SUBSET CURRENTS ON FREE GROUPS

ILYA KAPOVICH AND TATIANA NAGNIBEDA

ABSTRACT. We introduce and study the space $S\text{Curr}(F_N)$ of subset currents on the free group $F_N$, and, more generally, on a word-hyperbolic group. A subset current on $F_N$ is a positive $F_N$-invariant locally finite Borel measure on the space $\mathcal{C}_N$ of all closed subsets of $\partial F_N$ consisting of at least two points. The well-studied space $\text{Curr}(F_N)$ of geodesics currents, positive $F_N$-invariant locally finite Borel measures defined on pairs of different boundary points, is contained in the space of subset currents as a closed $\mathbb{R}$-linear $\text{Out}(F_N)$-invariant subspace. Much of the theory of $\text{Curr}(F_N)$ naturally extends to the $S\text{Curr}(F_N)$ context, but new dynamical, geometric and algebraic features also arise there. While geodesic currents generalize conjugacy classes of nontrivial group elements, a subset current is a measure-theoretic generalization of the conjugacy class of a nontrivial finitely generated subgroup in $F_N$. If a free basis $A$ is fixed in $F_N$, subset currents may be viewed as $F_N$-invariant measures on a “branching” analog of the geodesic flow space for $F_N$, whose elements are infinite subtrees (rather than just geodesic lines) of the Cayley graph of $F_N$ with respect to $A$. Similarly to the case of geodesics currents, there is a continuous $\text{Out}(F_N)$-invariant “co-volume form” between the Outer space $\mathbb{CV}_N$ and the space $S\text{Curr}(F_N)$ of subset currents. Given a tree $T \in \mathbb{CV}_N$ and the “counting current” $\eta_H \in S\text{Curr}(F_N)$ corresponding to a finitely generated nontrivial subgroup $H \leq F_N$, the value $(T, \eta_H)$ of this intersection form turns out to be equal to the co-volume of $H$, that is the volume of the metric graph $T_H/H$, where $T_H \subseteq T$ is the unique minimal $H$-invariant subtree of $T$. However, unlike in the case of geodesic currents, the co-volume form $\mathbb{CV}_N \times S\text{Curr}(F_N) \to [0, \infty)$ does not extend to a continuous map $\mathbb{CV}_N \times S\text{Curr}(F_N) \to [0, \infty)$.

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1. Introduction

Geodesic currents were introduced in the context of hyperbolic surfaces by Bonahon in the papers [11, 12] where they were used to study the Teichmüller space and mapping class groups, with various applications to 3-manifolds. In the context of free groups, geodesic currents were first investigated by Reiner Martin in his 1995 PhD thesis [57], but have only become the object of systematic study in the last five years, leading to a number of interesting recent applications and developments.

A geodesic current may be thought of as a measure-theoretic analog of the notion of a conjugacy class in the group, or of a free homotopy class of a closed curve in the surface. More formally, a geodesic current on a free group $F_N$ is a positive Borel measure on $\partial^2 F_N := \{(x, y) \in \partial F_N \times \partial F_N : x \neq y\}$ which is locally finite (i.e., finite on compact subsets), $F_N$-invariant and invariant with respect to the “flip map” $\partial^2 F_N \to \partial^2 F_N$, $(x, y) \mapsto (y, x)$. Equivalently, it is a positive Borel locally finite $F_N$-invariant measure on the space of 2-element subsets of $\partial F_N$.

This paper is devoted to the study of a natural generalization of the space of geodesic currents obtained by replacing the space of 2-element subsets of $\partial F_N$ by the space $\mathcal{C}_N$ of all closed subsets $S \subseteq \partial F_N$ such that $S$ consists of at least two points. The space $\mathcal{C}_N$ has a natural topology given by the Hausdorff distance for the subsets of $\partial F_N$, where $\partial F_N$ is endowed with the standard visual metric provided by any choice of basis in $F_N$ (this topology is independent of the choice of the basis and coincides with the Vietoris topology). Similarly to the Cantor set $\partial F_N$, the space $\mathcal{C}_N$ is locally compact and totally disconnected. See Subsection 3.1 for details about the space $\mathcal{C}_N$. A subset current on $F_N$ is a positive locally finite $F_N$-invariant Borel measure on $\mathcal{C}_N$, and we consider the space $\mathcal{S} \text{Curr}(F_N)$ of all subset currents on $F_N$ equipped with the weak-$*$ topology and with the $\mathbb{R}_{\geq 0}$-linear structure.
This definition naturally extends to the case where $F_N$ is replaced by an arbitrary word-hyperbolic group, for details see Problem 10.1 below.

The space $\text{SCurr}(F_N)$ of subset currents admits a natural $\text{Out}(F_N)$-action by $\mathbb{R}_{\geq 0}$-linear homeomorphisms, and the space of geodesic currents $\text{Curr}(F_N)$ is canonically embedded in $\text{SCurr}(F_N)$ as a closed $\text{Out}(F_N)$-invariant $\mathbb{R}_{\geq 0}$-linear subspace. We show below that subset currents are measure-theoretic analogs of conjugacy classes of nontrivial finitely generated subgroups in $F_N$, with geodesic currents corresponding to cyclic subgroups, and we study this connection in detail.

The space $\text{Curr}(F_N)$ of all geodesic currents on $F_N$ is a natural counterpart for the Culler-Vogtmann Outer space $\text{cv}_N$, and the present paper explores deeper levels of this interaction in a more general context of subset currents. The Outer space $\text{cv}_N$, introduced in [25], is a free group “cousin” of the Teichmüller space and consists of $F_N$-equivariant isometry classes of free minimal discrete isometric actions of $F_N$ on $\mathbb{R}$-trees. Points of $\text{cv}_N$ may also be thought of as marked metric graph structures on $F_N$, see Section 2 below for details. The space $\text{cv}_N$ comes equipped with a natural $\text{Out}(F_N)$-action that factors through to the action on the projectivized Outer space $\text{CV}_N$ (whose points are homothety classes of elements of $\text{cv}_N$).

The closure $\overline{\text{cv}}_N$ of $\text{cv}_N$ with respect to the equivariant Gromov-Hausdorff convergence topology is an important object in the study of dynamics of $\text{Out}(F_N)$ and is known to consist of $F_N$-equivariant isometry classes of all very small minimal isometric actions of $\text{Out}(F_N)$ on $\mathbb{R}$-trees. The projectivization $\text{CV}_N$ of $\overline{\text{cv}}_N$ is a compact finite-dimensional space analog to Thurston’s compactification of the Teichmüller space. Compared to the Teichmüller space, the geometry of the Outer space remains much less understood.

In this regard studying the interaction between the Outer space and the space of currents proved to be quite useful. This interaction is primarily given by the geometric intersection form or length pairing: in [15] Kapovich and Lustig proved that there exists a unique continuous map

$$\langle \cdot, \cdot \rangle : \overline{\text{cv}}_N \times \text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$$

which is $\text{Out}(F_N)$-invariant, $\mathbb{R}_{>0}$-homogeneous with respect to the first argument, $\mathbb{R}_{>0}$-linear with respect to the second argument, and has the property that for every nontrivial element $g \in F_N$ and every $T \in \overline{\text{cv}}_N$ one has

$$\langle T, \eta_g \rangle = \|g\|_T.$$

Here $\|g\|_T = \inf_{x \in T} d_T(x, gx)$ is the translation length of $g$ with respect to $T$, and $\eta_g \in \text{Curr}(F_N)$ is the counting current associated to $g$ (see Subsection 2.4 for the definition). The set of scalar multiples of all counting currents is a dense subset of $\text{Curr}(F_N)$, which in particular justifies thinking about the notion of a current as generalizing that of a conjugacy class. This approach provided a number of useful recent applications to the study of the dynamics and geometry of $\text{Out}(F_N)$, such as the results of Kapovich and Lustig about various analogs of the curve complex in the free group case [45], a construction by Bestvina and Feighn, for a given finite collection of iwip elements in $\text{Out}(F_N)$, of a Gromov-hyperbolic graph with an isometric $\text{Out}(F_N)$-action, where these iwip automorphisms act as hyperbolic isometries; a result of Hamenstädt [45] about “lines of minima” in Outer space; the work of Clay and Pettet [21] on realizability of an arbitrary matrix from $GL(N, \mathbb{Z})$ as the abelianization action of a hyperbolic iwip element of $\text{Out}(F_N)$, and others (see, for example, [41, 42, 44, 45, 46, 47, 48, 24, 29, 50, 51, 18]).

As noted above, in this paper we extend this framework in a way that allows us to study the dynamics of the action of $\text{Out}(F_N)$ on conjugacy classes of finitely generated subgroups of $\text{Out}(F_N)$ that are not necessarily cyclic.
In the context of subset currents, we similarly define, given a finitely generated non-trivial subgroup \( H \leq F_N \), the \emph{counting current} \( \eta_H \in \mathcal{S} \text{Curr}(F_N) \). For the case where \( H = \langle g \rangle \leq F_N \) is infinite cyclic, we actually get \( \eta_H = \eta_g \). For a finitely generated non-trivial \( H \leq F_N \), \( \eta_H \in \mathcal{S} \text{Curr}(F_N) \) is defined (see Definition 4.3) as a sum of atomic measures on the limit set of \( H \) and its conjugates. It is then shown (see Theorem 4.19), that it can equivalently be understood in terms of the corresponding Stallings core graphs.

If a free basis \( A \) is fixed in \( F_N \), a convenient basis of topology is formed by cylinder sets. A subset cylinder \( \mathcal{S} \text{Cyl}_A(K) \subseteq \mathcal{C}_N \) (see Definition 3.2 below) is determined by a finite non-degenerate subtree \( K \) of the Cayley tree \( X_A \) of \( F_N \) with respect to \( A \), and it consists of all those \( S \in \mathcal{C}_N \) such that the convex hull of \( S \) in \( X_A \) contains \( K \) and such that every bi-infinite geodesic in \( X_A \) with both endpoints in \( S \) which intersects \( K \) in a non-degenerate segment, enters and exits \( K \) through vertices of degree 1 in \( K \).

We develop the appropriate notion of an \emph{occurrence} of a finite subtree \( K \subseteq X_A \) in a Stallings core graph \( \Delta \) (see Definition 4.13). Namely, an occurrence of \( K \) in \( \Delta \) is a locally injective label-preserving morphism from \( K \) into \( \Delta \) which is a local homeomorphism at every point of \( K \) except for the terminal vertices of leaves of \( K \). Apart from being a useful tool in dealing with counting currents, the language of occurrences in core graphs produces an interesting model of non-linear (that is, not based on a segment of \( \mathbb{Z} \) words, with a good notion of a subword (or “factor”) in such a word. Studying such models is an active subject of research in combinatorics of words (see, for example, [2]).

As noted earlier, in the context of \( \text{Curr}(F_N) \), a basic fact of the theory states that the set of all \emph{rational currents}, that is of all \( \mathbb{R}_{\geq 0} \)-scalar multiples of the counting currents \( \eta_g \), where \( g \in F_N, g \neq 1 \), is a dense subset of \( \text{Curr}(F_N) \). Proofs of this fact are usually relying, in an essential way, on the “commutative” nature of the dynamical systems associated with \( \text{Curr}(F_N) \). Thus, given a free basis \( A \) of \( F_N \), one can naturally view a geodesic current on \( F_N \) as a positive shift-invariant Borel finite measure on the space of bi-infinite freely reduced words over \( A^{\pm 1} \). Under this correspondence, the counting currents correspond exactly to the shift-invariant measures supported by periodic orbits of the shift map, and the density of the set of rational currents is then a consequence of classical results about density of periodic orbits. In the context of \( \mathcal{S} \text{Curr}(F_N) \) the situation is considerably more complicated, since the symbolic dynamical systems corresponding to subset currents are no longer “commutative” in nature. Geometrically, a 2-element subset of \( \partial F_N \) determines an infinite unparameterized geodesic line in the Cayley graph \( X_A \) of \( F_N \) with respect to a free basis \( A \), so that a geodesic current may be thought of as an \( F_N \)-invariant measure on the space of unparameterized geodesic lines in \( X_A \). Similarly, an element \( S \in \mathcal{C}_N \) determines an infinite subtree \( Y \subseteq X_A \) without degree-one vertices, namely, the convex hull of \( S \) in \( X_A \). Thus, once \( A \) is chosen, a subset current translates into a positive finite Borel measure on the space \( \mathcal{T}_1(X_A) \) of infinite subtrees of \( X_A \) containing \( 1 \in F_N \) and without degree-one vertices. \( F_N \)-invariance of the current implies that the corresponding measure on \( \mathcal{T}_1(X_A) \) is invariant with respect to the root change. The space \( \mathcal{T}_1(X_A) \) may be viewed as a “branching” analog of the geodesic flow space for \( X_A \), since its elements are infinite trees rather than lines.

Recent results of Bowen [14, 15] and Elek [28] about approximability of such measures on the space of rooted trees are applicable to our set-up and allow us to conclude in Theorem 5.8 that the set \( \{r \eta_H | r \geq 0, H \leq F_N \text{ is nontrivial and finitely generated} \} \) of \emph{rational subset currents} is dense in \( \mathcal{S} \text{Curr}(F_N) \).

The notion of a subset current is related to the study of “invariant random subgroups”. If \( G \) is a locally compact group, an \emph{invariant random subgroup} is a probability measure...
on the space $S(G)$ of all closed subgroups of $G$, such that this measure is invariant with respect to the conjugation action of $G$ on $S(G)$.

The study of invariant random subgroups in various contexts goes back to the work of Stuck and Zimmer [62] and has recently become an active area of research, see for example [1, 10, 17, 26, 27, 30, 60, 63, 64]. If $G$ is a free basis of $F_N$, one can view an invariant random subgroup on $F_N$ as a measure on the space of all rooted Stallings core graphs labelled by example [1, 16, 17, 26, 27, 30, 60, 63, 64]. If $G$ of Stuck and Zimmer [62] and has recently become an active area of research, see for $G$ respect to the conjugation action of max $\text{cv}$. 

Consider a nontrivial finitely generated subgroup $H$ of $F_N$ of Stallings core graphs corresponding to a nontrivial subgroup is never a tree, and, moreover, every edge in this graph is contained in some nontrivial immersed circuit. By contrast, as noted above and as we explain in greater detail in Subsection 5.2 below, a subset current on $F_N$ can be viewed as a measure on the space $T_1(X_A)$ of infinite trees which is invariant with respect to root change. We plan to investigate deeper connections between these two notions in a future work.

In this paper we also construct (see Section 7) a continuous $\text{Out}(F_N)$-invariant co-volume form

$$\langle , \rangle : \text{cv}_N \times \mathcal{S}\text{Curr}(F_N) \to \mathbb{R}_{\geq 0}.$$ 

It has properties similar to that of the geometric intersection form on geodesic currents, except that instead of translation length it computes the co-volume: for any $T \in \text{cv}_N$ and a nontrivial finitely generated subgroup $H \leq F_N$, we have $\langle T, \eta_H \rangle = \text{vol}(H \setminus T_H)$ where $T_H$ is the convex hull of the limit set of $H$, and is also the unique minimal $H$-invariant subtree of $T$. For an infinite cyclic $H = \langle g \rangle \leq F_N$ we have $||g||_T = \text{vol}(H \setminus T_H)$ since in this case the quotient graph $H \setminus T_H$ is a circle. By contrast to the result of Kapovich and Lustig [13] for ordinary geodesic currents, we prove (Theorem 7.6 below) that for $N \geq 3$ the co-volume form $\text{cv}_N \times \mathcal{S}\text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$ does not extend to a continuous map $\text{cv}_N \times \mathcal{S}\text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$.

For a finitely generated subgroup $H \leq F_N$, the reduced rank, $\text{rk}(H)$, is defined as $\max\{\text{rk}(H) - 1, 0\}$, where $\text{rk}(H)$ is the cardinality of a free basis of $H$. It turns out that reduced rank uniquely extends to a continuous $\text{Out}(F_N)$-invariant $\mathbb{R}_{\geq 0}$-linear functional $\text{rk} : \mathcal{S}\text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$, such that for every nontrivial finitely generated $H \leq F_N$ we have $\text{rk}(\eta_H) = \text{rk}(H)$. As we note in Section 10 there are likely deeper connections arising here with the study of intersections of finitely generated subgroups of free groups.

More open problems regarding subset currents are formulated in Section 10. We believe that subset currents exhibit considerably more interesting and varied geometric and dynamical behavior than geodesic currents, and can provide new and interesting information about $\text{Out}(F_N)$. Indeed, we hope this paper to serve as a starting point for investigating the space of subset currents in its different aspects, such as $\text{Out}(F_N)$-related questions, connections with invariant random subgroups, connections to the study of ergodic properties of subgroup actions on $\partial F_N$ and their Schreier graphs (see [21]), etc.

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2. Outer space and the space of geodesic currents

We give here only a brief overview of basic facts related to Outer space and the space of geodesic currents. We refer the reader to [25, 40] for more detailed background information.

2.1. Conventions regarding graphs. A graph is a 1-complex. The set of 0-cells of a graph $\Delta$ is denoted $V\Delta$ and its elements are called vertices of $\Delta$. The closed 1-cells of
a graph $\Delta$ are called topological edges of $\Delta$. The set of all topological edges is denoted $E_{\text{top}}\Delta$. We will sometimes call the open 1-cells of $\Delta$ open edges of $\Delta$. The interior of every topological edge is homeomorphic to the interval $(0, 1) \subseteq \mathbb{R}$ and thus admits exactly 2 orientations (when considered as a 1-manifold). We call a topological edge endowed with the choice of an orientation on its interior an oriented edge of $\Delta$. The set of all oriented edges of $\Delta$ is denoted $E\Delta$. For an oriented edge $e \in E\Delta$ changing its orientation to the opposite produces another oriented edge of $\Delta$ denoted $e^{-1}$ and called the inverse of $e$. Thus $^{-1} : E\Delta \to E\Delta$ is a fixed-point-free involution. For every oriented edge $e$ of $\Delta$ there are naturally defined (and not necessarily distinct) vertices $o(e) \in V\Delta$, called the origin of $e$, and $t(e) \in V\Delta$, called the terminus of $e$, satisfying $o(e^{-1}) = t(e)$, $t(e^{-1}) = o(e)$. An orientation on a graph $\Delta$ is a partition $E\Delta = E^+\Delta \sqcup E^-\Delta$, where for every $e \in E\Delta$ one of the edges $e, e^{-1}$ belongs to $E^+\Delta$ and the other edge belongs to $E^-\Delta$.

An edge-path $\gamma$ of simplicial length $|\gamma| = n \geq 1$ in $\Delta$ is a sequence of oriented edges $\gamma = e_1, \ldots, e_n$ such that $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n - 1$. We say that $o(\gamma) := o(e_1)$ is the origin of $\gamma$ and that $t(\gamma) = t(e_n)$ is the terminus of $\gamma$. An edge-path is called reduced if it does not contain a back-tracking, that is a sub-path of the form $ee^{-1}$, where $e \in E\Delta$.

To every edge-path $\gamma = e_1, \ldots, e_n$ in $\Delta$ there is a naturally associated continuous map $\tilde{\gamma} : [0, n] \to \Delta$ with $\tilde{\gamma}(0) = o(\gamma)$ and $\tilde{\gamma}(n) = t(\gamma)$.

A graph morphism $f : \Delta \to \Delta'$ is a continuous map from a graph $\Delta$ to a graph $\Delta'$ that maps vertices of $\Delta$ to vertices of $\Delta'$ and such that each open edge of $\Delta$ is mapped homeomorphically to an open edge of $\Delta'$. Thus $f$ induces natural maps $f : E\Delta \to E\Delta'$ and $f : V\Delta \to V\Delta'$ such that $f(e^{-1}) = (f(e))^{-1}$, $o(f(e)) = f(o(e))$ and $t(f(e)) = f(t(e))$ for every $e \in E\Delta$.

2.2. Markings. Let $N \geq 2$. We fix a free basis $A = \{a_1, \ldots, a_N\}$ of $F_N$. We also let $R_N$ to denote the wedge of $N$ loop-edges at a vertex $x_0$. We identify $F_N$ with $\pi_1(R_N, x_0)$ by mapping each $a_i \in B$ to one of the loop edges in $R_N$. We fix this identification $F_N = \pi_1(R_N, x_0)$ for the remainder of the paper. The graph $R_N$ will be referred to as the standard $N$-rose.

A marking on $F_N$ is an isomorphism $\alpha : F_N \cong \pi_1(\Gamma)$, where $\Gamma$ is a finite connected graph without degree-1 vertices.

To each marking $\alpha$ we also associate a continuous map $\hat{\alpha} : R_N \to \Gamma$ such that $\hat{\alpha}$ is a homotopy equivalence and such that $\hat{\alpha}_# = \alpha$. Two markings $\alpha_1 : F_N \cong \pi_1(\Gamma_1)$ and $\alpha_2 : F_N \cong \pi_1(\Gamma_2)$ are equivalent if there exists a graph isomorphism $j : \Gamma_1 \to \Gamma_2$ such that for the associated continuous maps $\hat{\alpha}_1 : R_N \to \Gamma_1$ and $\hat{\alpha}_2 : R_N \to \Gamma_2$, the maps $j \circ \hat{\alpha}_1$ and $\hat{\alpha}_2$ are freely homotopic. We will usually only be interested in the equivalence class of a marking, and for that reason an explicit mention of a base-point in the graph $\Gamma$ in the definition of a marking will almost always be omitted.

If $\alpha : F_N \cong \pi_1(\Gamma)$ is a marking, then $\alpha$ defines an $F_N$-equivariant quasi-isometry between $F_N$ and $\Gamma$ (endowed with the simplicial metric, giving every edge of $\Gamma$ length 1). This quasi-isometry induces an $F_N$-equivariant homeomorphism $\partial F_N \to \partial \Gamma$. When talking about markings, we will usually implicitly assume that $\partial F_N$ is identified with $\partial \Gamma$ via this homeomorphism and write $\partial F_N = \partial \Gamma$.

Each marking $\alpha : F_N \cong \pi_1(\Gamma)$ provides a Hausdorff metric $d_\alpha$ on $\partial F_N$ as follows. For $\xi, \zeta \in \partial F_N$, put $d_\alpha(\xi, \zeta) = M/2$ where $M$ is the length of the maximal common initial segment of the geodesic rays $[x_0, \xi]$ and $[x_0, \zeta]$ in $\bar{\Gamma}$.

