On quantum advantage in dense coding

M Horodecki\(^1\) and M Piani\(^2\)

\(^1\) Institute of Theoretical Physics and Astrophysics, University of Gda\’nsk, 80–952 Gda\’nsk, Poland
\(^2\) Institute for Quantum Computing & Department of Physics and Astronomy, University of Waterloo, N2L 3G1 Waterloo ON, Canada

E-mail: mpiani@iqc.ca

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Abstract
The quantum advantage of dense coding—the increase in the rate of classical information transmission due to shared entanglement—is studied. General encoding quantum operations are considered. Particular attention is devoted to the case of many senders and one receiver. It is shown that restrictions on the possible operations among the many senders can strongly affect the usefulness of a shared quantum state for dense coding. It is shown, e.g., that there are states that do not provide any quantum advantage if communication among the senders is not allowed, but are useful for dense coding as soon as the senders can communicate classically. These results are actually independent of the particular quantification of the quantum advantage, being valid for any reasonable definition of such a rate increase. It is further shown that the quantum advantage of dense coding satisfies a monogamy relation with the so-called entanglement of purification.

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1. Introduction
Entanglement [1] plays a central role in quantum information theory [2, 3], especially in quantum communication. It is a physical resource exploited in tasks such as teleportation [4] and dense coding [5]. In the last communication problem, the sender, Alice, and the receiver, Bob, shared a pair of two-level systems, or qubits, in a maximally entangled state

\[
|\psi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.
\] (1)

Alice can transmit two classical bits of information sending her qubit to Bob, i.e. with the exchange of just one qubit. To achieve this result, Alice sends her qubit after having applied an appropriate unitary rotation, corresponding for example to the identity \(\sigma_0\) and the Pauli matrices \(\sigma_i\), \(i = 1, 2, 3\). The resulting states \(|\psi_{\mu}\rangle = (\sigma_\mu \otimes 1)|\psi_0\rangle\) (Bell states) are orthogonal, so that when Bob receives Alice’s subsystem they can be unambiguously distinguished.
The previous result is possible because the two parties share initially an entangled state. Indeed, the Holevo bound [6] implies that one qubit may carry at most one classical bit of information, if no pre-established (quantum) correlations between the parties exist.

Unfortunately, in real-world applications we have to deal with imperfect knowledge and noisy operations; therefore, the resulting (shared) quantum states are mixed and described in terms of density matrices. From the point of view of quantum information, this is most relevant in the distant-labs paradigm. In this case, two (or more) parties may want to share a maximally entangled system, but their actions are limited to local operations and classical communication (LOCC), so that they cannot create shared entangled states, but only process pre-existing (imperfect) quantum correlated resources by LOCC. One possible way out is entanglement distillation [7], which can be realized by LOCC. On the other hand, the task in dense coding is exactly that of communicating classical information in the most efficient way with the exchange of a quantum system; hence, we should not allow classical communication between the parties3.

Therefore, it is interesting to study coding protocols for Alice to send classical information to Bob, directly acting on copies of a shared mixed state $\rho_{AB}$. The problem was further generalized in [8, 9], where the notion of distributed quantum dense coding was introduced: in that picture, many senders, called Alices, share states with many receivers, called Bobs. In [8, 9] the encoding is purely unitary, as in the standard pure two-parties setting, i.e. a letter in an alphabet is associated with a unitary operation. In [8, 9] it was shown that, with such a protocol, in the case of many senders and one receiver the possibility of dense coding does not depend on the allowed operations among the senders. On the other hand, it was shown that in the two-receiver case it does depend on the allowed operations among the receivers, i.e. on the allowed decoding processes.

In the case global operations are allowed among the Bobs, i.e. there is essentially only a single receiver, dense coding with unitary encoding is possible if and only if the coherent information [10] between the Alices (here jointly denoted by $A$) and the Bobs (here jointly denoted by $B$),

$$I(A:B) = S(\rho_B) - S(\rho_{AB}),$$

with $S(\sigma) = -\text{Tr}\rho \log \rho$ being the von Neumann entropy, is strictly positive. Both $I(A:B)$ and $I(B:A)$ are less or equal to zero for separable or bound entangled states [11]. Indeed, if $I(A:B)$ or $I(B:A)$ is strictly positive, we know that the state $\rho_{AB}$ is distillable across the cut $A : B$, thanks to the hashing inequality [12]. Therefore, neither separable nor bound entangled states (across the cut $A : B$) are useful for dense coding.

In this paper, we consider a more general encoding scheme: each letter in an alphabet is associated with a completely positive trace preserving (CPTP) map. Such a scheme was already presented in [13, 14], in the one-sender–one-receiver context, where it was found that the optimal encoding is still unitary, but only after a ‘pre-processing’ operation which optimizes the coherent information $I(A:B)$ between the parties and is independent from the letter of the alphabet to be sent. Indeed, while no CPTP operation by Bob can increase $I(A:B)$, an operation by Alice can. The simplest example was given in [13]: the sender discards a noisy subsystem of hers, which is factorized with respect to the remnant state (see (8) in section 2) and as such cannot increase the capacity, i.e. the maximal rate of transmission.

3 Another possibility is that of counting communication, considering consumed and total communication, and computing the achieved communication as the difference, but we will not consider such a framework.

4 Mathematically, this is immediately proved by means of strong subadditivity of von Neumann entropy. Moreover, it is consistent with the fact that in the decoding process the most general operations by Bob are allowed, so that any pre-processing operation by Bob can be thought to happen after he has received the other half of the state from Alice.
In view of the described previous results two problems arise. The first question is of quantitative type. Namely, the existing expressions of dense coding capacity of mixed states even in the bipartite case do not have information theoretic character, i.e. they depend not only on the state itself, but also on some additional parameter—the dimension, of the Hilbert space (either the one describing sender’s subsystem or noiseless channel input). The second question is whether the restriction to local encoding in the multiparty case can affect the capacity. Actually, in [8, 9] it is suggested that even under general encoding the restriction will not diminish the information transfer.

