HÉNON–HEILES HAMILTONIAN FOR COUPLED UPPER–HYBRID AND MAGNETOACOUSTIC WAVES IN MAGNETIZED PLASMAS

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Abstract

We show that the coupled mode equations for the stationary propagation of upper–hybrid and magnetoacoustic waves in magnetized electron–ion plasmas with negative group dispersion can be exactly derived from the generalized Hénon–Heiles Hamiltonian. The parameter regimes for the integrable cases of the coupled mode equations have been explicitly obtained. For positive group dispersion of the upper–hybrid waves, the relevant governing equations lead to a novel Hamiltonian where the kinetic energy is not positive definite.
1. Introduction

Amplitude modulated upper hybrid waves in a magnetized electron–ion plasma are known to be governed by a Schrödinger–like equation wherein the potential is given in terms of the associated low–frequency number density perturbations [1–3]. For small amplitudes, the latter are governed by a linear wave equation driven by the ponderomotive force of the high–frequency upper–hybrid waves. On the other hand, for finite amplitudes, the density perturbations are governed by a driven nonlinear Boussinesq equation [4] which is coupled to the Schrödinger equation. For uni–directional propagation, the (driven) Boussinesq equation reduces to the well–known (driven) Korteweg–de Vries (K–dV) equation. For stationary propagation of the coupled waves, the time–dependent Schrödinger–Boussinesq (or, K–dV) equations give rise to a coupled system of nonlinear ordinary differential equations which can be derived from a Hamiltonian. While special analytical solutions valid in some specific regions of the allowed parameter space have been obtained [5], the question of the complete integrability of the associated Hamiltonian for arbitrary boundary conditions has not yet been answered.

We show, in this Letter, that the coupled mode equations for the upper–hybrid and magnetoacoustic waves with negative group dispersion can be exactly reduced to the generalized Hénon–Heiles equations which are extensively studied in the field of Nonlinear Dynamics [6–8]. We thereby obtain explicitly the parameter regimes for the integrability of the associated Hamiltonian. For the case of positive group dispersion, the coupled mode equations are derivable from a novel kind of Hamiltonian having indefinite kinetic energy.

2. Governing Equations

We consider the one–dimensional propagation along \( \hat{x} \)-direction of the high–frequency upper–hybrid waves in a homogeneous, magnetized electron–ion plasma with the external magnetic field (\( \vec{B}_0 \)) along the \( \hat{z} \)-direction, that is, \( \vec{B}_0 = B_0 \hat{z} \). For normal modes, the upper–hybrid wave frequency (\( \omega_0 \)) and the wavenumber (\( k_0 \)) are related by the linear dispersion relation [9],

\[
\omega_0^2 = \omega_{\mu 0}^2 + \frac{3 \omega_{pe 0}^2 k_0^2 v_{te}^2}{\omega_0^2 - 4 \Omega_{e 0}^2},
\]

(1)

where \( \omega_{pe 0} = (4 \pi n_0 e^2/m_e)^{1/2} \) is the electron plasma frequency, \( \Omega_{e 0} = e B_0/m_e c \) is the electron gyro–frequency, \( v_{te} = (T_e/m_e)^{1/2} \) is the electron thermal speed, \( \omega_{\mu 0} = (\omega_{pe 0} + \Omega_{e 0}^2)^{1/2} \) is the upper–hybrid frequency, and all the other symbols have their usual meanings [4]. In the long wavelength limit, Eq. (1) can be approximated by
\[ \omega_0 = \omega_{h0} + \frac{1}{2} D_0 k_0^2, \quad (2) \]

where \( D_0 \equiv \frac{\partial^2 \omega_0}{\partial k_0^2} = \frac{3 \omega_{pe0}^2 v_{te}^2}{\omega_{h0} (\omega_{pe0}^2 - 3 \Omega_{e0}^2)} , \quad (3) \)

denotes the group dispersion coefficient for the upper–hybrid waves. Clearly, the latter have positive (negative) dispersion for plasma parameters such that \( \omega_{pe0}^2 > 3 \Omega_{e0}^2 \) (\( \omega_{pe0}^2 < 3 \Omega_{e0}^2 \)).

