The chromatic index of finite projective spaces

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Abstract
A line coloring of $\text{PG}(n, q)$, the $n$-dimensional projective space over GF$(q)$, is an assignment of colors to all lines of $\text{PG}(n, q)$ so that any two lines with the same color do not intersect. The chromatic index of $\text{PG}(n, q)$, denoted by $\chi'(\text{PG}(n, q))$, is the least number of colors for which a coloring of $\text{PG}(n, q)$ exists. This paper translates the problem of determining the chromatic index of $\text{PG}(n, q)$ to the problem of examining the existences of $\text{PG}(3, q)$ and $\text{PG}(4, q)$ with certain properties. In particular, it is shown that for any odd integer $n$ and $q \in \{3, 4, 8, 16\}$, $\chi'(\text{PG}(n, q)) = (q^n - 1)/(q - 1)$, which implies the existence of a parallelism of $\text{PG}(n, q)$ for any odd integer $n$ and $q \in \{3, 4, 8, 16\}$.

KEYWORDS
chromatic index, parallelism, projective space, resolvable Steiner system, spread

1  |  INTRODUCTION

A Steiner system $S(2, k, v)$ is a pair $\mathcal{D} = (V, B)$ where $V$ is a set of $v$ points, and $B$ is a collection of $k$-subsets of $V$ called blocks, such that every pair of distinct elements of $V$ is contained in exactly one block of $B$. A set of pairwise disjoint blocks in $\mathcal{D}$ is called a partial parallel class. A partial parallel class containing all the points of $\mathcal{D}$ is called a parallel class. $\mathcal{D}$ is said to be resolvable if $B$ can be partitioned into parallel classes. A block coloring of $\mathcal{D}$ is a mapping $c : B \to C$ where $C$ is a set of colors such that blocks with the same color do not pairwise intersect. If $C$ has cardinality $r$ that $\mathcal{D}$ is $r$-colorable. A set of blocks in $\mathcal{D}$ with the same color is called a color class and each color class is a partial parallel class. The chromatic index of $\mathcal{D}$, denoted by $\chi'(\mathcal{D})$, refers to the minimum number of colors needed to color all blocks. Since the number of blocks contained in each color class is no more than $|v/k|$, $\chi'(\mathcal{D}) \geq |B|/|v/k|$ (cf. [17, Section 3.2]). $\mathcal{D}$ is resolvable if and only if $\chi'(\mathcal{D}) = k|B|/v = (v - 1)/(k - 1)$. 
A special case of a well-known conjecture of Erdös, Faber, and Lovász [10] can be formulated as the statement that the chromatic index of any \( S(2, k, v) \) is no more than \( v \). This conjecture has been shown to hold for any cyclic \( S(2, k, v) \) by Colbourn and Colbourn [8], and for any \( S(2, k, v) \) with sufficiently large \( v \) by Kang, Kelly, Kühn, Methuku, and Osthus [12].

Let \( q \) be a prime power and \( V(n + 1, q) \) be the \((n + 1)\)-dimensional vector space over the finite field \( \mathbb{F}_q \). \( PG(n, q) \) denotes the \( n \)-dimensional projective space over \( \mathbb{GF}(q) \). A \( d \)-subspace of \( PG(n, q) \) corresponds to a \((d + 1)\)-dimensional subspace of \( V(n + 1, q) \). The 0-subspaces and 1-subspaces of \( PG(n, q) \) are called points and lines, respectively. The points of \( PG(n, q) \) together with the lines of \( PG(n, q) \) as blocks and incidence by natural containment form an 

\[
S\left(2, q + 1, \binom{n + 1}{1}_q\right) \quad \text{(cf. [4, Proposition 2.16])},
\]

where \( \binom{a}{b}_q := \prod_{i=1}^{b} \frac{q^{a-i+1} - 1}{q_i - 1} \), called the Gaussian binomial coefficient, is the number of distinct \( b \)-dimensional subspaces of \( V(a, q) \). Two subspaces are called disjoint if their intersection is the trivial subspace. Similar to the concept of partial parallel classes, a set of pairwise disjoint lines in \( PG(n, q) \) is called a \textit{partial spread}. A partial spread is said to be a \textit{spread} if it is a partition of all points of \( PG(n, q) \). A spread exists if and only if \( n \) is odd [1]. A \textit{parallelism} of \( PG(n, q) \) is a partition of all lines into spreads. A parallelism of \( PG(n, q) \) forms a resolvable \( S\left(2, q + 1, \binom{n + 1}{1}_q\right) \). A \textit{line coloring} of \( PG(n, q) \) is a mapping from the set of lines of \( PG(n, q) \) to a set of colors such that any two intersecting lines have different colors. The \textit{chromatic index} of \( PG(n, q) \), denoted by \( \chi'(PG(n, q)) \), is the least number of colors for which a coloring of \( PG(n, q) \) exists. A parallelism of \( PG(n, q) \) exists if and only if \( \chi'(PG(n, q)) = \binom{n}{1}_q \).

\( PG(1, q) \) has only one line, so \( \chi'(PG(1, q)) = 1 \). In \( PG(2, q) \) every two lines intersect, so

\[
\chi'(PG(2, q)) = \left[ \frac{3}{2} \right]_q = q^2 + q + 1.
\]

For \( n \geq 3 \), Beutelspacher, Jungnickel, and Vanstone [7] showed that \( \chi'(PG(n, q)) \leq 4\binom{n-1}{1}_q + 2q^{n-1} \), which confirms the conjecture of Erdös, Faber, and Lovász [10] in the case of certain projective spaces. To give a lower bound for \( \chi'(PG(n, q)) \) with \( n \geq 3 \), let \( \mu_q(n) \) denote the maximum size of any partial spread of \( PG(n, q) \). Since there are \( l := \binom{n + 1}{2}_q \) lines in \( PG(n, q) \), \( \chi'(PG(n, q)) \geq l/\mu_q(n) \). If \( n \) is odd, a spread of \( PG(n, q) \) exists, so \( \mu_q(n) = \binom{n + 1}{1}_q/(q + 1) \), which yields \( \chi'(PG(n, q)) \geq \binom{n}{1}_q \). If \( n \) is even, it is known from [6, 11] (also, see [15] for a generalization) that

\[
\mu_q(n) = \frac{(q^n + 1)(q^n - 1)}{(q^{n+1} - q^3 + q^2 - 1)(q - 1)} \quad \text{and} \quad \frac{q^n - 1}{q - 1} + q + \frac{q^4 - q^3 + q - q^2}{q^{n+1} - q^3 + q^2 - 1} = \frac{q^n - 1}{q - 1} + q + 1,
\]

where the last equality holds because of \( n \geq 3 \), we have \( \chi'(PG(n, q)) \geq \binom{n}{1}_q + q + 1 \). Therefore, we obtain a lower bound for \( \chi'(PG(n, q)) \).
Lemma 1.1. For any integer $n \geq 3$ and any prime power $q$,

$$
\chi'(\text{PG}(n, q)) = \begin{cases} 
\left\lfloor \frac{n}{1} \right\rfloor, & \text{if } n \text{ is odd}, \\
\left\lfloor \frac{n}{1} \right\rfloor + q + 1, & \text{if } n \text{ is even}.
\end{cases}
$$

