On the density of ratios of Chern numbers
of embedded threefolds

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Let $X$ be a 3-fold of general type with Chern class $c_i(X) = c_i(T_X) \in H^{2i}(X, \mathbb{Z})$. Then cup product gives the Chern numbers $c_3^1(X), c_1c_2(X), c_3(X) \in \mathbb{Z}$. One natural question to ask is: Which point in $\mathbb{P}^2(\mathbb{Q})$ does correspond to the Chern numbers $[c_3^1(X), c_1c_2(X), c_3(X)]$ for some 3-fold $X$?

By using Fermat cover (desingularization of the branched cover of $\mathbb{C}P^3$ over an arrangement of planes), Hunt [H] was able to show that there are two triangles inside which every point corresponds to a 3-fold. Extending Hunt’s idea, Liu [C3, L] obtained a larger region in the affine chart $c_1c_2 \neq 0$ (LMCN in the chart attached). In [C2], the author gave an explicit description of the “SCI zone”, the limit points of the Chern ratios $(c_3^1/c_1c_2, c_3/c_1c_2)(X)$ of all the complete intersection 3-folds $X$. However the only known bound of the region of all possible Chern numbers, under the assumption that either $X$ is minimal or the canonical divisor $K_X$ is ample, is $0 \leq c_3^1/c_1c_2 \leq 8/3$. (The right inequality is Yau’s)

In this paper, we study the bounds of the Chern ratios of 3-folds in $\mathbb{P}^5$. Let $X \subset \mathbb{P}^5$ be a 3-fold with hyperplane class $H$, canonical class $K$. What we can show is the following

**Theorem.** The limit points of $\{(x, y) = (c_3^1/c_1c_2, c_3/c_1c_2)(X) \mid X \subset \mathbb{P}^5 \text{ has } H^iK^j > 0\}$ lie on the line segment $x + y = 2, 1 \leq x \leq 2$.

**Remark 1.** In [C1], we showed that the limit points of Chern ratios of determinantal 3-folds in $\mathbb{P}^5$ are the line segment $x + y = 2, 1 \leq x \leq 17/12$.

**Remark 2.** Note that $x < 2$ means the third Segre class of the cotangent bundle $\Omega_X^1$ is negative. As Robert Braun commented that when the degree $d$ goes to infinity, 3-folds in $\mathbb{P}^5$ have small positive $\Omega^1$.

**Remark 3.** The line $x + y = 2$ is part of the boundary of the SCI zone, the limit points of Chern ratios of all complete intersection 3-folds. In fact, the first piece of the "lower" curve of the boundary is $x + y = 2, 1 \leq x \leq 4/3$.

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Our approach to the problem is expressing the Chern numbers in terms of the intersection numbers $H^iK^j$ (Property (1)), and reducing the problem to whether $K^3$ is greater than a linear combination of $d^2, dH^2K,$ and $HK^2$. (cf. Proposition 6). For 3-fold $X$ contained in hypersurfaces of fixed degree $s$, when the degree $d$ goes to infinity, $K^3$ (which is greater than $d^4/s^3$) dominates, and the ratios $(c_1^3/c_1c_2, c_3/c_1c_2)$ approximate to the line $x + y = 2$ arbitrarily. On the other hand, when $s$ goes to infinity, we use the positivity of the third Segre class of $N_{X/P_s}(-1)$ (Proposition 1(ii)), and reduce the problem further to an inequality between $dH^2K$ and $HK^2$ (cf. Proposition 5), and use the genus bound.

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Let $X$ be a 3-fold of degree $d$ in $\mathbb{P}^5$. We use the following notation:

$H =$ hyperplane class
$K =$ canonical class
$Y =$ general hyperplane section
$C =$ general curve section
$s(X) =$ smallest degree of hypersurfaces containing $X$

$$(x, y) := (c_1^3(X)/c_1c_2(X), c_3(X)/c_1c_2(X))$$

$$= (K^3/(K^3 - \Delta_1), (K^3 - \Delta_2)/(K^3 - \Delta_1)), \text{ where}$$

$\Delta_1 := (d - 15)H^2K - 6HK^2$

$\Delta_2 := (6d - 70)d + (2d - 51)H^2K - 12HK^2.$

The last equality follows from

$$c_2(X) = (15 - d)H^2 + 6HK + K^2$$

$$(1) \quad c_3(X) = (6d - 70)d + (2d - 51)H^2K - 12HK^2 - K^3,$$

which is the generalized double point formula gotten from the normal-tangent sequence.