A metric graph structure on a graph $\Delta$ is a function $L : E\Delta \to (0, \infty)$ such that $L(e) = L(e^{-1})$ for every $e \in E\Delta$. Equivalently, we may think of a metric graph structure
on $\Delta$ as a function $L : E_{\text{top}} \Delta \to (0, \infty)$. For an edge-path $\gamma = e_1, \ldots, e_n$ in $\Delta$ the $L$-length of $\gamma$ is $L(\gamma) := \sum_{i=1}^n L(e_i)$.

A marked metric graph structure on $F_N$ is a pair $(\alpha, L)$ where $\alpha : F_N \cong \pi_1(\Gamma)$ is a marking on $F_N$ and $L$ is a metric graph structure on $\Gamma$. Two marked metric graph structures $(\alpha_1 : F_N \cong \pi_1(\Gamma_1), L_1)$ and $(\alpha_2 : F_N \cong \pi_1(\Gamma_2), L_2)$ are equivalent if there exists a graph isomorphism $j : \Gamma_1 \to \Gamma_2$ such that $j : (\Gamma_1, L_1) \to (\Gamma_2, L_2)$ is an isometry and such that for the associated continuous maps $\hat{\alpha}_1 : R_N \to \Gamma_1$ and $\hat{\alpha}_2 : R_N \to \Gamma_2$, the maps $j \circ \hat{\alpha}_1$ and $\hat{\alpha}_2$ are freely homotopic. Note that if $(\alpha_1 : F_N \cong \pi_1(\Gamma_1), L_1)$ and $(\alpha_2 : F_N \cong \pi_1(\Gamma_2), L_2)$ are equivalent marked metric graph structures on $F_N$ then the markings $\alpha_1, \alpha_2$ are equivalent.

2.3. Outer space. Let $N \geq 2$. The Outer space $cv_N$ consists of all minimal free and discrete isometric actions on $F_N$ on $\mathbb{R}$-trees (where two such actions are considered equal if there exists an $F_N$-equivariant isometry between the corresponding trees). There are several different topologies on $cv_N$ that are known to coincide, in particular the equivariant Gromov-Hausdorff convergence topology and the so-called length function topology.

Every $T \in cv_N$ is uniquely determined by its translation length function $|| \cdot ||_T : F_N \to \mathbb{R}$, where $||g||_T$ is the translation length of $g$ on $T$. Two trees $T_1, T_2 \in cv_N$ are close if the functions $|| \cdot ||_{T_1}$ and $|| \cdot ||_{T_2}$ are close point-wise on a large ball in $F_N$. The closure $\overline{cv}_N$ of $cv_N$ in either of these two topologies is well-understood and known to consist precisely of all the so-called very small minimal isometric actions of $F_N$ on $\mathbb{R}$-trees, see [2] and [19].

The automorphism group $\text{Aut}(F_N)$ has a natural continuous right action on $\overline{cv}_N$ (that leaves $cv_N$ invariant) defined as follows: for $T \in cv_N$ and $\varphi \in \text{Aut}(F_N)$ we have $||g||_{T \varphi} = ||\varphi(g)||_T$, where $g \in F_N$. In terms of tree actions, $T \varphi$ is equal to $T$ as a metric space, but the action of $F_N$ is modified as: $g \cdot x = \varphi(g) \cdot x$ where $x \in T$, $g \in F_N$. It is not hard to see that the subgroup $\text{Im}(F_N) \leq \text{Aut}(F_N)$ of inner automorphisms is contained in the kernel of the action of $\text{Aut}(F_N)$ on $\overline{cv}_N$. Hence this action quotients through to the action of $\text{Out}(F_N)$ on $\overline{cv}_N$, where $cv_N \subseteq \overline{cv}_N$ is an $\text{Out}(F_N)$-invariant dense subset. The right action of $\text{Out}(F_N)$ on $\overline{cv}_N$ can be converted into a left action as follows: for $T \in \overline{cv}_N$ and $\varphi \in \text{Out}(F_N)$ put $\varphi T := T \varphi^{-1}$.

The projectivized Outer space $CV_N = \mathbb{P}cv_N$ is defined as the quotient $cv_N / \sim$ where for $T_1 \sim T_2$ whenever $T_2 = cT_1$ in $cv_N$ for some $c > 0$. One similarly defines the projectivization $\overline{CV}_N = \mathbb{P}\overline{cv}_N$ of $\overline{cv}_N$ as $\overline{cv}_N / \sim$ where $\sim$ is the same as above. The space $\overline{CV}_N$ is compact and contains $CV_N$ as a dense $\text{Out}(F_N)$-invariant subset. The compactification $\overline{CV}_N$ of $CV_N$ is a free group analog of the Thurston compactification of the Teichmüller space. For $T \in \overline{cv}_N$ its $\sim$-equivalence class is denoted by $[T]$, so that $[T]$ is the image of $T$ in $\overline{CV}_N$.

Every marked metric graph structure $(\alpha : F_N \cong \pi_1(\Gamma), L)$ defines a point in $cv_N$ as follows. Consider the universal covering tree $X = \overline{T}$ and let $d_L$ be the metric on $X$ obtained by giving every edge of $\overline{T}$ the same length as the $L$-length of its projection in $\Gamma$. Then $X$ is an $\mathbb{R}$-tree, and the action of $F_N$ on $X$ via $\alpha$ by covering transformations is a free minimal discrete isometric action on $(X, d_L)$. Thus $(X, d_L)$, equipped with this action of $F_N$, is a point of $cv_N$.

It is well-known that for two marked metric graph structures $(\alpha_1 : F_N \cong \pi_1(\Gamma_1), L_1)$ and $(\alpha_2 : F_N \cong \pi_1(\Gamma_2), L_2)$ we have $(\overline{\Gamma}_1, d_{L_1})_{\alpha_1} = (\overline{\Gamma}_2, d_{L_2})_{\alpha_2}$ in $cv_N$ if and only if $(\alpha_1, L_1)$ is equivalent to $(\alpha_2, L_2)$. Moreover, every point of $cv_N$ comes from some marked metric graph structure on $F_N$. Namely, if $T \in cv_N$, take $\Gamma = F_N \setminus T$ and endow the edges of $\Gamma$ with the same lengths as their lifts in $T$. Since the action of $F_N$ on $T$ is free and discrete, there is a natural identification $F_N$ with $\pi_1(\Gamma)$, giving us a marking
2.4. Geodesic currents. Let \( \partial^2 F_N := \{(x, y) \mid x, y \in \partial F_N, x \neq y \} \). The action of \( F_N \) by translations on its hyperbolic boundary \( \partial F_N \) defines a natural diagonal action of \( F_N \) on \( \partial^2 F_N \). A geodesic current on \( F_N \) is a positive Borel measure on \( \partial^2 F_N \), which is locally finite (that is, finite on all compact subsets), \( F_N \)-invariant and is also invariant under the “flip” map \( \partial^2 F_N \to \partial^2 F_N, (x, y) \mapsto (y, x) \).

The space \( \text{Curr}(F_N) \) of all geodesic currents on \( F_N \) has a natural \( \mathbb{R}_{\geq 0} \)-linear structure and is equipped with the weak-* topology of point-wise convergence on continuous functions. Any choice of a marking on \( F_N \) allows one to think about geodesic currents as systems of nonnegative weights satisfying certain Kirchhoff-type equations; see [40] for details. We briefly recall this construction for the case where \( X_A \in \text{cv}_N \) is the Cayley tree corresponding to a free basis \( A \) of \( F_N \). For a non-degenerate geodesic segment \( \gamma = [p, q] \) in \( X_A \) the two-sided cylinder \( \text{Cyl}_A(\gamma) \) \( \subseteq \partial^2 F_N \) consists of all \( (x, y) \in \partial^2 F_N \) such that the geodesic from \( x \) to \( y \) in \( X_A \) passes through \( \gamma = [p, q] \). Given a nontrivial freely reduced word \( v \in F(A) = F_N \) and a current \( \mu \in \text{Curr}(F_N) \), the “weight” \( (v; \mu)_A \) is defined as \( \mu(\text{Cyl}_A(\gamma)) \) where \( \gamma \) is any segment in the Cayley graph \( X_A \) labelled by \( v \) (the fact that the measure \( \mu \) is \( F_N \)-invariant implies that a particular choice of \( \gamma \) does not matter). A current \( \mu \) is uniquely determined by a family of weights \( \{(v; \mu)_A \}_{v \in F_N - \{1\}} \). The weak-* topology on \( \text{Curr}(F_N) \) corresponds to point-wise convergence of the weights for every \( v \in F_N, v \neq 1 \).

There is a natural left action of \( \text{Out}(F_N) \) on \( \text{Curr}(F_N) \) by continuous linear transformations. Specifically, let \( \mu \in \text{Curr}(F_N) \), \( \varphi \in \text{Out}(F_N) \) and let \( \Phi \in \text{Aut}(F_N) \) be a representative of \( \varphi \) in \( \text{Aut}(F_N) \). Since \( \Phi \) is a quasi-isometry of \( F_N \), it extends to a homeomorphism of \( \partial F_N \) and, diagonally, defines a homeomorphism of \( \partial^2 F_N \). The measure \( \varphi \mu \) on \( \partial^2 F_N \) is defined as follows. For a Borel subset \( S \subseteq \partial^2 F_N \) we have \( (\varphi \mu)(S) := \mu(\Phi^{-1}(S)) \). One then checks that \( \varphi \mu \) is a current and that it does not depend on the choice of a representative \( \Phi \) of \( \varphi \).

The space of projectivized geodesic currents is defined as \( \mathbb{P}\text{Curr}(F_N) = \text{Curr}(F_N) - \{0\} / \sim \) where \( \mu_1 \sim \mu_2 \) whenever there exists \( c > 0 \) such that \( \mu_2 = c \mu_1 \). The \( \sim \)-equivalence class of \( \mu \in \text{Curr}(F_N) - \{0\} \) is denoted by \( [\mu] \). The action of \( \text{Out}(F_N) \) on \( \text{Curr}(F_N) \) descends to a continuous action of \( \text{Out}(F_N) \) on \( \mathbb{P}\text{Curr}(F_N) \). The space \( \mathbb{P}\text{Curr}(F_N) \) is compact.

For every \( g \in F_N, g \neq 1 \) there is an associated counting current \( \eta_g \in \text{Curr}(F_N) \). If \( A \) is a free basis of \( F_N \) and the conjugacy class \([g]\) of \( g \) is realized by a “cyclic word” \( W \) (that is a cyclically reduced word in \( F(A) \) written on a circle with no specified base-vertex), then for every nontrivial freely reduced word \( v \in F(A) = F_N \) the weight \( (v; \eta_g)_A \) is equal to the total number of occurrences of \( v^{\pm 1} \) in \( W \) (where an occurrence of \( v \) in \( W \) is a vertex on \( W \) such that we can read \( v \) in \( W \) clockwise without going off the circle). We refer the reader to [40] for a detailed exposition on the topic. By construction, the counting current \( \eta_g \) depends only on the conjugacy class \([g]\) of \( g \) and it also satisfies \( \eta_g = \eta_{g^{-1}} \). One can check [40] that for \( \varphi \in \text{Out}(F_N) \) and \( g \in F_N, g \neq 1 \) we have \( \varphi \eta_g = \eta_{\varphi(g)} \). Scalar multiples \( c \eta_g \in \text{Curr}(F_N) \), where \( c \geq 0, g \in F_N, g \neq 1 \) are called rational currents. A key fact about \( \text{Curr}(F_N) \) states that the set of all rational currents is dense in \( \text{Curr}(F_N) \). The set \( \{[\eta_g] \mid g \in F_N, g \neq 1 \} \) is dense in \( \mathbb{P}\text{Curr}(F_N) \).

2.5. Intersection form. In [45] Kapovich and Lustig constructed a natural geometric intersection form that pairs trees and currents:
Proposition 2.1. \cite{[15]} Let \( N \geq 2 \). There exists a unique continuous map \( \langle , \rangle : \mathcal{C} \mathcal{N} \times \text{Curr}(F_N) \to \mathbb{R}_{\geq 0} \) with the following properties:

1. We have \( \langle T, c_1\mu_1 + c_2\mu_2 \rangle = c_1\langle T, \mu_1 \rangle + c_2\langle T, \mu_2 \rangle \) for any \( T \in \mathcal{C} \mathcal{N}, \mu_1, \mu_2 \in \text{Curr}(F_N), c_1, c_2 \geq 0 \).
2. We have \( \langle cT, \mu \rangle = c\langle T, \mu \rangle \) for any \( T \in \mathcal{C} \mathcal{N}, \mu \in \text{Curr}(F_N) \) and \( c \geq 0 \).
3. We have \( \langle T\varphi, \mu \rangle = \langle T, \varphi \mu \rangle \) for any \( T \in \mathcal{C} \mathcal{N}, \mu \in \text{Curr}(F_N) \) and \( \varphi \in \text{Out}(F_N) \).
4. We have \( \langle T, \eta_g \rangle = ||g||_T \) for any \( T \in \mathcal{C} \mathcal{N} \) and \( g \in F_N, g \neq 1 \).

3. The space of subset currents

3.1. The space \( \mathcal{C} \mathcal{N} \). Recall that for a Hausdorff topological space \( Y \), the so-called hyper space \( \mathcal{H}(Y) \) consists of all non-empty closed subsets of \( Y \). The space \( \mathcal{H}(Y) \) comes equipped with the Vietoris topology, which has the basis consisting of all sets of the form

\[
\{ U_1, \ldots, U_n \} = \{ B \in \mathcal{H}(Y) \mid B \subset U_1 \cup \ldots \cup U_n; B \cap U_i \neq \emptyset, i = 1, \ldots, n \},
\]

where \( \{ U_1, \ldots, U_n \} \) is a family of open subsets in \( Y \).

If \( Y \) is a compact metrizable space, then the Vietoris topology coincides with the Hausdorff topology given by the Hausdorff distance between closed subsets of \( Y \), and in this case \( \mathcal{H}(Y) \) is also compact (see e.g. \cite{[30]}, Chapter b-6). If \( Y \) is totally disconnected then \( \mathcal{H}(Y) \) is also totally disconnected.

Definition 3.1 (Space of closed subsets of the boundary). Let \( N \geq 3 \). We denote by \( \mathcal{C} \mathcal{N} \) the set of all closed subsets \( S \subseteq \partial F_N \) such that \( |S| \geq 2 \). Thus \( \mathcal{C} \mathcal{N} \subseteq \mathcal{H}(\partial F_N) \) and we endow \( \mathcal{C} \mathcal{N} \) with the subspace topology inherited from the Vietoris topology on \( \mathcal{H}(\partial F_N) \).

In view of the above remarks, the topology on \( \mathcal{C} \mathcal{N} \) coincides with the Hausdorff topology given by the Hausdorff distance \( D_\alpha \) on \( \mathcal{C} \mathcal{N} \) with respect to the metric \( d_\alpha \) on \( \partial F_N \) corresponding to any marking \( \alpha \) of \( F_N \) (see \cite{[22]}). It is also not difficult to check directly that the topology on \( \mathcal{C} \mathcal{N} \) defined by the metric \( D_\alpha \) does not depend on the choice of the marking.

The definition also straightforwardly implies that \( \mathcal{C} \mathcal{N} \) is locally compact and totally disconnected.

The condition that \( |S| \geq 2 \) is an important non-degeneracy condition for setting up the notion of a subset current, analogous to the assumption that \( \xi \neq \zeta \) for \( (\xi, \zeta) \in \partial^2 F_N \) in the definition of a geodesic current.

The space \( \mathcal{C} \mathcal{N} \) that interests us in this paper, is the complement in \( \mathcal{H}(\partial F_N) \) of the closed subspace consisting of 1-element subsets \( \{ y \}, y \in \partial F_N \).

It will be useful in our study of the space \( \text{SCurr}(F_N) \) to understand the topology on \( \mathcal{C} \mathcal{N} \) more explicitly. For this reason we describe the construction of “cylindrical” subsets in \( \mathcal{C} \mathcal{N} \).

Definition 3.2 (Cylinders). Let \( \alpha : F_N \to \pi_1(\Gamma) \) be a marking on \( F_N \), and let \( X = \tilde{\Gamma} \) be the universal cover of \( \Gamma \).

We give each edge of \( \Gamma \) and of \( X \) length 1, so that \( X \) can also be considered an element of \( cv_N \).

Let \( K \subseteq X \) be a non-degenerate finite simplicial subtree of \( X \), that is a finite simplicial subtree with at least two distinct vertices of degree 1. Let \( e_1, \ldots, e_n \) be all the terminal edges of \( K \), that is oriented edges whose terminal vertices are precisely all the vertices of \( K \) of degree 1. (Note that \( n \geq 2 \) by the assumption on \( K \).) For each of the edges \( e_i \) denote by \( \text{Cyl}_X(e_i) \subseteq \partial F_N \) the homeomorphic image of the subset in \( \partial X \) consisting of all equivalence classes of geodesic rays in \( X \) beginning with the edge \( e_i \).
Define the subset cylinder \( SCyl_\alpha(K) \) to be the set \( \langle Cyl_X(e_1), \ldots, Cyl_X(e_n) \rangle \subset C_N \). Thus for a closed subset \( S \subseteq \partial F_N \) with \( \# S \geq 2 \) we have \( S \in SCyl_\alpha(K) \) if and only if the following hold:

1. The subset \( S \subseteq \partial F_N \) is closed.
2. We have \( S \subseteq \cup_{i=1}^n Cyl_X(e_i) \).
3. For each \( i = 1, \ldots, n \) we have \( S \cap Cyl_X(e_i) \neq \emptyset \).

The following key basic fact is a straightforward exercise in unpacking the definitions:

**Proposition 3.3.** Let \( \alpha, \Gamma \) and \( X \) be as in Definition 3.2. Then

1. For every non-degenerate finite subtree \( K \subseteq X \) the subset \( SCyl_\alpha(K) \subset SCurr(F_N) \) is compact and open.
2. The collection of all \( SCyl_\alpha(K) \), where \( K \) varies over all non-degenerate finite subtrees of \( X \), forms a basis for the topology on \( C_N \) given in Definition 3.2.

**Notation 3.4.** Let \( X \) be a simplicial tree and let \( e \) be an oriented edge of \( X \). Denote by \( q(e) \) the set of all oriented edges \( e' \) in \( X \) such that \( e, e' \) is a reduced edge-path in \( X \).

For any set \( B \) we denote by \( P_+(B) \) the set of all nonempty subsets of \( B \).

The following simple lemma is key for the Kirchhoff-type formulas for subset currents (see Proposition 3.11 below).

**Lemma 3.5.** Let \( \Gamma, X, K \subseteq X \) be as in Definition 3.2 and let \( e_1, \ldots, e_n \) be the terminal edges of \( K \), as in Definition 3.2. Then for every \( i = 1, \ldots, n \) we have

\[
SCyl_\alpha(K) = \cup_{U \in P_+(q(e_i))} SCyl_\alpha(K \cup U).
\]

**Proof.** Fix \( i, 1 \leq i \leq n \). It is obvious from the definitions that for every such nonempty \( U \) we have \( SCyl_\alpha(K \cup U) \subseteq SCyl_\alpha(K) \) and hence the union of \( SCyl_\alpha(K \cup U) \) over all such \( U \) is contained in \( SCyl_\alpha(K) \).

Let \( S \in SCyl_\alpha(K) \) be arbitrary. By the definition of \( SCyl_\alpha(K) \) we have \( S \cap Cyl_X(e_i) \neq \emptyset \). Let \( U \) be the set of all edges \( e \in EX \) such that there exists a point \( \xi \in S \) that contains the geodesic ray in \( X \) beginning with the edge-path \( (e_i, e) \). Then \( S \subseteq SCyl_\alpha(K \cup U) \).

It follows that \( SCyl_\alpha(K) \subseteq \cup U SCyl_\alpha(K \cup U) \)

with \( U \) as required. We leave it as an exercise to the reader verifying that the union on the right-hand side in the above formula is a disjoint union.

**3.2. Subset currents.** Observe that the left translation action of \( F_N \) on \( \partial F_N \) naturally extends to a left translation action by homeomorphisms on \( C_N \). We can now define the main notion of this paper:

**Definition 3.6** (Subset currents). A subset current on \( F_N \) is a positive Borel measure \( \mu \) on \( C_N \) which is \( F_N \)-invariant and locally finite (i.e., finite on all compact subsets of \( C_N \)).

The set of all subset currents on \( F_N \) is denoted \( SCurr(F_N) \). The space \( SCurr(F_N) \) is endowed with the natural weak-* topology of convergence of integrals of continuous functions with compact support.

Define an equivalence relation \( \sim \) on \( SCurr(F_N) - \{0\} \) as: \( \mu \sim \mu' \) if \( \mu = c\mu' \) for some \( c > 0 \), where \( \mu, \mu' \in SCurr(F_N) - \{0\} \). For a nonzero \( \mu \in SCurr(F_N) \) the \( \sim \)-equivalence class of \( \mu \) is denoted \( \langle \mu \rangle \) and is called the projective class of \( \mu \). Put \( PSCurr(F_N) = (SCurr(F_N) - \{0\}) / \sim \) and endow \( PSCurr(F_N) \) with the quotient topology.
It is not hard to check (c.f. the proof by Francaviglia [29] of a similar statement for ordinary geodesic currents on $F_N$) that the weak-* topology on $SCurr(F_N)$ can be described in more concrete terms:

**Proposition 3.7.** Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking on $F_N$, and let $X = \tilde{\Gamma}$ be the universal cover of $\Gamma$.

1. Let $\mu, \mu_n \in SCurr(F_N)$, where $n = 1, 2, \ldots$. Then $\lim_{n \to \infty} \mu_n = \mu$ in $SCurr(F_N)$ if and only if for every finite non-degenerate subtree $K$ of $X$ we have
   \[ \lim_{n \to \infty} \mu_n(Cyl_\alpha(K)) = \mu(Cyl_\alpha(K)). \]

2. For each finite non-degenerate subtree $K$ of $X$ the function
   \[ SCurr(F_N) \to \mathbb{R}, \ \mu \mapsto \mu(Cyl_\alpha(K)) \]
   is continuous on $SCurr(F_N)$.

3. Let $\mu \in SCurr(F_N)$. For $\varepsilon > 0$ and an integer $M \geq 1$ let $U(M, \varepsilon, \mu)$ be the set of all $\mu' \in SCurr(F_N)$ such that for every finite non-degenerate subtree $K$ of $X$ with at most $M$ edges we have
   \[ |\mu'(Cyl_\alpha(K)) - \mu(Cyl_\alpha(K))| < \varepsilon. \]
   Then the family $\{U(M, \varepsilon, \mu) : M \geq 1, 0 < \varepsilon < 1\}$ forms a basis of open neighborhoods for $\mu$ in $SCurr(F_N)$.

