CATEGORICAL ASPECTS OF BIVARIANT K-THEORY

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Abstract. This survey article on bivariant Kasparov theory and E-theory is mainly intended for readers with a background in homotopical algebra and category theory. We approach both bivariant K-theories via their universal properties and equip them with extra structure such as a tensor product and a triangulated category structure. We discuss the construction of the Baum–Connes assembly map via localisation of categories and explain how this is related to the purely topological construction by Davis and Lück.

1. Introduction

Non-commutative topology deals with topological properties of $C^\ast$-algebras. Already in the 1970s, the classification of AF-algebras by K-theoretic data [15] and the work of Brown–Douglas–Fillmore on essentially normal operators [6] showed clearly that topology provides useful tools to study $C^\ast$-algebras. A breakthrough was Kasparov’s construction of a bivariant K-theory for separable $C^\ast$-algebras. Besides its applications within $C^\ast$-algebra theory, it also yields results in classical topology that are hard or even impossible to prove without it. A typical example is the Novikov conjecture, which deals with the homotopy invariance of certain invariants of smooth manifolds with a given fundamental group. This conjecture has been verified for many groups using Kasparov theory, starting with [31]. The $C^\ast$-algebraic formulation of the Novikov conjecture is closely related to the Baum–Connes conjecture, which deals with the computation of the K-theory $K_\ast(C^\ast_{red}G)$ of reduced group $C^\ast$-algebras and has been one of the centres of attention in non-commutative topology in recent years.

The Baum–Connes conjecture in its original formulation [4] only deals with a single K-theory group; but a better understanding requires a different point of view. The approach by Davis and Lück in [14] views it as a natural transformation between two homology theories for $G$-CW-complexes. An analogous approach in the $C^\ast$-algebra framework appeared in [38]. These approaches to the Baum–Connes conjecture show the importance of studying not just single $C^\ast$-algebras, but categories of $C^\ast$-algebras and their properties. Older ideas like the universal property of Kasparov theory are of the same nature. Studying categories of objects instead of individual objects is becoming more and more important in algebraic topology and algebraic geometry as well.

Several mathematicians have suggested, therefore, to apply general constructions with categories (with additional structure) like generators, Witt groups, the centre, and support varieties to the $C^\ast$-algebra context. Despite the warning below, this seems a promising project, where little has been done so far. To prepare for this enquiry, we summarise some of the known properties of categories of $C^\ast$-algebras; we cover tensor products, some homotopy theory, universal properties, and triangulated structures. In addition, we examine the Universal Coefficient Theorem and the Baum–Connes assembly map.

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Despite many formal similarities, the homotopy theory of spaces and non-commutative topology have a very different focus.

On the one hand, most of the complexities of the stable homotopy category of spaces vanish for $C^*$-algebras because only very few homology theories for spaces have a non-commutative counterpart: any functor on $C^*$-algebras satisfying some reasonable assumptions must be closely related to K-theory. Thus special features of topological K-theory become more transparent when we work with $C^*$-algebras.

On the other hand, analysis may create new difficulties, which appear to be very hard to study topologically. For instance, there exist $C^*$-algebras with vanishing K-theory which are nevertheless non-trivial in Kasparov theory; this means that the Universal Coefficient Theorem fails for them. I know no non-trivial topological statement about the subcategory of the Kasparov category consisting of $C^*$-algebras with vanishing K-theory; for instance, I know no compact objects.

It may be necessary, therefore, to restrict attention to suitable “bootstrap” categories in order to exclude pathologies that have nothing to do with classical topology. More or less by design, the resulting categories will be localisations of purely topological categories, which we can also construct without mentioning $C^*$-algebras. For instance, we know that the Rosenberg–Schochet bootstrap category is equivalent to a full subcategory of the category of $BU$-module spectra. But we can hope for more interesting categories when we work equivariantly with respect to, say, discrete groups.

2. Additional structure in $C^*$-algebra categories

We assume that the reader is familiar with some basic properties of $C^*$-algebras, including the definition (see for instance [2, 13]). As usual, we allow non-unital $C^*$-algebras. We define some categories of $C^*$-algebras in §2.1 and consider group $C^*$-algebras and crossed products in §2.2. Then we discuss $C^*$-tensor products and mention the notions of nuclearity and exactness in §2.3. The upshot is that $\mathcal{C}^{\text{alg}}$ and $G\cdot\mathcal{C}^{\text{alg}}$ carry two structures of symmetric monoidal category, which coincide for nuclear $C^*$-algebras. We prove in §2.4 that $\mathcal{C}^{\text{alg}}$ and $G\cdot\mathcal{C}^{\text{alg}}$ are bicomplete, that is, all diagrams in them have both a limit and a colimit. We equip morphism spaces between $C^*$-algebras with a canonical base point and topology in §2.5; thus the category of $C^*$-algebras is enriched over the category of pointed topological spaces. In §2.6, we define mapping cones and cylinders in categories of $C^*$-algebras; these rudimentary tools suffice to carry over some basic homotopy theory.

2.1. Categories of $C^*$-algebras.

Definition 1. The category of $C^*$-algebras is the category $\mathcal{C}^{\text{alg}}$ whose objects are the $C^*$-algebras and whose morphisms $A \to B$ are the $\ast$-homomorphisms $A \to B$; we denote this set of morphisms by $\text{Hom}(A, B)$.

A $C^*$-algebra is called separable if it has a countable dense subset. We often restrict attention to the full subcategory $\mathcal{C}^{\text{sep}} \subseteq \mathcal{C}^{\text{alg}}$ of separable $C^*$-algebras.

Examples of $C^*$-algebras are group $C^*$-algebras and $C^*$-crossed products. We briefly recall some relevant properties of these constructions. A more detailed discussion can be found in many textbooks such as [44].

Definition 2. We write $A \in \mathcal{C}$ to denote that $A$ is an object of the category $\mathcal{C}$. The notation $f \in \mathcal{C}$ means that $f$ is a morphism in $\mathcal{C}$; but to avoid confusion we always specify domain and target and write $f \in \mathcal{C}(A, B)$ instead of $f \in \mathcal{C}$.

2.2. Group actions, and crossed products. For any locally compact group $G$, we have a reduced group $C^*$-algebra $C^*_\text{red}(G)$ and a full group $C^*$-algebra $C^*(G)$.
Both are defined as completions of the group Banach algebra $(L^1(G), \ast)$ for suitable $C^*$-norms and are related by a canonical surjective $\ast$-homomorphism $C^*(G) \to C^*_\text{red}(G)$, which is an isomorphism if and only if $G$ is amenable.

The norm on $C^*(G)$ is the maximal $C^*$-norm, so that any strongly continuous unitary representation of $G$ on a Hilbert space induces a $\ast$-representation of $C^*(G)$. The norm on $C^*_\text{red}(G)$ is defined using the regular representation of $G$ on $L^2(G)$; hence a representation of $G$ only induces a $\ast$-representation of $C^*_\text{red}(G)$ if it is weakly contained in the regular representation. For reductive Lie groups and reductive $p$-adic groups, these representations are exactly the tempered representations, which are much easier to classify than all unitary representations.

**Definition 3.** A $G$-$C^*$-algebra is a $C^*$-algebra $A$ with a strongly continuous representation of $G$ by $C^*$-algebra automorphisms. The category of $G$-$C^*$-algebras is the category $G\text{-}\mathcal{C}^*\text{alg}$ whose objects are the $G$-$C^*$-algebras and whose morphisms $A \to B$ are the $G$-equivariant $\ast$-homomorphisms $A \to B$; we denote this morphism set by $\text{Hom}_G(A, B)$.

**Example 4.** If $G = \mathbb{Z}$, then a $G$-$C^*$-algebra is nothing but a pair $(A, \alpha)$ consisting of a $C^*$-algebra $A$ and a $\ast$-automorphism $\alpha: A \to A$: let $\alpha$ be the action of the generator $1 \in \mathbb{Z}$.

Equipping $C^*$-algebras with a trivial action provides a functor
\[(1) \quad \tau: \mathcal{C}^*\text{alg} \to G\text{-}\mathcal{C}^*\text{alg}, \quad A \mapsto A_r.
\]
Since $\mathbb{C}$ has only the identity automorphism, the trivial action is the only way to turn $\mathbb{C}$ into a $G$-$C^*$-algebra.

The full and reduced $C^*$-crossed products are versions of the full and reduced group $C^*$-algebras with coefficients in $G$-$C^*$-algebras (see [44]). They define functors
\[G \ltimes_r G \ltimes_r \mathcal{C}^*\text{alg} \to \mathcal{C}^*\text{alg}, \quad A \mapsto G \ltimes_r A, \quad G \ltimes_r \mathcal{C} = C^*_\text{red}(G).
\]

**Definition 5.** A diagram $I \to E \to Q$ in $\mathcal{C}^*\text{alg}$ is an extension if it is isomorphic to the canonical diagram $I \to A \to A/I$ for some ideal $I$ in a $C^*$-algebra $A$; extensions in $G\text{-}\mathcal{C}^*\text{alg}$ are defined similarly, using $G$-invariant ideals in $G$-$C^*$-algebras. We write $I \Rightarrow E \Rightarrow Q$ to denote extensions.

Although $C^*$-algebra extensions have some things in common with extensions of, say, modules, there are significant differences because $\mathcal{C}^*\text{alg}$ is not Abelian, not even additive.

**Proposition 6.** The full crossed product functor $G \ltimes_r \mathcal{C}^*\text{alg} \to \mathcal{C}^*\text{alg}$ is exact in the sense that it maps extensions in $G\text{-}\mathcal{C}^*\text{alg}$ to extensions in $\mathcal{C}^*\text{alg}$.

**Proof.** This is Lemma 4.10 in [21]. □

**Definition 7.** A locally compact group $G$ is called exact if the reduced crossed product functor $G \ltimes_r \mathcal{C}^*\text{alg} \to \mathcal{C}^*\text{alg}$ is exact.

Although this is not apparent from the above definition, exactness is a geometric property of a group: it is equivalent to Yu’s property (A) or to the existence of an amenable action on a compact space [33].

Most groups you know are exact. The only source of non-exact groups known at the moment are Gromov’s random groups. Although exactness might remind you of the notion of flatness in homological algebra, it has a very different flavour. The difference is that the functor $G \ltimes_r \mathcal{C}$ always preserves injections and surjections. What may go wrong for non-exact groups is exactness in the middle (compare the
discussion before Proposition 18. Hence we cannot study the lack of exactness by derived functors.

Even for non-exact groups, there is a class of extensions for which reduced crossed products are always exact:

**Definition 8.** A section for an extension

\[ I \xrightarrow{i} E \xrightarrow{p} Q \]

in \( G\text{-}\mathcal{C}\text{-}\text{alg} \) is a map (of sets) \( s : Q \to E \) with \( p \circ s = \text{id}_Q \). We call (2) split if there is a section that is a \( G \)-equivariant *-homomorphism. We call (2) \( G \)-equivariantly cp-split if there is a \( G \)-equivariant, completely positive, contractive, linear section.

Sections are also often called lifts, liftings, or splittings.

**Proposition 9.** Both the reduced and the full crossed product functors map split extensions in \( G\text{-}\mathcal{C}\text{-}\text{alg} \) again to split extensions in \( \mathcal{C}\text{-}\text{alg} \) and \( G \)-equivariantly cp-split extensions in \( G\text{-}\mathcal{C}\text{-}\text{alg} \) to cp-split extensions in \( \mathcal{C}\text{-}\text{alg} \).

**Proof.** Let \( K \twoheadrightarrow E \xrightarrow{p} Q \) be an extension in \( G\text{-}\mathcal{C}\text{-}\text{alg} \). Proposition 6 shows that \( G \ltimes K \twoheadrightarrow G \ltimes E \to G \ltimes Q \) is again an extension. Since reduced and full crossed products are functorial for equivariant completely positive contractions, this extension is split or cp-split if the original extension is split or equivariantly cp-split, respectively. This yields the assertions for full crossed products.

Since a *-homomorphism with dense range is automatically surjective, the induced map \( G \ltimes r, p : G \ltimes E \to G \ltimes Q \) is surjective. It is evident from the definition of reduced crossed products that \( G \ltimes r, i \) is injective. What is unclear is whether the range of \( G \ltimes r, i \) and the kernel of \( G \ltimes r, p \) coincide. As for the full crossed product, a \( G \)-equivariant completely positive contractive section \( s : Q \to E \) induces a completely positive contractive section \( G \ltimes r, s \) for \( G \ltimes r, p \). The linear map

\[ \varphi := \text{id}_{G \ltimes r, E} - (G \ltimes r, s) \circ (G \ltimes r, p) : G \ltimes r, E \to G \ltimes r, E \]

is a retraction from \( G \ltimes r, E \) onto the kernel of \( G \ltimes r, p \) by construction. Furthermore, it maps the dense subspace \( L^1(G, E) \) into \( L^1(G, K) \). Hence it maps all of \( G \ltimes r, E \) into \( G \ltimes r, K \). This implies \( G \ltimes r, K = \ker(G \ltimes r, p) \) as desired. \( \square \)

### 2.3. Tensor products and nuclearity

Most results in this section are proved in detail in [72, 69]. Let \( A_1 \) and \( A_2 \) be two \( C^\ast \)-algebras. Their (algebraic) tensor product \( A_1 \otimes \mathbb{C} \) is still a *-algebra. A \( C^\ast \)-tensor product of \( A_1 \) and \( A_2 \) is a \( C^\ast \)-completion of \( A_1 \otimes \mathbb{C} \), that is, a \( C^\ast \)-algebra that contains \( A_1 \otimes \mathbb{C} \) as a dense *-subalgebra. A \( C^\ast \)-tensor product is determined uniquely by the restriction of its norm to \( A_1 \otimes \mathbb{C} \). A norm on \( A_1 \otimes \mathbb{C} \) is allowed if it is a \( C^\ast \)-norm, that is, multiplication and involution have norm 1 and \( \|x^*x\| = \|x\|^2 \) for all \( x \in A_1 \otimes \mathbb{C} \).

There is a maximal \( C^\ast \)-norm on \( A_1 \otimes \mathbb{C} \). The resulting \( C^\ast \)-tensor product is called maximal \( C^\ast \)-tensor product and denoted \( A_1 \otimes_{\text{max}} A_2 \). It is characterised by the following universal property:

**Proposition 10.** There is a natural bijection between non-degenerate *-homomorphisms \( A_1 \otimes_{\text{max}} A_2 \to \mathcal{B}(H) \) and pairs of commuting non-degenerate *-homomorphisms \( A_1 \to \mathcal{B}(H) \) and \( A_2 \to \mathcal{B}(H) \); here we may replace \( \mathcal{B}(H) \) by any multiplier algebra \( \mathcal{M}(D) \) of a \( C^\ast \)-algebra \( D \).

A *-representation \( A \to \mathcal{B}(H) \) is non-degenerate if \( A \cdot \mathcal{H} \) is dense in \( \mathcal{H} \); we need this to get representations of \( A_1 \) and \( A_2 \) out of a representation of \( A_1 \otimes_{\text{max}} A_2 \) because, for non-unital algebras, \( A_1 \otimes_{\text{max}} A_2 \) need not contain copies of \( A_1 \) and \( A_2 \).

