TWISTOR THEORY FOR CR QUATERNIONIC MANIFOLDS
AND RELATED STRUCTURES

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Abstract

We introduce the classes of CR quaternionic and co-CR quaternionic manifolds, both containing the class of quaternionic manifolds, whilst in dimension three they particularize to, essentially, give the conformal manifolds and the Einstein–Weyl spaces, respectively. We show that these manifolds have a rich natural Twistor Theory and, along the way, we obtain a heaven space construction for quaternionic(-Kähler) manifolds.

Introduction

It is well known that the (four-dimensional) anti-self-dual manifolds and the three-dimensional Einstein–Weyl spaces admit similar twistorial structures (see [14]). Furthermore, from the point of view of Twistor Theory, the anti-self-dual manifolds are just four-dimensional quaternionic manifolds (see [7]).

This raises the obvious question: is there a natural class of manifolds, endowed with twistorial structures, which contains both the quaternionic manifolds and the three-dimensional Einstein–Weyl spaces?

On the other hand, an apparently unrelated question is the following: is there a quaternionic version of CR Geometry?

In this paper we answer in the affirmative to these two questions by introducing the notions, essentially dual to each other, of co-CR quaternionic and CR quaternionic manifolds, respectively, and we initiate the study of their twistorial properties.

Noteworthy, although both CR quaternionic and co-CR quaternionic manifolds contain, as particular cases, the quaternionic manifolds, in dimension three, the former, essentially, gives the (three-dimensional) conformal manifolds endowed with the twistorial structure of [12].

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The twistor space of a CR quaternionic manifold $M$ is a CR manifold $Z$. We prove that the CR structure of $Z$ is induced by local immersions of $Z$ into complex manifolds if and only if the CR quaternionic structure of $M$ is induced by an embedding of $M$ in a (germ unique) quaternionic manifold $N$ such that $TN|_M$ is generated, as a quaternionic vector bundle, by $TM$; we call $N$ the heaven space of $M$. In particular, any real analytic CR quaternionic manifold admits a heaven space.

A manifold may be endowed with both a CR quaternionic and a co-CR quaternionic structure which are compatible. This gives the notion of $f$-quaternionic manifold, which, obviously, has two twistor spaces. The simplest example is provided by the three-dimensional Einstein–Weyl spaces, endowed with the twistorial structures of [12] and [6], respectively. Also, the quaternionic manifolds may be characterised as $f$-quaternionic manifolds for which the two twistor spaces coincide.

Let $N$ be the heaven space of a real analytic $f$-quaternionic manifold $M$, with $\dim N = \dim M + 1$. If the connection of the $f$-quaternionic structure on $M$ is induced by a torsion free connection on $M$ then the twistor space of $N$ is endowed with a natural holomorphic distribution of codimension one which is transversal to the twistor lines corresponding to the points of $N \setminus M$. Furthermore, this construction also works if, more generally, $M$ is a real analytic CR quaternionic manifolds which is a $q$-umbilical hypersurface of its heaven space $N$. Then, under a non-degeneracy condition, this distribution defines a holomorphic contact structure on the twistor space of $N$. Therefore, according to [13], it determines a quaternionic-Kähler structure on $N \setminus M$.

Moreover, we show that this heaven space construction for quaternionic-Kähler manifolds has the adequate level of generality. Consequently, the ‘quaternionic contact’ manifolds of [4] (see [5]) are nondegenerate $q$-umbilical CR quaternionic manifolds.

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5. $f$-Quaternionic linear maps
1. Linear CR and co-CR structures

Unless otherwise stated, all the vector spaces are assumed real and finite dimensional and all the linear maps are assumed real.

A linear complex structure on a vector space $U$ is a linear map $J : U \to U$ such that $J^2 = -\text{Id}_U$. Then $U^C = U^J \oplus \overline{U^J}$ where $U^J$ is the eigenspace of (the complexification of) $J$ corresponding to $-i$. Therefore a linear complex structure on $U$ is given by a complex vector subspace $C \subseteq U^C$ such that $U^C = C \oplus \overline{C}$.

More generally, we have the following definition.

**Definition 1.1.** Let $U$ be a vector space and let $C \subseteq U^C$ be a complex vector subspace of $U^C$.

1) $C$ is a linear CR structure on $U$ if $C \cap \overline{C} = \{0\}$.

2) $C$ is a linear co-CR structure on $U$ if $C + \overline{C} = U^C$.

Let $U$ be a vector space. If $C$ is a complex vector subspace of $U^C$, we shall denote by $C^0 = \{ \alpha \in (U^C)^* | \alpha|_C = 0 \}$ the annihilator of $C$. Then $C$ is a linear CR structure if and only if $C^0$ is a linear co-CR structure.

Let $U$ and $U'$ be vector spaces endowed with linear (co-)CR structures $C$ and $C'$, respectively. A (co-)CR linear map is a linear map $t : (U, C) \to (U', C')$ such that $t(C) \subseteq C'$.

Let $(E, J)$ be a complex vector space (that is, $J$ is a linear complex structure on $E$). If $\iota : U \to E$ is an injective (real) linear map then $C = \iota^{-1}(E^J)$ is a linear CR structure on $U$, where $E^J$ is the eigenspace of $J$ corresponding to $-i$.

Dually, if $\rho : E \to U$ is a surjective linear map then $\rho(E^J)$ is a linear co-CR structure on $U$.

Since we are interested only in ‘generically induced (co-)CR structures’, in the following definition, we omit the term ‘generic’ (cf. [16]):
Definition 1.2. Let $U$ be a vector space and let $(E, J)$ be a complex vector space.

1) Let $\iota : U \to E$ be an injective linear map. We say that $(\iota, J)$ induces a linear CR structure on $U$ if $\text{im} \iota + J(\text{im} \iota) = E$.

2) Let $\rho : E \to U$ be a surjective linear map. We say that $(\rho, J)$ induces a linear co-CR structure on $U$ if $\text{ker} \rho \cap J(\text{ker} \rho) = \{0\}$.

Let $(E, J)$ be a complex vector space. We endow $E^*$ with the complex structure $J^*$ given by the opposite of the transpose of $J$; equivalently, the eigenspace of $J^*$ corresponding to $-i$ is the annihilator of the eigenspace of $J$ corresponding to $-i$.

Proposition 1.3. Let $\iota : U \to E$ be an injective linear map and let $\iota^* : E^* \to U^*$ be its transpose. Then the following assertions are equivalent:

(i) $(\iota, J)$ induces a linear CR structure on $U$.

(ii) $(\iota^*, J^*)$ induces a linear co-CR structure on $U^*$.

Moreover, if (i) or (ii) holds then the linear co-CR structure induced by $(\iota^*, J^*)$ is the annihilator of the linear CR structure induced by $(\iota, J)$.

To prove Proposition 1.3 we need the following lemma.

Lemma 1.4. Let $t : E \to E'$ be a (real or complex) linear map and let $B \subseteq E'$ be a vector subspace.

Then $t^* (B^0) = (t^{-1}(B))^0$, where $t^* : E'^* \to E^*$ is the transpose of $t$ and $B^0 \subseteq E'^*$ is the annihilator of $B$.

Proof. Let $\alpha \in (t^{-1}(B))^0$. Then $\alpha|_{\text{ker} t} = 0$ and, hence, there exists $\beta \in (\text{im} t)^*$ such that $\alpha = \beta \circ t$. As $\alpha|_{t^{-1}(B)} = 0$ we have $\beta|_{B \cap (\text{im} t)} = 0$. Thus, we can extend $\beta$ to $E'$ such that $\beta|_{B} = 0$. Thus, $\alpha \in t^* (B^0)$.

The inclusion $t^* (B^0) \subseteq (t^{-1}(B))^0$ is obvious. \qed

Proof of Proposition 1.3. The equivalence (i) $\iff$ (ii) follows by duality.

If assertion (i) or (ii) holds then, by Lemma 1.4, we have $\iota^*((E^J)^0) = (\iota^{-1}(E^J))^0$; equivalently, $\iota^*((E^*)^{J^*}) = (\iota^{-1}(E^J))^0$. \qed

From Proposition 1.3 and Lemma 1.4 it follows quickly that a linear map is CR if and only if its transpose is co-CR linear.

The proof of the following proposition is omitted.

Proposition 1.5. Let $U$ be a vector space, $(E, J)$ a complex vector space and $\iota : U \to E$ an injective linear map; denote by $2k$ the (real) dimension of $E$ and by $l$ the codimension of $\text{im} \iota$ in $E$. 

Let $C = \iota^{-1}(E^J)$. The following assertions are equivalent:

(i) $(\iota, J)$ induces a linear CR structure on $U$.
(ii) $C \oplus \overline{C}$ has complex codimension $l$ in $U^\mathbb{C}$.
(iii) $C$ has complex codimension $k$ in $U^\mathbb{C}$.
(iv) $(\im \iota)C + E^J = E^\mathbb{C}$.
(v) $\iota$ induces a complex linear isomorphism $U^\mathbb{C}/C = E$.

In particular, if one of the assertions (i), . . . , (v) holds then $l \leq k$.

The next result follows quickly from the equivalence $(i) \iff (v)$ of Proposition 1.5.

**Corollary 1.6.** Let $\iota : U \to E$ and $\iota' : U' \to E'$ be injective (real) linear maps from the vector spaces $U$ and $U'$ to the complex vector spaces $(E, J)$ and $(E', J')$, respectively. Suppose that $(\iota, J)$ and $(\iota', J')$ induce linear CR structures on $U$ and $U'$, respectively.

Let $t : U \to U'$ be a map. Then the following assertions are equivalent:

(i) $t$ is a CR linear map.
(ii) There exists a complex linear map $\tilde{t} : (E, J) \to (E', J')$ with the property $\iota'^{-1} \circ t = \tilde{t} \circ \iota$.

Furthermore, if (i) holds then there exists a unique $\tilde{t}$ satisfying (ii).

The next two results are the duals of Proposition 1.5 and Corollary 1.6, respectively.

**Proposition 1.7.** Let $U$ be a vector space, $(E, J)$ a complex vector space and $\rho : E \to U$ a surjective linear map; denote by $2k$ the dimension of $E$ and by $l$ the dimension of $\ker \rho$.

Let $C = \rho(E^J)$. The following assertions are equivalent:

(i) $(\rho, J)$ induces a linear CR structure on $U$.
(ii) $C \cap \overline{C}$ has complex dimension $l$.
(iii) $C$ has complex dimension $k$.
(iv) $(\ker \rho)^C \cap E^J = \{0\}$.
(v) $\rho$ induces a complex linear isomorphism $\overline{C} = E$.

In particular, if one of the assertions (i), . . . , (v) holds then $l \leq k$.

**Corollary 1.8.** Let $\rho : E \to U$ and $\rho' : E' \to U'$ be surjective (real) linear maps from the complex vector spaces $(E, J)$ and $(E', J')$ to the vector spaces $U$ and $U'$, respectively. Suppose that $(\rho, J)$ and $(\rho', J')$ induce linear co-CR structures on $U$ and $U'$, respectively.

Let $t : U \to U'$ be a map. Then the following assertions are equivalent:

(i) $t$ is a co-CR linear map.
There exists a complex linear map $\tilde{t} : (E, J) \to (E', J')$ with the property $\rho' \circ \tilde{t} = t \circ \rho$.

Furthermore, if (i) holds then there exists a unique $\tilde{t}$ satisfying (ii).

Next, we give the definitions of (co-)CR vector subspaces.

**Definition 1.9.** Let $\iota : U \to E$ and $\iota' : U' \to E'$ be injective linear maps from the vector spaces $U$ and $U'$ to the complex vector spaces $(E, J)$ and $(E', J')$, respectively. Suppose that $(\iota, J)$ and $(\iota', J')$ induce linear CR structures on $U$ and $U'$, respectively.

Let $t : U \to U'$ be a CR linear map and let $\tilde{t} : E \to E'$ be the corresponding complex linear map such that $\iota' \circ t = \tilde{t} \circ \iota$.

We say that $U$ is a CR vector subspace of $U'$ if $t$ and $\tilde{t}$ are injective and $U = E \cap U'$, where we have identified $U$, $U'$ and $E$ with their images in $E'$ through the injective linear maps $\iota' \circ t$, $\iota'$ and $\tilde{t}$, respectively.

With the same notations as in Definition 1.9, the following assertions are equivalent:

1. $\tilde{t}$ is injective.
2. $t^{-1}(C') = C$, where $C = \iota^{-1}(E')$ and $C' = \iota'^{-1}(E'^J')$.

Note that, if (i) or (ii) holds then $t$ is injective. Furthermore, if $\tilde{t}$ is injective then $U$ is a CR vector subspace of $U'$ if and only if $\iota'' : U'/U \to E'/E$ is injective; in particular, $(\iota'', J'')$ induces a linear CR structure on $U'/U$, where $J''$ is the linear complex structure of $E'/E$.

**Definition 1.10.** Let $\rho : E \to U$ and $\rho' : E' \to U'$ be surjective (real) linear maps from the complex vector spaces $(E, J)$ and $(E', J')$ to the vector spaces $U$ and $U'$, respectively. Suppose that $(\rho, J)$ and $(\rho', J')$ induce linear co-CR structures on $U$ and $U'$, respectively.

Let $t : U \to U'$ be a co-CR linear map and let $\tilde{t} : E \to E'$ be the corresponding complex linear map such that $\rho' \circ \tilde{t} = t \circ \rho$.

We say that $U$ is a co-CR vector subspace of $U'$ if $t$ and $\tilde{t}$ are injective and $t^{-1}(C') = C$, where $C = \rho(E')$ and $C' = \rho'(E'^J')$.

With the same notations as in Definition 1.10, if $t$ and $\tilde{t}$ are injective then $U$ is a co-CR vector subspace of $U'$ if and only if $(\rho'', J'')$ induces a linear co-CR structure on $U'/U$, where $\rho'' : E'/E \to U'/U$ is the projection and $J''$ is the linear complex structure of $E'/E$.

**Remark 1.11.** Let $U' \to U$ be an injective CR (co-CR) linear map and let $(U/U')^\ast \to U^\ast$ be the transpose of the projection $U \to U/U'$. Then $U'$ is a
CR (co-CR) vector subspace of $U$ if and only if $(U/U'')^*$ is co-CR (CR) vector subspace of $U^*$.

2. Linear $f$-structures

Firstly, we prove the following result.

**Lemma 2.1.** Let $C$ and $D$ be linear CR and co-CR structures, respectively, on a vector space $U$. Then the following assertions are equivalent.

(i) $D = C \oplus (D \cap \overline{D})$.

(ii) $C = D \cap (C \oplus \overline{C})$.

Furthermore, if assertion (i) or (ii) holds then $U^C = (D \cap \overline{D}) \oplus C \oplus \overline{C}$.

**Proof.** By duality it is sufficient to prove (i)$\implies$(ii).

Suppose that (i) holds. Then

\[
\dim(D + \overline{D}) = \dim D + \dim \overline{D} - \dim(D \cap \overline{D}) = \dim C + \dim \overline{C} + \dim(D \cap \overline{D}).
\]

(2.1)

Also, we have

\[
D + \overline{D} = (C \oplus \overline{C}) + (D \cap \overline{D}),
\]

and, consequently,

\[
\dim(D + \overline{D}) = \dim C + \dim \overline{C} + \dim(D \cap \overline{D}) - \dim((C \oplus \overline{C}) \cap D \cap \overline{D}).
\]

(2.2)

From (2.1) and (2.3) we obtain

\[
(C \oplus \overline{C}) \cap D \cap \overline{D} = \{0\}.
\]

(2.4)

Now, by applying (2.4), we obtain

\[
D \cap (C \oplus \overline{C}) = (C \oplus (D \cap \overline{D})) \cap (C \oplus \overline{C}) = C.
\]

The last assertion follows from (2.2) and (2.4) and the fact that $D$ is a co-CR linear structure. □

Let $C$ and $D$ be linear CR and co-CR structures, respectively, on a vector space $U$. We say that $C$ and $D$ are compatible if they satisfy assertion (i) or (ii) of Lemma 2.1. Note that, $C$ and $D$ are compatible if and only if $D^0$ and $C^0$ are compatible (see, also, Proposition 2.2, below), where $C^0$ and $D^0$ are the annihilators of $C$ and $D$, respectively.

