A Family of the Inertial Manifolds for a Class of Generalized Kirchhoff-Type Coupled Equations

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Abstract

The paper considers the long-time behavior for a class of generalized high-order Kirchhoff-type coupled equations, under the corresponding hypothetical conditions, according to the Hadamard graph transformation method, obtain the equivalent norm in space $E_k (k = 1, 2, \cdots, 2m)$, and we obtain the existence of a family of the inertial manifolds while such equations satisfy the spectral interval condition.

Keywords

Kirchhoff-Type Coupled Equations, Spectral Interval Condition, A Family of the Inertial Manifolds

1. Introduction

This paper investigates the following primal value problems of a system of generalized Kirchhoff-type coupled equations:

$$
\begin{align*}
&u_t + M \left( \| D^m u_x \|_{p}^p + \| D^m v_x \|_{p}^p \right) (-\Delta)^{2m} u + \beta \Delta^{2m} u + f_1(u, v) = f_1(x), \quad (1) \\
v_t + M \left( \| D^m u_x \|_{p}^p + \| D^m v_x \|_{p}^p \right) (-\Delta)^{2m} v + \beta \Delta^{2m} v + g_1(u, v) = f_2(x), \quad (2) \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \quad (3) \\
v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), x \in \Omega, \quad (4) \\
\frac{\partial u}{\partial n'} = 0, \frac{\partial v}{\partial n'} = 0 (i = 0, 1, 2, \cdots, 2m). \quad (5)
\end{align*}
$$

where $\Omega$ is a bounded region with a smooth boundary in $R^n$, $\partial \Omega$ represents the boundary of $\Omega$, $u_0(x), u_1(x)$ and $v_0(x), v_1(x)$ are known functions, where $g_j(u, v), f_j(u, v) (j = 1, 2)$ are nonlinear terms and external interference terms, respectively, and are known functions on $\Omega \times (0, T)$, $\beta$ is the normal
number, \( M\left(\|D^m u\|_p^p + \|D^n v\|_p^p\right) \) is a non-negative first-order continuous derivative function, and \( m > 1 \) is the normal number, \( \|D^m u\|_p^p = \int_\Omega |D^m u|^p \, dx \).

In order to overcome the research difficulties, G. Foias, G. R. Sell and R. Temam [1] proposed the concept of inertial manifolds, which greatly promoted the study of infinite-dimensional dynamical systems. Where the inertial manifold is a positive, finite-dimensional Lipschitz manifold, and the existence of an inertial manifold depends on the establishment of a spectral interval condition. Therefore, the research on a family of inertial manifolds is of great significance from both theoretical and practical aspects, and the relevant theoretical achievements can be referred to [2]-[9].

Guoguang Lin, Lingjuan Hu [10] studied a system of coupled wave equations of higher-order Kirchhoff type with strong damping terms

\[
\begin{align*}
    u_{tt} + M\left(\nabla^n u\right)(-\Delta)^n u + \beta(-\Delta)^n u + g_1(u, v) &= f_1(x), \\
    v_{tt} + M\left(\nabla^m v\right)(-\Delta)^m v + \beta(-\Delta)^m v + g_2(u, v) &= f_2(x), \\
    u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \\
    v(x, 0) &= v_0(x), v_t(x, 0) = v_1(x), x \in \Omega, \\
    \frac{\partial u}{\partial n} = 0, \frac{\partial v}{\partial n} = 0 & \quad (i = 0, 1, 2, \cdots, 2m - 1) \quad x \in \partial \Omega.
\end{align*}
\]

where \( \Omega \) is a bounded region with a smooth boundary in \( \mathbb{R}^n \), \( \partial \Omega \) represents the boundary of \( \Omega \), \( g_j(u, v)(j = 1, 2) \) is a nonlinear source term, \( f_1(x), f_2(x) \) is an external force interference term, and \( \beta(-\Delta)^n u, \beta(-\Delta)^m v (\beta \geq 0) \) is a strong dissipation terms. Using the Hadamard graph transformation method, the Lipschitz constant \( I_F \) of \( F \) is further estimated, and the inertial manifolds that satisfies the spectral interval condition is obtained.

