PROPERTY (RD) FOR HECKE PAIRS

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Abstract. As the first step towards developing noncommutative geometry over Hecke \( C^* \)-algebras, we study property (RD) (Rapid Decay) for Hecke pairs. When the subgroup \( H \) in a Hecke pair \( (G, H) \) is finite, we show that the Hecke pair \( (G, H) \) has (RD) if and only if \( G \) has (RD). This provides us with a family of examples of Hecke pairs with property (RD). We also adapt Paul Jolissat’s works in \([19, 20]\) to the setting of Hecke \( C^* \)-algebras and show that when a Hecke pair \( (G, H) \) has property (RD), the algebra of rapidly decreasing functions on the set of double cosets is closed under holomorphic functional calculus of the associated (reduced) Hecke \( C^* \)-algebra. Hence they have the same \( K_0 \)-groups.

1. Introduction

Let \( H \) be a subgroup of an arbitrary group \( G \). It is called almost normal in \( G \) and is denoted by \( H \triangleleft_a G \) if for every \( g \in G \), the double coset \( HgH \) is a finite union of its left cosets. A pair \( (G, H) \) as above is called a Hecke pair. Elementary examples of almost normal subgroups are normal subgroups, finite subgroups and subgroups of finite index. Besides these examples, Hecke pairs have been appeared for the first time in the theory of modular forms by considering \( G = GL(2, \mathbb{Q})^+ := \{ g \in GL(2, \mathbb{Q}); \det(g) > 0 \} \) and \( H = SL(2, \mathbb{Z}) \). We refer the reader to Proposition 1.4.1 of \([8]\), for a proof of the fact that \( H \triangleleft_a G \). Hecke \( C^* \)-algebras were used by Jean-Benoît Bost and Alain Connes, in \([7]\), in order to construct a \( C^* \)-dynamical system illustrating the class field theory of the field \( \mathbb{Q} \) of rational numbers. Their Hecke pair consists of the group \( P^+_\mathbb{Q} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Q} \text{ and } a > 0 \right\} \) and its subgroup \( P^+_\mathbb{Z} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \). All groups in the above examples are considered as discrete groups. In the setting of locally compact topological groups, one easily observes that every compact open subgroup of a locally compact group is almost normal. In fact, Kroum Tzanev has shown that every Hecke pair can be “replaced” with a Hecke pair of this type and the associated enveloping Hecke \( C^* \)-algebra would not change, see Theorem 4.2 of \([20]\).

In this paper, we restrict our attention to the case that \( G \) is a discrete group. The set of all double cosets of a Hecke pair \( (G, H) \) is denoted by \( G//H \) and \( \mathcal{H}(G, H) \) denotes the vector space of finite support complex functions on \( G//H \). An arbitrary

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Let $(G, H)$ be a Hecke pair. The vector space $\mathcal{H}(G, H)$ equipped with the following convolution like product

$$ f_1 * f_2(g) := \sum_{\gamma \in <H\backslash G>} f_1(\gamma)f_2(\gamma^{-1}g), \quad f_1, f_2 \in \mathcal{H}(G, H). $$

and the involution

$$ f^*(g) := \overline{f(g^{-1})} $$

is called the Hecke algebra of $(G, H)$.

Let $C(H\backslash G)$ denote the vector space of finite support complex functions on the set of right cosets of $H$ in $G$. By extending the definition of the above convolution product, it is endowed with an $\mathcal{H}(G, H)$-module structure as follows:

$$ \mathcal{H}(G, H) \times C(H\backslash G) \to C(H\backslash G) $$

$$ f_1 * f_2(g) := \sum_{\gamma \in <H\backslash G>} f_1(\gamma)f_2(\gamma^{-1}g), $$

for all $g \in H\backslash G$. This gives rise to a $\ast$-representation $\lambda : \mathcal{H}(G, H) \to B(\ell^2(H\backslash G))$ defined by left convolutions;

$$(1.1) \quad \lambda(f)(\xi)(g) := (f \ast \xi)(g) = \sum_{\gamma \in <H\backslash G>} f(\gamma)\xi(\gamma^{-1}g),$$

for all $f \in \mathcal{H}(G, H)$, $\xi \in \ell^2(H\backslash G)$ and $g \in H\backslash G$. Due to the fact that each double coset is the union of only finitely many right cosets, one easily checks that every function in $\mathcal{H}(G, H)$ is mapped to a bounded operator by $\lambda$. We call this map the regular representation of the Hecke pair $(G, H)$. For $f \in \mathcal{H}(G, H)$, the norm of $\lambda(f)$ in $B(\ell^2(H\backslash G))$ is called the convolution norm of $f$ and is denoted by $\|\lambda(f)\|$.  

#### Definition 1.2.
The norm closure of the image of the regular representation of a Hecke pair $(G, H)$ is called the reduced Hecke $C^\ast$-algebra of $(G, H)$ (or shortly Hecke $C^\ast$-algebra of $(G, H)$) and is denoted by $C^*_r(G, H)$.

We refer the reader to Rachel Hall’s thesis, [16], for the definition of full Hecke $C^\ast$-algebras and a study of the problem of the existence of full Hecke $C^\ast$-algebras. We also refer the reader to [26] for the definition of the enveloping $C^\ast$-algebra of a Hecke pair. The Hecke algebra $\mathcal{H}(G, H)$ also acts on $\ell^2(G/H)$ by right convolution, see [10] for details.

It is clear that when $H$ is normal in $G$, the above definitions coincide correspondingly with the definitions of the group algebra, the convolution product and norm, the regular representation, and the reduced group $C^\ast$-algebra of the quotient group $G/H$. This point of view motivates our program to generalize the concepts and tools of noncommutative geometry over group $C^\ast$-algebras to the more general setting of Hecke $C^\ast$-algebras. We note that Kroum Tzanev started a similar program to reformulate the Baum-Connes conjecture in the setting of Hecke $C^\ast$-algebras in his thesis, see [25]. In the following, we explain why we begin with the study of property (RD) for Hecke pairs.

Many notions of noncommutative geometry over $C^\ast$-algebras are defined on a specific type of dense subalgebras of $C^\ast$-algebras, often called smooth subalgebras of $C^\ast$-algebras. The name comes from the commutative case that the algebra $C^\infty(M)$
of smooth complex functions on a compact smooth manifold \( M \) is dense in the \( C^\ast \)-algebra \( C(M) \) of continuous complex functions on \( M \). The main feature of these smooth subalgebras is that they are closed under holomorphic functional calculus of the containing \( C^\ast \)-algebras.

**Definition 1.3.** An involutive dense subalgebra \( A \) of a \( C^\ast \)-algebra \( A \) is called smooth if for every element \( a \in A \) and every holomorphic function \( f \) defined over an open set containing the spectrum of \( a \) in \( A \), \( f(a) \) belongs to \( A \). In this case, we also say that \( A \) is closed under holomorphic functional calculus of \( A \).

