On geometric properties of sets of positive reach in $\mathbb{E}^d$

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Abstract

Some geometric facts concerning sets with positive reach in $\mathbb{E}^d$ are proved. For $x_1$ and $x_2$ in $\mathbb{E}^d$ and $R > 0$ let us denote by $\mathcal{H}(x_1, x_2, R)$ the intersection of all closed balls of radius $R$ containing $x_1$ and $x_2$. We prove that $\text{reach}(K) \geq R$ if and only if for every $x_1, x_2 \in K$ such that $\|x_1 - x_2\| < 2R$, $\mathcal{H}(x_1, x_2, R) \cap K$ is connected. A corollary is that if $\text{reach}(K) \geq R > 0$ and $D$ is a closed ball of radius less than or equal to $R$ (intersecting $K$) then $\text{reach}(K \cap D) \geq R$. For $A \subset \mathbb{E}^d$ and $R > 0$ we say that $A$ admits $R$-hull if there exists $A \supset \hat{A}$, with $\text{reach}(\hat{A}) \geq R$ and such that $\hat{A}$ is the minimal set (with respect to inclusion) having these properties. A necessary and sufficient condition for a set $A \subset \mathbb{E}^d$ to admit a $R$-hull is provided.

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1 Introduction

Sets of positive reach were introduced by Federer in [2]. This class of sets can be viewed as an extension of that of convex sets. It is well known that every point $x$ external to a closed convex set $C$ in $\mathbb{E}^d$ admits a unique projection on $C$, i.e. a point which minimizes the distance from $x$ among all points in $C$. Sets of positive reach are those for which the projection is unique for the points of a parallel neighborhood of the set (and not necessarily for all external points).

Along with their definition, Federer provided the main fundamental properties of sets of positive reach. Namely, the validity of global and local Steiner formulas and consequently the existence of curvature measures and many relevant properties of such measures.

The study of properties of sets with positive reach has been continued by several authors and along various directions. Let us mention the contributions given by Zähle [7] and Rataj and Zähle [6] on integral representation of curvature measures, the results by Hug [4], and Hug and the first author [1] on singular points of sets with positive reach and the extensions of Steiner type formulas by Hug, Last and Weil [5]. Moreover, in [3] Fu proved several interesting connections between sets of positive reach and semi-convex functions.

As stated by Federer, closed convex sets represent a limit case of sets of positive reach, as the reach tends to $\infty$. The following question was at the origin of the research carried out in this paper. Is it possible to see (at least some of) the geometric properties of convex sets as limit case of suitable geometric properties of sets of positive reach?

The first property that we analyse is the very definition of convex set: if $x_1$ and $x_2$ belong to a convex set $C$, then the segment joining them is entirely contained in $C$. In §3
we prove a possible counterpart of this fact for sets of positive reach. For two points \( x_1 \) and \( x_2 \) in \( \mathbb{E}^d \) and \( R > 0 \) we denote by \( \delta(x_1, x_2, R) \) the intersection of all closed balls of radius \( R \) containing \( x_1 \) and \( x_2 \). The set \( \delta(x_1, x_2, R) \) is a rugby ball-shaped set with cusps in \( x_1 \) and \( x_2 \); moreover for \( R \to \infty \), \( \delta(x_1, x_2, R) \) tends to the segment with endpoints \( x_1 \) and \( x_2 \). Theorem 3.8 states that \( \text{reach}(K) \geq R \) if and only if for every \( x_1, x_2 \in K \) such that \( \|x_1 - x_2\| < 2R \), \( \delta(x_1, x_2, R) \cap K \) is connected. The proof of this result is geometric and does not require sophisticated techniques. As a corollary (see Theorem 3.10) we have the following fact: if \( \text{reach}(K) \geq R > 0 \) and \( D \) is a closed ball of radius less than or equal to \( R \), intersecting \( K \), then \( \text{reach}(K \cap D) \geq R \). The latter property can be seen as a counterpart, for sets with positive reach, of the well-known fact that the intersection of a convex set with an half-space is convex (if it is non-empty).

