Radiation of the blackbody in the external field

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Abstract

The blackbody is considered in the external general field. The additional coefficients of stimulated emission and stimulated absorption are introduced into the Einstein mechanism and the generalized Planck formula is derived. The Einstein and Debye formula for the specific heat is possible to generalize. The Bose-Einstein statistics is broken in the external field. The relation of the theory to the sonoluminescence, the relic radiation and solar spectrum is considered.

Key words. Planck formula, Einstein’s blackbody, phonons, Einstein’s and Debye’s specific heat, sonoluminescence, relic radiation.

1 Introduction

The distribution law of photons inside of the so called blackbody was derived in 1900 by Planck (1900, 1901), (Schöpf, 1978). The derivation was based on the investigation of the statistics of the system of oscillators. Later Einstein (1917) derived the Planck formula from the Bohr model of atom. Bohr created two postulates which define the model of atom: 1. every atom can exist in the discrete series of states in which electrons do not radiate even if they are moving at acceleration (the postulate of the stationary states), 2. transiting electron from the one stationary state to other, emits the energy according to the law
\[ \hbar \omega = E_m - E_n, \]  

where \( E_m \) is the energy of an electron in the initial state, and \( E_n \) is the energy of the final state of an electron to which the transition is made and \( E_m > E_n \).

Einstein introduced coefficients of spontaneous and stimulated emission \( A_{mn}, B_{mn}, B_{nm} \). In the case of spontaneous emission, the excited atomic state decays without external stimulus as an analogue of the natural radioactivity decay. The energy of the emitted photon is given by the Bohr formula (1). In the process of the stimulated emission the atom is induced by the external stimulus to make the same transition. The external stimulus is a blackbody photon that has an energy given by the Bohr formula (1).

If the number of the excited atoms is equal to \( N_m \), the emission energy per unit time conditioned by the spontaneous transition from energy level \( E_m \) to energy level \( E_m \) is

\[ P_{\text{spont. emiss.}} = N_m A_{mn} \hbar \omega, \]  

where \( A_{mn} \) is the coefficient of the spontaneous emission.

In case of the stimulated emission, the coefficient \( B_{mn} \) corresponds to the transition of an electron from energy level \( E_m \) to energy level \( E_n \) and coefficient \( B_{nm} \) corresponds to the transition of an electron from energy level \( E_n \) to energy level \( E_m \). So, for the energy of the stimulated emission per unit time we have two formulas:

\[ P_{\text{stimul. emiss.}} = \varrho \omega N_m B_{mn} \hbar \omega \]  
\[ P_{\text{stimul. absorption}} = \varrho \omega N_n B_{nm} \hbar \omega. \]

If the blackbody is in thermal equilibrium, then the number of transitions from \( E_m \) to \( E_n \) is the same as from \( E_n \) to \( E_m \) and we write:

\[ N_m A_{mn} \hbar \omega + N_m \varrho \omega B_{mn} \hbar \omega = N_n \varrho \omega B_{nm} \hbar \omega, \]

where \( \varrho \omega \) is the density of the photon energy of the blackbody.

Then, using the Maxwell statistics

\[ N_n = D e^{-\frac{E_n}{kT}}, \quad N_m = D e^{-\frac{E_m}{kT}}, \]  

we get:

\[ \varrho \omega = \frac{A_{mn} B_{mn}}{B_{mn} \varrho \omega \hbar \omega - 1}. \]  

The spectral distribution of the blackbody does not depend on the specific atomic composition of the blackbody and it means the formula (7) must be so called the Planck formula:

\[ \varrho \omega = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar \omega}{kT}} - 1}. \]  

After comparison of eq. (7) with eq. (8) we get:
\[ B_{mn} = B_{nm} = \frac{\pi^2 e^3}{\hbar \omega^3} A_{mn}. \]  

(9)

It means that the probabilities of the stimulated transitions from \( E_m \) to \( E_n \) and from \( E_n \) to \( E_m \) are proportional to the probability of the spontaneous transition \( A_{mn} \). So, it is sufficient to determine only one of the coefficient in the description of the radiation of atoms.

