OPTIMAL RELLICH-SOBOLEV CONSTANTS AND THEIR EXTREMALS

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Abstract. We prove that extremals for second order Rellich-Sobolev inequalities have constant sign. Then we show that the optimal constants in Rellich-Sobolev inequalities on a bounded domain Ω and under Navier boundary conditions do not depend on Ω.

1. Introduction and main results

Let \( n \geq 3 \) be an integer and \( p, q \) given exponents, such that
\[ 1 < p < q, \quad \text{and} \quad q \leq p^* = \frac{np}{n-2p} \text{ if } n > 2p. \]
Assume that \( \alpha, \beta \in \mathbb{R} \) are given in such a way that
\[
\begin{cases}
\beta = n - q \frac{n - 2p}{p} + \alpha \\
2p - n < \alpha < np - n.
\end{cases}
\]
In [16] it has been proved that there exists a best constant \( c > 0 \) such that
\[
\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p \, dx \geq c \left( \int_{\mathbb{R}^n} |x|^{-\beta} |u|^q \, dx \right)^{p/q}
\]
for any \( u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \), see also Corollary 2.12 in Subsection 2.2. A rescaling argument plainly shows that (1a) is a necessary condition for the validity of (2), while assumption (1b) can be weakened. For instance, if \( p = 2 \) then there exists an increasing sequence of integers \( j_k \geq n - 2 \) such that \( j_k \to \infty \) and such that (2) holds with a constant \( c > 0 \) if and only if \( (\alpha - 2)^2 \neq j_k \) for any \( k \geq 1 \), compare with [2, Theorem 1.1].

When \( q = p > 1, \beta = 2p - \alpha \) and (1b) hold, then (2) includes the sharp Rellich-type inequality
\[
\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta u|^p \, dx \geq \gamma_{p,\alpha} \int_{\mathbb{R}^n} |x|^{\alpha - 2p} |u|^p \, dx \quad \forall u \in C^2_c(\mathbb{R}^n),
\]
that has been proved by Mitidieri in [13]. Here we have put
\[
\gamma_{p,\alpha} := \frac{n - 2p + \alpha}{p} \frac{np - n - \alpha}{p}.
\]

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In particular, inequality (2) is naturally related to the function space
\[ D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) := \left\{ u \in L^p(\mathbb{R}^n; |x|^\alpha dx) \mid \Delta u \in L^p(\mathbb{R}^n; |x|^\alpha dx) \right\}, \]
and one is lead to study the minimization problem
\[ S_{p,q}(\alpha) = \inf_{\substack{u \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \\cup \ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-\beta} |u|^q dx \right)^{p/q}}. \]

If \( n > 2p, q = p^* \) and \( \alpha = \beta = 0 \), then the infimum in (5) equals the Sobolev constant
\[ S_p = \inf_{\substack{u \in D^{2,p}(\mathbb{R}^n) \\cup \ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{p/p^*}}, \]
that is relative to the critical embedding \( D^{2,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \). In [12], Corollary I.2, P.L. Lions proved that every bounded minimizing sequence is relatively compact up to dilations and translations, and in particular \( S_p \) is achieved. Moreover, by using Schwarz symmetrization he showed that, up to a change of sign, any extremal for \( S_p \) is spherically symmetric, positive and decreasing. Using this information, Hulshof and Van der Vorst [11] were able to prove uniqueness of extremals for \( S_p \), modulo dilations, translations in \( \mathbb{R}^n \) and change of sign.

As concerns general exponents \( p, q, \alpha \) and \( \beta \) satisfying (1), one can find sufficient conditions for the existence of extremals for \( S_{p,q}(\alpha) \) in the appendix of [16], see also [2] for the Hilbertian case \( p = 2 \).

In presence of weights rearrangement techniques are in general not applicable. As a matter of fact, breaking symmetry may occur, see Section 5 of [2], where \( p = 2 \) is assumed. Actually breaking positivity phenomena can not be a priori excluded as well: indeed, it may happen that no extremal for \( S_{p,q}(\alpha) \) has constant sign. From a merely technical point of view, the reason lies in the fact that truncation \( u \mapsto u^+ \) can not be used to prove the positivity of extremals for \( S_{p,q}(\alpha) \), since in general \( u^+ \notin D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \) for \( u \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \). Nevertheless, in Section 4 we prove the next result, that gives a positive answer to a query raised in [2].

**Theorem 1.1.** Assume that (1b) holds. Let \( u \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \) be an extremal for \( S_{p,q}(\alpha) \). Then (up to a change of sign), \( u \) is positive and superharmonic.

Assumption (1b) can not be neglected: by Theorem 4.1 in [2], breaking positivity does occur if \( p = 2, |\alpha - 2| \) is large enough and \( q \approx 2 \).

Next, let \( \Omega \) be a bounded and smooth domain containing the origin. Let
\[ D^{2,p}_N(\Omega; |x|^\alpha dx) = \left\{ u \in L^p(\Omega; |x|^{\alpha-2p} dx) \mid \Delta u \in L^p(\Omega; |x|^\alpha dx), u = 0 \text{ on } \partial \Omega \right\}, \]
(see Section 2.3 for details). The optimal Rellich-Sobolev constant under Navier boundary conditions is given by

$$S_{p,q}^{\text{Nav}}(\Omega; \alpha) = \inf_{\substack{u \in D^2_{N}(\Omega; |x|^\alpha dx) \setminus \{0\} \atop \int_{\Omega} \int \frac{|x|^\alpha |\Delta u|^p}{|x|^{-\beta} |u|^q \, dx}^{p/q} }. $$

A rescaling argument plainly shows that $S_{p,q}^{\text{Nav}}(\Omega; \alpha) \leq S_{p,q}(\alpha)$. The opposite inequality is not trivial at all, as in general a function $u \in D^2_{N}(\Omega; |x|^\alpha dx)$ can not be extended to $\overline{u} \in D^2(\mathbb{R}^n, |x|^\alpha dx)$ by putting $\overline{u} \equiv 0$ outside $\Omega$. In Section 5 we prove the next result.

**Theorem 1.2.** If (1) holds, and if $\Omega$ is a bounded domain of class $C^2$ containing the origin, then

$$S_{p,q}^{\text{Nav}}(\Omega; \alpha) = S_{p,q}(\alpha)$$

and in particular $S_{p,q}^{\text{Nav}}(\Omega; \alpha)$ is positive. Moreover, $S_{p,q}^{\text{Nav}}(\Omega; \alpha)$ is not achieved in $D^2_{N}(\Omega; |x|^\alpha dx)$.

If $n > 2p$, $q = p^{*\ast}$ and $\alpha = \beta = 0$, then the infimum in (7) coincides with

$$S_p^{\text{Nav}}(\Omega) := \inf_{\substack{u \in W^{2,p} \cap W^{1,p}_0(\Omega) \setminus \{0\} \atop \int_{\Omega} \int \frac{|\Delta u|^p}{|u|^{p^{*\ast}} \, dx}^{p/p^{*\ast}}}. $$

Hence, by Theorem 1.2 we have that

$$S_p^{\text{Nav}}(\Omega) = S_p$$

and $S_p^{\text{Nav}}(\Omega)$ is not achieved. In the Hilbertian case $p = 2$, equality (3) has been proved by Van der Vorst in [19] and by Ge in [7]. The general case $p > 1$ has been recently exploited by Gazzola, Grunau and Sweers in [6]. All the above mentioned papers are based again on a rearrangement argument that, in general, fails in presence of weights. Our arguments to check the more general equality (5) are simpler and self-contained.

In the last theorem we provide an unexpected result. We denote by $C^2_N(\Omega \setminus \{0\})$ the set of functions $u \in C^2(\Omega)$ such that $u = 0$ on $\partial \Omega$ and in a neighborhood of 0.

