On Conformal, $SL(4, \mathbb{R})$ and $Sp(8, \mathbb{R})$ Symmetries of 4d Massless Fields

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Abstract

The $sp(8, \mathbb{R})$ invariant formulation of free field equations of massless fields of all spins in $AdS_4$ available previously in terms of gauge invariant field strengths is extended to gauge potentials. As a by-product, free field equations for a massless gauge field are shown to possess both $su(2, 2) \sim o(4, 2)$ and $sl(4, \mathbb{R}) \sim o(3, 3)$ symmetry. The proposed formulation is well-defined in the $AdS_4$ background but experiences certain degeneracy in the flat limit that does not allow conformal invariant field equations for spin $s > 1$ gauge fields in Minkowski space. The basis model involves the doubled set of fields of all spins. It is manifestly invariant under $U(1)$ electric-magnetic duality extended to higher spins. Reduction to a single massless field contains the equations that relate its electric and magnetic potentials which are mixed by the conformal transformations for $s > 1$. We use the unfolded formulation approach recalled in the paper with some emphasis on the role of Chevalley-Eilenberg cohomology of a Lie algebra $g$ in $g$-invariant field equations. This method makes it easy to guess a form of the 4d $sp(8, \mathbb{R})$ invariant massless field equations and then to extend them to the ten dimensional $sp(8, \mathbb{R})$ invariant space-time. Dynamical content of the field equations is analyzed in terms of $\sigma_-$ cohomology.
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1 Introduction

It was argued by Fronsdal in the pioneering work [1] that the tower of free higher-spin (HS) fields of all spins in four dimensions admits $sp(8, \mathbb{R})$ symmetry. The conformal symmetry $su(2, 2)$, that acts individually on a massless field of a fixed spin, extends to $sp(8, \mathbb{R})$ that mixes states of different spins. Based on this observation Fronsdal addressed two fundamental questions: what is a minimal space where $sp(8, \mathbb{R})$ is geometrically realized and what are manifestly $sp(8, \mathbb{R})$ invariant field equations that describe $4d$ massless fields of all spins?

He answered the first question by showing that the relevant space is ten dimensional with real symmetric matrices $X_{AB} = X_{BA}$ as local coordinates [1] ($A, B \ldots = 1, \ldots 4$ are $4d$ Majorana spinor indices). This is Lagrangian Grassmannian $M_4 \sim Sp(8, \mathbb{R})/P$ where $P$ is the parabolic subgroup of $Sp(8, \mathbb{R})$ that results from crossing out the right node of the Dynkin diagram of $sp(8, \mathbb{R})$. Using two-component spinor notation, $A = (\alpha, \alpha')$, $\alpha, \beta \ldots = 1, 2$, $\alpha', \beta' \ldots = 1, 2$, the ten dimensional matrix space extends Minkowski coordinates $x^{\alpha\alpha'}$ to $X^{AB} = (X^{\alpha\alpha'}, X^{\alpha\beta}, \overline{X}^{\alpha'\beta'})$ where $X^{\alpha\beta}$ and $\overline{X}^{\alpha'\beta'}$ are additional six coordinates that form an antisymmetric Lorentz tensor. Note that the relevance of this space to the description of massless fields was rediscovered by Bandos, Lukierski and Sorokin in [2]. More generally $M_M$ denotes the Lagrangian Grassmannian with local coordinates $X^{AB} = X^{BA}$, $A, B = 1, \ldots M$ (in this paper we do not distinguish between the parabolic space and its big cell $R^{M(M+1)/2}$).

The form of the dynamical variables and field equations in $M_4$ was obtained later in [3] where it was shown that the tower of all $4d$ massless integer and half-integer spins can be described, respectively, by a single scalar $C(X)$ and spinor $C_A(X)$ in $M_4$ that satisfy the field equations

\[
\left( \frac{\partial^2}{\partial X^{AB}\partial X^{CD}} - \frac{\partial^2}{\partial X^{CB}\partial X^{AD}} \right) C(X) = 0, \tag{1.1}
\]

\[
\left( \frac{\partial}{\partial X^{AB}} C_{C}(X) - \frac{\partial}{\partial X^{CB}} C_A(X) \right) = 0. \tag{1.2}
\]

These equations possess no gauge symmetry because the fields $C(X)$ and $C_A(X)$ describe gauge invariant objects like scalar (spin 0), Maxwell field strength (spin 1), Weyl tensor (spin 2) and their HS generalizations.
The infinite towers of fields that appear in the $Sp(8, \mathbb{R})$ invariant consideration are precisely the HS multiplets of the $4d$ nonlinear HS gauge theory [4, 5]. This is not accidental because the original argument of Fronsdal in favor of $Sp(8, \mathbb{R})$ was based on the prominent Flato-Fronsdal theorem [6] stating that the tensor product of two singletons, where $Sp(8, \mathbb{R})$ acts in a natural way, is equivalent to the set of massless fields of all spins as a $sp(4, \mathbb{R}) \sim o(3, 2)$-module. On the other hand, the states of the HS gauge theories can also be understood as resulting from tensoring singletons [7]. (Recall that singletons are conformal scalar and spinor fields in three dimensions.)

The fields $C(X)$ and $C_A(X)$ can be interpreted as “hyperfields” in the “hyperspace” $M_4$ that allow to describe all $4d$ massless fields at once. $M_4$ plays for a HS multiplet a role analogous to that of superspace for supersymmetric theories. The concise form of the equations (1.1) and (1.2) makes it tempting to look for a formulation of the full nonlinear theory in this formalism. Note that, as mentioned in [3], the $4d$ nonlinear HS models of [4, 5] can indeed be interpreted as possessing a spontaneously broken $Sp(8, \mathbb{R})$ symmetry.

Theories in $M_4$ have been studied in a number of papers from different perspectives [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In particular, an attempt to formulate a nonlinear theory in this framework was undertaken in [15]. One reason why it was hard to go beyond free theory is that $C(X)$ and $C_A(X)$ describe gauge invariant curvature tensors. It was not clear so far how to describe gauge field potentials like spin two metric tensor in the $sp(8, \mathbb{R})$ covariant way and in $M_4$. As gauge potentials play the key role in any nonlinear field theory including Yang-Mills theory, Einstein gravity, supergravity and nonlinear HS gauge theories [22, 23, 24, 25, 4, 5, 26, 27] (see also [28, 29, 30] for reviews of nonlinear HS theories and more references), to proceed towards a nonlinear HS theory in $M_4$ one has to introduce the gauge potentials in the $Sp(8, \mathbb{R})$ invariant framework. This is the primary aim of this paper.

For the first sight, apart from being interesting, the problem may look unsolvable. Indeed, at any rate the project is to find an $sp(8, \mathbb{R})$ invariant formulation of free massless fields described in terms of gauge potentials. Since $sp(8, \mathbb{R})$ contains the conformal symmetry $su(2, 2)$, this should result in a conformal invariant formulation of free massless field equations in terms of potentials. There is a lot of studies of conformal field equations in the literature starting from the seminal work of Dirac [31] (see, e.g., [32, 33, 34, 35, 36, 37, 38] and references therein). It is known however that the free field equations in terms of potentials are not conformal for spins $s > 1$ in flat space. Actually, the complete list of free conformal invariant equations in flat space available in [38] does not contain the $4d$ massless equations in terms of gauge potentials except for the $4d$ Maxwell equations, i.e., spin one.

On the other hand this looks unnatural because the equations in terms of gauge invariant field strengths are conformal invariant and the space of states of $4d$ massless equations admits the action of the conformal symmetry. Aiming at preserving the $sp(8, \mathbb{R})$ symmetry, we have to find out what goes wrong with conformal symmetry in terms of potentials.

The results of this paper show that the $sp(8, \mathbb{R})$ covariant formulation is consistent in the $AdS_4$ background but experiences certain degeneracy in the flat limit to Minkowski space, in which either the form of the equations breaks down leading to (anti)selfdual equations...
(insisting on the symmetry) or the symmetry transformation law on the fields in Minkowski space blows up (insisting on the full massless equations). The same happens with the conformal symmetry that can be defined in $AdS_4$ but not in Minkowski space. Note that this is not the first time when the $AdS$ curvature resolves a no-go statement. Analogous phenomenon occurs for HS interactions in HS gauge theories [23].

Our model exhibits manifest electric-magnetic (EM) duality symmetry extended to all HS fields as a $u(1)$ subalgebra of $sp(8, \mathbb{R})$. A closely related property is that it contains two sets of fields of all spins related by the EM duality symmetry. This doubling also plays a role in the conformal transformations that mix the two species of gauge fields. The reduction to the undoubled set of fields in which every massless field of spin $s > 0$ appears in one copy is also possible. In this case, the field equations relate electric and magnetic potentials of spins $s \geq 1$. An interesting feature of this dynamical system is that the conformal transformations mix electric and magnetic potentials for $s > 1$.

Apart from the conformal embedding of the $AdS_4$ symmetry $sp(4, \mathbb{R}) \sim o(3, 2) \subset su(2, 2) \sim o(4, 2) \subset sp(8)$ a different embedding $sp(4, \mathbb{R}) \sim o(3, 2) \subset sl(4, \mathbb{R}) \sim o(3, 3) \subset sp(8, \mathbb{R})$ exists. This simple observation has a surprising output that the gauge theories in $AdS_4$ exhibit $sl(4, \mathbb{R}) \sim o(3, 3)$ symmetry at the free field level even for a single massless field of a fixed spin. This raises an intriguing question whether the $sl(4, \mathbb{R})$ extends to nonlinear 4d models, including both HS theories and lower spin (super)gravity-like theories.

The organization of the rest of the paper is as follows. The Sections 2-9 remind the reader some known facts about HS field equations and unfolded dynamics approach extensively used throughout this paper. Namely, in Section 2 we recall the $Sp(8, \mathbb{R})$ invariant formulation of the dynamics of massless fields in terms of gauge invariant field strengths. In Section 3 the unfolded formulation of the field equations of HS gauge fields in $AdS_4$ is summarized. The flat limit that reproduces the standard on-shell formulation of HS dynamics in Minkowski space is discussed in Section 4. In Section 5 we summarize relevant elements of the unfolded dynamics approach with some emphasize on the role of Chevallay-Eilenberg cohomology. General strategy of searching unfolded formulation of a $g$-symmetric field-theoretical model is outlined in Section 6. In Section 7 we interpret Minkowski and $AdS_4$ geometries in terms of flat connections of $o(d-1,2)$. In Section 8 we extend this analysis to $Sp(8, \mathbb{R})$, focusing main attention on the group manifold $Sp(4, \mathbb{R})$ as a ten dimensional generalization of $AdS_4$. In Section 9 we introduce the star-product formalism underlying the unfolded formulation of the HS dynamics and recall the pure gauge representation of the flat $Sp(2M)$ connection found in [10].

The original part of the paper starts in Section 10 where the Fock modules appropriate for the $Sp(8, \mathbb{R})$ invariant description of HS gauge fields are introduced. Conformal invariant unfolded field equations for massless fields of all spins are analyzed in Section 11 where we prove the formal consistency of the conformal field equations, analyze their dynamical content, global symmetries and specificities of the flat limit. In particular we show in this Section that the proposed field equations are invariant under the EM duality transformation and that the conformal symmetry cannot be preserved in Minkowski space because the special conformal part of the field transformation blows up in the flat limit. In Section 12 these
results are extended to $sp(8, \mathbb{R})$ while the $gl(4, \mathbb{R})$ symmetry of the equations is considered in Section 13. The detailed study of the dynamical content of the proposed equations within the $\sigma$-cohomology approach is done first for the 4d case in Section 14 and then for the case of matrix space in Section 15. Sections 14 and 15 can be skipped by the reader not interested in details of the formalism. Conclusions and perspectives are discussed in Section 16.

2 4d massless fields and $Sp(8, \mathbb{R})$ symmetry

The key observation is [39, 40] (see also [41]) that the generating function

$$C(b|x) = \sum_{k=0}^{\infty} \frac{1}{k!} C_{A_1...A_k}(x) b^{A_1} \ldots b^{A_k} = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} C_{\alpha_1...\alpha_n,\alpha'_1...\alpha'_m}(x) b^{\alpha_1} \ldots b^{\alpha_n} \bar{b}^{\alpha'_1} \ldots \bar{b}^{\alpha'_m}$$

can be used to describe all 4d massless fields by virtue of the equation

$$\left( \frac{\partial}{\partial x^{\alpha\alpha'}} + \frac{\partial^2}{\partial b^{\alpha} \partial \bar{b}^{\alpha'}} \right) C(b|x) = 0,$$

where $x^{\alpha\alpha'}$ are Hermitian coordinates of Minkowski space-time ($x^{\alpha\alpha'} = \sigma^{\alpha\alpha'} n x^n$, $n = 0, 1, 2, 3$, $\sigma^{\alpha\alpha'}_n$ are four $2 \times 2$ Hermitian matrices) and $b^{\alpha}, \bar{b}^{\alpha'}$ are auxiliary commuting spinor variables.

The interpretation of the components $C_{\alpha_1...\alpha_n,\alpha'_1...\alpha'_m}(x)$ is as follows. Those that carry both primed and unprimed indices (i.e., $mn \neq 0$) are expressed by (2.1) via space-time derivatives of the holomorphic and antiholomorphic components that carry only primed or only unprimed indices, respectively, i.e.,

$$C_{\alpha_1...\alpha_{m+k},\alpha'_1...\alpha'_{m}}(x) = (-1)^m \partial_{\alpha_1\alpha'_1} \ldots \partial_{\alpha_m\alpha'_m} C_{\alpha_{m+1}...\alpha_{m+k}}(x), \quad \partial_{\alpha\alpha'} = \frac{\partial}{\partial x^{\alpha\alpha'}};$$

where the indices $\alpha_k$ and (independently) $\alpha'_k$ are symmetrized. The formula in the conjugated antiholomorphic sector is analogous.

The holomorphic and antiholomorphic fields describe a scalar field $C(x)$ for $s = 0$, selfdual and anti-selfdual components of the spin one Maxwell field strength, $C_{\alpha\beta}(x)$ and $C_{\alpha',\beta'}(x)$, selfdual and anti-selfdual components of the Weyl tensor for spin two, $C_{\alpha_1...\alpha_4}(x)$ and $C_{\alpha'_1...\alpha'_4}(x)$, and so on. The system (2.1) decomposes into an infinite set of subsystems for the fields of definite helicities according to different eigenvalues of the helicity operator $H = \frac{1}{2} \left( b^{\alpha} \frac{\partial}{\partial b^{\alpha'}} - \bar{b}^{\alpha'} \frac{\partial}{\partial \bar{b}^{\alpha'}} \right)$.

Apart from expressing auxiliary fields in terms of space-time derivatives of the (anti)holomorphic fields via (2.2), the equation (2.1) imposes the massless field equations on the latter

$$\frac{\partial}{\partial b^{[\alpha}} \frac{\partial}{\partial x^{\beta]\alpha'}} C(b,0|x) = 0, \quad \frac{\partial}{\partial \bar{b}^{\alpha'}} \frac{\partial}{\partial x^{\alpha}[\beta}} C(0,\bar{b}|x) = 0,$$

and

$$\Box C(0,0|x) = 0.$$
It is easy to check that (2.2), (2.3) and (2.4) exhaust the content of (2.1).

The equation (2.1) can be interpreted as the covariant constancy condition

\[ D|C(b|x)\rangle = 0 \]  

for the field

\[ |C(b|x)\rangle = C(b|x)|0\rangle, \]  

that takes values in the Fock module generated from the vacuum state

\[ a_A|0\rangle = 0 \]  

of the algebra of oscillators

\[ [a_A, b^B] = \delta_A^B, \quad [a_A, a_B] = 0, \quad [b^A, b^B] = 0. \]  

The generators of \( sp(8, \mathbb{R}) \) are realized in this module as

\[ P_{AB} = \frac{1}{2} a_A a_B, \quad L_A^B = \frac{1}{2} (a_A b^B + b^B a_A), \quad K^{AB} = \frac{1}{2} b^A b^B. \]  

Note that \( L_A^B \) form \( gl(4|\mathbb{R}) \).

The covariant derivative

\[ D = d + d x^{\alpha\alpha'} P_{\alpha\alpha'} \]  

is a particular flat \( sp(8, \mathbb{R}) \) connection, i.e.

\[ D = d + w, \quad w = h^{AB} P_{AB} + \omega_B^A L_A^B + f_{AB} K^{AB}, \quad D^2 = 0. \]  

The connection (2.10) is flat because the generators \( P_{\alpha\alpha'} \) commute to themselves. This choice of the flat connection corresponds to Cartesian coordinates in Minkowski space.

As explained in Subsection 5.1, that the 4d massless equations (2.1) have the form of a covariant constancy condition with the covariant derivative in a \( sp(8, \mathbb{R}) \)-module \( V \) (here \( V \) is the space of functions of \( b \)) implies their invariance under the global \( sp(8, \mathbb{R}) \) symmetry. The generators of \( sp(8, \mathbb{R}) \) act on the dynamical fields as differential operators with coefficients polynomial in \( x \) (see [3] and Section 9 for explicit field transformations).

The conformal symmetry \( su(2, 2) \) extended to \( u(2, 2) \) by the helicity generator is the subalgebra of \( sp(8, \mathbb{R}) \) spanned by the generators

\[ P_{\alpha\beta} = a_\alpha a_\beta, \quad L_\alpha^\beta = \frac{1}{2} \{ a_\alpha, b_\beta \}, \quad L_\alpha^{\beta'} = \frac{1}{2} \{ a_\alpha', b_\beta' \}, \quad K^{\alpha\beta} = b^\alpha b^{\beta'} \]  

with the respective gauge fields \( h^{\alpha\beta'}, \omega_\beta^\alpha, \omega_\beta^{\alpha'}, f_{\alpha\beta'} \). The dilatation and helicity generators \( \mathcal{D} \) and \( \mathcal{H} \) are

\[ \mathcal{D} = \frac{1}{2} \left( L_\alpha^\alpha + L_\alpha^{\alpha'} \right), \]  

\[ \mathcal{H} = \frac{1}{2} \left( L_\alpha^\alpha - L_\alpha^{\alpha'} \right). \]
The helicity operator $H$ is central in $u(2,2)$. This is expected because it takes a fixed value on any $su(2,2)$-module with definite helicity. $H$ is the generator of EM duality transformations which is the manifest symmetry of this formulation [3].

The extension of the dynamical equations to $M_4$ is achieved by replacing the unfolded equation (2.1) with
\[
\left( \frac{\partial}{\partial X^{AB}} + \frac{\partial^2}{\partial b^A \partial b^B} \right) C(b|X) = 0, \tag{2.15}
\]
where $X^{AB} = X^{BA}$ are symmetric matrix coordinates associated with the generalized momentum $P_{AB}$. This is the equivalent extension of the unfolded 4d massless equations to $M_4$. Indeed, the part of the equations (2.15) with $A = \alpha$, $B = \beta'$ and $X^{\alpha\beta'} = x^{\alpha\beta'}$ is just the equation (2.1) while the equations that contain extra six coordinates $X^{\alpha\beta}$ and $X^{\alpha'\beta'}$ reconstruct the dependence on these coordinates in terms of the generating function of 4d massless fields $C(b|x)$. (See also Subsection 5.5.)

On the other hand, one can interpret the equation (2.15) differently by observing that they express all components of $C(b|X)$ that contain two or more oscillators $b^A$ via derivatives over the hyperspace coordinates $X^{AB}$ of the two dynamical fields which are polynomials of zeroth and first degree of $b^A$. Thus $C(X) + C_A(X)b^A$ are dynamical fields in $M_4$ while all other components are auxiliary fields expressed by (2.15) via $X$-derivatives of the dynamical fields. Namely, for
\[
C(b|X) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{A_1...A_n}(X)b^{A_1}...b^{A_n},
\]
\[
C_{A_1...A_n}(X) = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \partial_{A_1} A_2 ... \partial_{A_{\left\lfloor \frac{n}{2} \right\rfloor - 1}} A_{\left\lfloor \frac{n}{2} \right\rfloor} C_{A_{\left\lfloor \frac{n}{2} \right\rfloor}...A_{\left\lfloor \frac{n}{2} \right\rfloor}}(X), \quad \partial_{AB} = \frac{\partial}{\partial X^{AB}}, \tag{2.16}
\]
where $\left\lfloor \frac{n}{2} \right\rfloor$ is the integer part of $\frac{n}{2}$ and $C_{A_{\left\lfloor \frac{n}{2} \right\rfloor}...A_{\left\lfloor \frac{n}{2} \right\rfloor}}(X)$ is either $C(X)$ or $C_A(X)$.

The equations (1.1) and (1.2) are consequences of (2.15). Eqs. (1.1), (1.2) and (2.16) exhaust all restrictions on $C(b|X)$ imposed by (2.15). This proves that the equations (1.1) and (1.2) in $M_4$ are equivalent to the 4d massless field equations for all spins. That all 4d massless fields are described by only two hyperfields is because spin is carried by the spinning coordinates $X^{\alpha\beta}$ and $X^{\alpha'\beta'}$ in the hyperspace $M_4$ (see also [16].)

3 Higher spin gauge fields in $AdS_4$

In this Section we recall the unfolded form of 4d free HS field equations proposed in [39, 40]. It is based on the frame-like approach to HS gauge fields [42, 43] where a spin $s$ HS gauge field is described by the set of 1-forms
\[
\omega_{\alpha_1...\alpha_k,\alpha_1'...\alpha_l'} = dx^\alpha \omega_{\alpha_1...\alpha_k,\alpha_1'...\alpha_l'}, \quad k + l = 2(s - 1).
\]
The HS gauge fields are self-conjugated $\omega_{\alpha_1...\alpha_k,\beta_1'...\beta_l'} = \omega_{\beta_1...\beta_l,\alpha_1'...\alpha_k'}$. This set is equivalent to the real 1-form $\omega_{A_1...A_{2(s-1)}}$ symmetric in the Majorana indices $A$, that carries an irreducible module of the $AdS_4$ symmetry algebra $sp(4,\mathbb{R}) \sim o(3,2)$. 

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The $AdS_4$ space is described by the Lorentz connection $\omega^{\alpha\beta}$, $\varpi^\gamma{}_{\beta}$ and vierbein $e^{\alpha\prime}$.

Altogether they form the $sp(4,\mathbb{R})$ connection $w^{AB} = w^{BA}$ that satisfies the $sp(4,\mathbb{R})$ zero curvature conditions

$$R^{AB} = 0, \quad R^{AB} = dw^{AB} + w^{AC} \wedge w_C^B,$$

where indices are raised and lowered by a $sp(4,\mathbb{R})$ invariant form $C_{AB} = -C_{BA}$

$$A_B = A^A C_{AB}, \quad A^A = C^{AB} A_B, \quad C_{AC} C^{BC} = \delta^A_B. \tag{3.2}$$

In terms of Lorentz components $w^{AB} = (\omega^{\alpha\beta}, \varpi^\gamma{}_{\beta}, \lambda e^{\alpha\beta}, \lambda e^{\beta\alpha})$ where $\lambda^{-1}$ is the $AdS_4$ radius, the $AdS_4$ equations (3.1) read as

$$R_{\alpha\beta} = 0, \quad \overline{R}_{\alpha\prime\beta'} = 0, \quad R_{\alpha\alpha'} = 0, \tag{3.3}$$

where

$$R_{\alpha\beta} = d\omega_{\alpha\beta} + \omega_{\alpha}{}^{\gamma} \wedge \omega_{\beta\gamma} + \lambda^2 e_{\alpha}{}^{\gamma'} \wedge e_{\beta\gamma'}, \quad \overline{R}_{\alpha\prime\beta'} = d\varpi_{\alpha\prime\beta'} + \varpi_{\alpha\prime}{}^{\gamma'} \wedge \varpi_{\beta'\gamma'} + \lambda^2 e_{\alpha\prime}{}^{\gamma} \wedge e_{\beta'\gamma'}, \quad R_{\alpha\beta'} = de_{\alpha\beta'} + \omega_{\alpha}{}^{\gamma} \wedge e_{\beta'\gamma'} + \varpi_{\beta'\gamma'} \wedge e_{\alpha\beta'} \tag{3.4}\tag{3.5}$$

(Two-component indices are raised and lowered as in (3.2) with $C_{AB}$ replaced by the two-component symplectic forms $\epsilon_{\alpha\beta}$ or $\epsilon_{\alpha\prime\beta'}$.)

The unfolded equations of motion of a spin-$s$ massless field read as

$$R_{\alpha_1...\alpha_{n}s-1,\alpha_{n}s} = \delta^{0}_n H_{\alpha_{1}...\alpha_{n}s-1,\alpha_{n}s} + \delta^{0}_{m} H_{\alpha_{1}...\alpha_{2}s-1,\alpha_{2}s} C_{\alpha_{1}...\alpha_{2}s}, \quad n + m = 2(s - 1) \tag{3.6}$$

and

$$D_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}} = 0, \quad n - m = 2s, \quad D_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}} = 0, \quad m - n = 2s. \tag{3.7}$$

Here the HS field strength and twisted adjoint covariant derivative have the form

$$R_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}} = D_{\omega_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}} + n\lambda e_{\alpha_{1}}{}^{\alpha_{1}+1} \wedge \lambda e_{\alpha_{2}}...e_{\alpha_{1}'}{}^{\alpha_{1}+1} + m\lambda e_{\alpha_{1}}{}^{\alpha_{2}+1} \wedge \lambda e_{\alpha_{1}+1}...e_{\alpha_{2}'}{}^{\alpha_{1}+1},$$

$$D_{\omega_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}}} = D_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}} + \lambda (e_{\gamma'}{}^{\gamma'} e_{\alpha_{1}}...e_{\alpha_{1}'}{}^{\gamma'} + n e_{\alpha_{1}}{}_{\alpha_{2}}...e_{\alpha_{2}'}{}^{\gamma'}),$$

$$D_{\omega_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}}} = D_{\alpha_1...\alpha_{n},\alpha'_{1}...\alpha'_{m}} + \lambda (e_{\gamma'}{}^{\gamma'} \overline{e}_{\alpha_{1}}...e_{\alpha_{1}'}{}^{\gamma'} + n e_{\alpha_{1}}{}_{\alpha_{2}}...e_{\alpha_{2}'}{}^{\gamma'}), \tag{3.8}\tag{3.9}$$

where the indices $\alpha$ and $\alpha'$ are (separately) symmetrized and $D_{L}$ is the Lorentz covariant derivative

$$D_{L} \psi_{\alpha} = d\psi_{\alpha} + \omega_{\alpha}{}^{\beta} \psi_{\beta}, \quad D_{L} \overline{\psi}_{\alpha'} = d\overline{\psi}_{\alpha'} + \varpi_{\alpha'}{}^{\beta'} \overline{\psi}_{\beta'}. \tag{3.10}$$

$H^{\alpha\beta} = H^{\beta\alpha}$ and $\overline{H}^{\alpha'\beta'} = \overline{H}^{\beta'\alpha'}$ are the basis 2-forms built of the vierbein 1-form $e^{\alpha\alpha'}$

$$H^{\alpha\beta} = e^{\alpha}{}_{\alpha'} \wedge e^{\beta\alpha'}, \quad \overline{H}^{\alpha'\beta'} = e^{\alpha'}{}_{\alpha} \wedge e^{\beta'}{}_{\alpha'} \tag{3.11}$$
Formulae simplify in terms of the generating functions
\[ A(y, \bar{y} \mid x) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{\alpha_1} \cdots y_{\alpha_n} \bar{y}_{\beta_1} \cdots \bar{y}_{\beta_m} \omega^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}(x) \] (3.12)

with \( A = \omega, C, \bar{C}, R \) etc. In particular, we have
\[ R(y, \bar{y} \mid x) = D^{ad} \omega(y, \bar{y} \mid x) = D^L \omega(y, \bar{y} \mid x) - \lambda e^{\alpha \beta}(y_\alpha \frac{\partial}{\partial y^{\beta}} + \frac{\partial^2}{\partial y^\alpha \partial y^{\beta}}) \omega(y, \bar{y} \mid x), \] (3.13)
\[ D^{tw} C(y, \bar{y} \mid x) = D^L C(y, \bar{y} \mid x) + \lambda e^{\alpha \beta}(y_\alpha \bar{y}^{\beta} + \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta}}) C(y, \bar{y} \mid x), \] (3.14)

where the Lorentz covariant derivative \( D^L \) takes the form
\[ D^L A(y, \bar{y} \mid x) = dA(y, \bar{y} \mid x) - (\omega^{\alpha \beta} y_\alpha \frac{\partial}{\partial y^{\beta}} + \bar{\omega}^{\alpha \beta} \bar{y}_\alpha \frac{\partial}{\partial \bar{y}^{\beta}}) A(y, \bar{y} \mid x). \] (3.15)

As a consequence of the zero curvature equation (3.1) which is true for AdS_4 geometry, the covariant derivatives \( D^{ad} \) and \( D^{tw} \) are flat, i.e.,
\[ (D^{ad})^2 = (D^{tw})^2 = 0. \]

These conditions are necessary for the consistency of the equations (3.6) and (3.7) (i.e., the compatibility with \( d^2 = 0 \)) and guarantee the gauge invariance of the field strength (3.13) and, therefore, the free HS field equations (3.6) under Abelian HS gauge transformations
\[ \delta \omega(y, \bar{y} \mid x) = D^{ad} \epsilon(y, \bar{y} \mid x). \] (3.16)

It is important that the consistency of the equations is not spoiled by the \( C \)-dependent terms in (3.6). As explained in more detail in Subsection 5.4 this means that these terms correspond to a Chevalley-Eilenberg cohomology of \( sp(4, \mathbb{R}) \).

