A Riemannian gossip approach to
decentralized subspace learning on Grassmann manifold

Bamdev Mishra 1  Hiroyuki Kasai 2  Pratik Jawanpuria 1  Atul Saroop 1

Abstract

In this paper, we propose novel gossip algorithms for decentralized subspace learning problems that are modeled as finite sum problems on the Grassmann manifold. Interesting applications in this setting include low-rank matrix completion and multi-task feature learning, both of which are naturally reformulated in the considered setup. To exploit the finite sum structure, the problem is distributed among different agents and a novel cost function is proposed that is a weighted sum of the tasks handled by the agents and the communication cost among the agents. The proposed modeling approach allows local subspace learning by different agents while achieving asymptotic consensus on the global learned subspace. The resulting approach is scalable and parallelizable. Our numerical experiments show the good performance of the proposed algorithms on various benchmarks, e.g., the Netflix dataset.

1. Introduction

Learning a low-dimensional subspace is a fundamental problem in many machine learning applications, e.g., in control systems and system identification (Markovsky & Usevich, 2013), subspace identification (Balzano et al., 2010), similarity learning (Meyer et al., 2011), collaborative filtering (Rennie & Srebro, 2005), multitask learning (Argyriou et al., 2008; Amit et al., 2007; Ando & Zhang, 2005), and information theory (Shi & Letaief, 2016), to name a few. Consequently, the study of the Grassmann manifold, which is the set of subspaces, has attracted much attention in machine learning lately and has led to a growth of large-scale Grassmann algorithms, both in batch (Absil et al., 2008) and online variants (Bonnabel, 2013; Zhang et al., 2016; Sato et al., 2017) which adapt well to large-scale data.

In this paper, we are interested in a decentralized setting, where we assume that our data is inherently distributed among many agents. Additionally, in order to minimize the communication overhead between the agents, we constrain each agent to communicate with only one other agent as in the standard gossip framework (Boyd et al., 2006). This scenario is not uncommon in situations where there are privacy concerns of sharing sensitive data (Lin & Ling, 2015). Another motivation is that the gossip framework is robust to scenarios where certain agents may be inactive at certain time slots.

To this end, we propose a novel optimization formulation on the Grassmann manifold that combines together a weighted sum of tasks (accomplished by agents individually) and consensus terms (that couples subspace information transfer among agents). The formulation allows to readily propose a stochastic gradient algorithm on the Grassmann manifold (Bonnabel, 2013) and allows a straightforward parallel implementation. We show the performance of the proposed approach on two popular subspace learning problems: low-rank matrix completion (Cai et al., 2010; Keshavan et al., 2010; Balzano et al., 2010; Boumal & Absil, 2015; 2011) and rank-constrained multitask feature learning (Argyriou et al., 2008).

The present work extends and completes the unpublished technical report (Mishra et al., 2016).

The organization of the paper is as follows. Two problems of interest are motivated in Section 2 as minimization of finite sum problems on the Grassmann manifold. In Section 3, we discuss the decentralized problem setup and propose a novel problem formulation. In Section 4, we discuss the proposed stochastic gradient Riemannian gossip algorithm for finite sum problems. A preconditioned variant of the Riemannian gossip algorithm is motivated in Section 4.3. Additionally,
we discuss a way to parallelize the proposed algorithms in Section 4.4. Numerical comparisons in Section 5 show that the proposed algorithms compete effectively with state-of-the-art on various benchmarks.

The Matlab codes for the proposed algorithms are available at https://bamdevmishra.com/codes/gossipMC/.

2. Problem setup and motivation

We look at the subspace learning problem of type

$$\min_{U \in \text{Gr}(r,m)} \sum_{i=1}^{N} f_i(U),$$

(1)

where $\text{Gr}(r,m)$ is the Grassmann manifold, i.e., the set of $r$-dimensional subspaces in $\mathbb{R}^m$. We assume that the functions $f_i : \mathbb{R}^{m \times r} \to \mathbb{R}$ for all $i = \{1, \ldots, N\}$ are smooth.

Following (Absil et al., 2008), an element $U$ of the Grassmann manifold is characterized by a matrix $U$ of size $m \times r$ with orthonormal columns. The set of orthonormal matrices of size $m \times r$ is called the Stiefel manifold $\text{St}(r,m)$.

Mathematically, the relationship between the elements of $\text{Gr}(r,m)$ and the elements of $\text{St}(r,m)$ is captured as

$$U := \{UO : O \text{ is a } r \times r \text{ orthogonal matrix}\}.$$

Equivalently, $U$ is an abstract set representing the column space of $U$. Numerically, optimization algorithms on the Grassmann manifold are implemented with matrices $U \in \text{St}(r,m)$, e.g., the cost computations in (1).

Below we look at two specific problems and formulate them as subspace learning problem (1) on the Grassmann manifold.

2.1. Low-rank matrix completion

The problem of low-rank matrix completion amounts to completing a matrix from a small number of entries by assuming a low-rank model for the matrix. Rank constrained matrix completion problem is formulated as

$$\min_{Y \in \mathbb{R}^{m \times n}} \frac{1}{2}\|P_{\Omega}(Y) - P_{\Omega}(Y^*)\|_F^2 + \lambda\|Y - P_{\Omega}(Y)\|_F^2,$$

subject to $\text{rank}(Y) = r,$

(2)

where $\| \cdot \|_F$ is the Frobenius norm, $\lambda$ is the regularization parameter (Boumal & Absil, 2015; 2011), and $Y^* \in \mathbb{R}^{n \times m}$ is a matrix whose entries are known for indices if they belong to the subset $(i,j) \in \Omega$ and $\Omega$ is a subset of the complete set of indices $\{(i,j) : i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}\}$. The operator $P_{\Omega}(Y_{ij}) = Y_{ij}$ if $(i,j) \in \Omega$ and $P_{\Omega}(Y_{ij}) = 0$ otherwise is called the orthogonal sampling operator and is a mathematically convenient way to represent the subset of known entries. The rank constraint parameter $r$ is usually set to a low value, i.e., $r \ll (m, n)$, which implies that we seek low-rank completion.

