Smoothness of anisotropic wavelets, frames and subdivision schemes

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Abstract

This paper presents a detailed regularity analysis of anisotropic wavelet frames and subdivision. In the univariate setting, the smoothness of wavelet frames and subdivision is well understood by means of the matrix approach. In the multivariate setting, this approach has been extended only to the special case of isotropic refinement with the dilation matrix all of whose eigenvalues are equal in the absolute value. The general anisotropic case has resisted to be fully understood: the matrix approach can determine whether a refinable function belongs to $C(\mathbb{R}^s)$ or $L_p(\mathbb{R}^s)$, $1 \leq p < \infty$, but its Hölder regularity remained mysteriously unattainable.

In this paper we show how to compute the Hölder regularity in $C(\mathbb{R}^s)$ or $L_p(\mathbb{R}^s)$, $1 \leq p < \infty$. In the anisotropic case, our expression for the exact Hölder exponent of a refinable function reflects the impact of the variable moduli of the eigenvalues of the corresponding dilation matrix. In the isotropic case, our results reduce to the well-known facts from the literature. We provide an efficient algorithm for determining the finite set of the restricted transition matrices whose spectral properties characterize the Hölder exponent of the corresponding refinable function. We also analyze the higher regularity, the local regularity, the corresponding moduli of continuity, and the rate of convergence of the corresponding subdivision schemes. We illustrate our results with several examples.

Keywords: multivariate wavelets and frames, refinable functions, Hölder regularity, modulus of continuity, local regularity, anisotropic dilation matrix, transition matrix, joint spectral radius, invariant polytope algorithm

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1 Introduction

We study multivariate refinement equation

\[ \varphi(x) = \sum_{k \in \mathbb{Z}^s} c_k \varphi(Mx - k), \quad x \in \mathbb{R}^s, \]

with a compactly supported sequence of coefficients \( c_k \in \mathbb{R} \) and with a general integer dilation matrix \( M \in \mathbb{Z}^{s \times s} \) all of whose eigenvalues are larger than one in the absolute value. We do not make any assumptions on the stability of the integer shifts of \( \varphi \).

In this paper, we characterize continuous and \( L^p \) solutions of (1). Our main contribution is the exact expression for the Hölder exponent of \( \varphi \) in \( C(\mathbb{R}^s) \) and in \( L^p(\mathbb{R}^s) \), \( 1 \leq p < \infty \), see Theorems 1 and 7. In the anisotropic case, the Hölder exponent of \( \varphi \) reflects the influence of the invariant subspaces of \( M \) corresponding to its different by modulus eigenvalues. In the isotropic case, when all the eigenvalues of \( M \) are equal in the absolute value, our results reduce to the well-known ones from the literature. We also estimate the modulus of continuity and analyze Lipschitz, local and higher regularity of continuous \( \varphi \).

It is well known that compactly supported solutions (refinable functions) of (1) can generate systems of multivariate wavelets or frames, see e.g. [8, 9, 14, 44]. Refinable functions are building blocks for the limits of subdivision algorithms widely used in approximation and for generating curves and surfaces, see e.g. [4, 10, 21, 45, 62]. Refinable functions naturally appear in recent applications in probability, number theory, and combinatorics [17, 24, 39, 48, 53].

In the univariate case, \( M \geq 2 \) is an integer, there are several efficient methods for determining the regularity of refinable functions. In [11, 18, 23] the authors compute precisely the Sobolev exponent of \( \varphi \in L^2(\mathbb{R}) \). The so-called matrix approach yields the Hölder exponent of \( \varphi \in C(\mathbb{R}) \) and, in addition, provides a detailed analysis of its moduli of continuity and of its local regularity [13, 16, 49, 56]. An obstacle to the practical use of the matrix approach is the NP-hardness of the joint spectral radius computation. This problem, however, was successfully resolved for a large class of problems by recent results in [28, 29, 42] where the authors presented fast and efficient methods of the joint spectral radius computation. Indeed, the invariant polytope algorithm [28] estimates the joint spectral radius for the corresponding transition matrices of size up to 20 and, in most cases, even determines its precise value.

The generalization of the matrix approach to the multivariate case turned out to be a difficult task in the case of general dilation matrices. The special case of isotropic dilation is currently fully understood, see [4, 5, 6, 11, 18, 21, 23, 30, 32, 36, 38, 57]. Several partial results in the anisotropic case are also available: for characterizations of continuity and \( L^p \), \( 1 \leq p < \infty \), regularity of \( \varphi \) see e.g. [3, 33]; for estimates for the Hölder exponent of \( \varphi \) see e.g. [3, 33].

The reason for the difficulty of the anisotropic case is natural and hardly avoidable. In the univariate case, say \( M = 2 \), the distance between two points \( x, y \in \mathbb{R} \) can be expressed in terms of their binary expansions. The distance between the values \( \varphi(x) \) and \( \varphi(y) \) depends on the behavior of the products of certain square matrices derived from \( c_k, k \in \mathbb{Z} \). These two observations establish a correlation between \( |x - y| \) and \( |\varphi(x) - \varphi(y)| \), which leads to the
formula for the Hölder exponent of \( \varphi \). This summarizes the essence of the matrix approach. In the multivariate case, one can similarly estimate the distance between \( \varphi(x) \) and \( \varphi(y) \) by suitable matrix products. The problem occurs at an unexpected point: the expression for the distance between \( x, y \in \mathbb{R}^s \). One can try to use the corresponding \( M \)-adic expansions with a certain set of digits from \( \mathbb{Z}^s \), but such expansions do not provide a clear estimate for the distance between \( x \) and \( y \). Indeed, unless the matrix \( M \) is isotropic, multiplication by a high power of \( M \) can enlarge distances differently in different directions. Hence, the points \( M^\ell x \) and \( M^\ell y \), \( \ell \in \mathbb{N} \), whose \( M \)-adic expansions are essentially the same, may have different asymptotic behavior as \( \ell \to \infty \). Remarkably simple examples show that a direct analogue of the isotropic formula for the Hölder exponent does not hold in the anisotropic case. Moreover, unless \( M \) is isotropic, this formula never holds for Lipschitz refinable functions, see section 3.1.

Nevertheless, there are ways of treating the anisotropic case. In [12], the authors consider special anisotropic Sobolev spaces. We emphasize on incorporating the spectral properties of the dilation matrix \( M \) into the expression for the Hölder exponent of \( \varphi \). Furthermore, we get rid of the \( M \)-adic expansions and base our analysis on geometric properties of tilings generated by \( M \).

Our paper is organized as follows. In section 3 we characterize the continuity and determine the Hölder regularity of multivariate refinable functions, see Theorems 1 and 2. In subsection 3.2 we provide an algorithm for construction of continuous solutions of (1). We consider several examples and list several important special cases of Theorems 1 in subsection 3.1. The crucial steps of the proofs and actual proofs of Theorems 1 and 2 are given in subsections 3.3 and 5.5. We illustrate our results by numerical examples in subsection 3.6. In section 4 we show how to factorize smooth refinable functions and compute the Hölder exponents of their directional derivatives. Sections 5 and 6 deal with the moduli of continuity of continuous refinable functions and with determining their local regularity. In section 7 we analyze the existence of \( L_p \)-solutions of (1). We show that a direct analogue of the formula for the Hölder exponent (i.e. replacing the joint spectral radius by the \( p \)-radius) does not hold in \( L_p \), \( 1 \leq p < \infty \). To characterize the \( L_p \) Hölder exponent of \( \varphi \), we consider extended transition matrices, see subsections 7.3 and 7.4. In section 8 we derive the expression for the convergence rate of subdivision. We show that, in the anisotropic case, the convergence rate of subdivision and the Hölder exponent of the corresponding refinable function \( \varphi \) cannot be related similarly to the isotropic case, even if \( \varphi \) is stable.

### 2 Background and notation

We consider the standard notation for the function spaces \( C, C^k, L_p, 1 \leq p < \infty \), the space of vector-valued functions \( f : X \to \mathbb{R}^n \) with components belonging to \( L_p \) is denoted by \( L_p(X, \mathbb{R}^n) \). We simply write \( L_p(X) \), if the range space is fixed. The Schwartz space of smooth rapidly decreasing functions over \( \mathbb{R}^s \) is denoted by \( \mathcal{S} \), and \( \mathcal{S}' \) is the space of tempered distributions (distributions over \( \mathcal{S} \) or distributions of slower growth); by \( \mu(X) \) we denote the Lebesgue measure of a set \( X \subset \mathbb{R}^n \), by \( |\cdot| \) we denote either a modulus of a complex number or
the cardinality of a finite set. The norm $\| \cdot \|$ in finite dimensional spaces is always Euclidean, unless stated otherwise.

### 2.1 Spectral properties of the dilation matrix

We make a standard assumption that the integer dilation matrix $M \in \mathbb{Z}^{s \times s}$ is expansive, i.e., all its eigenvalues are larger than 1 in the absolute value. Hence, $m = |\det M| \geq 2$. Among the eigenvalues $1 < |\lambda_1| \leq \cdots \leq |\lambda_s|$ of $M$, exactly $n_i$ of them are in the absolute value equal to $r_i$, $i = 1, \ldots, q(M) \leq s$. If $M$ is isotropic, then $q(M) = 1$. For $i = 1, \ldots, q(M)$, let $J_i \subset \mathbb{R}^s$ be the root subspaces of $M$ corresponding to the eigenvalues of modulus $r_i$. Thus, $\dim(J_i) = n_i$ and the operator $M|_{J_i}$ has all its eigenvalues equal to $r_i$ in the absolute value.

The space $\mathbb{R}^s$ is a direct sum $\mathbb{R}^s = \bigoplus_{i=1}^{q(M)} J_i$ of the subspaces $J_1, \ldots, J_{q(M)}$. There exists an invertible transformation $B : \mathbb{R}^s \to \mathbb{R}^s$ such that $M$ has the following block diagonal structure

$$B^{-1}MB = \begin{pmatrix} M|_{J_1} & 0 & \cdots & 0 \\ 0 & M|_{J_2} & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 & M|_{J_q} \end{pmatrix}.$$  \hspace{1cm} (2)

### 2.2 Dilation matrix and tiles

The matrix $M$ splits the integer lattice $\mathbb{Z}^s$ into $m$ equivalence (quotient) classes defined by the relation $x \sim y \iff y - x \in M\mathbb{Z}^s$. Choosing one representative $d_i \in \mathbb{Z}^s$ from each equivalence class, we obtain a set of digits $D(M) = \{d_i : i = 0, \ldots, m - 1\}$. We always assume that $0 \in D(M)$. The standard choice is to take $D(M) = \mathbb{Z}^s \cap M[0,1)^s$.

For every integer point $d \in \mathbb{Z}^s$, we denote by $M_d$, the affine operator $M_d x = Mx - d$, $x \in \mathbb{R}^s$.

We use the notation $0.d_1d_2\ldots = \sum_{i=1}^{\infty} M^{-i}d_i$, $d_i \in D(M)$. Consider the following set

$$G = \left\{ \sum_{i=1}^{\infty} M^{-i}d_i : d_i \in D(M) \right\}. \hspace{1cm} (3)$$

By [26, 27], for every expansive integer matrix $M$ and for an arbitrary set of digits $D(M)$, the set $G$ is a compact set with a nonempty interior and possesses the properties:

a) the Lebesgue measure $\mu(G) \in \mathbb{N}$;

b) $G = \bigcup_{d \in D(M)} M_d^{-1}G$, the sets $M_d^{-1}G$ have intersections of zero measure;

c) the indicator function $\chi = \chi_G(x)$ of $G$ satisfies the refinement equation

$$\chi(x) = \sum_{d \in D(M)} \chi(Mx - d) \quad x \in \mathbb{R}^s;$$
d) \( \sum_{k \in \mathbb{Z}^s} \chi(x + k) \equiv \mu(G) \), i.e., integer shifts of \( \chi \) cover \( \mathbb{R}^s \) with \( \mu(G) \) layers;

e) \( \mu(G) = 1 \) if and only if the function system \( \{\chi(\cdot + k)\}_{k \in \mathbb{Z}^s} \) is orthonormal.

If \( \mu(G) = 1 \), then \( G \) is called a tile. The integer shifts of a tile define a tiling.

**Definition 1** A tiling generated by an integer expansive matrix \( M \) and by a set of digits \( D(M) \) is a collection of sets \( G = \{k + G\}_{k \in \mathbb{Z}^s} \) such that

a) the union of the sets in \( G \) covers \( \mathbb{R}^s \) and \( \mu((\ell + G) \cap (k + G)) = 0, \ell \neq k; \)

b) \( G = \bigcup_{d \in D(M)} M_d^{-1} G \).

Not every \( M \) possesses a digit set \( D(M) \) such that \( G \) is a tile. Those situations, however, are rare. For instance, a digit set generating a tile always exists in cases \( s = 2, 3 \) and also for arbitrary \( s \) with an extra assumption \( |\det M| > s \), which is quite general for integer expansive matrices [40]. See [3, 41] for more details. Thus, in this paper, we assume that \( G \) is a tile.

We denote

\[ G_{d_1\ldots d_n} = M_{d_1}^{-1} \cdots M_{d_n}^{-1} G, \quad d_1, \ldots, d_n \in D(M). \]

Then \( G^n = M^{-n} G = \{M^{-n}(k + G)\}_{k \in \mathbb{Z}^s} = \{G_{d_1\ldots d_n} : d_1, \ldots, d_n \in D(M)\}, n \in \mathbb{N} \).

### 2.3 Refinable functions and the transition operator

A compactly supported distribution \( \varphi \in S'(\mathbb{R}^s) \) satisfying equation (1) is called a refinable function. It is well known that the solution of (1) such that \( \int_{\mathbb{R}^s} \varphi(x) \, dx \neq 0 \) exists if and only if \( \sum_{k \in \mathbb{Z}^s} c_k = m \). We assume further that the coefficients of (1) satisfy sum rules of order one

\[ \sum_{k \in \mathbb{Z}^s} c_{Mk - d} = 1, \quad d \in D(M). \]  

These conditions arise naturally in the context of subdivision and are necessary for existence of stable refinable functions [4]. Consider the transition operator \( T : S'(\mathbb{R}^s) \to S'(\mathbb{R}^s) \) defined by

\[ T f(x) = \sum_{k \in \mathbb{Z}^s} c_k f(Mx - k), \quad x \in \mathbb{R}^s. \]  

For every compactly supported function \( f \in S' \) such that \( \int_{\mathbb{R}^s} f(x) \, dx = 1 \), the sequence \( \{T^j f\}_{j \in \mathbb{N}} \) converges to \( \varphi \) in the space \( S' \) [3]. The space of distributions supported on the set

\[ K = \left\{ x \in \mathbb{R}^s : x = \sum_{j=1}^{\infty} M^{-j} \gamma_j, \quad \gamma_j \in \text{supp}(c), \quad c = \{c_k\}_{k \in \mathbb{Z}^s} \right\} \]  

is invariant under \( T \). Hence, for \( f \in S'(K) \), we have \( T^j f \in S'(K) \) for all \( j \in \mathbb{N} \). Therefore, the limit \( \varphi \in S'(K) \). Thus, supp\( \varphi \subset K \), see [3, Proposition 2.2].
Definition 2  A finite set \( \Omega \subset \mathbb{Z}^s \) is a minimal subset of \( \mathbb{Z}^s \) with the property
\[
K \subset \Omega + G = \bigcup_{k \in \Omega} (k + G).
\]

We denote \( N = |\Omega| \).

It is shown easily that \( M_d^{-1}(\Omega + G) \subset \Omega + G \) for every \( d \in D(M) \).

The main idea of the matrix approach is to pass from a function \( f : \mathbb{R}^s \to \mathbb{R} \) supported on \( K \)

to the vector-valued function
\[
v : G \to \mathbb{R}^N, \quad v(x) = v_f(x) = (f(x + k))_{k \in \Omega}, \quad x \in G.
\]

Then the transition operator \([5]\) restricted to the space
\[
\left\{ f \in L_1(\mathbb{R}^s) \ : \ \text{supp} \, f \subset \Omega + G, \right\}
\]
becomes the self-similarity operator \( A : L_1(G, \mathbb{R}^N) \to L_1(G, \mathbb{R}^N) \) defined by
\[
(Av)(x) = T_d v(Mx - d), \quad x \in M^{-1}(d + G), \quad d \in D(M),
\]
where \( T_d \) are the \( N \times N \) transition matrices defined by
\[
(T_d)_{ab} = c_{Ma-b+d}, \quad a, b \in \Omega, \quad d \in D(M).
\]

The rows and columns of the matrices \( T_d \) are enumerated by elements from the set \( \Omega \). We denote
\[
T = \{ T_d : d \in D(M) \}.
\]
The refinement equation becomes the self-similarity equation \( Av = v \) for the vector-valued function
\( v(x) = v_\varphi(x) \) defined by \([7]\) with \( f = \varphi \), i.e.
\[
v(x) = T_d v(Mx - d), \quad x \in M^{-1}(d + G), \quad d \in D(M).
\]

2.4 Important subspaces of \( \mathbb{R}^N \)

We consider the following affine subspace of the space \( \mathbb{R}^N \)
\[
V = \left\{ w = (w_1, \ldots, w_N) \in \mathbb{R}^N \ : \ \sum_{j=1}^N w_j = 1 \right\}.
\]

It is well known that every compactly supported refinable function \( \varphi \in \mathcal{S}' \) such that
\[
\int_{\mathbb{R}^s} \varphi(x) \, ds = 1
\]
possesses the partition of unity property:
\[
\sum_{k \in \mathbb{Z}^s} \varphi(x + k) = 1
\]
(see, e.g. [3, 1]). Hence, if \( \varphi \) is continuous, then \( v(x) \in V \) for all \( x \in G \). In particular \( v(0) \in V \). For summable refinable function, \( v(x) \in V \) for almost all \( x \in G \).