Next, we consider the following problem: given a set \( A \) and a number \( R > 0 \) is it possible to find the minimal set (with respect to inclusion) containing \( A \) and having reach greater than or equal to \( R \)? The corresponding problem in the context of convexity (\( R = \infty \)) has an affirmative answer: every set admits a least convex cover, i.e. its convex hull. We will see through simple examples that this is not the case for arbitrary \( A \) and \( R \) and we will find necessary and sufficient conditions so that \( A \) admits a minimal cover of reach greater than or equal to \( R \).

The paper is organized as follows: in §2 we introduce some notations; in §3 we prove Theorem 3.8 and some related results; in §4 we deal with the least cover with prescribed reach of a given set.

## 2 Notations

Let \( \mathbb{E}^d \) be the \( d \)-dimensional Euclidean space; for \( a, b \in \mathbb{E}^d \), let \( \|b - a\| \) be their distance and let \((\cdot, \cdot)\) denote the usual scalar product.

If \( A \) is a subset of \( \mathbb{E}^d \), then \( \text{int}(A) \), \( \text{cl}(A) \) and \( A^c \) will denote the interior, the closure and the complement set of \( A \), respectively. For \( x_0 \in \mathbb{E}^d \) and \( r > 0 \) we set

\[
B(x_0, r) = \{ x \in \mathbb{E}^d : \|x - x_0\| < r \}, \quad \text{and} \quad D(x_0, r) = \text{cl}(B(x_0, r)).
\]

For \( A \subset \mathbb{E}^d \) and \( a \in \mathbb{E}^d \), the distance of \( a \) from \( A \) is given by

\[
\delta_A(a) = \inf\{ \|a - x\| : x \in A \}.
\]

Let us recall the definition of sets of positive reach, introduced in [2]. Let \( K \subset \mathbb{E}^d \) be closed; let \( \text{Unp}(K) \) be the set of points having a unique projection (or foot point) on \( K \):

\[
\text{Unp}(K) := \{ a \in \mathbb{E}^d : \exists! x \in K \text{ s.t. } \delta_K(x) = \|a - x\| \}.
\]

This definition implies the existence of a projection mapping \( \xi_K : \text{Unp}(K) \to K \) which assigns to \( x \in \text{Unp}(K) \) the unique point \( \xi_K(x) \in K \) such that \( \delta_K(x) = \|x - \xi_K(x)\| \). For a point \( a \in K \) we set:

\[
\text{reach}(K, a) = \sup\{ r > 0 : B(a, r) \subset \text{Unp}(K) \}.
\]

The reach of \( K \) is then defined by:

\[
\text{reach}(K) = \inf_{a \in K} \text{reach}(K, a),
\]

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and $K$ is said to be of positive reach if $\text{reach}(K) > 0$.

If $K \subset \mathbb{E}^d$ is compact and $x \in K$, the tangent and the normal spaces to $K$ at $a$ are:

$$\text{Tan}(K, a) = \{0\} \cup \left\{ u : \forall \epsilon > 0 \exists b \in K \text{ s.t. } 0 < \|b - a\| < \epsilon, \left\| \frac{b - a}{\|b - a\|} - \frac{u}{\|u\|} \right\| < \epsilon \right\},$$

$$\text{Nor}(K, a) = \{v : (u, v) \leq 0, \forall u \in \text{Tan}(K, a)\}.$$

Notice in particular that $\text{Nor}(K, a)$ is a closed convex cone. Let $\text{reach}(K) > 0$; for $a \in K$ we set:

$$P_a = \{v : \xi_K(a + v) = a\}, \quad Q_a = \{v : \delta_K(a + v) = \|v\|\}.$$

### 3 Characterization and geometrical properties of sets with positive reach

The following definition will be useful later.

