathbb{R}))$ and $(H_C, H) = (\text{SO}(1, 1; \mathbb{C}), \text{SO}(1, 1; \mathbb{R}))$ already illustrates the situation.

With $G$ and $H$ we form the object of our concern

$$Y = G/H;$$

we refer to $Y$ as a real (semi-simple) symmetric variety (or space). Henceforth we will denote by $y_0 = H$ the standard base point in $Y$. We write $Y_C = G_C/H_C$ for the affine complexification of $Y$ and view, whenever convenient, $Y$ as a subspace of $Y_C$ via the embedding

$$Y \hookrightarrow Y_C, \quad gH \mapsto gH_C.$$

At this point it is useful to introduce infinitesimal notation. Lie groups will always be denoted by upper case Latin letters, e.g. $G$, $H$, $K$ etc., and the corresponding Lie algebras by lower case German letters, eg. $g$, $h$, $\mathfrak{k}$ etc. It is convenient to use the same symbol $\tau$ for the derived automorphism $d\tau(1)$ of $g$. Let us denote by $\mathfrak{q}$ the $-1$-eigenspace
of $\tau$ on $\mathfrak{g}$. Note that $\mathfrak{q}$ is an $H$-module which naturally identifies with the tangent space $T_{y_0}Y$ at the base point.

From now we will request that $Y$ is irreducible, i.e. we assume that the only $\tau$-invariant ideals in $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. In practice this means that $G$ is simple except for the group case $G/H = H \times H/H \simeq H$.

We recall that maximal compact subgroups $K < G$ are in one-to-one correspondences with Cartan involutions $\theta : G \to G$. The correspondence is given by $K = G^\theta$. We form the Riemann symmetric space

$$X = G/K$$

of the non-compact type and denote by $x_o = K$ the standard base point. As before we write $\theta$ for the derived involution on $\mathfrak{g}$. We let $\mathfrak{p} \subset \mathfrak{g}$ be the $-1$-eigenspace $\theta$ and note that the $K$-module $\mathfrak{p}$ identifies with $T_{x_o}X$.

According to Berger, we may (and will) assume that $K$ is $\tau$-invariant. This implies that both $\mathfrak{h}$ and $\mathfrak{q}$ are $\tau$-stable. Let us fix a maximal abelian subspace

$$\mathfrak{a} \subset \mathfrak{q} \cap \mathfrak{p}.$$ 

We wish to point out that $\mathfrak{a}$ is unique modulo conjugation by $H \cap K$, see [12], Lemma 7.1.5. Set $A = \exp(\mathfrak{a})$.

Our next concern is the centralizer $Z_G(A)$ of $A$. We first remark that $Z_G(A)$ is reductive and admits a natural splitting

$$Z_G(A) = A \times M,$$

(cf. [9], Prop. 7.82 (a)). The Lie algebra of $M$ is given by

$$\mathfrak{m} = \mathfrak{z}_G(\mathfrak{a}) \cap \mathfrak{a}^\perp$$

where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $\kappa$ of $\mathfrak{g}$. If $M_0$ denotes the connected component of $M$, then

$$M = M_0 F$$

where $F \subset M \cap K$ is a finite 2-group (cf. [9], Prop. 7.82 (d) and Th. 7.52).

**Remark 2.1.** If $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, then $F \subset H$ as follows from the explicit description of $F$ in [9], Th. 7.52. In general however, the $\tau$-stable group $F$ is not contained in $H$ and does not even admit a factorization $F = F^\tau F^{-\tau}$ in $\tau$-fixed and $\tau$-anti-fixed points.

We write $\mathfrak{m}_{ns}$ for the non-compact semisimple part of $\mathfrak{m}$ and note that
(2.1) \[ m_{ns} \subset h \]

(cf. [12], Lemma 7.1.4). Set \( M_H = M \cap H = Z_H(A) \) and let \( m = m_h + m_q \) be the splitting of \( m \) into +1 and −1-eigenspace. Note that \( m_h \) is the Lie algebra of \( M_H \). Then (2.1) implies that \( m_q \subset \mathfrak{t} \) and consequently \( M_q = \exp(m_q) \) is compact. Moreover:

\[ M/F = M_H M_q/F \quad \text{with} \quad M_H \cap M_q \quad \text{discrete}. \]

We turn our attention to the root space decomposition of \( g \) with respect to \( a \). For \( \alpha \in a^* \), let

\[ g^\alpha = \{ X \in g \mid (\forall Y \in a) [Y, X] = \alpha(Y)X \} \]

and set

\[ \Sigma = \{ \alpha \in a^* \setminus \{0\} \mid g^\alpha \neq \{0\} \}. \]

It is a fact that \( \Sigma \) is a (possibly reduced) root system, cf. [12], Prop. 7.2.1. Hence we may fix a positive system \( \Sigma^+ \subset \Sigma \) and define a corresponding nilpotent subalgebra

\[ n := \bigoplus_{\alpha \in \Sigma^+} g^\alpha. \]

Set \( N := \exp(n) \). Note that \( \tau(n) = \theta(n) \). We record the decomposition

\[ g = a \oplus m \oplus n \oplus \tau(n). \]

We shift our focus to the real flag manifold of \( G \) associated to \( A \) and \( \Sigma^+ \). We define

\[ P_{\text{min}} := M A N \]

and note that \( P_{\text{min}} \) is a minimal \( \theta \tau \)-stable parabolic subgroup of \( G \).

The open \( H \)-orbit decomposition on the flag manifold \( G/P_{\text{min}} \) is essential in the theory of \( H \)-spherical representations of \( G \). In order to describe this decomposition we have to collect some facts on Weyl groups first.

Let us denote by \( W \) the Weyl group of the root system \( \Sigma \). The Weyl group admits an analytic realization:

\[ W = N_K(a)/Z_K(a). \]

The group \( W \) features a natural subgroup

\[ W_H := N_{H \cap K}(a)/Z_{H \cap K}(a). \]

Knowing \( W \) and \( W_H \), we can quote the decomposition of \( G \) into open \( H \times P_{\text{min}} \)-cosets (cf. [11]):
\[ G \doteq \coprod_{w \in W \setminus W_H} HwP_{\text{min}}, \]

where \( \doteq \) means equality up to a finite union of strictly lower dimensional \( H \times P_{\text{min}} \)-orbits.

