FLOPS, FLIPS AND PERVERSE POINT SHEAVES ON THREEFOLD STACKS

DAN ABRAMOVICH AND JIUN C. CHEN

1. Introduction

We work with varieties over $\mathbb{C}$.

1.1. Threefold flops as Moduli of perverse point sheaves. In [Br00], T. Bridgeland considered a morphism $X \to Y$ of complex projective varieties satisfying

(B.1) $\mathbb{R}f_* O_X = O_Y$, and
(B.2) $\dim f^{-1}(z) \leq 1$ for every $z \in Y$.

Note that our notation differs from [Br00], in that $X$ and $Y$ are switched. Bridgeland defined an abelian subcategory $\text{Per}(X/Y) \subset D^b(X)$ of the derived category $D^b(X) := D^b_{\text{coh}}(\text{Qcoh}X)$ of bounded complexes with coherent cohomology, and in this subcategory he identifies certain objects called \textit{perverse point sheaves}. He proved:

\textbf{Theorem 1.1.1} (Bridgeland, [Br00], Theorem 3.8). \textit{There exists a fine moduli space $M(X/Y)$ of perverse point sheaves, which is projective over $Y$. It contains a distinguished component $W \subset M(X/Y)$ which is birational to $Y$.}

Bridgeland further proved the following remarkable result:

\textbf{Theorem 1.1.2} (Bridgeland, [Br00], Theorem 1.1). \textit{Assume that $X$ is a smooth threefold and $X \to Y$ is a flopping contraction. Then

1. $W$ is smooth,
2. The Fourier-Mukai type transformation induced by the universal perverse point sheaf is an isomorphism, and
3. $W \cong X^+$, the flop of $X \to Y$.}

In [Ch02], Theorem 1.1.2 is generalized to the case where $X$ is a threefold with Gorenstein terminal singularities and $X \to Y$ is a flopping contraction.

1.2. Non-Gorenstein threefolds as stacks. In this paper we are concerned with generalizing the results of [Br00], [Ch02] to some $\mathbb{Q}$-Gorenstein threefolds, using algebraic stacks, as commented in [Ch02], Section 1.8. It should be pointed out that Kawamata obtained some very general results [Ka01] [Ka02], also using algebraic stacks, concentrating on equivalences (or embeddings) of derived categories in birational transformations. To avoid excessive overlap with Kawamata’s work, we emphasize the moduli construction of birational transformations.

The underlying idea is the following: in all considerations of Fourier–Mukai transforms, smoothness is an essential assumption. Thus, if one is to prove results analogous to 1.1.2 for singular varieties, some “hidden smoothness” would be desirable. In the terminal Gorenstein

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\end{footnotesize}
case, a deformation \( X \to Y \) of \( X \to Y \) with \( X \) smooth is used. In the \( \mathbb{Q} \)-Gorenstein case, the singularities can be simplified back to the Gorenstein case, by taking the canonical Gorenstein stack (this is also the main idea behind Kawamata’s result [Ka01]).

1.3. Threefold terminal flops. By way of comparison, we consider two very different cases here. The first is that of a threefold terminal flopping contraction \( X \to Y \), with flop \( X^+ \to Y \). In this case Kawamata [Ka02], Theorem 6.5, has proven an equivalence of derived categories of the canonical covering stacks \( X \to X \) and \( X^+ \to X^+ \). Here we add a footnote to Kawamata’s result: the flop can in some sense be constructed \textit{a priori} as a moduli space. Indeed the entire program of [BKR99, Br00, Ch02] works:

**Proposition 1.3.1.** Let \( X \to Y \) be a flopping contraction of terminal threefolds, and let \( X \to Y \) be the contraction of associated canonical covering stacks. Then

1. the distinguished component \( \mathcal{W}(X/Y) \) of the moduli stack of perverse point sheaves has terminal Gorenstein singularities,
2. the Fourier–Mukai type transform
   \[
   D^b(\mathcal{W}) \to D^b(X)
   \]
   given by the universal object is an equivalence, and
3. \( \mathcal{W}(X/Y) \simeq X^+ \), the flop of \( X \to Y \).

This in particular gives a slightly different approach to [Ka02], Theorem 6.5.

While our result extends the ideas of [Br00] to this case, there is still an unsatisfactory point which we do not know how to resolve: our moduli construction relies on a presentation of the canonical covering stack; it would be desirable to have a construction directly in terms of the stack.

1.4. Threefold Francia flips. The other case we consider is that of the simplest sequence of flips in dimension 3, the so-called Francia flips, obtained as the quotient of the standard threefold flop by a particular action of a cyclic group. In this case something new and very different happens: consider the canonical covering stack \( X \) of \( X \); in this case there is no canonical covering stack of \( Y \) since \( Y \) is not \( \mathbb{Q} \)-gorenstein. The usual moduli space of perverse point sheaves \( M(X/Y) \) is not isomorphic to \( X^+ \). Instead we construct a new abelian category \( \text{Per}(-1,0)(X/Y) \) and consider the analogous notion of perverse point sheaves. We then show:

**Theorem 1.4.1.** Let \( X \to Y \) be a flipping contraction which is locally analytically of the Francia type. Let \( X \to X \) be the canonical covering stack. Then there is a separated moduli space \( M(X/Y) \) of \((-1,0)\)-perverse point sheaves on \( X/Y \), whose distinguished component \( W(X/Y) \) is projective and birational over \( Y \).

**Theorem 1.4.2.**

1. the distinguished component \( W(X/Y) \) is smooth,
2. the Fourier–Mukai type transform
   \[
   D^b(W) \to D^b(X)
   \]
   given by the universal object is fully faithful, and
3. \( W(X/Y) \simeq X^+ \), the flip of \( X \to Y \).

It should be again pointed out that the existence of a fully faithful embedding \( D^b(W) \to D^b(X) \) is a special case of the main theorem of [Ka01].
Unlike the previous cases, we do not provide an *a priori* construction of the flip as a moduli space: the present proof of these theorems uses the existence of $X^+$ in a very explicit way, in two points. First, our proof of projectivity of $X/Y$ relies on the existence of a morphism $X^+ \to W$, i.e. the existence of a family of $(-1,0)$-perverse point sheaves for $X'/Y$ parametrized by $X^+$. Second, in our proof of the fully faithful embedding we used the same morphism to show that $W \to Y$ does not contract a surface to a point. This is needed when applying the method of [BKR99] with the intersection theorem.

1.5. **Some generalities.** We use the notation $D(X)$ exclusively for the derived category of coherent sheaves on $X$, and $D^b(X)$ for the bounded category.

For the definition and properties of perverse sheaves, perverse ideal and point sheaves, and their properties we rely on [Br00] and [Ch02]. We cannot improve here much on the presentation there.

For a Deligne–Mumford stack $\mathcal{X}$ with coarse moduli space $\pi: \mathcal{X} \to X$ we note that $\pi_* \mathcal{O}_X = \mathcal{O}_X$, and that $\pi_*$ is exact on quasi-coherent sheaves.