**Definition 3.1** Let $a, b \in \mathbb{E}^d$, $a \neq b$, $R > 0$, $\|a - b\| < 2R$. Let

$$\mathcal{D}(a, b, R) = \{D(x, R) : \|a - x\|, \|b - x\| \leq R\}.$$

We set

$$\mathcal{F}(a, b, R) = \bigcap_{D \in \mathcal{D}(a, b, R)} D.$$

It is clear from the definition that $\mathcal{F}(a, b, R)$ is a compact convex set, containing $a$ and $b$. The boundary of $\mathcal{F}(a, b, R)$ is obtained rotating an arc of circle of radius $R$ joining $a$ and $b$, about the line through $a$ and $b$.

**Lemma 3.2** Let $a, b \in \mathbb{E}^d$ be such that $0 < \|a - b\| < 2R$ where $R > 0$. If $c, d \in \mathcal{F}(a, b, R)$, then $\mathcal{F}(c, d, R) \subset \mathcal{F}(a, b, R)$.

**Proof.** If $D \in \mathcal{D}(a, b, R)$, then $c, d \in D$ so that $D \in \mathcal{D}(c, d, R)$. The conclusion follows from Definition 3.1. □

A set is convex if and only if given any two points belonging to it, it contains the line segment joining them. In this section we prove (see Theorem 3.8) a characterization of sets of positive reach that somehow resembles the above characterization of convex sets. The proof of this result requires various lemmas. The next proposition is Theorem 4.8 (7) of [2].

**Proposition 3.3** Let $K \subset \mathbb{E}^d$ be closed, $x \in \text{Unp}(K)$ and $\text{reach}(K, \xi_K(x)) > 0$. Then, for every $b \in K$

$$(x - \xi_K(x), \xi_K(x) - b) \geq -\frac{\|\xi_K(x) - b\|^2 \|x - \xi_K(x)\|}{2 \text{reach}(K, \xi_K(x))}. \tag{1}$$

Let $R > 0$ and $a, b \in \mathbb{E}^d$ be such that $0 < \|a - b\| < 2R$. We define the cone

$$\mathcal{C}(a, b, R) = \left\{ v \neq 0 : \left( v, \frac{b - a}{\|b - a\|} \right) > \frac{\|b - a\|}{2R} \right\}.$$

A geometric version of the above proposition follows.
Corollary 3.4 Let $K$ be a closed subset of $\mathbb{E}^d$ such that reach($K$) $\geq R > 0$. Let $x \in \text{Unp}(K) \setminus K$, $a = \xi_K(x) \in \partial K$ and $b \in K$ such that $0 < \|a - b\| < 2R$. Then

$$x - a \notin C(a, b, R).$$

We proceed with some geometric considerations in the plane. Given $v$ and $w$ vectors in $\mathbb{E}^2$, $v, w \neq 0$, we set

$$S(v, w) = \{z : z = tv + \tau w, t, \tau > 0\}.$$

Remark 3.5 Let $R > 0$ and $z_1, z_2, z_3, z_4 \in \mathbb{E}^2$ be such that

$$\|z_1 - z_2\| = \|z_2 - z_3\| = \|z_3 - z_4\| = \|z_4 - z_1\| = R, \quad 0 < \|z_1 - z_3\| < 2R.$$

We have

$$C(z_1, z_3, R) = S(z_2 - z_1, z_4 - z_1).$$

Lemma 3.6 Let $R > 0$, $b_1, b_2 \in \mathbb{E}^2$ with $0 < \|b_1 - b_2\| < 2R$, $\Gamma_j = \partial B(b_j, R)$, $j = 1, 2$, $b_3, b_4 \in \mathbb{E}^2$ such that $\{b_3, b_4\} = \Gamma_1 \cap \Gamma_2$. Let

(i) $\Sigma \subset \Gamma_1$ be the closed arc joining $b_3$ and $b_4$ of smaller length;

(ii) $\Sigma' \subset \Gamma_1$ be the closed arc having length $\pi R$ and such that $\Sigma \cap \Sigma' = \{b_4\}$.

