Production of primordial black holes induced by the modification of gravity

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The enhancement of the spectrum of primordial curvature perturbations \(\mathcal{R}\) can induce the production of primordial black holes (PBH) which could account for part of present day dark matter. We investigate the effects on the spectrum of \(\mathcal{R}\) produced by the modification of gravity in the case of KGB models, deriving the relation between the unitary gauge \(\zeta\) and comoving curvature perturbation \(\mathcal{R}\), identifying a background dependent enhancement function \(E\) which can induce large differences between the two gauge invariant variables. We then use this relation to derive an equation for \(\mathcal{R}\), and use it to study its super-horizon behavior, finding a new conserved quantity, whose conservation implies a possible super-horizon evolution of \(\mathcal{R}\). We identify two mechanisms which can enhance the power spectrum of \(\mathcal{R}\), acting respectively on sub-horizon and super-horizon scale, cause the production of PBH, and which could be used to constrain modified gravity theories or explain the production of PBHs.

Introduction

The study of primordial perturbations is fundamental in any cosmological model, since it allows to make predictions of the conditions which provided the seeds for the anisotropies of the cosmic microwave background (CMB) radiation or for the process of structure formation. Among the different theoretical scenarios proposed to explain the accelerated expansion of the Universe, Horndeski’s theory \(\text{[1]}\) has received a lot of attention, both in the contest of inflation and dark energy.

The calculation of the equation for cosmological perturbations for these theories have been so far performed in the so called unitary gauge, also known as uniform field gauge. While this gauge has some computational convenience when only a scalar field is present, it is not directly related to observations, which depend on the comoving curvature perturbations \(\zeta\). Another example are the numerical codes developed for the solution of the Boltzmann’s equations in a perturbed Friedman-Lematre-Robertson-Walker (FLRW) Universe, which are using equations in the synchronous gauge \([3]\), which for adiabatic perturbations coincides approximately with the comoving gauge \([3]\), justifying the use of the comoving slices gauge for early Universe calculations.

The comoving gauge can differ form the unitary gauge in modified gravity theories because the effective energy momentum tensor arising from the modification of gravity can produce some effective entropy terms, which are absent in \(K(X)\) theories, but are present in any more complicated Horndeski’s theory. The general form of the equation of curvature perturbation in comoving gauge \(\mathcal{R}\) was derived in \([3]\) assuming an arbitrary form of the total effective energy-stress tensor (EST), but no explicit calculation was given in the case of modified gravities. In this letter we compute the general relation between \(\mathcal{R}\) and \(\zeta\) and use it to derive an equation for \(\mathcal{R}\) for KGB theories, confirming the general form predicted in \([3]\), and use it determine the effects on the power spectrum of \(\mathcal{R}\), and its implications on the production of PBHs.

KGB models

In KGB inflation the scalar field \(\Phi\) is minimally coupled to gravity according to the action \([6]\)

\[
S = \int d^4x \sqrt{-g} \left( \frac{M_{Pl}^2}{2} R + L_{KGB}(\Phi, X) \right),
\]

where \(X = g^\mu\nu \partial_\mu \Phi \partial_\nu \Phi\), \(R\) is the Ricci scalar and \(L_{KGB}(\Phi, X)\) is the Lagrangian density of the scalar field corresponding to the KGB action \([6]\], and use it to derive an equation for \(\mathcal{R}\) for KGB theories, identifying a background dependent enhancement function \(E\).

The enhancement of the spectrum of primordial curvature perturbations \(\mathcal{R}\) is fundamental in any cosmological model, since it determine the effects on the power spectrum of \(\mathcal{R}\), and its implications on the production of PBHs.

The perturbed effective energy-stress-momentum tensor

The most general scalar perturbations with respect to a flat FLRW background can be written as

\[
ds^2 = a^2 \left\{ - (1 + 2\Lambda) d\tau^2 + 2\partial_i B dx^i d\tau + [\delta_{ij}(1 - 2\Lambda) + 2\partial_i \partial_j E] dx^i dx^j \right\}
\]

where \(\Lambda\) is the cosmological constant, \(B\) is a real, and \(E\) is a complex scalar.