Lin Guoguang, Liu Xiaomei [11] studied a family of inertial manifolds for a class of generalized higher-order Kirchhoff equations with strong dissipation terms

\[
\begin{align*}
    u_{tt} + M\left(\nabla^m u\right)(-\Delta)^m u + \beta(-\Delta)^m u + |u|^\rho (u + u) &= f(x), \\
    u(x, t) &= 0, \frac{\partial u}{\partial n} = 0, i = 1, 2, \cdots, 2m - 1, x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), x \in \Omega \subset \mathbb{R}^n.
\end{align*}
\]

where \( m \in \mathbb{N}^* \), \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with a smooth boundary in \( \partial \Omega \), \( f(x) \) is an external force term, \( M\left(\nabla^m u\right) \) is the stress term of Kirchhoff equation, \( \beta(-\Delta)^m u \) is a strong dissipative term, \( |u|^\rho (u + u) \) is a nonlinear source term. Based on appropriate assumptions and the Hadamard graph transformation method, the spectral interval condition is verified, and the existence of a family of the inertial manifolds of the equation is obtained.

On the basis of previous research, rigid term strengthening becomes
and this paper seeks a family of inertial manifolds. When defining the equivalence norm in space $E_k$, by making reasonable assumptions, it is obtained that the equation satisfies the spectral interval condition so that there is a family of inertial manifolds.

2. Preliminaries

For narrative convenience, we introduce the following symbols and assumptions:

Set $\mathbb{D} = \{\alpha \in \mathbb{R} : \alpha \in (0,1)\}$. Consider Hilbert space family $(V_\alpha)_{\alpha \in \mathbb{D}}$, whose inner product and norm are $(\cdot, \cdot)_{V_\alpha} = \left( (-\Delta)^{\alpha/2}, (-\Delta)^{\alpha/2} \right)$ and $\| \cdot \|_{V_\alpha} = \left\| (-\Delta)^{\alpha/2} \right\|$, respectively. Apparently there are $(\cdot, \cdot)_{V_{k_m}} = \left( (-\Delta)^{\alpha/2}, (-\Delta)^{\alpha/2} \right)$ and $\| \cdot \|_{V_{k_m}} = \left\| (-\Delta)^{\alpha/2} \right\|$, respectively. The assumption is as follows:

Let $M(s)$ be a continuous function on interval $D_1(D_0 \in \mathbb{D})$, and $M(s) \in C^1(R^+)$:

$$1 \leq \mu_0 \leq M(s) \leq \mu_0,$$ set $M(s) = M\left( \left\| D^\alpha u \right\|_{V_{k_m}} + \left\| D^\alpha v \right\|_{V_{k_m}} \right)$.

3. A Family of Inertial Manifolds

Definition 1 [12] lets $\mathcal{S} = \{S(t)\}_{t \geq 0}$ be the solution semigroup on Banach space $E_k = H_0^{2m+1}(\Omega) \times H_0^1(\Omega) (k = 1,2,\cdots,2m)$, and a subset $\mu \subset E_k$ satisfies:

1) $\mu_k$ is finite-dimensional Lipschitz popular;
2) $\mu_k$ is positively unchanging, $\{S(t)\}_{t \geq 0} : \forall t \in \mu_k$, $S(t)u_0 \subset \mu_k, t \geq 0$;
3) $\mu_k$ attracts the solution orbit exponentially, i.e. for any $u \in E_k$, the existence constant $\eta > 0, c > 0$ makes $\text{dist}(S(t)u, \mu) \leq ce^{-\eta t}$, $t \geq 0$.

Then $\mu_k$ is called $E_k$ is a family of inertial manifolds.

In order to describe the spectral interval condition, first consider that the nonlinear term $F : E_k \rightarrow E_k$ is integrally bounded and continuous, and has a positive Lipschitz constant $l_F$, and its operator $A$ has several eigenvalues and eigenfunctions of the positive real part.

