Dynamical quasitilings of amenable groups

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Abstract

We prove that for any compact zero-dimensional metric space \( X \) on which an infinite countable amenable group \( G \) acts freely by homeomorphisms, there exists a dynamical quasitiling with good covering, continuity, Følner and dynamical properties, i.e. to every \( x \in X \) we can assign a quasitiling \( \mathcal{T}_x \) of \( G \) (with all the \( \mathcal{T}_x \) using the same, finite set of shapes) such that the tiles of \( \mathcal{T}_x \) are disjoint, their union has arbitrarily high lower Banach Density, all the shapes of \( \mathcal{T}_x \) are large subsets of an arbitrarily large Følner set, and if we consider \( \mathcal{T}_x \) to be an element of a shift space over a certain finite alphabet, then the mapping \( x \mapsto \mathcal{T}_x \) is a factor map.

1 Introduction

Many constructions in symbolic and zero-dimensional dynamics with the action of \( \mathbb{Z} \) rely on partitioning a sequence of symbols into disjoint blocks (i.e. partitioning \( \mathbb{Z} \) into disjoint intervals) and performing some operations on such blocks. Analogous constructions in systems with the action of an arbitrary amenable group \( G \) require partitioning \( G \) into disjoint, finite subsets with good invariance properties — in other words, constructing an appropriate (quasi-)tiling of \( G \). Such quasitilings, in which tiles are „almost” disjoint and whose union is „almost” all of \( G \) were first developed by Ornstein and Weiss in [OW], and have later been improved to tilings in [DHZ].

The (quasi-)tilings of [OW] and [DHZ] are algebraic in nature and their construction does not rely on any „external” dynamics. However, proofs in
traditional symbolic and zero-dimensional dynamics often require that the partitioning of symbolic sequences into blocks be continuous and consistent with the shift actions, i.e. that two sequences that agree on a long interval should also be identically partitioned over this (or slightly shorter) interval, and that shifting a sequence should result in shifting the corresponding partition. In the present paper we prove that given a free action of a countable, discrete, amenable group \( G \) on a zero-dimensional compact metric space \( X \), it is possible to construct a quasitiling \( \mathcal{T}_x \) for every \( x \in X \) such that all such quasitilings have arbitrarily good invariance and covering properties, and if we view \( \mathcal{T}_x \) as an element of an appropriate shift space, then the mapping \( x \mapsto \mathcal{T}_x \) is a factor map. A careful reader will notice that large portions of this note are copied from [DHZ]. We do so in order to make this paper self-contained.

2 Preliminaries

2.1 Basic notions

Throughout this paper \( G \) denotes an infinite countable amenable group, i.e., a group in which there exists a sequence of finite sets \( F_n \subset G \) (called a Følner sequence, or a sequence of Følner sets), such that for any \( g \in G \) we have

\[
\lim_{n \to \infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,
\]

where \( gF = \{gf : f \in F\} \), \( |\cdot| \) denotes the cardinality of a set, and \( \triangle \) is the symmetric difference. Since \( G \) is infinite, the sequence \( |F_n| \) tends to infinity. Without loss of generality (see [N, Corollary 5.3]) we can assume that the sets in the Følner sequence are symmetric (i.e. \( F_n^{-1} = F_n \) for every \( n \)) and contain the unit.

**Definition 2.1.** If \( T \) and \( K \) are nonempty, finite subsets of \( G \) and \( \varepsilon < 1 \), we say that \( T \) is \((K, \varepsilon)-invariant\) if

\[
\frac{|KT \triangle T|}{|T|} < \varepsilon,
\]

where \( KT = \{gh : g \in K, h \in T\} \).

Observe that if \( K \) contains the unit of \( G \), then \((K, \varepsilon)-invariance\) is equivalent to the simpler condition

\[
|KT| < (1 + \varepsilon)|T|.
\]
The following facts are not difficult to see, so we skip the proofs, referring the reader to [DHZ].

**Lemma 2.2.** A sequence of finite sets \((F_n)\) is a Følner sequence if and only if for every finite set \(K\) and every \(\varepsilon > 0\) the sets \(F_n\) are eventually \((K, \varepsilon)\)-invariant.

**Lemma 2.3.** Let \(K \subset G\) be a finite set and fix some \(\varepsilon > 0\). There exists \(\delta > 0\) such that if \(T \subset G\) is \((K, \delta)\)-invariant and \(T'\) satisfies \(|T' \triangle T| \leq \delta\) then \(T'\) is \((K, \varepsilon)\)-invariant.

