Inverse-temperature 4-vector in special relativity

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Abstract – There exist several prescriptions for identifying the notion of temperature in special relativity. We argue that the inverse-temperature 4-vector $\beta$ is the only viable option from the laws of thermodynamics, and $\beta$ is a future-directed timelike 4-vector. Using a superfluidity thought experiment, one can show that $\beta$ is not necessarily along the time direction of the comoving frame of the system, as is usually thought. It is conjectured that, for an isolated system, the 4-vector is determined from the entropy-maximum principle.

No consensus has been reached in the treatment of thermodynamics in the context of special relativity, even a whole century after the formulation of special relativity [1–11].

In relativistic thermodynamics the most imminent problem concerns the transformation laws of heat and temperature under the Lorentz group. There are several published options in the literature:

(a) $\delta Q = \delta Q_0 \gamma^{-1}, \quad T = T_0 \gamma^{-1}$, \hspace{1cm} (1)
(b) $\delta Q = \delta Q_0 \gamma, \quad T = T_0 \gamma$, \hspace{1cm} (2)
(c) $\delta Q = \delta Q_0, \quad T = T_0$, \hspace{1cm} (3)

where $\delta Q$ and $T$ represent heat and temperature, respectively, the variables with (without) subscript 0 denote those observed in the comoving (laboratory) frame, and $\gamma$ is the Lorentz factor $(1 - u^2)^{-1/2}$, where $u$ is the relative velocity of the comoving frame with respect to the laboratory frame. In addition to options (a), (b) and (c), some authors claimed (d) that “there is no meaningful law of temperature under boosts” [7,8].

Options (a), (b) and (c) are held by the authors of [1,2], [3,4] and [5,6], respectively. It is noted that, in principle, the temperature in (a) and (b) can be defined operationally using a relativistic Carnot cycle [12–14].

One of the earliest attempts to find a covariant form of thermodynamics was made by Israel and collaborators [9–11]. They proposed a 4-vector $S^\mu$ for the flux of entropy, in a similar way to the 4-vector for the flux of particle number. The particle number in a comoving frame is a scalar. Likewise, we will show that entropy in its comoving frame is a scalar as well, and so Israel’s proposal is supported. It is known that in a wide framework [15] a path integral for a system in the Euclidean regime can be identified by its partition function. The entropy of the system is the logarithm of the partition function in a microcanonical ensemble. For this ensemble the right representation should be chosen. In particular, at the WKB level, the entropy of the system is the negative of its instanton action [16,17]. Since the path integral and action are scalars, the entropy should be so too.

About other thermodynamic variables, various authors hold very diversified opinions. In this letter we shall concentrate on the temperature issue in special relativity. We shall use Planck units in which $c = \hbar = k = G = 1$. The metric signature is $(-,+,+,+)$.

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If one takes the zeroth component of a 4-vector $T$ as the “temperature” $T$, and assume that $T$ has component $(T_0, 0, 0, 0)$ in the comoving frame, then we should easily obtain $T = T_0 \gamma$ in the laboratory frame, that agrees with the opinion (b) for the zeroth component. Option (c) implies that the temperature would be a scalar.

In the Israel covariant formulation of thermodynamics [9–11], not only the equilibrium problem in the presence of gravity was investigated, but also the off-equilibrium problem was studied, using the entropy flux $S^\mu$ to formulate the First Law. However, it was expected that the controversy about the transformation law of thermodynamic quantities would never lead anywhere [10].

Earlier van Kampen [21] considered the temperature as a scalar $T_0$, and proposed a covariant form of the First Law, using $\beta_\mu = u_\mu / T_0$. It seems that the inverse-temperature 4-vector is reducible. Our proposal is distinct from his and other similar arguments; in our case, one cannot always identify $\beta_\mu$ as $u_\mu / T_0$, as mentioned earlier. All other arguments on an inverse-temperature 4-vector are based on the existence of a rest frame and that the vector $\beta$ is a priori oriented along the comoving 4-velocity of the system. However, as was pointed out by Israel, the notion of a rest frame is not always well defined, therefore the Lorentz invariance as applied to thermodynamics cannot be devoid of physical content [10].

Our proposal is distinct from these arguments. We argue that the notion of temperature should be replaced by the inverse-temperature 4-vector $\beta$. It will be shown that the inverse-temperature 4-vector $\beta$ is the only viable option, with $\beta$ being a future-directed timelike 4-vector which, however, is not necessarily along the direction of the comoving frame. That is, there always exists a frame in which $\beta$ takes the form $(T_0^{-1}, 0, 0, 0)$, but this frame is not necessarily identical with the comoving one.

