OPERATOR LOG-CONVEX FUNCTIONS AND $f$-DIVERGENCE FUNCTIONAL

MOHSEN KIAN

Abstract. We present a characterization of operator log-convex functions by using positive linear mappings. More precisely, we show that the continuous function $f : (0, \infty) \to (0, \infty)$ is operator log-convex if and only if $f(\Phi(A)) \leq (\Phi(f(A)^{-1}))^{-1}$ for every strictly positive operator $A$ and every unital positive linear map $\Phi$. Moreover, we study the non-commutative $f$-divergence functional of operator log-convex functions. In particular, we prove that $f$ is operator log-convex if and only if the non-commutative $f$-divergence functional is operator log-convex in its first variable and operator convex in its second variable.

1. Introduction and Preliminaries

Throughout this paper, assume that $B(H)$ is the $C^*$-algebra of all bounded linear operators on a Hilbert space $H$ and $I$ denote the identity operator. An operator $A \in B(H)$ is called positive (denoted by $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for every $x \in H$. If in addition $A$ is invertible, then it is called strictly positive (denoted by $A > 0$). A linear map $\Phi$ on $B(H)$ is said to be positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ and is called unital if $\Phi(I) = I$.

A continuous real function $f$ defined on an interval $J$ is said to be operator convex if

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$$

for all self-adjoint operators $A, B$ with spectra contained in $J$ and every $\lambda \in [0, 1]$. If $-f$ is operator convex, then $f$ is said to be operator concave. If $f : J \to \mathbb{R}$ is operator convex, then the celebrated Jensen operator inequality $f(C^*AC) \leq C^*f(A)C$ holds true for every self-adjoint operator $A$ with spectrum contained in $J$ and every isometry $C$ (see e.g. [4]). Another variant of this inequality, the Choi–Davis–Jensen inequality asserts that if $f$ is operator convex, then

$$f(\Phi(A)) \leq \Phi(f(A)) \quad (1)$$

for every unital positive linear map $\Phi$ on $B(H)$. The reader is referred to [4, 5, 6] and references therein for more information about operator convex functions and the Jensen operator inequality.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras of Hilbert space operators and $T$ be a locally compact Hausdorff space with a bounded Radon measure $\mu$. A field $(A_t)_{t \in T}$ of operators in $\mathfrak{A}$ is said to be continuous if the function $t \mapsto A_t$ is norm continuous on $T$. Moreover, if the

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function $t \mapsto A_t$ is integrable on $T$, then the Bochner integral $\int_T A_t d\mu(t)$ is defined to be the unique element of $\mathfrak{A}$ for which
\[
\rho \left( \int_T A_t d\mu(t) \right) = \int_T \rho(A_t) d\mu(t),
\]
for any linear functional $\rho$ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$.

A field $(\Phi_t)_{t \in T} : \mathfrak{A} \to \mathfrak{B}$ of positive linear mappings is said to be continuous if the function $t \mapsto \Phi_t(A)$ is continuous on $T$ for every $A \in \mathfrak{A}$. If the $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are unital and the function $t \mapsto \Phi_t(I)$ is integrable on $T$ with integral $I$, then we say that the field $(\Phi_t)_{t \in T}$ is unital.

By the well-known Kubo–Ando theory [7], an operator mean $\sigma$ is a binary operation on the set of positive operators which satisfies the following conditions:

1. monotonicity: if $A \leq C$ and $B \leq D$, then $A\sigma B \leq C\sigma D$;
2. Transformer inequality: $C(A\sigma B)C \leq (CAC)\sigma(CBC)$. If $C$ is invertible, then equality holds.
3. Continuity: if $A_n$ and $B_n$ are two decreasing sequences of positive operators which are converging respectively to $A$ and $B$ in the strong operator topology, then $A_n\sigma B_n$ converges to $A\sigma B$.