In the equations (3.6) and (3.7), a spin \( s \) field is described by the set of gauge 1-forms \( \omega^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}(x) \) with \( n + m = 2(s - 1) \) (for \( s \geq 1 \)) and 0-forms \( C^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}(x) \) with \( n - m = 2s \) along with their conjugates \( \bar{C}^{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m}(x) \) with \( m - n = 2s \). Indeed it is easy to see that the field equations (3.6) and (3.7) for such a set of fields with some \( s \) form an independent subsystem.

The dynamical massless fields are
- \( C(x) \) and \( \bar{C}(x) \) for two spin zero fields,
- \( C_\alpha(x) \) and \( \bar{C}_\alpha(x) \) for a massless spin 1/2 field,
- \( \omega^{\alpha_1 \cdots \alpha_{s-1} \beta_1 \cdots \beta_{s-1}}(x) \) for an integer spin \( s \geq 1 \) massless field,
- \( \omega^{\alpha_1 \cdots \alpha_{s-3/2} \beta_1 \cdots \beta_{s-3/2}}(x) \) and its complex conjugate \( \omega^{\alpha_{s-1/2} \cdots \alpha_{s-1} \beta_{s-3/2}}(x) \) for a half-integer spin \( s \geq 3/2 \) massless field.
All other fields are auxiliary, being expressed via derivatives of the dynamical massless fields by the equations (3.6) and (3.7).

The key fact of the unfolded form of free massless field equations is the so called Central On-Shell Theorem [40] that states that the content of the equations (3.6) and (3.7) is just that they express all auxiliary fields in terms of derivatives of the dynamical fields and impose the massless field equations on the latter in the standard form of Fronsdal [44] and Fang and Fronsdal [45]. To make the paper as self-contained as possible we sketch the proof of Central On-Shell Theorem in Section 14 using the \( \sigma \)-cohomology technics.

The meaning of the equations (3.6) and (3.7) is as follows. The equations (3.7) provide the \( \text{AdS}_4 \) deformation of (2.1). They remain independent for spins \( s = 0 \) and \( s = \frac{1}{2} \) and partially independent for spin one but become consequences of (3.6) for \( s > 1 \). The equations (3.6) express the holomorphic and antiholomorphic components of spin \( s \geq 1 \) 0-forms \( C(y, \bar{y} | x) \) via derivatives of the massless field gauge 1-forms described by \( \omega(y, \bar{y} | x) \). This identifies the spin \( s \geq 1 \) holomorphic and antiholomorphic components of the 0-forms \( C(y, \bar{y} | x) \) with the Maxwell tensor, on-shell Rarita-Schwinger curvature, Weyl tensor and their HS generalizations. In addition, the equations (3.6) impose the standard field equations on the spin \( s > 1 \) massless gauge fields so that the field equations (2.3) become their consequences by virtue of Bianchi identities. The dynamical equations for spins \( s \leq 1 \) are still contained in the equations (3.7). (For more detail see e.g. [28] and also Section 14).

Although the system (3.6) and (3.7) is consistent at the free field level, to extend it to the nonlinear case one has to double the set of HS fields [7, 40, 4, 5]. This can be achieved by introducing the fields

\[
\omega^{ii}(y, \bar{y} | x), \quad C^{i1-i}(y, \bar{y} | x), \quad i = 0, 1
\]

such that \( \omega^{ii}(y, \bar{y} | x) \) are selfconjugated, while \( C^{01}(y, \bar{y} | x) \) and \( C^{10}(y, \bar{y} | x) \) are conjugated to one another,

\[
\omega^{ii}(y, \bar{y} | x) = \omega^{ii}(\bar{y}, y | x), \quad C^{i1-i}(y, \bar{y} | x) = C^{1-i i}(\bar{y}, y | x).
\]

The unfolded system for the doubled set of fields is

\[
R^{ii}(y, \bar{y} | x) = \mathcal{H}^{i'i'} \frac{\partial^2}{\partial y^{i'} \partial \bar{y}^i} C^{1-i i}(0, \bar{y} | x) + H^{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} C^{n1-i}(y, 0 | x), \quad (3.17)
\]

\[
D^{tw} C^{i1-i}(y, \bar{y} | x) = 0. \quad (3.18)
\]

Note that now all components of the expansions of \( C^{n1-i}(y, \bar{y} | x) \) contribute to the equations (3.17) and (3.18), while in (3.6) and (3.7) with the single HS 1-form \( \omega(y, \bar{y}) \) only parts of the components of \( C(y, \bar{y}) \) and \( \bar{C}(y, \bar{y}) \) contributed.

\footnote{Note that, as discussed in [7] (see also [28]), the full \( N = 2 \) supersymmetric nonlinear HS system can be truncated to subsystems with reduced sets of fields. In particular, truncating out fermions, it is possible to consider a system with bosonic fields of all spins in which every integer spin appears once and further to the minimal system considered in some detail in [46], in which every even spin appears just once.}
In the standard formulation of the 4d nonlinear HS gauge theory \([5]\) (see \([28]\) for a review) the doubling of the fields is due to the dependence on the Klein operators \(k\) and \(\bar{k}\) that flip chirality \([47]\)

\[
k^2 = 1, \quad ky_\alpha = -y_\alpha k, \quad k\bar{y}_{\alpha'} = \bar{y}_{\alpha'} k,
\]

\[
\bar{k}^2 = 1, \quad \bar{k}y_\alpha = y_\alpha \bar{k}, \quad \bar{k}\bar{y}_{\alpha'} = -\bar{y}_{\alpha'} \bar{k}, \quad [k, \bar{k}] = 0.
\]

The fields are 1-forms

\[
\omega(k, \bar{k}; y, \bar{y} | x) = \sum_{ij=0,1} (k)^i (\bar{k})^j \omega^{ij}(y, \bar{y} | x)
\]

and 0-forms

\[
C(k, \bar{k}; y, \bar{y} | x) = \sum_{ij=0,1} (k)^i (\bar{k})^j C^{ij}(y, \bar{y} | x).
\]

Now both the adjoint and twisted adjoint covariant derivative result from different sectors of the adjoint covariant derivative in the Weyl algebra extended by the Klein operators \([47]\).

Massless fields are those with

\[
\omega(-k, -\bar{k}; y, \bar{y} | x) = \omega(k, \bar{k}; y, \bar{y} | x), \quad C(-k, -\bar{k}; y, \bar{y} | x) = -C(k, \bar{k}; y, \bar{y} | x).
\]

The fields with the opposite oddness in the Klein operators are topological, carrying at most a finite number of degrees of freedom per an irreducible subsystem \([48]\). We will see in Section 11 how this pattern of HS fields emerges in the \(sp(8, \mathbb{R})\) invariant formulation. In particular the topological field sector also plays a role in the model we focus on in this paper.

### 4 Flat limit

To take the flat limit it is necessary to perform certain rescalings. To this end let us introduce notations \(A_\pm\) and \(A_0\) so that the spectrum of the operator \(\left(y^\alpha \partial_{y^{\alpha'}} - \bar{y}_{\alpha'} \partial_{\bar{y}^{\alpha'}}\right)\) is positive on \(A_+(y, \bar{y} | x)\), negative on \(A_-(y, \bar{y} | x)\) and zero on \(A_0(y, \bar{y} | x)\). Having the decomposition

\[
A(y, \bar{y} | x) = A_+(y, \bar{y} | x) + A_-(y, \bar{y} | x) + A_0(y, \bar{y} | x),
\]

we introduce a new field

\[
\tilde{A}(y, \bar{y} | x) = A_+(\lambda y, \bar{y} | x) + A_-(\lambda y, \lambda \bar{y} | x) + A_0(\lambda y, \bar{y} | x).
\]

(Note that \(A_0(\lambda y, \bar{y} | x) = A_0(y, \lambda \bar{y} | x)\)). For the rescaled variables, the flat limit \(\lambda \to 0\) of the adjoint and twisted adjoint covariant derivatives \((3.13)\) and \((3.14)\) gives

\[
D_{fl}^{ad} \tilde{A}(y, \bar{y} | x) = D^L \tilde{A}(y, \bar{y} | x) - e^{\alpha \beta'} \left(y_\alpha \frac{\partial}{\partial y^{\beta'}} \tilde{A}_-(y, \bar{y} | x) + \frac{\partial}{\partial \bar{y}^{\alpha'}} \bar{y}_{\beta'} \tilde{A}_+(y, \bar{y} | x)\right),
\]

\[
D_{fl}^{tw} \tilde{A}(y, \bar{y} | x) = D^L \tilde{A}(y, \bar{y} | x) + e^{\alpha \beta'} \frac{\partial^2}{\partial y^{\alpha} \partial \bar{y}^{\beta'}} \tilde{A}(y, \bar{y} | x).
\]
The flat limit of the unfolded massless equations results from (3.17) and (3.18) via the substitution of $D^L$ and $e$ of Minkowski space and the replacement of $D^{ad}$ and $D^{tw}$ by $D^{ad}_{fl}$ and $D^{tw}_{fl}$, respectively (recall that $R^{ii} = D^{ad}_{fl} \omega^{ii}$). The resulting field equations describe free HS fields in Minkowski space. Let us stress that the flat limit prescription (4.2), that may look somewhat unnatural in the two-component spinor notation, is designed just to give rise to the theory of Fronsdal [44] and Fang and Fronsdal [45] (for more detail see Section 14).

Note that, although the contraction $\lambda \rightarrow 0$ with the rescaling (4.2) is consistent with the free HS field equations, it turns out to be inconsistent in the nonlinear HS theory because negative powers of $\lambda$ survive in the full nonlinear equations upon the rescaling (4.2), not allowing the flat limit in the nonlinear theory. This is why the Minkowski background is unreachable in the non-linear HS gauge theories of [25, 4, 26]. The reason why conformal symmetry blows up in the flat limit is that the translation generators in the sector of HS gauge fields $\omega$, that respect the described limiting procedure, disagree with their standard identification in the conformal algebra (for more detail see Subsection 11.3).

The flat limit of the equation $D^{tw} C = 0$ just reproduces the equation (2.1) which underlies the original extension of the HS dynamics from four to ten dimensions. The main question addressed in this paper is how to include the HS gauge 1-forms into the manifestly $sp(8, \mathbb{R})$ invariant formalism and then into the ten dimensional formulation. This is most naturally achieved within the unfolded formulation. In fact, the 4d equations (3.17) and (3.18) do have the unfolded form. To proceed, we now summarize relevant properties of the unfolded formulation using the 4d HS system as the basis example.

5 Unfolded dynamics

5.1 Unfolded equations

Let $M^d$ be a $d$-dimensional manifold with coordinates $x^n$ ($n = 0, 1, \ldots d - 1$). (For $d = 4$ we use the Hermitian coordinates $x^{\alpha \alpha'}$.) By unfolded formulation of a linear or nonlinear system of differential equations and/or constraints in $M^d$ we mean its equivalent reformulation in the first-order form

$$dW^\Phi(x) = G^\Phi(W(x)),$$

(5.1)

where $d = dx^n \frac{\partial}{\partial x^n}$ is the exterior differential on $M^d$, $W^\Phi(x)$ is a set of degree $p_\Phi$-differential forms and $G^\Phi(W)$ is some degree $p_\Phi + 1$ function of the differential forms $W^\Phi$

$$G^\Phi(W) = \sum_{n=1}^{\infty} f^\Phi_{\Omega_1 \ldots \Omega_n} W^{\Omega_1} \wedge \ldots \wedge W^{\Omega_n},$$

---

2By constraints we mean equations like $dA = B$ that express auxiliary fields like $B$ in terms of derivatives of other fields like $A$ imposing no differential equations on the latter.

3The idea of this approach was suggested and applied to the analysis of interacting HS gauge theory in [39, 40] while the name unfolded formulation was given somewhat later in [49].
where the coefficients $f_{\Omega_1 \ldots \Omega_n}$ satisfy the (anti)symmetry condition $f_{\Omega_1 \ldots \Omega_k \Omega_{k+1} \ldots \Omega_n} = (-1)^{p_1 + \ldots + p_k} f_{\Omega_1 \ldots \Omega_k \Omega_{k+1} \ldots \Omega_n}$ (extension to the supersymmetric case with an additional boson-fermion grading is straightforward) and $G^\Phi$ satisfies the condition

$$G^\Omega(W) \wedge \frac{\partial G^\Phi(W)}{\partial W^\Omega} = 0 \quad (5.2)$$

(the derivative $\frac{\partial}{\partial W^\Omega}$ is left) equivalent to the generalized Jacobi identity on the structure coefficients

$$\sum_{n=0}^{m} (n+1) f^{\Lambda}_{\Phi_1 \ldots \Phi_{m-n}} f^\Phi_{\Lambda \Phi_{m-n+1} \ldots \Phi_m} = 0 \quad (5.3)$$

where the brackets $\{\}$ denote appropriate (anti)symmetrization of indices $\Phi_i$. Strictly speaking, formal consistency demands (5.3) to be satisfied only at $p_\Phi < d$ for a $d$-dimensional manifold $\mathcal{M}^d$ where any $d + 1$-form is zero. Given solution of (5.3) it defines a free differential algebra (FDA) [50, 51, 52, 53]. We call a free differential algebra universal if the generalized Jacobi identity is true independently of $d$. The HS FDAs that appear in HS gauge theories and, in particular, those discussed in this paper belong to the universal class. Unfolded formulation is a covariant multidimensional extension of the first-order reformulation of ordinary differential equations.

Universal FDAs have the distinguished property that the operator $\frac{\partial}{\partial W^\Omega}$ is well-defined irrespectively of on what it acts, i.e. $\frac{\partial F(W)}{\partial W^\Omega}$ is defined for any $F(W)$ built of wedge products of differential forms. For non-universal FDAs this is not true. Actually, the condition

$$W_1 \wedge \ldots \wedge W_k = 0, \quad p_1 + \ldots + p_k > d \quad (5.4)$$

may lead to a contradiction upon formal differentiation. For instance, differentiating (5.4) over all $W_i$ involved leads to the contradiction $1 = 0$. In other words, for nonuniversal FDAs the space of $W^\Omega$ is constrained by the relations (5.4). Correspondingly, only vector fields tangent to the constraint surface are allowed.

For universal FDAs, the equation (5.1) is invariant under the gauge transformation

$$\delta W^\Phi(x) = d\varepsilon^\Phi(x) + \varepsilon^\Omega(x) \frac{\partial G^\Phi(W(x))}{\partial W^\Omega(x)} \quad (5.5)$$

where the gauge parameter $\varepsilon^\Phi(x)$ is an arbitrary $(p_\Phi - 1)$-form. (0-forms $W^\Phi(x)$ do not give rise to gauge symmetries.) This property of universal FDAs makes the unfolded formulation an efficient tool for the study of gauge invariant dynamical systems. Since unfolded equations are formulated in terms of the exterior algebra, this approach respects diffeomorphisms thus providing a natural framework for the study of models that contain gravity.

### 5.2 Vacuum

An important class of universal FDAs is in the one-to-one correspondence with Lie algebras. Indeed, let $w^a$ be a set of 1-forms. If no other forms are involved (e.g., all of them are
consistently set equal to zero in a larger system) the most general expression for \( G^\alpha(w) \), that has to be a 2-form, is \( G^\alpha(w) = -\frac{1}{2} f^\alpha_{\beta\gamma} w^\beta \wedge w^\gamma \). The consistency condition (5.3) then becomes the Jacobi identity for the structure coefficients \( f^\alpha_{\beta\gamma} \) of a Lie algebra \( g \). The unfolded equations (5.1) impose the flatness condition on the connection \( w^\alpha \)

\[
dw^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} w^\beta \wedge w^\gamma = 0 .
\]  

The transformation law (5.5) gives the usual gauge transformation of the connection \( w \)

\[
\delta w^\alpha(x) = D\varepsilon^\alpha(x) = d\varepsilon^\alpha(x) + f^\alpha_{\beta\gamma} w^\beta(x) \varepsilon^\gamma(x) .
\]  

A flat connection \( w(x) \) is invariant under the global transformations with the covariantly constant parameters

\[
D\varepsilon^\alpha(x) = 0 .
\]  

This equation is consistent by (5.6). Therefore, locally, it reconstructs \( \varepsilon^\alpha(x) \) in terms of its values \( \varepsilon^\alpha(x_0) \) at any given point \( x_0 \). \( \varepsilon^\alpha(x_0) \) are the moduli of the global symmetry \( g \) that is now recognized as the stability algebra of a given flat connection \( w(x) \).

This example is of key importance because this is how \( g \)-invariant vacuum fields appear in the unfolded formulation. In particular, the equation (3.1) of \( AdS_4 \) space-time is of this type. The same happens for the general case. Typically, an unfolded system that contains 1-forms \( w^\alpha \) associated with some Lie algebra \( g \) admits a flat connection \( w^\alpha \) as its natural \( g \)-symmetric vacuum solution. In the perturbative analysis, \( w^\alpha \) is assumed to be of the zeroth order because it contains the background metric that is usually non-degenerate, being of order zero. The flat connection \( w^\alpha \) is then referred to as vacuum connection.

Let us stress that this way of description of the background geometry is coordinate independent. A particular form of \( w^\alpha(x) \) is not needed for the analysis unless one is interested in explicit solutions in a specific coordinate system. The only important condition is that a flat vacuum connection has to be chosen so that its part associated with the generators of translations in \( g \) (i.e., vielbein sometimes also called soldering form, which is the 1-form \( e^{\alpha\alpha'} \) in the discussion of Section 3) is nondegenerate. The ambiguity of the choice of a particular vacuum form \( w^\alpha \) is up to the gauge transformations (5.7). In particular, the coordinate choice ambiguity, which of course preserves the flatness property, is reproduced by the gauge transformations (5.7). In fact, this is the general property of the unfolded dynamics where a diffeomorphism generated by an infinitesimal vector field \( \xi^\alpha(x) \) can be realized as the field-dependent gauge transformation (5.5) with the gauge parameter of the form

\[
\varepsilon^\Omega(x) = \xi^\alpha(x) \frac{\partial}{\partial dx^n} W^\Omega(dx, x) .
\]  

5.3 Free fields and Chevalley-Eilenberg cohomology

Let us now linearize the unfolded equation (5.1) around some vacuum flat connection \( w \) of a Lie algebra \( g \), that solves (5.1). To this end we set

\[
W^\Omega = w^\Omega + \omega^\Omega ,
\]  

\[ (5.10) \]
where \( \omega^\Omega \) are differential forms of various degrees that are treated as small perturbations and enter the equations linearly. Consider first the sector of forms \( \omega^i(x) \) of a given degree \( p \) (e.g., 0-forms) within the set \( \omega^\Omega(x) \). Then \( G^i \) is bilinear in \( w \) and \( \omega \), i.e., \( G^i = -w^a(T_a)^i_j \wedge \omega^j \). In this case the condition (5.2) implies that the matrices \( (T_a)_i^j \) form a representation \( T \) of \( g \) in a vector space \( V \) where \( \omega^i(x) \) takes its values (index \( i \)). The corresponding equation (5.1) is the covariant constancy condition

\[
D_w \omega^i = 0 \tag{5.11}
\]

with \( D_w \equiv d + w \) being the covariant derivative in the \( g \)-module \( V \).

The equations (5.6) and (5.11) are invariant under the gauge transformations (5.5)

\[
\delta \omega^i(x) = -\epsilon^a(x)(T_a)^i_j \omega^j(x). \tag{5.12}
\]

Once the vacuum connection is fixed, the system (5.11) is invariant under the global symmetry \( g \) with the parameters satisfying (5.8). This simple analysis allows useful applications.

Firstly, one observes that, by unfolding, any \( g \)-invariant linear dynamical system turns out to be reformulated in terms of \( g \)-modules. This allows for full classification and explicit derivation of \( g \)-invariant equations by studying various \( g \)-modules. In particular, the full list of conformal invariant equations in flat space-time of any dimension has been obtained this way in [38]. In this paper we will derive the manifestly \( sp(8, \mathbb{R}) \) covariant form of the massless field equations in terms of gauge potentials just by guessing appropriate \( sp(8, \mathbb{R}) \)-modules.

Secondly, it follows that if \( \tilde{g} \) is a larger Lie algebra that acts in \( V \), \( g \subset \tilde{g} \subset \text{End} \ V \), it is also a symmetry of (5.11) simply because any flat \( g \)--connection is the same time a flat \( \tilde{g} \)--connection. As a result, \( \tilde{g} = \text{End} \ V \) is the maximal symmetry of (5.11) (of course, modulo possible subtleties in the infinite dimensional case).

Thirdly, this gives an efficient tool for the derivation of the explicit form of the symmetry transformation laws via (5.8) and (5.12).

For example, once the equation (2.1) is reunderstood in the form (2.5), (2.10), (2.11), from the general analysis it follows that it is \( sp(8, \mathbb{R}) \) invariant. To derive the explicit form of the global \( sp(8, \mathbb{R}) \) transformation we solve the covariant constancy condition for the global symmetry parameter \( \epsilon(x) \)

\[
\epsilon_{gl}(a, b|x) = \exp[-x^{\alpha\alpha'}a_\alpha a_{\alpha'}] \epsilon_0(a, b) \exp[x^{\alpha\alpha'} a_\alpha a_{\alpha'}], \quad \epsilon_0(a, b) = \frac{1}{2} \epsilon_{AB} b^A b^B + \epsilon_A b^A a_B + \frac{1}{2} \epsilon^{AB} a_A a_B,
\]

where \( b^A = (b^\alpha, \bar{b}^{\alpha'}) \) and \( a_A = (a_\alpha, \bar{a}_{\alpha'}) \). This gives

\[
\epsilon_{gl}(a, b|x) = \frac{1}{2} \epsilon_{AB} b^A b^B + \epsilon_A b^A a_B + \frac{1}{2} \epsilon^{AB} a_A a_B - x^{\alpha\alpha'} \epsilon_0^{\alpha'} a_\alpha a_B - x^{\alpha\alpha'} \epsilon_0^\alpha a_B a_{\alpha'} - x^{\alpha\alpha'} \epsilon_0 a_\alpha a_{\alpha'} - b^D a_\alpha \epsilon_{\alpha D} x^{\alpha'}. \]

The desired transformation law is then obtained by restricting the variation

\[
\delta|C(b|x)\rangle = -\epsilon_{gl}(a, b|x)|C(b|x)\rangle \tag{5.13}
\]
to the dynamical holomorphic fields $C(b,0|x)$ or $C(0,\bar{b}|x)$ and using (2.2) for the auxiliary fields that appear on the r.h.s. of (5.13) to derive terms with $x$-derivatives of the dynamical fields in the transformation law. Note that this way it is possible to derive explicit form of the transformation law of the full HS algebra in a far more complicated AdS case (for more detail see [3, 10] and Section 9).

Suppose now that $\omega^a(x)$ and $\omega^i(x)$ are forms of different degrees, say, $p_a - p_i = k > 0$. Then, in the linearized approximation, one can consider functions $G^a$ polylinear in the vacuum field $w^\alpha$ but still linear in the dynamical fields $\omega$

$$G^a(w, \omega) = -f^a_{\alpha_1...\alpha_{k+1}i} w^{\alpha_1} \wedge ... \wedge w^{\alpha_{k+1}} \omega^i.$$  \hspace{1cm} (5.14)

(Note that the case of $k = -1$ with $G^a(w, \omega)$ independent of $w^\alpha$ is also possible. It corresponds to the so-called contractible FDAs [50] and is dynamically empty because the corresponding unfolded equation just expresses a $p_i$-form $\omega^i$ via the lower degree forms and their derivatives. As such it is not considered in this paper.)

Let $\omega^i$ be a 0-form. The equation for $\omega^i$ is then always a covariant constancy condition (5.11). The consistency condition (5.2) applied to (5.14) then literally implies that $f^a_{\alpha_1...\alpha_{k+1}i} w^{\alpha_1} \wedge ... \wedge w^{\alpha_{k+1}} \omega^i$ is a Chevalley-Eilenberg cocycle of $g$ with coefficients in $V_1 \otimes V_r^*$ where $V_1$ is the module where $G^a$ takes values while $V_r^*$ is the module conjugated to that of $\omega^i$. Coboundaries are dynamically empty because, as is easy to see, they can be removed by a field redefinition. Thus, in the unfolded formulation, the Chevalley-Eilenberg cohomology classifies possible nontrivial mixings of higher form fields with 0-forms. This type of mixing is of most importance in the context of known HS theories.

More generally, one can imagine the equations with the terms of the type of (5.14) that involve $p > 0$ forms on the r.h.s. The consistency condition then is that the cohomology algebra with respect to the natural product of the elements $f(V_1, V_2)$ and $\tilde{f}(\tilde{V}_1, \tilde{V}_2)$ with $V_2 = \tilde{V}_1$ is nilpotent in the sense that (5.2) is true.

The free HS system (3.1), (3.6) and (3.7) has unfolded form. It is consistent in the sense of (5.2). The terms on the r.h.s. of the equations (3.6) describe the Chevalley-Eilenberg cohomology of $sp(4, \mathbb{R})$ with coefficients in the corresponding infinite dimensional modules. Let us note that without these cohomological terms, i.e., relaxing the r.h.s. of the equation (3.6), the sector of 1-forms would become dynamically trivial (any solution of the zero curvature equation is pure gauge in the topologically trivial situation). The gluing with 0-forms via (3.6) and (3.7) makes the 1-form gauge potential dynamically nontrivial and, the same time, expresses the 0-forms in terms of derivatives of the gauge potentials (except for spin zero and spin 1/2 fields that have no associated gauge potentials because of the second derivatives over $y$ and $\bar{y}$ on the r.h.s. of (3.17)).

Note that the statement that $g$ extends to a larger symmetry algebra $\tilde{g}$ that acts in the $g$-modules $\omega^i$, which is true in the 0-form sector, may not be true in the other sectors in presence of the cohomological terms because it is not a priori guaranteed that a $g$-cohomology extends to a $\tilde{g}$-cohomology. To large extent our analysis of the $sp(8, \mathbb{R})$ symmetry in Section 12 amounts to the analysis of the extension of the $sp(4, \mathbb{R})$ HS cohomology to $sp(8, \mathbb{R})$.

It is important that in presence of the cohomological terms (5.14) the system remains
invariant under the global symmetry $g$. Indeed, the system (5.6) along with

$$d\omega^\Omega = G^\Omega_{\Phi}(w)\omega^\Phi$$

is formally consistent and therefore gauge invariant. The global symmetry $g$ is still the part of the gauge symmetry that leaves invariant the vacuum fields $w^\alpha$. Its action on $\omega^\Phi$ is deformed however by the cohomological terms (5.14) according to (5.3). In addition the system is invariant under the Abelian gauge transformations (5.5) associated with $p > 0$ form fields among $\omega^\Phi$.