**Definition 2.4.** We say that \(T' \subset T\) (\(T\) finite) is a \((1-\varepsilon)\)-subset of \(T\) if \(|T'| \geq (1-\varepsilon)|T|\).

Like in [DHZ], we will use the following definition of lower Banach density:

**Definition 2.5.** For \(S \subset G\) and a finite, nonempty \(F \subset G\) denote

\[
D_F(S) = \inf_{g \in G} \frac{|S \cap Fg|}{|F|}.
\]

If \((F_n)\) is a Følner sequence then define

\[
D(S) = \limsup_{n \to \infty} D_{F_n}(S),
\]

which we call the lower Banach density of \(S\).

The proof of the following standard fact can again be found in [DHZ]:

**Lemma 2.6.** Regardless of the set \(S\), the value of \(D(S)\) does not depend on the Følner sequence, the limit superior in the definition is fact a limit, and moreover

\[
D(S) = \sup \{D_F(S) : F \subset G, F\text{ is finite}\}
\]

Let \(X\) be a compact metric space and let \(G\) be a discrete amenable group. We say that \(G\) acts on \(X\) by homeomorphisms, if for every \(g \in G\) there exists a homeomorphism \(h_g : X \to X\) such that the mapping \(h \mapsto h_g\) is a group isomorphism between \(g\) and a subgroup of \(\text{Homeo}(X)\). In a slight abuse of notation, \(h_g(x)\) is often written just as \(g(x)\), i.e. \(G\) is identified with a subgroup of \(\text{Homeo}(X)\). We say that the action of \(G\) is free, if the identity \(g(x) = x\) for some \(x \in X, g \in G\) implies that \(g = e\).

Let \(\Lambda\) be a finite set with the discrete topology. There exists a standard action of \(G\) on \(\Lambda^G\) (called the shift action), defined as follows: \((gx)(h) = x(hg)\). \(\Lambda^G\) with the product topology and the shift action of \(G\) becomes a zero-dimensional dynamical system, called the full shift over \(\Lambda\). A symbolic dynamical system over \(\Lambda\) is any closed, \(G\)-invariant subset \(X\) of the full shift.
3 Quasitilings

The first two definitions below are the same as in [DHZ]

Definition 3.1. A quasitiling is determined by two objects:

1. a finite collection $S(T)$ of finite subsets of $G$ containing the unit $e$, called the shapes.

2. a finite collection $C(T) = \{C(S) : S \in S(T)\}$ of disjoint subsets of $G$, called center sets (for the shapes).

The quasitiling is then the family $T = \{(S, c) : S \in S(T), c \in C(S)\}$. We require that the map $(S, c) \mapsto Sc$ be injective.$^1$ Hence, by the tiles of $T$ (denoted by the letter $T$) we will mean either the sets $Sc$ or the pairs $(S, c)$ (i.e., the tiles with defined centers), depending on the context.

Note that every quasitiling $T$ can be represented in a symbolic form, as a point $x_T \in \Lambda^G$, with the alphabet $\Lambda = S(T) \cup \{\emptyset\}$, as follows: $x_T(g) = S \iff g \in C(S), \emptyset$ otherwise.

Definition 3.2. Let $\varepsilon \in [0, 1)$ and $\alpha \in (0, 1]$. A quasitiling $T$ is called

1. $\varepsilon$-disjoint if there exists a mapping $T \mapsto T^\circ$ ($T \in T$) such that
   
   - $T^\circ$ is a $(1 - \varepsilon)$-subset of $T$, and
   - $T \neq T' \implies T^\circ \cap T'^\circ = \emptyset$;

2. disjoint if the tiles of $T$ are pairwise disjoint;

3. $\alpha$-covering if $D(\bigcup T) \geq \alpha$.

Definition 3.3. If $X$ is a zero-dimensional compact metric space and $G$ is an amenable group acting on $X$ by homeomorphisms, then a dynamical quastiling is a map $x \mapsto T_x$ which assigns to every $x \in X$ a quasitiling of $G$ such that the set of all shapes $S = \bigcup_{x \in X} S(T_x)$ is finite, and $x \mapsto T_x$ is a factor map from $X$ onto a symbolic dynamical system over the alphabet $\Lambda = S \cup \{\emptyset\}$. We say that a dynamical quastiling is $\varepsilon$-disjoint, disjoint or $\alpha$-covering, if $T_x$ has the respective property for every $x$.

