SECOND EIGENVALUE OF THE YAMABE OPERATOR AND APPLICATIONS

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Abstract. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). In this paper, we give various properties of the eigenvalues of the Yamabe operator \(L_g\). In particular, we show how the second eigenvalue of \(L_g\) is related to the existence of nodal solutions of the equation \(L_g u = \varepsilon |u|^N-2 u\), where \(\varepsilon = +1, 0,\) or \(-1\).

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1. INTRODUCTION

This paper is part of a Phd thesis whose purpose is to study the relationships between the eigenvalues of the Yamabe operator, in particular their sign, and analytic, geometrical or topological properties of compact manifolds of dimension \(n \geq 3\):

Let \((M, g)\) be a \(n\)-dimensional compact Riemannian manifold \((n \geq 3)\). The Yamabe operator or conformal Laplacian operator \(L_g\) is defined by

\[ L_g(u) := c_n \Delta_g u + S_g u, \]

where \(\Delta_g\) is the Laplace-Beltrami operator, \(c_n = \frac{4(n-1)}{n-2}\) and \(S_g\) the scalar curvature of \(g\). The Yamabe operator \(L_g\) has discrete spectrum

\[ \text{spec}(L_g) = \{ \lambda_1(g), \lambda_2(g), \cdots \}, \]

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where the eigenvalues are such that
\[ \lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \leq \lambda_k(g) \rightarrow +\infty. \]

The \( i \)-th eigenvalue \( \lambda_i(g) \) is characterized by
\begin{equation}
\lambda_i(g) = \inf_{V \in Gr_i(H^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_g v \, dv_g}{\int_M v^2 \, dv_g},
\end{equation}
where \( Gr_i(H^2(M)) \) stands for the set of all \( i \)-dimensional subspaces of \( H^2(M) \).

Our project is to understand what we can deduce from the sign of \( \lambda_i \). Now, we summarize what is known about this question and explain our motivations. At first, it is straightforward to see that the sign of \( \lambda_1(g) \) is the same as the sign of the Yamabe constant \( \mu(M, g) \) of \((M, g)\) (and as a consequence is conformally invariant). See Section 4 for more informations. Hence the positivity of \( \lambda_1(g) \) has many consequences usually stated in terms of positivity of the Yamabe constant. For instance, we obtain

**Proposition 1.1.** A compact manifold \( M \) of dimension \( n \geq 3 \) carries a metric with positive scalar curvature if and only if it carries a metric \( g \) such that \( \lambda_1(g) > 0 \).

We recall that classifying such compact manifolds is a challenging open problem, only solved for \( n = 3 \) using Perelman’s techniques. We also mention [BD03] where M. Dahl and C. Bär deduce many topological properties of compact manifolds from a careful study of the eigenvalues \( \lambda_i \) of the Yamabe operator \( L_g \).

The sign of \( \lambda_1 \) can also be read in terms of existence or non-existence of positive solutions of the Yamabe equation:
\begin{equation}
L_g u = \epsilon |u|^{N-2} u,
\end{equation}
where \( N := \frac{2n}{n-2} \) and \( \epsilon \in \{-1; 0; 1\} \). Inspired by this observation, B. Ammann and E. Humbert [AH06] enlightened the role of \( \lambda_2 \) in the existence of nodal solutions (i.e. having a changing sign) of the Yamabe equation (2). See again Section 4 for more explanations.

In this paper, we establish various properties of the eigenvalues of the Yamabe operator. First of all, we extend their definition to what we call generalized metrics when possible (see Paragraph 2) and prove that their sign is a conformal invariant (see Paragraph 3.1). This paper initiates the study of the relationships between these conformal invariants and the topology of the manifold by showing that their negativity is not topologically obstructed (see Paragraph 2.2). These investigations will be treated much more deeply in [ES]. The main point of this article is to complete the results of B. Ammann and E. Humbert [AH06] and to study how the sign of the second eigenvalue of the Yamabe operator can be related to the existence of nodal solutions of the Yamabe equation (2), in particular when the Yamabe constant of \((M, g)\) is negative. Our main result is to prove that under this condition, such a solution always exists with \( \epsilon = \text{sign}(\lambda_2(g)) \). This is the object of Theorem 4.1.

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2. Eigenvalues in conformal metrics

In the whole paper, we will deal with the behavior of the eigenvalues of the Yamabe operator in a fixed conformal class. It will be useful to express their definition relatively to a fixed metric. This is the goal of this section.
2.1. Smooth metrics. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\), we keep the notations of the introduction and for any metric \(\tilde{g}\), we will denote by
\[\lambda_1(\tilde{g}) < \lambda_2(\tilde{g}) \leq \lambda_3(\tilde{g}) \leq \cdots \leq \lambda_k(\tilde{g}) \cdots \to +\infty,\]
the eigenvalues of the Yamabe operator. We will deal with the case where \(\tilde{g}\) is conformal to \(g\), i.e. when \(\tilde{g} = u^{N-2}g\), where \(u\) is a positive function of class \(C^\infty\). By referring to [AH06], one sees that the \(i\)-th eigenvalue \(\lambda_i(\tilde{g})\) is given by
\[
\lambda_i(\tilde{g}) = \inf_{v \in Gr_i(H^2(L))} \sup_{v \in V \setminus \{0\}} \frac{\int_M c_n |v|^2 + S_g v^2 \, dv_g}{\int_M v^2 u^{N-2} \, dv_g},
\]
where \(\tilde{g} = u^{N-2}g; u \in C^\infty(M), u > 0\) and \(Gr_i(H^2(L))\) stands for the set of all \(i\)-dimensional subspaces of \(H^2(L)\).

2.2. Generalized metrics. Reducing to smooth metrics will too restrictive for our investigations. We will need to work with generalized metrics, i.e. metrics of the form \(\tilde{g} = u^{N-2}g\) with \(u \in L^N(M), u \geq 0\) and \(u \not\equiv 0\). The Yamabe operator \(L_\tilde{g}\) has no meaning any more but the definition of \(\lambda_i(\tilde{g})\) can anyway be extended to this case by using (3) as it was done in [AH06] when the Yamabe constant was non-negative i.e. when \(\lambda_1(g) \geq 0\). When \(\lambda_1(g) < 0\), the situation is a little bit different: \(\lambda_i(\tilde{g})\) defined by (3) can be \(-\infty\) as proved by the following proposition.

**Proposition 2.1.** Assume that \(\lambda_1(g) < 0\), then there exists \(u \in L^N(M), u \not\equiv 0, u \geq 0\) such that \(\lambda_1(\tilde{g}) = -\infty, \text{where} \ g = u^{N-2}\tilde{g}\).

This proposition will be proved in Paragraph 2.2.1. To make sure that \(\lambda_1(\tilde{g})\) is finite, one has to assume in addition that \(u\) is positive.

**Proposition 2.2.** Let \(u\) be a positive function in \(L^N(M)\). Suppose that \(\lambda_1(g) < 0\). Then, we have
\[\lambda_1(\tilde{g}) > -\infty.\]

The proposition is proved in Paragraph 2.2.2.

**Notation 2.3.** The \(i\)-th eigenvalue of \(L_\tilde{g}\), \(\lambda_i(\tilde{g}) = \lambda_i(u^{N-2}g)\) will be denoted by \(\lambda_i(u)\) when there is no ambiguity about \(g\).

2.2.1. Proof of Proposition 2.1. We have \(\lambda_1(g) < 0\), this implies that there exists a function \(v \in C^\infty(M)\) such that
\[
\int_M (L_g v) v \, dv_g < 0.
\]
Let \(P\) be a point of \(M\). For \(\varepsilon > 0\), we define \(\eta_\varepsilon\) as follows
\[
\begin{cases}
0 \leq \eta_\varepsilon \leq 1, \\
\eta_\varepsilon = 0 \text{ on } B_\varepsilon(P), \\
\eta_\varepsilon = 1 \text{ on } M \setminus B_{2\varepsilon}(P), \\
|\nabla \eta_\varepsilon| \leq \frac{\varepsilon}{2},
\end{cases}
\]
where \(B_\delta(P)\) stands for the ball of center \(P\) and radius \(\delta\) in the metric \(g\). Then one easily checks
\[
\lim_{\varepsilon \to 0} \int_M (L_g (\eta_\varepsilon v))(\eta_\varepsilon v) \, dv_g = \int_M (L_g v) v \, dv_g.
\]
We define \(w := \eta_\varepsilon v\). Therefore, for a fixed small \(\varepsilon > 0\), we have
\[
\int_M (L_g w) w \, dv_g < 0.
\]
Let $u \geq 0$, $u \neq 0$ of class $C^\infty$ with support in $B_\varepsilon(P)$. For $\alpha > 0$, since $(w + \alpha)u \neq 0$, we can write
\[
\lambda_1(\bar{g}) = \inf_{v'} \frac{\int_M (L_g v')^2 dv_g}{\int_M u^{N-2}v'^2 dv_g} \leq \lim_{\alpha \to 0^+} \frac{\int_M (L_g(w + \alpha))(w + \alpha) dv_g}{\int_M u^{N-2}(w + \alpha)^2 dv_g}.
\]
Moreover, we have
\[
\lim_{\alpha \to 0^+} \int_M (L_g(w + \alpha))(w + \alpha) dv_g = \int_M (L_g(w))w dv_g < 0,
\]
and
\[
\lim_{\alpha \to 0^+} \int_M u^{N-2}(w + \alpha)^2 dv_g = 0
\]
which gives that
\[
\lim_{\alpha \to 0} \frac{\int_M (L_g(w + \alpha))(w + \alpha) dv_g}{\int_M u^{N-2}(w + \alpha)^2 dv_g} = -\infty.
\]
This ends the proof of Proposition 2.1.