Denniston [9] employed the Klein correspondence for $\text{PG}(3, q)$ to demonstrate that there is a parallelism of $\text{PG}(3, q)$ for any prime power $q$. Later, Beutelspacher [5] gave a different proof for this result and, by using induction on $m$, showed that a parallelism of $\text{PG}(2m - 1, q)$ exists for any positive integer $m$ and any prime power $q$. Baker [2] investigated the partition problem for planes in the $2m$-dimensional affine geometry over $\text{GF}(2)$ and found that there is a parallelism of $\text{PG}(2m - 1, 2)$ for any positive integer $m$. Such a parallelism was also found in the context of Preparata codes [3]. Wettl [18] provided an explicit proof for the theorem of Baker by prescribing a collineation group. Since the existence of a parallelism of $\text{PG}(n, q)$ implies $\chi'(\text{PG}(n, q)) = \left\lfloor \frac{n}{1} \right\rfloor$, we have the following theorem.

Theorem 1.1. For any positive integer $m$ and any prime power $q$,

1. (Beutelspacher [5]) $\chi'(\text{PG}(2m - 1, q)) = \left\lfloor \frac{2m - 1}{1} \right\rfloor$, and
2. (Baker [2]) $\chi'(\text{PG}(2m - 1, 2)) = \left\lfloor \frac{2m - 1}{2} \right\rfloor$.

When $n$ is even, the exact value of $\chi'(\text{PG}(n, 2))$ was determined by Meszka [13].

Theorem 1.2 (Meszka [13]). For any even integer $n \geq 4$, $\chi'(\text{PG}(n, 2)) = \left\lfloor \frac{n}{1} \right\rfloor + 3$.

This paper generalizes Meszka’s work in [13] to establish a framework to count the exact value of $\chi'(\text{PG}(n, q))$ for any integer $n \geq 5$ and any prime power $q$. A special property of $\text{PG}(3, q)$ is introduced and explored in Section 2. Making use of this property, we translate the problem of determining the chromatic index of $\text{PG}(n, q)$ to the problem of examining the existences of $\text{PG}(3, q)$ and $\text{PG}(4, q)$ with certain properties in Section 3 (see Theorem 3.1). In particular, we obtain the following result, which implies the existence of a parallelism of $\text{PG}(n, q)$ for any odd integer $n$ and $q \in \{3, 4, 8, 16\}$.

Theorem 1.3. For any odd integer $n \geq 3$ and $q \in \{3, 4, 8, 16\}$, $\chi'(\text{PG}(n, q)) = \left\lfloor \frac{n}{1} \right\rfloor$.

2 PROPERTY E OF $\text{PG}(3, q)$

Meszka [13, Lemma 2] pointed out that $\text{PG}(3, 2)$ has a special property, which can be used to determine the exact value of $\chi'(\text{PG}(n, 2))$ for any $n > 3$. In this section, we generalize this property to $\text{PG}(3, q)$ for any prime power $q$. This property plays a significant role in the determination of $\chi'(\text{PG}(n, q))$ for any integer $n \geq 5$ in Theorem 3.1.
Definition 2.1. PG(3, q) is said to have Property E if it contains a special spread P and a set S of \( \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = (q^4 - 1)/(q - 1) = (q^2 + 1)(q + 1) \) spreads such that: (1) for each line \( l \) in \( P \), there are exactly \( q + 1 \) spreads in \( S \) containing \( l \), (2) for each line \( l \) not in \( P \), there are exactly \( q \) spreads in \( S \) containing \( l \), and (3) each spread in \( S \) cannot contain two lines of \( P \) at the same time.

To better understand the concept of PG(3, q) with Property E, we count the number of lines covered by \( S \) in two different ways. There are \( (q^4 - 1)/(q - 1) \) spreads in \( S \) and a spread contains \( q^2 + 1 \) lines, so \( S \) covers \( (q^2 + 1) \cdot (q^4 - 1)/(q - 1) \) lines. On the other hand, each line in the spread \( P \) occurs in exactly \( q + 1 \) spreads in \( S \) and each line not in \( P \) occurs in exactly \( q \) spreads in \( S \), so \( S \) covers \( (q^2 + 1) \cdot (q + 1) + \left( \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q - (q^2 + 1) \right) \cdot q \) lines. These two numbers are equal.

We shall present a sufficient condition in Lemma 2.2 to examine Property E of PG(3, q). First we recall a basic fact concerning the lines of PG(n, q).

Since \( F_{q^{n+1}} \) can be regarded as an \((n + 1)\)-dimensional vector space over \( \mathbb{F}_q \), each element in the cyclic group \( F_{q^{n+1}}^*/\mathbb{F}_q^* \) of order \( \begin{bmatrix} n + 1 \\ 1 \end{bmatrix}_q \) can be seen as a point of PG(n, q). Let \( \beta \) be a primitive element of \( F_{q^{n+1}} \) and \( \alpha = \beta^{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q} \) be a primitive element of \( \mathbb{F}_q \). We can identify the set of points in PG(n, q) with

\[
V = \left\{ 1, \beta, \beta^2, ..., \beta^{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q - 1} \right\},
\]

which is a complete set of distinct representatives for the equivalence classes of \( F_{q^{n+1}}^*/\mathbb{F}_q^* \) and is closed under addition and multiplication defined over \( F_{q^{n+1}}^*/\mathbb{F}_q^* \). The line through two distinct points \( x \) and \( y \) of \( V \) is \( \{ x, y, x + y, x + \alpha y, ..., x + \alpha^{q-2} y \} \). Let \( \sigma : x \mapsto \beta x \) be a permutation on \( V \) and let \( \langle \sigma \rangle \) be the cyclic group generated by \( \sigma \). Then the lines of PG(n, q) can be partitioned into orbits under the action of \( \langle \sigma \rangle \). By the orbit-stabilizer theorem, one can obtain the following result (see, e.g., [14, Lemma 1.1]).

Lemma 2.1. The lines in PG(n, q) under the action of \( \langle \sigma \rangle \) can be partitioned into

(i) \( \frac{q^n - 1}{q^2 - 1} \) orbits of length \( \begin{bmatrix} n + 1 \\ 1 \end{bmatrix}_q \) if \( n \) is even, or

(ii) a single orbit of length \( \begin{bmatrix} n + 1 \\ 1 \end{bmatrix}_q / (q + 1) \) and \( \frac{q^n - q}{q^2 - 1} \) orbits of length \( \begin{bmatrix} n + 1 \\ 1 \end{bmatrix}_q \) if \( n \) is odd.

By Lemma 2.1, under the action of \( \langle \sigma \rangle \), all lines in PG(3, q) can be partitioned into a short orbit of length \( q^2 + 1 \) and \( q \) full orbits of length \( \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \). The following lemma gives a sufficient condition to examine Property E of PG(3, q). For convenience, throughout this section, we identify \((V, \cdot)\) with the cyclic group \( \mathbb{Z}_{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q}^+ \), and so \( \langle \sigma \rangle \) can be seen as \( \mathbb{Z}_{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q}^+ \).
Lemma 2.2. \(\text{PG}(3, q)\) has Property \(E\) if under the action of \(\langle \sigma \rangle\), one can take \(q\) lines from every full orbit and one line from the short orbit such that the collection of \(q^2 + 1\) lines form a spread, written as \(P_0^0\).