If $\mathcal{X} = [V/G]$, with $V$ a variety and $G$ a finite group, then $\text{Coh}(\mathcal{X}) \simeq \text{Coh}^G(V)$. The moduli space is simply $X = V/G$. Write $q: V \to X$ for the schematic quotient morphism. If $\mathcal{F}$ is a sheaf on $\mathcal{X}$ corresponding to a $G$-sheaf $\mathcal{G}$ on $V$, then $\pi_* \mathcal{F} = (q_* \mathcal{G})^G$.

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2. **Canonical covering stacks and the moduli space for $\mathbb{Q}$-Gorenstein flops**

2.1. **The canonical covering stack.** Recall that a variety $X$ is $\mathbb{Q}$-Gorenstein if it is Cohen-Macaulay and there is a integer $m$, such that the saturation $\omega^m_X$ is invertible. The minimal positive $m$ satisfying this is called the **canonical index** of $X$.

**Definition 2.1.1.** Let $X$ be a normal quasi-projective $\mathbb{Q}$-Gorenstein variety. The **canonical covering stack** $\mathcal{X} \to X$ is defined as the stack-theoretic quotient

$$\mathcal{X} = \left[ P / \mathbb{G}_m \right],$$

where

$$P = \text{Spec}_Y \left( \bigoplus_{i \in \mathbb{Z}} \omega^{|i|}_X \right).$$

The variety $P$ is the canonical $\mathbb{G}_m$-space of $X$, and is Gorenstein. The canonical covering stack is Gorenstein, and automatically a Deligne–Mumford stack, since the stabilizer of any point of $P$ is contained in the finite group $\mu_m$.

More traditionally, a different but equivalent construction has been used. For each point $x \in X$, one can find an open neighborhood $U_x$ such that $m_x K_X$ is a Cartier divisor for a minimum positive integer $m_x$. The canonical covering $\pi_x: \tilde{U}_x \to U_x$ is a finite morphism of degree $m_x$ where $\tilde{U}_x$ is a normal variety and $\pi_x$ is étale in codim = 1 such that $K_{\tilde{U}_x} = \pi_x^*(m_x K_X)$ is Cartier. The canonical coverings are étale locally unique. Thus one can define the canonical covering stack $\mathcal{X}$ by the atlas given by the collection of canonical coverings.

Consider the natural morphism $\pi: \mathcal{X} \to X$. The stacky points of $\mathcal{X}$ are the preimage of points where $K_X$ fails to be Cartier. When $X$ is a terminal $\mathbb{Q}$-Gorenstein variety, the Weil
divisor $K_X$ is Cartier in codimension 2; in particular, when $\dim X = 3$ this locus consists of isolated points.

2.2. Perverse moduli stack for $\mathbb{Q}$-Gorenstein flops. Let $X$ and $Y$ be terminal $\mathbb{Q}$-Gorenstein threefolds and $f : X \to Y$ be a flopping contraction. Under these assumptions, the varieties $P_X$ and $P_Y$ are Gorenstein fourfolds. There is a $\mathbb{C}^*$-equivariant lifting $f_{P_Y} : P_X \to P_Y$. This morphism satisfies the conditions (B.1) and (B.2). Therefore we can define the moduli space of perverse point sheaves as in [Br00]. There is a distinguished component $W(P_X/P_Y)$, which is birational to $P_Y$. By its universal property, the action of $\mathbb{C}^*$ on $P_X$ and $P_Y$ induces an action on $W(P_X/P_Y)$.

We note that $W(P_X/P_Y)$ is the flop of $P_X \to P_Y$. This follows from the results of [Ch02], as follows: let $H_Y$ be a general hyperplane section of $P_Y$, with inverse image $H_X \subset P_X$. The morphism $H_X \to H_Y$ is a Gorenstein flopping contraction. The restriction of $W(P_X/P_Y)$ to $H_Y$ is isomorphic to $W(H_X/H_Y)$ by Proposition 4.4 of [Ch02]. By Theorem 1.1 of [Ch02] the latter is the flop of $H_X \to H_Y$, which gives the claim.

Consider the stack 

$$\mathcal{W} := \left[ W(P_X/P_Y) / \mathbb{C}^* \right].$$

By definition, we can give an interpretation of this as a moduli stack. For each scheme $S$, the quotient stack $[W(P_X/P_Y)/\mathbb{C}^*]$ gives a category $\mathcal{W}(S) = [W(P_X/P_Y)/\mathbb{C}^*](S)$. An object in this category consists of a $\mathbb{C}^*$-bundle $Q \to S$ with a $\mathbb{C}^*$ equivariant morphism $\alpha : Q \to W(P_X/P_Y)$. For any two objects $(Q_1, \alpha_1)$ and $(Q_2, \alpha_2)$ in $\mathcal{W}(S)$, an arrow is a $\mathbb{C}^*$-equivariant morphism $\beta : Q_1 \to Q_2$ such that $\alpha_1 = \alpha_2 \circ \beta$.

Using the definition of $W(P_X/P_Y)$, we can get a more concrete interpretation of objects. Since $W(P_X/P_Y)$ is a fine moduli space, an element in $W(P_X/P_Y)(Q)$ is equivalent to a family of perverse point sheaves for $P_X \to P_Y$ parametrized by $Q$. Thus an object in the category $\mathcal{W}(S)$ is a pair $(Q, E)$ where $Q$ is a $\mathbb{C}^*$-bundle over $S$ and $E$ is a $\mathbb{C}^*$-equivariant family of perverse point sheaves for $P_X \to P_Y$ parametrized by $Q$.

This interpretation relies on the particular presentation of $\mathcal{W}$ we chose. A similar interpretation in terms of perverse point sheaves on a more general presentation with descent data can be obtained.

By [Ch02], Proposition 4.2, the universal perverse point sheaf over $P_X \times W(P_X/P_Y)$ is the structure sheaf of the fibered product $P_X \times_{P_Y} W(P_X/P_Y)$. The natural $\mathbb{C}^*$ action defines a corresponding sheaf $\mathcal{E}$ on the quotient stack $\mathcal{W}$, which is evidently the structure sheaf of the fibered product $\mathcal{X} \times_Y \mathcal{W}$.

Can we view $\mathcal{W}$ as a moduli stack of perverse point sheaves on $\mathcal{X}/\mathcal{Y}$? The answer is “yes” if we are careful about the definition, and still a bit unsatisfactory. The fibers of $\mathcal{E}$ over geometric points of $\mathcal{W}$ are indeed elements of $\text{Per}(\mathcal{X}/\mathcal{Y})$, defined just as in [Br00]. However the universal perverse point sheaf is not a quotient of $\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}$, for the same reason that the universal non-pervasive “point sheaf” $\mathcal{O}_Y$ of a non-representable stack $\mathcal{Y}$ is not a quotient of $\mathcal{O}_{\mathcal{Y} \times \mathcal{Y}}$. Only after pulling back to an étale cover of $\mathcal{Y}$ can we write it as a quotient. (One might suspect that any moduli construction of “Hilbert scheme” type on a stack should requires a nontrivial game with descent data, as we presented above, in contrast with the “Quot scheme” construction of [OS02].)

