AFFINE REPRESENTATIONS OF FRACTIONAL PROCESSES  
WITH APPLICATIONS IN MATHEMATICAL FINANCE

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ABSTRACT. Fractional processes have gained popularity in financial modeling  
due to the dependence structure of their increments and the roughness of their  
sample paths. The non-Markovianity of these processes gives, however, rise  
to conceptual and practical difficulties in computation and calibration. To  
address these issues, we show that a certain class of fractional processes can  
be represented as linear functionals of an infinite dimensional affine process.  
We demonstrate by means of several examples that the affine structure allows  
one to construct tractable financial models with fractional features.

1. INTRODUCTION

Empirical evidence suggests that certain financial time series may not be captured  
well by low-dimensional Markovian models. In particular, this applies to short-term  
interest rates, which tend to have long-range dependence [1], and to volatilities of  
stock prices, which have rough sample paths and behave essentially as fractional  
Brownian motion with small Hurst index [10]. Dependent increments and rough  
sample paths are, however, characteristic features of fractional processes.  
The wide-spread adoption of fractional processes in financial modeling was impeded  
by several difficulties. Conceptually, one of the major challenges is the lack of the  
Markov property. In the absence of the Markov property, it is unclear what the  
states of the model are. This makes it difficult to talk about calibration in a sensible  
way and to compare the model across time. Moreover, PDE methods for option  
pricing cannot be used.

In this paper we introduce a class of fractional processes which can be represented  
as linear functionals of an infinite-dimensional affine process. The key idea, which  
goes back to Carmona and Coutin [4], is to express the fractional integral in the  
Mandelbrot-Van Ness representation of fractional Brownian motion by a Laplace  
transform: for each $H < 1/2$, by the stochastic Fubini’s theorem,

$$
\int_0^t (t - s)^{H - \frac{1}{2}} dW_s \propto \int_0^t \int_0^\infty e^{-x(t-s)} \frac{dx}{x^{H + \frac{1}{2}}} dW_s = \int_0^\infty \int_0^t e^{-x(t-s)} dW_s \frac{dx}{x^{H + \frac{1}{2}}}.
$$

The right-hand side is a superposition of infinitely many Ornstein-Uhlenbeck (OU)  
processes with varying speed of mean reversion. Extensions and numerical approximations  
of this representation can be found in Carmona, Coutin, and Montseny [5],  
Muravlev [10]. We show that the collection of OU processes, indexed by the speed of  
mean reversion, is a Banach-space valued affine process. Linear functionals of this  

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process are in general not semimartingales. Instead, they are fractional processes with positively or negatively correlated increments and are closely related to fractional Brownian motion. More precisely, fractional Brownian motion is obtained by randomizing the initial condition of the OU processes according to the stationary distribution.

Our result is relevant in mathematical finance for the following reasons. First, it is within reach to solve some simple fractional models where the affine structure is preserved. We demonstrate this by means of several examples in this paper. In particular, we construct interest rate models where either the short rate or the bank account process is modeled by a fractional process. In contrast to [18] and [2], we build the model such that discounted zero-coupon bond prices are martingales. This implies absence of arbitrage by construction, while certain quantities of the model such as the short rate may very well be non-semimartingales. We also build a fractional version of the stochastic volatility model by Stein and Stein [21].

Second, there is recently a high interest in non-affine fractional volatility models such as the fractional Bergomi and SABR models [15, 10]. It is a major challenge to derive short-time, large-time, and wing asymptotics for these models, as well as to develop numerical schemes for pricing and calibration. Hopefully, the Markovian point of view and the affine structure will be helpful for achieving these goals.

Third, the Markovian structure is useful for characterizing the behavior of fractional Brownian motion after a stopping time. This is crucial for characterizing arbitrage opportunities in models with fractional price processes (c.f. the stickiness property in [12, 6] and the notion of arbitrage times in [19]).

The paper is structured as follows. In Section 2 we prove that the collection of OU processes is indeed a Banach-space valued affine process. In Section 3 we deduce the affine representation of fractional Brownian motion. Section 4 is dedicated to applications in interest rate modeling and Section 5 to a fractional version of the stochastic volatility model of Stein and Stein [21].

2. Infinite-dimensional Ornstein-Uhlenbeck process

2.1. Setup and notation. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{Q})\) be a filtered probability space satisfying the usual conditions and let \(W\) be two-sided \((\mathcal{F}_t)\)-Brownian motion on \(\Omega\). The probability measure \(\mathbb{Q}\) plays the role of a risk-neutral measure.

**Definition 2.1 (OU processes).** Given a collection of \(\mathcal{F}_0\)-measurable \(\mathbb{R}\)-valued random variables \(Y^x_0, Z^x_0\) indexed by \(x \in (0, \infty)\), let for each \(t \geq 0\)

\[
Y^x_t = Y^x_0 e^{-tx} + \int_0^t e^{-(t-s)x} dW_s,
\]

\[
Z^x_t = Z^x_0 e^{-tx} + \int_0^t e^{-(t-s)x} Y^x_s ds.
\]

**Remark 2.2.** For each \(x \in (0, \infty)\), the process \((Y^x_t, Z^x_t)_{t \geq 0}\) solves the SDE

\[
dY^x_t = -xY^x_t dt + dW_t, \quad dZ^x_t = (-xZ^x_t + Y^x_t) dt.
\]

Therefore, it is a bi-variate OU process, and the variable \(x\) is related to the speed of mean reversion of the process (see Lemma D.1 in the Appendix for details).
2. Ornstein-Uhlenbeck process in $L^1$. Let $Y_t = (Y^x_t)_{x>0}$ and $Z_t = (Z^x_t)_{x>0}$ denote the collection of OU processes indexed by the speed of mean reversion $x$. We show in this section that the process $(Y_t, Z_t)_{t \geq 0}$ takes values in $L^1(\mu) \times L^1(\nu)$, where the measures $\mu$ and $\nu$ are subject to the following conditions.

**Assumption 2.3** (Integrability condition). $\mu$ and $\nu$ are sigma-finite measures on $(0, \infty)$ such that $\nu$ has a density $p$ with respect to $\mu$ and for each $t > 0$,

$$
\int_0^\infty (1 \wedge x^{-\frac{1}{2}}) \mu(dx) < \infty, \quad \int_0^\infty (1 \wedge x^{-\frac{1}{2}}) \nu(dx) < \infty, \quad \sup_{x \in (0, \infty)} p(x) e^{-tx} < \infty.
$$

The pairing between $L^1(\mu)$ and $L^\infty(\mu)$ is denoted by $(\cdot, \cdot)_\mu$, and similarly for $L^1(\nu)$ and $L^\infty(\nu)$. The complexification of these spaces is denoted by $L^1(\mu; \mathbb{C})$, etc.

**Theorem 2.4** (OU process in $L^1$). Let $\mu, \nu$ satisfy Assumption 2.3 and let $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$. Then the process $(Y_t, Z_t)_{t \geq 0}$ has a predictable $L^1(\mu) \times L^1(\nu)$-valued version and is Gaussian.

**Remark 2.5.** Carmona and Coutin [4] show the weaker statement that for each fixed $t \geq 0$, the random variable $Y_t$ lies a.s. in $L^1(\mu)$.

Proof. It is shown in Lemma D.1 that for each $x \in (0, \infty)$ the process $(Y^x_t, Z^x_t)_{t \geq 0}$ can be represented as

$$
Y^x_t = Y^x_0 e^{-tx} + \int_0^t e^{-(t-s)x} dW_s, \quad Z^x_t = Z^x_0 e^{-tx} + Y^x_0 e^{-tx} + \int_0^t (t-s)e^{-(t-s)x} dW_s.
$$

By Assumption 2.3, the deterministic parts in the above representation are $L^1(\mu)$- and $L^1(\nu)$-valued functions, respectively. Therefore, we can assume without loss of generality that $Y_0$ and $Z_0$ are zero.

In Lemma D.2 it is shown that for each fixed $t \geq 0$, $(Y_t, Z_t) \in L^1(\mu) \times L^1(\nu)$ holds almost surely. Moreover, for any $(u, v) \in L^\infty(\mu) \times L^\infty(\nu)$, the random variables $(Y_t, u)_\mu$ and $(Z_t, v)_\mu$ are centered Gaussian, as shown in Lemma D.3. Let $P_t \colon L^\infty(\mu) \to L^1(\mu)$ and $Q_t \colon L^\infty(\mu) \to L^1(\nu)$ be the associated covariance operators, which are calculated explicitly in Lemma D.4.

We now show that $Y_t$ is a version of an $L^1(\mu)$-valued stochastic convolution. To this aim, let $H_t \subseteq L^1(\mu)$ be the reproducing kernel Hilbert space of $P_t$ (see Appendix B). The inclusion of $H_t$ in $L^1(\mu)$ is $\gamma$-radonifying because $Y_t$ provides an instance of a Gaussian random variable with covariance operator $P_t$ [17, Theorem 7.4]. For each $s > 0$ define $\Theta_1(s) \in L^1(\mu)$ and $\Theta^*_1(s) : L^\infty(\mu) \to \mathbb{R}$ by

$$
\Theta_1(s)(x) = e^{-sx}, \quad \Theta^*_1(s)(u) = (\Theta_1(s), u)_\mu.
$$

Then $\Theta^*_1$ satisfies for each $t \geq 0$ and any $u \in L^\infty(\mu)$

$$
\int_0^t (\Theta^*_1(t-s)(u))^2 ds = \int_0^t \left( \int_0^\infty e^{-x(t-s)} u(x) \mu(dx) \right)^2 ds = (P_t u, u)_\mu < \infty,
$$

where the order of integration can be exchanged because condition (A.1) is satisfied by Equation (C.22). By [3, Theorem 3.3], the bound on $\Theta^*_1$ and the $\gamma$-radonifying property of the inclusion of $H_t$ in $L^1(\mu)$ imply that the stochastic convolution of $\Theta_1$
with \( W \) exists as an \( L^1(\mu) \)-valued \((\mathcal{F}_t)_{t \geq 0}\)-predictable process \( \tilde{Y} \) such that for each \( t \geq 0 \) and any \( u \in L^\infty(\mu) \),

\[
\langle \tilde{Y}_t, u \rangle_\mu = \int_0^t \Theta^*_1(t-s)(u)dW_s
\]

holds almost surely. The same equation is also satisfied by \( Y_t \). As stochastic convolutions are unique up to modifications \([3, \text{ Theorem 3.3}]\), \( \mathbb{Q}(Y_t = \tilde{Y}_t) = 1 \) holds for each \( t \geq 0 \). This proves that \( Y \) has a predictable, \( L^1(\mu) \)-valued version.

We use the same argument to show that \( Z \) has a predictable, \( L^1(\nu) \)-valued version. For each \( s > 0 \) define \( \Theta_2(s) \in L^1(\nu) \) and \( \Theta^*_2(s) : L^\infty(\nu) \to \mathbb{R} \) by

\[
\Theta_2(s)(x) = se^{-sx}, \quad \Theta^*_2(s)(v) = \langle \Theta_2(s), v \rangle_\nu.
\]

Then \( \Theta^*_2 \) satisfies for each \( t \geq 0 \) and any \( v \in L^\infty(\nu) \)

\[
\int_0^t (\Theta^*_2(t-s)(v))^2 \, ds = \int_0^t \left( \int_0^\infty (t-s)e^{-z(t-s)}v(x)\nu(dx) \right)^2 \, ds = \langle Q_t v, v \rangle_\nu < \infty,
\]

where the order of integration can be exchanged because condition \([\text{A.1}]\) is satisfied by \( \text{Equation (C.23)} \). By the same argument as above there exists an \( L^1(\nu) \)-valued \((\mathcal{F}_t)_{t \geq 0}\)-predictable process \( \tilde{Z} \) such that for each \( t \geq 0 \) and any \( v \in L^\infty(\nu) \),

\[
\langle \tilde{Z}_t, v \rangle_\nu = \int_0^t \Theta^*_2(t-s)(u)dW_s,
\]

holds almost surely. As \( Z \) satisfies the same equation and stochastic convolutions are unique up to modifications \([3, \text{ Theorem 3.3}]\), \( \tilde{Z} \) is a version of \( Z \). \( \square \)

2.3. Affine structure. We derive an infinite-dimensional affine transformation formula for the conditional exponential moments of \( \langle Y, u \rangle_\mu \) and \( \langle Z, v \rangle_\nu \) for test functions \( u \in L^\infty(\mu; \mathbb{C}) \) and \( v \in L^\infty(\nu; \mathbb{C}) \).

**Theorem 2.6** (Affine structure). Let \( \mu, \nu \) satisfy Assumption 2.3 and let \( (Y_0, Z_0) \in L^1(\mu) \times L^1(\nu) \). Then the process \((Y, Z)\) is affine in the sense that for each \( 0 \leq t \leq T \) and \((u, v) \in L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C}) \), the relation

\[
\mathbb{E} \left[ \langle Y_T, u \rangle_\mu + \langle Z_T, v \rangle_\nu \bigg| \mathcal{F}_T \right] = e^{\phi_0(T-t, u, v) + \phi_1(T-t, u, v)_\mu + \phi_2(T-t, u, v)_\nu}
\]

holds with probability one, where the functions

\[
(\phi_0, \phi_1, \phi_2) : [0, \infty) \times L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C}) \to \mathbb{C} \times L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C})
\]

are given by

\[
\begin{align*}
\phi_0(\tau, u, v) &= \frac{1}{2} \int_0^\tau \left( \int_0^\infty \phi_1(s, u, v)(x)\mu(dx) \right)^2 \, ds, \\
\phi_1(\tau, u, v)(x) &= e^{-\tau x} (u(x) + \tau v(x)p(x)), \\
\phi_2(\tau, u, v)(x) &= e^{-\tau x} v(x).
\end{align*}
\]

**Proof.** Lemma D.3 states that for each \( 0 \leq t \leq T \), the random variable \( \langle Y_T, u \rangle_\mu + \langle Z_T, v \rangle_\nu \) is Gaussian, given \( \mathcal{F}_t \), with mean

\[
\begin{align*}
\int_0^\infty Y_t^x e^{-(T-t)x}u(x)\mu(dx) + \int_0^{\infty} \left( Z_t^x e^{-x(T-t)} + Y_t^x(T-t)e^{-x(T-t)} \right) v(x)\nu(dx) \\
= \langle Y_t, \phi_1(T-t, u, v) \rangle_\mu + \langle Z_t, \phi_2(T-t, u, v) \rangle_\nu.
\end{align*}
\]
By Itô’s isometry, the conditional variance of \( \langle Y_T, u \rangle_\mu + \langle Z_T, v \rangle_\nu \) given \( F_t \) is

\[
\int_t^T \left( \int_0^\infty e^{-(T-s)x}u(x)\mu(dx) + \int_0^\infty (T-s)e^{-x(T-s)}v(x)\nu(dx) \right)^2 ds,
\]

which equals \( 2\phi_0(T-t, u, v) \). Thus,

\[
\mathbb{E} \left[ e^{\langle Y_T, u \rangle_\mu + \langle Z_T, v \rangle_\nu} \bigg| F_t \right] = e^{\frac{1}{2} \text{Var} \left( \langle Y_T, u \rangle_\mu + \langle Z_T, v \rangle_\nu \big| F_t \right)} + \mathbb{E} \left[ (Y_T, u)_{\mu} + (Z_T, v)_{\nu} \right] \bigg| F_t \right]
\]

\[
= e^{\phi_0(T-t, u, v) + \langle Y_t, \phi_1(T-t, u, v) \rangle_\mu + \langle Z_t, \phi_2(T-t, u, v) \rangle_\nu}.
\]

The coefficient functions \( (\phi_0, \phi_1, \phi_2) \) are solutions of an infinite-dimensional system of Riccati equations. To formulate the equations, we need to introduce some topology. We endow the spaces \( L^\infty(\mu; \mathbb{C}) \) and \( L^\infty(\nu; \mathbb{C}) \) with the weak-star topology. Then they are locally convex separable Hausdorff vector spaces. In particular, differentiability of curves with values in these spaces is well-defined.

**Definition 2.7** (Riccati equations). Mappings \( \phi_0, \phi_1, \phi_2 \) as in (2.6) are called solutions of the Riccati equations if they are continuous in \( t \) on the interval \([0, \infty)\), differentiable in \( t \) on the interval \((0, \infty)\), and satisfy

\[
\begin{align*}
\partial_t \phi_0(\phi_1, \phi_2)(\tau, u, v) &= (R_0, R_1, R_2)(\phi_1(\tau, u, v), \phi_2(\tau, u, v)), \\
(\phi_0, \phi_1, \phi_2)(0, u, v) &= (0, u, v),
\end{align*}
\]

where the mappings

\[
(R_0, R_1, R_2): L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C}) \to \mathbb{C} \times L^0(\mu; \mathbb{C}) \times L^0(\nu; \mathbb{C})
\]

are given by

\[
\begin{align*}
R_0(u, v) &= \frac{1}{2} \left( \int_0^\infty u(x)\mu(dx) \right)^2, \\
R_1(u, v)(x) &= -xu(x) + p(x)v(x), \\
R_2(u, v)(x) &= -xv(x).
\end{align*}
\]

**Lemma 2.8** (Riccati equations). The functions \( (\phi_0, \phi_1, \phi_2) \) defined in Equation (2.7) are the unique solution of the Riccati equations (2.8).

**Proof.** It is straightforward to verify that the functions \( (\phi_0, \phi_1, \phi_2) \) given by Equation (2.7) solve the Riccati equations in the sense of Definition 2.7. Let \( (\phi_0, \phi_1, \phi_2) \) be any other solution. Then \( e^{xt}(\phi_2 - \phi_2) \) has vanishing derivative and initial condition, implying that it is constant and \( \phi_2 = \phi_2 \). The same applies to \( e^{xt}(\phi_1 - \phi_1) \), showing that \( \phi_1 = \phi_1 \), and to \( \phi_0 - \phi_0 \), showing that \( \phi_0 = \phi_0 \).

2.4. **Continuity of sample paths.** Under the following conditions on the measures \( \mu \) and \( \nu \), the process \( (Y, Z) \) has continuous sample paths in \( L^1(\mu) \times L^1(\nu) \) with respect to the norm topology.

**Assumption 2.9** (Integrability condition). \( \mu \) and \( \nu \) are sigma-finite measures on \((0, \infty)\) satisfying

\[
\int_0^\infty \log(1 + tx)x^{-\frac{3}{2}}\mu(dx) < \infty, \quad \int_0^\infty \log(1 + tx)x^{-\frac{3}{2}}\nu(dx) < \infty.
\]

Moreover, \( \nu \) has a density \( p \) with respect to \( \mu \), such that for each \( t > 0 \)

\[
\sup_{x \in (0, \infty)} p(x)e^{-tx} < \infty.
\]
Remark 2.10. Compared to Assumption 2.3, Assumption 2.9 is weaker near zero and stronger near infinity, as can be seen from the limits
\[ \forall t > 0: \lim_{x \to 0^+} \frac{\log(1 + tx)x^{-\frac{3}{2}}}{1 \wedge x^{-\frac{3}{2}}} = 0, \quad \lim_{x \to \infty} \frac{\log(1 + tx)x^{-\frac{3}{2}}}{1 \wedge x^{-\frac{3}{2}}} = \infty. \]

Theorem 2.11 (Continuity of sample paths). Under Assumption 2.9, the process \((Y, Z)\) has continuous sample paths in \(L^1(\mu) \times L^1(\nu)\); this follows from Theorem 2.4 under Assumption 2.3.

Proof. The expressions \(Y_0^x e^{-tx}\) and \(Z_0^x e^{-tx} + Y_0^x t e^{-tx}\) define continuous \(L^1(\mu)\)- and \(L^1(\nu)\)-valued functions, respectively. Thus, it follows from the representation of \((Y, Z)\) in Equation (2.3) that we may assume \((Y_0, Z_0) = 0\) without loss of generality.

