Equations of motion as projectors and the gyromagnetic factor $g_s = \frac{1}{s}$ from first principles.

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Abstract

In this work we adopt the point of view that the equations of motion satisfied by a field are just a consequence of the representation space which the field belongs to, and the discrete symmetries we impose on it. We illustrate this viewpoint by rederiving Dirac and Proca equations as projectors over the subspaces with well defined parity of $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ representations respectively. We formulate the equation of motion corresponding to the identification of elementary systems with states in the invariant subspaces of the squared Pauli-Lubanski operator and couple minimally to electromagnetism the corresponding equation for the $(s, 0) \oplus (0, s)$ representation space using the gauge principle. We obtain $g = \frac{1}{s}$ for particles with arbitrary spin $s$ as conjectured by Belinfante long ago.

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The spin is a physical quantity whose nature is not completely clear in spite of its long history [1]. The existence of this quantity was originally formulated to explain the atomic fine structure, and conceived as a classical “intrinsic” angular momentum. However, the gyromagnetic factor required by the spectral lines ($g_s = 2$) was in conflict with this classical picture. The notion of spin as an angular momentum formally found a place in the non-relativistic quantum description of rotations. Indeed, as explicitly done in almost every textbook on quantum mechanics, the irreducible representations (irreps) of the rotation group can be constructed from the corresponding Lie algebra as $2j + 1$—dimensional subspaces characterized by the eigenvalue $j$ of the Casimir operator $J^2$ of this group. The eigenvalues $j$ can take any value integer or semi-integer. It is in the quantum realm that representations with semi-integer values of $j$ are realized as the representation spaces for the physically relevant property called spin. The concept of spin was clarified further with the formulation by Dirac.
of his famous equation describing point-like spin $\frac{1}{2}$ particles with the correct
gyromagnetic factor $g_s = 2$ which was claimed to be a consequence of special
relativity. As emphasized in [2] this value is actually a consequence of the
relativity principle either Galilean or Einsteinian.

A remarkable property of the Dirac states is that they are eigenstates of $S^2$
in every reference frame. In other words, they describe particles with truly
well defined spin. Notice that this additional property is not necessary from
the point of view of the classification of elementary systems according to the
irreducible representations of the Poincaré group since, in general, the relevant
quantum numbers are the eigenvalues of the Casimir operators for the Poincaré
group, namely the squared four-momentum, $P^2$, and the squared Pauli-Lubanski
operator, $W^2$. In general, the last operator is proportional to $S^2$ only in the
rest frame.

The gyromagnetic factor for a system with spin $s$ minimally coupled to
electromagnetism was conjectured by Belinfante to be $g_s = 1/s$ [3] and an
explicit proof was given for a spin $\frac{3}{2}$ system using the Fierz-Pauli formalism
[4]. Although partial proofs exist for this conjecture under restricted conditions
[5], a general proof is still lacking.

The aim of the present paper is to reformulate the equations of motion as
restrictions due to the irreducible representation of the Homogeneous Lorentz
Group we use and the discrete symmetries we impose in this space. Follow-
ing this philosophy we re-derive Dirac and Proca equations as projectors over
parity-invariant subspaces of the corresponding representations. We apply this
formalism to the Poincaré group and obtain the equation of motion corresponding
to the projection over invariant subspaces of the casimir operators of the
Poincaré Group for an arbitrary representation. Using the gauge principle in
the corresponding equation for the $(s, 0) \oplus (0, s)$ representation, we give a proof
of Belinfante’s conjecture based only on the structure of space-time and on
minimal coupling, i.e. $U(1)_{em}$ gauge structure for electromagnetism.

1 Dirac and Proca equations as projectors over
subspaces with well defined parity.

In this work we take the point of view that the equation of motion satisfied by a
field is just as a kinematical statement of the representation space which it be-
longs to and the discrete symmetries we impose on it [6, 7]. In order to illustrate
the point, in this section we will obtain Dirac and Proca equations as simple
projectors over parity eigen-subspaces of the corresponding representations of
the HLG.
1.1 Dirac Equation

The Dirac field belongs to the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation of the Homogeneous Lorentz Group (HLG) and is an eigenstate of parity operator. This last condition can be imposed in the rest frame as

$$\Pi \psi(0) = \eta \psi(0),$$  \hspace{1cm} (1)

where \(\Pi\) denotes parity operator in the rest frame (I-parity in the following to distinguish it from the full parity operation which requires also the change \(p \rightarrow -p\)) and \(\eta = \pm 1\).

