Abstract—In this paper we propose distributed dynamic controllers for sharing both frequency containment and restoration reserves of asynchronous AC systems connected through a multi-terminal HVDC (MTDC) grid. The communication structure of the controller is distributed in the sense that only local and neighboring state information is needed, rather than the complete state. We derive sufficient stability conditions, which guarantee that the AC frequencies converge to the nominal frequency. Simultaneously, a global quadratic power generation cost function is minimized. The proposed controller also regulates the voltages of the MTDC grid, asymptotically minimizing a quadratic cost function of the deviations from the nominal DC voltages. The results are valid for distributed cable models of the HVDC grid (e.g. π-links), as well as AC systems of arbitrary number of synchronous machines, each modeled by the swing equation. We also propose a decentralized, communication-free version of the controller. The proposed controllers are tested on a high-order dynamic model of a power system consisting of asynchronous AC grids, modelled as IEEE 14 bus networks, connected through a six-terminal HVDC grid. The performance of the controller is successfully evaluated through simulation.

I. INTRODUCTION

Power transmission over long distances with low losses is an important challenge as the distances between generation and consumption increase. As the share of fluctuating renewables rises, so does the need to balance generation and consumption mismatches, often over large geographical areas, for which high-voltage direct current (HVDC) power transmission is a commonly used technology. In addition to offering lower cost solutions for longer overhead lines and cable transmission [1], the controllability of the HVDC converters offers flexibility and means to mitigate problems due to power fluctuations from renewables. Increased use of HVDC technologies for electrical power transmission suggests that future HVDC transmission systems are likely to consist of multiple terminals connected by HVDC transmission lines, so called multi-terminal HVDC (MTDC) systems [2].

The fast operation of the DC converters enables frequency regulation of one of the AC grids connected to the HVDC link. One such example is the frequency regulation of the island of Gotland in Sweden, which is connected to the much stronger Nordic grid through an HVDC cable [3]. By connecting multiple AC grids by an MTDC system, enables frequency regulation of one or more of the AC grids connected. Traditional AC frequency controllers and HVDC voltage controllers do however not take advantage which the increased connectivity of the grids brings. Rather than sharing control reserves, each AC area is responsible for maintaining its own frequency in an acceptable range [4], which reduces the need for frequency regulation reserves in the individual AC systems [5], [6]. A challenge is to bring back the HVDC grid, e.g. the DC voltages, to a normal operation state after a contingency have happens.

Stability analysis of combined AC and MTDC systems was performed in [7]. In [8] and [9], decentralized controllers are employed to share frequency control reserves. In [9] no stability analysis is performed, whereas [8] guarantees stability provided that the connected AC areas have identical parameters and the voltage dynamics of the HVDC system are neglected. [10] considers an optimal decentralized controller for AC grids connected by HVDC systems.

By connecting the AC areas with a communication network supporting the frequency controllers, the performance can be further improved, compared to a decentralized controller structure without such communication. In this paper, we seek to explore controllers which improve performance of existing controllers. For this, we first propose a controller performance measure.

Several distributed and decentralized controllers for sharing frequency control reserves have been proposed in the literature. In [11], a distributed controller, relying on a communication network, was developed to share frequency control reserves of asynchronous AC transmission systems connected through an MTDC system. However, the controller requires a slack bus to control the DC voltage, and is thus only able to share the generation reserves of the non-slack AC areas. Another distributed controller is proposed in [12]. Stability is guaranteed, and the need for a slack bus is eliminated. The voltage dynamics of the MTDC system are however neglected. Moreover the implementation of the controller requires every controller to access measurements of the DC voltages of all MTDC terminals. In [13], [14] distributed secondary generation controllers are proposed.
where the MTDC dynamics are explicitly modeled, and the DC voltages are controlled in addition to the frequencies. The controller does not rely on a slack bus for controlling the DC voltages. The distributed control architecture is more scalable than a centralized architecture where information from all controllers has to be processed simultaneously. By using local and neighboring state information, we propose controllers, which can be implemented even when communication is unavailable. This paper builds on the results in [15], [13], [14], but significantly generalizes the models of the power system.

The remainder of this paper is organized as follows. In Section II, the system model and the control objectives are defined. In Section III, a distributed secondary frequency controller for sharing frequency control and restoration reserves is presented, and is shown to satisfy the control objectives. In Sections IV and V, the results are generalized to general AC networks and π-link models of the HVDC lines, respectively. In Section VI simulations of the distributed controller on a six-terminal MTDC test system are provided, showing the effectiveness of the proposed controller. The paper ends with concluding remarks in Section VII.

II. MODEL AND PROBLEM SETUP

A. Notation

Let \( \mathcal{G} \) be a static, undirected graph. Denote by \( \mathcal{V} \) and \( \mathcal{E} \) the set of vertices and edges of \( \mathcal{G} \), respectively. Let \( \mathcal{N}_i \) be the set of neighboring vertices to \( i \in \mathcal{V} \). Denote by \( \mathcal{L}_W \) the weighted Laplacian matrix of \( \mathcal{G} \), with edge-weights given by the elements of the diagonal matrix \( W \). Let \( e_i \) denote the \( i \)th Cartesian unit vector. Let \( \mathcal{C}^- \) denote the open left half complex plane, and \( \mathcal{C}^+ \) its closure. We denote by \( c_{n \times m} \) an \( n \times m \)-matrix, whose elements are all equal to \( c \). For simplifying notation, we write \( c_n \) for \( c_{n \times 1} \).

