Abstract

After providing an overview of $\theta$-expansions introduced by Chakraborty and Rao, we focus on the Gauss-Kuzmin problem for this new transformation. Actually, we complete our study on these expansions by proving a two-dimensional Gauss-Kuzmin theorem. More exactly, we obtain such a theorem related to the natural extension of the associated measure-dynamical system. Finally, we derive explicit lower and upper bounds of the error term which provide interesting numerical calculations for the convergence rate involved.

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1 Introduction

Chakraborty and Rao [1] have introduced the continued fraction expansion of a number in terms of an irrational $\theta \in (0,1)$. This new expansion of
positive reals is called $\theta$-expansion. We mention that the case $\theta = 1$ refers to regular continued fraction (RCF) expansions. The study initiated by Chakraborty and Rao on the analogous transformation of the Gauss map, was completed by Chakraborty and Dasgupta. Actually, in [2] was identified the absolutely continuous invariant probability measure of this new transformation only in the particular case $\theta^2 = 1/m, \ m \in \mathbb{N}_+$.

It is only recently that Sebe and Lascu [13] proved the first Gauss-Kuzmin theorem for $\theta$-expansions, applying the method of random systems with complete connections (RSCC) by Iosifescu and Grigorescu [6]. Following the treatment in the case of the RCF, the Gauss-Kuzmin problem for this new transformation can be approached in terms of the associated Perron-Frobenius operator under the invariant measure induced by the limit distribution function. Moreover, using a Wirsing type approach Sebe [12] obtained a near-optimal solution for the Gauss-Kuzmin problem. The strategy was to restrict the domain of the Perron-Frobenius operator to the Banach space of all functions which have a continuous derivative on $[0, \theta]$.

The aim of this paper is to show a two-dimensional Gauss-Kuzmin theorem for $\theta$-expansions. Note that in the literature there are known similar results for other types of expansions (see [3, 5, 4, 10, 11]).

The paper is organized as follows. In the next section we gather prerequisites needed to prove our results in sections 3 and 4. More exactly, in Section 3 we obtain a Gauss-Kuzmin theorem related to the natural extension [9] of the measure-dynamical system corresponding to these expansions. In Section 4 we try to get close to the optimal convergence rate. Here, the characteristic properties of the Perron-Frobenius operator on the Banach space of functions of bounded variations allows us to derive explicit lower and upper bounds of the error term which provide a more refined estimate of the convergence rate involved. In the last section we conclude by giving numerical calculations.

2 Prerequisites

For a fixed $\theta \in (0, 1)$, Chakraborty and Rao [1] showed that any $x \in (0, \theta)$ can be written in the form

$$x = \frac{1}{a_1 \theta + \frac{1}{a_2 \theta + \frac{1}{a_3 \theta + \cdots}}} := [a_1 \theta, a_2 \theta, a_3 \theta, \ldots], \quad (2.1)$$
which is called the $\theta$-expansion of $x$. Here $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$. Such $a_n$’s are called $\theta$-expansion digits and they are obtained using the transformation

$$T_\theta : [0, \theta] \to [0, \theta]; \quad T_\theta(x) := \begin{cases} \frac{1}{x} - \theta \lfloor \frac{1}{x\theta} \rfloor & \text{if } x \in (0, \theta], \\ 0 & \text{if } x = 0. \end{cases}$$

Thus, if we define the quantized index map $\eta : [0, \theta] \to \mathbb{N} := \{0, 1, 2, \ldots\}$ by

$$\eta(x) := \begin{cases} \lfloor \frac{1}{x\theta} \rfloor & \text{if } x \neq 0, \\ \infty & \text{if } x = 0 \end{cases}$$

then the sequence $(a_n)_{n \in \mathbb{N}_+}$ in (2.1) is obtained as follows:

$$a_n(x) = \eta(T_\theta^{n-1}(x)), \quad n \geq 1,$$

with $T_\theta^0(x) = x$.

Figure 1: The transformation $T_\theta$ for $\theta = \frac{1}{\sqrt{6}}$

This new expansion of positive reals, different from the regular continued fraction expansion, was also studied in $[2, 8, 12, 13]$. 

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In [1] it was shown that $\theta$-expansions are convergent. To this end, define real functions $p_n(x)$ and $q_n(x)$, for $n \in \mathbb{N}_+$, by

$$p_n(x) := a_n(x)\theta p_{n-1}(x) + p_{n-2}(x), \quad (2.5)$$

$$q_n(x) := a_n(x)\theta q_{n-1}(x) + q_{n-2}(x), \quad (2.6)$$

with $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$ and $q_0(x) := 1$. It follows that $p_n(x)/q_n(x) = [a_1 \theta, a_2 \theta, \ldots, a_n \theta]$ which is called the $n$-th order convergent of $x \in [0, \theta]$. One easily shows that for any $x \in [0, \theta]$ it follows

$$|x - p_n(x)/q_n(x)| \leq \frac{1}{q_n(x)q_{n+1}(x)} \leq \frac{1}{(1 + \theta^2)^{2[n/2]} \theta^2}, \quad n \in \mathbb{N}_+. \quad (2.7)$$

Then $p_n(x)/q_n(x) \to x, n \to \infty$.

In [1], Chakraborty and Rao showed that for $\theta^2 = 1/m$, $m \in \mathbb{N}_+$, $T_\theta$ is ergodic with respect to the measure $\gamma_\theta$ defined by

$$\gamma_\theta(A) := \frac{1}{\log (1 + \theta^2)} \int_A \frac{\theta dx}{1 + \theta x}, \quad A \in \mathcal{B}_{[0,\theta]}.$$

(2.8)

Let us note that $\gamma_\theta$ is $T_\theta$-invariant, that is, $\gamma_\theta(T_\theta^{-1}(A)) = \gamma_\theta(A)$ for any $A \in \mathcal{B}_{[0,\theta]}$. Therefore, $(a_n)_{n \in \mathbb{N}_+}$ is a strictly stationary sequence on $(I, \mathcal{B}_{[0,\theta]}, \gamma_\theta)$ and $a_n \geq m$ for any $m \in \mathbb{N}_+$.

Put $N_m := \{m, m+1, \ldots\}, m \in \mathbb{N}_+$. For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}_m^n$ we will say that

$$I(i^{(n)}) = \{x \in [0, \theta] : a_k(x) = i_k \text{ for } k = 1, \ldots, n\} \quad (2.9)$$

is the $n$-th order cylinder and make the convention that $I(i^{(0)}) = [0, \theta]$. For example, for any $i \in \mathbb{N}_m$ we have

$$I(i) = \{x \in [0, \theta] : a_1(x) = i\} = \left\{ \frac{1}{(i+1)\theta}, \frac{1}{i\theta} \right\}. \quad (2.10)$$

2.1 Natural extension, extended random variables and Perron-Frobenius operators

Let $m \in \mathbb{N}_+$ and an irrational $\theta \in (0, 1)$ with $\theta^2 = 1/m$. In this section, we introduce the natural extension $T_\theta$ of $T_\theta$ in (2.2) and its extended random variables according to Chap. 1.3 of [7], and we consider the Perron-Frobenius operator of $T_\theta$. 

