Wavelet analysis as a $p$–adic spectral analysis

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Abstract

New orthonormal basis of eigenfunctions for the Vladimirov operator of $p$–adic fractional derivation is constructed. The map of $p$–adic numbers onto real numbers ($p$–adic change of variable) is considered. $p$–Adic change of variable maps the Haar measure on $p$–adic numbers onto the Lebesgue measure on the positive semiline. $p$–Adic change of variable (for $p = 2$) provides an equivalence between the constructed basis of eigenfunctions of the Vladimirov operator and the wavelet basis in $L^2(\mathbb{R}_+)$ generated from the Haar wavelet. This means that the wavelet analysis can be considered as a $p$–adic spectral analysis.

1 Introduction

In the present paper we construct a new orthonormal basis of eigenfunctions of the Vladimirov operator of $p$–adic fractional derivation. The example of such a basis one can find in [1]. Different basises of eigenvectors of the Vladimirov operator were built in [2], [3], [4]. The basis constructed in the present paper consists of locally constant functions with support on $p$–adic discs.

We also check that the constructed basis of eigenfunctions of the Vladimirov operator (for $p = 2$) is equivalent to the wavelet basis in $L^2(\mathbb{R}_+)$ generated from the Haar wavelet. This equivalence is given by $p$–adic change of variables: the map of $p$–adic numbers onto real numbers that conserves the measure. This means that the wavelet analysis can be considered as a $p$–adic
harmonic analysis (decomposition of functions over the eigenfunctions of the Vladimirov operator of $p$–adic fractional derivation).

For introduction to $p$-adic analysis see [1]. $p$-Adic analysis and $p$-adic mathematical physics attract great interest, see [1], [3], [4]. For instance, $p$-adic models in string theory were introduced, see [4], [8], and $p$-adic quantum mechanics [1] and $p$-adic quantum gravity [10] were investigated. $p$–Adic analysis was applied to investigate the spontaneous breaking of the replica symmetry, cf. [11], [12], [13], [14].

Let us make here a brief review of $p$-adic analysis. The field $\mathbb{Q}_p$ of $p$-adic numbers is the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm on $\mathbb{Q}$. This norm is defined in the following way. An arbitrary rational number $x$ can be written in the form $x = p^\gamma \frac{m}{n}$ with $m$ and $n$ not divisible by $p$. The $p$-adic norm of the rational number $x = p^\gamma \frac{m}{n}$ is equal to $|x|_p = p^{-\gamma}$.

The most interesting property of the field of $p$-adic numbers is ultrametricity. This means that $\mathbb{Q}_p$ obeys the strong triangle inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

We will consider disks in $\mathbb{Q}_p$ of the form $\{ x \in \mathbb{Q}_p : |x - x_0|_p \leq p^{-k} \}$. For example, the ring $\mathbb{Z}_p$ of integer $p$-adic numbers is the disk $\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}$ which is the completion of integers with the $p$-adic norm. The main properties of disks in arbitrary ultrametric space are the following:

1. Every point of a disk is the center of this disk.
2. Two disks either do not intersect or one of these disks contains the other.

The $p$-adic Fourier transform $F$ of the function $f(x)$ is defined as follows

$$F[f](\xi) = \tilde{f}(\xi) = \int_{\mathbb{Q}_p} \chi(\xi x) f(x) d\mu(x)$$

Here $d\mu(x)$ is the Haar measure. The inverse Fourier transform have the form

$$F^{-1}[\tilde{g}](x) = \int_{\mathbb{Q}_p} \chi(-\xi x) \tilde{g}(\xi) d\mu(\xi)$$

Here $\chi(\xi x) = \exp(2\pi i \xi x)$ is the character of the field of $p$-adic numbers. For example, the Fourier transform of the indicator function $\Omega(x)$ of the disk of
radius 1 with center in zero (this is a function that equals to 1 on the disk and to 0 outside the disk) is the function of the same type:

\[ \tilde{\Omega}(\xi) = \Omega(\xi) \]

In the present paper we use the following Vladimirov operator \( D_\alpha^x \) of the fractional \( p \)-adic differentiation, that is defined [1] as

\[
D_\alpha^x f(x) = F^{-1} \circ |\xi|^\alpha_p \circ F[f](x) = \frac{p^\alpha - 1}{1 - p^{1-\alpha}} \int_{Q_p} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}_p} d\mu(y) \quad (1)
\]

Here \( F \) is the (\( p \)-adic) Fourier transform, the second equality holds for \( \alpha > 0 \).

For further reading on the subject of \( p \)-adic analysis see [1].

