Field Definitions, Spectrum and Universality in Effective String Theories

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Abstract: It is shown, by explicit calculation, that the third-order terms in inverse string length in the spectrum of the effective string theories of Polchinski and Strominger are also the same as in Nambu-Goto theory, in addition to the universal Lüscher terms. While the Nambu-Goto theory is inconsistent outside the critical dimension, the Polchinski-Strominger theory is by construction consistent for any space-time dimension. In the analysis of the spectrum, care is taken not to use any field redefinition, as it is felt that this has the potential to obscure important points. Nevertheless, as field redefinition is an important tool and the definition of the field should be made precise, a careful analysis of the choice of field definition leading to the terms in the action is also presented. Further, it is shown how a choice of field definition can be made in a systematic way at higher orders. To this end the transformation of measure involved is calculated, in the context of effective string theory, and thereby a quantum evaluation made of equivalence of theories related by a field redefinition. It is found that there are interesting possibilities resulting from a redefinition of fluctuation field.

Keywords: Sigma Models, Bosonic Strings, QCD, Long strings.
1. Introduction

Fundamental string theories can only be consistently quantised in the so-called critical dimension which is $D = 26$ for bosonic and $D = 10$ for supersymmetric theories. On the other hand string-like defects or solitons occur in a wide variety of physical circumstances, the most well-known being vortices in superfluids, the Nielsen-Olesen vortices of quantum field theories, vortices in Bose-Einstein condensates and QCD strings. These objects clearly exist in dimensions other than the previously mentioned critical dimensions. The
challenge then is to find consistent quantum descriptions of such objects without restriction on the dimension.

Polchinski and Strominger (PS) [1] indeed showed how to do this. Their proposal is in spirit very close to that of chiral perturbation theory [2], which is an effective description of QCD at low energies. While requiring the symmetries of QCD to be maintained, it is otherwise unconstrained by requirements like polynomial lagrangians and renormalisability. Likewise PS advocated including all possible terms in the action that preserve the constraint and symmetry structure of string theories. The action terms they propose are not polynomial and in fact can become singular for certain string configurations. However, understood as terms in an effective action, they are to be used in a long-string vacuum, which allows perturbation in the small parameter \( R^{-1} \) where \( 2\pi R \) is the length of the (closed) string. Then the dominant term in the action is the usual quadratic action. In their construction, PS dropped terms in the action which are proportional to equations of motion and constraints, to appropriate orders in \( 1/R \). The quadratic action is equivalent to the well known Nambu-Goto (NG) theory when suitable constraints are imposed. The spectrum of the NG theory is exactly calculable as shown first by Arvis [3].

In the present work, we extend the original results of Polchinski and Strominger to the next order in \( 1/R \). In doing this, we find that the action need not be modified beyond what PS originally proposed [1]. We work first within the PS scheme and prove that there are no corrections to the PS action at order \( R^{-3} \). Using this result, we find that there are no subsequent further corrections to the spectrum, which thus still does not deviate from that of Nambu-Goto theory. Of course, this comparison is made whilst noting that the effective string theory construction is valid in any dimension. In this analysis of the spectrum, (and in our earlier presentation in [4]), we have stayed entirely within the PS formulation and we carefully avoid the use of any additional ingredients such as field redefinitions.

Field redefinitions bring with them a number of new issues like associated changes in measures and intrinsic arbitrariness; after our analysis of the absence of corrections to the spectrum, we thus turn to the definition of the theory itself, in the sense of a definition of the field. We first carefully define and then investigate the effects of field redefinitions, when considering further higher corrections to the action.

With regard to the field definition, the PS prescription involves a particular choice in which terms proportional to the leading-order equation of motion do not appear in the action. While this may always be chosen, we show how other choices may be made, and cast some light on the question of which terms are important in the action in a general expansion, concluding with the possibility that even the PS term is unnecessary, to certain orders, provided the transformation laws leaving the action invariant are chosen suitably. In general, there is a delicate interplay between the transformation law and the field definition; only the combined specification of the field variables, the action and the transformation laws defines a theory.

