Sharp subcritical and critical $L^p$ Hardy inequalities on the sphere

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Abstract
We obtain sharp inequalities of Hardy type for functions in the Sobolev space $W^{1,p}$ on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. We achieve this in both the subcritical and critical cases. The method we use to show optimality takes into account all the constants involved in our inequalities. We apply our results to obtain lower bounds for the the first eigenvalue of the $p$-Laplacian on the sphere.

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1 Introduction
Hardy inequalities have been studied extensively on various types of manifolds (see for example [3, 4, 8, 11], the references therein, and the recent cited papers that are too many to mention). To our best knowledge, Xiao [15] was the first to look at Hardy inequalities in the particular case of the Euclidean sphere. He developed the idea used in [9] to obtain sharp inequalities of Hardy and Rellich type on Riemannian manifolds. The Laplacian of the geodesic distance on the sphere changes sign (see formula (3.13) of Lemma 3.4 below). This makes known results on compact manifolds not easy to apply directly. Xiao [15] obtained $L^2$ inequalities of the Hardy type on the sphere $S^n$, $n \geq 3$. These results were complemented in [1] in the limiting case where optimal $L^2$ inequalities of the Hardy type were proved on $S^2$. The results in [15] were also extended in [12] to $L^p(S^n)$, $1 < p < n$, $n \geq 3$.

In [1, 12, 15], the singularity is assumed to be at either the north or south pole so that the geodesic distance will be simply the polar angle. Hence, if the singularity is not polar, we must rotate the local axes in order to apply these inequalities. However, we should not need to rotate the axes. It is not physically plausible as we could be dealing with a punctured sphere missing a closed connected piece, or a sphere with a crack missing an open simple curve. This motivates us to look for $L^p$ Hardy inequalities in which the singularity is the geodesic distance from an arbitrary point.

The general geodesic distance was very recently considered in [2, 16]. The proofs in [2, 16] are based on a formula for the Laplacian of the geodesic distance. No reference was provided for that formula, and no proof of it was given either. We also noted that...
the definition of the geodesic distance on $\mathbb{S}^n$ adopted in [2, 16] is not specified. Such a definition is important to understand the set up of the inequalities. This is also technically important since the singularities in the inequalities involve trigonometric functions. That in turn necessitates determining whether the range of the geodesic distance is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ or $[0, \pi]$.

The results in [2] are supposed to generalize the $L^2$ Hardy inequality presented in [16] to an $L^p$ inequality on $\mathbb{S}^n$ where $1 < p < n$ and $n \geq 3$. We revisit the proof presented in [2] in [17], where we additionally prove the limiting case $L^n$ Hardy-type inequalities on the sphere $\mathbb{S}^n$, $n \geq 2$, with optimal coefficients, considering the general geodesic distance and adopting Xiao’s method.

When it comes to the sharpness of the coefficients, all the results in [1, 2, 12, 15, 16] are based on the same principle that we find insufficient. The method implemented is also unnecessarily involved at times. Inequalities of Hardy type obtained in [1, 2, 12, 15, 16] on $\mathbb{S}^n$ take the generic form

$$An_p \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^p} \, d\sigma_n \leq B_{n,p} \int_{\mathbb{S}^n} |\nabla u|^p \, d\sigma_n + C_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^{p-2}} \, d\sigma_n, \quad 2 \leq p \leq n,$$

where $u \in C^\infty(\mathbb{S}^n)$, and $f$ is a continuous function of the geodesic distance $\rho$. Sharpness of the constants $A_{n,p}$, $B_{n,p}$ and $C_{n,p}$ is claimed to be proved by showing that

$$\sup_{u \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{A_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^p} \, d\sigma_n}{B_{n,p} \int_{\mathbb{S}^n} |\nabla u|^p \, d\sigma_n + C_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^{p-2}} \, d\sigma_n} = 1. \quad (1.1)$$

However, the latter does not prove that the constants $B_{n,p}$ and $C_{n,p}$ are both the smallest possible.

We prove sharp $L^p$ Hardy inequalities on the sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ in both the subcritical and critical exponent cases. We follow a method of proof different from that used in [1, 2, 12, 15, 16]. The method we adopt is fairly simple and requires fewer computations. Before delving into the derivation of the inequalities, we use explicit formulas for the geodesic distance, the surface gradient and the Laplace–Beltrami operator on the $n$-dimensional sphere to demonstrate some basic properties of the geodesic distance on which we rely heavily in obtaining our results.

In addition to proving (1.1), we show the optimality of all the constants in our inequalities by proving that

$$\sup_{u \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{A_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^p} \, d\sigma_n - C_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^{p-2}} \, d\sigma_n}{B_{n,p} \int_{\mathbb{S}^n} |\nabla u|^p \, d\sigma_n} = 1,$$

$$\sup_{u \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{A_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^p} \, d\sigma_n - B_{n,p} \int_{\mathbb{S}^n} |\nabla u|^p \, d\sigma_n}{C_{n,p} \int_{\mathbb{S}^n} \frac{|u|^p}{f(\rho)^{p-2}} \, d\sigma_n} = 1.$$
2 Preliminaries

Let $n \geq 2$ and let $\theta_j \in [0, \pi], j = 1, \ldots, n - 2$, and $\theta_{n-1} \in [0, 2\pi]$. Denote $(\theta_1, \ldots, \theta_{n-1})$ by $\Theta_{n-1}$. Any point on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ has the spherical coordinates parametrization $(x_1(\Theta_{n-1}), \ldots, x_n(\Theta_{n-1}))$, where

$$x_m(\Theta_{n-1}) := \begin{cases} 
\cos \theta_1, & m = 1; \\
\prod_{j=1}^{m-1} \sin \theta_j \cos \theta_m, & 2 \leq m \leq n - 1; \\
\prod_{j=1}^{n-1} \sin \theta_j, & m = n. 
\end{cases} \tag{2.1}$$

The surface gradient $\nabla_{S^{n-1}}$ on the sphere $S^{n-1}$ is then given by

$$\nabla_{S^{n-1}} = \frac{\partial}{\partial \theta_1} \hat{\theta}_1 + \frac{1}{\sin \theta_1} \frac{\partial}{\partial \theta_2} \hat{\theta}_2 + \cdots + \frac{1}{\sin \theta_1 \cdots \sin \theta_{n-2}} \frac{\partial}{\partial \theta_{n-1}} \hat{\theta}_{n-1},$$

where $\{ \hat{\theta}_j \}$ is an orthonormal set of tangential vectors with $\hat{\theta}_j$ pointing in the direction of increasing $\theta_j$. Moreover, the Laplace–Beltrami operator $\Delta_{S^{n-1}}$ is given by

$$\Delta_{S^{n-1}} = \frac{1}{\sin^{n-2} \theta_1} \frac{\partial}{\partial \theta_1} \left( \sin^{n-2} \theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{1}{\sin^2 \theta_1 \sin^{n-3} \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin^{n-3} \theta_2 \frac{\partial}{\partial \theta_2} \right) + \cdots \tag{2.2}$$

Identifying each point $(x_m(\Theta_{n-1}))_{m=1}^n \in S^{n-1}$ with its parameters $\Theta_{n-1}$, we can express the geodesic distance $d(\Theta_{n-1}, \Phi_{n-1})$ from a point $\Phi_{n-1} \in S^{n-1}$ as

$$d(\Theta_{n-1}, \Phi_{n-1}) = \arccos \lambda(\Theta_{n-1}, \Phi_{n-1}), \tag{2.3}$$

where

$$\lambda(\Theta_{n-1}, \Phi_{n-1}) := \sum_{m=1}^n x_m(\Theta_{n-1}) x_m(\Phi_{n-1}). \tag{2.4}$$

2.1 A useful formula for integration over the sphere

Let $v \in \mathbb{R}^n \setminus \{0\}$ and let $F \in L^1(S^{n-1} \rightarrow \mathbb{R})$ be such that $F(\Theta_{n-1}) := f(v \cdot \Theta_{n-1})$. Then,

$$\int_{S^{n-1}} F(\Theta_{n-1}) \, d\sigma_{n-1} = C_n \int_1^1 f(|v| t) \left( \sqrt{1 - t^2} \right)^{n-3} \, dt, \tag{2.5}$$

where $C_n = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$. (See [7], Appendix D).
2.2 The Sobolev space $W^{1,p}(S^{n-1})$

It is useful to define the weak Laplace–Beltrami gradient of a function $f \in L^1(S^{n-1})$. Let $f \in C^\infty(S^{n-1} \to \mathbb{R})$. Then, by the divergence theorem, we have

$$\int_{S^{n-1}} \nabla f \cdot V \, d\sigma = -\int_{S^{n-1}} f \nabla \cdot V \, d\sigma$$

for any vector field $V \in C^\infty(S^{n-1} \to T(S^{n-1}))$, where $T(S^{n-1})$ is the tangent bundle on the smooth manifold $S^{n-1}$. Therefore, $f$ is weakly differentiable if there exists a vector field $\Psi_f \in L^1(S^{n-1} \to T(S^{n-1}))$ such that

$$\int_{S^{n-1}} \Psi_f \cdot V \, d\sigma_{n-1} = -\int_{S^{n-1}} f \nabla \cdot V \, d\sigma_{n-1}, \quad \forall V \in C^\infty(S^{n-1} \to T(S^{n-1})).$$

Such a vector field $\Psi_f$, if it exists, is called the weak surface gradient of $f$. The weak surface gradient is unique up to a set of measure zero. As shown in ([5], Proposition 3.2, page 15)

$$W^{1,p}(S^{n-1}) := \left\{ f \in L^p(S^{n-1}) : |\Psi_f| \in L^p(S^{n-1}) \right\}. \quad (2.6)$$

The definition (2.6) is equivalent to defining $W^{1,p}(S^{n-1})$ as the completion of the space $C^\infty(S^{n-1})$ in the usual Sobolev norm.