By Lemma D.5 and Assumption 2.9 on \(\mu\), integration with respect to \(\mu\) yields
\[ \mathbb{E} \left[ \sup_{s \in [0, t]} |Y_s^x| \mu(dx) \right] \leq C \int_0^\infty \log(1 + tx)x^{-\frac{3}{2}} \mu(dx) < \infty, \]
where we are allowed to exchange the order of integration since the integrand is positive. This implies that \(Q[\forall t: Y_t \in L^1(\mu)] = 1\). Moreover, by the dominated convergence theorem with the sup process of \(Y\) as majorant, \(Q[Y \in C([0, \infty); L^1(\mu))] = 1\). For the process \(Z\), the estimate of Lemma D.5 and Assumption 2.9 on \(\nu\) show that \(Q[\forall t: Z_t^x \in L^1(\nu)] = 1\). As before, the dominated convergence theorem with the sup process of \(Z\) as majorant implies \(Q[Z \in C([0, \infty); L^1(\nu))] = 1\).

2.5. Semimartingale property. In this section we investigate under which conditions linear functionals of the process \((Y, Z)\) are semimartingales. We consider time-dependent linear functionals as this will be needed later in applications.

Theorem 2.13 (Semimartingale property). Let Assumption 2.3 be in place. Let \(f_t^x\) and \(g_t^x\) be real-valued, deterministic, jointly measurable in \((x, t) \in (0, \infty) \times [0, \infty)\), differentiable in \(t\) and satisfy
\[ \forall t \geq 0: \|f_t\|_{L^\infty(\mu)} < \infty \text{ and } \|g_t\|_{L^\infty(\nu)} < \infty. \]

Assume \((Y_0, Z_0) \in L^1(\mu) \times L^1(\nu), \text{ a.s., and for each } t \geq 0\)
\begin{align*}
\int_0^\infty \int_0^t |\partial_s f_s^x - x f_s^x| (1 \wedge x^{-\frac{3}{2}}) ds \mu(dx) &< \infty, \\
\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds} \mu(dx) &< \infty, \\
\int_0^\infty \int_0^t |\partial_s g_s^x - x g_s^x| (1 \wedge x^{-\frac{3}{2}}) ds \nu(dx) &< \infty, \\
\int_0^\infty \int_0^t |g_s^x| (1 \wedge x^{-\frac{3}{2}}) ds \nu(dx) &< \infty.
\end{align*}
Then \((Y_t, f_t)_{t \geq 0}\) and \((Z_t, g_t)_{t \geq 0}\) are semimartingales with decompositions

\[
\langle Y_t, f_t \rangle = \langle Y_0, f_0 \rangle + \int_0^t \int_0^\infty (\partial_s f_s^x - x f_s^x) Y_s^x \mu(dx) ds + \int_0^t \int_0^\infty f_s^x \mu(dx) dW_s,
\]

\[
\langle Z_t, g_t \rangle = \langle Z_0, g_0 \rangle + \int_0^t \int_0^\infty (\partial_s g_s^x - x g_s^x) Z_s^x \nu(dx) ds + \int_0^t \int_0^\infty g_s^x Z_s^x \nu(dx) ds.
\]

(2.13)

Proof. First observe that

\[
\langle Y_t, f_t \rangle = \langle Y_t - Y_0 e^{-xt}, f_t \rangle + \langle Y_0 e^{-xt}, f_t \rangle,
\]

\[
\langle Z_t, g_t \rangle = \langle Z_t - Z_0 e^{-xt} - Y_0 te^{-xt}, g_t \rangle + \langle Z_0 e^{-xt}, g_t \rangle + \langle Y_0 e^{-xt}, g_t \rangle.
\]

Since \((Y_0 e^{-xt}, f_t)\), \((Z_0 e^{-xt}, g_t)\) and \((Y_0 e^{-xt}, g_t)\) are finite variation processes we assume without loss of generality that \(Y_0 = Z_0 = 0\). By SDE (2.2) for \((Y, Z)\) and Itô’s formula, the semimartingale decomposition of the process \((f_t Y_t^x, g_t Z_t^x)\) is given by

\[
f_t^x Y_t^x = \int_0^t (\partial_s f_s^x - x f_s^x) Y_s^x ds + \int_0^t f_s^x dW_s,
\]

\[
g_t^x Z_t^x = \int_0^t (\partial_s g_s^x - x g_s^x) Z_s^x ds + \int_0^t g_s^x Y_s^x ds.
\]

Therefore,

\[
\langle Y_t, f_t \rangle = \int_0^\infty \int_0^t (\partial_s f_s^x - x f_s^x) Y_s^x ds \mu(dx) + \int_0^t \int_0^\infty f_s^x dW_s \mu(dx),
\]

\[
\langle Z_t, g_t \rangle = \int_0^\infty \int_0^t (\partial_s g_s^x - x g_s^x) Z_s^x ds \nu(dx) + \int_0^t \int_0^\infty g_s^x Y_s^x ds \nu(dx).
\]

By Theorem A.1 one obtains the semimartingale decompositions of \(\langle Y_t, f_t \rangle\) and \(\langle Z_t, g_t \rangle\). By Lemma D.6 and Equations (2.9)-(2.12) conditions (A.1) and (A.2) are satisfied.

2.6. Stationary distribution. We show that the stationary distribution of \((Y, Z)\) is in general not a Gaussian distribution on \(L^1(\mu) \times L^1(\nu)\), but only on a larger space \(L^1(\mu_\infty) \times L^1(\nu_\infty)\) corresponding to stronger integrability conditions on the measures \(\mu_\infty\) and \(\nu_\infty\).

Assumption 2.14 (Integrability condition). \(\mu_\infty, \nu_\infty\) are sigma-finite measures on \((0, \infty)\) such that \(\nu_\infty\) has a density \(p_\infty\) with respect to \(\mu_\infty\) and

\[
\int_0^{\infty} x^{-1/2} \mu_\infty(dx) < \infty, \quad \int_0^{\infty} x^{-3/2} \nu_\infty(dx) < \infty, \quad \sup_{x \in (0, \infty)} p_\infty(x) e^{-tx} < \infty.
\]

Remark 2.15. Assumption 2.14 is more stringent than Assumption 2.3. The difference is the decay of the measures near zero: \(\mu, \nu\) satisfy Assumption 2.3 if and only if the measures

\[
\mu_\infty(dx) = (1 \wedge x^{1/2}) \mu(dx), \quad \nu_\infty(dx) = (1 \wedge x^{1/2}) \nu(dx)
\]

satisfy Assumption 2.14. In this case, \(L^1(\mu) \times L^1(\nu) \subset L^1(\mu_\infty) \times L^1(\nu_\infty)\).
Theorem 2.16 (Stationary distribution). The random variables \( Y_\infty = (Y_\infty^x)_{x>0} \) and \( Z_\infty = (Z_\infty^x)_{x>0} \) defined by

\[
(2.15) \quad Y_\infty^x = \int_{-\infty}^{0} e^{sx}dW_s, \quad Z_\infty^x = -\int_{-\infty}^{0} se^{sx}dW_s
\]

are normally distributed on \( L^1(\mu_\infty) \times L^1(\nu_\infty) \). Their distribution is stationary in the sense that \( (Y_t, Z_t) \) is equal in distribution to \( (Y_\infty, Z_\infty) \) if \( (Y_0, Z_0) \) is equal in distribution to \( (Y_\infty, Z_\infty) \).

Proof. \( (Y_\infty, Z_\infty) \in L^1(\mu_\infty) \times L^1(\nu_\infty) \) holds almost surely because

\[
\mathbb{E} \left[ \|Y_\infty\|_{L^1(\mu_\infty)} \right] = \int_0^\infty \mathbb{E} \left[ \left( \int_{-\infty}^{0} e^{sx}dW_s \right)^2 \right] \mu_\infty(dx) = \int_0^\infty \sqrt{\frac{1}{\pi x}} \mu_\infty(dx) < \infty,
\]

\[
\mathbb{E} \left[ \|Z_\infty\|_{L^1(\nu_\infty)} \right] = \int_0^\infty \mathbb{E} \left[ \left( \int_{-\infty}^{0} se^{sx}dW_s \right)^2 \right] \nu_\infty(dx) = \int_0^\infty \sqrt{\frac{1}{2\pi x^3}} \nu_\infty(dx) < \infty.
\]

For each \( u, v \in L^\infty(\mu_\infty) \times L^\infty(\nu_\infty) \), the random variable \( \langle Y_\infty, u \rangle_{\mu_\infty} + \langle Z_\infty, v \rangle_{\nu_\infty} \) can be expressed by Fubini (Theorem A.1) as

\[
\langle Y_\infty, u \rangle_{\mu_\infty} + \langle Z_\infty, v \rangle_{\nu_\infty} = \int_{-\infty}^{0} \int_{-\infty}^{\infty} e^{sx}u(x)\mu_\infty(dx)dW_s + \int_{-\infty}^{0} \int_{-\infty}^{\infty} se^{sx}v(x)\nu_\infty(dx)dW_s.
\]

Condition [A.2] of Fubini’s theorem is satisfied because

\[
\int_{-\infty}^{\infty} \sqrt{\int_{-\infty}^{0} e^{2sx}u(x)^2ds} \mu_\infty(dx) \leq \|u\|_{L^\infty(\mu_\infty)} \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\pi x}} \mu_\infty(dx) < \infty,
\]

\[
\int_{-\infty}^{\infty} \sqrt{\int_{-\infty}^{0} s^2e^{2sx}v(x)^2ds} \nu_\infty(dx) \leq \|v\|_{L^\infty(\nu_\infty)} \int_{-\infty}^{\infty} \sqrt{\frac{1}{4\pi x^3}} \nu_\infty(dx) < \infty.
\]

Therefore, \( \langle Y_\infty, u \rangle_{\mu_\infty} + \langle Z_\infty, v \rangle_{\nu_\infty} \) is a centered Gaussian random variable on \( L^1(\mu_\infty) \times L^1(\nu_\infty) \). To show that the distribution of \( (Y_\infty, Z_\infty) \) is stationary, let us assume that \( (Y_0, Z_0) = (Y_\infty, Z_\infty) \). Then Lemma D.1 implies

\[
Y_t^x = \int_{-\infty}^{t} e^{-(t-s)x}dW_s, \quad Z_t^x = \int_{-\infty}^{t} (t-s)e^{-(t-s)x}dW_s,
\]

which is equal in distribution to \( Y_\infty \) and \( Z_\infty \), respectively. \( \square \)

Theorem 2.17 (Convergence to the stationary distribution). For any initial condition \( (Y_0, Z_0) \in L^1(\mu_\infty) \times L^1(\nu_\infty) \) and any \( t \geq 0 \), we consider \( (Y_t, Z_t) \) as a random variable with values in the space \( L^1(\mu_\infty) \times L^1(\nu_\infty) \), which we endow with the weak topology. Then \( (Y_t, Z_t) \) converges in distribution to \( (Y_\infty, Z_\infty) \) as \( t \to \infty \).

Proof. Let \( u, v \in L^\infty(\mu_\infty) \times L^\infty(\nu_\infty) \). By Equation (2.16) and Itô’s isometry the variance of the centered Gaussian random variable \( \langle Y_\infty, u \rangle_{\mu_\infty} + \langle Z_\infty, v \rangle_{\nu_\infty} \) is

\[
\mathbb{E} \left[ (\langle Y_\infty, u \rangle_{\mu_\infty} + \langle Z_\infty, v \rangle_{\nu_\infty})^2 \right] = \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} e^{sx}(u(x) + sv(x)p_\infty(x))\mu_\infty(dx) \right)^2 ds.
\]
Assume for a moment that \((Y_0, Z_0) = 0\). As the measures \(\mu_\infty\) and \(\nu_\infty\) satisfy the conditions of Theorem 2.6

\[
\lim_{t \to \infty} E \left[ e^{(Y_t, u)_{\mu_\infty} + (Z_t, v)_{\nu_\infty}} \right] = \lim_{t \to \infty} e^{\phi_0(t, u, v) + (Y_0, \phi_1(t, u, v))_{\mu_\infty} + (Z_0, \phi_2(t, u, v))_{\nu_\infty}}
\]

\[
= e^{\frac{1}{2} \int_0^\infty (\int_0^\infty e^{-tx}(u(x) + sv(x)p_\infty(x))\mu_\infty(dx))^2\,ds}
\]

\[
= e^{\frac{1}{2} \text{Var}((Y_t, u)_{\mu_\infty} + (Z_t, v)_{\nu_\infty})}
\]

\[
= E \left[ e^{(Y_\infty, u)_{\mu_\infty} + (Z_\infty, v)_{\nu_\infty}} \right].
\]

This shows point-wise convergence of the characteristic functions of \((Y_t, Z_t)\) to the characteristic functions of \((Y_\infty, Z_\infty)\). By Lemma D.7 the laws of the random variables \((Y_t, Z_t)\) are tight on the space \(L^1(\mu_\infty) \times L^1(\nu_\infty)\) with the weak topology. It follows that \((Y_t, Z_t)\) converges in distribution on \(L^1(\mu_\infty) \times L^1(\nu_\infty)\) to \((Y_\infty, Z_\infty)\) (see e.g. [3] Theorem 9).

To account for arbitrary initial conditions \((Y_0, Z_0) \in L^1(\mu_\infty) \times L^1(\nu_\infty)\), we need to add the deterministic functions \(Y_0^x e^{-tx}\) and \(Z_0^x e^{-tx} + Y_0^x te^{-tx}\) to the processes \(Y^x\) and \(Z^x\) considered above (see Lemma D.1). For \(t \to \infty\), these functions converge to zero in the corresponding \(L^1\) spaces. It follows that convergence in distribution to \((Y_\infty, Z_\infty)\) holds regardless of the initial condition.

2.7. Ornstein-Uhlenbeck process with values in \(L^2\). We defined \((Y, Z)\) as an \(L^1\)-valued process because the construction of fractional Brownian motion in Section 3 involves a pairing of \((Y, Z)\) with the constant function 1. Nevertheless, it is good to know that \((Y, Z)\) can also be understood as an \(L^2\)-valued process.

**Assumption 2.18** (Integrability condition). \(\mu\) and \(\nu\) are sigma-finite measures on \((0, \infty)\) such that \(\nu\) has a density \(p\) with respect to \(\mu\) and for each \(t > 0\),

\[
\int_0^\infty (1 \land x^{-1})\mu(dx) < \infty, \quad \int_0^\infty (1 \land x^{-3})\nu(dx) < \infty, \quad \sup_{x \in (0, \infty)} e^{-tx}p(x) < \infty.
\]

**Theorem 2.19** (OU process in \(L^2\)). Let \(\mu, \nu\) satisfy **Assumption 2.18** and let \((Y_0, Z_0) \in L^2(\mu) \times L^2(\nu)\). Then the process \((Y_t, Z_t)_{t \geq 0}\) has a predictable \(L^2(\mu) \times L^2(\nu)\)-valued version and is a Gaussian affine process on \(L^2(\mu) \times L^2(\nu)\).

The theorem can be proven along the lines of Theorems 2.4 and 2.6. Here we present an alternative proof, which uses the theory of Hilbert-space valued stochastic convolutions.

**Proof.** We want to construct the stochastic convolutions in Equation (2.3) as \(L^2(\mu)\)- and \(L^2(\nu)\)-valued processes, respectively. The setting of [3] Sections 5.1.1–5.1.2 does not apply directly because the integrand in the expression \(\int_0^t e^{-(t-s)x}\,dW_s\) cannot be written as a strongly continuous semigroup acting on \(L^2(\mu)\). Nevertheless, we can adapt the arguments of [3] Theorem 5.2 and Proposition 3.6 to our setting. Let

\[
S_t: \mathbb{R} \to L^2(\mu), \quad \lambda \mapsto (x \mapsto e^{-tx}\lambda).
\]
Then the $L^2(\mu)$-valued convolution $\int_0^t S_{t-s} dW_s$ exists by [2, Theorem 5.2] because

$$\int_0^t \|S_s\|_{L^2(\mu)}^2 ds = \int_0^t \|S_s 1\|_{L^2(\mu)}^2 ds = \int_0^t \int_0^\infty e^{-2sx} \mu(dx) ds$$

$$= \int_0^\infty \frac{1 - e^{-2tx}}{2x^2} \mu(dx) < \infty$$

by [Equation (C.4)] and [Assumption 2.18]. It is mean-square continuous by the same arguments as in the proof of [2, Theorem 5.2]. Therefore, it is predictable [2, Proposition 3.6]. Similarly, it can be shown that $Z$ has a predictable, $L^2(\nu)$-valued version. The affine structure can be derived as in Section 2.3.

**Assumption 2.20** (Integrability condition). $\mu$ and $\nu$ are sigma-finite measures on $(0, \infty)$ such that $\nu$ has a density $p$ with respect to $\mu$. There is $\epsilon \in (0, 1)$ such that for each $t > 0$,

$$\int_0^\infty (1 \wedge x^{-1+\epsilon}) \mu(dx) < \infty, \quad \int_0^\infty (1 \wedge x^{-3+\epsilon}) \nu(dx) < \infty, \quad \sup_{x \in (0, \infty)} e^{-tx} p(x) < \infty.$$

**Theorem 2.21** (Continuity of sample paths). Under [Assumption 2.20], the process $(Y, Z)$ has continuous sample paths in $L^2(\mu) \times L^2(\nu)$ if the initial condition $(Y_0, Z_0)$ lies in this space.

**Proof.** Let $S$ be as in the proof of Theorem 2.19. Then the estimate

$$\int_0^t s^{-\epsilon} \|S_s\|_{L^2(\mu)}^2 ds = \int_0^t \int_0^\infty s^{-\epsilon} e^{-2sx} \mu(dx) ds$$

$$\leq \int_0^\infty \left(2^{x-1} \Gamma(1 - \epsilon) \vee \frac{t^{1-\epsilon}}{1 - \epsilon}\right) (1 \wedge x^{-1}) \mu(dx) < \infty$$

holds by [Equation (C.7)] for $\epsilon \in (0, 1)$ as in Assumption 2.20. Therefore, [2, Theorem 5.11] may be applied, showing that $Y$ has continuous sample paths in $L^2(\mu)$. (While the stochastic convolution $Y$ is not covered by the setting of [2, Section 5.1.1–5.1.2], the same arguments as in the proof of Theorem 2.19 show that [2, Theorem 5.11] holds.) Similarly, it may be shown that the process $Z$ given by [Equation (2.3)] has continuous sample paths in $L^2(\nu)$.

**2.8. Smoothness in the spatial dimension.** We show in the following theorem that $(Y^x_t, Z^x_t)$ varies smoothly in $x$. To this aim, we extend [Definition 2.1] of $(Y^x_t, Z^x_t)$ to $x \leq 0$ in the obvious way. The space $C^k(\mathbb{R})$, $k \in \mathbb{N} \cup \{\infty\}$, is the Fréchet space with the topology of uniform convergence of derivatives up to order $k$ on compact sets.

**Theorem 2.22** (Smoothness in the spatial dimension). For each $k \in \mathbb{N} \cup \{\infty\}$ and initial value $(Y_0, Z_0) \in C^k(\mathbb{R}) \times C^k(\mathbb{R})$, the process $(Y, Z)$ is a Gaussian process on $C^k(\mathbb{R})^2$ with continuous sample paths.

**Proof.** The deterministic parts in [Equation (2.3)] are smooth in $t$ and $x$. We set them to zero by assuming without loss of generality that $(Y_0, Z_0) = 0$. By partial integration, the stochastic integrals in [Equation (2.3)] can be transformed into
The goal in this section is to obtain a Markovian representation of fractional Brownian motion (fBM) in terms of \((Y, Z)\). To show that \((Y_t, Z_t)_{t \geq 0}\) has continuous sample paths in \(C^\infty(\mathbb{R})\). The integrands, seen as functions of \((s, t)\), are continuous with values in \(C^\infty(\mathbb{R})\). This shows that \((Y_t, Z_t)_{t \geq 0}\) has continuous sample paths in \(C^\infty(\mathbb{R})^2\). The \(k\)-th spatial derivative, expressed as a stochastic integral, is given by

\[
\partial^k Y_t^x = \int_0^t (s-t)^k e^{-t(s-t)} dW_s, \quad \partial^k Z_t^x = -\int_0^t (s-t)^{k+1} e^{-t(s-t)} dW_s.
\]

To show that \((Y, Z)\) is a Gaussian process, it suffices to test with linear functionals on \(C^k([-K, K])\) for \(K \in \mathbb{N}\). By the Riesz representation theorem, the dual of \(C^k([-K, K])\) is \(\mathbb{R}^k \times M([-K, K])\), where \(M\) stands for the space of signed regular Borel measures endowed with the total variation norm \([3, \text{IV.13.36}]\). The pairing of \(Y_t \in C^k([-K, K])\) with an element \((m, \mu)\) of the dual space \(\mathbb{R}^k \times M([-K, K])\) reads as

\[
\langle Y_t, (m, \mu) \rangle = \sum_{j=0}^{k-1} m_j \partial^j_x |_{x=0} Y_t^x + \int_{-K}^K \partial^k_x Y_t^x \mu(dx)
\]

By the stochastic Fubini theorem (Theorem A.1), the order of the integrals in the last expression can be exchanged. The assumptions of Theorem A.1 are satisfied because \(\mu\) is a finite measure and the integrand is bounded. This shows that \(\langle Y_t, (m, \mu) \rangle\) is Gaussian. As \((m, \mu)\) was arbitrary, \(Y_t\) is Gaussian on \(C^k([-K, K])\), for each fixed \(t\). A similar argument shows that \(Z_t\) is Gaussian on the same space. \(\square\)

3. Fractional Brownian motion as a functional of a Markov process

The goal of this section is to obtain a Markovian representation of fractional Brownian motion (fBM) in terms of \((Y, Z)\). We use the representation of Mandelbrot and Van Ness \([14]\) to define fBM.

