On the infimum convolution inequality \*†‡

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Abstract

In the paper we study the infimum convolution inequalities. Such an inequality was first introduced by B. Maurey to give the optimal concentration of measure behaviour for the product exponential measure. We show how IC inequalities are tied to concentration and study the optimal cost functions for an arbitrary probability measure \( \mu \). In particular, we show the optimal IC inequality for product log–concave measures and for uniform measures on the \( \ell_p^n \) balls. Such an optimal inequality implies, for a given measure, in particular the Central Limit Theorem of Klartag and the tail estimates of Paouris.

1 Introduction and Notation

In the seminal paper [16], B. Maurey introduced the so called property \((\tau)\) for a probability measure \( \mu \) with a cost function \( \varphi \) (see Definition 2.1 below) and established a very elegant and simple proof of Talagrand’s two level concentration for the product exponential distribution \( \nu_n \) using \((\tau)\) for this distribution and an appropriate cost function \( w \).

It is natural to ask what other pairs \((\mu, \varphi)\) have property \((\tau)\)? As any \( \mu \) satisfies \((\tau)\) with \( \varphi \equiv 0 \), one will rather ask how big a cost function one can take. In this paper we study the probability measures \( \mu \) that have property \((\tau)\) with respect to the largest (up to a multiplicative factor) possible convex cost function \( \Lambda^*_\mu \). This bound comes from checking property \((\tau)\) for linear functions. We say a measure satisfies the infimum convolution inequality (IC for short) if the pair \((\mu, \Lambda^*_\mu)\) satisfies \( \tau \).

It turns out that such an optimal infimum convolution inequality has very strong consequences. It gives the best possible concentration behaviour, governed by the so–called \( L_\mu \)-centroid bodies (Corollary 3.10). This, in turn, implies in particular a weak–strong moment comparison (Proposition 3.12), the Central Limit Theorem of Klartag [12] and the tail estimates estimates of Paouris [18] (Proposition 3.15). We believe that IC holds for any log–concave probability measure, which is the main motivation for this paper.

\*Partially supported by the Foundation for Polish Science and MEiN Grant 1 PO3A 012

†Keywords: infimum convolution, concentration, log–concave measure, isoperimetry, \( \ell_p^n \) ball

‡2000 Mathematical Subject Classification: 52A20 (52A40, 60E15)
Maurey’s inequality for the exponential measure is of this optimal type. We transport this to any log–concave measure on the real line, and as the inequality tensorizes, any product log–concave measure satisfies IC (Corollary 2.19). However, the main challenge is to provide non–product examples of measures satisfying IC. We show how such an optimal result can be obtained from concentration inequalities, and follow on to prove IC for the uniform measure on any \( \ell^n_p \) ball for \( p \geq 1 \) (Theorem 5.30).

With the techniques developed we also prove a few other results. We give a proof of the log–Sobolev inequality for \( \ell^n_p \) balls, where \( p \geq 2 \) (Theorem 5.31) and provide a new concentration inequality for the exponential measure for sets lying far away from the origin (Theorem 4.6).

Organization of the paper. This section, apart from the above introduction, defines the notation used throughout the paper. The second section is devoted to studying the general properties of the inequality IC. In subsection 2.1 we recall the definition of property \((\tau)\) and its ties to concentration from [16]. In subsection 2.2 we study the opposite implication — what additional assumptions one needs to infer \((\tau)\) from concentration inequalities. In subsection 2.3 we show that \( \Lambda^\star_{\mu} \) is indeed the largest possible cost function and define the inequality IC. In subsection 2.4 we show that product log–concave measures satisfy IC.

In the third section we give more attention to the concentration inequalities tied to IC. In subsection 3.1 we show the connection to \( Z_p \) bodies. In subsection 3.2 we continue in this vein with the additional assumption our measure is \( \alpha \)–regular. In subsection 3.3 we show how IC implies a comparison of weak and strong moments and the results of [12] and [18].

In the fourth section we give a modification of the two–level concentration for the exponential measure, in which for sets lying far away from the origin only an enlargement by \( tB^n_p \) is used. This will be used in the fifth section, which focuses on the uniform measure on the \( B^n_p \) ball. In subsection 5.1 we define and study two rather standard transports of measure used further on. In subsection 5.2 we use these transports along with the concentration from section 4 and Cheeger inequality from [19] to give a proof of IC for \( p \leq 2 \). In section 5.3 we show a proof of IC for \( p \geq 2 \) and a proof of the log–Sobolev inequality for \( p \geq 2 \).

We conclude with a few possible extensions of the results of the paper in the sixth section.

Notation. By \( \langle \cdot, \cdot \rangle \) we denote the standard scalar product on \( \mathbb{R}^n \). For \( x \in \mathbb{R}^n \) we put \( \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \) for \( 1 \leq p < \infty \) and \( \|x\|_\infty = \max_i |x_i| \); we also use \( |x| \) for \( \|x\|_2 \). We set \( B^n_p = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\} \).

By \( \nu \) we denote the symmetric exponential distribution on \( \mathbb{R} \), i.e. the probability measure with the density \( \frac{1}{2} \exp(-|x|) \). For \( p \geq 1 \), \( \nu_p \) is the probability distribution on \( \mathbb{R} \) with the density \( (2\gamma_p)^{-1} \exp(-|x|^p) \), where \( \gamma_p = \Gamma(1 + 1/p) \), in particular \( \nu_1 = \nu \). For a probability measure \( \mu \) we write \( \mu^n \) for a product measure \( \mu \otimes^n \), thus \( \nu^n_p \) has the density \( (2\gamma_p)^{-n} \exp(-\|x\|_p^p) \).

For a Borel set \( A \) in \( \mathbb{R}^n \) by \( |A| \) or \( \lambda_n(A) \) we mean the Lebesgue measure of \( A \). We choose numbers \( r_{p,n} \) such that \( |r_{p,n}B^n_p| = 1 \) and by \( \mu_{p,n} \) denote
the uniform distribution on $B^n_p$.

The letters $c, C$ denote absolute numerical constants, which may change from line to line. By $c(p), C(p)$ we mean constants dependent on $p$ (or, formally, a family of absolute constants indexed by $p$), these also may change from line to line. Other letters, in particular greek letters, denote constants fixed for a given proof or section. For any sets of positive real numbers $a_i$ and $b_i$, $i \in I$, by $a_i \sim b_i$ we mean there exist absolute numerical constants $c, C > 0$ such that $ca_i < b_i < Ca_i$ for any $i \in I$. Similarly for collections of sets $A_i$ and $B_i$ by $A_i \sim B_i$ we mean $cA_i \subset B_i \subset CA_i$ for any $i \in I$, where again $c, C > 0$ are absolute numerical constants. By $\sim_p$ we mean the constants above can depend on $p$.

## 2 Infimum convolution inequality

### 2.1 Property $(\tau)$

**Definition 2.1.** Let $\mu$ be a probability measure on $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to [0, \infty]$ be a measurable function. We say that the pair $(\mu, \varphi)$ has property $(\tau)$ if for any bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$
\int_{\mathbb{R}^n} e^{f \square \varphi} d\mu \int_{\mathbb{R}^n} e^{-f} d\mu \leq 1,
$$

(1)

where for two functions $f$ and $g$ on $\mathbb{R}^n$,

$$
f \square g(x) := \inf \{f(x) + g(y) : y \in \mathbb{R}^n\}
$$

denotes the infimum convolution of $f$ and $g$.

The following two easy observations are almost immediate (c.f. [16]):

**Proposition 2.2 (Tensorization).** If pairs $(\mu_i, \varphi_i), i = 1, \ldots, k$ have property $(\tau)$ and $\varphi(x_1, \ldots, x_k) = \varphi_1(x_1) + \ldots + \varphi_k(x_k)$, then the couple $(\otimes_{i=1}^k \mu_i, \varphi)$ also has property $(\tau)$.

**Proposition 2.3 (Transport of measure).** Suppose that $(\mu, \varphi)$ has property $(\tau)$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ is such that

$$
\psi(Tx - Ty) \leq \varphi(x - y) \text{ for all } x, y \in \mathbb{R}^n.
$$

Then the pair $(\mu \circ T^{-1}, \psi)$ has property $(\tau)$.

Maurey noticed that property $(\tau)$ implies $\mu(A + B_\varphi(t)) \geq 1 - \mu(A)^{-1}e^{-t}$, where

$$
B_\varphi(t) := \{x \in \mathbb{R}^n : \varphi(x) \leq t\}.
$$

We will need a slight modification of this estimate.
**Proposition 2.4.** Property (τ) for \((φ,μ)\) implies for any Borel set \(A\) and \(t \geq 0\),

\[
\mu(A + B_φ(t)) \geq \frac{e^t μ(A)}{(e^t - 1)μ(A) + 1}, \quad (2)
\]

In particular for all \(t > 0\),

\[
\mu(A) > 0 \Rightarrow \mu(A + B_φ(t)) > \min\{e^{t/2}μ(A), 1/2\}, \quad (3)
\]

\[
\mu(A) \geq 1/2 \Rightarrow 1 - μ(A + B_φ(t)) < e^{-t/2}(1 - μ(A)), \quad (4)
\]

and

\[
μ(A) = ν(−∞, x) \Rightarrow μ(A + B_φ(t)) ≥ ν(−∞, x + t/2). \quad (5)
\]

**Proof.** Take \(f(x) = t1_{R^n \setminus A}\). Then \(f(x)\) is non-negative on \(R^n\), so \(f ▽ φ\) is non-negative (recall that by definition we consider only nonnegative cost functions). For \(x \notin A + B_φ(t)\) we have \(f ▽ φ(x) = \inf_y (f(y) + φ(x − y)) ≥ t\), for either \(y \notin A\), and then \(f(y) = t\), or \(y \in A\), and then \(φ(x − y) ≥ t\) as \(x \notin A + B_φ(t)\).

Thus from property (τ) for \(f\) we have

\[
1 ≥ \int e^{f ▽ φ(x)}dμ(x) ≥ \int e^{-f(x)}dμ(x) ≥ \left[μ(A + B_φ(t)) + e^t(1 - μ(A + B_φ(t)))\right] \left[μ(A) + e^{-t}(1 - μ(A))\right],
\]

from which, extracting the condition upon \(μ(A + B_φ(t))\) by direct calculation, we get (2).

Let \(f_t(p) : = e^t p / ((e^t - 1)p + 1)\), notice that \(f_t\) is increasing in \(p\) and for \(p ≤ e^{-t/2}/2\),

\[
(e^t - 1)p + 1 ≤ e^{t/2} + 1 - \frac{1}{2}(e^{t/2} + e^{-t/2}) < e^{t/2},
\]

hence \(f_t(p) > \min(e^{t/2}p, 1/2)\) and (3) follows. Moreover for \(p ≥ 1/2\)

\[
1 - f_t(p) = \frac{1 - p}{(e^t - 1)p + 1} ≤ \frac{1 - p}{(e^t + 1)/2} < e^{-t/2}(1 - p)
\]

and we get (4).

Let \(F(x) = ν(−∞, x)\) and \(g_t(p) = F(F^{-1}(p) + t)\). Previous calculations show that for \(t, p > 0\), \(f_t(p) ≥ g_t(p)\) if \(F^{-1}(p) + t/2 ≤ 0\) or \(F^{-1}(p) ≥ 0\). Since \(g_{t+s} = g_t ∘ g_s\) and \(f_{t+s} = f_t ∘ f_s\), we get that \(f_t(p) ≥ g_t(p/2)\) for all \(t, p > 0\), hence (2) implies (4).

The main theorem of (16) states that \(ν\) satisfies (τ) with a sufficiently chosen cost function.

**Theorem 2.5.** Let \(w(x) = \frac{1}{36}x^2\) for \(|x| ≤ 4\) and \(w(x) = \frac{2}{9}(|x| - 2)\) otherwise. Then the pair \((ν^n, \sum_{i=1}^n w(x_i))\) has property (τ).
Theorem 2.5 together with Proposition 2.4 immediately gives the following two-level concentration:
\[ \nu^n(A) = \nu(-\infty, x] \Rightarrow \forall t \geq 0 \quad \nu^n(A + 6\sqrt{2t}B_{2B}^n + 18tB_1^n) \geq \nu(-\infty, x + t], \quad (6) \]
that was first established (with different universal, rather large constants) by Talagrand [21].

2.2 From concentration to property \((\tau)\)
Proposition 2.4 shows that property \((\tau)\) implies concentration, the next result presents the first approach to the converse implication.

Corollary 2.6. Suppose that the cost function \(\varphi\) is radius-wise nondecreasing, \(\mu\) is a Borel probability measure on \(\mathbb{R}^n\) and \(\beta > 0\) is such that for any \(t > 0\) and \(A \in B(\mathbb{R}^n)\),
\[ \mu(A) = \nu(-\infty, x] \Rightarrow \mu(A + \beta B_{\varphi}(t)) \geq \nu(-\infty, x + \max\{t, \sqrt{t}\}). \quad (7) \]
Then the pair \((\mu, \frac{1}{36}\varphi(\frac{\cdot}{\beta}))\) has property \((\tau)\). In particular if \(\varphi\) is convex, symmetric and \(\varphi(0) = 0\) then (7) implies property \((\tau)\) for \((\mu, \varphi(\frac{\cdot}{36\beta}))\).

Proof. Let us fix \(f: \mathbb{R}^n \to \mathbb{R}\). For any measurable function \(h\) on \(\mathbb{R}^k\) and \(t \in \mathbb{R}\) we put
\[ A(h, t) := \{x \in \mathbb{R}^k : h(x) < t\}. \]
Let \(g\) be a nondecreasing right-continuous function on \(\mathbb{R}\) such that \(\mu(A(f, t)) = \nu(A(g, t))\). Then the distribution of \(g\) with respect to \(\nu\) is the same as the distribution of \(f\) with respect to \(\mu\) and thus
\[ \int_{\mathbb{R}^n} e^{-f(x)}d\mu(x) = \int_{\mathbb{R}} e^{-g(x)}d\nu(x). \]
To finish the proof of the first assertion, by Theorem 2.5 it is enough to show that
\[ \int_{\mathbb{R}^n} e^{f(\frac{1}{36}\varphi(\frac{\cdot}{\beta}))}d\mu \leq \int_{\mathbb{R}} e^{g(36\beta)}d\nu. \]
We will establish stronger property:
\[ \forall u \quad \mu\left(A\left( f \square \frac{1}{36} \varphi\left( \frac{\cdot}{\beta} \right), u \right) \right) \geq \nu(A(g \square w, u)). \]
Since the set \(A(g \square w, u)\) is a halfline, it is enough to prove that
\[ g(x_1) + w(x_2) < u \Rightarrow \mu\left(A\left( f \square \frac{1}{36} \varphi\left( \frac{\cdot}{\beta} \right), u \right) \right) \geq \nu(-\infty, x_1 + x_2]. \quad (8) \]
Let us fix \(x_1\) and \(x_2\) with \(g(x_1) + w(x_2) < u\) and take \(s_1 > g(x_1)\) \(s_2 = w(x_2)\) with \(s_1 + s_2 < u\). Put \(A := A(f, s_1)\), then \(\mu(A) = \nu(A(g, s_1)) \geq \nu(-\infty, x_1]. \)
By the definition of $w$ it easily follows that $x_2 \leq \max\{6\sqrt{s_2}, 9s_2\}$, hence by (7),
$$\mu(A + \beta B_f(36s_2)) \geq \nu(-\infty, x_1 + x_2).$$
Since
$$A + \beta B_f(36s_2) = A(f, s_1) + B_{\varphi(s_1)/36}(s_2) \subseteq A\left(f \bigcap_{n}^{1} \varphi\left(\frac{s_1}{\beta}\right), s_1 + s_2\right),$$
we obtain the property (8).

