Measurement-induced dynamics of many-body systems at quantum criticality

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We consider a dynamic protocol for quantum many-body systems, which enables to study the interplay between unitary Hamiltonian driving and random local projective measurements. While the unitary dynamics tends to increase entanglement, local measurements tend to disentangle, thus favoring decoherence. Close to a quantum transition where the system develops critical correlations with diverging length scales, the competition of the two drivings is analyzed within a dynamic scaling framework, allowing us to identify a regime (dynamic scaling limit) where the two mechanisms develop a nontrivial interplay. We perform a numerical analysis of this protocol in a measurement-driven Ising chain, which supports the scaling laws we put forward. The local measurement process generally tends to suppress quantum correlations, even in the dynamic scaling limit. The power law of the decay of the quantum correlations turns out to be enhanced at the quantum transition.

One of the greatest challenges of modern statistical mechanics is understanding and controlling the quantum dynamics of many-body systems. The recent progress in atomic physics and quantum optical technologies has provided a great opportunity for a thorough investigation of the interplay between the coherent quantum dynamics and the interaction with the environment, from both experimental and theoretical viewpoints [11–13]. The competition of such mechanisms may originate a subtle interplay, likely representing the most intricate dynamic regime of quantum systems where complex many-body phenomena may appear. In this respect, it is worth focusing on situations close to a quantum phase transition, where quantum critical fluctuations emerge and correlations develop a diverging length scale [7, 8].

In general, while the unitary time evolution gives rise to a growth of entanglement, measurements of observables disentangle degrees of freedom and thus tend to decrease quantum correlations, similarly to decoherence. A quantum measurement is physically realized when the interaction with a macroscopic classical object makes a quantum mechanical system rapidly collapse into an eigenstate of a specific operator, and the resulting time evolution appears to be a non-unitary projection. Such process is referred to as a projective measurement [9, 10]. When the system is projected into an eigenstate of a local operator, the corresponding local degree of freedom is disentangled from the rest of the system. Moreover, if measurements are performed frequently, the quantum state gets localized in the Hilbert space near a trivial product state, leading to the quantum Zeno effect [11, 12].

Inspired by recent pioneering studies of the entanglement dynamics in measurement-induced random unitary quantum circuits [13–15], we introduce a framework to address the interplay of unitary and projective dynamics in experimentally viable many-body systems at quantum transitions, such as quantum spin networks. For this purpose, we consider dynamic problems arising from protocols combining the unitary Hamiltonian and local measurement drivings (for a cartoon, see Fig. 1). In such conditions, it is not clear how the presence of projective measurements modifies the quantum critical behavior of a purely unitary system. One can easily imagine that different regimes emerge, depending on the measurement protocols and their parameters. If every site were measured during each projective step, then the system would be continually reset to a tensor product state. A more intriguing scenario should hold when the local measurements are spatially dilute.

Most of the work done so far in this context focused on the investigation of entanglement transitions genuinely driven by local measurements, either in random circuits [13–22], or in the Bose-Hubbard model [23, 24], and recently on measurement-induced state preparation [25]. In noninteracting models, continuous local measurements were shown to largely suppress entanglement [26]. Here we focus on a substantially different dynamic problem:

![FIG. 1: Sketch of the protocol: A quantum spin system, initially frozen in its ground state at quantum criticality (t = 0), is perturbed with local projective measurements (stars) occurring every time interval t_m, with a homogeneous probability p per site. In between two measurement steps, the system evolves unitarily according to its Hamiltonian. Red stars denote the occurrence of a measurement on a given site (for the sake of clarity, in the figure we considered σ^z-type measures: spins colored in red are projected along the z-axis).](image-url)
understanding and predicting the effects of local random measurements on the quantum critical dynamics of many-body systems, i.e., when a quantum transition is driven by the Hamiltonian parameters.

Specifically, we consider quantum lattice spin systems, assuming that only one relevant Hamiltonian parameter can deviate from the critical point, which we generically call \( \mu \) and assume the critical point to be at \( \mu_c = 0 \), with corresponding renormalization-group (RG) dimension \( y_\mu > 0 \). The system is initialized, at \( t = 0 \), from the ground state close to the critical point, thus \( |\mu| \ll 1 \). Random local measurements are then performed at every time interval \( t_m \), such that each site has a (homogeneous) probability \( p \) to be measured. In between two measurement steps, the system evolves according to the unitary probability \( p \), as sketched in Fig. 1, where \( H(\mu) \) is its Hamiltonian and we fix \( h = k_B = 1 \). If \( p \to 1 \), each spin gets measured every \( t_m \), and the effects of projections are expected to dominate over those of the unitary evolution. In contrast, for \( p \) sufficiently small, the time evolution may result unaffected by the measurements. In between these two regimes, we unveil the existence of a competing unitary vs. projective dynamics, characterized by controllable dynamic scaling behaviors associated with the universality class of the quantum transition.

