A Note On The Chern-Simons
And Kodama Wavefunctions

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Yang-Mills theory in four dimensions formally admits an exact Chern-Simons wavefunction. It is an eigenfunction of the quantum Hamiltonian with zero energy. It is known to be unphysical for a variety of reasons, but it is still interesting to understand what it describes. We show that in expanding around this state, positive helicity gauge bosons have positive energy and negative helicity ones have negative energy. Some of the negative energy states have negative norm. We also show that the Chern-Simons state is the supersymmetric partner of the naive fermion vacuum in which one does not fill the fermi sea. Finally, we give a sort of explanation of “why” this state exists. Similar properties can be expected for the analogous Kodama wavefunction of gravity.
Four-dimensional Yang-Mills theory has the surprising property of admitting an exact zero energy eigenfunction of the Schrodinger equation, the wave-function being the exponential of the Chern-Simons form. This wavefunction, which has been known for a long time (the only original reference I know of is \[1\], where it is presented as an exercise!), is highly unnormalizable. It is constructed without the asymptotic freedom and coupling constant renormalization that are needed for the standard quantization of Yang-Mills theory, which makes contact both with nonperturbative lattice calculations and with real strong interaction and weak interaction experiments. Finally, it is not invariant under CPT (as we discuss more fully later), so on general grounds it could not be the ground state of a quantum field theory. For all these reasons and more we will find later, the Chern-Simons wave function of Yang-Mills theory is not the physical ground state of the theory. Nonetheless, one would like to know how it should be interpreted, and in some sense, “why” it exists. Answering these questions will be the goal of the present paper.

It is easy to describe directly the Chern-Simons wavefunction of Yang-Mills theory. The Hamiltonian of Yang-Mills theory is

\[
H = \frac{1}{2g^2} \int d^3x Tr (E^2 + B^2) = \frac{1}{2} \int d^3x Tr \left( -g^2 \frac{\delta^2}{\delta A(x)^2} + \frac{1}{g^2} B^2 \right). \tag{1}
\]

Here \(g\) is the gauge coupling, and \(E_i = F_{0i}\) and \(B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}\) are the electric and magnetic fields. The canonical momentum is \(\Pi = E/g^2\), and quantum mechanically it becomes \(-i\delta/\delta A\), whence the second formula in (1). Given the expression for the Hamiltonian, it is clear that any wavefunction \(\Psi\) with

\[
0 = (E + iB)\Psi = i \left( -g^2 \frac{\delta}{\delta A} + B \right) \Psi \tag{2}
\]

is also an eigenfunction of the Hamiltonian with \(H\Psi = 0\). Moreover, if \(I\) is the Chern-Simons functional, \(I = \frac{1}{4\pi} \int d^3xe^{ijk} (A_i \delta_j A_k + \frac{2}{3} A_i A_j A_k)\), then \(\delta I/\delta A = B/2\pi\), so that

\[
\Psi = \exp \left( (2\pi/g^2)I(A) \right) \tag{3}
\]

obeys (2) and hence is an eigenstate of the Hamiltonian with zero energy. This is what we call the Chern-Simons state. It is far from being normalizable, since \(I(A)\) has no properties of positivity – it changes sign under parity. It would be equally good to have a state annihilated by \(E - iB\), and clearly \(\widetilde{\Psi} = \exp(- (2\pi/g^2)I(A))\) does this job.\footnote{In the nonabelian case, \(\Psi\) and \(\widetilde{\Psi}\) are not invariant under homotopically non-trivial gauge transformations. We ignore this. Along with the unnormalizability, lack of CPT invariance, etc., and additional properties that we will see below, this is one more reason that the Chern-Simons state is formal and does not really correspond to a sensible physical theory.}
The Chern-Simons wavefunction of Yang-Mills theory has an even more surprising gravitational analog, commonly called the Kodama state [2]. Some authors have proposed the Kodama wavefunction as a starting point for understanding the real universe; for a review and references, see [3]. Our discussion here will make it clear how the Kodama state should be interpreted. For example, in the Fock space that one can build (see [3]) in expanding around the Kodama state, gravitons of one helicity will have positive energy and those of the opposite helicity will have negative energy.