For nonlinear propagations, the slowly varying complex amplitude \( E(x, t) \) of the upper–hybrid wave electric field is governed by a Schrödinger equation of the form \([2,4]\),

\[ i \left( \frac{\partial E}{\partial t} + V_g \frac{\partial E}{\partial x} \right) + \frac{D_0}{2} \frac{\partial^2 E}{\partial x^2} = \mu \omega_{h0} N E , \quad (4) \]

where, \( N = \delta n_e/n_0 \) is the normalized low–frequency density perturbation, \( V_g \equiv \partial \omega_0/\partial k_0 = k_0 D_0 \) denotes the group velocity and \( \mu = \frac{1}{2}(\omega_{pe0}^2 + 2 \Omega_{e0}^2)/(\omega_{pe0}^2 + \Omega_{e0}^2) \).

For finite amplitudes, the low–frequency density perturbation \( (N) \) is governed by a Boussinesq or a K–dV type of nonlinear equation which is driven by the ponderomotive force of the high–frequency upper–hybrid waves. The driven Boussinesq or the K–dV equation can be derived from the low–frequency fluid equations (for the electrons and the ions) which are coupled to the Maxwell equations. Omitting the details of the derivation \([4]\), we write below the driven Boussinesq equation in the form,

\[ \frac{\partial^2 N}{\partial t^2} - V_M^2 \frac{\partial^2 N}{\partial x^2} - \theta^2 \frac{\partial^4 N}{\partial x^4} - a^2 \frac{\partial^2}{\partial x^2} \left( N^2 \right) = \eta^2 \frac{\partial^2}{\partial x^2} \left( \frac{|E|^2}{16 \pi n_0 T_e} \right) , \quad (5) \]

where, \( V_M = (V_A^2 + C_s^2)^{1/2} \) is the magnetoacoustic speed, \( V_A = (B_0^2/4 \pi n_0 m_i)^{1/2} \) is the Alfvén speed, \( C_s = (T_e/m_i)^{1/2} \) is the ion–acoustic speed, \( \theta = c V_A/\omega_{pe0} \), \( a^2 = (3 V_A^2 + 2 C_s^2)/2 \), and \( \eta = \omega_{h0} C_s/\omega_{pe0} \)

Note that in the linear limit for normal modes, Eq. \( (5) \) gives,

\[ \omega^2 = V_M^2 k^2 - \theta^2 k^4 , \quad (6) \]

which is the linear dispersion relation for the magnetoacoustic modes in the long wavelength regime. Equation \( (5) \) describes the bi–directional propagation of the nonlinear magnetoacoustic waves driven by the upper–hybrid ponderomotive force.

On the other hand, for uni–directional propagation, Eq. \( (5) \) can be reduced to the form,
\[
\frac{\partial N}{\partial t} + V_M \frac{\partial N}{\partial x} + \frac{\theta^2}{2V_M} \frac{\partial^2 N}{\partial x^2} + \frac{a^2}{V_M} N \frac{\partial N}{\partial x} = -\frac{\eta^2}{2V_M} \frac{\partial}{\partial x} \left( \frac{|E|^2}{16\pi n_0 T_e} \right),
\]

which is the driven K–dV equation. For normal modes, Eq. (7) yields the dispersion relation, \( \omega = V_M k - \theta^2 k^3 / 2V_M \) which follows also from Eq. (6) in the small wavenumber limit.

For stationary propagation, the wave fields are expressed in form,

\[
E(x, t) = E(\xi) \exp \left\{ i \{ X(x) + T(t) \} \right\},
\]

\( N(x, t) = N(\xi), \)

where \( \xi = x - Mt \) represents the coordinate in the stationary frame whose speed is determined by the free parameter \( M \); the functions \( X(x) \) and \( T(t) \) are introduced in Eq. (8.1) to account for the possible shifts in the wavenumber as well as in the wave frequency due to the nonlinear interactions. Using Eqs. (8.1) and (8.2) in Eq. (4), we obtain,

\[
D_0 \frac{d^2 E}{d\xi^2} = \lambda E + b^2 NE,
\]

where \( \lambda = 2\delta + (M^2 - V^2_g) / D_0 \) is the nonlinear shift parameter, \( \delta = dT/dt \) denotes the shift in the wave frequency and \( b^2 = 2\mu \omega_{n0} \).