Definition 2 [12] Set operator $A : X \rightarrow X$ has several eigenvalues of positive real numbers, and $F \in C_k(X, X)$ satisfies the Lipschitz condition:

$$\left\| F(u) - F(v) \right\|_X \leq l_F \left\| u - v \right\|_X, \quad u, v \in X,$$

The point spectrum of the operator $A$ can be divided into two parts $\sigma_1$ and $\sigma_2$, and $\sigma_1$ is finite,
\[\Lambda_1 = \sup \{ \text{Re} \lambda | \lambda \in \sigma_1 \},\]
\[\Lambda_2 = \sup \{ \text{Re} \lambda | \lambda \in \sigma_2 \},\]
\[X_i = \text{span} \{ \omega_j | \lambda_j \in \sigma_i \}, i = 1, 2.\]

and conditions

\[\Lambda_2 - \Lambda_1 > 4l_f\]  \hspace{1cm} (6)

are satisfied.

Where the continuous projection \( P_1 : X \to X_1, P_2 : X \to X_2 \), there is orthogonal decomposition \( X = X_1 \oplus X_2 \), then the operator \( A \) satisfies the spectral interval condition.

**Lemma 1** \( g_i : V_{2m+1} \times V_{2m+1} \to V_{2m+1} \times V_{2m+1} \) \((i = 1, 2)\) is a uniform bounded and integral Lipschitz continuous function.

Proof: \( \forall (\bar{u}, \bar{v}), (u, v) \in V_{2m+1} \times V_{2m+1} \) \((k = 1, 2, \cdots, 2m)\),

\[\|g_i(\bar{u}, \bar{v}) - g_i(u, v)\|_{2m+1} \leq l\|\bar{u} - u\|_{2m+1} + \|\bar{v} - v\|_{2m+1},\]

Similarly, there are

\[\|g_i(\bar{u}, \bar{v}) - g_i(u, v)\|_{2m+1} \leq l\|\bar{u} - u\|_{2m+1} + \|\bar{v} - v\|_{2m+1},\]

where \( l \) is the Lipschitz constant of \( g_i \), \( \theta \in (0, 1) \).

**Lemma 2** \([12]\) lets the sequence of eigenvalues \( \{ \mu_j \}_{j=1}^\infty \) is a non-subtractive sequence, then \( \exists N_0 \in \mathbb{N}_+ \) for \( \forall N \geq N_0 \), \( \mu_N \) and \( \mu_{N+1} \) are consecutive adjacent values.

In order to verify that the operator satisfies the spectral interval condition, so as to draw the conclusion that there is a family of inertial manifolds in questions (1)-(5), the following definitions and assumptions can be made first.

Based on the above relevant conditions, consider the first-order development equation equivalent to Equations (1)-(5), as follows:

\[U_j + A'U = F(U)\]  \hspace{1cm} (7)

Of which \( U = (u, z, v, q) \),

\[A' = \begin{pmatrix}
0 & -I & 0 & 0 \\
M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m} & 0 & 0 \\
0 & 0 & -I & 0 \\
0 & 0 & M(s)(-\Delta)^{2m} & \beta(-\Delta)^{2m}
\end{pmatrix},\]
In order to determine the eigenvalue of matrix operator $A'$, first consider graph module 

$$D(A') = \{(u,v) \in V_{2m+1} \times V_{2m+1} \mid \langle u,v \rangle \in V_0 \times V_0, \langle D^{2m+1} u, D^{2m+1} v \rangle \in V_0 \times V_0\} \times V_f \times V_f,$$

where

$$s = \|D_u u\|_p^p + \|D_v v\|_p^p.$$

In order to determine the eigenvalue of matrix operator $A'$, first consider graph module

$$(U,V)_{E_k} = \left(M(s) D^{2m+1} u, D^{2m+1} u\right) + \left(D^z u, M(s) (-\Delta)^{2m} u + \beta (-\Delta)^{2m} z\right)
+ \left(D^v v, M(s) (-\Delta)^{2m} v + \beta (-\Delta)^{2m} q\right)$$

generated by inner product in $E_k$.