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2.1.1 Natural extension

Let \(( [0, \theta], B_{[0, \theta]}, T_\theta) \) be as in Section 2.1. Define \((u_i)_{i \in \mathbb{N}_m}\) by

\[
u_i : [0, \theta] \to [0, \theta]; \quad u_i(x) := \frac{1}{x + i \theta}, \quad x \in [0, \theta]. \tag{2.11}\]

For each \(i \in \mathbb{N}_m\), \(u_i\) is a right inverse of \(T_\theta\), that is,

\[
(T_\theta \circ u_i)(x) = x, \quad \text{for any } x \in [0, \theta]. \tag{2.12}\]

Furthermore, if \(\eta(x) = i\), then \((u_i \circ T_\theta)(x) = x\) where \(\eta\) is as in (2.3).

**Definition 2.1.** The natural extension \(\left( [0, \theta]^2, B_{[0, \theta]}, \overline{T_\theta} \right)\) of \(( [0, \theta], B_{[0, \theta]}, T_\theta)\) is the transformation \(\overline{T_\theta}\) of the square space \(\left( [0, \theta]^2, B_{[0, \theta]}^2 \right) := \left( [0, \theta], B_{[0, \theta]} \right) \times \left( [0, \theta], B_{[0, \theta]} \right)\) defined as follows [9]:

\[
\overline{T_\theta} : [0, \theta]^2 \to [0, \theta]^2; \quad \overline{T_\theta}(x, y) := (T_\theta(x), u_{\eta(x)}(y)), \quad (x, y) \in [0, \theta]^2. \tag{2.13}\]

From (2.12), we see that \(\overline{T_\theta}\) is bijective on \([0, \theta]^2\) with the inverse

\[
(\overline{T_\theta})^{-1}(x, y) = (u_{\eta(y)}(x), T_\theta(y)), \quad (x, y) \in [0, \theta]^2. \tag{2.14}\]

Iterations of (2.13) and (2.14) are given as follows for each \(n \geq 2\):

\[
(T_\theta)^n(x, y) = (T_\theta^n(x), [x_n, x_{n-1}, \ldots, x_2, x_1 + y]), \tag{2.15}\]

\[
(\overline{T_\theta})^{-n}(x, y) = ([y_n, y_{n-1}, \ldots, y_2, y_1 + x], T_\theta^n(y)) \tag{2.16}\]

where \(x_i := \eta(T_\theta^{i-1}(x))\) and \(y_i := \eta(T_\theta^{i-1}(y))\) for \(i = 1, \ldots, n\).

For \(\gamma_\theta\) in (2.8), define its extended measure \(\tilde{\gamma}_\theta\) on \([0, \theta]^2, B_{[0, \theta]}^2\) as

\[
\tilde{\gamma}_\theta(B) := \frac{1}{\log(1 + \theta^2)} \int_B \frac{dxdy}{(1 + xy)^2}, \quad B \in B_{[0, \theta]^2}. \tag{2.17}\]

Then \(\tilde{\gamma}_\theta(A \times [0, \theta]) = \gamma_\theta([0, \theta] \times A) = \gamma_\theta(A)\) for any \(A \in B_{[0, \theta]}\).

The measure \(\tilde{\gamma}_\theta\) is preserved by \(\overline{T_\theta}\) [13], i.e., \(\tilde{\gamma}_\theta((\overline{T_\theta})^{-1}(B)) = \tilde{\gamma}_\theta(B)\) for any \(B \in B_{[0, \theta]}^2\). Since \(\overline{T_\theta}\) is invertible on \([0, \theta]^2\), the last equation is equivalent to

\[
\tilde{\gamma}_\theta(\overline{T_\theta}(B)) = \tilde{\gamma}_\theta(B), \quad \text{for any } B \in B_{[0, \theta]}^2. \tag{2.18}\]
2.1.2 Extended random variables

Define the projection \( E : [0, \theta]^2 \to [0, \theta] \) by \( E(x, y) := x \). With respect to \( T_\theta \) in (2.2), define extended incomplete quotients \( \overline{a}_l(x, y) \), \( l \in \mathbb{Z} := \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \) at \((x, y) \in [0, \theta]^2\) by

\[
\overline{a}_l(x, y) := (\eta \circ E)((T_\theta)^{-l}(x, y)), \quad l \in \mathbb{Z}. \tag{2.19}
\]

Remark that \( \overline{a}_l(x, y) \) in (2.19) is also well-defined for \( l \leq 0 \) because \( T_\theta \) is invertible. For any \( n \in \mathbb{N}_+ \) and \((x, y) \in [0, \theta]^2\), by (2.15) and (2.16), we have

\[
\overline{a}_n(x, y) = x_n, \quad \overline{a}_0(x, y) = y_1, \quad \overline{a}_{-n}(x, y) = y_{n+1}, \tag{2.20}
\]

where we use notations in (2.15) and (2.16).

Since \( \gamma_\theta \) is preserved by \( T_\theta \), the doubly infinite sequence \((\overline{a}_l(x, y))_{l \in \mathbb{Z}}\) is strictly stationary (i.e., its distribution is invariant under a shift of the indices) under \( \gamma_\theta \). The stochastic property of \((\overline{a}_l)_{l \in \mathbb{Z}}\) follows from the fact that

\[
\gamma_\theta([0, x] \times [0, \theta] | \overline{a}_0, \overline{a}_{-1}, \ldots) = \frac{(a\theta + 1)x}{(ax + 1)\theta} \quad \text{\(\gamma_\theta\)-a.s.}, \tag{2.21}
\]

for any \( x \in [0, \theta] \), where \( a := [\overline{a}_0, \overline{a}_{-1}, \ldots] \) with \( \overline{a}_l := \overline{a}_l(x, y) \) for \( l \in \mathbb{Z} \) and \((x, y) \in [0, \theta]^2\). Hence

\[
\gamma_\theta(\overline{a}_1 = i | \overline{a}_0, \overline{a}_{-1}, \ldots) = P_i(a) \quad \text{\(\gamma_\theta\)-a.s.}, \tag{2.22}
\]

where

\[
P_i(x) := \frac{x \theta + 1}{(x + i\theta)(x + (i + 1)\theta)}. \tag{2.23}
\]

The strict stationarity of \((\overline{a}_l)_{l \in \mathbb{Z}}\) under \( \gamma_\theta \) implies that

\[
\gamma_\theta(\overline{a}_{l+1} = i | \overline{a}_l, \overline{a}_{l-1}, \ldots) = P_i(a) \quad \text{\(\gamma_\theta\)-a.s.} \tag{2.24}
\]

for any \( i \in \mathbb{N}_m \) and \( l \in \mathbb{Z} \).

The last equation emphasizes that \((\overline{a}_l)_{l \in \mathbb{Z}}\) is an infinite-order-chain in the theory of dependence with complete connections (see [6], Section 5.5).