2.2. Proof of Proposition 2.2. Let $(v_m)_m$ be a minimizing sequence for $\lambda_1(u)$, i.e. $v_m \in H^2_1(M)$ such that
\[
\lim_{m \to \infty} \frac{\int_M c_n |\nabla v_m|^2 + S_g v_m^2 dv_g}{\int_M u^{N-2}v_m^2 dv_g} = \lim_{m \to \infty} \lambda_m = \lambda_1(u) < 0.
\]
Since $(|v_m|)_m$ is also a minimizing sequence for $\lambda_1(u)$, we can assume that $v_m \geq 0$. We normalize $v_m$ by $\int_M |u|^{N-2}v_m^2 dv_g = 1$. Here we show that $(v_m)_m$ is bounded in $H^2_1(M)$. Indeed, suppose that $(v_m)_m$ is not bounded in $H^2_1(M)$ and let
\[
v_m' = \frac{v_m}{\|v_m\|_{H^2_1(M)}}.
\]
$(v_m')_m$ is bounded in $H^2_1(M)$, and his norm is equal to 1, then there exists $v' \in H^2_1(M)$, (after restriction to a subsequence) such that
\[
v_m' \to v' \text{ in } H^2_1(M),
\]
\[
v_m' \to v' \text{ in } L^2(M).
\]
We have
\[
c_n \int_M |\nabla v_m'|^2 dv_g + \int_M S_g v_m'^2 dv_g = \lambda_m \int_M |u|^{N-2}v_m'^2 dv_g.
\]
Moreover,
\[
\int_M |u|^{N-2}v_m'^2 dv_g \leq \int_M |u|^{N-2}v_m'^2 dv_g \to m \to \infty 0
\]
since
\[
\|v_m\|_{H^2_1(M)} \to \infty.
\]
It follows that
\[
\int_M |u|^{N-2}v_m'^2 dv_g = 0
\]
and since $u$ is positive,
\[
v' = 0.
\]
Now, we write
\[
1 = \int_M |\nabla v_m'|^2 dv_g + \int_M |v_m'|^2 dv_g.
\]
We deduce that
\[
\lim_{m \to \infty} \int_M |\nabla v_m'|^2 dv_g = 1,
\]
Moreover, we can normalize the functions
\[ \frac{c_n \int_M |\nabla v_m|^2 \, dv_g + \int_M S_2 v_m^2 \, dv_g}{\int_M u^{N-2} v_m^2 \, dv_g} = \lambda_m \int_M |u|^{N-2} v_m^2 \, dv_g \leq 0. \]

This proves that \((v_m)_m\) is bounded in \(H^2_1(M)\), and implies that \(\lambda_m \geq C\). We finally get \(\lambda_1(\tilde{g}) > -\infty\).

2.3. PDE associated to \(\lambda_i\).

**Proposition 2.4.** For any non-negative function \(u \in L^N(M)\), such that \(\lambda_1(u) > -\infty\), there exists functions \(v_1 > 0, v_2, \ldots, v_k \in H^2_1(M)\) having a changing sign, such that in the sense of distributions, we have
\[ L_g v_1 = \lambda_1(u) |u|^{N-2} v_1, \]
and
\[ L_g v_k = \lambda_k(u) |u|^{N-2} v_k. \]
Moreover, we can normalize the \(v_k\) by
\[ \int_M |u|^{N-2} v_k^2 \, dv_g = 1 \text{ and } \int_M |u|^{N-2} v_i v_j \, dv_g = 0 \text{ } \forall i \neq j. \]

**Proof:** Let \((v_m)_m\) be a minimizing sequence for \(\lambda_1(u)\), i.e. \(v_m \in H^2_1(M)\) such that
\[ \lim_{m \to \infty} \frac{\int_M c_n |\nabla v_m|^2 + S_2 v_m^2 \, dv_g}{\int_M |u|^{N-2} v_m^2 \, dv_g} = \lambda_1(u). \]
According to the Paragraph 2.2.2, we get that \((v_m)_m\) is bounded in \(H^2_1(M)\) and there exists \(v \geq 0\) in \(H^2_1(M)\) such that \(v_m\) converges to \(v\) weakly in \(H^2_1(M)\) and strongly in \(L^N(M)\) (after restriction to a subsequence). We now want to prove
\[ \int_M |u|^{N-2} v^2 \, dv_g = \lim_{m \to \infty} \int_M |u|^{N-2} v_m^2 \, dv_g = 1. \] (4)

If \(u\) is smooth, this relation is clear. So let us assume that \(u \in L^N(M)\), let \(A\) be a large real number and set \(u_A = \inf \{u, A\}\). By Hölder inequality, we write
\[
\left| \int_M u^{N-2} (v_m^2 - v^2) \, dv_g \right| = \left| \int_M (u^{N-2} - u_A^{N-2} + u_A^{N-2}) (v_m^2 - v^2) \, dv_g \right| \\
\leq \left( \int_M u_A^{N-2} |v_m^2 - v^2| \, dv_g \right) + \int_M (u^{N-2} - u_A^{N-2}) (|v_m| + |v|)^2 \, dv_g \\
\leq A^{N-2} \int_M |v_m^2 - v^2| \, dv_g \\
+ \left( \int_M (u^{N-2} - u_A^{N-2}) \frac{N}{N-2} \, dv_g \right) \left( \int_M (|v_m| + |v|)^N \, dv_g \right)^{\frac{N-2}{N}}.
\]

\((v_m)_m\) is bounded in \(H^2_1(M)\), it is bounded in \(L^N(M)\). Hence there exists a constant \(C\) such that
\[ \int_M (|v_m| + |v|)^N \, dv_g \leq C. \]
The convergence in \(L^2(M)\) gives
\[ \lim_{m \to \infty} \int_M |v_m^2 - v^2| \, dv_g = 0. \]
By dominated convergence theorem, we have
\[ \lim_{A \to \infty} \int_M (u^{N-2} - u_A^{N-2}) \frac{N}{N-2} \, dv_g = 0. \]
Hence, we get (1). Since
\[
\lim_m \int_M \langle \nabla v_m, \nabla \varphi \rangle \, dv_g = \int_M \langle \nabla v, \nabla \varphi \rangle \, dv_g,
\]
\[
\lim_m \int_M S_g v_m \varphi \, dv_g = \int_M S_g v \varphi \, dv_g
\]
and
\[
\lim_m \int_M |u|^{N-2} v_m \varphi \, dv_g = \int_M |u|^{N-2} v \varphi \, dv_g,
\]
(by strong convergence in $L^2(M)$), we obtain that in the sense of distributions $v$ verifies
\[
L_g v = \lambda_1(u) |u|^{N-2} v.
\]
Now we define
\[
\lambda'_k(u) = \inf_{\lambda_i \neq \lambda_k} \frac{\int_M |\nabla v_k|^2 + S_g v_k^2 \, dv_g}{\int_M |u|^{N-2} |v_k|^2 \, dv_g},
\]
we remark that $\lambda'_k(u) = \lambda_k(u)$ and $v_k$ is constructed by induction using the same method. This ends the proof of Proposition 2.4. □

3. Sign of $\lambda_i$

3.1. The sign of $\lambda_i$ is conformally invariant.

**Proposition 3.1.** The sign of $\lambda_i$ is independent of the metric selected in the conformal class. More precisely, for any conformal metric $\tilde{g} = u^{N-2} g$, where $u$ is a non-negative function in $L^N(M)$, $\lambda_i(u)$ and $\lambda_i(1)$ have same sign.