In the next section, we show some interesting properties of the geodesic distance on the sphere that carry on to all dimensions.

3 The gradient and Laplacian of the geodesic distance on the sphere

The geodesic distance $d$ on the sphere $S^{n-1}$ has a gradient and Laplacian analogous to those of the Euclidean metric. We demonstrate that $|\nabla_{S^{n-1}} d|_{S^{n-1}} = 1$ and that $\Delta_{S^{n-1}} d = (n-2) \cos d / \sin d$, in any dimension $n \geq 2$. Unlike with the Euclidean distance, the Laplacian of the geodesic distance $d$ changes sign on the sphere. We start with showing that

$$x_j(\Theta_{n-1}) x_k(\Theta_{n-1}) + \nabla_{S^{n-1}} x_j(\Theta_{n-1}) \cdot \nabla_{S^{n-1}} x_k(\Theta_{n-1}) = \delta_{jk},$$

the Kronecker delta.

**Lemma 3.1** Let $n \geq 2$ and let $\nabla_{S^{n-1}}$ be the gradient on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. Then,

$$x_m^2(\Theta_{n-1}) + \left| \nabla_{S^{n-1}} x_m(\Theta_{n-1}) \right|^2 = 1, \quad (3.1)$$

$$x_{\ell}(\Theta_{n-1}) x_m(\Theta_{n-1}) + \nabla_{S^{n-1}} x_{\ell}(\Theta_{n-1}) \cdot \nabla_{S^{n-1}} x_m(\Theta_{n-1}) = 0, \quad \ell \neq m. \quad (3.2)$$

**Proof** Lemma 3.1 is trivial in the dimension $n = 2$ and similarly easily verifiable when $n = 3$ by the computation

$$\nabla_{S^2} x_m(\Theta_2) = \begin{cases} -\sin \theta \widehat{\theta}_1, & m = 1; \\ \cos \theta \cos \theta_2 \widehat{\theta}_1 - \sin \theta_2 \widehat{\theta}_2, & m = 2; \\ \cos \theta \sin \theta_2 \widehat{\theta}_1 + \cos \theta_2 \widehat{\theta}_2, & m = 3. \\ \end{cases}$$
Suppose \( n \geq 4 \). Again, the identity (3.1) is easy to prove when \( m = 1, 2 \), and so is the identity (3.2) when \( 1 \leq \ell, m \leq 2 \). Observe that, for all \( n \geq 4 \),

\[
x_m(\Theta_{n-1}) = x_m(\Theta_2), \quad \nabla_{\Theta_{n-1}} x_m(\Theta_{n-1}) = \nabla_{\Theta_2} x_m(\Theta_2), \quad m = 1, 2.
\]

Fix \( m \geq 3 \). We obtain (3.1) from the calculation

\[
\nabla_{\Theta_{n-1}} x_m(\Theta_{n-1}) = \begin{cases} 
\sum_{j=1}^{m-2} \cos \theta_j \prod_{k=j+1}^{m-1} \sin \theta_k \cos \theta_m \widehat{\theta}_j + \cos \theta_{m-1} \cos \theta_m \widehat{\theta}_{m-1} - \sin \theta_m \widehat{\theta}_m, & 3 \leq m \leq n-1; \\
\sum_{j=1}^{n-2} \cos \theta_j \prod_{k=j+1}^{n-1} \sin \theta_k \widehat{\theta}_j + \cos \theta_{n-1} \widehat{\theta}_{n-1}, & m = n,
\end{cases}
\]

and the orthonormality of the set \( \{\widehat{\theta}_j\}_{j=1}^{n-1} \) along with the identity

\[
U(\Theta_{m-1}) := \sum_{j=1}^{m} x_j^2(\Theta_{m-1}, \ldots, \Theta_1) = 1.
\]

Indeed, one can write

\[
x_m^2(\Theta_{n-1}) + |\nabla_{\Theta_{n-1}} x_m(\Theta_{n-1})|^2 = \begin{cases} 
U(\Theta_{m-1}) \cos^2 \theta_m + \sin^2 \theta_m, & m \leq n-1; \\
U(\Theta_{n-1}), & m = n.
\end{cases}
\]

Now, we turn to the identity (3.2). Assume, losing no generality, that \( 1 \leq \ell < m \). Then, tedious yet straightforward computation uncovers that

\[
- \nabla_{\Theta_{n-1}} x_1(\Theta_{n-1}) \cdot \nabla_{\Theta_{n-1}} x_m(\Theta_{n-1}) = \begin{cases} 
\cos \theta_1 \prod_{j=1}^{m-1} \sin \theta_j \cos \theta_m, & 3 \leq m \leq n-1; \\
\cos \theta_1 \prod_{j=1}^{n-1} \sin \theta_j, & m = n,
\end{cases}
\]

and when \( 2 \leq \ell \leq m - 1 \) we have

\[
- \nabla_{\Theta_{n-1}} x_\ell(\Theta_{n-1}) \cdot \nabla_{\Theta_{n-1}} x_m(\Theta_{n-1}) = \begin{cases} 
\prod_{j=1}^{\ell-2} \sin^2 \theta_j \prod_{j=\ell+1}^{m-1} \sin \theta_k - \sin^2 \theta_{\ell+1}, & \ell \leq m \leq n-1; \\
\prod_{j=1}^{m-1} \sin \theta_j \cos \theta_m, & m \leq n-1; \\
\prod_{j=1}^{n-1} \sin \theta_j, & m = n
\end{cases}
\]

\[
= \prod_{j=1}^{\ell-1} \sin^2 \theta_j \cos \theta_\ell \begin{cases} 
\prod_{j=1}^{m-1} \sin \theta_j \cos \theta_m, & m \leq n-1; \\
\prod_{j=1}^{n-1} \sin \theta_j, & m = n
\end{cases}
\]

\[
= -x_\ell(\Theta_{n-1}) x_m(\Theta_{n-1}). \quad \square
\]
The next lemma shows that the components $x_m$ are eigenfunctions of the Laplace–Beltrami operator \((2.2)\).

**Lemma 3.2**

\[
\Delta_{g^{n-1}} x_m(\Theta_{n-1}) = -(n-1)x_m(\Theta_{n-1}), \quad n \geq 2. 
\] \tag{3.5}

**Proof** Write

\[
\Delta_{g^{n-1}} = \sum_{\ell=1}^{n-1} \Delta_{\ell},
\]

where $\Delta_{\ell}$ are the differential operators

\[
\Delta_1 := \frac{1}{\sin^{n-2} \theta_1} \frac{\partial}{\partial \theta_1} \left( \sin^{n-2} \theta_1 \frac{\partial}{\partial \theta_1} \right),
\]

\[
\Delta_{\ell} := \frac{1}{\prod_{j=1}^{\ell-1} \sin^2 \theta_j \sin^{n-\ell-1} \theta_\ell} \frac{\partial}{\partial \theta_\ell} \left( \sin^{n-\ell-1} \theta_\ell \frac{\partial}{\partial \theta_\ell} \right), \quad 2 \leq \ell \leq n-1.
\]

Then, to prove (3.5), it suffices to establish that

\[
\sum_{\ell=1}^{m} \Delta_{\ell} x_m(\Theta_{n-1}) = -(n-1)x_m(\Theta_{n-1}), \quad 1 \leq m \leq n-1, \tag{3.6}
\]

\[
\sum_{\ell=1}^{n-1} \Delta_{\ell} x_n(\Theta_{n-1}) = -(n-1)x_n(\Theta_{n-1}). \tag{3.7}
\]

Straightforward calculations affirm (3.6) when $m = 1$. We prove (3.6) by induction. Assume (3.6) holds true for some $1 \leq m \leq n-2$. Let us define

\[
\delta_m(\Theta_{n-1}) := \frac{x_{m+1}(\Theta_{n-1})}{x_m(\Theta_{n-1})} = \frac{\sin \theta_m \cos \theta_{m+1}}{\cos \theta_m}.
\]

Now, since

\[
\sum_{\ell=1}^{m-1} \Delta_{\ell} x_{m+1} = \delta_m \sum_{\ell=1}^{m-1} \Delta_{\ell} x_m \\
= \delta_m \sum_{\ell=1}^{m} \Delta_{\ell} x_m - \delta_m \Delta_m x_m \\
= -(n-1)x_{m+1} - \delta_m \Delta_m x_m,
\]

then we have

\[
\sum_{\ell=1}^{m+1} \Delta_{\ell} x_{m+1} = -(n-1)x_{m+1} - \delta_m \Delta_m x_m + (\Delta_m + \Delta_{m+1}) x_{m+1}.
\]

Consequently, what remains to prove is

\[-\delta_m \Delta_m x_m + (\Delta_m + \Delta_{m+1}) x_{m+1} = 0. \tag{3.8}\]
Calculating further, we find
\[
\Delta_m x_{m+1} = \delta_m \Delta_m x_m + x_m \Delta_m \delta_m + \frac{2}{\prod_{j=1}^{m-1} \sin^2 \theta_j \partial \theta_m \partial \theta_m},
\]
\[
\Delta_{m+1} x_{m+1} = x_m \Delta_{m+1} \delta_m.
\]
Therefore, (3.8) is equivalent to
\[
(\Delta_m + \Delta_{m+1}) \delta_m = \frac{2 \sin \theta_m}{\prod_{j=1}^{m-1} \sin^2 \theta_j \cos \theta_m \partial \theta_m \partial \theta_m},
\]
which is easy to verify. Having proved (3.6), we can exploit its validity for \(m = n - 1\) in particular to prove (3.7). Write \(x_n = x_{n-1} \delta_{n-1}\) with \(\delta_{n-1} : = \sin \theta_{n-1}/ \cos \theta_{n-1}\). Arguing as above, we discover that
\[
\sum_{k=1}^{n-1} \Delta_k x_n = -(n - 1) x_n + x_{n-1} \Delta_{n-1} \delta_{n-1} + \frac{2}{\prod_{j=1}^{n-2} \sin^2 \theta_j \partial \theta_{n-1} \partial \theta_{n-1}}.
\]
This reduces (3.7) to
\[
\frac{\partial^2 \delta_{n-1}}{\partial \theta_{n-1}^2} = \frac{2 \sin \theta_{n-1}}{\cos \theta_{n-1}} \frac{\partial \delta_{n-1}}{\partial \theta_{n-1}},
\]
which is simple to check. □

