Symmetries of the free Schrödinger equation in the non-commutative plane

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Abstract

We study the symmetries of the free Schrödinger equation in the non-commutative plane. These symmetries generate an infinite two dimensional Weyl algebra that appears naturally from a two dimensional Heisenberg algebra generated by boosts and momenta. A finite dimensional subalgebra is the Schrodinger algebra which apart from the Galilei generators has dilatation and expansion. We consider the quantizations in both the reduced and extended phase spaces.
1 Introduction and results

The symmetries of a free massive non-relativistic particle and the associated Schrödinger equation have been investigated. The projective symmetries of the Schrödinger equation induced by the transformation on the coordinates $(t, \vec{x})$ are well known. They form the Schrödinger group \cite{1} \cite{2} \cite{3} \cite{4} that, apart from the Galilei symmetries, contains the dilatation and the expansion. Recently Valenzuela \cite{5} (see also \cite{6}) has found all the symmetries of the free Schrödinger equation. These symmetries generate an infinite dimensional Weyl algebra constructed from the generators of space-translation and the ordinary commuting Galilean boost. The extra symmetries correspond to higher spin symmetries. These transformations are not induced by the transformations on the coordinates but they map solutions into solutions of the Schrödinger equation.

In the case of 2+1 dimensions, the Galilei group admits two central extensions \cite{7, 8, 9}, one associated to the exotic non-commuting boost and other appearing in the commutator of the ordinary boost and spatial translations. The non-relativistic particle in the non-commutative plane was introduced in \cite{10} by considering a higher order Galilean invariant lagrangian for the coordinates $(t, \vec{x})$ of the particle. A first order lagrangian depending on the coordinates $(t, \vec{x})$ and extra coordinates $\vec{v}$ was introduced in \cite{11}. For these lagrangians there are two possible realizations, one with non-commuting (exotic) boosts, and the order with ordinary commuting boosts \cite{9}

In this paper we study all the classical symmetries of a massive free particle in the non-commutative plane. As we will see they are constructed from the Heisenberg algebra of commuting boost $X_i$ and the generators of translations $P_i$, \{\{X_i, P_j\} = \delta_{ij}, (i, j = 1, 2)\}, all of which are constants of motion. The algebra of these symmetries is the infinite dimensional Weyl algebra associated with that Heisenberg algebra.

The subset of generators constructed up to terms quadratic on $(X_i, P_j)$ form a finite dimensional subalgebra, which in turn contains the 9-dimensional Schrödinger algebra generated by $X_i, P_j, J, Z, H, D, C$. A general element of the Weyl algebra is given by $\mathcal{G}(X_i, P_j)$. We study the realization of these algebras in the classical unreduced phase-space, as well as in the reduced one, which appears due to the presence of second class constraints.

We have also studied all the symmetries of the free Schrödinger equation in the non-commutative plane, which also form a Weyl infinite dimensional algebra. The generators of the Schrödinger algebra are constructed in the quantum reduced phase space, as well as in the extended one. In the extended space we impose a non-hermitean combination of the second class constraints. In this case we consider two representations for the physical states, namely a Fock representation \cite{12} and a coordinate representation. We show the equivalence between the reduced and extended space formulations.

The organization of the paper is as follows. In section 2 we study the classical symmetries of the massive particle in the non-commutative plane. Section 3 is devoted to study the quantum symmetries of the Schrödinger equation.
2 Classical symmetries of the non-relativistic particle Lagrangian in the non-commutative plane

The first order Lagrangian of a non-relativistic particle in the non-commutative plane, see for example [11], is given by, \((i,j = 1,2)\),
\[
L_{nc} = m \left( v_i \dot{x}_i - \frac{v_i^2}{2} \right) + \frac{\kappa}{2} \epsilon_{ij} v_i \dot{v}_j.
\] (2.1)

This Lagrangian can be obtained using the NLR method [13] applied to the exotic Galilei group in 2+1 dimensions\(^1\), see [14] for the case of exotic Newton-Hooke whose flat limit gives (2.1). The coordinates \(x_i\)'s are the Goldstone bosons of the transverse translations and \(v_i\)'s are the Goldstone bosons of broken boost. The \(v_i\)'s and \(\kappa\) are dimensionless.

The lagrangian (2.1) gives two primary second class constraints
\[
\Pi_i = \pi_i + \frac{\kappa}{2} \epsilon_{ij} v_j \approx 0, \\
V_i = p_i - mv_i \approx 0,
\] (2.2)
where \(p_i\) and \(\pi_i\) are the momenta canonically conjugate to \(x_i\) and \(v_i\). The constraints (2.2) satisfy relations
\[
\{\Pi_i, \Pi_j\} = \kappa \epsilon_{ij}, \\
\{V_i, V_j\} = 0, \\
\{\Pi_i, V_j\} = m \delta_{ij},
\] (2.3)
and the Dirac hamiltonian is
\[
H = \frac{p_i^2}{2m},
\] (2.4)
up to quadratic terms in the constraints.

