Exclusive double $B_c$ meson production from $e^+e^-$ annihilation into two virtual photons

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We calculate cross sections of a pair $B_c$ meson production on the basis of two-photon mechanism from electron-positron annihilation. We investigate the production cross sections in nonrelativistic approximation and with the account of relativistic corrections. Relativistic production amplitudes of S-wave pair pseudoscalar, vector and pseudoscalar+vector $B_c$-mesons are constructed on the basis of relativistic quark model. Numerical values of the production cross sections are obtained at different center-of-mass energies. The comparison of one-photon and two-photon annihilation contributions is presented.

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I. INTRODUCTION

The production of bound states of heavy quarks (double-heavy mesons and baryons) in such fundamental processes as electron-positron annihilation and proton-proton interaction is of great importance for testing the perturbative and non-perturbative aspects of quantum chromodynamics and the theory of bound states. Among these reactions stands out the process of exclusive pair production of heavy quarkonia and double heavy baryons, since a pair of bound states of heavy quarks is produced in it and the binding effects of quarks are significantly enhanced. The activity in the theoretical study of the pair charmonium production in electron-positron annihilation was largely connected with the experimental results obtained by the Belle and BaBar collaborations [1–4], which differed significantly from the predictions of non-relativistic quantum chromodynamics [5–7]. During a short period of research, it has been shown that the theoretical results for the pair production of charmonium can be reconciled with experimental data, taking into account perturbative corrections of order $O(\alpha_s)$ to the production amplitudes and relativistic corrections, due to the relative motion of heavy quarks [8–20]. Along with the mechanism of one-photon annihilation, the production process of a pair of charmoniums in two-photon annihilation was also studied at this time [21]. Despite the fact that such contributions contain an additional small factor $\alpha^2/\alpha_s^2$, nevertheless, the fragmentation contribution has a structure that compensates for the suppression of the production amplitude by coupling constants.

In our work [22], we used the developed methods of studying the reactions of pair exclusive quarkonia production in the case of $B_c$-mesons. Our results [22] show that, as in the case of $(c\bar{b})$ mesons, relativistic effects substantially change the magnitude of the cross sections...
for the production of a pair of $B_c$-mesons. In [23], the calculation was performed for one-loop corrections in the pair-production of $B_c$-mesons in a one-photon annihilation. An approximation was used in which the relative motion of heavy quarks was not taken into account. In this paper, we explore the contribution of two-photon annihilation process in reactions of the production of a pair of $B_c$-mesons. We calculate the cross sections for the production of a pair of $B_c$-mesons in the nonrelativistic approximation and with the account of relativistic corrections. A comparison of numerical contributions of one-photon and two-photon mechanisms in total cross section is performed.

While extensive literature is devoted to the issues of the production of single $B_c$ mesons [24], the problem of the production of a pair of $B_c$ mesons or heavy diquarks ($bc$) in different reactions has been discussed to a much lesser extent [22, 25–27]. This is mainly due to the lack of a clear experimental perspective to observe a sufficient number of such events in existing experiments. This work continues our research of exclusive double heavy meson production in $e^+e^-$ annihilation. Our approach to calculating the observed cross sections for the production of a pair of mesons is based on methods of relativistic quark model (RQM) and perturbative quantum chromodynamics [15, 20, 22, 28–30]. In this approach we can take into account relativistic effects in the construction of relativistic production amplitudes, relativistic production cross sections, and in the description of bound states of heavy quarks through the use of the corresponding quark interaction operator. We can say that in this approach we have microscopic picture of the photon, quark and gluon interaction at different stages of meson production. The approach based on relativistic quark model allows you to perform a self-consistent calculation of various theoretical parameters, which ultimately determine the total numerical values of the production cross sections.

II. GENERAL FORMALISM

Two-photon annihilation amplitudes leading to the production of a pair of $B_c$ mesons, at leading order are presented in Fig. 1. The production process can be divided into two stages. At the first stage, a pair of virtual photons is formed. At the second stage, each virtual photon produces a pair of quark-antiquark, which then with some probability combine into $B_c$ mesons with definite spin. To properly take into account the quark binding effects and relativistic corrections, we express quark four-momenta in terms of the total and relative four-momenta in the form:

\[ p_1 = \eta_1 P + p, \quad p_2 = \eta_2 P - p, \quad (p \cdot P) = 0, \quad \eta_{1,2} = \frac{M_{B_{bc}}^2 \pm m_1^2 \pm m_2^2}{2M_{B_{bc}}^2}, \]  
\[ q_1 = \rho_1 Q + q, \quad q_2 = \rho_2 Q - q, \quad (q \cdot Q) = 0, \quad \rho_{1,2} = \frac{M_{B_{bc}}^2 \pm m_1^2 \pm m_2^2}{2M_{B_{bc}}^2}, \]

where $M_{B_{bc}}$ is the mass of pseudoscalar or vector $B_c^+ (B_c^{*+})$ meson consisting of $\bar{b}$-antiquark and $c$-quark. $P(Q)$ are the total four-momenta of mesons $B_c^+$ and $B_c^{*-}$, relative quark four-momenta $p = L_P(0, p)$ and $q = L_P(0, q)$ are obtained from the rest frame four-momenta $(0, p)$ and $(0, q)$ by the Lorentz transformation to the system moving with the momenta $P$ and $Q$. It can be noted that a good choice of coefficients in the formulas (1) leads to the orthogonality of the total and relative four-momenta. In turn, the virtual photon momenta
\( k_{1,2} \) can also be expressed through \( P, Q, p, q \) in the form:

