ON SEMICONTINUITY OF MULTIPLICITIES IN FAMILIES

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ABSTRACT. We investigate the behavior of Hilbert–Samuel multiplicity and Hilbert–Kunz multiplicity in families of ideals. We show that Hilbert–Samuel multiplicity is upper semicontinuous and that Hilbert–Kunz multiplicity is upper semicontinuous in families of finite type. Our machinery can be applied for families over \( \mathbb{Z} \) and yields a partial solution to the question posed by Brenner, Li, and Miller. We also apply our methods to show that over an algebraically closed field the infimum in the definition of F-rational signature, an invariant defined by Hochster and Yao, is attained.

1. Introduction

Hilbert–Kunz multiplicity is a multiplicity theory native to positive characteristic. Its definition mimics the definition of Hilbert–Samuel multiplicity but replaces regular powers \( I^n \) with Frobenius powers \( I[p^e] = \{ x^{p^e} \mid x \in I \} \). The Hilbert–Kunz multiplicity of an \( m \)-primary ideal \( I \) of a local ring \((R, m)\) is the limit

\[
e_{HK}(I) = \lim_{e \to \infty} \frac{\lambda_R(R/I[p^e])}{p^e \dim R}.
\]

It is not easy to see that the above limit exists. Existence was shown by Monsky, who introduced the concept in [Mon83] as a continuation of earlier work of Kunz [Kun69, Kun76].

Hilbert–Kunz multiplicity is very hard to compute and Paul Monsky was a driving force behind most of the known examples. There were several interesting families computed, for example, plane cubics ([Mon97, Mon11, BC97, Par94]), quadrics in characteristic two ([Mon98a, Mon98b]), and another family in [Mon05]. The most famous of these families is the one appearing in [Mon98b].

Example 1.1. Let \( K \) be an algebraically closed field of characteristic 2. For \( \alpha \in K \) let \( R_\alpha = K[x, y, z]/(z^4 + xyz^2 + (x^3 + y^3)z + \alpha x^2 y^2) \) localized at \((x, y, z)\). Then

1. \( e_{HK}(R_\alpha) = 3 + \frac{1}{2} \), if \( \alpha = 0 \),
2. \( e_{HK}(R_\alpha) = 3 + 4^{-m} \), if \( \alpha \neq 0 \) is algebraic over \( \mathbb{Z}/2\mathbb{Z} \), where \( m = [\mathbb{Z}/2\mathbb{Z}(\lambda) : \mathbb{Z}/2\mathbb{Z}] \) for \( \lambda \) such that \( \alpha = \lambda^2 + \lambda \)
3. \( e_{HK}(R_\alpha) = 3 \) if \( \alpha \) is transcendental over \( \mathbb{Z}/2\mathbb{Z} \).

Monsky’s computations were later used by him and Brenner in [BM10] to give a counter-example to an outstanding problem in the field: localization of tight closure, originating from the foundational treatise of Hochster and Huneke [HH90]. For this result, it is better to think about the example as a family of rings parametrized by \( \text{Spec } K[t] \) and the necessary phenomenon is the jump in the values between the generic fiber, corresponding to transcendental values, and special fibers.

Another consequence of Monsky’s example was found by the author in [Smi], where it was shown that Hilbert–Kunz multiplicity takes infinitely many values as a function on

\[
\text{Spec } K[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2 y^2)
\]

by developing a technique of lifting this phenomenon from special fibers to the corresponding maximal ideals \( m_\alpha = (x, y, z, t - \alpha) \).

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Semicontinuity in Hilbert–Kunz theory was already studied by Kunz, in [Kun76], where it was shown for individual terms of the sequence (also, see [SB79]), but the real momentum was given by Enescu and Shimomoto in [ES05], where they investigated both semicontinuity of Hilbert–Kunz multiplicity as a function on the spectrum and in a one-parameter family. In both settings, they established weaker forms of semicontinuity [ES05, Theorem 2.5, Theorem 2.6]. For the spectrum a complete solution was obtained by the author in [Smi16, Smi], and the goal of this article is to establish semicontinuity for a class families similar to the situation in Example 1.1 (see Definition 3.8).

Our definition of a family is versatile enough to relate to another outstanding problem in the field: the behavior of Hilbert–Kunz multiplicity in reduction mod $p$. To illustrate this problem, let us consider a family

$$\mathbb{Z} \to R := \mathbb{Z}[x, y, z]/(z^4 + xyz^2 + (x^3 + y^3)z + x^2y^2).$$

Then one may want to define the Hilbert–Kunz multiplicity of the general fiber, $\mathbb{Q}[x, y, z]/(z^4 + xyz^2 + (x^3 + y^3)z + x^2y^2)$, the ring of characteristic zero, by taking the limit of Hilbert–Kunz multiplicities of special fibers, $\lim_{p \to \infty} e_{HK}(R(p))$.

The only general case where this problem was solved is in graded dimension two [Tri07, BLM12]. In an attempt to simplify the problem, in [BLM12] Brenner, Li, and Miller asked whether it is possible to replace $\lim_{p \to \infty} e_{HK}(R(p))$ by the limit of the individual terms $R(p)/\mathfrak{m}^e\mathfrak{m}^e$ for a fixed $e$. A positive answer to this question was recently announced by Pérez, Tucker, and Yao ([PTY]). The methods of this paper provide an easy proof of this result in special case and essentially generalize a recent result of Trivedi ([HY]) which was established in a graded case.