Kubo and Ando [7] showed that for every operator mean $\sigma$ there exists an operator monotone function $f : (0, \infty) \to (0, \infty)$ such that
\[
A\sigma B = B^{1/2}f \left( B^{-1/2}AB^{-1/2} \right) B^{1/2}
\]
for all positive operators $A, B$. Conversely, they proved that if $f : (0, \infty) \to (0, \infty)$ is operator monotone, the binary operation defined by (2) is an operator mean. Some of the most familiar operator means are
\[
A\triangledown B = \frac{A + B}{2} \quad \text{Arithmetic mean}
\]
\[
A\sharp B = B^{1/2} \left( B^{-1/2}AB^{-1/2} \right)^{1/2} B^{1/2} \quad \text{Geometric mean}
\]
\[
A!B = \left( \frac{A^{-1} + B^{-1}}{2} \right)^{-1} \quad \text{Harmonic mean}.
\]

A continuous real function $f : (0, \infty) \to (0, \infty)$ is called operator log-convex function if
\[
f(A\triangledown B) \leq f(A)\sharp f(B)
\]
for all positive operators $A$ and $B$. This notion was considered by Ando and Hiai [1]. They presented the following result.

**Theorem A.** [1, Theorem 2.1] Let $f : (0, \infty) \to (0, \infty)$ be continuous. The following conditions are equivalent:

1. $f$ is operator decreasing;
2. $f$ is operator log-convex;
3. $f(A\triangledown B) \leq f(A)\sigma f(B)$ for all positive operators $A$ and $B$ and every operator mean $\sigma$;
(4) $f(A \nabla B) \leq f(A)\sigma f(B)$ for all positive operators $A$ and $B$ and some operator mean $\sigma \neq \nabla$.

If $f$ is operator convex, then (2) defines [2, 3] the perspective of $f$ denoted by $g$, i.e.,

$$g(A, B) = B^{\frac{1}{2}} \left( B^{\frac{1}{2}} A B^{\frac{1}{2}} \right) B^{\frac{1}{2}}.$$

It is known that $f$ is operator convex if and only if $g$ is jointly operator convex [2, 3]. A more general version of $g$, the non-commutative $f$-divergence functional $\Theta$ was defined in [8] to be

$$\Theta(\tilde{A}, \tilde{B}) = \int_{\mathcal{T}} B^{\frac{1}{2}} \left( B^{\frac{1}{2}} A_{i} B^{\frac{1}{2}} \right) B^{\frac{1}{2}} d\mu(t),$$

where $\tilde{A} = (A_{i})_{i \in T}$ and $\tilde{B} = (B_{i})_{i \in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$.

In section 2, we give some properties of operator log-convex functions. We present a characterization of operator log-convex functions by using positive linear mappings. More precisely, we show that the continuous function $f : (0, \infty) \to (0, \infty)$ is operator log-convex if and only if $f(\Phi(A)) \leq (\Phi(f(A))^{-1})^{-1}$ for every strictly positive operator $A$ and every unital positive linear map $\Phi$.

In section 3, we study the non-commutative $f$-divergence functional of operator log-convex functions. In particular, we prove that $f$ is operator log-convex if and only if $\Theta$ is operator log-convex in its first variable and operator convex in its second variable.

2. Main Result

If $f$ is operator log-convex, then a sharper variant of the Jensen operator inequality holds true. The proof of the next lemma is based on that of [4, Theorem 1.9].

**Lemma 2.1.** If $f : (0, \infty) \to (0, \infty)$ is an operator log-convex function, then

$$f \left( C^* AC \right) \leq \left( C^* f(A)^{-1} C \right)^{-1}$$

for every strictly positive operator $A$ and every isometry $C$ provided that $C^* f(A)^{-1} C$ is invertible.

**Proof.** If $A$ and $B$ are two strictly positive operators in $\mathfrak{B}(\mathcal{H})$, then $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is regarded as a strictly positive operator in $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$. Set $D = \sqrt{T - CC^*}$ so that operators $U$ and $V$ defined by

$$U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}, \quad V = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix},$$

