A modification of the Chen-Nester quasilocal expressions

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Abstract

Chen and Nester proposed four boundary expressions for the quasilocal quantities using the covariant Hamiltonian formalism. Based on these four expressions, there is a simple generalization that one can consider, so that a two parameter set of boundary expressions can be constructed. Using these modified expressions, a nice result for gravitational energy-momentum can be obtained in holonomic frames.

1 Introduction

For any field theory, Chen and Nester [1, 2] proposed certain boundary expressions for the quasilocal quantities. They found four special expressions. Two are unique under the property of their three requirements: the well defined requirement, the symplectic structure requirement and the covariant requirement. The other two correspond to boundary conditions imposed on a physically meaningful but non-covariant set of variables. Generalizing the latter, in the present paper we make some modification by a simple adjustment of the four boundary expressions. Using these four expressions as the basis, one can obtain a two parameter set of boundary expressions. The modified two parameter set of expressions for general relativity includes one case with a nice “positive energy” result in holonomic frames in small regions.

2 Chen-Nester’s 4 boundary expressions

Following [1], begin with a first order Lagrangian density (i.e., a 4-form)

\[ \mathcal{L} = dq \wedge p - \Lambda(q, p), \]  

where \( q \) and \( p \) are the canonical conjugate form fields, \( \Lambda \) is a potential, we suppose that \( q \) is a \( f \)-form and let \( \epsilon = (-1)^f \). The corresponding Hamiltonian 3-form is defined as

\[ \mathcal{H}(N) := \mathcal{L}_N q \wedge p - i_N \mathcal{L}. \]
Taking the interior product of the Lagrangian density \( i_N \mathcal{L} = i_N dq \wedge p - \epsilon dq \wedge i_N p - i_N \Lambda = \mathcal{L}_N q \wedge p - \epsilon i_N q \wedge dp - \epsilon dq \wedge i_N p - i_N \Lambda - d(i_N q \wedge p), \) (3)

where the Lie derivative or form component is \( \mathcal{L}_N := i_N d + di_N \). We see that the Hamiltonian 3-form (density) can be put in the form

\[ \mathcal{H}(N) = N^\mu \mathcal{H}_\mu + dB(N), \] (4)

where

\[ N^\mu \mathcal{H}_\mu = i_N q \wedge dp + \epsilon dq \wedge i_N p + i_N \Lambda, \] (5)

and the natural boundary term is

\[ B(N) = i_N q \wedge p. \] (6)

This is called the boundary expression, because when one integrates the Hamiltonian density over a finite region to get the Hamiltonian, the boundary term leads to an integral over the boundary of the region. However this boundary term is not unique, one can add to it whatever he likes without changing the Hamilton equations. Indeed this boundary term can be removed by introducing the new Hamiltonian 3-form as follows

\[ \mathcal{H}'(N) = \mathcal{H}(N) + d(-i_N q \wedge p) = N^\mu \mathcal{H}_\mu. \] (7)

The variation of the Hamiltonian (5) has the form (for details see (3))

\[ \delta \mathcal{H}'(N) = -i_N (\text{F.E.}) - \delta q \wedge \mathcal{L}_N p + \mathcal{L}_N q \wedge \delta p + dC(N), \] (8)

where the field equation term

\[ \text{F.E.} := \frac{\delta \mathcal{L}}{\delta q} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p, \] (9)

is proportional to the first order Lagrangian field equations and can be assumed to vanish. The boundary variation term is

\[ C(N) = -i_N q \wedge \delta p + \epsilon \delta q \wedge i_N p. \] (10)

