D3-brane – D-instanton configuration
and $\mathcal{N} = 4$ super YM theory
in constant self-dual background

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Abstract

We consider $SO(4) \times SO(6)$ invariant type IIB string solution describing D3-branes superposed with D-instantons homogeneously distributed over D3-brane world-volume. In the near D3-brane horizon limit this background interpolates between $AdS_5 \times S^5$ space in UV and flat space (with non-constant dilaton and RR scalar) in IR. Generalizing the AdS/CFT conjecture we suggest that type IIB string in this geometry is dual to $\mathcal{N} = 4$ SYM theory in a state with a constant self-dual gauge field background. The semiclassical string representation for the Wilson factor implies confinement with effective string tension depending on constant D-instanton density parameter. This provides a simple example of type IIB string – gauge theory duality with clear D-brane and gauge theory interpretation.

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1. Introduction

Recent discovery of a remarkable connection between gravity and gauge theory descriptions of D-branes gives a hope of eventual string-theoretic description of properties of strongly coupled gauge theories such as confinement [1] (see [2] for reviews). The best understood example of this duality [3,4,5] based on D3-branes relates $\mathcal{N} = 4$ super YM theory and type IIB string in $AdS_5 \times S^5$ space. To clarify the meaning of the duality in the context of type IIB string theory and to extend it to non-conformal (and less supersymmetric) cases one may study various modifications of the pure D3-brane background, in particular:

(i) add extra energy to the branes, i.e. consider near-extremal D3-branes; the corresponding gauge theory is then in a thermal state (see, e.g., [6,7]); (ii) add D7-branes breaking supersymmetry on the gauge theory side to $\mathcal{N} = 2$ [8,9]; (iii) break supersymmetry completely by adding dilaton charge as in the solution of [10,11] (which is a special case of the solutions in [12] with tachyon field turned off) or, more generally, exciting other scalar fields in $S^5$-compactified 5d theory, see [14] and references there.

Here we propose another example which is similar to the one in [10,11] and [12] in that the background preserves $SO(4) \times SO(6)$ symmetry but is simpler having a clear interpretation on both supergravity and gauge theory sides. We consider a homogeneous distribution of D-instantons on a large number of coincident D3-branes. This corresponds to modifying the D3-brane background by adding a RR scalar charge together with (equal) dilaton charge in a way that preserves 1/2 of supersymmetry. The resulting supergravity solution is a limit of superposition [15] of D3-brane [16] and D(-1)-brane [17,18] backgrounds. The D3+D(-1) configuration is T-dual to D4+D0 or D5+D1 marginal bound state backgrounds and preserves 1/4 (1/2 near the horizon of D3-brane) of supersymmetry.

In the present case the D-instanton charge is homogeneously smeared over the D3-brane world-volume instead of being localized as in [19,20] The corresponding gauge theory is not simply the vacuum $\mathcal{N} = 4$ SYM theory ‘perturbed’ by a single instanton (with perturbations of the supergravity D3-brane and gauge field backgrounds related as in [19,21]) but rather is a different $\mathcal{N} = 2$ supersymmetric state described by a homogeneous self-dual gauge field background. The latter is assumed to be averaged in some way to preserve the Euclidean $SO(4)$ and translational invariance. Its order parameter (instanton density) is kept fixed in the large $N$ limit.

Following [3] we conjecture that the type IIB string theory in near-horizon D3+D(-1) geometry is dual to $\mathcal{N} = 4$ SYM theory in such constant self-dual gauge field background.

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1 Similar dilatonic solutions were discussed also in [13].

2 A limiting case when D-instanton is put at the center of the Euclidean $AdS_5$ space but is still localized in the boundary coordinates and thus corresponds to a very large (i.e., approximately homogeneous) instanton gauge field in the boundary SYM theory was recently discussed in [20].
This conjecture is consistent with analogous correspondence between the supergravity and
gauge theory descriptions of D3-branes in flat space with additional gauge field distributions
on them. Adding a self-dual gauge field on a D3-brane in flat space is equivalent to
adding D-instanton charge [22,23]. The remaining supersymmetry ensures that the leading
$\frac{1}{2} F^4$ long- and short- distance interaction potentials between branes (computed in classi-
cal supergravity and in one-loop SYM theory) agree [24] like they do in the D0-brane case
[25]. This implies that the (smeared) D3+D(-1) supergravity configuration corresponds to
a D3-brane with a (homogeneous) self-dual background gauge field.

Though this example is somewhat unphysical (being based on a Euclidean background
which usually has a well-defined meaning only as a virtual configuration in the path in-
tegral) our main motivation is that it provides a clear setting for testing certain aspects
of string theory – gauge theory duality. As in other similar cases [12,10,11,14] here the
background interpolates between a regular UV $AdS_5 \times S^5$ space (conformal gauge theory)
and a singular IR background. A remarkable simplification which happens in D3+D(-1)
case (in addition to a transparent analytic form of the supergravity solution) is that the
IR limit of the string-frame metric is actually flat with the singularity of the background
residing only in the dilaton field. Assuming the string-theory representation for the Wil-
son factor [26] that immediately leads (in the semiclassical approximation) to the area law
behaviour.

This prediction of the ‘string in curved space – gauge theory’ duality is consistent with
expectations that gauge theory in a constant self-dual background [27,28,29,30] should be
(partially) confining. The confinement should be simply a consequence of the presence of
a non-trivial background field (implying, e.g., that propagators do not have poles in the
complex energy plane [30]). We find that the effective string tension following from the
curved space picture depends indeed only on the background gauge field order parameter
and not on the gauge coupling.

We shall start in section 2 by presenting the D3+D(-1) supergravity background. The
corresponding gauge theory state described by a self-dual background will be considered
in section 3. The relation between with the supergravity solution and gauge theory state
will be illustrated in section 4 by SYM effective action interpretation of the D3-brane
probe action in the D3+D(-1) background. In section 5 we shall use the dual string theory
description to compute the Wilson loop factor using the semiclassical fundamental string
action and demonstrate that it exhibits area law behaviour. In section 6 we shall present
analogous discussion of the ‘t Hooft loop factor based on the D-string action and show
that magnetic monopoles are screened at large separations. Section 7 will contain some
concluding remarks. In Appendix we shall consider perturbations of the background fields
near the D3-brane + D-instanton supergravity background and demonstrate that scalar
field perturbations do not, in general, decouple from the metric one. While one linear
combination of the dilaton and RR scalar perturbations still satisfies the massless scalar
equation in $AdS_5 \times S^5$ (so there are still gapless excitations in this background), the
second combination mixes with graviton and should have complicated dynamics, reflecting
the presence of the self-dual gauge field background on the gauge theory side.

