Abstract. We classify Dehn surgeries on \((p, q, r)\) pretzel knots resulting in a manifold \(M(\alpha)\) having cyclic fundamental group and analyze those leading to a finite fundamental group. The proof uses the theory of cyclic and finite surgeries developed by Culler, Shalen, Boyer, and Zhang. In particular, Culler-Shalen seminorms play a central role.

1. Introduction

Thurston [Th] has shown that all but a finite number of Dehn surgeries on a hyperbolic knot result in manifolds which are again hyperbolic. Those which produce a manifold having cyclic or finite fundamental group are important examples of exceptional (i.e., non-hyperbolic) surgeries. The theory of Culler-Shalen seminorms has proven to be a useful tool for understanding these kinds of exceptional surgeries. These seminorms were first introduced as part of the proof of the Cyclic Surgery Theorem [CGLS] and later extended by Boyer and Zhang [BZ1] to the study of finite surgeries, eventually leading to a proof of the Finite Filling Conjecture [BZ4].

While those results establish global bounds on the number of cyclic or finite surgeries a knot may have, the current paper shows how they may be refined by focusing on a particular family of knots, the \((p, q, r)\) pretzel knots. To be specific, we have the following theorems.

Theorem 1.1. The only non-torus pretzel knot which admits a non-trivial cyclic surgery is the \((-2, 3, 7)\) pretzel knot. The non-trivial cyclic surgeries on this knot are of slope 18 and 19.

Theorem 1.2. If a non-torus pretzel knot \(K\) admits a non-trivial finite surgery, then one of the following holds.

- \(K\) is a \((-2, p, q)\) pretzel knot with \(5 \leq p \leq q\) odd and the filling is not cyclic.
- \(K\) is the \((-2, 3, 7)\) pretzel knot and the filling is along slope 17, 18, or 19.
- \(K\) is the \((-2, 3, 9)\) pretzel knot and the filling is along slope 22 or 23.

Note that a torus knot admits an infinite number of finite cyclic fillings and that the torus pretzel knots are well understood [K]. In particular, if \(|p|, |q|, |r| > 1\), then a \((p, q, r)\) pretzel knot is torus only if \(\{p, q, r\} = \{-2, 3, 3\}\) or \(\{-2, 3, 5\}\). (We consider the \((p, q, r)\) and \((-p, -q, -r)\) knots equivalent as they are mirror images of one another.)

The surgeries listed for the \((-2, 3, 7)\) and \((-2, 3, 9)\) pretzel knots were discovered by Fintushel and Stern [FS] and Bleiler and Hodgson [BH]. The content here is
that there are no other cyclic surgeries and that the possibilities for a finite surgery are restricted. On the other hand, we know of no instances of a finite surgery on a knot \((-2,p,q)\) with \(5 \leq p \leq q\). Indeed, we expect that there are none.

Our results can be seen as complementing the work of Delman \cite{De} and others who have been using laminations to study Dehn surgery. In particular, if a hyperbolic arborescent knot admits a non-trivial cyclic or finite surgery, then it must either be a \((p,q,r)\) pretzel knot or else belong to a certain family of 3-tangle Montesinos knots (see \cite{Wu2}, especially Theorem 2.1). The current article deals with the first case and naturally suggests the following:

**Problem:** Complete the analysis of cyclic and finite surgeries on hyperbolic arborescent knots by investigating Montesinos knots of the form \(M(x,1/p,1/q)\) where \(x \in \{-1 \pm 1/2n, -2 + 1/2n\}\) and \(p, q, n\) are positive integers.

Our main theorems are consequences of the following two results and work of Delman \cite{De}.

**Theorem 1.3.** Suppose \(K\), a \((-2,p,q)\) pretzel knot \((p,q\) odd and positive), admits a non-trivial cyclic surgery. Then one of the following holds.

1. \(K\) is a torus knot and therefore admits an infinite number of cyclic surgeries. In this case either \(\{p,q\} = \{3,3\}\) or \(\{3,5\}\) or \(\{1,n\}\) for some \(n > 0\).
2. \(K\) is the \((-2,3,7)\) pretzel knot and the surgery is 18 or 19.

**Theorem 1.4.** A \((p,q,-r)\) pretzel knot, with \(4 \leq r\) even and \(3 \leq p \leq q\) odd admits no non-trivial finite surgeries.

**Proof:** (of Theorem 1.1) Delman \cite{De} has shown that if such a knot admits a cyclic filling, then it is of the form \((p,q,-r)\), with \(2 \leq r\) even and \(3 \leq p \leq q\) odd. As only the trivial knot admits a \(\mathbb{Z}\) filling \cite{Ga}, Theorem 1.4 implies further that \(r\) must be 2. Theorem 1.3 completes the proof.

**Proof:** (of Theorem 1.2) Again, Delman \cite{De} allows us to reduce to the case of a \((p,q,-r)\) pretzel knot and Theorem 1.3 further shows that \(r = 2\). The finite surgeries on \((-2,3,n)\) pretzel knots are classified in \cite{M2}. That a non-trivial finite filling of \((-2,p,q)\) with \(p \geq 5\) is not cyclic follows from Theorem 1.1.

Thus, the main part of this paper is given over to proving Theorems 1.3 and 1.4. The latter in turn depends largely on

**Theorem 1.5.** If \(K\) is a \((p,q,r)\) pretzel knot with \(p,q\) odd, \(r\) even and \(1/|p| + 1/|q| + 2/|r| < 1\), then \(K\) admits at most one non-trivial finite surgery. Moreover such a surgery slope \(u\) is odd integral and there is a non-integral boundary slope in \((u-1,u+1)\).

