BISMUT’S GRADIENT FORMULA FOR VECTOR BUNDLES

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ABSTRACT. We prove a general Bismut’s formula for the gradient of a class of smooth Wiener functionals over vector bundles of a compact Riemannian manifold. This general formula can be used repeatedly for obtaining probabilistic representation of higher order covariant derivatives of solutions of the heat equation similar to the classical Bismut’s representation for the covariant gradient of the heat kernel.

(preliminary version)

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1. INTRODUCTION

Let \( M \) be a compact Riemannian manifold of dimension \( n \) and \( P_x \) the law of Brownian motion on \( M \) starting from \( x \). The now classical Bismut’s formula ([1]) is a probabilistic representation of the gradient of the heat semigroup

\[
P_t f(x) = \mathbb{E}_x f(X_t) = \int_M p(t, x, y) f(y) \, dy,
\]

where \( p(t, x, y) \) is the transition density function of the Brownian \( X \) on \( M \) (the heat kernel). Several interesting objects are involved in this formula. On the orthonormal frame bundle \( \mathcal{O}(M) \) of \( M \) the scalarized Ricci curvature tensor is realized as an \( O(n) \)-invariant, \( \text{End}(\mathbb{R}^n) \)-valued function \( \text{Ric} : u \mapsto \text{Ric}_u \in \text{End}(\mathbb{R}^n) \). Let \( \{U_i\} \) be a horizontal lift of the Brownian

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motion \( X = \{X_t\} \) to the frame bundle \( \mathcal{O}(M) \). The first object of interest is the multiplicative Feynman-Kac functional \( M = \{M_t\} \) defined by

\[
dM_t + \frac{1}{2} \text{Ric}_{U_t} M_t = 0, \quad M_0 = I.
\]

The second object of interest is the so-called stochastic anti-development \( W = \{W_t\} \) of the Brownian motion \( M \). It is the euclidean Brownian motion which drives the stochastic differential equation for the horizontal Brownian motion \( U \), namely,

\[
dU_t = \sum_{i=1}^n H_i(U_t) \circ dW^i_t,
\]

where \( \{H_i\} \) are the canonical horizontal vector fields on \( \mathcal{O}(M) \). The classical Bismut’s formula is given by

\[
T \nabla E_x f(X_T) = E_x \left[ f(X_T) \int_0^T M_s dW_s \right].
\]

Since \( u(t, x) = E_x f(X_t) \) is the solution of the heat equation on \( M \) with the initial function \( u(0, x) = f(x) \), the above identity gives a probabilistic representation of the covariant gradient of the solution. This representation is equivalent to the following probabilistic representation of the gradient of the heat kernel:

\[
T \nabla_x \log p(t, x, y) = E_{x, y; T} \left[ \int_0^t M_s dW_s \right],
\]

where \( E_{x, y; T} \) is the expectation with respect to the law \( \mathbb{P}_{x, y; T} \) of a Brownian bridge from \( x \) to \( y \) in time \( T \). It can be regarded as a form of integration by parts formula because of the absence of the gradient operator on the right side; see Hsu [2] for a detailed exposition for this formula.

Ever since its appearance 30 years ago, this explicit representation has found applications in many areas. Besides its obvious significance in stochastic analysis on manifolds, it has also played important roles in certain problems in financial mathematics involving computations of option greeks. One naturally considers possible generalizations of this beautiful formula in several directions. Two possible directions come to mind immediately: higher order derivatives and functions in more general spaces. It is an important observation due to Norris [5] that these two possible generalizations can be combined into one framework, in which the function \( f \) is replaced by a smooth map on a vector bundle and the process \( X \) is lifted to the vector bundle by introducing, in addition to a compatible connection on the vector bundle, also a vertical motion driven by \( X \) itself. In this way, the formula for higher derivative can simply be obtained by from the general formula by a judicious choice of the bundle map and the vertical motion.

