Correlated correlation functions in random-bond ferromagnets

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Abstract

The two-dimensional random-bond $Q$-state Potts model is studied for $Q$ near 2 via the perturbative renormalisation group to one loop. It is shown that weak disorder induces cross-correlations between the quenched-averages of moments of the two-point spin/spin and energy/energy correlation functions, which should be observable numerically in specific linear combinations of various quenched correlation functions. The random-bond Ising model in $(2 + \epsilon)$ dimensions is similarly treated. As a byproduct, a simple method for deriving the scaling dimensions of all moments of the local energy operator is presented.

1 Introduction

When studying disordered systems at or near their critical points, the notion of a single scaling dimension governing the behaviour of an observable field at large distances becomes inadequate. Randomness gives rise to a broad probability distribution for the scaling dimension of a two-point correlation function $\ln G / \ln r$, so the quenched average of $n$th moments of the two-point function will have a non-linear dependence on $n$:

$$G_n(0, r) \sim r^{-2x_n} \text{ with } x_n \neq nx_1$$

where the over-bar indicates the average over quenched disorder. Such multiscaling behaviour has been both predicted analytically and observed numerically in a wide range of systems, but we will primarily concern ourselves with the random-bond $Q$-state Potts model in 2D (the random-bond Ising model in $(2 + \epsilon)$ dimensions is briefly considered in section 3.4). For $Q$ near 2, the critical behaviour of this system is obtainable via an $\epsilon$-expansion about the pure Ising-model fixed point, with an expansion parameter proportional to the specific heat exponent $\alpha$. (We recall the Harris criterion [1], which states that bond randomness will be relevant if and only if the specific heat exponent is positive: hence the disorder is marginal for the Ising model ($Q=2$) and relevant for all $Q$ greater than this value). Following initial work by Ludwig [2,3], multiscaling behaviour of moments of the two-point spin-spin correlation function has been well established in this model both theoretically and numerically (see e.g., [4,5]), and multiscaling in the corrections to scaling for the energy-energy correlator has been predicted. (The energy-sector calculation is complicated by the fact that scaling operators correspond to irreducible representations of the permutation group of replica indices, $S_n$).

In this paper we examine some further structure induced by the disorder, which shows up as correlations between the two-point spin/spin and energy/energy functions. (In fact, the non-trivial replica structure of the energy sector means that a distinction must be made between the
connected and disconnected two-point energy correlators: these may have different behaviour with the spin/spin function, and also be mutually correlated). For example, does
\[ \langle \sigma(0) \sigma(R) \rangle_p \langle \varepsilon(0) \varepsilon(R) \rangle_q \sim R^{-2(x_{\sigma,p}+x_{\varepsilon,q})} \]
or is a more complex behaviour to be expected? We show that this is dependent on both the model in question, and whether the connected or full \( \langle \varepsilon \varepsilon \rangle \) correlator is taken.

In section 2 we shall introduce the \( Q \)-state Potts model and the replica method used to carry out the average over quenched disorder, before describing results already obtained for this model. In section 3.1 we present calculations of scaling dimensions (to one loop) of moments of the energy-operator, and then show (section 3.2) that a non-trivial behaviour of the mixed spin-energy moments is to be expected. This indicates that the quantities \( \ln G_{\sigma\sigma}/\ln R \) and \( \ln G_{\varepsilon\varepsilon}/\ln R \) should be considered as being drawn from a joint probability distribution function that is not equal to the product of the individual marginal distributions, i.e., the two quantities are not statistically independent. We then make contact with quantities available via numerical calculations, by expressing two-point functions of the irrep scaling operators as linear combinations of various quenched averaged energy correlation functions (section 3.3). Finally we describe a similar calculation for the random-bond Ising model in \( (2 + \epsilon) \) dimensions (section 3.4).

2 Model and previous results

We write the reduced lattice hamiltonian for the \( Q \)-state Potts model with weak bond disorder as

\[ H = \sum_{\langle i,j \rangle} \left( -J_0 + \delta J_{i,j} \right) \delta_{s_i,s_j} \]

where \( s_i \in \{1 \ldots Q\} \) are the Potts spin variables, \( J_0 \) is the average bond strength and \( \delta J_{i,j} \ll J_0 \) are the local fluctuations about \( J_0 \), assumed to be completely uncorrelated. The sum is taken over all nearest-neighbour pairs. Taking the local energy density

\[ e(r) = e_{(i,j)} = (\langle -J_0 + \delta J \rangle/kT_c) \delta_{s_i,s_j} \]