1.1. Summary of the results

In this paper, we propose an elegant, information theoretic formula for the advantage of the dense coding capacity over the capacity not supported by entanglement. We also show that, in the basic one-sender–one-receiver scenario, our quantity satisfies a monogamy relation with the so-called entanglement of purification [15]. The latter is a measure of total correlations, where all correlations—even those of separable states—are thought as being due to entanglement. Furthermore, we show that in the case of many senders, the capacity does depend on the allowed operations among the senders, because these may restrict the pre-processing to be non-optimal. Here, optimality is understood with respect to the case where the senders are allowed to perform global operations, i.e. to the case where there is effectively only one sender. This may be understood considering the case where the noise to be traced out to increase the coherent information is not concentrated in some factorized subsystem, but spread over many subsystems; thus, it may be possible to concentrate and discard such noise with some global (on Alices’ side) operation, but not with local ones. This intuition was the starting point of this work, and is clearly illustrated in example 2 of section 4.

In [8, 9], a classification of quantum states according to their usefulness for distributed dense coding with respect to the allowed operation on the receivers’ side was depicted. Considering pre-processing, a similar classification can be made with respect to the allowed operations on the senders’ side. We remark that in the case of simple unitary encoding—i.e. no pre-processing—and one receiver, such a classification collapses, because the protocol performs as well with local unitaries as it does with global unitaries [8]. Allowing pre-processing, the classification becomes non-trivial, and we provide concrete examples, exhibiting states that are useful for dense coding only if, e.g., global operations are allowed among the senders—i.e. there is only one sender—but not if they are limited to LOCC (similarly, there are states that are useful only if LOCC are allowed, but not if the senders are limited to local operations). We remark that the constraints on the usefulness of a given multi-partite state, based on the allowed operations, persist even in the most general scenario of pre-processing on many copies.

The paper is organized as follows. In section 2, we define the terms of the problem and present a formula for the capacity of two-party dense coding with pre-processing. In section 3, we move to the many-senders–one-receiver setting, specifying the various classes of operations that we will allow among the senders. In section 4, we analyze the many-senders–one-receiver setting, giving examples of how the dense codeability of multi-partite states depends on the allowed pre-processing operations among the senders, particularly providing sufficient conditions for non-dense codeability with restricted operations. In section 5, we discuss how dense codeability is related to distillability and the concept of symmetric extensions. In section 6, we briefly consider the asymptotic setting, and argue that the limits to dense codeability presented in the previous sections remain valid. In section 7, we elaborate on
the monogamy relation between what we call the quantum advantage of dense coding and entanglement of purification. Finally, we discuss our results in section 8.

2. Two-party dense coding with pre-processing

We rederive here some known results for the two-party case, i.e. we allow global operations on both sides (sender’s and receiver’s). Such results will be the backbone of our further discussion. Moreover, we discuss some subtle points of dense coding which arise already at the level of the two-party setting, and stress the importance of a quantity defined in terms of a maximization of coherent information, that we call quantum advantage of dense coding.

Alice and Bob share a mixed state \( \rho^{AB} \) in dimension \( d_A \otimes d_B \), i.e. \( \rho^{AB} \in \mathcal{M}(\mathbb{C}^{d_A}) \otimes \mathcal{M}(\mathbb{C}^{d_B}) \), with \( \mathcal{M}(\mathbb{C}^{d}) \) the set of \( d \times d \) matrices with complex entries. The protocol that we want to optimize is the following. (i) Alice performs a local CPTP map \( \Lambda_i : \mathcal{M}(\mathbb{C}^{d_i}) \rightarrow \mathcal{M}(\mathbb{C}^{d_i}) \) (note that the output dimension \( d'_A \) may be different from the input one \( d_A \)) with \( \Lambda_i \) a priori probability \( p_i \) on her part of \( \rho^{AB} \). She therefore transforms \( \rho^{AB} \) into the ensemble \( \{(p_i, \rho_i^{AB})\} \), with \( \rho_i^{AB} = (\Lambda_i \otimes \text{id})[\rho^{AB}] \). (ii) Alice sends her part of the ensemble state to Bob. (iii) Bob, having at his disposal the ensemble \( \{(p_i, \rho_i^{AB})\} \), extracts the maximal possible information about the index \( i \). Note that for the moment we allow only one-copy actions, i.e. Alice acts partially only on one copy of \( \rho^{AB} \) a time. On the other hand, we analyze the asymptotic regime where long sequences are sent. Moreover, note that stated as it is, the protocol requires a perfect quantum channel of dimension equal to the output dimension \( d'_A \), i.e. the ability to send perfectly a quantum system characterized by the mentioned dimension. We now prove the following.

**Theorem 1.** The one-shot dense coding capacity, that is, the optimal rate of reliable information transmission per shared state used, corresponds for this protocol to

\[
\chi^{d_A}(\rho^{AB}) = \log d'_A + S(\rho^B) - \min_{\Lambda} S((\Lambda_A \otimes \text{id}_B)[\rho^{AB}])
\]

(3)

\[
= \log d'_A + \max_{\Lambda} I'(A|B),
\]

(4)

where \( I'(A|B) \) is the coherent information of the transformed state \( (\Lambda_A \otimes \text{id}_B)[\rho^{AB}] \).

Note that the quantity depends both on the shared state \( \rho^{AB} \) and on the output dimension \( d'_A \) of the maps \( \{\Lambda_i\} \), not only through the first logarithmic term, but also because the minimum runs over maps \( \Lambda : \mathcal{M}(\mathbb{C}^{d_A}) \rightarrow \mathcal{M}(\mathbb{C}^{d'_A}) \).

We reproduce here essentially the same proof used in [13], but our attention will ultimately focus on the rate of communication per copy of the state used, not on the capacity of a perfect quantum channel of a given dimension assisted by an unlimited amount of noisy entanglement. In this sense, our approach is strictly related to the one pursued in [14], as we will discuss in the following. We anticipate that there will be an important difference: in [14] there is no optimization over the output dimension \( \Lambda_A \) (i.e. of the quantum channel), when it comes to consider the rate per copy of the state, with one use of the channel per copy.