The last part of the statement immediately follows since any symmetric convex function $\varphi$ is radius-wise nondecreasing and if additionally $\varphi(0) = 0$, then $\varphi(x/36) \leq \varphi(x)/36$ for any $x$.

The next proposition shows that inequalities (3) and (4) are strongly related.

**Proposition 2.7.** The following two conditions are equivalent for any Borel set $K$ and $\gamma > 1$,

$$\forall A \in B(\mathbb{R}^n) : \mu(A) > 0 \Rightarrow \mu(A + K) > \min\left\{\gamma \mu(A), \frac{1}{2}\right\},$$

(9)

$$\forall A \in B(\mathbb{R}^n) : \mu(A) \geq \frac{1}{2} \Rightarrow 1 - \mu(A - K) < \frac{1}{\gamma}(1 - \mu(A)).$$

(10)

**Proof.** $\mathbf{(9) \Rightarrow (10)}$. Suppose that $\mu(A) \geq 1/2$ and $1 - \mu(A - K) \geq \gamma^{-1}(1 - \mu(A))$. Let $A := \mathbb{R}^n \setminus (A - K)$, then $(A + K) \cap A = \emptyset$, so $\mu(A + K) \leq 1/2$ and

$$\mu(A + K) \leq 1 - \mu(A) \leq \gamma(1 - \mu(A - K)) = \gamma \mu(A)$$

and this contradicts (9).

$\mathbf{(10) \Rightarrow (9)}$. Let us take $A \in \mathbb{R}^n$ with $\mu(A) > 0$ such that $\mu(A + K) \leq \min\{\gamma \mu(A), 1/2\}$. Let $A := \mathbb{R}^n \setminus (A + K)$, then $\mu(A) \geq 1/2$. Moreover $(A - K) \cap A = \emptyset$, thus

$$1 - \mu(A - K) \geq \mu(A) \geq \frac{1}{\gamma} \mu(A + K) = \frac{1}{\gamma}(1 - \mu(A))$$

and we get the contradiction with (10).

**Corollary 2.8.** Suppose that $t > 0$ and $K$ is a symmetric convex set in $\mathbb{R}^n$ such that

$$\forall A \in B(\mathbb{R}^n) : \mu(A) > 0 \Rightarrow \mu(A + K) > \min\{e^t \mu(A), 1/2\}.$$

Then for any Borel set $A$,

$$\mu(A) = \nu(-\infty, x] \Rightarrow \mu(A + 2K) > \nu(-\infty, x + t].$$

**Proof.** Let us fix the set $A$ with $\mu(A) = \nu(-\infty, x]$. Notice that $A + 2K = A + K + K \supseteq A + K$. If $x + t \leq 0$, then $\mu(A + K) > e^t \mu(A) = \nu(-\infty, x + t]$. If $x \geq 0$, Proposition 2.7 gives

$$\mu(A + K) > 1 - e^{-t}(1 - \mu(A)) = \nu(-\infty, x + t].$$
Finally, if $x \leq t$, we get $\mu(A + K) \geq 1/2 = \nu(-\infty, 0]$, hence by the previous case,

$$\mu(A + 2K) = \mu((A + K) + K) > \nu(-\infty, t] \geq \nu(-\infty, x + t).$$

Corollary 2.8 shows that if the cost function $\varphi$ is symmetric and convex, condition (7) (with $2\beta$ instead of $\beta$) for $t \geq 1$ is implied by the following:

$$\forall A \in \mathcal{B}(\mathbb{R}^n) \quad \mu(A) > 0 \quad \Rightarrow \quad \mu(A + \beta B_2^n) > \min\{e^t \mu(A), 1/2\}. \quad (11)$$

To treat the case $t \leq 1$ we will need Cheeger’s version of the Poincaré inequality.

We say that a probability measure $\mu$ on $\mathbb{R}^n$ satisfies Cheeger’s inequality with constant $\kappa$ if for any Borel set $A$

$$\mu^+(A) := \liminf_{t \to 0^+} \frac{\mu(A + tB_2^n) - \mu(A)}{t} \geq \kappa \min\{\mu(A), 1 - \mu(A)\}. \quad (12)$$

It is not hard to check that Cheeger’s inequality (cf. [6, Theorem 2.1]) implies

$$\mu(A) = \nu(-\infty, x] \quad \Rightarrow \quad \mu(A + tB_2^n) \geq \nu(-\infty, x + \kappa t].$$

Finally, we may summarize this section with the following statement.

**Proposition 2.9.** Suppose that the cost function $\varphi$ is convex, symmetric with $\varphi(0) = 0$ and $1 \land \varphi(x) \leq (\alpha|x|^2)$ for all $x$. If the measure $\mu$ satisfies Cheeger’s inequality with the constant $\beta = 1/\delta$ and the condition (11) is satisfied for all $t \geq 1$ and $C = \gamma$ then $(\mu, \varphi(\cdot/C))$ has property $(\tau)$ with the constant $C = 36 \min\{2\gamma, \alpha \delta\}$.

**Proof.** Notice that $\alpha B_2^n(t) \supset \sqrt{t}B_2^n$ for all $t < 1$, hence Cheeger’s inequality implies that condition (7) holds for $t < 1$ with $C = \alpha \delta$. Therefore (7) holds for all $t \geq 0$ with $C = \min\{2\gamma, \alpha \delta\}$ and the assertion follows by Corollary 2.6. \qed

### 2.3 Optimal cost functions

A natural question arises: what other pairs $(\mu, \varphi)$ have property $(\tau)$? First we have to choose the right cost function. To do this let us recall the following definitions.

**Definition 2.10.** Let $f : \mathbb{R}^n \to (\neg \infty, \infty]$. The Legendre transform of $f$, denoted $Lf$ is defined by $Lf(x) := \sup_{y \in \mathbb{R}^n} \{\langle x, y \rangle - f(y)\}$.

The Legendre transform of any function is a convex function. If $f$ is convex and lower semi-continuous, then $L(Lf) = f$, and otherwise $L(Lf) \leq f$. In general, if $f \geq g$, then $Lf \leq Lg$. The Legendre transform satisfies $L(Cf)(x) = C \cdot Lf(x/C)$ and if $g(x) = f(x/C)$, then $Lg(x) = Lf(Cx)$. For this and other properties of $L$, cf. [15]. The Legendre transform has been previously used in the context of convex geometry, see for instance [2] and [12].
Definition 2.11. Let \( \mu \) be a probability measure on \( \mathbb{R}^n \). We define
\[
M_\mu(v) := \int_{\mathbb{R}^n} e^{(v,x)}d\mu(x), \quad \Lambda_\mu(v) := \log M_\mu(v)
\]
and
\[
\Lambda_\mu^*(v) := \mathcal{L}\Lambda_\mu(v) = \sup_{u \in \mathbb{R}^n} \left\{ \langle v, u \rangle - \ln \int_{\mathbb{R}^n} e^{(u,x)}d\mu(x) \right\}.
\]
The function \( \Lambda_\mu^* \) plays a crucial role in the theory of large deviations cf. [9].

Remark 2.12. Let \( \mu \) be a symmetric probability measure on \( \mathbb{R}^n \) and let \( \varphi \) be a convex cost function such that \( (\mu, \varphi) \) has property (\( \tau \)). Then
\[
\varphi(v) \leq 2\Lambda_\mu^*(v/2) \leq \Lambda_\mu^*(v).
\]
Proof. Take \( f(x) = \langle x, v \rangle \). Then
\[
f \square \varphi(x) = \inf_y (f(x - y) + \varphi(y)) = \inf_y (\langle x - y, v \rangle + \varphi(y)) = \langle x, v \rangle - \mathcal{L}\varphi(v).
\]
Property (\( \tau \)) yields
\[
1 \geq \int e^{f \cdot \varphi}d\mu \int e^{-f}d\mu = e^{-\mathcal{L}\varphi(v)} \int e^{(x,v)}d\mu \int e^{-\langle x,v \rangle}d\mu = e^{-\mathcal{L}\varphi(v)}M_\mu^2(v),
\]
where the last equality uses the fact that \( \mu \) is symmetric. Thus by taking the logarithm we get \( \mathcal{L}\varphi(v) \geq 2\Lambda_\mu(v) \), and by applying the Legendre transform we obtain \( \varphi(v) = \mathcal{L}\mathcal{L}\varphi(v) \leq 2\Lambda_\mu^*(v/2) \). The inequality \( 2\Lambda_\mu^*(v/2) \leq \Lambda_\mu^*(v) \) follows by the convexity of \( \Lambda_\mu^* \).

The above remark motivates the following definition.

Definition 2.13. We say that a symmetric probability measure \( \mu \) satisfies the infimum convolution inequality with constant \( \beta \) (IC(\( \beta \)) in short), if the pair \( (\mu, \Lambda_\mu^*(\cdot/\beta)) \) has property (\( \tau \)).

Proposition 2.14. If \( \mu_i \) are symmetric probability measures on \( \mathbb{R}^n \), \( 1 \leq i \leq k \) satisfying IC(\( \beta_i \)), then \( \mu = \otimes_{i=1}^k \mu_i \) satisfies IC(\( \beta \)) with \( \beta = \max_{1 \leq i \leq k} \beta_i \).

Proof. By independence, \( \Lambda_\mu(x_1, \ldots, x_k) = \sum_{i=1}^k \Lambda_{\mu_i}(x_i) \) and \( \Lambda_\mu^*(x_1, \ldots, x_k) = \sum_{i=1}^k \Lambda_{\mu_i}^*(x_i) \). Since IC(\( \beta \)) implies IC(\( \beta' \)) with any \( \beta' \geq \beta \), the result immediately follows by Proposition 2.2.

Proposition 2.15. For \( v = (v_0, v_1, \ldots, v_n) \) in \( \mathbb{R}^{n+1} \) let \( \tilde{v} \) denote the vector \( (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \). A probability measure \( \mu \) on \( \mathbb{R}^n \) satisfies IC(\( \beta \)) if and only if for any nonempty \( V \subset \mathbb{R}^{n+1} \) and a bounded measurable function \( f \) on \( \mathbb{R}^n \),
\[
\int_{\mathbb{R}^n} e^{f \cdot \psi_V}d\mu \mid_{\mathbb{R}^n} \int e^{-f}d\mu \leq \sup_{v \in V} \left\{ e^{v_0} \int_{\mathbb{R}^n} e^{\beta(x,\tilde{v})}d\mu(x) \right\},
\]
where
\[
\psi_V(x) := \sup_{v \in V} \{v_0 + \langle x, \tilde{v} \rangle\}.
\]
Proof. If we put \( V = \{(v_0, \tilde{v}):(v_0 = -\Lambda_\mu(\beta \tilde{v}))\} \), then the right-hand side is equal to 1 and \( \psi_V(x) = \Lambda^*_\mu(x/\beta) \), so if \( \mu \) satisfies (13) for this \( V \), it satisfies IC(\( \beta \)).

On the other hand, suppose \( \mu \) satisfies IC(\( \beta \)). Take an arbitrary nonempty set \( V \). If the right-hand side supremum is infinite, the inequality is obvious, so we may assume it is equal to some \( s < \infty \). This means that for any \((v_0, \tilde{v}) \in V\) we have \( v_0 + \Lambda_\mu(\beta \tilde{v}) \leq \log s \), that is \( v_0 \leq \log s - \Lambda_\mu(\beta \tilde{v}) \). Thus

\[
\psi_V(x) = \sup_{v \in V} \{v_0 + (x, \tilde{v})\} \leq \log s + \sup_{v \in V} \{(x, \tilde{v}) - \Lambda_\mu(\beta \tilde{v})\}
\]

\[
\leq \log s + \sup_{\tilde{v} \in \mathbb{R}^n} \{(x, \tilde{v}) - \Lambda_\mu(\beta \tilde{v})\} = \log s + \Lambda^*_\mu(x/\beta),
\]

which in turn means from IC(\( \beta \)) that the left hand side is no larger than \( s \).

**Proposition 2.16.** Let \( L: \mathbb{R}^n \rightarrow \mathbb{R}^k \) be a linear map and suppose that a probability measure \( \mu \) on \( \mathbb{R}^n \) satisfies IC(\( \beta \)). Then the probability measure \( \mu \circ L^{-1} \) satisfies IC(\( \beta \)).

**Proof.** For any set \( V \subset \mathbb{R} \times \mathbb{R}^k \) and any function \( f: \mathbb{R}^k \rightarrow \mathbb{R} \) put \( \tilde{f}(x) := f(L(x)) \) and \( \bar{V} := \{(v_0, L^*(\tilde{v})): (v_0, \tilde{v}) \in V\} \), where \( L^* \) is the Hermitian conjugate of \( L \). Then direct calculation shows \( \psi_{\tilde{V}}(L(x)) = \psi_{V}(x) \) and \( f\psi_{\tilde{V}}(L(x)) = f\psi_{V}(x) \), thus

\[
\int_{\mathbb{R}^k} e^{\tilde{f}\psi_{V}} d(\mu \circ L^{-1}) \leq \int_{\mathbb{R}^n} e^{f\psi_{V}} d\mu
\]

and

\[
\int_{\mathbb{R}^k} e^{-\tilde{f}} d(\mu \circ L^{-1}) = \int_{\mathbb{R}^n} e^{-f} d\mu
\]

and finally

\[
\sup_{v \in V} \left\{ e^{v_0} \int_{\mathbb{R}^k} e^{\beta(x, \tilde{v})} d(\mu \circ L^{-1}) \right\} = \sup_{v \in \bar{V}} \left\{ e^{v_0} \int_{\mathbb{R}^n} e^{\beta(x, \tilde{v})} d\mu \right\},
\]

which substituted into (13) gives the thesis.

**Proposition 2.17.** For any \( x \in \mathbb{R} \),

\[
\frac{1}{5} \min(x^2, |x|) \leq \Lambda^*_\nu(x) \leq \min(x^2, |x|),
\]

in particular the measure \( \nu \) satisfies IC(9).

**Proof.** Direct calculation shows that \( \Lambda_\nu(x) = -\ln(1 - x^2) \) for \( |x| < 1 \) and \( \Lambda^*_\nu(x) = \sqrt{1 + x^2} - 1 - \ln \left( \frac{\sqrt{1 + x^2} + 1}{2} \right) \).

Since \( a/2 \leq a - \ln(1 + a/2) \leq a \) for \( a \geq 0 \), we get \( \frac{1}{2}(\sqrt{1 + x^2} - 1) \leq \Lambda^*_\nu(x) \leq \sqrt{1 + x^2} - 1 \). Finally

\[
\min(x, |x|^2) \geq \sqrt{1 + x^2} - 1 = \frac{x^2}{\sqrt{1 + x^2} + 1} \geq \frac{1}{\sqrt{2} + 1} \min(|x|, x^2).
\]

The last statement follows by Theorem 2.5, since \( \min((x/9)^2, |x|/9) \leq w(x) \).
2.4 Logarithmically concave product measures

A measure \( \mu \) on \( \mathbb{R}^n \) is logarithmically concave (log–concave for short) if for all nonempty compact sets \( A, B \) and \( t \in [0,1] \),

\[
\mu(tA + (1-t)B) \geq \mu(A)^t \mu(B)^{1-t}.
\]

By Borell’s theorem \([7]\) a measure \( \mu \) on \( \mathbb{R}^n \) with a full–dimensional support is logarithmically concave if and only if it is absolutely continuous with respect to the Lebesgue measure and has a logarithmically concave density, i.e. \( d\mu(x) = e^{h(x)} dx \) for some concave function \( h : \mathbb{R}^n \to [-\infty, \infty) \).

Note that if \( \mu \) is a probabilistic, symmetric and log–concave measure on \( \mathbb{R}^n \), then both \( \Lambda_\mu \) and \( \Lambda_\mu^* \) are convex and symmetric, and \( \Lambda_\mu(0) = \Lambda_\mu^*(0) = 0 \).