The perturbed effective energy-stress-momentum tensor is<br>the most general form of the equation of perturbation in this case. As in the flat FLRW case, the perturbed energy-stress-momentum tensor can be written as

\[
T^\mu_\nu = \delta^\mu_\nu \rho + \epsilon \eta^\mu_\nu + \Psi \partial^\mu \Phi \partial_\nu \Phi + \partial^\mu \Phi \partial_\nu \Phi - \eta^\mu_\nu \partial^\alpha \partial_\alpha \Phi,
\]

where \(\rho\) is the energy density, \(\epsilon\) is the equation of state, \(\eta^\mu_\nu\) is the fluid energy-momentum tensor, \(\Psi\) is the trace of the energy-momentum tensor, \(\partial^\mu \Phi \partial_\nu \Phi\) is the effective energy density, and \(\partial^\mu \Phi \partial_\nu \Phi\) is the effective energy-momentum tensor.

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For the decomposition of the scalar field and the EST into their background and perturbation parts we use the notation
\[ \Phi(x^\mu) = \phi(\tau) + \delta \phi(x^\mu) \] (6)
\[ T^{\mu \nu} = T_c^{\mu \nu} + \delta T^{\mu \nu} \] (7)
. The background components of the EST are
\[ T_0^0 = - \rho = K(\phi, \chi) + \frac{3 \mathcal{H} \phi^3}{a^4} G_\chi(\phi, \chi) + \frac{\phi''}{a^2} \left[ K_\chi(\phi, \chi) + G(\phi, \chi) \right], \] (8)
\[ T_{ij} = T_0 = 0, \] (9)
\[ \mathcal{T}_j = \delta j \mathcal{T}, \]
\[ \mathcal{T} = K(\phi, \chi) - \frac{\mathcal{H} \phi'}{a^2} G_\chi(\phi, \chi) + \frac{\phi''}{a^2} \left[ G_\chi(\phi, \chi) + \phi''/a^2 G_\chi(\phi, \chi) \right], \] (10)
where the prime stands for derivatives with respect to \( \tau \), the subscript \( \chi \) stands for derivative with respect to \( \chi = \phi'/a \); and the subscripts \( \phi \) and \( \chi \) denote partial derivatives with respect to these quantities, i.e. \( G_\phi(\phi, \chi) = \partial_\phi G(\phi, \chi) \) and \( G_\chi(\phi, \chi) = \partial_\chi G(\phi, \chi) \). In order to define the comoving slices gauge we need this component of the perturbed EST
\[ \delta T_{0i} = - \left( K_i + 2 G_\phi \cdot \frac{3 \mathcal{H} \phi'}{a^4} G_\chi \right) \frac{\phi''}{a^2} \partial_i \delta \phi + \frac{\phi''}{a^2} G_\chi \partial_i \left( \delta \phi' - \phi' A \right), \] (11)
where \( \mathcal{H} = a'/a \). Under a gauge transformation of the form \( (\tau, x^i) \rightarrow (\tau + \delta \tau, x^i + \delta x^i) \) the perturbations \( \delta \phi, A, B, C, \) and \( E \) transform according to 
\[ \delta \phi \rightarrow \delta \phi - \phi' \delta \tau, \] (12)
\[ A \rightarrow A - \mathcal{H} \delta \tau - \delta \tau', \] (13)
\[ B \rightarrow B + \delta \tau - \delta x', \] (14)
\[ C \rightarrow C + \mathcal{H} \delta \tau, \] (15)
\[ E \rightarrow E - \delta x. \] (16)
The freedom in making an infinitesimal space translation can be used to impose the condition \( E = 0 \) using eq. (11), which we will use in the rest of the letter, while the time translation freedom will be used later to choose the comoving slices gauges.

