Multisymplectic formalism for cubic Horndeski theories

Mauricio Doniz\(^1\)\(^,\)\(^2\)\(^,\)\(^\ast\) and Jordi Gaset\(^1\)\(^,\)\(^2\)

\(^1\) Department of Mathematics, Universitat Politècnica de Catalunya, Spain
\(^2\) Department of Applied Mathematics, Universidad Politècnica de Madrid, Spain

\(^\ast\) Author to whom any correspondence should be addressed.

E-mail: mauricio.doniz@upc.edu and jordi.gaset@unir.net

Keywords: higher-order field theories, covariant Hamiltonian field theory, multisymplectic structure, Horndeski theory

Abstract

We present the covariant multisymplectic formalism for the so-called cubic Horndeski theories and discuss the geometrical and physical interpretation of the constraints that arise in the unified Lagrangian-Hamiltonian approach. We analyse in more detail the covariant Hamiltonian formalism of these theories and we show that there are particular conditions that must be satisfied for the Poincaré-Cartan form of the Lagrangian to project onto \(J^1\pi\). From this result, we study when a formulation using only multimomenta is possible. We further discuss the implications of the general case, in which the projection onto \(J^1\pi\) conditions are not met.

1. Introduction

The multisymplectic formalism is a generalisation of symplectic geometry for field theories. It provides a covariant framework of the Lagrangian, Hamiltonian and the Hamilton-Jacobi formulations for field theories. The geometrical aspects of the multisymplectic formalism for first order theories, and its manifolds and forms have been studied in detail in [1–7]. Throughout this work we go through the features and results of the formalism that we will need. To understand the formalism at a deeper level, [8–10] constitute a good starting point. Relevant references regarding the mathematical precursors of the formalism include [11] and [12]. These works, along with [13] contain explicit application of the formalism and its precursors to relativistic theories.

It is possible study field theories of up to second-order with this formalism. In particular, [14] covers the most relevant features of the second-order multisymplectic formalism. However, third and higher order field theories do not have unique Poincaré-Cartan forms, although some efforts in that direction have been discussed in [15, 16]. Besides this, more problems regarding the non-uniqueness of the geometrical structures appear in the definition of the Legendre maps associated with higher-order Lagrangians and also problems arise while trying to impose a multimomentum phase space for the Hamiltonian formalism of such theories.

The best way to overcome the aforementioned problems is to use the unified Lagrangian-Hamiltonian formalism. It was first introduced by Skinner and Rusk in [17–19], and the basic idea is to merge the Lagrangian and Hamiltonian formalisms into one. Even though Skinner and Rusk’s idea tames some of the problems, still some arbitrary parameters that appear in the solutions of the higher-order field equations and in the definition of the Legendre maps must be fixed to guarantee its consistency. A modification of this framework that clarifies the choice of the jet and the multimomentum bundles and removes all ambiguity for second-order field theories was developed in [20], and this is the approach that we shall use throughout this work.

The unified Lagrangian-Hamiltonian formalism allows us to extract all the relevant physical information of a given system. First, we identify the geometry, manifolds and bundles of the theory and set the Lagrangian-Hamiltonian problem. For regular Lagrangians, the field equations that arise are compatible and have solutions on the jet-multimomentum bundle. This is not the case for singular Lagrangians, where it is needed to implement a constraint algorithm to be able to find the corresponding submanifolds of the jet-multimomentum bundle on which the field equations are compatible and have solutions. The constraint algorithm that we will apply was developed in [21].
In regular first-order theories, the holonomy condition is recovered from the local coordinate expression of the field equations [22]. That is not the case for second-order field theories (even for regular Lagrangians) and, therefore, it is required to impose it \textit{a priori}. Moreover, in the unified formalism the singularity of the Lagrangian appears also as constraints, in particular, as the definition of the Legendre transform. This is convenient as the implementation of the constraint algorithm is straightforward.

Another advantage of the unified Lagrangian-Hamiltonian formalism is that one can derive a covariant Hamiltonian formulation, as long as some regularity conditions are meet. Another common construction of a Hamiltonian formulation for field theories consists on performing a space + time decomposition of the covariant Lagrangian formalism and then perform an instantaneous Legendre transform. This was originally performed by Arnowitt, Deser and Misner for General Relativity [23]. This ADM-like approach, also called the instantaneous Hamiltonian formulation, has been studied from a geometric point of view [24, 25]. We shall delve into the relation and equivalence between these Hamiltonian formalisms for theories of gravity in a future work.

For all these reasons the multisymplectic formalism, and particularly the unified Lagrangian-Hamiltonian formalism, is suitable for studying singular second-order field theories such as certain string theories, the Korteweg–de Vries model and some of the most relevant theories of gravity, including General Relativity.

It is known that General Relativity (GR) is one of the most successful theories in the history of physics. For over a hundred years now, it has been tested and shown to be the standard model of gravity. However, it is also known that it is a low-energy effective theory, incomplete in the sense that it is non-renormalisable [26]. There are different motivations for studying modified models of gravity. From the phenomenological perspective, the relatively recent detection of the first gravitational waves opens a new way of testing generalised models of gravity that predict something different from what GR predict [27]. The problem of the fine-tuned cosmological constant needed to explain the accelerated expansion of the Universe in General Relativity is a strong incentive for physicists to explore generalised models of gravity as well. If we turn to a theoretical point of view, studying modified gravity grants a deeper understanding of GR. Finally, mathematically speaking, understanding the geometric structure of generalised models of gravity could impose strong constraints on the theory that could be used by physicists to discard models or, on the other hand, make them turn their attention to a certain model.

That being said, Horndeski’s theory is an interesting candidate for being the generalisation of GR since it is the most general diffeomorphism invariant, scalar-tensor theory that leads to second order equations of motion [28]. It is strongly hyperbolic, at least at weak coupling, and therefore admits a well-posed initial value problem [29]. Most importantly, this type of theories are causal and hence allow for the existence of dynamical black holes [30] which could be potentially observed with the current techniques.

We will be focusing on the construction of the multisymplectic formalism for the cubic subclass of Horndeski’s theory. This subclass leads to strongly hyperbolic equations [31] and recently has gained attention among cosmologists. It was first introduced in the cosmology community in [32]. The model can describe a non-singular bouncing Universe as it was noticed in [33] and later in [34]. Additionally, general comments on bouncing cosmology, along with a discussion regarding the regularity of [34], can be found in [35]. Besides the importance of this subclass of theories as a physical model, its covariant Hamiltonian formulation arises relevant geometric and physical consequences, as we shall discuss in the last section.

The multisymplectic formalism of relevant theories of gravity can be found in the literature. In fact, one of the most relevant examples is that of General Relativity [36]. In the cited work, a covariant Hamiltonian formulation of GR is obtained from the unified Lagrangian-Hamiltonian formalism. In contrast, there are more approaches to obtain a covariant Hamiltonian formulation of GR such as [37], [38] and [39]. Relevant mathematical generalisations via variational approaches of General Relativity are presented in [40, 41].

Other relevant examples of applications of the multisymplectic formalism are the metric-affine gravity with [42, 43] and without vielbein [44], Lovelock Gravity [45] and Chern–Simons gravity and the bosonic string [46]. There is also a proposal for a covariant renormalizable field theory of gravity [47].

The aim of the present work is to present the multisymplectic framework for the cubic subclass of Horndeski theories. A major feature of this formalism is that it provides a recipe for obtaining the Hamiltonian formulation of generalised theories of Gravity. Hence, we will first establish the geometric framework of the theory and introduce a suited change of co-ordinates that simplifies the calculations of the constraint algorithm. The quantity and nature of the constraints depend on the specific Horndeski model we consider. At each step, the resulting conditions become increasingly more complicated, therefore, we apply the algorithm as long as it provides us with relevant information. Finally we will show how to obtain the Hamiltonian formulation of the theory and briefly discuss its implications.

All the manifolds are real, second countable and of class \( C^\infty \). Manifolds and mappings are assumed to be smooth. Sum over crossed repeated indices is understood. Comas denote partial derivatives and semicolons covariant derivatives.
2. Setting up the problem

In this section, we will introduce the geometrical structures and the manifolds and bundles that we need to construct the formalism for cubic Horndeski’s theories. To do so, we will present the Horndeski’s Lagrangian, explain its main features and justify the importance of the cubic subclass of Horndeski’s theories. Finally we will set up the Lagrangian-Hamiltonian problem for this theories.

2.1. Geometry, manifolds and bundles of the theory

Let $M$ be an oriented 4-dimensional spacetime with coordinates $x^\mu$, $\mu = 0,1,2,3$, and whose volume form is denoted by $\eta \in \mathcal{F}^4(M)$. A scalar field is a map $\phi : M \rightarrow \mathbb{R}$ or, equivalently, its graph is a section of the product bundle $M \times \mathbb{R}$ over $M$.

The covariant configuration bundle $^\ast\mathcal{F}^4$ for this system is a fiber bundle $\pi : E \rightarrow M$, with $E$ being the manifold $(M \times \mathbb{R}) \times_M \mathcal{S}^2_{1,4}(M)$, where $\mathcal{S}^2_{1,4}(M)$ denotes the bundle of symmetric covariant two-tensors $g$ of Lorentz signature $(-,+,+,+)$ acting on $T_xM$.

The adapted fiber coordinates in $E$ are $(x^\mu, g_{\alpha\beta}, \phi, \omega_{\alpha\beta\mu}, \phi_{\mu}, \phi_{\alpha\beta\mu\nu}, \phi_{\alpha\beta\mu\nu\lambda})$, with $0 \leq \mu \leq \nu \leq \lambda \leq 3$. We shall use all the possible permutations, although only the ordered ones are actual coordinates.

The explicit expression of the total derivative $D_\tau$ in these local co-ordinates is

$$D_\tau = \frac{\partial}{\partial x^\tau} + \sum_{\alpha \leq \beta, \mu \leq \nu \leq \lambda} \left( g_{\alpha\beta,\tau} \frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\nu} \frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\lambda} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} ight.$$

$$\left. + g_{\alpha\beta,\mu\nu\lambda\theta} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} + \phi_{\alpha\beta,\tau} \frac{\partial}{\partial \phi_{\alpha\beta}} + \phi_{\alpha\beta,\mu\nu} \frac{\partial}{\partial \phi_{\alpha\beta,\mu}} + \phi_{\alpha\beta,\mu\nu\lambda} \frac{\partial}{\partial \phi_{\alpha\beta,\mu\nu}} + \phi_{\alpha\beta,\mu\nu\lambda\theta} \frac{\partial}{\partial \phi_{\alpha\beta,\mu\nu\lambda}} \right).$$

Notice that, if $f \in C^\infty(J^k\pi)$, then $D_\tau f \in C^\infty(J^{k+1}\pi)$.

Next, consider the bundle $J^1\pi$ and let $M \equiv \Lambda^2_1(J^1\pi)$ be the bundle of 4-forms over $J^1\pi$ vanishing by the action of two $\pi^1$-vertical vector fields; with the canonical projections

$$\pi_{x^\tau} : \Lambda^1_1(J^1\pi) \rightarrow J^1\pi; \; \gamma_{x^\tau} = \pi_{x^\tau} \pi_z : \Lambda^1_1(J^1\pi) \rightarrow M.$$

The induced local coordinates in $\Lambda^1_1(J^1\pi)$ are $(x^\mu, \phi, \omega_{\alpha\beta\mu}, \phi_{\mu}, P_g, \phi_{g,\alpha\beta\mu}, P_g^{\alpha\beta,\mu}, P_g^{\alpha\beta\mu\nu}, F_g^{\alpha\beta,\mu\nu}, F_g^{\alpha\beta\mu\nu\lambda})$, with $0 \leq \alpha \leq \beta \leq 3$ and $\mu, \nu = 0, 1, 2, 3$.