2.3. Proof of Proposition 1.3.1 We claim that $\mathcal{W}$ is the flop of $\mathcal{X} \to \mathcal{Y}$ and the sheaf $\mathcal{E}$ gives an equivalence of derived categories.
Consider a scheme $Y'$ and an étale morphism $Y' \to Y$. Write $X' = X \times_Y Y'$. The scheme $Y'$ is a terminal Gorenstein threefold, and $X' \to Y'$ is a flopping contraction. Write $W' = W(X'/Y')$. We have

**Lemma 2.3.1.** $W' = \mathcal{W} \times_Y Y'$

**Proof.** The formation of the category $\text{Per}$ commutes with flat base change (see e.g., [VdB02], Proposition 3.1.4). Consider the fiber diagram

$$
\begin{array}{ccc}
P_X \times_X X' & \longrightarrow & P_X \\
| & | & | \\
P_Y \times_Y Y' & \longrightarrow & P_Y \\
| & | & | \\
X' & \longrightarrow & X' \\
| & | & | \\
Y' & \longrightarrow & Y
\end{array}
$$

where all the horizontal arrows are flat, and the diagonal arrows are $\mathbb{G}_m$-bundles. We therefore have morphisms

$$W(P_X/P_Y) \times_Y Y' \longrightarrow W(P_X \times_Y Y' / P_Y \times_Y Y') \leftarrow W' \times_Y (P_Y \times_Y Y').$$

These are isomorphisms since all these varieties are the relevant flops. Also

$$W(P_X \times_Y Y' / P_Y \times_Y Y') \to W'$$

is a $\mathbb{C}^*$-bundle. Taking quotients by $\mathbb{C}^*$ gives the result. □

By definition the sheaf $\mathcal{E}$ pulls back to the structure sheaf of $X' \times_Y W'$, which is the universal perverse point sheaf for $X'/Y'$.

Applying [Ch02], Theorem 1.1, we have that the Fourier–Mukai type transform $D^b(W') \to D^b(X')$ is an equivalence and $W' \simeq X'^+$, the flop of $X' \to Y'$. Applying [Ch02], Proposition 3.2 we have that the transform

$$F_{\mathcal{E}} : D^b(W) \longrightarrow D^b(X)$$

is an equivalence.

Let $D$ be a negative divisor for $f : X/Y$, let $\mathcal{D}$ be its pullback to $X$ and $D'$ the pullback to $X'$. Since $W' \simeq X'^+$, we have that

$$W' = \text{Proj}_{Y'} \bigoplus_{n \geq 0} f'_* \mathcal{O}_{X'}(nD').$$

It follows that

$$\mathcal{W} = \text{Proj}_Y \bigoplus_{n \geq 0} f'_* \mathcal{O}_X(n\mathcal{D}).$$

This completes the proof. □
3. Threefold Francia Flips

3.1. Local Models. In this section, we study the Francia flip case. Recall the simplest sequence of flips (see [KM98], p.39). Let \( Y_1 \) be the quadratic singularity \( \{xy = uv\} \in \mathbb{C}^4 \). Denote by \( X_1 \) the variety obtained by blowing up the ideal \((x,v)\). Denote by \( X_1^+ \) the variety obtained by blowing up the ideal \((x,u)\). Let \( \mathbb{Z}/n\mathbb{Z} \) be the cyclic group of \( n \) elements, acting on \( Y_1 \) via \( (x,y,u,v) \mapsto (\zeta x, y, \zeta u, v) \), which lifts to an action on \( X_1 \) and on \( X_1^+ \). The corresponding quotients are denoted by \( X_n, Y_n, \) and \( X_n^+ \). Consider the diagram

\[
\begin{array}{ccc}
X_n & \rightarrow & X_n^+ \\
\downarrow & & \nearrow \\
Y_n & & \\
\end{array}
\]

Note that the Picard numbers satisfy \( \rho(X_n^+/Y_n) = \rho(X_n/Y_n) = 1 \) and the variety \( X_n^+ \) is smooth. Denote by \( C_n \) the exceptional curve in \( X_n \) and by \( C_n' \) the exceptional curve in \( X_n^+ \). A standard computation shows that

\[
C_n \cdot K_{X_n} = \frac{-(n-1)}{n},
\]

and

\[
C_n' \cdot K_{X_n^+} = n - 1.
\]

Thus \( X_n^+ \to Y_n \) is the flip of \( X_n \to Y_n \) when \( n \geq 2 \).

Definition 3.1.1. Let \( f : X \to Y \) be a flipping contraction between two quasi-projective threefolds. We call \( f : X \to Y \) a local Francia flipping contraction if \( f \) is étale locally isomorphic to some \( X_n \to Y_n \).

By “étale locally isomorphic to \( X_n \to Y_n \)” we mean that every point \( y \in Y \) has an étale neighborhood \( \phi : U \to Y \) with another étale morphism \( \psi : U \to Y_n \) and an isomorphism

\[
X \times_Y U \simeq X_n \times_{Y_n} U.
\]

This induces an isomorphism

\[
X^+ \times_Y U \simeq X_n^+ \times_{Y_n} U.
\]

Étale locally we have the following diagram

\[
\begin{array}{ccc}
X_1 & \rightarrow & X_1^+ \\
\downarrow & & \nearrow \\
[ X_1 / (\mathbb{Z}/n\mathbb{Z}) ] & & \left[ X_1^+ / (\mathbb{Z}/n\mathbb{Z}) \right] \\
\downarrow & & \downarrow \\
X_n & \rightarrow & X_n^+ \\
\downarrow & & \downarrow \\
[ Y_1 / (\mathbb{Z}/n\mathbb{Z}) ] & & X_n^+.
\end{array}
\]
Dropping the subscripts \( n \) and concentrating on the left side, we have

\[
\begin{align*}
\text{(1)} \\
X_1 &\xrightarrow{q} X' \\
\xrightarrow{\tau} X &\xrightarrow{\pi} X_1
\end{align*}
\]

The quotient morphism

\[ q: X_1 \rightarrow \left[ X_1 / (\mathbb{Z}/n\mathbb{Z}) \right] \]

is, by definition, étale. The morphism

\[ \sigma: \left[ X_1 / (\mathbb{Z}/n\mathbb{Z}) \right] \rightarrow X' \]

is the natural morphism “forgetting the stacky structure” along a \( \mathbb{Z}/n\mathbb{Z} \) divisor \( [D / (\mathbb{Z}/n\mathbb{Z})] \). The divisor \( D \subset X_1 \) is given, in the local coordinates introduced earlier, by the function \( \frac{x}{v} \) in the affine chart

\[ V = \text{Spec} \mathbb{C} \left[ \frac{x}{v}, y, v \right] . \]

We denote by \( D' \) the image of \( D \) in \( X \), so \( q'^* D' = nD \).

The exceptional curve in \( X_1 \) is given in this chart by \( y = v = 0 \).

Note that the morphism \( \sigma \) is flat since \( [X_1/(\mathbb{Z}/n\mathbb{Z})] \rightarrow X' \) is surjective, proper and quasi-finite, \( X' \) is smooth and \( [X_1/(\mathbb{Z}/n\mathbb{Z})] \) is Cohen-Macaulay (in fact, it is smooth). The coarse moduli space of \( [X_1/(\mathbb{Z}/n\mathbb{Z})] \) is also \( X \). We also have that \( \sigma_* \sigma^* = id \) on any quasi-coherent sheaf, since \( \sigma_* \mathcal{O}_{X_1/(\mathbb{Z}/n\mathbb{Z})} = \mathcal{O}_X \).

