Schematic Harder-Narasimhan Stratification

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Abstract

For any flat family of pure-dimensional coherent sheaves on a family of projective schemes, the Harder-Narasimhan type (in the sense of Gieseker semistability) of its restriction to each fiber is known to vary semicontinuously on the parameter scheme of the family. This defines a stratification of the parameter scheme by locally closed subsets, known as the Harder-Narasimhan stratification.

In this note, we show how to endow each Harder-Narasimhan stratum with the structure of a locally closed subscheme of the parameter scheme, which enjoys the universal property that under any base change the pullback family admits a relative Harder-Narasimhan filtration with a given Harder-Narasimhan type if and only if the base change factors through the schematic stratum corresponding to that Harder-Narasimhan type.

The above schematic stratification induces a stacky stratification on the algebraic stack of pure-dimensional coherent sheaves. We deduce that coherent sheaves of a fixed Harder-Narasimhan type form an algebraic stack in the sense of Artin.

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1. Introduction

Let $X$ be a projective scheme over a locally noetherian base scheme $S$, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$. Let $E$ be a coherent sheaf on $X$ which is flat over $S$, such that the restriction $E_s = E|_{X_s}$ of $E$ to the schematic fiber $X_s$ of $X$ over each $s \in S$ is a pure-dimensional sheaf of a fixed dimension $d \geq 0$. For any $s \in S$, let $\text{HN}(E_s)$ denote the Harder-Narasimhan type of $E_s$ in the sense of Gieseker semistability. With respect to a certain natural partial order on the set $\text{HNT}$ of all possible Harder-Narasimhan types $\tau$, the Harder-Narasimhan function $s \mapsto \text{HN}(E_s)$ is known to be upper semicontinuous on $S$.

In this note, we prove that each level set $S^\tau(E)$ of the Harder-Narasimhan function has a natural structure of a locally closed subscheme of $S$, with the following universal property: a morphism $T \to S$ factors via $S^\tau(E)$ if and only if the pullback $E_t$ on $X_t$ for each $t \in T$ is of type $\tau$ and the pullback family $E_T$ on $X \times_S T$ admits a relative Harder-Narasimhan filtration, that is, a filtration $0 \subset F_1 \subset \ldots \subset F_\ell = E_T$ by coherent subsheaves such that the graded pieces $F_i/F_{i-1}$ are flat over $T$, which for each $t \in T$ restricts to the Harder-Narasimhan filtration of $E_t$ on $X_t$. 

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As a corollary, we deduce that sheaves of a fixed Harder-Narasimhan type form an algebraic stack in the sense of Artin.

In Section 2 we recall the basic definitions and results of Harder-Narasimhan-Shatz that we need. In Section 3 we prove our main result (Theorem 5), which gives natural schematic structures on the Harder-Narasimhan strata. In Section 4, we show (Theorem 8) that sheaves of a given Harder-Narasimhan type form an Artin algebraic stack.

This work had its origin in questions arising from the proposal of Leticia Brambila-Paz to construct a moduli scheme for indecomposable unstable rank 2 vector bundles on a curve, fixing their Harder-Narasimhan type and the dimension of their vector space global endomorphisms. A construction of such a moduli scheme is given in [B-M-Ni], which uses special cases of the results proved here.

2. The Harder-Narasimhan filtration and stratification

Let \( \mathbb{Q}[\lambda] \) be the polynomial ring in the variable \( \lambda \). An element \( f \in \mathbb{Q}[\lambda] \) is called a numerical polynomial if \( f(\mathbb{Z}) \subset \mathbb{Z} \). If a nonzero numerical polynomial \( f \) has degree \( d \), it can be uniquely expanded as \( f = (r(f)/d!)(\lambda^d) + \text{lower degree terms} \), where \( r(f) \in \mathbb{Z} \). If \( f = 0 \) we put \( r(f) = 0 \). There is a total order \( \leq \) on \( \mathbb{Q}[\lambda] \) under which \( f \leq g \) if \( f(m) \leq g(m) \) for all sufficiently large integers \( m \). Let the set of all \textbf{Harder-Narasimhan types}, denoted by HNT, be the set consisting of all finite sequences \((f_1, \ldots, f_p)\) of numerical polynomials in \( \mathbb{Q}[\lambda] \), where \( p \) is allowed to vary over all integers \( \geq 1 \), such that the following three conditions are satisfied.

