The Second Regularized Trace of Even Order Differential Operators with Operator Coefficient

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Abstract.
In this paper, we investigate the spectrum of the self adjoint operator $L$ defined by

$$L := (-1)^r \frac{d^r}{dx^r} + A + Q(x),$$

where $A$ is a self adjoint operator, $Q(x)$ is a nuclear operator in a separable Hilbert space, and we derive asymptotic formulas for the sum of eigenvalues of the operator $L$.

1. Introduction
The theory of regularized traces of differential operators began with the study of Gelfand and Levitan [12]. They calculated the trace formula for the sum of substraction of eigenvalues of two self adjoint operators. After this primary work, many mathematicians concentrated on this theory in a large scale. Dikiy [13], Halberg and Kramer [4], Levitan [2] and some others studied the regularized traces of scalar differential operators. The list of works on the subject was given in the works Levitan and Sargsyan [3] and Fulton and Pruess [16], but a few of these works were about the regularized trace of differential operators with operator coefficient. Chalilova [15] calculated regularized trace of Sturm Liouville operator with bounded operator coefficient. Adıgüzeloğlu [5] computed regularized trace of the difference of two Sturm-Liouville operators with bounded operator coefficient given in the semi-axis. Maksudov, Bayramoglu and Adıgüzeloğlu [10] found a formula for the regularized trace of Sturm-Liouville operators with unbounded operator coefficient under the Dirichlet boundary conditions. Bayramoglu and Adıgüzeloğlu [14] obtained the regularized trace of second order singular differential operator with bounded operator coefficient. Furthermore Adıgüzeloğlu and Bakşi [6], Adıgüzeloğlu and Sezer [7, 8] and Sen, Bayramoglu and Orucoglu [9] investigated the regularized trace formulas of differential operator with operator coefficient. Although most of the previous researches on the subject dealt with regularized trace of second order differential operators, we focused on higher order differential operators. It is clear that our study advances the formulation of regularized trace that the prior manuscripts has proved. This paper aims to explore the second regularized trace of higher order differential operators with operator coefficient.
Let us begin by recalling some definitions and properties:

Let $H$ be an infinite dimensional separable Hilbert space. We denote the inner products in $H$ by $(.,.)$ and the norm in $H$ by $\|\|$.

Let $H_1 = L_2(0, \pi; H)$ denote the set of all functions $f$ from $[0, \pi]$ into $H$ which are strongly measurable and satisfy the condition $\int_0^\pi |f(x)|^2 \, dx < \infty$. The space $H_1$ is a linear space. If the inner product of arbitrary two elements $f$ and $g$ of the space $H_1$ is defined as $(f, g)_{H_1} = \int_0^\pi f(x) g(x) \, dx$, then $H_1$ becomes an infinite dimensional separable Hilbert space [1]. The norm in the space $H_1$ is denoted by $\|\|_1$.

$s_\infty(H)$ denotes the set of all compact operators from $H$ to $H$. If $T \in s_\infty(H)$, then $T^* T$ is a nonnegative self adjoint operator and $(T^* T)^{\frac{1}{2}} \in s_\infty(H)$ [11]. Let the nonzero eigenvalues of the operator $(T^* T)^{\frac{1}{2}}$ be $\{s_j\}_{j=1}^\infty$ such that $s_1 \geq s_2 \geq \ldots \geq s_k$ according to its multiplicity number. Since $(T^* T)^{\frac{1}{2}}$ is non negative, $s_k$’s are positive numbers. The numbers $s_k$ are called s-numbers of the operator $T$. If $k < \infty$, then $s_j = 0$ ($j = k + 1, k + 2, \ldots$) will be accepted. s-numbers of the operator $T$ are also denoted by $s_j(T)$ ($j = 1, 2, \ldots$). Here $s_1(T) = \|T\|_1$.

If $T$ is a normal operator, then $s_j(T) = |\lambda_j(T)|$ ($j = 1, 2, \ldots, k$) [11]. Here, $\{\lambda_1(T), \lambda_2(T), \ldots, \lambda_k(T)\}$ is an ordering of all nonzero eigenvalues of the operator $T$ according to $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_k(T)|$. $\sigma_p$ or $\sigma_p(H)$ denotes the set of all compact operators, the s-numbers of which satisfy the condition $\sum_{j=1}^\infty s_j(T) < \infty$ ($p \geq 1$). The set $\sigma_p$ ($p \geq 1$) is a separable Banach space with respect to the norm $\|T\|_{\sigma_p(H)} = \left[\sum_{j=1}^\infty s_j^p(T)\right]^{\frac{1}{p}}$, for $T \in \sigma_p(H)$ [11].

