Research of the nonuniform layers by a reflectometry method near the Brewster’s angle

V V Shagaev
Bauman Moscow State Technical University, Kaluga Branch, Russia
E-mail: shagaev_vv@rambler.ru

Abstract. The mathematical model of the electromagnetic wave reflection with $p$-polarization from the non-sharp boundary between two transparent media is built and studied. The feature of the model consisted in the fact that the thickness of the transition layer was much less than the electromagnetic wave length. The analytical ratio expressing the reflection coefficient through the coordinate dependence of permittivity is deduced. It is shown that the logarithmic derivative from the reflection coefficient with respect to angle can be used as the inhomogeneity indicator. The interval of the angles located near the Brewster’s angle is the most informative in this case. The developed theoretical representations are illustrated by calculations executed for the terahertz range wave reflection from an implanted layer.

1. Introduction
It is known that the angular dependence of a reflection coefficient of the $p$-polarised electromagnetic wave from a homogeneous medium has the zero value minimum. The corresponding angle is named as Brewster's angle. The presence of thin transitive layer near a partition boundary results in the fact that the reflection coefficient, as before, has the minimum in angular dependence, which, however, doesn't reach zero. It is convenient in this case to use a logarithmic derivative instead of angular dependence of the reflection coefficient $R_p(\theta)$ [1]:

$$\eta(\theta) = \frac{1}{R_p(\theta)} \frac{dR_p(\theta)}{d\theta}.$$  \hspace{1cm} (1)

The function $\eta(\theta)$ unlike the function $R_p(\theta)$ has not only the minimum but the maximum near the Brewster's angle. Besides that the experimental dependence $\eta(\theta)$ can be built in the values of the reflection coefficient expressed by the relative units. This circumstance simplifies execution of the necessary measurements because it is possible to ignore the change of the sounding radiation intensity as well as reflections from the input and the exit windows of the measuring system.

It is possible to deduce the analytical expression for electromagnetic wave reflection coefficient from a non-homogeneous medium only in the exclusive cases [2, 3]. The simplest example is the Drude model in which the permittivity changes by the jump at some depth. The reflection coefficient in this case can be expressed through the amplitude reflection coefficients from the semi-infinite homogeneous media with the sharp boundaries of partition (Fresnel’s coefficients). The numerical methods for arbitrary profile of the permittivity go out on the first plan. The method in which the non-homogeneous part of a medium breaks into the thin homogeneous layers is the most often used [4].
The number of layers depends on the permittivity profile, and the reflection coefficient is calculated on the recurrent formula expressed through Fresnel's coefficients.

In this work the reflection coefficient was calculated by a perturbation theory method. It was necessary that the thickness of non-homogeneous layer was much smaller than the electromagnetic wave length. As an example the possibility of identification of the implanted layer by means of terahertz radiation is considered.

2. The development of a mathematical model

The electromagnetic wave with $p$-polarization is completely defined by strength of its magnetic field:

$$\mathbf{H} = \begin{bmatrix} 0 \\ h(z) \end{bmatrix} \cdot \exp(i\omega t - k_x x).$$

It was believed that the medium is homogeneous in the $xy$ plane and the permittivity is a function of the $z$ coordinate (Figure 1). Besides that the function $\varepsilon(z)$ approaches at $z \to \pm \infty$ to the permanent values $\varepsilon_1 = \varepsilon(-\infty)$ and $\varepsilon_2 = \varepsilon(+\infty)$. $k_x$ is the $x$-component of a wave vector, and $k_x = \left(\frac{\omega \varepsilon_1 \sin \theta}{c}\right)$. $\omega$ is the wave circular frequency, $\theta$ is a wave angle of incidence on the media partition boundary, and $c$ is the light velocity in vacuum. It was assumed that the magnetic permeability is equal to unit.

![Figure 1. A structure of medium.](image)

The differential equation follows from Maxwell's equations:

$$\frac{1}{\varepsilon(z)} \frac{\partial}{\partial z} \left( \frac{1}{\varepsilon(z)} \frac{\partial h(z)}{\partial z} \right) + p^2(z) h(z) = 0,$$

where

$$p^2(z) = \frac{1}{\varepsilon^2(z)} \left[ \frac{\omega^2}{c^2} \varepsilon(z) - k_x^2 \right].$$

The function $h(z)$ was presented in the expression form:

$$h(z) = A(z) \exp[-i\varphi(z)] + B(z) \exp[i\varphi(z)],$$

where

$$\varphi(z) = \int_0^z \rho(z') \varepsilon(z') dz'.$$

The values $A(-\infty)$, $A(+\infty)$ and $B(-\infty)$ are the amplitudes of the incident, passed and reflected waves respectively.

