Self organized criticality in a sandpile model with threshold dissipation

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We study a nonconservative sandpile model in one dimension, in which, if the height at any site exceeds a threshold value, the site topples by transferring one particle along each bond connecting it to its neighbours. Its height is then set to one, irrespective of the initial value. The model shows nontrivial critical behavior. We solve this model analytically in one dimension for all driving rates.

We calculate all the two point correlation functions in this model, and find that the average local height decreases as inverse of the distance from the nearest boundary and the power spectrum of fluctuations of the total mass varies as $1/f$.

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Self-organized Criticality (SOC) was proposed by Bak, Tang and Wiesenfeld [1] to explain the widespread occurrence of fractal structures and $1/f$ noise in nature. They suggested that extended dissipative systems evolve to a critical state by a self-organizing process and proposed the sandpile model as a prototype of SOC systems. Since then several sandpile-like stochastic cellular automata models with threshold dynamics have been studied to understand the origin of criticality. In models with bulk mass conservation [2–5], the balance of rate of driving and the rate of loss of particles through the boundary leads to a divergence of the average avalanche size in the steady state. However, the origin of criticality in nonconservative sandpile models is still not well understood [6,7]. In particular, the nature of nonconservation which can preserve the criticality of the model are not known. Analytic proofs of criticality in nonconservative sandpile models are lacking even in one dimension.

In this paper we study a discrete nonconservative sandpile model [9] in which mass dissipation (loss of a particles) occurs if the height at a site exceeds a threshold value. In the steady state, our model organizes such that mass dissipation is minimized. As a result, it shows nontrivial critical behavior. We calculate its spatial and temporal correlation functions analytically in one dimension. The average height profile in the steady state of this model has a nontrivial power law dependence on the distance from the nearest boundary. We find that the power spectrum of the fluctuations of the total mass of the sandpile shows $1/f$ behaviour. These power laws are robust and do not depend on the details of the model. The model is critical even for finite driving rate (unlike the forest fire model [8], which is critical only for infinitesimal driving rate). To our knowledge, this is the first nonconservative model of SOC which has been shown to be critical for all driving rates.

The model is defined as follows: There is an integer variable $h_i$ (the number of particles), at each site $i$. The particles are added to the system at randomly selected sites. If $h_i$ exceeds the threshold height $h_i^c$, then $h_i$ is set to 1 and one particle is transferred to each neighbor of $i$. This process occurs at all sites in parallel. We choose $h_i^c$ to be equal to the coordination number for $i$ inside the bulk and to be greater than the coordination number for $i$ on the boundary. Note that this model is not abelian [2]. In the Abelian sandpile model (ASM), $h_i$ decreases by $h_i^c$ after the toppling, whereas in this model $h_i$ is set to 1 irrespective of its initial value.

The dynamics of this model resembles the dynamics of the continuous stick-slip model discussed by Feder and Feder (FF) [6]. The FF model has a real variable $u_i$ at each site $i$, which grows slowly with time. If $u_i$ exceeds $U_c$, then $u_i$ is set to zero and $u_j$ is increased by 1 for all neighbors $j$ of site $i$. Let us define $h_i$ as the integer part of $(u_i + 1)$ and $h_i^c$ as $(U_c + 1)$. Then clearly, the updating rule can be written in terms of $h_i$ and it is identical to the toppling rule of our model. However, the external driving force in FF model is deterministic and not noisy as in our model. Note that the argument given above crucially depends on the fact that the transfer to the neighbours does not depend on the local variable. Therefore, it does not hold for the earthquake model [7], where the transfer is proportional to the local variable.

To illustrate the difference between this model and the ASM, consider the model on a square lattice and choose an initial configuration in which the four corners of a given plaquette have all heights equal to 4 (threshold height). Now add a particle at the lower left corner of the plaquette, say at $(i, j)$. First $(i, j)$ topples then $(i + 1, j)$ and $(i, j + 1)$ topple. After this the height at $(i + 1, j + 1)$ becomes 6. So far there is no dissipation (just as in the ASM). When $(i + 1, j + 1)$ topples, unlike in the ASM, 1 particle is lost. Note that this model is the same as the ASM if the underlying graph does not have a loop.