Proof. Let \(v = \left[ \begin{array}{c} 1 \\ \vdots \\ q \end{array} \right]\) and \(P = \{L_0, L_1, \ldots, L_q\}\) be the short orbit of \(\text{PG}(3, q)\) under the action of \(\langle \sigma \rangle \cong (\mathbb{Z}_q, +)\), where \(L_i = \{j(q^2 + 1) + i \pmod v : 0 \leq j \leq q\}\) for \(0 \leq i \leq q^2\).

We take \(P\) as the special spread in Definition 2.1. Under the action of \(\langle \sigma \rangle\), without loss of generality, we can assume that \(L_0 \in P_0^0\).

For \(0 \leq i \leq q^2\) and \(0 \leq j \leq q\), let

\[P^j_i = P_0^0 + j(q^2 + 1) + i.\]

Note that \(j(q^2 + 1) + i\) runs over \(\mathbb{Z}_q\) when \(i\) and \(j\) run over the integer intervals \([0, q^2]\) and \([1, q]\), respectively. Then \(S = \{P^j_i : 0 \leq i \leq q^2, 0 \leq j \leq q\}\) is a set of \((q^4 - 1)/(q - 1)\) spreads. It is readily checked that \(S\) satisfies the three conditions in Definition 2.1: (1) for each line \(L_i \in P\), one can check that \(L_i \in P^j_i\) for each \(0 \leq j \leq q\); (2) for each line \(L \notin P\), \(L\) occurs in exactly one full orbit \(\mathcal{O}\) of \(\text{PG}(3, q)\); since \(P_0^0\) contains \(q\) lines from \(\mathcal{O}\), there are exactly \(q\) spreads in \(S\) containing \(L\); (3) each spread in \(S\) cannot contain two lines of \(P\) at the same time.

\(\blacksquare\)

Example 2.1. \(\text{PG}(3, 2)\) has Property \(E\).

Proof. Although Meszka [13, Lemma 2] provided an example of \(\text{PG}(3, 2)\) having Property \(E\), we here provide a different construction by using Lemma 2.2. Let \(\mathbb{F}_2^4 = \mathbb{F}_2[x]/\langle x^4 + x + 1 \rangle\) where \(x^4 + x + 1\) is an irreducible polynomial over \(\mathbb{F}_2\) and \(x \in \mathbb{F}_2[x]\) is a fixed primitive element of \(\mathbb{F}_2^4\). Each point in \(\text{PG}(3, 2)\) can be represented by \(x^i\) or simply \(i\) for some \(0 \leq i \leq 14\). By Lemma 2.1, \(\text{PG}(3, 2)\) has a unique short orbit

\[P = \{[0, 5, 10] + i : i \in \{0, 1, 2, 3, 4\}\},\]

and two full orbits \(\mathcal{O}_1 = \{[0, 11, 12] + i : i \in \mathbb{Z}_{15}\}\) and \(\mathcal{O}_2 = \{[0, 2, 8] + i : i \in \mathbb{Z}_{15}\}\). We can take two disjoint lines \([0, 11, 12] + 1 = [1, 12, 13]\) and \([0, 11, 12] + 7 = [3, 4, 7]\) from \(\mathcal{O}_1\) and two disjoint lines \([0, 2, 8] + 6 = [6, 8, 14]\) and \([0, 2, 8] + 9 = [2, 9, 11]\) from \(\mathcal{O}_2\) such that

\[P_0^0 = \{[1, 12, 13], [3, 4, 7], [6, 8, 14], [2, 9, 11], [0, 5, 10]\}\]

forms a spread satisfying the condition in Lemma 2.2. Therefore,

\[S = \left\{P^j_i = P_0^0 + 5j + i : 0 \leq i \leq 4, 0 \leq j \leq 2\right\}\]

is a set of 15 spreads which ensure \(\text{PG}(3, 2)\) has Property \(E\). We list all \(P^j_i\) explicitly as follows to facilitate the reader to check it.
The underlined lines are from the spread \( P \).

In this example, to find \( P^0 \) one can, without loss of generality, choose the line \([0, 5, 10]\) first and then search for the other four lines. A way to speed up the search is the use of Frobenius maps of \( 2^{4} \times 2^{4} \). It is readily checked that \( l\{\{1, 12, 13\} \times 2 (\text{mod } 15) : 0 \leq i \leq 3\} = \{\{1, 12, 13\}, \{2, 9, 11\}, \{4, 3, 7\}, \{8, 6, 14\}\} = P^0\{\{0, 5, 10\}\} \). Thus it suffices to find the initial line \([1, 11, 12]\) and then develop it through multiplying by all powers of 2 to obtain all the other lines except for the fixed one \([0, 5, 10]\).

**Lemma 2.3.** PG(3, 4) has Property E.

**Proof.** Let \( F_{2^{5}} = F_{2}[x]/(x^{8} + x^{6} + x^{5} + x^{2} + 1) \) where \( x^{8} + x^{6} + x^{5} + x^{2} + 1 \) is an irreducible polynomial over \( F_{2} \) and \( x \in F_{2}[x] \) is a fixed primitive element of \( F_{2^{5}} \cong F_{4} \).

Each point in PG(3, 4) can be represented by \( x^{i} \) or simply \( i \) for some \( 0 \leq i \leq 84 \). By Lemma 2.1, PG(3, 4) has a unique short orbit \( P = \{0, 17, 34, 51, 68\} + i : 0 \leq i \leq 16\) and four full orbits

\[ O_1 = \{0, 1, 13, 70, 23, i : i \in \mathbb{Z}_{83}\}, \quad O_2 = \{0, 2, 26, 55, 13, 46, i : i \in \mathbb{Z}_{83}\}, \quad O_3 = \{0, 3, 21, 81, 48, 32, i : i \in \mathbb{Z}_{83}\}, \quad O_4 = \{0, 5, 36, 79, 71, 2, i : i \in \mathbb{Z}_{83}\}. \]

We take two initial lines \([2, 12, 59, 74, 75]\) and \([14, 29, 30, 42, 52]\), and then develop them through multiplying by \( 2^{l}, 0 \leq i \leq 7\), to obtain 16 lines such that each full orbit contains exactly 4 lines. Specifically the 16 lines are

\[ D_1 = \{0, 1, 13, 23, 70, j : j \in \{74, 29, 9, 54\}\} \subseteq O_1, \]
\[ D_2 = \{0, 2, 26, 55, j : j \in \{63, 58, 18, 23\}\} \subseteq O_2, \]
\[ D_3 = \{0, 3, 21, 81, j : j \in \{45, 35, 40, 50\}\} \subseteq O_3, \]
\[ D_4 = \{0, 5, 36, 79, j : j \in \{11, 76, 21, 71\}\} \subseteq O_4. \]
Then \( \{0, 17, 34, 51, 68\} \cup (\bigcup_{i=1}^{4} D_i) \) forms a spread \( P_0^0 \) satisfying the condition in Lemma 2.2. Therefore, PG(3, 4) has Property E. □