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3 The IR limit of the Einstein-frame metric is again $AdS_5 \times S^5$. 
2. Supergravity picture: D3+D(-1) background

A simple example of a ‘deformation’ of the D3-brane solution of type IIB supergravity by non-trivial scalar field backgrounds is obtained by switching on a RR scalar charge and balancing it by a dilaton charge to preserve 1/2 of supersymmetry. The resulting exact solution representing a marginal ‘bound state’ of D3-brane and a (smeared) D-instanton is a non-trivial example of a type IIB brane intersection configuration parametrised by two harmonic functions. It has the following string-frame metric [15]

$$ds^2_{10} = (H_{-1})^{1/2}(H_{3}^{-1/2}dx_m dx_m + H_{3}^{1/2}dx_s dx_s) ,$$

where \( m = 1, \ldots, 4 \), \( s = 5, \ldots, 10 \), and \( H_{-1} \) and \( H_{3} \) are the harmonic functions depending only on \( x_s \). In addition to the self-dual 5-form background \((C_{1234} = H_{3}^{-1})\) representing the D3-brane charge we have the non-trivial dilaton \( \Phi = \phi_0 + \phi \) and RR scalar backgrounds

$$e^{\phi} = H_{-1} \text{ , } C = (H_{-1})^{-1} - 1 .$$

This solution is T-dual to \( D_4 + D_0 \) or \( D_5 + D_1 \) configurations. The total stress tensor of the two scalar fields is zero so that the Einstein frame metric remains the same as in the D3-brane case.

Since the D-instanton is smeared over the 3-brane world-volume the two harmonic functions have the same structure \((r^2 = x_s x_s)\)

$$H_3 = 1 + \frac{Q_3}{r^4} \text{ , } H_{-1} = 1 + \frac{Q_{-1}}{r^4} ,$$

where the coefficients are normalized as follows (see, e.g., [31,24]):

$$Q_3 = 4\pi N g_s \alpha'^2 = \lambda \alpha'^2 \text{ , } Q_{-1} = \frac{K (2\pi)^4 \alpha'^2}{N V_4} = q \lambda \alpha'^4 ,$$

$$q \equiv \frac{K (2\pi)^4}{N V_4} , \text{ } \lambda = 4\pi N g_s \text{ , } e^{\phi_0} = g_s .$$

4 We absorb the asymptotic value of the dilaton \((e^{\phi_0} = g_s)\) into the overall coefficient \( \frac{1}{2\pi^2} = \frac{1}{(2\pi)^2 g_s^2 \alpha'^4} \) of the action (rescaling also the RR fields by \( g_s \)-factor). Here \( C = iC \), where \( C \) is the original RR scalar. In the type IIB theory with the Euclidean time \( x_4 = it \) the self-dual RR 5-form is imaginary, so this solution is complex in both Euclidean and Minkowski versions. We shall view D-instanton or D3+D(-1) backgrounds as formal complex solutions of type IIB theory.

5 In general, T-duality implies that for a Dp-brane background which is smeared in \( n \) transverse toroidal directions \( H_p = 1 + \frac{Q_p^{(n)}}{r^{p-3}} \), where \( Q_p = N_p g_s (2\pi)^{(5-p)/2} T^{(p-7)/2} (\omega_{6-p})^{-1} \), \( \omega_{k-1} = 2\pi \frac{\omega_p}{\Gamma(\frac{p}{2})} \), \( Q_p^{(n)} = N_p N_{p+n} Q_{p+n} (2\pi)^{\frac{p}{2}} T^{-\frac{p}{2}} V_n^{-1} \). Here \( V_n \) is the volume of the flat internal torus and \( T = \frac{1}{2\pi \alpha'} \).
The integers $N$ and $K$ are the D3-brane and D-instanton numbers.

The supergravity approximation applies when derivatives of the fields remain small in units of $\alpha'$, i.e.

$$\lambda \gg 1, \quad r \gg (q\lambda)^{1/5} \alpha'^{9/10},$$

and when the effective string coupling is small, i.e. $r \gg (q\lambda)^{1/4} \alpha'$.

Let us now consider the following generalization of the decoupling limits of [32,33]

$$\alpha' \to 0, \quad \{u = \frac{r}{\alpha'}, \lambda, q\} = \text{fixed}.$$  \hspace{1cm} (2.7)

This limit is equivalent to taking a special solution without 1 in the 3-brane harmonic function $H_3$ (and making a coordinate redefinition). The dilaton and RR scalar are expressed (2.2) in terms of the function $H_{-1}$ which remains finite in this limit:

$$H_{-1} = 1 + \frac{q\lambda}{u^4},$$

while the string-frame metric takes the following form

$$ds^2_{10} = \alpha'^{1/2}(1 + \frac{q\lambda}{u^4})^{1/2} \left( \lambda^{-1} u^2 dx_m dx_m + \frac{du^2}{u^2} + d\Omega_5^2 \right).$$

(2.9)

The corresponding Einstein-frame metric $ds_E^2 = e^{-\phi/2}ds^2_{10}$ is still $AdS_5 \times S^5$ as in the D3-brane case. As in the pure D3-brane case [3], the factor $\alpha'$ cancels out in the classical string action which thus remains finite in the limit (2.7).

Like the $AdS_5 \times S^5$ Einstein frame metric, the string frame metric (2.9) is completely regular: near the core $r \to 0$ (i.e. in the IR region $u \to 0$) it becomes flat

$$ds^2_{10} = \alpha' q^{1/2} \left( dx_m dx_m + dz^2 + z^2 d\Omega_5^2 \right), \quad z \equiv \frac{\lambda^{1/2}}{u}.$$  \hspace{1cm} (2.10)

Note that the overall coefficient (effective string tension) is just $q^{1/2}$, i.e. does not depend on $\lambda$. The full D3+D(-1) background is, however, singular because of the blow-up of the dilaton at $u = 0$. This is the same behaviour as in the case of the pure D-instanton background [17].

The background (2.9) thus interpolates between $AdS_5 \times S^5$ at $u = \infty$ (UV) and flat space with singular dilaton near $u = 0$ (IR), while in the absence of D-instantons ($q = 0$) the $AdS_5 \times S^5$ space describes both regions. The constant instanton density parameter $q$ breaks conformal invariance in the IR region. Similar interpolating backgrounds (with singular metrics in the IR) were found in [12,10,11,14]. An attractive feature of the present case is its simplicity (not unrelated to remaining supersymmetry) and clear identification

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6 This limit was considered also by N. Itzhaki.
of the corresponding dual gauge theory – $\mathcal{N} = 4$ SYM with a constant self-dual background (section 3). This interpretation is supported by the relation between the classical action of a D3-brane probe moving in this geometry and the SYM one-loop effective action in the corresponding gauge field background (see section 4).

