Renormalization of Hamiltonians

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Abstract. A matrix model of an asymptotically free theory with a bound state is solved using a perturbative similarity renormalization group for hamiltonians. An effective hamiltonian with a small width, calculated including the first three terms in the perturbative expansion, is projected on a small set of effective basis states. The resulting small hamiltonian matrix is diagonalized and the exact bound state energy is obtained with accuracy of order 10%. Then, a brief description and an elementary illustration are given for a related light-front Fock space operator method which aims at carrying out analogous steps for hamiltonians of QCD and other theories.

1. INTRODUCTION

This lecture has two aims. The first aim is to show a simple example of a new kind of calculation of effective hamiltonians, based on the perturbative similarity renormalization group. The second aim is to show how one can generalize the simple example and start systematic perturbative calculations for quantum field theoretic hamiltonians in the light-front Fock space.

Although the methods we present are quite general, the main motivation came from QCD. QCD is asymptotically free and its perturbative running coupling constant grows at small momentum transfers beyond limits. This rise invalidates usual perturbative expansions in the region of scales where the bound states are formed.

Ref. outlined a light-front hamiltonian approach to this problem in QCD, using the perturbative similarity renormalization group. Independently, Wegner proposed a flow equation for hamiltonians in solid state physics. He introduced an explicit expression for the generator of the similarity transformation which leads to a Gaussian similarity factor of a uniform width.

Wilson and I have solved numerically a simple matrix model to gain quantitative experience with the similarity scheme using Wegner's equation. We also made perturbative studies. This lecture is based on those works in the
part describing the model. The remaining part contains an outline of how one can attempt to make similar steps for light-front hamiltonians in quantum field theory using creation and annihilation operators. [7]

2. MODEL

Consider a quantum theory which is characterized by a large range of energy scales as measured by certain $H_0$. QCD has this feature. It extends in energies from $\infty$ (asymptotic freedom) down to the infrared energy region. We represent the theory by a model with a hamiltonian $H = H_0 + H_I$ acting in a space spanned by a finite discrete set of nondegenerate eigenstates of the hamiltonian $H_0$, 

$$H_0|i\rangle = E_i|i\rangle.$$  \hspace{1cm} (2.1)

Matrix elements of the interaction are assumed to be 

$$<i|H_I|j\rangle = -g\sqrt{E_iE_j}.$$  \hspace{1cm} (2.2)

g is a dimensionless coupling constant.

We choose $E_i = 2^i$ and $M \leq i \leq N$. $M$ is large and negative and $N$ is large and positive. We use $M = -21$ and $N = 16$ in our numerical example. Let the energy equal 1 correspond to 1 GeV. Then, the ultraviolet cutoff corresponds to 65 TeV and the infrared cutoff corresponds to 0.5 eV.

The same model can be alternatively derived by discretization of the 2-dimensional Schrödinger equation with a potential of the form a coupling constant times a $\delta$-function. [8]

For $g > 1/38$, the hamiltonian matrix has one negative eigenvalue and 37 positive eigenvalues. $g$ is adjusted to obtain the negative eigenvalue equal $-1$ GeV; $g \sim 0.06$. This eigenvalue corresponds to the s-wave bound state energy in the 2-dimensional Schrödinger equation.

We calculate effective hamiltonians, $\mathcal{H} \equiv \mathcal{H}(\lambda)$, using the similarity renormalization group equations in the differential form. The effective hamiltonians are parametrized by their energy width $\lambda$. The notion of the hamiltonian width will become clear shortly. We use Wegner’s flow equation [4]

$$\frac{d\mathcal{H}}{d\lambda^2} = -\frac{1}{\lambda^4}[[\mathcal{D}, \mathcal{H}], \mathcal{H}],$$  \hspace{1cm} (2.5)

with the initial condition $\mathcal{H}(\infty) = H$. The matrix $\mathcal{D}$ is the diagonal part of $\mathcal{H}$ with elements $\mathcal{D}_{mn} = \mathcal{H}_{mm}\delta_{mn}$. Thus, $\mathcal{H}(\lambda)$ is a unitary transform of $H$ and both have the same spectrum (see Wegner’s lecture in this volume).

Equation (2.5) can be approximately solved for a small $g$ keeping only terms order 1 and $g$. One obtains 

$$\mathcal{H}_{mn} = E_m\delta_{mn} - g\sqrt{E_mE_n}\exp \left[\frac{-(E_m - E_n)^2}{\lambda^2}\right].$$  \hspace{1cm} (2.6)
Here, $D_{mm} = (1 - g)E_m$. The Gaussian factor of width $\lambda$ is the similarity function. This explains the notion of the hamiltonian width. Ref. [5] demonstrated that the Wegner flow equation has a renormalization group interpretation. Including terms order $g^2$, we let $g$ depend on $\lambda$ and we introduce $\tilde{g}(\lambda) \equiv \tilde{g}$. It follows from equations satisfied by the matrix elements $H_{mn}$ with the indices $m$ and $n$ close to $M$ that, neglecting small energies,

