A multi-component electron model on a lattice is constructed whose ground state exhibits a spontaneous ordering which follows the rule of map-coloring used in the solution of the four color problem. The number of components is determined by the Euler characteristics of a certain surface into which the lattice is embedded. Combining the concept of chromatic polynomials with the Heawood-Ringel-Youngs theorem, we derive an index theorem relating the degeneracy of the ground state with a hidden topology of the lattice. The system exhibits coloring transition and hidden-topological structure transition. The coloring phase exhibits a topological order.

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Physics and Mathematics are two disciplines in Exact Science which are known to nurture each other. There are famous problems in Mathematics which are physically realizable. An example is how to determine the configuration of $N$ repulsive point particles confined inside a spherical shell. It is then natural to ask whether there is a possible realization of other famous mathematical challenges. Among them we focus here on the celebrated four-color problem, formulated by F. Guthrie in 1852. This problem remained unsolved for more than hundred years until an affirmative answer was proved by Appel and Haken. Alongside, a plethora of new mathematical concepts were introduced, such as a computer-aided proof and Non-deterministic Polynomial-time (NP)-hard problems. In the framework of chromatic problems in Mathematics, coloring is a passive procedure to the map. Namely, the colors are artificially assigned to the regions by try-and-error so as to satisfy mathematical rules. A more focused question is whether there is an active coloring as a phenomenon in Nature, by which we mean a spontaneous emergence of an ordering such as molecular crystallization, magnetization, and orbital, whose ordering pattern follows the map-coloring rule.

In this work we give an affirmative answer to this question in terms of an electron model with several components (colors) defined on an arbitrary lattice. The pertinent ground states exhibits spontaneous ordering which follows the map-coloring rule. (Although the Potts model is known to exhibit a similar state, the mechanism is trivial due to the antiferromagnetic exchange interaction between classical spin variables. While ours is a quantum mechanical one.) Employing the concept of chromatic polynomials and combining it with the Heawood-Ringel-Youngs theorem, we derive an index theorem, relating the degeneracy of the electronic ground state with a hidden topology of the random lattice. Conversely, for a given number of components, we propose a constructive method for building a lattice on which the exact ground state can be realized. To this end we modify the Hajós construction of a random graph. The ground state exhibits coloring transition where the coloring phase is characterized by a topological order. This electronic model, hence, provides a meeting point for condensed matter physics, manifold embedding and topological graph theory. It is also another route to a realization of a topological order in quantum system.

The model Hamiltonian (see below) is composed of local non-negative operators. The procedure of constructing its ground state is very similar to the one employed in other models. Electronic realization of such Hamiltonian is presented in Refs. Here we consider a multi-component model. The number of components, e.g. spin or orbital projections, is denoted by $N_c$.

To expose the complete Hamiltonian, the “cell construction” technique is employed. Consider the usual finite random lattice sites $\Lambda$, which can be written as $\Lambda = \mathcal{C} \cup \mathcal{B}$ where $\mathcal{C} \cap \mathcal{B} = \emptyset$. (The usual regular and

![FIG. 1](image-url) (a) Example of sites and cells. The cells, $C_1$, $C_2$, $C_3$, and $C_4$ are composed of two, three, four, and five sites respectively. The open and bold circles denote the capital

![FIG. 1](image-url) (b) The lattice associated with (a). The lines denote possible hopping of electron. (c) The dual graph associated with (a).
periodic lattices are allowed.) For each site $i \in \mathcal{C}$, we define a cell $C_i = \{i\} \cup B_i$, where $B_i$ is a subset of $\mathcal{B}$ with $|B_i| \geq 1$ (see Fig. 1(a)). The cells $C_i$ and $C_j$ are said to be directly connected if they share at least one site (i.e., $C_i \cap C_j \neq \emptyset$). We assume that every site in $\mathcal{B}$ is contained in at least one cell and, for any pair of cells $C_i$ and $C_j$, there is a path of directly connected cells from $C_i$ to $C_j$. A given set $u \in \mathcal{B}$ may be shared by several cells and the number of cells to which the site $u$ belongs is the coordination number of cell, $N_u$. We assume that $\max(N_u) \leq N_c$. As usual, we denote by $c_{x,\sigma}(c^\dagger_{x,\sigma})$ the fermion annihilation (creation) operator at the site $x \in \Lambda$ with the component $\sigma$, and introduce fermion operators,

$$
\alpha_{i,\sigma} = \lambda^*_i c_{i,\sigma} + \sum_{u \in B_i} \lambda^*_u c_{u,\sigma} \quad \text{for } i \in \mathcal{C} \quad (1)
$$

