Efficient estimation in the Topp-Leone distribution

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Abstract

In the current paper, the estimation of the probability density function and the cumulative distribution function of the Topp-Leone distribution is considered. We derive the following estimators: maximum likelihood estimator, uniformly minimum variance unbiased estimator, percentile estimator, least squares estimator and weighted least squares estimator. A simulation study shows that the maximum likelihood estimator is more efficient than the others estimators.

Keywords: Topp-Leone distribution, maximum likelihood estimator, uniformly minimum variance unbiased estimator, percentile estimator, least squares estimator, weighted least squares estimator.

MSC class 2010 Subject Classification: 62E15, 62N05.

1 Introduction

The univariate continuous Topp-Leone distribution with bounded support was originally proposed by Topp and Leone (1955) and applied it as a model for some failure data. The probability density function (PDF) and cumulative distribution function (CDF) are respectively given, for \( 0 < x < 1 \) and \( \alpha > 0 \), by

\[
    f (x) = \alpha (2 - 2x) (2x - x^2)^{\alpha - 1}
\]

and

\[
    F (x) = (2x - x^2) ^ \alpha .
\]
In recent years, the Topp-Leone distribution has received a huge attention in the literature. For instance, Nadarajah and Kotz (2003) derived the structural properties of this distribution including explicit expressions for the moments, hazard rate function and characteristic function. Kotz and Van Dorp (2004) proposed a generalized version of the Topp-Leone distribution for modeling some financial data and studied its properties. Ghitany et al. (2005) discussed some reliability measures of the Topp-Leone distribution and their stochastic orderings. Kotz and Nadarajah (2006) gave a bivariate generalization of this distribution. Ghitany (2007) derived the asymptotic distribution of order statistics of this model. Vicari et al. (2008) introduced a two-sided generalized version of the distribution and discussed some of its properties. The moments of order statistics from this distribution were discussed by Genç (2012) and MirMostafaei (2014). Genç (2013) considered the estimation of the stress-strength parameter for this distribution. Admissible minimax estimates for the shape of this distribution were derived by Bayoud (2016). Bayesian and non-Bayesian estimation of Topp-Leone distribution based lower record values were obtained by MirMostafaei et al. (2016).

In many situations we need to estimate the PDF, CDF or both. For example, we use the PDF to estimate the differential entropy, Kullback-Leibler divergence, Rényi entropy and Fisher information; we use the CDF to estimate the quantile function, the cumulative residual entropy, Bonferroni and Lorenz curves whereas we can use the PDF and CDF together for estimating the probability weighted moments, the hazard rate function, the reverse hazard rate function and the mean deviation about mean.

In this paper, we discuss the efficient estimation of the PDF and the CDF for the Topp-Leone distribution by considering the following estimators: maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE), percentile estimator (PCE), least squares estimator (LSE) and weighted least squares estimator (WLSE).

There are several similar studies for other distributions. We mention: Pareto distribution by Dixit and Jabbari (2010, 2011), generalized exponential-Poisson distribution by Bagheri et al. (2014), generalized exponential distribution by Alizadeh et al.(2014), exponentiated Weibull distribution by Alizadeh et al. (2015), Weibull extension model by Bagheri et al. (2016), exponentiated Gumbel distribution by Bagheri et al. (2016).

The rest of the paper is organized as follows. In Sections 2 and 3, we derive the
MLE and the UMVUE of the PDF and the CDF with their mean squared errors (MSEs), respectively. The PCE, LSE and WLSE are considered in Sections 4 and 5, respectively. Section 6 includes a simulation study in order to compare the different proposed estimators.

2 MLE of the PDF and the CDF

Suppose \(X_1, \ldots, X_n\) is a random sample from the Topp-Leone distribution given by (1) and (2). The MLE of \(\alpha\), denoted by \(\hat{\alpha}\), is given by

\[
\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \ln (2x_i - x_i^2)}.
\]

Therefore, using the invariance property of MLE, we obtain the MLEs of the PDF (1) and the CDF (2) as follows

\[
\hat{f}(x) = \hat{\alpha} (2 - 2x) (2x - x^2)^{\hat{\alpha} - 1}
\]

and

\[
\hat{F}(x) = (2x - x^2)^{\hat{\alpha}},
\]

respectively for \(0 < x < 1\).

