Higher order Soliton Complexes in Coupled Nonlinear Schrödinger Equation with Variable Coefficients

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Abstract

We present the explicit dark-bright three soliton solution and the associated spectral problem for the variable coefficient integrable coupled NLS equation. Using asymptotic analysis as well as graphical analysis we study the interactions in soliton complexes. We present a correlation between the soliton parameters and the interaction pattern in three soliton complexes. Using asymptotic analysis, we present a few interesting features of complex three soliton bound state and interaction of dark-bright two soliton complex with a regular soliton. Using three soliton interactions we have shown that the energy sharing take place between soliton even when the soliton do not collide with each other. The results found by us might be useful for the development of soliton control, all optical gates as well as all optical switching devices. We hope that the analysis of three soliton complexes would be useful for a better understanding of soliton interactions in nonlinear fiber as well as in a bulk medium.

05.45.Yv, 42.81Dp, 42.65.Tg

1 Introduction

Since it’s discovery in 1965 by Kruskal [1], the potential application of soliton has been explored in many areas of physical sciences, such as in fluid mechanics and plasma physics[2], in cold atoms [4, 3], in high energy physics[5] etc. Among the integrable systems perhaps nonlinear Schrödinger equation (NLS) has the widest applications. Optical soliton is one such application. In early seventies of twentieth century Hasegawa through a scientific communication [6], explained that there exist an entity called optical soliton which maintains a balance between dispersion and nonlinearity as it propagates through a dispersive dielectric

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medium and the dynamics of the optical soliton can be described by NLS equation. A uniform optical medium with anomalous dispersion supports a bright soliton whereas a dark soliton (described as a dip in a continuous background) is supported by a medium with normal dispersion. Incidentally dark solitons are more stable in noisy background and under perturbations and they interact weakly compared to bright solitons[7]. Over the past three decades there have been remarkable advances in both theoretical and experimental research on soliton and its applications [8, 9] (and the references therein). A dispersion managed (DM) soliton[8] however, is a more advanced concept, where solitons may exist in a normal and anomalous dispersion coexisting media, and is described by the standard nonlinear Schrödinger equation model with varying dispersion and nonlinear coefficients and gain[10]. A DM soliton can be accelerated or retarded and amplified while preserving its shape. Recently there have been many important publications on the dynamics of a bright soliton [11, 12, 13] and dark soliton [14, 15, 16, 17, 18] based on the model given in [10] and a more recently developed non-autonomous soliton model[19], where the authors predicted various applications of DM soliton.

In comparison to scalar soliton a vector soliton (with more than one mutually self-trap components), proposed by Manakov [20] demonstrates interesting additional features. For instance, bright-bright soliton pair (bright soliton in all the modes), bright-dark soliton pair (modes are shared between bright solitons and dark solitons) [21, 22, 23], dark-dark soliton pair (dark solitons in all the modes) [24] obtained in a normal dispersion and anomalous dispersion coexisting dielectric medium, soliton shape changing in an inelastic soliton interaction [25], simulation of electronic logic gates using elastic and inelastic soliton interactions [26, 27, 28]. All these discoveries has made the field of applications of vector solitons more exciting and vibrant.

It is now natural to ask whether a DM vector soliton is also possible and if possible what would be the dynamics, how do they interact and their possible applications. Secondly whether the mathematical techniques available for the scalar soliton and vector solitons are applicable to DM vector soliton also. In the literature other than few notable publications [29, 30, 31, 32, 33], study in this direction is comparatively sparse. Present authors in one of their recent works investigated the two soliton interactions using the variable coefficient NLS model[34].

A three soliton solution while being an important signature of integrability [37], shows many interesting features. An interesting aspect of three soliton solution is the bound state soliton interactions, many of which are unlike normal soliton interactions, as found in two soliton bound state interactions[22, 34]. A three soliton bound state where a two soliton complex interact with a third soliton, is expected to bring forth more interesting facts about the soliton interaction. Keeping in context the above facts, here we shall investigate three soliton solution of a Lax integrable variable coefficient 2-coupled NLS model. We shall also analyze interactions in dark-bright soliton complexes using asymptotic analysis, wherever applicable and the remaining cases using graphical analysis. Such analysis might contain useful information that would be necessary for further
development in the applications of optical solitons.

The organization of the paper is as follows. Section 1 contains introduction. In section 2 the model for the proposed study is described and the associated Lax pair is presented. Bilinear form and soliton solutions of the proposed model is also included in this section. In section 3 soliton interactions are studied using asymptotic analysis. In section 4 three soliton complexes are studied using asymptotic as well as graphical analysis. Section 5 is the concluding one.

2 THE MODEL EQUATION

We consider the following generalized NLS equation with variable coefficients [34]:

\[ i q_j + \frac{D(t)}{2} q_{jzz} + \gamma R(t) \sum_{l=1}^{n} \sigma_l |q_l|^2 q_j = i \Gamma(t) q_j \]  

(1)

where \( q_j \) (\( j = 1, \cdots, n \)) are complex amplitude of the \( j^{th} \) field component, of an inhomogeneous dispersive and nonlinear medium, subscript \( t \) and \( z \) are the dimensionless parameters, and denote the partial derivatives with respect to \( t \) and \( z \) respectively, \( D(t) \) and \( R(t) \) denote the variable dispersion coefficient and nonlinearity coefficient respectively. \( \Gamma(t) = \frac{\partial D(t) R(t) - D(t) \partial R(t)}{2 R(t) D(t)} \) and \( \sigma_l (= \pm 1) \) define the sign of the nonlinearity. \( \sigma_l = -1(+1) \) stands for a defocusing (focusing) type nonlinearity. If \( \sigma_l = +1 \) (for \( l = 1, 2, \cdots, n \)) nonlinearity is only focusing type and if \( \sigma_l = -1 \) (for \( l = 1, 2, \cdots, n \)) nonlinearity is defocusing type. If \( \sigma_l = +1 \) (for \( l = 1, 2, \cdots, k \)) and \( \sigma_l = -1 \) (for \( l = k + 1, k + 2, \cdots, n \)) then both focusing (for \( k \) components) and defocusing (for \( (n-k) \) components) type nonlinearity occur at the same time.

Bright-dark type soliton solution for eq.1, with bright solitons in \( k \) modes and dark solitons in \((n-k)\) modes can be obtained with \((2^n-1)\) possible combinations of \( \sigma_l = \pm 1 \) [34]. Notice that the choice \( D(t) = R(t) = 1 \) and \( \sigma_l = 1 \) lead eq. 1 to standard Manakov model [20].

In the present paper we shall obtain dark-bright type 3-soliton for eq.1 using the Hirota [38] method. Subsequently using asymptotic and graphical analysis we present a correlation between soliton parameters and the pattern of interaction in a 3-soliton complex. The analysis here can be generalized to system with \((k>2)\) modes for bright soliton and \((n-k>1)\) modes for dark soliton. However, it is worthwhile to mention here that in Manakov system, adding number of modes do not add more intricacies in to the system, as pairwise collision in \( N(>2)\)-coupled(NLS equation can be reduced to pairwise collision in 2-coupled NLS equation [35].