The boost operator for fields transforming in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation, can be constructed from first principles [8, 9, 10]. We work in the Dirac basis for the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) space which is related to the Weyl basis by the transformation matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$  

In this basis parity operator is diagonal \(\Pi = \mathrm{Diag}(1, -1)\) whereas the boost operator reads

$$B(p) = \exp(iK \cdot \varphi) = \begin{pmatrix} \cosh(S \cdot \hat{n} \varphi) & \sinh(S \cdot \hat{n} \varphi) \\ \sinh(S \cdot \hat{n} \varphi) & \cos(S \cdot \hat{n} \varphi) \end{pmatrix} \hspace{1cm} (2)$$

$$= \frac{1}{\sqrt{2m(E + m)}} \begin{pmatrix} E + m & \sigma \cdot p \\ \sigma \cdot p & E + m \end{pmatrix}.$$  

Applying the boost operator to Eq. (1) and using \(\Pi B(p) \Pi = B^{-1}(p)\) we obtain

$$[B^2(p) - \eta] \psi(p) = \begin{pmatrix} \cosh(2S \cdot \hat{n} \varphi) - \eta & -\sinh(2S \cdot \hat{n} \varphi) \\ \sinh(2S \cdot \hat{n} \varphi) & \cos(2S \cdot \hat{n} \varphi) - \eta \end{pmatrix} \psi(p)$$

$$= \begin{pmatrix} E - \eta & -\frac{\sigma \cdot p}{m} \\ \frac{\sigma \cdot p}{m} & E - \eta \end{pmatrix} \psi(p) = 0.$$  

If we now define the matrices

$$\gamma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hspace{1cm} \gamma^i \equiv \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix},$$  

this equation can be rewritten in Dirac form

$$[\gamma^\mu p_\mu - \eta m] \psi(p) = 0.$$  \hspace{1cm} (3)

States satisfying this condition for \(\eta = 1\) describes particles while anti-particles correspond to the subspace with \(\eta = -1\). So far we have just identified the \(\gamma^\mu\) matrices and assumed they transform as a four-vector. It can be explicitly shown that indeed these matrices transform as a four-vector by calculating \(\gamma^\nu = B(p) \gamma^\mu B^{-1}(p)\) with the boost operator in Eq.(2).
1.2 Proca equation

The very same philosophy can be used to show that Proca equation is just a projector over the subspaces with well defined (negative) parity of the representation space \((\frac{1}{2},0) \otimes (0,\frac{1}{2})\). Indeed, a basis for the states living in the \((\frac{1}{2},0) \otimes (0,\frac{1}{2})\) (denoted in the following simply as \((\frac{1}{2},\frac{1}{2})\)) is \(\{ |\frac{1}{2}m\rangle_R \otimes |\frac{1}{2}m\rangle_L \}\). In the following we refer to this basis of the representation space as the tensor product basis (TPB). The specific representation of these states in the \(|\frac{1}{2}m\rangle\) basis for the rest frame state reads

\[
|+\rangle_R \otimes |+\rangle_L \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
|+\rangle_R \otimes |-\rangle_L \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},

|\rangle_R \otimes |+\rangle_L \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
|\rangle_R \otimes |-\rangle_L \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

(4)

In the latter equation we used the customary notation \( |\frac{1}{2}\rangle \equiv |+\rangle\) and \( |\frac{1}{2}\rangle \equiv |\rangle\), and the \(L,R\) subindices to denote the different transformation properties under boosts of spinors belonging to \((\frac{1}{2},0)\) and \((0,\frac{1}{2})\) respectively. A general rest frame state residing in the \((\frac{1}{2},0) \otimes (0,\frac{1}{2})\) space can be written as

\[
\phi^{TPB} \equiv \phi_R \otimes \phi_L = l.c.\{ |+\rangle_R \otimes |+\rangle_L, \ |+\rangle_R \otimes |-\rangle_L, \ |-\rangle_R \otimes |+\rangle_L, \ |-\rangle_R \otimes |-\rangle_L \},
\]

where \(l.c.\) stands for linear combination. Notice that under I-parity

\[
\phi_R \otimes \phi_L \rightarrow \phi_L \otimes \phi_R,
\]

thus formally I-parity take us from the \((\frac{1}{2},0) \otimes (0,\frac{1}{2})\) to the \((0,\frac{1}{2}) \otimes (\frac{1}{2},0)\) space, a different one. However, there exists a unitary transformation connecting these spaces thus they are unitarily equivalent. Indeed, in general it is possible to show that

\[
A \otimes B = U(B \otimes A)U^\dagger,
\]