We denote the linear part of the subspace \( V \) by

\[
W = \left\{ w = (w_1, \ldots, w_N) \in \mathbb{R}^N : \sum_{j=1}^N w_j = 0 \right\}.
\]

Finally, every continuous refinable function defines the space of differences of the vector-valued function \( v = v_\varphi \)

\[
U = \text{span} \left\{ v(y) - v(x) : y, x \in G \right\}, \quad n = \dim U. \tag{12}
\]

Since \( v(x) \in V \) for all \( x \in G \), we have \( U \subseteq W \), and, therefore, \( n \leq N - 1 \). The sum rules [3, 1] imply that the column sums of each matrix \( T_d \) are equal to one. Therefore, \( T_dV \subset V \) and \( T_dW \subset W \). Thus, \( V \) is a common affine invariant subspace of the family \( T \) and \( W \) is its common linear invariant subspace.

Since \( U \) is invariant under all \( T_d \), \( d \in D(M) \), the restrictions \( A_d = T_d|_U \) of the operators \( T_d \), \( d \in D(M) \), to the subspace \( U \) are well defined. For a fixed basis of \( U \), we denote by

\[
\mathcal{A} = \mathcal{T}|_U = \{ A_d : d \in D(M) \} \tag{13}
\]

the set of the associated \( n \times n \) matrices. If the family \( \mathcal{T} \) is irreducible on \( W \), then \( \mathcal{A} = \mathcal{T}|_W \).

We also consider the following subspaces of the space \( U \)

\[
U_i = \text{span} \left\{ v(y) - v(x) : x, y \in G, y - x \in J_i \right\}, \quad i = 1, \ldots, q(M). \tag{14}
\]

Note that \( U_i \) are nonempty, due to the interior \( \text{int}(G) \) of \( G \) being nonempty. It is seen easily that the spaces \( \{ U_i \}_{i=1}^{q(M)} \) span the whole space \( U \), but their sum may not be direct. Indeed, the subspaces \( \{ U_i \}_{i=1}^{q(M)} \), unlike the subspaces \( \{ J_i \}_{i=1}^{q(M)} \), may have nontrivial intersections. For example, they can all coincide with \( U \). The following result shows that all \( U_i \) are invariant under \( \mathcal{A} \).

**Lemma 1** If \( J \) is an invariant subspace of \( M \), then \( L = \text{span} \{ v(y) - v(x) : y - x \in J \} \) is a common invariant subspace for \( \mathcal{A} \).

**Proof.** If \( u \in L \), then \( u \) is a linear combination of several vectors of the form \( v(y) - v(x) \) with \( y - x \in J \). For every \( d \in D(M) \) we define \( x' = M^{-1}(x + d), y' = M^{-1}(y + d) \) and have

\[
v(y') - v(x') = A_d \left( v(My' - d) - v(Mx' - d) \right) = A_d \left( v(y) - v(x) \right).
\]

Hence, \( A_d(v(y) - v(x)) \in L \) for each pair \( (x, y) \), and, therefore, \( A_d u \in L \) for all \( u \in L \). \( \square \)
2.5 Joint spectral radius

Definition 3 The joint spectral radius of a finite family $A$ of linear operators $A_d$ is defined by

$$\rho(A) = \lim_{k \to \infty} \max_{A_d \in A, i=1,\ldots,k} \|A_{d_1} \cdots A_{d_k}\|^{1/k}.$$ 

This limit always exists and does not depend on the operator norm. The joint spectral radius measures the simultaneous contractibility of the operators from $A$. Indeed, $\rho(A) < 1$ if and only if there exists a norm in $\mathbb{R}^n$ in which all $A \in A$ are contractions. In general,

$$\rho(A) = \inf \left\{ \beta \geq 0 : \exists \| \cdot \| \text{ in } \mathbb{R}^n \text{ such that } \|A\| < \beta, A \in A \right\}.$$ 

We denote

$$\rho_i = \rho(A|_{U_i}).$$

3 Continuous solutions and Hölder regularity

In this section, in Theorem 1, we characterize the continuity of a solution $\varphi$ of the refinement equation (1) in terms of the spectral properties of $A$ and determine the exact Hölder exponent $\alpha_\varphi = \sup \left\{ \alpha \geq 0 : \|\varphi(\cdot + h) - \varphi\|_{C(\mathbb{R}^s)} \leq C\|h\|^\alpha, h \in \mathbb{R}^s \right\}$ of $\varphi$. Although the definition of $U$ in (12) depends on $\varphi$, the result of Proposition 1 and Definition 4 remove this dependency. Moreover, the space $U$ can be determined explicitly using Algorithm 1 in subsection 3.2 without the knowledge of $\varphi$. If this algorithm fails, then there exists no continuous solution of the corresponding refinement equation. The special cases of Theorem 1 are considered in subsection 3.1 for its summary see Remark 3. The crucial result for the proof of Theorem 1 is Theorem 2. The main steps of the proof of Theorem 2 are summarized in subsection 3.3 and the proofs of Theorems 1 and 2 are given in subsection 3.5. We illustrate our results with examples in subsection 3.6. For the readers convenience, we start by listing shortly the crucial results of this section. The proof of Proposition 1 is given in subsection 3.2.

Proposition 1 \textit{Let }$v_0 \in V$ \textit{be an eigenvector of }$T_0$ \textit{associated to the eigenvalue }$1$. \textit{If }$\varphi \in C(\mathbb{R}^s)$, \textit{then }$U$ \textit{is the smallest by inclusion common invariant subspace of the matrices }$T_d$, $d \in D(M)$, \textit{that contains }$m$ \textit{vectors }$T_d v_0 - v_0$, $d \in D(M)$.

Remark 1 \textit{Recall that }$0 \in D(M)$, \textit{which justifies the notation }$T_0$. \textit{The existence of the eigenvector }$v_0 \in V$ \textit{of }$T_0$ \textit{associated to the eigenvalue }$1$ \textit{follows by the continuity }$\varphi$ (which implies, by (11), that }$T_0 v(0) = v(0)$ \textit{and by the fact that }$v(0) \in V$.

Proposition 1 yields an equivalent definition of $U$, which we use in the sequel.
Definition 4 Let $v_0 \in V$ be an eigenvector of $T_0$ associated to the eigenvalue 1. The space $U$ is the minimal common invariant subspace of $m$ matrices $T_d$, $d \in D(M)$, that contains $m$ vectors $T_d v_0 - v_0$, $d \in D(M)$.

Remark 2 Since $T_0 v_0 - v_0 = 0$, it suffices to use $T_d v_0 - v_0$, $d \in D(M) \setminus \{0\}$ in Definition 4. Note that due to the sum rules (4), the column sums of each $T_d$ are equal to one. Hence, each $T_d$ has an eigenvalue one. Even if the eigenvalue 1 is not simple, Proposition 2 in subsection 3.2 guarantees that there exists at most one eigenvector $v_0 \in V$ such that $\rho(A) < 1$ for $U$ as in Definition 3. Thus, the subspace $U$ is always well defined, unless the refinement equation does not possess a continuous solution. For the sake of simplicity, we make the following assumption.

Assumption 1 The matrix $T_0$ has a simple eigenvalue 1.

Recall that $\rho_i = \rho(A|U_i)$, where $U_1, \ldots, U_{q(M)}$ are the subspaces defined in (14). Now we are ready to formulate the main result of this section.

Theorem 1 A refinable function $\varphi$ belongs to $C(\mathbb{R}^s)$ if and only if $\rho(A) < 1$. In this case,
\begin{equation}
\alpha_{\varphi} = \min_{i=1, \ldots, q(M)} \frac{\log \rho_i}{r_i}. \tag{15}
\end{equation}

The proof of (15) is based on Theorem 2. To state it, we define the Hölder exponent of $\varphi$ along a linear subspace $J \subset \mathbb{R}^s$ by
\begin{equation}
\alpha_{\varphi,J} = \sup \{ \alpha \geq 0 : \| \varphi(y) - \varphi(x) \| \leq C \| y - x \|^\alpha, y - x \in J \}.
\end{equation}

Theorem 2 If $\varphi \in C(\mathbb{R}^s)$, then for each $i = 1, \ldots, q(M)$, we have
\begin{equation}
\alpha_{\varphi,J_i} = \log \frac{1}{r_i} \rho_i. \tag{16}
\end{equation}

Remark 3 The identity (15) emphasizes the influence of the spectral structure of the dilation matrix $M$ on the regularity of the solution $\varphi$. Recall that, in the univariate case, the Hölder exponent is given by $\alpha_{\varphi} = \log_1 r \rho(A)$, where $M = r \geq 2$ is the corresponding dilation factor. In the multivariate case, the Hölder exponent is equal to the minimum of several such values taken over different dilation coefficients $r_i$ on the corresponding subspaces $J_i$ of $M$. In special, favorable multivariate cases, the expression in (15) becomes $\alpha_{\varphi} = \log_1 (\rho(M)) \rho(A)$ and, thus, resembles the univariate case. This happens, for instance, when the matrix $M$ is isotropic, i.e. $|\lambda_1| = \ldots = |\lambda_s| = \rho(M)$, in particular, when $M = r I$, $r \geq 2$. Another favorable situation is when the matrices in $\mathcal{A}$ do not possess any common invariant subspace. However, the need for the minimum in (15) is not exceptional. It is of crucial importance e.g. for anisotropic refinable Lipschitz continuous functions $\varphi$, see Corollary 3 in subsection 3.1.
3.1 Special cases of Theorem 1 and examples

To compare the result of Theorem 1 with the known results from the wavelet and subdivision literature, we need to define the stability of \( \varphi \).

**Definition 5** A compactly supported \( f \in L_\infty(\mathbb{R}^s) \) is stable, if there exists \( 0 < C_1 \leq C_2 < \infty \) such that for all \( c \in \ell_\infty(\mathbb{Z}^s) \),

\[
C_1\|c\|_{\ell_\infty} \leq \left\| \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) f(-\alpha) \right\|_{\infty} \leq C_2\|c\|_{\ell_\infty}.
\]

**The univariate case** \( (s = 1) \). In this case, the dilation factor is \( M \geq 2 \) and \( M = m = r \). Theorem 1 becomes a well-known statement that \( \alpha_\varphi = \log_{1/r} \rho(A) \). If \( \varphi \) is stable, then we have \( \rho(A) = \rho(T|_U) = \rho(T|_W) \) even if \( U \neq W \) (see [4]). The space \( U \) was completely characterized in [50] and it was shown that every refinement equation can be factorized to the case \( U = W \). In the multivariate case, however, there is no factorization procedure and some equations, even with stable solutions, cannot be reduced to the case \( U = W \), see Example 1 below.

**The case \( s \geq 2 \) with isotropic dilation matrix.** Since \( q(M) = 1 \), it follows that \( U_1 = U \). Theorem 1 then implies the following well-known fact.

**Corollary 1** If \( M \) is isotropic, then \( \alpha_\varphi = \log_{1/\rho(M)} \rho(A) \).

**The irreducible case with \( s \geq 2 \).** The dilation matrix \( M \) can be anisotropic, i.e. the number of different in modulus eigenvalues of \( M \) is \( q(M) > 1 \). We say that the set of matrices \( A = T|_U \) is irreducible, if they do not possess any common invariant subspace. Another corollary of Theorem 1 states the following.

**Corollary 2** If the family \( A \) is irreducible, then \( \alpha_\varphi = \log_{1/\rho(M)} \rho(A) \).

The irreducibility assumption fails however in many important cases. For instance, if \( \varphi \) is a tensor product of two refinable functions, then \( A \) is always reducible.

**Example 1** Let \( \varphi_1, \varphi_2 \in C^1(\mathbb{R}) \) be two univariate refinable function with dilations \( M_1 = 2 \) and \( M_2 = 3 \) and refinement coefficients \( c_1 \in \ell_0(\mathbb{Z}) \) and \( c_2 \in \ell_0(\mathbb{Z}) \), respectively. Then the function \( \varphi = \varphi_1 \otimes \varphi_2 \) satisfies the refinement equation with \( M = \text{diag}(2, 3) \) and \( c = c_1 \otimes c_2 \). Due to \( \varphi_1, \varphi_2 \in C^1(\mathbb{R}) \), we have \( \rho_1 = \rho(A|_{U_1}) = \frac{1}{2}, \rho_2 = \rho(A|_{U_2}) = \frac{1}{3} \). By Theorem 1, \( \alpha_\varphi = \min \{\log_{1/2} \rho_1, \log_{1/3} \rho_2\} = 1 > \frac{1}{2} \). Hence, \( \log_{1/\rho(M)} \rho(A) = \log_{1/3} \frac{1}{2} = 0.630092 \ldots \). Thus, in this case, \( \alpha_\varphi > \log_{1/\rho(M)} \rho(A) \). Note that, if \( \varphi_1 \) and \( \varphi_2 \) are both stable, then so is \( \varphi \). Nevertheless, unlike in the univariate case, the Hölder exponent of \( \varphi \) is not determined by the value \( \log_{1/\rho(M)} \rho(A) \).
After Example 1 one may hope that the case of reducible family $\mathcal{A}$ is exceptional, and the equality $\alpha_\varphi = \log_{1/\rho(M)} \rho(\mathcal{A})$ actually holds for most refinable functions. On the contrary, the result of Corollary 3 shows that the situation when the isotropic formula fails is rather generic.

**Corollary 3** If the matrix $M$ is anisotropic and the refinable function $\varphi \neq 0$ is Lipschitz continuous, then $1 = \alpha_\varphi > \log_{1/\rho(M)} \rho(\mathcal{A})$ and the family $\mathcal{A}$ is reducible.

**Proof.** The inequality $1 > \log_{1/\rho(M)} \rho(\mathcal{A})$ is equivalent to $\rho(\mathcal{A}) > 1/\rho(M)$. Assume that $\rho(\mathcal{A}) \leq 1/\rho(M)$. Since $M$ is anisotropic, factorization (2) contains $q(M) \geq 2$ blocks, and, hence, $r_i < \rho(M)$ for some $i \in \{1, \ldots, q(M)\}$. By Theorem 2 we have $\alpha_{\varphi, r_i} = \log_{1/r_i} \rho(\mathcal{A}) > \log_{1/\rho(M)} \rho(\mathcal{A}) \geq 1$. Therefore, $\varphi$ is constant on every affine subspace $u + U_i$, $u \in \mathbb{R}^n$. Hence, $\varphi \equiv 0$, because it is compactly supported. The reducibility of $\mathcal{A}$ follows by Corollary 2. □

Thus, we see that at least for all anisotropic smooth refinable functions, the simple formula for the Hölder exponent fails and the minimum in (15) is significant.

**The case of a dominant invariant subspace.** In practice, this case is much more generic than the irreducible case.

**Definition 6** A subspace $U' \subset U$ is called dominant for a family of operators $\mathcal{A}$ if

(i) $U'$ is a common invariant subspace of $\mathcal{A}$,

(ii) $U'$ is contained in all common invariant nontrivial subspaces of $\mathcal{A}$ and

(iii) $\rho(\mathcal{A}|_{U'}) = \rho(\mathcal{A})$.

Take a basis of a dominant subspace $U'$ and complement it to a basis of $U$. Let $B$ be the $n \times n$ matrix containing these basis elements of $U$. Then every matrix $A_d \in \mathcal{A}$ in this basis has a block lower triangular form

$$B^{-1} A_d B = \begin{pmatrix} \hat{A}_d & 0 \\ \ast & A_d|_{U'} \end{pmatrix}, \quad d \in D(M).$$

(17)

By Definition 6

$$\rho(\mathcal{A}|_{U'}) = \max \{ \rho(\hat{A}), \rho(\mathcal{A}|_{U'}) \} = \rho(\mathcal{A}), \quad \hat{A} = \{ \hat{A}_d : d \in D(M) \}.$$ 

Furthermore, since any common invariant subspace of $\mathcal{A}$ contains $U'$, it follows that the joint spectral radius of $\mathcal{A}$ restricted to any common invariant subspace is equal to $\rho(\mathcal{A})$. Therefore, we have proved the following result.

**Corollary 4** If the family $\mathcal{A}$ possesses a dominant subspace, then $\alpha_\varphi = \log_{1/\rho(M)} \rho(\mathcal{A})$. 

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3.2 Construction of the space $U$ and of the continuous refinable function $\varphi$.

The continuity of the refinable function $\varphi$ is characterized in terms of the joint spectral radius of the matrices $T_d$, $d \in D(M)$ restricted to the common invariant subspace $U$ in Definition 4. In this section, we answer two crucial questions: how to determine the space $U$ and how to construct the corresponding continuous refinable function $\varphi$. In many cases $U$ coincides with $W$. In the univariate case, the algorithm for determining the space $U$ was elaborated in [13]. In this section, we present its multivariate analogue and explain several significant unavoidable modifications.

Algorithm for construction of the space $U$

Algorithm 1: For a given set $\mathcal{T} = \{T_d \in \mathbb{R}^{N \times N} : d \in D(M)\}$ of transition matrices

1. Step: Compute the eigenvector $v_0$ of $T_0v_0 = v_0$ and normalize it so that $(1, v_0) = 1$, where $1 = (1, \ldots, 1)^T \in \mathbb{R}^N$.

2. Step: Define $U^{(1)} = \text{span}\{T_dv_0 - v_0 : d \in D(M) \setminus \{0\}\}$.

3. Step: Repeat

$$U^{(k+1)} = U^{(k)} \cup \text{span}\{T_du^{(k)} : u^{(k)} \in U^{(k)}, \ d \in D(M)\}, \ 1 \leq k \leq N - 1,$$

until $\dim(U^{(k)}) < \dim(U^{(k+1)})$.