**Proof:** We assume for example that $\lambda_i(u) = 0$ and $\lambda_i(1) > 0$, we know that
\[
\lambda_i(u) = \inf_{u_1, \ldots, u_i} \sup_{\lambda_1, \ldots, \lambda_i} \frac{\int_M L_g \left( \lambda_1 u_1 + \cdots + \lambda_i u_i \right) \left( \lambda_1 u_1 + \cdots + \lambda_i u_i \right) \, dv_g}{\int_M \left( \lambda_1 u_1 + \cdots + \lambda_i u_i \right)^2 u^{N-2} \, dv_g},
\]
and
\[
\lambda_i(1) = \inf_{u_1, \ldots, u_i} \sup_{\lambda_1, \ldots, \lambda_i} \frac{\int_M L_g \left( \lambda_1 u_1 + \cdots + \lambda_i u_i \right) \left( \lambda_1 u_1 + \cdots + \lambda_i u_i \right) \, dv_g}{\int_M \left( \lambda_1 u_1 + \cdots + \lambda_i u_i \right)^2 \, dv_g}.
\]
Suppose that $\lambda_i(u)$ is attained by $v_1, \ldots, v_i$. Since denominators of this expressions are positive, then
\[
\sup_{\lambda_1, \ldots, \lambda_i} \int_M L_g \left( \lambda_1 v_1 + \cdots + \lambda_i v_i \right) \left( \lambda_1 v_1 + \cdots + \lambda_i v_i \right) \, dv_g = 0.
\]
So
\[
\lambda_i(1) \leq \sup_{\lambda_1, \ldots, \lambda_i} \frac{\int_M L_g \left( \lambda_1 v_1 + \cdots + \lambda_i v_i \right) \left( \lambda_1 v_1 + \cdots + \lambda_i v_i \right) \, dv_g}{\int_M \left( \lambda_1 v_1 + \cdots + \lambda_i v_i \right)^2 u^{N-2} \, dv_g} = 0,
\]
which gives a contradiction. The remaining cases are treated similarly.

3.2. The negativity of $\lambda_k$ is not topologically obstructed. In this paragraph, we will see that on each manifold, there exists a metric which has a negative $k^{th}$-eigenvalue.

**Proposition 3.2.** On any compact Riemannian manifold $M$, and for all $k \geq 1$ there exists a metric $g$ such that
\[
\lambda_k(g) < 0.
\]
Proof: Let $M$ be a compact Riemannian manifold of dimension $n$, and we take $k$ spheres of dimension $n = \dim(M)$. We equip each sphere $\mathbb{S}^n$ by the same metric $g$, such that $\mu(g) < 0$. We can do this by referring to [Aub98] (Theorem [1] page 38). Let $P \in M$, since $\mu(g) < 0$, for all $\varepsilon, \delta > 0$ we can find a function $u$ supported in $\mathbb{S}^n \setminus B_{\varepsilon}(P)$ such that

$$\frac{\int_M c_n |\nabla u|^2 + S_g u^2 \, dv_g}{\int_M u^2 \, dv_g} < -\delta.$$ 

Indeed, let $\eta_{\varepsilon}$ be a smooth cut-off function such that $0 \leq \eta_{\varepsilon} \leq 1$, $\eta_{\varepsilon}(B_{\varepsilon}(P)) = 0$, $\eta_{\varepsilon}(\mathbb{S}^n \setminus B_{2\varepsilon}(P)) = 1$, $|\nabla \eta_{\varepsilon}| \leq \frac{2}{\varepsilon}$ and a function $v$ satisfying

$$I_g(v) = \frac{\int_M c_n |\nabla v|^2 + S_g v^2 \, dv_g}{\int_M v^2 \, dv_g} < -2\delta.$$ 

Note that the existence of $v$ is given by the fact that $\mu(g) < 0$. The desired function $u$ will be given by $\eta_{\varepsilon} v$, where $\varepsilon > 0$ is sufficiently small. Indeed, it suffices to notice that, as easily checked,

$$\lim_{\varepsilon \to 0} I_g(\eta_{\varepsilon} v) = I_g(v).$$

Let $P_1, \ldots, P_k$ be points of $M$. We consider the following connected sum

$$M' = M \# (\mathbb{S}^n)_1 \# \ldots \# (\mathbb{S}^n)_k,$$

where the $(\mathbb{S}^n)_i$ are attached at $P$ on the spheres $\mathbb{S}^n$ and at $P_i$ on $M$ so that the handles are attached in $B_{\varepsilon}(P)$ and $B_{\varepsilon}(P_i)$. Note that $M'$ is diffeomorphic to $M$. Moreover, the above construction allows to see $(\mathbb{S}^n)_i \setminus B_{\varepsilon}(P_i)$ as a part of $M'$. We take on $M'$ any metric $h$ satisfying

$$h|_{(\mathbb{S}^n)_i \setminus B_{\varepsilon}(P_i)} = g.$$

On $M'$, we define the following function

$$u_i = \begin{cases} u & \text{on } (\mathbb{S}^n)_i \setminus B_{\varepsilon}(P_i) \\ 0 & \text{otherwise.} \end{cases}$$

Since the $u_i$ have disjoint supports, we get

$$\lambda_k(1) \leq \sup_{\lambda_1, \ldots, \lambda_k} \frac{\int_M L_g(\lambda_1 u_1 + \cdots + \lambda_k u_k)(\lambda_1 u_1 + \cdots + \lambda_k u_k) \, dv_h}{\int_M (\lambda_1 u_1 + \cdots + \lambda_k u_k)^2 \, dv_h}$$

$$\leq \sup_{\lambda_1, \ldots, \lambda_k} \frac{\int_M (L_g u)(u \, dv_g)}{(\lambda_1^2 + \cdots + \lambda_k^2) \int_M u \, dv_g}$$

$$\leq \frac{\int_M (L_g u) \, dv_g}{\int_M u^2 \, dv_g} < -\delta.$$ 

4. Nodal solutions of the Yamabe equations

A famous problem in Riemannian geometry is the Yamabe problem, solved between 1960 and 1984 by Yamabe, Trüdinger, Aubin and Schoen, [Yam60, Tru68, Aub76, Sch84]. The reader can also refer to [LP87, Heb97, Aub98]. The Yamabe problem consists in finding a metric $\tilde{g}$ conformal to $g$ such that the scalar curvature $S_g$ of $\tilde{g}$ is constant. Solving this problem is equivalent to finding a positive smooth function and a number $C_0 \in \mathbb{R}$ such that

$$L_g(u) = C_0 |u|^{N-2} u,$$ 

(5)

where $N = \frac{2n}{n-2}$. In order to obtain solutions of the Yamabe equation we define the Yamabe invariant by

$$\mu(M, g) := \inf_{u \neq 0, u \in C^\infty(M)} Y(u),$$
where

\[ Y(u) = \frac{\int_M c_n |\nabla u|^2 + S_g u^2 \, dv_g}{\left( \int_M |u|^N \, dv_g \right)^{\frac{2}{N}}}. \]

The works of Yamabe, Trüdinger, Aubin and Schoen provides a positive smooth minimizer \( u \) of \( Y \), satisfying, if normalized by \( \|u\|_{L^N(M)} = 1 \),

\[ L_g u = \mu(M, g) |u|^{N-2} u. \]

The metric \( \tilde{g} = u^{N-2} g \) is the desired metric: its scalar curvature is constant equal to \( \mu(M, g) \). If we set \( u' = \mu(M, g)^{\frac{4}{N-2}} u \), we obtain a positive solution of

\[ L_g u' = \varepsilon |u'|^{N-2} u' \]

where \( \varepsilon = \text{sign} (\mu(M, g)) = \text{sign} (\lambda_1(g)) \).

Now, if \( \mu(M, g) \geq 0 \), it is easy to check that

\[ \mu(M, g) = \inf_{\tilde{g} \in [g]} \lambda_1(\tilde{g}) \text{vol}(M, \tilde{g})^{\frac{2}{N}}, \]

where \([g]\) is the conformal class of \( g \) and \( \lambda_1 \) is the first eigenvalue of the Yamabe operator \( L_g \). Inspired by this approach, in their paper [AH06], B. Ammann et E. Humbert introduced the second Yamabe invariant defined by

\[ \mu_2(M, g) = \inf_{\tilde{g}} \lambda_2(\tilde{g}) \text{vol}(M, \tilde{g})^{\frac{2}{N}}, \]

\[ = \inf_u \lambda_2(u^{N-2} g) \left( \int_M u^N \, dv_g \right)^{\frac{2}{N}}. \]

They studied this invariant in the case where \( \mu(M, g) \geq 0 \), and they proved that \( \mu_2 \) is attained by a generalized metric, (i.e. a metric of the form \( u^{N-2} g \) where \( u \in L^N(M), u \geq 0 \) which may vanish), in the following two cases

- \( \mu(M, g) > 0 \), \( (M, g) \) is not locally conformally flat and \( n \geq 11 \).
- \( \mu(M, g) = 0 \), \( (M, g) \) is not locally conformally flat and \( n \geq 9 \).

In this context, they proved that \( u \) is the absolute value of a changing sign function \( w \) of class \( C^{3,\alpha}(M) \), which verifies the following equation

\[ L_g w = \mu_2(M, g) |w|^{N-2} w. \]

Many works are devoted to the study of this kind of solutions, for example [AH06], [DJ02], [Hol09], [HV94], [Véto07]. See also [BB10] for an analogue study for the Paneitz-Branson operator. Setting again \( u' = \mu_2(M, g)^{\frac{4}{N-2}} w \), we obtain a solution of

\[ L_g u' = \varepsilon |u'|^{N-2} u' \]

with \( \varepsilon = 1 = \text{sign} (\mu_2(M, g)) = \text{sign} (\lambda_2(g)) \). The goal of this section is to study if this result extends to metrics where the sign of \( \lambda_2(g) \) is arbitrary.