$$
\beta_{u,\sigma} = \mu^*_u c_{u,\sigma} + \sum_{i \in D_u} \mu^*_i c_{i,\sigma} \quad \text{for } u \in \mathcal{B}, \quad (2)
$$

where $D_u$ is the set of sites $i$ in $\mathcal{C}$ such that $u \in C_i$. (See Fig. 1(a).) The $\alpha_{i,\sigma}^\dagger$ and $\beta_{u,\sigma}^\dagger$ create the localized single-electron states with the component $\sigma$ in the cell $C_i$ and $\{u\} \cup D_u$, respectively. Here, we assume that the real parameters $\lambda^*_i$ and $\mu^*_u$ are nonvanishing and they satisfy $\lambda^*_i \mu^*_i + \lambda^*_u \mu^*_u = 0$ for any pair of $i \in \mathcal{C}$ and $u \in \mathcal{B}$ contained in the same cell. We consider the Hamiltonian

$$
H = \sum_{\sigma=1}^{N_c} \sum_{u \in \mathcal{B}} \beta_{u,\sigma}^\dagger \beta_{u,\sigma} + \sum_{i \in \mathcal{C}} \sum_{\sigma=1}^{N_c} n_{i,\sigma} n_{i,\tau} + V \sum_{i \in \mathcal{C}} \sum_{u \in B_i} \sum_{\sigma=1}^{N_c} n_{i,\sigma} n_{u,\sigma}, \quad (3)
$$

where $n_{i,\sigma} = c^\dagger_{i,\sigma} c_{i,\sigma}$ is the number operator, and $U > 0$ and $V > 0$. The first term is the hopping term (see Fig. 1(b)), the second one is the onsite Coulomb repulsion, and the last one is the density correlation. For $N_c = 2$ with $\sigma = (\uparrow, \downarrow)$ the system describes fermions with spin $1/2$. To construct a model with $f$ orbitals one set $N_c = 2f$. With the further assumption such that $\lambda^*_i$ are positive for any $i \in \mathcal{C}$ and $u \in B_i$, we can also consider another Hamiltonian

$$
H = \sum_{\sigma=1}^{N_c} \sum_{u \in \mathcal{B}} \beta_{u,\sigma}^\dagger \beta_{u,\sigma} + \sum_{i \in \mathcal{C}} \sum_{\sigma=1}^{N_c} n_{i,\sigma} n_{i,\tau} + V \sum_{i \in \mathcal{C}} \sum_{\sigma=1}^{N_c} n_{i,\sigma} n_{i,\sigma}, \quad (4)
$$

where $\alpha^\dagger_{i,\sigma} = \sum_{u \in B_i} c^\dagger_{u,\sigma}$. In the following we fix the total electron number $N_e = \sum_{i,\sigma} n_{i,\sigma}$ to $|\mathcal{C}|$ and find the exact ground states of $H$. Consider the states of the form

$$
\Phi(\{\sigma\}) = \prod_{i \in \mathcal{C}} \alpha_{i,\sigma}^\dagger \{0\}, \quad (5)
$$

where $\{\sigma\}$ represents a configuration of components $(\sigma_j)_{j \in \mathcal{C}}$ of the $\sigma$-operators $\mathbf{1}$. Let $\{\sigma_G\}$ be a configuration satisfying that $\sigma_i \neq \sigma_j$ if the cells $C_i$ and $C_j$ are directly connected, and denote by $\mathcal{S}_G$ the collection of these configurations. Note that a site in $\mathcal{B}$ may be occupied by several (not more than $N_e$) electrons. Using the anticommutation relation $\{\alpha^\dagger_{i,\sigma}, \beta_{u,\sigma}\} = 0$ for $i \in \mathcal{C}$ and $u \in \mathcal{B}$, we find that the state $\Phi(\{\sigma_G\})$ is a zero mode of the Hamiltonians $\mathbf{1}$ and $\mathbf{1}$ and satisfies the relation $H \Phi(\{\sigma_G\}) = 0$. It is evident that the Hamiltonians are non-negative (positive semidefinite) and hence, $\Phi(\{\sigma_G\})$ is an exact ground state. It is possible to prove that any ground state can be represented as

$$
\sum_{\{\sigma_G\} \in \mathcal{S}_G} \phi(\{\sigma_G\}) \Phi(\{\sigma_G\}) \quad (6)
$$

with coefficients $\phi(\{\sigma_G\})$.