Output: $U = U^{(k)}$ \hfill (18)

Note that the choice of $1 \leq k \leq N - 1$ is imposed by the fact that $\dim U \leq N - 1$ and that, by construction, at least one extra element is added to $U^{(k+1)}$ before the algorithm terminates.

Remark 4: In practice, one would first determine a basis of $U^{(1)}$. Then, in 3. Step for $1 \leq k \leq N - 1$, this basis will be consequently extended by $T_du^{(k)}$ as long as the extended set stays linearly independent. This extended set provides a basis $\{u^{(k)} : k = 1, \ldots, r^{(k)}\}$ for $U^{(k)}$. The algorithm terminates, if, in the $k$th iteration, for every vector from $\{T_du^{(k)} : u^{(k)} \in U^{(k)}, \ d \in D(M)\}$ we have

$$\text{rank}\left(u^{(1)}, \ldots, u^{(r^{(k)}), T_du^{(k)}}\right) = \text{rank}\left(u^{(1)}, \ldots, u^{(r^{(k)})}\right).$$

Then $U^{(k+1)} = U^{(k)}$ and we set $U = U^{(k)}$. 

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Note first that the existence of the eigenvector \( v_0 \in V \) of the matrix \( T_0 \) with the eigenvalue one (in 1.\textit{Step} of the Algorithm) follows from continuity of the refinable function (see Remark 1). Hence, if 1.Step is impossible, i.e., the eigenvector \( v_0 \) does not exist, then the solution of the refinement equation (11) is not continuous.

Secondly, we show that the space \( U \) in Algorithm 1 coincides with the space in (12), i.e., we prove Proposition 1 stated at the beginning of section 3. To do that we define

\[
Q_k = \{0.d_1 \ldots d_k = \sum_{j=1}^{k} M^{-j} d_j : d_j \in D(M) \}, \quad k \geq 1.
\]  

(19)

Every point from \( Q_k \) belongs to the set \( G_{d_1 \ldots d_k} = M_{d_1}^{-1} \cdots M_{d_k}^{-1} G \). Thus, the set \( Q_k \) contains \( m^k \) points. The sets in (19) are nested, i.e. \( Q_k \subset Q_{k+1} \) for all \( k \geq 1 \). The nestedness follows due to each point \( 0.d_1 \ldots d_k \in Q_k \) being equal to \( 0.d_1 \ldots d_k 0 \in Q_{k+1} \). Moreover, each set \( Q_k \) is an \( \varepsilon_k \)-net for the set \( G \) with \( \varepsilon_k = \text{diam}(G_{d_1 \ldots d_k}) \leq C (\rho(M^{-1}) + \varepsilon)^k \) going to 0 as \( k \) goes to \( \infty \). Therefore, the set

\[
Q = \bigcup_{k \geq 1} Q_k
\]  

(20)

is dense in \( G \).

**Proof of Proposition 1.** Let \( U \) be as in Definition 4. It suffices to show that \( v(y) - v(x) \in U \) for all \( x, y \in G \). Equivalently, since \( Q \) is dense in \( G \), it suffices to show that \( v(y) - v(x) \in U \) for all \( x, y \in Q \). Equivalently, due to the definition of \( Q \), it suffices to establish by induction on \( k \in \mathbb{N} \) that \( v(y) - v(x) \in U \) for all \( x, y \in Q_k \).

For \( k = 1 \), the set \( Q_1 \) consists of \( m \) points \( 0.d = M^{-1}d, \ d \in D(M) \). For each \( d \neq 0 \), we have \( v(0.d) - v(0) = T_d v(0) - v(0) \in U \), by Definition 4. Hence \( v(0.d_1) - v(0.d_2) = (v(0.d_1) - v(0)) - (v(0.d_2) - v(0)) \in U \).

Assume the claim is true for some \( k \in \mathbb{N} \). Take arbitrary \( x, y \in Q_{k+1} \). For \( x = 0.d_1 \ldots d_k d_{k+1} \) and \( a = 0.d_1 \ldots d_k 0 \in Q_k \), we have that \( M_{d_1} x \) and \( M_{d_1} a \) are both in \( Q_k \). Hence, by the inductive assumption, \( v(M_{d_1} x) - v(M_{d_1} a) \in U \). Recall that \( U \) is invariant under \( T \), thus

\[
v(x) - v(a) = T_{d_1} \left( v(M_{d_1} x) - v(M_{d_1} a) \right) \in U.
\]

Similarly we take the corresponding point \( b \in Q_k \) for the point \( y \) and prove that \( v(y) - v(b) \in U \). Since \( a, b \in Q_k \) it follows that \( v(b) - v(a) \in U \) and, therefore,

\[
v(y) - v(x) = \left( v(y) - v(b) \right) + \left( v(b) - v(a) \right) + \left( v(a) - v(x) \right) \in U.
\]

This completes the proof.

The proof of Proposition 1 also implies that the spaces \( U^{(k)} \) defined in the algorithm above are of the form \( U^{(k)} = \text{span}\left\{ v(y) - v(x) : x, y \in Q_k \right\} \). From \( Q_k \subset Q_{k+1} \), we have \( U^{(k)} \subset U^{(k+1)} \subset \mathbb{R}^N \) for all \( 1 \leq k \leq N - 1 \).
Algorithm for construction of a continuous $\varphi$.

Due to the fact that the set $Q$ in \cite{22} is dense in $G$, the slight modification of Algorithm 1 yields a method for the step-by-step construction of the vector-valued function $v = v_x$ defined on $G$ or, equivalently, of the function $\varphi$.

**Algorithm 2:** For a given set $T = \{T_d \in \mathbb{R}^{N \times N} : d \in D(M)\}$ of transition matrices

1. **Step:** Compute $(0, v_0)$ such that $T_0 v_0 = v_0$ and normalize $v_0$ so that $(1, v_0) = 1$, where $1 = (1, \ldots, 1)^T \in \mathbb{R}^N$.

2. **Step:** Define $V^{(0)} = \{(0, v_0)\}$.

3. **Step:** For $k = 1, \ldots$

   $$V^{(k)} = V^{(k-1)} \cup \{(x, v_x) : x = M^{-1}_d y, v_x = T_d \tilde{v}_y, (y, v_y) \in V^{(k-1)}, d \in D(M)\}$$

If the function $v$ is continuous, this algorithm determines $V^{(k)}$ consisting of $(x, v(x))$, $x \in Q_k$, in a unique way. The piecewise constant function $v_k : \mathbb{R}^s \to \mathbb{R}^N$ such that

$$v_k(x) = v(x), \quad x \in Q_k,$$

is an approximation of $v$ and the difference $\|v - v_k\|_{\infty}$ can be efficiently estimated by the joint spectral radius of the family $A$. This yields a linear rate of convergence of Algorithm 2. See section 8 for more details.

The last result of this section, Proposition 2, ensures that $U$ is well defined even if the eigenvalue 1 of the matrix $T_0$ is not simple.

**Proposition 2** For an arbitrary refinement equation, the matrix $T_0$ has at most one, up to normalization, eigenvector $v_0 \in V$ associated with the eigenvalue 1 such that, for $U$ in Definition 4, we have $\rho(T|U) < 1$. If such $v_0 \in V$ exists, then $\varphi$ is continuous and $v_0 = v_\varphi(0)$.

**Proof.** If such an eigenvector $v_0$ exists, then by Theorem 1 the refinable function is continuous and, by Proposition 1, $U = \text{span} \{v(y) - v(x) : x, y \in G\}$. By Algorithm 2, there exists a refinable function $\varphi$ such that we get $v_0 = v_\varphi(0)$. If there is another eigenvector $\tilde{v}_0 \in V$ with this property, then, by Algorithm 2, it generates another refinable function $\tilde{\varphi}$ for which $\tilde{v}_0 = v_{\tilde{\varphi}}(0)$. By the uniqueness of the solution of the refinement equation, these two solutions may only differ by a constant, hence, the vectors $v_0$ and $\tilde{v}_0$ are collinear. □
3.3 Road map of our main results

We would like to emphasize that, to tackle the anisotropic case, we use geometric properties of tilings rather than the $M$-adic expansions of points in $\mathbb{R}^q$ (the latter being a successful strategy in the isotropic case). Our key contribution is Theorem 2 that finally reveals the delicate dependency of the Hölder exponent of a refinable function on its Hölder exponents along the subspaces $J_i$, $i = 1, \ldots, q(M)$. Due to the importance of Theorem 2, we would like to give here a preview of its proof.

Step 1. Extend the vector-valued function $v$ in (1) defined on the tile $G$ to the whole $\mathbb{R}^{q}$, see [22]. Lemma 3 yields, for $x,y \in G$ (i.e. $x,y \in G - j$ for some $j \in \mathbb{N}$), the estimate $\|\tilde{v}(y) - \tilde{v}(x)\| \leq \|v(y + j) - v(x + j)\|$. The extension of $v$ is motivated by the fact that parts of the line segment $[x,y]$ can lie outside of $G$, due to its possible fractal structure.

Step 2. Lemma 3 shows that, for a tiling $G$ of $\mathbb{R}^q$, the total number of the subsets of the tiling intersected by a line segment is proportional to the length of that segment.

Step 3. Due to Step 2, Lemma 2 and Proposition 3 imply that, for $k \in \mathbb{N}$, any line segment $[x,y]$ in $\mathbb{R}^q$, $y - x \in J_i$, consists of several line segments such that 1) the endpoints of each of those line segments belong to one subset of the tiling $G^k$; 2) the total number of those line segments is bounded by $C \|M^k(x - y)\| \geq r^k_1\|x - y\|$.

Step 4. The difference between the values of the function $v$ at the endpoints of each of those subsegments of $[x,y]$ is bounded from above by $C_1 (\rho_i + \varepsilon)^k$ for some $\varepsilon > 0$. Hence, by Step 1, the same is true for $\tilde{v}$. Therefore, by the triangle inequality, $\|\tilde{v}(y) - \tilde{v}(x)\| \leq C_2 (\rho_i + \varepsilon)^k$. For $k$ such that $r^k_1 \geq 1/\|x - y\|$, we obtain $\|\tilde{v}(y) - \tilde{v}(x)\| \leq C_3 \|x - y\| \alpha(\varepsilon)$, where $\alpha(\varepsilon)$ approaches $\log_{1/r^k} \rho_i$ as $\varepsilon$ goes to 0.

3.4 Auxiliary results for Theorems 1 and 2

The proofs of our main results, Theorems 1 and 2, are based on an important observation formulated in Proposition 3. We also make use of the following basic properties of the joint spectral radius and two auxiliary lemmas.

Theorem A1. For a family of operators $A$ acting in $\mathbb{R}^n$ and for any $\varepsilon > 0$, there exists a norm $\| \cdot \|_\varepsilon$ in $\mathbb{R}^n$ such that $\|A\|_\varepsilon < \rho(A) + \varepsilon$ for all $A \in A$.

Theorem A2. For a family of operators $A$ acting in $\mathbb{R}^n$ there exists $u \in \mathbb{R}^n$ and a constant $C(u) > 0$ such that $\max_{A_{d_1} \in A} \|A_{d_1} \cdots A_{d_k} u\| \geq C(u) \rho(A)^k$, $k \in \mathbb{N}$. Moreover, if $A$ is irreducible, then $\max_{A_{d_1} \in A} \|A_{d_1} \cdots A_{d_k}\| \leq C \rho(A)^k$, $k \in \mathbb{N}$, for some constant $C > 0$.

Lemma 2. Assume that the segment $[0,1]$ is covered with $\ell$ distinct closed sets. Then there exist $\ell + 1$ points $0 = a_0 \leq \ldots \leq a_\ell = 1$, such that for each $i = 0, \ldots, \ell - 1$, the points $a_i, a_{i+1}$ belong to one of these sets.

Proof. Let the first set contain the point $a_0 = 0$. Choose $a_1$ to be the maximal (in the natural ordering of the real line) point of the first set. If $a_1 \neq 1$, then $a_1$ must belong to another set of the tiling. Choose $a_3$ to be the maximal point of this set. Repeat until $a_{\ell_0} = 1$.
for some $\ell_0 \in \mathbb{N}$. We have $\ell_0 \leq \ell$, since the sets are distinct. If $\ell_0 < \ell$, we extend the sequence $a_0 \leq \ldots \leq a_{\ell_0}$ by the points $a_{\ell_0+1}, \ldots, a_{\ell} = 1$.

Next we show that a segment of a given length intersects a finitely many sets of the tiling $G$.

**Lemma 3** For a tiling $G$, there exists a constant $C > 0$ such that every line segment $[x, y] \in \mathbb{R}^s$ intersects at most $C \max\{1, \|y - x\|\}$ sets of $G$.

**Proof.** It suffices to prove that the number of sets $G + k \subset G$ intersected by a segment of length one is bounded above by some constant $C$. It will imply that the number of sets $G + k \subset G$ intersected by any segment of length $\|y - x\| > 1$, is bounded by $C \|y - x\|$, and the claim follows. Thus, let a segment $[x, y]$ be of length one. If $(G + k) \cap [x, y] \neq \emptyset$ for some $k \in \mathbb{N}$, then the set $G + k$ is contained in $[x, y] + B_r(0)$, where $B_r(0)$ is the Euclidean ball of radius $r = \text{diam}(G)$. Denote by $V$ the volume of $[x, y] + B_r(0)$, then the total number of sets $G + k \subset G$ intersecting $[x, y]$ is bounded by $C = \frac{V}{\mu(G)} = V < \infty$, due to $\mu(G) = 1$. This completes the proof.

To deal with line segments $[x, y], x, y \in G$, that do not completely belong to $G$, we extend the continuous vector-valued function $v = v_\varphi$ in (22) which is defined on $G$ to the whole $\mathbb{R}^s$. Define

\[ \tilde{v}: \mathbb{R}^s \to \mathbb{R}^N, \quad \tilde{v}(x) = (\varphi(x + k))_{k \in \Omega}. \] (22)

In Lemma 4 and in Proposition 3 we compare the properties of $v$ and $\tilde{v}$.

**Lemma 4** Let $x, y \in G - j, \ j \in \mathbb{Z}^s$. Then $\|\tilde{v}(y) - \tilde{v}(x)\| \leq \|v(y + j) - v(x + j)\|.$

**Proof.** Let $j \in \mathbb{Z}^s$. By (7) and due to the compact support of $\varphi$, the $k$-th component of $\tilde{v}(y) - \tilde{v}(x)$ is given by

\[ (\tilde{v}(y) - \tilde{v}(x))_k = \begin{cases} (v(y + k) - v(x + k)), & k \in \Omega, \\ 0, & \text{otherwise}, \end{cases} \]

Hence, in the Euclidean norm we have $\|\tilde{v}(y) - \tilde{v}(x)\| \leq \|v(y + j) - v(x + j)\|.$

**Proposition 3** Let $\varphi \in C(\mathbb{R}^s)$ be refinable, $x, y \in \mathbb{R}^s$ and $k \in \mathbb{N}$. There exist

\[ \ell \leq \max C \{1, \|M^k(x - y)\|\} \]

(with $C > 0$ from Lemma 2), integers $\{d^{(i)}_1, \ldots, d^{(i)}_k\}_{i=0}^{\ell-1}$ from $D(M)$, positive numbers $\{\alpha_i\}_{i=0}^{\ell-1}$ whose sum is equal to one, and sets of points $\{x_i\}_{i=0}^{\ell-1}, \{y_i\}_{i=0}^{\ell-1}$ from $G$ such that $y_i - x_i = \alpha_i M^k(y - x)$ for all $i = 0, \ldots, \ell - 1$, and

\[ \|\tilde{v}(y) - \tilde{v}(x)\| \leq \sum_{i=0}^{\ell-1} \left\| T_{d^{(i)}_1} \cdots T_{d^{(i)}_k} (v(y_i) - v(x_i)) \right\|. \] (23)
Proof. We have \([x, y] \subset G^k\). By Lemma 2, there exist \(\ell + 1\) points \(\{x = a_0 \leq a_1 \leq \ldots \leq a_\ell = y\} \subset [x, y]\) such that each pair of successive points \(a_i, a_{i+1}\) belongs to only one set \(G_{d_1^{(i)} \ldots d_k^{(i)}} - j^{(i)}\), \(j^{(i)} \in \mathbb{N}\), of the tiling \(G^k\). First we give an estimate for \(\ell\). Since \(\ell\) elements of the tiling \(G^k = M^{-k}G\) cover a segment of length \(\|y - x\|\), the same number of elements of the tiling \(G\) cover a segment of length \(\|M^k(y - x)\|\). Therefore, Lemma 2 yields \(\ell \leq C \max\{1, \|M^k(y - x)\|\}\). Furthermore, the set \(G_{d_1^{(i)} \ldots d_k^{(i)}} - j^{(i)} \subset G - j^{(i)}\), \(i = 0, \ldots, \ell - 1\). Thus, by Lemma 3, we obtain

\[
\left\| \tilde{v}(y) - \tilde{v}(x) \right\| \leq \sum_{i=0}^{\ell-1} \left\| \tilde{v}(a_{i+1}) - \tilde{v}(a_i) \right\| \leq \sum_{i=0}^{\ell-1} \left\| v(a_{i+1} + j^{(i)}) - v(a_i + j^{(i)}) \right\|.
\]

Due to \(a_i + j^{(i)}, a_{i+1} + j^{(i)} \in G_{d_1^{(i)} \ldots d_k^{(i)}}\), \(i = 0, \ldots, \ell - 1\), the points

\[
x_i = M_{d_1^{(i)} \ldots d_k^{(i)}} \cdots M_{d_1^{(i)}} (a_i + j^{(i)}) \quad \text{and} \quad y_j = M_{d_1^{(i)} \ldots d_k^{(i)}} (a_{i+1} + j^{(i)})
\]

belong to \(G\). Thus, by (11), we obtain

\[
\left\| \tilde{v}(y) - \tilde{v}(x) \right\| \leq \sum_{i=0}^{\ell-1} \left\| T_{d_1^{(i)} \ldots d_k^{(i)}} (v(y_i)) - v(x_i) \right\|. \tag{24}
\]

For each \(i = 0, \ldots, \ell - 1\), we define the number \(\alpha_i\) from the equality \(\|a_{i+1} - a_i\| = \alpha_i \|y - x\|\).