are unitary operators in $\mathfrak{B}(\mathcal{H} \oplus \mathcal{H})$ and

$$U^* X U = \begin{pmatrix} C^* AC & C^* AD \\ DAC & DAD + CBC^* \end{pmatrix}, \quad V^* X V = \begin{pmatrix} C^* AC & -C^* AD \\ -DAC & DAD + CBC^* \end{pmatrix}.$$
Therefore
\[
\left( \begin{array}{cc} f(C^*AC) & 0 \\ 0 & f(DAD + CBC^*) \end{array} \right) = f \left( \begin{array}{cc} C^*AC & 0 \\ 0 & DAD + CBC^* \end{array} \right)
= f \left( \frac{U^*XU + V^*XV}{2} \right)
\leq f(U^*XU)!f(V^*XV) \quad \text{(since } f \text{ is operator log-convex)}
= (U^*f(X)U)(V^*f(X)V) \quad \text{(since } U, V \text{ are unitary)}
= \left( \frac{(U^*f(X)U)^{-1} + (V^*f(X)V)^{-1}}{2} \right)^{-1}
= \left( \frac{U^*f(X)^{-1}U + V^*f(X)^{-1}V}{2} \right)^{-1}
= \left( \begin{array}{cc} C^*f(A)^{-1}C & 0 \\ 0 & Df(A)^{-1}D + Cf(B)^{-1}C^* \end{array} \right)^{-1}.
\]
Hence \( f(C^*AC) \leq (C^*f(A)^{-1}C)^{-1} \).

Note that the operator convexity of \( f(x) = x^{-1} \) implies that
\[
\frac{f(C^*AC)}{C^*f(A)^{-1}C} \leq f(A)^{-1} \leq C^*f(A)C.
\]

**Corollary 2.2.** If \( f : (0, \infty) \to (0, \infty) \) is an operator log-convex function and \( A_1, \cdots, A_n \) are strictly positive operators, then
\[
f \left( \sum_{i=1}^{n} C_i^*A_iC_i \right) \leq \left( \sum_{i=1}^{n} C_i^*f(A_i)^{-1}C_i \right)^{-1}
\]
for all operators \( C_i \ (i = 1, \cdots, n) \) with \( \sum_{i=1}^{n} C_i^*C_i = I \).

**Proof.** Apply Lemma 2.1 to the strictly positive operator \( A = A_1 \oplus \cdots \oplus A_n \) and the isometry \( C = \left( \begin{array}{c} C_1 \\ \vdots \\ C_n \end{array} \right) \). \[\square\]

We can present the following characterization of operator log-convex functions using positive linear mappings.

**Theorem 2.3.** A continuous function \( f : (0, \infty) \to (0, \infty) \) is operator log-convex if and only if
\[
f(\Phi(A)) \leq \Phi \left( f(A)^{-1} \right)^{-1}
\]
for every unital positive linear map \( \Phi \) and every strictly positive operator \( A \).

**Proof.** Suppose that \( A \) is a strictly positive operator an a finite dimensional Hilbert space \( \mathcal{H} \) with the spectral decomposition \( A = \sum_{i=1}^{n} \lambda_i P_i \). If \( \Phi \) is a unital positive linear map
on $\mathbb{B}(\mathcal{H})$, then $\Phi(A) = \sum_{i=1}^{n} \lambda_i \Phi(P_i)$ and $\sum_{i=1}^{n} \Phi(P_i) = I$. Therefore

$$f(\Phi(A)) = f \left( \sum_{i=1}^{n} \lambda_i \Phi(P_i) \right) = f \left( \sum_{i=1}^{n} \Phi(P_i)^{\frac{1}{2}} \lambda_i \Phi(P_i)^{\frac{1}{2}} \right)$$

$$\leq \left( \sum_{i=1}^{n} \Phi(P_i)^{\frac{1}{2}} f(\lambda_i)^{-1} \Phi(P_i)^{\frac{1}{2}} \right)^{-1} \quad \text{(by Corollary 2.2)}$$

$$= \left( \sum_{i=1}^{n} f(\lambda_i)^{-1} \Phi(P_i) \right)^{-1}$$

$$= \Phi \left( f(A)^{-1} \right)^{-1}.$$

If $A$ is a strictly positive operator on an infinite dimensional Hilbert space, then (3) follows by using a continuity argument. For the converse assume that (3) holds true. Put

$$\mathcal{D}(\mathcal{H} \oplus \mathcal{H}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \ A, B \in \mathbb{B}(\mathcal{H}) \right\}.$$

Then $\mathcal{D}(\mathcal{H} \oplus \mathcal{H})$ is a unital closed $*$-subalgebra of $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. Let the unital positive linear map $\Psi$ be defined on $\mathcal{D}(\mathcal{H} \oplus \mathcal{H})$ by

$$\Psi \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) = \frac{A + B}{2}.$$

