Estimating mixing properties of local Hamiltonian dynamics and continuous quantum random walks is PSPACE-hard

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Abstract

A major topic of (classical) ergodic theory is to examine qualitatively how the phase space of dynamical systems is penetrated by the orbits of their dynamics. We consider interacting qubit systems with dynamics according to 4-local Hamiltonians and continuous quantum random walks. For these systems one could use the von Neumann entropy of the time-average to characterize the mixing properties of the corresponding orbits, i.e., what portion of the state space and how uniformly it is filled out by the orbits. We show that the problem of estimating this entropy is PSPACE-hard.

1 Introduction

Ergodic theory studies the long-term average behavior of dynamical systems. The continuous time-evolution is described by a one-parameter family \( \{ T_t : t \in \mathbb{R} \} \) of maps of the phase space \( X \) into itself. A major topic of classical ergodic theory is to examine qualitatively how the orbits \( \{ T_t x \} \) (where \( x \) is the initial state) penetrate the phase space \( X \). If the orbits “spread uniformly” over \( X \), then the system is called ergodic. Ergodicity of dynamical systems plays an important role in statistical mechanics and thermodynamics because in the ergodic case the time averages and ensemble averages of physical variables are equal (see \cite{H} for a rigorous study of these questions).

In this paper we consider the case that \( X \) is the Hilbert space of many qubits, and \( \{ T_t : t \in \mathbb{R} \} \) is the unitary one-parameter group \( \{ e^{-iHt} : t \in \mathbb{R} \} \) generated by the system Hamiltonian \( H \). We study the computational complexity of estimating the von Neumann entropy generated by Hamiltonian time evolutions if we average over

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the time. The motivation for considering the entropy of the time average is that it quantifies in a certain sense what portion of the state space and how uniformly it is filled out by the trajectory according to the time evolution. A second interesting property of the entropy of the time-average is that it could be used to quantify how far a pure state $|\Phi\rangle$ is from all eigenvectors of a Hamiltonian (this question was raised in [2]). The reason is that the entropy of the time-average of the initial pure state $|\Phi\rangle\langle\Phi|$ can only be large if the initial state is a superposition of many eigenvectors (corresponding to different eigenvalues) of $H$ with equal amplitudes. We explain this later in more detail.

The time-average is defined as follows:

**Definition 1 (Time average)**

Let $H$ be the Hamiltonian of a quantum system $\mathcal{H}$. For an arbitrary state $\rho$ on $\mathcal{H}$ its time average is the state $\bar{\rho}$ defined by

$$\bar{\rho} := \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} e^{-iHt} \rho e^{iHt} dt.$$  \hspace{1cm} (1)

It is well-known that the time average depends on the spectral decomposition of the Hamiltonian (a proof of the following lemma is included in the appendix for completeness).

**Lemma 1** The time-average $\bar{\rho}$ of $\rho$ is given by

$$\bar{\rho} = \sum_{\lambda} P_{\lambda} \rho P_{\lambda},$$

where $P_{\lambda}$ are the orthogonal projections onto the eigenspaces of $H$ for different eigenvalues $\lambda$.

Before we can formulate question on the complexity of estimating certain properties of time averages we have to restrict the class of considered Hamiltonians. This is because physically realistic Hamiltonians must satisfy some locality condition. One way to formalize this is the following definition.

**Definition 2 (local operator)**

Let $\mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be the tensor product Hilbert space of $n$ Hilbert spaces. We call an operator $k$-local if it is a sum of operators that act on at most $k$ tensor components non-trivially.

If each tensor component is the state space of a physical particle, then the natural interactions are usually pair-interactions, that is, 2-local operators in the sense of the above definition.