The answer is yes when \( \lambda_2 < 0 \) without any other condition, we obtain this result by a method different than the one of [AH06]. Notice that this situation occurs for a large number of metrics (see Proposition 3.2). When \( \lambda_2 \geq 0 \), we show that the methods in [AH06] can be extended to the case where \( \mu(M, g) < 0 \). Namely, the main result of this paper is:

**Theorem 4.1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 3 \) whose Yamabe invariant \( \mu(M, g) \) is strictly negative, we denote by \( \lambda_2 \) the second eigenvalue of \( L_g \). Then, if \( \lambda_2 \leq 0 \) or if \( \lambda_2 > 0 \), \( (M, g) \) not locally conformally flat and \( n \geq 6 \):

There exists a function \( w \) changing sign, solution of the equation

\[ L_g w = \varepsilon |w|^{N-2} w, \]

where
where \( \varepsilon = +1 \) if \( \lambda_2 > 0 \), \( \varepsilon = -1 \) if \( \lambda_2 < 0 \) and \( \varepsilon = 0 \) if \( \lambda_2 = 0 \). Moreover, \( w \in C^{3,\alpha}(M) \), for all \( \alpha < N - 2 \).

4.1. The case \( \lambda_2 = 0 \). This case is obvious: indeed, Proposition 2.4 provides the existence of a nodal solution \( v \) of \( L_g v = 0 = \varepsilon |v|^{N-2} v \) where \( \varepsilon = \text{sign}(\lambda_2(g)) \).

4.2. The case \( \lambda_2 > 0 \). As in [AH06], we introduce the second Yamabe invariant given by

\[
\mu_2(M, g) = \inf_{\varphi} \lambda_2(g) \text{vol}(M, g)^{\frac{2}{n}}
\]

By Proposition 4.2 below, the problem reduces to finding a minimizer of \( \mu_2(M, g) \). The case where \( \mu(M, g) \geq 0 \) have been treated in [AH06]. We will then focus on the case where \( \mu(M, g) < 0 \) (i.e. \( \lambda_1(g) < 0 \)). We will see that the method of Ammann and Humbert remains valid in this case and the following three propositions answer our questions.

**Proposition 4.2.** Let \( (M, g) \) be a compact Riemannian manifold of dimension \( n \geq 3 \), such that \( \lambda_2 > 0 \). If

\[
\mu_2(M, g) < \mu(S^n),
\]

with \( \mu(S^n) = n(n-1)\omega_n^\frac{2}{n} \), where \( \omega_n \) stands for the volume of the standard sphere \( S^n \), then the second Yamabe invariant is attained by a non-negative function \( u \in L^N(M) \) that we normalize by \( \int_M u^N \, dv_g = 1 \). There exists a function \( w \) having a changing sign which verifies in the sense of distributions the following equation

\[
L_g w = \mu_2(M, g) |u|^{N-2} w.
\]

The functions \( u \) and \( w \) will be normalized by

\[
\int_M u^N \, dv_g = 1, \quad \int_M u^{N-2} w^2 \, dv_g = 1.
\]

**Proposition 4.3.** The two functions \( u \) and \( w \) given by Proposition 4.2 satisfy

\[
|u| = |w|.
\]

Finally, we give a condition under which assumption (\ref{eq6}) is satisfied:

**Proposition 4.4.** Let \( (M, g) \) be a compact Riemannian manifold of dimension \( n \geq 6 \), suppose that \( M \) is not locally conformally flat and his Yamabe invariant \( \mu(M, g) < 0 \), then

\[
\mu_2(M, g) < \mu(S^n).
\]

4.2.1. **Proof of Proposition 4.2** The case where \( \mu(M, g) \geq 0 \) is done in [AH06], hence we consider here the case where \( \mu(M, g) < 0 \). By the solution of the Yamabe problem, we can assume without loss of generality, that \( S_g = -1 \). Let \( (u_m)_m \) be a minimizing sequence for \( \mu_2(M, g) \), i.e., \( u_m \) is positive, smooth and

\[
\lim_{m \to \infty} \lambda_2(u_m) \left( \int_M u_m^N \, dv_g \right)^\frac{2}{n} = \mu_2(M, g).
\]

The sequence \( (u_m)_m \) will be choosen such that \( \int_M u_m^N \, dv_g = 1 \), hence \( \mu_2(M, g) = \lim_{m \to \infty} \lambda_2(u_m) \). For each \( u_m \), Proposition 2.4 provides the existence of a function \( w_m \in H^2_1(M) \) such that

\[
L_g w_m = \lambda_2(u_m) |u_m|^{N-2} w_m.
\]

Moreover, the sequence \( (w_m)_m \) can be normalized by

\[
\int_M |u_m|^{N-2} w_m^2 \, dv_g = 1.
\]
Since $\int_M u_m^N dv_g = 1$, $(u_m)_m$ is bounded in $L^N(M)$ which is a reflexive space, there exists $u \in L^N(M)$ such that $u_m$ converges weakly to $u$ in $L^N(M)$, we have

$$u_m \rightharpoonup u$$ in $L^N(M)$.

- The sequence $(w_m)_m$ is bounded in $H^2_1(M)$.

We proceed by contradiction and assume that $\|w_m\|_{H^2_1(M)} \to \infty$. Let

$$w'_m = \frac{w_m}{\|w_m\|_{H^2_1(M)}}.$$ 

$\|w'_m\|_{H^2_1(M)} = 1$, hence $(w'_m)_m$ is bounded in $H^2_1(M)$. Since $H^2_1(M)$ is a reflexive space, this implies using Kondrakov and Banach-Alaoglu theorems, that there exists a subsequence $(w'_m)_m$ and $w' \in H^2_1(M)$ such that

$$w'_m \rightharpoonup w'$$ in $H^2_1(M)$, and

$$w'_m \to w'$$ in $L^2(M)$.

Equation (8) is linear, so $w'_m$ satisfies

$$L_g w'_m = \lambda_2(u_m) |u_m|^{N-2} w'_m.$$ 

Hence for all $\varphi \in C^\infty(M)$, we have:

$$c_n \int_M \langle \nabla w'_m, \nabla \varphi \rangle dv_g + \int_M S_g w'_m \varphi dv_g = \int_M \lambda_2(u_m) |u_m|^{N-2} w'_m \varphi dv_g.$$ 

Since $w'_m \to w'$ in $H^2_1(M)$ and $w \mapsto \langle \nabla w, \nabla \varphi \rangle$ is a linear form on $H^2_1(M)$, then

$$c_n \int_M \langle \nabla w'_m, \nabla \varphi \rangle dv_g \to c_n \int_M \langle \nabla w', \nabla \varphi \rangle dv_g.$$ 

The sequence $w'_m$ converges strongly to $w'$ in $L^2(M)$. This gives that

$$\int_M S_g w'_m \varphi dv_g \to \int_M S_g w \varphi dv_g.$$ 

Using Hölder inequality, we obtain that $\int_M |u_m|^{N-2} w'_m \varphi dv_g \to 0$. Indeed,

$$\left| \int_M |u_m|^{N-2} w'_m \varphi dv_g \right| \leq \|\varphi\|_\infty \int_M |u_m|^{\frac{N-2}{2}} |w'_m| |u_m|^{\frac{N-2}{2}} dv_g$$

$$\leq \|\varphi\|_\infty \left( \int_M |u_m|^{N-2} w'_m^2 dv_g \right)^{\frac{1}{2}} \left( \int_M |u_m|^N dv_g \right)^{\frac{1}{2}}$$

$$\leq \|\varphi\|_\infty \frac{1}{\|w'_m\|_{H^2_1(M)}} \left( \int_M |u_m|^{N-2} w'_m^2 dv_g \right)^{\frac{1}{2}} \left( \int_M |u_m|^N dv_g \right)^{\frac{N-2}{2}} \left( \text{vol}(M,g) \right)^{\frac{N-2}{2}} \to m \to +\infty 0.$$ 

Then

$$c_n \int_M \langle \nabla w', \nabla \varphi \rangle dv_g + \int_M S_g w' \varphi dv_g = 0,$$

which means that in the sense of distributions, we have

$$L_g w' = 0.$$ 

Since $\lambda_1(1) < 0$ and $\lambda_2(1)$ is positive, $0 \notin Sp(L_g)$. It follows that $w' = 0$. Now, we also have

$$\int_M c_n |\nabla w'_m|^2 dv_g + \int_M S_g w'_m^2 dv_g = \lambda_2(u_m) \int_M |u_m|^{N-2} w'_m^2 dv_g,$$
with
\[
\lambda_2(u_m) \int_M |u_m|^{N-2} w \, dv_g = \frac{\lambda_2(u_m)}{\|w_m\|_{H^2(M)}} \to 0
\]
and
\[
\int_M S_g w \, dv_g \to \int M S_g w \, dv_g = 0.
\]
Hence
\[
\int_M |\nabla w'|^2 \, dv_g \to 0.
\]
Finally, we get that
\[
\|w_m\|^2_{H^2(M)} = 1 = \int_M |\nabla w'|^2 \, dv_g + \int M w_m^2 \, dv_g \to 0,
\]
which gives the desired contradiction. We obtain that \((w_m)_m\) is a bounded sequence in \(H^2(M)\). Then there exists \(w \in H^2(M)\) such that:
\[
w_m \to w \text{ in } H^2(M),
\]
\[
w_m \to w \text{ in } L^2(M).
\]
It follows that in the sense of distributions, we have
\[
L_g w = \mu_2(M, g) |w|^{N-2} w.
\]
It remains to show that \(w\) changes sign and is different from zero.