**Upside Down Wave Function Of The Harmonic Oscillator**

Consider a simple harmonic oscillator with Hamiltonian $H = (p^2 + x^2)/2$. We have set $\hbar = 1$ and normalized the frequency and mass to be 1. (Accordingly, when we get back to gauge theory, we will set $g = 1$.) The usual ground state wave function is $\psi = \exp(-x^2/2)$. It is annihilated by the annihilation operator $a$, and has energy $1/2$. In expanding around it, one can make a Fock space of states $(a^*)^n\psi$, of energy $n + 1/2$. One could also start with the wave function $\psi' = \exp(+x^2/2)$. For our present purposes, we will not worry about normalization of the wavefunction (the Chern-Simons wavefunction of Yang-Mills theory is just as badly behaved as this upside-down Gaussian). We will just proceed algebraically. One can easily see that $\psi'$ is annihilated by the creation operator $a^*$ and (therefore) is an eigenfunction of $H = a^*a + 1/2 = aa^* - 1/2$ with energy $-1/2$. Starting with $\psi'$, one can build a Fock space of states $a^n\psi'$, with energy $-n - 1/2$. The only thing wrong with this Fock space, apart from the unnormalizability of the wavefunctions, is that the energies are negative.

Obviously, one cannot define inner products of the states $a^n\psi'$ by the usual formula $\langle \psi_1 | \psi_2 \rangle = \int dx d\psi_1 \psi_2$, because the integrals will not converge. Might there be some other way to define suitable inner products? Let us assume there is some inner product relative to which $x$ and $p$ are hermitian, and hence $a^*$ and $a$ are adjoints. We can always normalize the inner product so that $\langle \psi' | \psi' \rangle = 1$. Then the norm of the first “excited” state $a\psi'$ is $\langle a\psi' | a\psi' \rangle = \langle \psi' | a^*a\psi' \rangle = -\langle \psi' | \psi' \rangle = -1$. We used the fact that $a^*a = aa^* - 1$ and that $a^*\psi' = 0$. Continuing in this way, one finds that the sign of the norm of $a^n\psi'$ is $(-1)^n$. Similarly, in all of the other Fock spaces we consider below which contain negative energy bosonic excitations, half of the states would have negative norm.

Now suppose one has two harmonic oscillators, with coordinates $x, y$ and $H = (p_x^2 + p_y^2 + x^2 + y^2)/2$. Combining the standard construction for $x$ with the upside-down wave function for $y$, we take the wave function $\exp(-(x^2 - y^2)/2)$ for the combined system.
Clearly its energy is 0, as the ground state energies cancel between $x$ and $y$. Starting with this state one can make a Fock space of states, acting with creation operators in $x$ and annihilation operators in $y$. The only unusual property is that the $y$ excitations have negative energy.

One can make a 45 degree rotation of the $x-y$ plane and then this wavefunction becomes $\exp(xy)$, an indefinite Gaussian similar to the Chern-Simons wavefunction.

More generally, suppose one has $s$ harmonic oscillators (for any positive integer $s$) with coordinates $x_i$ and $H = (\sum_i p_i^2 + (x, Mx))/2$, where $M$ is any symmetric positive definite matrix and $(x, Mx)$ is the corresponding quadratic function of $x$. If $N$ is any matrix such that $N^2 = M$, then

$$\psi = \exp(-(x, Nx)/2) \tag{4}$$

is an eigenfunction of $H$, the ground state energy being $\text{Tr}N/2$. If $N$ is the (unique) positive square root of $M$, then one gets the standard ground state. In this case, one can proceed to construct the usual Hilbert space of excitations with positive energy. In general, for any square root $N$, one can construct a Fock space, the only oddity being that some of the modes have negative energy.

**Abelian Gauge Theory In Four Dimensions**

Now let us consider the case of $U(1)$ gauge theory in $3+1$ dimensions. For the moment, we work in Coulomb gauge. The role of $x$ is played by $A_T$, the transverse part of the vector potential $A$ (thus, $A_T$ is a divergence-free one-form on $\mathbb{R}^3$). The matrix $M$ is $*d*d$ where $d$ is the exterior derivative and $*$ is the Hodge star operator. The positive definite square root of $M$ can be represented by an integral kernel in $\mathbb{R}^3$. Taking this to define the wave function, we get the usual ground state for the free photons. The ground state energy is positive and divergent (requiring the standard subtraction) and the excitations have the standard positive energies.

Instead, $M$ has an obvious local square root, $N = *d$ (or $-*d$). If one uses this, then $N$ is positive for positive helicity photons and negative for negative helicity photons. So the zero-point energy cancels out, analogous to what happens for the wave function $\exp(xy)$ that was the toy example above. Moreover, in expanding around this vacuum, one can construct a Fock space of states; clearly, the positive helicity photons have positive energy and the negative helicity photons have negative energy. If we use $-N$ instead of
in constructing the wave functions, it is positive helicity photons that have negative energy.