More complex protocols may be devised. For example, the initial ground state might be replaced with a Gibbs state at a finite temperature \( T \). One may also consider a quench of the control parameter at \( t = 0 \), starting from the ground state for a given value \( \mu_0 \) (so that \( |\mu_0| \ll 1 \)), to a different value \( \mu \) which characterizes the unitary evolution between the measurement steps. In this case, the out-of-equilibrium evolution arises from both the initial quench and the measurement protocol. For the sake of clarity in our presentation, we will focus on the simpler version discussed before, even though an extension to such more complex scenarios is not difficult (see Appendix A).

As for the model, we consider the paradigmatic \( d \)-dimensional quantum Ising Hamiltonian,

\[
H_{\text{Is}} = -J \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} \sigma_{\mathbf{x}}^{(3)} \sigma_{\mathbf{y}}^{(3)} - g \sum_{\mathbf{x}} \sigma_{\mathbf{x}}^{(1)} - h \sum_{\mathbf{x}} \sigma_{\mathbf{x}}^{(3)},
\]

where \( \sigma \equiv (\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}) \) are the spin-1/2 Pauli matrices, the first sum is over the bonds connecting nearest-neighbor sites \( \langle \mathbf{x}, \mathbf{y} \rangle \), while the other sums are over the sites. We fix \( J = 1 \) as the energy scale. At \( g = g_c \) and \( h = 0 \), the model undergoes a continuous quantum transition belonging to the two-dimensional Ising universality class, separating a disordered phase \( (g > g_c) \) from an ordered \( (g < g_c) \) one [7,8]. Such transition is characterized by a diverging length scale \( \xi \) of the critical correlations, and the suppression of the energy gap \( \Delta \) as \( \Delta \approx \xi^{-\gamma} \), where \( z = 1 \) is the dynamic exponent. The power-law divergence of \( \xi \) is related to the RG dimensions of the relevant parameters \( \delta \equiv g - g_c \) and \( h \): it behaves as \( \xi \sim |\delta|^{-1/\delta} \) at \( h = 0 \), and \( \xi \sim |h|^{-1/\delta} \) for \( \delta = 0 \) [27].
The quantum critical Ising chain (Fig. 3: The susceptibility ratio $R_\chi$ vs. rescaled time $tL^{-z}$ for the quantum critical Ising chain $(g = 1, h = 0)$ for various $L$. Measurements occur along the longitudinal direction, with $t_m = 0.1$ and $p$ being either constant (left: $c = 0.005$) or equal to $p = cL^{-a}$ (middle: $a = 1, c = 0.05$; right: $a = 3, c = 5$).

We note that, in the cases $a = 0$ and $a = 1$, the time scale $\tau_m$ of the suppression of the quantum correlations significantly decreases with increasing $L$ as a function of the scaling time $tL^{-z}$; on the other hand it clearly increases for $a = 3$.

The arbitrariness of the scale parameter $b$ in Eq. 3 can be fixed by setting $b = \lambda \equiv |\mu|^{-1/b}$, where $\lambda \sim \xi$ is the length scale of the critical modes. The scaling variable associated with the time interval $t_m$ should be given by the ratio $t_m/\tau$, where $\tau \sim \Delta^{-1} \sim \lambda^z$ is the time scale of the critical models (this implies $\zeta = -z$). Keeping $t_m$ fixed in the large-$\lambda$ limit, the dependence on $t_m$ disappears asymptotically, giving only rise to $O(\lambda^{-z})$ scaling corrections. Moreover, noticing that the parameter $p$ is effectively a probability per unit of time and space, a reasonable guess would be that its correct scaling to compete with the critical modes is that $p \sim \lambda^{-z-d}$, thus

$$\varepsilon = z + d.$$  \hspace{1cm} (4)

This leads to the dynamic scaling equation $B(\mu, t, t_m, p) \approx \lambda^{-y_B} B(\mu L^{y_B}, tL^{-z}, \mu L^\zeta)$. We stress that the value of the exponent $\varepsilon$ in Eq. 4 is crucial, because it allows to separate the measurement-irrelevant regime $p = O(\lambda^{-\varepsilon})$ (right panel of Fig. 3) from the measurement-dominant regime $pL^\varepsilon \to \infty$ (left and middle panels of Fig. 3). Note that, since $p \sim \lambda^{-z}$ and $t \sim \lambda^z$, the dynamic scaling ansatz predicts that the time scale $\tau_m$ associated with the suppression of the quantum correlations behaves as $\tau_m \sim p^{-\varepsilon}$ with $\kappa = z/\varepsilon < 1$.

The above scaling theory holds in the thermodynamic limit $L/\lambda \to \infty$, that is expected to be well defined for any $\mu \neq 0$, for which $\lambda$ is finite. Nonetheless, for most practical purposes, both experimental and numerical, one typically has to face with systems of finite length. Such situations can be framed in the FSS framework, where the scale parameter in Eq. 3 is fixed to $b = L$.