For another application of our work we turn to F-rational signature, an invariant introduced by Hochster and Yao in [HY]. If $(R, \mathfrak{m})$ is a local ring, then it is defined by

$$s_{rat}(R) = \inf_u \{e_{HK}(\mathfrak{z}) - e_{HK}(\mathfrak{z}, u)\}$$

where the infimum is taken over socle elements $u$ modulo a system of parameters $\mathfrak{z}$. In Proposition 4.13 we show that if the residue field is algebraically closed, then the infimum in the definition is attained. This recovers the main property of F-rational signature ([HY, Theorem 4.1]): its positivity determines F-rational singularity.

Last, we want to mention that using results in [PTY] Carvajal-Rojas, Schwede, and Tucker [CRST] recently obtained results in the spirit of this work. However, their motivation is to study the behavior of Hilbert–Kunz multiplicity on a family of varieties, while this work focuses on a family of ideals which may not be maximal.

The methods and the structure of the paper. We employ the uniform convergence techniques pioneered by Tucker in [Tuc12] and furthered by Polstra and Tucker in [PT18] where it was shown that the discriminant technique in tight closure theory developed by Hochster and Huneke [HH90, Section 6] also provides a more “functorial” approach to the uniform convergence constants. This approach was recently used by Polstra and the author [PS] to study Hilbert–Kunz multiplicity under small perturbations. The uniform convergence machinery of this paper are largely a mix of the techniques developed in [PS] and [Smi16]. It should be noted that [CRST, Proposition 4.5] can be used to get a version of Theorem 4.7 but will require stronger assumptions. Moreover, our treatment of constants allow us to vary the characteristic and obtain in Corollary 4.9 a uniform convergence statement for fibers even if the base ring has characteristic zero.

In Section 2 we slightly expand on [PT18] by further incorporating ideas from [HH90]. In Section 3 we present old and new results on the behavior of Hilbert–Samuel function in families. Definition 3.8 introduces the assumptions of this work. The main results are presented in Section 4 and we finish with questions coming from this work.
2. Discriminants and separability

**Definition 2.1.** Let $A$ be a ring and $R$ a finite $A$-algebra which is free as an $A$-module with a basis $e_1, \ldots, e_n$. The discriminant of $R$ over $A$ is

$$D_A(R) = \det \begin{pmatrix} \text{Tr}(e_1^2) & \text{Tr}(e_1e_2) & \cdots & \text{Tr}(e_1e_n) \\ \text{Tr}(e_2e_1) & \text{Tr}(e_2^2) & \cdots & \text{Tr}(e_2e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Tr}(e_ne_1) & \text{Tr}(e_ne_2) & \cdots & \text{Tr}(e_n^2) \end{pmatrix},$$

where $\text{Tr}(r)$ denotes the trace of the multiplication map $x \mapsto r$ on $R$. Up to multiplication by a unit of $A$, the discriminant is independent of choice of basis. Discriminants are also functorial, for example, see [PS].

We start with a fundamental lemma provided by Hochster and Huneke in [HH90, Lemma 6.5].

**Lemma 2.2.** Let $A$ be a normal domain and $R$ be a module-finite, torsion-free, and generically separable $A$-algebra. Let $L$ be the fraction field of $A$, $L' = R \otimes_A L$, and $d = D_L(L')$ computed using a basis of elements in $R$. Then $0 \neq d \in A$ and $dR^{1/p} \subseteq A^{1/p}[R]$.

The lemma also provides a way to define a discriminant of a non-free algebra. We will abuse the notation and still denote it as $D_A(R)$. If $R$ is not torsion-free, we will use the ideal $T_A(R) = \{ r \in R \mid ar = 0 \text{ for some } 0 \neq a \in A \}$.

**Corollary 2.3.** Let $A$ be a normal domain and $R$ be module-finite and generically separable $A$-algebra. Let $L$ be the fraction field of $A$, $L' = R \otimes_A L$, and $d = D_A(R)$ computed as in Lemma 2.2. If $c \in A$ such that $cT_A(R) = 0$, then we have exact sequences of $R$-modules

$$A^{1/p} \otimes_A R \to R^{1/p} \to C_1 \to 0$$

and

$$R^{1/p} \to A^{1/p} \otimes_A R \to C_2 \to 0$$

with $cdC_1 = 0$ and $cdC_2 = 0$.

**Proof.** We have a map $R' := R/T_A(R) \xrightarrow{\times c} R$ induced by multiplication by $c$ on $R$. Observe that $R'$ is still generically separable over $A$, since $L' = R \otimes_A L = R' \otimes_A L$. Hence $dF_s R' \subseteq A^{1/p}[R'] \cong A^{1/p} \otimes_A R'$.

Now we may compose maps:

$$F_s R \to F_s R' \xrightarrow{\times d} A^{1/p} \otimes_A R' \xrightarrow{\times c} A^{1/p} \otimes_A R,$$

$$A^{1/p} \otimes_A R \to A^{1/p} \otimes_A R' \to F_s R' \xrightarrow{\times F_s c} F_s R.$$

In the first map, we note that $F_s R \to F_s R'$ is surjective, $A^{1/p}[R'] \subseteq R'$, and $cR = cR'$, so it follows that the cokernel is annihilated by $cd$. For the second map, we note that $cdF_s R \subseteq dF_s R' \subseteq A^{1/p}[R']$, which is the image of $A^{1/p} \otimes_A R \to F_s R'$. \[\square\]

The corollary becomes especially powerful after combining it with another result of Hochster and Huneke [HH90, Lemma 6.15].

**Lemma 2.4.** Let $R$ be a reduced ring, module-finite over a regular ring $A$ of characteristic $p > 0$. Then for all sufficiently large $e$, $R \otimes_A A^{1/p^e}$ is module-finite and generically separable over $A^{1/p^e}$.