$s_1(H)$ is the set of all operators $T \in s_\infty(H)$, the s-numbers of which satisfy the condition $\sum_{j=1}^\infty s_j(T) < \infty$. If an operator belongs to $s_1(H)$, then it is called a nuclear operator. If the operators $T \in \sigma_p(H)$ and $B \in B(H)$, then $TB, BT \in \sigma_p(H)$ and $\|TB\|_{\sigma_p(H)} \leq \|B\|\|T\|_{\sigma_p(H)}, \|BT\|_{\sigma_p(H)} \leq \|B\|\|T\|_{\sigma_p(H)}$.

If $T$ is a nuclear operator and $\{e_j\}_{j=1}^\infty \subset H$ is any orthonormal basis, then the series $\sum_{j=1}^\infty (Te_j, e_j)$ is convergent and the sum of the series does not depend on the choice of the basis $\{e_j\}_{j=1}^\infty$. The sum of this series is said to be matrix trace of the operator $T$ denoted by $trT$. Moreover

$$trT = \sum_{j=1}^{\nu(T)} \lambda_j(T)$$

[11]. Here, each eigenvalue counted according to its algebraic multiplicity number. $\nu(T)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of the operator $T$ [11]. The sum of the series $\sum_{j=1}^{\nu(T)} \lambda_j(T)$ is called spectral trace of the operator $T$.

Now, let us return to our problem. Consider the differential expression

$$\ell_0(y) = (-1)^ry^{(2r)}(x) + Ay(x)$$

in the space $H_1 = L_2(0, \pi; H)$. Here, the densely defined operator $A : D(A) \to H$ satisfies the conditions $A = A^* \geq I$ (I is unit operator in $H$) and $A^{-1} \in s_\infty(H)$. Let $\{\gamma_n\}_{n=1}^\infty$ be an ordering of all eigenvalues of $A$ according to $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq \cdots$ and $\varphi_n$ the corresponding orthonormal eigenfunctions. Here, each eigenvalue counted according to its multiplicity number.

Let $D_0$ be a subset of the space $H_1$. A function $y(x) \in D_0$, if $y(x)$ satisfies the following conditions:

(y1) $y(x)$ has continuous derivative of the $(2r)$th-order with respect to the norm in the space $H$ for every $x \in [0, \pi]$,

(y2) $Ay(x)$ is continuous with respect to the norm of the space $H$ on $[0, \pi]$,

(y3) $y^{(r)}(0) = y^{(r)}(\pi) = \cdots = y^{(2r-1)}(0) = y^{(r)}(\pi) = \cdots = y^{(2r-2)}(\pi) = 0$ ($r = 1, 2, \ldots, m$).

Here, $D_0 = H_1$. Define the linear operator $L_0' : D_0 \to H_1$ as $L_0' y := \ell_0(y)$. 

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The construction above gives that \( L_0^j \) is symmetric. The eigenvalues of \( L_0^j \) are \((k + \frac{1}{2})^2 + \gamma_j (k = 0, 1, 2, \ldots; j = 1, 2, \ldots)\) and \( \sqrt{\gamma_j} \cos(k + \frac{1}{2})x\) the corresponding orthonormal eigenvectors.

We can see that the orthonormal eigenvector system of the symmetric operator \( L_0^j \) is an orthonormal basis in the space \( H_1 \). We denote the closure of \( L_0^j \) by \( L_0 : D(L_0) \rightarrow H_1 \). Since the orthonormal eigenvector system of the operator \( L_0^j \) is an orthonormal basis in the space \( H_1 \), \( L_0 \) is a self adjoint operator.

Let \( Q(x) \) defined on \([0, \pi]\) be an operator function satisfying the following conditions:

**Q1** \( Q(x) \) has weak derivative of \((2r + 2)\)th order and \( Q^{(2r+1)}(0) = Q^{(2r+1)}(\pi) = 0 \) \((i = 0, 1, 2, \ldots, r)\)

**Q2** \( Q^{(i)}(x) : H \rightarrow H \) \((i = 0, 1, 2, \ldots, 2r+2)\) are self-adjoint operators for every \( x \in [0, \pi] \), \( AQ''(x), Q^{(2r+2)}(x) \in \sigma_1(H) \) and the functions \( \|AQ''(x)\|_{\sigma_1(H)}, \|Q^{(2r+2)}(x)\|_{\sigma_1(H)} \) are bounded and measurable in the interval \([0, \pi]\).

Define the operator \( L : D(L_0) \rightarrow H_1 \) as follows

\[
L = L_0 + Q.
\]

The operators \( L_0 \) and \( L \) are self adjoint operators and have purely discrete spectrum [6]. We denote the resolvent sets of \( L_0, L \) by \( \rho(L_0), \rho(L) \) and the resolvent operators of \( L_0, L \) by \( R_0^\lambda = (L_0 - \lambda I)^{-1}, R_1 = (L - \lambda I)^{-1} \), respectively. Also, we denote the eigenvalues of the operators \( L_0 \) and \( L \) by \( \{\mu_n\}_1^{\infty} \) and \( \{\lambda_n\}_1^{\infty} \) satisfying the inequalities \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots \) and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \).

