Abstract. The paper is the second of two and shows that (assuming large cardinals) set theory is a tractable (and we dare to say tame) first order theory when formalized in a first order signature with natural predicate symbols for the basic definable concepts of second and third order arithmetic, and appealing to the model-theoretic notions of model completeness and model companionship.

Specifically we use the general framework linking generic absoluteness results to model companionship introduced in the first paper to show that strong forms of Woodin’s axiom ($\ast$) entail that any theory $T$ extending $\text{ZFC}$ by suitable large cardinal axioms has a model companion $T^*$ with respect to certain signatures $\tau$ containing symbols for $\Delta^0_0$-relations and functions, constant symbols for $\omega$ and $\omega_1$, a predicate symbol for the nonstationary ideal on $\omega_1$, symbols for certain lightface definable universally Baire sets.

Moreover $T^*$ is axiomatized by the $\Pi^2_2$-sentences $\psi$ for $\tau$ such that $T$ proves that $L(\text{UB}) \models (P_{\text{max}} \models \psi)$, where $L(\text{UB})$ denotes the smallest transitive model containing the universally Baire sets.

Key to our results is the recent breakthrough of Asperò and Schindler establishing that a strong form of Woodin’s axiom ($\ast$) follows from $\text{MM}^{++}$.

Throughout this paper we assume the reader is familiar with the results and terminology of [9]. We will give detailed references of where to find inside [9] the notations, theorems, and definitions we will use here.

Let us start rightaway stating the main results.

Let $\tau^*_\text{ST}$ be a signature containing predicate symbols $R_\psi$ of arity $m$ for all bounded $\in$-formulae $\psi(x_1, \ldots, x_m)$, function symbols $f_\theta$ of arity $k$ for for all bounded $\in$-formulae $\theta(y, x_1, \ldots, x_k)$, constant symbols $\omega$ and $\emptyset$. $\text{ZFC}_{\tau^*_\text{ST}} \supseteq \text{ZFC}^\omega$ is the $\tau^*_\text{ST}$-theory obtained adding axioms which force in each of its $\tau^*_\text{ST}$-models $\emptyset$ to be interpreted by the empty set, $\omega$ to be interpreted by the first infinite ordinal, each $R_\psi$ as the class of $k$-tuples defined by the bounded formula $\psi(x_1, \ldots, x_k)$, each $f_\theta$ as the $l$-ary class function whose graph is the extension of the bounded formula $\theta(x_1, \ldots, x_l, y)$ (whenever $\theta$ defines a functional relation), see [9, Notation 2] for details.

We supplement [9, Notation 2] with another piece of notation that will be used throughout the paper.

Notation 1.

- $\tau^\omega_{\omega_1}$ is the signature $\tau^*_\text{ST} \cup \{\omega_1\} \cup \{\text{NS}_{\omega_1}\}$ with $\omega_1$ a constant symbol, $\text{NS}_{\omega_1}$ a unary predicate symbol.
- $T^\omega_{\omega_1}$ is the $\tau^\omega_{\omega_1}$-theory given by $T^*_\text{ST}$ together with the axioms $\omega_1$ is the first uncountable cardinal,
  \[ \forall x [(x \subseteq \omega_1 \text{ is non-stationary}) \leftrightarrow \text{NS}_{\omega_1}(x)] \].
- $\text{ZFC}^\omega_{\omega_1}$ is the $\tau^\omega_{\omega_1}$-theory
  \[ \text{ZFC}^\omega_{\text{ST}} + T^\omega_{\omega_1} \].

Accordingly we define $\text{ZFC}^\omega_{\omega_1}$.

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Let $\text{UB}$ denote the family of universally Baire sets (see for details [9, Section 4.2]), and $L(\text{UB})$ denote the smallest transitive model of $\text{ZF}$ which contains $\text{UB}$.

**Theorem 1.** Let $\mathcal{V} = (V, \in)$ be a model of $\text{ZFC} + \text{MAX}(\text{UB})$; there is a supercompact cardinal and class many Woodin cardinals, and $\text{UB}$ denote the family of universally Baire sets in $V$.

$\text{TFAE}$

(1) $(V, \in)$ models $(*)\text{-UB}$;

(2) $\text{NS}_{\omega_1}$ is precipitous\(^1\) and the $\tau_{\text{NS}_{\omega_1}} \cup \text{UB}$-theory of $V$ has as model companion the $\tau_{\text{NS}_{\omega_1}} \cup \text{UB}$-theory of $H_{\omega_2}$.

(1) implies (2) does not need the supercompact cardinal.

We give rightaway the definitions of $\text{MAX}(\text{UB})$ and $(*)\text{-UB}$.

**Definition 2.** $\text{MAX}(\text{UB})$: There are class many Woodin cardinals in $V$, and for all $G$ $V$-generic for some forcing notion $P \in V$:

(1) Any subset of $(2^\omega)^V[G]$ definable in $(H_{\omega_1}^V[G] \cup \text{UB}^V[G], \in)$ is universally Baire in $V[G]$.

(2) Let $H$ be $V[G]$-generic for some forcing notion $Q \in V[G]$. Then\(^2\):

$$(H_{\omega_1}^V[G] \cup \text{UB}^V[G], \in) \prec (H_{\omega_1}^V[G][H] \cup \text{UB}^V[G][H], \in).$$

We will comment more on $\text{MAX}(\text{UB})$ in Section 2; for now we observe that $\text{MAX}(\text{UB})$ is a form of sharp for the family of universally Baire sets which holds if $V$ has class many Woodin cardinals and is a generic extension obtained by collapsing a supercompact cardinal to become countable ($\text{MAX}(\text{UB})$ is a weakening of the conclusion of [7, Thm 3.4.17]). Moreover if $\text{MAX}(\text{UB})$ holds in $V$, it remains true in all further set forcing extensions of $V$. It is open whether $\text{MAX}(\text{UB})$ is a direct consequence of suitable large cardinal axioms.

We now turn to the definition of $(*)\text{-UB}$, a natural maximal strengthening of Woodin’s axiom $(*)$. Key to all results of this paper is an analysis of the properties of generic extensions by $P_{\max}$ of $L(\text{UB})$. In this analysis $\text{MAX}(\text{UB})$ is used to argue (among other things) that all sets of reals definable in $L(\text{UB})$ are universally Baire, so that most of the results established in [6] on the properties of $P_{\max}$ for $L(\mathbb{R})$ can be also asserted for $L(\text{UB})$.

We will use various forms of Woodin’s axiom $(*)$ each stating that $\text{NS}_{\omega_1}$ is saturated together with the existence of $P_{\max}$-filters meeting certain families of dense subsets of $P_{\max}$ definable in $L(\text{UB})$. However in this paper we will not define the $P_{\max}$-forcing. The reason is that in the proof of all our results, we will use equivalent characterizations of the proper forms of $(*)$ which do not mention at all $P_{\max}$. We will give at the proper stage the relevant definitions. Meanwhile we assume the reader is familiar with $P_{\max}$ or can accept as a blackbox its existence as a certain forcing notion; our reference on this topic is [6].

**Definition 3.** Let $\mathcal{A}$ be a family of dense subsets of $P_{\max}$.

- $(*)\mathcal{A}$ holds if $\text{NS}_{\omega_1}$ is saturated\(^3\) and there exists a filter $G$ on $P_{\max}$ meeting all the dense sets in $\mathcal{A}$.

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\(^1\)See [7, Section 1.6, pag. 41] for a definition of precipitousness and a discussion of its properties. A key observation is that $\text{NS}_{\omega_1}$ being precipitous is independent of $\text{CH}$ (see for example [7, Thm. 1.6.24]), while $(*)\text{-UB}$ entails $2^{\aleph_0} = \aleph_2$ (for example by the results of [6, Section 6]).

Another key point is that we stick to the formulation of $P_{\max}$ as in [6] so to be able in its proof to quote verbatim from [6] all the relevant results on $P_{\max}$-preconditions we will use. It is however possible to develop $P_{\max}$ focusing on Woodin’s countable tower rather than on the precipitousness of $\text{NS}_{\omega_1}$ to develop the notion of $P_{\max}$-precondition. Following this approach in all its scopes, one should be able to reformulate Thm. 1(2) omitting the request that $\text{NS}_{\omega_1}$ is precipitous. We do not explore this venue any further.

\(^2\)Elementarity is witnessed via the map defined by $A \mapsto A^{V[G][H]}$ for $A \in \text{UB}^V[G]$ and the identity on $H_{\omega_1}^V[G]$ (See [9, Notation 4.6] for the definition of $A^{V[G][H]}$).

\(^3\)See [7, Section 1.6, pag. 39] for a discussion of saturated ideals on $\omega_1$. 
\bullet (\ast)\text{-UB} holds if \( \text{NS}_{\omega_1} \) is saturated and there exists an \( L(\text{UB}) \)-generic filter \( G \) on \( \mathbb{P}_{\text{max}} \).

Woodin’s definition of \( (\ast) \) [6, Def. 7.5] is equivalent to \((\ast)\text{-A}+\text{there are class many Woodin cardinals} \) for \( \mathcal{A} \) the family of dense subsets of \( \mathbb{P}_{\text{max}} \) existing in \( L(\mathbb{R}) \).

A key role in all proofs is played by the following generic absoluteness result:

**Theorem 4.** Assume \(^4\) \((V, \tau_{\text{NS}_{\omega_1}}^V) \) models \( \text{ZFC}_{\text{NS}_{\omega_1}}^+ \) there are class many Woodin cardinals. Then the \( \Pi_1 \)-theory of \( V \) for the language \( \tau_{\text{NS}_{\omega_1}} \cup \text{UB} \) is invariant under set sized forcings.

An objection to Thm. 1 is that it subsumes the Platonist standpoint that there exists a definite universe of sets. At the price of introducing another bit of notation, we can prove a version of Thm. 1 which makes perfect sense also to a formalist.

**Notation 2.**
\begin{itemize}
\item \( \sigma_{\text{ST}} \) is the signature containing a predicate symbol \( S_\phi \) of arity \( n \) for any \( \tau_{\text{ST}} \)-formula \( \phi \) with \( n \)-many free variables.
\item \( \sigma_{\omega,\text{NS}_{\omega_1}} \) is the signature \( \tau_{\text{ST}} \cup \sigma_{\text{ST}} \).
\item \( T_{\text{UB}} \) is the \( \sigma_{\omega,\text{NS}_{\omega_1}} \)-theory given by the axioms
\begin{align*}
\forall x_1 \ldots x_n [S_\psi(x_1, \ldots, x_n) \leftrightarrow (\bigwedge_{i=1}^n x_i \subseteq \omega^\omega \land \psi^{L(\text{UB})}(x_1, \ldots, x_n))]
\end{align*}
as \( \psi \) ranges over the \( \tau_{\text{ST}} \)-formulae.
\item \( \text{ZFC}^+_{\text{UB}} \) is the \( \sigma_\omega \)-theory \( \text{ZFC}^-_{\text{ST}} \cup T_{\text{UB}} \);
\item \( \text{ZFC}^+_{\text{UB},\text{NS}_{\omega_1}} \) is the \( \sigma_{\omega,\text{NS}_{\omega_1}} \)-theory \( \text{ZFC}^-_{\text{NS}_{\omega_1}} \cup T_{\text{UB}} \);
\item Accordingly we define \( \text{ZFC}^+_{\text{UB},\text{NS}_{\omega_1}} \).
\end{itemize}

A key observation is that \( \text{ZFC}^-_{\text{ST}}, \text{ZFC}^-_{\text{NS}_{\omega_1}}, \text{ZFC}^-_{\text{UB},\text{NS}_{\omega_1}}, \text{ZFC}^+_{\text{UB},\text{NS}_{\omega_1}} \) are all definable extensions of \( \text{ZFC} \); more precisely any \( \varepsilon \)-structure \((M, E)\) of \( \text{ZFC}^- \) admits a unique extension to a \( \tau \)-structure satisfying the extra axioms outlined in the above items for \( \tau \) among the signatures written above (for \( \tau_{\text{ST}} \cup \{\omega_1, \text{NS}_{\omega_1}\} \) the \( \varepsilon \)-model must satisfy the sentence stating the existence of a smallest uncountable cardinal). The same considerations apply to \( \text{ZFC}^-_{\text{ST}}, \text{ZFC}^-_{\text{NS}_{\omega_1}}, \text{ZFC}^+_{\text{UB},\text{NS}_{\omega_1}} \).

**Theorem 5.** Let \( T \) be any \( \sigma_{\omega,\text{NS}_{\omega_1}} \)-theory extending \( \text{ZFC}^+_{\text{UB},\text{NS}_{\omega_1}} \). Then \( T \) has a model companion \( T^* \).

Moreover TFAE for any \( \Pi_2 \)-sentence \( \psi \) for \( \sigma_{\omega,\text{NS}_{\omega_1}} \):
\begin{itemize}
\item (A) \( T^* \vdash \psi \);
\item (B) For any complete theory \( S \supseteq T \)
\[ S \vdash \exists P (P \text{ is a partial order } \land \vDash_P \psi^{H_{\omega_2}}) ; \]
\end{itemize}

\(^4\) We follow the convention introduced in [9, Notation 2.1] to define \((V, \tau_{\text{NS}_{\omega_1}})\).