$^1$This requirement is stronger than asking that different tiles have different centers. Two tiles $Sc$ and $S'c'$ may be equal even though $c \neq c'$ (this is even possible when $S = S'$). However, when the tiles are disjoint, then the (stronger) requirement follows automatically from the fact that the centers belong to the tiles.
The following lemma is very similar to the one in [DHZ] (which in turn largely uses the techniques developed in [OW, I.82, Theorem 6]), with the major difference concerning the dynamical properties of the obtained tilings.

Lemma 3.4. Let $X$ be a compact, zero-dimensional metric space and let $G$ be a countable amenable group acting freely on $X$ by homeomorphisms, with a Følner sequence $(F_n)$ of symmetric sets containing the unit. Given $\varepsilon > 0$, there exists a positive integer $r = r(\varepsilon)$ such that for each positive integer $n_0$ there exists a dynamical quastitiling $x \mapsto T_x$ which is $\varepsilon$-disjoint and $(1 - \varepsilon)$-covering, and the set of shapes $\bigcup_{x \in X} S(T_x)$ consists of $r$ shapes $\{F_{n_1}, \ldots, F_{n_r}\}$, where $n_0 < n_1 < \cdots < n_r$.

Proof. Find $r$ such that $(1 - \frac{\varepsilon}{2})^r < \varepsilon$. This is going to be the cardinality of the family of shapes. Choose integers $n_1 = n_0 + 1, n_2, \ldots, n_r$ so that they increase and for each pair of indices $j < i$, $i, j \in \{1, 2, \ldots, r\}$ the set $F_{n_i}$ is $(F_{n_j}, \delta_j)$-invariant, where $\delta_j$ will be specified later. For every $x$, we let $S(T_x) = \{F_{n_j} : j = 1, \ldots, r\}$ be our family of shapes. With this choice, the assertions about the shapes and their number are fulfilled. It remains to construct the corresponding center sets $C_x(F_{n_j})$ for every $x$ so as to satisfy $\varepsilon$-disjointness and $(1 - \varepsilon)$-covering of $T_x$, and to ensure that the mapping $x \mapsto T_x$ is a factor map (to which end $T_x$ must depend continuously on $X$, and for every $g \in G$ we must have $T_{g x} = g(T_x)$).

We proceed by induction over $j$ decreasing from $r$ to 1. Begin with $j = r$. Since $G$ acts freely on $X$, every $x \in X$ has a clopen neighborhood $U_x$ such that the sets $g(U_x)$ are pairwise disjoint for $g \in F_{n_j}$. By compactness, we can choose a finite number of such neighborhoods whose union covers $X$. We will label these neighborhoods $U_1, U_2, \ldots, U_m$.

Fix $x \in X$, and let

$$C^{(r)}_1(x) = \{g \in G : gx \in U_1\}.$$  

Then for $i = 2, \ldots, m$ let

$$C^{(r)}_i(x) = C^{(r)}_{i-1}(x) \cup \left\{g \in G : gx \in U_i, \text{ and } \left|F_{n_i} g \cap F_{n_i} C^{(r)}_{i-1}(x)\right| < \varepsilon |F_{n_i}|\right\}.$$ 

We can show that for every $x \in X, h \in G$ and $i = 1, \ldots, m$, $C^{(r)}_i(hx) = C^{(r)}_i(x)h^{-1}$. Indeed, for $i = 1$ we have $g \in C^{(r)}_1(hx)$ if and only if $ghx \in U_1$, which is equivalent to stating that $gh \in C^{(r)}_1(x)$, i.e. $g \in C^{(r)}_1(x)h^{-1}$.

Now assuming that $C^{(r)}_i(hx) = C^{(r)}_i(x)h$, we have the following equivalent statements:

$$g \in C^{(r)}_i(hx)$$
Also note that since the set \( C_i^{(r)}(x) \) is determined by the visits of \( x \) in clopen sets of \( X \) under the action of \( G \), for every \( F \in G \) there exists an \( \eta \) such that if \( d(x, y) < \eta \), then \( C_i^{(r)}(x) \cap F = C_j^{(r)}(y) \cap F \). Now, let \( C_i^{(r)}(x) = C_m^{(r)}(x) \) and let \( T_x^{(r)} \) be a point in \( \{ \emptyset, F_{n_1}, \ldots, F_{n_m} \}^G \) such that \( T_x^{(r)} = F_{n_r} \) if \( g \in C_i^{(r)}(x) \) and \( \emptyset \) otherwise. By the remarks made earlier, the mapping \( x \mapsto T_x^{(r)} \) is continuous, and also \( (T_x^{(r)})_g = F_{n_r} \) if and only if \( g \in C_i^{(r)}(hx) \), if and only if \( g \in C_i^{(r)}(x)h^{-1} \) if and only if \( gh \in C_i^{(r)}(x) \), if and only if \( (T_x^{(r)})_{gh} = F_{n_r} \), which means that \( T_x^{(r)} = h(T_x^{(r)}) \), and thus the mapping \( x \mapsto T_x^{(r)} \) commutes with the shift action.