The following key observation follows directly from the definition of subset cylinders (see Definition 3.2 above) and from the fact that subset currents are $F_N$-invariant.

**Proposition-Definition 3.8** (Weights). Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking on $F_N$, let $X = \tilde{\Gamma}$, and let $K$ be a finite non-degenerate subtree of $X$. Then for every element $g \in F_N$ we have
   \[ gSCyl_\alpha(K) = SCyl_\alpha(gK) \]
   and
   \[ \mu(SCyl_\alpha(K)) = \mu(gSCyl_\alpha(K)). \]

We denote
   \[ (K; \mu)_\alpha := \mu(SCyl_\alpha(K)) \]
and call it the weight of $K$ in $\mu$. For a given finite subtree $K$ of $X$, we denote the $F_N$-translation class of $K$ by $[K]$ (so that $[K]$ consists of all the translates of $K$ by elements of $F_N$). We put
   \[ ([K]; \mu)_\alpha := (K; \mu)_\alpha \]
and call it the weight of $[K]$ in $\mu$. In view of $(*)$, the weight $([K]; \mu)_\alpha$ is well-defined and does not depend on the choice of $K$ in $[K]$. It defines a continuous function on $SCurr(F_N)$.

**Corollary 3.9.** Let $\alpha, \Gamma$ and $X$ be as in Proposition 3.8. Let $K$ be a finite non-degenerate subtree of $X$. Then the function $f : SCurr(F_N) \to \mathbb{R}$ given by $f(\mu) = (K; \mu)_\alpha$, where $\mu \in SCurr(F_N)$, is continuous.

**Notation 3.10.** Let $\alpha, \Gamma$ and $X$ be as in Proposition 3.8. For every topological edge $e \in E_{top}\Gamma$ denote by $\tilde{e}$ the subgraph of $X = \tilde{\Gamma}$ consisting of any lift of $e$. By $F_N$-invariance of subset currents, the weight $([\tilde{e}]; \mu)_\alpha$ depends only on $\mu, \alpha$ and $e$ and does not depend on the choice of a lift $\tilde{e}$ of $e$ to $X$. We thus denote $(e; \mu)_\alpha := ([\tilde{e}]; \mu)_\alpha$. 
Proposition 3.11 (Kirchhoff formulas for weights). Let \( \alpha : F_N \to \pi_1(\Gamma) \) be a marking on \( F_N \), let \( X = \overline{\Gamma} \), and let \( K \) be a finite non-degenerate subtree of \( X \) with terminal edges \( e_1, \ldots, e_n \), as in Definition 3.2. Let \( \mu \in SCurr(F_N) \). Then for every \( i = 1, \ldots, n \) we have
\[
(K; \mu)_\alpha = \sum_{U \in F_{\alpha}(q(e_i))} (K \cup U; \mu)_\alpha,
\]
in notations of [3.4].

Proof. The statement follows directly from Lemma 3.5 and from the fact that \( \mu \) is finite-additive. \( \square \)

Since the Borel \( \sigma \)-algebra on \( \mathcal{C}_N \) is generated by the collection of all subset cylinders, it follows, by Kolmogorov Extension Theorem, that any \( F_N \)-invariant system of weights on all the subset cylinders satisfying the Kirchhoff formulas actually defines a subset current:

Proposition 3.12. Let \( \alpha, \Gamma \) and \( X \) be as in Proposition 3.11. Let \( K_\Gamma \) be the set of all finite non-degenerate simplicial subtrees of \( X \) and let
\[
\vartheta : K_\Gamma \to [0, \infty)
\]
be a function satisfying the following conditions:
1. For every \( K \in K_\Gamma \) and every \( g \in F_N \) we have \( \vartheta(gK) = \vartheta(K) \).
2. For every \( K \in K_\Gamma \) and every terminal edge \( e \) of \( K \) we have
\[
\vartheta(K) = \sum_{U \in F_{\alpha}(q(e))} \vartheta(K \cup U).
\]

Then there exists a unique \( \mu \in SCurr(F_N) \) such that for every \( K \in K_\Gamma \) we have
\[
\vartheta(K) = (K; \mu)_\alpha.
\]

Proposition 3.13. The space \( SCurr(F_N) \) is locally compact and the space \( \mathbb{P}SCurr(F_N) \) is compact.

Proof. It follows from the definition of \( SCurr(F_N) \) as the space of \( F_N \)-invariant locally finite positive Borel measures on \( \mathcal{C}_N \) that \( SCurr(F_N) \) is metrizable. Then to show local compactness it suffices to establish sequential local compactness of \( SCurr(F_N) \).

Consider a marking \( \alpha : F_N \to \pi_1(\Gamma) \). It is not hard to show, using Proposition 3.11 Proposition 3.5 and a standard diagonalization argument for individual weights, that for every \( C > 0 \) the sets
\[
\{ \mu \in SCurr(F_N) : \sum_{e \in B_{\log \Gamma}} (\hat{e}; \mu)_\alpha \leq C \}
\]
and
\[
\{ \mu \in SCurr(F_N) : \sum_{e \in B_{\log \Gamma}} (\hat{e}; \mu)_\alpha = C \}
\]
are sequentially compact. This implies local compactness of \( SCurr(F_N) \) and compactness of \( \mathbb{P}SCurr(F_N) \). \( \square \)

Remark 3.14. Recall also that elements of \( Curr(F_N) \) are positive locally finite \( F_N \)-invariant Borel measures on the space of 2-element subsets in \( \partial F_N \) which is clearly a closed \( F_N \)-invariant subset of \( \mathcal{C}_N \). Hence the space \( Curr(F_N) \) can be thought of as canonically embedded in \( SCurr(F_N) \), \( Curr(F_N) \subseteq SCurr(F_N) \), and it is not hard to see that \( Curr(F_N) \) is a closed subset of \( SCurr(F_N) \). Moreover, once the action of \( Out(F_N) \) on \( SCurr(F_N) \) is defined in Section 5 below, it will be obvious that \( Curr(F_N) \) is an \( Out(F_N) \)-invariant subset of \( SCurr(F_N) \). For similar reasons \( \mathbb{P}Curr(F_N) \subseteq \mathbb{P}SCurr(F_N) \) is a closed \( Out(F_N) \)-invariant subset.
4. Rational subset currents

4.1. Counting and rational subset currents. Recall that for a subgroup \( H \leq G \) of a group \( G \) the commensurator or virtual normalizer \( \text{Comm}_G(H) \) of \( H \) in \( G \) is defined as

\[
\text{Comm}_G(H) := \{ g \in G : [H : H \cap gHg^{-1}] < \infty, \text{ and } [gHg^{-1} : H \cap gHg^{-1}] < \infty \}.
\]

It is easy to see that \( \text{Comm}_G(H) \) is again a subgroup of \( G \) and that \( H \leq \text{Comm}_G(H) \).

Suppose now that \( N \geq 2 \) and \( H \leq F_N \) is a nontrivial subgroup of \( F_N \). The limit set \( \Lambda(H) \) of \( H \) in \( \partial F_N \) is the set of all \( \xi \in \partial F_N \) such that there exists a sequence \( h_n \in H, n \geq 1 \) satisfying

\[
\lim_{n \to \infty} h_n = \xi \quad \text{in} \quad F_N \cup \partial F_N.
\]

We recall some elementary properties of limit sets. We see in particular that for every \( H \leq F_N, H \neq 1 \) we have \( \Lambda(H) \in \mathcal{E}_N \).

**Proposition 4.1.** Let \( H \leq F_N \) be a nontrivial subgroup. Then:

1. The limit set \( \Lambda(H) \) is a closed \( H \)-invariant subset of \( \partial F_N \). If \( H \) is infinite cyclic, \( \Lambda(H) \) consists of two distinct points; if \( H \) is not cyclic, \( \Lambda(H) \) is infinite.
2. For any \( \xi \in \Lambda(H) \) the closure of the orbit \( H\xi \) in \( \partial F_N \) is equal to \( \Lambda(H) \).
3. For every \( g \in G \)

\[
\Lambda(gHg^{-1}) = g\Lambda(H).
\]
4. If \( H \leq Q \leq F_N \) then \( \Lambda(H) \subseteq \Lambda(Q) \).
5. Either \( H \) is infinite cyclic and \( \Lambda(H) \) consists of exactly two distinct points or \( H \) contains a nonabelian free subgroup and \( \Lambda(H) \) is uncountable.
6. Let \( T \in cv_N \) and let \( \text{Conv}_T(\Lambda(H)) \) be the convex hull of \( \Lambda(H) \) in \( T \), that is, the union of all bi-infinite geodesics in \( T \) with endpoints in \( \Lambda(H) \). Then \( \text{Conv}_T(\Lambda(H)) = T_H \) is the unique minimal \( H \)-invariant subtree of \( T \).

Another useful basic fact relates limit sets of finitely generated subgroups and their commensurators (see [52][19] for details).

**Proposition 4.2.** Let \( H \leq F_N \) be a nontrivial finitely generated subgroup. For a subset \( Z \subseteq \partial F_N \) denote \( \text{Stab}_{F_N}(Z) := \{ g \in F_N : gZ = Z \} \). Then

1. \( \text{Stab}_{F_N}(\Lambda(H)) = \text{Comm}_{F_N}(H) \) and \( [\text{Comm}_{F_N}(H) : H] < \infty \).
2. \( \Lambda(H) = \Lambda(\text{Comm}_{F_N}(H)) \).
3. For \( H_1 = \text{Comm}_{F_N}(H) \) we have \( \text{Comm}_{F_N}(H_1) = H_1 \).
4. Let \( L \leq F_N \) such that \( H \leq L \). Then \( [L : H] < \infty \) if and only if \( L \leq \text{Comm}_G(H) \).
5. Suppose \( H = \langle g \rangle \), where \( g \in F_N, g \neq 1 \). Then \( H = \text{Comm}_{F_N}(H) \) if and only if \( g \) is not a proper power in \( F_N \), that is, if and only if \( H \) is a maximal infinite cyclic subgroup of \( F_N \).

**Definition 4.3.** Let \( H \leq F_N \) be a nontrivial finitely generated subgroup.

1. Suppose first that \( H = \text{Comm}_{F_N}(H) \). Define the measure \( \eta_H \) on \( \mathcal{E}_N \) as

\[
\eta_H := \sum_{H_i \in [H]} \delta_{\Lambda(H_i)},
\]

where \( [H] \) is the conjugacy class of \( H \) in \( F_N \).

2. Now let \( H \leq F_N \) be an arbitrary nontrivial finitely generated subgroup. Put \( H_0 := \text{Comm}_{F_N}(H) \) and let \( m := [H_0 : H] \). Proposition 4.2 implies that \( m < \infty \) and that \( H_0 = \text{Comm}_{F_N}(H_0) \). Then put \( \eta_H := m\eta_{H_0} \).

**Lemma 4.4.** Let \( H \leq F_N \) be a nontrivial finitely generated subgroup. Then \( \eta_H \in \text{SCurr}(F_N) \).
Proof. It is enough to show that $\eta_H \in SCurr(F_N)$ for the case where $H = Comm_{F_N}(H)$. Thus we assume that $H \leq F_N$ is a nontrivial finitely generated subgroup with $H = Comm_{F_N}(H)$. Fix a free basis $A$ of $F_N$ and let $X$ be the Cayley graph of $F_N$ with respect to $A$.

Part (3) of Proposition 4.5 implies that $\eta_H$ is an $F_N$-invariant positive Borel measure on $\mathcal{C}_N$. Thus it remains to check that $\eta_H(C) < \infty$ for every compact $C \subseteq \mathcal{C}_N$. Every compact subset of $\mathcal{C}_N$ is the union of finitely many cylinder subsets. Thus we only need to check that $\eta_H(SCyl_X(K)) < \infty$ for every finite non-degenerate subtree $K$ of $X$. After replacing $K$ by its $F_N$-translate, we may assume that the element $1 \in F_N$ is a vertex of $K$. Also, since $\eta_H$ depends only on the conjugacy class of $H$, after replacing $H$ by a conjugate we may assume that $1 \in X_H = Conv_X(\Lambda(H))$.

Recall from Proposition 4.1 that $Conv_X(\Lambda(H)) = X_H$ is the unique minimal $H$-invariant subtree of $X$. Whenever $g \in F_N$ is such that $g\Lambda(H) = \Lambda(gHg^{-1}) \subseteq SCyl_X(K)$, we have $1 \in Conv_X(\Lambda(H)) = X_H$. Thus it suffices to show that the number of distinct translates $gX_H$ of $X_H$ that contain $1 \in F_N$ is finite.

It is not hard to see, since by assumption $1 \in X_H$, that for $g \in F_N$ we have $1 \in gX_H$ if and only if $g \in V(X_H) \subseteq F_N$.

Since $X_H$ is the minimal $H$-invariant subtree and $H$ is finitely generated, the quotient $H \setminus X_H$ is a finite graph. In particular $H \setminus V(X_H)$ is a finite set. Every $H$-orbit of a vertex of $X_H$ is a coset class $uH$ for some $u \in F_N$. Thus there exists a finite set $u_1, \ldots, u_m \in F_N$ such that $V(X_H) = \{u_i h | h \in H, 1 \leq i \leq m\}$. For $g = u_i h$, where $h \in H$, $1 \leq i \leq m$, we have $gX_H = u_i hX_H = u_i X_H$. Thus indeed there are only finitely many translates of $X_H$ that contain $1 \in F_N$. Therefore $\eta_H(SCyl_X(K)) < \infty$, as required.

□

Definition 4.5 (Rational and counting currents). For a nontrivial finitely generated $H \leq F_N$, we call $\eta_H \in SCurr(F_N)$ given by Definition 4.3 and Lemma 4.4 the counting subset current associated to $H$.

A subset current $\mu \in SCurr(F_N)$ is called rational if $\mu = r\eta_H$ for some $r \geq 0$ and some nontrivial finitely generated subgroup $H \leq F_N$.

Definition 4.3 directly implies:

Proposition 4.6. Let $H \leq F_N$ be a nontrivial finitely generated subgroup and let $H' = gHg^{-1}$ for some $g \in F_N$. Then $\eta_H = \eta_{H'}$.

The above statement has a partial converse:

Proposition 4.7. Let $H, H' \leq F_N$ be nontrivial finitely generated subgroups such that $H = Comm_{F_N}(H)$ and $H' = Comm_{F_N}(H')$. Then $\eta_H = \eta_{H'}$ if and only if $[H] = [H']$.

Proof. We have already seen that if $[H] = [H']$ then $\eta_H = \eta_{H'}$.

Suppose now that $\eta_H = \eta_{H'}$. Choose a marking $\alpha : F_N \to \pi_1(\Gamma)$ on $F_N$. Let $X = \Gamma$.

Let $K \in K_\Gamma$ be such that $\Lambda(H) \in SCyl_\alpha(K)$. Since $\eta_H(SCyl_\alpha(K)) < \infty$ and $\eta_{H'}(SCyl_\alpha(K)) < \infty$, Definition 4.3 implies that only finitely many distinct $F_N$-translates of $\Lambda(H)$ and $\Lambda(H')$ belong to $SCyl_\alpha(K)$. Let these translates be $g_1 \Lambda(H), \ldots, g_m \Lambda(H)$ and $f_1 \Lambda(H'), \ldots, f_l \Lambda(H')$, with $g_1 = 1$. Thus we have $m + t$ distinguished points in the open set $SCyl_\alpha(K)$. Since $\mathcal{C}_N$ is metrizable and Hausdorff, we can find an open subset $V$ of $SCyl_\alpha(K)$ such that $V$ contains exactly one of these $m + t$ points, namely the point $g_1 \Lambda(H) = \Lambda(H)$. Since the cylinder subsets form a basis of open sets for $\mathcal{C}_N$, there exists a finite subtree $K_1$ of $X$ such that $\Lambda(H) \in SCyl_\alpha(K_1) \subseteq V$. Thus $\Lambda(H) \in SCyl_\alpha(K_1)$ and no $F_N$-translate of $\Lambda(H), \Lambda(H')$, distinct from $\Lambda(H)$, belongs to $SCyl_\alpha(K_1)$. Then $\eta_H(SCyl_\alpha(K_1)) = 1$. By assumption $\eta_H = \eta_{H'}$ and hence $\eta_{H'}(SCyl_\alpha(K_1)) = 1$. Thus, by
There is a unique translate $g \Lambda(H')$, where $g \in F_N$, such that $g \Lambda(H') \in SCyl_s(K_1)$. By the choice of $K_1$, the only $F_N$-translate of $\Lambda(H), \Lambda(H')$ contained in $SCyl_s(K_1)$ is $\Lambda(H)$. Therefore $\Lambda(H) = g \Lambda(H')$ so that $\Lambda(H) = g \Lambda(H') = \Lambda(gH'g^{-1})$.

Since $H = \Comm_{F_N}(H)$ and $gH'g^{-1} = \Comm_{F_N}(gH'g^{-1})$, Proposition 4.2 implies that $H = \Stab_{F_N}(\Lambda(H)) = \Stab_{F_N}(\Lambda(gH'g^{-1})) = gH'g^{-1}$, and $[H] = [H']$, as required. 

4.2. $\Gamma$-graphs. We need to introduce some terminology slightly generalizing the standard set-up of Stallings foldings for subgroups of free groups (see [H149]). That set-up usually involves choosing a free basis $A$ of $F_N$ and considering graphs whose edges are labelled by elements of $A^\pm 1$. Choosing a free basis $A$ of $F_N$ corresponds to a marking on $F_N$ that identifies $F_N$ with the fundamental group of the standard $N$-rose $R_N$. We need to relax the requirement that the graph in question be $R_N$.

Definition 4.8 ($\Gamma$-graphs). Let $\Gamma$ be a finite connected graph without degree-one and degree-two vertices. A $\Gamma$-graph is a graph $\Delta$ together with a graph morphism $\tau : \Delta \to \Gamma$.

For a vertex $x \in V\Delta$ we say that the type of $x$ is the vertex $\tau(x) \in V\Gamma$. Similarly, for an oriented edge $e \in E\Gamma$ the type of $e$, or the label of $e$ is the edge $\tau(e)$ of $\Gamma$.

Note that every covering of $\Gamma$ has a canonical $\Gamma$-graph structure. In particular, $\Gamma$ itself is a $\Gamma$-graph and so is the universal cover $\tilde{\Gamma}$ of $\Gamma$. Also, every subgraph of a $\Gamma$-graph is again a $\Gamma$-graph.

Note that for a graph $\Delta$ a graph-morphism $\tau : \Delta \to \Gamma$ can be uniquely specified by assigning a label $\tau(e) \in E\Gamma$ for every $e \in E\Delta$ in such a way that $\tau(e^{-1}) = (\tau(e))^{-1}$ and such that for every pair of edges $e', e'' \in E\Delta$ with $o(e) = o(e')$ we have $o(\tau(e)) = o(\tau(e')) \in V\Gamma$. Thus we usually will think of a $\Gamma$-graph structure on $\Delta$ as such an assignment of labels $\tau : E\Delta \to E\Gamma$. Also, by abuse of notation, we will often refer to a graph $\Delta$ as a $\Gamma$-graph, assuming that the graph morphism $\tau : \Delta \to \Gamma$ is implicitly specified.

Definition 4.9 ($\Gamma$-graph morphism). Let $\tau_1 : \Delta_1 \to \Gamma$ and $\tau_2 : \Delta_2 \to \Gamma$ be $\Gamma$-graphs. A graph morphism $f : \Delta_1 \to \Delta_2$ is called a $\Gamma$-graph morphism, if it respects the labels of vertices and edges, that is if $\tau_1 = \tau_2 \circ f$.

Definition 4.10 (Folded $\Gamma$-graph). Let $\Delta$ be a $\Gamma$-graph. For a vertex $x \in V\Delta$ denote by $Lk_\Delta(x)$ (or just by $Lk(x)$) the link of $x$, namely the function $Lk_\Delta(x) : E\Gamma \to \mathbb{Z}_{\geq 0}$ where for every $e \in E\Gamma$ the value $(Lk_\Delta(x))(e)$ is the number of edges of $\Delta$ with origin $x$ and label $e$.

A $\Gamma$-graph $\Delta$ is said to be folded if for every vertex $x \in V\Delta$ and every $e \in E\Gamma$ we have $(Lk_\Delta(x))(e) \leq 1$.

If $\Delta$ is folded, we will also think of $Lk_\Delta(x)$ as a subset of $E\Gamma$ consisting of all those $e \in E\Gamma$ with $(Lk_\Delta(x))(e) = 1$, that is, of all $e \in E\Gamma$ such that there is an edge in $\Delta$ with origin $x$ and label $e$.

The following is an immediate corollary of the definitions.

Lemma 4.11. Let $\Delta$ be a $\Gamma$-graph and let $\tau : \Delta \to \Gamma$ be the associated graph morphism. Then:

1. $\Delta$ is folded if and only if $\tau$ is an immersion.
2. Suppose $\Delta$ is connected. Then $\Delta$ is a covering of $\Gamma$ if and only if $Lk_\Delta(x) = Lk_\Gamma(\tau(x))$ for every $x \in V\Delta$. 

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(3) Let $\Delta_1, \Delta_2$ be two $\Gamma$-graphs such that $\Delta_1$ is connected and $\Delta_2$ is folded. Let $x_1 \in V\Delta_1$, $x_2 \in V\Delta_2$ be vertices of the same type $x \in V\Gamma$. Then there exists at most one $\Gamma$-graph morphism $f : \Delta_1 \to \Delta_2$ such that $f(x_1) = x_2$.

**Definition 4.12** (Γ-core graph). Referring to Stallings’ notion of core graphs, [61], we say that a finite $\Gamma$-graph $\Delta$ is a $\Gamma$-core graph if $\Delta$ is cyclically reduced, that is, it is folded and has no degree-one and degree-zero vertices.

Informally, we think of $\Gamma$-core graphs as generalizations of cyclic words, over the “alphabet” $\Gamma$. We need the following analog of the notion of a subword in this context:

**Definition 4.13** (Occurrence). Let $K \subseteq \tilde{\Gamma}$ be a finite non-degenerate subtree, considered together with the canonical $\Gamma$-graph structure inherited from $\tilde{\Gamma}$. Let $\Delta$ be a finite $\Gamma$-core graph.

An occurrence of $K$ in $\Delta$ is a $\Gamma$-graph morphism $f : K \to \Delta$ such that for every vertex $x$ of $K$ of degree at least 2 in $\Delta$ we have $Lk_K(x) = Lk_\Delta(f(x))$.