\(^5\) \( H_{\omega_2} \) denotes a canonical \( P \)-name for \( H_{\omega_2} \) as computed in generic extension by \( P \).
(D) $T$ proves
\[ L(\text{UB}) \models [\mathbb{P}_{\text{max}} \forces \psi^H_{\omega_2} ]; \]

(E) \[ T_v + \text{ZFC}^\ast_{1,\text{UB},\text{NS}_{\omega_1}} + \text{MAX}(\text{UB}) + (\ast)\text{-UB} \models \psi^H_{\omega_2}. \]

Crucial to the proof of Theorems 5 and 1 is the recent breakthrough of Asperó and Schindler [2] establishing that $(\ast)$-UB follows from $\text{MM}^{++}$.

The paper is organized as follows:

- Section 1 shows that for many natural signatures $\sigma_A = \tau_{ST} \cup A$ given by certain families $A$ of universally Baire sets, the the $\sigma_A$-theory of $H_{\aleph_1}$ is the model companion of the $\sigma_A$-theory of $V$. These results are preliminary to the proofs of Thm. 5, 1.
- Section 2 proves Theorems 1, 4, 5.

Our objective is to make this paper accessible to the widest possible audience (which is however limited to scholars with a strong background in forcing and large cardinals), this has been done at the expenses of its brevity. We tried as much as possible to make the reading of Section 1 accessible also to readers unfamiliar with the stationary tower forcing and with $\mathbb{P}_{\text{max}}$. We also tried to formulate the main results of in such a way that the use of stationary tower forcing is confined to their proofs, and does not hamper the comprehension of the key ideas. This is unfortunately not possible for many of the results in Section 2, where a great familiarity with the content of [6, 7] is needed and assumed. We also decided to give (overly?) detailed arguments for all non-trivial proofs. Almost all proofs in Section 2 employ the key results on the properties of $\mathbb{P}_{\text{max}}$-forcing presented in [6]. The unique proof containing mathematical ideas not at all present in [6, 7] is that of Thm 2.16, in this case we are inspired by [2, Lemma 3.2].

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1. Model companionship versus generic absoluteness for the theory of $H_{\aleph_1}$

1.1. Model companionship for the theory of $H_{\aleph_1}$

Definition 1.1. Let $(V, \in)$ be a model of ZFC and $N \subseteq V$ be a transitive class (or set) which is a model of ZFC$^\ast$. $A \subseteq \text{UB}^V$ is $N$-closed if whenever $B \subseteq (2^\omega)^k$ is such that for some $\in$-formula $\phi(x_0, \ldots, x_n)$
\[ B = \left\{ (r_0, \ldots, r_{k-1}) \in (2^\omega)^k : (N, \in, A_0, \ldots, A_{n-k}) \models \phi(r_0, \ldots, r_{k-1}, A_0, \ldots, A_{n-k}) \right\}, \]
with $A_0, \ldots, A_{n-k} \in A$, we have that $B \in A$.

Example 1.2. Given a model $(V, \in)$ of ZFC+$\text{there are class many Woodin cardinals}$, simple examples of $H_{\omega_1}$-closed families (which we will use) are:

1. UB$^V$, i.e. the family of all universally Baire sets of $V$.
2. l-UB$^V$, i.e. the subsets of $(2^\omega)^k$ (as $k$ varies in the natural) which are the extension of some $\in$-formula relativized to $L(\text{UB})$.
3. The family UB$\cap X$ for some $X \prec V_\theta$ with $\theta$ inaccessible.

Theorem 1.3. Let $(V, \in)$ be a model of ZFC, and assume $A$ is $H_{\omega_1}$-closed.
Let $\tau_A = \tau_{ST} \cup A$. Then the $\tau_A$-theory of $H_{\omega_1}$ is model complete and is the model companion of the $\tau_A$-theory of $V$.
Proof. Let $T$ be the $\tau_A$-theory of $V$ and $T^*$ be the $\tau_A$-theory of $H_{\omega_1}$.

By Levy’s absoluteness Lemma [9, Lemma 4.1]

$$ (H_{\omega_1}, \tau_A^V) \prec_1 (V, \tau_A^V), $$

hence the two structures share the same $\Pi_1$-theory. Therefore (by the standard characterization of model companionship — [9, Thm. 3.18]) it suffices to prove that $T^*$ is model complete.

By Robinson’s test [9, Lemma 3.14(c)], it suffices to show that any existential $\tau_A$-formula is $T^*$-equivalent to a universal $\tau_A$-formula.

Let $A_1, \ldots, A_k$ be the predicates in $A$ appearing in $\phi$.

Let

$$ B = \{ (r_1, \ldots, r_n) \in (2^\omega)^n : (H_{\omega_1}, \tau^V_{ST}, A_1, \ldots, A_k) \models \phi(\text{Cod}_\omega(r_1), \ldots, \text{Cod}_\omega(r_n)) \}. $$

Then $B$ belongs to $A$, since $A$ is $H_{\omega_1}$-closed. Now for any $a_1, \ldots, a_n \in H_{\omega_1}$:

$$ (H_{\omega_1}, \tau_A^V) \models \phi(a_1, \ldots, a_n) $$

if and only if

$$ \forall r_1 \ldots r_n \bigwedge_{i=1}^n \text{Cod}_\omega(r_i) = a_i \rightarrow B(r_1, \ldots, r_n). $$

This yields that

$$ T^* \vdash \forall x_1, \ldots, x_n (\phi(x_1, \ldots, x_n) \leftrightarrow \theta_\phi(x_1, \ldots, x_n)). $$

where $\theta_\phi(x_1, \ldots, x_n)$ is the $\Pi_1$-formula in the predicate $B \in A$

$$ \forall y_1, \ldots, y_n [\bigwedge_{i=1}^n x_i = \text{Cod}_\omega(y_i) \rightarrow B(y_1, \ldots, y_n)]. $$

We leave to the reader to check that the above proof yields the following:

**Corollary 1.4.** Let $T \supseteq \text{ZFC}_{1-\text{UB}}$ be a $\sigma_\omega$-theory. Then $T$ has as model companion the $\Pi_2$-sentences $\psi$ for $\sigma_\omega$ such that

$$ T \vdash \psi_{H_{\omega_1}}. $$

Finally we will need the following observation:

**Fact 1.5.** Assume $(V, \in)$ models that there are class many Woodin cardinals and $A \subseteq \text{UB}$ is $H_{\omega_1}$-closed in $V$. Let $G$ be $V$-generic for some forcing $P \in V$.

Then $\{ A^V[G] : A \in A \}$ is $H_{\omega_1}$-closed in $V[G]$.

Proof. The assumptions grant that

$$ (H_{\omega_1}^V, \tau^V_{ST}, A : A \in A) \prec_1 (H_{\omega_1}^V[G], \tau^V_{ST}[G], A : A \in A) $$

(by [9, Thm. 4.7]). The very definition of being $H_{\omega_1}$-closed gives that the same sentences holding in $(H_{\omega_1}^V, \tau^V_{ST}, A : A \in A)$ granting in $V$ that $A$ is $H_{\omega_1}$-closed, also grant that $\{ A^V[G] : A \in A \}$ is $H_{\omega_1}$-closed in $V[G]$.

6See [9, Def. 2.2] for a definition of Cod$_\omega$. 

1.2. \textbf{MAX}(UB). From now on we will need in several occasions that \textbf{MAX}(UB) holds in \(V\) (recall Def. 2). We will always explicitly state where this assumption is used, hence if a statement does not mention it in the hypothesis, the assumption is not needed for its thesis.

We will use both properties of \textbf{MAX}(UB) crucially: (1) is used in the proof of Lemma 2.10; (2) in the proof of Fact 2.12. Similarly they are essentially used in Remark 2.15. Specifically we will need \textbf{MAX}(UB) to prove that certain subsets of \(H_{\omega_1}\) are coded by a universally Baire set, and that this coding is absolute between generic extensions, i.e. if

\[
x \in H_{\omega_1}^V : (H_{\omega_1} \cup \text{UB}, \tau_{7\text{ST}}^V) \models \phi(x)
\]

is coded by \(A \in \text{UB}^V\),

\[
x \in H_{\omega_1}^{V[G]} : (H_{\omega_1}^{V[G]} \cup \text{UB}^{V[G]}, \tau_{7\text{ST}}^{V[G]}) \models \phi(x)
\]

is coded by \(A^{V[G]} \in \text{UB}^{V[G]}\) for \(\phi\) some \(\tau_{7\text{ST}}\)-formula\footnote{Note that the structures \((H_{\omega_1} \cup \text{UB}, \in), (H_{\omega_1} \cup \text{UB}, \tau_{7\text{ST}}^V)\) have the same algebra of definable sets, hence we will use one or the other as we deem most convenient, since any set definable by some formula in one of these structures is also defined by a possibly different formula in the other. The formulation of \textbf{MAX}(UB) is unaffected if we choose any of the two structures as the one for which we predicate it.}

It is useful to outline what is the different expressive power of the structures \((H_{\omega_1}, \tau_{7\text{ST}}^V, A : A \in \text{UB}^V)\) and \((H_{\omega_1} \cup \text{UB}^V, \tau_{7\text{ST}}^V)\). The latter can be seen as a second order extension of \(H_{\omega_1}\), where we also allow formulae to quantify over the family of universally Baire subsets of \(2^{\omega_1}\); in the former quantifiers only range over elements of \(H_{\omega_1}\), but we can use the universally Baire subsets of \(H_{\omega_1}\) as parameters. This is in exact analogy between the comprehension scheme for the Morse-Kelley axiomatization of set theory (where formulae with quantifiers ranging only over classes are allowed) and the comprehension scheme for Gödel-Bernays axiomatization of set theory (where just formulae using classes as parameters and quantifiers ranging only over sets are allowed). To appreciate the difference between the two set-up, note that that the axiom of determinacy for universally Baire sets is expressible in

\[
(H_{\omega_1} \cup \text{UB}, \tau_{7\text{ST}}^V)
\]

by the \(\tau_{7\text{ST}}\)-sentence

\[
\text{For all } A \subseteq 2^{\omega_1} \text{ there is a winning strategy for one of the players in the game with payoff } A,
\]

while in

\[
(H_{\omega_1}^V, \tau_{7\text{ST}}^V, A : A \in \text{UB}^V)
\]

it is expressed by the axiom schema of \(\Sigma_1\)-sentences for \(\tau_{7\text{ST}} \cup \{A\}\)

\[
\text{There is a winning strategy for some player in the game with payoff } A
\]

as \(A\) ranges over the universally Baire sets.

We will crucially use the stronger expressive power of the structure \((H_{\omega_1} \cup \text{UB}, \in)\) to define certain universally Baire sets as the extension in \((H_{\omega_1} \cup \text{UB}, \tau_{7\text{ST}}^V)\) of lightface \(\Sigma_1\)-properties (according to the Levy hierarchy): properties which require an existential quantifier ranging over all universally Baire sets.

2. \textbf{Model companionship versus generic absoluteness for the theory of } \(H_{\aleph_2}\)

This section is devoted to the proofs of Theorems 5 and 1. Along the way we will also prove (and use) Theorem 4.

Let us give a general outline of these proofs before getting into details. From now on we assume the reader is familiar with the basic theory of \(P_{\text{max}}\) as exposed in [6].
Notation 2.1. For a given family of universally Baire sets \( A \), \( \sigma_A \) is the signature \( \tau_{\mathcal{ET}} \cup A \), \( \sigma_{A,\mathcal{NS}_{\omega_1}} \) is the signature \( \tau_{\mathcal{NS}_{\omega_1}} \cup A \).

The key point is to prove (just on the basis that \( (V, @) \models \text{MAX}(\text{UB}) + (\ast)\cdot\text{UB} \)) the model completeness of the \( \sigma_{\text{UB},\mathcal{NS}_{\omega_1}} \)-theory of \( H_{\omega_2} \) assuming (\( \ast \))-UB. To do so we use Robinson’s test and we show the following:

Assuming \( \text{MAX}(\text{UB}) \) there is a special universally Baire set \( D_{\text{UB},\mathcal{NS}_{\omega_1}} \) defined by an \( \epsilon \)-formula (in no parameters) relativized to \( L(\text{UB}) \) coding a family of \( \mathbb{P}_{\text{max}} \)-preconditions with the following fundamental property:

For any \( \sigma_{\text{UB},\mathcal{NS}_{\omega_1}} \)-formula \( \psi(x_1, \ldots, x_n) \) mentioning the universally Baire predicates \( B_1, \ldots, B_k \), there is an algorithmic procedure which finds a universal \( \sigma_{\text{UB},\mathcal{NS}_{\omega_1}} \)-formula \( \theta_{\psi}(x_1, \ldots, x_n) \) mentioning just the universally Baire predicates \( B_1, \ldots, B_k, D_{\text{UB},\mathcal{NS}_{\omega_1}} \) such that

\[
(H_{\omega_2}^{L(\text{UB})}[G], \sigma_{\text{UB},\mathcal{NS}_{\omega_1}}^L[G]) \models \forall \vec{x} \left( \psi(x_1, \ldots, x_n) \leftrightarrow \theta_{\psi}(x_1, \ldots, x_n) \right)
\]

whenever \( G \) is \( L(\text{UB}) \)-generic for \( \mathbb{P}_{\text{max}} \).