Secondly, using asymptotic analysis we shall explicitly show that there is never an energy sharing between the dark solitons during an interaction. Thus in the proposed study, we consider a minimum, yet sufficient number of modes, in NLS eq.1.
2.1 Associated spectral problem

The Lax pair associated with the eq. 1 is:

\[
\begin{align*}
\partial_z \Psi(z,t) &= U \Psi(z,t) \\
\partial_t \Psi(z,t) &= V \Psi(z,t)
\end{align*}
\] (2)

where \(U\) and \(V\) respectively are \((n+1) \times (n+1)\) matrices. The explicit form of \(U\) and \(V\) are:

\[
U = -i\lambda \Sigma + \sqrt{\frac{R(t)}{D(t)}} \Lambda
\]

\[
V = -i \Sigma \left\{ \frac{R(t)}{2} \Lambda^2 + \int^t \lambda_0 ds + D(t) \lambda^2 - \gamma_0(t) - \sqrt{\frac{R(t)}{D(t)}} \frac{1}{2} \Lambda_z \right\} + \sqrt{\frac{R(t)}{D(t)}} D(t) \lambda \Lambda
\] (3)

\[
\Sigma = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & 0 & -1
\end{pmatrix}_{(n+1) \times (n+1)}
\]

\[
\Lambda = \begin{pmatrix}
0 & \cdots & 0 & \sqrt{\sigma_1 q_1} \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -\sqrt{\sigma_n q_n^*}
\end{pmatrix}_{(n+1) \times (n+1)}
\]

where \(\lambda\) is the spectral parameter. The compatibility condition namely, \(U_t - V_z + UV - VU = 0\) gives the eq. 1 provided \(\int_0^t \lambda_0 ds = \gamma_0(t)\).

2.2 Bilinearization and Soliton Solutions using Hirota’s Method

In order to write eq. 1 in the bilinear form, we make the following bilinear transformation,

\[
q_j = \frac{g^{(j)}(t,z)}{f(t,z)}; \text{(bright soliton)}
\]

\[
q_i = \frac{g^{(i)}(t,z)}{f(t,z)}; \text{(dark soliton)}
\] (6)
where \( g^{(j)}(t, z) \) and \( g^{(l)}(t, z) \) are complex and \( f(t, z) \) is real. Consequently in the new set of variables we have the following set of bilinear equations:

\[
(iD_t + \frac{D_t(t)}{2} D_z^2 - \lambda)(g^{(j)} f) = i\Gamma(t) (g^{(j)} f) \\
\frac{D_t(t)}{2} D_z^2 - \lambda)(f f) = \gamma R(t) \sum_{l=1}^{n} \sigma_l g^{(l)} g^{(l)*}
\]

which follows from eq.1. \( D_t \) and \( D_z^2 \) are Hirota derivatives \([38, 39]\) and are defined by,

\[
D^n_z D^m_t u(z, t) v(z, t) = \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n u(z, t) v(z', t') \bigg|_{z'=z; t'=t}
\]

In order to obtain the soliton solutions \( g^{(j)} (j = 1, 2, \cdots k) \), \( g^{(l)} (l = k + 1, k + 2, \cdots n) \) and \( f \) are expanded with respect to an arbitrary parameter \( \epsilon \) as follows,

\[
g^{(j)} = \epsilon g^{(j)}_1 + \epsilon^3 g^{(j)}_3 + \cdots \\
g^{(l)} = g^{(l)}_0 (1 + \epsilon^2 g^{(l)}_2 + \cdots) \\
f = 1 + \epsilon^2 f_2 + \cdots
\]

Dark-Bright one soliton, two soliton, and three soliton solutions are obtained from the following expressions,

\[
q_{j1} = \frac{\epsilon g^{(j)}_1}{1 + \epsilon^2 f_2} \bigg|_{\epsilon=1}; \ \text{bright soliton} \\
q_{l1} = \frac{g^{(l)}_0 (1 + \epsilon^2 g^{(l)}_2)}{1 + \epsilon^2 f_2} \bigg|_{\epsilon=1}; \ \text{dark soliton}
\]

\[
q_{j2} = \frac{\epsilon g^{(j)}_1 + \epsilon^3 g^{(j)}_3}{1 + \epsilon^2 f_2 + \epsilon^4 f_4} \bigg|_{\epsilon=1}; \ \text{bright 2-soliton} \\
q_{l2} = \frac{g^{(l)}_0 (1 + \epsilon^2 g^{(l)}_2 + \epsilon^4 g^{(l)}_4)}{1 + \epsilon^2 f_2 + \epsilon^4 f_4} \bigg|_{\epsilon=1}; \ \text{dark 2-soliton}
\]

\[
q_{j3} = \frac{\epsilon g^{(j)}_1 + \epsilon^3 g^{(j)}_3 + \epsilon^5 g^{(j)}_5}{1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \epsilon^6 f_6} \bigg|_{\epsilon=1}; \ \text{bright 3-soliton} \\
q_{l3} = \frac{g^{(l)}_0 (1 + \epsilon^2 g^{(l)}_2 + \epsilon^4 g^{(l)}_4 + \epsilon^6 g^{(l)}_6)}{1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \epsilon^6 f_6} \bigg|_{\epsilon=1}; \ \text{dark 3-soliton}
\]
Explicit one soliton and two soliton solutions of eq. 1 are obtained in [34]. However, three soliton solution is one of the important criteria for the existence of N-soliton solutions [37]. Substituting eq. 11 in eq. 7 we obtain the three soliton solution of eq. 1. In this paper we consider $j = 1, 2$ and $l = 3$, that is bright $3$-soliton $3SS$ is in component 1, 2 and dark $3SS$ is in component 3. Explicit solution is given in the Appendix.

### 3 Soliton Interaction and Asymptotic Analysis

Energy sharing in Manakov solitons during an interaction is an well established fact. In the present system however, the interaction energy sharing is a periodic phenomena. Let us analyze the asymptotic behavior of the 3-soliton solution eq. 11 (see Appendix ), where a bright three soliton is in two components and a dark three soliton is in one component in a three components system. Asymptotically solitons, irrespective of their coordinates are sufficiently separated, such that there is no interaction among them. Consider for instance, at an instant when soliton $q_3^{(j)}$ is at $z \to \infty$, $q_1^{(j)}$ and $q_2^{(j)}$ are at $z \to -\infty$ and they are moving with relative velocities as they are approaching each other. Note that the sign before $z$ are interchangeable as the solitons change their positions periodically. Then the asymptotic expressions of bright soliton are, as $z \to -\infty$

\[
q_1^{(j)} \rightarrow \frac{\alpha_1^{(j)}(t)}{\sqrt{\delta_1}} \text{Sech}(\frac{\theta_1 + \theta_1^* + \ln(\delta_1)}{2})e^{\frac{\theta_1 - \theta_1^*}{2}}; \; \theta_1 + \theta_1^* + \theta_3 + \theta_3^* \to -\infty
\]

\[
q_2^{(j)} \rightarrow \frac{\alpha_2^{(j)}(t)}{\sqrt{\delta_2}} \text{Sech}(\frac{\theta_2 + \theta^*_2 + \ln(\delta_2)}{2})e^{\frac{\theta_2 - \theta_2^*}{2}}; \; \theta_1 + \theta_1^* + \theta_3 + \theta_3^* \to -\infty \tag{12}
\]

\[
q_3^{(j)} \rightarrow \frac{\alpha_3^{(j)}(t)}{\sqrt{\delta_3}} \text{Sech}(\frac{\theta_3 + \theta_3^* + \ln(\delta_3)}{2})e^{\frac{\theta_3 - \theta_3^*}{2}}; \; \theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to +\infty
\]

and as $z \to +\infty$

\[
q_1^{(j)} \rightarrow \frac{\alpha_3^{(j)}(t)}{\sqrt{\delta_3}} \text{Sech}(\frac{\theta_1 + \theta_1^* + \ln(\delta_1)}{2})e^{\frac{\theta_1 - \theta_1^*}{2}}; \; \theta_2 + \theta_2^* + \theta_3 + \theta_3^* \to +\infty
\]

\[
q_2^{(j)} \rightarrow \frac{\alpha_2^{(j)}(t)}{\sqrt{\delta_2}} \text{Sech}(\frac{\theta_3 + \theta_3^* + \ln(\delta_2)}{2})e^{\frac{\theta_3 - \theta_3^*}{2}}; \; \theta_1 + \theta_1^* + \theta_3 + \theta_3^* \to +\infty \tag{13}
\]

\[
q_3^{(j)} \rightarrow \frac{\alpha_3^{(j)}(t)}{\sqrt{\delta_3}} \text{Sech}(\frac{\theta_3 + \theta_3^* + \ln(\delta_3)}{2})e^{\frac{\theta_3 - \theta_3^*}{2}}; \; \theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to -\infty
\]