To calculate \(E \left( [\hat{f}(x)]^r \right)\) and \(E \left( [\hat{F}(x)]^r \right)\), we need to find the PDF of random variable \(\hat{\alpha}\). Let \(Z_i = -\ln (2X_i - X_i^2)\), \(i = 1, \ldots, n\) and \(T = \sum_{i=1}^{n} Z_i\). Then, the distribution of \(Z_1\) is exponential with PDF given by

\[
f_{Z_1}(z) = \alpha e^{-\alpha z}, \text{ for } z > 0,
\]

and then \(T\) is a gamma random variable with the PDF given by

\[
f_T(t) = \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t}, \text{ for } t > 0.
\]

Therefore, using some elementary algebra, the PDF of \(\hat{\alpha} = S = n/T\) is

\[
f_S(s) = \frac{n^n \alpha^n e^{-\alpha n}}{\Gamma(n) s^{n+1}}, \text{ for } s > 0. \tag{3}
\]

In the following Theorem, we give \(E \left( [\hat{f}(x)]^r \right)\) and \(E \left( [\hat{F}(x)]^r \right)\).
Theorem 1 We have

\[
E\left(\left[\hat{f}(x)\right]^r\right) = \frac{2 (n \alpha)^{\frac{r+n}{2}}}{\Gamma(n)} \frac{(2 - 2x)^r}{(2x - x^2)^r} \left(\frac{-1}{r \ln(2x - x^2)}\right)^\frac{r+n}{2} K_{r-n}\left(2 \sqrt{-n \alpha r \ln(2x - x^2)}\right)
\]

and

\[
E\left(\left[\hat{F}(x)\right]^r\right) = \frac{2 (n \alpha)^{\frac{r+n}{2}}}{\Gamma(n)} \left(\frac{-1}{r \ln(2x - x^2)}\right)^\frac{r+n}{2} K_{-n}\left(2 \sqrt{-n \alpha r \ln(2x - x^2)}\right),
\]

where \(K_v(\cdot)\) is the modified Bessel function of the second kind of order \(v\).

Proof. From Equation (3), we can write

\[
E\left(\left[\hat{f}(x)\right]^r\right) = \int_0^\infty \left[ s (2 - 2x) (2x - x^2)^{s-1}\right]^r \frac{n^a \alpha^a e^{-\frac{n\alpha}{r}}}{\Gamma(n) s^{n+1}} ds
\]

\[
= \frac{n^a \alpha^a (2 - 2x)^r}{\Gamma(n) (2x - x^2)^r} \int_0^\infty s^{r-n-1} (2x - x^2)^{rs} e^{-\frac{n\alpha}{r}} ds
\]

\[
= \frac{n^a \alpha^a (2 - 2x)^r}{\Gamma(n) (2x - x^2)^r} \int_0^\infty s^{r-n-1} e^{rs \ln(2x-x^2)} e^{-\frac{n\alpha}{r}} ds.
\]

Using Formula (3.471.9) in Gradshteyn and Ryzhik (2000), we obtain

\[
\int_0^\infty s^{r-n-1} e^{rs \ln(2x-x^2)} e^{-\frac{n\alpha}{r}} ds = 2 \left(\frac{-n\alpha}{r \ln(2x - x^2)}\right)^\frac{r+n}{2} K_{r-n}\left(2 \sqrt{-n \alpha r \ln(2x - x^2)}\right).
\]

Therefore

\[
E\left(\left[\hat{f}(x)\right]^r\right) = \frac{2 (n \alpha)^{\frac{r+n}{2}} (2 - 2x)^r}{\Gamma(n) (2x - x^2)^r} \left(\frac{-1}{r \ln(2x - x^2)}\right)^\frac{r+n}{2} K_{r-n}\left(2 \sqrt{-n \alpha r \ln(2x - x^2)}\right).
\]

In a similar manner, one can obtain \(E\left(\left[\hat{F}(x)\right]^r\right)\) and then the proof of Theorem 2.1 is completed.

Remark 2 From Theorem 2.1, we observe (when \(r = 1\)) that the estimators \(\hat{f}(x)\) and \(\hat{F}(x)\) are biased for \(f(x)\) and \(F(x)\), respectively.

