Kaleidoscopic Symmetries and Self-Similarity of Integral Apollonian Gaskets

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Abstract

We describe various kaleidoscopic and self-similar aspects of the integral Apollonian gaskets - fractals consisting of close packing of circles with integer curvatures. Self-similar recursive structure of the whole gasket is shown to be encoded in transformations that forms the modular group $SL(2, \mathbb{Z})$. The asymptotic scalings of curvatures of the circles are given by a special set of quadratic irrationals with continued fraction $[n+1 : 1, n]$ - that is a set of irrationals with period-2 continued fraction consisting of 1 and another integer $n$. Belonging to the class $n = 2$, there exists a nested set of self-similar kaleidoscopic patterns that exhibit three-fold symmetry. Furthermore, the even $n$ hierarchy is found to mimic the recursive structure of the tree that generates all Pythagorean triplets.
Integral Apollonian gaskets (IAG) such as those shown in figure (I) consist of close packing of circles of integer curvatures (reciprocal of the radii), where every circle is tangent to three others. These are fractals where the whole gasket is like a kaleidoscope reflected again and again through an infinite collection of curved mirrors that encodes fascinating geometrical and number theoretical concepts[2]. The central themes of this paper are the kaleidoscopic and self-similar recursive properties described within the framework of Möbius transformations that maps circles to circles[3].

![Diagram of Apollonian gaskets](image)

**FIG. 1:** Integral Apollonian gaskets. The gaskets shown in panels (A) and (B) are dual to each other, as illustrated in panel (C) where we overlay the two gaskets. The duality described by the matrix $D$ (displayed in lower left) corresponds to each circle in the dual set passing through three of the tangency points of the original set of circles, and the reverse holds as well. For example, $(-1, 2, 2, 3)^T = D(4, 1, 1, 0)^T$ and $(4, 1, 1, 0) = D(-1, 2, 2, 3)^T$.

Named in honor of Apollonius of Perga who studied the geometrical problem of mutually tangent circles before 300BC, the building block of Apollonian gaskets are configurations of four mutually tangent circles whose curvatures $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ satisfy the following relation[4][5]:

$$2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2) = (\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)^2.$$

(1)
Here, the curvature of the outer circle that encloses all inner circles has to be taken negative to satisfy an equation. Eq. (1)), known as the Descartes theorem, can also be written as $v^T D v = 0$. Here $v = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and $v^T$ is its transpose and the matrix $D$ is defined in Fig. 1. Configuration of circles satisfying Descartes theorem will be referred as the “Descartes configurations”. Originally discussed by French philosopher and mathematician René Descartes in 1643 in a letter to Princess Elizabeth of Bohemia, Apollonian circles were rediscovered in 1936 by Chemistry Nobel laureate Frederick Soddy who published in Nature a poetic version of Descartes’ theorem, which he called “The Kiss Precise”.

For a given integral Apollonian packing, the investigations about its Diophantine properties such as what integers appear as curvatures and the number of circles with prime curvatures have piqued many mathematicians. The formula for the number of primes in an integral Apollonian gasket bears a striking resemblance to the Prime Number Theorem. These questions have been intimately linked to the hidden symmetries and consequently the deceptively simple geometrical construction of close packing of circles is strongly tied with sophisticated mathematics involving group theory. Furthermore, recent studies have shown that these abstract fractals are related to a quantum fractal known as the Hofstadter butterfly which models all possible integer quantum Hall states - the exotic topological states of matter. A detailed investigation of the recursive nature of the integral Apollonian gasket using Möbius transformations that form modular group $SL(2, \mathbb{Z})$ as described here shows that in close analogy to the integer curvatures of the gasket, the recursive structure of the topological quantum numbers of the butterfly, explained earlier using quantum mechanics are in fact rooted in pure geometry and number theory.

An exercise in geometry as illustrated in Figure 2, shows how to obtain a configuration of four mutually tangent circles starting with any three points. As every Descartes configuration is fully determined by three distinct points in a plane, this implies that any two such configurations are related by a unique conformal map - a Möbius transformation of the complex plane of the form,

$$z \mapsto f(z) = \frac{az + b}{cz + d}$$

A brief review of some of the properties of these transformations is given in Appendix A. Section 1 illustrates kaleidoscopic properties of the IAG. In Section 2, we introduce the Ford Apollonian gasket in which each circle’s curvature is the square of some non-negative integer. Its recursive structure is described in Section 3, by a Möbius transformation that forms $SL(2, \mathbb{Z})$ group where $(a, b, c, d)$ in the map are integers and Section 4 discusses its scaling properties. In section
5, Möbius maps that belong to the group $SL(2, C)$ where $(a, b, c, d)$ are in general complex numbers, are shown to map the entire $IAG$ to the Ford Apollonian gasket. Section 6 shows one of a kind self-similar kaleidoscopic patterns of the gasket. In Section 7, the hierarchical pattern of the Pythagorean tree is shown to encode part of the recursions of the $IAG$. In section (8) we discuss the Apollonian group and its relation to modular group $SL(2, Z)$. Appendix B gives a proof of Descartes theorem for Ford Apollonian configurations where it is shown to be a consequence of a property of the homogeneous functions.

![Diagram](image)

**FIG. 2:** (A) Given any three points $(P_1, P_2, P_3)$, one can construct a configuration of three mutually tangent circles of curvatures $(\kappa_1, \kappa_2, \kappa_3)$: we first draw a circle (dotted blue) through these points and then draw tangents to this circle. Three tangents meet at $C_1, C_2$ and $C_3$ and using $(C_1, C_2, C_3)$ as the centers, we now draw three circles which are mutually tangent as the distance between $P_1$ and $C_1$ is same as the distance between $P_2$ and $C_2$ and so on. The curvature $\kappa_4$ and hence the radius of the fourth circle is determined by Descartes theorem (Eq. (1)). The center $C_4$ is determined in terms of the curvatures of the four circles[13].

