The Shatashvili-Vafa $G_2$ superconformal algebra as a Quantum Hamiltonian Reduction of $D(2, 1; \alpha)$.

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Abstract

We obtain the superconformal algebra associated to a sigma model with target a manifold with $G_2$ holonomy, i.e., the Shatashvili-Vafa $G_2$ algebra as a quantum Hamiltonian reduction of the exceptional Lie superalgebra $D(2, 1; \alpha)$ for $\alpha = 1$. We produce the complete family of $W$-algebras $SW(\frac{3}{2}, \frac{3}{2}, 2)$ (extensions of the $N = 1$ superconformal algebra by two primary supercurrents of conformal weight $\frac{3}{2}$ and 2 respectively) as a quantum Hamiltonian reduction of $D(2, 1; \alpha)$. As a corollary we find a free field realization of the Shatashvili-Vafa $G_2$ algebra, and an explicit description of the screening operators.

1 Introduction

The Shatashvili-Vafa $G_2$ algebra\textsuperscript{[1]} is a superconformal vertex algebra with six generators $\{L, G, \Phi, K, X, M\}$. It is an extension of the $N = 1$ superconformal algebra of central charge $c = 21/2$ (formed by the super-partners $\{L, G\}$) by two fields $\Phi$ and $K$, primary of conformal weight $\frac{3}{2}$ and 2 respectively, and their superpartners $X$ and $M$ (of conformal weight 2 and $\frac{5}{2}$ respectively). Their OPEs can be found in Appendix A in the language of lambda brackets of\textsuperscript{[2]}.

This superconformal algebra appeared as the chiral algebra associated to the sigma model with target a manifold with $G_2$ holonomy in\textsuperscript{[1]} its classical counterpart had been studied by Howe and Papadopoulos in\textsuperscript{[3]}. In fact this algebra is a member of a two-parameter family $SW(\frac{3}{2}, \frac{3}{2}, 2)$ previously studied in\textsuperscript{[4]} where the author found the family of all superconformal algebras which are extension of the super-Virasoro algebra, i.e., the $N = 1$ superconformal algebra, by two primary supercurrents of conformal weights $\frac{3}{2}$ and 2 respectively. It is a family parametrized by $(c, \varepsilon)$ ($c$ is the central charge and $\varepsilon$ the coupling constant) of non-linear $W$-algebras. Its generators and relations are recalled in Appendix B.

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The Shatashvili-Vafa $G_2$ algebra is a quotient of $SW(\frac{3}{2}, \frac{3}{2}, 2)$ with $c = \frac{21}{2}$ and $\varepsilon = 0$, in other words is the only one among this family which has central charge $c = \frac{21}{2}$ and contains the tri-critical Ising model as a subalgebra. It is precisely the fact that the Shatashvili-Vafa $G_2$ algebra appears as a $W$-algebra that motivated the authors to try to obtain this algebra as a quantum Hamiltonian reduction of some Lie superalgebra using the method developed in [7].

That $D(2, 1; \alpha)$ is the right Lie superalgebra candidate to be used in the Hamiltonian reduction is known from scattered results in the physics literature. It was shown in [10] that $SW(\frac{3}{2}, \frac{3}{2}, 2)$ is the symmetry algebra of the quantized Toda theory corresponding to $D(2, 1; \alpha)$ (in [9] was worked a classical version of this result in the case $\alpha = 1$ ($D(2, 1; \alpha) = osp(4|2)$) and from the well established connection between the theory of nonlinear integrable equations and $W$-algebras, see for example [12].

A coset realization of the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra and therefore of the Shatashvili-Vafa algebra can be found in [14]. In [15] was shown that the Hamiltonian reduction of $D(2, 1; \alpha)$ coincides with this coset model (the authors however restrict their attention to the even part of the superalgebra).

Some representations of the Shatashvili-Vafa $G_2$ superconformal algebra can be found in [14], but the character formulae remains unknown. It was observed in [11] that in order to systematically study the representation theory and the character formula for this algebra one should construct the Shatashvili-Vafa algebra using the quantum Drinfeld-Sokolov reduction developed in [8, 16]. This step is accomplished in this article.

In section 2 we review how to perform the quantum Hamiltonian reduction of a Lie superalgebra as introduced in [7]. We recap some of the main theorems as well as under which conditions this Hamiltonian reduction process induces a free field realization.

In section 3 we prove that the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ superconformal algebra is the quantum Hamiltonian reduction of the Lie superalgebra $D(2, 1; \alpha)$, and obtain a free field realization of the $SW(\frac{3}{2}, \frac{3}{2}, 2)$ algebra on a space of three free Bosons and three free Fermions. As particular cases $\alpha \in \{1, -\frac{1}{2}, -2\}$ we obtain the Shatashvili-Vafa $G_2$ algebra as a quantum Hamiltonian reduction of the Lie superalgebra $osp(4|2)$, and also the corresponding free field realizations. We summarize our main result as (see Theorem 3.1 its remark)

**Theorem.** Let $\mathfrak{h}$ be the Cartan subalgebra of $D(2, 1; \alpha)$. It is a three dimensional vector space with a non-degenerate bilinear form $(,)$ given by the Cartan matrix. Consider $\Pi \mathfrak{h}^*$ the odd vector space ($\Pi$ denotes parity change) with its natural bilinear form $-(,)$.

Let $V_k(\mathfrak{h}_{\text{super}})$ be the super affine vertex algebra generated by three Bosons from $\mathfrak{h}$ and three Fermions from $\Pi \mathfrak{h}^*$ and lambda brackets

$$[h_\lambda h''] = k\lambda(h, h''), \quad [\phi_\lambda \phi''] = -(\phi, \phi''), \quad h, h' \in \mathfrak{h}, \phi, \phi' \in \Pi \mathfrak{h}^*.$$
2. The Shatashvili-Vafa $G_2$ superconformal algebra is a quotient of this algebra by an ideal generated in conformal weight $7/2$ \([3, 2]\).

3. For each $\alpha_i$ of the three odd simple roots of $D(2, 1, \alpha)$ there exists a module $M_i$ of $V_{\kappa}(\mathfrak{h}_{\text{super}})$ generated by a vector $|\alpha_i\rangle$ such that $h_n|\alpha_i\rangle = 0$ for $n > 0$, $h_0|\alpha_i\rangle = (h, \alpha_i)|\alpha_i\rangle$ and $\phi_n|\alpha_i\rangle = 0$ for $n > 0$ (for all $h \in \mathfrak{h}$ and $\phi \in \mathfrak{h}^*$). Let $\Gamma_i(z)$ be the unique intertwiner of type $(V_{\kappa}(\mathfrak{h}_{\text{super}}), M_i)$ and $Q_i \in \text{Hom}(V_{\kappa}(\mathfrak{h}_{\text{super}}), M_i)$ its zero mode. Then for generic values of $(c, \varepsilon)$ we have $SW(\frac{3}{2}, \frac{3}{2}, 2) = \bigcap_{i} Q_i \subset V_{\kappa}(\mathfrak{h}_{\text{super}})$.

In Section 3 the reader can find a stronger version of this Theorem as the generators for $SW(\frac{3}{2}, \frac{3}{2}, 2)$ are found for any values of the parameters $(c, \varepsilon)$.

2 Quantum reduction of Lie superalgebras

In this section we recall the construction of the W-algebras $W_{\kappa}(\mathfrak{g}, x, f)$ introduced in [7]. We follow the presentation in [8].

To construct the vertex algebra $W_{\kappa}(\mathfrak{g}, x, f)$ we need a quadruple $(\mathfrak{g}, x, f, k)$ where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a simple finite-dimensional Lie superalgebra with a non-degenerate even invariant supersymmetric bilinear form $\langle.,.\rangle$, and $x, f \in \mathfrak{g}_0$ such that $ad_x$ is diagonalizable on $\mathfrak{g}$ with half-integer eigenvalues, $[x, f] = -f$, the eigenvalues of $ad_x$ on the centralizer $\mathfrak{g}'$ of $f$ in $\mathfrak{g}$ are non-positive, and $k \in \mathbb{C}$.

We recall that a bilinear form $\langle ., . \rangle$ on $\mathfrak{g}$ is called even if $\langle \mathfrak{g}_0|\mathfrak{g}_0\rangle = 0$, supersymmetric if $\langle ., . \rangle$ is symmetric (resp. skew-symmetric) on $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$), invariant if $\langle [a, b]|c\rangle = [\langle a|b\rangle, c]$ for all $a, b, c \in \mathfrak{g}$.

A pair $(x, f)$ satisfying the above properties can be obtained when $x, f$ are part of an $\mathfrak{sl}_2$ triple, i.e., $[x, e] = e$, $[x, f] = -f$ and $[e, f] = x$. As this will be the case in the quantum reduction performed in section 3 we assume for the rest of this section that we are working with such a pair. Let $\mathfrak{g} = \oplus_{j \in \frac{1}{2} \mathbb{Z}} \mathfrak{g}_j$, be the eigenspace decomposition with respect to $ad_x$. Denote

$$
\mathfrak{g}^+ = \bigoplus_{j > 0} \mathfrak{g}_j, \quad \mathfrak{g}^- = \bigoplus_{j < 0} \mathfrak{g}_j, \quad \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{g}^-.
$$

Let $V_k(\mathfrak{g})$ denote the affine vertex algebra of level $k$ associated to $\mathfrak{g}$. Denote by $F(A)$ the vertex algebra of free superfermions associated to a vector superspace $A$ with an even skew-supersymmetric non-degenerate bilinear form $\langle ., . \rangle$, i.e., the $\lambda$-bracket is given by $[\varphi|\psi] = \langle \varphi|\psi \rangle$, $\varphi, \psi \in A$.