\[
    k_1^2 = (p_1 + q_1)^2 = (\eta_1 P + \rho_1 Q + p + q)^2, \quad k_2^2 = (p_2 + q_2)^2 = (\eta_2 P + \rho_2 Q - p - q)^2,
\]

and the virtuality of each photon is large. We note that in our approach, quarks are not in an intermediate state on the mass shell, since after their production there is always an interaction between them, and there is a symmetric escape of particles beyond the mass shell: \( p_{1,2}^2 = \eta_{1,2}^2 P^2 - p^2 = \eta_{1,2}^2 M_{B_{c\bar{c}}}^2 - p^2 \neq m_{1,2}^2, \) \( p_1^2 - m_1^2 = p_2^2 - m_2^2. \) Since we are discussing the creation of a pair of S-wave states, in the case of a pair of pseudoscalar or pair of vector mesons \( \eta_{1,2} = \rho_{1,2} \) and at the production of pseudoscalar and vector mesons \( \eta_{1,2} \approx \rho_{1,2} \) with good accuracy. Indeed, the numerical values of these coefficients can be obtained by choosing the masses of the mesons \( M_P = 6.2749 \text{ GeV} \) \[31\], \( M_V = 6.332 \text{ GeV} \) \[32\]: \( \eta_1 = 0.228, \rho_1 = 0.233, \eta_2 = 0.772, \rho_2 = 0.767 \) (hereinafter, the indices denote pseudoscalar \( \mathcal{P} \) and vector \( \mathcal{V} \) states.).

\[
    \mathcal{T}_1(p, q, P, Q) = \frac{16\pi^2 Q_c Q_b}{k_1^2 k_2^2} \bar{u}(p_+) \gamma^\alpha (\hat{p}_- - \hat{k}_1 + m_e) \gamma^\beta u(p_-) \frac{\gamma^\alpha v_1^i(q_1) \gamma^\beta v_2^j(q_2) \gamma^\alpha \gamma^\beta}{(p_- - k_1)^2 - m_e^2},
\]

where \( p_-, p_+ \) are four momenta of electron and positron, \( u_{1,2}^i, v_{1,2}^j \) are wave functions of quarks and anti-quarks playing the role of projection operators on positive energy states. Accounting for color part of the meson wave function \( (\delta_{ij}/\sqrt{3}) \) we obtain total color factor in the amplitude \[3\] equal to 1. In quasipotential approach we can express the amplitude of the reaction \( e^+ + e^- \rightarrow B_{c\bar{c}}^+(B_{c\bar{c}}^{*+}) + B_{\bar{c}c}(B_{\bar{c}c}^{*-}) \) as a convolution of \( \mathcal{T}_1(p, q, P, Q) \) and quasipotential wave functions of produced quark bound states:

\[
    \mathcal{M}_1(p_-, p_+, P, Q) = \int \frac{dP}{(2\pi)^3} \bar{\Psi}_P(p, P) \int \frac{dq}{(2\pi)^3} \bar{\Psi}_Q(q, Q) \mathcal{T}_1(p, q, P, Q).
\]
function transformation looks as follows:

$$\Psi_P^{\rho\omega}(p) = D_{1/2}^{1/2, \omega}(R_{L_p}^W) D_{2}^{1/2, \omega}(R_{L_p}^W) \Psi_0^{\alpha\beta}(p)$$

(5)

$$\bar{\Psi}_P^{\lambda\sigma}(p) = \bar{\Psi}_0^{\epsilon\tau}(p) D_{1}^{+ 1/2, \epsilon}(R_{L_p}^W) D_{2}^{+ 1/2, \epsilon}(R_{L_p}^W),$$

(6)

where $R^W$ is the Wigner rotation, $L_p$ is the Lorentz rest frame to a moving one, and the rotation matrix $D^{1/2}(R)$ is defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_{1/2}^{1/2}(R_{L_p}^W) = S^{-1}(p_{1,2}) S(P) S(p),$$

(7)

where the explicit form for the Lorentz transformation matrix of the four-spinor is

$$S(p) = \sqrt{\frac{\epsilon(p) + m}{2m}} \left( 1 + \frac{\alpha p}{\epsilon(p) + m} \right).$$

Further transformations of the amplitude $\Psi$ can be carried out by means of the following relations:

$$S_{\alpha\beta}(\Lambda) u_{\alpha}^{\lambda}(p) = \sum_{\sigma = \pm 1/2} u_{\sigma}^{\alpha}(\Lambda p) D_{\sigma\lambda}^{1/2}(R_{\Lambda p}),$$

(8)

$$\bar{u}_{\beta}^{\alpha}(p) S_{\beta\alpha}(\Lambda) = \sum_{\sigma = \pm 1/2} D_{\alpha\sigma}^{+ 1/2}(R_{\Lambda p}) \bar{u}_{\sigma}^{\beta}(\Lambda p).$$

Using also the transformation property of the Dirac bispinors to the rest frame

$$\bar{u}_{1}(p) = \bar{u}_{1}(0) \frac{(p'_{1} + m_1)}{\sqrt{2\epsilon_1(p)(\epsilon_1(p) + m_1)}}, \quad p'_{1} = (\epsilon_1, p),$$

$$v_{2}(-p) = \frac{(p'_{2} - m_2)}{\sqrt{2\epsilon_2(p)(\epsilon_2(p) + m_2)}} v_{2}(0), \quad p'_{2} = (\epsilon_2, -p),$$