(5)

where \(U\) is a permutation matrix which is also a unitary transformation. For the case at hand, we have the following representation for the TPB of \((0,\frac{1}{2}) \otimes (\frac{1}{2},0)\) representation space

\[
|+\rangle_L \otimes |+\rangle_R \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
|+\rangle_L \otimes |-\rangle_R \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},

|-\rangle_L \otimes |+\rangle_R \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
|-\rangle_L \otimes |-\rangle_R \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

(6)
Thus in this case

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (7)

Clearly, I-parity has the representation \( \Pi^{TPB} = \eta U \) where \( \eta \) is a phase which is restricted to \( \eta = \pm 1 \) because \((\Pi^{TPB})^2 = 1\). Thus the space \((\frac{1}{2}, 0) \otimes (0, \frac{1}{2})\) span an irreducible representation of the Lorentz group (boosts and rotations) extended by parity. The boost and rotation operators for this space can be constructed (in the TPB) as

\[
R(\vec{\theta}) = e^{i\vec{\sigma} \cdot \vec{\theta}} \otimes e^{i\vec{\sigma} \cdot \vec{\theta}}, \quad B(\vec{\phi}) = e^{i\vec{\sigma} \cdot \vec{\phi}} \otimes e^{-i\vec{\sigma} \cdot \vec{\phi}}.
\] (8)

The corresponding generators read

\[
S^{TPB} = \frac{1}{2}(\sigma \otimes 1 + 1 \otimes \sigma), \quad iK^{TPB} = \frac{1}{2}(\sigma \otimes 1 - 1 \otimes \sigma).
\] (9)

In terms of \( E \) and \( \vec{p} \), the boost operator for rest frame states reads

\[
B^{TPB}(\vec{p}) = \frac{1}{2m(p_0 + m)}[p_0 + m + \vec{\sigma} \cdot \vec{p}] \otimes [p_0 + m - \vec{\sigma} \cdot \vec{p}].
\] (10)

Transforming the boost operator under I-parity we obtain

\[
\Pi^{TPB}B(\vec{p})\Pi^{TPB} = B(-\vec{p}).
\] (11)

So far, we have worked in the tensor product basis \( \{|\frac{1}{2}m\rangle_R \otimes |\frac{1}{2}m'\rangle_L\} \) for the \((\frac{1}{2}, 0) \otimes (0, \frac{1}{2})\) space. It is interesting to work in the total angular momentum basis in the rest frame (TAMB) or ”physical basis”. This basis is related to the TPB as \(|TAMB\rangle = M |TPB\rangle\). Explicitly

\[
\begin{pmatrix}
|0, 0\rangle \\
|1, 1\rangle \\
|1, 0\rangle \\
|1, -1\rangle
\end{pmatrix} = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
1 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
|+\rangle_R |+\rangle_L \\
|+\rangle_R |-\rangle_L \\
|-\rangle_R |+\rangle_L \\
|-\rangle_R |-\rangle_L
\end{pmatrix}.
\] (12)

Under this change of basis the operators transform as:

\[
\Pi = M\Pi^{TPB} M^\dagger = \eta \text{Diag}(-1, 1, 1, 1),
\] (13)

\[
S = M S^{TPB} M^\dagger = \begin{pmatrix}
0_{1\times 1} & 0_{1\times 3} \\
0_{1\times 1} & S^{(1)}
\end{pmatrix},
\] (14)

\[
iK = M iK^{TPB} M^\dagger = \begin{pmatrix}
0_{1\times 1} & V^\dagger \\
V & 0_{3\times 3}
\end{pmatrix},
\] (15)

where

\[
V^\dagger_1 = \frac{1}{\sqrt{2}}(-1, 0, 1), \quad V^\dagger_2 = -\frac{i}{\sqrt{2}}(1, 0, 1), \quad V^\dagger_3 = (0, 1, 0),
\]
and $S^{(1)}$ denote the angular momentum operators for spin 1

$$S_1^{(1)} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad S_2^{(1)} = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad S_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{16}$$

In this representation $S^2 = Diag(0, 2, 2)$ and $K^2 = Diag(-3, -1, -1)$. These relations make clear that fields in this representation space in general describe a multiplet composed of a spin zero field and a spin 1 field with opposite intrinsic parities. The choice $\eta = -1$ (which we use in the following) yields $(\Pi)_{\mu\nu} = g_{\mu\nu}$ and with this choice we have a description for a spin 0 field with positive intrinsic parity and a spin 1 field with with negative intrinsic parity.