Where $U=(u,z,v,q), V=(u',z',v',q'),$ and $\overline{u}, \overline{z}, \overline{v}, \overline{q}$ represent the conjugation of $u', z', v', q'$ respectively. In addition, operator $A'$ is monotonic, and for $U \in D(A')$, there is

$$(A'U,U)_{E_k} = -(M(s) D^{2m+1} u, D^{2m+1} u) + \left(D^z u, M(s) (-\Delta)^{2m} u + \beta (-\Delta)^{2m} z\right)
- \left(M(s) D^{2m+1} q, D^{2m+1} u\right)
= \left(D^{2m+1} z, \beta D^{2m+1} u\right) - \left(M(s) D^{2m+1} q, D^{2m+1} u\right)
+ \left(D^{2m+1} v, M(s) D^{2m+1} v\right) + \left(D^{2m+1} q, \beta D^{2m+1} q\right)
= \beta \left(\|D^{2m+1} z\|_p^p + \|D^{2m+1} q\|_p^p\right) \geq 0.$$

Therefore, $(A'U,U)_{E_k}$ is a nonnegative real number.

In order to further determine the eigenvalue of the matrix operator $A'$, the following characteristic equation can be considered,

$$(A'U,U) = \langle u, z, v, q \rangle \in E_k, \quad (8)$$

That is

$$\begin{cases}
-z = \lambda u, \\
M(s)(-\Delta)^{2m} u + \beta (-\Delta)^{2m} z = \lambda z, \\
-q = \lambda v, \\
M(s)(-\Delta)^{2m} v + \beta (-\Delta)^{2m} q = \lambda q.
\end{cases}$$

Thus $u, v$ meet the eigenvalue problem

$$\begin{cases}
\lambda^2 u - \lambda \beta (-\Delta)^{2m} u + M(s)(-\Delta)^{2m} u = 0, \\
\lambda^2 v - \lambda \beta (-\Delta)^{2m} v + M(s)(-\Delta)^{2m} v = 0, \\
\frac{\partial^i u}{\partial n^i} |_{\partial \Omega} = \frac{\partial^i v}{\partial n^i} |_{\partial \Omega} = 0, i = 0, 1, 2, \ldots, 2m - 1,
\end{cases}$$

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Take the inner product of \((-\Delta)^k u, (-\Delta)^k v\) and Equations (1) and (2) above respectively, with

\[
\begin{align*}
\left[\lambda^2 \left\|D^k u\right\|^2 - \lambda \beta \left\|D^{2m+k} u\right\|^2 + M(s) \left\|D^{2m+k} u\right\|^2\right] &= 0, \\
\left[\lambda^2 \left\|D^k v\right\|^2 - \lambda \beta \left\|D^{2m+k} v\right\|^2 + M(s) \left\|D^{2m+k} v\right\|^2\right] &= 0,
\end{align*}
\]

That is

\[
\lambda^2 \left(\left\|D^k u\right\|^2 + \left\|D^k v\right\|^2\right) - \lambda \beta \left(\left\|D^{2m+k} u\right\|^2 + \left\|D^{2m+k} v\right\|^2\right) + M(s) \left(\left\|D^{2m+k} u\right\|^2 + \left\|D^{2m+k} v\right\|^2\right) = 0. \tag{9}
\]

Equation (9) is a univariate quadratic equation about \(\lambda\). Replace \(u, v\) with \(u_j, v_j\). For each positive integer \(j\), Equation (8) has paired eigenvalues

\[
\lambda_j^\pm = \beta \mu_j \pm \sqrt{\left(\beta \mu_j\right)^2 - 4M(s)\mu_j},
\]

where \(\mu_j\) is the characteristic root of \((-\Delta)^{2m}\) in \(V_{2m} \times V_{2m}\), then \(\mu_j = \lambda_j^{2m}\).

If \(\left(\beta \mu_j\right)^2 \geq 4M(s)\mu_j\), then \(\mu_j \geq \frac{4M(s)}{\beta^2}\), the eigenvalues of operator \(A'\) are all real numbers, and the corresponding eigenfunction form is

\[
U_j^\pm = \left(u_j, -\lambda_j^+ u_j, v_j, -\lambda_j^+ v_j\right).
\]

For convenience, mark for any \(j \geq 1\), there are

\[
\begin{align*}
\left\|D^{2m+k} u_j\right\|^2 + \left\|D^{2m+k} v_j\right\|^2 &= \mu_j, \\
\left\|D^k u_j\right\|^2 + \left\|D^k v_j\right\|^2 &= \frac{1}{\mu_j}.
\end{align*}
\]