The Planck law (8) can be also written as

\[ \varrho_\omega = G(\omega) < E_\omega > = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar \omega}{e^{\frac{\hbar \omega}{kT}} - 1}. \]  

(10)

where the term

\[ < E_\omega > = \frac{\hbar \omega}{e^{\frac{\hbar \omega}{kT}} - 1} \]  

(11)

is the average energy of photons in the blackbody and

\[ G(\omega) = \frac{\omega^2}{\pi^2 c^3} \]  

(12)

is the number of electromagnetic modes in the interval \( \omega, \omega + d\omega \).

Let us remark that coefficients \( A_{mn} \) of the so called spontaneous emission cannot be specified in the framework of the classical thermodynamics, or, statistical physics. They can be determined only by the methods of quantum electrodynamics as the consequences of the so called radiative corrections. So, the radiative corrections are hidden external stimulus, which explains the spontaneous emission.

## 2 N-dimensional blackbody

The problem of the N-dimensional blackbody is related to the dimensionality of space, or, of space-time and some ideas on the dimensionality was also expressed and analyzed in recent time by author (Pardy, 2005).

The experimental facts following from QED experiments, galaxy formation and formation of the molecules DNA, prove that the external space is 3-dimensional. With regard to the Russell philosophy of mathematics, there is no possibility to prove the dimensionality of space, or, space-time by means of the pure mathematics, because the statements of mathematics are nonexistental. The existence of the external world cannot be also proved by pure mathematics. However, if there is an axiomatical system related to the external world and reflecting correctly the external world, then, it is possible to do many predictions on the external world by pure logic. This is the substance of exact sciences. We know for instance that the success of special theory of relativity is is based on the adequate axiomatical system and on logic.

In case of the n-dimensional blackbody, the number of modes can be determined (Al-Jaber, 2003). We use here alternative and elementary derivation. We know, that in case that the electromagnetic field is in a box of the volume \( L^n \), the wave vector \( \mathbf{k} \) is quantized and the elementary volume in the k-space is \( \Delta_0 = (2\pi)^n/L^n \).

The elementary volume of the n-dimensional k-space is evidently the volume \( dV_n \) between spheres with radius \( k \) and \( k + dk \):
\[ dV_n = d \left\{ \frac{2\pi^{n/2}}{n\Gamma \left( \frac{n}{2} \right)} k^n \right\} = \frac{2\pi^{n/2}}{\Gamma \left( \frac{n}{2} \right)} k^{n-1} dk, \]  
(13)

where \( \Gamma(n) \) is so called Euler gamma-function defined in the internet mathematics (http://mathworld.wolfram.com/GammaFunction.html) as

\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt; \quad \Gamma\left( \frac{n}{2} \right) = \frac{(n-2)!! \sqrt{\pi}}{2^{(n-1)/2}}.
\]  
(14)

The number of electromagnetic modes involved inside the spheres between \( k \) and \( k + dk \) is then, with \( \omega = ck \), or \( k = \omega/c \) and \( dk = d\omega/c \),

\[
2 \times \frac{dV_n}{\Delta_0 n} = 2 \times \frac{1}{\Gamma \left( \frac{n}{2} \right)} \frac{1}{2^{(n-1)/2}} \pi^{n/2} \frac{L_n}{c^n} \omega^{n-1} \frac{d\omega}{c^n},
\]  
(15)

where isolated number 2 expresses the fact that light has 2 polarizations. It means that the number of modes \( G_n \), in volume \( V \), involved between spheres \( k \) and \( k + dk \), or, \( \omega \) and \( \omega + d\omega \), is

\[
G_n(\omega) d\omega = 2 \times \frac{1}{2^{(n-1)/2}} \frac{1}{\pi^{n/2}} L_n \frac{\omega^{n-1} d\omega}{c^n}.
\]  
(16)

For the spectral distribution, we have

\[
\varrho_{n\omega} d\omega = \langle E_n \rangle \frac{G(\omega) d\omega}{L_n} = \langle E_n \rangle g(\omega) d\omega,
\]  
(17)

where the average value of energy \( \langle E_n \rangle \) is given by eq. (11).

