The cohomology of $\mathcal{M}_{0,n}$ as an FI-module

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Abstract In this paper we revisit the cohomology groups of the moduli space of $n$-pointed curves of genus zero using the FI-module perspective introduced by Church-Ellenberg-Farb. We recover known results about the corresponding representations of the symmetric group.

1 Introduction

Our space of interest is $\mathcal{M}_{0,n}$, the moduli space of $n$-pointed curves of genus zero. It is defined as the quotient

$$\mathcal{M}_{0,n} := \mathcal{F}(\mathbb{P}^1(\mathbb{C}),n)/\text{Aut}(\mathbb{P}^1(\mathbb{C})),$$

where $\mathcal{F}(\mathbb{P}^1(\mathbb{C}),n)$ is the configuration space of $n$-ordered points in the projective line $\mathbb{P}^1(\mathbb{C})$ and the automorphism group of the projective line $\text{Aut}(\mathbb{P}^1(\mathbb{C})) = \text{PGL}_2(\mathbb{C})$ acts componentwise on $\mathcal{F}(\mathbb{P}^1(\mathbb{C}),n)$. For $n \geq 3$, $\mathcal{M}_{0,n}$ is a fine moduli space for the problem of classifying smooth $n$-pointed rational curves up to isomorphism (see for example [16, Proposition 1.1.2]).

The space $\mathcal{M}_{0,n}$ carries a natural action of the symmetric group $S_n$. The cohomology ring of $\mathcal{M}_{0,n}$ is known and the $S_n$-representations $H^i(\mathcal{M}_{0,n};\mathbb{C})$ are well-understood (see for example [9], [15], [11]).

In this paper, we will consider the sequence of $S_n$-representations $H^i(\mathcal{M}_{0,n};\mathbb{C})$ as a single object, an FI-module over $\mathbb{C}$. Via this example, we introduce the basics of the FI-module theory developed by Church, Ellenberg and Farb in [3]. We then use a well-known description of the cohomology ring of $\mathcal{M}_{0,n}$ to show in Theorem 4.5 that...
a finite generation property is satisfied which allows us to recover information about the $S_n$-representations in Theorem 5.1. Specifically, we obtain a stability result concerning the decomposition of $H^i(M_{0,n}; \mathbb{C})$ into irreducible $S_n$-representations, we exhibit a bound on the lengths of the representations and show that their characters have a highly constrained "polynomial" form.

2 The co-FI-spaces $\mathcal{M}_{0, \bullet}$ and $\mathcal{M}_{0, \bullet+1}$

Let $\text{FI}$ be the category whose objects are natural numbers $n$ and whose morphisms $m \to n$ are injections from $[m] := \{1, \ldots , m\}$ to $[n] := \{1, \ldots , n\}$.

We are interested in the co-FI-space $\mathcal{M}_{0, \bullet}$: the functor from $\text{FI}^{op}$ to the category $\textbf{Top}$ of topological spaces given by $n \mapsto \mathcal{M}_{0,n}$ that assigns to $f : [m] \to [n]$ in $\text{Hom}_{\text{FI}}(m,n)$ the morphism $f^* : \mathcal{M}_{0,n} \to \mathcal{M}_{0,m}$ defined by $f^*([(p_1, p_2, \ldots , p_n)]) = [(p_{f(1)}, p_{f(2)}, \ldots , p_{f(m)})]$. This is a particular case of the co-FI-space $\mathcal{M}_{g, \bullet}$ considered in [12] which is the functor given by $n \mapsto \mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus $g$ with $n$ marked points.

An FI-module over $\mathbb{C}$ is a functor $V$ from $\text{FI}$ to the category of $\mathbb{C}$-vector spaces $\textbf{Vec}_\mathbb{C}$. Below, we denote $V(n)$ by $V_n$. Church, Ellenberg and Farb used FI-modules in [3] to encode sequences of $S_n$-representations in single algebraic objects and with this added structure significantly strengthened the representation stability theory introduced in [5]. FI-modules translate the representation stability property into a finite generation condition.

By composing the co-FI-space $\mathcal{M}_{0, \bullet}$ with the cohomology functor $H^i(-; \mathbb{C})$, we obtain the FI-module $H^i(\mathcal{M}_{0, \bullet}) := H^i(\mathcal{M}_{0, \bullet}; \mathbb{C})$. We can also consider the graded version $H^i(\mathcal{M}_{0, \bullet}) := H^i(\mathcal{M}_{0, \bullet}; \mathbb{C})$, we call this a graded FI-module over $\mathbb{C}$.

The co-FI-space $\mathcal{F}(\mathbb{C}, \bullet)$ given by $n \mapsto \mathcal{F}(\mathbb{C}, n)$, the configuration space of $n$ ordered points in $\mathbb{C}$, and the corresponding FI-modules $H^i(\mathcal{F}(\mathbb{C}, \bullet))$ are key in our discussion below. In the expository paper [17], representation stability and FI-modules are motivated mainly through this example. A formal discussion of FI-modules and their properties is given in [3]. In [4] the theory of FI-modules is extended to modules over arbitrary Noetherian rings.