Recall also that a probability measure \( \mu \) on \( \mathbb{R}^n \) is called isotropic if

\[
\int \langle u, x \rangle d\mu(x) = 0 \quad \text{and} \quad \int \langle u, x \rangle^2 d\mu(x) = |u|^2 \quad \text{for all} \quad u \in \mathbb{R}^n.
\]

It is easy to check that for any measure \( \mu \) with a full–dimensional support there exists a linear map \( L \) such that \( \mu \circ L^{-1} \) is isotropic.

The next theorem (with a different universal, but rather large constant) may be deduced from the results of Gozlan \([10]\). We give the following, relatively short proof for the sake of completeness.

**Theorem 2.18.** Any symmetric log-concave measure on \( \mathbb{R} \) satisfies IC(48).

**Proof.** Let \( \mu \) be a symmetric log–concave probability measure on \( \mathbb{R} \), we may assume \( \mu \) is isotropic by Proposition \([2,10]\). Denote the density of \( \mu \) by \( g(x) \) and let the tail function be \( \mu[x,\infty) = e^{-h(x)} \).

Let

\[
a := \inf \{ x > 0 : g(x) \leq e^{-1}g(0) \},
\]

then \( g(x) \leq e^{-x/a}g(0) \) for \( x > a \) and \( g(x) \geq e^{-x/a}g(0) \) for \( x \in [0,a) \). Therefore

\[
\int_0^\infty g(x)dx \leq ag(0) + \int_a^\infty g(0)e^{-x/a}dx,
\]

that is

\[
1/2 \leq ag(0)(1 + e^{-1}). \quad (14)
\]

We also have

\[
\int_0^a x^2g(0)e^{-x/a}dx \leq \int_0^\infty x^2g(x)dx,
\]

so

\[
a^3g(0)(2 - 5e^{-1}) \leq 1/2. \quad (15)
\]

From (14) and (15) we get in particular

\[
\frac{1}{8} \leq \sqrt{\frac{e^2(2e - 5)}{4(e + 1)^3}} \leq g(0).
\]
Let $T: \mathbb{R} \to \mathbb{R}$ be a function such that $\nu(-\infty, x) = \mu(-\infty, Tx)$. Then $\mu = \nu \circ T^{-1}$, $T$ is odd and concave on $[0, \infty)$. In particular, $|Tx - Ty| \leq 2|T(x - y)|$ for all $x, y \in \mathbb{R}$.

Notice that $T'(0) = 1/(2g(0)) \leq 4$, thus by concavity of $T$, $Tx \leq 4x$ for $x \geq 0$. Moreover for $x \geq 0$, $h(Tx) = x + \ln 2$.

Define

$$\tilde{h}(x) := \begin{cases} x^2 & \text{for } |x| \leq 2/3 \\ \max\{4/9, h(x)\} & \text{for } |x| > 2/3. \end{cases}$$

We claim that $(\mu, \tilde{h}(\cdot))$ has property $(\tau)$. Notice that $\tilde{h}(\frac{Tx}{48}) \leq \tilde{h}(\frac{T|x|}{24})$ so by Proposition 2.3 it is enough to check that

$$\tilde{h}\left(\frac{Tx}{24}\right) \leq w(x) \text{ for } x \geq 0,$$

where $w(x)$ is as in Theorem 2.5. We have two cases.

i) $Tx \leq 16$, then

$$\tilde{h}\left(\frac{Tx}{24}\right) = \left(\frac{Tx}{24}\right)^2 \leq \min\left\{\frac{4}{9}, \left(\frac{x}{6}\right)^2\right\} \leq w(x).$$

ii) $Tx \geq 16$, then $x \geq 4$ and

$$\tilde{h}\left(\frac{Tx}{24}\right) = \max\left\{\frac{4}{9}, h\left(\frac{Tx}{24}\right)\right\} \leq \max\left\{\frac{4}{9}, h\left(\frac{Tx}{24}\right)\right\} = \max\left\{\frac{4}{9}, \frac{x + \ln 2}{24}\right\} \leq \frac{x}{9} \leq w(x).$$

So (16) holds in both cases.

To conclude we need to show that $\Lambda^*_\mu(x) \leq \tilde{h}(x)$. For $|x| \leq 2/3$ it follows from the more general Proposition 3.3 below. Notice that for any $t, x \geq 0$, $\Lambda_\mu(t) \geq tx + \ln \mu(x, \infty) = tx - h(x)$, hence

$$\Lambda^*_\mu(x) = \Lambda^*_\mu(|x|) = \sup_{t \geq 0} \left\{t|x| - \Lambda_\mu(t)\right\} \leq h(|x|) \leq \tilde{h}(x)$$

for $|x| > 2/3$.

Using Corollary 2.14 we get

**Corollary 2.19.** Any symmetric, log–concave product probability measure on $\mathbb{R}^n$ satisfies IC$(48)$.

We expect that in fact a more general fact holds.

**Conjecture 1.** Any symmetric log–concave probability measure satisfies IC(C) with a uniform constant C.
3 Concentration inequalities.

3.1 $L_p$-centroid bodies and related sets

**Definition 3.1.** Let $\mu$ be a probability measure on $\mathbb{R}^n$, for $p \geq 1$ we define the following sets

$$M_p(\mu) := \{ v \in \mathbb{R}^n : \int |\langle v, x \rangle|^p d\mu(x) \leq 1 \}.$$ 

$$Z_p(\mu) := (M_p(\mu))^p = \{ x \in \mathbb{R}^n : |\langle v, x \rangle|^p \leq \int |\langle v, y \rangle|^p d\mu(y) \text{ for all } v \in \mathbb{R}^n \}.$$ 

and for $p > 0$ we put

$$B_p(\mu) := \{ v \in \mathbb{R}^n : \Lambda^*_{\mu}(v) \leq p \}.$$ 

Sets $Z_p(\mu_K)$ for $p \geq 1$, when $\mu_K$ is the uniform distribution on the convex body $K$ are called $L_p$-centroid bodies of $K$, their properties were investigated in [18].

**Proposition 3.2.** For any symmetric probability measure $\mu$ on $\mathbb{R}^n$ and $p \geq 1$, $Z_p(\mu) \subset 2^{1/p} B_p(\mu)$.

**Proof.** Let us take $v \in Z_p(\mu)$, we need to show that $\Lambda^*_{\mu}(v/(2^{1/p}e)) \leq p$, that is

$$\langle u, v \rangle \leq \Lambda_{\mu}(u) \leq p \text{ for all } u \in \mathbb{R}^n.$$ 

Let us fix $u \in \mathbb{R}^n$ with $\int |\langle u, x \rangle|^p d\mu(x) = \beta^p$, then $u/\beta \in M_p(\mu)$. We will consider two cases.

i) $\beta \leq 2^{1/p} e$. Then, since $\Lambda_{\mu}(u) \geq \int \langle u, x \rangle d\mu(x) = 0$,

$$\frac{\langle u, v \rangle}{2^{1/p} e} - \Lambda_{\mu}(u) \leq \frac{\beta}{2^{1/p} e} \left\langle \frac{u}{\beta}, v \right\rangle \leq p \cdot 1.$$ 

ii) $\beta > 2^{1/p} e$. We have

$$\int e^{\langle u, x \rangle} d\mu(x) \geq \int \left| e^{\langle u, x \rangle} \right|^p I_{\{\langle u, x \rangle \geq 0\}} d\mu(x) \geq \int \frac{\langle u, x \rangle}{p} I_{\{\langle u, x \rangle \geq 0\}} d\mu(x) \geq \frac{1}{2} \int \left| \frac{\langle u, x \rangle}{p} \right|^p d\mu(x),$$ 

thus

$$\int e^{2^{1/p} e \langle u, x \rangle / \beta} d\mu(x) \geq \frac{1}{2} \int \left| \frac{2^{1/p} e \langle u, x \rangle}{\beta} \right|^p d\mu(x) = e^p.$$ 

Hence $\Lambda_{\mu}(2^{1/p} e u / \beta) \geq p$ and $\Lambda_{\mu}(u) \geq \frac{\beta}{2^{1/p} e} \Lambda_{\mu}(2^{1/p} e u / \beta) \geq \frac{\beta}{2^{1/p} e}$. Therefore

$$\frac{\langle u, v \rangle}{2^{1/p} e} - \Lambda_{\mu}(u) \leq \frac{\beta}{2^{1/p} e} \left\langle \frac{u}{\beta}, v \right\rangle - \frac{\beta}{2^{1/p} e} \leq 0.$$ 

$\square$
Proposition 3.3. If $\mu$ is a symmetric, isotropic probability measure on $\mathbb{R}^n$, then $\min\{1, \Lambda^*_\mu(u)\} \leq |u|^2$ for all $u$, in particular
\[ \sqrt{p}B_2^n \subset B_p(\mu) \text{ for } p \in (0,1). \]

Proof. Using the symmetry and isotropicity of $\mu$, we get
\[ \int e^{\langle u, x \rangle} d\mu(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \int \langle u, x \rangle^{2k} d\mu(x) \geq 1 + \sum_{k=1}^{\infty} \frac{|u|^{2k}}{(2k)!} = \cosh(|u|). \]
Hence for $|u| < 1$,
\[ \Lambda^*_\mu(u) \leq \mathcal{L}(\ln \cosh)(|u|) = \frac{1}{2} \left[ (1 + |u|) \ln(1 + |u|) + (1 - |u|) \ln(1 - |u|) \right] \leq |u|^2, \]
where to get the last inequality we used $\ln(1 + x) \leq x$ for $x > -1$.

3.2 $\alpha$-regular measures.
To establish inclusions opposite to those in the previous subsection, we introduce the following property:

Definition 3.4. We say that a measure $\mu$ on $\mathbb{R}^n$ is $\alpha$-regular if for any $p \geq q \geq 2$ and $v \in \mathbb{R}^n$,
\[ \left( \int |\langle v, x \rangle|^p d\mu(x) \right)^{1/p} \leq \alpha \frac{p}{q} \left( \int |\langle v, x \rangle|^q d\mu(x) \right)^{1/q}. \]

Proposition 3.5. If $\mu$ is $\alpha$-regular for some $\alpha \geq 1$, then for any $p \geq 2$,
\[ B_p(\mu) \subset 4e\alpha \mathcal{Z}_p(\mu). \]

Proof. First we will show that
\[ u \in \mathcal{M}_p(\mu) \Rightarrow \Lambda\mu\left( \frac{pu}{2e\alpha} \right) \leq p. \quad (17) \]
Indeed if we fix $u \in \mathcal{M}_p(\mu)$ and put $\tilde{u} := \frac{pu}{2e\alpha}$, then
\[ \left( \int |\langle \tilde{u}, x \rangle|^k d\mu(x) \right)^{1/k} \leq \frac{p}{2e\alpha} \left( \int |\langle u, x \rangle|^k d\mu(x) \right)^{1/k} \leq \left\{ \begin{array}{ll} \frac{p}{2e\alpha} & k \leq p \\ \frac{k}{2e\alpha} & k > p. \end{array} \right. \]
Hence
\[ \int e^{\langle \tilde{u}, x \rangle} d\mu(x) \leq \int e^{\langle \tilde{u}, x \rangle} d\mu(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \int |\langle \tilde{u}, x \rangle|^k d\mu(x) \\ \leq \sum_{k \leq p} \frac{1}{k!} \left| \frac{p}{2e\alpha} \right|^k + \sum_{k > p} \frac{1}{k!} \left| \frac{k}{2e\alpha} \right|^k \leq e^{\frac{p}{2e\alpha}} + 1 \leq e^p \]
and (17) follows.

Take any \( v \notin 4 \alpha Z_p(\mu) \), then we may find \( u \in M_p(\mu) \) such that \( \langle v, u \rangle > 4 \alpha \) and obtain

\[
\Lambda^*_\mu(v) \geq \langle v, \frac{p u}{2e\alpha} \rangle - \Lambda_\mu \left( \frac{p u}{2e\alpha} \right) > \frac{p}{2e\alpha} - p = p.
\]

\[ \Box \]

**Proposition 3.6.** If \( \mu \) is symmetric, isotropic \( \alpha \)-regular for some \( \alpha \geq 1 \), then

\[
\Lambda^*_\mu(u) \geq \min \left\{ \frac{|u|}{2e\alpha}, \frac{|u|^2}{2e^2} \right\},
\]

in particular

\[
B_p(\mu) \subset \max \{2\alpha e, \alpha e \sqrt{2p} \} B_2^n \text{ for all } p > 0.
\]

**Proof.** We have by the symmetry, isotropicity and regularity of \( \mu \),

\[
\int e^{\langle u, x \rangle} d\mu(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int \langle u, x \rangle^{2k} d\mu(x) \leq 1 + \frac{|u|^2}{2} + \sum_{k=2}^{\infty} \frac{(\alpha |u|)^{2k}}{(2k)!} \leq 1 + \frac{|u|^2}{2} + \sum_{k=2}^{\infty} \left( \frac{\alpha |u|}{2} \right)^{2k}.
\]

Hence if \( \alpha e |u| \leq 1, \)

\[
\int e^{\langle u, x \rangle} d\mu(x) \leq 1 + \frac{|u|^2}{2} + \frac{4}{3} \left( \frac{\alpha |u|}{2} \right)^4 \leq 1 + \alpha^2 e^2 |u|^2 + \frac{\alpha |u|^4}{2} \leq e^{\alpha^2 e^2 |u|^2/2}
\]

so \( \Lambda_\mu(u) \leq \alpha^2 e^2 |u|^2/2 \) for \( \alpha e |u| \leq 1 \). Thus \( \Lambda^*_\mu(u) \geq \min \left\{ \frac{|u|}{2e\alpha}, \frac{|u|^2}{2e^2} \right\} \) for all \( u \).

\[ \Box \]

**Remark 3.7.** We always have for \( p \geq q, M_p(\mu) \subset M_q(\mu) \) and \( Z_p(\mu) \subset Z_q(\mu) \).

If the measure \( \mu \) is \( \alpha \)-regular, then \( M_q(\mu) \subset \frac{q}{p} M_p(\mu) \) and \( Z_p(\mu) \subset \frac{q}{p} Z_q(\mu) \) for \( p \geq q \geq 2 \). Moreover for any symmetric measure \( \mu \), \( \Lambda^*_\mu(0) = 0 \), hence by the convexity of \( \Lambda^*_\mu \), \( B_q(\mu) \subset B_p(\mu) \subset \frac{p}{q} B_q(\mu) \) for all \( p \geq q > 0 \).