A way to handle the rank constraint in (2) is by using the parameterization $Y = UW^T$, where $U \in \text{St}(r,m)$ and $W \in \mathbb{R}^{n \times r}$, where $\text{St}(r,m)$ is the set of $m \times r$ orthonormal matrices, i.e., the columns are orthonormal (Boumal & Absil, 2015; 2011; Mishra et al., 2014). The problem (2) reads

$$\min_{U \in \text{St}(r,m)} \min_{W \in \mathbb{R}^{n \times r}} \frac{1}{2}\|P_{\Omega}(UW^T) - P_{\Omega}(Y^*)\|_F^2 + \lambda\|UW^T - P_{\Omega}(UW^T)\|_F^2.$$

(3)

The inner least-squares optimization problem in (3) is solved in closed form by exploiting the least-squares structure. Consequently, it is readily verified that the outer problem only depends on the column space of $U$, and therefore, is on the Grassmann manifold (Dai et al., 2012; Boumal & Absil, 2015; 2011). The problem is

$$\min_{U \in \text{Gr}(r,m)} \frac{1}{2}\|P_{\Omega}(UW^T_U) - P_{\Omega}(Y^*)\|_F^2 + \lambda\|UW^T_U - P_{\Omega}(UW^T_U)\|_F^2,$$

(4)

where $W_U$ is the solution to the inner optimization problem in (3).
Consider the case when \( \mathbf{Y}^* = [\mathbf{Y}_1^*, \mathbf{Y}_2^*, \ldots, \mathbf{Y}_N^*] \) is partitioned along the columns such that the size of \( \mathbf{Y}_t^* \) is \( m \times n_t \) with \( \sum n_t = n \) for \( t = \{1, 2, \ldots, N\} \). \( \Omega_t \) is the local set of indices for each of the partitions. A straightforward reformulation of (4) is

\[
\min_{\mathbf{U} \in \text{Gr}(r,m)} \sum_{t=1}^{N} \frac{1}{2} \| \mathbf{P}_{\Omega_t} (\mathbf{UW}_t\mathbf{U}^T) - \mathbf{P}_{\Omega_t} (\mathbf{Y}_t^*) \|_F^2 + \lambda \| \mathbf{UW}_t\mathbf{U}^T - \mathbf{P}_{\Omega_t} (\mathbf{UW}_t\mathbf{U}^T) \|_F^2,
\]

where \( \mathbf{W}_t\mathbf{U} \) is the least-squares solution to \( \arg \min_{\mathbf{w}_t \in \mathbb{R}^m} \| \mathbf{P}_{\Omega_t} (\mathbf{UW}_t\mathbf{U}^T) - \mathbf{P}_{\Omega_t} (\mathbf{Y}_t^*) \|_F^2 + \lambda \| \mathbf{UW}_t\mathbf{U}^T - \mathbf{P}_{\Omega_t} (\mathbf{UW}_t\mathbf{U}^T) \|_F^2 \).

Problem (5) is a particular case of (1).

### 2.2. Low-dimensional multitask feature learning

We next transform an important problem in the multitask learning setting (Caruana, 1997; Baxter, 1997; Evgeniou et al., 2005) as a subspace learning problem on the Grassmann manifold. The paradigm of multitask learning advocates joint learning of related learning problems (such as classification or regression problems). This is because related problems share feature information which can be shared during the training phase. This may help in obtaining better generalization performance compared to the standard single-task learning setup, where different problems are learned independently.

A common notion of task-relatedness among different tasks (problems) is as follows: tasks share a latent low-dimensional feature representation (Ando & Zhang, 2005; Argyriou et al., 2008; Zhang et al., 2008; Kang et al., 2011). In the following, we propose to learn this shared feature subspace. We first introduce a few notations related to multitask setting. Let \( T \) be the number of given tasks, with each task \( t \) having \( d_t \) training examples. Let \( (\mathbf{X}_t, y_t) \) be the training instances and corresponding labels for task \( t = 1, \ldots, T \), where \( \mathbf{X}_t \in \mathbb{R}^{d_t \times m} \) and \( y_t \in \mathbb{R}^{d_t} \). (Argyriou et al., 2008) proposed the following formulation to learn a shared latent feature subspace:

\[
\min_{\mathbf{O} \in \mathbb{R}^{m \times m}, \mathbf{w}_t \in \mathbb{R}^m} \frac{1}{2} \sum_{t=1}^{T} \| \mathbf{X}_t \mathbf{Ow}_t - y_t \|_F^2 + \lambda \| \mathbf{W}^T \|_{2,1}^2,
\]

where \( \lambda \) is the regularization parameter, \( \mathbf{O} \) is an orthogonal matrix of size \( m \times m \) that is shared among \( T \) tasks, \( w_t \) is the weight vector (also know as task parameter) for task \( t \), and \( \mathbf{W} := [w_1, w_2, \ldots, w_T]^T \). The term \( \| \mathbf{W}^T \|_{2,1} \) is the \((2,1)\) norm over the matrix \( \mathbf{W}^T \). It enforces the group sparse structure (Yuan & Lin, 2006) across the columns of \( \mathbf{W} \), i.e., only a few columns in the matrix \( \mathbf{W} \) will have non-zero entries. The sparsity across columns in \( \mathbf{W} \) ensures that we learn a low-dimensional latent feature representation for the tasks. The basis vectors of this low-dimensional latent subspace are the columns of \( \mathbf{O} \) corresponding to non-zeros columns of \( \mathbf{W} \). Hence, (6) learns a full rank \( m \times m \) latent feature space \( \mathbf{O} \) and performs feature selection (via spare regularization) in this latent space. This is computationally expensive especially in large scale applications desiring a low \((r)\) dimensional latent feature representation where \( r \ll m \). In addition, the sparsity inducing 1-norm is non-smooth which poses additional optimization challenges.