Assuming again that $t_m$ is kept fixed, straightforward manipulations lead to the following scaling law

$$B(\mu, t, t_m, p, L) \approx L^{-y_B} B(\mu L^{y_B}, tL^{-z}, pL^\zeta).$$  \hspace{1cm} (5)

The proper dynamic FSS behavior is obtained by taking $L \to \infty$, while keeping $t_m$ and the arguments of the scaling function $B$ fixed.
It is worth mentioning that analogous scaling ansatzes for more general observables, such as fixed-time correlation functions of two operators, can be obtained using the same arguments and assumptions (see Appendix A). They can be extended to include an initial quench of the Hamiltonian parameter $\mu_0 \rightarrow \mu$ (by adding a further dependence on $\mu_0 b^{h_0}$ in Eq. (5), to consider finite-temperature initial Gibbs states (by adding a dependence on $T b^2$), and allowing for weak dissipation [43]. We note that the scaling arguments do not depend on the type of local measurement, therefore they are expected to be somehow independent of them. Further investigations are called for to classify the extension of such independence.

The above phenomenological scaling theory has been checked on the quantum Ising chain. The dynamic FSS laws for the magnetization $m$ and its susceptibility $\chi$ follow Eq. (5), in which the parameter $\mu$ corresponds to either $\delta = g - g_c$ or $h$ in Eq. (1). In particular, for $\delta = h = 0$, one obtains $m(t) = 0$ by symmetry, and

$$R_\chi(t, t_m, p, L) \equiv \frac{\chi(t) - 1}{\chi(t = 0) - 1} \approx R_\chi(tL^{-z}, pL^\kappa).$$

Further details are provided in Appendix B. Results for a system at the quantum critical point, with random local longitudinal and transverse spin measurements, are shown in Fig. 1. The data of $R_\chi$ versus $tL^{-z}$ nicely agree with Eq. (6). Corrections to the scaling are consistent with a $L^{-3/4}$ approach, as expected (see the insets). An analogous agreement has been obtained for the magnetization at $h \neq 0$, keeping $hL^{6k}$ constant (see Appendix C).

We finally focus on systems which are not close to a phase transition (for example, $g > g_c$ in the case of quantum Ising models). In this case, the system lies in the disordered phase, where the length scale $\xi$ of the quantum correlations and the gap $\Delta$ remain finite with increasing $L$. The data reported in Fig. 5 for $R_\chi$ at fixed size $L$ suggest that, away from criticality, the characteristic time $\tau_m$ of the measurement process scales as $\tau_m \sim p^{-1}$, unlike the critical behavior, where $\tau_m \sim p^{-\kappa}$ with $\kappa = z/\varepsilon < 1$.

Summarizing, we showed the emergence of different dynamic regimes arising from the interplay between unitary and projective dynamics in many-body systems at quantum transitions, where quantum correlations develop a diverging length scale. One of them is characterized by the dominance of the local random measurements, for example for any finite probability $p$ of making the local measurement. In contrast, for sufficiently small values of $p$ (i.e., decreasing as a sufficiently large power of the inverse diverging length scale $\xi$), the measurements turn out to be irrelevant. We conjectured these two regimes to be separated by dynamic conditions imposing suitable scaling behaviors for the characterizing parameters of the protocol, such as the local measurement probability $p$, which is controlled by the universality class of the quantum transition. For a $d$-dimensional critical system, this occurs when $p \sim \xi^{-\epsilon}$, for which we argue $\epsilon = z + d$, where $z$ is the dynamic exponent of the transition.

This general scenario is supported by numerical results for the quantum Ising chain, for protocols involving the measurements of either transverse or longitudinal component of the local spin operator. Local measurements generally tend to suppress quantum correlations, even in the dynamic scaling limit. The corresponding time scale
is expected to behave as $\tau_m \sim p^{-\kappa}$ with $\kappa = z/\varepsilon < 1$, to be compared with the noncritical case $\tau_m \sim p^{-1}$. The smaller power $\kappa$ at the critical point can be explained by the fact that the relevant probability ($p_r$) driving the measurement process is the probability to perform a local measurement within the critical volume $\xi^d$, therefore $p_r = p\xi^d$. The time rate thus behaves as $\tau_m \sim p_r^{-1}$, similarly to the noncritical case, where $\xi = O(1)$.

Additional checks are called for, in order to achieve a definite validation of our scaling conjectures, such as the study of protocols with quenches of the Hamiltonian parameters, higher dimensions, other quantum transitions (in particular characterized by different values of the dynamic exponent $z$) and measurement schemes (non necessarily strictly onsite, but still sufficiently local). Furthermore we expect that arguments similar to those employed here could be used for measurements that are localized in restricted regions of space, and also to derive possible peculiar scaling phenomena in proximity of first-order quantum transitions, where the boundary conditions could play a more relevant role [33, 49].

Given the relatively small sizes required to reach the scaling limit, a direct experimental realization of our protocol can be reasonably considered as a near-future target for quantum simulations. Promising platforms are superconducting quantum circuits [51–53], nuclear spins [54, 55], trapped ions [56–58] and ultracold atomic systems [59–62].

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### Appendix A: Phenomenological scaling theory of the out-of-equilibrium dynamics induced by local measurements

We work out a phenomenological scaling theory for the out-of-equilibrium dynamics arising from random local projective measurements during the evolution of a many-body system at a quantum transition [7]. For simplicity, we assume that the quantum transition is driven by a relevant parameter $\mu$ of the Hamiltonian $H(\mu)$, whose critical value is $\mu_c = 0$. At the critical point, the low-energy unitary Hamiltonian dynamics develops long-distance correlations, characterized by a diverging length scale $\xi \sim |\mu|^{-\nu}$, where $\nu = 1/y_\mu$ and $y_\mu$ is the renormalization-group (RG) dimension of the relevant parameter.