This boundary variation term cannot be removed because it comes from \( \delta \mathcal{H}' \) directly. Boundary conditions are obtained by requiring that the boundary term in the variation of the Hamiltonian vanish. We have to add a suitable boundary term to the Hamiltonian 3-form (5)

\[ \mathcal{H}'(N) \rightarrow \mathcal{H}_k(N) = N^\mu \mathcal{H}_\mu + dB_k(N), \] (11)
to modify the variational boundary term, in order to get nice components like \( i_N(\delta q \wedge \Delta p) \) or \( i_N(\Delta q \wedge \delta p) \). In order to achieve such forms, there are just four simple boundary expressions that can be added, as can be seen by referring to (10). The variation of the four Hamiltonians including these different boundary expressions are

\[
\begin{align*}
\delta \mathcal{H}_q(N) &= K + di_N(\delta q \wedge \Delta p) = K + d(i_N \delta q \wedge \Delta p + \epsilon \delta q \wedge i_N \Delta p), \\
\delta \mathcal{H}_p(N) &= K - di_N(\Delta q \wedge \delta p) = K - d(i_N \Delta q \wedge \delta p + \epsilon \Delta q \wedge i_N \delta p), \\
\delta \mathcal{H}_d(N) &= K + d(-i_N \Delta q \wedge i_N \delta p + \epsilon \delta q \wedge i_N \Delta p), \\
\delta \mathcal{H}_c(N) &= K + d(i_N \delta q \wedge \Delta p - \epsilon \Delta q \wedge i_N \delta p),
\end{align*}
\]

where \( K \) is defined as

\[
K := -i_N(F.E.) - \delta q \wedge \mathcal{L}_N p + \mathcal{L}_N q \wedge \delta p.
\]

Thus we recover the Chen-Nester 4 boundary expressions

\[
\begin{align*}
\mathcal{B}_q(N) &= i_N q \wedge \Delta p - \epsilon \Delta q \wedge i_N \overline{p}, \\
\mathcal{B}_p(N) &= i_N \overline{q} \wedge \Delta p - \epsilon \Delta q \wedge i_N p, \\
\mathcal{B}_d(N) &= i_N \overline{q} \wedge \Delta p - \epsilon \Delta q \wedge i_N \overline{p}, \\
\mathcal{B}_c(N) &= i_N q \wedge \Delta p - \epsilon \Delta q \wedge i_N p,
\end{align*}
\]

(refer to [2, 3] for an explanation of the terminology) where \( \mathcal{B}_d \) stands for \( \mathcal{B}_{\text{dynamics}} \), \( \mathcal{B}_c \) for \( \mathcal{B}_{\text{constraint}} \), here \( \Delta q = q - \overline{q}, \Delta p = p - \overline{p}, \overline{q} \) and \( \overline{p} \) are the background reference values. Note that the above four boundary expressions can be written in a compact form

\[
\mathcal{B}_{k_1,k_2}(N) = \mathcal{B}_p(N) + k_1 i_N \Delta q \wedge \Delta p + \epsilon k_2 \Delta q \wedge i_N \Delta p,
\]

where \( (k_1, k_2) = (0, 0), (0, 1), (1, 0) \) and \( (1, 1) \). In detail

\[
\mathcal{B}_{0,0} = \mathcal{B}_p, \quad \mathcal{B}_{0,1} = \mathcal{B}_d, \quad \mathcal{B}_{1,0} = \mathcal{B}_c, \quad \mathcal{B}_{1,1} = \mathcal{B}_q.
\]

The boundary conditions associated with \( \mathcal{B}_q \) and \( \mathcal{B}_p \) are the simplest. For the case of \( \mathcal{B}_q \), we want the variation boundary term to satisfy \( i_N(\delta q \wedge \Delta p) = 0 \). There are two ways to achieve this requirement. First control \( q \), then \( \delta q = 0 \). The second is to freely vary \( q \) and then \( \delta q \) can be anything which implies that we want \( \Delta p = 0 \). This is the “natural boundary condition”, it forces \( p = \overline{p} \), where \( \overline{p} \) is the background reference. Similarly for the variation boundary term \( i_N(\Delta q \wedge \delta p) = 0 \).