**Proof.** Let $L' = R \otimes_A L$. Since $R$ is reduced, $L'$ is a product of fields. Tensoring with the fraction field $L$ of $A$ we get that

$$R \otimes_A A^{1/p^e} \otimes_A L = (R \otimes_A L) \otimes_L (A^{1/p^e} \otimes_A L) = L' \otimes_L L^{1/p^e}.$$ 

Hence the statement is reduced to the field case. \[\square\]
Corollary 2.5. Let $R$ be a reduced ring, module-finite over a regular ring $A$ of characteristic $p > 0$. Let $c \in A$ such that there exists a free $A$-module $F \subseteq R$ such that $cR \subseteq F$. Then for large $e$ we have exact sequences of $R$-modules

$$A^{1/p^{e+1}} \otimes_A R \to F_*(R \otimes_A A^{1/p^e}) \to C_1 \to 0$$

and

$$F_*(R \otimes_A A^{1/p^e}) \to A^{1/p^{e+1}} \otimes_A R \to C_2 \to 0,$$

where the cokernels are annihilated by $cD_{A^{1/p^e}}(R \otimes_A A^{1/p^e})$.

Proof. Let $R' = R \otimes_A A^{1/p^e}$, $A' = A^{1/p^e}$, and $F' = F \otimes_A A^{1/p^e}$. Because $A^{1/p^e}$ is flat by [Kun69], $cR' \subseteq F'$, so $cT_{A'}(R') \subseteq cR' \subseteq F'$ and $cT_{A'}(R') = 0$ because $F'$ is torsion-free. Now, we may use Corollary 2.3 for $A'$ and $R'$.

\[\square\]

3. Families and semicontinuity

We adopt the following notion of a family from [Lip82]. Let $S \to R$ be a homomorphism of rings and $I \subseteq R$ be an ideal such that $R/I$ is a finitely generated $S$-module. For any prime ideal $p \in \text{Spec } S$ define $R(p) = R \otimes_S k(p)$ and $I(p) = IR(p)$. By the assumption, $R(p)/I(p) = R/I \otimes_S k(p)$ has finite length. Thus, $I(p)$ is a family of finite colength ideals in a family of rings $R(p)$ parametrized by Spec $S$. Similarly, if $M$ is a finite $R$-module, then $M(p) := M \otimes_R k(p)$ is a finite $R(p)$-module for all $p$.

We now may use Hilbert–Kunz multiplicity, or Hilbert–Samuel multiplicity, as a real-valued function on Spec $S$ by $p \mapsto e_{HK}(I(p), R(p))$. In particular, Example 1.1 is a family given by $K[t] \to K[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2)$ with $I = (x, y, z)$.

We also fix the following definition of semicontinuity.

Definition 3.1. Let $X$ be a topological space and $(\Lambda, \prec)$ be a partially ordered set. We say that a function $f : X \to \Lambda$ is upper semicontinuous if for each $\lambda \in \Lambda$ the set

$$X_{\prec \lambda} = \{x \in X \mid f(x) \prec \lambda\}$$

is open.

Another common way to define semicontinuity is to require the sets $X_{\preceq \lambda} = \{x \in X \mid f(x) \preceq \lambda\}$ to be open. As it was observed by Enescu and Shimomoto ([ES05, Theorem 2.7]), Monsky’s example shows that Hilbert–Kunz multiplicity is not an upper semicontinuous function in this sense.

Remark 3.2. Nagata’s criterion of openness ([Mat80, 22.B]) is very useful for showing that a function is semicontinuous. Namely, a function $f : \text{Spec } S \to \Lambda$ is upper semicontinuous if and only if the following two conditions hold:

1. If $p \subseteq q$ then $f(p) \preceq f(q)$,
2. If $f(p) \prec \lambda$ then there exists an element $s \notin p$ such that for every $s \notin q \supseteq p$ we have $f(q) \prec \lambda$.

3.1. Hilbert–Samuel function in families. The theory of families of ideals originates from the work of Teissier ([Tei80]) on the principle of specialization of integral closure and was further developed by Lipman in [Lip82]. The author is not aware whether Theorem 3.3 presenting semicontinuity of the Hilbert–Samuel function, is new.

We start with a lemma found in the proof of [FM00, Proposition 4.2].
Lemma 3.3. Let $S \to R$ be a map of rings and $I$ be an ideal of $R$ such that $S \to R/I$ is finite. Suppose $M$ is a finite $R$-module. If $\text{Gr}_I(M)$ is flat over $S$, then for every finite $S$-module $N$ the canonical map

$$\text{Gr}_I(M) \otimes_S N \to \text{Gr}_I(M \otimes_S N)$$

is an isomorphism.

Proof. It is sufficient to show that the natural map $I^nM \otimes_S N \to I^n(M \otimes_S N)$ is an isomorphism for all $n$. Because $R$ acts on $M \otimes_S N$ by multiplication on $M$, the map is surjective, so it remains to check injectivity.

By flatness of $I^nM/I^{n+1}M$ there is an exact sequence

$$0 \to I^{n+1}M \otimes_S N \to I^nM \otimes_S N \to (I^nM/I^{n+1}M) \otimes_S N \to 0.$$ 

Using induction on $n$ it is now easy to verify the natural map $I^nM \otimes_S N \to I^n(M \otimes_S N)$ is injective.

□

Using this lemma we are able to expand [Lip82, Proposition 3.1].