Bismut [1] derived his formula by a perturbation method in path space, which was very much in line with the techniques available at the time when
Malliavin calculus was studied intensively. This method was followed by
and large by Norris [5] in his generalization of Bismut’s formula in the set-
ting we have just mentioned. The technical difficulties involved in this
method is tremendous. In fact, up to now Norris’ paper has not been fully
digested by the probabilistic community. In view of the new ideas and tech-
nical tools introduced during the intervening decades (mainly gradient-
heat semigroup commutation relations and Itô calculus for diffusion pro-
cesses on manifolds) and importance of these results, it is highly appro-
piate that we revisit these results with the goal of finding a proof more in line
with the current state of art of stochastic analysis on manifolds. In addi-
tion, in our recent investigation of functional inequalities in path and loop
spaces of a Riemannian manifold, we feel the need to extend the framework
adopted in Norris [5] by considering maps between two different vector
bundles instead a self-map on a single vector bundle. Thus the purpose
of this current work is to use Itô calculus to prove a probabilistic represen-
tation of the gradient of the solution of a class of heat equations between
two vector bundles. The basic idea of the current approach is explained in
Hsu [3] for the simplest case of a trivial vector bundle.

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2. NOTATIONS AND THE STATEMENT OF THE PROBLEM

We work on a Riemannian manifold $M$ of dimension $m$. Let $\tau_1 : E_1 \to M$
and $\tau_2 : E_2 \to M$ be two Riemannian vector bundles over $M$ of dimensions
$m_1$ and $m_2$, respectively. Let $x = \{x_t, t \geq 0\}$ be a Riemannian Brownian
motion on $M$ and $W = \{W_t, t \geq 0\}$ its anti-development, which is a stan-
dard Brownian motion in $\mathbb{R}^m$ adapted to the filtration of $x$. The horizontal
lift of $x$, i.e. the horizontal Brownian motion $U = \{U_t, t \geq 0\}$, is defined by

$$\partial U_t = H_i(U_t) \circ \partial W_t^i,$$

where $\{H_i\}$ are canonical horizontal vector fields on $\mathcal{F}(M)$. The (stochas-
tic) parallel transport along the path $x[0, t]$ is $\tau_t = U_0 U_t^{-1}$. 
Following Norris [5], we introduce a diffusion process on $E_1$ driven by $x$ (or equivalently, by $W$) in the following manner,

$$\partial y_t = v_0 \partial t + v_1 \partial W_t,$$

where $v_0, v_1$ are vector fields on $E_1$. This precise interpretation of the equation for $y$ is

$$\partial \tau_t^{-1} y_t = \tau_t^{-1} v_0 \partial t + \tau_t^{-1} v_1 \partial W_t.$$

Consider a smooth map between the two vector bundles $f : E_1 \to E_2$ which respects the fibres. Then $\tau_t^{-1} f(x_t, y_t)$ is an $E_{2,x}$-valued random variable and the expectation $\mathbb{E}_{x,y} \tau_t^{-1} f(x_t, y_t)$ is an element in $E_{2,x}$ which varies smoothly as a function of the initial point $(x_0, y_0) = (x, y)$. The focus of this paper is to give a probabilistic representation of the covariant gradient $\nabla \mathbb{E}_{x,y} \tau_t^{-1} f(x_t, y_t)$ (with respect to the $x$ variable).

For a section of a vector bundle, it will be much more convenient for our exposition if we deal with its scalarization, which is a euclidean space valued function on the frame bundle of that vector bundle. We introduce the frame bundle $\mathcal{F}(M) = \cup_{x \in M} U_x$, where

$$U = (U_0, U_1, U_2) = \begin{cases} U_0 : \mathbb{R}^m \to T_x M \\ U_1 : \mathbb{R}^{n_1} \to E_{1,x} \\ U_2 : \mathbb{R}^{n_2} \to E_{2,x} \end{cases}$$

Each $U_i$ is an isometry between its respective domain and target spaces. The scalarization of the bundle function $f : E_1 \to E_2$ is defined by

$$F : \mathcal{F}(M) \times \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$$

$$F(U, Y) := U_2^{-1} f(x, U_1 Y).$$

Let $Y = \{Y_t, t \geq 0\}$ be the scalarization of the process $y$. Then it is the solution of the stochastic differential equation

$$\partial Y_t = V_0(U_t, Y_t) \partial t + \sum_i V_{1,i}(U_t, Y_t) \circ \partial W^i_t,$$

where $V_0(U_t, Y_t) := U_{1,j}^{-1} v_0(x_t, U_{1,i} Y_t)$ and $V_{1,i}(U_t, Y_t) := U_{1,j}^{-1} v_{1,i}(x_t, U_{1,i} Y_t)$ are the scalarization of $v_0(x_t, y_t)$ and $v_{1,i}(x_t, y_t)$, respectively.