The plan of the paper is as follows. In section 2 we briefly review the original PS scheme which is valid to order \( R^{-2} \). We then prove in section 3, in very general terms, the absence of additional terms in the action which are of order \( R^{-3} \). This is crucial in establishing our results in 4.
Using this proof we carry out in section 4 an analysis of the spectrum to higher orders, where we show the absence of both order-$R^{-2}$ terms and corrections to order-$R^{-3}$ terms of the spectrum of NG theory. These results have already been presented in [4], which forms the basis for sections 3 and 4. This result for the spectrum has also been shown in [5]¹, although using somewhat different methods, and making use of field redefinition. Also, the absence of $R^{-3}$ terms in the action was merely asserted in [5] without systematic proof. The authors have had some debate [6, 7] with the author of that paper, and we shall point out differences and issues as appropriate, particularly in sections 7 and 8.

Field definitions and equivalence of theories in the effective-string context are investigated in sections 6 and 7. We finally conclude in section 10 with some remarks, observations, and ideas for further investigation.

2. Leading-order analysis

Here, we review the analysis given by Polchinski and Strominger [1]. They begin with the action

$$S = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left\{ \frac{1}{a^2} \partial_+ X^\mu \partial_- X_\mu + \beta \frac{\partial^2 X \cdot \partial_- X_\mu \partial_+ X \cdot \partial^2 X}{(\partial_+ X \cdot \partial_- X)^2} + \mathcal{O}(R^{-3}) \right\}.$$  \tag{2.1}

They show that this action is invariant, i.e $\delta S < \mathcal{O}(R^{-2})$, under the modified conformal transformations

$$\delta_- X^\mu = e^-(\tau^-) \partial_- X^\mu - \frac{\beta a^2}{2} \partial_-^2 e^- (\tau^-) \frac{\partial_+ X^\mu}{\partial_+ X \cdot \partial_- X},$$ \tag{2.2}

(and another; $\delta_+ X$ with + and − interchanged). Actually it turns out that if we do not truncate the action in eqn(2.1) to $\mathcal{O}(R^{-2})$, $\delta S < \mathcal{O}(R^{-3})$. It can also be shown that the PS transformation law also closes to this higher order. Thus both the PS action and their transformation law can be considered consistently to include order-$R^{-3}$ terms. The full equation of motion (EOM), $E^\mu = 0$, from the untruncated action is

$$E^\mu = -\frac{1}{2\pi a^2} \partial_+ X^\mu + \frac{\beta}{4\pi} \left\{ \partial^2 \frac{\partial_- X^\mu (\partial^2 X \cdot \partial_+ X)}{L^2} \right\}$$

$$+ 2 \partial_+ \left\{ \frac{\partial_- X^\mu (\partial^2 X \cdot \partial_+ X) (\partial^2 X \cdot \partial_+ X)}{L^3} \right\}$$

$$- \partial_- \left\{ \frac{\partial^2 X^\mu (\partial^2 X \cdot \partial_+ X)}{L^2} \right\} + \{ + \leftrightarrow - \},$$ \tag{2.3}

where we have used the notation $L = \partial_+ X \cdot \partial_- X$. It is easy to see that

$$X^\mu_{\text{cl}} = e^\mu_+ R \tau^+ + e^\mu_- R \tau^-;$$ \tag{2.4}

where $e_- = e_+ = 0$ and $e_+ \cdot e_- = -1/2$ satisfies the full EOM. Fluctuations around the classical solution are denoted by $Y^\mu$, so that

$$X^\mu = X^\mu_{\text{cl}} + Y^\mu.$$ \tag{2.5}

¹This unpublished work came to our attention while preparing the manuscript [4].
When fluctuations are considered around the long-string vacuum characterised by large \( R \), it becomes meaningful to use \( R^{-1} \) as an expansion parameter. Then the leading order EOM is \( \partial_{+-} X^\mu \). The untruncated transformation law leads to the energy momentum tensor (which agrees with eqn(11) of [1] to the relevant order)

\[
T_{-\bar{r}} = -\frac{1}{2a^2} \partial_- X \cdot \partial_- X + \frac{\beta}{2L^2} (L \partial^2 L - (\partial_- L)^2) + \partial_- X \cdot \partial_- X \partial^2 X - \partial_- L \partial_- X \cdot \partial^2 X
\]

(2.6)

where we have omitted terms proportional to the leading-order equation of motion. This is justified because these terms are at most proportional to \( R^{-1} \) and from eqn(2.3) the leading order EOM is itself of order-\( R^{-2} \) at the most and the neglected terms are therefore order-\( R^{-3} \), whereas for the higher-order analysis to be considered later, it suffices to keep up to order-\( R^{-2} \) terms in \( T_{-\bar{r}} \).