For example, the system (3.1), (3.6) and (3.7) is invariant under the HS gauge transformations (5.16) associated with the 1-form connections $\omega(y, \bar{y}|x)$, which are Abelian gauge transformations for free massless fields of spins $s \geq 1$ whose form depends on the vacuum $AdS$ fields $w^{AB} = (\omega^{\alpha\bar{\beta}}, \varpi^{\alpha'}\beta', e^{\alpha'\beta'})$ that enter via the flat covariant derivative $D^{ad}$. Also the system (3.1), (3.6) and (3.7) is invariant under the global $sp(4, \mathbb{R})$ symmetry that leaves invariant the $AdS_4$ vacuum fields $w^{AB}$. The formula (5.3) applied to the equation (3.6) gives the $sp(4, \mathbb{R})$ transformation law

$$\delta^g\omega(y, \bar{y}|x) = \left( \varepsilon^{\alpha\beta}(x)y_\alpha \frac{\partial}{\partial y^\beta} + \varepsilon^{\alpha'\beta'}(x)\bar{y}_\alpha' \frac{\partial}{\partial \bar{y}^{\beta'}} + \lambda \varepsilon^{\gamma\delta}(x)\left(y_\alpha \frac{\partial}{\partial y^\gamma} + \frac{\partial}{\partial y^\gamma}\bar{y}_\delta\right) \right)\omega(y, \bar{y}|x)$$

$$+ 2\varepsilon^{\alpha'\alpha}(x)e^{\beta\gamma}(x)\frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}} C^{1-i}(0, \bar{y}|x) + 2\varepsilon^{\alpha'\alpha}(x)e^{\beta\gamma}(x)\frac{\partial^2}{\partial y^\gamma \partial \bar{y}^{\beta'}} C^{1-i}(y, 0|x),$$

$$\delta^g C(y, \bar{y}|x) = \left( \varepsilon^{\alpha\beta}(x)y_\alpha \frac{\partial}{\partial y^\beta} + \varepsilon^{\alpha'\beta'}(x)\bar{y}_\alpha' \frac{\partial}{\partial \bar{y}^{\beta'}} + \lambda \varepsilon^{\gamma\delta}(x)\left(y_\alpha \frac{\partial}{\partial y^\gamma} + \frac{\partial}{\partial y^\gamma}\bar{y}_\delta\right) \right)C(y, \bar{y}|x),$$

where the parameters of global $sp(4, \mathbb{R})$ transformations $e^{\alpha\beta}(x)$, $e^{\alpha'\beta'}(x)$ and $e^{\alpha\alpha'}(x)$ satisfy the $sp(4, \mathbb{R})$ covariant constancy conditions (5.3). The $C$-dependent terms in (5.16) generalize to HS fields the well-known formula that represents diffeomorphisms in gravity as the deformation of the $o(d-1, 2)$ (or Poincare') gauge transformations by the Riemann tensor dependent term (see e.g. [54]). Thus, the unfolded form of field equations reproduces necessary deformation terms automatically via the Chevalley-Eilenberg cohomology deformation.

Finally, let us note that the condition that the mixture of differential forms via Chevalley-Eilenberg cohomology is perturbatively nontrivial, i.e., that the terms (5.14) do not accidentally vanish because of unlucky choice of the vacuum connection $w$, may impose an additional restriction on the latter. We shall see below that it is this condition that gives preference to $AdS_4$ geometry in the models of interest because the relevant terms (5.14) turn out to be nondegenerate in the $AdS_4$ case but trivialize in the Minkowski case.

### 5.4 Dynamical content via $\sigma_-$ cohomology

In the unfolded dynamics approach, dynamical fields (i.e., those that are neither pure gauge nor auxiliary (see footnote 2)), their differential gauge symmetries (i.e., those that are not Stueckelberg (=shift) symmetries) and differential field equations (i.e., those that are not constraints), are characterized by the so-called $\sigma_-$ cohomology [55] (see also [3, 29]). The
aim of this Subsection is to recall briefly the main idea of this method to make it possible to explain in Section 6 the general strategy of the search and investigation of the \( sp(8, \mathbb{R}) \) invariant equations. To keep it short, the consideration of this Subsection is general and formal. It will be applied to the analysis of the unfolded HS field equations first in Section 14, explaining in some detail the dynamical content of the 4d unfolded HS equations and, in particular, the Central On-Shell Theorem, and then in Section 15 to the case of \( \mathcal{M}_4 \). Sections 14 and 15 will provide examples clarifying the formal scheme sketched in this Subsection.

\( \sigma_- \)-cohomology is a perturbative concept that emerges in the linearized analysis. The equation (5.18) linearized by (5.10) gives

\[
\mathcal{D} \omega^\Omega(x) = 0 ,
\]

where \( \mathcal{D} \) is some differential built of the order zero flat vacuum connection \( w^\alpha \). To fulfill the consistency condition (5.2), \( \mathcal{D} \) should square to zero

\[
\mathcal{D}^2 = 0 .
\]

Generally, \( \mathcal{D} \) is the generalized covariant derivative that, in addition to usual connection-like terms linear in the vacuum connection \( w^\alpha \), contains all Chevalley-Eilenberg-type terms polylinear in the \( w^\alpha \). For a \( p_\Omega \)-form \( \omega^\Omega \) with \( p_\Omega \geq 1 \), the linearized gauge transformation is

\[
\delta \omega^\Omega = \mathcal{D} \epsilon^\Omega ,
\]

where \( \epsilon^\Omega(x) \) is a \( (p_\Omega - 1) \)-form gauge parameter.

For a meaningful dynamical interpretation of the equation (5.18), a space \( V \), where fields \( \omega^\Omega \) take their values, should be endowed with some grading \( G \) such that its spectrum is bounded from below. Typically \( G \) counts a rank of a tensor (equivalently, a power of an appropriate generating polynomial) and eventually is associated with the order of space-time derivatives of dynamical fields. Suppose that

\[
\mathcal{D} = \mathcal{D}_0 + \sigma_- + \sigma_+ ,
\]

where \( [G, \sigma_-] = -\sigma_-, [G, \mathcal{D}_0] = 0 \) and \( \sigma_+ \) is a sum of some operators of positive grade. From (5.19) it follows that \( \sigma_-^2 = 0 \). Provided that \( \sigma_- \) acts vertically (i.e., does not differentiate \( x^n \)), cohomology of \( \sigma_- \) determines the dynamical content of the dynamical system at hand. Namely, as shown in [55], for a \( p \)-form \( \omega^\Omega \) that takes values in a vector space \( V \), \( H^{p+1}(\sigma_-, V) \), \( H^p(\sigma_-, V) \) and \( H^{p-1}(\sigma_-, V) \) characterize, respectively, differential equations, dynamical fields and differential gauge symmetries encoded by the equation (5.18).

The meaning of this statement is quite simple. From the level-by-level analysis of the equations (5.18) and (5.21) it follows that all fields that do not belong to \( \text{Ker} \sigma_- \) are auxiliary, being expressed by (5.18) via derivatives of the lower grade fields. Those that are \( \sigma_- \) exact can be gauged away by the Stueckelberg part of the gauge transformation (5.20), associated with the \( \sigma_- \) part of \( \mathcal{D} \) in (5.21). The fields that remain belong to the cohomology of \( \sigma_- \).

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4 Let us note that higher \( H^p(\sigma_-, V) \) describe so called syzygies of the field equations.
These are dynamical fields. Analogously one analyzes the dynamical content of the gauge transformations and field equations. (For more detail see e.g. [29].)

Usually $\sigma_-$ originates from the part of the covariant derivative of a space-time symmetry algebra, that contains vielbein required to be nondegenerate thus providing a frame of 1-forms at any point $x_0$ of space-time. The nondegeneracy of the vielbein allows to express as many as possible auxiliary fields via space-time derivatives of the dynamical fields.

The role of the Chevalley-Eilenberg cohomology terms in the unfolded equations is that they fill in unwanted $\sigma_-$ cohomologies with auxiliary Weyl tensor-like variables to avoid too strong differential consequences of the unfolded system. For example, relaxing the $C$-dependent terms on the r.h.s. of (3.17) would imply that all HS gauge 1-forms are pure gauge, which condition is too strong to describe nontrivial dynamics.

In principle, the leftover $\sigma_-$ cohomology responsible for the HS field equations can also be glued with the additional 0-form fields. This corresponds to the situation with $H^{p+1}(\sigma_-, V) = 0$ where no differential equations on the dynamical variables are imposed at all. In this case, the unfolded equation (5.11) just expresses the Bianchi identities for some constraints on auxiliary fields. Unfolded systems of this type are referred to as off-shell. As discussed in [56], they are useful for the Lagrangian formulation of a dynamical system.

Let us stress that the $\sigma_-$ cohomology analysis applies both to linear and to non-linear systems treated perturbatively. In particular, nonlinear equations are off-shell once their linearization is off-shell.

5.5 Properties

Let us summarize briefly the main properties of unfolded dynamics. First of all, the method is universal in the sense that any dynamical system can in principle be unfolded. This statement is analogous to the text-book fact that any system of ordinary differential equations can be reformulated in the first-order form.

Indeed, let $w = e_0^a P_a + \frac{1}{2} \omega_0^{ab} M_{ab}$ be a vacuum gravitational gauge field taking values in some space-time symmetry algebra $g$. Let $C^{(0)}(x)$ be a given space-time field satisfying some dynamical equations to be unfolded. Consider for simplicity the case where $C^{(0)}(x)$ is a 0-form. The general procedure of unfolding goes schematically as follows. For a start, one writes the equation $D^L_0 C^{(0)} = e_0^a C^{(1)}_a$, where $D^L_0$ is the covariant Lorentz derivative and the field $C^{(1)}_a$ is auxiliary. Next, one checks whether the original field equations for $C^{(0)}$ impose any restrictions on the first derivatives of $C^{(0)}$. More precisely, some part of $D^L_{0m} C^{(0)}$ might vanish on-mass-shell (e.g. for Dirac spinors). These restrictions in turn impose some restrictions on the auxiliary fields $C^{(1)}_a$. If these constraints are satisfied by $C^{(1)}_a$, then these fields parameterize all on-mass-shell nontrivial components of first derivatives. One continues by writing analogous equation for the first-level auxiliary fields $D^L_0 C^{(1)}_a = e_0^b C^{(2)}_{a,b}$, where the new fields $C^{(2)}_{a,b}$ parameterize the second derivatives of $C^{(0)}$. Once again one checks (taking into account the Bianchi identities) which components of the second level fields $C^{(2)}_{a,b}$ are nonvanishing provided that the original equations of motion are satisfied. This process continues indefinitely, leading to a chain of equations having the form of some covariant
constancy condition for the chain of fields $C^{(m)}_{a_1,a_2,...,a_m}$ ($m \in \mathbb{N}$) parameterizing all on-mass-shell nontrivial derivatives of the original dynamical field. By construction, this leads to a particular unfolded equation. The set of fields $C^{(m)}_{a_1,a_2,...,a_m}$ realizes some $g$-module $V$. The full infinite chain of equations becomes a single covariant constancy condition $D_0 C = 0$, where $D_0$ is the $g$-covariant derivative in $V$.

If one starts with some gauge field like, for example, the fluctuational part of the metric tensor, analogous analysis determines a form of the Stueckelberg shift gauge transformations that subtract Stueckelberg field components to be introduced to describe a system in terms of differential forms. (For instance in gravity, the local Lorentz symmetry results this way as the Stueckelberg symmetry that removes the extra components of the vielbein 1-form compared to the metric tensor.) The correspondence between $p \geq 1$ forms and gauge symmetries in the unfolded dynamics approach uncovers the pattern of local and global symmetries associated with a given gauge field. In particular, the pattern of the linearized 4d HS algebras was deduced this way in [43]. These results were then used in [57, 58, 47] to find infinite dimensional non-Abelian HS algebras that underly the nonlinear 4d HS theories and in [40, 4, 5] to construct full nonlinear HS field equations as a nonlinear deformation of the unfolded system (3.17) and (3.18).

Other important properties of the unfolded formulation include:

- Manifest gauge invariance and invariance under diffeomorphisms (i.e., coordinate independence) due to using the exterior algebra formalism is perfectly suited for the study of gauge invariant theories in the framework of gravity and, in particular, HS gauge theories.

- In the topologically trivial situation, degrees of freedom are concentrated in 0-forms $\omega^i_0(x_0)$ at any $x = x_0$. This is a consequence of the Poincare’ lemma: the unfolded equations express all exterior derivatives in terms of the values of fields themselves modulo exact forms that can be gauged away by the gauge transformation (5.5). What is left is the “constant part” of the 0-forms.

This simple observation has a consequence that, to describe a system with an infinite number of degrees of freedom, it is necessary to work with an infinite set of 0-forms that form an infinite dimensional module of the space-time symmetry $g$. In fact, the module carried by 0-forms turns out to be dual (complex equivalent) to the space of single-particle states in the respective QFT.

On the other hand, if the unfolded formulation of a system operates with a finite set of 0-forms, the system is topological, describing at most a finite number of degrees of freedom. In particular, the topological dynamical systems of [48] mentioned in the end of Section 3 are just of this type. A typical example of such a system is the equation (5.8) on the global symmetry parameters. In fact, the covariant constancy condition on the 0-forms in the topological sector of Section 3 coincides with the equation (5.8) on the HS global symmetry parameters.
• Unfolded formulation admits natural realization of higher derivative infinite symmetries as endomorphisms of the infinite-dimensional modules of 0-forms.

• Chevalley-Eilenberg cohomology of a Lie algebra $g$ underlying the unfolded formulation of one or another system is responsible for nontrivial mixture of differential forms of different degrees, thus making gauge fields associated with $(p > 0)$-forms dynamically nontrivial. Note that the corresponding cohomology turns out to be nontrivial even for simple Lie algebras associated with the space-time symmetries like $o(d,2)$, $sp(M,\mathbb{R})$, etc. just because it has coefficients in infinite-dimensional $g$-modules.

• Unfolded formulation unifies various dual versions of the same system. The difference results from the ambiguity in what is chosen to be dynamical or auxiliary fields, the nomenclature governed by the choice of the grading $G$ and $\sigma_-$. Different gradings lead to different interpretations of the same unfolded system in terms of different dynamical fields that satisfy seemingly unrelated differential equations. The key point is that if two dynamical systems give rise to the same unfolded system, they are equivalent.

We conjecture that all dual descriptions of a given dynamical system $D$ are contained in its maximally extended projective unfolded version $P(D)$. By a $P(D)$-projective unfolded system we mean such a maximal unfolded formulation of $D$ that (i) any unfolded description of $D$ is a subsystem of $P(D)$ and (ii) $P(D)$ does not decompose into two independent subsystems one of which is an unfolded formulation of $D$. Note that $P(D)$ may require a larger set of differential form variables.

For example, the extension of the set of Weyl 0-forms $C(y, \bar{y}\lvert x)$ by the HS gauge 1-forms $\omega(y, \bar{y}\lvert x)$ leads to the unfolded system (3.17), (3.18) that extends the HS equations (2.3) in terms $C(y, \bar{y}\lvert x)$ to those in terms of HS gauge potentials. The unfolded system (3.17), (3.18) is not projective however. It can be further extended without changing its dynamical content by replacing 1-forms $\omega^{ii}$ and 0-forms $C^{i1-i}$ by forms of all odd and even degrees, respectively. The resulting system is likely to be projective for the doubled set of 4$d$ massless fields of all spins.

The concept of $P(D)$-projective unfolded system is homological in nature. General analysis of this interesting and important issue lies beyond the scope of this paper and will be given elsewhere.

• One of the striking features of the unfolded formulation based on universal FDAs is that, to some extent, it makes the space-time $x$-dependence artificial. The dynamics is entirely encoded in the form of the function $G^i(W)$. In particular, unfolded formulation allows one to extend space-time without changing dynamics simply by letting the

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5It is worth to note that, as shown in [59], differential duality relations between dual systems in flat space may sometimes become algebraic in $AdS$ geometry. This phenomenon is analogous to the $AdS$ resolution of the flat space degeneracy discussed below: some operators to be interpreted as components of $\sigma_-$ that degenerate in flat background may be non-degenerate in the $AdS$ background.
differential \( d \) and differential forms \( W^\Phi \) to live in a larger space

\[
d = dx^n \frac{\partial}{\partial x^n} \rightarrow \hat{d} = dx^n \frac{\partial}{\partial x^n} + dx^n \frac{\partial}{\partial \hat{x}^n}, \quad dx^n w_n \rightarrow dx^n w_n + d\hat{x}^n \hat{w}_n,
\]

where \( \hat{x}^n \) are some additional coordinates. For a universal unfolded system such a substitution neither spoils the consistency nor changes the local dynamics still determined by the 0-forms at any point of (any) space-time. Alternatively, one observes that the unfolded system in the \( x \) space remains a subsystem of that in the enlarged space while the additional equations reconstruct the dependence on the additional coordinates in terms of solutions of the original system (of course, this consideration is local).

This property is not only practically useful allowing to introduce easily appropriate hyperspaces [3, 60], but is likely to have deep meaning encouraging to reconsider a role of such fundamental concepts as local event and metric tensor in a fundamental theory. An illuminating manifestation of this issue comes from the analysis of 4d HS physics formulated in \( M_4 \) in [8], where it was argued that the concepts of local event, space-time dimension and metric tensor have dynamical origin. In this respect, the unfolded dynamics [40] has some similarity with the matrix model approach to string theory [61, 62, 63], having however the advantage of being covariant. (See also [64, 65] where matrix models are linked to HS theory in a somewhat different fashion.)

- The unfolded formulation approach provides an efficient tool for the analysis of gauge invariant interactions. The problem reduces to the unification of the zero-order vacuum field \( w \) and first-order dynamical gauge fields \( \omega \) into a single field \( W^\Omega \) by (5.10) and to searching for a nontrivial deformation of \( G^\alpha(W) \) that respects the consistency condition (5.2) and reproduces correctly the linearized dynamics. In fact, the results of this paper form a basis for the future search of nonlinear \( sp(8, \mathbb{R}) \) invariant HS gauge theories.

Note that there is a great similarity between the unfolded formulation approach and the prolongation technics in the analysis of partial differential equations (see e.g., textbook [66]) that operates with jet spaces designed to describe higher derivatives to make it possible to rewrite a partial differential system in the first order form. The important novelty is due to the extension to differential forms as dynamical variables in the unfolded dynamics approach, that results in the properties listed above. Note also that the unfolded formulation admits a nice interpretation [56, 67] in terms of \( L_\infty \) strong homotopy algebra [68].

6 Strategy

Equipped with the unfolded dynamics approach, the strategy of the analysis of free \( g \)-symmetric dynamical equations may be as follows.

- Fix a flat (vacuum) connection \( w \) of \( g \) that gives a nondegenerate vielbein as the 1-form connection associated with the generators of translations in the respective space-time symmetry subalgebra \( s \subset g \) (e.g., Poincare’, \( (A)dS_d \), or conformal).
• Guess a set of $g$-modules where the variables $\omega^\Phi$, that are differential forms of different degrees, take their values and find Chevalley-Eilenberg cohomology \([5.14]\) of $g$ with the coefficients in the respective $g$-modules. It is this step that fixes a particular dynamical system encoded by the unfolded equations \([5.18]\).

• Introduce a grading $G$ of $g$-modules where the $\omega^\Phi$ take their values, that gives rise to the decomposition \([5.21]\) of the generalized covariant derivative. Since, the ambiguity in the choice of $G$ results in dual descriptions of the same model, to describe a system in terms of some preferable dynamical variables one has to choose $G$ appropriately. Analyze the $\sigma_-$ cohomology to figure out what are differential gauge symmetries, dynamical fields and differential field equations encoded by the unfolded equations \([5.15]\).

• Let $M$ be some $s$-invariant manifold and $w$ be a vacuum connection of $s$ with a nondegenerate vielbein. A larger symmetry $g$ of an unfolded system at hand may admit no geometric interpretation in $M$. To find an equivalent formulation of the same unfolded dynamics in a larger space-time $\mathcal{M}$ where $g$ acts geometrically it suffices to extend the exterior differential $d$ from $M$ to $\mathcal{M}$ and to replace $w$ by a flat connection of $g$ that contains nondegenerate basis 1-forms (generalized vielbein) in $\mathcal{M}$. Provided that the Chevalley-Eilenberg cohomology terms were defined with respect to $g$, the resulting unfolded equations in $\mathcal{M}$ remain consistent and describe the same dynamics. Keeping the grading $G$ unchanged, one finds the new $\sigma_-$ operator in $\mathcal{M}$. The dynamical content (i.e., symmetries, field variables and field equations) of the unfolded system in $\mathcal{M}$ is uncovered via the analysis of the cohomology of the new $\sigma_-$. Although the resulting dynamical variables and field equations in $\mathcal{M}$ may differ from those of the original system in $M$ (pretty much as superfields and superfield equations of supersymmetric theories look differently from their component counterparts), the resulting system in $\mathcal{M}$ is guaranteed to be $g$-invariant and (locally) equivalent to that in $M$.

In the rest of the paper we systematically implement this approach. Although it still has some research freedom in guessing appropriate $g$-modules, for distinguished systems like the one explored in this paper this part of the project is not too hard because the choice is usually quite limited if not obvious. The benefit is that the $g$-symmetry is guaranteed and the rest of the analysis is straightforward. Most notably, the $g$-invariant differential equations are derived (rather than guessed) via the analysis of $\sigma_-$ cohomology and the equivalence of the equations in different spaces is automatic.

7 Conformal geometry

To describe a conformal system in $d$ dimensions in the unfolded dynamics approach one should first of all fix a nondegenerate flat connection of the conformal algebra $o(d,2)$. In terms of Lorentz (i.e., $o(d-1,1)$) irreducible components, the $o(d,2)$ has the generators $P_n, L_{nm} = -L_{mn}, D, K^n$. The $o(d,2)$ connection 1-form $w$ and curvature 2-form $R$ are

$$w = h^n P_n + \omega^{nm} L_{nm} + f_n K^n + b D, \quad R = R^n P_n + R^{nm} L_{nm} + r_n K^n + r D,$$

(7.1)
where
\[
R^n = dh^n + \omega^n_m h^m - b \land h^n, \\
R^{nm} = d\omega^{nm} + \omega^n_k \land \omega^k_m - h^n \land f^m + h^m \land f^n, \\
r = db + h^n \land f_n, \\
r^n = df^n + \omega^n_m \land f^m + b \land f^n.
\]

Here the 1-form \( h^n = dx^n h^m_m \) is identified with vielbein. It is required to be nondegenerate in the sense that \( \det |h^m_m| \neq 0 \). \( \omega^{nm} \) is Lorentz connection. \( f_n \) and \( b \) are gauge fields for special conformal transformations and dilatation, respectively.

The conformal gauge transformations are
\[
\delta h^n = D^L e^n - e^n_m h^m + \epsilon h^n - \epsilon^n b, \\
\delta \omega^{nm} = D^L e^{nm} - h^n e^m + e^n f^m + h^m e^n - e^m f^n, \\
\delta b = d\epsilon + h^n \land \bar{\epsilon}_n - e^n f_n, \\
\delta f^n = D^L \bar{e}^n - e^n_m h^m - \epsilon f^n + \bar{\epsilon}^n b,
\]
where \( e^n(x) \), \( e^{mn}(x) \), \( \bar{\epsilon}_n(x) \) and \( \epsilon(x) \) are gauge parameters of translations, Lorentz transformations, special conformal transformations and dilatations, respectively.

The interpretation of these fields is as follows (for more detail see, e.g., [69, 35]). The 1-form \( b \) can be gauge fixed to zero
\[
b = 0 \quad (7.2)
\]
by a special conformal gauge transformation with the parameter \( \bar{\epsilon}^n(x) \). (Here one uses that \( h^n \) is nondegenerate.) The leftover gauge symmetries are local translations, Lorentz transformations and dilatations.

Imposing the condition that the \( o(d, 2) \) curvatures are all zero
\[
R = 0 \quad (7.3)
\]
has the following consequences. \( R^n = 0 \) in the gauge (7.2) is the usual zero torsion condition that expresses the Lorentz connection \( \omega^{nm} \) in terms of the vielbein \( h^n \). The condition \( R^{nm} = 0 \) requires the Weyl tensor \( C^{nm,kl}(h) \) to be zero and expresses the symmetric part \( f_{(nm)} \) of \( f^n = dx^n f_m^n \) in terms of the Ricci tensor of \( h \). The antisymmetric part of \( f_m^n \) is zero by virtue of \( r = 0 \) in the gauge (7.2). \( r^n = 0 \) holds by virtue of Bianchi identities. Thus, in terms of the vielbein, (7.3) implies that Weyl tensor is zero. All other equations contained in (7.3) are either constraints or consequences of the other equations.

The translation of these results into the \( \sigma_- \) cohomology language is as follows. The grading \( G \) is just the conformal dimension in \( o(d, 2) \) induced by \( D \), i.e., \( h^n \) has grading \(-1\), \( \omega^{nm} \) and \( b \) have grading zero and \( f_n \) has grading \(+1\). \( \sigma_- \) is the \( h \)-dependent part the covariant derivative where \( h^n \) is treated as the respective part of the vacuum connection. \( (h^n = dx^n \) in Cartesian coordinates.) \( H^0(\sigma_-) \) describes linearized diffeomorphisms (all other gauge transformations are Stueckelberg). \( H^1(\sigma_-) \) describes the conformal class of metrics.
(perturbatively, second rank traceless tensors). $H^2(\sigma_-)$ describes the Weyl tensor (i.e., the only nontrivial differential equation in (7.3) is that the Weyl tensor is zero). We leave it to the reader to check details of this correspondence as a useful exercise.

Taking into account that, locally, any two $o(d,2)$ flat connections are related by a $o(d,2)$ gauge transformation and that $o(d,2)$ gauge transformations contain local dilatations of the metric, a simple consequence of this analysis is that the metric tensor is conformally flat iff the Weyl tensor is zero.

In Cartesian coordinates, the Minkowski space solution of (7.3) is $w = dx^n P_n$. Another important example of conformally flat space is provided by the anti-de Sitter geometry. Indeed, the $AdS_d$ algebra $o(d-1,2)$ can be realized as the subalgebra of $o(d,2)$ spanned by the generators $P_n = P_n + \lambda^2 K_n$ and $L_{nm}$. Choosing a flat connection of $o(d-1,2)$, namely $e^n$ and $\omega^{nm}$, gives us a flat connection of $o(d,2)$ with $h^n = \lambda e^n$, $f_n = \lambda e_n$, $b = 0$ and the same Lorentz connection $\omega^{nm}$. This Ansatz solves (7.3) for the conformal algebra once $e^n$ and $\omega^{nm}$ solve the zero curvature equations for $o(d-1,2)$. As a by-product, this gives a coordinate independent proof of the fact that $AdS_d$ is conformally flat.

In the 4d case one can use two-component spinor notation. In these terms, the $su(2,2) \sim o(4,2)$ connections are $h^\alpha{}^\beta, \omega^{\alpha}{}_{\beta}, \bar{\omega}^{\alpha}{}_{\beta}, b$ and $f_{\alpha\beta}$. Extending $su(2,2)$ to $u(2,2)$ by adding a central helicity generator with the gauge connection $\tilde{b}$, the $u(2,2)$ flatness conditions read as

\begin{align*}
R^{\alpha\beta} &= dh^{\alpha\beta} - \omega^{\alpha}{}_{\gamma} h^{\gamma\beta} - \bar{\omega}^{\gamma}{}_{\beta} h^{\alpha\gamma} = 0, \quad (7.4) \\
R_{\alpha\beta} &= df_{\alpha\beta} + \omega^{\alpha}{}_{\gamma} f^{\gamma\beta} + \bar{\omega}^{\gamma}{}_{\beta} f_{\alpha\gamma} = 0, \quad (7.5) \\
\bar{R}_{\alpha'\beta'} &= d\omega^{\alpha}{}_{\beta} + \omega^\gamma{}_{\beta} \omega^{\alpha}{}_{\gamma} - f_{\alpha\gamma} h^{\gamma\beta} = 0, \quad (7.6) \\
\bar{R}^{\alpha'\beta'} &= d\bar{\omega}^{\alpha}{}_{\beta} + \bar{\omega}^{\gamma}{}_{\beta} \bar{\omega}^{\alpha}{}_{\gamma} - f_{\gamma\alpha'} h^{\gamma\beta} = 0. \quad (7.7)
\end{align*}

Here the traceless parts $\omega^{L\alpha}{}_{\beta}$ and $\bar{\omega}^{L\alpha}{}_{\beta}$ of $\omega^{\alpha}{}_{\beta}$ and $\bar{\omega}^{\alpha}{}_{\beta}$ describe the Lorentz connection while their traces contain the gauge fields $b$ and $\tilde{b}$ according to

\begin{equation}
\begin{align*}
b &= \frac{1}{2} \left( \omega^\alpha{}_{\alpha'} + \bar{\omega}^{\alpha}{}_{\alpha'} \right), \\
\tilde{b} &= \frac{1}{2} \left( \omega^\alpha{}_{\alpha'} - \bar{\omega}^{\alpha}{}_{\alpha'} \right). \quad (7.8)
\end{align*}
\end{equation}

The $AdS_4$ geometry is described by

\begin{equation}
\begin{align*}
h^{\alpha\alpha'} &= \lambda e^{\alpha\alpha'}, \\
f_{\alpha\beta} &= \lambda e_{\alpha\beta}, \\
\bar{b} &= \tilde{b} = 0. \quad (7.9)
\end{align*}
\end{equation}

The $u(2,2)$ flatness conditions are solved by the vierbein $e^{\alpha\alpha'}$ and Lorentz connection $\omega^{\alpha\beta}$ and $\bar{\omega}^{\alpha}{}_{\beta}$ that satisfy the zero curvature conditions (3.3). This Ansatz for the vacuum connection will be used later on to describe 4d conformal invariant systems in the unfolded dynamics approach.

8 Generalized conformal geometry

The example of $o(d,2)$ conformal symmetry admits a natural generalization to $sp(2M,\mathbb{R})$ treated as generalized conformal symmetry with the group manifold $Sp(M,\mathbb{R})$ as an analog
of $AdS_d$. (Note that the case of $M = 2$ reproduces the usual 3d case with $AdS_3 \sim Sp(2, \mathbb{R})$ and $sp(4, \mathbb{R}) \sim so(3, 2)$.) This interpretation of $Sp(M, \mathbb{R})$ was discussed in [70, 3, 10, 11].