We will now show that for every \( x \in X \), the family \( \{ F_{n_r} : c \in C_i^{(r)}(x) \} \) is an \( \varepsilon \)-covering quasitiling of \( G \) which is \( \varepsilon \)-disjoint.

As for the \( \varepsilon \)-disjointness, note that for every \( i \) and every \( c \in C_i^{(r)}(x) \) the set \( F_{n_r}c \setminus F_{n_r}C_{i-1}^{(r)}(x) \) is a \((1 - \varepsilon)\) subset of \( F_{n_r}c \). Such subsets of \( F_{n_r}c_1 \) and \( F_{n_r}c_2 \) are pairwise disjoint by definition if \( c_1 \) and \( c_2 \) belong to \( C_i^{(r)}(x) \) for different \( i \)'s, and if they both belong to the same \( C_i^{(r)}(x) \) then if \( F_{n_r}c_1 \cap F_{n_r}c_2 \) is nonempty, then it has an element of the form \( t_1c_1 = t_2c_2 \) for \( t_1, t_2 \in F_{n_r} \) and \( c_1, c_2 \in C_i^{(r)}(x) \). Thus \( t_1c_1(x) = t_2c_2(x) \), but as \( c_1(x) \) and \( c_2(x) \) are both in \( U_i \), this implies that the images of \( U_i \) under \( t_1 \) and \( t_2 \) are not disjoint, which contradicts the definition of \( U_i \).

Finally, for every \( x \in X \) and every \( g \in G \) there exists an \( i \) such that \( gx \in U_i \). Now either \( g \in C_i^{(r)}(x) \), or \( |F_{n_r}g \cap F_{n_r}C_i^{(r)}(x)| \geq |F_{n_r}g \cap F_{n_r}C_{i-1}^{(r)}(x)| \geq \varepsilon |F_{n_r}| \) — in both cases, \( F_{n_r}g \cap F_{n_r}C_i^{(r)}(x) \geq \varepsilon |F_{n_r}| \), which means that the lower Banach density of \( F_{n_r}C_i^{(r)}(x) \) is at least \( \varepsilon \).

Fix some \( j \in \{1, 2, \ldots, r - 1\} \) and suppose that for every \( x \in X \) we have constructed an \( \varepsilon \)-disjoint quasitiling \( T_x^{(j+1)} = \{ F_{n_r}c : j + 1 \leq i \leq r, c \in C^{(j)}(x) \} \), such that for every \( x \) the union

\[
H^{(j+1)}(x) = \bigcup_{i=j+1}^{r} F_{n_r}C^{(i)}(x)
\]
has lower Banach density strictly larger than $1 - (1 - \varepsilon)^{r-j}$ (this is our inductive hypothesis on $H^{(j+1)}(x)$ and it is fulfilled for $H^{(j)}(x)$), and that $x \mapsto T^{(j+1)}_x$ is a factor map. We need to go one step further in our “decreasing induction”, i.e., add a center set $C^{(j)}(x)$ for the shape $F_n^j$.

As before, we can cover $X$ by a finite family of clopen sets $U_1, U_2, \ldots, U_m$, such that for every $i$ the sets $g(U_i)$ are pairwise disjoint for $g \in F_n^j$. Let $C^{(j)}_0(x) = \emptyset$, and for $i = 1, \ldots, m$, let

$$C^{(j)}_i(x) = C^{(j)}_{i-1}(x) \cup \left\{ g \in G : gx \in U_i, \text{ and } \left| F_n^j \cap \left( F_n^j C^{(j)}_{i-1}(x) \cup \bigcup_{l=j+1}^k F_n^l C^{(l)}(x) \right) \right| < \varepsilon |F_n^j| \right\}.$$ 

Let $C^{(j)}(x) = C^{(j)}_m(x)$. By the same arguments as before, we easily obtain the following properties of this set:

- For any $x \in X$ and $h \in G$ we have $C^{(j)}(hx) = C^{(j)}(x)h^{-1}$.
- The mapping $x \mapsto T^{(j)}_x$ is a factor map.
- For every $x \in X$, the quasitiling $T^{(j)}(x)$ is $\varepsilon$-disjoint.
- For every $x$, if we denote by $H^{(j)}(x)$ the union of the above family, then for every $g \in G$ we have

$$\frac{|H^{(j)}(x) \cap F_n^j g|}{|F_n^j|} > \varepsilon. \tag{1}$$

The rest of the proof is nearly identical as the analogous proof in [DHZ] for “static” quasitilings. We copy it here for completeness, adapting it to our dynamical situation. Since the arguments below involve only algebraic operations and combinatorics (and not the dynamics and topology on $X$), and are the same for every $x$, for the next few paragraphs we will omit references to $x$ and write just $H^{(j)}$ rather than $H^{(j)}(x)$ and $C^{(j)}$ rather than $C^{(j)}(x)$, implicitly stating that the estimates provided are true for every $x$.