Theorem 3.4. Let $S \to R$ be a map of rings and $I$ be an ideal in $R$ such that $R/I$ is a finite $S$-module. Let $M$ be a finitely generated $R$-module. Then the following functions are upper semicontinuous:

1. $p \mapsto \lambda(M(p)/I^nM(p))$ for any $n$,
2. $p \mapsto \lambda(I^nM(p)/I^{n+1}M(p))$ for any $n$,
3. $p \mapsto (\lambda(M(p)/IM(p)), \lambda(M(p)/IM^2(p)), \ldots)$ (with lex-order),
4. $p \mapsto (\lambda(M(p)/IM(p)), \lambda(IM(p)/IM^2(p)), \ldots)$ (with lex-order).

Proof. It can be shown by induction that for all $n$ $M/I^nM$ and $I^nM/I^{n+1}M$ are finitely generated $S$-modules. But for any finite $S$-module, $\dim N(p) = \dim N \otimes_S k(p)$ is the minimal number of generators of $N(p)$, which is clearly an upper semicontinuous function. In particular, we obtain that the first condition of Nagata's criterion from Remark 3.2 is satisfied.

For the second condition, we provide a neighborhood of $p$ where all four functions are constant. Observe that $\text{Gr}_I(M)$ is a finitely generated module over a finitely generated $S$-algebra, because it is a finite $\text{Gr}_I(R)$-module and $\text{Gr}_I(R)$ is a finitely generated module over $R/I[x_1, \ldots, x_n]$ where $I$ is generated by the images of $x_1, \ldots, x_n$ in $\text{Gr}_I(R)$. For a fixed prime ideal $p \in \text{Spec } S$, we may apply generic freeness ([Mat80, 22.A]) over $S/p$ for the module $\text{Gr}_I(M/pM)$.

In the resulting neighborhood $D_s$ where $\text{Gr}_I(M/pM)$ is free, by Lemma 3.3 and flatness of localization, for all $q \in D_s \cap V(p)$ we have the isomorphism

$$\text{Gr}_I(M/pM) \otimes_R k(q) \cong \text{Gr}_I(M \otimes_R k(q)).$$

Because each $(I^n + p)M/(I^{n+1} + p)M$ is projective, it follows that $\dim k(q) I^nM/I^{n+1}M \otimes_R k(q)$ is constant on $V(p) \cap D_s$ for all $n$. □

Corollary 3.5 ([Lip82]). Let $S \to R$ be a map of rings and $I \subset R$ be an ideal such that $R/I$ is a finite $S$-module. If $p \subseteq q \subset S$ are prime ideals and $M$ is a finitely generated $R$-module, then $\dim M(p) \leq \dim M(q)$ and if $\dim M(p) = \dim M(q)$ then $e(IM(p)) \leq e(IM(q))$.

The following result of Lipman ([Lip82, Proposition 3.3]) provides a natural sufficient condition for equidimensionality of a family.

Lemma 3.6. Let $S \to R$ be a map of Noetherian rings and $I$ an ideal of $R$ such that $R/I$ is a finite $S$-module and $S \cap I = 0$. Furthermore, assume that

1. $\text{ht } q + \dim R/q = \dim R$ for every ideal prime ideal $q \supseteq I$ in $R$,
2. $\dim R/mR + \dim S = \dim R$ for every maximal ideal $m$ of $S$.
Then for every prime ideal \( p \) of \( S \) we have \( \dim S(p) = \dim R - \dim S = \text{ht} I \).

Due to the fundamental nature of Lemma 3.6, we would like to call the map \( S \to R \) satisfying its assumptions an \( I \)-family. We note the following corollary of the proof and Theorem 3.4.

**Corollary 3.7.** Let \( S \to R \) be a map of Noetherian rings and \( I \) be an ideal such that \( R/I \) is a finite \( S \)-module. Let \( d = \max_{m \in \text{Max} S} \dim R/mR \). Then there exists a constant \( C \) such that for all \( p \in \text{Spec} S \) and all \( n \)

\[
\dim_{k(p)} R(p)/I^n(p) < Cn^d.
\]

**Proof.** First, note that if \( p \subseteq m \) then \( \dim R/mR = \dim R(m) \geq \dim R(p) \). So, for every \( p \), there is some constant \( C(p) \) that will work for all \( n \). Given any \( C \) the set

\[
U(C) = \{ p \mid \dim_{k(p)} R(p)/I^n(p) < Cn^d \text{ for all } n \} = \bigcap_{n=1}^{\infty} \{ p \mid \dim_{k(p)} R(p)/I^n(p) < Cn^d \}
\]

is open by Theorem 3.4. Thus we can build \( C \) by Noetherian induction: we first choose \( C \) to be the maximum \( C(p) \) over the generic points and then keep increasing \( C \) by considering generic points of the complement of \( U(C) \) until \( U(C) = \text{Spec} S \). \( \square \)

Following Lemma 3.6 we introduce the following definition.

**Definition 3.8.** We say that \( S \to R \) is an affine \( I \)-family if \( R \) is a finitely generated \( S \)-algebra and \( I \subseteq R \) is an ideal such that

1. \( R/I \) is a finite \( S \)-module,
2. \( S \cap I = 0 \),
3. \( \text{ht} q + \dim R/q = \dim R \) for every ideal prime ideal \( q \supseteq I \) in \( R \),
4. \( \dim R/mR + \dim S = \dim R \) for every maximal ideal \( m \) of \( S \).

The second condition guarantees that \( I(p) \neq R(p) \) for every \( p \). We can always pass to such a family by factoring \( I \cap S \). If \( R \) is formally equidimensional then it satisfies (3), if \( R \) is a flat \( S \)-algebra, then it satisfies (4). In particular, Example 1.1 is coming from an affine \((x, y, z)\)-family: localization does not change the Hilbert–Kunz multiplicity because the Frobenius powers are be \((x, y, z)\)-primary.