Similarly the asymptotic expression of dark solitons are, as $z \to -\infty$
Let $A_1$, $A_2$, $A_3$ denote the amplitudes and $\Phi_1^+, \Phi_2^+, \Phi_3^+$ denote the phase of soliton 1, soliton 2 and soliton 3 respectively before interaction and $A_1^{+1}$, $A_2^{+1}$ and $A_3^{+1}$ denote the amplitudes and $\Phi_1^{+1}$, $\Phi_2^{+1}$ and $\Phi_3^{+1}$ denote the phase of the three solitons after interaction. Then under one of the following conditions, 

$$\frac{\alpha_1^{(1)}(t)}{\alpha_2^{(1)}(t)} = \frac{\alpha_2^{(1)}(t)}{\alpha_3^{(1)}(t)} = \frac{\alpha_3^{(1)}(t)}{\alpha_3^{(1)}(t)}$$

and as $z \to +\infty$

$$\left| q_1^{(3)} \right|^+ = \chi^{(3)}(t) \left( 1 + \frac{\gamma_1^{(3)}}{\delta_1} + \left( \frac{\gamma_1^{(3)}}{\delta_1} - 1 \right) \right) \right| \times \text{Tanh}(\frac{\theta_1 + \theta_1^* + \text{ln}(\delta_1)}{2}) e^{\phi^{(3)} + \frac{\gamma_1^{(3)} - \gamma_1^*}{\delta_1}}; \quad \theta_2 + \theta_2^* + \theta_3 + \theta_3^* \to -\infty

$$q_2^{(3)} - \chi^{(3)}(t) \frac{\rho_1^{(3)}}{\rho_2^{(3)}} \left( 1 + \frac{\gamma_2^{(3)}}{\delta_2} + \left( \frac{\gamma_2^{(3)}}{\delta_2} - 1 \right) \right) \right| \times \text{Tanh}(\frac{\theta_2 + \theta_2^* + \text{ln}(\delta_2)}{2}) e^{\phi^{(3)} + \frac{\gamma_2^{(3)} - \gamma_2^*}{\delta_2}}; \quad \theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to -\infty

$$q_3^{(3)} - \chi^{(3)}(t) \frac{\rho_3^{(3)}}{\rho_3^{(3)}} \left( 1 + \frac{\gamma_3^{(3)}}{\delta_3} + \left( \frac{\gamma_3^{(3)}}{\delta_3} - 1 \right) \right) \right| \times \text{Tanh}(\frac{\theta_3 + \theta_3^* + \text{ln}(\delta_3)}{2}) e^{\phi^{(3)} + \frac{\gamma_3^{(3)} - \gamma_3^*}{\delta_3}}; \quad \theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to +\infty

(15)
\[
\begin{align*}
\alpha_2^{(1)}(t) &= \alpha_1^{(2)}(t) = \alpha_3^{(2)}(t) = 0; \\
\alpha_1^{(1)}(t) &\neq 0; \alpha_2^{(2)}(t) \neq 0; \alpha_3^{(1)}(t) \neq 0 
\end{align*}
\] (18)

\[
\begin{align*}
\alpha_1^{(2)}(t) &= \alpha_2^{(2)}(t) = \alpha_3^{(1)}(t) = 0; \\
\alpha_1^{(1)}(t) &\neq 0; \alpha_2^{(1)}(t) \neq 0; \alpha_3^{(2)}(t) \neq 0 
\end{align*}
\] (19)

\[
\begin{align*}
\alpha_1^{(2)}(t) &= \alpha_2^{(1)}(t) = \alpha_3^{(1)}(t) = 0; \\
\alpha_1^{(1)}(t) &\neq 0; \alpha_2^{(2)}(t) \neq 0; \alpha_3^{(2)}(t) \neq 0 
\end{align*}
\] (20)

the amplitude of each soliton in each component remain same before and after interaction, That is
\[
A_j^{1-} = A_j^{1+}; \quad A_j^{2-} = A_j^{2+}; \quad A_j^{3-} = A_j^{3+}. 
\] (21)

This is the case of an elastic interaction [34]. However, each solitons in each component suffer a phase-shift due to interaction, that is \(\Phi_j^{1-} - \Phi_j^{1+} \neq 0\) for \(l = 1, 2, 3\). We use MATHEMATICA to plot the results of our analysis. To plot the figures we have chosen \(D(t) = e^{\sigma t} \cos(kt)\) and \(R(t) = \cos(kt)\); consequently the gain coefficient \(\Gamma(t)\) is nonvanishing. Fig.1(a, b) and Fig.2(a, b) show an example of periodic elastic interactions occurring at three points between three bright solitons moving with different velocities, in components \(|q_1|, |q_2|\) and between three dark solitons in component \(|q_3|\) Fig.1(1, 2) (c), with \(\alpha_1^{(1)}(t), \alpha_1^{(2)}(t), \alpha_2^{(1)}(t), \alpha_2^{(2)}(t), \alpha_3^{(1)}(t)\) and \(\alpha_3^{(2)}(t)\), satisfying the condition 16. The trajectories of solitons resemble to three synchronized damped simple harmonic oscillators, oscillating with their velocity changing periodically between maxima (the point where the two soliton interact) and minima. The soliton amplitude of each soliton remains same after each interaction but changes in phases occur. However, after two successive cycle of interactions net phase-shift of each soliton turns out to be zero. The trajectories of solitons in both the components remain identical.

On the other hand if none of the eqs. 16 - 20 is satisfied then, soliton amplitude changes after interaction. That is,
\[
A_j^{1-} \neq A_j^{1+}; \quad A_j^{2-} \neq A_j^{2+}; \quad A_j^{3-} \neq A_j^{3+} 
\] (22)

which is an inelastic interaction [34]. However, the asymptotic analysis of dark solitons eqs. 14, 15 show that the amplitudes of solitons do not change after interaction. That is, dark soliton interactions in multi-component NLS model is similar to scalar NLS model.

Fig.(3, 4) are two examples of inelastic interactions between three bright solitons in components \(|q_1|, |q_2|\) between three dark solitons in components \(|q_3|\), moving with arbitrarily chosen relative velocities, and \(\alpha_1^{(1)}(t), \alpha_1^{(2)}(t), \alpha_2^{(1)}(t), \alpha_2^{(2)}(t), \alpha_3^{(1)}(t), \alpha_3^{(2)}(t)\) are in accordance with the condition eq.22. In component \(|q_1|\), one of the solitons gains energy after interaction and in component \(|q_2|\) it
looses energy after interaction. The energy sharing and phase-shift are however, temporary. In a cycle of interactions net inter-component energy transfer and soliton phase shift become zero, which is different from the inelastic interaction in a Manakov two soliton interaction [25, 36]. This is interesting and we may describe this as if an input signal is passing through two successive 'NOT' gates and ultimately remained unchanged.