**Proof of theorem 1.** The capacity (attained in the asymptotic limit where Alice sends long strings of states \( \rho_i^{AB} \) of the dense coding protocol depicted above) is given by the Holevo quantity [16, 17]

\[
\chi^{d_A}(\rho^{AB}) = \max_{\{(p_i, \Lambda_i)\}} \left( S\left( \sum_i p_i \rho_i^{AB} \right) - \sum_i p_i S(\rho_i^{AB}) \right).
\]

(5)
We bound the capacity from above considering an optimal set \( \{(\hat{p}_i, \Lambda_i)\} \). Since the entropy is subadditive and no operation by Alice can change the reduced state \( \rho^B \), we have

\[
\chi^d_k(\rho^{AB}) \leq S \left( \sum_i p_i \hat{p}_i^A \right) + S(\rho^B) - \sum_i p_i S(\hat{p}_i^{AB}) \leq \log d^A_k + S(\rho^B) - \min_{\Lambda_A} S((\Lambda_A \otimes \text{id}_B)[\rho_{AB}]).
\]

(6) \( \quad (7) \)

The quantity in the last line of the previous inequality, corresponding to (3), can be actually achieved by an encoding with \( p_i = 1/d^A_k \) and \( \Lambda_i[X] = U_i \Lambda_A[X] U_i^\dagger \). The unitaries \( \{U_i\}_{i=1}^{d_A} \) can be chosen to be orthogonal, \( \text{Tr}(U_i^\dagger U_j) = d^A_k \delta_{ij} \), and to satisfy \( 1/d^A_k \sum_i U_i X U_i^\dagger = \text{Tr}(X) I_1 \), for all \( X \in \mathcal{M}(\mathbb{C}^{d^A_k}) \). An example of a set of unitaries with such properties is given by the generalized Pauli operators \( X^k Z^l \), for \( k, l = 0, \ldots, d^A_k - 1 \), with \( X|j\rangle = |(j + 1) \text{mod}(d^A_k)\rangle \) and \( Z|j\rangle = e^{2\pi i / d^A_k}|j\rangle \), and \( \{|j\rangle \mid j = 0, \ldots, d^A_k - 1\} \), an orthonormal basis for \( A \). The CPTP map \( \Lambda_A \) corresponds to the pre-processing operation. Indeed, with this choice \( \sum_i p_i \rho_i^{AB} = 1/d^A_k \otimes \rho^B \), so that \( S(\sum_i \rho_i^{AB}) = \log d^A_k + S(\rho^B) \) and \( \sum_i p_i S(\rho_i^{AB}) = S((\Lambda_A \otimes \text{id}_B)[\rho_{AB}]). \)

We note in particular that Alice may always choose to substitute her part of the shared state with a fresh ancilla in a pure state. This corresponds to a pre-processing \( \Lambda_{\text{sub}}[X] = \text{Tr}(X)|\psi\rangle\langle \psi| \) and gives a rate \( \log d^A_k \), corresponding to the classical transmission of information with a \( d^A_k \)-long alphabet, i.e. without a quantum advantage. A quantum effect is present if \( \chi > \log d^A_k \), i.e. if a local operation on Alice side is able to reduce the entropy of the global state strictly below the local entropy of Bob, or if \( I(A;B) > 0 \) from the very beginning.

An almost trivial case where pre-processing has an important role is

\[
\rho^{AAB} = \rho^{AB} \otimes \rho^A,
\]

(8)

with \( S(\rho^{AB}) < S(\rho^B) \) but \( S(\rho^{AB}) + S(\rho^A) \geq S(\rho^B) \). Here, we consider \( AA' \) as a composite system Alice can globally act on. Then, a possible pre-processing operation is \( \text{id}_A \otimes \Lambda_{\text{sub}}[\rho^{AAB}] = \rho^{AB} \otimes \text{id}_A \) (which can be realized acting on \( A' \) only).

Since the log \( d^A_k \) contribution in (3) can be considered purely classical, we choose a different way of counting the rate of transmission: indeed, it appears natural to subtract the logarithmic contribution in order to define the following.

**Definition 1.** The quantum advantage of dense coding is defined as

\[
\Delta(A|B) \equiv S(\rho^B) - \inf_{\Lambda_A} S((\Lambda_A \otimes \text{id}_B)[\rho_{AB}])
\]

(9)

\[
= \sup_{\Lambda_A} I(A;B).
\]

(10)

The infimum (or the supremum) is over all maps \( \Lambda_A \), with whatever output dimension.

Since a possible map is \( \Lambda_{\text{sub}} \), we have \( \Delta(A|B) \geq 0 \). We say that a state is dense-codeable (DC) if \( \Delta(A|B) \) is strictly positive. It may be that \( \Delta \) is not additive; hence, to ensure that the state is not useful at all for dense coding, one has to consider its regularization (see section 6).

We remark that the classification of states in terms of their dense-codeability for different classes of encoding operations, will not depend on such redefinition. Moreover, the redefined quantity (9) appears more information-theoretical and depends only on the state.

Since the von Neumann entropy is concave, in the optimization over \( \Lambda_A \) in (9) it is sufficient to consider extremal maps. The input of the map \( \Lambda_A \) is an operator acting on a \( d_A \)-dimensional system; thus, according to [18], if \( \Lambda_A \) is extremal it can be written by means of at most \( d_A \) Kraus operators, i.e.

\[
\Lambda_A[X] = \sum_{i=1}^{d_A} A_i X A_i^\dagger,
\]

(11)
The range of the operator $\Lambda_1(X)$ is given by all the columns of the Kraus operators $A_i$, and each operator $A_i$ has $d_i$ (the input dimension) columns. Therefore, the optimal output dimension $d_A$ can be taken to be $d_A^2$, and the infimum in (9) is actually a minimum. This analysis is the same that can be done in a similar optimization problem which occurs in the study of entanglement of purification [15]. It is possible to further relate the quantum advantage of dense coding with entanglement of purification, as we do in section 7.

