A Bestiary of Higher Dimensional Taub-NUT-AdS Spacetimes

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Preprint MIT-CTP-3053

November 9, 2017

Abstract

We present a menagerie of solutions to the vacuum Einstein equations in six, eight and ten dimensions. These solutions describe spacetimes which are either locally asymptotically adS or locally asymptotically flat, and which have non-trivial topology. We discuss the global structure of these solutions, and their relevance within the context of M-theory.

1 Introduction: The NUTs and bolts of higher dimensional spaces

As is well known, every odd dimensional sphere $S^{2k+1}$ may be expressed, via the ‘Hopf fibration’, as a $U(1)$ bundle over $\mathbb{CP}^k$:

$$S^{2k+1} \equiv U(1) \rightarrow \mathbb{CP}^k$$

What is perhaps less well known, is that this fibration allows us to place a canonical Lorentz metric (which is non-singular and time-orientable) on any odd dimensional sphere. Explicitly, one writes the Lorentz metric $g^L$ as

$$g^L = g^R - 2\hat{V} \otimes \hat{V},$$

where $g^R$ is the standard round Riemannian metric and $\hat{V}$ is the one-form dual to the unit vector field $V$ which is tangent to the Hopf fibration. If $Z^a$, $a = 1, \ldots k + 1$ are complex coordinates for $\mathbb{R}^{2k+2} \equiv \mathbb{C}^{k+1}$ then the Hopf fibration corresponds to the $SO(2) \subset SO(2k + 2)$ action:

$$Z^a \rightarrow \exp(it)Z^a$$
The case $k = 1$ should be familiar because it is encountered in the four dimensional Taub-NUT and Taub-NUT-adS solutions of Einstein’s equations, which play a central role in the construction of diverse and interesting M-theory configurations. Indeed, Taub-NUT is central to the supergravity realization of the D6-brane of type IIA string theory, and Taub-NUT-adS in four dimensions provided the first testbed for the adS/CFT correspondence in spacetimes where the asymptotic structure was only locally asymptotically adS [3], [4]. It is therefore natural to suppose that the higher dimensional generalizations of these spacetimes might provide us with a window on some interesting new corners of the M-theory moduli space.

Our construction of these higher-dimensional spacetimes is based on the construction of Bais and Batenberg [5], who found higher dimensional Riemannian metrics with the same topological structure. In [6] the Lorentzian sections for the Bais-Batenberg metrics were written down. While the general form of these solutions was constructed in [7], the global structure of these solutions has not been discussed. One purpose of this note is to discuss the global topology of these solutions (in particular the issue of spin structure), and also to discuss the behaviour of curvature invariants.

Following Bais and Batenberg, let $\{B, g^B, F^B\}$ be a $2k$-dimensional Einstein-Kähler manifold with Kähler form $F^B$ which obeys the Dirac quantization condition, i.e., it represents an integral class

$$\left[\frac{1}{2\pi} F^B\right] \in H_2(B; \mathbb{Z})$$

Then $F^B$ may be realized as the curvature of an $S^1$ bundle over $B$. Let

$$e^0 = dt + A$$

where $0 \leq t < 4\pi$ is a coordinate on the $S^1$ fibre and $A$ the ‘potential’ for the field $F^B$, so that

$$dA = F^B$$

Then we will say that a $(2k + 2)$-dimensional time-orientable Lorentzian metric is a Taub-NUT-type metric if it can be written in the following form:

$$F^{-1}(r)dr^2 + (r^2 + N^2)g^B - N^2 F(r)e^0 \otimes e^0$$ (1.1)

where $F(r)$ is some function of $r$ only and $N$ is the ‘NUT charge’. Clearly, $N$ measures the size of the $S^1$ fibre, and so in general it measures how ‘squashed’ the sphere is. We will often refer to $B$ as the ‘base space’, since it is the base space for the fibre bundle. In the usual way, a solution describes a ‘NUT’ if the fixed point set of $e^0$ (i.e., the points where $F(r)$ vanishes) is zero dimensional, and it describes a ‘bolt’ if the fixed point set is higher dimensional. Thus, if a metric of the form (1.1) solves the vacuum Einstein equations with a negative cosmological constant, then we will say that it is a Taub-NUT-adS (TN-adS) solution when the fixed point set of $e^0$ is zero dimensional, and that it is a Taub-bolt-adS (TB-adS) solution when the fixed point set is higher dimensional. Likewise, if it solves the vacuum equations with vanishing Ricci scalar then we will say it is a Taub-NUT (TN) or Taub-bolt (TB) metric accordingly.