A linear $f$-structure on a vector space $U$ is an endomorphism $F \in \text{End}(U)$ such that $F^3 + F = 0$. 
Proposition 2.2. Let $U$ be a vector space. There exists a natural bijective correspondence $F \mapsto (C,D)$ between linear $f$-structures $F$ on $U$ and pairs $(C,D)$ of compatible linear CR and co-CR structures, respectively, on $U$; the correspondence is given by the condition that $D \cap \overline{D}$ and $C$ be the eigenspaces of $F$ corresponding to $0$ and $-i$, respectively.

Proof. This is an immediate consequence of the last assertion of Lemma 2.1. □

Let $U$ and $U'$ be vector spaces endowed with linear $f$-structures $F$ and $F'$, respectively. An $f$-linear map is a linear map $t : (U,F) \to (U',F')$ such that $t \circ F = F' \circ t$.

A map between vector spaces, endowed with linear $f$-structures, is $f$-linear if and only if it is both CR and co-CR linear with respect to the corresponding linear CR and co-CR structures.

Furthermore, a map $t : (U,F) \to (U',F')$ is injective and $f$-linear if and only if $U$ is both a CR and a co-CR vector subspace of $U'$ with respect to the corresponding linear CR and co-CR structures.

Let $U$ be a vector space endowed with a linear $f$-structure $F$. Let $V = \ker F$ and $E = U \oplus V$. Denote by $\iota : U \to E$ and $\rho : E \to U$ the corresponding inclusion and projection, respectively.

Then there exists a unique linear complex structure $J$ on $E$ such that $(\iota,J)$ and $(\rho,J)$ induce the linear CR and co-CR structures, respectively, which correspond to $F$.

Conversely, let $U$ be a vector space and let $(E,J)$ be a complex vector space. Also, let $\iota : U \to E$ and $\rho : E \to U$ be injective and surjective linear maps, respectively, such that $\rho \circ \iota = \Id_U$. Suppose, further, that $(\iota,J)$ and $(\rho,J)$ induce on $U$ the CR and co-CR linear structures $C$ and $D$, respectively. If $C$ and $D$ are compatible we say that $(\iota,\rho,J)$ induces a linear $f$-structure on $U$.

Proposition 2.3. Let $U$ and $U'$ be vector spaces and let $(E,J)$ and $(E',J')$ be complex vector space. Let $\iota : U \to E$ and $\rho : E \to U$ be injective and surjective linear maps, respectively, such that $\rho \circ \iota = \Id_U$. Suppose, further, that $(\iota,J)$ and $(\rho,J)$ induce the linear $f$-structure $F$ on $U$. Also, let $\iota' : U' \to E'$ and $\rho' : E' \to U'$ be injective and surjective linear maps, respectively, such that $(\iota',\rho',J')$ induces the linear $f$-structure $F'$ on $U'$.

Let $t : U \to U'$ be a map. Then the following assertions are equivalent:

(i) $t$ is an $f$-linear map.

(ii) There exists a complex linear map $\tilde{t} : (E,J) \to (E',J')$ with the properties $\iota' \circ t = \tilde{t} \circ \iota$ and $\rho' \circ \tilde{t} = t \circ \rho$.

Furthermore, if (i) holds then there exists a unique $\tilde{t}$ satisfying (ii).
Proof. Let \( C \) and \( D \) be the linear CR and co-CR structures, respectively, corresponding to \( F \). Let \( C' \) and \( D' \) be the linear CR and co-CR structures, respectively, corresponding to \( F' \).

Suppose that \( t \) is \( f \)-linear and let \( \tilde{t}_C \) and \( \tilde{t}_D \) be the complex linear maps from \( E \) to \( E' \) induced by the CR linear map \( t : (U, C) \rightarrow (U', C') \) and the co-CR linear map \( t : (U, D) \rightarrow (U', D') \), respectively.

By Corollaries 1.6 and 1.8 it is sufficient to prove that \( \tilde{t}_C = \tilde{t}_D \).

Now, \( \tilde{t}_D = t^C \big|_{\mathcal{D}} \), under the complex linear isomorphism \( E = \mathcal{D} \) induced by \( \rho \), whilst \( \tilde{t}_C \) is the map induced by \( t^C \) through the complex linear isomorphisms \( E = U^C / C \) and \( E' = U'^C / C' \). As \( U^C = (D \cap \overline{D}) \oplus C \oplus \overline{C} \) and \( \overline{D} = (D \cap \overline{D}) \oplus C \oplus \overline{C} \) (and, similarly, for \( U'^C \) and \( \overline{D}' \)), we have \( \tilde{t}_C = \tilde{t}_D \). \( \square \)

3. CR QUATERNIONIC AND CO-CR QUATERNIONIC VECTOR SPACES

Let \( \mathbb{H} \) be the associative algebra of quaternions. The automorphism group of \( \mathbb{H} \) is \( \text{SO}(3) \) acting trivially on \( \mathbb{R} \) and canonically on \( \text{Im} \mathbb{H} \). \( (= \mathbb{R}^3) \).

A linear hypercomplex structure on a vector space \( E \) is a morphism of associative algebras from \( \mathbb{H} \) to \( \text{End}(E) \). A hypercomplex vector space is a vector space endowed with a linear hypercomplex structure \([1]\).

A linear quaternionic structure on a vector space \( E \) is an equivalence class of morphisms of associative algebras from \( \mathbb{H} \) to \( \text{End}(V) \) where two such morphisms \( \sigma \) and \( \tau \) are equivalent if there exists \( a \in \text{SO}(3) \) such that \( \tau = \sigma \circ a \). A quaternionic vector space is a vector space endowed with a linear quaternionic structure (see \([7]\)).

Let \( E \) be a quaternionic vector space. Any representant of the linear quaternionic structure of \( E \) is called an admissible linear hypercomplex structure on \( E \). Let \( \sigma \) be an admissible linear hypercomplex structure on \( E \). We denote \( Q = \sigma(\text{Im} \mathbb{H}) \); obviously, \( Q \) depends only of the linear quaternionic structure of \( E \). Let \( Z \) be the unit sphere of \( Q \); then any \( J \in Z \) is an admissible linear complex structure on \( E \).

Let \( \sigma \) be an admissible linear hypercomplex structure on a quaternionic vector space \( E \). Define \( \sigma^* : \mathbb{H} \rightarrow \text{End}(E^*) \) by \( \sigma^*(q) \) is the transpose of \( \sigma(q) \), \( (q \in \mathbb{H}) \). Then \( \sigma^* \) is a linear hypercomplex structure on \( E^* \). Furthermore, the induced linear quaternionic structure on \( E^* \) depends only of the linear quaternionic structure of \( E \).

**Definition 3.1.** A linear CR quaternionic structure on a vector space \( U \) is a pair \((E, \iota)\) where \( E \) is a quaternionic vector space and \( \iota : U \rightarrow E \) is an injective linear map such that \((\iota, J)\) induces a linear CR structure on \( U \), for any \( J \in Z \).
If, further, $E$ is hypercomplex then $(E, \iota)$ is a linear hyper-CR structure on $U$.

A CR quaternionic vector space (hyper-CR vector space) is a vector space endowed with a linear CR quaternionic structure (linear hyper-CR structure).

The following definition is the dual of Definition 3.1.

**Definition 3.2.** A linear co-CR quaternionic structure on a vector space $U$ is a pair $(E, \rho)$ where $E$ is a quaternionic vector space and $\rho : E \to U$ is a surjective linear map such that $(\rho, J)$ induces a linear co-CR structure on $U$, for any $J \in \mathbb{Z}$. If, further, $E$ is hypercomplex then $(E, \rho)$ is a linear hyper-co-CR structure on $U$.

A co-CR quaternionic vector space (hyper-co-CR vector space) is a vector space endowed with a linear co-CR quaternionic structure (linear hyper-co-CR structure).

Let $(E, \iota)$ and $(E, \rho)$ be linear CR quaternionic and co-CR quaternionic structures, respectively, on $U$. Then $(E, \iota)$ and $(E, \rho)$ are compatible if $\rho \circ \iota = \text{Id}_U$ and, for any $J \in \mathbb{Z}$, the linear CR and co-CR structures induced on $U$ by $(\iota, J)$ and $(\rho, J)$, respectively, are compatible.

**Definition 3.3.** A linear $f$-quaternionic structure on a vector space $U$ is a pair $(E, V)$, where $E$ is a quaternionic vector space such that $U, V \subseteq E$, $E = U \oplus V$ and $J(V) \subseteq U$, for any $J \in \mathbb{Z}$. If, further, $E$ is hypercomplex then $(E, V)$ is a linear hyper-$f$-structure on $U$.

An $f$-quaternionic vector space (hyper-$f$ vector space) is a vector space endowed with a linear $f$-quaternionic structure (linear hyper-$f$-structure).

Let $(U, E, V)$ be an $f$-quaternionic vector space; denote by $\iota : U \to E$ the inclusion and by $\rho : E \to U$ the projection determined by the decomposition $E = U \oplus V$. Then $(E, \iota)$ and $(E, \rho)$ are linear CR-quaternionic and co-CR quaternionic structures, respectively; moreover, they are compatible.

Any pair of compatible linear CR-quaternionic and co-CR quaternionic structures arise this way from a linear $f$-quaternionic structure (cf. Proposition 2.2).

**Proposition 3.4.** Let $(U, E, V)$ be an $f$-quaternionic vector space, $\dim E = 4k$, $\dim V = l$.

(i) The quaternionic vector subspace of $E$ generated by $V$ has dimension $4l$; in particular, $k \geq l$.

(ii) $\bigcap_{J \in \mathbb{Z}} J(U)$ has dimension $4k - 3l$ and $U \cap \bigcap_{J \in \mathbb{Z}} J(U)$ is a quaternionic vector subspace of $E$, of real dimension $(k - l)$.

(iii) We have $U = Q(V) \oplus W$ and $\bigcap_{J \in \mathbb{Z}} J(U) = V \oplus W$, where $Q(V)$ is the vector space generated by $\sum_{J \in \mathbb{Z}} J(V)$ and $W = U \cap \bigcap_{J \in \mathbb{Z}} J(U)$.
Proof. Let \((I, J, K)\) be a positive orthonormal frame of \(Q\). Define \(\psi : V^3 \to U\) by \(\psi(a, b, c) = Ia + Jb + Kc\), for any \((a, b, c) \in V^3\). From \(U \cap V = \{0\}\) and \(J(V) \subseteq U\), \((J \in Z)\), we obtain \(\psi\) injective. As the quaternionic vector subspace of \(E\) generated by \(V\) is equal to \(V \oplus \text{im}\ \psi\), this implies assertion (i).

Let \((I, J, K)\) be a positive orthonormal basis of \(Q\). It is easy to prove that \(I(U) \cap J(U) \cap K(U) = \bigcap_{J' \in Z} J'(U)\).

By applying Proposition 1.5 we obtain

\[
\dim(J(U) \cap K(U)) = \dim(U \cap I(U)) = 4k - 2l.
\]

Note that, we have

\[
(3.2) \quad U = I(V) \oplus (U \cap I(U));
\]
equivalently,

\[
I(U) = V \oplus (U \cap I(U)).
\]

Similarly,

\[
J(U) = V \oplus (U \cap J(U)),
\]

\[
K(U) = V \oplus (U \cap K(U)).
\]

Therefore, we have

\[
(3.3) \quad J(U) \cap K(U) = V \oplus (U \cap J(U) \cap K(U)),
\]

\[
I(U) \cap J(U) \cap K(U) = V \oplus W.
\]

As \(I(I(U) \cap J(U) \cap K(U)) = U \cap J(U) \cap K(U)\), from (3.1) and the first relation of (3.3) it follows quickly that \(\dim(I(U) \cap J(U) \cap K(U)) = 4k - 3l\). Together with the second relation of (3.3) this implies that \(\dim W = 4k - 4l\).

It is easy to prove that \(W\) is the largest quaternionic vector subspace of \(E\) contained by \(U\). The proof of (ii) is complete.

From (3.3) it follows that \(U \cap I(U) = J(V) \oplus K(V) \oplus W\) which together with (3.2) implies (iii). \(\square\)

Remark 3.5. With the same notations as in Proposition 3.4, we also have \(Q(V) = I(V) \oplus J(V) \oplus K(V)\).

Next, we give examples of (co-)CR quaternionic vector spaces.

Example 3.6. 1) Let \(\mathbb{H}^2\) be endowed with its canonical linear quaternionic and Euclidean structures. Let \(V = \mathbb{R}^2 + \mathbb{R}(i, j) \subseteq \mathbb{H}^2\) and denote \(U = V^\perp\); obviously, \(\dim U = 5\).

Then \(V \cap qV = \{0\}\) for any \(q \in S^2 \subseteq \text{Im}\ \mathbb{H}\) and it follows that \((U, \mathbb{H}^2)\) is a CR quaternionic vector space. We, also, have \(iU \cap jU \cap kU = \{0\}\).
2) Similarly, if \( U \) is the orthogonal complement of \( \mathbb{R}^3 + \mathbb{R}(i,j,k) \) in \( \mathbb{H}^3 \) then \((U,\mathbb{H}^3)\) is a CR quaternionic vector space such that \( iU \cap jU \cap kU = \{0\} \).

Note that, if we identify \( \mathbb{H}^2 = \mathbb{H}^2 \times \{0\} \subseteq \mathbb{H}^3 \) then \((U \cap \mathbb{H}^2,\mathbb{H}^2)\) is the CR quaternionic vector space of Example 3.6(1).

3) Let \( E = \mathbb{H}^k \) and let \( U = (\text{Im}\mathbb{H})^l \times \mathbb{H}^{k-l} \), for some \( k \geq l \). Then \((U,E,V)\) is an \( f\)-quaternionic vector space, where \( V = \mathbb{R}^l \times \{0\} \subseteq E \).

The next result gives, in particular, a necessary and sufficient condition for a CR quaternionic vector space to be \( f\)-quaternionic (with respect to a suitable complement).

**Proposition 3.7.** Let \((U,E,i)\) be a CR quaternionic vector space and let \( W \) be the largest quaternionic vector subspace of \( E \) contained by \( U \); denote \( 4k = \dim E \), \( 4k' = \dim W \) and by \( l \) the codimension of \( U \) in \( E \).

Then \( k - k' \leq l < 2(k - k') \). Moreover, \( l = k - k' \) if and only if there exists \( V \subseteq E \) such that \((U,E,V)\) is an \( f\)-quaternionic vector space.

**Proof.** We may suppose \( k' = 0 \). Otherwise, we replace \( E \) and \( U \) by \( E' \) and \( U \cap E' \), respectively, where \( E' \) is a quaternionic complement of \( W \) in \( E \).

Then we, firstly, show that \( l < 2k \). Indeed, this is equivalent to \( \dim U > \frac{1}{2} \dim E \) which if it does not hold then from the diagram of Section 4, below, it follows that \( E^{0,1} \) is (isomorphic to) a trivial holomorphic vector bundle, a contradiction.

Let \((I,J,K)\) be a positive orthonormal basis of \( Q \). As \( U \cap I(U) \cap J(U) \cap K(U) = W = \{0\} \), we have \( \dim(I(U) \cap J(U) \cap K(U)) \leq l \) and a similar inequality applies to \( U \cap I(U) \cap J(U) = K(U \cap I(U) \cap J(U)) \). Together with the obvious relation \((U \cap I(U)) + (U \cap J(U)) \subseteq U \) this gives

\[
4k - l \geq \dim \left( (U \cap I(U)) + (U \cap J(U)) \right) \\
= \dim(U \cap I(U)) + \dim(U \cap J(U)) - \dim(U \cap I(U) \cap J(U)) \\
\geq (4k - 2l) + (4k - 2l) - l \\
= 8k - 5l .
\]

This proves that \( l \geq k \). Moreover, if \( l = k \) then \( \dim(U \cap I(U) \cap J(U)) = l \); equivalently, \( V = I(U) \cap J(U) \cap K(U) \) is a complement of \( U \) in \( E \). As, obviously, \( I(V), J(V), K(V) \subseteq U \), we obtain that \((U,E,V)\) is an \( f\)-quaternionic vector space.

Now, by Proposition 3.4, if \((U,E,V)\) is an \( f\)-quaternionic vector space and \( k' = 0 \) then \( l = k \). The proof is complete. \( \square \)
Note that, with the same notations as in Proposition 3.7, we have \( \dim U \geq 2k + 1 \). Moreover, this lower bound is attained as the following example shows.