The type IIB string theory in the background (2.1), (2.2) is thus conjectured to be dual to a certain $\mathcal{N} = 2$ supersymmetric, and $SO(4) \times SO(6)$ and translationally invariant state of $U(N) \mathcal{N} = 4$ SYM theory (which may be equivalent to some $\mathcal{N} = 2$ supersymmetric YM theory with matter which is conformal in the UV). Symmetries of the string theory should therefore imply certain symmetries on the gauge theory side. The string theory and gauge theory are parametrised by $\lambda$, $N$, $q$, $V_4$ (or $g_s$, $N$, $K$, $V_4$).

One obvious string symmetry is $T$-duality. The original D3+D(-1) background (2.1), (2.2) with standard ($r \to \infty$) asymptotically flat region is covariant under $T$-duality along all four 3-brane world-volume directions: $T$-duality interchanges $H_3$ with $H_{-1}$, i.e. $Q_3$ with $Q_3^{(4)}$. The limiting background (2.1), (2.2), (2.8) (and $C_{1234} = H_3^{-1}$) is parametrised by ‘asymmetric’ combination of harmonic functions ($H_3' \equiv H_3 - 1 = \frac{\lambda}{\alpha'^2u^4}$, $H_{-1} = 1 + \frac{q\lambda}{u^4}$) and thus changes its form under the $T$-duality. Assuming that all four directions are compact with equal radii $a$, $x_m = a\theta_m$, $\theta_m \equiv \theta_m + 2\pi$ (i.e. $V_4 = (2\pi \alpha')^4$) and that the dual angular coordinate is $\tilde{\theta}_m \equiv \tilde{\theta}_m + 2\pi$, the string metric $T$-dual to (2.9) is found to be (in terms of the string Lagrangian

$$d\tilde{s}_{10}^2 = \alpha' \lambda^{1/2} (1 + \frac{q\lambda}{u^4})^{1/2} \left[ \frac{d\tilde{\theta}_m d\tilde{\theta}_m}{a^2u^2 (1 + \frac{q\lambda}{u^4})} + \frac{du^2}{u^2} + dQ_5^2 \right].$$

(2.11)

The effective string coupling $e^{\tilde{\phi}}$ dual to $e^{\Phi}$ is

$$e^{\tilde{\phi}} = \tilde{g}_s' H_{-1} = g_s (1 + \frac{q\lambda}{u^4}).$$

(2.12)

The large $u$ limit of the dual metric (2.11) still has the same $AdS_5 \times S^5$ form (with $\tilde{x}_m = a\tilde{\theta}_m$) when written in terms of the ‘dual coordinate’ (or ‘dual energy scale’) $\tilde{u} = \frac{\lambda^{1/2}}{\alpha'^2u^4}$.

The small $u$ limit of (2.11) is again a flat space region (cf.(2.10)). Using the expression for $q$ (2.3) the fundamental string Lagrangian in the near $u = 0$ region (2.9) can be put

7 More precisely, here under $T$-duality $V_4 \leftrightarrow \tilde{V}_4^{(0)}$, $g_s \leftrightarrow \tilde{g}_s^{(0)}$, $N \leftrightarrow K$, where $V_4 \tilde{V}_4^{(0)} = (2\pi)^8 \alpha'^4$, $V_4/g_s^2 = \tilde{V}_4^{(0)}/\tilde{g}_s^{(0)2}$, and $\tilde{g}_s^{(0)} = \frac{(2\pi)^3}{V_4^4}g_s$ is the dual coupling in the standard asymptotically flat region. As a result (see (2.4)) $\tilde{Q}_3 = Q_3^{(4)}$, $\tilde{Q}_3^{(4)} = Q_3$.

8 This is the same conclusion as in the near-horizon D3-brane case which is mapped by $T$-duality into the near-horizon limit of smeared D-instanton [14]. Though the $AdS_5$ metric with compact $x_m$ is singular at $u = 0$, string theory is supposed to resolve this singularity and should be dual to SYM theory on a 4-torus.
into the form $L = \frac{1}{4\pi} \sqrt{\frac{K}{N}} \partial_b \theta_m \partial^b \theta_m + \ldots$ (the radius $a$ cancels out). The $T$-dual string Lagrangian is then

$$L = \frac{1}{4\pi} \sqrt{\frac{N}{K}} \partial_b \tilde{\theta}_m \partial^b \tilde{\theta}_m + \ldots ,$$  \tag{2.13}$$
i.e. the duality corresponds to interchanging $K \leftrightarrow N$. Measured in terms of the string metric the dual radius is $\tilde{a}_s = \frac{1}{\sqrt{qa}}$ (near $u = 0$ the effective string tension is $\sqrt{a}/2\pi$, see (2.10)). The gauge theory volume is defined, however, as the volume of the boundary theory in the $u \to \infty$ $AdS_5 \times S^5$ space limit and remains unchanged, $\tilde{V}_4 = (2\pi a)^4$ (we consider the simplest rectangular 4-torus). Note that near $u = 0$ and $u = \infty$ the ratio of the effective string couplings $e^{\Phi} / e^{\Phi} \equiv \tilde{g}_s(u) / g_s(u)$ becomes $\frac{\tilde{g}_s(0)}{g_s(0)} = \frac{N}{K}$, $\frac{\tilde{g}_s(\infty)}{g_s(\infty)} = \frac{\lambda}{a^4 u^4} |_{u \to \infty} \to 0$.

We are thus led to the conclusion that string $T$-duality implies an equivalence relation between $U(N)$ gauge theory state parametrized by $K$ (and $g_s(u)$) and $U(K)$ gauge theory state parametrized by $N$ (and $\tilde{g}_s(u)$). Given that $K$ may be interpreted (section 3) as an instanton number in $U(N)$ gauge theory, this suggests a relation to the Nahm duality which maps a charge $K$, $U(N)$ self-dual solution on a 4-torus to a charge $N$, $U(K)$ self-dual solution on a dual 4-torus $^{35}$. Nahm duality was previously discussed in connection with $T$-duality in string theory in flat background in, e.g., $^{36,37,38}$. In the present context of the “curved space string theory – gauge theory” duality conjecture one indeed seems to see why “YM theory (via the Nahm transformation) still knows about $T$-duality”, cf. $^{37}$. Here we are getting not just a relation between two classical self-dual Yang-Mills backgrounds, but an equivalence between two quantum gauge theories (which is reminiscent of some known dualities between supersymmetric gauge theories $^{33}$).

