Gaussian Radial Basis Function for Unsteady Groundwater Flow

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Abstract. The radial basis function (RBF) collocation methods are meshless with spectrally accurate; its approximation is an extremely powerful tool for solving various types of partial differential equations (PDEs). In this paper, the Gaussian RBF (G-RBF) collocation method is applied to solving unsteady groundwater flow governed by nonlinear elliptic PDEs. During the whole solution process, no iterations are required for a kind of nonlinear PDEs. Numerical results are compared with the well-known Kansa’s method, also namely as Multiquadrics radial basis function (MQ-RBF). The results obtained show good performance of the G-RBF as solution to unsteady groundwater flow problems.

1. Introduction
As is known to all, most of the problems in groundwater flow can be represented by partial differential equations (PDEs). In recent years, investigation on numerical approximate solutions of PDEs has become a hot topic for engineers and mathematicians. The finite element method (FEM) [1], which is based on mesh-generation, has been an intense research subject due to its stable numerical solutions in many areas. Although these kinds of mesh-based methods have made great progress, the mesh-generation requires expensive computation cost for a large scope of practical problems. Thus, the mature software only copes with the problems in regular domains or lower-dimensional cases. The aforementioned issues facilitated the investigation on the other numerical methods.

The radial basis function (RBF) collocation methods are mathematically simple and truly meshless, which avoid troublesome mesh-generation for high-dimensional problems involving irregular or moving boundary. The RBF methodology for interpolation problems derived from the early work of Hardy in 1971 [2]. Franke made a comparison and found that the multiquadric RBF, a globally supported interpolation, performed the best results [3]. Kansa first used the multiquadric radial basis function (MQ-RBF) to approximate the numerical solution of PDEs [4]. It is also called the Kansa’s method to calculate the approximate solution of various PDEs thereafter. Because of its good performance, the MQ-RBF has been triumphantly applied to solving different PDEs, such as 2D Possion’s and parabolic equations, PDEs with variable coefficients, shallow water equation for tide and currents simulation, nonlinear algebraic PDEs [5].

Meanwhile, the Gaussian radial basis function (G-RBF) is also frequently investigated because of its good error bound and numerical tractability [6]. J. Rashidinia et al. [7] reported a new stable method based on the eigen-function expansion for the evaluation of G-RBF. In addition, the G-RBF
has promising prospects in the numerical solutions of high-dimensional problems [8-9]. In this work, the G-RBF is applied to handling with unsteady groundwater flow governed by nonlinear PDEs. During the whole solution process, no iterations are required for nonlinear PDEs. There is hardly any literature to deal with aforementioned problems using G-RBFs. The accuracy of the approximate solution to PDEs is related to a free parameter, which controls the shape of the RBF known as shape parameter. However, the determination of an optimal shape parameter \( c \) has been an outstanding problem for many years. In practice, the most common empirical approaches to choosing a good value have been used, such as the power law formulation and the curvature formulation [10]. Here, we will choose good values of the constant \( c \) from a simple approach.

The structure of the paper is as follows. In Section 2, brief introduction to the RBF is given. In Section 3, we extend the method to variable coefficient, linear problems. Nonlinear PDEs will be dealt with in Section 4. Extensive numerical comparisons of G-RBF and MQ-RBF for solving 2D elliptic equations are illustrated by four cases in Section 5. Numerical results illustrate that G-RBF provides an optimal accuracy in PDEs numerically.

2. Radial Basis Function
A multivariate real-valued function \( \Phi: \mathbb{R}^t \rightarrow \mathbb{R} \) is called radial if there exists a univariate function \( \varphi:[0,\infty) \rightarrow \mathbb{R} \) such that \( \Phi(x) = \varphi(\|x\|) \), where \( x \in \mathbb{R}^t \), \( t \) represents the dimension and \( \| \| \) is the Euclidean norm.

Interpolation with RBFs takes the form:

\[
S(x) = \sum_{i=1}^{N} \alpha_i \varphi(\|x-x_i\|) \tag{1}
\]

where \( N \) is the number of total collocation points. This approximation is solved for the \( \alpha_i \) unknowns from the system of \( N \) linear equations of the type

\[
s(x_i) = f(x_i), i = 1,2,...,N \tag{2}
\]

where \( X = \{x_1,x_2,...,x_N\} \) are a given set of distinct centre points in \( \mathbb{R}^t \) and \( f(x_i) \) in \( \mathbb{R} \) is known. Equations (1)-(2) lead to the following

\[
A\alpha = f \tag{3}
\]

where

\[
A_{ij} = \|x_i-x_j\|, i,j = 1,2,...,N
\]

\[
f = [f(x_1), f(x_2),...,f(x_N)]^T
\]

\[
\alpha = [\alpha_1, \alpha_2,...,\alpha_N]^T
\]

The matrix \( A \) is called the interpolation matrix. Although many basis functions \( \varphi(r) \) have been used, in this research, we emphasis on the G-RBF

\[
\varphi(r) = e^{-c r^t}
\]

where \( c > 0 \) is a free parameter known as the shape parameter, and \( r = \|x-x_i\| \). In the following sections, we will focus on solving different types of PDEs with G-RBFs.