Consider now the other affine chart

\[ U = \text{Spec} \mathbb{C} \left[ x, u, \frac{v}{x} \right] \]

on \( X_1 \). The exceptional curve is defined by \( x = u = 0 \). Denote by \( E \) the divisor defined by \( \frac{v}{x} \) - it meets the exceptional curve properly. Again we write \( E' = q'_* E \), so \( q'^* E' = nE \).

On \( X_1 \) we have an equality of divisors

\[ \text{div} \left( \frac{v}{x} \right) = E - D. \]

The function \( \frac{v}{x} \) is a \( \mathbb{Z}/n\mathbb{Z} \)-semi-invariant with respect to the character

\[ \frac{v}{x} \mapsto \zeta^{-1} \cdot \frac{v}{x} . \]

Therefore we can identify the \( \mathbb{Z}/n\mathbb{Z} \)-sheaf \( L_{\chi_i} \), isomorphic to \( \mathcal{O}_{X_1} \) twisted with action given by a character \( \chi_i \) as

\[ L_{\chi_i} \cong \mathcal{O}_{X_1}(j(E - D)) \]
for suitable $i$.

3.2. A perversity for Francia flips. Let $X$ be as in Section 3.1. In this case the natural canonical covering stack $\mathcal{X}$ is a smooth Deligne-Mumford stack, whose coarse moduli space is $X$. Consider the natural morphism $\pi : \mathcal{X} \to X$. The stacky points of $\mathcal{X}$ are the preimage of points where $K_X$ fails to be Cartier. This loci consists of isolated points.

Let $f : \mathcal{X} \to Y$ be the morphism as above. We write $f = \bar{f} \pi$ with $\pi : \mathcal{X} \to X$ and $\bar{f} : X \to Y$.

Following [Br00], we can define for any $m$ a perverse $t$-structure $t(m)$ for the morphism of schemes $\bar{f} : X \to Y$. The heart of this $t$-structure is denoted by $\text{Per}^m(X/Y)$.

We proceed by defining a perverse $t$-structure for the stack $\mathcal{X}$. We define two sub categories of $D(\mathcal{X})$:

$$B = \{L\pi^*C \in D^b(\mathcal{X}) | C \in D(X)\};$$

and

$$C_2 = \{C \in D(\mathcal{X}) | R\pi_*C = 0\}.$$ 

The pair $(B, C_2)$ gives a semiorthogonal decomposition on $D(\mathcal{X})$.

On the category $C_2$, the standard $t$-structure induced by the standard $t$-structure on $D(X)$ gives a $t$-structure on $C_2$.

Since $R\pi_*$ has the right adjoint $\pi^!$ and the left adjoint $L\pi^*$, we can glue any $t$-structures on $C_2$ and $D(X)$. We define new $t$-structures $t(m, n)$ on $D(\mathcal{X})$:

**Definition 3.2.1.** Denote by $t(m, n)$ the $t$-structure obtained by gluing: the perverse $t$-structure $t(m)$ on $D(X)$, and the standard $t$-structure shifted by $n$ on $C_2$. We denote the heart of this $t$-structure by $\text{Per}^{m,n}(\mathcal{X}/Y)$.

Inspired by [Br00], we shall only be interested in the case that $(m, n) = (-1, 0)$ and denote $\text{Per}^{-1,0}(\mathcal{X}/Y)$ simply by $\text{Per}(\mathcal{X}/Y)$.

By definition, an object $E \in D(\mathcal{X})$ is in $\text{Per}(\mathcal{X}/Y)$ if and only if the following conditions are satisfied:

1. $R\pi_*E$ is a perverse sheaf for the morphism $\bar{f} : X \to Y$, and
2. (a) $\text{Hom}(E, C) = 0$ for all $C$ in $C_2^{>0}$, and
   (b) $\text{Hom}(D, E) = 0$ for all $D$ in $C_2^{<0}$.

**Definition 3.2.2.** A perverse sheaf $I$ is called a perverse ideal sheaf if there is an injection $I \hookrightarrow \mathcal{O}_X$ in the abelian category $\text{Per}(\mathcal{X}/Y)$. A perverse sheaf $E$ is called a perverse structure sheaf if there is a surjection $\mathcal{O}_X \to E$ in the abelian category $\text{Per}(\mathcal{X}/Y)$. A perverse point sheaf is a perverse structure sheaf which is numerically equivalent to the structure sheaf of a point $x \in \mathcal{X}$.

3.3. Characterization of perverse sheaves and perverse ideal sheaves. The next lemma is an analogue of Lemma 3.2 in [Br00] and can be proved in a similar manner.

**Lemma 3.3.1.** An object $E$ of $D(\mathcal{X})$ is a perverse sheaf if and only if the following four conditions are satisfied:

1. $H_i(E) = 0$ unless $i = 0$ or 1,
We continue following [Br00]:

**Lemma 3.3.2.** A perverse ideal sheaf is a sheaf.

**Proof.** Let \( F \) be a perverse ideal sheaf on \( X \) and \( E \) be the corresponding perverse structure sheaf. Consider the exact sequence in \( \text{Per}(X/Y) \)

\[
0 \to F \to \mathcal{O}_X \to \pi_*F \to 0.
\]

By Lemma 3.3.1 all the terms in this sequence have homology only in degrees 0 and 1, and \( H_1(\mathcal{O}_X) = 0 \). Applying the homology functor to this exact sequence, we get

\[
0 \to H_1F \to 0 \to H_1(E) \to \cdots
\]

so \( H_1(F) = 0 \).

**Lemma 3.3.3.** A sheaf \( F \in \text{Coh}(X) \) is a perverse ideal sheaf if and only if it satisfies the following two conditions:

\begin{enumerate}[(PIS.1)]
    
    \item \( R^i\pi_*F \) is a perverse ideal sheaf for \( \bar{f} : X \to Y \), and
    
    \item \( \text{Hom}(D, F) = 0 \) for any sheaf \( D \in C_2 \).
\end{enumerate}

In addition, a perverse ideal sheaf satisfies the following property:

\begin{enumerate}[(PIS.3)]
    
    \item the cokernel of the natural morphism \( f^*f_*(F) \to F \) is in the category \( C_2 \).
\end{enumerate}

**Proof.** Let \( F \in D(X) \) be a perverse ideal sheaf and \( E \) be the corresponding perverse point sheaf. Since \( \pi_* \) is exact on the abelian category \( \text{Per}(X/Y) \), there is an exact sequence in \( \text{Per}(X/Y) \)

\[
0 \to \pi_*F \to \mathcal{O}_X \to \pi_*E \to 0.
\]

It follows that \( \pi_*F \) is a perverse ideal sheaf for \( \bar{f} : X \to Y \). Since \( \pi_*F \) is a perverse ideal sheaf for \( \bar{f} : X \to Y \), it follows that \( f^*\bar{f}_*(\pi_*F) \to \pi_*F \) is surjective. Applying the right-exact functor \( \pi_* \), we have \( f^*f_*(F) \to \pi^*\pi_*F \) is surjective. Therefore, to show (PIS.3), it suffices to show that the cokernel \( D \) of \( \pi^*\pi_*F \to \pi_*F \) is in \( C_2 \). But \( \pi_*\pi^*\pi_*F \to \pi_*F \) is an isomorphism, therefore its cokernel \( \pi_*D \) vanishes.