I. APOLLONIAN GASKET - A KALEIDOSCOPE

Apollonian gaskets exhibit numerous kaleidoscopic symmetries - a rare and fascinating feature among known functions. In various figures below, the circular mirrors describing kaleidoscopic symmetries will be shown as dotted blue circles.

The existence of a simplest kaleidoscope in a gasket follows from the Descartes formula (1). Given any three mutually tangent circles of curvatures $(\kappa_1, \kappa_2, \kappa_3)$, there are exactly two possible circles $\kappa_4$ or $\overline{\kappa}_4$ that are tangent to these three circles:

$$\kappa_4(\overline{\kappa}_4) = (\kappa_1 + \kappa_2 + \kappa_3) \pm 2\sqrt{\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_2}$$  

(3)
Equation (3) leads to a linear equation connecting the two solutions:

\[ \kappa_4 + \bar{\kappa}_4 = 2(\kappa_1 + \kappa_2 + \kappa_3). \]  

(4)

This linear equation implies that if the original four circles have integer curvature, all of the circles in the packing will have integer curvatures. This is because starting with three mutually tangent circles, we can construct two distinct quadruplets \((\kappa_1, \kappa_2, \kappa_3, \kappa_4)\) and \((\kappa_1, \kappa_2, \kappa_3, \bar{\kappa}_4)\) and thus adding additional circles to the gasket. In fact we obtain the entire gasket as the equation (4) can also be written as \(\kappa_2 + \bar{\kappa}_2 = 2(\kappa_1 + \kappa_4 + \kappa_3)\) and so on.

\[ \kappa_2 + \bar{\kappa}_2 = 2(\kappa_1 + \kappa_4 + \kappa_3) \]  

FIG. 3: Examples of circular mirrors shown as dotted blue lines. Upper panel shows that the two solutions of (4) are mirror image. The lower two panels show two examples: Pappus chain - chain of (blue) circles that are tangent to two mutually tangent (black) circles \((\kappa_1, \kappa_2)\) which are mirror images of each other through Pappus mirror \((\text{dotted blue circles})\). Pappus mirror passes through the tangency points of Pappus chain and it reflects \(\kappa_1\) and all the circles that are tangent to it \(\text{(red circles)}\) to \(\kappa_2\) and all the circles that are tangent to it \(\text{(green circles)}\).

A remarkable aspect of the pair of solutions such as \((\kappa_4, \bar{\kappa}_4)\) is that they are mirror images of each other through a circular mirror that passes through the tangency points of \((\kappa_1, \kappa_2, \kappa_3)\) as shown in Fig. (3). Therefore, packing of circles in the whole gasket is via kaleidoscopic images in the curvilinear triangular spaces in between the circles.

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Lower caption in Fig. (3) shows a different kind of mirror due to global symmetries where an entire hierarchical pattern and its mirror image is part of the gasket. Given two tangent circles of curvatures $\kappa_1$ and $\kappa_2$, there exists a chain of (blue) circles where two consecutive circles of the chain are tangent to each other as seen in the figure. These are examples of Pappus chains, investigated by Pappus of Alexandria in the 3rd century [14]. The tangency points of the chains of circles lie on a circle. The circular mirror at these tangency points - dubbed Pappus mirror” reflect $\kappa_1$ into $\kappa_2$ and also the entire hierarchical set of circles that are tangent to $\kappa_1$ (red circles) to the hierarchical set that are tangent to $\kappa_2$ (green circles). In other words, for the hierarchical configuration, where all circles share a common circle, which we refer as the “boundary circle”, there exists twin configuration - its mirror image. We emphasize that the hierarchical sets under consideration here consist of self-similar Descartes configuration that share the boundary circle. Two mutually tangent circles of curvatures $\kappa_1$ and $\kappa_2$ are respectively the boundary circles for the red and the green hierarchy and all the circles in the Pappus chains are tangent to both these boundary circles.

FIG. 4: Self-similar kaleidoscopic symmetry in a hierarchy where the outer circle (green) becomes the inner circle (red) at the next level of the hierarchy. Inner and outer circles are mirror images through (dotted) blue circle. As we zoom into the innermost circles (whose curvatures are shown below the arrows), we see a scaled version of the original pattern. The sequence of ratios of curvatures (15/1, 209/15, 2911/209, 40545/2911, ...) approaches a constant $(2 + \sqrt{3})^2$.

A rather exotic kind of hierarchical kaleidoscopic phenomena prevalent in packing of circles is shown in Figure (4) where the object and its mirror image are self-similar. In other words, there exists some special set of recursive patterns in the gasket that are captured by kaleidoscopes that reproduce the exact replica of the original set after appropriate scaling. Relationship of this
self-similar kaleidoscope that exhibits three-fold symmetry with the gasket in figure 1 will be discussed later.

II. FORD CIRCLES

FIG. 5: Gasket made up of Ford circles where each circle with center at \( \left( \frac{p}{q}, \frac{1}{2q^2} \right) \) represents a primitive fraction \( \frac{p}{q} \). After scaling by a factor of 2, curvatures of all Ford circles are integer-squared.