More specifically, we consider the dynamic problem associated with the following protocol: (a) The system starts at $t = 0$ from the ground state close to the critical point, thus $|\mu| \ll 1$; (b) Random local measurements are performed every time interval $t_m$, with a homogeneous probability $p$ per site. Between two measurement steps, the system evolution is driven by the unitary operator $e^{-iH(\mu)t}$. Hereafter we adopt units of $\hbar = k_B = 1$.

The out-of-equilibrium critical dynamics at continuous quantum transitions has been shown to obey homogeneous scaling laws [30–41], even in the presence of dissipation [42, 43]. For example, after an instantaneous quench from $\mu_0 = 0$ to $\mu$, a generic observable $B$ at fixed time $t$ after the quench, is generally expected to behave as [35]

$$B(\mu, t, L) \approx b^{-y_B} B(\mu b^y, tb^{-\zeta}, L/b), \quad (A1)$$

where $b$ is an arbitrary positive parameter, $L$ is the linear size of the $d$-dimensional system under investigation, and $B$ is a universal scaling function apart from normalizations. The exponent $y_B$ denotes the RG dimension of the operator associated to $B$, while the dynamic exponent $z$ characterizes the behavior of the energy differences of the lowest-energy states and, in particular, the ground-state gap $\Delta \sim L^{-z}$. Equation (A1) is expected to provide the asymptotic power-law behavior in the large-$b$ limit.

We now extend the dynamic scaling arguments leading to Eq. (A1), by allowing for the dependence on the parameters $t_m$ and $p$ which characterize the measurement procedure of the protocol. As a working hypothesis, we assume that an asymptotic scaling behavior is achieved by appropriately rescaling $t_m$ and $p$, such as

$$B(\mu, t, t_m, p, L) \approx b^{-y_B} B(\mu b^y, tb^{-\zeta}, t_m b^\zeta, pb^\varepsilon, L/b), \quad (A2)$$

where $\zeta$ and $\varepsilon$ are appropriate exponents whose relevance is discussed below.

### 1. Dynamic finite-size scaling

It is possible to exploit the arbitrariness of the scale parameter $b$. For example, by setting $b = L$, we obtain the dynamic finite-size scaling (FSS) equations, extending those holding for closed systems [8, 33, 46–48]. Analogously as for the time $t$ that has passed after the quench, the scaling variable associated with the time interval $t_m$ should be given by the ratio $t_m/\tau$ where $\tau \sim \Delta^{-1} \sim L^z$ is the time scale of the critical models, thus

$$\zeta = -z. \quad (A3)$$

If one keeps $t_m$ fixed in the large-$L$ dynamic FSS limit, the dependence on $t_m$ disappears asymptotically, giving only rise to $O(L^{-z})$ scaling corrections. Moreover, noting that the parameter $p$ is effectively a probability per unit of time and space, a reasonable guess is that we must have the power-law scaling behavior $p \sim L^{-\varepsilon d}$ to achieve a nontrivial competition with the critical modes. Therefore,

$$\varepsilon = z + d. \quad (A4)$$

We stress that the value of the exponent $\varepsilon$ is crucial, because it allows us to separate the regime $p = o(L^{-\varepsilon})$ in
which the random measurements are irrelevant for the asymptotic dynamic scaling, from that where they drive the evolution overwhelming the unitary Hamiltonian dynamics, when \( pL^\varepsilon \to \infty \).

On the basis of these scaling arguments, from Eq. (A2) we conjecture that, keeping \( t_m \) and the arguments of the scaling function \( B \) fixed, the dynamic FSS law associated with the random-measurement protocol reads

\[
B(\mu, t, t_m, p, L) \approx L^{-y_B} B(\mu L^{y_B}, tL^{-z}, \mu t_m L^{-z}, pL^\varepsilon). \tag{A5}
\]

The scaling function \( B \) is expected to be largely universal with respect to the Hamiltonian of the system, within a given universality class, and also with respect to the details of the protocol. Of course, like any scaling function a quantum transition, such universality is expected modulo a multiplicative overall constant and normalizations of the scaling variables. Note that, in this case, the asymptotic scaling behavior does not depend on \( t_m \) and therefore it is expected to hold also in the limit \( t_m \to 0 \).

Alternatively, one may rescale the time interval \( t_m \), as \( \tau \sim L^z \), thus keeping the ratio \( t_m/L^z \) fixed. In this case we expect the probability \( p \) to scale as the inverse volume only, i.e.

\[
B(\mu, t, t_m, p, L) \approx L^{-y_B} B(\mu L^{y_B}, tL^{-z}, t_m L^{-z}, pL^d). \tag{A6}
\]

Note that, analogously to Eq. (A5), the FSS limit requires that \( p/t_m \sim L^{-z} \). Similar scaling ansatzes for more general observables, such as fixed-time correlation functions of two operators, can be straightforwardly obtained using the same assumptions and scaling arguments.

