T-Duality and Space-Time Supersymmetry

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We analyze in detail the possible breaking of space-time supersymmetry under T-duality transformations. We find that when appropriate world-sheet effects are taken into account apparent puzzles concerning supersymmetry in space-time are solved. We study T-duality in general heterotic σ-models analyzing possible anomalies, and we find some modifications of Buscher’s rules. We then work out a simple but representative example which contains most of the difficulties. We also consider the issue of supersymmetry versus duality for marginal deformations of WZW models and present a mechanism that restores supersymmetry dynamically in the effective theory.

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1. Introduction

Target-space duality (T-duality) \[1\] is an important example of discrete string symmetries showing the equivalence of strings propagating on background space-times with different geometry and (sometimes) topology. In some recent papers \[2\] it was argued that from the point of view of the low energy effective action, the original and T-dual theories do not share the same number of space-time supersymmetries. If we identify them with the number of Killing spinors in the given space-time background \[3\], it is not difficult to find examples where such numbers come out to be different. There are several reasons why this conclusion is hard to accept. First if we take the point of view that a duality transformation is a change of variables, or a canonical transformation \[4\], the symmetries of the original theory should be preserved although they may become non-local \[5\]. Second, there are general theorems \[3\] relating space-time supersymmetries to symmetries on the world-sheet. This connection is of a very general nature, and if we can show that the manipulations involved in carrying out a T-duality transformation preserve them, we should expect space-time supersymmetry to be maintained, although we may have to revise the relationship between world-sheet properties and those of the the associated low energy effective action.

T-duality can be defined when the target space geometry admits isometries. We can classify the isometries into those with and without fixed points. A typical example of isometries without fixed points are the translational symmetries on tori. If the isometry has a fixed point at \(x_0\), the associated Killing vector \(k^i\) vanishes at \(x_0\). Since \(\nabla_i k_j + \nabla_j k_i = 0\), on the tangent space at \(x_0\), \(k^i\) acts via the rotation matrix \(\nabla_i k_j - \nabla_j k_i\). In the \(\sigma\)-model formulation of a string propagating on a background, we assume that its size and curvature are respectively large and small compared with the inverse string tension \(\alpha'\). Hence, close to the fixed point, \(x_0\), we can approximate the target manifold by flat space. If for simplicity we consider the case where coordinates can be chosen so that \(\nabla_i k_j - \nabla_j k_i\) only acts on a two-plane \((x, y)\) on the tangent space, locally close to \(x_0\) the metric in the \((x, y)\) directions takes the form

\[
d s^2 \sim d x^2 + d y^2 \sim d \rho^2 + \rho^2 d \theta^2,
\]

where the Killing vector corresponding to translations in \(\theta\) is \(k \sim \frac{\partial}{\partial \theta}\). Under T-duality \[4\], \((1.1)\) is transformed to

\[
d \tilde{s}^2 \sim d \rho^2 + \frac{\alpha' }{\rho^2} d \tilde{\theta}^2
\]

(1.2)
and the dual dilaton receives a contribution \( \tilde{S} = S - \frac{1}{2} \log \frac{\rho^2}{\alpha'} \). The dual geometry has a curvature singularity at \( \rho = 0 \). It is then clear that the usual \( \sigma \)-model formulation of strings in this background breaks down close to the singularity, and more powerful techniques should be used in order to understand the behavior of strings close to this point.

If one insists on exploring the metric (1.2) from the low energy point of view, one easily finds that together with the dilaton above it satisfies the \( \beta \)-function equations to first order in \( \alpha' \) \( ^8 \). It is not difficult to verify that (1.2) does not admit Killing spinors, although (1.1) admits two. This is the example studied in the last paper of \( ^2 \). In this reference one considers the heterotic string on flat ten-dimensional Minkowski space. This theory has obviously \( N = 1 \) space-time supersymmetry in \( D = 10 \) (with a Majorana-Weyl spinor generator). If we select two spatial coordinates, say \( (x, y) \), and perform a duality rotation, as in (1.2), the supersymmetric variation of the dilatino does not vanish \( ^2 \)

\[
\delta_\epsilon \lambda = (\gamma^\mu \partial_\mu S) \epsilon \quad (1.3)
\]

and one would be tempted to conclude that the dual theory has no supersymmetries. Although this example is particularly simple, it is clear from the previous arguments that it is representative of any target space manifold with a rotational Killing vector close to its fixed point. Hence we should understand what happens to space-time supersymmetry on the flat ten-dimensional heterotic string under T-duality with respect to one of its rotational Killing symmetries. We should also clarify how the apparent puzzle raised in \( ^2 \) can be understood from the point of view of the low energy effective action.

The outline of this paper is as follows. In section 2 we work out the duality transformation for a general \( (1, 0) \) heterotic \( \sigma \)-model \( ^4 \) with arbitrary connection and background gauge field. We find that if one does not want to have a world-sheet non-local T-dual action due to anomalies which appear when implementing the duality transformation, one has to transform under the isometry the right-moving fermions (which on the heterotic string generate the \( \text{Spin}(32)/\mathbb{Z}_2 \) or \( \text{SO}(16) \times \text{SO}(16) \) currents). This yields a non-trivial transformation of the background gauge field under T-duality. We find also that if in the original model the gauge and the spin connections match (the simplest and most widely used condition necessary to guarantee conformal invariance to order \( \alpha' \) and anomaly cancellation on the ten-dimensional effective theory \( ^{10} \)), the change in the gauge field under T-duality ensures the same matching in the dual theory. Furthermore if the original theory
had (2, 0) or (2, 2) superconformal invariance, the dual theory also has these properties; a prerequisite for the application of the theorems mentioned above concerning world-sheet conditions to get space-time supersymmetry.

In section 3 we present how space-time supersymmetry is restored from the point of view of the low energy effective action. Here we concentrate in the simplest example of the heterotic string considered previously, but the generalization to other cases is straightforward.

In section 4 we present a detailed analysis, within the framework of conformal field theory, of the duality transformations \( \mathcal{L}_2 \), and show that there are indeed world-sheet operators on the dual theory associated to the space-time supersymmetry charges, although some world-sheet non-locality is generated. We will find an interesting interplay between the picture-changing operator \( \mathcal{L}_2 \) and T-duality.

In section 5 we consider the issue of supersymmetry versus duality for continuous \( O(d, d) \) transformations and in particular for marginal \( J\bar{J} \)-deformations of a given conformal field theory. We find that space-time supersymmetry exhibits the same “anomalous” behaviour for generic values of the modulus field, when the Killing vector fields have fixed points in their action. We also consider a mechanism that restores dynamically the supersymmetry in the framework of Kazama-Suzuki models \( \mathcal{L}_2 \), and for \( SU(2) \times U(1) \) in particular.

Finally, in section 6 we present our conclusions and directions for some further work.