Motivated by (2.21), we shall consider the one-parameter family \( \{ \gamma_{\theta, a} : a \in [0, \theta] \} \) of (conditional) probability measures on \(([0, \theta], \mathcal{B}_{[0, \theta]})\) defined by their distribution functions

\[
\gamma_{\theta, a}([0, x]) := \frac{(a\theta + 1)x}{(ax + 1)\theta}, \quad x \in [0, \theta], \quad a \in [0, \theta]. \tag{2.25}
\]

Note that \( \gamma_{\theta, 0} = \lambda_\theta \).
Let $a_n$'s be as in (2.4). For each $a \in [0, \theta]$, define $(s_{n,a})_{n \in \mathbb{N}_+}$ by
\[ s_{0,a} := a, \quad s_{n,a} := \frac{1}{a_n \theta + s_{n-1,a}}, \quad n \in \mathbb{N}_+. \]
Then we have
\[ s_{1,a} = \frac{1}{a_1 \theta + a}, \quad s_{n,a} = [a_n \theta, \ldots, a_2 \theta, a_1 \theta + a], \quad n \geq 2. \]
Note that
\[ \gamma_{\theta,a}(A|a_1, \ldots, a_n) = \gamma_{\theta,s_{n,a}}(T^n_\theta(A)), \]
for all $a \in [0, \theta]$, $A \in \sigma(a_{n+1}, \ldots)$ and $n \in \mathbb{N}_+$. In particular, it follows that for any $a \in [0, \theta]$
\[ \gamma_{\theta,a}(T^n_\theta < x|a_1, \ldots, a_n) = \frac{(s_{n,a} \theta + 1)x}{(s_{n,a} x + 1) \theta} \quad (2.26) \]
for any $x \in [0, \theta]$, $n \in \mathbb{N}_+$.

### 2.1.3 Perron-Frobenius operator of $T_\theta$

Let $([0, \theta], B_\theta, \gamma_\theta, T_\theta)$ be as in Section 2.1. and let $L^1([0, \theta], \gamma_\theta) := \{ f : [0, \theta] \to \mathbb{C} : \int_0^\theta |f|d\gamma_\theta < \infty \}$. The Perron-Frobenius operator of $([0, \theta], B|_{[0, \theta]}, \gamma_\theta, T_\theta)$ is defined as the bounded linear operator $U$ on the Banach space $L^1([0, \theta], \gamma_\theta)$ such that the following holds [13]:
\[ Uf(x) = \sum_{i \in \mathbb{N}_m} P_i(x) f(u_i(x)), \quad f \in L^1([0, \theta], \gamma_\theta) \quad (2.27) \]
where $P_i$ and $u_i$ are as in (2.23) and (2.11), respectively.

For a function $f : [0, \theta] \to \mathbb{C}$, define the variation $\text{var}_A f$ of $f$ on a subset $A$ of $[0, \theta]$ by
\[ \text{var}_A f := \sup \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)|, \quad (2.28) \]
where the supremum being taken over $t_1 < \cdots < t_k$, $t_i \in A$, $i = 1, \ldots, k$ and $k \geq 2$. We write simply $\text{var} f$ for $\text{var}_{[0, \theta]} f$. Let $BV([0, \theta]) := \{ f : [0, \theta] \to \mathbb{C} : \text{var} f < \infty \}$ and let $L^\infty([0, \theta])$ denote the collection of all bounded measurable functions $f : [0, \theta] \to \mathbb{C}$. It is known that $BV([0, \theta]) \subset L^\infty([0, \theta]) \subset L^1([0, \theta], \gamma_\theta)$. In [3] we showed that
\[ \text{var}(Uf) \leq \frac{1}{m+1} \text{var} f \quad (2.29) \]
where $U$ is as in (2.27) and $f \in BV([0, \theta])$ is a real-valued function.

If $f \in L^\infty([0, \theta])$ define the linear functional $U_\infty$ by

$$U_\infty : L^\infty([0, \theta]) \to \mathbb{C}; \quad U_\infty f = \int_0^\theta f(x) \gamma_\theta(dx). \quad (2.30)$$

Then we have

$$U_\infty U^n f = U_\infty f \quad \text{for any } n \in \mathbb{N}_+. \quad (2.31)$$

**Proposition 2.2.** For any $f \in BV([0, \theta])$ and for all $n \in \mathbb{N}_+$ we have

$$\text{var } U^n f \leq \frac{1}{(m + 1)^n} \cdot \text{var } f, \quad (2.32)$$

$$|U^n f - U_\infty f| \leq \frac{1}{(m + 1)^n} \cdot \text{var } f. \quad (2.33)$$

**Proof.** Note that for any $f \in BV([0, \theta])$ and $u \in [0, \theta]$ we have

$$|f(u)| - \left| \int_0^\theta f(x) \gamma_\theta(dx) \right| \leq \left| f(u) - \int_0^\theta f(x) \gamma_\theta(dx) \right| = \left| \int_0^\theta (f(u) - f(x)) \gamma_\theta(dx) \right| \leq \text{var } f,$$

whence

$$|f| \leq \left| \int_0^\theta f(x) \gamma_\theta(dx) \right| + \text{var } f, \quad f \in BV([0, \theta]). \quad (2.34)$$

Finally, (2.31) and (2.34) imply that

$$|U^n f - U_\infty f| \leq \text{var } (U^n f - U_\infty f) = \text{var } U^n f.$$

for all $n \in \mathbb{N}$ and $f \in BV([0, \theta])$, which leads to (2.33). \qed

By induction with respect to $n \in \mathbb{N}$ we get

$$U^n f(x) = \sum_{i_1, \ldots, i_n \in \mathbb{N}_m} P_{i_1 \ldots i_n}(x) f(u_{i_n \ldots i_1}(x)), \quad x \in [0, \theta]$$

where

$$u_{i_n \ldots i_1} = u_{i_n} \circ \ldots \circ u_{i_1} \quad (2.35)$$

$$P_{i_1 \ldots i_n}(x) = P_{i_1}(x)P_{i_2}(u_{i_1}(x)) \ldots P_{i_n}(u_{i_{n-1} \ldots i_1}(x)), \quad n \geq 2. \quad (2.36)$$
Here the functions \( u_i \) and \( P_i \) are defined in (2.11) and (2.23), respectively, for all \( i \in \mathbb{N}_m \).

Putting

\[
\frac{p_n(i_1, \ldots, i_n)}{q_n(i_1, \ldots, i_n)} = [i_1 \theta, \ldots, i_n \theta], \quad n \in \mathbb{N}_+,
\]

for arbitrary indeterminates \( i_1, \ldots, i_n \), we get

\[
P_{i_1\ldots i_n}(a) = \frac{1 + a \theta}{q_n(i_2, \ldots, i_n)(a + i_1 \theta) + p_{n-1}(i_2, \ldots, i_n)} \times \frac{1}{q_n(i_2, \ldots, i_n, m)(a + i_1 \theta) + p_n(i_2, \ldots, i_n, m)} \quad (2.37)
\]

for all \( n \geq 2, i_n \in \mathbb{N}_m \), and \( a \in [0, \theta] \).