The “shifted” co-FI-space $\mathcal{M}_{0, \bullet+1}$. Consider the functor $\mathcal{Z}_1$ from $\text{FI}$ to $\text{FI}$ given by $[n] \mapsto [n] \sqcup \{0\}$. Notice that this functor induces the inclusion of groups

$$J_n : S_n = \text{End}_{\text{FI}}([n]) \hookrightarrow \text{End}_{\text{FI}}([n+1]) = S_{n+1}$$

that sends the generator $(i, i+1)$ of $S_n$ to the transposition $(i+1, i+2)$ of $S_{n+1}$. In our discussion below we are interested in the “shifted” co-FI-space $\mathcal{M}_{0, \bullet+1}$ obtained by $\mathcal{M}_{0, \bullet} \circ \mathcal{Z}_1$. Notice that this co-FI-space is given by $n \mapsto \mathcal{M}_{0,n+1}$. In the notation from [4, Section 2], this means that the FI-module

$$H^i(\mathcal{M}_{0, \bullet+1}) = S_{i+1}(H^i(\mathcal{M}_{0, \bullet})),$$
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where $S_{+1} : \text{FI-Mod} \rightarrow \text{FI-Mod}$ is the shift functor given by $S_{+1} := - \otimes \mathbb{Z}_1$. The functor $S_{+1}$ performs the restriction, from an $S_{n+1}$-representation to an $S_n$-representation, consistently for all $n$ so that the resulting sequence of representations still has the structure of an FI-module. Comparing the $S_n$-representation $H^i(M_{0,n+1})$ with the $S_{n+1}$-representation $H^i(M_{0,\bullet})_{n+1} = H^i(M_{0,n+1})$ we have an isomorphism of $S_n$-representations

$$H^i(M_{0,n+1}) \cong \text{Res}_{S_n}^{S_{n+1}} H^i(M_{0,n+1}).$$

3 Relation with the configuration space

We will understand the cohomology ring of $\mathcal{M}_{0,n}$ through its relation with the configuration space $\mathcal{F}(\mathbb{C}, n)$ of $n$ ordered points in $\mathbb{C}$.

In our descriptions below, we consider $\mathbb{P}^1(\mathbb{C})$ with coordinates $[t : z]$ and the embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$, given by $z \mapsto [1 : z]$ and let $[0 : 1] = \infty$. We use the brackets to indicate “equivalence class of”. Since there is a unique element in PGL$_2(\mathbb{C})$ that takes any three distinct points in $\mathbb{P}^1(\mathbb{C})$ to $([0 : 1], [1 : 0], [1 : 1]) = (\infty, 0, 1)$, every element in $\mathcal{M}_{0,n+1}$ can be written canonically as $\left(\left([0 : 1], [1 : 0], [1 : 1], [t_1 : z_1], \ldots, [t_{n-2} : z_{n-2}]\right)\right)$. Hence, $\mathcal{M}_{0,4} \cong \mathbb{P}^1(\mathbb{C}) \backslash \{\infty, 0, 1\}$ and $\mathcal{M}_{0,n+1} \cong \mathcal{F}(\mathcal{M}_{0,4}, n - 2)$.

Define the map $\psi : \mathcal{F}(\mathbb{C}, n) \rightarrow \mathcal{M}_{0,n+1}$ by

$$\psi(z_1, z_2, \ldots, z_n) = \left(\infty, 0, 1, \frac{z_3 - z_1}{z_2 - z_1}, \frac{z_4 - z_1}{z_2 - z_1}, \ldots, \frac{z_n - z_1}{z_2 - z_1}\right).$$

The symmetric group $S_n$ acts on $\mathcal{F}(\mathbb{C}, n)$ by permuting the coordinates. Let $(1 \ 2), (2 \ 3), \ldots, (n - 1 \ n)$ be transpositions generating $S_n$ and notice that

$$\psi((1 \ 2) \cdot (z_1, z_2, \ldots, z_n)) = \left(\infty, 0, 1, \frac{z_3 - z_2}{z_1 - z_2}, \frac{z_4 - z_2}{z_1 - z_2}, \ldots, \frac{z_n - z_2}{z_1 - z_2}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \cdot \left(\left([0 : 1], [1 : 0], [1 : 1], [1 : \frac{z_3 - z_2}{z_1 - z_2}], [1 : \frac{z_4 - z_2}{z_1 - z_2}], \ldots, [1 : \frac{z_n - z_2}{z_1 - z_2}]\right)\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left(\left([0 : 1], [1 : 1], [1 : 0], [1 : \frac{z_3 - z_1}{z_2 - z_1}], [1 : \frac{z_4 - z_1}{z_2 - z_1}], \ldots, [1 : \frac{z_n - z_1}{z_2 - z_1}]\right)\right)$$

$$= (2 \ 3) \cdot \psi(z_1, z_2, \ldots, z_n)$$

and in general

$$\psi((i \ i + 1) \cdot (z_1, z_2, \ldots, z_n)) = (i + 1) \cdot \psi(z_1, z_2, \ldots, z_n) \text{ for } i \geq 2.$$