The above predictions can be extended to the more complex protocol including an initial quench of the Hamiltonian parameter from \( \mu_0 \) to \( \mu \); this is achieved by adding the further dependence on \( \mu_0 L^{y_B} \). Moreover, one may also consider an initial Gibbs state for a small temperature \( T \), and this can be taken into account by adding a further scaling variables \( TL^z \).

We finally note that our scaling arguments do not apparently depend on the type of local measurement, thus they are expected to be somehow independent on them.

2. Dynamic scaling in the thermodynamic limit

To derive a dynamic scaling theory for infinite-volume systems, we may restart from the general homogeneous power law in Eq. (A2) and set

\[
b = \lambda \equiv |\mu|^{1/y_B}, \tag{A7}
\]

where \( \lambda \) is the length scale of the critical modes, and consider the limit \( L/\lambda \to \infty \), assuming that it is well defined (this limit corresponds to the so-called thermodynamic limit, which is expected to be well defined for any \( \mu \neq 0 \), for which \( \lambda \) is finite). Then, keeping again \( t_m \) fixed such that \( t_m/\lambda^z \to 0 \) in the \( \lambda \to \infty \) limit, and using the fact that the power law associated with \( p \) is expected to be characterized by the same exponent \( \varepsilon \) given in Eq. (A4), one obtains the dynamic scaling ansatz

\[
B(\mu, t, t_m, p) \approx \lambda^{-y_B} B(\lambda^{-z}, \mu \lambda^\varepsilon), \tag{A8}
\]

where \( \varepsilon = z+d \) is given as in Eq. (A4). Note that, strictly speaking, one has two scaling functions \( B \), depending on the sign of \( \mu \).

Appendix B: Dynamic scaling within the quantum Ising model

The one-dimensional (1D) quantum Ising model in a transverse field is one of the simplest paradigmatic quantum many-body systems exhibiting a nontrivial zero-temperature phase diagram. The corresponding Hamiltonian reads

\[
H_{1s} = -J \sum_x \sigma_x^{(3)} \sigma_{x+1}^{(3)} - g \sum_x \sigma_x^{(1)} - h \sum_x \sigma_x^{(3)}, \tag{B1}
\]

where \( \sigma \equiv (\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}) \) are the spin-1/2 Pauli matrices, the first sum is over all bonds of the chain connecting nearest-neighbor sites, while the other sums are over the sites of the chain. In our numerical studies, we set \( J = 1 \) as the energy scale and consider chains of size \( L \) with periodic boundary conditions \( (\sigma_{L+1} \equiv \sigma_1) \).

At \( g = g_c \) and \( h = 0 \) (in 1D, \( g_c = 1 \)), the model undergoes a continuous quantum transition belonging to the two-dimensional Ising universality class, separating a disordered phase \( (g > 1) \) from an ordered \( (g < 1) \) one (see e.g. Refs. 17, 8). Such transition is characterized by a diverging length scale \( \xi \) of the critical correlations and the suppression of the ground-state energy gap \( \Delta \) as \( \Delta \approx \xi^{-z} \) with \( z = 1 \). The power-law divergence of \( \xi \) is related to the RG dimensions, \( y_s \) and \( y_h \), of the relevant parameters \( \delta \equiv g - g_c \) and \( h \), respectively. For the transverse field it is given by

\[
y_s = 1/\nu, \tag{B2}
\]

while for the longitudinal field

\[
y_h = \frac{1}{2}(d+z+2-\eta), \tag{B3}
\]

\( \eta \) being the exponent which describes the critical behavior of the correlation function of the order parameter \( \sigma^{(3)} \). Therefore, the critical length scale diverges as \( \xi \sim |\delta|^{-1/y_s} \) for \( h = 0 \), and \( \xi \sim |h|^{-1/y_h} \) for \( \delta = 0 \). Specializing to the 1D case, one has \( d = 1, \nu = 1 \), and \( \eta = 1/4 \), therefore \( y_s = 1 \) and \( y_h = 15/8 \).

For dynamic protocols using the quantum Ising Hamiltonian in Eq. (B1), the evolution of the system can be effectively characterized by the time dependent magnetization along the coupling direction

\[
M(t) = \frac{1}{L} \sum_x \langle \sigma_x^{(3)} \rangle_t, \tag{B4a}
\]
the fixed-time longitudinal correlation function
\[ G(x, y, t) = \langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_t , \tag{B4b} \]
and the corresponding susceptibility
\[ \chi(t) = \frac{1}{L} \sum_{x,y} G(x, y, t) . \tag{B4c} \]

Here \( \langle \cdot \rangle_t \) indicates the expectation value of a given observable at time \( t \). Note that translation invariance, which also applies in finite-size systems with periodic boundary conditions, implies \( G(x, y, t) = G(x - y, t) \).