2. Duality in Heterotic \( \sigma \)-models

A heterotic \( \sigma \)-model is best formulated in (1,0) superspace \( \mathcal{L}_2 \). (1,0) superfields have the simple form

\[
\Phi^i(\sigma, \theta) = x^i(\sigma) + \theta \lambda^i, \\
\Psi^A(\sigma, \theta) = \psi^A + \theta F^A.
\]

We use light-cone coordinates on the world-sheet, \( \sigma^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}, \) \( \partial_\pm = \frac{\partial_0 \pm \partial_1}{\sqrt{2}} \), and it is useful to introduce the operator:

\[
D = \frac{\partial}{\partial \theta} + i \theta \frac{\partial}{\partial \sigma^+}; \quad D^2 = i \partial_.
\]

If we consider a manifold \( M \) with metric \( g_{ij} \), antisymmetric tensor field \( b_{ij} \) and a background gauge connection \( V_{iAB} \) associated to a gauge group \( G \in O(32) \) (for simplicity
we consider the $O(32)$ heterotic string), we need two types of superfields: coordinate superfields $\Phi^i$ (2.1) and gauge superfields. The fermions $\lambda$ and $\psi$ have opposite world-sheet chirality, $x^i(\sigma)$ are the fields embedding the world-sheet in the target space, and the $F^A$ are auxiliary fields. The Lagrangian density is given by

$$L = \int d\theta \left( -i(g_{ij} + b_{ij})D\Phi^i\partial_\tau\Phi^j - \delta_{AB}\Psi^A D\Psi^B \right), \quad (2.4)$$

with

$$D\Psi^A = D\Psi^A + V_i^A B(\Phi) D\Phi^i \Psi^B. \quad (2.5)$$

Eliminating the auxiliary fields one obtains

$$L = (g_{ij} + b_{ij})\partial_\tau x^i \partial_\tau x^j + ig_{ij}\lambda^i D_- \lambda^j + i\psi^A D_+ \psi^A + \frac{1}{2} F_{ijAB} \lambda^i \lambda^j \psi^A \psi^B, \quad (2.6)$$

with

$$D_+ \psi^A = \partial_\tau \psi^A + V_i^A B \partial_+ x^i \psi^B, \quad (2.7)$$

$$D_- \lambda^i = \partial_\tau \lambda^i + (\Gamma^i_{jk} + \frac{1}{2} H^i_{jk}) \partial_\tau x^j \lambda^k, \quad (2.8)$$

$$H_{ijk} = \partial_i b_{jk} + \partial_j b_{ki} + \partial_k b_{ij}, \quad (2.9)$$

$$F_{ij}^A^B = \partial_i V_j^A B - \partial_j V_i^A B + [V_i, V_j]^A_B. \quad (2.10)$$

The world-sheet supercurrent is of type (1,0):

$$G_+ = (2g_{ij} + b_{ij})\partial_\tau x^i \lambda^j - \frac{1}{2} H_{ijk} \lambda^i \lambda^j \lambda^k. \quad (2.11)$$

To carry out a duality transformation in (2.4) we need to assume that the metric has an isometry under which (2.4) and (2.6) are invariant. Following the procedure outlined in [13] we next gauge the isometry, with some gauge fields $A_\pm$ and add an extra term with a Lagrange multiplier making the gauge field strength vanish. If we integrate out the Lagrange multiplier, $A_\pm$ become pure gauge, $A_\pm = \partial_\pm \alpha$. Using the invariance of the action we can change variables to remove all presence of $\alpha$ and recover the original action. If instead we integrate first over $A_\pm$ and then fix the gauge, we obtain the dual theory. In our case we will carry out these steps in a manifestly (1,0)-invariant formalism. The conditions that need to be satisfied by (2.6) to be able to gauge an isometry have been studied in [14]. If $k^i$ is the Killing vector of the metric $g_{ij}$, the first term of (2.7) is invariant provided

$$k^i H_{ijk} = \partial_j v_k - \partial_k v_j \quad (2.12)$$
and
\[ \delta_k b_{ij} = \partial_i (k^l b_{lj} + v_j) - \partial_j (k^l b_{li}). \]  \hfill (2.13)

The conserved (1,0)-supercurrent for the first term of (2.4) is:
\[ J_+ = (k_i - v_i) \partial_- \Phi^i, \quad J_- = (k_i + v_i) D\Phi^i, \]  \hfill (2.14)
so that
\[ D J_- + \partial_- J_+ = 0. \]

We now introduce (1,0) gauge fields \( A_- \), \( A_+ \) of bosonic and fermionic character respectively
\[ A_- = A_- + \theta \chi, \quad A_+ = \chi + i \theta A_+. \]  \hfill (2.15)

If \( \epsilon(\sigma, \theta) \) is the gauge parameter, we can take
\[ \delta_\epsilon A_- = -\partial_- \epsilon, \quad \delta_\epsilon A_+ = -D \epsilon. \]  \hfill (2.16)

Then, to first order in \( \epsilon \),
\[ (g_{ij} + b_{ij}) D\Phi^i \partial_- \Phi^j + J_+ A_- + J_- A \]
is gauge invariant, with
\[ \delta_\epsilon \Phi^i = \epsilon k^i(\Phi). \]  \hfill (2.17)

Full gauge invariance can be achieved if we assume that \( k^i v_i = \text{const.} \); in which case the gauge invariant Lagrangian is
\[ (g_{ij} + b_{ij}) D\Phi^i \partial_- \Phi^j + J_+ A_- + J_- A + k^2 A_- A. \]

The left-moving part
\[ \Psi (D\Psi + V_i D\Phi^i) \Psi \]
is invariant under the global transformation (2.17) when the isometry variation can be compensated by a gauge transformation:
\[ \delta \Phi^i = \epsilon k^i(\Phi), \quad \delta \Psi = -\kappa \Psi, \]  \hfill (2.18)

with
\[ \delta_k V_i = D_i \kappa = \partial_i \kappa + [V_i, \kappa], \]  \hfill (2.19)
which implies
\[ k^i F_{ij} = D_j \mu; \quad \mu = \kappa - k^i V_i. \tag{2.20} \]

Making \( \epsilon \) a function of \((1,0)\) superspace one obtains after some algebra:
\[ \delta_\epsilon (\Psi^T \mathcal{D} \Psi) = D\epsilon \Psi^T \mu \Psi. \tag{2.21} \]

Hence, adding the coupling
\[ A \Psi^T \mu \Psi, \]
we achieve gauge invariance, because \( \Psi^T \mu \Psi \) is gauge invariant:
\[ k^i F_{ij} = \partial_j \mu + \{V_j, \mu\}, \quad 0 = k^i k^j F_{ij} = k^j \partial_j \mu + [k^j V_j, \mu] = k^j \partial_j \mu + [\kappa, \mu]. \tag{2.22} \]

Then,
\[ \delta_\epsilon \Psi^T \mu \Psi = \Psi^T (k^j \partial_j \mu + [\kappa, \mu]) \Psi = 0. \tag{2.23} \]

The full gauge invariant Lagrangian is:
\[ L = -i \left( (g_{ij} + b_{ij}) D \Phi^i \partial_\Phi^j + J_+ A_+ + J_- A + k^2 A_+ A \right) - (\Psi^T \mathcal{D} \Psi + A \Psi^T \mu \Psi). \tag{2.24} \]

Add now the Lagrange multiplier superfield term
\[ i \Lambda (D A_- - \partial \Lambda). \tag{2.25} \]

Integrating over \( \Lambda \) implies that \( \Lambda = D \alpha, \ A_- = \partial_\alpha \), and using the invariance of \( \text{(2.23)} \) we obtain the original theory. Classical duality is obtained by integrating out \( A, A_- \). Since \( \text{(2.23)} \) is at most quadratic in \( A, A_- \), we can solve their equations of motion to obtain the dual Lagrangian:
\[ \tilde{L}_{cl} = -i \left( (\tilde{g}_{ij} + \tilde{b}_{ij}) D \tilde{\Phi}^i \partial_\tilde{\Phi}^j + (\tilde{J}_+ + D \Lambda) \frac{1}{k^2} (\partial_\Lambda + i \Psi^T \mu \Psi - J_-) \right) - \Psi^T \mathcal{D} \Psi. \tag{2.26} \]