If \( \gamma_j \sim a^{-q} \) \((a > 0, 0 < a < \infty)\) as \( j \rightarrow \infty \), then

\[
\mu_n, \lambda_n \sim d_1 n^{\frac{2}{a^q}}, \tag{1.1}
\]

as \( n \rightarrow \infty \) [7]. Here, \( d_1 \) is a positive constant. By using the asymptotic formula (1.1), there exists a subsequence \( n_p \) of positive integers such that

\[
\mu_q - \nu_q \geq d_2 (n^{\frac{2}{a^q}} - n_p^{\frac{2}{a^q}}), \quad (q = n_p + 1, n_p + 2, \ldots)
\]

where \( d_2 \) is a positive constant.

In the work [8], the formula in the form

\[
\text{lim}_{p \rightarrow \infty} \sum_{q=1}^{n_p} \lambda_q - \mu_q - \frac{1}{\pi} \int_0^\pi \left( Q(x) \varphi_{j_q}^+, \varphi_{j_q}^- \right) dx = \frac{1}{4} \left( \text{tr}Q(0) + \text{tr}Q(\pi) \right) - \frac{1}{2\pi} \int_0^\pi \text{tr}Q(x) dx
\]

is obtained for the first regularized trace of the operator \( L \). In this present work, we find a formula in the form

\[
\text{lim}_{p \rightarrow \infty} \sum_{q=1}^{n_p} \left( \lambda_q^2 - \mu_q^2 - 2 \sum_{s=2}^{m} \left( -1 \right)^{s-1} s^{-1} \text{Re} s_{\lambda=\mu} \text{tr} \left[ \lambda (QR_0^\lambda )^s \right] - \frac{2\mu_q}{\pi} \int_0^\pi \left( Q(x) \varphi_{j_q}^+, \varphi_{j_q}^- \right) dx \right)
\]

\[
= (-1)^{s-1} 2^{-2s} \left[ \text{tr}Q^{(2s)}(0) - \text{tr}Q^{(2s)}(\pi) \right] + \frac{1}{2} \left[ \text{tr}AQ(0) - \text{tr}AQ(\pi) \right]. \tag{1.2}
\]

The left hand side of equality (1.2) is called the second regularized trace of the differential operator \( L \).

2. Main Results

The main purpose of this section is to obtain the second trace formula for the operator \( L \). Now, we find the relations between resolvents and eigenvalues of the operators \( L_0 \) and \( L \).
If $\alpha > \frac{2\pi}{3}$ and $\lambda \neq \lambda_q, \mu_q$ ($q = 1, 2, \ldots$), then by (2.1), $R_0^0$ and $R_3$ are trace class operators. Hence

$$tr(R_3 - R_3^0) = trR_3 - trR_3^0 = \sum_{q=1}^{\infty} \left( \frac{1}{\lambda_q} - \frac{1}{\mu_q} \right).$$

If this equality is multiply with $\frac{\lambda^2}{2\pi i}$ and integrated on the circle $|\lambda| = b_p = \frac{1}{2}(\mu_{p+n} + \mu_{n+1})$, then we have

$$\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr(R_3 - R_3^0)d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{q=1}^{\infty} \left( \frac{\lambda^2}{\lambda_q} - \frac{\lambda^2}{\mu_q} \right)d\lambda - \frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{q=1}^{\infty} \left( \frac{\lambda^2}{\mu_q} - \lambda \right)d\lambda. \quad (2.1)$$

We can see that for the large values of $p$,

$$\{\lambda_q, \mu_q\} \in B(0, b_p) = \{\lambda : |\lambda| < b_p\}$$

$$\lambda_q, \mu_q \not\in B[0, b_q] = \{\lambda : |\lambda| \leq b_p\} \quad (q \geq n_p + 1).$$

Therefore by (2.1), we have

$$\sum_{q=1}^{n_p} (\lambda^2_q - \mu^2_q) = -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr(R_3 - R_3^0)d\lambda. \quad (2.2)$$

This is well known formula for the resolvents of the operators $L_0$ and $L$:

$$R_3 = R_3^0 - R_3 QR_3 \quad (\lambda \in \rho(L) \cap \rho(L_0)).$$

By using the last formula, we obtain

$$R_3 - R_3^0 = \sum_{s=1}^{m} (-1)^s R_3^0 (QR_3^0)^s + (-1)^{m+1} R_3 (QR_3^0)^{m+1},$$

for every positive integer $m$. By (2.2) and the last equality, we have

$$\sum_{q=1}^{n_p} (\lambda^2_q - \mu^2_q) = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr\left( \sum_{s=1}^{m} (-1)^s R_3^0 (QR_3^0)^s + (-1)^{m+1} R_3 (QR_3^0)^{m+1} \right)d\lambda$$

or

$$\sum_{q=1}^{n_p} (\lambda^2_q - \mu^2_q) = \sum_{s=1}^{m} D_{ps} + D^{(m)}_{p}, \quad (2.3)$$

