A new model of turbulent relative dispersion: a self-similar telegraph equation based on persistently separating motions

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Turbulent relative dispersion is studied theoretically with a focus on the evolution of probability distribution of the relative separation of two passive particles. A finite separation speed and a finite correlation of relative velocity, which are crucial for real turbulence, are implemented to a master equation by multiple-scale consideration. A telegraph equation with scale-dependent coefficients is derived in the continuous limit. Unlike the conventional case, the telegraph equation has a similarity solution bounded by the maximum separation. The evolution is characterized by two parameters: the strength of persistency of separating motions and the coefficient of the drift term. These parameters are connected to Richardson’s constant and, thus, expected to be universal. The relationship between the drift term and coherent structures is discussed for two 2-D turbulences.

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Turbulent transport and mixing underlie wide range of phenomena from star formations to coffee in a cup. However, the mechanisms of their significant deviation from molecular counterparts are not well understood yet even in such a simple case as relative dispersion of passive particles.

In the inertial subrange, reflecting the scaling law of turbulent velocity fluctuation, well-known Richardson’s $r^3$ law is realized. For the probability density function (PDF) of separation $r$, $P(r,t)$, quite a few models and theories consistent with Richardson’s law have been proposed. Besides recent progresses of particle tracking techniques and numerical simulations enable direct experimental investigations of the separation PDF. Although accuracies of the experiments are not enough, most of their results are so far close to the prediction of Richardson’s diffusion equation:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left[ K(r)^{d-1} \frac{\partial}{\partial r} \left( \frac{P}{r^{d-1}} \right) \right], \tag{1}$$

where $K(r) \propto r^{4/3}$ in the Kolmogorov scaling and $d$ is the spatial dimension. This closeness indicates Eq. (1) is a good basis for description of turbulent relative dispersion. Eq. (1) ought to be exact only if the relative velocity is $\delta$-correlated in time. However, there are spatio-temporal correlations in turbulent flows as implied by existence of coherent structures. This indicates relative velocity cannot be $\delta$-correlated in time. To resolve the inconsistency, Eq. (1) has to be extended to include these correlations. As mentioned in detail below, we treat these correlations as those not in time but in scale-space. That is, we focus on correlations between scales $r$ and $\rho r$, where $\rho$ is a scale multiplier. This treatment is appropriate for describing self-similarity. Employing multiple-scale consideration with correlation in scale, we derive a telegraph equation with scale-dependent coefficients, Eq. (11), and obtain a similarity solution, which never coincides with Richardson’s one markedly in the tail part even in the long time limit.

Coherent structures observed in turbulence must share their origin with finite correlation and self-similarity, so that particles appear to be advected according to coherent structures. For example, in two-dimensional (2D) inverse cascade (IC) turbulence, particles separate step-by-step through nested cat’s eye vortices with scattering by stagnation points; in 2D free convection (FC) turbulence, particles separate through advection by stretching and folding plumes (Fig. 1). Clearly, separation processes in these two systems are different. However, despite these differences, effects of coherent structures on relative dispersion appear in the same way, i.e., persistent separation: persistent expansion and compression of a relative separation. Sokolov et al. model such motions based on Lévy walk. Their model consists of persistent separation ceased by probabilistic turn in direction. They introduced also the persistent parameter $P_s$, the ratio of the correlation length to the scale.

The telegraph model was introduced to implement (I) a finite diffusion speed, and (II) a finite correlation time to...
the diffusion process. It has been widely applied to various diffusion phenomena from molecular diffusion to population dynamics.

To satisfy the self-similarity of turbulence, (I) and (II) are extended to be scale-dependent according to the following scaling assumptions: \( v(r) = Ar^{1-g} \) and \( T_c(r) = r/v(r) = A^{-1}r^g \), where \( v(r) \) and \( T_c(r) \) are relative velocity and characteristic time, \( A \) a constant, and \( g \) a scaling exponent.