The other properties we need are the following: (please see [BOSS] as a general reference.)

(2) (GHIT, generalized Hodge index theorem) $dHK^2 \leq (H^2K)^2$

(3) (special genus formula) $2g - 2 \leq d^2/s + d(s - 4)$

(4) (Castelnuovo)

$$p_g(X) \leq 2 \left( \binom{M}{4} \right) + \left[ \frac{M}{3} \right]$$

$$p_g(Y) \leq 2 \left( \binom{M}{3} \right) + \left[ \frac{M}{2} \right], \quad \text{where} \quad M = [(d - 1)/2]$$

(5) [BOSS] $-24\chi(O_X) \geq d^4/s^3 + l.t. \text{ in } \sqrt{d}.$

(6) $\chi(O_X) = c_1c_2/24$, combined with property (1), this is

(6') $K^3 = (d - 15)H^2K - 6HK^2 - 24\chi(O_X)$

**Proposition 1.** The following hold

(i) $9H^2K + HK^2 \geq d(d - 21)$

(ii) $K^3 \geq 8d(d - 13) + 2(d - 33)H^2K - 14HK^2$

**Proof.** Let $N$ be the normal bundle of $X$ in $\mathbb{P}^5$. Then $N(-1)$ is generated by global sections and has $c_1(N(-1)) = 4H + K$ and $c_2(N(-1)) = (d - 5)H^2 - HK$. Now

(i) is $(c_1^2 - c_2)N(-1) \cdot H \geq 0$, and (ii) is $(c_1^3 - 2c_1c_2)N(-1) \geq 0$. \qed

**Notation 1.** $l.t.$ means terms of lower degrees (possibly fractional) in $d$. 

Proposition 2. The following hold

(i) \( H^2K < d^2/s + \text{l.t.} \)

(ii) \( HK^2 < d^3/s^2 + \text{l.t.} \)

(iii) \( K^3 > d^4/s^3 + \text{l.t.} \)

Proof. (i) follows from the adjunction formula and Property (3), the genus formula.
(ii) follows from (i) and Property (2), GHIT. Applying (i), (ii) and Property (5) to Property (6'), we have (iii). □

Proposition 3. for any integer \( d_0 \), there are finitely many points \((x, y)\) corresponding to \((c_1^3/c_1c_2, c_3/c_1c_2)(X)\) such that \( \deg X \leq d_0 \).

Proof. for each degree \( d \) fixed, (i) and (ii) in the previous proposition imply that \( H^2K \) and \( HK^2 \) have only finitely many possible values. The finiteness of \( K^3 \) follows from Propoer (6'). (The finiteness of \( \chi(O_X) \) follows from Property (4) and that \( h^1(O_X) = 0 \).) In Property (1), we see that the Chern ratios are functions of \( H^iK^j \). □

Since we are interested in the limit points of the Chern ratios, in our statement we are free to exclude finitely many of them. By Proposition 1, we are free to exclude finitely many degrees in our inequalities. So we use the following

Notation 2. \( A < . \ B \) means that except for 3-folds of finitely many possible degrees, \( A \) is less than \( B \).

Proposition 4. \( d > cs^{5/3} \), where \( c > 0 \) is a constant independent of the 3-fold.

Proof. Let \( k = s - 1 \), then \( H^0(I_X(k)) = 0 \) and we have

\[
h^0(O_{\mathbb{P}_5}(k)) \leq h^0(O_X(k)) \leq \chi(O_X(k)) + h^1(O_X(k)) + h^3(O_X(k))
\]

Riemann-Rock gives, for \( d \geq 15 \)

\[
\chi(O_X(k)) \leq k^3d/6 + kH^2K/2 + kHK^2/6 + h^0(O_X) - h^1(O_X) + h^2(O_X) - h^3(O_X).
\]

The \( H^3 \)-cohomologies of the sequence

\[
(*) \quad 0 \rightarrow O_X(t - 1) \rightarrow O_X(t) \rightarrow O_Y(t) \rightarrow 0
\]

give that

\[
h^3(O_X(h)) \leq h^3(O_X).
\]
Similarly, the $H^1$-cohomologies of the same sequence (·) and its restriction $0 \to \mathcal{O}_Y(t-1) \to \mathcal{O}_Y(t) \to \mathcal{O}_C(t) \to 0$ give

\[(d) \quad h^1(\mathcal{O}_X(k)) \leq h^1(\mathcal{O}_X) + \sum_{t=0}^{k} (k + 1 - t)h^1(\mathcal{O}_C(t)).\]