If we use adapted coordinates to the Killing vector, \( k^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^\alpha} \), split the coordinates as \( i = (0, \alpha) \) (with \( i = 0, 1, \ldots, D-1; \alpha = 1, \ldots, D-1 \)), and choose locally the gauge \( \Phi^0 = 0 \), the dual values for \( \tilde{g}, \tilde{b}, \tilde{V} \) are:
\[
\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{k_{\alpha}k_{\beta} - v_{\alpha}v_{\beta}}{k^2}, \\
\tilde{b}_{\alpha\beta} = b_{\alpha\beta} + \frac{k_{\alpha}b_{0\beta} - k_{\beta}b_{0\alpha}}{k^2}, \\
\tilde{V}_{0AB} = -\frac{1}{k^2}\mu_{AB}, \\
\tilde{V}_{\alpha AB} = V_{\alpha AB} - \frac{1}{k^2}(k_{\alpha} + v_{\alpha})\mu_{AB}.
\]

(2.27)

Since in adapted coordinates we can choose \( v_{\alpha} = -b_{0\alpha} \), (2.27) is equivalent to Buscher’s formulae \([7]\); but we find a change in the background gauge field as well. \(\square\)

The preceding formulae (2.27) were obtained using only classical manipulations. In general, however, there will be anomalies and the dual action (2.26) may not have the same properties as the original one. Depending on the choice for \( \mu \) and the gauge group \( G \in O(32) \), the theory (2.26) may be afflicted with anomalies. In this case (2.26) and (2.24) are not equivalent. Equivalence would follow provided we include some Wess-Zumino-Witten terms \([16]\) generated by the quantum measure. If we want the local Lagrangians (2.24) and (2.26) to be equivalent, we must find the conditions on \( g, b, V, \mu \) in order to cancel the anomalies. To better understand the origin of the anomalies, consider first the simpler case where we ignore the manifest (1,0) supersymmetry and take \( b_{ij} = 0 \) as well. Hence \( A_- = A_- \), and \( A = i\theta A_+ \). We will show later how the formulae are modified to go back to the general case.

Under these simplifications the kinetic term for the fermions is:

\[
i g_{ij} \lambda^i D_- \lambda^j,
\]

(2.28)

where

\[
D_- \lambda^j = (\partial_- \delta^i_j + \Gamma^i_{jk} \partial_- x^k - A_- \Omega^i_j)\lambda^j,
\]

\[
\Omega_{ij} = \frac{1}{2}(\nabla_i k_j - \nabla_j k_i).
\]

\(^{1}\) It came to our attention that a similar result was independently derived in \([15]\).
Note that
\[ [D_+, D_-]^i_j = R^i_{jkli} \partial_+ x^k \partial_- x^l - F_{+-} \Omega^i_j, \]  
where
\[ F_{+-} = \partial_+ A_- - \partial_- A_+. \]

Working in orthonormal frames, \( \delta_{ab} e^a_i e^b_j = g_{ij} \), (2.28) becomes:
\[ i \delta_{ab} \lambda^a D_- \lambda^b = i \lambda_a (\partial_- \delta^a b + \omega_-^a b - A_- \Omega^a b) \lambda^b, \]  
where \( \omega_-^a b = \omega_i^a b \partial_- x^i \) is the pull-back of the target space spin connection to the worldsheet and \( \lambda^a = e^a_i \lambda^i \).

The variation of \( \lambda^a \) under the gauged isometry is
\[ \delta \epsilon \lambda = - (\kappa_L \lambda)^a, \]  
where
\[ \kappa_L^a b = \epsilon (k^i \omega_i + \Omega)^a b. \]

Defining the effective \( SO(D) \) gauge field
\[ V_-^a b = \omega_-^a b - A_- \Omega^a b, \]  
(2.30) is invariant under
\[ \delta \lambda = - \kappa_L \lambda, \]
\[ \delta V_- = \partial_- \kappa_L + [V_-, \kappa_L]. \]

However, the fermionic effective action
\[ i \Gamma^L_{eff} [V_-] = \log \det \frac{1}{2} D_- (V) \]  
is anomalous under (2.31). The determinant (2.34) can be computed along the lines of [17] in terms of the Wess-Zumino-Witten Lagrangian for the field \( g \) defined by \( V_- = g^{-1} \partial_- g \).

Then the variation of (2.34) under (2.31) is
\[ \delta \Gamma^L_{eff} [V_-] = - \frac{1}{4 \pi} \int Tr V_- \partial_+ \kappa_L. \]  

A similar computation can be carried out for the \( \psi \) fermions. The corresponding quadratic term in the Lagrangian is:
\[ L = i \psi^T (\partial_+ + V_i \partial_+ x^i - A_+ \mu) \psi. \]  
(2.36)
Defining
\[ V_+ = V_i \partial_+ x^i - A_+ \mu, \]  
we obtain an effective action:
\[ i \Gamma_{\text{eff}}^R [V_+] = -\frac{1}{4\pi} \int Tr \partial_- \kappa R V_+. \]  
Adding up the two effective actions, (2.38) and (2.39), we arrive at the following result,
\[ \delta \Gamma_{\text{eff}} = -\frac{1}{4\pi} \int Tr (\omega_i \partial_- x^i - A_- \Omega) \partial_+ (\epsilon (K^j \omega_j + \Omega)) d^2 \sigma \]
\[ - \frac{1}{4\pi} \int Tr (V_i \partial_+ x^i - A_+ \mu) \partial_-(\epsilon (k^j V_j + \kappa)) d^2 \sigma. \]  
The generalization to the (1,0) supersymmetric case and to a $b_{ij} \neq 0$ is quite simple. First, when $b_{ij} \neq 0$ the spin connection contains a contribution from the torsion. Furthermore, the $\pm v_i$ contribution to the $J_\pm$ currents complete the covariant derivatives in $\Omega_{ij} = \frac{1}{2} (\nabla_i k_j - \nabla_j k_i)$ to be covariant with respect to the full connection with torsion: $\Omega_{ij} \rightarrow \frac{1}{2} (D_i k_j - D_j k_i)$. The supersymmetric extension follows if we use the (1,0)-WZW Lagrangians [18]. If $d^3 Z = d^2 \sigma d\theta$, (2.39) becomes:
\[ \delta \Gamma_{\text{eff}} = -\frac{1}{4\pi} \int Tr (\omega_i \partial_- \Phi^i - A_- \Omega) D(\epsilon (k^j \omega_j + \Omega)) d^3 Z \]
\[ - \frac{1}{4\pi} \int Tr (V_i D\Phi^i - A\Omega) \partial_-(\epsilon (k^j V_j + \mu)) d^3 Z. \]  
Note that unless we cancel the anomaly, the dual theory will contain non-local contributions. The anomaly (2.40) is a mixture between $U(1)$ and $\sigma$-model anomalies [19]. The simplest way to cancel the anomaly (2.40) is to assume that the spin and gauge connection match in the original theory, a condition that also makes space-time anomalies cancel [3], [10]. In this case if $\mu = \Omega$ with matching quadratic Casimirs in (2.40), the anomalous variation (2.40) can be cancelled by a local counterterm. Depending on the case considered there may be other ways to cancel the anomalies which do not require matching $\omega$ with $V$, however we have not pursued an exhaustive analysis.