In fact, in [6] it was shown that four dimensional Taub-NUT space is its own anti-particle, in the sense that there exists a diffeomorphism in the identity component Diff$_0$ of the diffeomorphism group the reverses the light cones everywhere. Surprisingly, there is a topological obstruction to the existence of such a diffeo for the six dimensional Taub-NUT solution.

We thank M. Taylor-Robinson for pointing this out.
2 Six Dimensional Solutions

2.1 Taub-NUT-adS solution

For simplicity, we will mainly consider the Euclidean sections for these metrics. The Lorentzian sections may be obtained by analytically continuing the coordinate $\tau$ and also the parameter $n$ (i.e., one replaces $n^2$ with $-N^2$). With this in mind, using $S^2 \times S^2$ as a base space, the Taub-NUT-adS solution has the following form

\[ ds^2 = F(r)(d\tau + 2n \cos \theta_1 d\phi_1 + 2n \cos \theta_2 d\phi_2)^2 + F(r)^{-1} dr^2 + (r^2 - n^2)(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2), \]  

(2.2)

where $F$ is given by

\[ F(r) = \frac{1}{3l^2(r^2 - n^2)^2} \left[ 3r^6 + (l^2 - 15n^2)r^4 - 3n^2(2l^2 - 15n^2)r^2 - 6mr^2 - 3n^4(l^2 - 5n^2) \right] \]  

(2.3)

In order for this to describe a NUT solution, it must be the case that $F(r = n) = 0$, so that all of the extra dimensions collapse to zero size at the fixed-point set of $\partial_\tau$. This can only happen when the mass parameter is fixed to be

\[ m_n = \frac{4n^3(6n^2 - l^2)}{3l^2} \]  

(2.4)

If the mass parameter is not fixed in this way, then $F(r_b > n) = 0$ and we will recover a bolt solution. We will say more about these solutions below. Fixing the mass at the value (2.4), we find that $F(r)$ is given as

\[ F(r) = \frac{(r - n)(3r^3 + 9mr^2 + (l^2 + 3n^2)r + 3n(l^2 - 5n^2))}{3(r + n)^2l^2} \]  

(2.5)

In order to avoid conical singularity the fiber has to close smoothly at $r = n$, and consequently $\beta F'(r = n) = 4\pi$ where $\beta = \Delta \tau$ is the period of $\tau$. This constraint then implies that

\[ \beta = 12\pi n \]  

(2.6)

However, even if we avoid a coordinate singularity in this way, there is still a curvature singularity at the location of the nut, $r = n$. Indeed, we find that the Riemann curvature invariant diverges as $R_{ijkl}R_{ijkl} \sim (r^2 - n^2)^{-2}$. As first discussed by Bais and Batenberg, we expect such a singularity whenever we take $B \neq \mathbb{C}P^k$. We will have more to say about such singularities later in the paper.

2.2 Taub-bolt-adS solution

In order to have a regular bolt at $r = r_b > n$, we must satisfy the following two conditions simultaneously:

(i) $F(r_b) = 0$

(ii) $F'(r_b) = \frac{1}{3l^2}$

Condition (ii) follows from the fact that we still want to avoid a conical singularity at the bolt, together with the fact that the period of $\tau$ will still be $12\pi n$. Obviously, in this case $r^2 - n^2$ will not vanish at $r = r_b$ and so the fixed point set of the Killing field $\partial_\tau$ will be a four-dimensional bolt with the topology $S^2 \times S^2$. If we impose condition (i), then we find that the mass is fixed:

\[ m = m_b = \frac{-1}{6l^2} \left[ 3r_b^5 + (l^2 - 15n^2)r_b^3 - 3n^2(2l^2 - 15n^2)r_b - 3n^4(l^2 - 5n^2)/r_b \right] \]  

(2.7)
From which we may deduce that

$$F'(r_b) = \frac{5(r^2 - n^2) + l^2}{rl^2} \quad (2.8)$$

If we now impose condition (ii), we recover $r_b$ as a function of $n$ and $l$:

$$r_b \pm = \frac{1}{30n} \left( l^2 \pm \sqrt{l^4 - 180n^2l^2 + 900n^4} \right) \quad (2.9)$$

To have a real value for $r_b$ the discriminant in the above equation must be non-negative. Adding the previous condition to the requirement that $r_b > n$ this leads to

$$n \leq \left( \frac{3 - 2\sqrt{2}}{30} \right) \frac{1}{l} \quad (2.10)$$

Unlike the NUT, this solution has no curvature singularities and is everywhere regular.