**Remark 1.4.** If \( A \) is a smooth subalgebra of a \( C^\ast \)-algebra \( A \), then it has the property of spectral permanence, see [2], namely the spectrum of each element of \( A \) in \( A \) is the same as its spectrum in \( A \), see proposition 3.1.3 in [4]. The converse is true if \( A \) is endowed with a topology for which the Cauchy integral of holomorphic functional calculus converges. For instance, if \( A \) is an involutive dense Banach algebra and has the property of spectral permanence in \( A \), then it is a smooth subalgebra of \( A \). This was noted by Jean-Benoît Bost in the discussion after Theorem 1.3.1 of [6].

If \( M_n(A) \) is a smooth subalgebra of \( M_n(A) \) for every positive integer \( n \), then \( A \) is called a local \( C^\ast \)-algebra. This property ensures that \( A \) is similar enough to the \( C^\ast \)-algebra \( A \) to carry many features of \( A \). For example, both have the same \( K_0 \)-groups, see [4].

**Remark 1.5.** In the appendix of [6], J.-B. Bost introduced an interesting method to show under some minor conditions spectral permanence implies the equality of \( K \)-groups too. Clearly, his method is more general than using stability under holomorphic functional calculus, because it works in the setting of Fréchet algebras. Reader can find more details in Theorem A.2.1 of [6] about his approach.

The main reason for constructing smooth subalgebras inside \( C^\ast \)-algebras is that working with smooth subalgebras instead of \( C^\ast \)-algebras has the advantage that they are more capable of algebraic constructions like Connes’ cyclic cohomology and geometric constructions like Connes’ spectral triple, [11] [12]. Therefore, in many situations, the first step to study the noncommutative geometry of a \( C^\ast \)-algebra is to define a smooth subalgebra.

There are many ways to define smooth subalgebras of \( C^\ast \)-algebras. For example, the subalgebra of smooth elements with respect to an action of a Lie group on a \( C^\ast \)-algebra is smooth. For more general constructions of smooth subalgebras using differential seminorms and derivations, see [5]. Whenever the \( C^\ast \)-algebra under consideration is related to a group, for example group \( C^\ast \)-algebras and crossed product \( C^\ast \)-algebras, harmonic analysis provides us with a method to define smooth subalgebras. From Fourier analysis one knows that the algebra of rapidly decreasing (also called Schwartz) functions on \( Z \) is isomorphic to the algebra \( C^\infty(\mathbb{T}) \) of smooth functions on the unit circle \( \mathbb{T} \subset \mathbb{C} \). Since \( C^\infty(\mathbb{T}) \) is a smooth subalgebra of \( C(\mathbb{T}) \) and this \( C^\ast \)-algebra is isomorphic to \( C^\ast(\mathbb{Z}) \), one can consider the algebra of rapidly decreasing functions on \( Z \) as a smooth subalgebra of \( C^\ast(\mathbb{Z}) \). The idea of considering the subalgebra of rapidly decreasing functions on a group as the smooth subalgebra of its reduced group \( C^\ast \)-algebra was generalized by Paul Jolissaint for groups possessing property (RD) in [19] [20].

The main purpose of this article is to show, in details, that this idea works similarly for Hecke pairs and Hecke \( C^\ast \)-algebras. Therefore, our main result is that
when a Hecke pair \((G, H)\) possesses property (RD) with respect to a length function \(L\), the subalgebra of rapidly decreasing functions on the set of double cosets with respect to \(L\) is a smooth subalgebra of \(C^*_r(G, H)\), see Proposition 3.7 and results after that. As the first application of this fact, we show that these algebras have the same \(K_0\)-groups, see Corollary 3.12. These are done in Section 3. Our proofs are a modification of the original proofs, given by Paul Jolissaint, for the general framework of Hecke \(C^*\)-algebras.

Section 2 is devoted to the definition of property (RD) and equivalent definitions. Some remarks and complementary discussions are also given in Section 2. In order to give some examples of non-trivial Hecke pairs with property (RD), we show, in Theorem 2.11, that if \(H\) is a finite subgroup of a group \(G\), then the Hecke pair \((G, H)\) has (RD) if and only if the group \(G\) has (RD).

2. Property (RD)

The definition of property (RD) for a group is based on the notion of a length function. For a discrete group \(G\), a length function on \(G\) is a function \(L : G \rightarrow [0, \infty]\) which satisfies following conditions for all \(g, h \in G\):

(i) \(L(gh) \leq L(g) + L(h)\),
(ii) \(L(g) = L(g^{-1})\),
(iii) \(L(1) = 0\).

A length function \(L\) on \(G\) gives rise to a weighted \(\ell^2\)-norm, for every \(s > 0\), as follows:

\[
\| f \|_{s,L} := \left( \sum_{g \in G} |f(g)|^2 (1 + L(g))^{2s} \right)^{\frac{1}{2}}, \quad \forall f \in \mathbb{C}G.
\]

Let \(\lambda : \mathbb{C}G \rightarrow \mathcal{B}(\ell^2(G))\) be the left regular representation of \(G\). The convolution norm of a function \(f\) in \(\mathbb{C}G\) is defined by \(\| \lambda(f) \|\).

**Definition 2.1.** We say a group \(G\) has property (RD) if there exist a length function \(L\) on \(G\) and positive real numbers \(C\) and \(s\) such that the Haagerup inequality:

\[
\| \lambda(f) \| \leq C \| f \|_{s,L}
\]

holds for all \(f \in \mathbb{C}G\).

Uffe Haagerup introduced and proved Inequality 2.1 for \(C = s = 2\), for free groups of rank \(n \geq 2\) equipped with the word length function in \([15]\). Afterwards, Jolissaint gave the formal definition of property (RD) and proved several statements related to this property and discussed several examples of groups possessing this property in \([20]\). He also showed that in the presence of property (RD) the subalgebra of rapidly decreasing functions is smooth in the reduced group \(C^*\)-algebra, \([19]\). Pierre de la Harpe proved property (RD) for hyperbolic groups in \([17]\). As it was noted in \([13, 19]\), when a discrete group \(G\) has property (RD) a cyclic co-cycle of \(C\), (an element of the cyclic cohomology group of \(\mathbb{C}G\)), satisfying some additional conditions extends to an \(n\)-trace over \(C^*_r(G)\). Thus, it defines an index map \(K_0(C^*_r(G)) \rightarrow \mathbb{C}\), (in the terminology of noncommutative geometry and cyclic cohomology). This phenomenon was used by Alain Connes and Henri Moscovici to prove the Novikov conjecture for hyperbolic groups in \([13]\). Property (RD) also appears in Vincent Lafforgue’s proof of the Baum-Connes conjecture for a new family of groups, see \([22]\) for more details. More recently, property (RD) has been applied.
to define and study noncommutative metrics over the state spaces of several reduced group $C^*$-algebras by Cristina Antonescu and Erik Christensen in [1]. This latter application of property (RD) is another motivation of the present work and will be discussed in our subsequent paper. We also note that property (RD) for discrete quantum groups was studied by Roland Vergnioux in [28].

**Remark 2.2.** Jolissaint has shown that when $G$ is an amenable group, $G$ has (RD) if and only if $G$ is of polynomial growth, see Corollary 3.1.8 of [20]. This is the only obstruction known for property (RD) yet. In Proposition 6 of [27], Alain Valette refined this result to investigate a family of groups which are not amenable.