On the vector superspace $\mathfrak{g}_{1/2}$ the element $f$ defines an even skew-supersymmetric non-degenerate bilinear form $\langle ., . \rangle_{ne}$ by the formula:

$$
\langle a|b \rangle = \langle f|[a, b] \rangle.
$$
The associated vertex algebra $F(\mathfrak{g}_{1/2})$ is called the vertex algebra of neutral free superfermions. Similarly on the vector superspace $\Pi\mathfrak{g}_+ \oplus \Pi\mathfrak{g}_+^*$ (where $\Pi$ denotes parity-reversing), define an even skew-supersymmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{ch}$ by:

$$
\langle \Pi \mathfrak{g}_+|\Pi \mathfrak{g}_+ \rangle_{ch} = 0 = \langle \Pi \mathfrak{g}_+^*|\Pi \mathfrak{g}_+^* \rangle_{ch},
$$

$$
\langle a|b^* \rangle_{ch} = -(-1)^{p(a)p(b^*)} \langle b^*|a \rangle_{ch} = b^*(a), \quad a \in \Pi \mathfrak{g}_+, b^* \in \Pi \mathfrak{g}_+^*,
$$

where $p(a)$ denotes the parity of the element $a$. The associated vertex algebra $F(\Pi \mathfrak{g}_+ \oplus \Pi \mathfrak{g}_+^*)$ is called the vertex algebra of charged free superfermions. This vertex algebra carries an extra $\mathbb{Z}$-grading by charge by assigning: charge $\varphi = 1$ and charge $\varphi^* = -1$, $\varphi \in \Pi \mathfrak{g}_+$, $\varphi^* \in \Pi \mathfrak{g}_+^*$. Consider the vertex algebra

$$
C(\mathfrak{g}, x, f, k) = V_k(\mathfrak{g}) \otimes F(\Pi \mathfrak{g}_+ \oplus \Pi \mathfrak{g}_+^*) \otimes F(\mathfrak{g}_{1/2}).
$$

The charge decomposition of $F(\Pi \mathfrak{g}_+ \oplus \Pi \mathfrak{g}_+^*)$ induces a charge decomposition on $C(\mathfrak{g}, x, f, k)$ by declaring charge $V_k(\mathfrak{g}) = 0$ and charge $F(\mathfrak{g}_{1/2}) = 0$. This makes $C(\mathfrak{g}, x, f, k)$ a $\mathbb{Z}$-graded vertex algebra. We introduce a differential $d_{(0)}$ that makes $(C(\mathfrak{g}, x, f, k), d_{(0)})$ a $\mathbb{Z}$-graded complex as follows. Let $\{u_\alpha\}_{\alpha \in S_j}$ be a basis of each $\mathfrak{g}_j$, an let $S := \bigsqcup_{j \in \frac{1}{2} \mathbb{Z}} S_j$, $S_+ := \bigsqcup_{j > 0} S_j$. Put $m_\alpha := j$ if $\alpha \in S_j$. The structure constants $c_{\alpha\beta}^\gamma$ are defined by $[u_\alpha, u_\beta] = \sum_{\gamma} c_{\alpha\beta}^\gamma u_\gamma$, for $(\alpha,\beta,\gamma \in S)$. Denote by $\{\varphi_\alpha\}_{\alpha \in S_+}$ the corresponding basis of $\Pi \mathfrak{g}_+$ and by $\{\varphi^\alpha\}_{\alpha \in S_+}$ the basis of $\Pi \mathfrak{g}_+^*$ such that $\langle \varphi_\alpha, \varphi^\beta \rangle_{ch} = \delta_{\alpha}^\beta$. Similarly denote by $\{\Phi_\alpha\}_{\alpha \in S_{1/2}}$ the corresponding basis of $\mathfrak{g}_{1/2}$, and by $\{\Phi^\alpha\}_{\alpha \in S_{1/2}}$ the dual basis with respect to $\langle \cdot, \cdot \rangle_{nc}$, i.e., $\langle \Phi_\alpha, \Phi^\beta \rangle_{nc} = \delta_{\alpha}^\beta$. It is useful to define $\Phi_u$ for any $u = \sum_{\alpha \in S} c_\alpha u_\alpha \in \mathfrak{g}$ by $\Phi_u := \sum_{\alpha \in S_{1/2}} c_\alpha \Phi_\alpha$. Define the odd field

$$
d = \sum_{\alpha \in S_+} (-1)^{p(u_\alpha)} : u_\alpha \varphi^\alpha : - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in S_+} (-1)^{p(u_\alpha)p(u_\gamma)} c_{\alpha\beta}^\gamma \varphi_\gamma \varphi^\alpha \varphi^\beta :
$$

$$
+ \sum_{\alpha \in S_+} (f|u_\alpha) \varphi^\alpha + \sum_{\alpha \in S_{1/2}} : \varphi^\alpha \Phi_\alpha :.
$$

Its Fourier mode $d_{(0)}$ is an odd derivation of all products of the vertex algebra $C(\mathfrak{g}, x, f, k)$, such that $d_{(0)}^2 = 0$ and that $d_{(0)}$ decreases the charge by 1. Thus $(C(\mathfrak{g}, x, f, k), d_{(0)})$ becomes a $\mathbb{Z}$-graded homology complex. Define the affine $W$-algebra $W_k(\mathfrak{g}, x, f)$ to be: as vector superspace the homology of this complex $W_k(\mathfrak{g}, x, f) := H(C(\mathfrak{g}, x, f, k), d_{(0)})$ together with the vertex algebra structure induced from $C(\mathfrak{g}, x, f, k)$. The vertex algebra $W_k(\mathfrak{g}, x, f)$ is also called the

quantum reduction

associated to the quadruple $(\mathfrak{g}, x, f, k)$. Define the Virasoro field of $C(\mathfrak{g}, x, f, k)$ by

$$
L = L^g + \partial x + L^{ch} + L^{nc},
$$

where
\[ L^{g} = \frac{1}{2(k + h^{\vee})} \sum_{\alpha \in S} (-1)^{p(u_{\alpha})} : u_{\alpha} u^{\alpha} : , \]

is given by the Sugawara construction, where \( \{ u^{\alpha} \}_{\alpha \in S} \) is the dual basis to \( \{ u_{\alpha} \}_{\alpha \in S} \), i.e., \( (u_{\alpha} | u^{\beta}) = \delta_{\alpha}^{\beta} \). Here we are assuming that \( k \neq -h^{\vee} \), where \( h^{\vee} \) denotes the dual Coxeter number of \( g \).

\[ L^{ch} = - \sum_{\alpha \in S_{+}} m_{\alpha} : \varphi^{\alpha} \partial \varphi_{\alpha} : + \sum_{\alpha \in S_{+}} (1 - m_{\alpha}) : (\partial \varphi_{\alpha}) \varphi_{\alpha} :, \]

\[ L^{nc} = \frac{1}{2} \sum_{\alpha \in S_{1/2}} : (\partial \varphi_{\alpha}) \varphi_{\alpha} :, \]

The central charge of \( L \) is given by

\[ c(g, x, f, k) = \frac{ksdim g}{k + h^{\vee}} - 12k(x|x) \]

\[ - \sum_{\alpha \in S_{+}} (-1)^{p(u_{\alpha})} (12m_{\alpha}^{2} - 12m_{\alpha} + 2) - \frac{1}{2} sdim g_{1/2}. \]

With respect to \( L \) the fields \( a (a \in g_{j}) \), \( \varphi_{\alpha}, \varphi^{\alpha} (\alpha \in S_{+}) \) and \( \Phi_{\alpha} (\alpha \in S_{1/2}) \) are primary vectors except for \( a (a \in g_{0}) \) such that \( (a | a) \neq 0 \), and the conformal weights are as follows: \( \Delta(a) = 1 - j (a \in g_{j}) \), \( \Delta(\varphi_{\alpha}) = 1 - m_{\alpha} \), \( \Delta(\varphi^{\alpha}) = m_{\alpha} \) and \( \Delta(\Phi_{\alpha}) = \frac{1}{2} \). In [7] is proved that \( d_{(0)} L = 0 \), then the homology class of \( L \) (which does not vanish) defines the Virasoro field of \( W_{k}(g, x, f) \), which is again denoted by \( L \).

To construct other fields of \( W_{k}(g, x, f) \) define for each \( v \in g_{j} \)

\[ J^{(v)} = v + \sum_{\alpha, \beta \in S_{+}} (-1)^{p(u_{\alpha})} c_{\alpha}^{\beta}(v) : \varphi_{\alpha} \varphi^{\beta} :, \]

where the numbers \( c_{\alpha}^{\beta}(v) \) are given by \( [v, u_{\alpha}] = \sum_{\alpha \in S} c_{\alpha}^{\beta}(v) u_{\alpha} \). The field \( J^{(v)} \in C(g, x, f, k) \) has the same charge, the same parity and the same conformal weight as the field \( v \). The \( \lambda \)-bracket between these fields is as follows:

\[ [J^{(v)} \lambda J^{(v')}'] = J^{([v | v'])} + \lambda \left( k(v | v') + \frac{1}{2} (\kappa_{g}(v, v') - \kappa_{g_{0}}(v, v')) \right) , \]

if \( v \in g_{i}, v' \in g_{j} \) and \( ij \geq 0 \) where \( \kappa_{g} (\text{resp.} \kappa_{g_{0}}) \) denotes the Killing form on \( g \) (resp. \( g_{0} \)).