Returning to the leading order-analysis of PS, the energy-momentum tensor in terms of the fluctuation field is then

\[
T_{-\bar{r}} = -\frac{R}{a^2} e_- \partial_- Y - \frac{1}{2a^2} \partial_- Y \cdot \partial_- Y - \frac{\beta}{R} e_+ \cdot \partial^3 Y + \ldots
\]

(2.7)

with the OPE of \( T_{-\bar{r}} (\tau-) T_{-\bar{r}} (0) \) given by

\[
\frac{D + 12\beta}{2(\tau^-)^4} + \frac{2}{(\tau^-)^2} T_{-\bar{r}} + \frac{1}{\tau^-} \partial_- T_{-\bar{r}} + O(R^{-1}).
\]

(2.8)

It should be noted that due to the \( -\frac{R}{a^2} e_- \cdot \partial_- Y \) term in \( T_{-\bar{r}} \), in principle the order-\( R^{-2} \) term in the \( Y-Y \) propagator could contribute. It turns out that for the PS field definition it does not. When eqn(2.3) is restricted to terms linear in \( Y^\mu \) we get an equation from which the two-point function can be computed;

\[
\langle Y^\mu Y^\nu \rangle = -a^2 \log(\tau^+ \tau^-) \delta^{\mu\nu} + 2\frac{\beta a^4}{R^2} e_+ ^\nu \delta^2(\tau)
\]

(2.9)

Consequently the potential contribution to the central charge \( \frac{R^2}{a^4} e_- ^\mu e_- ^\nu (Y^\mu Y^\nu) \) vanishes, as \( e_- \cdot e_- = 0 \). This is not always true as can be checked by redefining the \( X^\mu \) field. Of course the total central charge does not change. One must add the contribution \(-26\) from the ghosts, leading to the total central charge \( D + 12\beta - 26 \). Vanishing of the conformal anomaly thus requires

\[
\beta_c = -\frac{D - 26}{12},
\]

(2.10)

valid for any dimension \( D \).

Using standard techniques the spectrum of this effective theory can be worked out. PS have shown how to do this at the leading order. We briefly reproduce their results here in order to set the stage for the rest of the paper. The Virasoro generators operate on the Fock space basis provided by \( \partial_- Y^\mu = a \sum_{m=-\infty}^{\infty} \alpha_m e^{-im\tau^-} \) and are given by

\[
L_n = \frac{R}{a} e_- \cdot \alpha_n + \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_n \cdot \alpha_m : + \frac{\beta_c}{2} \delta_n - \frac{a \beta_c n^2}{R} e_+ \cdot \alpha_n + O(R^{-2}).
\]

(2.11)
The quantum ground state is $|k, k; 0\rangle$ which is also an eigenstate of $\alpha_0^\mu$ and $\tilde{\alpha}_0^\mu$ with common eigenvalue $ak^\mu$. This state is annihilated by all $\alpha_n^\mu$ for positive-definite $n$. The ground state momentum is $p_{\text{gnd}}^\mu = \frac{R}{2a}(e_+^\mu + e_-^\mu) + k^\mu$ while the total rest energy is

$$(-p^2)^{1/2} = \sqrt{\left(\frac{R}{2a^2}\right)^2 - k^2 - \frac{R}{a^2}(e_+ + e_-) \cdot k}.$$  

(2.12)

The physical state conditions $L_0 = \tilde{L}_0 = 1$ fix $k$, so that

$$k^1 = 0, \quad k^2 + \frac{R}{a^2}(e_+ + e_-) \cdot k = \frac{(2 - \beta_c)}{a^2}.$$  

(2.13)

The first follows from the periodic boundary condition for the closed string which gives $e_+^\mu - e_-^\mu = \delta_1^\mu$. Substituting the critical value $\beta_c = (26 - D)/12$ one arrives at

$$(-p^2)^{1/2} = \frac{R}{2a^2} \sqrt{1 - \frac{D - 2}{12} \left(\frac{2a}{R}\right)^2},$$  

(2.14)

which is the precise analog of the result obtained by Arvis for open strings [3]. Expanding this and keeping only the first correction, one obtains for the static potential

$$V(R) = \frac{R}{2a^2} - \frac{D - 2}{12} \frac{1}{R} + \cdots.$$  

(2.15)

3. Absence of additional terms at order $R^{-3}$

It is of crucial importance for the arguments of this paper that the next possible candidate term in the action is not $R^{-3}$ order. PS have stated without proof in [1] that the next such term is actually of order $R^{-4}$. Drummond has stated, again without proof, that further terms do not appear until order $R^{-6}$ [5]. However, as it is such a vital point, we give here the most general proof for absence of terms at order $R^{-3}$, which we had already presented in [4].

