How Overlap Determines the Macronuclear Genes in Ciliates

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Abstract

Formal models for gene assembly in ciliates have been developed, in particular the string pointer reduction system (SPRS) and the graph pointer reduction system (GPRS). The reduction graph is a valuable tool within the SPRS, revealing much information about how gene assembly is performed for a given gene. The GPRS is more abstract than the SPRS and not all information present in the SPRS is retained in the GPRS. As a consequence the reduction graph cannot be defined for the GPRS in general, but we show that it can be defined (in an equivalent manner as defined for the SPRS) if we restrict ourselves to so-called realistic overlap graphs. Fortunately, only these graphs correspond to genes occurring in nature. Defining the reduction graph within the GPRS allows one to carry over several results within the SPRS that rely on the reduction graph.

1 Introduction

Gene assembly is a biological process that takes place in a large group of one-cellular organisms called ciliates. The process transforms one nucleus, called the micronucleus, through a large number of splicing operations into another nucleus, called the macronucleus. The macronucleus is very different from the micronucleus, both functionally and in terms of differences in DNA. Each gene occurring in the micronucleus is transformed into a corresponding gene in the macronucleus. Two models that are used to formalize this process are the string pointer reduction system (SPRS) and the graph pointer reduction system (GPRS). The former consist of three types of string rewriting rules operating on strings, called legal strings, while the latter consist of three types of graph rewriting rules operating on graphs, called overlap graphs. The GPRS can be seen as an abstraction of the SPRS, however it is not fully equivalent with the SPRS: some information present in the SPRS is lost in the GPRS.

Legal strings represent genes in their micronuclear form. The reduction graph, which is defined for legal strings, is a notion that describes the corresponding gene in its macronuclear form (along with its waste products). Moreover, it
has been shown that the reduction graph retains much information on which string negative rules (one of the three types of string rewriting rules) can be or are used in this transformation \[3, 2, 1\]. Therefore it is natural to define an equivalent notion for the GPRS. However, as we will show, since the GPRS loses some information concerning the application of string negative rules, there is no unique reduction graph for a given overlap graph. We will show however, that when we restrict ourselves to ‘realistic’ overlap graph then there is a unique reduction graph corresponding to this graph. These overlap graphs are called realistic since non-realistic overlap graphs cannot correspond to (micronuclear) genes. Moreover, we explicitly define the notion of reduction graph for these overlap graphs (within the GPRS) and show the equivalence with the definition for legal strings (within the SPRS). Finally, we show some immediate results due to this equivalence, including an open problem formulated in Chapter 13 in \[4\].

In Section 2 we recall some basic notions and notation concerning sets, strings and graphs. In Section 3 we recall notions used in models for gene assembly, such as legal strings, realistic strings and overlap graphs. In Section 4 we recall the notion of reduction graph within the framework of SPRS and we prove a few elementary properties of this graph for legal strings. In particular we establish a calculus for the sets of overlapping pointers between vertices of the reduction graph. In Section 5 we prove properties of the reduction graph for a more restricted type of legal strings, the realistic strings. It is shown that reduction graphs of realistic strings have a subgraph of a specific structure, the root subgraph. Moreover the existence of the other edges in the reduction graph is shown to depend directly on the overlap graph, using the calculus derived in the Section 4. In Section 6 we provide a convenient function for reduction graphs (but not only reduction graphs) which simplifies reduction graphs without losing any information. In Section 7 we define the reduction graph for realistic overlap graphs, and prove the main theorem of this paper: the equivalence of reduction graphs defined for realistic strings and reduction graphs defined for realistic overlap graphs. In Section 8 we show immediate consequences of this theorem.

## 2 Notation and Terminology

In this section we recall some basic notions concerning functions, strings, and graphs. We do this mainly to set up the basic notation and terminology for this paper.

The cardinality of set \(X\) is denoted by \(|X|\). The symmetric difference of sets \(X\) and \(Y\), \((X\setminus Y) \cup (Y\setminus X)\), is denoted by \(X \oplus Y\). Being an associative operator, we can define the symmetric difference of a family of sets \((X_i)_{i \in A}\) and denote it by \(\bigoplus_{i \in A} X_i\). The composition of functions \(f : X \to Y\) and \(g : Y \to Z\) is the function \(gf : X \to Z\) such that \((gf)(x) = g(f(x))\) for every \(x \in X\). The restriction of \(f\) to a subset \(A\) of \(X\) is denoted by \(f|A\).

We will use \(\lambda\) to denote the empty string. For strings \(u\) and \(v\), we say that \(v\) is a substring of \(u\) if \(u = w_1vw_2\), for some strings \(w_1, w_2\); we also say that \(v\) occurs in \(u\). Also, \(v\) is a cyclic substring of \(u\) if either \(v\) is a substring of \(u\) or \(u = v_2wv_1\) and \(v = v_1v_2\) for some strings \(v_1, v_2, w\). We say that \(v\) is a conjugate of \(u\) if \(u = w_1w_2\)
and \( v = w_2w_1 \) for some strings \( w_1 \) and \( w_2 \). For a string \( u = x_1x_2 \ldots x_n \) over \( \Sigma \) with \( x_i \in \Sigma \) for all \( i \in \{1, \ldots, n\} \), we say that \( v = x_nx_{n-1} \ldots x_1 \) is the reversal of \( u \). A homomorphism is a function \( \varphi : \Sigma^* \to \Delta^* \) such that \( \varphi(uv) = \varphi(u)\varphi(v) \) for all \( u, v \in \Sigma^* \).

We move now to graphs. A labelled graph is a 4-tuple

\[
G = (V, E, f, \Gamma),
\]

where \( V \) is a finite set, \( E \subseteq \{\{x, y\} \mid x, y \in V, x \neq y\} \), and \( f : V \to \Gamma \).

The elements of \( V \) are called vertices and the elements of \( E \) are called edges. Function \( f \) is the labelling function and the elements of \( \Gamma \) are the labels. We say that \( G \) is discrete if \( E = \emptyset \). Labelled graph \( G' = (V', E', f|V', \Gamma) \) is a subgraph of \( G \) if \( V' \subseteq V \) and \( E' \subseteq E_{V'} = E \cap \{\{x, y\} \mid x, y \in V', x \neq y\} \). If \( E' = E_{V'} \), we say that \( G' \) is the subgraph of \( G \) induced by \( V' \).

A string \( \pi = e_1e_2 \ldots e_n \in E^* \) with \( n \geq 1 \) is a path in \( G \) if there is a \( v_1v_2 \ldots v_{n+1} \in V^* \) such that \( e_i = \{v_i, v_{i+1}\} \) for all \( 1 \leq i \leq n \). Labelled graph \( G \) is connected if there is a path between every two vertices of \( G \). A subgraph \( H \) of \( G \) induced by \( V_H \) is a component of \( G \) if both \( H \) is connected and for every edge \( e \in E \) we have either \( e \subseteq V_H \) or \( e \subseteq V \setminus V_H \).

As usual, labelled graphs \( G = (V, E, f, \Gamma) \) and \( G' = (V', E', f', \Gamma) \) are isomorphic, denoted by \( G \cong G' \), if there is a bijection \( \alpha : V \to V' \) such that \( f(v) = f'(\alpha(v)) \) for \( v \in V \), and

\[
\{x, y\} \in E \text{ iff } \{\alpha(x), \alpha(y)\} \in E'
\]

for \( x, y \in V \). Bijection \( \alpha \) is then called an isomorphism from \( G \) to \( G' \).

In this paper we will consider graphs with two sets of edges. Therefore, we need the notion of 2-edge coloured graphs. A 2-edge coloured graph is a 5-tuple

\[
G = (V, E_1, E_2, f, \Gamma),
\]

where both \( (V, E_1, f, \Gamma) \) and \( (V, E_2, f, \Gamma) \) are labelled graphs.

