A Reply to “Comment on ‘Big Bang Nucleosynthesis and Active-Sterile Neutrino Mixing: Evidence for Maximal $\nu_\mu \leftrightarrow \nu_\tau$ Mixing in Super Kamiokande’”

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In the paper “Big Bang Nucleosynthesis and Active-Sterile Neutrino Mixing: Evidence for Maximal Muon-Neutrino/Sterile-Neutrino Mixing in Super Kamiokande” (http://xxx.lanl.gov/abs/astro-ph/9810075), we suggested that to evade the Big Bang Nucleosynthesis exclusion of the muon neutrino to sterile neutrino oscillation explanation of the Super Kamiokande data, the tau neutrino must have a mass over about 15 eV and it must mix with a lighter sterile neutrino. A stable tau neutrino with this mass is inconsistent with cosmological structure formation. In a comment on our paper (http://xxx.lanl.gov/abs/astro-ph/9811067), Foot and Volkas argued that our result is incorrect and that the required tau neutrino mass should be much lower. Here we back up our original result with a more detailed calculation. We show that the argument of Foot and Volkas is invalid, most likely due to an insufficient energy resolution in the low energy part of the neutrino spectrum.

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The Super Kamiokande atmospheric neutrino data have provided evidence for muon neutrino oscillation as a result of a maximal or near maximal mixing between \( \nu_\mu \) and either \( \nu_\tau \) or a sterile neutrino \( \nu_s \) [1]. The mass-squared-difference involved in the mixing is \( \sim 10^{-3} \text{ eV}^2 \) [1]. The maximal or near maximal \( \nu_\mu \leftrightarrow \nu_s \) mixing explanation, however, would violate the Big Bang Nucleosynthesis (BBN) bound by nearly completely equilibrating a fourth neutrino flavor \( \nu_s \) in the early universe and, hence, increasing the primordial \( ^4\text{He} \) yield above the observed level [2,3]. One mechanism proposed to evade this bound is to invoke a large lepton number asymmetry (\( \gtrsim 10^{-5} \)) at the BBN epoch which acts to suppress the \( \nu_\mu \leftrightarrow \nu_s \) transformation by matter effects. A clever scheme along these lines suggested by Foot and Volkas [4] is to have a massive \( \nu_\tau \) mix with a lighter sterile neutrino \( \nu_{s'} \) (which could in principle be the same \( \nu_s \)). The resonant transformation of tau neutrinos to sterile neutrinos via matter-enhanced mixing at the BBN epoch would generate a tau lepton number asymmetry \( L_{\nu_\tau} \) that grows with time [4,5]. The \( \nu_\tau \leftrightarrow \nu_{s'} \) resonant transformation must occur before any significant \( \nu_\mu \leftrightarrow \nu_s \) transformation can occur, so that the \( L_{\nu_\tau} \) generated can subsequently suppress the \( \nu_\mu \leftrightarrow \nu_s \) transformation. However, one twist in this scheme is that while \( L_{\nu_\tau} \) grows, it crosses a parameter region where the MSW (Mikheyev-Smirnov-Wolfenstein) matter effect causes either \( \nu_\mu \) (if \( L_{\nu_\tau} > 0 \)) or \( \bar{\nu}_\mu \) (if \( L_{\nu_\tau} < 0 \)) to resonantly transform into \( \nu_s \) or its antiparticle [2,5]. The result is a newly generated \( L_{\nu_\mu} \), with a sign opposite to that of \( L_{\nu_\tau} \). This \( L_{\nu_\mu} \) acts to counter the suppression effect of \( L_{\nu_\tau} \) on the \( \nu_\mu \leftrightarrow \nu_s \) transformation. This is because the matter-antimatter asymmetry contribution to the effective potential of the \( \nu_\mu \leftrightarrow \nu_s \) system is \( \propto 2L_{\nu_\mu} + L_{\nu_\tau} \) [2]. The countering effect of \( L_{\nu_\mu} \) is immaterial only if \( L_{\nu_\tau} \) is sufficiently large at the time of the \( \nu_\mu \) or \( \bar{\nu}_\mu \) resonant conversion. In turn, this requires the mass-squared-difference of the \( \nu_\tau \leftrightarrow \nu_{s'} \) mixing to satisfy \( m_{\nu_\tau}^2 - m_{\nu_{s'}}^2 \gtrsim 300 \text{ eV}^2 \), which implies \( m_{\nu_\tau} \gtrsim 15 \text{ eV} \) [2]. Tau neutrinos this massive are incompatible with cosmological structure formation, and therefore would have to be unstable [2].