2 \( p \)-Adic spectral analysis

Let us construct an orthonormal basis of eigenvectors of the Vladimirov operator. The following lemma gives the basic technical result.

**Lemma 1.** The function

\[
\psi(x) = \chi(p^{-1}x) \Omega \left( |x|_p \right)
\]

is an eigenvector of the Vladimirov operator:

\[
D^\alpha \psi(x) = p^\alpha \psi(x)
\]

**Remark.** The eigenvalue for \( \psi(x) \) is the same as for the character \( \chi(p^{-1}x) \):

\[
D^\alpha \chi(p^{-1}x) = p^\alpha \chi(p^{-1}x)
\]

**Proof** Let us check that \( \psi(x) \) is an eigenvector of the Vladimirov operator. We get

\[
D^\alpha \psi(x) = \frac{p^\alpha - 1}{1 - p^{1-\alpha}} \int_{Q_p} \frac{\psi(x) - \psi(y)}{|x - y|^{1+\alpha}_p} d\mu(y) = \]

3
\[
\frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \int \frac{\chi(p^{-1}x)\Omega(|x|_p) - \chi(p^{-1}y)\Omega(|y|_p)}{|x - y|_p^{1+\alpha}} d\mu(y) = \\
= \chi(p^{-1}x) \frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \int \frac{\Omega(|x|_p) - \chi(p^{-1}(y - x))\Omega(|y|_p)}{|x - y|_p^{1+\alpha}} d\mu(y) \quad (3)
\]

Let us calculate the integral over \(y\) in (3) in two cases.

1) Let \(|x|_p \leq 1\). In this case the integral in (3) is given by

\[
\int \frac{1 - \chi(p^{-1}(y - x))\Omega(|y|_p)}{|x - y|_p^{1+\alpha}} d\mu(y)
\]

Using that every point of \(p\)-adic disk is its center we get that for \(|x|_p \leq 1\) we have \(\Omega(|y|_p) = \Omega(|x - y|_p)\) and therefore the integral is equal to

\[
\int \frac{1 - \chi(p^{-1}z)\Omega(|z|_p)}{|z|_p^{1+\alpha}} d\mu(z) = \int \frac{1 - \chi(p^{-1}z)\Omega(|z|_p)}{|z|_p^{1+\alpha}} d\mu(z)
\]

2) Let \(|x|_p > 1\). For the integral in (3) we get

\[
-\frac{1}{|x|_p^{1+\alpha}} \int_{|y|_p \leq 1} \chi(p^{-1}(y - x)) d\mu(y) = 0
\]

This proves that \(\psi(x)\) is an eigenvector of the Vladimirov operator with the following eigenvalue

\[
D^\alpha \psi(x) = \psi(x) \frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \int \frac{1 - \chi(p^{-1}z)\Omega(|z|_p)}{|z|_p^{1+\alpha}} d\mu(z) = \\
= \psi(x) \frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \left( p^{-1} \sum_{i=0}^{p-1} (1 - \chi(p^{-1}i)) + (1 - p^{-1}) \sum_{\gamma=1}^{\infty} p^\gamma p^{-(1+\alpha)\gamma} \right) = p^\alpha \psi(x)
\]

that finishes the proof of the lemma.

The lemma implies

\[
D^\alpha \psi(ax + b) = |a|_p^\alpha p^\alpha \psi(ax + b)
\]

(4)

One can check that the set of (integrable) functions \(\psi(ax + b)\) (for different \(a, b\)) is a complete system in a Hilbert space \(L^2(Q_p)\). Moreover,
**Theorem 2.** The set of functions \( \{ \psi_{\gamma jn} \} \):

\[
\psi_{\gamma jn}(x) = p^{-\frac{\gamma}{2}} \chi(p^{\gamma - 1} j x) \Omega(|p^\gamma x - n|_p), \quad \gamma \in \mathbb{Z}, \quad n \in Q_p/Z_p, \quad j = 1, \ldots, p-1
\]

is an orthonormal basis in \( L^2(Q_p) \) of eigenvectors of the operator \( D^\alpha \):

\[
D^\alpha \psi_{\gamma jn} = p^{\alpha(1-\gamma)} \psi_{\gamma jn}
\]

The group \( Q_p/Z_p \) in (5) is parameterized by

\[
n = \sum_{i=1}^{m} n_i p^{-i}, \quad n_i = 0, \ldots, p-1
\]