**Definition 3.1** (fBM). Fractional Brownian motion \(W^H\) with initial value \(w_0^H \in \mathbb{R}\) and Hurst index \(H \in (0, 1)\) is defined for each \(t \geq 0\) as

\[
W_t^H = w_0^H + \frac{1}{\Gamma(H+\frac{1}{2})} \int_{-\infty}^t \left( (t-s)^H - (-s)^H \right) dW_s,
\]

where \(W = (W_t)_{t \in \mathbb{R}}\) is two-sided Brownian motion as defined in Section 2.1.

**3.1. Markovian representation of fBM on \(L^1\)-spaces.**

**Definition 3.2** (Markovian representation). Let \((Y, Z)\) be the process in Definition 2.1 with initial value \((Y_0, Z_0)\) equal to the random variable \((Y_\infty, Z_\infty)\) defined in Equation [2.15]. Furthermore, let \(\mu, \nu\) be measures on \((0, \infty)\) given by

\[
\mu(dx) = \frac{dx}{x^{\frac{1}{2}+H}\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}, \quad \nu(dx) = \frac{dx}{x^{H-\frac{1}{2}}\Gamma(\frac{1}{2}+H)\Gamma(\frac{1}{2}-H)}.
\]
Remark 3.3. The measures $\mu, \nu$ in the definition above satisfy Assumption 2.3 but not Assumption 2.14. It follows by Theorem 2.11 that $(Y, Z)$ has continuous paths in $L^1(\mu_{\infty}) \times L^1(\nu_{\infty})$ with $(\mu_{\infty}, \nu_{\infty})$ as in Equation (2.14) but not necessarily in $L^1(\mu) \times L^1(\nu)$. Nevertheless, $(Y - Y_0, Z - Z_0)$ has continuous paths in $L^1(\mu) \times L^1(\nu)$, as shown in the proof of Theorem 3.4.

Theorem 3.4 (Markovian representation). Under the specification of Definition 3.2, fBM has representation

$$W^H_t = \begin{cases} w^H_0 + \int_0^\infty (Y^x - Y_0^x) \mu(dx), & \text{if } H < \frac{1}{2}, \\ w^H_0 + \int_0^\infty (Z^x - Z_0^x) \nu(dx), & \text{if } H > \frac{1}{2}, \end{cases}$$

where $(Y - Y_0, Z - Z_0)$ is a continuous process in $L^1(\mu) \times L^1(\nu)$.

Remark 3.5. The fractional integral in Definition 3.1 can be decomposed as

$$W^H_t = w^H_0 + \frac{1}{\Gamma(H + \frac{1}{2})} \int_{-\infty}^0 (t-s)^{H-\frac{1}{2}} (-s)^{H-\frac{1}{2}} dW_s,$$

$$+ \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}} dW_s.$$ (3.1)

Markovian representations of the integral $\int_0^t$ were found by Carmona and Coutin [4] for $H < 1/2$ and by Carmona, Coutin, and Montseny [3] for $H > 1/2$. Muravev [16] incorporated also the integral $\int_{-\infty}^0$ in his representation and interpreted it as a random initial value. Moreover, in contrast to [3], his representation is time-homogeneous also in the case $H > 1/2$. Our representation can be seen as a modification of [16] which allows us to identify an infinite-dimensional state space for the Markov process (c.f. Section 2).

Proof of Theorem 3.4 for $H < \frac{1}{2}$. The function $\tau \mapsto \tau^{H-\frac{1}{2}}/\Gamma(H + \frac{1}{2})$ on $(0, \infty)$ appearing in the definition of $W^H$ is the Laplace transform of $\mu$, i.e., for each $\tau > 0$ and $H < \frac{1}{2}$

$$\mathcal{L}(\mu)(\tau) = \int_0^\infty e^{-\tau x} \mu(dx) = \frac{\tau^{H-\frac{1}{2}}}{\Gamma(H + \frac{1}{2})}.$$ Therefore,

$$W^H_t = w^H_0 + \int_{-\infty}^0 \int_0^\infty (e^{-x(t-s)} - e^{-x(-s)}) \mu(dx) dW_s + \int_0^t \int_0^\infty e^{-x(t-s)} \mu(dx) dW_s.$$

By the stochastic Fubini’s theorem A.1

$$W^H_t = w^H_0 + \int_0^\infty \int_{-\infty}^0 (e^{-x(t-s)} - e^{-x(-s)}) dW_s \mu(dx) + \int_0^\infty \int_0^t e^{-x(t-s)} dW_s \mu(dx).$$
Condition [A.2] of Fubini’s theorem is satisfied because
\[
\int_0^\infty \int_{-\infty}^{\infty} (e^{-x(t-s)} - e^{-x(-s)})^2 \, ds \mu(dx) = \int_0^\infty \frac{1 - e^{-tx}}{\sqrt{2x}} \mu(dx)
\]
\[
\leq \int_0^\infty \frac{1 - e^{-tx}}{x} \mu(dx) < \infty,
\]
\[
\int_0^\infty \int_0^t e^{-2x(t-s)} \mu(dx) \leq \int_0^\infty \sqrt{\frac{1 - e^{-2tx}}{x}} \mu(dx) < \infty,
\]
where we use \(1 - e^{-tx} \leq \sqrt{1 - e^{-tx}}\) and Equation (C.12). By the definition of \(Y^x_t\),

\[
W^H_t = w^H_0 + \int_0^t (e^{-xt} - 1) Y^x_0 \mu(dx) + \int_0^t \int_0^\infty e^{-x(t-s)} dW_s \mu(dx)
\]
\[
= w^H_0 + \int_0^\infty (Y^x_t - Y^x_0) \mu(dx).
\]

The expressions
\[
(e^{-xt} - 1) Y^x_0 \quad \text{and} \quad \int_0^t e^{-x(t-s)} dW_s
\]
define continuous \(L^1(\mu)\)-valued processes: the first expression has majorant \((1 \lor t)(1 \land x) Y^x_0\) in \(L^1(\mu)\), which allows one to apply the dominated convergence theorem, and the second expression is treated in Theorem 2.11. □

Proof of Theorem 3.4 for \(H > \frac{1}{2}\). As the function \(\tau^{H-\frac{3}{2}} / \Gamma(H + \frac{3}{2})\) is the Laplace transform of the measure \(\nu\), the relation
\[
\tau \mathcal{L}(\nu)(\tau) = \tau \int_0^\infty e^{-\tau x} \nu(dx) = \frac{\tau^{H-\frac{3}{2}}}{\Gamma(H + \frac{3}{2})},
\]
holds for each \(\tau > 0\) and \(H \in (\frac{1}{2}, 1)\). Therefore,

\[
W^H_t = w^H_0 + \int_0^t \int_0^\infty (t-s)e^{-x(t-s)} + se^{x} \nu(dx)dW_s
\]
\[
+ \int_0^\infty \int_0^t (t-s)e^{-x(t-s)} \nu(dx)dW_s.
\]

By the stochastic Fubini’s theorem [A.1]

\[
W^H_t = w^H_0 + \int_0^\infty \int_0^t (t-s)e^{-x(t-s)} + se^{x} \nu(dx)dW_s \nu(dx)
\]
\[
+ \int_0^\infty \int_0^t (t-s)e^{-x(t-s)} dW_s \nu(dx).
\]
Condition [A.2] of Fubini’s theorem is satisfied because
\[
\int_0^\infty \int_{-\infty}^0 \left( (t-s)e^{-xt} + s e^{xs} \right)^2 \, d\nu(dx) \\
= \int_0^\infty \sqrt{1 - 2e^{-tx}(tx + 1) + 2txe^{-2tx}(tx + 1) + e^{-2tx}} \, d\nu(dx) \\
\leq \int_0^{1/t} \frac{\sqrt{2}}{6x} \, d\nu(dx) + \int_1^{\infty} \frac{\sqrt{2}}{x^2} \, d\nu(dx) \\
\leq \sqrt{2} (t \vee 1) \int_0^\infty (x^{-\frac{3}{2}} \vee x^{-\frac{5}{2}}) \, d\nu(dx) < \infty,
\]
where we used Equations [C.13] and [C.14]. Using the definition of \( (Y^x, Z^x) \) in Equation (3.2) can be expressed as
\[
W^H_t = u^H_0 + \int_0^t \int_{-\infty}^0 e^{xs} \left( te^{-xt} + s(1 - e^{-xt}) \right) \, dW_s \, d\nu(dx) \\
\quad + \int_0^\infty \int_0^t (t-s)e^{-xt}dW_s \, d\nu(dx) \\
= u^H_0 + \int_0^\infty \left( te^{-xt} \int_{-\infty}^0 e^{xs} \, dW_s + (1 - e^{-xt}) \int_{-\infty}^0 se^{xs} \, dW_s \right) \, d\nu(dx) \\
\quad + \int_0^\infty (Z^x_t - Z^x_0 e^{-xt} - Y^x_0 t e^{-xt}) \, d\nu(dx) \\
= u^H_0 + \int_0^\infty (Z^x_t - Z^x_0) \, d\nu(dx).
\]
By Lemma D.1, \( Z^x_t - Z^x_0 \) can be written as the sum of the following expressions:
\[
Z^x_t (e^{-xt} - 1), \quad Y^x_t e^{-tx}, \quad \int_0^t (t-s)e^{-(t-s)x} \, dW_s.
\]
All three expressions define continuous \( L^1(\nu) \)-valued processes: the first and second expression have \( |Z^x_0| (1 \vee t) (1 \wedge x) \) and \( |Y^x_0| (1 \vee t) (1 \wedge x^{-1}) \) as majorants in \( L^1(\nu) \), which allows one to apply the dominated convergence theorem, and the third expression is treated in Theorem 2.11. \( \square \)

Remark 3.6. The representation in Theorem 3.4 lends itself to numerical implementation. Indeed, the integrals can be approximated by finite sums as described in \( \mathbb{B} \). Alternatively, aiming for a more parsimonious representation, one has in the case \( H > 1/2 \)
\[
W^H_t = u^H_0 - \int_0^\infty \partial_x (Y^x_t - Y^x_0) \, d\nu(dx).
\]
This follows from the following deterministic relationship between \( Y \) and \( Z \) (c.f. Theorem 2.22)
\[
Z^x_t = -\partial_x Y^x_t + (\partial_x Y^x_0 + Z^x_0) e^{-tx}, \quad t \geq 0.
\]
Remark 3.7. The case $H = 1/2$ fits into the framework of Theorem 3.4 with $\mu$ equal to the Dirac measure. Indeed, the process $(Y^0_t - Y^0_0)_{t \geq 0}$ is Brownian motion, as can be seen from the definition of $Y$. Moreover, the choice of $\mu$ as a Dirac measure is in line with the proof of Theorem 3.4 where $\mu$ is defined as the inverse Laplace transform of the integrand in Definition 3.1. Note that the representing Markov process $Y_t \in L^1(\mu)$ is one-dimensional and can be identified with Brownian motion.

3.2. Filtrations. The filtration generated by $W^H$ is essentially the same as the one generated by $(Y,Z)$, as shown in the following lemmas. Therefore, the law of fractional Brownian motion after a stopping time can be characterized using the strong Markov property of $(Y,Z)$. This is important for understanding the existence of arbitrage opportunities in models with fractional price processes (see, e.g. the stickiness property in [12, 6] and the notion of arbitrage times in [19]).

Lemma 3.8 (Filtrations). Let $H < 1/2$. Then the completed filtrations generated by the processes $W - W_0$, $W^H - W^H_0$, and $Y - Y_0$ are equal. The same statement holds for $H > 1/2$ with $Y$ replaced by $Z$.

Proof. Let $\mathcal{N}$ denote the $\mathbb{Q}$-null sets. Then the following sigma algebras are equal for each $T \geq 0$:

$$\sigma(W_t - W_0, 0 \leq t \leq T) \vee \mathcal{N} = \sigma(W^H_t - W^H_0, 0 \leq t \leq T) \vee \mathcal{N}$$

$$\subseteq \sigma(Y_t - Y_0, 0 \leq t \leq T) \vee \mathcal{N}$$

$$\subseteq \sigma(W_t - W_0, 0 \leq t \leq T) \vee \mathcal{N}.$$  

The first equality above follows from [20, Proposition 1]. The proof for $H > 1/2$ is similar.

From a Markovian point of view, the canonical definition of fractional Brownian motion is $V^H_t = \langle Y_t, 1 \rangle_\mu$ or $V^H_t = \langle Z_t, 1 \rangle_\nu$, depending on whether $H$ is smaller or greater than $1/2$. Here the initial value $(Y_0, Z_0)$ is fixed and deterministic. Moreover, the initial value $W_0$ can be normalized to zero. Then the following lemma holds.

Lemma 3.9 (Filtrations). If $H > 1/2$, then the completed filtrations generated by the processes $W$, $V^H$, and $Y$ are equal. The same statement holds for $H > 1/2$ with $Y$ replaced by $Z$.

Proof. As before, $\mathcal{N}$ denotes the $\mathbb{Q}$-null sets. Let us assume for a moment that the initial value $(Y_0, Z_0)$ is zero. Then one has for each $T \geq 0$

$$\sigma(W_t, 0 \leq t \leq T) \vee \mathcal{N} = \sigma(V^H_t, 0 \leq t \leq T) \vee \mathcal{N}$$

$$\subseteq \sigma(Y_t, 0 \leq t \leq T) \vee \mathcal{N}$$

$$\subseteq \sigma(W_t, 0 \leq t \leq T) \vee \mathcal{N}.$$  

The first equality above follows from [21, Proposition 1] applied to a Brownian path which is set to zero for all $t \leq 0$, noting that the relevant integrals are defined pathwise. To get rid of the assumption on $(Y_0, Z_0)$, note that the process $(Y,Z)$ depends on the initial condition $(Y_0,Z_0)$ only via a deterministic function, which is $\mathcal{N}$-measurable. The proof for $H > 1/2$ is similar.
3.3. Markovian representation of fBM on $L^2$-spaces. There is also an $L^2$-version of the results of Section 3.1.

**Definition 3.10** (Markovian representation). Let $(Y, Z)$ be the process in Definition 2.1 with initial value $(Y_0, Z_0)$ equal to the random variable $(Y_\infty, Z_\infty)$ defined in Equation (2.15). Furthermore, let $f(x) = 1 \wedge x^{-1/2}$ and

$$
\mu(dx) = \frac{dx}{f(x)x^{H+\frac{1}{2}}\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}, \quad \nu(dx) = \frac{dx}{f(x)x^{H-\frac{1}{2}}\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}.
$$

**Remark 3.11.** The measures $\mu$ and $\nu$ in the definition above satisfy Assumptions 2.18 and 2.20, but $(Y_\infty, Z_\infty)$ does not take values in $L^2(\mu) \times L^2(\nu)$ (c.f. Remark 3.3). Nevertheless, the process $(Y - Y_0, Z - Z_0)$ does, as the following theorem shows.

**Theorem 3.12** (Markovian representation). Under the specification of Definition 3.10, fBM has representation

$$
W^H_t = \begin{cases}
  w_0^H + \int_0^\infty (Y^x_t - Y^x_0)f(x)\mu(dx), & \text{if } H < \frac{1}{2}, \\
  w_0^H + \int_0^\infty (Z^x_t - Z^x_0)f(x)\nu(dx), & \text{if } H > \frac{1}{2},
\end{cases}
$$

where $(Y - Y_0, Z - Z_0)$ is a continuous $L^2(\mu) \times L^2(\nu)$-valued process and $f \in L^2(\mu) \cap L^2(\nu)$.

This can be shown along the lines of the proof of Theorem 3.4.

4. Applications to interest rate modeling

In this section we construct two interest rate models: one with fractional short rate and another one with fractional bank account process. In both models, the affine structure gives rise to explicit formulas for zero-coupon bond (ZCB) prices, forward rates, and calls and puts on ZCB’s.

4.1. Essentials of interest rate modeling. We refer to [9] for further details.

**Definition 4.1** (Interest rates). The bank account is given by a positive process $B = (B_t)_{t \geq 0}$ such that $B_t^{-1}$ is integrable for all $t \geq 0$. Zero-coupon bond (ZCB) prices are given by

$$
P(t, T) = \mathbb{E} \left[ \frac{B_t}{B_T} \right]_{\mathcal{F}_t}, \quad T \geq t \geq 0,
$$

and the (instantaneous) forward rates are given by

$$
h(t)(\tau) = -\partial_T|_{T=t+\tau} \log P(t, T), \quad t, \tau \geq 0.
$$

**Remark 4.2.** Note that for each $T > 0$ the process $B^{-1}P(\cdot, T)$ is by definition a martingale. This means that $Q$ is a risk-neutral measure by construction, and that the model is free of arbitrage.

**Definition 4.3** (Call and put options). The prices at time $t \geq 0$ of European call and put options with expiry date $T < S$ and strike $K$ on the ZCB with maturity
date $S > T$ are given by
\[
\pi_t^\text{Call}(T, S, K) = \mathbb{E} \left[ \frac{B_0}{B_T} (P(T, S) - K)^+ | \mathcal{F}_t \right], \\
\pi_t^\text{Put}(T, S, K) = \mathbb{E} \left[ \frac{B_0}{B_T} (K - P(T, S))^+ | \mathcal{F}_t \right].
\]

**Definition 4.4** (Caps and floors). Consider interest rate cap and floor with maturity $T_n$, strike rate $\kappa$ and payment dates $0 < T_0 < T_1 < \ldots < T_n$ where $T_k - T_{k-1} = \Delta$, $k \in \mathbb{N}$. At time $t < T_0$ the cap and floor prices are given by
\[
\text{Cp}_t = \sum_{k=1}^{n} \text{Cp}_t(T_{k-1}, T_k, \kappa), \quad \text{and} \quad \text{Fl}_t = \sum_{k=1}^{n} \text{Fl}_t(T_{k-1}, T_k, \kappa),
\]
where
\[
\text{Cp}_t(T_{k-1}, T_k, \kappa) = (1 + \Delta \kappa) \pi_t^\text{Put}(T_{k-1}, T_k, (1 + \Delta \kappa)^{-1}), \\
\text{Fl}_t(T_{k-1}, T_k, \kappa) = (1 + \Delta \kappa) \pi_t^\text{Call}(T_{k-1}, T_k, (1 + \Delta \kappa)^{-1}).
\]
In order to calculate prices of call and put options on ZCB’s it is convenient to consider forward measure changes.

**Definition 4.5** (Forward measure). For $0 \leq t \leq T$ define the $T$-forward measure $\mathbb{Q}^T$ by the following Radon-Nikodým derivative
\[
\xi(t, T) = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \bigg|_{\mathcal{F}_t} = \mathbb{E} \left[ \frac{B_0}{P(0, T) B_T} | \mathcal{F}_t \right] = B_t^{-1} P(t, T) \mathbb{E} \left[ \frac{B_0}{B_0} P(0, T) \right],
\]
where the last equality follows directly from **Definition 4.1**.

The following property is useful for computations. The symbol $\mathcal{E}$ denotes the stochastic exponential, see e.g. [3, Section 4.1].