Here,

$$D_{ps} = \frac{(-1)^{s+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr(R_3^0 (QR_3^0)^s)d\lambda, \quad (s = 1, 2, \ldots) \quad (2.4)$$

$$D^{(m)}_{p} = \frac{(-1)^{m}}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 tr(R_3 (QR_3^0)^{m+1})d\lambda. \quad (2.5)$$
Theorem 2.1. If $\gamma_j \sim a^j$ ($0 < a, \alpha > \frac{2r}{\pi} - 1$) as $j \to \infty$, then
\[
D_p = \frac{(-1)^{y}}{\pi is} \int_{|\lambda|=\beta_{y}} \lambda \text{tr}((QR_{\lambda}^{0})^{y})d\lambda \quad (s = 1, 2, \ldots).
\]

Theorem 2.2. If the operator function $Q(x)$ satisfies the conditions (Q1) and (Q2), then the series
\[
\sum_{j=0}^{\infty} \sum_{j=1}^{\infty} \left(\frac{k + \frac{1}{2}}{2^{j} + \frac{1}{2}} + \gamma\right) \int_{0}^{\infty} (Q(x)\psi_{j}, \phi_{j}) \cos((2k + 1)x)dx
\]
is absolutely convergent.

We are at the position to give the main result:

Theorem 2.3. If the operator function $Q(x)$ satisfies the conditions (Q1), (Q2), and $\gamma_j \sim a^j$ as $j \to \infty$ ($a > 0, \frac{2r}{\pi} - 1 < \alpha$), then we have
\[
\lim_{p \to \infty} \sum_{q=1}^{m} \left(\lambda_{q}^{2} - \mu_{q}^{2} - 2 \sum_{s=2}^{m} (-1)^{s-1} \text{Res}_{\lambda=\mu_{s}} \text{tr}\left(\lambda(QR_{\lambda}^{0})^{y}\right) - \frac{2\mu_{q}}{\pi} \int_{0}^{\infty} (Q(x)\psi_{j}, \phi_{j})dx\right)
\]
\[
= (-1)^{y} 2^{-1-2r} (\text{tr}Q^{2r}(0) - \text{tr}Q^{2r}(\pi)) + \frac{1}{2} (\text{tr}AQ(0) - \text{tr}AQ(\pi)).
\]
where $m = \left\lceil \frac{2\alpha + 6x + 3\lambda}{2\alpha - 6x - 3\lambda} \right\rceil$. Here, $\lfloor \cdot \rfloor$ shows the greatest integer function whose value at any number $x$ is the greatest integer less than or equal to $x$.

3. Proofs

Proof of Theorem 1. We can show that the operator function $(QR_{\lambda}^{0})^{y}$ is analytic with respect to the norm in the space $\sigma_{1}(H_{1})$ in the region $\rho(L_{0})$ and
\[
\text{tr}([QR_{\lambda}^{0}]^{y}) = \text{str}([QR_{\lambda}^{0}]^{y}(QR_{\lambda}^{0})^{y-1}) = (QR_{\lambda}^{0})^{y} = Q(R_{\lambda}^{0})^{2}.
\]
Therefore, we have
\[
\text{tr}([QR_{\lambda}^{0}]^{y}) = \text{str}(R_{\lambda}^{0}(QR_{\lambda}^{0})^{y}).
\]
From (2.4) and the last equality we obtain
\[
D_{p} = \frac{(-1)^{y+1}}{2\pi is} \int_{|\lambda|=\beta_{y}} \lambda^{2} \text{tr}\left([QR_{\lambda}^{0}]^{y}\right)d\lambda.
\]
We can also write the last formula in the following form:
\[
D_{p} = \frac{(-1)^{y+1}}{2\pi is} \int_{|\lambda|=\beta_{y}} \text{tr}\left([\lambda^{2}(QR_{\lambda}^{0})^{y}] - 2\lambda(QR_{\lambda}^{0})^{y}\right)d\lambda
\]
\[
= \frac{(-1)^{y}}{\pi is} \int_{|\lambda|=\beta_{y}} \text{tr}(QR_{\lambda}^{0})d\lambda + \frac{(-1)^{y+1}}{2\pi is} \int_{|\lambda|=\beta_{y}} \text{tr}\left([\lambda^{2}(QR_{\lambda}^{0})^{y}]\right)d\lambda.
\]
We can see

\[
\int_{|\lambda|=b_p} \text{tr}\left(\lambda^2 (QR^0_\lambda y)\right) d\lambda = \int_{|\lambda|=b_p} \left[\text{tr}\left(\lambda^2 (QR^0_\lambda y)\right)\right] d\lambda. \tag{3.2}
\]

We write the right hand side of above equality in the following way:

\[
\int_{|\lambda|=b_p} \left[\text{tr}\left(\lambda^2 (QR^0_\lambda y)\right)\right] d\lambda = \int_{|\lambda|=b_p} \left[\text{tr}\left(\lambda^2 (QR^0_\lambda y)\right)\right] d\lambda + \int_{|\lambda|=b_p} \left[\text{tr}\left(\lambda^2 (QR^0_\lambda y)\right)\right] d\lambda. \tag{3.3}
\]

Let \( \varepsilon_0 \) be a positive number satisfying the inequality \( b_p + \varepsilon_0 < \mu_{n+1} \).