The basic notions and notation for labelled graphs carry over to 2-edge coloured graphs. However, for the notion of isomorphism care must be taken that the two sorts of edges are preserved. Thus, if \( G = (V, E_1, E_2, f, \Gamma) \) and \( G' = (V', E'_1, E'_2, f', \Gamma') \) are 2-edge coloured graphs, then it must hold that for an isomorphism \( \alpha \) from \( G \) to \( G' \),

\[
(x, y) \in E_i \text{ iff } (\alpha(x), \alpha(y)) \in E'_i
\]

for \( x, y \in V \) and \( i \in \{1, 2\} \).

### 3 Gene Assembly in Ciliates

Two models that are used to formalize the process of gene assembly in ciliates are the string pointer reduction system (SPRS) and the graph pointer reduction system (GPRS). The SPRS consist of three types of string rewriting rules operating on legal strings while the GPRS consist of three types of graph rewriting...
rules operating on overlap graphs. For the purpose of this paper it is not necessary to recall the string and graph rewriting rules; a complete description of SPRS and GPRS, as well as a proof of their “weak” equivalence, can be found in [3]. We do recall the notions of legal string and overlap graph, and we also recall the notion of realistic string.

We fix $\kappa \geq 2$, and define the alphabet $\Delta = \{2, 3, \ldots, \kappa\}$. For $D \subseteq \Delta$, we define $\tilde{D} = \{\tilde{a} \mid a \in D\}$ and $\Pi_D = D \cup \tilde{D}$; also $\Pi = \Pi_\Delta$. The elements of $\Pi$ will be called pointers. We use the “bar operator” to move from $\Delta$ to $\tilde{\Delta}$ and back from $\tilde{\Delta}$ to $\Delta$. Hence, for $p \in \Pi$, $\bar{p} = p$. For $p \in \Pi$, we define $p = \begin{cases} p & \text{if } p \in \Delta, \\ \bar{p} & \text{if } p \in \tilde{\Delta}, \end{cases}$ i.e., $p$ is the “unbarred” variant of $p$.

For a string $u = x_1x_2 \cdots x_n$ with $x_i \in \Pi$ ($1 \leq i \leq n$), the complement of $u$ is $\bar{x}_1\bar{x}_2 \cdots \bar{x}_n$. The inverse of $u$, denoted by $\bar{u}$, is the complement of the reversal of $u$, thus $\bar{u} = \bar{x}_n \bar{x}_{n-1} \cdots \bar{x}_1$. The domain of $u$, denoted by $\text{dom}(u)$, is $\{p \mid p \text{ occurs in } v\}$. We say that $u$ is a legal string if for each $p \in \text{dom}(u)$, $u$ contains exactly two occurrences from $\{p, \bar{p}\}$.

We define the alphabet $\Theta_\kappa = \{M_i, \bar{M}_i \mid 1 \leq i \leq \kappa\}$. We say that $\delta \in \Theta^*_\kappa$ is a micronuclear arrangement if for each $i$ with $1 \leq i \leq \kappa$, $\delta$ contains exactly one occurrence from $\{M_i, \bar{M}_i\}$. With each string over $\Theta_\kappa$, we associate a unique string over $\Pi$ through the homomorphism $\pi_\kappa : \Theta^*_\kappa \rightarrow \Pi^*$ defined by:

$$\pi_\kappa(M_i) = 2, \quad \pi_\kappa(M_i) = \kappa, \quad \pi_\kappa(M_i) = i(i+1) \quad \text{for } 1 < i < \kappa,$$

and $\pi_\kappa(\bar{M}_j) = \pi_\kappa(\bar{M}_j)$ for $1 \leq j \leq \kappa$. We say that string $u$ is a realistic string if there is a micronuclear arrangement $\delta$ such that $u = \pi_\kappa(\delta)$. We then say that $\delta$ is a micronuclear arrangement for $u$.

Note that every realistic string is a legal string. However, not every legal string is a realistic string. For example, a realistic string cannot have “gaps” (missing pointers): thus 2244 is not realistic while it is legal. It is also easy to produce examples of legal strings which do not have gaps but still are not realistic — 3322 is such an example. Realistic strings are most useful for the gene assembly models, since only these legal strings can correspond to genes in ciliates.

For a pointer $p$ and a legal string $u$, if both $p$ and $\bar{p}$ occur in $u$ then we say that both $p$ and $\bar{p}$ are positive in $u$; if on the other hand only $p$ or only $\bar{p}$ occurs in $u$, then both $p$ and $\bar{p}$ are negative in $u$. So, every pointer occurring in a legal string is either positive or negative in it. Therefore, we can define a partition of $\text{dom}(u) = \text{pos}(u) \cup \text{neg}(u)$, where $\text{pos}(u) = \{p \in \text{dom}(u) \mid p \text{ is positive in } u\}$ and $\text{neg}(u) = \{p \in \text{dom}(u) \mid p \text{ is negative in } u\}$.

Let $u = x_1x_2 \cdots x_n$ be a legal string with $x_i \in \Pi$ for $1 \leq i \leq n$. For a pointer $p \in \Pi$ such that $\{x_i, x_j\} \subseteq \{p, \bar{p}\}$ and $1 \leq i < j \leq n$, the $p$-interval of $u$ is the substring $x_ix_{i+1} \cdots x_j$. Substrings $x_i \cdots x_{j_1}$ and $x_{j_2} \cdots x_j$ overlap in $u$ if $i_1 < i_2 < j_1 < j_2$ or $i_2 < i_1 < j_2 < j_1$. Two distinct pointers $p, q \in \Pi$ overlap in $u$ if the $p$-interval of $u$ overlaps with the $q$-interval of $u$. Thus, two distinct pointers $p, q \in \Pi$ overlap in $u$ if there is exactly one occurrence from $\{p, \bar{p}\}$ in the $q$-interval, or equivalently, there is exactly one occurrence from $\{q, \bar{q}\}$ in the $p$-interval of $u$. Also, for $p \in \text{dom}(u)$, we denote $O_u(p) = \{q \in \text{dom}(u) \mid p \text{ and } q \text{ overlap in } u\}$.
For 0 ≤ i ≤ j ≤ n, we denote by \( O_u(i, j) \) the set of all \( p \in \text{dom}(u) \) such that there is exactly one occurrence from \( \{p, \bar{p}\} \) in \( x_{i+1}x_{i+2} \cdots x_j \). Also, we define \( O_u(j, i) = O_u(i, j) \). Intuitively, \( O_u(i, j) \) is the set of \( p \in \text{dom}(u) \) for which the substring between “positions” \( i \) and \( j \) in \( u \) contains exactly one representative from \( \{p, \bar{p}\} \), where position \( i \) for \( 0 < i < n \) means the “space” between \( x_i \) and \( x_{i+1} \), and for \( i = 0 \) it is the “space” on the left of \( x_1 \), and for \( i = n \) it is the “space” on the right of \( x_n \). A few elementary properties of \( O_u(i, j) \) follow. We have \( O_u(i, n) = O_u(0, i) \) for \( i \) with \( 0 ≤ i ≤ n \). Moreover, for \( i, j, k \in \{0, \ldots, n\} \), \( O_u(i, j) \oplus O_u(j, k) = O_u(i, k) \); this is obvious when \( i < j < k \), but it is valid in general. Also, for \( 0 ≤ i ≤ j ≤ n \), \( O_u(i, j) = \emptyset \) iff \( x_{i+1} \cdots x_j \) is a legal string.

**Definition 1**
Let \( u \) be a legal string. The **overlap graph** of \( u \), denoted by \( \gamma_u \), is the labelled graph

\[
(\text{dom}(u), E, \sigma, \{+, -\}),
\]

where

\[
E = \{\{p, q\} \mid p, q \in \text{dom}(u), p \neq q, \text{ and } p \text{ and } q \text{ overlap in } u\},
\]

and \( \sigma \) is defined by:

\[
\sigma(p) =
\begin{cases} + & \text{if } p \in \text{pos}(u) \\
- & \text{if } p \in \text{neg}(u) \end{cases}
\]

for all \( p \in \text{dom}(u) \). □

**Example 1**
Let \( u = 24535423 \) be a legal string. The overlap graph of \( u \) is

\[
\gamma = (\{\{2, 3, 4, 5\}, \{\{2, 3\}, \{4, 3\}, \{5, 3\}\}, \sigma, \{+, -\}),
\]

where \( \sigma(v) = - \) for all vertices \( v \) of \( \gamma \). The overlap graph is depicted in Figure 1.

