Universal notion of classicality based on ontological framework

Shubhayan Sarkar

*Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotnikow 32/46, 02-668 Warsaw, Poland*

Existence of physical reality in the classical world is a well-established fact from day-to-day observations. However within quantum theory, it is not straightforward to reach such a conclusion. A framework to analyse how observations can be described using some physical states of reality in a theory independent way was recently developed, known as ontological framework. Different principles when imposed on the ontological level give rise to different observations in physical experiments. Using the ontological framework, we formulate a novel notion of classicality termed “universal classicality” which is based upon the physical principles that in classical theories pure states are physical states of reality and every projective measurement just observes the state of the system. We construct a communication task in which the success probability is bounded from above for ontological models satisfying the notion of universal classicality. Contrary to previous notions of classicality which either required systems of dimension strictly greater than two or atleast three preparations, a violation of “universal classicality” can be observed using just a pair of qubits and a pair of incompatible measurements. We further show that violations of previously known notions of classicality such as preparation non-contextuality and Bell’s local causality is a violation of universal classicality.

I. INTRODUCTION.

The operational picture of physical theories aim at understanding the physical world in terms of statistics generated by experiments performed on physical systems. The novelty of such picture relies on the idea that observable facts are enough to predict outcomes of some physical experiments without much emphasis on the underlying physical states of reality, commonly referred to as ontological states. However, there are limitations of the operational picture not just towards predictive power but also that such a picture lacks any deeper understanding of nature. As a consequence from a realist perspective, any operational prediction needs to be supported by some ontological states of reality even if they might be unobservable or hidden. Interestingly, observations from classical world exhibit that observable facts corresponds to physical realities. However, in any general theory it might also be possible that observable facts are just a reflection of some underlying physical reality even when the physical reality can not be directly observed. As a consequence in classical physics, any laws of physics applicable on the observable level is valid at the ontological level too. Building on this viewpoint, Bell in 1964 [1] could prove that quantum theory is inherently different from classical physics.

One of the interesting avenues in foundations of quantum theory is to understand how quantum theory diverges from classical theories. To facilitate such understanding, different notions of classicality have been suggested like Bell’s local causality [1], Kochen-Specker non-contextuality [2, 3], and preparation non-contextuality [4] to name a few. All such notions of classicality differ based on the assumptions imposed on the ontological level. The strength of such notions lies in the amount of quantumness which can be captured by these notions. If quantum theory violates any notion of classicality, then such a notion of classicality can also be understood as no-go theorem for quantum theory, i.e., quantum theory can not be compatible with all the assumptions which define the notion of classicality. For example, Bell’s theorem shows that any ontological description of quantum theory must violate the principle of local causality. Apart from their foundational importance all such notions of classicality have led to various tasks in cryptography, communication and computation which show quantum advantage over classical strategies. For example, the security of the well-known cryptographic scheme E91 protocol is based on Bell’s theorem [5]. Infact, every device independent scheme is based upon violations of some notion of classicality [6–10].

In this work, we construct a novel notion of classicality which is named as “universal classicality” based on the idea that in classical theories, any pure state is a physical state of reality and any projective measurement just observes the state of the system. Then, we construct an operational task which can be violated using only two pure qubit states and two incompatible measurements. We further show that the notion of universal classicality implies all the other known notions of classicality like preparation non-contextuality, Kochen-Specker non-contextuality and Bell’s local causality.

II. PRELIMINARIES

Before moving on to results, we would like to introduce the notations and relevant concepts required for this work.
A. Operational picture

Any experiment performed on some physical system can be understood as prepare, transform and measure experiment. In each run of the experiment, a preparation (the prepared system) undergoes a transformation (some dynamical process) after which the system is measured. The experiment is repeated for large number of times to gather enough statistics. In any such experiment, it is always assumed that each run of the experiment is statistically independent of other runs and the preparation, transformation and measurement remain consistent throughout the runs of the experiment.