The maximal tensor product is natural, that is, it defines a bifunctor

\[ \otimes_{\text{max}} : \mathcal{C}\text{-}\text{alg} \times \mathcal{C}\text{-}\text{alg} \to \mathcal{C}\text{-}\text{alg} . \]
If $A_1$ and $A_2$ are $G$-$C^*$-algebras, then $A_1 \otimes_{\text{max}} A_2$ inherits two group actions of $G$ by naturality; these are again strongly continuous, so that $A_1 \otimes_{\text{max}} A_2$ becomes a $G \times G$-$C^*$-algebra. Restricting the action to the diagonal in $G \times G$, we turn $A_1 \otimes_{\text{max}} A_2$ into a $G$-$C^*$-algebra. Thus we get a bifunctor

$$\otimes_{\text{max}} : G\mathcal{C}^*\text{-alg} \times G\mathcal{C}^*\text{-alg} \to G\mathcal{C}^*\text{-alg}.$$ 

The following lemma asserts, roughly speaking, that this tensor product has the same formal properties as the usual tensor product for vector spaces:

**Lemma 11.** There are canonical isomorphisms

$$\begin{align*}
(A \otimes_{\text{max}} B) \otimes_{\text{max}} C &\cong A \otimes_{\text{max}} (B \otimes_{\text{max}} C), \\
A \otimes_{\text{max}} B &\cong B \otimes_{\text{max}} A, \\
C \otimes_{\text{max}} A &\cong A \otimes_{\text{max}} C
\end{align*}$$

for all objects of $G\mathcal{C}^*\text{-alg}$ (and, in particular, of $\mathcal{C}^*\text{-alg}$). These define a structure of symmetric monoidal category on $G\mathcal{C}^*\text{-alg}$ (see [35, 52]).

A functor between symmetric monoidal categories is called symmetric monoidal if it is compatible with the tensor products in a suitable sense [52]. A trivial example is the functor $\tau : \mathcal{C}^*\text{-alg} \to G\mathcal{C}^*\text{-alg}$ that equips a $C^*$-algebra with the trivial $G$-action.

It follows from the universal property that $\otimes_{\text{max}}$ is compatible with full crossed products: if $A \in G\mathcal{C}^*\text{-alg}$, $B \in \mathcal{C}^*\text{-alg}$, then there is a natural isomorphism

$$G \times (A \otimes_{\text{max}} \tau(B)) \cong (G \times A) \otimes_{\text{max}} B.$$ 

(3)

Like full crossed products, the maximal tensor product may be hard to describe because it involves a maximum of all possible $C^*$-tensor norms. There is another $C^*$-tensor norm that is defined more concretely and that combines well with reduced crossed products.

Recall that any $C^*$-algebra $A$ can be represented faithfully on a Hilbert space. That is, there is an injective $^*$-homomorphism $A \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$; here $\mathbb{B}(\mathcal{H})$ denotes the $C^*$-algebra of bounded operators on $\mathcal{H}$. If $A$ is separable, we can find such a representation on the separable Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$. The tensor product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ carries a canonical inner product and can be completed to a Hilbert space, which we denote by $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ support faithful representations of $C^*$-algebras $A_1$ and $A_2$, then we get an induced $^*$-representation of $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

**Definition 12.** The minimal tensor product $A_1 \otimes_{\text{min}} A_2$ is the completion of $A_1 \otimes A_2$ with respect to the operator norm from $\mathbb{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

It can be checked that this is well-defined, that is, the $C^*$-norm on $A_1 \otimes A_2$ does not depend on the chosen faithful representations of $A_1$ and $A_2$. The same argument also yields the naturality of $A_1 \otimes_{\text{min}} A_2$. Hence we get a bifunctor

$$\otimes_{\text{min}} : G\mathcal{C}^*\text{-alg} \times G\mathcal{C}^*\text{-alg} \to G\mathcal{C}^*\text{-alg};$$

it defines another symmetric monoidal category structure on $G\mathcal{C}^*\text{-alg}$.

We may also call $A_1 \otimes_{\text{min}} A_2$ the spatial tensor product. It is minimal in the sense that it is dominated by any $C^*$-tensor norm on $A_1 \otimes A_2$ that is compatible with the given norms on $A_1$ and $A_2$. In particular, we have a canonical surjective $^*$-homomorphism

$$A_1 \otimes_{\text{max}} A_2 \twoheadrightarrow A_1 \otimes_{\text{min}} A_2.$$ 

(4)

**Definition 13.** A $C^*$-algebra $A_1$ is nuclear if the map in (4) is an isomorphism for all $C^*$-algebras $A_2$. 
The name comes from an analogy between nuclear $C^*$-algebras and nuclear locally convex topological vector spaces (see [20]). But this is merely an analogy: the only $C^*$-algebras that are nuclear as locally convex topological vector spaces are the finite-dimensional ones.

Many important $C^*$-algebras are nuclear. This includes the following examples:

- commutative $C^*$-algebras;
- $C^*$-algebras of type I and, in particular, continuous trace $C^*$-algebras;
- group $C^*$-algebras of amenable groups (or groupoids);
- matrix algebras and algebras of compact operators on Hilbert spaces.

If $A$ is nuclear, then there is only one reasonable $C^*$-algebra completion of $A \otimes B$. Therefore, if we can write down any, it must be equal to both $A \otimes_{\min} B$ and $A \otimes_{\max} B$.

**Example 14.** For a compact space $X$ and a $C^*$-algebra $A$, we let $C(X, A)$ be the $C^*$-algebra of all continuous functions $X \to A$. If $X$ is a pointed compact space, we let $C_0(X, A)$ be the $C^*$-algebra of all continuous functions $X \to A$ that vanish at the base point of $X$; this contains $C(X, A)$ as a special case because $C(X, A) \cong C_0(X_+, A)$, where $X_+ = X \cup \{\ast\}$ with base point $\ast$. We have

$$C_0(X, A) \cong C_0(X) \otimes_{\min} A \cong C_0(X) \otimes_{\max} A.$$  

**Example 15.** There is a unique $C^*$-norm on $M_n \otimes A = M_n(A)$ for all $n \in \mathbb{N}$.

For a Hilbert space $H$, let $K(H)$ be the $C^*$-algebra of compact operators on $H$. Then $K(H) \otimes A$ contains copies of $M_n(A)$, $n \in \mathbb{N}$, for all finite-dimensional subspaces of $H$. These carry a unique $C^*$-norm. The $C^*$-norms on these subspaces are compatible and extend to the unique $C^*$-norm on $K(H) \otimes A$.

The class of nuclear $C^*$-algebras is closed under ideals, quotients (by ideals), extensions, inductive limits, and crossed products by actions of amenable locally compact groups. In particular, this covers crossed products by automorphisms (see Example [1]).

$C^*$-subalgebras of nuclear $C^*$-algebras need not be nuclear any more, but they still enjoy a weaker property called*exactness*:  

**Definition 16.** A $C^*$-algebra $A$ is called exact if the functor $A \otimes_{\min} -$ preserves $C^*$-algebra extensions.

It is known [33,43] that a discrete group is exact (Definition [7]) if and only if its group $C^*$-algebra is exact (Definition [16]), if and only if the group has an amenable action on some compact topological space.

**Example 17.** Let $G$ be the non-Abelian free group on 2 generators. Let $G$ act freely and properly on a tree as usual. Let $X$ be the ends compactification of this tree, equipped with the induced action of $G$. This action is known to be amenable, so that $G$ is an exact group. Since the action is amenable, the crossed product algebras $G \ltimes_c C(X)$ and $G \ltimes C(X)$ coincide and are nuclear. The embedding $\mathbb{C} \to C(X)$ induces an embedding $C^*_{\text{red}}(G) \to G \ltimes C(X)$. But $G$ is not amenable. Hence the $C^*$-algebra $C^*_{\text{red}}(G)$ is exact but not nuclear.

As for crossed products, $\otimes_{\min}$ respects injections and surjections. The issue with exactness in the middle is the following. Elements of $A \otimes_{\min} B$ are limits of tensors of the form $\sum_{i=1}^n a_i \otimes b_i$ with $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$. If an element in $A \otimes_{\min} B$ is annihilated by the map to $(A/I) \otimes_{\min} B$, then we can approximate it by such finite sums for which $\sum_{i=1}^n (a_i \mod I) \otimes b_i$ goes to 0. But this does not suffice to find approximations in $I \otimes B$. Thus the kernel of the projection map $A \otimes_{\min} B \twoheadrightarrow (A/I) \otimes_{\min} B$ may be strictly larger than $I \otimes_{\min} B$.  


Proposition 18. The functor \( \otimes_{\max} \) is exact in each variable, that is, \( \cdots \otimes_{\max} D \) maps extensions in \( G \cdot \mathcal{C}^* \text{alg} \) again to extensions for each \( D \in G \cdot \mathcal{C}^* \text{alg} \).

Both \( \otimes_{\min} \) and \( \otimes_{\max} \) map split extensions to split extensions and (equivariantly) \( cp \)-split extensions again to (equivariantly) \( cp \)-split extensions.

The proof is similar to the proofs of Propositions \( \square \) and \( \square \).

If \( A \) or \( B \) is nuclear, we simply write \( A \otimes B \) for \( A \otimes_{\max} B \equiv A \otimes_{\min} B \).

2.4. Limits and colimits.

Proposition 19. The categories \( \mathcal{C}^* \text{alg} \) and \( G \cdot \mathcal{C}^* \text{alg} \) are bicomplete, that is, any (small) diagram in these categories has both a limit and a colimit.

Proof. To get general limits and colimits, it suffices to construct equalisers and coequalisers for pairs of parallel morphisms \( f_0, f_1 : A \to B \), direct products and coproducts \( A_1 \times A_2 \) and \( A_1 \sqcup A_2 \) for any pair of objects and, more generally, for arbitrary sets of objects.

The equaliser and coequaliser of \( f_0, f_1 : A \to B \) are

\[
\ker(f_0 - f_1) = \{ a \in A \mid f_0(a) = f_1(a) \} \subseteq A
\]

and the quotient of \( A_1 \) by the closed \( * \)-ideal generated by the range of \( f_0 - f_1 \), respectively. Here we use that quotients of \( C^* \)-algebras by closed \( * \)-ideals are again \( C^* \)-algebras. Notice that \( \ker(f_0 - f_1) \) is indeed a \( C^* \)-subalgebra of \( A \).

The direct product \( A_1 \times A_2 \) is the usual direct product, equipped with the canonical \( C^* \)-algebra structure. We can generalise the construction of the direct product to infinite direct products: let \( \prod_{i \in I} A_i \) be the set of all \( \text{norm-bounded} \) sequences \( (a_i)_{i \in I} \) with \( a_i \in A_i \) for all \( i \in I \); this is a \( C^* \)-algebra with respect to the obvious \( * \)-algebra structure and the norm \( \| (a_i) \| := \sup_{i \in I} |a_i| \). It has the right universal property because any \( * \)-homomorphism is norm-contracting. (A similar construction with \( \text{Banach} \) algebras would fail at this point.)

The coproduct \( A_1 \sqcup A_2 \) is also called free product and denoted \( A_1 \ast A_2 \); its construction is more involved. The free \( \mathcal{C} \)-algebra generated by \( A_1 \) and \( A_2 \) carries a canonical involution, so that it makes sense to study \( C^* \)-norms on it. It turns out that there is a maximal such \( C^* \)-norm. The resulting \( C^* \)-completion is the free product \( C^* \)-algebra. In the equivariant case, \( A_1 \sqcup A_2 \) inherits an action of \( G \), which is strongly continuous. The resulting object of \( G \cdot \mathcal{C}^* \text{alg} \) has the correct universal property for a coproduct.

An inductive system of \( C^* \)-algebras \( (A_i, \alpha^i_j)_{i \in I} \) is called reduced if all the maps \( \alpha^i_j : A_i \to A_j \) are injective; then they are automatically isometric embeddings. We may as well assume that these maps are identical inclusions of \( C^* \)-subalgebras. Then we can form a \( * \)-algebra \( \bigcup A_i \), and the given \( C^* \)-norms piece together to a \( C^* \)-norm on \( \bigcup A_i \). The resulting completion is \( \text{lim}(A_i, \alpha^i_j) \). In particular, we can construct an infinite coproduct as the inductive limit of its finite sub-coproducts. Thus we get infinite coproducts.

The category of commutative \( C^* \)-algebras is equivalent to the opposite of the category of pointed compact spaces by the GELFAND–NAIMARK Theorem. It is frequently convenient to replace a pointed compact space \( X \) with base point \( * \) by the locally compact space \( X \setminus \{ * \} \). A continuous map \( X \to Y \) extends to a pointed continuous map \( X_+ \to Y_+ \) if and only if it is proper. But there are more pointed continuous maps \( f : X_+ \to Y_+ \) than proper continuous maps \( X \to Y \) because points in \( X \) may be mapped to the point at infinity \( \infty \in Y_+ \). For instance, the zero homomorphism \( C_0(Y) \to C_0(X) \) corresponds to the constant map \( x \mapsto \infty \).

Example 20. If \( U \subseteq X \) is an open subset of a locally compact space, then \( C_0(U) \) is an ideal in \( C_0(X) \). No map \( X \to U \) corresponds to the embedding \( C_0(U) \to C_0(X) \).
Example 21. Products of commutative $C^*$-algebras are again commutative and correspond by the Gelfand–Naimark Theorem to coproducts in the category of pointed compact spaces. The coproduct of a set of pointed compact spaces is the Stone–Čech compactification of their wedge sum. Thus infinite products in $\mathcal{C}^*\text{alg}$ and $G\text{-}\mathcal{C}^*\text{alg}$ do not behave well for the purposes of homotopy theory.

The coproduct of two non-zero $C^*$-algebras is never commutative and hence has no analogue for (pointed) compact spaces. The smash product for pointed compact spaces corresponds to the tensor product of $C^*$-algebras because

$$C_0(X \wedge Y) \cong C_0(X) \otimes_{\min} C_0(Y).$$

2.5. Enrichment over pointed topological spaces. Let $A$ and $B$ be $C^*$-algebras. It is well-known that a $^*$-homomorphism $f \colon A \to B$ is automatically norm-contracting and induces an isometric embedding $A/\ker f \to B$ with respect to the quotient norm on $A/\ker f$. The reason for this is that the norm for self-adjoint elements in a $C^*$-algebra agrees with the spectral radius and hence is determined by the algebraic structure; by the $C^*$-condition $\|a\|^2 = \|a^*a\|$, this extends to all elements of a $C^*$-algebra.

It follows that $\text{Hom}(A,B)$ is an equicontinuous set of linear maps $A \to B$. We always equip $\text{Hom}(A,B)$ with the topology of pointwise norm-convergence. Its subbasic open subsets are of the form

$$\{f \colon A \to B \mid \|(f - f_0)(a)\| < 1 \forall a \in S\}$$

for $f_0 \in \text{Hom}(A,B)$ and a finite subset $S \subseteq A$. Since $\text{Hom}(A,B)$ is equicontinuous, this topology agrees with the topology of uniform convergence on compact subsets, which is generated by the corresponding subsets for compact $S$. This is nothing but the compact-open topology on mapping spaces. But it differs from the topology defined by the operator norm. We shall never use the latter.

Lemma 22. If $A$ is separable, then $\text{Hom}(A,B)$ is metrisable for any $B$.

Proof. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $A$ with $\lim a_n = 0$ whose closed linear span is all of $A$. The metric

$$d(f_1, f_2) = \sup\{\|f_1(a_n) - f_2(a_n)\| \mid n \in \mathbb{N}\}$$

defines the topology of uniform convergence on compact subsets on $\text{Hom}(A,B)$ because the latter is equicontinuous. \qed

There is a distinguished element in $\text{Hom}(A,B)$ as well, namely, the zero homomorphism $A \to 0 \to B$. Thus $\text{Hom}(A,B)$ becomes a pointed topological space.