- Suppose that \(w\) does not change sign. Without loss of generality, we can assume that \(w \geq 0\). In the sense of distributions, we have
\[
c_n \Delta_g w + S_g w = \mu_2(M, g) |w|^{N-2} w. \tag{9}
\]
It was already mentioned at the beginning of this section that we can assume that \(S_g < 0\), because \(\mu(M, g) < 0\). Integrating (9) over \(M\), we get:
\[
\int_M c_n \Delta_g w \, dv_g + \int M S_g w \, dv_g = \mu_2(M, g) \int M |w|^{N-2} w \, dv_g.
\]
This gives a contradiction unless \(w \equiv 0\) which is prohibited by what follows.

- Assume that \(w = 0\). By referring to [Heb97] and [Aub76] we have the following theorem:
If \((M, g)\) is a Riemannian manifold of dimension \(n \geq 3\), for all \(\epsilon > 0\), there exists \(B_\epsilon\) such that for any \(u \in H^2(M)\), we have
\[
\left( \int_M |u|^N \, dv_g \right)^{\frac{1}{N}} \leq (\mu(S^n)^{-1} + \epsilon) \left( \int_M c_n |\nabla u|^2 \, dv_g + B_\epsilon \int M u^2 \, dv_g \right).
\]
We obtain
\[
c_n \int_M |\nabla w_m|^2 \, dv_g + S_g \int M w_m^2 \, dv_g = \mu_2(M, g) \int M |u_m|^{N-2} w_m^2 \, dv_g
\]
\[
\leq \mu_2(M, g) \left( \int M |u_m|^N \, dv_g \right)^{\frac{N-2}{N}} \left( \int M |w_m|^N \, dv_g \right)^{\frac{2}{N}}
\]
\[
\leq \mu_2(M, g) (\mu(S^n)^{-1} + \epsilon) \left( \int_M c_n |\nabla w_m|^2 \, dv_g + B_\epsilon \int M w_m^2 \, dv_g \right).
\]
Hence
\[ c_n \left[ 1 - \mu_2(M, g)(\mu(S^n)^{-1} + \epsilon) \right] \int_M |\nabla w_m|^2 \, dv_g \leq c \int_M w_m^2 \, dv_g, \]
then \( \int_M |\nabla w_m|^2 \, dv_g \to 0 \), so \( w_m \to 0 \) in \( H^1(M) \). We finally get that
\[ 1 = \int_M |u_m|^{N-2} w_m^2 \, dv_g \leq \left( \int_M |u_m|^N \, dv_g \right)^{\frac{N-2}{2}} \int_M w_m^N \, dv_g. \]
This gives a contradiction, then \( w \neq 0 \).

4.2.2. Proof of Proposition 4.3. Since \( \lambda_2(g) > 0 \), then
\[ \mu_2(M, g) = \inf_{u \geq 0} \lambda_2(u) \left( \int_M u^N \, dv_g \right)^{\frac{2}{N}} = \inf_{u \geq 0} \lambda_2(u) \left( \int_M u^N \, dv_g \right)^{\frac{2}{N}}. \]
We mimic the proof of Lemma 3.3 in [AH06] by taking \( w_1 = w_+ = \sup \{0, w\} \) and \( w_2 = w_- = \sup \{0, -w\} \). This gives that
\[ u = aw_+ + bw_-, \]
where \( a, b > 0 \). By Lemma 3.1, \( w \in C^{2,\alpha} \), \( u \in C^{0,\alpha} \) and Step 4 of the proof of Theorem 3.4 in [AH06] then shows that
\[ u = |w|. \]
Since \( w \) is in \( H^1_1(M) \), Lemma 3.1 of [AH06] says that \( w \in L^{N+\epsilon}(M) \), because \( w \) satisfies the equation
\[ L_\mu w = \mu_2 |w|^{N-2} w, \]
and standard bootstrap arguments gives that \( w \in C^{3,\alpha}(M) \) for all \( \alpha < N - 2 \).

4.2.3. Proof of Proposition 4.4. In this paragraph, we will see that if \( M \) is not locally conformally flat of dimension \( n \geq 6 \), then we obtain that
\[ \mu_2(M, g) < \mu(S^n). \]
We still consider the case where \( \mu(M, g) < 0 \). Then there exists a positive function \( v \) solution of the Yamabe equation
\[ L_\mu v = \mu(M, g) v^{N-1}. \] (10)
Let \( x_0 \) be a point of \( M \) at which the Weyl tensor is not zero (such a point exists because the manifold is not locally conformally flat and \( n \geq 4 \)) and \((x_1, \ldots, x_n)\) be a system of normal coordinates at \( x_0 \). For \( x \in M \), denote by \( r = d(x, x_0) \) the distance to the point \( x_0 \). If \( \delta \) is a small fixed number, let \( \eta \) be a cut-off function of class \( C^\infty \) defined by
\[ \left\{ \begin{array}{l}
0 \leq \eta \leq 1, \\
\eta = 1 \text{ on } B_\delta(x_0), \\
\eta = 0 \text{ on } M \setminus B_{2\delta}(x_0), \\
|\nabla \eta| \leq \frac{2}{\delta}.
\end{array} \right. \]
For all \( \epsilon > 0 \) we define the following function
\[ v_\epsilon = c_\epsilon \eta(\epsilon + r^2)^{\frac{2-n}{4}}, \]
where \( c_\epsilon \) is choosen such that
\[ \int_M v_\epsilon^N \, dv_g = 1. \]
By referring to [Aub76],
\[
\lim_{\varepsilon \to 0} Y(v_{\varepsilon}) = \mu_1(S^n),
\]
where \( Y(u) \) is the Yamabe functional defined by
\[
Y(u) = \frac{\int_M c_n |\nabla u|^2 + S_g u^2 \, dv_g}{(\int_M u^N \, dv_g)^{2/N}}.
\]
If \((M, g)\) is not locally conformally flat, by a calculation made in [Aub76], there exists a constant \( C(M) > 0 \) such that
\[
Y(v_{\varepsilon}) = \begin{cases} 
\mu_1(S^n) - C(M)\varepsilon^2 + o(\varepsilon^2) & \text{if } n > 6 \\
\mu_1(S^n) - C(M)|\ln(\varepsilon)| + o(\varepsilon^2|\ln(\varepsilon)|) & \text{if } n = 6.
\end{cases}
\]
Again from [Aub76], there exists constants \( a, b, C_1, C_2 > 0 \), such that
\[
a\varepsilon^{\frac{n-2}{4}} \leq c_\varepsilon \leq b\varepsilon^{\frac{n-2}{4}},
\]
and
\[
C_1 a_{p,\varepsilon} \leq \int_M u^p \, dv_g \leq C_2 a_{p,\varepsilon}
\]
where
\[
\alpha_{p,\varepsilon} = \begin{cases} 
\varepsilon^{\frac{2n-(n-2)p}{4}} & \text{if } p > \frac{n}{n-2}; \\
|\ln(\varepsilon)|\varepsilon^{\frac{p}{4}} & \text{if } p = \frac{n}{n-2}; \\
\varepsilon^{\frac{(n-2)p}{4}} & \text{if } p < \frac{n}{n-2}.
\end{cases}
\]
We have
\[
\mu_2(M, g) = \inf_u \lambda_2(u) \left( \int_M u^N \, dv_g \right)^{\frac{2}{N}} = \inf_u \sup_{\lambda, \mu} \frac{\int_M L_g(\lambda w + \mu w')(\lambda w + \mu w') \, dv_g}{\int_M u^{N-2}(\lambda w + \mu w')^2 \, dv_g} \left( \int_M u^N \, dv_g \right)^{\frac{2}{N}} = \inf_u \sup_{\lambda, \mu} F(u, \lambda w + \mu w').
\]
Let \( \lambda_{\varepsilon}, \mu_{\varepsilon} \) such that
\[
\lambda_{\varepsilon}^2 + \mu_{\varepsilon}^2 = 1
\]
and
\[
F(v_{\varepsilon}, \lambda_{\varepsilon} v + \mu_{\varepsilon} v_{\varepsilon}) = \sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}} F(v_{\varepsilon}, \lambda v + \mu v_{\varepsilon}),
\]
where \( v \) is the function defined in the equation (10).
Calculating \( F(v_{\varepsilon}, \lambda_{\varepsilon} v + \mu_{\varepsilon} v_{\varepsilon}) \), we get
\[
F(v_{\varepsilon}, \lambda_{\varepsilon} v + \mu_{\varepsilon} v_{\varepsilon}) = \frac{\int_M L_g(\lambda_{\varepsilon} v + \mu_{\varepsilon} v_{\varepsilon})(\lambda_{\varepsilon} v + \mu_{\varepsilon} v_{\varepsilon}) \, dv_g}{\int_M v_{\varepsilon}^{N-2}(\lambda_{\varepsilon} v + \mu_{\varepsilon} v_{\varepsilon})^2 \, dv_g} \left( \int_M v_{\varepsilon}^N \, dv_g \right)^{\frac{2}{N}} = \frac{\lambda_{\varepsilon}^2 \mu(M, g) + \mu_{\varepsilon}^2 Y(v_{\varepsilon}) + 2\lambda_{\varepsilon} \mu_{\varepsilon} \mu(M, g) \int_M v_{\varepsilon}^{N-1} v_{\varepsilon} \, dv_g}{\lambda_{\varepsilon} \int_M v_{\varepsilon}^{N-2} v_{\varepsilon}^2 \, dv_g + \mu_{\varepsilon}^2 + 2\lambda_{\varepsilon} \mu_{\varepsilon} \int_M v_{\varepsilon}^{N-1} v \, dv_g} = \frac{A_{\varepsilon}}{B_{\varepsilon}}.
\]
If 
\[
\lambda_\varepsilon \rightarrow \lambda \neq 0, \quad \mu_\varepsilon \rightarrow \mu \neq 0,
\]
then
\[
F(v_\varepsilon, \lambda_\varepsilon v + \mu_\varepsilon v_\varepsilon) \rightarrow \frac{\lambda^2 \mu(M, g) + \mu^2 \mu(S^n)}{\mu^2} < \mu(S^n).
\]
Similarly, if \( \mu = 0, \lambda^2 = 1 \), then the numerator \( A_\varepsilon \sim \mu(M, g) \) remains negative, which gives again that
\[
F(v_\varepsilon, \lambda_\varepsilon v + \mu_\varepsilon v_\varepsilon) \leq 0 < \mu(S^n),
\]
which gives the desired inequality. Then, in the sequel, we assume that \( \lambda_\varepsilon \rightarrow 0 \) and \( \mu_\varepsilon \rightarrow \pm 1 \).