Any preparation in the operational picture is denoted by $P$, transformation by $T$ and measurements are denoted by $M$ with outcomes denoted by $k$. The statistics are denoted by $p(k|M,P,T)$ which specifies the probability of obtaining the outcome $k$ of some measurement $M$ when the preparation $P$ undergoes a transformation $T$. For different physical theories, the probabilities are obtained in different ways. For example, in quantum theory the preparations are given by density matrices $\rho$ belonging to some $d-$dimensional Hilbert space $\mathcal{H}_d$, the transformations are unitary matrices $U$ and the measurements are given by positive operator valued measure (POVM) $\{M_k\}$. The probabilities are obtained using the Born rule, $p(k|M,P,T) = \text{Tr}(U\rho U^\dagger M_k)$. For the rest of this manuscript, we would not consider systems which evolve. As a consequence, the scenario can be simplified to only prepare and measure experiments where the probabilities are denoted by $p(k|M,P)$.

B. Ontological framework

The idea of ontic states or hidden variables was put forward by Bell in his seminal work in 1964 [1]. The idea was put in a more rigorous way by Speckens and Harrigan [11] and subsequently by Leifer [12]. The ontological framework, provides a basis for realist extensions of any physical theory and the ontic states can be understood as some “real physical states” which in general might not be observed directly. The ontic state is denoted by $\lambda$ which belongs to the ontological state space denoted by $\Lambda$. In general, any preparation procedure $P$ prepares a distribution of such ontic states denoted by $\mu(\lambda|P)$ which can be understood as probability of preparing the ontic state $\lambda$ using a preparation procedure $P$. It is required that $\Lambda$ is a measurable space along with the condition that $\int_{\Lambda} \mu(\lambda|P)d\lambda = 1$ for all preparation $P$ which signifies that every preparation must always prepare some state $\lambda \in \Lambda$ along with the constraint $\mu(\lambda|P) \geq 0$. The set $\Omega_P \in \Lambda$ represents the set of $\lambda$’s for some preparation $P$ such that the probability density $\mu(\lambda|P) \neq 0$. Thus, we also have $\int_{\Omega_P} \mu(\lambda|P)d\lambda = 1$.

Any measurement in the ontological framework just measures the ontic state. The measurements are represented as response functions denoted by $\xi(k|M,\lambda)$ which specifies the probability of obtaining the outcome $k$ of some measurement $M$ given the ontic state $\lambda$. It is also required that the response function is measurable along with the condition $\xi(k|M,\lambda) \geq 0$ and $\sum_{\lambda} \xi(k|M,\lambda) = 1$ for all $\lambda \in \Lambda$.

Using the above prescription, any probabilities obtained in the prepare and measure experiments can be represented as,

$$p(k|M,P) = \int_{\Lambda} \xi(k|M,\lambda)\mu(\lambda|P)d\lambda \quad (1)$$

Note that we assume that the preparation and measurement are related via the ontic state $\lambda$. This is known as $\lambda-$mediation.

C. Ontic-Epistemic distinction

One of the most intriguing questions in foundations of quantum theory is whether quantum state is a physical state of reality or epistemological knowledge of some underlying reality. To answer this question, the first requirement is to have a realist perspective and an ontological framework which can describe quantum theory. If every pure quantum state corresponds to some unique set of ontological states, then quantum state is ontic or corresponds to real physical state of reality. On the other hand, if two pure quantum state share some ontological states, quantum state is epistemic or just a representation of underlying physical states of reality. It was proven that for infinite dimensional systems, quantum states are necessarily ontic [13]. With some additional assumptions, it could be proven that quantum state is ontic [14, 15] even for finite dimensional systems. However, without such additional assumptions the question still remains open whether quantum states are ontic or epistemic for finite dimensional systems. Further, there has been a large number of ontological models, in some of which the quantum states are ontic like Bohmian mechanics [16, 17], Beltrametti–Bugajski model [18] and Bell model [19] to name a few. Some other ontological models of quantum theory in which the quantum state not a physical reality but just an epistemic knowledge are Kochen-Specker model [2], Spekkens toy theory [20], LBR model [21] and ABCL models [22] to name a few.