### 2.3 Taub-NUT and Taub-Bolt solutions

We may recover a NUT or bolt solution of the vacuum Einstein equations with a vanishing cosmological constant\(^3\) simply by taking the limit $l \rightarrow \infty$ in the metric (2.2), from which we recover the following form for $F$:

$$F(r) = \frac{r^4/3 - 2n^2r^2 - 2mr - n^4}{(r^2 - n^2)^2} \quad (2.11)$$

For the NUT case the mass is

$$m_n = -\frac{4n^3}{3} \quad (2.12)$$

In case of bolt solution, the mass has the form

$$m_b = \frac{1}{2}(r_b^3/3 - 2n^2r_b - n^4/r_b) \quad (2.13)$$

where $r_b$ in this case is

$$r_b = 3n \quad (2.14)$$

Notice that the only branch from the adS solution that contributes here (i.e. $l \rightarrow \infty$) is $r_b+$. Again there is a curvature singularity for the NUT (at $r = n$), but no singularity for the bolt.

Since this solution is Ricci flat in six dimensions, we may take the product of this metric with five-dimensional Minkowski space in order to obtain a solution of the IIA supergravity theory. The size of the $S^1$ fibre is then the size of the eleventh dimension, and therefore determines the string coupling constant.

### 2.4 Solutions with $B = \mathbb{CP}^2$

The following six dimensional solution generalizes the Taub-NUT metric found by Bais and Batenberg to include a negative cosmological constant. This metric is given by

$$ds^2 = F(r)(d\tau + A)^2 + F(r)^{-1}dr^2 + (r^2 - n^2)d\Sigma_2^2, \quad (2.15)$$

\(^3\)We think that solutions with $B \neq \mathbb{CP}^k$ in the limit $l \rightarrow \infty$ were not known before. However, it is worth pointing out that higher dimensional locally asymptotically flat solutions were studied in \[3\].
where $d\Sigma_2^2$ is the metric over $\mathbb{CP}^2$ which has the following form

$$d\Sigma_2^2 = \frac{du^2}{(1 + u^2/6)^2} + \frac{u^2}{4(1 + u^2/6)^2} (d\psi + \cos \theta d\phi)^2 + \frac{u^2}{4(1 + u^2/6)} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.16)$$

also $A$ is given by

$$A = \frac{u^2}{2(1 + u^2/6)} (d\psi + \cos \theta d\phi) \quad (2.17)$$

In this case $F(r)$ has exactly the same form as for the choice $B = S^2 \times S^2$. We will say more about that later. If we take the $l \to \infty$ limit of (2.3), in the NUT case, we recover the form for $F(r)$ discovered by Bais and Batenberg (2.11) [3]. In accordance with the Bais-Batenberg prediction, this solution is free of curvature singularities.

### 3 Eight Dimensional Solutions

#### 3.1 Taub-NUT-adS solution

The following metric is a result of a $U(1)$ fibration over $S^2 \times S^2 \times S^2$:

$$ds^2 = F(r)(dr + 2n \cos \theta_1 d\phi_1 + 2n \cos \theta_2 d\phi_2 + 2n \cos \theta_3 d\phi_3)^2 + F(r)^{-1} dr^2 + (r^2 - n^2)(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 + d\theta_3^2 + \sin^2 \theta_3 d\phi_3^2), \quad (3.18)$$

where $F(r)$ has the form

$$F(r) = \frac{5r^8 + (l^2 - 28n^2)r^6 + 5n^2(14n^2 - l^2)r^4 + 5(3l^2 - 28n^2)r^2 - 10ml^2 + 5n^6(l^2 - 7n^2)}{5l^2(r^2 - n^2)^3} \quad (3.19)$$

