A SHORT PROOF OF HULANICKI’S THEOREM

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Abstract. We outline a simple proof of Hulanicki’s theorem, that a locally compact group is amenable if and only if the left regular representation weakly contains all unitary representations. This combines some elements of the literature which have not appeared together, before.

Let $G$ be a locally compact group. For $p = 1, 2, \infty$ we let $L^p(G)$ denote the $L^p$-space with respect to the left Haar measure $m\,(dx = dm(x))$. For a unitary representation $\pi : U(H)$ (we assume continuous with respect to weak or strong operator topology on unitary operators $U(H)$), the integrated form is given by $\pi : L^1(G) \to B(H)$, $\pi(f) = \int_G f(x)\pi(x)\,dx$ (integral understood in weak operator sense), and is well-known to be contractive and satisfy $\pi(f^*) = \pi(f)^*$. The left regular representation $\lambda : G \to U(L^2(G))$ is given by $\lambda(x)f = x*f$, where $x*f(y) = f(x^{-1}y)$ for $m$-a.e. $y$ in $G$. Its integrated form is given by $\lambda : L^1(G) \to B(L^2(G))$, $\lambda(f)h = f*h$.

We recall that $G$ is amenable if $L^\infty(G)$ admits a left invariant mean, a linear functional $\mu$ such that $\mu(\varphi) \geq 0$ provided $\varphi \geq 0$, $\mu(1) = 1$, and $\mu(x*\varphi) = \mu(\varphi)$ for all $x$ in $G$ and $\varphi$ in $L^\infty(G)$. This is well-known to be equivalent to Reiter’s condition $(P_1)$, i.e. the existence of a Reiter net $(r_\alpha)$ in $L^1(G)$: each $r_\alpha \geq 0$ m-a.e., $\int_G r_\alpha \,dm = 1$ and $\lim_\alpha \|x*r_\alpha - r_\alpha\|_1 = 0$ uniformly for $x$ in compact sets. See §3.2 of the classic book of Greenleaf [5].

Theorem 0.1. (Hulanicki [6,7]) A locally compact group $G$ is amenable if and only if for any unitary representation $\pi : G \to U(H)$, we have $\|\lambda(f)\| \geq \|\pi(f)\|$ for all $f$ in $L^1(G)$.

The necessity condition above is the property that $\lambda$ weakly contains all unitary representations. This is equivalent to having, for every $\pi,$
the existence of a representation \( \pi_\lambda : C^*_\lambda \to C^*_\pi \) for which \( \pi_\lambda \circ \lambda = \pi \)
on
on \( L^1(G) \), where \( C^*_\pi = \overline{\pi(L^1(G))} \) (norm closure in \( B(H) \)). None of the
on elements of this direction of the proof are novel, but are combined in a
on manner which does not seem to appear in the existant literature.

**Proof of necessity.** If \( (r_\alpha) \) is a Reiter net in \( L^1(G) \), then \( k_\alpha = r_\alpha^{1/2} \)
non defines a net in \( L^2(G) \) such that matrix coefficients \( \langle \pi(\cdot)k_\alpha, k_\alpha \rangle \) tend
on uniformly on compacta to 1. (This is the easy direction of Theorem 3.5.2 in [5].) Since compactly supported elements are dense in \( L^2(G) \),
we conclude that amenablity of \( G \) is entails having a net \( (u_\alpha) \) of compactly supported positive definite elements which converges uniformly
on compacta to 1. A theorem of Godement [4] (see 13.8.6 the book
on of Dixmier [3]) shows that any compactly supported positive definite
on function on \( G \) is of the form \( \langle \pi(\cdot)h, h \rangle \) for some \( h \) in \( L^2(G) \).

Let \( f \in L^1(G) \). By density of such elements, we may assume that \( f \)
non is compactly supported. Given a unitary representation \( \pi \) and \( \varepsilon > 0 \),
let \( \xi \) be a unit vector in \( H \) for which \( \| \pi(f)\xi \|^2 + \varepsilon > \| \pi(f) \|^2 \). Then
given the net \( (u_\alpha) \), promised above, we find elements \( h_\alpha \) in \( L^2(G) \) for
which

\[ \langle \lambda(\cdot)h_\alpha, h_\alpha \rangle = \langle \pi(\cdot)\xi, \xi \rangle u_\alpha \to \langle \pi(\cdot)\xi, \xi \rangle \text{ uniformly on compacta} \]

so we compute

\[ \| \pi(f) \|^2 - \varepsilon < \int_G f^* f(x) \langle \pi(x)\xi, \xi \rangle \, dx \]

\[ = \lim_\alpha \int_G f^* f(x) \langle \pi(x)\xi, \xi \rangle u_\alpha(x) \, dx \]

\[ = \lim_\alpha \int_G f^* f(x) \langle \lambda(x)h_\alpha, h_\alpha \rangle \, dx \]

\[ \leq \lim_\alpha \| \lambda(f) \|^2 \langle \lambda(e)h_\alpha, h_\alpha \rangle = \| \lambda(f) \|^2 \]

which establishes the desired inequality. \( \square \)

To prove the sufficiency condition, we shall use a specialization of a
*multiplicative domain* result of Choi [2] to states. We recall that a state
on a unital C*-algebra \( B \) is any functional \( \tau \) which satisfies \( \tau(B^*B) \geq 0 \)
for \( B \) in \( B \) and \( \tau(I) = 1 \).