To check the property (PIS.2), we use the condition (PS.4) on \( E \) and the exact sequence

\[
0 \to H_1(E) \to F \to \mathcal{O}_X \to H_0(E) \to 0.
\]

It follows that \( \text{Hom}(D, F) = 0 \) for all sheaf \( D \in C_2 \) since we have \( \text{Hom}(D, H_1(E)) = 0 \) and \( \text{Hom}(D, \mathcal{O}_X) = 0 \).

For the converse, consider a sheaf \( F \) satisfying (PIS.1) and (PIS.2). Then (PS.1) is automatic as \( H_i(F) = 0 \) for \( i \neq 0 \). Since \( \pi_* \) is exact we have \( R^i\pi_*F = R^i\bar{f}_*(\pi_*F) \), and (PS.2) follows since \( \pi_*F \) is in \( \text{Per}(X/Y) \). For the same reason (PS.3) holds. And (PS.4) is exactly (PIS.2). Thus \( F \) is a perverse sheaf.

We now show that \( \text{Hom}(F, \mathcal{O}) = \text{Hom}(\pi_*F, \mathcal{O}) \), hence nonzero. Consider the exact sequence of sheaves

\[
0 \to A \to \pi^*\pi_*F \to F \to D \to 0,
\]
and denote the image of \( \eta : \pi^* \pi_* F \to F \) by \( C \). Since \( \eta \) is an isomorphism away from the singular locus of \( X \), the sheaf \( A \) is torsion, therefore \( \text{Hom}(C, \mathcal{O}) \subset \text{Hom}(\pi^* \pi_* F, \mathcal{O}) \). Also the sheaf \( D \) is torsion, and we have an exact sequence
\[
0 \to \text{Hom}(F, \mathcal{O}) \to \text{Hom}(C, \mathcal{O}) \to \text{Ext}^1(D, \mathcal{O}),
\]
but since \( D \) is supported in dimension 0,
\[
\text{Ext}^1(D, \mathcal{O}) = H^2(D \otimes \omega_X)^\vee = 0.
\]

Fix a nonzero element in \( \text{Hom}(F, \mathcal{O}) \) and consider the triangle \( F \to \mathcal{O} \to E \to F[1] \). It suffices to show that \( E \) is perverse. The long exact sequence of homology (sequence 2 above) gives (PS.1). Clearly \( \mathbf{R} \pi_* E \) is the perverse quotient of the corresponding element of \( \text{Hom}(\pi_* F, \mathcal{O}) \), so (PS.2) and (PS.3) follow. And (PS.4) follows again because \( \text{Hom}(D, H_1(E)) \subset \text{Hom}(D, F) = 0 \) by (PS.2). \( \square \)

Since \( X \to Y \) is an isomorphism outside the singular points of \( Y \), we can compactify \( X \) and \( Y \) to get \( \bar{f} : \bar{X} \to \bar{Y} \). It is clear from the definition that perverse point sheaves are local objects over \( \bar{Y} \), i.e. \( W(\bar{X}/\bar{Y})_\gamma \cong W(X/Y) \). Abusing the notation, we shall still denote this new morphism by \( f : X \to Y \).

### 3.4. Simplicity of perverse point and perverse point-ideal sheaves.

**Lemma 3.4.1.** Let \( E_1 \) and \( E_2 \) be two perverse point sheaves. Then,
\[
\text{Hom}(E_1, E_2) = \begin{cases} 
0 & : \text{if } E_1 \not\cong E_2, \\
\mathbb{C} & : \text{if } E_1 \cong E_2.
\end{cases}
\]

**Proof.** See Lemma 3.6 in [Br00], using the following:

**Lemma 3.4.2.** Let \( E \in D^b(X/Y) \) be a perverse sheaf with \( \text{Supp}(E) \subset f^{-1}(y) \) for a point \( y \in Y \). If \( E \) is numerically equivalent to 0, then \( E \cong 0 \).

**Proof.** This follows immediately from the following Lemma 3.4.3 (which is implicit in [Br00]) and property (PS.4) in Lemma 3.4.1. \( \square \)

**Lemma 3.4.3.** Let \( E \in D^b(X/Y) \) be a perverse sheaf with \( \text{Supp}(E) \subset f^{-1}(y) \) for a point \( y \in Y \). If \( E \) is numerically equivalent to 0, then \( E \cong 0 \).

**Proof.** Let \( E \) be a perverse sheaf which is numerically equivalent to 0. It suffices to show that all homology groups vanish, that is \( H_0(E) = H_1(E) = 0 \).

First note that \( \chi(H_0(E)) = \chi(H_1(E)) \). Since \( E \) is a perverse sheaf, we have \( \mathbf{R}^1 \bar{f}_*(H_0(E)) = \bar{f}_*(H_1(E)) = 0 \). Therefore, \( \chi(H_0(E)) \) is a nonnegative integer and \( \chi(H_1(E)) \) is a nonpositive integer. This implies \( \chi(H_0(E)) = \chi(H_1(E)) = 0 \). By the support assumption, \( f_*(H_0(E)) \) is supported at a point, and this implies that \( H^0 \bar{f}_*(H_0(E)) = 0 \), which means \( H_0(E) = 0 \). The same holds for \( \mathbf{R}^1 \bar{f}_*(H_1(E)) \).

We thus have \( \mathbf{R} \bar{f}_*(H_1(E)) = 0 \) and therefore the two sheaves \( H_1(E) \) have support in pure dimension 1. For \( H_0(E) \) we have \( \text{Hom}(H_0(E), H_0(E)) = 0 \) by (PS.3) in Lemma 3.2 in [Br00]. This implies \( H_0(E) \cong 0 \). Therefore \( H_1(E) \) is a numerically trivial sheaf.

Now consider \( L \) a sufficiently ample bundle. Then \( H^1(L \otimes H_1(E)) = 0 \) and \( L \otimes H_1(E) \) is generated by global sections, therefore \( \chi(L \otimes H_1(E)) > 0 \), so \( H_1(E) \) is not numerically trivial, giving a contradiction. \( \square \)
Lemma 3.4.4.  
(1) Let $F_1$ be a perverse point-ideal sheaf. Then $\dim \text{Hom}(F_1, \mathcal{O}_X) = 1$.
(2) Let $F_1$ and $F_2$ be perverse point-ideal sheaves. Then

$$\text{Hom}(F_1, F_2) = \begin{cases} 0 & \text{if } F_1 \not\cong F_2, \\ \mathbb{C} & \text{if } F_1 \cong F_2. \end{cases}$$

Proof. Let $F_1$ be a perverse point-ideal sheaf. Consider the following exact sequence (in the usual abelian category)

$$0 \to A \to f^* f_* F_1 \to F_1 \to C \to 0$$

where $C$ is an object in $C_2$ since $\pi_* F_1$ is a perverse point-ideal sheaf for $\bar{f}: X \to Y$. It is also clear that $C$ is a torsion sheaf. Taking $\text{Hom}(\cdot, \mathcal{O}_X)$, we get

$$0 \to \text{Hom}(F_1, \mathcal{O}_X) \to \text{Hom}(f^* f_* F_1, \mathcal{O}_X).$$

Since

$$\dim(\text{Hom}(f^* f_* F_1, \mathcal{O}_X)) = \dim(\text{Hom}(f_* F_1, f_* \mathcal{O}_X)) = 1,$$

this shows that $\dim(\text{Hom}(F_1, \mathcal{O}_X)) = 1$.