B. Model and objective

We will give here a unified model for an MTDC system interconnected with several asynchronous AC systems. We consider an MTDC transmission system consisting of \( n \) converters, denoted \( i = 1, \ldots, n \), each connected to an AC system, i.e., there are no pure DC nodes of the MTDC grid. The converters are assumed to be connected by an MTDC transmission grid, i.e. there exist only one connected MTDC grid and not several MTDC grids. The node connected to converter \( i \) is modelled by

\[ C_i \dot{V}_i = - \sum_{j \in \mathcal{N}_i} \frac{1}{R_{ij}}(V_i - V_j) + I_{ij}^{\text{inj}}, \]

where \( V_i \) is the DC voltage of converter node \( i \), \( C_i > 0 \) the total capacitance of the converter and the HVDC line connected to the considered converter, and \( I_{ij}^{\text{inj}} \) the injected current from the DC converter to the DC node. The constant \( R_{ij} \) denotes the resistance of the HVDC transmission line connecting the converters \( i \) and \( j \). The MTDC transmission grid is assumed to be connected. Note that the converter model (1) of the MTDC system does not take the dynamics of the HVDC lines into account, caused by the inductance and capacitance of the lines. In Section VII, however, we show that the model (1) can be generalized to a π-link model, where each HVDC line consists of an arbitrary number of resistors, inductors, and capacitors in series. Only HVDC nodes which are connected to a converter are considered in our model (1). This implies that intermediate nodes are not captured by the model. Modelling intermediate nodes would result in differential-algebraic equations, resulting in a far more complex analysis. While systems with intermediate nodes can be transformed into systems without intermediate nodes by Kron reduction [17], this is beyond the scope of this paper. Each AC system is assumed to consist of a single generator which is connected to a DC converter, representing an aggregated model of an AC grid. The dynamics of the AC system are given by [18]:

\[ m_i \dot{\omega}_i = P_i^{\text{gen}} + P_i^{\text{m}} - P_i^{\text{inj}}, \]

where \( m_i > 0 \) is its moment of inertia. The constant \( P_i^{\text{gen}} \) is the generated power, \( P_i^{\text{m}} \) is the power load, and \( P_i^{\text{inj}} \) is the power injected to the DC system through converter \( i \), respectively. While the model (2) is restricted to single-generator AC systems, we show in Section IV that this model can be generalized to a network of arbitrary many generators. The control objective can now be stated as follows.

Objective 1. The frequency deviations are asymptotically equal to zero, i.e.,

\[ \lim_{t \to \infty} \omega_i(t) - \omega^{\text{ref}} = 0 \quad i = 1, \ldots, n, \]

where \( \omega^{\text{ref}} \) is the nominal frequency. The total quadratic cost of the power generation is minimized asymptotically, i.e.,

\[ \lim_{t \to \infty} P_i^{\text{gen}} = P_i^{\text{gen}^*}, \forall i = 1, \ldots, n, \]

where

\[ \left[ P_1^{\text{gen}^*}, \ldots, P_n^{\text{gen}^*} \right] = \arg\min_{P_1, \ldots, P_n} \frac{1}{2} \sum_{i=1}^n f_i^P \left( P_i^{\text{gen}} \right)^2 \]

subject to (3), i.e., \( P_i^{\text{gen}} + P_i^{\text{m}} - P_i^{\text{inj}} = 0, \forall i = 1, \ldots, n \) and \( \sum_{i=1}^n P_i^{\text{inj}} = 0 \), i.e., power balance both in the AC grids and in the MTDC grid. The positive constants \( f_i^P \) represent the local cost of generating power. Finally, the DC voltages are such that the a quadratic cost function of the voltage deviations is minimized asymptotically, i.e.,

\[ \lim_{t \to \infty} V_i = V_i^* = V_i^{\text{ref}}, \forall i = 1, \ldots, n, \]

where

\[ \left[ V_1^*, \ldots, V_n^* \right] = \arg\min_{V_1, \ldots, V_n} \frac{1}{2} \sum_{i=1}^n f_i^V \left( V_i - V_i^{\text{ref}} \right)^2 \]

subject to (5)–(4), and where the \( f_i^V \) is a positive constant reflecting the local cost of DC voltage deviations and \( V_i^{\text{ref}} \) is the nominal DC voltage of converter \( i \).

Remark 1. Note that the order in which the optimization problems (3)–(5) are solved is crucial, as (3) and the optimal solution of (4) are constraints of (5).

Remark 2. The minimization of (4) is equivalent to power sharing, where the generated power of AC area \( i \) is asymptotically inverse proportional to the cost \( f_i^P \). The cost \( f_i^P \)
can be chosen to reflect the available generation capacity of area $i$.

**Remark 3.** It is in general not possible that $\lim_{t \to \infty} V_i(t) = V_{i,ref} \forall i = 1, \ldots, n$, since this does not allow for the currents between the HVDC converters to change by (1). Note that the optimal solution to (4) fixes the relative DC voltages, leaving only the ground voltage as a decision variable of (3). Note also that the reference DC voltages $V_{i,ref}$, $i = 1, \ldots, n$, are generally not uniform, as is the reference frequency $\omega_{ref}$.

**Remark 4.** Note that Objective (7) does not include constraints of, e.g., generation and line capacities. This requires that the perturbations from the operating point are sufficiently small, to guarantee that these constraints are not violated.

### III. DISTRIBUTED FREQUENCY CONTROL

#### A. Controller structure

In this section we propose a distributed secondary frequency controller. In addition to the generation controller proposed in [14], we also propose a secondary controller for the power injections into the HVDC grid. We implement the controllers for single AC generators. In Section IV, we generalize the controller for AC grids of arbitrary size.