Proposition 23. The above construction provides an enrichment of $\mathcal{C}^*\text{alg}$ over the category of pointed Hausdorff topological spaces.

Proof. It is clear that $0 \circ f = 0$ and $f \circ 0 = 0$ for all morphisms $f$. Furthermore, we must check that composition of morphisms is jointly continuous. This follows from the equicontinuity of $\text{Hom}(A,B)$. \qed

This enrichment allows us to carry over some important definitions from categories of spaces to $\mathcal{C}^*\text{alg}$. For instance, a homotopy between two $^*$-homomorphisms $f_0, f_1 \colon A \to B$ is a continuous path between $f_0$ and $f_1$ in the topological space $\text{Hom}(A,B)$. In the following proposition, $\text{Map}_+(X,Y)$ denotes the space of morphisms in the category of pointed topological spaces, equipped with the compact-open topology.

Proposition 24 (compare Proposition 3.4 in [28]). Let $A$ and $B$ be $C^*$-algebras and let $X$ be a pointed compact space. Then

$$\text{Map}_+(X, \text{Hom}(A,B)) \cong \text{Hom}(A, C_0(X,B))$$
as pointed topological spaces.

Proof. If we view \( A \) and \( B \) as pointed topological spaces with the norm topology and base point 0, then \( \text{Hom}(A, B) \subseteq \text{Map}_+ (A, B) \) is a topological subspace with the same base point because \( \text{Hom}(A, B) \) also carries the compact-open topology. Since \( X \) is compact, standard point set topology yields homeomorphisms

\[
\text{Map}_+ (X, \text{Map}_+ (A, B)) \cong \text{Map}_+ (X \land A, B)
\cong \text{Map}_+ (A, \text{Map}_+ (X, B)) = \text{Map}_+ (A, \text{C}_0 (X, B)).
\]

These restrict to the desired homeomorphism. \( \square \)

In particular, a homotopy between two \(^*\)-homomorphisms \( f_0, f_1 : A \Rightarrow B \) is equivalent to a \(^*\)-homomorphism \( f : A \Rightarrow \mathcal{C}([0, 1], B) \) with \( \text{ev}_t \circ f = f_t \) for \( t = 0, 1 \), where \( \text{ev}_t \) denotes the \(^*\)-homomorphism

\[
\text{ev}_t : \mathcal{C}([0, 1], B) \to B, \quad f \mapsto f(t).
\]

We also have

\[
\text{C}_0 (X, \text{C}_0 (Y, A)) \cong \text{C}_0 (X \land Y, A)
\]

for all pointed compact spaces \( X, Y \) and all \( G\text{-C}^*\)-algebras \( A \). Thus a homotopy between two homotopies can be encoded by a \(^*\)-homomorphism

\[
A \Rightarrow \text{C}([0, 1], \text{C}([0, 1], B)) \cong \text{C}([0, 1]^2, B).
\]

These constructions work only for pointed \textit{compact} spaces. If we enlarge the category of \( C^*\)-algebras to a suitable category of projective limits of \( C^*\)-algebras as in \([28]\), then we can define \( \text{C}_0 (X, A) \) for any pointed \textit{compactly generated} space \( X \). But we lose some of the nice analytic properties of \( C^*\)-algebras. Therefore, I prefer to stick to the category of \( C^*\)-algebras itself.

2.6. Cylinders, cones, and suspensions. The following definitions go back to \([21]\), where some more results can be found. The description of homotopies above leads us to define the \textit{cylinder} over a \( C^*\)-algebra \( A \) by

\[
\text{Cyl} (A) := \mathcal{C}([0, 1], A).
\]

This is compatible with the cylinder construction for spaces because

\[
\text{Cyl} (\text{C}_0 (X)) \cong \mathcal{C} ([0, 1], \text{C}_0 (X)) \cong \text{C}_0 ([0, 1]_\land X)
\]

for any pointed compact space \( X \); if we use locally compact spaces, we get \([0, 1] \times X\) instead of \([0, 1]_\land X\).

The universal property of \( \text{Cyl} (A) \) is dual to the usual one for spaces because the identification between pointed compact spaces and commutative \( C^*\)-algebras is contravariant.

Similarly, we may define the \textit{cone} \( \text{Cone} (A) \) and the \textit{suspension} \( \text{Sus} (A) \) by

\[
\text{Cone} (A) := \text{C}_0 ([0, 1] \setminus \{0\}, A), \quad \text{Sus} (A) := \text{C}_0 ([0, 1] \setminus \{0, 1\}, A) \cong \text{C}_0 (S^1, A),
\]

where \( S^1 \) denotes the pointed 1-sphere, that is, circle. These constructions are compatible with the corresponding ones for spaces as well, that is,

\[
\text{Cone} (\text{C}_0 (X)) \cong \text{C}_0 ([0, 1] \land X), \quad \text{Sus} (\text{C}_0 (X)) \cong \text{C}_0 (S^1 \land X).
\]

Here \([0, 1] \) has the base point 0.

Definition 25. Let \( f : A \to B \) be a morphism in \( \mathfrak{C}^*\text{alg} \) or \( G\mathfrak{C}^*\text{alg} \). The \textit{mapping cylinder} \( \text{Cyl} (f) \) and the \textit{mapping cone} \( \text{Cone} (f) \) of \( f \) are the limits of the diagrams

\[
A \xrightarrow{f} B \xleftarrow{\text{ev}_1} \text{Cyl} (B), \quad A \xrightarrow{f} B \xleftarrow{\text{ev}_1} \text{Cone} (B).
\]
More concretely,
\[
\text{Cone}(f) = \{(a, b) \in A \times C_0((0, 1], B) \mid f(a) = b(1)\},
\]
\[
\text{Cyl}(f) = \{(a, b) \in A \times C_0([0, 1], B) \mid f(a) = b(1)\}.
\]

If \( f : X \to Y \) is a morphism of pointed compact spaces, then the mapping cone and mapping cylinder of the induced \( \ast \)-homomorphism \( C_0(f) : C_0(Y) \to C_0(X) \) agree with \( C_0(\text{Cyl}(f)) \) and \( C_0(\text{Cone}(f)) \), respectively.

The cylinder, cone, and suspension functors are exact for various kinds of extensions: they map extensions, split extensions, and cp-split extensions again to extensions, split extensions, and cp-split extensions, respectively. Similar remarks apply to mapping cylinders and mapping cones: for any morphism of extensions
\[
I \quad \xrightarrow{\alpha} \quad E \quad \xrightarrow{\beta} \quad Q
\]
we get extensions
\[
I' \quad \xrightarrow{\alpha'} \quad E' \quad \xrightarrow{\beta'} \quad Q',
\]
if the extensions in (5) are split or cp-split, so are the resulting extensions in (6).

The familiar maps relating mapping cones and cylinders to cones and suspensions continue to exist in our case. For any morphism \( f : A \to B \) in \( G\text{-}C^\ast\text{-alg} \), we get a morphism of extensions
\[
\text{Sus}(B) \quad \xrightarrow{\text{Cone}(f)} \quad A
\]
\[
\text{Cone}(B) \quad \xrightarrow{\text{Cyl}(f)} \quad A
\]
The bottom extension splits and the maps \( A \leftrightarrow \text{Cyl}(f) \) are inverse to each other up to homotopy. By naturality, the composite map \( \text{Cone}(f) \to A \to B \) factors through \( \text{Cone}(\text{id}_B) \cong \text{Cone}(B) \) and hence is homotopic to the zero map.

3. Universal functors with certain properties

When we study topological invariants for \( C^\ast \)-algebras, we usually require homotopy invariance and some exactness and stability conditions. Here we investigate these conditions and their interplay and describe some universal functors.

We discuss homotopy invariant functors on \( G\text{-}\mathcal{C}^\ast\text{-alg} \) and the homotopy category \( \text{Ho}(G\text{-}\mathcal{C}^\ast\text{-alg}) \) in \( \S 3.1 \) This is parallel to classical topology. We turn to Morita–Rieffel invariance and \( C^\ast \)-stability in \( \S 3.2, 3.3 \). We describe the resulting localisation using correspondences. By the way, in a \( C^\ast \)-algebra context, \emph{stability} usually refers to algebras of compact operators instead of suspensions. \( \S 3.4 \) deals with various exactness conditions: split-exactness, half-exactness, and additivity.

Whereas each of the above properties in itself seems rather weak, their combination may have striking consequences. For instance, a functor that is both \( C^\ast \)-stable and split-exact is automatically homotopy invariant and satisfies Bott periodicity.

Throughout this section, we consider functors \( \mathfrak{S} \to \mathfrak{C} \) where \( \mathfrak{S} \) is a full subcategory of \( \mathcal{C}^\ast\text{-alg} \) or \( G\text{-}\mathcal{C}^\ast\text{-alg} \) for some locally compact group \( G \). The target category \( \mathfrak{C} \) may be arbitrary in \( \S 3.1, 3.3 \) to discuss exactness properties, we require \( \mathfrak{C} \) to be an exact category or at least additive. Typical choices for \( \mathfrak{S} \) are the categories of separable or separable nuclear \( C^\ast \)-algebras, or the subcategory of all separable nuclear \( G\text{-}C^\ast \)-algebras with an amenable (or a proper) action of \( G \).
Definition 26. Let $P$ be a property for functors defined on $\mathcal{S}$. A universal functor with $P$ is a functor $u : \mathcal{S} \to \text{Univ}_P(\mathcal{S})$ such that

- $F \circ u$ has $P$ for each functor $F : \text{Univ}_P(\mathcal{S}) \to \mathcal{C}$;
- any functor $F : \mathcal{S} \to \mathcal{C}$ with $P$ factors uniquely as $F = F \circ u$ for some functor $F : \text{Univ}_P(\mathcal{S}) \to \mathcal{C}$.

Of course, a universal functor with $P$ need not exist. If it does, then it restricts to a bijection between objects of $\mathcal{S}$ and $\text{Univ}_P(\mathcal{S})$. Hence we can completely describe it by the sets of morphisms $\text{Univ}_P(A, B)$ from $A$ to $B$ in $\text{Univ}_P(\mathcal{S})$ and the maps $\mathcal{S}(A, B) \to \text{Univ}_P(A, B)$ for $A, B \in \mathcal{S}$; the universal property means that for any functor $F : \mathcal{S} \to \mathcal{C}$ with $P$ there is a unique functorial way to extend the maps $\text{Hom}_C(A, B) \to \mathcal{C}(F(A), F(B))$ to $\text{Univ}_P(A, B)$. There is no a priori reason why the morphism spaces $\text{Univ}_P(A, B)$ for $A, B \in \mathcal{S}$ should be independent of $\mathcal{S}$; but this happens in the cases we consider, under some assumption on $\mathcal{S}$.

3.1. Homotopy invariance. The following discussion applies to any full subcategory $\mathcal{S} \subseteq \text{Gr}^{\ast\alg}$ that is closed under the cylinder, cone, and suspension functors.

Definition 27. Let $f_0, f_1 : A \Rightarrow B$ be two parallel morphisms in $\mathcal{S}$. We write $f_0 \sim f_1$ and call $f_0$ and $f_1$ homotopic if there is a homotopy between $f_0$ and $f_1$, that is, a morphism $f : A \to \text{Cyl}(B) = \mathcal{C}([0, 1], B)$ with $e_{t \ast} f = f_t$ for $t = 0, 1$. It is easy to check that homotopy is an equivalence relation on $\text{Hom}_C(A, B)$. We let $[A, B]$ be the set of equivalence classes. The composition of morphisms in $\mathcal{S}$ descends to maps

$$[B, C] \times [A, B] \to [A, C], \quad ([f], [g]) \mapsto [f \circ g],$$

that is, $f_1 \sim f_2$ and $g_1 \sim g_2$ implies $f_1 \circ f_2 \sim g_1 \circ g_2$. Thus the sets $[A, B]$ form the morphism sets of a category, called homotopy category of $\mathcal{S}$ and denoted $\text{Ho}(\mathcal{S})$. The identity maps on objects and the canonical maps on morphisms define a canonical functor $\mathcal{S} \to \text{Ho}(\mathcal{S})$. A morphism in $\mathcal{S}$ is called a homotopy equivalence if it becomes invertible in $\text{Ho}(\mathcal{S})$.

Lemma 28. The following are equivalent for a functor $F : \mathcal{S} \to \mathcal{C}$:

(a) $F(e_{0 \ast}) = F(e_{1 \ast})$ as maps $F(\mathcal{C}([0, 1], A)) \to F(A)$ for all $A \in \mathcal{S}$;
(b) $F(e_{t \ast})$ induces isomorphisms $F(\mathcal{C}([0, 1], A)) \to F(A)$ for all $A \in \mathcal{S}$, $t \in [0, 1]$;
(c) the embedding as constant functions $\text{const} : A \to \mathcal{C}([0, 1], A)$ induces an isomorphism $F(A) \to F(\mathcal{C}([0, 1], A))$ for all $A \in \mathcal{S}$;
(d) $F$ maps homotopy equivalences to isomorphisms;
(e) if two parallel morphisms $f_0, f_1 : A \Rightarrow B$ are homotopic, then $F(f_0) \sim F(f_1)$;
(f) $F$ factors through the canonical functor $\mathcal{S} \to \text{Ho}(\mathcal{S})$.

Furthermore, the factorisation $\text{Ho}(\mathcal{S}) \to \mathcal{C}$ in (f) is necessarily unique.

Proof. We only mention two facts that are needed for the proof. First, we have $e_{t \ast} \circ \text{const} = \text{id}_A$ and $\text{const} \circ e_{t \ast} \sim \text{id}_{\mathcal{C}([0, 1], A)}$ for all $A \in \mathcal{S}$ and all $t \in [0, 1]$. Secondly, an isomorphism has a unique left and a unique right inverse, and these are again isomorphisms.

The equivalence of (d) and (f) in Lemma 28 says that $\text{Ho}(\mathcal{S})$ is the localisation of $\mathcal{S}$ at the family of homotopy equivalences.

Since $\mathcal{C}([0, 1], \mathcal{C}_0(X)) \cong \mathcal{C}_0([0, 1] \times X)$ for any locally compact space $X$, our notion of homotopy restricts to the usual one for pointed compact spaces. Hence the opposite of the homotopy category of pointed compact spaces is equivalent to a full subcategory of $\text{Ho}(\text{Gr}^{\ast\alg})$.

The sets $[A, B]$ inherit a base point $[0]$ and a quotient topology from $\text{Hom}(A, B)$; thus $\text{Ho}(\mathcal{S})$ is enriched over pointed topological spaces as well. This topology on $[A, B]$ is not so useful, however, because it need not be Hausdorff.
A similar topology exists on KASPAROV groups and can be defined in various ways, which turn out to be equivalent [12].

Let \( F : G-\mathcal{C}^\ast \text{Alg} \to H-\mathcal{C}^\ast \text{Alg} \) be a functor with natural isomorphisms
\[
F(\mathcal{C}([0, 1], A)) \cong \mathcal{C}([0, 1], F(A))
\]
that are compatible with evaluation maps for all \( A \). The universal property implies that \( F \) descends to a functor \( \text{Ho}(G-\mathcal{C}^\ast \text{Alg}) \to \text{Ho}(H-\mathcal{C}^\ast \text{Alg}) \). In particular, this applies to the suspension, cone, and cylinder functors and, more generally, to the functors \( A \otimes_{\text{max}} - \) and \( A \otimes_{\text{min}} - \) on \( G-\mathcal{C}^\ast \text{Alg} \) for any \( A \in G-\mathcal{C}^\ast \text{Alg} \) because both tensor product functors are associative and commutative and
\[
\mathcal{C}([0, 1], A) \cong \mathcal{C}([0, 1]) \otimes_{\text{max}} A \cong \mathcal{C}([0, 1]) \otimes_{\text{min}} A.
\]
The same works for the reduced and full crossed product functors \( G-\mathcal{C}^\ast \text{Alg} \to \mathcal{C}^\ast \text{Alg} \).