Figure 2: Contour plot of elastic interactions occurring at three points between bright 3-solitons in two components, $|q_1|$ and $|q_2|$, and dark 3-solitons in component, $|q_3|$, with same set of parameters as in Fig 1

4 THREE SOLITON BOUND STATE

In three soliton bound state we consider two cases. In case one solitons have no relative velocities among themselves and in case two, a two solitons complex interact with a third soliton moving with a relative velocity. We analyze the bound soliton interactions using asymptotic analysis, wherever applicable as
Figure 3: Inelastic Soliton interactions between bright 3-solitons in two components, $|q_1|$ and $|q_2|$, satisfying the conditions of a "NOT" like logic gate repeatedly and dark 3-solitons in component, $|q_3|$, with $\eta_1 = 1.2 + i$, $\eta_2 = 1.01$, $\eta_3 = -1 - i$, $\alpha_{11} = \alpha_{12} = \alpha_{23} = 1$, $\alpha_{21} = \alpha_{22} = 2$, $\alpha_{13} = 8$.

Figure 4: Inelastic Soliton interactions between bright 3-solitons in two components, $|q_1|$ and $|q_2|$, satisfying the conditions of a "NOT" logic gate repeatedly and dark 3-solitons in component, $|q_3|$, with $\eta_1 = 1.2 + i$, $\eta_2 = 1.01$, $\eta_3 = -1 - i$, $\alpha_{11} = \alpha_{12} = 2$, $\alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\alpha_{23} = 8$.

Figure 5: Plot of bound bright 3-solitons in two components, $|q_1|$ and $|q_2|$ and dark 3-solitons in component, $|q_3|$, with $k = \pi/6$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 1$, $\eta_2 = 1.01$, $\eta_3 = 2$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $\alpha_{11} = \alpha_{12} = \alpha_{23} = \alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\sigma = 0.1$, $\Gamma(t) = 0.05$. 
Figure 6: Plot of bound bright 3-solitons in two components, $|q_1|$ and $|q_2|$ and dark 3-solitons in component, $|q_3|$, with $k = \frac{\pi}{36}$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 1$, $\eta_2 = 3$, $\eta_3 = 3.1$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $\alpha_{11} = \alpha_{12} = \alpha_{23} = \alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\sigma = 0.1$, $\Gamma(t) = 0.05$.

Figure 7: Plot of bound bright 3-solitons in two components, $|q_1|$ and $|q_2|$ and dark 3-solitons in component, $|q_3|$, with $k = \frac{\pi}{36}$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 2$, $\eta_2 = 2.1$, $\eta_3 = 2.01$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $\alpha_{11} = \alpha_{12} = \alpha_{23} = \alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\sigma = 0.1$, $\Gamma(t) = 0.05$.

Figure 8: Plot of energy sharing bound bright 3-solitons in two components, $|q_1|$ and $|q_2|$ and dark 3-solitons in component, $|q_3|$, with $k = \frac{\pi}{36}$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 1$, $\eta_2 = 1.01$, $\eta_3 = 2.01$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $-\alpha_{11} = \alpha_{12} = \alpha_{23} = \alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\sigma = 0.1$, $\Gamma(t) = 0.05$. 

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Figure 9: Plot of energy sharing bound bright 3-solitons in two components, $|q_1|$ and $|q_2|$ and dark 3-solitons in component, $|q_3|$, with $k = \frac{\pi}{36}$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 1$, $\eta_2 = 2.2$, $\eta_3 = 2.21$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $\alpha_{11} = \alpha_{12} = -\alpha_{23} = \alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\sigma = 0.1$, $\Gamma(t) = 0.05$.

Figure 10: Plot of energy sharing bound bright 3-solitons in two components, $|q_1|$ and $|q_2|$ and dark 3-solitons in component, $|q_3|$, with $k = \frac{\pi}{36}$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 2$, $\eta_2 = 2.1$, $\eta_3 = 2.01$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $\alpha_{11} = \alpha_{12} = -\alpha_{23} = \alpha_{21} = \alpha_{22} = \alpha_{13} = 1$, $\sigma = 0.1$, $\Gamma(t) = 0.05$.

Figure 11: Plot of Soliton interactions between a bright 2-soliton complex, with an accelerating soliton, in components, $|q_1|$ and $|q_2|$,(bright soliton), with $k = \frac{\pi}{36}$, $\gamma = -1$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $\eta_1 = 2$, $\eta_2 = 2.1$, $\eta_3 = 3.01 + i$, $\chi_3 = 1 + i$, $\xi_3 = 1$, $\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \alpha_{13} = \alpha_{23} = 1$, $\sigma = 0.01$, $\Gamma(t) = 0.005$. 

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Figure 12: Plot of Soliton interactions between a 2-soliton complex, which changes to a breather type soliton after each interaction and an accelerating soliton, in components, \(|q_1|, |q_2|\) (bright solitons) and \(|q_3|\) (dark soliton), with 
\[k = \frac{\pi}{12}, \gamma = -1, \sigma_1 = \sigma_2 = \sigma_3 = 1, \eta_1 = -1, \eta_2 = -1.6, \eta_3 = -1 + 0.2i, \]
\[\chi_3 = 1 + i, \xi_3 = 1, \delta_2 = \delta_4 = 0, \] that is, \(\alpha_{11} = \alpha_{12} = -\alpha_{21} = \alpha_{22} = 1, \alpha_{13} = 2, \]
\(\alpha_{23} = 1, \sigma = 0.006, \Gamma(t) = 0.003.\)

Figure 13: Contour Plot of Soliton interactions between a 2-soliton complex, with an accelerating soliton, in components, \(|q_1|, |q_2|\) (bright solitons) and \(|q_3|\) (dark soliton), with same set of parameters as in Fig.12.

Figure 14: Plot of Soliton interactions between 2-soliton complex and an accelerating soliton, in components, \(|q_1|, |q_2|\) (bright solitons) and \(|q_3|\) (dark soliton), with 
\[k = \frac{\pi}{12}, \gamma = 1, \sigma_1 = \sigma_2 = \sigma_3 = 1, \eta_1 = 2.1, \eta_2 = 2, \eta_3 = 2.5 - i, \chi_3 = 1 - i, \]
\[\xi_3 = 1, \delta_3 = \delta_7 = \delta_5 = \delta_8 = 0, \] that is, \(\alpha_{11} = \alpha_{12} = 1, \alpha_{21} = \alpha_{22} = 0, \alpha_{13} = 0, \]
\(\alpha_{23} = 1, \sigma = 0.006, \Gamma(t) = 0.003.\)
Figure 15: Contour Plot of Soliton interactions between a 2-soliton complex and an accelerating soliton, in components, \(|q_1|, |q_2|\) (bright solitons) and \(|q_3|\) (dark soliton), with same set of parameters as in Fig.14.

Figure 16: Soliton interactions between a bright 2-soliton complex with an accelerating soliton in \(|q_1|, |q_2|\) and two dark solitons complex with an accelerating soliton in \(|q_3|\), with \(k = \frac{\pi}{12}, \gamma = -1, \sigma_1 = \sigma_2 = \sigma_3 = 1, \eta_1 = 1 + 2i, \eta_2 = 2.01, \eta_3 = 1, \chi_3 = 1 - i, \xi_3 = 1, \alpha_{11} = \alpha_{22} = \alpha_{23} = 1, \alpha_{21} = \alpha_{13} = \alpha_{12} = -1, \sigma = 0.005, \Gamma(t) = 0.0025.