The components of the $sp(2M, \mathbb{R})$ gauge connection (2.11) are real. Generalized “conformally flat” background geometry is described by a $sp(2M, \mathbb{R})$ flat connection that satisfies

$$R^{AB} = dh^{AB} - \omega^A_C \wedge h^{CB} - \omega^B_C \wedge h^{CA} = 0,$$

$$R_{AB} = df_{AB} + \omega^A_C \wedge f_{CB} + \omega^B_C \wedge f_{CA} = 0,$$

$$R_A^B = d\omega^B_A + \omega^A_C \wedge \omega^B_C - f_{AC} \wedge h^{CB} = 0.$$  

(8.1) (8.2) (8.3)

The “Cartesian coordinates” in $\mathcal{M}_M$ are associated with the particular flat connection

$$h^{AB} = dX^{AB}, \quad \omega^A_B = 0, \quad f_{AB} = 0.$$  

(8.4)

A different choice of the $sp(2M, \mathbb{R})$ flat connection describes the group manifold $Sp(M, \mathbb{R})$. The group of motions $Sp(M, \mathbb{R}) \times Sp(M, \mathbb{R})$ of $Sp(M, \mathbb{R})$ results from the left and right action of the group on itself. Let two flat $sp(M, \mathbb{R})$ connections satisfy

$$R^\pm_{AB} = dw^\pm_{AB} \mp w^\pm_A \wedge w^\pm_C = 0,$$

(8.5)

where indices are raised and lowered by a $Sp(M, \mathbb{R})$ invariant symplectic form $C_{AB} = -C_{BA}$ according to (3.2). Locally, they admit the pure gauge representation

$$w^\pm_{AB}(X) = \mp U^\pm_{AC}(X)dU^\pm_B(X),$$

where $U^\pm_B(X)$ is an arbitrary $Sp(M, \mathbb{R})$ valued matrix that satisfies

$$U^\pm_A(X)U^\pm_B(X)C_{CD} = C_{AB}, \quad U^\pm_A(X)U^\pm_B(X)C^{AB} = C^{CD}.$$  

(8.6)

Setting

$$\omega_{AB} = \frac{1}{2}(w^-_{AB} - w^+_{AB}), \quad \lambda e_{AB} = \frac{1}{2}(w^+_{AB} + w^-_{AB})$$

and using (8.4), we observe that

$$R_{AB} = d\omega_{AB} + \omega^A_C \wedge \omega_{CB} + \lambda^2 e^A_C \wedge e_{CB} = 0,$$

$$r_{AB} = de_{AB} + \omega^A_C \wedge e_{CB} + \omega^B_C \wedge e_{CA} = 0.$$  

(8.7)

To embed $sp(M, \mathbb{R}) \oplus sp(M, \mathbb{R})$ into $sp(2M, \mathbb{R})$ we express the connections of $sp(2M, \mathbb{R})$ in terms of those of $sp(M, \mathbb{R}) \oplus sp(M, \mathbb{R})$ as follows

$$h^{AB} = \lambda e^{AB}, \quad f_{AB} = \lambda e_{AB}, \quad \omega^B_A = \omega^A_B.$$  

(8.8)

Then the $sp(2M, \mathbb{R})$ flatness conditions (8.1)-(8.3) hold as a consequence of (8.5) and (8.6).

We will use the $sp(2M, \mathbb{R})$ flat vacuum connection (8.7) in the analysis of the $Sp(2M, \mathbb{R})$ invariant field equations in $Sp(M, \mathbb{R})$. The important difference between the Minkowski-like geometries and their $AdS$-like counterparts is that in the former case the special conformal connections $f^n$ and $f^{AB}$ vanish while in the latter case they are nondegenerate by virtue of (7.9) and (8.7). This property will be of crucial importance for the physical interpretation of the usual and generalized HS conformal equations because they contain some $f$-dependent terms that should be non-degenerate for a consistent perturbative interpretation.
9 Star-product and vacuum symmetry

Instead of working in terms of oscillators (2.8), it is convenient to use the star-product in the algebra of polynomials of commuting variables $a_A$ and $b^A$

$$(f \star g)(a, b) = \frac{1}{\pi^{2M}} \int f(a + u, b + t)g(a + s, b + v)e^{2(s_Au^A - a^Av^A)}d^M u d^M t d^M s d^M v. \quad (9.1)$$

The star-product defined this way, often called Moyal product, describes the product of symmetrized (i.e., Weyl ordered) polynomials of oscillators in terms of their symbols. The integral is normalized so that 1 is the unit element of the algebra

$$\frac{1}{\pi^{2M}} \int e^{2(s_Au^A - a^Av^A)}d^M u d^M t d^M s d^M v = 1.$$  

Eq. (9.1) defines the associative algebra with the defining relations

$$[a_A, b^B]_\star = \delta_A^B, \quad [a_A, a_B]_\star = 0, \quad [b^A, b^B]_\star = 0$$

($$[a, b]_\star = a \star b - b \star a$$). The following useful formulae hold

$$a_A^\star = a_A + \frac{1}{2} \frac{\partial}{\partial b_A}, \quad b^A_\star = b^A - \frac{1}{2} \frac{\partial}{\partial a_A}, \quad (9.2)$$

$$a_A^\star = a_A - \frac{1}{2} \frac{\partial}{\partial b_A}, \quad b^A_\star = b^A + \frac{1}{2} \frac{\partial}{\partial a_A}. \quad (9.3)$$

The star-product realization of the generators of $sp(2M, \mathbb{R})$ is

$$L_{AB} = a_A b^B, \quad P_{AB} = \frac{1}{2} a_A a_B, \quad K^{AB} = \frac{1}{2} b^A b^B. \quad (9.4)$$

$sp(2M, \mathbb{R})$ extends to the superalgebra $osp(1\vert 2M, \mathbb{R})$ by adding the supergenerators

$$Q_A = a_A, \quad S^B = b^B. \quad (9.5)$$

Using the oscillator realization of $sp(M, \mathbb{R}) \oplus sp(M, \mathbb{R}) \subset sp(2M, \mathbb{R})$ we can set

$$w(a, b\vert X) = \omega_B^A(X)a_A b^B + \frac{1}{2} \epsilon_{AB}(X)(a^A a^B + \lambda^2 b^A b^B), \quad (9.6)$$

where $\omega_{AB}$ and $\epsilon_{AB}$ satisfy the flatness conditions (8.5) and (8.6) to ensure that $w(a, b\vert X)$ satisfies the vacuum flatness condition

$$dw + w \star \wedge w = 0. \quad (9.7)$$

Let us introduce the oscillators $a^\pm_A = a_A \pm b^C C_{CA}$ with the commutation relations

$$[a^A_+, a^B_-]_\star = \pm 2 C_{AB}, \quad [a^A_+, a^B_+]_\star = 0.$$
Then $T_{AB}^{\pm} = \frac{1}{2} \alpha_A^\pm \alpha_B^\pm$ are the generators of $sp^+(M, \mathbb{R}) \oplus sp^-(M, \mathbb{R}) \subset sp(2M, \mathbb{R})$.

A useful viewpoint is that $w(a, b|X)$ takes values in the infinite dimensional star product algebra of various polynomials of $a_A$ and $b_B$, which is the HS symmetry algebra as a Lie superalgebra. (In accordance with the spin-statistics relationship, the boson-fermion $Z_2$ grading $\pi$ counts the oddness of a number of spinor indices, i.e., $w(a, b|X) = (-1)^{\pi(w)}w(-a, -b|X)$.) As explained in Subsection 5.2 any fixed vacuum solution $w_0$ of (9.7) breaks the local HS symmetry to its global stability subalgebra with the infinitesimal parameters $\epsilon_0(a, b|X)$ that satisfy the equation

$$d\epsilon_0 + [w_0, \epsilon_0]_* = 0.$$  \hspace{1cm} (9.8)

Solving (9.7) in the pure gauge form

$$w_0(a, b|X) = g^{-1}(a, b|X) \star dg(a, b|X),$$  \hspace{1cm} (9.9)

where $g(a, b|X)$ is some invertible element of the star-product algebra, $g \star g^{-1} = g^{-1} \star g = 1$, we solve (9.8) as

$$\epsilon_0(a, b|X) = g^{-1}(a, b|X) \star \xi(a, b) \star g(a, b|X),$$  \hspace{1cm} (9.10)

where an arbitrary $X$-independent star-product element $\xi(a, b)$ describes free parameters of the global HS symmetry. Note that the explicit form of the HS transformations depends on a chosen coordinate system encoded by $w_0(a, b|X)$.

As shown in [10], the star product realization of the pure gauge representation (8.5) is given by the formula

$$g(X) = g^{+}(X) \star g^{-}(X) = g^{+}(X)g^{-}(X),$$

where

$$g^{\pm}(X) = \frac{2\mathcal{M}}{\sqrt{\det \|U^{\pm} + 1\|}} \exp \left( -2f^{AB}[U^{\pm}]_{|aA^\pm, aB^\pm} \right),$$  \hspace{1cm} (9.11)

$$(g^{\pm})^{-1}(X) = \frac{2\mathcal{M}}{\sqrt{\det \|U^{\pm} + 1\|}} \exp \left( 2f^{AB}[U^{\pm}]_{|aA^\pm, aB^\pm} \right)$$  \hspace{1cm} (9.12)

and we have set $\lambda = 1$ for simplicity. Here $U^{AB}(X)$ is some $sp(M, \mathbb{R})$ valued function of local coordinates and we use the notations

$$f^{AB}[U] = \left( \frac{U - 1}{U + 1} \right)^{AB}, \quad U^{AB}[f] = \left( \frac{1 + f}{1 - f} \right)^{AB}.$$ \hspace{1cm}  

Note that the gauge functions $g^{\pm}(X)$ (9.11) are chosen so [10] that the corresponding flat connection $w^{\pm}$ is bilinear in the oscillators $a_A, b_B$, i.e., it indeed takes values in $sp^{\pm}(M, \mathbb{R})$. A particularly useful choice is that with $U^{+} = (U^{-})^{-1} = U$, giving

$$g(X) = \frac{2\mathcal{M}}{\det \|U + 1\|} \exp \left[ -f^{AB}[U](a_Aa_B + b_Ab_B) \right].$$

As noted in [10], the ambiguity in the function $f^{AB}(X)$ parameterizes the ambiguity in the choice of local coordinates of $Sp(M, \mathbb{R})$. (For particular coordinate choices see [10].)
Once the vacuum solution \( w_0 \) is fixed in the pure gauge form (9.9) with some gauge function \( g \), it is easy to find the gauge parameter \( \epsilon_0(a,b;X) \) of the leftover global symmetry. Indeed, let the generating parameter \( \xi(a,b;\mu,\eta) \) in (9.10) be of the form \( \xi = \xi_0 \exp(a_A \mu^A - b^A \eta_A) \) where \( \xi_0 \) is an infinitesimal constant while \( \mu^A \) and \( \eta_A \) are constant parameters. Substitution of (9.11) into (9.10) gives (10)

\[
\epsilon_0(a,b;\mu,\eta|X) = g^{-1} \star \xi \star g = \xi_0 \exp(a_A \hat{\mu}^A - b^A \hat{\eta}_A),
\]

where

\[
\hat{\mu}_A = \left( \frac{1 + \lambda^2 f^2(X)}{1 - \lambda^2 f^2(X)} \right)_A \hat{\mu}_B - \left( \frac{2f(X)}{1 - \lambda^2 f^2(X)} \right)_A \eta_B,
\]

\[
\hat{\eta}_A = \left( \frac{1 + \lambda^2 f^2(X)}{1 - \lambda^2 f^2(X)} \right)_A \eta_B - \lambda^2 \left( \frac{2f(X)}{1 - \lambda^2 f^2(X)} \right)_A \hat{\mu}_B.
\]

Now, any HS global symmetry parameter, which is some \( x \)-dependent star-product polynomial, can be obtained by differentiation of \( \epsilon_0(a,b;\mu,\eta|X) \) over \( \mu^A \) or/and \( \eta_A \).

This simple procedure demonstrates the efficiency of the unfolded dynamics approach. Analogously to the flat space example (5.13), to obtain the form of the HS transformation of massless fields in \( Sp(M,\mathbb{R}) \) one has to act by the parameter \( \epsilon_0(a,b;\mu,\eta|X) \) on a module where the fields take their values. The corresponding \( sp(8,\mathbb{R}) \)-modules, which at the same time form the HS symmetry-modules, are introduced in the next section.

10 \( Sp(8,\mathbb{R}) \) Fock modules

To extend the free \( sp(8,\mathbb{R}) \) invariant equation (2.15) on HS Weyl field strengths to the sector of gauge fields we have to identify a \( sp(8,\mathbb{R}) \)-module where 4d HS gauge fields of Section 3 take their values. Since the HS Weyl 0-forms were described in Section 2 in terms of the Fock module (2.6), (2.7) in which the \( sp(8,\mathbb{R}) \) is realized by bilinears of oscillators, a natural option is to use the same realization of \( sp(8,\mathbb{R}) \), changing however the Fock module by changing its vacuum.

Fock vacua projectors can be realized in terms of the star-product algebra. For example, with the help of (9.12) and (9.13) one finds that the Fock vacuum \( |0,0\rangle \) defined by \( a_A \star |0,0\rangle = |0,0\rangle \star b^A = 0 \) is realized as the exponential \( |0,0\rangle = 2^M \exp 2a_A b^A \), where the normalization factor is chosen so that \( |0,0\rangle \star |0,0\rangle = |0,0\rangle \). Alternatively, one can consider Fock vacua projectors defined with respect to different sets of creation and annihilation operators. Demanding their Lorentz invariance and definite scaling dimension with respect to the Lorentz generators in (2.12) and the dilatation generator (2.13), respectively, we set

\[
|1\rangle \langle 0| = 4 \exp -2a_\alpha b^\alpha : b^\beta \star |1\rangle \langle 0| = |1\rangle \langle 0| \star a_\beta = 0,
\]

\[
|0\rangle \langle 1| = 4 \exp 2a_\alpha b^\alpha : a_\beta \star |0\rangle \langle 1| = |0\rangle \langle 1| \star b^\beta = 0,
\]

\[
|1\rangle \langle 0| = 4 \exp -2\bar{a}_\alpha \bar{b}^{\alpha'} : \bar{b}^{\alpha'} \star |1\rangle \langle 0| = |1\rangle \langle 0| \star \bar{a}_{\beta'} = 0,
\]

\[
|0\rangle \langle 1| = 4 \exp 2\bar{a}_\alpha \bar{b}^{\alpha'} : \bar{a}_{\beta'} \star |1\rangle \langle 0| = |1\rangle \langle 0| \star \bar{b}^{\beta'} = 0.
\]
We have
\[ |0, 0| = |0 \rangle \langle 1| \star |\overline{0}\rangle \langle \overline{1}| = 16 \exp 2a_A b^A : a_A \star |0, 0| = |0, 0| \star b^A = 0 , \quad (10.5) \]
\[ |1, 0| = |1 \rangle \langle 0| \star |\overline{0}\rangle \langle \overline{1}| = 16 \exp -2(a_\alpha \beta^\alpha - \overline{a}_\alpha \overline{\beta}^\alpha) : \quad b^\beta \star |1, 0| = \overline{\alpha}_\beta \star |1, 0| = |1, 0| \star a_\beta = |1, 0| \star b^\beta = 0 , \quad (10.6) \]
\[ |0, 1| = |0 \rangle \langle 1| \star |\overline{0}\rangle \langle \overline{1}| = 16 \exp 2(a_\alpha \beta^\alpha - \overline{a}_\alpha \overline{\beta}^\alpha) : \quad a_\beta \star |0, 1| = \overline{\beta}^\gamma \star |0, 1| = |0, 1| \star b^\beta = |0, 1| \star \overline{a}_\beta = 0 , \quad (10.7) \]
\[ |1, 1| = |1 \rangle \langle 0| \star |\overline{1}\rangle \langle \overline{\overline{0}}| = 16 \exp -2a_A b^A : \quad b^B \star |1, 1| = |1, 1| \star a_A = 0 . \quad (10.8) \]

Correspondingly, we introduce two-component oscillators \( \alpha_\alpha, \overline{\alpha}_\alpha \) and \( \beta_i, \overline{\beta}_i \) that are, respectively, the annihilation and creation operators of the vacuum \( |i, j| \)
\[ \alpha_0 = a_\alpha , \quad \alpha_1 = b^\alpha , \quad \overline{\alpha}_0 = \overline{a}_\alpha' , \quad \overline{\alpha}_1 = \overline{\beta}^\alpha , \quad (10.9) \]
\[ \beta_0 = b^\alpha , \quad \beta_1 = a_\alpha , \quad \overline{\beta}_0 = \overline{\beta}^\alpha , \quad \overline{\beta}_1 = \overline{a}_\alpha . \quad (10.10) \]

Note that
\[ \beta_i = \alpha_{1-i} , \quad \overline{\beta}_i = \overline{\alpha}_{1-i} . \quad (10.11) \]

It should be noted that the Fock vacua \( |i, j| \) cannot be star-multiplied with \( |i', j'| \) in the class of regular functions if \( i \neq i' \) and/or \( j \neq j' \). This is not accidental. Indeed, if, say, \( T = |0, 0| \star |1, 1| \) existed, it would satisfy \( a_A \star T = T \star a_A = 0 \). Taking into account (9.2) and (9.3), from here it follows that \( T = \delta(a) \). (Of course, this can be directly derived from (9.1).) However \( T \) does not belong to the star-product algebra because \( T \star T = \delta(0) \).

On the other hand, different Fock spaces originating from \( |i, j| \) form well-defined modules of the star-product algebra. This is sufficient for the analysis of the free HS dynamics of this paper. To go beyond the free field level one has to handle potential difficulties of co-existence of different sectors of the star-product algebra associated with the different Fock modules. We hope to come back to this interesting question elsewhere.

Now we introduce four 0-form modules
\[ |C_{ij}(\beta_i, \overline{\beta}_j)X) = C_{ij}(\beta_i, \overline{\beta}_j)X) \star |i, j| \quad (10.12) \]
and four 1-form modules
\[ |\omega_{ij}(\beta_i, \overline{\beta}_j)X) = \omega_{ij}(\beta_i, \overline{\beta}_j)X) \star |i, j| , \quad \omega_{ij}(\beta_i, \overline{\beta}_j)X) = dX^U \omega_{ij}(\beta_i, \overline{\beta}_j)X) , \quad (10.13) \]
where the meaning of the coordinates \( X^U \) will be specified later on depending on the problem under study. More precisely, \( |C_{ij}(\beta_i, \overline{\beta}_j)X) \) and \( |\omega_{ij}(\beta_i, \overline{\beta}_j)X) \) are sections of the Fock fiber bundles over a space-time manifold with local coordinates \( X \), that can be either a 4d space-time or one of the ten dimensional space-times \( M_4 \) or \( Sp(4, \mathbb{R}) \).

The generating fields \( |C_{ij}(\beta_i, \overline{\beta}_j)X) \) and \( |\omega_{ij}(\beta_i, \overline{\beta}_j)X) \) form \( sp(8, \mathbb{R})\)-modules with the generators (9.4). This allows us to define the \( sp(8, \mathbb{R}) \) covariant derivatives \( D_{ij} \) in the module induced from the vacuum \( |i, j| \). Note that \( |C_{ij}(\beta_i, \overline{\beta}_j)X) \) and \( |\omega_{ij}(\beta_i, \overline{\beta}_j)X) \) also form
modules of $su(2,2)$ with the generators $(2.13), (2.12)$, of $osp(1|8, \mathbb{R})$ with the supergenerators $(9.5)$ and of the infinite dimensional HS superalgebra whose generators are various (i.e., not only bilinear) polynomials of $a_A$ and $b^A$. Let us stress that, because the vacua $|i, j|$ are Lorentz invariant and have definite scaling dimensions, so defined $sp(8, \mathbb{R})$-modules consist of towers of Lorentz multispinor fields with definite scaling dimensions. (If vacua were not Lorentz invariant, the resulting Lorentz algebra–modules could be infinite dimensional.)

In our construction we postulate that the oscillators $a_\alpha$ and $b^\alpha$ are complex conjugated to $\bar{a}_{\alpha'}$ and $\bar{b}^{\alpha'}$, respectively. With this convention the conjugation $\sigma$, that singles out the real form $sp(8|\mathbb{R})$ of $sp(8|\mathbb{C})$, acts as follows

$$\sigma(P_{AB}) = P_{AB}, \quad \sigma(L_A^B) = L_A^B, \quad \sigma(K^{AB}) = K^{AB}.$$  

From the definition of the Fock modules $|i, j|$ it follows then that $|\bar{i}, \bar{j}| = |j, i|$, $\bar{\alpha} = \bar{\alpha'}$, $\bar{\beta}_i = \bar{\beta}_{i'}$. Correspondingly, the following reality conditions are imposed

$$\omega_{ij}(\beta_i, \bar{\beta}_j) = \omega_{ji}(\beta_j, \bar{\beta}_i), \quad C_{ij}(\beta_i, \bar{\beta}_j) = C_{ji}(\beta_j, \bar{\beta}_i).$$

The original $sp(8, \mathbb{R})$ invariant form of the HS equations $(2.13)$ is the covariant constancy condition $D_{00}C_{00}(b|X) = 0$ which is the analogue of flat space limit of the $4d$ equation $(3.18)$ in Cartesian coordinates. Our aim is to extend the equations $(3.17)$ and $(3.18)$ first to $su(2,2)$ and then to $sp(8,\mathbb{R})$ symmetric formulations. To make it possible to use general properties of the unfolded formulation, this will be done for generic $su(2,2)$ and $sp(8,\mathbb{R})$ flat connections in Sections 11.1 and 11.2 respectively. As a result, the obtained systems will be proven to have global $su(2,2)$ and $sp(8,\mathbb{R})$ symmetries that act both on the 1-form HS gauge fields and on the 0-form field strengths.

11 Conformal invariant unfolded massless equations

11.1 Consistent equations

The $u(2, 2) \subset sp(8, \mathbb{R})$ covariant derivatives in the various Fock modules are defined by $D^c|C_{ij}(\beta_i, \bar{\beta}_j|X)$ where

$$D^c = d + w, \quad w = h^{\alpha\alpha'}a_\alpha \bar{a}_{\alpha'} + \omega_\beta^\alpha a_\alpha b^\beta + \bar{\omega}_\beta^{\alpha'}a_{\alpha'} \bar{b}^{\beta'} + f_{\alpha\beta'} b^\alpha \bar{b}^{\beta'}.$$  

Here the traceless parts $\omega^L_{\alpha\beta}$ and $\bar{\omega}^L_{\alpha'\beta'}$ of $\omega_{\alpha\beta}$ and $\bar{\omega}_{\alpha'\beta'}$, respectively, describe the Lorentz connection while the traces describe the gauge fields $b$ and $\bar{b}$ $(7.8)$. For the generating functions $(10.12)$ this defines the the covariant derivatives $D^c_{ij}$,

$$\left(D^c_{ij}C_{ij}(\beta_i, \bar{\beta}_j|X)\right) * |i, j| = D^c|C_{ij}(\beta_i, \bar{\beta}_j|X),$$

which have the form

$$D^c_{10} = D^L - \frac{1}{2} \omega_\alpha^{\alpha'} (a_\beta \frac{\partial}{\partial a_\beta} + 1) + \frac{1}{2} \bar{\omega}^{\alpha'}_{\alpha'} (\bar{b}^{\beta'} \frac{\partial}{\partial b^{\beta'}} + 1) + h^{\alpha\beta} a_\alpha \frac{\partial}{\partial b^{\beta}} - f_{\alpha\beta'} b^\alpha \frac{\partial}{\partial a_\alpha} \bar{b}^{\beta'}. \quad (11.1)$$
\[ D_{01}^{\text{con}} = D^L + \frac{1}{2} \omega_\alpha (b_\beta \frac{\partial}{\partial b_\beta} + 1) - \frac{1}{2} \omega_\alpha' (\bar{a}_\beta' \frac{\partial}{\partial \bar{a}_\beta'} + 1) + h^{\alpha \beta} \bar{a}_{\beta'} \frac{\partial}{\partial \bar{a}_{\beta'}} - f_{\alpha \beta'} \frac{\partial}{\partial b_{\beta'}} b_\alpha , \quad (11.2) \]

\[ D_{00}^{\text{con}} = D^L + \frac{1}{2} \omega_\alpha (b_\beta \frac{\partial}{\partial b_\beta} + 1) + \frac{1}{2} \omega_\alpha' (\bar{b}_{\beta'} \frac{\partial}{\partial \bar{b}_{\beta'}} + 1) + h^{\alpha \beta} \frac{\partial^2}{\partial b_\alpha \partial b_{\beta'}} + f_{\alpha \beta'} b_\alpha \bar{b}_{\beta'} , \quad (11.3) \]

\[ D_{11}^{\text{con}} = D^L - \frac{1}{2} \omega_\alpha (a_\beta \frac{\partial}{\partial a_\beta} + 1) - \frac{1}{2} \omega_\alpha' (\bar{a}_\beta' \frac{\partial}{\partial \bar{a}_\beta'} + 1) + h^{\alpha \beta} a_{\alpha} \bar{a}_{\beta'} + f_{\alpha \beta'} \frac{\partial^2}{\partial a_\alpha \partial \bar{a}_{\beta'}} , \quad (11.4) \]

where \( D^L \) is the Lorentz covariant derivative \((3.15)\).

From the form of covariant derivatives \( D_{ij}^{\text{con}} \), it follows in particular that the operators of helicity \( \mathcal{H} \ (2.14) \) and dilatation \( \mathcal{D} \ (2.13) \) act on the respective modules \( \phi_{ij} = \omega_{ij} \) or \( \phi_{ij} = C_{ij} \) as follows

\[ \mathcal{H} \phi_{10}(a, \bar{b}) = \frac{1}{2} \left( \frac{\partial}{\partial a_\alpha} + b_\alpha \frac{\partial}{\partial b_\alpha} \right) \phi_{10}(a, \bar{b}) , \quad (11.5) \]

\[ \mathcal{H} \phi_{01}(b, \bar{a}) = \frac{1}{2} \left( \frac{\partial}{\partial b_\alpha} + \bar{a}_\alpha \frac{\partial}{\partial \bar{a}_\alpha} \right) \phi_{01}(b, \bar{a}) , \quad (11.6) \]

\[ \mathcal{H} \phi_{00}(b, \bar{b}) = \frac{1}{2} \left( \frac{\partial}{\partial b_\alpha} - \bar{b}_\alpha \frac{\partial}{\partial \bar{b}_\alpha} \right) \phi_{00}(b, \bar{b}) , \quad (11.7) \]

\[ \mathcal{H} \phi_{11}(a, \bar{a}) = \frac{1}{2} \left( \frac{\partial}{\partial a_\alpha} - \bar{a}_\alpha \frac{\partial}{\partial \bar{a}_\alpha} \right) \phi_{11}(a, \bar{a}) , \quad (11.8) \]

\[ \mathcal{D} \phi_{10}(a, \bar{b}) = \frac{1}{2} \left( \frac{\partial}{\partial a_\alpha} - \bar{a}_\alpha \frac{\partial}{\partial \bar{a}_\alpha} \right) \phi_{10}(a, \bar{b}) , \quad (11.9) \]

\[ \mathcal{D} \phi_{01}(b, \bar{a}) = \frac{1}{2} \left( a_\alpha \frac{\partial}{\partial b_\alpha} + \bar{b}_\alpha \frac{\partial}{\partial \bar{b}_\alpha} \right) \phi_{01}(b, \bar{a}) , \quad (11.10) \]

\[ \mathcal{D} \phi_{00}(b, \bar{b}) = \frac{1}{2} \left( b_\alpha \frac{\partial}{\partial b_\alpha} - \bar{b}_\alpha \frac{\partial}{\partial \bar{b}_\alpha} \right) \phi_{00}(b, \bar{b}) , \quad (11.11) \]

\[ \mathcal{D} \phi_{11}(a, \bar{a}) = \frac{1}{2} \left( a_\alpha \frac{\partial}{\partial a_\alpha} + \bar{a}_\alpha \frac{\partial}{\partial \bar{a}_\alpha} \right) \phi_{11}(a, \bar{a}) . \quad (11.12) \]

Now we are in a position to write the conformal invariant unfolded system of equations

\[ D_{10}^{\text{con}} \omega_{10}(a, \bar{b}) = h_{\alpha}^{\alpha'} \wedge h^{\alpha \beta} \frac{\partial^2}{\partial b_\alpha \partial b_{\beta'}} C_{00}(b, 0) + f_{\alpha \alpha'} \wedge f_{\beta}^{\alpha'} \frac{\partial^2}{\partial a_\alpha \partial a_{\beta'}} C_{11}(a, 0) , \quad (11.13) \]

\[ D_{01}^{\text{con}} \omega_{01}(b, \bar{a}) = h_{\alpha}^{\alpha'} \wedge h^{\beta \alpha'} \frac{\partial^2}{\partial b_{\beta} \partial b_{\alpha'}} C_{00}(b, 0) + f_{\alpha \alpha'} \wedge f_{\beta}^{\beta'} \frac{\partial^2}{\partial a_{\alpha'} \partial a_{\beta'}} C_{11}(0, \bar{a}) , \quad (11.14) \]

\[ D_{00}^{\text{con}} \omega_{00}(b, \bar{b}) = f_{\alpha \beta} \wedge h_{\beta}^{\alpha'} b_{\beta'} \bar{b}_{\beta'} C_{10}(0, \bar{b}) + f_{\alpha \beta} \wedge f_{\beta}^{\alpha'} b_{\beta} \bar{b}_{\beta} C_{01}(b, 0) , \quad (11.15) \]

\[ D_{11}^{\text{con}} \omega_{11}(a, \bar{a}) = h_{\alpha}^{\alpha'} \wedge h_{\beta}^{\beta'} \bar{a}_{\alpha'} \bar{a}_{\beta'} C_{10}(0, \bar{a}) + h_{\alpha}^{\alpha'} \wedge h_{\beta}^{\beta'} a_{\alpha} a_{\beta} C_{01}(a, 0) , \quad (11.16) \]

\[ D_{ij}^{\text{con}} C_{ij} = 0 . \quad (11.17) \]
This system decomposes into two independent subsystems. One contains the 1-forms $\omega_{ii}$ and 0-forms $C_{i}$ while another one contains $\omega_{1-i}$ and $C_{ii}$. As will be explained in Subsection 11.2, the subsystem (11.13), (11.16) and (11.17) with $i = j$ is topological while the subsystem (11.13), (11.14) and (11.17) with $i + j = 1$ describes massless fields of all spins.