**Example 3.8.** Let \( k \geq 1 \) and let \( q_1, \ldots, q_{k+1} \in S^2 \) be such that \( q_i \neq \pm q_j \), if \( i \neq j \). For \( j = 1, \ldots, k \) let \( e_j = (0, \ldots, 0, q_j, q_{j+1}, 0, \ldots, 0) \).

Denote \( V_k = \mathbb{R}^{k+1} + \mathbb{R}e_1 + \ldots + \mathbb{R}e_k \) and let \( U_k = V_k^\perp \). Then \( (U_k, \mathbb{H}^{k+1}) \) is a CR quaternionic vector space.

Note that, \( \dim U_k = 2k + 3 \) and \( U_1 = U \) of Example 3.6(1). Similar generalizations can be, also, given for Example 3.6(2).

Next, we prove the following result.

**Corollary 3.9.** Let \( (U, E, \iota) \) be a CR quaternionic vector space with \( \text{codim}_E U \leq 2 \). Then there exists \( V \subseteq E \) such that \( (U, E, V) \) is an \( f \)-quaternionic vector space.

**Proof.** With the same notations as in Proposition 3.7, we deduce \( l = k - k' \). \( \square \)

**Proposition 3.10.** Let \( (U, E, V) \) be an \( f \)-quaternionic vector space. Then for any positive orthonormal frame \( (I, J, K) \) of \( Q \) we have
\[
\rho(E^I) \cap \rho(E^J) \cap \rho(E^K) = \{0\},
\]
where \( \rho : E \rightarrow U \) is the projection whilst \( E^I, E^J \) and \( E^K \) are eigenspaces of \( I, J \) and \( K \), respectively, corresponding to \( -i \).

**Proof.** Let \( \dim E = 4k, \dim V = l, (k \geq l) \). As \( \dim_C(E^I) = \dim_C(E^J) = 2k \) and \( E^I \cap E^J = \{0\} \) we have \( E^C = E^I \oplus E^J \). Hence, we also have \( \rho(E^I) + \rho(E^J) = U^C \).

It follows quickly that \( F = \rho(E^I) \cap \rho(E^J) \) has complex dimension \( l \). Moreover, as a similar statement holds for the \( f \)-quaternionic vector space \( (Q(V), V \oplus Q(V), V) \) we have that \( F \) is contained by the complexification of \( V \oplus Q(V) \). Hence, it is sufficient to consider the case \( k = l \).

If \( U = Q(V) \) then \( U \cap E^I = (J + iK)(V) \) and similarly for \( J \) and \( K \). Thus, we have to show that
\[
(I(V) \oplus (J + iK)(V)) \cap (J(V) \oplus (K + iI)(V)) \cap (K(V) \oplus (I + iJ)(V)) = \{0\}.
\]
By using the fact that \( Q(V) = I(V) \oplus J(V) \oplus K(V) \), the proof follows. \( \square \)

We end this section with the following immediate consequence of Proposition 3.10.

**Corollary 3.11.** Let \( (U, E, V) \) be an \( f \)-quaternionic vector space. Then
\[
\bigcap_{J \in \mathbb{Z}} \rho(E^J) = \{0\},
\]
where $E^J$ is the eigenspace of $J$ corresponding to $-i$, $(J \in Z)$.

4. CR quaternionic and co-CR quaternionic linear maps

We start this section by recalling (see [7]) the definition of quaternionic linear maps.

**Definition 4.1.** Let $E$ and $E'$ be quaternionic vector spaces. Let $t : E \to E'$ be a linear map and let $T : Z \to Z'$ be a map, where $Z$ and $Z'$ are the spaces of admissible linear complex structures of $E$ and $E'$, respectively.

Then $t : E \to E'$ is a quaternionic linear map, with respect to $T$, if for any $J \in Z$ we have $t \circ J = T(J) \circ t$.

The following two definitions are natural generalizations of Definition 4.1 (see, also, Definition 5.1, below).

**Definition 4.2.** Let $(U, E, \iota)$ and $(U', E', \iota')$ be CR quaternionic vector spaces and let $t : U \to U'$ and $T : Z \to Z'$ be maps, where $Z$ and $Z'$ are the spaces of admissible linear complex structures of $E$ and $E'$, respectively.

Then $t : (U, E, \iota) \to (U', E', \iota')$ is CR quaternionic linear, with respect to $T$, if there exists a linear map $\tilde{t} : E \to E'$ which is quaternionic, with respect to $T$, and such that $\iota' \circ t = \tilde{t} \circ \iota$.

Dually, we have the following definition.

**Definition 4.3.** Let $(U, E, \rho)$ and $(U', E', \rho')$ be co-CR quaternionic vector spaces and let $t : U \to U'$ and $T : Z \to Z'$ be maps.

Then $t : (U, E, \rho) \to (U', E', \rho')$ is co-CR quaternionic linear, with respect to $T$, if there exists a linear map $\tilde{t} : E \to E'$ which is quaternionic, with respect to $T$, and such that $\rho' \circ \tilde{t} = t \circ \rho$.

Note that, if $(U, E, \iota)$ is a CR quaternionic vector space then the inclusion $\iota : U \to E$ is CR quaternionic linear (with respect to the identity map $Z \to Z$).

Dually, if $(U, E, \rho)$ is a co-CR quaternionic vector space then the projection $\rho : E \to U$ is co-CR quaternionic linear.

Let $(U, E, \iota)$ be a CR quaternionic vector space and let $Z(= S^2)$ be the space of admissible linear complex structures of $E$. We denote $U^J = \iota^{-1}(E^J)$ where $E^J$ is the eigenspace of $J$ corresponding to $-i$, $(J \in Z)$.

**Proposition 4.4.** Let $(U, E, \iota)$ and $(U', E', \iota')$ be CR quaternionic vector spaces. Let $t : U \to U'$ be a nonzero linear map and let $T : Z \to Z'$ be a map.

Then the following assertions are equivalent:
(i) $t$ is CR quaternionic, with respect to $T$.

(ii) $T$ is a holomorphic diffeomorphism and

$$t(U^J) \subseteq (U')^T(J),$$

for any $J \in Z$.

Furthermore, if assertion (i) or (ii) holds then there exists a unique linear map $	ilde{t} : E \to E'$ which is quaternionic, with respect to $T$, and such that $t' \circ t = \tilde{t} \circ t$.

Proof. It is obvious that if (i) holds then (4.1) holds, for any $J \in Z$. If, further, $t \neq 0$, the fact that $T$ is a holomorphic diffeomorphism is an immediate consequence of [7, Proposition 1.5]. This proves (i)$\Rightarrow$(ii).

To prove the converse, let $E = Z \times E \to Z$ be the complex vector bundle whose fibre over each $J \in Z$ is $(E, J)$. Alternatively, $E$ is the quotient of the trivial holomorphic vector bundle $Z \times E^C$ through the holomorphic vector bundle $E^{0,1} \to Z$, whose fibre over each $J \in Z$ is $E^J$. It follows that $E$ is a holomorphic vector bundle, over $Z = \mathbb{C}P^1$, isomorphic to $2k\mathcal{O}(1)$, where $\dim E = 4k$ and $\mathcal{O}(1)$ is the dual of the tautological line bundle over $\mathbb{C}P^1$. Moreover, from the long exact sequence of cohomology groups of $0 \to E^{0,1} \to Z \times E^C \to E \to 0$ we obtain a natural complex linear isomorphism $E^C = H^0(Z, \mathcal{E})$.

Obviously, $U = Z \times U$ is a CR submanifold of $E$ (equivalently, for any $(J, u) \in U$, the complex structure of $T_{(J, u)}E$ induces a linear CR structure on $T_{(J, u)}U$).

Define, similarly, $E'$ and $U'$ and note that (4.1) holds, for any $J \in Z$, if and only if there exists a (necessarily unique) morphism of complex vector bundles $T : E \to E'$, over $T$, such that $T|_U = T \times t$ (this follows from Proposition 1.5).

Thus, if assertion (ii) holds then, on identifying $Z = Z' (= \mathbb{C}P^1)$, $T = \text{Id}_Z$, there exists a holomorphic section $T$ of $\text{Hom}_C(E, E')$ such that $T|_U = \text{Id}_Z \times t$. Then $T$ induces a complex linear map $\tau : E^C \to (E')^C$ such that $\tau|_U = t$ and, for any $J \in Z$, we have $\tau(E^J) \subseteq (E')^J$; moreover, $\tau$ descends to the complex linear map $T_J : (E, J) \to (E', J)$.

It follows that $\tau = T_J \oplus T_{-J}$, $(J \in Z)$, and, in particular, $\tau$ is the complexification of some (real) linear map $\tilde{t} : E \to E'$. This completes the proof of (ii)$\Rightarrow$(i).

Remark 4.5. Let $(U, E)$ and $(U', E')$ be CR quaternionic vector spaces. Let $t : U \to U'$ be a linear map such that (4.1) holds, for any $J \in Z$, with respect to some map $T : Z \to Z'$.

With the same notations as in the proof of Proposition 4.4, the following assertions are equivalent:

(a) $t$ is CR quaternionic, with respect to $T$.  

(b) $T_{J_1} = T_{J_2}$, for any $J_1, J_2 \in \mathbb{Z}$.

Furthermore, if assertion (a) or (b) holds then $\bar{t} = T_J$, $(J \in \mathbb{Z})$, is a linear map from $E$ to $E'$ which is quaternionic, with respect to $T$, and such that $\bar{t} |_{U} = t$.

Suppose, further, that $T$ is holomorphic. Then $T = \psi \circ T_1$ for some unique holomorphic section $T_1$ of $\text{Hom}_{\mathbb{C}}(\mathcal{E}, T^*(\mathcal{E}'))$, where $\psi : T^*(\mathcal{E}') \to \mathcal{E}'$ is the canonical bundle map.

Then the following assertion can be added to the above list:

(c) $T_1(H^0(Z, \mathcal{E})) \subseteq \psi^*(H^0(Z', \mathcal{E}'))$.

Let $(U, E, \rho)$ be a CR quaternionic vector space and let $Z (= S^2)$ be the space of admissible linear complex structures of $E$. We denote $U^J = \rho(E^J)$ where $E^J$ is the eigenspace of $J$ corresponding to $-i$, $(J \in \mathbb{Z})$.

The next result is the dual of Proposition 4.4.

**Proposition 4.6.** Let $(U, E, \rho)$ and $(U', E', \rho')$ be co-CR quaternionic vector spaces. Let $t : U \to U'$ be a nonzero linear map and let $T : Z \to Z'$ be a map.

Then the following assertions are equivalent:

(i) $t$ is co-CR quaternionic, with respect to $T$.

(ii) $T$ is a holomorphic diffeomorphism and

\[ t(U^J) \subseteq (U')^{T(J)}, \]

for any $J \in \mathbb{Z}$.

Furthermore, if assertion (i) or (ii) holds then there exists a unique linear map $\bar{t} : E \to E'$ which is quaternionic, with respect to $T$, and such that $\rho' \circ \bar{t} = t \circ \rho$.

Remark 4.5 can be easily reformulated for co-CR quaternionic linear maps.

Let $E$ be a quaternionic vector space. Let $E^{0,1}$ be the holomorphic vector subbundle of $Z \times E^\mathbb{C}$ whose fibre over each $J \in \mathbb{Z}$ is $E^J$. Denote by $\mathcal{E}$ the quotient of $Z \times E^\mathbb{C}$ through $E^{0,1}$.

**Definition 4.7.**

1) Let $(U, E, \iota)$ be a CR quaternionic vector space. We say that $\mathcal{U} = E^{0,1} \cap (Z \times U^\mathbb{C})$ is the *holomorphic vector bundle of $(U, E, \iota)$*.

2) Let $(U, E, \rho)$ be a co-CR quaternionic vector space. The *holomorphic vector bundle of $(U, E, \rho)$* is the dual of the holomorphic vector bundle of $(U^*, E^*, \rho^*)$.

Next, we explain how the holomorphic vector bundle $\mathcal{U}$ of a co-CR quaternionic vector space $(U, E, \rho)$ can be constructed directly, without passing to the dual CR quaternionic vector space.

For this, note that, with the same notations as above, $\rho$ induces an injective morphism of holomorphic vector bundles $E^{0,1} \to Z \times U^\mathbb{C}$. Then $\mathcal{U}$ is the quotient
of $Z \times U^C$ through $E^{0,1}$.

Furthermore, we have the following commutative diagram, where $\mathcal{R}$ is the ‘twistorial representation’ of $\rho$:

$$
\begin{array}{ccc}
0 & \rightarrow & Z \times \ker \rho^C \\
\downarrow & & \downarrow \\
Z \times \ker \rho^C & \rightarrow & Z \times \ker \rho^C \\
\downarrow & & \downarrow \\
0 & \rightarrow & E^{0,1} \\
\downarrow & & \downarrow \\
E^{0,1} & \rightarrow & Z \times E^C \\
\downarrow & & \downarrow \text{Id}_{Z \times E^C} \\
0 & \rightarrow & Z \times U^C \\
\downarrow & & \downarrow \mathcal{R} \\
0 & \rightarrow & U \\
\end{array}
$$

Then, similarly to the proof of Proposition [4.4], there exists a natural complex linear isomorphism $U^C = H^0(Z, \mathcal{U})$. Moreover, the conjugation of $U^C$ ($E^C$) induces a conjugation (that is, an antiholomorphic diffeomorphism) on $\mathcal{U}$ ($\mathcal{E}$) which descends to the antipodal map on $Z$. Then there exists a natural isomorphism between $U$ and the vector space of holomorphic sections of $\mathcal{U}$ which intertwine the conjugations.

Also, note that $H^1(Z, \mathcal{U}) = 0$ (this follows, for example, from the long exact sequence of cohomology groups determined by the second row of the above diagram).

Let $(U, E, \rho)$ and $(U', E', \rho')$ be co-CR quaternionic vector space and let $\mathcal{U}$ and $\mathcal{U}'$, respectively, be their holomorphic vector bundles.

Let $t : (U, E, \rho) \rightarrow (U', E', \rho')$ be a co-CR quaternionic linear map, with respect to some map $T : Z \rightarrow Z'$. Then $t$ induces a morphism of holomorphic vector bundles $T : \mathcal{U} \rightarrow \mathcal{U}'$ which intertwines the conjugations. Furthermore, the above diagram can be easily generalized with $t$ and $T$ instead of $\rho$ and $\mathcal{R}$.

**Theorem 4.8.** There exists a covariant functor $\mathcal{F}$ from the category of co-CR quaternionic vector spaces, whose morphisms are the co-CR quaternionic linear maps, to the category of holomorphic vector bundles over $\mathbb{C}P^1$, given by $\mathcal{F}(U) = \mathcal{U}$ and $\mathcal{F}(t) = T$.

Moreover, if $\mathcal{F}(U) = \mathcal{U}$ and $\mathcal{F}(U') = \mathcal{U}'$ then for any morphism of holomorphic vector bundles $T : \mathcal{U} \rightarrow \mathcal{U}'$, which intertwines the conjugations, there exists a unique co-CR quaternionic linear map $t : U \rightarrow U'$ such that $\mathcal{F}(t) = T$. 
Therefore, two co-CR quaternionic vector spaces are isomorphic if and only if there exists an isomorphism, which intertwines the conjugations, between their holomorphic vector bundles. Furthermore, for any positive integer $k$ there exists a nonempty finite set of isomorphism classes of co-CR quaternionic vector spaces $(U, E, \rho)$ with $\dim E = 4k$.

Proof. Let $(U, E, \rho)$ and $(U', E', \rho')$ be co-CR quaternionic vector spaces and let $U$ and $U'$, respectively, be their holomorphic vector bundles. Let $T : U \to U'$ be a morphism of holomorphic vector bundles which intertwines the conjugations. Then $T$ induces, through the isomorphisms $U^C = H^0(Z, U)$, $U'^C = H^0(Z', U')$, a linear map $t : U \to U'$. Furthermore, $t$ satisfies assertion (ii) of Proposition 4.6 (with respect to the identity map of $\mathbb{C}P^1$). Thus, $t$ is co-CR quaternionic linear and, obviously, $F(t) = T$.

The last assertion follows from Proposition 4.9, below. □

Next, we prove the following:

**Proposition 4.9.** Let $(U, E, \rho)$ be a co-CR quaternionic vector space and let $U$ be its holomorphic vector bundle, $\dim E = 4k$, $\dim(\ker \rho) = l$.