For ease of notation, we combine the two processes $U$ and $Y$ into a single process $Z_t = (U_t, Y_t)$. It is clear that

$$\partial Z_t = (0, V_0) \partial t + \sum_i (H_i, V_{1,i}) \circ \partial W^i_t.$$

To further simplify the notation, we will write $(H_i, 0)$, $(0, V_0)$, and $(0, V_{1,i})$ simply as $H_i$, $V_0$, and $V_{1,i}$, respectively. Whenever there is not possibility of confusion. Then the generator of the diffusion process $Z_t$ is

$$L = \frac{1}{2} \sum_i (H_i + V_{1,i})^2 + V_0.$$
And \( F(z, t) = \mathbb{E}_z\{F(Z_t)\} \) is the solution to equation

\[
\frac{\partial F}{\partial t} = LF.
\]

The goal of this paper is to find a probabilistic representation of the gradient of the solution to this heat equation over the vector bundles. Our main object of interest \( \nabla \mathbb{E}_{x,y}^{\tau_T^{-1}} f(x_T, y_T) \), becomes \( \nabla^H F(z, T) \). Here \( \nabla^H \) is the horizontal gradient defined by

\[
\nabla^H = \sum_j (H_j F) e_j,
\]

where \( \{e_j\}_{j=1}^{n_1} \) is the canonical orthonormal basis in \( \mathbb{R}^{n_1} \).

3. Commutation of the Gradient and the Heat Semigroup

The first step towards establishing the probabilistic representation of the gradient of a solution to the heat equation is to commute the gradient operator with the heat semigroup. In this way we pass the gradient operator under the expectation on the bundle map. This will be followed by an integration by parts argument to remove the gradient operator from the bundle map.

Consider the equation for \( \nabla^H F \)

\[
\frac{\partial \nabla^H F}{\partial t} = \nabla^H LF
= L \nabla^H F + [\nabla^H, L] F
\]

We have

\[
L = \frac{1}{2} \sum_i H_i^2 + \frac{1}{2} \sum_i V_i^2 + \frac{1}{2} \sum_i (H_i V_{1,i} + V_{1,i} H_i) + V_0.
\]

By further expand the commutator bracket, we obtain

\[
\frac{\partial \nabla^H F}{\partial t} = L \nabla^H F - \frac{1}{2} \text{Ric} \nabla^H F + (1) + (2) + (3) + (4) + (5)
\]

where

\[
(1) = \sum_j \left( \sum_i -\Omega_{ji}^{(2)} (H_i F) + \sum_i (DH_i F) (\Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
+ \frac{1}{2} \sum_j \left( \sum_i - (H_i \Omega_{ji}^{(2)}) F + \sum_i (DF) (H_i \Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
(2) = \sum_j \left( \sum_i -\Omega_{ji}^{(2)} (V_i F) + \sum_i (DV_i F) (\Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
+ \frac{1}{2} \sum_j \left( \sum_i - (V_i \Omega_{ji}^{(2)}) F + \sum_i (DF) (V_i \Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
(3) = \sum_j \left( \sum_i -\Omega_{ji}^{(2)} (H_i V_j) F + \sum_i (DH_i V_j) (\Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
+ \frac{1}{2} \sum_j \left( \sum_i - (H_i \Omega_{ji}^{(2)}) (H_i V_j) F + \sum_i (DH_i V_j) (H_i \Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
(4) = \sum_j \left( \sum_i -\Omega_{ji}^{(2)} (V_i F) + \sum_i (DV_i F) (\Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
+ \frac{1}{2} \sum_j \left( \sum_i - (V_i \Omega_{ji}^{(2)}) F + \sum_i (DF) (V_i \Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
(5) = \sum_j \left( \sum_i -\Omega_{ji}^{(2)} (H_i V_j) F + \sum_i (DH_i V_j) (\Omega_{ji}^{(1)} Y) \right) e_j
\]

\[
+ \frac{1}{2} \sum_j \left( \sum_i - (H_i \Omega_{ji}^{(2)}) (H_i V_j) F + \sum_i (DH_i V_j) (H_i \Omega_{ji}^{(1)} Y) \right) e_j
\]
\begin{align}
(2) \quad & \sum_j \left( \sum_i (D^2 F)(H_j V_{1,i})(V_{1,i}) \right) e_j \\
& + \frac{1}{2} \sum_j DF \left( \sum_i (DH_j V_{1,i})(V_{1,i}) + (DV_{1,i})(H_j V_{1,i}) \right) e_j \\
(3) \quad & \sum_j \left( \sum_i (DH_j F)(H_j V_{1,i}) \right) e_j + \frac{1}{2} \sum_j DF \left( \sum_i H_i H_j V_{1,i} \right) e_j \\
(4) \quad & \sum_j \left( \sum_i (D^2 F)(V_{1,i})(\Omega_{ji}^{(1)} Y) \right) e_j \\
& + \sum_j \left( \sum_i -\Omega_{ji}^{(2)}(DF)(V_{1,i}) \right) e_j \\
& + \frac{1}{2} \sum_j DF \left( \sum_i (DV_{1,i})(\Omega_{ji}^{(1)} Y) \right) e_j \\
(5) \quad & \sum_j (DF)(H_j V_0) e_j
\end{align}