Now if $A$ and $B$ are two strictly positive operators on $\mathcal{H}$, then it follows from (3) that

$$f(A \nabla B) = f \left( \Psi \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) \right) \leq \Psi \left( f \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right)^{-1} \right)^{-1}$$

$$= \Psi \left( f(A)^{-1} 0 \\ 0 f(B)^{-1} \right)^{-1} = \left( \frac{f(A)^{-1} + f(B)^{-1}}{2} \right)^{-1}$$

$$= f(A)^{\frac{1}{2}} f(B) \leq f(A)^{\frac{1}{2}} f(B),$$

Which implies that $f$ is operator log-convex.

Note that it follows from the operator convexity of $x \to x^{-1}$ that

$$f(\Phi(A)) \leq \Phi \left( f(A)^{-1} \right)^{-1} \leq \Phi(f(A)).$$

Let us give an example to show that in the case of operator log-convex functions, inequality (3) is really sharper than the Choi–Davis–Jensen inequality.

**Example 2.4.** The function $f(x) = x^{-\frac{1}{2}}$ is operator log-convex on $(0, \infty)$. Assume that the unital positive linear map $\Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ is defined by

$$\Phi((a_{ij})) = (a_{ij})_{2 \leq i,j \leq 3}.$$
If $A \in \mathcal{M}_3(\mathbb{C})$ is the positive matrix
\[
A = \begin{pmatrix}
2 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 3
\end{pmatrix},
\]
then by a simple calculation we have
\[
f(\Phi(A)) = \begin{pmatrix}
1.1945 & -0.2706 \\
-0.2706 & 0.6533
\end{pmatrix}, \quad \Phi \left( f(A)^{-1} \right)^{-1} = \begin{pmatrix}
1.2192 & -0.2933 \\
-0.2933 & 0.6760
\end{pmatrix}
\]
\[
\Phi(f(A)) = \begin{pmatrix}
1.2420 & -0.3261 \\
-0.3261 & 0.7234
\end{pmatrix}
\]
and so
\[
f(\Phi(A)) \preceq \Phi \left( f(A)^{-1} \right)^{-1} \preceq \Phi(f(A)).
\]

**Corollary 2.5.** If $\Phi$ is a unital positive linear map and $A$ is a strictly positive operator, then
\[
\begin{align*}
(1) \quad & \Phi(A)^{-\alpha} \leq \Phi(A^\alpha)^{-1} \leq \Phi(A^{-\alpha}) \quad \text{for all } 0 \leq \alpha \leq 1. \\
(2) \quad & \Phi(A^\alpha)^{\frac{1}{\alpha}} \leq \Phi(A^{-1})^{-1} \leq \Phi(A) \quad \text{for all } \alpha \leq -1.
\end{align*}
\]

**Proof.** (1): follows from the operator log-convexity of $t^{-\alpha}$.
(2): the function $t \rightarrow t^{\frac{1}{\alpha}}$ is operator log-convex. So it follows from Theorem 2.3 that $\Phi(A)^{\frac{1}{\alpha}} \leq \left( \Phi \left( A^{-1}_\alpha \right) \right)^{-1}$. Replacing $A$ by $A^\alpha$ we get desired inequality. \(\square\)

**Corollary 2.6.** Let $\Phi_1, \cdots, \Phi_n$ be positive linear mappings on $\mathbb{B}(\mathcal{H})$ such that $\sum_{i=1}^n \Phi_i(I) = I$. If $f : (0, \infty) \to (0, \infty)$ is an operator log-convex function, then
\[
f \left( \sum_{i=1}^n \Phi_i(A_i) \right) \leq \left( \sum_{i=1}^n \Phi_i(f(A_i)^{-1}) \right)^{-1}
\]
for all strictly positive operators $A_1, \cdots, A_n$.

**Proof.** Apply Theorem 2.3 to the strictly positive operator $A = A_1 \oplus \cdots \oplus A_n$ and the unital positive linear map $\Phi : \mathbb{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H}) \to \mathbb{B}(\mathcal{H})$ defined by $\Phi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^n \Phi_i(A_i)$.

The next result shows that every operator log-convex function is sub-additive.