We instead learn only the basis vectors of the low-dimensional latent subspace, by restricting the dimension of the subspace (Ando & Zhang, 2005; Lapin et al., 2014). We formulate the \( r \)-dimensional multitask feature learning problem as follows:

\[
\min_{\mathbf{U} \in \text{St}(r,m)} \sum_{t=1}^{T} \min_{\mathbf{w}_t \in \mathbb{R}^r} \frac{1}{2} \| \mathbf{X}_t \mathbf{Uw}_t - y_t \|_F^2 + \lambda \| \mathbf{w}_t \|_2^2,
\]

where \( \mathbf{U} \) is an \( m \times r \) matrix representing the low-dimensional latent subspace. Similar to the earlier case in Section 2.1, the inner least-squares optimization problem in (7) is solved in closed form by exploiting the least-squares structure. It is straightforward to verify that the outer problem (7) is on the Grassmann manifold, i.e., it depends on the column space of \( \mathbf{U} \). To this end, the problem is

\[
\min_{\mathbf{U} \in \text{Gr}(r,m)} \sum_{t=1}^{T} \frac{1}{2} \| \mathbf{X}_t \mathbf{Uw}_{t\mathbf{U}} - y_t \|_F^2,
\]

where \( \mathbf{w}_{t\mathbf{U}} \) is the least-squares solution to \( \arg \min_{\mathbf{w}_t \in \mathbb{R}^r} \| \mathbf{XU} \mathbf{w}_t - y_t \|_F^2 + \lambda \| \mathbf{w}_t \|_2^2 \). More generally, we distribute the \( T \) tasks into \( N \) groups such that \( \sum n_t = T \). This leads to the formulation

\[
\min_{\mathbf{U} \in \text{Gr}(r,m)} \sum_{i=1}^{N} \sum_{t \in T_i} \frac{1}{2} \| \mathbf{X}_t \mathbf{Uw}_{t\mathbf{U}} - y_t \|_F^2,
\]
3. Decentralized problem formulation on Grassmann manifold

We exploit the finite sum (sum of $N$ sub cost functions) structure of the problem (1) by distributing the tasks among $N$ agents, which perform certain computations, e.g., computation of the functions $f_i$ given $U$, independently. Although the computational workload gets distributed among the agents, all agents require the knowledge of $U$ (to compute matrices $W_i(U)$). To circumvent this issue, instead of one shared subspace $U$ for all agents, each agent $i$ stores a local subspace copy $U_i$, which it then updates based on information from its neighbors. For minimizing the communication overhead between agents, we additionally put the constraint that at any time slot only two agents communicate, i.e., each agent has exactly only one neighbor. This is the basis of the gossip framework (Boyd et al., 2006).

To this end, the agents are numbered according to their proximity, e.g., for $i \leq N - 1$, agents $i$ and $i + 1$ are neighbors. Equivalently, agents 1 and 2 are neighbors and can communicate. Similarly, agents 2 and 3 communicate, and so on. This communication between the agents allows to reach a consensus on the subspaces $U_i$.

Our proposed approach to handle the finite sum problem (1) in a decentralized fashion is to solve the problem

$$
\min_{U_1, \ldots, U_N \in Gr(r, m)} \sum_{i=1}^{N} f_i(U_i) + \frac{\rho}{2} \left( d_i^2(U_i, U_i) + d_2^2(U_2, U_3) + \ldots + d_{N-1}^2(U_{N-1}, U_N) \right),
$$

where $d_i$ is the geometric distance measure (its matrix characterization is shown in Table 1) between subspaces $U_i$ and $U_{i+1}$ for $i \leq N - 1$ and $\rho \geq 0$ is a parameter that trades off individual (per agent) task solving with consensus.

For a large $\rho$, the consensus term in (9) dominates, minimizing which allows the agents to arrive at consensus, i.e., their subspaces converge. For $\rho = 0$, the optimization problem (9) solves $N$ independent tasks and there is no consensus among the agents. For a sufficiently large $\rho$, the problem (9) achieves the goal of approximate task solving along with approximate consensus.

It should be noted that the consensus term in (9) has only $N - 1$ pairwise distances. For example, Bonnabel (2013) uses this consensus term structure for covariance matrix estimation. Furthermore, it allows to parallelize the updates in a trivial way (discussed in Section 4.4).

4. The Riemannian gossip algorithm for (9)

We exploit the stochastic gradient algorithm framework on manifolds (Bonnabel, 2013; Sato et al., 2017; Zhang et al., 2016) to propose gossip algorithms for (9). As a first step, we reformulate the problem (9) as a single sum problem, i.e.,

$$
\min_{U_1, \ldots, U_N \in Gr(r, m)} \sum_{i=1}^{N-1} g_i(U_i, U_{i+1}),
$$

where $g_i(U_i, U_{i+1}) := \alpha_i f_i(U_i) + \alpha_{i+1} f_{i+1}(U_{i+1}) + 0.5 \rho d_i^2(U_i, U_{i+1})$. Here, $\alpha_i$ is a scalar that ensures that $\sum g_i = f_1 + \ldots + f_N + 0.5 \rho (d_1^2(U_1, U_2) + d_2^2(U_2, U_3) + \ldots + d_{N-1}^2(U_{N-1}, U_N))$, i.e., $\alpha_i = 1$ if $i = \{1, N\}$, else $\alpha_i = 0.5$.

We endow the Grassmann manifold $Gr(r, m)$ with a Riemannian structure. This allows to characterize the updates on the subspaces into concrete matrix updates. Notions such as the Riemannian gradient grad (first order derivative of a cost function), the exponential map $\text{Exp}$ (the mapping from tangent vectors onto the manifold, so that all updates are strictly feasible), and logarithm mapping $\text{Log}$ (“difference” between subspaces) have closed-form expressions (Absil et al., 2008). The distance measure $d_i$ in (9) is specifically chosen as the Riemannian distance (the shortest distance) between the subspaces $U_i$ and $U_{i+1}$. The basics of optimization on the Grassmann manifold are provided as supplementary material.

At each iteration of the stochastic gradient algorithm, we sample a sub cost function $g_i$ from the cost function in (10) uniformly at random (we stick to this sampling process for simplicity). Based on the chosen sub cost function, the subspaces $U_i$ and $U_{i+1}$ are updated by following the negative Riemannian gradient of the sub cost function $g_i$, with a stepsize. The stepsize sequence over the iterations satisfies the conditions that it is square integrable and its summation is divergent.