A test of the validity of the duality transformation (2.27) is that if we start with a theory with matching spin and gauge connection $\omega = V$, (2.27) guarantees that in the dual theory also $\tilde{\omega} = \tilde{V}$. This is also important for the consistency of the model with respect to global world-sheet and target-space anomalies [10], and it implies that if the original theory is conformally invariant to $O(\alpha')$, so is the dual theory.
If for simplicity we take $b_{ij} = 0$, the original metric is
\[ ds^2 = k^2(dx^\alpha + A_\alpha dx^\alpha)^2 + g^{tr}_{\alpha\beta}dx^\alpha dx^\beta, \]  
(2.41)
where
\[ g^{tr}_{\alpha\beta} = g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}, \quad A_\alpha = -\frac{k_\alpha}{k^2} \]
and we can choose frames:
\[ e^0 = k(dx^0 + A_\alpha dx^\alpha), \quad e^{tr\, a} = e^{tr\, a\, \alpha}dx^\alpha. \]  
(2.42)
The spin connection has components:
\[ \omega^0_a = \partial_a \log ke^0 - \frac{k}{2} F_{ab} e^{tr\, b}, \]
\[ \omega_{ab} = \omega^{tr\, ab} + \frac{k}{2} F_{ab} e^0, \]  
(2.43)
where $a, b = 1, ..., D - 1$. For the dual theory we have
\[ d\tilde{s}^2 = \frac{1}{k^2}(d\tilde{x}^0)^2 + g^{tr}_{\alpha\beta}dx^\alpha dx^\beta, \]  
(2.44)
where
\[ \tilde{e}^0 = \frac{1}{k}d\tilde{x}^0, \quad \tilde{e}^a = e^{tr\, a}. \]
Including the contribution from the torsion, the total spin connection is:
\[ \tilde{\omega}_{oa} = -\frac{\partial_a k}{k^2}dx^0 - \frac{k}{2} F_{ab} e^{\, b\, \mu}dx^\mu, \]
\[ \tilde{\omega}_{ab} = \frac{1}{2} F_{ab} dx^0 + \omega^{tr\, \mu ab} dx^\mu. \]  
(2.45)
Using the explicit forms of (2.27) with $A = (0, a), B = (0, b)$ (as frame indices) it is not difficult to verify that $\tilde{\omega}_{oa} = \tilde{V}_{oa}$ and $\tilde{\omega}_{ab} = \tilde{V}_{ab}$. Hence the duality transformation preserves this condition.

There is yet one more possible source of anomalies under duality if the original model is (2,0) or (2,2)-superconformal invariant. In the (2,2) case for instance we have a $U(1)_L \times U(1)_R$ current algebra. The manifold has a covariantly constant complex structure, $\nabla_k J^i_j = 0$, $J^i_k J^k_j = -\delta^i_j$. The R-symmetry is generated by the rotation $\delta \lambda^i = \epsilon J^i_j \lambda^j$ with current $J_+ = i J_{ij} \lambda^i \lambda^j$. This current has an anomaly
\[ \partial_- J_+ = -\frac{1}{4\pi} R_{ijk} \, ^l J^i_k \partial_+ x^i \partial_- x^j, \]  
(2.46)
which can be removed only if the right-hand side of (2.46) is cohomologically trivial. From [13] we know that T-duality preserves $N = 2$ global supersymmetry (cf. also [20]), hence, we should be able to improve the dual R-current so that the $U(1)_L \times U(1)_R$ current algebra is preserved, as needed for the application of the theorem in [5]. Since generically a T-duality transformation generates a non-constant dilaton, the energy-momentum tensor of the dual theory contains an improvement term due to the dilaton $S$ of the form $\partial^2 S$. As a consequence of $N = 2$ global supersymmetry there should also be an improvement term in the fermionic currents and in the $U(1)$ currents. Since the one-loop $\beta$-function implies (in complex coordinates) $R_{\alpha\bar{\beta}} \sim \partial_\alpha \partial_{\bar{\beta}} S$ [21], we can improve the $U(1)_L \times U(1)_R$ currents so that they are chirally conserved. In the (2,2) case the improvements are:

$$\Delta J_+ = \partial_\alpha S \partial_+ Z^\alpha - \partial_{\bar{\alpha}} S \partial_+ Z^{\bar{\alpha}},$$
$$\Delta J_- = - (\partial_\alpha S \partial_- Z^\alpha - \partial_{\bar{\alpha}} S \partial_- Z^{\bar{\alpha}}).$$

(2.47)

For instance in the example (1.1), (1.2) the complex coordinates for the metric (1.2) are

$$z = \frac{1}{2} \rho^2 + i \tilde{\theta}$$

and

$$ds^2 = d\frac{z dz}{z + \bar{z}}.$$  

Then:

$$\Delta J_+ = - \frac{i}{\rho^2} \partial_+ \tilde{\theta}, \quad \Delta J_- = \frac{i}{\rho^2} \partial_- \tilde{\theta}.$$  

(2.49)

With these improvements the currents are chirally conserved to order $\alpha'$ (and presumably to all orders, since the higher loop counterterms are cohomologically trivial for a (2,2) supersymmetric $\sigma$-model), hence we conclude that under duality the (2,0) or (2,2) superconformal algebra is preserved [5]. As we will argue later in more detail, the direct correspondence of operators under T-duality may map local into non-local operators, and the structure of the associated low energy effective action has to be understood with some care. The mapping of local into non-local states is familiar from the case of toroidal duality, where momentum states are mapped into winding states.

\[\text{Hence we meet the conditions to apply the theorem, in [5] implying that the theory is space-time supersymmetric. More on this in section 4.}\]
3. The effective action point of view

In this section we want to give an answer to some of the puzzles raised in BKO [2] and analyze the example (1.1), (1.2) from the effective action point of view. The original background describes the motion of the heterotic string in flat Minkowski space. Hence we have full $ISO(1,9)$ Lorentz invariance and $O(32)$ gauge symmetry (the same arguments apply to the $E_8 \times E_8$ string). Since we perform duality in (1.1) with respect to rotations in the $(x, y)$ plane, only the subgroup of $ISO(1,9)$ commuting with them will be a manifest local symmetry of the effective action. Similarly if we preserve manifest $(1,0)$ supersymmetry on the world-sheet and avoid anomalies, we embed the isometry group $SO(2) \subset G \equiv SO(32)$. The subgroup of $G$ commuting with $SO(2)$ is $SO(30) \times SO(2)$ and once again this will be a manifest symmetry in the low energy theory. It is well known [5], [1], [4] that under T-duality, symmetries not commuting with the ones generating duality are generally realized non-locally. Hence although the dual background (1.2) still contains all the original symmetries from the CFT point of view, the low energy theory does not seem to exhibit them. The theory will be explicitly symmetric under $ISO(1,7) \times SO(30) \times SO(2)$ only. We want to make sure nevertheless that the original space-time supersymmetry is preserved.

The world-sheet supersymmetry (2.1) commutes with $(1,0)$ T-duality, and, from the arguments in the previous section we expect the dual theory to exhibit the equivalent of the full $N = 1$ space-time supersymmetry of the original space, although not necessarily in a manifest $O(1,9)$-covariant formalism. To find the graviton, gravitino, etc vertex operators in the dual background (1.2) we should solve the anomalous dimension operators constructed in the metric (1.2) including the dilaton and background gauge field, as was done for tachyons in [22]. We can proceed differently. The dual background is

$$d\tilde{s}^2 = d\rho^2 + \frac{1}{\rho^2}d\tilde{\phi}^2 - (dx^0)^2 + (dx^i)^2 ; \quad i = 1, ..., 7,$$

$$(3.1)$$

$$S = -\log \rho, \quad V_\mu dx^\mu = \frac{1}{\rho^2}d\tilde{\phi}M,$$

where $M$ is the matrix describing the embedding of the spin connection in the gauge group, which we take to be the standard one acting only on two of the right-moving fermions. (3.1) satisfies the heterotic $\beta$-function equations [8] to $O(\alpha')$. We can consider the variation of the fermionic degrees of freedom in a formalism adapted to the $ISO(1,7) \times SO(30) \times SO(2)$ symmetry, and look for which combination of the $O(1,9)$ fermions are annihilated by
supersymmetry. We believe these combinations are the ones that we would obtain if we followed the method in [22]. We will find that the number of space-time fermionic symmetries does not change.