It follows that \(\sum_{i=1}^{\ell-1} \alpha_i = 1\) and that \(y_i - x_i = M^k(a_{i+1} - a_i) = \alpha_i M^k(y - x)\).

\[\square\]

### 3.5 Proofs of Theorems 1 and 2

In this subsection we prove Theorems 1 and 2. We start with Theorem 2 as its proof is a crucial part of the proof of Theorem 1. Note that for both Theorems 1 and 2 the assumption that \(\varphi \in C(\mathbb{R}^q)\) implies, e.g. by 3, that \(\rho(A) < 1\). We will not reprove this result here.

**Proof of Theorem 2** Let \(\varepsilon \in (0, 1 - \rho_i)\) and \(i \in \{1, \ldots, q(M)\}\). We first show that \(\alpha_{\varphi, J_i} \geq \log_{1/r_i} \rho_i\). For arbitrary points \(x, y \in G\) such that \(y - x \in J_i\) and \(\|y - x\| < 1\), define \(k\) to be the smallest integer such that \(\|M^k(y - x)\| \geq 1\). Since \(y - x \in J_i\), it follows that

\[
1 \leq \|M^k(y - x)\| \leq C \left( r_i + \varepsilon \right)^k \|y - x\|, \tag{25}
\]

where the constant \(C > 0\) depends only on \(M\). By (22), Theorem A1 and by Proposition 3 for these \(x, y\) and \(k\), there exist a constant \(C_1 > 0\) depending on \(G\) and the integer

\[
\ell \leq C_1 \max \{1, \|M^k(y - x)\|\} = C_1 \|M^k(y - x)\|
\]

such that (note that \(y - x \in J_i\) implies, by Proposition 3 that \(x_j, y_j\) in (24) satisfy \(y_j - x_j \in J_i\), \(j = 0, \ldots, \ell - 1\))

\[
\left\| v(y) - v(x) \right\| \leq 2 \ell C_2 (\rho_i + \varepsilon)^k \|v\|_{C(G)} \leq 2 C_3 \|M^k(y - x)\| \|v\|_{C(G)} (\rho_i + \varepsilon)^k
\]

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with the constant $C_3$ independent of $k$. By the choice of $k$, we have $\|M^{k-1}(y - x)\| < 1$ and, hence, $\|M^k(y - x)\| \leq \|M\|\|M^{k-1}(y - x)\| < \|M\|$. Thus,
\[ \|v(y) - v(x)\| \leq 2C_5 \|M\| \|v\|_{C(G)} (\rho_i + \varepsilon)^k. \] (26)
Combining the above estimate with (25) (i.e. $k \geq \frac{\log \|y - x\|}{\log (r_i + \varepsilon)} + C_4$), we get, due to $\rho_i + \varepsilon < 1$,
\[ \|v(y) - v(x)\| \leq C \|y - x\|^\alpha(\varepsilon) \]
with $\alpha(\varepsilon) = \log_{1/(r_i + \varepsilon)}(\rho_i + \varepsilon)$ and with some constant $C$ depending on $\varepsilon$. Letting $\varepsilon \to 0$, we obtain the claim.

Next we establish the reverse inequality $\alpha_{\varphi,J_i} \leq \log_{1/r_i} \rho_i$. Let $\varepsilon \in (0, r_i)$ and $d_1, \ldots, d_k \in D(M)$, $k \in \mathbb{N}$. By Theorem A2, there exist $u \in U_i$ and a constant $C(u) > 0$ such that $\|A_{d_1} \cdots A_{d_k} u\| \geq C(u) \rho_i^k$. Since the subspace $U_i$ is spanned by the differences $v(y) - v(x)$, $y - x \in J_i$, $y, x \in G$, there exist $x_j, y_j \in G$, $y_j - x_j \in J_i$, $j = 1, \ldots, n_i$ ($n_i$ dimension of $U_i$), such that $u = \sum_{j=1}^{n_i} \gamma_j (v(y_j) - v(x_j))$, $\gamma_j \in \mathbb{R}$. Denote $x_j^{(k)} = M_{d_1}^1 \cdots M_{d_k}^{1-k} x_j$ and $y_j^{(k)} = M_{d_1}^{-1} \cdots M_{d_k}^{-1} y_j$. Thus, $x_j^{(k)}, y_j^{(k)} \in G_{d_1 \cdots d_k}$ and there exists $C(\varepsilon) > 0$ such that
\[ \|y_j^{(k)} - x_j^{(k)}\| \leq C(\varepsilon) (r_i - \varepsilon)^{-k} \|y_j - x_j\|, \quad k \in \mathbb{N}. \] (27)
Moreover, we have
\[ \sum_{j=1}^{n_i} |\gamma_j| \cdot \|v(y_j^{(k)}) - v(x_j^{(k)})\| = \sum_{j=1}^{n_i} |\gamma_j| \cdot \|A_{d_1} \cdots A_{d_k} (v(y_j) - v(x_j))\| \geq \sum_{j=1}^{n_i} |\gamma_j| A_{d_1} \cdots A_{d_k} (v(y_j) - v(x_j)) \cdot \|A_{d_1} \cdots A_{d_k} u\| \geq C(u) \rho_i^k, \quad k \in \mathbb{N}. \]
Consequently, at least one of the $n_i$ numbers $\|v(y_j^{(k)}) - v(x_j^{(k)})\|$, $j = 1, \ldots, n_i$, is larger than or equal to $C(u) \sum_{j=1}^{n_i} |\gamma_j| \rho_i^k$. Combining this estimate with (27) (i.e. $k \leq -\frac{\log \|y_j^{(k)} - x_j^{(k)}\|}{\log (r_i - \varepsilon)} + C_1$ with the constant $C_1$ independent of $k$), we obtain
\[ \|v(y_j^{(k)}) - v(x_j^{(k)})\| \geq C \|y_j^{(k)} - x_j^{(k)}\|^\alpha(\varepsilon), \]
where $\alpha(\varepsilon) = \log_{1/(r_i - \varepsilon)} \rho_i$ and the constant $C > 0$ does not depend on $k$. Since $\|y_j^{(k)} - x_j^{(k)}\| \to 0$ as $k \to \infty$, there are arbitrary small segments $[x_j^{(k)}, y_j^{(k)}]$ on which the variation of the function $v$ is at least a constant times the length of that segment to the power of $\log_{1/(r_i - \varepsilon)} \rho_i$. Therefore, $\alpha_{\varphi,J_i} \leq \log_{1/(r_i - \varepsilon)} \rho_i$. Since $\varepsilon$ is arbitrary, the claim follows.
orthogonal to each other. Using a natural expansion $h$, $\alpha$ where $\varepsilon > 0$ for arbitrary $\rho$, $\varepsilon_{\varphi}$ we obtain

\[ \| v(0.d_1 \ldots d_j) - v(0) \| \leq \sum_{i=1}^{j-1} \| T_{d_i} \cdots T_{d_1} \left( v(0.d_{i+1}) - v(0) \right) \|, \quad j \in \mathbb{N}. \]

Note that, by construction, $v(0.d_{i+1}) - v(0) = T_{d_{i+1}} v_0 - v_0 \in U$ for $i = 1, \ldots, j - 1$. Therefore, by Theorem A1, $\| T_{d_i} \cdots T_{d_1} (v(0.d_{i+1}) - v(0)) \| \leq C_1 (\rho + \varepsilon)^i$, $i = 1, \ldots, j - 1$. Thus, we obtain

\[ \| v(0.d_1 \ldots d_j) - v(0) \| \leq C_1 \sum_{i=1}^{j-1} (\rho(\mathcal{A}) + \varepsilon)^i \leq C_1 \sum_{i=0}^{\infty} (\rho(\mathcal{A}) + \varepsilon)^i = \frac{C_1}{1 - \rho(\mathcal{A}) - \varepsilon}, \]

where the constant $C_1 > 0$ is independent of $j \in \mathbb{N}$. Hence, $\| v(x) \| \leq \| v(0) \| + \frac{C_1}{1 - \rho(\mathcal{A}) - \varepsilon}$ which proves the uniform boundedness of $v$ on $\mathcal{A}$.

The values of $v$ on $\mathcal{A}$ define the function $\varphi$ on $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} = \bigcup_{k \in \mathbb{Z}} (k + Q)$ is the set of all rational $M$-adic points of $\mathbb{R}^s$. The so constructed $\varphi : \tilde{\mathcal{A}} \rightarrow \mathbb{R}$ is supported on $K \cap \tilde{\mathcal{A}}$. Using $\varphi$, define the extension $\tilde{\varphi} : \tilde{\mathcal{A}} \rightarrow \mathbb{R}$ of $\mathbf{3}$. We show next that $\tilde{\varphi}$ is uniformly continuous on $\tilde{\mathcal{A}}$, which implies that its extension to $\mathbb{R}^s$ is continuous. Take arbitrary points $x, y \in \tilde{\mathcal{A}}$. By Proposition 3 and the same argument as in the first part of the proof of Theorem 2 we obtain

\[ \| \tilde{\varphi}(y) - \tilde{\varphi}(x) \| \leq 2 C_2 M \| M \|_{C(\mathcal{A})} (\rho(\mathcal{A}) + \varepsilon)^k, \]

where $k$ is the smallest number such that $\| M^k (y - x) \| \geq 1$. Note that the value $\| v \|_{C(\mathcal{A})}$ is finite because $v$ is bounded $\mathcal{A}$. Since $\| M^k \| \geq 1/\| y - x \|$, i.e. $k$ goes to $\infty$ as $\| y - x \|$ goes to zero, $\tilde{\varphi}$ is uniformly continuous on $\tilde{\mathcal{A}}$, which completes the proof of continuity. Thus, if $\rho(\mathcal{A}) < 1$, then $\varphi \in C(\mathbb{R}^s)$.

By Theorem 2 the Holder exponent $\alpha_{\varphi}$ of $\varphi$ on shifts along the subspace $J_i$ is equal to $\alpha_i = \log_{1/r_i} \rho_i$. We pass to a basis in the space $\mathbb{R}^s$, in which all the subspaces $J_i$ are orthogonal to each other. Using a natural expansion $h = h_1 + \ldots + h_{q(M)}$, $h_i \in J_i$ we obtain for arbitrary $\varepsilon > 0$

\[ \| \varphi(\cdot + h) - \varphi(\cdot) \| \leq \sum_{i=1}^{q(M)} \| \varphi(\cdot + h_i) - \varphi(\cdot) \| \leq \sum_{i=1}^{q(M)} C \| h_i \|^{\alpha_i - \varepsilon} \leq C d^{\alpha - \varepsilon} \| h \|^{\alpha - \varepsilon}, \]

where $\alpha = \min_{i=1, \ldots, q(M)} \alpha_i$. Consequently, $\alpha_{\varphi} = \min_{i=1, \ldots, q(M)} \alpha_i$. \hfill \Box
3.6 Examples
We consider the dilation matrix \( M = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \) and the refinement equation with five nonzero coefficients
\[
c_{(0,0)} = \frac{1}{2}, \quad c_{(1,-1)} = \frac{1}{2}, \quad c_{(1,0)} = \frac{1}{4}, \quad c_{(2,0)} = \frac{1}{4} \quad \text{and} \quad c_{(1,1)} = \frac{3}{4}.
\]
The dilation matrix has eigenvalues \( \lambda_1 = \frac{1 - \sqrt{13}}{2} \) and \( \lambda_2 = \frac{1 + \sqrt{13}}{2} \). By [20], for the digit set \( D(M) = \{(0,0), (1,0), (2,0)\} \), the set \( G \) in [3] is a tile. Using the result of [3], we determine
\[
\Omega = \{-1, 0, 1\}^2 \cup \{(0, \pm 2), (1, -2), (-1, 2)\}.
\]
The corresponding transition matrices \( T_{(0,0)}, T_{(1,0)} \) and \( T_{(2,0)} \) are given by
\[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 3 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 4 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
4 & 2 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
respectively. The matrix \( T_{(0,0)} \) has one eigenvalue 1 with the corresponding eigenvector
\[
v_0 = \frac{1}{5} \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in V, \quad (1, v_0) = 1.
\]
Using the algorithm for construction of \( U \) from subsection 3.2 we obtain
\[
U^{(1)} = \{u_1^{(1)}, u_2^{(1)}\} = \{T_{(0,0)}v_0 - v_0, T_{(2,0)}v_0 - v_0\},
\]
\[
U^{(2)} = \{u_1^{(2)}, \ldots, u_4^{(1)}\} = U^{(1)} \cup \{T_{(0,0)}u_1^{(1)}, T_{(1,0)}u_1^{(1)}\},
\]
\[
U^{(3)} = \{u_1^{(3)}, \ldots, u_7^{(3)}\} = U^{(2)} \cup \{T_{(0,0)}u_3^{(2)}, T_{(2,0)}u_3^{(2)}, T_{(2,0)}u_4^{(2)}\},
\]
\[
U^{(4)} = \{u_1^{(4)}, \ldots, u_{11}^{(4)}\} = U^{(3)} \cup \{T_{(0,0)}u_5^{(3)}, T_{(1,0)}u_5^{(3)}, T_{(2,0)}u_5^{(3)}, T_{(2,0)}u_6^{(3)}\},
\]
\[
U = U^{(5)} = U^{(4)} \cup \{T_{(1,0)}u_1^{(1)}\}.
\]
Due to \( \dim(U) = 12 \), we have \( W = U \) and, therefore, \( \mathcal{A} = \{ T_d|_W : d \in D(M) \} \). Denote \( A_0 = T_{[0,0]}|_W, A_1 = T_{(1,0)}|_W, A_2 = T_{(2,0)}|_W \). We computed the joint spectral radius of the set \( \mathcal{A} = \{ A_0, A_1, A_2 \} \) using the invariant polytope algorithm from [28] and obtained that the joint spectral radius is attained at the finite product \((A_0A_1)^2A_0^2A_2\) of length 7, i.e.

\[
\rho(\mathcal{A}) = \left[ \rho \left( (A_0A_1)^2A_0^2A_2 \right) \right]^{1/7} = 0.93816\ldots
\]

The algorithm constructs an invariant polytope of the operators \( A_0, A_1, A_2 \) in \( \mathbb{R}^{12} \). That polytope has 434 vertices.

Since \( M \) has two eigenvalues of different moduli, \( q(M) = 2 \), and there exist two corresponding non-zero subspaces \( U_1, U_2 \subset U \). On the other hand, we verified that the matrix family \( \mathcal{A} \) is irreducible in this case. Hence, the only non-zero common invariant subspace of the matrices in \( \mathcal{A} \) is \( U \). Thus, \( U_1 = U_2 = U = W \). Therefore, \( \rho_1 = \rho_2 = \rho(\mathcal{A}) \) and Theorem 1 implies that

\[
\alpha_\varphi = \log_{1/\rho(M)} \rho(\mathcal{A}) = \frac{1}{7} \log_2 \frac{1}{1+\sqrt{13}} \rho \left( (A_0A_1)^2A_0^2A_2 \right) = 0.07652\ldots
\]

4 Higher order regularity

It is well known that in the univariate case, if the solution \( \varphi \) of the refinement equation belongs to \( C^1(\mathbb{R}) \), then \( \varphi \) is a convolution of a piecewise-constant function and of a continuous solution of a refinement equation of a smaller order \([16, 50, 61]\). This observation resolves the question of the differentiability of refinable functions and classifies all smooth refinable functions. In particular, every \( C^\ell \)-refinable function is a convolution of a refinable spline of order \( \ell - 1 \) and of a continuous refinable function. This recursive decomposition technique, however, cannot be extended to the multivariate case, see e.g. \([12, 35, 43]\).

In this section, we show that the derivatives of the multivariate refinable function \( \varphi \in \mathcal{S}'(\mathbb{R}^s) \) satisfy a system of nonhomogeneous refinement equations. The differentiability of \( \varphi \in C(\mathbb{R}^s) \) is then equivalent to continuity of the solutions of all these equations, see Theorem \( \Box \). The main idea is that the directional derivatives of \( \varphi \) along the eigenvectors of the dilation matrix \( M \) satisfy certain refinement equations and the directional derivatives along the generalized eigenvectors (of the Jordan basis) of \( M \) satisfy nonhomogeneous refinement equations, see Proposition \( \Box \).

**Definition 7** A multivariate nonhomogeneous refinement equation is a functional equation of the form

\[
\varphi = T \varphi + g,
\]

where \( T \) is the transition operator in \( \mathcal{E} \) and \( g \) is a given compactly supported function or distribution.
For more details on nonhomogeneous refinement equations see e.g. \[19, 37, 60\] and references therein.