In order to have a NUT solution in this case, the mass parameter $m$ must be

$$m = m_n = \frac{8n^5(l^2 - 8n^2)}{5l^2} \quad (3.20)$$

Again by fixing the mass at the above value, we find that $F(r)$ is given as

$$F(r) = \frac{(r - n)(5r^4 + 20nr^3 + (l^2 + 22n^2)r^2 + (4nl^2 - 12n^3)r - 35n^4 + 5l^2n^2)}{5(r + n)^3l^2} \quad (3.21)$$

In order to avoid a conical singularity the fiber has to close smoothly at $r = n$, and consequently $\beta F'(r = n) = 4\pi$ where $\beta = \Delta \tau$ is the period of $\tau$. This constraint then implies that

$$\beta = 16\pi n \quad (3.22)$$

Again, even though we have managed to avoid a conical defect there is still a curvature singularity at the NUT, where we find that $R_{ijkl} R_{ijkl} \sim (r^2 - n^2)^{-2}$.

#### 3.2 Taub-bolt-adS solution

In order to have a regular bolt solution we must impose the following conditions simultaneously:

(i) $F(r_b) = 0$

(ii) $F'(r_b) = \frac{1}{4n}$ whence the mass is fixed to be

$$m = m_b = \frac{1}{10l^2} [r_b^7 + (l^2 - 28n^2)r_b^5 + 5n^2(14n^2 - l^2)r_b^3 + 5(3l^2 - 28n^2)r_b + 5n^6(l^2 - 7n^2)/r_b] \quad (3.23)$$
From this we have

$$F'(r_b) = \frac{7(r^2 - n^2) + l^2}{rl^2}$$

To satisfy (ii), \( r_b \) must be

$$r_{b\pm} = \frac{1}{56n} \left( l^2 \pm \sqrt{l^4 - 448n^2l^2 + 3136n^4} \right)$$

Requiring that \( r_b \) is real and greater than \( n \) implies that

$$n \leq \left( \frac{4 - \sqrt{15}}{56} \right) l$$

Again, this solution possesses no curvature singularities.

### 3.3 Taub-NUT and Taub Bolt solutions

Again, if we take the limit \( l \to \infty \) in the function (3.12) we will obtain a Ricci flat solution. Doing this we obtain the form for \( F \):

$$F(r) = \frac{r^6 - 5n^2r^4 + 15r^2 - 10mr + 5n^6}{5(r^2 - n^2)^3}$$

In this limit the NUT mass is

$$m_n = \frac{8n^5}{5}$$

One can obtain a bolt solution in this limit by requiring \( r_b > n \). The mass of this solution is

$$m_b = \frac{1}{10}(r_b^6 - 5n^2r_b^3 + 15r_b^2 + 5n^6/r_b),$$

where

$$r_b = 4n$$

There is a curvature singularity at the NUT, and the bolt is everywhere regular. Again, since this is Ricci flat we may take the product of this solution with three-dimensional Minkowski space to obtain a solution of eleven-dimensional SUGRA with vanishing four-form field strength (or equivalently IIA SUGRA with the dilaton VEV set by the size of the \( S^1 \) fibre).

### 3.4 Solutions with \( B = S^2 \times \mathbb{CP}^2 \)

The following metric is a result of a \( U(1) \) fibration over \( S^2 \times \mathbb{CP}^2 \):

$$ds^2 = F(r)(d\tau + A)^2 + F(r)^{-1} dr^2 + (r^2 - n^2)(d\Sigma_2^2 + d\Omega_2^2),$$

where \( d\Sigma_2^2 \) is again the metric over \( \mathbb{CP}^2 \) and \( d\Omega_2^2 \) is the metric over \( S^2 \)

$$d\Omega_2^2 = d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2,$$

also \( A \) is given by

$$A = \cos \theta_1 d\phi_1 + \frac{u^2}{2(1 + u^2 / 6)}(d\psi + \cos \theta d\phi).$$

Again the form of \( F(r) \) is exactly the same as for the choice \( B = S^2 \times S^2 \times S^2 \). As a result these solutions will share the same properties of the \( B = S^2 \times S^2 \times S^2 \) solutions. In this case we have also curvature singularity for both the NUT and NUT–adS cases (Again \( R^{ijkl} R_{ijkl} \sim (r^2 - n^2)^{-2} \)), but the bolt solutions are everywhere regular.
4 Ten Dimensional Solutions