**Lemma 2.4.** PG(3, 8) has Property E.

**Proof.** Let \( F_{2^{12}} = \mathbb{F}_2[x]/(x^{12} + x^{10} + x^2 + x + 1) \) where \( x^{12} + x^{10} + x^2 + x + 1 \) is an irreducible polynomial over \( \mathbb{F}_2 \) and \( x \in \mathbb{F}_2[x] \) is a fixed primitive element of \( \mathbb{F}_{2^{12}} \cong \mathbb{F}_g^4 \). Each point in PG(3, 8) can be represented by \( x^i \) or simply \( i \) for some \( 0 \leq i \leq 584 \). By Lemma 2.1, PG(3, 8) has a unique short orbit \( P = \{0, 65, 130, 195, 260, 325, 390, 455, 520\} + i : 0 \leq i \leq 64 \) and eight full orbits

\[
\begin{align*}
O_1 &= \{0, 1, 37, 84, 328, 404, 490, 558\} + i : i \in \mathbb{Z}_{SSS},
O_2 &= \{0, 2, 71, 74, 168, 178, 223, 395, 531\} + i : i \in \mathbb{Z}_{SSS},
O_3 &= \{0, 4, 142, 148, 205, 336, 356, 446, 477\} + i : i \in \mathbb{Z}_{SSS},
O_4 &= \{0, 7, 16, 29, 153, 174 , 235, 254, 568\} + i : i \in \mathbb{Z}_{SSS},
O_5 &= \{0, 8, 87, 127, 284, 296, 307, 369, 410\} + i : i \in \mathbb{Z}_{SSS},
O_6 &= \{0, 14, 32, 58, 306, 348, 470, 508, 551\} + i : i \in \mathbb{Z}_{SSS},
O_7 &= \{0, 15, 155, 204, 255, 354, 389, 414, 467\} + i : i \in \mathbb{Z}_{SSS},
O_8 &= \{0, 30, 123, 193, 243, 310, 349, 408, 510\} + i : i \in \mathbb{Z}_{SSS}.
\end{align*}
\]

We take six initial lines:

\[
\begin{align*}
&\{1, 77, 163, 231, 258, 295, 342, 347\}, &\{7, 12, 251, 327, 413, 481, 508, 509, 545\},
&\{15, 42, 43, 79, 126, 131, 370, 446, 532\}, &\{53, 90, 137, 142, 381, 457, 543\},
&\{22, 121, 156, 181, 234, 352, 367, 507, 556\}, &\{18, 136, 151, 291, 340, 391, 490, 525, 550\},
\end{align*}
\]

and then develop them through multiplying by \( 2^l \), \( 0 \leq l \leq 11 \), to obtain 64 lines such that each full orbit contains exactly 8 lines (note that \( \{22, 121, 156, 181, 234, 352, 367, 507, 556\} \) can only generate 4 distinct lines). Specifically the 64 lines are

\[
\begin{align*}
D_1 &= \{0, 1, 37, 84, 328, 404, 490, 558\} + j : j \in \{258, 159, 508, 364, 42, 375, 53, 494\},
D_2 &= \{0, 2, 71, 74, 168, 178, 223, 395, 531\} + j : j \in \{516, 318, 431, 143, 84, 165, 106, 403\},
D_3 &= \{0, 4, 142, 148, 205, 336, 356, 446, 477\} + j : j \in \{447, 51, 277, 286, 168, 330, 212, 221\},
D_4 &= \{0, 7, 16, 29, 153, 174, 235, 254, 568\} + j : j \in \{33, 204, 523, 559, 87, 150, 263, 299\},
D_5 &= \{0, 8, 87, 127, 284, 296, 307, 369, 410\} + j : j \in \{309, 102, 554, 572, 336, 75, 424, 442\},
D_6 &= \{0, 14, 32, 58, 306, 348, 470, 508, 551\} + j : j \in \{66, 408, 461, 533, 174, 300, 526, 13\},
D_7 &= \{0, 15, 155, 204, 255, 354, 389, 414, 467\} + j : j \in \{46, 100, 136, 190, 352, 406, 415, 469\},
D_8 &= \{0, 30, 123, 193, 243, 310, 349, 408, 510\} + j : j \in \{92, 119, 200, 227, 245, 272, 353, 380\}.
\end{align*}
\]

Then \( \{0, 65, 130, 195, 260, 325, 390, 455, 520\} \cup (\bigcup_{i=1}^{8} D_i) \) forms a spread \( P_0^0 \) satisfying the condition in Lemma 2.2. Therefore, PG(3, 8) has Property E. □
Lemma 2.5. \( \text{PG}(3, 16) \) has Property E.

Proof. Let \( F_{2^{16}} = F_2[x]/(x^{16} + x^{14} + x^{12} + x^7 + x^6 + x^4 + x^2 + x + 1) \) where \( x^{16} + x^{14} + x^{12} + x^7 + x^6 + x^4 + x^2 + x + 1 \) is an irreducible polynomial over \( F_2 \) and \( x \in F_2[x] \) is a fixed primitive element of \( F_{2^{16}} \). Each point in \( \text{PG}(3, 16) \) can be represented by \( x^i \) or simply \( i \) for some \( 0 \leq i \leq 4368 \). By Lemma 2.1, \( \text{PG}(3, 16) \) has a unique short orbit and 16 full orbits. We take 16 initial lines:

\[
\begin{align*}
\{0, 1, 727, 733, 997, 1133, 1877, 1917, 2477, 2864, 3070, 3172, 3453, 3594, 3672, 3928, 4279\} + i \\
\{0, 7, 898, 1060, 1146, 1362, 1392, 1817, 2054, 2287, 2323, 2504, 2875, 3723, 3749, 4205, 4227\} + i
\end{align*}
\]

where \( i \in \{117, 135, 550, 660, 782, 2563, 3254, 3529\} \); \( i \in \{183, 818, 1495, 1554, 1754, 1981, 2088, 3979\} \), and then develop them through multiplying by \( 2^l \), \( 0 \leq l \leq 15 \), to obtain 256 lines such that each full orbit contains exactly 16 lines. The 256 lines together the line \( \{257 : 0 \leq j \leq 16\} \) form a spread \( P^0_0 \) satisfying the condition in Lemma 2.2. Therefore, \( \text{PG}(3, 16) \) has Property E. \( \square \)

It only took 4 seconds by a common personal computer to search for Property E of \( \text{PG}(3, 16) \). We believe that one can use computer and apply Lemma 2.2 together with the use of Frobenius maps of \( F_{2^m} \) over \( F_2 \) to confirm Property E of \( \text{PG}(3, 2^m) \) for bigger \( m \). It is reasonable to conjecture that \( \text{PG}(3, 2^m) \) has Property E for any positive integer \( m \).