Let

\[ E = \{e_1, \ldots, e_s\} \]

be the Jordan basis of the matrix \( M \) in \( \mathbb{R}^s \). The Jordan basis consists of the eigenvectors of \( M \), which satisfy \( Me_i = \lambda e_i \), and of the generalized eigenvectors, which satisfy \( Me_i = \lambda e_i + e_{i-1} \). Consider an \( \ell \times \ell \) Jordan block of \( M \) corresponding to an eigenvalue \( \lambda \). With a slight abuse of notation, denote by

\[ E_\lambda = \{e_1, \ldots, e_\ell\} \subset E, \quad Me_1 = \lambda e_1, \quad Me_i = \lambda e_i + e_{i-1}, \quad i = 2, \ldots, \ell, \]

the Jordan basis corresponding to this Jordan block. In the following we study the properties of the directional derivatives of the refinable function \( \varphi \in S'(\mathbb{R}^s) \), which belong to the following subspaces of \( S'(\mathbb{R}^s) \).

**Definition 8** For a vector \( a \in \mathbb{R}^s \), we denote by

\[ S'_{a}(\mathbb{R}^s) = \left\{ \varphi \in S'(\mathbb{R}^s) : \int_{x = at + b} \varphi(x) dx = 0, \quad b \in \mathbb{R}^s, \ parallel \ to \ a \right\} \]

the space of compactly supported distributions, whose mean along every straight line \( \{x = at + b : t \in \mathbb{R}\} \), \( b \in \mathbb{R}^s \), parallel to \( a \), is equal to zero.

By \( \nabla \varphi = (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_s}) \) we denote the total derivative (gradient) of \( \varphi \) and by \( \frac{\partial \varphi}{\partial a} = (a, \nabla \varphi) \), its directional derivative along a nonzero vector \( a \in \mathbb{R}^s \). Due to the compact support of \( \varphi \in S'(\mathbb{R}^s) \), its directional derivative \( \frac{\partial \varphi}{\partial a} \) belongs to \( S'_{a}(\mathbb{R}^s) \).

The next result shows that a directional derivative of a refinable function \( \varphi \) along an eigenvector of the dilation matrix \( M \) is also a refinable function and satisfies the refinement equation (28). A directional derivative of \( \varphi \) along a generalized eigenvector of \( M \) satisfies the nonhomogeneous refinement equation (29).

**Proposition 4** Let \( \varphi \in S'(\mathbb{R}^s) \) and \( \lambda \) be an eigenvalue of \( M \). If \( \varphi = T\varphi \), then \( \varphi_i = (e_i, \nabla \varphi) \in S'_{e_i} := S'_{e_i}(\mathbb{R}^s) \), \( e_i \in E_\lambda \), satisfy the refinement equation

\[ \varphi_1 = \lambda T\varphi_1 \]  \hspace{1cm} (28)

and the nonhomogeneous refinement equations

\[ \varphi_i = \lambda T\varphi_i + \sum_{j=1}^{i-1} (-1)^{i-j} \lambda^{-j} \varphi_{i-j}, \quad i = 2, \ldots, \ell = \dim(E_\lambda). \]  \hspace{1cm} (29)

Conversely, the system of equations (28)-(29) possesses a unique up to a normalization solution \((\varphi_1, \ldots, \varphi_\ell) \in S'_{\ell} \times \cdots \times S'_{\ell} \). Moreover, if \( \varphi \in S'(\mathbb{R}^s) \) satisfies \((e_i, \nabla \varphi) = \varphi_i, \ e_i \in E_\lambda \), then \( \varphi = T\varphi \) along the lines parallel to \( e_i \in E_\lambda \).
Proof. By induction on \( \ell \), we show that, if \( \varphi \in S'(\mathbb{R}^s) \) satisfies the refinement equation \( \varphi = T \varphi \), then \( \varphi_i = (e_i, \nabla \varphi) \), \( i = 1, \ldots, \ell \), satisfy \( (28)-(29) \). For \( \ell = 1 \), due to \( Me_1 = \lambda e_1 \), we have

\[
\varphi_1 = (e_1, \nabla \varphi) = \left( e_1, \sum_{k \in \mathbb{Z}^s} c_k M^T \nabla \varphi \right) (M \cdot -k) = \left( M e_1, \sum_{k \in \mathbb{Z}^s} c_k \nabla \varphi (M \cdot -k) \right) = \lambda \sum_{k \in \mathbb{Z}^s} c_k \left( e_1, \nabla \varphi \right) (M \cdot -k) = \lambda \sum_{k \in \mathbb{Z}^s} c_k \varphi_1 (M \cdot -k) = \lambda T \varphi_1.
\]

For \( i \in \{2, \ldots, \ell\} \), due to \( Me_i = \lambda e_i + e_i-1 \) and \( (29) \), we obtain

\[
\varphi_i = \lambda T \varphi_i + T \varphi_{i-1} = \lambda T \varphi_i + \lambda^{-1} \varphi_{i-1} - \sum_{j=1}^{i-2} (-1)^{j-1} \lambda^{-j} \varphi_{i-1-j}.
\]

And the claim follows.

Conversely, by \( \mathbf{[37]} \), the system \( (28)-(29) \) possess a unique up to normalization solution. We show that the compactly supported primitive \( \varphi \) of \( \varphi_i \), \( i = 1, \ldots, \ell \), along \( e_i \in E_\lambda \) satisfies \( \varphi = T \varphi \). Indeed, since \( Me_1 = \lambda e_1 \) and by \( (28) \), we obtain

\[
(e_1, \nabla \varphi) = \lambda T (e_1, \nabla \varphi) = (e_1, M^T T \nabla \varphi) = (e_1, \nabla T \varphi),
\]

which implies that the gradient of the function \( T \varphi - \varphi \) is orthogonal to \( e_1 \). Hence, the function \( T \varphi - \varphi \) is constant along all lines parallel to \( e_1 \in E_\lambda \). On the other hand, \( \varphi \) is compactly supported, consequently \( T \varphi - \varphi = 0 \) along all lines parallel to \( e_1 \in E_\lambda \). For \( i \in \{2, \ldots, \ell\} \), due to \( Me_i = \lambda e_i + e_i-1 \) and \( \varphi_i = \lambda T \varphi_i + T \varphi_{i-1} \), we obtain

\[
(e_i, \nabla \varphi) = \lambda T (e_i, \nabla \varphi) + T (e_{i-1}, \nabla \varphi) = (\lambda e_i + e_{i-1} T \nabla \varphi) = (e_i, \nabla T \varphi).
\]

Analogous considerations about \( T \varphi - \varphi \) along \( e_i \in E_\lambda \) imply the claim. \( \square \)

Remark 5 If, for the eigenvalue \( \lambda \) of the dilation matrix \( M \), the set \( E_\lambda \) does not contain any generalized eigenvectors, then the system \( (28)-(29) \) reduces to homogeneous refinement equations \( \varphi_i = \lambda T \varphi_i, e_i \in E_\lambda \).

The main result of this section, Theorem 3, states that \( \varphi \in C^1(\mathbb{R}^s) \) if and only if the (nonhomogeneous) refinement equations in Proposition 4 corresponding to the Jordan basis \( E \subset \mathbb{R}^s \) of the dilation matrix \( M \) have continuous solutions \( \varphi_i \in S'_c(\mathbb{R}^s), e_i \in E \). The directional derivatives \( \varphi_i, i = 1, \ldots, s \), determine the total derivative \( \nabla \varphi \) of \( \varphi \). Moreover, \( \varphi_i, i = 1, \ldots, s \), can be constructed and their Hölder exponents can be computed as described in section 3 (see Remark 6). Thus, the higher regularity of any refinable function \( \varphi \) can be analyzed by this recursive reduction to a set of continuous refinable functions.

Theorem 3 Let \( \varphi \in S'(\mathbb{R}^s) \). There exist continuous solutions \( \varphi_i \in S'_c(\mathbb{R}^s), e_i \in E_\lambda \), of \( (28)-(29) \) for each eigenvalue \( \lambda \) of the dilation matrix \( M \) if and only if \( \varphi \in C^1(\mathbb{R}^s) \) satisfies \( \varphi = T \varphi \) and \( \frac{\partial \varphi}{\partial e_i} = \varphi_i, e_i \in E \).
Proof. If $\varphi \in C^1(\mathbb{R}^s)$ is a compactly supported solution of the refinement equation $\varphi = T\varphi$, then, by Proposition 4 its directional derivatives $\varphi_i := \frac{\partial \varphi}{\partial e_i} \in C(\mathbb{R}^s)$ along $e_i \in E$ satisfy equations (28)-(29). Conversely, if the equations (28)-(29) possess continuous solutions, then, by Proposition 4 $\varphi$ is in $C^1(\mathbb{R}^s)$, $\frac{\partial \varphi}{\partial e_i} = \varphi_i$, $e_i \in E$, and satisfies $\varphi = T\varphi$. \hfill \Box

Corollary 5 Suppose that $E$ does not contain any generalized eigenvectors, i.e., the matrix $M$ has a basis of eigenvectors; then

(i) If $\varphi \in C^1(\mathbb{R}^s)$ is refinable, then $\varphi_i = \frac{\partial \varphi}{\partial e_i} \in C(\mathbb{R}^s)$ satisfy $\varphi_i = \lambda_i T\varphi_i$, $i = 1, \ldots, s$.

(ii) Conversely, if the solutions $\varphi_i \in S'_{\alpha_i}$ of $\varphi_i = \lambda_i T\varphi_i$, $i = 1, \ldots, s$, are continuous, then the solution of $\varphi = T\varphi$ belongs to $C^1(\mathbb{R}^s)$. Moreover, $\frac{\partial \varphi}{\partial e_i} = \varphi_i$, $i = 1, \ldots, s$.

Remark 6 The system of refinement equations (28)-(29) is solved and analysed in the same way described in subsection 3.2 for the usual refinement equation (11). First we solve the equation $\varphi_1 = \lambda_1 T\varphi_1$. We find $v_{\varphi_1}(0)$ as an eigenvector of the matrix $T_0$ with the eigenvalue $1/\lambda$. If $T_0$ does not have this eigenvalue, then equation $\varphi_1 = \lambda_1 T\varphi_1$ does not have a solution, and hence $\varphi \notin C^1(\mathbb{R}^s)$. Then we compute $\varphi_1(x)$ at $M$-adic points by the formula $v_{\varphi_1}(0.d_1 \ldots d_k) = \lambda^k T_{d_1} \cdots T_{d_k} v_{\varphi_1}(0)$, and extend it by continuity to the whole set $K$ (as in Algorithm 2, subsection 3.2). Then we define the space $U_{\alpha_1}$ as the minimal common invariant subspace of the transition matrices containing the vectors $\lambda T_d v_{\varphi_1}(0) = v_{\varphi_1}(0)$, $d \in D(M)$. This can be done by Algorithm 1 from subsection 3.2. Then $\varphi_1 \in C(\mathbb{R}^s)$ if and only if the joint spectral radius of the matrices $\lambda T_d$, $d \in D(M)$, restricted to the subspace $U_{\alpha_1}$ is smaller than one. The Hölder regularity of $\varphi_1$ is computed by formula (15) for the matrices $A_d = \lambda T_d|_{U_{\alpha_1}}$. Similarly, we solve the other equations of the system (29) successively for $i = 2, \ldots, \ell$.

5 Modulus of continuity and Lipschitz continuity

Apart from the computing of the exact Hölder exponent of a refinable function $\varphi \in C(\mathbb{R}^s)$, the matrix approach allows for a refined analysis of its modulus of continuity

$$\omega(\varphi, t) = \sup_{\|\cdot + h\| \leq t} \|\varphi(\cdot + h) - \varphi\|_{C(\mathbb{R}^s)}, \quad t > 0,$$

also in the case of a general dilation matrix $M$. In Theorem 4 we show how the asymptotic behavior of $\omega(\varphi, t)$ as $t \to 0$ depends on the spectral properties of $M$. Corollary 8 states under which conditions on $M$ the Hölder exponent $\alpha_\varphi = 1$ of $\varphi$ guarantees its Lipschitz continuity. Indeed, the condition $\alpha_\varphi = 1$ on the Hölder exponent

$$\alpha_\varphi = \sup \{ \alpha \geq 0 : \omega(\varphi, t) \leq C t^\alpha, \quad t > 0 \}$$

is not sufficient to guarantee the Lipschitz continuity of $\varphi$. The Lipschitz continuity takes place if and only if the exponent $\alpha_\varphi = 1$ is sharp.
Definition 9 The Hölder exponent $\alpha_\varphi$ of $\varphi \in C(\mathbb{R}^s)$ is sharp if there exists a constant $C > 0$ such that $\omega(\varphi, t) \leq C t^\alpha$, $t \in (0, 1)$.

Remark 7 Even in the univariate case the Hölder exponent of a refinable function may not be sharp. For example, the derivative of the refinable function generated by the four-point interpolatory subdivision scheme with the parameter $w = \frac{1}{16}$ is “almost Lipschitz” with factor 1, i.e., $\omega(\varphi', t) \asymp t |\log t|$ as $t \to 0$. See [29]. It has been shown recently [29], that in the bivariate case with the dilation matrix $M = 2I$, the derivatives of the refinable function generated by the butterfly subdivision scheme with the parameter $w = \frac{1}{16}$ is “almost Lipschitz” with factor 2, i.e., $\omega(\varphi', t) \asymp t |\log t|^2$ as $t \to 0$.

To formulate the main result of this section we need to introduce some further notation.

Definition 10 The resonance degree of a compact set $\mathcal{A}$ of $n \times n$ matrices is defined by

$$\nu(\mathcal{A}) = \min \{\nu \in \mathbb{N} \cup \{0\} : \max_{A_{d_1},...,A_{d_k} \in \mathcal{A}} \|A_{d_1} \cdots A_{d_k}\| \leq C \rho(\mathcal{A})^k k^\gamma, \ k \in \mathbb{N}\}.$$ 

Remark 8 (i) Note that the resonance degree $\nu(\mathcal{A})$ of one square matrix $A$ is less by one than the size of the largest Jordan block of $A$ corresponding to one of the largest eigenvalues in the absolute value. Thus, the resonance degree of one matrix can be computed efficiently. (ii) In general, $\nu(\mathcal{A}) \leq n - 1$. Indeed, by [43], the resonance degree does not exceed the valency of $\mathcal{A}$ minus one, determined from the lower triangular Frobenius factorization of the family $\mathcal{A}$, i.e. there exists an invertible $B \in \mathbb{C}^{n \times n}$

$$B^{-1}AB = \begin{pmatrix} A^{(1)} & 0 & \cdots & 0 \\ \ast & A^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \ast & \cdots & \ast & A^{(r)} \end{pmatrix}, \quad A \in \mathcal{A}, \quad (31)$$

where each family $\mathcal{A}^{(j)} = \{A^{(j)} : A \in \mathcal{A}\}, \ j = 1, \ldots, r \leq n$, is irreducible. The valency of $\mathcal{A}$ is defined as the number of $\mathcal{A}^{(j)} = \{A^{(j)} : A \in \mathcal{A}\}$ such that $\rho(\mathcal{A}^{(j)}) = \rho(\mathcal{A})$. In particular, if the family $\mathcal{A}$ is irreducible, then the Frobenius factorization is trivial with $r = 1$. In this case, the valency of $\mathcal{A}$ is equal to one, which is stated in Theorem A2. Thus, in this case, $\nu(\mathcal{A}) = 0$. (iii) For a sharper estimate on the valency of $\mathcal{A}$ see [47]. (iv) The resonance degree is an integer, by definition. By [51], there exist finite matrix families (even pairs of matrices $\mathcal{A} = \{A_0, A_1\}$) for which $\max_{A_{d_1},...,A_{d_k} \in \mathcal{A}} \|A_{d_1} \cdots A_{d_k}\| \asymp \rho(\mathcal{A})^k k^\beta$ with a non-integer $\beta$.

Now we are ready to formulate the main result of this section.

Theorem 4 Let $\varphi \in C(\mathbb{R}^s)$ with the Holder exponent $\alpha = \alpha_\varphi$. For

$$\alpha \nu(M|_{U_j}) + \nu(\mathcal{A}|_{U_j}) = \max\{\alpha \nu(M|_{U_j}) + \nu(\mathcal{A}|_{U_j}) : \log_{1/\rho_j} \rho_j = \alpha, \ j = 1, \ldots, q(M)\}$$

the modulus of continuity of $\varphi$ satisfies

$$\omega(\varphi, t) \leq C t^\alpha |\log t|^{\alpha \nu(M|_{U_j}) + \nu(\mathcal{A}|_{U_j})}, \quad t \in (0, 1/2). \quad (32)$$
Proof. Let \( j \in \{ 1, \ldots, q(M) \} \) and \( y - x \in J_j, \, \| y - x \| < 1 \). The proof is similar to the first part of the proof of Theorem 2 where, using Definition 10 we replace the estimate (25) by
\[
1 \leq \| M^k (y - x) \| \leq C \gamma^k j \nu(M, j) \| y - x \|. \tag{33}
\]
Then in the estimate (26), by Definition 10 we have
\[
\| A_{d_1} \cdots A_{d_h} | U_j \| \leq C \rho_j^k k^{\nu(A, U_j)} \tag{34}
\]
Let \( v = v_\varphi \) be defined in (7). Combining (33) and (34), we obtain
\[
\| v(y) - v(x) \| \leq C \| y - x \| \alpha \log \| y - x \| | ^{\alpha(1/\nu(M, j) + \nu(A, U_j))}.
\]
Since this estimate holds for each \( j \in \{ 1, \ldots, q(M) \} \), the claim follows.

The first corollary of Theorem 4 lists the conditions sufficient for the sharpness of the Hölder exponent of \( \varphi \).

**Corollary 6** Let \( \varphi \in C(\mathbb{R}^s) \) with the Hölder exponent \( \alpha = \alpha_\varphi \). If for each \( i \in \{ 1, \ldots, q(M) \} \) such that \( \log \gamma_j \rho_i = \alpha \), the matrix \( M | J_i \) has only trivial Jordan blocks and \( \nu(A, U_j) = 0 \), then the Hölder exponent \( \alpha_\varphi \) of \( \varphi \) is sharp.