The distributed generation controller of the AC systems is given by

$$P_i^{\text{gen}} = -K_i^{\text{droop}}(\omega_i - \omega_{\text{ref}}) - K_i^V \dot{\phi}_i$$

where $K_i^{\text{droop}}$ and $K_i^V$ are positive controller parameters.

Moreover, $\phi_i$ are controller variables.

The state variable $\phi_i$ is the difference between the generator angle $\theta_i$ and the reference angle $\theta_{i,ref}$, i.e., the communication graph $G_i$ is directed and the states $\phi_i$ are directed variables.

The distributed generation controller of the AC systems is given by

$$P_i^{\text{gen}} = -K_i^{\text{droop}}(\omega_i - \omega_{\text{ref}}) - K_i^V \dot{\phi}_i$$

where $K_i^{\text{droop}}$ and $K_i^V$ are positive controller parameters, and $P_i^{\text{inj}}$ is the nominal injected power.

**Remark 3.** Note that Objective (7) does not include constraints of, e.g., generation and line capacities. This requires that the perturbations from the operating point are sufficiently small, to guarantee that these constraints are not violated.

#### B. Stability analysis

We now analyze the stability of the closed-loop system. Define the state vectors $\hat{\omega} = \omega - \omega_{\text{ref}}$ and $\hat{V} = V - V_{\text{ref}}$, where $\omega = [\omega_1, \ldots, \omega_n]^T$, $V = [V_1, \ldots, V_n]^T$, $V_{\text{ref}} = [V_{1,\text{ref}}, \ldots, V_{n,\text{ref}}]^T$, $\eta = [\eta_1, \ldots, \eta_n]^T$, and $\phi = [\phi_1, \ldots, \phi_n]^T$.

Combining the MTDC (11), the AC dynamics (2) with the power-current relationship (11), we obtain the closed-loop
dynamics
\[
\hat{\omega} = M\left( - (K^{\text{droop}} + K^\omega) \hat{\omega} + K^V \dot{V} - K^V (K^\omega)^{-1} K^{\text{droop}}.1 \eta - \mathcal{L}_\phi \phi + P^m \right) \\
\dot{V} = \frac{1}{V_{\text{nom}}} E K^\omega \hat{\omega} - E \left( \mathcal{L}_R + \frac{K^V}{V_{\text{nom}}} \right) \dot{V} + \frac{1}{V_{\text{nom}}} E \mathcal{L}_\phi \phi \\
\dot{\eta} = K^{\text{droop}}.1 \hat{\omega} - \mathcal{L}_\eta \eta \\
\dot{\phi} = (K^\omega)^{-1} K^\omega \hat{\omega} - \gamma \phi, 
\]
where \( M = \text{diag}(m_1^{-1}, \ldots, m_n^{-1}) \) is a matrix of inverse generator inertia, \( E = \text{diag}(C_1^{-1}, \ldots, C_n^{-1}) \) is a matrix of electrical elasticities, \( \mathcal{L}_R \) is the weighted Laplacian matrix of the MTDC grid with edge-weights \( 1/R_{ij}, \mathcal{L}_\eta \) and \( \mathcal{L}_\phi \) are the weighted Laplacian matrices of the communication graphs with edge-weights \( c^q_{ij} \) and \( c^p_{ij} \), respectively, and \( P^m = [P^m_1, \ldots, P^m_n]^T \). We define the diagonal matrices of the controller gains as \( K^\omega = \text{diag}(K^\omega_1, \ldots, K^\omega_n) \), etc.

Let \( y = [\hat{\omega}^T, V^T]^T \) define the output of (12). Clearly the linear combination \( \hat{\omega}_n^T \phi \) is unobservable and marginally stable with respect to the dynamics (12), as it lies in the nullspace of \( \mathcal{L}_\phi \). In order to facilitate the stability analysis, we will perform a state-transformation to this unobservable mode. Consider the following state-transformation:
\[
\phi' = \left[ \frac{1}{\sqrt{n}} \hat{\omega}^T \right] S \phi \quad \phi = \left[ \frac{1}{\sqrt{n}} \hat{\omega} \right] S \phi'
\]
where \( S \) is an \( n \times (n-1) \) matrix such that \( \left[ \frac{1}{\sqrt{n}} \hat{\omega}^T \right] S \) is orthonormal. By applying the state-transformation (13) to (12), we obtain dynamics where it can be shown that the state \( \phi'_1 \) is unobservable with respect to the defined output. Hence, omitting \( \phi'_1 \) does not affect the output dynamics. Thus, we define \( \phi'' = [\phi'_2, \ldots, \phi'_n] \), and obtain the dynamics
\[
\hat{\omega} = M\left( - (K^{\text{droop}} + K^\omega) \hat{\omega} + K^V \dot{V} - K^V (K^\omega)^{-1} K^{\text{droop}}.1 \eta - \mathcal{L}_\phi \phi'' + P^m \right) \\
\dot{V} = \frac{1}{V_{\text{nom}}} E K^\omega \hat{\omega} - E \left( \mathcal{L}_R + \frac{K^V}{V_{\text{nom}}} \right) \dot{V} + \frac{1}{V_{\text{nom}}} E \mathcal{L}_\phi \phi'' \\
\dot{\eta} = K^{\text{droop}}.1 \hat{\omega} - \mathcal{L}_\eta \eta \\
\dot{\phi''} = S^T (K^\omega)^{-1} K^\omega \hat{\omega} - \gamma \phi''. 
\]
We are now ready to show the main stability result of this section. The following assumptions are later used as sufficient conditions for closed-loop stability.