Nevertheless, for the Hilbert space $\mathcal{H} := (\mathbb{C}^2)^{\otimes n}$ of $n$ interacting qubits it may also be physically reasonable to allow $k$-local interaction for $k > 2$. One reason is that they may describe effective Hamiltonians. A second reason is that there is not necessarily a one-to-one correspondence between qubits and physical particles (for instance, several qubits could describe the state of one particle).
We consider time-evolutions according to 4-local Hamiltonians and continuous quantum random walks. The time evolution $e^{-iHt}$ is called a continuous quantum random walk if $H$ is the adjacency matrix of some graph $\mathcal{G}$. Recall that a matrix $A$ is called an adjacency matrix if it is symmetric, has only 0s and 1s as entries, and all its entries on the diagonal are 0. Furthermore, we say that a matrix is a $k$-local adjacency matrix if it is a sum of $k$-local operators that are adjacency matrices themselves. We will consider 5-local adjacency matrices.

We show that the problem estimating the von-Neumann entropy of time-averages of computational basis states, where the time-evolution is described by 4-local Hamiltonians and 5-local quantum random walks, is PSPACE hard. The problem of estimating the density matrix of the time-average reduced to a qubit is also PSPACE-hard.

Due to Lemma 1 it is clear that the entropy generation depends on the whole spectrum of the Hamiltonians. Therefore, these results should be compared to the results on the complexity of estimating the minimal eigenvalue (ground state energy) of local Hamiltonians. The problem of determining the lowest energy value of a (classical) spin-spin interaction of Ising type is known to be NP-complete [4, 5]. For interacting qubits the problem of determining the lowest energy value is even QMA-complete (“Quantum-NP”) if one allows 3-local interactions only [6, 7]. Recall that in these NP and Quantum-NP problems the task is not to determine the lowest eigenvalues with high precision. The demanded accuracy is only inverse polynomially in the number $n$ of interacting qubits.

Note that the entropy of the time-average of a pure state $\rho := |\Phi\rangle\langle\Phi|$ could be used to measure how “far away” $|\Phi\rangle$ is from all eigenstates of $H$. This question was studied in [2]; the authors constructed quantum states that are far away from any eigenstate of any non-trivial local Hamiltonian, in the sense that $||\Phi\rangle - |\Psi\rangle||$ is greater than some constant lower bound for all eigenstates $|\Psi\rangle$ of all $k$-local (for a fixed $k$) Hamiltonians, independent of the form of the Hamiltonians. The measure for the distance of a pure quantum state $|\Phi\rangle$ from the spectrum of a Hamiltonian used in [2] is

$$d(|\Phi\rangle, H) := \min_{\{|\Psi\rangle\}} \min_\lambda \| |\Phi\rangle - |\Psi_\lambda\rangle \|,$$

where $\{|\Psi\rangle\}$ runs over all sets consisting of eigenvectors of $H$ corresponding to different eigenvalues $\lambda$. One easily verifies that the distance is equivalent to

$$\min_{\{|\Psi\rangle\}} \min_\lambda 2(1 - |\alpha_\lambda|^2),$$

where $|\Phi\rangle = \sum_\lambda \alpha_\lambda |\Psi_\lambda\rangle$. Therefore, the bound $d(|\Phi\rangle, H)$ is maximal if and only if $|\Phi\rangle$ is an equally weighted superposition of eigenvectors corresponding to different eigenvalues of $H$. Using Lemma 1 one see that this is equivalent that the entropy of the time average of $|\Phi\rangle$ is maximal; the maximally possible entropy is the logarithm of the number of different eigenvalues of $H$. Based on these observations, our results may be also interpreted as a proof of the computational difficulty of deciding for a given quantum state how far it is from all eigenstates of a local Hamiltonian.
2 Stating the problem

The techniques we use here to prove PSPACE-hardness of estimating the entropy of time-averages are closely related to those in [3] where we studied the complexity of measuring local observables. The general idea is to describe the computational steps carried out by a (reversible) space-bounded Turing machine by a quantum circuit consisting of elementary quantum gates or Toffoli gates and then to encode the circuit into a local Hamiltonian (this will be explained in Sections 3 and 4).