First, we focus on a scale \( \hat{r} \) in the inertial subrange. The correlation length and time for a separation to be expanded or compressed persistently are defined as \( P^\pm_{s} \hat{r} \) and \( P^\pm_{s} T_c(\hat{r}) \), respectively. We assume \( P^\pm_{s} \) is the order of unity, and then, \( \rho_{\text{out}} = 1 + P^\pm_{s} \). We call this scale outer.

We consider the evolution of probability density of the separation in a small region around \( \hat{r} \) where its extent is much smaller than \( P^\pm_{s} \hat{r} \). Since \( P^\pm_{s} \hat{r} \) and \( P^\pm_{s} T_c(\hat{r}) \) are regarded as constants, we can apply the approach deriving the telegraph equation to this small region. We call this scale inner.

We divide the inner region into shells defined as \( [r_n, r_{n+1}] \) where \( r_n = \hat{r} \xi^n \), \( n \) is integer, and \( \xi \) is close to unity, that is, \( \rho_n \approx \xi^n \). Since \( \rho_n \approx 1 \), we deal with two scales in scale-space. To take into a finite speed, pass-through time \( \tau_n \) of the \( n \)-th shell, the time for a relative separation to expand (compress) through the \( n \)-th shell, is defined as follows:

\[
\tau_n = \frac{\Delta r_n}{v(r_n)} = \frac{r_{n+1} - r_n}{v(r_n)} \approx \gamma r_n / v(r_n),
\]

where \( \gamma = \log(\xi) (\xi \ll 1) \), and \( v(r_n) \) is the relative velocity at a spatial scale \( r_n \). This relation means \( O(\tau_n) = O(\gamma) \), which is the key difference from the diffusion equation case including Richardson’s equation.

We introduce probabilities \( Q^\pm_n \): the probability for a relative separation to be expanding (\( Q^+_n \)) or compressing (\( Q^-_n \)) in the \( n \)-th shell. Transition probabilities \( \Delta p^\pm_n \), the probability from expansion to compression (+) and the opposite (−) during \( \tau_n \), must be self-similar in outer scale. Therefore the simplest form of \( \Delta p^\pm_n \) is

\[
\Delta p^\pm_n(r_n) = \frac{\tau_n}{P^\pm_{s} T_c(\hat{r})} = \frac{\lambda^\pm}{T_c(\hat{r})} \tau_n,
\]

where \( P^\pm_{s} T_c(\hat{r}) \) are correlation times depending on the direction and \( \lambda^\pm = 1/P^\pm_{s} \). Note that \( P^\pm_{s} T_c(\hat{r}) \) are outer scale and, thus, constants in the inner region. This form was first given by Sokolov et al. [11]. Unlike their model, we assign different values to \( \lambda^\pm \) to represent difference of persistency between expansion and compression.

Based on above considerations, we construct master equations for \( Q^+_n(t) \) and \( Q^-_n(t) \) as follows:

\[
Q^+_n(t + \tau_{n-1}) = \left( 1 - \frac{\lambda^+}{T_c(\hat{r})} \tau_{n-1} \right) Q^+_n(t) + \frac{\lambda^+}{T_c(\hat{r})} \tau_{n-1} Q^-_n(t) + \left( 1 - \frac{\tau_{n-1}}{\tau_n} \right) Q^n(t),
\]

\[
Q^-_n(t + \tau_{n-1}) = \left( 1 - \frac{\lambda^-}{T_c(\hat{r})} \tau_{n-1} \right) Q^-_n(t) + \frac{\lambda^-}{T_c(\hat{r})} \tau_{n-1} Q^+_n(t) + \left( 1 - \frac{\tau_{n-1}}{\tau_n} \right) Q^n(t).
\]

The last terms in the r.h.s. of Eqs. [11] and [12] denote the remainder of particle pairs leaving the \( n \)-th shell during \( \tau_{n-1} \) and \( \tau_{n+1} \), respectively. Because \( \Delta r_n \ll 1 \) and \( \tau_n \ll 1 