### 2.2. Horospheres

This paragraph is devoted to horospheres on the symmetric variety \( Y \). By a \textit{(generic) horosphere} on \( Y \) we understand an orbit of a conjugate of \( N \) of maximal dimension (i.e. \( \dim N \)). The entity of all horospheres will be denoted by \( \text{Hor}(Y) \). We remark that \( G \) acts naturally on \( \text{Hor}(Y) \) from the left.

Our goal is to show that \( \text{Hor}(Y) \) is a connected analytic manifold. For that we define

\[ G_h := \{ x \in G \mid Nx \cdot y_o \in \text{Hor}(Y) \} \]

and note the following immediate things:

- \( G_h \) is open, right \( H \)-invariant and left \( P_{\text{min}} \)-invariant.
- \( G_h \) contains the open \( P_{\text{min}} \times H \)-cosets \( P_{\text{min}}wH \) where \( w \in W/W_H \), see (2.2). In particular, \( G_h \) is dense.
- \( \text{Hor}(Y) = \{ gNx \cdot y_o \mid g \in G, x \in G_h \} \).
- (Infinitesimal characterization) \( G_h = \{ x \in G \mid \text{Ad}(x^{-1})n \cap h = \{0\} \} \).

**Remark 2.2.** The set \( G_h \) is in general bigger then the open dense disjoint union \( \bigcup_{w \in W \setminus W_H} P_{\text{min}}wH \). It is in particular connected as we show below. To see an example, consider \( G = \text{Sl}(2, \mathbb{R}) \), \( H = \text{SO}(1, 1; \mathbb{R}) \) and \( P_{\text{min}} \) the upper triangular matrices with determinant one. Then \( h \) and \( n \), both one-dimensional, can never be conjugate. Thus, by the infinitesimal characterization from above, one has \( G_h = G \) in this case.

Next we provide charts for \( \text{Hor}(Y) \). For that we introduce the \( G \)-manifold

\[ \Xi = G/MH \]

and define the \( G \)-equivariant map

\[ E : \Xi \to \text{Hor}(Y), \quad \xi = gMH \mapsto E(\xi) = gN \cdot y_o. \]

As in [8], Prop. 2.1, one verifies that \( E \) is an injection. Now we move our chart by elements \( x \in G_h \). Set \( L := MA, L^x := x^{-1}Lx, \)

\[ L^x_H := L^x \cap H, N^x := x^{-1}Nx \]

and \( \Xi_x := G/L^x_HN^x \). Then the map
\[
E_x : G/L^x_H N^x \to \text{Hor}(Y), \quad g L^x_H N^x \mapsto g N^x \cdot y_o.
\]
is \(G\)-equivariant and injective. It is immediate that \((E_x, \Xi_x)_{x \in G_h}\) forms an analytic atlas for \(\text{Hor}(Y)\).

**Lemma 2.3.** \(\text{Hor}(Y)\) is connected.

**Proof.** It is sufficient to show that \(G_h\) is connected. We know that \(G_h\) is an open and dense \(P_{\text{min}} \times H\)-invariant subset of \(G\). Now there are only finitely many orbits of \(P_{\text{min}} \times H\) on \(G\) and those are described explicitly; see [11].

As \(G_h\) contains all open orbits, it is sufficient to show that \(G_h\) contains all codimension one orbits. This in turn follows from the explicit description of all orbits in [11], Th. 3 (i): if \(HcP_{\text{min}} \subset G\) is not open, then [11] implies that \(\text{Ad}(c)a \cap \mathfrak{h} \neq \{0\}\). In particular if \(HcP_{\text{min}}\) is of codimension one, then \(\text{Ad}(c)(m + a + n) \cap \mathfrak{h} = \text{Ad}(c)a \cap \mathfrak{h}\) and therefore \(\text{Ad}(c)n \cap \mathfrak{h} = \{0\}\). Our infinitesimal characterization completes the proof. \(\square\)

Finally we discuss polar coordinates on \(\Xi\).

\[
(2.3) \quad K/(M_H \cap K) \times A \to \Xi, \quad (k(M_H \cap K), a) \mapsto kaM_H N
\]
is a diffeomorphism. Often we view \(A\) as subspace of \(\Xi\) via
\[
A \hookrightarrow \Xi, \quad a \mapsto aM_H N.
\]

3. Function spaces and the definition of the Radon transform

We consider the left regular representation \(L\) of \(G\) on \(L^1(Y)\), i.e. for \(g \in G\) and \(f \in L^1(Y)\) we look at
\[
[L(g)f](y) = f(g^{-1}y) \quad (y \in Y).
\]
Then we focus on the subspace
\[
\mathcal{A} := L^1(Y)^\omega
\]
of analytic vectors for \(L\). We note that \(f \in \mathcal{A}\) means that \(f \in C^\omega(Y)\) such that there exists an open neighborhood \(U\) of \(1\) in \(G_\mathbb{C}\) such that \(f\) extends holomorphically to
\[
UG \cdot y_o \subset Y_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C}
\]
(here we view \(Y\) embedded in \(Y_\mathbb{C}\) via \(gH \mapsto gH_\mathbb{C}\)) such that for all compact \(C \subset U:\)
(3.1) \[ \sup_{c \in C} \| f(c) \|_{L^1(Y)} < \infty, \]
(cf. [3], Prop. A.2.1). For an open neighborhood \( U \) of 1 in \( G_C \) we denote by \( \mathcal{A}_U \) the space of holomorphic functions on \( UG \cdot y_o \) which satisfy (3.1) for all compact \( C \subset U \). Note that \( \mathcal{A}_U \) can be seen as a closed subspace of \( \mathcal{O}(U, L^1(Y)) \) and hence is a Fréchet space. Moreover
\[ \mathcal{A} = \bigcup_U \mathcal{A}_U \]
with continuous inclusions
\[ \mathcal{A}_U \to \mathcal{A}_V, \quad f \mapsto f|_{VG \cdot y_o} \]
for \( V \subset U \). In this way we can endow \( \mathcal{A} \) with a structure of of a locally convex space.

We observe that \( G, \) via \( L_\cdot \), acts on \( \mathcal{A} \) in an analytic manner.

3.1. Definition of the Radon transform

We begin with the crucial technical fact.