The bundle $\Lambda^1_1(J^1\pi)$ is endowed with the tautological 4-form $\Theta_1 \in \Omega^4(\Lambda^1_1(J^1\pi))$ and the canonical 5-form $\Omega_5 = -\partial \Theta_1 \in \Omega^5(\Lambda^1_1(J^1\pi))$ (also called the Liouville 4 and 5 form, respectively). The canonical form $\Omega_5$ is a multisymplectic form; meaning it is closed and 1-nondegenerate. Their local expression is

$$\Theta_1 = \rho d^4x + \sum_{\alpha \leq \beta} (P_{g,\alpha\beta,\mu} dg_{\alpha\beta} \wedge d^4x_\mu + P_g^{\alpha\beta,\mu} dg_{\alpha\beta,\mu} \wedge d^4x_\mu)$$

$$+ P_g d\phi + P_g^{\mu} d\phi_{\mu},$$

3 Covariant configuration bundle will be just referred to as configuration bundle from now on. In contrast, there is an instantaneous configuration bundle that appears in the ADM-like formulation of the theory.
\[ \Omega_1 = -d\rho \land d^4x \]

\[ - \sum_{\alpha \leq \beta} (d\phi^\alpha_{\beta} \land d\phi_{\beta} \land d^3x) + d\phi^\alpha_{\beta} \land d^3x_{\mu} + d\phi^\alpha_{\beta} \land d^3x_{\nu} \]

\[ + d\phi_{\mu} \land d^3x_{\mu} + d\phi_{\nu} \land d^3x_{\nu} \]

\[ - d\phi_{\mu} \land d\phi_{\nu} \land d^3x_{\nu} \]

where \(d^3x_{\nu} = \left(\frac{\partial}{\partial y^\nu}\right) d^4x\)

The bundle \(\Lambda^3_1(J^1\pi)\) has more momenta than desired. Therefore, we introduce the extended 2-symmetric multimomentum bundle, \(J^2_{\pi^2} \rightarrow \Lambda^3_1(J^1\pi)\) defined locally by the constraints \(F_{\mu}^{\alpha^3_{\beta_\mu}} = \pi_{\mu}^{\alpha^3_{\beta_\mu}}\) and \(\pi_{\mu_\nu} = \pi_{\mu_\nu}^{\alpha^3_{\beta_\mu}}\). Even though it is defined in local coordinates, it is a canonical construction \([49]\). Let \(\pi_{J^2_{\pi^2}}: J^2_{\pi^2} \rightarrow J^1\pi; \pi_{\mu_\nu} = \pi_{\mu_\nu}^{\alpha^3_{\beta_\mu}}: J^2_{\pi^2} \rightarrow M\)

be the canonical projections. The coordinates in \(J^2_{\pi^2}\) are

\[ (x^\mu, g_{\alpha^3_{\beta_\mu}}, \phi, \phi^\alpha_{\beta}, \phi^\alpha_{\beta}, \rho_\mu, \rho_\nu, \pi_{\alpha^3_{\beta_\mu}}, \pi_{\mu_\nu}, \pi_{\mu_\nu}^{\alpha^3_{\beta_\mu}}), \]

with (0 \(\leq \alpha \leq \beta \leq 3; 0 \leq \mu \leq \nu \leq 3\), and \(j_\mu F_{\mu}^{\alpha^3_{\beta_\mu}} = \frac{1}{n(\mu\nu)} j_\mu F_{\mu}^{\alpha^3_{\beta_\mu}}, j_\mu F_{\mu}^{\mu_\nu} = \frac{1}{n(\mu\nu)} j_\mu F_{\mu}^{\mu_\nu}\), where \(n(\mu\nu)\) is a combinatorial factor defined by \(n(\mu\nu) = 1\) for \(\mu = \nu\), and \(n(\mu\nu) = 2\) for \(\mu \neq \nu\).

Denote \(\Theta_{\mu_\nu} = j_\mu \Theta_\mu \in \Omega^{\mu_\nu}(J^2_{\pi^2})\) and the multisymplectic form \(\Theta_{\mu_\nu} = \pi_{\mu_\nu}^{\alpha^3_{\beta_\mu}}\), which are called symmetrised Liouville 1 and \((m+1)\)-forms. In this case, the local expressions are

\[ \Theta_{\mu_\nu} = \rho d^4x + \sum_{\alpha \leq \beta} \frac{1}{n(\mu\nu)} p_{\mu}^{\alpha^3_{\beta_\mu}} d\phi_{\alpha^3_{\beta_\mu}} \land d^3x_{\mu} \]

\[ + \sum_{\alpha \leq \beta} \frac{1}{n(\mu\nu)} p_{\mu}^{\alpha^3_{\beta_\mu}} d\phi_{\alpha^3_{\beta_\mu}} \land d^3x_{\mu} \]

\[ + \sum_{\alpha \leq \beta} \frac{1}{n(\mu\nu)} p_{\mu}^{\mu_\nu} d\phi_{\mu_\nu} \land d^3x_{\mu} \]

\[ - \sum_{\alpha \leq \beta} \frac{1}{n(\mu\nu)} p_{\mu}^{\alpha^3_{\beta_\mu}} d\phi_{\alpha^3_{\beta_\mu}} \land d^3x_{\mu} \]

\[ - \sum_{\alpha \leq \beta} \frac{1}{n(\mu\nu)} p_{\mu}^{\mu_\nu} d\phi_{\mu_\nu} \land d^3x_{\mu} \]

Last, consider the quotient bundle \(J^2_{\pi^2} \rightarrow J^1\pi; \pi_{\mu_\nu} : J^2_{\pi^2} \rightarrow J^1\pi; \pi_{\mu_\nu} : J^2_{\pi^2} \rightarrow M\), called the restricted 2-symmetric multimomentum bundle, endowed with the following projections

\[ \mu: J^2_{\pi^2} \rightarrow J^1\pi; \pi_{\mu_\nu} : J^2_{\pi^2} \rightarrow J^1\pi; \pi_{\mu_\nu} : J^2_{\pi^2} \rightarrow M. \]

Notice that \(J^2_{\pi^2}\) is also the submanifold of \(\Lambda^3_1(J^1\pi)/\Lambda^3_1(J^1\pi)\) defined by the local constraints \(F_{\mu}^{\alpha^3_{\beta_\mu}} = \pi_{\mu}^{\alpha^3_{\beta_\mu}}\) and \(\pi_{\mu_\nu} = \pi_{\mu_\nu}^{\alpha^3_{\beta_\mu}}\). Therefore, the coordinates in \(J^2_{\pi^2}\) are

\[ (x^\mu, g_{\alpha^3_{\beta_\mu}}, \phi, \phi^\alpha_{\beta}, \phi^\alpha_{\beta}, \rho_\mu, \rho_\nu, \pi_{\alpha^3_{\beta_\mu}}, \pi_{\mu_\nu}, \pi_{\mu_\nu}^{\alpha^3_{\beta_\mu}}), \]

with (0 \(\leq \alpha \leq \beta \leq 3; 0 \leq \mu \leq \nu \leq 3\)).

2.2. Horndeski’s theory

As things stand, General Relativity is the currently accepted theory of gravity, valid in a low-energy, macroscopic scale. It has been tested over and over by astronomical observations and experiments and has shown to be consistent every single time \([50]\). However, there are certain situations in which it appears to have some flaws.

For starters, in the extreme conditions of the early Universe, or near the event horizon of a black hole, the gravitational field becomes so strong that it is expected to interact with quantum fields. A consistent theory of quantum gravity is needed to fully understand the physics in these regimes, however, General Relativity fails to provide answers, mostly due to UV divergences and its non-renormalizability \([51]\). Furthermore, it predicts the existence of singularities, regions of spacetime where the curvature becomes infinite. The theory also does not provide a full description of the physics at these regions \([52]\). Additionally, in its purest form, it fails to explain the phenomena of dark matter and dark energy, which have been observed to make up a large fraction of the matter-energy content of the Universe \([53]\). Moreover, there is the so-called cosmological constant problem, firstly introduced by Einstein to counterbalance the effect of gravity and try to achieve a static Universe, then removed and reintroduced to be interpreted as the vacuum energy of spacetime, has a measured value that is significantly smaller than what is predicted by theory \([54]\). It is thought to be related to dark energy, as it could model the accelerated
expansion of the Universe. Finally, the observations of an exponential expansion of space in the early Universe, usually referred to as cosmic inflation, are failed to be explained solely by General Relativity as additional constants or fields need to be added to the theory [55].

Alternative theories of gravity have been proposed in an attempt to provide a satisfactory explanation of the aforementioned phenomena and also to potentially serve as a precursor for a quantum theory of gravity. These theories could be classified as scalar field theories, quasilinear theories, tensor theories, scalar—tensor theories, vector—tensor theories, bimetric theories, among other.

Horndeski’s theory is the most general scalar—tensor theory of gravity in four dimensions that leads to second order equations of motion. In that sense, it generalises Einstein’s theory of General Relativity, and it allows for a wide range of cosmological models to be explored. It’s relevance lies in the fact that it could be used to investigate the dynamics of the early Universe, the formation of galaxies, the behaviour of black holes, as well as other physical phenomena that remain not well understood such as dark matter and dark energy, as it includes terms that might describe them as a scalar field that is coupled with the metric.

Furthermore, it has been shown to be consistent with experimental observations, such as the behaviour of gravitational waves and the cosmic microwave background radiation. Additionally, predictions of the theory could, in principle, be tested by observations [36].

The Horndeski Lagrangian density is a $\pi^2$-semibasic form $\mathcal{L}_{\text{Q}} \in \Omega^1(J^2\pi)$. Hence $\mathcal{L}_{\text{Q}} = \mathcal{L}_{\text{G}}(\pi^2)^{p_\eta}$, where $\mathcal{L}_{\text{G}} \in C^\infty(J^2\pi)$ is the Horndeski Lagrangian function

$$\mathcal{L}_{\text{G}} = \frac{1}{16\pi G} \sqrt{|g|} \left( \sum_{i=1}^{3} L_i \right),$$

where, in the coordinates of the manifold:

$$L_1 = R + X; \quad L_2 = G_2(\phi, X); \quad L_3 = G_3(\phi, X) \Box \phi;$$

$$L_4 = G_4(\phi, X) R + G_{4,\chi}(\phi, X) [(\Box \phi)^2 - g^{\mu \nu} g^{\alpha \beta} \chi_{\mu \alpha \beta}]$$

$$L_5 = G_5(\phi, X) g_{\mu \nu} g^{\alpha \beta} \chi_{\mu \alpha \beta} - \frac{1}{6} G_{5,\chi}(\phi, X) [(\Box \phi)^3 + 2 g^{\mu \nu} g^{\alpha \beta} \chi_{\mu \alpha \beta} \chi_{\gamma \sigma} - 2 g^{\mu \nu} g^{\alpha \beta} \chi_{\mu \alpha \beta} \Box \phi].$$

Here $g^{\mu \nu}$ are the components of the inverse of the metric tensor, that is, $g_{\alpha \beta} g^{\mu \nu} = \delta_{\alpha \beta}^\mu \nu$ the determinant of the metric tensor; $X = -\frac{1}{2} g^{\mu \nu} \phi_{\mu \nu}$, and $\Box \phi = g^{\mu \nu} \phi_{\mu \nu}$. The Ricci tensor is given by:

$$R_{\alpha \beta} = D_\alpha D_\beta - D_\beta D_\alpha + \Gamma^\gamma_\alpha_\beta_\gamma - \Gamma^\gamma_\beta_\alpha_\gamma .$$

Hence, the Ricci scalar has the local expression:

$$R = g^{\alpha \beta} R_{\alpha \beta} = g^{\alpha \beta} (D_\beta D_\alpha - D_\alpha D_\beta + \Gamma^\gamma_\alpha_\beta_\gamma - \Gamma^\gamma_\beta_\alpha_\gamma),$$

By setting the scalar field constant throughout spacetime, the Einstein–Hilbert Lagrangian is recovered, hence Horndeski’s theory includes General Relativity. The term $X$ can be understood as the kinetic energy of the scalar field. This term, along with the special case in which $L_2 = G_2(\phi, X) = V(\phi)$ constitute what is usually referred to as the minimal coupling of a scalar field to gravity.

There are a few other well-known theories included within Horndeski’s theory. For instance, setting $L_2 = 0$, $G_2 = G_3 = G_4 = G_5 = 0$, yields Brans–Dicke theory [57]. Similarly, setting $G_2 = V(\phi) + 2\frac{\epsilon(\phi)}{\phi} X$, yields a dilaton theory in four dimensions. The dilaton field emerges naturally in some candidates for a quantum theory of gravity, such as M-theory [58].

Other important special cases contained within Horndeski’s theory, are the so-called gallileon, quintessence, and chameleon theories [28]. Remarkably, it has been shown that Horndeski’s theory is linked to Hofava—Lifshitz gravity [59]. Particularly, modified versions of it could be regarded as special cases of Horndeski’s theory [60]. These theories of gravity are an attempt to make gravity renormalisable by abandoning Lorentz symmetry in the UV regime and hence are candidates for a quantum theory of gravity.

Observations have constrained some of the free parameters of the theory. Particularly $L_4$ and $L_5$ have been strongly constrained by the direct measurement of the speed of gravitational waves [61]. Throughout this work we shall consider a subclass of Horndeski’s theories, often called cubic Horndeski theories, where $G_4 = G_5 = 0$.