The important property of the system (11.13)-(11.17) is that it is consistent for any flat $u(2, 2)$ connection. Let us for definiteness consider the sector of $\omega_{10}(a, \bar{b})$, i.e., the equation (11.13) along with

$$D_{00}^{\text{con}} C_{00}(b, \bar{b}) = 0, \quad D_{11}^{\text{con}} C_{11}(a, \bar{a}) = 0.$$  \hfill (11.18)

Because vacuum connections are such that $D_{ij}^{\text{con}}$ squares to zero, the proof of consistency is equivalent to checking that the application of $D_{10}^{\text{con}}$ to the r.h.s. of (11.13) gives zero by virtue of the equations (11.18). The analysis of the two terms on the r.h.s. of (11.13) is independent of each other. Since they are exchanged by the Chevalley automorphism that exchanges translations and special conformal transformations, we only consider the $h$–dependent term in (11.13), which is not a coboundary because the $h$–dependent terms in $D_{10}^{\text{con}}$ (11.1) are proportional to $a_{\alpha}$ while the $h$–dependent terms on the r.h.s. of (11.13) are $a$–independent. The proof of cocyclicity is elementary and consists of the following observations:

- Since the whole setup is Lorentz covariant, the $\omega^{L}, \varpi^{L}$–dependent terms in the consistency conditions cancel out.

- The $f_{\alpha\beta'}$–dependent terms cancel out because the one in $D_{10}^{\text{con}}$ contains derivative over $a_{\alpha}$ of the $a_{\alpha}$-independent expression on the r.h.s. of (11.13) while the other one in $D_{00}^{\text{con}} C_{00}(b, \bar{b})$ disappears upon setting $b = 0$.

- The $h^{3}$– terms vanish because

$$h^{\alpha\alpha'} \wedge h^{\beta\beta'} \wedge h^{\gamma\gamma'} \frac{\partial^{3}}{\partial b^{\alpha'} \partial b^{\beta'} \partial b^{\gamma'}} = 0$$  \hfill (11.19)

as a result of antisymmetrization of the three two-component indices $\alpha, \beta, \gamma$.

- The terms with $\varpi^{\alpha'}_{\alpha}$ are the same in $D_{10}^{\text{con}}$ and $D_{00}^{\text{con}}$ and cancel each other while the terms that result from the differentiation of $h_{\alpha}^{\alpha'} \wedge h_{\alpha}^{\beta'}$ by virtue of (7.4) compensate those that result from the commutator of the covariant derivative with $\frac{\partial^{2}}{\partial b^{\alpha'} \partial b^{\beta'}}$ (equivalently, the $L_{\alpha'}^{\alpha}$ weight of $h_{\alpha}^{\alpha'} \wedge h_{\alpha}^{\beta'}$ compensates that of $\frac{\partial^{2}}{\partial b^{\alpha'} \partial b^{\beta'}}$).

- Finally, the terms with $\omega_{\alpha}^{\alpha}$ also cancel out. Namely the differential parts trivialize either because of differentiating a constant $(D_{1,0}^{\text{con}})$ or setting $b^{3} = 0 (D_{0,0}^{\text{con}})$. The constant terms, which are different, then exactly cancel the result of differentiation of $h_{\alpha}^{\alpha'} \wedge h_{\alpha}^{\beta'}$ (Equivalently, the $L_{\alpha}^{\alpha}$ weight of $h_{\alpha}^{\alpha'} \wedge h_{\alpha}^{\beta'}$ is compensated by the difference of the weights of the Fock vacua $|1, 0|$ and $|0, 0|$).

The cocyclicity of all other terms on the r.h.s. of (11.13)-(11.16) is checked analogously. The essentials of the construction include
(i) The matching between the number of values of two-component indices and the form degree on the r.h.s. of \((11.13)-(11.16)\). This allows us to use the identity \((11.19)\).

(ii) The twist to \(C_{1-i} - i j\) or \(C_{i1} - j\) of the modules glued to \(\omega_{ij}\) via the r.h.s. of \((11.13)-(11.16)\) leads to the shifts of the vacuum helicities and conformal dimensions (i.e., the constant terms in the operators acting on \(\phi_{ij}\) in \((11.5)-(11.12)\)) of the respective Fock vacua that compensate those carried by \(a, b\) or \(\frac{\partial}{\partial a}\) and \(\frac{\partial}{\partial b}\) on the r.h.s. of \((11.13)-(11.16)\).

11.2 Dynamical content

First of all we observe that the covariant derivatives \(D^{con}_{i1} - i\) and \((11.2)\) preserve a degree of a polynomial on which it acts. This means that the Fock modules induced from the vacua \(|i, 1 - i\rangle\) decompose into infinite sums of finite dimensional \(u(2, 2)\)-modules carried by homogeneous polynomials. On the other hand, from the form of covariant derivatives \(D^{con}_{ii} (11.3)\) and \((11.4)\) it follows that the Fock modules induced from the vacua \(|i, i\rangle\) decompose into infinite sums of infinite dimensional \(u(2, 2)\)-modules.

As explained in Subsection 5.5, local degrees of freedom of a system are carried by 0-forms. We conclude that, the fields \(C_{i1} - i\) and, therefore, by virtue of \((11.13)\) and \((11.16)\), \(\omega_{ii}\) describe an infinite set of topological systems each carrying at most a finite number of degrees of freedom equal to the dimension of the space of polynomials of an appropriate degree. Note that the sector of gauge fields of this type was originally analyzed by far more complicated Hamiltonian methods in \([48]\) with the same conclusion that these fields are of topological type. (In \([48]\) these fields were called auxiliary to emphasize that they do not carry field-theoretical degrees of freedom.)

On the other hand, each irreducible subsystem in the sector of \(C_{ii}\) and \(\omega_{i1} - i\) describes an infinite number of degrees of freedom. These are massless fields of various spins. Since the gauge massless fields \(\omega_{10}(a_\alpha, \bar{b}^{\alpha'})\) are complex conjugated to \(\omega_{01}(b^{\alpha}, \bar{a}_{\alpha'})\), the system of equations \((11.13), (11.14)\) and \((11.17)\) at \(i = j\) describes the set of massless fields in which every spin appears twice. This pattern matches that of the \(AdS_4\) HS theories as discussed in Section 2 although the mechanism of the doubling is different.

The realization of the helicity operator \(\mathcal{H}\) on different Fock modules is given in \((11.5)-(11.8)\). Correspondingly, the eigenvalues of the helicity operator on the HS gauge 1-forms are

\[
\mathcal{H}\omega_{10} = -sw_{10}, \quad \mathcal{H}\omega_{01} = sw_{01},
\]

where spin \(s\) is defined according to \((3.6)\), i.e.,

\[
\left( y^\alpha \frac{\partial}{\partial y^\alpha} + \bar{y}^{\alpha'} \frac{\partial}{\partial \bar{y}^{\alpha'}} \right) \omega(y, \bar{y}|x) = 2(s - 1)\omega(y, \bar{y}|x).
\]

(We assume that spin is non-negative while helicities \(\pm s\) may have any sign.) On the HS Weyl 0-forms, the eigenvalues of the helicity operator are

\[
\mathcal{H}C_{00} = \pm sC_{00}, \quad \mathcal{H}C_{11} = \mp sC_{11},
\]

35
where signs are determined by those of the corresponding eigenvalues of the operators (11.7) and (11.8) (cf. (3.6), (3.7)). We see that the $u(1)$ symmetry generated by $\mathcal{H}$ rotates two species of fields of all spins with the spin-dependent phases.

Let us expand $\omega_{1-i}(y, \bar{y}|x)$ into the real and imaginary parts

$$
\omega_{10}(y, \bar{y}|x) = \omega_1(y, \bar{y}|x) + i\omega_2(y, \bar{y}|x), \quad \omega_{01}(y, \bar{y}|x) = \omega_1(y, \bar{y}|x) - i\omega_2(y, \bar{y}|x),
$$

(11.20)

where $\omega_i$ are real in the sense $\omega_i(y, z|x) = \omega_i(z, \bar{y}|x)$. In the $AdS_4$ case, where the $u(2,2)$ connections are realized by those of $sp(4, \mathbb{R}) \subset u(2,2)$ according to (7.9), both $h^{aa'} \neq 0$ and $f_{aa'} \neq 0$, allowing to express the 0-forms $C_{ii}$ via the space-time derivatives of the dynamical massless fields. In this case, the equations (11.13) and (11.14) take the form

$$
R_1(y, \bar{y}|x) = \mathcal{H}^\alpha{}_{\beta'} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}} C_1(0, \bar{y}|x) + \mathcal{H}^\alpha{}_{\beta} \frac{\partial^2}{\partial y^\alpha \partial y^{\beta}} C_1(y, 0|x),
$$

(11.21)

$$
R_2(y, \bar{y}|x) = \mathcal{H}^\alpha{}_{\beta'} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\beta'}} C_2(0, \bar{y}|x) - \mathcal{H}^\alpha{}_{\beta} \frac{\partial^2}{\partial y^\alpha \partial y^{\beta}} C_2(y, 0|x),
$$

(11.22)

where $R_{1,2}(y, \bar{y}|x)$ have the form of the linearized HS curvatures (3.13) and

$$
C_1(y, \bar{y}|x) = \frac{\lambda^2}{2}(C_{00}(y, \bar{y}|x) + C_{11}(y, \bar{y}|x)), \quad C_2(y, \bar{y}|x) = \frac{\lambda^2}{2i}(C_{00}(y, \bar{y}|x) - C_{11}(y, \bar{y}|x))
$$

(11.23)

satisfy the twisted adjoint covariant constancy condition

$$
D^{tu}C_i(y, \bar{y}|x) = 0
$$

(11.24)

with $D^{tu}$ (3.14) and the reality conditions $\overline{C_1(y, z)} = C_1(z, \bar{y})$, $\overline{C_2(y, z)} = -C_2(z, \bar{y})$.

The equations (11.21), (11.22) and (11.24) are equivalent to the unfolded massless equations (3.17) and (3.18) in the $AdS_4$ background with the identification

$$
\omega^{11} = \omega_2, \quad \omega^{00} = \omega_1, \quad C^{10} = C_{1-} - C_{2+} + C_{10} - C_{20}, \quad C^{01} = C_{1+} + C_{2-} + C_{10} + C_{20},
$$

where the labels $+$, $-$ and 0 refer to the decomposition (4.1). Thus, in the $AdS_4$ background, the conformal invariant equations (11.13), (11.14) and (11.17) with $i = j$ amount to the standard unfolded field equations for the doubled set of free massless fields of all spins.

### 11.3 Global symmetries and EM duality

As a consequence of general properties of unfolded equations, the massless equations (11.13), (11.14) and (11.17) are invariant under the $u(2,2)$ global symmetry that consists of the $su(2,2)$ conformal symmetry and $u(1)$ EM duality transformation generated by the helicity operator $\mathcal{H}$. Since $\mathcal{H}$ is central in $u(2,2)$, from (5.8) it follows that the global symmetry parameter of EM duality transformation remains $x$-independent.
The transformation law of the gauge 1-forms consists of the Lie-algebraic transformation in the module carried by the gauge fields and the additional terms resulting from the Chevalley-Eilenberg cohomology terms via (5.5). From the equations (11.13) and (11.14) it follows that the transformation law is

\[
\delta \omega_{10}(a, \bar{b}) = -\epsilon_{gl}(a, \bar{a}, b, \bar{b}|x) \star |\omega_{10}(a, \bar{b})| + 2\left(\epsilon^{\alpha^\prime}(x)\omega^{|\alpha^\prime}| \right) 2|1,0| \tag{11.25}
\]

\[
\delta \omega_{11}(b, \bar{a}) = -\epsilon_{gl}(a, \bar{a}, b, \bar{b}|x) \star |\omega_{11}(a, \bar{b})| + 2\left(\epsilon^{\alpha^\prime}(x)\omega^{|\alpha^\prime}| \right) 2|1,0| \tag{11.26}
\]

where \(\epsilon^{\alpha^\prime}(x)\) and \(\tilde{\epsilon}_{\alpha^\prime}(x)\) are parameters of global translations and special conformal transformations, respectively. All other symmetry parameters of the conformal algebra enter only through the original \((i.e., C\text{-independent})\) module transformation law \(\epsilon \star |\omega\). The transformation law of the Weyl 0-forms does not deform

\[
\delta|C_{00}(b, \bar{b}|x)| = -\epsilon_{gl}(a, \bar{a}, b, \bar{b}|x) \star |C_{00}(b, \bar{b}|x)|, \quad \delta|C_{11}(a, \bar{a}|x)| = -\epsilon_{gl}(a, \bar{a}, b, \bar{b}|x) \star |C_{11}(a, \bar{a}|x)|.
\]

The precise form of the transformation law depends on a chosen vacuum connection and, in particular, on a coordinate system parameterized by the function \(U^{AB}(X)\) of Section 9. For a broad class of vacuum connections, the global symmetry parameter \(\epsilon_{gl}(a, \bar{a}, b, \bar{b}|X)\) can be obtained from (9.13).

It is important to note that the full conformal symmetry does not act individually on every massless field, mixing together the two copies of fields of equal spins. This happens because of the gauge field sector where the complex conjugated gauge fields \(\omega_{i+1} - \omega_{i}\), with \(i = 0,1\), transform differently. This in particular implies that the real fields \(\omega_i\) (11.20) are mixed by the \(su(2,2)\) transformations as well as by the EM duality transformations. Correspondingly, the 0-forms \(C_i\) (11.23) are also mixed by the \(u(2,2)\) transformations. This agrees with the fact that the EM duality transformations cannot act locally on a single gauge potential. On the other hand, that the \(AdS_4\) symmetry \(sp(4, \mathbb{R})\) acts individually on the each of the two subsets of \(\omega_i\) and \(C_i\) with \(i = 1\) or 2 leads to two decoupled \(sp(4, \mathbb{R})\) invariant HS systems in (11.21), (11.22) and (11.24). Somewhat surprisingly, we shall see in Subsection 13 that, for any spin, the \(sp(4, \mathbb{R})\) symmetry of each of these subsystems extends to \(sl(4, \mathbb{R}) \sim o(3,3) \subset sp(8, \mathbb{R})\).

Conformal invariant truncation to the undoubled set of massless fields can be obtained as follows. Since the generalized Weyl tensors \(C_{00}\) and \(C_{11}\) are self-conjugated, each carrying a \(u(2,2)\) (in fact, \(sp(8, \mathbb{R})\)) module, it is consistent with the \(u(2,2)\) symmetry to set \(C_{ii} = 0\) for some \(i\). Let us, for example, set

\[
C_{11} = 0. \tag{11.27}
\]

This implies that the fields \(C_i\) (11.23) are linearly dependent, namely \(C_2(y, \bar{y}) = -ic_1(y, \bar{y})\).

Either of the equations (11.21) or (11.22) along with the respective covariant constancy condition (11.24) for \(C_1\) or \(C_2\) describes the set of massless fields of all spins. However, the
other one then has different interpretation. Let, say, \( \omega_1(y, \bar{y}|x) \) be chosen as independent electric HS gauge connection. The equation (11.21) expresses the generalized Weyl tensors in terms of derivatives of \( \omega_1(y, \bar{y}|x) \). Then the meaning of the equation (11.22) with \( C_{11} = 0 \) is that it defines the gauge field \( \omega_2(y, \bar{y}|x) \) in terms of \( C_1(y, \bar{y}|x) \). If the form of the r.h.s. of the equations (11.21) and (11.22) was the same, the corresponding equations meant that a linear combination of \( \omega_1(y, \bar{y}|x) \) and \( \omega_2(y, \bar{y}|x) \) would be pure gauge. However, the r.h.s. of the equations (11.21) and (11.22) have different relative signs of the holomorphic and antiholomorphic parts. As a result, (11.22) expresses \( \omega_2(y, \bar{y}|x) \) as the EM dual of \( \omega_1(y, \bar{y}|x) \), i.e., the potential \( \omega_2(y, \bar{y}|x) \) is magnetic.

In terms of the complex gauge fields \( \omega_{i1-i} \), the (anti)holomorphic components \( (C_{11}(0, \bar{a})) \) \( C_{11}(a, 0) \) describe (anti)selfdual components of the complexified \( s = 1 \) Maxwell field strength \( (\bar{C}_{a'\bar{a}'}) C_{a\beta}, s = 2 \) Weyl tensor \( (\bar{C}_{a'1...a'4}) C_{a_1...a_4} \) and their HS counterparts. As a result, the condition (11.27) implies that the HS field strength is (anti)selfdual. The (anti)selfduality condition imposed in the Minkowski signature is consistent because the gauge field \( \omega_{10} \) is complex, having \( \omega_{01} \) as its complex conjugate. This complex (anti)selfduality condition relates the field strength of one of the two spin \( s \) real fields contained in \( \omega_{10} \) to the dual field strength of the other one. Let us stress that this relationship is non-local in terms of gauge potentials, being expressed by a differential duality equation that follows from (11.27).

For example, for the spin one case in tensor notation, (11.27) implies the condition

\[
F_{nm}^i = \pm \frac{1}{2} \epsilon^{ij} \varepsilon_{nm}^{pq} F_{pq}^j, \quad F_{nm}^i = \partial_n \omega_m^i - \partial_m \omega_n^i, \tag{11.28}
\]

where \( i, j = 1, 2 \) label two real components of the complex spin one potential part \( \omega_{10}(0, 0) \) of \( \omega_{10}(a, \bar{b}) \) \((\epsilon^{ij} = -\epsilon^{ji}, \epsilon^{12} = 1)\). The relationship (11.28) between \( F_{nm}^1 \) and \(* F_{nm}^2\) is consistent in Minkowski case due to the factor of \( \epsilon^{ij} \) which becomes the imaginary unit in complex notation.

Thus, the unfolded equations (11.13), (11.14) and (11.17) with \( C_{11} = 0 \) or \( C_{00} = 0 \) describe the undoubled set of all spins in conformal and duality invariant way. In this case, the unfolded HS equations describe both electric and magnetic HS potentials. The conformal transformations resulting from the \( \omega \)-dependent terms in (11.25) and (11.26) mix the HS electric and magnetic potentials. In agreement with the known conformal invariance of (distinguished) 4d spin one models, the spin one case is degenerate, allowing conformal transformation that acts only on the electric potential. Technically, this happens because the spin one connection carries trivial \( su(2, 2) \)-module so that the \( \omega \)-dependent part of the transformation (11.25) and (11.26) is absent in this case, i.e., the conformal transformation of spin one results only from the \( C \)-dependent terms in (11.25) and (11.26).

It is tempting to speculate that the restrictions \( C_{11} = 0 \) or \( C_{00} = 0 \) may result from some dynamical mechanism in a full theory with spontaneously broken HS and \( sp(8, \mathbb{R}) \) symmetries, that freezes degrees of freedom of the magnetic phase at least below some mass scale, leaving us only with the half of degrees of freedom corresponding to the electric phase. The EM duality can then be expected to be a true local symmetry of a full \( sp(8, \mathbb{R}) \) invariant nonlinear HS gauge theory. This can even be true in the HS models with nonAbelian Yang-Mills symmetries considered in [7, 10] and in the sector of spin two massless fields that can
appear in the HS theories in many copies, carrying color indices. It would be interesting to interpret from this perspective the gravitational duality studied e.g. in [77, 78, 79, 80, 81, 82].

11.4 Flat limit

From (11.9)-(11.12) it follows that the full chain of field variables in the unfolded formulation of a massless field of definite spin contains components with the conformal weights from \(-\infty\) to \(\infty\). The dynamical HS fields are those with minimal absolute values of conformal weights, namely \(D = \pm 1\) for spin zero, \(D = \pm 3/2\) for spin one-half, \(D = 0\) for spin \(s \geq 1\) bosonic fields and \(D = \pm \frac{1}{2}\) for spin \(s \geq 3/2\) fermionic gauge fields (for more detail see the \(\sigma\_\)-cohomology analysis of Section 14). All other fields in the chain are auxiliary, being expressed via derivatives of the dynamical fields by the unfolded equations. Namely, the difference of the modules of conformal dimensions of a certain auxiliary field \(A\) and related dynamical field \(D\) equals to the highest number of derivatives in the resulting expression \(A(\partial^k(D))\). The property that the module of the conformal dimension rather than the conformal dimension itself matters, is intimately related to the relevance of the \(AdS\) background: the mismatch of dimension is compensated by the powers of the \(AdS\) radius \(\lambda\)^{-1}.

Let us consider more closely the translation and special conformal transformations of the gauge field \(\omega_{10}\). From (11.11) it follows that, discarding the \(C\)-dependent terms,

\[
\delta \omega_{10}(a, \bar{b}|x) = -2(\epsilon^{\alpha\beta'}(x)a_{\alpha} \frac{\partial}{\partial b^{\beta'}} - \bar{\epsilon}_{gl, \alpha\beta'}(x) \frac{\partial}{\partial a_{\alpha}} \bar{b}^{\beta'}) \omega_{10}(a, \bar{b}|x) + O(C).
\]

In terms of components (3.12) this gives

\[
\delta \omega_{10}^{\alpha_1...\alpha_n, \beta'_1...\beta'_m}(x) = -2\left(n\epsilon_{gl, \alpha\gamma}(x)\omega_{10}^{\alpha_1...\alpha_{n-1}, \gamma', \beta'_1...\beta'_m}(x) + m\bar{\epsilon}_{gl, \gamma}(x)\omega_{10}^{\alpha_1...\alpha_n, \beta'_1...\beta'_{m-1}}(x)\right).
\]

With the identification (7.9), the l.h.s. of the unfolded equation (11.13) has the form (3.13). Abusing notation, the expressions for auxiliary fields resulting from the unfolded system are

\[
\omega_{10}(n, m)(x) = (\lambda^{-1}D^L)^{[\alpha_0m]}\omega_{10}^{\text{dyn}}(n_0, m_0)(x), \quad |n_0 - m_0| \leq 1, \quad n_0 + m_0 = n + m,
\]

where \(\omega(n, m)(x)\) symbolizes \(\omega_{10}^{\alpha_1...\alpha_n, \beta'_1...\beta'_m}(x)\) and we only keep track of the highest derivative terms. In particular, in the case of spin two, the vierbein \(\omega^{\alpha\beta}\) is the dynamical field while the Lorentz connection \(\omega^{\alpha\beta}, \omega^{\alpha'\beta'}\) is auxiliary.

The transformation law (11.29) then implies

\[
\delta \omega_{10}^{\text{dyn}}(x) \sim \lambda^{-1}(\epsilon_{gl}(x)D^L\omega_{10}^{\text{dyn}}(x) + \bar{\epsilon}_{gl}(x)D^L\omega_{10}^{\text{dyn}}(x)).
\]

This transformation law is ill-defined in the flat limit \(\lambda \to 0\). It is possible to rescale the generators of the conformal algebra \(P_{\alpha\alpha'} \to \mu^{-1}P_{\alpha\alpha'}, K^{\alpha\alpha'} \to \mu K^{\alpha\alpha'}\) without affecting

\[\text{The no-go statements on the existence of models with several interacting spin two fields [71, 72] are avoided analogously to the no-go statements on HS interactions [73, 74] due to using the AdS}_4\text{ background and infinite sets of HS fields.}\]
their commutation relations. This induces the rescaling of the parameters $\epsilon^{\alpha\alpha'} \to \mu \epsilon^{\alpha\alpha'}$, $\bar{\epsilon}_{\alpha\alpha'} \to \mu^{-1} \bar{\epsilon}_{\alpha\alpha'}$ as well as of the corresponding vacuum connections. This ambiguity can be used to compensate the factor of $\lambda^{-1}$ in front of one of the two terms on the r.h.s. of (11.30). In particular, choosing $\mu = \lambda$ we obtain instead of (11.30)

$$\delta \omega_{i10}^\text{dyn}(x) \sim \epsilon_{gl}(x) D^L \omega_{i10}^\text{dyn}(x) + \lambda^{-2} \epsilon_{gl}(x) D^L \omega_{i10}^\text{dyn}(x).$$

(11.31)

In the flat limit, this transformation law is well defined in the sector of translations but blows up in the sector of special conformal transformations. This is why spin $s > 1$ massless field equations formulated in terms of gauge fields in Minkowski space (in particular, linearized gravity) respect Poincaré’ symmetry but are not conformal invariant.

A closely related fact is that the rescalings procedure of Section 4, that leads to the conventional flat limit description of HS fields in terms of potentials, breaks down conformal invariance. The algebraic reason for this is that the Poincaré’ covariant derivative in (4.3) is defined so that Poincaré’ translations on $\tilde{A}_+$ and on $\tilde{A}_-$ are generated, respectively, by the translation and special conformal transformations of the original conformal module. Clearly, this realization of translations is incompatible with the standard realization of the Poincaré’ algebra as a proper subalgebra of the conformal algebra. On the other hand, the Lorentz, duality and dilatation transformations survive in so defined flat limit because they act on homogeneous polynomials of the oscillators and commute with the rescalings.

On the other hand, the naive flat limit of the equations (11.13), (11.14) and (11.17) with $f_{\alpha\alpha'} = 0$ at $\lambda \to 0$ gives rise to conformal invariant equations that is hard to interpret. Actually, in this system $C_{11}$ is not any longer expressed via the gauge fields $\omega_{i1-i}$, thus becoming an independent field. As a result, the system decomposes into two parts. One contains the 1-forms $\omega_{i1-i}$ along with the 0-forms $C_{00}$ while another one is the covariant constancy condition (11.17) on the 0-form $C_{11}$. The latter however does not make sense in terms of any conventional formulation of massless fields because it does not express higher components of the expansion of $C_{11}(a, \bar{a})$ via the space-time derivatives of the lower ones, imposing the condition that the lowest component is a constant. From perspective of $\sigma_-$-cohomology analysis, this awkward picture results from the degeneracy of the operator $\sigma_-$ which is zero in this case. As a result, the system gets a form of an infinite set of topological-like field equations which, in fact, is hard to interpret. This makes the naive limit ill-defined even though the subsystem that contains $\omega_{i1-i}$ and $C_{00}$ is more tractable if the antiholomorphic field $\omega_{10}(0, \bar{a}|x)$ (which is the antiholomorphic part of the Lorentz connection in the spin two case) is chosen as dynamical.

As mentioned in Subsection 11.3, the spin one case is degenerate because there are no auxiliary gauge connections, i.e., both terms on the r.h.s. of (11.29) are absent. The nontrivial transformation originates from the $C$-dependent terms on the r.h.s. of (11.23) and (11.26) where the spin one components of the 0-forms $C_{ii}$ identify with the Maxwell tensor. These can be rescaled independently to get rid of the $\lambda$-dependence, preserving conformal invariance in the flat limit in agreement with the fact of conformal invariance of the 4d spin one gauge theory in Minkowski space.
12 \( sp(8, \mathbb{R}) \) invariant massless equations

The \( u(2, 2) \) invariant unfolded equations (11.13)-(11.17) admit an extension to the \( sp(8, \mathbb{R}) \) invariant form. The fields are still described in terms of the modules \(|\omega_{1-i}(x)\rangle\) and \(|C_{1i}(x)\rangle\).