1. We have \( 0 < f_1 < \ldots < f_p \) in \( \mathbb{Q}[\lambda] \).
2. The polynomials \( f_i \) are all of the same degree, say \( d \), and
3. the following inequalities are satisfied
   \[
   \frac{f_1}{r(f_1)} > \frac{f_2 - f_1}{r(f_2) - r(f_1)} > \ldots > \frac{f_p - f_{p-1}}{r(f_p) - r(f_{p-1})}.
   \]

Given any \( x = (a, f) \) and \( y = (b, g) \) in \( \mathbb{Z} \times \mathbb{Q}[\lambda] \), the segment joining \( x \) and \( y \) is the subset \( xy \subset \mathbb{Z} \times \mathbb{Q}[\lambda] \), consisting of all \( (c, h) \) such that \( (c, h) = t(a, f) + (1-t)(b, g) \) for some \( t \in \mathbb{Q} \) with \( 0 \leq t \leq 1 \). For any \((f_1, \ldots, f_p)\) in HNT, we define the corresponding \textbf{Harder-Narasimhan polygon} to be the subset

\[
\text{HNP}(f_1, \ldots, f_p) \subset \mathbb{Z} \times \mathbb{Q}[\lambda]
\]

which is the union of the segments \( x_0x_1 \cup x_1x_2 \cup \ldots \cup x_{p-1}x_p \) where \( x_0 = (0, 0) \) and \( x_i = (r(f_i), f_i) \) for \( 1 \leq i \leq p \).

A point \((a, f)\) in \( \mathbb{Z} \times \mathbb{Q}[\lambda] \) is said to lie under another point \((b, g)\) in \( \mathbb{Z} \times \mathbb{Q}[\lambda] \) if \( a = b \) in \( \mathbb{Z} \) and \( f \leq g \) in \( \mathbb{Q}[\lambda] \). A point \((a, f)\) in \( \mathbb{Z} \times \mathbb{Q}[\lambda] \) is said to lie under the
Grassmannian subsheaves unique strictly increasing filtration $0 = HN_0 \subset HN_1 \subset \ldots \subset HN_\ell E$ of the support of $E$ is called the Harder-Narasimhan filtration of the support of $E$. The first step $HN_1 E$ is a subobject of $E$ that is destabilizing. This filtration is called the Harder-Narasimhan filtration of $E$ in the sense of Gieseker semistability. The first step $HN_1 E$ is the maximal destabilizing subsheaf of $E$. The integer $\ell$ (also written as $\ell(E)$) is called the length of the Harder-Narasimhan filtration of $E$. In these terms, a nonzero pure-dimensional coherent sheaf is semistable if and only if its Harder-Narasimhan filtration is of length $\ell(E) = 1$. The ordered $\ell(E)$-tuple

$$HN(E) = (P(HN_1(E)), \ldots, P(HN_\ell(E))) \in \text{HNT}$$

is called the Harder-Narasimhan type of $E$.

In his path-breaking paper [Sh], S.S. Shatz addressed the question of the variation of the Harder-Narasimhan type in a family. The set-up for this is as follows. Let $S$ be a locally noetherian scheme, and let $\pi : X \to S$ be a projective scheme over $S$, with a relatively ample line bundle $\mathcal{O}_X(1)$. Let $E$ be a coherent sheaf of $\mathcal{O}_X$-modules such that each restriction $E_s$ to the schematic fiber $X_s = \pi^{-1}(s)$ is pure-dimensional. The Harder-Narasimhan function of $E$ is the function

$$|S| \to \text{HNT} : s \mapsto HN(E_s)$$
where $|S|$ denotes the underlying topological space of the scheme $S$. Shatz proved in [Sh] that $\text{HN}(E_s)$ is upper-semicontinuous w.r.t. the partial order $\leq$ on $\text{HNT}$ defined above (actually, $\text{HN}$-filtrations in the sense of $\mu$-semistability rather than Gieseker semistability are considered in [Sh], but the proofs in the Gieseker semistability case are similar with obvious changes).