(i) the Kramers-Moyal expansion and (ii) addition and subtraction of Eqs. [11] and [12] lead to

\[
\frac{\partial}{\partial t}(Q^+_n + Q^-_n) + \frac{\partial}{\partial n} \left( \frac{Q^+_n}{\tau_n} - \frac{Q^-_n}{\tau_n} \right) = 0,
\]

\[
\frac{\partial}{\partial t}(Q^+_n - Q^-_n) + \frac{\partial}{\partial n} \left( \frac{Q^+_n}{\tau_n} + \frac{Q^-_n}{\tau_n} \right) = -2\lambda^+ \frac{Q^+_n}{T_c(\hat{r})} + 2\lambda^- \frac{Q^-_n}{T_c(\hat{r})}.
\]

In order to combine Eqs. [11] and [12], first, divided by \( \tau_n \) and differentiated with respect to \( n \), then

\[
\frac{\partial^2}{\partial n \partial t} \left( \frac{Q^+_n}{\tau_n} - \frac{Q^-_n}{\tau_n} \right) + \frac{\partial}{\partial n} \left[ \frac{1}{\tau_n} \frac{\partial}{\partial n} \left( \frac{Q^+_n}{\tau_n} + \frac{Q^-_n}{\tau_n} \right) \right] = \frac{\lambda}{T_c(\hat{r})} \frac{\partial Q_n}{\partial t} - \frac{\delta}{\lambda \partial n} \left( \frac{Q_n}{\tau_n} \right),
\]

where \( Q_n = Q^+_n + Q^-_n \), \( \lambda \equiv \lambda^+ + \lambda^- \), and \( \delta \equiv \lambda^+ - \lambda^- \).

Next, differentiating [14] with respect to \( t \) and substituting [15] to it, we obtain an equation for \( Q_n \) around \( \hat{r} \):

\[
\frac{T_c(\hat{r})}{\lambda} \frac{\partial^2 P}{\partial t^2} + \frac{\partial P}{\partial t} = \frac{T_c(\hat{r})}{\lambda} \frac{\partial}{\partial \hat{r}} \left( \frac{v(r)}{\hat{r}} P(r) \right) + \frac{\delta}{\lambda \partial \hat{r}} \left( v(r) \right) P(r).
\]

Then Eq. [16] with the scaling assumptions yields

\[
\frac{T_c(\hat{r})}{\lambda} \frac{\partial^2 P}{\partial t^2} + \frac{\partial P}{\partial t} = D(\hat{r}) \frac{\partial^{d-1}}{\partial \hat{r}^{d-1}} \left( \frac{P}{\hat{r}^{d-1}} \right) + \sigma \frac{\partial}{\partial \hat{r}} \left( v(r) \right) P(r),
\]

where \( D(\hat{r}) \) is Richardson’s diffusion coefficient, \( \lambda^{-1} r^2 g \), and \( \sigma \equiv (d - 2g + \delta) \lambda^{-1} \equiv \delta \lambda^{-1} \). The conservation of probability is satisfied at least for the similarity solution.
The limit of infinite speed and δ-correlation, the first term of Eq. (11) disappears, which is the same form as Palm’s equation (see p.575 of [3]) [20]. Non-Richardson terms, the first term in the l.h.s. and the last one in the r.h.s. of Eq. (11), describe effects of persistent separation. The last term in the r.h.s. of Eq. (11) is a drift term consistent with the scaling assumptions. The drift velocity is \( -\sigma v(r) \) and, hence, the direction is determined by \( \delta \), which consists of the “scaling-determined” part, \( d - 2g \), and the “dynamics-determined” part, \( \delta \).