Exploiting the convexity of entropy, it is immediate to find the following upper bound for $\Delta_1(A \rangle B)$:

$$\Delta_1(A \rangle B) \leq S(\rho_B) - \min_X S(\rho^{AB}(X)),$$

(12)

with

$$\rho^{AB}(X) = \frac{X \otimes \mathbb{1} \rho^{AB} X^\dagger \otimes \mathbb{1}}{\text{Tr}(X \otimes \mathbb{1} \rho^{AB} X^\dagger \otimes \mathbb{1})},$$

where, according to the reasoning of the previous paragraph, $X$ can be taken as a $d_A \times d_A$ square matrix. We remark that this is only an upper bound: local filtering is not allowed in our framework, because it requires postselection and therefore classical communication. Moreover, with a true local filtering, the reduced density matrix $\rho_B$ changes, while we keep it fixed in (12).

Example 1. In [9], the dense-codeability by unitaries of the Werner states equivalent to $\rho_p = p|\psi_0\rangle\langle\psi_0| + (1 - p)\mathbb{1}_4$, (13)

with $\psi_0$ defined in (1), was studied. The state $\rho_p$ is entangled for $p > 1/3$, but unitarily DC only for $p > p_{U-DC} = 0.7476$. Werner states are among the most interesting and studied classes of states, and we will address the following two natural questions.

(i) Is the state $\rho_p$ DC for some $p < p_{U-DC}$ if we allow general encoding operations, i.e. pre-processing?

(ii) Can $\Delta_1(A \rangle B)$ be greater than $I(A \rangle B)$ when the latter is strictly positive?

Unfortunately, we are unable to reach definite conclusions, but the study of this example illustrates the use of the concepts and tools developed so far. We remark that question (i) addresses the problem of deciding whether a state can change from being ‘useless’ for dense coding to being ‘useful’, if we consider more general encoding maps. On the other hand, question (ii) addresses the issue of whether a state can become more useful when we allow more encoding operations.

In [13], numerical evidence was found that no pre-processing $\Lambda_A : \mathcal{M}(\mathbb{C}^2) \to \mathcal{M}(\mathbb{C}^2)$ can enhance $I(A \rangle B)$ in the case of a shared two-qubit state. However, note that, as previously discussed, optimal pre-processing maps are in principle of the form $\Lambda_A : \mathcal{M}(\mathbb{C}^2) \to \mathcal{M}(\mathbb{C}^2)$, i.e. with a larger output. Here, we focus instead on the bound (12), for which, as discussed, we can consider the matrix $X$ as $X : \mathbb{C}^2 \to \mathbb{C}^2$. We observe that, thanks to the $U \otimes U^*$ symmetry of the state [19] and to the invariance under unitaries of both the entropy and the trace, we can take $X$ to be diagonal in Alice’s Schmidt basis for $\psi_0$, i.e. to be of the form $X = \begin{pmatrix} r & 0 \\ 0 & 1-r \end{pmatrix}$, with $0 \leq r \leq 1$. It is possible to compute analytically the entropy $S(\rho^{AB}(X))$. One finds that the optimal choice is $r = 0$, and the bound is $1 - H_2\left(\frac{1-2p}{2}\right)$, where $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy. Thus, in this example we see that the bound (12) is far from being tight, since it is strictly positive for every $p > 0$. 

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i.e. even for separable states. Therefore, it is not possible to use it to conclude something as regards question (i). Anyway, it constitutes a non-trivial limit on $\Delta(A)B$, and provides some information about question (ii).

3. Pre-processing with many senders

If there are many senders, it may happen that the operations among them are restricted, for example, no communication may be allowed, or they may collaborate only through LOCC. Both latter situations do not affect the unitary encoding part of the dense coding. Indeed, it is possible to realize the optimal unitary operations locally: each Alice acts with unitaries satisfying the optimal condition on her subsystem [8, 9].

We note though that in the case of many senders and many receivers, even considering just unitary encoding, it happens that in certain cases local unitary encoding is not enough to take advantage of quantum correlations. This may be due, for example, to the fact that with restricted operations at the senders’ and at the receivers’ side, the question of who sends what to whom is important. A very simple case where this is evident is that of two senders $A$ and $A'$, and two receivers $B$ and $B'$, with $A$ and $B'$ ($A'$ and $B$) sharing an Einstein–Podolsky–Rosen (EPR) pair, $A$ ($A'$) sending her quantum system to $B$ ($B'$), and both the senders and the receivers restricted to act by local operations. It is then clear that although the senders and the receivers share two EPR pairs, these are useless for the sake of dense coding, because they happen to pertain to the ‘wrong’ pairs of senders and receivers, so that the latter could as well share maximally mixed states. It is also obvious that the availability of global operations—or even just of the swap operation—on the senders’ side would make dense coding possible, by letting each sender–receiver pair share a maximally entangled state.

Considering the case of many senders and one receiver, but allowing encoding by general operations, one realizes that it might not be possible to apply the optimal pre-processing map. Indeed, just repeating the considerations which led to (3), it is clear that, in the case of many senders and one receiver, the dense coding capacity may be expressed as

$$\chi_{d_{\text{G}}}^{d_{A}} = \log d_{A} + S(\rho^{B}) - \min_{\Lambda \in O}(\Lambda \otimes \text{id})[\rho^{AB}],$$

where $O$ is the set of allowed operations on the senders’ side, for example, global$^5$ (G), LOCC or LO. This is true as long as the class of operations considered allows the use of unitaries with the properties described in the paragraph after (6). This certainly holds as long as the class of operations is larger than LO, as one can consider the use of independent generalized Pauli unitaries for every sender. The only subtle point is the compatibility of the choice of the target output dimension $d_{A}^\prime$: we will suppose it is always of the factorized form $d_{A}^\prime = d_{A_{1}}^\prime \cdots d_{A_{N}}^\prime$, so that it can be achieved exactly by an optimal local unitary encoding. Obviously,

$$\chi_{G}^{d_{A}} \geq \chi_{\text{LOCC}}^{d_{A}} \geq \chi_{\text{LO}}^{d_{A}} \geq \log d_{A}^{\prime}.$$ 