**Proposition 2.7.** If $f : (0, \infty) \to (0, \infty)$ is an operator log-convex function, then $f$ is sub-additive. More precisely
\[
f(A + B) \leq f(A)f(B) \leq f(A) + f(B)
\]
for all strictly positive operators $A, B$. 
Proof. Assume that $A$ and $B$ are strictly positive operators. Then

$$f(A + B) = f((2A)\nabla (2B)) \leq f(2A)^{\#} f(2B)$$
$$\leq f(A)^{\#} f(B) \quad \text{(by (1) of Theorem A)}$$
$$\leq f(A) \nabla f(B) \quad \text{(by the A-G inequality)}$$
$$\leq f(A) + f(B).$$

3. THE NON-COMMUTATIVE $f$-DIVERGENCE FUNCTIONAL

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be $n$-tuples of positive operators on $\mathcal{M}$. If follows from the jointly operator concavity of the operator geometric mean that

$$\sum_{i=1}^{n} X_i^{\#} Y_i \leq \left( \sum_{i=1}^{n} X_i \right)^{\#} \left( \sum_{i=1}^{n} Y_i \right). \quad (4)$$

This inequality is known as the operator version of the Cauchy–Schwarz inequality.

To achieve our result, we need a more general version of (4). Assume that $(A_t)_{t \in T}$ and $(B_t)_{t \in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$. We can generalize (4) as

$$\int_T (A_t^{\#} B_t) d\mu(t) \leq \left( \int_T A_t d\mu(t) \right)^{\#} \left( \int_T B_t d\mu(t) \right).$$

Lemma 3.1. If $(A_t)_{t \in T}$ and $(B_t)_{t \in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$, then

$$\int_T (A_t^{\#} B_t) d\mu(t) \leq \left( \int_T A_t d\mu(t) \right)^{\#} \left( \int_T B_t d\mu(t) \right).$$

Proof. Put $A = \int_T A_t d\mu(t)$ and $B = \int_T B_t d\mu(t)$. We have

$$\left( B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \right)^{\#} = \left( \left( \int_T B_s d\mu(s) \right)^{-\frac{1}{2}} \int_T A_t d\mu(t) \left( \int_T B_s d\mu(s) \right)^{-\frac{1}{2}} \right)^{\frac{1}{2}}$$
$$= \left( \int_T \left( \int_T B_s d\mu(s) \right)^{-\frac{1}{2}} B_t^\frac{1}{2} (B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}) B_t^\frac{1}{2} \left( \int_T B_s d\mu(s) \right)^{-\frac{1}{2}} d\mu(t) \right)^{\frac{1}{2}}$$
$$= \left( \int_T C^*_t (B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}) C_t d\mu(t) \right)^{\frac{1}{2}}, \quad (5)$$
where \( C = B_t^{1/2} \left( \int_T B_s d\mu(s) \right)^{-1/2} \) so that \( \int_T C_t^* C_t d\mu(t) = I \). It follows from the operator concavity of the function \( t^{1/2} \) that

\[
\left( \int_T C_t^* \left( B_t^{1/2} A_t B_t^{1/2} \right) C_t d\mu(t) \right)^{1/2} \\
\geq \int_T C_t^* \left( B_t^{1/2} A_t B_t^{1/2} \right)^{\frac{2}{3}} C_t d\mu(t) \quad \text{(by the operator Jensen inequality)}
\]

\[
= \left( \int_T B_s d\mu(s) \right)^{-\frac{1}{2}} \int_T B_t^{\frac{1}{2}} \left( B_t^{1/2} A_t B_t^{1/2} \right)^{\frac{2}{3}} B_t^{\frac{1}{2}} d\mu(t) \left( \int_T B_s d\mu(s) \right)^{-\frac{1}{2}}.
\]

(6)

It follows that from (5) and (6) that

\[
\left( B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \right)^{\frac{2}{3}} \geq B^{-\frac{1}{2}} \left( \int_T B_t^{\frac{1}{2}} \left( B_t^{1/2} A_t B_t^{1/2} \right)^{\frac{2}{3}} B_t^{\frac{1}{2}} d\mu(t) \right) B^{-\frac{1}{2}}
\]

from which we get the desired result. \( \square \)

Now we present a property of the non-commutative \( f \)-divergence functional of an operator log-convex function.

