Abstract

We give a generalized definition of stretch that simplifies the efficient construction of low-stretch embeddings suitable for graph algorithms. The generalization, based on discounting highly stretched edges by taking their $p^{th}$ power for some $0 < p < 1$, is directly related to performances of existing algorithms. This discounting of high-stretch edges allows us to treat many classes of edges with coarser granularity. It leads to a two-pass approach that combines bottom-up clustering and top-down decompositions to construct these embeddings in $O(m \log \log n)$ time. Our algorithm parallelizes readily and can also produce generalizations of low-stretch subgraphs.

1 Introduction

Over the last few years substantial progress has been made on a large class of graph theoretic optimization problems. We now know substantially better asymptotic running time bounds and parallelizations for approximate undirected maximum flow/minimum cut [Mad10, CKM+11, LRS13, KLOS13, She13], bipartite matching [Mad13], minimum cost maximum flow [DS08], minimum energy flows [ST04, KMP11, KOSZ13, CFM+14], and graph partitioning [She09, OSV12]. One commonality of all these new algorithms is that they either explicitly find low-stretch spanning trees or call an algorithm that at least at present uses these trees.

The fastest known algorithm for generating these trees, due to Abraham and Neiman runs in $O(m \log n \log \log n)$ time [AN12]. Among the problems listed above, this running time is only the bottleneck for the minimum energy flow problem and its dual, solving symmetric diagonally dominant linear systems. However, there is optimism that all of the above problems can be solved in $o(m \log n)$ time, in which case finding these trees becomes a bottleneck as well. The main question

*Work supported in part by NSF grants CCF-1018463 and CCF-1065106.
†Part of this work was done while at CMU
‡Part of this work was done while at CMU and was supported by a Microsoft Research PhD Fellowship
we address in this paper is finding algorithms for constructing even better trees in $O(m)$ time. Unfortunately, this remains an open question.

This paper removes the tree construction obstacle from $o(m \log n)$ time algorithms for solving SDD systems, as well as other graph optimization problems. We give two modifications to the definition of low stretch spanning trees that can simplify and speed up their construction. Firstly, we allow additional vertices in the tree, leading to a Steiner tree. This avoids the need for the complex graph decomposition scheme of [AN12]. Secondly, we discount the cost of high-stretch edges in ways that more accurately reflect how these trees are used. This allows the algorithm to be more “forgetful,” and is crucial to our speedup.

Throughout this paper we let $G = (V, E, l)$ be a graph with edge lengths $l(e)$, and $T = (V_T, E_T, l_T)$ to denote the trees that we consider. In previous works on low stretch spanning trees, $T$ was required to be a subgraph of $G$ in the weighted sense. In other words, $E_T \subseteq E$, and $l_T(e) = l(e)$ for all $e \in E_T$. We relax this condition by only requiring edge lengths in $T$ to be not too short with respect to $G$ through the notion of embeddability, which we formalize in Section 2.

For a tree $T = (V_T, E_T, l_T)$, the stretch of an edge $e = uv$ with respect to $T$ is

$$\text{STR}_T(e) \overset{\text{def}}{=} \frac{l_T(u, v)}{l(e)},$$

where $l_T(u, v)$ is the length of the unique path between $u$ and $v$ in $T$. Previous tree embedding algorithms aim to pick a $T$ such that the total stretch of all edges $e$ in $G$ is small [AKPW95, AN12]. A popular alternate goal is to show that the expected stretch of any edge is small, and these two definitions are closely related [AKPW95, CCG+98]. Our other crucial definition is the discounting of high stretches by adopting the notion of $\ell_p$-stretch:

$$\text{STR}^p_T(e) \overset{\text{def}}{=} (\text{STR}_T(e))^p.$$

These two definitional changes greatly simplify the construction of low stretch embeddings. It also allows the combination of existing algorithms in a robust manner. Our algorithm is based on the bottom-up clustering algorithm used to generate AKPW low-stretch spanning trees [AKPW95], combined with the top-down decompositions common in recent algorithms [Bar96, EEST08, ABN08, AN12]. Its guarantees can be stated as follows:

**Theorem 1.1** Let $G = (V, E, d)$ be a weighted graph with $n$ vertices and $m$ edges. For any parameter $p$ strictly between 0 and 1, we can construct a distribution over trees embeddable in $G$ such that for any edge $e$ its expected $\ell_p$-stretch in a tree picked from this distribution is $O((\frac{1}{1-p})^2 \log^p n)$. Furthermore, a tree from this distribution can be picked in expected $O(\frac{1}{1-p} m \log \log n)$ time in the RAM model.

We will formally define embeddability, as well as other notations, in Section 2. An overview of our algorithm for generating low $\ell_p$-stretch embeddable trees is in Section 3. We expand on it using existing low-stretch embedding algorithms in mostly black-box manners in Section 4. Then in Section 5 we show a two-stage algorithm that combines bottom-up and top-down routines that gives our main result.

Although our algorithm runs in $O(m \log \log n)$ time, the running time is in the RAM model, and our algorithm calls a sorting subroutine. As sorting is used to approximately bucket the edge
weights, this dependency is rather mild. If all edge lengths are between 1 and Δ, this process can be done in \( O(m \log(\log \Delta)) \) time in the pointer machine model, which is \( O(m \log \log m) \) when \( \Delta \leq m^{\text{poly}(\log m)} \). We suspect that there are pointer machine algorithms without even this mild dependence on Δ, and perhaps even algorithms that improve on the runtime of \( O(m \log \log n) \).

Less speculatively, we also believe that our two-stage approach of combining bottom-up and top-down schemes can be applied with the decomposition scheme of [AN12] to generate actual spanning trees (as opposed to merely embeddable Steiner trees) with low \( \ell_p \)-stretch. However, we do not have a rigorous analysis of this approach, which would presumably require a careful interplay with the radius-bounding arguments in that paper.

1.1 Related Works

Alon et al. [AKPW95] first proposed the notion of low stretch embeddings and gave a routine for constructing such trees. They showed that for any graph, there is a distribution over spanning trees such that the expected stretch of an edge is \( \exp(O(\sqrt{\log n \log \log n})) \). Subsequently, results with improved expected stretch were obtained by returning an arbitrary tree metric instead of a spanning tree. The only requirement on metric on these tree metrics is that they don’t shorten distances from the original graph, and they may also include extra vertices. However, in contrast to the objects constructed in this paper, they do not necessarily fulfill the embeddability property.

Bartal gave trees with expected stretch of \( O(\log^2 n) \) [Bar96], and \( O(\log n \log \log n) \) [Bar98]. Optimal trees with \( O(\log n) \) stretches are given by Fakcharoenphol et al. [FRT04], and are known as the FRT trees. This guarantee can be written formally as

\[
E_T[\text{STR}_T(e)] \leq O(\log n).
\]

Recent applications to SDD linear system solvers has led to renewed interest in finding spanning trees with improved stretch over AKPW trees. The first LSSTs with \( \text{poly}(\log n) \) stretch were given by Elkin et al. [EEST08]. Their algorithm returns a tree such that the expected stretch of an edge is \( O(\log^2 n \log \log n) \), which has subsequently been improved to \( O(\log n \log \log n (\log \log \log n)^3) \) by Abraham et al. [ABN08] and to \( O(\log n \log \log n) \) by Abraham and Neiman [AN12].

Notationally our guarantee is almost identical to the expected stretch above when \( p \) is a constant strictly less than 1:

\[
E_T[\text{STR}_T^p(e)] \leq O(\log^p n).
\]

The power mean inequality implies that our embedding is weaker than those with \( \ell_1 \)-stretch bounds. However, at present, \( O(\log n) \) guarantees for \( \ell_1 \)-stretch are not known—the closest is the result by Abraham and Neiman [AN12], which is off by a factor of \( \log \log n \).

Structurally, the AKPW low-stretch spanning trees are constructed in a bottom-up manner based on repeated clusterings [AKPW95]. Subsequent methods are based on top down decompositions starting with the entire graph [Bar96]. Although clusterings are used implicitly in these algorithms, our result is the first that combines these bottom-up and top-down schemes.

1.2 Applications

The \( \ell_p \)-stretch embeddable trees constructed in this paper can be used in all existing frameworks that reduce the size of graphs using low-stretch spanning trees. In Appendix A, we check that the larger graph with Steiner trees can lead to linear operators close to the graph Laplacian of the
original graph. It allows us to use these trees in algorithms for solving linear systems in graph Laplacians, and in turn SDD linear systems. This analysis also generalizes to other convex norms, which means that our trees can be used in approximate flow [LS13, She13] and minimum cut [Mad10] algorithms.

Combining our algorithm with the recursive preconditioning framework by Koutis et al. [KMP11] leads to an algorithm that runs solves such a system to constant accuracy in $O(m \log n)$ time. They are also crucial for the recent faster solver by Cohen et al. [CKP14], which runs in about $m \log^{1/2} n$ time. Parallelizations of it can be used can also lead to work-efficient parallel algorithms for solving SDD linear systems with depth of about $m^{1/3}$ [BGK+13], and in turn for spectral sparsification [SS08, KLP12]. For these parallel applications, ignoring a suitable fraction of the edges leads to a simpler algorithm with lower depth. This variant is discussed in Section 5.3. On the other hand, these applications can be further improved by incorporating the recent polylog depth, nearly-linear work parallel solver by Peng and Spielman [PS13]. Consequently, we omit discussing the best bounds possible with the hope of a more refined parallel algorithm.

2 Background

Before we describe our algorithm, we need to formally specify the simple embeddability property that our trees satisfy. The notion used here is the same as the congestion/dilation definition widely used in routing [Lei92, LMR94]. It was used explicitly in earlier works on combinatorial preconditioning [Vai91, Gre96], and is implicit in the more recent algorithms.

Informally, an embedding generalizes the notion of a weighted subgraph in two ways. First, in an embedding of $H$ into $G$, edges in $H$ may correspond to paths in $G$, rather than just edges. Second, $H$ may contain Steiner vertices that can be seen as “shadow copies” of vertices in $G$. Edges in $G$ can be apportioned between different paths and connect to different Steiner vertices, but their weight must be reduced proportionally.

Formally, an embedding can be viewed as a weighted mapping from one graph to another. Splitting an edge will make it lighter, and therefore easier to embed. However, it will also make it harder to traverse, and therefore longer. As a result, for embeddings it is convenient to view an edge $e$ by both its length $l(e)$ and weight $w(e)$, which is the reciprocal of its length:

$$w(e) \overset{\text{def}}{=} \frac{1}{l(e)}.$$

A path embedding is then a weighted mapping from the edges of a graph to paths in another. Such a mapping from a graph $H = (V_H, E_H, l_H)$ to a graph $G = (V_G, E_G, l_G)$ is given by the following three functions:

1. A mapping from vertices of $H$ to those in $G$, $\pi : V_H \rightarrow V_G$.
2. A function from each edge $e_H \in E_H$ to a weighted path of $G$, denoted by $\text{Path}(e_H = x_G y_G)$ that goes from $\pi(x_G)$ to $\pi(y_G)$.
3. We let $W_{\text{path}}(e_H, e_G)$ denote the weight of the edge $e_G$ on path $\text{Path}(e_H)$. This value is zero if $e_G \notin \text{Path}(e_H)$.