(9)

we can introduce the projection operators $\hat{\Pi}^{P,V}$ onto the states $(\bar{c}b)$, $(b\bar{c})$ with total spin 0 and 1 as follows:

$$\hat{\Pi}^{P,V} = [v_{2}(0) \bar{u}_{1}(0)]_{s=0,1} = \gamma_5(\epsilon^*) \frac{1 + \gamma^0}{2\sqrt{2}}.$$

(10)

As a result of such transformations the total amplitude of pair $B_c$ meson production can be written in the form:

$$\mathcal{M}(p_{-}, p_{+}, P, Q) = \int \frac{dp}{(2\pi)^3} \int \frac{dq}{(2\pi)^3} \frac{4\pi^2\alpha^2 Q_{c} Q_{b}}{k^2_1 k^2_2} \sqrt{M_{B_{c}} M_{B_{c}}^*} \bar{v}(p_{+}) \left[ \gamma^{\alpha}(\hat{p}_{-} - \hat{k}_{1} + m_{c}) \gamma^{\beta} \right] +$$

$$\gamma^{\beta}(\hat{p}_{-} - \hat{k}_{2} + m_{c}) \gamma^{\alpha} \frac{(p_{-} - k_{2})^{2}}{(p_{-} - k_{2})^{2} - m^{2}_{c}} u(p_{-}) S p \left\{ \Psi^{P}_{B_{c}}(p, P) \gamma^{\beta} \Psi^{V}_{B_{c}}(q, Q) \gamma^{\alpha} \right\},$$

(11)

where $s$ is the center-of-mass energy, a superscript $P$ indicates a pseudoscalar $B_c$ meson, a superscript $V$ indicates a vector $B_c$ meson, $\alpha$ is the fine structure constant. The transition
wave functions \( \Psi_{B_{bc}}^V(q, Q) \) and \( \Psi_{B_{bc}}^P(p, P) \) (form factors) have the following form:

\[
\Psi_{B_{bc}}^P(p, P) = \frac{\Psi_{B_{bc}}^0(p)}{\sqrt{\frac{\epsilon_1(p)(\epsilon_1(p)+m_1)}{m_1} \frac{\epsilon_2(p)(\epsilon_2(p)+m_2)}{m_2}}} \left[ \hat{v}_1 + \frac{1}{2} + \hat{v}_1 \frac{\text{c}}{2m_1(\epsilon_1(p)+m_1)} + \hat{p} \frac{\text{c}}{2m_1} \right],
\]

\[
\Psi_{B_{bc}}^V(q, Q) = \frac{\Psi_{B_{bc}}^0(q)}{\sqrt{\frac{\epsilon_1(q)(\epsilon_1(q)+m_1)}{m_1} \frac{\epsilon_2(q)(\epsilon_2(q)+m_2)}{m_2}}} \left[ \hat{v}_2 + \frac{1}{2} + \hat{v}_2 \frac{\text{c}}{2m_2(\epsilon_2(q)+m_2)} - \hat{q} \frac{\text{c}}{2m_2} \right],
\]

where the symbol hat denotes convolution of four-vector with the Dirac gamma matrices, \( v_1 = P/M_{B_{bc}}, v_2 = Q/M_{B_{bc}} \); \( \epsilon_V(Q, S_c) \) is the polarization vector of the \( B_{c}^*(-1^+) \) meson, relativistic quark energies \( \epsilon_{1,2}(p) = \sqrt{p^2 + m_{1,2}^2} \) and \( m_{1,2} \) are the masses of \( c \) and \( b \) quarks.

In (11) we have complicated factor including the bound state wave function in the rest frame. Therefore instead of the substitutions \( M_{B_{bc}} = \epsilon_1(p) + \epsilon_2(p) \) and \( M_{B_{bc}}^* = \epsilon_1(q) + \epsilon_2(q) \) in the production amplitude we carry out the integration over the quark relative momenta \( p \) and \( q \). Relativistic wave functions in (12) and (13) are equal to the product of wave functions in the rest frame \( \Psi_{B_{bc}}^0(p) \) and spin projection operators that are accurate at all orders in \( |p|/m \). An expression of spin projector in different form for \((c\bar{c})\) system was obtained in [35, 36] where spin projectors are written in terms of heavy quark momenta \( p_{1,2} \) lying on the mass shell. We can consider (12) and (13) as a transition form factors for heavy quark-antiquark pair from free state to bound state.