For the sake of presentation and without loss of generality, we fix \( g = g_c \) and only vary \( h \), so that \( h \) corresponds to the parameter \( \mu \) of the above-reported scaling equations (analogous equations would hold if \( g \) were varied, with the substitution \( h \rightarrow \delta \) and \( y_h \rightarrow y_b \)). The dynamic FSS laws of the observables \( \chi, t_m \), keeping \( t_m \) fixed, thus read
\begin{align*}
M(h, t, t_m, p, L) & \approx L^{-y_m} M(hL^{y_b}, tL^{-z}, pL^\epsilon), \tag{B5a} \\
G(x, h, t, t_m, p, L) & \approx L^{-2y_m} G(\frac{x}{L}, hL^{y_b}, tL^{-z}, pL^\epsilon), \tag{B5b} \\
\chi(h, t, t_m, p, L) & \approx L^{d-2y_m} C(hL^{y_b}, tL^{-z}, pL^\epsilon). \tag{B5c}
\end{align*}

Corrections to scaling are generally expected to be \( \mathcal{O}(1/L) \), see for example Refs. \[38, 41, 46\]. However we note that, in the case of the susceptibility \( \chi \) defined as in Eq. \[\text{(B4c)}\], \( \mathcal{O}(L^{-d+2y_m}) \) corrections are also present, already at the level of the equilibrium ground-state values of \( \chi \), due to analytic contributions to the critical behavior, as explained in Ref. \[46\]. Therefore, in the case of the Ising chain, we expect that the leading scaling corrections to the asymptotic dynamic scaling of the evolution of \( \chi \) are \( \mathcal{O}(L^{-3/4}) \).

Our numerical results show that the measurement process generally tends to suppress quantum correlations, therefore \( \chi(t) \) is a monotonic decreasing function. In particular, the numerics provides evidence of the fact that
\[ \lim_{t \to \infty} \chi(t) = 1 . \tag{B8} \]
The asymptotic value corresponds to a fully disordered state with vanishing correlations, \( G(x, y, t \to \infty) = 0 \) (for \( x \neq y \)), and where the only non-zero contributions entering the sum \[\text{(B4c)}\] are those for \( x = y \), which trivially sum up to one. To monitor the suppression of quantum correlations due to the measurement process, it is thus convenient to introduce the ratio
\[ R_x = \frac{\chi(t) - 1}{\chi(t = 0) - 1} , \tag{B9} \]
which goes from one (for \( t = 0 \)) to zero (for \( t \to \infty \)). In the dynamic scaling limit at the critical point, using Eq. \[B7\], we can immediately derive the asymptotic behavior
\[ R_x(t, t_m, p, L) \approx R_x(tL^{-z}, pL^\epsilon) . \tag{B10} \]
Note that, in the dynamic scaling limit, \( R_x \approx \chi(t)/\chi(0) \), i.e. the finite subtraction of one in the numerator and denominator of the definition of \( R_x \) turns out to be irrelevant. Therefore, like for the susceptibility, the approach to the asymptotic dynamic FSS behavior \[\text{(B10)}\] is expected to be characterized by \( \mathcal{O}(L^{-3/4}) \) corrections for the quantum Ising chain (see results reported in Fig. \[4\].

In the infinite-volume limit, at \( g = g_c \), we expect to have
\begin{align*}
M(h, t, t_m, p) & \approx \lambda^{-y_m} M(t^{1-\epsilon}, pL^\epsilon), \tag{B11a} \\
G(x, h, t, t_m, p) & \approx \lambda^{-2y_m} G(x/\lambda, t^{1-\epsilon}, pL^\epsilon), \tag{B11b} \\
\chi(h, t, t_m, p) & \approx \lambda^{d-2y_m} C(t^{1-\epsilon}, pL^\epsilon). \tag{B11c}
\end{align*}

where \( \lambda = |h|^{-1/y_b} \). Such dynamic scaling behaviors are expected to be approached asymptotically for \( L \to \infty \), keeping fixed the scaling variables of the functions \( M \) and \( C \).

As already noted above, the dynamic scaling arguments that we have outlined do not apparently depend on the type of local measurement. In particular, in the case of the quantum Ising model, they should apply to protocols based on both \( \sigma_z \) or \( \sigma_x^{(3)} \) local measurements.

The time scale \( \tau_m \) of the suppression of the quantum correlations may be estimated from the halving time of \( R_x(t) \). Its power-law scaling behavior in terms of the probability \( p \) can be easily derived in the dynamic scaling limit, by noting that \( p \sim \xi^{-\epsilon} \) and \( t \sim \xi^\xi \) where \( \xi \) is the length scale of the critical modes (that is \( \xi \sim L \) at the critical point and \( \xi \sim \lambda \) around it). Therefore, the dynamic scaling predicts that the time scale \( \tau_m \) associated with the suppression of the quantum correlations behaves as
\[ \tau_m \sim p^{-\kappa} , \quad \kappa = \frac{z}{\xi} = \frac{z}{z + d} . \tag{B12} \]
Note that \( \kappa < 1 \), thus the time scale in terms of \( p \) turns out to be accelerated with respect the noncritical behavior \( \tau_m \sim p^{-1} \) which has been obtained numerically, see Fig. \[5\]. This apparently counterintuitive behavior can be explained by the nontrivial fact that the relevant probability \( p_r \), which drives the measurement process, is the probability to perform a local measurement within the critical volume \( \xi^d \), therefore \( p_r = p \xi^d \). In terms of \( p_r \), the time scale thus behaves as
\[ \tau_m \sim p_r^{-1} , \quad p_r = p \xi^d , \tag{B13} \]
similarly to the noncritical case, where \( \xi = O(1) \).
Appendix C: Some details on the numerical computations