The congestion-dilation notion of embeddability can then be formalized as follows:
Definition 2.1 A graph $H$ is path embeddable, or simply embeddable, into a graph $G$, if there exists a path embedding $(\pi, \text{Path})$ of $H$ into $G$ such that:

- for all edges $e \in E_G$, $\sum_{e_H \in E_H} W_{\text{Path}}(e_H, e_G) \leq w_G(e_G)$: congestion is at most one, and
- for all edges $e_H \in E_H$, $\sum_{e_G \in \text{Path}(e_H)} \frac{1}{w_{\text{Path}}(e_H, e_G)} \leq l_H(e) = \frac{1}{w_H(e)}$: dilation is at most one.

Note that since $G$ has no self-loops, the definition precludes mapping both endpoints of an edge in $H$ to the same point in $G$. Also note that if $H$ is a subgraph of $G$ such that $l_H(e) \geq l_G(e)$, setting $\pi$ to be the identity function and $\text{Path}(e) = e$ and $W_{\text{Path}}(e, e) = w_H(e)$ is one way to certify embeddability.

3 Overview

We now give an overview of our main results. Our algorithm follows the decomposition scheme taken by Bartal for generating low stretch embeddings [Bar96]. This scheme partitions the graph repeatedly to form a laminar decomposition, and then constructs a tree from the laminar decomposition. However, our algorithm also makes use of spanning trees of the decomposition itself. As a result we start with the following alternate definition of Bartal decompositions where these trees are clearly indicated.

Definition 3.1 Let $G = (V, E, l)$ be a connected multigraph. We say that a sequence of forests $B$, where

$$B = (B_0, B_1, \ldots, B_t),$$

is a Bartal decomposition of $G$ if all of the following conditions are satisfied:

1. $B_0$ is a spanning tree of $G$ and $B_t$ is an empty graph.
2. For any $i \leq t$, $B_i$ is a subgraph of $G$ in the weighted sense.
3. For any pair of vertices $u, v$ and level $i < t$, if $u$ and $v$ are in the same connected component of $B_{i+1}$, then they are in the same connected component of $B_i$.

Condition 2 implies that each of the $B_i$s is embeddable into $G$. A strengthening of this condition would require the union of all the $B_i$s to be embeddable into $G$. We will term such decompositions embeddable Bartal decompositions.

Bartal decompositions correspond to laminar decompositions of the graphs: if any two vertices $u$ and $v$ are separated by the decomposition in level $i$, then they are also separated in all levels $j > i$. If $u$ and $v$ are in the same partition in some level $i$, but are separated in level $i + 1$, we say that $u$ and $v$ are first cut at level $i$. This definition is useful because if the diameters are decreasing, the stretch of an edge can be bounded using only information related to level at which it is first cut.

We will work with bounds on diameters, $d = (d_0, \ldots, d_t)$. We say that such a sequence is geometrically decreasing if there exists some constant $0 < c < 1$ such that $d_{i+1} \leq cd_i$. Below we formalize a condition when such sequences can be used as diameter bounds for a Bartal decomposition.
Definition 3.2 A geometrically decreasing sequence $d = (d_0 \ldots d_t)$ bounds the diameter of a Bartal decomposition $B$ if for all $0 \leq i \leq t$,

1. The diameter of any connected component of $B_i$ is at most $d_i$, and
2. any edge $e \in B_i$ has length $l(e) \leq \frac{d_i}{\log n}$.

Given such a sequence, the bound $d_i$ for the level where an edge is first cut dominates its final stretch. This motivates us to define the $\ell_p$-stretch of an edge w.r.t. a Bartal decomposition as follows:

Definition 3.3 Let $B$ be a Bartal decomposition with diameter bounds $d$, and $p$ a parameter such that $p > 0$. The $\ell_p$-stretch with respect to $B, d$ of an edge $e$ with length $l(e)$ that is first cut at level $i$ is

$$\text{STR}_{B,d}^p(e) \overset{\text{def}}{=} \left( \frac{d_i}{l(e)} \right)^p.$$

In Section 4, we will check rigorously that it suffices to generate (not necessarily embeddable) Bartal decompositions for which edges are expected to have small $\ell_p$-stretch. We will give more details on these transformations later in the overview as well.

The decomposition itself will be generated using repeated calls to variants of probabilistic low-diameter decomposition routines [Bar96]. Such routines allow one to partition a graph into pieces of diameter $d$ such that the probability of an edge being cut is at most $O(\log n/d)$. At a high level, our algorithm first fixes a geometrically decreasing sequence of diameter bounds, then repeatedly decomposes all the pieces of the graph. With regular ($\ell_1$) stretch, such routines can be shown to give expected stretch of about $\log^2 n$ per edge [Bar96], and most of the follow-up works focused on reducing this factor. With $\ell_p$-stretch on the other hand, such a trade-off is sufficient for the optimum bounds when $p$ is a constant bounded away from 1.

Lemma 3.4 Let $B$ be a distribution over Bartal decompositions. If $d$ is a geometrically decreasing sequence that bounds the diameter of any $B \in \mathcal{B}$, and the probability of an edge with length $l(e)$ being cut on level $i$ of some $B \in \mathcal{B}$ is

$$O\left(\left(\frac{l(e) \log n}{d_i}\right)^q\right)$$

for some $0 < q < 1$. Then for any $p$ such that $0 < p < q$, we have

$$\mathbb{E}_{B \in \mathcal{B}} \left[ \text{STR}_{B,d}^p(e) \right] \leq O\left(\frac{1}{q - p} \log^p n\right)$$

Its proof relies on the following fact about geometric series, which plays a crucial role in all of our analyses.

Fact 3.5 There is an absolute constant $c_{geo}$ such that if $c$ and $\epsilon$ are parameters such that $c \in [e, e^2]$ and $\epsilon > 0$

$$\sum_{i=0}^{\infty} c^{-i} = c_{geo} \epsilon^{-1}.$$
**Proof** Since \(0 < c^{-1} < 1\), the sum converges, and equals
\[
\frac{1}{1 - c^{-\epsilon}} = \frac{1}{1 - \exp(-\epsilon \ln c)}.
\]
Therefore it remains to lower bound the denominator. If \(\epsilon \geq 1/4\), then the denominator can be bounded by a constant. Otherwise, \(\epsilon \ln c \leq 1/2\), and we can invoke the fact that \(\exp(-t) \leq 1 - t/2\) when \(t \leq 1/2\) to obtain
\[
1 - \exp(-\epsilon \ln c) \geq \epsilon \ln c.
\]
Substituting in the bound on \(c\) and this lower bound into the denominator then gives the result. ■

**Proof of Lemma 3.4:** If an edge is cut at a level with \(d_i \leq l(e) \log n\), its stretch is at most \(\log n\), giving an \(\ell_p\)-stretch of at most \(\log n\). It remains only to consider the levels with \(d_i \geq l(e) \log n\). Substituting the bounds of an edge cut on level \(i\) and the probability of it being cut into the definition of \(\ell_p\)-stretch gives:
\[
E_B[ STR_{B,d}^p(e)] \leq \sum_{i,d_i \geq \log n} \left( \frac{d_i}{l(e)} \right)^p O \left( \left( \frac{l(e) \log n}{d_i} \right)^q \right) = O \left( \log n \sum_{i,d_i \geq \log n} \left( \frac{l(e) \log n}{d_i} \right)^{q-p} \right).
\]
Since an edge \(e\) is only cut in levels where \(d_i \geq l(e) \log n\) and the \(d_i\)s are geometrically increasing, this can be bounded by
\[
O \left( \log n \sum_{i=0}^{c^{-i(q-p)}} \right)
\]
Invoking Fact 3.5 then gives a bound of \(O \left( \frac{1}{q-p} \log n \right)\). ■

This is our approach for showing that a Bartal decomposition has small \(\ell_p\)-stretch, and it remains to convert them into embeddable trees. This conversion is done in two steps: we first show how to obtain a decomposition such that all of the \(B_i\)s are embeddable into \(G\), and then we give an algorithm for converting such a decomposition into a Steiner tree. To accomplish the former, we first ensure that each \(B_i\) is embeddable by choosing them to be subgraphs. Then we present pre-processing and post-processing procedures that converts such a guarantee into embeddability of all the \(B_i\)s simultaneously.

In order to obtain a tree from the decomposition, we treat each cluster in the laminar decomposition as a Steiner vertex, and join them using parts of \(B_i\)s. This step is similar to Bartal trees in that it identifies centers for each of the \(B_i\)s, and connects the centers between one level and the next. However, the need for the final tree to be embeddable means that we cannot use the star-topology from Bartal trees [Bar96]. Instead, we must use part of the \(B_i\)s between the centers. As each \(B_i\) is a forest with up to \(n\) edges, a tree obtained as such may have a much larger number of Steiner vertices. As a result, the final step involves reducing the size of this tree by contracting the paths connecting the centers. This process is illustrated in Figure 1.

In Section 4, we give the details on these steps that converts Bartal decompositions to embeddable trees. Furthermore, we check that Bartal’s algorithm for generating such trees meets the good cutting probability requirements of Lemma 3.4. This then gives the following result:
Lemma 3.6 Given a graph $G$ with weights are between $[1, \Delta]$, for the diameter sequence $d$ where $d_0 = 2n\Delta, d_1 = 2^{-1}n\Delta, \ldots d_t < 1$, we can create a distribution over Bartal decompositions with diameters bounded by $d$ such that for any edge $e$ and any parameter $0 < p < 1$,

$$\mathbb{E}_B[STR^p_{B,d}(e)] \leq O\left(\frac{1}{1-p}\log^p n\right).$$

Furthermore, a random decomposition from this distribution can be sampled with high probability in $O(m \log (n \Delta) \log n)$ time in the RAM model.

This routine plus the transformations gives a simple algorithm for constructing low $\ell_p$-stretch embeddable trees. with expected stretch matching the bound stated our main result, Theorem 1.1. However, the running time of $O(m \log (n \Delta) \log n)$ is more than the current best for finding low-stretch spanning trees [AN12], as well as the $O(m \log^2 n)$ running time for finding Bartal trees.

Our starting point towards a faster algorithm is the difference between our simplified routine and Bartal’s algorithm. Bartal’s algorithm, as well as subsequent algorithms [EEST08] ensure that an edge participates in only $O(\log n)$ partitions. At each step, they work on a graph obtained by contracting all edges whose lengths are less than $d_i/poly(n)$. This coupled with the upper bound of edge lengths from Definition 3.2, Part 2 and the geometric decrease in diameter bounds gives that each edge is involved in $O(\log(poly(n))) = O(\log n)$ steps of the partition.

As a path in the tree has at most $n$ edges, the additive increase in stretch caused by these shrunken edges is negligible. Furthermore, the fact that the diameter that we partition upon de-
creases means that once we uncontract an edge, it remains uncontracted in all future steps. Therefore, these algorithms can start from the initial contraction for $d_0$, and maintain all contractions in work proportional to their total sizes.

When viewed by itself, this contraction scheme is almost identical to Kruskal’s algorithm for building minimum spanning trees (MSTs). This suggests that the contraction sequence can be viewed as another tree underlying the top-down decomposition algorithm. This view also leads to the question of whether other trees can be used in place of the MST. In Section 5, we show that if the AKPW low-stretch spanning tree is used instead, each edge is expected to participate in $O(\log \log n)$ levels of the top-down decomposition scheme. Combining this with a $O(m \log \log n)$ time routine in the RAM model for finding the AKPW low-stretch spanning tree and a faster decomposition routine then leads to our faster algorithm.