The forms of the dilaton and RR scalar (2.2),(2.8) suggest the following expressions for the IR flow of the SYM couplings

$$\lambda(u) = \lambda(1 + \frac{q\lambda}{u^4}), \quad \theta(u) = (1 + \frac{q\lambda}{u^4})^{-1} - 1 .$$  \tag{2.14}$$

Similar power-like running of the dilaton was found also in $^{10,11}$. In the $\mathcal{N} = 2$ gauge theory related to $D7 + D3$ brane configuration $^8$ analogous IR running of dilaton may be explained by instanton corrections (see $^1$ and refs. there). Here it should be induced by the presence of a constant self-dual field condensate.$^9$

The flatness of the $u \to 0$ limit of the metric suggests that the IR limit of gauge theory should be described by type IIB string theory in flat space (with additional dilaton and RR scalar and 4-tensor backgrounds being invisible in a semiclassical approximation). That obviously leads to the area law behaviour of the Wilson factor (see section 5).
The major question, however, is whether the background (2.3), (2.2), (2.8) may be extrapolated all the way into the IR region $u \to 0$. We expect that due to its remaining supersymmetry, this background, like $AdS_5 \times S^5$, is actually an exact (all order in $\alpha'$) solution of the classical type IIB string theory. Then one may relax the conditions (2.6) but there is still the condition that the effective string coupling $e^\phi$ should remain small, i.e.

$$u \geq u_* , \quad u_* \gg (q\lambda)^{1/4} = 2\pi \left(\frac{K\lambda}{NV_4}\right)^{1/4}.$$  

(2.15)

If $q\lambda$ is not constrained, we may allow $u_*$ to be sufficiently small to probe the IR limit and still have weak string coupling to ignore string loop corrections. This will be our assumption below. $u_* \sim (V_4)^{-1/4}$ may be interpreted as an IR energy cutoff on the gauge theory side.

3. Gauge theory picture

The gauge field theory counterpart of the supergravity solution discussed above is the $\mathcal{N} = 4$ $SU(N)$ SYM theory in a certain (non-vacuum) state which may be described as a stochastic averaging of a constant homogeneous self-dual background (see, e.g., [27,28,29,30]). Namely, we shall demand that in such state

$$<F_{mn}> = 0, \quad <\text{tr}(F_{mn}F_{mn} - F_{mn}^*F_{mn}^*)> = 0, \quad <\text{tr}(F_{mn}F_{mn})> = \frac{16\pi^2}{V_4^2} K ,$$

(3.1)

where a finite volume $V_4$ will be used as an IR cutoff. We shall assume that this $N = 2$ supersymmetric state may obtained by some averaging over different 4d directions so that $K$ is the only order parameter, i.e. that the Euclidean group – $SO(4)+$translations – is unbroken (while $P$- and $T$- invariances are obviously broken). Indeed, the supergravity solution has the same 4d Euclidean group symmetry and depends only on one extra constant $q$ which is related to $K$.

For the purpose of comparing with supergravity such state may be represented simply by (a stochastic averaging of) a constant self-dual abelian background such that

$$<F_{mn}F_{mn}^*> = <\text{tr}(F_{mn}F_{mn}^*)> = \frac{1}{N} <\text{tr}(F_{mn}F_{mn})> = \frac{16\pi^2}{V_4^2} K .$$  

(3.2)

We shall consider the large $N$ limit defined by

$$N, K, g_{YM}^{-1} \to \infty , \quad \lambda \equiv Ng_{YM}^2 = \text{fixed} ,$$

(3.3)

10 Related homogeneous distributions of instantons were considered also in [41].

11 Here tr is in the fundamental representation, $\text{tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$. The classical action is

$$S = \frac{1}{2g_{YM}^2} \int d^4x \text{tr}(F_{mn}F_{mn}) - \frac{i\theta}{16\pi^2} \int d^4x \text{tr}(F_{mn}F_{mn}^*).$$
Again, $q$ will be the only order parameter ‘visible’ on the supergravity side, i.e. the gauge theory state is defined by averaging over all fields satisfying $F_{mn} = F^*_{mn}$ and (3.2). Introducing a background field does not change UV finiteness property of the $\mathcal{N} = 4$ SYM theory, but may, of course, modify IR behaviour. Indeed, the propagators of massless fermions and ‘off-diagonal’ gluons in the self-dual background exhibit confinement-type (no-pole) behaviour [30]. The confinement on the SYM side is caused by the background field [29,30]. The dual string/gravity description will lead to an area law for the Wilson factor with effective string tension proportional simply to $\sqrt{q}$ (section 5). The supergravity description also predicts [214] that the gauge coupling constants ($g_{YM}, \theta$) should have power-like running in the IR.

4. D3 brane probe in D3+D(-1) background and SYM effective action

To support the gauge theory interpretation of the above supergravity background let us consider the action of a D3-brane probe moving in the D3+D(-1) background and demonstrate its relation to the quantum effective action of $\mathcal{N} = 4$ SYM theory. The latter will be computed in a constant gauge field background containing self-dual component on the stack of $N$ D3-branes representing a source and a constant abelian field on a separated D3-brane probe (see, e.g., [14,24,42,43,13,15]). We shall choose the static gauge on the Euclidean D3-brane world volume action. Then the action for a D3-brane probe moving in the background (2.9), (2.2), (2.8) and $C_{1234} \sim u^4$ takes the form

$$I_3 = \frac{N}{2\pi^2} \int d^4x \left[ u^4 \sqrt{\det \left( \delta_{mn} + \frac{\lambda}{u^4} \partial_m u^s \partial_n u^s + \frac{\lambda^{1/2}}{(u^4 + \lambda q)^{1/2}} F_{mn} \right) - u^4 + \frac{\lambda q}{4(u^4 + \lambda q)} F_{mn} F^*_{mn} \right] ,$$

where we have used (2.2), (2.8) in the RR scalar coupling term $-iCF_{mn}F^*_{mn} = -C F_{mn} F^*_{mn}$. Expanding in powers of derivatives we get

$$I_3 = \frac{N}{2\pi^2} \int d^4x \left[ \frac{1}{2} \partial_m u^s \partial_n u^s + \frac{1}{4} F_{mn} F_{mn} - \frac{q}{4(u^4 + \lambda q)} (F_{mn} F_{mn} - F^*_{mn} F^*_{mn}) + \ldots \right] .$$

12 Similar averaged gauge field configurations were discussed in the context of comparing with supergravity solutions in, e.g., [14,24]. The idea is to ‘engineer’ a gauge field background that has certain global characteristics (energy, momentum, instanton number, etc.) which are the only parameters present in the supergravity solution.