and using replicas to average over the disorder, we end up with an effective hamiltonian in terms of \( n \) replicas of the system coupled together. We assume that \( n \) is large, and take the \( n \to 0 \) limit to perform the quenched average at the end of the calculation (see [3] or [3], chapter 8 for an overview of the replica method in field-theoretic calculations):

\[ H_{\text{eff}} = \sum_{a=1}^{n} \left( H^*_a(Q) + t \sum_r e_a(r) \right) - \sum_r \Delta \sum_{a,b}^{n} e_a(r) e_b(r) \]

where \( H^*_a(Q) \) is the reduced hamiltonian for the critical pure Potts model, \( t \) is the reduced temperature \( (T - T_c)/T_c \) and \( \Delta \) is proportional to the second cumulant of the distribution for disorder (power-counting arguments show that higher cumulants are irrelevant at the pure fixed point for \( Q \) near 2). We note further that the expectation value of \( e \) only shifts the critical temperature and cumulants, without affecting the critical exponents in any way; so we can write \( e(r) = \langle e \rangle + \varepsilon(r) \) and absorb the first term into the definitions of the other parameters. Also, terms with \( a = b \) in the double sum can also be neglected since they are either irrelevant by power-counting or yield disconnected diagrams which do not contribute to the renormalisation [3, appendix B]. Finally, moving to the continuum limit, we find

\[ H_{\text{eff}} = \sum_{a=1}^{n} H^*_a(q) + t \int d^2r \sum_{a=1}^{n} \varepsilon_a(r) - \Delta \int d^2r \sum_{a \neq b} \varepsilon_a(r) \varepsilon_b(r). \]
Correlation functions calculated against this effective, ‘replicated’ hamiltonian will correspond to correlators averaged against the initial hamiltonian with quenched disorder:

$$\langle \varepsilon(r) \rangle_H \leftrightarrow \lim_{n \to 0} \langle \varepsilon_i(r) \rangle_{\text{rep}}$$

$$\langle \varepsilon(0) \varepsilon(r) \rangle_H \leftrightarrow \lim_{n \to 0} \langle \varepsilon_i(0) \varepsilon_i(r) \rangle_{\text{rep}}$$

$$\langle \varepsilon(0) \rangle_H \langle \varepsilon(r) \rangle_H \leftrightarrow \lim_{n \to 0} \langle \varepsilon_i(0) \varepsilon_j(r) \rangle_{\text{rep}}, \ i \neq j$$

where $i$ and $j$ label the replicas the operators are lying in. Note that it is assumed that the above correlators are independent of which replicas are actually taken — this is the assumption of replica symmetry, which appears to be valid for weakly disordered ferromagnets.

The quenched average of $p^{th}$ moments of the spin operator was calculated by Ludwig for all $p$, to one loop. The scaling dimension of the operator $\sigma_1 \sigma_2 \ldots \sigma_p$ was found to be

$$X_p = px_\sigma - \frac{y}{16}p(p-1) + O(y^2)$$

where $x_\sigma$ is the scaling dimension of the spin operator at the pure fixed point and $y$ is the RG eigenvalue of $\Delta$, vanishing proportional to $(Q-2)$. For typical realisations of disorder, the behaviour of the two-point function at large $r$ is governed by a multifractal exponent (c.f. [9]). This is given by the saddle point of the Legendre transformation of $X_p$, and was found to be

$$a_0 \equiv \left. \frac{\partial X_p}{\partial p} \right|_{p=0} = x_\sigma + y/16 + O(y^2).$$

In general a given moment of the energy operators does not have a pure scaling behaviour. In the replica formalism, our perturbing operator $\Delta \sum_{a \neq b} \varepsilon_a \varepsilon_b$ is a singlet under the group of permutations of the replica indices, $S_n$ (in this context the permutation group is often named the ‘replica’ group instead). Thus scaling dimensions of operators will only necessarily be constant on subspaces of operators corresponding to irreducible representations (irreps) of $S_n$. Degeneracies between irreps may arise — in fact all irreps of moments of the spin operator are degenerate, due to their different fusion rules with the perturbing operator — but in general a given operator $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_q$ will have a ‘sum-of-powers’ scaling behaviour

$$\langle \varepsilon_1 \varepsilon_2 \ldots \varepsilon_q(0) \varepsilon_1 \varepsilon_2 \ldots \varepsilon_q(r) \rangle \sim \sum_{\mu \in \mathcal{R}} A_\mu r^{-2x_\mu}$$

where $\mu$ runs over the different irreps of $S_n$ present in the decomposition of $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_q$.