D. Pure states and projective measurements

Most of the ontological models aim to characterise pure states and projective measurements. The reason being every mixed state is just a classical mixture of pure states and within classical and quantum theory can be realised as a pure state in some higher dimensional system. Similarly, non-projective measurements within classical and quantum theory can be realised as a projective measurement acting on some higher dimensional system [23].
Any preparation \( \mathcal{P} \) which generate pure states can not be described as a convex mixture of two other preparation \( \mathcal{P}' \) and \( \mathcal{P}'' \). Mathematically, if
\[
\mathcal{P} = z \mathcal{P}' + (1 - z) \mathcal{P}''
\]
then, \( z = 1 \) and \( \mathcal{P}' = \mathcal{P} \). Similarly, any projective measurement \( \mathcal{M} \) can not be realised as some convex combination of two different measurements \( \mathcal{M}' \) and \( \mathcal{M}'' \). Mathematically, if
\[
\mathcal{M} = z \mathcal{M}' + (1 - z) \mathcal{M}''
\]
then, \( z = 1 \) and \( \mathcal{M}' = \mathcal{M} \). Now, we proceed towards the main result of this manuscript.

### III. RESULTS

We present here the novel measure of classicality termed as “universal classicality” which is based upon two physical assumptions well-supported by experiments till date. In this section, first we introduce the assumptions defining “universal classicality” which are imposed on the ontological state space. Next, we construct a simple operational task and show that the success probability of the task using classical strategies is bounded from above. We find states and measurements within quantum theory which gives a higher success probability than achievable using classical strategies. We begin by stating the assumptions.

#### A. Assumptions

The first assumption defining “universal classicality” is given as,

**Definition 1 (No overlap).** For two preparations, \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) which generate pure states, the corresponding ontic distributions \( \mu(\lambda | \mathcal{P}_1) \) and \( \mu(\lambda | \mathcal{P}_2) \) do not overlap. Mathematically,
\[
\frac{1}{2} \int_\lambda |\mu(\lambda | \mathcal{P}_1) - \mu(\lambda | \mathcal{P}_2)| \, d\lambda = 1
\]

This assumption stems from the fact in classical theories, the knowledge about the preparation producing a pure state imposes that it is a physical state of reality and as discussed before physical states of reality do not overlap. For simplicity, let us consider classical Hamiltonian dynamics. In classical Hamiltonian dynamics, any state that lies within the phase space is described by the position and momentum of the system \( (x, p) \). Now, position and momentum are physically observable quantities and can be attributed as physical states of reality or the ontic states. Going back to the ontological framework, consider a preparation which prepares a state with some fixed energy \( E \) which is a distribution over the phase space states. It is known that energy is a physically measurable quantity. As a consequence, if we are able to know the position and momentum of the system, we can predict the energy of the system. However, if the same phase space state belonged to two different energy states, then it would not have been a physically measurable quantity. Using a similar argument, it can concluded that for any preparation which is a distribution over ontic states must have some unique set of ontic states to be a physical state of reality upto classical mixture. Note that classical mixture represent lack of knowledge or ignorance which can be regarded as epistemic.

The second assumption involves measurements,

**Definition 2 (Strong duality).** For all pure state preparations \( \mathcal{P}_i \), such that \( p(k|M, \mathcal{P}_i) = 1 \) for some outcome \( k \) of projective measurement \( M \) imposes that the corresponding response function \( \xi(k|M, \lambda) = 1 \) for every \( \lambda \in \bigcap_i \Omega_i \), and 0 for all other \( \lambda \notin \bigcap_i \Omega_i \).

The above assumptions stems from the fact that any projective measurement in the classical world is just observing the state of the system. Likewise, considering classical Hamiltonian dynamics, projective measurements on the phase space just reads the state of the system \( (x, p) \). In general, the measurements might not exactly suggest the state of the system but specify some region of phase space where the system resides. Such scenarios can be understood as measurements whose different outcomes are classically coarse grained to a single outcome. Any non-projective measurement within classical theories are just probabilistic mixture of projective measurements. As a consequence within classical theories, without loss of generality it can always be assumed that measurements are projective on some larger dimension system. Now, we proceed towards constructing the operational task whose success probability is upper bounded for any classical strategy.