**Theorem 4.6** (Black-Scholes formula). Assume that there is a process $(v(t, T))_{t \geq 0}$ such that for each $0 \leq t \leq T$,
\[
\xi(t, T) = \mathcal{E} \left( \int_0^t v(s, T) dW_s \right)_t.
\]
Then, for any $S, T > 0$ the process $W^T = W - \int_0^T v(s, T) ds$ is $\mathbb{Q}^T$-Brownian motion and the price process of the ZCB with maturity date $S$ discounted by the ZCB with maturity $T$
\[
\frac{P(t, S)}{P(t, T)} = \mathbb{P} \left( \int_0^T (v(s, S) - v(s, T)) dW_s^T \right)_t, \quad t \in [0, S \wedge T],
\]
is a $\mathbb{Q}^T$-martingale. Moreover, assuming that $v(\cdot, T)$ is deterministic, call and put option prices are given by the following version of the Black-Scholes formula
\[
\pi_t^\text{Call} = P(t, S) \Phi_0^{\text{Gauss}}(d_1) - KP(t, T) \Phi_0^{\text{Gauss}}(d_2), \\
\pi_t^\text{Put} = KP(t, T) \Phi_0^{\text{Gauss}}(-d_2) - P(t, S) \Phi_0^{\text{Gauss}}(-d_1),
\]
where $\Phi_0^{\text{Gauss}}$ is the standard Gaussian cumulative distribution function and
\[
d_{1,2} = \log \left( \frac{P(t, S)}{KP(t, T)} \right) \pm \frac{1}{2} \int_0^T (v(\cdot, S) - v(\cdot, T))^2 ds \sqrt{\int_0^T (v(\cdot, S) - v(\cdot, T))^2 ds}.
\]
Proof. The derivation of [8, Section 7] can also be used in this setting because the discounted ZCB price process \( B^{-1} P(\cdot, T) \) is a martingale by construction. \( \square \)

4.2. Fractional short rate process. In this section, we construct an interest rate model with a fractional short rate. To this aim, we fix measures \( \mu, \nu \) on \((0, \infty)\) satisfying the following slightly strengthened version of Assumption 2.3.

**Assumption 4.7.** \( \mu \) and \( \nu \) are sigma-finite measures on \((0, \infty)\). The measure \( \nu \) has a density \( p \) with respect to \( \mu \), and there exists \( \beta \in (0, 2) \) such that for each \( t > 0 \),

\[
\int_0^\infty (1 \wedge x^{-\frac{1}{2}}) \mu(dx) < \infty, \quad \int_0^\infty (1 \wedge x^{-\frac{1}{2}}) \nu(dx) < \infty, \quad \sup_{x \in (0, \infty)} p(x)(1 \wedge x^{-\beta}) < \infty.
\]

Moreover, we fix \((u, v) \in L^\infty(\mu) \times L^\infty(\nu), \ell \in \mathbb{R}, \) and an initial value \((Y_0, Z_0) \in L^1(\mu) \times L^1(\nu) \) for the process \((Y, Z)\) defined in Section 2. Given these model parameters, we define the short rate and bank account as

\[
r_t = \ell + \langle Y_t, u \rangle_\mu + \langle Z_t, v \rangle_\nu, \quad B_t = \exp \left( \int_0^t r_s \, ds \right).
\]

**Example 4.8.** Set \( u = v = 1 \) and consider measures of the form \( \mu(dx) \propto x^{-\alpha}dx \) and \( \nu(dx) \propto x^{\frac{1}{2}-\alpha}dx \) for \( \alpha \in \left(\frac{1}{4}, 1\right) \). Then Assumption 4.7 is satisfied. The process \((Y, Z)\) takes values in \( L^1(\mu) \times L^1(\nu) \) and has continuous sample paths by Theorem 2.11. Therefore, it can be used to construct fractional processes as in Section 3. In particular, \((Y, u)_\mu\) is a fractional process of the same roughness as fBM with Hurst index \( H = \alpha - \frac{1}{2} \in (0, \frac{1}{2}) \), and \((Z, v)_\nu\) has the same roughness as fBM with \( H = \alpha \in (\frac{1}{2}, 1) \). None of the two processes are semimartingales.

**Remark 4.9.** While the short rate may take negative values, the probability of yields becoming negative can be reduced by shifting the parameter \( \ell \) and scaling the parameters \( u, v \). Often times, either \( u \) or \( v \) will be set to zero, unless one is interested in mixing processes with long- and short-range dependence.

**Theorem 4.10** (Bond prices and forward rates). In the fractional short rate model (4.3) ZCB prices and forward rates are given by

\[
P(t, T) = e^{-\ell(T-t)+\Phi_0(T-t,u,v)+\langle Y_t, \Phi_1(T-t,u,v) \rangle_\mu+\langle Z_t, \Phi_2(T-t,u,v) \rangle_\nu}, \quad 0 \leq t \leq T,
\]

\[
h(t, \tau) = \ell - \partial_\tau \Phi_0(\tau, u, v) - \langle Y_t, \partial_\tau \Phi_1(\tau, u, v) \rangle_\mu - \langle Z_t, \partial_\tau \Phi_2(\tau, u, v) \rangle_\nu, \quad t, \tau \geq 0,
\]

where for each \( \tau \geq 0 \) and \( x \in (0, \infty) \)

\[
\Phi_0(\tau, u, v) = \frac{1}{2} \int_0^\tau \langle \Phi_1(s, u, v) \rangle_\mu ds,
\]

\[
\Phi_1(\tau, u, v)(x) = e^{-\tau x} - \frac{1}{x} u(x) + \left( \frac{e^{-\tau x} - 1}{x^2} + x e^{-\tau x} \right) p(x)v(x),
\]

\[
\Phi_2(\tau, u, v)(x) = e^{-\tau x} - \frac{1}{x} v(x).
\]

**Proof.** Lemma E.2 implies that the random variable \( \int_t^T \left( \langle Y_s, u \rangle_\mu + \langle Z_s, v \rangle_\nu \right) ds \) is Gaussian, given \( \mathcal{F}_t \), with mean

\[
- \langle Y_t, \Phi_1(T-t, u, v) \rangle_\mu - \langle Z_t, \Phi_2(T-t, u, v) \rangle_\nu.
\]
and variance $2\Phi_0(T-t,u,v)$. Thus, the formula for ZCB prices follows from the formula of the moment generating function of the normal distribution. The expression for forward rates follows by differentiation with respect to the time to maturity.

**Remark 4.11.** The functions $\Phi_0, \Phi_1, \Phi_2$ are the unique solution of the Riccati equations

\[
\begin{align*}
\partial_t \Phi_0(\tau, u, v) &= R_0(\Phi_1(\tau, u, v), \Phi_2(\tau, u, v)), \quad \Phi_0(0, u, v) = 0, \\
\partial_t \Phi_1(\tau, u, v) &= R_1(\Phi_1(\tau, u, v), \Phi_2(\tau, u, v)) - u, \quad \Phi_1(0, u, v) = 0, \\
\partial_t \Phi_2(\tau, u, v) &= R_2(\Phi_1(\tau, u, v), \Phi_2(\tau, u, v)) - v, \quad \Phi_2(0, u, v) = 0,
\end{align*}
\]

with $R_0, R_1, R_2$ as in Lemma 2.8. Here, solutions are defined in analogy to Definition 2.7 and Lemma 2.8.

**Theorem 4.12 (HJM equation).** In the fractional short rate model (4.3) bond prices $(P(t,T))_{0 \leq t \leq T}$ and forward rates $(h(t)(\tau))_{t \geq 0}$ are semimartingales for each fixed $T, \tau > 0$. The forward rate process $h = (h(t)(\tau))_{t \geq 0}$ is a solution of the HJM equation

\[
dh(t) = (Ah(t) + \mu^\text{HJM}) \, dt + \sigma^\text{HJM} \, dW_t,
\]

where $A$ denotes differentiation with respect to time to maturity $\tau$ and $\mu^\text{HJM}, \sigma^\text{HJM}$ are measurable functions on $(0,\infty)$ given by

\[
\mu^\text{HJM}(\tau) = \partial^2_t \Phi_0(\tau, u, v), \quad \sigma^\text{HJM}(\tau) = -\langle \partial_\tau \Phi_1(\tau, u, v), 1 \rangle_{\mu}.
\]

**Proof.** The semimartingale property of prices and forward rates follows from Lemmas 2.3 and 2.4 which are based on Theorem 2.13. The semimartingale decomposition of $h(\cdot)(\tau)$ is obtained by collecting the terms in Equation (2.13)

\[
dh(t)(\tau) = -d \langle Y_i, \partial_\tau \Phi_i(\tau, u, v) \rangle_{\mu} - d \langle Z_i, \partial_\tau \Phi_2(\tau, u, v) \rangle_{\nu} = \left( \langle Z_i, x \partial_\tau \Phi_1(\tau, u, v) - \partial_\tau \Phi_2(\tau, u, v)p \rangle_{\mu} + \langle Z_i, x \partial_\tau \Phi_2(\tau, u, v) \rangle_{\nu} \right) dt - \langle \partial_\tau \Phi_1(\tau, u, v), 1 \rangle_{\mu} dW_t.
\]

Note that by abuse of notation, we wrote $x \partial_\tau \Psi_i(\tau, u, v)$ to designate the function $x \mapsto \partial_\tau \Psi_i(\tau, u, v)(x)$ for $i = 1, 2$. The second derivatives of $\Psi_i$ are

\[
\begin{align*}
\partial^2_t \Phi_0(\tau, u, v) &= -x \partial_\tau \Phi_1(\tau, u, v) + \partial_\tau \Phi_2(\tau, u, v)p, \\
\partial^2_t \Phi_2(\tau, u, v) &= -x \partial_\tau \Phi_2(\tau, u, v).
\end{align*}
\]

Therefore, we have for all $t \geq 0$ and $\tau > 0$

\[
Ah(t)(\tau) = -\partial^2_t \Phi_0(\tau, u, v) + \langle Y_i, x \partial_\tau \Phi_1(\tau, u, v) - \partial_\tau \Phi_2(\tau, u, v)p \rangle_{\mu} + \langle Z_i, x \partial_\tau \Phi_2(\tau, u, v) \rangle_{\nu}.
\]

It follows that

\[
dh(t)(\tau) = (Ah(t)(\tau) + \partial^2_t \Phi_0(\tau, u, v)) \, dt - \langle \partial_\tau \Phi_1(\tau, u, v), 1 \rangle_{\mu} dW_t,
\]

which allows one to identify $\mu^\text{HJM}$ and $\sigma^\text{HJM}$.

**Remark 4.13.** The HJM drift condition is satisfied because

\[
\mu^\text{HJM} = \partial^2_t \Phi_0(\tau, u, v) = \langle \partial_\tau \Phi_1(\tau, u, v), 1 \rangle_{\mu} \langle \Phi_1(\tau, u, v), 1 \rangle_{\mu} = \sigma^\text{HJM}(\tau) \int_0^T \sigma^\text{HJM}(s) \, ds.
\]
Corollary 4.14 (Covariances). For each $\tau_1, \tau_2 > 0$ the following relation holds:

$$d[h(\cdot)(\tau_1), h(\cdot)(\tau_2)]_t = \langle \partial_x \Phi_1(\tau_1, u, v), 1 \rangle_\mu (\partial_x \Phi_1(\tau_2, u, v), 1)_\mu dt.$$  

To show that the Black-Scholes formula of Theorem 4.6 holds, we verify that the $T$-forward density process is the stochastic exponential of $\int_0^T v(s, T)dW_s$ for a deterministic function $v(\cdot, T)$.

Corollary 4.15 (Forward measure). For $0 \leq t \leq T$ the density process $\xi(t, T)$ takes the form \((4.1)\) with deterministic $v(t, T) = \langle \Phi_1(T - t, u, v), 1 \rangle_\mu$.

Proof. In Lemma E.3 we verified that the expressions $\langle Y, \Phi_1(T - \cdot, u, v) \rangle_\mu$ and $\langle Z, \Phi_2(T - \cdot, u, v) \rangle_\nu$ are semimartingales. Their semimartingale decompositions are given by Equation (2.13):

$$d \langle Y_t, \Phi_1(T - t, u, v) \rangle_\mu = \int_0^\infty \left( u(x) - e^{-(T-t)x} - 1 \right) x p(x) v(x) Y_t^x \mu(dx) dt + \langle \Phi_1(T - t, u, v), 1 \rangle_\mu dW_t,$$

$$d \langle Z_t, \Phi_2(T - t, u, v) \rangle_\nu = \left( \langle v, Z_t \rangle_\nu + \langle \Phi_2(T - t, u, v), Y_t \rangle_\nu \right) dt.$$  

By the formula for bond prices in Theorem 4.10 $\log(\xi(t, T))$ satisfies

$$d(\log(\xi(t, T))) = \left( -\langle Y_t, u \rangle_\mu - \langle Z_t, v \rangle_\nu - \partial_x \Phi_0(T - t, u, v) \right) dt + d \langle Y_t, \Phi_1(T - t, u, v) \rangle_\mu + d \langle Z_t, \Phi_2(T - t, u, v) \rangle_\nu.$$  

Applying Itô’s formula and canceling out terms yields

\begin{align*}
    d\xi(t, T) &= \xi(t, T) \left( d(\log(\xi(t, T))) + \frac{1}{2} d(\log(\xi(\cdot, T)))_t \right) \\
    &= \xi(t, T) \langle \Phi_1(T - t, u, v), 1 \rangle_\mu dW_t,
\end{align*}

which implies that $\xi$ is a stochastic exponential of the form \((4.1)\) with $v(t, T) = \langle \Phi_1(T - t, u, v), 1 \rangle_\mu$. \qed

Remark 4.16. We summarize the results of Section 4.2. We considered a model with fractional short rate process constructed as a superposition of infinitely many OU processes. We derived closed-form expressions for ZCB prices and forward rates. Bond prices and forward rates are semimartingales, and HJM equation \((4.5)\) holds. It follows that prices of interest rate derivatives can be calculated as in the standard HJM framework (see e.g. [4, Section 6 and 7]), even though the short rate is not a semimartingale. Our results provide two ways of identifying model parameters: either, they could be calibrated to interest rate caps and floors using Black-Scholes formula \((4.2)\) (c.f. Corollary 4.15), or they could be estimated from realized covariances of forward rates (c.f. Corollary 4.14).

4.3. Fractional bank account process. In this section, we construct an interest rate model with a fractional bank account process. To this aim, we fix measures $\mu, \nu$ on $(0, \infty)$ satisfying Assumption 2.3. Moreover, we fix $(u, v) \in L^\infty(\mu) \times L^\infty(\nu)$, $\ell \in \mathbb{R}$, and an initial value $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ for the process $(Y, Z)$ defined in Section 2. Given these model parameters, we define the bank account process as

$$B_t = e^{\ell t + \langle Y_t, u \rangle_\mu + \langle Z_t, v \rangle_\nu}.$$
Theorem 4.17 (Bond prices and forward rates). In the fractional bank account model \((4.7)\), ZCB prices and forward rates are given by

\[
P(t,T) = e^{-\int_0^T \phi(t,T-t,u,-v) + (Y_t, \phi_t(T-t,u,-v) + \mu(\tau, \phi_2(T-t,u,-v) + v) \nu}_v,
\]

\[
h(t)(\tau) = \ell - \partial_\tau \phi_0(\tau, -u, -v) - (Y_t, \partial_\tau \phi_1(\tau, -u, -v))_\mu - (Z_t, \partial_\tau \phi_2(\tau, -u, -v))_\nu
\]

for each \(0 \leq t \leq T\) and \(\tau > 0\), where \(\phi_0, \phi_1, \) and \(\phi_2\) are given by \(\text{Theorem 2.6}\).

\[\text{Proof.}\] The formula for the ZCB prices follows directly from \(\text{Theorem 2.6}\) and Equation \((4.7)\) and the formula for the forward rates follows by definition. \(\square\)

Theorem 4.18 (HJM equation). Discounted bond prices \((B_t^{-1}P(t,T))_{t \geq 0}\) and forward rates \((h(t)(\tau))_{\tau \geq 0}\) are semimartingales for each \(T, \tau > 0\). The forward rates solve HJM equation \((4.5)\) with \(\mu^{\text{HJM}}\) and \(\sigma^{\text{HJM}}\) given by

\[
\mu^{\text{HJM}}(\tau) = \partial_\tau^2 \phi_0(\tau, -u, -v), \quad \sigma^{\text{HJM}}(\tau) = -\langle \partial_\tau \phi_1(\tau, -u, -v), 1 \rangle_{\mu}.
\]

Remark 4.19. In contrast to discounted bond prices and forward rates, undiscounted bond prices \((P(t,T))_{0 \leq t \leq T}\) are not semimartingales, in general. For example, they are fractional processes if \(\mu, \nu\) are chosen as in \(\text{Section 3}\) and \(u = v = 1\).

\[\text{Proof.}\] Discounted bond prices are martingales by definition. Forward rates are semimartingales because \((Y, \partial_\tau \phi_1(\tau, -u, -v))_\mu\) and \((Z, \partial_\tau \phi_2(\tau, -u, -v))_\nu\) are semimartingales by \(\text{Lemma F.5}\). The semimartingale decomposition of the forward rate process is given by \(\text{Equation (2.13)}\) and reads as

\[
dh(t)(\tau) = -d(Y_t, \partial_\tau \phi_1(\tau, -u, -v))_\mu - d(Z_t, \partial_\tau \phi_2(\tau, -u, -v))_\nu
\]

\[
= (Y_t, x\partial_\tau \phi_1(\tau, -u, -v) - \partial_\tau \phi_2(\tau, -u, -v))_\mu dt
\]

\[
+ (Z_t, x\partial_\tau \phi_2(\tau, -u, -v))_\nu dt - (\partial_\tau \phi_1(\tau, -u, -v), 1)_\mu dW_t,
\]

where by abuse of notation we wrote \(x\partial_\tau \phi_1(\tau, -u, -v)\) to designate the function \(x \mapsto x\partial_\tau \phi_1(\tau, -u, -v)(x)\). The second derivatives of \(\phi_1, \phi_2\) are

\[
\partial_\tau^2 \phi_1(\tau, -u, -v) = -x\partial_\tau \phi_1(\tau, -u, -v) + p\partial_\tau \phi_2(\tau, -u, -v),
\]

\[
\partial_\tau^2 \phi_2(\tau, -u, -v) = -x\partial_\tau \phi_2(\tau, -u, -v).
\]

Hence, for all \(t \geq 0\) and for all \(\tau > 0\) we have

\[
Ah(t)(\tau) = -\partial_\tau^2 \phi_0(\tau, -u, -v) + (Y_t, x\partial_\tau \phi_1(\tau, -u, -v))_\mu + (Z_t, x\partial_\tau \phi_2(\tau, -u, -v))_\nu
\]

\[
- (Y_t, \partial_\tau \phi_2(\tau, -u, -v))_\nu.
\]

Therefore, the semimartingale decomposition of \(h(\cdot)(\tau)\) can be written as

\[
dh(t)(\tau) = (Ah(t)(\tau) + \partial_\tau^2 \phi_0(\tau, -u, -v)) dt - (\partial_\tau \phi_1(\tau, -u, -v), 1)_\mu dW_t,
\]

which allows one to identify \(\mu^{\text{HJM}}\) and \(\sigma^{\text{HJM}}\). \(\square\)

Remark 4.20. The HJM drift condition is satisfied:

\[
\mu^{\text{HJM}}(\tau) = \partial_\tau^2 \phi_0(\tau, -u, -v) = \langle \partial_\tau \phi_1(\tau, -u, -v), 1 \rangle_{\mu} \langle \phi_1(\tau, -u, -v), 1 \rangle_{\mu}
\]

\[
= \sigma^{\text{HJM}}(\tau) \int_0^\tau \sigma^{\text{HJM}}(s) ds.
\]

Corollary 4.21 (Covariations). For fixed \(\tau_1, \tau_2 > 0\)

\[
d[h(\cdot)(\tau_1), h(\cdot)(\tau_2)]_\mu = (\partial_\tau \phi_1(\tau_1, -u, -v), 1)_\mu (\partial_\tau \phi_1(\tau_2, -u, -v), 1)_\mu dt.
\]
To show that the Black-Scholes formula of Theorem 4.6 holds, we verify that the $T$-forward density process is the stochastic exponential of $\int_0^T v(s,T)\,dW_s$ for a deterministic function $v(\cdot,T)$.