Calculations for $q \leq 3$ were given in [4]. In each case the most antisymmetric irrep was the most relevant, and the scaling dimensions of these most relevant irreps were linear in $q$, suggesting no multiscaling behaviour to this order. A more complex structure was however present in the corrections to scaling.

3 Calculation

3.1 Energy Sector: irreducible representations of $S_n$ and $SU(2)$

First we shall exploit the connection between the irreps. of $S_n$ and $SU(2)$ to obtain complete sets of scaling dimensions for moments of the energy-energy correlation function, $\langle \varepsilon(0) \varepsilon(r) \rangle^q$, to one loop. At this order, the effect of a shift in the short-distance cutoff on the couplings is most easily treated in terms of the operator product expansion (OPE) (see e.g., [3] chapter 5). Consider a general partition function $Z$ for a critical hamiltonian $H^*$, perturbed by a set of scaling fields $\phi_i$ with corresponding couplings $g_i$,

$$Z = \text{Tr} \ e^{-H^* - \sum_i g_i \int d^d r \ a_i(r) \phi_i(r)},$$
where \( a \) is the microscopic cutoff. The couplings flow (to first order) as

\[
g_k / dl = (d - x_k)g_k - \sum_{ij} c_{ijk}g_ig_j + \ldots
\]

where \( c_{ijk} \) are the OPE coefficients

\[
\phi_i(r_1) \cdot \phi_j(r_2) \sim \sum_k c_{ijk} \phi_k \left( \frac{r_1 + r_2}{2} \right).
\]

Before the randomness is introduced, the operators \( \varepsilon_a \varepsilon_d \ldots \) with \( q \) distinct replica labels are degenerate. We need to calculate the OPE coefficients of the disorder operator \( O = \sum_{a \neq b} \varepsilon_a \varepsilon_b \) within this subspace. In \( SU(2) \) language let us denote the presence of an energy operator in replica \( i \) by \( |\uparrow_i\rangle \) and its absence by \( |\downarrow_i\rangle \). We can write a general operator by the action of a series of \( SU(2) \) raising operators on a ‘vacuum’ \( |\downarrow_1 \downarrow_2 \ldots \downarrow_n\rangle \):

\[
\varepsilon_a \varepsilon_b \ldots \leftrightarrow \tau^+_a \tau^+_b \ldots |\downarrow_1 \downarrow_2 \ldots \downarrow_n\rangle
\]

with

\[
q = \frac{1}{2} \sum_{i=1}^{n} (1 + 2\tau^z_i) = \frac{1}{2} n + \sum_i \tau^z_i.
\]

In the subspace corresponding to the energy sector, the OPE of one of these operators with the disorder operator \( O \) is equivalent to the action of the matrix

\[
\mathcal{M} = 2 \sum_{i \neq j} \tau^+_i \tau^-_j
\]

on the corresponding state, with \( [\mathcal{M}, \sum_i \tau^z_i] = 0 \) as we would expect. (The extra factor of two arises from the two different ways of performing the operator contraction). If we define the total spin vector \( \vec{S} = \sum_{i=1}^{n} \vec{\tau}_i \) then we can write

\[
\mathcal{M} = 2 \sum_{i \neq j} \tau^+_i \tau^-_j
\]

\[
= 2 \left( \sum_i \tau^+_i \right) \left( \sum_j \tau^-_j \right) - 2 \sum_i \tau^+_i \tau^-_i
\]

\[
= 2(S^+S^- - q)
\]

\[
= 2(S^2 - (S^z)^2 + S^z - q)
\]

Using \( q = S^z + \frac{n}{2} \) we can thus write

\[
\mathcal{M} = 2 \left( S(S + 1) - (S^z)^2 - \frac{n}{2} \right).
\]

However, we now have the problem of interpreting the total spin angular momentum \( S \). One solution is to ask which values of \( S \) are consistent with the known value of \( S^z \) — clearly, the possible values are

\[
S = \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, \frac{n}{2} - q.
\]

We can support this by comparing the irreducible representations of \( SU(2) \) with the irreducible representations of \( S_n \) considered by Ludwig. In the \( S_n \) case, we have to consider the representations
given by all Young tableaux of \( n \) boxes and at most two rows, with up to \( q \) boxes in the second row. E.g., for \( q = 2, n = 6 \) the possible irreps are [11).

\[
\begin{array}{c}
\varepsilon \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\end{array}
\quad
\begin{array}{c}
\varepsilon \\
\varepsilon \\
\varepsilon \\
\varepsilon \\
\end{array}
\quad
\begin{array}{c}
\varepsilon \\
\varepsilon \\
\end{array}
\]