Explicitly, $(A, NA) = \frac{1}{2} \int d^3 x \epsilon^{ijk} A_i \partial_j A_k$, so the wavefunctions $\exp(\pm(A, NA))$ predicted by this analysis are precisely the Chern-Simons wavefunctions $\Psi$ and $\tilde{\Psi}$. (Our analysis really leads to the Coulomb gauge wavefunctions $\exp(\pm(A_T, NA_T))$, but as the gauge-invariant generalization of this is merely $\exp(\pm(A, NA))$, there is no problem in expressing our result in a gauge-invariant language, and we have done so.)

Since CPT exchanges positive and negative helicities while commuting with the energy, these results imply that CPT must exchange $\Psi$ with $\tilde{\Psi}$, as one can indeed verify directly. CPT acts by complex conjugation, which leaves both $\Psi$ and $\tilde{\Psi}$ invariant, combined with a reflection of space, which reverses the sign of the Chern-Simons functional and so exchanges $\Psi$ and $\tilde{\Psi}$.

**Nonabelian Gauge Theory**

We have now understood the existence of the Chern-Simons wavefunction for abelian gauge theory, as well as its physical interpretation. What about the nonabelian case? The explicit computation that we reviewed at the beginning of this paper showed that the Chern-Simons wavefunction of Yang-Mills theory has a simple extension to the nonabelian case. This computation was so simple that it is hard to simplify it further, but I want to explain from a different point of view “why” the Chern-Simons state of nonabelian gauge theory exists.

In general, consider a classical mechanical system with phase space $\mathcal{M}$, and with a Lagrangian submanifold $\mathcal{N}$. At least formally, one can always associate with $\mathcal{N}$ a quantum state $\Psi_{\mathcal{N}}$: $\Psi_{\mathcal{N}}$ is the state annihilated by all operators obtained by quantization of functions that vanish on $\mathcal{N}$. To see how this works, consider a classical system with canonical variables $p_i$ and $x^i$, $i = 1, \ldots, s$. Define a Lagrangian submanifold $\mathcal{N}$ by the equations

$$p_i = \frac{\partial F}{\partial x^i},$$

for any function $F(x^1, \ldots, x^s)$. The corresponding quantum state should be annihilated by $p_i - \partial F/\partial x^i$, and, in a representation in which the $x^i$ act by multiplication and $p_i = -i\partial/\partial x^i$, it is clearly $\Psi_{\mathcal{N}} = \exp(iF)$.

As this example shows, if $\mathcal{N}$ is a real Lagrangian submanifold, then $\Psi_{\mathcal{N}}$ is an oscillatory state (and is normalizable or delta-function normalizable depending on the global behavior of $\mathcal{N}$). If one is willing to proceed more formally, one can replace $\mathcal{M}$ by its complexification
\(M_C\) and let \(\mathcal{N}\) be a complex Lagrangian submanifold of \(M_C\). The wavefunction is then a holomorphic function of the (complexified) coordinates. The same formal discussion applies, though the considerations of normalizability may be quite different.

For example, let us go back to the case of the simple harmonic oscillator, with phase space variables \(x\) and \(p\). In the case of a two-dimensional phase space \(M\), any codimension-one submanifold is Lagrangian. So (upon complexification), we can define a Lagrangian submanifold by \(p = ix\), or in other words \(p = dF/dx\) with \(F = ix^2/2\). The wavefunction \(\Psi_\mathcal{N} = \exp(iF)\) is then the conventional harmonic oscillator ground state \(\exp(-x^2/2)\). Alternatively, we could use the Lagrangian submanifold \(p = -ix\), and then we get the highly unnormalizable wavefunction \(\exp(x^2/2)\) that we considered as a step to explaining the abelian Chern-Simons state. In general, as long as we work formally and do not worry about normalizability, any complex Lagrangian submanifold of the complexified phase space will do.

Now consider in this spirit nonabelian Yang-Mills theory in four dimensions. The phase space \(\mathcal{M}\) is the space of classical solutions of the Yang-Mills equations \(D_\mu F^{\mu\nu} = 0\) (with reasonable behavior at spatial infinity), modulo gauge transformations. In Minkowski space, a non-trivial solution of the self-dual or anti-self-dual equations cannot be real. So we cannot define a Lagrangian submanifold of \(\mathcal{M}\) by taking self-dual or anti-self-dual solutions. Let us, however, complexify \(\mathcal{M}\). The complexified space \(\mathcal{M}_C\) is the space of complex-valued solutions of the Yang-Mills equations (or if you wish, solutions for a connections that takes values in the complexification of the Lie algebra). Since our considerations are somewhat formal, we do not need to worry about precise existence theorems for \(\mathcal{M}_C\) in what follows. In \(\mathcal{M}_C\), self-dual or anti-self-dual solutions do exist. To get an anti-self-dual solution, we simply work in the gauge \(A_0 = 0\) and solve the evolution equation \(\partial A_i/\partial x^0 = -(i/2)\epsilon^{ijk}F_{jk}\). So a solution exists for arbitrary initial values of \(A_i\) at time zero.