From the canonical pairs \((x, v, p, \pi)\) we get a new set of canonical pairs \((\tilde{x}, \tilde{v}, \tilde{p}, \tilde{\pi})\) as
\[
\begin{pmatrix}
\tilde{x} \\
\tilde{p} \\
\tilde{v} \\
\tilde{\pi}
\end{pmatrix} =
\begin{pmatrix}
1 & -\frac{\kappa}{2m} \epsilon & \frac{\kappa}{2m} \epsilon & -\frac{1}{m} \\
1 & 0 & 0 & 0 \\
-\frac{1}{m} & 0 & 1 & 0 \\
\frac{\kappa}{2m} \epsilon & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
p \\
v \\
\pi
\end{pmatrix}.
\] (2.5)

In terms of new variables the constraints (2.2) become a canonical pair,
\[
\tilde{v}_i = -\frac{1}{m} V_i \approx 0, \\
\tilde{\pi}_i = \Pi_i + \frac{\kappa}{2m} \epsilon_{ij} V_j \approx 0.
\] (2.6)

The position and momentum of the particle are expressed as
\[
x_i = \tilde{x}_i - \frac{\kappa}{2m^2} \epsilon_{ij} \tilde{p}_j - \frac{\kappa}{2m} \epsilon_{ij} \tilde{v}_j + \frac{1}{m} \tilde{\pi}_i, \\
p_i = \tilde{p}_i,
\] (2.7)

\(^1\)Note that this Lagrangian is not dynamically equivalent to the higher order lagrangian for a non-relativistic particle proposed in [10]. It can be obtained from (2.1) using the inverse Higgs mechanism [15].
and the Dirac Hamiltonian is (2.4) is written as

\[ H = \frac{1}{2m} \tilde{p}_i^2. \] (2.8)

The phase space is a direct product of two spaces. One is spanned by \((\tilde{v}, \tilde{\pi})\) with the constraints (2.6)

\[ \tilde{v}_i \approx 0, \quad \tilde{\pi}_i \approx 0 \] (2.9)

and thus classically trivial. The other one is spanned by \((\tilde{x}, \tilde{p})\) with the Hamiltonian (2.8). It is a system of free non-relativistic particle in 2D but with the coordinates \(\tilde{x}_i\). We will see how the Schrödinger, or the more general symmetry algebras, are realized on it.

Classical Noëther symmetries are generated by constants of motion which are arbitrary functions of

\[ X_i = \tilde{x}_i(0) = \tilde{x}_i(t) - \frac{t}{m} \tilde{p}_i(t), \quad P_i = \tilde{p}_i(0) = \tilde{p}_i(t), \] (2.10)

verifying

\[ \{P_i, P_j\} = 0, \quad \{X_i, P_j\} = \delta_{ij}, \quad \{X_i, X_j\} = 0. \] (2.11)

The Lagrangian (2.1) is quasi-invariant under the transformation generated by \(G(X_i, P_j)\).

The canonical variation of \((x, v)\) are

\[ \delta x_i = \frac{\partial G}{\partial p_i}, \quad \frac{\partial G}{\partial P_i} - \frac{t}{m} \frac{\partial G}{\partial X_i}, \quad \frac{\kappa}{2m^2} \epsilon_{ij} \frac{\partial G}{\partial X_j}, \]

\[ \delta v_i = \frac{\partial G}{\partial \pi_i}. \] (2.12)

When we consider the variation of the Lagrangian (2.1) \((\delta x, \delta v)\) are (2.12) in which \((p_i, \pi_i)\) are replaced by, using the definition of momenta (2.2),

\[ p_i \rightarrow mv_i, \quad \pi_i \rightarrow -\frac{\kappa}{2} \epsilon_{ij} v_j, \quad X_i \rightarrow x_i - tv_i + \frac{\kappa}{2m} \epsilon_{ij} v_j, \] (2.13)

It follows that the variation of the Lagrangian becomes a total derivative,

\[ \delta L_{nc} = \frac{d}{dt} \mathcal{F}(x, v, t), \]

\[ \mathcal{F}(x, v, t) = [p_i \delta x_i + \pi_i \delta v_i - \mathcal{G}]_{p_i = mv_i, \pi_i = -\frac{\kappa}{2} \epsilon_{ij} v_j} = \left[ mv_i \left( \frac{\partial \mathcal{G}}{\partial P_i} - \frac{t}{m} \frac{\partial \mathcal{G}}{\partial X_i} \right) - \mathcal{G} \right]_{p_i = mv_i, \pi_i = -\frac{\kappa}{2} \epsilon_{ij} v_j}. \] (2.14)

### 2.1 Galilean symmetries

We start by considering the Galilean symmetries of (2.1), the action is invariant under translations and boost

\[ x'_i = x_i + \alpha_i \quad v'_i = v_i \] (2.15)

\[ x'_i = x_i - \beta_i t, \quad v'_i = v_i - \beta_i \] (2.16)

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and an under rotations and time translations,

\[ x'_i = x_i \cos \varphi + \epsilon_{ij} x_j \sin \varphi, \quad v'_i = v_i \cos \varphi + \epsilon_{ij} v_j \sin \varphi, \]
\[ t' = t - \gamma. \]  

(2.17)  

(2.18)  

The corresponding Noether charges of translations and boosts are given as

\[ P_i = p_i, \quad K_i = m x_i - p_i t - \pi_i + \frac{\kappa}{2} \epsilon_{ij} v_j = m X_i + \frac{\kappa}{2m} \epsilon_{ij} P_j, \]  

(2.19)  

while the angular momentum is

\[ J = \epsilon_{ij} (x_i p_j + v_i \pi_j) = \epsilon_{ij} (X_i P_j + \tilde{\alpha}_i \tilde{\pi}_j). \]  

(2.20)  

Together with total Hamiltonian (2.4). They generate the exotic Galilei algebra [7, 8, 9]

\[ \{H, J\} = 0, \]  

(2.21)  

\[ \{H, K_i\} = -P_i, \quad \{H, P_i\} = 0, \]  

(2.22)  

\[ \{J, P_i\} = \epsilon_{ij} P_j, \quad \{J, K_i\} = \epsilon_{ij} K_j, \]  

(2.23)  

\[ \{K_i, P_j\} = m \delta_{ij}, \quad \{K_i, K_j\} = -\kappa \epsilon_{ij}, \]  