Theorems 1.3, 1.3, and 1.4 will be proved in Sections 3, 4, and 5. Section 2, which follows, introduces notation and provides a brief review of Culler-Shalen theory which will play a central role in our arguments.
2. Notation and Culler-Shalen Theory

In this section let \( K \) denote a \((p, q, r)\) pretzel knot. Our sign conventions are illustrated by Figure 1 which shows the \((-3, 3, 4)\) pretzel knot. Pretzel knots whose indices \( p, q, r \) agree up to permutation are ambient isotopic. Moreover, taking the mirror reflection corresponds to changing the signs of all the indices. As this reduces to an isomorphism of the knot group \( \pi \), we will consider the knots \((p, q, r)\) and \((-p, -q, -r)\) equivalent.

By \([Oe]\), \( K \) is small in the sense that the knot complement \( M = S^3 \setminus K \) contains no closed essential surfaces. The knot is therefore either torus or hyperbolic. The torus pretzel knots are classified in \([K]\).

The fundamental group of the 2-fold branched cyclic cover \( \pi_1(\Sigma_2) \) of \( K \) is a central extension of the triangle group \( \Delta(p, q, r) \). Let \( \mu \) denote the class of a meridian in \( \pi \) and \( \lambda \) that of a preferred longitude. We will use \( \{\mu, \lambda\} \) coordinates to identify the surgery slopes on \( K \) with \( \mathbb{Q} \cup \{1/0\} \) and \((-p, -q, -r)\) equivalent.

The distance between two surgery slopes \( a/b \) and \( c/d \) is the minimal geometric intersection number \( \Delta(a/b, c/d) = |ad - bc| \). We will say that \( a/b \) is a boundary slope if there is an essential surface in \( M \) which meets \( \partial M \) in a non-empty set of curves having slope \( a/b \). An essential surface is one which is properly embedded, orientable, incompressible, \( \partial \)-incompressible, and non-\( \partial \)-parallel.

We now briefly introduce Culler-Shalen theory under the assumption that \( K \) is a small, hyperbolic knot. A more detailed account may be found in \([CGLS, Chapter 1]\) and \([BZ2]\).

Let \( R = \text{Hom}(\pi, \text{SL}_2(\mathbb{C})) \) denote the set of \( \text{SL}_2(\mathbb{C}) \)-representations of the fundamental group of \( M \). Then \( R \) is an affine algebraic set, as is \( X \), the set of characters of representations in \( R \). Since \( M \) is small, the irreducible components of \( X \) are curves \([CCGLS, Proposition 2.4]\). Moreover, for each component \( R_i \) of \( R \) which contains an irreducible representation, the corresponding curve \( X_i \) induces a non-zero seminorm \( \| \cdot \|_i \) on \( V = H_1(\partial M; \mathbb{R}) \) \([BZ2, Proposition 5.7]\) via the following construction.

For \( \gamma \in \pi \), define the regular function \( I_\gamma : X \to \mathbb{C} \) by \( I_\gamma(\chi_\rho) = \chi_\rho(\gamma) = \text{trace}(\rho(\gamma)) \). By the Hurewicz isomorphism, a class \( \gamma \in L = H_1(\partial M; \mathbb{Z}) \) determines an element of \( \pi_1(\partial M) \), and therefore an element of \( \pi \) well-defined up to conjugacy. The function \( f_\gamma = I_\gamma^2 - 4 \) is again regular and so can be pulled back to \( \tilde{X}_i \), the smooth projective variety birationally equivalent to \( X_i \). For \( \gamma \in L \), \( \|\gamma\|_i \) is the degree of \( f_\gamma : \tilde{X}_i \to \mathbb{C} \mathbb{P}^1 \). In practice, this degree can often be calculated using
\(Z_x(f_\gamma),\) the degree of zero of \(f_\gamma\) at a point \(x\) in the character variety. The seminorm is extended to \(V\) by linearity. We will call a seminorm constructed in this manner a Culler-Shalen seminorm.

Let \(\| \cdot \|_T\) denote the sum of the Culler-Shalen seminorms, i.e., \(\|v\|_T = \sum_i \|v_i\|\) (here \(v \in V\)) and let \(S = \min\{\|\gamma\|_T : \gamma \in L, \|\gamma\|_T > 0\}\) be the minimal norm. Note that, as \(K\) is hyperbolic, \(\| \cdot \|_T\) will be a norm (and not just a seminorm) \[CGLS,\] Chapter 1.

3. Proof of Theorem 1.3

In this section, let \(K\) be a \((-2, p, q)\) pretzel knot \((p, q\) odd and positive, \(p \leq q)\).

The following lemma can be proved using the methods of Hatcher and Oertel \[HO\].

**Lemma 3.1.** Assume \(3 \leq p \leq q.\) If \(p \geq 7\) (respectively \(q \geq 7\)) then
\[
\frac{p^2 - p - 5}{p - 3} \quad \text{(resp. } \frac{q^2 - q - 5}{q - 3} \text{)}
\]
is a non-integral boundary slope of \(K.\) Moreover, these are the only non-integral boundary slopes of \(K.\)

**Lemma 3.2.** \(M(2(p + q))\) contains an incompressible torus.

**Proof:** Use the double cover of the “obvious” spanning surface of the knot (which is a punctured Klein bottle).

**Theorem 1.3.** Suppose \(K\) admits a non-trivial cyclic surgery. Then one of the following holds.

1. \(K\) is a torus knot and therefore admits an infinite number of cyclic surgeries. In this case either \(\{p, q\} = \{3, 3\}\) or \(\{p, q\} = \{3, 5\}\) or \(\{p, q\} = \{1, n\}\) for some \(n > 0.\)
2. \(K\) is the \((-2, 3, 7)\) pretzel knot and the surgery is 18 or 19.