Theorem 1 (PSPACE-hardness)

Let \((A_l)\) be a uniformly generated family of 4-local Hamiltonians (or 5-local adjacency matrices) on the Hilbert spaces \(H_l\) consisting of \(\text{poly}(l)\) qubits. Then the problems of (1) estimating the von-Neumann entropy of the time-average \(S(\bar{\rho})\) and (2) estimating the reduced state of \(\bar{\rho}\) to single qubits are PSPACE-hard.

More precisely: (1) Deciding if the entropy of the time-average of a computational basis state of \(H_l\) according to the dynamics given by \(A_l\) is smaller than \(a_l\) or greater than \(b_l\), where \(b_l - a_l > \text{poly}(l)\), is PSPACE-hard.

Let \(\hat{\rho}_i\) denote the reduced state of the time-average \(\bar{\rho}\) to the \(i\)th qubit of \(H_l\). (2) Deciding if \(|1\rangle \langle 1| \hat{\rho}_i|1\rangle < 1/d_l\) or \(|1\rangle \langle 1| \hat{\rho}_i|1\rangle > 1/2 - 1/d_l\), where \(d_l = \text{poly}(l)\), is PSPACE-hard.

We prove this theorem in Section 4. The proof is based on a characterization of the complexity class PSPACE with quantum circuits that is explained in the next section.

3 Characterizing PSPACE by circuits

The complexity class PSPACE is usually defined with respect to the Turing machine model [9]. PSPACE is the class of all languages recognizable by polynomial space bounded deterministic Turing machines that halt on all inputs [10].

In the following we work with a characterization of PSPACE with respect to circuits consisting of elementary quantum gates (Toffoli gates only). This characterization is proved in Theorem 1 in [3].

Theorem 2 (PSPACE)

For every language \(L\) in PSPACE there is a polynomial-time uniformly generated family of quantum circuits \((V_l)_{l\in\mathbb{N}}\). Here \(l\) denotes the length of the input \(x\). Each \(V_l\) consists of \(s_l = \text{poly}(l)\) elementary quantum gates (or Toffoli gates\(^1\)) and acts on \(m_l = \text{poly}(l)\) many qubits. The circuit \(V_l\) decides whether an input string \(x\) is an element of \(L\) in the following sense.

There is a polynomial-time computable natural number \(r_l\) such that the \(r_l\)-fold concatenation of \(V_l\) solves the corresponding PSPACE problem, i.e.

\[
V_l^{r_l}(|x\rangle \otimes |y\rangle \otimes |00\ldots0\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle \otimes |00\ldots0\rangle,
\]

\(^1\)In [3] we have used elementary quantum gates instead of Toffoli gates. But it is clear that the transitions performed reversible Turing machines can be simulated by a circuit consisting of Toffoli gates only. This is because the Toffoli gate is universal for reversible computation. The reason why we use the Toffoli gates is that it is possible to construct from such a circuit a Hamiltonian is an adjacency matrix.
where \( f \) is the characteristic function of \( L \). That is \( f(x) = 1 \) if \( x \in L \) and \( f(x) = 0 \) otherwise. The vector \( \ket{x} \) is the basis state given by the binary word \( x \in \{0,1\}^l \), the vector \( \ket{y} \) is the state of the output qubit and \( \ket{00\ldots0} \) is the initial state of \( m_1 - l - 1 \) ancilla qubits.

The intuition behind this construction is as follows. Recall that irreversible TMs (used to define PSPACE) can be simulated space-efficiently by reversible TMs \([11,12]\). The computational steps of a reversible TM can be simulated efficiently by a quantum circuit \( U \). This circuit acts on registers representing the state of the head of TM, the current tape position, and a polynomial portion of the tape (see \([3]\) for an explicit construction of the quantum circuit \( U \)) and realizes permutations of basis states corresponding to the transitions of the TM. By augmenting \( U \) with some control logic, we can achieve that once the TM moves into a final state, that the new circuit performs some idle cycles, flips the output qubit if and only if the answer is yes after exactly \( r_l/2 \) steps, then performs some idle cycles, and reverses the computation. This is schematically illustrated in Figure 1. It is necessary to perform idle cycles because we sometime do not know exactly after how many steps the TM moves into a final step, but only know a lower bound. By including the idle steps we guarantee the computational time depends only on the input length.