(2.24)  

\[ \{P_i, P_j\} = 0. \]  

(2.25)  

It seems that the Lagrangian (2.1) gives a phase space realization of the 2+1 Galilei group with two central charges \( m, \kappa \) however one of the central charges is trivial, since, if we modify the generator of boost as in [9],

\[ \tilde{K}_i = K_i - \frac{\kappa}{2m} \epsilon_{ij} P_j = m X_i = m x_i - \pi_i + \frac{1}{2} \kappa \epsilon_{ij} v_j - \frac{\kappa}{2m} \epsilon_{ij} p_j - p_i t \]  

(2.26)  

\((H, P, \tilde{K}, J)\) verifies standard Galilean algebra without \( \kappa \). The physical change of the boost generators is to shift the parameter of translations

\[ \alpha_i \rightarrow \alpha_i + \frac{\kappa}{2m} \epsilon_{ij} \beta_j. \]  

(2.27)  

Note that modified boost generators \( \tilde{K}_i \) are proportional to the coordinates at \( t = 0 \), \( X_i = \tilde{x}^i(0) \), that verify \( \{X_i, X_j\} = 0 \) and we have a realization with only one non-trivial central charge associated to the mass of the particle\(^2\).

\(^2\)Note however that \( \delta_K L = \delta_{\tilde{K}_i} L = \frac{d}{dt} (-m x_i - \frac{\kappa}{2} \epsilon_{ij} v_j) \beta_i \), where \( \beta_i \) is boost parameter.
2.2 Schrödinger symmetries

Note that $X_i, P_j$ in (2.11) form a two dimensional Heisenberg algebra

$$\mathfrak{h}_2 = \{ 1, X_i, P_i \}; \quad \{X_i, P_j\} = \delta_{ij}, \quad i, j = 1, 2.$$

The Weyl algebra, denoted by $[\mathfrak{h}_2^*]$, can be defined as one generated by (the Weyl ordered) polynomials of the Heisenberg algebra generators, $(X_i, P_i)$. $[\mathfrak{h}_2^*]$ is the infinite higher dimensional algebra of a particle in the non-commutative plane. There are finite dimensional subalgebras of the higher spin algebra whose generators are constructed from the product of generators $X_i, P_j$ up to second order [5];

$$\mathfrak{h}_2 \subset \text{Galilei} \subset \text{Sch}(2) \subset \mathfrak{h}_2 \oplus \mathfrak{sp}(4) \subset [\mathfrak{h}_2^*]. \quad (2.28)$$

$\text{Sch}(2)$ is the Schrödinger algebra in 2D, whose generators are those of the Galilean algebra $X_i, P_i, H, J$, and the dilatations, $D$, and the expansions, $C$, given by

$$D = X_i P_i = x_ip_i - \frac{t}{m} p_i^2 - \frac{1}{m} \pi_i p_i + \frac{\kappa}{2m} \epsilon_{ij} p_i p_j, \quad (2.29)$$

$$C = mX_i X_i = mx_i^2 + \frac{1}{m} x_i^2 + \frac{1}{m} \pi_i^2 + \frac{\kappa^2}{4m} v_i^2 + \frac{\kappa^2}{4m} p_i^2$$

$$- 2tx_ip_i - 2x_i \pi_i + \kappa \epsilon_{ij} x_i v_j - \frac{\kappa}{m} \epsilon_{ij} x_i p_j + \frac{2}{m} t p_i \pi_i$$

$$- \frac{\kappa}{m} t \epsilon_{ij} p_i v_j - \frac{\kappa}{m} \epsilon_{ij} \pi_i v_j + \frac{\kappa}{m^2} \epsilon_{ij} v_i v_j - \frac{\kappa^2}{2m^2} v_i p_i. \quad (2.30)$$

In the same spirit, we also redefine the generator of rotations as

$$J = \epsilon_{ij} X_i P_j = \epsilon_{ij} x_i p_j - \frac{\kappa}{m^2} p_i^2 + \frac{\kappa}{2m} v_i p_i + \frac{1}{m} \epsilon_{ij} p_i \pi_j \quad (2.31)$$

which, up to constraints, coincides with (2.20).

The new, non-zero Poisson brackets are

$$\{D, C\} = -2C, \quad \{D, H\} = 2H, \quad \{H, C\} = -2D,$$

$$\{D, P_i\} = P_i, \quad \{D, X_i\} = -X_i, \quad \{C, P_i\} = 2m X_i. \quad (2.32)$$

The transformation of the coordinates $x_i, v_i$ under dilatations and expansions is obtained from (2.12) as

$$\delta_D x_i = \frac{\alpha}{m} (mx_i - 2mt v_i + \kappa \epsilon_{ij} v_j), \quad (2.33)$$

$$\delta_D v_i = -\alpha v_i,$$

$$\delta_C x_i = \frac{\lambda}{m} (2m t^2 v_i - 2m t x_i + \kappa \epsilon_{ij} x_j - 2 \kappa t \epsilon_{ij} v_j - \frac{\kappa^2}{2m} v_i), \quad (2.34)$$

$$\delta_C v_i = \frac{\lambda}{m} (-2m x_i + 2mt v_i - \kappa \epsilon_{ij} v_j), \quad (2.35)$$

where $\alpha$ and $\lambda$ are the corresponding infinitesimal parameters.
2.3 Reduction of second class constraints