Let \( \gamma \) be an overlap graph. Similar to legal strings, we define \( \text{dom}(\gamma) \) as the set of vertices of \( \gamma \), \( \text{pos}(\gamma) = \{p \in \text{dom}(\gamma) \mid \sigma(p) = +\} \), \( \text{neg}(\gamma) = \{p \in \text{dom}(\gamma) \mid \sigma(p) = -\} \) and for \( q \in \text{dom}(u) \), \( O_\gamma(q) = \{p \in \text{dom}(\gamma) \mid \{p, q\} \in E\} \). An overlap graph \( \gamma \) is **realistic** if it is the overlap graph of a realistic string. Not every overlap graph of a legal string is realistic. For example, it can be shown that the overlap graph \( \gamma \) of \( u = 24535423 \) depicted in Figure 1 is not realistic. In fact, one can show that it is not even **realizable** — there is no isomorphism \( \alpha \) such that \( \alpha(\gamma) \) is realistic.

![Figure 1: The overlap graph of legal string u = 24535423.](image-url)
4 The Reduction Graph

We now recall the (full) reduction graph, which was first introduced in [3].

Remark

Below we present this graph in a slightly modified form: we omit the special vertices $s$ and $t$, called the source vertex and target vertex respectively, which did appear in the definition presented in [3]. As shown in Section 5, in this way a realistic overlap graph corresponds to exactly one reduction graph. Fortunately, several results concerning reduction graphs do not rely on the special vertices, and therefore carry over trivially to reduction graphs as defined here.

Definition 2

Let $u = p_1 p_2 \cdots p_n$ with $p_1, \ldots, p_n \in \Pi$ be a legal string. The reduction graph of $u$, denoted by $R_u$, is a 2-edge coloured graph

$$(V, E_1, E_2, f, \text{dom}(u)),$$

where

$V = \{I_1, I_2, \ldots, I_n\} \cup \{I'_1, I'_2, \ldots, I'_n\},$

$E_1 = \{e_1, e_2, \ldots, e_n\}$ with $e_i = \{I'_i, I_{i+1}\}$ for $1 \leq i \leq n-1$, $e_n = \{I'_n, I_1\}$,

$E_2 = \{\{I'_i, I'_j\}, \{I_i, I'_j\} | i, j \in \{1, 2, \ldots, n\} \text{ with } i \neq j \text{ and } p_i = p_j\} \cup \{\{I_i, I_j\}, \{I'_i, I'_j\} | i, j \in \{1, 2, \ldots, n\} \text{ and } p_i = \bar{p}_j\},$ and

$f(I_i) = f(I'_i) = p_i$ for $1 \leq i \leq n.$

The edges of $E_1$ are called the reality edges, and the edges of $E_2$ are called the desire edges. Intuitively, the “space” between $p_i$ and $p_{i+1}$ corresponds to the reality edge $e_i = \{I'_i, I_{i+1}\}$. Hence, we say that $i$ is the position of $e_i$, denoted by $\text{posn}(e_i)$, for all $i \in \{1, 2, \ldots, n\}$. Note that positions are only defined for reality edges. Since for every vertex $v$ there is a unique reality edge $e$ such that $v \in e$, we also define the position of $v$, denoted by $\text{posn}(v)$, as the position of $e$. Thus, $\text{posn}(I'_i) = \text{posn}(I_{i+1}) = i$ (while $\text{posn}(I_1) = n$).

Example 2

Let $u = 32 \bar{4} 3 \bar{2} 4$ be a legal string. Since $\bar{4} 3 \bar{2}$ cannot be a substring of a realistic string, $u$ is not realistic. The reduction graph $R_u$ of $u$ is depicted in Figure 2. The labels of the vertices are also shown in this figure. Note the desire edges corresponding to positive pointers (here 2 and 4) cross (in the figure), while those for negative pointers are parallel. Since the exact identity of the vertices in a reduction graph is not essential for the problems considered in this paper, in order to simplify the pictorial representation of reduction graphs we will omit this in the figures. We will also depict reality edges as “double edges” to distinguish them from the desire edges. Figure 3 shows the reduction graph in this simplified representation.
Figure 2: The reduction graph of $u$ of Example 2.

Figure 3: The reduction graph of $u$ of Example 2 in the simplified representation.

Figure 4: The reduction graph of $u$ of Example 3.
Thus in both cases we have $O_p$.

First, assume that $v_1 = 1^2$.

Then, $v_2 = 2^1$.

Next we consider sets of overlapping pointers corresponding to pairs of vertices in reduction graphs, and start to develop a calculus for these sets that will later enable us to characterize the existence of certain edges in the reduction graph, cf. Theorem 13.

Example 4
We again consider the legal string $u = 324324$ and its reduction graph $R_u$ from Example 3. Desire edge $e = \{I'_2, I'_3\}$ is connected to reality edges $e_1 = \{I_2, I_3\}$ and $e_2 = \{I'_5, I_5\}$ with positions 2 and 5 respectively. We have $O_u(2, 5) = \{2, 3, 4\}$. Also, reality edges $\{I'_1, I_2\}$ and $\{I'_2, I_3\}$ have positions 1 and 2 respectively. We have $O_u(1, 2) = \{2\}$.

Lemma 3
Let $u$ be a legal string. Let $e = \{v_1, v_2\}$ be a desire edge of $R_u$ and let $p$ be the label of both $v_1$ and $v_2$. Then

$$O_u(\text{posn}(v_1), \text{posn}(v_2)) = \begin{cases} O_u(p) & \text{if } p \text{ is negative in } u, \\ O_u(p) \oplus \{p\} & \text{if } p \text{ is positive in } u. \end{cases}$$

Proof
Let $u = p_1p_2\ldots p_n$ with $p_1, p_2, \ldots, p_n \in \Pi$ and let $i$ and $j$ be such that $i < j$ and $p = p_i = p_j$. Without loss of generality, we can assume $\text{posn}(v_1) < \text{posn}(v_2)$. Then, $v_1 \in \{I_i, I'_i\}$ and $v_2 \in \{I_j, I'_j\}$, hence $\text{posn}(v_1) \in \{i - 1, i\}$ and $\text{posn}(v_2) \in \{j - 1, j\}$.

First, assume that $p$ is negative in $u$. By the definition of reduction graph, the following two cases are possible:

1. $e = \{I_i, I'_j\}$, thus $O_u(\text{posn}(I_i), \text{posn}(I'_j)) = O_u(i - 1, j) = O_u(p)$,

2. $e = \{I'_i, I_j\}$, thus $O_u(\text{posn}(I'_i), \text{posn}(I_j)) = O_u(i, j - 1) = O_u(p)$,

Thus in both cases we have $O_u(\text{posn}(v_1), \text{posn}(v_2)) = O_u(p)$.

Finally, assume that $p$ is positive in $u$. By the definition of reduction graph, the following two cases are possible:

1. $e = \{I_i, I'_j\}$, thus $O_u(\text{posn}(I_i), \text{posn}(I'_j)) = O_u(i - 1, j - 1) = O_u(p) \oplus \{p\}$,

2. $e = \{I'_i, I_j\}$, thus $O_u(\text{posn}(I'_i), \text{posn}(I_j)) = O_u(i, j) = O_u(p) \oplus \{p\}$,

Thus in both cases we have $O_u(i_1, i_2) = O_u(p) \oplus \{p\}$.

