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Vienna, Preprint ESI 1584 (2005) January 26, 2005

Supported by the Austrian Federal Ministry of Education, Science and Culture
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Equatorial Podleś Sphere

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Abstract
We propose a slight modification of the properties of a spectral geometry a la
Connes, which allows for some of the algebraic relations to be satisfied only mod-
ulo compact operators. On the equatorial Podleś sphere we construct $U_q(su(2))$-
equivariant Dirac operator and real structure which satisfy these modified proper-
ties.

Key words and phrases: Noncommutative geometry, spectral triple, quantum $SU(2)$.
Mathematics Subject Classification: Primary 58B34; Secondary 17B37.

*Partially supported by Polish State Committee for Scientific Research (KBN) under grant 2 P03B 022 25.
1 Introduction

Recent examples of noncommutative spectral geometries on spaces coming from quantum groups \([2, 6, 4, 13, 10, 7]\) have opened a new interesting and promising area of research. Along these lines, by introducing a slight modification of the defining properties of a noncommutative geometry, we present a construction of a spectral geometry for the equatorial Podleś sphere \(S^2_q\). Both the Dirac operator \(D\) and the real structure \(J\) will be equivariant under the action of \(\mathcal{U}_q(su(2))\) on \(S^2_q\).

We briefly recall some basic facts about the equatorial Podleś sphere and its symmetry \([12]\). With \(q\) a real number, \(0 < q \leq 1\), we denote by \(\mathcal{A}(S^2_q)\) the algebra of polynomial functions generated by operators \(a, a^*\) and \(b = b^*\), which satisfy the following commutation rules:

\[
b^2 = 1, \quad q^2a a^* + q^{-2}b^2 = q^2.
\]

This algebra contains a \(S^1\)-worth of classical points (the `equator') given by the one dimensional representations \(b = 0, a = \lambda\) with \(\lambda \in S^1\).

The symmetry, which we shall use for the equivariance, is the Hopf algebra module structure with respect to the \(\mathcal{U}_q(su(2))\) Hopf algebra and derived from the canonical \(\mathcal{U}_q(su(2))\) action on the \(\mathcal{A}(SU_q(2))\) algebra. Explicitly, the generators \(e, f, k\) of \(\mathcal{U}_q(su(2))\) act on the generators of \(\mathcal{A}(S^2_q)\) in the following way:

\[
k \triangleright a = qa, \quad e \triangleright a = -(1 + q^2)q^{-\frac{5}{2}}b, \quad f \triangleright a = 0,
\]

\[
k \triangleright a^* = \frac{1}{q}a^*, \quad e \triangleright a^* = 0, \quad f \triangleright a^* = (1 + q^2)q^{-\frac{3}{2}}b, \quad (1)
\]

\[
k \triangleright b = b, \quad e \triangleright b = q^{\frac{1}{2}}a^*, \quad f \triangleright b = -q^2 a.
\]

We shall use the fact that the irreducible finite dimensional representations of the Hopf algebra \(\mathcal{U}_q(su(2))\) are labelled by a positive half-integers (see \([9]\), for example) and each representation space \(V_I\) has a basis \(\{|I, m\}, m \in \{-l, -l+1, \ldots, l\}\) declared to be orthonormal.

2 Variations on spectral geometry

A spectral geometry (a la Connes) are data \((\mathcal{A}, \pi, \mathcal{H}, \gamma, J, D)\) fulfilling a series of requirements \([3]\).

On the equatorial Podleś sphere \(\mathcal{A} = \mathcal{A}(S^2_q)\) we construct an equivariant spectral geometry \([15]\), starting from an equivariant representation \(\pi\) on a suitable Hilbert space \(\mathcal{H}\). On the latter there are an equivariant real structure \(J\) and an equivariant Dirac operator \(D\). However, with such a \(J\) it is not possible to satisfy all the requirements of \([3]\). Nevertheless, as we shall see, the algebraic requirements shall be obeyed up to compact operators. In particular, the antilinear isometry \(J\) which provides the real structure, will map \(\pi(\mathcal{A})\) to its commutant only modulo compact operators.

\[
\forall x, y \in \mathcal{A} : \quad [\pi(x), J\pi(y)J^{-1}] \in \mathcal{K}, \quad (2)
\]

and the first order condition is valid only modulo compact operators,

\[
\forall x, y \in \mathcal{A} : \quad [J\pi(x)J^{-1}, [D, \pi(y)]] \in \mathcal{K}. \quad (3)
\]
The essentially unique Dirac operator, which comes out as the solution of the above condition, shall have the crucial property that all commutators \([D, \pi(x)], x \in \mathcal{A}(S^2_q)\), are bounded.