These irreps can be placed in one-to-one correspondence with the representations formed by the tensor product of \( n \) spin-half representations of \( SU(2) \) with precisely \( q \) of them spin-up. Of these latter tableaux, a tableau with \( (n - r) \) boxes in the first row and \( r \) boxes in the second row has a dimensionality of \( D = n - 2r + 1 \) corresponding to a total spin of \( S = \frac{D - 1}{2} = \frac{n}{2} - r \), as was proposed by the consistency condition. Hence, in general the \( q^{th} \) moment of the energy operator will have a series of irreps \( \mathcal{I}_{q,r} \) with scaling dimensions

\[
X_{q,r}(n) = qx_c - y \cdot \frac{1}{4} \cdot 2 \left( \left( \frac{n}{2} - r \right) \left( \frac{n}{2} - r + 1 \right) - \left( \frac{n}{2} - q \right)^2 - \frac{n}{2} \right) + O(y^2)
\]

where \( r \in \{0, 1, \ldots, q\} \), \( y \) is the RG eigenvalue of the coupling to disorder and \( x_c \) is the scaling dimension of the energy operator at the pure fixed point,

\[
x_c = 1 - \frac{y}{2} + O(y^2).
\]

The factor of \( \frac{1}{4} \) arises from the position of the disordered fixed point, set (to one loop) by

\[
d\Delta/dl = \alpha \Delta + 4(n-2)\Delta^2 + O(\Delta^3).
\]

After taking the \( n \to 0 \) limit we obtain

\[
X_{q,r}(0) = q - \frac{y}{2} (r(r-1) - q^2 + q) + O(y^2).
\]

The most relevant scaling dimension belongs to the most antisymmetric irrep \( \mathcal{I}_{q,q} \) and has a linear dependence on \( q \). Note that the irreps \( \mathcal{I}_{q,0} \) and \( \mathcal{I}_{q,1} \) are always degenerate at \( n = 0 \). This must happen because for \( n \neq 0 \) they have degeneracies 1 and \( (n-1) \) respectively, while as \( n \to 0 \) all operators must arrange themselves to have degeneracies \( \alpha n \), so that the torus partition function is \( 1 + O(n) \). In fact, this collision of the scaling dimensions at \( n = 0 \) will give rise to a logarithmic operator [1].

The operators \( \varepsilon_1 \ldots \varepsilon_q \) span a vector space of dimension \( \binom{n}{q} \). In general, a Young tableau of shape \([n-r, r] \) corresponds to a vector space of dimensionality \( \binom{n-r}{n-r-r+1} \), and it can be shown that the sets of permitted irreps \( (r \in [0, q]) \) exhaust this vector space. The dimensionality of a given irrep does not depend on the number of energy operators in the tableau, and is preserved under the addition of extra \( \varepsilon \)-operators via \( S^+ \) — irreps related via the action of \( S^\pm \) are joined in the plot below (figure [1]):

The two-point function of the most antisymmetric irrep for a given \( q, \mathcal{I}_{q,q} \), corresponds to the quenched average of the \( q^{th} \) moment of the connected correlation function,

\[
\overline{G^2_q} = \langle \varepsilon_1 - \varepsilon_2 \ldots (\varepsilon_{2q-1} - \varepsilon_{2q})(0)(\varepsilon_1 - \varepsilon_2) \ldots (\varepsilon_{2q-1} - \varepsilon_{2q}) \rangle .
\]

This can be seen by noting that the state corresponding to the operator \( (\varepsilon_1 - \varepsilon_2) \ldots (\varepsilon_{2q-1} - \varepsilon_{2q}) \) is annihilated by the step-down operator \( S^- = \sum_{i=1}^n \tau_i^- :
\]

\[
\left( \sum_{i=1}^n \tau_i^- \right) (\tau_1^+ - \tau_2^+) \ldots (\tau_{2q-1}^+ - \tau_{2q}^+)(\downarrow\downarrow\downarrow\downarrow) = 0.
\]

All \( \tau_i^- \) with \( i > 2q \) give zero when acting on everything to their right, so we only need consider those \( \tau_i^- \) with \( i \leq 2q \). Take the terms of the step-down operator in pairs: the first pair \( (\tau_1^- + \tau_2^-) \) will act on everything to the right to give

\[
(\tau_1^- \tau_1^+ - \tau_2^- \tau_2^+)(\tau_3^+ - \tau_4^+) \ldots (\tau_{2q-1}^+ - \tau_{2q}^+)(\downarrow\downarrow\downarrow\downarrow) = 0.
\]
The action of all other pairs similarly vanishes, so the state corresponding to the connected correlation function is annihilated by $S^-$ and is therefore also the state corresponding to the most antisymmetric irrep.