13 We use that the D3-brane tension is $T_3 = \frac{1}{2\pi^2} \frac{1}{2\pi \alpha'(2\pi \alpha')^2}$ and absorb the factor $2\pi$ into $F_{mn}$.

14 The leading interaction term here vanishes in the case of the self-dual $F_{mn}$ which corresponds to adding D-instanton charge to the D3-brane probe. This is the expected cancellation of the static potential between the same-type marginal bound states of branes like $D3 + D(-1)$ and $D3 + D(-1)$ or $D4 + D0$ and $D4 + D0$ [10,24].
This action can indeed be interpreted as a result of integrating out open string modes connecting the probe and the source separated by distance $u$ in flat space with $F_{mn}$ background on the probe and a self-dual background parametrized by $q$ on the source. Let us assume that $N$ coinciding D3-branes carry a constant $U(1)$ self-dual background $\mathcal{F}_{mn} = \mathcal{F}_{mn}^*$, so that the total $U(N + 1)$ background is block-diagonal, $\hat{F}_{mn} = \text{diag}(\mathcal{F}_{mn}I_{N \times N}, F_{mn}I_{1 \times 1})$. The leading $F^4$ one-loop correction in the $\mathcal{N} = 4$ SYM effective action has the following structure

$$
\frac{1}{u^4} \text{STr} \left[ (\hat{F}_{mn}\hat{F}_{mn})^2 - (\hat{F}_{mn}\hat{F}_{mn}^*)^2 \right],
$$

(4.3)

where $\hat{F}$ is the total background field, $u$ is the effective mass scale set by the separation between branes and STr is the symmetrized trace in the adjoint representation. For a commuting block-diagonal background $\hat{F} = \text{diag}(\mathcal{F}, F)$ we may simply replace $\hat{F}_{mn}$ by $F_{mn} - \mathcal{F}_{mn}$ under the adjoint trace adding the overall factor (number of non-zero eigenvalues) $2N$. Then the $\mathcal{F}^4$ term in (4.3) vanishes because of the self-duality of $\mathcal{F}$ and the $O(F^2F^2)$ cross-term becomes proportional to

$$
\frac{N}{u^4} \mathcal{F}_{mn} \mathcal{F}_{mn} \left( F_{mn} F_{mn} - F_{mn} F_{mn}^* \right).
$$

(4.4)

Assuming that $\mathcal{F}_{mn} \mathcal{F}_{mn} = q/\pi^2$ as in (3.4) and expanding (4.2) in powers of $q$ we find the (precise numerical) agreement between the supergravity (4.2) and the SYM (4.4) expressions (see [24] for details of similar calculations). A non-trivial feature of this example is that while $q$ and $F_{mn}$ have very different interpretation on the supergravity side (one is a parameter of the background and another is a ‘coordinate’ of the probe) they have the same background gauge field interpretation on the gauge theory side.

5. String representation for the Wilson loop

Let us now consider a type IIB string propagating in the D3+D(-1) supergravity background and compute, following [26], the potential between far separated ‘quark’ and ‘anti-quark’ (W-bosons) by evaluating the string partition function representation for the

---

15 Note that the $O(FF^3)$ term $\frac{N}{u^4} \mathcal{F}_{mn} \mathcal{F}_{mn} \left( F_{mn} F_{mn} - F_{mn} F_{mn}^* \right)$ absent in the supergravity expression vanishes after averaging over the directions of the self-dual field $\mathcal{F}$, implying $< \mathcal{F}_{mn} > = 0$. As was already mentioned above, this averaging is necessary in order for the brane source gauge theory state to represent the $SO(4)$ invariant supergravity background. The $O(F^4)$ term in (4.3) is also reproduced by the $F^4$ term in the expansion of (4.1).

16 From the ‘D-branes in flat space’ perspective the reason why the supergravity (long-distance) and SYM (short-distance) interaction potentials agree [25] is in the non-renormalization theorem for $F^4$ terms [18].
Wilson factor in the semiclassical approximation. This amounts to computing the value of the bosonic Nambu string action on the classical string configuration. Assuming that the string penetrates the IR \((u = 0)\) region the flatness of the string metric in this region (2.10) implies that the Nambu action will be proportional to the area. This implies the confinement behaviour similar to what is found in the non-extremal (finite temperature) case \([49]\) and in the case of the dilaton-charge deformed 3-brane solution \([11]\).

We shall take the fundamental string action in the form \(\frac{1}{2\pi\alpha} \int d^2\sigma \sqrt{\det G_{ab}}\) and fix the static gauge \(x^0 = \tau, \ x^1 = \sigma\), where the world-sheet coordinates \(\tau\) and \(\sigma\) run from 0 to \(\mathcal{T}\) and \(-L/2\) to \(L/2\). As in \([26]\), we shall consider a static string solution which has only the radial coordinate \(u\) changing with \(\sigma\). The stationary point is found with the boundary condition that \(u\) runs to infinity at \(-L/2\) and \(L/2\) and takes the minimal value in between.

Since the dilaton and RR field couplings can be ignored in the semiclassical approximation, the string action is determined simply by the metric (2.9)

\[
I = \frac{\mathcal{T}}{2\pi} \int d\sigma \sqrt{H_{-1}(u) \left(u'^2 + \lambda^{-1} u^4\right)} , \quad H_{-1} = 1 + \frac{q\lambda}{u^4},
\]

or, equivalently,

\[
I = \frac{\mathcal{T}}{2\pi} \int d\sigma \ n(z) \sqrt{z'^2 + 1},
\]

\[
n(z) = \sqrt{q + \frac{\lambda}{z^4}}, \quad z = \frac{\lambda^{1/2}}{u}.
\]

Since \(f^2(u) \equiv H_{-1}\lambda^{-1} u^4 = \lambda^{-1} u^4 + q\), has a minimum at \(u = 0\) it follows from the general analysis of \([50]\) that the action (5.1) leads to confinement with the string tension proportional to \(f(0) = \sqrt{q}\). The same conclusion follows from the light-ray analogy used in \([11]\) to argue that for a large separation \(L\) the potential is dominated by the minimum of the ‘refraction index’ \(n(z)\), i.e. by the limit \(z \gg (\lambda/q)^{1/4}\).