Using these spanning trees to contract parts of the graph leads to additional difficulties in the post-processing steps where we return embeddable Steiner trees. A single vertex in the contracted graph may correspond to a large cluster in the original graph. As a result, edges incident to it in the decomposition may need to be connected by long paths. Furthermore, the total size of these paths may be large, which means that they need to be treated implicitly. In Section 5.5, we leverage the tree structure of the contraction to implicitly compute the reduced tree. Combining it with the faster algorithm for generating Bartal decompositions leads to our final result as stated in Theorem 1.1.

4 From Bartal Decompositions to Embeddable Trees

In this section, we show that embeddable trees can be obtained from Bartal decompositions using the process illustrated in Figure 1. We do this in three steps: exhibiting Bartal’s algorithm in Section 4.1, showing that a decomposition routine that makes each $B_i$ embeddable leads to a routine that generates embeddable decompositions in Section 4.2, and giving an algorithm for finding a tree from the decomposition in Section 4.3. We start by formally describing Bartal’s algorithm for decomposing the graph.

4.1 Bartal’s Algorithm

Bartal’s algorithm in its simplest form can be viewed as repeatedly decomposing the graph so the pieces have the diameter guarantees specified by $d$. At each step, it calls a low-diameter probabilistic decomposition routine with the following guarantees.

Lemma 4.1 (Probabilistic Decomposition) There is an algorithm Partition that given a graph $G$ with $n$ vertices and $m$ edges, and a diameter parameter $d$, returns a partition of $V$ into $V_1 \cup V_2 \cup \ldots \cup V_k$ such that:

1. The diameter of the subgraph induced on each $V_i$ is at most $d$ with high probability, certified by a shortest path tree on $V_i$ with diameter $d$, and

2. for any edge $e = uv$ with length $l(e)$, the probability that $u$ and $v$ belong to different pieces is at most $O\left(\frac{l(e) \log n}{d}\right)$.

Furthermore, Partition can be implemented using one call to finding a single source shortest path tree on the same graph with all vertices connected to a super-source by edges of length between 0 and $d$.
**B = DecomposeSimple**(\(G, d\)), where \(G\) is a multigraph, and \(d\) are diameter bounds.

1. Initialize \(B\) by setting \(B_0\) to a shortest path tree from an arbitrary vertex in \(V_G\).

2. For \(i = 1 \ldots t\) do
   
   (a) Initialize \(B_i\) to empty.
   
   (b) Remove all edges \(e\) with \(l(e) \geq \frac{d_i}{\log n}\) from \(G\).
   
   (c) For each subgraph \(H\) of \(G\) induced by a connected component of \(B_{i-1}\) do
      
      i. \(G_1 \ldots G_k \leftarrow \text{Partition}(H, d_i)\).
      
      ii. Add shortest path trees in each \(G_j\) to \(B_i\).

3. Return \(B\).

---

**Figure 2: Bartal’s Decomposition Algorithm**

This routine was first introduced by Bartal to construct these decompositions. It and the low diameter decompositions that it’s based on constructed each \(V_i\) in an iterative fashion. Miller et al. [MPX13] showed that a similar procedure can be viewed globally, leading to the implementation-independent view described above. Dijkstra’s algorithm (Chapter 24 of [CSRL01]) then allows one to obtain a running time of \(\mathcal{O}((m+n) \log n)\). It can be further sped up to \(\mathcal{O}(m+n \log n)\) using Fibonacci heaps due to Fredman and Tarjan [FT87], and to \(\mathcal{O}(m)\) in the RAM model by Thorup [Tho00]. In this setting where approximate answers suffice, a running time of \(\mathcal{O}(m + n \log \log \Delta)\) was also obtained by Koutis et al. [KMP11]. As our faster algorithm only relies on the shortest paths algorithm in a more restricted setting, we will use the most basic \(\mathcal{O}(m \log n)\) bound for simplicity.

We can then obtain Bartal decompositions by invoking this routine recursively. Pseudocode of the algorithm is given in Figure 2. The output of this algorithm for a suitable diameter sequence gives us the decomposition stated in Lemma 3.6.

**Proof of Lemma 3.6:** Consider the Bartal distributions produced by running \(\text{DecomposeSimple}(G, d)\). With high probability it returns a decomposition \(B\). We first check that \(B\) is a Bartal decomposition. Each tree in \(B_i\)’s is a shortest path tree on a cluster of vertices formed by the partition. As these clusters are disjoint, \(B_i\) is a subgraph in the weighted sense. Since the algorithm only refines partitions, once two vertices are separated, they remain separated for any further partitions. Also, the fact that \(B_0\) is spanning follows from the initialization step, and \(B_i\) cannot contain any edge since any edge has length at least 1 and \(d_t < 1\).

We now show that \(d\) are valid diameter bounds for any decomposition produced with high probability. The diameter bounds on \(d_i\) follow from the guarantees of \(\text{Partition}\) and the initialization step. The initialization of \(d_0 = 2n\Delta\) also ensures that no edge’s length is more than \(\frac{d}{\log n}\), and this invariant is kept by discarding all edges longer than \(\frac{d}{\log n}\) before each call to \(\text{Partition}\).

The running time of the algorithm follows from \(t \leq \mathcal{O}(\log(n\Delta))\) and the cost of the shortest path computations at all the steps. It remains to bound the expected \(\ell_p\)-stretch of an edge \(e\) w.r.t. the
decomposition. When $l(e) \geq \frac{d_{i}}{c \log n}$, a suitable choice of constants allows us to bound the probability of $e$ being cut by 1. Otherwise, $e$ will not be removed unless it is already cut. In case that it is in the graph passed onto PARTITION, the probability then follows from Lemma 4.1. Hence the cutting probability of edges satisfies Lemma 3.4, which gives us the bound on stretch.

4.2 Embeddability by Switching Moments

We now describe how to construct embeddable Bartal decomposition by using a routine that returns Bartal decompositions. This is done in three steps: pre-processing the graph to transform the edge lengths of $G$ to form $G'$, running the decomposition routine on $G'$ for a different parameter $q$, and post-processing its output.

Pseudocode of this conversion procedure is given in Figure 3. Both the pre-processing and post-processing steps are deterministic, linear mappings. As a result, we can focus on bounding the expected stretch of an edge in the decomposition given by $\text{DECOMPOSE}$.

\begin{verbatim}
B, d = EMBEDDABLEDECOMPOSE(G, p, q, DECOMPOSE_q) where G is a graph, p, q are exponents for stretch, and DECOMPOSE is a routine that generates a decomposition B' along with diameter bounds d'.
1. Create graph G' with edge lengths $l'(e) = l(e)^{\frac{q}{p}}$.
2. B', d' ← DECOMPOSE_q(G').
3. Create decomposition B and diameter bounds d scaling lengths in B_i' and d_i by
   \[ \frac{c_{geo}}{q - p} \left( \frac{d_i'}{log n} \right)^{\frac{q - p}{p}} \]
   where $c_{geo}$ is the constant given by Fact 3.5.
4. Return B, d.
\end{verbatim}

Figure 3: Using a generic decomposition routine to generate an embeddable decomposition

We first verify that $d$ is a geometrically decreasing sequence bounding the diameters of $B$.

\textbf{Lemma 4.2} If $B'$ is a Bartal decomposition of $G'$ whose diameters are bounded by $d'$, then $d$ is geometrically decreasing sequence that bound the diameter of $B$.

\textbf{Proof}

The post-processing step scales the difference between adjacent $d'$s by an exponent of $\frac{q}{p}$, which is at least 1 since $q > p$. Therefore $d$ is also a geometrically decreasing sequence. As the lengths in $B_i'$ and $d_i'$ are scaled by the same factor, $d_i$ remains an upper bound for the diameter of $B_i$. Also, since $d_i' \geq l'(e) \log n = l(e)^{\frac{q}{p}} \log n$, we have

\[ d_i = \frac{c_{geo}}{q - p} \left( \frac{d_i'}{\log n} \right)^{\frac{q - p}{p}} d_i' \geq l(e) \log n. \]
Therefore $d$ upper bounds the diameters of $B$ as well.

We now check that $B$ is a subgraph in the weighted case, which makes it an embeddable Bartal decomposition.

**Lemma 4.3** For any edge $e$ we have

$$\sum_i w_{B_i}(e) \leq w(e).$$

**Proof** Combining the pre-processing and post-processing steps gives that the total weight of $e$ in all the layers is:

$$\sum_i w_{B_i}(e) = \sum_{i,e \in B_i} \frac{1}{l_{B_i}(e)} = \frac{p-q}{c_{geo}} \sum_{i,e \in B_i} \left( \frac{\log n}{d'_i} \right)^{\frac{q-p}{p}} w(e)^{\frac{q}{p}}.$$

Showing that this is at most $w(e)$ is therefore equivalent to showing

$$\frac{p-q}{c_{geo}} \sum_{i,e \in B_i} \left( \frac{\log n}{d'_i w(e)^{\frac{q}{p}}} \right)^{\frac{q-p}{p}} \leq 1.$$

Here we make use of the condition that the levels in which edge $e$ appears have $d'_i \geq l'(e) \log n$. Substituting in $l'(e) = w(e)^{-\frac{q}{p}}$ into this bound on $d'_i$ gives:

$$d'_i \geq w(e)^{-\frac{q}{p}} \log n$$
$$d'_i w(e)^{\frac{q}{p}} \geq \log n.$$

As $d'_i$s are decreasing geometrically, this means that these terms are a geometrically decreasing sequence whose first term can be bounded by 1. Fact 3.5 then gives that the summation is bounded by $\frac{c_{geo}}{q-p} \leq \frac{c_{geo}}{q-p}$, which cancels with the coefficient in front of it.

We can also check that the stretch of an edge $e$ w.r.t. $B, d$ is comparable to its stretch in $B', d'$.

**Lemma 4.4** For parameters $0 < p < q < 1$, the $\ell_p$-stretch of an edge $e$ in $G$ w.r.t. $B, d$ and its $\ell_q$-stretch in $G'$ w.r.t. $B', d'$ are related by

$$\text{STR}_{B,d}^{\ell_p}(e) = O\left( \frac{1}{q-p} \log^{p-q} n \cdot \text{STR}_{B',d'}^{\ell_q}(e) \right).$$

**Proof** Rearranging scaling on $d'_i$ used to obtain $d_i$ gives

$$d_i = \frac{c_{geo}}{q-p} \left( \frac{d'_i}{\log n} \right)^{\frac{q-p}{p}} d'_i = \frac{c_{geo}}{q-p} \log^{p-q} n \cdot d'_i^{\frac{q}{p}}.$$

We can then relate the stretch of an edge in the new decomposition with that of its $\ell_q$-stretch in $G'$. For an edge cut at level $i$, we have

$$\text{STR}_{B,d}(e) = \frac{d_i}{l(e)} = \frac{c_{geo}}{q-p} \log^{p-q} n \cdot d'_i^{\frac{q}{p}} = \frac{c_{geo}}{q-p} \log^{p-q} n \left( \text{STR}_{B',d'}^{\ell_q}(e) \right)^{\frac{q}{p}}.$$
Taking both sides to the $p$-th power, and using the fact that $p < 1$, then gives the desired bound.

It’s worth noting that when $p$ and $q$ are bounded away from 1 by constants, this procedure is likely optimal up to constants. This is because the best $t_p$-stretch that one could obtain in these settings are $O(\log^p n)$ and $O(\log^q n)$ respectively.