**Remark 1** In particular, for any $\tau \in \text{HNT}$, the corresponding level set

$$|S|^\tau(E) = \{ s \in |S| \text{ such that } \text{HN}(E_s) = \tau \}$$

is locally closed in $|S|$, the subset $|S|^\leq \tau(E) = \bigcup_{\alpha \leq \tau} |S|^{\alpha}(E) \subset |S|$ is open in $|S|$, and $|S|^\tau(E)$ is closed in $|S|^\leq \tau(E)$.

**Remark 2** If $(f_1, \ldots, f_p) \in \text{HNT}$, then $(f_2 - f_1, \ldots, f_p - f_1)$ is again in $\text{HNT}$. Let $E$ be pure-dimensional on $Y$ with $\text{HN}(E) \leq (f_1, \ldots, f_p) \in \text{HNT}$. If $E' \subset E$ is a coherent subsheaf with $P(E') = f_1$, then we must have $\text{HN}_1(E) = E'$, that is, such an $E'$ is automatically the maximal destabilizing subsheaf of $E$. The quotient $E'' = E/E'$ is pure-dimensional, with $\text{HN}(E'') \leq (f_2 - f_1, \ldots, f_p - f_1)$. Moreover, we have $\text{Hom}_Y(E', E'') = 0$.

**Remark 3** If $(Y, \mathcal{O}_Y(1))$ is a projective scheme over a field $k$ and if $K$ any extension field of $k$, then a coherent sheaf $E$ on $Y$ is semistable w.r.t. $\mathcal{O}_Y(1)$ if and only if its base-change $E_K = E \otimes_k K$ to $Y_K$ is semistable w.r.t. $\mathcal{O}_{Y_K}(1) = \mathcal{O}_Y(1) \otimes_k K$. Consequently, if $E$ is any pure-dimensional sheaf on $Y$ then the Harder-Narasimhan filtration $\text{HN}_i(E_K)$ is just the pullback $\text{HN}_i(E) \otimes_k K$ of the Harder-Narasimhan filtration of $E$.

### 3. Scheme structures on HN strata

For basic facts that we need from Grothendieck’s theory of Quot schemes and their deformation theory, the reader can consult [Hu-Le], [F-G], [Ni 1] and [Ni 2].

**Lemma 4** A morphism $f : T \to S$ between locally noetherian schemes is a closed embedding if (and only if) $f$ is proper, injective, unramified and induces an isomorphism $k(f(t)) \to k(t)$ of residue fields for all $t \in T$.

**Proof** Note that $f(T)$ is closed in $S$, and $f_* \mathcal{O}_T$ is coherent. It only remains to show that the homomorphism $f^# : \mathcal{O}_S \to f_* \mathcal{O}_T$ is surjective. It is enough to show it stalk-wise at all points of $f(T)$, so we can assume that $S = \text{Spec} A$ where $A$ is a
Let $X$ be a projective scheme over a locally noetherian base scheme $S$, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$. Let $E$ be a coherent sheaf on $X$ which is flat over $S$, such that the restriction $E_s = E|_{X_s}$ of $E$ to the schematic fiber $X_s$ of $X$ over each $s \in S$ is a nonzero pure-dimensional sheaf of a fixed HN type $\tau = (f_1, \ldots, f_\ell)$. A relative Harder-Narasimhan filtration of $E$ is a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E$ by coherent subsheaves on $X$, such that for each $i$ with $1 \leq i \leq \ell$, the quotient $E_i/E_{i-1}$ is flat $S$, and for each $s \in S$ this filtration restricts to give the Harder-Narasimhan filtration $\text{HN}_i(E_s)$ of $E_s$.

We now come to the main result of this note.

\textbf{Theorem 5} (Main Theorem) Let $X$ be a projective scheme over a locally noetherian scheme $S$, with a relatively ample line bundle $\mathcal{O}_X(1)$. Let $E$ be a coherent sheaf on $X$ which is flat over $S$, such that the restriction $E_s = E|_{X_s}$ of $E$ to the schematic fiber $X_s$ of $X$ over each $s \in S$. Let $\tau = (f_1, \ldots, f_\ell) \in \text{HNT}$. Then we have the following.