**Remark 1** We can consider an output register instead of an output qubit. It is also possible to construct a quantum circuit such that

\[
V^0_l(|x\rangle \otimes |y\rangle \otimes |00\ldots0\rangle) = |x\rangle \otimes |y + f(x) \mod 2^c\rangle \otimes |00\ldots0\rangle,
\]

for any \( c \) that is polynomial in \( l \).

### 4 Constructing the Hamiltonians / the quantum random walks

Starting from the family \( \{V_l\} \) of circuits we construct a uniformly generated family of 4-local Hamiltonians (5-local adjacency matrices) \( \{A_l\}_{l \in \mathbb{N}} \) such that the time-average of the state

\[
|x\rangle \otimes |0\rangle \otimes |00\ldots0\rangle \otimes |100\ldots0\rangle
\]

and its entropy encodes the solution the instance $x$ of a PSPACE problem; the first three tensor components are as in Theorem 2 and the last one corresponds to the new register clock that is needed in our construction. The construction is based on Feynman’s construction of a computer whose dynamics is an autonomous time-evolution \[13, 14\].

To explain the construction of the Hamiltonian (adjacency matrix $A$) let $V$ be a circuit as in Theorem 2 and $s$ be its size, that is, the number of elementary gates (Toffoli gates). We need a register clock indicating which gate is applied. It consists of $s$ qubits. The allowed states of the register clock are of the form $|0\cdots010\cdots0\rangle$ indicating which gate of $V$ is applied currently. We denote by $T_k$ the elementary quantum gates (Toffoli gates) of $V$.

We first define the forward-time operator
\[
F = T_0 \otimes |1\rangle_2 \langle 0|_1 \otimes |0\rangle_1 \langle 1|_0 + T_1 \otimes |1\rangle_3 \langle 0|_2 \otimes |0\rangle_2 \langle 1|_1 + \cdots + T_{s-1} \otimes |1\rangle_1 \langle 0|_0 \otimes |0\rangle_s \langle 1|_{s-1},
\]
where the operators $|0\rangle_k \langle 1|_k$ and $|1\rangle_k \langle 0|_k$ are the annihilation $a$ and creation operators $a^\dagger$, respectively, acting on the $i$th qubit of the clock. The backward-time operator is defined as the adjoint of $F$. The Hamiltonian is defined as
\[
A := F + F^\dagger.
\]

Now we show that if $V$ consists of Toffoli gates only, then $A$ is a 5-local adjacency matrix of some graph in the computational basis, that is, $A$ is symmetric, has only 0s and 1s as entries, and its entries on the diagonal are all 0s. Since the matrix describing the action of the Toffoli gate (in the computational basis) contains only 0s and 1s, the sums
\[
T_k \otimes |1\rangle_k \langle 0|_k \otimes |0\rangle_{k+1} \langle 1|_{k+1} + T_k \otimes |0\rangle_k \langle 1|_k \otimes |1\rangle_{k+1} \langle 0|_{k+1}
\]
are adjacency matrices that are 5-local operators. $A$ is the sum of the above matrices. Therefore, it is symmetric and has only 0s on the diagonal. It remains to show that no 1s of these matrices can meet when summing. This is done by checking that the operators
\[
a_k^\dagger a_{k+1} \mod s, a_k a_{k+1}^\dagger \mod s,
\]
lead to orthogonal states for all $k = 0, \ldots, s-1$ when applied to any computational basis state. We see this by observing that if $k - l \mod s \geq 2$ then we have
\[
\langle b| a_k^\dagger a_{k+1} a_l^\dagger a_{l+1}^\dagger |b\rangle = 0 \quad \text{and} \quad \langle b| a_k a_{k+1}^\dagger a_l a_{l+1}^\dagger |b\rangle = 0
\]
because the operators $a_k^\dagger a_{k+1} a_l^\dagger a_{l+1}^\dagger$ and $a_k a_{k+1}^\dagger a_l a_{l+1}^\dagger$ either map $|b\rangle$ onto the zero vector or change the bits of $|b\rangle$ at the non-overlapping positions ($k, k+1$) and ($l, l+1$). Similarly, if $l = k + 1$ then we have
\[
\langle b| a_k^\dagger a_{k+1} a_{k+2} a_{k+2}^\dagger |b\rangle = 0 \quad \text{and} \quad \langle b| a_k a_{k+1}^\dagger a_{k+1} a_{k+2}^\dagger |b\rangle = 0
\]
because $a_k^\dagger a_{k+1} a_k a_{k+1}^\dagger a_{k+2}^\dagger$ is the zero operator and $a_k^\dagger a_{k+1} a_k a_{k+1} a_{k+2}^\dagger$ either maps $|b\rangle$ onto the zero vector or changes the bits of $|b\rangle$ at the non-overlapping positions $k$ and $k+2$.