A simple example of the idea is the one dimensional spring with a mass block, control the external force \( \delta F \) and the length of the spring \( \Delta x \) changes or response.
Likewise, control the position of the mass block by $\delta x$, there must be a force $\Delta F$ from outside the system as response. However, the variation boundary expressions from $B_d$ and $B_c$ are more complicated, because they both have mixed control-$q$, $p$-response and control-$p$, $q$-response type terms at the same time.

3 The modification of the Chen-Nester 4 boundary expressions

Consider equation (12), as long as the variational boundary terms pieces $i_N \delta q \wedge \Delta p$ and $\delta q \wedge i_N \Delta p$ vanish. It seems that there is no restriction on the magnitude. In other words, it can be any real number that one can impose. Consider the variation of the boundary expression in a simple way

$$\delta \tilde{H}_q = K + d(k_1 i_N \delta q \wedge \Delta p + \epsilon k_2 \delta q \wedge i_N \Delta p).$$

However, one could construct other similar forms proceeding from $\delta H_p$, $\delta H_d$ or $\delta H_c$. The general form of the variational Hamiltonian of the four different expressions, the modified equations (12) to (15) in one general form all together is

$$\delta \tilde{H} = K + b_1 i_N \delta q \wedge \Delta p + b_2 \delta q \wedge i_N \Delta p + b_3 i_N \Delta p \wedge \delta q + b_4 \Delta p \wedge i_N \delta q,$$

where $b_1$ to $b_4$ are real numbers (it will turn out that they cannot be all independent). All of these extra terms can be obtained by adding a suitable multiple of $i_N \Delta q \wedge \Delta p$ or $\Delta q \wedge i_N \Delta p$. Actually one can rewrite the modified Chen-Nester 4 boundary expressions in a simple way by adding $c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p$ for the individual boundary expressions, where $c_1, c_2 \in \mathbb{R}$. In detail, the 4 equivalent forms are

$$\tilde{B}_q = i_N q \wedge \Delta p - \epsilon \Delta q \wedge i_N \Delta p + c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p,$$
$$\tilde{B}_p = i_N q \wedge \Delta p - \epsilon \Delta q \wedge i_N \Delta p + c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p,$$
$$\tilde{B}_d = i_N q \wedge \Delta p - \epsilon \Delta q \wedge i_N \Delta p + c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p,$$
$$\tilde{B}_c = i_N q \wedge \Delta p - \epsilon \Delta q \wedge i_N \Delta p + c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p.$$

They are equivalent because a suitable choice for $c_1, c_2$ in (25) can reproduce (26) to (28) with new values for $c_1, c_2$. For example in (26), when $c_1 \rightarrow (c_1 - 1)$ gives (27). In short, one can rewrite the above four expressions in a compact form

$$B_{c_1,c_2}(N) = B_p(N) + c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p.$$
If the modification can be adjusted by the above 2 parameters of \( c_1 \) and \( c_2 \), then what is the role they are playing? Let’s consider the asymptotic order magnitude of the variables \( q, \overline{q}, p \) and \( \overline{p} \) as follows

\[
q \approx O(1) + O\left(\frac{1}{r}\right), \quad \overline{q} \approx O(1), \quad p \approx O\left(\frac{1}{r^2}\right), \quad \overline{p} \approx 0.
\]  
(30)

Since

\[
\Delta q \wedge \Delta p \approx O\left(\frac{1}{r^3}\right),
\]  
(31)
then the terms with the coefficients of \( c_1 \) and \( c_2 \) asymptotically have \( O(1/r^3) \) fall off. Consequently they do not change the dominate main asymptotic value. This shows that the modified expression of \( c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p \) adjusts the higher order terms; but not the lowest order terms, the dominate contribution. Indeed the modified expressions are really doing a modified job. Consider the variation of the Hamiltonians