The capacity corresponds at least to the classical one with many senders and one receiver, because it is always possible for the Alices to apply locally the substitution map $\Lambda_{\text{sub}}^{A} = \Lambda_{\text{sub}}^{A_{1}} \otimes \cdots \otimes \Lambda_{\text{sub}}^{A_{N}}$. Thus, we can define the corresponding (non-negative) quantum advantages

$$\Delta_{G} \geq \Delta_{\text{LOCC}} \geq \Delta_{\text{LO}} \geq 0.$$ 

$^5$ It corresponds to the case where there is only one sender.
We can obtain an upper bound for $\Delta_{\text{LOCC}}$—and therefore valid also for $\Delta_{\text{LO}}$—similar to that presented in (12):

$$\Delta_{\text{LOCC}}(A)B) \leq S(\rho^B) - \min_{X_{\text{prod}}} S \left( \frac{X_{\text{prod}} \otimes I_{AB} X_{\text{prod}}^\dagger \otimes I_{AB}}{\text{Tr}(X_{\text{prod}} \otimes \rho_{AB} X_{\text{prod}}^\dagger \otimes I_{AB})} \right),$$

where $X_{\text{prod}} = X_1 \otimes \cdots \otimes X_N$, with each $X_i$ being a $d_{A_i} \times d_{A_i}$ square matrix.

4. Examples of the hierarchy of capacities for multi-senders

We provide examples of the hierarchy (15), more precisely of shared states that are not DC for certain classes of allowed operations among senders, but are DC for more general operations.

4.1. LOCC-DC but not LO-DC

We first analyze the case where the state is not LO-DC but it is LOCC-DC: $\Delta_{\text{LO}} = 0$ while $\Delta_{\text{LOCC}} > 0$. We will need the following proposition.

**Proposition 1.** Consider a tripartite state $\rho_{AAB}$ such that (i) it is separable under the $A' : AB$ cut, and (ii) its reduction $\rho_{AB}$ is also separable. Then, after any bilocal operation $\Lambda_{AB} = \Lambda_A \otimes \Lambda_{A'}$ of parties $A$ and $A'$, we have $I(AA'B) \leq 0$.

**Proof.** For separable states coherent information is always non-positive [11]. For any state separable under $A' : AB$ cut we then have

$$S(A'AB) \geq S(AB).$$

Now, if the state $\rho_{AB}$ is also separable, then

$$S(AB) \geq S(B).$$

Thus, for a tripartite state separable along the $A' : AB$ cut and such that its $AB$ reduction is separable, $I(AA'B) \leq 0$. Moreover, after any bilocal operation $\Lambda_A \otimes \Lambda_{A'}$ the state still satisfies the above separability features so that $I(AA'B) \leq 0$. \[\Box\]

Note that the separability properties used in the previous proposition may not be preserved when the parties $AA'$ can communicate classically.

**Example 2.** Consider the state

$$\rho_{AAB} = \frac{1}{2}(|\phi_0\rangle \langle \phi_0| + |\phi_1\rangle \langle \phi_1|),$$

where

$$|\phi_0\rangle = |0\rangle_A \otimes \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}),$$

$$|\phi_1\rangle = |1\rangle_A \otimes \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}),$$

which has initial entropies $S(AA'B) = S(B) = 1$ and $I(AA'B) = 0$. The state is explicitly separable with respect to the $A' : AB$ cut. The trace over $A'$ gives an equal mixture of two-qubit orthogonal maximally entangled states; hence, it is separable [20]. Thus, according to proposition 1, parties $A$ and $A'$ cannot locally decrease the total entropy below $S(B)$. However, by LOCC they can. Namely, $A'$ can measure in the $\{0, 1\}$ basis, communicate the result to $A$, and further substitute the subsystem $A'$ with one in a pure state. Then, after a suitable local unitary rotation, $A$ will share a maximally entangled state with $B$ and $S'(AA'B) = 0$, so that $I'(AA'B) = 1$. 8
The previous example is the simplest possible one that illustrates how some ‘spread’ noise which conflicts with (unitary) dense coding can be undone only allowing operations among the senders that are more general than local operations. Indeed, in the system A\text{A}' one can single out a virtual qubit, carrying the whole noise. The noisy qubit is however encoded non-locally into the system A\text{A}', so that the senders do not have local access to it. Effective tracing out of the unwanted noise (prior to unitary encoding) is possible only if A and A' communicate. Indeed, one can go from \(\rho_{A\text{A}B}\) to \(\rho_{A''\text{B}} = \|A\|/2 \otimes \psi_0^{A''}\) by an invertible \(A : A'-\text{LOCC}\) operation, but not by an \(A : A'-\text{LOCC}\) operation.

The previous tripartite (two senders, one receiver) case can be generalized straightforwardly. Following the definition of multi-partite mutual information \(I(A_1 : \cdots : A_n)\) which can be represented as (cf \([21]\))

\[
I(A_1 : \cdots : A_n) = I(A_1 ; A_2) + I(A_1A_2 ; A_3) + \cdots + I(A_1A_2 \cdots A_{n-1} ; A_n),
\]

one can define a quantity, which is not an entanglement measure, but may be useful (see \([22]\) in this context):

\[
D(B : A_1 : \cdots : A_n) \equiv E(B : A_1) + E(BA_1 : A_2) + \cdots + E(BA_1A_2 \cdots A_{n-1} : A_n),
\]

where \(E\) is any entanglement parameter, i.e. it is positive and \(E( X : Y) = 0\) if and only if the state \(\rho_{X:Y}\) is separable. Similarly as in tripartite case, we obtain that if \(D\) is zero, then parties \(A_1, \ldots, A_n\) cannot make the global entropy be less than \(S(B)\) by LO (but not necessarily by LOCC), so that the state is useless for dense coding from \(A_1 \cdots A_n\) to \(B\).