Perhaps the simplest example of an hierarchical set consisting of packing of circles is a gasket made up circles where the curvatures of all the circles are integer-squared. These circles are tangent to the \( x \)-axis and provide a pictorial representation of fractions as discovered by an American mathematician Lester Ford in 1938. Ford showed\[15\] that at each rational point \( \frac{p}{q} \) where \( p \) and \( q \) are relatively prime, one can draw a circle of radius \( \frac{1}{2q^2} \) and whose center is the point \( (x, y) = \left( \frac{p}{q}, \frac{1}{2q^2} \right) \). Tangent to the \( x \)-axis in the upper half of the \( xy \)-plane, the curvatures of the Ford circles are \( 2q^2 \). Circles in the figure (5) have their curvatures \( q^2 \), scaled by a factor of half from Ford circles. We will continue to refer them as the Ford circle representing the fraction \( \frac{p}{q} \) and label them as \( \kappa_{\frac{p}{q}} \).

The key characteristic of the Ford circles is the fact that two Ford circles representing two distinct fractions never intersect. The closest they can come is being tangent to each other. This happens when the two fractions are “Farey neighbors”, neighboring fractions in the Farey tree. Three mutually tangent Ford circles representing three (left, center and right ) fractions \( \phi_L = \frac{p_L}{q_L} \), \( \phi_c = \frac{p_c}{q_c} \), \( \phi_R = \frac{p_R}{q_R} \), all tangent to the \( x \)-axis form a special case of Descartes configuration
obeying the Farey relation \[15\],

\[
\frac{p_c}{q_c} = \frac{p_L + p_R}{q_L + q_R} \tag{5}
\]

Consequently, the triplet of Farey neighbors, referred as the friendly triplet satisfy the follow identities:

\[
|q_L p_R - q_R p_L| = 1, \quad |q_L p_c - q_c p_L| = 1, \quad |q_R p_c - q_c p_R| = 1. \tag{6}
\]

Figure (5) shows an Apollonian gasket made up of the Ford circles where the curvatures are integer squared and we will refer such gaskets as the “Ford Apollonian gasket”. The dual of the Ford Apollonian gasket a subset of circles seen in figure (1) that exhibit reflection symmetry about the \(x\)-axis satisfying \(\kappa_1 + \kappa_2 = \kappa_3 + \kappa_4\). In the rest of the paper, we will denote Descartes configuration of Ford circles representing the friendly triplet \([p_L, p_c, p_R]\) as \((\kappa_c, \kappa_R, \kappa_L)\) and its dual as \((-\kappa_0, \kappa_1, \kappa_2, \kappa_3)\). Here \(\kappa_c > \kappa_R > \kappa_L\) and \(\kappa_0 < \kappa_1 < \kappa_2 < \kappa_3\).

It was pointed out by Richard Friedberg\[16\] that the Descartes’s theorem when applied to Ford circles is a consequence of a property of homogeneous functions of any three variables \((\lambda_1, \lambda_2, \lambda_3)\) with \(\lambda_1 \pm \lambda_2 \pm \lambda_3 = 0\). In view of the Farey relation \(q_c = q_L + q_R\), as shown in the Appendix B, this gives \(2(q_1^2 + q_2^2 + q_3^2) = (q_1^2 + q_2^2 + q_3^2)^2\) which is the simplified version (\(\kappa_4 = q_4^2 = 0\)) of the Descartes theorem with three mutually tangent Ford circles, tangent to the \(x\)-axis.

The Ford Apollonian gasket has certain properties that play an important role in determining the self-similar scalings of the entire Apollonian gasket even through the Ford circles form only a small subset of the fractal. Central result of this paper is that packing of the Ford circles determines the self-similar properties of the entire gasket. Further, there is a dichotomy in the recursive structure of the gasket as half of the gasket seems to follow the recursive pattern of the Pythagorean tree\[17\]. In addition, Ford circles also have a partner - a symmetric Descartes configuration where two of the three inner circles have same curvature and there exists a very special configuration that evolves into three-fold symmetry\[9\] as shown in Fig. (4). These three-fold symmetric configurations that exhibit a self-similar hierarchy with Kaleidoscopic symmetries lay scattered throughout the whole gasket.

III. SELF-SIMILARITY OF FORD APOLLONIAN GASKET

To establish recursion relations associated with the nesting of the Ford circles, we relate two sets of triplets: namely \((z_1, z_2, z_3)\) and \((w_1, w_2, w_3)\) as shown in figure (6). The self-similar
FIG. 6: With the root configuration \((\kappa_c, \kappa_R, \kappa_L) = (4, 1, 1)\) representing three fractions \((z_1, z_2, z_3) = \left(\frac{0}{1}, \frac{1}{2}, 1\right)\), we pick any other configuration of three mutually tangent Ford circles labeling three fractions as \((w_1, w_2, w_3) = \left(\frac{p_L}{q_L}, \frac{p_c}{q_c}, \frac{p_R}{q_R}\right)\) representing left, central and right circle. In principle, one can choose any of the three tangency points and their corresponding image of the Descartes configurations, that is, \(z_i\) and \(w_i\) need not lie on the \(x\)-axis. The choice indicated here simplifies the derivation of the recursion relation.

hierarchy preserves this relationship at all levels and we will denote two successive sets as \([\phi_L(l), \phi_c(l), \phi_R(l)]\) and \([\phi_L(l + 1), \phi_c(l + 1), \phi_R(l + 1)]\). The recursion relation underlying the self-similar pattern is a fixed Möbius transformation determined by mapping the triplets \((z_1, z_2, z_3) = \left(0, \frac{1}{2}, 1\right)\) to the triple \((w_1, w_2, w_3) = \left(\frac{p_L}{q_L}, \frac{p_c}{q_c}, \frac{p_R}{q_R}\right) = (\phi_L^*(1), \phi_c^*(1), \phi_R^*(1))\).