2.3. The higher-order jet multimomentum bundles

The unified Lagrangian–Hamiltonian formalism is set in a bundle that encompasses the jets and bundles described in the previous section and hence the manifolds $M$ and $E$. First, we construct the symmetric higher-order jet multimomentum bundle $\mathcal{W}^*$ and the restricted symmetric higher-order jet multimomentum bundle $\mathcal{W}_r$, as described in [14, 20]

$$\mathcal{W}^* = J^3\pi \times \rho_\pi \rightarrow J^2\pi,$$
\[ \mathcal{H}_r = J^2 \pi \times J^2 \pi. \]

Here \( J^2 \pi \times J^2 \pi \) are the extended and the restricted 2-symmetric multimomentum bundle respectively, as discussed in the previous section. The symmetric higher-order jet multimomentum bundles have the following natural local coordinates

\[ (x^\mu, \xi^\alpha, \phi, \xi^\beta, \gamma, \xi^\nu, \xi^\lambda, \xi^\rho, \xi^\sigma, \xi^\tau, p_{\phi}, p_{\gamma}, p_{\rho}, p_{\sigma}, p_{\tau}) , \]

and

with \((0 \leq \alpha \leq \beta \leq 3; 0 \leq \mu \leq \nu \leq 3)\), and are endowed with the following projections

\[ \rho: \mathcal{W} \to J^2 \pi, \rho^*: \mathcal{W} \to J^2 \pi, \rho_M: \mathcal{W} \to M \]

Moreover, the quotient map \( \mu \circ J^2 \pi \to J^2 \pi \) induces a natural submersion \( \mathcal{W} \to \mathcal{W}_r. \)

Now, we define the canonical pairing which will help us determine the Hamiltonian function.

\[ \mathcal{E}: J^2 \pi \times J^2 \pi \to \Lambda_2(J^1 \pi) \]

which can be written as

\[ \mathcal{E}((J^2 \phi, \omega)) = (J^2 \phi)^\alpha_\beta \omega. \]

hence we have can define a new pairing \( \mathcal{E}': J^2 \pi \times J^2 \pi \to \Lambda_2(J^1 \pi) \) as

\[ \mathcal{E}'(J^2 \phi, \omega) = \mathcal{E}(J^2 \phi, j_\omega) = (J^2 \phi)^\alpha_\beta j_\omega. \]

From here we get the second-order coupling 4-form in \( \mathcal{W} \), which is the \( \rho_M \)-semibasic 4-form \( \hat{\mathcal{E}} \in \Omega^4(\mathcal{W}) \) defined by

\[ \hat{\mathcal{E}}(J^2 \phi, \omega) = \mathcal{E}'(\pi^2(J^2 \phi), \omega), (J^2 \phi, \omega) \in \mathcal{W}. \]

Since \( \hat{\mathcal{E}} \) is a \( \rho_M \)-semibasic 4-form, there exists a function \( \hat{C} \in C^\infty(\mathcal{W}) \) such that \( \hat{\mathcal{E}} = \hat{C} \rho_M \eta. \) In co-ordinates this is written as

\[ \hat{\mathcal{E}} = p + \sum_{\alpha \leq \beta} p_{\alpha \beta} \eta_{\alpha \beta} \eta + \sum_{\alpha \leq \beta} p_{\alpha \beta \mu} \xi^\mu \eta_{\alpha \beta} \eta + \sum_{\mu \leq \nu} p_{\mu \nu} \phi_{\mu \nu} \eta. \]

A 4-form \( \hat{\mathcal{L}} = (\pi^2 \circ \rho_M) \mathcal{L}_{\mathcal{W}} \in \Omega^4(\mathcal{W}) \), which can be written as \( \hat{\mathcal{L}} = \hat{\mathcal{L}} \rho_M \eta \), where \( \hat{\mathcal{L}} = (\pi^2 \circ \rho_M) \mathcal{L}_{\mathcal{W}} \in C^\infty(\mathcal{W}) \), can be used to define the Hamiltonian submanifold \( \mathcal{W}' \)

\[ \mathcal{W}' = \{ w \in \mathcal{W}: \hat{\mathcal{L}}(w) = \hat{\mathcal{E}}(w) \} \]

which is ultimately defined by the following constraint

\[ \hat{\mathcal{L}} - \hat{\mathcal{E}} \equiv p + \sum_{\alpha \leq \beta} p_{\alpha \beta} \eta_{\alpha \beta} \eta + \sum_{\alpha \leq \beta} p_{\alpha \beta \mu} \xi^\mu \eta_{\alpha \beta} \eta + \sum_{\mu \leq \nu} p_{\mu \nu} \phi_{\mu \nu} = 0. \]

This submanifold is \( \mu_M \)-transverse and diffeomorphic to \( \mathcal{W}' \), \( \mathcal{W}' \to \mathcal{W} \), induces a Hamiltonian section \( \hat{h} \in \Gamma(\mu_M) \) by \( \hat{h} = j_\omega \phi^{-1} \): \( \mathcal{W}' \to \mathcal{W} \), specified by the local Hamiltonian function

\[ \hat{\mathcal{H}} = \sum_{\alpha \leq \beta} p_{\alpha \beta} \eta_{\alpha \beta} \eta + \sum_{\alpha \leq \beta} p_{\alpha \beta \mu} \xi^\mu \eta_{\alpha \beta} \eta + \sum_{\mu \leq \nu} p_{\mu \nu} \phi_{\mu \nu} - \hat{\mathcal{L}}. \]
that is,

\[
\hat{h}(x^\mu, g_{\alpha\beta}, \phi, g_{\alpha\beta, \mu}, \phi_{,\mu}, g_{\alpha\beta, \mu, \lambda}) \\
\phi_{,\mu;\lambda}, P_g^{\alpha,\mu}, P_\phi^{\alpha,\mu}, P_\phi^{\mu}\bigg) \\
= (x^\mu, g_{\alpha\beta}, \phi, g_{\alpha\beta, \mu}, \phi_{,\mu}, g_{\alpha\beta, \mu, \lambda} \\
\phi_{,\mu;\lambda}, -\hat{h}, P_g^{\alpha,\mu}, P_\phi^{\alpha,\mu}, P_\phi^{\mu}).
\]

This is all summarised in the following commutative diagram

The Liouville forms in \( W, \Theta = (\rho_1, \hat{h})^* \Theta_1 \in \Omega^2(W) \) and \( \Omega_r = -d\Theta_r = (\rho_2, \hat{h})^* \Omega_1 \in \Omega^2(W) \), for second order field theories [20], in these specific co-ordinates, are

\[
\Theta_r = -\hat{h} d^4x + \sum_{\alpha \leq \beta} p^{\alpha,\beta,\mu} d g_{\alpha\beta} \wedge d^2 x_\mu \\
+ \sum_{\alpha \leq \beta} \frac{1}{n(\mu)} p^{\alpha,\beta,\mu,\nu} d g_{\alpha\beta,\mu} \wedge d^3 x_\nu \\
+ p^{\mu\nu} d\phi \wedge d^2 x_\mu + \frac{1}{n(\mu)} p^{\mu\nu} d\phi_{,\mu} \wedge d^2 x_\nu \\
\Omega_r = d\hat{h} \wedge d^4x - \sum_{\alpha \leq \beta} d p_g^{\alpha,\beta,\mu} \wedge d g_{\alpha\beta} \wedge d^2 x_\mu \\
- \sum_{\alpha \leq \beta} \frac{1}{n(\mu)} d p_g^{\alpha,\beta,\mu,\nu} \wedge d g_{\alpha\beta,\mu} \wedge d^3 x_\nu \\
- d p_g^{\mu\nu} \wedge d\phi \wedge d^2 x_\mu - \frac{1}{n(\mu)} d p_g^{\mu\nu} \wedge d\phi_{,\mu} \wedge d^2 x_\mu (1)
\]

To obtain the form (1) explicitly, we need to calculate the exterior derivative of the Hamiltonian function. The exterior derivative of a function is the differential of the function. Specifically we have

\[
d\hat{h} = \sum_{\alpha \leq \beta} (g_{\alpha\beta,\mu} d p_g^{\alpha,\beta,\mu} + p_g^{\alpha,\beta,\mu} d g_{\alpha\beta}) \\
+ \sum_{\alpha \leq \beta} \sum_{\mu \leq \nu} (g_{\alpha\beta,\mu,\nu} d p_g^{\alpha,\beta,\mu,\nu} + p_g^{\alpha,\beta,\mu,\nu} d g_{\alpha\beta,\mu,\nu}) \\
+ \phi_{,\mu} d p_\phi^{\mu} + p^{\mu} d\phi_{,\mu} + \sum_{\mu \leq \nu} (\phi_{,\mu\nu} d p_\phi^{\mu\nu} + p_\phi^{\mu\nu} d\phi_{,\mu\nu}) - d\hat{L}.
\]

The differential of the Lagrangian, provided its dependency upon the metric, the first and second order derivatives of the metric, the scalar field and the first and second order derivatives of the scalar field, is
\[ d\tilde{L} = \sum_{\alpha < \beta} \frac{\partial}{\partial \theta_{\alpha\beta}} d\theta_{\alpha\beta} + \sum_{\alpha, \beta} \frac{\partial}{\partial \theta_{\alpha\beta\gamma}} d\theta_{\alpha\beta\gamma} + \sum_{\alpha, \beta, \mu} \frac{\partial}{\partial \theta_{\alpha\beta\gamma\mu}} d\theta_{\alpha\beta\gamma\mu} + \sum_{\alpha, \beta} \frac{\partial}{\partial \phi_{\alpha\beta}} d\phi_{\alpha\beta} + \sum_{\alpha, \beta} \frac{\partial}{\partial \phi_{\alpha\beta\mu}} d\phi_{\alpha\beta\mu}. \]

The Liouville forms are degenerate; this is
\[ \ker \Theta_r = \ker \Theta_r \]
\[ \geq \left[ \frac{\partial}{\partial g_{\alpha\beta\mu\nu}}, \frac{\partial}{\partial g_{\alpha\beta\mu\lambda}}, \frac{\partial}{\partial \phi_{\mu\nu}} \right]_{0 \leq \alpha < \beta \leq \lambda, 0 \leq \mu < \lambda}. \]

For a premultisymplectic form \( \Omega \), we call (geometric) gauge vector fields to those vector fields belonging to \( \ker \Omega \) (see [6, 62] for more details). Furthermore, \( \Theta_r \) is \( (\pi_1^* \circ \rho_2^*) \)-projectable.

### 2.4. The lagrangian-hamiltonian problem

The Lagrangian-Hamiltonian problem associated with the system \( (\mathcal{W}, \Omega_r) \) defined in previous sections is described in more detail in [20]. One can also find its development to Hilbert-Einstein gravity in [36].

**Definition 1.** A section \( \psi \in \Gamma(\pi^4) \) is holonomic if \( j^4(\pi^4 \circ \psi) = \psi; \) that is, \( \psi \) is the \( k \)th prolongation of a section \( \phi = \pi^4 \circ \psi \in \Gamma(\pi) \), and an integrable and \( \pi_M \)-transverse multivector field \( X \in \mathcal{X}(J^k \pi) \) is holonomic if its integral sections are holonomic.

A section \( \psi \in \Gamma(\pi^4) \) is holonomic in \( J^k \pi \) if \( \pi_j^4 \circ \psi \in \Gamma(\pi^4) \) is holonomic in \( J^k \pi \), and an integrable and \( \pi_M \)-transverse multivector field \( X \in \mathcal{X}(J^k \pi) \) is holonomic if its integral sections are holonomic.

Finally, a section \( \psi \in \Gamma(\pi_M) \) is holonomic in \( \mathcal{W} \); if \( \rho_M^j \circ \psi \in \Gamma(\pi_M) \) is holonomic in \( J^k \pi \), and an integrable and \( \rho_M^j \)-transverse multivector field \( X \in \mathcal{X}(\mathcal{W}) \) is holonomic if its integral sections are holonomic.

It is important to point out that the fact that a multivector field in \( \mathcal{W} \) has the local expression (5) (and then being locally decomposable and \( \rho_M^j \)-transverse) is just a necessary condition to be holonomic, since it may not be integrable. However, if such a multivector field admits integral sections, then its integral sections are holonomic. In general, a locally decomposable and \( \rho_M^j \)-transverse multivector field which has (5) as coordinate expression, is said to be semiholonomic in \( \mathcal{W} \).

The Lagrangian-Hamiltonian problem associated with the system \( (\mathcal{W}, \Omega_r) \) consists in finding holonomic sections \( \psi \in \Gamma(\pi^4_M) \) satisfying any of the following equivalent conditions:

1. \( \psi \) is a solution to the equation
   \[ \psi^* i(X) \Omega_r = 0, \text{ for every } X \in \mathcal{X}(\mathcal{W}), \]

2. \( \psi \) is an integral section of a multivector field contained in a class of holonomic multivector fields \( \{X\} \subset \mathcal{X}(\mathcal{W}) \) satisfying the equation
   \[ i(X) \Omega_r = 0. \]

As the form \( \Omega_r \) is 1-degenerate we have that \( (\mathcal{W}, \Omega_r) \) is a premultisymplectic system, and solutions to (3) or (4) do not exist everywhere in \( \mathcal{W} \).

### 3. A suitable change of coordinates

At this point, for the sake of facilitating calculations, it is useful to introduce a new set of co-ordinates on \( \mathcal{W} \). The argument is simple; most interesting Lagrangians contain covariant derivative terms, making it necessary to transform them to terms with partial derivatives and components of the Levi-Civita connection. However, if we choose the velocity, acceleration and jerk co-ordinates of the scalar field on \( \mathcal{W} \), to be the covariant derivatives of first, second and third order respectively, instead of the partial derivatives, we will reduce significantly the difficulty in further computations. The cost of this co-ordinate transformation is that the Poincaré-Cartan forms must be transformed to the new co-ordinates.
The suited coordinates on $\mathcal{M}$ in this case are:

\[
(\vec{x}^\mu, \vec{a}_{\alpha 3}, \vec{\phi}, \vec{g}_{\alpha 3, \mu}, \vec{\phi}_\mu, \vec{g}_{\alpha 3, \mu, \nu}, \vec{\phi}_{\mu \nu}).
\]

They are related to the previous chart by a set of maps. For the coordinates of spacetime we just set $\tilde{x}^\mu = x^\mu$.