The MSEs of \(\hat{f}(x)\) and \(\hat{F}(x)\) are given in the following Theorem.
Theorem 3 The MSEs of \( \hat{f}(x) \) and \( \hat{F}(x) \) are respectively given by

\[
MSE(\hat{f}(x)) = 2 \left( \frac{n\alpha}{\Gamma(n)} \right)^{\frac{1}{n}} \frac{(2x-2)^2}{(2x-2)^2} \left( \frac{-1}{2\ln(2x-2)} \right) \frac{2}{\Gamma(n)} \frac{n}{2} \times K_{2-n} \left( 2\sqrt{-2n\alpha \ln(2x-2)} \right) - 8 \frac{(n\alpha)^{\frac{1}{n}}}{\Gamma(n)} \frac{(1-x)f(x)}{2\ln(2x-2)} \times \left( \frac{-1}{\ln(2x-2)} \right)^{\frac{1}{n}} K_{1-n} \left( 2\sqrt{-n\alpha \ln(2x-2)} \right) + f^2(x),
\]

and

\[
MSE(\hat{F}(x)) = 2 \left( \frac{n\alpha}{\Gamma(n)} \right)^{\frac{1}{n}} \frac{1}{2\ln(2x-2)} \left( \frac{-2n\alpha}{\ln(2x-2)} \right)^{-n/2} K_{-n} \left( 2\sqrt{-2n\alpha \ln(2x-2)} \right) - 4 \frac{(n\alpha)^{n/2}}{\Gamma(n)} \frac{1}{\ln(2x-2)} \left( \frac{-2n\alpha}{\ln(2x-2)} \right)^{-n/2} \times K_{-n} \left( 2\sqrt{-n\alpha \ln(2x-2)} \right) + F^2(x).
\]

Proof. We have

\[
MSE(\hat{f}(x)) = E \left( \left[ \hat{f}(x) \right]^2 \right) - 2f(x)E(\hat{f}(x)) + f^2(x),
\]

then, by setting \( r = 1 \) and \( r = 2 \) in Theorem 2.1, we obtain \( MSE(\hat{f}(x)) \). In a similar manner, we can find \( MSE(\hat{F}(x)) \).

3 UMVUE of the PDF and the CDF

In this section, the UMVUEs of the PDF and the CDF of the Topp-Leone distribution are derived. The MSEs of these estimators are also obtained.

Suppose \( X_1, \ldots, X_n \) is a random sample from the Topp-Leone distribution. Then \( T = -\sum_{i=1}^{n} \ln(2X_i - X_i^2) \) is a complete sufficient statistic for \( \alpha \). The UMVUE of \( f(x) \), denoted by \( f^*(t) \), is given by Lehmann-Scheffé theorem

\[
E(f^*(T)) = \int f_{X_1|T}(x_1|t) f_T(t) \, dt = \int f_{X_1,T}(x_1,t) \, dt = f_{X_1}(x_1),
\]

where \( f_{X_1|T}(x_1|t) = f^*(t) \) is the conditional PDF of \( X_1 \) given \( T \) and \( f_{X_1,T}(x_1,t) \) is the joint PDF of \( X_1 \) and \( T \). Then, we need the following Lemma to find \( f_{X_1|T}(x_1|t) \).
Lemma 4  The conditional PDF of \( X_1 \) given \( T \) is

\[
f_{X_1|T}(x|t) = \frac{(n-1)(2-2x)(t + \ln(2x - x^2))^{n-2}}{(2x-x^2)tn^{-1}}, \quad -\ln(2x - x^2) < t < \infty
\]

Proof. We have \( T = \sum_{i=1}^{n} Z_i \) is a gamma random variable with the PDF given by (3). Therefore, using some elementary algebra, the conditional PDF of \( Z_1 \) given \( T \) is

\[
h_{Z_1|T}(z|t) = \frac{(n-1)(t-z)^{n-2}}{t^{n-1}}, \quad 0 < z < t.
\]

Then

\[
f_{X_1|T}(x|t) = \frac{2 - 2x}{2x - x^2} h_{Z_1|T}(-\ln(2x - x^2)|t), \quad -\ln(2x - x^2) < t < \infty,
\]

and the proof is completed.