Relative momenta of heavy quarks \( p \) and \( q \) enter in the propagators of photons and electron in intermediate state as well as in transition wave functions (12) and (13). The ratios of relative momenta to \( s \) or quark masses are small so we can use an expansion of all quantities depending on \( p \) and \( q \). In the decomposition of the corresponding factors, we take into account the terms of the second order in \( p \) and \( q \) in the form:

\[
\frac{1}{k_{1,2}^2} = \frac{1}{\eta_{1,2}^2 s^2} \left[ 1 \mp \frac{2(pQ + qP)}{\eta_{1,2}^2 s^2} \mp \frac{(p + q)^2}{\eta_{1,2}^2 s^2} + \cdots \right],
\]

\[
\frac{1}{(k_{1,2} - p)^2 - m_c^2} = \frac{1}{\eta_1 \eta_2 s^2} \left[ 1 \mp \frac{2(pQ + qP)}{\eta_{2,1} s^2} \pm \frac{2p - (p + q)}{\eta_{1} \eta_2 s^2} \mp \frac{(p + q)^2}{\eta_{1} \eta_2 s^2} + \cdots \right].
\]

The coefficients \( \eta_{1,2} \) include also bound state effects which we express in terms of bound state energies setting \( M_P = m_1 + m_2 + B_p, M_V = m_1 + m_2 + B_V \). Despite expansions (14)-(15) with respect to relative momenta, the integrals in (11) remain convergent when taking corrections of second-order in \( p \) or \( q \) and can be calculated if the \( \Psi_{B_{bc}}^0(p) \) functions are known. It is convenient to perform further transformations using the Form package [37] immediately when calculating the cross section for the production of a pair.

When calculating the differential cross section, we introduce the angle \( \theta \) between the electron momentum \( p_e \) and momentum \( P \) of \( B_c \) meson. After all the simplifications and the
explicit separation of relativistic corrections of the second order and the bound state effects, we obtain the differential cross section $d\sigma/d\cos\theta$ ($z = \cos\theta$) as a function of center-of-mass energy $s$ with a number of parameters containing quark masses, binding energies, and relativistic corrections. The differential cross sections for the production of pair pseudoscalar mesons, pair vector mesons and pair pseudoscalar plus vector mesons can be written in the following form:

$$\frac{d\sigma_{B_{bc},B_{bc}}}{dz} = \frac{8\pi^3 \alpha_q^2 M_{B_{bc}}, M_{B_{bc}} |P|^4}{f_{11, 21}^0 s_{15}} |\Psi^0_{B_{bc}}(0)|^2 |\Psi_{B_{bc}}(0)|^2 \left[ f_{B_{bc},B_{bc}}^{\text{LO}}(z, s) + \frac{(B_{B_{bc}} + B_{B_{bc}})}{2M} f_{B_{bc},B_{bc}}^{\text{bind}}(z, s) + \frac{1}{\sqrt{2}} f_{B_{bc},B_{bc}}^{1,\text{rel}}(z, s) + \frac{1}{2} f_{B_{bc},B_{bc}}^{2,\text{rel}}(z, s) + \omega f_{B_{bc},B_{bc}}^{3,\text{rel}}(z, s) \right],$$

where $B_{B_{bc}}$ is the binding energy, $|P| = \sqrt{s^2 - (M_{B_{bc}} + M_{B_{bc}})^2} [s^2 - (M_{B_{bc}} - M_{B_{bc}})^2]/4s^2$ is the meson three momentum in center-of-mass frame, $r_{1,2} = m_{1,2}/(m_1 + m_2) = m_{1,2}/M$.

The value of bound state wave function at the origin is equal

$$\Psi^0_{B_{bc}}(0) = \int \sqrt{\frac{(\epsilon_1(p) + m_1)(\epsilon_2(p) + m_2)}{2\epsilon_1(p) \cdot 2\epsilon_2(p)}} \Psi^0_{B_{bc}}(p) \frac{dP}{(2\pi)^3}. \quad (17)$$

In the integral function (17), we have identified the relativistic factor that usually arises in the relativistic quark model. Explicit analytical expressions for the functions $f_{B_{bc},B_{bc}}^{1,\text{rel}}(z, s), f_{B_{bc},B_{bc}}^{\text{bind}}(z, s), f_{B_{bc},B_{bc}}^{2,\text{rel}}(z, s)$ are presented in Appendix A. At the first stage of transformations of the production cross sections in the Forms package, we decompose the integrand function in powers of relativistic factors $C_{nk} = [(m_1 - \epsilon_1(p))/(m_1 + \epsilon_1(p))]^n[(m_2 - \epsilon_2(q))/(m_2 + \epsilon_2(q))]^k$, where $n, k$ are integers and half-integers with $n + k \leq 1$. To preserve the symmetry of the expression on the quark masses we make following substitution in some expansion terms: $p^2/4m_1m_2 \approx \sqrt{(\epsilon_1 - m_1)(\epsilon_2 - m_2)/(\epsilon_1 + m_1)(\epsilon_2 + m_2)}[1 + (\epsilon_1 - m_1)/(\epsilon_1 + m_1) + (\epsilon_2 - m_2)/(\epsilon_2 + m_2) + ...]$. On the second stage after simplifications based on the symmetry properties of integral functions when replacing $p \rightarrow q$ we extract in the cross sections specific relativistic parameters $\omega_{nk}$. These parameters can be expressed in terms of momentum integrals $I_{nk}$ and calculated in the quark model:

$$I_{nk}^{B_{bc},B_{bc}} = \int_0^\infty q^2 R_{B_{bc},B_{bc}}(q) \sqrt{\frac{(\epsilon_1(q) + m_1)(\epsilon_2(q) + m_2)}{2\epsilon_1(q) \cdot 2\epsilon_2(q)}} \left( \frac{\epsilon_1(q) - m_1}{\epsilon_1(q) + m_1} \right)^n \left( \frac{\epsilon_2(q) - m_2}{\epsilon_2(q) + m_2} \right)^k dq, \quad (18)$$