The low energy approximation to the heterotic string is given by $N = 1$ supergravity coupled to $N = 1$ super Yang-Mills in $d = 10$. In ten dimensions we can impose simultaneously the Majorana and Weyl conditions [23]. In terms of $SO(1,7)$, a Majorana-Weyl spinor of $SO(1,9)$ becomes a Weyl spinor. Write the Dirac algebra (in an orthonormal frame) as

$$\Gamma_\mu = \tau_3 \otimes \gamma_\mu; \quad \mu = 0, 1, ..., 7,$$

$$\Gamma_{7+i} = i\tau_i \otimes 1; \quad i = 1, 2,$$

$$\bar{\Gamma} = \tau_3 \otimes \gamma_9,$$

where $\gamma_9$ is the analogous of the four-dimensional $\gamma_5$ in eight dimensions. Ten dimensional indices will be hatted. The supersymmetric variation of the ten-dimensional fermions is given by:

$$\delta \hat{\Psi}_\mu = (\partial_\mu - \frac{1}{4} \omega_{\mu ab} \Gamma^{\dot{a} \dot{b}}) \hat{\epsilon},$$

$$\delta \hat{\lambda} = (\Gamma^\dot{b} \partial_\dot{b} S - \frac{1}{6} H_{\mu \nu \rho} \Gamma^{\mu \nu \rho}) \hat{\epsilon},$$

$$\delta \chi^A = -\frac{1}{4} F^A_{\mu \nu} \Gamma^{\mu \nu} \hat{\epsilon}$$

for the gravitino, dilatino and gluino, respectively. The background gauge field strength is:

$$F = -\frac{2}{\rho^3} d\rho \wedge d\tilde{\phi}.$$

Decomposing (3.3) with respect to $SO(1,7)$ we find:

$$\delta \Psi_\mu = \partial_\mu \epsilon,$$

$$\delta \Psi_{\{\rho\}} = \partial_{\{\rho\}} \epsilon,$$

$$\delta \Psi_{\{\tilde{\phi}\}} = (\partial_{\tilde{\phi}} + \frac{i}{4\rho^2} \tau_3 \otimes 1) \epsilon$$

for the gravitino $^3$,

$$\delta \lambda = -\frac{i}{\rho} (\tau_1 \otimes 1) \epsilon$$
for the dilatino, and
\[
\delta \chi^A = 0; \quad A \in SO(30),
\]
\[
\delta \chi = -\frac{i}{\rho^2} (\tau_3 \otimes 1) \epsilon; \quad \text{along the embedded } SO(2),
\]

for the gluino. In the preceding formulas, \( \epsilon \) is now an \( SO(1, 7) \) Weyl spinor with the same number of independent components as a ten-dimensional Majorana-Weyl spinor.

Next, if we define
\[
\tilde{\Psi}_\mu = \Psi_\mu,
\]
\[
\tilde{\Psi}_{\{\rho\}} = \Psi_{\{\rho\}},
\]
\[
\tilde{\Psi}_{\{\tilde{\phi}\}} = \Psi_{\{\tilde{\phi}\}} + \frac{i}{4} e^S (\tau_2 \otimes 1) \lambda,
\]

it is easy to see that the fields on the left hand side transform as \( \delta \tilde{\Psi}_\mu = \partial_\mu \epsilon \). Similarly,
\[
\tilde{\lambda} = \lambda + ie^{-S} (\tau_2 \otimes 1) \chi,
\]
\[
\tilde{\chi} = \chi - ie^{S} (\tau_2 \otimes 1) \lambda
\]

have vanishing variation under space-time supersymmetry. Furthermore, they have the correct chiralities as dictated by the ten-dimensional multiplet.

Thus if we use a formalism covariant only under the explicit \( SO(1, 7) \times SO(30) \times SO(2) \) symmetry of the background (3.4) we recover the full number of supersymmetric charges. From the low-energy effective action this is the most we could expect since at the level of the world-sheet CFT the full symmetry \( SO(1, 9) \times SO(32) \) is only realized non-locally. If we want to consider the complete symmetry and the complete massless spectrum in the dual theory it seems that the only reasonable thing to do is to go back to the two-dimensional point of view. This is not unreasonable because at \( \rho = 0 \) the dual background has a curvature singularity and the naive \( \sigma \)-model and effective action considerations should not be trusted. We will see in the next section how one can obtain in principle the vertex operators for the full massless spectrum in the dual theory.

An interesting exercise would be to verify that indeed (3.8), (3.9) are the vertex operators one would obtain for the dual background (3.1) if we followed reference [22].
4. Conformal field theory analysis

As suggested in the introduction, a representative example of duality transformations with respect to isometries having fixed points is to analyse the heterotic string in flat ten-dimensional Minkowski space and perform duality with respect to rotations. This example can be studied explicitly in detail using conformal field theory (CFT) techniques, and there are a number of things that can be learned. In the previous section we analyzed this example from the low-energy effective action point of view, and we have learned that only part of the symmetries of the original theory manifest locally in the dual transform. In this section we want to investigate in more detail the way the full symmetry is realized.

If we perform duality in the \((x, y)\)-plane, the part of the free heterotic Lagrangian of interest is

\[
L = \partial_+ \vec{x} \cdot \partial_- \vec{x} + i \vec{\lambda} \cdot \partial_- \vec{\lambda} + i \bar{\psi}^A \partial_+ \psi^A + ..., \tag{4.1}
\]

where the vector quantities are two-dimensional. The isometry we consider is

\[
x \to e^{\epsilon \alpha} x, \tag{4.2}
\]

where

\[
\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

For the time being we work in frames not adapted to the isometry. Hence for \((4.1)\) we can perform duality only in the bosonic sector. The world-sheet supercurrent is

\[
\mathcal{G}_+ = \vec{\lambda} \cdot \partial_+ \vec{x} = \vec{\lambda} \cdot \vec{P}_+ , \tag{4.3}
\]

where \(\vec{P}_+\) is a chiral current generating translations in the target space.

It is convenient to work in canonical pictures \([11]\) \((-\frac{1}{2} \text{ for fermion vertices, } -1 \text{ for boson vertices})\). The space-time supersymmetry charge is

\[
Q_\alpha (-\frac{1}{2}) = \oint e^{-\frac{\phi}{4}} S_\alpha , \tag{4.4}
\]

where \(\phi\) is the scalar which bosonizes the superconformal ghost current and \(S_\alpha\) is the spin-field associated to the \(\lambda\)-fermions. The translation operator in the \(-1\) picture is

\[
P_\mu (-1) = \oint e^{-\phi} \lambda_\mu . \tag{4.5}
\]
Note that in (4.3), (4.4) only the space-time fermion and the \((\beta, \gamma)\)-ghosts appear. Hence

\[
\{ Q_\alpha \left(-\frac{i}{2}\right), Q_\beta \left(-\frac{i}{2}\right) \} = \Gamma^\mu P_\mu \left(-1\right) \tag{4.6}
\]

is satisfied, and if we choose to perform duality for the bosonic part of the Lagrangian only, the same relationships (4.4), (4.5), (4.6) should still hold.