The classical symmetry algebra is also realized in the reduced phase space defined by the constraints \( \Pi_i = V_i = 0 \). The Dirac bracket is

\[
\{A, B\}^* = \{A, B\} + \{A, \Pi_i\} \frac{1}{m} \{V_i, B\} - \{A, V_i\} \frac{1}{m} \{\Pi_i, B\} - \{A, V_i\} \frac{\kappa \epsilon_{ij}}{m^2} \{V_j, B\}
\]

and yields

\[
\{x_i, x_j\}^* = \frac{\kappa}{m^2} \epsilon_{ij}, \quad \{x_i, p_j\}^* = \delta_{ij}, \quad \{p_i, p_j\}^* = 0.
\]

In this space, the symmetry transformations are generated using the Dirac bracket and the reduced generators, which can be obtained by substituting \( v_i = p_i/m, \, \pi_i = -\kappa/(2m) \epsilon_{ij} p_j \) into the standard ones. In particular the Schrödinger generators are given by

\[
P_i^{(R)} = p_i, \quad K_i^{(R)} = mx_i - tp_i + \frac{\kappa}{m} \epsilon_{ij} p_j, \quad (\text{exotic Gal})
\]

\[
\bar{K}_i^{(R)} = K_i^{(R)} - \frac{\kappa}{2m} \epsilon_{ij} P_j^{(R)} = mx_i - tp_i + \frac{\kappa}{2m} \epsilon_{ij} p_j, \quad (\text{standard Gal})
\]

\[
H^{(R)} = \frac{1}{2m} p_i^2,
\]

\[
J^{(R)} = \epsilon_{ij} x_i p_j + \frac{\kappa}{2m^2} p_i^2,
\]

\[
D^{(R)} = p_i x_i - \frac{1}{m} t p_i^2,
\]

\[
C^{(R)} = mx_i^2 + \frac{1}{m} t^2 p_i^2 + \frac{\kappa^2}{4m^3} p_i^2 - 2tx_i p_i + \frac{\kappa}{m} \epsilon_{ij} x_i p_j.
\]

They generate the Schrödinger algebra with the Dirac bracket, since \( \bar{K}_i^{(R)}, P_i^{(R)} \) generate a Heisenberg algebra:

\[
\left\{ \bar{K}_i^{(R)}, P_j^{(R)} \right\}^* = m \delta_{ij}, \quad \left\{ P_i^{(R)}, P_j^{(R)} \right\}^* = 0, \quad \text{and} \quad \left\{ \bar{K}_i^{(R)}, K_j^{(R)} \right\}^* = 0.
\]

Symmetry transformations are generated either using the Poisson brackets in the original phase space or using the Dirac brackets with the reduced generators, (2.38)-(2.43). For example the "exotic Galilei" generators \( K_i \) satisfying

\[
\{K_i, K_j\} = \left\{ K_i^{(R)}, K_j^{(R)} \right\}^* = -\kappa \epsilon_{ij},
\]

generate "standard(covariant) Galilei" transformation of \((x_i, p_i)\) as

\[
\delta x_i = \{x_i, \beta \cdot K\} = \{x_i, \beta \cdot K^{(R)}\}^* = -t \beta_i,
\]

\[
\delta p_i = \{p_i, \beta \cdot K\} = \{p_i, \beta \cdot K^{(R)}\}^* = -m \beta_i.
\]

The "standard Galilei" generators \( \bar{K}_i \) satisfying

\[
\left\{ \bar{K}_i, \bar{K}_j \right\} = \left\{ \bar{K}_i^{(R)}, \bar{K}_j^{(R)} \right\}^* = 0.
\]
generate "exotic(non-covaraint) Galilei" transformation of \((x_i, p_i)\)

\[
\begin{align*}
\delta x_i &= \{ x_i, \beta \cdot K \} = -P_i = \{ x_i, \beta \cdot \tilde{K}^{(R)} \}^* = -t\beta_i + \frac{\kappa}{2m} \epsilon_{ij}\beta_j, \\
\delta p_i &= \{ p_i, \beta \cdot \tilde{K} \} = \{ p_i, \beta \cdot \tilde{K}^{(R)} \}^* = -m\beta_i.
\end{align*}
\]

\[\text{(2.48)}\]

3 Quantum symmetries of free Schrödinger equation in the non-commutative plane

In this section we will study the quantization of the model at the level of the Schrödinger equation and their symmetries. We will quantize it in two approaches, one in the reduced phase space and the other in the extended phase space.

3.1 Quantization in the reduced phase space

In the classical theory \(x_i\) has a nonzero Dirac bracket \(\{ x_i, x_j \}\) as in (2.37) in the reduced phase space. Since Dirac brackets are replaced by commutators in the canonical quantization one cannot have a \(x_i\)-coordinate representation of quantum states\(^3\). To discuss symmetries of Schrödinger equations we introduce new coordinates

\[y_i \equiv x_i + \frac{\kappa}{2m^2} \epsilon_{ij} p_j, \quad q_i = p_i,\]

such that

\[\{ y_i, y_j \}^* = 0, \quad \{ y_i, q_j \}^* = \delta_{ij}, \quad \{ q_i, q_j \}^* = 0.\]

(3.2)

The Schrödinger equation \((i\partial_t - H)|\Psi(t)\rangle = 0\) takes a form of free particle for the wave function

\[\Psi(y, t) = \langle y | \Psi(t) \rangle, \quad \hat{y}_i | y \rangle = y_i | y \rangle, \quad \langle y | y' \rangle = \delta^2(y - y'),\]

(3.3)

as

\[\left( i\partial_t - \frac{1}{2m}(-i\partial_y)^2 \right) \Psi(y, t) = 0,\]

(3.4)

and the inner product is

\[\langle \Psi | \Psi \rangle = \int dy \overline{\Psi(y, t)} \Psi(y, t).\]

(3.5)

