Partial Differential Equations/Mathematical Physics

On two-particle Anderson localization at low energies

Localisation d'Anderson pour un système à deux particules, à basses énergies

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A R T I C L E   I N F O

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A B S T R A C T

We prove exponential spectral localization in a two-particle lattice Anderson model, with a short-range interaction and an external i.i.d. random potential, at sufficiently low energies. The proof is based on the multi-particle multi-scale analysis developed earlier in Chulaevsky and Suhov (2009) [4] in the case of high disorder. Our method applies to a larger class of random potentials than in Aizenman and Warzel (2009) [2] where dynamical localization was proved with the help of the fractional moment method.

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RÉSUMÉ

On démontre la localisation spectrale exponentielle pour un modèle d’Anderson discret, avec interaction à courte portée dans un champ de potentiel aléatoire i.i.d., à basses énergies. La démonstration utilise l’analyse multi-échelle multi-particule développée dans Chulaevsky et Suhov (2009) [4] dans le cas de grand désordre. Cette méthode s’applique à une classe de potentiels aléatoires plus large que dans Aizenman et Warzel (2009) [2], où la localisation dynamique a été démontrée par la méthode des moments fractionnaires.

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Version française abrégée

On étudie un système de deux particules quantiques avec interaction dans un milieu désordonné. Ce système est décrit par un Hamiltonien $H_{V,U}(\omega)$ agissant dans l’espace de Hilbert $H := \ell^2(\mathbb{Z}^d)$, et de la forme suivante

$$H_{V,U} = \Delta + \sum_{j=1}^{2} V(x_j, \omega) + U,$$

où $\Delta$ est le Laplacien discret relatif au réseau $\mathbb{Z}^d \times \mathbb{Z}^d \cong \mathbb{Z}^{2d}$, i.e.,

$$\Delta \Psi(x) = \sum_{y \in \mathbb{Z}^{2d}, |y|=1} \Psi(x+y), \quad x = (x_1, x_2) \in \mathbb{Z}^{2d}$$

avec $|y| := \|y\|_{\infty}$. De plus, $V : \mathbb{Z}^{2d} \times \Omega \to \mathbb{R}$ est un champ aléatoire i.i.d. sur $\mathbb{Z}^d$ relatif à un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$, et $U$ est l’opérateur de multiplication par une fonction bornée $U(x) = U(x_1, x_2)$, non nécessairement symétrique.

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Le résultat principal de cette Note est le théorème suivant :

**Théorème 0.1.** On suppose que $V$ est un champ aléatoire réel, à valeurs indépendantes identiquement distribuées, et vérifiant la condition (1). On suppose que le potentiel d’interaction $U$ est borné et vérifie la condition (2). On note $E^0 = \inf \sigma(H)$.

Il existe alors un nombre réel $E^* > 0$ tel que le spectre de l’opérateur $H(\omega)$ dans $(-\infty, E^*)$ soit purement ponctuel, et que toutes ses fonctions propres $\Psi_n(\omega)$ relatives aux valeurs propres $E_n(\omega) \leq E^*$ soient à décroissance exponentielle à l’infini :

$$|\Psi_n(\mathbf{x})| \leq C_n(\omega)e^{-m|x|},$$

pour un $m > 0$ non aléatoire.

1. Introduction and main result

Consider the lattice $(\mathbb{Z}^d)^2 \cong \mathbb{Z}^{2d}$, $d \geq 1$. We will use the notations $\mathcal{D} = \{ \mathbf{x} \in \mathbb{Z}^{2d} : \mathbf{x} = (x, x) \}$, $[a, b] : = [a, b] \cap \mathbb{Z}$. Vectors $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$ will be identified with configurations of two distinguishable quantum particles in $\mathbb{Z}^d$. We denote by $|\cdot|$ the max-norm $\| \cdot \|_{\infty}$.