**Lemma 3.3** Let \(\Phi_{n-1} \in S^{n-1}\), and let \(\lambda(. , \Phi_{n-1}) : S^{n-1} \to [-1,1]\) be the function defined in (2.4). Then,
\[
|\nabla_{S^{n-1}} \lambda| = \sqrt{1 - \lambda^2}, \tag{3.10}
\]
\[
\Delta_{S^{n-1}} \lambda = -(n - 1) \lambda. \tag{3.11}
\]

**Proof** Using Lemma 3.1, we obtain
\[
|\nabla_{S^{n-1}} \lambda(\Theta_{n-1}, \Phi_{n-1})|^2
\]
\[
= \sum_i x_i^2(\Phi_{n-1})|\nabla_{S^{n-1}} x_i(\Theta_{n-1})|^2
\]
\[
+ \sum_{i \neq j} x_i(\Phi_{n-1})x_j(\Phi_{n-1}) \nabla_{S^{n-1}} x_i(\Theta_{n-1}) \cdot \nabla_{S^{n-1}} x_j(\Theta_{n-1})
\]
\[
= \sum_i x_i^2(\Phi_{n-1}) - \sum_i x_i^2(\Phi_{n-1}) \lambda_i^2(\Theta_{n-1})
\]
\[
- \sum_{i \neq j} x_i(\Phi_{n-1})x_j(\Theta_{n-1})x_i(\Phi_{n-1})x_j(\Theta_{n-1})
\]
\[
= 1 - \lambda^2(\Theta_{n-1}, \Phi_{n-1}).
\]
This shows (3.10). We also obtain (3.11) as a direct consequence of Lemma 3.2, since \(\lambda(\Theta_{n-1}, \Phi_{n-1})\) is a linear combination of eigenfunctions of \(\Delta_{S^{n-1}}\) that all correspond to the eigenvalue \(-(n - 1)\). □
Lemma 3.4 Let \( \Phi_{n-1} \) be a point on the sphere \( \mathbb{S}^{n-1} \), and let \( d(\cdot, \Phi_{n-1}) : \mathbb{S}^{n-1} \to [0, \pi] \) be the geodesic distance from \( \Phi_{n-1} \) on \( \mathbb{S}^{n-1} \) defined in (2.3). Then,

\[
|\nabla_{\mathbb{S}^{n-1}} d| = 1, \tag{3.12}
\]

\[
\Delta_{\mathbb{S}^{n-1}} d = (n-2) \frac{\cos d}{\sin d}. \tag{3.13}
\]

Proof From (2.3), we find

\[
\nabla_{\mathbb{S}^{n-1}} d(\Theta_{n-1}, \Phi_{n-1}) = -\nabla_{\mathbb{S}^{n-1}} \lambda(\Theta_{n-1}, \Phi_{n-1}) \sqrt{1-\lambda^2(\Theta_{n-1}, \Phi_{n-1})}. \tag{3.14}
\]

Hence, (3.12) follows from (3.10). Taking the divergence of both sides of (3.14), then substituting for \( \nabla_{\mathbb{S}^{n-1}} \lambda \) from (3.10) and for \( \Delta_{\mathbb{S}^{n-1}} \lambda \) from (3.11), we deduce that

\[
\Delta_{\mathbb{S}^{n-1}} d(\Theta_{n-1}, \Phi_{n-1}) = -\frac{\Delta_{\mathbb{S}^{n-1}} \lambda(\Theta_{n-1}, \Phi_{n-1})}{\sqrt{1-\lambda^2(\Theta_{n-1}, \Phi_{n-1})}} \frac{\lambda(\Theta_{n-1}, \Phi_{n-1}) \nabla_{\mathbb{S}^{n-1}} \lambda(\Theta_{n-1}, \Phi_{n-1})}{(1-\lambda^2(\Theta_{n-1}, \Phi_{n-1}))^{\frac{3}{2}}}
\]

\[
= \frac{(n-2) \lambda(\Theta_{n-1}, \Phi_{n-1})}{\sqrt{1-\lambda^2(\Theta_{n-1}, \Phi_{n-1})}},
\]

which yields (3.13) in the light of (2.3). \( \square \)

4 Subcritical \( L^p \) Hardy inequalities

Let \( \mathbb{S}^{n-1} \) be the unit sphere in \( \mathbb{R}^n, n \geq 4 \). Let \( 1 < p < n-1 \) and consider the following nonlinear positive functionals on \( W^{1,p}(\mathbb{S}^{n-1} \to \mathbb{R}) \):

\[
S_p(u) := \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^p d} \, d\sigma_{n-1},
\]

\[
\tilde{S}_p(u) := \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^{p-2} d} \, d\sigma_{n-1},
\]

\[
T_p(u) := \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{|\tan d|^p} \, d\sigma_{n-1},
\]

\[
\tilde{T}_p(u) := \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{|\tan d|^{p-2}} \, d\sigma_{n-1},
\]

\[
F_p(u) := \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} u|^p \, d\sigma_{n-1},
\]

\[
G_p(u) := \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} u|^p \cos d^p \, d\sigma_{n-1}.
\]

Define also the constant

\[
\alpha_{n,p} := \frac{n-p-1}{p}.
\]

Remark 4.1 Formula (2.5) makes it clear that both integrals \( \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{|\tan d|^p} \, d\sigma_{n-1} \) and \( \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^p d} \, d\sigma_{n-1} \) are convergent when \( u \) is continuous. Indeed, recalling that \( d(\Theta_{n-1}, \Phi_{n-1}) \),
\( \Phi_{n-1} = \arccos(\Theta_{n-1} \cdot \Phi_{n-1}), \Phi_{n-1} \in S^{n-1}, \) we immediately see that
\[
\int_{S^{n-1}} \frac{d\sigma_{n-1}}{|\tan d|^p} < \int_{S^{n-1}} \frac{d\sigma_{n-1}}{\sin^p d} = \int_{S^{n-1}} \frac{d\sigma_{n-1}}{(1 - (\Theta_{n-1} \cdot \Phi_{n-1})^2)^{p/2}}
\]
\[= 2C_n \int_0^1 \frac{t - t^p}{(1 - t^2)^{p/2}} dt,
\]
which exists for \( p < n - 1 \).

We show that the functionals \( T_p, \tilde{T}_p, S_p, \) and \( \tilde{S}_p \) are all well defined and related by the following \( L^p \) inequalities of Hardy type:

**Theorem 4.1** (Subcritical \( L^p \) Hardy inequalities) Suppose \( u \in W^1p(S^{n-1} \rightarrow \mathbb{R}), n \geq 4 \). Then, \( \frac{u}{\sin d} \in L^p(S^{n-1}), \) when \( 1 < p < n - 1 \), and \( \frac{u}{|\tan d|} \in L^p(S^{n-1}), \) when \( 2 \leq p < n - 1 \). Moreover,

\[
\alpha_{n,p}^p S_p(u) \leq G_p(u) + (n - p)\alpha_{n,p}^{-1} \tilde{S}_p(u), \quad 1 < p < n - 1, \quad (4.1)
\]
\[
\alpha_{n,p}^p T_p(u) \leq F_p(u) + (p - 1)\alpha_{n,p}^{-1} \tilde{T}_p(u), \quad 2 \leq p < n - 1. \quad (4.2)
\]

**Proof** Let us start with the inequality (4.1). Using a density argument, we may assume \( u \in C^\infty(S^{n-1}). \) Recalling the identities (3.12) and (3.13) in Lemma 3.4, we can compute

\[
\Delta_{S^{n-1}} \sin d = -\sin d + (n - 2)\frac{\cos^2 d}{\sin d}.
\]

Integrating both sides of (4.3) against \( |u|^p / \sin^{p-1} d \) over \( S^{n-1} \), then employing the divergence theorem, we obtain

\[
(n - 2) \int_{S^{n-1}} \frac{|u|^p}{\sin^p d} \cos^2 d \, d\sigma_{n-1} = \int_{S^{n-1}} \frac{|u|^p}{\sin^{p-2} d} \, d\sigma_{n-1}
\]
\[
= \int_{S^{n-1}} \frac{|u|^p}{\sin^{p-1} d} \Delta_{S^{n-1}} \sin d \, d\sigma_{n-1}
\]
\[
= - \int_{S^{n-1}} \nabla_{S^{n-1}} \left( \frac{|u|^p}{\sin^{p-1} d} \right) \cdot \nabla_{S^{n-1}} \sin d \, d\sigma_{n-1}
\]
\[
= \int_{S^{n-1}} \frac{-p|u|^{p-2}u \nabla_{S^{n-1}} u \cdot \nabla_{S^{n-1}} \sin d}{\sin^{p-2} d} \, d\sigma_{n-1}
\]
\[
+ (p - 1) \int_{S^{n-1}} \frac{|u|^p}{\sin^{p-1} d} \cos^2 d \, d\sigma_{n-1}. \quad (4.4)
\]