Note that \(y_i\) are not covariant under exotic Galilei transformation generated by \(K_i\)

\[\delta y_i = \{ y_i, \beta \cdot K \} = \{ y_i, \beta \cdot K^{(R)} \}^* = -\beta_i t - \frac{\kappa}{2m} \epsilon_{ij}\beta_j,\]

(3.6)

but covariant under the Galilei transformation generated by \(\tilde{K}_i\)

\[\delta y_i = \{ y_i, \beta \cdot \tilde{K} \} = \{ y_i, \beta \cdot \tilde{K}^{(R)} \}^* = -\beta_i t.\]

(3.7)

\(^3\)Since \(q_i\)'s are commuting the momentum representation is possible[11].
The position operators, covariant under $K_i$, are

$$\hat{x}_i = y_i - \frac{\kappa}{2m^2} \epsilon_{ij}(\partial y_j). \tag{3.8}$$

They are Hermitian since $\hat{y}_i = y_i$, $\hat{q}_i = -i\partial y_i$ are Hermitian in appropriate boundary conditions on $\Psi(y,t)$.

Although in our free theory we are able to work with the operators $\hat{y}_i$, $\hat{q}_i$, the non-commutative position operator $\hat{x}_i = y_i - \frac{\kappa}{2m^2} \epsilon_{ij}(\partial y_j)$ may be necessary to describe interactions, for example with the electromagnetic field, which introduce couplings with the Schrödinger generators given by

$$W = \epsilon_{ij} y_i \hat{q}_j.$$ 

If we denote generically by $\mathfrak{g}(R)(t,x,p) = \mathfrak{g}(X,P)|_{P=V=0}$ the generators of the Weyl algebra in the reduced classical space the generators in this quantization are given by

$$\hat{G}_i^{(1)}(t,y,\hat{q}) = \mathfrak{g}_i^{(R)}|_{x_j = y_j - \frac{\kappa}{2m^2} \epsilon_{ij}(\partial y_j)} = \mathfrak{g}_i(y - \hat{q}t, \hat{q}), \tag{3.9}$$

with $\hat{q}_i = -i\partial y_i$ and with the appropriate dealing of operator ordering. In particular the Schrödinger generators are

$$\hat{P}^{(1)}_i = \hat{q}_i = -i\frac{\partial}{\partial y_i}, \tag{3.10}$$

$$\hat{K}^{(1)}_i = my_i - t\hat{q}_i = my_i + it\frac{\partial}{\partial y_i}, \tag{3.11}$$

$$\hat{H}^{(1)} = \frac{1}{2m} \hat{q}_i^2 = -\frac{1}{2m} \partial^2, \tag{3.12}$$

$$\hat{J}^{(1)} = \epsilon_{ij} y_i \hat{q}_j = -i\epsilon_{ij} y_i \frac{\partial}{\partial y_j}, \tag{3.13}$$

$$\hat{D}^{(1)} = y_i \hat{q}_i - i - \frac{1}{m} t \hat{q}_i^2 = -iy_i \frac{\partial}{\partial y_i} + \frac{1}{m} t \frac{\partial^2}{\partial y_i^2} - i, \tag{3.14}$$

$$\hat{C}^{(1)} = my_i^2 - 2ty_i \hat{q}_i + 2it + \frac{1}{m} t \hat{q}_i^2 = my_i^2 + 2it y_i \frac{\partial}{\partial y_i} - \frac{1}{m} t^2 \frac{\partial^2}{\partial y_i^2} + 2it, \tag{3.15}$$

where a Weyl ordering has been used for $\hat{D}^{(1)}$ and $\hat{C}^{(1)}$. These generators are Hermitian operators when acting on the wave functions $\Psi(t,y)$. Furthermore, they obey the abstract quantum Schrödinger algebra off shell, with non-zero commutators given by

$$[\hat{K}_i, \hat{P}_j] = im\delta_{ij}, \quad [\hat{J}, \hat{P}_1] = i\epsilon_{ij} \hat{P}_j, \quad [\hat{J}, \hat{K}_i] = i\epsilon_{ij} \hat{K}_j, \quad [\hat{H}, \hat{K}_i] = -i\hat{P}_i,$$

$$[\hat{D}, \hat{H}] = 2i\hat{H}, \quad [\hat{D}, \hat{P}_i] = i\hat{P}_i, \quad [\hat{D}, \hat{K}_i] = -i\hat{K}_i,$$

$$[\hat{D}, \hat{C}] = -2i\hat{C}, \quad [\hat{H}, \hat{C}] = -2i\hat{D}, \quad [\hat{C}, \hat{P}_i] = 2i\hat{K}_i. \tag{3.16}$$

Using these, together with

$$[i\partial_t, \hat{K}^{(1)}_i] = -i\hat{P}^{(1)}_i, \quad [i\partial_t, \hat{D}^{(1)}] = -2i\hat{H}^{(1)}, \quad [i\partial_t, \hat{C}^{(1)}] = -2i\hat{D}^{(1)},$$

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one can show that
\[
\left[ i\partial_t - \hat{H}^{(1)}, \hat{\mathcal{G}}_i^{(1)} \right] = 0
\] (3.17)
for all the generators \( \hat{\mathcal{G}}_i^{(1)} \), which proves the invariance of the Schrödinger equation under the Schrödinger transformations in this reduced space quantization.

The wave functions transform as
\[
\Psi'(y, t) = e^{i\alpha_i \hat{\mathcal{G}}_i^{(1)}(t, y, (-i\partial_y))} \Psi(y, t),
\] (3.18)
where \( \alpha_i \) are the parameters of the transformations.