The overall algorithm is listed as Algorithm 1, which converges to a critical point of (10) almost surely (Bonnabel, 2013; Anonymous, 2017). The gradient updates require the computation of the Riemannian gradient of a sub cost function in (10).
A Riemannian gossip approach to decentralized subspace learning on Grassmann manifold

Algorithm 1: Proposed online gossip algorithm for (10).

1. At each time slot \( t \), pick \( g_i \) with \( i \leq N - 1 \) randomly with uniform probability. This is equivalent to picking up the agents \( i \) and \( i + 1 \).
2. Compute the Riemannian gradients \( \nabla \mathcal{U}_i g_i \) and \( \nabla \mathcal{U}_{i+1} g_i \) of the function \( g_i \) at \( \mathcal{U}_i \) and \( \mathcal{U}_{i+1} \), respectively.
3. Given a stepsize \( \gamma_t \) (e.g., \( \gamma_t = a/(1 + bt) \), where \( a \) and \( b \) are constants), update \( \mathcal{U}_i \) and \( \mathcal{U}_{i+1} \) as
   
   \[
   (\mathcal{U}_i)_+ = \exp_{\mathcal{U}_i}(-\gamma_t \nabla \mathcal{U}_i g_i)
   \]
   \[
   (\mathcal{U}_{i+1})_+ = \exp_{\mathcal{U}_{i+1}}(-\gamma_t \nabla \mathcal{U}_{i+1} g_i),
   \]
   
   where \( (\mathcal{U}_i)_+ \) and \( (\mathcal{U}_{i+1})_+ \) are the updated subspaces and \( \exp_{\mathcal{U}_i} (\xi_{\mathcal{U}_i}) \) is the exponential mapping that maps the tangent vector \( \xi_{\mathcal{U}_i} \in T_{\mathcal{U}_i} \text{Gr}(r, m) \) onto \( \text{Gr}(r, m) \).
4. Repeat.

and moving along the geodesics with the exponential mapping. The matrix characterizations of the required ingredients are shown in Table 1.

4.1. Computational complexity

For an update of \( \mathcal{U}_i \) with the formulas shown in Table 1, the computational complexity depends on the computation of partial derivatives of the cost functions in (2) and (6), e.g., the gradient \( \nabla \mathcal{U}_i f_i \) computation of agent \( i \). Particularly, in the context of the matrix completion problem (2), the computational cost is \( O(|\Omega| r^2 + n_i r^3 + mr^2) \). In the context of multitask feature learning problem (6), the computational cost is \( O(m|T_i| r^2 + |T_i| r^3 + mr^2 + \sum_{t=1}^{T_i} d_t m) \), where \( T_i \) is the group of tasks assigned to agent \( i \). The Grassmann manifold related operations, e.g., \( \exp \) and \( \log \), cost \( O(mr^2 + r^3) \).

4.2. Convergence analysis

Asymptotic convergence analysis of Algorithm 1 follows directly from the analysis in (Bonnabel, 2013). In particular, we have the following proposition.

**Proposition 4.1.** Algorithm 1 converges to a first-order critical point of (10).

**Proof.** The problem (10) can be modeled as

\[
\min_{\mathcal{V} \in \mathcal{M}} \frac{1}{N - 1} \sum_{i=1}^{N-1} h_i(\mathcal{V}),
\]

where \( \mathcal{V} := (\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_N) \), \( \mathcal{M} \) is the Cartesian product of \( N \) Grassmann manifolds \( \text{Gr}(r, m) \), i.e., \( \mathcal{M} := \text{Gr}^N(r, m) \), and \( h_i : \mathcal{M} \to \mathbb{R} : \mathcal{V} \mapsto h_i(\mathcal{V}) = g_i(\mathcal{U}_i, \mathcal{U}_{i+1}) \). The updates shown in Algorithm 1 precisely correspond to stochastic gradients updates for (11).

It should be noted that \( \mathcal{M} \) is compact and has a Riemannian structure, and consequently, the problem (11) is an empirical risk minimization problem of on a compact manifold.

The key idea of the proof is that for a compact Riemannian manifold, all continuous functions of the parameter are bounded. In particular, the Riemannian Hessian of \( h(\mathcal{V}) \) is bounded for all \( \mathcal{V} \in \mathcal{M} \). We assume that 1) the stepsize sequence satisfies the condition that \( \sum \gamma_t = \infty \) and \( \sum (\gamma_t)^2 < \infty \) and 2) at each time slot \( t \), the stochastic gradient estimate \( \nabla g_i \) is an unbiased estimator of the batch Riemannian gradient \( \sum \nabla g_i h_i \). Under those assumptions, Algorithm 1 converges to a first-order critical point of (10).

A rigorous convergence analysis of stochastic gradients on manifolds is presented in (Anonymous, 2017).
4.3. Preconditioned variant

The performance of first order algorithms (including stochastic gradients) often depends on the condition number of the Hessian of the cost function (at the minimum). For the matrix completion problem (2), the issue of ill-conditioning arises when data $Y^*$ are drawn power law distributed singular values. Additionally, a large value of $\rho$ in (9) leads to convergence issues for numerical algorithms.

The recent works (Ngo & Saad, 2012; Mishra & Sepulchre, 2014; Boumal & Absil, 2015) exploit the concept of manifold preconditioning for the matrix completion problem (2). In particular, the Riemannian gradients are scaled by computationally cheap matrix terms that arise from the second order curvature information of the cost function. Matrix scaling of the gradients is equivalent to multiplying an approximation of the inverse Hessian to gradients. This operation on a manifold requires special attention. In particular, the matrix scaling must be a positive definite operator on the tangent space of the manifold (Mishra & Sepulchre, 2014; Boumal & Absil, 2015).