Then $U = \bigoplus_{j=1}^{l+1} a_j \mathcal{O}(j)$, where $a_1, \ldots, a_{l+1}$ are nonnegative integers such that $\sum_{j=1}^{l+1} a_j = 2k - l$ and $\sum_{j=1}^{l+1} ja_j = 2k$; moreover, $a_j$ is even if $j$ is odd.

Proof. As $H^1(Z, U) = 0$, from the theorem of Birkhoff and Grothendieck it follows that $U = \bigoplus_{j=-r}^r a_j \mathcal{O}(j)$ for some nonnegative integers $a_{-1}, a_0, \ldots, a_r$.

Let $J_1, \ldots, J_p \in \mathbb{Z}$ be distinct, $(p \geq 1)$. Under the isomorphism $U^C = H^0(Z, U)$ the complex vector space $\bigcap_{j=1}^p \rho(E_{J_j})$ corresponds to the space of holomorphic sections of $U$ which are zero at $J_1, \ldots, J_p$. Consequently, the complex dimension of $\bigcap_{j=1}^p \rho(E_{J_j})$ is equal to $h_0(Z, U \otimes \mathcal{O}(-p))$, where $h_0 = \dim \mathbb{C}H^0$. Hence,

$$h_0(Z, U \otimes \mathcal{O}(-1)) = \sum_{j=1}^r ja_j = 2k,$$

$$h_0(Z, U \otimes \mathcal{O}(-2)) = \sum_{j=2}^r (j-1)a_j = l.$$  \hspace{1cm} (4.3)

The second relation of (4.3) implies that $r - 1 \leq l$.

By combining the first relation of (4.3) with the fact that the degree of $U$ is $\sum_{j=1}^r ja_j = 2k$ we obtain $a_{-1} = 0$.

Now, from (4.3) and the fact that the complex rank of $U$ is $\sum_{j=0}^r a_j = 2k - l$ we obtain $a_0 = 0$.

The last statement follows from [15 (10.7)] and the proof is complete. □
The duals of Theorem 4.8 and Proposition 4.9 can be easily formulated.

The Definitions 1.9 and 1.10 can be easily extended to give the corresponding notions CR quaternionic and co-CR quaternionic vector subspaces, respectively.

**Remark 4.10.** Let $k$ and $l$ be positive integers, $l < 2k$. The set of CR quaternionic vector spaces $(U, \mathbb{H}^k, i)$ with $\text{codim} U = l$ is a Zariski open set (possibly empty) of the (real) Grassmannian $\text{Gr}_{4k-l}(\mathbb{H}^k)$.

Next, we give examples of holomorphic vector bundles of co-CR quaternionic vector spaces.

**Example 4.11.** 1) Let $(U, E, V)$ be an $f$-quaternionic vector space, $\dim E = 4k$, $\dim V = l$; let $\mathcal{U}$ be the holomorphic vector bundle of the associated co-CR quaternionic structure. From Proposition 3.10 and the proof of Proposition 4.9 we obtain $h_0(Z, \mathcal{U} \otimes \mathcal{O}(-3)) = \sum_{j=3}^{l+1} (j-2)a_j = 0$. Consequently, $a_3 = \ldots = a_{l+1} = 0$, $a_2 = l$ and $a_1 = 2(k-l)$.

We have thus proved that $\mathcal{U} = 2(k-l)\mathcal{O}(1) \oplus l\mathcal{O}(2)$.

2) Let $(U, E, \rho)$ be a co-CR quaternionic vector space whose holomorphic vector bundle $\mathcal{U}$ has complex rank 1; equivalently, $2k - l = 1$. Then $\mathcal{U} = \mathcal{O}(2k)$.

Thus, the holomorphic vector bundle of the CR quaternionic vector space $(U_{k-1}, \mathbb{H}^k)$ of Example 3.8 is $\mathcal{O}(-2k)$.

Moreover, from Theorem 4.8 and [15, (10.7)], we obtain that $(U, E, \rho)$ is isomorphic, as a co-CR quaternionic vector space, to the dual of $(U_{k-1}, \mathbb{H}^k)$.

3) In Example B.7, below, we shall prove that the holomorphic vector bundle of the CR quaternionic vector spaces of Example 3.6(2) is $2\mathcal{O}(-3)$.

4) Let $U_1$ and $U_2$ be co-CR quaternionic vector spaces, and let $Z_1$ and $Z_2$ be the spaces of admissible linear complex structures of the corresponding quaternionic vector spaces, respectively. Then any orientation preserving isometry $Z_1 = Z_2$ determines a unique linear co-CR quaternionic structure on $U_1 \oplus U_2$ which restricts to the given linear co-CR quaternionic structures on $U_1$ and $U_2$. The resulting co-CR quaternionic vector space is called the direct sum (product) of the co-CR quaternionic vector spaces $U_1$ and $U_2$, with respect to the isometry $Z_1 = Z_2$ (see Appendix A, below, for a generalization of this notion).

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the holomorphic vector bundles of $U_1$ and $U_2$, respectively. Then $\mathcal{U}_1 \oplus \mathcal{U}_2$ is the holomorphic vector bundle of $U_1 \oplus U_2$.

For example, $2\mathcal{O}(1) \oplus \mathcal{O}(4)$ is the holomorphic vector bundle of $\mathbb{H} \oplus U^*$, where $U$ is the CR quaternionic vector space of Example 3.6(1). Similarly, $\mathcal{O}(2) \oplus \mathcal{O}(4)$ is the holomorphic vector bundle of $\text{Im}\mathbb{H} \oplus U^*$.

**Remark 4.12.** 1) With the same notations as in Example 3.6(1), we have that $(\text{Im}\mathbb{H})^3$ and $\mathbb{H} \times U$ are nonisomorphic CR quaternionic vector subspaces of $\mathbb{H}^3$,
of the same dimension.

2) The fairly well known notion of hypercomplex linear map (see [1]) admit natural generalizations to the notions of hyper-CR and hyper-co-CR linear map. For example, the inclusion $\text{Im}\mathbb{H} \to \mathbb{H}$ is hyper-CR whilst the projection $\mathbb{H} \to \text{Im}\mathbb{H}$ is hyper-co-CR.

The following result will be used later on.

**Proposition 4.13.** Let $(U, E, \iota)$ be a CR quaternionic vector space. Then for any $\alpha \in U^* \setminus \{0\}$ there exist $J \in Z$ and $u, v \in U \cap J(U)$ such that $\alpha(u)\alpha(v) < 0$.

**Proof.** Obviously, $U \cap J(U) = U^J + U^{-J}$, $(J \in Z)$. Hence, we have to prove that $\bigcup_{J \in Z}(U^J + U^{-J})$ is not contained by a subspace of codimension one of $U$.

Note that, by duality, this is equivalent to the fact that for any co-CR quaternionic vector space $(U, E, \rho)$ we have $\bigcap_{J \in Z}(U^J \cap U^{-J}) = \{0\}$. Now, if $U$ is the holomorphic vector bundle of $(U, E, \rho)$ then $u \in \bigcap_{J \in Z}(U^J \cap U^{-J})$ if and only if $u$ corresponds to a holomorphic section of $U$ which is zero at any $J \in Z$; equivalently, $u = 0$. \qed

We end this section with the following:

**Remark 4.14.** The co-CR quaternionic and CR quaternionic vector spaces define augmented and strengthen $\mathbb{H}$-modules, in the sense of [8] and [15], respectively. However, the holomorphic vector bundles introduced by us are different from the sheaves introduced in [15].

5. **f-QUATERNIONIC LINEAR MAPS**

Firstly, we define the notion of $f$-quaternionic linear map.

**Definition 5.1.** Let $(U, E, V)$ and $(U', E', V')$ be $f$-quaternionic vector spaces and let $t : U \to U'$ and $T : Z \to Z'$ be maps.

Then $t : (U, E, V) \to (U', E', V')$ is $f$-quaternionic linear, with respect to $T$, if there exists a linear map $\tilde{t} : E \to E'$ which is quaternionic, with respect to $T$, and such that $\iota' \circ t = \tilde{t} \circ \iota$ and $\rho' \circ \tilde{t} = t \circ \rho$, where $\iota : U \to E$ and $\iota' : U' \to E'$ are the inclusions whilst $\rho : E \to U$ and $\rho' : E' \to U'$ are the projections.

Let $(U, E, V)$ be an $f$-quaternionic vector space. Denote by $\iota : U \to E$ the inclusion and by $\rho : E \to U$ the projection; also, let $F^J$ be the linear $f$-structure on $U$ induced by $(\iota, \rho, J)$, $(J \in Z)$.

The following result follows quickly from Propositions 2.3, 4.4 and 4.6.
Proposition 5.3. Let $(U, E, V)$ and $(U', E', V')$ be $f$-quaternionic vector spaces. Let $t : U \rightarrow U'$ be a nonzero linear map and let $T : Z \rightarrow Z'$ be a map.

Then the following assertions are equivalent:

(i) $t$ is $f$-quaternionic, with respect to $T$.

(ii) $T$ is a holomorphic diffeomorphism and $t \circ F^J = F^{t(J)} \circ t$, for any $J \in \mathbb{Z}$.

Furthermore, if assertion (i) or (ii) holds then there exists a unique linear map $\tilde{t} : E \rightarrow E'$ which is quaternionic, with respect to $T$, such that $t' \circ t = \tilde{t} \circ \iota$ and $\rho' \circ \tilde{t} = t \circ \rho$.

Let $t : (U, E, V) \rightarrow (U', E', V')$ be an $f$-quaternionic linear map, with respect to some map $T : Z \rightarrow Z'$. If $t$ is injective we say that $(U, E, V)$ is an $f$-quaternionic vector subspace of $(U', E', V')$.

Let $(U, E, \iota)$ and $(U, E, \rho)$ be the CR and co-CR quaternionic vector spaces, respectively, corresponding to $(U, E, V)$. Also, let $(U', E', \iota')$ and $(U', E', \rho')$ be the CR and co-CR quaternionic vector spaces, respectively, corresponding to $(U', E', V')$. Then $(U, E, V)$ is an $f$-quaternionic vector subspace of $(U', E', V')$ if and only if $(U, E, \iota)$ and $(U, E, \rho)$ are CR and co-CR quaternionic vector subspaces of $(U', E', \iota')$ and $(U', E', \rho')$, respectively.

Next, we prove the following result.

Corollary 5.2. Let $(U, E, V)$ and $(U', E', V')$ be $f$-quaternionic vector spaces. Let $t : U \rightarrow U'$ be a nonzero linear map and let $T : Z \rightarrow Z'$ be a map.

Then the following assertions are equivalent:

(i) $t$ is $f$-quaternionic, with respect to $T$.

(ii) $T$ is a holomorphic diffeomorphism and $t \circ F^J = F^{t(J)} \circ t$, for any $J \in \mathbb{Z}$.

Furthermore, if assertion (i) or (ii) holds then there exists a unique linear map $\tilde{t} : E \rightarrow E'$ which is quaternionic, with respect to $T$, such that $t' \circ t = \tilde{t} \circ \iota$ and $\rho' \circ \tilde{t} = t \circ \rho$.

Let $t : (U, E, V) \rightarrow (U', E', V')$ be an $f$-quaternionic linear map, with respect to some map $T : Z \rightarrow Z'$. If $t$ is injective we say that $(U, E, V)$ is an $f$-quaternionic vector subspace of $(U', E', V')$.

Let $(U, E, \iota)$ and $(U, E, \rho)$ be the CR and co-CR quaternionic vector spaces, respectively, corresponding to $(U, E, V)$. Also, let $(U', E', \iota')$ and $(U', E', \rho')$ be the CR and co-CR quaternionic vector spaces, respectively, corresponding to $(U', E', V')$. Then $(U, E, V)$ is an $f$-quaternionic vector subspace of $(U', E', V')$ if and only if $(U, E, \iota)$ and $(U, E, \rho)$ are CR and co-CR quaternionic vector subspaces of $(U', E', \iota')$ and $(U', E', \rho')$, respectively.

Next, we prove the following result.

Proposition 5.3. Let $(U, E, V)$ be an $f$-quaternionic vector space, $\dim E = 4k$, $\dim V = l$. Then there exists an $f$-quaternionic linear isomorphism from $U$ onto $(\text{Im} \mathbb{H})_l \times \mathbb{H}^{k-l}$.

Proof. With the same notations as in Proposition 3.4 we have $U = Q(V) \oplus W$. Then $(Q(V), V \oplus Q(V), V)$ and $(W, W, \{0\})$ are $f$-quaternionic vector subspaces of $(U, E, V)$.

Obviously, any basis of $V$ and positive orthonormal basis of $Q$ determine an $f$-quaternionic linear isomorphism $Q(V) = (\text{Im} \mathbb{H})_l$. Also, we have $W = \mathbb{H}^{k-l}$. □

Next, we describe the Lie group $G$ of $f$-quaternionic linear isomorphisms of $(\text{Im} \mathbb{H})_l \times \mathbb{H}^{k-l}$. For this, let $\rho_k : \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H}) \rightarrow \text{SO}(3)$ be the Lie group morphism defined by $\rho_k(q \cdot A) = \pm q$, for any $q \cdot A \in \text{Sp}(1) \cdot \text{GL}(k, \mathbb{H})$, $(k \geq 1)$. Denote

$$H = \{(A, A') \in (\text{Sp}(1) \cdot \text{GL}(l, \mathbb{H})) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})) \mid \rho_l(A) = \rho_m(A')\}. $$

Then $H$ is a closed subgroup of $\text{Sp}(1) \cdot \text{GL}(l+m, \mathbb{H})$ and $G$ is the closed subgroup of $H$ formed of those elements $(A, A') \in H$ such that $A$ preserves $\mathbb{H}^{l} \subseteq \mathbb{H}^{l}$. This follows from the fact that there are no nontrivial $f$-quaternionic linear maps from $\text{Im} \mathbb{H}$ to $\mathbb{H}$ (and from $\mathbb{H}$ to $\text{Im} \mathbb{H}$).
Now, the canonical basis of $\text{Im} \mathbb{H}$ induces a linear isomorphism $(\text{Im} \mathbb{H})^l = (\mathbb{R}^l)^3$ and, therefore, an effective action $\sigma$ of $\text{GL}(l, \mathbb{R})$ on $(\text{Im} \mathbb{H})^l$.

We define an effective action of $\text{GL}(l, \mathbb{R}) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H}))$ on $(\text{Im} \mathbb{H})^l \times \mathbb{H}^m$ by

$$(A, q \cdot B)(X, Y) = \left( q(\sigma(A)(X))q^{-1}, qYB^{-1} \right),$$

for any $A \in \text{GL}(l, \mathbb{R})$, $q \cdot B \in \text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})$, $X \in (\text{Im} \mathbb{H})^l$ and $Y \in \mathbb{H}^m$.

**Proposition 5.4.** There exists an isomorphism of Lie groups

$$G = \text{GL}(l, \mathbb{R}) \times (\text{Sp}(1) \cdot \text{GL}(m, \mathbb{H})),$$

given by $(A, A') \mapsto (A|_{\mathbb{R}^l}, A')$, for any $(A, A') \in G$.

In particular, the group of $f$-quaternionic linear isomorphisms of $(\text{Im} \mathbb{H})^l$ is isomorphic to $\text{GL}(l, \mathbb{R}) \times \text{SO}(3)$.

Note that, the group of $f$-quaternionic linear isomorphisms of $\text{Im} \mathbb{H}$ is $\text{CO}(3)$.

We end this section by listing all the CR quaternionic vector spaces $(U, E, \iota)$ such that $\dim E \leq 8$.

**Proposition 5.5.** Let $(U, E, \iota)$ be a CR quaternionic vector space such that $\dim E \leq 8$. Then, up to a CR quaternionic linear isomorphism, $U$ is either given by Example 3.6(1) or is one of the following $f$-quaternionic vector spaces \{0\}, $\text{Im} \mathbb{H}$, $\mathbb{H}$, $(\text{Im} \mathbb{H})^2$, $\mathbb{H} \times \text{Im} \mathbb{H}$, $\mathbb{H}^2$.

**Proof.** By Propositions 3.7 and 5.3, and Corollary 3.9 it is sufficient to consider the case $\dim U = 5$. Then, by Example 4.11(2), the holomorphic vector bundle of $U$ is $\mathcal{O}(-4)$. Therefore $(U^\mathbb{C}, E^\mathbb{C}, \iota^\mathbb{C})$ is isomorphic with the complexification of the CR quaternionic vector space given by Example 3.6(1).