To determine the constants \((a, b, c, d)\) of the map \(w = f(z) = \frac{az + b}{cz + d}\), we note that:

\[
f(0) = \frac{b}{d} = \frac{p_L^*}{q_L}, \quad f(1/2) = \frac{a + b}{c + d} = \frac{p_R^*}{q_R^*}, \quad f(1) = \frac{a + b}{c + d} = \frac{p_R^*}{q_R}
\]

Using the Farey relation (Eq. 6), we get

\[
a = p_R^* - p_L^*, \quad b = p_L^*, \quad c = q_R^* - q_L^*, \quad d = q_L^*
\]

Therefore, the Möbius map that underlies the recursive structure of the Ford Apollonian gasket is given by a matrix, which we denote as \(\mathcal{F}^*\). The corresponding \(\phi = \frac{p}{q}\) recursions can be written as,

\[
\phi(l + 1) = \frac{(p_R^* - p_L^*)\phi(l) + p_L^*}{(q_R^* - q_L^*)\phi(l) + q_L^*} \equiv \frac{ap + bq}{cp + dq} \equiv \mathcal{F}^*
\]
fraction $\frac{p_x}{q_x}$ where $x = L, c, R,$

$$\begin{pmatrix} p_x(l + 1) \\ q_x(l + 1) \end{pmatrix} = F^* \begin{pmatrix} p_x(l) \\ q_x(l) \end{pmatrix}$$

(10)

FIG. 7: Upper and lower panels respectively illustrate $n^* = 1$ and $n^* = 2$ self-similar hierarchies where three panels in each case represent three levels of the hierarchy consisting of Ford circles. Starting from the root Apollonian (black circles), we pick an Apollonian (red circles). The self-similar hierarchy corresponds to continuing this process iteratively where the relationship between two successive iterations is preserved. That is the relation between $(4, 1, 1)$ and $(25, 9, 4)$ is the same as the relation between $(25, 9, 4)$ and $(169, 64, 25)$ and so on. Asymptotic scale invariance is signaled as $\frac{25}{4} = 6.25, \frac{169}{25} = 6.76 \rightarrow \left(\frac{3+\sqrt{5}}{2}\right)^2 \approx 6.85$ (golden-hierarchy, upper panels) and $\frac{121}{9} = 13.44, \frac{1681}{121} = 13.89 \rightarrow (2+\sqrt{3})^2 \approx 13.93$ (diamond hierarchy, lower panels). As described later, the lower hierarchy where the parity of $q_c$ is conserved, mimics the tree structure of the Pythagorean tree.

IV. SCALING

With unit determinant, the eigenvalues of the transformation matrix $F^*$ are real. Denoting the pair of these eigenvalues as $(\zeta, \zeta^{-1}), \zeta > 1$ determines[9] the asymptotic scaling factors as:

$$\zeta = \lim_{l \to \infty} \frac{p_x(l + 1)}{p_x(l)} = \lim_{l \to \infty} \frac{q_x(l + 1)}{q_x(l)} = (q^*_L + p^*_R - p^*_L) \pm \sqrt{(q^*_L + p^*_R - p^*_L)^2 - 1}, \quad (11)$$
The ratio of curvatures of the circle at two successive levels is given by,

$$\lim_{l \to \infty} \frac{\kappa(l + 1)}{\kappa(l)} \to \zeta^2$$

(12)

Expressed as a continued fraction expansion, these irrationals satisfy the quadratic equation $$\zeta^2 + n^* \zeta - n^*$$ whose solutions are given by,

$$\zeta = [n^* + 1; 1, n^*], \quad n^* = q^*_L + p^*_R - p^*_L - 2$$

(13)

Figure (7) shows two examples of self-similar Descartes configurations of Ford circles corresponding to $$n^* = 1$$ (upper panel) and $$n^* = 2$$ (lower panel). Respectively referred as the golden and the diamond hierarchies[9], they correspond to the Möbius transformations

$$f(z) = \frac{1}{-z+3} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$$

and

$$f(z) = \frac{z+1}{2z+3} \equiv \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$ The eigenvalues of these two matrices $$\frac{3+\sqrt{5}}{2}$$ and $$2 \pm \sqrt{3}$$ determine the scaling factor $$\zeta$$. We emphasize that the Möbius map (9) captures the self-similarity of the whole gasket provided the root configuration is $$(4, 1, 1)$$. General case with arbitrary root is described later.

V. SELF-SIMILAR HIERARCHIES WITH NON-FORD CIRCLES

The Möbius transformations map circles to circles, preserving tangencies. Therefore, every Descartes configurations consisting of “non-Ford circles” - circles that are not tangent to $$x$$-axis, can be related to the Ford circles. The process of describing the self-similarity of the non-Ford hierarchies involves first finding the conformal image of the hierarchy of the non-Ford circles to the corresponding Ford circles. One can then use the equation (9) to describe the self-similar characteristics. Fig. (8) illustrates this process when the non-Ford circle hierarchical pattern is tangent to $$\kappa_{\frac{4}{3}}$$. In this example, we first obtain the Möbius transformation that maps the $$x$$-axis to $$\kappa_{\frac{4}{3}}$$ in the complex plane. As shown in the left panel, the three points on the $$x$$-axis and their conformal images on $$\kappa_{\frac{4}{3}}$$ can be chosen as: $$z_1 = 0 \to w_1 = 0, \quad z_2 = \frac{1}{2} \to w_2 = \frac{2}{5} + i\frac{1}{5}, \quad z_3 = \frac{1}{3} \to w_3 = \frac{3}{10} + i\frac{1}{10}$$. The resulting map is given by $$B : z \to w = f(z) = \frac{z}{-z+1}$$. Its inverse transforms the green circles to the red circles, and therefore, the hierarchical structure of the non-Ford circles is given by the general map $$B^{-1}F^*B$$. Eq. (9) determines $$F^*$$ where $$p^*_L = 1, q^*_L = 3, p^*_R = 1$$ and $$q^*_R = 2$$.