Given the Riemannian gradient, e.g., $\text{grad}_{U_i} g_i$ for agent $i$, the proposed preconditioner leads to the transformation

$$\text{grad}_{U_i} g_i \mapsto \text{grad}_{U_i} g_i (W^\top_{iU_i} W_{iU_i} + \rho I)_{\text{from tasks}}^{-1},$$

(12)

where $I$ is the $r \times r$ identity matrix. The use of preconditioning (12) costs $O(n_i r^2 + r^3)$. The term $W^\top_{iU_i} W_{iU_i}$ is motivated by the fact that it is computationally cheap to compute and captures a block diagonal approximation of the Hessian of the simplified (but related) cost function $\|U_iW^\top_{iU_i} - Y^*_i\|_F^2$, which is an approximation for the problems (2) and (6). For matrix completion, the works (Ngo & Saad, 2012; Mishra & Sepulchre, 2014; Boumal & Absil, 2015) use such preconditioners with superior performance. The term $\rho I$ is motivated as an approximation of the second order derivative of the square of the Riemannian geodesic distance. Finally, it should be noted that the matrix scaling is positive definite, i.e., $W^\top_{iU_i} W_{iU_i} + \rho I > 0$ and that the transformation (12) is on the tangent space.

4.4. Parallel variant

The particular structure (also known as the red-black ordering structure in domain decomposition methods) of the cost terms in (10), allows for a straightforward parallel update strategy for solving (10). We look at the following separation of
A Riemannian gossip approach to decentralized subspace learning on Grassmann manifold

Figure 1. Performance of the proposed algorithms in different scenarios.

the costs, i.e., the problem is

$$\min_{U_1, \ldots, U_N \in \text{Gr}(r, m)} g_{\text{odd}} + g_3 + \ldots + g_2 + g_4 + \ldots,$$

where the subspace updates corresponding to $g_{\text{odd}}$ (and similarly $g_{\text{even}}$) are parallelizable.

We apply Algorithm 1, where we pick the sub cost function $g_{\text{odd}}$ (or $g_{\text{even}}$) with uniform probability. The key idea is that sampling is on $g_{\text{odd}}$ and $g_{\text{even}}$ and not on the sub cost functions $g_i$ directly. This strategy allows to perform $\lfloor (N - 1)/2 \rfloor$ updates in parallel.

5. Numerical comparisons

Our proposed algorithm (Online Gossip) presented as Algorithm 1 and its preconditioned (Precon Online Gossip) and parallel (Parallel Gossip and Precon Parallel Gossip) variants are compared on various different benchmarks for the low-rank matrix completion and multitask feature learning problems. Our implementations are based on the Manopt toolbox (Boumal et al., 2014). For the matrix completion problem, certain operations rely on the mex files supplied with RTRMC (Boumal & Absil, 2015). We fix the number of agents $N$ to 6. Online algorithms are run for a maximum of 1000 iterations. The parallel variants are run for 400 iterations. Overall, agents 1 and $N$ perform a maximum of 200 updates (rest all perform 400 updates). The stepsizes sequence is defined as $\gamma_t = a/(1 + bt)$, where $t$ is the time slot and $a$ and $b$ are set using cross validation. For simplicity, all figures only show the plots for a few agents.

All simulations are performed in Matlab on a 2.7 GHz Intel Core i5 machine with 8 GB of RAM. The comparisons on Netflix and MovieLens-10M datasets are performed on a cluster with larger memory.

5.1. Matrix completion comparisons

For each example considered here, an $m \times n$ random matrix of rank $r$ is generated as in (Cai et al., 2010). Two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times r}$ are generated according to a Gaussian distribution with zero mean and unit standard deviation.
The matrix product $A B^\top$ gives a random matrix of rank $r$. A fraction of the entries are randomly removed with uniform probability and noise (sampled from the Gaussian distribution with mean zero and standard deviation $10^{-6}$) is added to each entry to construct the training set $\Omega$ and $Y^*$. The over-sampling ratio (OS) is the ratio of the number of known entries to the matrix dimension, i.e., $OS = |\Omega|/(mr + nr - r^2)$. We also create a test set by randomly picking a small set of entries from $A B^\top$. The matrices $Y^*_i$ are created by distributing the number of $n$ columns of $Y^*$ equally among the agents. The train and test sets are also partitioned similarly among $N$ agents. All the algorithms are initialized randomly with rank $r$ and $\lambda = 0$.

**Case 1: benefit of the Grassmann geometry.** In contrast to the proposed formulation (9), an alternative is to consider the formulation

$$
\min_{U_1, \ldots, U_N \in \mathbb{R}^{n \times r}} \sum_i f_i(U_i) + \frac{\rho}{2} \left( \|U_1 - U_2\|_F^2 + \ldots + \|U_{N-1} - U_N\|_F^2 \right),
$$

where the problem is in the Euclidean space and the consensus among the agents is with respect to the Euclidean distance. Although this alternative choice is appealing for its numerical simplicity, the benefit of exploiting the geometry of the problem is shown in Figure 1(a). Here, we consider a problem instance of size $10000 \times 100000$ of rank 5 and OS 6. The parameter $\rho$ is set to $10^3$ that achieves completion with consensus of agents. As shown in Figure 1(a), the algorithm with the Euclidean formulation (14) flattens out due to a very slow rate of convergence, especially for achieving consensus among the agents.

**Case 2: effect of $\rho$.** We consider the problem instance in Case 1. Two scenarios with $\rho = 10^3$ and $\rho = 10^{10}$ are considered. Figure 1(b) shows the performance of Online Gossip. Not surprisingly, for $\rho = 10^{10}$, we only see consensus (the distance between agents 1 and 2 tends to zero). For $\rho = 10^3$, we observe both completion and consensus of agents, which validates the theory.

**Case 3: performance of online versus parallel.** We consider Case 1 with $\rho = 10^3$. Figure 1(b) shows the performance of Online Gossip and Parallel Gossip, both of which show a similar behavior on the training set (as well as on the test, which is not shown here) sets.

**Case 4: ill-conditioned instances.** We consider a problem instance of size $5000 \times 50000$ of rank 5 and impose an exponential decay of singular values with condition number 500 and OS 6. Figure 1(c) shows the performance of Online Gossip and its preconditioned variant for $\rho = 10^3$. During the initial updates, the preconditioned variant aggressively minimizes the completion term of (9), which shows the effect of the preconditioner (12). Eventually, consensus among the agents is achieved.