**Theorem 1.3.** Let $\Omega$ be a bounded domain of class $C^2$ containing the origin. Let $q \geq p > 1$, $\alpha \in \mathbb{R}$ and define $\beta$ as in (1a). If $\alpha \geq np - n$, then

$$S_{p,q}^{\text{Nav}}(\Omega; \alpha) := \inf_{\substack{u \in C^2_N(\Omega \setminus \{0\}) \setminus \{0\} \atop \int_{\Omega} \int \frac{|x|^\alpha |\Delta u|^p}{|x|^{-\beta} |u|^q \, dx}^{p/q} = 0}. $$
To comment Theorem 1.3 we define also

\[ S_{\text{Dir}}^{p,q}(\alpha; \Omega) := \inf_{u \in C^2_c(\Omega \setminus \{0\}) \atop u \neq 0} \frac{\int_{\Omega} |x|^\alpha |\Delta u|^p dx}{\left( \int_{\Omega} |x|^{-\beta} |u|^q dx \right)^{p/q}} \]

\[ S_{p,q}(\alpha) := \inf_{u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \atop u \neq 0} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-\beta} |u|^q dx \right)^{p/q}}. \]

Thanks to Theorems 1.2 and 1.3 (see also Section 2.2), we have that

\[ S_{\text{Nav}}^{p,q}(\Omega; \alpha) = S_{p,q}^{\text{Dir}}(\alpha; \Omega) = S_{p,q}(\alpha) \text{ if } \alpha \in (2p-n, np-n) \]

\[ 0 = S_{p,q}^{\text{Nav}}(\Omega; \alpha) \leq S_{p,q}^{\text{Dir}}(\alpha; \Omega) = S_{p,q}(\alpha) \text{ if } \alpha \geq np-n. \]

In general, the strict inequality holds in (10). Assume \( q \leq p^{*} \) if \( n > 2p \). By Corollary 2.12 below, the infimum \( S_{p,q}(\alpha) \) is positive if and only if \(-\gamma_{p,\alpha}\) is not an eigenvalue of the Laplace-Beltrami operator on the sphere. Therefore, if in addition it holds that \( \alpha > np - n \), then

\[ 0 = S_{p,q}^{\text{Nav}}(\Omega; \alpha) < S_{p,q}^{\text{Dir}}(\alpha; \Omega) = S_{p,q}(\alpha). \]

In particular the optimal Rellich-Sobolev constant under Navier boundary conditions never depends on the domain, but it may not coincide with the Rellich-Sobolev constant under Dirichlet boundary conditions, nor with the Rellich-Sobolev constant on the whole space.

The picture is not complete if \( \alpha < 2p-n \); see Proposition 6.1 for a partial result.

Our proofs are based on some knowledge of the weighted Sobolev space involved, compare with Section 2 and on two basic facts. The first one concerns inequality (3) and its generalization to Rellich type inequalities on bounded domains with Navier boundary conditions. Actually Lemma 2.9 in Section 2.2 has been already proved in [15], but it has never been explicitly stated in the form we need for our purposes. The second basic fact is the core of Theorem 3.2 in Section 3 that is concerned with non-homogeneous equations of the form

\[ -\Delta v = f \text{ in } \mathbb{R}^n, \]

for \( f \) varying in weighted \( L^p \) spaces. The proofs of the main Theorems 1.1, 1.2 and 1.3 can be found in the last three sections.

2. Notation and weighted Sobolev spaces

The characteristic function of a domain \( \Omega \) in \( \mathbb{R}^n \) is denoted by \( \chi_\Omega \). If \( f : \Omega \to \mathbb{R} \) is given, then \( \chi_\Omega f \) denotes the extension of \( f \) by the null function outside \( \Omega \).
We will use several function spaces on $\Omega$. In addition to the usual spaces $C^k_c(\Omega)$ and $C^\infty_c(\Omega)$, let us denote
\[
C_N^2(\overline{\Omega}) = \{ u \in C^2(\overline{\Omega}) \mid \text{supp}(u) \text{ is compact, and } u \equiv 0 \text{ on } \partial \Omega \}\,
C_N^2(\overline{\Omega} \setminus \{0\}) = \{ u \in C_N^2(\overline{\Omega}) \mid u \equiv 0 \text{ in a neighborhood of } 0 \}.
\]

Let $a \in \mathbb{R}$. The weighted Lebesgue space $L^q(\Omega; |x|^a \, dx)$ is the space of measurable $u$ on $\Omega$ having finite norm $(\int_\Omega |x|^a |u|^q \, dx)^{1/q}$. For $a = 0$ we write $L^q(\Omega)$, as usual.

A proper function $u : \Omega \to \mathbb{R} \cup \{ \infty \}$ is superharmonic if it is lower semicontinuous and
\[
u(x) \geq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y) \, d\sigma(y) \quad \text{for any } x \in \Omega, \ r < \text{dist}(x, \partial \Omega).
\]

Every superharmonic function in $\Omega$ belongs to $L^1_{\text{loc}}(\Omega)$. Moreover, $u \in L^1_{\text{loc}}(\Omega)$ is superharmonic if and only if $-\Delta u \geq 0$ in $\mathcal{D}'(\Omega)$, that is,
\[
\int_\Omega u(-\Delta \varphi) \, dx \geq 0 \quad \text{for any nonnegative } \varphi \in C^\infty_c(\Omega).
\]

Let $\Omega$ be a bounded domain with $\partial \Omega$ of class $C^2$. By definition, $W^{k,p}_0(\Omega)$ is the closure of $C^k(\Omega)$ in the standard Sobolev space $W^{k,p}(\Omega)$. Hence, $W^{0,p}_0(\Omega)$ is isometrically embedded into $W^{k,p}(\mathbb{R}^n)$ via the null extension $u \mapsto \chi_\Omega u$.

We adopt the notation
\[
W^2_{\infty,p}(\Omega) = W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega).
\]

It turns out that $W^2_{\infty,p}(\Omega)$ is a Banach space with respect to the norm
\[
\|u\| = \left( \int_\Omega |\Delta u|^p \, dx \right)^{1/p},
\]
which is equivalent to norm induced by $W^{2,p}(\Omega)$. For $p = 2$ we will use the simplified notation $H^k$ instead of $W^{k,2}$.

Any function $u \in W^2_{\infty,p}(\Omega)$ is the limit in $W^{2,p}(\Omega)$ of a sequence $u_h \in C^2_N(\overline{\Omega})$. For instance, one can define $u_h$ to be the projection of $(\chi_\Omega u)^\ast \ast \rho_\varepsilon \in H^1(\mathbb{R}^n)$ on $H^1_0(\Omega)$, where $\varepsilon_h \to 0^+$ and $\rho_\varepsilon$ is the standard $\varepsilon$-mollifier.

For $n > 2p$ let $\mathcal{D}^{2,p}(\mathbb{R}^n)$ be the completion of $C_{\text{loc}}^\infty(\mathbb{R}^n)$ with respect to the norm
\[
\|u\|^p = \int_{\mathbb{R}^n} |\Delta u|^p \, dx.
\]

It is well known that the space $\mathcal{D}^{2,p}(\mathbb{R}^n)$ is continuously embedded into $L^{p^*}(\mathbb{R}^n)$. The Sobolev constant $S_p$ in [13] is positive and achieved in $\mathcal{D}^{2,p}(\mathbb{R}^n)$, see for instance [12], [18].

The next lemma is based on a standard trick.

**Lemma 2.1.** Let $\Omega, \Omega'$ be bounded domains such that $\overline{\Omega} \subset \Omega'$. In addition, assume that $\partial \Omega$ is Lipschitz and $\partial \Omega'$ is of class $C^2$. Let $u \in W^2_{\infty,p}(\Omega)$. The problem
\[
\begin{cases}
-\Delta v = \chi_\Omega |v| - \Delta u & \text{in } \Omega', \\
v = 0 & \text{on } \partial \Omega',
\end{cases}
\]
(11)
Moreover, the constant \( v \in W^{2,p}_N(\Omega') \), and
\[
v \geq \chi_\Omega |u| \quad \text{in } \Omega'.
\]

**Proof.** We can assume that \( u \neq 0 \). The existence of a unique \( v \in W^{2,p}_N(\Omega') \) solving (11) is indeed a well known fact. Since \( v \) is superharmonic then \( v > 0 \) in \( \Omega' \). Now \( v \pm u \in L^1(\Omega) \) and \( -\Delta(v \pm u) \geq 0 \) in \( \Omega \). Since \( v \pm u > 0 \) on \( \partial \Omega \) in the sense of traces, then \( v \geq \mp u \) in \( \Omega \). Thus \( v \geq |u| \) in \( \Omega \). \( \square \)

One of the main tools in our arguments is the Hardy inequality in [9], [10]. For any \( p > 1 \), \( a \in \mathbb{R} \) it holds that
\[
\int_{\mathbb{R}^n} |x|^a |\nabla u|^p \, dx \geq |H_{1,a}|^p \int_{\mathbb{R}^n} |x|^{a-p} |u|^p \, dx \quad \forall u \in C^1_c(\mathbb{R}^n \setminus \{0\}),
\]
where
\[
H_{1,a} := \frac{n + a}{p} - 1.
\]
Moreover, the constant \( |H_{1,a}|^p \) can not be improved and it is not achieved in any reasonable function space.

In the remaining part of this section we describe the weighted Sobolev spaces that are needed to prove our main results.