The spectral distribution of the \( n \)-dimensional blackbody has the final form following from eq. (17):

\[
\varrho_{n\omega} = \frac{\hbar \omega}{e^{\frac{\hbar \omega}{kT}} - 1} \frac{1}{2^{(n-2)/2}} \frac{1}{\pi^{n/2}} \frac{\omega^{n-1}}{c^n}.
\]  
(18)

3 Blackbody in the external field

The Einstein derivation does not consider the situation where the blackbody is influenced by some external field. Einstein was not motivated by the case of so called sonoluminescence, or by relic radiation and so on, which play at present time in physics the fundamental role. Now, if we include some external nonspecified field, it is necessary to introduce transition coefficients \( C_{mn}, C_{nm} \). From the equilibrium condition

\[
N_mA_{mn} + N_m\varrho_{\omega}B_{mn} + N_mC_{mn} = N_n\varrho_{\omega}B_{nm} + N_nC_{nm}
\]  
(19)

and Maxwell statistics (6) we get:

\[
A_{mn} + \varrho_{\omega}B_{mn} + C_{mn} = e^{\frac{\hbar \omega}{kT}} \varrho_{\omega}B_{nm} + e^{\frac{\hbar \omega}{kT}} C_{nm},
\]  
(20)

which can be modified using the definitions
as follows:

\[ 1 + \beta_{mn} - e^{\frac{\hbar\omega}{kT}} \beta_{mn} = \frac{2\pi^2 c^3}{\hbar\omega^3} \left( e^{\frac{\hbar\omega}{kT}} - 1 \right) \]  

(22)

The generalized Planck law follows from the equation (22):

\[ \varrho_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \left( \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} + \frac{P(\omega) - Q(\omega)e^{\frac{\hbar\omega}{kT}}}{\pi^2 c^3 e^{\frac{\hbar\omega}{kT}} - 1} \right), \]

(23)

where \( P = \beta_{mn}, Q = \beta_{nm} \) are some function which must be calculated using the advanced solid state physics and advanced quantum mechanics. In order to get the second term in eq. (23) finite for \( \omega \to \infty \), it is possible to postulate the mathematical structure of \( Q(\omega) \) as follows: \( Q(\omega) = R \exp(-S\omega) \), where \( R, S \) are some constants.

4 Discussion

The original Planck derivation of the blackbody radiation was based on the relation between the entropy of the system and the internal energy of the blackbody denoted by Planck as \( U \).

While from the postulation of the relation

\[ \frac{d^2 S}{dU^2} = -\frac{\text{const}}{U} \]

(24)

the Wien law follows, the a priori generalization of eq. (24) gives new physics. The generalization was postulated by Planck in the following form:

\[ \frac{d^2 S}{dU^2} = -\frac{k}{U(\varepsilon + U)}. \]

(25)

The first integration can be performed using the integral

\[ \int \frac{dx}{x(a + bx)} = -\frac{1}{a} \ln \left|\frac{a}{x} + b\right|. \]

(26)

We get the following result

\[ \frac{1}{T} = \frac{dS}{dU} = \frac{k}{\varepsilon} \ln \left( \frac{\varepsilon}{U} + 1 \right). \]

(27)

The solution of eq. (27) is

\[ U = \frac{\varepsilon}{e^{\varepsilon/kT} - 1}. \]

(28)

The general validity of the Wien law

\[ \frac{dS}{dU} = \frac{1}{\nu} f \left( \frac{U}{\nu} \right) \]

(29)
confronted with the equation (27) gives the famous Planck formula

$$\varepsilon = h\nu.$$  \hspace{1cm} (30)

The next step of Planck was to find the appropriate physical statistical system which led to the correct power spectrum of the blackbody. This model was the thermal reservoir of the independent electromagnetic oscillators with the discrete energies.

We know that Einstein later identified the reservoir with the oscillators in the crystalline medium in order to get his famous heat capacity with the high temperature limit via the Dulong-Petit law. He supposed that the frequency of the crystalline medium is \(\omega_E\), \(\omega_E\) being the Einstein frequency. Then \(U = 3N < E >\), \(N\) being the number of oscillators. Then,

$$C_V = \frac{\partial U}{\partial T} = \frac{3Nk\left(\frac{h\omega_E}{kT}\right)^2 \exp(h\omega_E/kT)}{\left(\exp(h\omega_E/kT) - 1\right)^2}. \hspace{1cm} (31)$$

For \(kT >> h\omega_E\) it is \(\exp(h\omega_E/kT) \approx 1 + h\omega_E/kT\) and we get

$$C_V \approx 3Nk. \hspace{1cm} (32)$$

For \(kT << h\omega_E\) we get from the formula \(U\) the following formula the approximation

$$U \approx 3N\hbar\omega_E e^{-\frac{\hbar\omega_E}{kT}}, \hspace{1cm} (33)$$

from which

$$C_V = \frac{\partial U}{\partial T} = \frac{3Nk\hbar^2\omega_E^2}{(kT)^2}e^{-\frac{\hbar\omega_E}{kT}} \hspace{1cm} (34)$$

and evidently \(C_V \to 0\), when \(T \to 0\), which is in agreement with experiment.