$$\omega_{10}^{B_{bc},B_{bc}} = \frac{I_{10}^{B_{bc},B_{bc}}}{I_{00}^{B_{bc},B_{bc}}}, \quad \omega_{01}^{B_{bc},B_{bc}} = \frac{I_{01}^{B_{bc},B_{bc}}}{I_{00}^{B_{bc},B_{bc}}}, \quad \omega_{11}^{B_{bc},B_{bc}} = \frac{I_{11}^{B_{bc},B_{bc}}}{I_{00}^{B_{bc},B_{bc}}}, \quad (19)$$

where $R_{B_{bc},B_{bc}}(q)$ is the radial wave function of the mesons $B_{bc},B_{bc}$ in momentum space. The expansion in (17) can extend and take into account terms of higher order in $p$ and $q$. In the process of obtaining the necessary integrand functions we use different substitutions for $p^2$ and $q^2$. All of them can be obtained using the following expansion

$$|p| = 2m_1 \left[ \frac{\epsilon_1 - m_1}{\epsilon_1 + m_1} + \left( \frac{\epsilon_1 - m_1}{\epsilon_1 + m_1} \right)^{3/2} + \left( \frac{\epsilon_1 - m_1}{\epsilon_1 + m_1} \right)^{5/2} + ... \right]. \quad (20)$$
\[ 2m_2 \left[ \sqrt{\frac{\epsilon_2 - m_2}{\epsilon_2 + m_2}} + \left(\frac{\epsilon_2 - m_2}{\epsilon_2 + m_2}\right)^{3/2} + \left(\frac{\epsilon_2 - m_2}{\epsilon_2 + m_2}\right)^{5/2} + \cdots \right], \quad (21) \]

which allows you to save, if necessary, the symmetry of the particles.

There are two groups of relativistic corrections to the production cross sections \[ (16) \] connected with relative quark momenta \( p \) and \( q \). First group is connected with different relativistic factors in the production amplitude \[ (11) \] containing relative momenta of heavy quarks \( p \) and \( q \). They are presented in cross section \[ (16) \] in terms of parameters \( \omega_{nk} \). It is important to emphasize that all these parameters can be calculated numerically within the quark model itself. In this sense, the proposed approach is self-consistent. Despite the convergence of the integrals \[ (18) \] determining the relativistic parameters, when calculating them, we introduce an additional cutoff for relativistic momenta near the mass of \( c \)-quark \( m_c \). The reason for this is that in the field of such relativistic momenta, the wave function is already very small and is not defined accurately enough, since its calculation uses the non-relativistic Shrödinger equation. Relativistic corrections of second group are determined by bound state wave functions of pseudoscalar and vector \( B_c \) mesons \( \Psi_{B_c}^0 (q) \). For their calculation with the account of relativistic corrections we use corresponding QCD generalization of the Breit Hamiltonian in the center-of-mass reference frame \[ (38)-(43) \]:
for $\alpha_s$. The typical momentum transfer scale in a quarkonium is of order of double reduced mass, so we set the renormalization scale $\mu = 2m_1m_2/(m_1 + m_2)$ and $\Lambda = 0.168$ GeV, which gives $\alpha_s = 0.265$ for $(\bar{b}c)$ meson. The coefficients $b_i$ are written explicitly in [17]. The parameters of the linear confinement potential $A = 0.18$ GeV$^2$ and $B = -0.16$ GeV were obtained in quark models and lattice calculations [48–51].

TABLE I: Numerical values of relativistic parameters (19) in pair $B_c$ meson production cross section (16).

| $B_{\bar{c}c}$ meson | $n^{2S+1}L_J$ | $M_{B_{\bar{c}c}}$, GeV | $\psi_{B_{\bar{c}c}}^0(0)$, GeV$^{3/2}$ | $\omega_{10}$ | $\omega_{01}$ | $\omega_{\frac{1}{2} \frac{1}{2}}$ |
|---------------------|--------------|-----------------|----------------|-------------|-----------|----------------|
| $B_{\bar{c}c}$     | $1^1S_0$     | 6.276           | 0.250          | 0.0489      | 0.0060    | 0.0171       |
| $B_{\bar{c}c}^*$   | $1^3S_1$     | 6.317           | 0.211          | 0.0540      | 0.0066    | 0.0188       |

TABLE II: Nonrelativistic and relativistic production cross sections of $B_c$ mesons.

| Final state | Center-of-mass energy $s$ | Nonrelativistic cross section $\sigma_{nr}$ | Relativistic cross section $\sigma_r$ |
|-------------|---------------------------|---------------------------------------------|-------------------------------------|
| $B_{\bar{c}c} + B_{\bar{c}c}$ | 22.0 GeV | $0.03 \times 10^{-3}$ fb | $0.02 \times 10^{-3}$ fb |
| $B_{\bar{c}c}^+ + B_{\bar{c}c}^-$ | 22.0 GeV | $0.18 \times 10^{-3}$ fb | $0.11 \times 10^{-3}$ fb |
| $B_{\bar{c}c}^+ + B_{\bar{c}c}^-$ | 22.0 GeV | $1.43 \times 10^{-3}$ fb | $0.46 \times 10^{-3}$ fb |