Denote by \( C^{-} \) the vertex subalgebra of the vertex algebra \( C(g, x, f, k) \) generated by the fields \( J^{(u)} \) for all \( u \in g_{\leq} \), the fields \( \varphi^{\alpha} \) for all \( \alpha \in S_{+} \) and the fields \( \Phi_{\alpha} \) for all \( \alpha \in S_{1/2} \). One of the main theorems on the structure of the vertex algebra \( W_{k}(g, x, f) \) is the following:
**Theorem 2.1.** [8] Theorem 4.1] Let \( g \) be a simple finite-dimensional Lie superalgebra with an invariant bilinear form \( (\cdot,\cdot) \) and let \( x, f \) be a pair of even elements of \( g \) such that \( ad\ x \) is diagonalizable with eigenvalues in \( \frac{1}{2} \mathbb{Z} \) and \( [x,f] = -f \). Suppose that all eigenvalues of \( ad\ x \) on \( g^f \) (the centralizer of \( f \)) are non-positive: \( g^f = \oplus_{j\leq 0} g_j^f \). Then

a) For each \( a \in g_{-j}^f (j \geq 0) \) there exists a \( d_{(0)} \)-closed field \( J^{(a)} \) in \( C^- \) of conformal weight \( 1+j \) (with respect to \( L \)) such that \( J^{(a)} - J^{(a)} \) is a linear combination of normal ordered products of the fields \( J^{(b)} \), where \( b \in g_{-s} \), \( 0 \leq s < j \), the fields \( \Phi_\alpha \), where \( \alpha \in S_{1/2} \), and the derivatives of these fields.

b) The homology classes of the fields \( J^{(a)} \), where \( a_1, a_2, \ldots \) is a basis of \( g^f \) compatible with its \( \frac{1}{2} \mathbb{Z} \)-gradation, strongly generate the vertex algebra \( W_k(g,x,f) \).

c) \( H_0 \left( C(g,x,f,k),d_{(0)} \right) = W_k(g,x,f) \) and \( H_j \left( C(g,x,f,k),d_{(0)} \right) = 0 \) if \( j \neq 0 \).

**Remark 2.1.** The complex \( \left( C(g,x,f,k),d_{(0)} \right) \) is formal, that is, the vertex algebra \( W_k(g,x,f) \) is a subalgebra of \( C(g,x,f,k) \) consisting of \( d_{(0)} \)-closed charge 0 elements of \( C^- \), furthermore the \( J^{(a)} \) can be computed recursively, for example in the case \( a \in g_{-1/2}^f \) the solution is unique an is given by:

**Theorem 2.2.** [8] Theorem 2.1 (d)\]

For \( v \in g_{-1/2} \) let

\[
G^{(v)} = J^{(v)} + \sum_{\beta \in S_{1/2}} J^{(v,u_\beta)}\Phi_\beta : + \frac{(-1)^{p(v)+1}}{3} \sum_{\alpha,\beta \in S_{1/2}} \Phi^{\alpha}\Phi^{\beta}\Phi_{[u_\alpha,u_\beta,u]} : - \sum_{\beta \in S_{1/2}} (k(v|u_\beta) + \text{str}_{g^+} (ad\ v)(ad\ u_\beta)) \partial\Phi^{\beta},
\]

Then provided that \( v \in g_{-1/2}^f \), we have \( d_{(0)}(G^{(v)}) = 0 \), hence the homology class of \( G^{(v)} \) defines a field of the vertex algebra \( W_k(g,x,f) \) of conformal weight \( \frac{2}{2} \). This field is primary.

**Remark 2.2.** In the case \( g^f \subset g \leq \) Theorem 2.1 and the identity (2.2) provides a construction of the vertex algebra \( W_k(g,x,f) \) as a subalgebra of \( V_k(g,g) \otimes F(g_{1/2}) \) where \( \nu_k \) is the 2-cocycle on \( g \leq [t, t^{-1}] \) given by

\[
\nu_k(at^m, bt^n) = m\delta_{m,-n} \left( k(a|b) + \frac{1}{2} (\kappa_g(a,b) - \kappa_g(a,b)) \right) \quad (2.3)
\]

for \( a, b \in g \leq \) and \( m, n \in \mathbb{Z} \).

**Remark 2.3.** Furthermore if this 2-cocycle is trivial outside \( g_0[t, t^{-1}] \), the canonical homomorphism \( g \leq \to g_0 \) induces a homomorphism from \( V_k(g,g) \otimes F(g_{1/2}) \) to \( V_k(g_0) \otimes F(g_{1/2}) \), obtaining in this way a free field realization of \( W_k(g,x,f) \) inside \( V_k(g_0) \otimes F(g_{1/2}) \).
3 Quantum Hamiltonian Reduction of $D(2, 1; \alpha)$

In this section we prove that the family $SW(\frac{3}{2}, \frac{3}{2}, 2)$ of W-algebras which has generators \{G, H, L, M, W, U\} of conformal weights $(\frac{3}{2}, \frac{3}{2}, 2, 2, 2, \frac{3}{2})$ and relations as given in Appendix B can be obtained as the quantum Hamiltonian reduction of $D(2, 1; \alpha)$. As a corollary we obtain a free field realization of this family. As a particular case we obtain a free-field realization of the Shatashvili-Vafa $G_2$ algebra on a space of three free Bosons and three free Fermions.

The Lie superalgebra $D(2, 1; \alpha)$ where $\alpha \in \mathbb{C} \setminus \{-1, 0\}$ is a one-parameter family of exceptional Lie superalgebras of rank 3 and dimension 17, which contains $D(2, 1) = osp(4, 2)$ as special cases (when $\alpha \in \{1, -\frac{1}{2}, -2\}$), see [13].

We present $\mathfrak{g} = D(2, 1; \alpha)$ as the contragradient Lie superalgebra associated to the Cartan matrix $A = (a_{ij})_{i,j}$ and $\tau = \{1, 2, 3\}$

$$
(a_{ij})_{i,j=1}^3 = \begin{pmatrix}
0 & 1 & \alpha \\
1 & 0 & -1 - \alpha \\
\alpha & -1 - \alpha & 0
\end{pmatrix}.
$$

(3.1)

We have generators $\{h_1, h_2, h_3, e_1, e_2, e_3, f_1, f_2, f_3\}$, $h_i$ being even for all $i$ and $e_i, f_i$ being odd for all $i$ and relations

$$[e_i, f_j] = \delta_{ij}h_i, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j.$$  

Introduce the elements:

$$[e_1, e_2] = :e_{12}, \quad [e_1, e_3] = :e_{13}, \quad [e_2, e_3] = :e_{23}, \quad [e_1, e_{23}] = :e_{123},$$

$$[f_1, f_2] = :f_{12}, \quad [f_1, f_3] = :f_{13}, \quad [f_2, f_3] = :f_{23}, \quad [f_1, f_{23}] = :f_{123}.$$  

Recall that $\mathfrak{g}$ has vanishing Killing form and consequently the dual Coxeter number $h^\vee = 0$. Fix the following non-degenerate even supersymmetric invariant bilinear form $(,)$$

(h_i, h_j) = a_{ij}, \quad (e_i, f_j) = \delta_{ij}, \quad (e_{12}, f_{12}) = (f_{12}, e_{12}) = -1, \quad (e_{13}, f_{13}) = (f_{13}, e_{13}) = -\alpha, \quad (e_{23}, f_{23}) = (f_{23}, e_{23}) = 1 + \alpha, \quad (e_{123}, f_{123}) = -(f_{123}, e_{123}) = (1 + \alpha)^2.$$

To perform the quantum Hamiltonian reduction we take the pair $(x, f)$:

$$x := \frac{(\alpha + 1)}{2\alpha}h_1 + \frac{\alpha}{2(\alpha + 1)}h_2 + \frac{1}{2\alpha(\alpha + 1)}h_3, \quad f := f_{12} + f_{13} + f_{23}.$$  

This pair together with $e = -(\frac{1}{2})e_{12} + (\frac{1}{2\alpha})e_{13} + (-\frac{1}{2(\alpha + 1)})e_{23}$ forms an $sl_2$ triple. We have the following eigenspace decomposition of the algebra with
respect to $ad x$:

$$
\begin{array}{cccccccc}
g_{-3/2} & g_{-1/2} & g_0 & g_{1/2} & g_1 & g_{3/2} \\
f_{123} & f_{12} & f_1 & h_1 & e_1 & e_{12} & e_{123} \\
f_{13} & f_2 & h_2 & e_2 & e_{13} \\
f_{23} & f_3 & h_3 & e_3 & e_{23}
\end{array}
$$

Furthermore $g' = g_{-1/2}^{f_1} \oplus g_{-1/2}^{f_2} \oplus g_{-1/2}^{f_3}$ with $\dim g_{-1/2}^{f_1} = 2$, $\dim g_{-1/2}^{f_2} = 3$, and $\dim g_{-1/2}^{f_3} = 1$. This shows that the algebra $W_k(g, x, f)$ has six generators with the expected conformal weights.