\[
\delta \tilde{\mathcal{H}}_q = K + d[(c_1 + 1)i_N \delta q \wedge \Delta p + \epsilon(c_2 + 1)\delta q \wedge i_N \Delta p + c_1 i_N \Delta q \wedge \delta p + \epsilon c_2 \Delta q \wedge i_N \delta p],
\]  
(32)

\[
\delta \tilde{\mathcal{H}}_p = K + d[c_1 i_N \delta q \wedge \Delta p, + \epsilon c_2 \delta q \wedge i_N \Delta p + (c_1 - 1)i_N \Delta q \wedge \delta p + \epsilon(c_2 - 1)\Delta q \wedge i_N \delta p],
\]  
(33)

\[
\delta \tilde{\mathcal{H}}_d = K + d[c_1 i_N \delta q \wedge \Delta p + \epsilon(c_2 + 1)\delta q \wedge i_N \Delta p + (c_1 - 1)i_N \Delta q \wedge \delta p + \epsilon c_2 \Delta q \wedge i_N \delta p],
\]  
(34)

\[
\delta \tilde{\mathcal{H}}_c = K + d[(c_1 + 1)i_N \delta q \wedge \Delta p + \epsilon c_2 \delta q \wedge i_N \Delta p + c_1 i_N \Delta q \wedge \delta p + \epsilon(c_2 - 1)\Delta q \wedge i_N \delta p].
\]  
(35)

Basically all of these four boundary expressions both have terms of the form control-\( q \), \( p \)-response and control-\( p \), \( q \)-response. The boundary conditions associated with these four expressions are somewhat like the expressions of \( \mathcal{B}_d \) and \( \mathcal{B}_c \).

4 On the physical interpretation of the modified Chen-Nester’s 4 boundary expressions

The modified expressions look a little strange, is this just a mathematical completeness game or they do have some real physical contribution? What is the idea here? Does it have any application? In the Sturm-Liouville theory of 2nd order linear differential equation. The necessary condition for self-adjoint is (see e.g. [4])

\[
p(y_1 y_2' - y_2 y_1') \big|_a^b = 0,
\]  
(36)
if \( p(a) \neq 0 \neq p(b) \) one needs to impose boundary conditions on the solutions. The simplest is Dirichlet \( y(a) = 0 = y(b) \) or Neumann \( y'(a) = 0 = y'(b) \). But a more
general choice is to take some linear combination \( \alpha y + \beta y' \) to vanish on the boundary. This is an example of the sort of thing we are considering. Such a boundary condition is appropriate for some practical applications such as an elastic cord which is held and shook. Then the slope at the end point will be proportional to the end point amplitude.

But when we consider field theory, the idea seems not very suitable. For example for an electrostatic system, such as a parallel plate capacitor, we know how to physically impose Dirichlet or Neumann boundary conditions, by fixing the voltage or the charge density, but it seems not at all simple to construct a device to fix some linear combination of the voltage and charge density, and it is hard to imagine why one would want to so such a thing.

Turning from the well understood electrodynamics boundary problem to the not yet so well understood gravitational field boundary value problem, we might at first think that the new combined boundary condition expressions would have little use. However a little more thought suggests a different view. Unlike the electrodynamic case we do not know physically how to arrange for Dirichlet or Neumann boundary conditions. When I imagine trying to change the metric on the boundary by moving a mass outside, it seems that there will be associated changes in the normal derivative of the metric. Maybe it is physically easier to fix a combination of the metric and its derivatives. In any case, we found an interesting analytic application for our idea to gravitational energy.