**Corollary 4.22** (Forward measure). For $0 \leq t \leq T$ the density process $\xi(t,T)$ takes the form (4.1) with deterministic $v(t,T) = \langle \phi_1(t,-u,-v), 1 \rangle_\mu$.

**Proof.** By Theorem 4.17 the density process $\xi(t,T)$ can be expressed equivalently as
\[
d(\log \xi(t,T)) = -\partial_t \phi_0(T-t,-u,-v) + d \langle Y_t, \phi_1(T-t,-u,-v) \rangle_\mu \\
+ d \langle Z_t, \phi_2(T-t,-u,-v) \rangle_\nu.
\]
The processes $(\langle Y_t, \phi_1(T-t,-u,-v) \rangle_\mu)_{t \geq 0}$ and $(\langle Z_t, \phi_2(T-t,-u,-v) \rangle_\nu)_{t \geq 0}$ are semimartingales with decompositions given by
\[
d \langle Y_t, \phi_1(T-t,-1,-1) \rangle_\mu = - \langle Y_t, \phi_2(T-t,-u,-v) \rangle_\nu \, dt \\
+ \langle \phi_1(T-t,-u,-v), 1 \rangle_\mu \, dW_t, \\
d \langle Z_t, \phi_2(T-t,-u,-v) \rangle_\nu = \langle Y_t, \phi_2(T-t,-u,-v) \rangle_\nu \, dt.
\]
By Itô’s formula, using ODE (2.8) for $\phi_0$, one obtains
\[
d\xi(t,T) = \xi(t,T) \left( d(\log \xi(t,T)) + \frac{1}{2} d[\log \xi(\cdot,T)]_t \right) \\
= \xi(t,T) \langle \phi_1(T-t,-u,-v), 1 \rangle_\mu \, dW_t. \quad \Box
\]

**Remark 4.23.** We summarize the results of Section 4.3. We defined an interest rate model where the logarithmic bank account is a fractional process constructed as a superposition of infinitely many OU processes. We derived closed-form expressions for ZCB prices and forward rates. While ZCB prices are typically not semimartingales, discounted ZCB prices and forward rates are. In the same way as in the fractional short rate model, the model parameters can be identified by calibration to caps and floors using Black-Scholes formula (4.2) (c.f. Corollary 4.22), or alternatively by estimation from forward rate realized covariances (c.f. Corollary 4.21).

## 5. Fractional Stein & Stein model

In this section we generalize an affine stochastic volatility model by Stein and Stein [21] to fractional volatility. In the original model, the volatility process is a single OU process. In our model, it is a fractional process constructed as a superposition of infinitely many OU processes. In accordance with empirical facts about realized volatility [10] we restrict ourselves to fractional processes with roughness and dependence structure similar to fBM of Hurst index $H < 1/2$.

### 5.1. Setup and notation.

Let $\tilde{W}$ be $(\mathcal{F}_t)_{t \geq 0}$-Brownian motion with correlation $d(W,\tilde{W})_t = \rho \, dt$ for some $\rho \in (-1,1)$. We fix a measure $\mu$ on $(0,\infty)$ satisfying Assumption 2.3, a function $v \in L^\infty(\mu)$, and an initial value $Y_0 \in L^1(\mu)$ for the process $Y$ defined in Section 2. Given these model parameters, the price process $S = (S_t)_{t \geq 0}$ is defined by the SDE
\[
dS_t = S_t(Y_t,v)_\mu \, d\tilde{W}_t.
\]
To bring the SDE for the process $S$ into an affine form, we introduce the following spaces of simple symmetric tensors:

\[
L^1(\mu) \otimes_s L^1(\mu) = \{y^{\otimes 2} : y \in L^1(\mu)\} \subset L^1(\mu^{\otimes 2}) \subset L^1(\mu^{\otimes 2}),
\]

\[
L^\infty(\mu) \otimes_s L^\infty(\mu) = \{v^{\otimes 2} : v \in L^\infty(\mu)\} \subset L^\infty(\mu^{\otimes 2}) \subset L^\infty(\mu^{\otimes 2}).
\]

For each $t \geq 0$ we set $\Pi_t = Y_t^{\otimes 2} \in L^1(\mu) \otimes_s L^1(\mu)$. Then the relation $(Y_t, v)_\mu^2 = \langle Y_t^{\otimes 2}, v^{\otimes 2}\rangle_{\mu^{\otimes 2}}$ holds. Therefore, the log-price process $X = \log(S)$ satisfies

\[
dX_t = -\frac{1}{2} \langle \Pi_t, v^{\otimes 2} \rangle_{\mu^{\otimes 2}} dt + \sqrt{\langle \Pi_t, v^{\otimes 2} \rangle_{\mu^{\otimes 2}}} d\tilde{W}_t.
\]

5.2. Affine structure of $\Pi$. The following theorem characterizes $\Pi$ as an affine process with values in $L^1(\mu) \otimes_s L^1(\mu)$.

**Theorem 5.1** (Affine structure). Let $v^{\otimes 2} \in L^\infty(\mu) \otimes_s L^\infty(\mu)$. Then, with probability one,

\[
E \left[ e^{\langle \Pi_t, v^{\otimes 2} \rangle_{\mu^{\otimes 2}}} \big| F_t \right] = e^{\psi_0(T-t, v^{\otimes 2}) + \langle \Pi_t, \psi_1(T-t, v^{\otimes 2}) \rangle_{\mu^{\otimes 2}}}, \quad 0 \leq t \leq T,
\]

where $\psi_0(t, v^{\otimes 2}) \in \mathbb{C}$ and $\psi_1(t, v^{\otimes 2}) \in L^\infty(\mu; \mathbb{C}) \otimes_s L^\infty(\mu; \mathbb{C})$ are given by

\[
\psi_0(t, v^{\otimes 2}) = -\frac{1}{2} \log(1 - 4\phi_0(t, v, 0)),
\]

\[
\psi_1(t, v^{\otimes 2}) = \frac{\phi_1(t, v, 0)^{\otimes 2}}{1 - 4\phi_0(t, v, 0)}.
\]

**Remark 5.2.** An immediate observation is that for each $(x, y) \in (0, \infty)^2$, the tuple $(\Pi_t^{x,x}, \Pi_t^{x,y}, \Pi_t^{y,y})$ is an affine process. This can be seen from the following SDE for $\Pi_t^{x,y} = Y_t^x Y_t^y$, which follows from Ito’s rule:

\[
d\Pi_t^{x,y} = (1 - (x + y)\Pi_t^{x,y}) dt + \sqrt{\Pi_t^{x,x} + 2\Pi_t^{x,y} + \Pi_t^{y,y}} dW_t.
\]

More generally, for any finite set of points $x_i$, the process $\Pi^{x_i,x_j}$ is affine. **Theorem 5.1** generalizes this observation to infinitely many points $x^1, x^j \in (0, \infty)$. A version of **Theorem 5.1** with $v^{\otimes 2}$ replaced by arbitrary symmetric test functions is given in **Lemma G.3**.

**Proof.** By **Lemma D.3** the random variable $\frac{1}{\sqrt{2\phi_0(T-t, v, 0)}} \langle Y_T, v \rangle_\mu$ is Gaussian, given $F_t$, with mean

\[
\langle Y_t, \phi_1(T-t, v, 0) \rangle_\mu \sqrt{2\phi_0(T-t, v, 0)}
\]

and unit variance. Hence, the random variable

\[
\frac{\langle \Pi_T, v^{\otimes 2} \rangle_{\mu^{\otimes 2}}}{2\phi_0(T-t, v, 0)} = \left( \frac{\langle Y_T, v \rangle_\mu}{\sqrt{2\phi_0(T-t, v, 0)}} \right)^2,
\]

is non central $\chi^2$-distributed, given $F_t$, with one degree of freedom and non centrality parameter

\[
\frac{\langle Y_t, \phi_1(T-t, v, 0) \rangle_\mu^2}{2\phi_0(T-t, v, 0)} = \frac{\langle \Pi_t, \phi_1(T-t, v, 0)^{\otimes 2} \rangle_{\mu^{\otimes 2}}}{2\phi_0(T-t, v, 0)}.
\]

\[1\] All tensor products are algebraic; we do not complete the tensor products.
Theorem 5.4 (Affine structure)

The coefficient functions $(\psi_0, \psi_1)$ of Theorem 5.1 are solutions of an infinite dimensional version of the Riccati ODE's in the sense of Definition 2.7.

Lemma 5.3 (Riccati equations). For any $\varphi \in L_0^\infty(\mu) \otimes_s L_0^\infty(\mu)$, the functions $\psi_0(\cdot, \varphi)$ and $\psi_1(\cdot, \varphi)$ given by Theorem 5.1 solve the following system of differential equations

$$
\partial_t \psi_0(t, \varphi) = F_0(\psi_1(t, \varphi)), \quad \psi_0(0, \varphi) = 0,
$$

$$
\partial_t \psi_1(t, \varphi) = F_1(\psi_1(t, \varphi)), \quad \psi_1(0, \varphi) = \varphi,
$$

where for any $w \in L_0^\infty(\mu; \mathbb{C}) \otimes_s 2$, $F_0(w)$ is a complex number given by

$$
F_0(w) = \int_0^\infty \int_0^\infty w(x, y) \mu(dx)\mu(dy),
$$

and $F_1(w)$ is a measurable function on $(0, \infty)^2$ given by

$$
F_1(w)(x, y) = -(x + y)w(x, y) + 2 \int_0^\infty \int_0^\infty w(x, x')w(y, y') \mu(dx')\mu(dy').
$$

Proof. The initial conditions are satisfied by Lemma 2.8. We differentiate with respect to $\tau$ and use Lemma 2.8

$$
\partial_{\tau} \psi_0(\tau, \varphi) = \frac{2}{1 - 4\psi_0(\tau, \varphi)} \partial_{\tau} \psi_0(\tau, \varphi) = F_0(\psi_1(\tau, \varphi)),
$$

$$
\partial_{\tau} \psi_1(\tau, \varphi)(x, y) = -x\phi_1(\tau, \varphi)(y) - y\psi_1(\tau, \varphi)(y) - y\phi_1(\tau, \varphi)(x)\psi_1(\tau, \varphi)(y) - y\phi_1(\tau, \varphi)(x)\psi_1(\tau, \varphi)(y)
$$

$$
\partial_{\tau} \psi_1(\tau, \varphi)(x, y) = \frac{2\phi_1(\tau, \varphi)(y)}{1 - 4\phi_1(\tau, \varphi)} \left( \int_0^\infty \phi_1(\tau, \varphi)(z) \mu(dz) \right)^2
$$

$$
= F_1(\psi_1(\tau, \varphi))(x, y).
$$

5.3. Affine structure of $(X, \Pi)$. The following theorem shows that $(X, \Pi)$ is an affine process with values in $\mathbb{R} \times L_1(\mu) \otimes_s L_1(\mu)$. The proof is based on an approximation of $(Y, u)^\mu$ going back to Carmona, Coutin, and Montseny [5]. This approximation also provides a mean for simulating the fractional Stein and Stein model.

Theorem 5.4 (Affine structure). Let $\mu$ satisfy Assumption 2.3 and $(X_0, \Pi_0) \in \mathbb{R} \times L_1(\mu) \otimes_s L_1(\mu)$. Then $(X, \Pi)$ is an affine process in the sense that for each $0 \leq t \leq T$, $u \in i\mathbb{R}$, and $\varphi \in L_0^\infty(\mu) \otimes_s L_0^\infty(\mu)$, the logarithmic conditional characteristic function

$$
\log \mathbb{E} \left[ e^{X_{T-u^+} + (\Pi_{T-u^+} \varphi)_{s^2}} \left| \mathcal{F}_t \right] \right],
$$

is affine in $(X_t, \Pi_t)$.

Proof. We approximate the measure $\mu$ by a sequence $\mu^n$ of atomic measures. If $\mu^n$ are suitably chosen, it follows from [5] that $(Y, \mu)^n$ converges uniformly on compacts in probability (ucp) to $(Y, \mu)^\mu$. It follows that $(\Pi, \varphi_{s^2})_{(\mu^n)_{s^2}} = (Y, \mu)^n_{s^2}$ converges ucp to $(\Pi, \varphi_{s^2})_{\mu_{s^2}} = (Y, \mu)^\mu_{s^2}$. Let $X^n$ be the corresponding process solving Equation (5.1) with $\mu$ replaced by $\mu^n$. As stochastic integrals are continuous in the ucp topology, it follows that $X^n_T$ converges in probability to $X_T$. This implies
convergence of the logarithmic characteristic function in Theorem 5.4. For each \( n \), the logarithm characteristic function is affine by Remark 5.2 and the affine nature of Equation (5.1). The result follows. \( \square \)

5.4. The uncorrelated case. By “uncorrelated” we mean \( d(W,W)_t = \rho dt = 0 \). In the uncorrelated case, the distribution of \( X_T \) depends immediately on the integrated variance, which is defined as

\[
IV(t, T) = \frac{1}{T - t} \int_t^T \langle Y_s, v \rangle_\mu^2 ds = \frac{1}{T - t} \int_t^T \langle \Pi_s, \nu^2 \rangle_\mu^2 ds.
\]

This dependence is made precise in the following lemma.

Lemma 5.5 (Conditional CDF). In the uncorrelated case \( \rho = 0 \), the \( \mathcal{F}_t \)-conditional cumulative distribution function of \( X_T \) is

\[
\mathbb{Q}[X_T \leq x | \mathcal{F}_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \mathbb{E} \left[ \exp \left( -\frac{(y - X_t + \frac{T - t}{2T} IV(t, T))^2}{2(T - t)IV(t, T)} \right) \right] dy,
\]

and the \( \mathcal{F}_t \)-conditional characteristic function is

\[
\mathbb{E} \left[ e^{X_T u + (\Pi_T, \nu^2)} | \mathcal{F}_t \right] = e^{X_t u + (\Pi_0, \nu^2)} \mathbb{E} \left[ e^{\frac{1}{2}(u^2 - u)IV(t, T)} | \mathcal{F}_t \right],
\]

where \( 0 \leq t \leq T, u \in \mathbb{R}, \nu^2 \in \mu \otimes L^\infty(\mu) \otimes L^\infty(\mu) \).

Proof. This can be seen as in [21] by conditioning on the sigma algebra generated by \( (Y_t, v)_{0 \leq t \leq T} \) and by using the independence of \( W \) and \( \tilde{W} \). \( \square \)

The Fourier transform of the integrated variance process can be calculated explicitly using the affine structure of the process \( \Pi \). Thus, in theory, it is possible to characterize the conditional distribution of the integrated variance. An example is given in the next corollary.

Corollary 5.6 (Conditional moments). For each \( 0 \leq t \leq T \), the first and second \( \mathcal{F}_t \)-conditional moments of the integrated variance \( IV(t, T) \) are given by

\[
\mathbb{E}[IV(t, T) | \mathcal{F}_t] = \int_t^T \left( 2\phi_0(s - t, v, 0) + \langle \Pi_t, \phi_1(s - t, v, 0) \otimes s \rangle \right) ds,
\]

\[
\mathbb{E}[IV(t, T)^2 | \mathcal{F}_t] = 4 \int_t^T \int_t^T \left( \phi_0(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) \phi_0(s_1 \wedge s_2 - t, v, 0)
\right.
\]
\[+ 2\phi_0(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) \langle \Pi_t, \phi_1(s_1 \wedge s_2 - t, v, 0) \otimes s \rangle
\]
\[+ \frac{1}{4} \mathbb{E}\left[ \langle \Pi_{s_1 \wedge s_2}, v(s_1, s_2) \rangle_\mu^2 \right] ds_1 ds_2
\]

where \( w(s_1, s_2) = v \otimes \phi_1(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) + \phi_1(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) \otimes v \) is symmetric two tensor and the last expectation is given by Lemma G.5.

Proof. We obtain the formula for the conditional mean using Lemma G.4. Note that we are allowed to exchange the conditional expectation and integration because the integrand is positive. For the second moment we use the tower property of
conditional expectations and Lemma G.4 for the conditional mean:

\[
\mathbb{E} \left[ IV(t, T)^2 \mid \mathcal{F}_t \right] = \int_t^T \int_t^T \mathbb{E} \left[ (\Pi_{s_1}, v^{\otimes 2})_{\mu \otimes 2} \langle \Pi_{s_2}, v^{\otimes 2} \rangle_{\mu \otimes 2} \mid \mathcal{F}_t \right] ds_2 ds_1 \\
= \int_t^T \int_t^T \mathbb{E} \left[ (\Pi_{s_1 \wedge s_2}, v^{\otimes 2})_{\mu \otimes 2} \mathbb{E} \left[ (\Pi_{s_2 \vee s_2}, v^{\otimes 2})_{\mu \otimes 2} \mid \mathcal{F}_{s_1 \wedge s_2} \right] \right] ds_2 ds_1 \\
= 4 \int_t^T \int_t^T \phi_0(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) \phi_0(s_1 \wedge s_2 - t, v, 0) \\
+ 2 \phi_0(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) \langle \Pi_t, \phi_1(s_1 \wedge s_2 - t, v, 0) \rangle_{\mu \otimes 2}^{\otimes 2} \\
+ \mathbb{E} \left[ (\Pi_{s_1 \wedge s_2}, v^{\otimes 2})_{\mu \otimes 2} \langle \Pi_{s_1 \vee s_2}, \phi_1(s_1 \vee s_2 - s_1 \wedge s_2, v, 0) \rangle_{\mu \otimes 2}^{\otimes 2} \mathbb{E} \left[ \mathcal{F}_t \right] \right] ds_2 ds_1.
\]

We set \( s = s_1 \wedge s_2 \) and \( \tau = s_1 \vee s_2 - s_1 \wedge s_2 \). Observe that

\[
\langle \Pi_s, v^{\otimes 2} \rangle_{\mu \otimes 2} \langle \Pi_s, \phi_1(\tau, v, 0) \rangle_{\mu \otimes 2}^{\otimes 2} = \langle Y_s^{\otimes 2}, v^{\otimes 2} \rangle_{\mu \otimes 2} \langle Y_s^{\otimes 2}, \phi_1(\tau, v, 0) \rangle_{\mu \otimes 2}^{\otimes 2} \\
= \left( \langle Y_s, v \rangle_\mu \langle Y_s, \phi_1(\tau, v, 0) \rangle_\mu \right)^2 = \langle \Pi_s, v \otimes \phi_1(\tau, v, 0) \rangle_{\mu \otimes 2}^2 = \frac{1}{4} \langle \Pi_s, w \rangle_{\mu \otimes 2}^2,
\]

where \( w = v \otimes \phi_1(\tau, v, 0) + \phi_1(\tau, v, 0) \otimes v \) is a symmetric two tensor. The result follows from Lemma G.5.

\begin{remark}
We summarize the results of Section 5. We generalized the stochastic volatility model by Stein and Stein \cite{SteinStein1992} to fractional volatility. We introduced an affine framework for formulating the model. Namely, we showed that \( (X, \Pi) \) is affine, where \( X \) is the logarithmic price and \( \Pi = Y^{\otimes 2} \). The model can be approximated by finite-dimensional affine models as shown in the proof of Theorem 5.4.
\end{remark}

\section*{Appendix A. Stochastic Fubini’s theorem}

We refer to the version of the theorem proved in \cite{LuschgyZeghal1997}. Let \( \mu \) be a \( \sigma \)-finite measure on \( (0, \infty) \). Fix \( T \geq 0 \) and denote by \( \mathcal{P} \) the \( \sigma \)-algebra on \( [0, T] \times \Omega \) generated by all progressively measurable processes.

\begin{theorem}[Stochastic Fubini Theorem]
Let \( G : (0, \infty) \times [0, T] \times \Omega \to \mathbb{R} \) be measurable with respect to the product \( \sigma \)-algebra \( \mathcal{B}(0, \infty) \otimes \mathcal{P} \). Define processes \( \zeta_{1,2} : (0, \infty) \times [0, T] \times \Omega \to \mathbb{R} \) and \( \eta : [0, T] \times \Omega \to \mathbb{R} \) by

\[
\zeta_1(x, t, \omega) = \int_0^t G(x, s, \omega) ds, \\
\zeta_2(x, t, \omega) = \left( \int_0^t G(x, s, \cdot) dW_s \right)(\omega), \quad \text{and} \\
\eta(t, \omega) = \int_0^\infty G(x, t, \omega) \mu(dx).
\]

(i) Assume \( G \) satisfies for almost all \( \omega \in \Omega \)

\begin{equation}
\int_0^\infty \int_0^T |G(x, s, \omega)| ds \mu(dx) < \infty.
\end{equation}

Then, for almost all \( \omega \in \Omega \) and for all \( t \in [0, T] \) we have \( \zeta_1(\cdot, t, \omega) \in L^1(\mu) \) and

\[
\int_0^\infty \zeta_1(x, t, \omega) \mu(dx) = \int_0^t \eta(s, \omega) ds.
\]

\end{theorem}
(ii) Assume $G$ satisfies for almost all $\omega \in \Omega$

\[(A.2) \quad \int_0^\infty \int_0^T G(x, s, \omega)^2 ds \mu(dx) < \infty.\]

Then, for almost all $\omega \in \Omega$ and for all $t \in [0, T]$ we have $\zeta_2(\cdot, t, \omega) \in L^1(\mu)$ and

\[\int_0^\infty \zeta_2(x, t, \omega) \mu(dx) = \left( \int_0^t \eta(s, \cdot) dW_s \right)(\omega).\]

Remark A.2. Note that

\[\int_0^\infty \int_0^T \mathbb{E} \left[ |G(x, s)| \right] ds \mu(dx) < \infty \quad \text{and} \quad \int_0^\infty \mathbb{E} \left[ \sqrt{\int_0^T G(x, s)^2 ds} \right] \mu(dx) < \infty\]

implies that conditions \[(A.1)\] and \[(A.2)\] hold with probability one.