The set of vectors $\{e_1, e_2, e_3\}$ is a basis of $g_{1/2}$, denote by $\Phi_1 := e_1$, $\Phi_2 := e_2$ and $\Phi_3 := e_3$ the corresponding free neutral fermions. The non-zero values of the (symmetric) bilinear form $\langle .| . \rangle_{ne}$ on $g_{1/2}$ are given by:

$$
\langle \Phi_1 | \Phi_2 \rangle_{ne} = -1, \quad \langle \Phi_1 | \Phi_3 \rangle_{ne} = -\alpha, \quad \langle \Phi_2 | \Phi_3 \rangle_{ne} = 1 + \alpha,
$$

(note that this is exactly minus the Cartan matrix of $D(2, 1; \alpha)$). Then the free neutral fermions satisfy the following non-zero $\lambda$-brackets:

$$
[\Phi_1, \Phi_2] = -1, \quad [\Phi_1, \Phi_3] = -\alpha, \quad [\Phi_2, \Phi_3] = 1 + \alpha,
$$

and the dual free neutral fermions with respect to $\langle .| . \rangle_{ne}$ are:

$$
\Phi^1 = (\frac{1+\alpha}{2\alpha})\Phi_1 + (-\frac{1}{2})\Phi_2 + (-\frac{1}{2\alpha})\Phi_3,
$$

$$
\Phi^2 = (-\frac{1}{2})\Phi_1 + (-\frac{\alpha}{2+2\alpha})\Phi_2 + (\frac{1}{2+2\alpha})\Phi_3,
$$

$$
\Phi^3 = (-\frac{1}{2\alpha})\Phi_1 + (\frac{1}{2+2\alpha})\Phi_2 + (-\frac{1}{2+2\alpha})\Phi_3.
$$

We fix the basis $\{h_1, h_2, h_3, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}, f_{123}\}$ of $g_{\leq}$ compatible with the $\frac{1}{2}\mathbb{Z}$ and $\mathbb{Z}$ gradation of $g$. We consider the building blocks $J^{(v)}$ for each $v$ that belongs to the above basis, (2.2) reduces to

$$
J^{(v)} \lambda J^{(v')} = J^{(|v, v'|)} + \lambda k(v|v'),
$$

because the Killing form $\kappa_0$ of $g$ is zero and $g_0$ equals the Cartan subalgebra $\mathfrak{h}$ of $g$, that is, the generators $J^{(v)}$ obey the same commutation relations as the generators of $V_k(D(2, 1; \alpha))$. Using Remark 2.2 we obtain that $W_k(g, x, f)$ is a subalgebra of $V_k(g_{\leq}) \otimes F(g_{1/2})$. For this reason and to simplify the notation we denote $J^{(v)}$ simply by $v$. Furthermore as the cocycle (2.3) is the original cocycle of $V_k(D(2, 1; \alpha))$ and this cocycle is trivial in $g_{<}$ outside $g_0 = \mathfrak{h}$, Remark 2.3 gives a free field realization of $V_k(g, x, f)$ inside $V_k(\mathfrak{h}) \otimes F(g_{1/2})$.

Let $J^{(f_i)}$ denote the $d_0$-closed fields associated to $\{f_1\}_{i=1}^3$ provided by Theorem 2.4. Using Theorem 2.2 we can compute $J^{(f_i)}$ explicitly:

$$
J^{(f_1)} = f_1 + \left(\frac{\alpha^2 - 1}{3}\right) : \Phi^1 \Phi^2 \Phi^3 : + : \Phi^1 h_1 : + k \partial \Phi^1,
$$

$$
J^{(f_2)} = f_2 + \left(\frac{\alpha(\alpha + 2)}{3}\right) : \Phi^1 \Phi^2 \Phi^3 : + : \Phi^2 h_2 : + k \partial \Phi^2,
$$

$$
J^{(f_3)} = f_3 + \left(\frac{2\alpha + 1}{3}\right) : \Phi^1 \Phi^2 \Phi^3 : + : \Phi^3 h_3 : + k \partial \Phi^3.
$$
We can compute the other fields \( J^{(f_1.2)} , J^{(f_1.3)} , J^{(f_2.3)} , J^{(f_1.2.3)} \) given by Theorem 2.1 that jointly with \( \{ J^{(f_i)} \}_{i=1}^3 \) strongly generate \( W_k(\mathfrak{g}, x, f) \), but in the \( SW(\frac{3}{2}, \frac{3}{2}, 2) \) superconformal algebra we can recover (using \( \lambda \)-brackets) all the fields from the generators in conformal weight \( \frac{3}{2} \), i.e., \( G \) and \( H \) (see Appendix B). Thus we only need to construct \( G \) and \( H \) from \( \{ J^{(f_i)} \}_{i=1}^3 \).

In order to do that observe that:

\[
a_1 f_1 + a_2 f_2 + a_3 f_3 \in \mathfrak{g}_{-1/2}^f \iff a_1 + a_2 (-\frac{a}{a+1}) + a_3 (-\frac{1}{a+1}) = 0, \tag{3.2}
\]

and that the central charge of the Virasoro field of \( W_k(\mathfrak{g}, x, f) \) given by formula (2.11) is \( c(\alpha, k) = \frac{9}{2} - 12 k(x|x) = \frac{9}{2} - \frac{6k(1+\alpha^2)}{\alpha(1+\alpha)} \).

We want to define a field \( G \) such that \( \{ G, L := \frac{1}{2} G_0 \} \) generate an \( N = 1 \) superconformal algebra with the above central charge, this is accomplished taking \( a_1 = a_2 = a_4 = \frac{1}{\sqrt{k}} \), i.e.,

\[
G := \frac{1}{\sqrt{k}} \left( J^{(f_1)} + J^{(f_2)} + J^{(f_3)} \right).
\]

We are looking for a vector \( H \) of conformal weight \( \frac{3}{2} \), such that:

\[
G_{(j)} H = 0, \quad j > 0, \tag{3.3}
\]

The most general vector of conformal weight \( \frac{3}{2} \) given by (3.2) is

\[
\left( \frac{\alpha}{\alpha+1} a_2 + \frac{1}{\alpha+1} a_3 \right) J^{(f_1)} + a_2 J^{(f_2)} + a_3 J^{(f_3)},
\]

which imposes the condition \( a_2 \alpha (-1 + 2\alpha) (1 + 2\alpha) + a_3 (2\alpha - \alpha) (2 + \alpha) = 0 \), which has as solution

\[
a_1' := \alpha (-1 + \alpha) (1 + 2\alpha),
\]

\[
a_2' := (-1) (2\alpha - \alpha) (2 + \alpha) (1 + \alpha),
\]

\[
a_3' := \alpha (-1 + 2\alpha) (1 + 2\alpha) (1 + \alpha).
\]

It follows from \( H(2) H = \frac{2\alpha^3}{3} \) (cf. (15.1)) that we need to rescale this solution to define \( H = \sum_{i=1}^3 a_i J^{(f_i)} \) with

\[
a_i := \left( -\frac{3}{2} (-1 + 2\alpha) \alpha^2 (1 + \alpha)^2 (2 + 4\alpha - \alpha (1 + \alpha)) \right)^{1/2} a_i'.
\]

We can obtain all other generators from \( G \) and \( H \), to perform this computations we use Thielemans's software [18]. Listed below are the explicit expressions of all the generators of \( W_k(\mathfrak{g}, x, f) \) as a subalgebra of \( V_k(\mathfrak{g}_- \otimes F(\mathfrak{g}_1/2)) \):

\[
G = \frac{i}{\sqrt{k}} f_1 + \frac{i}{\sqrt{k}} f_2 + \frac{i}{\sqrt{k}} f_3 + \frac{i}{\sqrt{k}} \Phi^1 h_1 : + \frac{i}{\sqrt{k}} : \Phi^2 h_2 : + \frac{i}{\sqrt{k}} : \Phi^3 h_3 :
\]

\[
+ i \sqrt{k} \partial \Phi^1 + i \sqrt{k} \partial \Phi^2 + i \sqrt{k} \partial \Phi^3,
\]

9
\[
\begin{align*}
L &= -\frac{1}{\kappa} f_{12} - \frac{1}{\kappa} f_{13} - \frac{1}{\kappa} f_{23} + \frac{(1 + \alpha)}{2\kappa} h_1 h_1 : + \frac{1}{2\kappa} : h_1 h_2 : + \frac{1}{2\kappa} : h_1 h_3 : \\
&\quad + \frac{\alpha}{4k + 4\kappa} : h_2 h_2 : - \frac{1}{2k + 2\kappa} : h_2 h_3 : + \frac{1}{4k + 4\kappa} : h_3 h_3 : + \frac{1}{2\kappa} : \Phi^1 f_2 : \\
&\quad + \frac{1}{2} : \Phi^1 f_3 : + \frac{1}{2} : \Phi^1 \partial \Phi^2 : + \sqrt{\alpha} : \Phi^1 h_2 : + \frac{1}{2} : \Phi^2 f_1 : - \frac{(1 + \alpha)}{2\kappa} h_1 : \Phi^2 f_2 : \\
&\quad + \sqrt{\alpha} : \Phi^1 h_3 : + \frac{(1 + \alpha)}{2\kappa} : \partial \Phi^1 : \\
&\quad + \frac{1}{2} \alpha : \partial \Phi^1 h_3 : + \frac{1}{2} (1 + \alpha) : \partial \Phi^2 : + (1 + \alpha) : \partial \Phi^3 : + (1 + \alpha) : \partial h_1 : + \frac{\alpha}{2 + 2\alpha} \partial h_2 \\
&\quad + \frac{1}{2 + 2\alpha} \partial h_3 :
\end{align*}
\]