#### B. Operational Task

The task simply consists of two players Alice and Bob who are not allowed to communicate with each other during the run of the experiment. Alice has access to a preparation box which consists of two inputs \( y = \{0, 1\} \) which generate two different preparations \( P_y \). Bob has access to a measurement box with two inputs \( x = \{0, 1\} \) specifying two different measurements \( M_x \) with any number of outputs strictly greater than 1, labelled as \( a = \{0, 1, 2, \ldots\} \). Now, Alice and Bob choose their inputs independent of each other. As a consequence, a system is prepared by the preparation box with Alice which is then sent to Bob who measures the system as specified by the measurement box. The task is repeated for a large number of times to collect statistics.
Now, we move onto describing the success probability $P_S$ which can be understood as a quantifier of the probability of winning in the above described task.

$$P_S = \sum_{a,x,y} c_{a,x,y} p(a|x,y)$$  \hspace{1cm} (5)

where $c_{a,x,y} = \frac{1}{4}$ if $a = x,y$ and 0 otherwise.

Recalling from Bell’s theorem, to observe a quantum advantage it is required that the correlations shared between Alice and Bob are no-signalling [24]. Similarly, to observe an advantage using quantum theory and constrain classical theories in the above described operational task (5), we are required to impose the condition that the measurements are of rank-one, which can be operationally defined as

**Definition 3** (Rank-one measurements). For any measurement $M$, if two preparations $P$ and $P'$ give the same outcome with certainty, then $P$ and $P'$ can not be distinguished.

This means for any measurement $M$ and preparation $P$ and $P'$, if

$$p(k|M, P) = p(k|M, P') = 1 \text{ for some } k$$  \hspace{1cm} (6)

then $p(k|M, P) = p(k|M, P')$ for all outcomes $k$ of all measurements $M$. Note that, for projective measurements if $p(k|M, P) = 1$, then $P$ is a pure state preparation.

Constraining the strategies using the condition of measurements being rank-one def-3, we establish the following theorem on the success probability as defined in (5). Note that along with the above assumptions defining "universal classicality", in any operational task the implicit assumption of free will is required. The assumption of free will ensures that Alice and Bob can freely choose respective inputs $(y, x)$ independent of each other’s choices.

**Theorem.** For any theory which satisfies the assumptions of no overlap def-1 and strong duality def-2, the success probability (5) is bounded from above as $P_S \leq \frac{3}{4}$.

**Proof.** Expanding the success probability (5), we have

$$P_S = \frac{1}{4} (p(0|0, 0) + p(0|0, 1) + p(0|1, 0) + p(1|1, 1))$$  \hspace{1cm} (7)

Since, the measurements consists of more than one outcome, we have $\sum_k p(k|1, 1) = 1$, which imposes that $p(0|1, 1) + p(1|1, 1) \leq 1$. Using this, we have

$$P_S \leq \frac{1}{4} (p(0|0, 0) + p(0|1, 0) + p(0|0, 1) - p(0|1, 1)) + \frac{1}{4}$$

Since there is no restriction on dimension, as discussed above any measurement can be realized as a projective measurement on some larger system. Using this fact and the ontological framework, we expand the above expression as

$$p(0|0, 0) + p(0|1, 0) + p(0|0, 1) - p(0|1, 1)$$

$$= \int_\Lambda \xi(0|\Omega_0, \lambda) (\mu(\lambda|P_0) + \mu(\lambda|P_1)) d\lambda$$

$$+ \int_\Lambda \xi(0|\Omega_1, \lambda) (\mu(\lambda|P_0) - \mu(\lambda|P_1)) d\lambda$$  \hspace{1cm} (8)