By using the fact that any automorphism of $\mathcal{O}(-4)$ is of the form $\lambda \text{Id}_{\mathcal{O}(-4)}$, for some nonzero complex number $\lambda$, we obtain that the conjugation induced by $(U, E, \iota)$ on $\mathcal{O}(-4)$ is equal to $\lambda \tau$, where $\tau$ is the conjugation induced by Example 3.6(1). As $(\lambda \tau)^2 = 1$, we obtain $\lambda = 1$ and the proof is complete. \qed

Note that, from Example 3.6(1) and the proof of Proposition 5.5 it follows that, up to a CR quaternionic linear isomorphism, there exists at most one CR quaternionic vector space $(U, E, \iota)$ such that $2k - l = 1$, where $\dim E = 4k$ and $\dim U = 4k - l$.

6. **CR QUATERNIONIC MANIFOLDS**

Unless otherwise stated, all the manifolds are connected and smooth and all the maps are smooth.
**Definition 6.1.** A (fibre) bundle of associative algebras is a vector bundle whose typical fibre is a (finite-dimensional) associative algebra and whose structural group is the group of automorphisms of the typical fibre.

Let $A$ and $B$ be bundles of associative algebras. A morphism of vector bundles $\rho : A \to B$ is called a morphism of bundles of associative algebras if $\rho$ restricted to each fibre is a morphism of associative algebras.

**Definition 6.2.** A quaternionic vector bundle over a manifold $M$ is a real vector bundle $E$ over $M$ endowed with a pair $(A, \rho)$ where $A$ is a bundle of associative algebras, over $M$, with typical fibre $\mathbb{H}$ and $\rho : A \to \text{End}(E)$ is a morphism of bundles of associative algebras; we say that $(A, \rho)$ is a linear quaternionic structure on $E$. If, further, $A = M \times \mathbb{H}$ then $E$ is called a hypercomplex vector bundle.

Standard arguments (see [7]) apply to show that a quaternionic vector bundle of (real) rank $4k$ is just a (real) vector bundle endowed with a reduction of its structural group to $\text{Sp}(1) \cdot \text{GL}(k, \mathbb{H})$. Similarly, a hypercomplex vector bundle of rank $4k$ is a vector bundle endowed with a reduction of its structural group to $\text{GL}(k, \mathbb{H})$.

Let $(A, \rho)$ be a linear quaternionic structure on the vector bundle $E$. We denote $Q = \rho(\text{Im}A)$, $\tilde{Q} = \rho(A)$ and by $Z$ the sphere bundle of $Q$.

An almost CR structure on a manifold $M$ is a complex vector subbundle $C$ of $T^cM$ such that $C \cap \overline{C} = \{0\}$. An (integrable almost) CR structure is an almost CR structure whose space of sections is closed under the usual bracket.

**Definition 6.3.** Let $E$ be a quaternionic vector bundle on a manifold $M$ and let $\iota : TM \to E$ be an injective morphism of vector bundles.

We say that $(E, \iota)$ is an almost CR quaternionic structure on $M$ if $(E_x, \iota_x)$ is a linear CR quaternionic structure on $T_xM$, for any $x \in M$. If, further, $E$ is a hypercomplex vector bundle then $(E, \iota)$ is called an almost hyper-CR structure on $M$.

An almost CR quaternionic manifold (almost hyper-CR manifold) is a manifold endowed with an almost CR quaternionic structure (almost hyper-CR structure).

If in Definition 6.3 we have that $\iota$ is an isomorphism then we obtain the well known notions (see [7]) of almost quaternionic and almost hypercomplex manifold.

**Example 6.4.** 1) Let $(M, c)$ be a three-dimensional conformal manifold and let $L$ be the line bundle of $M$; that is, $L = (\Lambda^3TM)^{1/3}$. Then, $E = L \oplus TM$ is an oriented vector bundle of rank four endowed with a (linear) conformal structure.
such that $L = (TM)^\perp$. Therefore $E$ is a quaternionic vector bundle and $(M, E, \iota)$ is an almost CR quaternionic manifold, where $\iota : TM \to E$ is the inclusion.

Moreover, any almost CR quaternionic structure on a three-dimensional manifold $M$ is obtained this way from a conformal structure on $M$.

2) Let $M$ be a hypersurface in an almost quaternionic manifold $N$. Then $(TN|_M, \iota)$ is an almost CR quaternionic structure on $M$, where $\iota : TM \to TN|_M$ is the inclusion.

3) More generally, let $N$ be an almost quaternionic manifold. Let $M$ be a submanifold of $N$ such that, at some point $x \in M$, we have that $(T_xM, E_x, \iota_x)$ is a CR quaternionic vector space, where $\iota$ is the inclusion $TM \to TN|_M$. Then, from Remark 4.10 it follows that by passing, if necessary, to an open neighborhood of $x$ in $M$ we have that $(TN|_M, \iota)$ is an almost CR quaternionic structure on $M$.

Next, we make the following:

**Definition 6.5.** Let $(M, E, \iota)$ be an almost CR quaternionic manifold. An almost quaternionic connection on $(M, E, \iota)$ is a connection $\nabla$ on $E$ which preserves $Q$. A quaternionic connection is a torsion-free (that is, $d\nabla \iota = 0$) almost quaternionic connection.

Next, we are going to introduce a natural almost twistorial structure on any almost CR quaternionic manifold $(M, E, \iota)$ endowed with an almost quaternionic connection $\nabla$ (see [14] for the definition of almost twistorial structures).

For any $J \in Z$, let $B_J \subseteq T^{C,J}_Z$ be the horizontal lift, with respect to $\nabla$, of $\iota^{-1}(E^J)$, where $E^J \subseteq E^C_{\pi(J)}$ is the eigenspace of $J$ corresponding to $-i$.

Define $C_J = B_J \oplus (\ker d\pi)^{0,1}_J$, $(J \in Z)$. Then $C$ is an almost CR structure on $Z$ and $(Z, M, \pi, C)$ is the almost twistorial structure of $(M, E, \iota, \nabla)$.

The following definition is motivated by [7, Remark 2.10(2)].

**Definition 6.6.** An (integrable almost) CR quaternionic structure on $M$ is a triple $(E, \iota, \nabla)$, where $(E, \iota)$ is an almost CR quaternionic structure on $M$ and $\nabla$ is an almost quaternionic connection of $(M, E, \iota)$ such that the almost twistorial structure of $(M, E, \iota, \nabla)$ is integrable (that is, $C$ is integrable). If, further, $E$ is a hypercomplex vector bundle and $\nabla$ induces the trivial flat connection on $Q$ then $(E, \iota, \nabla)$ is called an (integrable almost) hyper-CR structure on $M$.

A CR quaternionic manifold (hyper-CR manifold) is a manifold endowed with a CR quaternionic structure (hyper-CR structure).

From [7, Remark 2.10] it follows that a CR quaternionic (hyper-CR) structure $(E, \iota, \nabla)$ for which $\iota$ is an isomorphism is a quaternionic (hypercomplex) structure.

The following result is an immediate consequence of Theorem C.3.
Corollary 6.7. Let $M$ be endowed with an almost CR quaternionic structure $(E, \iota)$ and let $\nabla$ be an almost quaternionic connection of $(M, E, \iota)$; denote by $E^J$ the eigenspace of $J$ with respect to $-i$ and let $T^J M = \iota^{-1}(E^J)$, $(J \in \mathbb{Z})$.

The following assertions are equivalent:

(i) The almost twistorial structure of $(M, E, \iota, \nabla)$ is integrable.

(ii) $T(\Lambda^2(T^J M)) \subseteq E^J$ and $R(\Lambda^2(T^J M))(E^J) \subseteq E^J$, for any $J \in \mathbb{Z}$, where $T$ and $R$ are the torsion and curvature forms, respectively, of $\nabla$.

Next, we are going to prove the main result of this section.

Theorem 6.8. Let $M$ be endowed with an almost CR quaternionic structure $(E, \iota)$, rank $E = 4k$, dim $M = 4k - l$, $(0 \leq l \leq 2k - 1)$. Let $\nabla$ be a quaternionic connection on $(M, E, \iota)$.

If $2k - l \neq 2$ then the almost twistorial structure of $(M, E, \iota, \nabla)$ is integrable.

Proof. If $2k - l = 1$ then, as $T^J M$ is one-dimensional, for any $J \in \mathbb{Z}$, the proof is an immediate consequence of Corollary 6.7.

Assume $2k - l \geq 3$ and note that, as the complexification of the structural group of $E$ is $GL(2k, \mathbb{C}) \cdot SL(2, \mathbb{C})$, we have that, locally, $E^C = E' \otimes_{\mathbb{C}} E''$ where $E'$ and $E''$ are complex vector bundles of ranks $2$ and $2k$, respectively. Moreover, we have $\nabla^C = \nabla' \otimes \nabla''$, where $\nabla'$ and $\nabla''$ are connections on $E'$ and $E''$, respectively. Also, $Z = P(E')$ such that if $J \in Z_x$ corresponds to the line $[u] \in P(E'_x)$ then the eigenspace of $J$ corresponding to $-i$ is equal to $\{ u \otimes v \mid v \in E''_x \}$, $(x \in M)$.

Then assertion (ii) of Corollary 6.7 holds if and only if $R'(X, Y) \in [u]$, for any $u \in E'_x \setminus \{0\}$ and $X, Y \in T_x M$, such that $\iota(X), \iota(Y) \in \{ u \otimes v \mid v \in E''_x \}$, $(x \in M)$, where $R'$ is the curvature form of $\nabla'$.

The proof now follows quickly from the Bianchi identity $R \wedge \iota = 0$ and the fact that the dimension of $\iota^{-1}(\{ u \otimes v \mid v \in E''_x \})$ is at least $3$, $(x \in M)$.

Example 6.9. 1) Let $(M, c, D)$ be a three-dimensional Weyl space; that is, $(M, c)$ is a conformal manifold and $D$ is a torsion free conformal connection on it. With the same notations as in Example 6.4(1), let $D^L$ be the connection induced by $D$ on $L$. Then $\nabla = D^L \oplus D$ is a quaternionic connection on $(M, E, \iota)$; in particular, $(M, E, \iota, \nabla)$ is a CR quaternionic manifold.

2) If in Examples 6.4(2) and 6.4(3) we assume $N$ quaternionic then we obtain examples of CR quaternionic structures.

7. Quaternionic manifolds as heaven spaces

Firstly, we recall the following definition (see [2], [16]).
Definition 7.1. Let \((M, \mathcal{C})\) be a CR manifold, \(\dim M = 2k - l\), \(\operatorname{rank} \mathcal{C} = k - l\).

1) We say that \((M, \mathcal{C})\) is realizable if \(M\) is an embedded submanifold, of codimension \(l\), of a complex manifold \(N\) such that \(\mathcal{C} = T^C M \cap (T^{0,1} N)|_M\).

2) We say that \((M, \mathcal{C})\) is locally realizable if each point of \(M\) has an open neighbourhood \(U\) such that \((U, \mathcal{C}|_U)\) is realizable.

Note that, by [2, Proposition 1.10], in Definition 7.1(2) the codimension condition is superfluous.

We make the following:

Definition 7.2. Let \((M, E, \iota, \nabla)\) be a CR quaternionic manifold and let \((Z, \mathcal{C})\) be its twistor space.

We say that \((M, E, \iota, \nabla)\) is realizable if \(M\) is an embedded submanifold of a quaternionic manifold \(N\) such that \(E = TN|_M\), as quaternionic vector bundles, and \(\mathcal{C} = T^C Z \cap (T^{0,1} Z_N)|_M\), where \(Z_N\) is the twistor space of \(N\).

Next, we prove the main result of this section.

Theorem 7.3. Let \((M, E, \iota, \nabla)\) be a CR quaternionic manifold and let \((Z, \mathcal{C})\) be its twistor space.

Then the following assertions are equivalent:

(i) \((M, E, \iota, \nabla)\) is realizable.

(ii) \((Z, \mathcal{C})\) is locally realizable.

Proof. As the implication (i)\(\implies\)(ii) is trivial, it is sufficient to prove (ii)\(\implies\)(i).

If (ii) holds then, by using Proposition 4.13 and an argument as in the proof of Theorem C.3, we obtain that the criterion mentioned in [16, Theorem 3.7] can be applied to obtain that any (local) CR function on \((Z, \mathcal{C})\) can be locally extended to a holomorphic function. Then the proof of [2, Theorem 1.12] can be easily adapted to obtain that \((Z, \mathcal{C})\) is a closed embedded submanifold, of codimension \(l\), of a complex manifold \(Z_1\) such that \(\mathcal{C} = T^C Z \cap (T^{0,1} Z_1)|_Z\), where \(\dim M = 4k - l\), \(\operatorname{rank} E = 4k\); in particular, \(\dim_C Z_1 = 2k + 1\).

Furthermore, the fibres of \(\pi : Z \to M\) are complex projective lines embedded in \(Z_1\) as complex submanifolds with normal bundle \(2k\mathcal{O}(1)\). Then, [10] and [16, Proposition 2.5] implies that \(M\) is a submanifold of a complex manifold \(N_1\), \(\dim_C N_1 = 4k\), which parametrizes a holomorphic family of complex projective lines which contains \(\{\pi^{-1}(x)\}_{x \in M}\).

Let \(\tau : Z \to Z\) be defined by \(\tau(J) = -J\), \((J \in Z)\). Then \(\tau\) is a diffeomorphism and \(\tau(\mathcal{C}) = \overline{\mathcal{C}}\). Therefore, by passing, if necessary, to an open neighbourhood of \(Z\) in \(Z_1\), we may suppose that \(\tau\) extends to an anti-holomorphic diffeomorphism \(\tau_1 : Z_1 \to Z_1\). Furthermore, by applying [10] we obtain that we may suppose
that $\tau_1$ induces an anti-holomorphic diffeomorphism $\sigma$ of $N_1$. Moreover, as the fixed point set $N$ of $\sigma$ is nonempty ($M \subseteq N$) from [10] it, also, follows that $N$ is real analytic and its complexification is $N_1$; in particular, $\dim N = 4k$. Then $N$ is a quaternionic manifold whose twistor space is $Z_1$.

To complete the proof we have to show that the map $M \hookrightarrow N$ is a continuous embedding. This is an immediate consequence of the facts that $Z$ is embedded in $Z_1$ and that the topologies of $M$ and $N$ are equal to the quotient topologies of $Z$ and $Z_1$ with respect to the (open) projections $Z \to M$ and $Z_1 \to N$, respectively. □

From the proof of Theorem 7.3 it follows that if a CR quaternionic manifold $(M, E, \iota, \nabla)$ is realizable then the corresponding quaternionic manifold is germ unique, up to a choice of a connection; we call, this quaternionic manifold, the heaven space of $(M, E, \iota, \nabla)$.

**Corollary 7.4.** Any a real-analytic CR quaternionic manifold is realizable.

*Proof.* As any real analytic CR manifold is realizable [2], this is an immediate consequence of Theorem 7.3. □

We end this section with the following result.

**Corollary 7.5.** Let $(M, E, \iota, \nabla)$ be a CR quaternionic manifold and let $(Z, \mathcal{C})$ be its twistor space.

If $(Z, \mathcal{C})$ is locally realizable then $(M, E, \iota)$ admits quaternionic connections and for any such connection $\nabla'$ the twistor space of $(M, E, \iota, \nabla')$ is equal to $(Z, \mathcal{C})$.

*Proof.* This is an immediate consequence of Theorem 7.3. □

8. Co-CR quaternionic manifolds

An almost co-CR structure on a manifold $M$ is a complex vector subbundle $\mathcal{C}$ of $T^cM$ such that $\mathcal{C} + \overline{\mathcal{C}} = T^cM$. An (integrable almost) co-CR structure is an almost co-CR structure whose space of sections is closed under the bracket.

**Example 8.1.** Let $\varphi : M \to (N, J)$ be a submersion onto a complex manifold. Then $(d\varphi)^{-1}(T^{0,1}N)$ is a co-CR structure on $M$.

Moreover, any co-CR structure is, locally, of this form.

The following definition is the dual of Definition 6.3.

**Definition 8.2.** Let $E$ be a quaternionic vector bundle on a manifold $M$ and let $\rho : E \to TM$ be a surjective morphism of vector bundles.

Then $(E, \rho)$ is called an almost co-CR quaternionic structure, on $M$, if $(E_x, \rho_x)$
is a linear co-CR quaternionic structure on $T_{\pi(j)} M$, for any $x \in M$. If, further, $E$ is a hypercomplex vector bundle then $(E, \rho)$ is called an almost hyper-co-CR structure on $M$.