From this point of view there is clearly no problem with space-time supersymmetry. However, in constructing scattering amplitudes we need to use vertex operators in different pictures. Hence any problem should come from the interplay with the picture changing operator \(\mathcal{P}\). The picture changing operator acting on a vertex operator \(V_q(z)\) in the \(q\)-picture can be represented as \[11\]

\[
\mathcal{P}V_q(z) = \lim_{w \to z} e^{\phi(w)} \mathcal{G}_+(w)V_q(z). \tag{4.7}
\]

Hence the only possible difficulties may appear in anomalies in the world-sheet supercurrent under duality. Since \(\mathcal{G}_+\) does not commute with purely bosonic rotations, after duality \(\mathcal{G}_+\) will become non-local in the world-sheet. To guarantee that there are no problems with \(\mathcal{G}_+\) we want to make sure that the dual world-sheet supercurrent still has the form \(\vec{\lambda} \cdot \tilde{\mathcal{P}}_+\), where \(\tilde{\mathcal{P}}_+\) is the representation of the translation current in the dual theory, and it is here that non-locality resides. In fact the full theory in (4.1) can be constructed out of the knowledge that \(P_+^i\ (i = 1, 2)\) is chirally conserved and that its operator product expansion (OPE) is \(P^i(z)P^j(w) \sim \frac{\delta^{ij}}{(z-w)^2}\). It is hard to believe that the existence of the chiral currents is going to be lost under duality. To make sure that this is not the case, the simplest thing to do is to include sources for these currents and then follow their transformation under duality.

Following [13] we gauge the symmetry (4.2) and concentrate only on the bosonic part of (4.1), the only one relevant due to the previous arguments. Thus our starting point is:

\[
L = D_+ x^T D_- x + \lambda F_+-, \tag{4.8}
\]

There are two procedures presented in [13]. One, which we follow, is to gauge the isometry and impose the vanishing field strength constraint. The second procedure consists of writing a \(\sigma\)-model in \(D+1\) dimensions, where \(D\) is the dimension of the original manifold so that the original isometry is promoted to a full \(U(1)_L \times U(1)_R\) Kac-Moody algebra. Then the original and the dual theories are obtained by gauging respectively vector or axial vector combinations of the two \(U(1)\). This procedure however has to be modified to include also a dilaton in \(D+1\) dimensions to preserve conformal invariance (as can be easily checked by working out the \(\beta\)-function equations), and to guarantee that the dilaton transforms as in [3], [24] under duality. Since we do not want to keep track of extra dilaton contributions, we follow the first procedure.
where
\[ D_{\pm} x = \partial_{\pm} x + \epsilon x A_{\pm}, \quad F_{+-} = \partial_+ A_- - \partial_- A_+. \]

Using the \( \lambda \)-equation of motion, \( A_{\pm} = \partial_{\pm} \alpha \), \( D_{\pm} x = e^{-\epsilon \alpha} \partial_{\pm} (e^{\epsilon \alpha} x) \), and changing variables \( x \to e^{-\epsilon \alpha} x \), the original theory is recovered. It proves convenient to parametrize locally
\[ A_+ = \partial_+ \alpha_L, \quad A_- = \partial_- \alpha_R. \quad (4.9) \]

Then (4.8) has the symmetries:
\[ \delta x = e^{-\epsilon \alpha_R} a_R, \quad \delta \lambda = -x^T \epsilon e^{-\epsilon \alpha_R} a_R; \quad (4.10) \]
and
\[ \delta x = e^{-\epsilon \alpha_L} a_L, \quad \delta \lambda = x^T \epsilon e^{-\epsilon \alpha_L} a_L; \quad (4.11) \]
yielding respectively the conserved currents:
\[ \partial_-(e^{\epsilon \alpha_R} D_+ x) = 0, \quad \partial_+(e^{\epsilon \alpha_L} D_- x) = 0. \quad (4.12) \]

When \( \alpha_R = \alpha_L = \alpha \), we recover the original currents \( \partial_{\pm}(e^{\epsilon \alpha} x) \). Acting on \( x \), the symmetries (4.10) commute.

The sources to be added to (4.8) should be gauge invariant:
\[ J_- e^{\epsilon \phi_R} D_+ x + J_+ e^{\epsilon \phi_L} D_- x. \quad (4.13) \]

The exponents are non-local in \( A_{\pm} \), and they make the coupling gauge invariant. This also guarantees that the currents in (4.13) satisfy the OPE of the original theory as expected. The coupling (4.13) is quite different from the classical toroidal model where duality is performed with respect to translational symmetries. In this case the analogue of (4.8), (4.13) is
\[ R J_- D_+ x + R J_+ D_- x + R^2 \partial_+ x \partial_- x + \lambda F_{+-}, \quad (4.14) \]
where
\[ D_{\pm} x = \partial_{\pm} x + A_{\pm}. \]

After duality, we obtain:
\[ \frac{1}{R^2} \partial_+ \lambda \partial_- \lambda + \frac{1}{R} (\partial_+ \lambda J_- - \partial_- \lambda J_+) + \partial_- J_+ - \frac{1}{\partial_+ \partial_-} \partial_+ J_- , \quad (4.15) \]
exhibiting quite clearly the exchange between momentum and winding modes. In our example, let us take for simplicity $J_+ = 0$. The most straightforward way to integrate out the gauge fields is to work in the light-cone gauge $A_- = 0$. Then the integral over $A_+$ becomes a $\delta$-function which can be solved in two ways. If we choose to solve it in order to write the Lagrange multiplier $\lambda$ as a function of the other fields, we recover the original theory.

On the other hand if we choose to solve the adapted coordinate to the isometry in terms of $\lambda$ and the other variables, we obtain the dual theory. Furthermore we also obtain a determinant, which when properly evaluated \[7\], \[1\] yields the transformation of the dilaton. We use complex coordinates,

$$z = x + iy = \rho e^{i\theta}, \quad (4.16)$$

and

$$J_- = J_{-1} + iJ_{-2}, \quad (4.17)$$

where $J_{-1,2}$ are the components of $J_-$ in \([4.13]\); the complex conjugate of $J_-$ will be denoted by $J_-^*$. Then, we obtain the equation

$$\partial_- w = w \frac{\lambda}{\rho^2} \partial_- \lambda - \frac{\rho}{2i\alpha'} (J_- - J_-^* w^2), \quad (4.18)$$

where

$$w = e^{i\theta},$$

ie. a Riccati equation which can be solved order by order in $J_-$. Restoring powers of $\alpha'$,

$$\partial_- w = i w \frac{\alpha'}{\rho^2} \partial_- \lambda - \frac{\rho}{2i\alpha'} (J_- - J_-^* w^2). \quad (4.19)$$

The lowest order solution is:

$$w = \exp \int \frac{\alpha'}{\rho^2} \partial_- \lambda d\sigma^- \quad (4.20)$$

and to this order the current looks like

$$\partial_+ \left( \rho e^{i\theta[\lambda,J_-]} \right) = \partial_+ \left( \rho e^{i\alpha'} \int \frac{1}{\rho^2} \partial_- \lambda d\sigma^- \right). \quad (4.21)$$
The extra terms depending on $J_-, J^*_-$ are required to guarantee the equality between the correlation functions before and after the duality transformation. To leading order the currents are:

$$\partial_+ \left( \rho e^{\pm i\alpha'} \int \frac{1}{\rho^2} \partial_- \lambda d\sigma^- \right),$$

$$\partial_- \left( \rho e^{\pm i\alpha'} \int \frac{1}{\rho^2} \partial_+ \lambda d\sigma^+ \right).$$

(Note however that in solving (4.19) there will be corrections to all orders in $\alpha'$ in order to obtain the correct OPE's for dual currents $\tilde{P}_\pm$. These currents can be used to write the emission vertex operators in the dual theory and they are almost always non-local. Since the OPE's of $\tilde{P}_\pm$ are preserved, the spectrum of the original and the dual theories are equivalent. Nevertheless, we have to be careful regarding the operator mapping.