Figure 17: Contour Plot of Soliton interactions between a 2-soliton complex with an accelerating soliton, effecting a large phase shift in all components, \(|q_1|, |q_2|\) (bright solitons) and \(|q_3|\) (dark soliton), with same set of parameters as in Fig.16.
well as graphical analysis. When all three solitons are comoving, that is the solitons do not move sufficiently apart from each other asymptotically and hence an effective asymptotic analysis is difficult. We identify a correlations, among soliton parameters, which distinguishes different types of interactions among the solitons. First we consider the three comoving solitons, with components satisfying one of the following set of conditions.

Case A,

\[ \alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{22} = \alpha_{13} = \alpha_{23} = a \quad (23) \]

First suffix on \( \alpha \) denotes component and second suffix denotes soliton number (see Appendix A). "a" is a real constant. The condition eq.23 comply with the condition for elastic collision eq.16, given in the previous section. Fig.5 (a,b) shows an instance when the solitons are satisfying 23 as well as the following condition, \( Re(\eta_3) > Re(\eta_1), Re(\eta_2) \) (interchangeable), where \( |\eta_1| \) and \( |\eta_2| \) are comparable. The figure shows the trajectories of three bright solitons in an induced medium (induced by the combined power of three solitons). Two of the solitons of comparable intensities (solitons with \( \eta_1, \eta_2 \)) interact periodically. The remaining soliton, with higher intensity, however, remains isolated and sticks to a straight trajectory until the gain becomes very high. Dark solitons Fig.5(c) maintain nearly a similar profile as that of bright solitons.

Fig.6(a,b) shows another instance of bound soliton interaction, where the solitons satisfy eq.23 as well as follow the condition, \( Re(\eta_1) < Re(\eta_2), Re(\eta_3) \), where \( \eta_1, \eta_2, \eta_3 \) are comparable. In this case two of the solitons of stronger intensities (soliton with \( \eta_2, \eta_3 \)) are seen to interact periodically in this case. The interactions become stronger with gain. The trajectory of the soliton with lower intensity, however remains straight. The dark solitons Fig.6(c), once again maintain nearly a similar profile as that of the bright solitons. Fig.7 displays the third example of bound soliton interactions of three solitons when the soliton components satisfy eq.23 and have nearly same intensities. The bright as well as the dark solitons in this case show no sign of interactions even though the intensity increases due to gain and they maintain a similar straight trajectories.

The above analysis shows that in all the three examples bright soliton interactions and and their trajectories in first component are identical to that in second component. That is, just like scalar solitons the energy of each soliton is conserved in each component.

Case B.

\[ \alpha_{21} = \alpha_{12} = \alpha_{22} = \alpha_{13} = \alpha_{23} = a \]
\[ \alpha_{11} = -a; \quad \text{or} \]
\[ \alpha_{11} = \alpha_{12} = \alpha_{22} = \alpha_{13} = \alpha_{23} = a \quad (24) \]
\[ \alpha_{21} = -a; \]

These conditions also imply that the following parameters become zero, thus
making the three soliton expression simpler (see Appendix A),

\[
\begin{align*}
\delta_2 = \delta_3^* = \delta_3 = \delta_3^* = 0 \\
\gamma_2 = \gamma_3^* = \gamma_3 = \gamma_3^* = 0 \\
\varrho_3 = \varrho_3^* = \varrho_6 = \varrho_6^* = 0 \\
\rho_3 = \rho_3^* = \rho_3 = \rho_3^*; \quad \beta_j = 0 \\
\end{align*}
\]

(25)

Notice that, eq. 24 does not match with any one of the conditions eqs.16-20 for elastic interactions. A similar restriction are imposed on \(\eta_1, \eta_2\) and \(\eta_3\) as earlier but now along with eq. 24. First we consider the case, \(\text{Re}(\eta_3) > \text{Re}(\eta_1), \text{Re}(\eta_2)\) (interchangeable), where \(|\eta_1|\) and \(|\eta_2|\) are comparable. The bound bright solitons, in Fig.8(a) are not only interacting, but forming a breather like localized structure, alternately shifting position between the trajectories of each soliton. Third soliton maintain a straight trajectory until the gain \((e^{\sigma t})\) becomes too high, then all three solitons are subjected to attraction and repulsion. Fig.8(b) showing the soliton trajectories in the second component but in this case the trajectories rather than being identical to the first component are complementary in nature. This suggests that there is an energy sharing process involved in this interaction. The bound dark solitons in Fig.8(c) are also interacting but no energy exchange process is involved. The remaining (non interacting) dark soliton maintains a straight trajectory. Similarly when the condition eq.24 along with the condition, \(\text{Re}(\eta_1) < \text{Re}(\eta_2), \text{Re}(\eta_3)\), where \(|\eta_2|\) and \(|\eta_3|\) are comparable are imposed on the solitons another instance of energy sharing between the components is noticed in Fig.9(a,b). The trajectories of the interacting solitons are similar to that in Fig.6(a,b), but the intercomponent energy sharing is more distinct. When all the solitons are of comparable intensities and they follow 24 we notice an interesting phenomena. Fig. 10(a, b) shows that despite the fact, that the soliton trajectories remain parallel and there is no sign of interaction, but, the energy exchange between the components still occur. The solitons in second component once again happen to be complementary to the solitons in the first component. Dark solitons in Fig.10(c), on the other hand follow straight trajectories similar to that in Fig.7.

Similar to Case B two more cases may be obtained when \((\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{22} = -\alpha_{13} = \alpha_{23} = a)\), or \((\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{22} = \alpha_{13} = -\alpha_{23} = a)\). Then the following parameters become zero,

\[
\begin{align*}
\delta_3 = \delta_3^* = \delta_6 = \delta_6^* = 0 \\
\gamma_3 = \gamma_3^* = \gamma_6 = \gamma_6^* = 0 \\
\varrho_3 = \varrho_3^* = \varrho_3 = \varrho_3^* = 0 \\
\rho_3 = \rho_3^* = \rho_3 = \rho_3^*; \quad \beta_j = 0 \\
\end{align*}
\]

(26)

and \((\alpha_{11} = \alpha_{21} = -\alpha_{12} = \alpha_{22} = \alpha_{13} = \alpha_{23} = a)\), or \((\alpha_{11} = \alpha_{21} = \alpha_{12} = a)\), or \((\alpha_{11} = \alpha_{21} = \alpha_{12} = a)\).
\(-\alpha_{22} = \alpha_{13} = \alpha_{23} = \alpha\) Then the following parameters become zero,

\[
\delta_2 = \delta_5^* = \delta_6 = \delta_6^* = 0 \\
\gamma_2 = \gamma_4^* = \gamma_6 = \gamma_6^* = 0 \\
\varrho_2 = \varrho_4^* = \varrho_6 = \varrho_6^* = 0 \\
\rho_2 = \rho_4^* = \rho_6 = \rho_6^*; \quad \beta_{35} = 0
\]

(27)

Then also we expect a similar trajectories of solitons as described earlier (Case B). Notice that the interactions among the solitons in bound state is medium induced interactions and not a collision arising out of relative velocity of the solitons. However, the inter component energy exchange, a phenomena normally mentioned in the literature for colliding solitons moving with a relative velocity is prevalent also in case of bound solitons.