Theorem 5  Let \( T = t \) be given. Then, the UMVUEs of \( f(x) \) and \( F(x) \) are respectively given, for \(-\ln(2x - x^2) < t < \infty \), by

\[
\tilde{f}(x) = \frac{(n-1)(2-2x)(t + \ln(2x - x^2))^{n-2}}{(2x-x^2)tn^{-1}}
\]

and

\[
\tilde{F}(x) = \left( \frac{t + \ln(2x - x^2)}{t} \right)^{n-1}.
\]

Proof. From Lemma 3.1, we see immediately that \( \tilde{f}(x) \) is the UMVUE of \( f(x) \). Also, we obtain \( \tilde{F}(x) \) by integrating \( \tilde{f}(x) \).

The MSEs of \( \tilde{f}(x) \) and \( \tilde{F}(x) \) are given in the following Theorem.

Theorem 6  The MSES of \( f(x) \) and \( F(x) \) are respectively given by

\[
MSE(\tilde{f}(x)) = \frac{A^2}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} b^i \alpha^{i+3} \Gamma(n-i-2,-ab) - f^2(x)
\]

and

\[
MSE(\tilde{F}(x)) = \frac{1}{\Gamma(n)} \sum_{i=1}^{2n-2} \binom{2n-2}{i} b^i \alpha^{i+1} \Gamma(n-i,-ab) - F^2(x),
\]

where \( A = \frac{(n-1)(2-2x)}{2x-x^2} \), \( b = \ln(2x - x^2) \) and \( \Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t} dt \) is the complementary incomplete gamma function.
Proof. We have
\[
\tilde{f}(x) = \frac{A(t + b)^{n-2}}{t^{n-1}},
\]
and
\[
MSE(\tilde{f}(x)) = E\left(\left[\tilde{f}(x)\right]^2\right) - f^2(x),
\]
where, from Equation (3),
\[
E\left(\left[\tilde{f}(x)\right]^2\right) = \frac{\alpha^2A^2}{\Gamma(n)} \int_{-b}^{\infty} \left[\frac{(t + b)^{n-2}}{t^{n-1}}\right]^2 t^{n-1}e^{-\alpha t}dt
\]
\[
= \frac{\alpha^2A^2}{\Gamma(n)} \int_{-b}^{\infty} \frac{(t + b)^{2n-4}}{t^{2n-2}} t^{n-1}e^{-\alpha t}dt
\]
\[
= \frac{\alpha^2A^2}{\Gamma(n)} \int_{-b}^{\infty} \frac{(t + b)^{2n-4}}{t^{2n-4}} t^{n-3}e^{-\alpha t}dt
\]
\[
= \frac{\alpha^2A^2}{\Gamma(n)} \int_{-b}^{\infty} \left(1 + \frac{b}{t}\right)^{2n-4} t^{n-3}e^{-\alpha t}dt.
\]
Using
\[
\left(1 + \frac{b}{t}\right)^{2n-4} = \sum_{i=1}^{2n-4} \binom{2n-4}{i} \left(\frac{b}{t}\right)^i,
\]
we obtain
\[
E\left(\left[\tilde{f}(x)\right]^2\right) = \frac{\alpha^2A^2}{\Gamma(n)} \int_{-b}^{\infty} \left[\sum_{i=1}^{2n-4} \binom{2n-4}{i} \left(\frac{b}{t}\right)^i\right] t^{n-3}e^{-\alpha t}dt
\]
\[
= \frac{\alpha^2A^2}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} b^i \int_{-b}^{\infty} t^{n-i-3}e^{-\alpha t}dt.
\]
The change of variables \(u = \alpha t\), yields
\[
E\left(\left[\tilde{f}(x)\right]^2\right) = \frac{\alpha^2A^2}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} b^i \frac{\alpha^i}{\alpha^{n-i-3}} \int_{-\alpha b}^{\infty} u^{n-i-3}e^{-u}du
\]
\[
= \frac{A^2}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} b^i \alpha^{i+3} \Gamma(n - i - 2, -\alpha b).
\]
In a similar manner, we can find \(MSE\left(\tilde{F}(x)\right)\).
4 PCE of the PDF and the CDF