### 4. Semicontinuity

We want to show that \( e_{HK}(I(p)) \) is an upper semicontinuous function on \( \text{Spec} S \). To build the uniform convergence machinery, we start with auxiliary lemmas.

**Lemma 4.1.** Let \( S \to R \) be a map of rings and \( I \) be an ideal in \( R \) such that \( R/I \) is a finite \( S \)-module. For each \( e \) the function \( p \to \dim_{k(p)} (S(p)/I(p)[p^e]) \) is upper semicontinuous on \( \text{Spec} R \).

**Proof.** If \( I \) can be generated by \( h \) elements, then \( I^{hp} \subseteq I[p^e] \), so \( R/I[p^e] \) is a finite \( S \)-module as in Theorem 3.4. Thus \( p \to \dim_{k(p)} (S(p)/I(p)[p^e]) \) is the minimal number of generators of that module at \( p \) and is an upper semicontinuous function, see for example [PT18, Lemma 2.2]. \( \square \)

**Corollary 4.2.** Let \( S \to R \) be an \( I \)-family as in Lemma 3.6. Then for every \( p \subseteq q \) we have \( e_{HK}(I(p)) \leq e_{HK}(I(q)) \).

**Proof.** Observe that \( \dim S(p) = \text{ht} I \) by the assumption, so we may pass to the limit. \( \square \)

**Lemma 4.3.** Let \( S \) be a Noetherian ring and let \( R \) be an intersection flat \( S \)-algebra, i.e., \( \cap_{\lambda \in \Lambda} I_\lambda R = (\cap_{\lambda \in \Lambda} I_\lambda)R \) for arbitrary \( \Lambda \) and ideals \( I_\lambda \subseteq S \). Then for any element \( f \in R \) the set

\[
V_S(f) := \{ p \in \text{Spec} S \mid f \in pR \}
\]

is closed.
Proof. Let $I$ be the intersection of all primes in $V_S(f)$. Then $f \in \cap_{p \in V_S(f)}pR = (\cap_{p \in V_S(f)}p)R = IR$. Hence $V_S(f) = V(I)$. □

Last, we record a crucial lemma that provides a uniform upper bound to be used in the proof. Note that polynomial extensions are intersection flat.

**Lemma 4.4.** Let $S$ be a Noetherian domain, $R = S[T_1, \ldots, T_d]$, and $I$ be an $(T_1, \ldots, T_d)$-primary ideal. Let $M$ be a finite $R$-module annihilated by $0 \neq f \in R$. Then there exists a constant $D$ with the following property: for any $e \geq 0$ and $p$ in an open subset $\text{Spec } S \setminus V_S(f)$ with $p := \text{char } k(p)$ we have

$$\dim_{k(p)} M(p)/I^{[p^e]}M(p) < Dp^{e(d-1)},$$

where $p = \text{char } k(p)$ may vary.

**Proof.** For every maximal ideal $m \not\in V_S(f)$

$$\dim R/(f, m)R = \dim S/m[T_1, \ldots, T_d]/(f) \leq d - 1.$$ 

Let $N$ be such that $(T_1, \ldots, T_d)^N \subseteq I$. Then we have containments

$$(T_1, \ldots, T_d)^{Nd^e} \subseteq ((T_1, \ldots, T_d)^{[p^e]})^N \subseteq I^{[p^e]}.$$ 

Suppose that $M$ can be (globally) generated by $\nu$ elements. We note that $\text{Spec } S \setminus V_S(f)$ is a finite union of principal open set $D_c$ and for each $c$ we may apply Corollary 3.7 to the map $S_c \to R_c$ and estimate

$$\dim_{k(p)} M(p)/I^{[p^e]}M(p) \leq \nu \dim_{k(p)} R(p)/I(p)^{[p^e]} < \nu C(Nd^e)^{d-1} = (\nu CN^{d-1}d^{d-1})p^{e(d-1)}.$$ □

4.1. **Main result.** Before proceeding to the proof of the main theorem we recall two lemmas. The first is due to Kunz [Kun70].

**Lemma 4.5.** Let $R$ be a Noetherian ring of characteristic $p > 0$. Then for every $p \subseteq q$

$$[k(q)^{1/p^e} : k(q)] = p^{e \dim R_q/p}[k(p)^{1/p^e} : k(p)].$$

Second, we will need the following form of the Noether normalization theorem from [Nag62 14.4].

**Theorem 4.6.** Let $D$ be a domain and $R$ be a finitely generated $D$-algebra. Then there exists an element $0 \neq c \in D$ such that $R_c$ is module-finite over a polynomial subring $D_c[z_1, \ldots, z_d]$.

**Theorem 4.7.** Let $S$ be a regular $F$-finite ring of characteristic $p > 0$ and $S \to R$ be an affine $I$-family with reduced fibers of dimension $h = \text{ht } I$. Then there exists an open set $U$ and a constant $D$ such that for all $q \in U$

$$\left| \frac{\dim_{k(q)} R(q)/I^{[p^{e+1}]}}{p^{e(h+1)}} - \frac{\dim_{k(q)} R(q)/I^{[p^e]}}{p^{eh}} \right| < \frac{D}{p^{e-h}}.$$ 

**Proof.** Because $R(0)$ is reduced, after inverting an element of $S$ we may assume that $R$ is reduced. Next, by Theorem 4.6 we invert another element and assume that $R$ has a normalization $A = S[T_1, \ldots, T_h]$ over $S$.