The following result follows by iteratively applying the previous lemma.
Corollary 4
Let $u$ be a legal string. Let

$$p_0 \underbrace{p_1 \cdots p_1}_{p_2} \cdots p_n \underbrace{p_{n+1}}$$

be a subgraph of $R_u$, where (as usual) the vertices in the figure are represented by their labels, and let $e_1$ ($e_2$, resp.) be the leftmost (rightmost, resp.) edge. Note that $e_1$ and $e_2$ are reality edges and therefore $posn(e_1)$ and $posn(e_2)$ are defined. Then $O_u(posn(e_1), posn(e_2)) = (pos(u) \cap P) \oplus (\bigoplus_{t \in P} O_u(t))$ with $P = \{p_1, \ldots, p_n\}$.

By the definition of the reduction graph the following lemma holds.

Lemma 5
Let $u$ be a legal string. If $I_i$ and $I'_i$ are vertices of $R_u$, then $O_u(posn(I_i), posn(I'_i)) = \{p\}$, where $p$ is the label of $I_i$ and $I'_i$.

Example 5
We again consider the legal string $u$ and desire edge $e$ as in the previous example. Since $e$ has vertices labelled by positive pointer 2, by Lemma 3 we have (again) $O_u(2, 5) = O_u(2) \oplus \{2\} = \{2, 3, 4\}$. Also, since $I_2$ and $I'_2$ with positions 1 and 2 respectively are labelled by 2, by Lemma 5 we have (again) $O_u(1, 2) = \{2\}$.

5 The Reduction Graph of Realistic Strings

The next theorem asserts that overlap graph $\gamma$ for realistic string $u$ retains all information of $R_u$ (up to isomorphism). In the next few sections, we will give a method to determine $R_u$ (up to isomorphism), given $\gamma$. Of course, the naive method is to first determine a legal string $u$ corresponding to $\gamma$ and then to determine the reduction graph of $u$. However, we present a method that is able to construct $R_u$ in a direct way from $\gamma$.

Theorem 6
Let $u$ and $v$ be realistic strings. If $\gamma_u = \gamma_v$, then $R_u \approx R_v$.

Proof
By Theorem 1 in [6] (or Theorem 10.2 in [4]), we have $\gamma_u = \gamma_v$ iff $v$ can be obtained from $u$ by a composition of reversal, complement and conjugation operations. By the definition of reduction graph it is clear that the reduction graph is invariant under these operations (up to isomorphism). Thus, $R_u \approx R_v$.

The previous theorem is not true for legal strings in general — the next two examples illustrate that legal strings having the same overlap graph can have different reduction graphs.

Example 6
Let $u = 2653562434$ and $v = h(u)$, where $h$ is the homomorphism that interchanges 5 and 6. Thus, $v = 2563652434$. Note that both $u$ and $v$ are not realistic, because substrings 535 of $u$ and 636 of $v$ can obviously not be substrings of realistic strings. The overlap graph of $u$ is depicted in Figure 5. From Figure 5 and the fact that $v$ is obtained from $u$ by renaming 5 and 6, it follows that the overlap graphs of $u$ and $v$ are equal. The reduction graph $R_u$ of $u$ is depicted in Figure 6. The reduction graph $R_v$ of $v$ is obtained from $R_u$ by renumbering the labels of the vertices according to $h$. Clearly, $R_u \not\approx R_v$. 9
Figure 5: The overlap graph of both legal strings $u$ and $v$ of Example 6.

Figure 6: The reduction graph of $u$ of Example 6.

Figure 7: The reduction graph of $u$ of Example 7.

Figure 8: The reduction graph of $v$ of Example 7.
Example 7
Let \( u = \pi_\kappa(M_1 M_2 M_3 M_4) = 223344 \) be a realistic string and let \( v = 234432 \) be a legal string. Note that \( v \) is not realistic. Legal strings \( u \) and \( v \) have the same overlap graph \( \gamma \) (\( \gamma = \{(2, 3, 4), \varnothing, \sigma, \{+,-\} \)\), where \( \sigma(v) = - \) for \( v \in \{2, 3, 4\} \)).

The reduction graph \( R_u \) of \( u \) is depicted in Figure 7 and the reduction graph \( R_v \) of \( v \) is depicted in Figure 8. Note that \( R_u \) has a component consisting of six vertices, while \( R_v \) does not have such a component. Therefore, \( R_u \neq R_v \).

For realistic strings the reduction graph has a special form. This is seen as follows. For \( 1 < i < \kappa \) the symbol \( M_i \) or \( \bar{M}_i \) in the micronuclear arrangement defines two pointers \( p_i \) and \( p_{i+1} \) (or \( \bar{p}_i \) and \( \bar{p}_{i+1} \)) in the corresponding realistic string \( u \). At the same time the substring \( p_i p_{i+1} \) (or \( \bar{p}_i \bar{p}_{i+1} \), resp.) of \( u \) corresponding to \( M_i \) (or \( \bar{M}_i \), resp.) defines four vertices \( I_j, I'_j, I_{j+1}, I'_{j+1} \) in \( R_u \). It is easily verified (cf. Theorem 8 below) that the “middle” two vertices \( I'_j \) and \( I_{j+1} \), labelled by \( p_i \) and \( p_{i+1} \) respectively, are connected by a reality edge and \( I'_j \) (\( I_{j+1} \), resp.) is connected by a reality edge to a “middle vertex” resulting from \( M_{i-1} \) or \( M_{i+1} \) \((M_{i+1} \text{ or } M_{i+1}, \text{ resp.})\). This leads to the following definition.

Definition 7
Let \( u \) be a legal string and let \( \kappa = |\text{dom}(u)| + 1 \). If \( R_u \) contains a subgraph \( L \) of the following form:

\[
\begin{align*}
2 & \quad 2 \quad 3 \quad 3 \quad \vdots \quad \kappa \quad \kappa
\end{align*}
\]

where the vertices in the figure are represented by their labels, then we say that \( u \) is rooted. Subgraph \( L \) is called a root subgraph of \( R_u \).

Example 8
The realistic string \( u \) with \( \text{dom}(u) = \{2, 3, \ldots, 7\} \) in Example 3 is rooted because the reduction graph of \( u \), depicted in Figure 4 contains the subgraph

\[
\begin{align*}
2 & \quad 2 \quad 3 \quad 3 \quad \vdots \quad 7 \quad 7
\end{align*}
\]

The next theorem shows that indeed every realistic string is rooted.

Theorem 8
Every realistic string is rooted.

Proof
Consider a micronuclear arrangement for a realistic string \( u \). Let \( \kappa = |\text{dom}(u)| + 1 \). By the definition of \( \pi_\kappa \), there is a reality edge \( e_i \) (corresponding to either \( \pi_\kappa(M_i) = i(i+1) \) or \( \pi_\kappa(\bar{M}_i) = (i+1)i \)) connecting a vertex labelled by \( i \) to a vertex labelled by \( i+1 \) for each \( 2 \leq i < \kappa \). It suffices to prove that there is a desire edge connecting \( e_i \) to \( e_{i+1} \) for each \( 2 \leq i < \kappa - 1 \). This can easily be seen by checking the four cases where \( e_i \) corresponds to either \( \pi_\kappa(M_i) \) or \( \pi_\kappa(\bar{M}_i) \) and \( e_{i+1} \) corresponds to either \( \pi_\kappa(M_{i+1}) \) or \( \pi_\kappa(\bar{M}_{i+1}) \).

In the remaining of this paper, we will denote \( |\text{dom}(u)| + 1 \) by \( \kappa \) for rooted strings, when it is clear which rooted string \( u \) is meant. The reduction graph of a realistic string may have more than one root subgraph: it is easy to verify that realistic string 234\cdot\cdot\cdot\kappa234\cdot\cdot\cdot\kappa \) for \( \kappa \geq 2 \) has two root subgraphs.

Example 2 shows that not every rooted string is realistic. The remaining results that consider realistic strings also hold for rooted strings, since we will not be
using any properties of realistic string that are not true for rooted strings in general.

For a given root subgraph \( L \), it is convenient to uniquely identify every reality edge containing a vertex of \( L \). This is done through the following definition.