The PCE was first introduced by Kao (1958, 1959) and used it when some probability distribution has the quartile function in a closed-form expression. In this section, the PCEs of the PDF and the CDF of the Topp-Leone distribution are derived. Let $X_1, \ldots, X_n$ be a random sample of size $n$ from the Topp-Leone distribution in (2), and let $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics. Then the PCE of $\alpha$, denoted by $\tilde{\alpha}_{PCE}$, is obtained by minimizing

$$\sum_{j=1}^{n} \left( p_i^{1/\alpha} - \left( 2x_{(i)} - x_{(i)}^2 \right) \right)^2,$$

where $p_i = \frac{i}{n+1}$. So, the PCEs of the PDF and the CDF are respectively given, for $0 < x < 1$, by

$$\tilde{f}_{PCE}(x) = \tilde{\alpha}_{PCE} \left( 2 - 2x \right) \left( 2x - x^2 \right) \tilde{\alpha}_{PCE}^{-1},$$

and

$$\tilde{F}_{PCE}(x) = \left( 2x - x^2 \right) \tilde{\alpha}_{PCE}.$$

Since it’s difficult to find the MSEs of these estimators analytically, we shall calculate them by simulation.

5 LSE and WLSE of the PDF and the CDF

The LSE and WLSE were originally proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. In this section, we use the same methods for the Topp-Leone distribution. Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics of a size $n$ random sample from a distribution with CDF $G(\cdot)$. It is well known that

$$E\left( G(X_{(i)}) \right) = \frac{i}{n+1}$$

and

$$Var\left( G(X_{(i)}) \right) = \frac{i(n - i + 1)}{(n + 1)^2 (n + 2)}.$$

Using the expectations and the variances, two variants of the least squares method follow.

5.1 LSE of the PDF and the CDF

The LSE of the unknown parameter is obtained by minimizing the function

$$\sum_{i=1}^{n} \left( F(X_{(i)}) - \frac{i}{n+1} \right)^2,$$
with respect to the unknown parameters. Therefore, for the Topp-Leone distribution the LSE of $\alpha$, denoted by $\tilde{\alpha}_{LS}$, is obtained by minimizing

$$\sum_{i=1}^{n} \left( \left( 2x_{(i)} - x_{(i)}^2 \right)^\alpha - \frac{i}{n+1} \right)^2.$$ 

Then, the LSEs of the PDF and the CDF are respectively given, for $0 < x < 1$, by

$$\tilde{f}_{LS} (x) = \tilde{\alpha}_{LS} \left( 2 - 2x \right) \left( 2x - x^2 \right) \tilde{\alpha}_{LS}^{-1},$$

and

$$\tilde{F}_{LS} (x) = \left( 2x - x^2 \right) \tilde{\alpha}_{LS}.$$ 

Since it’s difficult to find the MSEs of these estimators analytically, we shall calculate them by simulation.

### 5.2 WLSE of the PDF and the CDF

The WLSE of the unknown parameter is obtained by minimizing

$$\sum_{i=1}^{n} w_i \left( F(X_{(i)}) - \frac{i}{n+1} \right)^2$$

with respect to the unknown parameters, where

$$w_i = \frac{1}{Var (F(X_{(i)}))} = \frac{(n+2)(n+1)^2}{i(n-i+1)}.$$ 

Therefore, for the Topp-Leone distribution the WLSE of $\alpha$, denoted by $\tilde{\alpha}_{WLS}$, is obtained by minimizing

$$\sum_{i=1}^{n} w_i \left( \left( 2x_{(i)} - x_{(i)}^2 \right)^\alpha - \frac{i}{n+1} \right)^2.$$ 

Then, the WLS estimators of the PDF and the CDF are respectively given, for $0 < x < 1$, by

$$\tilde{f}_{WLS} (x) = \tilde{\alpha}_{WLS} \left( 2 - 2x \right) \left( 2x - x^2 \right) \tilde{\alpha}_{WLS}^{-1}$$

and

$$\tilde{F}_{WLS} (x) = \left( 2x - x^2 \right) \tilde{\alpha}_{WLS}.$$ 

Since it’s difficult to find the MSEs of these estimators analytically, we shall calculate them by simulation.
6 Simulation study