Therefore the map $\psi : \mathcal{F}(\mathbb{C}, n) \rightarrow \mathcal{M}_{0,n+1}$ is equivariant with respect to the inclusion $J_n : S_n \hookrightarrow S_{n+1}$. In other words, $\psi : \mathcal{F}(\mathbb{C}, \bullet) \rightarrow \mathcal{M}_{0,\bullet+1}$ is a map of co-FI-spaces.
**Relation with the Coxeter arrangement of type** $A_{n−1}$. The complement of the complexified Coxeter arrangement of hyperplanes type $A_{n−1}$ is $M(A_{n−1})$, the image of $\mathcal{F}(\mathbb{C}, n)$ under the quotient map $\mathbb{C}^n \to \mathbb{C}^n/N$, where $N = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i = z_j \text{ for } 1 \leq i, j \leq n\}$.

As explained in [9], it turns out that the moduli space $\mathcal{M}_{0,n+1}$ is also in bijective correspondence with the projective arrangement

$$M(dA_{n−1}) := \pi(M(A_{n−1})) \cong M(A_{n−1})/\mathbb{C},$$

where $\pi : \mathbb{C}^{n−1}\setminus\{0\} \to \mathbb{P}^{n−2}(\mathbb{C})$ is the Hopf bundle projection, which takes $z \in \mathbb{C}^{n−1}\setminus\{0\}$ to $\lambda z$ for $\lambda \in \mathbb{C}^*$. Moreover, the map $\psi$ factors through $M(A_{n−1})$ and $M(dA_{n−1})$.

In [5], Gaiffi extends the natural $S_n$-action on $H^*(M(A_{n−1}); \mathbb{C})$ to an $S_{n+1}$-action using the vertical map in the diagram above and the natural $S_{n+1}$-action on $H^*(\mathcal{M}_{0,n+1}; \mathbb{C})$.

**The cohomology rings.** As proved in [9, Prop. 2.2 & Theorem 3.2] , the map $\psi$ allows us to relate the cohomology rings of $\mathcal{M}_{0,n+1}$ and $\mathcal{F}(\mathbb{C}, n)$. See also [11, Cor. 3.1].

**Proposition 3.1.** The maps $\psi$ induces an isomorphism of cohomology rings

$$H^*(\mathcal{F}(\mathbb{C}, n); \mathbb{C}) \cong H^*(\mathcal{M}_{0,n+1}; \mathbb{C}) \otimes H^*(\mathbb{C}^*; \mathbb{C})$$

as $S_n$-modules. The symmetric group $S_n$ acts trivially on $H^*(\mathbb{C}^*; \mathbb{C})$ and acts on $H^*(\mathcal{M}_{0,n+1}; \mathbb{C})$ through the inclusion $J_n : S_n \hookrightarrow S_{n+1}$ that sends the generator $(i+1)$ of $S_n$ to the transposition $(i+1)$ of $S_{n+1}$.

This means that the map of co-FI-spaces $\psi : \mathcal{F}(\mathbb{C}, \bullet) \to \mathcal{M}_{0,\bullet+1}$ induces an isomorphism of graded FI-modules

$$H^*(\mathcal{F}(\mathbb{C}, \bullet)) \cong H^*(\mathcal{M}_{0,\bullet+1}) \otimes H^*(\mathbb{C}^*),$$

where $H^*(\mathbb{C}^*)$ is the trivial graded FI-module given by $n \mapsto H^*(\mathbb{C}^*; \mathbb{C})$.

Furthermore, Arnol’d obtained a presentation of the cohomology ring of $\mathcal{F}(\mathbb{C}, n)$ in (11).

**Theorem 3.2.** The cohomology ring $H^*(\mathcal{F}(\mathbb{C}, n); \mathbb{C})$ is isomorphic to the $\mathbb{C}$-algebra $\mathcal{R}_n$ generated by 1 and forms $\omega_{i,j} := \frac{d\log|z_i - z_j|}{2\pi i}$, $1 \leq i \neq j \leq n$, with relations
\[ \omega_{i,j} = \omega_{i,j}, \quad \omega_{i,j} \cdot \omega_{k,l} = -\omega_{k,l} \cdot \omega_{i,j} \text{ and } \omega_{i,j} \cdot \omega_{j,k} + \omega_{j,k} \cdot \omega_{k,l} + \omega_{k,l} \cdot \omega_{i,j} = 0. \]

The action of \( S_n \) is given by \( \sigma \cdot \omega_{i,j} = \omega_{\sigma(i) \sigma(j)} \) for \( \sigma \in S_n \).