Now, let us calculate the similarity solution of Eq. (11). To obtain it, we introduce the similarity variable \( \xi \),

\[
\xi \equiv \left( \frac{r}{\langle r^2 \rangle^{1/2}} \right)^g = C_R^{-g} \frac{\lambda}{At} r^g,
\]

where we set \( \langle r^2 \rangle^{1/2} = C_R (At/\lambda)^{1/2} \) [21]. Substituting \( P(r, t) = F(\xi) \langle r^2 \rangle^{-1/2} \) into Eq. (11), we obtain an equation of \( F(\xi) = F(\eta/C_R^g) = F(\eta) \):

\[
\frac{d^2 F}{d\eta^2} + H_1(\eta) \frac{dF}{d\eta} + H_2(\eta) F = 0,
\]

where \( \eta = C_R^{-g} \xi \),

\[
H_1(\eta) = \frac{2(1 + g) \eta^2 - \lambda^2 g \eta - \lambda^2 g^2 (2 - \tilde{d})}{(\eta^2 - \lambda^2 g^2 \eta) \eta},
\]

\[
H_2(\eta) = \frac{(1 + g) \eta^2 - \lambda^2 g \eta - \lambda^2 g^2 (1 - \tilde{d})}{(\eta^2 - \lambda^2 g^2 \eta) \eta^2},
\]

and \( \tilde{d} = d - \sigma \). Eq. (13) has three fixed singular points: \( \xi = 0 \) and \( \xi = \xi_\pm = \pm \lambda_0 \eta / C_R^g \). Starting from an initial condition, where all particles occupy a small and compact region, the solution is confined in \([0, \xi_+]\). This finiteness of the similarity solution is due to the assumption of a finite speed and, thus, the maximum relative separation, \( r_{\text{max}} = \xi_+ \langle r^2 \rangle^{1/2} = (At/\lambda)^{1/2}, \) exists [22].

To see asymptotic behaviors of the similarity solution, we consider two limits: the small \( \xi \ll 1 \) and the maximum \( (\xi \sim \xi_+) \) separation regimes. In the small separation regime, \( \xi \ll 1 \) or \( r \ll \langle r^2 \rangle^{1/2} \), we obtain the following solution:

\[
F(\xi) \propto (\beta_1 \xi)^{\beta_2} \exp(\beta_2 \xi),
\]

where \( \beta_1 = (d - \sigma - 1)/g \) and \( \beta_2 = C_R^g / g^2 \). This solution is also a similarity solution of Palm’s equation, because the first term in Eq. (11) can be neglected under the condition \( r \ll \langle r^2 \rangle^{1/2} \). We call this limit the diffusive regime.

On the other hand, in the maximum separation regime, the frontal edge of the PDF is abrupt at \( r_{\text{max}} \). The functional form of the edge is approximated to the first order by

\[
F(\xi) \propto (\xi_+ - \xi)^{-\sigma_2 / g^2}.
\]

We call this limit the telegraphic regime. From this expression, it is clear that if \( \lambda < d \), \( P(r, t) \rightarrow \infty \) as \( r \rightarrow r_{\text{max}} \). That is, most of particle pairs are accumulated at the frontal edge, where relative separations expand away without changing their directions. However, this situation is unrealistic in real turbulence, so that \( \lambda \) is considered to be greater than \( d \).

Control parameters of our model are \( \lambda \) and \( \sigma \). The functional form of the PDF, \( F(\xi) \), is determined by these parameters: \( \sigma \) controls mainly the diffusive regime, and \( \lambda \) does the telegraphic regime. As \( \lambda = 1/P_\sigma^+ + 1/P_\sigma^- \), it represents the strength of persistency of moving direction. On the other hand, as \( \sigma \lambda^{-1} \) is the coefficient of the drift term, \( \sigma \) represents total effects of persistent separations and probabilistic transitions. Because persistent motions model the advection by coherent and self-similar flows, \( \sigma \) seems to characterize the average effects of flow structures such as coherent structures on dispersion processes. In order to calculate the value of \( \sigma \), we have to estimate “dynamics-determined” part of it, \( \delta \).

To estimate \( \delta \) from direct numerical simulations (DNS), we use the PDF of exit-time [3]. Exit-time for particle-pairs experienced many turns form an exponential tail (for details see [6]). In our case, the slope is evaluated with Eq. (11) in the limit of infinite time, i.e., Palm’s equation, where the slope is related to \( \delta \) [23].