In papers and in a comment to our aforementioned paper, Foot et al. investigated a
similar problem and argued for a much lower limit $m_{\nu_e}^2 - m_{\nu_s}^2 \gtrsim 16 \text{eV}^2$\cite{3}. The difference between the required threshold $\nu_\tau$ mass in our calculations and theirs may stem from differences in the energy resolution employed. Namely, we believe that an accurate treatment of the counter-suppression effect of $L_{\nu_\mu}$ requires a very fine energy resolution in the low energy part of the mu neutrino spectrum.

We agree with Foot and Volkas that $L_{\nu_\tau}$ grows exponentially at the initial stage of lepton number generation but then subsequently approaches an asymptotic $T^{-4}$ growth (where $T$ is the temperature of the universe). Foot and Volkas argued that since most of $\nu_\mu$ or $\bar{\nu}_\mu$ underwent resonances during the exponential growth phase, there was not enough time for any significant generation of $L_{\nu_\mu}$. Our results, on the other hand, show that the growth rate of $L_{\nu_\tau}$ is not the central issue in the problem. Rather, since $L_{\nu_\tau}$ itself is small, especially during its initial phase of exponential growth (e.g., $\lesssim 10^{-8}$), even the resonant conversion of a tiny fraction of $\nu_\mu$ or $\bar{\nu}_\mu$ into sterile neutrinos may generate a competing and significant $L_{\nu_\mu} \approx -L_{\nu_\tau}/2$. Once such an $L_{\nu_\mu}$ has been generated, the potential responsible for the $\nu_\mu \leftrightarrow \nu_s$ transformation is driven close to zero, rendering the suppression from $L_{\nu_\tau}$ ineffective. Therefore, in order to suppress the $\nu_\mu \leftrightarrow \nu_s$ mixing required for Super Kamiokande, the $\nu_\tau \leftrightarrow \nu_{s'}$ mixing must generate an $L_{\nu_\tau}$ that is much larger than its induced $L_{\nu_\mu}$ at any moment.

There are two resonances involved in the problem: the $\nu_\tau \leftrightarrow \nu_{s'}$ resonance that generates $L_{\nu_\tau}$; and the $L_{\nu_\tau}$-induced $\nu_\mu \leftrightarrow \nu_s$ resonance that generates a competing $L_{\nu_\mu}$. Because the effective potentials of active-sterile neutrino mixings are neutrino energy dependent, at any given temperature each resonance occurs only in a narrow energy bin in the neutrino energy spectra. As the temperature of the universe drops, a particular resonance energy bin gradually moves to higher neutrino energy, eventually sweeping across the entire neutrino energy spectrum. The narrow widths of the resonance energy bins and the rapid decrease of the effective mixing angles outside the resonance energy bins enable a simple analytical/semi-analytical calculation.

In our semi-analytical numerical approach, we track the neutrino transformations in
the resonant parts of the neutrino energy spectra. Again, these resonant regions in energy space consist of only narrow energy bins and are functions of the temperature and the lepton number asymmetry. This semi-analytical approach offers two distinctive advantages: a very high energy resolution of the neutrino spectrum \((\Delta E/T \lesssim 10^{-6})\); and ease in understanding the physical processes involved, especially the interplay between the two coupled mixing systems. A high energy resolution is essential because the initial exponential growth of \(L_{\nu_e}\) comes from minuscule differences between the resonance energies of the \(\nu_e \leftrightarrow \nu_{s'}\) system and the \(\bar{\nu}_e \leftrightarrow \bar{\nu}_{s'}\) system, and the tiny \(L_{\nu_e}\) at this stage can be easily matched by a competing \(L_{\nu_{\mu}}\). Energy bins that are too coarse may therefore result in an incorrect account of the \(L_{\nu_{\mu}}\) growth and an omission of the counter-balancing effect of \(L_{\nu_{\mu}}\) when \(L_{\nu_{e}}\) is small.

Figure 1 shows our result based on the same mixing parameters assumed in the figure of Foot and Volkas’ comment (astro-ph/9811067) to our paper. For the purpose of comparison, no simultaneous \(\nu_{\mu} \leftrightarrow \nu_s\) transformation is assumed in this figure. Figure 1 is for the most part similar to that of Foot and Volkas. There are minor differences that are readily identifiable: (1) our result tracks \(T^{-4}\) more closely in the “power-law growth” epoch; (2) \(L_{\nu_{e}}\) in our results does not switch sign at the initial point of growth. The sign difference is not surprising because of the chaotic character of the growth, which introduces a sign ambiguity to the problem [3].