**Definition 9**

Let \( u \) be a rooted string and let \( L \) be a root subgraph of \( \mathcal{R}_u \). We define \( rspos_{L,k} \) for \( 2 \leq k < \kappa \) as the position of the edge of \( L \) that has vertices labelled by \( k \) and \( k + 1 \). We define \( rspos_{L,1} \) as the position of the edge of \( \mathcal{R}_u \) not in \( L \) containing a vertex of \( L \) labelled by \( 2 \) (\( \kappa \), resp.). When \( \kappa = 2 \), to ensure that \( rspos_{L,1} \) and \( rspos_{L,\kappa} \) are well defined, we additionally require that \( rspos_{L,1} \neq rspos_{L,\kappa} \).

Thus, \( rspos_{L,k} \) (for \( 1 \leq k \leq \kappa \)) uniquely identifies every reality edge containing a vertex of \( L \). If it is clear which root subgraph \( L \) is meant, we simply write \( rspos_k \) instead of \( rspos_{L,k} \) for \( 1 \leq k \leq \kappa \).

The next lemma is essential to prove the main theorem of this paper.

**Lemma 10**

Let \( u \) be a rooted string. Let \( L \) be a root subgraph of \( \mathcal{R}_u \). Let \( i \) and \( j \) be positions of reality edges in \( \mathcal{R}_u \) that are not edges of \( L \). Then \( O_u(i, j) = \emptyset \) iff \( i = j \).

**Proof**

The reverse implication is trivially satisfied. We now prove the forward implication. The reality edge \( e_k \) (for \( 2 \leq k < \kappa \)) in \( L \) with vertices labelled by \( k \) and \( k + 1 \) corresponds to a cyclic substring \( \bar{M}_k \in \{p_1p_2p_3 | p_1 \in \{k, \bar{k}\}, \bar{p}_2 \in \{k + 1, \bar{k} + 1\}\} \) of \( u \). Let \( k_1 \) and \( k_2 \) with \( 2 \leq k_1 < k_2 < \kappa \). If \( k_1 + 1 = k_2 \), then \( e_{k_1} \) and \( e_{k_2} \) are connected by a desire edge (by the definition of \( L \)). Therefore, if the reality edges \( \bar{M}_{k_1} \) and \( \bar{M}_{k_2} \) originate from two different occurrences in \( u \). If on the other hand \( k_1 + 1 \neq k_2 \), then \( \bar{M}_{k_1} \) and \( \bar{M}_{k_2} \) do not have a letter in common. Therefore, in both cases, \( \bar{M}_{k_1} \) and \( \bar{M}_{k_2} \) disjoin cyclic substrings of \( u \). Thus the \( \bar{M}_k \) for \( 2 \leq k < \kappa \) are pairwise disjoint cyclic substrings of \( u \).

Without loss of generality assume \( i \leq j \). Let \( u = u_1u_2\cdots u_n \) with \( u_i \in \Pi \). Since \( u \) is a legal string, every \( u_l \) for \( 1 \leq l \leq n \) is either part of a \( \bar{M}_k \) (with \( 2 \leq k < \kappa \)) or in \( \{2, 2, \kappa, \bar{\kappa}\} \). Consider \( u' = u_{i+1}u_{i+2}\cdots u_j \). Since \( i \) and \( j \) are positions of reality edges in \( \mathcal{R}_u \) that are not edges of \( L \), we have \( u' = \bar{M}_{k_1}\bar{M}_{k_2}\cdots\bar{M}_{k_m} \) for some distinct \( k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, \kappa\} \), where \( \bar{M}_1 \in \{2, 2\} \) and \( \bar{M}_m \in \{\kappa, \bar{\kappa}\} \).

It suffices to prove that \( u' = \lambda \). Assume to the contrary that \( u' \neq \lambda \). Then there is a \( 1 \leq l \leq \kappa \) such that \( \bar{M}_l \) is a substring of \( u' \). Because \( O_u(i, j) = \emptyset \), we know that \( u' \) is legal. If \( l > 1 \), then \( \bar{M}_{l-1} \) is also a substring of \( u' \), otherwise \( u' \) would not be a legal string. Similarly, if \( l < \kappa \), then \( \bar{M}_{l+1} \) is also a substring of \( u' \). By iteration, we conclude that \( u' = u \). Therefore, \( i = 0 \). This is a contradiction, since \( 0 \) cannot be a position of a reality edge. Thus, \( u' = \lambda \).

**Lemma 11**

Let \( u \) be a rooted string. Let \( L \) be a root subgraph of \( \mathcal{R}_u \). If \( I_i \) and \( I_i' \) are vertices of \( \mathcal{R}_u \), then exactly one of \( I_i \) and \( I_i' \) belongs to \( L \).

**Proof**

By the definition of reduction graph, \( I_i \) and \( I_i' \) have a common vertex label \( p \) but are not connected by a desire edge. Therefore, \( I_i \) and \( I_i' \) do not both belong
to $L$. Now, if $I_i$ and $I'_i$ both do not belong to $L$, then the other vertices labelled by $p$, which are $I_j$ and $I'_j$ for some $j$, both belong to $L$ – a contradiction by the previous argument. Therefore, either $I_i$ or $I'_i$ belongs to $L$, and the other one does not belong to $L$.

The next result provides the main idea to determine the reduction graph given (only) the overlap graph as presented in Section 7. It relies heavily on the previous lemmas.

**Theorem 12**

Let $u$ be a rooted string, let $L$ be a root subgraph of $\mathcal{R}_u$, and let $p, q \in \text{dom}(u)$ with $p < q$. Then there is a reality edge $e$ in $\mathcal{R}_u$ with both vertices not in $L$, one vertex labelled by $p$ and the other labelled by $q$ iff

$$\bigoplus_{t \in P} O_u(t) = (\text{pos}(u) \cap P) \oplus \{p\} \oplus \{q\},$$

where $P = \{p + 1, \ldots, q - 1\} \cup P'$ for some $P' \subseteq \{p, q\}$.

**Proof**

We first prove the forward implication. Let $e = \{v_1, v_2\}$ with $v_1$ labelled by $p$, $v_2$ labelled by $q$, and $\text{pos}(e) = i$. Thus $e = \{I'_i, I_{i+1}\}$. We assume that $v_1 = I'_i$ and $v_2 = I_{i+1}$, the other case is proved similarly. Let $i_1 = \text{posn}(I_i)$ and $i_2 = \text{posn}(I_{i+1})$. By Lemma 5 $O_u(i, i_1) = \{p\}$ and $O_u(i_2, i) = \{q\}$. By Lemma 11, $I_i$ (labelled by $p$) and $I'_{i+1}$ (labelled by $q$) belong to $L$. Thus $i_1 \in \{r\text{pos}_{p-1}, r\text{pos}_p\}$ and $i_2 \in \{r\text{pos}_{q-1}, r\text{pos}_q\}$. By applying Corollary 4 on $L$, we have $O_u(i_1, i_2) = (\text{pos}(u) \cap P) \oplus (\bigoplus_{t \in P} O_u(t))$ with $P = \{p + 1, \ldots, q - 1\} \cup P'$ for some $P' \subseteq \{p, q\}$. By definition of $O_u(i, j)$ we have

$$\emptyset = O_u(i, i) = O_u(i, i_1) \oplus O_u(i_1, i_2) \oplus O_u(i_2, i)$$

Thus the desired result follows.

We now prove the reverse implication. By applying Corollary 4 on $L$, we have $O_u(i_1, i_2) = (\text{pos}(u) \cap P) \oplus (\bigoplus_{t \in P} O_u(t))$ for some $i_1 \in \{r\text{pos}_{p-1}, r\text{pos}_p\}$ and $i_2 \in \{r\text{pos}_{q-1}, r\text{pos}_q\}$ (depending on $P'$). By Lemma 5, there is a vertex $v_1$ (resp.) labelled by $p$ (resp.) with position $i$ (resp.) such that $O_u(i, i_1) = \{p\}$ and $O_u(i_2, j) = \{q\}$. By Lemma 11, these vertices are not in $L$. We have now

$$\emptyset = O_u(i, i_1) \oplus O_u(i_1, i_2) \oplus O_u(i_2, j) = O_u(i, j)$$

By Lemma 10, $O_u(i, j) = \emptyset$ implies that $i = j$. Thus, there is a reality edge $\{v_1, v_2\}$ in $\mathcal{R}_u$ (with position $i$), such that $v_1$ is labelled by $p$ and $v_2$ is labelled by $q$ and both are not vertices of $L$.