**Evolution of curvature perturbations in the unitary gauge** In single scalar field models the unitary gauge is defined by the condition \( \delta \phi_a = 0 \). From the gauge transformation in eq. (12) we can see that the time translation \( \delta \tau_a \) necessary to go to the unitary gauge is given by
\[ \delta \tau_a = \frac{\delta \phi}{\phi'}. \] (17)
Using eq. (15) we can compute the curvature perturbation in the unitary gauge \( \zeta \)
\[ \zeta \equiv - C_u = - C - \mathcal{H} \delta \tau_u = - C - \mathcal{H} \frac{\delta \phi}{\phi'} \] (18)
which is by construction gauge invariant. We can also define other gauge invariant quantities such as the unitary gauge lapse function
\[ A_u \equiv A - \mathcal{H} \delta \tau_u - \delta \tau' = A - \mathcal{H} \frac{\delta \phi}{\phi'} - \left( \frac{\delta \phi}{\phi'} \right)'. \] (19)
The second order action for \( \zeta \) in Horndeski’s theories was computed in [8]
\[ S^{(2)}_\zeta = \int dtd^3 x a^3 \left[ G_S \zeta'^2 - \frac{F_S}{a^2} (\partial_\zeta)^2 \right], \] (20)
where \( G_S \) and \( F_S \) are functions of \( K(\phi, \chi) \) and \( G(\phi, \chi) \) and their derivatives. The Lagrange equations for this action give the equation of motion of \( \zeta \)
\[ \zeta'' + \left( 2 \mathcal{H} + \frac{G'_S}{G_S} \right) \zeta' - c_s^2 \Delta \zeta = 0, \] (21)
where \( c_s^2(\tau) = F_S/G_S \). For the Fourier transform of the above equation we use the notation
\[ \zeta''_k + \left( 2 \mathcal{H} + \frac{G'_S}{G_S} \right) \zeta_k' + c_s^2 k^2 \zeta_k = 0. \] (22)

**Comoving slices gauge in KGB models** The comoving slices gauge is defined by the condition \( \delta T_{0i} = 0 \). In KGB theory, combing eqs. (21-23) with eq. (11) we have that under an infinitesimal time translation
\[ \delta T_{0i} \rightarrow \delta T_{0i} + \partial_i \left( \frac{\phi''}{a^2} D \delta \tau \right), \] (23)
where
\[ D = a^2 (2 G_\phi + K_\chi) + G_\chi (-4 \mathcal{H} \phi' + \phi''), \] (24)
from which we get the time translation \( \tau_c \) required to go to the comoving slices gauge
\[ \delta \tau_c = \frac{1}{\phi'} D \left[ - \phi' G_\chi (3 \mathcal{H} \delta \phi + \phi' A - \phi'') + a^2 (2 G_\phi + K_\chi) \delta \phi \right]. \] (25)

Note that in the particular case in which \( G \) does not depend explicitly on \( \chi \), i.e. \( G(\phi, \chi) = G(\phi) \) the above transformation reduces to
\[ \delta \tau_c = \frac{\delta \phi}{\phi'}, \] (26)
and the comoving gauge coincides with the unitary gauge, since in this case the system is equivalent to a \( K(X) \) theory [2-11].
We can now define the comoving curvature perturbation $\mathcal{R}$ according to

$$ \mathcal{R} \equiv - C_c = - C - \mathcal{H} \delta \tau_c . $$  

(27)

Our goal is to derive the equation of motion of $\mathcal{R}$ from eq. (22), and we can achieve this by performing the gauge transformation between the unitary and comoving slices gauge. This is just a special case, when $\delta \theta = 0$ and $A = A_u$, of the general gauge transformation defined by the time translation in eq. (25), giving

$$ \delta \tau_{uc} = - \frac{\phi G}{D} A_u $$  

(28)

from which we can finally get

$$ \mathcal{R} = \zeta + \mathcal{H} \frac{\phi G}{D} A_u . $$  

(29)

We can then eliminate $A_u$ in the above equation using the perturbed Einstein’s equation $\delta G^0_0 = \delta T^0_0 / M_p^2$ in the unitary gauge, which using eq. (11) gives

$$ - \zeta' + \mathcal{H} A_u = - \frac{\phi G}{2 M_p^2 a^2} A_u . $$  

(30)

We can then combine eq. (26) and eq. (30) to obtain a relation between $\mathcal{R}$ and $\zeta$ only

$$ \mathcal{R} = \zeta + \mathcal{H} \frac{\phi G}{D} \left( \frac{\phi G}{2 M_p^2 a^2} + \mathcal{H} \right) \zeta' $$  

(31)

$$ \mathcal{R} = \zeta + \mathcal{E}(\tau) \zeta' . $$

where we have defined the enhancement factor $\mathcal{E}(\tau)$, a quantity depending only on the background, which can induce a significant difference between the curvature on comoving slice and uniform field slices. The relation between the power spectrum of $\zeta$ and $\mathcal{R}$ is then given by

$$ P_{\mathcal{R}} = \frac{k^3}{2 \pi^2} | \mathcal{R}(k) |^2 = P_\zeta + \frac{k^3}{2 \pi^2} \Delta $$  