Observe that we simplified the latter integral using the fact \( |\nabla_{S^{n-1}} d| = 1 \). So far, it suffices to require that \( p > 1 \) to make sense of the gradient of \( |u|^p \). Invoking Hölder’s inequality then applying Young’s inequality and using (3.12) once more, we can bound

\[
\int_{S^{n-1}} \frac{-p|u|^{p-2}u \nabla_{S^{n-1}} u \cdot \nabla_{S^{n-1}} \sin d}{\sin^{p-1} d} \, d\sigma_{n-1}
\]
\[
\leq p \left( \int_{S^{n-1}} \frac{|u|^p}{\sin^{p-1} d} \, d\sigma_{n-1} \right)^{p-1} \left( \int_{S^{n-1}} \left| \nabla_{S^{n-1}} u \cdot \nabla_{S^{n-1}} \sin d \right|^p \, d\sigma_{n-1} \right)^{1/p}
\]
\[ \leq (p-1) \beta^{\frac{n}{p}} \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^d d} d\sigma_{n-1} + \frac{1}{\beta^p} \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} u|^p \cos d^p d\sigma_{n-1}, \]  

(4.5)

with \( \beta > 0 \) as yet undetermined. Inserting the estimate \( (4.5) \) into the inequality \( (4.4) \) then rearranging gives

\[ \begin{align*}
(n-p-1) & \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^d d} \cos^2 d d\sigma_{n-1} - (p-1) \beta^{\frac{n}{p}} \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^d d} d\sigma_{n-1} \\
& \leq \frac{1}{\beta^p} \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} u|^p \cos d^p d\sigma_{n-1} + \beta^p (n-p) \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^{p-2} d} d\sigma_{n-1}.
\end{align*} \]

(4.6)

Note here that Remark 4.1 justifies this manipulation of the terms of \( (4.4) \). We proceed from \( (4.6) \) by simply replacing the factor \( \cos^2 d \) by \( 1 - \sin^2 d \) in the first integral to obtain

\[ \begin{align*}
(n-p-1 - (p-1) \beta^{\frac{n}{p}}) & \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^d d} d\sigma_{n-1} \\
& \leq \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} u|^p \cos d^p d\sigma_{n-1} + \beta^p (n-p) \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\sin^{p-2} d} d\sigma_{n-1}.
\end{align*} \]

(4.7)

The optimal value of \( \beta \) for \( (4.7) \) is easily determined through finding the maximum point \( t_\ast \) of the function \( t \mapsto t(n-p-1 - (p-1) t^{\frac{1}{p}}) \) on \([0, +\infty[\). We find \( t_\ast = (\frac{n-p-1}{p})^{\frac{1}{p}} \). Hence, the inequality \( (4.7) \) takes the form \( (4.1) \).

With the exception of some technical details, the proof of \( (4.2) \) is similar to that of \( (4.1) \). Instead of using \( (4.3) \), we capitalize on \( (3.13) \). Let \( 2 \leq p < n - 1 \). Integration by parts on \( \mathbb{S}^{n-1} \) yields

\[ \begin{align*}
\int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\tan^d d} d\sigma_{n-1} \\
& = \frac{1}{n-2} \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\tan^d d} \tan \Delta_{\mathbb{S}^{n-1}} d d\sigma_{n-1} \\
& = \frac{-1}{n-2} \int_{\mathbb{S}^{n-1}} \nabla_{\mathbb{S}^{n-1}} d \cdot \nabla_{\mathbb{S}^{n-1}} \left( \frac{|u|^p}{\tan^d d} \tan d \right) d\sigma_{n-1} \\
& = \frac{1}{n-2} \int_{\mathbb{S}^{n-1}} \frac{-p|u|^{p-2} \nabla_{\mathbb{S}^{n-1}} u \cdot \nabla_{\mathbb{S}^{n-1}} d}{\tan^d d} d\sigma_{n-1} \\
& \quad + \frac{p-1}{n-2} \int_{\mathbb{S}^{n-1}} \frac{|u|^p}{\tan^d d} \cos^2 d d\sigma_{n-1}.
\end{align*} \]

(4.8)

Observe that the restriction \( 2 \leq p < n - 1 \) is necessary to make sense of \( \nabla_{\mathbb{S}^{n-1}} \cos d^{p-2} \cos d \). It also guarantees the convergence of the integral \( \int_{\mathbb{S}^{n-1}} \frac{1}{\tan^d d} \frac{1}{\cos^d d} d\sigma_{n-1} \). This is inferred by formula \( (2.5) \) that asserts

\[ \int_{\mathbb{S}^{n-1}} \frac{d\sigma_{n-1}}{\sin^d d \cos^p d} = 2C_n \int_0^1 \frac{ds}{s^{2-p}(1-s^2)^{-\frac{n-p-2}{2}}} . \]

Since \( |\nabla_{\mathbb{S}^{n-1}} d| = 1 \), then, applying Hölder’s inequality followed by Young’s inequality analogously to \( (4.5) \) gives

\[ \int_{\mathbb{S}^{n-1}} \frac{-p|u|^{p-2} \nabla_{\mathbb{S}^{n-1}} u \cdot \nabla_{\mathbb{S}^{n-1}} d}{\tan^d d} d\sigma_{n-1} \]
The optimal value of $\beta$ for (4.11) is $\frac{n-1}{p}$. This proves the inequality (4.2).

**Theorem 4.2** (Sharpness of the inequalities (4.1) and (4.2)) The constants on both sides of the inequality (4.1) are sharp. Precisely, we have

\[
\sup_{u \in W^{1,p}(S^{n-1}) \setminus \{0\}} \frac{\alpha_{n,p}^p S_p(u)}{G_p(u) + (n-p)\alpha_{n,p}^{p-1} S_p(u)} = 1, \quad 1 < p < n - 1, \tag{4.12}
\]

\[
\sup_{u \in W^{1,p}(S^{n-1}) \setminus \{0\}} \frac{\alpha_{n,p}^p T_p(u)}{F_p(u) + (p-1)\alpha_{n,p}^{p-1} T_p(u)} = 1, \quad 1 < p < n - 1, \tag{4.15}
\]

\[
\sup_{u \in W^{1,p}(S^{n-1}) \setminus \{0\}} \frac{\alpha_{n,p}^p \bar{T}_p(u)}{\bar{F}_p(u) + (p-1)\alpha_{n,p}^{p-1} \bar{T}_p(u)} = 1, \quad 1 < p < n - 1, \tag{4.16}
\]

All the constants involved in the inequality (4.2) are sharp for all $2 \leq p < n - 1$. Precisely,

\[
\sup_{u \in W^{1,p}(S^{n-1}) \setminus \{0\}} \frac{\alpha_{n,p}^p T_p(u)}{F_p(u) + (p-1)\alpha_{n,p}^{p-1} \bar{T}_p(u)} = 1, \tag{4.13}
\]

\[
\sup_{u \in W^{1,p}(S^{n-1}) \setminus \{0\}} \frac{\alpha_{n,p}^p \bar{T}_p(u)}{\bar{F}_p(u) + (p-1)\alpha_{n,p}^{p-1} \bar{T}_p(u)} = 1, \tag{4.14}
\]

\[
\sup_{u \in W^{1,p}(S^{n-1}) \setminus \{0\}} \frac{\alpha_{n,p}^p \bar{T}_p(u)}{\bar{F}_p(u) + (p-1)\alpha_{n,p}^{p-1} \bar{T}_p(u)} = 1, \tag{4.17}
\]

**Remark 4.2** The values $\frac{n}{2} \leq p < n - 1$ can be admitted in (4.14) if the supremum is taken over nontrivial functions in $L^p(S^{n-1})$ with a weak gradient in the weighted space $L^p(S^{n-1}| \cos d(\Theta_{n-1}, \Phi_{n-1})^p \ d\sigma_{n-1})$.

**Proof** Fix $n \geq 4$ and $1 < p < n - 1$. Consider the function

\[
u_\epsilon(\Theta_{n-1}) := |\cot d(\Theta_{n-1}, \Phi_{n-1})|^{(p-1)/2} \cos d(\Theta_{n-1}, \Phi_{n-1})
\]
on $S^{n-1} \setminus \{\pm \Phi_{n-1}\}$. Verifiably, $\nu_\epsilon \in W^{1,p}(S^{n-1})$ for every $\epsilon > 0$. Moreover,

\[
\int_{S^{n-1}} \frac{|\nu_\epsilon|^p}{\sin^p d} \ d\sigma_{n-1} = I_{n,p}(\epsilon), \tag{4.18}
\]
where

\[ I_{n,p}(\epsilon) := \int_{S^{p-1}} \frac{|\cos d|^{n-2\epsilon}}{(\sin d)^{n-2\epsilon}} d\sigma_{n-1}. \]

Exploiting formula (2.5) we find

\[ I_{n,p}(\epsilon) = C_n \int_{-1}^{1} t^{n-1-2\epsilon} \left( \frac{\sqrt{1-t^2}}{1-t^2} \right)^{n-3} dt = 2C_n \int_{0}^{1} t^{n-1-2\epsilon} (1-t^2)^{-1+\epsilon} dt = C_n \frac{\Gamma(\epsilon)\Gamma(\frac{n}{2} - \epsilon)}{\Gamma(\frac{n}{2})}. \]  

(4.19)

Note that \( I_{n,p}(\epsilon) \) is finite for every \( \epsilon > 0 \) but blows up in the limit. Additionally,

\[ \int_{S^{p-1}} |u_e|^p \sin^{n-2\epsilon} d\sigma_{n-1} = \int_{S^{p-1}} \frac{|\cos d|^{n-2\epsilon}}{(\sin d)^{n-3+2\epsilon}} d\sigma_{n-1} = 2C_n \int_{0}^{1} t^{n-1-2\epsilon} (1-t^2)^{\epsilon} dt = C_n \frac{\epsilon \Gamma(\epsilon)\Gamma(\frac{n}{2} - \epsilon)}{\Gamma(\frac{n}{2})}. \]  

(4.20)