Therefore, only computational basis states appear in $A|b\rangle$ with coefficients all equal to 1. Consequently, we have either $\langle b|A|b\rangle = 0$ or $\langle b|A|b\rangle = 1$. This show that $A$ is a 5-local adjacency matrix if $V$ contains Toffoli gates only.

We denote the linear span of the vectors

$$F^j|\Psi_0\rangle \quad \text{for} \quad j \in \mathbb{N}$$

where $|\Psi_0\rangle := |x00\cdots0\rangle \otimes |100\cdots0\rangle$ as $O$. Here the first part of the tensor product denotes the register where $V$ acts on (input and ancilla registers) and the second component is the register clock. All states of this orbit are orthogonal until one has a recurrence to the initial state $|\Psi_0\rangle$. This can be seen as follows: If the register clock is in an allowed state there is only one summand of $F$ that is relevant. Its action on the clock is simple since it moves the symbol 1 to the next qubit (this dynamics of the clock may be interpreted as a propagation of a spin-wave). Therefore it is clear that the first $s-1$ states are orthogonal. The whole circuit $V$ is a classical logical operation which permutes basis states. Therefore the state $F^s|\Psi_0\rangle$ is either orthogonal to $|\Psi_0\rangle$ or both states coincide. Along the same line we argue that all states of the orbit are orthogonal until a state coincides with the initial state. Hence $F$ acts as a cyclic shift on $O$.

For our construction is is essential that the dimension of the orbit depends on the solution of the PSPACE problem. It is $2sr$ if $f(x) = 1$ and $sr$ if $f(x) = 0$, where $f(x)$ is the answer for the instance $x$ of the considered PSPACE problem. We denote the dimension of the orbit $O$ by $d$. Let $\omega$ be a primitive complex $d$-th root of unity. The eigenvalues of $F$ restricted to $O$ are

$$\omega^0, \omega^1, \omega^2, \ldots, \omega^{d-1}.$$ 

Furthermore, $|\Psi_0\rangle$ is a superposition of all eigenvectors of $F$ restricted to $O$ with equal amplitudes. All this follows from properties of the cyclic shift operator. Since $F$ and $F^\dagger$ commute on $O$ the eigenvalues of $A$ restricted to $O$ are $\omega^k + \bar{\omega}^k = 2\cos(2\pi j/d)$ for $k = 0, \ldots, d-1$.