As an illustration, also plotted in Figure 1, are the \(|L_{\nu_{e}}|\) required for the \(\nu_{\mu} \leftrightarrow \nu_s\) or \(\bar{\nu}_{\mu} \leftrightarrow \bar{\nu}_s\) resonance to occur at \(\nu_{\mu}\) or \(\bar{\nu}_{\mu}\) energies \(\epsilon_{\text{res}} \equiv p_{\text{res}}(\nu_{\mu})/T = 0.01, 1, 10\). (The parameters for the \(\nu_{\mu} \leftrightarrow \nu_s\) mixing are \(\delta m^2 = 10^{-3}\) eV\(^2\) and \(\sin 2\theta = 1\).) It can be seen from the figure that the lower energy component of the \(\nu_{\mu}\) or \(\bar{\nu}_{\mu}\) neutrinos encounters the resonance first when \(|L_{\nu_{e}}|\) is very small, and the resonance region moves through the \(\nu_{\mu}\) or \(\bar{\nu}_{\mu}\) spectrum to higher neutrino energies as \(|L_{\nu_{e}}|\) becomes larger.

For the potential proportional to \(L_{\nu_{e}} + 2L_{\nu_{\mu}}\) to successfully suppress the \(\nu_{\mu} \leftrightarrow \nu_s\) transformation, the \(L_{\nu_{\mu}}\) generated by the \(\nu_{\mu}\) or \(\bar{\nu}_{\mu}\) resonance has to be much smaller than \(L_{\nu_{e}}\) in magnitude at any temperature as the \(\nu_{\mu}\) or \(\bar{\nu}_{\mu}\) resonance sweeps through the entire spectrum,
\[ f(\epsilon_{\text{res}}) \delta \epsilon_{\text{res}} R \left| \frac{\delta \epsilon_{\text{res}}}{\epsilon_{\text{res}}} \right| < \frac{4}{3} (L_{\nu_\tau} + 2L_{\nu_\mu}) \]  
\( (1) \)

for any \( \epsilon_{\text{res}} \). In the equation, \( f \) is the Fermi-Dirac distribution function. \( \delta \epsilon_{\text{res}} \) is the energy width of the resonance. \( f(\epsilon_{\text{res}}) \delta \epsilon_{\text{res}} \) is therefore the fraction of mu neutrinos in resonance. \( R \) is the resonant transition rate. \( |\delta \epsilon_{\text{res}}/\epsilon_{\text{res}}| \) is the duration of the resonance at \( \epsilon_{\text{res}} \). The energy width of the resonance depends on whether the resonant transition is collision-dominated (with the quantum damping rate \( D \sim 0.5G_F^2 T^5 \epsilon_{\text{res}} > V_x \)) or oscillation-dominated \((D < V_x)\):

\[ \delta \epsilon_{\text{res}} \sim \begin{cases} 2 |D \partial \epsilon_{\text{res}}/\partial V_x| & \text{if } D > V_x \\ 2 |V_x \partial \epsilon_{\text{res}}/\partial V_x| & \text{if } D < V_x \end{cases} \]  
\( (2) \)

So does the resonant transition rate:

\[ R \approx \begin{cases} V_x^2/D & \text{if } D > V_x \\ V_x & \text{if } D < V_x \end{cases} \]  
\( (3) \)

The effective potentials for the \( \nu_\mu \leftrightarrow \nu_\tau \) transformation are \[ V_x = \frac{|m_{\nu_\mu}^2 - m_{\nu_\tau}^2|}{2\epsilon T} \sin 2\theta, \quad V_z \approx 22G_F^2 T^5 \epsilon \pm 0.35G_FT^3 (L_{\nu_\tau} + 2L_{\nu_\mu}), \]  
\( (4) \)

where \( G_F \) is the Fermi constant, the “+” (“−”) sign is for \( \nu_\mu \) (\( \bar{\nu}_\mu \)), and we employ natural units. (Here \( V_y = 0 \).) The \( \nu_\mu \) or \( \bar{\nu}_\mu \) resonance occurs at an energy \( \epsilon_{\text{res}} \approx (L_{\nu_\tau} + 2L_{\nu_\mu})/63G_FT^2 \), with \( \partial \epsilon_{\text{res}}/\partial V_x \approx (22G_F^2 T^5)^{-1} \). The temperature \( T_{\text{res}} \) at which the resonance occurs is almost independent of \( \epsilon_{\text{res}} \): \( T_{\text{res}} \approx 22[(m_{\nu_\tau}^2 - m_{\nu_\mu}^2)/1 \text{ eV}^2]. \) \( (T_{\text{res}} \text{ is an insensitive function of the } \nu_\tau \leftrightarrow \nu_\mu \text{ vacuum mixing angle.}) \) Furthermore, we only consider the case where \( \epsilon_{\text{res}} \ll 1 \).