We follow PS and construct actions that are (1,1) in the naïve sense; that is, the net number of $(+, -)$ indices is (1,1). We include no terms proportional to the leading order constraints $\partial_\pm X \cdot \partial_\pm X$ or to the leading order equations of motion $\partial_{+-} X^\mu$; otherwise they can be of arbitrary form. Clearly such actions can be constructed out of skeletal forms of the type

$$\frac{X_{\mu_1 \nu_1, m_1} X_{\mu_2 \nu_2, m_2} \cdots X_{\mu_N \nu_N, m_N}}{L^M},$$  

(3.1)

by contracting the Lorentz indices $\mu_1, \mu_2, \ldots, \mu_N$ with the help of invariant tensors $\eta_{\mu \nu}$ and $\epsilon_{\mu_1 \mu_2 \cdots \mu_D}$. Let us consider the potentially parity-violating terms involving the Levi-Civita symbols later. Here $X_{\mu, m}$ stands for $m$ derivatives of type $s = \pm$ acting on $X^{\mu}$. The numbers $\{m_i\}, M$ are adjusted to achieve the (naïve) (1,1) nature.

The PS lagrangian is not strictly a (1,1) form as can be checked explicitly. However the PS action, to the desired accuracy, is invariant under the transformation laws of eqn.(2.2). It is (1,1) only in the naïve sense mentioned above. The naïve criterion is necessary but
not sufficient, thus it suffices to prove the absence of action terms that are $R^{-3}$ using this criterion. In fact, it is desirable to have a formulation that is manifestly covariant. This will be presented elsewhere [8].

Only powers of $L$ have been used in the denominator to get a $(1,1)$ form. It may appear that any scalar in target space would have sufficed. However, the action should not become singular on any fluctuation. Thus a scalar, say, of the type $\partial^2_X \cdot \partial_X$ would not be permissible as it vanishes with $Y$. Whatever is in the denominator must be of the form $\partial_X \cdot \partial_X + \cdots$; this can always be expanded around the dominant $L$ term to produce forms as in eqn.(3.1). A covariant formulation [8] gives a natural explanation for this as well as for the forms considered in eqn.(3.1).

For now we will adhere closely to the PS prescription and discard all irrelevant terms, that is, terms proportional to the leading-order equation of motion. With regard to the meaning and consistency of this procedure, we return later in section 7.1 where we explain that this choice of irrelevant terms (i.e. none of them) in fact represents a particular choice of field definition. This should be compared with [6] where it is suggested that the correct way of handling irrelevant terms is to first eliminate them by a suitable redefinition of the field and then work out the consequences for the action and transformation laws.

### 3.1 Parity-conserving Sector

All those cases where the Lorentz contractions produce additional factors of $L$ can be reduced to forms with lower $N$; we therefore need not consider cases where the number of factors with higher derivatives ($m \geq 2$) is smaller than the number with only single derivatives. On the other hand cases with more higher-derivative factors than single-derivative factors are less dominant. Thus for the even-$N$ case considered first (taken as $2N$ from now onwards) we need to consider the maximal case of exactly $N$ single-derivative terms and $N$ terms with all possible higher derivatives.

Among the single-derivative terms, let $n_+$ be the number with $+$-derivatives; then there are $N-n_+$ single derivative terms with $-$-derivatives. Among the higher derivative terms let $p_+$ be the number of terms with only $+$-derivatives, and likewise $p_-$. Let $m_+$ be the total number of higher $+$-derivatives and $m_-$ the corresponding number of higher $-$-derivatives. As

$$p_+ + p_- = N, \quad m_+ \geq 2p_+, \quad m_- \geq 2p_-,$$

it follows that $m_+ + m_- \geq 2N$,

$$m_+ + n_+ = m_- + N - n_+, \quad M = m_+ + n_+ - 1,$$

and subsequently that $2m_+ \geq 3N - 2n_+$.