\[
\begin{align*}
H &= \sqrt{-\frac{1}{2}(2 + 2\kappa)(1 + \alpha)} \left( -1 + \alpha \right) \left( 1 + 2k + \alpha \right) (f_1 + : \Phi^1 h_1 : \\
&\quad + k \partial \Phi^1) - (2k - \alpha) \left( 2 + 3\alpha + \alpha^2 \right) (f_2 + : \Phi^2 h_2 : + k \partial \Phi^2) \\
&\quad + (-1 + 2k) \alpha \left( 1 + 3\alpha + 2\alpha^2 \right) (f_3 + : \Phi^3 h_3 : + k \partial \Phi^3) \\
&\quad + \alpha (1 + \alpha) (-3 \alpha (1 + \alpha) + 4k (1 + \alpha + \alpha^2)) : \Phi^1 \partial \Phi^3 : ) ,
\end{align*}
\]

\[
\begin{align*}
\tilde{M} &= \sqrt{-\frac{1}{2}(2 + 2\kappa)(1 + \alpha)} \left( -1 + \alpha \right) \left( 1 + 2k + \alpha \right) (f_{12} - \frac{1}{2} : h_1 h_2 : \\
&\quad : \Phi^1 f_2 : - : \Phi^2 f_1 :) + \frac{(2 + \alpha)(1 + 1 + 4k(1 + \alpha))}{\sqrt{k}} \left( f_{13} - \frac{1}{2} h_3 h_3 : + \alpha^2 : \Phi^1 f_3 : \\
&\quad + (1 + \alpha) : \Phi^2 f_2 : + (1 + \alpha) : \Phi^2 f_1 :) \\
&\quad - i\sqrt{k} (-1 + \alpha)(1 + \alpha)(1 + 2k + \alpha) \left( \partial h_1 : + \frac{1}{2\kappa} : h_3 h_1 : \\
&\quad + i\sqrt{k} (2k - \alpha) \alpha (2 + \alpha) \left( \partial h_2 : + \frac{1}{2\kappa} : h_3 h_2 : \\
&\quad - i\sqrt{k} (-1 + 2k) (1 + 2\alpha) \left( \partial h_3 : + \frac{1}{2\kappa} : h_3 h_3 : \\
&\quad + \frac{(3 \alpha (1 + \alpha) + 4k (1 + \alpha + \alpha^2))}{2\sqrt{k}} (-1 + \alpha) : \Phi^1 \Phi^2 h_2 : + \alpha : \Phi^1 \Phi^3 h_3 : + \alpha : \Phi^1 \Phi^2 h_3 : \\
&\quad - \alpha : \Phi^1 \Phi^3 h_3 : - (1 + \alpha) : \Phi^2 \Phi^3 h_3 : \\
&\quad - i\sqrt{k} (-1 + \alpha) \alpha (1 + 2k + \alpha) : \Phi^1 \partial \Phi^2 : \\
&\quad - i\sqrt{k}(-1 + \alpha) \alpha^2 (1 + 2k + \alpha) : \Phi^1 \partial \Phi^3 : \\
&\quad - i\sqrt{k} (2k - \alpha) (1 + \alpha) (2 + \alpha) : \Phi^2 \partial \Phi^3 : \\
&\quad - i\sqrt{k} (2k - \alpha) (2 + 3\alpha + \alpha^2) : \partial \Phi^1 \Phi^2 : \\
&\quad + i\sqrt{k}(-1 + 2k) (1 + 33 + 2\alpha^2) : \partial \Phi^1 \Phi^3 : \\
&\quad - i\sqrt{k} (-1 + 2k) \alpha (1 + \alpha) (1 + 2\alpha) : \partial \Phi^2 \Phi^3 : ) ,
\end{align*}
\]
\[ W = \frac{-\alpha}{\mu^2 + 4k(1 + \alpha + \alpha^2)} \left( \frac{(2\mu + \alpha^2)}{k} \right) (-f_{12} + \frac{\mu}{2} : h_1 h_2 : + : \Phi^1 f_2 : \\
+ : \Phi^2 f_1 : + \frac{4\alpha + 2k}{k} \mu : \left( -\alpha f_{13} + \frac{\mu}{2} : h_1 h_3 : + \alpha^2 : \Phi^1 f_3 : + \alpha^2 : \Phi^3 f_1 : \\
+ \frac{1 + \mu}{k} (\alpha + 2k(1 + \alpha)) \right) \right) (-f_{23} + \mu : h_2 h_3 : = -\left( 1 + \alpha : \Phi^2 f_3 : - (1 + \alpha : \Phi^3 f_2) \right) \\
+ \frac{1 + \mu}{k} (\alpha + 2k + \alpha) : h_1 h_1 : + \frac{2k + 2\alpha}{k : 2k - \mu} : h_2 h_3 : \\
+ \frac{-2k + 1}{k : \alpha + \mu} : h_3 h_3 : = \left( -1 + \alpha^2 : \Phi^1 \Phi^2 h_1 : - \alpha (2 + \alpha : \Phi^3 \Phi^2 h_2 : \\
+ (1 - 2\alpha) : \Phi^1 \Phi^2 h_3 : + \left( \alpha - \alpha^3 : \Phi^1 \Phi^3 h_1 : - \alpha^2 (2 + \alpha : \Phi^3 \Phi^2 h_2 : \\
- \alpha (1 + 2\alpha) : \Phi^1 \Phi^3 h_3 : - \alpha (1 + 2k + \alpha) : \Phi^1 \Phi^2 : \\
- \alpha^2 (1 + 2k + \alpha) : \Phi^1 \Phi^2 : + (1 + \alpha^2) : \Phi^2 h_1 : \\
+ \alpha (2 + 3\alpha + \alpha^2) : \Phi^2 \Phi^1 h_2 : + (1 - 3\alpha - 2\alpha^2) : \Phi^3 \Phi^2 h_3 : \\
- (2k - \alpha) : \Phi^2 \Phi^3 : - (2k - \alpha) : \Phi^3 \Phi^2 : \\
+ \frac{4}{3} (1 + \alpha) (1 + 2k + \alpha) : h_1 h_1 : + \frac{4}{3} \alpha (2 + \alpha) : h_2 h_3 : + \frac{2k + 1}{k : 2k - \mu} : h_3 h_3 : , \right) \\
\]

\[ U = \frac{-\alpha}{\mu^2 + 4k(1 + \alpha + \alpha^2)} \left( \frac{(2\mu + \alpha^2)}{k} \right) (-f_{123} + \frac{3(1 + \alpha)}{\mu^2} : h_1 f_2 : + \frac{3(1 + \alpha)}{\mu^2} : h_3 f_2 : \\
- \frac{3\alpha}{\mu^2} : h_2 f_3 : + \frac{3(1 + \alpha)}{\mu^2} : h_2 f_3 : - \frac{3\alpha}{\mu^2} : h_3 f_1 : \\
+ \frac{3(1 + \alpha)}{\mu^2} : h_3 f_2 : + \frac{6\alpha + 1 + \alpha^2}{\mu^2} : \Phi^1 f_3 : - \frac{3\alpha}{\mu^2} : \Phi^1 h_1 h_2 : \\
- \frac{-3\alpha}{\mu^2} : \Phi^1 h_1 h_3 : + \frac{3(1 + \alpha)}{\mu^2} : \Phi^1 h_2 h_2 : + \frac{3(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^1 \Phi^2 f_3 : + i\sqrt{\alpha} (1 + 3\alpha + 2\alpha^2) : \Phi^1 \Phi^2 : \\
- \frac{-3\alpha^2 (1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^1 \Phi^2 f_1 : + \frac{3(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^1 \Phi^1 f_3 : \\
- \frac{-3\alpha^2 (1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^1 \Phi^1 f_1 : + \frac{3(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^1 \Phi^1 f_3 : \\
+ 3i\sqrt{\alpha} (1 + \alpha) : \Phi^1 \Phi^2 : \Phi^2 : - i\sqrt{\alpha} (2 + 3\alpha + \alpha^2) : \Phi^1 \Phi^2 : \Phi^1 \Phi^3 : \\
- 3i\sqrt{\alpha} (1 + \alpha) : \Phi^1 \Phi^3 : - \frac{1 + 3\alpha + 2\alpha^2}{\mu^2} : \Phi^1 \Phi^3 : \\
+ \frac{6(1 + \alpha)}{\mu^2} : \Phi^2 f_{13} : + \frac{3(1 + \alpha)}{\mu^2} : \Phi^2 f_{13} : + \frac{3(1 + \alpha)}{\mu^2} : \Phi^2 f_{13} : \\
- \frac{3\alpha (1 + \alpha)^2}{\mu^2} : \Phi^2 f_{13} : + \frac{3\alpha (1 + \alpha)^2}{\mu^2} : \Phi^2 f_{13} : - \frac{3\alpha (1 + \alpha)^2}{\mu^2} : \Phi^2 f_{13} : \\
+ 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : \Phi^3 : + \frac{2k - \alpha (1 + \alpha)}{\mu^2} : \Phi^2 f_{12} : + \frac{6(1 + \alpha)}{\mu^2} : \Phi^3 f_{12} : \\
- \frac{3\alpha (1 + \alpha)}{\mu^2} : \Phi^3 f_{12} : + \frac{3\alpha (1 + \alpha)}{\mu^2} : \Phi^3 f_{12} : + \frac{3\alpha (1 + \alpha)}{\mu^2} : \Phi^3 f_{12} : \\
- \frac{2(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^2 f_{13} : - \frac{2(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^2 f_{13} : - \frac{2(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^2 f_{13} : \\
- \frac{2(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^2 f_{13} : - \frac{2(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^2 f_{13} : - \frac{2(1 + 3\alpha + 2\alpha^2)}{\mu^2} : \Phi^2 f_{13} : \\
- 3i\sqrt{\alpha} (1 + \alpha) : \Phi^1 \Phi^1 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^1 \Phi^1 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^1 \Phi^1 : \\
- i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : \\
+ 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : \\
+ 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : \\
+ 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : - 3i\sqrt{\alpha} (1 + \alpha) : \Phi^2 : \Phi^2 : \\]
\[ +3\sqrt{k}\alpha(1 + \alpha) : \partial \Phi^3 h_2 : + \frac{2(1+k)\alpha(1+\alpha)}{\sqrt{k}} : \partial \Phi^3 h_3 : \]
\[ - \frac{\alpha(1+2k+\alpha)}{\sqrt{k}} \partial f_1 + \frac{(2k-\alpha)(1+\alpha)}{\sqrt{k}} \partial f_2 + \frac{(1+2k)\alpha(1+\alpha)}{\sqrt{k}} \partial f_3 \]
\[ - \frac{i}{2}\sqrt{k}(-1 + 4k - \alpha)\alpha \partial^2 \Phi^1 + \frac{i}{2}\sqrt{k}(1 + \alpha)(4k + \alpha)\partial^2 \Phi^2 \]
\[ + \frac{1}{2}\sqrt{k}(1 + 4k)\alpha(1 + \alpha)\partial^2 \Phi^3 \],