This is when \( L_{\nu_\tau} \) is in its initial stage of growth \((L_{\nu_\tau} \ll 10^{-7})\) and is most easily matched by a competing \( L_{\nu_\mu} \). Given the small \( \epsilon_{\text{res}} \) we have \( f(\epsilon_{\text{res}}) \approx \epsilon_{\text{res}}^2/1.8 \). Eq. \( (1) \) can then be rewritten in the form

\[ \frac{|m_{\nu_\mu}^2 - m_{\nu_\tau}^2|^2}{1600G_F^2 T_{\text{res}}^7 |\dot{\epsilon}_{\text{res}}|} < 80G_FT_{\text{res}}^2, \]  
\( (5) \)

if \( D > V_x \), or
\[
\frac{|m^2_{\nu_{\mu}} - m^2_{\nu_{s'}}|^3}{1800G_F^4 T_{\text{res}}^2 |\dot{\epsilon}_{\text{res}}|} < 80G_F T_{\text{res}}^2.
\]

(6)

if \(D < V_x\).

We can further rewrite \(|\dot{\epsilon}_{\text{res}}| \equiv H_{\text{res}}|\ln \epsilon_{\text{res}}/\ln T|\) where \(H = -d\ln T/dt \approx 5.5T^2/m_{\text{pl}}\) is the Hubble expansion rate. The Planck mass is \(m_{\text{pl}} \approx 1.22 \times 10^{28}\) eV. Then Eq. (5) becomes

\[
\left(\frac{m^2_{\nu_{\tau}} - m^2_{\nu_{s'}}}{1\text{ eV}^2}\right)^{11/6} > 2 \times 10^4 \epsilon^{-1}_{\text{res}} \left|\frac{d\ln \epsilon_{\text{res}}}{d\ln T}\right|^{-1} \left|\frac{m^2_{\nu_{\mu}} - m^2_{\nu_{s'}}}{10^{-3}\text{ eV}^2}\right|^2.
\]

(7)

for \(D > V_x\). And Eq. (6) becomes

\[
\left(\frac{m^2_{\nu_{\tau}} - m^2_{\nu_{s'}}}{1\text{ eV}^2}\right)^{17/6} > 10^3 \epsilon^{-3}_{\text{res}} \left|\frac{d\ln \epsilon_{\text{res}}}{d\ln T}\right|^{-1} \left|\frac{m^2_{\nu_{\mu}} - m^2_{\nu_{s'}}}{10^{-3}\text{ eV}^2}\right|^3.
\]

(8)

for \(D < V_x\).

The value of \(|d\ln \epsilon_{\text{res}}/d\ln T|\) is related to the growth of \(L_{\nu_{\tau}}\) by \(|d\ln \epsilon_{\text{res}}/d\ln T| \approx |d\ln L_{\nu_{\tau}}/d\ln T - 2| \approx |d\ln L_{\nu_{\mu}}/d\ln T|\) (with \(L_{\nu_{\mu}}\) safely ignored). Figure 2 shows \(|d\ln L_{\nu_{\tau}}/d\ln T|\) as a function of the \(\nu_{\tau} \leftrightarrow \nu_{s'}\) vacuum mixing parameters, \(m^2_{\nu_{\tau}} - m^2_{\nu_{s'}}\) and \(\sin^2 2\theta'\), in the initial exponential stage of \(L_{\nu_{\tau}}\) growth when \(L_{\nu_{\tau}} \ll 10^{-7}\). \(|d\ln L_{\nu_{\tau}}/d\ln T|\) can be approximately fit as

\[
|\frac{d\ln L_{\nu_{\tau}}}{d\ln T}| \approx 6 \times 10^6 \sin 2\theta',
\]

(9)

and is insensitive to \(m^2_{\nu_{\tau}} - m^2_{\nu_{s'}}\). The vacuum mixing parameter \(\sin 2\theta'\) of the \(\nu_{\tau} \leftrightarrow \nu_{s'}\) mixing must satisfy the BBN bound \((m^2_{\nu_{\tau}} - m^2_{\nu_{s'}}) \sin^4 2\theta' < 10^{-7}\text{ eV}^2\) (the cut-off at the upper-right corner of Figure 2) [4].