To calculate relativistic corrections of second group to the pseudoscalar and vector $B_c$-meson wave functions $\Psi_{B_{\bar{c}c}}^0(p)$ we take the Breit potential (22) and construct the effective potential model as in [28, 52] by means of the rationalization of kinetic energy operator. Numerical values of relativistic parameters entering the cross section (19) are calculated on the basis of (18) and by means of numerical solution of the Schrödinger equation [53, 54] and presented in Table I. On the basis of this model we calculate a number of observed quantities such as the masses of charmonium, bottomonium and $B_c$ mesons and compare the results with existing experimental data and other theoretical predictions. The obtained results are in good agreement with them (the accuracy amounts to about one percent). For example, in the case of low lying $(\bar{b}c)$ mesons we obtain $M(1^1S_0) = 6.276$ GeV and $M(1^3S_1) = 6.317$ GeV (compare with experimental value $M(1^1S_0) = 6.2749$ GeV [31] and $M(1^3S_1) = 6.332$ GeV [32]). The similar situation occurs for charmonium states [28]. Our estimates of the charmonium massess agree with experimental data with more than a per cent accuracy [28, 31]. Our nonrelativistic values of $B_c$ meson wave functions at the origin differs on 20 per cent from the values presented in [24, 48, 51]. Taking the values of parameters of the $B_c$ mesons, we calculate the production cross sections as functions of center-of-mass energy $s$. The plots of total production cross sections of pair $B_c$ mesons are presented in Fig. 2. In Table II we give numerical values of total production cross sections at $s = 22$ GeV and compare them with nonrelativistic result in our quark model. The effect of relativistic corrections to the bound state wave functions (the Breit potential) plays a key role in total decreasing of the production cross sections as compared with nonrelativistic results. The
FIG. 2: The cross section $\sigma$ in $10^{-3} fb$ of $e^+e^-$ annihilation into a pair of pseudoscalar and vector $B_c$ meson states as a function of the center-of-mass energy $s$ in GeV (solid line). The dashed line shows nonrelativistic result without bound state and relativistic corrections.

decreasing of nonrelativistic values of cross sections in our model with regard to relativistic corrections ranges from 40 to 70 percent.

III. NUMERICAL RESULTS AND CONCLUSION

The studies in many previous papers [8–20] have convincingly shown that in such reactions as the production of bound states of heavy quarks, relativistic corrections should be taken into account in order to obtain more accurate values of the observed production cross sections. Therefore, in this work we investigate not only another mechanism for the production of a pair of $B_c$-mesons, but also carried out an account of relativistic effects in the production cross section. We develop our formalism, which was used in previous work [22], in case of two-photon processes. In this case, the general structure of the relativistic amplitudes of the pair $B_c$ meson production remains the same, but the vertex functions change. The important role of relativistic effects in exclusive processes of $B_c$ pair production from two-photon electron-positron annihilation is confirmed in this case. It is useful to say once again that when constructing the relativistic production amplitude (4) we keep two types of relativistic corrections which act in different directions. The first type corrections can be qualified as relativistic corrections to the amplitude connected with the relative quark momenta $p$ and $q$. The corrections of second type appear from the perturbative and nonperturbative treatment of the quark-antiquark interaction operator which leads to the
modification of the quark bound state wave functions \( \Psi_{B_c}(p) \) as compared with nonrelativistic case. We also systematically account for the bound state corrections working with masses of \( B_c \) mesons. The calculated masses of \( B_c \) mesons agree well with previous theoretical results and experimental data \([24, 31, 48, 51]\). Note that the quark model, which we have used in the calculations is based on quantum chromodynamics and has certain characteristics of universality.

Total cross sections for the exclusive pair production of pseudoscalar and vector \( B_c \) mesons in \( e^+e^- \) annihilation can be obtained from (16) after angular integration in the form:

\[
\sigma_{PP} = \frac{32\pi^3\alpha^4q^2q_b^2M_P^2|P|M^4}{45r_1^6r_2^6s^{15}}|\Psi_P^0(0)|^2|\Psi_V^0(0)|^2\left[48s^4 - 24s^6 + 3s^8 - \frac{s}{M}\right]
\]

12\( \omega_{\frac{1}{2}^+} \left( 16s^4 - 8s^6 + \frac{s}{M} \right) + 4\omega_0(320s^2 - 144s^4 + 12s^6 + \frac{s}{M}) - 12\omega_10(64s^2 - 80s^4 + 28s^6 - \frac{s}{M}) + \]

\[
\tilde{B}_P \tilde{B}_V \left( -\frac{75r_2s^4}{2r_1} + \frac{75r_1s^4}{2r_2} + \frac{3s^4}{r_1r_2} + \frac{309r_2s^2}{2r_1} + \frac{309r_1s^2}{2r_2} + \frac{12s^2}{r_1r_2} - \frac{288r_2}{r_1} - \frac{288r_1}{r_2} + 12\omega_{\frac{3}{2}^+} - 309s^2 + 309s^2 \right)
\]

\[
\sigma_{PV} = \frac{32\pi^3\alpha^4q^2q_b^2M_P^2|P|M^4}{45r_1^6r_2^6s^{15}}|\Psi_P^0(0)|^2|\Psi_V^0(0)|^2\left[3s(24 - 96r_1 + 96r_2^2) + \frac{s}{M}\left( 16s^4 - 32s^6 - 8s^8 \right) + \omega_0(704 - 4288s_1 + 4352r_2^2) + \right.
\]