Set $\rho := |\Psi_0\rangle \langle \Psi_0|$ and $\lambda_k := 2\cos(2\pi k/d)$. Note that $d$ is always even; this is because $r$ in Theorem 2 is even. Therefore, there are two eigenvalues with multiplicity 1 (corresponding to the real eigenvalues $+1$ and $-1$ of $F$) and $d-2$ eigenvalues with multiplicity 2 (corresponding to the complex eigenvalues of $F$). We have

$$\rho = \frac{1}{d} \sum_{k,l=0}^{d-1} |k\rangle \langle l|,$$

where $|k\rangle$ denotes the $k$th eigenvalue of $F$ restricted to $O$. For the time-average we obtain with Lemma 1

$$\bar{\rho} = \frac{1}{d} \sum_{k,l=0}^{d-1} [\lambda_k = \lambda_l] |k\rangle \langle l|,$$
where $[\lambda_k = \lambda_l]$ is 1 if both eigenvalues are equal and 0 otherwise. If we permute the order of the eigenvectors $|k\rangle$ such that eigenvectors corresponding to the same eigenvalues are adjacent, then $\rho$ looks in this basis as follows:

$$
\frac{1}{d} \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
\vdots \\
1 \\
1
\end{pmatrix}
$$

Using the form of the time-average in (2), its entropy is easily computed

$$
S(\bar{\rho}) = \log_2 d - (d - 2)/d \approx \log_2 d - 1.
$$

Therefore, the difference in entropy of the time-average of the computational basis state $|\Psi_0\rangle$ encoding a “yes”-instance and a “no”-instance is at least $c/2$ bits, where $c$ is the size of output register (see Remark 1). This proves the PSPACE-hardness of estimating the entropy of the time-average as stated in Theorem 2.

Let the output register consist of only one qubit. Obviously, the reduced state of time-average to the output qubit is $|0\rangle\langle 0|$ for a “no”-instance. For a “yes”-instance we know that the output qubit of the states $|\Psi_j\rangle := F^j|\Psi_0\rangle$ is 1 for $j = d/4, \ldots, 3d/4 - 1$ and 0 for all other $j$. To see this, recall how $V$ works: at step $r_l/2$ the circuit $V$ changes the output qubit from 0 to 1, at step $r_l$ we obtain the initial state with the output flipped, at step $r_l + r_l/2$ the output is flipped from 1 to 0, and finally in step 2$r_l$ we obtain the original input state.

Therefore, the occupation probability of the level $|1\rangle$ of $\bar{\rho}_{\text{out}}$ ($\bar{\rho}$ restricted to the output qubit) is exactly

$$
\langle 1|\bar{\rho}_{\text{out}}|1 \rangle = \sum_{j=d/4}^{3d/4-1} \langle \Psi_j|\bar{\rho}|\Psi_j \rangle.
$$

Let us denote by $P(j)$ the probability to observe the state $|\Psi_j\rangle$ if we measure the time-average in the computational basis. The probability $P(j)$ is given by

$$
P(j) = \langle \Psi_j|\bar{\rho}|\Psi_j \rangle
$$

$$
= \frac{1}{d^2} \sum_{k,l=0}^{d-1} \omega^{(k-l)j} [\lambda_k = \lambda_l]
$$

$$
= \frac{1}{d} + \frac{1}{d^2} \sum_{k \neq l}^{d-1} \omega^{(k-l)j} [\lambda_k = \lambda_l]
$$

This seen by observing that $|\Psi_j\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{jk}|k\rangle$. Now we make use of the Diaconis-Shahshahani bound (see Appendix) applied to the cyclic group $\mathbb{Z}_d$. The characters of
\[ Z_d \] are given by \( \chi_a(j) = \omega^a j \). The coefficients of the Fourier transform of the probability density \( P = (P(0), \ldots, P(d-1)) \) are

\[
\hat{P}(a) = \sum_{j=0}^{d} P(j) \chi_a(j) = \frac{1}{d^2} \sum_{k \neq l: \lambda_k = \lambda_l} \sum_{j=0}^{d-1} \omega^{(k-l+a)j}.
\]

Note that the possible values for \( k - l \) are only the even numbers \( 2, 4, \ldots, d/2 \) since we sum over all pairs \( (k, l) \) satisfying \( k \neq l \) and the condition \( \cos(2\pi k/d) = \cos(2\pi l/d) \) being equivalent to \( k = -j \).