Since two functions were injected instead of one required, it is possible to impose an additional condition on them. The condition was expressed by the equation:
\[
\frac{dA(z)}{dz} \exp[-i\phi(z)] + \frac{dB(z)}{dz} \exp[i\phi(z)] = 0. \tag{6}
\]

For calculation of the reflection coefficient it is convenient to use a function

\[
V(z) = \frac{B(z)}{A(z)} \exp[2i\phi(z)].
\]

Then a wave reflection coefficient is connected with \(V(z)\) by equality

\[
R_p = |V(-\infty)|^2. \tag{7}
\]

From the Eqs. (2), (4), and (6) the equation follows:

\[
\frac{d[V \exp(-2i\phi)]}{dz} = \frac{1}{2} \frac{dp}{dz} (1 - V^2) \exp(-2i\phi). \tag{8}
\]

The boundary condition for \(V(z)\) can be deduced from the physical restriction: \(B(+\infty) = 0\). Then

\[
V(+\infty) = 0. \tag{9}
\]

Thus, the reflection coefficient from the media partition boundary with the non-uniform transition layer can be calculated from the differential Eq. (8) with the boundary condition (9) and from the formula (7).

The Eq. (8) generally hasn't an analytical solution. However approximate methods can be used [5]. The equation has the feature. Its right part approaches to zero at \(z \to \pm \infty\), for \(\varepsilon(z)\) approaches to the constant values, and according to Eq. (3) there is the limit \(dp(z)/dz \to 0\). Dependences of \(\varepsilon(z)\) with sharp change on one asymptotic value \(\varepsilon_1 = \varepsilon(-\infty)\) to another \(\varepsilon_2 = \varepsilon(+\infty)\) were considered. In this case the change interval of \(\varepsilon(z)\) is defined by the \(d\) size, and the coordinate \(z = 0\) was situated inside this interval. Then the derivative \(dp(z)/dz\) will have the greatest values nearby \(z = 0\). In this case for the reflection coefficient calculation it is possible to use the approximate solution of the Eq. (8), constructed in an interval \(\Delta z = d\).

Let the value of \(d\) is so small that the inequality is carried out

\[
d \max \{|p(z)|d(z)| \ll 1. \tag{10}
\]

The inequality expresses in essence a condition of the phase thickness smallness of the transition layer. Then the function \(\phi(z)\) will be small for the values of \(z\) lying in the \(\Delta z = d\) interval, and the function \(\exp(-2i\phi)\) can be approximated by a power series on \(\phi(z)\) in which the several first terms will be essential. The solution of the Eq. (8) can be also constructed in the form of a row which terms correspond to the \(\phi(z)\) powers taken into account. Besides at the large values of \(z\) the integral in expression of (5) can result in linear growth of \(\phi(z)\) dependence. In this case, for the profiles of \(\varepsilon(z)\) aiming to asymptotic values not more slowly than exponentially, integration of the right side of Eq. (8), presented in the row form, leads to finite values of each term of the row even when the limits of integration will be infinite. Such feature of the equation arises due to the \(dp/dz\) derivative [6]. It allows to replace in final expressions the integration interval \(\Delta z = d\) on integration with the infinite limits. Therefore the dependences \(\varepsilon(z)\) aiming to asymptotic values at least exponentially were considered.

Decomposition of the function \(V(z)\) into the row of the perturbations theory was used:

\[
V(z) = V_0(z) + iV_1(z) + V_2(z). \tag{11}
\]

The order of the used approximation is noted by the bottom index, and for convenience in corrections of an odd order the multiplier \(i = \sqrt{-1}\) is entered.

It is assumed that \(\exp[-2i\phi(z)] \approx 1\) in zero approximation, and then from the Eq.(8) follows:
\[
\frac{dV_0}{dz} = \frac{1}{2p} \frac{dp}{dz} (1-V_0^2), \quad V_0(+\infty) = 0,
\]
\[
V_0(z) = p(z) - p(+\infty) \quad \frac{p(z) + p(+\infty)}{p(z) + p(+\infty)}.
\]  
\(\text{(12)}\)

Equality \(\exp[-2i\varphi(z)] \approx 1 - 2i\varphi(z)\) was as a first approximation used, and the function \(V_1(z)\) is defined by the equation:
\[
\frac{dV_1}{dz} = -\frac{1}{p} \frac{dp}{dz} V_0 V_1 + 2peV_0, \quad V_1(+\infty) = 0,
\]
\[
V_1(z) = -2\left[1 - V_0^2(z)\right] \int_z^{z'} \frac{V_0(z')p(z') \kappa(z')}{1 - V_0^2(z')} dz'.
\]  
\(\text{(13)}\)

In the second approximation it was assumed that \(\exp[-2i\varphi(z)] \approx 1 - 2i\varphi(z) - 2\varphi^2(z)\), and the function \(V_2(z)\) is defined by the equation:
\[
\frac{dV_2}{dz} = \frac{1}{2p} \frac{dp}{dz} \left[V_1^2 - 2V_0 V_2\right] - 2peV_1, \quad V_2(+\infty) = 0,
\]
\[
V_2(z) = 4\left[1 - V_0^2(z)\right] \int_z^{z'} \frac{V_0(z') p(z') \kappa(z')}{1 - V_0^2(z')} dz' - \int_z^{z'} \left[1 + \frac{V_0^2}{1 - V_0^2}\right] p(z') dz',
\]  
\(\text{(14)}\)

where \(I\) designate the function
\[
I(z') = 4\int_z^{z'} \frac{V_0(z')p(z') \kappa(z')}{1 - V_0^2(z')} dz'.
\]