**Proof** Consider the scalar product

\[
\langle \psi_{\gamma jn}, \psi_{\gamma' j'n'} \rangle =
\]

\[
= \int_{Q_p} p^{-\frac{\gamma}{2}} \chi(-p^{\gamma - 1} j x) \Omega(|p^\gamma x - n|_p) p^{-\frac{\gamma'}{2}} \chi(p^{\gamma' - 1} j' x) \Omega(|p^{\gamma'} x - n'|_p) d\mu(x)
\]

Consider \( \gamma \leq \gamma' \). We have that the product of indicators is equal to the indicator or zero:

\[
\Omega(|p^\gamma x - n|_p) \Omega(|p^{\gamma'} x - n'|_p) = \Omega(|p^{\gamma - \gamma'} n - n'|_p)
\]

Since \( n' \in Q_p/Z_p \) the function \( \Omega(|p^{\gamma - \gamma'} n - n'|_p) \) is non–zero (and equal to one) for

\[
n' = p^{\gamma' - \gamma} n
\]

We get for (7)

\[
\int_{Q_p} p^{-\frac{\gamma}{2}} \chi(-p^{\gamma - 1} j x) \Omega(|p^\gamma x - n|_p) p^{-\frac{\gamma'}{2}} \chi(p^{\gamma' - 1} j' x) \Omega(|p^{\gamma'} x - n'|_p) d\mu(x) = 0
\]

Consider \( \gamma < \gamma' \). Then for the integral (8) we get

\[
\int_{Q_p} p^{-\frac{\gamma}{2}} p^{-\frac{\gamma'}{2}} \chi(-p^{\gamma - \gamma' - 1} j' n) \Omega(|p^\gamma x - n|_p) \Omega(|p^{\gamma' - \gamma} n - n'|_p) d\mu(x) = 0
\]
Therefore the scalar product (7) can be non-zero only for $\gamma = \gamma'$. For $\gamma = \gamma'$ the integral (8) is equal to

$$
\int_{Q_p} p^{-\gamma} \chi(-p^{\gamma-1} jx) \chi(p^{\gamma-1} j'x) \Omega(|p^{\gamma} x - n|_p) \Omega(|n - n'|_p) d\mu(x)
$$

Since $n, n' \in Q_p/Z_p$

$$
\Omega(|n - n'|_p) = \delta_{nn'}
$$

we get for (8)

$$
\langle \psi_{\gamma j n}, \psi_{\gamma' j' n'} \rangle = \delta_{\gamma \gamma'} \delta_{nn'} \int_{Q_p} p^{-\gamma} \chi(p^{\gamma-1} (j' - j)x) \Omega(|p^{\gamma} x - n|_p) d\mu(x) = 
$$

$$
= \delta_{\gamma \gamma'} \delta_{nn'} \delta_{jj'}
$$

that proves that the vectors $\psi_{\gamma j n}$ are orthonormal.

To prove that the set of vectors $\{\psi_{\gamma j n}\}$ is an orthonormal basis (is total in $L^2(Q_p)$) we use the Parsevaley identity.

Since the set of indicators (characteristic functions) of $p$–adic discs is total in $L^2(Q_p)$ and the set of vectors $\{\psi_{\gamma j n}\}$ is translationally invariant and invariant under dilations $x \mapsto p^n x, x \in Q_p$, to prove that $\{\psi_{\gamma j n}\}$ is a complete system it is enough to check the Parsevaley identity for the indicator $\Omega(|x|_p)$.

We have

$$
\langle \Omega(|x|_p), \psi_{\gamma j n} \rangle = p^{-\frac{\gamma}{2}} \theta(\gamma) \delta_{n0}, \quad \theta(\gamma) = 0, \gamma \leq 0, \quad \theta(\gamma) = 1, \gamma \geq 1 \quad (9)
$$

Formula (9) implies the Parsevaley identity for $\Omega(|x|_p)$:

$$
\sum_{\gamma j n} |\langle \Omega(|x|_p), \psi_{\gamma j n} \rangle|^2 = \sum_{\gamma = 1}^{\infty} (p - 1) p^{-\gamma} = 1 = |\langle \Omega(|x|_p), \Omega(|x|_p) \rangle|^2
$$

that proves that $\{\psi_{\gamma j n}\}$ is an orthonormal basis in $L^2(Q_p)$.

Formula (4) implies that the basis $\{\psi_{\gamma j n}\}$ is an orthonormal basis of eigenvectors of the operator $D^\alpha$ with eigenvalues (6). This finishes the proof of the theorem.
3 Wavelet interpretation

Let us discuss the connection between the constructed basis \( \{ \psi_{\gamma j n} \} \) and the basis of wavelets in the space of quadratically integrable functions \( L^2(\mathbb{R}_+) \) on positive semiline. The wavelet basis in \( L^2(\mathbb{R}_+) \) is a basis given by shifts and dilations of the so called mother wavelet function, cf. [15]. The simplest example of such a function is the Haar wavelet

\[
\Psi(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1]}(x) \tag{10}
\]

(difference of two characteristic functions).