**Example 3.** Consider the state

\[
\rho_{A_1 \cdots A_n B} = \frac{1}{2^n-1} \sum_{i_1, \ldots, i_n = 0}^{1} |i_1\rangle \langle i_1| \otimes \cdots \otimes |i_n\rangle \langle i_n| \otimes (\sigma_{A_1}^{i_1} \otimes \cdots \otimes \sigma_{A_n}^{i_n}) (\otimes \mathbb{I}^B) P_{A_1B}(\sigma_{A_1}^{i_1} \otimes \cdots \otimes \mathbb{I}^B)),
\]

where \(\sigma_0\) and \(\sigma_1\) are the identity and the flip operator, respectively, and \(P^A\) is the projector onto the maximally entangled state \(|\psi_0\rangle\). It is easily checked that \(\rho_{A_1 \cdots A_n B}\) satisfies \(D = 0\). The unitary rotation \(\sigma_a\) applied to the \(A_1\) part of the maximally entangled state depends on the ‘parity’ of the state of the other Alices. It is correctly identified if all \(A_2, \ldots, A_n\) measure their qubits and communicate their results to \(A_1\), and then \(A_1\) can share a singlet with \(B\).

**4.2. G-DC but not LOCC-DC**

To have an example of a state for which \(\Delta_{\text{LOCC}} = 0\) while \(\Delta_G > 0\), consider the Smolin state \([23]\)

\[
\rho_{A_1B: A_2A_3} = \sum_{\mu=0}^{3} |\psi_\mu\rangle \langle \psi_\mu|_{A_1B} \otimes |\psi_\mu\rangle \langle \psi_\mu|_{A_2A_3},
\]

where \(\psi_\mu\) are Bell states (see equation (1) and the paragraph below this equation). Note that states \(\psi_\mu\) are indistinguishable by LOCC [24]. Hence, it seems reasonable that the state cannot be used for dense coding, even if the parties \(A_1, A_2, A_3\) can use LOCC. For example, the parties \(A_2A_3\) cannot distinguish which Bell state they have, and hence cannot tell \(A_1\) what rotation to apply, in order to share the singlet with \(B\). Let us now prove that this is true.
The state is $A_1A_2 : BA_3$ separable (from equation (27) it is explicitly $A_1B : A_2A_3$ separable; however, it is permutationally invariant [23]). After any LOCC operation this will not change. Thus, the output state of systems $A_1B$ will be separable; hence, $S'(A_1B) \geq S(B)$. Moreover, the total output state will remain $A_1B : A_2A_3$ separable, which implies $S'(A_1BA_2A_3) \geq S'(A_1B)$. Combining the two inequalities we obtain

$$S'(A_1BA_2A_3) \geq S(B).$$

Of course, if for example $A_2$ and $A_3$ could meet and perform global operations, the state would become useful for dense coding, as they could help $A_1$ to share a singlet with $B$.

5. Limits on pre-processing from one-way distillability and symmetric extensions

The possibility of global pre-processing makes non-trivial the identification of states which, although $A_1 \cdots A_n : B$ is entangled, are not G-DC ($\Delta_G = 0$). One has to exclude that the coherent information can be made strictly positive by any action on the side of Alices. We will now see how this may be related to one-way distillation and the concept of symmetric extension. Since we will focus on global operations, we may as well consider a bipartite setting.

Loosely speaking, entanglement distillation consists of the process of obtaining $m$ copies of the highly entangled pure states (1), starting from $n$ copies of a mixed entangled state, by means of a restricted class of operations that cannot create entanglement [20]. The optimal rate, i.e. the optimal ratio $m/n$, of the conversion for $n$ that goes to infinity, is the distillable entanglement under the given constraint on operations. The class may be chosen to be LOCC operations—in such case we speak simply of distillable entanglement—or, more restrictively, one-way LOCC operations, for which classical communication is allowed only from one party to the other, and not in both directions. In the latter case, we speak of one-way distillable entanglement. If we suppose that the communication goes from Alice to Bob, it has been shown [12] that the one-way distillable entanglement $E_D(A)B$ of a state $\rho^A_B$ satisfies the hashing inequality

$$E_D(A)B \geq I(A)B.$$

Hence,

$$E_D(A)B \geq \Delta(A)B,$$

i.e. it is greater than the quantum advantage of dense coding. It follows that any DC state is not only distillable, but even one-way distillable. In turn, if a state is not one-way distillable, then it cannot be DC.

There are entangled states for which we know $E_D(A)B = 0$: states which admit $B$-symmetric extensions [25–27]. A state $\rho^A_B$ admits a $B$-symmetric extension if there exists a state $\sigma^{ABB'}$ such that its reductions satisfy

$$\sigma^{AB} = \sigma^{AB'} = \rho^A_B.$$

Suppose $\rho^A_B$ has a tripartite symmetric extension $\sigma^{ABB'}$ and is at the same time one-way distillable. A one-way distillation protocol consists of an Alice operation whose result—the index of the Kraus operator in (11)—is communicated to the other party. Bob can then perform an operation depending on the result received; no further action of Alice is required. The communication involved is classical, so it can be freely sent to many parties. If, having at our disposal $\sigma^{ABB'}$, we run the one-way LOCC protocol, which by hypothesis allows distillation, in parallel between $A$ and $B$, and $A$ and $B'$, we would end up with a subsystem $A$ which is at the same time maximally entangled both with $B$ and $B'$. However, this is impossible, because of
monogamy of entanglement [28]. We conclude that a one-way distillable state does not admit a symmetric $B$-extension [29]. As regards the case of the two-qubit Werner state (13), in [30], it was proved that it admits a symmetric extension for $p \leq 2/3$.