**Case 5: Comparisons with D-LMaFit (Lin & Ling, 2015).** We show comparisons with D-LMaFit, the decentralized algorithm proposed in (Ling et al., 2012; Lin & Ling, 2015). It builds upon the batch matrix completion algorithm in (Wen et al., 2012), which is adapted to decentralized updates of the low-rank factors. It requires an inexact dynamic consensus step at every iteration by performing an Euclidean averaging of low-rank factors (of all the agents). In contrast, our algorithms enforce soft averaging of only two agents at every iteration with the consensus term in (9). In Figure 1(e), we show comparisons on smaller instances as the D-LMaFit code (supplied by the authors) does not scale to large-scale instances. D-LMaFit is run for 400 iterations, i.e., each agent performs 400 updates. We consider a problem instance of size $500 \times 12000$, rank 5, and OS 6. D-LMaFit is run with the default parameters. For Online Gossip, we set $\rho = 10^3$. As shown in Figure 1(e), Online Gossip quickly outperforms D-LMaFit. Overall, Online Gossip takes fewer number of updates of the agents to reach a high accuracy.

**Case 6: comparisons on the Netflix and the MovieLens-10M datasets.** The Netflix dataset (obtained from the code of (Recht & Ré, 2013)) consists of 100 480 507 ratings by 480 189 users for 17 770 movies. We perform 10 random 80/20-train/test partitions. The training ratings are centered around 0, i.e., the mean rating is subtracted. We split both the train and test data among the agents along the number of users. We run Online Gossip with $\rho = 10^7$ (set with cross validation).
Table 3. Test RMSE scores on MovieLens 10M dataset for different ranks. Online Gossip is run with five and ten agents.

|                  | Rank 3 | Rank 5 | Rank 7 | Rank 9 |
|------------------|--------|--------|--------|--------|
| Online Gossip, N = 10 | 0.844  | 0.836  | 0.845  | 0.860  |
| Online Gossip, N = 5  | 0.834  | 0.821  | 0.829  | 0.841  |
| RTRMC-1 (batch)      | 0.829  | 0.814  | 0.812  | 0.814  |

Figure 2. The proposed approach allows the agents to learn a 5-dimensional subspace directly. Furthermore, the agents converge to the optimal subspace $U_\ast$. 

and for $400(N - 1)$ iterations and $N = \{2, 5, 10, 15, 20\}$ agents. We show the results for rank 10 (the choice is motivated in (Boumal & Absil, 2015)). Additionally for Online Gossip, we set the regularization parameter to 0.01. For comparisons, we show the best test root mean square error (RMSE) score obtained by RTRMC-1, which is a batch method for solving the matrix completion problem on the Grassmann manifold. In order to compute the test RMSE for Online Gossip on the full test set (not on the agent-partitioned test sets), we use the (Karcher) mean subspace of the subspaces obtained by the agents as the final subspace obtained by our algorithm. Table 2 shows the RMSE scores for Online Gossip and RTRMC-1 averaged over ten runs. Table 2 shows that the gossip approach allows to reach a reasonably good solution on the Netflix data with different number of agents (which interact minimally among themselves). It should be noted that as the number of agents increases, the consensus problem (9) becomes challenging. Similarly, consensus of agents at higher ranks is more challenging as we need to learn a larger subspace. Figure 1(e) shows the consensus of agents for the case $N = 10$.

We also show the results on the MovieLens-10M dataset of 10 000 054 ratings by 71 567 users for 10 677 movies (MovieLens, 1997). The setup is similar to the earlier case. We run Online gossip with $N = \{5, 10\}$ and $\rho = 10^3$. We show the RMSE scores for different ranks in Table 3.

5.2. Multitask comparisons

Case 7: synthetic datasets. We consider a toy problem instance with $T = 1000$ tasks. The number of training instance in each task $t$ is between 10 and 50 ($d_t$ chosen randomly). The input space dimension is $m = 100$. The training instances $X_t$ are generated according to the Gaussian distribution with zero mean and unit standard deviation. A 5-dimensional feature subspace $U_\ast$ for the problem instance is generated as a random point on $U_\ast \in St(5,100)$. The weight vector $w_t$ for the task $t$ is generated from the Gaussian distribution with zero mean and unit standard deviation. The labels for training instances for task $t$ are computed as $y_t = X_t U_\ast U_\ast^\top w_t$. The labels $y_t$ are subsequently perturbed with a random mean zero Gaussian noise with $10^{-6}$ standard deviation. The tasks are uniformly divided among $N = 6$ agents and Online Gossip is initialized with $r = 5$ and $\rho = 10^3$. Figure 2 shows that all the agents are able to converge to the optimal
A Riemannian gossip approach to decentralized subspace learning on Grassmann manifold

Figure 3. Parkinsons: Online Gossip learns a 3-dimensional subspace with NMSE scores (averaged over the agents) comparable to Alt-Min, which learns a 19-dimensional subspace.

Table 4. Average test NMSE scores obtained on multitask datasets. Online Gossip is run with six agents and learns a $r$-dimensional subspace. The NMSE scores obtained by Online Gossip are comparable to the the scores Alt-Min, which learns an $m$-dimensional subspace.

| Datasets | Online Gossip $N = 6$ | Alt-Min (batch method) |
|----------|-----------------------|------------------------|
|          | $r = 3$ | $r = 5$ | $r = 7$ | $r = 9$ | $r = m$ |
| Parkinsons | 0.345 | 0.339 | 0.342 | 0.341 | 0.340 |
| School    | 0.761 | 0.786 | 0.782 | 0.786 | 0.781 |

subspace $U_*$.