**Appendix B. Reproducing kernel Hilbert spaces**

We adapt the exposition of [17, Section 8] to our setting and refer to this reference for further details. Let $P: L^\infty(\mu; \mathbb{C}) \to L^1(\mu; \mathbb{C})$ be a positive and symmetric bounded linear operator, i.e., $(Pu, u)_\mu \geq 0$ and $(Pu, v)_\mu = (Pv, u)_\mu$ for all $u, v \in L^\infty(\mu; \mathbb{C})$. The bilinear form $(Pu, Pv) \mapsto (Pu, v)_\mu$ defines an inner product on the image of $P$. The completion of the image of $P$ with respect to this inner product is a Hilbert space, which we denote by $\text{im}(P)$. The inclusion of the image of $P$ in $L^1(\mu; \mathbb{C})$ extends to a bounded injective operator $i: \text{im}(P) \to L^1(\mu; \mathbb{C})$. The space $H = \text{im}(i) \subseteq L^1(\mu; \mathbb{C})$ with the Hilbert structure induced by the bijection $i: \text{im}(P) \to H$ is called the reproducing kernel Hilbert space of $P$. If $u, v \in L^\infty(\mu; \mathbb{C})$, then $Pu, Pv \in H$ and $(Pu, Pv)_H = (Pu, v)_\mu$, where the inclusion $i$ is dropped from our notation.

**Appendix C. Basic estimates**

We collect some inequalities and estimates which are used throughout the paper.

**Lemma C.1 (Elementary inequalities).** The following inequalities hold true for all $x, y > 0$

\[(C.1) \quad 1 \land xy \leq (1 \lor x)(1 \land y),\]
\[(C.2) \quad y \land x^{-1} \leq (1 \lor y)(1 \land x^{-1}),\]

\[2\text{In [17] the space } \text{im}(P) \text{ is called reproducing kernel Hilbert space of } P.\]
and for all $\alpha, \tau > 0$ and $0 < \epsilon < 1$,

\begin{align}
(C.3) \quad e^{-x\tau} &\leq \left(1 \vee \left(\frac{\tau}{\alpha}\right)^{-\alpha}\right) (1 \wedge x^{-\alpha}), \\
(C.4) \quad \frac{1 - e^{-x\tau}}{x} &\leq (1 \vee \tau) (1 \wedge x^{-1}), \\
(C.5) \quad \frac{1 - e^{-x\tau}(1 + \tau x)}{x^2} &\leq (1 \vee \tau^2) (1 \wedge x^{-2}), \\
(C.6) \quad \frac{1 - e^{-x\tau}(1 + \tau x + \frac{1}{2}\tau^2x^2)}{x^3} &\leq (1 \vee \tau^3) (1 \wedge x^{-3}), \\
(C.7) \quad \int_0^t s^{-\epsilon} e^{-2sx} ds &\leq \left(2^{\epsilon-1} \Gamma(1 - \epsilon) \vee \frac{(1 - \epsilon)}{1 - \epsilon}\right) (1 \wedge x^{-1}).
\end{align}

**Proof.** For the inequalities (C.1)-(C.2) consider the following four cases separately.

1. If $0 < x, y \leq 1$. Then, $1 \wedge xy = xy \leq y = (1 \vee x) (1 \wedge y)$ and $y \wedge x^{-1} = y \leq 1 = (1 \vee y) (1 \wedge x^{-1})$.
2. If $0 < x \leq 1 \leq y$. Then, $1 \wedge xy \leq 1 = (1 \vee x) (1 \wedge y)$ and $y \wedge x^{-1} \leq y = (1 \vee y) (1 \wedge x^{-1})$.
3. If $0 < y \leq 1 \leq x$. Then, $1 \wedge xy \leq xy = (1 \vee x) (1 \wedge y)$ and $y \wedge x^{-1} \leq x^{-1} = (1 \vee y) (1 \wedge x^{-1})$.
4. If $1 \leq x, y$. Then, $1 \wedge xy = 1 \leq x = (1 \vee x) (1 \wedge y)$ and $y \wedge x^{-1} \leq y x^{-1} = (1 \vee y) (1 \wedge x^{-1})$.

Consider the functions $f(x, \tau) = e^{-x\tau}$ and $g(x, \tau, \alpha) = x^\alpha f(x, \tau)$. Obviously, $f(x, \tau) \leq 1$ for all $x, \tau > 0$. Note that $\partial_\tau g(x, \tau, \alpha) = x^{\alpha-1} e^{-x\tau} (\alpha - \tau x)$ and $g$ attains its maximum in $x$ at $\frac{\tau}{\alpha}$. Hence, Equation (C.3) follows from

$$
f(x, \tau) = \frac{g(x, \tau, \alpha)}{x^\alpha} \leq \frac{g\left(\frac{\tau}{\alpha}, \tau, \alpha\right)}{x^\alpha} = \left(\frac{\tau}{\alpha}\right)^{-\alpha} e^{-\alpha} \leq \left(\frac{\tau}{\alpha}\right)^{-\alpha},$$

and Equation (C.1). Define $k_1(x, \tau) = \frac{1 - e^{-x\tau}}{x}$, $k_2(x, \tau) = \frac{1 - e^{-x\tau}(1 + \tau x)}{x^2}$ and $k_3(x, \tau) = \frac{1 - e^{-x\tau}(1 + \tau x + \frac{1}{2}\tau^2x^2)}{x^3}$.

Computing the derivatives with respect to $x$ shows that $k_{1,2,3}(\cdot, \tau)$ are decreasing functions in $x$ for all $\tau > 0$. The inequalities (C.4)-(C.6) follow from

$$
\lim_{x \to \infty} k_{1,2,3}(x, \tau) = 0, \quad \lim_{x \to 0^+} k_i(x, \tau) = \begin{cases} \tau, & i = 1, \\ \frac{\tau^2}{2}, & i = 2, \\ \frac{\tau^3}{6}, & i = 3,
\end{cases}
$$

and Equation (C.2). Equation (C.7) follows from the relation

$$
\int_0^t s^{-\epsilon} e^{-2sx} ds = \int_0^{2tx} (2x)^{\epsilon-1} s^{-\epsilon} e^{-s} ds,
$$

and from the following two estimates:

$$
\int_0^{2tx} (2x)^{\epsilon-1} s^{-\epsilon} e^{-s} ds \leq \int_0^{\infty} (2x)^{\epsilon-1} s^{-\epsilon} e^{-s} ds = (2x)^{\epsilon-1} \Gamma(1 - \epsilon),
$$

$$
\int_0^{2tx} (2x)^{\epsilon-1} s^{-\epsilon} e^{-s} ds \leq \int_0^{2tx} (2x)^{\epsilon-1} s^{-\epsilon} ds = \frac{t^{1-\epsilon}}{1 - \epsilon}. \quad \square
$$
Lemma C.2 (Integrability of elementary expressions). Let Assumption 2.3 be in place and let \( \tau, \alpha > 0 \). Then

\[(C.8) \quad \int_0^\infty e^{-x\tau} \mu(dx) < \infty,\]

\[(C.9) \quad \int_0^\infty e^{-x\tau} \nu(dx) < \infty,\]

\[(C.10) \quad \int_0^\infty x^\alpha e^{-x\tau} \mu(dx) < \infty,\]

\[(C.11) \quad \int_0^\infty x^\alpha e^{-x\tau} \nu(dx) < \infty,\]

\[(C.12) \quad \int_0^\infty \sqrt{\frac{1 - e^{-2x\tau}}{x}} \mu(dx) < \infty,\]

\[(C.13) \quad \int_0^\infty \sqrt{\frac{1 - e^{-2x\tau}(1 + 2\tau x + 2\tau^2 x^2)}{x^3}} \nu(dx) < \infty,\]

\[(C.14) \quad \int_0^\infty \sqrt{\frac{1 - 2e^{-\tau x}(\tau x + 1)(1 - \tau xe^{\tau x}) + e^{-2\tau x}}{x^3}} \mu(dx) < \infty.\]

Furthermore, for each \( 0 \leq t < T \) we have

\[(C.15) \quad \int_0^\infty \sqrt{\int_t^T e^{-2x(T-s)} ds} \mu(dx) < \infty,\]

\[(C.16) \quad \int_0^\infty \sqrt{\int_t^T (T-s)^2 e^{-2x(T-s)} ds} \nu(dx) < \infty,\]

\[(C.17) \quad \int_0^\infty \int_t^T e^{-x(T-s)} ds \mu(dx) < \infty,\]

\[(C.18) \quad \int_0^\infty \int_t^T (T-s) e^{-x(T-s)} ds \nu(dx) < \infty,\]

\[(C.19) \quad \int_0^\infty \int_t^T 1 - e^{-x(T-s)} x (1 \wedge x^{-\frac{1}{2}}) ds \nu(dx) < \infty,\]

\[(C.20) \quad \int_0^\infty \sqrt{\int_t^T \left(1 \wedge \frac{1 - e^{-x(T-s)}}{x}\right)^2} ds \mu(dx) < \infty,\]

\[(C.21) \quad \int_0^\infty \sqrt{\int_t^T \left(1 \wedge \frac{1 - e^{-x(T-s)}(1 + x(T-s))}{x^2}\right)^2} ds \mu(dx) < \infty,\]

\[(C.22) \quad \int_0^\infty \int_0^\infty \int_t^T e^{-(x+y)(T-s)} ds \mu(dx) \mu(dy) < \infty,\]

\[(C.23) \quad \int_0^\infty \int_0^\infty \int_t^T (T-s)^2 e^{-(x+y)(T-s)} ds \nu(dx) \nu(dy) < \infty.\]
Proof. Equations (C.8) and (C.9) follow directly from (C.3) for $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, respectively. Applying Equation (C.3) for $\beta > \alpha$ we obtain

$$\int_{0}^{\infty} \alpha e^{-x^{\alpha} \mu(dx)} \leq \int_{0}^{1} e^{-x^{\alpha} \mu(dx)} + \int_{1}^{\infty} \alpha e^{-x^{\alpha} \mu(dx)}$$

$$\leq \int_{0}^{1} \left( 1 \vee \left( \frac{\tau}{\beta} \right) \right)^{-\beta} \mu(dx) + \int_{1}^{\infty} \alpha e^{-x^{\alpha-\beta} \mu(dx)} \left( 1 \vee \left( \frac{\tau}{\beta} \right) \right)^{-\beta} \mu(dx)$$

$$= \left( 1 \vee \left( \frac{\tau}{\beta} \right) \right) \int_{0}^{\infty} \left( 1 \wedge x^{\alpha-\beta} \right) \mu(dx),$$

and in the same way $\int_{0}^{\infty} x^{\beta} e^{-x^{\beta} \nu(dx)} \leq \int_{0}^{1} e^{-x^{\beta} \nu(dx)} + \int_{1}^{\infty} x^{\beta} e^{-x^{\beta} \nu(dx)}$. Setting $\beta = \alpha + \frac{1}{2}$ and $\beta = \alpha + \frac{3}{2}$ one proves (C.10) and (C.11) respectively. By Equation (C.4) we obtain Equation (C.12)

$$\int_{0}^{\infty} \sqrt{1 - e^{-2\tau x}} x \mu(dx) \leq \left( 1 \vee \left( 2\tau \right) \right) \int_{0}^{\infty} \left( 1 \wedge x^{-\frac{1}{2}} \right) \mu(dx) < \infty.$$

By Equation (C.12) we obtain Equation (C.15)

$$\int_{0}^{\infty} \sqrt{\int_{t}^{T} e^{-2\tau (T-s)} ds} \nu(dx) = \int_{0}^{\infty} \sqrt{1 - e^{-2(T-t)x}} 2x \mu(dx) < \infty.$$

By Equation (C.6) we obtain Equation (C.13)

$$\int_{0}^{\infty} \sqrt{1 - e^{-2\tau x} \left( 1 + 2\tau x + 2\tau^{2}x^{2} \right)} \nu(dx)$$

$$\leq \left( 1 \vee \left( 2\tau \right) \right) \int_{0}^{\infty} \left( 1 \wedge x^{-\frac{1}{2}} \right) \nu(dx) < \infty.$$

Equation (C.14) follows from

$$\int_{0}^{\infty} \sqrt{1 - 2e^{-\tau x}(\tau x + 1) + 2\tau x e^{-2\tau x}(\tau x + 1) + e^{-2\tau x}} \nu(dx)$$

$$\leq \int_{0}^{1/\tau} \sqrt{\frac{\tau^{2}}{6x}} \nu(dx) + \int_{1/\tau}^{\infty} \sqrt{\frac{2}{x^{3}}} \nu(dx)$$

$$\leq \sqrt{2} (\tau \vee 1) \int_{0}^{\infty} (x^{-\frac{1}{2}} \wedge x^{-\frac{3}{2}}) \nu(dx) < \infty.$$

Equation (C.13) implies Equation (C.16)

$$\int_{0}^{\infty} \sqrt{\int_{t}^{T} (T-s)^{2} e^{-2\tau (T-s)} ds} \nu(dx)$$

$$= \int_{0}^{\infty} \sqrt{1 - e^{-2(T-t)x} \left( 1 + 2(T-t)x + 2(T-t)^{2}x^{2} \right)} \frac{4x^{3}}{2x^{3}} \nu(dx) < \infty.$$

Equation (C.17) is obtained using (C.3) for $\alpha = \frac{1}{2}$

$$\int_{0}^{\infty} \int_{t}^{T} e^{-x(T-s)} ds \mu(dx) \leq \int_{t}^{T} (1 \vee (T-s)^{-\frac{1}{2}}) ds \int_{0}^{\infty} (1 \wedge x^{-\frac{1}{2}}) \mu(dx)$$

$$= \left( t \vee (T-1) - t + 2\sqrt{T - (t \vee (T-1))} \right) \int_{0}^{\infty} (1 \wedge x^{-\frac{1}{2}}) \mu(dx) < \infty.$$
Equation (C.18) is obtained using (C.3) for $\alpha = \frac{3}{2}$

$$
\int_0^T \int_t^T (T-s)e^{-x(T-s)} ds \nu(dx)
\leq \int_t^T (T-s) \left( 1 \lor (T-s)^{-\frac{3}{2}} \right) ds \int_0^\infty (1 \lor x^{-\frac{3}{2}}) \mu(dx)
\leq \left( \int_t^T (T-s) ds \lor \int_t^T (T-s)^{-\frac{3}{4}} ds \right) \int_0^\infty (1 \lor x^{-\frac{3}{2}}) \mu(dx)
= \left( \frac{(T-t)^2}{2} \lor 2\sqrt{T-t} \right) \int_0^\infty (1 \lor x^{-\frac{3}{2}}) \mu(dx) < \infty.
$$

Equation (C.4) immediately implies Equation (C.19)

$$
\int_0^\infty \int_t^T \frac{1 - e^{-x(T-s)}}{x} \left( 1 \lor x^{-\frac{3}{4}} \right) ds \nu(dx)
\leq \int_0^\infty \left( 1 \lor x^{-\frac{3}{4}} \right) \nu(dx) \int_t^T (1 \lor (T-s)) ds < \infty,
$$

and Equation (C.20)

$$
\int_0^\infty \sqrt{\int_t^T \left( 1 - e^{-x(T-s)} \right)^2 \frac{1}{x} ds} \mu(dx)
\leq \sqrt{\int_t^T (1 \lor (T-s)^2) ds} \int_0^\infty \left( 1 \lor \frac{1}{x} \right) \mu(dx) < \infty.
$$

Equation (C.5) immediately implies Equation (C.21)

$$
\int_0^\infty \sqrt{\int_t^T \left( e^{-x(T-s)} \frac{1 + x(T-s)}{x^2} - 1 \right)^2 \frac{1}{x^2} ds} \mu(dx)
\leq \sqrt{\int_t^T (1 \lor (T-s)^4) ds} \int_0^\infty (1 \lor x^{-2}) \mu(dx) < \infty.
$$

Equation (C.22) follows from Equation (C.15) applying Cauchy-Schwarz inequality

$$
\int_0^\infty \int_0^T e^{-(x+y)(T-s)} ds \mu(dx) \mu(dy)
\leq \int_0^\infty \int_0^T e^{-2x(T-s)} ds \sqrt{\int_t^T e^{-2y(T-s)} ds} \mu(dx) \mu(dy)
= \left( \int_0^\infty \sqrt{\int_t^T e^{-2x(T-s)} ds} \mu(dx) \right)^2 < \infty.
$$
In the same way, Equation (C.23) follows from Equation (C.16)
\[
\int_0^\infty \int_0^\infty \int_t^T (T-s)^2 e^{-(x+y)(T-s)} ds \left( d\nu(dx) \nu(dy) \right) < \infty.
\]

\[\text{Appendix D. Auxiliary results for Section 2} \]

**Lemma D.1** (Conditional moments of \((Y, Z)\)). For each \(x \in (0, \infty)\) and \(0 \leq t \leq T\), the process \((Y^x, Z^x)\) can be represented as
\[
Y^x_t = Y^x_t e^{-(T-t)x} + \int_t^T e^{-(T-s)x} dW_s, \tag{D.1}
\]
\[
Z^x_t = Z^x_t e^{-(T-t)x} + Y^x_t (T-t) e^{-(T-t)x} + \int_t^T (T-s) e^{-(T-s)x} dW_s.
\]

The random variables \(Y^x_t\) and \(Z^x_t\) have conditional means given by
\[
E[Y^x_t | \mathcal{F}_t] = Y^x_t e^{-(T-t)x}, \quad E[Z^x_t | \mathcal{F}_t] = Z^x_t e^{-(T-t)x} + Y^x_t (T-t) e^{-(T-t)x}.
\]

Moreover, for \(x_1, x_2 \in (0, \infty)\) we have conditional covariances
\[
\text{Cov}(Y^x_t, Y^x_s | \mathcal{F}_t) = \frac{1 - e^{-(T-t)(x_1 + x_2)}}{x_1 + x_2},
\]
\[
\text{Cov}(Y^x_t, Z^x_s | \mathcal{F}_t) = \frac{1 - e^{-(T-t)(x_1 + x_2)} (1 + (T-t)(x_1 + x_2))}{(x_1 + x_2)^2},
\]
\[
\text{Cov}(Z^x_t, Z^x_s | \mathcal{F}_t) = \frac{2 - e^{-(T-t)(x_1 + x_2)} (2 + 2(T-t)(x_1 + x_2) + (T-t)^2(x_1 + x_2)^2)}{(x_1 + x_2)^3}.
\]

**Proof:** The representation in Equation (D.1) can be deduced from the SDE (2.2) for \((Y^x, Z^x)\) using Theorem A.1(ii)
\[
Z^x_t = Z^x_t e^{-(T-t)x} + \int_t^T e^{-(T-s)x} \left( Y^x_t e^{-(s-t)x} + \int_t^s e^{-(s-u)x} dW_u \right) ds
\]
\[
= Z^x_t e^{-(T-t)x} + Y^x_t (T-t) e^{-(T-t)x} + \int_t^T \int_t^s e^{-(T-u)x} dW_u ds
\]
\[
= Z^x_t e^{-(T-t)x} + Y^x_t (T-t) e^{-(T-t)x} + \int_t^T \int_u^T e^{-(t-u)x} ds dW_u
\]
\[
= Z^x_t e^{-(T-t)x} + Y^x_t (T-t) e^{-(T-t)x} + \int_t^T (T-u) e^{-(T-u)x} dW_u.
\]

The condition \([A.2]\) is satisfied because \(\int_t^T \sqrt{\int_t^s e^{-2(t-u)x} ds} duds < \infty\). The conditional means can be read off directly from the representation of \((Y^x, Z^x)\). The
formulas for the conditional covariances are obtained using Itô’s isometry by calculating the following integrals

\[ \text{Cov}(Y_t^x, Y_t^z | \mathcal{F}_t) = \int_0^t e^{-(t-s)(x_1+x_2)} ds, \]
\[ \text{Cov}(Y_t^x, Z_t^z | \mathcal{F}_t) = \int_0^t (T-s)e^{-(t-s)(x_1+x_2)} ds, \]
\[ \text{Cov}(Z_t^x, Z_t^z | \mathcal{F}_t) = \int_0^t (T-s)^2 e^{-(t-s)(x_1+x_2)} ds. \]

**Lemma D.2** (Integrability of \((Y, Z)\)). Let [Assumption 2.3](#) be in place and assume \((Y_0, Z_0) \in L^1(\mu) \times L^1(\nu) a.s.\) Then, for each \(t \geq 0\), \(Y_t \in L^1(\mu)\) and \(Z_t \in L^1(\nu)\) holds with probability one.