In order to elaborate upon the ground state and relation to the coloring problem, let us fix the terminology. A dual graph $G(\Lambda)$ is associated with the lattice $\Lambda$ in such a way that a vertex of $G(\Lambda)$ is defined by a cell $i$ of $\Lambda$. When the cells $C_i$ and $C_j$ are directly connected, an edge in $G(\Lambda)$ is defined between the vertices $i$ and $j$. (See Figs. 1(a) and (c).) Note carefully that the dual graph so defined is different from the usual dual lattice associated with the original lattice $\Lambda$. In standard Graph Theory, the lattice and its dual lattice have a one-to-one correspondence. In the present case, the dual graph is defined uniquely for a given lattice, but the inverse is not necessarily true. Due to this property, we have two kinds of redundancy as the lattice is constructed from a given dual graph. (See Fig. 2) (i) The number of sites composing a cell is arbitrary. (ii) The coordination number of cell, $N_u$, is arbitrary even when the coordination number of vertices is fixed. This plays a crucial role in the modification of the Hajós construction. As we argue below, the existence problem of the ground state is now transformed into that of vertex coloring of the dual graph.

For the case where the dual graph is planar, we can regard the cell $C_1$ as the country and the component $\sigma_i$ as its color. (In this case, the site $i$ is the capital and the

![FIG. 2: Two types of a configuration of cells (a) and (c), whose dual graph is the same (b). Then, for a given dual graph (b), we have two types of local configuration of the cells. The coordination number of the cell, $N_u$, of the site which is shared by $C_1$, $C_2$, and $C_3$ in (a) is 3. The number of such a site which is shared by $C_1$, $C_2$ in (c) is 2.](image-url)
sites in $B_2$ shared by several cells are the border lines.) The rule for generating configurations $\{\sigma_G\}$ is equivalent to the map-coloring rule. Therefore we call $\{\sigma_G\}$ as coloring. The answer to the question of whether a coloring $\{\sigma_G\}$ does exist depends on $N_c$ and the structure of the lattice. Finding coloring for a given lattice is recognized as one of the NP-hard problems. Fortunately, in the present coloring problem, there are a couple of routes to approach it. The first one tells us how to look for $N_c$ for a given lattice. Here the four color and Heawood-Ringel-Youngs theorems are applicable respectively for the two-dimensional planar graph and for higher-dimensional random graphs. The second route guides us how to construct the lattice for a given $N_c$. Here we need to modify the Hajós construction [1, 2] of random lattice.

As a first step, consider a lattice $\Lambda$ whose dual graph is planar. Employing the four color theorem [2], any random planar graph is at least four colorable, and there exists at least one ground state of the model with $N_c \geq 4$. For the model with $N_c < 4$, existence of a ground state depends on the structure of the lattice. For example, the ground state on a lattice whose dual graph is bipartite can be constructed for $N_c \geq 2$ because the bipartite graph is two-colorable. When a dual graph is $N_c$-colorable but not $(N_c - 1)$-colorable, we refer to $N_c$ as the chromatic number for constructing the ground state and denote it by $N_c^*$. (Note that if a graph is $N_c$-colorable, it is also colorable for the color $> N_c$. [1].)

Consider now a general random lattice. The problem of finding $S_G$ is then equivalent to coloring the random hyper-polyhedra. The dual graph $G$ is in general not planar and the four color theorem does not apply. Arbitrary dual graphs can be embedded into a higher surface with a proper genus $g$ [1]. They are classified into the torus with the Euler characteristic $\chi = 2 - 2g$, the projective plane with $\chi = 1 - 2g$, and the Klein bottle with $\chi = -2g$ [21]. We respectively denote them by $T^g$, $P^g$, and $K^g$. Now recall the Heawood-Ringel-Youngs theorem [1, 2]. Let $G$ be a dual graph on a higher surface $S$ of Euler characteristic $\chi$, and let each cell of the lattice have at most $m$ disjoint connected parts. Except for $(m, \chi) = (1, 2)$, an upper bound on $N_c^*$ is given by

$$N_c^* = \left[ \frac{6m + 1 + \sqrt{(6m + 1)^2 - 24\chi}}{2} \right],$$

(7)

where $[\ ]$ is the Gaussian symbol.

Conversely, for a given $N_c$, the lattice with $N_c^* = N_c$ is obtained by proper recursive use of the following procedures (i), (ii) and (iii): (i) Prepare the complete $N_c$-graph, consisting of $N_c$ vertices and edges connecting any two of vertices. Replace each vertex by an arbitrary cell and identify their sites in such a way that the lattice so constructed has the complete $N_c$-graph as its dual graph. (ii) In a dual graph of an already obtained lattice, identify two vertices not joined by an edge and form a new dual graph. (ii-a) Replace two cells associated with the two vertices identified above by a proper (arbitrary) cell, and identify sites so that the resulting lattice maintains the connectivity of the new dual graph. (ii-b) If the coordination number of cell, $N_u$, of the sites exceeds $N_c$, reduce it by the relation shown in Fig. 2(a) $\rightarrow$ (b) $\rightarrow$ (c). If any such reduction attempt fails, increase the number of sites in the replaced cell, or go back to (ii) and choose another pair of vertices for identification. (iii) (modification of the Hajós construction): For disjoint dual graphs $G_1$ and $G_2$ of lattices which are already obtained, remove an edge between the vertices $x_1$ and $y_1$ in $G_1$, which means the separation of two cells $C_{x_1}$ and $C_{y_1}$, and remove an edge between $x_2$ and $y_2$ in $G_2$. Identify $x_1$ and $x_2$, and add the edge between $y_1$ and $y_2$, which means an identification of site in the cells $C_{y_1}$ and $C_{y_2}$. Create a lattice by following the same ways as (ii-a) and (ii-b).