**Proof:** Theorem III of \[K\] shows that \(K\) is torus iff it is as characterized in 1.

Since \(K\) is small \[D\], we can assume that \(K\) is hyperbolic. The cyclic surgeries 18 and 19 of the \((-2, 3, 7)\) pretzel knot were first observed by Fintushel and Stern (see \[FS, Section 4\]). Our task is to show that there is no other choice for \(p\) and \(q\) leading to a cyclic surgery.

The case \(p = 3\) is the subject of \[M2\] where we show that there are no non-trivial cyclic surgeries when \(q \geq 9\) and that the cyclic surgeries of the \((-2, 3, 7)\) pretzel knot are as stated.

If \(p = 5,\) the boundary slopes \[HO\] are \(0, 14, 15, \frac{2^2 - p - 5}{2^2 - 3}, 2q + 10,\) and \(2q + 12.\) By \[D, Theorem 4.1\], a non-trivial cyclic surgery could occur only at \(2q + 4\) or \(2q + 5.\) However, as we explain below, a cyclic surgery would have to be within distance 5 of the toroidal surgery \(2q + 10\) (see Lemma \[BZ2\]). So the only candidate is \(2q + 5.\)

Now the \((-2, 5, 5)\) pretzel has no non-integral boundary slopes so \[D, Theorem 4.1\] it has no non-trivial cyclic surgeries. As for \((-2, 5, 7),\) SnapPea \[W\] shows that \(2q + 5 = 19\) surgery on this knot is hyperbolic. So we can assume \(q \geq 9.\)

Suppose (for a contradiction) that \(2q + 5\) is indeed a cyclic surgery. By \[BZ1,\] Lemma 6.2, the (total) norm can be written
\[
\|\gamma\|_T = 2[a_1 \Delta(\gamma_0, 0) + a_2 \Delta(\gamma, 14) + a_3 \Delta(\gamma, 15) + a_4 \Delta(\gamma, q^2 - q - 5) + a_5 \Delta(\gamma, 2q + 10) + a_6 \Delta(\gamma, 2q + 12)].
\]
where the $a_i$ are non-negative integers. If $2q + 5$ is cyclic it has minimal norm $S$, as does the meridian surgery $\mu$ (Corollary 1.1.4). The norm of $2q + 4$ will also be of interest, and it will be bounded by the minimal norm $S$.

\[
S = \|\mu\|_T = 2[a_1 + a_2 + a_3 + \frac{q - 3}{2}a_4 + a_5 + a_6]
\]

\[
S = \|2q + 5\|_T = 2[(2q + 5)a_1 + (2q - 9)a_2 + (2q - 10)a_3 + \frac{q - 5}{2}a_4 + 5a_5 + 7a_6]
\]

\[
S \leq \|2q + 4\|_T = 2[2(q + 4)a_1 + (2q - 10)a_2 + (2q - 11)a_3 + a_4 + 6a_5 + 8a_6]
\]

Subtracting the first two equations, we have

\[
a_4 = (2q + 4)a_1 + (2q - 10)a_2 + (2q - 11)a_3 + 4a_5 + 6a_6,
\]

while subtracting the second from the third leaves

\[
a_5 + a_6 \geq a_1 + a_2 + a_3 + \frac{q - 7}{2}a_4,
\]

\[
\Rightarrow \eta(a_5 + a_6 - a_1 - a_2 - a_3) \geq a_4,
\]

where $\eta = \frac{2}{q - 7} \leq 1$.

Combining this with Equation 1, we have

\[
0 \geq (2q + 4 + \eta)a_1 + (2q - 10 + \eta)a_2 + (2q - 11 + \eta)a_3 + (4 - \eta)a_5 + (6 - \eta)a_6.
\]

Since $a_i \geq 0$, this shows $a_1 = a_2 = a_3 = a_5 = a_6 = 0$. On the other hand, for a norm, at least two of the $a_i$ must be non-zero. This contradiction shows that there can be no non-trivial cyclic surgery when $p = 5$.

So let us assume $7 \leq p \leq q$. Dunfield [Du, Theorem 4.1] has shown that any non-trivial cyclic surgery on a knot such as $K$ must lie near a non-integral surgery. Combining this with Lemma 3.1, the only candidates for a non-trivial cyclic surgery are $2p + 4$, $2p + 5$, $2q + 4$, and $2q + 5$. Suppose that $u$ is one of these candidates slopes and $M(u)$ is a cyclic filling. Since $K$ is strongly invertible, the Orbifold Theorem implies that $M(u)$ admits a geometric decomposition (see CHK, Corollary 1.21)). Now, as $\Delta(u, 2(p + q)) > 5$, $M(u)$ is irreducible [Oh, Wu] and atoroidal [Go] and therefore has a geometric structure.