We study a system of two interacting lattice quantum particles in a disordered environment, described by a Hamiltonian $H_{V, U}(\omega)$ in the Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z}^{2d})$ of the form

$$H_{V, U} = \Delta + \sum_{j=1}^{2} V(x_j, \omega) + U,$$

where $\Delta$ is the nearest-neighbor lattice Laplacian on $(\mathbb{Z}^d)^2 \cong \mathbb{Z}^{2d}$, i.e.

$$\Delta \Psi(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{Z}^{2d} : |\mathbf{y}| = 1} \Psi(\mathbf{x} + \mathbf{y}), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{Z}^d,$$

$V : \mathbb{Z}^d \times \Omega \rightarrow \mathbb{R}$ is an i.i.d. random field on $\mathbb{Z}^d$ relative to some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $U$ is the multiplication operator by a function $\mathbf{U}(\mathbf{x}) = \mathbf{U}(x_1, x_2)$ which we assume bounded — but not necessarily symmetric.

Aizenman and Warzel [2] proved by the fractional moment method — introduced in [1] for single-particle systems — spectral and dynamical localization for such Hamiltonians under the assumption that the marginal probability distribution of the random field $V$ with i.i.d. (independent and identically distributed) values admits a bounded probability density $\rho_V$, satisfying some additional conditions.

In this note, using the Multi-Scale Analysis (MSA), we follow [4] and prove exponential localization under a much weaker assumption of log-Hölder continuity of the marginal distribution function $F_V$ of the field $V$. Specifically, we require that for some $\beta \in (0, 1)$, some large enough $q > 0$, and all sufficiently large $L > 0$,

$$\sup_{a \in \mathbb{R}} \mathbb{P}\{V(0, \omega) \in [a, a + e^{-L^q}]\} \leq L^{-\beta}. \quad (1)$$

The interaction potential is assumed to be bounded and to satisfy the following condition:

there exists $r_0 \in [0, +\infty)$ such that $|x_1 - x_2| > r_0 \Rightarrow \mathbf{U}(x_1, x_2) = 0$. \quad (2)

We denote by $\sigma(H(\omega))$ the spectrum of $H(\omega)$. It follows from well-known results that the quantity

$$E^0 := \inf \sigma(H(\omega))$$

is non-random, although it may be infinite, e.g., for Gaussian random potentials.

Given an arbitrary finite lattice cube $C_L(\mathbf{u}) := \{ \mathbf{x} \in \mathbb{Z}^{2d} : |\mathbf{x} - \mathbf{u}| \leq L \}$, we will consider a finite-volume approximation of the Hamiltonian $H$

$$H_{C_L(\mathbf{u})} = H|_{\mathcal{D}(C_L(\mathbf{u}))}$$

with Dirichlet boundary conditions on $\partial C_L(\mathbf{u})$,

where the boundary is

$$\partial C_L(\mathbf{u}) = \{ \mathbf{v} \in \mathbb{Z}^{2d} : \text{dist}(\mathbf{v}, \mathbb{Z}^{2d} \setminus C_L(\mathbf{u})) = 1 \}.$$

The main result of this note is the following:

**Theorem 1.1.** Suppose $V$ is a real i.i.d. random field satisfying condition (1). Suppose also the interaction potential $U$ is bounded and satisfies (2). Let $E^0 = \inf \sigma(H)$.

Then there exists $E^* > E^0$ such that the spectrum of $H(\omega)$ in $(-\infty, E^*)$ is pure point, and all its eigenfunctions $\Psi_n(\omega)$ with eigenvalues $E_n(\omega) \leq E^*$ are exponentially decaying at infinity:

$$|\Psi_n(\mathbf{x})| \leq C_n(\omega)e^{-m|x|},$$

where $m > 0$ is non-random.
2. Proof scheme

Following [4], we use an adaptation to the two-particle interacting systems of the multi-scale analysis (MSA) which was earlier developed for single-particle models [8]. Given a finite cube $C_L(u) \subseteq \mathbb{Z}^{2d}$, introduce the resolvent of the operator $H_{C_L(u)}$:

$$G_{C_L(u)}(E) := (H_{C_L(u)} - E)^{-1}, \quad E \in \mathbb{R}. $$

Its matrix elements $G_{C_L(u)}(x, y; E)$ in the canonical basis $\delta_x$ in $\ell^2(\mathbb{Z}^{2d})$ is usually called the (discrete) Green function of the operator $H_{C_L(u)}$.