Moreover the definition of \( D_{\text{UB},\mathcal{NS}_{\omega_1}} \) and the computation of \( \theta_{\psi}(x_1, \ldots, x_n) \) from \( \psi(x_1, \ldots, x_n) \) are just based on the assumption that \( (V, @) \) is a model of \( \text{MAX}(\text{UB}) \), hence can be replicated mutatis-mutandis in any model of \( \text{ZFC} + \text{MAX}(\text{UB}) \). We will need that \( (V, @) \) is a model of \( \text{MAX}(\text{UB}) + (\ast)\cdot\text{UB} \) just to argue that in \( V \) there is an \( L(\text{UB}) \)-generic filter \( G \) for \( \mathbb{P}_{\text{max}} \) such that \( H_{\omega_2}^{L(\text{UB})}[G] = H_{\omega_2}^V \). Since in all our arguments we will only use that \( (V, @) \) is a model of \( \text{MAX}(\text{UB}) \) and (in some of them also of (\( \ast \))-UB), we will be in the position to conclude easily for the truth of Theorem 5 and 1.

We condense the above information in the following:

Theorem 2.2. There is an \( \epsilon \)-formula \( \phi_{\text{UB},\mathcal{NS}_{\omega_1}}(x) \) in one free variable such that:

1. \( \text{ZFC}^{L(\text{UB})} + \text{MAX}(\text{UB}) \) proves that \( S_{\phi_{\text{UB},\mathcal{NS}_{\omega_1}}} \) is universally Baire.

2. Given predicate symbols \( B_1, \ldots, B_k \), consider the theory \( T_{B_1, \ldots, B_k} \) in signature \( \sigma_{\omega} \cup \{ B_1, \ldots, B_k \} \) extending \( \text{ZFC}^{L(\text{UB})} + \text{MAX}(\text{UB}) \) by the axioms:

\[
B_i \text{ is universally Baire}
\]

for all predicate symbols \( B_1, \ldots, B_k \).

There is a recursive procedure assigning to any existential formula \( \phi(x_1, \ldots, x_k) \) for \( \sigma_{\{B_1, \ldots, B_k\},\mathcal{NS}_{\omega_1}} \) a universal formula \( \theta_{\phi}(x_1, \ldots, x_k) \) for \( \sigma_{\{B_1, \ldots, B_k, S_{\phi_{\text{UB},\mathcal{NS}_{\omega_1}}}\}} \) such that \( T_{B_1, \ldots, B_k} \) proves that

\[
\mathbb{P}_{\text{max}} \models [ (H_{\omega_2}^{L(\text{UB})}[G], \sigma_{\text{UB},\mathcal{NS}_{\omega_1}}^L[G]) \models \forall \vec{x} \left( \phi(x_1, \ldots, x_k) \leftrightarrow \theta_{\phi}(x_1, \ldots, x_k) \right) ]
\]

where \( \dot{G} \in L(\text{UB}) \) is the canonical \( \mathbb{P}_{\text{max}} \)-name for the generic filter.

2.1. Proofs of Thm. 5, and of (1)\( \rightarrow \) (2) of Thm. 1. Theorem 5, (1)\( \rightarrow \) (2) of Theorem 1 are immediate corollaries of the above theorem combined with Asperò and Schindler’s proof that MM\( ^{+++} \) implies (\( \ast \))-UB, and with Theorem 4.

We start with the proof of (1)\( \rightarrow \) (2) of Thm. 1 assuming Thm. 2.2 and Thm. 4:

Proof. Assume \( (V, @) \) models (\( \ast \))-UB. Then there is a \( \mathbb{P}_{\text{max}} \)-filter \( G \in V \) such that \( H_{\omega_2}^{L(\text{UB})}[G] = H_{\omega_2}^V \). By Thm. 2.2 and Robinson’s test, we get that the first order \( \sigma_{\text{UB},\mathcal{NS}_{\omega_1}} \)-theory of \( H_{\omega_2}^{L(\text{UB})}[G] \) is model complete. By Levy’s absoluteness [9, Lemma 4.1], \( H_{\omega_2}^{L(\text{UB})}[G] \) is

\[\text{It is this part of our argument where the result of Asperò and Schindler establishing the consistency of (\( \ast \))-UB relative to a supercompact is used in an essential way. We will address again the role of Asperò and Schindler’s result in all our proofs in some closing remarks.}\]
a $\Sigma_1$-elementary substructure of $V$ also according to the signature $\sigma_{UB,NS_{\omega_1}}$. We conclude (by [9, Thm. 3.18]), since the two theories share the same $\Pi_1$-fragment. \hfill \square

The proof of the converse implication requires more information on $D_{UB,NS_{\omega_1}}$ then what is conveyed in Thm. 2.2. We defer it to a later stage.

We now prove Thm. 5:

Proof. By Thm. 2.2 and Robinson’s test, the $\Pi_2$-sentences for $\sigma_{UB,NS_{\omega_1}}$ which are $\text{ZFC}^*_{UB}+\text{MAX}(UB)$-provably forced to hold in the $H_{\omega_2}$ of the generic extension of $L(UB)$ by $\mathbb{P}_{\max}$ form a model complete theory.

Let us call $T^*_{UB,NS_{\omega_1}}$ this model complete theory.

We now show that any model of $\text{ZFC}^*_{UB,NS_{\omega_1}}+\text{MAX}(UB)+\text{there is a supercompact cardinal}$ has the same $\Pi_1$-theory of some model of $T^*_{UB,NS_{\omega_1}}$. This suffices by [9, Lemma 3.19].

($\ast$)-UB holds in any model of $\text{MM}^{++}$ by Schindler and Asperò’s breakthrough [2]. It is a standard result that one can force $\text{MM}^{++}$ over any model of $\text{ZFC}+\text{there is a supercompact cardinal}$ [3].

Let $\mathcal{M}$ be any model of $\text{ZFC}+\text{MAX}(UB)+\text{there is a supercompact cardinal}$ and $\mathcal{N}$ be a model of $\text{MM}^{++}$ obtained as a forcing extension of $\mathcal{M}$ by the methods of [3].

By Thm. 4, $\mathcal{N}$ has the same $\Pi_1$-theory of $\mathcal{M}$ according to the signature $\sigma_{UB,NS_{\omega_1}}$. Now $\mathcal{N}$ is a model of $\text{MM}^{++}$ and therefore of ($\ast$)-UB, by [2].

Hence $H^\mathcal{N}_{\omega_2}$ is also (according to $\mathcal{N}$) the $H_{\omega_2}$ of the generic extension of $L(UB)^\mathcal{N}$ by $\mathbb{P}_{\max}$. Since $H^\mathcal{N}_{\omega_2} \prec \mathcal{N}$ also according to the signature $\sigma_{UB,NS_{\omega_1}}$, we conclude that $H^\mathcal{N}_{\omega_2}$ and $\mathcal{M}$ share the same $\Pi_1$-theory. But $H^\mathcal{N}_{\omega_2}$ is a model of $T^*_{UB,NS_{\omega_1}}$.

We are left with the proof of the equivalence between (A), (B), (C), (D), (E).

(A) $\iff$ (B): By [9, Lemma 3.19] $T$ and $T^*$ have this property.

(A) $\implies$ (E): By Levy’s absoluteness if $\mathcal{M}$ models

$$T^*_V + \text{ZFC}^*_{UB,NS_{\omega_1}} + \text{MAX}(UB) + (\ast)$-UB

$H_{\omega_2} \models T^*$. Therefore if $T^* \models \psi$, $\mathcal{M} \models \psi^H_{\omega_2}$.

(E) $\implies$ (D): By definition of ($\ast$)-UB.

(D) $\implies$ (C): If $P$ forces $\text{MM}^{++}$, by Asperò and Schindler result $P \Vdash (\ast)$-UB, hence $P \Vdash \psi^H_{\omega_2}$ by (D).

(C) $\implies$ (B): Given some complete $S \supseteq T$, and a model $\mathcal{M}$ of $S$, find $\mathcal{N}$ forcing extension of $\mathcal{M}$ which models $\psi$. By Thm. 4 and Levy’s absoluteness [9, Lemma 4.1], $H^\mathcal{N}_{\omega_2} \models S_\psi$ and we are done. \hfill \square

2.2. Proofs of Thm. 2.2 and Thm. 4. The rest of this section is devoted to the proof of Thm. 2.2 and Thm. 4.

Let us first set up the proper language and terminology in order to deal with the $\mathbb{P}_{\max}$-technology.

2.2.1. Generic iterations of countable structures.

Definition 2.3. [6, Def. 1.2] Let $M$ be a transitive countable model of $\text{ZFC}$. Let $\gamma$ be an ordinal less than or equal to $\omega_1$. An iteration $\mathcal{J}$ of $M$ of length $\gamma$ consists of models $\langle M_\alpha : \alpha \leq \gamma \rangle$, sets $\langle G_\alpha : \alpha < \gamma \rangle$ and a commuting family of elementary embeddings

$$\langle j_{\alpha\beta} : M_\alpha \to M_\beta : \alpha \leq \beta \leq \gamma \rangle$$

such that:

- $M_0 = M$,
• each $G_\alpha$ is an $M_\alpha$-generic filter for $(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1})^{M_\alpha}$,
• each $j_{\alpha\alpha}$ is the identity mapping,
• each $j_{\alpha+1}$ is the ultrapower embedding induced by $G_\alpha$,
• for each limit ordinal $\beta \leq \gamma$, $M_\beta$ is the direct limit of the system $\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\}$,
and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

We adopt the convention to denote an iteration $\mathcal{J}$ just by $\langle j_{\alpha\beta} : \alpha \leq \beta \leq \gamma \rangle$, we also stipulate that if $X$ denotes the domain of $j_{\alpha\beta}$, $X_\alpha$ or $j_{\alpha\beta}(X)$ will denote the domain of $j_{\alpha\beta}$ for any $\alpha \leq \beta \leq \gamma$.

**Definition 2.4.** Let $A$ be a universally Baire sets of reals. $M$ is $A$-iterable if:

1. $M$ is transitive and such that $H^M_{\omega_1}$ is countable.
2. $M \models \text{ZFC + NS}_{\omega_1}$ is precipitous.
3. Any iteration $\{j_{\alpha\beta} : \alpha \leq \beta \leq \gamma \}$ of $M$ is well founded and such that $A \cap M_\beta = j_{\alpha\beta}(A \cap M_0)$ for all $\beta \leq \gamma$.

**2.2.2. Generic invariance of the universal fragment of the $\sigma_{\text{UB,NS}_{\omega_1}}$-theory of $V$.** We now prove Theorem 4.

**Proof.** Let $\phi$ be a $\Pi_1$-sentence for $\sigma_{A,\text{NS}_{\omega_1}}$ which holds in $V$. Assume that for some forcing notion $P$, $\phi$ fails in $V[h]$ with $h$ $V$-generic for $P$. By forcing over $V[h]$ with the appropriate stationary set preserving (in $V[h]$) forcing notion (using a Woodin cardinal $\gamma$ of $V[h]$), we may assume that $V[h]$ is extended to a generic extension $V[g]$ such that $V[g]$ models $\text{NS}_{\omega_1}$ is saturated$^9$. Since $V[g]$ is an extension of $V[h]$ by a stationary set preserving forcing and there are in $V[h]$ class many Woodin cardinals, we get that $V[h] \subseteq V[g]$ with respect to $\sigma_{\text{UB,NS}_{\omega_1}}$. Since $\Sigma_1$-properties are upward absolute and $\neg \phi$ holds in $V[h]$, $\phi$ fails in $V[g]$ as well.

Let $\delta$ be inaccessible in $V[g]$ and let $\gamma > \delta$ be a Woodin cardinal.

Let $G$ be $V$-generic for $\mathcal{T}_{\gamma}$ (the countable tower $\mathbb{Q}_{<\gamma}$ according to [7, Section 2.7]) and such that $g \in V[G]$. Let $j_G : V \to \text{Ult}(V,G)$ be the induced ultrapower embedding.

Now remark that $V_\delta[g] \in \text{Ult}(V,G)$ is $\mathcal{B}^{V[G]}$-iterable for all $B \in \mathcal{B}^{V} \ (\text{since } V_\eta[g] \in \text{Ult}(V,G) \text{ for all } \eta < \gamma)$, and this suffices to check that $V_\delta[g]$ is $\mathcal{B}^{V[G]}$-iterable for all $B \in \mathcal{B}^{V}$, see [6, Thm. 4.10]).