3 Gauss-Kuzmin theorem related to the natural extension

In this section a Gauss-Kuzmin theorem for \( ([0, \theta], \gamma_\theta, T_\theta) \) is given.

First we give a modified version of the Gauss-Kuzmin theorem for \( T_\theta \) proved in [8]. Then we show some important results used in the proof of the main theorem.

3.1 Gauss-Kuzmin theorems for \( T_\theta \) and \( \overline{T}_\theta \)

**Theorem 3.1.** (A Gauss-Kuzmin theorem for \( T_\theta \)) Let \( ([0, \theta], \gamma_\theta, T_\theta) \) as in (2.2) and (2.8). If \( \lambda_\theta \) is the Lebesgue measure on \( [0, \theta] \), then there exists a constant \( 0 < q < \theta \) such that for any \( A \in \mathcal{B}_{[0, \theta]} \) we have

\[
|\lambda_\theta(T_\theta^{-n}(A)) - \gamma_\theta(A)| < C \lambda_\theta(A)O(q^n)
\]

where \( C \) is an universal constant.

**Proof.** In [13] (Prop.14(ii)) we show that \( \mu \left( (T_\theta)^{-n}(A) \right) = \int_A U^n f(x) d\gamma_\theta(x) \) where \( \mu \) is a probability measure on \( ([0, \theta], \mathcal{B}_{[0, \theta]}) \) absolutely continuous with respect to the Lebesgue measure \( \lambda_\theta \), and \( f(x) := (\log(1 + \theta^2))^{1+\theta x} h(x) \) with \( h := d\mu/d\lambda_\theta \) a.e. in \( [0, \theta] \). In the special case \( \mu = \lambda_\theta \) we obviously have

\[
\lambda_\theta(T_\theta^{-n}(A)) = \frac{\theta}{\log(1 + \theta^2)} \int_A \frac{U^n f(x)}{1 + \theta x} dx
\]

(3.2)
with \( f(x) := (\log(1 + \theta^2))^{\frac{1 - \theta^2}{\theta^2}}, \ x \in [0, \theta] \). Thus, from (2.30) we have that \( U_{\infty}f = 1 \). Therefore,

\[
\gamma_{\theta}(A) = \frac{\theta}{\log(1 + \theta^2)} \int_A U_{\infty}f \frac{dx}{1 + \theta x}.
\]  

(3.3)

Using \[13\] (Prop.27) it follows that there exist two positive constants \( q < \theta \) and \( K \) such that

\[
\|U^n f - U_{\infty}f\|_L \leq Kq^n \|f\|_L, \quad f \in L([0, \theta]), n \in \mathbb{N}_+
\]  

(3.4)

where \( L([0, \theta]) \) denote the Banach space of all complex-valued Lipschitz continuous functions on \([0, \theta]\) with the following norm:

\[
\|f\|_L := \sup_{x \in [0, \theta]} |f(x)| + \sup_{x' \neq x''} \frac{|f(x') - f(x'')|}{|x' - x''|}.
\]

Therefore

\[
\left| \lambda_{\theta}(T^{\theta^{-n}}(A)) - \gamma_{\theta}(A) \right| \leq \frac{\theta}{\log(1 + \theta^2)} \int_A \left| \frac{U^n f(x) - U_{\infty}f}{1 + \theta x} \right| dx
\]

\[
< Kq^n \|f\|_L \frac{\theta}{\log(1 + \theta^2)} \int_A \frac{1}{1 + \theta x} dx
\]

\[
< Cq^n \gamma_{\theta}(A)
\]

and since

\[
\gamma_{\theta}(A) \leq \frac{\theta}{\log(1 + \theta^2)} \lambda_{\theta}(A), \quad A \in B_{[0, \theta]}
\]

then the proof is complete. \( \Box \)

In \[8\] the proof of Gauss-Kuzmin theorem is based on the Gauss-Kuzmin-type equation which in this case is

\[
F_{n+1}(x) = \sum_{i \geq m} \left\{ F_n \left( \frac{1}{i\theta} \right) - F_n \left( \frac{1}{i\theta + x} \right) \right\}
\]  

(3.5)

where the functions \((F_n)_{n \in \mathbb{N}}\) are defined for \( x \in [0, \theta] \) by

\[
F_0(x) := \lambda_{\theta}([0, x]) = \frac{x}{\theta}, \quad F_n(x) := \lambda_{\theta}(T^{\theta^{-n}} \leq x), \ n \in \mathbb{N}_+.
\]  

(3.6)

The measure \( \gamma_{\theta} \) defined in (2.8) is an eigenfunction of (3.5), namely, if we put \( F_n(x) = \log(1 + \theta x), \ x \in [0, \theta] \), we obtain \( F_{n+1}(x) = \log(1 + \theta x) \). The factor \( 1/(\log(1 + \theta^2)) \) is a normalizing constant.
We give now the main theorem of this section. First, define \( \Delta_{x,y} = [0,x] \times [0,y] \) for any \( x,y \in [0,\theta] \), and the functions \((\overline{F}_n)_{n \in \mathbb{N}_+}\) on \([0,\theta]^2\) by
\[
\overline{F}_n(x,y) := \lambda_{\theta} \left( (T_{\theta})^n (x,y) \in \Delta_{x,y} \right),
\]
where \( \lambda_{\theta} \) is the Lebesgue measure on \([0,\theta]^2\).

**Theorem 3.2.** (A Gauss-Kuzmin theorem for \( T_{\theta} \)) For every \( n \geq 2 \) and \((x,y) \in [0,\theta]^2\) one has
\[
\overline{F}_n(x,y) = \frac{\log(1 + xy)}{\log(1 + \theta^2)} + O(q^n)
\]
with \( 0 < q < \theta \).

### 3.2 Necessary results

In this subsection, we give necessary results used to prove the Gauss-Kuzmin theorem for \( T_{\theta} \). Like in the one-dimensional case, we need the Gauss-Kuzmin-type equation associated with the functions \((\overline{F}_n)_{n \in \mathbb{N}_+}\) defined in (3.7). Thus, for any \( 0 < y \leq \theta \), put \( \ell_1 := \eta(y) \), where \( \eta \) is as in (2.3). Then \((T_\theta)^n (x,y) \in \Delta_{x,y}\) is equivalent to
\[
(T_\theta)^n \in \left\{ \bigcup_{i \geq \ell_1 + 1} \left[ \frac{1}{i\theta + x}, \frac{1}{i\theta} \right] \times [0,\theta] \right\} \cup \left\{ \left[ \frac{1}{\ell_1 \theta + x}, \frac{1}{\ell_1 \theta} \right] \times \left[ \frac{1}{y} - \ell_1 \theta, \theta \right] \right\}.
\]

From this and (3.7) we get the Gauss-Kuzmin-type equation on \([0,\theta]^2\):
\[
\overline{F}_{n+1}(x,y) = \sum_{i \geq \ell_1} \left\{ \overline{F}_n \left( \frac{1}{i\theta}, \theta \right) - \overline{F}_n \left( \frac{1}{i\theta + x}, \theta \right) \right\}
\]
\[
- \left\{ \overline{F}_n \left( \frac{1}{\ell_1 \theta}, y - \ell_1 \theta \right) - \overline{F}_n \left( \frac{1}{\ell_1 \theta + x}, y - \ell_1 \theta \right) \right\}.
\]
A straightforward calculation shows that the measure \( \gamma_{\theta} \) defined in (2.17) is an eigenfunction of (3.9), namely, if we put \( \overline{F}_n(x,y) = \log(1 + xy) \), \( x,y \in [0,\theta] \), we obtain \( \overline{F}_{n+1}(x,y) = \log(1 + xy) \).