We consider that the function \( \text{tr}\left(\lambda^2 (QR^0_\lambda y)\right) \) is analytic in the simply connected regions:

\[ G_1 = \{ \lambda : b_p - \varepsilon_0 < |\lambda| < b_p + \varepsilon_0, \quad \text{Im}\lambda > -\varepsilon_0 \} \]

\[ G_2 = \{ \lambda : b_p - \varepsilon_0 < |\lambda| < b_p + \varepsilon_0, \quad \text{Im}\lambda < \varepsilon_0 \} \]

and

\[ \{ \lambda : |\lambda| = b_p, \quad \text{Im}\lambda \geq 0 \} \subset G_1, \]

\[ \{ \lambda : |\lambda| = b_p, \quad \text{Im}\lambda \leq 0 \} \subset G_2. \]

By using the Leibnitz Formula and (3.3), we get

\[
\int_{|\lambda|=b_p} \left[\text{tr}\left(\lambda^2 (QR^0_\lambda y)\right)\right] d\lambda = \text{tr}\left(b_p^2(QR^0_{b_p} y)^*\right) - \text{tr}\left(b_p^2(QR^0_{b_p} y)^*\right) + \text{tr}\left(b_p^2(QR^0_{b_p} y)^*\right) - \text{tr}\left(b_p^2(QR^0_{b_p} y)^*\right) = 0. \tag{3.4}
\]

From (3.1), (3.2) and (3.4), we have

\[ D_{ps} = \frac{(-1)^y}{\pi i s} \int_{|\lambda|=b_p} \text{tr}(QR^0_\lambda y)d\lambda. \]

**Proof of Theorem 2.** Let \( h_j(x) = (Q(x) \varphi_j \varphi_j) \). Using the integration by parts formula and the condition (Q1),
we get
\[
\int_0^\pi h_j(x) \cos((2k + 1)x) dx = \int_0^\pi h_j(x) \left( \frac{1}{2k + 1} \sin((2k + 1)x) \right) dx
\]
\[
= \frac{1}{2k + 1} \left[ h_j(x) \sin((2k + 1)x) \right]_0^\pi - \int_0^\pi h_j'(x) \sin((2k + 1)x) dx
\]
\[
= \frac{1}{(2k + 1)^2} \int_0^\pi h_j'(x) \cos((2k + 1)x) dx
\]
\[
= \frac{1}{(2k + 1)^2} \left[ h_j'(x) \sin((2k + 1)x) \right]_0^\pi - \int_0^\pi h_j''(x) \cos((2k + 1)x) dx
\]
\[
= -\frac{1}{(2k + 1)^3} \int_0^\pi h_j''(x) \sin((2k + 1)x) dx
\]
\[
= \ldots = \frac{(-1)^{k+1}}{(2k + 1)^{2k+2}} \int_0^\pi h_j^{(2k+2)}(x) \cos((2k + 1)x) dx. \tag{3.5}
\]

By (3.3), we find
\[
\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left\{ \left( k + \frac{1}{2} \right)^{2r} + \gamma_j \right\} \int_0^\pi h_j(x) \cos((2k + 1)x) dx
\]
\[
\leq \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} (2k + 1)^{-2} \int_0^\pi \left( |h_j^{(2k+2)}(x)| + \gamma_j |h_j''(x)| \right) dx
\]
\[
= \sum_{j=1}^{\infty} \int_0^\pi \left( \sum_{k=0}^{\infty} |Q^{(2k+2)}(x)\varphi_j, \varphi_j)| + \sum_{k=0}^{\infty} |AQ''(x)\varphi_j, \varphi_j)| \right) dx \sum_{k=0}^{\infty} (2k + 1)^{-2}
\]
\[
\leq \text{Const.} \int_0^\pi \left( \sum_{j=1}^{\infty} |Q^{(2k+2)}(x)\varphi_j, \varphi_j)| + \sum_{k=0}^{\infty} |AQ''(x)\varphi_j, \varphi_j)| \right) dx
\]
\[
\leq \text{Const.} \int_0^\pi \left( \|Q^{(2k+2)}(x)\|_{L_1} + \|AQ''(x)\|_{L_1} \right) dx. \tag{3.6}
\]