We denote the number of all occurrences of $K$ in $\Delta$ by $(K; \Delta)_\Gamma$, or just $(K; \Delta)$.

In topological terms, a $\Gamma$-graph morphism $f : K \to \Delta$ is an occurrence of $K$ in $\Delta$ if $f$ is an immersion and if $f$ is a covering map at every point $x \in K$ (including interior points of edges) except for the degree-1 vertices of $K$. That is, for every $x \in K$, other than a degree-1 vertex of $K$, $f$ maps a small neighborhood of $x$ in $K$ homeomorphically onto a small neighborhood of $f(x)$ in $\Delta$.

**Proposition 4.14.** Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking on $F_N$ and let $X = \tilde{\Gamma}$. As before (see [6, 12]), let $K_\Gamma$ denote the set of all non-degenerate finite simplicial subtrees of $X$. Let $\Delta$ be a finite $\Gamma$-core graph. Define $\vartheta_{\Delta} : K_\Gamma \to \mathbb{R}$ as

$$\vartheta_{\Delta}(K) := (K; \Delta)_\Gamma$$

for each $K \in K_\Gamma$.

Then the function $\vartheta_{\Delta}$ satisfies the conditions of Proposition 3.12. Thus there is a unique subset current $\mu_{\Delta} \in S\text{Curf}(F_N)$ such that for every $K \in K_\Delta$

$$(K; \mu_{\Delta})_\alpha = (K; \Delta)_\Gamma$$

**Proof.** We will see later on that, for a connected $\Delta$, the function $\vartheta_{\Delta}$ defines the counting current of a finitely generated subgroup of $F_N$. However, we think it is useful to have a direct argument for Proposition 4.14.

It easily follows from the definitions that for every $K \in K_\Gamma$ and every $g \in F_N$ we have $(K; \Delta)_\Gamma = (gK; \Delta)_\Gamma$. Thus we only need to verify that the condition (2) of Proposition 3.12 holds for $\vartheta_{\Delta}$.

Let $K \in K_\Gamma$ and let $e$ be a terminal edge of $K$, as in Definition 3.2. Consider the set of trees

$$Q(K, e) = \{K \cup U | U \in P_+(q(e))\}$$

as in Notation 3.2. Thus each element of $Q(K, e)$ is obtained by adding to $K$ a nonempty subset $U$ from the set of edges $q(e)$.

We now construct a function $D$ from the set $R$ of all occurrences of $K$ in $\Delta$ to the set $R'$ of all occurrences of elements of $Q(K, e)$ in $\Delta$. Put $x := t(e) \in V\Gamma$.

Let $f : K \to \Delta$ be an occurrence of $K$. Since $\Delta$ is a core graph, the vertex $y = f(x)$ has degree bigger than 1 in $\Delta$. Note that $f(e^{-1})$ is an edge of $\Delta$ with initial vertex $y$. Put $U_0$ to be the set of the labels of all the edges in $\Delta$ with origin $y$, excluding the edge $f(e^{-1})$. Thus $U_0$ is nonempty. Let $U$ be the set of all edges in $\tilde{\Gamma}$ with initial vertex $x$ and with label belonging to $U_0$. Put $K' = K \cup U$, so that $K' \in Q(K, e)$. We now extend $f$ to a morphism $f' : K' \to \Delta$ by sending every edge from $U$ to the unique edge in $\Delta$ with...
the same label and with origin \( y \). Then by construction, \( f' \) is a \( \Gamma \)-graph morphism whose restriction to \( K \) is \( f \) and, moreover, \( \text{Lk}_{K'}(x) = \text{Lk}_\Delta(y) \). Therefore \( f' \) is an occurrence of \( K' \) in \( \Delta \). We put \( D(f) := f' \).

This defines a function \( D : R \to R' \). By construction, this function is injective. Moreover, \( D \) is clearly onto. Indeed, if \( K' = K \cup U \in Q(K, e) \) and \( f' : K' \to \Delta \) is an occurrence of \( K' \) in \( \Delta \) then \( f := f'|_K \) is an occurrence of \( K \) in \( \Delta \) and \( D(f) = f' \). Thus \( D \) is a bijection, and hence \( R \) and \( R' \) have equal cardinalities. It follows that
\[
(K; \Delta)_\Gamma = \sum_{U \in P_\Gamma(g(e))} (K \cup U; \Delta)_\Gamma,
\]
as required. \( \square \)

Note that if \( \Delta \) is a finite \( \Gamma \)-core graph with connected components \( \Delta_1, \ldots, \Delta_k \), then each \( \Delta_i \) is again a \( \Gamma \)-core graph and we have
\[
\mu_{\Delta} = \mu_{\Delta_1} + \cdots + \mu_{\Delta_k}.
\]

We leave it to the reader to verify the following:

**Lemma 4.15.** Let \( \Delta \) be as in Proposition 4.14 and assume that \( \Delta \) is connected. Let \( \hat{\Delta} \) be an \( m \)-fold cover of \( \Delta \) for some \( m \geq 1 \). Then \( \mu_{\hat{\Delta}} = m \mu_{\Delta} \).

### 4.3. Counting currents and \( \Gamma \)-graphs

We now want to relate the current constructed in Proposition 4.14 to counting currents of finitely generated subgroups of \( F_N \).

Let \( \alpha : F_N \to \pi_1(\Gamma) \) be a marking on \( F_N \), and let \( X := \hat{\Gamma} \). Recall from Subsection 2.2 that \( \partial F_N \) and \( \partial X \) are identified via the homeomorphism induced by \( \alpha \). Note that if \( S \in \mathfrak{C}_N \), the fact that \( S \) has cardinality at least 2 implies that its convex hull \( \text{Conv}_X(S) \) is nonempty. Moreover, it is easy to see that \( \text{Conv}_X(S) \) is an infinite subtree of \( X \) without any degree-1 vertices. Let us denote by \( \mathcal{T}(X) \) the set of all infinite subtrees of \( X \) without degree-1 vertices.

The following statement is an elementary consequence of the definitions and we leave the details to the reader.

**Proposition 4.16.** Let \( \alpha, \Gamma, X \) be as above. Then

1. For any \( S \in \mathfrak{C}_N \) we have \( \partial \text{Conv}_X(S) = S \) and for any \( Y \in \mathcal{T}(X) \) we have \( Y = \text{Conv}_X(\partial Y) \).
2. The convex hull operation yields a bijection \( \text{Conv}_X : \mathfrak{C}_N \to \mathcal{T}(X) \) which is \( F_N \)-equivariant: for any \( S \in \mathfrak{C}_N \) and \( g \in F_N \) we have \( \text{Conv}_X(gS) = g \text{Conv}_X(S) \).
3. For any \( S \in \mathfrak{C} \) we have \( \partial \text{Conv}_X(S) = S \) and for any \( Y \in \mathcal{T}(X) \) we have \( Y = \text{Conv}_X(\partial Y) \).

Recall that if \( T \in \mathfrak{T}_N \) and if \( H \leq F_N \) is a nontrivial subgroup, there is a unique smallest \( H \)-invariant subtree of \( T \) denoted \( T_H \), and, moreover \( T_H \) has no degree-one vertices.

Propositions 4.10, 4.1 and 4.2 easily imply:

**Proposition 4.17.** Let \( \alpha, \Gamma, X \) be as above. Let \( H \leq F_N \) be a nontrivial finitely generated subgroup and let \( X_H \) be the minimal \( H \)-invariant subtree of \( X \). Then:

1. \( X_H \in \mathcal{T}(X) \) and \( X_H = \text{Conv}_X(\Lambda(H)) \).
2. For any \( g \in F_N \), \( gX_H = X_{gHg^{-1}} \).
3. \( H = \text{Comm}_{F_N}(H) \) if and only if \( \text{Stab}_{F_N}(X_H) = H \).

**Convention 4.18** (\( \Gamma \)-core graphs representing conjugacy classes of subgroups of \( F_N \)). Let \( \alpha : F_N \to \pi_1(\Gamma) \) be a marking on \( F_N \), and let \( X := \hat{\Gamma} \). Let \( H \leq F_N \) be a finitely generated subgroup and let \( X_H \) be the minimal \( H \)-invariant subtree of \( X \). Note that,
being a subgraph of $\tilde{\Gamma}$, the tree $X_H$ comes equipped with a canonical $\Gamma$-graph structure $X_H \to \Gamma$ (which is just the restriction to $X_H$ of the universal covering map $\tilde{\Gamma} \to \Gamma$) which is $F_N$-invariant. Put $\Delta = H \setminus X_H$. Then $\Delta$ is a finite connected graph without degree-1 vertices (by minimality of $X_H$); that is, $\Delta$ is a $\Gamma$-core graph. Moreover, $\Delta$ inherits a natural $\Gamma$-graph structure from $X_H$.

It is easy to see, in view of part (2) of Proposition 4.17, that the isomorphism type of $\Delta$ as a $\Gamma$-graph depends only on the conjugacy class $[H]$ of $H$ in $F_N$. Thus we say that $\Delta$ is the $\Gamma$-core graph representing $[H]$.

We can also describe $\Delta$ as follows: $\Delta$ is the core of $\tilde{\Gamma} = H \setminus X$, that is, $\Delta$ is the union of all immersed (but not necessarily simple) circuits of infinite tree “hanging branches”. Also, $\Delta$ is the smallest subgraph of $\tilde{\Gamma}$ whose inclusion in $\tilde{\Gamma}$ is a homotopy equivalence with $\tilde{\Gamma}$. The labelling map $\tau : \Delta \to \Gamma$ satisfies $\tau_\#(\pi_1(\Delta)) = H$.

In fact $\tau_\#$ is an isomorphism between $\pi_1(\Delta)$ and $H$ and sometimes, by abuse of notation, we will write $\pi_1(\Delta) = H$.

**Theorem 4.19.** Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking on $F_N$. Let $H \leq F_N$ be a nontrivial finitely generated subgroup such that $\text{Comm}_{F_N}(H) = H$. Let $\Delta$ be the $\Gamma$-core graph representing $[H]$. Then $\eta_H = \mu_\Delta$.

**Proof.** We need to show that for every finite non-degenerate subtree $K$ of $X = \tilde{\Gamma}$ we have $(K; \eta_H)_\alpha = (K; \mu_\Delta)_\alpha$.

Choose a vertex $x_0$ in $K$ and let $v_0$ be the projection of $x_0$ in $\Gamma$. We may assume that $v_0$ is the base-point of $\Gamma$ and that $\alpha : F_N \simeq \pi_1(\Gamma, v_0)$. Recall that $X_H = \text{Conv}_X(\Lambda H)$ is the smallest $H$-invariant subtree of $X$. By replacing $H$ by its conjugate if necessary, we may assume that $x_0 \in X_H$. Moreover, $\Delta = H \setminus X_H$, and $(X_H, x_0)$ is canonically identified with $(\Delta, y_0)$, where $y_0$ is the image of $x_0$ under the projection $p : X_H \to \Delta$.

We need to show that the number $(K; \Delta)_\alpha$ of occurrences of $K$ in $\Delta$ is equal to the number of distinct $F_N$-translates of $X_H$ that contain $K$. We will construct a function $Q$ from the set of occurrences of $K$ in $\Delta$ to the set of $F_N$-translates of $X_H$ that contain $K$.

Let $f : K \to \Delta$ be an occurrence of $K$ in $\Delta$. Choose an edge-path $\gamma$ from $y_0$ to $f(x_0)$ in $\Delta$. The fact that both $y_0$ and $f(x_0)$ project to $v_0$ in $\Gamma$ implies that the (unique) lift $\tilde{\gamma}$ of $\gamma$ to an edge-path in $X$ with origin $x_0$ has its terminal vertex of the form $g x_0$ for some (unique) $g \in F_N$. Then the definition of occurrence of $K$ in $\Delta$ and the fact that $\Delta$ is identified with $X_H$ imply that $g K \subseteq X_H$ and so $K \subseteq g^{-1}X_H$. We set $Q(f) := g^{-1}X_H$.

We need to check that $Q(f)$ is well-defined, that is, that the translate $g^{-1}X_H$ does not depend on the particular choice of an edge-path $\gamma$ from $y_0$ to $f(x_0)$ in $\Delta$. Indeed, if $\gamma'$ is another such edge-path, then $\gamma' \gamma^{-1} \in \pi_1(\Delta, y_0) = H$. Hence if $\gamma'$ lifts to an edge-path from $x_0$ to $g'x_0$ in $X$, we have $g'g^{-1} \in H$, so that $g^{-1} = (g')^{-1}h$ for some $h \in H$ and hence $g^{-1}X_H = (g')^{-1}X_H$. Thus the translate $g^{-1}X_H$ containing $K$ is uniquely determined by the occurrence $f : K \to \Delta$ of $K$ in $\Delta$, so that $Q(f)$ is well-defined. Hence we have constructed a map $Q$ from the set of occurrences of $K$ in $\Delta$ to the set of $F_N$-translates of $X_H$ containing $K$.

We claim that the map $Q$ is injective. Indeed, let $f_1 : K \to \Delta$ be another occurrence of $K$ in $\Delta$. Let $\gamma_1$ be an edge-path in $\Delta$ from $y_0$ to $f_1(x_0)$ and let $g_1 \in F_N$ be such that the lift to $X$ of $\gamma_1$ with origin $x_0$ has terminus $g_1x_0$. Suppose that $g_1^{-1}X_H = g^{-1}X_H$. Then $g_1g^{-1}X_H = X_H$ and hence $g_1g^{-1} \in \text{Comm}_{F_N}(H)$. By assumption, on $H = \text{Comm}_{F_N}(H)$, so that $g_1g^{-1} \in H$ and $g_1 = hg$ for some $h \in H$.

However, since $\pi_1(\Delta, y_0) = H$, it follows that $h$ labels a closed edge-path from $y_0$ to $y_0$ in $\Delta$ and hence, up to edge-path reductions, $\gamma_1 = \beta \gamma$ for some closed loop $\beta$ from $y_0$ to
As the sum of any two counting currents can be approximated by rational currents.

Consequently, since all the graphs under consideration are folded, that $f = f_1$. Thus indeed, the map $Q$ is injective.

We claim that $Q$ is also surjective. Suppose a translate $g^{-1}X_H$ contains $K$. Then $X_H$ contains $y_0$. Both $x_0$ and $g^{-1}x_0$ belong to $X_H$ and hence the geodesic edge-path $[x_0, g^{-1}x_0]$ is contained in $X_H$. This edge-path projects to an edge-path in $\Delta = H \setminus X_H$ from $y_0$ to the vertex $z_0$, which is the image of $g^{-1}x_0$ in $\Delta$ under the projection $p : X_H \to \Delta$. Since $p$ is a covering map, $f : K \to \Delta$ defined as $f(x) = p(g(x))$, $x \in K$, is an occurrence of $K$ in $\Delta$, with $f(x_0) = z_0$. The definition of $Q$ now gives $Q(f) = g^{-1}X_H$.

Thus $Q$ is a bijection between the set of occurrences of $K$ in $\Delta$ and the set of $F_N$-translates of $X_H$ containing $K$. It follows that $(K; \eta_H)_\alpha = (K; \eta_H)_\alpha$, as required. □

**Corollary 4.20.** Let $H \leq F_N$ be any nontrivial finitely generated subgroup. Let $\alpha : F_N \to \Gamma$ be a marking and let $\Delta$ be the $\Gamma$-core graph representing $[H]$. Then $\eta_H = \mu_\Delta$.

**Proof.** Put $H_1 = \text{Comm}_{F_N}(H)$. Then by Proposition 4.2 $H_1 = \text{Comm}_{F_N}(H_1)$, and $[H_1 : H] = m < \infty$. Let $\Delta_1$ be the $\Gamma$-core graph representing $[H_1]$. Therefore by Theorem 4.19 $\eta_{H_1} = \mu_{\Delta_1}$. Moreover, $\Delta$ is an $m$-fold cover of $\Delta_1$ and hence by Lemma 4.15 $\mu_\Delta = m \mu_{\Delta_1}$. Also, by definition, $\eta_H = m \eta_{H_1}$. Hence $\eta_H = m \eta_{H_1} = m \mu_{\Delta_1} = \mu_\Delta$, as required. □

5. **Rational currents are dense**

**Convention 5.1.** In order to avoid technical complications in the proof of the main result of this Section, Theorem 5.5 in Subsection 5.4 below, we will restrict ourselves in this section to only considering $\Gamma$-graphs for $\Gamma = R_N$, the standard $N$-rose. The universal cover $X = \overline{R_N}$ is then the Cayley graph of $F_N$ with respect to some basis. We will fix some basis $A$ in $F_N$ and we will think of $A$ as defining a marking $\alpha_A : F_N \to R_N$, that will also be fixed for the remainder of this Section.

Note that an $\Delta$-structure on a graph $\Delta$ can be specified by assigning every edge $e \in E\Delta$ a label $\tau(e) \in A^{\pm 1}$ so that $\tau(e^{-1}) = (\tau(e))^{-1}$.

5.1. **Linear span of rational currents.**

**Proposition 5.2.** Denote by $\text{SCurr}_r(F_N)$ the set of all rational subset currents. Let $\text{Span}(\text{SCurr}_r(F_N))$ denote the $\mathbb{R}_{\geq 0}$-linear span of $\text{SCurr}_r(F_N)$.

Let $N \geq 2$. Then the set $\text{SCurr}_r(F_N)$ is a dense subset of $\text{Span}(\text{SCurr}_r(F_N))$.

**Proof.** We need to show that an arbitrary linear combination $c_1\eta_{H_1} + \cdots + c_k\eta_{H_k}$ (where $k \geq 1$, $c_i \in \mathbb{R}_{\geq 0}$) can be approximated by currents of the form $c\eta_H$, where $c \in \mathbb{R}_{\geq 0}$. Arguing by induction on $k$, we see that it suffices to prove this statement for $k = 2$.

Every current of the form $c_1\eta_{H_1} + c_2\eta_{H_2}$, where $c_1, c_2 \in \mathbb{R}_{\geq 0}$ can be approximated by currents of the form $r_1\eta_{H_1} + r_2\eta_{H_2}$, where $r_1, r_2 > 0$ are rational numbers. Thus it suffices to approximate by rational currents every current of the form $r_1\eta_{H_1} + r_2\eta_{H_2}$ where $r_1, r_2 > 0$ are rational numbers. Taking $r_1, r_2$ to a common denominator, we have $r_1 = p_1/q$, $r_2 = p_2/q$ where $p_1, p_2, q > 0$ are integers. Since dividing a rational current by $q$ again yields a rational current, it is enough to approximate by rational currents all currents of the form $p_1\eta_{H_1} + p_2\eta_{H_2}$ where $p_1, p_2$ are positive integers. However, $p_1\eta_{H_1} = \eta_{L_1}$, $p_2\eta_{H_2} = \eta_{L_2}$, where $L_i$ is a subgroup of index $p_i$ in $H_i$ for $i = 1, 2$. Thus we only need to show that the sum of any two counting currents can be approximated by rational currents.
Suppose now that \( H_1, H_2 \leq F_N \) are two nontrivial finitely generated subgroups and let \( \mu = \eta_{H_1} + \eta_{H_2} \). For \( i = 1, 2 \) let \( \Delta_i \) be the \( N \)-core graph representing \([H_i]\), as in Convention \ref{con:core_graphs}. Thus, by Theorem \ref{thm:core_graphs}, \( \eta_{H_i} = \mu_{\Delta_i} \) for \( i = 1, 2 \).

For \( n = 1, 2, \ldots \) let \( \Lambda_{i,n} \) be a connected \( n \)-fold cover of \( \Delta_i \). We now define a sequence of finite connected \( R_N \)-core graphs as follows. Let \( n \geq 1 \). First assume that each of \( \Lambda_{1,n}, \Lambda_{2,n} \) has a vertex \( v_{i,n} \) of degree \( < 2N \), where \( i = 1, 2 \). Then, since \( A^{\pm 1} \) contains at least 4 distinct letters, there exists an \( R_N \)-graph \([u_1, u_2]\), which is a segment of three edges with origin denoted \( u_1 \) and terminus denoted \( u_2 \), such that identifying the origin of this segment with \( v_{1,n} \) and the terminus of this segment with \( v_{2,n} \) yields a folded \( R_N \)-graph:

\[
\Lambda_n := \Lambda_{1,n} \cup \Lambda_{2,n} \cup [u_1, u_2]/ \sim
\]

where \( u_1 \sim v_{1,n}, u_2 \sim v_{2,n} \). Note that by construction \( \Lambda_n \) is connected and cyclically reduced.

Suppose now that \( \Lambda_{1,n} \) has a vertex \( v_{1,n} \) of degree \( < 2N \) but that every vertex in \( \Lambda_{2,n} \) has degree \( 2N \).

Let \( v_{2,n} \) be any vertex of \( \Lambda_{2,n} \). Let \( \Lambda'_{2,n} \) be obtained from \( \Lambda_{2,n} \) by removing one of the edges incident to \( v_{2,n} \). Note that \( \Lambda'_{2,n} \) is still finite, connected and cyclically reduced.

Then, as in the previous case, there exists a simplicial segment \([u_1, u_2]\) consisting of three edges labelled by elements of \( A^{\pm 1} \) such that

\[
\Lambda_n := \Lambda_{1,n} \cup \Lambda'_{2,n} \cup [u_1, u_2]/ \sim
\]

is a folded \( R_N \)-graph, where \( u_1 \sim v_{1,n}, u_2 \sim v_{2,n} \). Again, by construction, \( \Lambda_n \) is connected and cyclically reduced. In the case where \( \Lambda_{2,n} \) has a vertex of degree \( < 2N \) but that every vertex in \( \Lambda_{1,n} \) has degree \( 2N \), we define \( \Lambda_n \) in a similar way.

Suppose now that for \( i = 1, 2 \) every vertex of \( \Lambda_{i,n} \) has degree \( 2N \). Then \( \Lambda_{i,n} \) is a finite cover of \( R_N \), so that \( \Delta_1, \Delta_2 \) are both finite covers of \( R_N \). Let \( m_i \geq 1 \) be such that \( \Delta_i \) is an \( m_i \)-fold cover of \( R_N \). Choose \( \Lambda_n \) to be any connected \( n(m_1 + m_2) \)-fold cover of \( R_N \).