The case \( n > 6 \)

Using (12) we have
\[
\int_M v^{N-1} v_\varepsilon dv_g \sim_{\varepsilon \rightarrow 0} C_\varepsilon \varepsilon^{\frac{n-2}{4}},
\]
\[
\int_M v_{\varepsilon}^{N-2} v^2 dv_g \sim_{\varepsilon \rightarrow 0} C_\varepsilon,
\]
and
\[
\int_M v_{\varepsilon}^{N-1} v dv_g \sim_{\varepsilon \rightarrow 0} C_\varepsilon \varepsilon^{\frac{n-2}{4}},
\]
where \( C \) denotes a constant that might change its value from line to line. We distinguish two cases

- there exists a constant \( C > 0 \) such that
  \[
  |\lambda_\varepsilon| \leq C \varepsilon^{\frac{n-2}{4}},
  \]  
  (13)
  or

- there exists \( \alpha_\varepsilon \) such that
  \[
  |\lambda_\varepsilon| = \alpha_\varepsilon \varepsilon^{\frac{n-2}{4}},
  \]  
  (14)

(possibly extracting a subsequence).

1. Suppose first that (13) is verified. Then we have
   \[
   |\lambda_\varepsilon| \leq C \varepsilon^{\frac{n-2}{4}},
   \]
   Hence \( \lambda_\varepsilon^2 = O(\varepsilon^{\frac{n-2}{2}}) \), so \( \mu_\varepsilon^2 = 1 - \lambda_\varepsilon^2 = 1 + O(\varepsilon^{\frac{n-2}{2}}) \). Therefore
   \[
   \mu_\varepsilon = 1 + O(\varepsilon^{\frac{n-2}{2}}).
   \]
   This gives
   \[
   A_\varepsilon = O(\varepsilon^{\frac{n-2}{2}}) + (1 + O(\varepsilon^{\frac{n-2}{2}})) \left( \mu(S^n) - C(M)\varepsilon^2 + o(\varepsilon^2) \right) + O(\varepsilon^{\frac{n-2}{2}})
   = \mu(S^n) - C(M)\varepsilon^2 + O(\varepsilon^{\frac{n-2}{2}}) + o(\varepsilon^2).
   \]
   Since \( \frac{n-2}{2} > 2 \),
   \[
   A_\varepsilon = \mu(S^n) - C(M)\varepsilon^2 + o(\varepsilon^2),
   \]
   and
   \[
   B_\varepsilon = O(\varepsilon^{\frac{n-2}{2}+1}) + 1 + O(\varepsilon^{\frac{n-2}{2}}) + O(\varepsilon^{\frac{n-2}{2}}) = 1 + o(\varepsilon^2).
   \]
   Then,
   \[
   \frac{A_\varepsilon}{B_\varepsilon} = \mu(S^n) - C(M)\varepsilon^2 + o(\varepsilon^2) < \mu(S^n).
   \]
(2) Assume now that \([14]\) is fulfilled. In this case

\[
\frac{A_\varepsilon}{B_\varepsilon} = \frac{\lambda_2^2 \mu(M, g) + (1 - \lambda_2^2) Y(\varepsilon \nu) + \lambda_\varepsilon O(\varepsilon^{n-2})}{\lambda_2^2 O(\varepsilon) + (1 - \lambda_2^2) + 2 \lambda_\varepsilon \mu_\varepsilon O(\varepsilon^{\eta})}
\]

\[
= \frac{\lambda_2^2 \mu(M, g) + (1 - \lambda_2^2) Y(\varepsilon \nu) + o(\lambda_2^2)}{o(\lambda_2^2) + (1 - \lambda_2^2) + o(\lambda_2^2)}
\]

\[
= \frac{\lambda_2^2 \mu(M, g)}{1 - \lambda_2^2 + o(\lambda_2^2)} + \frac{Y(\varepsilon \nu)}{1 + o(\lambda_2^2)} + o(\lambda_2^2)
\]

\[
\leq \mu(M, g) \lambda_2^2 + \mu(S^n)(1 + o(\lambda_2^2)) + o(\lambda_2^2)
\]

\[
\leq \mu(S^n) + \mu(M, g) \lambda_2^2 + o(\lambda_2^2)
\]

\[
< \mu(S^n),
\]

because \(\mu(M, g) < 0\) and \(Y(\varepsilon \nu) \leq \mu(S^n)\).

The case \(n = 6\)

Since

\[
\int_M \nu_\varepsilon^{N-2} \nu_\varepsilon^2 \nu_\nu \sim_{\varepsilon \to 0} C\varepsilon,
\]

\[
\int_M \nu_\varepsilon^{N-1} \nu_\varepsilon \nu_\nu \sim_{\varepsilon \to 0} C\varepsilon,
\]

\[
\int_M \nu_\varepsilon^{N-1} \nu \nu \sim_{\varepsilon \to 0} C\varepsilon,
\]

then

\[
A_\varepsilon = \lambda_2^2 \mu(M, g) + \mu_\varepsilon^2 Y(\varepsilon \nu) + 2 \lambda_\varepsilon \mu_\varepsilon O(\varepsilon),
\]

\[
B_\varepsilon = \lambda_2^2 O(\varepsilon) + \mu_\varepsilon^2 + 2 \lambda_\varepsilon \mu_\varepsilon O(\varepsilon).
\]

Again, we have two cases to study

(1) If \(|\lambda_\varepsilon| \leq C\varepsilon\), then

\[
\lambda_2^2 \leq C\varepsilon^2.
\]

This implies

\[
A_\varepsilon = \mu(S^n) - C\varepsilon^2|\ln(\varepsilon)| + o(\varepsilon^2|\ln(\varepsilon)|)
\]

and

\[
B_\varepsilon = 1 + O(\varepsilon^2) = 1 + o(\varepsilon^2|\ln(\varepsilon)|).
\]

Hence

\[
\frac{A_\varepsilon}{B_\varepsilon} < \mu(S^n).
\]

(2) If \(|\lambda_\varepsilon| = \alpha_\varepsilon \varepsilon\), with \(\alpha_\varepsilon \to +\infty\). Since \(Y(\varepsilon \nu) \leq \mu(S^n)\), therefore

\[
A_\varepsilon = \alpha_\varepsilon^2 \varepsilon^2 \mu(M, g) + \mu_\varepsilon^2 \mu(S^n) + o(\alpha_\varepsilon^2 \varepsilon^2),
\]

and

\[
B_\varepsilon = \mu_\varepsilon^2 + o(\alpha_\varepsilon^2 \varepsilon^2).
\]

Therefore

\[
\frac{A_\varepsilon}{B_\varepsilon} = \mu(M, g) \frac{\alpha_\varepsilon^2 \varepsilon^2}{1 + o(1)} + \frac{\mu(S^n)}{1 + o(\alpha_\varepsilon^2 \varepsilon^2)} + \frac{o(\alpha_\varepsilon^2 \varepsilon^2)}{1 + o(1)}
\]

\[
< \mu(S^n).
\]
This ends the proof of Proposition 4.4. □

So we get a solution $w$ having a changing sign of the equation

$$L_g w = \mu_2 |w|^{N-2} w.$$ 

Finally, to obtain the result announced in Theorem 4.1, it suffices to set

$$w' = \frac{\mu_2}{n-2} w,$$

then $w'$ verifies

$$L_g w' = \varepsilon |w'|^{N-2} w',$$

with $\varepsilon = 1 = \text{sign}(\lambda_2(g))$.