(32)

where

$$ \Delta = \left[ \mathcal{E} \zeta^2 \zeta' + \mathcal{E} \zeta \zeta'' (\zeta + \mathcal{E} \zeta') \right] $$  

(33)

Note that the above relations are valid on any scale, since they are just based on gauge transformations, without assuming any sub or super horizon limit. This implies that the spectra of $\mathcal{R}$ and $\zeta$ could be different due to a change in the evolution of both sub-horizon and super-horizon modes during the time interval when $\mathcal{E}(\eta)$ is large. On sub-horizon scales the effect is always present, since $\zeta$ is oscillating and $\zeta' \neq 0$, while for super-horizon scales the effect could be suppressed if $\zeta \approx 0$, but even for models conserving $\zeta$ there could be an effect, since the freezing does not happen immediately after horizon crossing. We will discuss later the implication on the production of PBHs.

**Conservation of $\mathcal{R}$ and $\zeta$.** From the above equation we can reach the important conclusion that

$$ \zeta = \text{const} \Rightarrow \zeta = \mathcal{R} = \text{const} $$  

(34)

however the opposite in not true, i.e.

$$ \mathcal{R} = \text{const} \nRightarrow \zeta = \text{const} . $$  

(35)

which can have important implications for conservation laws of $\mathcal{R}$ and non-Gaussianity consistency conditions [11]. As explained previously $\mathcal{R}$, not $\zeta$, is the quantity related to observations, so it would be inconsistent to infer constraints on $\zeta$ from CMB observations for example, since the latter depend on $\mathcal{R}$. From a theoretical point of view the models conserving $\zeta$ on super-horizon scales may be incompatible with observations, because $\mathcal{R}$ could not be conserved, implying for example a violation of the non-Gaussianity consistency condition or a miss-estimation of PBH production.

**Evolution of $\mathcal{R}$ in KGB inflation.** We can finally use eq. (22) and eq. (32) to derive the equation $\mathcal{R}$ in Fourier space

$$ \mathcal{R}'' + \alpha_k (\tau) \mathcal{R}' + \beta_k (\tau) k^2 \mathcal{R} = 0 , $$  

(36)

with the coefficients $\alpha_k$ and $\beta_k$ given by

$$ \alpha_k = \frac{1}{D_k} \left\{ \mathcal{E} k^2 c_s g_S \left[ - 2 \mathcal{E} g_S c_s' + c_s \left( 2 \mathcal{E} g_S + g_S' \right) + 
- 2 \mathcal{E} g_S' \right] + g_S' (2 \mathcal{E} - 4 \mathcal{E} \mathcal{H} + 1) g_S' + \mathcal{E} g_S' + 
- 2 \mathcal{E} g_S'^2 \right\} , $$  

(37)

$$ \beta_k = \frac{1}{D_k} \left\{ \mathcal{E} k^2 c_s^2 g_S^2 - \mathcal{E}^2 c_s g_S' + c_s g_S' \left[ 2 \mathcal{E} c_s' \left( \mathcal{E}' + 
- 2 \mathcal{E} \mathcal{H} + 1 \right) + c_s \left( 2 \mathcal{E}^2 \mathcal{H}' + 2 \mathcal{E}^2 \mathcal{H} + 3 \mathcal{E}' - \mathcal{E} \mathcal{E}'' + 
+ 2 \mathcal{H} \left( \mathcal{E}' + 1 \right) \right) \right] + \mathcal{E} c_s g_S \left[ - 2 \mathcal{E} c_s' g_S' + 
- c_s \left( (\mathcal{E}' + 1) g_S' - \mathcal{E} g_S' \right) \right] \right\} , $$  

(38)

where

$$ D_k = g_S (g_S (2 \mathcal{E}^2 c_s^2 + \mathcal{E}' - 2 \mathcal{E} \mathcal{H} + 1) - \mathcal{E} g_S') . $$  

(39)

Eq. (38) is the main result of this letter. Note that the above equation is in agreement with the general form of the equation in presence of entropy which was derived in [5].
Equation of $\mathcal{R}$ in K-inflation