On the other hand, by exhaustive search, we found that Lemma 2.2 cannot be applied for \( \text{PG}(3, 3) \) and \( \text{PG}(3, 5) \). To deal with the case of \( \text{PG}(3, 3) \), we searched for 40 spreads satisfying the conditions in Definition 2.1 without the use of any group action.

Lemma 2.6. \( \text{PG}(3, 3) \) has Property E.

Proof. Let \( F_{3^t} = F_3[x]/(x^4 + x + 2) \) where \( x^4 + x + 2 \) is an irreducible polynomial over \( F_3 \) and \( x \in F_3[x] \) is a fixed primitive element of \( F_{3^t} \). Each point in \( \text{PG}(3, 3) \) can be represented by \( x^i \) or simply \( i \) for some \( 0 \leq i \leq 39 \). By Lemma 2.1, the 130 lines of \( \text{PG}(3, 3) \) can be given below according to orbits:

\[
\begin{align*}
L_i &= \{0, 1, 4, 13\} + i \pmod{40}, \quad L_{40+i} = \{0, 2, 17, 24\} + i \pmod{40}, \\
L_{80+i} &= \{0, 5, 26, 34\} + i \pmod{40}, \quad L_{120+j} = \{0, 10, 20, 30\} + j \pmod{40},
\end{align*}
\]

where \( i \in \mathbb{Z}_{40} \) and \( 0 \leq j \leq 9 \). We take the spread

\[
P = \{L_0, L_2, L_{32}, L_{127}, L_8, L_{10}, L_{110}, L_{18}, L_{114}, L_{25}\}
\]

as the special spread in Definition 2.1, and a set \( S \) of 40 spreads of \( \text{PG}(3, 3) \) satisfying the conditions in Definition 2.1 is listed below. To save space we write \( i \) instead of the line \( L_i \) for \( 0 \leq i \leq 129 \).
3 | THE CHROMATIC INDEX OF PG(n, q)

**Definition 3.1.** Given a subspace $\text{PG}(n - 2, q)$ of $\text{PG}(n, q)$, let $\text{IPG}(n, q; n - 2)$ denote the geometry that is obtained from $\text{PG}(n, q)$ by removing all lines of the given $\text{PG}(n - 2, q)$.

**Lemma 3.1.** $\text{IPG}(3, q; 1)$ is $(q^2 + q + 1)$-colorable, where $q(q + 1)$ colors are used to color all lines incident with the given $\text{PG}(1, q)$.

**Proof.** By Theorem 1.1, $\chi'(\text{PG}(3, q)) = q^2 + q + 1$, so the lines of $\text{PG}(3, q)$ can be partitioned into $q^2 + q + 1$ spreads. There is exactly one spread $P$ containing the line of the given $\text{PG}(1, q)$, and the remaining lines in $P$ are not incident with the points of $\text{PG}(1, q)$. Thus $q(q + 1)$ colors are used to color all lines incident with the points of $\text{PG}(1, q)$. 

A group divisible design (GDD) $k$-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying that: (1) $\mathcal{G}$ is a partition of a finite set $X$ into subsets (called groups); (2) $\mathcal{A}$ is a set of $k$-subsets of $X$ (called blocks), such that every 2-subset of $X$ is either contained in exactly one block or in exactly one group, but not in both. If $\mathcal{G}$ contains $u_i$ groups of size $g_i$ for $1 \leq i \leq r$, then $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ is called the type of the GDD. A $k$-GDD of type $n^k$ is often called a transversal design and denoted by a TD($k$, $n$). It is known that a TD($q + 1, q$) exists for any prime power $q$ (cf. [4]).
The following theorem is a generalization of [13, Lemma 5] which only deals with the case \( q = 2 \).

**Theorem 3.1.**

1. Suppose that \( \text{PG}(3, q) \) has Property E. Then \( \chi'(\text{PG}(n, q)) = \left[ \frac{n}{1} \right]_q \) for any odd integer \( n \geq 5 \).
2. Suppose that \( \text{PG}(3, q) \) has Property E and \( \chi'(\text{PG}(4, q)) = \left[ \frac{4}{1} \right]_q + q + 1 \). If \( \text{IPG}(4, q; 2) \) is \( \left( \left[ \frac{4}{1} \right]_q + q + 1 \right) \)-colorable, where \( q^2(q+1) \) colors are used to color all lines incident with the given \( \text{PG}(2, q) \), then \( \chi'(\text{PG}(n, q)) = \left[ \frac{n}{1} \right]_q + q + 1 \) for any even integer \( n \geq 6 \).

**Proof.** Let \( \mathbb{F}_q^* = \langle \alpha \rangle \) where \( \alpha \) is a primitive element in \( \mathbb{F}_q \). Take two linearly independent elements \( x \) and \( y \) in \( \mathbb{F}_q^* \) over \( \mathbb{F}_q \) and let \( (x, y) = \{x, y, x + y, x + \alpha y, \ldots, x + \alpha^{q-2}y\} \) be the unique line in \( \text{PG}(1, q) \) defined on \( \mathbb{F}_q^* \).

First we give a construction for \( \text{PG}(n, q) \): This construction is standard from the point of view of combinatorial design theory. Roughly speaking, we take a \((q + 1)\)-GDD of type \((q + 1)^2+1\), and then give every point weight \( q^{n-3} \) to get a \((q + 1)\)-GDD of type \((q^{n-3}(q + 1))^{2+1}\). Fill in each group by using an \( S \left( 2, q + 1, \left[ \frac{n-1}{1} \right]_q \right) \) containing an \( S \left( 2, q + 1, \left[ \frac{n-3}{1} \right]_q \right) \) as a subdesign to obtain an \( S \left( 2, q + 1, \left[ \frac{n+1}{1} \right]_q \right) \), which happens to be our desired \( \text{PG}(n, q) \). The details are as follows.

Let \( (X_0, A) \) be an \( S \left( 2, q + 1, \left[ \frac{4}{1} \right]_q \right) \) formed by the points and lines of \( \text{PG}(3, q) \), where \( X_0 = \{(v_1, v_2) : v_1 \in \langle x, y \rangle, v_2 \in \mathbb{F}_q^* \} \cup \{(0, v_2) : v_2 \in \langle x, y \rangle \} \) (note that 0 \( \notin \mathbb{F}_q^* \)). Since \( \text{PG}(3, q) \) has a spread, assume that \( \mathcal{G}_0 = \{G_1, G_2, \ldots, G_{q+1}\} \) is a spread of \( \text{PG}(3, q) \). Write \( A_0 = A \setminus \mathcal{G}_0 \). Then \( (X_0, G_0, A_0) \) is a \((q + 1)\)-GDD of type \((q + 1)^2+1\). For each line \( L_{a,b} = \{a, b, a + b, a + \alpha b, \ldots, a + \alpha^{q-2}b\} \in A_0 \), construct a TD\((q + 1, q^{n-3})\) \((X_{L_{a,b}}, G_{L_{a,b}}, A_{L_{a,b}}) \) where \( X_{L_{a,b}} = \{(u_1, u_2) : u_1 \in L_{a,b}, u_2 \in \mathbb{F}_q^{n-3}\}, G_{L_{a,b}} = \{(u_1, u_2) : u_2 \in \mathbb{F}_q^{n-3} : u_1 \in L_{a,b}\}, \) and \( A_{L_{a,b}} = \{(a, u_2), (b, u'_2), (a + b, u_2 + u'_2), \ldots, (a + \alpha^{q-2}b, u_2 + \alpha^{q-2}u'_2) : u_2, u'_2 \in \mathbb{F}_q^{n-3}\} \). Note that each \( u_i \in L_{a,b} \) is either of the form \((v_1, v_2) \) where \( v_1 \in \langle x, y \rangle \) and \( v_2 \in \mathbb{F}_q^* \), or of the form \((0, v'_2) \) where \( v'_2 \in \langle x, y \rangle \), and consequently each block in \( A_{L_{a,b}} \) can be seen as a line of \( \text{PG}(n, q) \) whose underlying \((n + 1)\)-dimensional vector space \( V(n + 1, q) \) is defined on \( \mathbb{F}_q^* \oplus \mathbb{F}_q^* \oplus \mathbb{F}_q^{n-3} \).