2.1. The spaces \( \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) \).
Assume \( a \neq p - n \) and define the space \( \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) \) as the completion of \( C^1_c(\mathbb{R}^n \setminus \{0\}) \) with respect to the norm
\[
||u|| = \left( \int_{\mathbb{R}^n} |x|^a |\nabla u|^p \, dx \right)^{1/p}.
\]
Then \( \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) \leftrightarrow L^p(\mathbb{R}^n; |x|^{a-p} dx) \) by the Hardy inequality.

**Remark 2.2.** Assume \( a < pn - n - p \), \( a \neq p - n \). Then \( \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) \subset L^1_{\text{loc}}(\mathbb{R}^n) \).
Indeed, if \( u \in \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) \) then for any bounded domain \( \Omega \) we have that
\[
\int_\Omega |u| \, dx \leq \left( \int_{\mathbb{R}^n} |x|^{a-p} |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_\Omega |x|^{\frac{p}{p-a}} \, dx \right)^{\frac{a}{p}} < \infty.
\]
In order to simplify the proofs it is convenient to introduce the cylinder
\( \mathcal{Z}^n = \mathbb{R} \times S^{n-1} \),
whose points are denoted by \((s, \sigma)\), and the transform
\[
T_{1,a} : C^1_c(\mathcal{Z}^n) \to C^1_c(\mathbb{R}^n \setminus \{0\}) \quad (T_{1,a}g)(x) = |x|^{-H_{1,a}} g \left( -\log |x|, \frac{x}{|x|} \right).
\]
The next lemma has been already pointed out in [17], in a radial setting.

**Lemma 2.3.** If \( a \neq p - n \), then
\[
\|g\|_{1,a} := \left( \int_{\mathbb{R}^n} |x|^a |\nabla(T_{1,a}g)|^p \, dx \right)^{1/p} \quad \text{for } g \in C^1_c(\mathcal{Z}^n),
\]
is equivalent to the standard norm in $W^{1,p}(\mathbb{Z}^n)$. Thus $\mathcal{T}_{1,a}$ can be uniquely extended to an isomorphism $W^{1,p}(\mathbb{Z}^n) \to \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx)$, and
\[
\mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) = \{ u \in L^p(\mathbb{R}^n; |x|^{a-p} dx) \mid |\nabla u| \in L^p(\mathbb{R}^n; |x|^a dx) \}.
\]
If in addition $a > p - n$, then $C^1_c(\mathbb{R}^n) \subset \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx)$.

Proof. Notice that
\[
\int_{\mathbb{R}^n} |x|^{a-p} |T_{1,a} g|^p \, dx = \int_{\mathbb{Z}^n} |g|^p \, dsd\sigma
\]
\[
\int_{\mathbb{R}^n} |x|^a |\nabla (T_{1,a} g)|^p \, dx = \int_{\mathbb{Z}^n} (|g_s + H_{1,a} g|^2 + |\nabla g|^2)^{\frac{p}{2}} \, dsd\sigma
\]
for any $g \in C^1_c(\mathbb{Z}^n)$. In particular, the Hardy inequality and a density argument give
\[
(13) \quad \|g\|_{1,a}^p \geq |H_{1,a}| \int_{\mathbb{Z}^n} |g|^p \, dsd\sigma
\]
for any $g \in W^{1,p}(\mathbb{Z}^n)$. It is easy to prove that $\| \cdot \|_{1,a}$ is uniformly bounded from above by the standard norm in $W^{1,p}(\mathbb{Z}^n)$. To prove the converse take a sequence $g_h$ such that $\|g_h\|_{1,a} \to 0$. Then $g_h \to 0$ in $L^p(\mathbb{Z}^n)$ by (13) and since $H_{1,a} \neq 0$. Thus
\[
o(1) = \|g_h\|_{1,a}^p = \int_{\mathbb{Z}^n} (|g_h|^2 + |\nabla g_h|^2)^{\frac{p}{2}} \, dsd\sigma + o(1)
\]
\[
\geq \frac{1}{2} \int_{\mathbb{Z}^n} (|g_h|^p + |\nabla g_h|^p) \, dsd\sigma + o(1),
\]
hence $g_h \to 0$ in $W^{1,p}(\mathbb{Z}^n)$. The equivalence of the two norms is proved. To conclude, recall that $W^{1,p}(\mathbb{Z}^n) = \{ g \in L^p(\mathbb{Z}^n) \mid g_s, |\nabla g| \in L^p(\mathbb{Z}^n) \}$, and notice that the weights $|x|^a, |x|^{a-p}$ are locally integrable if $a > p - n$.

Remark 2.4. For $a \neq p - n$, $u \in C^1_c(\mathbb{R}^n \setminus \{0\})$ we put $\hat{a} = 2(p-n) - a$ and
\[
\hat{u}(x) = u \left( \frac{x}{|x|^2} \right),
\]
respectively. By direct computation one gets that
\[
\int_{\mathbb{R}^n} |x|^a |\nabla u|^p \, dx = \int_{\mathbb{R}^n} |x|^\hat{a} |\nabla \hat{u}|^p \, dx.
\]
Thus the functional transform $u \mapsto \hat{u}$ can be extended to a unique isometry
\[
\mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx) \to \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^{\hat{a}} dx).
\]

In the next result we provide an alternative proof of the celebrated Maz’ya and Caffarelli-Kohn-Nirenberg inequalities in \[13, 1\].

Lemma 2.5. Let $q > p$, and assume $q \leq p^* = \frac{np}{n-p}$ if $n > p$. If $a \neq p - n$ then there exists a positive best constant $s_{p,q}(a)$ such that
\[
\int_{\mathbb{R}^n} |x|^a |\nabla u|^p \, dx \geq s_{p,q}(a) \left( \int_{\mathbb{R}^n} |x|^{-n+qH_{1,a}} |u|^q \, dx \right)^{p/q}
\]
for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^n; |x|^a dx)$. In addition, $s_{p,q}(a) = s_{p,q}(2(p-n) - a)$.
Proof. Notice that
\[
\int_{\mathbb{R}^n} |x|^{-n+qH_1,a} |T_{1,a}g|^q \, dx = \int_{\mathbb{R}^n} |g|^q \, ds \sigma
\]
for any \( g \in C_c^1(\mathbb{Z}^n) \). Since \( W^{1,p}(\mathbb{Z}^n) \to L^q(\mathbb{Z}^n) \) by Sobolev embedding theorem, we readily infer that \( D^{1,p}(\mathbb{R}^n; |x|^a \, dx) \to L^q(\mathbb{R}^n; |x|^{-n+qH_1,a} \, dx) \) with a continuous embedding, and the desired inequality follows. Finally, \( s_{p,q}(a) = s_{p,q}(2(p-n) - a) \) by Remark 2.3.

Remark 2.6. Assume \( n > p \) and take \( a = 0 \). The above lemmata apply to the standard space \( D^{1,p}(\mathbb{R}^n) \). In particular, \( C^1(\mathbb{R}^n \setminus \{0\}) \) is dense in \( D^{1,p}(\mathbb{R}^n) \), and \( D^{1,p}(\mathbb{R}^n) \) can be identified with \( W^{1,p}(\mathbb{Z}^n) \) via the transform \( T_{1,a} \). These facts are well known when \( p = 2 \), see for instance [1].

The next maximum principle for superharmonic functions might have an independent interest.

Theorem 2.7. Assume \( p - n < a < pn - n - p \). If \( \omega \in D^{1,p}(\mathbb{R}^n; |x|^a \, dx) \setminus \{0\} \) is superharmonic in \( \mathbb{R}^n \setminus \{0\} \), then \( \omega \) is superharmonic and strictly positive on \( \mathbb{R}^n \).