The space \(\mathcal{N}\) of complex-valued anti-self-dual solutions is in fact a Lagrangian submanifold of \(\mathcal{M}_C\). To prove this, the main point is to show that the symplectic structure \(\omega\) of \(\mathcal{M}_C\) vanishes when restricted to \(\mathcal{N}\). For this, we use the covariant approach to the canonical formalism, as described for example in [4]. In the canonical formalism, we let \(\delta A\) denote a variation in a classical solution \(A\); we treat it as an anticommuting variable, representing a one-form on the space of solutions. We then define the symplectic current \(J_\mu = \text{Tr}\delta A^\nu(D_\mu \delta A_\nu - D_\nu \delta A_\mu)\). It is easily shown to be conserved, and its integral over an arbitrary initial value hypersurface gives the symplectic two-form \(\omega\). For example, if
we pick the initial value surface to be at \(x^0 = 0\) and work in the gauge \(\delta A_0 = 0\), we get a formula for \(\omega\):

\[
\omega = \int d^3 x \, \text{Tr} \delta A_i \frac{\partial}{\partial x^0} \delta A_i.
\]  

(6)

Now restricting to \(\mathcal{N}\) means taking \(\delta A_i\) to obey

\[
\frac{\partial \delta A_i}{\partial x^0} = -i \epsilon^{ijk} D_j \delta A_k,
\]

which is the linearization of the anti-self-dual equations. When we do this, we get \(\omega = -i \int d^3 x \epsilon^{ijk} \delta A_i D_j \delta A_k\), and this vanishes using integration by parts and Fermi statistics for \(\delta A\).

So a quantum state associated with the symplectic manifold \(\mathcal{N}\) should exist; it should be annihilated by \(F^+\), the self-dual part of \(F\). This quantum state is simply the Chern-Simons wavefunction \(\Psi\). Indeed, we already showed in (3) that \(\Psi\) is annihilated by \(F^+\) at time zero. It follows from this that \(\Psi\) is Poincaré invariant (that is, invariant under the connected part of the Poincaré group, though not, as we saw earlier, under CPT!) and hence is annihilated by \(F^+\) at all times. To prove Poincaré invariance of \(\Psi\), note that the stress tensor \(T_{\mu\nu} = \text{Tr} \left( F_{\mu \alpha} F_{\nu \alpha} - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right)\) of Yang-Mills theory transforms with spin \((1,1)\) under the Lorentz group, while \(F^-\) and \(F^+\) transform as \((1,0)\) and \((0,1)\), respectively. So \(T \sim F^+ F^-\), and hence any state annihilated by \(F^+\) at time zero is also annihilated at time zero by all components of \(T_{\mu\nu}\). Hence (as the Poincaré generators are certain integrals of components of \(T_{\mu\nu}\) at time zero), such a state is automatically Poincaré invariant. Poincaré invariance implies that the state is annihilated by \(F^+\) at all times, given that this is the case at time zero.

The covariance is illustrated by the dispersion relation that we found in the abelian theory (or equivalently in the weak coupling limit of a nonabelian theory). The dispersion relation can be written \(E = \epsilon |p|\) where \(E\) is the energy, \(p\) the three-momentum, and \(\epsilon = \pm 1\) is the sign of the helicity; this relation is covariant, though exotic. This form of the dispersion relation will be preserved when higher order corrections are considered (to the

\footnote{To complete the proof that \(\mathcal{N}\) is Lagrangian, one needs to show that it is a maximal subspace on which \(\omega\) vanishes. One simply uses the same formulas to show that if \(\delta A = \delta_1 A + \delta_2 A\), with \(\delta_1 A\) obeying the linearization of the self-dual Yang-Mills equations, then vanishing of \(\omega(\delta A)\) for any \(\delta_1 A\) implies that \(\delta_2 A\) obeys the same equation. So \(\omega\) would not vanish on any enlargement of \(\mathcal{N}\).}
extent that they make sense given the unnormalizable ground state and negative energy excitations), because it is protected by Poincaré symmetry.