**Lemma 3.1.** Let \( f \in \mathcal{A} \). Then the following assertions hold:

(i) \[ \sup_{a \in A} \int_N |f(kan \cdot y_o)| \ dn < \infty. \]
(ii) \[ \sup_{a \in A} a^{-2\rho} \int_N |f(kna \cdot y_o)| \ dn < \infty. \]

**Proof.**

(i) Let \( f \in \mathcal{A} \). Let \( B_a \subset \mathfrak{a} \) and \( B_n \subset \mathfrak{n} \) be balls around zero and set
\[ U_A := \exp(B_a + iB_a) \subset A_C, \]
\[ U_N := \exp(iB_n) \exp(B_n) \subset N_C. \]
If we choose \( B_a \) and \( B_n \) small enough, then \( f \) will extend to a holomorphic function in a neighborhood of
\[ KU_NU_A G \cdot y_o \]
such that
\[ \sup_{c \in KU_NU_A} \| f(c) \|_{L^1(Y)} < \infty. \]
To reduce notation let us assume that \( M \subset H \) – this is no loss as the complementary piece \( M_q F \) in \( M \) to \( M_H \) is compact, (2.1). Then
\[ NA \to Y, \quad na \mapsto na \cdot y_o \]
is an open embedding. In particular \( NA \cdot y_o \subset Y \) is open. It follows that
is open and we may assume that
\[ U_A U_N A \cdot y_o \subset Y_C \]
injects into \( Y_C \). Fix \( k \in K \) and define a holomorphic function \( F \) on \( U_N U_A A \cdot y_o \) by \( F(z) := f(kz \cdot y_o) \). In particular, we obtain a constant \( C > 0 \) such that for all \( an \in AN \) we get that

\[ |F(an)| \leq C \int_{U_N U_A} |F(n'a'an)| \, da' \, dn' \]

with \( da' \) and \( dn' \) Haar-measures on \( A_C \) and \( N_C \) (Bergman-estimate).

Let us write \( dy \) for a Haar measure on \( Y \) and observe that \( dy \) restricted to \( AN \) is just \( da \, dn \) with \( da \) and \( dn \) Haar measures on \( A \) and \( N \) respectively. Therefore

\[
\int_N |F(an)| \, dn \leq C \int_{U_N U_A} \int_N |F(n'a'an)| \, da' \, dn' \, \, dn \\
\leq C \int_{U_N} \int_{B_a} \int_{A} \int_N |F(n' \exp(iX)a'an)| \, \, dn' \, dX \, da' \, dn \\
\leq C \int_{U_N} \int_{B_a} \int_Y |f(kn' \exp(iX)y)| \, \, dn' \, dX \, dy \\
\leq C \cdot \text{vol}(U_N) \cdot \text{vol}(B_a) \sup_{c \in KU_A U_N} \|f(c\cdot)\|_{L^1(Y)} < \infty .
\]

We observe that the last expression does not depend on \( a \in A \) and \( k \in K \). This proves (i). Now (ii) is just a variable change of (i):

\[
\int_N |f(kn' \cdot y_o)| \, \, dn = \int_N |f(kaa^{-1}na \cdot y_o)| \, \, dn \\
= a^{2\rho} \int_N |f(kan \cdot y_o)| \, \, dn
\]

For \( f \in A \) we define a function \( R(f) \) on \( \Xi \) via

\[
R(f)(gMHN) := \int_N f(gn \cdot y_o) \, \, dn.
\]

According to our previous lemma the defining integrals are absolutely convergent.

Let us write \( C_0^\infty(\Xi) \) for the space of analytic functions on \( \Xi \) which vanish at infinity. In view of (2.3) vanishing at infinity for \( F \) means
\[
\lim_{a \to \infty} \sup_{a \in A, k \in K} |F(kaM_N)| = 0.
\]

**Proposition 3.2.** The following assertions hold:

(i) For all \( f \in A \) one has \( \mathcal{R}(f) \in C_0^\omega(\Xi) \).

(ii) The map \( \mathcal{R} : A \to L^1(\Xi)^\omega \subset C_0^\omega(\Xi) \) is continuous.

**Proof.** (i) As before it is no loss to assume that \( M \subset H \) – the piece of \( M \) not in \( H \) is compact by (2.1). Let us first show that \( F := \mathcal{R}(f) \vert_A \in C_0^\omega(\Xi) \).

In fact as \( f \) is in \( L^1(Y) \) and \( dy\vert_{AN} = da \, dn \), it follows that \( F \in L^1(A) \).

Moreover \( F \in L^1(A)^\omega \), i.e. it is an analytic vector for the regular representation of \( A \) on \( L^1(A) \). Therefore the standard Sobolev lemma implies that \( F \in C_0^\omega(A) \).

Finally, employing the additional compact parameter \( k \in K \) causes no difficulty.

(ii) This follows from (i) and the last (and crucial) estimate in the proof of Lemma 3.1(i). \qed

**Remark 3.3.** Actually one can define the Radon transform with image on the whole horosphere space \( \text{Hor}(Y) \). Recall the set \( G_h \subset G \) and for \( x \in G_h \) the parameter space \( \Xi_x = G/L^xH_Nx \). For \( f \in A \) one then defines

\[
\mathcal{R}_x(f)(gL^x_HN^x) = \int_N f(gx^{-1}nx \cdot y_o) \, dn.
\]

The resulting function \( \mathcal{R}_x(f) \) is then, as above seen to lie in \( C_0^\omega(\Xi_x) \cap L^1(\Xi_x) \). Patching matters together we thus obtain a well defined \( G \)-map

\[
\mathcal{R} : A \to C^\omega(\text{Hor}(Y))
\]

**4. The kernel of the Radon transform: discrete spectrum**

In this section we show that the discrete spectrum of \( L^2(Y) \), as far as it meets \( A \), lies in the kernel of \( \mathcal{R} \). In fact we show even more: namely that the trace of \( A \) in the orthocomplement of the most continuous spectrum lies in the kernel.

Recall our minimal \( \theta \tau \)-stable parabolic subgroup

\[
P_{\text{min}} = MAN.
\]

In the sequel we use the symbol \( Q \) for a \( \theta \tau \)-stable parabolic which contains \( P_{\text{min}} \). There are only finitely many. We write

\[
Q = MQ_AQ_NQ
\]
for its standard factorization and observe:

- \( M_Q \supset M \),
- \( A_Q \subset A \),
- \( N_Q \subset N \).