Indeed, the large \(L\) limit is dominated by the IR region \(u \to 0\) where the string-frame metric (2.9) becomes flat (2.10), so that the string action (5.2) takes the flat-space form

\[
I \to \frac{\mathcal{T}\sqrt{q}}{2\pi} \int d\sigma \sqrt{z'^2 + 1},
\]

and thus automatically leads to the area-law behaviour of the Wilson factor,

\[
W(C) \to e^{-T_{\text{eff}}\mathcal{T}L}, \quad T_{\text{eff}} = \frac{\sqrt{q}}{2\pi}.
\]

An interesting feature is that the effective string tension \(T_{\text{eff}}\) does not depend on the ‘t Hooft coupling \(\lambda\), but depends only on the background order parameter \(q\) (instanton

\[\text{Note that in contrast to the case discussed in \([11]\) here the effective tension does not depend on } \alpha'\]
density). This is consistent with the gauge theory picture (section 3) since confinement in a self-dual gauge field background should be caused simply by the gauge field condensate.

In the above discussion we have ignored the fact that near the true minimum of \( n(z) \), i.e. \( z = \infty, \; u = 0 \), the string coupling becomes strong and thus string loop corrections (higher topologies) may not be, in principle, ignored.\(^{18}\) For large \( L \) the string probes the far IR region where the string coupling is no longer small. A way to avoid this problem may be to impose the IR cutoff (2.15) \( u \geq u_* \), i.e. to restrict \( z \) to values \( z \leq z_* \ll (\lambda/q)^{1/4} \). Then the minimal value of \( n(z) \) will be at \( z = z_* \) and the potential will contain additional dependence on \( L/z_* \). One may qualitatively approximate the resulting effective string tension as \( T_{\text{eff}} = \frac{1}{2\pi} \sqrt{q + \frac{u_*^4}{\lambda}} \) which reduces to (5.3) in the limit \( u_* \rightarrow 0 \).

6. D-string action, S-duality and ‘t Hooft loop

Additional insight into the structure of the duality is obtained by probing D3+D(-1) background with a D-string. Following [51,52] the exponential of the D-string action determines (in the semiclassical approximation) the corresponding ‘t Hooft loop factor and thus the potential between magnetic monopoles on the gauge theory side.

In general, a D-string action in a background with a non-trivial metric, dilaton and RR scalar has the form\(^{19}\)

\[
I_1 = T_1 \int d^2 \sigma \left[ e^{-\phi} \sqrt{\det(G_{ab} + F_{ab})} + \frac{i}{2} \epsilon^{ab} C F_{ab} \right].
\]  

Solving for the 2d gauge field \( F_{ab} \), i.e. ‘integrating it out’ in the semiclassical approximation (as, e.g., in [53,54]) gives

\[
I'_1 = T_1 \int d^2 \sigma \sqrt{e^{-2\phi} + C^2 \sqrt{\det G_{ab}}}. \tag{6.2}
\]

The difference as compared to the fundamental string Lagrangian is thus in the ‘tension factor’ which in the case of the D-instanton background (2.2),(2.8) becomes \( (C = i\lambda) \)

\[
\frac{1}{g_s} \sqrt{e^{-2\phi} - C^2} = \frac{1}{g_s} \sqrt{2e^{-\phi} - 1} = \frac{1}{g_s} \sqrt{\frac{1 - \frac{\lambda^2}{u_*^4}}{1 + \frac{\lambda^2}{u_*^4}}}. \tag{6.3}
\]

\(^{18}\) This problem did not appear in the dilatonic charge solution [10,11] where the supergravity approximation was still valid near the minimum [11].

\(^{19}\) Here \( T_1 = \frac{T}{g_s} = \frac{1}{2\pi \alpha' g_s} \) and we absorb the \( 2\pi \alpha' \) factor into the Euclidean world sheet gauge field \( F_{ab} \).
The D-string counterpart of the fundamental string action (5.1) is thus is

\[ I = \frac{T}{2\pi g_s} \int d\sigma \sqrt{(1 - \frac{q\lambda}{u^4}) (u'^2 + \lambda^{-1}u^4)} . \]  

(6.4)

Remarkably, this action is exactly the same (up to the tension ratio factor \(1/g_s\)) as the fundamental string action (5.1) but with \(q \rightarrow -q\)! Since the F-strings and D-strings are interchanged by S-duality \([53,55,54]\), this suggests that \(q \rightarrow -q\) is simply a consequence of S-duality transformation applied to our background – D-string ‘sees’ S-dual geometry (2.9),(2.8),(2.2) with \(q \rightarrow -q\).

Indeed, under the basic S-duality transformation \(\tau \rightarrow -1/\tau\), \(\tau = C + i e^\phi\), which is a symmetry of the scalar \(SL(2,R)/SO(2)\) sigma model action \((\partial \phi)^2 + e^{2\phi}(\partial C)^2\) one has

\[ e^{\phi'} = (e^{-2\phi} + C^2) e^\phi , \quad C' = -(e^{-2\phi} + C^2)^{-1}C , \quad g'_{\mu\nu} = g_{\mu\nu} E . \]  

(6.5)

This target space transformation combined with world-volume duality is a symmetry of the D3-brane action \([54]\) and relates D-string and F-string actions. In the case of the D-instanton or D3+D(-1) backgrounds (2.2),(2.8) one finds from (6.5)

\[ e^{\phi'} = 2 - e^\phi = 1 - \frac{q\lambda}{u^4} , \quad C' = e^{-\phi'} - 1 , \]  

(6.6)

with the (unchanged) Einstein-frame metric again given by \(AdS_5 \times S^5\). Thus indeed the basic S-duality transformation corresponds to changing the sign of \(q\) (and inverting the string coupling). The selfduality of D3-brane action suggests that switching of the sign of \(q\) should be induced by the duality transformation on the SYM side.