A rough estimate by the genus bound is

\[(e) \quad \sum_{t=0}^{k} (k + 1 - t)h^1(\mathcal{O}_C(t)) \leq (k + 1)(k + 2)g/2 \leq kd^2/4 + l.t.\]

Property (4) implies

\[(f) \quad h^2(\mathcal{O}_X) = p_g(Y) \leq \frac{d^3}{24} + l.t.\]

So combining (a) ∼ (f) and applying Proposition 2, we have

\[
\binom{k + 5}{5} = h^0(\mathcal{O}_{P_5}(k)) \leq k^3d/6 + d^2/2 + d^3/6k + kd^2/4 + d^3/24 + l.t. \quad \square
\]

**Proposition 5.** If $H^2K, HK^2 > 0$, then $fHK^2 < (d - \delta)H^2K$ holds for $X$ with $s(X) > f$, where $\delta$ is any constant.

**Proof.** We may assume $f > 0$. Assuming the contrary that $fHK^2 > (d - \delta)H^2K$. Multiplying by $H^2K$, applying GHIT (Property (2)), and cancelling $HK^2$, we have

\[fH^2K > (d - \delta)d.\]

The adjunction formula and genus bound give

\[d^2/s + d(s - 4) \geq 2g - 2 > 2d + (d - \delta)d/f\]

This is

\[(g) \quad d < fs + f(f - 6 + \delta_1) + f^2(f - 6 + \delta_1)/(s - f),\]

where $\delta_1 = \delta/f$.

By Proposition 4, (g) is false except for finitely many $s \leq s_0$. For each $s \leq s_0$, there are only finitely many $d$ such that (g) is true. \square
Proposition 6. if $2 - a - 2b > 0$, then $a\Delta_1 + b\Delta_2 < . K^3$.

Proof. By Proposition 1 (ii), it suffices to show

$$a[(d - 15)H^2K - 6HK^2] + b[(6d - 70)d + (2d - 51)H^2K - 12HK^2] < . 8d(d - 15) + 2(d - 33)H^2K - 14HK^2.$$ 

By Proposition 5 and Proposition 1 (i), it suffices to show

$$(6b - 8)HK^2 + (14 - 6a - 12b)HK^2 < . (2 - a - 2b)dH^2K.$$ 

If $2 - a - 2b > 0$, then this is

(h) $HK^2(6 - 6a - 6b)/(2 - a - 2b) < . dH^2K$. 

Let $f = (6 - 6a - 6b)/(2 - a - 2b)$. Then Proposition 5 implies our claim for those 3-folds $X$ with $s(X) > f$.

For $X$ with $s(X) \leq f$, the claim follows from Proposition 2. □

Proof of the Theorem.

1. Claim. $y < . -x + 2 + \epsilon$, for any $\epsilon > 0$.

Proof. This is $K^3 - \Delta_2 < . -K^3 + (2 + \epsilon)(K^3 - \Delta_1)$. i.e. $(1 + 2/\epsilon)\Delta_1 - 1/\epsilon\Delta_2 < . K^3$.

We apply Proposition 6 with $a = 1 + 2/\epsilon$ and $b = -1/\epsilon$.

2. Claim. $-\alpha x + 1 + \alpha - \epsilon < . y$, for any $\alpha > 1$, $\epsilon > 0$.

Proof. This is

$$(1 - (1 + \alpha)/\epsilon)\Delta_1 + (1/\epsilon)\Delta_2 < . K^3$$

Again, we take $a = 1 - (1 + \alpha)/\epsilon$ and $b = 1/\epsilon$ in Proposition 6. Putting Claims 1 and 2 together, we have all limit points lie on the line $x + y = 2$.

Similarly, we can show that $1 - \epsilon < . x$ and $-\epsilon < . y$, for any $\epsilon > 0$. Therefore there are no limit point with $x < 1$ or $y < 0$. □

Remark. In [C1] we showed that the limit points of $\{(c_1^3/c_1c_2, c_3/c_1c_2) | X \}$ is the dependency locus of $\mathcal{O}_{\mathbb{P}_5} \rightarrow r+1 \mathcal{O}_{\mathbb{P}_5}(a_i)$ is the line segment connecting the points $(1, 1)$ and $(17/12, 7/12)$. 
References

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