**Assumption 1.** The Laplacian matrix satisfies \( \mathcal{L}_\phi = k_\phi \mathcal{L}_R \).

Assumption 1 can be interpreted as the emulated AC dynamics of the HVDC lines having the same susceptance ratios as the conductance ratios of the HVDC lines. Assumption 1 can always be satisfied by appropriate choices of the constants \( c_{ij} \) in (8).

**Assumption 2.** The gain \( \gamma \) satisfies \( \gamma > k_\phi / (AV_{\text{nom}}) \).

Assumption 2 provides lower bounds for the damping coefficient of the converter controllers. Note that the bound on \( \gamma \) is independent of the topology of the communication network. This is particularly desirable in a plug-and-play setting, where new nodes can be added to the system, without having to change \( \gamma \).

**Theorem 1.** If Assumptions 1 and 2 hold, the equilibrium of (14) is globally asymptotically stable.

Proof: The proof follows from Theorem 4 and is thus omitted.

**Corollary 2.** Let Assumption 1 hold and let \( \gamma, k_\phi \) be given such that Assumption 2 holds. Let \( K^V, K^\omega \) and \( K^{\text{droop}}.1 \) be such that \( (F^P)^{-1} = K^V (K^\omega)^{-1} K^{\text{droop}}.1 \) and \( F^V = K^V \), where \( F^P = \text{diag}(f^P_1, \ldots, f^P_n) \) and \( F^V = \text{diag}(f^V_1, \ldots, f^V_n) \). Then the dynamics (14) satisfy Objective 1 in the limit when \( \| (K^\omega)^{-1} K^V \|_\infty \rightarrow 0 \), provided that the disturbance \( P^m \) is constant.

Proof: By Theorem 1 the proof is immediate.

By inserting \( \hat{\omega} = k_1 n \) and premultiplying the above equation with \( 1^T_n \), we obtain that \( \hat{\omega} = 0_n \), so \( \hat{\omega} = 0_n \) is a solution of Objective 1 is thus satisfied. Thus \( \eta = k_2 n \) for some \( k_2 \in \mathbb{R} \). Finally, we let \( \dot{V} = 0_n \), in (14):
\[
K^\omega \hat{\omega} - \left( V_{\text{nom}} \mathcal{L}_R + K^V \right) \dot{V} + \mathcal{L}_\phi \phi'' = 0_n. 
\]
Inserting \( \hat{\omega} = 0_n \) and premultiplying (15) with \( 1^T_n \) yield
\[
1^T_n K^\omega \dot{V} = 0_n. 
\]
Inserting \( \hat{\omega} = 0_n \) and \( \eta = k_2 n \) in (6) yields
\[
P^\text{gen} = -k_2 K^V (K^\omega)^{-1} K^{\text{droop}}.1_n, 
\]
where \( P^\text{gen} = [P^\text{gen}_1, \ldots, P^\text{gen}_n]^T \). It now remains to show that the equilibrium of (14) minimizes the cost functions (4) and (5) of Objective 1. Consider first (4). We conclude that (4) is minimized. Since \( P^\text{gen} = -k_2 K^V (K^\omega)^{-1} K^{\text{droop}}.1_n \) and \( \hat{\omega} = 0_n \), premultiplying the first \( n \) rows of the equilibrium of (14) with \( M^{-1} \), and adding to the \((n + 1)\)th to \( 2n \)th rows premultiplied with \( V_{\text{nom}} E^{-1} \) yields
\[
- V_{\text{nom}} \mathcal{L}_R \dot{V} - k_2 K^V (K^\omega)^{-1} K^{\text{droop}}.1_n = P^m. 
\]
Premultiplying the above equation with \( 1^T_n \) yields \( k_2 = - \sum_{i=1}^n P^m_i \sum_{j=1}^n \frac{k^2_{ij} v^2_j}{R^2_{ij}} \). Additionally, \( \mathcal{L}_R \dot{V} \) is uniquely
determined. Now consider (5). Note that $P_i$ and hence $I_i$, are uniquely determined by (4). By the equilibrium of (1), $L_i V = I_i$, where $I_i = \{i_1, \ldots, i_n\}$. Thus, the KKT condition of (5) is
\[ F V \dot{V} = L_{RR}, \]
where $r \in \mathbb{R}^n$. Since $L_{RR} \dot{V}$ is uniquely determined, we premultiply (19) with $T_i$ and obtain the equivalent condition
\[ T_i F V \dot{V} = 0. \]
Since $F V = K V$, (16) and (20) are equivalent. Hence (5) is minimized, so Objective 1 is satisfied.

**Remark 5.** Corollary 2 provides insight in choosing the controller gains of (6) and (8), to satisfy Objective 1.

While the generation controller (6) and the converter controller (8) offer good performance in terms of satisfying Objective 1, it may not be possible to implement these distributed controllers, e.g., due to lack of communication infrastructure. For such MTDC systems where appropriate communication is lacking, it may be desirable to instead implement decentralized generation and converter controllers. In other situations it might be possible to implement the distributed generation controller (6), while it is more desirable to have the HVDC converters operating independently with decentralized controllers. In the following corollary, we show that the decentralized generation and converter controllers (7) and (9) also globally asymptotically stabilize the combined MTDC and AC system.

**Corollary 3.** Let Assumption 1 hold and let $\gamma$, $k_\phi$ be given such that Assumption 2 holds. Consider the dynamics of the MTDC dynamics (1) and the AC dynamics (2) with the generation controller (7) or (6), respectively, and the converter controller (9). The equilibria of the resulting closed-loop systems are globally asymptotically stable.