4.1 Taub-NUT-adS solution

The following metric is a result of a $U(1)$ fibration over $S^2 \times S^2 \times S^2 \times S^2$:

$$ds^2 = F(r)(dr + 2n \cos \theta_1 d\phi_1 + 2n \cos \theta_2 d\phi_2 + 2n \cos \theta_3 d\phi_3 + 2n \cos \theta_4 d\phi_4)^2 + F'(r)^{-1}dr^2 + (r^2 - n^2)(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 + d\theta_3^2 + \sin^2 \theta_3 d\phi_3^2 + d\theta_4^2 + \sin^2 \theta_4 d\phi_4^2),$$

where $F(r)$ has the form

$$F(r) = \frac{1}{35l^2(r^2 - n^2)^2} \left[ 35r^{10} + 5(l^2 - 45n^2)r^8 + 14n^2(45n^2 - 2l^2)r^6 + 70n^4(l^2 - 15n^2)r^4 
+ 35n^6(45n^2 - 4l^2)r^2 - 70nr^2 + 35n^8(9n^2 - l^2) \right]$$

In order to have a NUT solution in this case, the mass parameter $m$ must be

$$m = m_n = \frac{64n^2(10n^2 - l^2)}{35l^2}$$

Again by fixing the mass at the above value, we find that $F(r)$ is given as

$$F(r) = \frac{(r - n)(35r^5 + 175nr^4 + (300n^2 + 5l^2)r^3 + (25n^2 + 100n^2)r^2 + (47n^2l^2 - 295n^4)r - 315n^5 + 35l^2n^3)}{35(r + n)l^2}$$

In order to avoid a conical singularity the fiber has to close smoothly at $r = n$, and consequently $\beta F'(r = n) = 4\pi$ where $\beta = \Delta \tau$ is the period of $\tau$. This constraint then implies that

$$\beta = 20\pi n$$

The curvature singularity at the NUT is signalled by the divergence $R^{ijkl}R_{ijkl} \sim (r^2 - n^2)^{-2}$.

4.2 Taub-bolt-adS solution

In this case the conditions we must satisfy in order to have a regular bolt solution are

(i) $F'(r_b) = 0$

(ii) $\frac{m_b}{3n} = \frac{1}{70l^2}[35r_b^9 + (5l^2 - 225n^2)r_b^7 + n^2(630n^2 - 28l^2)r_b^5$

$$+ n^4(70l^2 - 1050n^2)r_b^3 + n^6(1575n^2 - 140l^2)r_b + n^8(315n^2 - 35l^2)]$$

From this we compute

$$F'(r_b) = \frac{9(r^2 - n^2) + l^2}{rl^2}$$

To satisfy (ii), $r_b$ must be

$$r_b = \frac{1}{90n} \left( l^2 \pm \sqrt{l^4 - 900l^2 + 8100n^4} \right)$$

As we mentioned previously conditions on $r_b$ lead to the following constraint on $n$

$$n \leq \left( \frac{5 - 2\sqrt{6}}{90} \right)^{\frac{1}{2}} l$$
4.3 Taub-NUT and Taub-Bolt solutions

If we take the $l \rightarrow \infty$ limit of (4.18) we recover the following form for $F(r)$:

$$F(r) = \frac{5r^8 - 28n^2r^6 + 70n^4 - 140n^6r^4 - 70nmr - 35n^8}{35(r^2 - n^2)^4} \tag{4.43}$$

The NUT mass for this solution becomes

$$m_n = \frac{-64n^7}{35} \tag{4.44}$$

In the case of bolt solution, the mass of has the form

$$m_b = \frac{1}{70}[5r_b^7 - 28n^2r_b^5 + 70n^4r_b^3 - 140n^6r_b - 35n^8/r_b], \tag{4.45}$$

where

$$r_b = 5n \tag{4.46}$$

As expected the NUT is singular (at $r = n$) and the bolt is not. This solution is Ricci flat in ten dimensions and so we obtain a solution of eleven dimensional SUGRA by multiplying with a trivial time direction $-dt^2$. It would be interesting to have a deeper understanding of the ‘braney’ interpretation of these solutions.