Let $\gamma_u$ be the overlap graph of some legal string $u$. Clearly we have $\text{pos}(u) = \text{pos}(\gamma_u)$ and for all $p \in \text{dom}(u) = \text{dom}(\gamma_u)$, $O_u(p) = O_{\gamma_u}(p)$. Thus by Theorem 12 we can determine, given the overlap graph of a rooted string $u$, if there is a reality edge in $\mathcal{R}_u$ with both vertices outside $L$ that connects a vertex labelled by $p$ to a vertex labelled by $q$. We will extend this result to completely determine the reduction graph given the overlap graph of a rooted string (or a realistic string in particular).
6 Compressing the Reduction Graph

In this section we define the cps function. The cps function simplifies reduction graphs by replacing the subgraph \( p \rightarrow p \) by a single vertex labelled by \( p \). In this way, one can simplify reduction graphs without “losing information”. We will define cps for a general family of graphs \( \mathcal{G} \) which includes all reduction graphs. The formal definitions of \( \mathcal{G} \) and cps are given below.

Let \( \mathcal{G} \) be the set of 2-edge coloured graphs \( G = (V, E_1, E_2, f, \Gamma) \) with the property that for all \( \{v_1, v_2\} \in E_2 \), it holds that \( f(v_1) = f(v_2) \). Note that for a reduction graph \( \mathcal{R}_u \), we have \( \mathcal{R}_u \in \mathcal{G} \) because both vertices of a desire edge have the same label. For all \( G \in \mathcal{G} \), \( \text{cps}(G) \) is obtained from \( G \) by considering the second set of edges as vertices in the labelled graph. Thus, for the case when \( G \) is a reduction graph, the function \( \text{cps} \) “compresses” the desire edges to vertices.

**Definition 13**

The function \( \text{cps} \) from \( \mathcal{G} \) to the set of labelled graphs is defined as follows. Let \( G = (V, E_1, E_2, f, \Gamma) \in \mathcal{G} \), then

\[
\text{cps}(G) = (E_2, E_1', f', \Gamma)
\]

is a labelled graph, where

\[
E_1' = \{\{e_1, e_2\} \subseteq E_2 : \exists v_1, v_2 \in V : v_1 \in e_1, v_2 \in e_2, e_1 \neq e_2 \text{ and } \{v_1, v_2\} \in E_1\},
\]

and for \( e \in E_2 \): \( f'(e) = f(v) \) with \( v \in e \).

Note that \( f' \) is well defined, because for all \( \{v_1, v_2\} \in E_2 \), it holds that \( f(v_1) = f(v_2) \).

**Example 9**

We are again considering the realistic string \( u \) defined in Example 3. The reduction graph of \( \mathcal{R}_u \) is depicted in Figure 4. The labelled graph \( \text{cps}(\mathcal{R}_u) \) is depicted in Figure 9. Since this graph has just one set of edges, the reality edges are depicted as ‘single edges’ instead of ‘double edges’ as we did for reduction graphs.
It is not hard to see that for reduction graphs \( R_u \) and \( R_v \), we have \( R_u \approx R_v \) iff \( \text{cps}(R_u) \approx \text{cps}(R_v) \). In this sense, function \( \text{cps} \) allows one to simplify reduction graphs without losing information.

7 From Overlap Graph to Reduction Graph

Here we define reduction graphs for realistic overlap graphs, inspired by the characterization of Theorem 12. In the remaining part of this section we will show its equivalence with reduction graphs for realistic strings.

**Definition 14**

Let \( \gamma = (\text{Dom}_\gamma, E_\gamma, \sigma, \{+, -\}) \) be a realistic overlap graph and let \( \kappa = |\text{Dom}_\gamma| + 1 \). The reduction graph of \( \gamma \), denoted by \( R_\gamma \), is a labelled graph

\[
R_\gamma = (V, E, f, \text{Dom}_\gamma),
\]

where

\[
V = \{ J_p, J'_p \mid 2 \leq p \leq \kappa \},
\]

\[
f(J_p) = f(J'_p) = p, \text{ for } 2 \leq p \leq \kappa, \text{ and}
\]

\( e \in E \) iff one of the following conditions hold:

1. \( e = \{ J'_p, J'_{p+1} \} \) and \( 2 \leq p < \kappa \).

2. \( e = \{ J_p, J_q \}, 2 \leq p < q \leq \kappa, \) and

\[
\bigoplus_{t \in P} O_\gamma(t) = (\text{pos}(\gamma) \cap P) \oplus \{p\} \oplus \{q\},
\]

where \( P = \{p + 1, \ldots, q - 1\} \cup P' \) for some \( P' \subseteq \{p, q\} \).

3. \( e = \{ J'_2, J_p \}, 2 \leq p \leq \kappa, \) and

\[
\bigoplus_{t \in P} O_\gamma(t) = (\text{pos}(\gamma) \cap P) \oplus \{p\},
\]

where \( P = \{2, \ldots, p - 1\} \cup P' \) for some \( P' \subseteq \{p\} \).

4. \( e = \{ J'_\kappa, J_p \}, 2 \leq p \leq \kappa, \) and

\[
\bigoplus_{t \in P} O_\gamma(t) = (\text{pos}(\gamma) \cap P) \oplus \{p\},
\]

where \( P = \{p + 1, \ldots, \kappa\} \cup P' \) for some \( P' \subseteq \{p\} \).

5. \( e = \{ J'_2, J'_\kappa \}, \kappa > 3, \) and

\[
\bigoplus_{t \in P} O_\gamma(t) = \text{pos}(\gamma) \cap P,
\]

where \( P = \{2, \ldots, \kappa\} \). \( \blacksquare \)
Example 10

The overlap graph $\gamma$ in Figure 10 is realistic. Indeed, for example realistic string $u = \pi_7(M_4M_3M_7M_5M_2M_1M_6) = 453475623267$ has this overlap graph. Clearly, the reduction graph $R_\gamma$ of $\gamma$ has the edges $\{J'_p, J'_{p+1}\}$ for $2 \leq p < 7$. The following table lists the remaining edges of $R_\gamma$. The table also states the characterizing conditions for each edge as stated in Definition 14. Note that $\text{pos}(\gamma) = \emptyset$, and consequently the right-hand side of the defining equations in points 2, 3 and 4 in Definition 14 are independent of the choice of $P'$.

| Edge      | $P$                      | Witness                      |
|-----------|--------------------------|------------------------------|
| $\{J_2, J_6\}$ | $\{3, 4, 5\}$          | $\{2, 4, 5, 6, 7\} \oplus \{3, 5\} \oplus \{3, 4, 7\} = \{2, 6\}$ |
| $\{J_2, J_6\}$ | $\{2, 3, 4, 5, 6\}$    | $\{3\} \oplus \{2, 4, 5, 6, 7\} \oplus \{3, 5\} \oplus \{3, 4, 7\} \oplus \{3\} = \{2, 6\}$ |
| $\{J_4, J_7\}$ | $\{5, 6\}$             | $\{3, 4, 7\} \oplus \{3\} = \{4, 7\}$ |
| $\{J_4, J_7\}$ | $\{4, 5, 6, 7\}$       | $\{3, 5\} \oplus \{3, 4, 7\} \oplus \{3\} \oplus \{3, 5\} = \{4, 7\}$ |
| $\{J_3, J_5\}$ | $\{4\}$              | $\{3, 5\} = \{3, 5\}$ |
| $\{J_5, J'_2\}$ | $\{6, 7\}$            | $\{3\} \oplus \{3, 5\} = \{5\}$ |
| $\{J'_2, J_3\}$ | $\{2\}$           | $\{3\} = \{3\}$ |

We have now completely determined $R_\gamma$; it is shown in Figure 11. As we have done for reduction graphs of legal strings, in the figures, the vertices of reduction graphs of realistic overlap graphs are represented by their labels.