The on-shell Schrödinger transformations on the wave functions \( \Psi(y, t) \) induce
\[
\Psi'(y, t) = e^{A + iB} \Psi(y', t');
\] (3.19)
where the coordinate transformations of \((y, t)\) is the (N=1) conformal Galilean transformation and in the multiplicative factor \( e^{A+iB} \), \( A \) and \( B \) are real. For each Schrödinger transformation we have (see, for instance, [3])
1. \( H \) (time translation),
   \[
t' = t + a, \quad y' = y, \quad A = B = 0,
\] (3.20)
2. \( D \) (dilatation),
   \[
t' = e^\lambda t, \quad y' = e^{\frac{\lambda}{2}} y, \quad A = \frac{\lambda}{2}, \quad B = 0,
\] (3.21)
3. \( C \) (expansion),
   \[
t' = \frac{t}{1 - \kappa t}, \quad y'_i = \frac{y_i}{1 - \kappa t}, \quad e^A = \frac{1}{(1 - \kappa t)}, \quad B = -\frac{\kappa my^2}{2(1 - \kappa t)},
\] (3.22)
4. \( \beta_0^0 P_i + \beta_1^1 X_i \), \( ([\beta_0^0] = L, \ [\beta_1^1] = L^{-1}) \), (spatial translations and boost)
   \[
t' = t, \quad y'_i = y_i + (\beta_0^0 + t\frac{\beta_1^1}{m})_i, \quad A = 0, \quad B = -2\pi \omega_1 = -m(y_i + \frac{1}{2}(\beta_0^0 + t\frac{\beta_1^1}{m})\frac{\beta_1^1}{m}.
\] (3.23)

The difference of the transformation from one of ordinary Schrödinger equation is that in the non-commutative case the coordinates that are transformed by conformal Galilean transformations are the canonical one \( y_i \) and not in the physical positions of the particle \( x_i \).

The invariance of the solutions of the Schrödinger equations under a general element of the Weyl algebra can be proved using the invariance under the generators of the Heisenberg algebra and commutator properties.
3.2 Quantization in the extended phase space

In order to quantize the model in the extended phase space the second class constraints (2.2) are imposed as physical state conditions by taking their non-hermitean combinations as in [14]. We first consider the canonical transformation (2.5) that separates the second class constraints as new coordinates. It is realized at quantum level as a unitary transformation

\[ \tilde{q} = U^\dagger q U, \quad U = e^{\frac{i}{m}p_i (\pi_i - \frac{k}{2} \epsilon_{ij} v_j)}, \]  

(3.24)

For example,

\[ \tilde{x}_i = U^\dagger x_i U = x_i - \frac{1}{m} (\pi_i - \frac{k}{2} \epsilon_{ij} v_j) + \frac{1}{2m} \epsilon_{ij} (-\frac{p_j}{m}). \]  

(3.25)

It is useful to introduce the complex combinations of the phase space variables \( \tilde{\pi}_\pm = \tilde{\pi}_1 \pm i \tilde{\pi}_2 \) and \( \tilde{v}_\pm = \tilde{v}_1 \pm i \tilde{v}_2 \), which allow us to introduce two pairs of annihilation operators

\[ \tilde{a}_\pm = \frac{i}{\sqrt{2k}} (\tilde{\pi}_\pm - i \frac{k}{2} \tilde{v}_\pm), \quad \tilde{a}_\pm^\dagger = \frac{-i}{\sqrt{2k}} (\tilde{\pi}_\pm + i \frac{k}{2} \tilde{v}_\pm), \]  

(3.26)

The creation operators \( \tilde{a}_\pm^\dagger \) are their hermitean conjugate satisfying

\[ [\tilde{a}_\pm, \tilde{a}_\pm^\dagger] = 1, \quad others = 0. \]  

(3.27)

Using the Fock representation for \( (\tilde{v}, \tilde{\pi}) \) and coordinate representation for \( (\tilde{x}, \tilde{p}) \), any state of this system is described by

\[ |\Psi\rangle = \sum_{n_+, n_- \geq 0} \int dy \, |n_+, n_-\rangle \otimes |y\rangle \Phi_{n_+, n_-}(y, t), \]  

(3.28)

where \( |n_+, n_-\rangle \) is the eigenstate of \( \tilde{N}_\pm = \tilde{a}_\pm^\dagger \tilde{a}_\pm \) with eigenvalues \( n_\pm \in \mathbb{N} \cup \{0\} \) and \( |y\rangle \) is the eigenstate of commuting operators \( \tilde{x}_i \) with eigenvalue \( y_i \). They are normalized as

\[ \langle n_+, n_-|n'_+, n'_-\rangle = \delta_{n_+ n'_+} \delta_{n_- n'_-}, \quad \langle y|y'\rangle = \delta^2(y - y'). \]  

(3.29)

The scalar product is given by

\[ \langle \Psi|\Psi'\rangle = \sum_{n_\pm} \int dy \, \Phi_{n_+, n_-}(y, t) \Phi_{n'_+, n'_-}(y, t). \]  

(3.30)

In the quantization in the extended phase space the second class constraints (2.2) are imposed as the physical state conditions by taking their non-hermitean combination,

\[ \tilde{a}_\pm |\Psi_{phys}(t)\rangle = 0. \]  