Now, we extend the definition of property (RD) to Hecke pairs, and later on we will use this notion to define smooth subalgebras in Hecke $C^*$-algebras.

**Remark 2.3.** Given a length function $L$ on $G$, it is easy to check that $N_L := \{g \in G; L(g) = 0\}$ is a subgroup of $G$. Let $H$ be a subgroup of $N_L$. Then, for every $h_1, h_2 \in H$ and $g \in G$, we have $L(g) = L(h_1^{-1}h_1g) \leq L(h_1^{-1}) + L(h_1g) = L(h_1g) \leq L(h_1) + L(g) = L(g)$, and consequently $L(g) = L(h_1g)$. Similarly, one can show that $L(g) = L(h_1gh_2)$.

This observation leads us to the following definition:

**Definition 2.4.** A length function on a Hecke pair $(G, H)$ is a length function $L$ on $G$ such that $H \leq N_L$.

Similar to group algebras, for every $s > 0$, we define the weighted $\ell^2$-norm associated with $s$ and $L$ on finite support functions on the set $G//H$ of double cosets as follows

$$\|f\|_{s,L} := \left( \sum_{g \in <H\backslash G>} |f(g)|^2 \left(1 + L(g)\right)^{2s} \right)^{\frac{1}{2}}, \quad \forall f : G//H \rightarrow \mathbb{C}. $$

**Definition 2.5.** We say a Hecke pair $(G, H)$ has property (RD) if there exist a length function $L$ on $(G, H)$ and positive real numbers $C$ and $s$ such that the Haagerup inequality;

$$\|\lambda(f)\| \leq C\|f\|_{s,L}$$

holds for all $f \in \mathcal{H}(G, H)$.

**Remark 2.6.** For $f \in \mathcal{H}(G, H)$, we defined property (RD) based on the norm $\|f\|_{s,L}$ which uses a sum over right cosets. Alternatively, one can define property (RD') using the norm $\|f\|'_{s,L} := \left( \sum_{g \in <G//H>} |f(g)|^2 \left(1 + L(g)\right)^{2s} \right)^{\frac{1}{2}}$. Clearly, we have $\|f\|'_{s,L} \leq \|f\|_{s,L}$, so property (RD') with respect to a length function $L$ implies property (RD) with respect to $L$.

**Proposition 2.7.** If a Hecke pair $(G, H)$ has property (RD') with respect to a length function $L$ then there are positive constants $D$ and $t$ such that $|HgH/Hg| \leq D(1 + L(g))^t$ for all $g \in G$.

First, we note that, in the presence of the above inequality, property (RD) clearly implies property (RD').
Proof. Let \((G, H)\) has property (RD') with respect to \(L\) and with some positive constants \(C\) and \(s\). For \(g \in G\), let \(\delta_g \in \mathcal{H}(G, H)\) denote the characteristic function of the double coset \(HgH\) and let \(\delta_1 \in \mathcal{C}(H\backslash G)\) denote the characteristic function of the right coset \(H\). Then we have \(|HgH/Hg| = \|\delta_g\|_2^2 = \|\delta_g \ast \delta_1\|_2^2 \leq \|\lambda(\delta_g)\|_2^2 \leq C^2\|\delta_1\|_2^{2s} = C^2(1 + L(g))^2s\), where the norm \(\| - \|_2\) is the norm of \(\ell^2(H\backslash G)\). \(\square\)

We do not try to verify all statements for both property (RD) and property (RD'). But one can easily check that the content of Section 3 holds similarly for property (RD'). To conform the content of the rest of this section with property (RD'), one should modify Proposition 2.10 according to this property.

Example 2.8. (i) When \(H\) is normal in \(G\), one notes that property (RD) for a Hecke pair \((G, H)\) coincides with property (RD) of \((G/H, H)\), in particular when \(H = \{e\}\).

(ii) Let \(H\) be a subgroup of \(G\) of finite index, say \(n\). Then, using Proposition 6.12 of [14], we have \(\| - \|_2 \leq n\| - \|_1\) and \(\| - \|_2\) are \(\ell^1\) and \(\ell^2\) norms of \(H\backslash G\), respectively. Given \(f \in \mathcal{H}(G, H)\), \(f\) is an element of \(\mathbb{C}(H\backslash G)\) too and by repeating the proof of Young’s inequality in our setting, see Proposition 8.7 of [14], we have \(\|\lambda(f)\| \leq \|f\|_1\). Therefore, in this case, the Haagerup inequality [22] holds for \(L = 0, C = \sqrt{n}\) and any positive real number \(s\).

One possible application of property (RD) for Hecke pairs is the following problem which is motivated by Remark 2.2.

Problem 2.9. Define the notion of the growth of a Hecke pair, compatible with the definition of growth for groups. Find a relationship between the amenability of Hecke pairs as defined by Tzanev in [26] and property (RD) of Hecke pairs. Look for examples of non-amenable Hecke pairs using this relationship.

There are several conditions equivalent to property (RD) which can be generalized to the setting of Hecke pairs and are necessary for our discussion. First, we need some notations. The subsets of non-negative real functions in \(\mathcal{H}(G, H)\), \(\mathbb{C}(H\backslash G)\), and \(\mathbb{C}G\) are denoted by \(\mathcal{H}_+(G, H)\), \(\mathbb{R}_+(H\backslash G)\), and \(\mathbb{R}_+(G)\), respectively. For a length function \(L\) on a Hecke pair \((G, H)\) and for every non-negative real number \(r\), we define

\[
B_{r,L}(G, H) := \{HgH \in < G/H ; L(g) \leq r\},
\]

\[
C_{r,L}(G, H) := \{HgH \in < G/H ; L(g) \leq r \in \mathbb{R} \}
\]

We also denote the similar sets in \(< H\backslash G >\) and \(G\) by \(B_{r,L}(H\backslash G)\), \(C_{r,L}(H\backslash G)\) and \(B_{r,L}(G)\), \(C_{r,L}(G)\), respectively. For \(f \in \mathbb{C}(H\backslash G)\) or \(f \in \mathcal{H}(G, H)\), the norm of \(f\) in \(\ell^2(H\backslash G)\) is denoted by \(\|f\|_2\).

Proposition 2.10. Let \(L\) be a length function on a Hecke pair \((G, H)\). Then the following statements are equivalent:

(i) The Hecke pair \((G, H)\) has property (RD) with respect to \(L\).

(ii) There exists a polynomial \(P\) such that for any \(r > 0\) and \(f \in \mathcal{H}_+(G, H)\) so that \(\text{supp} f \subseteq B_{r,L}(G, H)\), we have

\[
\|\lambda(f)\| \leq P(r)\|f\|_2.
\]
Proof. Let $s, C$ be positive real numbers such that $\|\lambda(-)\| \leq C\| \cdot s, L \|$ over $\mathcal{H}(G, H)$. For $r \geq 0$, assume $f \in \mathcal{H}_+(G, H)$ is supported in $B_{r, L}(G, H)$. Then we have

$$\|\lambda(f)\| \leq C\| f \|_{s, L} = C \left( \sum_{g \in B_{r, L}(H \setminus G)} |f(g)|^2 (1 + L(g))^{2s} \right)^{1/2} \leq C \left( \sum_{g \in B_{r, L}(H \setminus G)} |f(g)|^2 (1 + r)^{2s} \right)^{1/2} = C(1 + r)^s \| f \|_2.$$ 

Hence (ii) follows from (i) by considering $P(r) = C(1 + r)^s$.