1. Each Harder-Narasimhan stratum $|S|^\tau(E)$ of $E$ has a unique structure of a locally closed subscheme $S^\tau(E)$ of $S$, with the following universal property: a morphism $T \to S$ factors via $S^\tau(E)$ if and only if the pullback $E_T$ on $X \times_S T$ admits a relative Harder-Narasimhan filtration of type $\tau$.

2. A relative Harder-Narasimhan filtration on $E$, if it exists, is unique.

3. For any morphism $f : T \to S$ of locally noetherian schemes, the schematic stratum $T^\tau(E_T) \subset T$ for $E_T$ equals the schematic inverse image of $S^\tau(E)$ under $f$.

\textbf{Proof} If $\ell = 1$, then we take $S^\tau(E)$ to be the open subscheme of $S$ consisting of all $s$ such that $E_s$ is semistable with Hilbert polynomial $f_1$. We now argue by induction on $\ell \geq 2$. By Remark \[\text{all} s \text{ with } \text{HN}(E_s) \leq \tau \text{ form an open subset } |S|^\leq \tau(E) \text{ of } S, \]

and $|S|^\tau(E)$ is a closed subset of $|S|^\leq \tau(E)$. We give $|S|^\leq \tau(E)$ the unique structure of an open subscheme of $S$, which we denote by $S^\leq \tau(E)$. In what follows we will give the closed subset $|S|^\tau(E)$ a particular structure of a closed subscheme of $S^\leq \tau(E)$, which has the desired universal property.

Let $X^\leq \tau$ be the inverse image of $S^\leq \tau = S^\leq \tau(E)$ in $X$, and let $\mathcal{O}_{X^\leq \tau}(1)$ and $E^\leq \tau$ be the restrictions of $\mathcal{O}_X(1)$ and $E$ to $X^\leq \tau$. Consider the relative Quot scheme

$$Q = \text{Quot}^{|f_\tau - f_1|, \mathcal{O}_{X^\leq \tau}(1)}_{E^\leq \tau/X^\leq \tau/S^\leq \tau}$$
with projection \( \pi : Q \to S^{\leq \tau} \). Then \( \pi \) is projective, hence proper.

Let \( q \in Q \) represent a quotient \( q' : E_q \to \mathcal{F} \) on \( X_q \). Then \( \ker(q') = \text{HN}_1(E_q) \) by Remark 2. If \( q \mapsto s \in S^{\leq \tau} \), then by Remark 3 the quotient \( q' \) is the pullback of the quotient \( E_s \to E_s/\text{HN}_1(E_s) \) which is defined over \( X_s \). Hence the residue field extension \( k(s) \to k(q) \) is trivial. By the uniqueness of \( \text{HN}_1(E_s) \), there exists at most one such \( q \) over \( s \). The fiber of \( \pi : Q \to S^{\leq \tau} \) over \( s \) is the Quot scheme

\[
\pi^{-1}(s) = \text{Quot}_{E_s/X_s/k(s)}^{f_2-f_1, \mathcal{O}_{X_s}(1)}.
\]

By a standard fact in the deformation theory for Quot schemes (see, for example, Theorem 3.11.(2) in [Ni 2]), its tangent space at \( q \) is given by

\[
T_q(\pi^{-1}(s)) = \text{Hom}_{X_q}(\ker(q'), E_q/\ker(q')) = \text{Hom}_{X_q}(\text{HN}_1(E_q), E_q/\text{HN}_1(E_q))
\]

which is zero by Remark 2. Hence \( \pi : Q \to S^{\leq \tau} \) is unramified.

It now follows by Lemma 4 that \( \pi : Q \to S^{\leq \tau} \) is a closed imbedding.

Now consider the universal quotient sheaf \( E_q \to E'' \) on \( X_Q = X \times_S Q \). By Remark 2 for all \( q \in Q \) the sheaf \( E''_q \) on \( X_q \) is pure-dimensional, with

\[
\text{HN}(E''_q) \leq \tau'' = (f_2 - f_1, \ldots, f_{\ell} - f_1).
\]

In particular, we have \( Q^{\leq \tau''}(E'') = Q \). The Harder-Narasimhan type \( \tau'' \) has length \( \ell - 1 \), hence by induction on the length, the closed subset \( |Q|^\tau''(E'') \) of \( Q \) has the structure of a closed subscheme \( Q^{\tau''}(E'') \subset Q \) which has the desired universal property for \( E'' \). We regard \( Q \) as a closed subscheme of \( S^{\leq \tau} \) via \( \pi \), and we finally define the closed subscheme \( S^{\tau}(E) \subset S^{\leq \tau} \) by putting

\[
S^{\tau}(E) = Q^{\tau''}(E'') \subset Q \subset S^{\leq \tau}.
\]