In spite of the non-abelian character, the set recurrent configurations of this model $S_1$ is the same as that of the Abelian sandpile model $S_2$. To prove this, we first show $S_2 \subset S_1$. Let us add a particle in $C_0 \in S_2$ and let it relax by the rules of the ASM. Let the final configuration be $C_1$. Now we add a particle at the same site in $C_0$ and let...
it relax by the rules of this model. If there is no dissipation then the final configuration is the same as $C_1$. If there is dissipation, then we compensate for the dissipation by adding particles one by one at sites where dissipation has occurred and letting the system relax. This process is repeated until no site is left where dissipation has occurred. Since in the ASM the final configuration does not depend on the order of the topplings, this sequence of additions and relaxations in this model must lead to $C_1$. Thus, in this model, there is a finite probability of transitions from one recurrent configuration of the ASM to the another. This implies that $S_2 \subset S_1$. One can easily check using the argument of Ref. [2] that the forbidden configurations of the ASM are also forbidden in this model. Therefore, $S_1 = S_2$. However, unlike the abelian case, the probabilities of occurrence of different recurrent configurations in the steady state of this model need not be equal.

Let us analyse this model in one dimension. The model on a simple linear chain is not different from the ASM, because this lattice has no loops. Hence, we consider this model on decorated one dimensional chains, formed by joining unit cells which have loops (see Fig. 1). We have earlier studied ASM’s on this type of chain and found interesting finite-size scaling behavior [10], which differs from that seen in simple linear chains [11].

Consider first the chain of doublets (case A, Fig. 1). We label the doublets by integers $i = -l$ to $l$ and the sites inside the doublet by $j = 1$ to 2. The size of the chain $L = 2l + 1$. We generalize the rule such that after the toppling one particle is transferred along each bond connecting the site to its neighbors, and take the threshold height equal to 3 for all sites. The recurrent configurations of this model are characterized using the burning algorithm as in the ASM. This algorithm is specified by the following rule: A site is burnt if its height is greater than the number of bonds joining it to its unburnt neighbours (see [2], for detail). A stable configuration is recurrent if and only if all the sites are eventually burnt.

In this algorithm the sites can be burnt in any order. We let the burning start from the left boundary and hold right boundary unburnt. The point at which burning from this boundary stops is called the break point (BP). Afterwards, the right boundary is burnt and subsequently the remaining sites. Thus the allowed values of $(h_{11}, h_{12})$ for $i$ to the left of the BP are (3, 3) and (3, 2), and those for $i$ to the right of the BP are (3, 3) and (2, 3). The allowed values at the BP are (1, 3), (3, 1) and (2, 3).

In any side the avalanche spreads to the first doublet which cannot be burnt from that side. For example, if the particle is added to the left of the BP the avalanche spreads to the BP on the right and to the first doublet of type (3, 2) on the left. For a typical avalanche the distance between the site at which the particle is added and the BP is $O(L)$. Therefore, the spread of avalanche is $O(L)$.

When the avalanche crosses a doublet of type (3, 3), one particle is dissipated. Thus, each avalanche wipes out (3, 3) type of doublets from a region of order $L$, leaving a small density of the (3, 3) type doublets. This minimizes the possibility of dissipation. Also, since the steady state configurations are dominated by (3, 2) type doublets (2, 3 type doublets) on the left (right) of the BP, the leftward (rightward) spread of the avalanche starting from the left (right) of the BP is small (see Fig. 2).

The BP shows an interesting stochastic dynamics. After the avalanche, it moves towards the starting point of the avalanche, by a distance of order 1. If $i$ is the position of the BP then the probability that an avalanche starts from the left of the BP is $(l + i)/(2L)$ to leading order (ignoring the density of (3, 3) type of doublets), and that it starts from the right of the BP is $(l - i)/(2L)$. Hence, the mean displacement of the BP after the avalanche scales as $i/(2L)$. The fluctuation about this mean value is of order 1. The dynamics of the BP can thus be described by an equation of the following form

$$\frac{dx}{dt} = -\frac{x}{2L} + \eta(t)$$  (1)

where $x$ is the scaled variable $i/L$, and $dt$ is of the order of the time interval at which avalanches hit the break point. The noise $\eta(t) \sim 1/L$, and is $\delta$-correlated. From the above equation it follows that the number of avalanches required to reach the steady state is of order $L$, and the asymptotic distribution of the BP goes as $\exp(-cx^2 L)$, where $c$ is a constant. The width of this distribution can be ignored in the large $L$ limit. A typical avalanche then starts from the point at which particle is added (source point) and terminates at the center. If the source point is $i$ then the linear extension $s$ and the duration $t$ of an avalanche are equal to $|i - L/2|$. Averaging over $i$ we get

$$\text{Prob}(X) \sim 2/L \text{ for } X < L/2,$$

and 0 otherwise, where $X = t, s$.