**Theorem 3.2.** A continuous function \( f : (0, \infty) \to (0, \infty) \) is operator log-convex function if and only if

\[
\Theta \left( \mathbf{A} \nabla \mathbf{C}, \mathbf{B} \nabla \mathbf{D} \right) \leq \left( \Theta \left( \mathbf{A}, \mathbf{B} \right) \nabla \Theta \left( \mathbf{A}, \mathbf{D} \right) \right)^{\frac{2}{3}} \left( \Theta \left( \mathbf{C}, \mathbf{B} \right) \nabla \Theta \left( \mathbf{C}, \mathbf{D} \right) \right)^{\frac{1}{3}}
\]

(7)

for all continuous fields \( \mathbf{A} = (A_t)_{t \in T}, \mathbf{B} = (B_t)_{t \in T}, \mathbf{C} = (C_t)_{t \in T} \) and \( \mathbf{D} = (D_t)_{t \in T} \) of strictly positive operators in \( \mathfrak{A} \).

**Proof.** First we show that if \( f \) is operator log-convex on \( (0, \infty) \), then \( \Theta \) is operator log-convex in its first variable. Assume that \( \mathbf{A} = (A_t)_{t \in T}, \mathbf{B} = (B_t)_{t \in T}, \mathbf{C} = (C_t)_{t \in T} \) and \( \mathbf{D} = (D_t)_{t \in T} \) are continuous fields of strictly positive operators in \( \mathfrak{A} \). For every \( t \in T \)

\[
f \left( B_t^{-\frac{1}{2}} (A_t \nabla C_t) B_t^{-\frac{1}{2}} \right) = f \left( \left( B_t^{-\frac{1}{2}} A_t B_t^{\frac{1}{2}} \right) \nabla \left( B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}} \right) \right)
\]

\[
\leq f \left( B_t^{-\frac{1}{2}} A_t B_t^{\frac{1}{2}} \right) \# f \left( B_t^{-\frac{1}{2}} C_t B_t^{\frac{1}{2}} \right),
\]

(8)

where we use the operator log-convexity of \( f \). Multiplying both sides of (8) by \( B_t^{\frac{1}{2}} \) we get

\[
B_t^{\frac{1}{2}} f \left( B_t^{-\frac{1}{2}} (A_t \nabla C_t) B_t^{-\frac{1}{2}} \right) B_t^{\frac{1}{2}} \leq B_t^{\frac{1}{2}} \left( f \left( B_t^{-\frac{1}{2}} A_t B_t^{\frac{1}{2}} \right) \# f \left( B_t^{-\frac{1}{2}} C_t B_t^{\frac{1}{2}} \right) \right) B_t^{\frac{1}{2}}
\]

(9)

\[
= B_t^{\frac{1}{2}} f \left( B_t^{-\frac{1}{2}} A_t B_t^{\frac{1}{2}} \right) B_t^{\frac{1}{2}} \# B_t^{\frac{1}{2}} B_t^{\frac{1}{2}} f \left( B_t^{-\frac{1}{2}} C_t B_t^{\frac{1}{2}} \right) B_t^{\frac{1}{2}}.
\]
The last equality follows from the property of (geometric) means. Integrating (9) over $T$ and using Lemma 3.1 we obtain

$$
\int_T B_t^\frac{1}{2} f \left( B_t^{-\frac{1}{2}} (A_t \nabla C_t) B_t^{-\frac{1}{2}} \right) B_t^\frac{1}{2} d\mu(t)
\leq \int_T B_t^\frac{1}{2} f \left( B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}} \right) B_t^\frac{1}{2} d\mu(t) \left( \int_T B_t^\frac{1}{2} f \left( B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}} \right) B_t^\frac{1}{2} d\mu(t) \right)
\leq \left( \int_T B_t^\frac{1}{2} f \left( B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}} \right) B_t^\frac{1}{2} d\mu(t) \right) \left( \int_T B_t^\frac{1}{2} f \left( B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}} \right) B_t^\frac{1}{2} d\mu(t) \right),
$$

i.e.,

$$
\Theta (\tilde{A} \nabla \tilde{C}, \tilde{B}) \leq \Theta (\tilde{A}, \tilde{B}) \sharp \Theta (\tilde{C}, \tilde{B}). \tag{10}
$$