The stationary governing equation for the density perturbations \( N \) is obtained from Eq. (5) or (7) as,

\[
\theta^2 \frac{d^2 N}{d\xi^2} = f N - a^2 N^2 - \eta^2 \frac{E^2}{16\pi n_0 T_e},
\]

where,

\[
f = \begin{cases} 
M^2 - V^2_m, & \text{for the driven Boussinesq Eq. (5),} \\
2V_M(M - V_M), & \text{for the driven K–dV Eq. (7).}
\end{cases}
\]

Equations (9) and (10) are the relevant coupled mode equations for the stationary propagation of upper–hybrid and magnetoacoustic waves. We have presented elsewhere [5,10] different classes of exact analytical solutions which either use special boundary conditions or are valid in limited regions.
of the allowed parameter space. In the next section, we reduce these equations to the generalized Hénon–Heiles form and thereby determine the integrable parameter regimes.

3. Reduction to Hénon–Heiles equations

In order to reduce the coupled mode equations (9) and (10) to the generalized Hénon–Heiles form, we note that the parameters $\lambda$, $D_0$ and $f$ can have either signs. We normalize $N$ with respect to $-2D_0D/b^2$ and $E^2$ with respect to $16\pi n_0 T_e (2|D_0|/b^2)$, and obtain,

\[
\frac{d^2E}{d\xi^2} = \frac{\lambda}{D_0} E - \frac{2\eta^2}{\theta^2} E N ,
\]

(12)

\[
\frac{d^2N}{d\xi^2} = \frac{f}{\theta^2} N + p \frac{2 \alpha^2 \eta^2 |D_0|}{b^2 \theta^4} N^2 + p \frac{\eta^2}{\theta^2} E^2 ,
\]

(13)

where $p = -1$ for negative dispersion ($D_0 < 0$) and $p = +1$ for positive dispersion ($D_0 > 0$) of the upper–hybrid waves. We consider below these two cases separately.

(A) Negative dispersion \( (D_0 < 0) \)

For upper–hybrid waves with negative group dispersion \( (\omega_{pe0}^2 < 3\Omega_{ce0}^2) \), the coupled mode equations (12) and (13) become,

\[
\frac{d^2E}{d\xi^2} = -AE - 2DEN ,
\]

(14)

\[
\frac{d^2N}{d\xi^2} = -BN - CN^2 - DE^2 .
\]

(15)

where,

\[
A = \lambda |D_0| , \quad B = -\frac{f}{\theta^2} , \quad C = \frac{2 \alpha^2 \eta^2 |D_0|}{b^2 \theta^4} , \quad D = \frac{\eta^2}{\theta^2} .
\]

(16)

Equations (14) and (15) can be derived from the Hamiltonian,

\[
H_+ = \frac{1}{2} (\Pi_E^2 + \Pi_N^2) + \frac{1}{2} (AE^2 + BN^2) + \left( \frac{1}{3} CN^3 + DNE^2 \right) ,
\]

(17)

where $\Pi_E \equiv dE/d\xi$ and $\Pi_N \equiv dN/d\xi$ are, respectively, the “canonical momenta” conjugate to $E$ and $N$. 
Clearly, by treating $E$ and $N$ as the spatial coordinates and $\xi$ as the temporal coordinate, Eq. (17) may be considered as the Hamiltonian for the two-dimensional motion of a pseudo-particle of unit mass. In fact, $H_+$ is identically the same as the generalized Hénon–Heiles Hamiltonian [7,11] which has been extensively studied in the field of Nonlinear Dynamics. Since the stationary coordinate ($\xi$) does not explicitly appear in Eq. (17), the associated potential is conservative and hence the Hamiltonian $H_+$ is an integral of motion. For the two-dimensional motion, the system is completely integrable provided there exists the second integral of motion which is in involution with the Hamiltonian [12]. On the other hand, in recent years, the so-called “Painlevé Analysis” has been extensively used to obtain the parameter regimes wherein low-dimensional Hamiltonian systems may be completely integrable [7,8]. In particular, it is well-known that the Hamiltonian (17) is completely integrable for the following three sets of parameter values:

(a) $A = B, \quad C = D$ \hspace{1cm} (18.1)  
(b) $16A = B, \quad C = 16D$ \hspace{1cm} (18.2) 
(c) arbitrary $A$ and $B, \quad C = 6D$. \hspace{1cm} (18.3)

The associated second integrals of motion for the above parameters have been summarized in Ref. [8]. Furthermore, there are indications from general considerations that these are possibly the only integrable cases of the generalized Hénon–Heiles Hamiltonian [11]. Thus, a necessary condition for the integrability of Eqs. (14) and (15) seems to be that the nonlinear terms should have the same signs.