Table (1) gives some examples of Möbius transformations that relate $$x$$-axis to other circles in the $$I\{A\}G$$. The corresponding “mirrors” listed in the table (see Fig. (9)) provides geometrical
FIG. 8: The green and the red boxes in the left panel respectively show the locations of the nesting patterns of the green and the red circles - the two hierarchies shown in the upper and lower right panels. These two hierarchies exhibit same scaling exponents. There distinct Descartes configurations of red and green circles show three levels of the self-similar hierarchies.

representation of this mapping. We note that with the first three entries in the Table that show the mappings of the $x$-axis to the Ford circles $\kappa_{p/q}$ can be written as $g(z) = \frac{1}{q} f(q^2(z - \frac{p}{q})) + \frac{p}{q}$ where $f(z) = \frac{z}{iz + 1}$ maps the $x$-axis to $\kappa_{0/1}$. These transformations form a subgroup of $SL(2, C)$ with trace, $a + d = 2$. They are “parabolic as they have only one fixed point.

All the transformations described in the table have unit determinant and real trace. Such transformations, which we denote as $B$, that relate boundary circle $\kappa_{p/q}$ of an hierarchy to the $x$-axis, the self-similar recursive pattern is given by $BF^*B^{-1}$. Therefore, the scaling factor associated with the hierarchy is same as the scaling factor associated with their corresponding Ford circle hierarchy.

Finally, when the root configuration is not the $(4, 1, 1)$, the chosen root is conformally mapped to $(4, 1, 1)$ as illustrated in figure (10). In this example, using the triplet $(z_1, z_2, z_3)$ and its image $(w_1, w_2, w_3)$ leads to the map $B : w = f(z) = \frac{(3-i)z-1}{(-3i)z+4}$ that transforms the chosen root to $(4, 1, 1)$. We then use the map $F^* : z \rightarrow \frac{z+1}{2z+3}$ that describes the self-similar recursions in the Ford circles in the lower panel. In other words, the self-similar characteristics of the upper panel hierarchy is described by the map $BF^*B^{-1}$. 


TABLE I: Examples of Möbius transformations and the corresponding mirrors that map x-axis to the Ford circles $\frac{p}{q}$ or the conformal images of the Ford circle, tangent to $\kappa_0$, which we denote as $\bar{\kappa}_{\frac{p}{q}}$. Fig. (9) shows two examples of the mirrors, corresponding to the first two entries in this table.

| Nature of Map | $f(z)$ | Equation of the Mirror |
|---------------|--------|------------------------|
| $x^-$ axis $\rightarrow \kappa_{\frac{0}{1}}$ | $\frac{z}{iz+1}$ | $x^2 + (y - 1)^2 = 1$ |
| $x^-$ axis $\rightarrow \kappa_{\frac{1}{2}}$ | $\frac{(1-2i)z+i}{4iz+(1+2i)}$ | $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$ |
| $x^-$ axis $\rightarrow \kappa_{\frac{1}{2}}$ | $\frac{(1-3i)z+i}{9iz+(1+3i)}$ | $(x - \frac{1}{2})^2 + (y - \frac{2}{5})^2 = (\frac{2}{5})^2$ |
| $x^-$ axis $\rightarrow \kappa_{\frac{1}{3}}$ | $\frac{(2-5i)z+2i}{14iz+(2+5i)}$ | $(x - \frac{5}{17})^2 + (y - \frac{2}{7})^2 = (\frac{1}{7})^2$ |

FIG. 9: Left and right panels show the Pappus chains (blue circles) and the corresponding Pappus mirrors (dotted blue circles) that reflect the x-axis to $\kappa_{\frac{0}{1}}$ and $\kappa_{\frac{1}{2}}$ respectively. The red and the green boxes are the triangular regions hosting circles that are mirror images of each other.

VI. SELF-SIMILAR KALEIDOSCOPE

Among the infinity of self-similar hierarchies in the $IAG$, there is one type of a hierarchy that is characterized by a very unique symmetry shown in Fig. (4) and also in Fig. (11). These configurations are related to the Ford Apollonian as every Descartes configuration of the Ford circles can be mapped to another Descartes configuration of the non-Ford circles where two of the inner circles have the same curvatures[9]. In other words, given $(\kappa_c, \kappa_R, \kappa_L)$ or its dual $(-\kappa_0, \kappa_1, \kappa_2, \kappa_3)$, we can find a Descartes configuration which we denote as $(\kappa^s_a, \kappa^s_b, \kappa^s_c, \kappa^s_e)$ where,

$$\kappa^s_a = \eta \kappa_0, \quad \kappa^s_b = \frac{\eta}{2} (\kappa_1 + \kappa_2), \quad \kappa^s_c = \eta (2 \kappa_1 - \kappa_0)$$