Proof. First of all we recall that \( \omega \in L^1_{\text{loc}}(\mathbb{R}^n) \) by Remark 2.2. Fix any nonnegative \( \eta \in C^\infty_c(\mathbb{R}^n) \). Notice that \(-\frac{a}{p-1} > p' - n\), where \( p' = \frac{p}{p-1} \). Use Lemma 2.3 to infer that \( \eta \in D^{1,p'}(\mathbb{R}^n; |x|^{-\frac{n}{p-1}} \, dx) \). Thus there exists a sequence \( \eta_h \in C^\infty_c(\mathbb{R}^n \setminus \{0\}) \) such that \( \eta_h \to \eta \) in \( D^{1,p'}(\mathbb{R}^n; |x|^{-\frac{n}{p-1}} \, dx) \). Using truncation and a standard convolution argument we can assume that \( \eta_h \geq 0 \). Since \( \omega \in W^{1,p}(\mathbb{R}^n \setminus \{0\}) \) is superharmonic on \( \mathbb{R}^n \setminus \{0\} \), then
\[
0 \leq \int_{\mathbb{R}^n} \omega (-\Delta \eta_h) \, dx = \int_{\mathbb{R}^n} \nabla \omega \cdot \nabla \eta_h \, dx = \int_{\mathbb{R}^n} \left( |x|^{-\frac{n}{p-1}} \nabla \omega \right) \cdot \left( |x|^{-\frac{n}{p-1}} \nabla \eta_h \right) \, dx.
\]
Next notice that
\[
|x|^{-\frac{n}{p-1}} \nabla \omega \in L^p(\mathbb{R}^n)^n, \quad |x|^{-\frac{n}{p-1}} \nabla \eta_h \to |x|^{-\frac{n}{p-1}} \nabla \eta \quad \text{in} \quad L^{p'}(\mathbb{R}^n)^n.
\]
Therefore we can pass to the limit to infer
\[
0 \leq \int_{\mathbb{R}^n} \nabla \omega \cdot \nabla \eta \, dx = \int_{\mathbb{R}^n} \omega (-\Delta \eta) \, dx.
\]
Thus \(-\Delta \omega \geq 0\) as a distribution in \( D'(\mathbb{R}^n) \), as \( \eta \) was arbitrarily chosen.

To conclude the proof we only have to show that \( \omega \geq 0 \) almost everywhere in \( \mathbb{R}^n \). For sake of clarity we first assume \( p \geq 2 \), as the proof needs less computations in this case. Use Lemma 2.3 and known results on truncations to get that \( \omega^- := -\min\{\omega, 0\} \in D^{1,p}(\mathbb{R}^n; |x|^a \, dx) \). Then approximate \( \omega^- \) in \( D^{1,p}(\mathbb{R}^n; |x|^a \, dx) \) with a sequence of functions in \( C^1_c(\mathbb{R}^n \setminus \{0\}) \) to prove that we can test \(-\Delta \omega \geq 0\) with \( |x|^{a+2-p}(\omega^-)^{p-1} \). At the end one gets
\[
\int_{\mathbb{R}^n} (-\Delta \omega) |x|^{a+2-p}(\omega^-)^{p-1} \, dx = \int_{\mathbb{R}^n} \nabla \omega \cdot \nabla (|x|^{a+2-p}(\omega^-)^{p-1}) \, dx
\]
\[
= -\int_{\mathbb{R}^n} \nabla \omega^- \cdot \nabla (|x|^{a+2-p}(\omega^-)^{p-1}) \, dx \geq 0,
\]
Remark 2.8. solving a problem that has been left open for long time.

If (15) see also (4). In [14], Metafune, Sobajima and Spina proved that any

\[ H_{1,a}(2 + a - p) \int |x|^{a-p} |\omega|^p \, dx. \]

If \( a \leq p - 2 \) we readily get that \( \omega^- \equiv 0 \). Otherwise, we use the Hardy inequality

\[ \int R^n |x|^{2+a-p} |\nabla v|^2 \, dx \geq \frac{|n+a-p|^2}{2} \int R^n |x|^{a-p} |v|^2 \, dx \]

with \( v = (\omega^-)^{\frac{1}{2}} \in D^{1,2}(R^n; |x|^{2+a-p} \, dx) \), to infer

\[ (p-1) \int R^n |x|^{2+a-p} |\nabla \omega^-|^2 |\omega^-|^{p-2} \, dx \leq \frac{4(2+a-p)}{p(n+a-p)} \int R^n |x|^{2+a-p} |\nabla |\omega^-|^2 \, dx \]

\[ = \frac{2+a-p}{H_{1,a}} \int R^n |x|^{2+a-p} |\omega^-|^{p-2} |\nabla \omega^-|^2 \, dx \]

Since \( H_{1,a}^{-1}(2+a-p) < p-1 \) as \( a < np - p - n \), then necessarily \( \omega^- \equiv 0 \), that is, \( \omega \geq 0 \). If \( p \in (1,2) \) one repeats the same argument, with \( (\omega^-)^{p-2} \omega^- \) replaced by

\[ \omega_\varepsilon = \left( |\omega|^2 + \varepsilon^2 \right)^{\frac{p-2}{2}} \omega^- , \]

where \( \varepsilon \to 0^+ \), to get in similar way that \( \omega \geq 0 \) almost everywhere on \( R^n \). The proof is complete, as every superharmonic, nontrivial and nonnegative function on \( R^n \) is everywhere positive. \( \square \)

2.2. The spaces \( D^{2,p}(R^n; |x|^\alpha \, dx) \).

For any given exponent \( \alpha \in \mathbb{R} \) we introduce the weighted Rellich constant

\[ \mu_{p,\alpha} := \inf_{u \in C^1_0(R^n \setminus \{0\}) \setminus \{0\}} \frac{\int R^n |x|^{\alpha} |\Delta u|^p \, dx}{\int R^n |x|^{\alpha-2p} |u|^p \, dx} . \]

A crucial role is played by the constants

\[ H_{2,\alpha} = \frac{n+\alpha}{p} - 2 \, , \quad \gamma_{p,\alpha} := \left( n - \frac{n+\alpha}{p} \right) H_{2,\alpha} , \]

see also [14]. In [14], Metafune, Sobajima and Spina proved that

\[ \mu_{p,\alpha} > 0 \text{ if and only if } -\gamma_{p,\alpha} \notin \{ k(n-2+k) : k \in \mathbb{N} \cup \{0\} \} , \]

solving a problem that has been left open for long time.

Remark 2.8. It is easy to check that \( \mu_{p,\alpha} = 0 \) if \( -\gamma_{p,\alpha} = k(n-2+k) \) for an integer \( k \geq 0 \). For the proof, let \( \varphi_k \in H^1(S^{n-1}) \) be an eigenfunction of \( -\Delta_{\sigma} \) (the Laplace-Beltrami operator on the sphere) relative to the eigenvalue \( \lambda_k = k(n-2+k) \). Fix a nontrivial function \( \omega \in C^2_c(\mathbb{R}^n) \) and for any \( \varepsilon > 0 \) use polar coordinates \((r,\sigma) \in \mathbb{R}_+ \times S^{n-1} \) to define

\[ u_\varepsilon(r,\sigma) := r^{-H} \omega(r^\varepsilon) \varphi_k(\sigma) , \]
where we have put $H = H_{2,\alpha}$ to simplify notation. Then test $\mu_{p,\alpha}$ with $u_\varepsilon$. By direct computation one gets
\[ \mu_{p,\alpha} \leq \frac{\varepsilon^p}{\int_0^\infty s^{p-1}|\varepsilon s\omega'' + (n - 2 - 2H + \varepsilon)\omega'|^p \, ds} \int_0^\infty s^{-1}|\omega|^p \, ds. \]

The conclusion follows by taking the limit as $\varepsilon \to 0$.

The explicit value of $\mu_{2,\alpha}$ (case $p = 2$) has been computed in [5, 13] and [14]. The sharp value of $\mu_{p,\alpha}$ in case of general exponents $\alpha, p$ is not known yet, unless $\gamma_{p,\alpha}$ is positive (hence, $-\gamma_{p,\alpha}$ is below the spectrum of $-\Delta_\sigma$). The weighted Rellich inequality in the next Lemma has been essentially proved in [15]. We cite also [5], Lemma 2, where $p = 2$ is assumed. We sketch its proof for the convenience of the reader.

**Lemma 2.9.** Let $\Omega$ be a domain in $\mathbb{R}^n$. If $\partial\Omega$ is not empty, assume that $\partial\Omega$ is of class $C^2$. Let $p > 1$ and $\alpha \in \mathbb{R}$ such that $2p - n < \alpha < np - n$. Then
\[ (16) \quad \gamma_{p,\alpha} \int_{\Omega} |x|^{-2p+2} |u|^p \, dx \leq \frac{1}{\mu_{p,\alpha}} \int_{\Omega} |x|^\alpha |\Delta u|^p \, dx \quad \forall u \in C^2_N(\overline{\Omega}). \]

In particular, we have that $[5]$ holds. If $0 \in \Omega$ then the constant in the left hand side of (16) can not be improved.