Our goal is to estimate from below the lower Banach density of $H^{(j)}$. By Lemma 2.6, it suffices to estimate $D_F(H^{(j)})$ for just one finite set $F$ which we will define in a moment. Define $B = \left( \bigcup_{x=j+1}^k F^2_n \right) F_n^j$. Clearly, $B$ contains $F_n^j$ (hence the unit), and, as easily verified, it has the following property:
Whenever $F_{n_j} F_n c \cap A \neq \emptyset$, for some $i \in \{j+1, \ldots, r\}$, $c \in G$ and $A \subset G$, then $F_{n_j} c \subset BA$.

Let $n$ be so large that $F_n$ is $(B, \delta_j)$-invariant and that $\frac{D}{D F_n}(H_{j+1}) > 1 - (1 - \frac{\delta_j}{2})^{r-j}$ (the latter is possible due to the assumption on $D(H_{j+1})$). Now we define the aforementioned set $F$ as $F = F_{n_j} F_n$.

Fix some $g \in G$ and define
\[
\alpha_g = \frac{|H^{j+1} \cap F_n g|}{|F_n|} \quad \text{and} \quad \beta_g = \frac{|H^{j+1} \cap BF_n g|}{|BF_n|}.
\]

Notice that
\[
\alpha_g \geq \frac{D}{D F_n}(H^{j+1}) > 1 - (1 - \frac{\delta_j}{2})^{r-j}.
\] (3)

Also, we have
\[
\begin{align*}
\beta_g & \geq \frac{|H^{j+1} \cap F_n g|}{(1 + \delta_j)|F_n|} = \frac{\alpha_g}{1 + \delta_j}, \quad \text{and} \\
\beta_g & \leq \frac{|H^{j+1} \cap F_n g| + |BF_n g \setminus F_n g|}{|F_n|} \leq \alpha_g + \delta_j.
\end{align*}
\]

(4)

(5)

Note that since $F_{n_j} \subset B$ and $F_n$ is $(B, \delta_j)$-invariant, $F_n$ is automatically $(F_{n_j}, \delta_j)$-invariant. Thus
\[
\frac{|H^{j+1} \cap F_g|}{|F|} \geq \frac{|H^{j+1} \cap F_n g|}{(1 + \delta_j)|F_n|} = \frac{\alpha_g}{1 + \delta_j} \geq \frac{\beta_g - \delta_j}{1 + \delta_j}.
\] (6)

Consider only these finitely many component sets $F_n c$ of $H^{j+1}$ (i.e., with $i \in \{j+1, \ldots, r\}$, $c \in C^{(j)}$) for which $F_{n_j} F_n c$ has a nonempty intersection with $F_n g$, and denote by $E_g$ the union of so selected components $F_n c$. By the property (*) of $B$ (with $A = F_n g$), $E_g$ is a subset of $BF_n g$ (and also of $H^{j+1}$), so
\[
|E_g| \leq |H^{j+1} \cap BF_n g| = \beta_g |BF_n| \leq \beta_g (1 + \delta_j)|F_n|.
\] (7)

Each of the selected components $F_n c \subset E_g$ is $(F_{n_j}, \delta_j)$-invariant, hence, when multiplied on the left by $F_{n_j}$ it can gain at most $\delta_j |F_n c|$ new elements. Thus the set $E_g$, when multiplied on the left by $F_{n_j}$, can gain at most $\delta_j \sum_{F_n c \subset E_g} |F_n c|$ new elements. On the other hand, denoting by $(F_n c)^{\circ}$ the pairwise disjoint sets (contained in respective sets $F_n c$) as in the definition of $\varepsilon$-disjointness, we also have
\[
\sum_{F_n c \subset E_g} |F_n c| \leq \frac{1}{1 - \varepsilon} \sum_{F_n c \subset E_g} |(F_n c)^{\circ}| = \frac{1}{1 - \varepsilon} \left| \bigcup_{F_n c \subset E_g} (F_n c)^{\circ} \right| \leq \frac{1}{1 - \varepsilon} |E_g|.
\]
Combining this with the preceding statement, we obtain that the set $E_g$, when multiplied on the left by $F_{n_j}$, can gain at most $\frac{\delta_j}{1 - \delta_j} |E_g|$ new elements, which is less than $2\delta_j |E_g|$ (we can assume that $\varepsilon < \frac{1}{2}$). Denote $\hat{H}^{(j+1)} = F_{n_j} H^{(j+1)}$. By the choice of the components included in $E_g$, the set $F_{n_j} E_g$ contains all of $\hat{H}^{(j+1)} \cap F_n g$. Thus, using $(1 + 2\delta_j) \leq (1 + \delta_j)^2$ and (7), we obtain that