5. The case $\lambda_2 < 0$

In this section, we will show that in all cases, there exists a nodal solution of the equation

$$L_g w = C_0 |w|^{N-2} w,$$

where $C_0$ is a negative constant.

First, since $\mu < 0$, we assume in the whole section that the metric $g$ is such that $S_g = -1$. In this context, the approach will be different. Indeed, the second Yamabe invariant is not well defined as shown in the following proposition:

**Proposition 5.1.** Let $M$ be a compact Riemannian manifold of dimension $n \geq 3$. Suppose that $\lambda_2 < 0$, then

$$\inf_u \lambda_2(u) \left( \int_M u^N \, dv_g \right)^{\frac{n}{n+2}} = -\infty.$$ 

The proof will be detailed in Subsection 5.0.3.

We will use a new functional

$$I_g(u) = \frac{\left( \int_M |L_g u|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}}{\left( \int_M u L_g u \, dv_g \right)^{\frac{n+2}{n}}}.$$ 

We study $\alpha := \inf I_g(u)$ where the infimum is taken over the functions $u \in H^{\frac{2n}{n+2}}(M)$ such that

$$\int_M u L_g u \, dv_g < 0,$$

and with the following constraint

$$\int_M |u|^{N-2} u \, dv_g = 0,$$

for any function $v \in \ker L_g$.

We will show that $\alpha$ is a conformal invariant. We obtain also that the infimum of this functional is attained by a function $u$. We set

$$v = |L_g u|^{\frac{2}{n+2}} L_g u,$$

and we will observe that $v$ has the following properties:

- $v$ is a solution of the equation

$$L_g v = \alpha' |v|^{N-2} v,$$

where $\alpha' < 0$ (i.e. has same sign than $\lambda_2$).
- $v$ has a changing sign.
- $v$ is of class $C^{3,\alpha}(M)$ ($\alpha < N - 2$).
5.0.4. **Conformal invariance of $\alpha$.** Let $\tilde{g} = \varphi^{\frac{4}{n-2}}g$ be a conformal metric, $\varphi$ a smooth positive function. Then

$$dv_{\tilde{g}} = \varphi^{\frac{2n}{n-2}}dv_g,$$

and

$$L_{\tilde{g}}u = \varphi^{-\frac{n+2}{n-2}}L_g(u\varphi),$$

for all functions $u$.

1. Remark that $I_{\tilde{g}}(u) = I_g(u\varphi)$.

$$I_{\tilde{g}}(u) = \left(\frac{\int_M |L_{\tilde{g}}u|^\frac{2n}{n+2}dv_{\tilde{g}}}{\int_M uL_{\tilde{g}}u dv_{\tilde{g}}}\right)^{\frac{n+2}{n}}$$

$$= \left(\frac{\int_M |\varphi|^{\frac{2n}{n+2}}|L_g(u\varphi)|^{\frac{2n}{n+2}}\varphi^{\frac{2n}{n-2}}dv_g}{\int_M u\varphi^{-\frac{n+2}{n-2}}L_g(u\varphi)\varphi^{\frac{2n}{n-2}}dv_g}\right)^\frac{n+2}{n}$$

$$= \left(\frac{\int_M |L_g(u\varphi)|^{\frac{2n}{n+2}}dv_g}{\int_M u\varphi L_g(u\varphi)dv_g}\right)^\frac{n+2}{n}$$

$$= I_g(u\varphi),$$

where we have used

$$\int_M uL_{\tilde{g}}u dv_{\tilde{g}} = \int_M u\varphi^{-\frac{n+2}{n-2}}L_g(u\varphi)\varphi^{\frac{2n}{n-2}}dv_g$$

$$= \int_M (u\varphi)L_g(u\varphi)dv_g.$$

2. Assume that for any $v \in \ker L_{\tilde{g}}$, we have

$$\int_M |u|^{N-2}uv dv_{\tilde{g}} = 0.$$

Then, for any $v' \in \ker L_g$, we obtain

$$\int_M |u\varphi|^{N-2}(u\varphi)v' dv_g = \int_M |u|^{N-2}u(v'\varphi^{-1}) dv_{\tilde{g}} = 0,$$

since

$$L_{\tilde{g}}(v'\varphi^{-1}) = \varphi^{-\frac{n+2}{n-2}}L_g(v') = 0,$$

i.e.

$$v'\varphi^{-1} \in \ker L_{\tilde{g}}.$$

5.0.5. **Proof of Proposition 5.1.** Assume that $\lambda_2(g) < 0$, and choose $u > 0$.

By Lemma 2.7, there exists two functions $v_1$ and $v_2$ solutions of the following equations

$$L_g v_1 = \lambda_1(u)|u|^{N-2}v_1,$$

and

$$L_g v_2 = \lambda_2(u)|u|^{N-2}v_2,$$

such that

$$\int_M |u|^{N-2}v_1v_2 dv_g = 0.$$

Let $v_\varepsilon$ the function defined in Section 4.2 and let $V = \{v_1, v_2\}$. For all $v \in V$, we get

$$\lim_{\varepsilon \to 0} \int_M v_\varepsilon^{N-2}v dv_g = 0.$$
Since $\lambda_1(u) < 0$ and $\lambda_2(u) < 0$, then for $\varepsilon$ sufficiently small, we have
\[
\lim_{\varepsilon \to 0} \left( \sup_{v \in V} \int_M \left( L_g(v)(v) \right) dv \right) = -\infty,
\]
hence
\[
\lim_{\varepsilon \to 0} \left( \inf_u \left( \int_M u^N \frac{dv}{g} \right)^{\frac{2}{n}} \right) = -\infty.
\]

5.0.6. The infimum of the functional $I_g$ is attained. Let $(u_m)_m$ be a minimizing sequence, i.e.,
\[
\lim_{m \to \infty} I_g(u_m) = \alpha,
\]
with
\[
\int_M |u_m|^{N-2} u_m v \, dv_g = 0, \quad \forall \, v \in \ker L_g.
\]
We can assume that
\[
\int_M u_m \, dv_g = -1. \quad (15)
\]
Then
\[
\alpha = \lim_{m \to \infty} \left( \int_M |L_g u_m|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}.
\]
Now we show that $(u_m)_m$ is a bounded sequence in $H^{\frac{2n}{n+2}}(M)$.
We proceed by contradiction and we assume that, up to a subsequence, $\lim \|u_m\|_{H^{\frac{2n}{n+2}}(M)} = +\infty$. Let
\[
v_m = \frac{u_m}{\|u_m\|_{H^{\frac{2n}{n+2}}(M)}}.
\]
Since $\|v_m\|_{H^{\frac{2n}{n+2}}(M)} = 1$, $(v_m)_m$ is a bounded sequence in $H^{\frac{2n}{n+2}}(M)$, and therefore there exists $v \in H^{\frac{2n}{n+2}}(M)$ such that after restriction to a subsequence
\[
v_m \rightharpoonup v \quad \text{in} \quad H^{\frac{2n}{n+2}}(M),
\]
\[
v_m \to v \quad \text{in} \quad L^2(M).
\]
By standard arguments, we get
\[
\left( \int_M |L_g v|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}} \leq \liminf_{m \to 0} \left( \int_M |L_g v_m|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}.
\]
This gives
\[
L_g v = 0,
\]
hence
\[
v \in \ker L_g.
\]
We have for all function $v' \in \ker L_g$,
\[
\int_M |v_m|^{N-2} v_m v' \, dv_g = \int_M \frac{|u_m|^{N-2} u_m v' \, dv_g}{\|u_m\|_{H^{\frac{2n}{n+2}}}} = 0.
\]
In particular for $v' = v$,
\[
\int_M |v_m|^{N-2} v_m v \, dv_g = 0 \quad \text{implies} \quad \int_M v^N \, dv_g,
\]
so

\[ v = 0. \quad (16) \]

According to the regularity Theorem 3.75 in \[ \text{Aub98} \], we have

\[ 1 = \|v_m\|_{H^2_{2n}} \leq C \left[ \|Lg v_m\|_{L^2_{2n}} + \|v_m\|_{L^2_{2n}} \right]. \]

Passing to the limit, we obtain

\[ \int_M v_{m}^{2n} \, dg \geq \frac{1}{C}, \]

which gives a contradiction. We deduce that \((u_m)_m\) is a bounded sequence in \(H^2_{2n}(M)\). Then, after restriction to a subsequence, there exists \(u \in H^2_{2n}(M)\) such that

\[ u_m \rightharpoonup u \text{ in } H^2_{2n}(M), \]
\[ u_m \to u \text{ in } L^2(M). \]