Applying Lemma 2.4 to the pair $A \subseteq R$ we find $e_0$ such that $A^{1/p^{e_0}} \to R \otimes_A A^{1/p^{e_0}}$ is generically separable. Since $A$ is a domain, there exists a free module $F \subseteq R$ and an element $0 \neq c \in A$ such that $cR \subseteq F$. Because $A^{1/p^{e_0}}$ is flat, $F \otimes_A A^{1/p^{e_0}} \subseteq R \otimes_A A^{1/p^{e_0}}$ is a free submodule and $c$ still annihilates the cokernel. Let $d^{1/p^{e_0}}$ be a discriminant of $R \otimes_A A^{1/p^{e_0}}$ over $A^{1/p^{e_0}}$. 

Claim 1. Let $\mathfrak{q}$ be a prime ideal in the open set $\text{Spec } S \setminus V_S(cd)$. Then $F(\mathfrak{q})$ is a free submodule of $R(\mathfrak{q})$ such that $cR(\mathfrak{q}) \subseteq F(\mathfrak{q})$.

Proof of the claim. We have the induced map $F \otimes_R R(\mathfrak{q}) \to R \otimes_R R(\mathfrak{q})$ whose cokernel is annihilated by the image of $c$ in $R(\mathfrak{q})$. The image of $c$ is nonzero by the assumption, $F_c \cong R_c$, and $c \notin \mathfrak{q}A$, so $F(\mathfrak{q})$ and $R(\mathfrak{q})$ are still generically isomorphic as $A(\mathfrak{q})$-modules. Thus, since $F(\mathfrak{q})$ is a free $A(\mathfrak{q})$-module and $A(\mathfrak{q}) \cong k[\mathfrak{q}][T_1, \ldots, T_h]$ is a domain, the induced map $F \otimes_R R(\mathfrak{q}) \to R \otimes_R R(\mathfrak{q})$ is still an inclusion. □

By the functoriality of discriminants (as in [PS, Proposition 2.2]), the image of $d$ is still a discriminant of $R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}}$ over $A(\mathfrak{q})^{1/p^{e_0}}$. Since $d \notin \mathfrak{q}A$, the inclusion is still generically separable. Hence, by Lemma 2.5, we have sequences

\begin{equation}
R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0+1}} \to F_* \left( R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} \right) \to C_1 \to 0
\end{equation}

and

\begin{equation}
F_* \left( R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} \right) \to R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0+1}} \to C_2 \to 0,
\end{equation}

where $cdC_1 = 0 = cdC_2$. Tensoring these exact sequences with $R/I[p^e]$, we obtain that

\begin{equation}
| \dim_{k(\mathfrak{q})} R(\mathfrak{q})/I[p^e] R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0+1}} - \dim_{k(\mathfrak{q})} R/I[p^e] \otimes_R F_* \left( R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} \right) | \leq \max \left( \dim_{k(\mathfrak{q})} C_1/I[p^e] C_1, \dim_{k(\mathfrak{q})} C_2/I[p^e] C_2 \right).
\end{equation}

Claim 2. There is a constant $D$ independent of $\mathfrak{q}$ such that

\[
\dim_{k(\mathfrak{q})} C_1/I[p^e] C_1, \dim_{k(\mathfrak{q})} C_2/I[p^e] C_2 < D[k(\mathfrak{q})^{1/p^{e_0+1}} : k(\mathfrak{q})] p^{(e_0+1)_h} p^e(h-1).
\]

Proof. The exact sequence (4.1) induces a natural surjection on $C_1/I[p^e] C_1$ from

\[
F_* \left( R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} \right) \otimes_R R/(cd, I[p^e]) \cong F_* \left( R(\mathfrak{q})/(p^d p^e, I[p^{e+1}]) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} \right).
\]

Since $A(\mathfrak{q})$ is a polynomial ring of dimension $h$, by Lemma 4.3, $A(\mathfrak{q})^{1/p^{e_0}}$ is a free $A(\mathfrak{q})$-module of rank $[k(\mathfrak{q})^{1/p^{e_0}} : k(\mathfrak{q})] p^{e_0 h}$. Applying Lemma 4.3, we now bound

\[
\dim_{k(\mathfrak{q})}(C_1/I[p^e] C_1) \leq [k(\mathfrak{q})^{1/p^{e_0+1}} : k(\mathfrak{q})] p^{e_0 h} \dim_{k(\mathfrak{q})} R(\mathfrak{q})/(p^d p^e, I[p^{e+1}]) R(\mathfrak{q}).
\]

Because $R/I$ is a finite $S$-module, $I$ is $(T_1, \ldots, T_h)$-primary, so by Corollary 4.4 applied to $R/(cd)$ we may find a constant $D$ independent of $\mathfrak{q}$ such that

\[
\dim_{k(\mathfrak{q})}(R(\mathfrak{q})/(p^d p^e, I[p^{e+1}]) R(\mathfrak{q})) \leq p \dim_{k(\mathfrak{q})}(R(\mathfrak{q})/(cd, I[p^{e+1}]) R(\mathfrak{q})) \leq p D p^{(e+1)_h} = D p^h p^e(h-1),
\]

thus

\[
\dim_{k(\mathfrak{q})}(C_1/I[p^e] C_1) \leq D[k(\mathfrak{q})^{1/p^{e_0+1}} : k(\mathfrak{q})] p^{(e_0+1)_h} p^e(h-1).
\]