**Lemma 3.3.** Let \( n \in \mathbb{N}, n \geq 2 \) and let \( y \in [0,\theta] \cap \mathbb{Q} \) with \( y = [\ell_1 \theta, \ldots, \ell_d \theta] \), \( \ell_1, \ldots, \ell_d \in \mathbb{N}_m, \ell_d \geq m + 1 \), where \( d \leq \lfloor n/2 \rfloor \). Then for every \( x, x^* \in [0,\theta] \) with \( x^* < x \),
\[
|\overline{F}_n(x,y) - \overline{F}_n(x^*,y) - \frac{1}{\log(1 + \theta^2)} \log \left( \frac{1 + xy}{1 + x^*y} \right)| < C\lambda_{\theta}(\Delta_{x,y} \setminus \Delta_{x^*,y}) q^{n-d}
\]
where \( C \) is an universal constant and \( q \) is from Theorem 3.1.
Proof. Let $y_0 = y$, $y_i := \lfloor \ell_i + 1 \theta, \ldots, \ell_d \theta \rfloor$, $i = 1, \ldots, d$, with $y_d = 0$. Then $y_1 = \frac{1}{y} - \ell_1 \theta$. Applying (3.9) one gets

$$
\overline{F}_n(x, y) - \overline{F}_n(x^*, y) = \sum_{i \geq \ell_1} \left\{ \overline{F}_n - 1 \left( \frac{1}{i \theta + x}, \frac{1}{i \theta + x^*} \right) - \overline{F}_n - 1 \left( \frac{1}{\ell_1 \theta + x}, y_1 \right) \right\}.
$$

Now for each $B \in B_{[0, \theta]}^2$ one has

$$
\frac{1}{(1 + \theta^2) \log (1 + \theta^2)} \lambda_\theta(B) \leq \gamma_\theta(B) \leq \frac{1}{\log (1 + \theta^2)} \lambda_\theta(B).
$$

Now from (2.10) and (3.11), it follows that:

$$
\sum_{i \geq \ell_1} \lambda_\theta \left( \left[ \frac{1}{i \theta + x}, \frac{1}{x^* + i \theta} \right] \right) = \sum_{i \geq \ell_1} \frac{1}{\theta} \left( \frac{1}{i \theta + x} - \frac{1}{i \theta + x^*} \right)
$$

$$
= \sum_{i \geq \ell_1} \lambda_\theta \left\{ \left( \frac{1}{i \theta + x}, \frac{1}{i \theta + x^*} \right) \times [0, \theta] \right\}
$$

$$
\leq (1 + \theta^2) \log (1 + \theta^2) \sum_{i \geq \ell_1} \gamma_\theta \left\{ \left( \frac{1}{i \theta + x}, \frac{1}{i \theta + x^*} \right) \times [0, \theta] \right\}
$$

$$
= (1 + \theta^2) \log (1 + \theta^2) \sum_{i \geq \ell_1} \gamma_\theta \{(x^*, x) \times I(i)\}
$$

$$
\leq (1 + \theta^2) \log (1 + \theta^2) \frac{1}{\log (1 + \theta^2)} \sum_{i \geq \ell_1} \lambda_\theta \{(x^*, x) \times I(i)\}
$$

$$
= (1 + \theta^2) \sum_{i \geq \ell_1} \lambda_\theta \left\{ (x^*, x) \times \left( \frac{1}{(i + 1) \theta}, \frac{1}{i \theta} \right) \right\}
$$

$$
= (1 + \theta^2) \frac{x - x^*}{\theta} \sum_{i \geq \ell_1} \lambda_\theta \left\{ \left( \frac{1}{(i + 1) \theta}, \frac{1}{i \theta} \right) \right\}
$$

$$
= (1 + \theta^2) \frac{x - x^*}{\theta} \sum_{i \geq \ell_1} \left( \frac{1}{i \theta} - \frac{1}{(i + 1) \theta} \right)
$$

$$
\leq (1 + \theta^2) \frac{x - x^*}{\theta^2} \frac{1}{\ell_1 \theta} \leq (1 + \theta^2) \frac{x - x^*}{\theta} \frac{y}{\theta (m + 1) y}.
$$

$$
= \frac{(1 + \theta^2)^2}{\theta^2} \lambda_\theta (\Delta_{x,y} \setminus \Delta_{x^*,y}). \tag{3.12}
$$
For every $2 \leq k \leq d$, a similar analysis leads to

$$
\sum_{i \geq \ell_k} \lambda_{\theta} \{[i \ell_k, \ell_{k-1} \theta, \ldots, \ell_1 \theta + x], [i \ell_k, \ell_{k-1} \theta, \ldots, \ell_1 \theta + x^*] \} =
\frac{1}{\theta} \sum_{i \geq \ell_k} |[i \ell_k, \ell_{k-1} \theta, \ldots, \ell_1 \theta + x^*] - [i \ell_k, \ell_{k-1} \theta, \ldots, \ell_1 \theta + x]| =
\sum_{i \geq \ell_k} \overline{\lambda}_{\theta} \{[i \ell_k, \ell_{k-1} \theta, \ldots, \ell_1 \theta + x], [i \ell_k, \ell_{k-1} \theta, \ldots, \ell_1 \theta + x^*] \} \times [0, \theta] \leq (1 + \theta^2) \log (1 + \theta^2) \times
\sum_{i \geq \ell_k} \overline{\pi}_{\theta} \{(x^*, x) \times I(\ell_1, \ldots, \ell_{k-1})\} \leq
\frac{(1 + \theta^2) \log (1 + \theta^2)}{\theta^2} \sum_{i \geq \ell_k} \overline{\lambda}_{\theta} \{(x^*, x) \times I(\ell_1, \ldots, \ell_{k-1})\} \leq
\frac{(1 + \theta^2)^2}{\theta^2} \overline{\lambda}_{\theta} (\Delta_{x,y} \setminus \Delta_{x^*,y}).
(3.13)
$$