4.4 Solutions with $\mathcal{B} = S^2 \times S^2 \times \mathbb{C}P^2$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$

The following metric is a result of a $U(1)$ fibration over $S^2 \times S^2 \times \mathbb{C}P^2$:

$$ds^2 = F(r)(d\tau + A)^2 + F(r)^{-1}dr^2 + (r^2 - n^2)^2(d\Sigma_2^2 + d\Omega_2^2 + d\Omega'_{2}^2), \tag{4.47}$$

where $d\Sigma_2^2$ are again the metrics over $\mathbb{C}P^2$, $d\Omega_2^2$ and $d\Omega'_{2}^2$ are the metrics over $S^2$. $A$ is given by

$$A = \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 + \frac{u^2}{2(1 + u^2/6)}(d\psi + \cos \theta d\phi). \tag{4.48}$$

The coordinates with subscript “1” and “2” are the coordinates of the spheres but those without subscript are the coordinates of $\mathbb{C}P^2$. One can construct another solution using a $U(1)$ fibration over $\mathbb{C}P^2 \times \mathbb{C}P^2$ instead:

$$ds^2 = F(r)(d\tau + A)^2 + F(r)^{-1}dr^2 + (r^2 - n^2)^2(d\Sigma_2^2 + d\Sigma'_{2}^2), \tag{4.49}$$

where $d\Sigma_2^2$ is another metric over $\mathbb{C}P^2$. $A$ in this case is given by

$$A = \frac{u'^2}{2(1 + u'^2/6)}(d\psi' + \cos \theta' d\phi') + \frac{u^2}{2(1 + u^2/6)}(d\psi + \cos \theta d\phi). \tag{4.50}$$

Again the form of $F(r)$ for both cases is exactly the same as for the choice $\mathcal{B} = S^2 \times S^2 \times S^2 \times S^2$ and the three cases share the same properties. In these three cases there is a curvature singularity for both the NUT and NUT-adS solutions (i.e. $R^{ijkl}R_{ijkl} \sim (r^2 - n^2)^{-2}$), but the Bolt and Bolt-adS solutions are regular everywhere.
5 Global structure: adS/CFT on exotic manifolds?

We have presented solutions of the vacuum Einstein equations with a negative cosmological constant, which are only locally asymptotic to adS space. We now briefly digress on the global structure of these solutions, and what we might hope to learn within the context of the adS/CFT correspondence. As is well known, the adS/CFT correspondence asserts that the propagators for a large N supersymmetric CFT on the boundary $\partial M$ of some locally asymptotically adS space $M$, are actually equivalent to supergravity partition functions in the bulk of $M$. In spite that most of the studied cases in the adS/CFT context were only anti–de–Sitter spacetimes in 2, 3, 4, 5, and 7 dimensions, it has been shown that the massive type IIA theory admits a wrapped-product solution of adS$_6$ with $S^4$ which turns out to be the near-horizon geometry of a semi-localized D$_4$/D$_8$ brane intersection [13, 14, 15, 16]. Therefore, it is interesting to look for higher dimensional solutions--especially in six dimensions--such as Taub-NUT and Taub-Bolt spacetimes and study their thermodynamics in the context of the adS/CFT correspondence. Leaving thermodynamics of these solutions for a future work, here we would like to discuss the global structure of these solutions.

For our solutions the boundary is generically a $U(1)$ fibre bundle over $B$

$$\partial M = S^1 \rightarrow B$$

with a metric

$$ds^2_{\text{boundary}} = g^B + \frac{n^2}{l^2} (d\tau + A) \otimes (d\tau + A) \quad (5.51)$$

where $A$ is potential for the field $F^B$. Thus, these solutions provide a window of opportunity for the study of conformal field theories on (Euclidean) spaces with exotic topologies. Indeed, as in [3], [4] we should be able to understand the thermodynamic phase structure of a CFT on one of these spaces by working out the corresponding phase structure for the supergravity solutions in the bulk. [1]

Of course, one might be concerned by the appearance of naked singularities in the bulk of these manifolds. However, we would argue that from the point of view of the adS/CFT duality it may be possible to resolve these singularities. Indeed, if we invoke the criterion of Gubser [9] then these singularities are ‘good’, in the sense that a finite temperature deformation will yield a non-singular solution (i.e., a bolt). In fact, we would assert that our analysis leads to a rather interesting prediction: A CFT on a Euclidean manifold of the form (5.51) must have some phase which is dual to a singular bulk supergravity solution, if $B \neq \mathbb{C}P^k$. Put another way, changing the topology of the boundary where the CFT is defined can have dramatic consequences for the bulk geometry. It would be interesting to know if it is possible to explicitly resolve the bulk singularity, through some mechanism such as that discussed in [10].

