One-dimensional Josephson arrays as superlattices for single Cooper pairs

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We investigate uniform one-dimensional arrays of small Josephson junctions ($E_J \ll E_C$, $E_C = (2e)^2/2C$) with a realistic Coulomb interaction $U(x) = E_C\lambda \exp(-|x|/\lambda)$ (here $\lambda \gg 1$ is the screening length in units of the lattice constant of the array). At low energies this system can be described in terms of interacting Bose particles (extra single Cooper pairs) on the lattice. With increasing concentration $\nu$ of extra Cooper pairs, a crossover from the Bose gas phase to the Wigner crystal phase and then to the superlattice regime occurs. The phase diagram in the superlattice regime consists of commensurable insulating phases with $\nu = 1/l$ ($l$ is integer) separated by superconducting regions where the current is carried by excitations with fractional electric charge $q = \pm 2e/l$. The Josephson current through a ring-shaped array pierced by magnetic flux is calculated for all of the phases.

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The investigation of arrays of small Josephson junctions attracts growing interest of theoreticians and experimentalists (see [1] for a review). In such arrays the Coulomb energy $E_C$ increases with concentration $\nu$ of the interaction is large [1], although for typical experimental parameters the range of concentrations commensurable with the lattice of junctions. On the other hand, it is known [7] that at very small concentrations $\nu \approx 1/\lambda^2$ of extra Cooper pairs (ECP) can form a variety of configurations with $\nu = 1/l$ ($l$ is integer) separated by superconducting regions where the current is carried by excitations with fractional electric charge $q = \pm 2e/l$. The Josephson current through a ring-shaped array pierced by magnetic flux is calculated for all of the phases.

We consider first the limit of zero Josephson coupling. For $|\mu| < \mu_{tr} \equiv \lambda E_C/2$ there is no ECP in the array ($n_i = 0$ for all $i$). Just above the threshold, $0 < \mu - \mu_{tr} \ll \mu_{tr}$, the ground state is still characterized by the absence of ECP on most of the islands ($n_i = 0$). The rest of the islands are occupied by one ECP ($n_i = 1$). The configurations with $n_i \neq 0, 1$ (for some $i$) are separated from the ground state by a Coulomb gap $\Delta_C = E_C/\lambda$, which corresponds to the difference between the Coulomb energies of the configurations $\{n_i\} = (0, ..., 1, -1, 1, 0, ...)$. In what follows we restrict the space of states to low energy configurations with $n_i = 0$ or 1. These configurations can be fully characterized by the coordinates $x_j$ of ECP on the lattice ($x_j$ are integer numbers).

The Josephson term of the Hamiltonian has a standard form, $H_J = -E_J \sum_{j=1}^{L} \cos(\phi_{j+1} - \phi_j - \pi)$. Here $\phi_j$ is the operator of the Josephson phase of the island $j$ obeying the commutation relation $[n_j, \phi_j'] = -\delta_{j,j'} i$. The Josephson term represents a field-induced Coulomb interaction $U(x) = E_C\lambda \exp(-|x|/\lambda)$ (here $\lambda \gg 1$ is the screening length in units of the lattice constant of the array). The phase diagram in the superlattice regime consists of commensurable insulating phases with $\nu = 1/l$ ($l$ is integer) separated by superconducting regions where the current is carried by excitations with fractional electric charge $q = \pm 2e/l$. The Josephson current through a ring-shaped array pierced by magnetic flux is calculated for all of the phases.

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and \( a = (2\pi/L)\Phi/\Phi_0 \) is the vector potential (for circular array pierced by a magnetic flux \( \Phi \)). The Josephson term acting on the restricted space of states describes a hopping of ECP on the lattice with the amplitude \( E_J e^{i\pi a/2} \). Corrections to the tunneling amplitude due to the states with \( n \neq 0,1 \) are small for \( E_J \ll \Delta C \). Therefore, the original Hamiltonian \( H \) in the low-energy space is equivalent to the Hamiltonian of Bose particles (ECP) on the lattice \([3]\).

\[
H = -E_J \sum_{j=1}^{N} \cos(p_j - a) + \sum_{j=1}^{N} U(x_{j+1} - x_j) - \bar{\mu} \mathcal{N}, \quad (1)
\]

where \( p_j \) are quasimomenta of ECP and \( \bar{\mu} = \mu - \mu_{tr} \). Since the interaction potential \( U(x) \) is screened strongly on the interparticle distances \( \nu^{-1} \gg \lambda \) we take into account the interaction of the neighboring ECP only.