**Proposition 3.8.** Symmetric log-concave measures are 1-regular.

**Proof.** If \( X \) is distributed according to a symmetric, log-concave measure \( \mu \) and \( u \in \mathbb{R}^n \), then the random variable \( S = \langle u, X \rangle \) has a log-concave symmetric distribution on the real line. We need to show that \( (\mathbb{E}|S|^p)^{1/p} \leq \frac{2}{q} (\mathbb{E}|S|^q)^{1/q} \) for \( p \geq q \geq 2 \). The proof of Remark 5 in [14] shows that

\[
(\mathbb{E}|S|^p)^{1/p} \leq \frac{(\Gamma(p+1))^{1/p}}{(\Gamma(q+1))^{1/q}} (\mathbb{E}|S|^q)^{1/q},
\]
so it is enough to show that the function \( f(x) := \frac{1}{x}(\Gamma(x+1))^{1/x} \) is nonincreasing on \([2, \infty)\). Binet’s form of the Stirling formula (cf. [1, Theorem 1.6.3]) gives

\[
\Gamma(x+1) = x\Gamma(x) = \sqrt{2\pi x^{x+1/2}e^{-x+\mu(x)}},
\]

where \( \mu(x) = \int_0^\infty \arctg(t/x)(e^{2\pi t} - 1)^{-1}dt \) is decreasing function. Thus

\[
\ln f(x) = \frac{\mu(x)}{x} + \frac{\ln(2\pi x)}{2x} - 1
\]

is indeed nonincreasing on \([2, \infty)\).

\[\square\]

Let us introduce the following notion:

**Definition 3.9.** We say that a measure \( \mu \) satisfies the concentration inequality with constant \( \beta \) (CI(\( \beta \)) in short) if

\[
\forall p \geq 2, \forall A \in B(\mathbb{R}^n) \quad \mu(A) \geq \frac{1}{2} \implies 1 - \mu(A + \beta Z_p(\mu)) \leq e^{-p}(1 - \mu(A)). \tag{18}
\]

This definition is motivated by the following Corollary:

**Corollary 3.10.** Let \( \mu \) be an \( \alpha \)–regular symmetric and isotropic probability measure with \( \alpha \geq 1 \). Then

i) If \( \mu \) satisfies IC(\( \beta \)), then \( \mu \) satisfies CI(8e\( \alpha \beta \)),

ii) If \( \mu \) satisfies CI(\( \beta \)) and additionally satisfies Cheeger’s inequality (12) with constant \( 1/\gamma \), then \( \mu \) satisfies IC(36 min\{6e\( \beta \),\( \gamma \}\}).

**Proof.** By Remark 3.7, Proposition 2.4 and the definition of \( B_p(\mu) \) we have

\[
\mu(A + 2\beta B_p(\mu)) \geq \mu(A + \beta B_{2p}(\mu)) \geq 1 - e^{-p}(1 - \mu(A)),
\]

so the first part of the statement immediately follows by Proposition 3.5.

On the other hand, if \( \mu \) satisfies CI(\( \beta \)), then by Remark 3.7 and Proposition 3.2 we have for \( \mu(A) \geq 1/2 \) and \( p \geq 1 \)

\[
e^{-p}(1 - \mu(A)) > e^{-2p}(1 - \mu(A)) \geq 1 - \mu(A + \beta Z_{2p}(\mu)) \geq 1 - \mu(A + e^{2^{1/2}p} B_{2p}(\mu)) \geq 1 - \mu(A + 3\beta B_p(\mu)).
\]

By Proposition 2.7 this implies property (11). Additionally \( \Lambda^*_p \) is convex, symmetric and \( \Lambda^*_p(0) = 0 \). Finally, from Proposition 3.3 we have \( \min\{1, \Lambda^*_p(u)\} \leq |u|^2 \). Thus, from Proposition 2.9 we get the second part of the statement.

\[\square\]

By Proposition 2.7 in the definition 3.9 we could use the equivalent condition \( \mu(A + \beta Z_p(\mu)) \geq \min\{e^p \mu(A), 1/2\} \). The next proposition shows that for log-concave measures these conditions are satisfied for large \( p \) and for small sets.
Proposition 3.11. Let \( \mu \) be a symmetric log-concave probability measure on \( \mathbb{R}^n \) and \( c \in (0, 1] \). Then

\[
\mu \left( A + 40 \frac{c}{e} Z_p(\mu) \right) \geq \frac{1}{2} \min \{e^p \mu(A), 1\}
\]

for \( p \geq cn \) or \( \mu(A) \leq e^{-cn} \).

Proof. Using a standard volumetric estimate for any \( r > 0 \) we may choose \( S \subset M_r(\mu) \) with \( \# S \leq 5^n \) such that \( M_r(\mu) \subset \bigcup_{u \in S} (u + \frac{1}{2} M_r(\mu)) \). Then for \( t > 0 \),

\[
x \notin t Z_r(\mu) \Rightarrow \max_{u \in S} \langle u, x \rangle \geq \frac{t}{2}
\]

and by the Chebyshev inequality,

\[
\mu(\mathbb{R}^n \setminus t Z_r(\mu)) \leq \sum_{u \in S} \mu \left\{ x : \langle u, x \rangle \geq \frac{t}{2} \right\} \leq \sum_{u \in S} \left( \frac{2}{7} \right)^r \int \langle u, x \rangle^r \, d\mu \leq \frac{1}{2} 5^n \left( \frac{2}{7} \right)^r .
\]

Let \( \mu(A) = e^{-q} \), we will consider two cases.

i) \( p \geq \max \{q, cn\} \). Then by Remark 3.7,

\[
\mu(30 c - 1 Z_p(\mu)) > \mu(30 Z_{\max(p,n)}(\mu)) \geq 1 - \frac{1}{2} e^{-\max(p,n)} \geq 1 - \mu(A),
\]

so \( A \cap 30 c^{-1} Z_p(\mu) \neq \emptyset \), hence \( 0 \in A + 30 c^{-1} Z_p(\mu) \) and

\[
\mu(A + 40 c^{-1} Z_p(\mu)) \geq \mu(10 c^{-1} Z_p(\mu)) \geq 1/2.
\]

ii) \( q \geq \max \{p, cn\} \). Let \( \tilde{q} := \max \{q, n\} \)

\[
\tilde{A} := A \cap 30 c^{-1} Z_q(\mu),
\]

we have as in i), \( \mu(30 c^{-1} Z_q(\mu)) > 1 - \tilde{q}^{-1}/2 \), thus \( \mu(\tilde{A}) \geq \mu(A)/2 \). Moreover,

\[
\left( 1 - \frac{p}{q} \right) \tilde{A} \subset A - \frac{p}{q} 30 c^{-1} Z_q(\mu) \subset A + 30 c^{-1} Z_p(\mu)
\]

and

\[
\mu(A + 40 c^{-1} Z_p(\mu)) \geq \mu \left( \left( 1 - \frac{p}{q} \right) \tilde{A} + \frac{p}{q} 10 Z_q(\mu) \right)
\]

\[
\geq \mu \left( \left( 1 - \frac{p}{q} \right) \tilde{A} + \frac{p}{q} 10 Z_q(\mu) \right) \geq \mu(\tilde{A})^{1-\frac{p}{q}} \mu(10 Z_q)^{\frac{p}{q}}
\]

\[
\geq \left( \frac{1}{2} \mu(A) \right)^{1-\frac{p}{q}} \left( \frac{1}{2} \right)^{\frac{p}{q}} \geq \frac{1}{2} \mu(A) \mu(A)^{-\frac{p}{q}} = \frac{1}{2} e^{-p \mu(A)} .
\]

The previous facts motivate the following.

Conjecture 2. Any symmetric log–concave probability measure satisfies CI(C) for some universal constant \( C \).
Proposition 3.10 shows that Conjecture 1 implies Conjecture 2. Both hypotheses would be equivalent provided that the following conjecture of Kannan, Lovász and Simonovits holds.

**Conjecture 3** (Kannan–Lovász–Simonovits [11]). There exists an absolute constant $C$ such that any symmetric isotropic log-concave probability measure satisfies Cheeger’s inequality with constant $1/C$.

### 3.3 Comparison of weak and strong moments

**Proposition 3.12.** Suppose that a probability measure $\mu$ on $\mathbb{R}^n$ is $\alpha$-regular and satisfies $\text{CI}(\beta)$. Then for any norm $\| \cdot \|$ on $\mathbb{R}^n$ and $p \geq 2$,

$$
\left( \int \|x\| - \text{Med}_\mu(\|x\|)^p d\mu \right)^{1/p} \leq 2\alpha \beta \sup_{\|u\|_* \leq 1} \left( \int |\langle u, x \rangle|^p d\mu \right)^{1/p},
$$

where $\| \cdot \|_*$ denotes the norm dual to $\| \cdot \|$.

**Proof.** For $p \geq 2$ we define

$$
m_p := \sup_{\|u\|_* \leq 1} \left( \int |\langle u, x \rangle|^p d\mu \right)^{1/p}.
$$

Let $M := \text{Med}_\mu(\|x\|)$, $A := \{x: \|x\| \leq M\}$ and $\tilde{A} := \{x: \|x\| \geq M\}$. Then $\mu(A), \mu(\tilde{A}) \geq 1/2$ so by CI($\beta$) and Remark 3.7

$$
\forall t \geq p \quad 1 - \mu\left( A + \beta \frac{\alpha t}{p} Z_p(\mu) \right) \leq \frac{1}{2} e^{-t}, \quad 1 - \mu(\tilde{A} + \beta \frac{\alpha t}{p} Z_p(\mu)) \leq \frac{1}{2} e^{-t}.
$$

Let $y \in Z_p$, then there exists $u \in \mathbb{R}^n$ with $\|u\|_* \leq 1$ such that

$$
\|y\| = \langle u, y \rangle \leq \left( \int |\langle u, x \rangle|^p d\mu(x) \right)^{1/p} \leq m_p,
$$

hence $\|x\| \leq M + tm_p$ for $x \in A + tZ_p(\mu)$. Thus for $t \geq p$,

$$
\mu\{x: \|x\| \geq M + \frac{\alpha \beta t}{p} m_p \} \leq 1 - \mu\left( A + \beta \frac{\alpha t}{p} Z_p(\mu) \right) \leq \frac{1}{2} e^{-t}.
$$

In a similar way we show $\|x\| \geq M - tm_p$ for $x \in \tilde{A} + tZ_p(\mu)$ and $\mu\{x: \|x\| \leq M - \alpha \beta tm_p/p \} \leq e^{-t/2}$, therefore

$$
\mu\{x: \|x\| - M \geq \frac{\alpha \beta t}{p} m_p \} \leq e^{-t} \text{ for } t \geq p.
$$
Now integrating by parts,

\[
\left( \int \|x\| - M \| d\mu \right)^{1/p} \leq \frac{\alpha_2 m_p}{p} \left( p + \left( p \int_p^\infty t^{p-1} \mu \left\{ \|x\| - M \geq \frac{\alpha_2 t}{p} m_p \right\} dt \right)^{1/p} \right)
\]

\[
\leq \frac{\alpha_2 m_p}{p} \left( p + \left( p \int_p^\infty t^{p-1} e^{-t} dt \right)^{1/p} \right)
\]

\[
\leq \frac{\alpha_2 m_p (1 + \Gamma(p+1)^{1/p})}{p} \leq 2\alpha_2 m_p.
\]

Remark 3.13. Under the assumptions of Proposition 3.12 by the triangle inequality we get for \( \gamma = 4\alpha_2 \beta \), \( \forall p \geq q \geq 2 \)

\[
\left( \int \|x\| - \left( \int \|x\|^q d\mu \right)^{1/q} \right)^{1/p} \leq \gamma \sup \|u\| \leq 1 \left( \int |\langle u, x \rangle|^{p} d\mu \right)^{1/p}.
\]

This motivates the following definition.

Definition 3.14. We say that a probability measure \( \mu \) on \( \mathbb{R}^n \) has comparable weak and strong moments with the constant \( \gamma \) (CWSM(\( \gamma \)) in short) if (19) holds for any norm \( \| \cdot \| \) on \( \mathbb{R}^n \).

Conjecture 4. Every symmetric log-concave probability on \( \mathbb{R}^n \) measure satisfies CWSM(\( C \)).

Proposition 3.15. Let \( \mu \) be an isotropic, probability measure on \( \mathbb{R}^n \) satisfying CWSM(\( \gamma \)). Then

i) \( \int \|x\|_2 - \sqrt{\gamma} d\mu(x) \leq \gamma^2 \),

ii) if \( \mu \) is also \( \alpha \)-regular, then for all \( p > 2 \),

\[
\left( \int \|x\|_2^p d\mu \right)^{1/p} \leq \sqrt{\gamma} + \frac{\gamma\alpha_2}{2}.
\]

Proof. Notice that \( \int \|x\|_2^2 d\mu = n \) and \( \|u\|_2 = \|u\|_2 \). Hence i) follows directly from (19) with \( p = q = 2 \). Moreover (19) with \( q = 2 \) implies

\[
\left( \int \|x\|_2^p d\mu \right)^{1/p} \leq \sqrt{\gamma} + \sup_{\|u\|_2 \leq 1} \left( \int |\langle u, x \rangle|^{p} d\mu \right)^{1/p} \leq \sqrt{\gamma} + \frac{\gamma\alpha_2}{2}
\]

by the \( \alpha \)-regularity and isotropicity of \( \mu \). \( \square \)

Remark 3.16. Property i) plays the crucial role in the Klartag proof of the central limit theorem for convex bodies [12]. Paouris [18] showed that moments of the Euclidean norm for symmetric isotropic log-concave measures are bounded by \( C(p + \sqrt{n}) \). Thus Conjecture 4 would imply both Klartag CLT (with the optimal speed of convergence) and Paouris concentration.
We conclude this section with the estimate that shows comparison of weak and strong moments for any probability measure and $p > n/C$.

**Proposition 3.17.** For any $p > 0$ we have

$$\left( \int \left\| x \right\| - \operatorname{Med}_\mu \left( \left\| x \right\| \right) \right)^{1/p} \leq \left( \int \left\| x \right\|^p d\mu \right)^{1/p} \leq 2 \cdot 5^{n/p} \sup_{\left\| u \right\| \leq 1} \left( \int \left\langle u, x \right\rangle \right)^{1/p}. $$

**Proof.** As in the proof of Proposition 3.11 we can find $u_1, \ldots, u_N$ with $\left\| u_i \right\|_* \leq 1$, $N \leq 5^n$ such that $\left\| x \right\| \leq 2 \max_{i \leq N} \left\langle u_i, x \right\rangle$ for all $x$. Then

$$\int \left\| x \right\|^p d\mu \leq 2^p \sum_{i \leq N} \left\| \left\langle u_i, x \right\rangle \right\|^p d\mu \leq 2^p 5^n \sup_{\left\| u \right\| \leq 1} \int \left\| \left\langle u_i, x \right\rangle \right\|^p d\mu.$$

Moreover

$$\int_{\left\{ \left\| x \right\| \geq M \right\}} \left( \left\| x \right\| - M \right)^p d\mu(x) \leq \int_{\left\{ \left\| x \right\| \geq M \right\}} \left( \left\| x \right\|^p - MP \right) d\mu(x) \leq \int \left\| x \right\|^p d\mu(x) - \frac{1}{2} MP$$

and

$$\int_{\left\{ \left\| x \right\| < M \right\}} \left( M - \left\| x \right\| \right)^p d\mu(x) \leq MP \mu \left\{ \left\| x \right\| < M \right\} \leq \frac{1}{2} MP. $$

\qed

4 Modified Talagrand concentration for exponential measure

In this section we show that for a set lying far from the origin Talagrand’s two level concentration for the exponential measure may be somewhat improved, namely (for sufficiently large $t$) it is enough to enlarge the set by $tB^n_1$ instead of $tB^n_1 + \sqrt{t} B^n_2$.