$$|\hat{H}^{(j+1)} \cap F_n g| \leq |F_{n_j} E_g| \leq (1 + 2\delta_j) |E_g| \leq (1 + \delta_j)^3 \beta_g |F_n|.$$

Let $N_g = F_n g \setminus \hat{H}^{(j+1)}$. By the above inequality, we know that

$$|N_g| \geq (1 - (1 + \delta_j)^3 \beta_g) |F_n| \geq (1 - (1 + \delta_j)^3 \beta_g) \frac{|F|}{1 + \delta_j}, \hspace{1cm} (8)$$

where the last inequality follows from the $(F_{n_j}, \delta_j)$-invariance of $F_n$.

Earlier we have established (inequality (9)) that $|F_{n_j} g| \geq \varepsilon$ for every $c \in G$, in particular for every $c \in N_g$. This implies that there are at least $\varepsilon |N_g| |F_{n_j}|$ pairs $(f, c)$ with $f \in F_{n_j}, c \in N_g$ such that $fc \in H^{(j)}$. This in turn implies that there exists at least one $f \in F_{n_j}$ for which

$$|H^{(j)} \cap f N_g| \geq \varepsilon |N_g|. \hspace{1cm} (9)$$

Notice that $f N_g$ is contained in $F g$ (because $N_g \subset F_n g$ and $f \in F_{n_j}$) and disjoint from $H^{(j+1)}$ ($N_g$ is disjoint from $\hat{H}^{(j+1)}$ which contains $f^{-1} H^{(j+1)}$). Thus we can estimate, using (8), (3) and (9):

$$\frac{|H^{(j)} \cap F g|}{|F|} \geq \frac{|H^{(j+1)} \cap F g| + |H^{(j)} \cap f N_g|}{|F|} = \frac{|H^{(j+1)} \cap F g|}{|F|} + \frac{|H^{(j)} \cap f N_g| |N_g|}{|F|} \geq \beta_g - \delta_j + \varepsilon \frac{1 - (1 + \delta_j)^3 \beta_g}{1 + \delta_j}.$$ 

Both terms in the last expression are linear functions of $\beta_g$, the first one with positive and large slope $\frac{1}{1 + \delta_j}$, the other with negative but small slope $-\varepsilon(1 + \delta_j)^2$. Jointly, the function increases with $\beta_g$. So, we can replace $\beta_g$ by any smaller value, for instance, by $\frac{1 - (1 - \frac{\varepsilon}{2})^{r-2}}{1 + \delta_j}$ (see (3) and (11)), to obtain

$$\frac{|H^{(j)} \cap F g|}{|F|} > 1 - \frac{(1 - \frac{\varepsilon}{2})^{r-2}}{(1 + \delta_j)^2} - \frac{\delta_j}{1 + \delta_j} + \varepsilon \left( \frac{1}{1 + \delta_j} - (1 + \delta_j)(1 - (1 - \frac{\varepsilon}{2})^{r-2}) \right).$$
Now notice, that if we replace the undivided occurrence of \( \varepsilon \) by \( \frac{3\varepsilon}{4} \), we make the entire expression smaller by some positive value (independent of \( g \)). On the other hand, if \( \delta_j \) is very small and we remove it completely from the expression, we will perhaps enlarge it, but very little. We now specify \( \delta_j \) to be so small, that if we replace \( \varepsilon \) by \( \frac{3\varepsilon}{4} \) and remove \( \delta_j \) completely, then the expression will become smaller. With such a choice of \( \delta_j \) we have

\[
\frac{|H^{(j)} \cap Fg|}{|F|} > 1 - (1 - \frac{\varepsilon}{2})^{-j} + \frac{3\varepsilon}{4}(1 - \frac{\varepsilon}{2})^{-j} = 1 - (1 - \frac{\varepsilon}{2})^{-j+1} + \xi,
\]

where \( \xi > 0 \) does not depend on \( g \). Taking infimum over all \( g \in G \) we get, by Lemma 2.6

\[
\overline{D}(H^{(j)}) \geq \overline{D}_F(H^{(j)}) > 1 - (1 - \frac{\varepsilon}{2})^{-j+1},
\]

and the inductive hypothesis has been derived for \( j \).