To check our phenomenological dynamic scaling theory discussed before, we have performed some numerical simulations on the 1D quantum Ising chain \[ \text{\cite{B1}} \], based on exact diagonalization (ED). We are interested in the random-measurement protocol starting from the ground state of a system of size \( L \) (with periodic boundary conditions) for the Hamiltonian parameter \( h \) and with \( g = g_c = 1 \), which has been obtained by means of a Lanczos technique. The evolution, monitored through a fourth-order Suzuki-Trotter decomposition of the unitary-evolution operator with time step \( dt = 0.005 \), is essentially driven by the random measurements, which are performed at every time interval \( t_m \). We have considered either local longitudinal \( \sigma_x^{(3)} \) or transverse \( \sigma_x^{(1)} \) measurements, occurring with a probability \( p \) per site.

In Fig. 4 we showed results only for the susceptibility ratio \( R_x \). Here we provide some additional data, both for the magnetization \[ \text{\cite{B4a}} \] and for the susceptibility \[ \text{\cite{B4c}} \], in which we kept \( g = 1 \) and varied the longitudinal field \( h \) (note that the quantum Ising chain with \( h \neq 0 \) is not integrable).

The need of averaging over many different trajectories, typically \( O(10^4) \), together with the fact that the numerical results shown in this paper are nicely consistent with the dynamic FSS theory, prevented us from studying systems with more than \( L = 18 \) sites, although larger sizes would be easily addressable for a single trajectory or a few ones. Also note that we preferred to use conventional (and fully controllable) ED techniques over DMRG-based algorithms \[ \text{\cite{B, B33}} \], since with those latter methods it is more complicated to guarantee the required accuracy in order to carefully test our phenomenological scaling theory. Nonetheless, there are no conceptual limitations in using DMRG for analyzing the measurement-induced dynamics of quantum lattice models with finite degrees of freedom \[ \text{\cite{B23}} \]. In summary, ED techniques are more controllable, but suffer from severe limitations in the reachable system sizes; DMRG allows to study larger systems, although it requires more care in the choice of the bond-link dimension for the study of dynamical problems.

Figure 6 displays the numerical outcomes for the susceptibility at the Ising critical point \( (g = 1, h = 0) \), for random local measurements taken along the longitudinal or the transverse direction. Note that the results presented in this figure are the same as those reported in Fig. 4 but for the rescaled susceptibility \( \chi L^{-3/4} \) [instead of the ratio \( R_x \) in Eq. \[ \text{\cite{B9}} \]]. Similarly as for the susceptibility ratio, we observe a nice agreement with the predicted scaling behavior in Eq. \[ \text{\cite{B5c}} \]. Moreover, corrections to the scaling are consistent with a \( L^{-3/4} \) behavior, as expected (see the two insets).

Results for the magnetization \( m(t) \) are reported in Fig. 7. In that case, we considered \( g = 1 \) and a nonzero longitudinal field \( h \), since the latter is essential in order to start from an initially magnetized state \( m(0) \neq 0 \).

After a suitable rescaling of all the relevant parameters, the various curves approach an asymptotic scaling behavior, as indicated in Eq. \[ \text{\cite{B5a}} \]. Notice that we also rescaled the field \( h \) so to keep the scaling variable \( h L^{15/8} \) constant. The approach to the scaling is governed by corrections whose leading order appear to be consistent with a \( L^{-1} \) behavior, as witnessed by the two insets.

All the numerical data presented above correspond to fixing the time interval between two consecutive measurements equal to \( t_m = 0.1 \). We have checked that analogous scaling results can be obtained for arbitrary values of \( t_m \). In particular, in Fig. 8 we have considered the limit \( t_m \to 0 \). More precisely, random local measurements have been performed at every Trotter time step, so that \( t_m = 0.005 \). The upper panel displays results for the rescaled susceptibility \( \chi L^{-3/4} \) as a function of the rescaled time \( t/L \), keeping \( g = 1 \) and \( h = 0 \), for measurements performed along the longitudinal direction \( \sigma_x^{(3)} \).
along a given axis, for which it is possible to derive an expression for the magnetization in a magnetic field, subject to periodic measurements at finite time. The approach to the asymptotic curves for the magnetization and susceptibility at infinite magnetic field along two orthogonal directions, already at very small system sizes (L = 5), contrary to the case of larger system sizes, may get suppressed in the limit L → ∞.

Finally, we observe that the asymptotic curves for the magnetization and susceptibility at L → ∞ should coincide with those at finite L, after a proper rescaling of all the relevant parameters in the dynamic protocol.

The lower panels highlight that corrections to the scaling are O(L^{-3/4}), as expected. Notice that here the compatibility with a L^{-3/4} behavior (dashed red lines) is excellent already at very small system sizes (L = 5), contrary to the case of larger Lm values: compare with the insets of Fig. 6, where deviations from the expected trend emerge at smaller L. This hints at the fact that other subleading corrections (dashed red lines) that may enter the scaling corrections at finite L, may get suppressed in the limit Lm → 0, such as those which are O(tm/L).