Furthermore, for all \( \Theta_{n-1} \notin \{ \pm \Phi_{n-1} \} \cup \{ \Theta_{n-1} \in S^{n-1} : d(\Theta_{n-1}, \Phi_{n-1}) = \pi/2 \} \)

\[ \cos d \nabla_{S^{p-1}} u_e = -\frac{n-p-1-2\epsilon}{p} |\cot d|^{\frac{n-1-2\epsilon}{p}} \text{sign} (\cos d) \nabla_{S^{p-1}} d \]

\[ - |\cot d|^{\frac{n-1-2\epsilon}{p}} \sin d \cos d \nabla_{S^{p-1}} d. \]

Also, Minkowski’s inequality implies

\[ \| \cos d \nabla_{S^{p-1}} u_e \|^p_{L^p(S^{p-1})} \leq \left( \frac{n-p-1-2\epsilon}{p} \right) \| |\cot d|^{\frac{n-1-2\epsilon}{p}} \|^p_{L^p(S^{p-1})} \]

\[ + \left( |\cot d|^{\frac{n-1-2\epsilon}{p}} \sin d \cos d \right)^p, \]

with \( 0 < \epsilon < (n-p-1)/2 \). That is,

\[ \int_{S^{p-1}} |\nabla_{S^{p-1}} u_e|^p \cos d|^p d\sigma_{n-1} \leq \left( \frac{n-p-1-2\epsilon}{p} \right)^p I_{n,p}(\epsilon) \]

\[ \times \left( 1 + \left( \int_{S^{p-1}} |\cos d|^{n-1-2\epsilon} (\sin d)^{n-2p-1-2\epsilon} d\sigma_{n-1} \right)^{1/p} \left( \frac{n-p-1-2\epsilon}{p} I_{n,p}(\epsilon) \right)^{1/p} \right)^p. \]

Since, by (2.5), we have

\[ \int_{S^{p-1}} |\cos d|^{n-1-2\epsilon} (\sin d)^{n-2p-1-2\epsilon} d\sigma_{n-1} = C_n \frac{\Gamma(\frac{n}{2} - \epsilon)\Gamma(p+\epsilon)}{\Gamma(\frac{n}{2} + p)}, \]
then, we can simplify

\[
\int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} u_\epsilon|^p \, d\sigma_{n-1} \leq \left( \frac{\alpha_{n,p} - \frac{2}{p} \epsilon}{\alpha_{n,p}} \right)^p I_{n,p}(\epsilon)(1 + \Lambda_{n,p}(\epsilon))^p, \tag{4.21}
\]

where

\[
\Lambda_{n,p}(\epsilon) := \left( \frac{\Gamma(p + \epsilon)\Gamma\left(\frac{n}{p}\right)}{\Gamma\left(\frac{n}{p} + p\right)\Gamma\left(\frac{n}{p} - \frac{2}{p} \epsilon\right)} \right)^{1/p}. 
\]

Consequently, putting together the inequalities (4.1), (4.18), (4.19), and (4.20), and the estimate (4.21) implies

\[
1 \geq \frac{\alpha_{n,p}^p S_p(u_\epsilon)}{G_p(u_\epsilon) + (n - p)\alpha_{n,p}^{p-1} S_p(u_\epsilon)} \geq \frac{\alpha_{n,p}^p}{(\alpha_{n,p} - \frac{2}{p} \epsilon)^p (1 + \Lambda_{n,p}(\epsilon))^p} \to 1
\]

as \( \epsilon \to 0^+ \) by the continuity of the gamma function on \( ]0, \infty[ \), and the fact that \( \lim_{\epsilon \to 0^+} \Gamma(\epsilon) = +\infty \) that makes \( \lim_{\epsilon \to 0^+} \Lambda_{n,p}(\epsilon) = 0 \). This squeeze along with the inequality (4.1) proves (4.12). In the same fashion

\[
1 \geq \frac{\alpha_{n,p}^p S_p(u_\epsilon) - (n - p)\alpha_{n,p}^{p-1} S_p(u_\epsilon)}{G_p(u_\epsilon)} \geq \frac{\alpha_{n,p}^p}{(\alpha_{n,p} - \frac{2}{p} \epsilon)^p (1 + \Lambda_{n,p}(\epsilon))^p} \to 1
\]

when \( \epsilon \to 0^+ \). This shows (4.13) in the light of (4.1). We proceed to prove (4.14) that shows that the constant \( n - p \) on the right-hand side of (4.1) is the smallest possible for all \( 1 < p < \frac{n}{2} \). Define the function

\[
\tilde{u}_\epsilon(\Theta_{n-1}) := |\cot d(\Theta_{n-1}, \Phi_{n-1})|^{(n-p-1-2\epsilon)/p}, \quad \Theta_{n-1} \in \mathbb{S}^{n-1} \setminus \{\pm \Phi_{n-1}\}.
\]

For \( \epsilon > 0 \), we have \( \tilde{u}_\epsilon \in L^p(\mathbb{S}^{n-1}) \), \( 1 < p < n - 1 \) and \( \nabla_{\mathbb{S}^{n-1}} \tilde{u}_\epsilon \in L^p(\mathbb{S}^{n-1}) \), \( 1 < p < n/2 \) as shown by the following calculations:

\[
\begin{align*}
\int_{\mathbb{S}^{n-1}} \frac{|\tilde{u}_\epsilon|^p}{\sin^{p-2} d} \, d\sigma_{n-1} &= \int_{\mathbb{S}^{n-1}} \frac{\cos d^{p-1-2\epsilon}}{(\sin d)^{p-1-2\epsilon}} \, d\sigma_{n-1} \\
&= C_n \int_{-1}^1 t^{n-p-1-2\epsilon} (1 - t^2)^{1/2} \, dt = C_n \frac{\Gamma(\epsilon)\Gamma\left(\frac{n-p-\epsilon}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{4.22}
\end{align*}
\]

\[
\begin{align*}
\int_{\mathbb{S}^{n-1}} \frac{|\tilde{u}_\epsilon|^p}{\sin^{p-1} d} \, d\sigma_{n-1} &= \int_{\mathbb{S}^{n-1}} \frac{\cos d^{p-1-2\epsilon}}{(\sin d)^{p-3-2\epsilon}} \, d\sigma_{n-1} \\
&= C_n \int_{-1}^1 t^{n-p-1-2\epsilon} (1 - t^2)^{1/2} \, dt = C_n \frac{\epsilon \Gamma(\epsilon)\Gamma\left(\frac{n-p-\epsilon}{2}\right)}{\Gamma\left(\frac{n}{2}\right)^2}, \tag{4.23}
\end{align*}
\]
\[ \int_{S^{n-1}} |\nabla S^{n-1} \tilde{u}_\epsilon|^p |\cos d|^p \ d\sigma_{n-1} \]
\[ = \left( \alpha_{n,p} - \frac{2}{p} \epsilon \right)^p \int_{S^{n-1}} |\cos d|^{n-p-1-2\epsilon} \ d\sigma_{n-1} \]
\[ = C_n \left( \alpha_{n,p} - \frac{2}{p} \epsilon \right)^p \frac{\Gamma(\epsilon) \Gamma\left( \frac{\epsilon}{p} + \frac{1}{2} \right)}{\Gamma\left( \frac{\epsilon}{p} + \frac{1}{2} \right)} . \quad (4.24) \]

Combining (4.22)–(4.24) yields
\[ \frac{\alpha_{n,p}^p T_p(\tilde{u}_\epsilon) - G_p(\tilde{u}_\epsilon)}{\alpha_{n,p}^{p-1} T_p(\tilde{u}_\epsilon)} \equiv \frac{n - p \alpha_{n,p}^p - (\alpha_{n,p} - \frac{2}{p} \epsilon)^p}{\epsilon} \rightarrow n - p \]
as \( \epsilon \rightarrow 0^+ \). This convergence together with the inequality (4.2) confirms (4.17).

Now, we turn our attention to (4.15)–(4.17). Let \( 2 \leq p < n - 1, \ n \geq 4 \), and define the function
\[ v_\epsilon(\Omega_{n-1}) := (\sin d(\Theta_{n-1}, \Phi_{n-1}))^{-(n-p-1-2\epsilon)/p} \]
on \( S^{n-1} \setminus \{ \pm \Phi_{n-1} \} \). Evidently \( v_\epsilon \in W^{1, p}(S^{n-1}) \) for every \( \epsilon > 0 \). Furthermore, \( 0 < \epsilon < (n - p - 1)/2 \), we have
\[ \int_{S^{n-1}} \frac{|v_\epsilon|^p}{|\tan d|^p} d\sigma_{n-1} = J_{n,p}(\epsilon), \quad \int_{S^{n-1}} |\nabla S^{n-1} v_\epsilon|^p d\sigma_{n-1} = \left( \alpha_{n,p} - \frac{2}{p} \epsilon \right)^p J_{n,p}(\epsilon), \]
where
\[ J_{n,p}(\epsilon) := \int_{S^{n-1}} |\cos d|^p \ d\sigma_{n-1} \]
\[ = 2C_n \int_0^1 t^p \left( 1 - t^2 \right)^{-1+\epsilon} dt = C_n \frac{\Gamma(\epsilon) \Gamma\left( \frac{p+1}{2} \right)}{\Gamma\left( \frac{\epsilon}{p+1} + \frac{1}{2} \right)} , \]
using formula (2.5). We also have
\[ \int_{S^{n-1}} \frac{|v_\epsilon|^p}{|\tan d|^{p-2}} d\sigma_{n-1} = \int_{S^{n-1}} \frac{|\cos d|^{p-2}}{|\sin d|^{n-3-2\epsilon}} d\sigma_{n-1} \]
\[ = 2C_n \int_0^1 t^{p-2} \left( 1 - t^2 \right)^{\epsilon} dt = C_n \frac{\epsilon \Gamma(\epsilon) \Gamma\left( \frac{p+1}{2} \right)}{\Gamma\left( \frac{\epsilon}{p+1} + \frac{1}{2} \right)} . \]
Consequently,
\[ \frac{\alpha_{n,p}^p T_p(v_\epsilon)}{F_p(v_\epsilon) + (p-1)\alpha_{n,p}^{p-1} T_p(v_\epsilon)} \]
\[ = \frac{\alpha_{n,p}^p \Gamma\left( \frac{p+1}{2} \right)}{(\alpha_{n,p} - \frac{2}{p} \epsilon)^p \Gamma\left( \frac{p+1}{2} \right) + (p-1)\epsilon \alpha_{n,p}^{p-1} \Gamma\left( \frac{p+1}{2} \right)} \rightarrow 1 \]
as $\epsilon \to 0^+$. This, together with the inequality (4.2), proves (4.15). In addition,