**Proof.** By [Lemma D.1](#) we have for \((Y^x, Z^x)\)

\[ Y_t^x = Y_0^x e^{-tx} + \int_0^t e^{-(t-s)x} dW_s, \]
\[ Z_t^x = Z_0^x e^{-tx} + Y_0^x e^{-tx} + \int_0^t (t-s)e^{-(t-s)x} dW_s. \]

The deterministic parts are integrable because

\[ \int_0^\infty |Y_0^x| e^{-tx} \mu(dx) \leq \|Y_0\|_{L^1(\mu)} < \infty, \]
\[ \int_0^\infty |Z_0^x| e^{-tx} \nu(dx) \leq \|Z_0\|_{L^1(\nu)} < \infty, \]
\[ \int_0^\infty |Y_0^x| te^{-tx} \nu(dx) \leq \sup_{x \in (0, \infty)} (p(x)e^{-x}) \|Y_0\|_{L^1(\mu)} < \infty, \]

where [Assumption 2.3](#) is used in the last line. Therefore we can assume without loss of generality that \((Y_0, Z_0)\) vanish. Then for each \(t \geq 0\),

\[ \mathbb{E} \left[ \|Y_t\|_{L^1(\mu)} \right] = \int_0^\infty \mathbb{E} \left[ |Y_t^x| \right] \mu(dx) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sqrt{\text{Var}(Y_t^x)} \mu(dx) \]
\[ = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sqrt{\frac{1-e^{-2tx}}{2x}} \mu(dx) < \infty, \]
\[ \mathbb{E} \left[ \|Z_t\|_{L^1(\nu)} \right] = \int_0^\infty \mathbb{E} \left[ |Z_t^x| \right] \nu(dx) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sqrt{\text{Var}(Z_t^x)} \nu(dx) \]
\[ = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sqrt{\frac{1-e^{-2tx}(1+2tx+2t^2x^2)}{4x^3}} \nu(dx) < \infty, \]

which follows from Equations [C.12](#) and [C.14](#) Therefore, \(Y_t \in L^1(\mu)\) and \(Z_t \in L^1(\nu)\) holds almost surely. \(\square\)

**Lemma D.3** (Linear functionals of \((Y, Z)\)). Let [Assumption 2.3](#) be in place and assume \((Y_0, Z_0) \in L^1(\mu) \times L^1(\nu) a.s.\) Then the process \((Y, Z)\) satisfies for each
In particular, the random variable \( \langle Y_T, u \rangle_{\mu} + \langle Z_T, v \rangle_{\nu} \) is Gaussian, given \( \mathcal{F}_t \).

Proof. The statement follows from Lemmas \ref{lem:covariance-operator} and \ref{lem:covariance-operator-2} and from Theorem \ref{thm:stochastic-fubini-theorem} Condition \ref{cond:stochastic-fubini-theorem-1} of the stochastic Fubini theorem are satisfied by Equations \ref{eq:covariance-operator} and \ref{eq:covariance-operator-2}.

\[ \text{Lemma D.4 (Covariance operators). Let Assumption 2.3 be in place and } \langle Y_0, Z_0 \rangle \in L^1(\mu) \times L^1(\nu) \text{ a.s. Then for all } (u_1, u_2) \in L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C}), (v_1, v_2) \in L^\infty(\nu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C}) \text{ and for all } 0 \leq t \leq T \]

\[
\text{Cov} \left( \langle Y_T, u_1 \rangle_{\mu}, \langle Y_T, u_2 \rangle_{\mu} \mid \mathcal{F}_t \right) = \langle P_{T-t} u_1, u_2 \rangle_{\mu},
\text{Cov} \left( \langle Z_T, v_1 \rangle_{\nu}, \langle Z_T, v_2 \rangle_{\nu} \mid \mathcal{F}_t \right) = \langle Q_{T-t} v_1, v_2 \rangle_{\nu},
\]

where \( P_t : L^\infty(\mu; \mathbb{C}) \to L^1(\mu; \mathbb{C}) \) and \( Q_t : L^\infty(\nu; \mathbb{C}) \to L^1(\nu; \mathbb{C}) \) are bounded linear operators given by

\[
P_t u(x) = \int_0^\infty \frac{1 - e^{-\tau(x+y)}}{x+y} u(y) \mu(dy),
\]

\[
Q_t v(x) = \int_0^\infty \frac{2 e^{-\tau(x+y)}(2 + 2\tau(x+y) + \tau^2(x+y)^2)}{(x+y)^3} v(y) \nu(dy),
\]

for \( u \in L^\infty(\mu; \mathbb{C}), v \in L^\infty(\nu; \mathbb{C}) \) and \( \tau \geq 0 \). In particular, \( Y_T \) and \( Z_T \) are Gaussian random variables, given \( \mathcal{F}_t \), with covariance operators \( P_{T-t} \) and \( Q_{T-t} \), respectively.

Proof. For each \( t \geq 0 \) and any \( u_{1,2} \in L^\infty(\mu) \) and \( v_{1,2} \in L^\infty(\nu) \) we have using the representation of Lemma \ref{lem:covariance-operator}

\[
\text{Cov} \left( \langle Y_T, u_1 \rangle_{\mu}, \langle Y_T, u_2 \rangle_{\mu} \mid \mathcal{F}_t \right) = \int_0^\infty \int_0^\infty \text{Cov} \left( Y_T, Y_T^\nu \mid \mathcal{F}_t \right) u_1(x)u_2(y) \mu(dy) \mu(dx)
= \int_0^\infty \int_0^\infty \int_t^T e^{-(T-s)} e^{-(T-s)y} ds u_1(x)u_2(y) \mu(dy) \mu(dx) = \langle P_{T-t} u_1, u_2 \rangle_{\mu},
\]

\[
\text{Cov} \left( \langle Z_T, v_1 \rangle_{\nu}, \langle Z_T, v_2 \rangle_{\nu} \mid \mathcal{F}_t \right) = \int_0^\infty \int_0^\infty \text{Cov} \left( Z_T, Z_T^\nu \mid \mathcal{F}_t \right) v_1(x)v_2(y) \mu(dy) \mu(dx)
= \int_0^\infty \int_0^\infty \int_t^T (T-s)^2 e^{-(T-s)} e^{-(T-s)y} ds v_1(x)v_2(y) \nu(dy) \nu(dx) = \langle Q_{T-t} v_1, v_2 \rangle_{\nu}.
\]

By Equations \ref{eq:covariance-operator-22} and \ref{eq:covariance-operator-23} we have

\[
\|P_t u\|_{L^1(\mu)} = \int_0^\infty \int_0^\infty \int_t^\infty e^{-s(x+y)} |u(x)| ds \mu(dx) \mu(dy) \leq C \|u\|_{L^\infty(\mu)} < \infty,
\]

\[
\|Q_t v\|_{L^1(\nu)} = \int_0^\infty \int_0^\infty \int_t^\infty s^2 e^{-s(x+y)} |v(x)| ds \nu(dx) \nu(dy) \leq C \|v\|_{L^\infty(\nu)} < \infty,
\]
Lemma D.5 (Maximum inequality for OU processes). There exists a constant $C > 0$ such that for each $t \geq 0$ and $x > 0$

$$
\mathbb{E} \left[ \sup_{s \in [0,t]} |Y_s^x| \right] \leq C \log(1 + tx)x^{-\frac{1}{2}},
$$

$$
\mathbb{E} \left[ \sup_{s \in [0,t]} |Z_s^x| \right] \leq C \log(1 + tx)x^{-\frac{3}{2}}.
$$

Proof. The inequality for $Y^x$ follows from the maximal inequalities for OU processes developed by Graversen and Peskir [11]. For the process $Z^x$, we estimate for each $t \geq 0$ and $x > 0$

$$
\mathbb{E} \left[ \sup_{s \in [0,t]} |Z_s^x| \right] \leq \mathbb{E} \left[ \int_0^t e^{-(t-s)x}|Y_s^x|ds \right] \leq C \int_0^t e^{-(t-s)x} \log(1 + sx)x^{-\frac{1}{2}}ds
$$

$$
= C \left[ e^{-(t-s)x} \log(1 + sx)x^{-\frac{1}{2}} \right]_0^t - C \int_0^t e^{-(t-s)x}(1 + sx)^{-1}x^{-\frac{3}{2}}dx
$$

$$
= C \log(1 + tx)x^{-\frac{3}{2}} - C \int_0^t e^{-(t-s)x}(1 + sx)^{-1}x^{-\frac{3}{2}}dx
$$

$$
\leq C \log(1 + tx)x^{-\frac{3}{2}}.
$$

Lemma D.6 (Auxiliary estimates for semimartingale decomposition). Let $G(x,t)$ be deterministic and jointly measurable in $(x,t) \in (0,\infty) \times [0,\infty)$. Assume $Y_0 = Z_0 = 0$. Then, with probability one,

$$
\int_0^\infty \int_0^t |G(x,t)Y_s^x(\omega)|ds\mu(dx) \leq (1 \vee t^\frac{1}{2}) \int_0^\infty \int_0^t |G(x,t) (1 \wedge x^{-\frac{1}{2}})|ds\mu(dx),
$$

$$
\int_0^\infty \int_0^t |G(x,s)Z_s^x(\omega)|ds\nu(dx) \leq (1 \vee t^\frac{1}{2}) \int_0^\infty \int_0^t |G(x,s) (1 \wedge x^{-\frac{1}{2}})|ds\nu(dx).
$$

Proof. Note that for each $s \geq 0$ the random variables $|Y_s^x|$ and $|Z_s^x|$ are half-normal distributed with mean

$$
\mathbb{E} [|Y_s^x|] = \sqrt{\frac{1 - e^{-2sx}}{\pi x}}, \quad \text{and} \quad \mathbb{E} [|Z_s^x|] = \sqrt{\frac{1 - e^{-2sx} (1 + 2sx + 2s^2x^2)}{2\pi x^3}}.
$$

By [C.4] we have

$$
\int_0^\infty \int_0^t \mathbb{E} [|G(x,s)Y_s^x|] ds\mu(dx) = \int_0^\infty \int_0^t |G(x,s)| \sqrt{\frac{1 - e^{-2sx}}{\pi x}} ds\mu(dx)
$$

$$
\leq (1 \vee t^\frac{1}{2}) \int_0^\infty \int_0^t |G(x,s) (1 \wedge x^{-\frac{1}{2}})|ds\mu(dx).
$$
By (C.6) we have
\[ \int_0^\infty \int_0^t \mathbb{E} \left[ \left\| G(x, s) Z_s^x \right\| \right] ds \, dv(dx) \]
\[ = \int_0^\infty \int_0^t |G(x, s)| \sqrt{1 - e^{-2sx} (1 + 2sx + s^2x^2)} \, ds \, d\mu(dx) \]
\[ \leq (1 \vee t^\frac{3}{2}) \int_0^\infty \int_0^t |G(x, s)| (1 \wedge x^{-\frac{3}{2}}) ds \, dv(dx). \]

Then the inequalities hold true with probability one. \qed

**Lemma D.7 (Tightness).** Let \( \mu_\infty, \nu_\infty \) satisfy Assumption 2.14. Then the laws of the random variables \((Y_t, Z_t)_{t \geq 0}\) are tight on the space \(L^1(\mu_\infty) \times L^1(\nu_\infty)\) with the weak topology.

**Proof.** We generalize the proof of [4, Proposition 2] to our setting. We endow \(L^1(\mu_\infty) \times L^1(\nu_\infty)\) with the weak topology and assume that \((Y_0, Z_0) = 0\). We will show using [8, Theorem IV.8.9] that for any \(M \geq 0\), the set
\[ K_M = \left\{ (y, z) \in L^1(\mu_\infty) \times L^1(\nu_\infty) : \|y\|_{L^2(\mu_\infty)}^2 + \|z\|_{L^2(\nu_\infty)}^2 \leq M \right\} \]
is pre-compact in \(L^1(\mu_\infty) \times L^1(\nu_\infty)\). For any measurable set \(E \subseteq [0, \infty)\) and \((y, z) \in K_M\), the Cauchy-Schwarz inequality implies
\[ \|1_E y\|_{L^1(\mu_\infty)} \leq \|y\|_{L^2(\mu_\infty)} \|1_E\|_{L^2(\mu_\infty)} \leq \sqrt{M} \|1_E\|_{L^2(\mu_\infty)}, \]
\[ \|1_E z\|_{L^1(\mu_\infty)} \leq \|z\|_{L^2(\mu_\infty)} \|1_E\|_{L^2(\mu_\infty)} \leq \sqrt{M} \|1_E\|_{L^2(\mu_\infty)}. \]

Setting \(E = [0, \infty)\) shows that \(K_M\) is bounded in \(L^1(\mu_\infty) \times L^1(\nu_\infty)\). Moreover, if \(E_n \subset [0, \infty)\) is a sequence of measurable sets which decreases to the empty set, then the above estimate shows that
\[ \lim_{n \to \infty} \sup_{(y, z) \in K} \|1_{E_n} y\|_{L^1(\mu_\infty)} + \|1_{E_n} z\|_{L^1(\mu_\infty)} = 0. \]

Therefore, the conditions of [8, Theorem IV.8.9] are satisfied and \(K_M\) is pre-compact. By Prokhorov’s theorem, the laws of \((Y_t, Z_t)_{t \geq 0}\) are tight if
\[ \lim_{M \to \infty} \sup_{t \geq 0} \mathbb{Q}( (Y_t, Z_t) \notin K_M ) = 0. \]

This follows from the estimate
\[ \mathbb{Q}( (Y_t, Z_t) \notin K_M ) \leq \frac{1}{M} \mathbb{E} \left[ \left\| Y_t \right\|_{L^2(\mu_\infty)}^2 + \left\| Z_t \right\|_{L^2(\nu_\infty)}^2 \right] \]
\[ = \frac{1}{M} \left( \int_0^\infty \text{Cov}(Y_s^1) \sqrt{\mu_\infty}(dx) + \text{Cov}(Z_s^1) \sqrt{\nu_\infty}(dx) \right) \]
\[ = \frac{1}{M} \left( \int_0^\infty \text{Cov}(Y_s^1) \sqrt{\mu_\infty}(dx) + \text{Cov}(Z_s^1) \sqrt{\nu_\infty}(dx) \right) \]
\[ = \frac{1}{M} \left( \int_0^\infty \int_0^\infty \frac{1}{2x} \sqrt{\mu_\infty}(dx) + \int_0^\infty \frac{1}{4x^2} \sqrt{\nu_\infty}(dx) \right), \]
where the right-hand side is finite by Assumption 2.14. \qed
Lemma E.1 (Integrability condition). Under Assumption 4.7 the following condition is satisfied:

\[ \sup_{x \in (0, \infty)} p(x) \int_0^t s e^{-sx} ds < \infty. \]

Proof. By assumption, there is \( \beta \in (0, 2) \) such that \( p(x)(1 \wedge x^{-\beta}) \) is bounded in \( x \).

Then the lemma follows from the estimate

\[ \int_0^t s e^{-sx} ds \leq \int_0^t s \left( 1 \lor \left( \frac{s}{\beta} \right)^{-\beta} \right) (1 \wedge x^{-\beta}) ds \leq \int_0^t \left( \frac{s}{\beta} \right)^{1-\beta} ds (1 \wedge x^{-\beta}) = \left( \frac{t^2}{2} + \frac{1}{2-\beta} \left( \frac{t}{\beta} \right)^{2-\beta} \right) (1 \wedge x^{-\beta}). \]

\[ \square \]

Lemma E.2 (Time-integrals of \((Y, Z)\)). Let Assumption 4.7 be in place and assume \((Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)\) a.s. Then, for each \( 0 \leq t \leq T \) and for all \((u, v) \in L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C})\) one has

\[ \int_t^T \left( \langle Y_s, u \rangle_\mu + \langle Z_s, v \rangle_\nu \right) ds = -\langle Y_t, \Phi_1(T - t, u, v) \rangle_\mu - \langle Z_t, \Phi_2(T - t, u, v) \rangle_\nu - \int_t^T \langle \Phi_1(T - s, u, v), 1 \rangle_\mu dW_s \]

with \( \Phi_1, \Phi_2 \) as in Theorem 4.10. In particular, the random variable \( \int_t^T \left( \langle Y_s, u \rangle_\mu + \langle Z_s, v \rangle_\nu \right) ds \) is Gaussian, given \( F_t \).

Proof. The time-derivatives of \( \Phi_1, \Phi_2 \) are given by

\[ \partial_t \Phi_1(r, u, v)(x) = -e^{-r x} (u(x) + r p(x) v(x)), \quad \partial_t \Phi_2(r, u, v)(x) = -e^{-r x} v(x). \]

It follows from Lemma D.3 that for any \( 0 \leq t \leq s \),

\[ \langle Y_s, u \rangle + \langle Z_s, v \rangle = -\langle Y_t, \partial_t \Phi_1(s - t, u, v) \rangle_\mu - \langle Z_t, \partial_t \Phi_2(s - t, u, v) \rangle_\mu - \int_t^s \langle \partial_r \Phi_1(s - r, u, v), 1 \rangle_\mu dW_r. \]

The result follows by integrating over \( s \in [t, T] \) and applying Fubini’s theorem Theorem A.1 to each of the three summands above. For the first summand, Condition (A.1) of Theorem A.1 is satisfied by Lemma E.1 and the estimate

\[ \int_0^\infty \int_t^T |Y_t^x \partial_x \Phi_1(s - t, u, v)| ds \mu(dx) \leq \| u \|_{L^\infty(\mu)} \| Y_t \|_{L^1(\mu)} + \| v \|_{L^\infty(\nu)} \int_0^\infty |Y_t^x| \int_t^T (s - t) e^{-(s-t)x} ds p(x) \mu(dx) = \| u \|_{L^\infty(\mu)} \| Y_t \|_{L^1(\mu)} + \| v \|_{L^\infty(\nu)} \int_0^\infty |Y_t^x| \int_0^{T-t} s e^{-sx} ds p(x) \mu(dx) < \infty. \]

For the second summand, Condition (A.1) reads as

\[ \int_0^\infty \int_t^T |Z_t^x e^{-(s-t)x} v(x)| ds \nu(dx) \leq (T - t) \| v \|_{L^\infty(\nu)} \| Z_t \|_{L^1(\nu)} < \infty. \]
For the third summand, we first use Fubini’s theorem to exchange the order of integration with respect to \(\mu(dx)\) and \(dW_r\):

\[
\int_t^s \langle \partial_t \Phi_1(s-r, u, v), 1 \rangle_\mu dW_r = - \int_0^\infty \int_t^s e^{-(s-r)x} (u(x) + (s-r)p(x)v(x)) dW_r \mu(dx).
\]

This is allowed because Equation (A.2) is satisfied by Equations (C.12) and (C.16):

\[
\int_0^\infty \sqrt{\int_t^s e^{-2(s-r)x} |u(x)| \mu(dx)} < \infty,
\]

\[
\int_0^\infty \sqrt{\int_t^s (s-r)^2 e^{-2(s-r)x} |v(x)| \nu(dx)} < \infty,
\]

Then we interchange the order of integration with respect to \(dW_r\) and the product measure \(\mu(dx)ds\), which brings the third summand into the form

\[
- \int_t^T \int_0^\infty \int_t^s e^{-(s-r)x} (u(x) + (s-r)p(x)v(x)) dW_r \mu(dx) ds
\]

\[
= - \int_t^T \int_r^T \int_0^\infty e^{-(s-r)x} (u(x) + (s-r)p(x)v(x)) \mu(dx) ds dW_r.
\]

This is allowed because Condition (A.2) is satisfied by Equations (C.24) and (C.25):

\[
\int_t^T \int_0^\infty \sqrt{\int_t^s e^{-2(s-r)x} |u(x)| \mu(dx) ds} < \infty,
\]

\[
\int_t^T \int_0^\infty \sqrt{\int_t^s (s-r)^2 e^{-2(s-r)x} |v(x)| \nu(dx) ds} < \infty.
\]

Finally, we exchange the innermost integrals \(\mu(dx)\) and \(ds\), which is justified by Condition (A.1) and Equations (C.17) and (C.18). Then the third summand is given by

\[
- \int_t^T \int_0^\infty \int_r^T e^{-(s-r)x} (u(x) + (s-r)p(x)v(x)) d\mu(dx) dW_r
\]

\[
= - \int_t^T \langle \Phi_1(T-r, u, v), 1 \rangle_\mu dW_r. \quad \square
\]

**Lemma E.3** (Semimartingale property). Under Assumption 4.7, the expressions \(\langle Y_t, \Phi_1(T-t, u, v) \rangle_\mu\) and \(\langle Z_t, \Phi_2(T-t, u, v) \rangle_\nu\) are continuous semimartingales in \(t \in [0, T]\), for each fixed \(T > 0\) and \((u, v) \in L^\infty(\mu) \times L^\infty(\nu)\).