**Lemma 4.1.** If $u \geq t > 0$ then for any $i \in \{1, \ldots, n\}$ we have

$$\left| (A + tB^n_1) \cap nB^n_i \cap \{ x: |x_i| \geq u - t \} \right| \geq e^{t/2} \left| A \cap nB^n_1 \cap \{ x: |x_i| \geq u \} \right|.$$

**Proof.** Obviously we may assume that $i = 1$ and $u \leq n$. Let $A_1 := A \cap nB^n_1 \cap \{ x: x_1 \geq u \}$ and $B := \{ x \in B^n_1 : x_1 \geq \sum_{i \geq 2} |x_i| \}$. From the definition of $B$ and $A_1$ we have $A_1 - tB \subset nB^n_1$. On the other hand $B = \{ x : |x_1 - 1/2| + \sum_{i \geq 2} |x_i| \leq 1/2 \}$, so $|B| = 2^{-n} |B^n_1| = (2r_{1,n})^{-n}$. Thus

$$\left| (A_1 + tB^n_1) \cap nB^n_1 \right| \geq \left| (A_1 - tB) \cap nB^n_1 \right| = \left| A_1 - tB \right|.$$
Now let us take
\[ s := \frac{2|A_1|^{1/n}r_{1,n}}{t + 2|A_1|^{1/n}r_{1,n}}. \]
Then we easily check that \(|tB/(1-s)| = |A_1/s|\). Since \(A_1 \subset \{x \in nB^n_1 : x \geq t\}\) we get \(|A_1|^{1/n} \leq (n - t)/r_{1,n}\) and \(s \leq 2(n - t)/(2n - t)\). Now we can use the Brunn-Minkowski inequality to get
\[
|A_1 - tB| = |s \frac{A_1}{s} + (1 - s) \frac{-t}{1 - s} B| \geq |A_1|^{s} |B|^{1-s} = |A_1|^{s}|s| = s^{-n}|A_1|
\]
\[
\geq \left(\frac{2n - t}{2n - 2t}\right)^n |A_1| = \left(\frac{1}{1 - \frac{t}{2n - t}}\right)^n |A_1| \geq e^{\frac{2n-t}{t}}|A_1| \geq e^{t/2}|A_1|.
\]
Notice that \(A_1 + tB^n_1 \subset \{x : x \geq u - t\}\), so we obtain
\[
|(A + tB^n_1) \cap nB^n_1 \cap \{x : x \geq u - t\}| \geq e^{t/2}|A \cap nB^n_1 \cap \{x : x \geq u\}|
\]
in the same way we show
\[
|(A + tB^n_1) \cap nB^n_1 \cap \{x : x \leq -u + t\}| \geq e^{t/2}|A \cap nB^n_1 \cap \{x : x \leq -u\}|
\]
\[\square\]

**Remark 4.2.** A similar result (although with a constant multiplicative factor) can be obtained using the same technique and more calculations for \(n^{1/p}B^n_1\) instead of \(nB^n_1\) for \(p \in [1, 2]\).

**Lemma 4.3.** If \(u \geq t > 0\) then for any \(i \in \{1, \ldots, n\}\) we have
\[
\nu^n((A + tB^n_1) \cap \{x : |x_i| \geq u - t\}) \geq e^{t/2} \nu^n(A \cap \{x : |x_i| \geq u\}).
\]

**Proof.** Take an arbitrary \(k \in \mathbb{N}\). Let \(P : \mathbb{R}^{n+k} \to \mathbb{R}^n\) be the projection onto first \(n\) coordinates. Let \(\rho_k\) be the uniform probability measure on \((n+k)B^{n+k}_1\), and \(\hat{\nu}_k\) the measure defined by \(\hat{\nu}_k(A) = \rho_k(P^{-1}(A))\). Take an arbitrary set \(A \subset \mathbb{R}^n\). Notice that for any set \(C \subset \mathbb{R}^n\) we have
\[
C \cap \{x : |x_i| \geq s\} = P(P^{-1}(C) \cap \{x : |x_i| \geq s\})
\]
and also \(P^{-1}(A) + B^{n+k}_1 \subset P^{-1}(A + B^n_1)\). From Lemma 4.1 we have
\[
\rho_k((P^{-1}(A) + tB^{n+k}_1) \cap \{x : |x_i| \geq u - t\}) \geq e^{t/2} \rho_k(P^{-1}(A) \cap \{x : |x_i| \geq u\}),
\]
and thus
\[
\hat{\nu}_k((A + tB^n_1) \cap \{x : |x_i| \geq u - t\}) \geq e^{t/2} \hat{\nu}_k(A \cap \{x : |x_i| \geq u\}).
\]
When \(k \to \infty\), we have \(\hat{\nu}_k(C) \to \nu^n(C)\) for any set \(C \in \mathcal{B}(\mathbb{R}^n)\). Thus by going to the limit we get the assertion. \[\square\]
\textbf{Proposition 4.4.} For any $t > 0$ and any $n \in \mathbb{N}$ we have
\[\int_{A+tB_1^n} |x|^2 d\nu^n(x) \geq e^{t/2} \int_A (|x| - t\sqrt{n})^2 d\nu^n(x).\]

\textbf{Proof.} Let $A_t = A + tB_1^n$. By Lemma 4.3 we get for any $s \geq 0$ and any $i$:
\[\int_{A_i} I_{\{|x_i| \geq s\}} d\nu^n(x) \geq e^{t/2} \int_{A_i} I_{\{|x_i| \geq s+t\}} d\nu^n(x).\]

Thus
\[\int_{A_t} x_i^2 d\nu^n(x) = \int_{A_t} \int_0^\infty 2s I_{\{|x_i| \geq s\}} ds \, d\nu^n(x) = \int_0^\infty 2s \int_{A_t} I_{\{|x_i| \geq s+t\}} d\nu^n(x)ds \]
\[\geq e^{t/2} \int_0^\infty 2s \int_A I_{\{|x_i| \geq s+t\}} d\nu^n(x)ds = e^{t/2} \int_A 2s I_{\{|x_i| \geq s+t\}} ds \, d\nu^n(x) = e^{t/2} \int_A (|x| - t)^2 d\nu^n(x).\]

To get the assertion it is enough to take the sum over all $i$ and notice that the function $f(y) := (\sqrt{y} - t)_+$ is convex on $[0, \infty)$, hence
\[\sum_{i=1}^n (|x_i| - t)_+^2 = \sum_{i=1}^n f(x_i^2) \geq nf(\frac{1}{n} \sum_{i=1}^n x_i^2) = (|x| - t\sqrt{n})^2_+.\]

\qed

\textbf{Lemma 4.5.} Suppose that $A \subset \{x \in \mathbb{R}^n : |x| \geq 5t\sqrt{n}\}$. Then
\[\nu^n(A + tB_1^n) \geq \frac{1}{8} e^{t/2} \nu^n(A).\]

\textbf{Proof.} Let
\[A_k := A \cap \{x : 5t\sqrt{n} + 2t(k-1) \leq |x| < 5t\sqrt{n} + 2tk\}, \quad k = 1, 2, \ldots.\]

Then $A_k + tB_1^n \subset \{x : 5t\sqrt{n} + t(2k-3) \leq |x| < 5t\sqrt{n} + t(2k+1)\}$, hence
\[\nu^n(A + tB_1^n) \geq \frac{1}{2} \sum_{k \geq 1} \nu(A_k + tB_1^n).\]

From Proposition 4.4 applied for $A_k$ we have
\[\left(5t\sqrt{n} + t(2k+1)\right)^2 \nu^n(A_k + tB_1^n) \geq \int_{A_k + tB_1^n} |x|^2 d\nu^n(x) \]
\[\geq e^{t/2} \int_{A_k} (|x| - t\sqrt{n})^2 d\nu^n(x) \geq e^{t/2} \left(4t\sqrt{n} + 2t(k-1)\right)^2 \nu^n(A_k).\]
Thus

$$\nu^n(A_k + tB_1^n) \geq \left(\frac{4t\sqrt{n} + 2t(k - 1)}{5t\sqrt{n} + t(2k + 1)}\right)^2 e^{t/2} \nu^n(A_k) \geq \frac{1}{4} e^{t/2} \nu^n(A_k).$$

and

$$\nu^n(A + tB_1^n) \geq \frac{1}{2} \sum_{k \geq 1} \frac{1}{4} e^{t/2} \nu^n(A_k) = \frac{1}{8} e^{t/2} \nu^n(A).$$

\[\square\]

**Theorem 4.6.** For any \(A \in B(\mathbb{R}^n)\) and any \(t \geq 10\), either

$$\nu^n((A + tB_1^n) \cap 50\sqrt{n}B_2^n) \geq \frac{1}{2} \nu^n(A)$$

or

$$\nu^n(A + tB_1^n) \geq e^{t/10} \nu^n(A).$$

In particular (20) holds if \(A \cap (50\sqrt{n}B_2^n + tB_1^n) = \emptyset\).

**Proof.** Let \(A_k\) denote \(A + 10kB_1^n\) for \(k = 0, 1, \ldots\). If for any \(0 \leq k \leq \lfloor t/10\rfloor\) we have \(\nu^n(A_k \cap 50\sqrt{n}B_2^n) \geq \nu^n(A)/2\), the thesis is proved. Thus we assume otherwise. Let \(A'_k := A_k \setminus 50\sqrt{n}B_2^n\). From Lemma 4.5 we have

$$\nu^n(A_{k+1}) \geq \nu^n(A'_k + 10B_1^n) \geq \frac{1}{8} e^{5} \nu^n(A'_k) \geq \frac{1}{16} e^{5} \nu^n(A_k) \geq e^2 \nu^n(A_k).$$

By a simple induction we get \(\nu^n(A_k) \geq e^{2k} \nu^n(A)\) for any \(k \leq \lfloor t/10\rfloor\). Thus we get

$$\nu^n(A + tB_1^n) \geq \nu^n(A_{\lfloor t/10\rfloor}) \geq e^{2\lfloor t/10\rfloor} \nu^n(A) \geq e^{t/10} \nu^n(A).$$

\[\square\]

## 5 Uniform measure on \(B_p^n\)

In this section we will prove the infimum convolution property IC(C) for \(B_p^n\) balls. Recall that \(\nu^n_p\) is a product measure, while \(\mu_{p,n}\) denotes the uniform measure on \(r_{p,n}B_p^n\). We have

$$r_{p,n}^{-n} = |B_p^n| = \frac{2^n \Gamma(1 + 1/p)^n}{\Gamma(1 + n/p)} \sim \frac{(2\Gamma(1 + 1/p))^{n(p)} n^{n/p}}{n^{n/p}(\sqrt{n/p} + 1)},$$

where the last part follows from Stirling’s formula. Thus \(r_{p,n} \sim n^{1/p}\).

For \(\nu^n_p\) we have IC(48) by Corollary 2.13. Let us first try to understand what sort of concentration this implies, that is, how does the function \(\Lambda^*\) behave for \(\nu^n_p\).
Proof. We shall use the facts proved in Section 3 to approximate $B_t(\nu_p)$. Note that $\nu_p$ is log-concave (as its density is log-concave) and symmetric. It is 1–regular from Proposition 5.1. Also

$$\sigma_p^2 := \int_{\mathbb{R}} x^2 d\nu_p(x) = \frac{1}{2\gamma_p} \int_{\mathbb{R}} x^2 e^{-|x|^p} dx = \frac{\Gamma(1 + \frac{2}{p})}{3\Gamma(1 + \frac{1}{p})} \sim 1$$

for $p \in [1, \infty)$. The measure $\tilde{\nu}_p$ with the density $\sigma_p d\nu_p(\sigma_p x)$ is isotropic, hence Propositions 3.3 and 3.6 yield $B_t(\tilde{\nu}_p) \sim \sqrt{t} B_1$ for $t \leq 1$. Thus, as $B_t(\nu_p) = \sigma_p B_t(\tilde{\nu}_p)$, we get $B_t(\nu_p) \sim [-\sqrt{t}, \sqrt{t}]$ for $t \leq 1$.

For $t \geq 1$ we have

$$\mathcal{M}_t(\nu_p) = \left\{ u \in \mathbb{R} : \frac{1}{2\gamma_p} \int_{\mathbb{R}} |u|^t |x|^t e^{-|x|^p} dx \leq 1 \right\}$$

$$= \left\{ u \in \mathbb{R} : |u| \leq \left( \frac{\Gamma(1 + \frac{1}{p})}{\Gamma(1 + \frac{t+1}{p})} \right)^{1/t} \sim \{ u \in \mathbb{R} : |u| \leq t^{-1/p} \}. \right.$$}

Thus $Z_t(\nu_p) \sim [-t^{1/p}, t^{1/p}]$ for $t \geq 1$, so by Propositions 3.2 and 3.5, $B_t(\nu_p) \sim [-t^{1/p}, t^{1/p}]$. Hence, for all $t \geq 0$ we have $\{ x : f_p(|x|) \leq t \} \sim \{ x : \Lambda_{\nu_p}^*(x) \leq t \}$, so $\Lambda_{\nu_p}^*(t/C) \leq f_p(t) \leq \Lambda_{\nu_p}^*(Ct)$. As $\Lambda_{\nu_p}^*$ is symmetric, the proof is finished. 

**Corollary 5.2.** For any $t > 0$ and $n \in \mathbb{N}$ we have

$$B_t(\nu_p^n) \sim \left\{ \begin{array}{ll} \sqrt{t} B_2^n + t^{1/p} B_p^n & \text{for } p \in [1, 2] \\sqrt{t} B_2^n \cap t^{1/p} B_p^n & \text{for } p \geq 2. \end{array} \right.$$ 

**Proof.** By Proposition 5.1

$$B_t(\nu_p^n) = \left\{ x \in \mathbb{R}^n : \sum \Lambda_{\nu_p}^*(x_i) \leq t \right\} \sim \left\{ x \in \mathbb{R}^n : \sum f_p(|x_i|) \leq t \right\}.$$ 

Simple calculations show that $\{ x \in \mathbb{R}^n : \sum f_p(|x_i|) \leq t \} \sim t^{1/2} B_2^n + t^{1/p} B_p^n$ for $p \in [1, 2]$ and $\{ x \in \mathbb{R}^n : \sum f_p(|x_i|) \leq t \} \sim t^{1/2} B_2^n \cap t^{1/p} B_p^n$ for $p \geq 2$. 

**Proposition 5.3.** For any $t \in [0, n]$, $p \geq 1$ and $n \in \mathbb{N}$ we have $B_t(\mu_{p,n}) \sim B_t(\nu_p^n)$.

**Proof.** For $t < 1$ we use Propositions 3.3 and 3.6. Both $\mu_{p,n}$ and $\nu_p^n$ are symmetric, log-concave measures, and both can be rescaled as in Proposition 5.1 to be isotropic, thus $B_t(\mu_{p,n}) \sim \sqrt{t} B^n_2 \sim B_t(\nu_p^n)$.

Lemma 6 from [3] gives (after rescaling by $r_{p,n}$),

$$\left( \int |\langle a, x \rangle|^t d\mu_{p,n}(x) \right)^{1/t} \sim \frac{r_{p,n}}{\max\{n, t\}^{1/p}} \left( \int |\langle a, x \rangle|^t d\nu_p^n(x) \right)^{1/t} \quad (21)$$

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for any \( p, t \geq 1 \) and \( a \in \mathbb{R}^n \). Note that as \( r_{p,n} \sim n^{1/p} \), this simply means the equivalence of \( t \)-th moments of \( \mu_{p,n} \) and \( \nu_{p,n} \) for \( t \in [0, n] \). Thus \( \mathcal{M}_t(\mu_{p,n}) \sim \mathcal{M}_t(\nu_{p,n}) \) for \( t \leq n \) and therefore \( B_t(\mu_{p,n}) \sim B_t(\nu_{p,n}) \). \( \square \)

**Remark 5.4.** It is not hard to verify that \( B_t(\mu_{p,n}) \sim r_{p,n}B_p^n \) for \( t \geq n \).

### 5.1 Transports of measure

We are now going to investigate two transports of measure. They will combine to transport a measure with known concentration properties (\( \nu^n \) or \( \nu^n_2 \), that is the exponential or Gaussian measure) to the uniform measure \( \mu_{p,n} \). We will investigate the contractive properties of these transports with respect to various norms. Our motivation is the following:

**Remark 5.5.** Let \( U : \mathbb{R}^n \to \mathbb{R}^n \) be a map such that

\[
\|U(x) - U(y)\|_p^p \geq \delta \|x - y\|_q^q \quad \text{for all} \quad x, y \in A.
\]

Then

\[
U(A + t^{1/q}B_q^n) \supset U(\mathbb{R}^n) \cap (U(A) + \delta t^{1/p}B_p^n).
\]

Analogously if

\[
\|U(x) - U(y)\|_p^p \leq \delta \|x - y\|_q^q \quad \text{for all} \quad x, y \in A
\]

then

\[
U(A + t^{1/q}B_q^n) \subset U(A) + \delta t^{1/p}B_p^n.
\]

**Proof.** Let us prove the first statement, the second proof is almost identical. Suppose \( U(x) \in U(A) + \delta t^{1/p}B_p^n \). Then there exists \( y \in A \) such that \( \|U(x) - U(y)\|_p^p \leq \delta t \). From the assumption we have \( t \geq \|x - y\|_q^q \), which means \( x \in A + t^{1/q}B_q^n \), and \( U(x) \in U(A + t^{1/q}B_q^n) \). \( \square \)

The first transport we introduce is the radial transport \( T_{p,n} \) which transforms the product measure \( \nu^n_p \) onto \( \mu_{p,n} \) – the uniform measure on \( r_{p,n}B_p^n \). We will show this transport is Lipschitz with respect to the \( \ell_p \) norm and Lipschitz on a large set with respect to the \( \ell_2 \) norm for \( p \leq 2 \).