\[
\tilde{B}_P \tilde{B}_V \left( 272 - 864r_1 + 768r_1^2 \right) + \frac{s^2(8 + 4r_1 - 8r_1^2)}{M} + \omega_0(3s(-256 - 320r_1 + 1280r_2^2) + \right.
\]

\[
\tilde{B}_P \tilde{B}_V \left( 304 - 1184r_1 + 1152r_1^2 \right) + \frac{s^2(36 - 116r_1 + 88r_1^2)}{M} - 12\left( \tilde{B}_P + \tilde{B}_V \right)(r_1 - r_2)^2s^4(s^2 + 6)
\]

\[
\sigma_{VV} = \frac{32\pi^3\alpha^4q^2q_b^2M_P^2|P|M^4}{45r_1^6r_2^6s^{15}}|\Psi_P^0(0)|^2|\Psi_V^0(0)|^2\left[144s^4 + 168s^6 + 39s^8 + \right.
\]

\[
\omega_{\frac{3}{2}^+} \left( -3008s^4 - 1376s^6 + 52s^8 \right) + \omega_0(3840s^2 + 6272 - 320r_1^2) + \frac{s^2(2064 - 160r_1^2)}{M} + 52s^8 + \right.
\]

\[
\omega_{\frac{5}{2}^+} \left( -2304s^2 + s^4(640 - 320r_2^2) + 468s^6 \right) + \frac{s^2(2192 - 160r_2^2)}{M} + 12\tilde{B}_V(12 - 41s^2 - 13s^4)s^4 + \right.
\]

where \( \tilde{s} = s/M = s/(m_1 + m_2) \).

The plots of total cross sections for the production of two pseudoscalar, pseudoscalar and vector and two vector \( B_c \) mesons as functions of center-of-mass energy \( s \) are presented in Fig. 2. A comparison of numerical results of the two-photon mechanism of electron-positron annihilation with the one-photon mechanism shows that the two-photon contribution to the total cross sections for the production of a pair is suppressed by a small factor \( 10^{-3} \), which is associated with the ratio of interaction constants \( \alpha^2/\alpha_s^2 \) and qualitatively corresponds to the results for \( D^+D^- \) \([55]\). There are no factors that could lead to an increase in the cross sections for two-photon annihilation compared with one-photon annihilation. At the production of \( B_c \) mesons, there is only a recombination mechanism shown in Fig. 1 while the fragmentation mechanism of the production is absent in contrast to the production of a pair of charmonia. The developed technique will be applied to the \( e^+e^- \rightarrow B_cB_c e^+e^- \) process, which is not suppressed by the propagators, and for which it can be expected that its contribution will be comparable to the contribution of one-photon annihilation, as shown for pair production \( D^+D^- \) in \([56]\). Note also that we are discussing a comparison of the relativistic and nonrelativistic cross sections in the tail section of the plots in Fig. 2 in which the \( M^2/s^2 \) corrections of higher order are strongly suppressed. Accounting for the
interference effects from the amplitudes of one-photon and two-photon annihilation gives terms of order $\alpha^2 \alpha_s^2$ in the differential cross section, which essentially has the same order of smallness as the radiative corrections of order $O(\alpha_s)$ to the amplitudes of one-photon annihilation. But since such terms are proportional to odd powers of $z = \cos \theta$ in the differential cross section, their contribution to the total cross section is zero.

From the results presented in Fig. 2, it follows that an account of relativistic and bound state corrections decreases the values of nonrelativistic production cross sections. We would like to emphasize that the term nonrelativistic limit means that $M_{Bc} = m_1 + m_2$, and the wave function of the bound state of quarks was determined by solving the Schrödinger equation with a purely nonrelativistic Hamiltonian. In the resulting expressions for relativistic cross sections, there are various relativistic factors that affect the change in the original non-relativistic section in different ways. The greatest reduction in cross sections is given by the value of bound state wave function at the origin. Relativistic effects in the production amplitude, which are determined by the parameters $\omega_{10}$, $\omega_{01}$, $\omega_{11}^{\parallel}$ give an increase in the cross sections by several tens of percent. It is useful to recall that effects of order $O(\alpha_s)$ also increase the values of the cross sections.

As is well known [4, 57], calculations similar to those given in this paper contain a number of theoretical uncertainties. Our calculation of production cross sections of a $B_c$ meson pair via two-photon annihilation mechanism is based on relativistic quark model, which can be considered as a microscopic theory of quark-gluon and photon interactions. The quark model allows you to perform the calculation of observables and different parameters describing the formation of bound states of heavy quarks can be found within its framework. This property of the quark model represents its obvious advantage. In this work we take into account corrections of the second order in relative momenta $p$ and $q$. The used calculation method can be extended to include fourth-order corrections in accordance with $p$ and $q$. The arising new parameters can also be estimated within the framework of the quark model itself. But in this calculation, these corrections are included in the theoretical calculation error, which we define at 30 percent. Another important theoretical calculation uncertainty is associated with the determination of the wave functions of the bound states of quarks in the region of relativistic momenta $m_c$. We estimate the total error of this type at 5%. Then the corresponding error in determining the cross sections (15) for the production of a pair should not exceed 20%. In our opinion, this rather approximate estimate is consistent with calculations of the mass spectrum of heavy quarkonia, in which a large error in determining the wave functions in the field of relativistic momenta will give a discrepancy with the observed masses of more than one percent. An important part of the total theoretical error is connected with radiative corrections of order $\alpha_s$, which are not considered in this work. It can be said that the use of the Breit Hamiltonian in the calculation of the wave functions leads only to a partial account of corrections of this type. We assume that such radiative corrections may give a 20 percent change in the production cross sections. The total maximum theoretical error can be estimated in 40%. To get it, we add the above estimates in quadrature.