Now the leading-order behaviour of such a term is $R^{N-2(m_++n_+-1)}$. On noting that $N+2-2n_+-2m_+ \leq 2-2N$ we see that for $N \geq 3$ the leading behaviour of the action is $R^{-4}$ at most\textsuperscript{2}. The case $N = 2$ is precisely the PS action with $R^{-2}$ behaviour. The dominant case among the subdominant class for $N = 2$ (four factors) is where there are

\textsuperscript{2}Drummond has observed [8] that when $N$ is odd one actually has $2m_+ + 2n_+ \geq 3N + 1$; then the leading behaviour is actually $R^{-5}$.
three factors with only higher derivatives and one with a single derivative which we can take to be of \(+\)-type without loss of generality. If \(l_+\) denotes the total number of \(+\)-derivatives among higher derivatives and likewise \(l_-\), we must have \(l_- - l_+ = 1\). As before, if \(P_+\) denotes the number of terms with only \(+\)-derivatives and likewise \(P_-\), we have \(P_+ + P_- = 3\) and then \(l_+ \geq 2P_+, l_- \geq 2P_-\) and \(l_+ + l_- \geq 6\). These lead to \(l_+^{\text{min}} = 4, l_-^{\text{min}} = 3\), giving \(M = 3\) and the leading-order behaviour is then of order \(R^{-5}\). For \(N = 1\) (two factors) we can only have higher-derivative terms and it is easy to see that the dominant term is \(\partial^2_+ X \cdot \partial^2_+ X/L\), which in the context of this analysis is equivalent to the PS action.

We have therefore proven that there is no possibility of an order-\(R^{-3}\) term, with the next-order terms potentially at order \(R^{-4}\) and \(R^{-5}\). Drummond has further refined \(6\) our above analysis and shown that in fact these potential terms reduce via partial differentiation to terms of order \(R^{-6}\) and terms proportional to the leading-order EOM, validating his original claim in \(5\), at least in the parity-conserving case.

It is worth pointing out that though the demonstration \(5\) of the absence of \(R^{-4}, R^{-5}\) order terms is certainly important, it is not of much use until the PS transformation law is appropriately modified as it closes only up to \(R^{-4}\) order. In general, giving possible terms in the action without discussing the transformation laws that would leave them invariant is incomplete. It could well be that there are no transformation laws that leave some or all of them invariant.

Finally, one may note that just as the absence of additional terms in the action of order \(R^{-3}\) does not automatically imply, as is evident from both \(5\) and our analysis, the absence of \(R^{-3}\) corrections to the NG spectrum, absence of \(R^{-4}\) or \(R^{-5}\) terms in the action also does not translate immediately into any statement about the even higher order corrections to the spectrum.

3.2 Parity-violating sector

Finally we turn to parity-violating cases and first to the case where there is an \textit{odd} number of \(X\) fields present. This can only happen when \(D\) is odd, say, \(2n + 1\). The contraction must be between \(\epsilon_{\mu_1...\mu_{2n+1}}\) and an expression of the form

\[
\partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial^2_+ X^{\mu_3} \partial^2_- X^{\mu_4} \ldots \partial^{n+1}_+ X^{\mu_{2n+1}}. \tag{3.4}
\]

The total number of \(+\)-derivatives is \(n(n + 1)/2 + n + 1\), while the total number of \(-\)-derivatives is \(n(n + 1)/2\). The above expression multiplied by \(\partial^{n+2}_- X \cdot \partial_+ X\) balances the \(+\), \(-\) derivatives (terms with + and – interchanged are also allowed). This has to be divided by \((\partial_+ X \cdot \partial_- X)^{n(n+1)/2+n+1}\), producing a leading behaviour of \(R^{3-n^2-3n-2}\) or \(R^{-(n^2+3n-1)}\). Clearly, when \(D \geq 5\) the dominant behaviour is at most \(R^{-9}\). In \(D = 3\) one can have

\[
L^{-3} \epsilon_{\mu_1\mu_2\mu_3} \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial^2_+ X^{\mu_3} \partial^2_- X \cdot \partial^3 X \tag{3.5}
\]

which has potential \(R^{-3}\) behaviour. By partial integration we can recast \(3.5\) as