Note that $\pi_1(M(u)) \not\cong \mathbb{Z}$ (see [62]), so $\pi_1(M(u))$ is finite. The geometry is therefore $S^3$, and as $\pi_1(M(u))$ is finite cyclic, we deduce that $M(u)$ is a lens space. However, this contradicts [50e2 Theorem 1.1] which states that the distance between a lens space surgery such as $u$ and a toroidal surgery such as $2(p + q)$ is at most 5. We conclude that there are also no non-trivial cyclic surgeries in this case.

\[
4. \text{Proof of Theorem} \quad \square
\]

Let $K$ be a $(p, q, r)$ pretzel knot where $p = 2k + 1$, $q = 2l + 1$ and $r = 2m$. We will be assuming that $1/|p| + 1/|q| + 1/|m| \leq 1$, and this ensures that $K$ is hyperbolic [3].

**Lemma 4.1.** Let $1/|p| + 1/|q| + 1/|m| \leq 1$. If $b$ is odd, then the Dehn filling $M(2a/b)$ of the knot complement $M$ has infinite $\pi_1$.

**Proof:** As $\Delta(p, q, m)$ is infinite, our strategy is to construct a representation of $\pi_1(M(2a/b))$ with image $\Delta(p, q, m)$. Changing the sign of $p, q, r$ will not change the triangle groups $\Delta(p, q, m)$ and $\Delta(p, q, r)$, so we will assume $p, q, r > 0$ in order to simplify the notation. For the general case, use $|p|, |q|, \text{and } \text{and } r$ instead.
First note that there is a faithful PSL$_2(\mathbb{C})$-representation of $\Delta(p, q, m)$. Indeed, either $\{p, q, m\} = \{3, 3, 3\}$, and $\Delta(p, q, m)$ is a set of isometries of the Euclidean plane $\mathbb{E}^2$, or else $\frac{p}{q} + \frac{r}{m} < 1$ and $\Delta(p, q, m)$ represents isometries of the hyperbolic plane $\mathbb{H}^2$. Now, both $\mathbb{E}^2$ and $\mathbb{H}^2$ imbed isometrically into hyperbolic 3-space $\mathbb{H}^3$. For example, in the upper half space model, the set $\{z = 1\}$ is a Euclidean plane, while the $xz$ plane is $\mathbb{H}^2$. Moreover, isometries of these planes are restrictions of isometries of $\mathbb{H}^3$. Thus, in either case, $\Delta(p, q, m)$ embeds in PSL$_2(\mathbb{C})$, the set of orientation preserving isometries of $\mathbb{H}^3$. This provides the required faithful representation.

Let $\tilde{\rho}_0$ be a representation of $\Delta(p, q, r)$ obtained by composing the obvious homomorphism $\Delta(p, q, r) \to \Delta(p, q, m)$ with a faithful representation of $\Delta(p, q, m)$ in PSL$_2(\mathbb{C})$. Then (as in [M2, Proposition 1.1]) $\tilde{\rho}_0$ “extends” to a PSL$_2(\mathbb{C})$-representation $\tilde{\rho}$ of the knot group $\pi$ which in turn lifts to an SL$_2(\mathbb{C})$-representation $\rho$. (The obstruction to such a lift is in $H^2(\pi; \mathbb{Z}/2)$ [BZ2, Section 3]). For a knot in $S^3$, the second cohomology is trivial and there is no obstruction. Moreover, $\tilde{\rho}(\mu^2) = 1$.

On the other hand, we can determine the image of $\lambda$ in $\Delta(p, q, r) = \langle f, g, h \mid f^r, g^q, h^m, fgfh \rangle$ to be $\tilde{\lambda} = g^k f^m g^{k+1} h^l f^m h^{l+1}$ (compare [H]). Now, as $\tilde{\rho}_0$ factors through a representation of $\Delta(p, q, m)$, we have $\tilde{\rho}_0(f^m) = 1$ and consequently $\tilde{\rho}_0(\lambda) = 1$. Then $\tilde{\rho}(\lambda) = 1$ as well.

So, for any filling of the form $\alpha = 2a/b$, we have $\tilde{\rho}(\alpha) = 1$ whence $\tilde{\rho}$ factors through $\pi_1(M(\alpha))$. Since $\tilde{\rho}$ is an extension of $\tilde{\rho}_0$, which factors through a faithful representation of the infinite group $\Delta(p, q, m)$, we see that $\pi_1(M(\alpha))$ must also be infinite.

So under the hypothesis of the lemma, every $2a/b$ filling of $K$ is infinite. This means that any finite surgeries would have to be of the form $(2a + 1)/b$ and therefore would have norm $\|2a + 1\|_T \leq S + 8$ [BZ1, Theorem 2.3]. On the other hand, the $2a/b$ fillings will have norm larger than $S + 8$.

**Lemma 4.2.** Let $1/|p| + 1/|q| + 1/|m| < 1$. If $b$ is odd, then $\|2a/b\|_T \geq S + 12$.

**Proof:** Again, we assume $p, q, r > 0$.

We first observe that there are at least three irreducible PSL$_2(\mathbb{C})$-characters of $\Delta(p, q, m)$. Indeed, by [B13, Proposition D], the number of PSL$_2(\mathbb{C})$-characters of $\Delta(p, q, r)$ is