Appendix D: One-spin model subject to periodic measurements

Here we discuss the dynamics of a single spin-1/2 system in a magnetic field, subject to periodic measurements along a given axis, for which it is possible to derive an analytic solution. We consider the following Hamiltonian model:

\[ H_1 = -g \sigma^{(1)} - h \sigma^{(3)}, \]  

where g and h denote the intensity of an applied external magnetic field along two orthogonal directions \( \sigma \equiv (\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}) \) are the usual spin-1/2 Pauli matrices. Without loss of generality, we fix g = 1. We now suppose to initialize the system in its ground state, associated with the parameter h at t = 0. Then we perform a sequence of repeated measurements of the operator \( \sigma^{(3)} \) at every time interval \( t_m \) (the choice of the measurement axis is arbitrary).

The dynamics arising from this protocol can be described in terms of the system’s density matrix \( \rho \). The starting (t = 0) state is a pure state, given by

\[ \rho_0 = \rho(t = 0) = |0_h\rangle\langle 0_h|, \]

where

\[ E_0 = -\sqrt{1 + h^2}; \]

\[ |0_h\rangle = N \left( (-h - \sqrt{1 + h^2})|+\rangle + |-\rangle \right), \]

and \( |\pm\rangle \) are the eigenstates of \( \sigma^{(3)} \), while \( N \) is the normalization to obtain \( \langle 0_h |0_h\rangle = 1 \). Then, defining the density matrix after n measurements, at time \( t = n t_m \), as

\[ \rho_n = \rho(t = n t_m), \]
the subsequent dynamics can be described as a series of two-step operations:

(i) A unitary time evolution for a time $t_m$,

$$\hat{\rho}_{n+1} = e^{-iH_1 t_m} \rho_n e^{iH_1 t_m}; \quad (D6a)$$

(ii) The measurement of $\sigma^{(3)}$,

$$\rho_{n+1} = \text{Tr}(\hat{\rho}_{n+1} P_+) P_+ + \text{Tr}(\hat{\rho}_{n+1} P_-) P_-, \quad (D6b)$$

where $P_\pm$ are the projectors onto the eigenstates $|\pm\rangle$ of $\sigma^{(3)}$. Simple manipulations thus lead to

$$\rho_{n+1} = \frac{1}{2} I + W_{n+1} \sigma^{(3)}, \quad W_{n+1} = \hat{\rho}_{n+1}^{11} - \frac{1}{2}, \quad (D6c)$$

where $W$ quantifies the deviation from the trivial completely unpolarized density matrix $\rho_u = I/2$. Given the initial condition $|D2\rangle$, one finds

$$W_1 = \frac{h(\sqrt{1 + h^2} + h)}{2(1 + h^2 + h\sqrt{1 + h^2})}. \quad (D7)$$

Straightforward computations allow to obtain

$$W_{n+1} = f W_n, \quad f = 1 - \frac{2[\sin(t_m \sqrt{1 + h^2})]^2}{1 + h^2}. \quad (D8)$$

In particular, for $h = 1$, one gets $W_1 = (1 + \sqrt{2})/(4 + 2\sqrt{2}) \approx 0.35355$ and $f = 1 - \sin^2(\sqrt{2}t_m)$. Note that the factor $f$ is bounded, indeed

$$|f| \leq 1. \quad (D9)$$

Moreover, for arbitrary values of $h$ and $t_m$, one strictly finds $|f| < 1$. Indeed $f = -1$ for $h = 0$ only, while $f = 1$ for the specific values $t_m = m\pi/\sqrt{1 + h^2}$, with integer $m$.

Equation $\langle D8 \rangle$ implies

$$W_n = W_1 f^{n-1}. \quad (D10)$$

Therefore, by monitoring the expectation value of $\sigma^{(3)}$, one eventually gets

$$\langle \sigma^{(3)} \rangle_{t = nt_m} = \text{Tr}[\sigma^{(3)} \rho_n] = 2W_n = 2W_1 f^{n-1}. \quad (D11)$$

Since in general $|f| < 1$, this shows that for any $h$ the dynamical protocol tends to produce disorder the spin model, leading to a completely unpolarized matrix density.

For the specific case of sufficiently small $t_m \ll 1$, one finds

$$f = 1 - 2t_m^2 + O(t_m^4), \quad (D12)$$

and thus

$$W_n \sim (1 - 2t_m^2)^n \approx e^{-2t_m^2 n}. \quad (D13)$$

Therefore, the dependence on $h$ disappears in the leading $O(t_m^4)$ term. Finally we note that, in the limit $t_m \to 0$, the quantum Zeno effect [11, 12] can be recovered, indeed one simply has $f \to 1$ and $W_n \to 1$.

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For 3D systems, they assume mean-field values, but there are very accurate estimates [28], and in particular [50]: \( \xi = 1/\nu \) with \( \nu = 0.629971(4) \) and \( y_b = (5 - \eta)/2 \) with \( \eta = 0.0636298(2) \). For 3D systems, they assume mean-field values, \( y_b = 2 \) and \( y_a = 3 \), apart from logarithmic corrections.

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