Case 8: comparisons on multitask benchmarks. We compared the generalization performance of formulation (8) solved by the proposed gossip algorithm against state-of-the-art multitask feature learning formulation (6) (Argyriou et al., 2008). Argyriou et al. (2008) propose an alternate minimization algorithm (Alt-Min) to solve an equivalent convex problem of (6). Conceptually, Alt-Min alternates between the subspace learning step and task weight vector learning step. The former involves an eigenvalue decomposition of an $m \times m$ matrix, which costs $O(m^3)$. Algorithms with such high complexity are difficult to execute on large-scale problems. In contrast, we learn a low-dimensional subspace in (7) that keeps the computational cost linear with $m$ (shown in Section 4.1). The comparisons on benchmark multitask learning datasets show that we are able to obtain smaller normalized mean square errors (NMSE) at much smaller rank compared to Alt-Min on (6). To this end, we compare Online Gossip and Alt-Min on two datasets: Parkinsons and School. In the Parkinsons dataset, the aim is to predict the Parkinson’s disease symptom score of $T = 42$ patients with $m = 19$ bio-medical features and 5 875 readings (Frank & Asuncion). The School dataset consists of 15 362 students from $T = 139$ schools (Argyriou et al., 2008). The aim is to predict the performance of each student with $m = 28$ features.

We perform 10 random 80/20-train/test partitions. We run Online Gossip with $\rho = 10^6$, $N = 6$, and for 200($N - 1$) iterations. Alt-Min is run for 400 iterations. Table 4 shows the NMSE scores (averaged over all $T$ tasks and ten runs) for both the algorithms. Online Gossip obtains competitive NMSE scores on lower dimensional subspaces than Alt-Min. Figure 3 shows the NMSE scores obtained by all the agents.
6. Conclusion

We have proposed a decentralized Riemannian gossip approach to subspace learning problems with the finite sum structure. The tasks are distributed among a number of agents, which are then required to achieve consensus. Building upon the gossip framework, this is modeled as minimizing a weighted sum of task solving and consensus terms on the Grassmann manifold. The rich geometry of the Grassmann manifold allowed to propose a novel stochastic gradient algorithm for the problem with simple updates. Two variants – preconditioned and parallel – of the algorithm are proposed for dealing with different scenarios. Numerical experiments on two interesting applications – low-rank matrix completion and low-dimensional multitask feature learning – show the competitive performance of the proposed gossip approach.

References

Absil, P.-A., Mahony, R., and Sepulchre, R. Optimization Algorithms on Matrix Manifolds. Princeton University Press, Princeton, NJ, 2008.

Amit, Y., Fink, M., Srebro, N., and Ullman, S. Uncovering shared structures in multiclass classification. In Proceedings of the 24th International Conference on Machine Learning, pp. 17–24, 2007.

Ando, R. K. and Zhang, T. A framework for learning predictive structures from multiple tasks and unlabeled data. Journal of Machine Learning Research, 6(May):1817–1853, 2005.

Anonymous, A. An expectation-based convergence analysis of Riemannian stochastic gradient descent. Technical report, Anonymous, 2017.

Argyriou, A., Evgeniou, T., and Pontil, M. Convex multi-task feature learning. Machine Learning, 73(3):243–272, 2008.

Balzano, L., Nowak, R., and Recht, B. Online identification and tracking of subspaces from highly incomplete information. In The 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 704–711, June 2010.

Baxter, J. J. A bayesian/information theoretic model of learning to learn via multiple task sampling. Machine Learning, 28(1):7–39, 1997.

Bonnabel, S. Stochastic gradient descent on Riemannian manifolds. IEEE Transactions on Automatic Control, 58(9):2217–2229, 2013.

Boumal, N. and Absil, P.-A. RTRMC: A Riemannian trust-region method for low-rank matrix completion. In Advances in Neural Information Processing Systems 24 (NIPS), pp. 406–414, 2011.

Boumal, N. and Absil, P.-A. Low-rank matrix completion via preconditioned optimization on the Grassmann manifold. Linear Algebra and its Applications, 475:200–239, 2015.

Boumal, N., Mishra, B., Absil, P.-A., and Sepulchre, R. Manopt: a Matlab toolbox for optimization on manifolds. Journal of Machine Learning Research, 15(Apr):1455–1459, 2014.

Boyd, S., Ghosh, A., Prabhakar, B., and Shah, D. Randomized gossip algorithms. IEEE Transaction on Information Theory, 52(6):2508–2530, 2006.

Cai, J. F., Candès, E. J., and Shen, Z. A singular value thresholding algorithm for matrix completion. SIAM Journal on Optimization, 20(4):1956–1982, 2010.

Caruana, R. Multitask learning. Machine Learning, 28(1):41–75, 1997.

Dai, W., Kerman, E., and Milenkovic, O. A geometric approach to low-rank matrix completion. IEEE Transactions on Information Theory, 58(1):237–247, 2012.

Edelman, A., Arias, T.A., and Smith, S.T. The geometry of algorithms with orthogonality constraints. SIAM Journal on Matrix Analysis and Applications, 20(2):303–353, 1998.

Evgeniou, T., Micchelli, C. A., and Pontil, M. Learning multiple tasks with kernel methods. Journal of Machine Learning Research, 6(Apr):615–637, 2005.
A Riemannian gossip approach to decentralized subspace learning on Grassmann manifold

Frank, A. and Asuncion, A. UCI machine learning repository. URL http://archive.ics.uci.edu/ml.

Kang, Z., Grauman, K., and Sha, F. Learning with whom to share in multi-task feature learning. In International Conference on Machine learning (ICML), 2011.

Keshavan, R. H., Montanari, A., and Oh, S. Matrix completion from a few entries. *IEEE Transactions on Information Theory*, 56(6):2980–2998, 2010.

Lapin, M., Schiele, B., and Hein, M. Scalable multitask representation learning for scene classification. In Conference on Computer Vision and Pattern Recognition, 2014.

Lin, A.-Y. and Ling, Q. Decentralized and privacy-preserving low-rank matrix completion. *Journal of the Operations Research Society of China*, 3(2):189–205, 2015.

Ling, Q., Xu, Y., Yin, W., and Wen, Z. Decentralized low-rank matrix completion. In *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 2925–2928, 2012.

Markovsky, I. and Usevich, K. Structured low-rank approximation with missing data. *SIAM Journal on Matrix Analysis and Applications*, 34(2):814–830, 2013.

Meyer, G., Bonnabel, S., and Sepulchre, R. Regression on fixed-rank positive semidefinite matrices: a Riemannian approach. *Journal of Machine Learning Research*, 11(Feb):593–625, 2011.