As a consequence of the isomorphism in Proposition 3.1, we also have a concrete description of the cohomology ring \( H^*(\mathcal{M}_{0,n+1}; \mathbb{C}) \). We refer the reader to [11, Cor. 3.1], [9, Theorem 3.4] and references therein.

**Theorem 3.3.** The cohomology ring \( H^*(\mathcal{M}_{0,n+1}; \mathbb{C}) \) is isomorphic to the subalgebra of \( \mathcal{K}_n \) generated by 1 and elements \( \theta_{i,j} := \omega_{i,j} - \omega_{i,2} \) for \( \{i,j\} \neq \{1,2\} \). The \( S_n \)-action given by \( \sigma \cdot \theta_{i,j} = \theta_{\sigma(i) \sigma(j)} - \theta_{\sigma(1) \sigma(2)} \), for \( \sigma \in S_n \).

## 4 Finite generation

We can use the explicit presentations in Theorems 3.2 and 3.3 to understand the FI-modules \( H^*(\mathcal{F}(\mathbb{C}, \bullet)) \) and \( H^*(\mathcal{M}_{0, \bullet+1}) \).

An FI-module \( V \) over \( \mathbb{C} \) is said to be **finitely generated in degree** \( \leq m \) if there exist \( v_1, \ldots, v_s \), with each \( v_i \in V_{n_i} \) and \( n_i \leq m \), such that \( V \) is the minimal sub-FI-module of \( V \) containing \( v_1, \ldots, v_s \). Finitely generated FI-modules have strong closure properties: extensions and quotients of finitely generated FI-modules are still finitely generated and finite generation is preserved when taking sub-FI-modules.

Notice that from Theorem 3.2 it follows that \( H^1(\mathcal{F}(\mathbb{C}, n); \mathbb{C}) \) is generated as an \( S_n \)-module by the class \( \omega_{1,2} \). Therefore, the FI-module \( H^1(\mathcal{F}(\mathbb{C}, \bullet)) \) is finitely generated in degree 2 by the class \( \omega_{1,2} \) in \( H^1(\mathcal{F}(2; \mathbb{C})) \).

Similarly, from Theorem 3.3 we know that \( H^1(\mathcal{M}_{0,n+1}) \) is generated by the \( \theta_{i,j} \) classes and notice that \( \theta_{i,j} = (j 3) \cdot \theta_{1,3} \) for \( j \neq \{1,2\} \); \( \theta_{2,1} = (j 3) \cdot \theta_{2,3} \) for \( j \neq \{1,2\} \) and \( \theta_{2,3} = (i 3)(j 4) \cdot \theta_{3,4} \) for \( \{i,j\} \neq \{1,2\} \). Therefore, \( H^1(\mathcal{M}_{0,n+1}) \) is generated by \( \theta_{1,3}, \theta_{2,3} \) and \( \theta_{3,4} \) as an \( S_n \)-module. This means that the FI-module \( H^1(\mathcal{M}_{0, \bullet+1}) \) is finitely generated by the classes \( \theta_{1,3}, \theta_{2,3} \) and \( \theta_{3,4} \) in \( H^1(\mathcal{M}_{0, \bullet+1}) \), hence in degree 4.

An FI-module \( V \) encodes the information of the sequence \( V_n \) of \( S_n \)-representations. Finite generation of \( V \) puts strong constraints on the decomposition of each \( V_n \) into irreducible representations and its character.

**Notation for representations of** \( S_n \). The irreducible representations of \( S_n \) over \( \mathbb{C} \) are classified by partitions of \( n \). A partition \( \lambda \) of \( n \) is a set of positive integers \( \lambda_1 \geq \cdots \geq \lambda_l > 0 \) where \( l \in \mathbb{Z} \) and \( \lambda_1 + \cdots + \lambda_l = n \). We write \( |\lambda| = n \). The corresponding irreducible \( S_n \)-representation will be denoted by \( V_\lambda \). Every \( V_\lambda \) is defined over \( \mathbb{C} \) and any \( S_n \)-representation decomposes over \( \mathbb{C} \) into a direct sum of irreducibles.

If \( \lambda \) is any partition of \( m \), i.e. \( |\lambda| = m \), then for any \( n \geq |\lambda| + \lambda_1 \) the **padded partition** \( \lambda[n] \) of \( n \) is given by \( n - |\lambda| \geq \lambda_1 \geq \cdots \geq \lambda_l > 0 \). Keeping the notation from [3], we set \( V(\lambda)_n = V_\lambda[|\lambda|] \) for any \( n \geq |\lambda| + \lambda_1 \). Every irreducible \( S_n \)-representation is of the form \( V(\lambda)_n \) for a unique partition \( \lambda \). We define the **length** of an irreducible representation of \( S_n \) to be the number of parts in the corresponding partition of \( n \).
The trivial representation has length 1, and the alternating representation has length \( n \). We define the \( \text{length} \ell(V) \) of a finite dimensional representation \( V \) of \( S_n \) to be the maximum of the lengths of the irreducible constituents.