Therefore

$$
\Theta (\tilde{A} \nabla \tilde{C}, \tilde{B} \nabla \tilde{D}) \leq \Theta (\tilde{A}, \tilde{B} \nabla \tilde{D}) \sharp \Theta (\tilde{C}, \tilde{B} \nabla \tilde{D}) \quad \text{(by (10))}
\leq \left( \Theta (\tilde{A}, \tilde{B}) \nabla \Theta (\tilde{A}, \tilde{D}) \right) \sharp \left( \Theta (\tilde{C}, \tilde{B}) \nabla \Theta (\tilde{C}, \tilde{D}) \right).
$$

The last inequality follows from the joint operator convexity of $\Theta$ [8] and monotonicity of operator means.

Assume for the converse that $\Theta$ satisfies (7). Let $T = \{1\}$ and $\mu$ be the counting measure on $T$. Let $A$ and $C$ be strictly positive operators in $\mathfrak{A}$. Then

$$
f(A \nabla C) = \Theta (A \nabla C, I) \leq \Theta (A, I) \sharp \Theta (C, I) = f(A) \sharp f(C),
$$

which means that $f$ is operator log-convex. \qed

**Remark 3.3.** In fact Theorem 3.2 assert that $f$ is operator log-convex if and only if the non-commutative $f$-divergence functional $\Theta$ is operator log-convex in its first variable and operator convex in its second variable.

**Corollary 3.4.** A continuous non-negative function $f$ is operator log-convex function if and only if the associated perspective function $g$ is operator log-convex function in its first variable and operator convex in its second variable.

The next theorem provides a Choi–Davis–Jensen type inequality for perspectives of operator log-convex functions.

**Theorem 3.5.** Let $f : (0, \infty) \to (0, \infty)$ be a continuous function and $g$ be the associated perspective function. Then $f$ is operator log-convex if and only if

$$
g \left( \int_T \Phi_t(A_t \nabla C_t) d\mu(t), \int_T \Phi_t(B_t) d\mu(t) \right) \leq \left( \int_T \Phi_t(g(A_t, B_t)) d\mu(t) \right) \sharp \left( \int_T \Phi_t(g(C_t, B_t)) d\mu(t) \right)
$$

for all unital fields $\tilde{\Phi} = (\Phi_t)_{t \in T} : \mathfrak{A} \to \mathfrak{B}$ of positive linear maps and all continuous fields $\tilde{A} = (A_t)_{t \in T}, \tilde{B} = (B_t)_{t \in T}$ and $\tilde{C} = (C_t)_{t \in T}$ of strictly positive operators in $\mathfrak{A}$. 
Proof. Assume that \( f \) is operator log convex. Then \( f \) is operator convex and so \( g \) is jointly operator convex \([3, 2]\). Put \( X = \int_T \Phi_t(B_s) d\mu(s) \) and let the continuous filed of positive linear mappings \((\Psi_t)_{t \in T} : \mathfrak{A} \to \mathfrak{B}\) be defined by

\[
\Psi_t(Y) = X^{-\frac{1}{2}} \Phi_t \left( B_t^{-\frac{1}{2}} Y B_t^{-\frac{1}{2}} \right) X^{-\frac{1}{2}}
\]

so that \( \int_T \Psi_t(I) d\mu(t) = I \). Therefore

\[
g \left( \int_T \Phi_t(A_t \nabla C_t) d\mu(t), \int_T \Phi_t(B_t) d\mu(t) \right) = X^{\frac{1}{2}} f \left( X^{-\frac{1}{2}} \int_T \Phi_t(A_t \nabla C_t) d\mu(t) X^{-\frac{1}{2}} \right) X^{\frac{1}{2}}
\]

\[
= X^{\frac{1}{2}} f \left( \int_T \Psi_t \left( B_t^{-\frac{1}{2}} (A_t \nabla C_t) B_t^{-\frac{1}{2}} \right) d\mu(t) \right) X^{\frac{1}{2}}
\]

\[
\leq X^{\frac{1}{2}} \left( \int_T \Psi_t \left( f \left( B_t^{-\frac{1}{2}} (A_t \nabla C_t) B_t^{-\frac{1}{2}} \right) \right) d\mu(t) \right) X^{\frac{1}{2}} \quad \text{(by the Jensen operator inequality)}
\]