In the original theory the target space has a flat metric while in the dual manifold (1.2) we have curvature singularities at $\rho = 0$. In the original theory the vertex operators are expressed in terms of momentum states, which in the dual theory correspond to winding states. This is reasonable because operators like (4.22) would produce non-normalizable states acting on momentum states, but normalizable acting on winding states on the dual space, as one can easily show by looking at the solutions of the anomalous dimension operators on the dual background [22]. We also see that when there are curvature singularities the $\sigma$-model approach does not work. We can only trust this approximation when the manifold is large, or the curvature is small when expressed in units of $\alpha'$. Since in the dual theory near $\rho = 0$ we have a curvature singularity, the low energy field theory limit cannot be trusted close to this point, and we have to deal with the dual theory without relying on an expansion in powers of $\alpha'$. If there were a minimum value $\rho_{\text{min}} \neq 0$ we would still be able to arrange the solution of (4.19) in a power series of $\alpha'$. When this is not the case, we can only obtain reliable information if we can control the exact operator mapping. Equations (4.19)-(4.22) show that in order to recover the expected OPE for the chiral currents $\tilde{P}_\pm$ we need to include all orders in $\alpha'$, which then in turn imply that the world-sheet supercurrent and the picture changing operator have the expected properties.

In conclusion, there is no problem with space-time supersymmetry from the point of view of CFT, but the correct operators that need to be used to represent the emission vertices of low energy particles in the dual theory are often non-local, and do not admit a straightforward $\alpha'$ expansion unless we write the dual states in terms of those which follow from the correspondence as dictated by the duality transformation. When there are
curvature singularities the approach based on the effective low energy theory has many limitations and to obtain reliable information we should go back to the underlying string theory.

5. Marginal deformations and supersymmetry

The issue of space-time supersymmetry versus duality arises even infinitesimally, when considering $O(d, d)$ deformations of superconformal backgrounds with $d$ abelian isometries. If some of the Killing vector fields have fixed points in their action, the corresponding deformations will exhibit an “anomalous” supersymmetric behaviour in a way similar to the previous analysis. It is particularly interesting in this context to examine the special class of $O(d, d)$ transformations that describe marginal $J\bar{J}$-deformations of a given conformal field theory, although the generalization to all other $O(d, d)$ elements can be easily incorporated. The most characteristic class of examples for our present purposes is provided by the $N = 2$ superconformal string backgrounds on group manifolds $G$, or symmetric coset spaces, which are known as Kazama-Suzuki models [12]. We will investigate the effect of a $J\bar{J}$-deformation on the supersymmetric properties of the associated $G_k$ WZW models, with main emphasis on $G \equiv SU(2) \times U(1)$ as the simplest but representative example that contains most of the difficulties encountered in the general case.

We have typically the following deformation of the $\sigma$-model action

$$S_{R+\delta R} = S_R + \frac{\delta R^2}{4\pi k} \int J(R)\bar{J}(R), \quad (5.1)$$

where $R$ is a modulus field and $S_{R=1}$ is the corresponding action of the undeformed WZW model. The main point that makes relevant our previous analysis to the present case is the realization that all Kazama-Suzuki group manifolds admit some Killing vector fields with fixed points, which in turn enter into the description of the $J\bar{J}$-deformations as $O(d, d)$ transformations. The undeformed models at $R = 1$ admit a Killing spinor and hence they have manifest $N = 1$ space-time supersymmetry. The Killing spinors, however, depend explicitly on the (adapted) coordinates of the corresponding rotational Killing vector fields, in exact analogy with the $\theta$-dependence of the Killing spinors in flat space with polar coordinates $(\rho, \theta)$. Switching on the $J\bar{J}$-deformation, we find that there are no solutions to the Killing spinor equation neither infinitesimally away from $R = 1$ nor at the end points (say $R = 0$ and $\infty$), the latter being related to each other by factorized duality. Also, away from $R = 1$, the dilatino variation is not zero, as it would have been required.
from the supersymmetric properties of a bosonic solution, and it appears as if space-time supersymmetry is lost without including any appropriate world-sheet effects.

There is an interesting phenomenon that we will find at this end for the simplest background of this type, $SU(2) \times U(1)$. Namely, there is a dynamical restoration of space-time supersymmetry as soon as we are prepared to alter our model by making the modulus field $R$ into a dynamical variable, while preserving conformal invariance. We will present some details of this mechanism, since we believe that it is of general value and deserves systematic study for all Kazama-Suzuki models and perhaps further beyond. This phenomenon seems to indicate that there is a systematic way to include dynamically all the appropriate non-perturbative world-sheet effects into a new string vacuum with manifest space-time supersymmetry.

We consider for simplicity the $SU(2)$ WZW model in the following and parametrize its group elements as

$$g = e^{i(\tau - \psi)\sigma_3/2} e^{i\varphi\sigma_2} e^{i(\tau + \psi)\sigma_3/2}. \quad (5.2)$$

The action in these coordinates describes a 2-dim $\sigma$-model with target space metric

$$ds^2 = \sin^2 \varphi d\psi^2 + \cos^2 \varphi d\tau^2 + d\varphi^2 \quad (5.3)$$

and anti-symmetric tensor field with non-zero component

$$b_{\tau\psi} = \cos^2 \varphi, \quad (5.4)$$

whereas the dilaton field $S$ is zero.

The $J\bar{J}$-deformation of the $SU(2) \times U(1)$ WZW model can be easily induced from the corresponding deformation of the $SU(2)$ model. The $R$-line of deformed $SU(2)$ models is known to have the following structure [25]:

$$ds^2_{(R)} = \frac{1}{\cos^2 \varphi + R^2 \sin^2 \varphi} \left( \sin^2 \varphi d\psi^2 + R^2 \cos^2 \varphi d\tau^2 \right) + d\varphi^2, \quad (5.5)$$

$$b_{\tau\psi}^{(R)} = \frac{\cos^2 \varphi}{\cos^2 \varphi + R^2 \sin^2 \varphi} \quad (5.6)$$

and

$$S^{(R)} = \frac{1}{2} \log \frac{R}{\cos^2 \varphi + R^2 \sin^2 \varphi}. \quad (5.7)$$
We note for completeness that the conformal invariance along the $R$-line of marginal deformations requires that $e^{-2S^{(R)}} \sqrt{\det g(R)}$ remains the same for all $0 \leq R < \infty$ in the $\sigma$-model frame. Also, for $R = 1$, we readily obtain the original $SU(2)$ WZW model.

The $J\bar{J}$-deformation of the $SU(2) \times U(1)$ WZW model can be simply performed by adding an abelian $U(1)$ field $\rho$ with the appropriate background charge, i.e. a linear dilaton. Then, the resulting 4-dim backgrounds along the $R$-line have the following metric, antisymmetric tensor and dilaton fields respectively,

$$d\rho^2 + dS_{(R)}^2, \quad b^{(R)}_{\tau \psi}, \quad -\rho + S^{(R)}.$$

(5.8)

We note at this point that although at $R = 1$ the $SU(2) \times U(1)$ WZW model has manifest space-time supersymmetry, the supersymmetry appears to be broken even infinitesimally away from it. The simplest way to see this is by considering the dilatino variation at generic values of $R$. The vanishing condition $\delta \lambda = 0$ would imply the balance

$$\frac{1}{2}H_{\mu \nu \rho} = -\sqrt{\det g} \epsilon_{\mu \nu \rho}^{\sigma} \partial_{\sigma} S$$

(5.9)

in the $\sigma$-model frame of the 4-dim class of models (5.8). Straightforward calculation yields

$$\cos^2 \varphi + R^2 \sin^2 \varphi = R,$$

(5.10)

which is clearly satisfied only for $R = 1$, as advertised earlier.