### 3 The equivariant geometry of \(\mathcal{A}(S^2_q)\)

A fully equivariant approach, both for the real structure and the Dirac operator was worked out for the standard Podleś sphere in [6]. Here we build up on this and an earlier approach [8] [11]) to construct an equivariant spectral geometry of the Equatorial Podleś sphere.

#### 3.1 The equivariant representation

The starting ingredient of the equivariant spectral geometry, which we are about to construct is a equivariant representation of \(\mathcal{A}(S^2_q)\) on a Hilbert space \(H\). Let us recall that an \(H\)-equivariant representation of an \(H\)-module algebra \(\mathcal{A}\) on \(H\), is a representation \(\pi\) of \(\mathcal{A}\) on \(H\) and a representation \(\rho\) of \(H\) on a dense subspace of \(H\) such that for every \(a \in \mathcal{A}\) and \(h \in H\) and \(v\) from a dense subspace we have:

\[
\rho(h)\pi(a)v = \pi(h(1) \triangleright a)h(2)v.
\] (4)

In our case, this is the same as a representation of the crossed product of \(U_q(su(2)) \ltimes \mathcal{A}(S^2_q)\) on a dense subspace of the Hilbert space. The general theory for the entire family of Podleś spheres is in [14].

**Proposition 3.1.** There exists two irreducible \(U_q(su(2))\)-equivariant representations (denoted by \(\pi_\pm\)) of the algebra \(\mathcal{A}(S^2_q)\) on the Hilbert space \(\mathcal{H}_h = \bigoplus_{l=\frac{1}{2}, \frac{3}{2}, \ldots} V_l\) given by:

\[
\pi_\pm(a)|l, m\rangle = \pm (1 + q^2) \frac{q^{m-\frac{l}{2}}}{[2l][2l+2]} \sqrt{[l+m+1][l-m]} |l, m+1\rangle + \frac{q^{m-l-\frac{1}{2}}}{[2l+2]} \sqrt{[l+m+1][l+m+2]} |l+1, m+1\rangle
\]

\[
-\frac{q^{m+l+\frac{1}{2}}}{[2l]} \sqrt{[l-m][l-m-1]} |l-1, m\rangle,
\] (5)

with \(\pi(a^*)\) being the hermitian conjugate of \(\pi(a)\) and \([x] := (q - q^{-1})(q^x - q^{-x})\).

The proof is a long but straightforward calculation based on the covariance property (4) with the natural representation \(\rho\) of \(U_q(su(2))\) on \(\mathcal{H}_h\) and the \(U_q(su(2))\)-module structure of \(\mathcal{A}(S^2_q)\) given in (1). The representations \(\pi_{\pm}\) are equivalent to the left regular
representation of \( \mathcal{A}(SU_q(2)) \) on \( L^2(SU_q(2)) \) (with the Haar measure) when this representation is restricted to the subalgebra \( \mathcal{A}(S^2_q) \) and the representation space is restricted to the \( (L^2\text{-completion}) \) of certain vector spaces (left \( \mathcal{A}(S^2_q) \)-modules) constructed in [1].

### 3.2 The equivariant geometry

We take as the Hilbert space of our geometry \( \mathcal{H} = \mathcal{H}_h \oplus \mathcal{H}_h \), with the natural grading \( \gamma = \text{id} \oplus (-\text{id}) \) and the representation \( \pi(x) = \pi_+(x) \oplus \pi_-(x) \) for any \( a \in \mathcal{A}(S^2_q) \), which is equivariant with respect to the diagonal action (which we call again \( \rho \)) of \( \mathcal{U}_q(su(2)) \) on \( \mathcal{H} \).