To obtain explicitly the various plasma parameters for integrability, we shall first consider the case when Eq. (10) together with (11) corresponds to the driven Boussinesq equation (5). In terms of the dimensionless parameters defined by $\alpha = \omega_{pe0}/\Omega_{e0}, \quad \beta = (C_s/V_\Lambda)^2$ and $\gamma = v_{te}/c$, the above relations can be written in the form,

$$3\alpha^4\gamma^2(3 + 2\beta) = \nu(2 + \alpha^2)(3 - \alpha^2),$$ \hspace{1cm} (19)

and

$$3\alpha^4\gamma^2(1 + \beta)(1 - M^2) = \nu\Lambda(1 + \alpha^2)(3 - \alpha^2),$$ \hspace{1cm} (20)

where $\Lambda = \lambda/\omega_{pe0}$ is the normalized nonlinear shift parameter, $M$ is the Mach number normalized with respect to $V_M$ and $\nu$ takes values 1, 6, or 16. Note that the parameter $\beta$ is essentially the usual plasma beta, that is, the ratio of the thermal pressure to the magnetic field pressure. For integrability of the coupled Eqs. (14) and (15), both the conditions (19) and (20) should be simultaneously satisfied for the case when $\nu = 1$ or 16, whereas only the condition (19) needs to be satisfied when $\nu = 6$. 
Defining $\Gamma^2 \equiv \frac{\gamma^2}{\nu}$, Eq. (19) can be written in the form,

$$\Gamma^2 = \frac{(2 + \alpha^2)(3 - \alpha^2)}{3\alpha^4(3 + 2\beta)}. \quad (21)$$

Note that $\Gamma^2$ remains positive definite since $\alpha^2 < 3$ is satisfied for negative group dispersion. Substituting for $\gamma^2$ from Eq. (19) into Eq. (20), we get,

$$\Delta \equiv \frac{1 - M^2}{\Lambda} = \frac{(1 + \alpha^2)(3 + 2\beta)}{(2 + \alpha^2)(1 + \beta)}. \quad (22)$$

Figure (1) shows a plot of $\Gamma^2$ as a function of $\alpha^2$ from Eq. (21) for different values of $\beta$. The value $\beta = 0$ corresponds to the cold plasma case whereas large values of $\beta$ correspond to weakly magnetized plasmas. For $\nu = 6$, the system of equations (14) and (15) is completely integrable for parameters given by Figure 1 and which satisfy $\Gamma^2 = \beta/\alpha^2\nu$ since $\beta \equiv \alpha^2\gamma^2$. Note that for $\nu = 6$, any arbitrary values of $M$ and $\Lambda$ are admissible. On the other hand, Figure (2) gives a plot of $\Delta$ as a function of $\alpha^2$ from Eq. (22) for different values of $\beta$. For $\nu = 1$ and 16, the corresponding values of the parameter $\gamma$ for integrability are given by Figure 1 whereas the parameters $M$ and $\Lambda$ are no longer arbitrary but are related by Eq. (22). Since the right–hand side of Eq. (22) is positive definite always, it follows that for positive (negative) values of the frequency shift parameter $\Lambda$, the governing equations (14) and (15) are integrable for sub–magnetoacoustic, that is, for $M < 1$ (super–magnetoacoustic, $M > 1$) values of the Mach number $M$.

The parameter values for integrability for the driven K–dV case in Eqs. (10) and (11) can similarly be obtained. In fact, it follows by inspection that Eqs. (21) and (22) hold good in this case also provided the factor $(1 - M^2)$ in the latter is replaced by $2(1 - M)$. The Mach number ($M$) regimes for integrability are, therefore, qualitatively the same as in the driven Boussinesq case.