We now show that \( S^{\tau}(E) \) so defined has the desired universal property. As \( \tau'' \) has length \( \ell - 1 \), by induction on the length, the restriction \( E''_{S^{\tau}(E)} \) of \( E'' \) to \( X \times_S S^{\tau}(E) \) has a unique relative Harder-Narasimhan filtration

\[
0 \subset E''_1 \subset \ldots \subset E''_{\ell-1} = E''_{S^{\tau}(E)}
\]

with Harder-Narasimhan type \( \tau'' \). For \( 2 \leq i \leq \ell \), let \( E_i \) be the inverse image of \( E''_{i-1} \) under the restriction of universal quotient \( E_q \to E'' \) to \( X_{S^{\tau}(E)} \). This defines a relative Harder-Narasimhan filtration \( 0 \subset E_1 \subset \ldots \subset E_\ell = E_{S^{\tau}(E)} \) of \( E_{S^{\tau}(E)} \) over the base \( S^{\tau}(E) \). In particular, if a morphism \( T \to S \) factors via \( S^{\tau}(E) \) then the pullback of this filtration gives a relative Harder-Narasimhan filtration over \( T \).

Conversely, let \( f : T \to S \) be a morphism such that the pullback \( E_T \) on \( X_T \) has a relative Harder-Narasimhan filtration \( 0 = F_0 \subset F_1 \subset \ldots \subset F_\ell = E_T \) of type
τ. The quotient $E_T \to E_T/F_1$ has Hilbert polynomial $f_i - f_1$ over all $t \in T$, so by the universal property of the Quot scheme $Q$, the morphism $T \to S$ factors via $Q \hookrightarrow S$, inducing a morphism $f' : T \to Q$. By Remark 2, the restriction of $E_T/F_1$ is pure-dimensional on $X_t$ for all $t \in T$, and $0 = (F_1/F_1) \subset (F_2/F_1) \subset \ldots \subset (F_\ell/F_1) = E_T/F_1$ is a relative Harder-Narasimhan filtration of $E_T/F_1 = (f')^*(E''')$ over the base $T$, with type $\tau''$ which has length $\ell - 1$. Hence by induction, $f' : T \to Q$ factors via $Q_{\tau''}(E'') = S^{\tau}(E)$, as desired. This completes the proof of (1).

Next we show the uniqueness of a relative Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E$ over a base $S$, assuming such a filtration exists. As $|S|^\tau(E) = |S|$, we at least have $S = S^{\le \tau}(E)$. With notation as above, we have shown that $\pi : Q \to S^{\le \tau}$ is a closed imbedding, therefore $\pi$ admits at most one global section, which shows that $E_1$ is unique. By inductive assumption on $\ell$, the quotient family $E/E_1$ admits a unique relative Harder-Narasimhan filtration $F_i$, so defining $E_i \subset E$ to be the inverse image of $F_{i-1}$ under $E \to E/E_1$ for $2 \le i \le \ell$, we see that $0 = E_0 \subset E_1 \subset \ldots \subset E_\ell = E$ is the only possible relative Harder-Narasimhan filtration on $E$. This proves the statement (2).

The base-change property (3) for the schematic strata is a direct consequence of the universal property (1). This completes the proof of the theorem. □

The following immediate implication of Theorem 5 shows that when the HN type is constant over a reduced base scheme, the outcome is as nice as can be expected.

**Corollary 6 (Case of constant HN type over a reduced base)** Let $X$ be a projective scheme over a locally noetherian base scheme $S$, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$. Let $E$ be a coherent sheaf on $X$ which is flat over $S$, such that the restriction $E_s = E|_{X_s}$ of $E$ to the schematic fiber $X_s$ of $X$ over each $s \in S$ is a pure-dimensional sheaf of a fixed Harder-Narasimhan type $\tau \in HNT$. Suppose moreover that $S$ is reduced. Then $S = S^\tau$, that is, $E$ admits a unique relative Harder-Narasimhan filtration.