Conversely, let (ii) hold for some polynomial $P$. One can easily find two positive numbers $C, s$ such that $P(n) \leq Cn^{s-1}$ for all $n \in \mathbb{N}$. Let $r$ be a non-negative real number and let $f \in \mathcal{H}(G, H)$ so that $\text{supp} f \subseteq B_{r, L}(G, H)$. It is easy to check that $\|\lambda(f)\| \leq \|\lambda([f])\|$. Combining this inequality with (ii) and the fact that $\|f\|_2 = \|f\|_2$, we get $\|\lambda(f)\| \leq P(r)\|f\|_2$. Now, let $f \in \mathcal{H}(G, H)$ be arbitrary and let $\chi_n$ denote the characteristic function of $C_{n-1, L}(H \setminus G)$ for all $n \in \mathbb{N}$. Then, $\|\lambda(f)\| = \left| \sum_{n=1}^{\infty} \lambda(f \chi_n) \right| \leq \sum_{n=1}^{\infty} \|\lambda(f \chi_n)\| \leq \sum_{n=1}^{\infty} P(n)\|f \chi_n\|_2 \leq \sum_{n=1}^{\infty} Cn^{s-1}\|f \chi_n\|_2 = C \left( \sum_{n=1}^{\infty} n^{s-1} \right) \|f \chi_n\|_2$. By applying Cauchy-Schwarz inequality of $L^2(\mathbb{N})$, we obtain $\|\lambda(f)\| \leq C \left( \sum_{n=1}^{\infty} n^{-2} \right)^{1/2} \left( \sum_{n=1}^{\infty} n^{2s-2} \|f \chi_n\|_2^2 \right)^{1/2}$. Setting $C' := C \left( \sum_{n=1}^{\infty} n^{-2} \right)^{1/2}$, we have $\|\lambda(f)\| \leq C' \left( \sum_{n=1}^{\infty} n^{2s} \|f \chi_n\|_2^2 \right)^{1/2} \leq C' \left( \sum_{n=1}^{\infty} (1 + L(g))^{2s} \sum_{g \in C_{n-1, L}(H \setminus G)} |f(g)|^2 \right)^{1/2}$, where the last inequality follows from the facts that $n \leq L(g) + 1$ for every $g \in C_{r-1, L}(H \setminus G)$ and $\|f \chi_n\|_2^2 = \sum_{g \in C_{n-1, L}(H \setminus G)} |f(g)|^2$. Finally, since $\bigcup_{n=1}^{\infty} C_{n-1, L}(H \setminus G) = H \setminus G$, we obtain $\|\lambda(f)\| \leq C' \left( \sum_{g \in H \setminus G} (1 + L(g))^{2s} |f(g)|^2 \right)^{1/2} = C'\|f\|_{s, L}$. The equivalence between (ii) and (iii) is easy and is left to the reader.

The above proposition and its proof, in the setting of groups, appeared implicitly in [15], for instance, see Lemmas 1.4 and 1.5 in there. Condition (ii) in the above also appeared as the definition of property (RD) in [21]. The proof of this proposition is an adaptation of the proof for the case of groups which was taken from [9], see also proposition 1.2 of [10].

For our discussion in the rest of this section, we need to recall some more definitions from [20] and set some notations. Let $L_1$ and $L_2$ be two length functions on some group $G$. We say $L_1$ dominates $L_2$ if there exist positive real numbers $a, b$ such that $L_2 \leq aL_1 + b$. In this case $1 + L_2 \leq (b+1)+aL_1$ and setting $M := \max\{b+1, a\}$, we get $1 + L_2 \leq M(1 + L_1)$. This implies that $||-||_{s, L_2} \leq M||-||_{s, L_1}$ for all $s > 0$. Thus if $G$ has property (RD) with respect to $L_2$ then it has it with respect to $L_1$ as well. For instance, the word length function of a finitely generated group $G$ dominates other length functions of $G$, see Lemma 1.1.4 of [20]. So, if $G$ does not have (RD) with respect to the word length function it does not have (RD) with respect to other length functions either. If two length functions $L_1$ and $L_2$ dominate each
other, we say $L_1$ and $L_2$ are equivalent. It is clear that the above discussion holds for Hecke pairs too. In particular, if a Hecke pair has property (RD) with respect to a length function $L$ it has property (RD) with respect to any length function equivalent to $L$.

The following theorem relates the property (RD) of groups and Hecke pairs consisting of groups and their finite subgroups.

**Theorem 2.11.** Let $H$ be a finite subgroup of a group $G$. Then $G$ has property (RD) if and only if the Hecke pair $(G, H)$ has property (RD).

**Proof.** Let $n$ be the order of $H$. Suppose $G$ has (RD) with respect to a length function $L$. Using Lemma 2.1.3 of [20], we can replace $L$ with an equivalent length function which is zero on $H$. Let $P$ be the polynomial appeared in Part (iii) of Proposition 2.10. Let $H = \{h_1, \cdots, h_n\}$. For $f \in H^+(G, H)$, define $\tilde{f} \in H^+(G, H)$ by $\tilde{f}(x) = k(Hx)$ (resp. $\tilde{f}(x) = f(HxH)$) for all $x \in G$. Then, we have

$$\|\tilde{k}\|_2^2 = n\|k\|_2^2,$$

where the norms are taken in $\ell^2(G)$ and $\ell^2(H \setminus G)$, respectively. Also, we note that $\tilde{f} \in B_{r,L}(G)$ and we have

$$\|\tilde{f}\|_2 = n\|f\|_2,$$

where the norms are taken in $\ell^2(G)$ and $\ell^2(H \setminus G)$, respectively. Now, we have

$$\|f \ast k\|_2^2 = \sum_{y \in H \setminus G} \left( \sum_{x \in H \setminus G} f(x)k(x^{-1}y) \right)^2$$

$$= \sum_{y \in H \setminus G} \left( \frac{1}{n} \sum_{x \in G} \tilde{f}(x)\tilde{k}(x^{-1}y) \right)^2$$

$$= \frac{1}{n^3} \sum_{y \in G} (\tilde{f} \ast \tilde{k}(y))^2$$

$$= \frac{1}{n^3}\|\tilde{k}\|_2^2$$

$$\leq \frac{1}{n^3}P(r)^2\|\tilde{f}\|_2^2\|\tilde{k}\|_2^2$$

$$= \frac{1}{n}P(r)^2\|f\|_2^2\|k\|_2^2.$$

Thus $(G, H)$ has (RD) with respect to $L$.