This defines the sequence \( \Lambda_n, n = 1, 2, \ldots \) of finite, connected \( R_N \)-core graphs. We claim that

\[
(!!) \quad \lim_{n \to \infty} \frac{1}{n} \mu_{\Lambda_n} = \mu_{\Delta_1} + \mu_{\Delta_2} \quad \text{in} \quad \text{SCurr}(F_N).
\]

We need to show that for every finite non-degenerate subtree \( K \) of \( X \) we have

\[
(!!) \quad \lim_{n \to \infty} \frac{1}{n} \left( K; \Lambda_n \right) = \left( K; \Delta_1 \right) + \left( K; \Delta_2 \right).
\]

Let us fix such a subtree \( K \). Let \( n \geq 1 \) be arbitrary. If \( \Lambda_n \) is of the last described type, where both \( \Lambda_{1,n} \) and \( \Lambda_{2,n} \) are finite covers of \( R_N \), then by construction \( \mu_{\Lambda_n} = n(\mu_{\Delta_1} + \mu_{\Delta_2}) \), and (!!) obviously holds. Suppose now that one of the other cases in the construction of \( \Lambda_n \) occurs. Since the graph \( \Lambda_n \) is folded, and has a vertex degree \( \leq 2N \), all but a bounded number (in terms of some constant depending on \( K \) but independent of \( n \)), occurrences of \( K \) in \( \Lambda_n \) are disjoint from the segment \([v_{1,n}, v_{2,n}]\) and thus come from occurrences of \( K \) in \( \Lambda_{1,n} \sqcup \Lambda_{2,n} \). Hence

\[
\left| \left( K; \Lambda_n \right) - \left( K; \Lambda_{1,n} \right) - \left( K; \Lambda_{2,n} \right) \right| \leq C_K
\]

for some constant \( C_K \) independent of \( n \). Since \( \Lambda_{i,n} \) is an \( n \)-fold cover of \( \Delta_i \), it follows that

\[
\left| \left( K; \Lambda_n \right) - n(\left( K; \Delta_1 \right) - n(\left( K; \Delta_2 \right) \right) \leq C_K.
\]

Dividing the above inequality by \( n \) and passing to the limit as \( n \to \infty \), we get (!!), as required. This implies (!!). By Theorem \ref{thm:core_graphs}, \( \mu_{\Lambda_n} = \eta_{\Lambda_n} \) for some finitely generated
nontrivial $L_n \leq F_N$. Hence (!) implies that
\[
\lim_{n \to \infty} \frac{1}{n} \eta_{L_n} = \eta_{H_1} + \eta_{H_2} \text{ in } SCurr(F_N),
\]
which completes the proof. \hfill \Box

5.2. Subset currents as measures on the space of rooted trees. Recall that $T(X)$ denotes the set of infinite subtrees of $X$ without degree-1 vertices. Denote by $T_1(X)$ the set of all those $T \in T(X)$ that contain the vertex 1 in $F_N$. Thus all elements of $T_1(X)$ come equipped with a base-point (or root), namely 1 $\in F_N$. Note also that since every element $Y \in T_1(X)$ is folded, $Y$ does not admit any nontrivial $\Gamma$-graph automorphisms that fix the root vertex 1. Hence we can also think of $T_1(X)$ as the set of rooted $R_N$-graphs that are folded infinite trees without degree-1 vertices.

We equip $T_1(X)$ with local topology, namely we say that $Y, Y' \in T_1(X)$ are close if for some large $n \geq 1$ we have $Y \cap B_X(n) = Y' \cap B_X(n)$, where $B_X(n)$ is the ball of radius $n$ in $X$. Equivalently, $Y$ and $Y'$ are close if $\partial Y, \partial Y' \subseteq \partial X$ have small Hausdorff distance as closed subsets of $\partial X$. This makes $T_1(X)$ a compact totally disconnected topological space.

Recall that for any finite non-degenerate subtree $K$ of $X$ we have defined a subset cylinder set $SCyl_\alpha(K) \subseteq \mathcal{C}_N$, Definition 3.2. For every such $K$ containing 1 $\in F_N$, put $TCyl_\alpha(K)$ to be the set of all $Y \in T_1(X)$ such that $\partial Y \in SCyl_\alpha(K)$. Thus $TCyl_\alpha(K)$ consists of all $Y \in T_1(X)$ such that $K \subseteq Y$ and such that whenever $\xi \in \partial Y$, then $\{1, \xi\} \cap K = \{1, v\}$ where $v$ is a vertex of degree one in $K$. The sets $TCyl_\alpha(K)$ are compact and open, and form a basis of open sets for the topology on $T_1(X)$, when $K$ varies over all finite non-degenerate subtrees of $X$ containing 1 $\in F_N$.

The space $T_1(X)$ has a natural (partially defined) root-change operation. For $g \in F_N$ denote by $T_{1,g}(X)$ the set of all $Y \in T_1(X)$ that contain the vertex $g \in F_N$. Define $r_g : T_{1,g}(X) \rightarrow T_1(X)$ by $r_g(Y) := g^{-1}Y$ for $Y \in T_{1,g}(X)$. In other words, $r_g$ moves the root vertex from 1 to $g$ in $Y$. Denote by $M_1(X)$ the set of all finite positive Borel measures on $T_1(X)$ that are invariant under all $r_g, g \in F_N$.

**Proposition 5.3.** There is a canonical $\mathbb{R}_{\geq 0}$-linear homeomorphism $t : M_1(X) \rightarrow SCurr(F_N)$.

*Proof.* Let $\mu \in M_1(X)$. Define a measure $t(\mu)$ on $\mathcal{C}_N$ as follows. For a finite non-degenerate subtree $K$ of $X$ containing 1 put
\[
(\clubsuit) \quad ([K] ; t(\mu))_\alpha := \mu(TCyl_\alpha(K)) .
\]
Invariance of $\mu$ with respect to the root-change implies that if $K, K'$ are two finite non-degenerate subtrees of $X$ that contain 1 and such that $K' = gK$ for some $g \in F_N$, then $\mu(TCyl_\alpha(K)) = \mu(TCyl_\alpha(K'))$, so that $([K] ; t(\mu))_\alpha$ is well-defined.

The assumption that $\mu$ is invariant with respect to $\{r_g : g \in F_N\}$ translates into the fact that $(\clubsuit)$ defines a collection of weights satisfying the requirements of Proposition 3.12 so that $t(\mu)$ is indeed a subset current. It is then easy to check that $t$ is an $\mathbb{R}_{\geq 0}$-linear homeomorphism and we leave the details to the reader. \hfill \Box

5.3. Weak approximation by finite graphs. Consider the set $U_N$ of all $R_N$-graphs with vertices of degree $\leq 2N$. For every integer $R \geq 1$ denote by $U_{N,R} \subset U_N$ the set of all rooted $R$-balls in graphs from $U_N$. Finally, denote by $U^f_{N,R}$ the set of all $K \in U_{N,R}$ such that $K$ is a folded tree, where the root vertex is not of degree 1 and such that the distance from the root to every vertex of degree 1 of $K$ is equal to $R$. 
Lemma 5.5. Let $R$ finite connected $K$. We also have $\Delta$ denote by $J(\Delta, R, \Delta)$ the set of all those vertices $v$ in $\Delta$ such that the $R$-ball centered at $v$ in $\Delta$ is isomorphic as a rooted $R_N$-graph to $\Delta$.

We say that a subset current $\mu \in SCurr(F_N)$ is normalized if the corresponding measure $\mu' = (t)^{-1}(\mu) \in M_1(X)$ is of total mass 1, $\mu'(\mathcal{T}_1(X)) = 1$. Note that if $\mu \in SCurr(F_N)$ is a nonzero subset current and $\mu' = (t)^{-1}(\mu) \in M_1(X)$ then for $c = \mu'(\mathcal{T}_1(X))$ the current $\frac{1}{c}\mu \in SCurr(F_N)$ is normalized.

The following definition is an adaptation, to our notations and to our context, of the definition of random weak limit of finite rooted graphs introduced by Benjamini and Schramm. [4].

**Definition 5.4 (Weak approximation).** Let $\mu \in SCurr(F_N)$ be a normalized subset current. We say that a sequence of $R_N$-graphs $\Delta_n$ weakly approximates $\mu \in SCurr(F_N)$ if the following conditions hold:

1. $\Delta_n \in U_N$ for every $n \geq 1$.
2. $\#V\Delta_n \to \infty$ as $n \to \infty$.
3. For every $R \geq 1$ and every $K \in U_{N,R}^f$ we have
   $$\lim_{n \to \infty} \frac{\#J(K, R, \Delta_n)}{\#V\Delta_n} = (K; \mu).$$
4. For every $R \geq 1$ and every $\Upsilon \in U_{N,R}$ such that $\Upsilon \notin U_{N,R}^f$ we have
   $$\lim_{n \to \infty} \frac{\#J(\Upsilon, R, \Delta_n)}{\#V\Delta_n} = 0.$$

In this case we write $\Delta_n \xrightarrow[w.a.]{\mu} \mu$.

**Lemma 5.5.** Let $\mu \in SCurr(F_N)$ be a normalized current and let $\Delta_n$ be a sequence of finite connected $R_N$-core graphs such that $\Delta_n \xrightarrow[w.a.]{\mu} \mu$. Then

$$\lim_{n \to \infty} \frac{1}{\#V\Delta_n} \mu_{\Delta_n} = \mu \text{ in } SCurr(F_N).$$

**Proof.** Let $R \geq 1$ and let $K \in U_{N,R}^f$. Recall that we think of $K$ as a subtree of $X$ with the root of $K$ being the vertex 1 $\in V\Delta$. Denote by $(K; \Delta_n)_{bad}$ the number of all occurrences $f : K \to \Delta_n$ such that $f$ is not an embedding. Then the $R$-ball in $\Delta_n$ around the $f$-image of the base-point of $K$ is not a tree. Since the set $U_{N,R}$ is finite and $\Delta_n \xrightarrow[w.a.]{\mu} \mu$, the definition of weak approximation then implies that

$$\lim_{n \to \infty} \frac{(K; \Delta_n)_{bad}}{\#V\Delta_n} = 0.$$ 

We also have $(K; \Delta_n) = \#J(K, R, \Delta_n) + (K; \Delta_n)_{bad}$. Also, since $\Delta_n \xrightarrow[w.a.]{\mu} \mu$, we have

$$\lim_{n \to \infty} \frac{\#J(K, R, \Delta_n)}{\#V\Delta_n} = (K; \mu).$$

Therefore

$$\lim_{n \to \infty} \frac{(K; \Delta_n)}{\#V\Delta_n} = (K; \mu),$$
that is,

\[
\lim_{n \to \infty} \frac{(K; \mu_{\Delta_n})}{\#V\Delta_n} = (K; \mu).
\]

Now let \( K \) be an arbitrary finite non-degenerate subtree of \( X \) containing 1 \( \in V_X \). Put \( R \) to be the maximum of the distances in \( K \) from 1 to other vertices of \( K \). Then \( K \) is precisely the \( R \)-ball in \( K \) centered at 1, so that \( K \in U^f_{N,R} \). Therefore (\( \spadesuit \)) holds for every finite subtree of \( X \) containing 1 and hence, by \( F_N \)-invariance of subset currents, for every finite subtree \( K \) of \( X \). Hence

\[
\lim_{n \to \infty} \frac{1}{\#V\Delta_n} \mu_{\Delta_n} = \mu \text{ in } S\text{Curr}(F_N),
\]
as required. \( \square \)

**Lemma 5.6.** Let \( \mu \in S\text{Curr}(F_N) \) be a normalized current and let \( \Delta_n \) be a sequence of \( R_N \)-graphs such that \( \Delta_n \xrightarrow{\text{w.a.}} \mu \). Then there exists a sequence of folded \( R_N \)-graphs \( \Delta'_n \) such that \( \Delta'_n \xrightarrow{\text{w.a.}} \mu \).

**Proof.** Let us say that a vertex \( v \) of \( \Delta_n \) is folded if the ball of radius 1 in \( \Delta_n \) around \( v \) is a folded \( R_N \)-graph.

Let \( \Upsilon_N \) be the \( R_N \)-graph which is a simplicial circle of length \( N^2 \) with the label \( a_1^N a_2^N \ldots a_N^N \). For every non-folded vertex \( v \) of \( \Delta_n \) choose a copy \( \Upsilon_{N,v} \) of \( \Upsilon_N \). We now perform a “blow-up” operation on \( \Delta_n \) as follows. Simultaneously, for every non-folded vertex \( v \) of \( \Delta_n \) we cut \( \Delta_n \) open at \( v \), so that \( v \) gives \( \text{deg}_{\Delta_n}(v) \leq 2N \) new vertices, and then we attach these new vertices to \( \Upsilon_{N,v} \) in a way that the resulting graph is folded. The resulting folded \( R_N \)-graph is denoted by \( \Delta'_n \).

The definition of weak approximation implies that \( \lim_{n \to \infty} b_n/\#V\Delta_n = 0 \), where \( b_n \) is the number of non-folded vertices of \( \Delta_n \). Since the number of vertices in the graph \( \Upsilon_N \) (used to blow-up non-folded vertices of \( \Delta_n \)) is equal to \( N^2 \) and does not depend on \( n \), the definition of weak approximation now implies that \( \Delta'_n \xrightarrow{\text{w.a.}} \mu \), as required. \( \square \)

**Lemma 5.7.** Let \( \mu \in S\text{Curr}(F_N) \) be a normalized current and let \( \Delta_n \) be a sequence of folded \( R_N \)-graphs such that \( \Delta_n \xrightarrow{\text{w.a.}} \mu \). Then there exists a sequence of \( R_N \)-core graphs \( \Delta'_n \) such that \( \Delta'_n \xrightarrow{\text{w.a.}} \mu \).

**Proof.** For every vertex \( v \) of degree 1 in \( \Delta_n \) attach a new loop-edge at \( v \) with label \( a \in A \) such that \( a^{\pm 1} \) is different from the label of the unique edge of \( \Delta_n \) starting at \( v \). Denote the resulting graph by \( \Delta'_n \).

Then \( \Delta'_n \) is a folded and cyclically reduced \( R_N \)-graph. It is easy to see, from the definition of weak approximation, that \( \Delta'_n \xrightarrow{\text{w.a.}} \mu \). \( \square \)

### 5.4. Rational currents are dense.

**Theorem 5.8.** The set \( S\text{Curr}_r(F_N) \) of all rational currents is a dense subset of \( S\text{Curr}(F_N) \).

**Proof.** Let \( \mu \in S\text{Curr}(F_N) \) be an arbitrary nonzero current. Then for some \( c > 0 \) the current \( \overline{\mu} := \frac{1}{c} \mu \) is normalized.

A slight modification of the main result of [28], together with Proposition 5.3 imply that there exists a sequence \( \Delta_n \) of \( R_N \)-graphs such that \( \Delta_n \xrightarrow{\text{w.a.}} \overline{\mu} \). By Lemma 5.6 and Lemma 5.7 we may modify the sequence \( \Delta_n \) to get a new sequence of \( R_N \)-graphs, which we
again denote $\Delta_n$, such that each $\Delta_n$ is a core graph and that $\Delta_n \xrightarrow{\text{w.a.}} \overline{\Phi}$. Now Lemma \[5,5\] implies that
\[ \lim_{n \to \infty} \frac{1}{\#V \Delta_n} \mu_{\Delta_n} = \overline{\Phi} \in S\text{Curr}(F_N). \]
Let $\Delta_{n,1}, \ldots, \Delta_{n,k_n}$ be the connected components of $\Delta_n$. Then
\[ \mu_{\Delta_n} = \mu_{\Delta_{n,1}} + \cdots + \mu_{\Delta_{n,k_n}} \]
and we see that $\overline{\Phi}$ belongs to the closure of the $\mathbb{R}_{\geq 0}$-span of $S\text{Curr}_r(F_N)$. Proposition \[5,2\] now implies that $\overline{\Phi}$ belongs to the closure of $S\text{Cur}_{r}(F_N)$. Hence $\mu = c\overline{\Phi}$ also belongs to the closure of $S\text{Cur}_{r}(F_N)$. Since $\mu \in S\text{Curr}(F_N)$ was an arbitrary nonzero current, this completes the proof.

\begin{remark}
In the proof of Theorem \[5,8\] instead of the result of Elek \[28\] invoked above, alternatively one can use the results of Bowen \[14,15\].
\end{remark}

6. The action of $\text{Out}(F_N)$

6.1. Defining the $\text{Out}(F_N)$-action. If $\varphi \in \text{Aut}(F_N)$ is an automorphism, then $\varphi$ is a quasi-isometry of $F_N$ and hence $\varphi$ extends to a homeomorphism (which we will still denote by $\varphi$) $\varphi : \partial F_N \to \partial F_N$. Thus for $S \in \mathcal{C}_N$ we have $\varphi(S) \in \mathcal{C}_N$. For a subset $U \subseteq \mathcal{C}_N$ we write $\varphi(U) := \{\varphi(S) : S \in U\}$. Thus $\text{Aut}(F_N)$ has a natural action on $\mathcal{C}_N$ and on the set of subsets of $\mathcal{C}_N$.

\begin{definition}[The action of $\text{Aut}(F_N)$]
Let $\varphi \in \text{Aut}(F_N)$ and let $\mu \in S\text{Cur}_{r}(F_N)$.

Define a measure $\varphi\mu$ on $\mathcal{C}_N$ as follows:

For a Borel subset $U \subseteq \mathcal{C}_N$ we put
\[ \varphi\mu(U) := \mu(\varphi^{-1}(U)). \]

\end{definition}

\begin{proposition}
Let $N \geq 2$.

(1) For any $\varphi \in \text{Aut}(F_N)$ and $\mu \in S\text{Cur}_{r}(F_N)$ we have $\varphi\mu \in S\text{Cur}_{r}(F_N)$.

(2) For any $\varphi_1, \varphi_2 \in \text{Aut}(F_N)$ and $\mu \in S\text{Cur}_{r}(F_N)$ we have $(\varphi_1 \varphi_2)(\mu) = \varphi_1(\varphi_2 \mu)$.

(3) For any $\varphi \in \text{Aut}(F_N) \varphi : S\text{Cur}_{r}(F_N) \to S\text{Cur}_{r}(F_N)$ is an $\mathbb{R}_{\geq 0}$-linear homeomorphism.

(4) If $\varphi \in \text{Aut}(F_N)$ is an inner automorphism then $\varphi\mu = \mu$ for every $\mu \in S\text{Cur}_{r}(F_N)$.

\end{proposition}

\begin{proof}
To see (1) note that for any $g \in F_N$ and $\xi \in \partial F_N$ we have $\varphi^{-1}(g\xi) = \varphi^{-1}(g)\varphi^{-1}(\xi)$. Hence if $U \subseteq \mathcal{C}_N$ and $g \in F_N$ then, in view of $F_N$-invariance of $\mu$, we have:
\[ (\varphi \mu)(gU) = \mu(\varphi^{-1}(gU)) = \mu(\varphi^{-1}(g)\varphi^{-1}(U)) = \mu(\varphi^{-1}(U)) = (\varphi \mu)(U). \]

Thus $\varphi\mu$ is an $F_N$-invariant measure, and hence $\varphi\mu \in S\text{Cur}_{r}(F_N)$ and (1) is established.

Now (2) and (3) follow directly from (1) and Definition \[6,1\].

Part (4) follows from the fact that if $\psi \in \text{Aut}(F_N)$ is inner, that is $\psi(g) = hgh^{-1}$ for every $g \in F_N$, then $\psi(\xi) = h\xi$ for every $\xi \in \partial F_N$.

Proposition \[6,2\] shows that Definition \[6,1\] defines a left action of $\text{Aut}(F_N)$ on $S\text{Cur}_{r}(F_N)$ by $\mathbb{R}_{\geq 0}$-linear homeomorphisms. Moreover, inner automorphisms of $F_N$ lie in the kernel of this action, and therefore this action factors through to the action of $\text{Out}(F_N)$ on $S\text{Cur}_{r}(F_N)$.

Also, by $\mathbb{R}_{\geq 0}$-linearity, we have $\varphi(r \mu) = r \varphi(\mu)$ for $r \geq 0$, $\mu \in S\text{Cur}_{r}(F_N)$ and $\varphi \in \text{Aut}(F_N)$. Therefore the action of $\text{Aut}(F_N)$ on $S\text{Cur}_{r}(F_N)$ also yields an action of $\text{Aut}(F_N)$ on $\mathbb{P}\text{Cur}_{r}(F_N)$ given by $\varphi[\mu] := [\varphi\mu]$, where $\mu \in S\text{Cur}_{r}(F_N), \mu \neq 0$ and $\varphi \in \text{Aut}(F_N)$. As before, the last action factors through to the action of $\text{Out}(F_N)$ on $\mathbb{P}\text{S\text{Cur}_{r}(F_N}$.
Proposition 6.3. Let \( H \leq F_N \) be a nontrivial finitely generated subgroup and let \( \varphi \in \text{Aut}(F_N) \). Then \( \varphi(\eta_H) = \eta_{\varphi(H)} \).

Proof. For \( m \geq 1 \), if \( [H : H_1] = m \), then \([\varphi(H) : \varphi(H_1)] = m\) and \( \eta_H = m \eta_{H_1}, \eta_{\varphi(H)} = m \eta_{\varphi(H_1)} \). Hence, by linearity, it suffices to establish the proposition for the case \( H = \text{Comm}_{F_N}(H) \). Thus assume that \( H \leq F_N \) is a nontrivial finitely generated subgroup with \( H = \text{Comm}_{F_N}(H) \).

Recall that

\[
\eta_H = \sum_{H' \sim H} \delta_{\Lambda(H')},
\]

It is easy to see that for any subgroup \( G \leq F_N \) we have \( \varphi(\Lambda(G)) = \Lambda(\varphi(G)) \).

Let \( U \subseteq \mathcal{C}_N \). By Definition 6.1, \( \varphi \eta_H(U) = \eta_H(\varphi^{-1}(U)) \) is equal to the number of elements of the form \( \Lambda(H') \) (where \( H' \sim H \)) that belong to \( \varphi^{-1}(U) \), which, in turn, is equal to the number of elements of the form \( \varphi(\Lambda(H')) \) (where \( H' \sim H \)) that belong to \( U \).

Hence

\[
\varphi(\eta_H) = \sum_{H' \sim H} \delta_{\varphi(\Lambda(H'))} = \sum_{H' \sim H} \delta_{\Lambda(\varphi(H'))} = \}

\[
= \sum_{K \sim \varphi(H)} \delta_{\Lambda(K)} = \eta_{\varphi(H)},
\]

as required. \( \square \)

Proposition 6.3 implies that the action of \( \text{Aut}(F_N) \) has a global nonzero fixed point, namely the current \( \eta_{F_N} = \delta_{B_{F_N}} \). Indeed, for any \( \varphi \in \text{Aut}(F_N) \) we have

\[
\varphi \eta_{F_N} = \eta_{\varphi(F_N)} = \eta_{F_N}.
\]

Moreover, all scalar multiples \( r \eta_{F_N} \), where \( r \geq 0 \), are also fixed by \( \text{Aut}(F_N) \). In particular, since applying an automorphism to a subgroup of \( F_N \) preserves the index of the subgroup, if \( [F_N : H] < \infty \) then \( \varphi \eta_H = \eta_H \) for every \( \varphi \in \text{Aut}(F_N) \).