We say that an FI-module \( V \) over \( \mathbb{C} \) has \textit{weight} \( \leq d \) if for every \( n \geq 0 \) and every irreducible constituent \( V(\lambda)_n \) we have \( |\lambda| \leq d \). The degree of generation of an FI-module \( V \) gives an upper bound for the weight (\cite{3} Prop. 3.2.5)). The weight of an FI-module is closed under subquotients and extensions. Moreover, if a finitely generated FI-module \( V \) has weight \( \leq d \), by definition, \( \ell(V_n) \leq d + 1 \) for all \( n \) and the alternating representation cannot not appear in the decomposition into irreducibles of \( V_n \) once \( n > d + 1 \).

Notice that the FI-module \( H^1(\mathcal{F}(\mathbb{C}, \bullet)) \) has weight at most 2 and so does \( H^1(\mathcal{M}_0, \bullet + 1) \), since it is a sub-FI-module of \( H^1(\mathcal{F}(\mathbb{C}, \bullet)) \).

An FI-module \( V \) has \textit{stability degree} \( \leq N \), if for every \( a \geq 0 \) and \( n \geq N + a \) the map of coinvariants

\[
(I_n)_*: (V_n)_{S_n} \rightarrow (V_{n+1})_{S_{(n+1)-a}}
\]

induced by the standard inclusion \( I_n : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, n, n + 1\} \), is an isomorphism of \( S_n \)-modules (see \cite{3} Definition 3.1.3 for a more general definition). Here, \( S_{n-a} \) is the subgroup of \( S_n \) that permutes \( \{a+1, \ldots, n\} \) and acts trivially on \( \{1, 2, \ldots, a\} \). The coinvariant quotient \( (V_n)_{S_{n-a}} \) is the \( S_n \)-module \( V_n \otimes_{\mathbb{C}[S_{n-a}]} \mathbb{C} \), the largest quotient of \( V_n \) on which \( S_{n-a} \) acts trivially.

The finite generation properties of the FI-modules \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \) have already been discussed in \cite{3} Example 5.1.A.

**Proposition 4.1.** The FI-module \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \) is finitely generated with weight \( \leq 2i \) and has stability degree \( \leq 2i \).

**Proof.** From Theorem 3.2 the graded FI-module \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \) is generated by the FI-module \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \) that has weight \( \leq 2 \). It follows by \cite{3} Theorem 4.2.3 that \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \) is finitely generated with weight \( \leq 2i \). Moreover, in \cite{3} it is shown that \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \) has the additional structure of what \cite{3} calls an FI#-module, which implies that it has stability degree bounded above by the weight (see proof of \cite[Cor. 4.1.8]{3}).

Finite generation for the FI-modules \( H^i(\mathcal{M}_0, \bullet + 1) \) follows from Theorem 3.3 and Proposition 4.1.

**Theorem 4.2.** The FI-module \( H^i(\mathcal{M}_0, \bullet + 1) \) is finitely generated in degree \( \leq 4i \), with weight \( \leq 2i \) and has stability degree \( \leq 2i \).

**Proof.** By Theorem 3.3 the graded FI-module \( H^i(\mathcal{M}_0, \bullet + 1) \) is generated by the FI-module \( H^i(\mathcal{M}_0, \bullet + 1) \), which is finitely generated in degree \( \leq 4 \) and has weight \( \leq 2 \). It follows from \cite{3} Proposition 2.3.6] that the FI-module \( H^i(\mathcal{M}_0, \bullet + 1) \) is finitely generated in degree \( \leq 4i \). By \cite{3} Corollary 4.2.1 it has weight \( \leq 2i \).

Moreover, from \cite{3} Lemma 3.1.6 we have that the stability degree of \( H^i(\mathcal{M}_0, \bullet + 1) \) is bounded above by the stability degree of \( H^i(\mathcal{F}(\mathbb{C}, \bullet)) \).
We can also relate the weights and stability degrees using the classical branching rule (see e.g. [8]).

The case when

\[ \text{Theorem 4.5.} \]

\[ \text{Proposition 4.3, it comes from the restriction of some} \]

\[ \text{of} \]

\[ \text{V}_{n+1} \]

\[ \text{if} \]

\[ \text{M} \]

\[ \text{generated FI-module generated in degree} \]

\[ \text{over those partitions} \]

\[ \text{The cohomology of} \]

\[ \text{H} \]

\[ \text{From} \]

\[ \text{H}^i(M_{g, \bullet}) \]

\[ \text{to the FI-module} \]

\[ \text{H}^i(M_{g, \bullet}) \]

\[ \text{The relation between the degree of} \]

\[ \text{generation of an FI-module} \]

\[ \text{and its “shift”} \]