The boost operator in the "physical basis" reads

$$B^{TAMB}(p) = \begin{pmatrix} E & -p_+ \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3 p_+}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle \\ -p_+ \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & p_3 \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3 p_+}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle \\ \frac{p_3}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \sqrt{2} E+m & \frac{p_3 p_+}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle \\ \frac{p_3}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \frac{p_3 p_+}{E+m} \langle E+m\rangle^2-p_+^2 \langle E+m\rangle & \sqrt{2} E+m \end{pmatrix} \tag{17}$$

Fields transforming in the $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ representation are commonly written in terms of a four-vector $A^\mu$. We establish our results in this basis (hereafter "Cartesian basis" (CB)) also. The CB is related to the TAMB as $|A\rangle = M_{CB}|s, m_s\rangle$. Explicitly

$$\begin{pmatrix} |A^0\rangle \\ |A^1\rangle \\ |A^2\rangle \\ |A^3\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} |0, 0\rangle \\ |1, 1\rangle \\ |1, 0\rangle \\ |1, -1\rangle \end{pmatrix}. \tag{18}$$

In this basis the operators so far discussed read

$$\Pi^{CB} = Diag(1, -1, -1, -1), \tag{19}$$

$$S_1^{CB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad S_2^{CB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad S_3^{CB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

$$iK_1^{CB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad iK_2^{CB} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad iK_3^{CB} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$
and still $S_{CB}^2 = \text{Diag}(0, 2, 2, 2)$, $K_{CB}^2 = \text{Diag}(-3, -1 - 1)$.

The casimir operators of the HLG (actually "internal" HLG, (IHLG) see next section) can be easily calculated from this explicit representations for the IHLG as

$$C_1 = \frac{1}{2}(S^2 - K^2) = \frac{3}{4}I_{4 \times 4}, \quad C_2 = iS \cdot G = 0. \quad (22)$$

It is worth to remark that, in contrast to the $(s, 0) \otimes (0, s)$ representation where $S^2$ is proportional to one of the casimir operators of the IHLG, for the $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ representation space $S^2$ ceases to be a good quantum number and it does not make sense to speak about the spin of fields transforming in this representation except as the eigenvalues of the squared PL operator (see below) which is proportional to $S^2$ only in the rest frame.

The boost operator in the "Cartesian basis" for this representation reads

$$B^{CB}(p) = \begin{pmatrix}
\frac{E}{m} & \frac{p_+}{m} & \frac{p_0}{m} & \frac{p_0}{m} \\
\frac{p_0}{m} & 1 + \frac{p_0 p_+}{m(E+m)} & \frac{p_0}{m} & \frac{p_0}{m} \\
\frac{p_0}{m} & \frac{p_0 p_+}{m(E+m)} & 1 + \frac{p_0 p_+}{m(E+m)} & \frac{p_0}{m} \\
\frac{p_0}{m} & \frac{p_0 p_+}{m(E+m)} & \frac{p_0}{m} & 1 + \frac{p_0^2}{m(E+m)}
\end{pmatrix} \quad (23)$$

The boosted states in each representation can be obtained acting with the corresponding boost operator on the rest frame states. For the "physical basis" we obtain

$$A^{0,0}(p) = \frac{1}{m} \begin{pmatrix} E \\ -p_- \\ p_+ \\ p_3 \end{pmatrix}, \quad A^{1,1}(p) = \frac{1}{m} \begin{pmatrix} \frac{-p_+}{m(E+m)} \\ \frac{(E+m)^2 - p_+^2}{2m(E+m)} \\ \frac{m}{p_3} \\ \frac{m}{E+m} \end{pmatrix} \quad (24)$$