Note that by (PIS.2) we have $\text{Hom}(C, F_2) = 0$ for any sheaf $C \in C_2$. Taking $\text{Hom}(\cdot, F_2)$, we get

$$0 \to \text{Hom}(F_1, F_2) \to \text{Hom}(f^* f_* (F_1), F_2).$$

Since $\text{Hom}(f^* f_* (F_1), F_2) = \text{Hom}(f_* (F_1), f_* (F_2))$, which has dimension $\leq 1$, it follows that $\dim(\text{Hom}(F_1, F_2)) \leq 1$, and if the dimension is 1 then the map factors $\text{Hom}(F_1, \mathcal{O})$. Since we have $F_1 \to \mathcal{O}$ is an injection in $\text{Per}(X/Y)$, it follows that $\theta: F_1 \to F_2$ is also an injection in $\text{Per}(X/Y)$. The cokernel of $\theta$ is in $\text{Per}(X/Y)$ and numerically equivalent to 0. Therefore it is isomorphic to 0 by Lemma 3.4.2.

3.5. Perversity and the dualizing sheaf.

Lemma 3.5.1. If $B$ is a sheaf on $X$ such that $B[1]$ is a perverse sheaf, then $f_* (B \otimes \omega) = 0$

Remark. The idea is that $\omega_X$ is negative along the exceptional curve, so tensoring by $\omega_X$ should not give more sections. This is not quite correct as it is - it fails for torsion sheaves supported at the non-representable point, so we need to be a bit careful and use the structure of the sheaf $B$.

Proof. We have the properties:

- $f_* B = 0$,
- $B$ has pure support on the exceptional loci.

This in particular implies that the problem is étale local over $Y$. Thus we may assume that we have a diagram as in Diagram 1, Section 3.1.

For any $\mathbb{Z}/n\mathbb{Z}$-sheaf $F$ on $X_1$ we have an isotypical decomposition

$$q'_* F = \bigoplus_{\chi_i: \mathbb{Z}/n\mathbb{Z} \to \mathbb{G}_m} (q'_* F)^{\chi_i}.$$

If $F$ is supported along the exceptional locus then there is an isomorphism $F \subset F(iE - iD)$ given locally by multiplication by a power of $q'$. In particular, for $i > 0$ we get a $\chi_i$-twisted embedding $F \subset F(nE - D)$. Thus, for any nontrivial character $\chi_i$ we can write

$$(q'_* F)^{\chi_i} \subset (q'_* F(nE - D))^\mathbb{Z}/n\mathbb{Z}.$$
We now analyze the sheaf $q'_*\tau^*(B \otimes \omega_{X_1})$. First notice that
\[
\tau^*(B \otimes \omega_{X_1}) \subset \tau^*B \otimes \omega_{X_1}(-(n-1)D) \subset \tau^*B \otimes \omega_{X_1},
\]
which is isomorphic to $\tau^*B$ as a sheaf, but with a different action given by twisting with a character $\chi$. This twisting only has the effect of permuting the $\chi_i$-isotypical components of the direct image by $q'$.

We thus get
\[
q'_*\tau^*(B \otimes \omega_{X_1}) = q'_*(\tau^*B(-(n-1)D)).
\]
Note also that $nE - nD$ is the pullback of the principal divisor $E' - D'$ on $X_n$.

Applying the discussion above we get
\[
q'_*(\tau^*B(-(n-1)D)) \hookrightarrow (\pi_*B \oplus \bigoplus_{\chi \neq 1} (q'_*(\tau^*B(nE - nD)))) \mathbb{Z}/n\mathbb{Z}.
\]
In particular we get an embedding
\[
\pi_*(B \otimes \omega) \hookrightarrow (\pi_*B)^{\oplus n}.
\]
Applying the left exact functor $\bar{f}_*$ we get
\[
f_*(B \otimes \omega) \hookrightarrow (f_*B)^{\oplus n} = 0.
\]

Lemma 3.5.2. The functor $E \mapsto E \otimes \omega$ is perverse-left exact, i.e. sends $\text{Per}(\mathcal{X}/Y)$ to $D^{\geq 0}(\mathcal{X})$.

We use the next lemma to prove Lemma 3.5.2.

Lemma 3.5.3. Suppose $B$ is a sheaf on $\mathcal{X}$ such that $B[1]$ is perverse. Then $B \otimes \omega[1]$ is also perverse.

Proof. Let $\pi : \mathcal{X} \to X$ and $f : \mathcal{X} \to Y$ be as above. First note that $f_*B = 0$ and $\text{Hom}(C, B) = 0$ for $C$ a sheaf in $C_2$, so $B$ contains no sky-scraper sheaves. Hence the same is true for $B \otimes \omega$.

We use Lemma 3.3.1. The only non-trivial conditions are the second part of (PS.2) and (PS.4). The second part of (PS.2) follows from Lemma 3.5.1. For (PS.4), take an object $D$ such that $\pi_*D = 0$. Since $X$ has only isolated singularities, any such $D$ must be supported in dimension 0. Note that $H_1(B \otimes \omega[1]) = B \otimes \omega$. Thus we have $\text{Hom}(D, B \otimes \omega) = 0$ since $B \otimes \omega$ contains no sky-scraper sheaves.

Proof of Lemma 3.5.2. This is an easy consequence of the above lemma. Assume the contrary. We can take a perverse sheaf $E$ and another object $D$ such that $\text{Hom}(D[1], E \otimes \omega)$ is nonzero. Taking homology in the standard t-structure, there must be a non-zero map $H_0(D) \to H_1(E) \otimes \omega = C$. Since $H_1(E)[1]$ is a perverse sheaf, it follows that $C[1]$ is also
a perverse sheaf by Lemma 3.5.3. The sheaf $H_0(D)$ is also a perverse sheaf since $D$ is a perverse sheaf. This then gives a homomorphism in $\text{Hom}(H_0(D), C)$, a contradiction. □

**Lemma 3.5.4.** Let $E_i$, $i = 1, 2$, be perverse point sheaves. Then $\text{Hom}^i(E_1, E_2) = 0$ unless $0 \leq i \leq 3$.

**Proof.**
Since $E_i$ are objects in the abelian category $\text{Per}(\mathcal{X}/Y)$, we have that $\text{Hom}^i(E_1, E_2) = 0$ for $i < 0$.

If $R{f}_*(E_1) \neq R{f}_*(E_2)$, then $\text{Hom}^i(E_1, E_2) = 0$ for all $i$ since their supports are disjoint, so we only need to consider the case $R{f}_*(E_1) = R{f}_*(E_2)$, and thus the problem is local over $Y$.