$$\begin{align*}
(p - \left\lfloor \frac{p}{2} \right\rfloor - 1)(q - \left\lfloor \frac{q}{2} \right\rfloor - 1)(r - \left\lfloor \frac{r}{2} \right\rfloor - 1) + \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor \\
+ \left\lfloor \frac{\gcd(p, q)}{2} \right\rfloor + \left\lfloor \frac{\gcd(p, r)}{2} \right\rfloor + \left\lfloor \frac{\gcd(q, r)}{2} \right\rfloor + 1
\end{align*}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$. This count includes characters of reducible representations. The character of a reducible representation is also the character of a diagonal (hence abelian) representation. So, to count the characters of reducible representations we can look at representations of $H_1(\Delta(p, q, r))$. Let $a = \gcd(p, q, r)$, $b = \gcd(q, p, r)$. Then $H_1(\Delta(p, q, r)) = \mathbb{Z}/a \oplus \mathbb{Z}/(b/a)$ and hence $|H_1(\Delta(p, q, r))| = b$. Consequently, the number of characters of $H_1(\Delta(p, q, r))$ is

$$\begin{align*}
\left\lfloor \frac{b}{2} \right\rfloor + 1, & \quad \text{if } a \equiv 1 \pmod{2}, \\
\left\lfloor \frac{b}{2} \right\rfloor , & \quad \text{if } a \equiv 0 \pmod{2}.
\end{align*}$$

Thus, by taking the difference of (3) and (3), we see that $\Delta(p, q, m)$ has at least three irreducible PSL$_2(\mathbb{C})$-characters. Using the method of the previous lemma,
these can be used to construct irreducible $\text{PSL}_2(\mathbb{C})$-characters of $\pi_1(M(\alpha))$ when $\alpha$ is of the from $2a/b$. None of these characters are dihedral, so each is covered twice in $\text{SL}_2(\mathbb{C})$ (see [BZ1, Lemma 5.5]). As they are the characters of irreducible representations of a triangle group, they are smooth points of $X(M)$ (see [BZ3, Proposition 7]).

Moreover, they are zeros of $f_\alpha$ which are not zeros of $f_\mu$. (As in the previous lemma, these are characters of representations which take $\mu$ to an element of order two.) It follows from [BB, Theorem A] that $Z_x(f_\alpha) = Z_x(f_\mu) + 2$, and since we have six such characters $x$, we see that $||2a/b||_T = ||\alpha||_T > ||\mu||_T + 12 = S + 12$.

**Theorem 1.5.** If $K$ is a $(p, q, r)$ pretzel knot with $p, q$ odd, $r$ even and $1/|p| + 1/|q| + 2/|r| < 1$, then $K$ admits at most one non-trivial finite surgery. Moreover such a surgery slope $u$ is odd integral and there is a non-integral boundary slope in $(u - 1, u + 1)$.

**Proof:** The conditions on $p, q, r$ ensure that $K$ is hyperbolic [K].

Let $\alpha$ be a finite surgery of such a knot. We have already observed (Lemma 1.1) that $\alpha = (2a + 1)/b$. Since meridional surgery is cyclic, we can apply [BZ1, Theorem 1.1] to see that $b \leq 2$.

If $\alpha = (2a + 1)/2$ were a finite filling, then, by [BZ1, Theorem 2.3], $||\alpha||_T \leq S + 8$. At the same time, $||\mu||_T = ||-\mu||_T = S$. The line joining $\alpha = (2a + 1, 2)$ and $\mu = (1, 0)$ in surgery space $V \cong H_1(\partial M; \mathbb{R}) \cong \mathbb{R}^2$ passes through $(a + 1, 1)$ while the line through $\alpha$ and $-\mu$ passes through $(a, 1)$. It follows that $||a + 1||_T$ and $||\alpha||_T$ are both less than $S + 4$. Since one of them is even, this contradicts Lemma 4.2.

So any non-trivial finite fillings must be odd integral. Suppose there were two such. Each would have norm at most $S + 8$. The line joining them would necessarily pass through some even integral surgeries which would therefore also have norm at most $S + 8$. This again contradicts Lemma 4.3.

Now suppose that $2a + 1$ is a non-trivial finite filling. Then $||2a + 1, 1||_T \leq S + 8$ while $||2a, 1||_T \geq S + 12$ by Lemma 1.2. Let $P \subset V$ denote the norm-ball of radius $S + 8$. By [CGLS, Proposition 1.1.2], $P$ is a finite-sided convex polygon whose vertices are multiples of boundary slopes. In particular, $(2a + 1, 1)$ is not a vertex of $P$ (otherwise [CGLS, Theorem 2.0.3], $M(2a + 1) \cong S^2 \times S^1$ which is absurd).

We now construct the non-integral boundary slope $c/d$ and show that it lies in the interval $(2a, 2a + 2)$. Since $(2a + 1, 1)$ is inside $P$ and $(2a, 1)$ is not, there is a segment of $\partial P$ which intersects the line $y = 1$ between them. Let $k(c, d)$ be the vertex of this segment which lies on or above $y = 1$, i.e., $k \in \mathbb{Q}$ and $c/d$ is a boundary slope. Consider the segment from the origin to $k(c, d)$. As both endpoints are in $P$, this segment is also. It crosses $y = 1$ at $(c/d, 1)$ which must lie between $(2a, 1)$ and $(2(a + 1), 1)$. (Otherwise, the segment joining $(2a + 1, 1)$ and $(c/d, 1)$ passes through $(2a, 1)$, say. Since both endpoints are in $P$, this segment is in $P$ and in particular $(2a, 1)$ is in $P$, a contradiction.)

Thus $|2a + 1 - c/d| < 1$, as required.

**Corollary 4.3.** If a knot satisfies the conditions of the theorem and has no non-integral boundary slopes, then it admits no non-trivial finite surgeries.