The vacuum covariant derivative (2.11) is defined in star-product notation by (9.4), i.e.,

\[
D = d + W, \quad W = \frac{1}{2} h^{AB} a_A a_B + \omega_B^A a_A b^B + \frac{1}{2} i f_{AB} b^A b^B . \tag{12.1}
\]

A particular form of the \( sp(8, \mathbb{R}) \) covariant derivative depends on a chosen Fock vacuum

\[
D|C_{ij}(\beta_i, \overline{\beta}_j|X)\rangle = (D_{ij} C_{ij}(\beta_i, \overline{\beta}_j|X)) \ast |i, j\rangle .
\]

For example,

\[
D_{00} = d + \frac{1}{2} h^{AB} \frac{\partial^2}{\partial b^A \partial b^B} + \omega_B^A b^B \frac{\partial}{\partial b^A} + \frac{1}{2} \omega_C^C + \frac{1}{2} i f_{AB} b^A b^B . \tag{12.2}
\]

Here \( \omega_B^A \) are gauge fields of \( gl(4, \mathbb{R}) \) that act on homogeneous polynomials of \( b^A \). The vacuum \( |0, 0\rangle \) forms a one dimensional \( gl(4, \mathbb{R}) \)-module. The \( gl(4, \mathbb{R}) \subset sp(8, \mathbb{R}) \) is an extension of the conformal embedding \( gl(2, \mathbb{C}) \subset u(2, 2) \), where \( gl(2, \mathbb{C}) \) contains Lorentz transformations, dilatations and duality transformations.

Note that, every vacuum \( |i, j\rangle \) has the \( gl_{ij}(4, \mathbb{R}) \) invariance with the generators bilinear in the respective creation and annihilation generators. The subalgebras \( gl_{ij}(4, \mathbb{R}) \subset sp(8, \mathbb{R}) \) are different for different \( i, j \), having \( gl(2, \mathbb{C}) \) as the maximal common subalgebra. As a result, \( gl(2, \mathbb{C}) \) remains the maximal manifest symmetry of the construction.

The explicit form of the derivatives \( D_{ij} \), that extend the conformal covariant derivatives (11.11)-(11.14) to the \( sp(8, \mathbb{R}) \) case, can be obtained from (12.2) by renaming the oscillators according to (10.9) and (10.10). The equation (11.17) still has the covariant constancy form

\[
D_{ij} C_{ij} = 0 . \tag{12.3}
\]

The extension of (11.13)-(11.16) is

\[
D \omega_{ij}(\beta_i, \overline{\beta}_j) = (\Delta_{ij} \otimes C_{1-i-j}(\beta_i, \overline{\beta}_{1-j}))\big|_{\beta_{1-j}=0} + (\Delta_{ij} \otimes C_{1-i-j}(\beta_{1-i}, \overline{\beta}_j))\big|_{\beta_{1-i}=0} , \tag{12.4}
\]

where

\[
\Delta_{ij} = \frac{1}{4} \epsilon^{\alpha\beta} \frac{\partial}{\partial \beta_i^\alpha} W(0, \beta_i, \overline{\alpha}_j, \overline{\beta}_j) \wedge \frac{\partial}{\partial \beta_{i}^\beta} W(0, \beta_i, \overline{\alpha}_j, \overline{\beta}_j) , \tag{12.5}
\]

\[
\overline{\Delta}_{ij} = \frac{1}{4} \epsilon^{\alpha'\beta'} \frac{\partial}{\partial \beta_j^\alpha} W(\alpha_i, \beta_i, 0, \overline{\beta}_j) \wedge \frac{\partial}{\partial \beta_{j}^\beta} W(\alpha_i, \beta_i, 0, \overline{\beta}_j) . \tag{12.6}
\]

with \( W(a_A, b^B) \) (12.1) represented as \( W(\alpha_i, \beta_i, \overline{\alpha}_j, \overline{\beta}_j) \). The tensor product symbol \( \otimes \) in (12.4) means that \( W(\alpha_i, \beta_i, \overline{\alpha}_j, \overline{\beta}_j) \) is realized as a sum of operators that act both on the
in the \( \Xi_{ij} \) term and

\[
\beta_{i} \otimes = \beta_{i} + \alpha_{1-i}, \quad \beta_{j} \otimes = 2\beta_{j}, \quad \alpha_{i} \otimes = 2\alpha_{i}
\]

in the \( \Delta_{ij} \) term in (12.4). Here \( \alpha_{k} \) and \( \alpha_{l} \) are understood as derivatives over \( \beta_{k} \) and \( \beta_{l} \) (precise signs follow from the definitions (10.9) and (10.10)) with the convention that the differentiation is done before \( \beta_{1-j} \) or \( \beta_{1-i} \) are set to zero in (12.4).

The proof of consistency of the unfolded system (12.3) and (12.4) is relatively simple but still miraculous. First of all we observe that the operator \( \Delta_{ij} \) (12.5) contains connections that have positive \( \beta_{i} - \alpha_{i} (\beta_{j} - \alpha_{j}) \) grading. Let us for definiteness, consider the case \( i = 1, j = 0 \).

Using (10.9) and (10.10) we find that the connections that contribute to \( \Delta_{10} \) are

\[
\frac{1}{2} h^{\alpha\beta} a_{\alpha} a_{\beta}, \quad h^{\alpha\beta} a_{\alpha} \bar{a}_{\beta'}, \quad \omega_{\beta'}^{\alpha} a_{\alpha} \bar{b}_{\beta'}, \quad (12.7)
\]

and

\[
\Delta_{10} \otimes C_{00}(b, \bar{b}) \bigg|_{b=0} = \left( \frac{1}{2} h^{\gamma\beta}(a_{\beta} + \frac{\partial}{\partial b^{\beta}}) + h^{\gamma\alpha} \frac{\partial}{\partial b^{\alpha}} + \omega_{\beta'}^{\gamma} \bar{b}_{\beta'} \right) C_{00}(b, \bar{b}) \bigg|_{b=0}. \quad (12.8)
\]

The analysis is greatly simplified in terms of the antiholomorphic Fock space vector

\[
|\Delta_{10} \rangle = \left( \Delta_{10} \otimes C_{00}(b, \bar{b}) \bigg|_{b=0} \right) \overline{\pi}|\bar{0} \rangle \langle \bar{1}|,
\]

where \( \overline{\pi} \) denotes the restriction of the star-product to the subalgebra generated by the oscillators carrying primed indices. It can be equivalently rewritten as

\[
|\Delta_{10} \rangle = (g^{\gamma} \wedge g_{\gamma}) \overline{\pi} C_{00}(b, \bar{b}) \overline{\pi}|\bar{0} \rangle \langle \bar{1}| \bigg|_{b=0},
\]

where

\[
g^{\gamma} = \frac{1}{2} h^{\gamma\beta}(a_{\beta} + \frac{\partial}{\partial b^{\beta}}) + h^{\gamma}, \quad h^{\gamma} = h^{\gamma\alpha} \overline{\pi}_{\alpha'} + \omega_{\beta'}^{\gamma} \overline{b}_{\beta'}'. \quad (12.10)
\]

Note that, because of the wedge product, the antisymmetrization with respect to the indices \( \gamma \) implies that \( g^{\gamma} \wedge \overline{\pi} g_{\gamma} \) is symmetrized in the primed indices carried by \( h^{\gamma} \) (12.11). This allowed us to replace \( g^{\gamma} \wedge \overline{\pi} g_{\gamma} \) by its \( \overline{\pi} \)-Weyl symbol \( g^{\gamma} \wedge g_{\gamma} \) in (12.10).
In the sector where only the fields \((12.7)\) are present, the consistency requires that

\[
H = \left( \frac{1}{2} h^{\alpha\beta} \wedge (a_\alpha a_\beta - \frac{\partial}{\partial b^\alpha} \frac{\partial}{\partial b^\beta}) g^\gamma \wedge g_\gamma + h^\gamma a_\gamma \wedge \overline{\nu} (g^\gamma \wedge g_\gamma) - (g^\gamma \wedge g_\gamma) \overline{\nu} \wedge h^\gamma \frac{\partial}{\partial b^\gamma} \right) \overline{\nu} C_{00}(b, \overline{b})
+ \left( dh^\gamma (a_\beta + \frac{\partial}{\partial b^\beta}) h_\gamma \right) \overline{\nu} C_{00}(b, \overline{b})
\]

(12.12)

should vanish (the terms resulting from \(dh^\gamma\) do not contribute to this sector). Taking into account that \(dh^\alpha = -2h^{(\alpha} \wedge \overline{\nu} h^{\beta)} + \ldots\), where dots denote terms that contain fields of non-positive \(a - b\) grading, we find that the terms resulting from the noncommutativity of the star-product in the second and third terms in the first line of (12.12) compensate the last term. As a result, we obtain that

\[
H \sim \left( g^\alpha \wedge g^\gamma \wedge g_\gamma (a_\alpha - \frac{\partial}{\partial b^\alpha}) \right) \overline{\nu} C_{00}(b, \overline{b}) = 0
\]

by antisymmetrization of three two-component indices of \(g^\beta\) due to the wedge product.

It remains to consider the part of the consistency condition that contains vacuum connections of non-positive \(a - b\) grading. These include

\[
\omega^\alpha_\beta a_\alpha b^\beta, \quad \frac{1}{2} h^{\alpha\beta} a_\alpha \overline{\nu} b_\beta, \quad \overline{\nu} b^{\alpha\beta} a_\alpha \overline{\nu} b_\beta, \quad \frac{1}{2} f_{\alpha^\beta^\gamma} b^{\alpha^\prime} b^{\beta^\prime}, \quad f_{\alpha^\beta^\gamma} b^{\alpha^\prime} b^{\beta^\prime}, \quad \omega^\alpha_\alpha a_\alpha b^\alpha, \quad \frac{1}{2} f_{\alpha^\beta^\gamma} b^{\alpha^\prime} b^{\beta^\prime}.
\]

Since \((12.8)\) is independent of these fields, they contribute to the consistency condition at most linearly. All such terms cancel out trivially except for the dilatation field contained in \((12.8)\). This concludes the analysis of the consistency of the term with \(\Delta\).

The analysis of the terms with other \(\Delta_{ij}\) and \(\Delta_{ij}\) is analogous modulo renaming the oscillators.

Since the consistency of the unfolded equations \((12.4)\) and \((12.3)\) has been verified for arbitrary flat \(sp(8, \mathbb{R})\) connection, from the general argument of Subsection 5.3 it follows that the system \((12.3), (12.4)\) is invariant under the global \(sp(8, \mathbb{R})\) transformations.

If vacuum connection is chosen to belong to \(su(2, 2) \subset sp(8, \mathbb{R})\), the system \((12.3), (12.4)\) amounts to the conformal system \((11.13)-(11.17)\) of Section 11, which is therefore also shown to be \(sp(8, \mathbb{R})\) invariant. As shown in Section 11, with this choice of the vacuum fields it describes the doubled set of field equations for all massless fields described by \(\omega_{i1-i}\) and \(C_{ii}\) plus an infinite set of topological fields, each carrying a finite number of degrees of freedom, described by \(\omega_{ii}\) and \(C_{i1-i}\). The interpretation of the \(sp(8, \mathbb{R})\) invariant equations \((12.3), (12.4)\) depends, however, on the choice and interpretation of the vacuum fields associated with a chosen flat \(sp(8, \mathbb{R})\) connection. In Section 13 we show that the roles of the massless and topological fields are exchanged if the nonzero vacuum fields are associated with the subalgebra \(sl(4, \mathbb{R}) \sim o(3, 3) \subset sp(8, \mathbb{R})\).

By the general argument of Subsection 5.5 to extend the obtained \(sp(8, \mathbb{R})\) invariant unfolded equations to \(\mathcal{M}_4\) one has to replace the four dimensional exterior differential by the ten dimensional one

\[
dx{\alpha^\beta^\gamma} \frac{\partial}{\partial x^{\alpha^\beta^\gamma}} \rightarrow dX^{AB} \frac{\partial}{\partial X^{AB}}
\]
simultaneously extending a 4d flat \( sp(8, \mathbb{R}) \) connection 1-form to \( \mathcal{M}_4 \). As in the 4d case, the flat limit degeneracy in the “Cartesian coordinates” with \( f_{AB} = 0 \) is resolved in the \( AdS \)-like ten dimensional space \( Sp(4, \mathbb{R}) \). The dynamical interpretation of the resulting equations in \( Sp(4, \mathbb{R}) \), \textit{i.e.}, which field components are dynamical, auxiliary, Stueckelberg etc, is most conveniently elucidated by the analysis of \( \sigma_- \) cohomology. This is done in Section 13.

As discussed in the beginning of this section, the manifest \( (i.e., \text{linearly acting}) \) symmetry of the system \((12.3), (12.4)\) consists of the usual Lorentz symmetry plus dilatation and duality transformations which altogether form \( gl(2, \mathbb{C}) = sl(2, \mathbb{C}) \oplus \mathbb{R} \oplus u(1) \). Because it is small enough, Minkowski coordinates \( X^{\alpha \alpha'} \) and spinning coordinates \( X^{\alpha \beta} \) and \( X^{\alpha' \beta'} \) have different appearance in the full \( sp(8, \mathbb{R}) \) invariant system lifted to \( \mathcal{M}_4 \) or \( Sp(4, \mathbb{R}) \), \textit{i.e.}, the equations for the gauge fields break the manifest \( gl(4, \mathbb{R}) \) symmetry of the equation on the generalized Weyl tensor \( C_{00} \) down to \( gl(2, \mathbb{C}) \). A related point is that the analysis of the role of different \( sp(8, \mathbb{R}) \) curvatures along the lines of the analysis of conformal field strengths sketched in Section 7, which answers the question which of the component of the HS field strengths can be set to zero as constraints and which are zero by virtue of field equations or/and Bianchi identities, turns out to be more complicated in \( \mathcal{M}_4 \). (This is analyzed in Section 15 in terms of \( \sigma_- \) cohomology.) Correspondingly, an \( \mathcal{M}_4 \) (or \( Sp(4, \mathbb{R}) \)) analog of the holonomy group is not expected to be larger than \( GL(2, \mathbb{C}) \).

13 \( gl(4, \mathbb{R}) \) invariant massless equations

The generators \( L_{AB} \) \((2.9)\) span \( gl(4, \mathbb{R}) \subset sp(8, \mathbb{R}) \). \( L_{\alpha}^{\beta} \) and \( L_{\alpha'}^{\beta'} \) include the Lorentz generators, \( \mathcal{H} \) \((2.14)\) and \( \mathcal{D} \) \((2.13)\). These span \( gl(2, \mathbb{C}) = gl(4, \mathbb{R}) \cap u(2, 2) \). Note that the generators \( \mathcal{H} \) and \( \mathcal{D} \) exchange their roles compared to the conformal case of \( su(2, 2) \). Now \( \mathcal{D} \) \((2.13)\) is the central element of \( gl(4, \mathbb{R}) \), that characterizes different irreducible subsystems in the \( gl(4, \mathbb{R}) \) invariant equations.

In addition, \( gl(4, \mathbb{R}) \) contains the generators

\[
L_{\alpha'}^{\alpha} = a_{\alpha} b_{\alpha'}, \quad L_{\alpha'}^{\alpha} = \bar{a}_{\alpha} b^{\alpha}.
\]

These are analogues of the translation and special conformal transformation generators \( P_{\alpha \alpha'} \) and \( K^{\alpha \alpha'} \) of the conformal algebra. The important difference is, however, that \( P_{\alpha \alpha'} \) and \( K^{\alpha \alpha'} \) are self-conjugated while \( L_{\alpha}^{\beta} \) and \( L_{\alpha'}^{\beta'} \) are conjugated to each other. This implies in particular that although

\[
[L_{\alpha'}^{\alpha}, L_{\beta}^{\beta'}] = 0, \quad [L_{\alpha}^{\alpha'}, L_{\beta'}^{\beta}] = 0,
\]

neither \( L_{\alpha}^{\alpha'} \) nor \( L_{\alpha'}^{\alpha} \) are translation generators of a Poincaré subalgebra of \( gl(4, \mathbb{R}) \).

One reason why \( gl(4, \mathbb{R}) \) symmetry might have been missed in field-theoretical models is that it does not allow lowest weight unitary modules because neither \( L_{\alpha}^{\alpha'} \) nor \( L_{\alpha'}^{\alpha} \) can serve as step operators. A related property is that \( GL(4, \mathbb{R}) \) admits no induced modules to define induced \( GL(4, \mathbb{R}) \) action on tensor fields. On the other hand, \( gl(4, \mathbb{R}) \) does act on the
lowest weight unitary $sp(8, \mathbb{R})$–modules and therefore can act on relativistic fields and their single-particle quantum states.

Now we observe that

$$P_{a\alpha'} = L_{a\alpha'} + L_{\alpha' a}, \quad (L_{a\alpha'} = L_\alpha^\beta \epsilon_{\beta\alpha'}, \quad L_{\alpha' a} = L_{\alpha}^\beta \epsilon_{\beta\alpha'})$$

along with the Lorentz generators span the $AdS_4$ subalgebra $o(3, 2) \sim sp(4, \mathbb{R}) \subset gl(4, \mathbb{R})$

(for simplicity we set $\lambda = 1$ in the rest of this section). This observation leads to the alternative interpretation of the proposed $sp(8, \mathbb{R})$ invariant equations, as $gl(4, \mathbb{R})$ invariant equations in $AdS_4$ associated with the embedding $o(3, 2) \sim sp(4, \mathbb{R}) \subset gl(4, \mathbb{R}) \subset sp(8, \mathbb{R})$.

In the oscillator realization we have $P_{a\alpha'} = a_\alpha b_{\alpha'} + \bar{a}_{\alpha'} b_\alpha$. So defined $P_{a\alpha'}$ acts in finite dimensional spaces of homogeneous polynomials of the modules $\omega_{ii}$ and $C_{ii}$. On the other hand, $\omega_{i1-i}$ and $C_{i1-i}$ now decompose into the infinite sum of infinite dimensional $sp(4, \mathbb{R})$-modules. This means that the subsystem of field equations for $\omega_{ii}$ and $C_{i1-i}$ now describes massless fields in $AdS_4$ while that for $\omega_{i1-i}$ and $C_{ii}$ describes an infinite set of topological fields. We see that the massless and topological fields exchange their roles depending on which $sp(4, \mathbb{R})$ subalgebra of $sp(8, \mathbb{R})$ is identified with the $AdS_4$ symmetry.

Note that the original form of the nonlinear massless field equations of [5] (see also [28]) in which the massless fields are self-conjugated as presented in Section 3 is most naturally related to the $gl(4, \mathbb{R})$ invariant version of the equations. The free $gl(4, \mathbb{R})$ invariant system admits a reduction to the subsystem that describes a single massless field of any spin. Algebraically, this is because, in the $gl(4, \mathbb{R})$ invariant case, spin is characterized by the eigenvalues of the non-compact generator $D$ (cf (11.9)-(11.12)) that admits one dimensional real modules rather than by the compact generator $H$ with minimal two dimensional modules as in the conformal case. Flat limit can be taken using the rescaling procedure of Section 4. However, analogously to $su(2, 2)$, the $gl(4, \mathbb{R})$ symmetry does not survive in the flat limit, thus being invisible in Minkowski space.

Equivalently, the $gl(4, \mathbb{R})$ covariant description can be obtained by keeping the same oscillator generators as in the conformal case but changing the conjugation conditions to $\overline{a_\alpha} = \overline{b_{\alpha'}}, \overline{b_\alpha} = -\overline{a_{\alpha'}}$, which implies that $[i, j] = [1 - j, 1 - i]$ and, therefore, $\overline{\phi_{ij}} = \phi_{1-j,1-i}$ for $\phi_{ij} = \omega_{ij}$ or $C_{ij}$. With these reality conditions we obtain that $\overline{\phi_{00}} = \phi_{11}, \overline{\phi_{01}} = \phi_{01}$ and $\overline{\phi_{10}} = \phi_{10}$. In this setup, the sets of higher spin and topological fields remain the same as in the $su(2, 2)$ case but the reality conditions change.

As shown in Subsection 11.3, the free equations for a single massless field of a fixed spin admit the action of $su(2, 2)$ that involves dual gauge potentials in the transformation law. This means that, allowing nonlocal field transformations of this kind, free field equations of a massless field of a given spin are invariant under both $su(2, 2)$ and $gl(4, \mathbb{R})$ and, therefore, under their closure $gl(4, \mathbb{C})$. Note that $gl(4, \mathbb{C})$ acts locally on the doubled sets of massless fields with the natural realization of a pair of real fields as a single complex field. Let us stress that $gl(4, \mathbb{C})$ algebra does not belong to $sp(8, \mathbb{R})$ that acts individually on every Fock module. Rather, $gl(4, \mathbb{C}) \subset sp(8, \mathbb{C})$ where $sp(8, \mathbb{C})$ mixes two Fock modules that describe massless fields in a chosen vacuum realization.

An interesting project for the future is to look for nonlinear $gl(4, \mathbb{R})$ invariant models in $AdS_4$. Since $gl(4, \mathbb{R})$ acts individually on fields of different spins, the problem can be...
analyzed, e.g. for spin two, i.e., $AdS_4$ gravity or supergravity. We hope to come back to this intriguing question elsewhere.

14 $\sigma_-$ analysis in Minkowski space

14.1 Grading

An appropriate grading $G$ of the Fock modules in the unfolded HS equations is $G = |D| = \frac{1}{2} [a_A b^A]$, where $|D|$ results from the dilatation generator $D$ via replacing its eigenvalues by their absolute values. In other words, the grading $G$ of a field equals to the absolute value of its scaling dimension. So defined $G$ is diagonalizable and bounded from below. Abusing notation, from (11.9)-(11.12) we obtain for $\phi_{ij} = C_{ij}$ or $\omega_{ij}$

$$G\phi_{00}(b, \bar{b}) = \frac{1}{2} \left( b^\alpha \frac{\partial}{\partial b^\alpha} + \bar{b}^{\alpha'} \frac{\partial}{\partial \bar{b}^{\alpha'}} + 2 \right) \phi_{00}(b, \bar{b}), \quad (14.1)$$

$$G\phi_{11}(a, \bar{a}) = \frac{1}{2} \left( a_\alpha \frac{\partial}{\partial a_\alpha} + \bar{a}_{\alpha'} \frac{\partial}{\partial \bar{a}_{\alpha'}} + 2 \right) \phi_{11}(a, \bar{a}), \quad (14.2)$$

$$G\phi_{10}(a, \bar{b}) = \frac{1}{2} \left| a_\alpha \frac{\partial}{\partial a_\alpha} - \bar{b}^{\alpha'} \frac{\partial}{\partial \bar{b}^{\alpha'}} \right| \phi_{10}(a, \bar{b}), \quad (14.3)$$

$$G\phi_{01}(b, \bar{a}) = \frac{1}{2} \left| b^\alpha \frac{\partial}{\partial b^\alpha} - \bar{a}_{\alpha'} \frac{\partial}{\partial \bar{a}_{\alpha'}} \right| \phi_{01}(b, \bar{a}). \quad (14.4)$$

The grading $G$ treats symmetrically the parts $A_+$ and $A_-$ in the decomposition (4.1) underlying the flat limit and leads to the standard description \cite{44,45} of $4d$ massless fields.

14.2 0-forms

Let us analyze the equation (2.1) on the 0-form $C(b|x)$. The grading (14.1) implies that

$$\sigma_- = dx^{\alpha\alpha'} \frac{\partial^2}{\partial b^\alpha \partial b^{\alpha'}}.$$ 

That $\sigma_-^2 = 0$ is the consequence of anticommutativity of the differentials $dx^{\alpha\alpha'}$. As expected, the dynamical fields in $H^0(\sigma_-)$ are holomorphic and antiholomorphic

$$H^0(\sigma_-) : \quad C(b) + \bar{C}(\bar{b}).$$

$H^1(\sigma_-)$ is of the form

$$H^1(\sigma_-) : \quad dx^{\alpha\alpha'} \left( b_\alpha E_\alpha(b) + \bar{b}_{\alpha'} \bar{E}_{\alpha'}(\bar{b}) + b_\alpha \bar{b}_{\alpha'} E \right), \quad (14.5)$$
where \( E_\alpha(b), \overline{E}_\alpha(\overline{b}) \) are arbitrary polynomials of their arguments and \( E \) is a constant. It is easy to see that the elements in (14.3) are \( \sigma_- \) closed but not exact. We leave it as an exercise to the reader to check that (14.3) describes full \( H^1(\sigma_-) \). \( E_\alpha(b), \overline{E}_\alpha(\overline{b}) \) and \( E \) parameterize the l.h.s. of the equations (2.3), (2.4). Indeed, the unfolded equations (2.1) demand all derivatives of the 0-form dynamical fields, that turn out to be \( \sigma_- \) closed because the dynamical fields themselves are \( \sigma_- \) closed, to be zero except for those that are \( \sigma_- \) exact to be absorbed by \( \sigma_- C^\text{aux} \) with some auxiliary fields \( C^\text{aux} \) (for more detail see e.g. [29]). This has the consequence that the equations (2.3), (2.4) follow from (2.1). Since (14.5) describes the full cohomology, they provide the full list of differential equations imposed by the unfolded equations (2.1) on the dynamical fields.

To analyze the content of the 4d massless field equations (3.17) and (3.18) for gauge potentials first of all we observe that all fields \( C(b,0|x) \) and \( C(0,\overline{b}|x) \), except for the scalar field \( C(0,0|x) \) and spinor field linear in \( b^4 \), are expressed via the gauge fields \( \omega \) by the equation (3.17) thus becoming auxiliary fields. To take this into account it is convenient to redefine \( \sigma_- \rightarrow \sigma'_- \) where \( \sigma'_- \) acts on the direct sum of spaces of 0-forms \( C \) and 1-forms \( \omega \) so that the terms on the r.h.s. of (3.17) become \( \sigma'_-(C) \). As a result, all auxiliary fields \( C(x) \) that correspond to spins \( s \geq 1 \) disappear from \( H^0(\sigma'_-) \). Correspondingly, their field equations become consequences of the Bianchi identities for the equations (3.17) for \( s > 1 \), thus disappearing from \( H^1(\sigma'_-) \).

The case of spin one is special. Here, the equation (3.17) is just the definition of the Maxwell field strength \( C_{\alpha\beta} \) and \( \overline{C}_{\alpha'\beta'} \) in terms of potentials while the spin one equation is still a part of the cohomology (14.3), \( dx^{\alpha\alpha'}(b_\alpha E_{\beta\alpha} b^\beta - \overline{b}_{\alpha'} E_{\alpha\beta'} b^{\beta'}) \), where \( E_{\beta\alpha'} \) is an arbitrary Hermitian bispinor (i.e., Lorentz vector) that parameterizes the l.h.s. of the second pair of the Maxwell equations. The first pair associated with \( dx^{\alpha\alpha'}(b_\alpha F_{\beta\alpha} b^\beta + \overline{b}_{\alpha'} F_{\alpha\beta'} b^{\beta'}) \) becomes the Bianchi identity and disappears from \( H^1(\sigma'_-) \).

The sector of spin 0 and 1/2 is unaffected by the transition from \( \sigma_- \) to \( \sigma'_- \). As a result, we conclude that the relevant part of the \( \sigma'_- \) cohomology in the sector of 0-forms \( C \) is

\[
H^0(\sigma'_-, C) = C + b_\alpha C_\alpha + \overline{b}_{\alpha'} \overline{C}_{\alpha'}, \tag{14.6}
\]

\[
H^1(\sigma'_-, C) = dx^{\alpha\alpha'}(b_\alpha E_{\beta\alpha'} + \overline{b}_{\alpha'} E_\alpha + b_\alpha b_{\alpha'} E + b_\alpha E_{\beta\alpha} b^\beta - \overline{b}_{\alpha'} E_{\alpha\beta'} b^{\beta'}), \tag{14.7}
\]

where \( C, C_\alpha, \overline{C}_{\alpha'} \) and \( E, E_\alpha, E_{\alpha'}, E_{\alpha\alpha'} \) parameterize, respectively, dynamical fields of spin 0 and 1/2 and the l.h.s. of the dynamical equations of spin 0, 1/2 and 1. Now we are in a position to analyze the sector of gauge fields.