Generically, the ground state is degenerate. The degree of degeneracy, $\Gamma$, is equivalent to that of the different coloring possibilities of the dual graph $G(\Lambda)$. In the coloring problem, the number of the different coloring possibilities of a graph $G$ is described by the chromatic polynomial $P(G, x)$ where $x$ is the number of colors [3]. Therefore, we obtain the number of zero modes,

$$\Gamma = P\left(G(\Lambda), N_c\right).$$

(8)

For some categories of graphs, the chromatic polynomials are known. For a general graph, however, derivation of its chromatic polynomial is an NP-hard problem. Our analysis then shows that the degree of degeneracy is determined through [7] and [8]. This index theorem is quite intriguing, because the algebraic and topological structures of the higher surface determine the properties of the interacting electron model.

As an example, we demonstrate the above procedures within a simple lattice. (See Fig. 3(a).) The dual graph is shown in Fig. 3(b) which is termed as $K_5$, the complete graph of degree five. $K_5$ is irreducibly embedded into $T^2$ whose Euler characteristics is $\chi = 2 - 2 \times 1 = 0$. From eq. (7), we obtain $N_c = 7$, which is an upper bound of $N_c^*$. The chromatic polynomial for $K_5$ is known [1] as

$$P(K_5, x) = \prod_{p=1}^{5} (x - p + 1).$$

(9)

The $N_c^*$ is given by the minimal integer which satisfies $P(K_5, x) > 0$, that is, $x = 5$. Hence, for a given $N_c \geq 5$ the number of zero modes is given by $\Gamma = \prod_{p=1}^{5} (N_c - p + 1)$. An extension of this procedure to the other lattices is straightforward.

We now discuss two types of exotic phase transitions which this system exhibits. Consider an arbitrary lattice with $N_c^*$. Denote by $\mathcal{M}$ the manifold into which the dual graph is embedded irreducibly.

The first type is a coloring transition [21] (For a context of the Potts model, see [22, 23, 24, 25, 26]). Assume
that the number of components is controllable by injection and absorption. For example, when the components are related to electron spin, one can control the number of components by applying a magnetic field. And when the components are related to an orbital degree of freedom, it is controlled by a lattice distortion. For $N_c > N^*_c$, the ground state is given by $\Phi(\{\sigma_G\})$ and it is a coloring phase. If we reduce the number of components and obtain a system with $(N_c-1)$ components on the same lattice, the ground state is also described by $\Phi(\{\sigma_G\})$ with $(N_c-1)$ components. Within this operation, the degeneracy of the ground state undergoes a macroscopic change. The corresponding change of the system entropy is accompanied by an emission or absorption of latent heat. When the number of components is reduced further, the coloring transition occurs at $N_c = N^*_c$ because for $N_c < N^*_c$ the ground state is not described by $\Phi(\{\sigma_G\})$. As the system goes through $N^*_c$ the pattern of orbital ordering should undergo a dramatic change. (Mathematically, we define the transition point by the smallest zero of the chromatic polynomial associated with the dual graph.) The coloring phase, $N_c > N^*_c$, is characterized by a global consistency of coloring. In this sense, the ground state has a topological order.

The second type is a hidden-topological structure transition. This is a novel structural transition realized for a fixed number of components and lattice sites $\Lambda$. In this transition there is a variation of the manifold $M$ into another manifold $M'$ which undergoes a modification of the hopping network. The transition is characterized by the set of manifolds $(M, M')$ and, as before, is accompanied by a transfer of latent heat. For a given random lattice, the ground state properties are essentially determined by the topological structure of the higher surface into which the dual graph is embedded irreducibly, and not by the original lattice structure in real space (hence the adjective "hidden"). The ground state is then classified by the hidden topology.

Future research directions should include detailed studies pertaining to numerous concrete lattices and their transitions. The case of infinite lattices is of special interest in view of the thermodynamic limit and its relation to Ramseys theories.

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