Let’s consider now the ground state of the system as a function of the chemical potential \( \bar{\mu} \) at fixed \( E_J \). The first ECP enters the array at \( \bar{\mu} = -E_J \). We denote the deviation from this threshold by \( \bar{\mu} = \bar{\mu} + E_J \). At low concentrations \( \nu \) (see Eq. \([3]\)) ECP form a Bose gas with hard core interaction \([3]\) (Fig. \([1]\)). The chemical potential \( \bar{\mu}^{BG} \) is determined by the kinetic energy of the particles in parabolic band \([10]\). Taking into account a finite range of the interaction we obtain by a variational calculation that the correction to \( \bar{\mu}^{BG} \) due to a finite size \( d = \ln(\lambda^3 E_C/E_J) \) of the core is small for

\[
\nu \lambda \ln(\lambda^3 E_C/E_J) \ll 1. \quad (2)
\]

With increasing concentration ECP get localized in coordinate space. We assume apriori that ECP form 1-D Wigner crystal. Expanding kinetic and potential energy \([1]\) up to quadratic terms we obtain the chemical potential \( \bar{\mu}^{WC} \) \([2]\) \( \equiv \bar{\mu}^{WC} \) \( = (2/\pi)(E_C E_J/\lambda)^{1/2} \exp(-1/2\nu\lambda) + \lambda E_C \exp(-1/\nu\lambda) \) and the mean-square displacement of neighboring particles \( \langle (x_{i+1} - x_i)^2 \rangle = [2E_J/U_0(\nu^{-1})]^{1/2}/\pi \), where

\[
U_0(l) \equiv (1/2)d^2 U(x)/dx^2|_{x=1} = (E_C/2\lambda) \exp(-l/\lambda). \quad (3)
\]

Note that the concentration of ECP \( \nu(\bar{\mu}) \) increases much more slowly than in the Bose gas regime. This expansion is legitimate if the fluctuations of displacement are small, \( \langle (x_{i+1} - x_i)^2 \rangle \ll \lambda^2 \), and the kinetic energy per ECP is much less than \( E_J \). These conditions determine respectively the lower and the upper bound of the range of concentrations

\[
\frac{E_J}{\lambda^3 E_C} \ll \exp\left(-\frac{1}{\nu\lambda}\right) \ll \frac{\lambda E_J}{E_C} \quad (4)
\]

in which the Wigner crystal phase exists (Fig. \([1]\)). For both the phases (Bose gas and Wigner crystal) the Hamiltonian is quadratic in momenta of the particles. Therefore, the vector potential \( a \) is coupled to the momenta of the center of mass only. For this reason the Josephson current through a ring-shaped array is given by a universal expression

\[
I_J = \frac{4e v J}{L} \Phi/\Phi_0, \quad v_J = \pi\nu E_J, \quad (5)
\]

(for \( |\Phi| < \Phi_0/2 \)). A similar result holds for the persistent current of interacting Fermions \([11]\) \( v_J \) being the Fermi velocity in that case.

For larger concentrations of ECP the kinetic energy per particle becomes comparable to the bandwidth \( E_J \). To investigate this case we start from the limit \( E_J \to 0 \). First we determine the range of the chemical potential \( \mu_{-0} < \bar{\mu} < \mu_{+0} \) where the configuration with equidistantly spaced (at a distance \( l \)) ECP is the ground state of the system. In order to add (subtract) a Cooper pair into (from) this configuration, one has to convert \( l \) bonds of length \( l \) between neighboring ECP into \( l \) bonds of length \( l - 1 \). The energy required for this conversion is given by \( \mu_{\pm0} = \pm l[U(l \mp 1) - U(l)] \simeq \epsilon \left(1 \pm 1/2\lambda\right) \) with \( \epsilon = E_C\exp(-l/\lambda) \). Clearly, addition of the second, third, etc. particle to the system requires the same energy \( \mu_{+0} \). For this reason, the ground state corresponds to a regular superlattice of ECP in the array; the period of the superlattice changes abruptly from \( l \to l - 1 \) at \( \bar{\mu} = \mu_{+0} = \mu_{-0} = \mu_{-0} = \mu_{-0} \), see Fig. \([2]\). This simple picture of the ground state is valid if one takes into account the interaction of neighboring ECP only (see Eq. \([3]\)). The interaction of next-nearest neighbors will lead to new ground states \([12]\) in exponentially narrow regions \( |\bar{\mu} - \mu_{+0}\rangle \sim \epsilon \exp(-l/\lambda) \) near the points \( \mu_{+0} \).