**Corollary 4.4.** Alternating $(p, q, r)$ pretzel knots with $1/|p| + 1/|q| + 2/|r| < 1$ admit no non-trivial finite surgeries.

**Proof:** This follows since pretzel knots are Montesinos knots and alternating Montesinos knots have no non-integral boundary slopes (see [IO]).
Remark: Note that the second Corollary also follows from Delman and Robert’s [{DR}] proof that alternating knots satisfy strong property P.

5. Proof of Theorem 1.4

We turn now to the case of a \((p, q, -r)\) pretzel knot \(K\) where \(4 \leq r\) is even and \(p\) and \(q\) are both odd. We will assume \(3 \leq p \leq q\). The strategy is similar to that of Section 3. We begin with two lemmas based on the work of Hatcher and Oertel [{HO}].

Lemma 5.1. If \(p \geq 2r + 1\), then

\[
\frac{p(p-1) + 1 - 3r}{\frac{p-1}{2}-r} \quad \text{and} \quad \frac{q(q-1) + 1 - 3r}{\frac{q-1}{2}-r}
\]

are the non-integral boundary slopes of \(K\).

Lemma 5.2. If \(p < r\), and \(K\) has a non-integral boundary slope, then that slope is

\[
2(p + q + r - 1 - \frac{(p-1)(q-1)}{p-1+q-1}).
\]

The proof of the following lemma is similar to that of Lemma 3.2.

Lemma 5.3. \(M(2(p+q))\) contains an incompressible torus.

Proposition 5.4. If \(p > 2r + 1\), then \(K\) admits no non-trivial finite surgeries.

Proof: By Theorem 1.5, a non-trivial finite surgery would be close to one of the non-integral boundary slopes of Lemma 5.1. However,

\[
2(p+q) - \frac{p(p-1) + 1 - 3r}{\frac{p-1}{2}-r} = \frac{2(p+q) - 2(p+r) - \frac{(r-1)^2}{\frac{p-1}{2}-r}}{2} \\
= \frac{2q - 2r - \frac{(r-1)^2}{\frac{p-1}{2}-r}}{2} \\
\geq \frac{4r + 6 - 2r - \frac{(r-1)^2}{\frac{r}{2}+1}}{2} \\
= \frac{7r + 5}{\frac{r}{2}+1} \geq 11
\]

and similarly for the other slope of Lemma 5.1. Therefore, any non-trivial finite surgery would be of distance (in the sense of minimal geometric intersection) greater than 10 from the toroidal surgery \(2(p+q)\). However this contradicts work of Agol [{A}] and Lackenby [{L}] showing that the distance between exceptional surgeries is \(\leq 10\).

Proposition 5.5. If \(p \leq r - 5\), then \(K\) admits no non-trivial finite surgeries.

Proof: As in the previous proposition, we observe that

\[
|2(p+q + r - 1 - \frac{(p-1)(q-1)}{p-1+q-1}) - 2(p+q)| > 10.
\]

Thus the lone non-integral boundary slope of Lemma 5.2 is too far from the toroidal boundary slope \(2(p+q)\) (by Theorem 1.5 a finite filling could only occur at an odd-integral slope, which would therefore have to be within distance 9 of the even number \(2(p+q)\)).

\[\Box\]
We now have a fairly precise description of what a finite filling $s$ on a $(p, q, -r)$ pretzel knot would look like. By Theorem 5.6, if $s$ would have to be odd-integral and near a non-integral boundary slope and by Propositions 6.4 and 6.5, $p + 3 \geq r \geq (p - 1)/2$. We now propose to explicitly calculate the fundamental group of such a filling. We will then project onto a group $G$ and observe that $G$ is generically infinite.

The Wirtinger presentation of a $(p, q, -r)$ pretzel knot is (compare Equation 1):

$$
\pi_1(M) = \langle x, y, z \mid (zx)^{p-1/2}z(xz)^{1-p/2} = (yx)^{-q+1/2}y(xz)^q(x^{q+1}/2),
\rangle
$$

$$
(yz^{-1})^{-r/2}y(yz^{-1})^{r/2} = (yx)^{1-a/2}x(yz)^{q+1/2},
\rangle
$$

$$
(yz^{-1})^{-r/2}y(yz^{-1})^{r/2} = (yx)^{(p+1)/2}x(yz)^{(p+1)/2},
$$

The longitude being

$$
l = x^{-2(p+q)/2}y(xz)^{(q+1)/2}(yz^{-1})^{-r/2}(yx)^{(q+1)/2}(zx)^{(p+1)/2},
$$

filling along an odd integral slope $s$ results in

$$
\pi_1(M(s)) = \langle x, y, z \mid (zx)^{(p+1)/2}z(xz)^{(1-p)/2} = (yx)^{-q+1/2}y(xz)^q(x^{q+1}/2),
\rangle
$$

$$
(yz^{-1})^{-r/2}y(yz^{-1})^{r/2} = (yx)^{1-a/2}x(yz)^{q+1/2},
\rangle
$$

$$
(yz^{-1})^{-r/2}y(yz^{-1})^{r/2} = (yx)^{(p+1)/2}x(yz)^{(p+1)/2},
$$

We can obtain a more manageable factor group $G$ by adding the relators $(yz^{-1})^{r/2}$, $y^{-1}$, and $(zx)^r$:

$$
G = \langle y, z \mid (yz^{-1})^{r/2}, (zy)^r, z = (zy)^{(p+1)/2}y(yz)^{-(p+1)/2}, y^{s-2p} \rangle
$$

$$
= \langle w, y \mid (y^2w^2)^{r/2}, w^p, (wy)^2, y^{s-2p} \rangle,
$$

where $w = (yz)^{(p-1)/2}$. This is an example of a group which Coxeter has called

$$
(2, a, b, c) = \langle R, S \mid R^a, S^b, (RS)^2, (R^2S^2)^c \rangle.
$$

Thus $G = (2, p, |s - 2p|; r/2)$. Moreover, $\pi_1(M(s))$ will be infinite whenever $G$ is.