The wavelet basis in \( L^2(\mathbb{R}) \) (or basis of multiresolution wavelets) is the basis

\[
\Psi_{\gamma n}(x) = 2^{-\gamma} \Psi(2^{-\gamma} x - n), \quad \gamma \in \mathbb{Z}, \quad n \in \mathbb{Z} \tag{11}
\]

Consider the \( p \)-adic change of variables, i.e. the onto map

\[
\rho : Q_p \to \mathbb{R}_+
\]

\[
\rho : \sum_{i=\gamma}^{\infty} a_i p^i \mapsto \sum_{i=\gamma}^{\infty} a_i p^{-i-1}, \quad a_i = 0, \ldots, p - 1, \quad \gamma \in \mathbb{Z} \tag{12}
\]

This map is not a one–to–one map. The map \( \rho \) is continuous and moreover

**Lemma 3.** The map \( \rho \) satisfies the Holder inequality

\[
|\rho(x) - \rho(y)| \leq |x - y|_p \tag{13}
\]

*Proof* Consider

\[
x = \sum_{i=\alpha}^{\infty} x_i p^i, \quad y = \sum_{i=\beta}^{\infty} y_i p^i,
\]

where \( \alpha \leq \beta \). Then

\[
\rho(x) - \rho(y) = \sum_{i=\alpha}^{\beta-1} x_i p^i + \sum_{i=\beta}^{\infty} (x_i - y_i) p^i
\]

Consider the following two cases:
1) Let $\alpha < \beta$. Then
$$|\rho(x) - \rho(y)| \leq (p - 1) \sum_{i=\alpha}^{\infty} p^{-i-1} = |x - y|_p$$

2) Let $\alpha = \beta$. Then $|x - y|_p = p^{-\gamma}, \gamma > \alpha$.
$$\rho(x) - \rho(y) = \sum_{i=\gamma}^{\infty} (x_i - y_i)p^{-i-1}$$
$$|\rho(x) - \rho(y)| \leq (p - 1) \sum_{i=\gamma}^{\infty} p^{-i-1} = p^{-\gamma} = |x - y|_p$$

that finishes the proof of the lemma.

The following map is a one–to–one map:
$$\rho : Q_p/Z_p \rightarrow \mathbb{N}$$

where $\mathbb{N}$ is a set of natural numbers including zero.

Here $Q_p/Z_p$ is a group (with respect to addition modulo 1) of numbers of the form
$$x = \sum_{i=\gamma}^{-1} x_ip^i$$

**Lemma 4.** For $n \in Q_p/Z_p$ and $m, k \in \mathbb{Z}$ the map $\rho$ satisfies the conditions
$$\rho : p^m n + p^k Z_p \rightarrow p^{-m} \rho(n) + [0, p^{-k}] \quad (14)$$
$$\rho : Q_p \backslash \{ p^m n + p^k Z_p \} \rightarrow \mathbb{R}_+ \backslash \{ p^{-m} \rho(n) + [0, p^{-k}] \} \quad (15)$$

up to a finite number of points.

**Proof** We consider for simplicity the case $\rho_0 = \rho$ and $k = 0$. Consider $n \in Q_p/Z_p$:

$$n = \sum_{i=\gamma}^{-1} n_ip^i$$
$$n - 1 = \sum_{i=\gamma}^{-1} n_ip^i + \sum_{i=0}^{\infty} (p - 1)p^i$$
Then
\[ \rho(n) = \sum_{i=\gamma}^{-1} n_i p^{-i-1} \]
\[ \rho(n-1) = \rho \left( \sum_{i=\gamma}^{-1} n_i p^i + \sum_{i=0}^{\infty} (p-1)p^i \right) = \sum_{i=\gamma}^{-1} n_i p^{-i-1} + \sum_{i=0}^{\infty} (p-1)p^{-i-1} = \rho(n) + 1 \]

Application of (13) for \( y = n, y = n - 1 \) proves that \( n + Z_p \) maps into \( \rho(n) + [0, 1] \). Since the map \( Q_p/Z_p \to \mathbb{N} \) is one–to–one this proves the lemma.