Since the functions \( |Q^{(2k+2)}(x)\|_{L_1} \) and \( |AQ''(x)\|_{L_1} \) in (3.6) are measurable and bounded in the interval \([0, \pi]\), we get
\[
\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left| \left( k + \frac{1}{2} \right)^{2r} + \gamma_j \right| \int_0^\pi (Q(x)\varphi_j, \varphi_j) \cos((2k + 1)x) dx < \infty. \quad \square
\]

Let \( \{|\psi_{\mu}\|_{L_1}^{(n)} \) be the orthonormal eigenvectors system corresponding to eigenvalues \( \{|\mu_{i_j}\|_{L_1} \) of the operator \( L_0 \), respectively. Since the orthonormal eigenvectors corresponding to eigenvalues \( (k + \frac{1}{2})^{2r} + \gamma_j \) (k =
0, 1, 2, \cdots; j = 1, 2, \cdots) of the operator $L_0$ are $\sqrt{\frac{2}{\pi}} \cos((k + \frac{1}{2})x)\varphi_j$, respectively,

$$\mu_q = (k_q + \frac{1}{2})^2 + \gamma_j$$

and

$$\psi_q(x) = \sqrt{\frac{2}{\pi}} \cos((k_q + \frac{1}{2})x)\varphi_j.$$  \hspace{1cm} (3.7)

We prove the main theorem of the paper.

**Proof of Theorem 3.**

By using the Theorem 1, one can write $D_{ps}$ as follows:

$$D_{ps} = 2(-1)^s^{-1} \frac{1}{2\pi i} \int_{|\lambda| = b_p} \text{tr}(\lambda(QR^0_{\lambda})^t) d\lambda$$

By using last formula, we can rewrite (2.3) as follows:

$$\sum_{q=1}^{n_p} \left( \lambda^2_q - \mu^2_q - 2 \sum_{s=2}^{m} (-1)^s \text{Res}_{\lambda=\mu_q} \text{tr}(\lambda(QR^0_{\lambda})^t) \right) = D_{p1} + D_{p(m)},$$  \hspace{1cm} (3.8)

$$D_{p1} = \frac{-1}{\pi i} \int_{|\lambda| = b_p} \lambda \text{tr}(QR^0_{\lambda}) d\lambda.$$  \hspace{1cm} (3.9)

Since $(QR^0_{\lambda})$ is a nuclear operator for every $\lambda \in \rho(L_0)$ and $|\psi_q|_1^\infty$ is an orthonormal basis in the space $H_1$, we have

$$\text{tr}(QR^0_{\lambda}) = \sum_{q=1}^{\infty} (QR^0_{\lambda}\psi_q, \psi_q)_{H_1}.$$ 

Here, $R^0_{\lambda}\psi_q = (\mu_q - \lambda)^{-1}\psi_q$.

If we substitute the last two equalities into (3.9), then we get

$$D_{p1} = \frac{-1}{\pi i} \int_{|\lambda| = b_p} \lambda \sum_{q=1}^{\infty} (QR^0_{\lambda}\psi_q, \psi_q)_{H_1} d\lambda$$

$$= \frac{-1}{\pi i} \int_{|\lambda| = b_p} \lambda \sum_{q=1}^{\infty} \frac{1}{\mu_q - \lambda} (Q\psi_q, \psi_q)_{H_1} d\lambda$$

$$= \frac{1}{\pi i} \sum_{q=1}^{\infty} (Q\psi_q, \psi_q)_{H_1} \int_{|\lambda| = b_p} \frac{\lambda}{\lambda - \mu_q} d\lambda$$

By using the Cauchy Integral Formula

$$\frac{1}{2\pi i} \int_{|\lambda| = b_p} \frac{\lambda}{\lambda - \mu_q} d\lambda = \begin{cases} \mu_q & \text{if } q \leq n_p \\ 0 & \text{if } q > n_p \end{cases}$$
by (3.7), we obtain

\[ D_{p1} = 2 \sum_{q=1}^{n_p} \mu_q \langle Q \psi_q, \psi_q \rangle \nu_i, \]

\[ = 2 \sum_{q=1}^{n_p} \mu_q \int_0^{\pi} (Q(x) \psi_q(x), \psi_q(x)) dx \]

\[ = 2 \sum_{q=1}^{n_p} \mu_q \frac{2}{\pi} \int_0^{\pi} \cos(\frac{k_q + \frac{1}{2}}{2} x \psi_{j_q}) dx \]

\[ = \sum_{q=1}^{n_p} \mu_q \frac{2}{\pi} \int_0^{\pi} \cos(2k_q + 1)x \psi_{j_q} dx \]

\[ = \frac{2}{\pi} \sum_{q=1}^{n_p} \mu_q \int_0^{\pi} \cos(2k_q + 1)x \psi_{j_q} dx \]

We substitute (3.10) in (3.8):

\[ \sum_{q=1}^{n_p} \left( \lambda_q^2 - \mu_q^2 - 2 \sum_{r=2}^{m} (-1)^r s^{-1} \text{Res}_{z=\pm \mu} \lambda (Q R_{\lambda}^s) \right) \]

\[ = \frac{2}{\pi} \sum_{q=1}^{n_p} \mu_q \int_0^{\pi} h_{j_q}(x) \cos(2k_q + 1)x dx \]