3. Gaussian RBF for nonlinear PDEs
Consider a nonlinear elliptic PDE with a linear boundary condition as:

\[
N_{\partial}\mu(X) = f_{\partial}(X) \quad X \in \Omega \tag{5}
\]

\[
Bu(X) = g(X) \quad X \in \partial \Omega \tag{6}
\]

where, \( N_{\partial} \) represents a nonlinear differential operator, \( B \) is the boundary operator and \( \Omega \subset \mathbb{R}^t \). Take this form for example

\[
N_{\partial}\mu = \Delta u + u_x^2 u_{xx} + u_y^2 u_{yy} - 2u_x u_y u_{xy} - k(1+u_x^2 + u_y^2)^{3/2}
\]

\[
(7)
\]
where \( k = \sqrt{2/5} \). The approximate solution for the problem (7)-(8) can then be expressed as
\[
\tilde{u}(x, y) = \sum_{j=1}^{N} c_j \varphi_j(x, y)
\]
(9)
where \( \{c_j\}_{j=1}^{N} \) are the unknown coefficients to be determined. The following equation is the mixed partial differential terms
\[
u_{ij} = 4e^{2xy}e^{-\alpha x^2}
\]
(10)
Then, substituting the value from equations (4) into equations (7)-(8), we have
\[
\sum_{j=1}^{N} \alpha_j \{(4e^{4r^2} - 4e^{2})e^{-\alpha r^2} - 8e^{6} r^2 e^{-3\alpha r^2} - \frac{\sqrt{2}}{5}(1 + 4e^{4} r^2 e^{-2\alpha r^2})^{3/2}\} = 0
\]
(11)
\[
\sum_{j=1}^{N} \alpha_j e^{-\alpha x^2} = g(x_i, y_i)
\]
(12)
for \( i = 1, 2, ..., n \). We will get the unknown \( \alpha_j \) by solving the linear system of equations (11)-(12). Therefore, the numerical approximate solution \( \tilde{u} \) of the problem can be obtained from equation (9).

4. Numerical results and discussions
To illustrate the effectiveness of the numerical results obtained using the G-RBF, four numerical examples are separately presented in this section. For simplicity, the Dirichlet boundary conditions are prescribed using the exact solutions and the domain is distributed with uniformly interpolation points. For the nonlinear case, we consider the irregular domain with 574 interior points and 105 boundary points. In addition, the determination of an optimal shape parameter \( c \) is achieved by simple numerical tests, from which we found that the optimal \( c \) can be chose from the interval \([0.001, 2]\). In order to investigate the accuracy of numerical method proposed, the results of G-RBF are compared with that of MQ-RBF in the following examples.
To examine the accuracy and stability of the proposed methods given in the above sections, the relative average error (root mean square relative error: RMSE) in the following figures is defined as follows [11-12]:
\[
RMSE = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \left| \frac{u(x_j) - \tilde{u}(x_j)}{u(x_j)} \right|^2}
\]
for \( |u(x_j)| \geq 10^{-3} \) and
\[
RMSE = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \left| \frac{u(x_j) - \tilde{u}(x_j)}{u(x_j)} \right|^2}
\]
for \( |u(x_j)| < 10^{-3} \), where \( j \) is the index of inner point we are interested in, \( u(x_j) \) and \( \tilde{u}(x_j) \) the exact and numerical solutions respectively, and \( N \) denotes the total number of interior testing knots. The behaviours of the PDEs using two different RBF methods are shown in the given curves of the relative average error.
To demonstrate the effectiveness of proposed methods, we consider the above-mentioned nonlinear PDEs (7)-(8). In this research, a non-iterative numerical method deserves to mention for solving nonlinear elliptic equations of second order. Note that \( k = \sqrt{2/5} \), \( g(x, y) = (50 - x^2 - y^2)^{3/2} \). Thus, the exact solution of this problem is \( u(x, y) = (50 - x^2 - y^2)^{3/2} \). For convenience of calculations, we only take the Dirichlet boundary into account and the domain \( \Omega = \Omega_1 + \Omega_2 \) is formed by combining a square and circular domain.
\[ \Omega_1 = \{(x,y) | 0 \leq x \leq 2, 0 \leq y \leq 2\} \]
\[ \Omega_2 = \{(x,y) | x = \cos \theta + 2, y = \sin \theta + 2, -\frac{\pi}{2} \leq \theta \leq \pi\} \]

The total collocation points in domain \( \Omega \) are shown in Figure 1.

Here, we shall analyse the impacts of the variation of the shape parameter \( c \) with G-RBF and MQ-RBF numerically. It can be seen from Figure 2 that when \( c \) is smaller than 0.5, the performance of G-RBF is better than that of MQ-RBF. By contrast, MQ-RBF obtains more accurate approximation solutions since the shape parameter \( c \) is bigger than 0.5. Besides, we observe that the optimal errors of using G-RBF(with \( c = 0.141 \)) and G-RBF(with \( c = 1.721 \)) are \( 8.963 \times 10^{-7} \) and \( 3.051 \times 10^{-8} \), separately.
5. Conclusions
The Gaussian RBF (G-RBF) collocation method is applied to solving unsteady groundwater flow governed by nonlinear elliptic PDEs in this paper. During the whole solution process, no iterations are required for a kind of nonlinear PDEs. Comparisons between the results of using the G-RBF and the MQ-RBF reveal that very similar performances are obtained. However, the G-RBF seems to perform slightly better for the types of problems we studied. According to the results in this work, the proposed method, while being a simple, accurate and efficient method, is able to achieve very good performance in terms of a wide range of unsteady groundwater flow governed by PEDs in spite of some outstanding theoretical issues.

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