**Proof:** The proof is in line with the proof of Theorem 4, where we discard the variables $\eta$ and $\phi$. While the optimality results of Corollary 2 do not hold for any other controller combinations than (6) and (8), the following remark can be made about the average frequency errors.

**Lemma 1.** Consider the dynamics of the MTDC dynamics (1) and the AC dynamics (2) with the generation controller (6) and the converter controller (8). Any equilibrium of the resulting closed-loop system satisfies $\sum_{i=1}^n K_i^{\text{droop}}(\omega_i - \omega^{\text{ref}}) = 0$, i.e., the average frequency errors are zero.

**Proof:** Consider the closed-loop dynamics (12). Letting $\dot{\omega} = 0_n$ and premultiplying this equation with $1_n^T K_i^{\text{droop}} \omega - 1_n^T L_i \eta = \sum_{i=1}^n K_i^{\text{droop}}(\omega_i - \omega^{\text{ref}}).$

IV. GENERALISATION TO AC GENERATION NETWORK

In this section we generalize the single-generator model of Section II-B to an AC grid with arbitrary size.

A. Objective

Consider the AC transmission grid connected to converter $i$, and suppose it consists of $n_i$ generator buses. Without loss of generality, we may assume that converter $i$ of the MTDC grid is connected to generator $i_1$ of the AC system. Let $\delta_{i_1}$ be the phase angle of bus $i_1$. The dynamics of the power system are assumed to be given by the linearized swing equation (13), where the voltages are assumed to be constant. As before, we consider the incremental states with respect to their reference values:

\[ \frac{\dot{\delta}_{i_1}}{\delta_{i_1}} = \omega_{i_1} - \omega^{\text{ref}} \]

\[ m_{i_1} \dot{\omega}_{i_1} = -(K_i^{\text{droop}} + K_i^{\omega}) \delta_{i_1} - \sum_{j \in N_{i_1}} k_{i_1 j}(\delta_{i_1} - \delta_j) + P_{i_1}^{\text{gen}} + P_{i_1}^{\text{em}} - P_{i_1}^{\text{inj}}, \]

where $\delta_{i_1}$ is the phase angle and $\omega_{i_1} = \omega_i - \omega^{\text{ref}}$ is the incremental frequency at bus $i_1$. $m_{i_1} > 0$ is the inertia of bus $i_1$, $k_{i_1 j} = \{V_{i_1}, V_j\} b_{i_1 j}$, where $V_i$ is the constant voltage of bus $i$, and $b_{i_1 j}$ is the susceptance of the power line $(i_k, j)$. Moreover $K_i^{\text{droop}} = 0$ for $i \neq 1$, since power injection through the HVDC converter only takes place at bus $i_1$. The constant $P_{i_1}^{\text{gen}}$ is the generated power by the generation control, $P_{i_1}^{\text{inj}}$ is the uncontrolled deviation from the nominal generated power at generator $i_1$, respectively. The variable $P_{i_1}^{\text{em}} = 0$ for $i \neq 1$ is the power injected to the DC system through converter $i$. We assume that the AC voltages are constant, thus implying that $k_{i_1 j}$ is constant. In order to account for the additional generators, we need to slightly modify Objective 1.

**Objective 2.** The frequency deviations converge to zero, i.e.,

\[ \lim_{t \to \infty} \omega_{i_1}(t) - \omega^{\text{ref}} = 0 \quad k = 1, \ldots, n_i, \quad i = 1, \ldots, n. \] (22)

The total cost of the power generation is minimized asymptotically, i.e., $\lim_{t \to \infty} P_{i_1}^{\text{gen}} = P_{i_1}^{\text{gen}}$, $i = 1, \ldots, n$, where

\[ [P_{1}^{\text{gen}}, \ldots, P_{n}^{\text{gen}}] = \arg\min_{P_1^{\text{gen}}, \ldots, P_n^{\text{gen}}} \frac{1}{2} \sum_{i=1}^n P_i^T i_i^T P_i \] (23)

subject to $1^T_n P_i^{\text{gen}} + P_i^{\text{em}} - P_i^{\text{inj}} = 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^n P_i^{\text{inj}} = 0$, i.e., power balance both in the AC grids and in the MTDC grid. Here $P_i^{\text{gen}} = [p_1^{\text{gen}}, \ldots, p_n^{\text{gen}}]^T$, $i = 1, \ldots, n$. Finally, the DC voltages are such that a quadratic cost function of the voltage deviations is minimized asymptotically, i.e., $\lim_{t \to \infty} V_i = V_i^{\text{ref}}$, $i = 1, \ldots, n$, where

\[ [V_1^{\text{ref}}, \ldots, V_n^{\text{ref}}] = \arg\min_{V_1, \ldots, V_n} \frac{1}{2} \sum_{i=1}^n f_i^V (V_i - V_i^{\text{ref}})^2 \] (24)

subject to (3)-(4). Here $f_i^P$ and $f_i^V$ are positive constants.