Example 11

In the second example we construct the reduction graph of an overlap graph that contains positive pointers. The overlap graph $\gamma$ in Figure 12 is realistic. Indeed, for example realistic string $u = \pi_7(M_7M_1M_6M_3M_5M_2M_4) = 72634563245$ introduced in Example 3 has this overlap graph. Again, the reduction graph $R_\gamma$ of $\gamma$ has the edges $\{J'_p, J'_{p+1}\}$ for $2 \leq p < 7$. The remaining edges are listed in the table below.

Figure 10: The overlap graph $\gamma$ of a realistic string (used in Example 10).

Figure 11: The reduction graph $R_\gamma$ of the overlap graph $\gamma$ of Example 10. The vertices in the figure are represented by their labels.
Figure 12: The overlap graph $\gamma$ of a realistic string (used in Example 11).

Figure 13: The reduction graph $\mathcal{R}_\gamma$ of the overlap graph $\gamma$ of Example 11.
Figures [9] and [13] show, for $u = 726734563245$, that $\text{cps}(R_u) \approx R_\gamma$. The next theorem shows that this is true for every realistic string $u$.

**Theorem 15**

Let $u$ be a realistic string. Then, $\text{cps}(R_u) \approx R_{\gamma_u}$.

**Proof**

Let $\kappa = |\text{dom}(u)| + 1$, let $\gamma = \gamma_u$, let $R_\gamma = (V_\gamma, E_\gamma, f_\gamma, \text{dom}(u))$, let $R_u = \text{cps}(R_u) = (V_u, E_u, f_u, \text{dom}(u))$ and let $L$ be a root subgraph of $R_u$. Recall that the elements of $V_u$ are the desire edges of $R_u$.

Let $h : V_u \rightarrow V_\gamma$ defined by

$$h(v) = \begin{cases} 
J_{f_u(v)} & \text{if } v \text{ is not an edge of } L \\
J_{f_u(v)}' & \text{if } v \text{ is an edge of } L 
\end{cases}.$$ 

We will show that $h$ is an isomorphism from $R_u$ to $R_\gamma$. Since for every $l \in \text{dom}(u)$ there exists exactly one desire edge $v$ of $R_u$ that belongs to $L$ with $f_u(v) = l$ and there exists exactly one desire edge $v$ of $R_u$ that does not belong to $L$ with $f_u(v) = l$, it follows that $h$ is one-to-one and onto. Also, it is clear from the definition of $f_\gamma$ that $f_u(v) = f_\gamma(h(v))$. Thus, it suffices to prove that

$$\{v_1, v_2\} \in E_u \Leftrightarrow \{h(v_1), h(v_2)\} \in E_\gamma.$$ 

We first prove the forward implication $\{v_1, v_2\} \in E_u \Rightarrow \{h(v_1), h(v_2)\} \in E_\gamma$. Let $\{v_1, v_2\} \in E_u$, let $p = f_u(v_1)$ and let $q = f_u(v_2)$. Clearly, $v_1 \neq v_2$. By the definition of $\text{cps}$, there is a reality edge $\hat{e} = \{\hat{v}_1, \hat{v}_2\}$ of $R_u$ with $\hat{v}_1 \in v_1$ and $\hat{v}_2 \in v_2$ (and thus $v_1$ and $v_2$ are labelled by $p$ and $q$ in $R_u$, respectively). Let $i$ be the position of $\hat{e}$. We consider four cases (remember that $v_1$ and $v_2$ are both desire edges of $R_u$):

1. Assume that $\hat{e}$ belongs to $L$. Then clearly, $v_1$ and $v_2$ are edges of $L$.

   Without loss of generality, we can assume that $p \leq q$. From the structure of root subgraph and the fact that $\hat{e}$ is a reality edge of $R_u$ in $L$, it follows that $q = p + 1$. Now, $h(v_1) = J'_p$ and $h(v_2) = J'_q = J'_{p+1}$. By the first item of the definition of reduction graph of an overlap graph, it follows that $\{h(v_1), h(v_2)\} = \{J'_p, J'_{p+1}\} \in E_\gamma$. This proves the first case. In the remaining cases, $\hat{e}$ does not belong to $L$.

2. Assume that $v_1$ and $v_2$ are both not edges of $L$ (thus $\hat{e}$ does not belong to $L$). Now by Theorem [12] and the second item of the definition of reduction graph of an overlap graph, it follows that $\{h(v_1), h(v_2)\} = \{J'_p, J'_{p+1}\} \in E_\gamma$. This proves the second case.

3. Assume that $v_1$ is an edge of $L$ and $v_2$ is not an edge of $L$.

   Without loss of generality, we can assume that $p \leq q$. From the structure of root subgraph and the fact that $\hat{e}$ is a reality edge of $R_u$ in $L$, it follows that $q = p + 1$. Now, $h(v_1) = J'_p$ and $h(v_2) = J'_q = J'_{p+1}$. By the first item of the definition of reduction graph of an overlap graph, it follows that $\{h(v_1), h(v_2)\} = \{J'_p, J'_{p+1}\} \in E_\gamma$. This proves the third case. In the remaining cases, $\hat{e}$ does not belong to $L$.

4. Assume that $v_1$ is not an edge of $L$ and $v_2$ is an edge of $L$.

   Without loss of generality, we can assume that $p \leq q$. From the structure of root subgraph and the fact that $\hat{e}$ is a reality edge of $R_u$ in $L$, it follows that $q = p + 1$. Now, $h(v_1) = J'_p$ and $h(v_2) = J'_q = J'_{p+1}$. By the first item of the definition of reduction graph of an overlap graph, it follows that $\{h(v_1), h(v_2)\} = \{J'_p, J'_{p+1}\} \in E_\gamma$. This proves the fourth case.

We have now completely determined the reduction graph; it is shown in Figure [13].
graph of an overlap graph, it follows that \( \{h(v_1), h(v_2)\} = \{J_p, J_q\} \in E_\gamma \). This proves the second case.

3. Assume that either \( v_1 \) or \( v_2 \) is an edge of \( L \) and that the other one is not an edge of \( L \) (thus \( \tilde{e} \) does not belong to \( L \)). We follow the same line of reasoning as we did in Theorem 12. Without loss of generality, we can assume that \( v_1 \) is not an edge of \( L \) and that \( v_2 \) is an edge of \( L \). Clearly,

\[
\emptyset = O_u(i, i) = O_u(i, i_1) \oplus O_u(i_1, i)
\]

for each position \( i_1 \). By the structure of \( L \) we know that \( q = 2 \) or \( q = \kappa \). We prove it for the case \( q = 2 \) (\( q = \kappa \), resp.). By Lemma 5 and Lemma 11 we can choose \( i_1 \in \{rspos_{p-1}, rspos_p\} \) such that \( O_u(i_1, i) = \{p\} \). By applying Corollary 4 on \( L \), we have \( O_u(i, i_1) = (\text{pos}(u) \cap P) \oplus (\bigoplus_{t \in P} O_u(t)) \) with \( P = \{2, \ldots, p - 1\} \cup P' \) (\( P = \{p + 1, \ldots, \kappa\} \cup P', \) resp.) for some \( P' \subseteq \{p\} \). By the third (fourth, resp.) item of the definition of reduction graph of an overlap graph, it follows that \( \{h(v_1), h(v_2)\} = \{J'_2, J'_3\} \in E_\gamma \) (\( \{h(v_1), h(v_2)\} = \{J'_2, J'_3\} \in E_\gamma \), resp.). This proves the third case.