where \( \mu = \sqrt{\frac{9c(1+\varepsilon)}{11(1-\varepsilon)}} \) and \( \varepsilon(\alpha, k) = -\frac{4\sqrt{2}k^{3/2}(1+2\varepsilon)(-2+\alpha+\alpha^2)}{3\sqrt{(-1+2k)\alpha^2(1+\alpha)^2}(2k+4k^2-\alpha(1+\alpha))}. \)

One can check straightforwardly with the aid of [18] that the \( \lambda \)-brackets of the algebra \( W_k(g, x, f) \) coincides with the \( \lambda \)-brackets of the family of superconformal algebras \( SW(\frac{3}{2}, \frac{3}{2}, 2) \) with parameters \( (c, k, \varepsilon) \). Shatashvili-Vafa’s \( G_2 \) superconformal algebra is a quotient of this algebra for \( (c, \varepsilon) = (21/2, 0) \) modulo an ideal generated in conformal weight \( \frac{7}{2} \) (cf. Remark [11], in particular, the explicit commutation relations obtained in [14] are an artifact of the free field realization the authors used [3]. Solving \( (c, \varepsilon) = (21/2, 0) \) in terms of \( \alpha \) and \( k \) there are three solutions: \( \{ \alpha = 1, k = -2/3 \}, \{ \alpha = -2, k = -2/3 \} \) and \( \{ \alpha = -1/2, k = 1/3 \} \). Precisely for this values of \( \alpha \) the superalgebra \( D(2, 1; \alpha) \) is nothing but the superalgebra \( osp(4|2) \), then the Shatashvili-Vafa \( G_2 \) superconformal algebra is a quotient of the quantum Hamiltonian reduction of \( osp(4|2) \).

**Remark 3.1.** The existence of the ideal [15,2] can be guessed from the fact that the affine vertex algebra \( V_k(osp(4|2)) \) at level \( k \in \{ -\frac{2}{3}, \frac{1}{3} \} \) is not simple, i.e., contains a non-trivial ideal [17].

Listed below are the explicit expressions of all the generators of the Shatashvili-Vafa \( G_2 \) superconformal algebra in the case \( \{ \alpha = 1, k = -2/3 \} \). Note that we are using the change of basis [13,3].

\[ G = \sqrt{\frac{2}{3}} f_1 + \sqrt{\frac{2}{3}} f_2 + \sqrt{\frac{2}{3}} f_3 + \sqrt{\frac{2}{3}} : \Phi^1 h_1 : + \sqrt{\frac{2}{3}} : \Phi^2 h_2 : \]
\[ + \sqrt{\frac{2}{3}} : \Phi^3 h_3 : - \sqrt{\frac{2}{3}} \partial \Phi^1 - \sqrt{\frac{2}{3}} \partial \Phi^2 - \sqrt{\frac{2}{3}} \partial \Phi^3, \]

\[ L = \frac{\sqrt{2}}{2} f_{12} + \frac{\sqrt{2}}{2} f_{13} + \frac{\sqrt{2}}{2} f_{23} - \frac{\sqrt{2}}{2} h_1 h_1 : - \frac{\sqrt{2}}{2} : h_1 h_2 : - \frac{\sqrt{2}}{2} : h_1 h_3 : - \frac{1}{\sqrt{16}} : h_2 h_2 : \]
\[ + \frac{\sqrt{2}}{2} : h_2 h_3 : - \frac{\sqrt{2}}{2} : h_3 h_3 : - \frac{\sqrt{2}}{2} : \Phi^1 f_2 : - \frac{\sqrt{2}}{2} : \Phi^1 f_3 : + \frac{\sqrt{2}}{2} \Phi^1 \partial \Phi^2 : \]
\[ + \frac{\sqrt{2}}{2} : \Phi^1 \partial \Phi^3 : - \frac{\sqrt{2}}{2} : \Phi^2 f_1 : + 3 : \Phi^2 f_3 : - : \Phi^2 \partial \Phi^3 : - \frac{\sqrt{2}}{2} : \Phi^3 f_1 : \]
\[ + 3 : \Phi^3 f_2 : - \frac{1}{2} : \partial \Phi^1 \Phi^2 : - \frac{1}{2} : \partial \Phi^3 \Phi^3 : + : \partial \Phi^2 \Phi^3 : + \partial h_1 \]
\[ + \frac{1}{4} \partial h_2 + \frac{1}{4} \partial h_3, \]

\[ \Phi = 3 f_2 - 3 f_3 - 6 : \Phi^1 \Phi^2 \Phi^3 : + 3 : \Phi^2 h_2 : - 3 : \Phi^3 h_3 : - 2 \partial \Phi^2 + 2 \partial \Phi^3, \]
\[ K = 3 \sqrt{\frac{2}{3}f_{12} - 3 \sqrt{\frac{2}{3}} f_{13} - \frac{3}{2} \sqrt{\frac{1}{2}} : h_{1} h_{2} + \frac{3}{2} \sqrt{\frac{1}{2}} : h_{1} h_{3} - \frac{3}{2} \sqrt{\frac{1}{2}} : h_{2} h_{3} : \]
\[ + \frac{3}{2} \sqrt{\frac{1}{2}} : h_{3} h_{3} - \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{1} f_{2} : + 3 \sqrt{\frac{1}{2}} : \Phi^{1} f_{3} : + 3 \sqrt{\frac{1}{2}} : \Phi^{1} \Phi^{2} h_{1} : \]
\[ - \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{1} \Phi^{2} h_{2} : + \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{1} \Phi^{2} h_{3} : - \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{1} \Phi^{3} h_{1} : \]
\[ - \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{1} \Phi^{3} h_{2} : + \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{1} \Phi^{3} h_{3} : - \frac{3}{2} \sqrt{\frac{1}{2}} : \Phi^{2} \Phi^{3} h_{1} : \]
\[ + 3 \sqrt{\frac{1}{2}} : \Phi^{2} \Phi^{3} h_{2} : + 3 \sqrt{\frac{1}{2}} : \Phi^{2} \Phi^{3} h_{3} : - 2 \sqrt{\frac{1}{2}} : \Phi^{2} \partial \Phi^{3} : + 3 \sqrt{\frac{1}{2}} : \Phi^{3} f_{1} : \]
\[ - \sqrt{\frac{1}{2}} : \partial \Phi^{4} : + \sqrt{\frac{1}{2}} : \partial \Phi^{1} \Phi^{3} : - 2 \sqrt{\frac{1}{2}} : \partial \Phi^{2} \Phi^{3} : + \sqrt{\frac{3}{2}} \partial h_{2} - \sqrt{\frac{3}{2}} \partial h_{3}, \]

\[ X = - 3 f_{23} - \frac{3}{2} : h_{2} h_{2} : - \frac{3}{2} : h_{2} h_{3} - \frac{3}{2} : h_{3} h_{3} : - \frac{3}{2} : \Phi^{1} \Phi^{2} h_{2} : \]
\[ - \frac{3}{2} : \Phi^{1} \Phi^{2} h_{3} : - \frac{3}{2} : \Phi^{1} \Phi^{3} h_{2} : - \frac{3}{2} : \Phi^{1} \Phi^{3} h_{3} : - \frac{1}{2} : \Phi^{1} \partial \Phi^{2} : \]
\[ - \frac{3}{2} : \Phi^{1} \partial \Phi^{3} : - 6 : \Phi^{2} f_{3} : + 3 : \Phi^{2} \Phi^{3} h_{2} : - 3 : \Phi^{2} \Phi^{3} h_{3} : + 5 : \Phi^{2} \partial \Phi^{3} : \]
\[ - 6 : \Phi^{3} f_{2} : + \frac{3}{2} : \Phi^{1} \Phi^{2} : + \frac{3}{2} : \partial \Phi^{1} \Phi^{3} : - 5 : \partial \Phi^{2} \Phi^{3} : + \frac{3}{2} \partial h_{2} + \frac{3}{2} \partial h_{3}, \]