(14)
FIG. 10: (Upper) Starting with a root Apollonian \( (28, 9, 4, 1) \), we select a configuration \( (568, 217, 72, 9) \) (red circles) that bears the same relationship with the root as \( (64, 25, 9) \) (lower panel) bears with the main root \( (4, 1, 1) \) associated with the Ford Apollonian gasket. Using \( (z_1, z_2, z_3) \) from the upper graph and their corresponding \( (w_1, w_2, w_3) \) in the lower graph, we obtain the Möbius map \( B : w = \frac{(\frac{-9+6i}{2}+(4-i)}{(-14+16i)z+(7-4i)} \) that transforms the root \( (28, 9, 4, 1) \) to \( (4, 1, 1) \).

\[
\eta_a^s = \frac{\eta}{2} (\kappa_c - \kappa_R - \kappa_L), \quad \kappa_b^s = \frac{\eta}{2} \kappa_c, \quad \kappa_c^s = \frac{\eta}{2} (\kappa_c - \kappa_R + 3 \kappa_L)
\]

Here \( \eta = 1 \) for \( \kappa_0 \)-odd and \( \eta = 2 \) for \( \kappa_0 \)-even.

Among these symmetric configurations, there exists a class of hierarchies where the above configuration evolves towards an asymptotic three-fold symmetry\[9\] as shown in figures (4) and (11). This is due to an invariant \( \Delta \)[9],

\[
\Delta = \kappa_b^s - \kappa_c^s
\]

which remains unchanged under iteration of the recursions. In our detailed study of various such configurations, differing in \( \Delta \), which was always found to be a prime number, all such hierarchies belong to the nested configurations of the diamond hierarchy. Relationship of three fold symmetry with diamond hierarchy follows from Eq. [1][9] as with \( \kappa_1 = \kappa_2 = \kappa_3, \frac{\kappa_4}{\kappa_4} = (2 + \sqrt{3})^2 \).
FIG. 11: Symmetric partners corresponding to lower panel of Fig. (8) where $\Delta = 3$. The corresponding Fig. (4) describes the symmetric partners of the upper panel of the Fig. (8) corresponding to $\Delta = 1$. Differing in the initial root, these two hierarchies belong to same universality class characterized by scaling exponent $\zeta = (2 + \sqrt{3})$.

VII. APOLLONIAN-PYTHAGOREAN MEET

The self-similar recursions described by the Ford Apollonian gasket $(\kappa_c, \kappa_R, \kappa_L)$ are found to belong to two classes depending upon whether they preserve “parity”- that is even or oddness of $\kappa_c$ (or $q_c$). It turns out that this parity conserving feature is directly correlated with the parity of the integer $n^*$ (see Eq. (13)). For the recursions characterized by even $n^*$, the parity is conserved while for the recursions with odd-$n^*$, the parity is not conserved. Fig. (7) shows examples of both of these cases.

An interesting aspect of the parity conserving recursions is that they are described by the Pythagorean treeciteHall, J3, SatPT. As shown in Fig. (12), the Pythagorean tree provides orderly arrangements of all primitive Pythagorean triplets $(n_x, n_y, n_t)$ where $n_x^2 + n_y^2 = n_t^2$ using three matrices $(H_1, H_2, H_3)$:

$$
H_1 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 3 & 3 \end{pmatrix}, \quad H_3 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix} \quad (17)
$$

For example, with Pythagorean triple $v = (1, 0, 1)$, we see that $H_1 v^T = (5, 12, 13)^T$, and so on. There is also a two-dimensional representation of $H_i$ using 2000 year old Euclid parametrization of the Pythagorean triplets given in terms of pair of integers $(q_R, q_L)$, $q_R > q_L$. 

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FIG. 12: Upper panel shows mapping between two mutually tangent Ford circles to a Pythagorean triplet\[17\]. The lower panel shows the diamond hierarchy described by $H_1 H_3$, highlighted with thick dark lines along with the Descartes configurations. Starting with the root $(4, 1, 1)$ that corresponds to the Pythagorean triplet $(1, 0, 1)$, the triplets $(15, 8, 17)$ and $(209, 120, 241)$ represent the level 1 and level 2 of Descartes configurations shown in red circles.

where,

\[
\begin{align*}
  n_x &= \eta(q_R q_L), \\
  n_y &= \eta\left(\frac{q_R^2 - q_L^2}{2}\right), \\
  n_z &= \eta\left(\frac{q_R^2 + q_L^2}{2}\right).
\end{align*}
\]  \hspace{1cm} (18)

Here $\eta = 1$ when $q_c$ is even and hence $q_L$ and $q_R$ are both odd integers and $\eta = 2$ otherwise. The possibilities of generating three more triplets from a given triplet, expressed in terms a pair $(q_R, q_L)$ can be written as three $2 \times 2$ matrices,

\[
\begin{align*}
  h_1 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\
  h_2 &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \\
  h_3 &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]  \hspace{1cm} (19)

The hierarchical pattern of parity-conserving Ford Apollonian can be associated with the
hierarchical pattern of the Pythagorean tree[17]. Given a Ford Apollonian \((\kappa_c, \kappa_R, \kappa_L)\), a linear transformation relates the Apollonian to a Pythagorean triplet \((n_x, n_y, n_t)\) as:

\[
\begin{pmatrix}
  n_x \\
  n_y \\
  n_t
\end{pmatrix} = \frac{1}{\eta}
\begin{pmatrix}
  1 & -1 & -1 \\
  0 & 1 & -1 \\
  0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
  \kappa_c \\
  \kappa_R \\
  \kappa_L
\end{pmatrix},
\]

(20)