6.2. Local formulas. Similarly to the case of \( \text{Curr}(F_N) \), we can get more explicit “local formulas”, with respect to a marking, for the action of \( \text{Out}(F_N) \) on \( \mathcal{S}(\text{Curr}(F_N)) \).

Proposition 6.4. Let \( \alpha : F_N \cong \pi_1(\Gamma) \) be a marking on \( F_N \) and let \( X \) denote the universal cover \( \tilde{\Gamma} \). As before (see 5.12), let \( \mathcal{K}_\Gamma \) denote the set of all non-degenerate finite simplicial subtrees of \( X \). For any \( \varphi \in \text{Out}(F_N) \) and any \( K \in \mathcal{K}_\Gamma \) there exist \( m \geq 1 \) and \( K_1, \ldots, K_m \in \mathcal{K}_\Gamma \) with the following property: For every \( \mu \in \mathcal{S}(\text{Curr}(F_N)) \) we have

\[
(K; \varphi(\mu))_\alpha = \sum_{i=1}^{m} (K_i; \mu)_\alpha.
\]

Proof. Since inner automorphisms are contained in the kernel of the action of \( \text{Aut}(F_N) \) on \( \mathcal{S}(\text{Curr}(F_N)) \), it suffices to prove the statement of the proposition under the assumption that \( \varphi \in \text{Aut}(F_N) \).

Let \( \varphi \in \text{Aut}(F_N) \) be arbitrary. Since \( \mathcal{S}(\text{Cy}_\alpha)(K) \subseteq \mathcal{C}_N \) is compact, the set \( \varphi^{-1} \mathcal{S}(\text{Cy}_\alpha)(K) \) is also compact. Since subset cylinders are not just compact, but also open, \( \varphi^{-1} \mathcal{S}(\text{Cy}_\alpha)(K) \) is covered by finitely many subset cylinders. By Lemma 4.3, after subdividing them, we may assume that \( \varphi^{-1} \mathcal{S}(\text{Cy}_\alpha)(K) \) is covered by finitely many pairwise disjoint subset cylinders:

\[
\varphi^{-1} \mathcal{S}(\text{Cy}_\alpha)(K) = \bigsqcup_{i=1}^{m} \mathcal{S}(\text{Cy}_\alpha)(K_i).
\]

Put \( M := \max_{i=1}^{m} \#EK_i \). By definition of \( \varphi \mu \) we have
\[(K; \varphi \mu)_\alpha = \varphi \mu (SCyl_\alpha(K)) = \mu (\varphi^{-1}SCyl_\alpha(K)) =
\]
\[
m (\bigcup_{i=1}^m SCyl_\alpha(K_i)) = \sum_{i=1}^m \mu (SCyl_\alpha(K_i)) = \sum_{i=1}^m (K_i; \mu)_\alpha,
\]
as required.

\[\square\]

**Remark 6.5.** As in the case of \(\text{Curr}(F_N)\) [10], a more detailed argument for the proof of Proposition 6.4 shows that, given \(\varphi \in \text{Out}(F_N)\) and \(K \in K_T\), one can algorithmically find the trees \(K_1, \ldots, K_m\) satisfying the conclusion of Proposition 6.4.

7. The co-volume form

7.1. Constructing co-volume form. For \(T \in \text{cv}_N\) and a nontrivial finitely generated subgroup \(H \leq F_N\) put

\[
\|H\|_T := \text{vol}(H \setminus T_H)
\]

where \(T_H\) is the smallest \(H\)-invariant subtree of \(T\). It is also the convex hull of the limit set of \(H\), see Proposition 3.11.

Note that if \(H = \langle g \rangle\), where \(g \in F_N, g \neq 1\) then \(\|H\|_T = \|g\|_T\), the translation length of \(T\).

**Lemma 7.1.** Let \(T \in \text{cv}_N\), let \(H \leq F_N\) be a nontrivial finitely generated subgroup and let \(\varphi \in \text{Aut}(F_N)\). Then \(\|\varphi(H)\|_{\varphi T} = \|H\|_T\).

**Proof.** Recall, that as a metric space, \(\varphi T\) is equal to \(T\), but the action of \(F_N\) on \(\varphi T\) is defined as

\[
g \cdot x = \varphi^{-1}(g) \cdot x
\]

where \(x \in T, g \in F_N\). It follows that the tree \(T_H \subseteq T\) is \(\varphi(H)\)-invariant with respect to the \(F_N\)-action on \(\varphi T\), and, moreover it is the smallest \(\varphi(H)\)-invariant subtree of \(\varphi T\). Also, it is easy to see that, as metric graphs, \(H \setminus T_H\) is equal to \(\varphi(H) \setminus (\varphi T)_{\varphi(H)}\). Hence \(\|\varphi(H)\|_{\varphi T} = \|H\|_T\), as claimed.

\[\square\]

**Proposition-Definition 7.2** (Co-volume form). Let \(N \geq 2\). Define a map

\[
\langle , \rangle : \text{cv}_N \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}
\]
as follows. Let \(T \in \text{cv}_N\) and \(\mu \in \text{SCurr}(F_N)\). Let \(\alpha : F_N \to \pi_1(\Gamma)\) be a marking and let \(\mathcal{L}\) be a metric graph structure on \(\Gamma\) such that \(T = (\bar{\Gamma}, d_{\mathcal{L}})\) in \(\text{cv}_N\). Put

\[
\langle T, \mu \rangle := \sum_{e \in E_{top}(\Gamma)} (e; \mu)_\alpha \mathcal{L}(e)
\]

where \((e; \mu)_\alpha\) is as in Notation 3.10. Then the following hold:

(1) The map \(\langle , \rangle : \text{cv}_N \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}\) is continuous, \(\mathbb{R}_{\geq 0}\)-linear with respect to the second argument and \(\mathbb{R}_{\geq 0}\)-homogeneous with respect to the first argument.

(2) The map \(\langle , \rangle\) is \(\text{Out}(F_N)\)-equivariant, that is, for any \(T \in \text{cv}_N, \mu \in \text{SCurr}(F_N)\) and \(\varphi \in \text{Out}(F_N)\) we have

\[
\langle \varphi T, \varphi \mu \rangle = \langle T, \mu \rangle.
\]

In other words, in terms of using the right \(\text{Out}(F_N)\)-action on \(\text{cv}_N\), for all \(T \in \text{cv}_N\) and \(\varphi \in \text{Out}(F_N)\) we have

\[
\langle T\varphi, \mu \rangle = \langle T, \varphi \mu \rangle.
\]
3. For any finitely generated nontrivial subgroup $H \leq F_N$ and any $T \in \text{cv}_N$ we have

$$\langle T, \eta_H \rangle = \|H\|_T$$

where $T_H$ is the minimal $H$-invariant subtree of $T$.

We call the map $\langle , \rangle : \text{cv}_N \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$ the co-volume form.

**Proof.** The continuity of $\langle , \rangle$ is a straightforward consequence of its definition, and the argument is essentially identical to that used in [40] to show continuity of the intersection form $\text{cv}_N \times \text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$. We refer the reader to the proof of Proposition 5.9 in [40] for details.

The definition also directly implies that

$$\langle T, c_1 \mu_1 + c_2 \mu_2 \rangle = c_1 \langle T, \mu_1 \rangle + c_2 \langle T, \mu_2 \rangle$$

for $c_1, c_2 \geq 0$, $T \in \text{cv}_N$ and $\mu_1, \mu_2 \in \text{SCurr}(F_N)$ and that

$$\langle cT, \mu \rangle = c \langle T, \mu \rangle$$

for $c \geq 0$, $T \in \text{cv}_N$, $\mu \in \text{SCurr}(F_N)$. Thus (1) holds.

We now check that (3) holds. Let $H \leq F_N$ be a nontrivial finitely generated subgroup. Recall that, $\Delta = H \setminus T_H$ is exactly the $\Gamma$-core graph representing the conjugacy class $[H]$ (see Convention 4.18). By Theorem 4.19 we have $\eta_H = \mu_\Delta$. The edges of $\Delta$ in $\Delta = H \setminus T_H$ have the same lengths as the $\mathcal{L}$-lengths of the corresponding edges of $\Gamma$. Thus the $\mathcal{L}$-volume of $\Delta$ is the sum of the $\mathcal{L}$-lengths of all the edges of $\Delta$, so that

$$\|H\|_T = \text{vol}(H \setminus T_H) = \text{vol}_{\mathcal{L}}(\Delta) = \sum_{e \in E_{\text{top}}(\Gamma)} (e; \Delta) \mathcal{L}(e) = \sum_{e \in E_{\text{top}}(\Gamma)} (e; \mu_\Delta) \mathcal{L}(e) = \sum_{e \in E_{\text{top}}(\Gamma)} (e; \eta_H) \mathcal{L}(e) = \langle T, \eta_H \rangle,$$

and (3) is verified.

We can now show that (2) holds. Since inner automorphisms of $F_N$ act trivially on both $\text{cv}_N$ and $\text{SCurr}(F_N)$, it suffices to check (2) for elements of $\text{Aut}(F_N)$ (rather than of $\text{Out}(F_N)$). Let $\varphi \in \text{Aut}(F_N)$.

We need to show that for any $T \in \text{cv}_N$ and $\mu \in \text{SCurr}(F_N)$,

$$\langle \varphi T, \varphi \mu \rangle = \langle T, \mu \rangle.$$

By continuity of the intersection form, already established in (1) and by Theorem 5.8, it suffices to verify the above formula for the case where $T \in \text{cv}_N$ is arbitrary and where $\mu$ is a rational current. Thus let $\mu = c\eta_H$ where $c \geq 0$ and $H \leq F_N$ is a nontrivial finitely generated subgroup. We have

$$\langle \varphi T, \varphi c\eta_H \rangle = c \langle \varphi T, \varphi \eta_H \rangle = c \langle \varphi T, \eta_{\varphi(H)} \rangle = c \|\varphi(H)\|_{\varphi T} = c \|H\|_T = c \langle T, \eta_H \rangle = \langle T, c\eta_H \rangle,$$

as required. □

### 7.2. Non-existence of a continuous extension of the co-volume form to $\overline{\text{cv}}_N$.

It turns out that, unlike for ordinary currents, there does not exist a continuous extension of the co-volume form to $\overline{\text{cv}}_N \times \text{SCurr}(F_N)$. Before proving this statement, we need to recall a few background facts regarding the dynamics of the action of iwip elements and of Dehn twists elements of $\text{Out}(F_N)$ on $\overline{\text{cv}}_N$ and $\overline{\text{cv}}_N$. Recall that an element $\varphi \in \text{Out}(F_N)$ is called an iwip (which stands for “irreducible with irreducible powers”) or fully irreducible if there do not exist $m \neq 0$ and a proper free factor $L$ of $F_N$ such that $\varphi^m([L]) = [L]$, where $[L]$ is the conjugacy class of $L$. 

The following “North-South” dynamics result for iwips is well-known and was obtained by Levitt and Lustig in [56].

**Proposition 7.3.** Let $N \geq 2$ and let $\varphi \in \text{Out}(F_N)$ be an iwip. Then there exist unique $[T_+] = [T_+(\varphi)], [T_-] = [T_-(\varphi)] \in \overline{CV}_N$ and $\lambda_+ = \lambda_+(\varphi) > 1$, $\lambda_- = \lambda_-(\varphi) > 1$ with the following properties:

1. We have $T_+\varphi = \lambda_+T_+$, $T_-\varphi = \frac{1}{\lambda_-}T_-$, so that $[T_-]\varphi = [T_-]$ and $[T_+]\varphi = [T_+]$.
2. For any $[T] \in \overline{CV}_N$ such that $[T] \neq [T_-]$ we have $\lim_{n \rightarrow \infty}[T]\varphi^n = [T_+]$ and for any $[T] \in \overline{CV}_N$ such that $[T] \neq [T_+]$ we have $\lim_{n \rightarrow \infty}[T]\varphi^n = [T_-]$.
3. For any $T \in \text{cv}_N$ we have $\lim_{n \rightarrow \infty}[T]\varphi^n = [T_+]$. Moreover, given any $T \in \text{cv}_N$, we can choose a representative $T_+ \in \overline{cv}_N$ of $[T_+]$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_+^n}T\varphi^n = T_+ \quad \text{in} \quad \overline{cv}_N.$$ 

4. The action of $F_N$ on $T_+$ has dense $F_N$-orbits. Moreover, if $N \geq 3$ and $\varphi \in \text{Out}(F_N)$ is an atoroidal iwip then the action of $F_N$ on $T_+$ is also free.

5. If $\theta \in \text{Out}(F_N)$ is arbitrary then for $\psi = \theta^{-1}\varphi\theta$ we have $\lambda_+(\psi) = \lambda_+(\varphi)$ and $[T_+(\psi)] = [T_+(\varphi)]\theta$. Moreover, if $T \in \text{cv}_N$ and $T' := \lim_{n \rightarrow \infty} \frac{1}{\lambda_+^n}T\varphi^n$ and $T'' := \lim_{n \rightarrow \infty} \frac{1}{\lambda_-^n}T\varphi^n$ then $T'' = T\theta$.

The trees $[T_+(\varphi)]$ and $[T_-(\varphi)]$ are called the attracting and repelling trees of $\varphi$ respectively.

The attracting tree $T_+(\varphi)$ of an iwip $\varphi$ can be understood fairly explicitly in terms of a train-track representative of $\varphi$ and in fact $\lambda_+(\varphi)$ is the Perron-Frobenius eigenvalue of any train-track representative of $\varphi$.

Guirardel proved in [33] that for $N \geq 3$ the action of $\text{Out}(F_N)$ on $\partial \overline{CV}_N = \overline{CV}_N - CV_N$ is not topologically minimal and that there exists a unique proper closed $\text{Out}(F_N)$-invariant subset of $\partial \overline{CV}_N$. We only need the following limited version of his result:

**Proposition 7.4.** Let $N \geq 3$. Then there exists a unique minimal nonempty closed $\text{Out}(F_N)$-invariant subset $\mathcal{M}_N^0 \subseteq \partial \overline{CV}_N$ (so that for every $[T] \in \mathcal{M}_N^0$ the $\text{Out}(F_N)$-orbit of $[T]$ is dense in $\mathcal{M}_N^0$). Moreover, if $T_*$ is the Bass-Serre tree (with all edges given length 1) of any nontrivial free product decomposition $F_N = B \ast C$, then $[T_*] \in \mathcal{M}_N^0$.

Proposition 7.3 easily implies that for any iwip $\varphi \in \text{Out}(F_N)$ we have $[T_+(\varphi)] \in \mathcal{M}_N^0$.

**Proposition 7.5.** Let $N \geq 3$. Let $F_N = B \ast C$ be a nontrivial free product decomposition and let $T_* \in \overline{cv}_N$ be the corresponding Bass-Serre tree. Then there exist sequences $T_n, T'_n \in \text{cv}_N$ such that $\lim_{n \rightarrow \infty} T_n = \lim_{n \rightarrow \infty} T'_n = T_*$ in $\overline{cv}_N$ and that

$$\lim_{n \rightarrow \infty} \langle T'_n, \eta_{F_N} \rangle = 1$$

but

$$\lim_{n \rightarrow \infty} \langle T_n, \eta_{F_N} \rangle = 0.$$ 

**Proof.** Let $\varphi \in \text{Out}(F_N)$ be any iwip. Choose an arbitrary point $T \in \text{cv}_N$. Let $[T_+] = [T_+(\varphi)]$ be the attracting tree of $\varphi$. We may assume that $T_+(\varphi) = \lim_{k \rightarrow \infty} \frac{1}{\lambda_+^k}T\varphi^k$.

Since both $[T_*]$ and $[T_+]$ belong to $\mathcal{M}_N^0$, there exists a sequence $\theta_n \in \text{Out}(F_N)$ such that $\lim_{n \rightarrow \infty} [T_*] \theta_n = [T_*]$. Thus for some sequence $c_n > 0$ we have $\lim_{n \rightarrow \infty} c_n T_k \varphi^{c_n} = T_*$ in $\overline{cv}_N$. By Proposition 7.3 we know that $[T_+] \theta_n = [T_+(\psi_n)]$ where $\psi_n = \theta_n^{-1} \varphi \theta_n$. Moreover, for each $n \geq 1$ we have

$$T_+ \theta_n = \lim_{k \rightarrow \infty} \frac{1}{\lambda_+^k}T\psi_n^k$$
Put $T_+ (\psi_n) := T_+ \theta_n$, and we then have
$T_+ \theta_n = T_+ (\psi_n) = \lim_{k \to \infty} \frac{1}{\lambda_k} T^k \psi_n$ in $\text{cv}_N$. For every $k \geq 1$ and $n \geq 1$ we have
$$\langle \frac{1}{\lambda_k} T^k \psi_n, \eta_{F_N} \rangle = \frac{1}{\lambda_k} \langle T, \psi_n \eta_{F_N} \rangle = \frac{1}{\lambda_k} \langle T, \eta_{F_N} \rangle = \frac{\text{vol}(F_N \setminus T)}{\lambda_k}.$$ 
For each $n \geq 1$ choose $k_n \geq 1$ such that for all $k \geq k_n$ we have $c_n \langle \frac{1}{\lambda_k} T^k \psi_n, \eta_{F_N} \rangle = c_n \frac{\text{vol}(F_N \setminus T)}{\lambda_k} \leq \frac{1}{n}.
$

Since $\text{cv}_N$ is metrizable, and since $\lim_{n \to \infty} c_n T_+ (\psi_n) = T_*$ and $c_n T_+ (\psi_n) = \lim_{k \to \infty} \frac{c_n}{\lambda_k} T^k \psi_n$, by a standard diagonalization argument we can find a sequence $m_n \geq k_n$ such that $\lim_{n \to \infty} \frac{c_n}{\lambda_{m_n}} T^{m_n} \psi_n = T_*$ in $\text{cv}_N$. Put $T_n = \frac{c_n}{\lambda_{m_n}} T^{m_n} \psi_n$. Then by construction we have $\lim_{n \to \infty} T_n = T_*$ and $\langle T_n, \eta_{F_N} \rangle \leq \frac{1}{n} \to 0$ as $n \to \infty$, so that
$$\lim_{n \to \infty} \langle T_n, \eta_{F_N} \rangle = 0.$$

To construct the sequence $T_n\,'$, choose a free basis $\{b_1, \ldots, b_i\}$ of $B$ and a free basis $\{c_{i+1}, \ldots, c_N\}$ of $C$. Let $\Gamma$ be the “barbell” graph (with the obvious marking) consisting of a non-loop edge $e$ with $i$ loop-edges (corresponding to $b_1, \ldots, b_i$) attached at $o(e)$ and with $N - i$ loop-edges (corresponding to $c_{i+1}, \ldots, c_N$) attached at $t(e)$. Give the edge $e$ length $1 - \frac{1}{n}$ and give each of $N$ loop-edges in $\Gamma$ length $\frac{1}{\lambda_n}$. This defines a point $T_n\,' \in \text{cv}_N$ with $\text{vol}(F_N \setminus T_n\,) = 1$. Also, by construction, $\lim_{n \to \infty} T_n\,' = T_*$ in $\text{cv}_N$. Then
$$\lim_{n \to \infty} \langle T_n, \eta_{F_N} \rangle = \lim_{n \to \infty} \text{vol}(F_N \setminus T_n\,) = 1,$$

as required. \hfill $\square$

Proposition 7.6 immediately implies:

**Theorem 7.6.** Let $N \geq 3$. Then the co-volume form $\text{cv}_N \times \text{SCurr}(F_N) \to R_{\geq 0}$ does not admit a continuous extension to a map $\text{cv}_N \times \text{SCurr}(F_N) \to R_{\geq 0}$.

The proof of Theorem 7.6 can be modified to cover the case $N = 2$, but we only deal with the case $N \geq 3$ for simplicity.

**Remark 7.7.** Let $H \leq F_N$ be a nontrivial finitely generated subgroup. Recall that if $T \in \text{cv}_N$ then $\langle T, \eta_H \rangle = \|H\|_T = \text{vol}(H \setminus T_H)$. If $T \in \text{cv}_N \setminus \text{cv}_N$, the quotient $H \setminus T_H$ is, in general, not a nice object, and, in particular, it is not necessarily a finite metric graph. However, one can still define a reasonable notion of “volume” $\|H\|_T$ for $H \setminus T_H$. Namely, if $H$ fixes a point of $T$, put $\|H\|_T := 0$. Otherwise, there exists a unique minimal $H$-invariant subtree $T_H$ of $T$. In that case define $\|H\|_T$ as the infimum of $\text{vol}(K)$ taken over all finite subtrees $K \subseteq T_H$ such that $HK = T_H$. The proof of Proposition 7.6 exploits the fact that in general, given $H$, the function $f_H : \text{cv}_N \to \mathbb{R}$, $f_H : T \mapsto \|H\|_T$, is not continuous on $\text{cv}_N$. However, one can show that $f_H$ is upper-semicontinuous.

## 8. The reduced rank functional

Recall that for a finitely generated free group $F$ the rank $\text{rk}(F)$ is the cardinality of a free basis of $F$ and the reduced rank $\overline{\text{rk}}(F)$ is defined as
$$\overline{\text{rk}}(F) := \max \{ \text{rk}(F) - 1, 0 \}.$$ 
Thus is $F \neq \{1\}$ then $\overline{\text{rk}}(F) = \text{rk}(F) - 1$. If $\Delta$ is a finite connected graph and $F = \pi_1(\Delta)$ then $\overline{\text{rk}}(F) = -\chi(\Delta)$, where $\chi(\Delta)$ is the Euler characteristic of $\Delta$. Reduced rank appears naturally in the context of the Hanna Neumann Conjecture, recently proved by
Mineyev [58]. The conjecture (now Mineyev’s theorem) states that if \( H_1, H_2 \leq F \) are finitely generated subgroups of a free group \( F \) then
\[
\text{rk}(H_1 \cap H_2) \leq \text{rk}(H_1) \text{rk}(H_2).
\]
It turns out that reduced rank extends to a \( \mathbb{R}_{\geq 0} \)-linear functional on the space of subset currents:

**Theorem 8.1.** Let \( N \geq 2 \). Then there exists a unique continuous \( \mathbb{R}_{\geq 0} \)-linear functional
\[
\text{rk}: \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}
\]
such that for every nontrivial finitely generated subgroup \( H \leq F_N \) we have
\[
\text{rk}(\phi H) = \text{rk}(H).
\]
Moreover, \( \text{rk} \) is \( \text{Out}(F_N) \)-invariant, that is, for any \( \phi \in \text{Out}(F_N) \) and \( \mu \in \text{SCurr}(F_N) \) we have \( \text{rk}(\phi \mu) = \text{rk}(\mu) \).