Since $F_n(x, \theta) = F_n(x)$, from Theorem [3.1] it follows that

$$
\sum_{i \geq \ell_1} \left\{ F_{n-1} \left( \frac{1}{i \ell_1 + x^*}, \theta \right) - F_{n-1} \left( \frac{1}{i \ell_1 + x}, \theta \right) \right\} =
\sum_{i \geq \ell_1} \left\{ F_{n-1} \left( \frac{1}{i \ell_1 + x^*} \right) - F_{n-1} \left( \frac{1}{i \ell_1 + x} \right) \right\} =
\sum_{i \geq \ell_1} \left\{ \gamma_{\theta} \left( \left[ \frac{1}{i \ell_1 + x}, \frac{1}{i \ell_1 + x^*} \right] \right) + \lambda_{\theta} \left( \left[ \frac{1}{i \ell_1 + x}, \frac{1}{i \ell_1 + x^*} \right] \right) \circ (q^{n-1}) \right\} =
\frac{1}{\log (1 + \theta^2)} \log \frac{\ell_1 \theta + x}{\ell_1 \theta + x^*} + \frac{(1 + \theta^2)^2}{\theta^2} \overline{\lambda}_{\theta} (\Delta_{x,y} \setminus \Delta_{x^*,y}) \circ (q^{n-1}).
(3.14)
$$

Now from (3.10), (3.14), we have:

$$
F_n(x, y) - F_n(x^*, y) =
\frac{1}{\log (1 + \theta^2)} \log \frac{\ell_1 \theta + x}{\ell_1 \theta + x^*}
+ \frac{(1 + \theta^2)^2}{\theta^2} \overline{\lambda}_{\theta} (\Delta_{x,y} \setminus \Delta_{x^*,y}) \circ (q^{n-1})
+ \left( F_{n-1} \left( \frac{1}{\ell_1 \theta + x}, y \right) - F_{n-1} \left( \frac{1}{\ell_1 \theta + x^*}, y \right) \right).
$$
Applying (3.10) and Theorem 3.1 again, we have:

\[
F_{n-1} \left( \frac{1}{\ell_1 \theta + x}, y_1 \right) - F_{n-1} \left( \frac{1}{\ell_1 \theta + x^*}, y_1 \right) = \\
\sum_{i \geq \ell_2} \left\{ F_{n-2} \left( \frac{1}{i \theta + \frac{1}{\ell_1 \theta + x}}, \theta \right) - F_{n-2} \left( \frac{1}{i \theta + \frac{1}{\ell_1 \theta + x^*}}, \theta \right) \right\} \\
+ F_{n-2} \left( \frac{1}{\ell_2 \theta + \frac{1}{\ell_1 \theta + x^*}}, y_2 \right) - F_{n-2} \left( \frac{1}{\ell_2 \theta + \frac{1}{\ell_1 \theta + x}}, y_2 \right) \\
= \frac{1}{\log (1 + \theta^2)} \log \ell_2 \theta + [\ell_1 \theta + x] \\
+ \frac{(1 + \theta^2)^2}{\theta^2} \chi_{\theta} (\Delta_{x,y} \setminus \Delta_{x^*,y}) \mathcal{O} \left( q^{n-2} \right) \\
+ F_{n-2} \left( \frac{1}{\ell_2 \theta + \frac{1}{\ell_1 \theta + x^*}}, y_2 \right) - F_{n-2} \left( \frac{1}{\ell_2 \theta + \frac{1}{\ell_1 \theta + x}}, y_2 \right).
\]

Applying (3.10) and Theorem 3.1 \(d\)-times and taking into account that

\[ y_d = 0, \]

we get

\[
F_n(x, y) - F_n(x^*, y) = \frac{1}{\log (1 + \theta^2)} \times \\
\log \left( \frac{\ell_1 \theta + x}{\ell_1 \theta + x^*} \right) \cdot \left[ \frac{[\ell_1 \theta + x] + \ell_2 \theta}{[\ell_1 \theta + x^*] + \ell_2 \theta} \right] + \\
\mathcal{O} \left( q^{n-1} \right) \] + \ldots + \mathcal{O} \left( q^{n-d} \right).
\]

If \( p_d \) and \( q_d \) are as in (2.5) and (2.6) with \( a_1 = \ell_1 + \frac{\theta}{\theta} \) and \( a_i = \ell_i, \) \( i = 2, \ldots, d, \)

then \( \ell_1 \theta + x = q_1, \) \( \ell_2 \theta + [\ell_1 \theta + x] = q_2, \) \( \ell_2 \theta + [\ell_1 \theta + x] = q_d. \)

Let \( p_d^* \) and \( q_d^* \) are as in (2.5) and (2.6), with \( a_1 = \ell_1 + \frac{\theta}{\theta} \) and \( a_i = \ell_i, \)

\( i = 2, \ldots, d. \) Note that \( p_d = p_d^* q_d. \) Thus we have

\[
\frac{\ell_1 \theta + x}{\ell_1 \theta + x^*} \cdot \frac{[\ell_1 \theta + x] + \ell_2 \theta}{[\ell_1 \theta + x^*] + \ell_2 \theta} \cdot \frac{[\ell_{d-1} \theta, \ldots, \ell_2 \theta, [\ell_1 \theta + x] + \ell_d \theta}{[\ell_{d-1} \theta, \ldots, \ell_2 \theta, [\ell_1 \theta + x^*] + \ell_d \theta} = \frac{q_d}{q_d^*} = \frac{p_d^* q_d}{q_d^*} = \frac{p_d}{q_d^*}.
\]

Then we have

\[
\frac{x + \ell_1 \theta + [\ell_2 \theta, \ldots, \ell_d \theta]}{x^* + \ell_1 \theta + [\ell_2 \theta, \ldots, \ell_d \theta]} = \frac{x + \frac{1}{y}}{x^* + \frac{1}{y}} = \frac{1 + xy}{1 + x^* y}.
\]

Therefore,

\[
F_n(x, y) - F_n(x^*, y) = \frac{1}{\log (1 + \theta^2)} \log \left( \frac{1 + xy}{1 + x^* y} \right) + \frac{(1 + \theta^2)^2}{\theta^2} \chi_{\theta} (\Delta_{x,y} \setminus \Delta_{x^*,y}) \mathcal{O} \left( q^{n-d} \right)
\]

which completes the proof. □
3.3 Proof of Theorem 3.2

Let \((x, y) \in [0, \theta]^2, n \geq 2\) and \(y \notin \mathbb{Q}\). Since \(\Delta_{x,p/q} \subset \Delta_{x,y}\) and \(F_n(x, y) = \lambda_\theta (\left(\mathcal{T}_\theta\right)^{-n} (\Delta_{x,y}))\), from (3.11), (2.7) and the fact that \(\mathcal{T}_\theta\) is \(\pi_\theta\)-invariant, we find that

\[
F_n(x, y) - F_n(x, \frac{p_d}{q_d}) = \lambda_\theta (\left(\mathcal{T}_\theta\right)^{-n} (\Delta_{x,y}) \setminus \left(\mathcal{T}_\theta\right)^{-n} (\Delta_{x,p/q}))
\]

\[
\leq (1 + \theta^2) \log (1 + \theta^2) \pi_\theta (\left(\mathcal{T}_\theta\right)^{-n} (\Delta_{x,y}) \setminus \left(\mathcal{T}_\theta\right)^{-n} (\Delta_{x,p/q}))
\]

\[
= (1 + \theta^2) \log (1 + \theta^2) \frac{1}{\log (1 + \theta^2)} \lambda_\theta (\left[0, x\right] \times [p_d, q_d])
\]

\[
= (1 + \theta^2) \frac{x}{\theta^2} \frac{1}{\log (1 + \theta^2)} \frac{y - \frac{p_d}{q_d}}{1 + x \xi} \leq \frac{x}{(1 + \theta^2)^{2[n/2]} \theta^2}.
\]

(3.15)

Since for every fixed \(x \in [0, \theta]\) the function \(y \mapsto \log (1 + xy)\) is a differentiable on \([0, \theta]\), by the Mean Value Theorem we have

\[
\left| \log (1 + xy) - \log \left(1 + x \frac{p_d}{q_d}\right) \right| = \left| y - \frac{p_d}{q_d} \right| \cdot \left| \frac{x}{1 + x \xi} \right| \leq \frac{x}{(1 + \theta^2)^{2[n/2]} \theta^2}.
\]

(3.16)

where \(p_d/q_d \leq \xi \leq y\).