By [6, Lemma 2.8] applied in $\text{Ult}(V,G)$, there exists in $\text{Ult}(V,G)$ an iteration $\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \gamma = \omega_1^{\text{Ult}(V,G)}\}$ of $V_\delta[g]$ such that $\text{NS}_{\omega_1}^{X_\eta} = \text{NS}_{\omega_1}^{\text{Ult}(V,G) \cap X_\gamma}$, where $X_\alpha = j_{\alpha\alpha}(V_\alpha[g])$ for all $\alpha \leq \gamma = \omega_1^{\text{Ult}(V,G)}$.

This gives that $X_\gamma \subseteq \text{Ult}(V,G)$ for $\sigma_{\text{UB,NS}_{\omega_1}}$. Since $V_\delta[g] \models \neg \phi$, so does $X_\gamma$, by elementarity. But $\neg \phi$ is a $\Sigma_1$-sentence, hence it is upward absolute for superstructures, therefore $\text{Ult}(V,G) \models \neg \phi$. This is a contradiction, since $\text{Ult}(V,G)$ is elementarily equivalent to $V$ for $\sigma_{\text{UB,NS}_{\omega_1}}$, and $V \models \phi$.

A similar argument shows that if $V$ models a $\Sigma_1$-sentence $\phi$ for $\sigma_{\text{UB,NS}_{\omega_1}}$ this will remain true in all of its generic extensions:

Assume $V[h] \models \neg \phi$ for some $h$ $V$-generic for some forcing notion $P \in V$. Let $\gamma > |P|$ be a Woodin cardinal, and let $g$ be $V$-generic for $\mathcal{T}_\gamma$ with $h \in V[g]$ and $\text{crit}(j_g) = \omega_1^V$ (hence there is in $g$ some stationary set of $V_\gamma$ concentrating on countable sets). Then $V[g] \models \phi$ since:

• $V_\gamma \models \phi$, since $V_\gamma \prec_\gamma V$ for $\sigma_{\text{UB,NS}_{\omega_1}}$ by [9, Lemma 4.1];

$^9$A result of Shelah whose outline can be found in [8, Chapter XVI], or [10], or in an handout of Schindler available on his webpage.

$^{10}$A result of Shelah whose outline can be found in [8, Chapter XVI], or [10], or in an handout of Schindler available on his webpage.
\[ V^\text{Ult}(V,g) = V^V[g], \text{ since } V[g] \text{ models that } \text{Ult}(V,g) \prec V \subseteq \text{Ult}(V,g); \]
\[ V^\text{Ult}(V,g) \models \phi, \text{ by elementarity of } j_g, \text{ since } j_g(V^\text{Ult}(V,g)) = V^\text{Ult}(V,g); \]
\[ V^V[g] \prec \Sigma_1 V[g] \text{ with respect to } \sigma_{A,\text{NS}_{\omega_1}}, \text{ again by [9, Lemma 4.1] applied in } V[g]. \]

Now repeat the same argument as before to the \( \Pi_1 \)-property \( \neg \phi \), with \( V[h] \) in the place of \( V \) and \( V[g] \) in the place of \( V[h] \).

Asperó and Veličković provided the following basic counterexample to the conclusion of the theorem if large cardinal assumptions are dropped.

**Remark 2.5.** Let \( \phi(y) \) be the \( \Delta_1 \)-property in \( \tau_{\text{NS}_{\omega_1}} \)
\[ \exists y (y = \omega_1 \land L_{g+1} \models y = \omega_1). \]

Then \( L \) models this property, while the property fails in any forcing extension of \( L \) which collapses \( \omega_1 \) to become countable.

2.2.3. **Proof of Thm. 2.2.** We now turn to the proof of Thm. 2.2.

What we will do first is to sketch a different proof of Thm. 1.3. This will give us the key intuition on how to define \( \tilde{D}_{\text{UB,NS}_{\omega_1}} \).

2.2.4. **A different proof of Thm. 1.3.** Let \( M \) be a countable transitive model of \( \text{ZFC} + \) there are class many Woodin cardinals. Then it will have its own version of Thm. 1.3. In particular it will model that the theory of \( (H^M_{\omega_1}, \sigma^M_{\text{UB}M}) \) is model complete, and also that \( \text{UB}^M \) is an \( H_{\omega_1} \)-closed family of universally Baire sets in \( M \).

Now assume that there is a countable family \( \text{UB}_M \) of universally Baire sets in \( V \) which is \( H_{\omega_1} \)-closed in \( V \) and such that \( \text{UB}^M = \{ B \cap M : B \in \text{UB}_M \} \). Then
\[ (H^M_{\omega_1}, \sigma^M_{\text{UB}M}) = (H^M_{\omega_1}, \{ B \cap M : B \in \text{UB}_M \}) \subseteq (H^V_{\omega_1}, \sigma^V_{\text{UB}M}) \]

But \( \text{UB}_M \) being \( H_{\omega_1} \)-closed in \( V \) entails that the first order theory of \( (H^V_{\omega_1}, \sigma^V_{\text{UB}M}) \) is model complete. In particular if \( (H^M_{\omega_1}, \sigma^M_{\text{UB}M}) \) and \( (H^V_{\omega_1}, \sigma^V_{\text{UB}M}) \) are elementarily equivalent, then
\[ (H^M_{\omega_1}, \{ B \cap M : B \in \text{UB}_M \}) \prec (H^V_{\omega_1}, \sigma^V_{\text{UB}M}). \]

The setup described above is quite easy to realize (for example \( M \) could the transitive collapse of some countable \( X \prec V_\theta \) for some large enough \( \theta \)); in particular for any \( a \in H_{\omega_1} \) and \( B_1, \ldots, B_k \in \text{UB} \), we can find \( M \) countable transitive model of a suitable fragment of \( \text{ZFC} \) with \( a \in H^M_{\omega_1} \) and \( \text{UB}_M \supseteq \{ B_1, \ldots, B_k \} \) countable and \( H_{\omega_1} \)-closed family of \( \text{UB} \)-sets in \( V \), such that:
\[ \text{UB}^M = \{ B \cap M : B \in \text{UB}_M \}; \]
the first order theory \( T_{\text{UB}M} \) of \( (H^V_{\omega_1}, \sigma^V_{\text{UB}M}) \) is model complete;
\[ (H^M_{\omega_1}, \{ B \cap M : B \in \text{UB}_M \}) \models T_{\text{UB}M}. \]

Letting \( B_M = \prod \text{UB}_M, (H_{\omega_1} \cup \text{UB}, \in) \) is able to compute correctly whether \( B_M \) encodes a set \( \text{UB}_M \) such that the pair \( (\text{UB}_M, M) \) satisfies the above list of requirements; here we use crucially the fact that being a model complete theory is a \( \Delta_0 \)-property, and also that it is possible to encode the structure \( (H^V_{\omega_1}, \sigma^V_{\text{UB}M}) \) in a single universally Baire set\(^\text{11} \) (for example \( \text{WFE}_\omega \times B_M \)).

In particular \( (H_{\omega_1} \cup \text{UB}, \in) \) correctly computes the set \( D_{\text{UB}} \) of \( M \in H_{\omega_1} \) such that there exists a universally Baire set \( B_M = \prod \text{UB}_M \) with the property that the pair \( (M, \text{UB}_M) \) realizes the above set of requirements. By \( \text{MAX}(\text{UB}), \tilde{D}_{\text{UB}} = \text{Cod}_\omega^{-1}[D_{\text{UB}}] \) is a universally Baire set \( \tilde{D}_{\text{UB}} \).

Note moreover that \( \tilde{D}_{\text{UB}} \) is defined by a \( \in \)-formula \( \phi_{\text{UB}}(x) \) in no extra parameters; in particular for any model \( W = (W, E) \) of \( \text{ZFC} + \text{MAX}(\text{UB}) \), we can define \( \tilde{D}_{\text{UB}} \) in \( W \) and all its properties outlined above will hold relativized to \( W \).

\(^{\text{11}}\text{See [9, Def. 2.2] for the definition of } \text{WFE}_\omega \text{ and } \text{Cod}_\omega.\)
For fixed universally Baire sets $B_1, \ldots, B_k$ the set $D_{UB,B_1,\ldots,B_k}$ of $M \in D_{UB}$ such that there is a witness $UB_M$ of $M \in D_{UB}$ with $B_1, \ldots, B_k \in UB_M$ is also definable in $(H_{\omega_1} \cup UB, \in)$ in parameters $B_1, \ldots, B_k$. Hence by $\text{MAX}(UB)$ $\text{Cod}_\omega^{-1}[D_{UB,B_1,\ldots,B_k}] = D_{UB,B_1,\ldots,B_k}$ is universally Baire (note as well that $\overline{D}_{UB,B_1,\ldots,B_k}$ belongs to any $L(UB)$-closed family $A$ containing $B_1, \ldots, B_k$).

Now take any $\Sigma_1$-formula $\phi(\vec{x})$ for $\sigma_{UB}$ mentioning just the universally Baire predicates $B_1, \ldots, B_k$. It doesn’t take long to realize that for all $\vec{a}$ in $H_{\omega_1}$

$$(H_{\omega_1}^V, \sigma_{UB}^V) \models \phi(\vec{a})$$

if and only if

$$(H_{\omega_1}^M, \sigma_{UBM}^M) \models \phi(\vec{a}) \text{ for all } M \in D_{UB,B_1,\ldots,B_k} \text{ with } \vec{a} \in H_{\omega_1}^M.$$

But $\overline{D}_{UB,B_1,\ldots,B_k}$ is universally Baire, so the above can be formulated also as:

$$\forall r \in \overline{D}_{UB,B_1,\ldots,B_k} [\vec{a} \in H_{\omega_1}^{\text{Cod}(r)} \rightarrow (H_{\omega_1}^{\text{Cod}(r)}, \sigma_{UBM}^{\text{Cod}(r)}) \models \phi(\vec{a})].$$

The latter is a $\Pi_1$-sentence in the universally Baire parameter $\overline{D}_{UB,B_1,\ldots,B_k}$.

This is exactly a proof that Robinson’s test applies to the $\sigma_{UB}$-first order theory of $H_{\omega_1}$ assuming $\text{MAX}(UB)$; i.e. we have briefly sketched a different (and much more convoluted) proof of the conclusion of Thm. 1.3 (using as hypothesis Thm. 1.3 itself). What we gained however is an insight on how to prove Theorem 2.2.

We will consider the set $D_{NS_{\omega_1},UB}$ of $M \in D_{UB}$ such that:

- $(M, NS_{\omega_1}^M)$ is a $\mathbb{P}_{\text{max}}$-precondition which is $B$-iterable for all $B \in UB_M$ (according to $[6, \text{Def. 4.1}]$);
- $j_{0_{\omega_1}}$ is a $\Sigma_1$-elementary embedding of $H_{\omega_2}^M$ into $H_{\omega_2}^V$ for $\sigma_{UBM,NS_{\omega_1}}$ whenever $J = \{j_{0,\beta} : \alpha \leq \beta \leq \omega_1\}$ is an iteration of $M$ with $j_{0_{\omega_1}}(NS_{\omega_1}) = NS_{\omega_1}^V \cap j_{0_{\omega_1}}(H_{\omega_2}^M)$.

It will take a certain effort to prove that assuming $(*)$-$UB$:

- for any $A \in H_{\omega_2}$ and $B \in UB$, we can find $M \in D_{NS_{\omega_1},UB}$ with $B \in UB_M$, $a \in H_{\omega_2}^M$, and an iteration $J = \{j_{0,\beta} : \alpha \leq \beta \leq \omega_1\}$ of $M$ with $j_{0_{\omega_1}}(NS_{\omega_1}) = NS_{\omega_1}^V \cap j_{0_{\omega_1}}(H_{\omega_2}^M)$ such that $j_{0_{\omega_1}}(a) = A$.
- $D_{NS_{\omega_1},UB}$ is correctly computable in $(H_{\omega_1} \cup UB, \in)$.

But this effort will pay off since we will then be able to prove the model completeness of the theory

$$(H_{\omega_1}, \sigma_{UB,NS_{\omega_1}})^V$$

using Robinson’s test with $\text{Cod}_{\omega}^{-1}[D_{NS_{\omega_1},UB}]$ in the place of $\overline{D}_{UB}$ and replicating in the new setting what was sketched before for $(H_{\omega_1}, \sigma_{UB,NS_{\omega_1}})$.

We now get into the details.

2.2.5. UB-correct models.

**Notation 2.6.** Given a countable family $\mathcal{A} = \{B_n : n \in \omega\}$ of universally Baire sets with each $B_n \subseteq (2^{\omega})^{kn}$, we say that $B_\mathcal{A} = \prod_{n \in \omega} B_n \subseteq \prod_{n}(2^{\omega})^{kn}$ is a code for $\{B_n : n \in \omega\}$.

Clearly $B_\mathcal{A}$ is a universally Baire subset of the Polish space $\prod_{n}(2^{\omega})^{kn}$.