**Proof.** The uniqueness of \( \text{rk} \) follows from (\( \diamond \)) and from the requirement that \( \text{rk} \) be linear and continuous, since this uniquely defines \( \text{rk} \) on the set of rational subset currents, which is dense in \( \text{SCurr}(F_N) \) by Theorem [58]. Thus it suffices to prove the existence of a functional \( \text{rk} \) with the required properties.

Choose a free basis \( A \) of \( F_N \) and the corresponding marking \( \alpha_A : F_N \to \pi_1(R_N) \) as in Convention 5.1. Recall that \( X = \tilde{R}_N \) is the Cayley graph of \( F_N \) with respect to \( A \). For each \( a \in A \) let \( e_a \) be the topological edge in \( X \) with endpoints \( 1, a \in F_N \). Note that every subgraph in \( X \) consisting of a single edge is a translate of some \( e_a \) by an element of \( F_N \).

Let \( B_N \) be the set of all non-degenerate finite subtrees \( K \) contained in the ball of radius 1 in \( X \) with center \( 1 \in F_N \) such that the vertex \( 1 \in F_N \) has degree \( \geq 2 \) in \( K \). Note that every \( K \in B_N \) is uniquely specified by the set of its vertices of degree 1, which is a subset of \( A \cup A^{-1} \) of cardinality at least 2. Thus there are exactly \( 2^{2N} - 2N - 1 \) elements in \( B_N \).

Let also that if \( \Delta \) is a nontrivial finite \( R_N \)-core graph then for every vertex \( x \) of \( \Delta \) there exists a unique \( K \in B_N \) such that \( \text{Lk}_K(x) = \text{Lk}(1) \). Define the function \( \text{rk}: \text{SCurr}(F_N) \to \mathbb{R} \) as follows. For \( \mu \in \text{SCurr}(F_N) \)
\[
\text{rk}(\mu) := \sum_{a \in A} (e_a; \mu) - \sum_{K \in B_N} (K; \mu).
\]
By construction \( \text{rk}: \text{SCurr}(F_N) \to \mathbb{R} \) is a continuous \( \mathbb{R}_{\geq 0} \)-linear function.

Suppose now that \( H \leq F_N \) is a nontrivial finitely generated subgroup. Let \( \Delta \) be the finite \( R_N \)-core graph representing \([H]\). Then
\[
\text{rk}(\mu_\Delta) := \sum_{a \in A} (e_a; \mu_\Delta) - \sum_{K \in B_N} (K; \mu_\Delta) = \sum_{a \in A} (e_a; \Delta) - \sum_{K \in B_N} (K; \Delta).
\]
It is easy to see, from the definition of an occurrence (Definition 4.13), that \( \sum_{a \in A} (e_a; \Delta) = \#E_{\text{top}} \Delta \) and that \( \sum_{K \in B_N} (K; \Delta) = \#V \Delta \). Thus
\[
\text{rk}(\phi H) = \text{rk}(\mu_\Delta) = \#E_{\text{top}} \Delta - \#V \Delta = -\chi(\Delta) = \text{rk}(H).
\]
Note that, by linearity, for any \( c \geq 0 \) \( \text{rk}(c \phi H) = c \text{rk}(H) \geq 0 \). Thus \( \text{rk} \geq 0 \) on a dense subset of \( \text{SCurr}(F_N) \) and hence, by continuity, \( \text{rk}(\mu) \geq 0 \) for every \( \mu \in \text{SCurr}(F_N) \).

Suppose now that \( \phi \in \text{Aut}(F_N) \). We claim that \( \text{rk}(\phi \mu) = \text{rk}(\mu) \) for every \( \mu \in \text{SCurr}(F_N) \). Indeed, for any finitely generated subgroup \( H \leq F_N \), \( H \) is isomorphic to \( \phi(H) \) and hence \( \text{rk}(H) = \text{rk}(\phi(H)) \). Thus, in view of (\( \diamond \)), the claim holds for every
rational subset current and hence, since by Theorem 5.8 rational currents are dense in \( SCurr(F_N) \), the claim holds for every \( \mu \in SCurr(F_N) \). This shows that \( \bar{r} \mathcal{K} \) is \( \text{Aut}(F_N) \)-invariant, and hence \( \text{Out}(F_N) \)-invariant as well. \( \square \)

9. Analogos of uniform currents

In [30], given a free basis \( A \) of \( F_N \) we constructed a uniform current \( m_A \in \text{Curr}(F_N) \) associated to \( A \). Intuitively, the current \( m_A \) “splits equally” in all directions in the Cayley graph \( X \) of \( F_N \) with respect to \( A \).

In the setup of subset currents, given a free basis \( A \) of \( F_N \) one can define a natural family \( m_{A,d} \in SCurr(F_N) \), \( d = 2, \ldots, 2N \) of “uniform subset currents”, with \( m_{A,2} = m_A \) and \( m_{A,2N} = \eta_{F_N} \). The current \( m_{A,d} \) is “supported” on \( d \)-regular subtrees of \( X \). That is, \( m_{A,d} \) will have the property that \( (K; m_{A,d}) > 0 \) if and only if \( K \subseteq X \) is a finite non-degenerate subtree where every vertex has degree either 1 or \( d \) in \( K \).

Before giving the explicit definition of \( m_{A,d} \) we present the following computation as motivation. Since \( m_{A,d} \) is a subset current, we need it to satisfy the weights condition (\( \bigstar \)) from Proposition 3.11. Let \( K \) be a non-degenerate finite subtree of \( X \) where every vertex has degree 1 or \( d \) in \( K \). Let \( e \) be a terminal edge of \( K \), as in Definition 3.2 and let \( a \in A^\pm 1 \) be the label of \( e \). Then, in notations 3.4 \( q(e) \) consists of precisely \( 2N - 1 \) distinct edges, with labels from \( A^\pm 1 \setminus \{a^{-1}\} \). We want to choose a nonempty subset \( U \subseteq q(e) \) so that in the tree \( K' = K \cup U \) the terminus of \( e \) has degree \( d \). Thus the set \( U \) needs to have cardinality \( d - 1 \). Hence there are exactly \( \binom{2N-1}{d-1} \) choices for \( U \subseteq q(e) \). Since \( m_{A,d} \) is supposed to be the “most symmetric possible”, we assign each of these choices equal weight, so we want

\[
(K \cup U; m_{A,d}) = \frac{(K; m_{A,d})}{\binom{2N-1}{d-1}}
\]

for every subset \( U \subseteq q(e) \) of size \( d - 1 \). For nonempty subsets \( U \subseteq q(e) \) of size different from \( d - 1 \) we want \((K \cup U; m_{A,d}) = 0\). Then, using notations 3.7 we will have

\[
(K; m_{A,d}) = \sum_{U \in \mathcal{P}_+(q(e))} (K \cup U; m_{A,d})
\]

as required by (\( \bigstar \)). Assigning every one-edge subtree of \( X \) weight \( 1/N \) in \( m_{A,d} \) and iterating the above splitting formula yields the following:

**Proposition-Definition 9.1** (Uniform subset currents). Let \( 2 \leq d \leq 2N \). Then there exists a unique subset current \( m_{A,d} \in SCurr(F_N) \) such that

1. if \( K \subseteq X \) is a finite subtree with \( n \geq 1 \) edges and with every vertex of degree either 1 or \( d \) in \( K \), then

\[
(K; m_{A,d}) = \frac{1}{N \left( \binom{2N-1}{d-1} \right)^{n-1}};
\]

2. if \( K \subseteq X \) is a finite subtree where some vertex has degree different from \( d \) and from 1, then

\[
(K; m_{A,d}) = 0;
\]

3. We have \( \langle X, m_{A,d} \rangle = 1 \).

The current \( m_{A,d} \in SCurr(F_N) \) is called the uniform subset current of grade \( d \) on \( F_N \) corresponding to \( A \).
Proof. It is easy to check, via a direct computation, that the weights \((K;m_{A,d})\) specified above satisfy condition \((\bigstar)\) and hence, by Proposition 3.11 there does exist a unique subset current realizing these weights. Also, for every single-edge tree \(K\) we have \((K;m_{A,d}) = \frac{1}{N}\), and this normalization ensures that \((X,m_{A,d}) = 1\). For \(d = 2N\) the above formulas give \((K;m_{A,2N}) = 1\) if \(K \subseteq X\) is a finite \(2N\)-regular subtree and \((K;m_{A,2N}) = 0\) if \(K\) is not \(2N\)-regular. This shows that \(m_{A,2N} = \eta_{FN}\). Also, for \(d = 2\), the definition of \(m_{A,2}\) yields \((K;m_{A,2}) = \frac{1}{2N(2N-1)}\) if \(K \subseteq X\) is a linear segment of length \(n \geq 1\) and \((K;m_{A,2}) = 0\) if \(K\) is not \(2\)-regular. Thus \(m_{A,2} = m_A \in \text{Curr}(F_N)\), as claimed. \(\square\)

In a similar way, uniform subset currents can be defined with respect to any marking \(\alpha : F_N \to \pi_1(\Gamma)\), where \(\Gamma\) is a \(k\)-regular graph (and not necessarily the standard rose).

Perhaps a more interesting notion is that of the absolute uniform current \(m_A^S \in \text{SCurr}(F_N)\) associated to \(A\).

If \(K\) is a finite non-degenerate tree, we say that a vertex \(x\) of \(K\) is interior if \(x\) has degree \(\geq 2\) in \(K\). Denote by \(\iota(K)\) the number of interior vertices in \(K\). Before giving a formal definition of \(m_A^S\) let us again start with some motivation. We want \(m_A^S\) to have the property that for a finite non-degenerate subtree \(K\) of \(X\) the weight \((K;m_A^S)\) depends only on \(\iota(K)\).

We build finite subtrees of \(X\) step-by-step, starting with a tree \(K_0\) consisting of a single edge. Since there are exactly \(N\) distinct \(F_N\)-translation classes of topological edges in \(X\), we assign every one-edge subtree weight \(\frac{1}{N}\) in \(m_A^S\). Note that \(\iota(K_0) = 0\). Arguing inductively, let \(n \geq 0\) and suppose \(K_n\) is already constructed and that \(\iota(K_n) = n\). We choose \(e\) to be any (oriented) leaf of \(K\). Then \(q(e)\) (see notation 3.2) consists of precisely \(2N - 1\) distinct edges, with labels from \(A^{\pm 1} - \{a^{-1}\}\). The set \(P_+(q(e))\) of nonempty subsets of \(q(e)\) has exactly \(2^{2N-1} - 1\) elements. We choose any nonempty subset \(U\) of \(q(e)\) “with equal probability”, and put \(K_{n+1} = K \cup U\). Note that the terminus of \(e\) has become an interior vertex of \(K_{n+1}\), so that \(\iota(K_{n+1}) = n + 1\). Since we are supposed to choose \(U \in P_+(q(e))\) uniformly at random, we want

\[
(K \cup U;m_{A,d}) = \frac{(K;m_{A,d})}{2^{2N-1} - 1}
\]

for every nonempty subset \(U \subseteq q(e)\). This choice will assure that

\[
(K;m_{A,d}) = \sum_{U \in P_+(q(e))} (K \cup U;m_{A,d})
\]

as required by condition \((\bigstar)\) of Proposition 3.11. The above considerations lead to the following:

**Proposition-Definition 9.2** (Absolute uniform current). Let \(N \geq 2\), \(A\) be a free basis of \(F_N\) and let \(X\) be the Cayley graph of \(F_N\) with respect to \(A\).

Then there exists a unique subset current \(m_A^S \in \text{SCurr}(F_N)\) such that:

1. If \(K \subseteq X\) is a finite non-degenerate subtree with \(\iota(K) = n \geq 0\) then

\[
(K;m_A^S) = \frac{1}{N (2^{2N-1} - 1)^{n+1}}.
\]

2. We have \((X,m_{A,d}) = 1\).

The current \(m_A^S \in \text{SCurr}(F_N)\) is called the absolute uniform subset current corresponding to \(A\).

**Proof.** It is not hard to check that the weights given in the definition of \(m_A^S\) satisfy the “switch” condition \((\bigstar)\) of Proposition 3.11 which directly yields the above statement. \(\square\)
Note that, unlike the currents $m_{A,d}$ constructed earlier, the current $m_A^S$ has “full support”, meaning that $(K;m_A^S) > 0$ for every finite non-degenerate subtree $K$ of $X$. Recall that, by Proposition 5.3, there is a canonical $R_{\geq 0}$-linear homeomorphism between $SCurr(F_N)$ and the space $M_1(X)$ of finite positive Borel measures on the space $\mathcal{T}_1(X)$ of rooted subtrees of $X$ with root-vertex 1, which are invariant with respect to root-change. Under this homeomorphism $m_A^S$ corresponds to a probability measure $m_A^S$ on $\mathcal{T}_1(X)$ and $m_A^S$-random points of $\mathcal{T}_1(X)$ appear to represent an interesting class of subtrees of $X$.

10. Open problems

10.1. Subset currents on surface groups. As noted in the introduction, the notion of a subset current makes sense for any non-elementary word-hyperbolic group $G$. Apart from $F_N$, a particularly interesting case is that of a surface group. Namely, let $\Sigma$ be a closed oriented surface of negative Euler characteristic and let $G = \pi_1(\Sigma)$. Put a hyperbolic Riemannian metric $\rho$ on $\Sigma$ so that $\Sigma$ with the lifted metric becomes isometric to the hyperbolic plane $\mathbb{H}^2$. The action of $G$ on $\Sigma = \mathbb{H}^2$ by covering transformations is a free isometric discrete co-compact action, so that $G$ is quasi-isometric to $\mathbb{H}^2$ and $\partial G$ is $G$-equivariantly homeomorphic to $\partial \mathbb{H}^2 = S^1$. As for a free group we can consider the space $(S^1)$ of all closed subsets $S \subseteq S^1$ such that $\# S \geq 2$. A subset current on $G$ is a locally finite $G$-invariant positive Borel measure on $(S^1)$. The space $SCurr(G)$ of all subset currents on $G$ again comes equipped with a natural weak-* topology and a natural action of the mapping class group $Mod(\Sigma)$. In this context we can again define the notion of a counting current associated with a nontrivial finitely generated subgroup $H \leq G$. If $H = Comm_G(H)$ then we put

$$\eta_H := \sum_{H_1 \in [H]} \delta_{\Lambda(H_1)}$$

where $[H]$ is the conjugacy class of $H$ in $G$ and where for a subgroup $H_1 \in [H] \Lambda(H_1) \subseteq S^1$ is the limit set of $H_1$ in $S^1$. If $H \leq G$ is an arbitrary nontrivial finitely generated subgroup, then, by well-known results, $H$ has a finite index $m \geq 1$ in its commensurator $H_0 := Comm_G(H)$ and we put $\eta_H := m \eta_{H_0}$.

**Problem 10.1.** In the above set-up, is it true that the set

$$\{cn_H | c \geq 0, H \leq G \text{ is a nontrivial finitely generated subgroup} \}$$

is dense in $SCurr(G)$?

Note that for every $S \in (S^1)$ one can consider the convex hull $Conv(S) \subseteq \mathbb{H}^2$, and therefore one can geometrically view a subset current on $G$ as a $G$-invariant measure on the space of “nice” convex subsets of $\mathbb{H}^2$. If $H \leq G$ is a nontrivial finitely generated subgroup of infinite index, then $H$ is a free group of finite rank $N \geq 1$ and $Conv(\Lambda H) \subseteq \mathbb{H}^2$ is an $H$-invariant subset which is quasi-isometric to $F_N$ (i.e. it is a “quasi-tree”). However, the machinery of measures on rooted graphs that we used to show that rational currents are dense in $SCurr(F_N)$ is not directly applicable for tackling Problem 10.1.

10.2. Continuity of co-volume for a fixed tree $T \in cv_N$. In Remark 7.7 we defined the notion of co-volume $||H||_T$ where $H \leq F_N$ is any nontrivial finitely generated subgroup and where $T \in cv_N$. We have seen that for a fixed $H$, the function $cv_N \rightarrow [0,\infty)$, $T \rightarrow ||H||_T$, is not necessarily continuous on $cv_N$ (although it is, of course, continuous if $H$ is infinite cyclic).

One can still ask if, given a fixed $T \in cv_N$, the co-volume $||.||_T$ extends to a continuous function on $SCurr(F_N)$:
Problem 10.2. Let $T \in \mathfrak{cv}_N$. Suppose $\mu \in \mathcal{SCurr}(F_N)$ and that $c_n \eta H_n \in \mathcal{SCurr}(F_N)$ are rational currents such that $\lim_{n \to \infty} c_n \eta H_n = \mu$. Does this imply that
\[
\lim_{n \to \infty} c_n \|H_n\|_T
\]
exists and is independent of the sequence $c_n \eta H_n$ approximating $\mu$? If yes, we will denote the above limit by $\|\mu\|_T$.

10.3. Volume equivalence. Kapovich, Levitt, Schupp and Shpilrain [43] introduced and studied the notion of translation equivalence in free groups. Namely, two elements $g, h \in F_N$ are translation equivalent in $F_N$, denoted $g \equiv_t h$, if for every $T \in \mathfrak{cv}_N$ we have $||g||_T = ||h||_T$. It is easy to see that $g \equiv_t h$ in $F_N$ if and only if for every $T \in \mathfrak{cv}_N$ $||g||_T = ||h||_T$. Similarly, we say (see [40]) that two currents $\mu_1, \mu_2 \in \text{Curr}(F_N)$ are translation equivalent if for every $T \in \mathfrak{cv}_N (T, \mu_1) = (T, \mu_2)$. Again, it clear that replacing $\mathfrak{cv}_N$ by $\mathfrak{cv}_N$ in this definition yields the same notion. Several different sources of translation equivalence (in particular traces of $SL(2, \mathbb{C})$-representations of free groups), have been exhibited in [43], and further results were obtained in [53, 54, 55]. The paper [43] also defined the notion of volume equivalence in free groups. Two nontrivial finitely generated subgroups $H_1, H_2 \leq F_N$ are said to be volume equivalent in $F_N$, denoted $H_1 \equiv_v H_2$, if for every $T \in \mathfrak{cv}_N$ we have $||H_1||_T = ||H_2||_T$. Note that for nontrivial $g, h \in F_N$ we have $\langle g \rangle \equiv_v \langle h \rangle$ in $F_N$ if and only if $g \equiv_t h$ in $F_N$. It is clear that conjugate subgroups are volume equivalent and that if $H_1, H_2 \leq F_N$ are nontrivial finitely generated subgroups such that $\text{Comm}_{F_N}(H_1) = \text{Comm}_{F_N}(H_2)$ and such that $H_1$ and $H_2$ have the same index in $\text{Comm}_{F_N}(H_1)$, then $H_1 \equiv_v H_2$ in $F_N$. Similarly, the definition of volume equivalence easily implies that if $H_1 \equiv_v H_2$ in $F_N$ and $\psi : F_N \to F_M$ is an injective homomorphism then $\psi(H_1) \equiv_v \psi(H_2)$ in $F_M$. Some more interesting sources of volume equivalence were found by Lee and Ventura in [55], who also exhibited an example of a cyclic subgroup that is volume equivalent to a subgroup that is free of rank 2. The definition of volume equivalence naturally leads to the following question:

Problem 10.3. Suppose $H_1 \equiv_v H_2$ in $F_N$. Does this imply that for every $T \in \mathfrak{cv}_N$
\[
||H_1||_T = ||H_2||_T?
\]

By analogy with the translation equivalence case, we can also say that two subset currents $\mu_1, \mu_2 \in \mathcal{SCurr}(F_N)$ are volume equivalent in $F_N$, denoted $\mu_1 \equiv_v \mu_2$, if for every $T \in \mathfrak{cv}_N (T, \mu_1) = (T, \mu_2)$. Thus for finitely generated subgroups $H_1, H_2 \leq F_N$ we have $H_1 \equiv_v H_2$ if and only if $\eta H_1 \equiv_v \eta H_2$. The notion of volume equivalence for subset currents measures the degeneracy of the co-volume form $\langle \cdot, \cdot \rangle$ with respect to its second argument, and one can also pose an analog of Problem 10.3 for subset currents.

10.4. Generalizing the Stallings fiber product construction. Let $H, L \leq F_N$ be nontrivial finitely generated subgroups and let $\alpha : F_N \cong \pi_1(\Gamma)$ be a marking on $F_N$. Let $\Delta_H, \Delta_L$ be the finite connected $\Gamma$-core graphs representing $[H]$ and $[L]$ respectively. Then one can define (see [61, 49]) the “fiber product graph” $\Delta_H \times \Delta_L$. This graph is again a finite folded $\Gamma$-graph but not necessarily connected. The fundamental groups of the non-contractible connected components of $\Delta_H \times \Delta_L$ (if there are any such components) represent all the possible $F_N$-conjugacy classes of nontrivial intersections of the form $gHg^{-1} \cap L$, where $g \in F_N$. There are finitely many such non-contractible components and denote the conjugacy classes of subgroups of $F_N$ represented by them by $[U_1], \ldots, [U_k]$. We thus define
\[
\hat{\eta} (\eta H, \eta L) = \sum_{i=1}^k \eta U_i.
\]
Note that $\text{rk}(\cap (\eta_H, \eta_L)) = \sum_{i=1}^k \text{rk}(\eta_{U_i}) = \sum_{i=1}^k \text{rk}(U_i)$. If all connected components of $\Delta_H \times \Delta_L$ are contractible, define $\cap (\eta_H, \eta_L) = 0$. We can extend $\cap$ by homogeneity to the set of rational subset currents as

$$\cap (c_1 \eta_H, c_2 \eta_L) := c_1 c_2 \cap (\eta_H, \eta_L).$$

Note that $\cap (c_1 \eta_H, c_2 \eta_L) = \cap (c_2 \eta_L, c_1 \eta_H)$ and that $\cap (c_1 \eta_H, c_2 \eta_L) = 0$ if at least one of $H, L$ is infinite cyclic. A particularly intriguing question is the following:

**Problem 10.4.** Does the map $\cap$ extends to a continuous function

$$\cap: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \text{SCurr}(F_N)?$$

If yes, then composing $\cap$ with the reduced rank functional $\text{rk}$ would give us a continuous, bilinear and *symmetric* “intersection functional” $J := \text{rk} \circ \cap$

$$J : \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \mathbb{R}.$$ Moreover, in view of Mineyev’s recent proof of the Strengthened Hanna Neumann Conjecture [58], it would follow that for any $\mu_1, \mu_2 \in \text{SCurr}(F_N)$ we have

$$J(\mu_1, \mu_2) \leq \text{rk}(\mu_1) \text{rk}(\mu_2).$$

10.5. Random subgroup graphs and uniform currents. Let $A$ be a free basis of $F_N$. Let $\xi = x_1 x_2 \ldots x_n \in \partial F_N$ be a “random” geodesic ray over $A^{\pm 1}$, that is, $\xi$ is a random trajectory of the non-backtracking simple random walk on $F_N$ corresponding to $A$. Denote $\xi|_n = x_1 \ldots x_n \in F_N$. It is shown in [40] for a.e. $\xi \in \partial F_N$ we have

$$\lim_{n \to \infty} \frac{\eta_{\xi|_n}}{n} = m_A \text{ in } \text{Curr}(F_N)$$

where $m_A \in \text{Curr}(F_N)$ is the *uniform current* on $F_N$ corresponding to $A$. For a random $\xi \in \partial F_N$ we may think of $\langle \xi|_n \rangle \leq F_N$ as a “random” cyclic subgroup of $F_N$.