Supersymmetric Extension

Now we will, finally, consider the supersymmetric extension of the Chern-Simons state. Yang-Mills theory can be supersymmetrized, with $\mathcal{N} = 1$ supersymmetry, by simply adding a Weyl fermion field $\lambda$ that has positive chirality and transforms in the adjoint representation of the gauge group. It thus transforms with spin $(0, 1/2)$ under Lorentz transformations. The adjoint field $\bar{\lambda}$ transforms with spin $(1/2, 0)$. The $\lambda$-dependent part of the Lagrangian is simply the minimally coupled Dirac action $\int d^4x \bar{\lambda} i \Gamma \cdot D\lambda$.

Classically, there is a $U(1)$ charge (called in this context an $R$-symmetry) under which $\lambda$ has charge 1 and $\bar{\lambda}$ has charge $-1$. When the quantum theory is quantized in the usual way, there is an anomaly in the $R$-symmetry. From a Hamiltonian point of view, as explained for example in [1], the anomaly means that homotopically nontrivial gauge transformations do not commute with the $R$-symmetry. In the present context, we can ignore this issue because we are anyway not dividing by large gauge transformations (as discussed in connection with (2), we cannot divide by them as the Chern-Simons state is not invariant under them).

In a fixed gauge field background, the fermion state of maximum $R$-charge is the state $\chi$ that is annihilated by all components of $\lambda$, of either positive or negative frequency; it is not annihilated, therefore, by any components of $\bar{\lambda}$. Of course, there is also a conjugate state $\tilde{\chi}$ of minimum $R$-charge, annihilated by all components of $\bar{\lambda}$. The states $\chi$ and $\tilde{\chi}$ are automatically eigenstates of the Hamiltonian, since they are the unique states of their $R$-charge.

Of course, Dirac taught that the proper quantization of this theory is to fill the Dirac sea and find a state whose excitations all have positive energy. This Dirac state can be obtained from $\chi$ by filling the negative energy states created by half the modes of $\bar{\lambda}$, or from $\tilde{\chi}$ by filling the negative energy states created by half the modes of $\lambda$. For our purposes here, however, instead of studying the standard quantization with the Dirac state, we want to consider the naive quantization using $\chi$ or $\tilde{\chi}$.

In expanding around $\chi$, all excitations are created by components of $\bar{\lambda}$. The positive helicity excitations have positive energy, and the negative helicity excitations have negative
energy. This is so for a simple and essentially familiar reason that one can readily understand by recalling the single particle massless Dirac equation obeyed by the two-component spinor $\lambda$. This equation reads

$$i \frac{\partial}{\partial x^0} \lambda = i \vec{\sigma} \cdot \vec{\nabla} \lambda,$$

or in momentum space $E = \vec{\sigma} \cdot \vec{p}$, where $\vec{\sigma}$ are $2 \times 2$ Pauli matrices, $E$ is the energy, and $\vec{p}$ is the spatial momentum. Since $\vec{\sigma} \cdot \vec{p} = \epsilon |\vec{p}|$, where $\epsilon$ is the sign of the helicity, we get the same dispersion relation $E = \epsilon |\vec{p}|$ that we found in studying the Chern-Simons state. (In expanding around the Dirac state, the negative energy modes created by $\lambda$ are replaced by modes of positive energy but still negative helicity created by $\lambda$. If a negative energy particle has momentum $p$, angular momentum $J$, and helicity $\epsilon$, then a positive energy hole representing absence of this particle has momentum $-p$, angular momentum $-J$, and helicity $\epsilon$.)

This suggests that the naive fermion vacuum $\chi$ is related by supersymmetry to the Chern-Simons state $\Psi$ for gauge bosons. This can be seen directly. As the Chern-Simons state is annihilated by the $(0, 1)$ part of $F$, its supersymmetric extension should be annihilated by the $(0, 1/2)$ field $\lambda$, which is related to $F^+$ by supersymmetry. Thus, the supersymmetric extension of the Chern-Simons state should be annihilated by $\lambda$ as well as $F^+$. The fermionic part of such a state is simply $\chi$. So the supersymmetric extension of the Chern-Simons state is essentially $\Psi \otimes \chi$ (or $\bar{\Psi} \otimes \bar{\chi}$ for the state with helicities reversed).\(^3\)

Like the conventional supersymmetric vacuum, the state $\Psi \otimes \chi$ has zero energy because of supersymmetry. For the conventional vacuum, the vanishing of the zero-point contribution to the energy is obtained by a cancellation between bosons and fermions, while for the state $\Psi \otimes \chi$ the vanishing is ensured by a cancellation between states (either bosons or fermions) of positive helicity and states of negative helicity.

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\(^3\) There is actually a fiber bundle structure here, rather than a simple tensor product, as the definition of $\chi$ depends on the connection $A$. For our present purposes we ignore this.
References

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