### 14.3 1-forms

Dynamics of spins \( s \geq 1 \) is described by the gauge 1-forms \( \omega \). As shown in [43] (without using the cohomology language, however) the dynamical fields are \( \omega_{\alpha_1...\alpha_n,\alpha'_{1}...\alpha'_m} \) with \( n = m \) for bosons and \( |n - m| = 1 \) for fermions, which are the frame-like counterparts of the Fronsdal’s double traceless boson and triple \( \gamma \)-transverse fermion metric-like fields, respectively. Field equations for massless fields of all spins \( s > 1 \) are contained in the sector of gauge 1-forms. Let us show how these facts are reproduced in terms of \( \sigma_- \) cohomology.
In the case of 1-forms, the grading operator is of the type (14.3)

\[
G = \frac{1}{2} \left| n - \bar{n} \right| , \quad n = y^\beta \frac{\partial}{\partial y^\beta} , \quad \bar{n} = \bar{y}^\beta \frac{\partial}{\partial \bar{y}^\beta} .
\]  

(14.8)

The operator \( \sigma'_- \) is

\[
\sigma'_-(A) = \sigma_- A + \sigma^\text{weyl}_- A , \quad \sigma_- A = e^{\alpha \beta'} \left( y_\alpha \frac{\partial}{\partial y^{\beta'}} A_- (y, \bar{y} \mid x) + \frac{\partial}{\partial y_\alpha} \bar{y}^{\beta'} A_+ (y, \bar{y} \mid x) \right) ,
\]

(14.9)

where \( \sigma^\text{weyl}_- \) is the part of \( \sigma'_- \) responsible for gluing the Weyl 0-forms \( C \) to the field strengths of the gauge 1-forms via the terms on the r.h.s. of \((3.17)\). Note that \( \sigma_- \) defined as the \( e \)-dependent part of \((4.3)\) respects the decomposition \((4.1)\). Equivalently,

\[
\sigma'_- = \rho_- \theta (n - \bar{n} - 2) + \overline{\rho}_- \theta (\bar{n} - n - 2) + \sigma^\text{weyl}_- ,
\]

(14.10)

where

\[
\rho_- = e^{\alpha \beta'} \frac{\partial}{\partial y^{\beta'}} y_\alpha , \quad \overline{\rho}_- = e^{\alpha \beta'} \frac{\partial}{\partial \bar{y}^{\beta'}} y_\alpha
\]

(14.11)

and

\[
\theta (m) = 1 (0) , \quad m \geq 0 (m < 0) .
\]

(14.12)

Although \( \rho_- \) and \( \overline{\rho}_- \) do not anticommute, \( \sigma'_- \) squares to zero because \((\rho_-)^2 = (\overline{\rho}_-)^2 = 0\) and the step functions guarantee that the parts of \( \sigma'_- \) associated with \( \rho_- \) and \( \overline{\rho}_- \) act in different spaces. The nontrivial cohomology of \( \sigma'_- \) is concentrated in the subspaces of \( G \)-grades 0, 1/2 and 1. This follows from the fact that the operators \( \rho_- \) and \( \overline{\rho}_- \) (14.11) act as the exterior differentials \( \theta^\alpha \frac{\partial}{\partial y^\alpha} \) and \( \bar{\theta}^\alpha \frac{\partial}{\partial \bar{y}^\alpha} \) with \( \theta^\alpha = e^{\alpha \alpha'} \bar{y}^{\alpha'} \) and \( \bar{\theta}^\alpha = e^{\alpha \alpha'} y_\alpha \) in the spaces of functions of \( y^\alpha \) and \( \bar{y}^{\alpha'} \), respectively. As a result, by Poincare’s lemma, the cohomology is concentrated in the sectors where \( \sigma'_- \) differs from \( \rho_- \) or \( \overline{\rho}_- \), that is where the step functions differ from a constant. Also let us note that, in the gauge field sector, the difference between \( \sigma_- \) and \( \sigma'_- \) due to \( \sigma^\text{weyl}_- \) matters only in the computation of \( H^2(\sigma'_-) \) because the Weyl 0-forms in \((3.17)\) contribute to the sector of 2-forms.

\( H^0(\sigma'_-) \) is easy to compute. The nontrivial cohomology appears in the subspaces of \( G = 0 \) or 1/2 where \( \sigma_- \) acts trivially because of the step functions in (14.10). So,

\[
H^0(\sigma'_-) : \quad \epsilon (y, \bar{y}) = \sum_{|n-m| \leq 1} \frac{1}{2 n! m!} y_{\alpha_1} \ldots y_{\alpha_n} \bar{y}^{\beta_1} \ldots \bar{y}^{\beta_m} \epsilon^{\alpha_1 \ldots \alpha_n , \beta_1 \ldots \beta_m} .
\]

(14.13)

\( \epsilon^{\alpha_1 \ldots \alpha_n , \beta_1 \ldots \beta_m} (x) , |n-m| \leq 1 \) are parameters of differential gauge symmetry transformations of spin \( s = 1 + \frac{1}{2} (n + m) \) massless fields. For integer spins with \( n = m = s - 1 \), the corresponding spin \( s \) gauge symmetry parameter is equivalent to a rank \( s - 1 \) symmetric traceless Lorentz tensor. This agrees with the standard Fronsdal formulation \([44]\). For half-integer spins, \( n = s - 3/2 , m = s - 1/2 \) or \( m = s - 3/2 , n = s - 1/2 \). The corresponding spin \( s \) gauge symmetry parameter is equivalent to a rank \( s - 3/2 \) symmetric \( \gamma \)-transversal tensor-spinor in tensor-spinor notation. This agrees with the Fang-Fronsdal theory \([45]\).
Eq. (14.13) implies that all gauge parameters \( e^{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_m}(x) \) with \( |n - m| > 1 \) are of Stueckelberg type, i.e., the associated gauge field transformations contain algebraic shifts that gauge away some components of the HS connections. In particular, the linearized local Lorentz symmetry parameters \( e^{\alpha \beta}(x) \) and \( e^{\alpha' \beta'}(x) \), which allow to gauge away the antisymmetric part of the spin two vierbein, are of this class.

\( H^1(\sigma'_-) \) describes those components of the 1-form connections that are neither auxiliary (i.e., being expressed in terms of other connections by algebraic constraints that can be imposed in terms of HS curvatures, a la zero-torsion constraint in gravity) nor Stueckelberg (i.e., cannot be gauge fixed to zero by algebraic shift gauge symmetries). It is not hard to see that \( H^1(\sigma'_-) \) is concentrated in the subspace with \( G = 0 \) for bosons and \( G = 1/2 \) for fermions, where \( \sigma'_- \) acts trivially so that all elements are \( \sigma'_- \) closed in this sector. To compute \( H^1(\sigma'_-) \) it is thus enough to factor out the \( \sigma'_- \) exact part, which is equivalent to gauging away the Stueckelberg components of the gauge fields. This gives the following results.

In the bosonic case

\[
H^{1\text{bos}}(\sigma'_-) : \quad \omega(y, \bar{y}) = e^{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \phi(y, \bar{y}) + e^{\alpha \beta} y_\alpha \bar{y}_\beta \phi'(y, \bar{y}) , \quad G(\omega) = 0 , \quad (14.14)
\]

i.e., spin \( s \geq 1 \) dynamical fields identify with the 0-forms \( \phi(y, \bar{y}|x) \) and \( \phi'(y, \bar{y}|x) \) that satisfy

\[
n(\phi(y, \bar{y}|x) = \bar{n} \phi(y, \bar{y}|x) = s \phi(y, \bar{y}|x) , \quad n(\phi'(y, \bar{y}|x) = \bar{n} \phi'(y, \bar{y}|x) = (s - 2) \phi'(y, \bar{y}|x) .
\]

These describe two irreducible components of the spin \( s \) Fronsdal double traceless symmetric tensor field. In particular, in the spin two sector, \( H^{1\text{bos}}(\sigma'_-) \) describes a rank two symmetric tensor in terms of two-component spinors. This is the fluctuation part of metric equivalent to the fluctuation part of the vierbein modulo linearized local Lorentz gauge symmetry.

In the fermionic case

\[
H^{1\text{fer}}(\sigma'_-) : \quad \omega^+(y, \bar{y}) + \omega^-(y, \bar{y}) = (n - \bar{n}) \omega^+(y, \bar{y}) = \pm \omega^-(y, \bar{y}) , \quad (14.15)
\]

\[
\omega^+(y, \bar{y}) = e^{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \psi^+_1(y, \bar{y}) + e^{\alpha \beta} y_\alpha \bar{y}_\beta \psi^+_2(y, \bar{y}) + e^{\alpha \beta} y_\alpha \bar{y}_\beta \psi^+_3(y, \bar{y}) ,
\]

\[
\omega^-(y, \bar{y}) = e^{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \psi^-_1(y, \bar{y}) + e^{\alpha \beta} \bar{y}_\alpha y_\beta \psi^-_2(y, \bar{y}) + e^{\alpha \beta} \bar{y}_\alpha y_\beta \psi^-_3(y, \bar{y}) .
\]

Here \( \psi^+_1(y, \bar{y}|x) \), \( \psi^-_2(y, \bar{y}|x) \) and \( \psi^-_3(y, \bar{y}|x) \) and their conjugates \( \psi^-_1(y, \bar{y}|x) \), \( \psi^+_2(y, \bar{y}|x) \) and \( \psi^+_3(y, \bar{y}|x) \) describe three irreducible components of the Fang-Fronsdal triple \( \gamma \)-transverse symmetric tensor-spinor fermionic field. For a half-integer spin \( s \) we have

\[
n(\psi^+_1 = (s \pm 1/2) \psi^+_1 , \quad \bar{n} \psi^+_1 = (s \mp 1/2) \psi^+_1 ,
\]

\[
n(\psi^+ = (s - 1 \pm 1/2) \psi^+_2 , \quad \bar{n} \psi^+_2 = (s - 1 \mp 1/2) \psi^+_2 ,
\]

\[
n(\psi^+_3 = (s - 2 \pm 1/2) \psi^+_3 , \quad \bar{n} \psi^+_3 = (s - 2 \mp 1/2) \psi^+_3 .
\]

Finally, \( H^2(\sigma'_-) \) classifies differential equations on the dynamical fields contained in the 4d unfolded HS system. \( H^2(\sigma_-) \) consists of the generalized Weyl part parameterized by the
r.h.s. of (3.17) and the Einstein cohomology that represents the l.h.s. of the massless field equations. Since the generalized Weyl tensor part has already been taken into account by $\sigma^\text{weyl}_-$ in (14.9), $H^2(\sigma'_-)$ consists of the Einstein cohomology.

Let us start with the simpler fermionic case. $H^2\text{fer}(\sigma'_-)$ consists of the grade 1/2 2-forms $R$, which are automatically closed, modulo exact 2-forms

$$R^\text{exact} = \rho_- W, \quad (n - \bar{n}) W = 3 \bar{W}, \quad \bar{R}^\text{exact} = \bar{\rho}_- \bar{W}, \quad (\bar{n} - n) \bar{W} = 3 \bar{W}. $$

Elementary computation shows that $H^2\text{fer}(\sigma'_-)$ is

$$H^2\text{fer}(\sigma'_-) = E^+(y, \bar{y}) + E^-(y, \bar{y}), \quad (n - \bar{n}) E^{\pm}(y, \bar{y}) = \pm E^{\pm}(y, \bar{y}), \quad (14.16)$$

$$E^+(y, \bar{y}) = \left[ \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\alpha} E_1^-(y, \bar{y}) + \bar{y}^\alpha \frac{\partial}{\partial \bar{y}^\alpha} E_2^+(y, \bar{y}) \right] + H^{\alpha\beta} y_\alpha y_\beta E_3^-(y, \bar{y})$$

$$E^-(y, \bar{y}) = H^{\alpha\beta} \left( \frac{\partial^2}{\partial \bar{y}^\alpha \partial y^\alpha} E_1^+(y, \bar{y}) + y_\alpha \frac{\partial}{\partial y^\alpha} E_2^-(y, \bar{y}) \right) + \bar{H}^{\alpha\beta} \bar{y}_\alpha \bar{y}_\beta E_3^+(y, \bar{y}),$$

where $H^{\alpha\beta}$ and $\bar{H}^{\alpha\beta}$ are the basis 2-forms (3.11) and

$$n E_1^{\pm} = (s \mp 1/2) E_1^{\pm}, \quad \bar{n} E_1^{\pm} = (s \mp 1/2) E_1^{\pm},$$

$$n E_2^{\pm} = (s - 1 \pm 1/2) E_2^{\pm}, \quad \bar{n} E_2^{\pm} = (s - 1 \mp 1/2) E_2^{\pm},$$

$$n E_3^{\pm} = (s - 2 \pm 1/2) E_3^{\pm}, \quad \bar{n} E_3^{\pm} = (s - 2 \mp 1/2) E_3^{\pm}.$$

As a linear space, Einstein cohomology $H^2\text{fer}(\sigma'_-)$ is isomorphic to $H^1\text{fer}(\sigma'_-) (14.15).$ This is expected because the massless field equations are Lagrangian, i.e., there are as many equations as field variables.

Let us now consider the bosonic case. Here the sectors of $G = 0$ and 1 should be analyzed.

It is easy to see that

$$H^2(\sigma'_-) \big|_{G=0} = 0, \quad (14.17)$$

which means that any 2-form with $G = 0$, which is automatically $\sigma'_-$ closed, is $\sigma'_-$ exact. For spin one this means that the respective part of the equation (3.17) with $G = 0$ is a constraint that expresses the Maxwell stress tensor $C^{\alpha\beta}, C^{'\alpha\beta}$ via derivatives of the spin one gauge potential. For spins $s \geq 2$, (14.17) implies that the 4d 1-form gauge fields $\omega(y, \bar{y})$ with $G = 1$ are auxiliary being expressed (modulo pure gauge components) via derivatives of the dynamical fields by the “zero torsion condition” $R(y, \bar{y})$ $|_{G=0} = 0$ that imposes no differential equations on the dynamical Fronsdal fields. In particular, in the spin two sector, (14.17) allowed to impose the standard zero-torsion condition, that expresses Lorentz connection via derivatives of the vierbein imposing no restrictions on the latter.

Now consider the part of the cohomology $H^2(\sigma'_-) in the G = 1 sector. We have to find such 2-forms $\Phi$ with $n - \bar{n} = 2$ and $\bar{\Phi}$ with $n - \bar{n} = -2$ that

$$\rho_- \Phi + \bar{\rho}_- \bar{\Phi} = 0. \quad (14.18)$$

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The nontrivial cohomology consists of solutions of this condition with \( \rho_\Phi \neq 0 \) and \( \overline{\rho}_\Phi \neq 0 \) (otherwise, \( \Phi \) and \( \overline{\Phi} \) are \( \sigma'_t \) exact). The key point is that, to cancel out in (14.18), the terms coming from \( \Phi \) and \( \overline{\Phi} \) should carry equivalent representations of the Lorentz algebra even though they support polynomials of \( y \) and \( \bar{y} \) with \( n - \bar{n} = 2 \) and \( n - \bar{n} = -2 \), respectively.

It is not hard to see that this is possible if the cohomology space is spanned by the polynomials \( E_i(y, \bar{y}) \) that contain as many \( y^\alpha \) as \( \bar{y}^{\alpha'} \), i.e., \( G(E_i(y, \bar{y})) = 0 \). The appropriate Ansatz is

\[
\Phi = H_{\alpha\beta}y^\alpha y^\beta E_1(y, \bar{y}) + \overline{\Pi}_{\alpha'\beta'} \frac{\partial^2}{\partial \bar{y}_{\alpha'} \partial y_{\beta'}} E_2(y, \bar{y}),
\]

(14.19)

\[
\overline{\Phi} = H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} E_3(y, \bar{y}) + H^{\alpha'\beta'} \bar{y}_{\alpha'} \bar{y}_{\beta'} E_4(y, \bar{y}.
\]

(14.20)

Using the identities

\[
H^{\alpha\beta} \wedge e^{\gamma \alpha'} = e^{\gamma \gamma} H^{\beta \alpha'} + e^{\beta \gamma} H^{\alpha \alpha'},
\]

\[
H^{\alpha'\beta'} \wedge e^{\gamma \alpha'} = -e^{\gamma \gamma'} H^{\alpha \beta'} - e^{\beta \gamma'} H^{\alpha \alpha'},
\]

where \( H^{\alpha\beta} = -\frac{1}{2} e^{\alpha \alpha'} \wedge e^{\beta \alpha'} \wedge e_{\beta'} \) are 4d basis 3-forms, we obtain

\[
\rho_\Phi + \overline{\rho}_\Phi = -H^{\alpha\alpha'} (y_{\alpha} \bar{y}_{\alpha'} (y^\beta \frac{\partial}{\partial y^\beta} + 3)(E_1 - E_4) + \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\alpha} (y^\beta \frac{\partial}{\partial y^\beta} - 1)(E_2 - E_3)).
\]

(14.21)

This implies that (14.19) and (14.20) describe a nontrivial cohomology provided that \( E_2 = E_3 = E \) and \( E_1 = E_4 = E' \), where \( E'(y, \bar{y}) \) is an arbitrary grade zero polynomial while \( E(y, \bar{y}) \) should be at least of fourth order to contribute.

For any integer spin \( s \geq 2 \), the Einstein cohomology

\[
H^2_{bos}(\sigma_t) = \left( H^{\alpha'\beta'} \frac{\partial^2}{\partial \bar{y}_{\alpha'} \partial y_{\beta'}} + H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta} \right) E(y, \bar{y}) + \left( H_{\alpha\beta}y^\alpha y^\beta + H^{\alpha'\beta'} \bar{y}_{\alpha'} \bar{y}_{\beta'} \right) E'(y, \bar{y})
\]

(14.22)

is described by the polynomials \( E(y, \bar{y}) \) and \( E'(y, \bar{y}) \) that satisfy

\[
nE(y, \bar{y}) = \bar{n} E(y, \bar{y}) = sE(y, \bar{y}), \quad nE'(y, \bar{y}) = \bar{n} E'(y, \bar{y}) = (s - 2)E'(y, \bar{y}).
\]

Einstein cohomology is responsible for the dynamical field equations for any integer spin \( s \geq 2 \): the condition that the part of the HS curvature 2-form parameterized by \( E(y, \bar{y}) \) and \( E'(y, \bar{y}) \) is zero, which is true by the unfolded equations (3.17), imposes the Fronsdal field equations on the dynamical fields of spins \( s \geq 2 \). This agrees with the condition that there are as many dynamical equations as field variables. In terms of Lorentz tensors, \( E(y, \bar{y}) \) and \( E'(y, \bar{y}) \) are equivalent to rank \( s \) and rank \( s - 2 \) traceless tensors, respectively.

The spin one part of \( E \) is \( \sigma_t \) exact because from (14.21) it follows that \( \rho_\Phi(\Phi) = \overline{\rho}_\Phi(\overline{\Phi}) = 0 \) in this case. This conforms the fact that the spin one dynamical equations are described by the 0-form cohomology (14.7).

The pattern of \( H^p(\sigma'_t) \) proves the so-called Central-On-Shell theorem [40] that states that the equations (3.17) and (3.18) are equivalent to the standard 4d massless field equations.
plus an infinite set of constraints that express infinitely many auxiliary fields via derivatives of the dynamical massless fields. Let us stress that the Central-On-Shell theorem is true both in 4d Minkowski space and in AdS$_4$. Actually, although the frame 1-form $e^{a\alpha'}(x)$ that enters $\sigma'$ does depend on a chosen geometry and coordinate system, the analysis of the $\sigma'$ cohomology only uses that the vacuum connection is flat and the vacuum frame field is nondegenerate, forming a frame in the space of 4d 1-forms.

15 $\sigma_-$ analysis in $\mathcal{M}_4$

15.1 $\sigma_-$ and $\tau_-$

The grading operator $G$, which is independent of the choice of a base manifold, is given by (14.1)-(14.4). $\sigma_-$ is the grade $-1$ part of the covariant derivative. Let $\sigma_{-ij}$ denote $\sigma_-$ in the module $\phi_{ij}$. We obtain

$$\sigma_{-00} = \frac{1}{2} h^{AB} \frac{\partial^2}{\partial b^A \partial b^B} ,$$

$$\sigma_{-11} = \frac{1}{2} f_{AB} \frac{\partial^2}{\partial a_A \partial a_B} ,$$

$$\sigma_{-01} = \frac{1}{2} \left( h^\alpha{}^\beta \frac{\partial^2}{\partial b^\alpha \partial b^\beta} + h^\alpha{}'^\beta' \frac{\partial}{\partial b^\alpha} \bar{a}^{\beta'} + h^{\alpha' \beta'} \bar{a}^\alpha \bar{a}^{\beta'} \right) \theta(n - \bar{n} - 2) + \frac{1}{2} \left( f_{\alpha' \beta'} \frac{\partial^2}{\partial \bar{a}^\alpha \partial \bar{a}^{\beta'}} - f_{\alpha \beta'} \frac{\partial}{\partial \bar{a}^{\beta'}} b^\alpha + f_{\alpha \beta} b^\alpha b^\beta \right) \theta(\bar{n} - n - 2) ,$$

$$\sigma_{-10} = \frac{1}{2} \left( f_{\alpha \beta} \frac{\partial^2}{\partial a_\alpha \partial a_\beta} - f_{\alpha' \beta'} \frac{\partial}{\partial a_\alpha} \bar{b}^{\beta'} + f_{\alpha' \beta'} \bar{b}^{\alpha'} \bar{b}^{\beta'} \right) \theta(n - \bar{n} - 2) + \frac{1}{2} \left( h^{\alpha' \beta'} \frac{\partial^2}{\partial \bar{b}^{\alpha'} \partial \bar{b}^{\beta'}} + h^{\alpha \beta} \frac{\partial}{\partial \bar{b}^{\beta}} a_\alpha + h^{\alpha' \beta'} a_\alpha a_\beta \right) \theta(\bar{n} - n - 2) ,$$

where $n$, $\bar{n}$ and the step function $\theta(n)$ are defined in (14.8) and (14.12). Because $\sigma_{-i1-i}$ are asymmetric with respect to primed and unprimed indices, the manifest symmetry is $GL(2, \mathbb{C})$ which consists of the Lorentz symmetry $SL(2, \mathbb{C})$, dilatations and duality transformations.

Recall that in the case (8.7) of $Sp(4, \mathbb{R})$, both $h^{AB}$ and $f_{AB}$ are nondegenerate. The naive flat limit with $f_{AB} \to 0$ is degenerate. However, because the operators $\sigma_{-i1-i}$ are defined differently in the sectors with $n - \bar{n} > 0$ and $n - \bar{n} < 0$, the fields can be rescaled so that $\sigma_{-i1-i}$ remain nondegenerate in the flat limit. The result extends the $e$-dependent part of the Minkowski covariant derivative (14.3) to $\mathcal{M}_4$.

Once both $h^{AB}$ and $f_{AB}$ are expressed in terms the vielbein $e^{a\alpha'}$ by (8.7), from (15.1)-(15.4) it is clear that, up to renaming the oscillator variables, there are two essentially different $\sigma_-$ operators. One in the sector of $\phi_{00}$ and $\phi_{11}$ and another one in the sector of
\( \phi_{01} \) and \( \phi_{10} \). To simplify notations we call the former \( \sigma_- \) and the latter \( \tau_- \). Denoting the respective oscillators as \( y^A \) we have

\[
\sigma_- = e^{AB} \frac{\partial^2}{\partial y^A \partial y^B},
\]

\[
\tau_- = t_- \theta(n - \bar{n} - 2) + \bar{t}_- \theta(\bar{n} - n - 2),
\]

where

\[
t_- = \nu^+ + \nu^0 + \nu^-, \quad \bar{t}_- = \bar{\nu}^+ + \bar{\nu}^0 + \bar{\nu}^-
\]

\[
\nu^- = e^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta}, \quad \nu^0 = e^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \bar{y}^\beta, \quad \nu^+ = e^{\alpha'\beta'} y_{\alpha'} \bar{y}^\beta,
\]

\[
\bar{\nu}^- = e_{\alpha'\beta'} \frac{\partial^2}{\partial \bar{y}^\alpha \partial y^\beta}, \quad \bar{\nu}^0 = e_{\alpha'\beta'} \frac{\partial}{\partial \bar{y}^\alpha} y^\beta, \quad \bar{\nu}^+ = e_{\alpha\beta} y^\alpha y^\beta.
\]

Note that \( \{\nu^i, \nu^j\} = 0, \{\bar{\nu}^i, \bar{\nu}^j\} = 0, \quad i, j = -, 0, + \)

and, therefore \((t_-)^2 = (\bar{t}_-)^2 = 0\). Although \(\{t_-, \bar{t}_-\} \neq 0\), \((\tau_-)^2 = 0\) because the step functions in (15.6) imply that the parts of \(\tau_-\) associated with \(t_-\) and \(\bar{t}_-\) act in different subspaces.

The dependence on \(\lambda\) has been removed by a field redefinition along the lines of Section 3, which also was used to adjust convenient coefficients in (15.7). In the rescaled variables \(y^A, \lambda\) appears in front of the \(\sigma_{+ij}\) part of the covariant derivative. Correspondingly, the flat space field equations in \(M_4\) result from setting \(\lambda = 0\) and dropping the terms with \(\sigma_{+ij}\). In the flat case of \(M_4\) one can use “Cartesian coordinates” with

\[
e^{AB} = dX^{AB}, \quad D^{tw:fl} = d + \sigma_-, \quad D^{ad:fl} = d + \tau_-.\]

The dynamical content of the unfolded field equations in \(Sp(4, \mathbb{R})\) and \(M_4\) is determined by the cohomology of \(\sigma_-\) and \(\tau_-\). We have to calculate \(H^0(\sigma_-)\) and \(H^1(\sigma_-)\) to identify the independent fields and field equations in the twisted adjoint 0-form sector and \(H^0(\tau_-), H^1(\tau_-)\) and \(H^2(\tau_-)\) to identify the gauge parameters, dynamical fields and gauge invariant combinations of derivatives of the dynamical fields that either represent the l.h.s. of field equations or identify with the generalized Weyl tensors via the Chevalley-Eilenberg cohomology terms.

We start with \(H^p(\sigma_-)\) extending the results of [3] to the AdS-like case of \(Sp(M, \mathbb{R})\).

### 15.2 Weyl 0-form sector

In the case of \(M_M\), the unfolded equations in question are (2.15). They are equivalent to \(D^{tw:fl} C = 0\). \(H^0(\sigma_-) = C + y^A C_A\) describes the dynamical fields \(C(X)\) and \(C_A(X)b^A\) in \(M_M\). \(H^1(\sigma_-)\) is

\[
H^1(\sigma_-) = e^{AB} y^C y^D E_{AB,CD} + e^{AB} y^C E_{AB,C},
\]
where $E_{AB,CD}$ and $E_{AB,C}$ represent the l.h.s. of the field equations (1.1) and (1.2) and satisfy
\[
E_{AB,CD} = E_{BA,CD} = E_{AB,DC}, \quad E_{(AB,C)D} = 0, \quad E_{AB,C} = E_{BA,C}, \quad E_{(AB,C)} = 0.
\]

The equations (1.1) and (1.2) in $\mathcal{M}_4$ were originally derived this way in [3]. The analysis in the curved $Sp(M, \mathbb{R})$ background is analogous. $\sigma_-$ is still given by (15.5), where $e^{AB}$ is the generalized vielbein of $Sp(M, \mathbb{R})$ introduced in Section 8. The symmetry type of the equations and leading derivative terms remain the same as in $\mathcal{M}_4$. The exact form of the equation (1.1) is deformed by the $\lambda^2$-dependent lower-derivative terms. These appear due to the $\sigma_+$ part of the covariant derivative associated with the non-zero "special conformal" connection (8.7)
\[
\sigma_+ = \frac{1}{4} \lambda^2 e_{AB} b^A b^B, \quad e_{AB} = C_{FAC} G_{GB} e^{FG},
\]
where $C_{AB}$ is the $Sp(M, \mathbb{R})$ invariant antisymmetric form.

Also, usual derivatives have to be replaced by the Lorentz-like $Sp(M, \mathbb{R})$ covariant derivatives associated with the connection $\omega^A_B$.

\[
Df_A(X) = df_A(X) + \omega^B_A f_B(X), \quad D = dX^{AB} D_{AB}, \quad e^{AB} = dX^{AB} e_{AB}, \quad D_{AB} = e_{AB} D_{AB}.
\]

As a result, the $Sp(2M, \mathbb{R})$ invariant deformation of the equations (1.1) and (1.2) to $Sp(M, \mathbb{R})$ reads as
\[
\left( D_{AB} D_{CD} - D_{CB} D_{AD} \right) C(X) - \frac{1}{2} \lambda^2 \left( C_{AC} C_{BD} + C_{BC} C_{AD} \right) C(X) = 0,
\]
\[
D_{AB} C_B(X) - D_{CB} C_A(X) = 0.
\]

### 15.3 Gauge 1-form sector

In this subsection we analyze $H^p(\tau_-)$ that determines dynamical content of the field equations in the sector of 1-form connections.

#### 15.3.1 Auxiliary Lemmas

The following useful lemmas will be used in what follows.

**Lemma 1:**
$H^p(\overline{\nu}^+ , P) = 0$ at $p = 0, 1, 2$ if $P$ is the space of polynomials of $y^a$ with $a = 1, 2 \ldots m > 1$.