**Theorem 1:** Assumes that \(I\) is the Lipschitz constant of \(g_i(u, v)(i = 1, 2)\). When \(N_0 \in N_+\) is sufficiently large, for \(\forall N \geq N_0\), the following inequality holds

\[
\left(\mu_{N+1} - \mu_N\right)\left(\beta - \sqrt{\beta^2 \mu_t - 4M(s)}\right) \geq \frac{32I}{\sqrt{\beta^2 \mu_t - 4M(s)}} + 1. \tag{10}
\]

Then all operators \(A'\) satisfy the spectral interval condition (6).

**Proof.** Because \(\mu_j \geq \frac{4M(s)}{\beta^2}\) and the eigenvalues of \(A'\) are positive real numbers, \(\left\{\lambda_j^\pm\right\}_{j \geq 1}\) and \(\left\{\lambda_j^\pm\right\}_{j \geq 1}\) are single increment sequences.

The following four steps are taken to prove theorem 1:

Step 1: Because \(\left\{\lambda_j^\pm\right\}_{j \geq 1}\) and \(\left\{\lambda_j^\pm\right\}_{j \geq 1}\) are non subtractive columns, according to lemma 2, there are \(\exists N_0 \in N_+\), for \(\forall N \geq N_0\), \(\lambda_N\) and \(\lambda_{N+1}\) are continuous adjacent values.

Therefore, there is \(N\), so that \(\lambda_N\) and \(\lambda_{N+1}\) are continuous adjacent values, and the eigenvalue of \(A'\) can be decomposed into...
\[ \sigma_1 = \{ \lambda^-_r | 1 \leq r \leq N \}, \sigma_2 = \{ \lambda^+_r | 1 \leq r \leq j \} \]

Step 2: Consider the corresponding decomposition of \( E_k \) into
\[
E_{k_1} = \text{span}\{U_j' | \lambda^-_r \in \sigma_1 \}, \\
E_{k_2} = \text{span}\{U'_j, U_j' | \lambda^+_r, \lambda^-_r \in \sigma_2 \}
\]

The equivalent inner product \( ((U,V))_{E_k} \) given below makes \( E_{k_1}, E_{k_2} \) orthogonal.

Further decompose \( E_{k_2} = E_H \oplus E_R \), of which
\[
E_H = \text{span}\{U'_j | 1 \leq r \leq N \}, E_R = \text{span}\{U'_j | j \geq N \}
\]

Because \( E_{k_1} \) and \( E_H \) are finite dimensional subspaces, \( U_N \in E_{k_1}, U_{N+1} \in E_R \), and \( E_{k_2} \) and \( E_R \) are orthogonal, while \( E_{k_1} \) and \( E_H \) are not orthogonal, \( E_{k_1} \) and \( E_{k_2} \) are not orthogonal. So we need to redefine the equivalent norm on \( E_A \), so that \( E_{k_1} \) and \( E_H \) are orthogonal. Order \( E_N = E_{k_1} \oplus E_H \).

Construct two functions \( \Phi : E_N \rightarrow R, \Psi : E_R \rightarrow R \) of which,

\[
\Phi(U,V) = 2\beta(\beta - 1)(D^{2m+k}u, D^{-(2m+k)}\overline{u}) + 2\beta(D^{-(2m+k)}\overline{z}, D^{2m+k}u) \\
+ 2\beta(D^{-(2m+k)}\overline{v}, D^{2m+k}u') + 4(D^{-(2m+k)}\overline{z}, D^{2m+k}z) \\
- 4M(s)(D^ku, D^k\overline{u}) + 2\beta(D^{2m+k}\overline{u}, D^{2m+k}u') \\
+ 2\beta(\beta - 1)(D^{2m+k}v, D^{2m+k}\overline{v}) + 2\beta(D^{-(2m+k)}\overline{q}, D^{2m+k}v) \\
+ 2\beta(D^{-(2m+k)}\overline{q}, D^{2m+k}v') + 4(D^{-(2m+k)}\overline{q}, D^{2m+k}q)
\]