**Definition 2.7.** $T_{UB}$ is the $\in$-theory of

$$(H_{\omega_1}, \sigma_{UB}).$$

A transitive model of $ZFC$ $(M, \in)$ is UB-correct if there is a $H_{\omega_1}$-closed (in $V$) family $UB_M$ of universally Baire sets in $V$ such that:
• The map

\[ \Theta_M : UB_M \to M \]

\[ A \mapsto A \cap M \]

is injective.

• \((M, \in)\) models that \(\{A \cap M : A \in UB_M\}\) is the family of universally Baire subsets of \(M\).

• Letting \(T_{UB_M}\) be the theory of \((H_{\omega_1}, \tau^M_{ST}, UB_M)\)

\[ (H^M_{\omega_1}, \tau^M_{ST}, A \cap M : A \in UB_M) \models T_{UB_M}. \]

• If \(M\) is countable, \(M\) is \(A\)-iterable for all \(A \in UB_M\).

Remark (by Thm. 1.3) that if \(M\) is UB-correct, \(T_{UB_M}\) is model complete, since \(UB_M\) is (in \(V\)) a \(H_{\omega_1}\)-closed family of universally Baire sets.

**Notation 2.8.** \(D_{UB}\) denotes the set of countable UB-correct \(M\): \(D_{UB} = \text{Cod}_{\omega}^{-1}[D_{UB}]\).

For each \(M UB_M\) is a witness that \(M \in D_{UB}\) and \(B_{UB_M} = \prod UB_M\) is a universally Baire coding this witness\(^{12}\).

For universally Baire sets \(B_1, \ldots, B_k\), \(E_{UB,B_1,\ldots,B_k}\) denotes the set of \(M \in D_{UB}\) with \(B_1, \ldots, B_k \in UB_M\) for some witness \(UB_M\) that \(M \in D_{UB}\): \(E_{UB,B_1,\ldots,B_k} = \text{Cod}_{\omega}^{-1}[E_{UB,B_1,\ldots,B_k}]\).

**Fact 2.9.** \((V, \in)\) models \(M\) is countable and UB-correct as witnessed by \(UB_M\) if and only if so does \((H_{\omega_1} \cup UB, \in)\).

Consequently the set \(D_{UB}\) of countable UB-correct \(M\) is properly computed in \((H_{\omega_1} \cup UB, \in)\).

Therefore assuming \(\text{MAX}(UB)\)

\[ D_{UB} = \text{Cod}_{\omega}^{-1}[D_{UB}] \]

is universally Baire.

Moreover there is in \(L(UB)\) a definable map \(M \mapsto UB_M\) assigning to each \(M \in D_{UB}\) a countable family \(UB_M\) witnessing it.

The same holds for \(E_{UB,B_1,\ldots,B_k}\) for given universally Baire sets \(B_1, \ldots, B_k\).

**Proof.** The first part follows almost immediately by the definitions, since the assertion in parameters \(B, M\):

\[ B = \prod_{n \in \omega} B_n \text{ codes a } \omega_1 \text{-closed family } UB_M = \{B_n : n \in \omega\} \text{ of sets such that} \]

• \(M\) is \(A\)-iterable for all \(A \in UB_M\),
• \(M\) models that \(\{A \cap M : A \in UB_M\}\) is its family of universally Baire sets and is \(\omega_1\)-closed,
• \((H^M_{\omega_1}, \tau^M_{ST}, \{A \cap M : A \in UB_M\})\) models \(T_{UB_M}\).

gets the same truth value in \((V, \in)\) and in \((H_{\omega_1} \cup UB, \in)\).

We conclude that \(D_{UB}\) has the same extension in \((V, \in)\) and in \((H_{\omega_1} \cup UB, \in)\). By \(\text{MAX}(UB)\) \(D_{UB}\) is universally Baire.

The existence of class many Woodin cardinals grants that we can always find\(^{13}\) a universally Baire uniformization of the universally Baire relation on \(D_{UB} \times 2^\omega\) given by the pairs \((r, B)\) such that \(B = \prod \{B_n : n \in \omega\}\) witnesses \(\text{Cod}_{\omega}(r) \in D_{UB}\).

The same argument can be replicated for \(E_{UB,B_1,\ldots,B_k}\).

**Lemma 2.10.** Assume \(\text{NS}_{\omega_1}\) is precipitous and there are class many Woodin cardinals in \(V\). Let \(\delta\) be an inaccessible cardinal in \(V\) and \(G\) be \(V\)-generic for \(\text{Coll}(\omega, \delta)\). Then \(V_\delta\)
\(\text{is UB}^{13}[G]\)-correct in \(V[G]\) as witnessed by \(\{B^{V[G]} : B \in UB^V\}\).

\(^{12}\)The Fact below shows that the map \(M \mapsto (UB_M, B_{UB_M})\), can be chosen in \(L(UB)\).

\(^{13}\)For example by [5, Thm. 36.9] and [7, Thm. 3.3.14, Thm. 3.3.19].
Proof. Let in $V \{ (T_A, S_A) : A \in \mathcal{UB}^V \}$ be an enumeration of pairs of trees $S_A, T_A$ on $\omega \times \gamma$ for a large enough inaccessible $\gamma > \delta$ such that $T_A, S_A$ projects to complements in $V[G]$ and $A$ is the projection of $T$. Then $A^V[G]$ is correctly computed as the projection of $T_A$ in $V[G]$ for any $A \in \mathcal{UB}^V$.

By Fact 1.5 and [9, Thm. 4.7] 

$$(H^V_{\omega_1}, \tau^V_{ST}, \mathcal{UB}^V) \prec (H^V_{\omega_1}, \tau^V_{ST}[G], A^V[G] : A \in \mathcal{UB}^V),$$

$\{A^V[G] : A \in \mathcal{UB}^V\}$ is a $H_{\omega_1}$-closed family of universally Baire sets in $V[G]$, and $T_{\mathcal{UB}^V}$ is also the theory of $(H^V_{\omega_1}, \tau^V_{ST}[G], A^V[G] : A \in \mathcal{UB}^V)$.

To conclude that $\{A^V[G] : A \in \mathcal{UB}^V\}$ witnesses in $V[G]$ that $V_\delta$ is $\mathcal{UB}^V[G]$-correct in $V[G]$ it remains to argue that $V_\delta$ is $B^V[G]$-iterable for any $B \in \mathcal{UB}^V$.

Let $\mathcal{J}$ be any iteration of $V_\delta$ in $V[G]$. Then by standard results on iterations (see [6, Lemma 1.5, Lemma 1.6]) $\mathcal{J}$ extends uniquely to an iteration $\tilde{\mathcal{J}}$ of $V$ in $V[G]$ such that

- $\tilde{\mathcal{J}}_{\alpha \beta}$ is a proper extension of $j_{\alpha \beta}$ for all $\alpha \leq \beta \leq \gamma$ (i.e. letting $\tilde{V}_\alpha = \tilde{j}_{\alpha}(V)$, we have that $j_{\alpha}(V_\delta)$ is the rank initial segments of elements of $\tilde{V}_\alpha$ of rank less than $\tilde{j}_{\alpha}(\delta)$).
- $\tilde{\mathcal{J}}$ is a well defined iteration of transitive structures.

In particular this shows that $V_\delta$ is iterable in $V[G]$.

Now fix $B \in \mathcal{UB}^V$. We must argue that $\tilde{j}_{\alpha}(B) = B^V[G] \cap \tilde{j}_{\alpha}(V)$. To simplify notation we assume $B \subseteq 2^\omega$. Let $(T_B, S_B)$ be the pair of trees selected in $V$ to define $B^V[G]$.

Then

$$\tilde{j}_{\alpha}(V) \models (\tilde{j}_{\alpha}(T_B), \tilde{j}_{\alpha}(S_B))$$

projects to complements; clearly $\tilde{j}_{\alpha}(T_B) \subseteq \tilde{j}_{\alpha}(T_B), \tilde{j}_{\alpha}(S_B) \subseteq \tilde{j}_{\alpha}(S_B)$. Let $p : (\gamma \times 2^\omega) \to 2^\omega$ be the projection map.

This gives that

$$B^V[G] \cap \tilde{j}_{\alpha}(V) = p[[T_B]] \cap \tilde{j}_{\alpha}(V) = p[[\tilde{j}_{\alpha}(T_B)]][\tilde{j}_{\alpha}(V)] \subseteq p[[\tilde{j}_{\alpha}(T_B)]][\tilde{j}_{\alpha}(V) = \tilde{j}_{\alpha}(B)$.

Similarly

$$(2^\omega)^V \setminus B^V[G] \cap \tilde{j}_{\alpha}(V) = p[[S_B]] \cap \tilde{j}_{\alpha}(V) \subseteq p[[\tilde{j}_{\alpha}(S_B)][\tilde{j}_{\alpha}(V) = \tilde{j}_{\alpha}(B) \setminus B).$$

By elementarity

$$\tilde{j}_{\alpha}(B) \cup \tilde{j}_{\alpha}(B) = (2^\omega) \ast \tilde{j}_{\alpha}(V).$$

These three conditions can be met only if

$$B^V[G] \cap \tilde{j}_{\alpha}(V) = \tilde{j}_{\alpha}(B).$$

Since $\mathcal{J}$ and $B$ were chosen arbitrarily, we conclude that $V_\delta$ is $B^V[G]$-iterable in $V[G]$ for all $B \in \mathcal{UB}^V$.

Hence $V_\delta$ is $\mathcal{UB}^V[G]$-correct in $V[G]$ as witnessed by $\{A^V[G] : A \in \mathcal{UB}^V\}$. 

\[ \square \]

**Definition 2.11.** Given $M, N$ iterable structures, $M \geq N$ if $M \in (H_{\omega_1})^N$ and there is an iteration

$$\mathcal{J} = \{j_{\alpha \beta} : \alpha \leq \beta \leq \gamma = (\omega_1)^N\}$$

of $M$ with $\mathcal{J} \in N$ such that

$$\text{NS}^N_M = \text{NS}^N_M \cap M_\gamma.$$

**Fact 2.12.** (MAX(UB)) Assume $\text{NS}_{\omega_1}$ is precipitous and MAX(UB) holds. Then for any iterable $M$ and $B_1, \ldots, B_k \in \mathcal{UB}$, there is an UB-correct $N \geq M$ with $B_1, \ldots, B_k \in \mathcal{UB}_N$. 

Proof. The assumptions grant that whenever $G$ is $\text{Coll}(\omega, \delta)$-generic for $V$, in $V[G]$ $V_\delta$ is $\text{UB}^{V[G]}$-correct in $V[G]$ (i.e. Lemma 2.10).

By [6, Lemma 2.8], for any iterable $M \in H_{\omega_1}^V$ there is in $V$ an iteration $J = \{ j_{\alpha \beta} : \alpha \leq \beta \leq \omega_1^V \}$ of $M$ such that $\text{NS}^M_{\omega_1} \cap M_{\omega_1} = \text{NS}^M_{\omega_1}$.

By $\text{MAX}(\text{UB})$

$$(H_{\omega_1}^V \cup \text{UB}^V, \in) \prec (H_{\omega_1}^V \cup \text{UB}^{V[G]}_V, \in).$$

Therefore we have that in $V[G]$ $\bar{E}_{\text{UB}, B_1, \ldots, B_k}^{V[G]}$ is exactly $\bar{E}_{\text{UB}, B_1^G, B_2^G, \ldots, B_k^G}$.