6. Limits on many-copy processing

The examples of the classification we discussed in section 4 depend only on relations among entropies which rely on separability properties. As such, the action on many copies of the state at disposal cannot help. Indeed, following [13, 14], one can define the quantum advantage per copy when the encoding is allowed on $n$-copies of the state at the same time:

$$\Delta^{(n)}(A)B = \frac{1}{n} \Delta(A)B. \rho^{\otimes n}_{AB}$$

$$= S(\rho) - \frac{1}{n} \min_{\Lambda^{(n)}_A} S\big((\Lambda^{(n)}_A) \otimes \text{id}_B\big)\big[\rho^{\otimes n}_{AB}\big],$$

where now $\Lambda^{(n)}_A$ acts on $\mathcal{M}(\mathbb{C}^{d^2_A} \otimes \mathcal{M}(\mathbb{C}^{d_B} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_B}))$, and the asymptotic quantum advantage per copy:

$$\Delta^\infty(A)B = \lim_{n \to \infty} \Delta^{(n)}(A)B.$$  \hspace{1cm} (33)

Correspondingly, one has the multi-partite quantum advantages $\Delta^{O}(A)B$ and $\Delta^{\infty}_{\text{LOCC}}(A)B$, where the Alices are restricted to the class of operations $O$. It is clear that $\Delta^O(A)B = \Delta^\infty_{\text{LOCC}}(A)B = 0$ and $0 < \Delta^{(n)}_{\text{LOCC}}(A)B \leq \Delta^\infty_{\text{LOCC}}(A)B$ for the state (21), while $\Delta^{(n)}_{\text{LOCC}}(A)B = \Delta^\infty_{\text{LOCC}}(A)B = 0$ and $0 < \Delta^{(n)}_{G}(A)B \leq \Delta^\infty_{G}(A)B$ for the state (21) for the Smolin state (27). Thus, the derived separations survive the asymptotic limit of many copies.

7. Monogamy relation between entanglement of purification and the advantage of dense coding

We observed in section 2 that there are similarities in the calculation of the advantage of dense coding and in that of entanglement of purification [15]. In this section, we will see that this relation is more than a coincidence: there is in fact a monogamy relation between the advantage of dense coding and the entanglement of purification. Such a relation does not seem to have already been reported in the literature.

Here, by monogamy relation we mean something more fundamental and more general than, e.g., the standard monogamy relation $E(A : BC) \geq E(A : B) + E(A : C)$ potentially satisfied by some entanglement measure $E$ [28]. We rather mean a constraint on how correlations are distributed in a multi-partite scenario [31]. Such a constraint may involve, as in our case, more than one quantifier of correlations.

We start by recalling the definition of entanglement of purification.

**Definition 2.** The entanglement of purification for a bipartite state $\rho_{AB} \in \mathcal{M}(\mathbb{C}^{d_A} \otimes \mathcal{M}(\mathbb{C}^{d_B}))$ is

$$E_p(\rho_{AB}) = E_p(A : B) = \min_{\psi : T_{A,B}(\psi) = \rho_{AB}} S(\psi_{AA}).$$

where the minimum runs over all purifications $\psi = |\psi\rangle\langle\psi|_{AB}$. $|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_B}$ such that $T_{A,B}(\psi) = \rho_{AB}$.

Entanglement of purification is a measure of total correlations, where all correlations—even those of separable states—are somehow thought as being due to entanglement.
Indeed, in the bipartite pure-state case, the entropy of one subsystem is an entanglement measure [7]. For $\{|\lambda_i\rangle\langle\lambda_i|\}$, the spectral ensemble of $\rho_{AB}$, consider its purification $|\tilde{\psi}\rangle = \sum_i \sqrt{\lambda_i} |\lambda_i\rangle_{AB} |0\rangle_B = |\psi\rangle_{AB}|0\rangle_B$. Then any other purification can be obtained from $|\psi\rangle$ by means of an isometry $U_{AB}$ as $|\psi\rangle = U_{AB} \otimes I_{AB} |\psi\rangle$. Following [15], we find

$$
\psi_{AA'} = \text{Tr}_{BB'} (\tilde{\psi}_{AA'B'})
$$

and

$$
\rho = \text{Tr}_{BB'} (U_{AB}^\dagger \tilde{\psi}_{AA'B'} \otimes |0\rangle_B \langle 0| B)
$$

where $\Lambda_A[X_A] = \text{Tr}_B (U_{AB} X_A \otimes |0\rangle_B \langle 0| B)$, for all $X_A \in C$. By varying $U_{AB}$, that is the purification $\psi$, we vary $\Lambda_A$. Thus,

$$
E_p(\rho_{AB}) \equiv \min_{\Lambda_A} S(\Lambda_A \otimes \text{id}_B)[\tilde{\psi}_{AA'}],
$$

Comparing (9) and (39) we then arrive at the following.

**Theorem 2.** Given a pure tripartite state $\psi_{ABC}$, one has

$$
S(B) = \Delta(A|B) + E_p(B:C),
$$

where $\Delta$ is the quantum advantage of dense coding (9) and $E_p$ is the entanglement of purification (34).

For fixed entropy $S(B)$, this means that the more $B$ is correlated with $C$, the less dense coding is advantageous from $A$ to $B$. For a tripartite mixed state $\rho_{ABC}$, following [31] we may consider a purification $\psi_{ABC}$, and apply equation (40) to the three parties $(AD)$, $B$ and $C$ to find

$$
S(B) \geq \Delta(A|B) + E_p(B:C).
$$

Indeed, from the definition of advantage it is easy to check that $\Delta(AD|B) \geq \Delta(A|B)$ for all tripartite states $\rho_{ABD}$, in particular for the $ABD$ reduction of $\psi_{ABCD}$. Following [31] again, we may consider the asymptotic case, applying the just found relations to $\psi_{ABC}(\rho_{ABC})$, using the additivity of the von Neumann entropy, dividing by $n$, and taking the limit $n \to \infty$, to find

$$
S(B) = S^\infty(A|B) + E_{LO}(B:C)
$$

and

$$
S(B) \geq S^\infty(A|B) + E_{LO}(B:C),
$$

for the case of pure and mixed states, respectively. Here, $E_{LO}(A:B) = \lim_n \frac{1}{n} E_p(\rho_{AB}^{\otimes n})$ is the cost—in singlets—to create $\rho_{AB}$ in the asymptotic regime, allowing approximation, by means of local operations and asymptotically vanishing communication from an initial supply of EPR pairs [15].