In K-inflation $G = 0$, implying that the unitary and comoving slices gauge coincide, i.e. $\mathcal{R} = \zeta$, and $\mathcal{E} = 0$, which replaced into eqs. (37, 38) give

$$\alpha_k = \left(2\mathcal{H} + \frac{G_S'}{G_S}\right) = \frac{(a^2 G_S')'}{a^2 G_S},$$

$$\beta_k = c_s^2,$$

as we were expecting from eq. (22). Let us now compute this coefficients in order to show that eq. (30) reduces to the well known equation in K-inflation models. In these models we have

$$G_S = \frac{\chi_\beta}{a^2 H^2},$$

$$F_S = -\frac{M_p^2 (H^2 + H')}{a^2 H^2},$$

where

$$\chi_\beta = K_\chi(\phi, \chi) + 2\chi K_{\chi\chi}(\phi, \chi).$$

After combining eqs. (12, 13) with the background equation

$$\frac{1}{a} (a\mathcal{H})' = -\frac{\mathcal{P} + \mathcal{P}}{2M_p^2},$$

we obtain

$$\beta_k = c_s^2 = \frac{\mathcal{P} + \mathcal{P}}{2\chi_\beta},$$

which coincides with the sound speed defined in K-inflation [10]. The slow-roll parameter $\epsilon$ is defined according to

$$\epsilon \equiv -\frac{(a\mathcal{H})'}{a^3 H^2} = \frac{-\mathcal{P} + \mathcal{P}}{2M_p^2 a^2 H^2},$$

and combining this relation with eq. (40) and eq. (42) we find

$$z^2 = \frac{2a^2 \mathcal{P}}{c_s^2} = \frac{2\chi_\beta}{M_p^2 H^2} = \frac{2a^2 G_S}{M_p^2},$$

which implies

$$\alpha_k = \frac{(z^2)'}{z^2}.$$ 

Thus, in the case of K-inflation eq. (30) reduces to the well known Sasaki-Mukhanov equation

$$\mathcal{R}_k'' + \frac{(z^2)'}{z^2} \mathcal{R}_k' + c_s^2 k^2 \mathcal{R}_k = 0,$$  

Super-horizon conservation of $\mathcal{R}$

On super-horizon scales, assuming the gradient terms can be neglected, and according to eq. (37) $\alpha_k$ become a function of time only, which we denote $\alpha$ and takes the form

$$\alpha = \frac{1}{D}\left\{G_S^2 \left[ -\mathcal{E}' + \mathcal{H} (4\mathcal{E}' + 2) - 4\mathcal{E} \mathcal{H}' + 2 \mathcal{E} \mathcal{H}'' + \mathcal{S} \right] + \left[ (2\mathcal{E}' - 4\mathcal{E} \mathcal{H}) + \mathcal{E} \mathcal{G}_S' \right] - 2 \mathcal{E} \mathcal{G}_S' \right\},$$

where

$$D = G_S \left( \mathcal{E}' - 2\mathcal{E} \mathcal{H} + 1 \right) - \mathcal{E} \mathcal{G}_S'.$$

We can also re-write eq. (30) on super-horizon scales as

$$\left( \frac{\tilde{z}^2}{\mathcal{F}_k} \right)' \approx 0,$$

where we have defined $\left( \frac{\tilde{z}^2}{\mathcal{F}_k} \right)' / \tilde{z}^2 = \alpha$, which implies that the conserved quantity is not $\mathcal{R}_k$ but $R_k \tilde{z}^2$. Depending on the behavior of $\tilde{z}^2$, $\mathcal{R}_k$ may be conserved or not, implying a possible violation of the non-Gaussianity consistency condition [11]. The definition of $\tilde{z}$ implies

$$\tilde{z}^2 \propto \exp \left( \int d\tau \alpha \right),$$

and integrating eq. (53) we can obtain the super-horizon behavior of $\mathcal{R}_k$

$$\mathcal{R}_k \propto \int \frac{d\tau}{\tilde{z}^2} \propto \int d\tau \exp \left( -2 \int d\tau \alpha \right),$$

implying that $\mathcal{R}_k$ can increase when $\tilde{z}^2$ is decreasing. This is consistent with eq. (42), since the enhancement function $E(\tau)$ can induce a growth of $\mathcal{R}$ despite $\zeta$ being approximately constant.