4. **Moduli stack $\text{Coh}_{X/S}^\tau$**

For basic terminology and conventions about stacks, we will follow the book [L-MB] by Laumon and Moret-Bailly. In what follows, $X$ will be a projective scheme over a locally noetherian base scheme $S$, with a chosen relatively ample line bundle $\mathcal{O}_X(1)$, and $\tau = (f_1, \ldots, f_\ell)$ will be any element of HNT.

Let $\text{Coh}_{X/S}$ denote the Artin algebraic stack over $S$ of all flat families of coherent sheaves on $X/S$ (see [L-MB] 2.4.4). In any such family, pure-dimensionality of all restriction to fibers is an open condition on the parameter scheme, pure-dimensionality is preserved by arbitrary base changes, and the base-change under a surjection is
pure-dimensional on all fibers if and only if the original is so. Hence pure-dimensional coherent sheaves form an open algebraic substack $\text{Coh}_X^{\text{pure}} \subset \text{Coh}_X$.

We will define the moduli stack $\text{Coh}_X^\tau$ of pure-dimensional coherent sheaves of type $\tau$ as a strictly full sub $S$-groupoid of $\text{Coh}_X^{\text{pure}}$, as follows. For any $S$-scheme $T$, we say that an object $E \in \text{Coh}_X^\tau(T)$ lies in $\text{Coh}_X^\tau(T)$ if and only if $E$ admits a relative Harder-Narasimhan filtration with constant type $\tau$. This is clearly closed under pullbacks $f^* : \text{Coh}_X^\tau(T) \to \text{Coh}_X^\tau(T')$ for all $S$-morphisms $f : T' \to T$.

To prove that the $S$-groupoid $\text{Coh}_X^\tau$ thus defined is a stack, we need the following property of effective descent.

**Lemma 7** Let $T$ be an $S$-scheme and let $E$ be an object of $\text{Coh}_X^\tau(T)$. Let $f : T' \to T$ be a faithfully flat quasi-compact morphism. If the pullback $f^*E$ is in $\text{Coh}_X^\tau(T')$, then $E$ is in $\text{Coh}_X^\tau(T)$.

**Proof** Each $E_t$, where $t \in T$, is pure-dimensional with Harder-Narasimhan type $\tau$, as its pullback $E_t'$ is so for any $t' \in T'$ with $t' \mapsto t$, and as $T' \to T$ is surjective. It now only remains to construct a relative Harder-Narasimhan filtration of $E$. This we do by showing that the relative Harder-Narasimhan filtration $(F_i)$ of the pullback $E_{T'}$ descends under $T' \to T$.

Let $T'' = T' \times_T T'$ with projections $\pi_1, \pi_2 : T'' \rightrightarrows T'$. By Grothendieck’s result on effective fpqc descent for quasicoherent subsheaves of the pullback of a quasicoherent sheaf, to show that the filtration descends to $T$ we just have to show that the pullbacks of the filtration under the two projections $\pi_1, \pi_2 : T'' \rightrightarrows T'$ are identical. But note that we have an identification $\pi_1^*(E_{T'}) = \pi_2^*(E_{T'}) = E_{T''}$, under which the pullbacks $\pi_1^*(F_i)$ and $\pi_2^*(F_i)$ are relative Harder-Narasimhan filtrations of $E_{T''}$. Hence these filtrations coincide by Theorem 5. □

**Theorem 8** Let $X$ be a projective scheme over a locally noetherian scheme $S$, with a relatively ample line bundle $\mathcal{O}_X(1)$. Let $\tau$ be any Harder-Narasimhan type. Then all flat families of pure-dimensional coherent sheaves on $X/S$ with fixed Harder-Narasimhan type $\tau$ form an algebraic stack $\text{Coh}_X^\tau$ over $S$, which is a locally closed substack of the algebraic stack $\text{Coh}_X$ of all flat families of coherent sheaves on $X/S$.

**Proof** The inclusion 1-morphism of $S$-groupoids $\theta : \text{Coh}_X^\tau \hookrightarrow \text{Coh}_X^{\text{pure}}$ is fully faithful. Hence $\text{Coh}_X^\tau$ is a pre-stack over $S$. By Lemma 7, the pre-stack $\text{Coh}_X^\tau$ satisfies effective fpqc descent, so it is a stack over $S$. We next prove that it is algebraic.