5 Gravitation application of the new quasilocal expressions

For an application of the new modified Chen-Nester expressions consider gravity theory. With \( \kappa = 8\pi G \), let

\[
q \rightarrow \Gamma^\alpha_\beta, \quad p \rightarrow \frac{1}{2\kappa} \eta_\alpha^\beta.
\]  

(37)

Here \( \Gamma \) is the connection one form and \( \eta^{\alpha\beta} = *(\theta^\alpha \wedge \theta^\beta) \) where \( \theta^\alpha \) is the coframe. For gravity following [2, 3], rewrite (29) as

\[
2\kappa B_{c_1,c_2}(N) = 2\kappa B_p(N) + c_1 i_N \Delta \Gamma^\alpha_\beta \wedge \Delta \eta_\alpha^\beta - c_2 \Delta \Gamma^\alpha_\beta \wedge i_N \Delta \eta_\alpha^\beta,
\]

(38)
where
\[ 2\kappa B_p(N) = \Delta \Gamma^\alpha_{\beta} \wedge i_N \eta^\beta_\alpha + \overline{D}_\beta N^\alpha \Delta \eta^\beta_\alpha. \] (39)

Since we are interested in the energy-momentum components, the \(DN\) terms in (39) can be ignored (i.e. we can presume \(DN = 0\)). Rewriting (39)
\[ 2\kappa B_p(N) = \Delta \Gamma^\alpha_{\beta} \wedge i_N \eta^\beta_\alpha. \] (40)

Using (40), rewrite (38)
\[ 2\kappa B_{c_1,c_2}(N) = \Delta \Gamma^\alpha_{\beta} \wedge i_N \eta^\beta_\alpha + c_1 i_N \Delta \Gamma^\alpha_{\beta} \wedge \Delta \eta^\beta_\alpha - c_2 \Delta \Gamma^\alpha_{\beta} \wedge i_N \Delta \eta^\beta_\alpha. \] (41)

For all \(c_1\) and \(c_2\), this expression will give good values at the spatial infinity limit. We can find preferred values for \(c_1\) and \(c_2\) by considering the small region limit.

Consider first using Riemann normal coordinates and the adapted orthonormal frames. As mentioned in [7], if we use orthonormal frames and the appropriate reference \((\Gamma^\alpha_{\beta} = 0)\) with \(N^\alpha = \text{constant within matter}\) we get the expected material energy-momentum tensor. In vacuum to lowest non-vanishing order we get the Bel-Robinson tensor with a positive coefficient only for the case \(c_1 = c_2 = 0\). As explained in more detail in [6], we should get Bel-Robinson tensor because it has positive energy.

On the other hand we can consider the same formal expression, but using holonomic variables and reference so that \(\Gamma^\alpha_{\beta} = 0, N^\alpha = \text{constant in the holonomic frame}\). Again to lowest order inside matter we get the desired material limit. In vacuum however, the basic term \(B_p(N)\) (it is just the quasilocal expression of KBLB [8] in the limit it reduces to the Freud superpotential which gives the Einstein pseudotensor) gives the value \(\frac{1}{18}(4B_{a_{\beta}\lambda\sigma} - S_{a_{\beta}\lambda\sigma})x^\lambda x^\sigma\) as has long been known [9]. This is not so good because the vacuum energy is not positive. Can some choices of \(c_1, c_2\) save the day? The answer is yes. Rewrite (40)
\[ 2\kappa B_p(N) = \Gamma^\alpha_{\beta} \wedge i_N \eta^\beta_\alpha = -\frac{1}{2} \sqrt{-g} N^\alpha U_\alpha^{[\mu\nu]} \epsilon_{\mu\nu}, \] (42)
where the Freud superpotential is
\[ U_\alpha^{[\mu\nu]} = -\sqrt{-g} g^{\beta\sigma} \Gamma^\tau_{\lambda\beta\delta} \lambda^{\mu\nu}. \] (43)
Consider the \(c_1\) term in (41)
\[ i_N \Gamma^\alpha_{\beta} \wedge \Delta \eta^\beta_\alpha = -\frac{1}{2} \sqrt{-g} N^\alpha (h^{\mu\pi} \Gamma^{\nu}_{\alpha\pi} - h^{\nu\pi} \Gamma^{\mu}_{\alpha\pi}) \epsilon_{\mu\nu}, \] (44)
and the $c_2$ term