4.3 From Decompositions to Trees

It remains to show that an embeddable decomposition can be converted into an embeddable tree. Our conversion routine is based on the laminar-decomposition view of the decomposition. From the bottommost level upwards, we iteratively reduce the interaction of each cluster with other clusters to a single vertex in it, which we term the centers. Centers can be picked arbitrarily, but to enforce the laminar decomposition view, we require that if a vertex $u$ is a center on level $i$, it is also a center on level $i + 1$ and therefore all levels $j > i$. Once the centers are picked, we can connect the clusters starting at the bottom level, by connecting all centers of level $i + 1$ to the center of the connected component they belong to at level $i$. This is done by taking the part of $B_i$ involving these centers. We first show that the tree needed to connect them has size at most twice the number of centers.

Fact 4.5 Given a tree $T$ and a set of $k$ vertices $S$, there is a tree $T_S$ on $2k - 1$ vertices including these $k$ leaves such that:

- The distances between vertices in $S$ are the same in $T$ and $T_S$.
- $T_S$ is embeddable into $T$.

Proof The proof is by induction on the number of vertices in $T$. The base case is when $T$ has fewer than $2k - 1$ vertices, where it suffices to set $T_S = T$. For the inductive case suppose the result is true for all trees with $n$ vertices, and $T$ has $n + 1$ vertices. We will show that there is a tree $T'$ on $n$ vertices that preserves all distances between vertices in $S$, and is embeddable into $T$.

If $T$ has a leaf that’s not in $S$, removing it and the edge incident to it does not affect the distances between the vertices in $S$, and the resulting tree $T'$ is a subgraph and therefore embeddable into $T$. Otherwise, we can check via a counting argument that there is a vertex $u$ of degree 2 that’s not in $S$. Let this vertex and its two neighbors be $u$ and $v_1, v_2$ respectively. Removing $u$ and adding an edge between $v_1v_2$ with weight $l(uv_1) + l(uv_2)$ preserves distances. This new tree $T'$ is embeddable in $T$ by mapping $v_1v_2$ to the path $v_1 - u - v_2$ with weights equaling the weights of the two edges.

Since $T'$ has $n$ vertices, the inductive hypothesis gives the existence of a tree $T_S$ meeting the requirements. As $T'$ is embeddable into $T$, $T_S$ is embeddable into $T$ as well.

Invoking this routine repeatedly on the clusters then leads to the overall tree. Pseudocode of this tree construction algorithm is given in Figure 4.

Lemma 4.6 Given a graph $G$ and an embeddable Bartal decomposition $B$, BuildTree gives an embeddable tree $T$ with $O(n)$ vertices containing $V$ such that for any geometrically decreasing sequence $d$ that bounds the diameters of $B$ and any edge $e$ we have

$$STR_T(e) = O(STR_{B,d}(e)).$$
\[ T = \text{buildTree}(G, B) \] where \( B \) is a Bartal decomposition of \( G \).

1. Designate a center vertex for each connected component of each level of \( B \) such that if \( u \) is a center vertex on level \( i \), it is also a center vertex on level \( i + 1 \).

2. For each connected component on level \( i \).
   
   (a) Find all center vertices in level \( i + 1 \) contained in this piece.
   
   (b) Connect these vertices using the small sized equivalent of \( B_i \) given by Fact 4.5.

3. Return \( T \).

Figure 4: Constructing a Steiner tree from a Bartal decomposition

**Proof** We first bound the total size of \( T \). Note that the number of vertices added is proportional to the decrease in number of components. Since the initial number of clusters is \( n \), \( T \) has at most \( 2n - 1 \) vertices.

Fact 4.5 gives that the trees used to connect the level \( i + 1 \) clusters are embeddable into the corresponding connected component of \( B_i \). Since the vertices in these clusters are disjoint and \( \cup_i B_i \) is embeddable into \( G \), \( T \) is also embeddable into \( G \).

It remains to bound the stretch of edges w.r.t. \( T \). For an edge \( e = uv \) that’s cut at level \( i \), consider the path from \( u \) to the centers of the clusters levels \( t, t - 1, \ldots i \). The diameter bounds give that the distance traversed on level \( j \) is bounded by \( d_j \). As \( d \) is a geometrically decreasing sequence, the total length of this path is bounded by \( O(d_i) \). A similar argument can be applied to \( v \), and since \( u \) and \( v \) is cut at level \( i \), the centers on level \( i \) are the same. Therefore, the distance between \( u \) and \( v \) in the tree can be bounded by \( O(d_i) \), giving the bound on stretch.

Combining these pieces leads to an algorithm generating low \( \ell_p \)-stretch embeddable trees.

**Lemma 4.7** Let \( G = (V, E, w) \) be a weighted graph with \( n \) vertices and \( m \) edges and weights \( w : E \to [1, \Delta] \), and \( p \) be any parameter strictly between 0 and 1. We can construct a distribution over Bartal decompositions such that for any edge \( e \), its expected \( \ell_p \)-stretch in a decomposition picked from this distribution is \( O((\frac{1}{1-p})^2 \log^p n) \).

**Proof**

Consider running \texttt{EMBEDDABLEDECOMPOSE} with \( q = \frac{1+p}{2} \), and \texttt{DECOMPOSESIMPLE} with the parameters given by Lemma 3.6 as the decomposition procedure. By Lemmas 3.6 and 4.4, the expected stretch of an edge \( e \) in the post-processed decomposition \( B \) w.r.t. diameter bounds \( d \) is:

\[
O \left( \frac{1}{q - p} \log^{p-q} \frac{1}{1 - q} \log^q n \right) = O \left( \left( \frac{1}{1 - p} \right)^2 \log^p n \right).
\]

Running \texttt{BUILDTREE} on this decomposition then gives a tree where the expected stretch of edges are the same. The embeddability of this tree also follows from the embeddability of \( B \) given by Lemma 4.3.
To bound the running time, note that as $0 \leq \frac{p}{q} < 1$, the lengths of edges in the pre-processed graph $G'$ are also between 1 and $\Delta$. Both the pre and post processing steps consist of only rescaling edge weights, and therefore take linear time. The total running time then follows from Lemma 3.6.

5 Two-Stage Tree Construction

We now give a faster algorithm for constructing Bartal decompositions. The algorithm proceeds in two stages. We first quickly build a lower quality decomposition using the same scheme as the AKPW low stretch spanning tree [AKPW95]. Then we proceed in the same way as Bartal’s algorithm and refine the decompositions in a top-down manner. However, with the first stage decomposition, we are able to construct a Bartal decomposition much faster.

Both the AKPW decomposition and the way that our Bartal decomposition routine uses it relies on repeated clustering of vertices. Of course, in an implementation, such clusterings will be represented using various linked-list structures. However, from an analysis perspective, it is helpful to view them as quotient graphs. For a graph $G$ and a subset of edges $A$, we let the quotient graph $G/A$ be the graph formed by the connected components of $A$. Each of these components corresponding to subsets of vertices becomes a single vertex in $G/A$, and the edges have their vertices relabeled accordingly. For our algorithms, it is essential for us to keep multi-edges as separate copies. As a result, all the graphs that we deal with in this section are potentially multi-graphs, and we will omit this distinction for simplicity.

The main advantages offered by the AKPW decomposition are

- it is a bottom-up routine that can be performed in linear time, and
- each edge only participates in $O(\log \log n)$ steps of the refinement process in expectation, and
- all partition routines are done on graphs with diameter $\text{poly}(\log n)$.

The interaction between the bottom-up AKPW decomposition scheme and the top-down Bartal decomposition leads to some distortions. The rest of this section can be viewed as analyzing this distortion, and the algorithmic gains from having it. We will show that for an appropriately constructed AKPW decomposition, the probability of an edge being cut can be related to a quantity in the $\ell_q$ norm for some $p < q < 1$. The difference between these two norms then allows us to absorb distortions of size up to $\text{polylog } n$, and therefore not affecting the quality of the resulting tree. Thus we will work mostly with a different exponent $q$ in this section, and only bring things back to an exponent in $p$ at the very end.

Both the AKPW and the top-down routines will issue multiple calls to PARTITION. In both cases the granularity of the edge weights will be $\text{poly}(\log n)$. As stated in Section 3, PARTITION can be implemented in linear time in the RAM model, using the rather involved algorithm presented in [Tho00]. In practice, it is also possible to use the low granularity of edge weights and use Dial’s algorithm [Dia69], worsening the total running time of our algorithm to $O(m \log \log n + \log \Delta \text{poly}(\log n))$ when all edge lengths are in the range $[1, \Delta]$. Alternatively, we can use the weight-sensitive shortest path algorithm from [KMP11], which works in the pointer machine model, but would be slower by a factor of $O(\log \log \log n)$. 

15
5.1 The AKPW Decomposition Routine

We first describe the AKPW algorithm for generating decomposition. The decomposition produced is similar to Bartal decompositions, although we will not impose the strict conditions on diameters in our definition.

**Definition 5.1** Let $G = (V, E, l)$ be a connected multigraph. We say that a sequence of forests $A$, where

$$A = (A_0, A_1, \ldots, A_s),$$

is an AKPW decomposition of $G$ with parameter $\delta$ if:

1. $A_s$ is a spanning tree of $G$.
2. For any $i < t$, $A_i \subseteq A_{i+1}$.
3. The diameter of each connected component in $A_i$ is at most $\delta^{i+1}$.

Pseudocode for generating this decomposition is given in Figure 5. We first bound the diameters of each piece, and the probability of an edge being cut in $A_i$.

\[
\begin{align*}
\text{A} = \text{AKPW}(G, \delta), \text{ where } G \text{ is a connected multigraph.} \\
1. & \text{ Bucket the edges by length into } E_0, E_1, \ldots, \text{ where } E_i \text{ contains all edges of length in } [\delta^i, \delta^{i+1}) \\
2. & \text{ Initialize } A_0 := \emptyset, s := 0. \\
3. & \text{ While } A_s \text{ is not a spanning tree of } G: \\
   & (a) \text{ Let } E' \text{ be the set of all edges from } E_0, \ldots, E_s \text{ that connect different components of } A_s. \\
   & (b) \text{ Set } G_s := (V, E', \vec{1})/A_s, \text{ where } \vec{1} \text{ is a constant function that assigns all edges length 1.} \\
   & (c) \text{ Decompose } G \text{ by calling } \text{PARTITION}(G_s, \delta/3); \text{ let } T_1, T_2, \ldots, T_k \text{ be the edge sets of the corresponding low diameter spanning trees.} \\
   & (d) \text{ Set } A_{s+1} := A_s \cup T_1 \cup \ldots \cup T_k \\
   & (e) \text{ Set } s := s + 1. \\
4. & \text{ Return } A := (A_0, \ldots, A_s).
\end{align*}
\]

**Figure 5:** The routine for generating AKPW decompositions

**Lemma 5.2** AKPW($G, \delta$) generates with high probability an AKPW decomposition $A$ such that for an edge $e = uv$ with $l(e) \in [\delta^i, \delta^{i+1})$ and some $j \geq i$, the probability that $u$ and $v$ are not connected in $A_j$ is at most

$$\left(\frac{c_{\text{Partition}} \log n}{\delta}\right)^{j-i},$$
where $c_{\text{Partition}}$ is a constant associated with the partition routine. Furthermore, if $\delta \geq 2c_{\text{Partition}} \log n$, it runs in expected $O(m \log \log^{1/2} n)$ time in the RAM model.