With \(\eta = 1\) or \(2\), there are two Pythagorean trees related simply by swapping \(n_x\) and \(n_y\). The key point to be noted here is that the Pythagorean triplets in each of these two trees retain their parity along the tree. Consequently, only parity conserving recursions of the Ford Apollonian are described by the Pythagorean tree. As a consequence of this dichotomy in characterization of self-similarity of an Apollonian gasket, half of the hierarchies in the Ford Apollonian gaskets share the recursive structure of the Pythagorean tree. The even parity hierarchies are generated by \((3, 4, 5)\) using \((H_1, H_2, H_3)\) while the odd-parity hierarchy is generated by \((4, 3, 5)\) using \((H_3, H_2, H_1)\). The diamond hierarchies which conserves \(q_c\) parity as described above correspond to \(H_3 H_1\). The hierarchical aspect of the golden hierarchy is not described by the tree as the corresponding recursions do not conserve parity as seen in figure (7). Recursions correspond to zigzagging between the \(\eta = 1\) and \(\eta = 2\) Pythagorean trees.

As a generalization of the relation between the Pythagorean triplet and the Descartes configuration of the Ford circles, we note that the four curvatures \((\kappa_1, \kappa_2, \kappa_3, \kappa_4)\) of any Descartes configuration are related to the well known quadruplets - \((N_x, N_y, N_z, N_t)\) [7], known as the Lorentz quadruplets, where \(N_x^2 + N_y^2 + N_z^2 = N_t^2\) [6]. This relationship is given by,

\[
\begin{pmatrix}
  N_x \\
  N_y \\
  N_z \\
  N_t
\end{pmatrix} =
\begin{pmatrix}
  1 & -1 & -1 & -1 \\
  0 & 0 & 0 & 2 \\
  0 & 1 & -1 & 0 \\
  1 & 1 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
  \kappa_1 \\
  \kappa_2 \\
  \kappa_3 \\
  \kappa_4
\end{pmatrix},
\]

(21)

It turns out that the Lorentz quadruplets associated with the four curvatures need not be primitive as is found to be case with the golden hierarchy. Furthermore, to best of our knowledge, there is no known quadruplet tree that generates all the primitive or non-primitive Lorentz quadruplets. Therefore, the representation of the \(\mathcal{LAG}\) with a tree like structure of Lorentz quadruplets remains an open question.
VIII. FROM SIMPLE GEOMETRY TO ELEGANT GROUP THEORY

The geometric construction of kaleidoscopic images to describe Apollonian packing can be formulated using Apollonian group where adding additional circles is accomplished by applying four matrices to a root configuration[2]. Eq. (4) determines the four generators $S_i$, ($i = 1 - 4$) of the group as,

$$S_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

These generators do not describe self-similar characteristics of the gasket as the application of $S_i$ on a column matrix consisting of four curvatures in monotonic order does not result in a new set of curvatures that preserves the monotonicity of the curvatures. That is, a self-similar hierarchy cannot be associated with a string of $S_i$. However, It is possible to define an alternative set of four matrices $D_i$ that preserves the monotonicity of the curvatures:

$$D_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2 & -1 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 2 & 2 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 2 & 2 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of the product of such matrices determine the self-similar scaling[10]. The golden ($n = 1$) and the diamond ($n = 2$) hierarchies described above are characterized by $D_3^2$ and $D_3^2D_2$, encoded in the eigenvalues ($(\frac{3+\sqrt{5}}{2})^2, 1, 1$) and $(2 \pm \sqrt{3})^2, 1, 1$) of these string of matrices.

As described earlier the self-similar properties of the whole gasket are described by the Möbius transformations $w = f(z) = \frac{az+b}{cz+d}$ as shown in equation [9]. These transformations consisting of $2 \times 2$ matrices of integers with unit determinant form the “Modular group” $SL(2, Z)[20]$. Modular group is a sub group of $SL(2, C)$ - a group of $2 \times 2$ matrices with unit determinant, where matrix entries are complex numbers. $SL(2, C)$ describes the recursive structure of the entire IAG.

Finally, we note that with every self-similar hierarchy, we can associate a Möbius transformation as well as a string of $D_i$ matrices whose eigenvalues determine the asymptotic scaling. There is
FIG. 13: Without showing the Ford circles, the sequences of upper (nested triplets of black and red dots) and lower (chain of black and blue dots) fractions show two levels of two distinct hierarchies, corresponding to Descartes configurations described by the friendly triplets \([0, 1/2, 1] \rightarrow [1/3, 3/8, 2/5] \rightarrow [4/11, 11/50, 7/19] \ldots \) and \([0, 1/3, 1/4] \rightarrow [1/3, 5/14, 4/11] \rightarrow [4/11, 19/52, 15/51] \ldots \). Recursions for both hierarchies are characterized by the same Möbius transformation with scaling \((2 + \sqrt{3})^2\). However, they differ in \(D_i\) and \(H_i\) strings as described in the text.

However an important distinction in these two characterization of the self-similar hierarchy. As illustrated in Fig. [13], two very distinct hierarchies, one describing a nested set of circles while the other is a chain of circles, both exhibiting same scaling, are represented by the same Möbius transformation. In contrast, within the Apollonian group, these two hierarchies are respectively described by \(D_2^2D_2 \equiv h_3h_1\) and \(D_2D_3^2 \equiv h_1h_3\). In the first case, the nested set of circles exhibit self-similar kaleidoscopic symmetries. In contrast, the \(D_2D_3^2 \equiv h_1h_3\) hierarchy does not exhibit this special symmetry. These differences are reflected in the eigenvectors of \(h_3h_1\) and \(h_1h_3\) which are respectively given by \(\begin{pmatrix} \sin \frac{\pi}{6} \\ \cos \frac{\pi}{6} \end{pmatrix}\) and \(\begin{pmatrix} \sin \frac{\pi}{12} \\ \cos \frac{\pi}{12} \end{pmatrix}\).