Since the measurements are projective, the assumption of strong duality def-2 imposes that for all rank-one projective measurements, the response function $\xi(0|\Omega_0, \lambda)$ for $\lambda \in \Omega_0$ and $\xi(0|\Omega_1, \lambda) = 1$ for $\lambda \notin \Omega_0$ and $\xi(0|\Omega_1, \lambda) = 1$ for $\lambda \in \Omega_1$ and $\xi(0|\Omega_1, \lambda) = 0$ for $\lambda \notin \Omega_1$ for some preparations $\overline{P}_0$ and $\overline{P}_1$ generating pure states. This imposes that

$$p(0|0, 0) + p(0|1, 0) + p(0|0, 1) - p(0|1, 1)$$

$$= \int_{\Omega_0} (\mu(\lambda|P_0) + \mu(\lambda|P_1)) d\lambda$$

$$+ \int_{\Omega_1} (\mu(\lambda|P_0) - \mu(\lambda|P_1)) d\lambda$$  \hspace{1cm} (9)

Now, the assumption of no-overlap def-1 imposes that either $\Omega_0$ and $\Omega_1$ are disjoint for $\overline{P}_0$ and $\overline{P}_1$, being distinct or $\Omega_0$ and $\Omega_1$ are equivalent for $\overline{P}_0$ and $\overline{P}_1$ being same. Let’s first consider the case when $\overline{P}_0$ and $\overline{P}_1$ are distinct.

$$p(0|0, 0) + p(0|1, 0) + p(0|0, 1) - p(0|1, 1)$$

$$= p_{0,0} + p_{0,1} + p_{1,0} - p_{1,1} \leq 2$$  \hspace{1cm} (10)

where $p_{ij} = \int_{\Omega_0} \mu(\lambda|P_i) d\lambda$ and we used the fact that the ontological states generated from pure state preparations do not overlap, which imposes

$$p_{0,0} + p_{1,0} \leq 1 \text{ and,}$$

$$p_{0,1} + p_{1,1} \leq 1$$  \hspace{1cm} (11)

For the case when $\overline{P}_0$ and $\overline{P}_1$ being same, we have

$$p(0|0, 0) + p(0|1, 0) + p(0|0, 1) - p(0|1, 1)$$

$$= 2p_{0,0} \leq 2$$  \hspace{1cm} (12)
Thus, from (7), (10) and (12) we can conclude that

\[ P_s \leq \frac{3}{4}. \] (13)

This completes the proof. \(\square\)

Since any classical theory satisfies the assumptions of no-overlap def-1 and strong duality def-2, the above theorem shows that using classical strategies, the maximum attainable value of the success probability is bounded from above by the value \( P_s = \frac{3}{4} \). For a better understanding of the above bound, let’s consider a possible classical strategy. Let’s say Alice sends her inputs \( y = \{0, 1\} \) to Bob. This can be realised by just sending a classical bit in which case, the constraint (3) imposes that \( p(0|0, 0) + p(0|0, 1) \leq 1 \) which in turn imposes from (5) that \( P_s \leq \frac{3}{4} \). Now, we show that there exist strategies in quantum theory which can attain a success probability \( P_s = \frac{1}{2} + \frac{1}{2\sqrt{2}} \) which is strictly greater than \( \frac{3}{4} \).

C. Quantum advantage

To witness quantum advantage, the preparations are chosen by Alice as, \( P_0 \) generates the state \( |\psi_0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and \( P_1 \) generates the state \( |\psi_1\rangle = |0\rangle \). The measurements are chosen by Bob represented as observables are given by, \( M_0 = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x) \) and \( M_1 = \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x) \) where \( \sigma_z \) and \( \sigma_x \) corresponds to Pauli-z and Pauli-x matrices respectively. Calculating the necessary probabilities to evaluate the success probability (5) using the above chosen quantum states and measurements, we have

\[ p(0|0, 0) = p(0|1, 0) = p(0|0, 1) = p(1|1, 1) = \cos^2 \left( \frac{\pi}{8} \right) \] (14)

Thus, the success probability is given by

\[ P_s = \cos^2 \left( \frac{\pi}{8} \right) = \frac{1}{2} + \frac{1}{2\sqrt{2}}. \] (15)