**Proof** The termination condition on Line 3 implies that $A_s$ is a spanning tree, and the fact that we generate $A_{i+1}$ by adding edges to $A_i$ gives $A_i \subseteq A_{i+1}$. The bound on diameter can be proven inductively on $i$.

The base case of $i = 0$ follows from the vertices being singletons, and as a result having diameter 0. For the inductive case, suppose the result is true for $i$. Then with high probability each connected component in $A_{i+1}$ corresponds to a tree with diameter $\delta/3$ connecting connected components in $A_i$. The definition of $E_i$ gives that each of these edges have length at most $\delta^{i+1}$, and the inductive hypothesis gives that the diameter of each connected component in $A_i$ is also at most $\delta^{i+1}$. This allows us to bound the diameter of $A_{i+1}$ by $(\delta/3) \cdot \delta^{i+1} + (\delta/3 + 1)\delta^{i+1} \leq \delta^{i+2}$. Hence the inductive hypothesis holds for $i + 1$ as well.

The guarantees of the probabilistic decomposition routine from Lemma 4.1 gives that on any level, an edge has its two endpoints separated with probability $\frac{c_p \log n}{\delta}$. The assumption of the length of $e$ means that it is in $E_i$. So by the time $A_j$ is formed, it has gone through $j - i$ rounds of partition, and is present iff its endpoints are separated in each of these steps. Multiplying the probabilities then gives the bound.

If $\delta \geq 2c_p \log n$, then the probability of an edge in $E_i$ appearing in subsequent levels decrease geometrically. This means that the total expected sizes of $G_i$ processed is $O(m)$. Combining this with the linear running time of PARTITION gives the expected running time once we have the buckets $E_0, E_1$, etc. Under the RAM model of computation, these buckets can be formed in $O(m \log \log^{1/2} n)$ time using the sorting algorithm by Han and Thorup [Han04]. Incorporating this cost gives the overall running time.

Combining the bound on diameter and probability of an edge being cut leads to the bound on the expected $\ell_1$-stretch of an edge shown by Alon et al. [AKPW95]. For an edge on the $i$th level, the ratio between its length and the diameter of the $j$th level can be bounded by $\delta^{j-i+1}$. As $j$ increases, the expected stretch of $e$ then increases by factors of

$$\delta \cdot O\left(\frac{\log n}{\delta}\right) = O(\log n),$$

which leads to the more than logarithmic bound on the expected $\ell_1$-stretch. With $\ell_p$-stretch however, the $p$th power of the diameter-length ratio only increases by factors of $\delta^p$. This means that, as long as the probabilities of an edge being cut increases by factors of less than $\delta^p$, a better bound can be obtained.

### 5.2 AKPW meets Bartal

In this section, we describe how we combine the AKPW decomposition and Bartal’s scheme into a two-pass algorithm. At a high level, Bartal’s scheme repeatedly partitions the graph in a top-down fashion, and the choice of having geometrically decreasing diameters translates to a $O(m \log n)$ running time. The way our algorithm achieves a speedup is by contracting vertices that are close to each other, in a way that does not affect the top-down partition scheme. More specifically, we precompute an appropriate AKPW decomposition, and only expose a limited number of layers while running the top-down partition. This way we ensure that each edge only appears in $O(\log \log n)$ calls to the partition routine.
Let $A = (A_0, A_1, \ldots, A_s)$ be an AKPW decomposition with parameter $\delta$, so that $G/A_i$ is the quotient graph where each vertex corresponds to a cluster of diameter at most $\delta^{i+1}$ in the original graph. While trying to partition the graph $G$ into pieces of diameter $d$, where under some notion $d$ is relatively large compared to $\delta^{i+1}$, we observe that the partition can be done on the quotient graph $G/A_i$ instead. As the complexity of our partition routine is linear in the number of edges, there might be some potential gain. We use the term scope to denote the point at which lower levels of the AKPW decomposition are handled at a coarser granularity. When the top-down algorithm reaches diameter $d_i$ in the diameter sequence $d$, this cutoff point in the AKPW decomposition is denoted by scope$(i)$. The algorithm is formalized in Figure 6.

**Figure 6:** Pseudocode of two pass algorithm for finding a Bartal decomposition

We first show that the increase in edge lengths to $\delta^{\text{scope}(i)+1}$ still allows us to bound the diameter of the connected components of $B_i$.

**Lemma 5.3** The diameter of each connected component in $B_i$ is bounded by $d_i$ with high probability.

**Proof** By the guarantee of the partition routine, the diameter of each $G_i$ is at most $d_i$ with high probability. However, since we are measuring diameter of the components in $G$, we also need to account for the diameter of the components that were shrunk into vertices when forming $G'$. These components correspond to connected pieces in $A_{\text{scope}(i)}$, therefore the diameters of the corresponding trees are bounded by $\delta^{\text{scope}(i)+1}$ with high probability. Our increase of edge lengths in $G'$, on the other hand, ensures that the length of any edge is more than the diameter of its endpoints. Hence the total increase in diameter from these pieces is at most twice the length of a path in $G'$, and the diameter of these components in $G$ can be bounded by $d_i$. 

18
Once we established that the diameters of our decomposition is indeed geometrically decreasing, it remains to bound the probability of an edge being cut at each level of the decomposition. In the subsequent sections, we give two different analyses of the algorithm DecomposeTwoStage with different choices of scope. We first present a simple version of our algorithm which ignores a \(1/\text{poly}(\log n)\) fraction of the edges, but guarantees an expected \(\ell_1\)-stretch close to \(O(\log n)\) for rest of the edges. Then we present a more involved analysis with a careful choice of scope which leads to a tree with small \(\ell_p\)-stretch.

### 5.3 Decompositions that Ignore \(\frac{1}{k}\) of the Edges

In this section, we give a simplified algorithm that ignores \(O(\frac{1}{k})\) fraction of the edges, but guarantees for other edges an expected \(\ell_1\)-stretch close to \(O(\log n)\). We also discuss how this relates to the problem of generating low-stretch subgraphs in parallel and its application to parallel SDD linear system solvers.

In this simplified algorithm, we use a naive choice of scope, reaching a small power of \(k\log n\) into the AKPW decomposition.

Let \(d = (d_0, d_1, \ldots, d_t)\) be a diameter sequence and let \(A = (A_0, A_1, \ldots, A_s)\) be an AKPW decomposition constructed with parameter \(\delta = k\log n\). We let \(\text{scope}(i) = \max\{j \mid \delta^{j+3} \leq d_i\}\). Note that \(\delta^{\text{scope}(i)}\) is always between \(d_i\delta^4\) and \(d_i\delta^3\). We say an edge \(e \in E_i\) is AKPW-cut if \(e\) is cut in \(A_{i+1}\). Furthermore, we say an edge \(e\) is floating in level \(i\) if it exists in \(B_{i-1}/A_{\text{scope}(i)}\) and has length less than \(\delta^{\text{scope}(i)+1}\). Note that the floating edges are precisely the edges whose length is increased before running the Bartal decomposition. We say that an edge is floating-cut if it is not AKPW-cut, but is cut by the Bartal decomposition at any level in which it is floating.

The simplification of our analysis over bounding overall \(\ell_p\) stretch is that we can ignore all AKPW-cut or floating-cut edges. We start by bounding the expected number of edges ignored in these two ways separately.

**Lemma 5.4** Let \(A = \text{AKPW}(G, \delta)\) where \(\delta = k\log n\). The expected number of AKPW-cut edges in \(A\) is at most \(O\left(\frac{m}{k}\right)\).

**Proof** For an edge \(e \in E_i\), the probability that \(e\) is cut in \(A_{i+1}\) is at most

\[
\frac{c_{\text{Partition}} \log n}{\delta} = \frac{c_{\text{Partition}}}{k}
\]

by Lemma 5.2, where \(c_{\text{Partition}}\) is the constant associated with the partition routine. Linearity of expectation then gives that the expected number of AKPW-cut edges is at most \(O\left(\frac{m}{k}\right)\). \(\blacksquare\)

We now bound the total number of floating-cut edges:

**Lemma 5.5** The expected number of floating-cut edges is \(O\left(\frac{m}{k}\right)\).

**Proof** First, we note that only edges whose length is at least \(\frac{d_i}{\delta^4}\) may be floating-cut at level \(i\): any edge smaller than that length that is not AKPW-cut will not be contained in \(B_{i-1}/A_{\text{scope}(i)}\). Furthermore, by the definition of floating, only edges of lengths at most \(\frac{d_i}{\delta^2}\) may be floating. Therefore, each edge may only be floating-cut for levels with \(d_i\) between \(\delta^2\) and \(\delta^4\) times the length of the edge. Since the \(d_i\) increase geometrically, there are at most \(\log(\delta)\) such levels.

Furthermore, at any given level, the probability that a given edge is floating-cut at the level is at most \(O\left(\frac{\log n}{\delta^2}\right)\), since any floating edge is passed to the decomposition with length \(\frac{d_i}{\delta^2}\). Taking a
union bound over all levels with \( d_i \) between \( \delta^2 \) and \( \delta^4 \) times the length of the edge, each edge has at most a \( O(\frac{\log n \log \delta}{\delta^2}) \) probability of being cut. Since \( \frac{\log \delta}{\delta} = O(1) \), this is \( O(\frac{\log n}{\delta}) = O(\frac{1}{k}) \).

Again, applying linearity of expectation implies that the expected number of floating-cut edges is \( O(\frac{m}{k}) \).

Combining these two bounds gives that the expected number of ignored edges so far is bounded by \( O(\frac{m}{k}) \). We can also check that conditioned on an edge being not ignored, its probability of being cut on some level is the same as before.

**Lemma 5.6** Assume \( A = AKPW(G, \delta) \). We may associate with the output of the algorithm a set of edges \( S \), with expected size \( O(\frac{m}{k}) \), such that for any edge \( e \) with length \( l(e) \), conditioned on \( e \notin S \), is cut on the \( i^{\text{th}} \) level of the Bartal decomposition \( B \) with probability at most

\[
O\left(\frac{l(e) \log n}{d_i}\right)
\]

**Proof** We set \( S \) to the union of the sets of AKPW-cut and floating-cut edges.

Fix a level \( i \) of the Bartal decomposition: if an edge \( e \) that is not AKPW-cut or floating-cut appears in \( B_{i-1}/A_{\text{scope}(i)} \), then its length is unchanged. If \( e \) is removed from \( G' \) due to \( l(e) \geq d_i/\log n \), the bound becomes trivial. Otherwise, the guarantees of PARTITION then give the cut probability.

**Lemma 5.7** The simplified algorithm produces with high probability an embeddable Bartal decomposition with diameters bounded by \( d \) where all but (in expectation) \( O(\frac{m}{k}) \) edges satisfy \( \mathbb{E}_B[\text{STR}_{B,d}(e)] \leq \mathbb{O}(\log n(\log(k \log n))^2) \).

**Proof** Let \( p = 1 - 1/\log(k \log n) \) and \( q = (1 + p)/2 \). Applying Lemma 5.6 and Lemma 3.4 we get that for edges not in \( S \), \( \mathbb{E}_B[\text{STR}_{B,d}(e)] = \mathbb{O}(\log^q n \log(k \log n)) \). Then using \textsc{EmbeddableDecompose} as a black box we obtain an embeddable decomposition with expected \( l_p \)-stretches of \( \mathbb{O}(\log^q n(\log(k \log n))^2) \) for non-removed edges.