$$A^{1,0}(p) = \frac{1}{m} \begin{pmatrix} \frac{p_3}{E+m} \\ \frac{-p_+ p_3}{E+m} \\ \frac{E+m}{2(E+m)} \\ \frac{E+m}{2(E+m)} \end{pmatrix}, \quad A^{1,-1}(p) = \frac{1}{m} \begin{pmatrix} \frac{p_-}{m(E+m)} \\ \frac{p_+}{m(E+m)} \\ \frac{m}{p_3} \\ \frac{m}{E+m} \end{pmatrix} \quad (25)$$

whereas for the "Cartesian basis" we get

$$A^0(p) = \frac{1}{m} \begin{pmatrix} E \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad A^1(p) = \frac{1}{m(E+m)} \begin{pmatrix} (E+m)p_1 \\ m(E+m) + p_1^2 \\ p_1 \\ p_2 \end{pmatrix}$$

$$A^2(p) = \frac{1}{m(E+m)} \begin{pmatrix} (E+m)p_2 \\ p_1 p_2 \\ m(E+m) + p_2^2 \\ p_2 p_3 \end{pmatrix}, \quad A^3(p) = \frac{1}{m(E+m)} \begin{pmatrix} (E+m)p_3 \\ p_3 p_1 \\ p_3 p_2 \\ m(E+m) + p_3^2 \end{pmatrix}$$
A few remarks concerning the role of parity in the normalization of these states, the completeness relation and the definition of the scalar product in $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ space are in order here. Following the structure of Dirac theory we define the scalar product using the parity operator as

\[(A^a, A^b) \equiv \bar{A}^a A^b \equiv (A^a)\dagger \Pi A^b.\]

It is easily shown that rest frame states satisfy the normalization

\[\bar{A}^a(0) A^b(0) = \eta_a \delta_{ab}\]

where \(\eta_a\) is the parity eigenvalue corresponding to \(A^a\) i.e. \(\eta_0 = 1, \eta_i = -1, i = 1, 2, 3\). We can show that this is a scalar product using Eq.(11). Indeed, for the boosted states we obtain

\[\bar{A}^a(p) A^b(p) = (B(p)A^a(0))\dagger \Pi (B(p)A^b(0)) = A^a(0)\dagger B^\dagger(p) \Pi B(p) A^b(0) \quad (26)\]

\[= A^a(0)\dagger \Pi^2 B(p) \Pi B(p) A^b(0) = \bar{A}^a(0) A^b(0), \quad (27)\]

where we used \(B^\dagger(p) = B(p)\) coming from the anti-hermiticity of \(K\) and Eq.(11).

The projectors over well defined parity (\(\eta\)) subspaces for this representation, in the rest frame, reads

\[\Lambda_{\eta}(0) = \frac{1}{2}(1 + \eta \Pi). \quad (28)\]

For an arbitrary frame this projector transforms as \(\Lambda_{\eta}(p) = B(p)\Lambda_{\eta}(0)B^{-1}(p)\) which can be explicitly calculated using Eq.(23) as

\[\Lambda_+(p) = \frac{1}{2}(1 + B^2(p)\Pi) \equiv \frac{1}{m^2} \begin{pmatrix}
E_1 & -E_2 & -E_3 \\
E_2 & -E_3 & -E_1 \\
E_3 & -E_1 & -E_2
\end{pmatrix}, \quad (29)\]

\[\Lambda_-(p) = \frac{1}{2}(1 - B^2(p)\Pi) \equiv \frac{1}{m^2} \begin{pmatrix}
-E_1 & E_2 & E_3 \\
-E_2 & E_3 & E_1 \\
-E_3 & E_1 & E_2
\end{pmatrix}, \quad (30)\]

or in explicitly covariant terms

\[\Lambda_+(p)_{\mu \nu} = \frac{p_\mu p_\nu}{m^2}, \quad \Lambda_-(p)_{\mu \nu} = \frac{p^2}{m^2} g_{\mu \nu} - \frac{p_\mu p_\nu}{m^2}. \quad (30)\]