\[ M = - 3 \sqrt{\frac{2}{3}} f_{13} + 3 \sqrt{\frac{2}{3}} : h_{1} f_{2} : + 3 \sqrt{\frac{2}{3}} : h_{1} f_{3} : - \frac{3}{2} \sqrt{\frac{2}{3}} : h_{2} f_{1} : - 3 \sqrt{\frac{2}{3}} : h_{2} f_{3} : \]
\[ - \frac{3}{2} \sqrt{2} : h_{3} f_{1} : + 3 \sqrt{\frac{2}{3}} : h_{3} f_{2} : + 3 \sqrt{\frac{2}{3}} : h_{3} f_{3} : - \frac{3}{2} \sqrt{2} : \Phi^{1} h_{1} h_{2} : \]
\[ - \frac{3}{2} \sqrt{2} : \Phi^{1} h_{1} h_{3} : - 3 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{1} : + 3 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{3} : + 9 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{3} : \]
\[ - \sqrt{\frac{2}{3}} : \Phi^{2} \Phi^{3} \Phi^{4} : - 3 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{1} : + 9 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{1} : + 9 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{1} : + 3 \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{3} f_{1} : \]
\[ + \sqrt{\frac{2}{3}} : \Phi^{2} \Phi^{3} \Phi^{4} : + \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{4} \Phi^{3} : + \sqrt{\frac{2}{3}} : \Phi^{1} \Phi^{4} \Phi^{3} : - \frac{3}{2} \sqrt{\frac{2}{3}} : \Phi^{1} \partial h_{1} : \]
\[ + 3 \sqrt{\frac{2}{3}} : \Phi^{2} f_{13} : + 3 \sqrt{\frac{2}{3}} : \Phi^{2} h_{1} h_{2} : + 3 \sqrt{\frac{2}{3}} : \Phi^{2} h_{1} h_{3} : + 3 \sqrt{\frac{2}{3}} : \Phi^{2} \Phi^{3} f_{2} : \]
\[ - 3 \sqrt{\frac{2}{3}} : \Phi^{2} \Phi^{3} f_{3} : - 2 \sqrt{\frac{2}{3}} : \Phi^{2} \Phi^{3} f_{3} : - \frac{3}{2} \sqrt{\frac{2}{3}} : \Phi^{2} \partial h_{2} : + 3 \sqrt{\frac{2}{3}} : \Phi^{3} f_{13} : \]
\[ + 3 \sqrt{\frac{2}{3}} : \Phi^{3} h_{1} h_{3} : + 3 \sqrt{\frac{2}{3}} : \Phi^{3} h_{2} h_{3} : - \frac{3}{2} \sqrt{\frac{2}{3}} : \Phi^{3} h_{2} h_{3} : + \frac{3}{2} \sqrt{\frac{2}{3}} : \partial \Phi^{1} h_{1} : \]
\[ + \sqrt{\frac{2}{3}} : \partial \Phi^{1} h_{2} : + \sqrt{\frac{2}{3}} : \partial \Phi^{3} h_{1} : + \sqrt{\frac{2}{3}} : \partial \Phi^{4} \Phi^{1} : + \sqrt{\frac{2}{3}} : \partial \Phi^{4} \Phi^{1} : - \sqrt{\frac{2}{3}} : \partial \Phi^{2} \phi^{3} : + \sqrt{\frac{2}{3}} : \partial \Phi^{2} \phi^{3} : \]
\[ - \sqrt{\frac{2}{3}} : \partial \Phi^{3} h_{1} : - \sqrt{\frac{2}{3}} : \partial \Phi^{3} h_{2} : + \frac{3}{2} \sqrt{\frac{2}{3}} : \partial \Phi^{3} h_{3} : - \frac{3}{2} \sqrt{\frac{2}{3}} \partial f_{1} : \]
\[ - \frac{3}{2} \sqrt{\frac{2}{3}} \partial f_{2} - \frac{3}{2} \sqrt{\frac{2}{3}} \partial f_{3} - \sqrt{\frac{2}{3}} \partial \phi^{1} : + \sqrt{\frac{2}{3}} \partial \phi^{2} : + \sqrt{\frac{2}{3}} \partial \phi^{2} : \]

The free field realization of \( W_{k}(\mathfrak{g}, f) \) inside \( V_{k}(\mathfrak{b}) \otimes F(\mathfrak{g}_{1/2}) \) is induced by the canonical homomorphism \( \mathfrak{g}_{<} \rightarrow \mathfrak{g}_{0} \), then we simply obtain the free field realization by removing the terms that contain a current \( v \in \mathfrak{g}_{<} \setminus \mathfrak{g}_{0} \), i.e., the terms containing \( f \)’s.
For example in the case \( \{ \alpha = 1, k = -2/3 \} \) the generators \( G \) and \( \Phi \) look as:

\[
G = \sqrt{\frac{2}{3}} : \Phi^1 h_1 : + \sqrt{\frac{2}{3}} : \Phi^2 h_2 : + \sqrt{\frac{2}{3}} : \Phi^3 h_3 : - \sqrt{\frac{2}{3}} \partial \Phi^1 - \sqrt{\frac{2}{3}} \partial \Phi^2 - \sqrt{\frac{2}{3}} \partial \Phi^3,
\]

\[
\Phi = -6 : \Phi^1 \Phi^2 \Phi^3 : + 3 : \Phi^2 h_2 : - 3 : \Phi^3 h_3 : - 2 \partial \Phi^2 + 2 \partial \Phi^3. \quad (3.4)
\]

Therefore we have proved:

**Theorem 3.1.** Let \( V_{-2/3}(\mathfrak{h}) \) be the affine vertex algebra of level \(-2/3\) associated to \( \mathfrak{h} \) with bilinear form \( A \), and \( F(\mathfrak{g}_{1/2}) \) the vertex algebra of neutral free fermions as defined above. The vectors \( G \) and \( \Phi \) given by the expressions above generate the \( SW(\frac{2}{3}, \frac{2}{3}, 2) \) vertex algebra with \( c = 21/2 \) and \( \varepsilon = 0 \) inside \( V_{-2/3}(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2}) \). This vertex algebra is not simple and dividing by the ideal \((\mathfrak{g}_{1/2})\) we obtain the Shatashvili-Vafa \( G_2 \) superconformal algebra.

**Remark 3.2.** Note that \( V_{-2/3}(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2}) \) is isomorphic (by a linear transformation on the generators) to the vertex algebra of three free Bosons and three free Fermions with inner product minus the inverse of Cartan matrix \((3.1)\) of \( D(2, 1; 1) \simeq \mathfrak{osp}(4|2) \).

**Remark 3.3.** This free field realization was found by Mallwitz \([10]\) using the most general ansatz on three free superfields of conformal weights \( \frac{1}{2} \). By obtaining this realization from the quantum Hamiltonian formalism we can find explicitly the screening operators associated with the reduction as follows.

First we rescale the currents \( h \in V_k(\mathfrak{h}) \) and consider instead \( \tilde{h} := \frac{h}{\sqrt{k}} \), therefore \( V_k(\mathfrak{h}) \) is identified as a vertex algebra with the Heisenberg algebra \( V_1(\mathfrak{h}) \) associated to \( \mathfrak{h} \).

Let \( V_Q \) denote the lattice vertex algebra \([6]\) associated to the root lattice \( Q \) (that correspond to the Cartan matrix that we have fixed at the beginning of the section) of \( D(2, 1; 2) \), i.e., we have three odd simple roots \( \{ \alpha_1, \alpha_2, \alpha_3 \} \). Then for every lattice element \( \alpha \) we have a \( V_1(\mathfrak{h}) \)-module \( M_\alpha \) and a vertex operator \( \Gamma_\alpha \) which is an intertwiner of type \((M_0, M_\alpha)\), hence its zero mode maps \( V_1(\mathfrak{h}) = M_0 \to M_\alpha \).

Let \( M_{-\alpha_i/\sqrt{k}} \) be the \( V_1(\mathfrak{h}) \)-module with highest weight \(-\alpha_i/\sqrt{k}\) and \( \Gamma_{-\alpha_i/\sqrt{k}} \) the intertwiner constructed just as in the lattice case, so that

\[
[\tilde{h}_i \lambda \Gamma_{-\alpha_i/\sqrt{k}}] = -\frac{(\alpha_i, \alpha_j)}{\sqrt{k}} \Gamma_{-\alpha_j/\sqrt{k}}, \quad \partial \left( \Gamma_{-\alpha_i/\sqrt{k}} \right) = -\tilde{h}_j \Gamma_{-\alpha_j/\sqrt{k}}.
\]

Define the operators

\[
Q_i := \Phi_i \Gamma_{-\alpha_i/\sqrt{k}} : V_1(\mathfrak{h}) \otimes F(\mathfrak{g}_{1/2}) \to M_{-\alpha_i/\sqrt{k}} \otimes F(\mathfrak{g}_{1/2}), \quad i = 1, 2, 3.
\]

A straightforward computation using \([13]\) shows that
equals the free field realization of \( W_k(g, x, f) \) inside \( V_1(h) \otimes F(g_{1/2}) \) that we have produced above.