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Appendix A: The coefficient functions $f^{i,rel}$, $f^{bind}$ and $f^{LO}$ entering in the $B_c$ meson production cross section (16)

\[ e^+ + e^- \rightarrow B^+_c + B^-_c. \]

\[ f^{LO} = 16s^4z^2(1 - z^2) - 8s^6z^2(1 - z^2) + s^8z^2(1 - z^2), \]  
(A1)

\[ f^{1,rel} = -64s^4z^2(1 - z^2) + 32s^6z^2(1 - z^2) - 4s^8z^2(1 - z^2), \]  
(A2)

\[ f^{2,rel} = \frac{1280}{3} s^2z^2(1 - z^2) - 192s^4z^2(1 - z^2) + 6s^6z^2(1 - z^2) + \frac{4}{3} s^8z^2(1 - z^2), \]  
(A3)

\[ f^{3,rel} = -256s^2z^2(1 - z^2) + 320s^4z^2(1 - z^2) - 112s^6z^2(1 - z^2) + 128s^8z^2(1 - z^2), \]  
(A4)

\[ f^{bind} = \frac{s^4}{r_{12}} z^2(-96(r_1 - r_2)^2 + 4s^2 + 44(r_1 - r_2)z^2 - s^4 - 5(r_1 - r_2)^2s^4 + 96(r_1 - r_2)^2z^2 - 4s^2z^2 - 39(r_1 - r_2)^2s^2z^2 + s^4z^2)], \]  
(A5)

\[ e^+ + e^- \rightarrow B^{++}_c + B^{--}_c. \]

\[ f^{LO} = (1 - 4r_1 + 4r_1^2)[4s^6 + s^8 + 4s^6z^2 - s^8z^2], \]  
(A6)

\[ f^{1,rel} = \frac{256}{3} s^4 - \frac{8}{3} s^8 + \frac{128}{3} s^4z^2 - \frac{64}{3} s^6z^2 + \frac{8}{3} s^8z^2, \]  
(A7)

\[ f^{2,rel} = \frac{1}{3} s^4(256 - 1984r_1 + 2176r_1^2) + \frac{1}{3} s^6(224 - 800r_1 + 704r_1^2) + \frac{4}{3} s^8(2 + r_1 - 2r_1^2) + (A8) \]

\[ \frac{1}{3} s^4(640 - 2624r_1 + 2176r_1^2)z^2 + \frac{32}{3} s^6(-4 + 21r_1 - 18r_1^2)z^2 - \frac{4}{3} s^8(2 + r_1 - r_1^2)z^2, \]  
(A9)

\[ f^{3,rel} = \frac{64}{3} s^4(-1 - 5r_1 + 10r_1^2) + \frac{1}{3} s^6(128 - 608r_1 + 704r_1^2) + \frac{4}{3} s^8(9 - 29r_1 + 22r_1^2) + \frac{320}{3} s^4(-1 + r_1 + 2r_1^2)z^2 + \frac{32}{3} s^6(7 - 17r_1 + 6r_1^2)z^2 + \frac{4}{3} s^8(-9 + 29r_1 - 22r_1^2)z^2, \]

\[ f^{bind} = [-2(3 + 3z^2 + s^2(1 - z^2))](r_1 - r_2)^2s^4]. \]  
(A10)

\[ e^+ + e^- \rightarrow B^{++}_c + B^{--}_c. \]

\[ f^{LO} = 48s^4(1 - z^2)z^2 + 8s^6(1 - 2z^2 + 3z^4) + s^8(2 + z^2 - 3z^4), \]  
(A11)

\[ f^{1,rel} = \frac{64}{3} s^4(-10 + 13z^2 - 3z^4) + \frac{32}{3} s^6(-4 - 7z^2 + 3z^4) + \frac{4}{3} s^8(2 + z^2 - 3z^4), \]  
(A12)

\[ f^{2,rel} = 1280s^2z^2(1 - z^2) + \frac{1}{3} s^4(896 - 128r_1^{-1}z^2 - 1216z^2 + 1728z^4) + \frac{16}{3} s^6(18 - 2r_1^{-1} + 2r_1^{-1}z^2 + 3z^2 - 9z^4) + \frac{4}{3} s^8(2 + z^2 - 3z^4), \]  
(A13)
\[ f^{3,\text{rel}} = -768 \tilde{s}^2 z^2 (1 - z^2) + \frac{1}{3} \tilde{s}^4 (-128 - 128 r_2^{-1} z^2 + 2368 z^2 - 2880 z^4) + \]
\[ + \frac{1}{3} \tilde{s}^6 (288 - 32 r_2^{-1} + 32 r_2^{-1} z^2 - 592 z^2 + 1008 z^4) + 12 \tilde{s}^8 (2 + z^2 - 3 z^4), \]
\[ f^{\text{bind}} = \left[ (-16 \tilde{s}^2 - 8 \tilde{s}^4 + (-96 + 40 \tilde{s}^2 - 4 \tilde{s}^4) z^2 + (192 - 96 \tilde{s}^2 + 12 \tilde{s}^4) z^4) \right] \tilde{s}^4. \]
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