4. Assume that both \( v_1 \) and \( v_2 \) are edges of \( L \), but \( \tilde{e} \) does not belong to \( L \). Again, we follow the same line of reasoning as we did in Theorem 12. Without loss of generality, we can assume that \( p \leq q \). By the structure of \( L \), we know that \( p = 2 \) and \( q = \kappa > 3 \). By applying Corollary 4 on \( L \), we have \( \emptyset = O_u(i, i) = (\text{pos}(u) \cap P) \oplus (\bigoplus_{t \in P} O_u(t)) \) with \( P = \{2, \ldots, \kappa\} \). By the fifth item of the definition of reduction graph of an overlap graph, it follows that \( \{h(v_1), h(v_2)\} = \{J'_2, J'_3\} \in E_\gamma \). This proves the last case.

This proves the forward implication. We now prove the reverse implication \( \{v_1, v_2\} \in E_\gamma \Rightarrow \{h^{-1}(v_1), h^{-1}(v_2)\} \in E_u \), where \( h^{-1} \), the inverse of \( h \), is given by:

- \( h^{-1}(J_p) \) is the unique \( v \in V_u \) with \( f_u(v) = p \) that is not an edge of \( L \),
- \( h^{-1}(J'_p) \) is the unique \( v \in V_u \) with \( f_u(v) = p \) that is an edge of \( L \),

for \( 2 \leq p \leq \kappa \). Let \( e \in E_\gamma \). We consider each of the five types of edges in the definition of reduction graph of an overlap graph.

1. Assume \( e \) is of the first type. Then \( e = \{J'_p, J'_{p+1}\} \) for some \( p \) with \( 2 \leq p < \kappa \). Since \( h^{-1}(J'_p) \) is the desire edge of \( L \) with both vertices labelled by \( p \) and \( h^{-1}(J'_{p+1}) \) is the desire edge of \( L \) with both vertices labelled by \( p + 1 \), it follows, by the definition of root subgraph, that \( h^{-1}(J'_p) \) and \( h^{-1}(J'_{p+1}) \) are connected by a reality edge in \( L \). Thus, we have \( \{h^{-1}(J'_p), h^{-1}(J'_{p+1})\} \in E_u \). This proves the reverse implication when \( e \) is of the first type (in Definition 1).

2. Assume \( e \) is of the second type. Then \( e = \{J_p, J_q\} \) for some \( p \) and \( q \) with \( 2 \leq p < q \leq \kappa \) and

\[
\emptyset = (\text{pos}(u) \cap P) \oplus \{p\} \oplus \{q\} \oplus \left( \bigoplus_{t \in P} O_u(t) \right)
\]

with \( P = \{p + 1, \ldots, q - 1\} \cup P' \) for some \( P' \subseteq \{p, q\} \). By Theorem 12 there is a reality edge \( \{w_1, w_2\} \) in \( R_u \), such that \( w_1 \) has label \( p \) and \( w_2 \) has
label $q$ and both are not vertices of $L$. By the definition of cps, we have a $\{w'_1, w'_2\} \in E_u$ such that $f_u(w'_1) = p$ (resp.) and $w'_1 (w'_2$, resp.) is not an edge of $L$. Therefore $w'_1 = h^{-1}(J_p)$ and $w'_2 = h^{-1}(J_q)$.

This proves the reverse implication when $e$ is of the second type.

3. The last three cases are proved similarly.

This proves the reverse implication and we have shown that $h$ is an isomorphism from $R_u$ to $R_\gamma$.

Example 12
The realistic string $u = 453475623267$ was introduced in Example 10. The reduction graph $R_\gamma$ of the overlap graph of $u$ is given in Figure 11. The reduction graph $R_u$ of $u$ is given in Figure 14. It is easy to see that the result of applying cps to $R_u$ is a graph that is indeed isomorphic to $R_\gamma$. This makes clear why there were two proofs for both edges $\{J_2, J_6\}$ and $\{J_4, J_7\}$ in Example 10; each one corresponds to one reality edge in $R_u$ (outside $L$).

Formally, we have not yet (up to isomorphism) constructed the reduction graph $R_u$ of a realistic string $u$ from its overlap graph. We have ‘only’ constructed $\text{cps}(R_u)$ (up to isomorphism). However, it is clear that $R_u$ can easily be obtained from $\text{cps}(R_u)$ (up to isomorphism) by considering the edges as reality edges and replacing every vertex by a desire edge of the same label.

8 Consequences

Using the previous theorem and [5] (or Chapter 13 in [4]), we can now easily characterize successfulness for realistic overlap graphs in any given $S \subseteq \{Gnr, Gpr, Gdr\}$. The notions of successful reduction, string negative rule and graph negative rule used in this section are defined in [4].

Below we restate a theorem of [3].
Theorem 16
Let \( N \) be the number of components in \( \mathcal{R}_u \). Then every successful reduction of \( u \) has exactly \( N - 1 \) string negative rules.

Due to the ‘weak equivalence’ of the string pointer reduction system and the graph pointer reduction system, proved in Chapter 11 of \([4]\), we can, using Theorem 15, restate Theorem 16 in terms of graph reduction rules.

Theorem 17
Let \( u \) be a realistic string, and \( N \) be the number of components in \( \mathcal{R}_{\gamma_u} \). Then every successful reduction of \( \gamma_u \) has exactly \( N - 1 \) graph negative rules.

As an immediate consequence we have the following corollary. It provides a solution to an open problem formulated in Chapter 13 in \([4]\).

Corollary 18
Let \( u \) be a realistic string. Then \( \gamma_u \) is successful in \( \{Gpr, Gdr\} \) iff \( \mathcal{R}_{\gamma_u} \) is connected.

Example 13
Every successful reduction of the overlap graph of Example 10 has exactly two graph negative rules, because its reduction graph consist of exactly three components. For example \( gn_{r4} \ gdr_{5,7} \ gn_{r2} \ gdr_{3,6} \) is a successful reduction of this overlap graph.

Every successful reduction of the overlap graph of Example 11 has exactly one graph negative rule. For example \( gnr_{2} \ gpr_{4} \ gpr_{5} \ gpr_{7} \ gpr_{6} \ gpr_{3} \) is a successful reduction of this overlap graph.

With the help of \([5]\) (or Chapter 13 in \([4]\)) and Corollary 18, we are ready to complete the characterization of successfulness for realistic overlap graphs in any given \( S \subseteq \{Gnr, Gpr, Gdr\} \).

Theorem 19
Let \( u \) be a realistic string. Then \( \gamma_u \) is successful in:

- \( \{Gnr\} \) iff \( \gamma_u \) is a discrete graph with only negative vertices.
- \( \{Gnr, Gpr\} \) iff each component of \( \gamma_u \) that consists of more than one vertex contains a positive vertex.
- \( \{Gnr, Gdr\} \) iff all components of \( \gamma_u \) are negative.
- \( \{Gnr, Gpr, Gdr\} \).
- \( \{Gdr\} \) iff all vertices of \( \gamma_u \) are negative and \( \mathcal{R}_{\gamma_u} \) is connected.
- \( \{Gpr\} \) iff each component of \( \gamma_u \) contains a positive vertex and \( \mathcal{R}_{\gamma_u} \) is connected.
- \( \{Gpr, Gdr\} \) iff \( \mathcal{R}_{\gamma_u} \) is connected.
9 Discussion

We have shown how to directly construct the reduction graph of a realistic string $u$ (up to isomorphism) from the overlap graph $\gamma$ of $u$. From a biological point of view, this allows one to reconstruct a representation of the macronuclear gene (and its waste products) given only the overlap graph of the micronuclear gene. Moreover, this results allows one to (directly) determine the number $n$ of graph negative rules that are necessary to reduce $\gamma$ successfully. Along with some results in previous papers, it also allows us to give a complete characterization of the successfulness of $\gamma$ in any given $S \subseteq \{Gnr, Gpr, Gdr\}$.

It remains an open problem to find a (direct) method to determine this number $n$ for overlap graphs $\gamma$ in general (not just for realistic overlap graphs). That is, a better method than first determining a legal string $u$ corresponding with $\gamma$ and then determining the reduction graph of $u$.

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