The S-dual string-frame metric is proportional to (2.9) with \(q \rightarrow -q\),

\[ (ds^2_{10})' = g_s^{-1} \alpha' \lambda^{1/2}(1 - \frac{q\lambda}{u^4})^{1/2} \left( \lambda^{-1}u^2 dx_m dx_m + \frac{du^2}{u^2} + d\Omega^2_5 \right) , \]  

(6.7)

\(^{20}\) We stress that this S-duality is the standard transformation of the Minkowski type IIB supergravity with \(C\) formally replaced by \(iC\) in the case of the D-instanton background. This transformation is different from the one considered in the first reference in \([18]\), where S-duality was interpreted as a symmetry of the ‘rotated’ action \((\partial \phi)^2 - e^{2\phi}(\partial C)^2\). The basic \(SL(2,R)\) transformation there was \(S_\pm \rightarrow -1/S_\pm\), \(S_\pm = C \pm e^{-\phi}\), while (5.4) corresponds in this notation to \(S_\pm \rightarrow 1/S_\pm\). Both transformations are formally the symmetries of the scalar \(SL(2,R)/SO(1,1)\) action \(-\frac{\partial S_+ \partial S_-}{(S_+ - S_-)^2}\) (see also \([56]\)), but it is (5.3) that is the standard \(SL(2,Z)\) symmetry that has a D-brane interpretation. More general \(SL(2,Z)\) transformations (6.5) do not admit a continuation to the case of imaginary \(C\) which preserves reality of the dilaton. Note that under the ‘Euclidean’ transformation \(S_\pm \rightarrow -1/S_\pm\) one finds that \(e^{\phi'} = e^\phi - 2 = \frac{2\lambda}{u^4} - 1\), i.e. that the dilaton is defined in the ‘dual’ region \(u < u_*\) but it again blows up at \(u = 0\).
with $u = u_* = (q\lambda)^{1/4}$ being a curvature singularity. The dual effective string coupling in (5.6) remains small for all values of $u > u_*$ and vanishes at $u = u_*$. Thus under S-duality the singularity of the original background in the dilaton (2.2), (2.8) (at $u = 0$) gets ‘transformed’ into the singularity in the metric (at $u = u_*$).

The potential between monopole $m$ and anti-monopole $\bar{m}$ can be found by minimizing the D-string action (6.4). As in the case considered in \cite{51,52}, there exists a critical distance $L_{\text{crit}} \sim (\lambda q)^{1/4}$, (6.8) such that for $L < L_{\text{crit}}$, there is minimal D-string world sheet connecting $m$ and $\bar{m}$. For $L \ll L_{\text{crit}}$ we are effectively in the $AdS_5 \times S^5$ region and the potential is proportional to $1/L$ as in the case of the conformal $\mathcal{N} = 4$ SYM theory. For $L > L_{\text{crit}}$ there is no string solution $u(\sigma)$ which minimizes (6.4), i.e. the potential becomes constant – the magnetic monopoles are screened. Since the D-string action (6.4) may be interpreted as an action of the fundamental string propagating in the S-dual background (5.7), the screening behavior of the $mm\bar{m}$ potential may be understood also from the fact that around the ‘horizon’(singularity) $u = u_*$ in (5.7) the string world sheet is contractable and can split into two separate parts.

The resulting picture is consistent with the discussion in section 5: in the region $L \geq (\lambda q)^{1/4}$ the electric charges (W-bosons) are confined, while the magnetic monopoles are screened.

7. Concluding remarks

Our motivation in discussing the above D3-brane + D-instanton background is that it provides a simple generalization of the gravity – gauge theory correspondence in the pure D3-brane case. We have seen that the supergravity or string theory description predicts a (coupling-independent) confinement behaviour, essentially because the constant D-instanton density makes the string-frame metric flat in the IR region. This confinement does not apply to all states, and there are still some gapless excitations in this background (in particular, one linear combination of the dilaton and RR scalar fluctuations satisfies the massless Laplace equation in $AdS_5 \times S^5$, see Appendix).

This example is different from the case of localized D-instantons in $AdS_5 \times S^5$ which correspond to localized YM instantons in the boundary gauge theory \cite{19,21} and are virtual Euclidean configurations contributing to path integral and producing non-perturbative contributions to observables on both sides of the duality correspondence. Adding a localized instanton is a perturbation of the pure D3-brane case, while we interpreted the homogeneous self-dual gauge field background as a special $\mathcal{N} = 2$ supersymmetric state of the $\mathcal{N} = 4$ SYM theory.
Though this Euclidean self-dual background is somewhat artificial from physical point of view, it would be interesting to understand the reason for a power-like IR flow of the gauge coupling (2.14) from the gauge theory point of view.

The conformal invariance of the \( \mathcal{N} = 4 \) SYM theory is broken by the background field density parameter \( q \) in a ‘soft’ way. Since the Einstein frame metric remains that of the \( AdS_5 \times S^5 \) space, some of the supergravity modes which do not directly couple to the dilaton (like the \( S^5 \)-radius mode) do not feel the presence of \( q \), i.e. still satisfy the same massive \( AdS_5 \) Laplace equations as in the pure D3-brane case. The correlators of the corresponding gauge field operators (\( F^4 + \ldots \) in the \( S^5 \) radius case) are thus expected to be the same as in the vacuum \( \mathcal{N} = 4 \) SYM case. At the same time, the dilaton and RR scalar modes which satisfied the massless Laplace equation in the pure D3-brane case no longer do so in the D3+D(-1) background (the correlators of \( F^2 \) and \( FF^* \) operators are certainly different from the vacuum case). Since the scalar field background values are non-constant, their fluctuations mix with each other and also with graviton perturbations. The corresponding quadratic fluctuation action is derived in Appendix below. Such mixing is a general feature of solutions with non-constant dilaton backgrounds, like D4-brane \[57,58\] or the solution of \[10,11,12\].

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Appendix A. Perturbations of the supergravity background

Below we shall study small fluctuations of the metric, dilaton and RR scalar fields near the background (2.2),(2.8),(2.9). The equations for the dilaton and RR scalar perturbations are in general of interest in connection with spectrum of possible bound states on gauge theory side [7,58,57]. The scalar perturbations are not, in general, expected to decouple from the metric perturbations in the case of non-constant scalar background as we illustrate below on the present D3+D(-1) example.

We start with the $D = 5$ Einstein-frame Lagrangian for the metric-dilaton-axion system ($C = iC; \mu, \nu = 1,...,10$)

$$L = \sqrt{\hat{g}} \left[ \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \left( \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - e^{2\hat{\phi}} \partial_\mu \hat{C} \partial_\nu \hat{C} \right) \right], \quad (A.1)$$

and expand the fields near their background values

$$\hat{\phi} = \phi + \eta , \quad \hat{C} = C + e^{-\phi} \xi , \quad \hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} . \quad (A.2)$$

The background fields $\phi$ and $C$ satisfy

$$\partial_\mu C = -e^{-\phi} \partial_\mu \phi , \quad e^{-\phi} D^2 e^\phi = D^2 \phi + (\partial_\mu \phi)^2 = 0 , \quad (A.3)$$

where $D_\mu$ is the covariant derivative depending on the background metric $g_{\mu\nu}$. The Einstein-frame metric in (2.9) is that of the $AdS_5 \times S^5$ space.