The metric is left unchanged related, and the scalar field is changed as:

\[
\tilde{\phi} = \phi; \quad \tilde{\phi}_\mu = \phi_\mu; \quad \tilde{\phi}_{\mu \nu} = \phi_{\mu \nu} - \phi_{\gamma \mu} \Gamma_{\gamma \mu}.
\]

\[
\tilde{\phi}_{\mu \nu \lambda} = \phi_{\mu \nu \lambda} - \phi_{\gamma \mu} \Gamma_{\gamma \nu \lambda} - \phi_{\gamma \nu} \Gamma_{\gamma \mu \lambda} + \phi_{\gamma \mu} \Gamma_{\gamma \nu \lambda} + \phi_{\gamma \nu} \Gamma_{\gamma \mu \lambda}.
\]

The multimomenta coordinates are mapped via the identity, and they are also unchanged. These expressions are a mere change of coordinates and hence the manifold and all the geometric structure of the theory remains intact. From these relations we get

\[
\begin{align*}
\frac{\partial}{\partial \phi_{\alpha 3}} &= \frac{\partial}{\partial \phi_{\alpha 3}} \frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\alpha 3}} = \frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\alpha 3}} \\
\frac{\partial}{\partial \phi_{\alpha 3, \mu}} &= \frac{\partial}{\partial \phi_{\alpha 3, \mu}} \frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\alpha 3, \mu}} = \frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\alpha 3, \mu}} \\
\frac{\partial}{\partial \phi_{\mu \nu}} &= \frac{\partial}{\partial \phi_{\mu \nu}} \frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\mu \nu}} = \frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\mu \nu}}.
\end{align*}
\]

where

\[
\frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\alpha 3}} = n(\alpha \beta) \tilde{g}_\gamma \tilde{g}^{\gamma ( \alpha \Gamma_{\beta \gamma \lambda})}_{\mu \nu}
\]

\[
\frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\alpha 3, \mu}} = \frac{n(\alpha \beta)}{2} \tilde{g}_\gamma \tilde{g}^{\gamma \mu \nu \lambda \rho \sigma \tau \gamma \mu \nu \lambda \rho \sigma \tau} \left[ \delta^{(1)}_{\mu \nu} \delta^{(1)}_{\mu \nu} - \delta^{(1)}_{\mu \nu} \delta^{(1)}_{\mu \nu} \right]
\]

\[
\frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\mu \nu}} = -\Gamma^{\mu \nu}_{\mu \nu \lambda}
\]

\[
\frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\mu \nu \lambda}} = -3n(\mu \nu) \Gamma^{\mu \nu}_{\mu \nu \lambda}
\]

\[
\frac{\partial \tilde{\phi}_{\mu \nu \lambda}}{\partial \phi_{\mu \nu \lambda}} = \frac{n(\alpha \beta)}{2} \left[ \tilde{g}^{\mu \nu \lambda}_{\mu \nu \lambda} - \tilde{g}^{\mu \nu \lambda}_{\mu \nu \lambda} \right]
\]

It is important to clarify that we have used the standard notation for symmetrization and antisymmetrization of indices. To symmetrize on $n$ indices, we sum over all possible permutations of these indices and divide the
result by $n!$. To antisymmetrize, we go through the same procedure, but weighting each term in the sum by the sign of the permutation. For instance

$$T^{(\mu \nu \rho \lambda)} = \frac{1}{3!} (T^{\mu \nu \rho \lambda} + T^{\nu \mu \rho \lambda} + T^{\rho \mu \nu \lambda} + T^{\rho \nu \mu \lambda} + T^{\mu \rho \nu \lambda} + T^{\nu \rho \mu \lambda}),$$

$$T^{(\mu \nu)}_{(cyclic)} = \frac{1}{3!} (T^{\mu \nu} - T^{\nu \mu} + T^{\nu \mu} - T^{\mu \nu} + T^{\nu \mu} - T^{\mu \nu}).$$

Sometimes it is convenient to (anti)symmetrize over indices which are not adjacent. In this case, we use vertical bars to denote that some indices will be excluded. For instance,

$$T^{(\mu \nu)}_{(\mu | \nu |)} = \frac{1}{2} (T^{\mu \nu} + T^{\nu \mu}).$$

From now on, we shall drop the tildes over the names of the co-ordinates, but it must be understood that the following calculations are performed in the new chart.

The Liouville forms in $\mathcal{W}$, in these new coordinates, become

$$\Theta_r = -\dot{H}d^4x + \sum_{\alpha,\beta} p^{\alpha \beta \mu} \frac{1}{n(\mu \nu)} \delta_{\alpha \beta, \mu} \wedge d^3x_\mu + \sum_{\alpha,\beta} p^{\alpha \beta \mu} \frac{1}{n(\mu \nu)} \delta_{\alpha \beta, \mu} \wedge d^3x_\mu + \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} \wedge d^3x_\mu$$

$$\Omega_r = d\dot{H} \wedge d^4x - \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} \wedge d^3x_\mu - \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} \wedge d^3x_\mu - \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} \wedge d^3x_\mu$$

where

$$\dot{H} = \sum_{\alpha,\beta} p^{\alpha \beta \mu} \frac{1}{n(\mu \nu)} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} p^{\alpha \beta \mu} \frac{1}{n(\mu \nu)} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{1}{n(\mu \nu)} p^{\alpha \beta \mu} \delta_{\alpha \beta, \mu}$$

Provided that we are interested in second order theories, we have

$$d\dot{L} = \sum_{\alpha,\beta} \frac{\partial \dot{L}}{\partial x^\mu} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{\partial \dot{L}}{\partial x^\mu} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{\partial \dot{L}}{\partial x^\mu} \delta_{\alpha \beta, \mu} + \sum_{\alpha,\beta} \frac{\partial \dot{L}}{\partial x^\mu} \delta_{\alpha \beta, \mu}$$

The local expression of a holonomic multivector field $X \in \mathfrak{X}(\mathcal{W})$ in the new coordinates is
and, if  

$$\psi(x^i) = (x^1, x_2, x_3), \quad \psi(x^1), \quad \psi(x^2), \quad \psi(x^3),$$

\[\psi_\lambda(x^1), \quad \psi_\lambda(x^2), \quad \psi_\lambda(x^3)\]

is an integral section of $X$, its component functions satisfy the following system of partial differential equations

\[
\frac{\partial \psi_0}{\partial x^\lambda} = g_{0\lambda} \circ \psi, \quad \frac{\partial \psi_{\alpha\beta\mu}}{\partial x^\lambda} = g_{\alpha\beta\mu \lambda} \circ \psi, \quad \frac{\partial \psi_{\alpha\beta\mu \nu}}{\partial x^\lambda} = g_{\alpha\beta\mu\nu \lambda} \circ \psi,
\]

\[
\frac{\partial \psi_{0,\alpha\beta\mu}}{\partial x^\lambda} = F_{0\alpha\beta\mu \lambda} \circ \psi, \quad \frac{\partial \psi_{0,\alpha\beta\mu \nu}}{\partial x^\lambda} = G_{0\alpha\beta\mu \nu \lambda} \circ \psi.
\]

4. Unified lagrangian-hamiltonian formalism and the constraint algorithm

Now we explicitly calculate the Legendre maps and the corresponding field equations for the multivector fields in the new coordinates.

4.1. Legendre maps

The multimomenta are related to the jets of the components of the metric and the scalar field. This relation is given by the Legendre maps, whose graphs define a submanifold in $\mathcal{W}$. Solutions can only take values in this submanifold.

**Proposition 1** A section $\psi \in \Gamma(\xi^1)$ solution to the equation (3) takes values in a 140-codimensional submanifold $\mathcal{J}_\pi: \mathcal{W} \to \mathcal{W}'$, which is identified with the graph of a bundle map $\xi^3: \Gamma^3 \to \Gamma^3$, over $\Gamma^3$, defined locally by

\[
\mathcal{F}_{\xi^3} = \frac{\partial L}{\partial g_{0\beta\mu}} - \frac{3}{n(\mu)} X_\gamma \left( \frac{\partial L}{\partial g_{0\beta\mu \gamma}} \right) - \frac{1}{2} \left( \frac{\partial L}{\partial \phi_{\gamma \mu}} \phi_{\gamma \mu} \phi_{\gamma \mu} \right) \phi_{\gamma \mu} - \frac{\partial L}{\partial \phi_{\gamma \mu}} \phi_{\gamma \mu} \phi_{\gamma \mu} = \hat{L}_{\xi^3},
\]

\[
\mathcal{F}_{\xi^3} = \frac{\partial L}{\partial g_{0\beta\mu}} - \frac{1}{n(\mu)} X_\gamma \left( \frac{\partial L}{\partial g_{0\beta\mu \gamma}} \right) - \frac{\partial L}{\partial \phi_{\gamma \mu}} \Gamma_{\gamma \mu} = \hat{L}_{\phi},
\]

The submanifold $\mathcal{W}'$ is the graph of a bundle morphism $\xi: \Gamma^3 \to \Gamma^3$ over $\Gamma^3$ defined locally by

\[
\mathcal{F}_{\phi:} \mathcal{F}_{\xi^3} = \frac{\partial L}{\partial g_{0\beta\mu}} - \frac{3}{n(\mu \nu)} X_\gamma \left( \frac{\partial L}{\partial g_{0\beta\mu \gamma}} \right) - \frac{1}{2} \left( \frac{\partial L}{\partial \phi_{\gamma \mu}} \phi_{\gamma \mu} \phi_{\gamma \mu} \right) \phi_{\gamma \mu} - \frac{\partial L}{\partial \phi_{\gamma \mu}} \phi_{\gamma \mu} \phi_{\gamma \mu} = \hat{L}_{\phi},
\]

\[
\mathcal{F}_{\phi:} \mathcal{F}_{\xi^3} = \frac{\partial L}{\partial g_{0\beta\mu}} - \frac{1}{n(\mu \nu)} X_\gamma \left( \frac{\partial L}{\partial g_{0\beta\mu \gamma}} \right) - \frac{1}{2} \left( \frac{\partial L}{\partial \phi_{\gamma \mu}} \phi_{\gamma \mu} \phi_{\gamma \mu} \right) \phi_{\gamma \mu} - \frac{\partial L}{\partial \phi_{\gamma \mu}} \phi_{\gamma \mu} \phi_{\gamma \mu} = \hat{L}_{\phi}.
\[ \mathcal{FL}_{\alpha}^{\mu} = \sum_{\nu=0}^{n} \frac{1}{n(\mu\nu)} X_{\nu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} - \frac{\partial \mathcal{L}}{\partial \phi_{\nu\mu}} \Gamma_{\nu}^{\mu} \right) = f_{\phi}^{\mu}, \]

\[ \mathcal{FL}^{\mu} = \frac{\partial \mathcal{L}}{\partial \phi_{\mu}}, \]

\[ \mathcal{FL}_{\alpha}^{\mu} = L - \sum_{\alpha, \beta, \mu \leq \nu} g_{\alpha, \beta, \mu} \frac{\partial \mathcal{L}}{\partial g_{\alpha, \beta, \mu}} - \sum_{\phi_{\mu\nu} + \phi_{\nu\mu}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}}. \]

\[ \phi_{\mu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} - \sum_{\mu \leq \nu} \frac{1}{n(\mu\nu)} X_{\nu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} - \frac{\partial \mathcal{L}}{\partial \phi_{\nu\mu}} \Gamma_{\nu}^{\mu} \right) \right) \]

\[ - \sum_{\alpha, \beta} g_{\alpha, \beta} \left[ \frac{\partial \mathcal{L}}{\partial g_{\alpha, \beta}} \frac{\partial \mathcal{L}}{\partial g_{\alpha, \beta}} - \sum_{\mu \leq \nu} \frac{1}{n(\mu\nu)} X_{\nu} \left( \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} - \frac{\partial \mathcal{L}}{\partial \phi_{\nu\mu}} \Gamma_{\nu}^{\mu} \right) \right] \]

\[ - \frac{1}{2} \left( \frac{\partial L}{\phi_{\mu\nu}} \phi_{\nu\beta} + \frac{\partial L}{\phi_{\mu\beta}} \phi_{\beta\nu} + \frac{\partial L}{\phi_{\nu\beta}} \phi_{\nu\beta} \right). \]

The maps \( \mathcal{FL}_{\alpha} \) and \( \mathcal{FL}^{\mu} \) are called the restricted and the extended Legendre maps respectively, and they are related by \( \mathcal{FL}_{\alpha} = \mu \mathcal{FL}^{\mu} \). For every \( j_{\phi}^{3} \phi \in I^{3} \pi \), we have that rank\( (\mathcal{FL}_{\alpha}(j_{\phi}^{3} \phi)) = \text{rank}(\mathcal{FL}^{\mu}(j_{\phi}^{3} \phi)) \).

A second-order Lagrangian density \( \mathcal{L} \in \Omega^{4}(I^{3} \pi) \) is regular, \[ 49 \] if

\[ \text{rank}(\mathcal{FL}(j_{\phi}^{3} \phi)) = \text{rank}(\mathcal{FL}(j_{\phi}^{3} \phi)) = \dim J^{3} \pi + \dim J^{3} \pi - \dim E = \dim J^{3} \pi, \]

otherwise, the Lagrangian density is singular. Regularity is equivalent to demand that \( \mathcal{FL} : J^{3} \pi \to J^{3} \pi \) is a submersion onto \( J^{3} \pi \) and, therefore, there exist local sections of \( \mathcal{FL} \). If the Lagrangian density is regular, then the sections \( \psi \in \Gamma(\rho_{\alpha}, \pi) \) solution to the equation \( (3) \) lies in \( \mathcal{W} \). Otherwise, the sections \( \psi \) may take values only in a submanifold \( \mathcal{W} \to \mathcal{W} \). In order to obtain this final constraint submanifold, the best way is to work with the equation \( (4) \) instead of \( (3) \).