In a continuous medium the law of energy-momentum conservation reads

$$ T^\mu_{\gamma,\rho} = 0, \tag{4} $$

where $T^\mu_{\gamma,\rho}$ is the total energy-momentum stress tensor. In general, the conservation law includes the effects of both heat exchange and applied work. The heat exchange can be considered as the zeroth component of the heat vector. Its spatial components represent the effect of the heatlike force [23].

Now let us consider a superfluidity thought experiment. Below some critical temperature, liquid $^4$He, under thermal equilibrium conditions, is capable of two different states at the same instant, the normal and superfluid states [24,25]. The liquid $^4$He model and its generalized model have previously been studied by Israel [10]. Here for simplicity, we only assume that there are two weakly interacting components, and these two components mutually penetrate without viscosity. Their energy-momentum is additive, that is, the total energy-momentum of the medium is the sum of those of the two components. This means that their interaction energy-momentum is negligible, although the interaction between the two states still exists. Each of the two components (states) has its own local density $\rho_i$ and velocity $v_i (i = 1, 2)$.

For our model (4) is rewritten as

$$ \sum_i T^\mu_{\gamma,i,\rho} = 0. \tag{5} $$

In addition to the energy-momentum tensor $T^\mu_{\gamma,\rho}$, in general, there exist a number of 4-vectors $J^\mu_{M,i,\rho}$ representing the flux densities of conserved charges $M$. Their conservation laws are expressed as

$$ \sum_i J^\mu_{M,i,\rho} = 0. \tag{6} $$

Following Israel, using the entropy 4-flux $S^\mu$, from the First and Second Laws of thermodynamics one can write the following covariant equation [10]:

$$ \sum_i S^\mu_{i,\rho} = - \sum_i \left( \sum_M \alpha_{Mi} J^\mu_{M,i,\rho} + \beta_\nu T^\mu_{i,\nu} \right) \geq 0, \tag{7} $$

where $S^\mu_{i,\rho}$ represents the creation rate of entropy density, and

$$ \alpha_{Mi} \equiv \mu_{Mi} |\beta_i|, \tag{8} $$

where $\mu_{Mi}$ is the chemical potential of particle $B_{Mi}$, which satisfies the equilibrium condition

$$ \sum_M \alpha_M b_{Mi} = 0, \tag{9} $$

where $b_{Mi}$ are the stoichiometric coefficients appearing in the reaction equations

$$ \sum_M b_{Mi} B_{Mi} = 0. \tag{10} $$

It is important to emphasize that only the material part $T^\mu_{\gamma,(mat)}$ enters (7), ensuring that reversible flows of the energy-momentum do not contribute to the entropy flux (this applies to (12) and (15) below).

To avoid the effect of finite size, we use the differential form (7) of the Israel formula for the creation rate of entropy density, in which the term associated with the volume variation in the usual formula vanishes. The tradeoff is, that for a given unit volume, the particle number and other conserved charges are not fixed in the process. Therefore, the macrocanonical ensemble and the fluxes of the charges must be introduced.

If there is no interaction between the two motions, then each component can itself be in thermal equilibrium and the entropy creation rate vanishes [10],

$$ S^\mu_{i,\rho} = 0, \tag{11} $$
Now the interaction between the two components is switched on. In general, the transportation of energy-momentum and other conserved charges will increase the total entropy of the system. From (5)–(7) one obtains
\[
S_{\mu}^i = \sum_{i} S_{\mu}^i = - \sum_{M} (\alpha_{M1} - \alpha_{M2}) \delta J_{M1,\mu}^\rho - (\beta_{\nu1} - \beta_{\nu2}) \delta T_{M1,\mu}^{\rho
u} \geq 0,
\]
where \( \delta J_{M1,\mu}^\rho \) and \( \delta T_{M1,\mu}^{\rho
u} \) represent the arbitrary transfer of charges and energy-momentum from component 2 to component 1. The Second Law demands each term in the right-hand side of (12) to be non-negative. This means that the flux \( \delta J_{M1,\mu}^\rho \) is always transferred between components from higher to lower chemical potential, as in the traditional theory. In comparison with this, the heatlike flux \( \delta T_{M1,\mu}^{\rho
u} \) behavior is more complicated than that in the traditional scenario, in the latter heat is transferred from the component with higher temperature to that with lower temperature. Apparently, the necessary conditions for the two components to approach equilibrium, i.e. \( S_{\mu}^i = 0 \), are
\[
\alpha_{M1} = \alpha_{M2}
\]
and
\[
\beta_{\nu1} = \beta_{\nu2}.
\]
Equation (13) was obtained by Israel [10]. Equation (14) is the Zeroth Law of thermodynamics in the new framework with the notion of the inverse-temperature 4-vector.

Since the comoving frames for the two components are different, the same inverse-temperature 4-vector cannot be along the two time directions of both frames. One can conclude that, in general, \( \beta \) is not necessarily along the time direction of the comoving frame of a system, as is usually believed.