The existence of such quantum strategies shows that in the above presented operational task, quantum theory outperforms classical strategies. Also, such a violation shows that the assumptions of no-overlap def-1 and strong duality def-2 are not compatible with quantum theory. Thus, it is impossible to construct any such ontological model of quantum theory which have the features of of no-overlap def-1 and strong duality def-2 together. However, there can exist some ontological model of quantum theory which violate either of the two assumptions. Some notable ontological models of quantum theory which violate the assumption of strong duality def-2 but satisfy the assumption of no-overlap def-1 are Beltrametti–Bugajski model [18] and Bell model [19]. Another class of ontological models like the Kochen-Specker model [2] and Spekkens toy theory [20] violate the assumption of no-overlap def-1 but satisfy the assumption of strong duality def-2. In the next section, we show that how the notion of “universal classicality” implies all the other known notions of classicality.

IV. RELATIONS TO OTHER NOTIONS OF CLASSICALITY

Comparisons between different notions of classicality is based on the amount of non-classicality which can be captured by such notions. Any newly proposed notion of classicality can be considered significant if it can capture higher amount of non-classicality than the previously proposed notions. To show this, one of the ways is to prove that a certain notion of classicality implies the previously known notions of classicality.

To show that “universal classicality” imposes other known notions of classicality, we need an additional auxiliary assumption concerning preparations which generate mixed states.

Definition 4 (Convexity). For any preparation \( P \) that can be realised as a classical mixture of some preparations \( P_i \)’s such that \( P = \sum_i z_i P_i \), then the corresponding ontological distribution is given by \( \mu(\lambda) = \sum_i z_i \mu(\lambda P_i) \) where \( z_i \geq 0 \) and \( \sum_i z_i = 1 \).

The above definition holds true for any preparations \( P_i \). However, for this work we only consider \( P_i \)’s which are pure states as any preparation which generate mixed state, can be understood as some classical mixture of pure states. As pure states represent physical states of reality in classical theories, any mixture of such pure states would be equivalently reinstated at the ontological level.

First, we prove that universal classicality and convexity implies preparation non-contexuality [4]. For this, let us first recall the definition of preparation non-contexuality. Preparation non-contexuality imposes that if two preparations are indistinguishable, then their respective ontological distributions are same. Mathematically, if

\[ p(k|M, P_1) = p(k|M, P_2) \quad \forall k, M \] (16)

then, \( \mu(\lambda|P_1) = \mu(\lambda|P_2) \) for all \( \lambda \in \Lambda \).

Now, we would show that the assumptions of no overlap def-1 and strong duality def-2 along with the assumption of convexity def-4 when imposed on the ontological state space imposes preparation non-contexuality. For this, let us note that as discussed before any measurement can be realized as a projective measurement in some higher dimensional space. Expressing (16) in the ontological framework,

\[ \int_{\Lambda} \xi(k|M, \lambda) \mu(\lambda|P_1) d\lambda = \int_{\Lambda} \xi(k|M, \lambda) \mu(\lambda|P_2) d\lambda \] (17)
for all outcomes $k$ of all projective measurements $\Omega$. To show $\mu(\lambda|P_1) = \mu(\lambda|P_2)$, it is enough to consider rank-one measurements def-3. As discussed before, the assumption of strong duality def-2 imposes that $\xi(k|M, \lambda) = 1$ for $\forall \lambda \in \Omega$ and $0$ for all other $\lambda \not\in \Omega$ for some pure state preparation $P$, 

$$\int_{\Omega} \mu(\lambda|P_1)d\lambda = \int_{\Omega} \mu(\lambda|P_2)d\lambda$$  \hspace{1cm} (18)$$

Now, assuming convexity def-4 we have

$$\sum_i z_{i,1} \Omega \int \mu(\lambda|P_{i,1})d\lambda = \sum_i z_{i,2} \int \mu(\lambda|P_{i,2})d\lambda$$  \hspace{1cm} (19)$$

Assuming no overlap def-1, it can be concluded that the above equality (19) holds iff $P_{i,1} = P_{i,2} = P$ and $z_{i,1} = z_{i,2}$ for some $i$. Using a similar argument for all outcomes $k$ of all rank-one projective measurements $\Omega$ we can conclude that $\mu(\lambda|P_1) = \mu(\lambda|P_2)$ for all $\lambda \in \Lambda$.