(3.31)
It means physical states are minimum uncertainty states in \((\hat{\nu}, \hat{\pi})\). It selects out only 
\(n_+ = n_- = 0\) state and \(\Phi_{n_+n_-}(y, t) = 0\) except for \(\Phi_{0,0}(y, t) \equiv \Phi_0(y, t)\),
\[
|\Psi_{\text{phys}}(t)\rangle = \int dy \ |0, 0\rangle \otimes |y\rangle \Phi_0(y, t). \quad (3.32)
\]
The Schrödinger equation is
\[
(i\partial_t - H)|\Psi_{\text{phys}}(t)\rangle = 0, \quad H = \frac{\hat{p}^2}{2m},
\]
and thus
\[
(i\partial_t - H)\Phi_0(y, t) = 0, \quad H = \frac{1}{2m}(-i\partial_{y_\nu})^2. \quad (3.34)
\]
Generators of Schrodinger algebra \(G\) in the extended space are written in bilinear forms of
\[
X_i = \tilde{x}_i(0) = \tilde{x}_i(t) - t\tilde{p}_i(t), \quad P_i = \tilde{p}_i(0) = \tilde{p}_i(t), \quad \text{in (2.10)}^4.
\]
Since they commute with \(a_\nu \) and \(a_\nu^\dagger \) physical states remain invariant. They act on the physical states as
\[
|\Psi_{\text{phys}}(t)\rangle \rightarrow |\Psi'_{\text{phys}}(t)\rangle = e^{i\Theta(X,P)}|\Psi_{\text{phys}}(t)\rangle
\]
It turns out the transformation of the wave function \(\Phi_0(y, t)\)
\[
\Phi_0(y, t) = e^{i\Theta(X,P)}\Phi_0(y, t) = e^{i\Theta(y-t(-i\partial_\nu),(-i\partial_{y_\nu}))}\Phi_0(y, t). \quad (3.37)
\]
This transformation has the same form as one in the reduced phase space generated by (3.9)-(3.15). Then the wave function in the reduced space \(\Psi(y, t) = \langle y|\Psi(t)\rangle\) and \(\Phi_0(y, t) = \langle y|\otimes\langle 0|\Psi(t)\rangle\) that appear in the extended space quantization are identified. Note in the former \(\langle y|\) is eigenstate of \(\hat{y}_i = x_i + \frac{\kappa}{2m^2} \epsilon_{ij} p_j\) in (3.1) but \(\langle y|\) in the latter is eigenstate of \(\hat{x}_i\) that are commuting in the extended space.

We can see how the non-commutativity of the position operators appears. \(\hat{x}_\pm = x_1 \pm ix_2\) are commuting in the extended phase space. Using (2.5) we write
\[
x_+ = \tilde{x}_+ + i\frac{\kappa}{2m^2}\tilde{p}_+ + i\sqrt{\frac{2\kappa}{m^2}}\tilde{a}_+^\dagger,
\]
\[
x_- = \tilde{x}_- - i\frac{\kappa}{2m^2}\tilde{p}_- - i\sqrt{\frac{2\kappa}{m^2}}\tilde{a}_- = x_+^\dagger. \quad (3.38)
\]
In the reduced space quantization \(\tilde{a}_\nu \) are effectively put to zero and \(x_\pm\) becomes non-commutative operator on \(|\Psi(t)\rangle\). On the other hand in the quantization in the extended space expectation values of the position operators between two physical states are
\[
\langle \Psi|\tilde{x}_\pm|\Psi'\rangle = \int dy dy' \Phi_0(y, t)\langle y|0]\langle \tilde{x}_\pm \pm i\frac{\kappa}{2m^2}\tilde{p}_\pm \pm i\sqrt{\frac{2\kappa}{m^2}}\tilde{a}_\pm^\dagger\rangle\langle 0|y'\rangle \Phi_0(y', t) = \int dy \frac{\Phi_1}{\Phi_0}(y, t)(y_\pm \pm i\frac{\kappa}{2m^2}(-2i\partial_{y_\nu}))\Phi_2(y, t). \quad (3.39)
\]

---

\(^4\)Although the angular momentum \(J\) in (2.20) contains a term that rotates constrained sector, which is trivially invariant under rotation.
Commutative position operators \( \hat{x}_\pm \) on states \(|\Psi\rangle\) act as non-commutative operators 
\( (y_\pm \pm i\frac{\kappa}{2m^2}(-2i\partial y_\pm)) \) on the wave functions \( \Phi_0(y,t) \).

It is useful to consider the unitary transformation \( U \) on the creation and annihilation operators \( a_\pm, a_\pm^\dagger \),

\[
\begin{align*}
\hat{a}_+ &= U^\dagger a_+ U = a_+ , \\
\hat{a}_- &= U^\dagger a_- U = a_- - \sqrt{\frac{\kappa}{2m^2}} p_- .
\end{align*}
\] (3.40)

The quantization in the extended phase can be also done considering the constraints equation (3.31) in terms of the operators \( a_\pm, a_\pm^\dagger \). The physical state conditions (3.31) are

\[
a_+ |\Psi_{phys}(t)\rangle = 0, \quad (p_- - \sqrt{2m^2/\kappa}a_-) |\Psi_{phys}(t)\rangle = 0 .
\] (3.41)

It is a coherent state of \( a_- \) with eigenvalue \( \sqrt{2m^2/\kappa} \). The Schrödinger generators in this representation are