The completeness relation can also be worked out in term of the states. Indeed, for the rest frame states we get

\[\sum_a \eta_a A^a(0) \bar{A}^a(0) = A^0 \bar{A}^0 - A^i \bar{A}^i = 1.\]
Again, using $B_t^b(p) = B(p)$ and Eq.(11) it can be shown that this relation is frame independent and the boosted states satisfy the completeness relation
\[
\sum_a \epsilon_a A^a(p) \bar{A}^a(p) = A^0(p) \bar{A}^0(p) - A^i(p) \bar{A}^i(p) = 1. \tag{31}
\]
This relation exhibits the decomposition of the $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ space into Stuckelberg and Proca sectors when we use the "Cartesian basis" to represent the states. Indeed, in this basis $A^0_\mu = \frac{p_\mu}{m}$ thus the $A^0$ state is divergenceful whereas the remaining states satisfy Lorentz condition $p^\mu A^i_\mu = 0$. Furthermore, $A^0_\mu \bar{A}^0_\nu = \frac{p_\mu p_\nu}{m^2}$ and since $\Pi^{CB}_{\mu\nu} = g_{\mu\nu}$, Eq.(31) can be rewritten to
\[
\sum_i A^i_\mu(p)(A^i_\nu)^\dagger(p) = -\frac{p^2}{m^2} g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}. \tag{32}
\]
Notice that the projector over the negative parity sector
\[
[A_-(p)]_{\mu\nu} A^\nu = A_\mu,
\]
is just Proca equation in momentum space, namely
\[
p_\mu F^{\mu\nu} - m^2 A^\nu = 0,
\]
where $F^{\mu\nu} \equiv p^\mu A^\nu - p^\nu A^\mu$ denotes the strength tensor.

2 Projectors over invariant subspaces of Casimir operators of the Poincaré group.

In general, a free field will satisfy equations of motion which just reflect the representation space to which it belongs, and the discrete symmetries we impose on it. Since the primary classification of elementary systems is usually done by identifying them with the irreps of the Poincaré group there must be conditions (equations of motion) associated to this classification. In order to make clear our point we briefly recall the transformation properties of fields under the action of the Poincaré group.

A Poincaré transformation in coordinate space
\[
x'_\mu = \Lambda_\mu^\nu x_\nu + a_\mu,
\]
\[
\Lambda_\mu^\nu = \exp \left[ -\frac{i}{2} \theta^{\mu\nu} L_{\mu\nu} \right], \quad L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu.
\]
induces the following transformation for the field $\psi$
\[
\psi'(x') = \exp \left[ -\frac{i}{2} \theta^{\mu\nu} M_{\mu\nu} + e^\mu P_\mu \right] \psi(x).
\]
Here, $\epsilon^\mu, \theta^\mu$ are continuous parameters, $P_\mu$ and the $n \times n$ matrices $M_{\mu\nu}$ representing a totally antisymmetric 2nd rank Lorentz tensor are the generators of the Poincaré group in the representation space of interest. They satisfy the commutation relations of the Poincaré algebra:

\[
[M_{\mu\nu}, M_{\alpha\beta}] = -i(g_{\mu\alpha}M_{\nu\beta} - g_{\mu\beta}M_{\nu\alpha} + g_{\nu\beta}M_{\mu\alpha} - g_{\nu\alpha}M_{\mu\beta}), \quad (33)
\]

\[
[P_\mu, M_{\alpha\beta}] = i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha), \quad [P_\mu, P_\nu] = 0,
\]

where $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric tensor. In the standard convention, $P_\mu$ are the generators of the translation group, $T_{1,3}$ in 1+3 time-space dimensions. The $M_{\mu\nu}$ generators consist of

\[
M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad [L_{\mu\nu}, S_{\mu\nu}] = 0,
\]

where $L_{\mu\nu}$, and $S_{\mu\nu}$ in turn generate rotations in external coordinate– and internal representation spaces. The generators of boosts ($K_x, K_y, K_z$) and rotations ($J_x, J_y, J_z$) are related to $M_{\mu\nu}$ via

\[
K_i = M_{0i}, \quad J_i = \frac{1}{2}\epsilon_{ijk}M_{jk},
\]

respectively.