For every $a \in B(b_4, R) \setminus D(b_3, R)$ there exist $c \in \Sigma$, $c \neq b_3$, $c \neq b_4$, and $c' \in \Sigma'$, uniquely determined, such that

$$\|b_1 - c'\| = \|c' - a\| = \|a - c\| = \|c - b_1\| = R.$$

Proof. We have $\|a - b_3\| > R$ and $\|a - b_4\| < R$. Let us notice that $b_3$ and $b_4$ are the endpoints of $\Sigma$. By continuity, there exists $c \in \Sigma$ such that $\|a - c\| = R$. Let $b_5$ be the endpoint of $\Sigma'$ which does not coincide with $b_4$; we have $\|b_4 - b_5\| = 2R$ and $\|a - b_5\| + \|a - b_4\| \geq \|b_4 - b_5\| = 2R$; thus $\|a - b_5\| \geq 2R - \|a - b_4\| > R$. By continuity, there exists $c' \in \Sigma'$ such that $\|a - c'\| = R$. The points $c$ and $c'$ are uniquely determined as intersection of $\Gamma_1$ and $\partial B(a, R)$. □

Figure 1
Lemma 3.7 Let $R > 0$, $b_1, b_2 \in \mathbb{E}^2$, $0 < \|b_1 - b_2\| < 2R$, $B_i = B(b_i, R)$, $\Gamma_i = \partial B_i$, $i = 1, 2$. Let $b_3, b_4$ be such that $\{b_3, b_4\} = \Gamma_1 \cap \Gamma_2$, $B_i = B(b_i, R)$, $i = 3, 4$. Assume that $a \in B_3 \cup B_4 \setminus \delta(b_1, b_2, R)$ and $c_i, c_i'$ are such that
\[
\|b_i - c_i'\| = \|c_i' - a\| = \|a - c_i\| = \|c_i - b_i\| = R, \quad \text{for } i = 1, 2,
\]
and let $S_i = S(c_i - a, c_i' - a)$, for $i = 1, 2$. Then:
\[
S_1 \cup S_2 \supset S(b_2 - a, b_1 - a).
\]
In particular
\[
\frac{1}{2}(b_1 + b_2) \in \operatorname{int}(S_1 \cup S_2).
\]

Figure 2

Proof. $S_1, S_2$ and $S(b_2 - a, b_1 - a)$ are open convex sectors with apex in $a$; moreover $b_i - a \in S_i$ for $i = 1, 2$ so that $\{b_1 - a, b_2 - a\} \subset S_1 \cup S_2$. Let $\Sigma_1 = \Gamma_1 \cap D(b_2, R)$ and $\Sigma_2 = \Gamma_2 \cap D(b_1, R)$. By Lemma 3.6 we may assume that $c_i \in \Sigma_i \setminus \{b_3, b_4\}$ for $i = 1, 2$. This in turn implies $\|c_1 - c_2\| < 2R$ (as $c_1, c_2 \in \delta(b_3, b_4, R)$). Hence it is uniquely determined $a' \neq a$ such that $\{a, a'\} = \partial B(c_1, R) \cap \partial B(c_2, R)$. The straight line through $a$ and $a'$ bounds two open half-planes such that $b_2$ and $c_1$ (resp. $b_1$ and $c_2$) are in the same half-plane. Thus
\[
a' - a \in S_1 \cap S_2 \neq \emptyset.
\]
This implies that $S_1 \cup S_2$ is a convex cone and, since it contains $b_1$ and $b_2$, (2) follows. \Box

Theorem 3.8 If $K \subset \mathbb{E}^d$ is closed then $\operatorname{reach}(K) \geq R > 0$ if and only if for every $b_1, b_2 \in K$, $\|b_1 - b_2\| < 2R$, $K \cap \delta(b_1, b_2, R)$ is connected.