Production of primordial black holes

The super-horizon growth of $\mathcal{R}_k$ could produce primordial black holes which could possibly account for part of dark matter [2, 12, 13] and produce gravitational waves (GW) detectable with future GW detectors such as LISA [2, 10]. The mass $M$ of PBHs produced by mode $\mathcal{R}_k$ re-entering the horizon during the radiation dominated is given by [2]

$$M = \gamma M_H \left|_{F} \right.,$$

where $\gamma \approx 0.2$ is a correction factor, and $M_H \left|_{F} \right.$ is the horizon mass $M_H \equiv (4\pi/3)\rho(a \mathcal{H})^{-3}$ at the time of PBH formation, corresponding to the horizon crossing time

$$k = (a^2 \mathcal{H}) \left|_{F} \right..$$
The present time fraction \( f_{PBH} \) of PBHs of mass \( M \) against the total dark matter component can be approximated as \[ f = 2.7 \times 10^8 \left( \frac{\gamma}{0.2} \right)^{1/2} \left( \frac{g_{*}F}{106.75} \right)^{-1/4} \left( \frac{M}{M_\odot} \right)^{-1/2} \beta, \]
where \( g_{*}F \) is the number of relativistic degrees of freedom at formation. The quantity \( \beta \) is the energy density fraction of PBHs at formation time \[ \beta = \frac{\mathcal{P}_{PBH}}{\mathcal{P}}. \]
which can be written in terms of the probability of the density contrast \( P(\delta) \) as \[ \beta(M) = \gamma \int_{\delta_i}^{1} P(\delta) d\delta, \]
where and \( \delta_i \) is the threshold for PBH formation. Assuming the density perturbations follow a Gaussian distribution \( \beta \) is given by \[ \beta(M) \approx \frac{\gamma}{\sqrt{2\pi}\nu(M)} \exp \left[ -\frac{\nu(M)^2}{2} \right], \]
where \( \nu(M) \equiv \delta_i/\sigma(M) \), and \( \sigma(M) \) is an estimation of the standard deviation of the density contrast on scale \( R \) from the variance \[ \sigma^2(M) = \int d\ln k W^2(kR)\mathcal{P}_R(k) \]
\[ = \int d\ln k W^2(kR) \left( \frac{16}{81} \right)(kR)^4\mathcal{P}_R(k), \]
where \( W(kR) \) is a window function smoothing over the comoving scale \( R(M) = (a^2\mathcal{H})^{-1} \) \[ = 2GM/a_F \gamma^{-1}. \] The PBH fraction \( \beta \) is affected by the power spectrum of \( \mathcal{R} \) since this can increase the standard deviation \( \sigma(M) \), and consequently increase \( \beta \), since the perturbations have a higher probability to collapse into a black hole where they re-enter the horizon.

In the case of KGB models the spectrum \( P_\mathcal{R} \) at the time horizon re-enter can have a strong increase with respect to \( P_\zeta \) due to the enhancement factor \( \mathcal{E} \), an effect that was neglected until now, when studying the evolution of curvature perturbation in the unitary gauge. There can be two main mechanisms of enhancement :

- Super-horizon evolution of \( \mathcal{R} \) due to the behavior of the function \( \alpha \) defined in eq. \[51].

Specific models based on the specification of the function \( K \) and \( G \) will be studied separately, but these conclusions we have achieved are general an can be applied to any of those models.

**Conclusions** We have computed the effective energy-stress-tensor for KGB theories in the comoving slices gauge and have used it to derive a general relation between the unitary gauge curvature \( \zeta \) and the comoving curvature perturbations \( \mathcal{R} \), involving an enhancement function which depends on the evolution of the background, and which can cause a large difference between the two gauge invariant quantities. We have then used this relation to derive an equation for the \( \mathcal{R} \) and used it to determine the super-horizon behavior of \( \mathcal{R} \), finding a new conserved quantity different form \( \mathcal{R} \), and in identifying two different mechanisms which can produce an enhancement of the spectrum of \( \mathcal{R} \) on sub-horizon or super-horizon scales.

We expect similar results to hold for other modified gravity theories such as other Horndeski’s theories \[1\] since also for these theories there can be effective entropy or anisotropy terms which can modify the evolution of curvature perturbations. We have then shown how the enhancement of the power spectrum can produce PBHs, depending on the behavior of the enhancement factor \( \mathcal{E} \). In the future it will be interesting to extend this study to other modified gravity theories and to use observation to constraints the different types of theories.

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