### 4.2 Field equations for multivector fields

For a generic cubic Horndeski theory, we have, at least, the following constraints:

**Theorem 1.** Consider the Horndeski Lagrangian with \( G_{\phi}(\phi, X) = 0 \) and \( G_{\phi}(0, X) = 0 \), and let \( W_{J} \mapsto \mathcal{W} \), be the submanifold defined locally by the constraints

\[ p_{g}^{\alpha, \beta, \mu} = \frac{\partial L}{\partial g_{\alpha, \beta, \mu}} = 0, p_{g}^{\alpha, \beta, \mu} = 0, \]

\[ p_{\phi}^{\mu} - \frac{\partial L}{\partial \phi_{\mu}} = 0, p_{\phi}^{\mu} = 0, \]

for \( 0 \leq \alpha \leq \beta \leq 3; 0 \leq \mu \leq \nu \leq 3, \text{ and } 0 \leq \tau \leq 3 \). Then, there exist classes of semiholonomic multivector fields \( \{ \mathcal{X} \} \subset \mathcal{X}^{4}(W_{J}) \) which are tangent to \( \mathcal{W} \) and such that

\[ i(X) \Omega_{\pi} |_{\mathcal{W}} = 0, \forall \ X \in \{ \mathcal{X} \} \subset \mathcal{X}^{4}(W_{J}). \] (6)

**Proof.** In order to find the final submanifold \( \mathcal{W} \), we compute the constraint algorithm in local coordinates. Bearing in mind \( (5) \), the local expression of a representative of a class of a semiholonomic multivector fields, not necessarily integrable, is, in this case. \( (5) \). Then, equation \( (4) \) leads to

\[ G_{\phi}^{\beta} = \frac{\partial L}{\partial \phi} = 0, \]

\[ \sum_{\mu \leq \nu} \frac{1}{n(\mu\nu)} G_{\phi}^{\beta, \mu} = \frac{\partial L}{\partial g_{\alpha, \beta, \mu}} + p_{g}^{\alpha, \beta} \phi_{\beta, \mu}^{
u} \Gamma_{\nu}^{\mu} = 0, \]

\[ \sum_{\mu \leq \nu} \frac{1}{n(\mu\nu)} G_{\phi}^{\beta, \mu} = \frac{\partial L}{\partial g_{\alpha, \beta, \mu}} + p_{g}^{\alpha, \beta} \phi_{\beta, \mu}^{
u} \Gamma_{\nu}^{\mu} = 0, \]

\[ \sum_{\mu \leq \nu} \frac{1}{n(\mu\nu)} G_{\phi}^{\beta, \mu} = \frac{\partial L}{\partial g_{\alpha, \beta, \mu}} + p_{g}^{\alpha, \beta} \phi_{\beta, \mu}^{
u} \Gamma_{\nu}^{\mu} + p_{\phi}^{\mu} = 0, \] (10)
\[ p^{\rho\beta\mu\nu}_\alpha - \frac{\partial L}{\partial g_{\rho\alpha\beta\mu\nu}} = 0, \quad (11) \]
\[ p^{\mu\nu}_\phi - \frac{\partial L}{\partial \phi_{\mu\nu}} = 0. \quad (12) \]

Expressions (11) and (12) are constraints that define the compatibility submanifold \( \mathcal{H}_c \). If we require tangency of the multivector field to \( \mathcal{H}_c \),

\[ L(X) \left( p^{\rho\beta\mu\nu}_\alpha - \frac{\partial L}{\partial g_{\rho\alpha\beta\mu\nu}} \right) \bigg|_{\mathcal{H}_c} = 0, \]
\[ L(X) \left( p^{\mu\nu}_\phi - \frac{\partial L}{\partial \phi_{\mu\nu}} \right) \bigg|_{\mathcal{H}_c} = 0, \]

we get

\[ G^{\rho\beta\mu\nu}_{\alpha} = X_{\alpha} \left( \frac{\partial L}{\partial g_{\rho\alpha\beta\mu\nu}} \right); \quad (13) \]
\[ G^{\mu\nu}_{\beta\sigma} = X_{\beta} \left( \frac{\partial L}{\partial \phi_{\mu\nu}} \right); \quad (14) \]

Contracting \( \tau \) and \( \nu \) and combining these expressions with (9) and (10) leads to

\[ 0 = \sum_{\mu=0}^{3} \frac{1}{n(\mu\nu)} X_{\mu} \left( \frac{\partial L}{\partial g_{\rho\alpha\beta\mu\nu}} \right) - \frac{\partial L}{\partial g_{\rho\alpha\beta\mu}}, \]
\[ + p^{\rho\beta\mu\nu}_\alpha + \sum_{\rho\in\sigma} \phi_{\rho\gamma} p^{\sigma}_{\rho\gamma} \frac{\partial \Omega_{\gamma}}{\partial g_{\rho\alpha\beta\mu\nu}}; \quad (\text{on } \mathcal{H}_c), \]

\[ 0 = \sum_{\mu=0}^{3} \frac{1}{n(\mu\nu)} X_{\mu} \left( \frac{\partial L}{\partial \phi_{\mu\sigma}} \right) \]
\[ - \frac{\partial L}{\partial \phi_{\rho\sigma}} + p^{\sigma}_{\rho\mu} \Gamma^{\mu}_{\rho\sigma} + p^{\mu}_{\phi}; \quad (\text{on } \mathcal{H}_c), \]

which can be rewritten as

\[ p^{\rho\beta\mu\nu}_\alpha = L^{\rho\beta\mu\nu}_\alpha; \quad p^{\mu\nu}_\phi = L^{\mu\nu}_\phi, \]

respectively. These constraints define the submanifold \( \mathcal{H}_c \). Imposing tangency conditions on these, gives

\[ G^{\rho\beta\mu\nu}_{\alpha} = X_{\alpha} \left( L^{\rho\beta\mu\nu}_\alpha \right); \quad G^{\mu\nu}_{\beta\sigma} = X_{\beta} \left( L^{\mu\nu}_\phi \right); \quad (\text{on } \mathcal{H}_c) \]

If we contract \( \mu \) and \( \tau \), and use (7), (8), (11), (12), (13) and (14), we get

\[ 0 = \frac{\partial L}{\partial g_{\rho\alpha\beta}} - X_{\mu} \left( L^{\rho\beta\mu\nu}_\alpha \right) \]
\[ + \sum_{\mu<\nu} p^{\sigma}_{\rho\mu} \phi_{\rho\gamma} \Gamma^{\mu}_{\rho\sigma}; \quad (\text{on } \mathcal{H}_c), \]

\[ 0 = \frac{\partial L}{\partial \phi_{\rho\sigma}} - X_{\mu} \left( L^{\mu\nu}_\phi \right); \quad (\text{on } \mathcal{H}_c). \]

These results hold for the full Horndeski’s theory since up to this point we have only assumed that our Lagrangian is constructed out of the metric tensor, a scalar field and its first and second order derivatives. To provide a better insight of the physical meaning of expressions (17) and (18) we shall consider the Horndeski Lagrangian with \( G_4(\phi, X) = 0 \) and \( G_5(\phi, X) = 0 \) and after carefully computing these expressions explicitly, we get

\[ \hat{L}^{\mu\nu}_\phi = -\sqrt{-g} \left[ g^{\mu\nu}_\phi \phi_{\phi} \left( 1 + \frac{\partial G_2(\phi, X)}{\partial \phi} + \frac{\partial G_3(\phi, X)}{\partial \phi} \right) \phi \right] \]
\[ + \sum_{\mu=0}^{3} \frac{1}{n(\mu\nu)} g^{\mu\nu}_{\phi\phi} \phi_{\phi} \left( \phi_{\phi} X + \phi_{\phi} \Gamma^{\mu}_{\phi\phi} \right) \frac{\partial G_3(\phi, X)}{\partial X} \]
\[ \left(19\right) \]
\[ \hat{L}_g^{\alpha\beta,\mu} = \frac{1}{2} \sqrt{g} \left( g^{|\alpha\beta|,\mu} - 3g^{\mu|\alpha\beta|} + 2g^{\alpha|\beta\mu} + 2g^{\beta|\alpha\mu} \right) \]

\[ \hat{L}_g^{\alpha\beta} = -\sqrt{g} n(\alpha\beta) \left[ R^{\alpha\beta} - \frac{1}{2} \left( R + X + G_2 + 2X \frac{\partial G_2}{\partial \phi} \right) g^{\alpha\beta} - \frac{1}{2} \left( 1 + \frac{\partial G_2}{\partial \phi} + 2 \frac{\partial G_3}{\partial \phi} \right) g^{\alpha\beta} \right] \]

\[ \hat{L}_f = -\sqrt{g} n(\alpha\beta) \left[ \frac{\partial^2 G_2}{\partial X^2} + 2 \frac{\partial^2 G_3}{\partial X \partial \phi} \right] g^{\mu\nu} \phi_{\mu\nu} + \frac{\partial^2 G_3}{\partial X \partial \phi} \left( g^{\mu\nu} \phi_{\mu\nu} - g^{\alpha\beta} \phi_{\alpha\beta} \right) \]

\[ \text{Expresions} (21) \text{ and } (22) \text{ are the Euler–Lagrange equations, and when they are evaluated on sections in } W_{\mathcal{L}_g}, \text{ we recover the cubic Horndeski equations of motion.} \]

It is important to remark that \( \hat{L}_g^{\alpha\beta} \) and \( \hat{L}_f \) do not depend on any of the momenta. There is a dependence on the velocities and accelerations of both the metric and the scalar field, but not on higher-order velocities of any of them. Hence, \( \hat{L}_g^{\alpha\beta} \) and \( \hat{L}_f \) project onto \( J^f \pi \) and they can be regarded as new constraints defining locally a submanifold \( W_{\mathcal{L}_g} \rightarrow W_{\mathcal{L}_f} \). Once again, demanding tangency of the multivector field to this new manifold we get

\[ L(X_{\pi}) \hat{L}_g^{\alpha\beta} |_{W_{\mathcal{L}_g}} = 0, \]

\[ L(X_{\pi}) \hat{L}_f |_{W_{\mathcal{L}_g}} = 0, \]

which are just

\[ X_{\pi}(\hat{L}_g^{\alpha\beta}) = 0; \text{ (on } W_{\mathcal{L}_g}), \]

\[ X_{\pi}(\hat{L}_f) = 0; \text{ (on } W_{\mathcal{L}_g}). \]

These are new constraints again that project onto \( J^f \pi \). They define locally the submanifold \( W_{\mathcal{L}_g} \rightarrow W_{\mathcal{L}_f} \rightarrow W_{\mathcal{L}_f} \). This manifold \( W_{\mathcal{L}_f} \) is the final constraint submanifold because there exist holonomic multivector fields, solutions to (6). Finally, the new tangency conditions,

\[ L(X_{\pi}) X_{\pi}(\hat{L}_g^{\alpha\beta}) |_{W_{\mathcal{L}_g}} = 0, \]

\[ L(X_{\pi}) X_{\pi}(\hat{L}_f) |_{W_{\mathcal{L}_g}} = 0, \]

which are explicitly

\[ X_{\pi}(X_{\pi}(\hat{L}_g^{\alpha\beta})) = 0; \text{ (on } W_{\mathcal{L}_g}), \]

\[ X_{\pi}(X_{\pi}(\hat{L}_f)) = 0; \text{ (on } W_{\mathcal{L}_g}). \]

These allow us to determine the remaining components of (5), \( F_g^{\alpha\beta,\mu\nu} \) and \( F_g^{\alpha\beta,\mu\nu} \). Finally, the complete set of constraints that define the final constraint submanifold \( W_{\mathcal{L}_g} \rightarrow W_{\mathcal{L}_f} \) are

\[ p_{g,\alpha\beta,\mu\nu}^2 - \frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = 0, \]

\[ p_{g,\alpha\beta,\mu}^2 - \frac{\partial L}{\partial g_{\alpha\beta,\mu}} = 0, \]

\[ p_{g,\alpha\beta}^2 - \frac{\partial L}{\partial g_{\alpha\beta}} = 0, \]

\[ L_{\phi} = 0; \text{ (on } W_{\mathcal{L}_g}), \]

\[ L_{\phi} = 0; \text{ (on } W_{\mathcal{L}_g}). \]

This is the last step of the constraint algorithm that can be performed without specifying the concrete values of \( G_{\mathcal{L}_f}(\phi,X) \) and \( G_{\mathcal{L}_f}(\phi,X) \). In contrast with the Hilbert–Einstein case, we cannot assume that there exists a...
holonomic solution (that is, integrable) in $\mathcal{W}_f$. Depending on the expression of $G_2(f,X)$ and $G_3(f,X)$, new constraints may appear when demanding integrability of the multivector field in the constraint algorithm.

### 4.3. Field equations for sections

Provided that we now know the solution for the holonomic multivector fields, we can evaluate equation (??) to recover the field equations for sections.

\[
\frac{\partial \psi_{g,3}}{\partial x^\mu} = \psi_{g,3,\mu}, \quad \frac{\partial \psi_{g,3,\mu}}{\partial x^\nu} = \psi_{g,3,\mu,\nu}, \quad \nabla_\mu \psi_0 = \psi_{0,\mu}, \quad \nabla_\mu \psi_{g,\mu} = \psi_{g,\mu,\mu}, \tag{23}
\]

\[
\frac{\partial \psi^{g,\beta,\mu}}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial g_{g,3,\beta,\mu}} - \psi^{g,\beta,\mu} + \frac{1}{2} \psi_{0,\gamma} \psi^{g,\beta}_g \Gamma^{g,\beta}_{\mu,\gamma}, \tag{24}
\]

\[
\frac{\partial \psi^{0,\mu}}{\partial x^\nu} = \frac{\partial \mathcal{L}}{\partial \phi_{g,3,\mu,\nu}} - \psi^{0,\mu} \Gamma^{0,\mu}_{\nu,\gamma} - \psi^{0,\mu}_g, \tag{25}
\]

\[
\psi^{g,3,\mu,\nu} = \frac{\partial \mathcal{L}}{\partial g_{g,3,\beta,\mu,\nu}}, \tag{26}
\]

\[
\psi^{0,\mu} = \frac{\partial \mathcal{L}}{\partial \phi_{g,3,\mu}}. \tag{27}
\]

Equations (23)–(26) are the holonomy conditions for the multivector field. Equations (29)–(32) define the Legendre transformations.