$$
\frac{\omega_{n,p}^p T_p(v_\epsilon) - (p-1)\omega_{n,p}^{p-1} \tilde{T}_p(v_\epsilon)}{F_p(v_\epsilon)} = \frac{\alpha_{n,p}^p \Gamma\left(\frac{p+1}{2}\right) - (p-1)\epsilon \alpha_{n,p}^{p-1} \Gamma\left(\frac{p-1}{2}\right)}{(\alpha_{n,p} - \frac{2}{p} \epsilon)^p \Gamma\left(\frac{p+1}{2}\right)} \to 1
$$

when $\epsilon \to 0^+$, which proves (4.16). Finally, we show (4.17) that demonstrates that the constant $p-1$ on the right-hand side of (4.2) is optimal. We have

$$
\frac{\alpha_{n,p}^p T_p(v_\epsilon) - F_p(v_\epsilon)}{\alpha_{n,p}^{p-1} \tilde{T}_p(v_\epsilon)} = \frac{\alpha_{n,p}^p \Gamma\left(\frac{p+1}{2}\right) - (\alpha_{n,p} - \frac{2}{p} \epsilon)^p \Gamma\left(\frac{p-1}{2}\right)}{\epsilon \alpha_{n,p}^{p-1} \Gamma\left(\frac{p-1}{2}\right)} = \frac{(\alpha_{n,p} - (\alpha_{n,p} - \frac{2}{p} \epsilon)^p) p - 1}{2\alpha_{n,p}^{p-1}}
$$

which converges to $p-1$. \[\square\]

An important consequence of (4.14) is that, in any dimension $n \geq 4$, and for every $1 < p < n-1$, we can find $u \in C^\infty(S^{n-1})$ such that

$$
\alpha_{n,p}^p S_p(u) > G_p(u) + \alpha_{n,p}^{p-1} \tilde{S}_p(u).
$$

It similarly follows from (4.17) that the inequality

$$
\alpha_{n,p}^p T_p(u) \leq F_p(u) + \alpha_{n,p}^{p-1} \tilde{T}_p(u), \quad p > 2
$$

does not hold true on $C^\infty(S^{n-1})$, $n \geq 4$. More interestingly:

**Theorem 4.3** The inequality

$$
\alpha_{n,p}^p S_p(u) \leq F_p(u) + \alpha_{n,p}^{p-1} \tilde{S}_p(u)
$$

is generally false on $W^{1,p}(S^{n-1})$ for every $1 < p < n-1$, $n \geq 4$. In particular, there exists $u \in H^1(S^{n-1})$ such that

$$
\left(\frac{n-3}{2}\right)^2 \int_{S^{n-1}} \frac{u^2}{\sin^2 d} d\sigma_{n-1} > \int_{S^{n-1}} |\nabla S_{p-1} u|^2 d\sigma_{n-1} + \frac{n-3}{2} \int_{S^{n-1}} u^2 d\sigma_{n-1}
$$

for every $n > 4$.

**Proof** If we test the inequality (4.25) with a constant function we find it false for $1 < p < n/2$. We provide an explicit counterexample in $W^{1,p}(S^{n-1})$ for which (4.25) fails for all $1 < p < n-1$. Define

$$
w_\epsilon(\Theta_{n-1}) := \left(\frac{1 + \cos d(\Theta_{n-1}, \Phi_{n-1})}{\sin d(\Theta_{n-1}, \Phi_{n-1})}\right)^{\frac{n-p-1}{p-1}}, \quad 1 < p < n-1.
$$
When \( \epsilon \) is sufficiently small, we have

\[
L_{n,p}(\epsilon) := \left( \frac{n - p - 1}{p} \right)^p \int_{S^{n-1}} \frac{|w_e|^p}{|\sin d|^p} d\sigma_{n-1} - \int_{S^{n-1}} |\nabla_{S^{n-1}} w_e|^p d\sigma_{n-1}
\]

\[
= \left( \left( \frac{n - p - 1}{p} \right)^p - \left( \frac{n - p - 1}{p} - \frac{2}{p} \epsilon \right)^p \right) K_{n,p}(\epsilon), \tag{4.26}
\]

with

\[
K_{n,p}(\epsilon) := \int_{S^{n-1}} \frac{(1 + \cos d)^{n-p-1-2\epsilon}}{(\sin d)^{n-3-2\epsilon}} d\sigma_{n-1}.
\]

Formula (2.5) helps us determine the exact value of \( K_{n,p}(\epsilon) \). Indeed, upon translating \( t \to t - 1 \) then rescaling \( t \to 2t \), we discover that

\[
K_{n,p}(\epsilon) = C_n \int_{-1}^{1} (1 + t)^{n-p-1-2\epsilon} (1 - t^2)^{-1+\epsilon} dt
\]

\[
= 2^{n-p-2} C_n \int_{0}^{1} t^{n-p-2-\epsilon} (1 - t)^{-1+\epsilon} dt
\]

\[
= 2^{n-p-2} C_n \frac{\Gamma(\epsilon) \Gamma(n - p - 1 - \epsilon)}{\Gamma(n - p - 1)}. \tag{4.27}
\]

Substitute (4.27) into (4.26). It follows then from the continuity of the gamma function on \( [0, \infty[ \), and the fact that \( \lim_{\epsilon \to 0^+} \epsilon \Gamma(\epsilon) = 1 \), that

\[
\lim_{\epsilon \to 0^+} L_{n,p}(\epsilon) = 2^{n-p-2} C_n \lim_{\epsilon \to 0^+} \left( \frac{\Gamma(\epsilon) \Gamma(n - p - 1 - \epsilon)}{\Gamma(n - p - 1)} \right)^{-1}.
\]

\[
= 2^{n-p-1} C_n \left( \frac{n - p - 1}{p} \right)^{p-1}. \tag{4.28}
\]

On the other hand,

\[
\int_{S^{n-1}} \frac{|w_e|^p}{|\sin d|^p} d\sigma_{n-1} = \int_{S^{n-1}} \frac{(1 + \cos d)^{n-p-1-2\epsilon}}{(\sin d)^{n-3-2\epsilon}} d\sigma_{n-1}
\]

\[
= 2^{n-p} C_n \int_{0}^{1} t^{n-p-1-\epsilon} (1 - t)^{\epsilon} dt
\]

\[
= 2^{n-p} C_n \frac{\Gamma(1 + \epsilon) \Gamma(n - p - \epsilon)}{(n - p) \Gamma(n - p)}. \tag{4.29}
\]

Hence, defining

\[
R_{n,p}(\epsilon) := \left( \frac{n - p - 1}{p} \right)^{p-1} \int_{S^{n-1}} \frac{|w_e|^p}{|\sin d|^p} d\sigma_{n-1},
\]

we see from (4.29) that

\[
\lim_{\epsilon \to 0^+} R_{n,p}(\epsilon) = 2^{n-p} C_n \left( \frac{n - p - 1}{p} \right)^{p-1}. \tag{4.30}
\]
Comparing (4.28) against (4.30) we conclude that, given $1 < p < n - 1$, there exists $0 < \epsilon_0 < (n - p - 1)/2$ such that

$$\alpha_{n,p}^p S_p(w_e) > F_p(w_e) + \alpha_{n,p}^{p-1} S_p(w_e)$$

for each $\epsilon < \epsilon_0$. \hfill \Box

5 Critical $L^p$ Hardy inequalities

Let $n \geq 2$ and define the following nonlinear positive functionals on $W^{1,n}(S^n) \to \mathbb{R}$:

$$U_n(u) := \int_{S^n} \frac{|u|^n}{\sin^n d(\log \frac{e}{\sin d})} \, d\sigma_n,$$

$$\tilde{U}_n(u) := \int_{S^n} \frac{|u|^n}{\sin^{n-2} d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n,$$

$$V_n(u) := \int_{S^n} \frac{|u|^n}{\sin d^n (\log \frac{e}{\sin d})^n} \, d\sigma_n,$$

$$\tilde{V}_n(u) := \int_{S^n} \frac{|u|^n}{\sin d^n (\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n,$$

$$H_n(u) := \int_{S^n} |\nabla_{S^n} u|^n \, d\sigma_n,$$

$$Q_n(u) := \int_{S^n} |\nabla_{S^n} u|^n \cos d^n \, d\sigma_n.$$

Define also the constant

$$\gamma_n := \frac{n - 1}{n}.$$

**Theorem 5.1** Suppose $u \in W^{1,n}(S^n) \to \mathbb{R}$, where $n \geq 2$. Then, $\frac{u}{\sin d \log \frac{e}{\sin d}}$, $\frac{u}{\sin d \log \frac{e}{\sin d}}$ are in $L^n(S^n)$. Furthermore,

$$\gamma_n^n U_n(u) \leq Q_n(u) + n \gamma_n^{n-1} \tilde{U}_n(u),$$

(5.1)

$$\gamma_n^n V_n(u) \leq H_n(u) + (n - 1) \gamma_n^{n-1} \tilde{V}_n(u).$$

(5.2)

**Remark 5.1** Arguing as in Remark 4.1, the functionals $U_n$ and $V_n$ are bounded on continuous functions as the integrals

$$\int_{S^n} \frac{d\sigma_n}{\sin^n d(\log \frac{e}{\sin d})^n}, \quad \int_{S^n} \frac{d\sigma_n}{\sin d^n (\log \frac{e}{\sin d})^n}$$

are convergent. This is clear thanks to formula (2.5) that ascertains

$$\int_{S^n} \frac{d\sigma_n}{\sin d^n (\log \frac{e}{\sin d})^n} < \int_{S^n} \frac{d\sigma_n}{\sin^d d(\log \frac{e}{\sin d})^n}$$

$$= C_n \int_{-1}^{1} \frac{dt}{(1 - t^2)(\log \frac{e}{\sqrt{1-t^2}})^n}$$

whereas the latter integral exists for all $n > 1$. 