By repeatedly running this algorithm, in an expected constant number of iterations, we obtain an embeddable decomposition \( B \) with diameters bounded by \( d \) such that for a set of edges \( E' \subseteq E \) and \( |E'| \geq m - \mathbb{O}(\frac{m}{k}) \) we have:

\[
\sum_{e \in E'} \mathbb{E}_B[\text{STR}_{B,d}^q(e)] = \mathbb{O}(m \log^q n(\log(k \log n))^2).
\]

By Markov’s inequality, at most 1/k of the edges in \( E' \) can have \( \text{STR}_{B,d}^q(e) \geq \mathbb{O}(k \log^q n(\log(k \log n))^2) \). This gives a set of edges \( E'' \) with size at least \( m - \mathbb{O}(\frac{m}{k}) \) such that any edge \( e \in E'' \) satisfies \( \text{STR}_{B,d}^q(e) \leq \mathbb{O}(k \log^q n(\log(k \log n))^2) \leq \mathbb{O}(k \log n)^2 \).

But for each of these edges

\[
\text{STR}_{B,d}(e) = (\text{STR}_{B,d}^q(e))^{1/q} \\
\leq (\text{STR}_{B,d}^q(e))^{1+2/\log(k \log n)} \\
\leq \text{STR}_{B,d}^q(e) \cdot \mathbb{O}\left((k \log n)^{4/\log(k \log n)}\right) \\
= \mathbb{O}(\text{STR}_{B,d}^q(e)).
\]
Excluding these high-stretch edges, the $\ell_1$ stretch is thus at most a constant factor worse than the $\ell_q$ stretch, and can be bounded by $O(\log n(\log(k \log n))^2)$. ■

The total running time of DecomposeTwoStage is dominated by the calls to Partition. The total cost of these calls can be bounded by the expected number of calls that an edge participates in.

Lemma 5.8 For any edge $e$, the expected number of iterations in which $e$ appears is bounded by $O(\log(k \log n))$.

Proof As pointed out in the proof of 5.5, an edge that is not AKPW-cut only appears in level $i$ of the Bartal decomposition if $l(e) \in \left(\frac{d_i}{\alpha^2}, \frac{d_i}{\log n}\right)$. Since the diameters decrease geometrically, there are at most $O(\log(k \log n))$ such levels. AKPW-cut edges can appear sooner than other edges from the same weight bucket, but using an argument similar to the proof of Lemma 5.4 we observe that the edge propagates up $j$ levels in the AKPW decomposition with probability at most $\left(\frac{1}{k}\right)^j$. Therefore the expected number of such appearances by an APKW-cut edge is at most $\sum_j \left(\frac{1}{k}\right)^j = O(1)$. ■

Combining all of the above we obtain the following result about our simplified algorithm. The complete analysis of its running time is deferred to Section 5.5.

Lemma 5.9 For any $k$, given an AKPW decomposition $A$ with $\delta = k \log n$, we can find in $O(m \log(k \log n))$ time an embeddable Bartal decomposition such that all but expected $O(m \log n(\log(k \log n))^2)$ edges have expected total $\ell_1$-stretch of at most $O(m \log n(\log(k \log n))^2)$.

5.3.1 Parallelization

If we relax the requirement of asking for a tree, the above analysis shows that we can obtain low stretch subgraphs edges and total stretch of $O(\log n(\log(k \log n))^2)$ for all but $O(\frac{m}{k})$ edges. As our algorithmic primitive Partition admits parallelization [MPX13], we also obtain a parallel algorithm for constructing low stretch subgraphs. These subgraphs are used in the parallel SDD linear system solver by [BGK+13]. By observing that Partition is run on graphs with edge weights within $\delta$ of each other and hop diameter at most polynomial in $\delta = k \log n$, and invoking tree-contraction routines to extract the final tree [MR89], we can obtain the following result.

Lemma 5.10 For any graph $G$ with polynomially bounded edge weights and $k \leq \text{poly}(\log n)$, in $O(k \log^2 n \log \log n)$ depth and $O(m \log n)$ work we can generate an embeddable tree of size $O(n)$ such that the total $\ell_1$-stretch of all but $O(\frac{m}{k})$ edges of $G$ is $O(m \log n(\log(k \log n))^2)$.

5.4 Bounding Expected $\ell_p$-Stretch of Any Edge

In this section we present our full algorithm and bound the expected $\ell_p$-stretch of all edges. Since we can no longer ignore edges whose lengths we increase while performing the top-down partition, we need to choose the scope carefully in order to control their probability of being cut during the second stage of the algorithm. We start off by choosing a different $\delta$ when computing the AKPW decomposition.

Lemma 5.11 If $A$ is generated by a call to AKPW($G, \delta$) with $\delta \geq (c_p \log n)^{\frac{1}{1-\alpha}}$, then the probability of an edge $e \in E_i$ being cut in level $j$ is at most $\delta^{-q(j-i)}$. 

21
Proof Manipulating the condition gives \( c_P \log n \leq \delta^{1-q} \), and therefore using Lemma 5.2 we can bound the probability by

\[
\left( \frac{c_P \log n}{\delta} \right)^{j-i} \leq \left( \frac{\delta^{1-q}}{\delta} \right)^{j-i} = \delta^{-q(j-i)}.
\]

Since \( \delta \) is \( \text{poly}(\log n) \), we can use this bound to show that expected \( \ell_p \)-stretch of an edge in an AKPW-decomposition can be bounded by \( \text{poly}(\log n) \). The exponent here can be optimized by taking into account the trade-offs given in Lemma 3.4.

This extra factor of \( \delta \) can also be absorbed into the analysis of Bartal decompositions. When \( l(e) \) is significantly less than \( d \), the difference between \( \frac{l(e) \log n}{d} \) and \( \left( \frac{l(e) \log n}{d} \right)^q \) is more than \( \delta \). This means that for an floating edge that originated much lower in the bucket of the AKPW decomposition, we can afford to increase its probability of being cut by a factor of \( \delta \).

From the perspective of the low-diameter decomposition routine, this step corresponds to increasing the length of an edge. This increase in length can then be used to bound the diameter of a cluster in the Bartal decomposition, and also ensures that all edges that we consider have lengths close to the diameter that we partition into. On the other hand, in order to control this increase in lengths, and in turn to control the increase in the cut probabilities, we need to use a different scope when performing the top-down decomposition.

Definition 5.12 For an exponent \( q \) and a parameter \( \delta \geq \log n \), we let the scope of a diameter \( d \) be

\[
\text{scope}(i) := \max_i \left\{ \delta^{i+\frac{1}{q}-1} \leq d_i \right\}.
\]

Note that for small \( d \), \( \text{scope}(i) \) may be negative. As we will refer to \( A_{\text{scope}(i)} \), we assume that \( A_i = \emptyset \) for \( i < 0 \). Our full algorithm can then be viewed as only processing the edges within the scope using Bartal’s top-down algorithm. Its pseudocode is given in Figure 6.

Note that it is not necessary to perform explicit contraction and expansion of the AKPW clusters in every recursive call. In an effective implementation, they can be expanded gradually, as \( \text{scope}(i) \) is monotonic in \( d_i \).

The increase in edge lengths leads to increases in the probabilities of edges being cut. We next show that because the AKPW decomposition is computed using a higher norm, this increase can be absorbed, giving a probability that is still closely related to the \( p \)th power of the ratio between the current diameter and the length of the edge.

Lemma 5.13 Assume \( A = \text{AKPW}(G, \delta) \) with parameter specified as above. For any edge \( e \) with length \( l(e) \) and any level \( i \), the probability that \( e \) is cut at level \( i \) of \( B = \text{DECOMPOSETWO-STAGE}(G, d, A) \) is

\[
\mathcal{O} \left( \left( \frac{l(e) \log n}{d_i} \right)^q \right).
\]

Proof There are two cases to consider based whether the length of the edge is more than \( \delta^{\text{scope}(i)+1} \). If it is and it appears in \( G' \), then its length is retained. The guarantees of PARTITION then gives
that it is cut with probability
\[
\mathcal{O}\left(\frac{l(e) \log n}{d_i}\right) \leq \mathcal{O}\left(\left(\frac{l(e) \log n}{d_i}\right)^q\right),
\]
where the inequality follows from \(l(e) \log n \leq d_i\).

Otherwise, since we contracted the connected components in \(A_{\text{scope}(i)}\), the edge is only cut at level \(i\) if it is both cut in \(A_{\text{scope}(i)}\) and cut by the partition routine. Lemma 5.11 gives that if the edge is from \(E_j\), its probability of being cut in \(A_{\text{scope}(i)}\) can be bounded by \(\delta - q \cdot (\text{scope}(i) - j)\). Combining with the fact that \(\delta^j \leq l(e)\) allows us to bound this probability by
\[
\left(\frac{l(e)}{\delta_{\text{scope}(i)}}\right)^q.
\]

Also, since the weight of the edge is set to \(\delta_{\text{scope}(i)} + 1\) in \(G'\), its probability of being cut by \textsc{Partition} is
\[
\mathcal{O}\left(\frac{\delta_{\text{scope}(i)} + 1 \log n}{d_i}\right).
\]

As the partition routine is independent of the AKPW decomposition routine, the overall probability can be bounded by
\[
\mathcal{O}\left(\frac{\delta_{\text{scope}(i)} + 1 \log n}{d_i}\right) \cdot \left(\frac{l(e)}{\delta_{\text{scope}(i)}}\right)^q \cdot \delta \log^q n \cdot \left(\frac{\delta_{\text{scope}(i)}^{1-q}}{d_i}\right).
\]

Recall from Definition 5.12 that \(\text{scope}(i)\) is chosen to satisfy \(\delta_{\text{scope}(i)} + \frac{1}{1-q} + 1 \leq d_i\). This along with the assumption that \(\delta \geq \log n\) gives
\[
\delta \log^{1-q} n \cdot \left(\frac{\delta_{\text{scope}(i)}}{d_i}\right)^{1-q} \leq \delta^{2-q} \left(\delta^{\frac{2-q}{1-q}}\right)^{1-q} \leq 1.
\]

Therefore, in this case the probability of \(e\) being cut can also be bounded by \(\mathcal{O}\left(\left(\frac{l(e) \log n}{d_i}\right)^q\right)\). \(\blacksquare\)

Combining this bound with Lemma 3.4 and setting \(q = \frac{1+p}{2}\) gives the bound on \(l_p\)-stretch.

**Corollary 5.14** If \(q\) is set to \(\frac{1+p}{2}\), we have for any edge \(e\)
\[\mathbb{E}_B[\text{STR}_{B,d}(e)] \leq \mathcal{O}\left(\frac{1}{1-p} \log^p n\right).
\]

Therefore, we can still obtain the properties of a good Bartal decomposition by only considering edges in the scope during the top-down partition process. On the other hand, this shrinking drastically improves the performance of our algorithm.

**Lemma 5.15** Assume \(A = \text{AKPW}(G, \delta)\). For any edge \(e\), the expected number of iterations of \textsc{DecomposeTwoStage} in which \(e\) is included in the graph given to \textsc{Partition} can be bounded by \(\mathcal{O}(\frac{1}{1-p} \log \log n)\).
Note that for any level $i$ it holds that

$$\delta^{\text{scope}(i)} \geq d_i \delta^{-\frac{1}{1-q} - 2}.$$  

Since the diameters of the levels decrease geometrically, there are at most $O(\frac{1}{1-q} \log \log n)$ levels $i$ such that $l(e) \in [d_i \delta^{-\frac{1}{1-q} - 2}, \frac{d_i}{\log n})$.