Further, we have

\[ \left( \int_M |Lg u|^{\frac{2n}{n+2}} \, dg \right)^{\frac{n+2}{n}} \leq \liminf_m \left( \int_M |Lg u_m|^{\frac{2n}{n+2}} \, dg \right)^{\frac{n+2}{n}}. \]

Moreover,

\[ \int_M uL_g u \, dg = \int_M |\nabla u|^2 \, dg - \int_M u^2 \, dg \]
\[ \leq \liminf_m \int_M |\nabla u_m|^2 \, dg - \int_M u_m^2 \, dg \]
\[ = \liminf_m \int_M u_m L_g u_m \, dg = -1. \quad (17) \]

Therefore

\[ \int_M uL_g u \, dg < 0, \]

and

\[ u \neq 0. \]

Finally, with (17)

\[ I_g(u) = \frac{\left( \int_M |Lg u|^{\frac{2n}{n+2}} \, dg \right)^{\frac{n+2}{n}}}{\left| \int_M uL_g u \, dg \right|} \]
\[ \leq \liminf_m \frac{\left( \int_M |Lg u_m|^{\frac{2n}{n+2}} \, dg \right)^{\frac{n+2}{n}}}{\left| \int_M u_m L_g u_m \, dg \right|} = \liminf_m I_g(u_m) = \alpha. \]

Hence the result is proved i.e. \( I_g(u) = \alpha \).

Euler equation

Notice that

\[ \int_M |u|^{N-2} u \, \nu' \, dg = 0, \text{ for any function } \nu' \in \ker L_g. \]

In particular, \( \alpha \neq 0 \). Remark also that

\[ \int_M uL_g u \, dg = -1. \]
Indeed, the relation \( \int_M uL_g u \, dv_g < -1 \) would imply that \( I_g(u) < \lim I_g(u_m) = \alpha \). We now write Euler equation of \( u \). Let \( \{u_1, \ldots, u_k\} \) be a base of \( \ker L_g \). By the Lagrange multipliers theorem, there exists real numbers \( \lambda_1, \ldots, \lambda_k \) for which, for all function \( \varphi \in C^\infty(M) \), we get

\[
\frac{d}{dt}_{|t=0} I_g(u + t\varphi) = \sum_i \lambda_i \frac{d}{dt}_{|t=0} g_i(u + t\varphi),
\]

where

\[
g_i(u) = \int_M |u|^{N-2} u_i \, dv_g.
\]

Setting \( a = \left( \int_M |L_g u|^{\frac{2N}{N-2}} \, dv_g \right)^{\frac{N-2}{N}} \), one checks

\[
2a \int_M |L_g u|^{\frac{2N}{N-2}} L_g u L_g \varphi \, dv_g + 2a \int_M \varphi L_g u \, dv_g = (N - 1) \sum_i \lambda_i \int_M |u|^{N-2} \varphi u_i \, dv_g.
\]

If \( \varphi \in \ker L_g \), this last equation implies that

\[
\sum_i \lambda_i \int_M |u|^{N-2} \varphi u_i \, dv_g = 0.
\]

Then, for \( \varphi = \sum_i \lambda_i u_i \in \ker L_g \), we have

\[
\int_M |u|^{N-2} \varphi^2 \, dv_g = 0.
\]

Therefore

\[
|u|^{N-2} \varphi^2 = 0 \implies |u|^{N-2} \varphi = 0 \implies \sum_i \lambda_i |u|^{N-2} u_i = 0.
\]

This gives, for any function \( \varphi \) (in \( \ker L_g \) or not), that

\[
\sum_i \lambda_i \int_M |u|^{N-2} \varphi u_i \, dv_g = 0.
\]

Then \( u \) verifies in the sense of distributions the following equation

\[
L_g \left( |L_g u|^{\frac{2N}{N+2}} L_g u \right) = \alpha' L_g u,
\]

where

\[
\alpha' = -\alpha \frac{N}{N+2} = -\int_M |L_g u|^{\frac{2N}{N+2}} \, dv_g.
\]

We set

\[
v = |L_g u|^{\frac{N}{N+2}} L_g u \in L^N(M),
\]

then

\[
|v| = |L_g u|^{1-\frac{N}{N+2}} = |L_g u|^{\frac{N-2}{N+2}}.
\]

Hence,

\[
L_g u = |v|^{N-2} v.
\]

Replacing each term by its value in Equation (18), we obtain

\[
L_g v = \alpha' |v|^{N-2} v.
\]

Regularity of \( v \)
We have \( u \in H_2^{\frac{2N}{N+2}}(M) \), then \( L_g u \in L^{\frac{2N}{N+2}}(M) \). Therefore

\[
v \in L^N(M),
\]

since \( |v|^N = |L_g u|^{\frac{2N}{N+2}} \). Moreover, in the sense of distributions

\[
L_g v = \alpha' |v|^{N-2} v,
\]

this implies that

\[
|L_g v| = |\alpha'||v|^{N-1},
\]
hence \( L_g v \in L^\frac{N}{n+2}(M) = L^\frac{2n}{N-2}(M) \), therefore \( v \in H^2_2(M) \subset H^1_1(M) \).

Using Lemma 3.1 of \([AH06]\), we get

\[
v \in L^{N+\varepsilon}(M),
\]

By a standard bootstrap argument, we show that

\[
v \in C^3,\alpha(M) (\alpha < N - 2).
\]

Calculating now \( I_g(v) \), using (19), we have

\[
I_g(v) = \frac{\left( \int_M |L_g v|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}}{\int_M v L_g v \, dv_g}
\]

\[
= \frac{\alpha^2 \left( \int_M |v|^{(N-1)\frac{n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}}{\int_M |v|^{N-2} \, dv_g}
\]

\[
= \frac{\alpha^{\frac{n}{n+2}} \left( \int_M |v|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}}{\int_M |v|^{\frac{N-2}{2}} \, dv_g}
\]

\[
= \alpha^{\frac{n}{n+2}} \left( \int_M |L_g u|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{2}{n}}
\]

\[
= \alpha^{\frac{n}{n+2}} \alpha^{\frac{2}{n+2}} = \alpha.
\]

The function \( v \) satisfies that, for any function \( v' \in \ker L_g \),

\[
\int_M |v|^{N-2} v v' \, dv_g = 0.
\]

Indeed,

\[
\int_M |v|^{N-2} v v' \, dv_g = \int_M L_g u \, v' \, dv_g = \int_M u \, L_g v' \, dv_g = 0.
\]

• \( v \) has changing sign

We proceed by contradiction and assume that \( v \geq 0 \). Since \( v \neq 0 \), we deduce from the maximum principle that \( v > 0 \). In addition, Equation (19) says that there exists an \( i \) such that \( \alpha' = \lambda_i(v) \). The only positive eigenfunctions are the ones associated to \( \lambda_1 \) and hence \( \alpha' = \lambda_1(v) \). By Proposition 2.4 there exists a function \( w \) solution of the following equation

\[
L_g w = \lambda_2(v)|v|^{N-2} w.
\]

\[
I_g(w) = \frac{\left( \int_M |L_g w|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}}{\int_M w L_g w \, dv_g}
\]

\[
= \frac{\lambda_2(v)^2 \left( \int_M |v|^{(N-2)\frac{n}{n+2}} |w|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n+2}{n}}}{\lambda_2(v) \int_M |v|^{N-2} w^2 \, dv_g}.
\]
By applying the Hölder inequality with $p = \frac{2n}{n-2}$ and $q = \frac{4n}{2n-4}$, we get

$$\int_M |v|^{(N-2)\times \frac{2n}{n-2}} |w|^{\frac{2n}{n-2}} \, dv_g = \int_M |v|^{(N-2)\times \frac{2n}{n-2}} \left| w \right|^{\frac{2n}{n-2}} \, |v|^{(N-2)\times \frac{2n}{n-2}} \, dv_g \leq \left( \int_M |v|^{N-2} \, w^2 \, dv_g \right)^{\frac{n}{n-2}} \left( \int_M |v|^{\frac{2n}{n-2} \times \frac{2n}{2n-4}} \, dv_g \right)^{\frac{2n}{2n-4}}.$$ 

Therefore

$$I_g(w) \leq |\lambda_2(v)| \left( \int_M |v|^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n}{n-2}} = |\lambda_1(v)| \left( \int_M |L_g u|^{\frac{2n}{n+2}} \, dv_g \right)^{\frac{n}{n+2}} = \alpha^{\frac{n-2}{2}} \alpha^{\frac{2n}{2n-4}} = \alpha,$$ 

since by assumption $\lambda_1(v) = \alpha' = \alpha$ when $\lambda_2(v)$ is a nodal solution of the equation

$$L_g v = \alpha' |v|^{N-2} v,$$

where $\alpha' < 0$. Setting

$$v' := |\alpha|^{\frac{n-2}{2}},$$

we obtain that $v'$ is a solution of the equation

$$L_g v' = \varepsilon |v'|^{N-2} v'$$

with $\varepsilon = -1$ sign (or $\lambda_2(g)$). This ends the proof of Theorem 4.1.



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