Proof. Let us assume that $\operatorname{reach}(K) \geq R > 0$. By contradiction, assume that $K' := K \cap \delta(b_1, b_2, R)$ is not connected; then there exist $K_1, K_2 \subset K'$, closed, such that $K' = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. By compactness, there exist $c_i \in K_i$ for $i = 1, 2$ such that
\[
\rho := \|c_1 - c_2\| = \inf\{\|x - y\| : x \in K_1, y \in K_2\} > 0.
\]
As $c_1, c_2 \in \mathcal{H}(b_1, b_2, R)$, $\rho \leq R$. We have

$$B(c_1, \rho) \cap B(c_2, \rho) \cap K' = \emptyset.$$ 

On the other hand it is easy to check that $\mathcal{H}(c_1, c_2, R) \subset [B(c_1, \rho) \cap B(c_2, \rho)] \cup \{c_1, c_2\}$. By Lemma 3.8 $\mathcal{H}(c_1, c_2, R) \subset \mathcal{H}(b_1, b_2, R)$, so that

$$\mathcal{H}(c_1, c_2, R) \cap K' = \{c_1, c_2\}. \quad (5)$$

In particular, $c := \frac{c_1 + c_2}{2} \notin K$; as $\delta_K(c) < R$, $c \in \text{Unp}(K) \setminus K$. Let $c_3 = \xi_K(c) \in \partial K$. Notice that if $c_3 \in \mathcal{H}(c_1, c_2, R)$ then either $c_3 = c_1$ or $c_3 = c_2$ so that $\delta_K(c) = ||c - c_1|| = ||c - c_2||$ in contradiction with $c \in \text{Unp}(K)$. Consequently, $c_3 \in K \setminus \mathcal{H}(c_1, c_2, R)$. We also observe that, for $i = 1, 2$,

$$||c_i - c_3|| \leq ||c_i - c|| + ||c - c_3|| < 2R$$

as $||c - c_3|| = \delta_K(c) < R$. We recall the definitions of the cones:

$$C_i(c_3, c_i, R) = \left\{ v \neq 0 : \left( \frac{v}{||v||}, \frac{c_i - c_3}{||c_i - c_3||} \right) > \frac{||c_i - c_3||}{2R} \right\}, \quad i = 1, 2.$$ 

By Corollary 3.3 we have that

$$c - c_3 \notin C_1 \cup C_2. \quad (6)$$

Apply Remark 3.3 and Lemma 3.7 to the (uniquely determined) 2-dimensional plane containing $c, c_1, c_2, c_3$ to obtain a contradiction with $\mathcal{H}(c_1, c_2, R)$.

Vice-versa, assume that for every $b_1, b_2 \in K$, $||b_1 - b_2|| < 2R$, the set $K \cap \mathcal{H}(b_1, b_2, R)$ is connected. If, by contradiction, $\text{reach}(K) < R$, then there exists $x \in K^c$ such that $\delta_K(x) = r < R$ and $||x - b_1|| = ||x - b_2|| = r$ for some $b_1, b_2 \in K, b_1 \neq b_2$. As $||b_1 - b_2|| < 2R$, $\mathcal{H}(b_1, b_2, R) \cap K$ is connected. On the other hand, $r < R$ implies that $\mathcal{H}(b_1, b_2, R) \subset B(x, r) \cup \{b_1, b_2\}$ so that there exists $b \in K \cap B(x, r)$ i.e. a contradiction. \qed

**Remark 3.9** If $\text{reach}(K) \geq R$ and $b_1, b_2 \in K$ are such that $||b_1 - b_2|| = 2R$, then $K \cap \mathcal{H}(b_1, b_2, R)$ is not necessarily connected. Any set consisting of two points at distance $2R$ is an example.