The expected number of occurrences of $e$ in lower levels can be bounded using Lemma 5.11 in a way similar to the proof of the above Lemma. Summing over all the levels $i$ where $e$ is in a lower level gives:

$$\sum_{i: l(e) < d_i \delta^{-\frac{1}{1-q} - 2}} \left( \frac{l(e)}{\delta^{\text{scope}(i)}} \right)^q.$$  

Substituting in the bound on $\delta^{\text{scope}(i)}$ from above and rearranging then gives:

$$\leq \sum_{i: l(e) \leq d_i \delta^{-\frac{1}{1-q} - 2}} \left( \frac{l(e)}{d_i \delta^{-\frac{1}{1-q} + 2}} \right)^q.$$  

As $d_i$ increase geometrically, this is a geometric sum with the first term at most 1. Therefore the expected number of times that $e$ appears on some level $i$ while being out of scope is $O(1)$.

Recall that each call to \textsc{Partition} runs in time linear in the number of edges. This then implies a total cost of $O(m \log \log n)$ for all the partition steps. We can now proceed to extract a tree from this decomposition, and analyze the overall running time.

### 5.5 Returning a Tree

We now give the overall algorithm and analyze its performance. Introducing the notion of scope in the recursive algorithm limits each edge to appear in at most $O(\log \log n)$ levels. Each of these calls partitions $G'$ in time linear in its size, which should give a total of $O(m \log \log n)$. However, the goal of the algorithm as stated is to produce a Bartal decomposition, which has a spanning tree at each level. Explicitly generating this gives a total size of $\Omega(nt)$, where $t$ is the number of recursive calls. As a result, we will circumvent this by storing only an implicit representation of the Bartal decomposition to find the final tree.

This smaller implicit representation stems from the observation that large parts of the $B_i$s are trees from the AKPW decomposition, $A_i$. As a result, such succinct representations are possible if we have pointers to the connected components of $A_i$. We first analyze the quality and size of this implicit decomposition, and the running time for producing it.

**Lemma 5.16** There is a routine that for any graph $G$ and parameter $p < 1$, produces in expected $O(\frac{1}{1-p} m \log \log n)$ time an implicit representation of a Bartal decomposition $B$ with expected size $O(\frac{1}{1-p} m \log \log n)$ and diameter bounds $d$ such that with high probability:

- $B$ is embeddable into $G$, and
- for any edge $e$, $E_B(STR_B(e)) \leq O((\frac{1}{1-p})^2 \log^p n)$.
- $B$ consists of edges and weighted connected components of an AKPW decomposition.
\[ B, d = \text{Decompose}(G, p), \text{where } G \text{ is a graph, } p \text{ is an exponent} \]

1. Set \( q = \frac{1+p}{2}, \delta = (c \log n)^{\frac{1}{1-p}}. \)

2. Compute an AKPW decomposition of \( G, A = \text{AKPW}(G, \delta). \)

3. Let \( d = (d_0, d_1, \ldots, d_t) \) be a geometrically decreasing sequence diameters where \( d_0 \) is the diameter of \( A_s. \)

4. Set \( B := \text{DecomposeTwoStage}(G, d, A). \)

5. Set \( B_0 \) to \( A_s. \)

6. Return \( B, d. \)

**Figure 7: Overall decomposition algorithm**

**Proof**  Consider calling \( \text{EmbeddableDecompose} \) from Section 4.2 with the routine given in Figure 7. The properties of \( B \) and the bounds on stretch follows from Lemma 4.4 and Corollary 5.14.

Since the number of AKPW components implicitly referred to at each level of the recursive call is bounded by the total number of vertices, and in turn the number of edges, the total number of such references is bounded by the size of the \( G' \)'s as well. This gives the bound on the size of the implicit representation.

We now bound the running time. In the RAM model, bucketing the edges and computing the AKPW decomposition can be done in \( O(m \log \log n) \) time. The resulting tree can be viewed as a laminar decomposition of the graph. This is crucial for making the adjustment in \( \text{DecomposeTwoStage} \) in \( O(1) \) time to ensure that \( A_{\text{scope}(i)} \) is disconnected. As we set \( q \) to \( \frac{1+p}{2}, \) by Lemma 5.15, each edge is expected to participate in \( O\left(\frac{1}{1-p} \log \log n\right) \) recursive calls, which gives a bound on the expected total.

The transformation of the edge weights consists of a linear-time pre-processing, and scaling each level by a fixed parameter in the post-post processing step. This process affects the implicit decomposition by changing the weights of the AKPW pieces, which is can be done implicitly in \( O(1) \) time by attaching extra ‘flags’ to the clusters.

It remains to show that an embeddable tree can be generated efficiently from this implicit representation. To do this, we define the notion of a contracted tree with respect to a subset of vertices, obtained by repeating the two combinatorial steps that preserve embeddability described in Section 2.

**Definition 5.17**  We define the contraction of a tree \( T \) to a subset of its vertices \( S \) as the unique tree arising from repeating the following operations while possible:

- removal of a degree 1 vertex not in \( S, \) and
- contraction of a degree 2 vertex not in \( S. \)

We note that it is enough to find contractions of the trees from the AKPW decomposition to the corresponding sets of connecting endpoints in the implicit representation. Here we use the fact that the AKPW decomposition is in fact a single tree.
Fact 5.18 Let $A = A_0, \ldots, A_s$ be an AKPW decomposition of $G$. Let $S$ be a subset of vertices of $G$. For any $i$ in $\{0, \ldots, s\}$, if $S$ is contained in a single connected component of $A_i$, then the contraction of $A_i$ to $S$ is equal to the contraction of $A_s$ to $S$.

This allows us to use data structures to find the contractions of the AKPW trees to the respective vertex sets more efficiently.

Lemma 5.19 Given a tree $A_s$ on the vertex set $V$ (with $|V| = n$) and subsets $S_1, \ldots, S_k$ of $V$ of total size $O(n)$, we can generate the contractions of $A_s$ to each of the sets $S_i$ in time $O(n)$ in the RAM model and $O(n\alpha(n))$ in the pointer machine model.

Proof Root $A_s$ arbitrarily. Note that the only explicit vertices required in the contraction of $A_s$ to a set $S \subseteq V$ are

$$\Gamma(S) \overset{\text{def}}{=} S \cup \{\text{LCA}(u, v) : u, v \in S\}$$

where $\text{LCA}(u, v)$ denotes the lowest common ancestor of $u$ and $v$ in $A_s$. Moreover, it is easily verified that if we sort the vertices $v_1, \ldots, v_{|S|}$ of $S$ according to the depth first search pre-ordering, then

$$\Gamma(S) = S \cup \{\text{LCA}(v_i, v_{i+1}) : 1 \leq i < |S|\}.$$  

We can therefore find $\Gamma(S_i)$ for each $i$ simultaneously in the following steps:

1. Sort the elements of each $S_i$ according to the pre-ordering, using a single depth-first search traversal of $A_s$.
2. Prepare a list of lowest common ancestor queries for each pair of vertices adjacent in the sorted order in each set $S_i$.
3. Answer all the queries simultaneously using an off-line lowest common ancestor finding algorithm.

Since the total number of queries in the last step is $O(n)$, its running time is $O(n\alpha(n))$ in the pointer machine model using disjoint union [Tar79], and $O(n)$ in the RAM model [GT83].

Once we find the sets $\Gamma(S_i)$ for each $i$, we can reconstruct the contractions of $A_s$ as follows:

1. Find the full traversal of the vertices in $\Gamma(S_i)$ for each $i$, using a single depth first search traversal of $A_s$.
2. Use this information to reconstruct the trees [Vui80].

Applying this procedure to the implicit decomposition then leads to the final embeddable tree.

Proof of Theorem 1.1: Consider the distribution over Bartal decompositions given by Lemma 5.16. We will apply the construction given in Lemma 4.6, albeit in a highly efficient manner.

For the parts of the decomposition that are explicitly given, the routine runs in linear time. The more intricate part is to extract the smaller contractions from the AKPW components that
are referenced to implicitly. Since all levels of the AKPW decomposition are subtrees of \( A_s \), these are equivalent to finding contractions of \( A_s \) for several sets of vertices, as stated in Fact 5.18. The algorithm given in Lemma 5.19 performs this operation in linear time. Concatenating these trees with the one generated from the explicit part of the decomposition gives the final result.

\[ \text{Acknowledgments} \]

We thank Ittai Abraham, Anupam Gupta, Nick Harvey, Jon Kelner, Yiannis Koutis, Aaron Sidford, Dan Spielman, and Zeyuan Zhu for very helpful comments and discussions.

\[ \text{References} \]

[ABN08] Ittai Abraham, Yair Bartal, and Ofer Neiman. Nearly tight low stretch spanning trees. In *Proceedings of the 2008 49th Annual IEEE Symposium on Foundations of Computer Science*, FOCS ’08, pages 781–790, Washington, DC, USA, 2008. IEEE Computer Society. 1, 1.1

[AKPW95] N. Alon, R. Karp, D. Peleg, and D. West. A graph-theoretic game and its application to the \( k \)-server problem. *SIAM J. Comput.*, 24(1):78–100, 1995. 1, 1.1, 5, 5.1

[AN12] Ittai Abraham and Ofer Neiman. Using petal decompositions to build a low stretch spanning tree. In *Proceedings of the 44th Symposium on Theory of Computing*, STOC ’12, pages 395–406, New York, NY, USA, 2012. ACM. 1, 1, 1.1, 3

[Bar96] Y. Bartal. Probabilistic approximation of metric spaces and its algorithmic applications. In *Foundations of Computer Science, 1996. Proceedings., 37th Annual Symposium on*, pages 184–193, 1996. 1, 1.1, 3, 3

[Bar98] Yair Bartal. On approximating arbitrary metrics by tree metrics. In *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, STOC ’98, pages 161–168, New York, NY, USA, 1998. ACM. 1.1

[BGK+13] Guy E. Blelloch, Anupam Gupta, Ioannis Koutis, Gary L. Miller, Richard Peng, and Kanat Tangwongsan. Nearly-linear work parallel SDD solvers, low-diameter decomposition, and low-stretch subgraphs. *Theory of Computing Systems*, pages 1–34, March 2013. 1.2, 5.3.1

[CCG+98] Moses Charikar, Chandra Chekuri, Ashish Goel, Sudipto Guha, and Serge Plotkin. Approximating a finite metric by a small number of tree metrics. In *Proceedings of the 39th Annual Symposium on Foundations of Computer Science*, FOCS ’98, pages 379–388, Washington, DC, USA, 1998. IEEE Computer Society. 1

[CFM+14] Michael Cohen, Brittany Terese Fasy, Gary L. Miller, Amir Nayyeri, Richard Peng, and Noel Walkington. Solving 1-laplacians of convex simplicial complexes in nearly linear time: Collapsing and expanding a topological ball. 2014. 1

[CKM+11] Paul Christiano, Jonathan A. Kelner, Aleksander Madry, Daniel A. Spielman, and Shang-Hua Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In *Proceedings of the 43rd annual ACM symposium*
1.2 [CKP+14] Michael B. Cohen, Rasmus Kyng, Jakub W. Pachocki, Richard Peng, and Anup Rao. Preconditioning in expectation. CoRR, abs/1401.6236, 2014.

1 [CSRL01] Thomas H. Cormen, Clifford Stein, Ronald L. Rivest, and Charles E. Leiserson. Introduction to Algorithms. McGraw-Hill Higher Education, 2nd edition, 2001.