Mishra, B. and Sepulchre, R. R3MC: A Riemannian three-factor algorithm for low-rank matrix completion. In *Proceedings of the 53rd IEEE Conference on Decision and Control (CDC)*, pp. 1137–1142, 2014.

Mishra, B., Meyer, G., Bonnabel, S., and Sepulchre, R. Fixed-rank matrix factorizations and Riemannian low-rank optimization. *Computational Statistics*, 29(3–4):591–621, 2014.

Mishra, B., Kasai, H., and Saroop, A. A Riemannian gossip approach to decentralized matrix completion. Technical report, arXiv preprint arXiv:1605.06968, 2016.

MovieLens. MovieLens, 1997. URL http://grouplens.org/datasets/movielens/.

Ngo, T. T. and Saad, Y. Scaled gradients on Grassmann manifolds for matrix completion. In *Advances in Neural Information Processing Systems 25 (NIPS)*, pp. 1421–1429, 2012.

Recht, B. and Ré, C. Parallel stochastic gradient algorithms for large-scale matrix completion. *Mathematical Programming Computation*, 5(2):201–226, 2013.

Rennie, J. and Srebro, N. Fast maximum margin matrix factorization for collaborative prediction. In *International Conference on Machine learning (ICML)*, pp. 713–719, 2005.

Sato, H., Kasai, H., and Mishra, B. Riemannian stochastic variance reduced gradient. Technical report, arXiv preprint arXiv:1702.05594, 2017.

Shi, Y.and Zhang, J. and Letaief, K. B. Low-rank matrix completion for topological interference management by Riemannian pursuit. *IEEE Transactions on Wireless Communications*, PP(99), 2016.

Wen, Z., Yin, W., and Zhang, Y. Solving a low-rank factorization model for matrix completion by a nonlinear successive over-relaxation algorithm. *Mathematical Programming Computation*, 4(4):333–361, 2012.

Yuan, M. and Lin, Y. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society, Series B*, 68:49–67, 2006.

Zhang, H., Reddi, S. J., and Sra, S. Riemannian svrg: Fast stochastic optimization on Riemannian manifolds. In *Advances in Neural Information Processing Systems (NIPS)*, pp. 4592–4600, 2016.

Zhang, J., Ghahramani, Z., and Yang, Y. Flexible latent variable models for multi-task learning. *Machine Learning*, 73(3):221–242, 2008.
A. Grassmann manifold $\text{Gr}(r, m)$

This section briefly introduces the Grassmann manifold $\text{Gr}(r, m)$. Additional details are in (Edelman et al., 1998; Absil et al., 2008).

An element $\mathcal{U}$ on the Grassmann manifold is represented by an $m \times r$ orthogonal matrix $U$ with orthonormal columns, i.e., $U^T U = I$. Two orthogonal matrices represent the same element on the Grassmann manifold if they are related by right multiplication of a $r \times r$ orthogonal matrix $O \in O(r)$. Equivalently, an element of the Grassmann manifold is identified with a set of $m \times r$ orthogonal matrices $U := \{UO : O \in O(r)\}$. In other words, $\text{Gr}(r, m) := \text{St}(r, m)/O(r)$, where $\text{St}(r, m)$ is the Stiefel manifold that is the set of matrices of size $m \times r$ with orthonormal columns. The Grassmann manifold has the structure of a Riemannian quotient manifold (Absil et al., 2008, Section 3.4).

Since the Grassmann manifold is an abstract manifold, an iterative optimization algorithm on it, by necessity, is implemented in $\text{St}(r, m)$. The following ingredients are required to implement a first order (including the stochastic gradient) algorithm on $\text{Gr}(r, m)$.

1. The tangent space of $\text{Gr}(r, m)$ at an element $\mathcal{U}$ has the matrix characterization

$$T_{\mathcal{U}} \text{Gr}(r, m):= \{\xi_U \in \mathbb{R}^{m \times r} : U^T \xi_U = 0\},$$

where $U$ is the matrix characterization of $\mathcal{U}$ and $\xi_U$ is the matrix characterization of the abstract tangent vector $\xi_{\mathcal{U}}$ at $\mathcal{U} \in \text{Gr}(r, m)$.

2. The Riemannian gradient $\text{grad}_{\mathcal{U}} f$ of a cost function, say $f : \text{Gr}(r, m) \to \mathbb{R}$ has the matrix representation

$$\text{grad}_{\mathcal{U}} f = \text{Grad}_{\mathcal{U}} f - U(U^T \text{Grad}_{\mathcal{U}} f),$$

where $\text{Grad}_{\mathcal{U}} f$ is the (Euclidean) gradient of $f$ in the matrix space $\mathbb{R}^{m \times r}$ at $U$.

3. The Exponential mapping for the Grassmann manifold from $U(0) \in \text{Gr}(r, m)$ in the direction of $\xi_{\mathcal{U}(0)} \in T_{U(0)} \text{Gr}(r, m)$ is given in closed form as (Absil et al., 2008, Section 5.4)

$$U(t) = \left[U(0) V W \right] \begin{bmatrix} \cos \Sigma \\
\sin \Sigma \end{bmatrix} V^T, \quad \text{(A.1)}$$

where the matrix representation of the tangent vector $\xi_{\mathcal{U}(0)}$ has the rank-$r$ singular value decomposition $W \Sigma V^T$. Here, $U(0)$ and $U(t)$ are the matrix characterizations of $\mathcal{U}(0)$ and $\mathcal{U}(t)$, respectively. The $\cos(\cdot)$ and $\sin(\cdot)$ operations are only on the diagonal entries.

4. The Logarithm map of $\mathcal{U}(t)$ at $\mathcal{U}(0)$ on the Grassmann manifold is given by the tangent vector

$$\xi_{\mathcal{U}(0)} = \text{Log}_{\mathcal{U}(0)}(\mathcal{U}(t)) = P \arctan(S) Q^T, \quad \text{(A.2)}$$

where $PSQ^T$ is the rank-$r$ singular value decomposition of $(U(t) - U(0)U(0)^TU(t))(U(0)^TU(t))^{-1}$. 