In the search of a real spectral geometry we follow, like in [6], the method of equivariance to find first the real structure \( J \). Let us recall, that \( J \) is the antiunitary part of an antilinear operator \( T \) on \( \mathcal{H} \), which must then satisfy for any \( h \in \mathcal{U}_q(su(2)) \) (on a dense subspace of \( \mathcal{H} \))

\[
\rho(h)T = T\rho(Sh)^*. 
\]

By taking into account the required commutation relations with the grading \( \gamma \), that is \( \gamma J = -J\gamma \) and \( J^2 = -1 \), one easily obtain that \( J \) must be,

\[
J[l, m]_\pm = i^{2m} |l, -m\rangle_\mp ,
\]

where the label \( \pm \) refers to the two copies of \( \mathcal{H}_h \) which are marked by the eigenvalues of \( \gamma \). With this data we immediately meet an obstruction:

**Proposition 3.2.** The operator \( J \) defined above does not satisfy the “commutant” requirement of a real spectral triple, that is, there exist \( x \in \mathcal{A}(S^2_q) \) for which \( J\pi(x)J^{-1} \) is not in the commutant of \( \pi(\mathcal{A}(S^2_q)) \).

Then, we move on to look for a variation of spectral geometry up to infinitesimal, as introduced earlier. Let \( \mathcal{K} \) denotes the ideal of compact operators on \( \mathcal{H} \) and \( \mathcal{K}_q \subset \mathcal{K} \) be the ideal generated by operators \( L_q \) of the form \( L_q[l, m]_\pm = q^l |l, m\rangle_\pm \).

**Proposition 3.3.** The operator \( J \), defined in (7), maps \( \pi(\mathcal{A}(S^2_q)) \) to its commutant modulo compact operators (in fact modulo \( \mathcal{K}_q \)). More precisely, for any \( x, y \in \mathcal{A}(S^2_q) \),

\[
[\pi(x), J\pi(y)J^{-1}] \in \mathcal{K}_q .
\]

To prove this proposition, it is convenient to use compact perturbations of the representation \( \pi \). Exact formulæ shall be contained in the extended version of this note.

### 3.3 The equivariant Dirac operator

As a next step we derive the Dirac operator \( D \). Beside postulating that \( D \) anticommutes with \( \gamma \) and commutes with \( J \), we shall also require that \( D \) is equivariant, that is it commutes with the representation \( \rho \) of \( \mathcal{U}_q(su(2)) \) on \( \mathcal{H} \). Each operator satisfying these condition must be of the form

\[
D[l, m]_\pm = d_l^\pm |l, m\rangle_\pm , \quad d_l^\pm \in \mathbb{R}.
\]
Proposition 3.4. Up to rescaling and addition of a constant, there exists only one operator $D$ of the form (9) which satisfies the order-one condition up to compact operators (in fact modulo $K_q$), that is for all $x, y \in \mathcal{A}(S^2_q)$,

$$[J \pi(x) J^{-1}, [D, \pi(y)]] \in K_q.$$  

(10)

For this operator $D$, the parameters $d_l^\pm$ are given by $d_l^+ = d_l^- = l + \frac{1}{2}$.

The condition (10) has been verify explicitly for all pairs of generators of $\mathcal{A}(S^2_q)$ with the help of a symbolic computation program. Furthermore, we have,

Proposition 3.5. For any $x \in \mathcal{A}(S^2_q)$, the commutators $[D, \pi(x)]$ are bounded.

It is evident that the operator $D$ is self-adjoint on a natural domain in $\mathcal{H}$ and that its resolvent is compact. Since the spectrum of $|D|$ consists only of eigenvalues $k = l + \frac{1}{2} \in \mathbb{N}$ with multiplicity $4k$, we managed to realize the suggestion in [5]. Thus, the deformation being isospectral, the dimension requirement is satisfied with the spectral dimension of $(\mathcal{A}(S^2_q), \mathcal{H}, D)$ being $n = 2$.

With the above results we made some advancement in study of the $q$-geometry and expect that similar structures exist on other $q$-deformed spaces. There are still many points to be addressed, notable the existence of a volume form and the fulfillment of other axioms of spectral geometries. These points shall be addressed in the extended version of the note.

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