### 5. 5 Hamiltonian formalism

The covariant Hamiltonian formalism takes place in the image of the Legendre Transformation. For singular Lagrangian this space could be highly degenerate. The Legendre maps in our case are given by Proposition 1. Then

\[
T_{\psi, g, \mathcal{F} L_{\psi, g}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Notice that, in general, \( \text{rank}(T_{ij};_{\theta},\mathcal{L}_{G}) \geq 59 \), depending on the arbitrary function \( G_{3}(\phi,X) \). Also, locally

\[
\ker(\mathcal{F}L_{G})_{\mathcal{E}} = \ker \Omega_{\mathcal{E}} \supset \left\{ \frac{\partial}{\partial x_{\alpha \beta \mu \nu}}, \frac{\partial}{\partial g_{\alpha \beta \mu \nu}}, \frac{\partial}{\partial \phi_{\mu \nu \lambda}} \right\}_{0 \leq \alpha \leq 3; 0 \leq \mu, \nu, \lambda \leq 3},
\]

hence \( \mathcal{F}L_{G} \) is highly degenerated.

We denote \( \tilde{\mathcal{P}} = \mathcal{F}L_{G}(J^{\pi}) \) and \( \mathcal{P} = \mathcal{F}L_{G}(J^{\pi}) \), and let \( \mathcal{F}L_{G}^{0} \) be the map defined by \( \mathcal{F}L_{G} = j_{c} \mathcal{F}L_{G}^{0} \) and \( \pi \colon \mathcal{P} \rightarrow M \) the natural projection. In order to assure the existence of the Hamiltonian formalism it is needed that the Lagrangian density \( L_{G} \in \Omega_{\pi}(J^{\pi}) \) is, at least, almost-regular; i.e., \( \mathcal{P} \) is a closed submanifold of \( J^{\pi} \), \( \mathcal{F}L_{G}^{0} \) is a submersion onto its image and, for every \( j_{\phi}^{\pi} \in J^{\pi} \), the fibers \( \mathcal{F}L_{G}^{0}(\mathcal{F}L_{G}(j_{\phi}^{\pi})) \) are connected submanifolds of \( J^{\pi} \). For more details in almost-regular Lagrangians and how to recover the Hamiltonian formalism from the unified Lagrangian-Hamiltonian formalism, we recommend consulting references [14, 63].

The proof that the Hilbert-Einstein Lagrangian is almost-regular is based on the fact that \( \mathcal{P} \) is diffeomorphic to the first jet of the corresponding fiber bundle [36]. This property is closely related to the fact that the Euler-Lagrange equations (Einstein’s Field Equations) are second-order, although one expects fourth-order equations for a second-order Lagrangian. This topic is called order-reduction (or projectability) of a theory [64–66]. Horndenski Lagrangians are constructed such that the corresponding field equations are second-order, therefore, one hopes to proceed in a similar way as in the Hilbert-Einstein case. Nevertheless, the fact that the Euler-Lagrange equations projects to lower order doesn’t imply that the geometric structures also project to a lower order. The Horndenski theories that have this property are characterised by proposition 2.

A form \( \alpha \in \Omega^{q}(J^{\pi}) \) projects to \( J^{\pi} \), \( s = 1, 2 \), if it is \( \pi^{s} \)-basic, that is, \( L_{\alpha} = 0 \) for all vector fields \( Y \) vertical with respect to \( \pi^{s} \). The multisymplectic Lagrangian system is \( (J^{\pi}, \Omega_{\mathcal{E}}) \), where \( \Omega_{\mathcal{E}} = \mathcal{F}L_{G}^{0} \ast \Omega^{s} \) is the Poincaré-Cartan form.

**Proposition 2** The Poincaré-Cartan form \( \Omega_{\mathcal{E}} \) of a curved Horndeski Lagrangian projects to \( J^{\pi} \) if, and only if,

\[
\frac{\partial G_{3}(\phi,X)}{\partial X} = 0.
\]

Proof. The necessary and sufficient conditions for the associated Poincaré-Cartan form of a second-order theory to project on \( J^{\pi} \), according to [64] and [65], are that \( L \in C^{\infty}(J^{\pi}) \) is an affine function with respect to the affine structure of \( \mathcal{P}^{1} \colon J^{\pi} \rightarrow J^{\pi} \), i.e.,

\[
L = L_{0}^{0}\mathcal{Y}_{(\phi)} + L_{0}^{\beta} = L_{0}^{\beta} \in C^{\infty}(J^{\pi}), \quad L_{0} \in C^{\infty}(J^{\pi}),
\]

and the following equations hold:

\[
2\frac{\partial L_{0}^{a}}{\partial y_{a}^{i}} - \frac{\partial L_{a}}{\partial y_{h}^{i}} - \frac{\partial L_{ab}}{\partial y_{i}^{a}} = 0, \quad a, h, i = 1, ..., n, \quad \alpha, \beta = 1, ..., m,
\]

which in curved Horndeski’s theory translate into

\[
L = \sum_{\alpha \leq \beta}^{\alpha} g^{\alpha \beta \mu \nu} g_{\alpha \beta \mu \nu} + L_{0}^{\beta} \mathcal{Y}_{\phi} + L_{0},
\]

where

\[
L_{0} = \sqrt{-g} \left[ g^{\alpha \beta} \left( \delta_{\phi_{\alpha \beta}} + \Gamma_{\alpha \beta}^{\gamma} + \nabla_{\alpha \beta} \Gamma_{\gamma}^{\gamma} - \nabla_{\alpha} \Gamma_{\beta \gamma}^{\gamma} - \nabla_{\beta} \Gamma_{\alpha \gamma}^{\gamma} \right) + G(\phi, X) \right],
\]

\[
L_{0}^{\beta} = \sqrt{-g} \left[ \frac{m(\alpha \beta)}{2} \left( g^{\alpha \beta} g^{30} + g^{\alpha \beta} g^{30} - 2 g^{\alpha \beta} g^{30} \right),
\]

and the equation (35) hold if, and only if,

\[
\sqrt{-g} \frac{\partial G_{3}(\phi,X)}{\partial X} \left( -2 g^{\alpha \beta} g^{30} + g^{\alpha \beta} g^{30} + g^{\alpha \beta} g^{30} \right) = 0.
\]

**Equation (40)** only holds if \( \frac{\partial G_{3}(\phi,X)}{\partial X} = 0. \)

If \( \frac{\partial G_{3}}{\partial X} = 0 \), then we should expect that the systems behaves like a first-order system. We will study this particular case first, and then we present the general case.
5.1. Hamiltonian formalism for a particular case

Throughout this subsection we shall consider that \( \frac{\partial G_3(\phi, X)}{\partial X} = 0 \), i.e. \( G_3(\phi, X) = G_3(\phi) \). Hence
\[
L_{\Omega} = \frac{1}{16\pi G} \sqrt{|g|} \left[ R + X + G_2(\phi, X) + G_3(\phi) \Box \phi \right].
\]
(41)

In this situation, the system projects to \( J^1\pi \) as is the case for the Hilbert–Einstein Lagrangian [36, 66]: Proposition 3 \( L_{\Omega} \) is an almost-regular Lagrangian and \( \mathcal{P} \) is diffeomorphic to \( J^1\pi \).

**Proof.** The submanifold \( \mathcal{P} \) is defined by the constraints
\[
p^{0,\alpha,\beta,\mu}_g \frac{\partial L}{\partial g_{\alpha,\beta,\mu}} = 0; \quad p^{0,\beta,\mu}_g - \dot{L}^\beta_{\mu} = 0,
\]
\[
p^{\alpha,\beta,\mu}_\phi - \frac{\partial \dot{L}}{\partial \phi_{\beta,\mu}} = 0; \quad p^\mu_\phi - \dot{L}^\mu_\phi = 0.
\]

Therefore, it is a closed submanifold of \( \pi^*\mathcal{P} \) and it has dimension is \( 4\pi^2 \), since the dimension of \( \mathcal{P} \) is \( 10 + 1 + 40 + 4 = 59 \). Since \( \frac{\partial G_3(\phi, X)}{\partial X} = 0 \), then \( T\mathcal{F}\mathcal{L}_{\Omega} \) has the minimal possible rank at every point, which is 59. Therefore, \( T\mathcal{F}\mathcal{L}_{\Omega} \) is surjective and \( \mathcal{F}\mathcal{L}_{\Omega} \) is a submersion. The fiber of the Legendre map are the integral submanifolds of:
\[
\ker(\mathcal{F}\mathcal{L}_{\Omega}) = \left\{ \frac{\partial}{\partial g_{\alpha,\beta,\mu}}, \frac{\partial}{\partial g_{\beta,\mu}}, \frac{\partial}{\partial \phi_{\beta,\mu}} \right\}_{\alpha,\beta,\mu,\lambda}, \quad \alpha, \beta, \mu, \lambda \in \mathbb{Z}, \quad \alpha < \beta, \quad \mu < \lambda.
\]
(42)

In other words, the fibers of the Legendre map coincide with the fibers of the projection \( \pi^3 \). This fibers are connected because we assume metrics with fixed signature.

Taking any local section \( \phi \) of the projection \( \pi^3 \), the map \( \Phi = \mathcal{F}\mathcal{L}_{\Omega} \circ \phi: J^1\pi \rightarrow P \) is a local diffeomorphism and it does not depend on the chosen section. Therefore, \( P \) and \( J^1\pi \) are diffeomorphic.

\[ J^1\pi \circ \phi = \mathcal{F}\mathcal{L}_{\Omega} \circ \phi \rightarrow P \]

The fact that both the Lagrangian (41) and the standard Hilbert–Einstein Lagrangian project to \( J^1\pi \) implies that both systems share a very similar geometry. Therefore, the following reasoning is similar than in [36].

The restriction of \( \mu \) to \( P \), denoted by \( \hat{\mu}: P \rightarrow P \), is a diffeomorphism between \( P \) and \( P \). With it we can define a **Hamiltonian section** as \( h_{\Omega} = J^1\pi \hat{\mu}^{-1} \). This is summarised in the following diagram:
\[
\begin{align*}
\mathcal{P}[\tau^3] &\rightarrow [d\tau^3]J^1\pi \rightarrow [d\tau^3]\mu^\beta \rightarrow [d\tau^3]\mu^\beta \\
\mathcal{P} \cap \ker(\mathcal{F}\mathcal{L}_{\Omega}) &\rightarrow [d\tau^3]J^1\pi \hat{\mu} \rightarrow [d\tau^3]\mu^\beta \rightarrow [d\tau^3]\mu^\beta.
\end{align*}
\]

The Hamiltonian section is given by a local Hamiltonian function \( h_{\Omega} \in C^\infty(\mathcal{P}) \); that is,
\[
h_{\Omega}(x^\mu, g_{\alpha,\beta,\mu}, \phi, g_{\alpha,\beta}, \phi_{\alpha}, \phi^0_{\beta,\mu}, p_{\alpha,\beta,\mu}, p_{\alpha,\beta,\mu}, p_{\alpha,\beta}^0_{\mu}, p^0_{\alpha,\beta}, p^{0,\beta,\mu}, p_{\alpha,\beta}^0)
\]
\[
= (x^\mu, g_{\alpha,\beta,\mu}, \phi, g_{\alpha,\beta,\mu}, \phi_{\alpha}, -H_{\Omega}, p_{\alpha,\beta,\mu}, p_{\alpha,\beta,\mu}, p_{\alpha,\beta}^0_{\mu}, p^0_{\alpha,\beta}, p^{0,\beta,\mu}, p_{\alpha,\beta}^0).
\]

This function \( h_{\Omega} \) is the Hamiltonian function defined on \( \mathcal{P} \) and is given by \( H = (\mathcal{F}\mathcal{L}_{\Omega})^* h_{\Omega} \); where \( H \), which is \( \mathcal{F}\mathcal{L}_{\Omega} \)-projectable, is
\[
H = \sum_{\alpha,\beta,\mu} L^\alpha_{\beta,\mu} g_{\alpha,\beta,\mu} + L^\alpha_{\beta} (\phi_{\beta} + \phi_{\beta}^\gamma \Gamma^\gamma_{\beta,\mu}) + \sum_{\alpha,\beta,\mu} L^\alpha_{\beta,\mu} g_{\alpha,\beta,\mu} + L^\alpha_{\beta} \phi_{\beta} - L.,
\]
(43)

The Hamiltonian forms are defined as
\[
\Theta_{h_{\Omega}}: = h_{\Omega}^* \Theta^\mu_1 \in \Omega^\mu(P), \quad \Theta_{h_{\Omega}}: = -d\Theta_{h_{\Omega}} = h_{\Omega}^* \Omega^\mu_1 \in \Omega^\mu(P),
\]
and the Hamiltonian system is \( (P, \Theta_{h_{\Omega}}) \). The **Hamiltonian problem** of this system consists in finding holonomic sections \( \psi_{h_{\Omega}}: M \rightarrow \mathcal{P} \) satisfying any of the following equivalent conditions:

1. \( \psi_{h_{\Omega}} \) is a solution to the equation
\[
\psi_{h_{\Omega}}^\mu(X) \Theta_{h_{\Omega}} = 0, \quad \text{for every} X \in \mathcal{X}(\mathcal{P}).
\]
(44)

2. \( \psi_{h_{\Omega}} \) is an integral section of a multivector field contained in a class of holonomic multivector fields \( \{X_h\} \subset \mathcal{X}(\mathcal{P}) \) satisfying the equation
\[
i(X_h) \Theta_{h_{\Omega}} = 0, \quad \forall \ X_h \in \{X_h\} \subset \mathcal{X}(\mathcal{P}).
\]
(45)
Holonomic sections and multivector fields are defined as in $f^2\pi^2$. The solutions of the Hamiltonian formalism can be recovered geometrically from the unified formalism using the adequate projections (see [20] for more details). Nevertheless, we will continue by presenting the local expression of equation (45).