**Lemma 5.** The map \( \rho \) maps the Haar measure \( \mu \) on \( Q_p \) onto the Lebesgue measure \( l \) on \( \mathbb{R}_+ \): for measurable subspace \( X \subset Q_p \)
\[ \mu(X) = l(\rho(X)) \]
or in symbolic notations
\[ \rho : d\mu(x) \mapsto dx \]

**Proof** Lemma 4 implies that balls in \( Q_p \) map onto closed intervals in \( \mathbb{R}_+ \) with conservation of measure. The map \( \rho : Q_p \to \mathbb{R}_+ \) is surjective and moreover nonintersecting balls map onto intervals that do not intersect or have intersection of the measure zero (by lemma 4). This proves the lemma.

Therefore the corresponding map
\[ \rho^* : L^2(\mathbb{R}_+) \to L^2(Q_p) \]
\[ \rho^* f(x) = f(\rho(x)) \quad (16) \]
is an unitary operator.

Lemma 4 implies the following:

**Lemma 6.** The map \( \rho \) maps the Haar wavelet (14) onto the function (4) (for \( p = 2 \)):
\[ \rho^* : \Psi(x) \mapsto \psi(x) \quad (17) \]
(17) is an equality in \( L^2 \): on the set of zero measure (17) may not be true.

Moreover, we have the following theorem:
Theorem 7. For \( p = 2 \) the map \( \rho \) maps the orthonormal basis of wavelets in \( L^2(\mathbb{R}_+) \) (generated from the Haar wavelet) onto the basis \( (\mathfrak{3}) \) of eigenvectors of the Vladimirov operator:

\[
\rho^* : \Psi_{\gamma \rho(n)}(x) \mapsto (-1)^n \psi_{\gamma 1 n}(x)
\]  

Proof. We have

\[2^{-\gamma} \rho(x) = \rho(2^\gamma x)\]  

Lemma 4 implies for \( \rho(n) \in \mathbb{N} \)

\[\chi_{[0,1]}(\rho(2^\gamma x) - \rho(n)) = \chi_{[0,1]}(\rho(2^\gamma x - n))\]  

\((20)\) is true almost everywhere. Analogously

\[\chi_{[0,\frac{1}{2}]}(\rho(2^\gamma x) - \rho(n)) = \chi_{[0,\frac{1}{2}]}(\rho(2^\gamma x - n))\]  

\((21)\)

Formulas \((\mathfrak{11})\), \((\mathfrak{19})\), \((\mathfrak{20})\), \((\mathfrak{21})\) imply

\[
\Psi_{\gamma \rho(n)}(\rho(x)) = 2^{-\gamma} \Psi(2^{-\gamma} \rho(x) - \rho(n)) = \\
= 2^{-\gamma} \Psi(\rho(2^\gamma x - n)) = 2^{-\gamma} \psi(2^\gamma x - n)
\]

The last equality follows from \((\mathfrak{16})\).

Formula \((\mathfrak{5})\) implies that for \( p = 2 \)

\[
\psi_{\gamma 1 n}(x) = 2^{-\frac{\gamma}{2}} \chi(2^{\gamma-1}x) \Omega(|2^\gamma x - n|_2) = (-1)^n 2^{-\frac{\gamma}{2}} \psi(2^\gamma x - n)
\]

that proves \((\mathfrak{18})\) and finishes the proof of the theorem.

We get that after the \( p \)–adic change of variables \((\mathfrak{12})\) the wavelet analysis becomes the \( p \)–adic spectral analysis (expansion of a function over the eigenfunctions of the Vladimirov operator of \( p \)–adic derivation).

Using this interpretation we will call the basis \((\mathfrak{5})\) the wavelet basis (or \( p \)–adic wavelet basis).

Using map \((\mathfrak{16})\) it is possible to define the action of the Vladimirov operator in \( L^2(\mathbb{R}_+) \) by the formula

\[
\partial^\alpha_p f(x) = \rho^{\alpha-1} D^\alpha \rho^* f(x)
\]
(let us note that $D^\alpha$ and $\rho^*$ depend on $p$).

We can see that

$$\partial^\alpha_p f(x) = \frac{p^\alpha - 1}{1 - p^{-1-\alpha}} \int_0^\infty \frac{f(x) - f(y)}{|\rho^{-1}(x) - \rho^{-1}(y)|_{1+\alpha}^{1+\alpha}} dy$$  \hspace{1cm} (22)

where $\rho^{-1}$ is the inverse map to $\rho$. Since $\rho$ is not one–to–one map the map $\rho^{-1}$ is ambiguous but ambiguity is concentrated on the set of zero measure that makes definition (22) correct.

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