If one accepts the notion of the inverse-temperature 4-vector, then the two states with non-vanishing relative velocity in the superfluidity model can coexist in thermodynamic equilibrium, which is distinct from thermal equilibrium in the special He model [10].

It seems that there is an alternative approach. By using the temperature 4-vector \( T \), the First Law can be recast into the following covariant form:
\[
T_{\nu} S_{\mu}^\rho = - \sum_{M} T_{\nu} \alpha_{M} J_{M,\mu}^\rho + T_{\nu
u} \mu_{\mu},
\]
where \( \alpha_{M} \) is redefined as
\[
\alpha_{M} \equiv \frac{\mu_{M}}{|T|}.
\]

However, formula (15) is too restrictive. It is noted that here the energy-momentum flux is along the orientation of the vector \( T \) instead of its spacetime gradient if we temporally ignore the terms of fluxes \( J_{M,\mu}^\rho \). In contrast, in the traditional thermodynamics, the heat flux is parallel to the temperature gradient for isotropic media. Therefore, this prescription has to be abandoned.

Some authors claimed that the temperature must be invariant with respect to relative uniform motions [5,6]. Considering two equilibrium identical bodies, which are in uniform relative motion, they argued that the heat exchange can be carried out in the course of smooth contact of the bodies and the flow would be at right angles to the motion. The observer attached to one body would judge the temperature of the other body as lower, according to option (a). From the usual relation between heat flow and temperature, heat would be transferred to the other body. On the other hand, the observation from the other body would be vice versa. This causes contradiction. The situation is similar for option (b). Therefore, one has to adopt option (c).

The reason leading to the above consequence is that the First Law was not treated in a covariant way. Roughly speaking, since the entropy is a scalar and the heat flux is a vector [23], the temperature must obey a 4-vector form.

It is concluded that the relativistic formulation of the First Law demands the notion of the inverse-temperature 4-vector \( \beta \), which should take the role of the traditional scalar temperature in classical thermodynamics. How to measure its spatial components is another problem, since the relative speed in the laboratory is much smaller than the speed of light. Its effects might be found in relativistic astrophysics [26].

Let us turn to relativistic statistics. It is known that in the comoving frame the Maxwell probability distribution for one-particle velocity of an ideal gas is expressed as
\[
f_M(v; m, |\beta|) = |m|/\sqrt{(2\pi)} \exp(-|\beta|mv^2/2),
\]
where \( |\beta| = T_0^{-1} \), \( m \) is the mass of the particle, \( v \) is its 3-velocity.

Its relativistic version was proposed by Juettner as follows [27]:
\[
f_J(v; m, |\beta|, u) = m^3/\gamma(v)^5 \exp(-|\beta|m\gamma(v)/|Z_J|),
\]
where \( Z_J = Z_J(m, |\beta|) \) is the normalization constant. In the laboratory frame, the Juettner function becomes
\[
f_J(v'; m, |\beta|, u) = m^3/\gamma(v')^5 \gamma(u)^{-1} \exp(-|\beta|m\gamma(u)\gamma(v')(1 + u \cdot v'))/|Z_J|,
\]
where \( u \) is the relative velocity of the laboratory with respect to the comoving frame and \( v' \) is the particle velocity in the laboratory frame. The extra factor \( \gamma(u)^{-1} \) is due to Lorentz contraction in velocity space. \( f_J(v'; m, |\beta|, u) \) can be rewritten as
\[
f_J(v'; m, |\beta|) = m^3/\gamma(v')^5 \gamma(u)^{-1} \exp[\beta\epsilon(v')]/|Z_J|,
\]
where \( \epsilon(v') \) is the energy-momentum of the particle. If one accepts the notion of inverse temperature 4-vector \( \beta \), then the exponent of the probability density function has
covariant form. In general, the Boltzmann factor in a Gibbs state should take the same covariant form [22].

From (20) it follows that $\beta$ must be a timelike future-directed 4-vector, otherwise the distribution (20) cannot be normalized.

The Juettner distribution function (18)–(20) revised for 2-dimensional spacetime has been confirmed by numerical simulations very recently [28].

One might ask what orientation it should take. Our conjecture is as follows: the entropy of an isolated system is a function of temperature and other thermodynamic parameters. Under the same restrictions, the direction of the 4-vector is oriented in a way so that the entropy takes a maximum value.

In this letter we dealt with a modest problem: the notion of temperature in special relativity. The notion of temperature in general relativity is much more complicated [4], since one has to consider the group of general coordinate transformations, instead of the Lorentz group. Firstly, in the classical framework ($\hbar = 0$) there does not exist a local definition of gravitational energy-momentum. Secondly, in the quantum framework, there exist fluctuations in quantum fields [29]. In particular, there does not exist a unique vacuum state even in the non-inertial frame of Minkowski spacetime [30], let alone in a curved spacetime.

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