And indeed, these groups are usually infinite as Edjvet has shown:

**Theorem 5.6 (Main Theorem of [1]).** If $2 \leq a \leq b$, $2 \leq c$ and $(2, a, b, c) \neq (2, 3, 13, 4)$, then the group $(2, a, b, c)$ is finite if and only if it is one of the following:

(i)  $(2, 2, b, c)$ $(2 \leq b, 2 \leq c)$;
(ii) $(2, 3, b, c)$ $(3 \leq b \leq 6, 4 \leq c)$;
(iii) $(2, 3, 7, c)$ $(4 \leq c \leq 8)$;
(iv) $(2, 3, b, c)$ $(8 \leq b \leq 9, 4 \leq c \leq 5)$;
(v)  $(2, 3, b, 4)$ $(10 \leq b \leq 11)$;
(vi) $(2, 4, b, 2)$ $(4 \leq b)$;
(vii) $(2, 4, 4, c)$ $(3 \leq c)$;
(viii) $(2, 4, 5, c)$ $(3 \leq c \leq 4)$;
(ix)  $(2, 4, 7, 3)$;
(x)  $(2, 5, b, 2)$ $(5 \leq b \leq 6)$;
(xi) $(2, 6, 7, 2)$.

Since $p$ is odd, if $p \leq |s - 2p|$, then $G$ is infinite unless $p = 3$ or 5.

Similarly, if $|s - 2p| \leq p$, we see that $G$ is infinite unless $|s - 2p| = 3$ or 5 ($|s - 2p|$ is also odd) whence $s \leq 2p + 5$. On the other hand, by [1], the
finite filling $s$ and the toroidal filling $2(p + q)$ (Lemma 5.3) have distance at most 10. Since $s$ is odd and $2(p + q)$ is even, they in fact differ by at most 9. Thus, $9 \geq 2(p + q) - s \geq 2(p + q) - (2p + 5) = 2q - 5$. It follows that $2q \leq 14$, whence $3 \leq p \leq q \leq 7$.

So we can assume that $3 \leq p \leq 7$. That is, if $p \geq 9$, the knot admits no non-trivial finite surgeries.

Now, earlier work shows that there are no non-trivial surgeries unless \( \frac{p-1}{2} \leq r \leq p + 3 \). So, given $r \geq 4$, and assuming $3 \leq p \leq 7$, we see that we are left to investigate $4 \leq r \leq 10$. And since Theorem 1.3 does not apply to the knots $(−4, 3, 3)$, $(−4, 3, 5)$ and $(−6, 3, 3)$ these knots must also be examined. The details may be found in [M1]. Note that we again use SnapPea [We] to resolve a few difficult cases.

In summary then, Theorems 1.5 and 5.6 combine to show that a $(p, q, −r)$ pretzel knot, with $4 \leq r$ even and $3 \leq p \leq q$ odd admits no non-trivial finite surgeries. We have therefore proved

**Theorem 1.4.** A $(p, q, −r)$ pretzel knot, with $4 \leq r$ even and $3 \leq p \leq q$ odd admits no non-trivial finite surgeries.

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**References**

[A] I. Agol, *Volume and topology of hyperbolic 3-manifolds*, PhD Thesis, U.C. San Diego (1998).

[BH] S. Bleiler and C. Hodgson, ‘Spherical space forms and Dehn fillings,’ *Topology* 35 (1996) 809-833.

[BB] S. Boyer and L. Ben Abdelghani, ‘A calculation of the Culler-Shalen seminorms associated to small Seifert Dehn fillings,’ *Proc. London Math. Soc.* 83 (2001) 235-256.

[BZ1] S. Boyer and X. Zhang, ‘Finite Dehn surgery on knots,’ *J. Amer. Math. Soc.* 9 (1996) 1005-1050.

[BZ2] ———, ‘On Culler-Shalen seminorms and Dehn filling,’ *Ann. of Math.* 148 (1998) 737-781.

[BZ3] ———, ‘On simple points of character varieties of 3-manifolds,’ *Knots in Hellas ’98, Proceedings*, World Scientific (2000) 27-35.

[BZ4] ———, ‘A proof of the finite filling conjecture,’ (submitted).

[CCGLS] D. Cooper, M. Culler, H. Gillet, D.D. Long and P.B. Shalen, ‘Plane curves associated to character varieties of 3-manifolds’, *Invent. Math.* 118 (1994), 47-84.

[CHK] D. Cooper, C. Hodgson, S. Kerchoff, ‘Three-dimensional orbifolds and cone-manifolds,’ *MSJ Memoirs* 5 Mathematical Society of Japan (2000).

[C] H.S.M. Coxeter, ‘The abstract groups $G_{m,n,p}$,’ *Trans. Amer. Math. Soc.* 45 (1939) 73-150.

[CGLS] M. Culler, C.McA. Gordon, J. Luecke and P.B. Shalen, ‘Dehn surgery on knots’, *Ann. of Math.* 125 (1987) 237-300.