**Theorem 3.10** Let $K$ be a closed set such that $\text{reach}(K) \geq R > 0$. If $D$ is a closed ball of radius less than or equal to $R$ then $\text{reach}(K \cap D) \geq R$.

**Proof.** The argument is similar to the one used in the second part of the proof of Theorem 3.8. Let $a \in (K \cap D)^c$ such that $r = \delta_{K \cap D}(a) < R$; let us show that $a \in \text{Unp}(K \cap D)$. Assume by contradiction that there exist $b_1, b_2 \in (K \cap D)$ such that $b_1 \neq b_2$ and $||a - b_1|| = ||a - b_2|| = r$. In particular $||b_1 - b_2|| < 2R$. Clearly, $\mathcal{H}(b_1, b_2, R) \subset D$; consequently, by Theorem 3.8 $\mathcal{H}(b_1, b_2, R) \cap (K \cap D)$ is connected. Also, notice that

$$(\mathcal{H}(b_1, b_2, R) \setminus \{b_1, b_2\}) \subset B(a, r).$$

Then there exists $b' \in K \cap D$ such that $||a - b'|| < r$, i.e. a contradiction. \qed
Corollary 3.11 If $\text{reach}(K) \geq R > 0$, $a, b \in K$, \( \|a - b\| \leq 2R \), then $\text{reach}(K \cap \overline{S}(a, b, R)) \geq R$.

It is well known that, if $K$ is a closed convex set in $\mathbb{E}^d$ and $H$ is an open half space, satisfying $H \cap K = \emptyset$, then $\partial H \cap K$ is either empty or a convex subset of $\partial H$. Let us show that a similar property holds for sets of reach $\geq R > 0$.

**Definition 3.12** Let $S$ be a sphere of radius $R > 0$ in $\mathbb{E}^d$; let $K$ be a closed subset of $S$. We say that $K$ is convex in $S$ if $x_1 \in K$, $x_2 \in K$, $\text{dist}(x_1, x_2) < 2R$ imply that the arc of great circle of $S$ joining $x_1$ and $x_2$, and having smaller length, is contained in $K$.

**Theorem 3.13** Let $K$ be a closed set in $\mathbb{E}^d$ and $\text{reach}(K) \geq R > 0$. Let $B$ be an open ball of radius $R$ satisfying $B \cap K = \emptyset$. Then $\partial B \cap K$ is either empty or a convex subset of $\partial B$.

**Proof.** Theorem 3.10 implies that $(B \cup \partial B) \cap K = \partial B \cap K$ has reach $\geq R$. Then, by theorem 3.9 if $b_1, b_2 \in K \cap \partial B$, $\|b_1 - b_2\| < 2R$, then $K \cap \partial B \cap \overline{S}(b_1, b_2, R)$ is connected. Now $K \cap \partial B \cap \overline{S}(b_1, b_2, R)$ is exactly the arc of great circle of $\partial B$, joining $b_1$ and $b_2$ and having smaller length. $\square$

4 On the $R$-hull of a set

Let $A$ be a subset of $\mathbb{E}^d$ and let $R > 0$. In this section we analyze the problem of finding $K$ such that $\text{reach}(K) \geq R$, $K \supset A$ and $K$ is the minimal set (with respect to inclusion) having these properties. In other words we look for a sort of hull of reach $R$ of $A$. Intuitively, when $R = \infty$ we are dealing with the convex hull of $A$ which exists for every $A$. On the other hand, for finite $R > 0$ not every set $A$ admits a hull of reach $R$ (see the examples below). Our aim is to give necessary and sufficient conditions for $A$ to have this property (see Theorems 4.4 and 4.6).

**Definition 4.1** Let $A \subset \mathbb{E}^d$, $R > 0$. We say that $A$ admits a $R$-hull if there exists $\hat{A} \subset \mathbb{E}^d$ such that:

(i) $A \subset \hat{A}$;

(ii) $\text{reach}(\hat{A}) \geq R$;

(iii) if $\text{reach}(K) \geq R$ and $A \subset K$, then $\hat{A} \subset K$.