**Proposition 0.2.** Let \( B \) be a unital C*-algebra and \( M \) a unital C*-
subalgebra of \( B \). Suppose a state \( \tau \) on \( B \) satisfies \( \tau(A^*A) = |\tau(A)|^2 \)
for \( A \) in \( M \). Then \( \tau \) satisfies \( \tau(AB) = \tau(A)\tau(B) = \tau(BA) \) for \( A \) in \( M \)
and \( B \) in \( B \).
Proof. Let \((\mathcal{H}, \pi, \xi)\) be the Gelfand-Naimark-Segal triple associated with \(\tau\), i.e. \(\tau = \langle \pi(\cdot)\xi, \xi \rangle\). Let \(P = \langle \cdot, \xi \rangle\), so \(P\) is the orthogonal projection onto \(C\xi\), and \(P\pi(B)\xi = \tau(B)\xi\) for \(B\) in \(\mathcal{B}\). Then, if \(A \in \mathcal{M}\) we have

\[
\| (I - P)\pi(A)\xi \|^2 = \langle (I - P)\pi(A)\xi, \pi(A)\xi \rangle = \tau(A^*A) - \tau(A)\overline{\tau(A)} = 0.
\]

Hence \(C\xi\) is \(\pi(\mathcal{M})\)-invariant, and it follows that \(\pi(A)\xi = \tau(A)\xi\) for \(A\) in \(\mathcal{M}\). Hence, if \(B \in \mathcal{B}\) we have

\[
\tau(AB) = \langle \pi(B)\xi, \pi(A^*)\xi \rangle = \tau(A)\tau(B)
\]

and, similarly, \(\tau(BA) = \tau(A)\tau(B)\). \(\square\)

The technique below has been observed for discrete groups in the book of Brown and Ozawa [1]. The author is unsure of the origin of this trick for the purposes of this theorem.

Proof of sufficiency condition of Theorem [1,4]. Let \(M(G)\) be the measure algebra of \(G\), in which \(L^1(G)\) is the ideal of elements absolutely continuous with respect to \(m\). Consider the augmentation characters

\[
\alpha : L^1(G) \to \mathbb{C}, \; \alpha(f) = \int_G f \, dm
\]

\[
\tilde{\alpha} : M(G) \to \mathbb{C}, \; \tilde{\alpha}(\mu) = \mu(G) = \int_G 1 \, d\mu
\]

which are \(\ast\)-homomorphisms with \(\tilde{\alpha}|_{L^1(G)} = \alpha\). Hence, our assumptions entail that

\[
|\alpha(f)| \leq \|\lambda(f)\| \text{ for } f \in L^1(G).
\]

Now, fix \(f\) in \(L^1(G)\) with \(f \geq 0\) \(m\)-a.e. and \(\alpha(f) = 1\), so \(\|f\|_1 = 1\). Then if \(\mu \in M(G)\) we have

\[
|\tilde{\alpha}(\mu)| = |\alpha(\mu * f)| \leq \|\lambda(\mu * f)\| \leq \|\lambda(\mu)\|
\]

where \(\lambda(\mu)h = \mu * h\) for \(h\) in \(L^2(G)\). Hence \(\tilde{\alpha}\) extends to a multiplicative functional \(\tau\) on \(M^*_\lambda = \overline{\lambda(M(G))}\), satisfying \(\tau(\lambda(\mu)) = \tilde{\alpha}(\mu)\). This is clearly a state which satisfies \(\tau(A^*A) = |\tau(A)|^2\) on \(M^*_\lambda\). Let \(\tilde{\tau}\) denote any norm preserving extension to \(B(L^2(G))\).

We let \(M : L^\infty(G) \to B(L^2(G))\) denote the \(\ast\)-homomorphism into multiplication operators: \(M(\varphi)h = \varphi h\). It is well known, and standard to verify, that \(\lambda(x)M(\varphi)\lambda(x^{-1}) = M(x \ast \varphi)\). Since \(\lambda(x) = \lambda(\delta_x)\) (Dirac measure at \(x\)), we see from Proposition [1,2] that

\[
\tilde{\tau} \circ M(x \ast \varphi) = \tilde{\tau}(\lambda(\delta_x)M(\varphi)\lambda(\delta_{x^{-1}})) = \tilde{\alpha}(\delta_x)\tilde{\tau} \circ M(\varphi)\tilde{\alpha}(\delta_{x^{-1}}) = \tilde{\tau} \circ M(\varphi).
\]

Hence \(\mu = \tilde{\tau} \circ M\) is a left invariant mean on \(L^\infty(G)\). \(\square\)
The only claim of originality made by the author is the transferring of the multiplicative domain technique of Theorem 2.6.8 of [1] to $M_{\lambda}^*$ in the proof of sufficiency.

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