\[ \text{S}_{n+1}V \]

\[ \text{was established in [4 Cor. 2.13].} \]

\[ \text{We can also relate the weights and stability degrees using the classical branching rule (see e.g. [8]).} \]

\[ \text{Proposition 4.3. Let } \lambda \text{ be a partition of } n+1 \text{ and } V_\lambda \text{ the corresponding irreducible} \]

\[ \text{S}_{n+1} \]-representation, then as } S_n \text{-representations we have the decomposition} \]

\[ \text{Res}^{S_{n+1}}_{S_n} V_\lambda \cong \bigoplus_v V_v \]

\[ \text{over those partitions } v \text{ of } n \text{ obtained from } \lambda \text{ by removing one box from one of the columns of the corresponding Young diagram.} \]

\[ \text{Theorem 4.4 (Finite generation and “shifted” FI-modules). Let } V \text{ be a finitely generated FI-module generated in degree } d \text{, then } S_{n+1}V \text{ is finitely generated in degree } d \leq d. \]

\[ \text{Conversely, if the FI-module } S_{n+1}V \text{ is finitely generated in degree } d \leq d, \text{ then } V \text{ is finitely generated in degree } d+1. \]

\[ \text{Furthermore, if } S_{n+1}V \text{ has weight } M \text{ and stability degree } N, \text{ then } V \text{ has weight } M+1 \text{ and stability degree } N+1. \]

\[ \text{Conversely, if } V \text{ has weight } M \text{ and stability degree } N, \text{ then } S_{n+1}V \text{ has weight } M \text{ and stability degree } N. \]

\[ \text{Proof.} \text{ If } V \text{ has weight } M, \text{ then for all } n \geq 0, \text{ the irreducible components } V(\mu)_{n+1} \text{ of } V_{n+1} \text{ have } |\mu| \leq M. \]

\[ \text{From Proposition 4.3 it follows that } \text{Res}^{S_{n+1}}_{S_n} V(\mu)_{n+1} \text{ will be a direct sum of irreducibles } V(\lambda)_n, \text{ with } |\lambda| |\mu| \leq M. \]

\[ \text{Conversely, if } S_{n+1}V \text{ has weight } M, \text{ then each irreducible component } V(\lambda)_n \text{ of } S_{n+1}V_n \text{ has } |\lambda| \leq M. \]

\[ \text{By Proposition 4.3 it comes from the restriction of some } V(\mu)_{n+1} \text{ with } |\mu| \leq |\lambda|+1 \leq M+1. \]

\[ \text{On the other hand, the functor } \Xi_1 \text{ sends } \{1, \ldots, a\} \text{ into } \{2, \ldots, a+1\} \text{ and } \{a+1, \ldots, n\} \text{ into } \{a+2, \ldots, n+1\}. \]

\[ \text{Therefore, the inclusion } J_n : S_n \hookrightarrow S_{n+1} \text{ maps the subgroup } S_{n-a} \text{ of } S_n \text{ onto the subgroup } S_{n+1-(a+1)} \text{ of } S_{n+1} \text{ and we have that} \]

\[ \text{V}_{n+1})_{S_{n+1-(a+1)}} = V_{n+1} \otimes C(S_{(n+1)-(a+1)}) C = S_{n+1}(V)_n \otimes C(S_{n-a}) C = (S_{n+1}(V)_n)_{S_{n-a}}, \]

\[ \text{which implies the statement about stability degrees.} \]

\[ \text{In [12] we proved finite generation for the FI-modules } H^i(M_{g, \bullet}) \text{ when } g \geq 2. \]

\[ \text{The case when } g = 0 \text{ follows from Theorem 4.4 and Theorem 4.2} \]

\[ \text{Theorem 4.5. The FI-module } H^i(M_{0, \bullet}) \text{ is finitely generated with weight } \leq 2i+1 \text{ and has stability degree } \leq 4i. \]

\[ \text{The first cohomology group. Recall that } H^1(M_{0, \bullet+1})_n \text{ is generated by the classes} \]

\[ \text{of } \theta_{i,j} = \omega_{i,j} - \omega_{1,2} \text{ and it is a subrepresentation of } H^1(\mathcal{F}(C,n)) \text{ which has a basis given by the classes} \]

\[ \text{of } \omega_{i,j}. \text{In particular, notice that } \dim H^1(M_{0, \bullet+1})_n = \dim H^1(\mathcal{F}(C,n)) - 1. \]

\[ \text{Moreover for } n \geq 4, \text{ we have the decomposition} \]

\[ H^1(\mathcal{F}(C,n)) = V(0)_n \oplus V(1)_n \oplus V(2)_n. \]
Then, for \( n \geq 4 \) the \( S_n \)-representation

\[
H^1(M_{0,\bullet+1})_n = V(1)_n \oplus V(2)_n \cong \text{Res}^{S_{n+1}}_{S_n} H^1(M_{0,\bullet+1}).
\]

Proposition 4.3 implies that for \( n \geq 4 \), we have that \( H^1(M_{0,\bullet+1}) = V(2)_{n+1} \) as a representation of \( S_{n+1} \). Moreover, notice that \( H^1(M_{0,\bullet+1}) \) is finitely generated by the classes \( \theta_{1,3}, \theta_{2,3} \) and \( \theta_{3,4} \) in \( H^1(M_{0,5}) \) not only as an \( S_n \)-module, but also as an \( S_{n+1} \)-module. Therefore, the \( \text{FI-module} H^1(M_{0,\bullet}) \) is finitely generated in degree \( \leq 5 \) and has weight \( \leq 2 \).