In this section, by means of the statistical software \textbf{R}, a simulation study is carried out to compare the different proposed estimators: the MLE, the UMVUE, the PCE, the LSE and the WLSE of the PDF and the CDF. This comparison is based on the MSEs. To this end, we generate 1000 independent replicates of sizes $n = 10, 20, 50$ and 100 from the Topp-Leone distribution with $\alpha = 0.5, 1.0, 2.0$ and 3.0. The Topp-Leone random number generation is performed using the inversion method: $X = 1 - \sqrt{1 - U^{1/\alpha}}$, where $U$ is a standard uniform random variable. The results are summarized in Table 1. From this table, we observe that the MLEs of the PDF and the CDF have the smallest MSEs. Then, the MLE perform better than the others estimators in all the cases considered in terms of MSEs. Also, we observe that the MSEs for each estimator decrease with increasing sample size as expected.

Table 1: MSEs of the PDF estimators and the CDF estimators (MSEs of the CDF estimators are given in brackets).

| $n$ | Method | $\alpha = 0.5$ | $\alpha = 1.0$ | $\alpha = 2.0$ | $\alpha = 3.0$ |
|-----|--------|----------------|----------------|----------------|----------------|
| 10  | MLE    | 0.0702[0.0892] | 0.0837[0.0112] | 0.0238[0.1021] | 0.0511[0.0464] |
|     | UMVUE  | 0.0921[0.1193] | 0.1021[0.0920] | 0.0436[0.1355] | 0.0518[0.0678] |
|     | PCE    | 0.1296[0.1247] | 0.1313[0.1047] | 0.0982[0.1513] | 0.1099[0.0897] |
|     | LSE    | 0.1941[0.2158] | 0.1967[0.2001] | 0.1869[0.1901] | 0.2010[0.1649] |
|     | WLSE   | 0.1604[0.1847] | 0.1741[0.1699] | 0.1107[0.1318] | 0.1334[0.1121] |
| 20  | MLE    | 0.0314[0.1701] | 0.0716[0.0053] | 0.020[0.0921]  | 0.0341[0.0344] |
|     | UMVUE  | 0.0422[0.0985] | 0.0921[0.1192] | 0.0398[0.1040] | 0.0403[0.0516] |
|     | PCE    | 0.0956[0.1003] | 0.1296[0.1247] | 0.0772[0.1106] | 0.0885[0.0662] |
|     | LSE    | 0.1540[0.1816] | 0.1941[0.2158] | 0.1552[0.1661] | 0.1819[0.1398] |
|     | WLSE   | 0.1297[0.1147] | 0.1601[0.1841] | 0.1079[0.1282] | 0.1223[0.0997] |
| 50  | MLE    | 0.0095[0.0080] | 0.0102[0.0026] | 0.0010[0.0352] | 0.0028[0.0039] |
|     | UMVUE  | 0.0193[0.0248] | 0.0467[0.0788] | 0.0035[0.0502] | 0.0177[0.0207] |
|     | PCE    | 0.04562[0.0803]| 0.0816[0.1098] | 0.0225[0.0651] | 0.0412[0.0388] |
|     | LSE    | 0.10410[0.1558]| 0.1511[0.1772] | 0.0998[0.1136] | 0.1196[0.1006] |
|     | WLSE   | 0.0974[0.10471]| 0.1153[0.1380] | 0.0296[0.0441] | 0.0757[0.0567] |
| 100 | MLE    | 0.00021[0.00027]| 0.00049[0.00016]| 0.00016[0.0017] | 0.00018[0.00047]|
|     | UMVUE  | 0.0022[0.0012] | 0.0050[0.0042] | 0.00077[0.0123] | 0.0092[0.0084] |
|     | PCE    | 0.0033[0.0016] | 0.00708[0.0056] | 0.0087[0.0209] | 0.0104[0.0196] |
|     | LSE    | 0.0727[0.0801] | 0.0930[0.1004] | 0.0299[0.0674] | 0.0789[0.0817] |
|     | WLSE   | 0.0131[0.0015] | 0.0431[0.086]  | 0.0099[0.0376] | 0.0327[0.0229] |
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