Replacing $Y$ by an étale base change we may assume that $\mathcal{X}$ is a global quotient stack, and thus $E_i$ are quasi isomorphic to bounded complexes of vector bundles. Serre duality gives

$$\text{Hom}^i(E_1, E_2) = \text{Hom}^{3-i}(E_2, E_1 \otimes \omega)^\vee.$$ 

This gives the result. □

4. **Moduli of perverse point sheaves for the Francia flips**

4.1. **The moduli space.** A family of perverse point sheaves for $\mathcal{X}/Y$ parametrized by a scheme $S$ is an object $\mathcal{E}$ of $D^b(S \times \mathcal{X})$ such that for every point $i : s \in S$ the fiber $L^i\mathcal{E}$ is a perverse point sheaf. Two such families $\mathcal{E}, \mathcal{E}'$ are equivalent if there is a line bundle $M$ on $S$ and an isomorphism $\mathcal{E} \simeq \mathcal{E}' \otimes M$.

Define a functor 

$$\mathcal{M}(\mathcal{Y}/Y) : \text{Sch} \rightarrow \text{Sets}$$

which to a scheme $S$ assigns the set of equivalence classes of families of perverse point sheaves parametrized by $S$.

Since every perverse point sheaf determines and is determined by a perverse point-ideal sheaf, then we can view the functor $\mathcal{M}(\mathcal{X}/Y)$ as the moduli functor of equivalence classes of perverse point-ideal sheaves. By Lemma 3.4.4 the endomorphism group of such a sheaf is $\mathbb{C}$, and the automorphism group is therefore $\mathbb{C}^*$. It follows by standard representability theory (see, e.g., [Ar74]) that the étale sheaf associated to $\mathcal{M}(\mathcal{X}/Y)$ is represented by an algebraic space $M(\mathcal{X}/Y)$, locally of finite type over $\mathbb{C}$. An argument of Mukai (see, e.g., [Br00], proof of Theorem 5.5) shows that $M(\mathcal{X}/Y)$ is a fine moduli space for $\mathcal{M}(\mathcal{X}/Y)$, i.e. there is a universal perverse point-ideal sheaf over $M(\mathcal{X}/Y) \times \mathcal{X}$.

**Lemma 4.1.1.** The algebraic space $M(\mathcal{X}/Y)$ is separated.

**Proof.** We use the valuative criterion for separation. Let $R$ be a discrete valuation ring, with fraction field $K$. Fix a perverse ideal sheaf $F$ over $\text{Spec}(K)$. Let $F_1$ and $F_2$ be two extensions of $F$ to $\text{Spec}(R)$ with an isomorphism $s : F_1|_{\text{Spec}(K)} \cong F_2|_{\text{Spec}(K)}$. We can write it as $s = u^n \cdot h$, where $u$ is a uniformizer in $R$ and $h$ extends comes from a homomorphism $F_1 \rightarrow F_2$. Taking $n$ minimal, we may assume that the restriction of $\tilde{h} : F_{1s} \rightarrow F_{2s}$ of $h \in \text{Hom}(F_1, F_2)$, to the special fiber is nonzero. By Lemma 3.4.4 we have that $F_1 \cong F_2$ over $\text{Spec}(R)$. □
4.2. Projectivity. There is a distinguished component of $M(\mathcal{X}/Y)$, which is birational to $Y$. We shall denote this component by $W$. To complete Theorem 4.1.4, we need to show that the distinguished component $W$ is projective over $Y$.

Note that it suffices to consider the case that $Y \cong Y_n$ and $X \cong X_n$. By Lemma 4.2.1 below, there is a birational morphism $X^+_n \to W$; and the composition $X^+_n \to W \to W(X_n/Y_n)$ is a finite morphism, where $W(X_n/Y_n)$ is the moduli space for the usual perverse point sheaves. This implies that $W \to W(X_n/Y_n)$ is finite. Since $W(X_n/Y_n) \to Y_n$ is projective, it follows immediately that $W \to Y_n$ is also projective.

**Lemma 4.2.1.** There exists a birational morphism $X^+_n \to W$. The composition $X^+_n \to W \to W(X/Y)$ is finite.

**Proof.** To give a morphism $X^+_n \to W$, we exhibit a family of perverse point sheaves over $X^+_n$. Unlike the terminal Gorenstein case, the correct family of perverse point sheaves is not the fiber product $X^+_n \times_{Y_n} \mathcal{X}$. This fiber product has an extra embedded component. We shall show that the correct candidate is the structure sheaf of the reduction of the fiber product. We denote it by $(X^+_n \times_{Y_n} \mathcal{X}_n)$.

To show this is a family of perverse point sheaves, we use Lemma 3.3.3 and check the conditions (PIS) for the corresponding perverse ideal sheaves.

Consider the morphism $id \times \pi : X^+_n \times_{Y_n} \mathcal{X}_n \to X^+_n \times_{Y_n} X_n$. Condition (PIS) amounts to checking that $(id \times \pi)_! I_n(\mathcal{X}_n)$ is the ideal sheaf $I_n(\mathcal{X}_n)$ of the fibered product. In [R-C], Proposition 1.3.2, it is shown that this is indeed a perverse ideal sheaf.

To check the condition (PIS), we consider the flat base change $p : X_1 \to \mathcal{X}$. We have a natural morphism $p_n : X_1 \times_{Y_1} X^+_1 \to X_1 \times X^+_n$. Denote by $Z$ the image of this morphism in $X_1 \times X^+_n$. The morphism $p_n$ factors through

$$X_1 \times_{Y_1} X^+_1 \to X_1 \times_{Y_n} X^+_n \hookrightarrow X_1 \times X^+_n.$$ 

The first morphism is finite and the second morphism is a closed embedding. Therefore the composite is a finite morphism. Note also that this morphism is birational, since the projections to $X_1$ are birational.

**Lemma 4.2.2.** The morphism $p_n : X_1 \times_{Y_1} X^+_1 \to Z$ is an isomorphism.

**Proof.** It suffices to check that the image $Z$ in $X_1 \times X^+_n$ is normal by Zariski’s main theorem. Indeed, we shall prove that $Z$ is smooth. To check this, we do explicit computations on affine charts.

The variety $Y_1$ is given by $\{xy - uv = 0\} \subset \mathbb{C}^4$. The variety $X^+_1$ is obtained by blowing up the ideal $(x, u)$, and the variety $X_1$ is obtained by blowing up $(x, v)$. The variety $X^+_1$ can be cover by two affine pieces $U^+_{u \neq 0}$ and $U^+_{x \neq 0}$. Similarly, the variety $X_1$ can be cover by two affine pieces $U^-_{v \neq 0}$ and $U^-_{x \neq 0}$. The variety $X_1 \times X^+_1$ can be covered by four affine charts.

We first consider the affine chart $U^+_{u \neq 0}$ in $X^+_1$ and $U^-_{v \neq 0}$ in $X_1$. The structure sheaf $\mathcal{O}_{Y_1}$ is generated by $\{z = x^* u^{n-1}, y, v\}$. The structure sheaf $\mathcal{O}_1$ is generated by $\{y_-, v_-, s_+ = x_-/v_-\}$. The structure sheaf $\mathcal{O}_{X^+_1}$ is generated by $\{t_+, y_+, u^+_n\}$. The ideal of the fiber product $X^+_n \times_{Y_n} X_1$ is $X^+_n \times X_1$ is generated by $\{t_+ y_+ - v_-, y_-, t_+ z_+ - s^+ v^+_i y_{n-i}^\pm\}$. An easy calculation shows that this ideal is generated by $\{t_+ y_+ -
The variety defined by this ideal is smooth, and it is isomorphic to \( Z \) in this affine chart.