An almost co-CR quaternionic manifold (almost hyper-co-CR manifold) is a manifold endowed with an almost co-CR quaternionic structure (almost hyper-co-CR structure).

Any almost co-CR quaternionic (hyper-co-CR) structure $(E, \rho)$ for which $\rho$ is an isomorphism is an almost quaternionic (hypercomplex) structure.

With the same notations as in Example 6.4, if $(M, c)$ is a three-dimensional conformal manifold then $(M, E, \rho)$ is an almost co-CR quaternionic manifold, where $\rho : E \to TM$ is the projection. Moreover, any three-dimensional almost co-CR quaternionic manifold is obtained this way.

Next, we are going to introduce a natural almost twistorial structure on any almost co-CR quaternionic manifold $(M, E, \rho)$ for which $E$ is endowed with a connection $\nabla$ compatible with its linear quaternionic structure.

For any $J \in \mathbb{Z}$, let $C_J \subseteq T_C Z$ be the direct sum of $(\ker d\pi)_J^{0,1}$ and the horizontal lift, with respect to $\nabla$, of $\rho(E^J)$, where $E^J$ is the eigenspace of $J$ corresponding to $-i$. Then $C$ is an almost co-CR structure on $Z$ and $(Z, M, \pi, C)$ is the almost twistorial structure of $(M, E, \rho, \nabla)$.

**Definition 8.3.** An (integrable almost) co-CR quaternionic manifold is an almost co-CR quaternionic manifold $(M, E, \rho)$ endowed with a compatible connection $\nabla$ on $E$ such that the associated almost twistorial structure $(Z, M, \pi, C)$ is integrable (that is, $C$ is integrable).

If, further, $E$ is a hypercomplex vector bundle and the connection induced by $\nabla$ on $Z$ is trivial then $(M, E, \rho, \nabla)$ is an (integrable almost) hyper-co-CR manifold.

Next, we discuss the simplest examples of co-CR quaternionic and hyper-co-CR manifolds.

**Example 8.4.** Let $(M, c)$ be a three-dimensional conformal manifold and let $(E, \rho)$ be the corresponding almost co-CR structure, where $E = L \oplus TM$ with $L$ the line bundle of $M$. Let $D$ be a Weyl connection on $(M, c)$ and, as in Example 6.9(1), let $\nabla = D^L \oplus D$, where $D^L$ is the connection induced by $D$ on $L$. It follows that $(M, E, \rho, \nabla)$ is co-CR quaternionic if and only if $(M, c, D)$ is Einstein–Weyl (that is, the trace-free symmetric part of the Ricci tensor of $D$ is zero).

Furthermore, let $\mu$ be a section of $L^*$ such that the connection defined by

$$D^\mu_X Y = D_X Y + \mu X \times_c Y$$
for any vector fields \( X \) and \( Y \) on \( M \), induces a flat connection on \( L^* \otimes TM \). Then \((M, E, \iota, \nabla^\mu)\) is, locally, a hyper-co-CR manifold, where \( \nabla^\mu = (D^\mu)^L \oplus D^\mu \), with \((D^\mu)^L\) the connection induced by \( D^\mu \) on \( L \) (this follows from well-known results; see [14] and the references therein).

Let \( \tau = (Z, M, \pi, \mathcal{C}) \) be the twistorial structure of a co-CR quaternionic manifold \((M, E, \rho, \nabla)\). Recall [14] that \( \tau \) is simple if and only if \( \mathcal{C} \cap \overline{\mathcal{C}} \) is a simple foliation (that is, its leaves are the fibres of a submersion) whose leaves intersect each fibre of \( \pi \) at most once. Then \((T, d\varphi(C))\) is the twistor space of \( \tau \), where \( \varphi : Z \to T \) is the submersion whose fibres are the leaves of \( \mathcal{C} \cap \overline{\mathcal{C}} \).

**Example 8.5.** Any co-CR quaternionic vector space is a co-CR quaternionic manifold, in an obvious way; moreover, the associated twistorial structure is simple and its twistor space is just its holomorphic vector bundle.

Next, we prove the main result of this section.

**Theorem 8.6.** Let \((M, E, \rho, \nabla)\) be a co-CR quaternionic manifold, \( \operatorname{rank} E = 4k \), \( \operatorname{rank}(\ker \rho) = l \).

If the twistorial structure of \((M, E, \rho, \nabla)\) is simple then it is real analytic and its twistor space is a complex manifold of dimension \( 2k - l + 1 \) endowed with a locally complete family of complex projective lines \( \{Z_x\}_{x \in M^c} \). Furthermore, for any \( x \in M \), the normal bundle of the corresponding twistor line \( Z_x \) is the holomorphic vector bundle of \((T_xM, E_x, \rho_x)\).

*Proof.* Let \((Z, M, \pi, \mathcal{C})\) be the twistorial structure of \((M, E, \rho, \nabla)\). Let \( \varphi : Z \to T \) be the submersion whose fibres are the leaves of \( \mathcal{C} \cap \overline{\mathcal{C}} \). Obviously, \( d\varphi(C) \) defines a complex structure on \( T \) of dimension \( 2k - l + 1 \).

Furthermore, if for any \( x \in M \) we denote \( Z_x = \varphi(\pi^{-1}(x)) \) then \( Z_x \) is a complex submanifold of \( T \) whose normal bundle is the holomorphic vector bundle of \((T_xM, E_x, \rho_x)\).

The proof follows from [10], [16] Proposition 2.5 and Proposition 4.9. \( \Box \)

We end this section with the following result.

**Proposition 8.7.** Let \((M, E, \rho, \nabla)\) be a co-CR quaternionic manifold whose twistorial structure is simple; denote by \( \varphi : Z \to T \) the corresponding holomorphic submersion onto its twistor space.

Then \((M, E, \rho, \nabla)\) is hyper-co-CR if and only if there exists a surjective holomorphic submersion \( \psi : T \to \mathbb{C}P^1 \) such that the fibres of \( \psi \circ \varphi \) are integral manifolds of the connection induced by \( \nabla \) on \( Z \).
Proof. Denote by $H$ the connection induced by $\nabla$ on $Z$. Then $H$ is integrable if and only if $d\phi(H)$ is a holomorphic foliation on $T$; furthermore, this foliation is simple if and only if $E$ is hypercomplex and $H$ is the trivial connection on $Z$. □

9. $f$-QUATERNIONIC MANIFOLDS

Let $F$ be an almost $f$-structure on a manifold $M$; that is, $F$ is an endomorphism of $TM$ such that $F^3 + F = 0$. Denote by $C$ the eigenspace of $F$ with respect to $-i$ and let $D = C \oplus \ker F$. Then $C$ and $D$ are compatible almost CR and almost co-CR structures, respectively.

An (integrable almost) $f$-structure is an almost $f$-structure for which the corresponding almost CR and almost co-CR structures are integrable.

Definition 9.1. An almost $f$-quaternionic structure on a manifold $M$ is a pair $(E,V)$, where $E$ is a quaternionic vector bundle on $M$ and $TM$ and $V$ are vector subbundles of $E$ such that $E = TM \oplus V$ and $J(V) \subseteq TM$, for any $J \in Z$.

An almost hyper-$f$-structure on a manifold $M$ is an almost $f$-quaternionic structure $(E,V)$ on $M$ such that $E$ is a hypercomplex vector bundle.

An almost $f$-quaternionic manifold (almost hyper-$f$-manifold) is a manifold endowed with an almost $f$-quaternionic structure (almost hyper-$f$-structure).

With the same notations as in Definition 9.1, an almost $f$-quaternionic structure (almost hyper-$f$-structure) for which $V$ is the zero bundle is an almost quaternionic structure (almost hypercomplex structure).

Let $k$ and $l$ be positive integers, $k \geq l$, and denote by $G_{k,l}$ the group of $f$-quaternionic linear isomorphisms of $(\text{Im} \mathbb{H})^l \times \mathbb{H}^{k-l}$.

The next result follows from the description of $G_{k,l}$ given in Section 5.

Proposition 9.2. Let $M$ be a manifold of dimension $4k - l$. Then any almost $f$-quaternionic structure $(E,V)$ on $M$, with rank $E = 4k$ and rank $V = l$, corresponds to a reduction of the frame bundle of $M$ to $G_{k,l}$.

Furthermore, if $(P,M,G_{k,l})$ is the reduction of the frame bundle of $M$, corresponding to $(E,V)$, then $V$ is the vector bundle associated to $P$ through the canonical morphism of Lie groups $G_{k,l} \to \text{GL}(l,\mathbb{R})$.

Example 9.3. 1) A three-dimensional almost $f$-quaternionic manifold is just a (three-dimensional) conformal manifold.

2) Let $N$ be an almost quaternionic manifold endowed with a Hermitian metric and let $M$ be a hypersurface in $N$. Then $\left(TN\big|_M, (TM)^\perp\right)$ is an almost $f$-quaternionic structure on $M$. 
Obviously, any almost $f$-quaternionic structures $(E, V)$ on a manifold $M$ corresponds to a pair $(E, \iota)$ and $(E, \rho)$ of almost CR quaternionic and co-CR quaternionic structures on $M$, where $\iota : TM \to E$ and $\rho : E \to TM$ are the inclusion and projection, respectively.

**Definition 9.4.** Let $(M, E, V)$ be an almost quaternionic manifold. Let $(E, \iota)$ and $(E, \rho)$ be the almost CR quaternionic and co-CR quaternionic structures, respectively, corresponding to $(E, V)$.

Let $\nabla$ be a connection on $E$ compatible with its linear quaternionic structure and let $\tau$ and $\tau_c$ be the almost quaternionic structures of $(M, E, \iota, \nabla)$ and $(M, E, \rho, \nabla)$, respectively.

We say that $(M, E, V, \nabla)$ is an $f$-quaternionic manifold if the almost twistorial structures $\tau$ and $\tau_c$ are integrable. If, further, $E$ is hypercomplex and $\nabla$ induces the trivial flat connection on $Z$ then $(M, E, V, \nabla)$ is an (integrable almost) hyper-$f$-manifold.

Let $(M, E, V, \nabla)$ be an $f$-quaternionic manifold and let $Z$ and $Z_c$ be the twistor spaces of $\tau$ and $\tau_c$, respectively (we assume, for simplicity, that $\tau_c$ is simple). Then $Z$ is called the CR twistor space and $Z_c$ is called the twistor space of $(M, E, V, \nabla)$.

Let $(M, E, V, \nabla)$ be an almost $f$-quaternionic manifold and let $\nabla$ be a connection on $E$ compatible with its linear quaternionic structure. Let $\mathcal{C}$ and $\mathcal{D}$ be the almost CR and almost co-CR structures on $Z$ determined by $\nabla$ and the underlying almost CR quaternionic and almost co-CR quaternionic structures of $(M, E, V)$, respectively. Then $\mathcal{C}$ and $\mathcal{D}$ are compatible; therefore $(M, E, V, \nabla)$ is $f$-quaternionic if and only if the corresponding almost $f$-structure on $Z$ is integrable.

Let $(M, E, V)$ be an almost $f$-quaternionic manifold, rank $E = 4k$, rank $V = l$, and let $D$ be some compatible connection on $M$ (equivalently, $D$ is a linear connection on $M$ which corresponds to a principal connection on the reduction to $G_{k,l}$, of the frame bundle of $M$, corresponding to $(E, V)$).

Then $D$ induces a connection $D^V$ on $V$. Moreover, $\nabla = D^V \oplus D$ is a connection on $E$ compatible with its linear quaternionic structure.

**Corollary 9.5.** Let $(M, E, V, \nabla)$ be an $f$-quaternionic manifold, rank $E = 4k$, rank $V = l$, where $\nabla = D^V \oplus D$ for some compatible connection $D$ on $M$. Denote by $\tau$ and $\tau_c$ the associated twistorial structures.

Then, locally, the twistor space of $(M, \tau_c)$ is a complex manifold, of complex dimension $2k-l+1$, endowed with a locally complete family of complex projective lines each of which has normal bundle $2(k-l)\mathcal{O}(1) \oplus l\mathcal{O}(2)$. 
Furthermore, if \((M, E, V, \nabla)\) is real analytic then, locally, there exists a twistorial map from the corresponding heaven space \(N\), endowed with its twistorial structure, to \((M, \tau_c)\) which is a retraction of the inclusion \(M \subseteq N\).

Proof. By passing to a convex open set of \(D\), if necessary, we may suppose that \(\tau_c\) is simple. Thus, the first assertion is a consequence of Theorem 8.6 and Example 4.11.

The second statement follows from the fact that there exists a holomorphic submersion from the twistor space of \(N\), endowed with its twistorial structure, to the twistor space of \((M, \tau_c)\), which maps diffeomorphically twistor lines onto twistor lines. \(\square\)

Note that, if \(\dim M = 3\) then Corollary 9.5 gives results of [11] and [6].

Example 9.6. Let \(M^{3l} = \text{Gr}_{3}^{+}(l + 3, \mathbb{R})\) be the Grassmann manifold of oriented vector subspaces of dimension 3 of \(\mathbb{R}^{l+3}\), \((l \geq 1)\). Alternatively, \(M^{3l}\) can be defined as the Riemannian symmetric space \(\text{SO}(l + 3)/\left(\text{SO}(l) \times \text{SO}(3)\right)\).

As the structural group of the frame bundle of \(M^{3l}\) is \(\text{SO}(l) \times \text{SO}(3)\), from Proposition 9.2 we obtain that \(M^{3l}\) is canonically endowed with an almost \(f\)-quaternionic structure. Moreover, if we endow \(M^{3l}\) with its Levi-Civita connection then we obtain an \(f\)-quaternionic manifold. Its twistor space is the hyperquadric \(Q_{l+1}\) of isotropic one-dimensional complex vector subspaces of \(\mathbb{C}^{l+3}\), considered as the complexification of the (real) Euclidean space of dimension \(l + 3\). Further, the CR twistor space \(Z\) of \(M^{3l}\) can be described as the closed submanifold of \(Q_{l+1} \times M^{3l}\) formed of those pairs \((\ell, p)\) such that \(\ell \subseteq p^C\).

Under the orthogonal decomposition \(\mathbb{R}^{l+4} = \mathbb{R} \oplus \mathbb{R}^{l+3}\), we can embed \(M^{3l}\) as a totally geodesic submanifold of the quaternionic manifold \(\tilde{M}^{4l} = \text{Gr}_{4}^{+}(l + 4, \mathbb{R})\) as follows: \(p \mapsto \mathbb{R} \oplus p\), \((p \in M^{3l})\).

Recall (see [13]) that the twistor space of \(\tilde{M}^{4l}\) is the manifold \(\tilde{Z} = \text{Gr}_{2}^{\perp}(l+4, \mathbb{C})\) of isotropic complex vector subspaces of dimension 2 of \(\mathbb{C}^{l+4}\), where the projection \(\tilde{Z} \to \tilde{M}\) is given by \(q \mapsto p\), with \(q\) a self-dual subspace of \(p^C\) (in particular, \(p^C = q \oplus \overline{q}\)).

Consequently, the CR twistor space \(Z\) of \(M^{3l}\) can be embedded in \(\tilde{Z}\) as follows: \((\ell, p) \mapsto q\), where \(q\) is the unique self-dual subspace of \((\mathbb{R} \oplus p)^C\) which intersects \(p^C\) along \(\ell\).

In the particular case \(l = 1\) we obtain the well-known fact (see [3]) that the twistor space of \(S^3\) is \(Q_2\) (\(= \mathbb{C}P^1 \times \mathbb{C}P^1\)). Also, the CR twistor space of \(S^3\) can be identified with the sphere bundle of \(O(1) \oplus O(1)\).

Similarly, the dual of \(M^{3l}\) is, canonically, an \(f\)-quaternionic manifold whose twistor space is an open set of \(Q_{l+1}\).
Let \((M, E, V)\) be an almost \(f\)-quaternionic manifold, with \(\text{rank} V = l\), and let \((P, M, G_{k,l})\) be the corresponding reduction of the frame bundle of \(M\), where \(\text{rank} E = 4k\). From Proposition 3.4 it follows that \(TM = (V \otimes Q) \oplus W\), where \(W\) is the quaternionic vector bundle associated to \(P\) through the canonical morphisms of Lie groups \(G_{k,l} \rightarrow \text{Sp}(1) \cdot \text{GL}(k-1, \mathbb{H})\). Note that, \(W\) is the largest quaternionic vector subbundle of \(E\) contained by \(TM\).