It follows that we obtain a \((q + 1)\)-GDD of type \((q^{n-3}(q + 1))^{2+1}\), \((X_1, G_1, A_1) \), where \( X_1 = \{(v_1, v_2, u_2) : v_1 \in \langle x, y \rangle, v_2 \in \mathbb{F}_q^*, u_2 \in \mathbb{F}_q^{n-3} \} \cup \{(0, v_2, u_2) : v_2 \in \langle x, y \rangle, u_2 \in \mathbb{F}_q^{n-3}\}, G_1 = \{G_i = \{(w_1, w_2, u_2) : (w_1, w_2) \in G_i, u_2 \in \mathbb{F}_q^{n-3} : 1 \leq i \leq q^2 + 1\}, \) and \( A_1 = \bigcup_{L_{a,b} \in A_0} A_{L_{a,b}} \).

Construct \( \text{PG}(n - 4, q) \) such that its underlying \((n - 3)\)-dimensional vector space \( V(n - 3, q) \) is defined on \( \{(0, 0, u_2) : u_2 \in \mathbb{F}_q^{n-3}\} \) (note that 0 \( \notin \mathbb{F}_q^* \)). Let \( Y \) and \( C \) be the
sets of points and lines of the PG\((n - 4, q)\), respectively. Then \((Y, \mathcal{C})\) forms an S\(\left(2, q + 1, \left\lceil \frac{n - 3}{1} \right\rceil \right)\).

For each \(1 \leq i \leq q^2 + 1\), construct an IPG\((n - 2, q; n - 4)\) with the set of points \(G'_i \cup Y\) such that \(Y\) is the set of points of its underlying PG\((n - 4, q)\). Denote by \(B_i\) the set of lines of the IPG\((n - 2, q; n - 4)\). Then \(B_i \cup \mathcal{C}\) forms an S\(\left(2, q + 1, \left\lceil \frac{n - 1}{1} \right\rceil \right)\) that contains an S\(\left(2, q + 1, \left\lceil \frac{n - 3}{1} \right\rceil \right)\) as a subdesign.

Let \(X_2 = X_i \cup Y\) and \(A_2 = A_i \cup \mathcal{C} \cup (\bigcup_{j=1}^{q^2 + 1} B_j)\). Then \(X_2\) and \(A_2\) form the sets of points and lines of PG\((n, q)\), respectively. That is, \((X_2, A_2)\) is an S\(\left(2, q + 1, \left\lceil \frac{n + 1}{1} \right\rceil \right)\).

Next we color the lines of PG\((n, q)\): Let
\[
c(n, q) = \begin{cases} 
\left\lceil \frac{n}{1} \right\rceil, & \text{if } n \text{ is odd,} \\
\left\lceil \frac{n}{1} \right\rceil + q + 1, & \text{if } n \text{ is even.}
\end{cases}
\]

We introduce the notion of Property \(R\) of IPG\((n, q; n - 2)\). Recall that IPG\((n, q; n - 2)\) is the geometry that is obtained by removing all lines of a given subspace PG\((n - 2, q)\) from PG\((n, q)\). If IPG\((n, q; n - 2)\) is \(c(n, q)\)-colorable such that \(q^{n-2}(q + 1)\) colors are used to color all lines incident with points of the given PG\((n - 2, q)\), then IPG\((n, q; n - 2)\) is said to be with Property \(R\). By Lemma 3.1, IPG\((3, q; 1)\) has Property \(R\), and by assumption, IPG\((4, q; 2)\) has Property \(R\).

To color the lines of PG\((n, q)\), we proceed by induction on \(n\). By Theorem 1.1, \(\chi'(\text{PG}(3, q)) = c(3, q)\), and by assumption \(\chi'(\text{PG}(4, q)) = c(4, q)\). Let \(n \geq 5\). Assume that \(\chi'(\text{PG}(n - 2, q)) = c(n - 2, q)\) and IPG\((n - 2, q; n - 4)\) is \(c(n - 2, q)\)-colorable such that \(q^{n-4}(q + 1)\) colors are used to color all lines incident with the points of the underlying PG\((n - 4, q)\). We shall prove that \(\chi'(\text{PG}(n, q)) = c(n, q)\).

We color the lines in \(A_2\) by the following three steps:

**Step 1.** We color the lines in \(B_i \cup \mathcal{C}\), which forms PG\((n - 2, q)\) with the set of points \(G'_i \cup Y\). By the induction hypothesis, \(\chi'(\text{PG}(n - 2, q)) = c(n - 2, q)\), so PG\((n - 2, q)\) can be colored using a set of \(c(n - 2, q)\) colors. We split this color set into two disjoint subsets \(C_1\) and \(C^*\) of cardinalities \(c_1 = q^{n-4}(q + 1)\) and \(c_2 = c(n - 2, q) - c_1\), respectively.

**Step 2.** We color the lines in \(\bigcup_{i=2}^{q^2 + 1} B_i\). For each \(2 \leq i \leq q^2 + 1\), \(B_i\) forms IPG\((n - 2, q; n - 4)\) with the set of points \(G'_i \cup Y\). By the induction hypothesis, \(\chi'_i\) is \(c(n - 2, q)\)-colorable in such a way that \(c_1 = q^{n-4}(q + 1)\) colors are used to color all lines incident with the points of PG\((n - 4, q)\) which has the point set \(Y\).

We take a set \(C_i\) of \(c_1\) colors, which were not previously used, to color all lines of \(B_i\) incident with \(Y\) such that \(C_{i_1} \cap C_{i_2} = \emptyset\) for any \(1 \leq i_1 < i_2 \leq q^2 + 1\). To color the remaining lines of \(B_i\), we use \(c_2 = c(n - 2, q) - c_1\) colors that come from the color set \(C^*\). Thus the total number of colors is \((q^2 + 1) \cdot c_1 + c_2 = c(n, q)\).
Step 3. We color the lines in $A_l = \bigcup_{L_{a,b} \in A_o} A_{L_{a,b}}$ using the colors from $\bigcup_{i=1}^{q^2+1} C_i$. Recall that for each $L_{a,b} \in A_o$, $A_{L_{a,b}}$ forms a TD($q + 1$, $q^{n-3}$).