One calls two parabolics \( Q \) and \( Q' \) associated if there exists an \( n \in N_K(A) \) such that \( nA_Qn^{-1} = A_{Q'} \). This induces an equivalence relation \( \sim \) on our parabolics and we write \([Q]\) for the corresponding equivalence class. If the context is clear we simply omit the brackets and just write \( Q \) instead of \([Q]\).

It follows from the Plancherel theorem ([3], [5]) that

\[
L^2(Y) = \bigoplus_{Q \supset P_{\text{min}}/\sim} L^2(Y)[Q]
\]

where \( L^2(Y)_Q = L^2(Y)[Q] \) stands for the part corresponding to representations which are induced off from \( Q \) by discrete series of \( M_Q/M_Q \cap wHw^{-1} \) with \( w \) running over representatives of \( W/W_H \).

As \( A \subset C^\infty_0(Y) \), see [10], we observe that \( A \subset L^2(Y) \). However we note that \( A \) might not be dense in \( L^2(Y) \): it has no components in this part of \( L^2(Y)_Q \) which is induced from non-integrable discrete series of \( M_Q/M_Q \cap wHw^{-1} \).

Let \( A_Q = L^2(Y)_Q \cap A \). For the extreme choices of \( Q \) we use a special terminology:

\[
L^2(Y)_{\text{disc}} := L^2(Y)_G \quad \text{and} \quad L^2(Y)_{\text{mc}} := L^2(Y)_{P_{\text{min}}}
\]

and one refers to the \( \text{discrete} \) and \( \text{most continuous} \) part of the square-integrable spectrum. Likewise we declare \( A_{\text{disc}} \) and \( A_{\text{mc}} \). Let us mention that we believe that

\( A_{\text{mc}} \subset L^2(Y)_{\text{mc}} \) is dense

(the heuristic reason for that is that \( M/M \cap H \) is compact).

**Theorem 4.1.** \( \mathcal{R}(A_{\text{disc}}) = \{0\} \).

**Proof.** The proof is the same as for the group, see [13], Th. 7.2.2 for a useful exposition.

Let \( f \in A_{\text{disc}} \). We have to show that \( \mathcal{R}(f) = 0 \). As \( \mathcal{R} \) is continuous (Proposition [3,2]), standard density arguments reduce to the case where \( f \) belongs to a single discrete series representation and that \( f \) is \( K \)-finite. Let

\[
V = \mathcal{U}(\mathfrak{g}_C)f
\]

be the corresponding Harish-Chandra module and set \( T := [\mathcal{R}|_V]|_A \). Then \( T \) factors over the Jacquet module \( j(V) = V/nV \). We recall that
$j(V)$ is an admissible finitely generated $(M, a)$-module. Hence
\[
\dim \mathcal{U}(a)T(f) < \infty.
\]
Consequently
\[
T(f)(a) = \sum_{\mu} a^\mu p_\mu(\log a) \quad (a \in A)
\]
where $\mu$ runs over a finite subset in $a^*_C$ and $p_\mu$ is a polynomial (see [13], 8.A.2.10). From $T(f) \in C^p_0(A)$, we thus conclude that $T(f) = 0$ and hence $\mathcal{R}(f) = 0$ by the $K$-finiteness of $f$.

As a consequence of the previous theorem we obtain the main result of this subsection.

**Theorem 4.2.** Let $Q \supset P_{\text{min}}$. Then $\mathcal{R}(\mathcal{A}_Q) = \{0\}$.

**Proof.** If $Q = G$, then this part of the previous theorem. The general case will be reduced to that. So suppose that $P_{\text{min}} \not\subset Q \subset G$. Define $\Xi_Q = G/(M_Q \cap H)N_Q$ and like in (2.3) one has a diffeomorphic parameterization $[K \times_{M_Q \cap K} M_Q/M_Q \cap H] \times A \to \Xi_Q$.

As in Subsection 3.1 one obtains that the map
\[
\mathcal{R}_Q : A \to L^1(\Xi_Q)^\omega, \quad f \mapsto \left( g(M_Q \cap H)N_Q \mapsto \int_{N_Q} f(gnH) \ dn \right)
\]
is defined, $G$-equivariant and continuous.

Next observe that
\[
N = N_Q \rtimes N^Q
\]
with $\{1\} \neq N^Q \subset M_Q$. As before the map
\[
\mathcal{R}^Q : L^1(\Xi_Q)^\omega \to L^1(\Xi)^\omega
\]
\[
f \mapsto \left( gM_HN \mapsto \int_{N_Q} f(gn(M_Q \cap H)N_Q) \ dn \right)
\]
is defined, equivariant and continuous.

Now we note that
\[
(4.1) \quad \mathcal{R} = \mathcal{R}^Q \circ \mathcal{R}_Q.
\]

Let now $f \in \mathcal{A}_Q$. Without loss of generality we may assume that $f$ belongs to a wave packet induced from a discrete series $\sigma \subset L^2(M_Q/M_Q \cap H)$.

Note that $M_Q/M_Q \cap H$ naturally embeds into $\Xi_Q$ and that the restriction of $L^1(\Xi_Q)^\omega$ to $M_Q/M_Q \cap H$ stays integrable. Hence $F := \mathcal{R}_Q(f)$ restricted to $M_Q/M_Q \cap H$ is integrable as well.
We claim that $F|_{M_Q/M_Q \cap H}$ belongs to the $\sigma$-isotypical class. First note that $L^1(\Xi_Q)^\omega \subset \hat{L}^2(\Xi_Q)$. By induction on stages

$$L^2(\Xi_Q) = \text{Ind}_{M_Q/M_Q \cap H}^{G} \text{triv} \cong \text{Ind}_{M_Q}^{G} L^2(M_Q/M_Q \cap H).$$

Thus if $L^2(M_Q/M_Q \cap H) = \int_{\hat{M}_Q} \oplus m_\pi \mathcal{H}_\pi \, d\mu(\pi)$ is the Plancherel decomposition, then as $G$-modules:

$$L^2(\Xi_Q) \cong \int_{\hat{M}_Q} \oplus m_\pi \text{Ind}_{M_Q}^{G} \mathcal{H}_\pi \, d\mu(\pi).$$

Now note that $A_Q$ acts one right on $\Xi_Q$ and this action commutes with $G$ (see the next section for a detailed discussion for $Q = P_{\text{min}}$). This gives us a further disintegration of the left regular representation $L_Q$ of $G$ on $L^2(\Xi_Q)$:

$$L_Q \cong \int_{\hat{M}_Q} \oplus m_\pi \int_{i\mathcal{A}_Q} \text{Ind}_{M_Q A_Q N_Q}^{G} [\pi \otimes (-\lambda - \rho_Q) \otimes 1] \, d\lambda \, d\mu(\pi).$$

As $R_Q$ is $G$-equivariant, we thus conclude that $R_Q(f) \in \text{Ind}_{M_Q N_Q}^{G} \sigma$.