According to the general MSA approach, the exponential localization will be derived from Theorem 2.2 below. To formulate it, we need to introduce the following notion:

**Definition 2.1.** Let $m > 0$ and $E \in \mathbb{R}$. A cube $C_L(u)$ is called $(E, m)$-non-singular ($(E, m)$-NS, in short) if

$$\max_{v \in \partial C_L(u)} |G_{C_L(u)}(u, v; E)| \leq e^{-mL}. $$

Otherwise, it is called $(E, m)$-singular ($(E, m)$-S, in short).

Introduce the symmetry $S: (x_1, x_2) \mapsto (x_2, x_1)$ in the lattice $\mathbb{Z}^{2d}$ (here $x_1, x_2 \in \mathbb{Z}^{d}$) and define the “symmetrized” distance

$$d_S(x, y) = \min\{ |x - y|, |S(x) - y| \}. $$

We will say that two lattice subsets $A, B$ are $\ell$-distant if $d_S(A, B) > \ell$.

Further, given an integer $L_0 > 2$, define the sequence of integers $L_{k+1} = [L_k^{3/2}]$, $k \geq 0$. In the course of the MSA, it is required that $L_0$ be large enough.

**Theorem 2.2.** Let $m > 0$. For any $p > 0$ there exists $E^* = E^*(p) > E_0$ such that for all $k \geq 0$ and any pair of $8L_k$-distant cubes $C_{L_k}(u)$, $C_{L_k}(v)$ the following bound holds true:

$$\mathbb{P}\left\{ \text{there exists } E \in [E_0, E^*) \text{ such that } C_{L_k}(u), C_{L_k}(v) \text{ are } (E, m)\text{-singular} \right\} \leq L_0^{-2p}. $$

provided $L_0$ is large enough.

The proof is based on induction in $k$. Note that the initial scale bound (for $L_0$ sufficiently large) uses the Combes–Thomas estimate and the “Lifshitz tails” phenomenon, essentially in the same way as for single-particle models [7], for the multi-particle structure of the potential energy is not relevant for such a bound. The inductive step is performed almost in the same way as for single-particle models [7], for the multi-particle structure of the potential energy is not relevant for such a bound. The inductive step is performed almost in the same way as for single-particle models [7].

Unlike the single-particle case, the proof of (3) depends upon the geometry of the pair $C_{L_k}(u), C_{L_k}(v)$. Namely, introduce the subset $D_{r_0} := \{ x = (x_1, x_2) \in \mathbb{Z}^{2d} : |x_1 - x_2| \leq r_0 \}$, and the following:

**Definition 2.3.** A 2-particle cube $C_L(u)$ is called diagonal when $C_L(u) \cap D_{r_0} \neq \emptyset$. Otherwise, it is called non-diagonal.

Property (3) is established separately for the following three types of pairs $C_{L_{k+1}}(u), C_{L_{k+1}}(v)$ of separable cubes:

(i) Both are diagonal.

(ii) Both are non-diagonal.

(iii) One is diagonal, while the other is non-diagonal.

The next statement is a reformulation of [4, Theorem 1.2]; see the proof given in [4]. This theorem was earlier formulated in [8, Theorem 2.3] and [5, Section 1] for single-particle models.

**Theorem 2.4.** Suppose that the bound (3) holds true for $p$ large enough and some $E^* > E_0$.

Then the spectrum of $H(\omega)$ in $(-\infty, E^*)$ is pure point, and there exists a non-random number $m > 0$ such that all eigenfunctions $\Psi_n(\omega)$ of $H(\omega)$ with eigenvalues $E_n(\omega) \leq E^*$ decay exponentially fast at infinity with rate $m$:

$$|\Psi_n(x)| \leq C_n(\omega)e^{-m|x|}. $$

Theorem 2.4 combined with Theorem 2.2 implies the main Theorem 1.1.
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