If we use Theorem 2, then we know that

\[ \lim_{p \to \infty} \frac{2}{\pi} \sum_{q=1}^{n_p} \mu_q \int_0^{\pi} h_{j_q}(x) \cos(2k_q + 1)x dx \]

\[ = \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left( (k + \frac{1}{2})^{2r} + y_j \right) \int_0^{\pi} h_j(x) \cos(2k + 1)x dx. \]
\[ \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left[ (k + \frac{1}{2})^2 + \gamma_j \right] \int_{0}^{\pi} h_j(x) \cos(2k + 1) x dx \]

\[ = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{2}{\pi} \int_{0}^{\pi} \left[ -\frac{1}{4} \gamma_j (h_j^{(2)})(x) + \gamma_j h_j(x) \right] \cos(2k + 1) x dx \]

\[ = \frac{1}{\pi} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \left( \int_{0}^{\pi} \left[ -\frac{1}{4} \gamma_j (h_j^{(2)})(x) + \gamma_j h_j(x) \right] \cos(kx) dx \right) \]

\[ - (-1)^k \int_{0}^{\pi} \left[ -\frac{1}{4} \gamma_j (h_j^{(2)})(x) + \gamma_j h_j(x) \right] \cos(kx) dx \right). \]

The sums according to the \( k \) on the right hand side of the last relation are the values at 0 and \( \pi \) of the Fourier Series of the function \((-\frac{1}{4}) (h_j^{(2)})(x) + \gamma_j h_j(x)\) according to the functions \{ cos(kx) \}_{k=0}^{\infty} on the interval \([0, \pi]\).

Therefore,

\[ \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left[ (k + \frac{1}{2})^2 + \gamma_j \right] \int_{0}^{\pi} h_j(x) \cos(2k + 1) x dx \]

\[ = \frac{1}{2} \sum_{j=1}^{\infty} \left[ \left( \frac{-1}{4} \gamma_j (h_j^{(2)})(0) - h_j^{(2)}(\pi) \right) + \gamma_j (h_j(0) - h_j(\pi)) \right] \]

\[ = (-1)^2 \frac{1}{2} \left[ trQ^{(2)}(0) - trQ^{(2)}(\pi) \right] + \frac{1}{2} \left[ trAQ(0) - trAQ(\pi) \right] \quad (3.13) \]

From (3.12) and (3.13), we obtain

\[ \frac{2}{\pi} \lim_{p \to \infty} \sum_{q=1}^{n_p} \mu_q \int_{0}^{\pi} h_j(x) \cos(2k_q + 1) x dx \]

\[ = (-1)^2 \frac{1}{2} \left[ trQ^{(2)}(0) - trQ^{(2)}(\pi) \right] + \frac{1}{2} \left[ trAQ(0) - trAQ(\pi) \right] \quad (3.14) \]

Let us estimate of \( D_p^{(m)} \) for the large value of \( p \). By using (2.5), we get

\[ \left| D_p^{(m)} \right| \leq \int_{|\lambda|=b_p} |\lambda|^2 \left| tr(R_{A}(QR_{A}^{\delta}))^{m+1} \right| |d\lambda| \]

\[ \leq \frac{b_{p}^{2}}{2} \int_{|\lambda|=b_p} \left\| R_{A}(QR_{A}^{\delta})^{m+1} \right\|_{t, (h_{A})} |d\lambda| \]
\[ \leq b_p^2 \int_{|\lambda|=b_p} \| R_\lambda \|_1 \left\| (QR^{0}_\lambda)^{m+1} \right\|_{\sigma(H)} |d\lambda| \]
\[ \leq b_p^2 \int_{|\lambda|=b_p} \| R_\lambda \|_1 \left\| (QR^{0}_\lambda)^m \right\|_1 \left\| QR^{0}_\lambda \right\|_{\sigma(H)} |d\lambda| \]
\[ \leq b_p^2 \int_{|\lambda|=b_p} \| R_\lambda \|_1 \left\| Q \right\|_1 \left\| R^{0}_\lambda \right\|_{\sigma(H)} |d\lambda|. \] (3.15)

One can prove the following inequalities similarly in work [7]:

\[ \left\| R^{0}_\lambda \right\|_{\sigma(H)} \leq \text{const} n_p^{1-b}, \]
\[ \left\| R_\lambda \right\|_1 \leq \text{const} n_p^{-\delta} \quad (\delta = \frac{2\alpha}{2r + \alpha} - 1). \]

From last two inequalities and (3.15), we obtain

\[ \left| D_p^{(m)} \right| \leq \text{const} b_p^2 n_p^{-m-2b+1}. \] (3.16)