We may stabilise the homotopy category with respect to the suspension functor and consider a suspension-stable homotopy category with morphism spaces
\[
\{ A, B \} := \lim_{k \to \infty} \text{Sus}^k A, \text{Sus}^k B
\]
for all \( A, B \in G-\mathcal{C}^\ast \text{Alg} \). We may also enlarge the set of objects by adding formal desuspensions and generalising the notion of spectrum. This is less interesting for \( C^\ast \)-algebras than for spaces because most functors of interest satisfy BOTT periodicity, so that suspension and desuspension become equivalent.

3.2. Morita–Rieffel equivalence and stable isomorphism. One of the basic ideas of non-commutative geometry is that \( G \times C_0(X) \) (or \( G \times_r C_0(X) \)) should be a substitute for the quotient space \( G \backslash X \), which may have bad singularities. In the special case of a free and proper \( G \)-space \( X \), we expect that \( G \times C_0(X) \) and \( C_0(G \backslash X) \) are “equivalent” in a suitable sense. Already the simplest possible case \( X = G \) shows that we cannot expect an isomorphism here because
\[
G \times C_0(G) \not\cong G \times_r C_0(G) \cong \mathbb{K}(L^2 G).
\]
The right notion of equivalence is a \( C^\ast \)-version of MORITA equivalence due to MARC A. RIEFFEL [10, 11]; therefore, we call it MORITA–RIEFFEL equivalence.

The definition of MORITA–RIEFFEL equivalence involves HILBERT modules over \( C^\ast \)-algebras and the \( C^\ast \)-algebras of compact operators on them; these notions are crucial for KASPAROV theory as well. We refer to [33] for the definition and a discussion of their basic properties.

Definition 29. Two \( G-\mathcal{C}^\ast \)-algebras \( A \) and \( B \) are called MORITA–RIEFFEL equivalent if there are a full \( G \)-equivariant HILBERT \( B \)-module \( \mathcal{E} \) and a \( G \)-equivariant *-isomorphism \( \mathbb{K}(\mathcal{E}) \cong A \).

It is possible (and desirable) to express this definition more symmetrically: \( \mathcal{E} \) is an \( A, B \)-bimodule with two inner products taking values in \( A \) and \( B \), satisfying various conditions [10]. MORITA–RIEFFEL equivalent \( G-\mathcal{C}^\ast \)-algebras have equivalent categories of \( G \)-equivariant HILBERT modules via \( \mathcal{E} \otimes_B \_ \). The converse is unclear.

Example 30. The following is a more intricate example of a MORITA–RIEFFEL equivalence. Let \( \Gamma \) and \( P \) be two subgroups of a locally compact group \( G \). Then \( \Gamma \) acts on \( G/P \) by left translation and \( P \) acts on \( \Gamma \backslash G \) by right translation. The corresponding orbit space is the double coset space \( \Gamma \backslash G/P \). Both \( \Gamma \times C_0(G/P) \) and \( P \times C_0(\Gamma \backslash G) \) are non-commutative models for this double coset space. They are indeed MORITA–RIEFFEL equivalent; the bimodule that implements the equivalence is a suitable completion of \( C_c(G) \), the space of continuous functions with compact support on \( G \).
These examples suggest that Morita–Rieffel equivalent $C^*$-algebras describe the same non-commutative space. Therefore, we expect that reasonable functors on $\mathcal{C}^*\text{alg}$ should not distinguish between Morita–Rieffel equivalent $C^*$-algebras. (We will slightly weaken this statement below.)

**Definition 31.** Two $G$-$C^*$-algebras $A$ and $B$ are called stably isomorphic if there is a $G$-equivariant $^*$-isomorphism $A \otimes \mathbb{K}(H_G) \cong B \otimes \mathbb{K}(H_G)$, where $H_G := L^2(G \times \mathbb{N})$ is the direct sum of countably many copies of the regular representation of $G$; we let $G$ act on $\mathbb{K}(H_G)$ by conjugation, of course.

The following technical condition is often needed in connection with Morita–Rieffel equivalence.

**Definition 32.** A $C^*$-algebra is called $\sigma$-unital if it has a countable approximate identity or, equivalently, contains a strictly positive element.

**Example 33.** All separable $C^*$-algebras and all unital $C^*$-algebras are $\sigma$-unital; the algebra $\mathbb{K}(H)$ is $\sigma$-unital if and only if $H$ is separable.

**Theorem 34 ([7]).** $\sigma$-Unital $G$-$C^*$-algebras are $G$-equivariantly Morita–Rieffel equivalent if and only if they are stably isomorphic.

In the non-equivariant case, this theorem is due to Brown–Green–Rieffel [7]. A simpler proof that carries over to the equivariant case appeared in [11].

### 3.3. $C^*$-stable functors

The definition of $C^*$-stability is more intuitive in the non-equivariant case:

**Definition 35.** Fix a rank-one projection $p \in \mathbb{K}(\ell^2\mathbb{N})$. The resulting embedding $A \to A \otimes \mathbb{K}(\ell^2\mathbb{N})$, $a \to a \otimes p$, is called a corner embedding of $A$.

A functor $F: \mathcal{C}^*\text{alg} \to \mathcal{C}$ is called $C^*$-stable if any corner embedding induces an isomorphism $F(A) \cong F(A \otimes \mathbb{K}(\ell^2\mathbb{N}))$.

The correct equivariant generalisation is the following:

**Definition 36 ([56]).** A functor $F: G-\mathcal{C}^*\text{alg} \to \mathcal{C}$ is called $C^*$-stable if the canonical embeddings $\mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \leftarrow \mathcal{H}_2$ induce isomorphisms $F(A \otimes \mathbb{K}(\mathcal{H}_1)) \cong F(A \otimes \mathbb{K}(\mathcal{H}_2))$ for all non-zero $G$-Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Of course, it suffices to require $F(A \otimes \mathbb{K}(\mathcal{H}_1)) \cong F(A \otimes \mathbb{K}(\mathcal{H}_2))$. It is not hard to check that Definitions 35 and 36 are equivalent for trivial $G$.

**Remark 37.** We have argued in [3.2] why $C^*$-stability is an essential property for any decent homology theory for $C^*$-algebras. Nevertheless, it is tempting to assume less because $C^*$-stability together with split-exactness has very strong implications.

One reasonable way to weaken $C^*$-stability is to replace $\mathbb{K}(\ell^2\mathbb{N})$ by $M_n$ for $n \in \mathbb{N}$ in Definition 35 (see [57]). If two unital $C^*$-algebras are Morita–Rieffel equivalent, then they are also Morita equivalent as rings, that is, the equivalence is implemented by a finitely generated projective module. This implies that a matrix-stable functor is invariant under Morita–Rieffel equivalence for unital $C^*$-algebras.

Matrix-stability also makes good sense in $G-\mathcal{C}^*\text{alg}$ for a compact group $G$: simply require $\mathcal{H}_1$ and $\mathcal{H}_2$ in Definition 36 to be finite-dimensional. But we seem to run into problems for non-compact groups because they may have few finite-dimensional representations and we lack a finite-dimensional version of the equivariant stabilisation theorem.

Our next goal is to describe the universal $C^*$-stable functor. We abbreviate $A_G := \mathbb{K}(L^2G) \otimes A$. 

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Definition 38. A correspondence from $A$ to $B$ (or $A \rightarrow B$) is a $G$-equivariant Hilbert $B_k$-module $E$ together with a $G$-equivariant essential (or non-degenerate) $\ast$-homomorphism $f: A_k \rightarrow \mathbb{K}(E)$.

Given correspondences $E$ from $A$ to $B$ and $F$ from $B$ to $C$, their composition is the correspondence from $A$ to $C$ with underlying Hilbert module $E \otimes_{B_k} F$ and map $A_k \rightarrow \mathbb{K}(E) \rightarrow \mathbb{K}(E \otimes_{B_k} F)$, where the last map sends $T \mapsto T \otimes 1$; this yields compact operators because $B_k$ maps to $\mathbb{K}(F)$. See [34] for the definition of the relevant completed tensor product of Hilbert modules.

Up to isomorphism, the composition of correspondences is associative and the identity maps $A \rightarrow A = \mathbb{K}(A)$ act as unit elements. Hence we get a category $\mathcal{Corr}_G$ whose morphisms are the isomorphism classes of correspondences. It may have advantages to treat $\mathcal{Corr}_G$ as a 2-category.

Any $\ast$-homomorphism $\varphi: A \rightarrow B$ yields a correspondence $f: A \rightarrow \mathbb{K}(E)$ from $A$ to $B$, so that we get a canonical functor $\sharp: G\text{-C}^*\text{alg} \rightarrow \mathcal{Corr}_G$. We let $E$ be the right ideal $\varphi(A_k) \cdot B_k$ in $B_k$, viewed as a Hilbert $B$-module. Then $f(a) \cdot b := \varphi(a) \cdot b$ restricts to a compact operator $f(a)$ on $E$ and $f: A \rightarrow \mathbb{K}(E)$ is essential. It can be checked that this construction is functorial.

In the following proposition, we require that the category of $G$-$C^*$-algebras $\mathcal{S}$ be closed under Morita–Rieffel equivalence and consist of $\sigma$-unital $G$-$C^*$-algebras. We let $\mathcal{Corr}_\mathcal{S}$ be the full subcategory of $\mathcal{Corr}_G$ with object class $\mathcal{S}$.

Proposition 39. The functor $\sharp: \mathcal{S} \rightarrow \mathcal{Corr}_\mathcal{S}$ is the universal $C^*$-stable functor on $\mathcal{S}$; that is, it is $C^*$-stable, and any other such functor factors uniquely through $\sharp$.

Proof. First we sketch the proof in the non-equivariant case. We verify that $\sharp$ is $C^*$-stable. The Morita–Rieffel equivalence between $\mathbb{K}(\ell^2(N) \otimes A) \cong \mathbb{K}(\ell^2(N, A))$ and $A$ is implemented by the Hilbert module $\ell^2(N, A)$, which yields a correspondence $(\text{id}, \ell^2(N, A))$ from $\mathbb{K}(\ell^2(N) \otimes A)$ to $A$; this is inverse to the correspondence induced by a corner embedding $A \rightarrow \mathbb{K}(\ell^2(N) \otimes A)$.

A Hilbert $B$-module $E$ with an essential $\ast$-homomorphism $A \rightarrow \mathbb{K}(E)$ is countably generated because $A$ is assumed $\sigma$-unital. Kasparov’s Stabilisation Theorem yields an isometric embedding $E \rightarrow \ell^2(N, B)$. Hence we get $\ast$-homomorphisms $A \rightarrow \mathbb{K}(\ell^2(N) \otimes B) \leftarrow B$.

This diagram induces a map $F(A) \rightarrow F(\mathbb{K}(\ell^2(N) \otimes B)) \cong F(B)$ for any $C^*$-stable functor $F$. Now we should check that this well-defines a functor $\hat{F}: \mathcal{Corr}_\mathcal{S} \rightarrow \mathcal{C}$ with $\hat{F} \circ \sharp = F$, and that this yields the only such functor. We omit these computations.

The generalisation to the equivariant case uses the crucial property of the left regular representation that $L^2(G) \otimes \mathcal{H} \cong L^2(G \times \mathbb{N})$ for any countably infinite-dimensional $G$-Hilbert space $\mathcal{H}$. Since we replace $A$ and $B$ by $A_k$ and $B_k$ in the definition of correspondence right away, we can use this to repair a possible lack of $G$-equivariance; similar ideas appear in [50].

Example 40. Let $u$ be a $G$-invariant multiplier of $B$ with $u^* u = 1$; such $u$ are also called isometries. Then $b \mapsto u b u^*$ defines a $\ast$-homomorphism $B \rightarrow B$. The resulting correspondence $B \rightarrow B$ is isomorphic as a correspondence to the identity correspondence: the isomorphism is given by left multiplication with $u$, which defines a $G$-equivariant unitary operator from $B$ to the closure of $uBu^* \cdot B = u \cdot B$.

Hence inner endomorphisms act trivially on $C^*$-stable functors. Actually, this is one of the computations that we have omitted in the proof above; the argument can be found in [11].

Now we make the definition of a correspondence more concrete if $A$ is unital. We have an essential $\ast$-homomorphism $\varphi: A \rightarrow \mathbb{K}(E)$ for some $G$-equivariant Hilbert
$B$-module $\mathcal{E}$. Since $A$ is unital, this means that $\mathbb{K}(\mathcal{E})$ is unital and $\varphi$ is a unital $^\ast$-homomorphism. Then $\mathcal{E} = B ^\infty \cdot p$ for some projection $p \in M_{\infty}(B)$ and $\varphi$ is a $^\ast$-homomorphism $\varphi: A \to M_{\infty}(B)$ with $\varphi(1) = p$.

Two $^\ast$-homomorphisms $\varphi_1, \varphi_2: A \to M_{\infty}(B)$ yield isomorphic correspondences if and only if there is a partial isometry $v \in M_{\infty}(B)$ with $\varphi_1(x)v^* = \varphi_2(x)$ and $v^*\varphi_2(a)v = \varphi_1(a)$ for all $a \in A$.

Finally, we combine homotopy invariance and $C^\ast$-stability and consider the universal $C^\ast$-stable homotopy-invariant functor. This functor is much easier to characterise: the morphisms in the resulting universal category are simply the homotopy classes of correspondences $A \rightarrow B$. Alternatively, we get the same category if we use homotopy classes of correspondences $A \rightarrow B$ instead.

3.4. **Exactness properties.** Throughout this subsection, we consider functors $F: \mathfrak{S} \to \mathfrak{C}$ with values in an exact category $\mathfrak{C}$. If $\mathfrak{C}$ is merely additive to begin with, we can equip it with the trivial exact category structure for which all extensions split. We also suppose that $\mathfrak{S}$ is closed under the kinds of $C^\ast$-algebra extensions that we consider; depending on the notion of exactness, this means: direct product extensions, split extensions, $\mathbb{C}$-split extensions, or all extensions, respectively. Recall that split extensions in $G$-$\mathfrak{C}^\ast\mathfrak{alg}$ are required to split by a $G$-equivariant $^\ast$-homomorphism.

3.4.1. **Additive functors.** The most trivial split extensions in $G$-$\mathfrak{C}^\ast\mathfrak{alg}$ are the product extensions $A \mapsto A \times B \rightarrow B$ for two objects $A, B$. In this case, the coordinate embeddings and projections provide maps

\[ A \supseteq A \times B \supseteq B. \]

**Definition 41.** We call $F$ additive if it maps product diagrams (7) in $\mathfrak{S}$ to direct sum diagrams in $\mathfrak{C}$.

There is a partially defined addition on $^\ast$-homomorphisms: call two parallel $^\ast$-homomorphisms $\varphi, \psi: A \supseteq B$ orthogonal if $\varphi(a_1) \cdot \psi(a_2) = 0$ for all $a_1, a_2 \in A$.

Equivalently, $\varphi + \psi: a \mapsto \varphi(a) + \psi(a)$ is again a $^\ast$-homomorphism.