**Formulation using multimomentum coordinates.**

The natural coordinates of $f^2\pi^2$ are

$$ (x^\mu, g_\alpha^\beta, \phi, g_{\alpha\beta\mu}, \phi_{\mu}, p^{\alpha\beta\mu}_G, p^\alpha_{\mu}, p^{\alpha(\beta\mu)}_G, p^{\mu}_0), $$

which contain the multimomenta and the velocities. These are the expected coordinates for a Hamiltonian formulation of a second-order regular Lagrangian. Nevertheless, our Lagrangian is singular and the Hamiltonian formulation takes place in the submanifold $\mathcal{P}$. Since it is diffeomorphic to $f^2\pi$ by proposition 3, a natural set of coordinates is

$$ (x^\mu, g_\alpha^\beta, \phi, g_{\alpha\beta\mu}, \phi_{\mu}). $$

This is an uninteresting coordinate system, as the resulting equations are identical than the Lagrangian ones. It is customary to write the Hamiltonian in terms of the positions and multimomenta only, so we need to isolate the velocities to be able to write the Hamiltonian in these terms. The relation between momenta and velocities is generally true even in the particular case where $\frac{\partial G_3}{\partial X} = 0$. To isolate the velocities we would need to fully specify $G_3(\phi, X)$ and in some cases it would not even be possible to do so.

To illustrate this procedure, we will consider the case $\frac{\partial^2 G_2}{\partial X^2} = 0$ and $1 + \frac{\partial G_2(\phi, X)}{\partial X} + \frac{\partial G_3(\phi)}{\partial \phi} \neq 0$.

With this in mind, we isolate the velocities in terms of the positions and multimomenta only:

$$ \phi_{\mu} = -\frac{1}{\sqrt{-g}} \left[ \frac{p^\mu_{\phi}}{1 + \frac{\partial G_2(\phi, X)}{\partial X} + \frac{\partial G_3(\phi)}{\partial \phi}} \right] = U_{\mu}. $$

$$ g_{\alpha\beta\mu} = \frac{1}{3\sqrt{-g}} \frac{1}{n(\alpha\beta)} $$

$$ \left[ p^{\alpha\beta\mu}_G \frac{1}{2} p^\phi_{\delta\epsilon\mu} \left[ \frac{G_3(\phi)}{1 + \frac{\partial G_2(\phi, X)}{\partial X} + \frac{\partial G_3(\phi)}{\partial \phi}} \right] \right] \left[ g^{\epsilon\sigma\lambda} g^{\delta\sigma\lambda} - g^{\epsilon\lambda}\delta^{\lambda\epsilon} \right] $$

$$ \left\{ -2g_{\alpha\lambda}g_{\beta\epsilon}g_{\lambda\epsilon} + 2g_{\alpha\beta}g_{\phi\epsilon}g_{\lambda\epsilon} - 6g_{\alpha\lambda}g_{\beta\epsilon}g_{\lambda\epsilon} \right\} = V_{\alpha\beta\mu}. $$

Notice that we require that if $\frac{\partial G_2(\phi, X)}{\partial X} + \frac{\partial G_3(\phi)}{\partial \phi} = -1$, then $p^\mu_{\phi} = 0$ and there is no hope to use $p^\mu_{\phi}$ as a coordinate instead of $\phi_{\mu}$.

Now we can set $(x^\mu, g_\alpha^\beta, \phi, p^{\alpha\beta\mu}_G, p^\mu_{\phi})$ as coordinates of $\mathcal{P}$ and then rewrite the Hamiltonian function

$$ H_0(x^\mu, g_\alpha^\beta, \phi, p^{\alpha\beta\mu}_G, p^\mu_{\phi}) = H_0(x^\mu, g_\alpha^\beta, \phi, V_{\alpha\beta\mu}(p^{\alpha\beta\mu}_G, p^\mu_{\phi}, g_\alpha^\beta, \phi), U_{\mu}(p^\mu_{\phi}, g_\alpha^\beta, \phi)). $$
The Hamiltonian function is hence
\[
H_\phi = \sum_{\alpha,\beta} p^{\alpha,\beta,\mu}_\phi V_{\alpha,\beta,\mu} + \sqrt{-g} \mathcal{G}_3(\phi)g^{\alpha\beta}U_{\gamma\mu} + p^\phi U_{\mu} - \sqrt{-g}(X + G_2(\phi, X)) - \sqrt{-g} [\frac{1}{2} \mathcal{G}_3(\phi)g^{\alpha\beta}g_{\epsilon;\sigma}(g_{\alpha,\rho} + g_{\rho,\alpha} + g_{\rho,\alpha}) \phi^\epsilon \phi, + \frac{1}{2} \mathcal{G}_3(\phi)g^{\alpha\beta}g_{\epsilon;\sigma}(g_{\alpha,\rho} + g_{\rho,\alpha} + g_{\rho,\alpha}) \phi^\epsilon \phi, + \frac{1}{2} \mathcal{G}_3(\phi)g^{\alpha\beta}g_{\epsilon;\sigma}(\Gamma^\epsilon_{\alpha\beta\gamma} + \Gamma^\epsilon_{\alpha\beta\gamma})]
\]

The field equations are derived again from (45) expressed using the new coordinates. Now, the Hamilton-Cartan form \(\Omega_h\) has the local expression:
\[
\Omega_h = \mathcal{d}H_\phi - \mathcal{d}x \sum_{\alpha,\beta} d\mathbf{p}^{\alpha,\beta,\mu} + \mathbf{d}\phi \mathbf{d}\phi - \mathbf{d}U_{\mu} \mathbf{d}U_{\mu} - \mathbf{d}L_{\mu} \mathbf{d}L_{\mu}
\]
and the local expression of a representative of a class \(\{X_h\}\) of semi-holonomic multivector fields in \(\mathcal{P}\) is
\[
X_h = \frac{i_{\mathbf{v}}}{4} \left( \frac{\partial}{\partial \mathbf{v}} + F_{g,\alpha,\beta} \frac{\partial}{\partial g_{\alpha,\beta}} + F_{\phi,\mu} \frac{\partial}{\partial \phi} + G_{p,\mu} \frac{\partial}{\partial p^\mu} + G_{\phi,\mu} \frac{\partial}{\partial \phi} \right)
\]
with \(F_{g,\alpha,\beta} G_{p,\mu} G_{\phi,\mu} \in C^\infty(\mathcal{P})\).

From (45) we obtain
\[
\frac{\partial H_\phi}{\partial g_{\alpha,\beta}} = -G^{\alpha,\beta,\mu}_\phi + G_{\phi,\mu} \frac{\partial V_{\alpha,\beta}}{\partial p^{\mu}} \frac{\partial L_{\mu}^{K,\alpha}}{\partial g_{\alpha,\beta}} + F_{g,\alpha,\beta} \left( \frac{\partial V_{\alpha,\beta}}{\partial g_{\alpha,\beta}} - \frac{\partial V_{\alpha,\beta}}{\partial g_{\alpha,\beta}} \right)
\]
\[
\frac{\partial H_\phi}{\partial \phi} = -G^{\phi,\mu}_\phi \left( \frac{\partial V_{\phi,\mu}}{\partial g_{\phi,\mu}} + \frac{\partial U_{\mu}}{\partial g_{\phi,\mu}} \frac{\partial L_{\mu}^{K,\alpha}}{\partial g_{\phi,\mu}} + F_{g,\phi,\mu} \left( \frac{\partial V_{\phi,\mu}}{\partial g_{\phi,\mu}} - \frac{\partial V_{\phi,\mu}}{\partial g_{\phi,\mu}} \right) \right)
\]
\[
\frac{\partial H_\phi}{\partial p^\mu} = F_{g,\phi,\mu} \left( \frac{\partial V_{\phi,\mu}}{\partial g_{\phi,\mu}} + \frac{\partial U_{\mu}}{\partial g_{\phi,\mu}} \frac{\partial L_{\mu}^{K,\alpha}}{\partial g_{\phi,\mu}} + F_{g,\phi,\mu} \left( \frac{\partial V_{\phi,\mu}}{\partial g_{\phi,\mu}} - \frac{\partial V_{\phi,\mu}}{\partial g_{\phi,\mu}} \right) \right)
\]
\[
\frac{\partial H_\phi}{\partial \phi^\mu} = \frac{\partial U_{\phi}^\mu}{\partial \phi^\mu} \frac{\partial L_{\phi}^{K,\alpha}}{\partial \phi^\mu} + \frac{\partial U_{\phi}^\mu}{\partial \phi^\mu} \frac{\partial L_{\phi}^{K,\alpha}}{\partial \phi^\mu} + \frac{\partial U_{\phi}^\mu}{\partial \phi^\mu} \frac{\partial L_{\phi}^{K,\alpha}}{\partial \phi^\mu}
\]
Expressions (50) through (53) would be the classical Hamilton-De Donder-Weil equations for a first order field theory except by the fact that they contain extra-terms because the cubic Horndeski Lagrangian is a second order theory with respect to the metric and the scalar field, and neither \(L_{\mu}^{K,\alpha,\beta} \cdot \frac{\partial L}{\partial g_{\alpha,\beta}}\) nor \(L_{\phi}^{\mu,\nu} \cdot \frac{\partial L}{\partial \phi^\mu}\) vanish.
5.2. Hamiltonian formalism for the general case

On this subsection we consider the full cubic Horndeski Lagrangian

\[ L_G = \frac{1}{16\pi G} \sqrt{|g|} [R + X + G_2(\phi, X) + G_3(\phi, X) \Box \phi]. \] (54)

We will assume that \( L \) is almost-regular. The multimomenta for the general cubic case are

\[ p_\alpha^{\alpha, \beta, \mu} = -\frac{1}{2} \sqrt{-g} \left( g^{\rho \lambda} \partial_{\rho \lambda} \phi \gamma, (g^{\gamma \beta} g^{\alpha \mu} + g^{\alpha \gamma} g^{\beta \mu} - g^{\gamma \mu} g^{\alpha \beta}) - g_{\rho \mu} g^{\alpha \beta} \gamma, \right) \]

\[ - \frac{1}{2} \sqrt{-g} \left( \frac{3}{2} g^{\rho \gamma} g^{\mu} g^{\alpha \beta} - 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} + 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} \right) - g^{\rho \beta} g^{\mu} g^{\alpha \gamma} \gamma, \]

\[ - \frac{1}{2} \sqrt{-g} \left( \frac{3}{2} g^{\rho \gamma} g^{\mu} g^{\alpha \beta} - 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} + 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} \right) \}

\[ = L_0^{\alpha, \beta, \mu}, \] (55)

\[ p_\mu^\nu = -\frac{1}{\sqrt{-g}} \left( \frac{\partial G_2(\phi, X)}{\partial X} + \frac{\partial G_3(\phi, X)}{\partial \phi} + \Box \frac{\partial G_3(\phi, X)}{\partial X} \right) \]

\[ + (\phi_{\alpha \nu} + \phi \Gamma_{\alpha \nu}) (g^{\alpha \beta} g^{\beta \mu} + g^{\gamma \mu} g^{\alpha \beta} - g^{\gamma \mu} g^{\alpha \beta}) \] (56)

The multimomentum (56) is not a constraint for the general case, in which \( G_3 \neq 0 \). In contrast, (55) is indeed a constraint and we must demand tangency of the multivector field to the submanifold defined by this constraint.