In Section 9 we have defined a family of uniform subset currents $m_{A,d} \in \text{SCurr}(F_N)$, for $d = 2, \ldots, 2N$, where $m_{A,2} = m_A \in \text{C curr}(F_N)$. Recall that for a finite non-degenerate subtree $K$ of the Cayley graph $X$ of $F_N$ with respect to $A$ we have $(K; m_{A,d}) > 0$ if and only if $K$ is a $d$-regular tree.

**Problem 10.5.** Let $3 \leq d \leq 2N - 1$. Define a reasonable discrete-time random process such that at time $n$ it outputs a $d$-regular finite $R_N$-core graph $\Delta_n$ with $\lim_{n \to \infty} \# V\Delta_n = \infty$ and show that for a.e. trajectory of this process we have

$$\lim_{n \to \infty} \frac{\eta_{\Delta_n}}{\# V\Delta_n} = m_{A,d} \text{ in } \text{SCurr}(F_N).$$

The same problem is also interesting for the absolute uniform current $m_A^S \in \text{SCurr}(F_N)$.

**Problem 10.6.** Define a reasonable discrete-time random process such that at time $n$ it outputs a finite $R_N$-core graph $\Delta_n$ with $\lim_{n \to \infty} \# V\Delta_n = \infty$ and show that for a.e. trajectory of this process we have

$$\lim_{n \to \infty} \frac{\eta_{\Delta_n}}{\# V\Delta_n} = m_A^S \text{ in } \text{SCurr}(F_N).$$

In particular, what happens when $\Delta_n$ is chosen using the Bassino-Nicaud-Weil $\Xi$ process for generating random Stallings subgroup graphs?
10.6. **Dynamics of the Out($F_N$)-action on $\mathbb{P}SCurr(F_N)$**. For ordinary currents the dynamics of the Out($F_N$)-action on $\mathbb{P}Cur(F_N)$ and the interaction of this dynamics with that of the Out($F_N$)-action on $\mathbb{P}N$ turned out to be particularly useful. For $\mathbb{P}SCurr(F_N)$ the dynamics of the Out($F_N$)-action appears to be more complicated than in the $\mathbb{P}Cur(F_N)$-case, particularly because, even without projectivization, Out($F_N$) fixes the point $\eta_{F_N} \in S\text{Cur}(F_N)$.

In [57], R. Martin introduced the subset $\mathcal{M}_N \subseteq \mathbb{P}Cur(F_N)$ as the closure in $\mathbb{P}Cur(F_N)$ of the set

$$\{[\eta g] : g \in F_N \text{ is a primitive element}\}.$$ 

It is easy to see that for every $N \geq 2$, $\mathcal{M}_N \subseteq \mathbb{P}Cur(F_N)$ is a closed Out($F_N$)-invariant nonempty subset. In [44] it was shown that for $N \geq 3$ $\mathcal{M}_N$ is the unique smallest such subset, that is, whenever $Z \subseteq \mathbb{P}Cur(F_N)$ is nonempty, closed and Out($F_N$)-invariant, then $\mathcal{M}_N \subseteq Z$.

As noted in Remark 3.14 above, $\mathbb{P}Cur(F_N) \subseteq \mathbb{P}SCurr(F_N)$ is a closed Out($F_N$)-invariant subset and, similarly, Cur($F_N) \subseteq S\text{Cur}(F_N)$ is a closed Out($F_N$)-invariant subset. The set $\{[\eta_{F_N}]\}$ is also a closed Out($F_N$)-invariant subset of $\mathbb{P}SCurr(F_N)$, so that $\mathbb{P}SCurr(F_N)$ contains at least 2 minimal nonempty closed Out($F_N$)-invariant subsets.

**Problem 10.7.** Let $N \geq 3$. Characterize those $[\mu] \in \mathbb{P}SCurr(F_N)$ such that the closure of the Out($F_N$)-orbit in $\mathbb{P}SCurr(F_N)$ contains $\mathcal{M}_N$. Is it true that the closure of Out($F_N$)$[\mu]$ contains $\mathcal{M}_N$ if and only if $[\mu] \neq [\eta_{F_N}]$?

As we note below, we do know that for every nontrivial finitely generated subgroup $H \leq F_N$ of infinite index, the closure of Out($F_N$)$[\eta_H]$ does contain $\mathcal{M}_N$.

Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be an atoroidal iwip (irreducible with irreducible powers) element. In this case it is known that $\varphi$ has exactly two distinct fixed points in $\mathbb{P}Cur(F_N)$, the currents $[\mu_+]$ and $[\mu_-]$; moreover there are $\lambda_+, \lambda_- > 1$ such that $\varphi \mu_+ = \lambda_+ \mu_+$ and $\varphi^{-1} \mu_- = \lambda_- \mu_-$. R. Martin proved [57] that the action of $\varphi$ on $\mathbb{P}Cur(F_N)$ has “North-South” dynamics: If $[\mu] \in \mathbb{P}Cur(F_N)$, $[\mu] \neq [\mu_-]$ then $\lim_{n \to \infty} \varphi^n[\mu] = [\mu_+]$ and if $[\mu] \in \mathbb{P}Cur(F_N)$, $[\mu] \neq [\mu_+]$ then $\lim_{n \to \infty} \varphi^{-n}[\mu] = [\mu_-]$.

As observed above, by Remark 3.14 we have $\mu_+ \in Curr(F_N) \subseteq S\text{Cur}(F_N)$ and $[\mu_+] \in \mathbb{P}Cur(F_N) \subseteq \mathbb{P}SCurr(F_N)$.

The situation for $\mathbb{P}SCurr(F_N)$ is immediately complicated by the presence of the global fixed point $[\eta_{F_N}]$. For example, for any $c_1, c_2 > 0$ it is not hard to see that

$$\lim_{n \to \infty} \varphi^n[c_1 \mu_- + c_2 \eta_{F_N}] = [\eta_{F_N}] \quad \text{and} \quad \lim_{n \to \infty} \varphi^{-n}[c_1 \mu_+ + c_2 \eta_{F_N}] = [\eta_{F_N}].$$

**Problem 10.8.** Let $N \geq 3$ and let $\varphi \in \text{Out}(F_N)$ be an atoroidal iwip. Characterize those $[\mu] \in \mathbb{P}SCurr(F_N)$ for which

$$\lim_{n \to \infty} \varphi^n[\mu] = [\mu_+].$$

Is it true that the above convergence holds for every $[\mu] \in \mathbb{P}SCurr(F_N)$ which is not of the form $[\mu] = [c_1 \mu_- + c_2 \eta_{F_N}]$ where $c_1, c_2 \geq 0$ and $|c_1| + |c_2| > 0$?

Using the results of [8] we can show that if $H \leq F_N$ is a nontrivial finitely generated subgroup of infinite index, then

$$\lim_{n \to \infty} \varphi^n[\eta_H] = [\mu_+] \quad \text{and} \quad \lim_{n \to \infty} \varphi^{-n}[\eta_H] = [\mu_-].$$

Since $[\mu_+] \in \mathcal{M}_N$, this fact does imply that the closure in $\mathbb{P}SCurr(F_N)$ of the orbit $\text{Out}(F_N)[\eta_H]$ contains $\mathcal{M}_N$. 
10.7. **Subset algebraic laminations.** An algebraic lamination on $F_N$ is a closed $F_N$-invariant subset of the space of 2-elements subsets of $(\partial F_N)$. Algebraic laminations proved to be useful objects in the study of $\text{Out}(F_N)$ and of the Outer space. In particular, for every $T \in \mathcal{W}_N$ there is a naturally defined dual lamination $L^2(T)$ which records some essential information about the geometry of the action of $F_N$ on $T$.

For an arbitrary current $\mu \in \text{Curr}(F_N)$ its support is an algebraic lamination. It is proved in [46] that for $\mu \in \text{Curr}(F_N)$ and $T \in \mathcal{W}_N$ $(T, \mu) = 0$ if and only if $\text{Supp}(\mu) \subseteq L^2(T)$, and this fact plays a key role in understanding the interplay between the dynamics of $\text{Out}(F_N)$-actions on $\overline{\mathcal{W}}_N$ and $\mathbb{P}\text{Curr}(F_N)$.

By analogy with the set-up discussed above, we say that a subset algebraic lamination on $F_N$ is a closed $F_N$-invariant subset of $\mathcal{C}_N$. In particular, for any $\mu \in S\text{Curr}(F_N)$, the support $\text{Supp}(\mu)$ is a subset algebraic lamination. Here $\text{Supp}(\mu) := \mathcal{C}_N - U$ where $U$ is the largest open subset of $\mathcal{C}_N$ such that $\mu(U) = 0$.

**Problem 10.9.** Assuming that the answer to Problem [10.2] is positive, given $T \in \mathcal{W}_N$ does there exist a naturally defined subset lamination $L^2\mathcal{S}(T) \subseteq \mathcal{C}_N$ which captures the information about all $\mu \in S\text{Curr}(F_N)$ with $\|\mu\|_T = 0$? Or at least some naturally defined subset lamination $L^2\mathcal{S}(T) \subseteq \mathcal{C}_N$ which captures the information about all finitely generated $H \leq F_N$ with $\|H\|_T = 0$? If such a notion of $L^2\mathcal{S}(T)$ does exist, how does $L^2\mathcal{S}(T)$ look like for stable trees of various free group automorphisms?

Note that for any reasonable definition of $L^2\mathcal{S}(T)$ one expects to have $L^2(T) \subseteq L^2\mathcal{S}(T)$.

10.8. **Approximation and weight realizability problem.** We know by Theorem [4.8] that any $\mu \in S\text{Curr}(F_N)$ can be approximated by rational subset currents. It would be interesting to find explicit procedures (e.g. algorithmic or probabilistic) for producing such approximations in various specific contexts.

**Problem 10.10.** Let $A$ be a free basis of $F_N$ and let $m_A^n$ be the corresponding absolute uniform current. Find a natural probabilistic process producing a sequence of core graphs $\Delta_n$ such that $\lim_{n \to \infty} [\mu\Delta_n] = [m_A^n]$ in $\mathbb{P}\text{SCurr}(F_N)$.

In [47] Martin proves that for a marking $\alpha : F_N \to \pi_1(\Gamma)$ and a nonzero geodesic current $\mu \in \text{Curr}(F_N)$ we have $(v; \mu)_\alpha \in \mathbb{Z}$ for every nontrivial reduced edge-path $v$ in $\Gamma$ if and only if $\mu = \eta g_1 + \cdots + \eta g_m$ for some $m \geq 1$ and some nontrivial $g_1, \ldots, g_m \in F_N$. It is natural to ask a similar question for subset currents:

**Problem 10.11.** Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking and let $\mu \in S\text{Curr}(F_N)$ be a nonzero subset current such that for every non-degenerate finite subtree $K$ of $\Gamma$ we have $(K; \mu)_\alpha \in \mathbb{Z}$. Does this imply that $\mu = \eta_1 H_1 + \cdots + \eta_m H_m$ for some $m \geq 1$ and some nontrivial finitely generated subgroups $H_1, \ldots, H_m \leq F_N$?

In the case of $\text{Curr}(F_N)$, the proof uses Whitehead graphs for cyclic words. In the context of subset currents the corresponding objects turn out to be hyper-graphs, rather than graphs.

Recall that a hyper-graph $G$ is a pair $(V, E)$, where $V$ is the set of vertices and $E$ is the set of hyper-edges. Every hyper-edge $e$ is a nonempty subset of $V$, whose elements are said to be incident to $e$ in $G$. We can also think of $e$ as the characteristic function of this subset, so that $e : V \to \{0, 1\}$ with $e(v) = 1$ if and only if $v$ is incident to $e$.

Now let $\alpha : F_N \rightharpoonup \pi_1(\Gamma)$ be a marking, let $X = \Gamma$ and let $K \in K_T$ be a finite non-degenerate subtree of $X$. Recall from Section [9] that $\nu(K)$ denotes the number of interior vertices in $K$.

We say that an interior vertex $x$ of $K$ is a boundary-interior vertex of $K$ if $x$ is an interior vertex of $K$, $x$ is adjacent to a terminal edge of $K$ and removing from $K$ all
terminal edges of $K$ incident to $x$ produces a tree in which $x$ has degree 1. That is, $x$ is an interior vertex of $K$ and the degree of $x$ in $K$ is equal to $p + 1$ where $p$ is the number of terminal edges of $K$ incident to $x$. Note that every finite $K$ with $\iota(K) \geq 2$ always has at least one boundary-interior vertex. For each boundary-interior vertex $x$ of $K$ let $U_x$ be the set of topological edges of $K$ that connect $x$ to vertices of degree 1 of $K$. Let $K_x$ be obtained from $K$ by removing all the edges of $U_x$. Thus $K = K_x \cup U_x$. Note that $x$ is a vertex of degree 1 in $K_x$, so that $\iota(K_x) = m - 1$.

For each $m \geq 2$ define the level-$m$ initial hyper-graph $G_{\alpha,m}$ as follows. The vertex set of $G_{\alpha,m}(\mu)$ is the set $Z_{m-1}$ of all $F_N$-translation classes $[K]$ where $K \in K_\Gamma$ is a tree with $\iota(K) = m$. The hyper-edge set is $Z_m$. Every $[K'] \in Z_m$ defines the incidence function $e_{[K']}: Z_{m-1} \rightarrow \{0,1\}$ as follows. For $[K] \in Z_{m-1}$ $e_{[K']}([K]) = 1$ if and only if there is a boundary-interior vertex $x$ of $K'$ such that $[K'_x] = [K]$.

For $m = 1$ we can also define $G_{\alpha,1}$ in a similar way. The vertex set is $Z_0$ and the hyper-edge set is $Z_1$, but the incidence is defined slightly differently. Namely, for $[K'] \in Z_1$ and an element $[K] \in Z_0$ (note that $K'$ is necessarily a single topological edge of $X$) we have $e_{[K']}([K]) = 1$ if and only if there is a topological edge of $K'$ which is an $F_N$-translate of $K$.

For $m \geq 1$, a weighted level-$m$ initial hyper-graph consists of the hyper-graph $G_{\alpha,m}$ endowed with the weight functions $\theta: Z_{m-1} \rightarrow \mathbb{R}_{\geq 0}$ and $\theta: Z_m \rightarrow \mathbb{R}_{\geq 0}$.

Every current $\mu \in SCurr(F_N)$ defines a weighted hyper-graph $G_{\alpha,m}(\mu)$, with the weight functions $\theta([K']): = (K'; \mu)_\alpha$, $\theta([K]) := (K; \mu)_\alpha$, where $[K'] \in Z_m$, $[K] \in Z_{m-1}$.

**Problem 10.12.** Let $m \geq 1$ and let $(G_{\alpha,m}, \theta)$ be a weighted initial level-$m$ hyper-graph such that the weight function $\theta$ is $\mathbb{Z}_{\geq 0}$-valued and satisfies the Kirchoff condition (see Proposition 3.11) for every $[K] \in Z_{m-1}$. Does there exist a finite $\Gamma$-core graph $\Delta$ such that $(G_{\alpha,m}, \theta) = G_{\alpha,m}(\mu_\Delta)$?

As shown in [10], in the $Curr(F_N)$ context the analog of the above question has positive answer. In that case initial hyper-graphs are directed graphs and the Kirchoff condition implies the existence of an Euler circuit in every connected component of the graph, when the weights are treated as multiple edges. However, it is not clear what might serve as a substitute for Euler’s theorem in the hyper-graph context, and even the proof of Theorem 5.8 does not appear to help with Problem 10.12.

**References**

[1] M. Abert, Y. Glasner and B. Virag, Kesten’s theorem for Invariant Random Subgroups, preprint, 2012; arXiv:1201.3399

[2] P. Arnoux, V. Berthé, T. Fernique, D. Jamet, Functional stepped surfaces, flips, and generalized substitutions. Theoret. Comput. Sci. 380 (2007), no. 3, 251–265

[3] F. Bassino, C. Nicaud, and P. Weil, Random generation of finitely generated subgroups of a free group. Internat. J. Algebra Comput. 18 (2008), no. 2, 375–405

[4] I. Benjamini, O. Schramm, Recurrence of distributional limits of finite planar graphs. Electronic Journal of Probability 6 (2001), no.23, 1–23

[5] N. Bergeron, and D. Gaboriau, Asymptotique des nombres de Betti, invariants $L^2$ et laminations, Comment. Math. Helv. 79 (2004), no. 2, 362395

[6] M. Bestvina, and M. Handel, Train tracks and automorphisms of free groups. Ann. of Math. (2) 135 (1992), no. 1, 1–51

[7] M. Bestvina and M. Feighn, Outer Limits, preprint, 1993; [http://andromeda.rutgers.edu/~feighn/papers/outer.pdf](http://andromeda.rutgers.edu/~feighn/papers/outer.pdf)

[8] M. Bestvina, M. Feighn, and M. Handel, Laminations, trees, and irreducible automorphisms of free groups. Geom. Funct. Anal. 7 (1997), no. 2, 215–244

[9] M. Bestvina and M. Feighn, The topology at infinity of Out$(F_n)$. Invent. Math. 140 (2000), no. 3, 651–692
[42] I. Kapovich, Random length-spectrum rigidity for free groups, Proc. AMS 140 (2012), no. 5, 1549–1560
[43] I. Kapovich, G. Levitt, P. Schupp and V. Shpilrain, Translation equivalence in free groups, Transact. Amer. Math. Soc. 359 (2007), no. 4, 1527–1546
[44] I. Kapovich and M. Lustig, The actions of Out(F_k) on the boundary of outer space and on the space of currents: minimal sets and equivariant incompatibility, Ergodic Theory Dynam. Systems 27 (2007), no. 3, 827–847
[45] I. Kapovich and M. Lustig, Geometric Intersection Number and analogues of the Curve Complex for free groups, Geometry & Topology 13 (2009), 1805–1833
[46] I. Kapovich and M. Lustig, Intersection form, laminations and currents on free groups, Geom. Funct. Anal. (GAFA), 19 (2010), no. 5, 1426–1467
[47] I. Kapovich and M. Lustig, Domains of proper discontinuity on the boundary of Outer space, Illinois J. Math. 54 (2010), no. 1, pp. 89–108, special issue dedicated to Paul Schupp
[48] I. Kapovich and M. Lustig, Ping-pong and Outer space, J. Topol. Anal. 2 (2010), no. 2, 173–201
[49] I. Kapovich and A. Myasnikov, Stallings foldings and the subgroup structure of free groups, J. Algebra 248 (2002), no 2, 608–668
[50] I. Kapovich and T. Nagnibeda, The Patterson-Sullivan embedding and minimal volume entropy for Outer space, Geom. Funct. Anal. (GAFA) 17 (2007), no. 4, 1201–1236
[51] I. Kapovich and T. Nagnibeda, Geometric entropy of geodesic currents on free groups, Dynamical Numbers: Interplay Between Dynamical Systems and Number Theory, Contemporary Mathematics series, American Mathematical Society, 2010, pp. 149-176
[52] I. Kapovich, and H. Short, Greenberg’s theorem for quasiconvex subgroups of word hyperbolic groups. Canad. J. Math. 48 (1996), no. 6, 1224–1244
[53] D. Lee, Translation equivalent elements in free groups. J. Group Theory 9 (2006), no. 6, 809–814
[54] D. Lee, An algorithm that decides translation equivalence in a free group of rank two. J. Group Theory 10 (2007), no. 4, 561–569
[55] D. Lee, and E. Ventura, Volume equivalence of subgroups of free groups. J. Algebra 324 (2010), no. 2, 195-217
[56] G. Levitt and M. Lustig, Irreducible automorphisms of F_n have North-South dynamics on compactified outer space. J. Inst. Math. Jussieu 2 (2003), no. 1, 59–72
[57] R. Martin, Non-Uniquely Ergodic Foliations of Thin Type, Measured Currents and Automorphisms of Free Groups, PhD Thesis, 1995
[58] I. Mineyev, Submultiplicativity and the Hanna Neumann conjecture. Ann. of Math. (2) 175 (2012), no. 1, 393–414
[59] F. Paulin, The Gromov topology on R-trees. Topology Appl. 32 (1989), no. 3, 197–221
[60] D. Savchuk, Schreier Graphs of Actions of Thompsons Group F on the Unit Interval and on the Cantor Set, preprint, 2011; arXiv: 1105.4017.
[61] John Stallings, Topology of finite graphs. Invent. Math. 71 (1983), no. 3, 551-565
[62] G. Stuck and R. J. Zimmer, Stabilizers for ergodic actions of higher rank semisimple groups, Ann. of Math. (2) 139 (1994), no. 3, 723–747
[63] A. Vershik, Nonfree actions of countable groups and their characters, arXiv:1012.4604
[64] A. Vershik, Totally nonfree actions and infinite symmetric group, preprint, 2011; arXiv:1109.3413
[65] K. Vogtmann, Automorphisms of Free Groups and Outer Space, Geom. Dedicata 94 (2002), 1–31

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/
E-mail address: kapovich@math.uiuc.edu

Section de mathématiques, Université de Genève, 2–4, rue du Lièvre, c.p. 64, 1211 Genève, Switzerland
http://www.unige.ch/math/folks/nagnibeda
E-mail address: tatiana.smirnova-nagnibeda@unige.ch