The proof of Corollary 6 follows by Remark 8 (i), which implies that \( \nu(M | J_i) = 0 \) in Theorem 4. An important particular case of Corollary 6 is stated in the following result.

**Corollary 7** Let \( \varphi \in C(\mathbb{R}^s) \). If the dilation matrix \( M \) has a complete system of eigenvectors and the family \( \mathcal{A} \) is irreducible, then the Hölder exponent \( \alpha_\varphi \) of \( \varphi \) is sharp.

**Corollary 8** Under the assumptions of Corollary 6, if \( \alpha_\varphi = 1 \), then \( \varphi \) is Lipschitz continuous.

**Example 2** The refinable function \( \varphi \) from subsection 3.6 has a sharp Hölder exponent. Indeed, for that equation \( M \) has two different eigenvalues, and hence \( \nu(M) = 0 \); the family \( \mathcal{A} \) is irreducible, therefore \( \nu(\mathcal{A}) = 0 \). Thus, \( \omega(\varphi, t) \leq C t^{0.07652...} \).

**Remark 9** The moduli of continuity along the subspaces \( J_i, \, i = 1, \ldots, q(M) \):
\[
\omega(\varphi, t, J_i) = \sup \{ \| \varphi(\cdot + h) - \varphi\|_{C(\mathbb{R}^s)} \mid h \leq t, \, h \in J_i \} \quad t \in (0, 1/2),
\]
satisfy the estimate in (32).

**Remark 10** A careful analysis of the proof of Theorem 4 makes it possible to construct examples for which the upper bound (32) is attained. Thus, inequality (32) cannot be improved in that terms. In particular, it is shown easily that if \( M \) has the largest Jordan block of a given size \( \nu + 1 \geq 2 \) corresponding to the biggest by modulus eigenvalue and the family \( \mathcal{A} \) is irreducible, then \( \omega(\varphi, t) \geq C t^{\alpha} \log t^{\alpha} \) for a sequence of numbers \( t \) that tends to zero.

**Remark 11** It is known that in the univariate case, the Hölder exponent of a refinable function is always sharp, whenever it is not an integer. Remark 10 shows that in the multivariate case this does not hold. If \( M \) has the largest Jordan block of a size \( \nu + 1 \geq 2 \) corresponding to the largest by modulus eigenvalue, and \( \mathcal{A} \) is irreducible, then \( \omega(\varphi, t) \geq C t^{\alpha} \log t^{\alpha} \) for \( t \) tending to zero.
6 Local regularity of continuous solutions

The matrix approach (for the matrices $A_d, d \in D(M)$ in (13)) makes it possible to compute the local regularity of continuous refinable functions at concrete points and to study sets of points with the given local regularity. In the univariate case, the analysis of the local regularity of the refinable function generating the Daubechies wavelet $D_2$ was done in [16]. For the general theory of local regularity of univariate refinable functions see [49]. In particular, in [49], the authors show that all univariate refinable functions, except for refinable splines, have a varying local regularity, which explains their fractal properties. The matrix approach, see Theorem 5, extends the above mentioned local regularity analysis to the case of multivariate refinable functions with an arbitrary dilation matrix $M$.

**Definition 11** The local Hölder exponent of $f \in C(\mathbb{R}^s)$ at a point $x \in \mathbb{R}^s$ along a subspace $J \subset \mathbb{R}^s$ is defined by

$$\alpha_{f,J}(x) = \sup \left\{ \alpha \geq 0 : \|f(x + h) - f(x)\|_{C(\mathbb{R}^s)} \leq C \|h\|^\alpha, \ h \in J \right\}.$$  

If $J = \mathbb{R}^s$, then we omit the index $J$ and denote the local Hölder exponent by $\alpha_f(x)$.

**Remark 12** Similarly to Definition 11 we define the local Hölder exponent of the vector-valued function $v$ in (7) for $x \in G$. Note that in Theorem 5 we determine the local Hölder exponent of $v_\varphi$ which satisfies

$$\alpha_{v_\varphi,J}(x) = \min \{\alpha_{\varphi,J}(x + k) : k \in \Omega\}, \ x \in G.$$  

For the sake of simplicity, we formulate Theorem 5 for rational $M$-adic points $x$. The following definition allows us to avoid the possible non-uniqueness of such representations.

**Definition 12** Let $\ell, L \in \mathbb{N}$. A point $x \in \mathbb{R}^s$ is called rational $M$-adic with period $(\ell, L)$, if

$$x = x_{d_1, \ldots, d_{\ell-1}(\ell, L)} = \sum_{j=1}^{\ell-1} d_j M^{-j} + \sum_{j=\ell+1}^{\ell+j(L+1)-1} d_j \sum_{j' \in \mathbb{N}_0} M^{-j}, \ d_j \in D(M).$$

For purely periodic $x \in \mathbb{R}^s$ we write $x = x_{(\ell, L)}$.

**Remark 13** The point $x_{(\ell, L)}$ is the unique fixed point of the contraction $M_{\ell}^{-1} \cdots M_{\ell+L}^{-1}$. Thus, $x_{(\ell, L)} \in G_{(\ell, L)}$ for all $j \geq 0$. If $x_{(\ell, L)} \in \text{int}(G)$, then $x_{(\ell, L)} \in \text{int}(G_{(\ell, L)j})$ for all $j \geq 0$, and this point has a unique $M$-adic expansion. Since $M$ is nonsigular, $x_{d_1, \ldots, d_{\ell-1}(\ell, L)} \in \text{int}(G_{d_1, \ldots, d_{\ell-1}(\ell, L)})$ for some $d_1, \ldots, d_{\ell-1} \in D(M)$.

We are now ready to formulate the main result of this section. Recall that the subspaces $J_i, i = 1, \ldots, q(M)$, of $\mathbb{R}^s$ determined by the dilation matrix $M$ are defined in subsection 2.1.
Theorem 5 Let \( \varphi \in C(\mathbb{R}^r) \) be refinable with the dilation matrix \( M \) and \( x_{(\ell,L)} \in \text{int}(G) \), \( \ell, L \in \mathbb{N} \). Then for every rational \( M \)-adic point \( x = x_{d_1,\ldots,d_{q(L)}} \) with the period \((\ell,L)\), we have

\[
\alpha_{\varphi,(\ell,L)}(x) \geq \frac{1}{L} \log_{1/r_i} \rho(A_{d_1} \cdots A_{d_{q(L)}} | U_i), \quad i = 1, \ldots, q(M)
\]

and

\[
\alpha_{\varphi}(x) \geq \max_{i=1,\ldots,q(M)} \alpha_{\varphi,(\ell,L)}(x).
\]

If the operators \( A_d | U_i \), are nonsingular (in particular, if all \( A_d, d \in D(M) \), are nonsingular), then the inequalities in (35) and in (36) become equalities.

Thus, to compute the local regularity at a given \( M \)-rational point \( x \), one does not need the joint spectral radius. The local regularity is determined by the usual spectral radius of the matrix product corresponding to the period of the \( M \)-adic expansion of \( x \).

**Remark 14** The assumption that the operators \( A_d | U_i \), \( d \in D(M) \), are nonsingular cannot be avoided even in the univariate case, see [49, Example 7].

**Proof of Theorem 5.** Let \( i \in \{1, \ldots, q(M)\} \). Denote \( \rho_{\ell,L} = \rho(A_{d_1} \cdots A_{d_{q(L)}} | U_i) \). Note that \( \rho_{\ell,L} < 1 \), due to the continuity of \( \varphi \). Choose \( \epsilon \in (0, 1 - \rho_{\ell,L}) \). Since \( x_{(\ell,L)} \in \text{int}(G) \), an open ball with the center \( x_{(\ell,L)} \) and radius \( \delta > 0 \) belongs to \( \text{int}(G) \). Choose \( y = x + h \), \( x = x_{d_1,\ldots,d_{q(L)}}(\ell,L) \) and \( h \in J_i \) such that \( \|h\| < \delta \|M\|^{-\ell-L} \). Let \( j \in \mathbb{N}_0 \) be the maximal number such that \( \|M^{\ell+1+jL}h\| < \delta \). Then, for the points

\[
x' = (M_{d_{1+L}} \cdots M_{d_{j}})^j M_{d_{q-L}} \cdots M_{d_{j}}, \quad x = x_{(\ell,L)}
\]

we have \( \|y' - x'\| \leq \|M^{\ell+1+jL}h\| < \delta \). Therefore, due to \( x' \in \text{int}(G) \), we have \( y' \in G \) and, consequently, \( x, y \in G_{d_1,\ldots,d_{q(L)}}(\ell,L) \). Thus, due to \( \|v(y') - v(x')\| \leq 2 \|v\|_{C(G)} \) and (31), we have

\[
\|v(y) - v(x)\| = \|A_{d_1} \cdots A_{d_{q-L}} \cdot (A_{d_{1}} \cdots A_{d_{q-L+1}})^j (v(y') - v(x'))\|
\leq 2 \|A_{d_{1}} \cdots A_{d_{j}}(U_i) | (\rho_{\ell,L} + \epsilon)^j \|v\|_{C(G)} \leq C_1 (\rho_{\ell,L} + \epsilon)^j
\]

with \( C_1 > 0 \) independent of \( h \). On the other hand, the choice of \( j \) implies that \( \|M^{\ell+jL}h\| \geq \delta \). Since \( h \in J_i \), we have \( \delta \leq \|M^{\ell+jL}h\| \leq C_2 (r_i + \epsilon)^{\ell+jL} \|h\| \), where \( C_2 > 0 \) is independent of \( h \). Thus, due to \( r_i + \epsilon > 1 \), we obtain

\[
\|h\| \geq \frac{\delta}{C_2 (r_i + \epsilon)^{-\ell-jL}} \iff j \geq -\frac{1}{L} \log_{r_i + \epsilon} \|h\| + C_3
\]

with \( C_3 > 0 \) independent of \( h \). Combining (37) and (38), we get, due to \( \rho_{\ell,L} + \epsilon < 1 \),

\[
\|v(x + h) - v(x)\| = \|v(y) - v(x)\| \leq C \|h\|^{\alpha(\epsilon)}
\]

with \( \alpha(\epsilon) = \frac{1}{L} \log_{1/(r_i + \epsilon)} (\rho_{\ell,L} + \epsilon) \) and with some constant \( C > 0 \) depending on \( \epsilon \). Taking \( \epsilon \to 0 \), we conclude that \( \alpha_{\varphi,(\ell,L)}(x) \geq \frac{1}{L} \log_{1/r_i} \rho_{\ell,L} \).
The proof of the reverse inequality in [35] under the assumption that the operators $A_d|U_i$, $d \in D(M)$, are nonsingular is done similarly to the second part of the proof of Theorem 2. Let $\varepsilon \in (0, r_i)$. By Theorem A2, there exists $u \in U_i$ and $C(u) > 0$ such that $\| (A_d \cdots A_{d_{t+1}})^j u \| \geq C(u) \rho_{\ell, L}^j$, $j \in \mathbb{N}$. Since all $A_d|U_i$ are nonsingular, it follows that $\| A_d \cdots A_{d_{t-1}} (A_d \cdots A_{d_{t+1}})^j u \| \geq C_1 \rho_{\ell, L}^j$ with a constant $C_1 > 0$. Next, we observe that, for $x(\ell, L) \in G$, the vectors \( \{ v(y) - v(x(\ell, L)) \} \) span $U_i$. Recall that $\dim(U_i) = n_i$, hence, there exist $n_i$ vectors $v(y_k) - v(x(\ell, L))$, $y_k \in G$, $y_k - x(\ell, L) \in J_i$, $k = 1, \ldots, n_i$ that span $U_i$. Hence $u = \sum_{j=1}^{n_i} \gamma_k (v(y_k) - v(x(\ell, L)))$, $\gamma_k \in \mathbb{R}$. Therefore, for the points

$$y_k^{(j)} = M_{d_2}^{-1} \cdots M_{d_{t-1}}^{-1} \left[ M_{d_1}^{-1} \cdots M_{d_{t+1}}^{-1} \right]^j y_k, \quad k = 1, \ldots, n_i,$$

we have, due to $x, y_k^{(j)} \in G_{d_1 \cdots d_{t-1}(\ell, L)}$,

$$\sum_{k=1}^{n_i} |\gamma_k| \left\| v(y_k^{(j)}) - v(x) \right\| \geq C_1 \rho_{\ell, L}^j, \quad j \in \mathbb{N}.$$ 

Hence, there exists $k \in \{1, \ldots, n_i\}$ such that

$$\left\| v(y_k^{(j)}) - v(x) \right\| \geq \frac{C_1}{\sum |\gamma_k|} \rho_{\ell, L}^j, \quad j \in \mathbb{N}.$$ 

Consequently, due to

$$\left\| y_k^{(j)} - x \right\| \leq (r_i - \varepsilon)^{-\ell - 1 - j} \left\| y_k - x(\ell, L) \right\| \quad \Leftrightarrow \quad j \leq \frac{1}{L} \log_{r_i - \varepsilon} \left\| y_k^{(j)} - x \right\| + C_2,$$

we obtain

$$\left\| v(y_k^{(j)}) - v(x) \right\| \geq C_3 \left\| y_j^{(k)} - x \right\|^{\alpha(\varepsilon)}$$

with $\alpha(\varepsilon) = \frac{1}{L} \log_{1/(r_i - \varepsilon)} \rho_{\ell, L}$ and with some constant $C_3 > 0$ depending on $\varepsilon$. Note that $\left\| y_k^{(j)} - x \right\|$ goes to zero as $j$ goes to infinity, due to $x, y_k^{(j)} \in G_{d_1 \cdots d_{t-1}(\ell, L)}$. Taking $\varepsilon \to 0$ we conclude that $\alpha_{\varphi, J_i}(x) \leq \frac{1}{L} \log_{1/r_i} \rho_{\ell, L}$. 

\[ \square \]

7 Existence and smoothness in $\mathbf{L}_p(\mathbb{R}^s), 1 \leq p < \infty$

In this section we prove Theorem [34] that characterizes the existence of refinable functions in $\varphi \in \mathbf{L}_p(\mathbb{R}^s), 1 \leq p < \infty$, and Theorem [47] that provides a formula for the Hölder exponent of such $\varphi$ in terms of the $p$-radius ($p$-norm joint spectral radius [34] [47]) of a set of transition matrices.
Definition 13 \textit{For }1 \leq p < \infty, \textit{the }p\text{-radius (}\, p\text{-norm joint spectral radius)} \textit{of a finite family of linear operators }\mathcal{A} = \{ A_0, \ldots, A_{m-1} \} \textit{ is defined by}

\[ \rho_p = \rho_p(\mathcal{A}) = \lim_{k \to \infty} \left( m^{-k} \sum_{A_{d_1} \in \mathcal{A}, i=1, \ldots, k} \| A_{d_1} \cdots A_{d_k} \|_p \right)^{1/p}. \]

Note that, for \( \varphi \in L_p(\mathbb{R}^s) \), the difference space \( U \) is defined similarly to (12) by

\[ U = \text{span} \left \{ v(y) - v(x) : \text{for almost all } y, x \right \} \subseteq W. \tag{39} \]

In this section, we also determine the exact Hölder regularity of refinable functions in the spaces \( L_p(\mathbb{R}^s) \), \( 1 \leq p < \infty \), see Theorem 7. Although these estimates look familiar to us from section 3, the corresponding proofs require totally different techniques.

The Hölder exponent of a function \( \varphi \in L_p(\mathbb{R}^s) \) is defined by

\[ \alpha_{\varphi,p} = \sup \left \{ \alpha \geq 0 : \| \varphi(\cdot + h) - \varphi(\cdot) \|_p \leq C \| h \|^\alpha, \ h \in \mathbb{R}^s \right \}. \]

Here and below we use the short notation \( \| \cdot \|_{L_p(\mathbb{R}^s)} = \| \cdot \|_p \). To determine the influence of the dilation matrix \( M \) on the Hölder exponent of \( \varphi \in L_p(\mathbb{R}^s) \), in Theorem 7 we consider the Hölder exponents of \( \varphi \) along the subspaces determined by the Jordan basis of \( M \). The Hölder exponent of \( \varphi \) along a subspace \( J \subset \mathbb{R}^s \) is defined by

\[ \alpha_{\varphi,J,p} = \sup \left \{ \alpha \geq 0 : \| \varphi(\cdot + h) - \varphi(\cdot) \|_p \leq C \| h \|^\alpha, \ h \in J \right \}. \]

In the proofs of Theorems 6 and 7 we use the following auxiliary results.

7.1 Auxiliary results

The following analogues of Theorems A1 and A2 from section 3 were proved in [51].

\textbf{Theorem A3.} Let \( 1 \leq p < \infty \). For a finite family \( \mathcal{A} \) of \( m \) operators acting in \( \mathbb{R}^n \) and for any \( \varepsilon > 0 \), there exists a norm \( \| \cdot \|_\varepsilon \) in \( \mathbb{R}^n \) such that

\[ \left( m^{-1} \sum_{A \in \mathcal{A}} \| Au \|_\varepsilon^p \right)^{1/p} \leq (\rho_p + \varepsilon) \| u \|_\varepsilon, \quad u \in \mathbb{R}^n. \]

For \( 1 \leq p < \infty \) and for a finite family \( \mathcal{A} \) of \( m \) operators acting in \( \mathbb{R}^n \), we denote

\[ \mathcal{F}_p(k, u) = \left( m^{-k} \sum_{A_{d_1} \in \mathcal{A}, i=1, \ldots, k} \| A_{d_1} \cdots A_{d_k} u \|^p \right)^{1/p}, \quad k \in \mathbb{N}. \]

Since each norm \( \| \cdot \| \) in \( \mathbb{R}^n \) is equivalent to the norm \( \| \cdot \|_\varepsilon \), Theorem A3 yields the following result.
Corollary 9 Let $1 \leq p < \infty$. For every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that $\mathcal{F}_p(k, u) \leq C(\varepsilon)(\rho_p + \varepsilon)^k \|u\|$ for all $u \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

Theorem A4. Let $1 \leq p < \infty$ and $A$ be a finite family of $m$ linear operators in $\mathbb{R}^n$. Then for every $u \in \mathbb{R}^n$ that does not belong to any common invariant linear subspace of $A$, there exists a constant $C(u) > 0$ such that
\[
\mathcal{F}_p(k, u) \geq C(u)\rho_p^k, \quad k \in \mathbb{N}.
\] (40)

In the next result we relax assumptions of Theorem A4. We show that (40) holds for all points $u$ apart from the ones in a proper linear subspace of $\mathbb{R}^n$.