Now, we show that universal classicality and convexity also gives rise to a recently proposed notion of classicality termed bounded ontological distinctness [25]. For this, let us first recall the definition of bounded ontological distinctness (BOD). Bounded ontological distinctness imposes that for distinguishability of preparations at the ontological level is same as distinguishability at the operational level. Mathematically, for n-preparations $P_x$ when $x \in 0, 1, \ldots, n-1$ if

$$\max_{M} \frac{1}{n} \sum_{x} p(k = x|M, P_x) = p$$  \hspace{1cm} (20)$$

then,

$$\frac{1}{n} \int_{\Lambda} \max_{x} \{\mu(\lambda|P_x)\}d\lambda = p$$  \hspace{1cm} (21)$$

Now, we would show that the assumptions of no overlap def-1 and strong duality def-2 along with the assumption of convexity def-4 when imposed on the ontological state space imposes bounded ontological distinctness. Expressing (20) in the ontological framework,

$$\max_{M} \frac{1}{n} \sum_{x} \int_{\Lambda} \xi(k = x|M, \lambda)\mu(\lambda|P_x) = p$$  \hspace{1cm} (22)$$

Assuming convexity def-4, any ontological distribution can be written as $\mu(\lambda|P_x) = \sum_i z_{i,x}\mu(\lambda|P_i)$, we have

$$\max_{M} \frac{1}{n} \sum_{x} \sum_i z_{i,x} \int_{\Lambda} \xi(k = x|M, \lambda)\mu(\lambda|P_i) = p$$  \hspace{1cm} (23)$$

Assuming strong duality def-2 and no overlap def-1 any measurement $M$ that maximizes the distinguishing probability must have the response functions $\xi(k = x|M, \lambda)$ such that they pick up the probabilities $z_{i,x}$ which is maximised for each $x$. Thus, we have

$$\frac{1}{n} \sum_{i} \max_{x} z_{i,x} = p$$  \hspace{1cm} (24)$$

Now, evaluating (21) assuming convexity def-4 we have,

$$\frac{1}{n} \int_{\Lambda} \max_{x} \left\{ \sum_i z_{i,x}\mu(\lambda|P_i) \right\}d\lambda = \frac{1}{n} \sum_{i} \max_{x} z_{i,x} = p$$  \hspace{1cm} (25)$$

where we arrived at the above expression by assuming that pure states do not overlap def-1.

Using some well established results among different notions of classicality we have the following hierarchy,

Universal classicality $\implies$ BOD \hspace{1cm} Prep non-contextuality $\implies$ K-S non-contextuality \hspace{1cm} Bell local-causality  \hspace{1cm} (26)

Thus, any violation of Bell’s local causality, Kochen-Specker non-contextuality, preparation non-contextuality or bounded ontological distinctness imposes violation of universal classicality. For a note, the above conclusion could be reached assuming that ontological models do not violate convexity def-4 which is based on the fact that there is no successful ontological model of quantum theory which violates convexity def-4. However, for any general theory such ontological models can always exist which do not assume convexity def-4. However, any violation of the above notions using only preparations which generate pure states would directly impose the violation of universal classicality.

V. CONCLUSIONS

In the presented work, from a realist perspective we first constructed a novel notion of classicality termed “universal classicality” based on just two assumptions imposed on the ontological state space. We justified both the assumptions based on physical arguments which are satisfied for any known physical theories of the classical world. Then, we constructed an operational task which is bounded for any theories which satisfy the assumptions of “universal classicality”. Kochen-Specker non-contextuality and Bell’s local causality requires systems of atleast dimension three and four respectively to show a quantum advantage. Bounded ontological distinctness and preparation non-contextuality can be violated using only qubits, however they require atleast three and four qubit preparations respectively to show a quantum advantage. Contrary to this, “universal classicality” can be violated using just a pair of qubits which makes the above presented operational task much more applicable in practical scenarios. Then, we showed that the notion of “universal classicality” implies all the other known notions of classicality like Bell’s local causality.