Similarly, $C_2/I[p^e]$ is an image of $R(\mathfrak{q})/(cd, I[p^e]) R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0+1}}$, and we may derive that

\[
\dim_{k(\mathfrak{q})}(C_2/I[p^e] C_2) \leq D[k(\mathfrak{q})^{1/p^{e_0+1}} : k(\mathfrak{q})] p^{(e_0+1)_h} p^e(h-1)
\]

and the claim follows. □

As in the proof of Claim 2, we may compute

\[
\dim_{k(\mathfrak{q})} R(\mathfrak{q})/I[p^e] R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} = [k(\mathfrak{q})^{1/p^{e_0+1}} : k(\mathfrak{q})] p^{(e_0+1)_h} \dim_{k(\mathfrak{q})} R(\mathfrak{q})/I[p^e] R(\mathfrak{q})
\]

and

\[
\dim_{k(\mathfrak{q})} R/I[p^e] \otimes_R F_* \left( R(\mathfrak{q}) \otimes_{A(\mathfrak{q})} A(\mathfrak{q})^{1/p^{e_0}} \right) = [k(\mathfrak{q})^{1/p^{e_0+1}} : k(\mathfrak{q})] p^{e_0 h} \dim_{k(\mathfrak{q})} R(\mathfrak{q})/I[p^{e+1}] R(\mathfrak{q}).
\]
Now, dividing (4.3) by $p^{e_0 + h} p^{(e+1)h} [k(q)^1/p^e : k(q)]$, from Claim [2] we obtain that
\[
\left| \frac{\dim_{k(q)} R(q)/I[p^{e+1}] R(q)}{p^{(e+1)h}} - \frac{\dim_{k(q)} R(q)/I[p^e] R(q)}{p^h} \right| < \frac{D p^{(e_0 + 1)h}}{p^{e_0 + e}} \leq \frac{D}{p^{e-h}}.
\]

\[\square\]

4.2. Families over $\mathbb{Z}$. A careful analysis of the proof shows that it can be applied even when the characteristic varies in a family.

**Theorem 4.8.** Let $S$ be a regular ring of characteristic 0 and $S \to R$ be an affine $I$-family with reduced fibers of dimension $h$. Suppose that for every $p \in \text{Spec } S$ the residue field $k(p)$ is F-finite whenever it has positive characteristic. Then there exists an open set $U$ and a constant $D$ with the following property: if $q \in U$ and $p := \text{char } k(q) > 0$ then
\[
\left| \frac{\dim_{k(q)} R(q)/I[p^{e+1}] R(q)}{p^{(e+1)h}} - \frac{\dim_{k(q)} R(q)/I[p^e] R(q)}{p^h} \right| < \frac{D}{p^{e-h}}.
\]

Note that $p$ may vary in the family and $D$ is independent of $p$.

**Proof.** After inverting an element if necessary, we choose a Noether normalization $A = \mathbb{Z}^f[x_1, \ldots , x_d]$ of $R_f$. Note that $A \subseteq R$ is generically separable, because $R(0)$ has characteristic 0. So, we may proceed with the proof of Theorem 4.7 with $e_0 = 0$. The constant $D$ in claim Claim [2] comes from Lemma 4.4 and does not depend on characteristic as it arises from the Hilbert–Samuel theory. \[\square\]

**Corollary 4.9.** Let $S \to R$ be an affine $I$-family with reduced fibers of dimension $h$. Suppose that for every $p \in \text{Spec } S$ the residue field $k(p)$ is F-finite whenever it has positive characteristic (e.g., $R$ is F-finite or $R = \mathbb{Z}$). Then there exists an open set $U$ and a constant $D$ with the following property: if $q \in U$ and $p := \text{char } k(q) > 0$ then
\[
\left| e_{\text{HK}}(I(p)) - \frac{\dim_{k(q)} R(q)/I[p^e] R(q)}{p^h} \right| < \frac{2D}{p^{e-h}}.
\]

**Proof.** We may pass to $S/p \to R/p$ and assume that $p = 0$. An F-finite ring is excellent ([Kun76, Theorem 2.5]), so the regular locus of $S$ is open and, by inverting an element, we assume that $S$ is regular.

Let $D$ be the constant provided by Theorem 4.7 or Theorem 4.8, then the claim follows from the proof of [PT18, Lemma 3.5]. \[\square\]

**Corollary 4.10.** Let $S$ be an F-finite ring of characteristic $p > 0$ and $S \to R$ be an affine $I$-family with reduced fibers. Then the function $p \mapsto e_{\text{HK}}(I(p))$ is upper semicontinuous on $\text{Spec } S$.

**Proof.** We use uniform convergence to pass semicontinuity from the individual term to the limit as in [PT18, Smi16]. Each individual term, $\dim_{k(p)} R(p)/I[p^e] R(p)$ is the number of generators of $R/I[p^e]$ at $p$ and, thus, is naturally upper semicontinuous. \[\square\]

The following corollary provides a partial positive answer to the question of Brenner, Li, and Miller from [BLM12] and recovers their main result, [BLM12, Corollary 3.3], for $e \geq 3$. An affirmative answer at a much larger level of generality was announced in [PTY]. A similar result for a family of geometrically integral graded rings and $e \geq h - 1$ was recently proved by Trivedi in [Tri, Corollary 1.2].

**Corollary 4.11.** Let $Z \to R$ be an affine $I$-family with reduced fibers of dimension $h$. Then for every $e \geq h + 1$
\[
\lim_{p \to \infty} \left( e_{\text{HK}}(IR(p)) - \frac{\lambda(R(p)/I(p)[p^e])}{p^eh} \right) = 0.
\]


Proof. By Corollary 4.9, we obtain that for all sufficiently large $p$

$$
\left| e_{\text{HK}}(IR(p)) - \frac{\lambda(R(p)/I(p)[p^e])}{p^e h} \right| < \frac{2D}{p^{e-h}},
$$

and the theorem follows.  