Finally, from Lemma 3.3, (3.15) and (3.16), and since \(F_n(0, \frac{p_d}{q_d}) = 0\), we have:

\[
\left| \frac{F_n(x, y) - \log (1 + xy)}{\log (1 + \theta^2)} \right| \leq \left| F_n(x, y) - F_n\left(\frac{p_d}{q_d}\right) \right|
\]

\[
+ \left| F_n\left(\frac{p_d}{q_d}\right) - F_n\left(0, \frac{p_d}{q_d}\right) - \frac{1}{\log (1 + \theta^2)} \log \left(1 + x \frac{p_d}{q_d}\right) \right|
\]

\[
+ \frac{1}{\log (1 + \theta^2)} \left| \log (1 + xy) - \log \left(1 + x \frac{p_d}{q_d}\right) \right|
\]

\[
\leq \frac{1 + \theta^2}{\theta^2} \frac{x}{(1 + \theta^2)^{2[n/2]} \theta^2} + C q^{n-d} + \frac{x}{(1 + \theta^2)^{2[n/2]} \theta^2}
\]

which completes the proof. \(\square\)

4 A two-dimensional Gauss-Kuzmin theorem

In this section we shall estimate the error term

\[
e_{n,a}(x, y) = \gamma_{\theta, a} (\mathcal{T}_\theta^n \in [0, x], s_{n,a} \in [0, y]) - \frac{\log (1 + xy)}{\log (1 + \theta^2)}
\]
for any \( a \in [0, \theta] \), \( x, y \in [0, \theta] \) and \( n \in \mathbb{N}_+ \).

In the main result of this section, Theorem 4.4, we shall derive lower and upper bounds (not depending on \( a \in [0, \theta] \)) of the supremum

\[
\sup_{x, y \in [0, \theta]} |e_{n,a}(x, y)|, \quad a \in [0, \theta],
\]

which provide a more refined estimate of the convergence rate involved. First, we obtain a lower bound for the following approximation error.

**Theorem 4.1.** For any \( a \in [0, \theta] \) and \( n \in \mathbb{N}_+ \) we have

\[
\frac{1}{2} P_{m(n)}(\theta) \leq \sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} \leq y) - \gamma_{\theta}([0, y])|.
\]

**Proof.** The continuity of the function \( y \to \gamma_{\theta}([0, y]), \ y \in [0, \theta] \) and the equation

\[
\lim_{h \to 0} \gamma_{\theta,a}(s_{n,a} \leq y - h) = \gamma_{\theta,a}(s_{n,a} \leq y)
\]

imply that

\[
\sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} \leq y) - \gamma_{\theta}([0, y])| = \sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} < y) - \gamma_{\theta}([0, y])|
\]

for any \( a \in [0, \theta] \) and \( n \in \mathbb{N}_+ \). For any \( s \in [0, \theta] \) we then have

\[
\gamma_{\theta,a}(s_{n,a} = s) = \gamma_{\theta,a}(s_{n,a} \leq s) - \gamma_{\theta}([0, s])
- (\gamma_{\theta,a}(s_{n,a} < s) - \gamma_{\theta}([0, s]))
\leq \sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} \leq y) - \gamma_{\theta}([0, y])|
+ \sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} < y) - \gamma_{\theta}([0, y])|
= 2 \sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} \leq y) - \gamma_{\theta}([0, y])|.
\]

Hence

\[
\sup_{y \in [0, \theta]} |\gamma_{\theta,a}(s_{n,a} \leq y) - \gamma_{\theta}([0, y])| \geq \frac{1}{2} \sup_{s \in [0, \theta]} \gamma_{\theta,a}(s_{n,a} = s),
\]

for any \( a \in [0, \theta] \) and \( n \in \mathbb{N}_+ \). Next, using (2.36) we have

\[
\gamma_{\theta,a}(s_{n,a} = [i_n \theta, \ldots, i_2 \theta, i_1 \theta + a]) = P_{i_1 \ldots i_n}(a), \quad n \geq 2,
\]

16
\[ \gamma_{\theta,a} \left( s_{1,a} = \frac{1}{i_{1}\theta + a} \right) = P_{i_{1}}(a) \]

for any \( a \in [0, \theta] \) and \( i_1, \ldots, i_n \in \mathbb{N}_m \). By (2.37) we have

\[ \sup_{s \in [0,\theta]} \gamma_{\theta,a} (s_{n,a} = s) = P_{m(n)}(a), \quad a \in [0, \theta] \]

where we write \( m(n) \) for \( (i_1, \ldots, i_n) \) with \( i_1 = \ldots = i_n = m, \ n \in \mathbb{N}_+ \), \( m \in \mathbb{N}_+ \).

By the same equation we have

\[ P_{m(n)}(a) = \frac{1 + a\theta}{q_{n-1}(m,\ldots,m)(a + m\theta) + p_{n-1}(m,\ldots,m)} \times \frac{1}{q_n(m,m,m,m)(a + m\theta) + p_n(m,m,m,m)}. \]

It is easy to see that \( P_{m(n)}(\cdot) \) is a decreasing function. Therefore

\[ \sup_{s \in [0,\theta]} \gamma_{\theta,a} (s_{n,a} = s) \geq P_{m(n)}(\theta), \quad n \in \mathbb{N}_+ \]

for any \( a \in [0, \theta] \).

\[ \square \]

**Theorem 4.2. (The lower bound)** For any \( a \in [0, \theta] \) and \( n \in \mathbb{N}_+ \) we have

\[ \frac{1}{2} P_{m(n)}(\theta) \leq \sup_{x,y \in [0,\theta]} \left| \gamma_{\theta,a} (T^n_{\theta} \in [0,x], s_{n,a} \in [0,y]) - \frac{\log(1 + xy)}{\log (1 + \theta^2)} \right|. \]