\[ L(X_{\alpha \beta})(p_\alpha^{\alpha, \beta, \mu} - L_0^{\alpha, \beta, \mu}) |_{X_{\alpha \beta}} = 0, \]

which yields

\[ 0 = G_2^{\alpha, \beta, \mu} - \frac{1}{2} \sqrt{-g} \left( g^{\rho \lambda} \partial_{\rho \lambda} \phi \gamma, (g^{\gamma \beta} g^{\alpha \mu} + g^{\alpha \gamma} g^{\beta \mu} - g^{\gamma \mu} g^{\alpha \beta}) - g_{\rho \mu} g^{\alpha \beta} \gamma, \right) \]

\[ - \frac{1}{2} \sqrt{-g} \left( \frac{3}{2} g^{\rho \gamma} g^{\mu} g^{\alpha \beta} - 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} + 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} \right) - g^{\rho \beta} g^{\mu} g^{\alpha \gamma} \gamma, \]

\[ - \frac{1}{2} \sqrt{-g} \left( \frac{3}{2} g^{\rho \gamma} g^{\mu} g^{\alpha \beta} - 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} + 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} \right) \}

\[ = L_0^{\alpha, \beta, \mu}, \] (55)

\[ M^{\rho \lambda \alpha \beta \mu} = -\frac{3}{2} g^{\rho \gamma} g^{\mu} g^{\alpha \beta} \gamma, + 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} + 2 g^{\rho \gamma} g^{\mu} g^{\alpha \beta} g^{\lambda} \gamma, \]

\[ + \frac{1}{2} g^{\rho \lambda} g^{\mu} g^{\alpha \beta} + g^{\rho \mu} g^{\alpha \beta} g^{\lambda} - g^{\rho \beta} g^{\mu} g^{\alpha \gamma} \gamma, \]

\[ N^{\alpha \beta \mu} = G_3(\phi, X) \phi \gamma, (g^{\gamma \beta} g^{\alpha \mu} + g^{\alpha \gamma} g^{\beta \mu} - g^{\gamma \mu} g^{\alpha \beta}) - g_{\rho \mu} g^{\rho \lambda} g^{\alpha \beta} \gamma, \times \] (56)

The second-order multimomenta \( p_{\mu}^{\rho \lambda \alpha \beta \nu} \) and \( p_{\mu}^{\rho \lambda \alpha \beta \nu} \) are completely determined by the constraints defining \( \mathcal{P} \). It is impossible, in general, to isolate the velocities in terms of the momenta unless \( G_2(\phi, X) \) and \( G_3(\phi, X) \) are explicitly specified. Moreover, the momenta depend on the acceleration of the scalar field, i.e. \( \mathcal{P} \) does not project on \( J^1 \pi \) as expected. It is impossible, in general, to explicitly isolate the velocities purely in terms of the momenta, so we will use the mix coordinates \((x^\mu, g_{\alpha \beta}, \phi, g_{\alpha \beta, \mu}, \phi_{\alpha \beta}, p_\lambda^{\rho \lambda \alpha \beta, \mu}, p_\mu^{\rho \lambda \alpha \beta, \nu}, p_{\mu}^{\rho \lambda \alpha \beta \nu})\) for the Hamiltonian formulation, which contains a positions, momenta and velocities.
In terms of these coordinates, the Hamiltonian function is

\[
H_{\mathcal{B}} = \sum_{\alpha \leq \beta} p^{\alpha\beta,\mu} \phi_{\alpha\beta,\mu}
+ \sqrt{-g} G_{\mathcal{A}}(\phi, X) g^{\mu\nu}\phi_{,\mu,\nu}^\gamma
+ p^{\phi}_{,\phi,\mu} \phi_{,\mu} - \sqrt{-g} (X + G_{\mathcal{A}}(\phi, X))
- \sqrt{-g} g^{ab} \left[ \frac{1}{2} g^{abd} \phi_{\gamma,\phi,\mu} \left( g_{bd,\alpha} + g_{ad,b} - g_{ad,\alpha} \right)
+ \frac{1}{2} g^{a\phi} g^{b\phi} \left( g_{d,a} + g_{d,a} - g_{d,a} + g_{d,a} \right) + \Gamma_{\alpha\beta}^\mu \Gamma_{\alpha\beta}^\mu \right]
\]

The Hamilton–Cartan form \( \Omega_{\phi} \) has the local expression:

\[
\Omega_{\phi} = d H_{\mathcal{B}} \wedge d^4 x - \sum_{\alpha \leq \beta} d p^{\alpha\beta,\mu} \wedge d g_{\alpha\beta,\mu} \wedge d^4 x_{\mu}
- \sum_{\alpha \leq \beta} d L^{\alpha\beta,\mu}_{,\phi} \wedge d g_{\alpha\beta,\mu} \wedge d^4 x_{\mu}
- \sum_{\alpha \leq \beta} d L^{\alpha\beta,\mu}_{,\phi} \wedge d g_{\alpha\beta,\mu} \wedge d^4 x_{\mu},
\]

and the local expression of a representative of a class \( \{ X_{\phi} \} \) of semi-holonomic multivector fields in \( P \) is

\[
X_{\phi} = \frac{i-\nu}{4} \left( \frac{\partial}{\partial x^\nu} + F_{g,\alpha\beta,\nu} \frac{\partial}{\partial g_{\alpha\beta,\nu}} + F_{\phi,\alpha\beta,\nu} \frac{\partial}{\partial \phi_{\alpha\beta,\nu}} + F_{g,\mu,\nu} \frac{\partial}{\partial g_{\mu,\nu}} + F_{\phi,\mu,\nu} \frac{\partial}{\partial \phi_{\mu,\nu}} + G_{g,\alpha\beta,\mu,\nu} \frac{\partial}{\partial g^{\alpha\beta,\mu,\nu}} + G_{\phi,\mu,\nu} \frac{\partial}{\partial \phi^{\mu,\nu}} \right)
\]

with \( F_{g,\alpha\beta,\nu}, F_{\phi,\alpha\beta,\nu}, F_{g,\mu,\nu} \), and \( G_{g,\alpha\beta,\mu,\nu}, G_{\phi,\mu,\nu} \) \( \in \mathbb{C}^N (\mathcal{F}) \).

From (45) we get

\[
\frac{\partial H_{\mathcal{B}}}{\partial g_{\alpha\beta,\mu}} = -G_{\alpha\beta,\mu}^{\alpha\beta,\mu} + F_{g,\mu,\nu} \frac{\partial}{\partial g_{\mu,\nu}} + F_{\phi,\mu,\nu} \frac{\partial}{\partial \phi_{\mu,\nu}}
\]

(59)

\[
\frac{\partial H_{\mathcal{B}}}{\partial g_{\alpha\beta,\mu}} = F_{g,\mu,\nu} \frac{\partial}{\partial g_{\mu,\nu}} + F_{\phi,\mu,\nu} \frac{\partial}{\partial \phi_{\mu,\nu}}
\]

(60)

\[
\frac{\partial H_{\mathcal{B}}}{\partial \phi_{\alpha\beta,\nu}} = F_{g,\mu,\nu} \frac{\partial}{\partial \phi_{\mu,\nu}}
\]

(61)

\[
\frac{\partial H_{\mathcal{B}}}{\partial \phi_{,\phi,\mu}} = -G_{\mu,\phi} + F_{\phi,\nu,\phi} \frac{\partial}{\partial \phi_{\nu,\phi}} + F_{\phi,\mu,\phi} \frac{\partial}{\partial \phi_{\mu,\phi}}
\]

(62)

\[
\frac{\partial H_{\mathcal{B}}}{\partial \phi_{,\phi,\mu}} = F_{\phi,\mu,\phi} \frac{\partial}{\partial \phi_{\mu,\phi}} - g^{\mu\nu} \phi_{,\mu,\phi} \frac{\partial}{\partial X} + F_{g,\mu,\phi} \frac{\partial}{\partial g_{\mu,\phi}}
\]

(63)

\[
\frac{\partial H_{\mathcal{B}}}{\partial \phi_{,\phi,\mu}} = F_{\phi,\mu,\nu} \frac{\partial}{\partial \phi_{\nu,\phi}}
\]

(64)

These are the covariant Hamilton equations for the cubic Horndeski’s theory. This is the last step of the constraint algorithm that can be performed without specifying the concrete values of \( G_{\mathcal{A}}(\phi, X) \) and \( G_{\mathcal{A}}(\phi, X) \).

Depending on their expression, these equations may not be compatible and the constraint algorithm should be continued. The dynamics of the Hamiltonian theory are determined by the Hamilton equations (59)–(64) and the tangency condition of the multivector field to the constraint (57).

6. Conclusion

In this work, we presented a multisymplectic covariant description of the cubic Horndeski theory using the unified Lagrangian–Hamiltonian formalism. The constraint algorithm was employed to determine a submanifold of the higher-order jet-multimomentum bundle \( \mathcal{F}_\gamma \) and the corresponding constraints that provide the main features of the theory.

The constraints (11), (11), (15) and (16) appear as a consequence of this formalism and define the Legendre map which further allows to pose a covariant Hamiltonian formulation and the corresponding Hamilton–de Donder–Weyl–like equations of the theory. Although more constraints appear, they have no physical relevance and are a mere consequence of the projectability of the theory, as we would expect a second order Lagrangian to produce fourth-order equations of motion, but it does produce second-order equations of motion.
We showed that the Poincaré-Cartan form of the theory does not necessarily project onto $\mathfrak{f}^\mu\pi$, unless $\frac{\partial G_\lambda(\phi, X)}{\partial X} = 0$. This makes it impossible, in general, to obtain a covariant Hamiltonian formulation with first order equations of motion. Moreover, this is a counterexample that proves that the projectability of the equations does not imply the projectability of the geometric structures. Hence, the reciprocal of proposition 1 in [66] does not hold.

With extra assumptions on the Lagrangian, we provide the expression of the velocities in function of the momenta, providing a covariant formulation of the Hamiltonian formalism which involves only multimomenta. We also present a situation where this is not possible.

For a general cubic Horndeski’s theory, we provide the covariant Hamiltonian formalism and we present the field equations. In general, they involve velocities of the metric and the scalar field, as well as the accelerations of the scalar field.

One of the most promising features of the multisymplectic framework is that it facilitates the derivation of a covariant Hamiltonian formalism, which is something that cannot be easily obtained for a theory of gravity in general due to their singular nature. This particular work was focused on a subclass of theories, however, it might serve as a reference for similar calculations for other theories of gravity.

The Hamiltonian formulation of a theory is an important step towards successfully quantising a theory. However, in quantum field theory time has a special role different from space. Therefore, the instantaneous Hamiltonian formalism, or ADM-like, has been the preferred framework to attempt canonical quantisation of classical theories of gravity, in a similar fashion than in [23].

It is yet to be seen what is the relation between our covariant formulation and the instantaneous Hamiltonian formulation of these theories. The instantaneous approach for Horndeski gravity was already introduced in [31]. A proof of the equivalence of the symplectic forms derived from the canonical and the covariant phase space formalisms for first-order field theories was derived in [67]. Recently, a proof for higher-order field theories was presented in [25]. Horndenski gravity is a singular second-order field theory, nevertheless, in light of the previous works, one expects a similar equivalence between the results of this paper and [31]. It would be interesting to work on this relationship in detail, determining how to map the different objects. This will be especially interesting because, depending on the values of $G_2$ and $G_3$, the space of multimomenta has a different dimension. This will be explored in future work.

Acknowledgments

The authors acknowledge financial support from the Ministerio de Ciencia, Innovación y Universidades (Spain), projects PGC2018–098265-B-C33 and D2021–125515NB-21 and from the grant RED2022-134301-T funded by MCIN/AEI/10.13039/501100011033

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Mauricio Doniz @ https://orcid.org/0000-0002-0063-5976
Jordi Gaset @ https://orcid.org/0000-0001-8796-3149

References

[1] Cantrijn F, Ibort A and de León M 1996 Hamiltonian structures on multisymplectic manifolds Rend. Sem. Mat. Univ. Politec. Torino 54 225–36
[2] Cantrijn F, Ibort A and de León M 1999 On the geometry of multisymplectic manifolds J. Austral. Math. Soc. Ser. A 66 303–30
[3] Echeverría-Enríquez A, De León M, Muñoz-Lecanda M C and Román–Roy N 2007 Extended Hamiltonian systems in multisymplectic field theories J. Math. Phys. 48 112901
[4] Carriñena J F, Crampin M and Ibort L A 1992 On the multisymplectic formalism for first order field theories Differ. Geom. Appl. 1 345–74
[5] Forger M, Paufler C and Roemer H 2003 The poisson bracket for poisson forms in multisymplectic field theory Rev.Math.Phys. 15 705–44
[6] Gaset J, Prieto-Martínez P D and Román–Roy N 2016 Variational principles and symmetries on fibered multisymplectic manifolds Communications in Mathematical Physics 24 137–152
[7] Guerra A and Román–Roy N 2023 More insights into symmetries in multisymplectic field theories Symmetry 15 390
[60] Cognola G, Myrzakulov R, Sebastiani L, Vagnozzi S and Zerbini S 2016 Covariant hořava-like and mimetic hornedeski gravity: cosmological solutions and perturbations Class. Quant. Grav. 33 225014
[61] Bettoni D, Ezquiaga J M, Hinterbichler K and Zumalacárregui M 2017 Speed of gravitational waves and the fate of scalar-tensor gravity Phys. Rev. D 95 084029
[62] Gaset J 2022 Geometric gauge freedom in multisymplectic field theories arXiv:2209.11121
[63] Román-Roy N 2009 Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories Symm. Integ. Geom. Methods Appl. (SIGMA) 5 100 25
[64] Rosado-Maria E and Muñoz-Masqué J 2014 Integrability of second-order Lagrangians admitting a first-order Hamiltonian formalism Differ. Geom. Appl. 35 164–77
[65] Rosado M E and Muñoz-Masqué J 2017 Second-order Lagrangians admitting a first-order Hamiltonian formalism J. Annali di Matematica 197 1–41
[66] Gaset J and Román-Roy N 2016 Order Reduction, projectability and constraints of second-order field theories and higher-order mechanics Rep. Math. Phys. 78 327–37
[67] Forger M and Sandro Vieira R 2005 Covariant poisson brackets in geometric field theory Comm. Math. Phys. 256 373–410