Interaction of a two soliton complex and an accelerating soliton, on the other hand is more eventful as will be explained using the asymptotic analysis. Here, asymptotic analysis is applied to three soliton solutions, where a bright two soliton complex, \(S^{12}\) (say), is interacting with another soliton \(S^3\) (say), which has a relative velocity with respect to \(S^{12}\). \(S^{12}\) and \(S^3\) remain separated asymptotically.

Thus as \(z \to -\infty\), asymptotic expressions for \(S^{12}\) and \(S^3\) are,

\[
S^{12} = \frac{\alpha^{(j)}_1(t) e^{\theta_1} + \alpha^{(j)}_2(t) e^{\theta_2} + \beta_{21} e^{\theta_1 + \theta_1^* + \theta_2} + \beta_{22} e^{\theta_1 + \theta_2 + \theta_2^* + \theta_2^*} + \gamma_{23} e^{\theta_1 + \theta_1^* + \theta_2} + \delta_{25} e^{\theta_2 + \theta_2^*} + \theta_3 + \theta_3^* - \infty}
\]

\[\theta_3 + \theta_3^* \to -\infty\]

\[S^3 = \frac{\gamma^{(j)}_{12}}{\sqrt{\theta_3}} \text{Sech}(\theta_3 + \theta_3^* + \ln(\frac{\gamma}{\theta_3^*})); \quad \theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to +\infty\]

(28)

and as \(z \to +\infty\), the expressions are

\[
S^{12}_j = \frac{\beta_{23} e^{\theta_1 + \theta_2} + \gamma_{23}^{(j)} e^{\theta_1 + \theta_1^* + \theta_2^*} + \gamma_{23}^{(j)} e^{\theta_1 + \theta_2 + \theta_2^*} + \gamma_{23}^{(j)} e^{\theta_1 + \theta_1^* + \theta_2^*}}{\theta_3 + \theta_3^* \to +\infty}
\]

\[S^3_j = \alpha^{(j)}_{33}(t) \text{Sech}(\theta_3 + \theta_3^* + \ln(\theta_3)); \quad \theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to -\infty\]

(29)

Similarly consider the interaction of a dark two soliton complex, \(X^{12}\) (say) and a third dark soliton \(X^3\), which is moving at a relative speed with respect to \(X^{12}\).
Thus asymptotic expressions for the dark solitons are, as \( z \to -\infty \),

\[
X^{12} = \frac{1 + \gamma_1^{(3)} e^{\theta_1 + \theta_1^*} + \gamma_2^{(3)} e^{\theta_1 + \theta_2^*} + \gamma_4^{(3)} e^{\theta_1 + \theta_4^*} + e^{\theta_2 + \theta_2^*} (\gamma_5^{(3)} + \nu^{(3)} e^{\theta_1 + \theta_1^*})}{(1 + \delta_1 e^{\theta_1 + \theta_1^*} + \delta_2 e^{\theta_1 + \theta_2^*} + \delta_4 e^{\theta_1 + \theta_4^*} + \delta_5 e^{\theta_2 + \theta_2^*} + \delta_6 e^{\theta_2 + \theta_4^*} + \delta_7 e^{\theta_4 + \theta_4^*} + \delta_8 e^{\theta_4 + \theta_2^*} + \delta_9 e^{\theta_4 + \theta_1^*} + \delta_{10} e^{\theta_1 + \theta_2^* + \theta_4^* + \theta_2^*})} \times \chi^{(3)}(t) e^{\varphi^{(3)}}; \quad \theta_3 + \theta_3^* \to -\infty
\]

\[
X^3 = \chi^{(3)}(t) e^{\varphi^{(3)}(3)} + \frac{\varphi_1^{(3)}}{\theta_1} (1 + \frac{\varphi_2^{(3)}}{\theta_2} + \frac{\varphi_3^{(3)}}{\theta_3} + \frac{\varphi_4^{(3)}}{\theta_4} - 1) \tanh(\frac{\theta_3 + \theta_3^* + \ln(\frac{\varphi_1}{\varphi_1^{(3)}})}{2});
\]

\[
\theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to +\infty
\]

and as \( z \to +\infty \),

\[
X^{12} = \frac{\gamma_9^{(3)} + \rho_5^{(3)} e^{\theta_1 + \theta_1^*} + \rho_6^{(3)} e^{\theta_1 + \theta_2^*} + \rho_7^{(3)} e^{\theta_2 + \theta_2^*} + \rho_8^{(3)} e^{\theta_2 + \theta_4^*} + \rho_9^{(3)} e^{\theta_4 + \theta_4^*} + \rho_{10}^{(3)} e^{\theta_4 + \theta_2^*} + \rho_{11}^{(3)} e^{\theta_4 + \theta_1^*} + \rho_{12}^{(3)} e^{\theta_1 + \theta_2^* + \theta_4^* + \theta_2^*})}{(\delta_9 + \delta_5 e^{\theta_1 + \theta_1^*} + \delta_6 e^{\theta_1 + \theta_2^*} + \delta_7 e^{\theta_1 + \theta_4^*} + \delta_8 e^{\theta_2 + \theta_2^*} + \delta_9 e^{\theta_2 + \theta_4^*} + \delta_{10} e^{\theta_4 + \theta_4^*} + \delta_{11} e^{\theta_4 + \theta_2^*} + \delta_{12} e^{\theta_4 + \theta_1^*} + \delta_{13} e^{\theta_1 + \theta_2^* + \theta_4^* + \theta_2^*})} \times \chi^{(3)}(t) e^{\varphi^{(3)}}; \quad \theta_3 + \theta_3^* \to +\infty
\]

\[
X^3 = \chi^{(3)}(t) e^{\varphi^{(3)}(3)} + \frac{\varphi_1^{(3)}}{\theta_1} (1 + \frac{\varphi_2^{(3)}}{\theta_2} + \frac{\varphi_3^{(3)}}{\theta_3} + \frac{\varphi_4^{(3)}}{\theta_4} - 1) \tanh(\frac{\theta_3 + \theta_3^* + \ln(\frac{\varphi_1}{\varphi_1^{(3)}})}{2});
\]

\[
\theta_1 + \theta_1^* + \theta_2 + \theta_2^* \to -\infty
\]

Once again consider, the solitons are such that, the components are satisfying eq. 23. The values of \( \eta_1, \eta_2 \) and \( \eta_3 \) are chosen such that two of them form a comoving bright soliton complex \( S^{12} \) (dark soliton complex, \( X^{12} \) ) and interact with the other bright soliton \( S^3 \) (dark soliton \( X^3 \)). Fig. 11 (a) demonstrate a case where two bright solitons with parallel trajectories interact with an accelerating soliton. First interaction causes a shift in trajectories (phase shift) of both the interacting soliton. The moving soliton then decelerate and interact with the second soliton just when its velocity reached minimum. The point is marked with a bright spot. The maxima of two solitons just get merged at that point. Both the components \( |q_1| \) and \( |q_2| \) show identical profile of solitons. Fig. 11(b) shows the dark soliton interactions under the same conditions as stated in eq. 23 The first interaction causes a normal phase shift. Interestingly at the second interaction point the merging of two solitons is marked with an increase in intensity, which exceeds the background intensity.