\[
\Psi(U,V) = 2\beta(D^{2m+k}u, D^{2m+k}u') + 2\beta(D^{-(2m+k)}\overline{z}, D^{2m+k}u) \\
+ \beta(D^{-(2m+k)}\overline{z}, D^{2m+k}u') + 4(D^{-(2m+k)}\overline{z}, D^{2m+k}z) \\
- 2M(s)(D^k u, D^k\overline{u}) + 2\beta(\beta - 1)(D^{2m+k}u, D^{-(2m+k)}\overline{u}) \\
+ 2\beta(D^{2m+k}\overline{v}, D^{2m+k}v') + \beta(D^{-(2m+k)}\overline{q}, D^{2m+k}v) \\
+ \beta(D^{-(2m+k)}\overline{q}, D^{2m+k}v') + 4(D^{-(2m+k)}\overline{q}, D^{2m+k}q) \\
- 2M(s)(D^k v, D^k\overline{v}) + 2\beta(\beta - 1)(D^{2m+k}v, D^{2m+k}\overline{v})
\]

Among them \( U = (u, z, v, q), V = (u', z', v', q') \in E_N \) or \( E_R \).

For \( U = (u, z, v, q) \in E_N \), then

\[
\Phi(U,U) = 2\beta(\beta - 1)(D^{2m+k}u, D^{-(2m+k)}\overline{u}) + 2\beta(D^{-(2m+k)}\overline{z}, D^{2m+k}u) \\
+ 2\beta(D^{-(2m+k)}\overline{v}, D^{2m+k}u') + 4(D^{-(2m+k)}\overline{z}, D^{2m+k}z) \\
- 4M(s)(D^ku, D^k\overline{u}) + 2\beta(D^{2m+k}\overline{u}, D^{2m+k}u') \\
+ 2\beta(\beta - 1)(D^{2m+k}v, D^{2m+k}\overline{v}) + 2\beta(D^{-(2m+k)}\overline{q}, D^{2m+k}v) \\
+ 2\beta(D^{-(2m+k)}\overline{q}, D^{2m+k}v') + 4(D^{-(2m+k)}\overline{q}, D^{2m+k}q)
\]
\[-4M(s)(D^4_v, D^4_v) + 2\beta(D^{2m_k}v, D^{2m_k}v)\]
\[\geq 2\beta(\beta - 1)\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) - 4\beta\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[+ \|D^{-(2m_k)}\|\|D^{-(2m_k)}\|\right) + 4\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[= 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) + 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right)\]
\[\geq 2\beta(\beta - 1)\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) - 4\beta\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[+ \|D^{-(2m_k)}\|\|D^{-(2m_k)}\|\right) + 4\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[= 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) + 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right)\]
\[\geq \left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right)\]
\[\geq \left(\beta^2 \mu_i - 2M(s)\right)\left(\|D^4_u\|^2 + \|D^4_v\|^2\right).\]

For any \( k \), there is \( \beta^2 \mu_i \geq 4M(s) \). According to hypothesis

\[1 \leq \mu_0 \leq M(s) \leq \mu_i \leq \frac{\beta^2 \mu_i}{4}, \text{ then } \Phi(U, U) \geq 0, \text{ that is, } \Phi \text{ is positive definite.}\]

Similarly, for any \( U = (u, z, v, q) \in E_R \), there is
\[
\Psi(U, U) = 2\beta\left(D^{2m_k}u, D^{2m_k}u\right) + \beta\left(D^{-(2m_k)}z, D^{2m_k}u\right)
\]
\[+ \beta\left(D^{-(2m_k)}z, D^{2m_k}u\right) + 4\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[+ 2\beta\left(D^{2m_k}v, D^{2m_k}v\right) + 2\beta(\beta - 1)\left(D^{2m_k}u, D^{2m_k}v\right)\]
\[+ \beta\left(D^{2m_k}v, D^{2m_k}v\right) + 4\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[\geq 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) - 2\beta\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[+ \|D^{-(2m_k)}\|\|D^{-(2m_k)}\|\right) + 4\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[= 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) + 2\beta(\beta - 1)\left(D^{2m_k}u, D^{2m_k}v\right)\]
\[\geq 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) - 2\beta\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[+ \|D^{-(2m_k)}\|\|D^{-(2m_k)}\|\right) + 4\left(\|D^{-(2m_k)}z\|^2 + \|D^{-(2m_k)}q\|^2\right)\]
\[= 2\beta\left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right) + 2\beta(\beta - 1)\left(D^{2m_k}u, D^{2m_k}v\right)\]
\[\geq \left(\|D^{2m_k}u\|^2 + \|D^{2m_k}v\|^2\right)\]
\[\geq \left(\beta^2 \mu_i - 2M(s)\right)\left(\|D^4_u\|^2 + \|D^4_v\|^2\right).\]
So there are \( \forall U = (u, z, v, q) \in E_R, \quad \Psi(U, U) \geq 0, \) then \( \Psi \) is also positive definite.