However, the absolute harmony of theory with experiment was not achieved, because experimentally it is \(C_V \approx T^3\), which does not follow from the formula (34).

Later Debye in 1912 generalized the Einstein model in order to get more realistic formula for the heat capacity of every crystalline matter (Reissland, 1973). Debye assumed that the solid state can be represented by the continual medium with many modes. However, if we consider a continual medium, then there are the infinite number of frequencies. Debye resolved this contradiction by the normalization condition that the number of frequencies must be \(3N\). Or,

$$\int_0^{\omega_d} G(\omega)d\omega = 3N, \hspace{1cm} (35)$$

where \(G(\omega)\) is the number of phonon modes in the three-dimensional continuum. Or,

$$G(\omega)d\omega = \frac{V}{2\pi^2} \left(\frac{2}{v_t^3} + \frac{1}{v_l^3}\right)\omega^2d\omega = \frac{3\omega^2V}{2\pi^2c^3}d\omega, \hspace{1cm} (36)$$

where \(v_t\) is the transversal velocity with two polarization and \(v_l\) is the longitudinal velocity of sound and we have defined \(c\) by the equation
\[
\frac{3}{c^3} = \left( \frac{2}{v_i} + \frac{1}{v_i^3} \right).
\]

(37)

From equation (35) follows the Debye maximal frequency

\[
\omega_D^3 = 18\pi^2 N \left( \frac{2}{v_i} + \frac{1}{v_i^3} \right)^{-1}.
\]

(38)

The average energy is then of the form:

\[
U = \frac{3V\hbar}{2\pi^2c^3} \int_{0}^{\omega_D} \omega^3 d\omega e^{\frac{\hbar\omega}{kT}} - 1,
\]

(39)

which can be written using the substitution \( x = \hbar\omega/kT \) in the following form:

\[
U = \frac{3V(kT)^4}{2\pi^2c^3\hbar^3} \int_{0}^{\frac{\hbar\omega_D}{kT}} x^3 dx e^{x} - 1.
\]

(40)

The natural approximation is for \( kT << \hbar\omega_D \) where the upper limit of the integral is infinity and we have with

\[
\int_{0}^{\infty} \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}
\]

(41)

\[
U = \alpha T^4; \quad \alpha = \frac{\pi^2k^4}{10c^3\hbar^3}V.
\]

(42)

The corresponding specific heat in the last approximation is \( C_V = \partial U/\partial T = 4\alpha T^3 \).

It can be easily to show that if we introduce the debye temperature by the relation

\[
T_D = \frac{\hbar\omega_D}{k}, \quad \omega_D = c \left( \frac{6\pi^2N}{V} \right)^{1/3}
\]

(43)

then

\[
U = 3NkT D \left( \frac{T_c}{T} \right); \quad D(y) = \frac{3}{y^3} \int_{0}^{y} \frac{x^3 dx}{e^x - 1}.
\]

(44)

The corresponding heat capacity is then given by the formula

\[
C_V = 3Nk \left\{ D \left( \frac{T_c}{T} \right) - \frac{T_c}{T} D' \left( \frac{T_c}{T} \right) \right\}.
\]

(45)

The Debye formula is only an approximation because the frequency spectrum of the crystalline and noncrystalline medium is very complex. There are also many defects in the crystalline matter and it means that the determination of this spectrum is not easy mathematical problem. Some theory of the determination of such spectrum was discussed for instance by Maradudin (1966) and others.

Nevertheless, after some elementary approximation, Debye was able to derive the heat capacity of his model of crystal in the following form (Rohlf, 1994):

\[
C_V = \frac{12\pi^4}{5} Nk \left( \frac{T}{T_D} \right)^3; \quad T_D = \hbar\omega_D/k.
\]

(46)
Of course, the Debye model is in the better agreement with experiment that the Einstein model.