Hence for each iterable $M \in H_{\omega_1}^V$ and $B \in \text{UB}^V$

$$(H_{\omega_1}^V, \sigma_{\text{UB}^V}^B) \models \exists N \geq M \text{ UB}^{V[G]} \text{-correct with } B^V \text{ in } \text{UB}_N,$$

as witnessed by $N = V_\delta$, i.e.

$$(H_{\omega_1}^V, \sigma_{\text{UB}^V}^B) \models \exists N \geq M (\bar{E}_{\text{UB}, B_1, \ldots, B_k}(N)).$$

Since

$$(H_{\omega_1}^V, \sigma_{\text{UB}^V}) \prec (H_{\omega_1}^V, \sigma_{\text{UB}^V}^B),$$

we get that for every iterable $M \in H_{\omega_1}$ and $B \in \text{UB}^V$

$$(H_{\omega_1}^V, \sigma_{\text{UB}^V}) \models \exists N \geq M (\bar{E}_{\text{UB}, B_1, \ldots, B_k}(N)).$$

The conclusion follows. \hfill $\square$

**Lemma 2.13.** ($\text{MAX}(\text{UB})$)

Let $M \geq N$ be both $\text{UB}$-correct structures, with $\text{UB}_N$ a witness of $N$ being $\text{UB}$-correct such that $D_{\text{UB}} \in \text{UB}_N$. Then

$$(H_M^N, \tau_{STM}, A \cap M : A \in \text{UB}_M) \prec (H_N^N, \tau_{STM}, A \cap N : A \in \text{UB}_M).$$

**Proof.** Since $N \leq M$, and $N$ is $\text{UB}$-correct with $D_{\text{UB}} \in \text{UB}_N$ we get that

$$(H_N^N, \sigma_{\text{UB}_N}) \models M \in D_{\text{UB}} \cap N = \text{Cod}[D_{\text{UB}} \cap N],$$

since

$$(H_N^N, \sigma_{\text{UB}_N}^N) \prec (H_{\omega_1}^V, \sigma_{\text{UB}_N}^N)$$

and

$$(H_{\omega_1}^V, \sigma_{\text{UB}_N}^V) \models M \in D_{\text{UB}} = \text{Cod}[D_{\text{UB}}].$$

Therefore $N$ models that there is a countable set $\text{UB}_M^N = \{ B_n^N : n \in \omega \} \in N$ coded by the universally Baire set in $N$ $B_n^{\text{UB}_M} = \prod_{n \in \omega} B_n^N$ such that $\{ A \cap M : A \in \text{UB}_M^N \} \in M$ defines the family of universally Baire sets according to $M$, and such that $N$ models that $M$ is $B^N$ iterable for all $B^N \in \text{UB}_M^N$. Now $N$ models that

$$\prod_{n \in \omega} B_n^N$$

is a universally Baire set on the appropriate product space. Therefore there is $B \in \text{UB}_N$ such that $B \cap N = \prod_{n \in \omega} B_n^N$. Clearly $\text{UB}_M^N$ is computable from $B \cap N$. Since

$$(H_{\omega_1}^V, \sigma_{\text{UB}_M}^V) \prec (H_{\omega_1}^V, \sigma_{\text{UB}_N}^V),$$

we conclude that in $V$ $B = \prod_{n \in \omega} B_n$ codes a set $\text{UB}_M = \{ B_n : n \in \omega \}$ witnessing that $M$ is $\text{UB}$-correct.

This gives that $\text{UB}_M \subseteq \text{UB}_N$.

Therefore $(H_{\omega_1}^V, \sigma_{\text{UB}_M})$ is also a model of $T_{\text{UB}_M}$. By model completeness of $T_{\text{UB}_M}$ we conclude that

$$(H_{\omega_1}^V, \sigma_{\text{UB}_M}) \prec (H_{\omega_1}^V, \sigma_{\text{UB}_N}),$$

as was to be shown. \hfill $\square$
2.3. Three characterizations of \((*)\)-UB.

**Definition 2.14.** For a UB-correct \(M\) with witness \(UB_M\), \(T_{\text{NS}_{\omega_1}, UB_M}\) is the \(\sigma_{UB_M, \text{NS}_{\omega_1}}\)-theory of \(H^M_{\omega_2}\).

A UB-correct \(M\) is \((\text{NS}_{\omega_1}, \text{UB})\)-ec if \((M, \in)\) models that \(\text{NS}_{\omega_1}\) is precipitous and there is a witness \(UB_M\) that \(M\) is UB-correct with the following property:

Assume an iterable \(N \supseteq M\) is UB-correct with witness \(UB_N\) such that \(D_{UB_M} \in UB_N\) (so that \(UB_M \subseteq UB_N\)).

Then for all iterations

\[ J = \{ j_{\alpha \beta} : \alpha \leq \beta \leq \gamma = \omega_1^N \} \]

in \(N\) witnessing \(M \supseteq N\), we have that \(j_{0 \gamma}\) defines a \(\Sigma_1\)-elementary embedding of

\[ (H^M_{\omega_2}, T^M_{\omega_1}, B \cap M : B \in UB_M, \text{NS}^M_{\omega_1}) \]

into

\[ (H^N_{\omega_2}, T^N_{\omega_1}, B \cap N : B \in UB_M, \text{NS}^N_{\omega_1}). \]

**Remark 2.15.** A crucial observation is that “\(x\) is \((\text{NS}_{\omega_1}, \text{UB})\)-ec” is a property correctly definable in \((H_{\omega_1} \cup UB, \in)\). Therefore (assuming \(\text{MAX}(UB)\))

\[ D_{\text{NS}_{\omega_1}, UB} = \{ M \in H_{\omega_1} : M \text{ is } (\text{NS}_{\omega_1}, \text{UB})\text{-ec} \} \]

is such that \(D_{\text{NS}_{\omega_1}, UB} = \text{Cod}_{\omega}^{-1}[D_{\text{NS}_{\omega_1}, UB}]\) is a universally Baire set in \(V\). Moreover letting for \(V[G]\) a generic extension of \(V\)

\[ D_{\text{NS}_{\omega_1}, UB}^{V[G]} = \left\{ M \in H_{\omega_1}^{V[G]} : M \text{ is } (\text{NS}_{\omega_1}, \text{UB}^{V[G]})\text{-ec} \right\} , \]

we have that

\[ D_{\text{NS}_{\omega_1}, UB}^{V[G]} = \text{Cod}_{\omega}^{-1}[D_{\text{NS}_{\omega_1}, UB}]^{V[G]} \].

**Theorem 2.16.** Assume \(V\) models \(\text{MAX}(UB)\). The following are equivalent:

1. Woodin’s axiom \((*)\)-UB holds (i.e. \(\text{NS}_{\omega_1}\) is saturated, and there is an \(L(UB)\)-generic filter \(G\) for \(P_{\max}\) such that \(L(UB)[G] \supseteq P(\omega_1)^V\)).
2. Let \(\delta\) be inaccessible. Whenever \(G\) is \(V\)-generic for \(\text{Coll}(\omega, \delta)\), \(V_\delta\) is \((\text{NS}_{\omega_1}, \text{UB}^{V[G]})\)-ec in \(V[G]\).
3. \(\text{NS}_{\omega_1}\) is precipitous and for all \(\bar{A} \in H_{\omega_2}, B \in UB\), there is an \((\text{NS}_{\omega_1}, \text{UB})\)-ec \(M\) with witness \(UB_M\), and an iteration \(J = \{ j_{\alpha \beta} : \alpha \leq \beta \leq \omega_1 \}\) of \(M\) such that:
   - \(A \in M_{\omega_1}\),
   - \(B \in UB_M\),
   - \(\text{NS}^M_{\omega_1} = \text{NS}_{\omega_1} \cap M_{\omega_1}\).

Theorem 2.16 is the key to the proofs of Theorem 2.2 and to the missing implication in the proof of Theorem 1.

2.3.1. **Proof of Theorem 2.2.** The theorem is an immediate corollary of the following:

**Lemma 2.17.** Let \(B_1, \ldots, B_k\) be new predicate symbols and \(T_{B_1, \ldots, B_k, \text{NS}_{\omega_1}}\) be the \(\tau_{\text{NS}_{\omega_1}} \cup \{ B_1, \ldots, B_k \}\)-theory \(ZFC_{\text{NS}_{\omega_1}} + \text{MAX}(UB)\) enriched with the sentences asserting that \(B_1, \ldots, B_k\) are universally Baire sets.

Let \(E_{B_1, \ldots, B_k}\) consists of the set of \(M \in D_{\text{NS}_{\omega_1}, UB}\) such that:

- \(M\) is \(B_j\)-iterable for all \(j = 1, \ldots, k\);
- there is \(UB_M\) witnessing \(M \in D_{\text{NS}_{\omega_1}, UB}\) with \(B_j \in UB_M\) for all \(j\).
Let also $E_{B_1, \ldots, B_k} = \text{Cod}^{-1}[E_{B_1, \ldots, B_k}]$.

Then $T_{B_1, \ldots, B_k, \text{NS}_{\omega_1}}$ proves that $E_{B_1, \ldots, B_k}$ is universally Baire.

Moreover let $T_{B_1, \ldots, B_k, E_{B_1, \ldots, B_k}, \text{NS}_{\omega_1}}$ be the natural extension of $T_{B_1, \ldots, B_k, \text{NS}_{\omega_1}}$ adding a predicate symbol for $E_{B_1, \ldots, B_k}$ and the axiom forcing its interpretation to be its definition.

Then $T_{B_1, \ldots, B_k, E_{B_1, \ldots, B_k}, \text{NS}_{\omega_1}}$ models that every $\Sigma^1_1$-formula $\phi(\vec{x})$ for the signature $\text{NS}_{\omega_1} \cup \{B_1, \ldots, B_k\}$ is equivalent to a $\Pi^1_1$-formula $\psi(\vec{x})$ in the signature $\text{NS}_{\omega_1} \cup \{B_1, \ldots, B_k, E_{B_1, \ldots, B_k}\}$.

**Proof.** $E_{B_1, \ldots, B_k}$ is universally Baire by $\text{MAX}(\text{UB})$, since $E_{B_1, \ldots, B_k}$ is definable in $(H_{\omega_1} \cup \text{UB}, \in)$ with parameters the universally Baire sets $B_1, \ldots, B_k, D_{\text{NS}_{\omega_1}}$, $\text{UB}$.

Given any $\Sigma^1_1$-formula $\phi(\vec{x})$ for $\text{NS}_{\omega_1} \cup \{B_1, \ldots, B_k\}$ mentioning the universally Baire predicates $B_1, \ldots, B_k$, we want to find a universal formula $\psi(\vec{x})$ such that

$$T_{\{B_1, \ldots, B_k, E_{B_1, \ldots, B_k}\}, \text{NS}_{\omega_1}} \models \forall \vec{x}(\phi(\vec{x}) \leftrightarrow \psi(\vec{x})).$$

Let $\psi(\vec{x})$ be the formula asserting:

- For all $M \in E_{B_1, \ldots, B_k}$, for all iterations $J = \{j_\alpha \beta : \alpha \leq \beta \leq \omega_1\}$ of $M$ such that:
  - $\vec{x} = j_{0\omega_1}(\vec{a})$ for some $\vec{a} \in M$,
  - $\text{NS}_{j_{0\omega_1}}(M) = \text{NS}_{\omega_1} \cap j_{0\omega_1}(M)$,
    $$(H_{\omega_2}^{M}, \sigma_{\text{UB}^{j_{0\omega_1}}}^{M}, \text{NS}_{\omega_1}) \models \phi(\vec{a}).$$

More formally,

$$\forall r \forall J \{$$

$$\{r \in E_{B_1, \ldots, B_k}\}$$

$$\wedge J = \{j_\alpha \beta : \alpha \leq \beta \leq \omega_1\} \text{ is an iteration of } \text{Cod}(r) \wedge$$

$$\wedge \text{NS}_{j_{0\omega_1}}(\text{Cod}(r)) = \text{NS}_{\omega_1} \cap j_{0\omega_1}(\text{Cod}(r)) \wedge$$

$$\wedge \exists \vec{a} \in \text{Cod}(r) (\vec{x} = j_{0\omega_1}(\vec{a}))$$

$$\} \rightarrow$$

$$(H_{\omega_2}^{\text{Cod}(r)}, \sigma_{\text{UB}^{\text{Cod}(r)}}^{\text{Cod}(r)}, \text{NS}_{\omega_1}) \models \phi(\vec{a}).$$

The above is a $\Pi^1_1$-formula for $\text{NS}_{\omega_1} \cup \{B_1, \ldots, B_k, E_{B_1, \ldots, B_k}\}$.

(We leave to the reader to check that the property

$$J = \{j_\alpha \beta : \alpha \leq \beta \leq \omega_1\} \text{ is an iteration of } M \text{ such that } \text{NS}_{j_{0\omega_1}}(M) = \text{NS}_{\omega_1} \cap j_{0\omega_1}(M)$$

is definable by a $\Delta^1_1$-property in parameters $M, J$ in the signature $\text{NS}_{\omega_1}$).

Now it is not hard to check that:

**Claim 1.** For all $\vec{A} \in H_{\omega_2}$

$$(H_{\omega_2}^{V}, \text{NS}_{\omega_1}^{V}, B_1, \ldots, B_k) \models \phi(\vec{A})$$

if and only if

$$(H_{\omega_2}^{V}, \text{NS}_{\omega_1}^{V}, B_1, \ldots, B_k, E_{B_1, \ldots, B_k}) \models \psi(\vec{A}).$$

**Proof.**
ψ(\vec{A}) \rightarrow \phi(\vec{A})$: Take any \( M \) and \( J \) satisfying the premises of the implication in \( \psi(\vec{A}) \), then \( (H_{\omega_2}^{M}, \tau_{\mathbf{NS}_{\omega_1}}^{M}, UB^M) \models \phi(\vec{a}) \) for some \( \vec{a} \) such that \( j_{0, \omega_1}(\vec{a}) = \vec{A} \) and \( B_j \cap M_{\omega_1} = j_{0, \omega_1}(B_j \cap M) \) for all \( j = 1, \ldots, k \).

Since \( \Sigma_1 \)-properties are upward absolute and \( (M_{\omega_1}, \tau_{\mathbf{NS}_{\omega_1}}^{M}, B_j \cap M_{\omega_1} : j = 1, \ldots, k) \) is a \( \mathbf{NS}_{\omega_1} \cup \{B_1, \ldots, B_k\} \)-substructure of \( (H_{\omega_2}, \tau_{\mathbf{NS}_{\omega_1}}^{V}, B_j : j = 1, \ldots, k) \) which models \( \phi(\vec{A}) \), we get that \( \phi(\vec{A}) \) holds for \( (H_{\omega_2}, \tau_{\mathbf{NS}_{\omega_1}}^{V}, B_1, \ldots, B_k) \).