**Lemma 42.** The functor $F$ is additive if and only if, for all $A, B \in \mathfrak{S}$, the maps $\text{Hom}(A, B) \to \mathfrak{C}(F(A), F(B))$ satisfy $F(\varphi + \psi) = F(\varphi) + F(\psi)$ for all pairs of orthogonal parallel $^\ast$-homomorphisms $\varphi, \psi$.

Alternatively, we may also require additivity for coproducts (that is, free products). Of course, this only makes sense if $\mathfrak{S}$ is closed under coproducts in $G$-$\mathfrak{C}^\ast\mathfrak{alg}$. The coproduct $A \sqcup B$ in $G$-$\mathfrak{C}^\ast\mathfrak{alg}$ comes with canonical maps $A \supseteq A \sqcup B \supseteq B$ as well; the maps $\iota_A: A \to A \sqcup B$ and $\iota_B: B \to A \sqcup B$ are the coordinate embeddings, the maps $\pi_A: A \sqcup B \to A$ and $\pi_B: A \sqcup B \to B$ restrict to $(\text{id}_A, 0)$ and $(0, \text{id}_B)$ on $A$ and $B$, respectively.

**Definition 43.** We call $F$ additive on coproducts if it maps coproduct diagrams $A \supseteq A \sqcup B \supseteq B$ to direct sum diagrams in $\mathfrak{C}$.

The coproduct and product are related by a canonical $G$-equivariant $^\ast$-homomorphism $\varphi: A \sqcup B \to A \times B$ that is compatible with the maps to and from $A$ and $B$, that is, $\varphi \circ \iota_A = \iota_A$, $\pi_A \circ \varphi = \pi_A$, and similarly for $B$. There is no map backwards, but there is a correspondence $\psi: A \times B \to A \sqcup B$, which is induced by the $G$-equivariant $^\ast$-homomorphism

\[ A \times B \to M_2(A \sqcup B), \quad (a, b) \mapsto \begin{pmatrix} \iota_A(a) & 0 \\ 0 & \iota_B(b) \end{pmatrix}. \]
Theorem 46. Suppose the folding homomorphism. Hence we get a quasi-homomorphism kernel. The two canonical embeddings $A \to A$ and $B \to B$ by homotopic homomorphisms with orthogonal ranges. We can achieve the same effect by a suspension (shift $t_A$ and $t_B$ to the open intervals $(0,1/2)$ and $(1/2,1)$, respectively). Therefore, any homotopy invariant functor satisfies $F\left(\text{Sus}(A \sqcup B)\right) \cong F\left(\text{Sus}(A \times B)\right)$.

3.4.2. Split-exact functors.

Definition 45. We call $F$ split-exact if, for any split extension $K \to E \to Q$ with section $s: Q \to E$, the map $(F(i), F(s)): F(K) \oplus F(Q) \to F(E)$ is invertible.

It is clear that split-exact functors are additive.

Split-exactness is useful because of the following construction of Joachim Cuntz [3].

Let $B \triangleleft E$ be a $G$-invariant ideal and let $f_+, f_-: A \triangleleft E$ be $G$-equivariant $\ast$-homomorphisms with $f_+(a) - f_-(a) \in B$ for all $a \in A$. Equivalently, $f_+$ and $f_-$ both lift the same morphism $f: A \to E/B$. The data $(A, f_+, f_-, E, B)$ is called a quasi-homomorphism from $A$ to $B$.

Pulling back the extension $B \to E \to E/B$ along $\bar{f}$, we get an extension $B \to E' \to A$ with two sections $f'_+, f'_-: A \to E'$. The split-exactness of $F$ shows that $F(B) \to F(E') \to F(A)$ is a split extension in $\mathcal{C}$. Since both $F(f'_+)$ and $F(f'_-)$ are sections for it, we get a map $F(f'_+) - F(f'_-): F(A) \to F(B)$. Thus a quasi-homomorphism induces a map $F(A) \to F(B)$ if $F$ is split-exact. The formal properties of this construction are summarised in [11].

Given a $C^\ast$-algebra $A$, there is a universal quasi-homomorphism out of $A$. Let $Q(A) := A \sqcup A$ be the coproduct of two copies of $A$ and let $\pi_A: Q(A) \to A$ be the folding homomorphism that restricts to id$_A$ on both factors. Let $q(A)$ be its kernel. The two canonical embeddings $A \to A \sqcup A$ are sections for the folding homomorphism. Hence we get a quasi-homomorphism $A \cong Q(A) \to q(A)$. The universal property of the free product shows that any quasi-homomorphism yields a $G$-equivariant $\ast$-homomorphism $q(A) \to B$.

Theorem 46. Suppose $\mathcal{G}$ is closed under split extensions and tensor products with $\mathcal{C}([0,1])$ and $\mathbb{K}(\ell^2 \mathbb{N})$. If $F: \mathcal{G} \to \mathcal{C}$ is $C^\ast$-stable and split-exact, then $F$ is homotopy invariant.

This is a deep result of Nigel Higson [24]; a simple proof can be found in [11]. Besides basic properties of quasi-homomorphisms, it uses that inner endomorphisms act identically on $C^\ast$-stable functors (Example [10]).

Actually, the literature only contains Theorem 46 for functors on $\mathcal{C}^\ast\mathcal{A}lg$. But the proof in [11] works for functors on categories $\mathcal{G}$ as above.

3.4.3. Exact functors.

Definition 47. We call $F$ exact if $F(K) \to F(E) \to F(Q)$ is exact (at $F(E)$) for any extension $K \to E \to Q$ in $\mathcal{G}$. More generally, given a class $\mathcal{E}$ of extensions in $\mathcal{G}$ like, say, the class of equivariantly cp-split extensions, we define exactness for extensions in $\mathcal{E}$.

It is easy to see that the composite correspondence $\varphi \circ \psi$ is equal to the identity correspondence on $A \times B$. The other composite $\psi \circ \varphi$ is not the identity correspondence, but it is homotopic to it (see [23]). This yields:

Proposition 44. If $F$ is $C^\ast$-stable and homotopy invariant, then the canonical map $F(\varphi): F(A \sqcup B) \to F(A \times B)$ is invertible. Therefore, additivity and additivity for coproducts are equivalent for such functors.

The correspondence $\psi$ exists because the stabilisation creates enough room to replace $t_A$ and $t_B$ by homotopic homomorphisms with orthogonal ranges. We can achieve the same effect by a suspension (shift $t_A$ and $t_B$ to the open intervals $(0,1/2)$ and $(1/2,1)$, respectively). Therefore, any homotopy invariant functor satisfies $F(\text{Sus}(A \sqcup B)) \cong F(\text{Sus}(A \times B))$. 

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It is clear that split-exact functors are additive.

Split-exactness is useful because of the following construction of Joachim Cuntz [3].

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Given a $C^\ast$-algebra $A$, there is a universal quasi-homomorphism out of $A$. Let $Q(A) := A \sqcup A$ be the coproduct of two copies of $A$ and let $\pi_A: Q(A) \to A$ be the folding homomorphism that restricts to id$_A$ on both factors. Let $q(A)$ be its kernel. The two canonical embeddings $A \to A \sqcup A$ are sections for the folding homomorphism. Hence we get a quasi-homomorphism $A \cong Q(A) \to q(A)$. The universal property of the free product shows that any quasi-homomorphism yields a $G$-equivariant $\ast$-homomorphism $q(A) \to B$.

Theorem 46. Suppose $\mathcal{G}$ is closed under split extensions and tensor products with $\mathcal{C}([0,1])$ and $\mathbb{K}(\ell^2 \mathbb{N})$. If $F: \mathcal{G} \to \mathcal{C}$ is $C^\ast$-stable and split-exact, then $F$ is homotopy invariant.

This is a deep result of Nigel Higson [24]; a simple proof can be found in [11]. Besides basic properties of quasi-homomorphisms, it uses that inner endomorphisms act identically on $C^\ast$-stable functors (Example [10]).

Actually, the literature only contains Theorem 46 for functors on $\mathcal{C}^\ast\mathcal{A}lg$. But the proof in [11] works for functors on categories $\mathcal{G}$ as above.
It is easy to see that exact functors are additive. Most functors we are interested in satisfy homotopy invariance and Bott periodicity, and these two properties prevent a non-zero functor from being exact in the stronger sense of being left or right exact. This explains why our notion of exactness is much weaker than usual in homological algebra.

It is reasonable to require that a functor be part of a homology theory, that is, a sequence of functors \((F_n)_{n \in \mathbb{Z}}\) together with natural long exact sequences for all extensions [54]. We do not require this additional information because it tends to be hard to get \(a \text{ priori}\) but often comes for free \(a \text{ posteriori}\):

**Proposition 48.** Suppose that \(F\) is homotopy invariant and exact (or exact for equivariantly cp-split extensions). Then \(F\) has long exact sequences of the form

\[
\cdots \to F(\text{Sus}(K)) \to F(\text{Sus}(E)) \to F(\text{Sus}(Q)) \to F(K) \to F(E) \to F(Q) \to \cdots
\]

for any (equivariantly cp-split) extension \(K \to E \to Q\). In particular, \(F\) is split-exact.

See §21.4 in [5] for the proof.

There probably exist exact functors that are not split-exact. It is likely that the algebraic \(K_1\)-functor provides a counterexample: it exact but not split-exact on the category of rings [19]; but I do not know a counterexample to its split-exactness involving only \(C^*\)-algebras.

Proposition 48 and Bott periodicity yield long exact sequences that are infinite in both directions. Thus an exact homotopy invariant functor that satisfies Bott periodicity is part of a homology theory in a canonical way.

For universal constructions, we should replace a single functor by a homology theory, that is, a sequence of functors. The universal functors in this context are non-stable versions of \(E\)-theory and \(KK\)-theory. We refer to [27] for details.

A weaker property than exactness is the existence of Puppe exact sequences for mapping cones. The Puppe exact sequence is the special case of the long exact sequence of Proposition 48 for extensions of the form \(\text{Sus}(B) \to \text{Cone}(f) \to A\) for a morphism \(f: A \to B\). In practice, the exactness of a functor is often established by reducing it to the Puppe exact sequence. Let \(K \to E \to Q\) be an extension. A variant of the Puppe exact sequence yields the long exact sequence for the extension \(\text{Cone}(p) \to \text{Cyl}(p) \to Q\). There is a canonical morphism of extensions

\[
K \to E \to Q
\]

\[
\text{Cone}(p) \to \text{Cyl}(p) \to Q
\]

where the vertical map \(E \to \text{Cyl}(p)\) is a homotopy equivalence. Hence a functor with Puppe exact sequences is exact for \(K \to E \to Q\) if and only if it maps the vertical map \(K \to \text{Cone}(p)\) to an isomorphism.

4. Kasparov theory

We define \(KK^G\) as the universal split-exact \(C^*\)-stable functor on \(G-\mathcal{C}^*\text{-sep}\); since split-exact and \(C^*\)-stable functors are automatically homotopy invariant, \(KK^G\) is the universal split-exact \(C^*\)-stable homotopy functor as well. The universal property of Kasparov theory due to Higson and Cuntz asserts that this is equivalent to Kasparov’s definition. We examine some basic properties of Kasparov theory and, in particular, show how to get functors between Kasparov categories.

We let \(E^G\) be the universal exact \(C^*\)-stable homotopy functor on \(G-\mathcal{C}^*\text{-sep}\) or, equivalently, the universal exact, split-exact, and \(C^*\)-stable functor.
Kasparov’s own definition of his theory is inspired by previous work of Atiyah [1] on K-homology; later, he also interacted with the work of Brown–Douglas–Fillmore [5] on extensions of C*-algebras. A construction in abstract homotopy theory provides a homology theory for spaces that is dual to K-theory. Atiyah realized that certain abstract elliptic differential operators provide cycles for this dual theory; but he did not know the equivalence relation to put on these cycles. Brown–Douglas–Fillmore studied extensions of C_0(X) (and more general C*-algebras) by the compact operators and found that the resulting structure set is naturally isomorphic to a K-homology group.

Kasparov unified and vastly generalised these two results, defining a bivariant functor KK_s(A, B) that combines K-theory and K-homology and that is closely related to the classification of extensions B \otimes \mathbb{K} \rightarrow E \rightarrow A (see [20],[30]). A deep theorem of Kasparov shows that two reasonable equivalence relations for these cycles coincide; this clarifies the homotopy invariance of the extension groups of Brown–Douglas–Fillmore. Furthermore, he constructed an equivariant version of his theory in [31] and applied it to prove the Novikov conjecture for discrete subgroups of Lie groups.

The most remarkable feature of Kasparov theory is an associative product on KK called Kasparov product. This generalises various known product constructions in K-theory and K-homology and allows to view KK as a category.

In applications, we usually need some non-obvious KK-element, and we must compute certain Kasparov products explicitly. This requires a concrete description of Kasparov cycles and their products. Since both are somewhat technical, we do not discuss them here and merely refer to [5] for a detailed treatment and to [50] for a very useful survey article. Instead, we use Higson’s characterisation of KK by a universal property [23], which is based on ideas of Cuntz ([9],[10]). The extension to the equivariant case is due to Thom森 [55]. A simpler proof of Thom森’s theorem and various related results can be found in [30].

We do not discuss KK^G for \mathbb{Z}/2-graded G-C*-algebras here because it does not fit so well with the universal property approach, which would simply yield KK^G\times\mathbb{Z}/2 because \mathbb{Z}/2-graded G-C*-algebras are the same as G \times \mathbb{Z}/2-C*-algebras. The relationship between the two theories is explained in [30], following Ulrich Haag [22]. The graded version of Kasparov theory is often useful because it allows us to treat even and odd KK-cycles simultaneously.

Fix a locally compact group G. The Kasparov groups KK^G_0(A, B) for A, B ∈ G-C*-sep form the morphisms sets A → B of a category, which we denote by R^G; the composition in R^G is the Kasparov product. The categories G-C*-sep and R^G have the same objects. We have a canonical functor

\[ KK^G_0 : G-K^G \rightarrow R^G \]

that acts identically on objects. This functor contains all information about equivariant Kasparov theory for G.

Definition 49. A G-equivariant *-homomorphism f : A → B is called a KK^G-identity if KK^G(f) is invertible in R^G.

Theorem 50. Let S be a full subcategory of G-C*-sep that is closed under suspensions, G-equivariantly cp-split extensions, and Morita–Rieffel equivalence. Let R^G(S) be the full subcategory of R^G with object class S and let KK^G_S : S → R^G(S) be the restriction of KK^G.

The functor KK^G_S : S → R^G_S is the universal split-exact C*-stable functor; in particular, R^G(S) is an additive category. In addition, it has the following properties and is, therefore, universal among functors on S with some of these extra properties: it is
Theorem 53. \( (\text{UCT}) \), which computes \( KK \)

Proof. \( q \)

for the split extension \( \text{rem 50}. \) Proposition 44 yields that it is additive for coproducts. Split-exactness therefore homotopy invariant by Theorem 46 (this is already asser ted in Theo-

Corollary 51. Let \( F : \mathcal{S} \to \mathcal{C} \) be split-exact and \( C^* \)-stable. Then \( F \) factors uniquely through \( \text{KK}_0^G \), is homotopy invariant, and satisfies Bott periodicity. A \( KK^G \)-equivalence \( A \to B \) in \( \mathcal{S} \) induces an isomorphism \( F(A) \to F(B) \).