$$\Gamma^\alpha_\beta \wedge i_N \Delta \eta^\beta_\alpha = -\frac{1}{2} \sqrt{-g} N^\alpha \left\{ \begin{array}{l}
(h^{\mu \pi} \Gamma^\nu_{\alpha \pi} - h^{\nu \pi} \Gamma^\mu_{\alpha \pi}) \\
- (\delta^\mu_\alpha h^{\rho \pi} \Gamma^\nu_{\pi \rho} - \delta^\nu_\alpha h^{\rho \pi} \Gamma^\mu_{\pi \rho}) \\
+ (\delta^\mu_\alpha \Gamma^\lambda_{\lambda \nu} - \delta^\nu_\alpha \Gamma^\lambda_{\lambda \mu})
\end{array} \right\} \epsilon_{\mu \nu}. \quad (45)$$

Therefore (41) can be rewritten as

$$2\kappa B_{c_1, c_2}(N) = -\frac{1}{2} N^\alpha U^\beta_\alpha \left\{ \begin{array}{l}
(h^{\mu \pi} \Gamma^\nu_{\alpha \pi} - h^{\nu \pi} \Gamma^\mu_{\alpha \pi}) \\
- (\delta^\mu_\alpha h^{\rho \pi} \Gamma^\nu_{\pi \rho} - \delta^\nu_\alpha h^{\rho \pi} \Gamma^\mu_{\pi \rho})
\end{array} \right\} \epsilon_{\mu \nu} + \frac{1}{2} c_1 \sqrt{-g} N^\alpha (h^{\mu \pi} \Gamma^\nu_{\alpha \pi} - h^{\nu \pi} \Gamma^\mu_{\alpha \pi}) \epsilon_{\mu \nu}
$$

$$- \frac{1}{2} c_2 \sqrt{-g} N^\alpha \left\{ \begin{array}{l}
(h^{\mu \pi} \Gamma^\nu_{\alpha \pi} - h^{\nu \pi} \Gamma^\mu_{\alpha \pi}) \\
- (\delta^\mu_\alpha h^{\rho \pi} \Gamma^\nu_{\pi \rho} - \delta^\nu_\alpha h^{\rho \pi} \Gamma^\mu_{\pi \rho})
\end{array} \right\} \epsilon_{\mu \nu}. \quad (46)$$

From now on, the weighting factor $\sqrt{-g}$ will be dropped for convenience. Using (46), the pseudotensor can be obtained as

$$t^\alpha_\mu = \partial^\nu \left\{ -U^\mu_{\alpha \nu} + c_1 (h^{\mu \pi} \Gamma^\nu_{\alpha \pi} - h^{\nu \pi} \Gamma^\mu_{\alpha \pi}) \\
- c_2 (h^{\mu \pi} \Gamma^\nu_{\alpha \pi} - h^{\nu \pi} \Gamma^\mu_{\alpha \pi}) - \delta^\mu_\alpha h^{\rho \pi} \Gamma^\nu_{\pi \rho} + \delta^\nu_\alpha h^{\rho \pi} \Gamma^\mu_{\pi \rho} + \delta^\mu_\alpha \Gamma^\lambda_{\lambda \nu} - \delta^\nu_\alpha \Gamma^\lambda_{\lambda \mu}
\right\}. \quad (47)$$

Inside matter at the origin a short calculation gives

$$2\kappa t^\alpha_\beta(0) = 2G^\alpha_\beta(0) = 2\kappa T^\alpha_\beta(0). \quad (48)$$