Conversely, let $(G, H)$ has (RD) with respect to $L$ and let $P$ be the polynomial in Part (iii) of Proposition 2.10. Let $H = \{h_1, \cdots, h_n\}$. For $f \in H^+(G)$ with supp$f \subseteq B_{r,L}(G)$, define $\tilde{f} \in H^+(G, H)$ by $\tilde{f}(HgH) := \sum_{j=1}^nf(h_ig)$. For $m \in \mathbb{N}$, let $c(m)$ be the least constant for which $(\sum_{i=1}^m x_i)^2 \leq c(m)\sum_{i=1}^m x_i^2$ for all
some finite subgroup $H$.

Proposition 2.1.1 of [20].

Every group $G$ has property (RD), then the Hecke pair $(G, H)$ possesses property (RD). Constructions in Section 2 of [20] provide us with a number of examples. Definition 2.13.

Let $E$ be a group generated by a finite set $S = S^{-1}$ and let $l$ be the word length function on $E$ with respect to $S$. For $\alpha \in \text{Aut}(E)$, the amplitude of $\alpha$ with respect to $S$ is the number $a(\alpha) := \max_{s \in S} l(\alpha(s))$. Let $F$ be another finitely generated group with a word length function $L$. We say a map $\theta : F \to \text{Aut}(E)$ has polynomial amplitude if there exist positive constants $c$ and $r$ such that $a(\theta(f)) \leq c(1 + L(f))^r$ for all $f \in F$.

Proposition 2.14 ([20], Corollary 2.1.10). Let $F$ and $E$ be two finitely generated groups and let $\alpha$ be an action of $F$ on $E$ of polynomial amplitude. If $E$ and $F$ have property (RD), then so does the semidirect product $E \ltimes_{\alpha} F$.

Clearly, any action of a finite group on a finitely generated group is of polynomial amplitude. Thus we have the following corollary:

Corollary 2.15. Let $F$ be a finite group acting on a finitely generated group $G$. If $G$ has property (RD), then the Hecke pair $(G \ltimes F, F)$ has property (RD).
Example 2.16. Consider the infinite dihedral group $D_∞ = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The Hecke pair $(D_∞, \mathbb{Z}/2\mathbb{Z})$ has property (RD).

We note that the Hecke pair $(D_∞, \mathbb{Z}/2\mathbb{Z})$ is actually a Gelfand pair, namely, its Hecke algebra is commutative.

Remark 2.17. Theorem 2.11 can be applied to give non-examples too. One only needs to find a group $G$ not having (RD) with some non-normal finite subgroup. To find such a group, we use Corollaries 3.1.8 and 3.1.9 of [20]. For example, $SL_3(\mathbb{Z})$ does not have (RD). To create a situation as above, set $T := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and let $H$ denote the subgroup generated by $T$. Since $STS^{-1} \notin H$, $H$ is not normal in $SL_3(\mathbb{Z})$. Thus, the Hecke pair $(SL_3(\mathbb{Z}), H)$ is a non-trivial Hecke pair which does not have (RD). If $G$ does not have a non-normal finite subgroup or it is difficult to find such a subgroup, one can try the semidirect product of $G$ with a finite group $H$ which $H$ acts non-trivially on $G$. If there is such a finite group $H$, then the Hecke pair $(G \rtimes H, H)$ does not have (RD).

3. The smooth subalgebra of rapidly decreasing functions

In this section, $(G, H)$ is a Hecke pair equipped with a length function $L$. We extend basic definitions of [19, 20] to this setting. For $s \in \mathbb{R}$, the Sobolev space of order $s$ with respect to $L$ is defined as

$$H^s_L(G, H) := \{ f : G//H \to \mathbb{C}; \| f \|_{s, L} < \infty \}.$$

Remark 3.1. (i) The space $H^t_L(G, H)$ can be considered as a Hilbert space equipped with the inner product defined by

$$\langle f_1, f_2 \rangle_{s, L} := \sum_{g \in H\backslash G} f_1(g) \overline{f_2(g)} (1 + L(g))^{2s}, \quad \forall f_1, f_2 \in H^s_L(G, H).$$

The completeness of $H^t_L(G, H)$ follows from a similar argument as the proof of completeness of $L^2$ spaces because the above inner product has been defined by an integral (with respect to the counting measure).

(ii) One easily observes that $\| - \|_{s, L} \leq \| - \|_{t, L}$ for $s \leq t$. Therefore, the spaces $H^s_L(G, H)$ are decreasing with respect to the parameter $s$.

(iii) Our discussion before Theorem 2.11 shows that if $L_1$ and $L_2$ are two length functions on a Hecke pair $(G, H)$ and $L_1$ dominates $L_2$, then $H^s_{L_1}(G, H) \subseteq H^s_{L_2}(G, H)$ for all $s \geq 0$.

Definition 3.2. The space of rapidly decreasing functions associated with the Hecke pair $(G, H)$ with respect to $L$ is

$$H^\infty_L(G, H) := \bigcap_{s \geq 0} H^s_L(G, H).$$

The norms $\{ \| - \|_{s, L} \}_{s \geq 0}$ induce a locally convex topology on $H^\infty_L(G, H)$ and by Remark 3.1, it is the same topology as when the parameter $s$ runs through natural numbers. Therefore, $H^\infty_L(G, H)$ is a countably normed space and is complete by Proposition 2.4 of [9]. Thereby, $H^\infty_L(G, H)$ becomes a Fréchet space.
Remark 3.3. If a Hecke pair \((G, H)\) has property (RD) with respect to a length function \(L\), then \(H^\infty_L(G, H) \subseteq C^*_r(G, H)\). This follows from the fact that \(\mathcal{H}(G, H)\) generates \(H^1_L(G, H)\) as a Hilbert space for all \(s \geq 0\) and for \(s\) sufficiently large we have \(H^1_L(G, H) \subseteq C^*_r(G, H)\) because of the Haagerup inequality 2.2.

The aim of this section is to modify the contents of [19] according to the setting of Hecke pairs to show that if a Hecke pair \((G, H)\) possesses property (RD) with respect to a length function \(L\), then \(H^\infty_L(G, H)\) is a smooth subalgebra of the Hecke \(C^*\)-algebra \(C^*_r(G, H)\).

For \(r \geq 0\), let \(P_r\) be the orthogonal projection on the closed span of the set \(\{\delta_g; g \in B_{r,L}(H \setminus G)\}\) in \(\ell^2(H \setminus G)\), where \(\delta_g\) is the characteristic function of the right coset \(Hg\) in \(H \setminus G\). For \(0 < \alpha < 1\) and \(q, N \in \mathbb{N}\), we define a map \(\rho_{\alpha,q,N} : C^*_r(G, H) \to [0, \infty]\) by

\[
\rho_{\alpha,q,N}(a) := N^q \left( \| (1 - P_N) a P_{N-N^\alpha} \| + \| P_{N-N^\alpha} a (1 - P_N) \| \right),
\]

for all \(a \in C^*_r(G, H)\).