We will view the universal property of Theorem 50 as a definition of \( \text{KK}^G \) and thus of the groups \( \text{KK}_0^G(A, B) \). We also let

\[
\text{KK}_0^G(A, B) := \text{KK}^G(A, \text{Sus}^n(B));
\]
since the Bott periodicity isomorphism identifies \( \text{KK}_2^G \cong \text{KK}_0^G \), this yields a \( \mathbb{Z}/2 \)-graded theory.

Now we describe \( \text{KK}_0^G(A, B) \) more concretely. Recall \( A_K := A \otimes \mathbb{K}(L^2G) \).

Proposition 52. Let \( A \) and \( B \) be two \( G \)-\( C^* \)-algebras. There is a natural bijection between the morphism sets \( \text{KK}_0^G(A, B) \) in \( \text{KK}^G \) and the set \([q(A_K), B_K \otimes \mathbb{K}(\mathbb{L}^2)] \) of homotopy classes of \( G \)-equivariant \( * \)-homomorphisms from \( q(A_K) \) to \( B_K \otimes \mathbb{K}(\mathbb{L}^2) \).

Proof. The canonical functor \( G-\mathcal{C}^* \text{sep} \to \text{KK}^G \) is \( C^* \)-stable and split-exact, and therefore homotopy invariant by Theorem 46 (this is already asserted in Theorem 50). Proposition 44 yields that it is additive for coproducts. Split-exactness for the split extension \( q(A) \to Q(A) \to A \) shows that \( \text{id}_A \ast : Q(A) \to A \) restricts to a \( KK^G \)-equivalence \( q(A) \sim A \). Similarly, \( C^* \)-stability yields \( KK^G \)-equivalences \( A \sim A_K \) and \( B \sim B_K \otimes \mathbb{K}(\mathbb{L}^2) \). Hence homotopy classes of \( * \)-homomorphisms from \( q(A_K) \) to \( B_K \otimes \mathbb{K}(\mathbb{L}^2) \) yield classes in \( \text{KK}_0^G(A, B) \). Using the concrete description of Kasparov cycles, which we have not discussed, it is checked in [36] that this map yields a bijection as asserted.

Another equivalent description is

\[
\text{KK}_0^G(A, B) \cong [q(A_K) \otimes \mathbb{K}(\mathbb{L}^2), q(B_K) \otimes \mathbb{K}(\mathbb{L}^2)];
\]
in this approach, the Kasparov product becomes simply the composition of mor-

Proposition 52 suggests that \( q(A_K) \) and \( B_K \otimes \mathbb{K}(\mathbb{L}^2) \) may be the cofibrant and fibrant replacement of \( A \) and \( B \) in some model category related to \( KK^G \). But it is not clear whether this is the case. The model category structure constructed in [28] is certainly quite different.

By the universal property, K-theory descends to a functor on \( \text{KK}^G \), that is, we get canonical maps

\[
\text{KK}_0(A, B) \to \text{Hom}(\text{K}_*(A), \text{K}_*(B))
\]
for all separable \( C^* \)-algebras \( A, B \), where the right hand side denotes grading-preserving group homomorphisms. For \( A = \mathcal{C} \), this yields a map \( \text{KK}_0(\mathcal{C}, B) \to \text{Hom}(\mathbb{Z}, \text{K}_0(B)) \cong \text{K}_0(B) \). Using suspensions, we also get a corresponding map \( \text{KK}_1(\mathcal{C}, B) \to \text{K}_1(B) \).

Theorem 53. The maps \( \text{KK}_*(\mathcal{C}, B) \to \text{K}_*(B) \) constructed above are isomorphisms for all \( B \in \mathcal{C}^* \text{sep} \).

Thus Kasparov theory is a bivariant generalisation of K-theory. Roughly speaking, \( \text{KK}_*(A, B) \) is the place where maps between K-theory groups live. Most construc-

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4.1. Extending functors and identities to $\mathfrak{K}^G$. We can use the universal property to extend various functors $G{-\mathcal{C}^*}_{\text{sep}} \to H{-\mathcal{C}^*}_{\text{sep}}$ to functors $\mathfrak{K}^G \to \mathfrak{K}^H$. We explain this by an example:

**Proposition 54.** The full and reduced crossed product functors

$$G \ltimes_r G \ltimes_l : G{-\mathcal{C}^*}_{\text{alg}} \to \mathcal{C}^*_{\text{alg}}$$

extend to functors $G \ltimes_r G \ltimes_l : \mathfrak{K}^G \to \mathfrak{K}$ called descent functors.

**Proof.** We only write down the argument for reduced crossed products, the other case is similar. It is well-known that $G \ltimes_r (A \otimes K(H)) \cong (G \ltimes_r A) \otimes K(H)$ for any $G$-Hilbert space $H$. Therefore, the composite functor

$$G{-\mathcal{C}^*}_{\text{sep}} \xrightarrow{G \ltimes_r} \mathcal{C}^*_{\text{sep}} \xrightarrow{KK} \mathfrak{K}$$

is $C^*$-stable. Proposition 5 shows that this functor is split-exact as well (regardless of whether $G$ is an exact group). Now the universal property provides an extension to a functor $\mathfrak{K}^G \to \mathfrak{K}$.

Similarly, we get functors

$$A \otimes_{\text{min}} A \otimes_{\text{max}} A : \mathfrak{K}^G \to \mathfrak{K}^G$$

for any $G$-$C^*$-algebra $A$. Since these extensions are natural, we even get bifunctors

$$\otimes_{\text{min}}, \otimes_{\text{max}} : \mathfrak{K}^G \times \mathfrak{K}^G \to \mathfrak{K}^G.$$  

The associativity, commutativity, and unit constraints in $G{-\mathcal{C}^*}_{\text{alg}}$ induce corresponding constraints in $\mathfrak{K}^G$, so that both $\otimes_{\text{min}}$ and $\otimes_{\text{max}}$ turn $\mathfrak{K}^G$ into a symmetric monoidal category.

Another example is the functor $\tau : \mathcal{C}^*_{\text{alg}} \to G{-\mathcal{C}^*}_{\text{alg}}$ that equips a $C^*$-algebra with the trivial $G$-action; it extends to a functor $\tau : \mathfrak{K} \to \mathfrak{K}^G$.

The universal property also allows us to prove identities between functors. For instance, we have natural isomorphisms $G \ltimes_r (\tau(A) \otimes_{\text{min}} B) = A \otimes_{\text{min}} (G \ltimes_r B)$ for all $G$-$C^*$-algebras $B$. To begin with, naturality means that the diagram

$$G \ltimes_r (\tau(A_1) \otimes_{\text{min}} B_1) \xrightarrow{\cong} A_1 \otimes_{\text{min}} (G \ltimes_r B_1)$$

commutes if $A_1 \to A_2$ and $\beta : B_1 \to B_2$ are a *-homomorphism and a $G$-equivariant *-homomorphism, respectively. Two applications of the uniqueness part of the universal property show that this diagram remains commutative in $\mathfrak{K}$ if $\alpha \in KK_0(A_1, A_2)$ and $\beta \in KK_0^*(B_1, B_2)$. Similar remarks apply to the natural isomorphism $G \ltimes_r (\tau(A) \otimes_{\text{max}} B) \cong A \otimes_{\text{max}} (G \ltimes B)$ and hence to the isomorphisms $G \ltimes \tau(A) \cong C^*(G) \otimes_{\text{max}} A$ and $G \ltimes_r \tau(A) \cong C^*_\text{red}(G) \otimes_{\text{min}} A$.

Adjointness relations in Kasparov theory are usually proved most easily by constructing the unit and counit of the adjunction. For instance, if $G$ is a compact group then the functor $\tau$ is left adjoint to $G \ltimes_r = G \ltimes_r$ (that is, for all $A \in \mathfrak{K}$ and $B \in \mathfrak{K}^G$, we have natural isomorphisms

$$\text{(8)} \quad KK^G(\tau(A), B) \cong KK_*(A, G \ltimes B).$$
This is also known as the Green–Julg Theorem. For $A = C$, it specialises to a natural isomorphism $K^G_*(B) \cong K_*(G \ltimes B)$; this was one of the first appearances of non-commutative algebras in topological K-theory.

Proof of (8). We already know that $\tau$ and $G \ltimes \cdot$ are functors between $\mathcal{R}$ and $\mathcal{R}^G$. It remains to construct natural elements

$$\alpha_A \in KK_0(A, G \ltimes \tau(A)), \quad \beta_B \in KK_0^G(\tau(G \ltimes B), B)$$

for all $A \in \mathcal{R}$, $B \in \mathcal{R}^G$.

The main point is that $\tau(G \ltimes B)$ is the $G$-fixed point subalgebra of $B \rtimes \mathbb{K} = B \otimes \mathbb{K}(L^2G)$. The embedding $\tau(G \ltimes B) \to B \rtimes \mathbb{K}$ provides a $G$-equivariant correspondence $\beta_B$ from $\tau(G \ltimes B)$ to $B$ and thus an element of $KK^G_0(\tau(G \ltimes B), B)$. This construction is certainly natural for $G$-equivariant $*$-homomorphisms and hence for $KK^G$-morphisms by the uniqueness part of the universal property of $KK^G$.

Let $e_\tau : C \to C^*(G)$ be the embedding that corresponds to the trivial representation of $G$. Recall that $G \ltimes \tau(A) \cong C^*(G) \otimes A$. Hence the exterior product of the identity map on $A$ and $KK(e_\tau)$ provides $\alpha_A \in KK_0(A, G \ltimes \tau(A))$. Again, naturality for $*$-homomorphisms is clear and implies naturality for morphisms in $KK$.

Finally, it remains to check that

$$\tau(A) \xrightarrow{\tau(A)} \tau(G \ltimes \tau(A)) \xrightarrow{\beta_{\tau(A)}} \tau(A)$$

for all $A \in \mathcal{R}$, $B \in \mathcal{R}^G$. This yields natural $*$-homomorphisms in $KK^G$. Then we get the desired adjointness using a general construction from category theory (see [35]). In fact, both composites are equal to the identity already as correspondences, so that we do not have to know anything about Kasparov theory except its $C^*$-stability to check this. \qed

A similar argument yields an adjointness relation

$$KK^G_0(A, \tau(B)) \cong KK_0(G \ltimes A, B) \quad (9)$$

for a discrete group $G$. More conceptually, (9) corresponds via Baaj–Skandalis duality [3] to the Green–Julg Theorem for the dual quantum group of $G$, which is compact because $G$ is discrete. But we can also write down unit and counit of adjunction directly.

The trivial representation $C^*(G) \to C$ yields natural $*$-homomorphisms

$$G \ltimes \tau(B) \cong C^*(G) \otimes_{\max} B \to B$$

and hence $\beta_B \in KK_0(G \ltimes \tau(B), B)$. The canonical embedding $A \to G \ltimes A$ is $G$-equivariant if we let $G$ act on $G \ltimes A$ by conjugation; but this action is inner, so that $G \ltimes A$ and $\tau(G \ltimes A)$ are $G$-equivariantly Morita–Rieffel equivalent. Thus the canonical embedding $A \to G \ltimes A$ yields a correspondence $A \to \tau(G \ltimes A)$ and $\alpha_A \in KK^G_0(A, \tau(G \ltimes A))$. We must check that the composites

$$G \ltimes A \xrightarrow{G \ltimes \alpha_A} G \ltimes \tau(A) \xrightarrow{\beta_{G \ltimes A}} G \ltimes A,$$

$$\tau(B) \xrightarrow{\tau(B)} \tau(G \ltimes \tau(B)) \xrightarrow{\tau(B)\beta_B} \tau(B)$$

are identity morphisms in $KK$ and $KK^G$, respectively. Once again, this holds already on the level of correspondences.
4.2. Triangulated category structure. We turn $\mathcal{R}^G$ into a triangulated category by extending standard constructions for topological spaces [38]. Some arrows change direction because the functor $C_0$ from spaces to $C^*$-algebras is contravariant. We have already observed that $\mathcal{R}^G$ is additive. The suspension is given by $\Sigma^{-1}(A) := \text{Sus}(A)$. Since $\text{Sus}^2(A) \cong A$ in $\mathcal{R}^G$ by Bott periodicity, we have $\Sigma = \Sigma^{-1}$. Thus we do not need formal desuspensions as for the stable homotopy category.

**Definition 55.** A triangle $A \to B \to C \to \Sigma A$ in $\mathcal{R}^G$ is called exact if it is isomorphic as a triangle to the mapping cone triangle

$$\text{Sus}(B) \to \text{Cone}(f) \to A \xrightarrow{f} B$$

for some $G$-equivariant $^*$-homomorphism $f$.

Alternatively, we can use $G$-equivariantly cp-split extensions in $G\text{-C}^*\text{sep}$. Any such extension $I \to E \to Q$ determines a class in $\text{KK}^G_1(Q,I) \cong \text{KK}^G_0(\text{Sus}(Q),I)$, so that we get a triangle $\text{Sus}(Q) \to I \to E \to Q$ in $\mathcal{R}^G$. Such triangles are called extension triangles. A triangle in $\mathcal{R}^G$ is exact if and only if it is isomorphic to the extension triangle of a $G$-equivariantly cp-split extension [38].

**Theorem 56.** With the suspension automorphism and exact triangles defined above, $\mathcal{R}^G$ is a triangulated category. So is $\mathcal{R}^G(\mathcal{G})$ if $\mathcal{G} \subseteq G\text{-C}^*\text{sep}$ is closed under suspensions, $G$-equivariantly cp-split extensions, and Morita–Rieffel equivalence as in Theorem 57.

**Proof.** That $\mathcal{R}^G$ is triangulated is proved in detail in [38]. We do not discuss the triangulated category axioms here. Most of them amount to properties of mapping cone triangles that can be checked by copying the corresponding arguments for the stable homotopy category (and reverting arrows). These axioms hold for $\mathcal{R}^G(\mathcal{G})$ because they hold for $\mathcal{R}^G$. The only axiom that requires more care is the existence axiom for exact triangles; it requires any morphism to be part of an exact triangle. We can prove this as in [38] using the concrete description of $\text{KK}^G_0(A,B)$ in Proposition 52. For some applications like the generalisation to $\mathcal{R}^G(\mathcal{G})$, it is better to use extension triangles instead. Any $f \in \text{KK}^G_0(A,B) \cong \text{KK}^G_0(\text{Sus}(A),B)$ can be represented by a $G$-equivariantly cp-split extension $\mathbb{K}(H_B) \to E \to \text{Sus}(A)$, where $H_B$ is a full $G$-equivariant Hilbert $B$-module, so that $\mathbb{K}(H_B)$ is $G$-equivariantly Morita–Rieffel equivalent to $B$. The extension triangle of this extension contains $f$ and belongs to $\mathcal{R}^G(\mathcal{G})$ by our assumptions on $\mathcal{G}$. \qed

Since model category structures related to $C^*$-algebras are rather hard to get (compare [28]), triangulated categories seem to provide the most promising formal setup for extending results from classical spaces to $C^*$-algebras. An earlier attempt can be found in [34]. Triangulated categories clarify the basic bookkeeping with long exact sequences. Mayer–Vietoris exact sequences and inductive limits are discussed from this point of view in [35]. More importantly, this framework sheds light on more advanced constructions like the Baum–Connes assembly map. We will briefly discuss this below.