Claim 3.1. This TD is resolvable, that is, $A_{L_{a,b}}$ can be partitioned into parallel classes, each of which contains every point of $X_{L_{a,b}}$ exactly once.

Proof. Let $F^{(q^2+1)} = \langle \beta \rangle$ such that $\alpha = \beta^{q^2+1}$. Construct a TD($q + 2$, $q^{n-3}$) on $X_{L_{a,b}} \cup \{(a + \beta b, u_2) : u_2 \in F^{(q-3)}\}$ with the set of groups $G_{L_{a,b}} \cup \{(a + \beta b, u_2) : u_2 \in F^{(q-3)}\}$ and the set of blocks $A'_{L_{a,b}} = \{(a, u_2), (b, u_2'), (a + b, u_2 + u_2'), \ldots, (a + \alpha^{q-2}b, u_2 + \alpha^{q-2}u_2'), (a + \beta b, u_2 + \beta u_2') : u_2, u_2' \in F^{(q-3)}\}$. We shall show that for every $j \in F^{(q-3)}$, $N_j = \{B \setminus (a + \beta b, j) : (a + \beta b, j) \in B \in A'_{L_{a,b}}\}$ is a parallel class of $(X_{L_{a,b}}, G_{L_{a,b}}, A_{L_{a,b}})$.

Clearly $N_j \subseteq A_{L_{a,b}}$. Since $|N_j| = q^{n-3}$, it suffices to show that the blocks in $N_j$ are pairwise disjoint. Assume that there are two distinct blocks $B_1, B_2 \in N_j$ satisfying $|B_1 \cap B_2| \geq 1$, where $\{(a, u_2), (b, u_2') \} \in B_1$ and $\{(a, u_2''), (b, u_2''') \} \in B_2$. Then $j = u_2 + \beta u_2' = u_2'' + \beta u_2'''$, and either $u_2 = u_2''$ or $u_2' = u_2'''$ or there exists $0 \leq k \leq q - 2$ such that $u_2 + \alpha^k u_2' = u_2'' + \alpha^k u_2'''$. Clearly the first two cases are impossible. The last case is also impossible since $\alpha^k \neq \beta$ for any $0 \leq k \leq q - 2$. $\square$

Therefore, $A_{L_{a,b}}$ is $q^{n-3}$-colorable. We shall choose $q^{n-3}$ colors from $\bigcup_{i=1}^{q^2+1} C_i$ to color lines in $A_{L_{a,b}}$ for each $L_{a,b} \in A_o$.

By assumption, $PG(3, q)$ has Property E. Without loss of generality, let $P = G_0 = \{G_1, G_2, \ldots, G_{q^2+1}\}$ be the special spread of $PG(3, q)$ in Definition 2.1, and $S = \{P_{G_i} : 1 \leq j \leq q + 1, 1 \leq i \leq q^2 + 1\}$ be a set of $(q^2 - 1)/(q - 1)$ spreads satisfying the three conditions in Definition 2.1, where for each line $G_i \in P$, there are exactly $q + 1$ spreads $P_{G_i}^j (1 \leq j \leq q + 1)$ in $S$ containing $G_i$.

For each $1 \leq i \leq q^2 + 1$, we split the color set $C_i$ into $q + 1$ disjoint subsets $C_i^1, C_i^2, \ldots, C_i^{q+1}$, each of cardinality $q^{n-4}$, and assign each subset $C_i^j, 1 \leq j \leq q + 1$, to the spread $P_{G_i}^j$. It follows that each spread in $S$ owns $q^{n-4}$ colors and any two different spreads in $S$ own different colors. By Property E of $PG(3, q)$, every line $L_{a,b} \notin P = G_0$ (i.e., $L_{a,b} \notin A_0$) of $PG(3, q)$ occurs in exactly $q$ spreads in $S$, so the number of colors assigned to $L_{a,b}$ is $q^{n-4} \cdot q = q^{n-3}$. Using the $q^{n-3}$ colors, we can color all of the lines in $A_{L_{a,b}}$.

Finally we check the correctness of the coloring: Take any two different intersecting lines $L_1$ and $L_2$ from $A_2 = A_1 \cup C \cup (\bigcup_{i=1}^{q^2+1} B_i)$. It suffices to show that they have assigned different colors.

If $L_1$ and $L_2$ are both from $C \cup (\bigcup_{i=1}^{q^2+1} B_i)$, then it follows from Steps 1 and 2 that $L_1$ and $L_2$ have different colors.

If $L_1$ and $L_2$ are both from $A_1 = \bigcup_{L_{a,b} \in A_o} A_{L_{a,b}}$, then either they come from the same $A_{L_{a,b}}$, or from two different ones, say $A_{L_{a,b}}$ and $A_{L_{a',b'}}$, where $L_{a,b}$ and $L_{a',b'}$ are different lines in $A_0$. In the former case $L_1$ and $L_2$ are assigned different colors by Step 3. In the latter case, since $L_1$ and $L_2$ are intersecting, $L_{a,b}$ must intersect with $L_{a',b'}$, and so $L_{a,b}$ and $L_{a',b'}$ belong to different spreads in $S$; by Step 3, they have different colors.
If $L_1$ is from $A_1$ and $L_2$ is from $C$, since $A_1$ is defined on $X_1$ and $C$ is defined on $Y$, $L_1$ and $L_2$ are disjoint, a contradiction.

If $L_1$ is from $A_1$ and $L_2$ is from $\bigcup_{i=1}^{q^2+1} B_i$, then $L_1 \in A_{l,a,b}$ for some $l,a,b \in A_0$, and $L_2 \in B_i$ for some $1 \leq i \leq q^2+1$. Since $L_1$ and $L_2$ are intersecting, $A_{l,a,b}$ must intersect with $G_i$, and so $L_{a,b}$ and $G_i$ belong to different spreads in $S$; by Step 3, they have different colors.

Therefore, combining Lemma 1.1, we have $\chi'(PG(n, q)) = c(n, q)$. Note that $A_1 \cup (\bigcup_{i=1}^{q^2+1} B_i)$ forms IPG($n, q; n - 2$), and the above coloring procedure implies that this IPG($n, q; n - 2$) is with Property $R$, that is, this IPG($n, q; n - 2$) is $c(n, q)$-colorable such that $q^{n-2}(q + 1)$ colors from $\bigcup_{i=1}^{q^2+1} C_i$ are used to color all lines incident with points of the PG($n - 2, q$) which has the set of points $G_1 \cup Y$ and the set of lines $B_1 \cup C$. □

Algorithm 1. Construction for PG($n, q$) with $n \geq 5$.

Input: Let $\alpha$ be a primitive element in $\mathbb{F}_q$.

Step 1: Construct PG($3, q$) on $\mathbb{F}_q \times \{0\}$ (note that $0 \in \mathbb{F}_{q^n-3}$) with the set $A$ of lines.