[De] C. Delman, ‘Constructing essential laminations and taut foliations which survive all Dehn surgeries,’ (preprint).

[DR] C. Delman and R. Roberts, ‘Alternating knots satisfy strong property P,’ *Comment. Math. Helvetici* 74 (1999) 376-397.
N. Dunfield, ‘Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds,’ *Invent. Math.* **136** (1999) 623-657 [math.GT/9802022].

M. Edjvet, ‘On certain quotients of the triangle groups,’ *J. Algebra* **169** (1994) 367-391.

R. Fintushel and R. Stern, ‘Constructing lens spaces by surgery on knots,’ *Math. Z.* **175** (1980) 33-51.

D. Gabai, ‘Foliations and the topology of 3-manifolds. III,’ *J. Diff. Geom.* **26** (1987) 479-536.

C.McA. Gordon, ‘Boundary slopes of punctured tori in 3-manifolds,’ *Trans. Amer. Math. Soc.* **350** (1998) 1713-1790.

M. Lackenby, ‘Word hyperbolic Dehn surgery,’ *Invent. Math.* **140** (2000) 243-282 [math.GT/9808129].

T.W. Mattman, ‘The Culler-Shalen seminorms of pretzel knots,’ Ph.D. Thesis, McGill University, Montreal (2000) available at [http://www.csuchico.edu/math/mattman](http://www.csuchico.edu/math/mattman)

U. Oertel, ‘Closed incompressible surfaces in complements of star links,’ *Pac. J. Math.* **111** (1984) 209-230.

S. Oh, ‘Reducible and toroidal 3-manifolds obtained by Dehn fillings,’ *Topology Appl.* **75** (1997) 93-104.

D. Rolfsen, *Knots and Links* 2nd Edition, Publish or Perish (1990).

W. Thurston, ‘The geometry and topology of 3-manifolds,’ Lecture notes, Princeton University, (1977).

H.F. Trotter, ‘Non-invertible knots exist,’ *Topology* **2** (1963) 275-280.

Y-Q. Wu, ‘Dehn fillings producing reducible manifolds and toroidal manifolds,’ *Topology* **37** (1998) 95-108.

H.F. Trotter, ‘Non-invertible knots exist,’ *Topology* **2** (1963) 275-280.

Y-Q. Wu, ‘Dehn fillings producing reducible manifolds and toroidal manifolds,’ *Topology* **37** (1998) 95-108.

[Du] N. Dunfield, ‘Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds,’ *Invent. Math.* **136** (1999) 623-657 [math.GT/9802022].

[Ed] M. Edjvet, ‘On certain quotients of the triangle groups,’ *J. Algebra* **169** (1994) 367-391.

[FS] R. Fintushel and R. Stern, ‘Constructing lens spaces by surgery on knots,’ *Math. Z.* **175** (1980) 33-51.

[Ga] D. Gabai, ‘Foliations and the topology of 3-manifolds. III,’ *J. Diff. Geom.* **26** (1987) 479-536.

[Go1] C.McA. Gordon, ‘Boundary slopes of punctured tori in 3-manifolds,’ *Trans. Amer. Math. Soc.* **350** (1998) 1713-1790.

[Go2] , ‘Toroidal Dehn surgeries on knots in lens spaces,’ *Math. Proc. Camb. Phil. Soc.* **125** (1999) 433-440.

[Ho] A.E. Hatcher and U. Oertel, ‘Boundary slopes for Montesinos knots,’ *Topology* **28** (1989) 453-480.

[K] A. Kawauchi, ‘Classification of pretzel knots,’ *Kobe J. Math* **2** (1985) 11-22.

[L] M. Lackenby, ‘Word hyperbolic Dehn surgery,’ *Invent. Math.* **140** (2000) 243-282 [math.GT/9808129].

[M1] T.W. Mattman, ‘The Culler-Shalen seminorms of pretzel knots,’ Ph.D. Thesis, McGill University, Montreal (2000) available at [http://www.csuchico.edu/math/mattman](http://www.csuchico.edu/math/mattman)

[M2] , ‘The Culler-Shalen seminorms of the (-2,3,n) pretzel knot,’ [math.GT/9911083](http://www.csuchico.edu/math/mattman)

[Oe] U. Oertel, ‘Closed incompressible surfaces in complements of star links,’ *Pac. J. Math.* **111** (1984) 209-230.

[Oh] S. Oh, ‘Reducible and toroidal 3-manifolds obtained by Dehn fillings,’ *Topology Appl.* **75** (1997) 93-104.

[R] D. Rolfsen, *Knots and Links* 2nd Edition, Publish or Perish (1990).

[Th] W. Thurston, ‘The geometry and topology of 3-manifolds,’ Lecture notes, Princeton University, (1977).

[Tr] H.F. Trotter, ‘Non-invertible knots exist,’ *Topology* **2** (1963) 275-280.

[We] J. Weeks, *SnapPea computer program for studying hyperbolic 3-manifolds* available at [http://thames.northnet.org/weeks/index/SnapPea.html](http://thames.northnet.org/weeks/index/SnapPea.html)

[Wu1] Y-Q. Wu, ‘Dehn fillings producing reducible manifolds and toroidal manifolds,’ *Topology* **37** (1998) 95-108.

[Wu2] , ‘Dehn surgery on arborescent knots and links—a survey,’ *Chaos Solitons Fractals* **9** (1998) 671–679 [math.GT/9704221].