\[
\begin{align*}
X^{(2)} &= (x_\pm \pm i\frac{\kappa}{2m^2} p_\pm) - \frac{t}{m}p_\pm \pm i\frac{\kappa}{m^2} (p_\pm - \sqrt{2m^2/\kappa} a_\dagger a_-), \\
P^{(2)} &= p_\pm = -2i \partial x_\pm, \quad [x_\pm , p_\pm] = 2i ,
\end{align*}
\]
\[
\begin{align*}
D^{(2)} &= \frac{1}{2} \left( (x_+p_- + p_+x_- - \frac{2t}{m}p_+p_-) + i\frac{\kappa}{m^2}(p_- - \sqrt{2m^2/\kappa} a_\dagger a_-) - \frac{t}{m^2}p_+ (p_- - \sqrt{2m^2/\kappa} a_-) \right) , \\
C^{(2)} &= \frac{1}{2} \left( (x_- - i\frac{\kappa}{2m^2} p_+)(x_- + i\frac{\kappa}{2m^2} p_-) - \frac{t}{m}(x_+ - i\frac{\kappa}{2m^2} p_+)p_- + p_+(x_- + i\frac{\kappa}{2m^2} p_-) \right) \\
&\quad + \frac{t^2}{2m^2} p_+ p_- + \frac{1}{2} \left( (x_+ - i\frac{\kappa}{2m^2} p_+)(x_- + i\frac{\kappa}{2m^2} p_-) - \frac{t}{m}p_- \right) \\
&\quad + \frac{1}{2} \left( i\frac{\kappa}{m^2} (p_- - \sqrt{2m^2/\kappa} a_-) \right) - \frac{t}{m} p_- \\
&\quad + \frac{1}{2} \left( i\frac{\kappa}{m^2} (p_- - \sqrt{2m^2/\kappa} a_-) \right) - \frac{t}{m} p_- \\
J^{(2)} &= \frac{i}{2} \left( (x_+p_- - p_+ x_- - i\frac{\kappa}{m^2} p_+ p_-) + i\frac{\kappa}{m^2} (p_- - \sqrt{2m^2/\kappa} a_\dagger a_-) p_- + i\frac{\kappa}{m^2} p_+ (p_- - \sqrt{2m^2/\kappa} a_-) \right) .
\end{align*}
\] (3.42)

These generators commute with the constraints equation and the Schrödinger operator \( (i\partial_t - H) \). Notice that the set of generators do not depend on \( a_+, a_\dagger_+ \), therefore the transition to the Fock space used in [12] is recovered.

The Fock expression of a generic element the Weyl algebra \( \mathfrak{G}(X,P) \) can be obtained using the expression of the operators \( X \) and \( P \) given by (2.10).
3.2.1 Coordinate representation

In the representation of coordinates the time Schrodinger equation and the constraints equations (3.40) in the non-commutative plane becomes [14]

\[ \hat{S}_1 \Psi \equiv \left( \frac{\partial}{\partial v} + \frac{\kappa}{4} v_+ \right) \Psi (x, v, t) = 0, \]  
(3.43)

\[ \hat{S}_2 \Psi \equiv \left( \frac{\partial}{\partial x} - i \frac{m}{4} v_- - i \frac{m}{\kappa} \frac{\partial}{\partial v_+} \right) \Psi (x, v, t) = 0, \]  
(3.44)

\[ \hat{S}_3 \Psi \equiv \left( i \frac{\partial}{\partial t} + \frac{2}{m} \frac{\partial^2}{\partial x_+ \partial x_-} \right) \Psi (x, v, t) = 0. \]  
(3.45)

In this representation, the operators associated to the generators of the Heisenberg algebra are

\[ \hat{P}_1 = -i \frac{\partial}{\partial x_+} - i \frac{\partial}{\partial x_-}, \]  

\[ \hat{P}_2 = \frac{\partial}{\partial x_+} - \frac{\partial}{\partial x_-}, \]  

\[ \hat{K}_1 = \frac{m}{2} (x_+ + x_-) + (it - \frac{\kappa}{2m}) \frac{\partial}{\partial x_+} + (it + \frac{\kappa}{2m}) \frac{\partial}{\partial x_-} + \frac{\kappa}{4} (v_+ - v_-) + i \frac{\partial}{\partial v_+} + i \frac{\partial}{\partial v_-}, \]  

\[ \hat{K}_2 = \frac{m}{2i} (x_+ - x_-) - (t + i \frac{\kappa}{2m}) \frac{\partial}{\partial x_+} + (t - i \frac{\kappa}{2m}) \frac{\partial}{\partial x_-} - \frac{\kappa}{4} (v_+ - v_-) - \frac{\partial}{\partial v_+} + \frac{\partial}{\partial v_-}, \]  
(3.46)

or, in covariant form,

\[ \hat{P}_i = -i \frac{\partial}{\partial x_i}, \]  
(3.47)

\[ \hat{K}_i = m x_i + it \frac{\partial}{\partial x_i} + i \frac{\kappa}{2m} \epsilon_{ij} \frac{\partial}{\partial x_j} + \frac{\kappa}{2} \epsilon_{ij} v_j + i \frac{\partial}{\partial v_i}, \]  
(3.48)

which, indeed, satisfy \([\hat{P}_i, \hat{K}_j] = -im \delta_{ij}\), with all the other commutators equal to zero.

It is immediate to check that the operators \(\hat{P}_i, \hat{K}_i\) commute with all of \(\hat{S}_1, \hat{S}_2\) and \(\hat{S}_3\), and hence that they generate Schrödinger symmetries for the free particle in the non-commutative plane.

The rest of generators of the Schrödinger algebra are given by
Using these expressions, one can check explicitly the commutators (3.16), and also that these quadratic generators commute with $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ (this also follows from the derivation properties of the commutators and the corresponding commutation of the linear generators $\hat{P}_i$, $\hat{K}_i$, and this proves that the Schrödinger equation for the free particle in the noncommutative plane has the Schrödinger algebra as a symmetry. Notice, however, that in this coordinate representation of the non-reduced quantum space
the quadratic operators contain second order derivatives, and hence do not generate point transformations for the coordinates $x$, $v$. This is in agreement with the results obtained in the reduced space quantization and the Fock space representation.

In any case, the fact that the linear generators commute with $S_1$, $S_2$ and $S_3$ allows to prove that the quadratic ones also commute, and thus generate symmetries of the Schrödinger equation of the free particle in the non-commutative plane.

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