In the above expansion, $-\frac{1}{4} \text{Ric} \nabla^H F + (1)$ comes from $[\nabla^H, \sum H_i^2]$, (2) comes from $[\nabla^H, \sum V_i^2]$, then (3) + (4) comes from $[\nabla^H, \sum H_i V_{1,i} + V_{1,i} H_i]$ and (5) comes from $[\nabla^H, V_0]$. We also group (3) + (4) so that all terms in their corresponding commutator with curvature go to (4) and the remaining terms go to (3). The reason for doing so is to emphasize the effect of the curvature of the bundle $E_1$ and $E_2$.

For clarification, we say a few words about the notations used here concerning the curvature tensors (see Kobayashi and Nomizu [4]). Let $H_i$ be the canonical horizontal vector fields of the product bundle $TM \oplus E_1 \oplus E_2$. Then the commutator $\Omega_{ij} = [H_j, H_i]$ is a vertical vector field on the same product bundle with three components, i.e.,

$$\Omega_{ij} = (\Omega_{ji}^{(0)}, \Omega_{ji}^{(1)}, \Omega_{ji}^{(2)}) \in \mathfrak{o}(m) \times \mathfrak{o}(n_1) \times \mathfrak{o}(n_2),$$

where $\Omega_{ji}^{(l)}$ are the (scalarized) curvature tensors for $l = 0$ (on the manifold $M$), 1 (for the vector bundle $E_1$), and 2 (for the vector bundle $E_2$).

In the above computation, we have also adopted the following conventions. If $Q$ is a vector in $\mathbb{R}^{n_1}$, the notation $DQ$ denotes the row vector $(D_i Q)_{i=1}^{n_1}$, if $Q$ has also other coordinates, say it is a vector in $\mathbb{R}^m \times \mathbb{R}^{n_1}$, then $DQ$ will just be 0 in $\mathbb{R}^m$ and $(D_i Q)_{i=1}^{n_1}$ in $\mathbb{R}^{n_1}$. We use $(D^2 Q)(\alpha)(\beta)$, where $Q$ is a map from $\mathbb{R}^{n_1}$ and $\alpha, \beta$ are vectors in $\mathbb{R}^{n_1}$, to denote $\sum_{i,j} (D_i D_j Q)\alpha_i \beta_j$. 
We now want to apply Feynman-Kac technique to deal with \( \text{Ric} \nabla^H F \) as in Hsu [2]. Let \( M_t \) be the solution of the matrix-valued differential equation
\[
\partial M_t + \frac{1}{2} M_t \text{Ric}_{U_t} \partial t = 0, M_0 = I.
\]
Using Itô’s formula, it is straightforward to verify that
\[
\partial M_t \nabla^H F(Z_t, T - t) + M_t \Phi(Z_t, T - t) \partial t
\]
is a martingale, where
\[
\Phi = (1) + (2) + (3) + (4) + (5).
\]
Taking the expectation with respect to \( \mathbb{E}_z \), we have
\[
\nabla^H \mathbb{E}_z F(Z_T) = \mathbb{E}_z M_t \nabla^H F(Z_t, T - t)
+ \mathbb{E}_z \int_0^T M_s \Phi(Z_s, T - s) \partial s.
\]
Integrating over \( t \) from 0 to \( T \) and using integration by parts on the second term on the right side, we have
\[
\begin{align*}
T \nabla^H \mathbb{E}_z F(Z_T) &= \mathbb{E}_z \int_0^T M_t \nabla^H F(Z_t, T - t) \partial t \\
&+ \mathbb{E}_z \int_0^T N_t \Phi(Z_t, T - t) \partial t.
\end{align*}
\]
Here \( N_t = (T - t)M_t \). With this last identity, we have accomplished the task we have set ourselves at the beginning of this section, namely, we have passed the gradient operator through the expectation to act on the bundle map \( f \), here represented by its scalarization \( F \). As we pointed out above, the next step is to remove the expectation by an integration by parts argument in the path space.