Remark 5.2 It is noteworthy that the integral \( \int_{\mathbb{S}^n} \frac{d\sigma_n}{|\tan d|^m |\log c| |\tan d|^m} \) is divergent for every \( m \in \mathbb{R} \) and any \( c > 0 \). Observe that
\[
\int_{\mathbb{S}^n} \frac{d\sigma_n}{|\tan d|^m |\log c| |\tan d|^m} = 2C_n \int_0^1 \frac{s^n \, ds}{(1 - s^2) |\log (\frac{cs - 1}{s})|^m}.
\]

Remark 5.3 The inequality (5.2) is proved in [17] using a different method.

Proof The proof of (5.1) is analogous to that of (4.1). Let \( n \geq 2 \) and use density to assume \( u \in C^\infty(\mathbb{S}^n) \). Starting from (4.3), we integrate both sides against \( |u|^n/(\sin^{n-1}d \log(e/\sin d))^{n-1} \), then use the divergence theorem. We obtain,
\[
(n - 1) \int_{\mathbb{S}^n} \frac{|u|^n \cos^2 d}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n = \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^{n-2} d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n
\]
\[
= \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^{n-2} d(\log \frac{e}{\sin d})^{n-1}} |\Delta_{\mathbb{S}^n} \sin d| \, d\sigma_n
\]
\[
= - \int_{\mathbb{S}^n} \nabla_{\mathbb{S}^n} u \cdot \nabla_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} u| \sin d \, d\sigma_n + (n - 1) \int_{\mathbb{S}^n} \frac{|u|^n \cos^2 d}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n
\]
\[
- (n - 1) \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \cos^2 d \, d\sigma_n. \tag{5.3}
\]

Using Hölder’s inequality then applying Young’s inequality implies
\[
\int_{\mathbb{S}^n} \frac{-n|u|^{n-2} u \cos d |\nabla_{\mathbb{S}^n} u| \cdot |\nabla_{\mathbb{S}^n} d|}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n
\]
\[
\leq (n - 1) \beta \frac{\pi}{2} \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n + \frac{1}{\beta^n} \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} u|^n \cos d |\sigma_n, \tag{5.4}
\]
with \( \beta > 0 \). Returning with the estimate (5.4) to the inequality (5.3), we deduce that
\[
(n - 1)(1 - \beta \frac{\pi}{2}) \beta^n \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n
\]
\[
\leq \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} u|^n \cos d |\sigma_n + \beta^n \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n, \tag{5.5}
\]
where we estimated \( \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^m d(\log \frac{e}{\sin d})^{n-1}} \leq \int_{\mathbb{S}^n} \frac{|u|^n}{\sin^m d(\log \frac{e}{\sin d})^{n-1}}. \) The value \( \beta = (\frac{n - 1}{n})^{\frac{1}{n + 1}} \) optimizes (5.5) and produces (5.1).

Again, the inequality (5.2) can be proved in the same way as (4.2). Since \( \Delta_{\mathbb{S}^n} d = (n - 1)/\tan d \), then invoking the divergence theorem we obtain
\[
(n - 1) \int_{\mathbb{S}^n} \frac{|u|^n}{|\tan d|^m(\log \frac{e}{\sin d})^{n-1}} \, d\sigma_n
\]
\[
= \int_{\mathbb{S}^n} \frac{|u|^n}{|\tan d|^m} \, d\sigma_n \Delta_{\mathbb{S}^n} d \, d\sigma_n.
\[
\begin{align*}
\int_{\mathbb{R}^n} \nabla u \cdot \nabla (\frac{|u|^n}{\tan d^{n-2} \log (\frac{d}{\sin d})^{n-1}}) \, d\sigma_n \\
= - \int_{\mathbb{R}^n} \nabla u \cdot \nabla (\frac{|u|^n}{\tan d^{n-2} \log (\frac{d}{\sin d})^{n-1}}) \, d\sigma_n \\
= \int_{\mathbb{R}^n} -n|u|^{n-2} \nabla u \cdot \nabla d \, d\sigma_n \\
+ (n-1) \int_{\mathbb{R}^n} |u|^n \, d\sigma_n \\
- (n-1) \int_{\mathbb{R}^n} |u|^n \, d\sigma_n.
\end{align*}
\] (5.6)

Analogously to the inequality (4.9), utilizing Hölder’s inequality followed by Young’s inequality we obtain

\[
\begin{align*}
\int_{\mathbb{R}^n} -n|u|^{n-2} \nabla u \cdot \nabla d \, d\sigma_n \\
\leq (n-1)\beta^{\frac{n}{n-1}} \int_{\mathbb{R}^n} |u|^n \, d\sigma_n + \frac{1}{\beta^n} \int_{\mathbb{R}^n} |\nabla u|^n \, d\sigma_n,
\end{align*}
\] (5.7)

for any \( \beta > 0 \). Inserting the estimate (5.7) into (5.6) and rearranging while taking into account the identity \( \sec^2 d = 1 + \tan^2 d \) produces

\[
\begin{align*}
\beta^n (1 - \beta^{\frac{n}{n-1}})(n-1) \int_{\mathbb{R}^n} |u|^n \, d\sigma_n \\
\leq \int_{\mathbb{R}^n} |\nabla u|^n \, d\sigma_n + \beta^n (n-1) \int_{\mathbb{R}^n} |u|^n \, d\sigma_n.
\end{align*}
\] (5.8)

The value of \( \beta \) that optimizes the inequality (5.8) is \( \beta = \left(\frac{n-1}{n}\right)^{\frac{1}{n-1}} \).

**Theorem 5.2** (Sharpness of the inequalities (5.1) and (5.2)) The inequality (5.1) is optimal in the following sense:

\[
\begin{align*}
\sup_{u \in W^{1,\infty}(\mathbb{R}^n) \setminus \{0\}} \frac{\gamma_n u_n(u)}{Q_n(u) + n\gamma_n^{n-1} u_n(u)} &= 1, \\
\sup_{u \in W^{1,\infty}(\mathbb{R}^n) \setminus \{0\}} \frac{\gamma_n u_n(u) - n \tilde{u}_n(u)}{Q_n(u)} &= \gamma_n^{1-n}.
\end{align*}
\] (5.9), (5.10)

The inequality (5.2) is also sharp. We precisely have

\[
\begin{align*}
\sup_{u \in W^{1,\infty}(\mathbb{R}^n) \setminus \{0\}} \frac{\gamma_n V_n(u)}{H_n(u) + (n-1)\gamma_n^{n-1} V_n(u)} &= 1, \\
\sup_{u \in W^{1,\infty}(\mathbb{R}^n) \setminus \{0\}} \frac{\gamma_n V_n(u) - (n-1) \tilde{V}_n(u)}{H_n(u)} &= \gamma_n^{1-n}, \\
\sup_{u \in W^{1,\infty}(\mathbb{R}^n) \setminus \{0\}} \frac{\gamma_n V_n(u) - H_n(u)}{V_n(u)} &= (n-1)\gamma_n^{n-1}.
\end{align*}
\] (5.11), (5.12), (5.13)
Proof. We prove (5.9)–(5.13) via introducing a sequence of nonzero real functions in $W^{1,n}(S^n)$ for which the respective suprema are attained. Define

$$f_\epsilon(\theta_n) := \left(\log \frac{e}{\sin d(\theta_n, \Phi_n)}\right)^{\frac{n-1}{\epsilon}}, \quad \theta_n \in S^n \setminus \{\pm \Phi_n\}.$$ 

Then, using formula (2.5), we have

$$\int_{S^n} |f_\epsilon| \, d\sigma_n = \int_{S^n} \frac{d\sigma_n}{\sin^n d(\log \frac{e}{\sin d})^{n\epsilon}} = 2C_n \int_0^1 \frac{ds}{(1-s^2)(\log \frac{e}{\sqrt{1-s^2}})^{1+\epsilon}} = 2C_n \int_0^1 \frac{ds}{s \sqrt{1-s^2}(\log \frac{e}{s})^{1+\epsilon}} = 2C_n (I(\epsilon) + \tilde{I}(\epsilon)), \quad (5.14)$$

where

$$I(\epsilon) := \int_0^1 \frac{ds}{s(\log \frac{e}{s})^{1+\epsilon}} = \frac{1}{\epsilon}, \quad (5.15)$$

$$0 < \tilde{I}(\epsilon) := \int_0^1 \left(\frac{1}{\sqrt{1-s^2}} - 1\right) \frac{ds}{s(\log \frac{e}{s})^{1+\epsilon}}$$

$$= \int_0^1 s \, ds \left(\frac{1}{\sqrt{1-s^2}}\right) \frac{1}{s} \left(\log \frac{e}{s}\right)^{1+\epsilon}$$

$$\leq \int_0^1 s \, ds \left(\frac{1}{\sqrt{1-s^2}}\right) \frac{1}{s} \left(\log \frac{e}{s}\right)^{1+\epsilon}$$

uniformly in $\epsilon$. Moreover,

$$\int_{S^n} |\nabla e_\epsilon|^n \, d\sigma_n (\gamma_n - \frac{1}{n} \epsilon)^n \quad (5.17)$$

with

$$0 < \tilde{I}(\epsilon) := \int_0^1 \frac{1 - (1-s^2)^{2n-1}}{s(\log \frac{e}{s})^{1+\epsilon}} \, ds = \int_0^1 \frac{(1 - (1-s^2)^{2n-1})}{(1 + (1-s^2)^{2n-1})s(\log \frac{e}{s})^{1+\epsilon}}$$
\[
\leq \int_0^1 \frac{1 - (1 - s^2)^{2n-1}}{s} \, ds = \sum_{r=1}^{2n-1} \binom{2n-1}{r} \frac{(-1)^r}{2r+1}. 
\]