The Pauli–Lubanski (PL) vector is defined as

\[
W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M^{\nu\alpha}P^\beta \quad (34)
\]

where $\epsilon_{0123} = 1$. The eigenvalues of the squared PL operator are easily calculated in the rest frame as $-p^2s(s+1)$ where $s$ stands for an integer or semi-integer positive number. This operator can be shown to have the following commutator relations:

\[
[W_\alpha, M_{\mu\nu}] = i(g_{\alpha\mu}W_\nu - g_{\alpha\nu}W_\mu), \quad [W_\alpha, P_\mu] = 0, \quad (35)
\]

i.e. it transforms as a four-vector under Lorentz transformations. The remarkable point is that the “orbital” part of $M_{\mu\nu}$, namely $L_{\mu\nu}$, does not contribute to the PL operator due to the anti-symmetric Levi-Civita tensor. As a result, $W_\mu$ can be rewritten to

\[
W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\tau}S^{\nu\rho}P^\tau, \quad (36)
\]

and its squared (in covariant form) is calculated to be\[11\]

\[
W^2 = -\frac{1}{2}S_{\mu\nu}S^{\mu\nu}P^2 + G^2, \quad G_\mu := S_{\mu\nu}P^\nu. \quad (37)
\]

The operators $S_{\mu\nu}$ generate Lorentz transformations in the intrinsic representation space and satisfy the algebra

\[
[S_{\mu\nu}, S_{\alpha\beta}] = -i(g_{\mu\alpha}S_{\nu\beta} - g_{\mu\beta}S_{\nu\alpha} + g_{\nu\beta}S_{\mu\alpha} - g_{\nu\alpha}S_{\mu\beta}), \quad [P_\mu, S_{\alpha\beta}] = 0,
\]
i.e., they commute with $T_{1,3}$ and satisfy also the algebra of the HLG, but this is a set of transformations which are different from those of the HLG generated by $M^{\mu\nu}$. Hereafter we will refer to the group generated by $S_{\mu\nu}$ as the “Internal Homogeneous Lorentz Group” (IHLG) to distinguish it from the HLG spanned by $M_{\mu\nu}$. The IHLG generators are:

$$K_i = S_{0i}, \quad S_i = \frac{1}{2}\epsilon_{ijk}S_{jk}.$$  \hfill (38)

In terms of intrinsic boost and rotation generators, the Pauli-Lubanski vector and the $G_{\mu}$ vectors are now expressed as

$$W_\mu = (-S \cdot P, -SP_0 + K \times P), \quad G_\mu = (K \cdot P, -KP_0 - S \times P),$$ \hfill (39)

On the other hand, the IHLG has by itself two Casimir invariants given by

$$C_1 = \frac{1}{4}S_{\mu\nu}S^{\mu\nu}; \quad C_2 = S_{\mu\nu}\tilde{S}^{\mu\nu},$$

where $\tilde{S}_{\mu\nu} = \epsilon_{\mu\nu\rho\tau}S^{\rho\tau}$. Explicitly, in terms of the generators of boosts and rotations we obtain

$$C_1 = \frac{1}{2}(S^2 - K^2), \quad C_2 = iS \cdot K,$$ \hfill (40)

which allow us to cast $W^2$ into the form

$$W^2(P) = -2C_1P^2 + G(P)^2,$$ \hfill (41)

where we explicitly wrote the $P$-dependence of $W^2$ and $G$.

Now, if we adopt the point of view that the equation of motion satisfied by a field is just as a kinematical statement of the representation space it belongs to [6, 7], the identification of elementary systems with the irreps of the Poincaré group lead us to two primary conditions for any field: it must belong to the invariant subspaces of both $P^2$ and $W^2$. These subspaces are characterized by the corresponding quantum numbers $\{m^2, s\}$ and the belonging of the field to a given representation is ensured by the corresponding projector. The first condition lead us to Klein-Gordon equation

$$[P^2 - m^2]\psi(p) = 0,$$ \hfill (42)

whereas the restriction of fields to invariant subspaces of the squared Pauli-Lubanski operator lead us to a new condition, namely

$$[W^2(P) + P^2s(s+1)]\psi(p) = 0.$$ \hfill (43)

These are the two conditions to be satisfied by any field. The information on the particular representation space to which the field belongs, is encoded in the specific form of the generators $S^{\mu\nu}$ entering $W^2(P)$. We emphasize that the latter equation is a general condition which just specify the value of $s$. More

\footnote{This equation was firstly proposed for the Rarita-Schwinger representation in the second paper of Ref.[11] with $P^2$ replaced by $m^2$.}
stringent conditions are obtained when we enforce definite discrete properties like parity in these subspaces.