We can also estimate \( \delta \) directly from the separation PDF around \( r \ll \langle r^2 \rangle^{1/2} \). In the 2D-IC case, Goto and Vassilicos estimated \( \alpha = 2/(g - \delta) \) by fitting the similarity solution of Palm’s equation to the PDF [7].

Table I shows the estimated values of \( \delta \) and \( \sigma \) for 2D-IC and -FC turbulences. These results indicate that the drift term of (11) enhances diffusion in the 2D-IC case but suppresses diffusion in the 2D-FC case; Compression of relative separations in 2D-IC turbulence but expansion of them in 2D-FC turbulence are comparatively restricted, respectively. This remarkable feature is considered to be induced by the difference in flow structures between 2D-IC and -FC turbulences: “cat’s eye in a cat’s eye” structures [7] and string-like structures [8]. We, therefore, expect that \( \sigma \) can characterize coherent structures.

In Fig. 2, the similarity solution \( F(\xi) \) obtained numerically for various values of \( \lambda \) is shown. A cut-off scale corresponding to \( r_{\text{max}} \) can be seen. Our similarity solu-

| Type          | Method               | \( \delta \) | \( \sigma \) |
|---------------|----------------------|--------------|-------------|
| 2D-IC\(a\)   | Exit-time PDF        | -1.48        | -0.81       |
| 2D-IC\(b\)   | Separation PDF       | -0.87        | -0.20       |
| 2D-FC\(c\)   | Exit-time PDF        | -0.71        | 0.49        |

\( a \)From DNS results by Boffetta and Sokolov [6]. Slope \( \approx -0.3 \).

\( b \)From DNS results by Goto and Vassilicos [7]. \( \alpha \approx 1.3 \).

\( c \)From our DNS with resolution 2048 [13].
tion approaches Palm’s one as $\lambda$ gets larger. However, even for large $\lambda$, the difference between them in the tail part is so evident that effects of persistent separation are not negligible.

In summary, we derived a telegraph equation with scale-dependent coefficients, into which finite separation speed and self-similarity are incorporated, by employing multiple-scale consideration in scale-space. Then we obtained a similarity solution of it. In the diffusive regime, $\xi \ll 1$, the similarity solution coincides with that of Richardson’s diffusion equation with the drift term, i.e., Palm’s equation; in the telegraphic regime, $\xi \sim \xi_{\text{max}}$, the finiteness of separation speed is realized and the separation PDF is abrupt at $r_{\text{max}}$. Therefore, finite separation speed is crucial for description of the tail part of the separation PDF unless relative velocity is $\delta$-correlated in time.

The drift term of Eq. [11] is induced by the deviation of the difference of persistency between expansion and compression of relative separation, $\delta$, from “scaling-determined” value, $2g - d$. The direction of the drift is determined by the sign of $-\delta = (2g - d) - \delta$. We estimated the value for two 2-D turblences, inverse cascade (IC) and free convection (FC), and found that positive and negative drift is imposed in the 2D-IC and -FC cases, respectively. We conjecture that this remarkable difference corresponds to the different types of coherent structures in the background flow. We need more precise investigations to obtain an evidence for this.

We neglected two significant effects: distribution of separation speed and intermittency. However, intermittency of relative velocity is negligible in the two 2D turbulences dealt with in this letter. Besides, not the distribution but the finiteness and self-similarity of separation speed is crucial for the existence of the cut-off of the separation PDF. We are expecting experiments which can resolve the tail part of the separation PDF.

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FIG. 2: PDFs of relative separation for the 2D-IC case. Solid lines represent similarity solutions of our model with $\lambda = 4, 6, 8, 10, 15$ from the curve with the smallest cut-off scale to that of the largest. The dashed line represents $\lambda = \infty$, or the similarity solution of Palm’s equation [10]. $\tilde{\delta}$ is fixed for $1/3$: $d = 2$, $g = 2/3$, and $\delta = -1$. The inset is the same plot in linear scale. Lines correspond to $\lambda = 4, 6, 8, 10, 15$, and $\infty$ from the curve with the lowest peak value to that of the highest.