The reason for this occurrence can also be traced to the specific form of the Killing spinors at the $R = 1$ point. We have determined that the spinor with components

$$\xi_\pm = e^{i\varphi_3/2}e^{i(\psi + \tau)/2} \epsilon_\pm$$

(5.11)

solves the Killing spinor equation for the $SU(2) \times U(1)$ background, where $\epsilon_\pm$ are the Weyl components of a constant Killing spinor. According to our general framework, it is the existence of fixed points in the action of the Killing vector fields $\partial/\partial \psi$ and $\partial/\partial \tau$ that accounts for the $\psi$ and $\tau$ dependence of the Killing spinor (5.11). The $J\bar{J}$-deformation in question is generated by an appropriately chosen $O(2,2)$ transformation that is associated with these two Killing vector fields and hence the loss of space-time supersymmetry comes

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5 This model actually has $N = 4$ world-sheet supersymmetry and $N = 2$ space-time supersymmetry.

6 We thank K. Sfetsos for supplying this formula.
as no surprise in this case. As before, the inclusion of non-perturbative world-sheet effects could resolve the issue of supersymmetry versus duality, although it might not necessarily be so at the infinitesimal level we are considering here.

We turn now to the detailed description of a mechanism that restores dynamically the space-time supersymmetry and yields a new 4-dim string background based on $SU(2)$. Recall that in our previous discussion the field $\rho$ was added as a fourth space-time coordinate, but it was important that it remained a spectator, in the sense that the modulus field $R$ driving the deformation was $\rho$-independent. This was of course necessary for constructing a one-parameter family of 4-dim conformal backgrounds with $R = 1$ being the $SU(2) \times U(1)$ model. Next, we will abandon this parametric construction by altering the role of the field $\rho$. We will introduce instead a single 4-dim background having $R$ with a dynamical $\rho$-dependence, in that at any given instance of “time” $\rho$ the 3-dim slices are points in the moduli space of $JJ$-deformations of the $SU(2)$ WZW model\[. The point of this method is mainly the observation that space-time supersymmetry can be restored for such a background. Although this dynamical background appears to be disconnected from the moduli space of the $R$-deformed $SU(2) \times U(1)$ models, it might be relevant for describing some of their non-perturbative aspects in $\alpha'$. This possibility deserves further study.

The 4-dim metric in this case is taken $-d\rho^2 + ds^2_{(R)}$, accounting for the $- + + +$ signature, while the anti-symmetric tensor field is given again by (5.6). The condition of conformal invariance is sufficient to determine the dilaton field

$$S^{(R)} = \frac{1}{2} \log \frac{R'}{\cos^2 \varphi + R^2 \sin^2 \varphi},$$

(5.12)

where the prime denotes the derivative with respect to $\rho$. This model was originally considered as paradigm for topology change in string theory [26], where it was also found that $R$ satisfies the differential equation

$$\frac{R'''}{R''} = \frac{R''}{R'} + \frac{R'}{R}.$$  

(5.13)

Simple integration yields

$$R' = C_1 R^2 + C_2,$$

(5.14)

\[ We choose $\rho$ to have the signature of physical time in order to stress this analogy. The Euclidean version can be obtained by simple analytic continuation $\rho \to i \rho$.  

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where \( C_1 \) and \( C_2 \) are arbitrary constants for the time being.

There are three classes of solutions depending on whether \( C_1 = 0 \) (or \( C_2 = 0 \)), \( C_1C_2 > 0 \) or \( C_1C_2 < 0 \). The observation now is that not all of these solutions are space-time supersymmetric. Explicit calculation shows that only the third class exhibits space-time supersymmetry (to lowest order in \( \alpha' \)), as it can be verified by considering the vanishing condition of the dilatino variation. In this case we find \( C_1 = -C_2 = 1 \) and integration of (5.14) determines the relevant supersymmetric solution

\[
R(\rho) = \tanh \rho
\]

(5.15)

for the dynamical model. It is known that this is T-dual to the \( SU(2) / U(1) \times SL(2) / U(1) \) WZW coset model [26], which also has manifest space-time supersymmetry.

The method of the dynamical restoration of space-time supersymmetry can be generalized to any Kazama-Suzuki model, but we will not present the details here. An interesting question is to determine the class of solutions that restore supersymmetry in the general case. Also the deeper connection between the original \( R \)-family of deformed WZW models and the associated dynamical backgrounds remains to be clarified. We hope that this mechanism incorporates in some way the appropriate world-sheet effects that have to be taken into account to resolve the problem of supersymmetry versus duality that arises in the context of the effective theory.

6. Conclusions

To summarize, we stress again that the problem that seems to arise in the effective theory concerning the fate of space-time supersymmetry under T-duality with respect to rotational Killing vector fields, is entirely due to non-local world-sheet effects that have to be taken into account in string theory. Indeed, our analysis provides the resolution to this problem in the context of CFT, where the physically correct operators for the emission vertices of low energy particles are often non-local without a straightforward \( \alpha' \) expansion. We have rediscovered from a different prospective that the low energy effective theory is not trustworthy for the description of regions close to curvature singularities. The mechanism for the dynamical restoration of space-time supersymmetry that we considered in the last section could be used to incorporate the world-sheet effects that are needed to extend the validity of the effective theory. After all, the initial motivation for promoting various moduli fields into dynamical variables was the extension of the effective field theory approach to
string dynamics by patching together the regions where the low energy approximations are valid \[26\].

A very useful exercise would be to study the anomalous dimension operators in the general (1,0) supersymmetric σ-model along the lines of \[22\]. The most straightforward and logical attack to the problem would of course be to maintain manifest space-time supersymmetry all along the calculation, in the Green-Schwarz formalism. Due to our limited understanding of the quantization of this theory, however, we are forced for the time being to use NSR fermions. This implies in particular that in order to relate these σ-model calculations to space-time supersymmetry, one has presumably to solve the problem of properly including sources for the spin operators.

We would like to point out that in reference \[27\] a similar problem is found. They work with (1,0) and (2,0) WZW models, where they find a clash between the super-Kac-Moody symmetry and the second supersymmetry. They show that if the Kac-Moody transformations are accompanied by a compensating deformation of the complex structure, the problem is resolved and yields a (2,0) supersymmetric extension of the superconformal and Kac-Moody algebras. This compensating transformation is not the one that naturally follows from T-duality in the case of rotational (non-holomorphic) isometries. We suspect that the dynamical restoration of supersymmetry in section 5 is closely related to the need of having a compensating deformation of the complex structure.

Some of the topics of this paper may also be relevant for superstring phenomenology, where we mainly work in terms of the lowest order effective theory. The issue of space-time supersymmetry versus duality, which arises to lowest order in α′, demonstrates explicitly that an apparently non-supersymmetric background can qualify as a vacuum solution of superstring theory, in contrary to the “standard wisdom” that has been considered so far. So, whether supersymmetry is broken or not in various phenomenological applications cannot be decided, unless one knows how to incorporate the appropriate (non-local) world-sheet effects that might lead to its restoration at the string level. Also, various gravitational solutions, like black-holes, might enjoy some supersymmetric properties in the string context. This might also provide a better understanding of the way that string theory, through its world-sheet effects, can resolve the fundamental problems of the quantum theory of black-holes. We hope to return to these problems elsewhere.

Finally, from the low energy point of view the fact that T-duality relates supersymmetric with non-supersymmetric backgrounds could provide examples of the mechanism advocated by Witten \[28\] to shed new light in the cosmological constant problem.
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