A note on conserved charges of asymptotically flat and anti-de Sitter spaces in arbitrary dimensions

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Abstract  The calculation of conserved charges of black holes is a rich problem, for which many methods are known. Until recently, there was some controversy on the proper definition of conserved charges in asymptotically anti-de Sitter (AdS) spaces in arbitrary dimensions. This paper provides a systematic and explicit Hamiltonian derivation of the energy and the angular momenta of both asymptotically flat and asymptotically AdS spacetimes in any dimension $D \geq 4$. This requires as a first step a precise determination of the asymptotic conditions of the metric and of its conjugate momentum. These conditions happen to be achieved in ellipsoidal coordinates adapted to the rotating solutions. The asymptotic symmetry algebra is found to be isomorphic either to the Poincaré algebra or to the $\mathfrak{so}(D - 1, 2)$ algebra, as expected. In the asymptotically flat case, the boundary conditions involve a generalization of the parity conditions, introduced by Regge and Teitelboim, which are necessary to make the angular momenta finite. The charges are explicitly computed for Kerr and Kerr–AdS black holes for arbitrary $D$ and they are shown to be in agreement with thermodynamical arguments.

Keywords  Conserved charges · Hamiltonian formulation of general relativity · Higher dimensions
1 Introduction

Energy is a subtle issue in general relativity. Indeed, in accordance with the equivalence principle, the gravitational contribution to energy cannot be localized. Nevertheless, in the presence of asymptotic conditions, the total energy, as well as angular momentum, of the system can be defined. However, even in such a case, some confusion used to exist in the literature: there are many different methods and in some cases they did not all give the same charges. For example, as has already been emphasized [1], not even in four dimensions did all authors obtain the same expression for the energy of Kerr–AdS black holes and some of these expressions were in disagreement with the first law of black hole thermodynamics.

Nowadays, though, number of methods give reliable results. In the paper [1], Gibbons et al. compute the energy of Kerr–AdS black holes first indirectly by integrating the first law and then using the conformal definition of Ashtekar, Magnon and Das [2,3], and they show that both of them agree. Besides the Regge–Teitelboim method [4] adopted here, some of the various other definitions are the following: the approach of Abbott and Deser [5–7], the spinor definition [8,9] based on the electric Weyl tensor, covariant phase space methods [10–13], cohomological techniques [14,15], the KBL approach [16–18], Noether methods [19–24], the “counterterm subtraction method” [25,26] (and more references on this in [12] for example), improved surface integrals [27] and regularisation of the Euclidean action [28,29].

In view of the contradictory results that existed in the literature, our main purpose in this paper is to define the conserved quantities through a method that relates unambiguously charges to symmetries without having to make arbitrary choices¹ and hence stands on a firm footing from this point of view. This method is the Hamiltonian approach introduced by Regge and Teitelboim [4] for asymptotically flat spacetimes in dimension $D = 4$ and Cartesian coordinates. It associates a conserved charge to any asymptotic, i.e. not necessarily exact, Killing vector. The charges are expressed as surface integrals over a 2-sphere at infinity and reproduce, as it should, the ADM energy and angular momentum [30].

This paper’s aim is the generalisation of the method to arbitrary dimensions for both asymptotically flat and AdS spacetimes and the explicit calculation of the corresponding rotating solution. This requires as a first step a precise definition of asymptotically flat or AdS spacetimes as asymptotic conditions on the metric and its conjugate momenta. Let us focus for a while on asymptotically flat spacetimes although the following generalizes to the asymptotically AdS case. In $D$ spacetime dimensions, one expects the metric describing a rotating mass to differ from the flat metric by terms that decay generically as $r^{3-D}$ as this is the asymptotic behavior of the elementary solution of the Poisson equation. In standard polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2d\Omega_{D-2}^2 + h_{\mu\nu}dx^\mu dx^\nu$$

¹ Except for overall additive constants that can be fixed either by background adjustment or using the algebra.