**Proof.** If $p \geq 2$ then (16) is an immediate consequence of Hölder and Hardy inequalities and of the identity
\[ \int_{\Omega} (-\Delta u)|x|^{\alpha-2p+2} |u|^{p-2} u \, dx = \frac{4(p-1)}{p} \int_{\Omega} |x|^{\alpha-2p+2} \left( |\nabla|u|^\frac{p}{2}\right)^2 \, dx + \frac{1}{p} \int_{\Omega} |x|^{\alpha-2p+2} |\nabla|^p |u| \, dx. \]

Some care is needed in case $p \in (1, 2)$. We first prove (16) for a fixed function $u \in C^2_N(\overline{\Omega} \setminus \{0\})$. For any $\varepsilon > 0$, we define
\[ \varphi_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{p-2}{2}} u, \quad \varphi_\varepsilon = |x|^{\alpha-2p+2} \varphi_\varepsilon \]
\[ \Phi_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{p-2}{2}} - \varepsilon^{p-2}, \quad \Theta_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{p-2}{2}} - \varepsilon^p. \]

Notice that
\[ \nabla \varphi_\varepsilon \cdot \nabla u \geq (p-1) \left( |u|^2 + \varepsilon^2 \right)^{\frac{p-4}{2}} |u|^2 |\nabla u|^2 \geq \frac{4(p-1)}{p^2} |\nabla \Phi_\varepsilon|^2. \]

In addition, the Hardy inequality gives
\[ \int_{\Omega} |x|^{\alpha-2p+2} |\nabla \Phi_\varepsilon|^2 \, dx \geq \left( \frac{n-2p+\alpha}{2} \right)^2 \int_{\Omega} |x|^{\alpha-2p} |\Phi_\varepsilon|^2 \, dx \]
\[ = \left( \frac{n-2p+\alpha}{2} \right)^2 \int_{\Omega} |x|^{\alpha-2p} |u|^p \, dx + o(1) \]
as $\varepsilon \to 0$. Therefore
\[ \int_{\Omega} |x|^{\alpha-2p+2} \nabla \varphi_\varepsilon \cdot \nabla u \, dx \geq (p-1)H_{2,\alpha}^2 \int_{\Omega} |x|^{\alpha-2p} |u|^p \, dx + o(1). \]


We notice that \( p \varphi_x \nabla u = \nabla \Theta_x \) and we integrate by parts to compute
\[
\int_{\Omega} \nabla |x|^{\alpha-2p+2} \cdot (\varphi_x \nabla u) \, dx = -\frac{1}{p} \int_{\Omega} (\Delta |x|^{\alpha-2p+2}) \Theta_x \, dx
\]
\[
= -\frac{(\alpha - 2p + 2)(n - 2p + \alpha)}{p} \int_{\Omega} |x|^{\alpha-2p} \Theta_x \, dx
\]
\[
= -((\alpha - 2p + 2) H_{2, \alpha} \int_{\Omega} |x|^{\alpha-2p} |u|^p \, dx + o(1)).
\]

Now we use integration by parts and Hölder inequality to estimate
\[
\int_{\Omega} \nabla u \cdot \nabla (|x|^{\alpha-2p} \varphi_x) \, dx = \int_{\Omega} (-\Delta u)(|x|^{\alpha-2p} \varphi_x) \, dx
\]
\[
\leq \left( \int_{\Omega} |x|^\alpha |\Delta u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |x|^{\alpha-2p} |\varphi_x|^\frac{2}{p} \, dx \right)^{\frac{p-1}{p}}
\]
\[
\leq \left( \int_{\Omega} |x|^\alpha |\Delta u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |x|^{\alpha-2p} |u|^p \, dx \right)^{\frac{p-1}{p}} + o(1).
\]

Since
\[
\int_{\Omega} \nabla u \cdot \nabla (|x|^{\alpha-2p} \varphi_x) \, dx = \int_{\Omega} |x|^{\alpha-2p+2} \nabla \varphi_x \cdot \nabla u \, dx + \int_{\Omega} \nabla |x|^{\alpha-2p+2} \cdot (\varphi_x \nabla u) \, dx,
\]
by gluing all above information we infer
\[
\left( \int_{\Omega} |x|^\alpha |\Delta u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |x|^{\alpha-2p} |u|^p \, dx \right)^{\frac{p-1}{p}} \geq \gamma_{p, \alpha} \int_{\Omega} |x|^{\alpha-2p} |u|^p \, dx + o(1),
\]
and \((16)\) readily follows for \( u \in C^2_0(\Omega \setminus \{0\}) \), as \( \gamma_{p, \alpha} > 0 \). To prove \((16)\) for \( C^2_0(\overline{\Omega}) \) notice that \( |x|^\alpha, |x|^{\alpha-2p} \in L^1_{\text{loc}}(\Omega) \) and use an approximation argument.

It remains to check that the constant \( \gamma_{p, \alpha} \) can not be improved if \( 0 \in \Omega \). Take a nontrivial function \( \omega \in C^2_c(0, 1) \), and then use \( u_\varepsilon(x) = |x|^{-H} \omega(|x|^\varepsilon) \) as test function, where \( H = H_{2, \alpha} \) and \( \varepsilon > 0 \) is a small parameter, so that \( u_\varepsilon \in C^2_c(\Omega \setminus \{0\}) \). Then compute
\[
\int_{\Omega} |x|^\alpha |\Delta u_\varepsilon|^p \, dx = \int_0^1 s^{-1} |\varepsilon^2 s^2 \omega'' + \varepsilon s(n-2 - 2H + \varepsilon) \omega' - \gamma_{p, \alpha} \omega|^p \, ds
\]
\[
\int_{\Omega} |x|^{\alpha-2p} |u_\varepsilon|^p \, dx = \int_0^1 s^{-1} |\omega|^p \, ds
\]
let \( \varepsilon \to 0 \) and conclude. \( \square \)

Now let \( \alpha \in \mathbb{R} \) and assume \( \mu_{p, \alpha} > 0 \). Define the space \( D^{2,p}(\mathbb{R}^n; |x|^\alpha \, dx) \) as the completion of \( C^2_0(\mathbb{R}^n \setminus \{0\}) \) with respect to the norm
\[
\| u \|_{2, \alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p \, dx.
\]
Then \( D^{2,p}(\mathbb{R}^n; |x|^\alpha \, dx) \) is continuously embedded into \( L^p(\mathbb{R}^n; |x|^{\alpha-2p} \, dx) \) and
\[
\mu_{p, \alpha} = \inf_{u \in D^{2,p}(\mathbb{R}^n; |x|^\alpha \, dx)} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p \, dx}{\int_{\mathbb{R}^n} |x|^{\alpha-2p} |u|^p \, dx}.
\]
We introduce the transform
\[ \mathcal{T}_{2,\alpha} : C^1_{c}(\mathbb{Z}^n) \to C^1_{c}(\mathbb{R}^n \setminus \{0\}), \quad (\mathcal{T}_{2,\alpha}g)(x) = |x|^{-H_{2,\alpha}}g \left( -\log |x|, \frac{x}{|x|} \right). \]
The "radial version" of the next lemma has been crucially used in [17].

**Lemma 2.10.** Assume that \( -\gamma_{p,\alpha} \) is not an eigenvalue of the Laplace-Beltrami operator on the sphere. Then
\[ \|g\|_{2,\alpha} := \left( \int_{\mathbb{R}^n} |x|^\alpha |\Delta (\mathcal{T}_{2,\alpha}g)|^p dx \right)^{1/p} \quad \text{for} \quad g \in C^2_{c}(\mathbb{Z}^n), \]
is equivalent to the standard norm in \( W^{2,p}(\mathbb{Z}^n) \). Thus \( \mathcal{T}_{2,\alpha} \) can be uniquely extended to an isomorphism \( W^{2,p}(\mathbb{Z}^n) \to \mathcal{D}^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \), and
\[ \mathcal{D}^{2,p}(\mathbb{R}^n; |x|^\alpha dx) = \{ u \in L^p(\mathbb{R}^n; |x|^{\alpha-2p}dx) \mid -\Delta u \in L^p(\mathbb{R}^n; |x|^{\alpha}dx) \}. \]
If in addition \( a > 2p - n \), then \( C^2_{c}(\mathbb{R}^n) \subset \mathcal{D}^{2,p}(\mathbb{R}^n; |x|^{\alpha}dx) \).