Redefine the inner product of \( E_k \):

\[
((U, V))_{E_k} = \Phi(P_N U, P_N V) + \Psi(P_N U, P_N V)
\]

(11)

where \( P_N \) and \( P_R \) are projections of \( E_k \to E_N \) and \( E_k \to E_R \), respectively.

Here, Equation (11) is written as

\[
((U, V))_{E_k} = \Phi(U, V) + \Psi(U, V)
\]

Under the redefined inner product of \( E_k \), to prove that \( E_{k_1} \) and \( E_{k_2} \) are orthogonal, we only need to prove that \( E_{k_1} \) and \( E_{k_2} \) are orthogonal, that is,

\[
((U_j^-, U_j^+))_{E_k} = \Phi(U_j^-, U_j^+) = 0.
\]

Because there are \( U_j^- \in E_{k_1}, U_j^+ \in E_{k_2}, \) that is

\[
\Phi(U_j^-, U_j^+) = 2\beta(\beta - 1)\left(\|D^{2\mu + k} u_j\|^2 + \|D^{2\beta \mu + k} u_j\|^2\right) - 2\beta^2 \mu_j - 2\beta^2 \mu_j - 2\beta(\lambda_j^+ + \lambda_j^-) \cdot \frac{1}{\mu_j}.
\]

Because of Equation (9), there are

\[
\lambda_j^+ + \lambda_j^- = \beta \mu_j, \quad \lambda_j^+ \cdot \lambda_j^- = M(\mu_j) \mu_j.
\]

So \( \Phi(U_j^-, U_j^+) = -4M(\mu_j) + 2\beta^2 \mu_j - 2\beta(\lambda_j^+ + \lambda_j^-) + 4\lambda_j^+ \lambda_j^- \cdot \frac{1}{\mu_j} = 0. \)

Step 3: according to the orthogonal decomposition established above, let’s prove that \( A' \) satisfies the spectral interval condition. First estimate the Lipschitz constant \( L_F \) of \( F \), where

\[
F(U) = \left(0, f_1(x) - g_1(u, v), 0, f_2(x) - g_2(u, v)\right)^T
\]

According to lemma 1, \( g_i(u, v) : V_{2m+i} \times V_{2m+i} \to V_{2m+i} \times V_{2m+i} \) are uniformly bounded and Lipschitz continuous, if \( U = (u, z, v, q) \in E_i, \)

\( U_j = (u, z, v, q_j) \in P_U (i = 1, 2). \)
Then
\[ P_1u = u_1, P_1v = v_1, P_2u = u_2, P_2v = v_2. \]
\[ \|U\|_E = \Phi(PUP, P_2U) + \Psi(PUP, P_2U) \]
\[ \geq \left( \beta^2 \mu - 4M(s) \right) \left( \|D^4 P_1u\|^2 + \|D^4 P_1v\|^2 \right) \]
\[ + \left( \beta^2 \mu - 2M(s) \right) \left( \|D^4 P_2u\|^2 + \|D^4 P_2v\|^2 \right) \]
\[ \geq \left( \beta^2 \mu - 4M(s) \right) \left( \|D^4 u\|^2 + \|D^4 v\|^2 \right). \]