### 3.2 Mixed Sector: Exclusion argument

We now wish to consider the behaviour of cross-moments, i.e., investigate the renormalisation of the operators $\sigma_1 \ldots \sigma_p \varepsilon_1 \ldots \varepsilon_q$. Fortunately, given the behaviour of the $\varepsilon_1 \ldots \varepsilon_q$ operators considered above this is not a particularly difficult extension. To one loop, we again consider the effect of the disorder operator $\sum_{a \neq b} \varepsilon_a \varepsilon_b$ on the operator $\sigma_{i_1} \ldots \sigma_{i_p} \varepsilon_{j_1} \ldots \varepsilon_{j_q}$, and look for operators of the same form to be produced by all possible contractions. These contractions are given by the fusion rules of our theory at the pure fixed point:

$$
\varepsilon_a \cdot \varepsilon_b \sim \delta_{a,b}
$$

$$
\sigma_a \cdot \sigma_b \sim \delta_{a,b} \left(1 + \frac{1}{2} + O(Q-2)\right) \varepsilon_a
$$

Contractions are only possible within a given replica. Given the fusion rules, at this order there are only two possibilities:

1. $a \in \{j\}, b \notin \{i\} \cup \{j\}$: one of the energy operators in the disorder operator contracts against another energy operator ($\varepsilon \varepsilon \rightarrow 1$), the other lies in an ‘empty’ replica. This gives the same behaviour as in the pure energy sector, but with a shift in the effective number of replica indices: $n \rightarrow n-p$

2. $a, b \in \{i\}, a \neq b$: both energy operators in the disorder operator contract with spin operators, giving rise to further spin operators. As considered by Ludwig, this gives rise to a term $\frac{1}{4}p(p-1)$

Any other attempted contractions will evaluate to zero when the the trace over the fields is taken. Substituting these changes into our expression for the energy sector, after taking the $n \rightarrow 0$ limit we find

$$
X_{p,q,r} = px_\alpha + q - \frac{y}{4} \left[ 2 \left\{ \left( \frac{p}{2} + r \right) \left( \frac{p}{2} + r - 1 \right) - \left( q + \frac{p}{2} \right)^2 + \frac{p}{2} + q \right\} + \frac{1}{4} (p^2 - p) \right] + O(y^2)
$$
where as before \( r \in [0, q] \) and \( x_r \) is the scaling dimensions of the spin operator at the pure fixed point. The spin operators exclude the energy operators from their replicas, giving rise to an effective shift in the number of replicas \( n \rightarrow n - p \).

The scaling dimensions of the most antisymmetric irreps \( r = q \) are unaffected by this exclusion process, as they have no \( n \)-dependence. This means that, to one loop, there will be no extra structure in the quenched average \( \langle \sigma(0) \sigma(R) \rangle \langle \varepsilon(0) \varepsilon(R) \rangle_c \). In section 3.4, however, we shall demonstrate that this quantity does have extra structure for the case of the random-bond Ising model in \( 2 + \epsilon \) dimensions.

By analogy with the \( SU(2) \) argument, it is possible to consider the mixed sector by writing our operators \( \{ \varepsilon, \sigma, 1 \} \) as a fundamental representation of \( SU(3) \). This analysis leads to the same results, as the \( \{ 1, \varepsilon \} \) \( SU(2) \) subgroup effectively decouples from the spin operator.

As one check on these results, standard arguments in probability theory (see, e.g., [12]) require that \(-X_{p,q,q}\) should be convex for all increases in \( p \) or \( q \), both individually and jointly. Considering the second derivatives of this quantity w.r.t. \( p \) and \( q \), this requirement can be seen to be trivially satisfied for \( y > 0 \).

### 3.3 Replica Structure

Given that non-trivial structure in the spin-energy sector only appears to sub-leading order, it is useful to be able to write two-point functions of irreducible representations of the replica group in terms of numerically available quantities. We have already noted that the most relevant, most antisymmetric irreps correspond to moments of the connected part of the two-point correlation function (section [3.2]) and would like to be able to say something similar for a general irrep. In Appendix A we detail the combinatorics needed to decompose a two-point function of irreps into a linear combination of various other two-point functions: here we shall merely note a few results.