B. Controller structure

In this section we generalize the distributed secondary frequency controller (6) and the converter controller (8) to
Since the input-output dynamics of (27) and (28) are identical, we henceforth only consider the dynamics (28). By combining the dynamics (1) and (28) with the controllers (25) and (26), and considering the change of coordinates [13] and \( \phi'' = [\phi''_1, \ldots, \phi''_n] \) we obtain the dynamics

\[
\begin{align*}
\dot{\delta}' &= S_i^T \dot{\omega}_i \\
\dot{\omega}_i &= M_i \left( -(K_{i}^{\text{droop}} + K_{i}^{\omega}) \omega_i + L_{i}\omega_i \right) + P_{i}^{\text{gen}} + P_{i}^{m} - P_{i}^{\text{maj}}.
\end{align*}
\]

(28)

Finally we let \( \tilde{V} = 0 \), insert \( \tilde{\omega} = 0 \), and premultiply the equation with \( L_{i}^{-1} \) finally yields \( k = 0 \), and thus \( \hat{\omega}_i = 0 \), \( i = 1, \ldots, n \), i.e., (22) is satisfied. Letting \( \eta = 0 \), and inserting \( \hat{\omega}_i = 0 \), \( i = 1, \ldots, n \) yields \( \delta'_i = 0 \), which inserted in (25) yields

\[
P_{i}^{\text{gen}} = k_i \frac{K_i^{\text{AC}}}{K_i^{\omega}} K_i^{\text{droop}}, \quad k = 1, \ldots, n_i.
\]

(30)

By similar arguments as in the proof of Corollary 2 we can show that (30) and (31) are equivalent to the KKT conditions of (23) and (24), respectively. This concludes the proof.

V. GENERALIZATION TO \( \pi \)-LINK HVDC MODEL

In this section we extend the HVDC line model to consider the inductance and capacitance of the HVDC lines. We model the HVDC lines as series of \( \ell \) \( \pi \)-links consisting of resistors,
| (i, j) | R_{ij} [p.u.] | L_{ij} [10^{-3} \text{ p.u.}] | C_{ij} [\text{p.u.}] |
|-------|-------------|-----------------|-----------------|
| (1, 2) | 0.0586     | 0.2560          | 0.0085          |
| (2, 3) | 0.0878     | 0.3840          | 0.0127          |
| (2, 5) | 0.0732     | 0.3200          | 0.0106          |
| (2, 6) | 0.1464     | 0.6400          | 0.0212          |

**Theorem 6.** The equilibrium of the dynamics (33) is globally asymptotically stable under Assumptions 7 and 8.

**VI. Simulations**

In this section, simulations are conducted on a test system to validate the performance of the proposed controllers. The simulation was performed in Matlab, using a dynamic phasor approach based on [20]. The test system is illustrated in Fig. 1. We model the HVDC line as a single \( \pi \)-link, with parameters given in Table I. The terminal capacitances are \( C_i = 0.375 \times 10^{-3} \) p.u. The AC grid parameters were obtained from [21]. The generators are modeled as a 6th order synchronous machine model controlled by an automatic voltage controller and a governor [4]. The consumers in all AC grids are modeled as constant power loads. It is assumed the linear power-current relation (10) holds.

**Table I:** HVDC grid line parameters

| (i, j) | R_{ij} [p.u.] | L_{ij} [10^{-3} \text{ p.u.}] | C_{ij} [\text{p.u.}] |
|-------|-------------|-----------------|-----------------|
| (1, 2) | 0.0586     | 0.2560          | 0.0085          |
| (2, 3) | 0.0878     | 0.3840          | 0.0127          |
| (2, 5) | 0.0732     | 0.3200          | 0.0106          |
| (2, 6) | 0.1464     | 0.6400          | 0.0212          |

**Table II:** Controller parameters

| \( K_i \) | \( V_i \) | \( K_{droop} \) | \( \phi_i \) | \( \phi_j \) | \( \gamma \) |
|-----------|-----------|--------------|-----------|-----------|------|
| 1501      | 80        | 9            | 3.35      | 5/R_{ij}  | \_   |
The three different controllers proposed in this paper are applied to the test grid, i.e., (7) and (9), (6) and (9), and (6) and (8), with parameters given in Table II. The communication network of (6) and (8) is illustrated by the dashed lines in Fig. 1.

We set \( \gamma = 0 \), so Theorem 4 does not guarantee stability of the equilibrium. However, the closed-loop system matrix can easily be verified to be Hurwitz. At time \( t = 1 \) s the output of generator 2 in AC area 1 was reduced by 0.2 p.u. Fig. 2 shows the average frequencies of the AC grids for all three controllers. Immediately after the fault the frequencies of the AC area of the increased load drop. The frequency drop is followed by a DC voltage drop in all converter nodes, due to (8) or (9). We note that the frequencies are restored to the nominal frequency for (6) and (8) as predicted by (22). However, for the controller combination (6) and (9) the frequencies are not restored to the nominal values, despite that a secondary frequency controller is employed. An intuitive explanation to this is that the distributed frequency controller (6) requires that the frequencies in the different areas synchronize. This synchronization is achieved by (8), but not by (9).

Fig. 3 shows the DC voltages of the terminals. A similar behavior as from the frequencies can be seen. As foreseen, the voltage deviations can not be controlled back to zero, since this would result in zero current flows. For the controllers (6) and (8), the average voltage deviation is restored to zero as predicted by (31).

Fig. 4 shows the total increase of generated power in the AC areas. After a reduced generation of 0.2 p.u. at \( t = 1 \) s. The generated power is increased for all AC areas regardless of the controllers employed. When the distributed generator and converter controllers (6) and (9) are employed, all generators generate to the same power and have the same marginal generation costs asymptotically, thus minimizing (4) asymptotically. For (6) and (9) or (7) and (9), the marginal generation costs do not converge to the same value.

Fig. 5 shows the power set-points of the converters. For controller (6) and (8) the control in area 1 overcompensate the disturbance at the beginning, also the other areas have some minor oscillations. A stable operation point is found after 6 s. Area 2-6 have all an equal share of the disturbance.
controllers (7) and (9) were employed, the converter set-points stabilizing after 6 s to a point with perfect sharing of the disturbance. For controllers (6) and (9) or (7) and (9), it takes some more time to find a stable point and the converters differ from each other.