4.3. F-rational signature. In [HY] Hochster and Yao introduced the following definition.

**Definition 4.12.** Let $(R, m)$ be a local ring. The F-rational signature of $R$ is defined as

$$
s_{\text{rat}}(R) = \inf_u \{ e_{\text{HK}}(x) - e_{\text{HK}}(x, u) \}
$$

where the infimum is taken over all systems of parameters $x$ and socle elements $u$.

In [HY, Theorem 2.5], it was shown that one can fix an arbitrary $x$ in the definition.

**Proposition 4.13.** Let $k$ be an algebraically closed field $k$ characteristic $p > 0$, $R = k[t_1, \ldots, t_N]/I$ be a $k$-algebra of finite type, and $m = (t_1, \ldots, t_N)$ be its maximal ideal. Then the infimum in the definition of $s_{\text{rat}}(R_m)$ is achieved.

Proof. Let $u_1, \ldots, u_N$ be a basis of $(x) : m/((x))$ as a $k$-vector space. We may parametrize the socle ideals via two affine families: $(x, T_1 u_1 + \cdots + T_{N-1} u_{N-1} + u_N)$-family

$$
k[T_1, \ldots, T_{N-1}] \to R[T_1, \ldots, T_{N-1}]
$$

and, similarly, for $u_1 + T_2 u_2 + \cdots + T_N u_N$. By Corollary 4.10 $p \mapsto e_{\text{HK}}((x, T_1 u_1 + \cdots + u_N)R_p)$ is upper semicontinuous, so it attains maximum. The maximum must be achieved at a closed point, and the statement follows.

**Remark 4.14.** We want to note that Proposition 4.13 can be also applied when $R$ is given as a quotient of a power series ring by an ideal generated by polynomials, since the lengths do not change under completion.

As a consequence, we recover a special case of [HY, Theorem 4.1].

**Corollary 4.15.** Let $k$ be an algebraically closed field $k$ characteristic $p > 0$, $R = k[t_1, \ldots, t_N]/I$ be a $k$-algebra of finite type, and $m = (t_1, \ldots, t_N)$ be its maximal ideal. Then $s_{\text{rat}}(R_m) > 0$ if and only if $R$ is F-rational.

Proof. Recall that $R$ is F-rational if $x$ is tightly closed, which is equivalent to say that $e_{\text{HK}}(x) > e_{\text{HK}}(x, u)$ for every socle element $u$.

**Remark 4.16.** In a forthcoming joint work with Kevin Tucker, we investigate a variation of the Hochster–Yao definition called Cartier signature

$$
s_{\text{cart}}(R) = \inf_{x \in I} \frac{e_{\text{HK}}(x) - e_{\text{HK}}(I)}{\lambda(R/x) - \lambda(R/I)},
$$

where the infimum is taken over all $m$-primary ideals containing a system of parameters $x$. We show that the definition also does not depend on the choice of $x$ and has better properties than the original.

By considering higher degree Grassmanians of $(x) : m/(x)$, from the proof of Proposition 4.13 we may also get that the Cartier signature is a minimum.
5. Questions

5.1. Nilpotents. As in the preceding result of [PS], the main stumbling block for extending the current results is our inability to deal with non-reduced rings in a controllable way. While Hilbert–Kunz multiplicity exists for non-reduced rings, the original proof in [Mon83] and its extensions pass to $R_{\text{red}}$ by observing that $F^{e_0}R$ is an $R_{\text{red}}$-module for large $e_0$. This is not satisfactory for two reasons: the approach via discriminants does not adapt for modules and we do not see how to control the exponent $e_0$.

5.2. F-signature. F-signature is another measure of singularity in positive characteristic introduced by Huneke and Leuschke in [HL02]. Following [AE05] a relative version of F-signature, with respect to an $m$-primary ideal $J$ of a local ring $(R, m)$ of characteristic $p$ can be defined using the ideals $I_e(J) = \{ x \mid \phi(F^e_*x) \in J \text{ for all } \phi \in \text{Hom}(F^e_*R, R) \}$ as 

$$s(J) = \lim_{e \to \infty} \lambda(R/I_e(J))/p^{e \dim R}.$$ 

A natural question is to extend the results of this paper for F-signature and the natural setting for such extension is to consider ideals $I_e(I(p))$. However this notion is not functorial and $I_e(I)(p) \neq I_e(I(p))$, so we do not see that an individual term, i.e., the splitting number, is lower semicontinuous. Note that [CRST, Proposition 4.5] will provide uniform convergence for flat affine families with geometrically normal fibers. Moreover, [CRST, Theorem 4.9] proves semicontinuity for flat affine families with geometrically normal fibers over a regular ring and such that $I(I(p))$ is a maximal ideal.

5.3. Localization of tight closure. As it was mentioned above, Brenner and Monsky showed that tight closure does not localize. However, we do not understand the underlying reasons. In particular, how does it relate to the results of [HH00] and how typical is this phenomena? As [BM10] depends on an irregular behavior of Hilbert–Kunz multiplicity in a family, we hope that it should be possible to give a general procedure for producing counter-examples from such families, for example, the family in [Mon05]. The study of Hilbert–Samuel multiplicity was pioneered by Teissier ([Tei80]) to give a criterion of equimultiplicity: $e(I(p))$ is independent of $p$ if and only if $\ell(I) = \text{ht}(I)$. The author suspects that a study of equimultiplicity in families for Hilbert–Kunz multiplicity might explain the phenomenon presented in [BM10].

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