Let $\mathcal{G}_0 = \{G_1, G_2, \ldots, G_{q^2+1}\}$ be a spread of PG($3, q$). Write $A_0 = A \setminus \mathcal{G}_0$.

Step 2: For each line $L_{a,b} = \{a, b, a + b, a + ab, \ldots, a + \alpha^{q-2}b\} \in A_0$, let

$$A_{l,a,b} = \{(a, u_2), (b, u'_2), (a+b, u_2+u'_2), \ldots, (a+\alpha^{q-2}b, u_2+\alpha^{q-2}u'_2)\} : u_2, u'_2 \in \mathbb{F}_{q^n-3}\}
$$

and $A_1 = \bigcup_{l,a,b \in A_0} A_{l,a,b}$.

Step 3: Construct PG($n - 4, q$) on $\{0\} \times \mathbb{F}_{q^n-3}$ (note that $0 \in \mathbb{F}_{q^4}$) with the set $C$ of lines.

Step 4: For each $1 \leq i \leq q^2+1$, construct an IPG($n - 2, q; n - 4$) on $\langle G_i \rangle \times \mathbb{F}_{q^n-3}$ such that the underlying PG($n - 4, q$) is constructed on $\{0\} \times \mathbb{F}_{q^n-3}$. Denote by $B_i$ the set of lines of the IPG($n - 2, q; n - 4$).

Output: $A_2 = A_1 \cup C \cup (\bigcup_{i=1}^{q^2+1} B_i)$ forms the set of lines of PG($n, q$), which is constructed on $\mathbb{F}_q^4 \times \mathbb{F}_{q^n-3}$.

Proof of Theorem 1.3. By Lemmas 2.3, 2.4, 2.5, and 2.6, PG(3, 4), PG(3, 8), PG(3, 16), and PG(3, 3) have Property $E$, so applying Theorem 3.1(1), we have $\chi'(PG(n, q)) = \binom{n}{1}_q$ for any odd integer $n \geq 5$ and $q \in \{3, 4, 8, 16\}$. The case of $n = 1$ is trivial. For $n = 3$, by Theorem 1.1(2), $\chi'(PG(3, q)) = \binom{3}{1}_q$ for any prime power $q$. □

Finally we give a proof of Theorem 1.2 by employing Theorem 3.1. It is known that $\chi'(PG(4, 2)) = \binom{4}{1}_2 + 3 = 18$ (see [13, Example 2]) and IPG(4, 2; 2) is $\left(\binom{4}{1}_2 + 3\right)$-colorable where 12 colors are used to color all lines incident with the given PG(2, 2) (see [13, Lemma 4(b)]). Since PG(3, 2) has Property $E$ by Example 2.1, we can apply Theorem 3.1 to obtain that $\chi'(PG(n, 2)) = \binom{n}{1}_2 + 3$ for any even integer $n \geq 6$. □
4  |  CONCLUDING REMARKS

Theorems 1.1–1.3 motivate us to conjecture that the lower bound for $\chi'(PG(n, q))$ in Lemma 1.1 is tight. However, when $n$ is even, it is only known by Theorem 1.2 that this lower bound for $\chi'(PG(n, q))$ is tight for $q = 2$. It is meaningful to examine whether $\chi'(PG(4, 3)) = \left\lceil \frac{4}{1} \right\rceil + 4 = 44$ and whether IPG(4, 3; 2) is \left\lceil \frac{4}{1} \right\rceil + 4-colorable where 36 colors are used to color all lines incident with the given PG(2, q). If so, applying Theorem 3.1 would yield the exact value of $\chi'(PG(n, 3))$ for any even integer $n \geq 4$. We have no idea how to solve it so far. Actually the points and lines of PG(4, 3) form an $S(2, 4, 121)$. Very little is known about the chromatic index of block colorings of an $S(2, 4, v)$ with $v \equiv 1 \pmod{12}$ (see [16, Section 10]).

Algorithms 1 and 2 are refined from Theorem 3.1. Algorithm 1 gives all lines of PG($n, q$). Algorithm 2 counts the chromatic index of PG($n, q$) and outputs the resulting color classes as long as the data in the input area are provided. Take a line coloring of PG(5, 4) for example. Start from PG(5, 4) constructed by Algorithm 1. It follows from Theorem 1.1, Lemma 2.3, and Lemma 3.1 that PG(3, 4) is 21-colorable, PG(3, 4) has Property E and IPG(3, 4; 1) is 21-colorable where 20 colors are used to color all lines incident with the given PG(1, 4). Then apply Algorithm 2 to obtain a parallelism of PG(5, 4), which is 341-colorable.

An interesting question is whether PG($3, q$) has Property E for any prime power $q$. By Theorem 3.1, the solution to this problem will give the existence of a parallelism of PG($n, q$) for any odd integer $n$ and any prime power $q$.

---

**Algorithm 2.** Line coloring of PG($n, q$) with $n \geq 5$.

**Input:** PG($3, q$) has Property E. PG($n - 2, q$) is $c(n - 2, q)$-colorable.

IPG($n - 2, q; n - 4$) is $c(n - 2, q)$-colorable where $q^{n-4}(q + 1)$ colors are used to color all lines incident with the points of the underlying PG($n - 4, q$).

**Step 1:** Use $c(n - 2, q)$ colors to color the lines in $B_1 \cup C$. Split this color set into two disjoint subsets $C_1$ and $C^*$ of sizes $c_1 = q^{n-4}(q + 1)$ and $c_2 = c(n - 2, q) - c_1$, respectively.

**Step $i$: (2 $\leq i \leq q^2 + 1$)** Take a set $C_i$ of $c_i$ colors, which were not previously used, to color all lines of $B_i$ that are incident with $\{0\} \times F_{q^{n-3}}$ such that $C_{j_1} \cap C_{j_2} = \emptyset$ for any $1 \leq j_1 < j_2 \leq q^2 + 1$. Use $c_2$ colors that come from the color set $C^*$ to color all the remaining lines of $B_i$.

**Step $q^2 + 2$:** Let $P = G_0 = \{G_1, G_2, \ldots, G_{q^2+1}\}$ be the special spread of PG($3, q$), and $S = \{P_{G_i}^j : 1 \leq j \leq q + 1, 1 \leq i \leq q^2 + 1\}$ be a set of $(q^2 + 1)(q + 1)$ spreads satisfying the three conditions in Definition 2.1. For each $1 \leq i \leq q^2 + 1$, we split the color set $C_i$ into $q + 1$ disjoint subsets $C_{i,1}, C_{i,2}, \ldots, C_{i,q^2+1}$, each of cardinality $q^{n-4}$, and assign each subset $C_{i,j}$, $1 \leq j \leq q + 1$, to the spread $P_{G_i}^j$. Then for each $L_a, b \in A_0$, the number of colors assigned to $L_{a, b}$ is $q^{n-4} \cdot q = q^{n-3}$. Use the $q^{n-3}$ colors to color the lines in $A_{L_{a, b}}$ (apply Claim 3.1).

**Output:** $A_2 = A_1 \cup C \cup (\bigcup_{i=1}^{q^2+1} B_i)$ is $c(n, q)$-colorable and output the color classes.
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