Definition 3.4. Given a Hecke pair \((G, H)\) with a length function \(L\), the vector space \(T^\infty_L(G, H)\) associated with \(L\) is defined as

\[
T^\infty_L(G, H) := \left\{ a \in C^*_r(G, H); \forall a \in [0, 1], \forall q \in \mathbb{N}, \sup_{N \geq 1} \rho_{\alpha,q,N}(a) < \infty \right\}
\]

and is endowed with the locally convex topology induced by the norm of \(C^*_r(G, H)\) and seminorms

\[
\nu_{\alpha,q}(a) := \sup_{N \geq 1} \rho_{\alpha,q,N}(a),
\]

for \(0 < \alpha < 1\) and \(q \in \mathbb{N}\).

We note that the defining condition for an element \(a \in C^*_r(G, H)\) to be in \(T^\infty_L(G, H)\) is equivalent to the condition that \(\| (1 - P_N) a P_{N-N^\alpha} \| + \| P_{N-N^\alpha} a (1 - P_N) \| \) = \(O(N^{-q})\) when \(N\) tends to infinity for all \(\alpha \in (0, 1)\) and all positive integers \(q\). We also note that if \(f \in \ell^2(H \setminus G)\), then \(\sum_{h \in <H \setminus G, (1 - P_{N})f(h) > \in L(h) > N} f(h)\) and like wise for \(P_{N-N^\alpha}\). In the following remark, we show that \(T^\infty_L(G, H)\) is actually an algebra and discuss various features of \(T^\infty_L(G, H)\).

Remark 3.5. (i) If \(\alpha > \beta\), then \(P_{N-N^\beta} = P_{N-N^\alpha} + Q\), where \(Q\) is the orthogonal projection on the closed span of the set \(\{\delta_g; g \in < H \setminus G >, N-N^\alpha < L(g) \leq N-N^\beta\}\) of \(\ell^2(H \setminus G)\), and so \(\nu_{\alpha,q} \leq \nu_{\beta,q}\). Therefore, in order to generate the topology of \(T^\infty_L(G, H)\), one may choose the parameter \(\alpha\) from an arbitrary sequence in \([0, 1]\) approaching \(0\). In this way, we get a countable family of seminorms defining the topology of \(T^\infty_L(G, H)\).

Now let \(\{a_n\}\) be a Cauchy sequence in \(T^\infty_L(G, H)\). Since it is Cauchy with respect to the norm of \(C^*_r(G, H)\), it has a limit, say \(a\), in \(C^*_r(G, H)\). On the other hand, since \(\{a_n\}\) is Cauchy with respect to every seminorm \(\nu_{\alpha,q}\), for given \(\epsilon > 0\) and \(m, n\) large enough, we have \(\nu_{\alpha,q}(a_n - a_m) \leq \epsilon\). Now due to the facts that \(\| a_n - a \| \to 0\) and the seminorms are continuous with respect to the norm of \(C^*_r(G, H)\), by letting \(m\) tend to infinity, we get \(\nu_{\alpha,q}(a_n - a) \leq \epsilon\). This shows that \(a_n \to a\) with respect to every seminorm \(\nu_{\alpha,q}\) and consequently \(a \in T^\infty_L(G, H)\). Hence, \(T^\infty_L(G, H)\) is a Fréchet space.
(ii) Indeed, $T_1^\infty(G, H)$ is a Fréchet algebra in the terminology of [23], namely in addition to the above remark, the multiplication in $T_1^\infty(G, H)$ is jointly continuous. For all $a, b \in T_1^\infty(G, H)$, and all $\alpha$ and $q$, we have

$$
(1 - P_N)abP_{N^{-\alpha}} = (1 - P_N)aP_{N^{-\alpha/2}}bP_{N^{-\alpha}}
+ (1 - P_N)a(1 - P_{N^{-\alpha/2}})bP_{N^{-\alpha}}.
$$

Hence,

$$
\|(1 - P_N)abP_{N^{-\alpha}}\| \leq \|(1 - P_N)aP_{N^{-\alpha/2}}\| \|bP_{N^{-\alpha}}\|
+ \|(1 - P_N)a\| \|(1 - P_{N^{-\alpha/2}})bP_{N^{-\alpha}}\|
\leq \|(1 - P_N)aP_{N^{-\alpha/2}}\| \|b\|
+ \|a\| \|(1 - P_{N^{-\alpha/2}})bP_{N^{-\alpha/2}}\|
\leq \|(1 - P_N)aP_{N^{-\alpha/2}}\| \|b\|
+ \|a\| \|(1 - P_M)bP_{M^{-\alpha/2}}\|,
$$

where $M = N - N^{\alpha/2}$ and $M \to \infty$ whenever $N \to \infty$. In the last step, we used the fact that $(N - N^{\alpha/2}) - N^{\alpha/2} \leq (N - N^{\alpha/2}) - (N - N^{\alpha/2})^{\alpha/2}$ which can be easily checked. We have a similar estimation for $\|P_{N^{-\alpha}}ab(1 - P_N)\|$. One also notes that $O(N^q) = O(M^q)$. These facts imply that

$$
\nu_{\alpha,q}(ab) \leq \nu_{\alpha/2,q}(a)\|b\| + \nu_{\alpha/2,q}(b)\|a\|.
$$

This shows that $T_1^\infty(G, H)$ is closed under multiplication and the multiplication is jointly continuous in the Fréchet topology of $T_1^\infty(G, H)$.

(iii) It is clear that $T_1^\infty(G, H)$ is closed under involution, addition and scalar multiplication, thus it is an involutive subalgebra of $C^*_r(G, H)$. Now, let $g$ be an element of $G/H$ and $\delta_g$ be the characteristic function of the double coset containing $g$ as an element of $H(G, H)$. Given $\alpha \in (0, 1)$, It is clear that $(1 - P_N)\delta_gP_{N^{-\alpha}} = 0$ for all $N \geq L(g)^{1/\alpha}$. Thus, $\delta_g$ belong to $T_1^\infty(G, H)$. This proves that $H(G, H) \subseteq T_1^\infty(G, H)$.

(iv) For every length function $L$ on $(G, H)$, we have $T_1^\infty(G, H) \subseteq H_1^\infty(G, H)$. To see this, recall that every element $f$ of $C^*_r(G, H)$ can be regarded as an element of $\ell^2(H \setminus G)$ by its action on $\delta_1 \in \ell^2(H \setminus G)$, i.e. $f(\delta_1)$. Thus, for given $\varphi \in T_1^\infty(G, H)$, we can write $\|(1 - P_N)\varphi\|_2 = \|(1 - P_N)\varphi(\delta_1)\|_2 = \|(1 - P_N)\varphi(P_{N^{-\alpha/2}}(\delta_1))\|_2 \leq \|(1 - P_N)\varphi(P_{N^{-\alpha/2}})\|$. Since, for every positive integer $q$, the last term is equal to $O(N^{-q})$ when $N \to \infty$, the following series is convergent for every positive integer $m$:

$$
\sum_{k \geq 1} k^{2m} \|(1 - P_k)\varphi\|_2^2 = \sum_{k \geq 1} k^{2m} \sum_{h \in \langle H \setminus G, L(h) \rangle > k} |\varphi(h)|^2
=: *
$$

Letting $\chi_l$ be the characteristic function of the set $C_{l^{-1}, L}(H \setminus G)$, we obtain

$$
* = \sum_{k \geq 1} k^{2m} \sum_{l > k} |\varphi(\chi_l)|_2^2
= \sum_{k \geq 2} \left( \sum_{l = 1}^{k-1} l^{2m} \right) |\varphi(\chi_k)|_2^2.
$$
Now, for given $s > 0$, there exists an integer $m$ large enough such that $\sum_{k=1}^{m} l^{2m} \geq (1 + k)^{2s}$ and this shows $\|\varphi\|_{s, L} \leq \infty$. Since this is true for all $s > 0$, we conclude that $\varphi \in H^\infty_L(G, H)$.