$$\frac{d\tilde{g}}{d\lambda} = -\tilde{g}^2 \sum_\ell \exp\left[-2E_{\ell}^2/\lambda^2\right], \tag{2.7}$$

and $\tilde{g}(\infty) = g$. Analytic integration of Eq. (2.7) in the model gives, approximately,

$$\tilde{g}_a(\lambda) = (1.45 \log \lambda - 0.9)^{-1}. \tag{2.8}$$

$\tilde{g}_a(\lambda)$ grows when $\lambda$ gets smaller and it exhibits the asymptotic freedom behavior: it is smaller for more violent interactions (i.e. of wider range in energy). $\tilde{g}_a(\lambda)$ blows up to infinity for $\lambda \sim 1.9$ GeV. In this approximation, matrix elements of $\mathcal{H}$ for $E_m \sim E_n \ll \lambda$ can be written as

$$H_{mn}(\lambda) = E_m\delta_{mn} - \tilde{g}_a(\lambda) \sqrt{E_mE_n} \exp\left[-|E_m - E_n|^2/\lambda^2\right] + \text{corrections}. \tag{2.9}$$

![Figure 1](image-url)

**Figure 1.** The approximate running coupling $\tilde{g}_a(\lambda)$ from Eq. (2.8) and the exact running coupling $\tilde{g}(\lambda)$, plotted as functions of the effective hamiltonian width $\lambda$. The matrix element $\tilde{\mu}(\lambda) = \mathcal{H}_{-1,-1}(\lambda) - 0.5$ GeV is also plotted to show the width range where the bound state eigenvalue appears on the diagonal.
Now, $D_{mm}(\lambda) = [1 - \tilde{g}_a(\lambda)]E_m$. The energy order of low energy states is reversed when $\tilde{g}_a(\lambda)$ grows above 1.

The exact running coupling, $\tilde{g}(\lambda)$, is defined by writing $H_{M,M+1}(\lambda) = -\tilde{g}(\lambda) \sqrt{E_M E_{M+1}}$. Eq. (2.9) shows that $\tilde{g}_a = \tilde{g}$ for large $\lambda$. To find $\tilde{g}$ for all values of $\lambda$, we solved Eq. (2.5) numerically. Fig. 1. shows that the approximate solution blows up in the flow before the effective hamiltonian width is reduced to the scale where the bound state is formed. That scale, order 1 GeV, equals $\lambda$ at which the bound state eigenvalue appears on the diagonal. The diagonal matrix element is also shown in Fig. 1.

The key feature, visible in Fig. 1, is that the exact effective coupling constant does not grow unlimitedly. The similarity renormalization group for hamiltonians provides a new option for investigating bound state dynamics in asymptotically free theories. The question is how far down in $\lambda$ we can reach using perturbation theory instead of the exact solution. The answer is: down to 1 GeV in second order with 10% accuracy. This is illustrated in Fig. 2.

The remaining question of how small the space of states can be on which one

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The accuracy of the bound state eigenvalues obtained from effective hamiltonians whose renormalization group flow with the width $\lambda$ is calculated expanding in powers of the effective coupling constant $\tilde{g}(\lambda_0)$ and including terms order 1, $\tilde{g}(\lambda_0)$, and $\tilde{g}^2(\lambda_0)$. The accuracy is given as ratio of the bound state eigenvalue obtained by diagonalization of the effective hamiltonian of width $\lambda$ to the exact value, $-1$ GeV. The curves correspond to the indicated values of $\lambda_0$ (in units of GeV). The result of expansion in the initial coupling $g$ is denoted by $\infty$. The arrows show points where $\lambda = \lambda_0$.}
\end{figure}
Table I. Ratio of the bound state eigenvalue of the small window hamiltonian with indices limited by $\tilde{m}$ and $\tilde{n}$, to the eigenvalue of the whole effective hamiltonian at $\lambda = 1$ GeV calculated using expansion up to second power in the running coupling $\tilde{g}(1\text{GeV})$. 0.993 corresponds to the absolute accuracy of the bound state eigenvalue equal 12% and 0.908 to 19% (see the text).

The model study shows that the perturbative similarity renormalization group allows a calculation of a small width effective hamiltonian, which can be projected on a small space of states. The small hamiltonian can be solved exactly and the bound state eigenvalue of the full theory is obtained with 10% accuracy. The question is how to repeat these steps in quantum field theory.

The method we propose [7] is based on the idea that one can unitarily transform the creation and annihilation operators, i.e.

$$a_\lambda^\dagger = U_\lambda a_\infty^\dagger U_\lambda^\dagger,$$

and the same for $a$’s. $a_\infty^\dagger$ and $a_\infty$ appear in the initial hamiltonian $H$. We call them “bare”. $a_\lambda^\dagger$ and $a_\lambda$ appear in the effective hamiltonian $H_\lambda$. They create and annihilate effective particles. In a way, $U_\lambda$ is analogous to the Melosh transformation in the case of quarks. However, we are building the transformation using the similarity renormalization group idea, the transformation is fully dynamical and it can be applied to other particles than quarks, too.