For small but finite Josephson coupling, \( E_J \ll U_0(l) \), each bond of the length \( l \pm 1 \) can be considered as a mobile excitation above the ground state where all the bonds have the length \( l \). Since a shift of an ECP to the neighboring island of the array leads to a shift of the excitation by \( \mp l \) lattice cells, the excitations have fractional charge \( \mp 2e/l \) \([3]\). Tunneling of excitations decreases the energy by an amount \( E_J \) per excitation. Since addition (subtraction) of one Cooper pair to (from) the array is accompanied by the creation of \( l \) excitations, the threshold chemical potentials are given by \( \mu_{\pm0}(E_J) = \mu_{\pm0}(0) \mp l E_J \).

In the superlattice regime under consideration (see Fig. \([1]\)) the fluctuations of displacement of ECP are

| State          | Interactions | Band          |
|----------------|--------------|---------------|
| Bose gas       | unharmonic   | parabolic     |
| Wigner crystal | harmonic     | parabolic     |
| Superlattice   | harmonic     | cos p         |

FIG. 1. Schematic phase diagram of 1-D Josephson array.
small and we can again expand the interaction term of the Hamiltonian around the average distance $l$ between the particles. We introduce the (integer) deviation $n_j = x_j - x_{j-1} - l$ of the distance between neighboring ECP from $l$, and the canonically conjugated operator $\varphi_j$, so that $p_j = \varphi_{j+1} - \varphi_j$. The number of particles $N$ can be expressed via $n_j$, $N = (L - \sum n_j)/l$. In terms of the new variables the Hamiltonian can be written as

$$H = -E_J \sum_j \cos(\varphi_{j+1} - \varphi_j - a) + U_0(l) \sum_j (n_j + \delta \mu)^2,$$

(6)

where $\delta \mu = \lambda(\bar{\mu} - \epsilon_i)/\epsilon_i$. This Hamiltonian formally coincides with the Hamiltonian of 1-D Josephson array with on-site Coulomb interactions, which has been extensively studied in Refs. 1, 4, 5, 12.

The boundary of the commensurable phase can be determined from the comparison of the ground state energies of the Hamiltonian in two subspaces of states with $\sum n_j = 0$ and with $\sum n_j = \pm 1$ (the sign coincides with the sign of $\epsilon_i - \bar{\mu}$ in (6)). In the limit $E_J \to 0$ the ground states in these subspaces are given by $\Psi_0 = |0,0,...\rangle$ and $\Psi_{\pm 1} = N^{-1/2} \sum_j |0,...,n_j = \pm 1,0,...\rangle$. Evaluating the energies of these states up to the third order in $E_J$ (see Ref. 12) we obtain

$$\mu_{i,\pm}(E_J) = \epsilon_i \pm \left\{ \frac{\epsilon_i}{2\lambda} - l E_J \left[ 1 - \frac{E_J}{8U_0(l)} - \frac{E_J^2}{32U_0^2(l)} \right] \right\},$$

(7)

for $E_J \ll U_0(l)$, see Fig. 3b. Note that the term linear in $E_J$ coincides with that obtained above from a simple argument.

With increasing $E_J$ the range of the chemical potential corresponding to commensurable phase decreases and both phase boundaries tend to the critical point, $\mu_{i,\pm}(E_J^{(cr)}) \to \epsilon_i$, see Fig. 3b. Clearly, the true behavior near the critical point cannot be described by perturbation theory of finite order. To extent the perturbative approach, an extrapolation to infinite order in $E_J$ was proposed 12. Unfortunately, this (somewhat speculative) extrapolation fails to converge to a critical point for the 1-D system. To determine the behavior near the critical point $E_J^{(cr)}$ one can map the Hamiltonian (6) (with $\bar{\mu} = \epsilon_i$) onto 2-D XY model 14. The parameter $(2U_0(l)/E_J)^{1/2}$ plays a role of dimensionless temperature $k_B T/J$ in the XY model. The point of the Kosterlitz-Thouless transition 14 corresponds to $E_J^{(cr)} \simeq 2.5U_0(l)$, see Fig. 2b. Below the transition temperature ($E_J > E_J^{(cr)}$) spin-spin correlations in the XY model decay algebraically with distance. The Josephson array shows superconducting properties: the Josephson current is inversely proportional to $L$. It scales as $1 + c \sqrt{E_J - E_J^{(cr)}}$ at $E_J \to E_J^{(cr)} + 0$ (here $c$ is non-universal constant). Above the transition temperature ($E_J < E_J^{(cr)}$) the correlations in the XY model decay exponentially. Near the critical point ($E_J \to E_J^{(cr)} - 0$) the coherence length is given by $\xi \propto \exp\{-b([E_J^{(cr)}]/(E_J^{(cr)} - E_J))^{1/2}\}$, where $b \simeq 2$. As a result, the Josephson current through 1-D array decays as $\exp(-L/\xi)$ signalling the formation of the insulating phase.