Expanding to the second order in fluctuation fields we get for the $\eta$ and $\xi$ dependent part of $L$ (we omit the pure graviton $h^2$ part)

$$L_2 = \sqrt{\hat{g}} \left[ \hat{h}^{\mu\nu} \partial_\mu \phi \partial_\nu \eta - \partial_\nu \phi \partial_\nu \phi \right] - (\partial_\mu \eta)^2 + 2(\partial_\mu \phi)^2 \eta^2 + (\partial_\mu \xi)^2 + 4 \partial_\mu \phi \partial_\mu \eta \xi \right] , \quad (A.4)$$

where

$$\hat{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2} g^{\mu\nu} h , \quad h = g_{\mu\nu} h^{\mu\nu} .$$

Following [59] we choose the diffeomorphism gauge fixing term to be

$$L_{g,f.} = -\frac{1}{2} \sqrt{\hat{g}} g^{\mu\nu} \left[ D_\beta \hat{h}_\mu^{\beta} - \partial_\mu \phi \left( \eta + \xi \right) \right] [D_\alpha \hat{h}_\nu^{\alpha} - \partial_\nu \phi \left( \eta + \xi \right)]$$

$$= \sqrt{\hat{g}} \left[ -\frac{1}{2} g^{\mu\nu} D_\alpha \hat{h}_\mu^{\alpha} D_\beta \hat{h}_\nu^{\beta} + D_\nu h^{\mu\nu} \partial_\mu \phi \left( \eta + \xi \right) - \frac{1}{2} (\partial \phi)^2 (\eta + \xi)^2 \right] . \quad (A.5)$$

The $\eta, \xi$ dependent part of the gauge-fixed Lagragian is then (we again omit $O(h^2)$ terms)

$$L'_2 = L_2 + L_{g,f.} = -\sqrt{\hat{g}} \left( \partial_\mu \phi \partial_\nu \phi + D_\mu D_\nu \phi \right) \hat{h}^{\mu\nu} (\eta + \xi)$$

15
\[ + (\partial_\mu \eta)^2 - 2(\partial_\mu \phi)^2 \eta^2 - (\partial_\mu \xi)^2 - 4\partial^\mu \phi \partial_\mu \eta \xi + \frac{1}{2}(\partial_\mu \phi)^2 (\eta + \xi)^2 \]. \quad (A.6)\]

Note that because of the classical equation for \( \phi \) \((A.3)\) the trace of \( h_{\mu \nu} \) does not actually couple to \( \eta \) and \( \xi \).

Using the explicit form of the background Einstein-frame metric and the dilaton (see \((2.2),(2.8),(2.9)\))
\[
d s_E^2 = \frac{1}{z^2}(dx_m dx_m + dz^2 + z^2 d\Omega^5) , \quad e^\phi = 1 + \frac{q z^4}{\lambda} , \quad (A.7)\]
we get for the coupling function in the graviton – \((\eta + \xi)\) mixing term
\[
\partial_\mu \phi \partial_\nu \phi + D_\mu D_\nu \phi = \delta_{\mu z} \delta_{\nu z} [\partial_z^2 \phi + z^{-1} \partial_z \phi + (\partial_z \phi)^2] - z^{-1} \delta_{\mu m} \delta_{\nu n} \partial_z \phi . \quad (A.8)\]

This function is non-vanishing, so there is a non-trivial mixing between \( \eta + \xi \) and the traceless part of \( h_{\mu \nu} \).

Introducing
\[
\eta_\pm = \eta \pm \xi , \]
we may rewrite \((A.6)\) as
\[
L'_{2} = -\sqrt{g} \left[ \partial^\mu \eta_+ \partial_\mu \eta_- - \frac{1}{2}(\partial_\mu \phi)^2 \eta_+^2 - 2\partial^\mu \phi \partial_\mu \eta_+ \eta_- + (\partial_\mu \phi) \partial_\mu \eta_+ + (\partial_\mu \phi) \partial_\nu \phi + D_\mu D_\nu \phi \right] h^{\mu \nu} \eta_+ . \quad (A.9)\]

Then the equations of motion for \( \eta_- \) and \( \eta_+ \) are
\[
D^2 \eta_+ - 2\partial^\mu \phi \partial_\mu \eta_+ - 2D^2 \phi \eta_+ = 0 , \quad D^2 \eta_- + 2\partial^\mu \phi \partial_\mu \eta_- + (\partial_\mu \phi)^2 \eta_+ + (\partial_\mu \phi) \partial_\nu \phi + D_\mu D_\nu \phi)h^{\mu \nu} = 0 . \quad (A.10)\]

Using \((A.3)\) the equation for \( \eta_+ \) can be written also as
\[
D^2 (e^{-\phi} \eta_+) = 0 , \quad (A.11)\]
i.e. the field \( e^{-\phi} \eta_+ \) satisfies the massless scalar equation in the \( AdS_5 \times S^5 \) space.

Since the fields \( \phi \) and \( C \) couple to the boundary gauge theory operators as
\[
L_{int}(\phi, C) = e^{-\phi} \text{tr}(F_{mn} F_{mn}) - C \text{tr}(F_{mn} F^*_{mn}) , \quad (A.12)\]
their perturbations couple as
\[
L_{int}(\phi + \eta, C + e^{-\phi} \xi) = L_{int}(\phi, C) - e^{-\phi} \left[ \eta \text{ tr}(F_{mn} F_{mn}) + \xi \text{ tr}(F_{mn} F^*_{mn}) \right] + ... = L_{int}(\phi, C) - e^{-\phi} \eta_+ \text{ tr}(F_{mn}^{(+)}) - e^{-\phi} \eta_- \text{ tr}(F_{mn}^{(-)}) + ... , \quad (A.13)\]

16
where
\[ F_{mn}^{(\pm)} \equiv \frac{1}{2} (F_{mn} \pm F_{mn}^*) . \]
In interpreting this expression we should take into account that \( \eta_+ \) and \( \eta_- \) are conjugate to each other like light-cone variables (cf. the kinetic term in (A.9)). Thus the term that couples to \( \eta_+ \) is actually the source for \( \eta_- \) and vice versa. In view of (A.11) we then conclude that the operator \( \text{tr}(F_{mn}^- F_{mn}^-) \) is still counterpart of a massless mode in the \( AdS_5 \) space. This is perfectly consistent with the fact that the background gauge field corresponding to the supergravity solution satisfies (3.1), \( \langle \text{tr}(F_{mn}^- F_{mn}^-) \rangle = 0 \), i.e. does not have a condensate.

The second field \( \eta_- \) is non-trivially coupled to the graviton and thus has more complicated dynamics, reflecting the fact that the operator \( \text{tr}(F_{mn}^+ F_{mn}^+) \) should ‘feel’ the presence of the gauge field background.
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