**Proof:** $H^0(\overline{\nu}^+ , P) = 0$ because $\text{Ker} \overline{\nu}^+ = 0$ on the space of polynomials (product of any two nonzero polynomials never gives zero polynomial).

The case of $H^1(\overline{\nu}^+ , P)$ can be analyzed as follows. Let $\omega(y)$ be a 1-form $\omega(y) = e^{\alpha \beta} \omega^{\alpha \beta \beta_1 \ldots \beta_n} y^{\beta_1} \ldots y^{\beta_n}$. The $\overline{\nu}^+$ closedness condition $y^{\alpha} y^{\alpha} \omega_{\beta \beta} (y) - y^{\beta} y^{\beta} \omega_{\alpha \alpha} (y) = 0$ is equivalent to
\[
\delta^{\gamma(n)}_{\alpha \beta} \omega_{\beta \beta} - \delta^{\gamma(n)}_{\beta \beta} \omega_{\alpha \alpha} = 0,
\]

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where we use the convention that lower (upper) indices denoted by the same letter are symmetrized and a number of symmetrized indices is indicated in parentheses. Contracting twice the indices $\alpha$ with $\gamma$ we obtain

$$((n + m + 1)(n + m) - 2)\omega_{\beta\beta}^{\gamma(n)} = 4n\delta_{\beta}^{\gamma}\omega_{\delta\delta}^{\delta\gamma(n-1)} + n(n - 1)\delta_{\beta}^{\gamma}\omega_{\delta\delta}^{\delta\gamma(n-2)}. \quad (15.13)$$

One more contraction gives

$$(n + m - 2)\omega_{\beta\beta}^{\gamma(n-1)} = (n - 1)\delta_{\beta}^{\gamma}\omega_{\delta\delta}^{\delta\gamma(n-2)}. \quad (15.14)$$

Now one observes that since a $\mathcal{V}^+$-exact 1-form has the form $e^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \partial_{y^\beta} \xi(y)$, a cohomology class can be fixed by setting $\omega_{\delta\delta}^{\delta\gamma(n-2)} = 0$. Then from (15.14) and (15.13) it follows that $\omega(y) = 0$, i.e., $H^1(\mathcal{V}^+, P) = 0$. The proof that $H^2(\mathcal{V}^+, P) = 0$ is analogous. $\Box$

**Lemma 2:** $H^p(t_-, P) = H^p(\bar{t}_-, P) = 0$ at $p = 0, 1, 2$.

**Proof:** Consider the sector of $p$-forms $\omega_0$ expandable into the wedge product of $p$ 1-forms $e^{\alpha\beta}$. From the condition that $\omega$ is $t_-$ closed it follows that $\omega_0$ is $\mathcal{V}^+$ closed. By Lemma 1 it follows that $\omega_0$ is $\mathcal{V}^+$ exact. Therefore, one can choose a representative of $H^0(t_-, P) = 0$ with $\omega_0 = 0$ in the purely $e^{\alpha\beta}$ sector. Then one considers the sector of $p$-forms $\omega_1$ that contains $p - 1 e^{\alpha\beta}$, repeating the analysis. The process continues till one proves that $H^p(t_-, P) = 0$. Analogously one proves that $H^p(\bar{t}_-, P) = 0$. $\Box$

**Corollary:** As a consequence of Lemma 2 it follows that $H^p(\nu_-, P)$ is concentrated in the subspace with $G \leq 1$ where $\nu^+$ and $\mathcal{V}^+$ do not act independently.

Note that the case with $n + \bar{n} = \pm 2$ is still nontrivial because here both $Im t_-$ and $Im \bar{t}_-$ belong to the space with $G = 0$ and can cancel each other thus extending $Ker \tau_-$ compared to $Ker t_+ \oplus Ker \bar{t}_-$. (Analogous phenomenon occurred in the $4d$ analysis of the bosonic sector of $H^2(\sigma_\nu^\prime)$ in the end of Subsection 14.3.)

### 15.3.2 $H^0(\tau_-)$

$H^0(\tau_-)$ is easy to compute. Here the key observation is that the sectors with $n - \bar{n} = \pm 2$ do not talk to each other. As a result, $H^0(\tau_-)$ is described by the same formula (14.13) as in the $4d$ case, i.e., the true gauge symmetry parameters in the matrix space are described by the same set of multispinors as in Minkowski space. In particular, all gauge parameters $e^{\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_m}(X)$ with $|n - m| > 1$ are of Stueckelberg type with the field transformations that contain algebraic shifts gauging away some components of the HS 1-form connections.

### 15.3.3 $H^1(\tau_-)$

The computation of $H^1(\tau_-)$ is also based on the fact that it is concentrated in the subspace where $\tau_-$ is identically zero, i.e., $|n - \bar{n}| \leq 1$.

In the bosonic case, it is necessary to check that the possible extension of $H^1(\tau_-)$ due to extension of $Ker \tau_-$ compared to $Ker t_+ \oplus Ker \bar{t}_-$ does not take place. The proof of this
fact, which is elementary but somewhat lengthy, we leave to the reader. The idea of the proof is illustrated below by the analysis of $H^2(\tau_-)$ which results in nontrivial cohomology.

Just as in the 4d case, the ambiguity in adding $\tau_-$-exact 1-forms is used to get rid of Stueckelberg components of the 1-forms of the form $e^{a\alpha'}\omega_{a\alpha'}(y,\bar{y})$. As a result,

$$H^1(\tau_-): \quad \omega(y,\bar{y}) = e^{a\beta}\omega_{a\beta}(y,\bar{y}) + e^{a\beta'}\bar{\omega}_{a\beta'}(y,\bar{y}) + e^{a\beta'}\omega_{a\beta}(y,\bar{y}),$$

where $\omega_{a\beta}(y,\bar{y})$ and $\bar{\omega}_{a\beta'}(y,\bar{y})$ are arbitrary fields with $|n - \bar{n}| \leq 1$ while $e^{a\beta'}\omega_{a\beta}(y,\bar{y})$ has the same content (14.14) and (14.15) as in the 4d theory. We conclude that, apart from the 4d Fronsdal gauge fields, the formulation in $\mathcal{M}_4$ requires the additional fields $\omega_{a\beta}(y,\bar{y})$ and $\bar{\omega}_{a\beta'}(y,\bar{y})$, that describe components of the 1-form connection along the spinning directions in $\mathcal{M}_4$. Since the system in $\mathcal{M}_4$ is by construction equivalent to that in Minkowski space, the additional dynamical fields in $\mathcal{M}_4$ are related by their field equations to the 4d HS fields.

To figure out the form of nontrivial field equations we have to compute $H^2(\tau_-)$. We start with the simpler fermionic case.

### 15.3.4 $H^2(\tau_-):$ fermions

$\mathcal{H}^2_{\text{fer}}(\tau_-)$ consists of the 2-forms $R^\pm = e^{AB} \wedge e^{CD}R_{AB,CD}(y,\bar{y})$ with $n - \bar{n} = \pm 1$ (which are all $\tau_-$-closed) modulo $\tau_-$-exact 2-forms

$$R^{\text{exact}+} = t_+ W^+, \quad R^{\text{exact}^-} = t_- W^-,$$

where $W^\pm$ are arbitrary 1-forms such that $(n - \bar{n})W^\pm = \pm 3W^\pm$.

In the $e^{a\alpha'} \wedge e^{\beta'}$ sector the analysis repeats that of the 4d case. The nontrivial class is represented by the Einstein cohomology and Weyl cohomology. The Weyl cohomology is represented by the $\bar{y}$ independent 0-form $C^{3/2}_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma$ and its conjugate that describe spin 3/2 (cf. (15.6)) Thus, this part of $\mathcal{H}^2_{\text{fer}+}$ is

$$H^2_{\text{fer}+} = \mathcal{P}^{\alpha\beta'} \left( \frac{\partial^2}{\partial \bar{y}^\alpha \partial y^{\beta'}} \phi_{-1}(y,\bar{y}) + \bar{y}_{\alpha'} \frac{\partial}{\partial \bar{y}^{\beta'}} \phi_{+2}(y,\bar{y}) \right) + H^{\alpha\beta} \left( y_\alpha y_\beta \phi_{-3}(y,\bar{y}) + \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C^{3/2}(y) \right),$$

where $(n - \bar{n})(\phi_{\pm t}(y,\bar{y})) = \pm \phi_{\pm t}(y,\bar{y})$ and the 2-forms $H^{\alpha\beta}$ and $\mathcal{P}^{\alpha\beta'}$ are defined in (3.11). $H^2_{\text{fer}+}$ is given by the complex conjugated expression.

Important difference compared to the 4d case is that, by Lemma 1, the Weyl 0-forms associated with spins $s > 3/2$ do not correspond to a nontrivial cohomology in the case of $\mathcal{M}_4$. This means that although the generalized Weyl tensors appear on the r.h.s. of (12.4), in $\mathcal{M}_4$ this is a consequence of Bianchi identities applied to the lower spin field equations. This conclusion is consistent with the fact that, in $\mathcal{M}_4$, different spins correspond to modes of the hyperfields $C(X)$ and $C_4(X)$ with respect to extra spinning directions. In other words, it is not possible to restrict to zero $C(y|X)$ of some power in $y$ without restricting its dependence on the spinning coordinates.
Having fixed the representatives $H^2_{0 \text{fer}^+}$ and $H^2_{0 \text{fer}^-}$ in the form (15.16) and complex conjugated, we have fixed the part of $\tau$--exact 2-forms (15.15) with $W^\pm_0 = e^{\alpha \alpha'} W^\pm_{0 \alpha \alpha'}$. The other components $W^\pm_1 = e^{\alpha \beta} W^\pm_{1 \alpha \beta} + e^{\alpha' \beta'} W^\pm_{1 \alpha' \beta'}$ remain to be factored out in the analysis of the remaining sectors of $H^2_{\text{fer}^\tau}(\tau_-)$. The analysis of the sectors $e^{\alpha \gamma} \wedge e^{\beta \gamma'}$ and $e^{\beta \alpha'} \wedge e^{\alpha \gamma'}$ amounts effectively to the $4d$ analysis of $H^1(\sigma'_-)$. Where the leftover components of the 1-forms $e^{\alpha \beta} W^\pm_{1 \alpha \beta}$ and $e^{\alpha' \beta'} W^\pm_{1 \alpha' \beta'}$ are treated as the 0-form Stueckelberg parameters. As a result, this part of $H^2_{\text{fer}^\tau}(\tau_-)$ is

$$H^2_{\text{fer}^\tau} = e^{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \Psi^+_i(y, \bar{y}) + e^{\alpha \beta} y_\alpha \frac{\partial}{\partial y^\beta} \Psi^-_i(y, \bar{y}) + e^{\beta \gamma} y_\beta \bar{y} \gamma \Psi^+_i(y, \bar{y}),$$

(15.17)

where $\Psi^+_i(y, \bar{y}) = e^{\alpha \beta} \Psi^\pm_{i \alpha \beta}(y, \bar{y}) + e^{\alpha \beta} \Psi^\pm_{i \alpha \beta}(y, \bar{y})$, and analogously for the complex conjugated.

Finally, the sectors of $e^{\alpha \beta} \wedge e^{\gamma \delta}$ and $e^{\alpha' \beta'} \wedge e^{\gamma' \delta'}$ fully belong to $H^2(\tau_-)$

$$H^2_{\text{fer}^\pm} = e^{\alpha \gamma} \wedge e^{\beta \gamma'} y_\alpha \bar{y} \gamma \Psi^+_i(y, \bar{y}) + e^{\beta \gamma} \alpha \gamma \Psi^+_i(y, \bar{y}) \gamma \Psi^+_i(y, \bar{y}).$$

To summarize, in the fermionic case $H^2_{\text{fer}^\tau}(\tau_-) = H^2_{\text{fer}^+}(\tau_-) \oplus H^2_{\text{fer}^-}(\tau_-)$, where $H^2_{\text{fer}^+}(\tau_-)$ and $H^2_{\text{fer}^-}(\tau_-)$ are complex conjugated and

$$H^2_{\text{fer}^\pm}(\tau_-) = H^2_{\text{fer}^\pm}(\tau_-) \oplus H^2_{0 \text{fer}^\pm}.$$
by 1-forms with coefficients in $e^{\alpha\beta}$ and $e^{\alpha'\beta'}$. Finally, the part of $H_0^{2\text{bos}}(\tau_-)$ in the sector free of $e^{\alpha\alpha'}$ remains unrestricted. To summarize,

$$H_0^{2\text{bos}}(\tau_-) = \rho(y, \bar{y}) + e^{\alpha\beta'} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^\beta'} W(y, \bar{y}) + e^{\alpha\beta'} y_a \bar{y}_b W'(y, \bar{y}),$$

where

$$\rho(y, \bar{y}) = e^{\alpha\gamma} \wedge e^{\beta\rho} \rho(y, \bar{y}) + e^{\alpha'\gamma'} \wedge e^{\beta'\rho} \rho(y, \bar{y}) + e^{\alpha\beta} \rho_{\alpha\beta}(y, \bar{y}),$$

$$W'(y, \bar{y}) = e^{\alpha\beta'} W'(y, \bar{y}) + e^{\alpha'\beta'} W'(y, \bar{y}).$$

This part of $H^{2\text{bos}}(\tau_-)$ is responsible for the field equations that relate additional field components in $H^1(\tau_-)$ to the usual 4d massless fields associated with $H^1(\tau_-)$.

Now let us consider the part of $H^{2\text{bos}}(\tau_-)$ with $G = 1$ which we denote $H^{2\text{bos}}(\tau_-)$. It is non-zero due to the phenomenon discussed in Corollary. The analysis of the 4d sector $e^{\alpha\alpha'} \wedge e^{\beta\beta'}$ is identical to that of Subsection 14.3. The question is how this cohomology extends to $\mathcal{M}_4$. In principle, it might happen that the Einstein cohomology disappears or shrinks to a smaller vector space in $\mathcal{M}_4$ which would imply that most of the 4d field equations become consequences of the Bianchi identities in the larger space-time, applied to the extended zero-torsion condition and/or to a subsystem of the field equations. Indeed, such a phenomenon occurred in the analysis of the equations on the Weyl 0-forms in [3] where the infinite dimensional cohomology in 4d shrinks to a finite dimensional one in $\mathcal{M}_4$ in agreement with the fact that the infinite set of 4d field equations (2.3) and (2.4) for massless fields of all spins amounts to the finite system of equations (1.1) and (1.2) in $\mathcal{M}_4$. This does not happen to the Einstein cohomology, however. Namely it is still parameterized by $E(y, \bar{y})$ and $E'(y, \bar{y})$ as in 4d Minkowski space. The detailed analysis is straightforward although somewhat annoying. The final result is that the non-zero components $\Phi$ of $H^{2\text{bos}}(\tau_-)$ are

$$\Phi = \Phi^+ + \Phi^-,$$

where

$$\Phi^+ = \Phi_2^+ + \Phi_1^+ + \Phi_0^+ + \Phi_{4d}^+ + \Phi_{nE}^+ + \Phi_{nE}^-,$$

$$\Phi^- = \Phi_2^- + \Phi_1^- + \Phi_0^- + \Phi_{4d}^- + \Phi_{nE}^- + \Phi_{nE}^-$$

and

$$\Phi_{4d}^+ + \Phi_{4d}^- = \left(\overline{\nabla_{\alpha'\beta'}} \frac{\partial^2}{\partial \bar{y}^\alpha \partial \bar{y}^\beta'} - H^{\alpha\beta}_{\alpha'\beta'} \frac{\partial^2}{\partial y^\alpha \partial y^\beta'} \right) E'(y, \bar{y}) + \left(2 + e^{\alpha\beta} y^\alpha y^\beta - \overline{\nabla_{\alpha'\beta'}} \bar{y}^\alpha \bar{y}^\beta \right) E'(y, \bar{y}),$$

(15.18)

$$\Phi_1^+ = e^{\gamma\beta'} \wedge e^{\alpha\beta} \bar{y}^\beta y_\alpha y_\beta \frac{2}{n + 4} E' + e^{\beta'\gamma'} \wedge e^{\alpha\beta} \frac{\partial}{\partial y^\alpha} y_a \frac{4}{n} E'$$

$$+ e^{\gamma\beta'} \wedge e^{\alpha\beta} \frac{\partial^2}{\partial y^\gamma \partial \bar{y}^\beta} y_a y_\beta \frac{2}{n(n + 1)(n + 2)} (2(n + 2)E' - n(n - 1)E)$$

$$+ e_{\alpha\beta'} \wedge e^{\gamma\beta} y^\beta \frac{\partial^2}{\partial \bar{y}^\beta \partial y^\gamma} \frac{2}{n(n + 1)(n + 2)} ((n + 2)E' + nE),$$

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\[ \Phi^-_1 = -e^{\gamma\alpha'_r} \wedge e^{\alpha\beta} \frac{\partial^4}{\partial y^\gamma \partial y^\alpha \partial y^\beta \partial y^\gamma} \frac{2}{n-2} E - e^{\alpha \gamma} \wedge e^{\gamma\beta'y'} \frac{\partial}{\partial y^\alpha} \frac{4}{n+2} E + e^{\alpha\beta'} \wedge e^{\gamma\beta'} \frac{\partial^2}{\partial y_{\gamma'} \partial y_{\beta'}} \frac{2}{n(n+1)(n+2)} ((n+2)(n+3)E' + 2nE) \]
\[ + e^{\alpha\beta'} \wedge e^{\gamma\beta'} \frac{\partial^2}{\partial y_{\gamma'} \partial y_{\beta'}} \frac{2(n+3)}{n(n+1)(n+2)} (nE + (n+2)E') , \]
\[ \Phi^+_1 = -2e^{\gamma\alpha'_r} \wedge e^{\alpha'\beta'} \frac{\partial^4}{\partial y^\gamma \partial y_{\gamma'} \partial y_{\alpha'} \partial y_{\beta'}} \frac{1}{n+2} E + \epsilon_{\beta'\gamma'} \wedge e^{\gamma\beta'} \frac{\partial}{\partial y_{\gamma'}} \frac{4}{n+2} E \]
\[ + e^{\alpha'\beta'} \wedge \epsilon_{\beta'\gamma'} y_{\gamma'} \frac{\partial^2}{\partial y_{\gamma'} \partial y_{\alpha'}} \frac{2}{n(n+1)(n+2)} (2nE + (n+2)(n+3)E') \]
\[ + e^{\alpha'\beta'} \wedge \epsilon_{\beta'\gamma'} \frac{\partial^2}{\partial y_{\gamma'} \partial y_{\alpha'}} \frac{2(n+3)}{n(n+1)(n+2)} (nE + (n+2)E') , \]
\[ \Phi^-_1 = e^{\gamma\gamma'} \wedge e^{\alpha'\beta'} y_{\gamma'} y_{\alpha'} y_{\beta'} \frac{2}{n+4} E' - e^{\beta'\gamma} e^{\alpha'\beta'} \frac{\partial}{\partial y_{\beta'}} \frac{4}{n+2} E' \]
\[ + e^{\alpha\gamma} \wedge e^{\alpha'\gamma} \frac{\partial^2}{\partial y^\alpha \partial y_{\alpha'}} \frac{2}{n(n+1)(n+2)} (2n+2)E' - n(n-1)E \]
\[ + 2e^{\alpha\beta'} \wedge e^{\gamma\alpha'} \frac{\partial^2}{\partial y^\gamma \partial y_{\alpha'}} \frac{n-1}{n+1}(nE + (n+2)E') , \]
\[ \Phi^+_2 = e^{\alpha \gamma} \wedge e^{\beta \gamma} y_{\alpha} y_{\beta} \frac{n-1}{(n+1)(n+2)} ((n+2)E' + nE) , \]
\[ \Phi^-_2 = -e^{\alpha \gamma} \wedge e^{\beta \gamma} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \frac{n+3}{n+1} ((n+2)E' + nE) , \]
\[ \Phi^+_2 = \epsilon_{\alpha'\gamma'} \wedge \epsilon_{\beta'\gamma'} \frac{\partial^2}{\partial y_{\gamma'} \partial y_{\beta'}} \frac{n+3}{n+1} ((n+2)E' + nE) \]
\[ \Phi^-_2 = -\epsilon_{\alpha'\gamma'} \wedge \epsilon_{\beta'\gamma'} \frac{\partial^2}{\partial y_{\gamma'} \partial y_{\beta'}} \frac{n-1}{n+2}(n+2)(n+1)((n+2)E' + nE) , \]
\[ \Phi^+_0 = 2\epsilon_{\alpha'\beta'} \wedge e^{\gamma\beta} \frac{\partial^3}{\partial y_{\alpha'} \partial y_{\beta'} \partial y^\gamma} \frac{n+3}{n+2(n+1)} ((n+2)E' + nE) \]
\[ + 2e^{\alpha\beta} \wedge \epsilon_{\gamma\alpha'} \frac{\partial}{\partial y_{\gamma'}} \frac{n-1}{n+2}(n+2)(n+1)((n+2)E' + nE) , \]
\[ \Phi^-_0 = 2\epsilon_{\alpha'\beta'} \wedge e^{\gamma\beta} \frac{\partial^3}{\partial y_{\gamma'} \partial y_{\beta'} \partial y_{\alpha'}} \frac{n-1}{n+3} ((n+2)E' + nE) \]
\[ + 2e^{\alpha\beta} \wedge \epsilon_{\gamma\beta'} \frac{\partial^3}{\partial y^\alpha \partial y^\beta \partial y_{\beta'}} \frac{n+3}{n+2(n+1)} ((n+2)E' + nE) , \]
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To impose HS field equations one has to set $E$ and $E'$ to zero. The structure of $H^{2 \text{bos}}_1(\tau_-)$ suggests that this can be done in a variety of equivalent ways by setting to zero any of the components of the curvature 2-forms that contain $E$ and $E'$. All differently looking field equations imposed by setting to zero different components of the curvatures proportional to $E$ and $E'$ are equivalent by virtue Bianchi identities along with the grade zero field equation $R_0 = 0$, $G(R_0) = 0$.

Let stress that the $\sigma_-$ cohomology analysis in $Sp(4, \mathbb{R})$ is identical to that in $\mathcal{M}_4$ because the form of $\sigma_-$ at any given point does not change. This means in particular that one has as many symmetries, dynamical fields and field equations in $Sp(4, \mathbb{R})$ as in $\mathcal{M}_4$. The precise form of the field equations may of course be different because of the appearance of the $\sigma_+$ operators as discussed in Section 15.2.

16 Conclusion

The main result of this paper is that free equations for massless fields of all spins in $AdS_4$ admit $sp(8, \mathbb{R})$ covariant formulation not only in terms of gauge invariant field strengths \[3\], but also in terms of the gauge potentials. The key point is that the formulation is well-defined in the $AdS_4$ background but experiences certain degeneracy in the flat limit that does not allow $sp(8, \mathbb{R})$ and conformal invariant formulations of spin $s > 1$ gauge fields in flat Minkowski space. There are two alternatives for the flat limit procedure. One leads to standard flat space massless field equations but blowing up special conformal symmetry transformations. Another one keeps the conformal transformation well defined, but the limiting flat space field equations is hard to interpret.

The formulation in terms of gauge potentials is needed to reach the $sp(8, \mathbb{R})$ covariant formulation at the action level and/or at the interaction level (even on-shell). The obtained results provide the starting point for the $sp(8, \mathbb{R})$ covariant study of the interacting HS theory. That $sp(8, \mathbb{R})$ symmetry may play a role in the nonlinear HS theory has been already observed in \[3\] in the relation with the doubling of auxiliary spinor variables in the nonlinear HS field equations of \[5, 28\]. However, the formalism developed in this paper is different from that of \[5, 28\] because physical degrees of freedom are described here by Fock modules rather than by twisted adjoint module as in \[5, 28\]. In fact, the proposed formulation is close to that of the 2d HS model of \[75\] that was also formulated in terms of Fock modules. It is tempting to extend this analogy to the $sp(8, \mathbb{R})$ invariant formulation of 4d HS theory.

An interesting direction for the future investigation is to study the models in the matrix spaces $\mathcal{M}_M$ with $M > 4$. Note that the dynamics of 6d and 10d conformal free massless fields in terms of gauge invariant field strengths has been understood \[2, 3, 8, 16\] in the matrix spaces $\mathcal{M}_8$ and $\mathcal{M}_{16}$, respectively. The results of this paper indicate how this theory can be extended to the level of gauge fields, although the details remain to be elaborated.

Surprisingly, the obtained results may even shed some light on the structure of conventional field-theoretical models like gravity. In particular, we have shown that linearized grav-
ity in $AdS_4$ exhibits the $gl(4, \mathbb{R}) \sim o(3, 3)$ symmetry. It would be interesting to see whether and how the $gl(4, \mathbb{R})$ symmetry extends to the nonlinear gravity in $AdS_4$. A promising starting point is the MacDowell-Mansouri action \[76].

An interesting feature of the proposed model is that it has manifest EM duality symmetry along with its HS generalization as the $u(1)$ part of $sp(8, \mathbb{R})$. In the unreduced model, that describes two infinite sets of massless fields of all spins, this symmetry rotates two species of a spin $s$ gauge field as a complex field. The reduction to the system of massless fields in which every spin appears once also respects the duality transformation because it still has two sets of the gauge fields that are dual to one another by virtue of the field equations. Hopefully, the proposed formulation may be helpful for the further analysis of duality in the models that contain gravity like those studied in \[77, 78, 79, 80, 81, 82\].

The main tool for the study of HS fields applied in this paper is the unfolded formulation which is a covariant first-order reformulation of a dynamical theory in any dimension \[40, 49\]. The unfolded formulation is perfectly suited for elucidation of symmetries, dynamical content and equivalent formulations of a theory, including those in extended (super)spaces with extra (super)coordinates. In this paper, the unfolded formulation insures the $sp(8, \mathbb{R})$ invariance, determines the precise form of $sp(8, \mathbb{R})$ field transformations and allows the straightforward extension of the $sp(8, \mathbb{R})$ invariant HS gauge theory in $AdS_4$ to the ten dimensional space-time with the coordinates $X^{AB}$ ($A, B = 1, \ldots, 4$), which is the group manifold $Sp(4, \mathbb{R})$ in the $AdS$ like case.

The original ten dimensional formulation of HS theory in terms of gauge invariant Weyl 0-forms \[3\] was manifestly covariant under $GL(4, \mathbb{R})$ transformation of the spinor indices $A, B \ldots 1, \ldots, 4$ so that all coordinates $X^{AB}$ appeared on equal footing. The manifest symmetry of the proposed extension to the HS model formulated in terms gauge fields turns out to be reduced to $GL(2, \mathbb{C}) \subset GL(4, \mathbb{R})$. This happens because different sectors of fields in the theory respect different subalgebras $gl(4, \mathbb{R}) \subset sp(8, \mathbb{R})$ which have $gl(2, \mathbb{C})$ as the maximal common subalgebra. As a result, the space-time coordinates $X^{\alpha\alpha'}$ and spinning coordinates $X^{\alpha\bar{\alpha}}$ and $X^{\alpha'\bar{\alpha}'}$ have different appearance in the theory. Nevertheless, $sp(8, \mathbb{R})$ acts geometrically on $X^{AB}$ and remains a symmetry of the system. Let us note that the fact that the manifest symmetry between space-time and spinning coordinates is lost in the full theory indicates that a generalized holonomy group in the space-times with matrix coordinates should be $GL(2, \mathbb{C})$ rather than $GL(4, \mathbb{R})$ which case was tested in \[15\].

As a first step towards a new version of nonlinear HS theory it is interesting to check whether the proposed formulation exhibits an infinite dimensional conformal HS symmetry that contains $sp(8, \mathbb{R})$ as a finite dimensional subalgebra. As explained in Subsection 5.3.3 one way to check this is to extend the $sp(8, \mathbb{R})$ Chevalley-Eilenberg cohomology to the full HS symmetry. Technically, this is equivalent to the conformal extension of the $AdS_4$ analysis of \[40\] to the first order in the Weyl 0-forms, that accounts all HS 1-form connections.

More generally, a nonlinear extension of the free HS theory formulated in this paper can go far beyond the original $4d$ HS gauge theory we started with just because it will allow a formulation in the ten dimensional space-time. Actually, as observed in \[13\], different types of $sp(8, \mathbb{R})$ invariant fields in $\mathcal{M}_M$ are visualized as usual fields that live in space-times of
different dimensions $1 \leq 10$ so that the resulting theory may provide a dynamical theory of different types of branes in the ten dimensional space-time. Let us note that, for higher rank (brane) solutions, $sp(8, \mathbb{R})$ extends to higher $sp(2^n, \mathbb{R})$ including $sp(32, \mathbb{R})$ and $sp(64, \mathbb{R})$ that have been argued long ago to be symmetries of $M$ theory [83, 84, 85, 86, 87, 88]. A deep relationship between HS theories and $M$ theory is also indicated by the recent papers [20, 89].

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