Just what we expect from the equivalence principle. In detail the energy density inside matter at the origin, the zeroth order term is

$$\mathcal{E} = -t^0_0(0) = -\frac{G^0_0(0)}{\kappa} = -T^0_0(0) = \rho, \quad (49)$$

where $\kappa = 8\pi G$ and $\rho$ is the mass-energy density. The momentum density is

$$\mathcal{P}_k = -t^0_k = -\frac{G^0_k}{\kappa} = -T^0_k. \quad (50)$$

At the origin in vacuum, the zeroth and the first derivative are

$$t^\alpha_\beta(0) = 0 = \partial^\nu t^\alpha_\beta(0). \quad (51)$$

The first non-vanishing contribution appears at 2nd order. The non-vanishing second derivatives in vacuum at the origin, after a little lengthy computation, are

$$\partial^2_{\mu \nu} t^\alpha_\beta(c_1, c_2) = \frac{1}{9} \left\{ (4 + c_1 - 5c_2) B_{\alpha \beta \mu \nu} - (1 - 2c_1 + c_2) S_{\alpha \beta \mu \nu} + (c_1 - 3c_2) K_{\alpha \beta \mu \nu} \right\}. \quad (52)$$
The Bel-Robinson tensor has many nice properties \cite{5} including energy positivity \cite{6}. Consider (52), when \((c_1, c_2) = (0, 0), (0, 1), (1, 0)\) and \((1, 1)\), they are classified as the original Chen-Nester four boundary expressions. The results from \cite{7} are

\[
\begin{align*}
\partial^2_{\mu\nu} t_{\alpha\beta}(0, 0) &= \frac{1}{9} (4B_{\alpha\beta\mu\nu} - S_{\alpha\beta\mu\nu}), \\
\partial^2_{\mu\nu} t_{\alpha\beta}(0, 1) &= -\frac{1}{9} (B_{\alpha\beta\mu\nu} + 2S_{\alpha\beta\mu\nu} + 3K_{\alpha\beta\mu\nu}), \\
\partial^2_{\mu\nu} t_{\alpha\beta}(1, 0) &= \frac{1}{9} (5B_{\alpha\beta\mu\nu} + S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu}), \\
\partial^2_{\mu\nu} t_{\alpha\beta}(1, 1) &= -\frac{2}{9} K_{\alpha\beta\mu\nu}.
\end{align*}
\]  

None are of the desired pure Bel-Robinson form. The general form of the Taylor expansion for the Chen-Nester four expressions in compact form is

\[
t_{\alpha}(k_1, k_2) = 2G_{\alpha} + \frac{1}{18} \left\{ (4 + k_1 - 5k_2)B_{\alpha}^{\beta \xi \kappa} - (1 - 2k_1 + k_2)S_{\alpha}^{\beta \xi \kappa} + (k_1 - 3k_2)K_{\alpha}^{\beta \xi \kappa} \right\} x^\xi x^\kappa + O(\text{Ricci}, x) + O(x^3).
\]

Consider (52) again, we want the coefficients of \(S_{\alpha\beta\mu\nu}\) and \(K_{\alpha\beta\mu\nu}\) to vanish. Taking \((c_1, c_2) = \left(\frac{3}{5}, \frac{1}{5}\right)\) gives

\[
\partial^2_{\mu\nu} t_{\alpha\beta} \left(\frac{3}{5}, \frac{1}{5}\right) = \frac{2}{5} B_{\alpha\beta\mu\nu}.
\]

This result is good, because it only contains the Bel-Robinson tensor. Consequently the small region energy will be positive. The general form of the Taylor expansion of the expression is

\[
t_{\alpha} \left(\frac{3}{5}, \frac{1}{5}\right) = 2G_{\alpha} + \frac{1}{5} B_{\alpha}^{\beta \xi \kappa} x^\xi x^\kappa + O(\text{Ricci}, x) + O(x^3).
\]