Remark 3.4. In fact a similar result can be obtained for the quantum Hamiltonian reduction of any simple Lie superalgebra when the nilpotent \( f \) is super-principal, that is, there exists an odd nilpotent \( F \in g_{-1/2} \) with \([F,F] = f\) (\( f \in g_{-1} \) being a principal nilpotent) and these two vectors together with \( x \) form part of a copy of \( osp(1|2) \subset g \). Not all Lie superalgebras admit a superprincipal embedding, in particular, it is necessary to admit a root system with all odd simple roots. In this case, one takes \( F = \sum_i e_{-\alpha_i} \), the sum of all simple root vectors. The list of simple Lie superalgebras admitting an \( osp(1|2) \) superprincipal embedding consists of

\[
\text{sl}(n \pm 1|n), \quad osp(2n \pm 1|2n), \quad osp(2n|2n), \quad osp(2n + 2|2n), \quad D(2, 1; \alpha)
\]

In these case we see that \( g_{1/2} \) is naturally isomorphic to \( \Pi h^* \) and we can form the Boson-Fermion system and the screening charges as above. The intersection of their kernels coincides with the quantum Hamiltonian reduction for generic levels.

A \( \lambda \)-brackets of the Shatashvili-Vafa \( G_2 \) superconformal algebra

\[
[\Phi \lambda \Phi] = (-\frac{7}{2})\lambda^2 + 6X, \quad [\Phi \lambda X] = -\frac{15}{2} \Phi \lambda - \frac{5}{2} \Phi, \\
[X \lambda X] = \frac{35}{24} \lambda^3 - 10X \lambda - 5 \partial X, \quad [G \lambda \Phi] = K, \\
[G \lambda X] = -\frac{1}{2} G \lambda + M, \quad [G \lambda K] = 3 \Phi \lambda + \partial \Phi, \\
[G \lambda M] = -\frac{7}{12} \lambda^3 + (L + 4X) \lambda + \partial X, \quad [\Phi \lambda K] = -3G \lambda - 3 \left(M + \frac{1}{2} \partial G\right), \\
[\Phi \lambda M] = \frac{9}{2} K \lambda - \left(3 : G \Phi : -\frac{5}{2} \partial K\right), \quad [X \lambda K] = -3K \lambda + 3 \left(: G \Phi : -\partial K\right), \\
[X \lambda M] = -\frac{9}{4} G \lambda^2 - \left(5M + \frac{9}{4} \partial G\right) \lambda + \left(4 : GX : -\frac{7}{2} \partial M - \frac{3}{4} \partial^2 G\right),
\]
\[ [K_{\lambda}K] = -\frac{21}{6} \lambda^3 + 6 (X - L) \lambda + 3 \partial (X - L), \]
\[ [K_{\lambda}M] = -\frac{15}{2} \Phi \lambda^2 - \frac{11}{2} \partial \Phi \lambda + 3 (: GK : + 2 : L \Phi :), \]
\[ [M_{\lambda}M] = -\frac{35}{24} \lambda^4 + \frac{1}{2} (20X - 9L) \lambda^2 + \left( 10 \partial X - \frac{9}{2} \partial L \right) \lambda + \left( \frac{3}{2} \partial^2 X - \frac{3}{2} \partial^2 L - 4 : GM : + 8 : LX : \right), \]
\[ [L_{\lambda}X] = -\frac{7}{24} \lambda^3 + 2X \lambda + \partial X, \quad [L_{\lambda}M] = -\frac{1}{4} G \lambda^2 + \frac{5}{2} M \lambda + \partial M. \]

**B The \( SW(\frac{3}{2}, \frac{3}{2}, 2) \) superconformal algebra**

Here we follow the presentation in [14]. The \( SW(\frac{3}{2}, \frac{3}{2}, 2) \) superconformal algebra has six generators \{G, L, H, \tilde{M}, W, U\} where G and L generate the \( N = 1 \) superconformal algebra of central charge \( c \), and \( (H, \tilde{M}) \) and \( (W, U) \) are two superconformal multiplets of dimensions \( \frac{3}{2} \) and 2 respectively.

A superconformal multiplet \( \Phi = (\Phi, \Psi) \) of dimension \( \Delta \) is a pair of two primary fields of conformal weights \( \Delta \) and \( \Delta + \frac{1}{2} \) respectively, such that the \( \lambda \)-brackets with the supersymmetry generator \( G \) are as follow:

\[ [G_{\lambda} \Phi] = \Psi, \quad [G_{\lambda} \Psi] = (\partial + 2\Delta) \Phi. \]

The other \( \lambda \)-brackets between the generators are as follow:

\[ [H_{\lambda}H] = \frac{c}{3} \lambda^2 + \varepsilon \tilde{M} + 2L + \frac{4}{3} \mu W, \]  
(B.1)
\[ [H_{\lambda} \tilde{M}] = (3G + 3\varepsilon H) \lambda + \frac{-2}{3} \mu U + \partial G + \varepsilon \partial H, \]
\[ [\tilde{M}_{\lambda} \tilde{M}] = \frac{1}{3} c \lambda^3 + (4\varepsilon \tilde{M} + 8L + \frac{4}{3} \mu W) \lambda + 2\varepsilon \partial \tilde{M} + 4\partial L + \frac{2}{3} \mu \partial W, \]
\[ [H_{\lambda} W] = \mu H \lambda + \frac{\varepsilon}{2} U + \frac{\mu}{3} \partial H, \]
\[ [\tilde{M}_{\lambda} W] = (\frac{\mu}{3} \tilde{M} + 2\varepsilon W) \lambda + \frac{9\mu}{2c} : GH : + \frac{\mu(-27 + 2c)}{12c} \partial \tilde{M} + \varepsilon \partial W, \]
\[ [H \lambda U] = (−\frac{2}{3} \mu \tilde{M} + 2 \varepsilon W) \lambda + \frac{9 \mu}{2c} : GH : -\frac{\mu(27 + 2c)}{12c} \partial \tilde{M} + \frac{\varepsilon}{2} \partial W, \]

\[ [\tilde{M} \lambda U] = \mu H \lambda^2 + \left(\frac{5}{2} \varepsilon U + \frac{2}{3} \mu \partial H\right) \lambda - \frac{9 \mu}{2c} : G \tilde{M} : + \frac{9 \mu}{c} : LH : + \varepsilon \partial U + \frac{\mu(-27 + 2c)}{12c} \partial^2 H, \]

\[ [W \lambda W] = \frac{c}{12} \lambda^3 + \left(\frac{5}{2} \varepsilon \tilde{M} + \frac{\mu(10c - 27)}{6c} W\right) \lambda + \partial L + \frac{\varepsilon}{4} \partial \tilde{M} + \frac{\mu(10c - 27)}{12c} \partial W, \]

\[ [W \lambda U] = \left(-\frac{3}{2} G - \frac{3}{4} \varepsilon H\right) \lambda^2 + \left(\frac{\mu(-27 + 10c)}{12c} U - \partial G - \frac{\varepsilon}{2} \partial H\right) \lambda \\
- \frac{1}{48c} \left(162 \varepsilon : G \tilde{M} : + 432 \mu : GW : - 324 : H \tilde{M} : + 648 : LG : \\
+ 324 \varepsilon : LH : - 8 \mu(27 + 2c) \partial U + 6(-27 + 2c) \partial^2 G \\
+ 3(-27 + 2c) \varepsilon \partial^2 H\right), \]

\[ [U \lambda U] = -\frac{c}{12} \lambda^4 - \left(\frac{5}{4} \varepsilon \tilde{M} + 5 L + \frac{\mu(-27 + 10c)}{6c} W\right) \lambda^2 - \left(\frac{5}{4} \varepsilon \partial \tilde{M} + 5 \partial L \\
+ \frac{\mu(-27 + 10c)}{6c} \partial W\right) \lambda - \frac{1}{16c} \left(-144 \mu : GU : - 108 : G \partial G : \\
- 54 \varepsilon : G \partial H : + 108 : H \partial H : - 108 : \tilde{M} \tilde{M} : + 216 \varepsilon : L \tilde{M} : \\
+ 432 : LL : + 288 \mu : LW : + 54 \varepsilon : \partial GH : - 3(9 - 2c) \varepsilon \partial^2 \tilde{M} \\
+ 24 \varepsilon \partial^2 L - 4 \mu(27 - 2c) \partial^2 W\right), \]

where \( c, \varepsilon \in \mathbb{C} \) and \( \mu = \sqrt{\frac{3c(4 + \varepsilon^2)}{2(27 - 2c)}} \).

Remark B.1. For \((c, \varepsilon) = (\frac{2}{3}, 0)\) it was checked in [14] that \( SW(\frac{2}{3}, \frac{2}{3}, 2) \) coincides with the Shatashvili-Vafa \( G_2 \) algebra at central charge \( \frac{21}{2} \) modulo the ideal generated by:

\[ 2\sqrt{14} : GW : - 3 : H \tilde{M} : + 2 : LG : - 2\sqrt{14} \partial U. \]  

(B.2)

The existence of this ideal was first observed in [5]. The relations between the generators of \( SW(\frac{2}{3}, \frac{2}{3}, 2) \) in the case \((\frac{2}{3}, 0)\) and the generators of the Shatashvili-Vafa \( G_2 \) algebra as presented in the Appendix A are given by:

\[ \Phi = iH, \ K = i\tilde{M}, \ X = -(L + \sqrt{14} W)/3, \ M = -(\partial G + 2\sqrt{14} U)/6. \]  

(B.3)
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