We find explicitly that the most antisymmetric irrep at a given \( q \), \( \mathcal{I}_{q,q} \) has a two-point function

\[
\langle \mathcal{I}_{q,q}(0) \mathcal{I}_{q,q}(R) \rangle \sim \langle (\varepsilon_1 - \varepsilon_2) \ldots (\varepsilon_{2q-1} - \varepsilon_{2q}) (0) (\varepsilon_1 - \varepsilon_2) \ldots (\varepsilon_{2q-1} - \varepsilon_{2q}) (R) \rangle_{\text{rep}}
\]

\[
\sim \frac{G_{\text{rep}}^2(R)}{G_{\text{rep}}^2(0)}
\]

In the \( q = 1 \) case, both the possible irreps \( \mathcal{I}_{1,0} \) and \( \mathcal{I}_{1,1} \) correspond to the connected correlator \( \langle \varepsilon(0) \varepsilon(R) \rangle - \langle \varepsilon(0) \rangle \langle \varepsilon(R) \rangle \). This is in fact a general feature, since the lowest two irreps for arbitrary \( q \) are always degenerate after taking the \( n \rightarrow 0 \) limit.

As an example of a correlation function which has a non-trivial behaviour on addition of \( p \) spin-spin correlators, the irrep \( \mathcal{I}_{2,1} \) is a non-leading irrep which can be written as

\[
\langle \mathcal{I}_{2,1}(0) \mathcal{I}_{2,1}(R) \rangle \sim \langle \varepsilon(0) \varepsilon(R) \rangle \langle \varepsilon(0) \rangle - 4 \langle \varepsilon(0) \varepsilon(R) \rangle \langle \varepsilon(0) \rangle \langle \varepsilon(R) \rangle + 3 \langle \varepsilon(0) \rangle \langle \varepsilon(0) \rangle \langle \varepsilon(R) \rangle \langle \varepsilon(R) \rangle.
\]

If we take a factor of \( \langle \sigma(0) \sigma(R) \rangle^p \) inside each quenched average on the RHS, the resulting scaling dimension will be

\[
X_{p,2,1} = p x_\sigma + 2 x_\varepsilon - \frac{y}{4} (-8 - 2p + \frac{1}{4} (p^2 - p)) + O(y^2)
\]

\[
X_{p,0,0} + X_{0,2,1} = p x_\sigma + 2 x_\varepsilon - \frac{y}{4} (-8 + \frac{1}{4} (p^2 - p)) + O(y^2)
\]

Thus this correlation function directly exhibits the cross-structure between the two-point function of the spin operator and the full two-point function of the energy operator.

### 3.4 Ising model in \( 2 + \epsilon \) dimensions

As an alternative to perturbing the pure Ising model by changing the number of possible values for the spin, we can consider the Ising model in \( 2 + \epsilon \) dimensions. The specific heat exponent \( \alpha \) is zero at \( d = 2 \) and small for \( d = 3 \) (< 0.1), so we shall assume that \( \alpha = O(\epsilon) \) with \( \epsilon \ll 1 \). The bond
disorder is relevant for all \( \epsilon > 0 \), and we assume that the system will flow to a nearby disordered fixed point. The calculation of scaling dimensions at this fixed point is very similar to that for the disordered Potts model — the only difference is that the OPE rules are altered to

\[
\begin{align*}
\varepsilon_a \cdot \varepsilon_b & \sim \delta_{a,b}(1 + c\varepsilon) \\
\sigma_a \cdot \sigma_b & \sim \delta_{a,b}(1 + ((1/2) + O(\epsilon))\varepsilon)
\end{align*}
\]

The OPE coefficient \( c \) is zero for \( d = 2 \) due to the self-duality of the Ising model in two dimensions \cite{[5]}, chapter 8], but non-zero for \( d > 2 \). Repeating the analysis of section 3.2 we can obtain two more types of contraction between the disorder operator \( \sum_{a \neq b} \varepsilon_a \varepsilon_b \) and \( \sigma \varepsilon \):

1. Both energy operators in the disorder operator contract with energy operators, giving rise to two further energy operators in the same replicas. This gives a term \( c^2 q(q - 1) \).

2. The disorder operator contracts with one energy operator and one spin operator, giving rise to one operator of each type with unchanged replica indices. This produces a term \( 2 \cdot (1/2) \cdot c \cdot pq = cpq \).