It takes longer time for the other controllers to find a stable operation point, where controllers (6) and (9) are significantly faster than (7) and (9). The sharing of the disturbance are for both controller not equal, indicated by the different power set-points of the converters.

VII. CONCLUSIONS

In this paper we have studied distributed secondary controllers for sharing frequency control reserves of asynchronous AC systems connected through an MTDC system. The proposed controllers were shown to stabilize the interconnected AC systems and the MTDC grid. The AC grid frequencies were shown to converge to the nominal frequency. Furthermore, quadratic cost functions of the voltage grid frequencies were shown to converge to the nominal frequency. The proposed controllers were shown to stabilize the synchronous AC systems connected through an MTDC system. The AC systems were shown to converge to the nominal frequency. The proposed controllers were shown to stabilize the synchronous AC systems connected through an MTDC system. The AC systems were shown to converge to the nominal frequency. The proposed controllers were shown to stabilize the synchronous AC systems connected through an MTDC system. The AC systems were shown to converge to the nominal frequency.

APPENDIX

Proof of Theorem 2

Without loss of generality, let $P_{i}^m(t) = 0, i = 1, \ldots, n$. Consider the Lyapunov function candidate, which is positive definite and radially unbounded:

$$W = \sum_{i=1}^{n} \frac{K_i^T}{2K_i} \left( (\delta_i')^T S_i^T L_i^A C_i S_i \delta_i' + \omega_i^T M_i^{-1} \omega_i \right) + \frac{V_{\text{nom}}}{2} \hat{V}^T C \hat{V} + \frac{1}{2} \eta^T \eta + \frac{1}{2} \phi'^T (L_i S_i) \phi''.
$$

Differentiating $W$ along trajectories of (29) yields

$$W = \sum_{i=1}^{n} \frac{K_i^T}{2K_i} \left( (\delta_i')^T S_i^T L_i^A C_i S_i \delta_i' + \omega_i^T M_i^{-1} \omega_i \right) + \frac{V_{\text{nom}}}{2} \hat{V}^T C \hat{V} + \frac{1}{2} \eta^T \eta + \frac{1}{2} \phi'^T (L_i S_i) \phi''.
$$

$$W = \sum_{i=1}^{n} \frac{K_i^T}{2K_i} \left( (\delta_i')^T S_i^T L_i^A C_i S_i \delta_i' + \omega_i^T M_i^{-1} \omega_i \right) + \frac{V_{\text{nom}}}{2} \hat{V}^T C \hat{V} + \frac{1}{2} \eta^T \eta + \frac{1}{2} \phi'^T (L_i S_i) \phi''.
$$

Since $SS^T = I_n - \frac{1}{n} 1_n 1_n^T$ and $S_i S^T = I_n - \frac{1}{n} 1_n 1_n^T$, we have $L_i S_i S^T = L_i$ and $L_i^A C_i S_i S^T = L_i^A$. Furthermore

$$\sum_{i=1}^{n} \frac{K_i^T}{K_i} \omega_i^T e_i \in S \phi'' = \omega^T K(\omega)^{-1} L_i S_i \phi''$$

$$\sum_{i=1}^{n} \frac{K_i^T}{K_i} \omega_i^T e_i \in S \phi'' = \omega^T K(\omega)^{-1} L_i S_i \phi''$$

$$\sum_{i=1}^{n} \frac{K_i^T}{K_i} \omega_i^T e_i \in S \phi'' = \omega^T K(\omega)^{-1} L_i S_i \phi''$$

$$\sum_{i=1}^{n} \frac{K_i^T}{K_i} \omega_i^T e_i \in S \phi'' = \omega^T K(\omega)^{-1} L_i S_i \phi''$$

$$\sum_{i=1}^{n} \frac{K_i^T}{K_i} \omega_i^T e_i \in S \phi'' = \omega^T K(\omega)^{-1} L_i S_i \phi''$$

$$\sum_{i=1}^{n} \frac{K_i^T}{K_i} \omega_i^T e_i \in S \phi'' = \omega^T K(\omega)^{-1} L_i S_i \phi''$$

By defining

$$\tilde{V}' = \left[ \frac{1}{\sqrt{n}} 1_n S \right]^T \hat{V} \quad \tilde{V} = \left[ \frac{1}{\sqrt{n}} 1_n S \right] \hat{V},$$

we obtain

$$\tilde{V}^T L_R \tilde{V} = \tilde{V}^T S \tilde{V} + \tilde{V}^T L S \tilde{V}$$

and

$$\tilde{V}^T L S \tilde{V} = \tilde{V}^T S \tilde{V} + \tilde{V}^T L S \tilde{V}$$

Thus we obtain

$$\tilde{V}^T S \tilde{V} + \tilde{V}^T L S \tilde{V} = \tilde{V}^T S \tilde{V} + \tilde{V}^T L S \tilde{V}$$