**Proof.** It turns out that \( \mu_{p,\alpha} > 0 \) by [15]. By direct computation one has that
\[ \left\{ \begin{array}{l}
\int_{\mathbb{R}^n} |x|^{\alpha} |\Delta u|^p dx = \int_{\mathbb{Z}^n} |\Delta g + g_{ss} - 2A_{p,\alpha}g_{s} - \gamma_{p,\alpha}g|^p \ ds d\sigma \\
\int_{\mathbb{R}^n} |x|^{\alpha-2p} |u|^p dx = \int_{\mathbb{Z}^n} |g|^p \ ds d\sigma ,
\end{array} \right. \]
where
\[ A_{p,\alpha} = \frac{n+2}{2} - \frac{n+\alpha}{p}. \]
To conclude, adapt the arguments in the proof of Lemma 2.3. \( \Box \)

**Remark 2.11.** The function \( \alpha \mapsto \mu_{p,\alpha} \) is symmetric with respect to
\[ \alpha^*_{p} := p + n \frac{p-2}{2}. \]
In fact, for any \( u \in C^2_{c}(\mathbb{R}^n \setminus \{0\}) \), \( t \in \mathbb{R} \) it turns out that
\[ \int_{\mathbb{R}^n} |x|^\alpha_{p}^{t-t} |\Delta \hat{u}|^p dx = \int_{\mathbb{R}^n} |x|^\alpha_{p}^{t+t} |\Delta u|^p dx \]
\[ \int_{\mathbb{R}^n} |x|^\alpha_{p}^{t-2t} |\hat{u}|^p dx = \int_{\mathbb{R}^n} |x|^\alpha_{p}^{t+t-2p} |u|^p dx , \]
where
\[ \hat{u}(x) := |x|^\frac{p}{2} u \left( \frac{x}{|x|^2} \right). \]
The weighted second order Emden-Fowler transforms can be used to avoid boring computations. Indeed, putting \( \hat{g} := \mathcal{T}_{2,\alpha^*_{p}-t} \hat{u}, \quad g = \mathcal{T}_{2,\alpha^*_{p}+t} u \), one has that \( \hat{g}(s, \sigma) = g(-s, \sigma) \). To conclude, use (17) and notice that the function \( \alpha \mapsto \gamma_{p,\alpha} \) is even with respect to \( \alpha^*_{p} \) while the function \( \alpha \mapsto A_{p,\alpha} \) is odd with respect to \( \alpha^*_{p} \), that is, \( \gamma_{p,\alpha^*_{p}-t} = \gamma_{p,\alpha^*_{p}+t} \) and \( A_{p,\alpha^*_{p}-t} = -A_{p,\alpha^*_{p}+t} \) for any \( t \in \mathbb{R} \).

In particular, we have that \( \mu_{p,\alpha} \neq 0 \) if and only if \( \mu_{p,\alpha} \neq 0 \), where \( \hat{\alpha} = 2\alpha^*_{p} - \alpha \). In this case, the spaces \( \mathcal{D}^{2,p}(\mathbb{R}^n; |x|^{\alpha}dx) \) and \( \mathcal{D}^{2,p}(\mathbb{R}^n; |x|^{\hat{\alpha}}dx) \) can be identified through the isometry \( u \mapsto \hat{u} \).
The next corollary is an immediate consequence of Lemma 2.10 and of Sobolev embedding theorems for $W^{2,p}(\Bbb{R}^n)$.

**Corollary 2.12.** Let $p, q$ be given exponents, such that $1 < p \leq q < \infty$ and $q \leq p^*$ if $n > 2p$. Let $\alpha \in \Bbb{R}$ and assume that $-\gamma_{p,\alpha}$ is not an eigenvalue of the Laplace-Beltrami operator on the sphere. Then

1. $D^{2,p}(\Bbb{R}^n; |x|^\alpha dx)$ is continuously embedded into $L^p(\Bbb{R}^n; |x|^{n+q-2p+\alpha - \gamma_{p,\alpha}} dx)$.
2. $D^{2,p}(\Bbb{R}^n; |x|^\alpha dx)$ is continuously embedded into $D^{1,p}(\Bbb{R}^n; |x|^{-n+(q-p^*+\alpha - \gamma_{p,\alpha} + 1/2)} dx)$.

**Remark 2.13.** Assume $n > 2p$ and take $\alpha = 0$. From Lemma 2.10 we infer that $D^{2,p}(\Bbb{R}^n; |x|^0 dx)$ coincides with the standard space $D^{2,p}(\Bbb{R}^n)$. In particular, $D^{2,p}(\Bbb{R}^n)$ can be identified with the standard Sobolev space $W^{2,p}(\Bbb{R}^n)$.

2.3. Bounded domains.

Here we assume that $\Omega \subset \Bbb{R}^n$ is a bounded domain of class $C^2$ containing the origin. We start with a density lemma.

**Lemma 2.14.** Assume $\alpha > 2p - n$ and let $u \in C^2_N(\Omega)$. Then there exists a sequence $u_h \in C^2_N(\Omega \setminus \{0\})$ such that $\Delta u_h \to \Delta u$ in $L^p(\Omega; |x|^\alpha dx)$ and $u_h \to u$ in $L^q(\Omega; |x|^{-\beta} dx)$, for any $q \geq p$ and $\beta$ as in (1a).

**Proof.** Take a smooth function $\eta: \Bbb{R} \to \Bbb{R}$ such that $0 \leq \eta \leq 1$, $\eta(s) \equiv 1$ for $s \leq 1$ and $\eta \equiv 0$ for $s \geq 2$. We put $\eta_h(x) = \eta(-h^{-1}\log |x|)$ and we check by direct computation that the sequence $u_h = \eta_h u$ satisfies the desired requirements. Notice that $\eta_h \to 1$, $\nabla \eta_h, \Delta \eta_h \to 0$ pointwise on $\Bbb{R}^n \setminus \{0\}$ and that the sequences $\eta_h, |x|\nabla \eta_h$ and $|x|^2 \Delta \eta_h$ are uniformly bounded. In addition, the supports of $\eta_h - 1$, $\nabla \eta_h$ and $\Delta \eta_h$ are contained in the closed ball of radius $e^{-h}$ about the origin. Thus

$$
\eta_h \to 1 \text{ in } L^q(\Omega; |x|^{\nu} dx) \text{ for any } q \geq 1 \text{ and } \nu > n,
$$

$$
|\nabla \eta_h| \to 0 \text{ in } L^p(\Omega; |x|^{\nu} dx) \text{ for any } \nu > p - n , \Delta \eta_h \to 0 \text{ in } L^p(\Omega; |x|^{\alpha} dx).
$$

In particular, $u_h \to u$ in $L^q(\Omega; |x|^\nu dx)$ for any $q \geq 1$, $\nu > -n$, and since

$$
\Delta u_h = \eta_h \Delta u + 2\nabla \eta_h \cdot \nabla u + u \Delta \eta_h,
$$

then $\Delta u_h \to \Delta u$ in $L^p(\Omega; |x|^\alpha dx)$. 

Now assume that $\alpha$ satisfies (1b), and define $D^{2,p}_N(\Omega; |x|^\alpha dx)$ as the completion of $C^2_N(\Omega)$ with respect to the norm

$$
\|u\|_{2,\alpha} = \left( \int_{\Omega} |x|^{\alpha} |\Delta u|^p dx \right)^{1/p}.
$$

Then $D^{2,p}_N(\Omega; |x|^\alpha dx)$ is continuously embedded into $L^p(\Omega; |x|^{\alpha - 2p})$ by Lemma 2.10.
Since $\partial \Omega$ is smooth and compactly contained in $\mathbb{R}^n \setminus \{0\}$, and since $C^2_c(\mathbb{R}^n \setminus \{0\})$ is dense in $D^{2,p}_0(\Omega; |x|\alpha \, dx)$ by Lemma 2.14, then $D^{2,p}_0(\Omega; |x|\alpha \, dx)$ is the space of functions $u \in L^p(\Omega; |x|\alpha \, dx)$ such that $\Delta u \in L^p(\Omega; |x|\alpha \, dx)$ and $u = 0$ in the sense of traces on $\partial \Omega$, coherently with the definition already given in the introduction.

3. A linear problem on $\mathbb{R}^n$

Here we deal with the non-homogeneous equation
\begin{equation}
- \Delta v = f \quad \text{in } \mathbb{R}^n.
\end{equation}

In this section we will always assume that (11b) is satisfied. We start with a simple lemma and then we prove an existence result.

**Lemma 3.1.** If $v \in L^p(\mathbb{R}^n; |x|\alpha - 2p \, dx)$ is harmonic on $\mathbb{R}^n \setminus \{0\}$, then $v \equiv 0$.