The non-zero \( c \) will shift the position of the fixed points, via the RG equation for the disorder

\[
d\Delta/dl = \alpha\Delta + (4(n - 2) + 2c^2) \Delta^2 + O(\Delta^3, \epsilon\Delta^2)
\]

so we obtain scaling dimensions

\[
X_{p,q,r} = px_x + qx_x + \frac{y}{c^2 - 4} \left\{ 2 \left[ \left( \frac{p}{2} + r \right) \left( \frac{p}{2} + r - 1 \right) - \left( q + \frac{p}{2} \right)^2 + \frac{p}{2} \right] + \cdots \right. \\
\left. + \frac{1}{4} (p^2 - p) + c^2 (q^2 - q) + cpq \right\} + O(y^2)
\]

The \( cpq \) term indicates that, with a non-zero \( c \), we now have cross-structure between the two-point function of the spin operator and the connected two-point function of the energy operator, i.e. \( \langle \sigma(0)\sigma(R) \rangle^p \langle \varepsilon(0)\varepsilon(R) \rangle^q \neq \langle \sigma(0)\sigma(R) \rangle^p \cdot \langle \varepsilon(0)\varepsilon(R) \rangle^q \).

4 Conclusions

To one loop, the interaction between the spin and energy operators is an ‘exclusion effect’: the spin operators block replicas, shifting the \( n \)-dependence of the scaling dimensions of the different irreps in the energy sector. For the Potts model this does not have any effect on the leading behaviour, as the most relevant irrep irrep \( r = q \) has a scaling dimension with no \( n \)-dependence. The sub-leading terms are affected, however, and this may be picked up in numerical studies. By writing the irreps in terms of correlation functions we have shown how linear combinations of correlators could be used to demonstrate the existence of an underlying joint distribution function for the quenched average of spin/spin and energy/energy moments. In the Ising model in \((2 + \epsilon)\) dimensions the leading behaviour of mixed moments is affected non-trivially. Also, three- or higher- point functions should also exhibit this cross-sector behaviour, presumably with selection rules coming from the group structure of the energy sector.

There remains the question of what happens when these calculations are continued to two loops. \textit{A priori}, there seems no reason to expect that the scaling dimensions of the most antisymmetric irreps will continue to be protected against corrections from cross-correlations between the spin and energy two-point functions. We will content ourselves here with noting that a coulomb-gas calculation along the lines of \cite{[8], chapter 8} would give two-loop corrections to the scaling dimensions of the operator \( \sigma \varepsilon \) proportional to

\[
\int d^2y \langle \varepsilon(0)\varepsilon(y)\varepsilon(1)\varepsilon(\infty) \rangle \langle \varepsilon(0)\varepsilon(y)\sigma(1)\sigma(\infty) \rangle
\]

which we would expect to be non-zero, although a detailed calculation would of course be necessary to confirm this.
5 Acknowledgements

After this work was substantially completed, we heard of work by Jeng and Ludwig [13] who, in the course of their investigation of random defect lines in 2D systems, also derived (by an alternative method) the scaling dimensions of moments of the energy operator, to two-loop order. We would like to thank them for interesting discussions on this and other points. We also thank A. Cavagna and R. Stinchcombe for useful conversations. This work was supported in part by the Engineering and Physical Sciences Research Council under Grant GR/J78327 and Studentship 98311143.
A Decomposition of $\langle J_{q,r}(0)J_{q,r}(R) \rangle$ into quenched averaged energy correlators

Start with the most antisymmetric irrep for $q$ energy operators, $J_{q,q}$, corresponding to the Young tableau

$$
\begin{array}{ccc}
\varepsilon_1 & \cdots & \varepsilon_q \\
\varepsilon_{q+1} & \cdots & \\
\vdots & \ddots & \varepsilon_{n-2q}
\end{array}
$$

The correlator $\langle J_{q,q}(0)J_{q,q}(R) \rangle$ is proportional to $\left[ \varepsilon \right]^2$

$$(\varepsilon_1 - \varepsilon_n)(\varepsilon_2 - \varepsilon_{n-1}) \cdots (\varepsilon_q - \varepsilon_{n-q+1})(0) \sum_{a^i \neq b^j \neq \cdots \neq q'}^{n-q} (\varepsilon_{a'} - \varepsilon_n)(\varepsilon_{b'} - \varepsilon_{n-1}) \cdots (\varepsilon_{q'} - \varepsilon_{n-q+1})(R).$$

We shall start by expanding this into $2^{2q}$ monomials:

- Choose $N$ operators from the LHS with index in $[n-q+1,n]$, $0 \leq N \leq q$. There are $\sum_{N=0}^{q} \binom{q}{N}$ ways of doing this.
- Choose $M$ matching operators on the RHS, $0 \leq M \leq N$: $\sum_{M=0}^{N} \binom{N}{M}$ ways.
- Choose $O$ other operators with index in $[n-q+1,n]$ on the RHS, $0 \leq O \leq q-N$: $\sum_{O=0}^{q-N} \binom{q-N}{O}$ ways.