(5.18)

It follows from (5.18) that

\[
\lim_{\epsilon \to 0^+} \tilde{I}(\epsilon) = \tilde{I}(0) = O(1)
\]

(5.19)

with an implicit constant that depends only on \( n \). Furthermore,

\[
\int_{\mathbb{S}} |f^*|^{\alpha} \, d\sigma_n = \int_{\mathbb{S}} \frac{d\sigma_n}{\sin^{n-2} d \left( \frac{e}{\sin d} \right)^{1+\epsilon}} = 2C_n \int_0^1 \frac{ds}{(\log \frac{e}{\sqrt{s^2-1}})^{1+\epsilon}} \rightarrow 2C_n \int_0^1 \frac{ds}{\log \frac{e}{\sqrt{s^2-1}}},
\]

(5.20)

as \( \epsilon \to 0^+ \), by the dominated (or monotone) convergence theorem. Observe that

\[
\int_0^1 \frac{ds}{\log \frac{e}{\sqrt{s^2-1}}} \leq 1.
\]

Using (5.14) together with (5.15), and (5.17), we obtain that

\[
\gamma_n^\mu U_n(f_\epsilon) = \frac{\gamma_n^\mu (1 + \tilde{I}(\epsilon))}{(\gamma_n - \frac{1}{n} \epsilon)^\mu (1 - \tilde{I}(\epsilon))} \rightarrow 1,
\]

by the limits (5.19) and (5.20). This convergence proves (5.9). In the same manner

\[
\gamma_n^\mu U_n(f_\epsilon) = \frac{\gamma_n^\mu (1 + \tilde{I}(\epsilon)) - n\gamma_n^\mu \tilde{U}_n(f_\epsilon)}{(1 - \epsilon \tilde{I}(\epsilon))} \rightarrow 1,
\]

which proves (5.10). Proceeding, we have

\[
\int_{\mathbb{S}} |f^*|^{\alpha} \, d\sigma_n = J_n(\epsilon), \quad \int_{\mathbb{S}} |\nabla f^*|^{\alpha} \, d\sigma_n = \left( \gamma_n - \frac{1}{n} \epsilon \right)^\mu J_n(\epsilon),
\]

(5.21)

where

\[
J_n(\epsilon) := \int_{\mathbb{S}} \frac{d\sigma_n}{\tan d^{\alpha} \left( \frac{e}{\sin d} \right)^{\alpha}} = 2C_n \int_0^1 \frac{s^n \, ds}{(1 - s^2)(\log \frac{e}{\sqrt{s^2-1}})^{1+\epsilon}} = 2C_n \int_0^1 \frac{(1 - s^2)^{\frac{\alpha+1}{2}}}{s(\log \frac{s}{\sqrt{2}})^{1+\epsilon}} \, ds.
\]

(5.22)

Note here that

\[
\lim_{\epsilon \to 0^+} \tilde{J}_n(\epsilon) = \tilde{J}_n(0) = O(1),
\]

(5.23)
where the implicit constant depends solely on the dimension $n$. This follows from the dominated convergence theorem as we have the uniform bound
\[
\left| \tilde{J}_n(\epsilon) \right| \leq \int_0^1 \frac{1 - (1 - s^2)^{n-1}}{s(1 + (1 - s^2)^{\frac{n+1}{2}})(\log \frac{s}{\epsilon})^{1+\epsilon}} \, ds \\
\leq \int_0^1 \frac{1 - (1 - s^2)^{n-1}}{s} \, ds \\
= \frac{1}{2} \sum_{r=1}^{n-1} \left( n - 1 \right) \frac{(-1)^{r+1} r}{r}.
\]

We also have
\[
\int_{\mathbb{S}^n} |f_\epsilon|^n d\sigma_n = 2C_n \int_0^1 \frac{s^{n-2}}{(\log \frac{s}{\sqrt{-1}})^{\epsilon}} \, ds \rightarrow \frac{2C_n}{n-1},
\]
by the dominated convergence theorem. Now, using (5.21) and (5.22) we obtain
\[
\frac{\gamma_n^\alpha V_n(f_\epsilon)}{H_n(f_\epsilon) + (n - 1)\gamma_n^{\alpha-1} V_n(f_\epsilon)} = \frac{1}{(\gamma_n - \frac{\epsilon}{n})(1 + \epsilon \tilde{J}_n(\epsilon)) + (n - 1)\gamma_n^{\alpha-1} \epsilon \frac{V_n(f_\epsilon)}{2C_n}} \rightarrow 1
\]
as $\epsilon \rightarrow 0^+$ by (5.23) and the convergence in (5.24). This proves (5.11). We also have
\[
\frac{\gamma_n V_n(f_\epsilon) - (n - 1)\tilde{V}_n(f_\epsilon)}{H_n(f_\epsilon)} = \frac{1}{(\gamma_n - \frac{\epsilon}{n})(1 + \epsilon \tilde{J}_n(\epsilon))} \rightarrow \frac{1}{\gamma_n^{\alpha-1}},
\]
when $\epsilon \rightarrow 0^+$ using (5.23) and (5.24). This proves (5.12). Finally, we prove (5.13). Employing (5.21) and (5.22) one last time, we find
\[
\frac{\gamma_n^\alpha V_n(f_\epsilon) - H_n(f_\epsilon)}{V_n(f_\epsilon)} = \frac{(\gamma_n^\alpha - (\gamma_n - \frac{\epsilon}{n})(\frac{1}{n} + \tilde{J}_n(\epsilon)))}{\frac{V_n(f_\epsilon)}{2C_n}} \\
= \frac{(\gamma_n^\alpha - (\gamma_n - \frac{\epsilon}{n})(1 + \epsilon \tilde{J}_n(\epsilon)))}{\epsilon} \rightarrow (n - 1)\gamma_n^{\alpha-1}
\]
by (5.23) and the limit in (5.24). □

6 A lower bound for the first eigenvalue of the $p$-Laplacian on the sphere

Let $M$ be a compact connected manifold without boundary. The $p$-Laplacian on $M$ is the operator given by $\Delta_p u := -\text{div}_g(|\nabla_g u|^{p-2} \nabla_g u)$, where $\nabla_g$ is the gradient induced by the Riemannian metric $g$ on $M$ and $\text{div}_g$ is the adjoint of $\nabla_g$ for the $L^2$-norm induced by the metric $g$ on the space of differential forms. The $p$-Laplacian is associated with the $p$-energy functional $E_p(u) := \int_M |\nabla_p u|^p d_\epsilon$, where $d_\epsilon$ is the Riemannian volume element induced by $g$. 
A nonzero function \(u\) that satisfies \(\Delta_p u = \lambda |u|^{p-2} u\) for some \(\lambda \in \mathbb{R}\) is an eigenfunction of \(\Delta_p\) that corresponds to the eigenvalue \(\lambda\). The set of nonzero eigenvalues of \(\Delta_p\) is an unbounded subset of \([0, +\infty)\) (see [6]). The infimum \(\lambda_p(M)\) of this set is itself an eigenvalue that has the variational characterization ([10, 13, 14]):

\[
\lambda_p(M) = \inf \left\{ \int_M |\nabla f|^p \, d\sigma : f \in W^{1,p}(M) \setminus \{0\}, \int_M |f|^{p-2} f^2 \, d\sigma = 0 \right\}.
\]

Let \(M\) be the unit sphere \(S^{n-1}\), \(n \geq 4\), one can apply Theorem 4.1 to obtain lower bounds for the first nonzero eigenvalue \(\lambda_p(S^{n-1})\) of the \(p\)-Laplacian

\[
\Delta_p u := -\nabla_{S^{n-1}} \cdot (|\nabla_{S^{n-1}} u|^{p-2} \nabla_{S^{n-1}} u).
\]

It is well known ([2, 10]) that \(\lambda_p(S^{n-1}) \geq \left(\frac{n-2}{n-1}\right)^{p/2}, p \geq 2\). Let \(2 \leq p < n - 1\). Using the sharp Hardy inequality (4.1) of Theorem 4.1, we obtain

\[
\lambda_p(S^{n-1}) = \inf \int_{S^{n-1}} |\nabla_{S^{n-1}} u|^p \, d\sigma_{n-1} \int_{S^{n-1}} |u|^p \, d\sigma_{n-1}
\geq \alpha_{n,p}^p \inf \int_{S^{n-1}} |u|^p \sin d \, d\sigma_{n-1} \int_{S^{n-1}} |u|^p \, d\sigma_{n-1}
- (n-p)\alpha_{n,p}^{p-1} \sup \int_{S^{n-1}} |u|^p \sin d \, d\sigma_{n-1} \int_{S^{n-1}} |u|^p \, d\sigma_{n-1},
\]

where the infima and supremum are taken over all nontrivial functions in \(W^{1,p}(S^{n-1})\). It similarly follows from the inequality (4.2) of Theorem 4.1 that

\[
\lambda_p(S^{n-1}) \geq \alpha_{n,p}^p \inf \int_{S^{n-1}} |u|^p \tan d \, d\sigma_{n-1} \int_{S^{n-1}} |u|^p \, d\sigma_{n-1}
- (p-1)\alpha_{n,p}^{p-1} \sup \int_{S^{n-1}} |u|^p \tan d \, d\sigma_{n-1} \int_{S^{n-1}} |u|^p \, d\sigma_{n-1}.
\]

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