Given \( U = (u, z, v, q), V = (\tilde{u}, \tilde{z}, \tilde{v}, \tilde{q}) \in E_k \), we can get
\[ \|F(U) - F(V)\|_E \]
\[ = \|g_1(u, v) - g_1(\tilde{u}, \tilde{v})\|_{2n+4 \times 2n+4} + \|g_2(u, v) - g_2(\tilde{u}, \tilde{v})\|_{2n+4 \times 2n+4} \]
\[ \leq 2l \left( \|u - \tilde{u}\|_{2n+4} + \|v - \tilde{v}\|_{2n+4} \right) \]
\[ \leq \frac{4l}{\sqrt{\beta^2 \mu - 4M(s)}} \|U - V\|_E. \]

So
\[ l_F \leq \frac{4l}{\sqrt{\beta^2 \mu - 4M(s)}} \] (13)

From (13), if
\[ \Lambda_2 - \Lambda_1 = \hat{\lambda}_{N+1} - \hat{\lambda}_N > \frac{16l}{\sqrt{\beta^2 \mu - 4M(s)}} \] (14)

Then the spectral interval condition (6) holds.

Step 4: according to the above paired eigenvalues, there are
\[ \Lambda_2 - \Lambda_1 = \hat{\lambda}_{N+1} - \hat{\lambda}_N = \frac{R(N) - R(N+1)}{\sqrt{\beta^2 \mu - 4M(s)}} + \frac{R(N)}{\sqrt{\beta^2 \mu - 4M(s)}} \]
\[ = \frac{R(N) - R(N+1)}{\sqrt{\beta^2 \mu - 4M(s)}} + \frac{R(N)}{\sqrt{\beta^2 \mu - 4M(s)}} \] (15)

Of which, \( R(N) = \beta^2 \mu_N^2 - 4M(s) \mu_N \).

There are \( N_0 \in N \), for \( \forall N \geq N_0 \), let \( R_0(N) = \frac{R(N)}{\sqrt{\beta^2 \mu - 4M(s)}} \), there are
\[ R(N) - R(N+1) + \sqrt{\beta^2 \mu - 4M(s)} (\mu_{N+1} - \mu_N) \]
\[ = \sqrt{\beta^2 \mu - 4M(s)} \left( \mu_{N+1} - \frac{R(N+1)}{\sqrt{\beta^2 \mu - 4M(s)}} \right) - \left( \mu_N - \frac{R(N)}{\sqrt{\beta^2 \mu - 4M(s)}} \right) \] (16)
\[ = \sqrt{\beta^2 \mu - 4M(s)} \left( (\mu_{N+1} - R_0(N+1)) - (\mu_N - R_0(N)) \right) \]

And because of \( \lim_{N \to +\infty} (\mu_N - R_0(N)) = \lim_{N \to +\infty} \left( \mu_N - \frac{R(N)}{\sqrt{\beta^2 \mu - 4M(s)}} \right) = 0 \), there are
\[ \lim_{N \to +\infty} \sqrt{R(N)} - \sqrt{R(N+1)} + \sqrt{\beta^2 \mu - 4M(s)} (\mu_{N+1} - \mu_N) = 0. \] (17)
According to the hypothesis (10) of Theorem 1 and Equations (13)-(17), there are

\[ \Lambda_2 - \Lambda_1 \geq \frac{1}{2} \left( (\mu_{\nu+1} - \mu_\nu) \left( \beta - \sqrt{\beta^2 \mu_\nu - 4M(s)} \right) - 1 \right) \]

\[ \geq - \frac{16l}{\sqrt{\beta^2 \mu_\nu - 4M(s)}} \geq 4l_\nu. \]  

(18)

Theorem 1 is proved.

**Theorem 2** [12] Through theorem1, operator \( A' \) satisfies the spectral interval condition, and problems (1)-(5) have a family of inertial manifolds \( \mu_k \), and \( \mu_k \in E_k \). The form is as follows,

\[ \mu_k = \text{graph}(\Gamma) \in E_k := \left\{ \varphi + \Gamma(\varphi) : \varphi \in E_k \right\} \]

where \( \Gamma : E_k \to E_k \) is Lipschitz continuous and has Lipschitz constant \( l_\nu \), and \( \text{graph}(\Gamma) \) represents the graph of \( \Gamma \).

**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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