**Definition 5.6.** For \( p \in [1, \infty) \) and \( n \in \mathbb{N} \) let \( f_{p,n} : [0, \infty) \to [0, \infty) \) be given by the equation

\[
\int_0^s e^{-r^p/r^{n-1}}dr = (2\gamma_p)^n \int_0^{f_{p,n}(s)} r^{n-1}dr
\]

and \( T_{p,n}(x) := xf_{p,n}(|x|_p)/|x|_p \) for \( x \in \mathbb{R}^n \).

Let us first show the following simple estimate.

**Lemma 5.7.** For any \( q > 0 \) and \( 0 \leq u \leq q/2 \),

\[
q \int_0^u e^{-t^q/2}dt \leq e^{-u/2^q}(1 + 2u/q).
\]

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Proof. Let
\[ f(u) = e^{-u}u^q \left(1 + \frac{u^q}{q}\right) - q \int_0^u e^{-t}t^{q-1}dt. \]
Then \( f(0) = 0 \) and \( f'(u) = e^{-u}u^q(1 - 2u/q + 2/q) \geq 0 \) for \( 0 \leq u \leq q/2 \).

Now we are ready to state the basic properties of \( T_{p,n} \).

**Proposition 5.8.**

i) The map \( T_{p,n} \) transports the probability measure \( \nu^n_p \) onto the measure \( \mu_{p,n} \).

ii) For all \( t > 0 \) we have \( e^{-\gamma_p t}t^n \geq 2\gamma_p f_{p,n}(t) \) and \( f'_{p,n}(t) \leq (2\gamma_p)^{-1} \leq 1 \).

iii) For any \( t > 0, 0 \leq f_{p,n}(t)/t - f'_{p,n}(t) \leq \min\{1, 2pt^p/n\} \).

iv) The function \( t \mapsto f_{p,n}(t)/t \) is decreasing on \((0, \infty)\) and for any \( s, t > 0 \),
\[
|t^{-1}f_{p,n}(t) - s^{-1}f_{p,n}(s)| \leq (st)^{-1}|s-t|f_{p,n}(s \wedge t) \leq \frac{|s-t|}{\max\{s,t\}}.
\]

**Proof.** The definition of \( T_{p,n} \) directly implies i). Differentiation of (22) gives
\[
e^{-s^n} = (2\gamma_p)^nf_{p,n}^{-1}(s)f'_{p,n}(s).
\]

By (22),
\[
e^{-\gamma_p t^n} \leq n \int_0^t e^{-\gamma_p r^n}dr = (2\gamma_p)^n f_{p,n}(t) \leq n \int_0^t r^{n-1}dr = t^n,
\]
which, when the \( n \)-th root is taken, give the first part of ii).

For the second part of ii) we use (23) and the estimate above to get
\[
f'_{p,n}(s) = e^{-s^n} (2\gamma_p)^{-n} \left(\frac{s}{f_{p,n}(s)}\right)^{n-1} \leq e^{-s^n} (2\gamma_p)^{-n} \left(e^{s^n/n} 2\gamma_p\right)^{-n-1} \\
= e^{-s^n/n} (2\gamma_p)^{-1} \leq (2\gamma_p)^{-1} \leq 1.
\]

To show iii) first notice that by (23) and ii),
\[
\frac{tf'_{p,n}(t)}{f_{p,n}(t)} = \left(\frac{t}{f_{p,n}(t)}\right)^n e^{-\gamma_p t^n} \leq \left(e^{t^n/n} 2\gamma_p\right)^n e^{-\gamma_p t^n} (2\gamma_p)^{-n} = 1,
\]
thus \( f_{p,n}(t)/t - f'_{p,n}(t) \geq 0 \). Moreover by ii), \( f_{p,n}(t)/t - f'_{p,n}(t) \leq f_{p,n}(t)/t \leq 1 \),
so we may assume that \( 2pt^p/n \leq 1 \). By (22) and Lemma 5.7 we obtain
\[
(2\gamma_p)^n f_{p,n}(t) = \frac{n}{p} \int_0^{t^n} e^{-u u^{n/p-1}}du \leq e^{-t^n} t^n (1 + 2pt^p/n).
\]
Thus using again (23) and part ii) we get
\[
\frac{f_{p,n}(t)}{t} - f'_{p,n}(t) = \frac{f_{p,n}(t)}{t} \left(1 - \frac{e^{-t^n}}{(2\gamma_p)^n f_{p,n}(t)}\right) \leq 1 - \left(1 + 2pt^p/n\right)^{-1} \leq \frac{2pt^p}{n}.
\]
By iii) we get \((f_{p,n}(t)/t)\) \(\leq 0\), which proves the first part of iv). For the second part suppose that \(s > t > 0\), then
\[
0 \leq \frac{f_{p,n}(t)}{t} - \frac{f_{p,n}(s)}{s} \leq \frac{f_{p,n}(t)}{s} = \frac{s-t}{st}f_{p,n}(t) \leq \frac{s-t}{s}.
\]

The next Proposition may be also deduced (with different constant) from the more general fact proved in [17].

**Proposition 5.9.** For any \(x, y \in \mathbb{R}^n\) we have \(\|T_{p,n}x - T_{p,n}y\|_p \leq 2\|x - y\|_p\).

**Proof.** Assume \(s := \|x\|_p \geq t := \|y\|_p\), we apply Proposition 5.8 and get
\[
\|T_{p,n}x - T_{p,n}y\|_p = \left(\sum_i |(T_{p,n}x)_i - (T_{p,n}y)_i|^p\right)^{1/p}
\]
\[
= \left(\sum_i \left|\frac{f_{p,n}(t)}{t}(x_i - y_i) + \left(f_{p,n}(s) - \frac{f_{p,n}(t)}{t}x_i\right)\right|^p\right)^{1/p}
\]
\[
\leq \left(\sum_i \left(|x_i - y_i| + \frac{|s-t|}{s}|x_i|\right)^p\right)^{1/p}
\]
\[
\leq \left(\sum_i |x_i - y_i|^p\right)^{1/p} + \frac{|s-t|}{s} \left(\sum_i |x_i|^p\right)^{1/p}
\]
\[
= \|x - y\|_p + \frac{\|x\|_p - \|y\|_p}{\|x\|_p}\|x\|_p \leq 2\|x - y\|_p.
\]

**Proposition 5.10.** Let \(u \geq 0\), \(p \in [1,2]\) and \(x \in \mathbb{R}^n\) be such that \(\|x\|_2^{2n-1/2} \leq u\|x\|_p\|x\|_p^{-1/p}\), then
\(\|T_{p,n}x - T_{p,n}y\|_p \leq (1 + u)\|x - y\|_p\) for all \(y \in \mathbb{R}^n\).

**Proof.** Let \(s = \|x\|_p\) and \(t = \|y\|_p\), we use Proposition 5.8 as in the proof of Proposition 5.9 and the Hölder inequality,
\[
\|T_{p,n}x - T_{p,n}y\|_2 \leq \left(\sum_i \left|x_i - y_i| + \frac{|s-t|}{s}\right|^2\right)^{1/2}
\]
\[
\leq \|x - y\|_2 + \frac{|s-t|}{s}\|x\|_2 \leq \|x - y\|_2 + \frac{\|x - y\|_p\|x\|_p}{\|x\|_p}\|x\|_2
\]
\[
\leq \|x - y\|_2 + \frac{\|x\|_2}{\|x\|_p}n^{\frac{p}{2}-\frac{1}{2}}\|x - y\|_2 \leq (1 + u)\|x - y\|_2.
\]

\(\square\)
The second transport we will use is a simple product transport which transports the measure $v^n_p$ onto $v^n_q$. We shall be particularly interested in the cases $p = 1$ and $p = 2$, but most of the results can be stated in the more general setting.

**Definition 5.11.** For $1 \leq p, q < \infty$ we define the map $w_{p,q} : \mathbb{R} \to \mathbb{R}$ by

$$
\frac{1}{\gamma_p} \int_x^{\infty} e^{-t^p} dt = \frac{1}{\gamma_q} \int_{w_{p,q}(x)}^{\infty} e^{-t^q} dt.
$$

By $v_p$ we denote $w_{p,1}$. We also define $W^p_{p,q} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$
W^n_{p,q}(x_1, x_2, \ldots, x_n) = (w_{p,q}(x_1), w_{p,q}(x_2), \ldots, w_{p,q}(x_n)).
$$

Note that $w_{p,q}^{-1} = w_{q,p}$ and $(W^n_{p,q})^{-1} = W^n_{q,p}$. Differentiating equality (24) we get

$$
w'_{p,q}(x) = \frac{\gamma_q}{\gamma_p} e^{-x^p + w^q_{p,q}(x)}. \tag{25}
$$

We will prove that $w_{p,q}$ behaves very much like $x^{p/q}$ for large $x$, and is more or less linear for small $x$. We begin with the bound for $q = 1$.

**Lemma 5.12.** For $p \geq 1$ we have

i) $v_p(x) \geq x^p + \ln(p \gamma_p x^{p-1})$ and $v'_p(x) \geq px^{p-1}$ for $x \geq 0$,

ii) $v_p(x) \leq e + x^p + \ln(p \gamma_p x^{p-1})$ and $v'_p(x) \leq e^p x^{p-1}$ for $x \geq 1$,

iii) $|v_p(x) - v_p(y)| \geq 2^{1-p} |x - y|^p$.

**Proof.** Note that $\gamma_1 = 1$. We have for $x \geq 0$,

$$
e^{-v_p(x)} = \frac{1}{\gamma_p} \int_x^{\infty} e^{-t^p} dt \leq \frac{1}{p \gamma_p x^{p-1}} \int_x^{\infty} p t^{p-1} e^{-t} dt = \frac{e^{-x^p}}{p \gamma_p x^{p-1}}. \tag{26}
$$

and for $x \geq 1$, since $(1 + r/p)^p \leq e^r \leq 1 + er$ for $r \in [0, 1]$, we get

$$
e^{-v_p(x)} \geq \frac{1}{\gamma_p} \int_x^{x + 1^{1-p}/p} e^{-t^p} \geq \frac{1}{p \gamma_p x^{p-1}} e^{-(x + 1^{1-p}/p)^p} \geq e^{-e - x^p}.
$$

Notice that by (25), $v'_p(x) = e^{-x^p + v_p(x)}/\gamma_p$, hence we may estimate $v'_p$ using the just derived bounds on $v_p$.

The lower bound on $v'_p$ yields $|v_p(x) - v_p(y)| \geq |x - y|^p$ for $x, y \geq 0$. The same estimate holds for $x, y \leq 0$, since $v_p$ is odd. Finally for $x \geq 0 \geq y$ we have

$$
|v_p(x) - v_p(y)| = |v_p(x)| + |v_p(y)| \geq |x|^p + |y|^p \geq 2^{1-p} |x - y|^p.
$$

$\square$

**Lemma 5.13.** i) For $p \geq q \geq 1$, $|w_{p,q}(x)| \geq |x|^{p/q}$ and $w'_{p,q}(x) \geq (2^{1-p})^{1/q} \geq \frac{1}{2}$.

ii) For $p \geq 2$, $w'_{p,2}(x) \geq \frac{1}{8} \sqrt{p} |x|^{p/2-1}$.
Proof. Since the function $w_{p,q}$ is odd, we may and will assume that $x \geq 0$.

i) We have by the monotonicity of $u^{q-1}$ on $[0, \infty)$,
\[
\frac{1}{\gamma_p} \int_x^\infty e^{-t^p} dt = \frac{1}{\gamma_q} \int_{w_{p,q}(x)}^\infty e^{-t^q} dt = \frac{\int_{w_{p,q}(x)}^\infty e^{-t^q} dt}{\int_0^\infty e^{-t^q} dt} = \frac{\int_{w_{p,q}(x)}^\infty e^{-t^q} dt}{\int_0^\infty u^{p/q-1} e^{-u^p} du}
\]
thus $w_{p,q}(x)^{q/p} \geq x$ and $w_{p,q}(x) \geq x^{p/q}$. Formula (25) gives $w'(x) \geq \gamma_q/\gamma_p \geq 1/2$.

ii) We begin by the following Gaussian tail estimate for $z > 0$:
\[
\int_z^\infty e^{-t^2} dt \geq \frac{1}{2\sqrt{2\pi} + 1} e^{-z^2}. \tag{27}
\]
We have equality when $z \to \infty$, and direct calculation shows the derivative of the left–hand–side is no larger than the derivative of the right–hand–side.

Let $\kappa := 4\sqrt{\pi}$, we will now show that for all $x > 0$ and $p \geq 2$,
\[
w_{p,2}(x) \geq u_p(x) := \max \left\{ \frac{\sqrt{\pi}}{2} x, \sqrt{(x^p + \ln(\sqrt{\pi}x^{p/2-1}/\kappa))} \right\}. \tag{28}
\]
Suppose on the contrary that $w_{p,2}(x) < u_p(x)$ for some $p \geq 2$ and $x > 0$. Note that by i) we have $w'_{p,2} \geq \gamma_2/\gamma_p \geq \gamma_2 = \sqrt{\pi}/2$. Thus $u_p(x)$ is equal to the second part of the maximum. This in particular implies that $x \geq 2/3$, since for $x < 2/3$ we have
\[
x^p + \ln(\sqrt{\pi}x^{p/2-1}/\kappa) \leq \frac{4}{9} + \left( \frac{p}{2} - 1 \right) \ln \frac{2}{3} + \frac{\sqrt{p}}{\kappa} - 1 \leq 0.
\]
Therefore $u_p(x) \geq \sqrt{\pi}x/2 \geq 1/\sqrt{3}$. Now by (25), (24) and (27),
\[
\sqrt{\pi} \frac{1}{px^{p-1}} e^{-x^p} \geq \frac{\gamma_2}{\gamma_p} \frac{1}{px^{p-1}} e^{-x^p} \geq \frac{\gamma_2}{\gamma_p} \int_x^\infty e^{-t^p} dt = \int_{w_{p,2}(x)}^\infty e^{-t^p} dt > \int_{u_p(x)}^\infty e^{-t^p} dt \geq \frac{1}{2u_p(x)} \frac{e^{-u_p^2(x)}}{1} > \frac{1}{4u_p(x)} e^{-u_p^2(x)}
\]
\[
= \frac{1}{4u_p(x)} e^{-(x^p + \ln(\sqrt{\pi}x^{p/2-1}/\kappa))} = \frac{\sqrt{\pi}}{\sqrt{\pi}u_p(x)} x^{1-p/2} e^{-x^p}.
\]
After simplifying this gives $u_p(x) > \sqrt{\pi}x^{p/2}$. Hence
\[
px^p < u_p^2(x) = x^p + \frac{1}{2} \ln(px^p) + \ln \frac{1}{\kappa e} \leq \frac{p}{2} x^p + \frac{1}{2} px^p = px^p,
\]
which is impossible. This contradiction shows that (28) holds.
Thus we have \( w_{p,2}(x) \geq u_p(x) \) and by (29) we obtain
\[
w_{p,2}(x) \geq \frac{\gamma_2}{2\kappa} e^{-x^p + u_p^2(x)} \geq \frac{\sqrt{\pi}}{2} \sqrt[p]{x} p^{p/2-1} = \frac{1}{8} \sqrt[p]{x} p^{p/2-1}.
\]

\[\square\]

**Remark 5.14.** By taking \( u_p(x) = \max\{\sqrt[p]{x}/2, \sqrt[p]{x} p^{p/2-1}/(\kappa \ln p)\} \) for sufficiently large \( \kappa \) and estimating carefully one may arrive at the bound \( w_{p,2}(x) \geq C^{-1} p x^{p/2-1}/\ln p \). One cannot, however, receive a bound of the order of \( p x^{p/2-1} \).