4.3. The Universal Coefficient Theorem. There is a very close relationship between K-theory and Kasparov theory. We have already seen that $K_*(A) \cong \text{KK}^*_0(\mathcal{C},A)$ is a special case of KK. Furthermore, KK inherits deep properties of K-theory such as Bott periodicity. Thus we may hope to express $\text{KK}^*_p(A,B)$ using only the K-theory of $A$ and $B$ — at least for many $A$ and $B$. This is the point of the Universal Coefficient Theorem.
The Kasparov product provides a canonical homomorphism of graded groups
\[ \gamma: \text{KK}(A, B) \rightarrow \text{Hom}_\ast(\text{K}_\ast(A), \text{K}_\ast(B)), \]
where \( \text{Hom}_\ast \) denotes the \( \mathbb{Z}/2 \)-graded Abelian group of all group homomorphisms \( \text{K}_\ast(A) \rightarrow \text{K}_\ast(B) \). There are topological reasons why \( \gamma \) cannot always be invertible: since \( \text{Hom}_\ast \) is not exact, the bifunctor \( \text{Hom}_\ast(\text{K}_\ast(A), \text{K}_\ast(B)) \) would not be exact on cp-split extensions. A construction of Lawrence Brown provides another natural map
\[ \kappa: \ker \gamma \rightarrow \text{Ext}_\ast(\text{K}_{\ast+1}(A), \text{K}_\ast(B)). \]
The following theorem is due to Jonathan Rosenberg and Claude Schochet [51, 53]; see also [5].

**Theorem 57.** The following are equivalent for a separable \( C^\ast \)-algebra \( A \):

(a) \( \text{KK}_\ast(A, B) = 0 \) for all \( B \in \mathfrak{A} \) with \( \text{K}_\ast(B) = 0 \);

(b) the map \( \gamma \) is surjective and \( \kappa \) is bijective for all \( B \in \mathfrak{A} \);

(c) for all \( B \in \mathfrak{A} \), there is a short exact sequence of \( \mathbb{Z}/2 \)-graded Abelian groups
\[ \text{Ext}_\ast(\text{K}_{\ast+1}(A), \text{K}_\ast(B)) \rightarrow \text{KK}_\ast(A, B) \rightarrow \text{Hom}_\ast(\text{K}_\ast(A), \text{K}_\ast(B)). \]

(d) \( A \) belongs to the smallest class of \( C^\ast \)-algebras that contains \( \mathbb{C} \) and is closed under KK-equivalence, suspensions, countable direct sums, and cp-split extensions;

(e) \( A \) is KK-equivalent to \( C_0(X) \) for some pointed compact metrisable space \( X \).

If these conditions are satisfied, then the extension in (c) is natural and splits, but the section is not natural.

The class of \( C^\ast \)-algebras with these properties is also called the bootstrap class because of description (d). Alternatively, we may say that they satisfy the **Universal Coefficient Theorem** because of (c). Since commutative \( C^\ast \)-algebras are nuclear, (e) implies that the natural map \( A \otimes_{\text{max}} B \rightarrow A \otimes_{\text{min}} B \) is a KK-equivalence if \( A \) or \( B \) belongs to the bootstrap class [55]. This fails for some \( A \), so that the Universal Coefficient Theorem does not hold for all \( A \). Remarkably, this is the only obstruction to the Universal Coefficient Theorem known at the moment: we know no nuclear \( C^\ast \)-algebra that does not satisfy the Universal Coefficient Theorem. As a result, we can express \( \text{KK}_\ast(A, B) \) using only \( \text{K}_\ast(A) \) and \( \text{K}_\ast(B) \) for many \( A \) and \( B \).

When we restrict attention to nuclear \( C^\ast \)-algebras, then the bootstrap class is closed under various operations like tensor products, arbitrary extensions and inductive limits (without requiring any cp-sections), and under crossed products by torsion-free amenable groups. Remarkably, there are no general results about crossed products by finite groups.

The Universal Coefficient Theorem and the universal property of KK imply that very few homology theories for (pointed compact metrisable) spaces can extend to the non-commutative setting. More precisely, if we require the extension to be split-exact, \( C^\ast \)-stable, and additive for countable direct sums, then only K-theory with coefficients is possible. Thus we rule out most of the difficult (and interesting) problems in stable homotopy theory. But if we only want to study K-theory, anyway, then the operator algebraic framework usually provides very good analytical tools. This is most valuable for equivariant generalisations of K-theory.

Jonathan Rosenberg and Claude Schochet [50] have also constructed a spectral sequence that, in favourable cases, computes \( \text{KK}^G(A, B) \) from \( \text{K}_G^\ast(A) \) and \( \text{K}_G^\ast(B) \); they require \( G \) to be a compact Lie group with torsion-free fundamental group and \( A \) and \( B \) to belong to a suitable bootstrap class. This equivariant UCT is clarified in [50, 51].
4.4. E-Theory and asymptotic morphisms. Recall that KASPAROV theory is only exact for (equivariantly) cp-split extensions. E-Theory is a similar theory that is exact for all extensions.

**Definition 58.** We let $E^G: G\mathcal{C}^*\text{sep} \to \mathcal{C}^G$ be the universal $C^*$-stable, exact homotopy functor.

**Lemma 59.** The functor $E^G$ is split-exact and factors through $KK^G: G\mathcal{C}^*\text{sep} \to KK^G$. Hence it satisfies Bott periodicity.

**Proof.** Proposition 48 shows that any exact homotopy functor is split-exact. The remaining assertions now follow from Corollary 51.

The functor $E$ (for trivial $G$) was first defined as above by Nigel Higson [25]. Then Alain Connes and Nigel Higson [3] found a more concrete description using asymptotic morphisms. This is what made the theory usable. The equivariant generalisation of the theory is due to Erik Guentner, Nigel Higson, and Jody Trout [21].

We write $E^G(A, B)$ for the space of morphisms $A \to \text{Sus}^n(B)$ in $\mathcal{C}^G$. Bott periodicity shows that there are only two different groups to consider.

**Definition 60.** The asymptotic algebra of a $C^*$-algebra $B$ is the $C^*$-algebra

$$\text{Asymp}(B) := C_0(\mathbb{R}_+, B)/C_0(\mathbb{R}_+, B).$$

An asymptotic morphism $A \to B$ is a $^*$-homomorphism $f: A \to \text{Asymp}(B)$.

Representing elements of $\text{Asymp}(B)$ by bounded functions $[0, \infty) \to B$, we can represent $f$ by a family of maps $f_t: A \to B$ such that $f_t(a) \in C_0(\mathbb{R}_+, B)$ for each $a \in A$ and the map $a \mapsto f_t(a)$ satisfies the conditions for a $^*$-homomorphism asymptotically for $t \to \infty$. This provides a concrete description of asymptotic morphisms and explains the name.

If a locally compact group $G$ acts on $B$, then $\text{Asymp}(B)$ inherits an action of $G$ by naturality (which need not be strongly continuous).

**Definition 61.** Let $A$ and $B$ be two $G$-$C^*$-algebras for a locally compact group $G$. A $G$-equivariant asymptotic morphism from $A$ to $B$ is a $G$-equivariant $^*$-homomorphism $f: A \to \text{Asymp}(B)$. We write $[A, B]$ for the set of homotopy classes of $G$-equivariant asymptotic morphisms from $A$ to $B$. Here a homotopy is a $G$-equivariant $^*$-homomorphism $A \to \text{Asymp}(C([0, 1], B))$.

The asymptotic algebra fits, by definition, into an extension

$$C_0(\mathbb{R}_+, B) \to C_0(\mathbb{R}_+, B) \to \text{Asymp}(B).$$

Notice that $C_0(\mathbb{R}_+, B) \cong \text{Cone}(B)$ is contractible. If $f: A \to \text{Asymp}(B)$ is a $G$-equivariant asymptotic morphism, then we can use it to pull back this extension to an extension $\text{Cone}(B) \to E \to A$ in $G\mathcal{C}^*\text{alg}$; the $G$-action on $E$ is automatically strongly continuous. If $F$ is exact and homotopy invariant, then $F(\text{Sus}^n(E)) \to F(\text{Sus}^n(A))$ is an isomorphism for all $n \geq 1$ by Proposition 48.

The evaluation map $C_0(\mathbb{R}_+, B) \to B$ at some $t \in \mathbb{R}_+$ pulls back to a morphism $E \to B$, and these morphisms for different $t$ are all homotopic. Hence we get a well-defined map $F(\text{Sus}^n(A)) \cong F(\text{Sus}^n(E)) \to F(\text{Sus}^n(B))$ for each asymptotic morphism $A \to B$. This explains how asymptotic morphisms are related to exact homotopy functors. This observation leads to the following theorem:

**Theorem 62.** There are natural bijections

$$E^G_0(A, B) \cong \left[\text{Sus}(A_K \otimes \mathbb{K}(\ell^2\mathbb{N})), \text{Sus}(B_K \otimes \mathbb{K}(\ell^2\mathbb{N}))\right]$$

for all separable $G$-$C^*$-algebras $A, B$. 
An important step in the proof of Theorem 62 is the Connes–Higson construction, which to an extension $I \rightarrow E \rightarrow Q$ in $\mathbb{C}^*\text{sef}$ associates an asymptotic morphism $\text{Sus}(Q) \rightarrow I$. A $G$-equivariant generalisation of this construction is discussed in [59]. Thus any extension in $G\mathbb{C}^*\text{sef}$ gives rise to an exact triangle $\text{Sus}(Q) \rightarrow I \rightarrow E \rightarrow Q$ in $\mathcal{E}^G$.

This also leads to the triangulated category structure of $\mathcal{E}^G$. As for $\mathcal{R}^G$, we can define it using mapping cone triangles or extension triangles — both approaches yield the same class of exact triangles. The canonical functor $\mathcal{R}^G \rightarrow \mathcal{E}^G$ is exact because it evidently preserves mapping cone triangles.

Now that we have two bivariant homology theories with apparently very similar formal properties, we must ask which one we should use. It may seem that the better exactness properties of $E$-theory raise it above $KK$-theory. But actually, these strong exactness properties have a drawback: for a general group $G$, the reduced crossed product functor need not be exact, so that there is no guarantee that it extends to a functor $\mathcal{E}^G \rightarrow \mathcal{E}$. Only full crossed products exist for all groups by Proposition 6; the construction of $G \times \omega: \mathcal{E}^G \rightarrow \mathcal{E}$ is the same as in $KK$-theory. Similar problems occur with $\otimes$-algebras $[4]$. We shall compare the approach of Davis–Lück to the obvious map that forgets the additional constraints.

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Further, since $\mathcal{R}^G$ has a weaker universal property, it acts on more functors, so that results about $\mathcal{R}^G$ have stronger consequences. A good example of a functor that is split-exact but probably not exact is $\text{local cyclic cohomology}$ (see [37, 45]). Therefore, the best practice seems to prove results in $\mathcal{R}^G$ if possible.

Many applications can be done with either $E$ or $KK$, we hardly notice any difference. An explanation for this is the work of Houghton–Larsen and Thomsen [27, 59], which describes $KK^G(A, B)$ in the framework of asymptotic morphisms. Recall that asymptotic morphisms $A \rightarrow B$ generate extensions $\text{Cone}(B) \rightarrow E \rightarrow A$. If this extension is $G$-equivariantly cp-split, then the projection map $E \rightarrow A$ is a $KK^G$-equivalence. A $G$-equivariant completely positive contractive section for the extension exists if and only if we can represent our asymptotic morphism by a continuous family of $G$-equivariant, completely positive contractions $f_t: A \rightarrow B$, $t \in [0, \infty)$.

**Definition 63.** Let $[A, B]_{cp}$ be the set of homotopy classes of asymptotic morphisms from $A$ to $B$ that can be lifted to a $G$-equivariant, completely positive, contractive map $A \rightarrow C^*_\text{red}(A, B)$; of course, we only use homotopies with the same kind of lifting.

**Theorem 64 ([59]).** There are natural bijections

$$ KK^G_0(A, B) \cong \left[ \text{Sus}(A_G \otimes K(\ell^2\mathbb{N})), \text{Sus}(B_G \otimes K(\ell^2\mathbb{N})) \right]_{cp} $$

for all separable $G$-$C^*$-algebras $A, B$; the canonical functor $\mathcal{R}^G \rightarrow \mathcal{E}^G$ corresponds to the obvious map that forgets the additional constraints.

**Corollary 65.** If $A$ is a nuclear $C^*$-algebra, then $KK_*(A, B) \cong E_*(A, B)$.

**Proof.** The Effros–Choi Lifting Theorem asserts that any extension of $A$ has a completely positive contractive section. \hfill \square

In the equivariant case, the same argument yields $KK^G_0(A, B) \cong E^G_0(A, B)$ if $A$ is nuclear and $G$ acts properly on $A$ (see also [55]). It should be possible to weaken properness to amenability here, but I am not aware of a reference for this.

5. The Baum–Connes assembly map for spaces and operator algebras

The Baum–Connes conjecture is a guess for the $K$-theory $K_*(C^*\text{red}(G))$ of reduced group $C^*$-algebras [1]. We shall compare the approach of Davis and Lück [14] using homotopy theory for $G$-spaces and its counterpart in bivariant $K$-theory.
formulated in [38]. To avoid technical difficulties, we assume that the group \( G \) is discrete.

The first step in the Davis–Lück approach is to embed the groups of interest such as \( K_*(C^*_{\text{red}}G) \) in a \( G \)-homology theory, that is, a homology theory on the category of (spectra of) \( G \)-CW-complexes. For the Baum–Connes assembly map we need a homology theory for \( G \)-CW-complexes with \( F_*(G/H) \cong K_*(C^*_{\text{red}}H) \). This amounts to finding a \( G \)-equivariant spectrum with appropriate homotopy groups \([14]\) and is the most difficult part of the construction. Other interesting invariants like the algebraic \( K \) - and \( L \)-theory of group rings can be treated using other spectra instead.

In the world of \( C^* \)-algebras, we cannot treat algebraic \( K \)- and \( L \)-theory; but we have much better tools to study \( K_*(C^*_{\text{red}}G) \). We do not need a \( G \)-homology theory but a homological functor on the triangulated category \( \text{mod}^G \). More precisely, we need a homological functor that takes the value \( K_*(C^*_{\text{red}}H) \) on \( C_0(G/H) \) for all subgroups \( H \). The functor \( A \mapsto K_*(C^*_{\text{red}}H) \) works fine here because \( G \rtimes C_0(G/H,A) \) is Morita–Rieffel equivalent to \( H \rtimes_r A \) for any \( H \)-\( C^* \)-algebra \( A \). The corresponding assertion for full crossed products is known as Green’s Imprimitivity Theorem; reduced crossed products can be handled similarly. Thus a topological approach to the Baum–Connes conjecture forces us to consider \( K_*(G \rtimes_r A) \) for all \( G \)-\( C^* \)-algebras \( A \), which leads to the Baum–Connes conjecture with coefficients.

5.1. Assembly maps via homotopy theory. Recall that a homology theory on pointed \( G \)-CW-complexes is determined by its value on \( S^0 \). Similarly, a \( G \)-homology theory \( F \) is determined by its values \( F_*(G/H) \) on homogeneous spaces for all subgroups \( H \subseteq G \). This does not help much because these groups — which are \( K_*(C^*_{\text{red}}H) \) in the case of interest — are very hard to compute.

The idea behind assembly maps is to approximate a given homology theory by a simpler one that only depends on \( F_*(G/H) \) for \( H \in F \) for some family of subgroups \( F \). The Baum–Connes assembly map uses the family of finite subgroups here; other families like virtually cyclic subgroups appear in isomorphism conjectures for other homology theories. We now fix a family of subgroups \( F \), which we assume to be closed under conjugation and subgroups.