Fig. 12 and 13 demonstrate another instance of bound soliton interaction, where two bright comoving soliton complex \( S^{12} \) (dark soliton complex, \( X^{12} \) ) interact with \( S^3 \) (\( X^3 \)), having a relative velocity with respect to \( S^{12} \) (\( X^{12} \)). The soliton components satisfy, \( \alpha_{11} = -\alpha_{21} = \alpha_{12} = -\alpha_{22} = -\frac{1}{2} \alpha_{13} = \alpha_{23} = 1 \), that is, they do not satisfy any one of the conditions eqs.16-20. Fig. 12 (a, b), 13(a, b) show that, interaction brings the two comoving soliton complex nearer effecting a large phase shift. They form a breather like structure as a result of the mutual interactions. In the subsequent interaction the solitons are again separated sufficiently so that mutual interaction between them disappear and this phenomena continue in a periodic manner. The interaction of dark solitons
causes a similar effect on their phase (position), moving the bound solitons closer and farther at regular intervals, Fig. 12 (c), 13(c).

Fig. (14, 15) (a, b, c) shows a special case, where the soliton components ($\alpha_{11} = \alpha_{12} = \alpha_{23} = 1; \alpha_{21} = \alpha_{22} = \alpha_{13} = 0$) satisfy the conditions of an elastic collision, namely eq.19. The values of $\eta$ are chosen as shown in the figure. In this interactions soliton ($S^3$) completely disappears from component $|q_1|$ and the soliton complex $S^{12}$ disappears from component $|q_2|$. However, traces of interactions are noticed in their phase shifts. Dark soliton on the other hand exhibit a regular interaction of an accelerating soliton with mutually interacting bound solitons.

Finally we bring an interesting phenomena where the soliton interaction causes a very large phase shift. The component of the solitons are satisfying eq. 16, that is the interaction is an elastic one. Fig.16 17(c) show that during the phase change dark solitons shows a dramatic variation of intensities, maintaining a unique envelope shape. This produces equally spaced fringes of bright and dark lines. The values of $\eta'$ are as shown in the figure. Bright soliton Fig.16 (a, b), on the other hand demonstrate that the interaction pushed solitons out of the frame, they reappear only during next cycle of interaction.

5 Conclusion

Three soliton solution of a Lax integrable coupled NLS equation with variable coefficients is obtained using Hirota’s bilinear method. Soliton complexes are studied using asymptotic as well as graphical analysis. It is shown that two co-moving solitons of sufficient amplitude are subjected to periodic attraction and repulsion. The energy sharing phenomena which was earlier reported for colliding Manakov solitons has been noticed also in non colliding comoving bound state solitons. A correlation between the bound state soliton interactions and the soliton parameters is shown through graphical analysis. One of the interesting phenomena noticed in three soliton bound state interaction is the long phase shift of solitons. The same is explained on the basis of the asymptotic analysis. Such observations have not been reported earlier either for two soliton or for three soliton. This allows us to draw a conclusion that the intercomponent energy exchange soliton collision is not a necessary condition. The analysis done in this paper are expected to be useful for the development of all optical logic gates, soliton communication systems as well as to understand bound soliton complexes.

Appendix

Bright 3-soliton is in components $|q_1|$ and $|q_2|$ and dark 3-soliton is in component $|q_3|$.

$$q_j = \frac{g_1^{(j)} + g_3^{(j)} + g_5^{(j)}}{1 + f_2 + f_4 + f_6}; \quad (for \quad j = 1, 2)$$
\[ g_3 = \frac{g_0^{(3)} (1 + g_2^{(3)} + g_4^{(3)} + g_6^{(3)})}{1 + f_2 + f_4 + f_6}; \]

where,

\[ g_0^{(3)} = \chi^{(3)}(t) e^{\delta(t)}; \quad g_1^{(j)} = \sum_{k=1}^{3} c_k^{(j)}(t) e^{\delta_k}; \]

\[ g_2^{(3)} = \gamma_1^{(3)} e^\delta + \gamma_2^{(3)} e^\theta_1 + \gamma_3^{(3)} e^\theta_2 + \gamma_4^{(3)} e^\theta_3 + \gamma_5^{(3)} e^\theta_4 + \gamma_6^{(3)} e^\theta_5 + \gamma_7^{(3)} e^\theta_6; \]

\[ g_3^{(j)} = \beta_{j1} e^\delta + \beta_{j2} e^\theta_1 + \beta_{j3} e^\theta_2 + \beta_{j4} e^\theta_3 + \beta_{j5} e^\theta_4 + \beta_{j6} e^\theta_5 + \beta_{j7} e^\theta_6; \]

\[ g_4^{(3)} = \rho_1^{(3)} e^\delta + \rho_2^{(3)} e^\theta_1 + \rho_3^{(3)} e^\theta_2 + \rho_4^{(3)} e^\theta_3 + \rho_5^{(3)} e^\theta_4 + \rho_6^{(3)} e^\theta_5 + \rho_7^{(3)} e^\theta_6; \]

\[ g_5^{(j)} = \xi_1^{(j)} e^\delta + \xi_2^{(j)} e^\theta_1 + \xi_3^{(j)} e^\theta_2 + \xi_4^{(j)} e^\theta_3 + \xi_5^{(j)} e^\theta_4 + \xi_6^{(j)} e^\theta_5 + \xi_7^{(j)} e^\theta_6; \]

\[ g_6^{(3)} = \nu_1^{(3)} e^\delta + \nu_2^{(3)} e^\theta_1 + \nu_3^{(3)} e^\theta_2 + \nu_4^{(3)} e^\theta_3 + \nu_5^{(3)} e^\theta_4 + \nu_6^{(3)} e^\theta_5 + \nu_7^{(3)} e^\theta_6; \]

\[ f_2 = \delta_1 e^\delta + \delta_2 e^\theta_1 + \delta_3 e^\theta_2 + \delta_4 e^\theta_3 + \delta_5 e^\theta_4 + \delta_6 e^\theta_5 + \delta_7 e^\theta_6; \]

\[ f_4 = \phi_1 e^\delta + \phi_2 e^\theta_1 + \phi_3 e^\theta_2 + \phi_4 e^\theta_3 + \phi_5 e^\theta_4 + \phi_6 e^\theta_5 + \phi_7 e^\theta_6; \]

\[ f_6 = \chi e^\delta + \chi e^\theta_1 + \chi e^\theta_2 + \chi e^\theta_3; \]
\[
\begin{align*}
\theta_1 &= \frac{i}{2} \eta_3^2 \int D(t) dt + \eta_1 z - i \int \lambda dt; \quad \theta_2 = \frac{i}{2} \eta_2^2 \int D(t) dt + \eta_2 z - i \int \lambda dt \\
\theta_3 &= \frac{i}{2} \eta_3^2 \int D(t) dt + \eta_3 z - i \int \lambda dt \\
\phi^{(3)} &= -i \zeta_3^2 \int D(t) dt + i \zeta_3 z - i \int \lambda dt \\
\lambda &= -\gamma R(t) \sigma \left| \chi^{(3)}(t) \right|^2 \\
\end{align*}
\]

\[
\begin{align*}
\alpha_1^{(j)}(t) &= \alpha_1 \sqrt{\frac{D(t)}{\gamma R(t)}}; \quad \alpha_2^{(j)}(t) = \alpha_2 \sqrt{D(t)} \\
\alpha_3^{(j)}(t) &= \alpha_3 \sqrt{\frac{D(t)}{\gamma R(t)}}; \quad \chi^{(3)}(t) = \chi \sqrt{\frac{D(t)}{\gamma R(t)}} \\
\end{align*}
\]