**Proposition 5.15.** For \( p \geq q \geq 1 \) we have
i) \( \nu_p^n(W_{q,p}^n(A)) = \nu_q^n(A) \) for \( A \in B(\mathbb{R}^n) \).
ii) \( |w_{q,p}(x) - w_{q,p}(y)| \leq 2|x - y| \) for \( x, y \in \mathbb{R} \),
iii) for \( x, y \in \mathbb{R}^n \) and \( r \geq 1 \),
\[ ||W_{q,p}^n(x) - W_{q,p}^n(y)||_r \leq 2||x - y||_r, \]
iv) for \( x, y \in \mathbb{R} \),
\[ |w_{1,p}(x) - w_{1,p}(y)| \leq 2\min(||x - y||, ||x - y||^{1/p}) \leq 2|x - y|^{1/q}, \]
v) \( ||W_{1,p}^n(x) - W_{1,p}^n(y)||_q^q \leq 2^q ||x - y||_1 \) for \( x, y \in \mathbb{R}^n \).

**Proof.** Property i) follows from the definition of \( w_{q,p} \) and \( W_{q,p}^n \). Since \( w_{q,p} = W_{p,q}^{-1} \) we get ii) by Lemma 5.13 i). Property iii) is a direct consequence of ii).

By Lemma 5.12 iii),
\[ |w_{1,p}(x) - w_{1,p}(y)| = |v_p^{-1}(x) - v_p^{-1}(y)| \leq 2^{1-1/p} ||x - y||^{1/p}. \]
The above inequality together with ii) gives iv) and iv) yields v). \[\square\]

Now we define a transport from the exponential measure \( \nu^n \) to \( \mu_{p,n} \) for \( p \geq 2 \):

**Definition 5.16.** For \( n \in \mathbb{N} \) and \( 2 \leq p < \infty \) we define the map \( S_{p,n} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by \( S_{p,n}(x) := T_{p,n}(W_{1,p}^n(x)) \).

This transport satisfies the following bound:

**Proposition 5.17.** We have \( ||S_{p,n}(x) - S_{p,n}(y)||_2 \leq 4 ||x - y||_2 \) for all \( x, y \in \mathbb{R}^n \) and \( p \geq 2 \).

**Proof.** It is enough to show that \( ||D S_{p,n}(x)|| \leq 4 \), where \( D S_{p,n} \) is the derivative matrix, and the norm is the operator norm from \( \ell_2^n \) into \( \ell_2^n \).

Let \( s = ||W_{1,p}^n(x)||_p \). By direct calculation we get
\[
\frac{\partial (D S_{p,n})_i}{\partial x_i}(x) = \frac{\partial f_{p,n}(s) w_{i,p}^n(x_i)}{s} + \alpha(s) w_{1,p}(x_j) \beta(x_i) \quad (29)
\]
Thus we can bound
$$\|DS_{p,n}(x)\| \leq \frac{f_{p,n}(s)}{s} \max_i |w_i'(x_i)| + |\alpha(s)||W_{1,p}^n(x)| \left( \sum_{i=1}^n \beta^2(x_i) \right)^{1/2}.$$

Since $w_{1,p} = w_{p,1}^{-1}$, Proposition 5.13(i) implies $|w_{1,p}(x_j)| \leq 2$, while by Proposition 5.8 we have $f_{p,n}(s)/s \leq 1$. Thus the first summand can be bounded by 2.

For the second summand note that by Proposition 5.8 iii),
$$|\alpha(s)| = s^{-p}[f_{p,n}(s) - \frac{f_{p,n}(s)}{s}] \leq s^{-p} \min \left\{ 1, \frac{2ps^n}{n} \right\}. \quad (30)$$

Moreover, $\|W_{1,p}^n(x)\|_2 \leq n^{1/2-1/p}s$ by the Hölder inequality and
$$|\beta(t)| = |w_{1,p}(t)|^{p-1}|w_1'(t)| = \frac{|w_{1,p}(t)|^{p-1}}{w_1(p(t))} \leq \frac{1}{p},$$
by Lemma 5.12. Thus
$$\|DS_{p,n}(x)\| \leq 2 + s^{-p} \min \left\{ 1, \frac{2ps^n}{n} \right\} n^{1/2-1/p}s n^{1/2} \leq 2 + 2sn^{-1/p} \min \{ ns^{-p}, 1 \} \leq 4.$$

\[\square\]

**Proposition 5.18.** For any $y, z \in \mathbb{R}^n$ and $p \geq 2$ we have
$$\|S_{p,n}(y) - S_{p,n}(z)\|_2 \leq \|W_{1,p}^n(y) - W_{1,p}^n(z)\|_2 + 2n^{-1/2}\|y - z\|_1.$$ 

**Proof.** Let $u_i(t) = (y_1, y_2, \ldots, y_{i-1}, t, z_{i+1}, z_{i+2}, \ldots, z_n)$ for $i = 1, \ldots, n$. Note that $u_i(y_i) = u_{i+1}(z_{i+1})$, $u_1(z_1) = z$ and $u_n(y_n) = y$, hence
$$S_{p,n}(z) - S_{p,n}(y) = \sum_{i=1}^n \left( S_{p,n}(u_i(z_i)) - S_{p,n}(u_i(y_i)) \right).$$

Let $s_i(t) := \|w_{1,p}(u_i(t))\|_p$. By vector–valued integration and (29) we get
$$S_{p,n}(u_i(z_i)) - S_{p,n}(u_i(y_i)) = \int_{g_i}^{z_i} \frac{\partial S_{p,n}(u_i(t))}{\partial x_i} dt = a_i + b_i,$$
where
$$a_i := \int_{g_i}^{z_i} \frac{s_i(t)}{s_i(t)} w_{1,p}'(t) \epsilon_i dt.$$
Thus by Proposition 5.17 we show that

$$\|\alpha(s_i(t))\beta(t)W_{1,p}^n(u_i(t))\|_2 \leq 2n^{-1/2}s_i(t)n^{-1/p}\min\{ns_i(t)^{-p}, 1\} \leq 2n^{-1/2},$$

thus

$$\left\| \sum_{i=1}^n b_i \right\|_2 \leq \sum_{i=1}^n \|b_i\|_2 \leq 2n^{-1/2}\sum_{i=1}^n |y_i - z_i| = 2n^{-1/2}\|y - z\|_1.$$

To deal with the sum of $a_i$'s we notice that, since $f_{p,n}(s)/s \leq 1$ and $w_{1,p}'(x) \geq 0$, 

$$\left| \left\langle \sum_j a_j, e_i \right\rangle \right| = |\langle a_i, e_i \rangle| = \left| \int_{y_i}^{z_i} f_{p,n}(s_i(t)) \frac{w_{1,p}'(t)}{s_i(t)} dt \right|$$

$$\leq \left| \int_{y_i}^{z_i} w_{1,p}'(t) dt \right| = |w_{1,p}(z_i) - w_{1,p}(y_i)|.$$ 

Thus

$$\left\| \sum_i a_i \right\|_2 \leq \left\| \sum_i (w_{1,p}(z_i) - w_{1,p}(y_i))e_i \right\|_2 = \|W_{1,p}^n(z) - W_{1,p}^n(y)\|_2.$$ 

\[\square\]

**Corollary 5.19.** If $x - y \in tB_1^n + t^{1/2}B_2^n$ for some $t > 0$, then for all $p \geq 2$, $S_{p,n}x - S_{p,n}y \in 10(t^{1/2}B_2^n \cap t^{1/p}B_p^n)$.

**Proof.** Let us fix $x, y$ with $x - y \in tB_1^n + t^{1/2}B_2^n$. By Proposition 5.14 iv),

$$\|W_{1,p}^n(x) - W_{1,p}^n(y)\|_p^p = \sum_i |w_{1,p}(x_i) - w_{1,p}(y_i)|^p$$

$$\leq 2^p \sum_i \min(|x_i - y_i|^p, |x_i - y_i|)$$

$$\leq 2^p \sum_i \min(|x_i - y_i|^2, |x_i - y_i|) \leq 2^{p+2}t.$$ 

Thus by Proposition 5.9

$$\|S_{p,n}x - S_{p,n}y\|_p \leq 2\|W_{1,p}^n(x) - W_{1,p}^n(y)\|_p \leq 8t^{1/p}.$$ 

By Hölder’s inequality $\|S_{p,n}x - S_{p,n}y\|_2 \leq n^{1/2-1/p}\|S_{p,n}x - S_{p,n}y\|_p \leq 8t^{1/2}$ for $t \geq n$.

Assume now that $t \leq n$. Let $z$ be such that $x - z \in t^{1/2}B_2^n$ and $z - y \in tB_1^n$. Then $S_{p,n}x - S_{p,n}z \in 4t^{1/2}B_2^n$ by Proposition 5.17 and $\|W_{1,p}^n(z) - W_{1,p}^n(y)\|_2 \leq 2\sqrt{t}$ by Proposition 5.18. Thus by Proposition 5.18

$$\|S_{p,n}x - S_{p,n}z\|_2 \leq 4t^{1/2} + 2n^{-1/2}t \leq 6t^{1/2}.$$ 

Hence $S_{p,n}x - S_{p,n}y \in 10t^{1/2}B_2^n$. 

\[\square\]
The last function we define transports the Gaussian measure $\nu_2^p$ to $\mu_{p,n}$ for $p \geq 2$.

**Definition 5.20.** For $n \in \mathbb{N}$ and $2 \leq p < \infty$ we define the map $\tilde{S}_{p,n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\tilde{S}_{p,n}(x) := T_{p,n}(W_{2,p}^n(x))$.

**Proposition 5.21.** We have $\|\tilde{S}_{p,n}(x) - \tilde{S}_{p,n}y\|_2 \leq 18\|x - y\|_2$ for all $x, y \in \mathbb{R}^n$ and $p \geq 2$.

**Proof.** We argue in a similar way as in the proof of Proposition 5.17. We need to show that $\|D\tilde{S}_{p,n}(x)\| \leq 18$. Direct calculation gives

$$
(\partial \tilde{S}_{p,n})_i(x) = \frac{\delta_{ij} f_{p,n}(\tilde{s}) w'_{2,p}(x_i)}{\tilde{s}} + \alpha(\tilde{s}) w_{2,p}(x_j) \tilde{\beta}(x_i)
$$

(31)

where $\tilde{s} = \|W_{2,p}^n(x)\|_p$, $\alpha(s) := s^{-p-1}(s f'_{p,n}(s) - f_{p,n}(s))$ and $\tilde{\beta}(t) := |w_{2,p}(t)|^{p-1}\text{sgn}(w_{2,p}(t))w'_{2,p}(t)$.

Thus we can bound

$$
\|D\tilde{S}_{p,n}(x)\| \leq \frac{f_{p,n}(\tilde{s})}{\tilde{s}} \max_i |w'_{2,p}(x_i)| + |\alpha(\tilde{s})|\|W_{2,p}^n(x)\|_2 \left(\sum_{i=1}^n \tilde{\beta}^2(x_i)\right)^{1/2}.
$$

(32)

The first summand is bounded by 2 as in the proof of Proposition 5.17. Since $w_{2,p} = w'_{2,p}$ we get by Lemma 5.13 ii)

$$
|\tilde{\beta}(x)| = |w_{2,p}(x)|^{p-1}|w'_{2,p}(x)| = \frac{|w_{2,p}(x)|^{p-1}}{w'_{2,p}(w_{2,p}(x))} \leq \frac{8}{\sqrt{p}} |w_{2,p}(x)|^{p/2}
$$

hence

$$
\left(\sum_{i=1}^n \tilde{\beta}^2(x_i)\right)^{1/2} \leq \frac{8}{\sqrt{p}} \tilde{s}^{p/2}.
$$

Using (30) and $\|W_{2,p}^n(x)\|_2 \leq n^{1/2 - 1/p} \tilde{s}$ we bound the second summand in (32) by

$$
\tilde{s}^{-p} \min \left\{\frac{1}{n}, \frac{2p\tilde{s}^{p}}{n}\right\} n^{1/2 - 1/p} \tilde{s}^{8\sqrt{p}/p} = 8p^{-1/p} \min\{u^{-1/2}, 2u^{1/2}\} u^{1/p} \leq 16,
$$

where $u := p\tilde{s}^{p}/n$. 

**5.2 Applying $\nu_1$ results $- p \leq 2$**

We start with the version of Theorem 4.10 for $\nu_p$.

**Lemma 5.22.** For any $A \in \mathcal{B}(\mathbb{R}^n)$, $p \in [1, 2]$ and $t \geq 1$ at least one of the following holds:
\[
\nu_p^n((A + 20t^{1/p}B^n_p)) \geq e^t \nu_p^n(A) \quad \text{or} \\
\nu_p^n((A + 20t^{1/p}B^n_p) \cap 100\sqrt{n}B^n_2) \geq \frac{1}{2} \nu_p^n(A).
\]

**Proof.** We will use the transport \(W^n_{1,p}\) from \(\nu^n\) to \(\nu^n_p\). Proposition 5.14 gives \(\|W^n_{1,p}(x) - W^n_{1,p}(y)\|_p \leq 2p\|x - y\|_1\). By Remark 5.5 this means that \(A + 20t^{1/p}B^n_p \supset W^n_{1,p}(W^n_{p,1}(A) + 10tB^n_1)\). Let us fix \(t \geq 1\) and apply Theorem 4.6 to \(W^n_{p,1}(A)\) and \(10t\). If the second case occurs, we have

\[
\nu_p^n((A + 20t^{1/p}B^n_p) \cap 100\sqrt{n}B^n_2) \geq \nu_p^n(W^n_{1,p}(W^n_{p,1}(A) + 10tB^n_1)) = \nu^n(W^n_{p,1}(A) + 10tB^n_1) \\
\geq e^t \nu^n(W^n_{p,1}(A)) = e^t \nu_p^n(A).
\]

If the first case of Theorem 4.6 occurs, then due to Proposition 5.15 iii) we have \(\|W^n_{1,p}(x)\|_2 \leq 2\|x\|_1\), so \(2\alpha B^n_2 \supset W^n_{1,p}(\alpha B^n_2)\) for any \(\alpha > 0\). Thus

\[
\nu_p^n((A + 20t^{1/p}B^n_p) \cap 100\sqrt{n}B^n_2) \geq \nu_p^n(W^n_{1,p}(W^n_{p,1}(A) + 10tB^n_1) \cap 100\sqrt{n}B^n_2) \\
= \nu_p^n(W^n_{1,p}(W^n_{p,1}(A) + 10tB^n_1) \cap W^n_{p,1}(100\sqrt{n}B^n_2)) \\
\geq \nu_p^n(W^n_{1,p}((W^n_{p,1}(A) + 10tB^n_1) \cap 50\sqrt{n}B^n_2)) \\
= \nu^n((W^n_{p,1}(A) + 10tB^n_1) \cap 50\sqrt{n}B^n_2) \\
\geq \frac{1}{2} \nu^n(W^n_{p,1}(A)) = \frac{1}{2} \nu_p^n(A).
\]

\[\square\]

**Lemma 5.23.** There exists a constant \(C\) such that for any \(p \in [1, 2]\), \(t > 0\) and \(n \in \mathbb{N}\) we have

\[
\nu_p^n((A + C(t^{1/p}B^n_p + t^{1/2}B^n_2)) \geq \min\left\{\frac{1}{2}, e^t \nu_p^n(A)\right\}.
\]

**Proof.** Corollary 6.2 gives \(B_s(\nu^n_p) \subset C(s^{1/p}B^n_p + s^{1/2}B^n_2)\) for \(s > 0\). By Corollary 2.19 \(\nu^n_p\) satisfies IC(48), which, due to Proposition 2.4 implies \(\nu^n_p(A + 48B_2(\nu^n_p)) \geq \min\{1/2, e^t \nu_p^n(A)\}\) for any Borel set \(A\). Thus we have

\[
\nu_p^n((A + 96C(t^{1/p}B^n_p + t^{1/2}B^n_2)) \geq \min\{1/2, e^t \nu_p^n(A)\}.
\]

\[\square\]

**Proposition 5.24.** For any \(\alpha > 1\) there exists a constant \(c(\alpha)\) such that for any \(n \in \mathbb{N}\) and \(p \geq 1\) we have

\[
\nu_p^n(\{x : \|x\|_p < c(\alpha)n^{1/p}\}) < \alpha^{-n}.
\]

**Proof.** We have

\[
\nu_p^n(\{x : \|x\|_p < c(\alpha)n^{1/p}\}) = \frac{n}{\Gamma(1 + \frac{2}{p})} \int_0^{c(\alpha)n^{1/p}} e^{-r^p} r^{n-1} dr \\
\leq \frac{n}{\Gamma(1 + \frac{2}{p})} \int_0^{c(\alpha)n^{1/p}} r^{n-1} \frac{c(\alpha)n^{n/p}}{\Gamma(1 + \frac{2}{p})} \leq (Cc(\alpha))^n,
\]

\[33\]
where in the last step we use the Stirling approximation and \( C \) as always denotes a universal constant. Thus it is enough to take \( c(\alpha) < (C\alpha)^{-1} \).