A \( G,F \)-CW-complex is a \( G \)-CW-complex in which the stabilisers of cells belong to \( F \). The universal \( G,F \)-CW-complex is a \( G,F \)-CW-complex \( E(G,F) \) with the property that, for any \( G,F \)-CW-complex \( X \) there is a \( G \)-map \( X \to E(G,F) \), which is unique up to \( G \)-homotopy. This universal property determines \( E(G,F) \) uniquely up to \( G \)-homotopy. It is easy to see that \( E(G,F) \) is \( H \)-equivariantly contractible for any \( H \in F \). Conversely, a \( G,F \)-CW-complex with this property is universal.

Example 66. Let \( G = \mathbb{Z} \) and let \( F \) be the family consisting only of the trivial subgroup; this agrees with the family of finite subgroups because \( G \) is torsion-free. A \( G,F \)-CW-complex is essentially the same as a \( G \)-CW-complex with a free cellular action of \( \mathbb{Z} \). It is easy to check that \( \mathbb{R} \) with the action of \( \mathbb{Z} \) by translation and the usual cell decomposition is a universal \( G,F \)-CW-complex.

Given any \( G \)-CW-complex \( X \), the canonical map \( E(G,F) \times X \to X \) has the following properties:

- \( E(G,F) \times X \) is a \( G,F \)-CW-complex;
- if \( Y \) is a \( G,F \)-CW-complex, then any \( G \)-map \( Y \to X \) lifts uniquely up to \( G \)-homotopy to a map \( Y \to E(G,F) \times X \);
- for any \( H \in F \), the map \( E(G,F) \times X \to X \) becomes a homotopy equivalence in the category of \( H \)-spaces.

The first two properties make precise in what sense \( E(G,F) \times X \) is the best approximation to \( X \) among \( G,F \)-CW-complexes.
Definition 67. The assembly map with respect to \( F \) is the map \( F_\ast(\mathcal{E}(G,F)) \to F_\ast(\ast) \) induced by the constant map \( \mathcal{E}(G,F) = \mathcal{E}(G,F) \times \ast \to \ast \).

More generally, the assembly map with coefficients in a pointed \( G \)-CW-complex (or spectrum) \( X \) is the map \( F_\ast(\mathcal{E}(G,F)_+ \wedge X) \to F_\ast(S^0 \wedge X) = F_\ast(X) \) induced by the map \( \mathcal{E}(G,F)_+ \to \ast_+ = S^0 \).

In the stable homotopy category of pointed \( G \)-CW-complexes (or spectra), we get an exact triangle \( \mathcal{E}(G,F)_+ \wedge X \to X \to N \to S^1 \wedge \mathcal{E}(G,F)_+ \wedge X \), where \( N \) is \( H \)-equivariantly contractible for each \( H \in \mathcal{F} \). This means that the domain of the assembly map \( F_\ast(\mathcal{E}(G,F)_+ \wedge X) \) is the localisation of \( F_\ast \) at the class of all objects that are \( H \)-equivariantly contractible for each \( H \in \mathcal{F} \).

Thus the assembly map is an isomorphism for all \( X \) if and only if \( F_\ast(N) = 0 \) whenever \( N \) is \( H \)-equivariantly contractible for each \( H \in \mathcal{F} \). Thus an isomorphism conjecture can be interpreted in two equivalent ways. First, it says that we can reconstruct the homology theory from its restriction to \( G, F \)-CW-complexes. Secondly, it says that the homology theory vanishes for spaces that are \( H \)-equivariantly contractible for \( H \in \mathcal{F} \).

5.2. From spaces to operator algebras. We can carry over the construction of assembly maps above to bivariant Kasparov theory; we continue to assume \( G \) discrete to simplify some statements. From now on, we let \( \mathcal{F} \) be the family of finite subgroups. This is the family that appears in the Baum–Connes assembly map. Other families of subgroups can also be treated, but some proofs have to be modified and are not yet written down.

First we need an analogue of \( G,F \)-CW-complexes. These are constructible out of simpler “cells” which we describe first, using the induction functors

\[
\text{Ind}_{G/H}^G : \mathcal{R} R^H \to \mathcal{R} R^G
\]

for subgroups \( H \subseteq G \). For a finite group \( H \), \( \text{Ind}_{G/H}^G(A) \) is the \( H \)-fixed point algebra of \( \mathcal{C}_0(G,A) \), where \( H \) acts by \( h \cdot f(g) = \alpha_h(f(gh)) \). For infinite \( H \), we have

\[
\text{Ind}_{G/H}^G(A) = \{ f \in \mathcal{C}_0(G,A) \mid \alpha_h f(gh) = f(g) \text{ for all } g \in G, h \in H, \text{ and } gH \mapsto \|f(g)\| \text{ is in } \mathcal{C}_0(G/H) \};
\]

the group \( G \) acts by translations on the left.

This construction is functorial for equivariant \( * \)-homomorphisms. Since it commutes with \( C^\ast \)-stabilisations and maps split extensions again to split extensions, it descends to a functor \( \mathcal{R} R^H \to \mathcal{R} R^G \) by the universal property (compare [4.1]).

We also have the more trivial restriction functors \( \text{Res}_H^G : \mathcal{R} R^G \to \mathcal{R} R^H \) for subgroups \( H \subseteq G \). The induction and restriction functors are adjoint:

\[
\text{KK}^G(\text{Ind}_{G/H}^G(A,B)) \cong \text{KK}^H(A,\text{Res}_H^G B)
\]

for all \( A \in \mathcal{R} R^G \); this can be proved like the similar adjointness statements in [4.1] using the embedding \( A \to \text{Res}_H^G \text{Ind}_{G/H}^G(A) \) as functions supported on \( H \subseteq G \) and the correspondence \( \text{Ind}_{G/H}^G \text{Res}_H^G(A) \cong \mathcal{C}_0(G/H,A) \to \mathbb{K}(\mathcal{L}^2 G/H) \otimes A \sim A \). It is important here that \( H \subseteq G \) is an open subgroup. By the way, if \( H \subseteq G \) is a cocompact subgroup (which means finite index in the discrete case), then \( \text{Res}_H^G \) is the left-adjoint of \( \text{Ind}_{G/H}^G \) instead.

Definition 68. We let \( \mathcal{C} T \) be the subcategory of all objects of \( \mathcal{R} R^G \) of the form \( \text{Ind}_{G/H}^G(A) \) for \( A \in \mathcal{R} R^H \) and \( H \in \mathcal{F} \). Let \( \langle \mathcal{C} T \rangle \) be the smallest class in \( \mathcal{R} R^H \) that contains \( \mathcal{C} T \) and is closed under \( \text{KK}^G \)-equivalence, countable direct sums, suspensions, and exact triangles.
Equivalently, \( \langle CT \rangle \) is the localising subcategory generated by \( CT \). This is our substitute for the category of \((G,F)\)-CW-complexes.

**Definition 69.** Let \( CC \) be the class of all objects of \( \mathfrak{R}^G \) with \( \text{Res}_H^G(A) \cong 0 \) for all \( H \in F \).

**Theorem 70.** If \( P \in \langle CT \rangle \), \( N \in CC \), then \( \text{KK}^G(P,N) = 0 \). Furthermore, for any \( A \in \mathfrak{R}^G \), there is an exact triangle \( P \to A \to \Sigma P \) with \( P \in \langle CT \rangle \), \( N \in CC \).

Definitions 68–69 and Theorem 70 are taken from [38]. The map \( F_*(P) \to F_*(A) \) for a functor \( F: \mathfrak{R}^G \to \mathfrak{C} \) is analogous to the assembly map in Definition 67 and deserves to be called the Baum–Connes assembly map for \( F \).

We can use the tensor product in \( \mathfrak{R}^G \) to simplify the proof of Theorem 70 once we have a triangle \( P_C \to C \to N_C \to \Sigma P_C \) with \( P_C \in \langle CT \rangle \), \( N_C \in CC \), then

\[
A \otimes P_C \to A \otimes C \to A \otimes N_C \to \Sigma A \otimes P_C
\]

is an exact triangle with similar properties for \( A \). It makes no difference whether we use \( \otimes_{\text{min}} \) or \( \otimes_{\text{max}} \) here. The map \( P_C \to C \) in \( \text{KK}^G(P_C,C) \) is analogous to the map \( \mathcal{E}(G,F) \to \ast \). It is also called a Dirac morphism for \( G \) because the K-homology classes of Dirac operators on smooth spin manifolds provided the first important examples [31].

The two assembly map constructions with spaces and \( C^* \)-algebras are not just analogous but provide the same Baum–Connes assembly map. To see this, we must understand the passage from the homotopy category of spaces to \( \mathfrak{R} \). Usually, we map spaces to operator algebras using the commutative \( C^* \)-algebra \( C_0(X) \). But this construction is only functorial for proper continuous maps, and the functoriality is contravariant. The assembly map for, say, \( G = \mathbb{Z} \) is related to the non-proper map \( p: \mathbb{R} \to \ast \), which does not induce a map \( C \to C_0(\mathbb{R}) \); even if it did, this map would still go in the wrong direction. The wrong-way functoriality in KK provides an element \( p_1 \in \text{KK}_1(C_0(\mathbb{R}),C) \) instead, which is the desired Dirac morphism up to a shift in the grading. This construction only applies to manifolds with a Spin\(^c\)-structure, but it can be generalised as follows.

On the level of Kasparov theory, we can define another functor from suitable spaces to \( \mathfrak{R} \) that is a covariant functor for all continuous maps. The definition uses a notion of duality due to Kasparov [31] that is studied more systematically in [17]. It requires yet another version \( \text{RKK}^G_0(X;A,B) \) of Kasparov theory that is defined for a locally compact space \( X \) and two \( G \)-\( C^* \)-algebras \( A \) and \( B \). Roughly speaking, the cycles for this theory are \( G \)-equivariant families of cycles for \( K(X,Y) \) parametrised by \( X \). The groups \( \text{RKK}^G_0(X;A,B) \) are contravariantly functorial and homotopy invariant in \( X \) (for \( G \)-equivariant continuous maps).

We have \( \text{RKK}^G_0(\ast;A,B) = \text{KK}^G_0(A,B) \) and, more generally, \( \text{RKK}^G_0(X;A,B) \cong \text{KK}^G_0(A,\mathcal{C}(X,B)) \) if \( X \) is compact. The same statement holds for non-compact \( X \), but the algebra \( \mathcal{C}(X,B) \) is not a \( C^* \)-algebra any more: it is an inverse system of \( C^* \)-algebras.

**Definition 71** ([17]). A \( G \)-\( C^* \)-algebra \( P_X \) is called an abstract dual for \( X \) if, for all second countable locally compact \( G \)-spaces \( Y \) and all separable \( G \)-\( C^* \)-algebras \( A \) and \( B \), there are natural isomorphisms

\[
\text{RKK}^G(X \times Y;A,B) \cong \text{RKK}^G(Y;P_X \otimes A,B)
\]

that are compatible with tensor products.

Abstract duals exist for many spaces. For trivial reasons, \( C \) is an abstract dual for the one-point space. For a smooth manifold \( X \) with an isometric action of \( G \), both \( C_0(T^*X) \) and the algebra of \( C_0 \)-sections of the Clifford algebra bundle on \( X \) are
abstract duals for $X$; if $X$ has a $G$-equivariant Spin$^c$-structure — as in the example of $\mathbb{Z}$ acting on $\mathbb{R}$ — we may also use a suspension of $C_0(X)$. For a finite-dimensional simplicial complex with a simplicial action of $G$, an abstract dual is constructed by Gennadi Kasparov and Georges Skandalis in [22] and in more detail in [17]. It seems likely that the construction can be carried over to infinite-dimensional simplicial complexes as well, but this has not yet been written down.

There are also spaces with no abstract dual. A prominent example is the Cantor set: it has no abstract dual, even for trivial $G$ (see [17]).

Let $D$ be the class of all $G$-spaces that admit a dual. Recall that $X \mapsto \text{RK}^G(X \times Y; A, B)$ is a contravariant homotopy functor for continuous $G$-maps. Passing to corepresenting objects, we get a covariant homotopy functor

$$D \rightarrow \text{RK}^G, \quad X \mapsto P_X.$$ 

This functor is very useful to translate constructions from homotopy theory to bivariant K-theory. An instance of this is the comparison of the Baum–Connes assembly maps in both setups:

**Theorem 72.** Let $\mathcal{F}$ be the family of finite subgroups of a discrete group $G$, and let $\mathcal{E}(G, \mathcal{F})$ be the universal $(G, \mathcal{F})$-CW-complex. Then $\mathcal{E}(G, \mathcal{F})$ has an abstract dual $P$, and the map $\mathcal{E}(G, \mathcal{F}) \mapsto P$ induces a Dirac morphism in $\text{KK}^G(P, \mathbb{C})$.

Theorem 72 should hold for all families of subgroups $\mathcal{F}$, but only the above special case is treated in [17].

### 5.3. The Dirac-dual-Dirac method and geometry.

Let us compare the approaches in §5.1 and §5.2. The bad thing about the $C^*$-algebraic approach is that it applies to fewer theories. The good thing about it is that KASPAROV theory is so flexible that any canonical map between K-theory groups has a fair chance to come from a morphism in $\text{RK}^G$ which we can construct explicitly.

For some groups, the Dirac morphism in $\text{KK}^G(P, \mathbb{C})$ is a KK-equivalence:

**Theorem 73 (Higson–Kasparov [23]).** Let the group $G$ be amenable or, more generally, a-T-menable. Then the Dirac morphism for $G$ is a KK$^G$-equivalence, so that $G$ satisfies the Baum–Connes conjecture with coefficients.

The class of groups for which the Dirac morphism has a one-sided inverse is even larger. This is the point of the Dirac-dual-Dirac method. The following definition in [38] is based on a simplification of this method:

**Definition 74.** A dual Dirac morphism for $G$ is an element $\eta \in \text{KK}^G(\mathbb{C}, P)$ with $\eta \circ D = \text{id}_P$.

If such a dual Dirac morphism exists, then it provides a section for the assembly map $F_\ast(P \otimes A) \rightarrow F_\ast(A)$ for any functor $F: \text{RK}^G \rightarrow \mathcal{C}$ and any $A \in \text{RK}^G$, so that the assembly map is a split monomorphism. Currently, we know no group without a dual Dirac morphism. It is shown in [16][18][19] that the existence of a dual Dirac morphism is a geometric property of $G$ because it is related to the invertibility of another assembly map that only depends on the coarse geometry of $G$ (in the torsion-free case).

Instead of going into this construction, we briefly indicate another point of view that also shows that the existence of a dual Dirac morphism is a geometric issue. Let $P$ be an abstract dual for some space $X$ (like $\mathcal{E}(G, \mathcal{F})$). The duality isomorphisms in Definition 71 are determined by two pieces of data: a Dirac morphism $D \in \text{KK}^G(P, \mathbb{C})$ and a local dual Dirac morphism $\Theta \in \text{RK}^G(X; \mathbb{C}, P)$. The notation is motivated by the special case of a Spin$^c$-manifold $X$ with $P = C_0(X)$, where $D$ is the K-homology class defined by the Dirac operator and $\eta$ is defined by
a local construction involving pointwise Clifford multiplications. If $X = \mathcal{E}(G,F)$, then it turns out that $\eta \in \KK^G(\mathcal{C},P)$ is a dual Dirac morphism if and only if the canonical map $\KK^G(\mathcal{C},P) \to \RKK^G(X,\mathcal{C},P)$ maps $\eta \mapsto \Theta$. Thus the issue is to globalise the local construction of $\Theta$. This is possible if we know, say, that $X$ has non-positive curvature. This is essentially how Kasparov proves the Novikov conjecture for fundamental groups of non-positively curved smooth manifolds in [31].

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