The formula (23) can be easily written in the form

\[ \varphi = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar \omega}{e^{\frac{\hbar \omega}{kT}} - 1} (1 + P(\omega) - Q(\omega) e^{\frac{\hbar \omega}{kT}}), \tag{47} \]

which means that the spectrum of the electromagnetical modes inside of the blackbody is modified. If we want to involve the external field in the Einstein model, Debye model and more sophisticated models, then the formal operation consists in the following elementary transformation

\[ \frac{1}{e^{\frac{\hbar \omega}{kT}} - 1} \longrightarrow \frac{1}{e^{\frac{\hbar \omega}{kT}} - 1} \left(1 + P(\omega) - Q(\omega) e^{\frac{\hbar \omega}{kT}}\right) \tag{48} \]

If this formal procedure has the physical meaning, then it is the generalization of the Bose-Einstein statistics (BE). The known generalization of the BE statistics and Fermi-Dirac (FD) statistics is known as the parastatistics. It is defined as the statistics where in one energetical state can be \( p \) particles. If \( p \) is arbitrary, we get the BE statistics and if \( p = 1 \), we get so called FD statistics. The derivation of parastatistics is involved in physics as an exercise of combinatorics (Isihara, 1971).

Here, we consider the statistics which is generated by dynamics. The historical analogue of this approach is the Maxwell statistics based on the dynamics of particles in reservoir and Einstein statistics (BE) of photons inside of the blackbody, derived on the basis of the dynamics of emission and absorption of photons. Our derivation is also based on the dynamics of emission and absorption of photons with the additional terms.

The different generalization of the Bose-Einstein statistics was realized for instance by Bekenstein (Bekenstein et al., 1994) in order to explain so called the gray-body radiation.

However, the idea of modification of the Bose-Einstein statistics or is forbidden by the quantum field theory where there is the strong connection between spin and statistics. The particles with the integer spins are bosons and the particles with the half-integer spins are fermions (Berestetzkii et al., 1999). Only if we accept the idea that the statistics of photons and phonons is of the dynamical origin, then we can also write the generalized heat capacity of solid state.

The next possibility to modify the statistics is to write the Planck hypothesis (25) in the generalized form

\[ \frac{d^2 S}{dU^2} = \frac{\alpha}{\sum_n a_n U^n} \tag{49} \]

and then to determine the modified Planck law for the specific thermal systems. To our knowledge this way was not still realized.

The formula (23) can be also related to the solar spectrum which is according to the NASA expertise not absolutely identical with the spectrum of the blackbody (Thekaekara, 1994).

The formulas for the modified specific heat can be eventually considered in case of the investigation of the specific heat of the carbon nanostructures in some nonspecified external field. Some investigation was performed by Li and Chou (2005). The theory and experiment with the carbon nanonstructures is at present time very actual area of
investigation and it means that every new information of the physical properties of such structures is useful.

The formula (23) can be probably related to the spectrum of the sonoluminescence, because the external field is the ultrasound. Sonoluminescence is defined as the emission of short burst of light from imploding bubbles in a liquid when excited by ultrasound. The effect of sonoluminescence was discovered at the University of Cologne in 1934. Franzel and Schultes put an ultrasound transducer in a tank of photographic developer fluid. They hoped to speed up the development process. Instead, they noticed tiny dots on the film after developing, and realized that the bubbles in the fluid were emitting light with the ultrasound turned on. The bubbles are very small when they emit the light. About 1 micrometer in diameter. The high compression of a small bubble of fluid is similar to the explosive compression of a pellet of material by laser beams, one of the methods proposed for the nuclear fusion.

The situation in cosmology is such the “cosmic microwave background” (CMB) discovered in 1965 by Penzias and Wilson is of the Planck distribution of the temperature 2.7 Kelvin. This radiation was predicted by Gamow as a consequence of the Big Bang. By the early 1970’ it became clear that the CMB sky is of the dipole form (it is hotter in one direction and cooler in the opposite direction with the temperature difference being a few milli Kelvin). The dipole form can be explained by the motion our galaxy with regard to the rest of universe. However, at some level one expects to see irregularities, or anisotropies of CMB, and then it is not excluded that the generalized Planck formula will be appropriate for the description of CMB.

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