Our claim combined with the previous theorem implies that

$$\mathcal{R}^Q(F)|_{M_Q/M_Q \cap H} = 0.$$

By the equivariance properties of $\mathcal{R}_Q$ and $\mathcal{R}^Q$ we are free to replace $f$ (and hence $F$) by any $G$-translate. Consequently $\mathcal{R}^Q(F) = 0$, as was to be shown. \hfill $\square$

5. Restriction of the Radon transform to the most continuous spectrum

The objective of this section is to show that $\mathcal{R}$ is faithful on the most continuous spectrum.

We recall a few facts on the spectrum of $L^2(\Xi)$ and the most continuous spectrum on $Y$ and start with the "horocyclic picture". The homogeneous space $\Xi$ carries a $G$-invariant measure. Consequently left shifts by $G$ in the argument of a function on $\Xi$ yields a unitary representation, say $L$, of $G$ on $L^2(\Xi)$; in symbols

$$(L(g)f)(\xi) = f(g^{-1} \cdot \xi) \quad (f \in L^2(\Xi), g \in G, \xi \in \Xi).$$

It is important to note that the $G$-action on $\Xi$ admits a commutating action of $A$ from the right

$$\xi \cdot a = gaMN \quad (\xi = gMHN \in \Xi, a \in A);$$
this is because $A$ normalizes $M_H N$. Therefore the description

$$(R(a)f)(\xi) = a^\rho \cdot f(\xi \cdot a) \quad (f \in L^2(\Xi), a \in A, \xi \in \Xi)$$

defines a unitary representation $(R, L^2(\Xi))$ of $A$ which commutes with the $G$-representation $L$. Accordingly we define an $A$-Fourier transform for an appropriate function $f$ on $\Xi$ by

$$F_A(f)(\lambda, gM_H N) := \int_A [R(a)f](gM_H N)a^\lambda \, da \quad (\lambda \in i a^*) .$$

For $\lambda \in a^*_C$ let us set

$$L^2(\Xi)_\lambda := \{ f : G \to \mathbb{C} \mid \bullet \ f \text{ measurable},$$

$$\bullet \ f(\cdot \text{man}) = a^{-\rho - \lambda} f(\cdot) \ \forall \text{man} \in M_H A N,$$

$$\bullet \int_K |f(k)|^2 \, dk < \infty \}$$

Likewise we write $C^\infty(\Xi)_\lambda$ for the smooth elements of $L^2(\Xi)_\lambda$. The disintegration of $L^2(\Xi)$ is then given by

$$L^2(\Xi) \simeq \int_{i a^*} L^2(\Xi)_\lambda \, d\lambda$$

with isomorphism given by the $A$-Fourier transform

$$f \mapsto (\lambda \mapsto F_A(f)(\lambda, \cdot)) .$$

In the next step we recall the Plancherel decomposition for the most continuous spectrum (cf. [II]).

Some generalities upfront. For a representation $\pi$ of a group $L$ on some topological vector space $V$ we denote by $\pi^*$ the dual representation on the (strong) topological dual $V^*$ of $V$.

Let $\sigma \in \widehat{M/M_H}$ and $V_\sigma$ a unitary representation module for $\sigma$. For $\lambda \in a^*_C$ we define

$$\mathcal{H}_{\sigma, \lambda} := \{ f : G \to V_\sigma \mid \bullet \ f \text{ measurable},$$

$$\bullet \ f(\cdot \text{man}) = a^{-\rho - \lambda} \sigma(m)^{-1} f(\cdot) \ \forall \text{man} \in P_{\text{min}},$$

$$\bullet \int_K \langle f(k), f(k) \rangle_\sigma \, dk < \infty \} .$$

The group $G$ acts on $\mathcal{H}_{\sigma, \lambda}$ by displacements from the left and the so-obtained Hilbert representation will be denoted by $\pi_{\sigma, \lambda}$. 
Remark 5.1. The relationship between $\mathcal{H}_{\sigma,\lambda}$ and $L^2(\Xi)_\lambda$ is as follows. If $\mu_\sigma$ is an (up to scalar unique) $M_H$-fixed element in $V_\sigma^*$, then the mapping

$$\mathcal{H}_{\sigma,\lambda} \to L^2(\Xi)_\lambda, \ f \mapsto \mu_\sigma(f)$$

is a $G$-equivariant injection. The map can be made isometric by appropriate scaling of $\mu_\sigma$. Employing induction in stages one therefore obtains an isometric identification

$$\bigoplus_{\sigma \in M/M_H} \mathcal{H}_{\sigma,\lambda} = L^2(\Xi)_\lambda.$$

Sometimes it is useful to realize $\mathcal{H}_{\sigma,\lambda}$ as $V_\sigma$-valued functions on $\overline{N} := \theta(N)$; we speak of the non-compact realization then. Define a weight function on $\overline{N}$ by

$$w_\lambda(\overline{\pi}) = a^{2\text{Re}\lambda}$$

where $a \in A$ is determined by $\overline{\pi} \in KaN$. Then the map

$$\mathcal{H}_{\sigma,\lambda} \to L^2(\overline{N}, w_\lambda(\overline{\pi})d\overline{\pi}) \otimes V_\sigma, \ f \mapsto f|_{\overline{\pi}}$$

is an isometric isomorphism.