\[
- L^{-3} \epsilon_{\mu_1\mu_2\mu_3} \partial_- X^{\mu_2} \partial_- (\partial_+ X^{\mu_1} \partial^2_+ X^{\mu_3}) \partial_+ X \cdot \partial^2_+ X
- L^{-3} \epsilon_{\mu_1\mu_2\mu_3} \partial_+ X^{\mu_1} \partial^2_+ X^{\mu_2} \partial^2_- X \cdot \partial^2_- X
- 3L^{-4} \epsilon_{\mu_1\mu_2\mu_3} \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial^2_+ X^{\mu_3} (\partial^2_- X \cdot \partial_+ X)^2. \tag{3.6}
\]
The first line produces irrelevant terms, meaning terms proportional to leading EOM, of order $R^{-3}$ and higher. We will discuss later how these and other irrelevant terms are treated while discussing the use of field redefinitions. Both the remaining lines are of order $R^{-4}$ and higher. It should be noted that this has been done without recourse to fluctuation field and thus automatically obeys our principle of $X$-uniformity introduced later in section 3.

If we do use the $Y$ field, we may further recast the $R^{-4}$ terms as

$$
\frac{8}{R^4} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e_+^{\mu_1} \partial_+^2 Y^{\mu_2} \partial_- Y \{ \partial_-^2 Y^{\mu_3} - 6 e_+^{\mu_2} e_+ \cdot \partial_-^2 Y \}
$$

(3.7)

These can again be reduced to irrelevant terms by partial integration, and therefore $R^{-4}$ terms may also be eliminated, at the expense of $X$-uniformity. Nevertheless, the $R^{-5}$ terms remain and we see no way to get rid of them, at least as of now.

In the $D = 4$ parity-violating case

$$
L^{-2} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} \partial_+ X^{\mu_1} \partial_- X^{\mu_2} \partial_+^2 X^{\mu_3} \partial_- Y^{\mu_4}
$$

(3.8)

the order-$R^{-2}$ term $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e_+^{\mu_1} e_-^{\mu_2} \partial_-^2 Y^{\mu_3} \partial_- Y^{\mu_4}$, can be eliminated by partial integration as it reduces to

$$
\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e_+^{\mu_1} e_-^{\mu_2} \partial_+ Y^{\mu_3} \partial_+ Y^{\mu_4}
$$

(3.9)

due to the complete antisymmetry of the $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$.

We thus conclude that in the parity violating sector there are indeed $R^{-4}$- and $R^{-5}$-order terms. We return later to the issue of how the irrelevant terms at order $R^{-3}$ should be handled in the $D = 4$ context also.

4. Higher corrections to ground-state energy

From the expression for the ground-state momentum, it is clear that all higher corrections are determined by $A(k) = k^2 + \frac{R}{a^2} (e_+ + e_-) \cdot k$ (eqn. (2.12)) which was only calculated to leading order in eqn. (2.13). Thus an order-$R^{-n}$ correction to this would result in order-$R^{-n-1}$ and higher corrections to the spectrum. As the original PS analysis gives $R^{-1}$ terms in the spectrum, we need to investigate both $R^{-1}$ and $R^{-2}$ corrections to $A(k)$. As this quantity is just a sum of the $L_0$ and $\bar{L}_0$ conditions, we need to calculate up to order-$R^{-2}$ corrections to $L_0$ and $\bar{L}_0$, or equivalently to $T_{--}$.

As the transformation laws (2.2) have a leading part linear in $R$, additional terms in the action at order $R^{-3}$ would in principle have induced $R^{-2}$ corrections to $T_{--}$. That would in turn have changed the $R^{-3}$ terms in the spectrum. This is the reason why the absence of such terms in the action needs to be established so carefully. Absence of such terms also means that the expression for $T_{--}$ in eqn. (2.4) can be consistently expanded to keep order-$R^{-2}$ terms. We give here the on-shell expression to the desired order;

$$
T_{--} = -\frac{R}{a^2} e_- \cdot \partial_+ Y - \frac{1}{2a^2} \partial_- Y \cdot \partial_- Y - \beta \frac{R}{a^2} e_+ \cdot \partial_+^3 Y
\quad \text{−} \quad \beta \frac{R}{a^2} [2(e_+ \cdot \partial_+^2 Y)^2 + 2e_+ \cdot \partial_+^3 Y (e_+ \cdot \partial_- Y + e_- \cdot \partial_+ Y)]
\quad \text{+} \quad 2e_- \cdot \partial_+^2 Y e_- \cdot \partial_- Y + \partial_+ Y \cdot \partial_-^2 Y.
$$