IX. CONCLUSION

Apollonian circle packing as described here using various number theoretical and geometrical properties, relates this fascinating fractal to the Ford circles, the Farey tree that generates all rationals as well as the the Pythagorean tree for generating primitive Pythagorean triplets. The entire framework can be described elegantly using Möbius transformations and the group theory involving special linear groups \(SL(2, C)\), \(SL(2, Z)\) and the Apollonian group. The central result of this paper, that the self-similar properties of the entire Apollonian gasket are encoded in the Ford circles, can be visualized geometrically due to various kaleidoscopic symmetries of the gasket as
shown in Table (1) and further illustrated in Fig. (9). The self-similar recursions associated with 
the packing are given by a Möbius map (9) forming a modular group $SL(2, \mathbb{Z})$, characterized by 
the matrix $F^*$ whose eigenvalues describe the asymptotic scalings. Intriguingly, a very special 
family of irrational numbers are found to be lurking in the self-similar scaling factors. Number 
theory unites this family, revealing its special elite status as its members, identified with a single 
integer $n$ can be represented by $[n + 1 : 1, n]$. It includes the golden-mean class, that is a group 
of irrationals with continued fraction representation $[n; 1]$ as the golden mean is $\frac{1+\sqrt{5}}{2} = [1; 1]$. It 
excludes the silver mean class, namely the irrationals with continued fraction expansion as $[n; 2]$.

As a final comment, we would like to point out that the proof of the Descartes theorem for 
the Ford circles stated in the form of the property of the homogeneous functions as described in 
Appendix B sets the stage for an alternative proof of the Descartes theorem that is more elegant 
than the previously known proofs of the theorem[5]. This is because the Descartes configurations 
for Ford circles are related to an arbitrary Descartes configuration via a conformal transformation.

**Appendix A: Möbius Transformations**

Conformal maps are functions in complex plane that preserve the angles between curves. 
There is a specific family of conformal maps $w = \frac{az+b}{cz+d}$, known as Möbius transformation or linear 
fractional transformation. Such transformation maps lines and circles to lines and circles. These 
maps can be represented ( denoted by symbol $\cong$) with a matrix: The map

$$w = f(z) = \frac{az+b}{cz+d} \cong \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (A1)$$

This identification of Möbius map with a matrix is useful because if we consider two 
conformal maps: $w_1 = f_1(z)$ and $w_2 = f_2(z)$ that respectively correspond to the matrices $C_1$ 
and $C_2$. Then the composition $f_1f_2(z)$ corresponds to the matrix $C_1 \cdot C_2$.

The constants of Mobius maps $(a, b, c, d)$ can be determined in terms of two sets of triplets: 
$(z_1, z_2, z_3)$ and their conformal image $(w_1, w_2, w_3)$:
\[ a = \det \begin{pmatrix} z_1 w_1 & w_1 & 1 \\ z_2 w_2 & w_2 & 1 \\ z_3 w_3 & w_3 & 1 \end{pmatrix}, \quad b = \det \begin{pmatrix} z_1 w_1 & z_1 & w_1 \\ z_2 w_2 & z_2 & w_2 \\ z_3 w_3 & z_3 & w_3 \end{pmatrix}, \quad c = \det \begin{pmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{pmatrix}, \quad d = \det \begin{pmatrix} z_1 w_1 & z_1 & 1 \\ z_2 w_2 & z_2 & 1 \\ z_3 w_3 & z_3 & 1 \end{pmatrix} \]  

(A2)

In other words, there is a unique map that connects two distinct set of triplets. Since any three points determine a configuration of four mutually tangent circles, as illustrated in Fig. (2), any two Descartes configurations are related by a Möbius map. These relations are consequence of the invariance of “cross-ratio”, \[ R = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}. \] It turns out that the self-similar hierarchies that share a boundary circle correspond to real value of \( R \) that remains invariant as we zoom in the recursive structure.

Appendix B: Descartes’s Theorem for Ford Circles

Theorem: If three variables \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are related by,

\[ \lambda_1 + \lambda_2 + \lambda_3 = 0 \]  

(B1)

Then they also satisfy,

\[ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^2 = 2(\lambda_1^4 + \lambda_2^4 + \lambda_3^4). \]  

(B2)

Proof: Define the standard three homogeneous symmetric polynomials on three variables,

\[ S_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad S_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_2, \quad S_3 = \lambda_1 \lambda_2 \lambda_3, \]  

(B3)

and some additional ones that will be useful,

\[ T_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad T_4 = \lambda_1^4 + \lambda_2^4 + \lambda_3^4, \quad U = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \]  

(B4)

By direct multiplication, we have,

\[ S_1^2 = T_2 + 2S_2, \quad T_2^2 = T_4 + 2U, \quad S_2^2 = U + 2S_1 S_3. \]  

(B5)

We now set \( S_1 = 0 \), that gives \( T_2 = -2S_2 \). Squaring both sides, and then eliminating \( U \) we get,

\[ T_2^2 = 4S_2^2 = 4U = 2T_2^2 - 2T_4 = 2T_4 \]  

(B6)
$T_2^2 = 2T_4$ gives the required equation (B2). Applying the above formulas for the Farey relation $q_c = q_R + q_L$, we get the Descartes theorem for Ford circles.

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