Now we calculate the correlation functions in this model. The model has two intrinsic time scales, the time taken for one toppling (defined as one time step), and the time interval $T$ between two consecutive addition of particles. Let us first consider the case, $T > L$, such that there is no overlap between two consecutive avalanches (see Eq. (2)).

It is convenient to use a two state variable $s_i$, because there are only two allowed configurations of each doublet (except the break point). For the (3, 3) type doublet, $s_i = 1$, otherwise $s_i = 0$. Consider the evolution $s_i$ for
\( \sqrt{L} \ll i \ll l \). We can assume that the BP is on left of \( i \). If the avalanche starts from the right of \( i \), then it crosses \( i \) to reach the BP, setting \( s_i = 0 \). The probability of this transition goes as \((l - i)/(2L)\). The transition of \( s_i \) from 0 to 1 occurs, if the particle is added at the left site of the \( i \)-th doublet, or if it is added to the left of the \( i \)-th doublet, but the avalanche reaches the \((i-1)\)-th doublet so that the left site of \( i \)-th doublet receives a particle. The probability of this transition goes as \( 1/L \). The corresponding probabilities for \(-l \ll i \ll -\sqrt{L} \), can be obtained by replacing \( i \) by \((-i)\) and \( l \) by \((-l)\). Thus the probability \( P_n(i) \), that \( s_i = 1 \) after additions of \( n \) particles, satisfies the following equation (to leading order)

\[
P_{n+1}(i) = \frac{1}{L} [1 - P_n(i)] - \left(1 - \frac{r(i)}{2L}\right) P_n(i) ,
\]

where \( r(i) \) is the distance of \( i \) from the nearest boundary. A straightforward calculation using the above equation gives

\[
\langle s_i \rangle \sim 2/r(i) ,
\]

and the autocorrelation

\[
\langle s_i(t_0) s_i(t_0 + t) \rangle \sim \frac{1}{r(i)} (1 - r_i^{-1})^{t/T} ,
\]

for \( t \gg T \), where \( \tau_i = 2L/r(i) \) and \( \langle \ldots \rangle \) denotes average over \( t_0 \). The value of \( s_i \) at time \( t \) is denoted by \( s_i(t) \). Note that the correction to Eq.(3) coming from the nonzero density of \( s_i \) (Eq. (4)) vanishes in the large \( L \) limit. The steady state of this model is not translationally invariant, but is self similar in space, i.e. the density and the relaxation time \( \tau_i \) depend on its distance from the nearest boundary as a power law. In Fig. 3, we plot the density of \( s_i \) against the distance from the left boundary. Note the excellent agreement of our numerical data with Eq. (4).

To calculate the correlation between \( s_i(t) \) and \( s_j(t') \), consider the evolution of the joint probability of \( s_i \) and \( s_j \). There are four possible values of \((s_i, s_j)\). Let the four-vector, \( P_n(i, j) \), denote the joint probability of \((s_i, s_j)\) after addition of \( n \) particles. Its evolution can be described by the following linear equation (to leading order)

\[
P_{n+1}(i, j) = G(i, j)P_n(i, j) ,
\]

where \( G(i, j) \) is the \((4 \times 4)\) matrix giving the transition probabilities of \( s_i \) and \( s_j \). This can be calculated in exactly the same way as the transition probability of \( s_i \) calculated above. For example, \((1,1)\) goes to \((0,0)\) if the avalanche passes over both the doubles i and j. If \( \sqrt{L} \ll i \ll j \ll l \) then the probability of this transition goes as \( r(j)/(2L) \). Other transition probabilities can also be calculated in the same way. From Eq. (6) one can find the steady state correlation functions. For \( i \) and \( j \) on the same side of the center of the chain

\[
\langle s_is_j \rangle_c \sim \frac{1}{r(i)^2} ,
\]

\[
\langle s_i(t_0) s_j(t_0 + t) \rangle_c \sim \frac{1}{r(i)} (1 - r_j^{-1})^{t/T} ,
\]

for \( r(i) > r(j) \),

\[
\langle s_i(t_0) s_j(t_0 + t) \rangle_c \sim \frac{1}{r(i)r(j)} (1 - r_i^{-1})^{t/T} ,
\]

for \( r(i) < r(j) \).