We remark that:

- $\pi_{\sigma,\lambda}$ is irreducible for generic $\lambda$.
- $\pi_{\sigma,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}^*$.
- The dual representation of $\pi_{\sigma,\lambda}$ is canonically isomorphic to $\pi_{\sigma^*,-\lambda}$; the dual pairing is given by

$$\langle f, g \rangle := \int_{\overline{N}} (f(\overline{\pi}), g(\overline{\pi}))_\sigma \ d\overline{\pi}$$

for $f \in \mathcal{H}_{\sigma,\lambda}$, $g \in \mathcal{H}_{\sigma^*,-\lambda}$ and $(,)_\sigma$ the natural pairing between $V_\sigma$ and $V_\sigma^*$.

Next we recall the description of the $H$-fixed elements in the distribution module $(\mathcal{H}_{\sigma,\lambda}^\infty)^*$. We first set for each $w \in \mathcal{W}_H \setminus \mathcal{W}$

$$V^*(\sigma, w) := (V_\sigma^*)^{w^{-1}M_Hw}.$$

Note that this space is one-dimensional. Set

$$V^*(\sigma) := \bigoplus_{w \in \mathcal{W}_H \setminus \mathcal{W}} V^*(\sigma, w) \simeq \mathbb{C}^{|\mathcal{W}_H \setminus \mathcal{W}|}$$

and for $w \in \mathcal{W}_H \setminus \mathcal{W}$ we denote by

$$V^*(\sigma) \to V^*(\sigma, w), \ \eta \mapsto \eta_w.$$
the orthogonal projection. In the sequel we will use the terminology \( \text{Re} \lambda >> 0 \) if
\[
(\text{Re} \lambda - \rho)^{\Lambda} > 0 \quad \forall \alpha \in \Sigma^+.
\]
Then, for \( \text{Re} \lambda >> 0 \) the description
\[
j(\sigma^*, -\lambda)(\eta)(g) = \begin{cases} 
a^{-\rho+\lambda} \sigma^*(m^{-1}) \eta_w & \text{if } g = \text{hwman} \in HwMAN, \\
0 & \text{otherwise}, \end{cases}
\]
defines a continuous \( H \)-fixed element in \( H_{\sigma^*, -\lambda} \). We may meromorphically continue \( j(\sigma^*, \cdot) \) in the \( \lambda \)-variable and obtain, for generic values of \( \lambda \) the identity
\[
j(\sigma^*, -\lambda)(V^*(\sigma)) = ((\mathcal{H}_{\sigma, \lambda}^*\{)^H.
\]
For large \( \lambda \) the inverse map to \( j \) is given by
\[
((\mathcal{H}_{\sigma, \lambda}^*\{)^H \ni \nu \mapsto (\nu(w))_{w\in \mathcal{W}_H\backslash \mathcal{W}} \in V^*(\sigma).
\]
For a smooth vector \( v \in \mathcal{H}_{\sigma, \lambda} \) and \( \eta \in V(\sigma^*) \) we obtain a smooth function on \( Y = G/H \) by setting
\[
F_{v, \eta}(gH) = \langle \pi_{\sigma, \lambda}(g^{-1})v, j(\sigma^*, -\lambda)(\eta) \rangle.
\]
The Plancherel theorem for \( L^2(\mathcal{Y})_{\text{mc}} \), see for instance [1], then asserts the existence of a meromorphic assignment
\[
a^*_{C} \rightarrow \text{Gl}(V(\sigma^*)), \quad \lambda \mapsto C(\sigma, \lambda)
\]
such that with \( j^0(\sigma, \lambda) := j(\sigma, \lambda) \circ C(\sigma, \lambda) \) the map
\[
\Phi : \bigoplus_{\sigma \in M/M_H} \int_{i\mathfrak{a}^*_+}^\oplus \mathcal{H}_{\sigma, \lambda} \otimes V^*(\sigma) \ d\lambda \rightarrow L^2(\mathcal{Y})_{\text{mc}}
\]
which for smooth vectors on the left is defined by
\[
\sum_{\sigma} (v_{\sigma, \lambda} \otimes \eta)_\lambda \mapsto \left( gH \mapsto \sum_{\sigma} \int_{i\mathfrak{a}^*_+} F_{v_{\sigma, \lambda} \beta^0(\sigma^*, -\lambda)(\eta)}(gH) \ d\lambda \right)
\]
extends to a unitary \( G \)-equivalence. Here \( \mathfrak{a}^*_+ \) denotes a Weyl chamber in \( \mathfrak{a}^* \).

**Remark 5.2.** Suppose that \( \mathcal{W} = \mathcal{W}_H \) (this happens in the group case). Then \( V(\sigma^*) \) is one dimensional and we obtain with Remark 5.1 the following isomorphism:
\[
\bigoplus_{\sigma \in M/M_H} \int_{i\mathfrak{a}^*_+}^\oplus \mathcal{H}_{\sigma, \lambda} \otimes V(\sigma^*) \ d\lambda \simeq \int_{i\mathfrak{a}^*_+} L^2(\mathcal{Y})_{\Lambda} \ d\lambda.
\]
Hence we may view \( \Phi \) as defined on a subspace of \( L^2(\mathcal{Y}) \).
The inverse of the map $\Phi$ is the most continuous Fourier transform $\mathcal{F}$ (or $\mathcal{F}_{mc}$). For $f \in L^2(Y)_{mc} \cap L^1(Y)$ the Fourier-transform is given by

$$
\mathcal{F}(f)(\sigma, \lambda, \eta)(g) := \int_Y f(y) j^0(\sigma, \lambda)(\eta)(y^{-1}g) \, dy,
$$

where $\sigma \in \hat{M}/M_H$, $\lambda \in ia^*$ and $\eta \in (V^*(\sigma))^* \simeq V(\sigma^*)$. As a last piece of information we need to relate the Fourier-transform and the Radon-transform.

5.1. The relation between Fourier and Radon transform

Now we can determine the relation between $\mathcal{R}$ and $\mathcal{F}$. Let us write $\mathcal{F}_A^w$ for $\mathcal{F}_A$ on $\Xi_w = G/M_H N^w$. Let $f \in C^\infty_c(Y)$.