Eq. (7), (8) and (9) show that the most stringent requirement on \(m^2_{\nu_{\tau}} - m^2_{\nu_{s'}}\) does indeed come not from \(\nu_{\mu} \leftrightarrow \nu_{s}\) resonances at \(\epsilon_{\text{res}} \sim 3\), but from resonances centered at the smallest possible \(\epsilon_{\text{res}}\) as long as the \(\nu_{\mu}\) or \(\bar{\nu}_{\mu}\) transition probability in that resonance energy bin is \(\ll 1.\) This condition, expressed as

\[
R \frac{\delta\epsilon_{\text{res}}}{\dot{\epsilon}_{\text{res}}} \lesssim 0.1,
\]

(10)
can be rewritten as

\[ \epsilon_{\text{res}} \gtrsim \left( \frac{m_{\nu_e}^2 - m_{\nu_{\mu}}^2}{1 \text{ eV}^2} \right)^{-1/2} \left| \frac{\text{d} \ln \epsilon_{\text{res}}}{\text{d} \ln T} \right|^{-1/3} \frac{m_{\nu_{\mu}}^2 - m_{\nu_s}^2}{10^{-3} \text{ eV}^2}^{2/3}, \]  

(11)

regardless of the value of \( D/V_x \). It can be shown that \( \epsilon_{\text{res}} \) is in the oscillation-dominated regime if

\[ \epsilon_{\text{res}} \lesssim 0.25 \left| \frac{m_{\nu_{\mu}}^2 - m_{\nu_s}^2}{10^{-3} \text{ eV}^2} \right|^{1/2} \left( \frac{m_{\nu_e}^2 - m_{\nu_{\mu}}^2}{1 \text{ eV}^2} \right)^{-1/2}. \]  

(12)

Therefore, the most stringent requirement on \( m_{\nu_e}^2 - m_{\nu_{\mu}}^2 \) comes from the oscillation-dominated regime for \( \sin^2 2\theta' \gtrsim 10^{-8} \) (while \( |\text{d} \ln \epsilon_{\text{res}}/\text{d} \ln T| \gtrsim 10^3 \)), and from collision-dominated regime for \( \sin^2 2\theta' \lesssim 10^{-8} \) (while \( |\text{d} \ln \epsilon_{\text{res}}/\text{d} \ln T| \lesssim 10^3 \)).

Combining Eq. (7) or (8), Eq. (9) and Eq. (11) yields a requirement on the mass-squared-difference necessary to effect suppression of \( \nu_{\mu} \leftrightarrow \nu_s \) transformation at the Super Kamiokande level:

\[ m_{\nu_e}^2 - m_{\nu_{\mu}}^2 \gtrsim \begin{cases} 200 \left( |m_{\nu_{\mu}}^2 - m_{\nu_s}^2|/10^{-3} \text{ eV}^2 \right)^{3/4} \text{ eV}^2 & \text{for } \sin^2 2\theta' \gtrsim 10^{-8} \\
(sin 2\theta')^{-1/2} \left( |m_{\nu_{\mu}}^2 - m_{\nu_s}^2|/10^{-3} \text{ eV}^2 \right) \text{ eV}^2 & \text{for } \sin^2 2\theta' \lesssim 10^{-8} \end{cases} \]  

(13)

This is in agreement with our previous work.

Perhaps because of insufficient energy resolution at low energies, Foot and Volkas may have missed the crucial effect of small \( \epsilon_{\text{res}} \). Neutrinos with energies \( \sim 1\% \) of the temperature are an insignificant fraction (\( \sim 10^{-6} \)) of the overall neutrino number. However, in this problem, they are the driving force which frees the \( \nu_{\mu} \leftrightarrow \nu_s \) transformation process. This is simply because the suppressing lepton asymmetry from \( \nu_{\tau} \) in this case is itself minuscule (e.g., \( \ll 10^{-7} \)).
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**Figure Captions:**

Figure 1. The growth of the tau neutrino asymmetry as a result of the tau neutrino-sterile neutrino mixing, assuming $m_{\nu_\tau}^2 - m_{\nu_s}^2 = 50$ eV$^2$ and $\sin^2 2\theta' = 10^{-4}$. The intersections between the growth curve of $L_{\nu_\tau}$ and the dashed lines indicate when resonances occur for $\nu_\mu$ (if $L_{\nu_\tau} > 0$) or $\bar{\nu}_\mu$ (if $L_{\nu_\tau} < 0$) neutrinos with momentum $p$.

Figure 2. The initial rate of $L_{\nu_\tau}$ growth, $d\ln L/d\ln T$, as a function of the vacuum tau neutrino-sterile neutrino mixing parameters.