The effective hamiltonians satisfy the equation

$$\frac{d}{d\lambda} H_\lambda = [H_\lambda, T_\lambda],$$

where $T_\lambda = U_\lambda^\dagger dU_\lambda/d\lambda$. The unitary transformation generator $T$ is constructed so that the effective hamiltonians have width $\lambda$ in the relative momentum transfer,

$$H_\lambda = F_\lambda [\vec{g}_\lambda].$$
The operation \( F_\lambda \) on the interaction terms \( \mathcal{G}_\lambda \), inserts the similarity factors, \( f_\lambda \). They are most easy to think about as form factors in the interaction vertices. The smaller is \( \lambda \) the softer are the interactions and the effective particles get more dressed.

Following the general idea of the similarity scheme [2], one can find the equation satisfied by the vertex operators in the effective hamiltonians [7], i.e.

\[
\frac{d}{d\lambda} \mathcal{G}_\lambda = \left\{ f_\lambda \mathcal{G}_{2\lambda}, \left\{ \frac{d}{d\lambda} (1 - f_\lambda) \mathcal{G}_{2\lambda} \right\} \mathcal{G}_{1\lambda} \right\}. \quad (3.4)
\]

\( \mathcal{G}_\lambda = \mathcal{G}_{1\lambda} + \mathcal{G}_{2\lambda} \), \( \mathcal{G}_{1\lambda} \) is the \( a^+a \) part of the hamiltonian and \( \mathcal{G}_{2\lambda} \) is the remaining part which changes momenta of the individual particles. The curly bracket with subscript \( \mathcal{G}_{1\lambda} \) denotes the similarity energy denominator factor.

An elementary example illustrates how it works in Yukawa theory which is defined by the following initial hamiltonian

\[
H_Y = \int dx^- d^2x^\perp \left[ \bar{\psi}_m \gamma^\perp - \partial^\perp - \frac{m^2}{2i\partial^+} \bar{\psi}_m + \frac{1}{2} \phi(-\partial^\perp + \mu^2) \phi 
+ g \bar{\psi}_m \psi_m \phi + g^2 \bar{\psi}_m \phi \gamma^+ \phi \psi_m \right]_{x^+ = 0}. \quad (3.5)
\]

The one particle energy is obtained in the form,

\[
\mathcal{G}_{1\text{meson}} \lambda = \int [k] \frac{k^\perp}{k^+} a_{k^+}^+ a_k. \quad (3.6)
\]

In second order perturbation theory in the coupling constant \( g \), Eq. (3.4) implies

\[
\frac{d\mu^2}{d\lambda} = g^2 \int [x\kappa] \frac{df^2(z^2_\lambda)}{d\lambda} \frac{8(x - \frac{1}{2})^2 \mathcal{M}^2}{\mathcal{M}^2 - \mu^2} r_\epsilon (x, \kappa). \quad (3.7)
\]

Here \( \mathcal{M}^2 = (\kappa^2 + m^2)/x(1-x) \) and \( r_\epsilon (x, \kappa) \) denotes the regularization factor which is an analog of the number \( N \) in the matrix model. The similarity function \( f^2(z^2_\lambda) \) can be made as simple as, for example, \( \theta(\lambda^2 + 3\mu^2 - \mathcal{M}^2) \). In this case, integration of Eq. (3.7) gives the following effective meson mass term

\[
\mu^2 = \mu^2_1 + \frac{\alpha}{48\pi} \left[ \lambda^2 - \lambda_1^2 + (\mu^2 - 6m^2) \log \frac{\lambda^2}{\lambda_1^2} \right] + \mu^2_{\text{conv}} (\lambda, \lambda_1) + o(g^4). \quad (3.8)
\]

\( \mu^2_{\text{conv}}(\lambda, \lambda_1) \) denotes a finite term which has a limit when \( \lambda \to \infty \). It equals 0 for \( \lambda = \lambda_1 \). \( \mu_1 \) is the effective meson mass in the hamiltonian \( \mathcal{H}(\lambda_1) \). In the second order calculation, it is equal to the physical meson mass if \( \lambda_1^2 \leq 4m^2 - 3\mu^2 \).

The reason I show this example is that one can do similar calculations for other terms in the effective hamiltonians. [6] For example, in second order perturbation theory, effective interactions between quarks are partly similar to the results obtained by Perry and his collaborators. [9] [10] [11]
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The questions how many orders of perturbation theory are required in the calculation of the effective hamiltonian for constituent quarks and gluons in QCD and how large must be the subspace of the light-front Fock space to diagonalize the effective hamiltonian of QCD, require much more work to answer than in the matrix model.

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