If such a set exists, we call it the R-hull of $A$.

**Example 1.** For an arbitrary $R > 0$ we may construct an example of set which does not admit a $R$-hull. Let $n = 2$ and $A = \{a, b\}$ with $\|a - b\| = R/2$. Assume by contradiction that there exists the $R$-hull of $A$, and denote it by $A$. Let $\hat{A}_1$ be the closed line segment joining $a$ and $b$: $\text{reach}(\hat{A}_1) = \infty$ so that $\hat{A}_1 \supset \hat{A}$. Let $\Gamma$ be a circle of radius $R$ passing through $a$ and $b$ and let $\hat{A}_2 \subset \Gamma$ be the closed arc of smaller length joining $a$ and $b$. We have $\text{reach}(\hat{A}_2) = R$ so that $\hat{A}_2 \supset \hat{A}$. As $\hat{A}_1 \cap \hat{A}_2 = A$, we must have $\hat{A} = A$; on the other hand $\text{reach}(A) = R/2$ so we have a contradiction.
Example 2. In $E^d$ consider a half-line $L$ with end-point in the origin. For every $i = 1, 2, \ldots$, let $a_i$ be the point of $L$ such that $\|a_i\| = 1/i$. The set $A = \{a_1, a_2, \ldots\}$ does not admit a $R$-hull for any $R \in (0, \infty)$.

For an arbitrary set $A \subset E^d$ and $R > 0$, we set

$$A'_R = \{x \in E^d : \delta_A(x) \geq R\}.$$ 

The proof of the following proposition is an easy application of Theorem 3.8.

Proposition 4.2 Let $A \subset E^d$, $R > 0$; reach$(A'_R) \geq R$ if and only if for every $a$ and $b$ such that $\delta_A(a), \delta_A(b) \geq R$ and $B(a, R) \cap B(b, R) \neq \emptyset$, there exists a continuous arc $\Gamma$ joining $a$ and $b$, $\Gamma \subset \partial B(a, b, R)$, such that $\delta_A(x) \geq R$ for every $x \in \Gamma$.

Lemma 4.3 Let $K \subset E^d$, then

(i) $K \subset (K'_R) \subset \{z \in E^d : \delta_K(z) < R\}$,

(ii) if reach$(K) \geq R$, then reach$(K'_R) \geq R$ and $K = (K'_R)$. 

Proof. If $x \in K$, then $\|x - y\| \geq R$ for every $y \in K$ so that $\delta_K(x) \geq R$ and $x \in (K'_R)$. On the other hand, if $z \in (K'_R)$ then $z \notin K$ so that $\delta_K(z) < R$. Claim (i) is proved.

For $s \geq 0$ set $K'_s = \{x \in E^d : \delta_K(x) \geq s\}$. Corollary 4.9 in [2] implies that reach$(K'_{R-1/i}) \geq R - 1/i$ for every $i = 1, 2, \ldots$. Moreover, the sequence $K'_{R-1/i}$ converges to $K'$ in the Hausdorff metric. On the other hand, the by Remark 4.14 in [2], for every $\epsilon > 0$ the family

$$\{A \subset E^d : \text{reach}(A) \geq R - \epsilon\}$$ 

is closed with respect to the Hausdorff metric. Then reach$(K'_R) \geq R - \epsilon$ for every $\epsilon > 0$. Now let us prove that reach$(K) \geq R$ then $K'_R \backslash K$ is empty. Let $z \in (K'_R) \backslash K$; (i) implies that $z \notin \text{Unp}(K)$. Let $x = \xi_K(z)$ and $y = x + t\frac{x - z}{\|x - z\|}$, $t \geq 0$. Note that $\frac{x - z}{\|z - x\|} \in \text{Nor}(K, x)$ so that, by claim (12) of Theorem 4.8 of [2], if $0 < t < R$, then $\delta_K(y_R) = t$ and by continuity $\delta_K(y_R) = R$. Then $y_R \in K'_R$ and $\|z - y_R\| < R$, i.e. a contradiction. \(\square\)