where $K_droop = diag(\tilde{K}_1, \ldots, \tilde{K}_n)$. By applying the Schur complement condition for positive definiteness, we see that $Q_1$ is positive definite, since $K(\omega)^{-1} (K(\omega) + K_droop) - K(\omega) = K(\omega)^{-1} K_droop > 0$. By similar arguments $Q_2$ is positive definite if $\gamma k_\phi > k_\omega (4V_{\text{nom}}) S \tilde{L}_R S > 0$. Clearly the above matrix inequality holds under Assumption 2 since $\tilde{S} S \tilde{R} S \geq 0$, and $S x \neq k_1 n$ for $k \neq 0$. Thus $W \leq 0$ under Assumption 2 and the set where $W$ is non-decreasing is given by $G = \{ \delta_i' \in R^n, i = 1, \ldots, n, \eta = k_n \}$, for any $k \in R$. The largest invariant set in $G$ with respect to (29) is the origin, since $L_i S_i \delta_i'' + \frac{K_i V_{\text{nom}}}{\kappa_i} K_i \delta_i'' \eta_i = 0$ implies $\delta_i'' = 0$, and
\[ V = \frac{V_{\text{nom}}}{2} \left( \dot{V}^T C \dot{V} + \sum_{i=1}^{l} I_i^T L^{-1} I_i + \sum_{i=1}^{\ell-1} V_i^T C_{\text{line}} V_i \right) + \frac{1}{2} \bar{\omega}^T K \bar{\omega} (K^V)^{-1} M \bar{\omega} + \frac{1}{2} \bar{\sigma}^T S \bar{\sigma} + \frac{1}{2} \bar{\sigma}^T \mathcal{L}_\phi \text{diag}(C_{\text{line}}) \bar{\sigma} \]

is a Lyapunov function for (33), where \( C_{\text{line}} = \text{diag}(C_{\text{line}}^1, \ldots, C_{\text{line}}^{\ell}) \). Differentiating \( W \) along trajectories of (33), where \( P^{\text{in}} = 0_n \) yields \( W \leq 0 \). By LaSalle’s theorem, the equilibrium of (33) is globally asymptotically stable.

---

**REFERENCES**

[1] Z. Melhem, *Electricity Transmission, Distribution and Storage Systems*, ser. Woodhead Publishing Series in Energy. Elsevier Science, 2013.

[2] T. M. Haileselassie and K. Uhlen, “Power system security in a meshed North Sea HVDC grid,” *Proc. of the IEEE*, vol. 101, no. 4, pp. 978–990, 2013.

[3] J. Arrillaga, Y. H. Liu, and N. R. Watson, *Flexible power transmission: the HVDC options*. John Wiley & Sons, 2007.

[4] P. Kundur, *Power System Stability and Control*, ser. The EPRI Power System Engineering. McGraw-Hill Companies, Inc., 1994.

[5] R. Li, S. Bozhko, and G. Abate, “Frequency control design for offshore wind farm grid with LCC-HVDC link connection,” *IEEE Trans. on Power Electronics*, vol. 23, no. 3, pp. 1085–1092, 2008.

[6] T. M. Haileselassie and K. Uhlen, “Primary frequency control of remote grids connected by multi-terminal HVDC,” in *Power and Energy Society General Meeting, 2010 IEEE*, July 2010, pp. 1–6.

[7] N. R. Chaudhuri, R. Majumder, B. Chaudhuri, and J. Pan, “Stability analysis of VSC MTDC grids connected to multimachine AC systems,” *IEEE Trans. on Power Delivery*, vol. 26, no. 4, pp. 2774–2784, 2011.

[8] J. Dai, Y. Phulpin, A. Sarlette, and D. Ernst, “Voltage control in an HVDC system to share primary frequency reserves between non-synchronous areas,” in *Power systems computation conf.*, 2011.

[9] B. Silva, C. L. Moreira, L. Seca, Y. Phulpin, and J. A. Peas Lopes, “Providing of inertial and primary frequency control services using offshore multiterminal HVDC networks,” *IEEE Transactions on Sustainable Energy*, vol. 3, no. 4, pp. 800–808, 2012.

[10] J. A. Taylor and L. Scardovi, “Decentralized control of DC-segmented power systems,” in *Annual Allerton Conference*, 2014.

[11] J. Dai, Y. Phulpin, A. Sarlette, and D. Ernst, “Impact of delays on a consensus-based primary frequency control scheme for AC systems connected by a multi-terminal HVDC grid,” in *IEEE Bulk Power System Dynamics and Control*, 2010, pp. 1–9.

[12] J. Dai and G. Damm, “An improved control law using HVDC systems for frequency control,” in *Power systems computation conf.*, 2011.

[13] M. Andreasson, R. Wiget, D. V. Dimarogonas, K. H. Johansson, and G. Andersson, “Distributed secondary frequency control through multi-terminal HVDC transmission systems,” *arXiv:1503.07358*, 2015.

[14] ——, “Coordinated frequency control through MTDC transmission systems,” *arXiv preprint arXiv:1505.07402*, 2015.

[15] ——, “Distributed primary frequency control through multi-terminal HVDC transmission systems,” in *American Control Conf.*, 2015.

[16] N. Biggs, *Algebraic graph theory*. Cambridge university press, 1993.

[17] F. Dörfler and F. Bullo, “Kron reduction of graphs with applications to electrical networks,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 60, no. 1, pp. 150–163, Jan 2013.

[18] J. Machowski, J. W. Bialek, and J. R. Bumby, *Power System Dynamics: Stability and Control*. Wiley, 2008.

[19] J. Jovcic and K. Ahmed, *High Voltage Direct Current Transmission: Converters, Systems and DC Grids*. John Wiley & Sons, 2015.

[20] T. Demiray, “Simulation of power system dynamics using dynamic phasor models,” Ph.D. dissertation, ETH Zurich, 2008.

[21] F. Milano, *Power System Modelling and Scripting*. Springer, 2010.

[22] M. Andreasson, D. Dimarogonas, H. Sandberg, and K. Johansson, “Distributed control of networked dynamical systems: Static feedback, integral action and consensus,” *IEEE Transactions on Automatic Control*, vol. 59, no. 7, pp. 1750–1764, July 2014.