**Theorem 9.7.** Let \((M, E, V)\) be an almost \(f\)-quaternionic manifold and let \(D\) be a compatible torsion free connection, \(\text{rank} E = 4k\), \(\text{rank} V = l\); suppose that \((k, l) \neq (2, 2), (1, 0)\).

Then \((M, E, V, \nabla)\) is \(f\)-quaternionic, where \(\nabla = D^V \oplus D\). Moreover, \(W\) is integrable and geodesic, with respect to \(D\) (equivalently, \(D_X Y\) is a section of \(W\), for any sections \(X\) and \(Y\) of \(W\)).

**Proof.** Let \(\iota: TM \rightarrow E\) be the inclusion and let \(\rho: E \rightarrow TM\) be the projection.

By Proposition 3.4(1), we have \(k \geq l\). It quickly follows that we may apply Theorem 6.8 to obtain that \((M, E, \iota, \nabla)\) is CR quaternionic.

To prove that \((M, E, \rho, \nabla)\) is co-CR quaternionic we apply Theorem C.3 to \(D\). Thus, we obtain that it is sufficient to show that for any \(J \in Z\) and any \(X, Y, Z \in E^J\) we have \(R^D(\rho(X), \rho(Y))(\rho(Z)) \in \rho(E^J)\), where \(E^J\) is the eigenspace of \(J\), with respect to \(-i\), and \(R^D\) is the curvature form of \(D\); equivalently, for any \(J \in Z\) and any \(X, Y, Z \in E^J\) we have \(R^\nabla(\rho(X), \rho(Y))Z \in E^J\), where \(R^\nabla\) is the curvature form of \(\nabla\).

The proof of the fact that \((M, E, V, \nabla)\) is \(f\)-quaternionic follows, similarly to the proof of Theorem 6.8.

The last statement, follows quickly from the fact that \((\nabla_X J)(Y)\) is a section of \(W\), for any section \(J\) of \(Z\) and \(X, Y\) of \(W\). \(\square\)

From the proof of Theorem 9.7 we immediately obtain the following.

**Corollary 9.8.** Let \((M, E, V)\) be an almost \(f\)-quaternionic manifold and let \(D\) be a compatible torsion free connection, \(\text{rank} E \geq 8\).

Then \((M, E, \rho, \nabla)\) is co-CR quaternionic, where \(\rho: E \rightarrow TM\) is the projection and \(\nabla = D^V \oplus D\).

Next, we prove two realizability results for \(f\)-quaternionic manifolds.

**Proposition 9.9.** Let \((M, E, V, \nabla)\) be an \(f\)-quaternionic manifold, \(\text{rank} V = 1\), where \(\nabla = D^V \oplus D\) for some compatible connection \(D\) on \(M\).

Then \((M, E, \iota, \nabla)\) is realizable, where \(\iota: TM \rightarrow E\) is the inclusion.
Proof. By passing to a convex open set of $D$, if necessary, we may suppose that the twistorial structure $(Z, M, \pi, D)$ of the co-CR quaternionic manifold $(M, E, \rho)$ is simple, where $\rho : E \to TM$ is the projection. Thus, by Theorem 8.6, we have that $(Z, M, \pi, D)$ is real analytic.

It follows that $Q^C$ is real analytic which, together with the relation $TM = (V \otimes Q) \oplus W$, quickly gives that the twistorial structure $(Z, M, \pi, C)$ of $(M, E, \iota)$ is real analytic.

By Theorem 7.3 the proof is complete. \hfill $\square$

The next result is an immediate consequence of Theorem 9.7 and Proposition 9.9.

**Corollary 9.10.** Let $(M, E, V)$ be an almost $f$-quaternionic manifold, $\text{rank} V = 1$, $\text{rank} E \geq 8$, and let $\nabla$ be a torsion free connection on $E$ compatible with its linear quaternionic structure.

Then $(M, E, \iota, \nabla)$ is realizable, where $\iota : TM \to E$ is the inclusion.

We end this section with the following result.

**Proposition 9.11.** Let $(M, E, V, \nabla)$ be a real analytic $f$-quaternionic manifold, $\text{rank} V = 1$, where $\nabla = D^V \oplus D$ for some torsion free compatible connection $D$ on $M$. Let $N$ be the heaven space of $(M, E, \iota, \nabla)$, where $\iota : TM \to E$ is the inclusion, and denote by $Z_N$ its twistor space.

Then $Z_N$ is endowed with a nonintegrable holomorphic distribution $\mathcal{H}$ of codimension one, transversal to the twistor lines corresponding to the points of $N \setminus M$.

Proof. By passing to a complexification, we may assume all the objects complex analytic. Furthermore, excepting $Z$, we shall denote by the same symbols the corresponding complexifications. As for $Z$, this will denote the bundle of isotropic directions of $Q$. Then any $p \in Z$ corresponds to a vector subspace $E^p$ of $E$.

Let $\mathcal{F}$ be the distribution on $Z$ such that $\mathcal{F}_p$ is the horizontal lift, with respect to $\nabla$, of $\iota^{-1}(E^p)$, $(p \in Z)$. As $(M, E, V, \nabla)$ is (complex) $f$-quaternionic $\mathcal{F}$ is integrable. Moreover, locally, we may suppose that its leaf space is $Z_N$.

Let $\mathcal{G}$ be the distribution on $Z$ such that, at each $p \in Z$, we have that $\mathcal{G}_p$ is the horizontal lift of $(V_x \otimes p^\perp) \oplus W_x$, where $x = \pi(p)$. Define $\mathcal{K} = \mathcal{G} \oplus \ker d\pi$.

Then arguments as in the proof of Theorem 8.3, based on the complex analytic versions of Cartan’s structural equations and [9, Proposition III.2.3], show that $\mathcal{K}$ is projectable with respect to $\mathcal{F}$. Thus, $\mathcal{K}$ projects to a distribution $\mathcal{H}$ on $Z_N$ of codimension one. Furthermore, by using again [9, Proposition III.2.3], we obtain $\mathcal{H}$ is nonintegrable. \hfill $\square$
10. Quaternionic-Kähler manifolds as heaven spaces

A quaternionic-Kähler manifold is a quaternionic manifold endowed with a (semi-Riemannian) Hermitian metric with nonzero scalar curvature and whose Levi-Civita connection is quaternionic.

Let \((M, E, \iota, \nabla)\) be a CR quaternionic manifold with rank \(E = \dim M + 1\). Let \(W\) be the largest quaternionic vector subbundle of \(E\) contained by \(TM\) and denote by \(I\) the (Frobenius) integrability tensor of \(W\). From the integrability of the almost twistorial structure of \((M, E, \iota, \nabla)\) it follows that, for any \(J \in Z\), the two-form \(I|_{E^J}\) takes values in \(E^J/(E^J \cap W^C)\); as this is one-dimensional the condition \(I|_{E^J}\) nondegenerate has an obvious meaning.

**Definition 10.1.** A CR quaternionic manifold \((M, E, \iota, \nabla)\), with rank \(E = \dim M + 1\), is nondegenerate if \(I|_{E^J}\) is nondegenerate, for any \(J \in Z\).

Let \(M\) be a submanifold of a quaternionic manifold \(N\) and let \(Z\) be the twistor space of \(N\). Denote by \(B\) the second fundamental form of \(M\) with respect to some quaternionic connection \(\nabla\) on \(N\); that is, \(B\) is the (symmetric) bilinear form on \(M\), with values in \((TN|_M)/TM\), characterised by \(B(X, Y) = \sigma(\nabla_X Y)\), for any vector fields \(X, Y\) on \(M\), where \(\sigma : TN|_M \to (TN|_M)/TM\) is the projection.

**Definition 10.2.** We say that \(M\) is q-umbilical in \(N\) if for any \(J \in Z|_M\) the second fundamental form of \(M\) vanishes along the eigenvectors of \(J\) which are tangent to \(M\).

From [7, Propositions 1.8(ii) and 2.8] it quickly follows that the notion of q-umbilical submanifold, of a quaternionic manifold, does not depend of the quaternionic connection used to define the second fundamental form.

Note that, if \(\dim N = 4\) then we retrieve the usual notion of umbilical submanifold. Also, if a quaternionic manifold is endowed with a Hermitian metric then any umbilical submanifold of it is q-umbilical.

The notion of q-umbilical submanifold of a quaternionic manifold can be easily extended to CR quaternionic manifolds. Indeed, just define the second fundamental form \(B\) of \((M, E, \iota, \nabla)\) by \(B(X, Y) = \frac{1}{2} \sigma(\nabla_X Y + \nabla_Y X)\), for any vector fields \(X\) and \(Y\) on \(M\), where \(\sigma : E \to E/TM\) is the projection.

The next proposition shows that for q-umbilical CR quaternionic manifolds \((M, E, \iota, \nabla)\), with rank \(E = \dim M + 1\) and \(\nabla\) torsion free, the nondegeneracy can be expressed solely in terms of the second fundamental form. We omit the proof.
Proposition 10.3. Let \((M, E, \iota, \nabla)\) be a q-umbilical CR quaternionic manifold, with rank \(E = \dim M + 1\) and \(\nabla\) torsion free; denote by \(B\) the second fundamental form of \((M, E, \iota, \nabla)\).

The following assertions are equivalent:
(i) \((M, E, \iota, \nabla)\) is nondegenerate.
(ii) \(B|_W\) is nondegenerate.

Next, we prove the main result of this section.

Theorem 10.4. Let \(N\) be the heaven space of a real analytic CR quaternionic manifold \((M, E, \iota, \nabla)\), with rank \(E = \dim M + 1\).

If \(M\) is q-umbilical in \(N\) then the twistor space \(Z_N\) of \(N\) is endowed with a nonintegrable holomorphic distribution \(\mathcal{H}\) of codimension one, transversal to the twistor lines corresponding to the points of \(N \setminus M\). Furthermore, the following assertions are equivalent:
(i) \(\mathcal{H}\) is a holomorphic contact structure on \(Z_N\).
(ii) \((M, E, \iota, \nabla)\) is nondegenerate.

Proof. We may assume \(\nabla\) torsion free. Also, as rank \(E = \dim M + 1\), there exists a subbundle \(V\) of \(E\) such that \((M, E, V)\) is an almost \(f\)-quaternionic manifold.

Then the proof of Proposition 9.11 works in this more general case as well, up to the projectability of \(K\) with respect to \(F\). Here, by also involving [14, Proposition 2.6(b)], we obtain that \(\mathcal{H}\) is projectable with respect to \(F\) if and only if \(M\) is q-umbilical in \(N\). \(\square\)

The following result is an immediate consequence of [13] and (the proof of) Theorem 10.4.

Corollary 10.5. The following assertions are equivalent, for a real analytic hypersurface \(M\) embedded in a quaternionic manifold \(N\):
(i) \(M\) is nondegenerate and q-umbilical.
(ii) By passing, if necessary, to an open neighbourhood of \(M\), there exists a metric \(g\) on \(N \setminus M\) such that \((N \setminus M, g)\) is quaternionic-Kähler and the twistor lines determined by the points of \(M\) are tangent to the contact distribution, on the twistor space of \(N\), corresponding to \(g\).

If \(\dim M = 3\) then Corollary 10.5 gives the main result of [11]. Also, the ‘quaternionic contact’ manifolds of [4] (see [5]) are nondegenerate q-umbilical CR quaternionic manifolds.

Similarly to the umbilical submanifolds of conformal manifolds, the q-umbilical submanifolds have nice twistorial properties. However, we are not going to employ these properties, here.
Appendix A. Semidirect products of (co-)CR quaternionic vector spaces

Let \((U, E, \rho), (U', E', \rho')\) and \((U'', E'', \rho'')\) be co-CR quaternionic vector spaces. We say that \((U, E, \rho)\) is a semidirect product of \((U', E', \rho')\) and \((U'', E'', \rho'')\) if there exists a commutative diagram of vector spaces and linear maps, as follows,

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\ker \rho'' & \rightarrow & \ker \rho \\
\downarrow & & \downarrow \\
E'' & \rightarrow & E \\
\rho'' & \downarrow \pi' & \rho' \\
\downarrow & & \downarrow \\
U'' & \rightarrow & U \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

where \(\tilde{\iota}\) and \(\tilde{\pi}\) are quaternionic linear, with respect to some maps \(T'': Z'' \rightarrow Z\) and \(T': Z \rightarrow Z', \) respectively. Note that, this holds if and only if \((U'', E'', \rho'')\) is a co-CR quaternionic vector subspace of \((U, E, \rho)\) and \((U', E', \rho')\) is the corresponding co-CR quaternionic quotient.

Suppose that \((U, E, \rho)\) is a semidirect product of \((U', E', \rho')\) and \((U'', E'', \rho'')\), \(U' \neq 0 \neq U''\). Let \(\sigma: E' \rightarrow E\) be a quaternionic linear map which is a section of \(\tilde{\pi}\) and let \(\tau: U \rightarrow U''\) be a linear map which is a retract of \(\iota\). Thus, we may identify \(E = E' \times E''\), as quaternionic vector spaces (where we have used the isometries \(Z'' = Z = Z'\) corresponding to \(\tilde{\iota}\) and \(\tilde{\pi}\), respectively), and \(U = U' \times U''\), as vector spaces.

Define the linear map \(\alpha = \tau \circ \rho \circ \sigma: E' \rightarrow U''\). Then it follows that \(\rho : E' \times E'' \rightarrow U' \times U''\) is given by

\[
(A.1) \quad \rho(e', e'') = \left(\rho'(e'), \alpha(e') + \rho''(e'')\right),
\]

for any \(e' \in E'\) and \(e'' \in E''\).

Conversely, if \((U', E', \rho')\) and \((U'', E'', \rho'')\) are co-CR quaternionic vector spaces and \(\alpha: E' \rightarrow U''\) is a linear map then \((U, E, \rho)\) is a semidirect product of \((U', E', \rho')\) and \((U'', E'', \rho'')\), where \(E = E' \times E''\) (with respect to some fixed orientation preserving isometry \(Z' = Z''\)), \(U = U' \times U''\) and \(\rho\) is given by \((A.1)\).

Furthermore, the semidirect product induced by some other linear map \(\beta:\)
$E' \to U''$ is isomorphic, as a co-CR quaternionic vector space, to $(U, E, \rho)$ if and only if

$$\beta = \alpha + \varphi \circ \rho' + \rho'' \circ \psi,$$

where $\varphi : U' \to U''$ is a linear map and $\psi : E' \to E$ is quaternionic linear (with respect to the induced orientation preserving isometry $Z' = Z$).

In particular, $U = U' \times U''$, as co-CR quaternionic vector spaces, if and only if there exists a linear map $\varphi : U' \to U''$ such that $\alpha + \varphi \circ \rho'$ is co-CR quaternionic linear (with respect to the isometry $Z' = Z''$).

Next, we characterise the holomorphic vector bundle of a semidirect product of co-CR quaternionic vector spaces.

**Proposition A.1.** Let $U'$ and $U''$ be the holomorphic vector bundles of the co-CR quaternionic vector spaces $(U', E', \rho')$ and $(U'', E'', \rho'')$, respectively.

Let $U$ be holomorphic vector bundle over $Z (= \mathbb{C}P^1)$ endowed with a (real) vector bundle automorphism which is antiholomorphic and descends to the antipodal map.

Suppose that $U$ is an extension of $U'$ by $U''$ such that $U'' \to U$ and $U \to U'$ intertwine the conjugations and let $c$ be a 1-cocycle with coefficients in $\text{Hom}(U', U'')$, invariant under the conjugations, determined by $U$.

Then $U$ is the holomorphic vector bundle of a semidirect product of $(U', E', \rho')$ and $(U'', E'', \rho'')$ if and only if, up to a refinement preserved by the antipodal map, $c \circ R = \delta b$, for some 0-cocycle $b$ with coefficients in $\text{Hom}(E', U'')$, invariant under the conjugations, where $R$ is the twistorial representation of $\rho'$.

**Proof.** This is straightforward. \hfill \Box

The following result gives a necessary and sufficient condition for a semidirect product of co-CR quaternionic vector spaces to be trivial (that is, a direct product).