The purpose of calculation is the reflection coefficient \(R_p = |V(-\infty)|^2\). Since the functions \(\varepsilon(z)\), \(V_0(z)\), \(V_1(z)\), \(V_2(z)\) has only the real values in a transparent medium, the ratio follows from the Eq. (11):
\[
R_p = V_0^2(-\infty) + V_1^2(-\infty) + 2V_0(-\infty) V_2(-\infty).
\]

Substitutions of deduced expressions (12), (13), (14) and transformation allow to deduce an estimated formula [5]:
\[
R_p = \left[p_1 - p_2 \right]^2 + \frac{8p_1 p_2}{(p_1 + p_2)^2} \cdot \int_{-\infty}^{z} \left[p^2(z) - p^2_{-\infty}\right] \kappa(z') dz' \varepsilon(z) dz',
\]  
\(\text{(15)}\)

where \(p_1 = p(-\infty)\) and \(p_2 = p(+\infty)\). The function \(p(z)\) is connected to the function \(\varepsilon(z)\) by formula (3).

Thus the ratio (15) expresses the reflection coefficient of the \(p\)-polarised electromagnetic wave by the analytical kind. The dielectric permittivity profile is given by the dependence of a general view \(\varepsilon(z)\). In this case two restrictions were introduced. One assumed absence of the absorption and therefore the function \(\varepsilon(z)\) can have only the real values. Another restriction is defined by the inequality (10).

3. Results of calculations
By means of the formula (15) the angular dependence of the reflection coefficient for the chosen functions \(\varepsilon(z)\) was calculated. Further these dependences by means of the formula (1) were transformed to function \(\eta(\theta)\).

In the non-uniform medium the dependence \(\varepsilon(z)\) was simulated by Gauss' function:
The type of the function $e(z)$ corresponds to the properties of the materials which underwent radiation by ions. Parameters $\delta \varepsilon$, $d$, and $\Delta$ are also defined by a radiation dose, energy, and dispersion of the ions run. Besides, the sign $\delta \varepsilon$ can be both the positive and the negative.

Figures 1, 2 give an understanding about influence of each of the parameters on dependences $R_p(\theta)$ and $\eta(\theta)$. The domain of the angles located near the Brewster’s angle is the most sensitive to the parameters. In this case the features of the dependence $e(z)$ are shown in the position and in the extremums amplitudes of the $\eta(\theta)$ dependence. However the integral connection between $R_p(\theta)$ and $e(z)$ doesn’t allow to solve the reverse problem – it is impossible to construct dependence $e(z)$ proceeding from the dependences $R_p(\theta)$ or $\eta(\theta)$. At the same time results of calculations show a possibility of the implanted layers identification with the help of the reflectometric characteristics.

![Figure 2](image)

**Figure 2.** Angular dependences of the reflection coefficient $R_p(\theta)$. On the upper insertion the coordinate dependences of dielectric permittivity $e(z/\lambda)$ which were used in calculations are figured ($\lambda$ – the electromagnetic wave length), on lower – fragments of $R_p(\theta)$ dependences near the Brewster's angle.

It is necessary to note that the depth of penetration of the ions reaches several microns. Then the condition (10) can be satisfied if the electromagnetic wave has length into hundreds of microns. The frequencies of such waves have the values of $\sim 10^{12}$ Hz. Many dielectrics in this range are transparent. The transparency was supposed on deducing of formula (15).
Figure 3. Angular dependences of the reflection coefficient logarithmic derivative near Brewster's angle $\eta(\theta)$. On an insert the dependences $\varepsilon(z/\lambda)$ which were used in calculations are represented.

4. Conclusions

1. The analysis of the electromagnetic wave reflection with $p$-polarization allows to detect thin layers with the non-uniform profile of dielectric permittivity. In this case the length of the wave can be into hundreds of times greater than the thickness of layer.

2. Estimated basis of such analysis is the deduced formula (15).

3. The reflectometric analysis allows to control the technological operations which are realized in case of the materials properties modification by an ion implantation technique.

The practical use of the expounded results is connected with a possibility of the reflection coefficient measurement in the terahertz range. The greatest difficulty arises in the interval of angles near the Brewster's angle where values of the coefficient become especially small. The problem can be solved both by increase in sensitivity of receivers and by increase in power of the used radiation sources.

5. References

[1] Zinchenko S P, Kovtun A P and Tolmachev G N 2009 Technical Physics 54 1689
[2] Shvartsburg A B 2000 Physics-Uspekhi (Advances in Physical Sciences) 43 1201
[3] Shvartsburg A B, Agranat M B, and Chefonov O V 2009 Quantum electronics 39 948
[4] Bilenko D I, Polyanskaya V P, Getsman M A, Gorin D A, Neveshkin A A and Yaschenok A M 2005 Technical Physics 50 742
[5] Shagaev V V 2015 Technical Physics 60 1738
[6] Babikov V V 1976 Method of Phase Functions in Quantum Mechanics (Moscow: Nauka) 286