Note that $\sum_{N=0}^{q} \sum_{M=0}^{N} \sum_{O=0}^{q-N} \binom{q-N}{O} = 2^{2q}$, so we have accounted for all the terms.

For a typical monomial

$$\varepsilon_1 \cdots \varepsilon_{q-N} \varepsilon_{q-N+1} \cdots \sum \varepsilon_{a'} \cdots \varepsilon_{q-M-O} \varepsilon_{q} \cdots \varepsilon_{q-M+1} \varepsilon_{q-N} \cdots \varepsilon_{q-N-O+1},$$

the number of explicit free indices $\varepsilon_{a'} \varepsilon_{b'} \cdots$ on the RHS that are paired, $P$, runs over the range $0 \leq P \leq \text{Min}(q-N, q-M-O)$. For a given $P$ we have the following factors:

- A sign of $(-1)^{N+M+O}$
- The number of ways of choosing the pairings, $\binom{q-N}{P} \binom{q-M-O}{P} P!$
- $(q-M-O-P)$ explicit, unpaired free indices ranging over $(n-q)-(q-N)$ replicas, giving a factor of

$$\frac{(n+N-2q)!}{(n+N-3q+M+O+P)!}$$

- $(M+O)$ implicit free indices on the RHS (which must still be summed over) ranging over $(n-q)-(q-M-O)$ replicas, a factor of

$$\frac{(n-2q+M+O)!}{(n-2q)!}$$

A term with $P$ pairings will produce a correlator

$$C_{P+M,q-P-M} = \langle \varepsilon(0) \rangle^{q-P-M} \langle \varepsilon(0) \rangle P^M \langle \varepsilon(R) \rangle^{q-P-M}$$

Altogether, our decomposition into correlators becomes

$$\langle J_{q,q}(0)J_{q,q}(R) \rangle = \sum_{N=0}^{q} \sum_{M=0}^{N} \sum_{O=0}^{q-N} \sum_{P=0}^{\text{Min}(q-N,q-M-O)} (-1)^{N+M+O} \frac{(n+N-2q)!}{(n+N-3q+M+O+P)!} \frac{(n-2q+M+O)!}{(n-2q)!} C_{P+M,q-P-M}.$$
For various $q$, this formula gives

\begin{align*}
q = 1 & \quad n(C_{1,0} - C_{0,1}) \\
q = 2 & \quad (n - 1)(n - 2)(C_{2,0} - 2C_{1,1} + C_{0,2}) \\
q = 3 & \quad (n - 2)(n - 3)(n - 4)(C_{3,0} - 3C_{2,1} + 3C_{1,2} - C_{0,3}) \\
q = 4 & \quad (n - 3)(n - 4)(n - 5)(n - 6)(C_{4,0} - 4C_{3,1} + 6C_{2,2} - 4C_{1,3} + C_{0,4}) \\
q = 5 & \quad (n - 4)(n - 5)(n - 6)(n - 7)(n - 8)(C_{5,0} - 5C_{4,1} + 10C_{3,2} - 10C_{2,3} + 5C_{1,4} - C_{0,5})
\end{align*}

i.e., the most antisymmetric irreps correspond to moments of the connected two-point correlation function $G^{\mu}_{\mu}$.

The above argument can easily be extended to irreps which are not the most antisymmetric. If we consider now the irrep $\Sigma_{q,r}$ corresponding to Young tableau

\begin{center}
\begin{array}{cccc}
\varepsilon & \ldots & \varepsilon & \varepsilon \\
\ldots & & \ldots & \\
q-r & & n-q-r & \\
\end{array}
\end{center}

the following changes occur:

- The $(q - M - O - P)$ explicitly unpaired indices now range over $(n - r) - (q - N)$ replicas, giving a factor of

\[
\frac{(n + N - q - r)!}{(n + N - 2q - r + M + O + P)!}
\]

- The $(M + O)$ implicit free indices range over $(n - r) - (q - M - O)$ replicas, a factor of

\[
\frac{(n - q - r + M + O)!}{(n - q - r)!}
\]

Incorporating these changes, we obtain (for instance)

\begin{align*}
q = 1, r = 0 & \quad C_{1,0} + (n - 1)C_{0,1} \\
q = 2, r = 0 & \quad 2C_{2,0} + 4(n - 2)C_{1,1} + (n - 2)(n - 3)C_{0,2} \\
q = 2, r = 1 & \quad n(C_{2,0} + (n - 4)C_{1,1} + (3 - n)C_{0,2})
\end{align*}
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