**Corollary A.2.** Let $(U, E, \rho)$ be a co-CR quaternionic vector space which is a semidirect product of the co-CR quaternionic vector spaces $(U', E', \rho')$ and $(U'', E'', \rho'')$ (with respect to some orientation preserving isometries $Z' = Z = Z''$). Let $U, U'$ and $U''$ be the holomorphic vector bundles of $(U, E, \rho), (U', E', \rho')$ and $(U'', E'', \rho'')$, respectively.

Then the following assertions are equivalent:

(i) $(U, E, \rho)$ is the direct product of $(U', E', \rho')$ and $(U'', E'', \rho'')$ (with respect to the given orientation preserving isometry $Z' = Z''$).

(ii) The extension $0 \to U' \to U \to U'' \to 0$ is trivial.
Proof. Assertion (ii) holds if and only if there exists a section \( \sigma : U'' \to U \) of the projection \( U \to U'' \). By replacing, if necessary, \( \sigma \) with \( \frac{1}{2}(\sigma + \tau \circ \sigma \circ \tau'') \), where \( \tau \) and \( \tau'' \) are the conjugations of \( U \) and \( U'' \), respectively, we may suppose that \( \sigma \) is real. The proof follows from Theorem 4.8.

In the following result the \( f \)-quaternionic vector spaces are considered to be endowed with their underlying co-CR quaternionic linear structures.

**Corollary A.3.** Any semidirect product of an \( f \)-quaternionic vector space with a co-CR quaternionic vector space is trivial.

Proof. Let \( U_0 \) and \( U \) be the holomorphic vector bundles of an \( f \)-quaternionic vector space and a co-CR quaternionic vector space, respectively. By Theorem 4.8 and Corollary A.2, it is sufficient to prove that any extension of \( U_0 \) by \( U \) is trivial.

From Proposition 4.9 and Example 4.11(1) it follows that there exists an integer \( k \geq -1 \) such that \( \text{Hom}(U_0, U) = \bigoplus_{j=-1}^{k} a_j O(j) \), where \( a_{-1}, \ldots, a_k \) are nonnegative integers. Hence, \( H^1(Z, \text{Hom}(U_0, U)) = 0 \) and the proof is complete.

Dually, we define the notion of semidirect product of CR quaternionic vector spaces. Note that, if \( (U, E, \iota) \) is a semidirect product of the CR quaternionic vector spaces \( (U', E', \iota') \) and \( (U'', E'', \iota'') \) then the dual of \( (U, E, \iota) \) is a semidirect product of the duals of \( (U'', E'', \iota'') \) and \( (U', E', \iota') \), respectively.

See Example B.7, below, for a nontrivial example of a semidirect product of (co-)CR quaternionic vector spaces.

**APPENDIX B. A CHARACTERISATION OF \( f \)-QUATERNIONIC VECTOR SPACES**

Let \( E \) be a vector space endowed with a linear quaternionic structure determined by the morphism of associative algebras \( \sigma : \mathbb{H} \to \text{End}(E) \); denote by \( Q = \sigma(\text{Im}\mathbb{H}) \) and \( \tilde{Q} = \sigma(\mathbb{H}) \).

If \( U \) is a (real) vector subspace of \( E \) then \( \bigcap_{J \in \tilde{Q}} J(U) \) is the largest quaternionic vector subspace of \( E \) contained by \( U \). Dually, \( \sum_{J \in \tilde{Q}} J(U) \) is the smallest quaternionic vector subspace of \( E \) which contains \( U \).

We make the following notations

\[
E_U = \sum_{I \in \tilde{Q}} I \left( \bigcap_{J \in Q} J(U) \right),
\]

\[
E^U = \bigcap_{I \in \tilde{Q}} I \left( \sum_{J \in Q} J(U) \right).
\]

Obviously, \( E_U \) and \( E^U \) are quaternionic vector subspaces of \( E \) and if we denote by \( U^0 \subseteq E^* \) the annihilator of \( U \) then \( E_{U^0} = (E^U)^0 \).
One of the aims of this appendix is to prove the following two results (dual to each other).

**Theorem B.1.** Let $E$ be a quaternionic vector space and let $U \subseteq E$ be a vector subspace. Then the following assertions are equivalent:

(i) $E_U = E$.

(ii) There exists $V \subseteq E$ such that $(U, E, V)$ is an $f$-quaternionic vector space.

**Theorem B.2.** Let $E$ be a quaternionic vector space and let $V \subseteq E$ be a vector subspace. Then the following assertions are equivalent:

(i) $E^V = \{0\}$.

(ii) There exists $U \subseteq E$ such that $(U, E, V)$ is an $f$-quaternionic vector space.

Theorems B.1 and B.2 follow from the following result.

**Proposition B.3.** Let $E$ be a quaternionic vector space and let $U \subseteq E$ be a vector subspace.

(i) $E_U$ is the largest quaternionic vector subspace of $E$ with the property that there exists $V \subseteq E_U$ such that $(U \cap E_U, E_U, V)$ is an $f$-quaternionic vector space.

(ii) $E^U$ is the smallest quaternionic vector subspace of $E$ with the property that there exists $V \supseteq E^U$ such that $(V/E^U, E/E^U, (U+E^U)/E^U)$ is an $f$-quaternionic vector space.

**Proof.** As assertions (i) and (ii) are dual to each other, it is sufficient to prove assertion (i).

Let $V$ be a complement of $\bigcap_{J \in Q} J(U)$ in $\bigcap_{J \in Q} J(U)$. Then

\[ V \cap U = V \cap U \cap \bigcap_{J \in Q} J(U) = V \cap \bigcap_{J \in \tilde{Q}} J(U) = \{0\}. \]

Also, if $J \in Q$ then

\[ J(V) \subseteq J\left( \bigcap_{I \in Q} I(U) \right) \subseteq U, \]

and, as $V \subseteq E_U$, we also have $J(V) \subseteq E_U$. Hence, $J(V) \subseteq U \cap E_U$, for any $J \in Q$.

By definition, $E_U$ is generated, as a quaternionic vector space, by $V$ and $\bigcap_{J \in \tilde{Q}} J(U) \subseteq U \cap E_U$. Hence, $E_U$ is generated, as a (real) vector space, by $U \cap E_U$ and the quaternionic vector space generated by $V$. It follows that $E_U = V \oplus (U \cap E_U)$.

We have, thus, proved that $(U \cap E_U, E_U, V)$ is an $f$-quaternionic vector space.

Let $E' \subseteq E$ be a quaternionic vector subspace such that $(U \cap E', E', V')$ is an $f$-quaternionic vector space, for some complement $V'$ of $U \cap E'$ in $E'$; denote
Let \( U \) be a vector subspace of the quaternionic vector space \( E \) and let \( U' \) be a complement of \( U \cap E_U \) in \( U \).

Then \( E_{U'} = \{0\} \); equivalently, \( \bigcap_{J \in Q} J(U') = \{0\} \).

Indeed, as \( U' \subseteq U \), we have \( E_{U'} \subseteq E_U \). Hence,

\[ U' \cap E_{U'} \subseteq U' \cap E_U = U' \cap U \cap E_U = \{0\} . \]

As \((U' \cap E_{U'}, E_{U'}, V')\) is an \( f \)-quaternionic vector space, for some complement \( V' \) of \( U' \cap E_{U'} \) in \( E_{U'} \), this implies \( E_{U'} = \{0\} \).

Similarly, if \( U'' \supseteq U \) is such that \( U'' + E_U = E \) then \( E_{U''} = E \); equivalently, \( \sum_{J \in Q} J(U'') = E \).

**Remark B.4.** Let \( \{U_a\}_a \) be a family of vector subspaces of \( E \) such that the sum \( \sum_a E_a \) is direct, where \( E_a \) is the quaternionic vector space generated by \( U_a \). Then \( E_{\sum_a U_a} = \sum_a E_{U_a} \) and \( E_{\sum_a U_a} = \sum_a E_{U_a} \).

Next, we prove the following result in which the \( f \)-quaternionic vector spaces are considered to be endowed with their underlying CR quaternionic linear structures.

**Theorem B.5.** Any CR quaternionic vector space is the direct product of an \( f \)-quaternionic vector space and a CR quaternionic vector space \((U,E,\iota)\) such that \( E_U = \{0\} \).

**Proof.** From Proposition [B.3(i)] it follows that any CR quaternionic vector space is a semidirect product of a CR quaternionic vector space and an \( f \)-quaternionic vector space. By the dual of Proposition [A.3] this semidirect product must be direct. The proof now follows from Proposition [B.3(i)] and Remark [B.4].

Dually, we have the following result in which the \( f \)-quaternionic vector spaces are considered to be endowed with their underlying co-CR quaternionic linear structures.

**Theorem B.6.** Any co-CR quaternionic vector space is the direct product of an \( f \)-quaternionic vector space and a co-CR quaternionic vector space \((U,E,\rho)\) such that \( E_{\ker \rho} = E \).

We can now give a nontrivial example of a semidirect product of (co-)CR quaternionic vector spaces.
Example B.7. Recall (Example 3.6(2)) that if $U$ is the orthogonal complement of $\mathbb{R}^3 + \mathbb{R}(i,j,k)$ in $\mathbb{H}^3$ then $(U, \mathbb{H}^3)$ is a CR quaternionic vector space such that $iU \cap jU \cap kU = \{0\}$; equivalently, $E_U = \{0\}$.

Also, if $U'$ is the orthogonal complement of $\mathbb{R}^2 + \mathbb{R}(i,j)$ in $\mathbb{H}^2$ then $(U', \mathbb{H}^2)$ is a CR quaternionic vector space such that $U' = U \cap \mathbb{H}^2$, where we have identified $\mathbb{H}^2 = \mathbb{H}^2 \times \{0\} \subseteq \mathbb{H}^3$.

It follows that $(U, \mathbb{H}^3)$ is a semidirect product of the CR quaternionic vector space associated to $\text{Im}\mathbb{H}$ and $(U', \mathbb{H}^2)$. Moreover, as $E_U = \{0\}$, Theorem B.5 implies that this semidirect product is not trivial.

Together with Proposition 4.9 this implies that the holomorphic vector bundle $\mathcal{U}$ of $(U, \mathbb{H}^3)$ is either $2\mathcal{O}(-3)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-5)$.

On the other hand, from Proposition A.1, and Examples 4.11(1) and 4.11(2) it follows that $U$ is an extension of $\mathcal{O}(-2)$ by $\mathcal{O}(-4)$.

From the fact that the space of holomorphic sections of $\text{Hom}(\mathcal{O}(-1), \mathcal{O}(-2))$ is $\{0\}$ it follows quickly that $\mathcal{U} = 2\mathcal{O}(-3)$.

Appendix C. An integrability result

Let $E$ be a vector bundle on a manifold $M$ and let $\alpha : TM \to E$ be a morphism of vector bundles. Let $(P, M, \text{GL}(n, \mathbb{R}))$ be the frame bundle of $E$, where $n = \text{rank} E$. Define an $\mathbb{R}^n$-valued one-form $\theta$ on $P$ by $\theta(X) = u^{-1}(\alpha(d\pi(X)))$, for any $u \in P$ and $X \in T_uP$, where $\pi : P \to M$ is the projection. Then $\theta$ is the tensorial form which corresponds to the $E$-valued one-form $\alpha$.

Let $\nabla$ be a connection on $E$.

Definition C.1. The torsion (with respect to $\alpha$) of $\nabla$ is the $E$-valued two-form $T$ on $M$ defined by $T = d\nabla \theta$; if $T = 0$ then $\nabla$ is called torsion-free.

Note that, the tensorial two-form which corresponds to the torsion of $\nabla$ is $d\theta|_\mathscr{H}$, where $\mathscr{H} \subseteq TP$ is the principal connection corresponding to $\nabla$ (cf. [9]).

The next result will be useful later on.

Lemma C.2. Let $u_0 \in P$, $X \in \mathscr{H}_{u_0}$ and let $u$ be any (local) section of $P$ tangent to $X$ (in particular, $u_{\pi(u_0)} = u_0$). Then for any vector field $Y$ on $P$, we have

$$ u_0(X(\theta(Y))) = \nabla_{d\pi(X)}(\alpha(d\pi(Y|_{u(M)}))). $$

Proof. Let $\tilde{Y} = d\pi(Y|_{u(M)})$. Then $\alpha(\tilde{Y})$ is a section of $E$ and $\theta(Y|_{u(M)})$ gives the components of $\alpha(\tilde{Y})$ with respect to $u$. The proof quickly follows (cf. the relation of the Lemma from [9], vol 1, page 115)].
Let $F$ be a complex submanifold of the Grassmannian $\text{Gr}_q(\mathbb{C}^n)$ on which the complexification $G$ of the structural group of $E$ and $\nabla$ acts transitively ($q \leq n$).

From now on, in this section, we shall denote by $(P, M, G)$ the bundle of complex $G$-frames on $E$. Let $Z = P \times_G F$ and suppose that the dimension of $\alpha^{-1}(p)$ does not depend of $p \in Z$.

Let $\mathcal{C}_0 \subseteq T^C Z$ be horizontal (with respect to $\nabla$) and such that $d\pi_p(\mathcal{C}_0) = \alpha^{-1}(p)$, for any $p \in Z$. Define $\mathcal{C} = \mathcal{C}_0 \oplus (\ker d\pi)^0$, where $\pi : Z \to M$ is the projection.

Note that, if the dimension of $\alpha^{-1}(p) \cap \overline{\alpha^{-1}(p)}$ does not depend of $p \in Z$ then $(Z, M, \pi, \mathcal{C})$ is an almost twistorial structure, in the sense of [14].

**Theorem C.3.** The following assertions are equivalent:

(i) $\mathcal{C}$ is integrable.

(ii) $T(X, Y) \in p$ and $R(X, Y)(p) \subseteq p$, for any $p \in Z$ and $X, Y \in \alpha^{-1}(p)$, where $T$ and $R$ are the torsion and curvature forms, respectively, of $\nabla$.

**Proof.** Fix $V \in F$ and let $H$ be the closed subgroup of $G$ which preserve $V$. Then $Z = P/H$ and let $\psi : P \to Z$ be the projection. If we denote $\mathcal{D} = (d\psi)^{-1}(\mathcal{C})$ then, as $\psi$ is a surjective submersion, $\mathcal{C}$ is integrable if and only if $\mathcal{D}$ is integrable.

Let $\mathcal{H} \subseteq TP$ be the principal connection corresponding to $\nabla^C$ and let $\mathcal{B}(V) \subseteq \mathcal{H}^C$ be such that $\mathcal{B}(V)_{u}$ is the horizontal lift of $\alpha^{-1}(u(V))$, for any $u \in P$.

Then, we have $\mathcal{D} = \mathcal{B}(V) \oplus (P \times \mathfrak{h}) \oplus (P \times \overline{\mathfrak{g}})$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$, respectively, and we have used that $\ker d\pi = P \times \mathfrak{g}$, with $\pi : P \to M$ the projection, and $\mathfrak{g}^C = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ (as complex Lie algebras).

Note that, $R_a(\mathcal{B}(V)) = \mathcal{B}(a^{-1}(V))$ for any $a \in G$, where $R$ denotes the action of $G$ on $P$. Also, $A \in \mathfrak{h}$ if and only if $A(V) \subseteq V$ whilst if $A \in \overline{\mathfrak{g}}$ then $A(V) = \{0\}$. Thus, by using the same notation for elements of $\mathfrak{g}$ and the corresponding fundamental vector fields, we obtain the following relation (cf. [9 Proposition III.2.3]):

$$\left[A, \Gamma(\mathcal{B}(V))\right] \subseteq \Gamma(\mathcal{B}(A(V) + V)) = \Gamma(\mathcal{B}(V)),$$

for any $A \in \mathfrak{h} \oplus \overline{\mathfrak{g}}$.

We have thus shown that assertion (i) holds if and only if, for any sections $X$ and $Y$ of $\mathcal{B}(V)$, we have that $[X, Y]$ is a section of $\mathcal{D}$.

Let $\Theta$ and $\Omega$ be the tensorial forms on $P$ corresponding to $T$ and $R$, respectively. By applying [9 Corollary II.5.3], Lemma C.2 and [14 Proposition 2.6(b)], we obtain that assertion (i) is equivalent to the condition that for any $X, Y \in \mathcal{B}(V)$ we have $\Omega(X, Y) \in \mathfrak{h}$ and $\Theta(X, Y) \in V$. The proof follows quickly. □
**Remark C.4.** With the same notations as in the proof of Theorem 6.8, the bracket of any section of $P \times \mathfrak{h}$ and the sections of $\mathcal{A}(V)$ which are basic with respect to $\psi$ is zero.

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