Once the induction reaches \( j = 1 \) we get that the lower Banach density of \( H^{(1)}(x) \) is larger than \( 1 - (1 - \frac{\varepsilon}{2})^{r} \) which, by the choice of \( r \), is larger than \( 1 - \varepsilon \) and means that \( \mathcal{T}_x = \mathcal{T}^{(1)}(x) \) is the desired quasitiling. \( \square \)

At the cost of increasing the number of possible shapes (but without sacrificing the other properties) we can make our quasitilings disjoint:

**Corollary 3.5.** Let \( X \) be a compact, zero-dimensional metric space and let \( G \) be a countable amenable group acting freely on \( X \) by homeomorphisms, with a Følner sequence \((F_n)\) of symmetric sets containing the unit. Given \( \varepsilon > 0 \) and any positive integer \( n_0 \), there exists a dynamical quasitiling \( x \mapsto \mathcal{T}_x \) which is disjoint, and \( (1 - \varepsilon)\)-covering, and such that every shape \( S \) of every \( \mathcal{T}_x \) is a \( (1 - \varepsilon)\)-subset of some Følner set \( F_{n(S)} \) where \( n(S) > n_0 \).

**Proof.** For every \( x \), let \( \hat{T}_x \) be the quasitiling delivered by Lemma 3.4, with \( \varepsilon \) and \( n_0 \). Recall that for every \( x \), the set of shapes of \( \mathcal{T}_x \) is \( \{F_{n_1}, \ldots, F_{n_r}\} \), where \( n_0 < n_1 < \ldots < n_r \). Furthermore, every tile of \( \mathcal{T}_x \) has the form \( \hat{T} = F_{n_j}c \) for some \( j \in \{1, \ldots, r\} \) and \( c \in \hat{C}_i^{(j)}(x) \). Let \( T = F_{n_j}c \setminus \left( F_{n_j}C_{i-1}^{(j)}(x) \cup \bigcup_{i=j+1}^k F_{n_i}C^{(i)}(x) \right) \). By the definition of the sets \( C_i^{(j)}(x) \) and \( C^{(i)}(x) \), \( T \) is a \( (1 - \varepsilon)\)-subset of \( \hat{T} \). In addition, if \( \hat{T} \neq \hat{T}' \), then \( T \) and \( T' \) are disjoint: If we represent \( \hat{T} = F_{n_j}c \) and \( \hat{T}' = F_{n_j}c' \), then we have two possibilities: either \( j = j' \), and \( c \) and \( c' \) belong to the same \( C_i^{(j)}(x) \) — in this case \( \hat{T} \) and \( \hat{T}' \) are disjoint, therefore so are \( T \) and \( T' \) as their respective subsets. Otherwise we can without loss of generality assume that either \( j < j' \), or \( j > j' \) and \( i > i' \) — in this case, since \( T = F_{n_j}c \setminus \left( F_{n_j}C_{i-1}^{(j)}(x) \cup \bigcup_{i=j+1}^k F_{n_i}C^{(i)}(x) \right) \),
it is disjoint from $T'$, as $T'$ is a subset of $F_{n'}c'$ and thus is included in the set subtracted from $F_{n'}c$.

Let $\mathcal{T}_x$ denote the quasitiling obtained from $\hat{T}_x$ by these modifications. Note that due to the properties of the sets $C_i(x)$, we still have $\mathcal{T}_{gx} = g(\mathcal{T}_x)$, and if we set $F = \bigcup_{j=1}^n F_{n_j}$, then if $\hat{T}_x$ agrees with $\hat{T}_y$ on a subset of $G$ of the form $FB$, then $\mathcal{T}_x$ and $\mathcal{F}_y$ agree on $B$. This means that $\mathcal{T}_x$ depends on $x$ continuously, and thus the mapping $x \mapsto \mathcal{T}_x$ is a factor map. In addition, for every $x$ the union of all the tiles of $\mathcal{T}_x$ has not changed, so the new quasitiling is still $(1 - \varepsilon)$-covering.