The corresponding four momentum within a small coordinate sphere is

\[
P_\mu \left(\frac{3}{5}, \frac{1}{5}\right) = \frac{1}{2\kappa} \int \frac{1}{5} B^0_{\mu \xi \kappa} x^\xi x^\kappa d^3 x = -\frac{r^5}{300G} B_{\mu\nu\xi\kappa} = -\frac{r^5}{300G} B_{\mu000},
\]

where the Bel-Robinson tensor \(B_{\alpha\beta\mu\nu}\), tensors \(S_{\alpha\beta\mu\nu}\) and \(K_{\alpha\beta\mu\nu}\) are defined as follows

\[
\begin{align*}
B_{\alpha\beta\mu\nu} &:= R_{\alpha \lambda \mu \sigma} R_{\beta \lambda \nu \sigma} - \frac{1}{8} g_{\alpha \beta} g_{\mu \nu} R_{\lambda \sigma \rho \tau} R_{\lambda \sigma \rho \tau}, \\
S_{\alpha\beta\mu\nu} &:= R_{\alpha \mu \lambda \sigma} R_{\beta \nu \lambda \sigma} + \frac{1}{4} g_{\alpha \beta} g_{\mu \nu} R_{\lambda \sigma \rho \tau} R_{\lambda \sigma \rho \tau}, \\
K_{\alpha\beta\mu\nu} &:= R_{\alpha \lambda \beta \sigma} R_{\mu \lambda \nu \sigma} - \frac{3}{8} g_{\alpha \beta} g_{\mu \nu} R_{\lambda \sigma \rho \tau} R_{\lambda \sigma \rho \tau}.
\end{align*}
\]
where \( B_{\mu 0}^I = B_{\mu 000} \), \( P_\mu = (-E, P_i) \) and energy \( E > 0 \). Alternatively the four momentum can be written in terms of the electric and magnetic parts as follows

\[
P_\mu \left( \frac{3}{5}, \frac{1}{5} \right) = -\frac{r^5}{300G} (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{c}^{\ ab} E_{ad}H_{bd}), \quad (64)
\]

where \( B_{\mu 000} = (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{c}^{\ ab} E_{ad}H_{bd}) \), here \( E_{ab} = R_{0a0b}, H_{ab} = \frac{1}{2} C_{0amn} \epsilon_{b}^{\ mn} \).

In an earlier work we have constructed a 10 parameter class of new superpotentials that give rise to pseudotensors which have positive Bel-Robinson small vacuum limit \[10, 11\]. But they all seemed very artificial. Our \( c_1, c_2 \) expressions are special cases. However they, in contrast, have clear meanings in terms of the boundary conditions in the Hamiltonian formalism. Here we found a simple specific value for \( c_1, c_2 \) which produces the desired positive vacuum result. There is one unique holonomic Hamiltonian boundary expression with this property. It is not one of the 4 previously considered quasilocal expression but rather a certain specific combination which corresponds to fixing a special combination of Dirichlet and Neumann type boundary conditions. Obtained from the boundary term in the variation of the Hamiltonian, namely

\[
\oint \left( \frac{3}{5} i_N \delta \Gamma^\alpha_\beta \wedge \Delta \eta_\alpha^\beta - \frac{1}{5} \delta \Gamma^\alpha_\beta \wedge i_N \Delta \eta_\alpha^\beta - \frac{2}{5} i_N \Delta \Gamma^\alpha_\beta \wedge \delta \eta_\alpha^\beta + \frac{4}{5} \Delta \Gamma^\alpha_\beta \wedge i_N \delta \eta_\alpha^\beta \right).
\]

\[
(65)
\]

6 Conclusion

Chen and Nester proposed the four boundary expressions for the quasilocal quantities by using the covariant Hamiltonian formalism. Two of them are unique and correspond to imposing boundary conditions on a covariant combination, the other two are non-covariant. Generalizing the latter two cases, one can make some modification by a simple adjustment of the four boundary expressions. The basis is using their four expressions. Applying the modified expressions to gravity, we found using holonomic frames a nice Bel-Robinson result in vacuum for certain specific \( c_1, c_2 \).

Acknowledgments

This work was supported by grant from National Science Council of the Republic of China under the grant number NSC 94-2112-M-008-038.
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