Several follow up questions arises from this work. From quantum foundations perspective, universal
classicality might give better lower bounds to the overlap of the ontological states of quantum theory than previously known bounds [26, 27]. Also, the above theorem serves as a no-go theorem for quantum theory. As a consequence the notion presented in this work might help construction of newer ontological models. Further, the notion of universal classicality would give a better understanding of the set of quantum correlations. As was shown above, the violation of known notions of classicality would also imply violation of universal classicality. As a consequence, the set of quantum correlations is enlarged and this might give interesting results towards boundedness of quantum correlations [28]. From quantum information perspective, the above presented operational task might give rise to cryptographic and communication tasks which need less resource to execute as entanglement is not needed to show quantum advantage. Also, some device independent schemes might be possible to construct employing the idea of the presented operational task.

Acknowledgements

We would like to thank Remigiusz Augusiak and Debashish Saha for fruitful discussions. This work is supported by Foundation for Polish Science through the First Team project (No First TEAM/2017-4/31).

[1] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
[2] S. Kochen, and E. P. Specker, Journal of Mathematics and Mechanics 17, 1 (1967).
[3] A. Cabello, Phys. Rev. Lett. 101, 210401 (2008)
[4] R. W. Spekkens, Phys. Rev. A 71, 052108 (2005)
[5] A. K. Ekert Phys. Rev. Lett. 67, 661 (1991)
[6] D. Mayers, and A. Yao, Proc. 39th Ann. Symp. on Foundations of Computer Science (FOCS), 503 (1998).
[7] T. H. Yang and M. Navascués, Phys. Rev. A 87, 050102(R) (2013).
[8] C. Bamps, and S. Pironio, Phys. Rev. A 91, 052111 (2015).
[9] S. Sarkar, D. Saha, J. Kaniewski, R. Augusiak, Self-testing quantum systems of arbitrary local dimension with minimal number of measurements, arXiv:1909.12722
[10] I. Šupić, J. Bowles, Quantum 4, 335 (2020).
[11] N. Harrigan, R. W. Spekkens, Foundations of Physics 40, 125 (2010)
[12] M. S. Leifer, Quanta 3, 1 (2014)
[13] R. Colbeck and R. Renner, New J. Phys. 19 013016 (2017)
[14] M. F. Pusey, J. Barrett and T. Rudolph, Nature Physics 8, 475 (2012)
[15] L. Hardy, International Journal of Modern Physics 27, 1345012 (2013)
[16] L. De Broglie, J. Phys. Radium 8, 225-241 (1927).
[17] Bohm, Phys. Rev. 85, 166 (1952).
[18] E. G. Beltrametti and S. Bugajski, J. Phys. A: Math. Gen. 28 3329 (1995)
[19] J. S. Bell, Rev. Mod. Phys. 38, 447 (1966).
[20] R. W. Spekkens, Phys. Rev. A 75, 032110 (2007)
[21] P. G. Lewis, D. Jennings, J. Barrett, and T. Rudolph., Phys. Rev. Lett. 109, 150404 (2012)
[22] S. Aaronson, A. Bouland, L. Chua, and G. Lowther, Physical Review A 88, 032111 (2013)
[23] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press (Cambridge, 2000).
[24] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani and S. Wehner, Rev. Mod. Phys. 86, 419 (2014)
[25] A. Chaturvedi and D. Saha, Quantum 4, 345 (2020)
[26] M. S. Leifer, Phys. Rev. Lett. 112, 160404 (2014)
[27] J. Barrett, E. G. Cavalcanti, R. Lal, and O. J. E. Maroney, Phys. Rev. Lett. 112, 250403 (2014)
[28] A. Coladangelo, Quantum 4, 282 (2020).