\[
= \int_T \Phi_t \left( B_t^{\frac{1}{2}} f \left( B_t^{-\frac{1}{2}} (A_t \nabla C_t) B_t^{-\frac{1}{2}} \right) B_t^{\frac{1}{2}} \right) d\mu(t)
\]

\[
= \int_T \Phi_t(g(A_t \nabla C_t, B_t)) d\mu(t)
\]

\[
\leq \int_T \Phi_t(g(A_t, B_t)) \sharp g(C_t, B_t) d\mu(t) \quad \text{(by Corollary 3.4)}
\]

\[
\leq \int_T \Phi_t(g(A_t, B_t)) \sharp \Phi_t(g(C_t, B_t)) d\mu(t) \quad \text{(by operator concavity of \( \sharp \))}
\]

\[
\leq \left( \int_T \Phi_t(g(A_t, B_t)) d\mu(t) \right) \sharp \left( \int_T \Phi_t(g(C_t, B_t)) d\mu(t) \right) \quad \text{(by Lemma 3.1)}.
\]

For the converse, put \( T = \{1\} \) and let \( \mu \) be the counting measure on \( T \). If \( A \) and \( C \) are strictly positive, then with \( \Phi(A) = A \) and \( B = I \), inequality \((11)\) implies the operator log-convexity of \( f \). \( \square \)

Example 3.6. Let the operator log-convex function \( f : (0, \infty) \to (0, \infty) \) be defined by \( f(t) = t^{-1} \). It follows from Theorem 3.5 that

\[
g(\Phi(A \nabla C), \Phi(B)) \leq \Phi(g(A, B)) \sharp \Phi(g(C, B))
\]

or equivalently

\[
\Phi(B) \Phi(A \nabla C)^{-1} \Phi(B) \leq \Phi(BA^{-1}B) \sharp \Phi(BC^{-1}B).
\]

Therefore

\[
\Phi(A \nabla C)^{-1} \leq \Phi(B)^{-1} \left( \Phi(BA^{-1}B) \sharp \Phi(BC^{-1}B) \right) \Phi(B)^{-1}
\]

\[
= \Phi(B)^{-1} \Phi(BA^{-1}B) \Phi(B)^{-1} \sharp \Phi(B)^{-1} \Phi(BC^{-1}B) \Phi(B)^{-1} \quad \text{(12)}
\]
Note that it follows from the operator convexity of $f$ that
\[
\Phi(A \nabla C)^{-1} = \left( \frac{\Phi(A) + \Phi(C)}{2} \right)^{-1} \\
\leq \frac{\Phi(A)^{-1} + \Phi(C)^{-1}}{2} \quad \text{(by operator convexity of } f(t) = t^{-1}) \\
\leq \frac{\Phi(A^{-1}) + \Phi(C^{-1})}{2} \quad \text{(by operator convexity of } f(t) = t^{-1}),
\]
while with $B = I$, inequality (12) provides a sharper inequality:
\[
\Phi(A \nabla C)^{-1} \leq \frac{\Phi(A^{-1}) + \Phi(C^{-1})}{2}.
\]

**Corollary 3.7.** Let $A, B, C$ be strictly positive operators. If $f$ is an operator log-convex function and $g$ is its perspective function, then
\[
g\left( \frac{A + C}{2} x, x \right), \langle Bx, x \rangle \leq \sqrt{\langle g(A, B)x, x \rangle \langle g(C, B)x, x \rangle}
\]
for every unit vector $x$.

**Example 3.8.** Applying Corollary 3.7 to the operator log-convex function $f(t) = t^{-1}$ defined on $(0, \infty)$ we get
\[
\langle Bx, x \rangle \left( \frac{A + C}{2} x, x \right)^{-1} \langle Bx, x \rangle \leq \sqrt{\langle BA^{-1}Bx, x \rangle \langle BC^{-1}Bx, x \rangle} \\
\leq \left( B \left( \frac{A^{-1} + C^{-1}}{2} \right) Bx, x \right)
\]
for every unit vector $x$.

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Mohsen Kian: Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran.

E-mail address: kian@member.ams.org and kian@ub.ac.ir.