**Definition 3.6.** For a Hecke pair $(G, H)$ equipped with a length function $L$, the Fréchet algebra $T^\infty_L(G, H)$ is called the Jolissaint algebra of the Hecke pair $(G, H)$ with respect to $L$.

**Proposition 3.7.** If a Hecke pair $(G, H)$ has property (RD) with respect to a length function $L$, then $T^\infty_L(G, H) = H^\infty_L(G, H)$.

**Proof.** In the above remark, we showed that $T^\infty_L(G, H) \subseteq H^\infty_L(G, H) \cap C^*_r(G, H)$. Now, by proving the converse of this inequality, we obtain the equality $T^\infty_L(G, H) = H^\infty_L(G, H) \cap C^*_r(G, H)$. From this, it is obvious that $T^\infty_L(G, H) = H^\infty_L(G, H)$ if and only if $H^\infty_L(G, H) \subseteq C^*_r(G, H)$.

Let $\varphi$ be an element of $H^\infty_L(G, H)$. For all $\alpha \in (0, 1)$, natural number $N$ and $\xi \in \ell^2(\mathbb{H}/G)$, we have

$$
\|(1 - P_N)\varphi P_{N^{-\alpha}}(\xi)\|^2 = \sum_{g \in \mathbb{H}/G} \|(1 - P_N)\varphi P_{N^{-\alpha}}(\xi)(g)\|^2 = \sum_{g \in \mathbb{H}/G, L(g) > N} |\varphi P_{N^{-\alpha}}(\xi)(g)|^2
$$

$$
\quad = \sum_{L(g) > N} \left( \sum_{L(h) > \alpha N} \varphi(h) P_{N^{-\alpha}}(\xi)(h^{-1}g) \right)^2 =: \star.
$$

The last line is because of the fact that if $L(h) \leq N^\alpha$, then $N - N^\alpha < L(g) - L(h) \leq L(h^{-1}g)$, and so $P_{N^{-\alpha}}(\xi)(h^{-1}g) = 0$. Now, because of the condition $L(h) > N^\alpha$, for every positive integer $q$, we have

$$
\star \leq \frac{1}{(1 + N^\alpha)^{2q/\alpha}} \sum_{L(g) > N} \left( \sum_{L(h) > \alpha N} \varphi(h)(1 + L(h))^{q/\alpha} P_{N^{-\alpha}}(\xi)(h^{-1}g) \right)^2.
$$

By defining $f(h) := \varphi(h)(1 + L(h))^{q/\alpha}$ and going backward, we obtain

$$
\star \leq \frac{1}{(1 + N^\alpha)^{2q/\alpha}} \|(1 - P_N)f P_{N^{-\alpha}}(\xi)\|^2 \leq \frac{1}{(1 + N^\alpha)^{2q/\alpha}} \|\lambda(f)\|^2\|\xi\|_2.
$$

Since $\varphi \in H^\infty_L(G, H)$, we have $f \in H^\infty_L(G, H)$ and, for some positive real numbers $C, s$, we can write

$$
\star \leq \frac{C^2}{(1 + N^\alpha)^{2q/\alpha}} \|f\|^2_{s, L}\|\xi\|^2
\leq \frac{C^2}{(1 + N^\alpha)^{2q/\alpha}} \|\varphi\|^2_{s, q/\alpha, L}\|\xi\|^2_2.
$$

This shows that $\|(1 - P_N)\varphi P_{N^{-\alpha}}\| = O(N^{-q})$ when $N$ tends to infinity. A similar argument applies to $\|P_{N^{-\alpha}}\varphi(1 - P_N)\|$. Therefore, $\varphi \in T^\infty_L(G, H)$. \[\square\]
Lemma 3.8. Let $\mathcal{A}$ be a unital dense subalgebra of a Banach algebra $A$. Let there exist a $0 < \delta < 1$ such that the series $\sum_{n \geq 1} a^n$ belongs to $\mathcal{A}$ for all $a \in \mathcal{A}$ that $\|a\| < \delta$. Then $\mathcal{A}$ has the property of spectral permanence in $A$.

Proof. Let $x \in \mathcal{A}$ be invertible in $A$. Then there exists a $y \in \mathcal{A}$ such that $\|x^{-1}y\| < \delta/\|x\|$. This implies that $\|1-xy\| < \delta$. By assumption, $\sum_{n \geq 0} (1-xy)^n$ belongs to $\mathcal{A}$ and is a right inverse of $xy$. This shows $x$ has a right inverse (and similarly a left inverse) in $A$. Thus, $x$ is invertible in $A$. \qed

Proposition 3.9. Let $a \in T^\infty_L(G, H)$ and let $\|a\| \leq 1/2$. Then $\sum_{n \geq 1} a^n \in T^\infty_L(G, H)$ and consequently $T^\infty_{\mathcal{F}}(G, H)$ has the property of spectral permanence in $C^*_\mathcal{F}(G, H)$. Moreover, for all positive integers $q$ and $\alpha \in (0, 1)$, we have

$$\nu_{\alpha,q} \left( \sum_{n \geq 1} a^n \right) \leq c(\alpha, q) \left( \nu_{\alpha/2,q+1}(a) + \|a\| \right).$$

where $c(\alpha, q) > 0$ is independent of $a$.

Proof. Let fix positive integers $q, N$ and real number $\alpha \in (0, 1)$. For $1 \leq n \leq N^{\alpha/2}$, one can write

$$\begin{align*}
\|(1-P_N)a^n a^{P_{N-N_n/2}} a^{n-1} P_{N-N_n}\| &= \|(1-P_N) a(1-P_{N-N_n/2}) a^{n-1} P_{N-N_n}\| \\
&\leq \|(1-P_N) a P_{N-N_n/2}\| + \|(1-P_N) a^{n-1} P_{N-N_n}\|.
\end{align*}$$

By repeating the same procedure for the last term until $a$ has no power greater than 1, we get

$$\|(1-P_N)a^n P_{N-N_n}\| \leq \sum_{k=0}^{n-1} \|(1-P_{N_k}) a P_{N_{k+1}}\|,$n \leq N - kN^{\alpha/2}$ for $0 \leq k \leq n - 1$. For $n > N^{\alpha/2}$, clearly we have $\|(1-P_N)a^n P_{N-N_n}\| \leq 2\|a\|/2N^{\alpha/2}$. Combining these facts and noticing that $N_{k+1} \leq N - N_k^{\alpha/2}$, we have

$$\|(1-P_N)\left( \sum_{n \geq 1} a^n \right) P_{N-N_n}\| \leq \sum_{n \leq N^{\alpha/2}} \sum_{k=0}^{n-1} \|(1-P_{N_k}) a P_{N-N_k}\| + \frac{2\|a\|}{2N^{\alpha/2}}.$$n the number of terms in the above double sum is less than or equal $N$. We also note that $O(N^{-q}) = O(N_k^{-q})$ when $N$ tends to infinity for every $k$. Therefore, up to a constant, which we call it $c(\alpha, q)$, we obtain the desired inequality. \qed

Theorem 3.10. $T^\infty_L(G, H)$ is a smooth subalgebra of $C^*_\mathcal{F}(G, H)$.