Given any $E$ in $\text{Coh}_X^{\text{pure}}(T)$, let $T^\tau(E) \subset T$ be the corresponding schematic Harder-Narasimhan stratum as given by Theorem 5. Let $[E] : T \to \text{Coh}_X^{\text{pure}}$ be the
classifying 1-morphism of $E$. By Theorem 5 we have a natural isomorphism

$$T \times_{[E], \text{Coh}_{X/S}^\tau} \text{Coh}_{X/S}^\tau \cong T^\tau(E)$$

of $S$-groupoids, under which the projection of the fibered product to $T$ corresponds to the imbedding of $T^\tau(E)$ as a locally closed subscheme in $T$.

This shows the inclusion 1-morphism $\theta : \text{Coh}_{X/S}^\tau \hookrightarrow \text{Coh}_{X/S}$ of stacks is a representable locally closed imbedding. Hence $\text{Coh}_{X/S}^\tau$ is an algebraic stack over $S$, which is a locally closed substack of $\text{Coh}_{X/S}$.

We now come to the question of quasi-projectivity of $\text{Coh}_{X/S}^\tau$. For a given $X/S$, $\mathcal{O}_X(1)$ and $\tau = (f_1, \ldots, f_\ell)$, consider the following boundedness condition (*).

(*): There exists a natural number $N$ such that for any morphism $\text{Spec } K \to S$ where $K$ is a field and any semistable coherent sheaf $F$ on the base-change $X_K$ whose Hilbert polynomial is equal to $f_i$ for any $1 \leq i \leq \ell$, the sheaf $F(N) = F \otimes \mathcal{O}_X(K(N)$ is generated by global sections, and all its cohomology groups $H^j(X_K, F(N))$ vanish for $j \geq 1$.

By the boundedness theorems of Maruyama-Simpson [Si] and Langer [La], the condition (*) is indeed satisfied in many cases of interest, for example, when $S$ is of finite type over an algebraically closed field $k$ of arbitrary characteristic.

**Proposition 9 (Quasi-projectivity of $\text{Coh}_{X/S}^\tau$)** If the above boundedness condition (*) is satisfied, then the stack $\text{Coh}_{X/S}^\tau$ admits an atlas $U \to \text{Coh}_{X/S}^\tau$ such that $U$ is a quasi-projective scheme over $S$.

**Proof** If a coherent sheaf $E$ is of type $\tau$ on $X_K$ for an $S$-field $K$, then by (*), $E$ is a quotient of $\mathcal{O}_{X_K}(-N)^{f_i(N)}$. Let $Q$ be the relative Quot scheme over $S$ which parameterizes all coherent quotient sheaves of $\mathcal{O}_{X}(-N)^{f_i(N)}$ on fibers of $X/S$, with fixed Hilbert polynomial $f_i$. Let $E$ be the universal quotient sheaf on $X \times_S Q$. Let $Q_o \subset Q$ be the open subscheme consisting of all $q \in Q$ satisfying the conditions that $E_q$ is pure-dimensional, $E_q(N)$ is generated by global sections, the map $H^0(X_q, \mathcal{O}_{X_q}^{f_i(N)}) \to H^0(X_{q}, E_q(N))$ induced by $q$ is an isomorphism, and $H^i(X_{q}, E_q(N)) = 0$ for all $i \geq 1$ (each of these conditions is an open condition).

Let $E_o$ be the restriction of $E$ to $X \times_S Q_o$. Let $Q_o^\tau(E_o)$ be the locally closed subscheme of $Q_o$ corresponding to the Harder-Narasimhan type $\tau$, given by Theorem 5. The classifying 1-morphism $[E_o] : Q_o^\tau(E_o) \to \text{Coh}_{X/S}^\tau$ of $E_o$ is an atlas for $\text{Coh}_{X/S}^\tau$ (that is, $[E_o]$ is a representable smooth surjection), as follows from the proof of Theorem 4.6.2.1 in Laumon and Moret-Bailly [L-MB]. As $Q$ is projective over $S$, its locally closed subscheme $U$ is quasi-projective over $S$, as desired. □
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