Theorem 4.4 Let $A \subset E^d$ and $R > 0$. If reach$(A'_R) \geq R$ then $A$ admits $R$-hull $\hat{A}$ and

$$\hat{A} = (A'_R).$$

Proof. Let $A_1 = (A'_R)$; we prove that $A_1$ is the $R$-hull of $A$. The inclusion $A \subset A_1$ is part (i) in Lemma 4.3. By the same lemma, as reach$(A'_R) \geq R$ we have reach$(A_1) \geq R$. It remains to show that $A_1$ satisfies (iii) in Definition 4.1. Let $K$ be such that $K \supset A$ and reach$(K) \geq R$. Then $K'_R \subset A'_R$ and, by Lemma 4.3 $K = (K'_R) \supset (A'_R) = A_1$. \(\square\)

Corollary 4.5 Let $A \subset E^d$ and $R > 0$. If for every $a$ and $b$ such that $\delta_A(a), \delta_A(b) \geq R$ and $\|a - b\| < 2R$, there exists a continuous arc $\Gamma$, joining $a$ and $b$ such that $\delta_A(x) \geq R$ for every $x \in \Gamma$, $\Gamma \subset \partial B(a, b, R)$, then $A$ admits $R$-hull $\hat{A}$ and

$$\hat{A} = (A'_R).$$
Theorem 4.6 Let $K \subset \mathbb{E}^d$ and $R > 0$. Assume that $K$ admits $R$-hull $\hat{K}$. Then $\text{reach}(K'_R) \geq R$.

Proof. We argue by contradiction. By using Theorem 3.8 there exist $b_1$ and $b_2 \in K'_R$ satisfying $\|b_1 - b_2\| < 2R$ and such that $\delta(b_1, b_2, R) \cap K'_R$ is not connected. Then, as we saw in the proof of Theorem 3.8 there exist $c_1$ and $c_2 \in K'_R$ such that

$$\delta(c_1, c_2, R) \cap K'_R = \{c_1, c_2\}. \quad (7)$$

For $j = 1, 2$ we have $\text{reach}(B(c_j, R)c) = R$ and $B(c_j, R)c \supset K$ thus $B(c_j, R)c \supset \hat{K}$. This implies in particular that $c_1, c_2 \in (\hat{K})'_R$. As $\text{reach}(\hat{K}) \geq R$, by Lemma 1.3 $\text{reach}(\hat{K}'_R) \geq R$, then $\delta(c_1, c_2, R) \cap \hat{K}'_R$ is connected. Let $a \in [\delta(c_1, c_2, R) \setminus \{c_1, c_2\}] \cap \hat{K}'_R$. We have $B_a(R) \cap K \subset B_a(R) \cap \hat{K} = \emptyset$ then $a \in K'_R$ which contradicts (7). $\square$

From the above theorem another connection between convex sets and sets of positive reach can be deduced. The convex hull of a closed set $C$ is the intersection of all the closed half-spaces containing $C$. Let us prove that if $K$ admits $R$-hull $\hat{K}$, then $\hat{K}$ is the intersection of the complement sets of all open balls that do not meet $K$. Note that for an arbitrary, non-empty, subset $K$ of $\mathbb{E}^d$ we have

$$(K'_R)'_R = \bigcap_{\delta_K(x) \geq R} B_x(R)c.$$ 

This remark and Theorem 4.6 lead to the following result.

Corollary 4.7 Let $K \subset \mathbb{E}^d$, $R \geq 0$. Assume that $K$ admits an $R$-hull $\hat{K}$. Then

$$\hat{K} = \bigcap_{\delta_K(x) \geq R} B_x(R)c.$$ 

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