## 5 The \( S_n \)-representations \( H^i(M_{0,\bullet}; \mathbb{C}) \)

At this point we can apply the theory of \( \text{FI-modules} \) to the finitely generated \( \text{FI-modules} H^i(M_{0,\bullet}) \) and \( H^i(M_{0,\bullet+1}) \) to obtain information about the corresponding sequences of \( S_n \)-representations and their characters. The following result is a direct consequence from [3, Prop. 3.3.3 and Theorem 3.3.4] and Theorems 4.2 and 4.5.

**Theorem 5.1.** Let \( i \geq 0 \). For \( n \geq 4i + 2 \), the sequence \( \{H^i(M_{0,n})\}_n \) of representations of \( S_n \) and the sequence \( \{H^i(M_{0,\bullet+1})_n\}_{n=1}^{\infty} \) of \( S_n \)-representations satisfy the following:

(a) The decomposition into irreducibles of \( H^i(M_{0,n}; \mathbb{C}) \) and of \( H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1} \) stabilize in the sense of uniform representation stability ([3]) with stable range \( n \geq 4i + 2 \).

(b) The length of \( H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1} \) is bounded above by \( 2i \) and the length of \( H^i(M_{0,n}; \mathbb{C}) \) is bounded above by \( 2i + 1 \).

(c) The sequence of characters of the representations \( H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1} \) and \( H^i(M_{0,n}; \mathbb{C}) \) are eventually polynomial, in the sense that there exist character polynomials \( P_i(X_1, X_2, \ldots, X_r) \) and \( Q_i(X_1, X_2, \ldots, X_s) \) in the cycle-counting functions \( X_k(\sigma) := (\text{number of } k\text{-cycles in } \sigma) \) such that for all \( n \geq 4i + 2 \):

\[
\chi_{H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1}}(\sigma) = P_i(X_1, X_2, \ldots, X_r)(\sigma) \quad \text{for all } \sigma \in S_{n-1}, \text{ and}
\]

\[
\chi_{H^i(M_{0,n}; \mathbb{C})}(\sigma) = Q_i(X_1, X_2, \ldots, X_s)(\sigma) \quad \text{for all } \sigma \in S_n.
\]

Moreover, the degree of \( P_i \) is \( \leq 2i \) and the degree of \( Q_i \) is \( \leq 2i + 1 \), where we take \( \deg X_k = k \). In particular, \( r \leq 2i \) and \( s \leq 2i + 1 \).

If \( e \in S_{n-1} \) is the identity element, from Theorem 5.1(c), we obtain that the dimensions

\[
\dim_{\mathbb{C}}(H^i(M_{0,n}; \mathbb{C})) = \chi_{H^i(M_{0,\bullet+1}; \mathbb{C})_{n-1}}(e) = P_i(X_1(e), \ldots, X_r(e)) = P_i(n-1, \ldots, 0)
\]
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are polynomials in $n$ of degree $\leq 2i$. This agrees with the known Poincaré polynomial of $M_{0,n}$ (see \[13, Cor. 2.10\] and also \[11, 5.5(8)\]).

From Theorem \[15, 4.1\] and the definition of weight, we recover the fact that the alternating representation does not appear in the cohomology of $M_{0,n}$ (\[15, Prop. 2.16\]).

Theorem \[5, 5.1(a)\] implies that the dimensions of the vector spaces $H^i(M_{0,n}/S_n; \mathbb{C})$ and $H^i(M_{0,n+1}/S_n; \mathbb{C})$ are constant. For the sequence $\{M_{0,n}/S_n\}$, this is actually a trivial consequence from the fact that $M_{0,n}/S_n$ has the cohomology of a point as shown in \[15, Theorem 2.3\].