(B) Positive dispersion $(D_0 > 0)$

For $\omega_{pe0}^2 > 3\Omega_{e0}^2$, that is, for $\alpha^2 > 3$, the upper–hybrid waves have positive group dispersion and the coupled mode equations (12) and (13) become,

$$\frac{d^2E}{d\xi^2} = AE - 2DEN, \quad (23)$$

$$\frac{d^2N}{d\xi^2} = -BN + CN^2 + DE^2, \quad (24)$$

where the coefficients $A, B, C$ and $D$ are defined, as earlier, by Eqs. (16). The coupled equations (23)
and (24) are derivable from the Hamiltonian,

\[ H_- = \frac{1}{2} \left( \Pi_E^2 - \Pi_N^2 \right) - \frac{1}{2} \left( AE^2 + BN^2 \right) + \left( \frac{1}{3} CN^3 + DNE^2 \right), \]  

(25)

where the canonical momenta, for the present case, are given by \( \Pi_E \equiv dE/d\xi \) and \( \Pi_N \equiv -dN/d\xi \). As earlier, the Hamiltonian \( H_- \) is an integral of motion corresponding to Eqs. (23) and (24).

The Hamiltonians \( H_+ \) and \( H_- \) given, respectively, by Eqs. (17) and (25) differ from each other by sign changes for the terms containing \( \Pi_N^2 \), \( E^2 \) and \( N^2 \). The sign changes for the terms containing \( E^2 \) and \( N^2 \) are trivial and can, in fact, be absorbed by redefining the coefficients \( A \) and \( B \). However, the sign change for the quadratic term in \( \Pi_N \) is indeed significant. For, unlike the case of the Hamiltonian \( H_+ \), the “kinetic energy” term in the Hamiltonian \( H_- \) is not positive definite. The latter is in contrast to the usual Hamiltonian systems of classical dynamics where the kinetic energy is positive definite always. An important consequence of this difference is that systems with indefinite kinetic energy need not necessarily have bounded motions around those points where the potential energy has a minimum. In fact, for such systems, both the canonical momenta can simultaneously increase or decrease but still keeping the Hamiltonian an integral of motion. Furthermore, the usual stability theorems developed in classical dynamics may not be directly applicable to such cases. It is therefore expected that the qualitative nature of the solutions in both the cases should be different. Hamiltonians with indefinite kinetic energy [13] and having different kinds of potential functions are known to arise in many problems dealing with the nonlinear evolution of the modulational instability of an high–frequency wave coupled to a suitable low–frequency wave. We have reported earlier [5,10] some special classes of exact analytical solutions of the coupled equations (23) and (24). However, the question of the complete integrability of such Hamiltonians, in general, and that of \( H_- \) given by Eq. (25), in particular, by either the standard Painlevé analysis or otherwise is still open.
4. Conclusions

To conclude, we have shown that the stationary governing equations for modulated upper–hybrid waves coupled to magnetoacoustic waves in a magnetized plasma with negative group dispersion are derivable from the generalized Hénon–Heiles Hamiltonian of nonlinear dynamics. The parameter regimes for the integrability of the associated Hamiltonian have been explicitly obtained. For the case of upper–hybrid waves with negative group dispersion, the equations give rise to a novel type of Hamiltonian with indefinite kinetic energy. The Painlevé analysis as well as the integrability of the latter remains to be investigated.

The results of the present investigation should have a bearing on the possible parameter regimes for the occurrence of upper–hybrid turbulence in magnetized plasmas [3,14,15]. For example, magnetized plasma turbulence has been invoked as a possible source for the narrow–band, non–thermal continuum radiation observed in the Earth’s magnetosphere [16]. In the present work, we have explicitly obtained those parameter regimes where the stationary wave fields are non–chaotic and hence can give rise to soliton–like coherent nonlinear structures [17].

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Figure Captions

**Figure 1.** Parameters $\Gamma$ and $\alpha$ for the integrability of the coupled mode equations (14) and (15) with negative group dispersion. $\beta = 0$ corresponds to the case of cold plasmas whereas large $\beta$ corresponds to the weakly magnetized plasmas. For $\nu = 6$, parameters $M$ and $\Lambda$ can independently take arbitrary values for integrability.

**Figure 2.** Parameters $\Delta$ and $\alpha$ for the integrability of the coupled mode equations (14) and (15) with negative group dispersion. $\beta = 0$ corresponds to the case of cold plasmas whereas large $\beta$ corresponds to the weakly magnetized plasmas. For $\nu = 1$ or 16, the corresponding values of $\gamma = \sqrt{\nu} \Gamma$ are to be obtained from Figure 1. Note that unlike the $\nu = 6$ case, both the parameters $M$ and $\Lambda$ are not arbitrary for $\nu = 1$ or 16, but are related by Eq. (22).