5 [Dia69] Robert B. Dial. Algorithm 360: shortest-path forest with topological ordering [h]. Commun. ACM, 12(11):632–633, November 1969.

A [DS84] Peter G. Doyle and J. Laurie Snell. Random Walks and Electric Networks, volume 22 of Carus Mathematical Monographs. Mathematical Association of America, 1984.

3 [DS08] Samuel I. Daitch and Daniel A. Spielman. Faster approximate lossy generalized flow via interior point algorithms. In Proceedings of the 40th annual ACM symposium on Theory of computing, STOC ’08, pages 451–460, New York, NY, USA, 2008. ACM.

1 [EEST08] Michael Elkin, Yuval Emek, Daniel A Spielman, and Shang-Hua Teng. Lower-stretch spanning trees. SIAM Journal on Computing, 38(2):608–628, 2008.

1.1 [FRT04] Jittat Fakcharoenphol, Satish Rao, and Kunal Talwar. A tight bound on approximating arbitrary metrics by tree metrics. J. Comput. Syst. Sci., 69(3):485–497, November 2004.

4.1 [FT87] Michael L. Fredman and Robert Endre Tarjan. Fibonacci heaps and their uses in improved network optimization algorithms. J. ACM, 34:596–615, July 1987.

2 [Gre96] Keith D. Gremban. Combinatorial Preconditioners for Sparse, Symmetric, Diagonally Dominant Linear Systems. PhD thesis, Carnegie Mellon University, 1996.

5.5 [GT83] Harold N. Gabow and Robert Endre Tarjan. A linear-time algorithm for a special case of disjoint set union. In Proceedings of the 15th annual ACM symposium on Theory of computing (STOC), pages 246–251, New York, NY, USA, 1983. ACM.

5.1 [Han04] Yijie Han. Deterministic sorting in o(nlog log n) time and linear space. J. Algorithms, 50(1):96–105, January 2004.

1 [KLOS13] Jonathan A. Kelner, Yin Tat Lee, Lorenzo Orecchia, and Aaron Sidford. An almost-linear-time algorithm for approximate max flow in undirected graphs, and its multi-commodity generalizations. CoRR, abs/1304.2338, 2013.

1.2 [KLP12] Ioannis Koutis, Alex Levin, and Richard Peng. Improved Spectral Sparsification and Numerical Algorithms for SDD Matrices. In Christoph Dürr and Thomas Wilke, editors, 29th International Symposium on Theoretical Aspects of Computer Science (STACS 2012), volume 14 of Leibniz International Proceedings in Informatics (LIPIcs), pages 266–277, Dagstuhl, Germany, 2012. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
Ioannis Koutis, Gary L. Miller, and Richard Peng. A nearly-linear time solver for SDD linear systems. In Proceedings of the 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science, FOCS ’11, pages 590–598, Washington, DC, USA, 2011. IEEE Computer Society.

Jonathan A. Kelner, Lorenzo Orecchia, Aaron Sidford, and Zeyuan Allen Zhu. A simple, combinatorial algorithm for solving SDD systems in nearly-linear time. In Proceedings of the 45th annual ACM symposium on Symposium on theory of computing, STOC ’13, pages 911–920, New York, NY, USA, 2013. ACM.

F. Thomson Leighton. Introduction to Parallel Algorithms and Architectures: Array, Trees, Hypercubes. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1992.

Frank Thomson Leighton, Bruce M Maggs, and Satish B Rao. Packet routing and job-shop scheduling in O(congestion+ dilation) steps. Combinatorica, 14(2):167–186, 1994.

Yin Tat Lee, Satish Rao, and Nikhil Srivastava. A new approach to computing maximum flows using electrical flows. In Proceedings of the 45th annual ACM symposium on Symposium on theory of computing, STOC ’13, pages 755–764, New York, NY, USA, 2013. ACM.

Yin Tat Lee and Aaron Sidford. Efficient accelerated coordinate descent methods and faster algorithms for solving linear systems. CoRR, abs/1305.1922, 2013.

Aleksander Madry. Fast approximation algorithms for cut-based problems in undirected graphs. In FOCS, pages 245–254. IEEE Computer Society, 2010.

Aleksander Madry. Navigating central path with electrical flows: from flows to matchings, and back. CoRR, abs/1307.2205, 2013.

Gary L. Miller, Richard Peng, and Shen Chen Xu. Parallel graph decompositions using random shifts. In Proceedings of the 25th ACM symposium on Parallelism in algorithms and architectures, SPAA ’13, pages 196–203, New York, NY, USA, 2013. ACM.

Gary L. Miller and John H. Reif. Parallel tree contraction part 1: Fundamentals. In Silvio Micali, editor, Randomness and Computation, pages 47–72. JAI Press, Greenwich, Connecticut, 1989. Vol. 5.

Lorenzo Orecchia, Sushant Sachdeva, and Nisheeth K. Vishnoi. Approximating the exponential, the Lanczos method and an Õ(m)-time spectral algorithm for balanced separator. In Proceedings of the 44th symposium on Theory of Computing, STOC ’12, pages 1141–1160, New York, NY, USA, 2012. ACM.

Richard Peng and Daniel A. Spielman. An efficient parallel solver for SDD linear systems. CoRR, abs/1311.3286, 2013.
A Sufficiency of Embeddability

In the construction of our trees, we made a crucial relaxation of only requiring embeddability, rather than restricting to subgraphs. In this section, we show that linear operators on the resulting graph can be related to linear operators on the original graph. Our analysis is applicable to $\ell_\infty$ flows as well.

The spectral approximation of two graphs can be defined in terms of their Laplacians. As we will interpret these objects combinatorially, we omit their definition and refer the reader to Doyle and Snell [DS84]. For matrices, we can define a partial ordering $\preceq$ where $A \preceq B$ if $B - A$ is positive semidefinite. That is, for any vector $x$ we have

$$x^T A x \leq x^T B b.$$  

If we let the graph formed by adding the tree to $G$ be $H$, then our goal is to bound $L_G$ and $L_H$ with each other. Instead of doing this directly, it is easier to relate their pseudoinverses. This will be done by interpreting $x^T L^*_i x$ in terms of the energy of electrical flows. The energy of an electrical flow is defined as the sum of squares of the flows on the edges multiplied by their resistances, which
in our case are equal to the lengths of the edges. Given a flow \( f \in \mathbb{R}^E \), we will denote its electrical energy using
\[
\mathcal{E}_G(f) \overset{\text{def}}{=} \sum_e l_e f(e)^2.
\]

The residue of a flow \( f \) is the net in/out flow at each vertex. This gives a vector on all vertices, and finding the minimum energy of flows that meet a given residue is equivalent to computing \( x^T L^T G x \). The following fact plays a central role in the monograph by Doyle and Snell [DS84]:

**Fact A.1** Let \( G \) be a connected graph. For any vector \( x \) orthogonal to the all ones vector, \( x^T L^T_G x \) equals the minimum electrical energy of a flow with residue \( x \).

**Lemma A.2** Let \( G = (V_G, E_G, w_G) \) and \( H = (V_H, E_H, w_H) \) be graphs such that \( G \) is a subgraph of \( H \) in the weighted sense and \( H \setminus G \) is embeddable in \( G \). Furthermore, let the graph Laplacians of \( G \) and \( H \) be \( L_G \) and \( L_H \) respectively. Also, let \( \Pi \) be the \( |V_G| \times |V_H| \) matrix with one 1 in each row at the position that corresponds to \( V_H \) in \( H \) and 0 everywhere else, and \( \Pi_1 \) the orthogonal projection operator onto the part of \( \mathbb{R}^{|V_G|} \) that’s orthogonal to the all-ones vector. Then we have:
\[
\frac{1}{2} L_G^\dagger \leq \Pi_1 \Pi L_H^\dagger \Pi^T \Pi_1^T \leq L_G^\dagger.
\]

**Proof** Since \( \Pi_1^T = \Pi_1 \) projects out any part space spanned by the all ones vector, and is this precisely the null space of \( L_G \), it suffices to show the result for all vectors \( x_G \) orthogonal to the all-1s vector. These vectors are in turn valid demand vectors for electrical flows. Therefore, the statement is equivalent to relating the minimum energies of electrical flows routing \( x_G \) on \( G \) and \( \Pi^T x_G \) on \( H \).

We first show that flows on \( H \) take less energy than the ones in \( G \). Let \( x_G \) be any vector orthogonal to the all ones vector, and \( f_G^* \) the flow of minimum energy in \( G \) that meets demand \( x_G \). Setting the same flow on the edges of \( E(G) \) in \( H \) and 0 on all other edges yields a flow \( f_H \). The residue of this flow is the same residue in \( V_G \), and 0 everywhere else, and therefore equal to \( \Pi^T x_G \). Since \( G \) is a subgraph of \( H \) in the weighted sense, the lengths of these edges can only be less. Therefore the energy of \( f_H \) is at most the energy of \( f_G \) and we have
\[
x_G^T \Pi L_H^\dagger \Pi^T x_G \leq \mathcal{E}_H(f_H) \leq \mathcal{E}_G(f_G^*) = x_G^T L_G^\dagger x_G.
\]

For the reverse direction, we use the embedding of \( H \setminus G \) into \( G \) to transfer the flow from \( H \) into \( G \). Let \( x_G \) be any vector orthogonal to the all ones vector, and \( f_H^* \) the flow of minimum energy in \( H \) that has residue \( \Pi^T x_G \). This flow can be transformed into one in \( G \) that has residue \( x_G \) using the embedding. Let vertex/edge mapping of this embedding be \( \pi_V \) and \( \pi_E \) respectively.

If an edge \( e \in E_H \) is also in \( E_G \), we keep its flow value in \( G \). Otherwise, we route its flow along the path that the edge is mapped to. Formally, if the edge is from \( u \) to \( v \), \( f_H(e) \) units of flow is routed from \( \pi_V(u) \) to \( \pi_V(v) \) along \( \text{path}(e) \). We first check that the resulting flow, \( f_G \) has residue \( x_G \). The net amount of flow into a vertex \( u \in V_G \) is
\[
\sum_{uv \in E_G} f_H^*(e) + \sum_{uv' \in E_H \setminus E_G, \pi_V(u') = u} f_H^*(e) = \sum_{uv \in E_G} f_H^*(e) + \sum_{u' \in V_H, \pi_V(u') = u} \left( \sum_{uv' \in E_H \setminus E_G} f_H^*(e) \right).
\]
Reordering the summations and noting that $\Pi(u) = u$ gives

$$
= \sum_{u' \in V_H, \pi_V(u') = u} \sum_{u'' \in E_H} f_H(e) = \sum_{u' \in V_H, \pi_V(u') = u} (\Pi^T x_G) (e) = x_G(u).
$$

The last equality is because $\pi_V(u) = u$, and all vertices not in $V_G$ having residue 0 in $\Pi^T x_G$.

To bound the energy of this flow, the property of the embedding gives that if split the edges of $G$ into the paths that form the embedding, each edge is used at most once. Therefore, if we double the weights of $G$, we can use one copy to support $G$, and one copy to support the embedding. The energy of this flow is then the same. Hence there is an electrical flow $f_G$ in $G$ such that $\mathcal{E}_G(f_G) \leq 2\mathcal{E}_H(f_H)$.

Fact A.1 then gives that it is an upper bound for $x_G^T L_G^\dagger x_G$, completing the proof. ■