**Recursive relation for characters.** In \[15, Theorem 4.1\], Gaiffi obtained a recursive formula that connects the characters of the $S_n$-representations $H^i(M_{0,n+1}; \mathbb{C})$ and $H^i(M_{0,n})$ as follows

$$\chi_{H^i(M_{0,n+1}), n} = \chi_{H^i(M_{0,n})} + (X_1 - 1) \cdot \chi_{H^{i-1}(M_{0,n})} \quad \text{for } n \geq 3. \quad (2)$$

In particular, we know that $\chi_{H^i(M_{0,n+1}), n} = \chi_{H^i(M_{0,1})} - 1 = \binom{n}{2} + X_2 - 1$ when $n \geq 4$. Therefore, for $i = 1$, the recursive formula (2) gives us the character polynomial of degree 2

$$\chi_{H^1(M_{0,n})} = \chi_{H^1(M_{0,n+1}), n} - (X_1 - 1) \cdot \chi_{H^0(M_{0,n})} = \binom{n}{2} + X_2 - X_1 = \chi_{V(2)}$$

as expected since $H^1(M_{0,n}) = V(2)_n$.

Furthermore, if $P_i$ and $Q_i$ are the character polynomials of $H^i(M_{0,n+1})$ and $H^i(M_{0,n})$ from Theorem \[5, 5.1\] (c) for $n \geq 4i + 2$, then formula (2) can be written as $Q_i = P_i - (X_1 - 1) \cdot Q_{i-1}$ and $\deg Q_i \leq \max (\deg P_i, 1 + \deg Q_{i-1}) \leq 2i$. As a consequence of this and Theorem \[5, 5.1\] (c) we have that, for $n \geq 4i + 2$, the values of $\chi_{H^i(M_{0,n+1}; \mathbb{C})}(\sigma)$ and $\chi_{H^i(M_{0,n}; \mathbb{C})}(\sigma)$ depend only on “short cycles”, i.e. cycles on $\sigma$ of length $\leq 2i$.

**More is known about the $S_n$-representations.** In this paper we were mainly interested in highlighting the methods, since more precise information about the characters of the $S_n$-representations is known. The moduli space $M_{0,n}$ can be represented by a finite type $\mathbb{Z}$-scheme and the manifold $M_{0,n}(\mathbb{C})$ of $\mathbb{C}$-points of this scheme corresponds to the definition in Section \[1\]. In \[13\], Kisin and Lehrer used an equivariant comparison theorem in $\ell$-adic cohomology and the Grothendieck-Lefschetz’s fixed point formula to obtain explicit descriptions of the graded character of the $S_n$-action on the cohomology of $M_{0,n}(\mathbb{C})$ via counts of number of points of varieties over finite fields. With their techniques they obtain the Poincaré polynomial of a permutation in $S_n$ of a specific cycle type acting on $H^*(M_{0,n}; \mathbb{C})$ (\[15, Theorem 2.9\]) and a description of the top cohomology $H^{n-3}(M_{0,n}; \mathbb{C})$ (\[15, Proposition 2.18\]). Furthermore, Getzler uses the language of operads in \[11\] to obtain formulas for the characters of the $S_n$-modules $H^i(M_{0,n}; \mathbb{C})$. 
The cohomology of $\overline{M}_{0,n}$. A related space of interest is $\overline{M}_{0,n}$, the Deligne-Mumford compactification of $M_{0,n}$. It is a fine moduli space for stable $n$-pointed rational curves for $n \geq 3$ (see [16, Chapter 1] and reference therein). It can also be constructed from $M(dA_{n-1})$ using the theory of wonderful models of hyperplanes arrangements developed by De Concini and Procesi (see for example [10, Chapter 2]). The space $\overline{M}_{0,n}$ also carries a natural action of the symmetric group $S_n$. Hence, a natural question to ask is whether the FI-module theory could tell us something about its cohomology groups as $S_n$-representations.

Explicit presentations of the cohomology ring of the manifold of complex points $\overline{M}_{0,n}(\mathbb{C})$ have been obtained by Keel [14] and Yuzvinsky [19]. Moreover, several recursive and generating formulas for the Poincaré polynomials have been computed (for instance see [13], [11], [17], [2]). The sequence $H^i(\overline{M}_{0,n}(\mathbb{C});\mathbb{C})$ has the structure of an FI-module, however, the Betti numbers of $\overline{M}_{0,n}(\mathbb{C})$ grow exponentially in $n$, which precludes finite generation. Therefore an analogue of Theorem 5.1 cannot be obtained for this space.

On the other hand, as observed in [6], the manifold $\overline{M}_{0,n}(\mathbb{R})$ of real points of $\overline{M}_{0,n}$ is topologically similar to $\mathcal{F}(\mathbb{C},n-1)$, the configuration space of $n-1$ ordered points in $\mathbb{C}$, in the sense that both are $K(\pi,1)$-spaces, have Poincaré polynomials with a simple factorization and Betti numbers that grow polynomially in $n$. The cohomology ring of the real locus $\overline{M}_{0,n}(\mathbb{R})$ was completely determined in [6] and an explicit formula for the graded character of the $S_n$-action was obtained in [18]. The presentation of the cohomology ring given in [6] can be used to prove finite generation for the FI-modules $H^i(\overline{M}_{0,n}(\mathbb{R});\mathbb{C})$ and to obtain an analogue of Theorem 5.1 for this space (see [12]).

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