**Theorem 5.25.** There exists a universal constant \( C \) such that \( \mu_{p,n} \) satisfies \( \text{CI}(C) \) for any \( p \in [1, 2] \) and \( n \in \mathbb{N} \).

**Proof.** By Propositions 2.7, 3.11, 3.5 and 5.3 it is enough to show
\[
\mu_{p,n}(A + C(t^{1/p}B_p^n + t^{1/2}B_2^n)) \geq \min\{1/2, e^t\mu_{p,n}(A)\}.
\] (33)
for \( 1 \leq t \leq n \) and \( \mu_{p,n}(A) \geq e^{-n} \).

Recall that \( T_{p,n} \) denotes the map transporting \( \nu_p^n \) to \( \mu_{p,n} \). Apply Lemma 5.22 to \( T_{p,n}^{-1}(A) \) and \( t \). If the first case occurs, we have
\[
\nu_p^n(T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \geq e^t\nu_p^n(T_{p,n}^{-1}(A)) = e^t\mu_{p,n}(A).
\]
Proposition 5.9 gives \( \|T_{p,n}x - T_{p,n}y\|_p \leq 2\|x - y\|_p \), thus by Remark 5.5
\[
\mu_{p,n}(A + 40t^{1/p}B_p^n) = \nu_p^n(T_{p,n}(A + 40t^{1/p}B_p^n)) \\
\geq \nu_p^n(T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \geq e^t\mu_{p,n}(A)
\]
and we obtain (33) in this case.

Hence we may assume that the second case of Lemma 5.22 holds, that is
\[
\nu_p^n(A') \geq \frac{1}{2}\nu_p^n(T_{p,n}(A)) = \frac{1}{2}\mu_{p,n}(A),
\]
where
\[
A' := (T_{p,n}^{-1}(A) + 20t^{1/p}B_p^n) \cap 100\sqrt{n}B_2^n.
\]
In particular \( \nu_p^n(A') \geq e^{-n/2} \). Let
\[
A'' := A' \cap \{x : \|x\|_p \geq \tilde{c}n^{1/p}\},
\]
where \( \tilde{c} = c(4e) \) is a constant given by Proposition 5.24 for \( \alpha = 4e \). Then
\[
\nu_p^n(A'') \geq \nu_p^n(A') - (4e)^{-n} \geq \frac{1}{2}\nu_p^n(A') \geq \frac{1}{4}\mu_{p,n}(A).
\]
We apply Lemma 5.22 for \( A'' \) and \( 4t \) to get
\[
\mu_{p,n}(T_{p,n}(A'' + 4C(t^{1/p}B_p^n + t^{1/2}B_2^n))) \geq \nu_p^n(A'' + C((4t)^{1/p}B_p^n + (4t)^{1/2}B_2^n)) \\
\geq \min\left\{\frac{1}{2}, e^{4t}\nu_p^n(A')\right\} \geq \min\left\{\frac{1}{2}, e^{4t}\mu_{p,n}(A)\right\} \geq \min\left\{\frac{1}{2}, e^t\mu_{p,n}(A)\right\}.
\]
Proposition 5.9 and Remark 5.5 imply
\[
T_{p,n}(A'' + 4Ct^{1/2}B_2^n + 4Ct^{1/p}B_p^n) \subset T_{p,n}(A'' + 4Ct^{1/2}B_2^n) + 8Ct^{1/p}B_p^n.
\]

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Moreover, for \( x \in A'' \) we have \( \|x\|_2 \leq 100\sqrt{n} \) and \( \|x\|_p \geq \tilde{c}n^{1/p} \). Thus \( n^{-1/2}\|x\|_2 \leq 100\tilde{c}^{-1}n^{-1/p}\|x\|_p \), so we can use Proposition 5.10 along with Remark 5.5 to get

\[
T_{p,n}(A'' + 4Ct^{1/2}B^n_2) \subset T_{p,n}(A'') + \tilde{C}t^{1/2}B^n_2.
\]

Proposition 5.9, Remark 5.5 and the definitions of \( A' \) and \( A'' \) yield

\[
T_{p,n}(A'') \subset T_{p,n}(A') \subset T_{p,n}(T_{p,n}^{-1}(A) + 20t^{1/p}B^n_p) \subset A + 40t^{1/p}B^n_p.
\]

Putting the four estimates together, we can write

\[
\mu_{p,n}\left(A + (40 + 8C)t^{1/p}B^n_p + \tilde{C}t^{1/2}B^n_2\right)
\geq \mu_{p,n}\left(T_{p,n}(A'') + \tilde{C}t^{1/2}B^n_2 + 8Ct^{1/p}B^n_p\right)
\geq \mu_{p,n}\left(T_{p,n}(A'') + 4Ct^{1/2}B^n_2 + 8Ct^{1/p}B^n_p\right)
\geq \mu_{p,n}\left(T_{p,n}(A'' + 4C(t^{1/p}B^n_p + t^{1/2}B^n_2))\right) \geq \min\left\{\frac{1}{2}, e^\epsilon \mu_{p,n}(A)\right\},
\]

which gives (33) in the second case and ends the proof.

**Corollary 5.26.** There exists an absolute constant \( C \) such that the measure \( \mu_{p,n} \) satisfies IC\((C)\) for any \( p \in [1, 2] \) and \( n \in \mathbb{N} \).

**Proof.** Let

\[
\tilde{\mu}_{p,n}(A) := \mu_{p,n}(\sigma_{p,n}A), \text{ where } \sigma_{p,n}^2 := \int x_i^2 d\mu_{p,n},
\]

then \( \tilde{\mu}_{p,n} \) is isotropic. Both properties IC and CI are affine invariant, so by Theorem 5.24 \( \tilde{\mu}_{p,n} \) has property CI\((C)\) and we are to show that it satisfies IC\((C)\). By Corollary 3.10 we only need to show Cheeger’s inequality for \( \tilde{\mu}_{p,n} \) with uniform constant.

A recent result of S. Sodin ([19, Theorem 1]) states (after rescaling from \( B^n_p \) to \( r_{p,n}B^n_p \)) that

\[
\mu_{p,n}(A) \geq c \min\{\mu_{p,n}(A), 1 - \mu_{p,n}(A)\} \log^{1-1/p} \frac{1}{\min\{\mu_{p,n}(A), 1 - \mu_{p,n}(A)\}}
\]

for some universal constant \( c \). Thus \( \tilde{\mu}_{p,n} \) satisfies Cheeger’s inequality with constant \( c\sigma_{n,p} \) and it is enough to notice that by (21),

\[
\sigma_{p,n} \sim r_{p,n} \frac{n^{1/p}}{n} \left( \int x_i^2 d\nu_p(x) \right)^{1/2} \sim 1.
\]

\[\square\]
5.3 The easy case – $p \geq 2$

This case will follow easily from the exponential case and the facts from subsection 5.3.

**Theorem 5.27.** There exists a universal constant $C$ such that for any $A \subset \mathbb{R}^n$, any $t \geq 1$, $n \geq 1$ and $p \geq 2$ we have

$$\mu_{p,n} \left( A + C \left( t^{1/p} B_p^n \cap t^{1/2} B_2^n \right) \right) \geq \min\{1/2, e^{t} \mu_p(A)\}.$$ 

**Proof.** In this case we will again use the transport $S_{p,n}$. Assume $A \subset r_{p,n} B_p^n$, let $\tilde{A} := S_{p,n}^{-1}(A)$. By Talagrand’s inequality (6) we have $\nu^n(\tilde{A} + C t B_2^n + \sqrt{C t} B_2^n) \geq \min\{e^t \nu^n(\tilde{A}), 1/2\}$. However by Corollary 5.19 we have $S_{p,n}(\tilde{A} + C t B_2^n + \sqrt{C t} B_2^n) \subset S_{p,n}(\tilde{A}) + 10 C \left( \sqrt{t} B_2^n + t^{1/p} B_p^n \right)$.

Thus, as $S_{p,n}(\tilde{A}) = A$ and $S_{p,n}$ transports the measure $\nu^n$ to $\mu_{p,n}$, we get the thesis.

By Propositions 2.7, 3.11, 3.5 and 5.3, Theorem 5.27 yield the following.

**Corollary 5.28.** There exists an absolute constant $C$ such that $\mu_{p,n}$ satisfies CI($C$) for $p \geq 2$.

**Theorem 5.29.** For any $p \geq 2$ and $n \geq 1$ the measure $\mu_{p,n}$ satisfies Cheeger’s inequality (12) with the constant $1/20$.

**Proof.** Again we shall transport this result from the exponential measure. By [5] Cheeger’s inequality holds for $\nu^n$ with the constant $\kappa = 1/(2\sqrt{6})$, thus by Proposition 5.17 $\mu_{p,n}$ satisfies (12) with the constant $\kappa/4 \geq 1/20$.

As in the proof of Corollary 5.28 we show that Theorem 5.29 and Corollary 5.28 imply infimum convolution inequality for $\mu_{p,n}$, $p \geq 2$. Adding the two results together we get

**Theorem 5.30.** There exists a universal constant $C$ such that for any $p \in [1, \infty]$ and any $n \in \mathbb{N}$ the measure $\mu_{p,n}$ satisfies IC($C$).

We conclude this section with the proof of logarithmic Sobolev–type inequality for $\mu_{p,n}$.

**Theorem 5.31.** Let $\Phi(x) = (2\pi)^{-1/2} \int_{\mathbb{R}^n} \exp(-y^2/2)$ be a Gaussian distribution function, $A \in \mathcal{B}(\mathbb{R}^n)$ and $p \geq 2$, then

$$\mu_{p,n}(A) = \Phi(x) \Rightarrow \mu_{p,n}(A + 18\sqrt{2t} B_2^n) \geq \Phi(x + t) \text{ for all } t > 0.$$ 

In particular there exists a universal constant $C$ such that

$$\mu_{p,n}(A) \geq \frac{1}{C} \min \left\{ \mu_{p,n}(A) \sqrt{\ln \frac{1}{\mu_{p,n}(A)}}, (1 - \mu_{p,n}(A)) \sqrt{\ln \frac{1}{1 - \mu_{p,n}(A)}} \right\}.$$
Proof. By Proposition 5.21, $\tilde{S}_{p,n}(\sqrt{2})$ is $18\sqrt{2}$-Lipschitz and transports the canonical Gaussian measure on $\mathbb{R}^n$ onto $\mu_{p,n}$. Hence the first part of theorem follows by the Gaussian isoperimetric inequality of Borell [8] and Sudakov, Tsirel’son [20]. The last estimate immediately follows by a standard estimate of the Gaussian isoperimetric function.

6 Concluding Remarks

1. With the notion of the IC property one may associate IC–domination of symmetric probability measures $\mu$, $\tilde{\mu}$ on $\mathbb{R}^n$: we say that $\mu$ is IC–dominated by $\tilde{\mu}$ with a constant $\beta$ if $(\mu, \Lambda^*_{\tilde{\mu}}(\cdot))$ has property $\tau$. IC–domination has the tensorization property: if $\mu_i$ are IC($\beta$)–dominated by $\tilde{\mu}_i$, $1 \leq i \leq n$, then $\otimes \mu_i$ is IC($\beta$)–dominated by $\otimes \tilde{\mu}_i$. An easy modification of the proof of Corollary 3.10 gives that if $\mu$ is IC($\beta$)–dominated by an $\alpha$–regular measure $\tilde{\mu}$, then

$$\forall p \geq 2 \forall A \in B(\mathbb{R}^n) \mu(A) \geq \frac{1}{2} \Rightarrow 1 - \mu(A + c(\alpha)\beta Z_p(\tilde{\mu})) \leq e^{-p}(1 - \mu(A)).$$

Following the proof of Proposition 3.12 we also get for all $p \geq 2$,

$$\left( \int \|x\| - \text{Med}_\mu(\|x\|) \right)^p d\mu \leq (\int \|x\| - \text{Med}_\mu(\|x\|) \right)^p d\tilde{\mu})^{1/p} \leq \tilde{c}(\alpha)\beta \sup_{\|u\| \leq 1} \left( \int \langle u, x \rangle \right)^p d\tilde{\mu}^{1/p}. $$

2. One may consider convex versions of properties CI and IC. We say that a symmetric probability measure $\mu$ satisfies the convex infimum convolution inequality with a constant $\beta$ if the pair $(\mu, \Lambda^*_{\tilde{\mu}}(\cdot))$ has convex property $(\tau)$, i.e. the inequality (11) holds for all convex function and $f$ with $\varphi(x) = \Lambda^*_{\tilde{\mu}}(x/\beta)$. Analogously $\mu$ satisfies convex concentration inequality with a constant $\beta$, if (18) holds for all convex Borel sets $A$. We do not know if convex IC implies convex CI, however for $\alpha$–regular measures it implies a weaker version of convex CI, namely

$$\mu(A) \geq 1/2 \Rightarrow \mu(A + c_1(\alpha)\beta Z_p(\tilde{\mu})) \geq 1 - 2e^{-p}$$

and this property yields CWSM($c_2(\alpha)\beta$).

From the results of [16] one may easily deduce that the uniform distribution on $\{-1, 1\}^n$ satisfies convex IC($C$) with a universal constant $C$.

3. Property IC may be also investigated for nonsymmetric measures. However in this case the natural choice of the cost function is $\Lambda^*_{\tilde{\mu}}(x/\beta)$, where $\tilde{\mu}$ is the convolution of $\mu$ and the symmetric image of $\mu$.

4. We do not know if the infimum convolution property (at least for regular measures) implies Cheeger’s inequality. If so, we would have equivalence of IC and CI + Cheeger.
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