On the second chart \( U^+_n \neq 0 \) in \( X^+_1 \) and \( U^-_x \neq 0 \) in \( X_1 \). The ring \( O_{X^+_n} \) is generated by \( \{ t^+, y^+, u^+_n \} \) as above. The ring \( O^-_1 \) is generated by \( \{ u_-, x_-, s_- = v_-/x_- \} \). The ideal of \( Z \) is generated by \( \{ t^+_y + s_- x_-, s_- u_+ - y_+, t^+_n u^n - x^n \} \). The fiber product is not smooth, but \( Z \) is a subscheme of \( X_1 \times Y \times X^+_n \) and it is still smooth, since we get the ideal of the image of \( \{ X_1 \times Y \times X^+_n \} \) in \( X_1 \times X^+_n \) adding the element \( t^+_n u_- - x_- \) to the ideal of the fiber product \( X_1 \times Y \times X^+_n \).

For the affine chart \( U^+_x \neq 0 \) in \( X^+_1 \) and \( U^-_y \neq 0 \) in \( X_1 \), it is easy to check the variety is smooth. For the affine chart \( U^+_x \neq 0 \) in \( X^+_1 \) and \( U^-_x \neq 0 \) in \( X_1 \), one can write down a similar generators and equations. The ideal of the fiber product is generated by \( \{ v_+ - s_- x_-, t^+_y v_+ - s_- u_-, t^+_n u^n - x^n \} \). It is also not smooth. This time we need to add the element \( z_+ - x^n \) to get the ideal of the image \( Z \).

**Lemma 4.2.3.** The ideal sheaf \( I_{(X^+_n \times Y \times X^+_n)} \) is flat over \( O_{X^+_n} \).

**Proof** We can check this after pullback by the flat morphism \( X_1 \to X_n \). This amounts to checking that the ideal \( I_Z \) is flat over \( O_{X^+_n} \).

First note that \( I_{X^+_1 \times Y \times X^+_1} \) is flat over \( O_{X^+_1} \).

To check \( I_Z \) is flat over \( O_{X^+_n} \), it suffices to show that

\[
Tor^2_{O_{X^+_n}}(O_Z, M) = 0
\]

for all \( M \). Since

\[
Tor^2_{O_{X^+_1}}(O_Z, M) = 0,
\]

the desired result would follow if we can show \( O_{X^+_1} \) is flat over \( O_{X^+_n} \). This holds since \( X^+_n \) is smooth and \( X^+_1 \) is Cohen Macaulay. \( \square \)

Now we check the condition (PIS2) for the ideal sheaf \( I_{(X^+_n \times Y \times X^+_n)} \). Consider the natural group of \( Z/nZ \) on \( X_1 \). There is a group action of \( Z/nZ \) on \( X_1 \times X^+_n \). The group \( Z/nZ \) acts on \( X_1 \) via the natural action, and acts trivially on \( X^+_n \). Consider the morphism \( \tau : X_1 \to X \) (see diagram \( \square \)). Let \( F \) be a sheaf on \( X \). To check

\[
\text{Hom}(C, F) = 0
\]

for all \( C \in D(X) \) satisfying \( \tau^* C = 0 \), it suffices to check that \( \tau^* F \) has no sections supported on the preimage of stacky points of \( X \).

Take the family of sheaves \( I_Z \) over the variety \( X^+_n \). We have the exact sequence

\[
0 \to I_Z \to O_{X_1 \times X^+_n} \to O_Z \to 0.
\]

Let \( i : p \to X^+_n \) be a point of \( X^+_n \). Tensoring the exact sequence with the sheaf \( O_p = O_{X^+_n}/m_p \), we get the following

\[
0 \to Tor^1_{O_{X^+_n}}(O_p, O_Z) \to i^* I_Z \to O_{X_p} \to O_{i^* Z} \to 0.
\]

It is clear that the image of \( i^* I_Z \) in \( O_{X_p} \) is torsion free on the fiber. To show that \( i^* I_Z \) has torsion with support in pure dimension 1, it suffices to show that the sheaf \( Tor^1_{O_{X^+_n}}(O_p, O_Z) \) has support in pure dimension 1. To achieve this, we use the change of ring formula

\[
Tor^1_{O_{X^+_n}}(O_{X^+_n}/m_p, O_Z) = Tor^1_{O_{X^+_n}}(O_{X^+_n}/m_p O_{X^+_n}, O_Z).
\]
Since $Z$ is the universal perverse point sheaf for the morphism $X_1 \to Y_1$, it can be directly checked that the sheaf $\text{Tor}^1_{\mathcal{O}}(\mathcal{O}_{X_1^+}/m_p\mathcal{O}_{X_1^+}, \mathcal{O}_Z)$ has support of pure dimension 1. It follows that the sheaf $\text{Tor}^1_{\mathcal{O}}(\mathcal{O}_{X_1^+}/m_p, \mathcal{O}_Z)$ has support of pure dimension 1. (Alternatively, the automorphism group of $Z \to X_1^+$ acts transitively on the fiber, and the sheaf cannot have sections with support in a point which is not fixed.)

Finally we verify that the composite morphism $X^+ \to W(X/Y)$ is finite. This composition is given by the sheaf $\mathcal{O}_{X^+\times Y, X}$, which is a perverse sheaf for $X/Y$ by [N-C], Proposition 1.3.2. The same proposition shows that $X^+ \to W(X/Y)$ is the normalization, which is therefore finite. This completes the proof of Lemma 4.2.1.

4.3. **Proof of Theorem 1.4.2.** Since, by Lemma 4.2.1, we have a finite birational morphism $X^+ \to W$, once we show $W$ is smooth it follows that the morphism is an isomorphism. We prove that $W$ is smooth and the Fourier–Mukai transform is fully faithful at the same time, following the method of [BKR99].

We denote the functor induced by the universal perverse point sheaf

$$\Phi : D(W) \to D(X).$$

As in [BKR99], $\Phi$ has a left adjoint $\Psi$ and the composition $\Psi \circ \Phi$ is defined by a sheaf $Q$ on $W \times W$, which is supported in $W \times_Y W$. We consider the restriction of $Q$ to the complement of the diagonal. The support of $Q$ in this open set has dimension 2, since the fibers of $W \to Y$ are images of the fibers of $X^+ \to Y$ which have dimension $\leq 1$. Also, by Lemma 3.5.4, the homological dimension of $Q$ on the open set is $\leq 3$. By the intersection theorem ([BKR99], Corollary 5.2) it follows that $Q$ vanishes outside the diagonal. The argument of [BKR99] Section 6, step 5-6 shows that $W$ is nonsingular and $Q$ is a line bundle on the diagonal, which implies that $\Phi$ is fully faithful. The proof is complete.

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Department of Mathematics, Boston University, 111 Cummington, Boston, MA 02215, U.S.A.
Current address: Department of Mathematics, Box 1917, Brown University, Providence, RI, 02912
E-mail address: abrmovic@math.bu.edu, abrmovic@math.brown.edu

Department of Mathematics, Harvard University, 1 Oxford, Cambridge, MA 02138, U.S.A.
E-mail address: jcchen@math.harvard.edu