**Proof.** We verify the conditions of Theorem 2.13. In the following estimates it can be assumed without loss of generality that the functions \(u\) and \(v\) are equal to 1 because they are bounded. Conditions (2.9) and (2.10) for \(f_t^s = \Phi_1(T-t, u, v)(x)\)
are satisfied by Equations (C.19), (C.20) and (C.21):
\[
\int_0^t \int_0^\infty |\partial_x f_s^x - x f_s^x| (1 \wedge x^{-\frac{1}{2}}) ds\mu(dx)
\]
\[
= \int_0^\infty \int_0^t \left(1 + \frac{1 - e^{-(T-s)x}}{x}\right)(1 \wedge x^{-\frac{1}{2}}) ds\mu(dx) < \infty,
\]
\[
\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds\mu(dx)} \leq \int_0^\infty \sqrt{\int_0^t \left(\frac{e^{-(T-s)x} - 1}{x^2} + \frac{e^{-(T-s)x}}{x}\right)^2 dx} < \infty.
\]

Conditions (2.11) and (2.12) are satisfied for \(g_t^x = \Phi_2(T-t, u, v)(x)\) by Equation (C.19):
\[
\int_0^\infty \int_0^t |\partial_x g_s^x - x g_s^x| (1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t (1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty,
\]
\[
\int_0^\infty \int_0^t |g_s^x| (1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t \frac{1 - e^{-\tau x}}{x}(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty.
\]

Thus, we have verified the conditions of Theorem 2.13 and the statement of the lemma follows. \(\square\)

**Lemma E.4** (Semimartingale property). Under Assumption 4.7, the expressions \(\langle Y_t, \partial_x \Phi_1(\tau, u, v) \rangle_\mu\) and \(\langle Z_t, \partial_x \Phi_2(\tau, u, v) \rangle_\nu\) are continuous semimartingales in \(t \in [0, T]\), for each fixed \(\tau > 0\) and \((u, v) \in L^\infty(\mu) \times L^\infty(\nu)\).

**Proof.** We calculate \(\partial_x \Phi_1(\tau, u, v)(x) = -e^{-\tau x}(u(x) + \tau p(x)v(x))\), \(\partial_x \Phi_2(\tau, u, v)(x) = -e^{-\tau x}v(x)\).

We show the semimartingale property by verifying the conditions of Theorem 2.13.

In the following estimates it can be assumed without loss of generality that the functions \(u\) and \(v\) are equal to 1 because they are bounded. Conditions (2.9) and (2.10) for \(f_t^x = \partial_x \Phi_1(\tau, u, v)(x)\) are satisfied by Equations (C.8) (C.11):
\[
\int_0^\infty \int_0^t |\partial_x f_s^x - x f_s^x| (1 \wedge x^{-\frac{1}{2}}) ds\mu(dx)
\]
\[
= \int_0^\infty \int_0^t e^{-\tau x}(1 + \tau p(x))(1 \wedge x^{-\frac{1}{2}}) ds\mu(dx) < \infty,
\]
\[
\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds\mu(dx)} \leq \int_0^\infty \sqrt{\int_0^t 2e^{-2\tau x} ds\mu(dx)}
\]
\[
+ \int_0^\infty \sqrt{\int_0^t 2\tau e^{-2\tau x} ds\nu(dx)} < \infty.
\]

Conditions (2.11) and (2.12) for \(g_t^x = \partial_x \Phi_2(\tau, u, v)(x)\) are satisfied by Equation (C.11):
\[
\int_0^\infty \int_0^t |\partial_x g_s^x - x g_s^x| (1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t e^{-\tau x}(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty,
\]
\[
\int_0^\infty \int_0^t |g_s^x| (1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t e^{-\tau x}(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty.
\]
Thus, we have verified the conditions of Theorem 2.13 and the statement of the lemma follows. □

Appendix F. Auxiliary results for Section 4.3

Lemma F.1 (Semimartingale property). Under Assumption 2.3, the expressions \( \langle Y_t, \partial_x \phi_1(\tau, -u, -v) \rangle \mu \) and \( \langle Z_t, \partial_x \phi_2(\tau, -u, -v) \rangle \nu \) are continuous semimartingales in \( t \in [0, \infty) \) for each fixed \( \tau > 0 \) and \( (u, v) \in L^\infty(\mu) \times L^\infty(\nu) \).

Proof. We verify the conditions of Theorem 2.13 As \( u \) and \( v \) are bounded we may assume without loss of generality in the following estimates that \( u = v = 1 \). Conditions (2.9)–(2.12) for \( f^t = \partial_x \phi_1(\tau, -u, -v)(x) \) and \( g^t = \partial_x \phi_2(\tau, -u, -v)(x) \) are satisfied by Equations (C.8)–(C.11)

\[
\int_0^\infty \int_0^t |\partial_s f^s - x f^s| (1 \wedge x^{-\frac 12}) d\mu(dx) = t \int_0^\infty |x \partial_x \phi_1(\tau, -u, -v)|(1 \wedge x^{-\frac 12}) \mu(dx)
\]

\[
\leq t \int_0^\infty x^2 e^{-xt} \mu(dx) + t \int_0^\infty x e^{-xt} \nu(dx) + t \int_0^\infty x^2 e^{-xt} \nu(dx) < \infty,
\]

\[
\int_0^\infty \sqrt{t} \int_0^t (f^2)^{\frac 12} d\mu(dx) = \sqrt{t} \int_0^\infty |\partial_x \phi_1(\tau, -u, -v)| \mu(dx)
\]

\[
\leq \sqrt{t} \int_0^\infty x e^{-xt} \mu(dx) + \sqrt{t} \int_0^\infty e^{-xt} \nu(dx) + \sqrt{t} \int_0^\infty x e^{-xt} \nu(dx) < \infty,
\]

\[
\int_0^\infty \int_0^t |\partial_s g^s - x g^s| (1 \wedge x^{-\frac 12}) d\nu(dx) = t \int_0^\infty |x \partial_x \phi_2(\tau, -u, -v)|(1 \wedge x^{-\frac 12}) \nu(dx)
\]

\[
= t \int_0^\infty x^2 e^{-xt} \nu(dx) < \infty,
\]

\[
\int_0^\infty \int_0^t |g^2| (1 \wedge x^{-\frac 12}) d\nu(dx) = t \int_0^\infty |\partial_x \phi_2(\tau, -u, -v)| \nu(dx)
\]

\[
\leq t \int_0^\infty e^{-xt} \nu(dx) < \infty.
\]

□

Appendix G. Auxiliary results for Section 5

Lemma G.1 (Injectivity of the covariance operator). For any \( \tau > 0 \), the mapping \( P_\tau \) is an injective linear operator from \( L^\infty(\mu; \mathbb{C}) \) to the complexification of the Hilbert space \( \mathbb{H}_\tau \).

Proof. For simplicity, we write \( \mathbb{H}_\tau \) for the complexified space \( \mathbb{H}_\tau \otimes \mathbb{C} \) (see Appendix B). If \( P_\tau v = 0 \) for some \( v \in L^\infty(\mu; \mathbb{C}) \), then

\[
0 = \langle P_\tau v, P_\tau v \rangle_{\mathbb{H}_\tau} = \langle P_\tau v, v \rangle_\mu = \int_0^\tau \left( \int_0^\infty v(x) e^{-sx} \mu(dx) \right) ds.
\]

Therefore, the Laplace transform \( \mathcal{L}(v \mu)(s) \) of the complex measure \( v \mu \) vanishes at almost all \( s \in [0, \tau] \). As \( \mathcal{L}(v \mu)(s) \) is analytic in \( s \), it vanishes identically. By the injectivity of the Laplace transform \[ [13, Section 3.8], the complex measure \( v \mu \) vanishes, which is equivalent to \( v = 0 \) in \( L^\infty(\mu; \mathbb{C}) \). □
Lemma G.2 (Diagonalization of symmetric two-tensors). For each \( \tau \geq 0 \) any symmetric two-tensor \( w \in L^\infty(\mu; \mathbb{C})^\otimes 2 \) has a representation as a sum of squares

\[
w = \sum_{k=1}^{n} \vartheta_k v_k \otimes v_k, \quad \text{with } \vartheta_k \in \mathbb{C} \text{ and } v_k \in L^\infty(\mu; \mathbb{C}),
\]

such that the functions \( v_k \) are orthonormal with respect to the covariance operator \( P_\tau \) defined in \( \text{Lemma D.4} \), i.e., \( \langle P_\tau v_k, v_l \rangle_\mu = \delta_{kl} \).

Proof. For simplicity, we write \( \mathcal{H}_\tau \) for the complexified space \( \mathcal{H}_\tau \otimes_B \mathbb{C} \). Let \( w = \sum_{k=1}^{m} w_k \otimes w_k \in L^\infty(\mu; \mathbb{C})^\otimes 2 \) be any symmetric two-tensor and set \( V = \text{span}_{\mathbb{C}}\{w_1, \ldots, w_m\} \).

By \( \text{Lemma G.1} \) the bilinear form \( \langle P_\tau \cdot, \cdot \rangle \) is a scalar product on the finite-dimensional vector space \( V \). The desired representation of \( w \) is obtained by diagonalizing \( w \in V^\otimes 2 \) with respect to this scalar product. \qed

Lemma G.3 (Affine structure). Let \( \mu \) satisfy \( \text{Assumption 2.3} \) and \( Y_0 \in L^1(\mu) \). Let \( w = \sum_{k=1}^{n} \vartheta_k v_k^\otimes 2 \in iL^\infty(\mu; \mathbb{C})^\otimes 2 \) be a symmetric tensor with decomposition into sums of squares in the sense of \( \text{Lemma G.2} \) and \( 0 \leq t \leq T \),

\[
\mathbb{E} \left[ e^{\langle \Pi_T w \rangle_{\mu, \otimes 2}} \bigg| \mathcal{F}_t \right] = e^{\varphi_0(T-t,w) + \langle \Pi_T, \varphi_1(T-t,w) \rangle_{\mu, \otimes 2}},
\]

where \( (\varphi_0, \varphi_1) : [0, \infty) \times L^\infty(\mu; \mathbb{C})^\otimes 2 \to \mathbb{C} \times L^\infty(\mu; \mathbb{C})^\otimes 2 \) are given by

\[
\varphi_0(\tau, w) = -\frac{1}{2} \sum_{k=1}^{n} \log \left( 1 - 2\vartheta_k \right),
\]

\[
\varphi_1(\tau, w)(x, y) = \sum_{k=1}^{n} \frac{\vartheta_k}{1 - 2\vartheta_k} v_k(x)v_k(y) e^{-(T-\tau)(x + y)}.
\]

Proof. Let \( 0 \leq t \leq T \) be fixed and let \( w = \sum_{k=1}^{n} \vartheta_k v_k^\otimes 2 \) be a decomposition of \( w \) into sums of squares in the sense of \( \text{Lemma G.2} \). By \( \text{Lemmas D.3, D.4} \) and \( \text{G.2} \) the random variables \( \langle Y_T, v_k \rangle_{\mu} \), \( \cdots \), \( \langle Y_T, v_n \rangle_{\mu} \) are independent Gaussian, given \( \mathcal{F}_t \), with conditional means

\[
\mathbb{E} \left[ \langle Y_T, v_k \rangle_{\mu} \bigg| \mathcal{F}_t \right] = \langle Y_t, \phi_1(T - t, v_k, 0) \rangle_{\mu}, \quad k \in \{1, \ldots, n\},
\]

and unit variances. Hence, the random variables \( \langle Y, v_1 \rangle_{\mu}^2, \ldots, \langle Y, v_n \rangle_{\mu}^2 \) are independent non-central \( \chi^2 \), given \( \mathcal{F}_t \), with non centrality parameters

\[
\langle Y_t, \phi_1(T - t, v_k, 0) \rangle_{\mu}^2 \sim \langle \Pi_t, \phi_1(T - t, v_k, 0) \rangle_{\mu, \otimes 2}, \quad k \in \{1, \ldots, n\}.
\]

We obtain the affine transformation formula using independence and the characteristic function of the non-central \( \chi^2 \) distribution

\[
\mathbb{E} \left[ e^{\langle \Pi_T w \rangle_{\mu, \otimes 2}} \bigg| \mathcal{F}_t \right] = \prod_{k=1}^{n} \mathbb{E} \left[ e^{\vartheta_k \langle Y_T, v_k \rangle_{\mu}^2} \bigg| \mathcal{F}_t \right] = \exp \left( -\frac{1}{2} \sum_{k=1}^{n} \log \left( 1 - 2\vartheta_k \right) \right) \times \exp \left( \sum_{k=1}^{n} \frac{\vartheta_k}{1 - 2\vartheta_k} \langle \Pi_t, \phi_1(T - t, v_k) \rangle_{\mu, \otimes 2} \right).
\]

We recognize the functions \( \varphi_0 \) and \( \varphi_1 \) in the right-hand side above. \qed
Lemma G.4 (Conditional mean). For each $v^{\otimes 2} \in L^\infty(\mu) \otimes L^\infty(\mu)$ and $0 \leq t \leq T$, the $\mathcal{F}_t$-conditional mean of $\langle \Pi_T, v^{\otimes 2} \rangle_{\mu^{\otimes 2}}$ is given by

$$E \left[ \langle \Pi_T, v^{\otimes 2} \rangle_{\mu^{\otimes 2}} \bigg| \mathcal{F}_t \right] = 2\phi_0(\tau, v, 0) + \langle \Pi_t, \phi_1(\tau, v, 0)^{\otimes 2} \rangle_{\mu^{\otimes 2}}.$$

Proof. We use Theorem 5.1 to calculate

$$E \left[ \langle \Pi_T, v^{\otimes 2} \rangle_{\mu^{\otimes 2}} \bigg| \mathcal{F}_t \right] = \frac{1}{t} \partial_q |_{q=0} E \left[ e^{iq \langle \Pi_T, v^{\otimes 2} \rangle_{\mu^{\otimes 2}}} \bigg| \mathcal{F}_t \right]$$

$$= \frac{1}{t} \partial_q |_{q=0} \psi_0(T-t, v^{\otimes 2} iq) + \langle \Pi_t, \phi_1(T-t, v^{\otimes 2} iq) \rangle_{\mu^{\otimes 2}}$$

$$= \frac{1}{t} \partial_q |_{q=0} - \frac{1}{2} \log(1-4\phi_0(\tau,v,0)) + \langle \Pi_t, \phi(\tau,v,0) \rangle_{\mu^{\otimes 2}}$$

$$= 2\phi_0(\tau, v, 0) + \langle \Pi_t, \phi_1(\tau, v, 0)^{\otimes 2} \rangle_{\mu^{\otimes 2}}. \quad \Box$$

Lemma G.5 (Conditional second moment). Let $w \in L^\infty(\mu) \otimes L^\infty(\mu)$ be a symmetric tensor with sum-of-squares representation

$$w = \sum_{k=1}^n \vartheta_k v_k^{\otimes 2}.$$

as in Lemma G.3. Then for each $0 \leq t \leq T$, the $\mathcal{F}_t$-conditional second moment of $\langle \Pi_T, w \rangle_{\mu^{\otimes 2}}$ is given by

$$E \left[ \langle \Pi_T, w \rangle^2_{\mu^{\otimes 2}} \bigg| \mathcal{F}_t \right] = 2 \sum_{k=1}^n \vartheta_k^2 + 2 \sum_{k=1}^n \vartheta_k^2 \langle \Pi_t, \phi_1(T-t,v_k,0)^{\otimes 2} \rangle_{\mu^{\otimes 2}}$$

$$+ \left( \sum_{k=1}^n \vartheta_k + \sum_{k=1}^n \vartheta_k \langle \Pi_t, \phi_1(T-t,v_k,0)^{\otimes 2} \rangle_{\mu^{\otimes 2}} \right)^2.$$

Proof. As in the proof of Lemma G.3 we have

$$\psi_0(T-t, iq\vartheta, 0) = -\frac{1}{2} \sum_{k=1}^n \log(1-2i\vartheta_k),$$

$$\psi_1(T-t, iq\vartheta, 0)(x,y) = e^{-(T-t)(x+y)} \sum_{k=1}^n \vartheta_k v_k(x)v_k(y).$$

For the derivatives of $\psi_0(T-t, iq\vartheta, 0)$ with respect to $q$ we have

$$\partial_q \psi_0(T-t, iq\vartheta, 0) = i \sum_{k=1}^n \frac{\partial_k}{1-2i\vartheta_k},$$

$$\partial_q^2 \psi_0(T-t, iq\vartheta, 0) = -2 \sum_{k=1}^n \frac{\partial_k^2}{(1-2i\vartheta_k)^2}.$$
and for the derivatives of $\psi_1(T - t, iqw, 0)$ with respect to $q$ we have

$$\partial_q \psi_1(T - t, iqw, 0)(x, y) = ie^{-(T-t)(x+y)} \sum_{k=1}^{n} \frac{\partial}{\partial q} \frac{k}{(1 - 2iq\theta_k)} v_k(x)v_k(y),$$

$$\partial^2_q \psi_1(T - t, iqw, 0)(x, y) = -2e^{-(T-t)(x+y)} \sum_{k=1}^{n} \frac{\partial^2}{\partial q^2} \frac{k}{(1 - 2iq\theta_k)}^2 v_k(x)v_k(y).$$

Using the characteristic function we obtain

$$E \left[ \left( \Pi_T, w \right)_{\mu \otimes \mathbb{Z}}^2 \bigg| \mathcal{F}_t \right] = -\partial^2_q \psi_1|_{q=0} \mathbb{E} \left[ e^{i\Psi(T-t, v_k, 0)}_{\mu \otimes \mathbb{Z}} \bigg| \mathcal{F}_t \right]$$

$$= 2 \sum_{k=1}^{n} \partial_k^2 + 2 \sum_{k=1}^{n} \partial_k^2 \left( \Pi_t, \phi_1(T - t, v_k, 0)_{\mu \otimes \mathbb{Z}} \right)$$

$$+ \left( \sum_{k=1}^{n} \partial_k + \sum_{k=1}^{n} \partial_k \left( \Pi_t, \phi_1(T - t, v_k, 0)_{\mu \otimes \mathbb{Z}} \right) \right)^2.$$  □

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