It would be desirable to work with a single equation which incorporates the information of both Eqs. (42, 43). This fusion must be done in such a way that enforcing Eq. (43) ensures the Klein-Gordon condition is automatically satisfied. The corresponding equation is obtained by inspection as

\[ \frac{W^2(P)}{s} + sP^2 + m^2 \psi(p) = 0. \]  (44)

As shown below, this particular combination gives the correct gyromagnetic factor \( g = 2 \) for \( s = \frac{1}{2} \) and in general predicts \( g = \frac{1}{s} \) for arbitrary \( s \).

3 Gyromagnetic factor for fields in the \((s, 0) \oplus (0, s)\) representation.

Coupling Eq. (44) minimally to an external electromagnetic field \( A_\mu \) we obtain

\[ \frac{W^2(\pi)}{s} + s\pi^2 + m^2 \psi(p) = 0, \]  (45)

where \( \pi_\mu = P_\mu - eA_\mu \). A straightforward calculation yields the general relation

\[ W^2(\pi) \equiv \frac{1}{4} \epsilon_{\mu\nu\rho\tau} S^{\mu\nu} \pi^\tau \epsilon_{\beta\gamma\delta} S^{\beta\gamma} \pi^\delta = -\frac{1}{2} S_{\mu\nu} S^{\mu\nu} \pi^2 + S_{\mu\beta}(S^{\mu\nu} \pi_\nu) \pi^\beta. \]  (46)

Let us now specialize the above relations to the \((s, 0) \oplus (0, s)\) representation. In this case, \( S = i \Gamma_0 K \) with \( \Gamma_0 = \text{Diag}(1_{(2s+1)(2s+1)}, -1_{(2s+1)(2s+1)}) \). The matrix \( \Gamma_0 \) satisfy \( \Gamma_0^2 = 1_{2(2s+1)(2s+1)} \) and \( [\Gamma_0, S] = 0 \), thus \( K^2 = -S^2 \), hence

\[ -\frac{1}{2} S_{\mu\nu} S^{\mu\nu} = -2S^2 = -2s(s+1)1_{2(2s+1)(2s+1)}. \]  (47)

As to the second term in Eq. (46) it is convenient to separate the product of generators as

\[ S^{\mu\nu} S_{\mu\alpha} = \frac{1}{2} \{ S^{\mu\nu}, S_{\mu\alpha} \} + \frac{1}{2} [S^{\mu\nu}, S_{\mu\alpha}] = T^\nu_\alpha - i S^\nu_\alpha \]

where we defined the symmetric part \( T^\nu_\alpha \equiv \frac{1}{2} \{ S^{\mu\nu}, S_{\mu\alpha} \} \) and used the Lie algebra of the ZHHLG to calculate the anti-symmetric part. The anti-symmetric part does not contribute in the free case but under minimal coupling this term generates the magnetic interaction. A straightforward calculation using the generators for the \((s, 0) \oplus (0, s)\) space yields \( T^\nu_\alpha = S^2 \gamma^\nu_\alpha \). As a result, for these representations we obtain

\[ S_{\mu\beta} S^{\mu\nu} = S^2 g_\beta^{\, \nu} - i S_\beta^{\, \nu} = s(s+1)1_{2(2s+1)(2s+1)} g_\beta^{\, \nu} - i S_\beta^{\, \nu}. \]  (48)
Using Eqs.(48,47) in the equation of motion (45) we finally obtain

\[
(\pi^2 - m^2) \psi(p) = \frac{i}{s} S^{\mu\nu} \pi_\mu \pi_\nu \psi(p).
\]  \hspace{1cm} (49)

The term on the right hand side of this equation contains the magnetic interaction which for general \( s \) gives the gyromagnetic factor \( g_s = \frac{1}{s} \).

3.1 Summary

Summarizing, in this work we adopt the point of view that equations of motion are just a Lorentz invariant record of the representation space to which the field belongs, and the discrete symmetries we impose on it. We illustrate the point re-deriving Dirac and Proca equations from first principles. We remark that under this view point any field must satisfy two primary conditions (equations): Klein-Gordon equation and a new equation of motion related to eigenvalues of the squared Pauli-Lubanski operator. The latter is a very general condition in the sense that none discrete symmetry is imposed. We incorporate both equations into a single one and couple minimally to an external electromagnetic field finding a spin gyromagnetic factor \( g_s = \frac{1}{s} \). This way, from first principles, we prove Belinfante’s conjecture [3] for fields transforming in the \((s,0) \oplus (0,s)\) representation space.

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