Lemma 5 Let $1 \leq p < \infty$. Every finite family $A$ of $m$ linear operators possesses a common invariant linear subspace $L \subset \mathbb{R}^n$, $L \neq \mathbb{R}^n$, (possibly $L = \{0\}$) such that for every $u \notin L$ there exists a constant $C(u) > 0$ for which (40) holds.

**Proof.** Without loss of generality, after a suitable normalization, it can be assumed that $\rho_p = 1$. Let $L$ be the biggest by inclusion common invariant subspace of $A$ such that $\rho_p(A|_L) < 1$. Note that $L$ is a proper subspace of $\mathbb{R}^n$, otherwise we get a contradiction to $\rho_p = 1$. Hence, $\dim L \leq n - 1$. Take arbitrary $u \notin L$ and denote by $L_u$ the minimal common invariant subspace of $A$ that contains $u$. If $\rho_p(A|_{L_u}) < 1$, then the $p$-joint spectral radius of $A$ on the linear span of $L$ and $L_u$ is equal to $\max\{\rho_p(A|_L), \rho_p(A|_{L_u})\} < 1$, which contradicts the maximality of $L$. Hence, $\rho_p(A|_{L_u}) = 1$. Since $u$ does not belong to any common invariant subspace of the finite family $A|_{L_u}$, by Theorem A4, there exists a constant $C(u) > 0$ such that $\mathcal{F}_p(k, u) \geq C(u)\rho(A|_{L_u})^k, \quad k \in \mathbb{N}$. On the other hand, $\rho_p = \rho(A|_{L_u}) = 1$, and, hence, the claim follows.

For $1 \leq p < \infty$, in the rest of this section, we denote by $C(u) > 0$ the largest possible constant in inequality (40), i.e.,
\[
C(u) = \inf_{k \in \mathbb{N}} (\rho_p)^{-k} \mathcal{F}_p(k, u).
\] (41)

This function is upper semi-continuous and, therefore, is measurable.

7.1.1 Properties of the space $U$

Note that, by (11), the vector $z = \int_G v(x) \, dx \in V$ is an eigenvector of the operator $T = \frac{1}{m} \sum_{d \in D(M)} T_d$ associated with the eigenvalue 1. The following result is an analogue of Proposition 4.

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Proposition 5 If \( \varphi \in L_p(\mathbb{R}^s) \), \( 1 \leq p < \infty \), then the subspace \( U \) in (38) coincides with the smallest by inclusion common invariant subspace of matrices \( T_d \), \( d \in D(M) \), and contains \( m \) vectors \( T_d z - z \), \( d \in D(M) \), where \( z \in V \) is an eigenvector of the operator \( T = \frac{1}{m} \sum_{d \in D(M)} T_d \) corresponding to the eigenvalue one.

The proof of Proposition 5 is similar to the one of Proposition 1. Note that, since \( \sum_{k \in \mathbb{Z}^s} c_k = m \), the column sums of the matrix \( T \) are equal to one. Hence, \( T \) has at least one eigenvalue one. This eigenvalue does not have to be simple. Nevertheless, the following result guarantees the correctness of the choice of \( U \). The proof of Proposition 5 is similar to the one of Proposition 2.

Proposition 6 There exists at most one eigenvector \( z \in V \) of \( T \) associated to the eigenvalue one and such that the common invariant subspace \( U \) of the family of the matrices \( T_d \), \( d \in D(M) \), is spanned by the vectors \( T_d z - z \), \( d \in D(M) \), and such that \( \rho_p(A) < 1 \).

7.2 \( L_p \)-solutions of refinement equations

We are now ready to formulate the first result of this section.

Theorem 6 A refinable function \( \varphi \) belongs to \( L_p(\mathbb{R}^s) \), \( 1 \leq p < \infty \), if and only if \( \rho_p(A) < 1 \).

Proof. Assume first that \( \rho_p = \rho_p(A) < 1 \). Choose \( \varepsilon \in (0, 1 - \rho_p) \) and consider the norm \( \| \cdot \|_\varepsilon \) in \( U \) as in Theorem A4. Define the function space

\[
V_{U,p} = \{ f \in L_p(G + \Omega) : v_f(x) \in V, \ v_f(x) - v_f(y) \in U \quad \text{a.e.,} \quad x, y \in G \}
\]

with the norm \( \| f \| = \left( \int_G \| v_f(x) \|_\varepsilon^p \, dx \right)^{1/p} \). The space \( V_{U,p} \) is nonempty because it at least contains a piecewise constant function \( f \) such that \( v_f \equiv z \) a.e., where \( z \in V \) is the eigenvector of the operator \( \frac{1}{m} \sum_{d \in D(M)} T_d \) associated with the eigenvalue one. Note that, for \( f_1, f_2 \in V_{U,p} \), we have

\[
\| T(f_1 - f_2) \| = \left( \int_G \| A(v_{f_1} - v_{f_2})(x) \|_\varepsilon \, dx \right)^{1/p} \leq (\rho_p + \varepsilon) \| f_1 - f_2 \|.
\]

Therefore, due to \( \rho_p + \varepsilon < 1 \), \( T \) is a contraction on \( V_{U,p} \), and, hence, it has a unique fixed point \( \varphi \), which is the solution of the refinement equation \( \varphi = T \varphi \).

Assume next that \( \varphi \in L_p(\mathbb{R}^s) \). By Lemma 3 there exists a proper subspace \( \mathcal{L} \subset U \) (due to \( \dim U = n \), we associate \( \mathbb{R}^n \) with \( U \)) invariant under \( A \) such that \( \mathcal{F}_p(k, u) \geq C(u) \rho_p^k \), \( k \in \mathbb{N} \), whenever \( u \notin \mathcal{L} \). Since \( U \) is the smallest by inclusion subspace of \( \mathbb{R}^N \) invariant under \( A \) and containing the differences \( v(x_2) - v(x_1) \) for almost all \( x_1, x_2 \in G \), the set \( \{ (x_1, x_2) \in G^2 : v(x_2) - v(x_1) \notin \mathcal{L} \} \) has a positive Lebesgue measure \( \mu \) in \( \mathbb{R}^s \times \mathbb{R}^s \). Hence, the set

\[
\{ (x, h) \in \mathbb{R}^s \times \mathbb{R}^s : x, x + h \in G, \ v(x + h) - v(x) \notin \mathcal{L} \}
\]

\[ (42) \]
has a positive Lebesque measure. By the Fubini theorem, the set in (42) has sections of positive Lebesque measure. Thus, there exists \( h \in \mathbb{R}^s \) such that
\[
\mu \left\{ x \in G \, : \, x + h \in G, \, v(x + h) - v(x) \notin \mathcal{L} \right\} > 0.
\]
Therefore, there exist \( \delta > 0 \) and a set \( H \subset G \) of positive Lebesque measure such that the function in (41) satisfies
\[
\mathcal{F}_p \left( k, v(x + h) - v(x) \right) \geq \delta \rho_p^k, \quad k \in \mathbb{N}, \quad \text{for almost all } x \in H.
\]
Thus,
\[
\mathcal{F}_p \left( k, \left( v(y + h) - v(y) \right) \right) dy \geq \delta \rho_p^k \mu(H). \tag{43}
\]
Denote \( h_k = M^{-k}h, \, k \in \mathbb{N} \), then, by (11), we get
\[
\left\| v(\cdot + h_k) - v \right\|_{L^p(\mathbb{R}^s)}^p \geq \sum_{d_1, \ldots, d_k \in D(M)} \int_{H_{d_1 \cdots d_k}} \left\| v(x + h_k) - v(x) \right\|^p dx \geq \sum_{d_1, \ldots, d_k \in D(M)} m^{-k} \int_{H} \left\| A_{d_1} \cdots A_{d_k} \left( v(y + h) - v(y) \right) \right\|^p dy \geq \int_{H} \mathcal{F}_p \left( k, \left( v(y + h) - v(y) \right) \right) dy. \tag{44}
\]
Since \( y \in H \), by (13), we obtain
\[
\int_{H} \mathcal{F}_p \left( k, \left( v(y + h) - v(y) \right) \right) dy \geq \delta \rho_p^k \mu(H).
\]
Thus,
\[
\left\| v(\cdot + h_k) - v \right\|_{L_p(\mathbb{R}^s)} \geq \delta \rho_p^k \left[ \mu(H) \right]^{1/p}, \quad h_k = M^{-k}h, \quad k \in \mathbb{N}. \tag{45}
\]
Since \( v \in L_p(\mathbb{R}^s, \mathbb{R}^N) \) and \( h_k \) goes to 0 as \( k \to \infty \), we get \( \rho_p < 1 \).

\[\square\]

Remark 15 The proof of Theorem 6 is much simpler than that of Theorem 1 in \( C(\mathbb{R}^s) \). Indeed, an elegant argument with a contraction operator \( T \) on the affine subspace \( V(U, p) \) cannot be directly extended to prove the continuity of \( \varphi \) due to the following reason: the piecewise constant function \( f \) for which \( v_f \equiv z \) is not continuous. Thus, it is not clear how to show that \( V(U, p) \) is nonempty. We are not aware of any simple proof of this fact in the multivariate case. We, however, apply this argument once again in estimating the rate of convergence of the subdivision schemes in section 8.

### 7.3 Hölder regularity in \( L_p(\mathbb{R}^s) \)

To be able to determine the exact Hölder regularity of a refinable function \( \varphi \in L_p(\mathbb{R}^s) \), \( 1 \leq p < \infty \), we need to adjust the definitions of the transition matrices \( T_d, \, d \in D(M) \). To do so we replace the set \( \Omega \) in Definition 2 by the set \( \Omega \) in (46), the latter contains \( \Omega \) and is determined by a certain admissible absorbing set \( \Delta \).
Definition 14  Let $1 \leq p < \infty$. A set $\Delta \subset \mathbb{R}^s$ is called absorbing if, for all $f \in L_p(\mathbb{R}^s)$, the Hölder exponent of $f$ along $\Delta$ satisfies $\alpha_{f,\Delta,p} = \alpha_{f,p}$.

Remark 16  An arbitrary set that contains some neighborhood of the origin is absorbing. It is also easy to show that any convex body (convex set with a nonempty interior) that contains the origin is absorbing.

For the sake of simplicity, we choose $\Delta$ to be an arbitrary simplex with one of the vertices at the origin and such that its interior intersects all the spaces $J_i, i = 1, \ldots, q(M)$. In this case $\Delta$ is absorbing, and the sets $\Delta \cap J_i, i = 1, \ldots, q(M)$, are absorbing in the corresponding subspaces $J_i$. We call such a simplex $\Delta$ admissible. Note that for each $t > 0$, the set $t \Delta$ is also an admissible simplex.

Define $\tilde{\Omega} \subset \mathbb{Z}^s$ to be the minimal set such that $K_\Gamma + \Delta \subset \bigcup_{k \in \tilde{\Omega}} (k + G)$. (46) Such a set $\tilde{\Omega}$ always exists, due to $\bigcup_{k \in \mathbb{Z}^s} (k + G) = \mathbb{R}^s$. Note that $\Omega \subset \tilde{\Omega}$. In many cases $\tilde{\Omega} = \Omega$, but not always, see examples 3 and 4. Thus, supp $\varphi + \Delta \subset \tilde{\Omega} + G$. Using $\tilde{\Omega}$ we redefine $\tilde{v}_\varphi = (\varphi(\cdot + k))_{k \in \tilde{\Omega}}$ a.e. on $\mathbb{R}^s$, and $(\tilde{T}_d)_{a,b} = c_{Ma-b+d}, a,b \in \tilde{\Omega}, d \in D(M)$, are now of size $\tilde{N} = |\tilde{\Omega}|$. This leads to the appropriate modification $\tilde{A} = \{\tilde{T}_d|_{U_i} : d \in D(M)\}$ of the finite set $A$ in (13). The modified subspaces $\tilde{V}, \tilde{U}, \tilde{U}_i, i = 1, \ldots, q(M)$, and $\tilde{W}$ differ from the subspaces $V, U, U_i$ and $W$, respectively, only by the lengths of their corresponding elements. We are now ready to formulate the main result of this subsection.

Theorem 7  Let $1 \leq p < \infty$. For a refinable function $\varphi \in L_p(\mathbb{R}^s)$, we have

$$\alpha_{\varphi,J_i,p} = \log_{1/r_i} \rho_p(\tilde{A}|_{U_i}), \quad i = 1, \ldots, q(M)$$

and, consequently,

$$\alpha_{\varphi,p} = \min_{i=1,\ldots,q(M)} \log_{1/r_i} \rho_p(\tilde{A}|_{U_i})$$

Proof. We first show that $\alpha_{\varphi,J_i,p} \geq \log_{1/r_i} \rho_p(\tilde{A}|_{U_i})$. Set $\rho_{i,p} = \rho_p(\tilde{A}|_{U_i})$. Choose an arbitrary $\tilde{h} \in J_i \cap \Delta$ such that $\|\tilde{h}\| < \delta, \delta \in (0,1)$. Then the function $\psi = \varphi(\cdot + \tilde{h}) - \varphi$ is supported
on $K_T + \Delta$. Hence, the vector-valued function $\tilde{\psi}$ is well defined on $G$. Thus, for arbitrary $k \in \mathbb{N}$, we have

$$
\|\varphi(\cdot + M^{-k} \tilde{h}) - \varphi\|_{L^p(G)}^p = \|T^k(\varphi(\cdot + \tilde{h}) - \varphi)\|_{L^p(G)}^p = \|T^k\tilde{\psi}\|_{L^p(K_T + \Delta)}^p = \sum_{d_1, \ldots, d_k \in D(M)} \int_G A_{d_1} \cdots A_{d_k} \tilde{\psi}(M_{d_k} \cdots M_{d_1} x)^p \, dx = \sum_{d_1, \ldots, d_k \in D(M)} m^{-k} \int_G \tilde{A}_{d_1} \cdots \tilde{A}_{d_k} \tilde{\psi}(y)^p \, dy = \int_G \mathcal{F}_p(k, \tilde{\psi}(y)) \, dy.
$$

By Corollary 9, for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$
\int_G \mathcal{F}_p(k, \tilde{\psi}(y)) \, dy \leq C(\varepsilon) (\rho_{i,p} + \varepsilon)^k \|\tilde{\psi}\|_{L^p(G)}^p \leq C(\varepsilon) (\rho_{i,p} + \varepsilon)^k 2^p \|\tilde{\psi}\|_{L^p(G)}^p.
$$

Thus, we obtain

$$
\|\varphi(\cdot + M^{-k} \tilde{h}) - \varphi\|_{L^p(G)}^p \leq C(\varepsilon) (\rho_{i,p} + \varepsilon)^k, \quad k \in \mathbb{N},
$$

where $C(\varepsilon) > 0$ is independent of either $k$ or $\tilde{h}$. Choose an arbitrary $h \in J_i \cap \Delta$, $\|h\| < \delta$, and let $k \in \mathbb{N}$ be the largest integer such that $\|M^k h\| < \delta$. From $\|M^{k+1} h\| \geq \delta$ and substituting $\tilde{h} = M^k \tilde{h}$ in (49), we obtain

$$
\|\varphi(\cdot + h) - \varphi\|_{L^p(G)}^p \leq C(\varepsilon) (\rho_{i,p} + \varepsilon)^k, \quad k \geq C_1 + \log_{r_i}(\delta/h)
$$

for some constant $C_1 > 0$. Combining these estimates, we obtain $\alpha_{\varphi, J_i,p} \geq \log_{1/r_i}(\rho_{i,p} + \varepsilon)$. Taking the limit for $\varepsilon \to 0$, we obtain the claim.

To establish the reverse inequality $\alpha_{\varphi, J_i,p} \leq \log_{1/r_i}(\rho_{i,p})$, we argue as in the second part of the proof of Theorem 5. We show the existence of a vector $h \in J_i \cap \Delta$ and of a subset $H \subset J_i$ of positive Lebesque measure (on the space $J_i$) for which inequality (55) holds (with $\rho_{i,p}$ replaced by $\rho_{i,p}$). Taking a limit in that inequality as $k \to \infty$ and using the fact that $k \leq C_2 + \log_{r_i}(h/\|h\|)$, where $C_2 > 0$ independent of $k$, we complete the proof.

\[ \square \]

### 7.4 Examples

The following examples illustrate the need for the modifications of the set $\Omega$ in subsection 7.3.