**Proof.** First of all we notice that $v \in L^1_{\text{loc}}(\mathbb{R}^n)$, argue as in Remark 2.2. Next, fix any $\eta \in C^\infty_c(\mathbb{R}^n)$ and put
\[
\tilde{p}' = \frac{p}{p - 1}, \quad \tilde{\alpha} = \frac{2p - \alpha}{p - 1} = 2p' - \frac{\alpha}{p - 1}.
\]
Notice that $2p' - n < \tilde{\alpha} < np' - n$, as $2p - n < \alpha < np - n$. Since the weights $|x|^\tilde{\alpha}$ and $|x|^\tilde{\alpha}'$ are in $L^1_{\text{loc}}(\mathbb{R}^n)$, then clearly $\eta \in D^{2,p'}(\mathbb{R}^n; |x|\tilde{\alpha} \, dx)$, compare with Lemma 2.10. Thus there exists a sequence $\eta_h \in C^\infty_c(\mathbb{R}^n \setminus \{0\})$ such that $\eta_h \to \eta$ in $D^{2,p'}(\mathbb{R}^n; |x|^\tilde{\alpha} \, dx)$. Since
\[
0 = \int_{\mathbb{R}^n} v \Delta \eta_h \, dx = \int_{\mathbb{R}^n} \left( |x|^{\tilde{\alpha} - p'} v \right) \left( |x|^{\tilde{\alpha}'} \Delta \eta_h \right) \, dx = \int_{\mathbb{R}^n} v \Delta \eta \, dx + o(1),
\]
we readily infer that $-\Delta v = 0$ on $\mathbb{R}^n$, as $\eta$ was arbitrarily chosen. Thus $v$ is the null function in $D^{2,p}(\mathbb{R}^n; |x|\alpha \, dx)$ by Lemma 2.10.

**Theorem 3.2.** Let $p > 1$, $\alpha \in (2p - n, np - n)$ and let $f \in L^p(\mathbb{R}^n; |x|\alpha \, dx)$ be a given function. Then there exists a unique $v \in D^{2,p}(\mathbb{R}^n; |x|\alpha \, dx)$ that solves (18) in the distributional sense on $\Omega$. If in addition $f \neq 0$ and $f \geq 0$ almost everywhere on $\mathbb{R}^n$, then $v$ is superharmonic and strictly positive on $\mathbb{R}^n$.

**Proof.** For any $R > 1$ we denote by $A_R$ the annulus $\{R^{-1} < |x| < R\}$. Notice that $f \in L^p(A_R)$. Let $h > 1$ be an integer and let $v_h \in W^{2,p}_N(A_h)$ be the unique solution to
\begin{equation}
\begin{cases}
- \Delta v_h = f & \text{in } A_h \\
v_h = 0 & \text{on } \partial A_h.
\end{cases}
\end{equation}

Now we extend $v_h$ by the null function outside $A_h$ and we write $v_h$ instead of $\chi_{A_h} v_h$ to simplify notation. We use the Rellich inequality (16) to infer
\[
\int_{\mathbb{R}^n} |x|^{\alpha - 2p} |v|^{p} \, dx \leq c \int_{A_h} |x|^{\alpha} |\Delta v_h|^{p} \, dx \leq c \int_{\mathbb{R}^n} |x|^{\alpha} |f|^{p} \, dx,
\]
where $c = C_p, \alpha > 0$. Therefore, we have that the sequence $(v_h)$ is uniformly bounded in $L^p(\mathbb{R}^n, |x|\alpha - 2p \, dx)$, and we can assume that $v_h \to v$ weakly in $L^p(\mathbb{R}^n, |x|\alpha - 2p \, dx)$.
for some \( v \in L^p(\mathbb{R}^n, |x|^{\alpha-2p} dx) \). For every fixed \( R > 1 \) and for \( h > R \) we clearly have
\[
\int_{A_R} |\Delta v_h|^p dx \leq c_R \int_{\mathbb{R}^n} |x|^\alpha |f|^p dx,
\]
where \( c_R \) denotes any constant that might depend on \( R \) and \( \alpha \) but not on \( h \).

Since \( (v_h) \) is bounded in \( L^p(A_R) \), then \( (v_h) \) is bounded in \( W^{2,p}(A_R) \). In particular, from every subsequence we can extract a new subsequence \( v_{h_j} \) such that \( v_{h_j} \to v_R \) weakly in \( W^{2,p}(A_R) \) for some \( v_R \in W^{2,p}(A_R) \). Actually, since \( v_{h_j} \to v_R \) strongly in \( L^p(A_R) \), we have \( v_R = v \) a.e. on \( A_R \). Thus \( v \in W^{2,p}_0(\mathbb{R}^n \setminus \{0\}) \), \( v \) is weakly in \( W^{2,p}_0(\mathbb{R}^n \setminus \{0\}) \), and \( v \) solves (18) almost everywhere on \( \mathbb{R}^n \). Since \( v \in W^{2,p}_0(\mathbb{R}^n \setminus \{0\}) \), then \( v \) has a distributional Laplacian \( -\Delta v = f \) in the distributional sense on \( \mathbb{R}^n \setminus \{0\} \). Arguing as in the proof of Lemma 3.1, one gets that \( v \) solves (18) in the \( D'(\mathbb{R}^n) \) sense, and in particular \( -\Delta v \in L^p(\mathbb{R}^n; |x|^\alpha dx) \). Since in addition \( v \in L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx) \), from Lemma 3.10 we infer that \( v \in D^{2,p}_N(\Omega; |x|^\alpha dx) \). The uniqueness of \( v \) readily follows by Lemma 3.1.

The last conclusion in case \( f \geq 0 \) is immediate, as \( v \in L^1_{loc}(\mathbb{R}^n) \).

As a consequence of the above results we get a new characterization of the space \( D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \).

**Corollary 3.3.** If \( \omega \in L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx) \) solves \( -\Delta \omega = f \) in the distributional sense on \( \mathbb{R}^n \setminus \{0\} \) for some \( f \in L^p(\mathbb{R}^n; |x|^\alpha dx) \), then \( \omega \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \).

**Proof.** Use Theorem 3.2 to find \( v \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \) such that \( -\Delta v = f \) on \( \mathbb{R}^n \). Then \( v - \omega \in L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx) \) is harmonic on \( \mathbb{R}^n \setminus \{0\} \). Hence, \( \omega = v \) by Lemma 3.1. \( \square \)

### 4. Proof of Theorem 1.1

Use Theorem 3.2 to find the unique superharmonic and positive function \( v \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \) such that
\[
-\Delta v = |u| \quad \text{on } \mathbb{R}^n.
\]

Since \( v \pm u \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \), then \( v \pm u \in D^{1,p}(\mathbb{R}^n; |x|^{\alpha-p} dx) \) by Corollary 2.12.

In addition \( -\Delta (v \pm u) \geq 0 \) on \( \mathbb{R}^n \), that implies \( v \pm u \geq 0 \) on \( \mathbb{R}^n \) by Theorem 2.7.

Thus \( v \geq |u| \) a.e. on \( \mathbb{R}^n \), and therefore
\[
S_{p,q}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-\beta} v^q \, dx \right)^{p/q} \leq \int_{\mathbb{R}^n} |x|^\alpha |\Delta v|^p \, dx
\]
\[
= \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p \, dx = S_{p,q}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-\beta} |u|^q \, dx \right)^{p/q}
\]
\[
\leq S_{p,q}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-\beta} v^q \, dx \right)^{p/q},
\]
that readily gives \( |u| = v \), as \( v^q - |u|^q \geq 0 \). Since \( v > 0 \) then up to a change of sign we can assume that \( u = v \). In particular, \( u \) is superharmonic and positive. \( \square \)
5. Proof of Theorem 1.2

Fix any $u \in C^2_N(\Omega \setminus \{0\}) \setminus \{0\}$ and use Theorem 5.2 to find $v \in D^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$, $v > 0$, such that

$$-\Delta v = \chi_\Omega |\Delta u| \text{ on } \mathbb{R}^n.$$  

Since $v \pm u \in L^1(\Omega)$, $-\Delta(v \pm u) \geq 0$ in $\Omega$ and $v \pm u \geq 0$ on $\partial \Omega$, then $v \geq |u|$ on $\Omega$. In particular

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta v|^p dx \leq \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx = S_{p,q}(\alpha).$$  

Thus $S_{p,q}(\alpha) \leq S_{p,q}(\Omega; \alpha)$, as $u$ was arbitrarily chosen and thanks to the result in Subsection 2.3. Next notice that $S_{p,q}(\Omega; \alpha)$ was arbitrarily chosen and thanks to Sobolev embedding theorem, we can assume that $\bar{\Omega} \subset \Omega$ and $q \leq p^*$ if $n > 2p$. One can adapt the choice of test functions that has been made in Remark 2.8 to check that $S^N_{p,q}(\Omega; \alpha) = 0$ if $\alpha = np - n$. Thus it suffices to prove the result in case $\alpha > np - n$.