It is also worth commenting on the global topological structure of these solutions. Recall [3] that in four dimensions, Taub-bolt and Taub-bolt-adS do not admit a spin structure [5]. Basically, this is because the bolt itself is a two-cycle with odd self-intersection number, i.e., the second Stiefel-Whitney class is non-vanishing on the bolt [8].

A very similar thing happens for the solutions with $B = S^2 \times \ldots S^2 \times S^2$. For these solutions, it is straightforward to show that each $S^2$ factor generates an element of $H_2$, and that each of these elements has odd self-intersection. Thus, the Taub-bolt-(adS) spacetimes with this topology will not admit a spin structure. On the other hand, the Taub-NUT-adS solutions all admit spin structure. [5]

Skenderis [11] has also mentioned these solutions as non-trivial examples to which one may apply the construction of a regularised energy momentum tensor.

Of course, the spacetime may still admit a Spin$^C$ structure.
5.1 The General Form of Solutions in Higher Dimensions

As we have noticed the forms of $F(r)$ for different choices of the base space in certain dimension are exactly the same. This is not a coincidence, the reason is that the differential equation one can get for $F(r)$ does not depend on the Kähler metric or the potential $A$. As a result the function $F(r)$ depends only on the dimension, $2k$ of the base space. For any $2k$ we can integrate the Einstein equations to obtain the general expression for $F_{2k}(r)$:

$$F_{2k}(r) = \frac{r}{(r^2 - n^2)^k} \int r \left[ \frac{(s^2 - n^2)^k}{s^2} + \frac{2k + 1}{l^2} \frac{(s^2 - n^2)^{k+1}}{s^2} \right] ds.$$  (5.52)

Notice that the mass term here is the integration constant. As a result the form of $F(r)$ for $\mathcal{B} = \mathbb{C}P^3$ and $\mathcal{B} = \mathbb{C}P^4$ will be the same form of $F(r)$ which we found for solutions in eight and ten dimensions respectively (again, these solutions should be regarded as the generalization of the Bais-Batenberg solutions to negative cosmological constant in eight and ten dimensions).

6 Conclusion: Global structure of the Lorentzian sections

Here we have focussed on the Euclidean sections for the solutions. As we mentioned, we may recover the Lorentzian sections simply by analytically continuing $\tau$ and simultaneously sending $n^2$ to $-N^2$. We feel it is worth commenting on the causal structure of these solutions.

First of all, since we are in Lorentzian signature, roots of $g_{tt} = F(r)$ correspond to horizons in the spacetime. In fact, a NUT (or bolt) is no longer a zero dimensional (or two dimensional) fixed point set, but rather it is a chronology horizon in the spacetime. What this means is that we can move from a region containing no closed timelike curves (the region where the coordinate $r$ is timelike), to a region containing closed timelike curves (the region where the compact coordinate $t$ is timelike) by moving across this horizon. So the Lorentzian sections of these manifolds generically contain closed timelike curves in the bulk.

Furthermore, since $n^2 \rightarrow -N^2$ the metric component in front of the base space part of the metric is now

$$r^2 + N^2$$

This means that the horizon region is always non-singular (the curvature singularities only manifest themselves on the Euclidean section) and so we may extend through the horizons without obstruction. This also means of course that the ‘base space’ dimensions never collapse to zero size, and we do not need to impose any regularity condition in order to avoid a conical singularity. This means that the mass and NUT charge are no longer related (as in e.g. (2.4)), but they can be freely specified.

Acknowledgements

We would like to thank Gary Gibbons, Roberto Emparan, Al Shapere and Marika Taylor-Robinson for useful conversations, and especially RE, AS and GG for suggestions and comments on a preliminary draft of this paper. AC is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818. AW is supported by funds provided by ID No. DE-FG02-00ER45832 and the graduate school fellowship, University of Kentucky.

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