Finally, we can also obtain a quasitiling that is “compatible” with another quasitiling by smaller tiles:

**Lemma 3.6.** Let $G$ be an amenable group acting freely on a zero-dimensional metric space $X$ and let $x \mapsto \mathcal{T}_x$ be any disjoint dynamical quasitiling of $G$. For any $\varepsilon > 0$, any finite $K \subset G$ and any $\delta > 0$ there exists a disjoint, $(1 - \varepsilon)$-covering dynamical quasitiling $x \mapsto \mathcal{T}_x'$ such that every shape of $\mathcal{T}_x'$ is $(K, \delta)$-invariant, and every tile of $\mathcal{T}_x$ is either a subset of some tile of $\mathcal{T}_x'$ or is disjoint from all such tiles.

**Proof.** First of all observe that there exist constants $\delta'$ and $\eta$ such that if $T$ is $(K, \delta')$-invariant then any set $T'$ such that $|T' \setminus T| < \eta |T|$, is $(K, \delta)$-invariant. We can also assume that $(1 - \frac{\delta}{2})(1 - \eta) > 1 - \varepsilon$. Let $U$ be the union of all shapes of $\mathcal{T}_x$ over all $x \in X$; there exists a $\delta''$ such that if $T$ is $(U, \delta'')$-invariant, and we denote by $\mathcal{T}_U$ the set $\{t \in T : Ut \subset T\}$, then $|UT \setminus T_U| < \eta |T|$. Once these parameters are set, lemma 3.3 ensures the existence of a disjoint, dynamical, $(1 - \frac{\delta}{2})$-covering quasitiling $x \mapsto \mathcal{T}_x''$ whose shapes are $(E, \delta'')$ invariant for all shapes $E$ of $\mathcal{T}_x$.

Every tile $T \in \mathcal{T}_x$ has a unique representation in the form $Ec$, where $E$ is one of the finitely many shapes. If $c$ belongs to some tile $T''$ of $\mathcal{T}''_x$ (by disjointness of the quasitiling, there can be at most one such tile), let $\phi_x(T) = \emptyset$. This gives us a mapping from $\mathcal{T}_x$ to $\mathcal{T}_x'' \cup \emptyset$, which by its construction is continuous and commutes with the dynamics. Now we can modify $\mathcal{T}_x''$ as follows: If $T'' \in \mathcal{T}_x''$, add to $T''$ all the tiles $T$ of $\mathcal{T}_x$ such that $T'' = \phi_x(T)$, and remove from it all the $T$ of $\mathcal{T}_x$ such that $T'' \neq \phi_x(T)$.

Observe that if $T'$ denotes a tile obtained from $T''$ after all such modifications, then $T'_U \subset T' \subset UT''$. It follows that $|T'' \setminus T'| < \eta |T''|$, and thus $T'$ is $(K, \delta)$-invariant. Therefore the map $x \mapsto \mathcal{T}_x'$ obtained as a result of these modifications is a disjoint, dynamical quasitiling whose tiles are all $(K, \delta)$-invariant and every tile of $\mathcal{T}_x$ is either a subset of some tile of $\mathcal{T}_x'$ or
disjoint from all such tiles. Finally, to estimate the lower Banach density of the union of all tiles of $\mathcal{T}_x$, observe that for every tile $T''$ of $\mathcal{T}_x$ there exists a tile $T'$ of $\mathcal{T}_x$ which contains a $(1 - \eta)$ subset of $T''$. Since $\mathcal{T}_x$ is a $(1 - \frac{\varepsilon}{2})$-covering quasitiling, a straightforward argument (see e.g. lemma 3.4 of [DHZ]) implies that the lower Banach density of the union of all tiles of $\mathcal{T}_x$ is at least $(1 - \frac{\varepsilon}{2})(1 - \eta) > 1 - \varepsilon$.

As a final note, we remark that replacing our quasitilings with actual tilings (in which the union of tiles is all of $G$) remains an open problem. If the procedure of completing a disjoint quasitiling to a tiling used in [DHZ] was applied individually to $\mathcal{T}_x$ for every $x \in X$, there is no guarantee that such new tilings would still depend continuously on $x$, and it is currently unclear whether the technique can be suitably modified.

References

[DHZ] T. Downarowicz, D. Huczek, G. Zhang, Tilings of Amenable Groups, preprint (arXiv: 1502.02413)

[N] I. Namioka, Følner’s conditions for amenable semi-groups, Math. Scand. 15, 18–28 (1964). MR 0180832 (31 #5062)

[OW] Donald S. Ornstein and Benjamin Weiss, Entropy and isomorphism theorems for actions of amenable groups, J. Analyse Math. 48, 1–141 (1987). MR 910005 (88j:28014)