If \( a \) is odd, then \( k - l + a \not\equiv 0 \mod d \). But if \( a \) is even, then there is exactly one pair of \( (k, l) \) satisfying above properties and \( k - l + a \equiv 0 \mod d \). With these observations we obtain

\[
\hat{P}(a) = \begin{cases} 
0 & \text{if } a \text{ odd} \\
\frac{1}{d} & \text{if } a \text{ even}
\end{cases}
\]

Now we obtain with Diaconis-Shahshani bound a lower bound on the total variation distance between \( P \) and the uniform distribution \( U \) on the orbit states \( F_j |\Psi_0\rangle \)

\[
\|P - U\|_{TV}^2 \leq \frac{1}{4} (d/2 - 1) \frac{1}{d^2} \leq \frac{1}{8d}.
\]

Taking the square root and multiplying both by 2 we obtain

\[
\sum_{j=0}^{d-1} |P(j) - \frac{1}{d}| \leq \frac{1}{\sqrt{2d}}.
\]

Using the triangle inequality the above bound implies that

\[
\langle 1 | \bar{\rho}_\text{out} | 1 \rangle = \langle 1 | \bar{\rho} | 1 \rangle = \sum_{j=d/4}^{3d/4-1} P(j) \in \left[ \frac{1}{2} - \frac{1}{\sqrt{2d}}, \frac{1}{2} + \frac{1}{\sqrt{2d}} \right].
\]

(Note that this is also true if we sum over any set containing half of all possible \( j \).)

In summary, we have proved that for a “yes”-instance the occupation probability \( \langle 1 | \bar{\rho}_\text{out} | 1 \rangle \) is almost 1/2, whereas it is 0 for a “no”-instance.

5 Appendix

Proof. Lemma 1: Let \( \{|k\rangle\} \) be a basis consisting of eigenvectors of \( H \) with corresponding eigenvalues \( \lambda_k \). Express the state \( \rho \) in this basis

\[
\rho = \sum_{k,l} \gamma_{kl} |k\rangle \langle l|.
\]
For the time-average we obtain

\[ \bar{\rho} = \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \sum_{k,l=0}^{d-1} e^{-i(\lambda_k - \lambda_l)t} \gamma_{kl} |k\rangle \langle l| \]

\[ = \sum_{k,l=0}^{d-1} \left( \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} e^{-i(\lambda_k - \lambda_l)t} \right) \gamma_{kl} |k\rangle \langle l| \]

\[ = \frac{1}{d} \sum_{k,l=0}^{d-1} [\lambda_k = \lambda_l] \gamma_{kl} |k\rangle \langle l| \]

\[ = \sum_{\lambda} P_\lambda \rho P_\lambda, \]

where \([\lambda_k = \lambda_l]\) is 1 if both eigenvalues are equal and 0 otherwise, \(\lambda\) are the different eigenvalues of \(H\), and \(P_\lambda\) are the orthogonal projection onto the corresponding eigenspaces. \(\square\)

The proof of the lemma can be found in [15].

**Lemma 2 (Diaconis-Shahshahani bound)**

Let \(A\) be an arbitrary abelian group, \(P\) an arbitrary probability distribution on \(A\) and \(U\) the uniform distribution on \(A\). The Diaconis-Shahshahani bound on the total variation distance

\[ \|P - U\|_{TV} := \frac{1}{2} \sum_{g \in A} |P(g) - U(g)| \]

is

\[ \|P - U\|_{TV}^2 \leq \frac{1}{4} \sum_{\chi \neq \chi_0} |\hat{P}(\chi)|^2, \quad (4) \]

where the sum runs over all non-trivial irreducible characters of \(A\) and \(\hat{P}\) denotes the Fourier transform of \(P\), that is, \(\hat{P}(\chi) = \sum_{g \in A} \chi(a)P(g)\).

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