For the pure state case, it is fascinating to put together the results of theorem 1 of [31] and the present ones to find relations between different notions of correlations and entanglement measures/parameters:

$$
I_{HV}(A:B) - \Delta(A|B) = E_p(B:C) - E_F(B:C),
$$

and

$$
C_F(A:B) - \Delta^\infty(A|B) = E_{LO}(B:C) - E_C(B:C),
$$

where, for a bipartite state $\rho_{AB}$:
- $I_{HV}$ is the measure of correlations defined in [32] as

$$I_{HV}(A|B) = \max_{\{M_x\}} \left[ S(\rho_B) - \sum_x p_x S(\rho_B^x) \right],$$

where the maximum is taken over all the POVMs $\{M_x\}$ applied on system $A$, $p_x = \text{Tr}(M_x \otimes I_\rho_{AB})$ is the probability of the outcome $x$, $\rho_B^x = \text{Tr}_A(M_x \otimes I_\rho_{AB})/p_x$ is the conditional state on $B$ given the outcome $x$ on $A$, and $\rho_B = \sum_x p_x \rho_B^x = \text{Tr}_A(\rho_{AB})$;

- $C_D$ is the common randomness distillable by means of one-way classical communication from $A$ to $B$, that is the net amount of correlated classical bits that $A$ and $B$ can asymptotically share starting from an initial supply of copies of $\rho_{AB}$; it is equal to

$$C_D(A|B) = \lim_{n \rightarrow \infty} \frac{1}{n} I_{HV}(A|B)_{\rho_{AB}^n} [33];$$

- $E_F$ is the entanglement of formation

$$E_F(A : B) = \min_{\{(p_i, \psi_i^A)\}} \sum_i p_i S(\psi_i^A),$$

where the minimum runs over all pure ensembles such that $\sum_i p_i \psi_i^{AB} = \rho_{AB}$;

- $E_C$ is the entanglement cost, that is the cost—in singlets—to create $\rho_{BC}$ in the asymptotic regime, allowing approximation, from an initial supply of EPR pairs by means of LOCC; it is equal to

$$E_C(A : B) = \lim_{n \rightarrow \infty} \frac{1}{n} E_F(A : B)_{\rho_{AB}^n} [34].$$

Note that the differences appearing in (43) and (44) are positive [15].

8. Discussion

In conclusion, we considered the transmission of classical information by exploiting (many copies of) a shared quantum state, both in the bipartite and in the multi-partite—more specifically, in the many to one—setting. The coherent information of a state—the relevant quantity for dense coding with unitary encoding—can be increased by some pre-processing by the senders, which is aimed to reduce the entropy of the state.

Starting from this point, we discussed fundamental limits on the usefulness of states for multi-partite dense coding, for given constraints on the operations allowed among senders. The intuition behind our results is that there might be ‘noise’ in the state that can be eliminated only by operations that are non-local, i.e. that require classical or quantum communication to be performed. Therefore, contrary to the unitary-encoding case, where in the many-senders-to-one-receiver case local unitaries perform as well as global unitaries, constraints on the senders’ side are crucial in the general-encoding setting. We note that the distinction of usefulness of states for dense coding according to the allowed encoding operations holds also for the quantities presented in [13, 14], as it is evident, for example, from (45). Limits on the usefulness are not removed even if we allow the most general encoding over whatever number of copies of the shared state. Our analysis leads to a non-trivial classification of quantum states, parallel to the one suggested in [8, 9], where constraints on the operations allowed on the receivers side were considered. Indeed, one can depict a subdivision of multi-partite states into classes of states that are many-to-one dense-codeable if certain operations, for example LOCC, are allowed among the senders, but not if the senders are restricted to local operations.

We have also defined a new quantity describing usefulness of a state for dense coding—the quantum advantage of dense coding. The difference with the quantities presented in [13, 14] is twofold. Firstly, in defining the quantum advantage of dense coding in (9), we immediately consider a maximum over maps without restricting the dimension of the output. This means that we focus on the property of the state, rather than of a pair state+channel. Secondly, exactly for the same reason, we do not distinguish between many uses of the state and many uses of
the channel: the rate is always defined in terms of the number of copies of the state used, even when we allow encoding on many copies. These two facts make our quantities $\Delta^\infty$ different from all the ones presented in [13, 14]. In particular, we claim that the quantity $\Delta^\infty$ is more information-theoretical than the quantity

$$DC^\infty(\rho) = 1 + \sup_{\Lambda_A} \frac{nS(\rho^B) - S((\Lambda_A \otimes \text{id}^B)(\rho^A))}{S(\Lambda_A[\rho_A^\otimes n])}, \quad (45)$$

which, according to [13, 14], corresponds to the rate of classical communication per qubit sent, i.e. per use of a two-dimensional quantum channel. Indeed, in the latter case one considers the use of whatever number of copies of the shared state per use of the channel. In particular, we remark that for pure states one has $DC^\infty(\psi^{AB}) = 2$ as soon as the state $\psi^{AB}$ is entangled—whatever the degree of its entanglement—while $\Delta(\psi^{AB}) = S(\rho^B)$.

Finally, focusing on general properties of dense-codeability of states, we observed that there exist a monogamy relation between the quantum advantage of dense coding and the entanglement of purification. Such a relation puts in quantitative terms the fact that the quantum advantage of dense coding is (or can be) large (only) if the disorder—as quantified by the von Neumann entropy—of the receiver is due to correlations with the sender, rather than with a third party.

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