**Proof.** For any \( a \in [0, \theta] \) and \( n \in \mathbb{N}_+ \), by Theorem 4.1 we have

\[ \sup_{x,y \in [0,\theta]} \left| \gamma_{\theta,a} (T^n_{\theta} \in [0,x], s_{n,a} \in [0,y]) - \frac{\log(1 + xy)}{\log (1 + \theta^2)} \right| \geq \sup_{y \in [0,\theta]} \left| \gamma_{\theta,a} (s_{n,a} \in [0,y]) - \gamma_{\theta} ([0,y]) \right| \geq \frac{1}{2} P_{m(n)}(\theta). \]

\[ \square \]

In what follows we use the characteristic properties of the transition operator associated with the RSCC underlying \( \theta \)-expansions. By restricting this operator to the Banach space of functions of bounded variation on \([0, \theta]\), we derive an explicit upper bound for the supremum \([4,1]\).
Theorem 4.3. (The upper bound) For any \( a \in [0, \theta] \) and \( n \in \mathbb{N} \) we have

\[
\sup_{x, y \in [0, \theta]} \left| \gamma_{\theta, a} \left( T^n_{\theta} \in [0, x], s_{n, a} \in [0, y] \right) - \frac{\log(1 + xy)}{\log(1 + \theta^2)} \right| \leq \frac{1}{(m + 1)^n}.
\]

Proof. Let \( F_{n, a}(y) = \gamma_{\theta, a}(s_{n, a} \leq y) \) and \( G_{n, a}(y) = F_{n, a}(y) - \gamma_{\theta}([0, y]) \), \( a, y \in [0, \theta], \ n \in \mathbb{N} \). As we have noted \( U \) is the transition operator of the Markov chain \((s_{n, a})_{n \in \mathbb{N}}\) on \( ([0, \theta], B_{[0, \theta]}, \gamma_{\theta, a}) \) for any \( a \in [0, \theta] \). For any \( y \in [0, \theta] \) consider the function \( f_y \) defined on \([0, \theta]\) as

\[
f_y(a) := \begin{cases} 1 & \text{if } 0 \leq a \leq y, \\ 0 & \text{if } y < a \leq \theta. \end{cases}
\]

Hence

\[
U^n f_y(a) = E_a \left( f_y(s_{n, a}) \middle| s_0, a = a \right) = \gamma_{\theta, a}(s_{n, a} \leq y)
\]

for all \( a, y \in [0, \theta], \ n \in \mathbb{N} \). As it follows from Proposition 2.2 that

\[
\frac{\int_0^\theta f_y(a) \gamma_{\theta}(da) - \gamma_{\theta}([0, y])}{\var f_y} = \frac{1}{(m + 1)^n} \tag{4.2}
\]

for all \( a, y \in [0, \theta], \ n \in \mathbb{N} \). By (2.26), for all \( a, x, y \in [0, \theta] \) and \( n \in \mathbb{N} \) we have

\[
\gamma_{\theta, a} \left( T^n_{\theta} \in [0, x], s_{n, a} \in [0, y] \right) = \int_0^y \gamma_{\theta, a} \left( T^n_{\theta} \in [0, x] \middle| s_{n, a} = z \right) dF_{n, a}(z)
\]

\[
= \int_0^y \frac{(z\theta + 1)x}{(zx + 1)\theta} dF_{n, a}(z) = \int_0^y \frac{(z\theta + 1)x}{(zx + 1)\theta} \gamma_{\theta}(dz) + \int_0^y \frac{(z\theta + 1)x}{(zx + 1)\theta} dG_{n, a}(z)
\]

\[
= \frac{\log(1 + xy)}{\log(1 + \theta^2)} + \frac{(z\theta + 1)x}{(zx + 1)\theta} G_{n, a}(z) \bigg|_0^y - \int_0^y \frac{x(\theta - x)}{(zx + 1)^2\theta} G_{n, a}(z) dz.
\]

Hence, by (4.2)

\[
\left| \gamma_{\theta, a} \left( T^n_{\theta} \in [0, x], s_{n, a} \in [0, y] \right) - \frac{\log(1 + xy)}{\log(1 + \theta^2)} \right| \leq \frac{1}{(m + 1)^n} \left( \frac{(y\theta + 1)x}{(xy + 1)\theta} - \frac{(\theta - x)xy}{(xy + 1)\theta} \right) = \frac{x}{(m + 1)^n} \leq \frac{1}{(m + 1)^n}
\]

for all \( a, x, y \in [0, \theta] \) and \( n \in \mathbb{N} \). □

Combining Theorem 4.2 with Theorem 4.3 we obtain Theorem 4.4.
Theorem 4.4. For any \( a \in [0, \theta] \) and \( n \in \mathbb{N}_+ \) we have

\[
\frac{1}{2} P_{m(n)}(\theta) \leq \sup_{x,y \in [0,\theta]} \left| \gamma_{\theta,a} (T^n_\theta \in [0,x], s_{n,a} \in [0,y]) - \frac{\log(1+xy)}{\log(1+\theta^2)} \right| \leq \frac{1}{(m+1)^n}.
\]

5 Final remarks

To conclude this paper, we note that

\[
P_{m(n)}(\theta) = \frac{m+1}{q_{n+1}q_{n+2}}, \quad n \in \mathbb{N}_+,
\]

where \( q_n = m\theta q_{n-1} + q_{n-2} \), \( n \in \mathbb{N}_+ \), with \( q_{-1} = 0 \) and \( q_0 = 1 \). It is easy to see that

\[
q_n = \frac{\theta}{\sqrt{1+4\theta^2}} \left[ \left( \frac{1+\sqrt{1+4\theta^2}}{2\theta} \right)^{n+1} - \left( \frac{1-\sqrt{1+4\theta^2}}{2\theta} \right)^{n+1} \right].
\]

It should be noted that Theorem 4.4 in connection with the limits

\[
\lim \left( \frac{1}{2} P_{m(n)}(\theta) \right)^{1/n} = \frac{2\theta^2}{1+2\theta^2 + \sqrt{1+4\theta^2}},
\]

\[
\lim \left( \frac{1}{(m+1)^n} \right)^{1/n} = \frac{1}{m+1},
\]

leads to an estimate of the order of magnitude of the supremum (4.1). Actually, Theorem 4.4 implies that the convergence rate is \( O(\alpha^n) \), with

\[
\frac{2\theta^2}{1+2\theta^2 + \sqrt{1+4\theta^2}} \leq \alpha \leq \frac{1}{m+1}.
\]

For example, we have

| \( m \) | \( g^2 \) | \( \alpha \) |
|---|---|---|
| 1 | 0.381966 | \( \leq \alpha \leq 0.50000 \) |
| 2 | 0.267949192 | \( \leq \alpha \leq 0.33333 \ldots \) |
| 3 | 0.208711948 | \( \leq \alpha \leq 0.25000 \ldots \) |
| 10 | 0.083920216 | \( \leq \alpha \leq 0.090909 \ldots \) |
| 100 | 0.009804864 | \( \leq \alpha \leq 0.00990099 \) |
| 1000 | 0.000998004 | \( \leq \alpha \leq 0.000999 \) |
| 10000 | 0.00009998 | \( \leq \alpha \leq 0.00009999 \) |

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Figure 2: Graphs of lower and upper bounds

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