**Example 3** The solution of the simplest univariate refinement equation

$$
\varphi(x) = \varphi(2x) + \varphi(2x - 1), \quad x \in \mathbb{R},
$$

is the characteristic function of the unit segment: $\varphi = \chi_{[0,1]}$. The $L_p$-regularity of $\varphi$ is $\alpha_{\varphi,p} = \frac{1}{p}$. In this case, $M = 2$ and, for the standard set of dyadic digits $D(M) = \{0,1\}$, we have $G = [0,1]$ and $\Omega = \{0\}$. Hence, $N = 1$ and we get two one-dimensional operators $T_0 = \ldots = T_{N-1}$.
$T_1 = 1$. The common invariant subspace of $T_0$ and $T_1$ is trivial $U = \{0\}$, hence, by definition, $\rho_p(A) = 0$. Thus, $\alpha_{\varphi,p} = \frac{1}{p}$, while $\log_{1/2} \rho_p(A) = +\infty$. We see that $\alpha_{\varphi,p} \neq \log_{1/2} \rho_p(A)$.

On the other hand, for $\tilde{\Omega} = \{0, 1\}$, we get
\[
\tilde{T}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{T}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The corresponding common invariant subspace $\tilde{U} = \tilde{W} = \{u \in \mathbb{R}^2 : u_1 + u_2 = 0\}$ is one-dimensional, and $A_0 = A_1 = 1$. Clearly, $\rho_p(\{A_0, A_1\}) = 2^{-1/p}$. Applying Theorem 7, we obtain the correct Hölder exponent $\alpha_{\varphi,p} = \frac{1}{p}$.

The next example shows that in some cases $\Omega = \tilde{\Omega}$.

**Example 4** The solution of the univariate refinement equation
\[
\varphi(x) = \varphi(3x) + \varphi(3x - 1) + \varphi(3x - 5), \quad x \in \mathbb{R},
\]
is $\varphi = \chi_G$, where $G$ is the tile in $\mathbb{R}$ corresponding to the dilation $M = 3$ and to the digit set $D(M) = \Gamma = \{0, 1, 5\}$. Thus, $\text{supp}(\varphi) \subseteq [0, \frac{2}{3}]$.

For the standard set of triadic digits $D(M) = \{0, 1, 2\}$, we have $G = \{0, 1\}$ and $\Omega = \{0, 1, 2\}$, i.e. $N = 3$. In this case, $K + \Delta \subseteq [0, 3]$. Hence, one can take the complemented set $\tilde{\Omega} = \Omega = \{0, 1, 2\}$. Thus, in this case,
\[
T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Their common invariant subspace $U = W = \{u \in \mathbb{R}^3 : u_1 + u_2 + u_3 = 0\}$ is two-dimensional. In the basis $e_1 = (1, -1, 0)^T, e_2 = (0, 1, -1)^T$ of $U$, the matrices $A_d = T_d|_U, d \in D(M)$, are
\[
A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Since $\rho_p(\{A_0, A_1, A_2\}) = \rho_p(\{A_0, -A_1, -A_2\})$, we need to compute $\rho_1$ for a family of non-negative matrices. For such families, $\rho_1$ is equal to the Perron eigenvalue of the arithmetic mean of matrices. In our case this is the matrix $\frac{1}{3}(A_0 - A_1 + A_2)$, for which $\lambda_{\text{max}} = \sqrt[3]{2} - 1$. By Theorem 7, $\alpha_{\varphi,1} = -\log_2 \frac{\sqrt[3]{2} + 1}{3}$. Since $\varphi \equiv 1$ on its support, it follows that $\alpha_{\varphi,p} = -\frac{1}{p} \log_2 \frac{\sqrt[3]{2} + 1}{3}$.

For another digit set $D(M) = \{0, 1, 5\}$, we have $\Omega = \{0\}$, i.e. $N = 1$. As in Example 3, the common invariant subspace $U = \{0\}$ of $T_d, d \in D(M)$, is trivial, and we have $\rho_p(A) = 0$. Thus, for this set of digits, $\alpha_{\varphi,p} \neq -\log_2 \rho_p(A)$. For $\Omega = \{0, 1, 2\}$, we get the same $3 \times 3$ matrices $T_d$ as above. Hence, $\rho_1 = \frac{\sqrt[3]{2} + 1}{3}$, and we have $\alpha_{\varphi,1} = -\log_2 \rho_1(\tilde{A})$. 

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Remark 17 It is well known that, if $p$ is an even integer, then $\rho_p(A)$ can be efficiently computed as an eigenvalue of a certain matrix derived from the matrices in $A$. Hence, Theorem 7 allows us to find the Hölder $L_p$-regularity at least for even integers $p$, in particular, for $p = 2$, see Example 5.

Example 5 For the refinement equation from subsection 3.6, we have $\Omega = \Omega$ and $U = W$. Therefore, Theorem 7 yields $\alpha_{p,2} = \log_2/(\rho(M)) \rho_p(T|W)$ and, hence, is not applicable in $L_2$. Hence, $\alpha_{p,2} = \log_2/(\rho(M)) \rho_p(T|W) = 0.27148 \ldots$. Recall that the Hölder exponent in $C(\mathbb{R}^n)$ of $\varphi$ is $\alpha_{p,2} = 0.07652$.

7.5 Construction of the space $U$ and of $L_p$-refinable function $\varphi$.

The construction of a continuous refinable function described in Section 3 is realized point-wise and, hence, is not applicable in $L_p(\mathbb{R}^n)$. Moreover, the vectors $v(z_s)$ are not well defined if $v \in L_p(\mathbb{R}^s, \mathbb{R}^N)$, thus, the constructions of $U$ and of the function $\varphi$ are modified in the following way, using Proposition 5.

First, we find the eigenvector $z \in V$ of the operator $T = \frac{1}{m} \sum_{d \in D(M)} T_d$ associated with the eigenvalue $1$. If such a vector does not exist, then $\varphi \notin L_p(\mathbb{R}^s)$. If such $z \in V$ exists, then the subspace $U$ is the minimal common invariant subspace of the matrices $T_d$, $d \in D(M)$, that contains $m$ vectors $T_d z - z$, $d \in D(M)$.

If $\rho_p = \rho_p(A) < 1$, then the solution $\varphi \in L_p(\mathbb{R}^s)$. Numerically $\varphi$ can be computed as follows. For every $d_1, \ldots, d_k \in D(M)$, $k \in \mathbb{N}$, the value $\frac{1}{\text{Vol}(G_{d_1} \cdots d_k)} \int_{G_{d_1} \cdots d_k} v(x) \, dx$, which is simply the mean of the function $v = v_\varphi$ on the set of the tile $G_{d_1} \cdots d_k$ (let us recall that $\text{Vol}(G_{d_1} \cdots d_k) = m^{-k}$) is equal to $A_{d_1} \cdots A_{d_k} z$. So, we can compute the mean values of the solution $\varphi$ on all sets of the tiling $G^k$. For instance, if $\chi = \chi_G$ is the characteristic function of the tile $G$, then the function $\varphi_k = T^k \chi$ is a piecewise-constant approximation of the solution $\varphi$. On each set $G_{d_1} \cdots d_k$, $\varphi_k(x)$ is equal to $A_{d_1} \cdots A_{d_k} z$. The function $\varphi_k$ converges to $\varphi$ at the linear rate $||\varphi_k - \varphi||_{L_p(\mathbb{R}^s)} \leq C(\rho_p + \varepsilon)^k$ as $k$ goes to infinity. In rare cases, when the eigenvalue $1$ of $T$ is not simple, by Proposition 6 there exists at most one vector $z \in V$ for which the corresponding subspace $U$ yields $\rho_p = \rho_p(T_U) < 1$.

8 The rate of convergence of subdivision schemes

In this section, we use the matrix approach to compute the rate of convergence in $C(\mathbb{R}^s)$ of subdivision schemes with anisotropic dilations, see Theorem 8. Example 6 illustrates one more difference between isotropic and anisotropic cases. Similar analysis can be done in the case of $L_p$-convergence using the results of section 7 see Remark 19.

Subdivision schemes are recursive algorithms for linear approximation of functions by their values on a mesh in $\mathbb{R}^s$ [1, 21]. Refinable functions appear naturally in the context of subdivision and of corresponding wavelet frames. Let $\ell(\mathbb{Z}^s)$ be the space of all sequences and $\ell_{\infty}(\mathbb{Z}^s)$ of all bounded sequences over $\mathbb{Z}^s$, respectively. The subdivision operator on $\ell(\mathbb{Z}^s)$ is
defined by
\[ \left[ Sa \right]_i = \sum_{j \in \mathbb{Z}^s} c_{i-M_j} a_j \quad i \in \mathbb{Z}^s, \ a \in \ell(\mathbb{Z}^s). \tag{50} \]

The subdivision scheme (repeated application of \( S \)) converges in \( C(\mathbb{R}^s) \) if for every \( a \in \ell_\infty(\mathbb{Z}^s) \) there exists a function \( f_a \in C(\mathbb{R}^s) \) such that
\[ \lim_{k \to \infty} \| f_a(M^{-k} \cdot) - S^k a \|_\infty = 0. \tag{51} \]

The map \( a \mapsto f_a \) is a linear shift-invariant operator from \( \ell_\infty(\mathbb{Z}^s) \) to \( C(\mathbb{R}^s) \). The limit function \( f_\delta \) for \( a = \delta \) (with \( \delta_0 = 1 \) and zero otherwise) is the solution \( \phi \) of the refinement equation (1) normalized by \( \int_{\mathbb{R}^s} \varphi(x) dx = 1 \).

The rate of convergence \( \tau = \tau(S) \) is defined by
\[ \tau = \inf \left\{ t > 0 : \| f_a(M^{-k} \cdot) - S^k a \|_\infty \leq C t^k \| a \|_\infty, \ k \in \mathbb{N}, \ a \in \ell_\infty(\mathbb{Z}^s) \right\}. \tag{52} \]

By e.g. [3, 4], the subdivision scheme converges if and only if \( \tau < 1 \). Thus, the convergence is always linear, whenever it takes place. Necessary (but not sufficient) conditions for the convergence are that the refinable function is continuous and the sum rules (4) are satisfied.

**Remark 18** It is well known that the rate of convergence of a subdivision scheme is equal to that of the cascade algorithm (repeated application of the transition operator \( T \) to some initial function \( f_0 \)). We denote by \( \mathcal{V} \) the affine space of continuous compactly supported functions on \( \mathbb{R}^s \) such that \( \sum_{j \in \mathbb{Z}^s} f(x + j) \equiv 1 \), and \( \mathcal{W} \) its linear part, which consists of functions such that \( \sum_{j \in \mathbb{Z}^s} f(x + j) \equiv 0 \). The sum rules (4) imply that the subspaces \( \mathcal{V} \) and \( \mathcal{W} \) are invariant under the transition operator \( T \). Then, for \( f_0 \in \mathcal{V} \) and \( g_0 \in \mathcal{W} \), the sequences
\[ \{ \| T^k f_0 \|_{C(\mathbb{R}^s)} : k \in \mathbb{N} \} \quad \text{and} \quad \{ \| T^k g_0 \|_{C(\mathbb{R}^s)} : k \in \mathbb{N} \} \tag{53} \]
have the same rate of convergence as that of the corresponding subdivision scheme [3, 4]. In other words, \( \tau \) is equal to the spectral radius of the operator \( T|_\mathcal{W} \). Moreover, one can restrict \( \mathcal{W} \) to functions supported on the set \( K \) defined in (6), the rate of convergence stays the same [3, 4].

In the isotropic case, it is known that \( \tau = \rho(T|_\mathcal{W}) \) with \( T \) in [10]. We derive an analogous result in the anisotropic case.

**Theorem 8** If a subdivision scheme satisfies sum rules [4], then \( \tau = \rho(T|_\mathcal{W}). \)

**Proof.** Denote \( \rho = \rho(T|_\mathcal{W}) \). By Theorem A1, for arbitrary \( \varepsilon > 0 \), there exists a norm \( \| \cdot \|_\varepsilon \) on \( \mathcal{W} \) such that \( \| T_d u \|_\varepsilon \leq (\rho + \varepsilon) \| u \|_\varepsilon \) for all \( u \in \mathcal{W} \) and \( d \in D(M) \). Then, for an
arbitrary function \( g_0 \subset \mathcal{W} \) supported on \( K \), we denote \( v(x) = v_{g_0}(x) \) and, for every point \( x = 0.d_1 \ldots \in G \), have
\[
\left\| T^k v(x) \right\|_\varepsilon = \left\| T_{d_1} \cdots T_{d_k} v(0.d_{k+1} \ldots) \right\|_\varepsilon \leq (\rho + \varepsilon)^k \left\| v(0.d_{k+1} \ldots) \right\|_\varepsilon \leq C (\rho + \varepsilon)^k \left\| v(0.d_{k+1} \ldots) \right\|_\varepsilon \leq C \sqrt[N]{N} (\rho + \varepsilon)^k \max_{j \in \Omega} g_0 (j + 0.d_{k+1} \ldots) \leq C \sqrt[N]{N} (\rho + \varepsilon)^k \left\| g_0 \right\|_{C(\mathbb{R}^k)}, \quad k \in \mathbb{N}.
\]
Thus, \( \left\| T^k v \right\|_{C(\mathbb{R}^k)} \leq C_0 (\rho + \varepsilon)^k \left\| g_0 \right\|_{C(\mathbb{R}^k)} \) for all \( k \in \mathbb{N} \). Therefore, by Remark \([18]\), \( \tau \leq \rho + \varepsilon \) for every \( \varepsilon > 0 \), and so \( \tau \leq \rho \).

The proof of the reverse inequality \( \tau \geq \rho \) is similar to the proof of Theorem \([2]\). By Theorem A2, there exists \( u \in \mathcal{W} \) such that \( \max_{d_1, \ldots, d_k \in D(M)} \left\| T_{d_1} \cdots T_{d_k} u \right\| \geq C(u) \rho^k, \) \( k \in \mathbb{N} \). If we find \( N - 1 \) functions \( g_i \in \mathcal{W} \), \( \text{supp} \, g_i \subset K, \) \( i = 1, \ldots, N - 1 \), such that at some point \( z \in G \), the vectors \( \{v_{g_i}(z)\}_{i=1}^{N-1} \) constitute a basis of the space \( W \), then the claim follows.

Indeed, the vector \( u \) possesses a representation \( u = \sum_{i=1}^{N-1} \gamma_i v_{g_i}(z), \gamma_i \in \mathbb{R} \), hence
\[
C(u) \rho^k \leq \max_{d_1, \ldots, d_k \in D(M)} \left\| T_{d_1} \cdots T_{d_k} u \right\| \leq \sum_{i=1}^{N-1} \left| \gamma_i \right| \max_{d_1, \ldots, d_k \in D(M)} \left\| T_{d_1} \cdots T_{d_k} v_{g_i}(z) \right\|.
\]
The latter sum does not exceed \( \sum_{i=1}^{N-1} \left| \gamma_i \right| \sqrt[N]{N} \left\| T^k g_i \right\|_{C(\mathbb{R}^k)}. \) On the other hand, \( \left\| T^k g_i \right\|_{C(\mathbb{R}^k)} \leq \tau^k \left\| g_i \right\|_{C(\mathbb{R}^k)}. \) Therefore, we obtain
\[
C(u) \rho^k \leq \left( \sqrt[N]{N} \sum_{i=1}^{N-1} \left| \gamma_i \right| \left\| g_i \right\|_{C(\mathbb{R}^k)} \right) \tau^k
\]
The expression inside the brackets is independent of \( k \), hence taking \( k \to \infty \), we obtain \( \rho \leq \tau \).

To construct the functions \( g_i \) we take arbitrary \( g \in C(\mathbb{R}^k) \) such that \( \text{supp} \, g \subset K \cap G \) and \( g(z) = 1 \) at some point \( z \). Then for the functions \( g_i(\cdot) = g(\cdot + d_i) - g(\cdot), d_i \in D(M), i = 1, \ldots, N - 1 \) (as usual, \( 0 \in D(M) \)), the vectors \( \{v_{g_i}(z)\}_{i=1}^{N-1} \) form a basis of \( W \). Indeed, the vector \( v_{g_0}(z) \) has the component 1 at position 0 and the component 1 at the position \( i \); all other components are zeros. Clearly, those vectors for \( i = 1, \ldots, N - 1 \), constitute a basis of the space \( W \).

Moreover, in the isotropic case, if the refinable function \( \varphi \) is stable, then the rate of convergence is related to the the Hölder exponent of \( \varphi \) by \( \log_{1/\rho(M)} \tau = \alpha_{\varphi}[3] \). For unstable refinable function this may not be true, however, in the univariate case, the convergence analysis can be still done as shown in \([50]\). The following example shows that, in the anisotropic case, the equality \( \log_{1/\rho(M)} \tau = \alpha_{\varphi} \) may fail even if the refinable function is stable.
Example 6 Consider the refinable function $\varphi = \varphi_1 \otimes \varphi_2$ that satisfies the refinement equation with $M = \text{diag}(2, 3)$ from Example 1. If $\varphi_1$ and $\varphi_2$ are stable and both belong to $C^1(\mathbb{R}^s)$, then $\varphi \in C^1(\mathbb{R}^s)$ is stable. Hence, $\alpha_\varphi = 1$. On the other hand, since $\alpha_{\varphi_1} = \alpha_{\varphi_2} = 1$, we have $\rho(T_1|W_1) = \frac{1}{2}$ and $\rho(T_2|W_2) = \frac{1}{3}$. Therefore, $\rho(T|W) = \frac{1}{2}$. By Theorem 8 we have $\tau = \frac{1}{2}$. On the other hand, $\log_{1/\rho(M)} \tau = \log_2 2 \neq \alpha_\varphi$, although $\varphi$ is stable.

Remark 19 A similar result as Theorem 8 holds for subdivision schemes in $L_p(\mathbb{R}^s)$. Their rate of convergence is also equal to $\rho_p(T|W)$. The proof is essentially the same as the one of Theorem 8 with the joint spectral radius replaced by the $p$-radius.

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