Proof. Let $a \in T^\infty_L(G, H)$ and let $f$ be a holomorphic function defined on a neighborhood of $\sigma(a)$, the spectrum of $a$. If we show that the map $C\backslash \sigma(a) \to T^\infty_L(G, H)$ defined by $\lambda \mapsto (\lambda - a)^{-1}$ is continuous in the Fréchet topology of $T^\infty_L(G, H)$, then the Riemann sum of the integral defining $f(a)$ converges in $T^\infty_{\mathcal{F}}(G, H)$, and so $f(a)$ belongs to $T^\infty_{\mathcal{F}}(G, H)$. 

Fix \( \sigma_0 \in \mathbb{C}\setminus\sigma(a) \). For every \( \sigma \in \mathbb{C}\setminus\sigma(a) \), set \( x(\sigma) := (\sigma_0 - \sigma)(\sigma_0 - a)^{-1} \). If \( |\sigma_0 - \sigma| \leq \frac{1}{2}||\sigma_0 - a|| \), then \( ||x(\sigma)|| \leq 1/2 \) and by Proposition 3.9 \( \sum_{n \geq 1} x(\sigma)^n \in T_L^\infty(G,H) \). Moreover, for every \( \alpha \in (0,1) \) and positive integer \( q \), we have

\[
\nu_{\alpha,q} \left( ((\sigma_0 - a)^{-1} - (\sigma - a)^{-1}) \right) = \nu_{\alpha,q} \left( (\sigma_0 - a)^{-1} [1 - (1 - x(\sigma))^{-1}] \right)
\]

\[
= \nu_{\alpha,q} \left( (\sigma_0 - a)^{-1} \sum_{n \geq 1} x(\sigma)^n \right)
\]

by 3.1 \( \leq \nu_{\alpha/2,q} \left( (\sigma_0 - a)^{-1} \sum_{n \geq 1} x(\sigma)^n \right) \)

+ \( ||(\sigma_0 - a)^{-1}|| \nu_{\alpha/2,q} \left( \sum_{n \geq 1} x(\sigma)^n \right) \)

by 3.2 \( \leq \nu_{\alpha/2,q} \left( (\sigma_0 - a)^{-1} \sum_{n \geq 1} x(\sigma)^n \right) \)

+ \( ||(\sigma_0 - a)^{-1}|| \nu_{\alpha/2,q} \left( \sum_{n \geq 1} x(\sigma)^n \right) \)

Because of \( x(\sigma) \), each term in the right hand side of the above inequality has a factor \( |\sigma_0 - \sigma| \) and this completes the proof. \( \square \)

**Corollary 3.11.** \( K_0(T_L^\infty(G,H)) \simeq K_0(C_r^*(G,H)) \).

**Proof.** Due to the facts that \( T_L^\infty(G,H) \) is a Fréchet algebra, Remark 3.5 Parts (i) and (ii), and it is a smooth subalgebra of \( C_r^*(G,H) \), we can use the result of Larry B. Schweitzer in [23] and deduce that \( M_n(T_L^\infty(G,H)) \) is smooth in \( M_n(C_r^*(G,H)) \) for all positive integers \( n \). Now, the statement follows from [4]. \( \square \)

The above argument can be considered as another proof for the part 4 of the proof of Theorem 1.4 in [19].

**Corollary 3.12.** If \( (G,H) \) has property (RD) with respect to a length function \( L \), then the algebra \( H_H^\infty(G,H) \) of rapidly decreasing functions on \( G/H \) with respect to \( L \) is a smooth subalgebra of \( C_r^*(G,H) \) and the inclusion \( H_H^\infty(G,H) \subset C_r^*(G,H) \) gives rise to the isomorphism \( K_0(H_H^\infty(G,H)) \simeq K_0(C_r^*(G,H)) \).

**Other methods**

Since Jolissaint’s work in [19], some new methods have been invented to prove that \( H_H^\infty(G) \) is a smooth subalgebra of \( C_r^*(G) \) if \( G \) has property (RD). In this section, we discuss this methods.

The first method uses unbounded derivations to define a smooth subalgebra of \( C_r^*(G) \). It has been used in [13] as well as Alain Connes’ book and some recent articles. Here, we follow the paper [18] by Ronghui Ji which contains more details. Using a length functions \( L \) on \( G \), one defines an unbounded operator \( d_L : \ell^2(G) \to \ell^2(G) \) by \( d_L(f)(g) := L(g)f(g) \) for all \( f \in \ell^2(G) \). It is a closed self-adjoint unbounded operator. Therefore the map \( \delta_L : B(\ell^2(G)) \to B(\ell^2(G)) \) defined by

\[
\delta_L(T) := i[d_L,T], \quad \forall T \in B(\ell^2(G))
\]

is a closed unbounded *-derivation. Let us denote the domain of an arbitrary unbounded operator (derivation) \( d \) by \( D(d) \). Then it is easy to see that \( CG \) is
contained in $\bigcap_{k=1}^{\infty} D(\delta_k)$. Set $S_L(G) := \bigcap_{k=1}^{\infty} D(\delta_k) \cap C^*_r(G)$. Since $S_L(G)$ is a dense $*$-subalgebra of $C^*_r(G)$, the following theorem is applied to show that $S_L(G)$ is actually a smooth subalgebra of $C^*_r(G)$.

**Theorem 3.13.** (Theorem 1.2 of [18]) Let $B$ be a $C^*$-algebra and $A$ a $C^*$-subalgebra of $B$. Let $\delta : B \to B$ be a closed unbounded derivation. Then $\bigcap_{k=1}^{\infty} D(\delta_k) \cap A$ is a subalgebra of $A$. Moreover, it is a smooth subalgebra if it is dense.

Regarding this theorem, it is enough to show that when $G$ has property (RD), $S_L(G)$ and $H^\infty_L(G)$ are actually the same Fréchet algebras. This was done in Theorem 1.3 of [18].

In order to extend the above argument to the framework of Hecke pairs, one has to replace $\ell^2(G)$ by $\ell^2(H\backslash G)$ and define an unbounded operator $d_L$ and an unbounded derivation $\delta_L$ similarly. The rest is just an adaptation of Ronghui Ji's argument for Hecke pairs.

The second method is due to Vincent Lafforgue. In Proposition 1.2 of [21], it is shown directly that if $G$ has property (RD) with respect to a length function $L$, then there is a positive real number $s$ such that $H^s_L(G)$ is a Banach algebra and is a smooth subalgebra of $C^*_r(G)$. Again Lafforgue's argument can be generalized for Hecke pairs too.

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