For $X = W^{2,p}_N(\Omega)$, $X = C^2_N(\Omega)$ or $X = C^2_N(\Omega \setminus \{0\})$ and for $v \in X$ we put

$$m(X) = \inf_{\substack{v \in X \setminus \{0\}}} R(v), \quad R(v) := \frac{\int_{\Omega} |x|^\alpha |\Delta v|^p dx}{\left(\int_{\Omega} |x|^{-\beta} |v|^q dx\right)^{p/q}}.$$
Since $\alpha > np - n > 0$, then $\Delta v \in L^p(\Omega; |x|^{\alpha} dx)$ for any $v \in W^{2,p}(\Omega)$. In particular, the infima $m(X)$ are well defined. By trivial inclusions and thanks to Lemma 2.14, we have that
\[
m(W^{2,p}_N(\Omega)) \leq m(C^2_N(\Omega)) \leq m(C^2_N(\Omega \setminus \{0\})) = S_{p,q}^{\text{Nav}}(\Omega; \alpha).
\]
Actually $m(C^2_N(\Omega)) = m(C^2_N(\Omega \setminus \{0\}))$ by Lemma 2.14. Therefore, to conclude the proof we have to show that
\[
m(C^2_N(\Omega)) = 0.
\]
The main step consists in proving that
\[
m(W^{2,p}_N(\Omega)) = 0.
\]
Since $\alpha > np - n$, we have that $\gamma_\alpha := \gamma_{p,\alpha} < 0$; compare with (4). In particular, we can find a geodesic ball $B \subset \mathbb{S}^{n-1}$ such that
\[
-\gamma_\alpha = \inf_{\varphi \in H^1_0(B) \setminus \{0\}} \frac{\int_B |\nabla \varphi|^2 d\sigma}{\int_B |\varphi|^2 d\sigma}.
\]
Fix an eigenfunction $\varphi \in H^1_0(B)$ relative to the eigenvalue $-\gamma_\alpha$ and any nontrivial function $\omega \in C^2_c(\mathbb{R}_+)$. For any small $\varepsilon > 0$ use polar coordinates $(r, \sigma) \in \mathbb{R}_+ \times S^{n-1}$ to define
\[
u_\varepsilon(r) := r^{-H} \omega(r^\varepsilon) \varphi(\sigma),
\]
where $H = H_{2,\alpha}$ is defined in (14). Let $\Omega_\varepsilon$ be the support of $u_\varepsilon$. Then $\Omega_\varepsilon$ has a Lipschitz boundary and it is compactly contained in $\Omega$ for any $\varepsilon$ small enough. In addition, $u_\varepsilon \in W^{1,p}_0 \cap W^{2,p}(\Omega_\varepsilon)$. Let $v_\varepsilon \in W^{2,p}_N(\Omega)$ be the solution of
\[
\begin{aligned}
-\Delta v &= \chi_{\Omega_\varepsilon} |\nabla u_\varepsilon| \quad &\text{in } \Omega, \\
v &= 0 &\text{on } \partial \Omega.
\end{aligned}
\]
By Lemma 2.1 we have that $v_\varepsilon \geq |u_\varepsilon|$ in $\Omega_\varepsilon$, and thus
\[
m(W^{2,p}_N(\Omega)) \leq R(v_\varepsilon) \leq \frac{\int_{\Omega_\varepsilon} |x|^{\alpha} |\Delta u_\varepsilon|^p dx}{\left(\int_{\Omega_\varepsilon} |x|^{-\beta} |u_\varepsilon|^q dx\right)^{p/q}}.
\]
Since $\Delta_\sigma \varphi = \gamma_\alpha \varphi$ on $B$, we compute
\[
(\Delta u_\varepsilon)(r) = \left[\Delta_r \left(r^{-H} \omega(r^\varepsilon)\right) + \gamma_\alpha r^{-H-2} \omega(r^\varepsilon)\right] \varphi(\sigma)
\]
where $r > 0$, $\sigma \in B$ and $\Delta_r w = w'' + (n-1)r^{-1}w'$ for any $w \in C^2_c(\mathbb{R}_+)$. Therefore
\[
(\Delta u_\varepsilon)(x) = \varepsilon r^{-H-2+\varepsilon} \left[\varepsilon r^\varepsilon \omega''(r^\varepsilon) + (c + \varepsilon) \omega'(r^\varepsilon)\right] \varphi(\sigma)
\]
where $c$ denotes any constant independent on $\varepsilon$, and
\[
\begin{aligned}
\int_{\Omega_\varepsilon} |x|^\alpha |\Delta u_\varepsilon|^p dx &= c\varepsilon^{p-1} \int_0^\infty s^{p-1} |\varepsilon s \omega'' + (c + \varepsilon) \omega'|^p ds \\
\int_{\Omega} |x|^{-\beta} |u_\varepsilon|^q dx &= c\varepsilon^{-1} \int_0^\infty s^{-1} |\omega|^q ds.
\end{aligned}
\]
In particular, \( R(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and \((21)\) is proved.

We are in position to prove \((20)\). Fix any \( \delta > 0 \) and use \((21)\) to find a nontrivial function \( \hat{v} \in W^{2,p}_{N}(\Omega) \) such that \( R(\hat{v}) < \delta \). Take a sequence \( v_h \in C_c^2(\Omega) \) such that \( v_h \to \hat{v} \) in \( W^{2,p}_{N}(\Omega) \). Notice that \( |x|^\alpha \in L^\infty(\Omega) \) as \( \alpha > 0 \). Therefore

\[
\lim_{h \to \infty} \int_{\Omega} |x|^\alpha |\Delta v_h|^p \, dx = \int_{\Omega} |x|^\alpha |\hat{\Delta v}|^p \, dx
\]

by Lebesgue’s theorem. Then use Fatou’s lemma to get

\[
\lim_{h \to \infty} \int_{\Omega} |x|^{-\beta} |v_h|^q \, dx \geq \int_{\Omega} |x|^{-\beta} |\hat{v}|^q \, dx
\]

up to a subsequence \( v_{h_j} \). Thus we have that

\[
m(C^2_c(\Omega)) \leq \lim_{j \to \infty} R(v_{h_j}) \leq R(\hat{v}) < \delta,
\]

that proves \((20)\), as \( \delta \) was arbitrarily chosen. The theorem is completely proved. \(\square\)

We conclude this paper with partial result for lower exponents \( \alpha \). We take \( p = 2 \), and we put

\[
\gamma_\alpha = \gamma_{2,\alpha} = \left( \frac{n-2}{2} \right)^2 - \left( \frac{\alpha-2}{2} \right)^2, \quad \gamma_\alpha = \left( \frac{n-2}{2} \right)^2 + \left( \frac{\alpha+2}{2} \right)^2.
\]

If \( q = 2 \) then \( \beta = \alpha - 4 \) and hence

\[
S_{2,2}^{N}(\Omega; \alpha) := \inf_{u \in C_c^2(\Omega \setminus \{0\})} \frac{\int_{\Omega} |x|^\alpha |\Delta u|^p \, dx}{\int_{\Omega} |x|^{\alpha-4} |u|^q \, dx}
\]

In \([8, 3] \) and \([14] \) it has been proved that

\[
S_{2,2}(\alpha) = S_{2,2}(\mathbb{R}^n; \alpha) = \min_{k \in \mathbb{N} \cup \{0\}} |\gamma_\alpha + k(n-2+k)|^2.
\]

**Proposition 6.1.** Assume that \( \Omega \) is the unit ball in \( \mathbb{R}^n \) and that \( \alpha \leq n \). Then for every \( u \in C_c^2(\Omega \setminus \{0\}) \) one has

\[
(22) \int_{\Omega} |x|^\alpha |\Delta u|^2 \, dx - S_{2,2}(\alpha) \int_{\Omega} |x|^{\alpha-4} |u|^2 \, dx \geq \frac{\gamma_\alpha}{2} \int_{\Omega} |x|^{\alpha-4} \log |x|^{-2} |u|^2 \, dx.
\]

In particular, \( S_{2,2}^{N}(\Omega; \alpha) > 0 \) for any \( \alpha < n \), such that \( -\gamma_\alpha \) is not an eigenvalue of the Laplace-Beltrami operator on the sphere.

In \([3, Theorem 5.1(i)]\) a similar proposition has been stated. However the assumption \( \alpha \leq n \) has been neglected there. In view of Theorem \([13] \) the assumption \( \alpha \leq n \) is clearly needed to have \((22)\). To prove Proposition \( 6.1 \) follow the computations in \([3] \).
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