We unwind definitions:

$$
\mathcal{F}(f)(\sigma, \lambda, \eta)(g) = \int_Y f(gy) j^0(\sigma, \lambda)(\eta)(y^{-1}) \, dy
$$

$$
= \sum_{w \in W/W_H} \int_{AM/\omega M_H N_{w-1}} f(ganmw \cdot y_0) 
\cdot j^0(\sigma, \lambda)(\eta)(w^{-1}m^{-1}a^{-1}n^{-1}) \, da \, dn \, dm
$$

$$
= \sum_{w \in W/W_H} \int_{AM/wM_H N_{w-1}} \mathcal{R}_w(f)(gMawM_H N^w) 
\cdot a^0 j^0(\sigma, \lambda)(w^{-1}m^{-1}) \, da \, dm
$$

$$
= \sum_{w \in W/W_H} \int_{M/wM_H N_{w-1}} [\mathcal{F}_A^w \circ \mathcal{R}_w](f)(w^{-1}\lambda, gmwM_H N^w) 
\cdot j^0(\sigma, \lambda)(w^{-1}m^{-1}) \, dm
$$

$$
= \sum_{w \in W/W_H} \int_{M/wM_H N_{w-1}} [\mathcal{F}_A^w \circ \mathcal{R}_w](f)(w^{-1}\lambda, gmwM_H N^w) 
\cdot \sigma(m) j^0(\sigma, \lambda)(\eta)(w^{-1}) \, dm
$$

Let us remark that $j^0(\sigma, \lambda)(\eta)$ is a distribution and a priori the evaluation $j^0(\sigma, \lambda)(\eta)(w^{-1})$ has only meaning for $\text{Re}\, \lambda$ sufficiently small. This problem is overcome by the meromorphic continuation of $j(\sigma, \lambda)$. This meromorphic continuation is in fact obtainable by an iterative procedure starting with $\text{Re}\, \lambda$ small and larger values obtained by a differential operator with polynomial coefficients [4]. This fact allows
us to replace $C_c^\infty(Y)$ with the Schwartz space $\mathcal{S}(Y)$ of rapidly decreasing functions (see [2] Sect. 12, and not be confused with the Harish-Chandra Schwartz space $\mathcal{C}(Y)$ in Section 5 below). Thus we have shown:

**Lemma 5.3.** Let $f \in \mathcal{S}(Y)$. Then for all $\sigma \in \widetilde{M/M_H}$, $\lambda \in i\mathfrak{a}_+^*$

$$
\mathcal{F}(f)(\sigma, \lambda, \eta)(g) = \sum_{w \in \mathcal{W}/\mathcal{W}_H} \int_{M/\mathcal{W}_Hw} \left[ \mathcal{F}_A^w \circ \mathcal{R}_w \right](f)(w^{-1}\lambda, gmwM_HN^w) \\
\cdot \sigma(m)^j(\sigma, \lambda)(\eta)(w^{-1}) \, dm .
$$

**Remark 5.4.** The special case of $\mathcal{W} = \mathcal{W}_H$ is of particular interest. Then the formula from above simplifies to

$$
\mathcal{F}(f)(\sigma, \lambda, \eta)(g) = \int_{M/M_H} \left[ \mathcal{F}_A \circ \mathcal{R} \right](f)(\lambda, gmM_HN) \\
\cdot \sigma(m)^j(\sigma, \lambda)(\eta)(1) \, dm .
$$

**Theorem 5.5.** $\mathcal{R}$ restricted to $\mathcal{A}_{mc} \cap \mathcal{S}(Y)$ is injective.

**Proof.** Let $f \in \mathcal{A}_{mc} \cap \mathcal{S}(Y)$. Suppose that $\mathcal{R}(f) = 0$. With Remark 3.3 we conclude that $\mathcal{R}_w(f) = 0$ for all $w$. Hence the lemma from above implies that $\mathcal{F}(f) = 0$. As the Fourier transform is injective on $\mathcal{S}(Y)$, see [2] Cor. 12.7, we get that $f = 0$. □

**Remark 5.6.** It is very likely that $\mathcal{S}(Y) \cap \mathcal{A}_{mc}$ is dense in $\mathcal{A}_{mc}$, but there does not exist a reference at the moment. If this were established, then the theorem above would imply that $\mathcal{R}$ restricted to $\mathcal{A}_{mc}$ is injective.

5.2. Concluding remarks

5.2.1. The group case. It is instructive to see what the results in this paper mean for a semisimple group $G$ viewed as a symmetric space, i.e.:

$$
G \cong G \times G/\Delta(G)
$$

with $\Delta(G) = \{(g, g) \mid g \in G\}$ the diagonal group. If $P = MAN$ is a minimal parabolic of $G$ and $\overline{P} = MAN$ its standard opposite (i.e. the image under the corresponding Cartan involution), then the parameter space for the horospheres is given by

$$
\Xi = G \times G/\Delta(MA)(N \times N) .
$$

Our function space $\mathcal{A}$ are then the analytic vectors for the left-right regular representation of $G \times G$ on $L^1(G)$. For $f \in \mathcal{A}$ one then has
\[
\mathcal{R}(f)((g,h)\Delta(MA)(N \times \overline{N})) = \int_{N \times \overline{N}} f(gn\overline{n}h^{-1})
\]
\[
dn \, d\overline{n}.
\]

5.2.2. The next steps. Coming back to our more general situation of \( Y = G/H \) let us consider the double fibration
\[
G/M_H \quad \Xi \quad Y.
\]

With \( \mathcal{R} \) comes a dual transform \( \mathcal{R}^\vee \) between appropriate function spaces \( \mathcal{F}(\Xi) \) and \( \mathcal{F}(Y) \) on \( \Xi \) and \( Y \):
\[
\mathcal{F}(\Xi) \rightarrow \mathcal{F}(Y); \quad \mathcal{R}^\vee(\phi)(gH) = \int_{H/M_H} \phi(gh \cdot M_H N)
\]
\[
d(hM_H).
\]

For \( f \in A_{mc} \) one then might ask about the existence of a pseudo-differential operator \( D \) such that
\[
 f = \mathcal{R}^\vee(D\mathcal{R}(f))
\]
holds. For \( Y = \text{Sl}(2, \mathbb{R})/\text{SO}(1, 1) \) this was considered in [7] where it was shown that such a pseudo-differential operator \( D \) exists. For \( Y \) being a group one might expect that \( D \) is in fact a differential operator.