In the insulating phase, the energy gap for the mobile excitations scales as $E_J^{(cr)} \xi^{-1}$. For this reason, the boundary of insulating phase near the critical point is given by

$$\mu_{i,\pm}(E_J) = \epsilon_i \pm a U_0(l)/\xi,$$

(8)

with $a \sim 1$. The upper curve in Fig. 2b corresponds to an extrapolation of Eq. (8) from the neighborhood of the critical point to lower values of $E_J$ ($1.56U_0(l) < E_J < E_J^{(cr)}$).
This extrapolation is joined to the extrapolation of the perturbative result (6) to infinite order in $E_J$ (drawn in the range $E_J < 1.56 U_0(l)$). A smooth connection of the two curves occurs for $a \simeq 1.73$ (and $b = 2$).

For completeness, we present the mean-field result for the phase boundary (3),

$$
\mu_{l,\pm}(E_J) = \epsilon_l \pm \frac{\epsilon_l}{2\lambda} \left[ 1 - \frac{2E_J}{U_0(l)} \right],
$$

Although this expression coincides with Eq. (6) to first order in $E_J$, the overall shape of the boundary (Fig. 3) and the critical value of the Josephson energy $E_J^{(cr)}$ differ considerably from the results discussed above. The reason for the failure of the mean-field approach is the absence of long-range order in one dimension.

We return now to the consideration of the superconducting phase and concentrate on the case of small Josephson coupling, $E_J \ll U_0(l)$. The commensurable phases with $\nu = 1/l$ and $\nu = 1/(l-1)$ are separated by a narrow superconducting region, $\mu_{l,+}(E_J) < \mu < \mu_{l,-}(E_J)$, of the width $(2l-1)E_J$. In this region the ground state of the array can be viewed as a gas of fractionally charged mobile excitations (bonds of length $l-1$). The excitations interact with each other via a contact potential $U_{xc}(x) = 2U_0(l-1)\delta_x$. This is effectively a hard-core interaction provided that $E_J \ll U_0(l)$. From the consideration of the hard-core particles (bonds of length $l-1$) on a lattice formed by the bonds of length $l$ we obtain the chemical potential $\tilde{\mu}$ and the Josephson current $I_J$ as functions of the concentration $\nu = 1/(l-q)$ of ECP,

$$
\tilde{\mu} = \mu_{l,+}(0) - E_J\{(l-q)\cos\pi q + \pi^{-1}\sin\pi q\},
$$

where the parameter $q$ takes on values in the range $0 < q < 1$. These dependences are presented in Fig. 3.

Finally, we discuss effects which are specific for finite size circular arrays. If the size $L$ of the array is commensurable with the spacing $l$ of the superlattice of ECP, the Josephson current is exponentially small in the insulating phase. However, if $L/l$ is not an integer, a number of residual mobile excitations remain in the array in the insulating phase, $\mu_{l,-}(E_J) < \tilde{\mu} < \mu_{l,+}(E_J)$. In the lower part of this range, $\mu_{l,-}(E_J) < \tilde{\mu} < \mu_{l,0}(E_J)$, there are $m = \text{mod}(L,l)$ residual excitations (bonds of the length $l+1$) in the ground state. One ECP enters into the array at $\tilde{\mu} = \mu_{l,0}(E_J)$. As a result, for $\mu_{l,0}(E_J) < \tilde{\mu} < \mu_{l,+}(E_J)$ a new ground state will contain $l-m$ residual excitations (bonds of the length $l-1$). The threshold chemical potential is given by $\mu_{l,0}(E_J) = [m\mu_{l,-}(E_J) + (l-m)\mu_{l,+}(E_J)]/l$. Since each excitation contribute an amount $I_{1CP} = (4\pi eE_J/hL^2)(\Phi/\Phi_0)$ to the Josephson current (cf. Eq. (3) with $\nu = 1/L$), the latter shows a jump at $\tilde{\mu} = \mu_{l,0}(E_J)$.

There is clearly a need for future investigations, such as an analysis of effects of disorder due to the offset charges (potential disorder) and due to non-uniformity of the Josephson coupling (kinetic disorder).

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