\( \phi(\vec{A}) \rightarrow \psi(\vec{A}) \): Assume

\[
(H_{\omega_2}, \tau_{\mathbf{NS}_{\omega_1}}^{V}, B_1, \ldots, B_k) \models \phi(\vec{A}).
\]

Take any \( (\mathbf{NS}_{\omega_1}, UB) \)-ec \( M \in V \) and any iteration \( J = \{j_\alpha : \alpha \leq \beta \leq \omega_1\} \) of \( M \) witnessing the premises of the implication in \( \psi(\vec{A}) \), in particular such that:

- \( \vec{A} = j_{0, \omega_1}(\vec{a}) \in M_{\omega_1} \) for some \( \vec{a} \in M \),
- \( NS_{\omega_1}^{M} = NS_{\omega_1} \cap M_{\omega_1} \),
- \( M \) is \( B_j \)-iterable for \( j = 1, \ldots, k \).

Such \( M \) and \( J \) exists by Thm. 2.16(3) applied to \( \mathcal{E}_{B_1, \ldots, B_k} \) and \( \vec{A} \).

Let \( G \) be \( V \)-generic for Coll(\( \omega, \delta \)) with \( \delta \) inaccessible. Then in \( V[G] \), \( V_\delta \) is \( UB^{V[G]} \)-correct, by Lemma 2.10.

Therefore (since \( M \) is \( (\mathbf{NS}_{\omega_1}, UB^{V[G]} \)-ec also in \( V[G] \) by \( \mathbf{MAX}(UB) \)), \( V[G] \) models that \( j_{0, V} \) is a \( \Sigma_1 \)-elementary embedding of \n
\[
(H_{\omega_2}^{M}, \tau_{\mathbf{NS}_{\omega_1}}^{M}, B \cap M : B \in UB^M)
\]

into \n
\[
(H_{\omega_2}^{V}, \tau_{\mathbf{NS}_{\omega_1}}^{V}, B : B \in UB^M).
\]

This grants that \n
\[
(H_{\omega_2}^{M}, \tau_{\mathbf{NS}_{\omega_1}}^{M}, B \cap M : B \in UB^M) \models \phi(\vec{a}),
\]

as was to be shown. \( \square \)

The Lemma is proved. \( \square \)

2.3.2. Proof of (2)\( \rightarrow \) (1) of Theorem 1.

Proof. Assume \( \delta \) is supercompact, \( P \) is a standard forcing notion to force \( MM^{++} \) of size \( \delta \) (such as the one introduced in [3] to prove the consistency of Martin’s maximum), and \( G \) is \( V \)-generic for \( P \); then \( (*) \)-UB holds in \( V[G] \) by Asperó and Schindler’s recent breakthrough [2]. By Thm. 4 \( V \) and \( V[G] \) agree on the \( \Pi_1 \)-fragment of their \( \sigma_{UB^{V[G]}, \mathbf{NS}_{\omega_1}} \) theory, therefore so do \( H_{\omega_2}^{V} \) and \( H_{\omega_2}^{V[G]} \) (by [9], Lemma LEVABS applied in \( V \) and \( V[G] \) respectively).

Since \( P \in \mathbf{SSP} \)

\[
(H_{\omega_2}^{V}, \tau_{\mathbf{NS}_{\omega_1}}^{V}, A : A \in UB^V) \sqsubseteq (H_{\omega_2}^{V[G]}, \tau_{\mathbf{NS}_{\omega_1}}^{V[G]}, A^{V[G]} : A \in UB^{V[G]}).
\]

Now the model completeness of \( T_{\mathbf{NS}_{\omega_1}, UB} \)-grants that any of its models (among which \( H_{\omega_2}^{V} \)) is \( (T_{\mathbf{NS}_{\omega_1}, UB})^{\forall, \exists} \). This gives that:

\[
(H_{\omega_2}^{V}, \tau_{\mathbf{NS}_{\omega_1}}^{V}, UB^V) \preceq_{\Sigma_1} (H_{\omega_2}^{V[G]}, \tau_{\mathbf{NS}_{\omega_1}}^{V[G]}, A^{V[G]} : A \in UB).
\]

Therefore any \( \Pi_2 \)-property for \( \sigma_{UB, \mathbf{NS}_{\omega_1}} \) with parameters in \( H_{\omega_2}^{V} \) which holds in \( H_{\omega_2}^{V[G]} \) also holds in \( H_{\omega_2}^{V[G]} \) with \( \tau_{\mathbf{NS}_{\omega_1}}^{V[G]} \), that is:

\[
(H_{\omega_2}^{V[G]}, \tau_{\mathbf{NS}_{\omega_1}}^{V[G]}, A^{V[G]} : A \in UB).
\]

Therefore this completes the proof of \( (2) \rightarrow (1) \).
also holds in \((H^V_{\omega_2}, \tau^V_{\text{NS}_{\omega_1}}, \text{UB}^V)\).

Hence in \(H^V_{\omega_2}\) it holds characterization (3) of \((\ast)\)-UB given by Thm. 2.16 and we are done. \(\square\)

2.3.3. Proof of Theorem 2.16.

Proof. Schindler and Asperó [1, Def. 2.1] introduced the following:

**Definition 2.18.** Let \(\phi(\vec{x})\) be a \(\sigma_{\text{UB}, \text{NS}_{\omega_1}}\)-formula in free variables \(\vec{x}\), and \(\vec{A} \in H^V_{\omega_2}\). \(\phi(\vec{A})\) is \(\text{UB}-honestly\ \text{consistent}\) if for all universally Baire sets \(U \in \text{UB}^V\), there is some large enough cardinal \(\kappa \in V\) such that whenever \(G\) is \(V\)-generic for \(\text{Coll}(\omega, \kappa)\), in \(V[G]\) there is a \(\sigma_{\text{UB}, \text{NS}_{\omega_1}}\)-structure \(M = (M, \ldots)\) such that

- \(M\) is transitive and \(U\)-iterable,
- \(M \models \phi(\vec{A})\),
- \(\text{NS}^M_{\omega_1} \cap V = \text{NS}^V_{\omega_1}\).

They also proved the following Theorem [1, Thm. 2.7, Thm. 2.8]:

**Theorem 2.19.** Assume \(V\) models \(\text{NS}_{\omega_1}\) is precipitous and \(\text{MAX}(\text{UB})\) holds.

**TFAE:**

- \((\ast)\)-UB holds in \(V\).
- Whenever \(\phi(\vec{x})\) is a \(\Sigma_1\)-formula for \(\sigma_{\text{UB}, \text{NS}_{\omega_1}}\) in free variables \(\vec{x}\), and \(\vec{A} \in H^V_{\omega_2}\), \(\phi(\vec{A})\) is honestly consistent if and only if it is true in \(H^V_{\omega_2}\).

We use Schindler and Asperó characterization of \((\ast)\)-UB to prove the equivalences of the three items of Thm. 2.16 (the proofs of these implications import key ideas from [2, Lemma 3.2]).

(1) implies (2): Let \(G\) be \(V\)-generic for \(\text{Coll}(\omega, \delta)\). By Lemma 2.10, \(V_\delta\) is \(\text{UB}^{V[G]}\)-correct in \(V[G]\) as witnessed by \(\{B^{V[G]} : B \in \text{UB}^V\} = \text{UB}^V = \{B^n_{V[G]} : n \in \omega\}\).

**Claim 2.** \(V_\delta\) is \((\text{NS}_{\omega_1}, \text{UB}^{V[G]}\)-ec as witnessed by \(	ext{UB}^V\).

Proof. Let in \(V[G]\) \(B_V = B_{\text{UB}^V} = \prod_{n \in \omega} B^n_{V[G]}\) be the universally Baire set coding \(\text{UB}^V\).

Let \(N \leq V_\delta\) in \(V[G]\) be \(\text{UB}^{V[G]}\)-correct with \(B_V \in \text{UB}^V\) for some \(\text{UB}^V\) witnessing that \(N\) is \(\text{UB}^{V[G]}\)-correct. Then we already observed that \(B^{V[G]} \cap N : B^{V[G]} \in \text{UB}^V\) \(\subseteq\) \(\{B \cap N : B \in \text{UB}^V\}\). Therefore

\[
(H^V_{\omega_1}, \sigma^{V}_{\text{UB}^V}) = (H^V_{\omega_1}, \sigma^{V}_{\text{UB}^V}) < (H^N_{\omega_1}, \tau^N_{\text{ST}}, B^{V[G]} \cap N : B \in \text{UB}^V).
\]

Let 

\[
J = \{j_{\alpha, \beta} : \alpha \leq \beta \leq \gamma = (\omega_1)^V\} \in N
\]

be an iteration witnessing \(V_\delta \geq N\) in \(V[G]\).

We must show that 

\[
j_{0, \gamma} : H^V_{\omega_2} \rightarrow H^N_{\omega_2}
\]

is \(\Sigma_1\)-elementary for \(\tau^{\text{NS}_{\omega_1}, \text{UB}^V}\) between

\[
(H^V_{\omega_2}, \tau^{V}_{\text{ST}}, \text{UB}^V, \text{NS}^V_{\omega_1})
\]

and

\[
(H^N_{\omega_2}, \tau^N_{\text{ST}}, B^{V[G]} \cap N : B \in \text{UB}^V, \text{NS}^N_{\omega_1}).
\]

Let \(\phi(a)\) be a \(\Sigma_1\)-formula for \(\tau^{\text{NS}_{\omega_1}, \text{UB}^V}\) in parameter \(a \in H^V_{\omega_2}\) with \(B_1, \ldots, B_k \in \text{UB}^V\) the universally Baire predicates occurring in \(\phi\) such that

\[
(N, \tau^N_{\text{ST}}, B^{V[G]} \cap N : B \in \text{UB}^V, \text{NS}^N_{\omega_1}) \models \phi(j_{0, \gamma}(a)).
\]
We must show that

\((H^V_{\omega_2}, \tau^V_{\omega_2}, \text{UB}^V, N_{\omega_1}) \models \phi(a)\).

Remark that the iteration \(\mathcal{J}\) extends to an iteration \(\bar{\mathcal{J}} = \{\bar{j}_{\alpha, \beta} : \alpha \leq \beta \leq \gamma = (\omega_1)^N\}\) of \(V\) exactly as already done in the proof of Lemma 2.10.

Using this observation, let \(\bar{M} = \bar{j}_{0\gamma}(V)\); then \(N_{\omega_1}^\bar{M} = N_{\omega_1}^V \cap \bar{M}\).

Now let \(H\) be \(V\)-generic for \(\text{Coll}(\omega, \eta)\) with \(G \in V[H]\) for some \(\eta > \delta\) inaccessible in \(V[G]\).

By \(\text{MAX}(\text{UB})\) \(N\) is \(\text{UB}^V[H]\)-correct in \(V[H]\): on the one hand

\[D_{\text{UB}^V[H]} = \text{Cod}[\bar{D}^V_{\text{UB}^V[H]}],\]

on the other hand

\[N \in \text{Cod}[\bar{D}^V_{\text{UB}^V[H]}] \subseteq \text{Cod}[\bar{D}^V_{\text{UB}^V[H]}].\]

In particular for any \(B \in \text{UB}^V\), \(N\) is \(\text{UB}^V[H]\)-iterable in \(V[H]\).

Therefore in \(H^\omega_{\omega_2}\) for any \(B \in \text{UB}^V\), the statement

\(\text{There exists a } \tau_{\text{NS}_{\omega_1}} \cup \{B, B_1, \ldots, B_k\}\)-super-structure \(\bar{N}\) of \(\bar{j}_{0\gamma}(V)\) which

is \(\{B^V[H], B_1^V[H], \ldots, B_k^V[H]\}\)-iterable and which realizes \(\phi(j_{0\gamma}(a))\)

holds true as witnessed by \(N\).

The following is a key observation:

**Subclaim 1.** For any \(s \in (2^\omega)^\bar{M}[H]\) and \(B \in \text{UB}^V\)

\(s \in j_{0\gamma}(B)^{\text{NS}[H]}\) if and only if \(s \in (2^\omega)^{\text{UB}^V[H]} \cap \bar{M}[H]\).

**Proof.** For each \(B \in \text{UB}^V\) find in \(V\) trees \((T_B, S_B)\) which project to complement in \(V[H]\) and such that \(B = p[T_B]\). Now since \(j_{0\gamma}[T_B] \subseteq j_{0\gamma}(T_B)\) and \(j_{0\gamma}[S_B] \subseteq j_{0\gamma}(S_B)\), we get that

- \((2^\omega)^{V[H]} = p[j_{0\gamma}(T_B)] \cup p[j_{0\gamma}(S_B)]\) (since \((2^\omega)^{V[H]}\) is already covered by \(p[j_{0\gamma}(T_B)] \cup p[j_{0\gamma}(S_B)]\)).
- \(\emptyset = p[j_{0\gamma}(T_B)] \cap p[j_{0\gamma}(S_B)]\) by elementarity of \(j_{0\gamma}\).