5.2.3. Radon transform on Schwartz spaces. One might ask to what extend \( \mathcal{R} \) might be defined on the Schwartz space of \( Y \). For some classes of \( Y \) this seems to be possible and we will comment on this in more detail below. Let us first recall the definition of the Schwartz space.

One uses
\[
G = KAH
\]
often referred to as the polar decomposition of \( G \) (with respect to \( H \) and \( K \)). Accordingly every \( g \in G \) can be written as \( g = k_ga_gh_g \) with \( k_g \in K \) etc. It is important to notice that \( a_g \) is unique modulo \( \mathcal{W}_H \).

Therefore the prescription
\[
\|gH\| := |\log a_g| \quad (g \in G)
\]
is well defined for \( |\cdot| \) the Killing norm on \( \mathfrak{p} \). An alternative, and often useful, description of \( \|\cdot\| \) is as follows
\[
\|y\| = \frac{1}{4}|\log [y_\tau(y)^{-1} \theta(y_\tau(y)^{-1})^{-1}]| \quad (y \in Y).
\]
For $u \in \mathcal{U}(\mathfrak{g})$ we write $L_u$ for the corresponding differential operator on $Y$, i.e. for $u \in \mathfrak{g}$

$$(L_u f)(y) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tu)y),$$

whenever $f$ is a differentiable function at $y$. With these preliminaries one defines the Harish-Chandra Schwartz space of $Y$ by

$$\mathcal{C}(Y) = \{ f \in C^\infty(Y) \mid \forall u \in \mathcal{U}(\mathfrak{g}) \forall n \in \mathbb{N} \sup_{y \in Y} \Theta(y)(1 + \|y\|)^n| (L_u f)(y) | < \infty \}$$

where $\Theta(gH) = \phi_0(g\tau(g)^{-1})^{-1/2}$ and $\phi_0$ Harish-Chandra’s basic spherical function.

It is not to hard to see that $\mathcal{C}(Y)$ with the obvious family of defining seminorms is a Fréchet space. Moreover $\mathcal{C}(Y)$ is $G$-invariant and $G$ acts smoothly on it. We note that $\mathcal{C}(Y) \subset L^2(Y)$ is a dense subspace.

We write $BC^\infty(\Xi)$ for the space of bounded smooth functions on $\Xi$. In the context of defining $\mathcal{R}$ on $\mathcal{C}(Y)$ we focus we wish to discuss a basic example.

**Lemma 5.7.** Let $Y = \text{Sl}(2, \mathbb{R})/\text{SO}(1, 1)$, and $f \in \mathcal{C}(Y)$. Then the following assertions hold:

(i) The integral $\int_N f(nH) \, dn$ is absolutely convergent.

(ii) The prescription

$$gM_H N \mapsto \int_N f(gnH) \, dn$$

defines a function in $BC^\infty(\Xi)$.

**Proof.** Let $A$ be the diagonal subgroup of $G$ (with positive entries) and $N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}$.

(i) For $x \in \mathbb{R}$ and $n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we have to determine $a_x \in A$ such that $n_x \in Ka_x H$. We use [5.3] and start:

$$z_x := n_x \tau(n_x)^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 - x^2 & -x \\ x & 1 \end{pmatrix}$$

and hence

$$y_x := z_x \theta(z_x)^{-1} = \begin{pmatrix} 1 - x^2 & -x \\ x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 - x^2 & x \\ -x & 1 \end{pmatrix} = \begin{pmatrix} (1 - x^2)^2 + x^2 & * \\ * & 1 + x^2 \end{pmatrix}.$$
For $|x|$ large we have $\log |y_x| = |\log y_x|$. Furthermore up to an irrelevant constant

$$|y_x| = [\text{tr}(y_x y_x)]^{\frac{1}{2}} \geq \frac{1}{2}[(1 - x^2)^2 + x^2 + 1 + x^2]$$

$$\geq \frac{1}{2}[x^4 + 1]$$

Therefore, for $|x|$ large

$$\|n_x\| \geq \frac{1}{4} \log(x^4/2 + 1/2)$$

From Harish-Chandra’s basic estimates of $\phi_0$ and our computation of $z_x$ we further get that

$$\Theta(n_x) \geq |x|.$$ 

Therefore for $f \in C(Y)$ we obtain that $x \mapsto |f(n_x H)|$ grows slower than $\frac{1}{|x| |\log x|^{\infty}}$ for any fixed $N > 0$ and $|x|$ large. This shows (i).

(ii) Let $f \in C(Y)$ and set $F := \mathcal{R}(f)$. From the proof of (i) we know that $F$ is smooth. It remains to see that $F$ is bounded. From $G = KAH$ we deduce that it is enough to show that $F|_A$ is bounded. We do this by direct computation. For $t > 0$ we set

$$a_t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}.$$ 

Then

$$a_t n_x = \begin{pmatrix} t & tx \\ 0 & 1/t \end{pmatrix}$$

and thus

$$z_{t,x} := a_t n_x \tau(a_t n_x)^{-1} = \begin{pmatrix} t & tx \\ 0 & 1/t \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ -tx & 1/t \end{pmatrix}$$

$$= \begin{pmatrix} t^2(1 - x^2) & -x \\ x & 1/t^2 \end{pmatrix}.$$ 

With that we get

$$y_{t,x} = z_{t,x} \theta(z_{t,x})^{-1} = \begin{pmatrix} t^4(1 - x^2)^2 + x^2 & * \\ * & 1/t^4 + x^2 \end{pmatrix}.$$ 

For $t \geq 1$ we conclude that

$$\|a_t n_x\| \gtrsim \log \left( \begin{array}{ll} c_1 t^4 & \text{for } |x| \leq 1/2, \\ c_2 t^4 x^4 - c_3 & \text{for } |x| \geq 1/2. \end{array} \right)$$
and for $|t| < 1$ one has
\[\|a_t n_x\| \geq \log |x| .\]
From that we obtain (ii). □

This example is somewhat specific. One might expect that the Radon transform on $\mathcal{C}(Y)$ converges whenever the real rank of $G$ and of $Y$ coincide.

For groups it is not hard to show that $\mathcal{R}(f)$ does not converge for general $f \in \mathcal{C}(G)$; integrability of $f$ is needed.

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