For large values of \( p \)

\[ b_p = 2^{-1} \left( \mu_{n_p} + \mu_{n_p+1} \right) \leq \text{const} n_p^{1+\delta}. \] (3.17)

From (3.16) and (3.17), we obtain

\[ \left| D_p^{(m)} \right| \leq \text{const} n_p^{4-(m-1)\delta}. \]

Therefore, for \( m = \lfloor \frac{2\alpha+6\alpha+3\alpha}{2\alpha-2\alpha-\alpha} \rfloor + 1 \), we find

\[ \lim_{p \to \infty} D_p^{(m)} = 0. \] (3.18)

From (3.11), (3.14) and (3.18), we find the following formula for second regularized trace formula of the operator \( L \):

\[ \lim_{p \to \infty} \sum_{q=1}^{n_p} \left( \lambda^2_{q} - \mu^2_{q} - 2 \sum_{s=2}^{m} (-1)^{s-1} \text{Res}_{\lambda=\mu_{s}} \text{tr} \left( \lambda (QR^{0}_\lambda)^{s} \right) - \frac{2\mu_{q}}{\pi} \int_{0}^{\pi} (Q(x)\varphi_{j}, \varphi_{j}) dx \right) = (-1)^{r}2^{-1-2r} \left[ \text{tr} Q^{(2r)}(0) - \text{tr} Q^{(2r)}(\pi) \right] + \frac{1}{2} \left[ \text{tr} A Q(0) - \text{tr} A Q(\pi) \right]. \]

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References

[1] A.A. Kirillov, Elements of the Theory of Representations, Springer of Verlag, New York, (1976).
[2] B.M. Levitan, Calculation of the Regularized Trace for the Sturm Liouville Operator, Uspekhi Mat. Nauk, Vol 19, 1, (1964), 161–165.
[3] B.M. Levitan, I.S. Sargsyan, Sturm-Liouville and Dirac Operators, Kluwer, Dordrecht, (1991).
[4] C.J. Halberg, V.A. Kramer, A Generalization of the Trace Concept, Duke Math. Journal, Vol 27, No 4, (1960), 607–618.
[5] E. Adigüzelov, About the Trace of the Difference of Two Sturm-Liouville Operators with Operator Coefficient, Iz.AN AZ SSR, Seriya Fiz-Tekn. I Mat. Nauk, No 5, (1976), 20–24.
[6] E. Adigüzelov, O. Bakşı, On The Regularized Trace of The Differential Operator Equation Given in a Finite Interval, Journal of Engineering and Natural Sciences Sigma, 2004/1, (2004), 47–55.
[7] E. Adigüzelov, Y. Sezer, On Spektrum of a Self Adjoint Differential Operator of Higher Order with Unbounded Operator Coefficient and Asymptotic Behaviour of Eigenvalues, Journal of Engineering and Natural Sciences Sigma, 2006/4, (2006), 32–49.
[8] E. Adigüzelov, Y. Sezer, The Regularized Trace of a Self Adjoint Differential Operator of Higher Order with Unbounded Operator Coefficient, Applied Mathematics and Computation, Volume 218, Issue 5, (2011), 2113–2121.
[9] E. Sen, A. Bayramoglu, K. Orucoglu, Regularized Trace Formula For Higher Order Differential Operators with Unbounded Coefficient, Electronic Journal of Differential Equations, Vol.2016, No.31, (2016), pp.1–12.
[10] F.G. Maksudov, M. Bayramoglu, and E.E. Adigüzelov, On a Regularized Trace of Sturm-Liouville Operator on a Finite Interval with the Unbounded Operator Coefficient, Dokl. Akad. Nauk SSSR, English Translation: Soviet Math. Dokl. 30, No1, (1984), 169–173.
[11] I.C. Cohberg, M.G. Krein, Introduction to the Theory of Linear Non-Self Adjoint Operators, Translation of Mathematical Monograph, Volume 18, Amer. Math. Soc., Providence, R.I, (1969).
[12] I.M. Gelfand, B.M. Levitan, On a Formula for Eigenvalues of a Differential Operator of Second Order, Dokl. Akad. Nauk SSSR, T 88, No 4, (1953), 593–596.
[13] L.A. Dikiy, About a Formula of Gelfand-Levitan, Uspekhi Mat. Nauk, T 8, No 2, (1953), 119–123.
[14] M. Bayramoglu, E. Adigüzelov, On a Regularized trace formula for the Sturm-Liouville Operator with a singularity, Differential Equations, 32, 1996, no 12, (1997), 1581–1585.
[15] R.Z. Challilova, On Regularization of the Trace of the Sturm-Liouville Operator Equation, Funks. Analiz, Teoriya Funksi I Ik Pril. Mahachkala, No 3, Part 1, (1976), 154–161.
[16] T.C. Fulton, S.A. Pruess, Eigenvalue and Eigenfunction Asymptotics for Regular Sturm-Liouville Problems, J. Math. Anal. Appl. 188, (1994), 297–340.