Hence \(B^{V[H]}\) is also the projection of \(j_{0\gamma}(T_B)\) and the pair \((j_{0\gamma}(T_B), j_{0\gamma}(S_B))\) projects to complement in \(V[H]\).

But this pair belongs to \(\bar{M}\), and (by elementarity of \(j_{0\gamma}\))

\(\bar{M} \models (j_{0\gamma}(T_B), j_{0\gamma}(S_B))\) projects to complements for \(\text{Coll}(\omega, j_{0\gamma}(\eta))\).

Since \(\eta \leq j_{0\gamma}(\eta)\) we get that

\(\bar{M} \models (j_{0\gamma}(T_B), j_{0\gamma}(S_B))\) projects to complements for \(\text{Coll}(\omega, \eta)\).

Therefore in \(V[H]\) \(s \in j_{0\gamma}(B)^{\text{NS}[H]}\) if and only if \(s \in p[j_{0\gamma}(T_B)]^{V[H]} \cap \bar{M}[H]\) if and only if \(s \in p[T_B]^{V[H]} \cap \bar{M}[H]\) if and only if \(s \in (2^\omega)^{\text{UB}^V[H]} \cap \bar{M}[H]\). \(\square\)

This shows that

\((\bar{M}[H], \sigma_{\text{NS}[H]}) \subseteq (V[H], \sigma_{\text{UB}^V[H]}).\)

Moreover \(H^{\bar{M}[H]}_{\omega_1}\) and \(H^{\text{UB}^V[H]}_{\omega_1}\) both realize the theory \(T_{\text{UB}^V}\) of \(H^V_{\omega_1}\) in this language: on the one hand

\((H^V_{\omega_1}, \sigma_{\text{UB}^V}) \prec (H^{\bar{M}[H]}_{\omega_1}, \sigma_{\text{NS}[H]}^{\bar{M}[H]} ) \prec (H^{\text{UB}^V[H]}_{\omega_1}, \sigma_{\text{UB}^V[H]}^{\bar{M}[H]} )\)

(the leftmost \(\prec\) holds since \(j_{0\gamma} : V \rightarrow \bar{M}\) is elementary, the rightmost \(\prec\) holds since \(\bar{M}\) models \(\text{MAX}(\text{UB})\)); on the other hand

\((H^V_{\omega_1}, \sigma_{\text{UB}^V}) \prec (H^{\text{UB}^V[H]}_{\omega_1}, \sigma_{\text{UB}^V[H]}^{\text{UB}^V[H]} )\)
(applying \( \mathbf{MAX}(UB) \) in \( V \)).

Since \( T_{UBV} \) is model complete, we get that \( H_{\omega_1}^{M[H]} \) is an elementary \( \sigma_{UBV} \)-substructure of \( H_{\omega_1}^{V[H]} \); therefore \( H_{\omega_1}^{M[H]} \) models

There exists a \( \tau_{NS_{\omega_1}, B, B_1, \ldots, B_k} \)-super-structure \( \tilde{N} \) of \( j_{0, \gamma}(V_\delta) \) which is

\[
\{ \tilde{j}_{0, \gamma}(B)^{\tilde{M}[H]}, \tilde{j}_{0, \gamma}(B_1)^{\tilde{M}[H]}, \ldots, \tilde{j}_{0, \gamma}(B_k)^{\tilde{M}[H]} \}\text{-iterable and which realizes } \phi(j_{0, \gamma}(a)) \]

By homogeneity of \( \text{Coll}(\omega, \eta) \), in \( \tilde{M} \) we get that any condition in \( \text{Coll}(\omega, \eta) \) forces:

There exists a \( \tau_{NS_{\omega_1}, B, B_1, \ldots, B_k} \)-super-structure \( \tilde{N} \) of \( j_{0, \gamma}(V_\delta) \) which is

\[
\{ \tilde{j}_{0, \gamma}(B)^{\tilde{M}[H]}, \tilde{j}_{0, \gamma}(B_1)^{\tilde{M}[H]}, \ldots, \tilde{j}_{0, \gamma}(B_k)^{\tilde{M}[H]} \}\text{-iterable and which realizes } \phi(j_{0, \gamma}(a)) \]

By elementarity of \( \tilde{j}_{0, \gamma} \) we get that in \( V \) it holds that:

There exists an \( \eta > \delta \) such that any condition in \( \text{Coll}(\omega, \eta) \) forces:

"There exists a countable super structure \( \tilde{N} \) of \( V_\delta \) with respect to \( \tau_{NS_{\omega_1}, B, B_1, \ldots, B_k} \) which is \( \{ B^{V[H]}, B_1^{V[H]}, \ldots, B_k^{V[H]} \} \)-iterable and which realizes \( \phi(a) \)"

This procedure can be repeated for any \( B \in UB \), showing that \( \phi(a) \) is honestly consistent in \( V \).

By Schindler and Asperó characterization of \( * \) we obtain that \( \phi(a) \) holds in \( H_{\omega_1}^{V_\delta} \).

\( (2) \text{ implies } (3) \): Our assumptions grants that the set

\[
D_{UB} = \{ M \in H_{\omega_1}^{V_\delta} : M \text{ is } UB^{V_\delta} \text{-correct} \}
\]

is coded by a universally Baire set \( \tilde{D}_{UB} \) in \( V \). Moreover we also get that whenever \( G \) is \( V \)-generic for \( \text{Coll}(\omega, \delta) \), the lift \( D_{UB}^{V[G]} \) of \( D_{UB} \) to \( V[G] \) codes

\[
D_{UB}^{V[G]} = \{ M \in H_{\omega_1}^{V[G]} : M \text{ is } UB^{V[G]} \text{-correct} \}.
\]

By \( (2) \) we get that \( V_\delta \in D_{UB}^{V[G]} \).

By Fact 2.12

\[
(H_{\omega_1}^{V, \tau_{ST}^V, UB} \models \text{ for all iterable } M \text{ there exists an } UB \text{-correct structure } M \geq M.)
\]

Again since

\[
(H_{\omega_1}^{V, \tau_{ST}^V, UB} \models \text{ and the latter is first order expressible in the predicate } \tilde{D}_{UB} \in UB \text{, we get that })
\]

\[
(H_{\omega_1}^{V[G], \tau_{ST}^V[G], UB} \models \text{ for all iterable } M \text{ there exists an } UB^{V[G]} \text{-correct structure } M \geq M.)
\]

So let \( N \leq V_\delta \) be in \( V[G] \) an \( UB^{V[G]} \)-correct structure with \( V_\delta \in H_{\omega_2}^N \).

Let \( J = \{ j_{\alpha, \beta} : \alpha \leq \beta \leq \gamma = \omega_1^V \} \in H_{\omega_2}^N \) be an iteration witnessing \( N \leq V_\delta \).

Now for any \( A \in P(\omega_1)^V \) and \( B \in UB \)

\[
(H_{\omega_2}^N, \tau_{ST}^N, \text{NS}_N^\gamma, B^{V[G]} \cap N : B \in UB)
\]

models

There exists an \( (\text{NS}_{\omega_1}, UB^{V[G]}) \)-ec structure \( M \) with \( B^{V[G]} \cap N \subseteq UB_M \) and an iteration \( \tilde{J} = \{ \tilde{j}_{\alpha, \beta} : \alpha \leq \beta \leq \gamma \} \) of \( M \) such that \( \tilde{j}_{0, \gamma}(A) = j_{0, \gamma}(A) \).

This statement is witnessed exactly by \( V_\delta \) in the place of \( M \) (since \( B = B^{V[G]} \cap V_\delta \in UB \) and \( UB_{V_\delta}^{V[G]} = \{ B^{V[G]} : B \in UB \} \)), and \( \tilde{J} \) in the place of \( \tilde{J} \).
Since $V_δ$ is $({\text{NS}}_{ω_1}, UB^{V[G]})$-ec in $V[G]$ we get that $j_{0_γ} \upharpoonright H^V_{ω_2}$ is $Σ_1$-elementary between $H^V_{ω_2}$ and $H^N_{ω_2}$ for $τ_{NS_{ω_1}, UB^V}$.

Hence

$$(H^V_{ω_2}, τ^V, NS^V_{γ}, UB^V)$$

models

There exists an $({\text{NS}}_{ω_1}, UB^V)$-ec structure $M$ with $B ∈ UB_M$ and an iteration $\mathcal{J} = \{j_{αβ} : α ≤ β ≤ (ω_1)^V\}$ of $M$ such that $j_{0ω_1}(a) = A$ and $NS_{j_{0ω_1}}(M) = NS^V_{ω_1} ∩ j_{0ω_1}(M)$.

(3) implies (1): We use again Schindler and Asperó characterization of ($*$).

Assume $φ(\alpha)$ is honestly consistent for some $Σ_1$-property $φ(x)$ in the language $σ_{UB, NS_{ω_1}}$ and $A ∈ P(ω_1)^V$. Let $B_1, \ldots, B_k$ be the universally Baire predicates in $UB$ mentioned in $φ(x)$.

By (3) there is in $V$ an $({\text{NS}}_{ω_1}, UB)$-ec $M$ with $B_1, \ldots, B_k ∈ UB_M$ and $a ∈ P(ω_1)^M$, and an iteration $\mathcal{J} = \{j_{αβ} : α ≤ β ≤ ω_1\}$ of $M$ such that $j_{0ω_1}(a) = A$ and $NS_{j_{0ω_1}}(M) = NS^V_{ω_1} ∩ j_{0ω_1}(M)$.

Let $G$ be $V$-generic for $\text{Coll}(ω, δ)$. Find $N ∈ V[G]$ such that $N ⊨ φ(A)$, $N$ is $B_1^{V[G]}, \ldots, B_k^{V[G]}$-iterable in $V[G]$ and $NS^N_{ω_1} ∩ V = NS^V_{ω_1}$ (this $N$ exists by the honest consistency of $φ(x)$).

Notice that $\mathcal{J} ∈ V_δ ⊆ N$ witnesses that $M ≥ N$ as well.

Let $N \leq N$ in $V[G]$ be a $UB^{V[G]}$-correct structure with $B_{UBV} ∈ UB_N$ ($N$ exists by Fact 2.12 applied in $V[G]$ to $N$ and $B_{UBV}$). Let $K = \{k_{αβ} : α ≤ β ≤ γ = ω_1^N\} ∈ N$ be an iteration witnessing that $N ≤ N$.

Remark that $H^N_{ω_2} ⊨ φ(k_{0γ}(A))$, since $Σ_1$-properties are upward absolute and $k_{0γ}(N)$ is a $τ_{NS_{ω_1}} \cup \{B_1, \ldots, B_k\}$-substructure of $H^N_{ω_2}$.

Also $\{B^{V[G]} : B ∈ UB^V\} ⊆ UB_N$ entail that $B^{V[G]}_{UB-M} ∈ UB_N$.

Letting

$$\bar{J} = \{\bar{j}_{αβ} : α ≤ β ≤ \bar{γ}\} = k_{0γ}(J),$$

we get that $\bar{j}_{0γ}(a) = k_{0γ}(j_{0γ}(a)) = k_{0γ}(A)$, and $\bar{J}$ is such that $B_j^{V[G]} ∈ UB_N$ for all $j = 1, \ldots, k$ since $B^{V[G]}_{UB-M} ∈ UB_N$.

Since $M$ is $({\text{NS}}_{ω_1}, UB^{V[G]})$-ec in $V[G]$ by $\text{MAX}(UB)$, we get that $\bar{j}_{0γ}$ defines a $Σ_1$-elementary embedding of

$$(H^M_{ω_2}, σ^M_{UB-M, NS_{ω_1}})$$

into

$$(H^N_{ω_2}, σ^N_{UB-M, NS_{ω_1}}).$$

Hence

$$(H^M_{ω_2}, σ^M_{UB-M, NS_{ω_1}}) ⊨ φ(a).$$

This gives that

$$(H^M_{ω_2}, σ^M_{UB-M, NS_{ω_1}}) ⊨ φ(A)$$

(since $j_{0ω_1}(a) = A$), and therefore that

$$(H^V_{ω_2}, σ^V_{UB-M, NS_{ω_1}}) ⊨ φ(A),$$

since $M_{ω_1}$ is a substructure of $H^V_{ω_2}$ for $σ_{UB-M, NS_{ω_1}}$.

$\square$
Question 2.20. Is the use of $\text{MAX}(\text{UB})$ really necessary? It is not at all clear whether the chain of equivalences for $(\ast)$-$\text{UB}$ could be proved replacing it with the usual Woodin’s axiom $(\ast)$ as formulated in [6, Def. 7.5]; in all cases where the argument appealed to $\text{MAX}(\text{UB})$ one should find a different strategy to reach the desired conclusion.

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