4. INTEGRATION BY PARTS

The first term on the right side of (1) contains the gradient on \( F \). From the heat equation \( \frac{\partial F}{\partial t} = LF \) we have by Itô’s formula,
\[
\partial F(Z_t, T - t) = \sum_i (H_i + V_{1,i}) F \partial W_t^i.
\]
For any $v \in \mathbb{R}^m$ and $r \in \{1, 2, \ldots, n_2\}$, we have
\[
E_z \int_0^T \langle M_t \nabla^H F_r, v \rangle dt \\
= E_z \int_0^T \langle \nabla^H F_r, M_t^* v \rangle dt \\
= E_z \int_0^T \langle \nabla^H F_r, \partial W_i \rangle \int_0^T \langle M_t^* v, \partial W_i \rangle \\
= E_z \left( F_r(Z_T) - E_z F_r(Z_T) - \int_0^T \sum_i V_{1,i} F_r \partial W_i \right) \int_0^T \langle v, M_t \partial W_i \rangle \\
= E_z \left( F_r(Z_T) \int_0^T \langle v, M_t \partial W_i \rangle \right) - E_z \int_0^T \langle \sum_i V_{1,i} F_r e_i, M_t^* v \rangle dt \\
= E_z \left( F_r(Z_T) \int_0^T \langle v, M_t \partial W_i \rangle \right) - E_z \int_0^T \langle M_t \sum_i V_{1,i} F_r e_i, v \rangle dt
\]

This can be written more compactly as
\[
E_z \int_0^T M_t \nabla^H F dt \\
= E_z \left( F(Z_T) \int_0^T M_t \partial W_i \right) - E_z \int_0^T M_t \sum_i V_{1,i} F_r e_i \partial t.
\]
The left side above is just the first term on the right side of (1). Note that we have removed the gradient operator from $F$. The remaining differentiation on $F$ are in the fibre directions.

The dealings with the remaining terms in (1) and the second term on the right side of (2)
\[
E_z \int_0^T N_t \Phi(Z_t, T_t) - M_t \sum_j (DF)(V_{1,j}) e_j dt
\]
are admittedly technical, but in the process we will single out the terms involving some vertical derivatives on which we can perform a second commutation operation.

We will adopt the following notations: $\bar{N}_i^j$ stands for the horizontal lift of $N_i e_j$ and $K^{(l)}_{i,j}$ for the 2nd and 3rd coordinates of $[\bar{N}_i^j, H]$. Note that $N_i$ is symmetric, therefore in explicit components we have
\[
N_t e_j = \sum_k N_{t,i}^k e_k, \quad \bar{N}_i^j = \sum_k N_{i,j}^k H_k, \quad \text{and} \quad K^{(l)}_{i,j} = \sum_k \Omega^{(l)}_{i,k} N_{i,k}^j.
\]
We Recall that $\Phi = (1) + (2) + (3) + (4) + (5)$. We have
\[
N_t \Phi(Z_t, T - t) - M_t \sum_j (DF)(V_{1,j}) e_j = -A_1 + A_2 + A_3
\]
where

\[ A_1 = \sum_j \sum_i \left( K_{ij}^{(2)} (H_i F) + \frac{1}{2} (H_i K_{ij}^{(2)}) F + K_{ij}^{(2)} (DF) (V_{1,i}) \right) e_j \]

\[ A_2 = \sum_j \sum_i \left( (DH_i F + (D^2 F) (V_{1,i})) \left( \bar{N}_i^j V_{1,i} + K_{ij}^{(1)} Y_t \right) \right) e_j \]

\[ A_3 = \sum_j DF \left( B_{t,j} + (\bar{N}_i^j V_0) - \sum_i V_{1,i} M_{t,i} \right) e_j \]

\[ B_{t,j} = \sum_i \frac{1}{2} \left( (H_i K_{ij}^{(1)}) Y_t + (D \bar{N}_i^j V_{1,i}) (V_{1,i}) \right. \]

\[ \left. + (DV_{1,i}) (\bar{N}_i^j V_{1,i}) + (DV_{1,i}) (K_{ij}^{(1)} Y_t) + H_i \bar{N}_i^j V_{1,i} \right) \]

Here as will be in the sequel, we have omitted the ubiquitous \((Z_t, T - t)\) for the ease of notation. The rule for regrouping the terms here may appear to be mysterious. Basically everything related to the curvature on \(E_2\) is included in \(A_1\), and the remaining terms follow the rule that the second order terms of \(F\) go to \(A_2\), and the first order terms go to \(A_3\).