\[
\begin{align*}
\beta_{j1} &= (\eta_1 - \eta_2) \left\{ \frac{\alpha_1(t) t \delta_2}{\eta_1 + \eta_1^*} - \frac{\alpha_2(t) t \delta_1}{\eta_2 + \eta_1^*} \right\} \\
\beta_{j2} &= (\eta_1 - \eta_2) \left\{ \frac{\alpha_2(t) t \delta_1}{\eta_1 + \eta_2^*} - \frac{\alpha_2(t) t \delta_2}{\eta_2 + \eta_2^*} \right\} \\
\beta_{j3} &= (\eta_1 - \eta_2) \left\{ \frac{\alpha_1(t) t \delta_1}{\eta_1 + \eta_1^*} - \frac{\alpha_2(t) t \delta_1}{\eta_2 + \eta_1^*} \right\} \\
\beta_{j4} &= (\eta_1 - \eta_2) \left\{ \frac{\alpha_1(t) t \delta_2}{\eta_1 + \eta_1^*} - \frac{\alpha_2(t) t \delta_2}{\eta_2 + \eta_1^*} \right\} \\
\beta_{j5} &= (\eta_1 - \eta_3) \left\{ \frac{\alpha_1(t) t \delta_3}{\eta_1 + \eta_2^*} - \frac{\alpha_1(t) t \delta_2}{\eta_3 + \eta_1^*} \right\} \\
\beta_{j6} &= (\eta_1 - \eta_3) \left\{ \frac{\alpha_1(t) t \delta_2}{\eta_1 + \eta_2^*} - \frac{\alpha_1(t) t \delta_3}{\eta_3 + \eta_2^*} \right\} \\
\beta_{j7} &= (\eta_2 - \eta_3) \left\{ \frac{\alpha_2(t) t \delta_3}{\eta_2 + \eta_1^*} - \frac{\alpha_2(t) t \delta_3}{\eta_2 + \eta_2^*} \right\} \\
\beta_{j8} &= (\eta_2 - \eta_3) \left\{ \frac{\alpha_2(t) t \delta_3}{\eta_2 + \eta_1^*} - \frac{\alpha_2(t) t \delta_3}{\eta_3 + \eta_2^*} \right\} \\
\beta_{j9} &= (\eta_2 - \eta_3) \left\{ \frac{\alpha_2(t) t \delta_3}{\eta_2 + \eta_1^*} - \frac{\alpha_1(t) t \delta_3}{\eta_3 + \eta_3^*} \right\} \\
\end{align*}
\]
\[ \begin{aligned}
\rho_1^{(3)} &= \rho_1^{(3)}(\xi + i\eta_1)(\xi + i\eta_2); & \rho_2^{(3)} &= \rho_2^{(3)}(\xi + i\eta_1)(\xi + i\eta_2) \\
\rho_3^{(3)} &= \rho_3^{(3)}(\xi + i\eta_1)(\xi + i\eta_2); & \rho_4^{(3)} &= \rho_4^{(3)}(\xi + i\eta_1)(\xi + i\eta_3) \\
\rho_5^{(3)} &= \rho_5^{(3)}(\xi + i\eta_1)(\xi + i\eta_3); & \rho_6^{(3)} &= \rho_6^{(3)}(\xi + i\eta_1)(\xi + i\eta_3) \\
\rho_7^{(3)} &= \rho_7^{(3)}(\xi + i\eta_2)(\xi + i\eta_1); & \rho_8^{(3)} &= \rho_8^{(3)}(\xi + i\eta_2)(\xi + i\eta_3) \\
\rho_9^{(3)} &= \rho_9^{(3)}(\xi + i\eta_2)(\xi + i\eta_3); & \\
\gamma_1^{(3)} &= \frac{\gamma^R(t)\left(\sum_{j=1}^{2}\sigma_j\left|\alpha_j^{(3)}(t)\right|^2\right)}{(\eta_1 + \eta_1^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_1^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_2^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_1 + i\eta_1)(\zeta_1 - i\eta_1^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_1 + \eta_2^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_3^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_2 + \eta_1^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_4^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_2 + \eta_3^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_3^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_5^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_3 + \eta_1^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_3^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_6^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_3 + \eta_2^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_3^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_7^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_3 + \eta_2^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_3^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_8^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_3 + \eta_2^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_3^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)}; \\
\gamma_9^{(3)} &= \frac{\gamma^R(t)\left(\sigma_1\alpha_2^{(1)*}(t)\alpha_2^{(1)}(t) + \sigma_2\alpha_2^{(2)*}(t)\alpha_2^{(2)}(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_2^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2\right)}{(\eta_3 + \eta_2^*)^2(D(t)(\zeta_3 + i\eta_1)(\zeta_3 - i\eta_3^*) + \gamma R(t)\sigma_3 |\chi^{(3)}(t)|^2)};
\end{aligned} \]
\begin{align*}
\zeta_1^{(j)} &= \frac{\alpha_1^{(j)}(t)\delta_1^2(\eta_1 - \eta_2)(\eta_1 - \eta_3)}{(\eta_1 + \eta_1^*)(\eta_1 + \eta_2^*)} - \frac{\alpha_2^{(j)}(t)\delta_2^2(\eta_1 - \eta_2)(\eta_2 - \eta_3)}{(\eta_2 + \eta_1^*)(\eta_2 + \eta_3^*)} \\
\zeta_2^{(j)} &= \frac{\alpha_1^{(j)}(t)\delta_3^2(\eta_1 - \eta_3)}{(\eta_3 + \eta_1^*)(\eta_3 + \eta_2^*)} - \frac{\alpha_2^{(j)}(t)\delta_5^2(\eta_1 - \eta_2)(\eta_2 - \eta_3)}{(\eta_2 + \eta_1^*)(\eta_2 + \eta_3^*)} \\
\zeta_3^{(j)} &= \frac{\alpha_1^{(j)}(t)\delta_6^2(\eta_1 - \eta_3)}{(\eta_3 + \eta_2^*)(\eta_3 + \eta_3^*)} - \frac{\alpha_2^{(j)}(t)\delta_6^2(\eta_1 - \eta_2)(\eta_2 - \eta_3)}{(\eta_2 + \eta_2^*)(\eta_2 + \eta_3^*)}
\end{align*}

\begin{equation*}
\Upsilon = \frac{|(\eta_1 - \eta_2)(\eta_2 - \eta_3)(\eta_1 - \eta_3)|^2}{|(\eta_1 + \eta_2)(\eta_1 + \eta_3)(\eta_2 + \eta_3)|^2} \\
\times (\delta_1 \delta_5 \delta_6 + M + M^*) \\
- \frac{\delta_1 |\delta_6|^2|\eta_2 + \eta_3|^2}{(\eta_2 + \eta_2^*)(\eta_3 + \eta_3^*)} - \frac{\delta_5 |\delta_2|^2|\eta_1 + \eta_3|^2}{(\eta_1 + \eta_1^*)(\eta_2 + \eta_2^*)} - \frac{\delta_6 |\delta_3|^2|\eta_1 + \eta_3|^2}{(\eta_1 + \eta_1^*)(\eta_3 + \eta_3^*)}
\end{equation*}

\begin{equation*}
M = \delta_2 \delta_6 \delta_5^* \frac{(\eta_2 + \eta_3^*)(\eta_3 + \eta_1^*)(\eta_1 + \eta_2^*)}{(\eta_1 + \eta_1^*)(\eta_2 + \eta_2^*)(\eta_3 + \eta_3^*)}; \quad \nu^{(3)} = \frac{\Upsilon(\xi + i\eta_1)(\xi + i\eta_2)(\xi + i\eta_3)}{(\xi - i\eta_1^*)(\xi - i\eta_2^*)(\xi - i\eta_3^*)}
\end{equation*}

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