The subscript \( c \) refers to the connected part of the correlation function. If the \( i \) and \( j \) are on different sides of the center then the correlation can be ignored. Note that, in Eq. (8), the first factor gives the equal time correlation, and the second factor gives the relaxation of \( s_j \). In Eq. (9) the \( s_i(t_0) \) and \( s_j(t_0 + t) \) are anti-correlated because an avalanche which makes \( s_i = 1 \) sweeps over site \( j \), setting \( s_j = 0 \).

The autocorrelation of mass (sum of heights) can be obtained using Eq. (5), (8) and (9). The power spectrum of mass fluctuation \( S(f) \), is obtained by taking the real part of the Fourier transform of the autocorrelation

\[
S(f) \sim \frac{1}{f} , \text{ for } \frac{2\pi}{TL} \ll f \ll 2\pi .
\]
In Fig. 4, we show the log-log plot of the numerical value of $S(f)$ versus $f$. This is obtained by taking the square of the Fourier transform of the time sequence of length 5000, for a chain of size 5000. We find that at low frequencies the data shows $1/f$ behaviour. The scaling region increases with the increase in the lattice size. Since we have taken only the slowest relaxation mode in Eq. (8), (9) and (5), the agreement is not good in the high frequency regime.

Consider now the case in which particles are added at a faster rate, i.e., $1 \ll T \ll L$. We note that the interaction between the avalanches can be ignored because the avalanches propagate with the same velocity towards the BP (see Fig. 2). Thus the steady state properties of the model as described by Eq. (7) and (4) and the autocorrelation given in Eq. (5) remain unchanged in this case. Since the avalanche takes $2|j-i|$ time steps to travel from $i$ to $j$, there is no anti-correlation between $s_i(t_0)$ and $s_j(t_0 + t)$ for $t < 2|j-i|$ (see Eq. (9)). However, the contribution to the mass autocorrelation coming from the this term can be ignored for $|i-j| > \sqrt{L}$, or for $t > \sqrt{L}$. Thus for $f < 2\pi T/\sqrt{L}$, the power spectrum $S(f) \sim 1/f$.

To check the robustness of our results we have studied the model on the diamond chain (case B in Fig. 1). An avalanche in this case has the same structure as in case A, i.e., it spreads to the BP on one side, to a distance of $O(1)$ on other side and the BP is confined in a region of $O(\sqrt{L})$. As a result, the power laws of the average height and the power spectrum of mass fluctuation are also the same.

The two special doublet configurations in this model, which stop the avalanches, are reminiscent of the ‘trough’ and ‘trap’ of the ‘singular diffusion’ model studied by Carlson et al [5]. In that case too there is a power law decay of the height profile which follows from the singularity of the effective diffusion coefficient of the particles as a function of the coarse grained density [4,12]. However, in our model the mechanism of self-organization is different as the diffusion coefficient does not diverge.

To summarize, we have determined exactly the critical behavior of a non-abelian nonconservative sandpile model in the one dimension. The critical steady state shows spatial structures. We determine time dependent correlation functions for finite driving rate, and show that fluctuations of the mass of the sandpile have a $1/f$ spectrum.

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Captions

Fig. 1: The one dimensional chains formed by joining (A) doublets, (B) diamonds and (C) single sites.

Fig. 2: The evolution of the sandpile. The dot denotes an toppling event.

Fig. 3: The log-log plot of the mean height of a doublet (minimum height is subtracted) versus the distance from the nearest boundary $r$. The solid line shows the analytic expression.

Fig. 4: The log-log plot of power spectrum of total mass fluctuation $S(f)$ vs frequency $f$. For comparison the theoretical curve is shown as the solid line.

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mean doublet height vs. $r$