We consider the term \(A_1\). From

\[ \partial F = \sum_i (H_i F + (DF) (V_{1,i})) \partial W_i^j \]

we have

\[ A_1 \partial t = \sum_j \left( \sum_i \left( K_{ij}^{(2)} (H_i F) + (DF) (V_{1,i}) \right) \right) \partial e_j \]

\[ + \frac{1}{2} \sum_j \left( \sum_i (H_i K_{ij}^{(2)}) F \right) \partial e_j \]

\[ = \sum_j \left( \sum_i K_{ij}^{(2)} \partial W_i^j \right) \cdot (\partial F) e_j \]

\[ + \frac{1}{2} \sum_j \left( \sum_i (\partial K_{ij}^{(2)}) \cdot (\partial W_i^j) \right) Fe_j \]

\[ = \sum_j \left( \partial G_i^{(2)} \cdot \partial F \right) e_j + \sum_j (\partial G_i^{(2)}) Fe_j + \partial S_t \]

\[ = \sum_j \partial (G_i^{(2)} F)e_j + \partial S_t^j \]
Here \( G^{i(l)}_t \) are matrix valued process defined by by
\[
G^{i(l)}_t = \int_0^t \sum_i K_{t,i}^{i(l)} \, \partial W_i^t,
\]
and \( S_t \) and \( S_t' \) are martingales. Integrating and taking the expectation we have
\[
E_z \int_0^T A_1 \, \partial t = E_z \sum_j \left( G^{i(2)}_T F(Z_T) \right) e_j.\]

**Remark 4.1.** As we mentioned before this paragraph of calculation, the curvature on \( E_2 \) gives rise to a separate term in the formula, and this term turns out to only rely on the 0th order term of the terminal value of \( F \).

5. SECOND COMMUTATION AND DÉNOUEMENT

Before starting the calculations of \( A_2, A_3 \), it is perhaps helpful to point out that if the vector bundle \( E_2 \) flat, then \( A_1 \) vanishes, but \( A_2 \) and \( A_3 \) remain. We therefore anticipate terms involving the derivative \( DF \). Every term in \( A_2 \) and \( A_3 \) includes the 0th or 1st order term of \( DF \). Following Hsu [3], we carry out a second commutation computation. From the heat equation for \( F \) we have
\[
\frac{\partial DF}{\partial t} = LDF + [D, L]F
\]
where
\[
A_4 = \sum_i \left( (D^2F)(V_{1,i}) + DH_iF \right) (DV_{1,i}),
\]
\[
A_5 = DF \sum_i \left( \frac{1}{2} \left( (D^2V_{1,i})(V_{1,i}) + (DV_{1,i})(DV_{1,i}) + DH_iV_{1,i} + DV_0 \right) \right).
\]
Here \( (D^2P)Q \) stands for the row vector \( \sum_i P_{ij}Q_i \). As a consequence we have
\[
\partial(DF)(Z_t, T - t) = \sum_i \left( (D^2F)(V_{1,i}) + DH_iF \right) \partial W_i^t - A_4 \partial t - A_5 \partial t.
\]
We now use a method similar to the one used in dealing with \( A_1 \) to absorb \( A_4 \) and \( A_5 \).

for this purpose we introduce the processes \( \mathbb{R}^{n_1} \)-valued processes \( Y^i_0 = 0 \) called the derived processes in Norris [5]. They are determined by the stochastic differential equation
\[
\partial Y^i_t = C^i_{2,t} \partial W_t^i + C^i_{3,t} \partial t + C^i_{4,t} \partial W_t^i + C^i_{5,t} \partial t,
\]
where
\[
C^i_{2,t} = \bar{N}^i_t V_{1,i} + K_{t,i}^{(1)} F_t.
\]
Theorem 5.1. We have

\[ TV^H_z \mathbb{E}_z(F(Z_T)) = \mathbb{E}_z \left[ F(Z_T) \int_0^T M_t \partial W_t - G_{T} F(Z_T) + DF(Z_T) Y_T \right]. \]

Remark 5.2. In the setting of Norris [5], his representation contains one extra term not present in our work. We have not worked through his work diligently to locate the source of this discrepancy.

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