(4.1)
We see hence that $L_0$ and $\tilde{L}_0$ do not receive any order-$R^{-1}$ correction. At this point, the $T_{--}$ of eqn. (4.1) does not seem holomorphic as there are $\pm$-derivative terms occurring in $T_{--}$, while the Noether procedure necessarily gives a $T_{--}$ which satisfies $\partial_+ T_{--} = 0$. The resolution of this apparent contradiction lies in the fact that the solution of the full equation of motion (2.3) can no longer be split into a sum of holomorphic and antiholomorphic pieces.

Because of the absence of additional terms in the action with the leading $R^{-3}$ behaviour, the equation of motion (2.3) is sensible inclusive of $R^{-3}$ terms. Now we expand this expression and retain terms up to order $R^{-3}$;

$$
\frac{2}{a^2} \partial_+ Y^\mu = -4 \frac{\beta}{R^2} \partial_+^2 \partial_+^2 Y^\mu - 4 \frac{\beta}{R^3} \left[ \partial_+^2 \{ \partial_+^2 Y^\mu (e_+ \cdot \partial_+ Y + e_- \cdot \partial_+ Y) \} + \partial_+^2 \{ \partial_+^2 Y^\mu (e_+ \cdot \partial_- Y + e_- \cdot \partial_+ Y) \} - e_+^\mu \partial_- (\partial_+^2 Y - \partial_+^2 Y) - e_-^\mu \partial_+ (\partial_+^2 Y - \partial_+^2 Y) \right].
$$

(4.2)

We can solve this equation iteratively by writing $Y^\mu = Y^\mu_0 + Y^\mu_1$ where $Y^\mu_0$ is a solution of the leading order equation of motion. Keeping terms only up to order $R^{-3}$ we obtain an expression which can be readily integrated to yield

$$
\frac{2}{a^2} \partial_- Y^\mu_1 = 4 \frac{\beta}{R^3} (e_+^\mu \partial_+ Y_0 \cdot \partial_+^3 Y_0 + e_-^\mu \partial_+^2 Y_0 \cdot \partial_+^2 Y_0 - \partial_+^2 Y^\mu_0 e_- \cdot \partial_+^3 Y_0 - \partial_+ Y^\mu_0 e_+ \cdot \partial_+^3 Y_0).
$$

(4.3)

Examining eqn. (4.1) one sees that to order $R^{-2}$ only the first term linear in $R$ contributes additional non-holomorphic terms which exactly cancel the remaining non-holomorphic pieces. This immediately leads to the manifestly holomorphic representation of $T_{--}$ to order $R^{-2}$,

$$
T_{--} = - \frac{R}{a^2} e_- \cdot \partial_- Y_0 - \frac{1}{2a^2} \partial_- Y_0 \cdot \partial_- Y_0 - \frac{\beta}{R} e_+ \cdot \partial_+^3 Y_0 - 2 \frac{\beta}{R^2} (e_+ \cdot \partial_+^2 Y_0)^2,
$$

(4.4)

whence we obtain the Virasoro generators with higher-order corrections,

$$
L_n = \frac{R}{a} e_- \cdot \alpha_n + \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m} \alpha_m : + \frac{\beta e_+}{2} \delta_n - \frac{a \beta e_n^2}{R} e_n - \frac{\beta e_+ a^2 n^2}{R^2} \sum_{m=-\infty}^{\infty} : e_{n-m} e_m :,
$$

(4.5)

where $e_n \equiv e_+ \cdot \alpha_n$.

Thus we have established that $L_0$ and $\tilde{L}_0$ have no corrections at either $R^{-1}$ or $R^{-2}$ order. As mentioned earlier, this means all the terms in the ground state energy and the excited state energies, inclusive of the order-$R^{-3}$ term, are identical to those in the Nambu-Goto theory.
5. Fluctuation Field Y and X-uniformity

We have made use of an expansion about a classical vacuum in our arguments about the order of terms in the action, and further made use of this fluctuation field to calculate the spectrum, along the lines of PS [1]. Furthermore, we will below consider some issues arising in field redefinitions, and we here tentatively put forth some ideas involving the fluctuation field in this context.

With no loss of generality one can make what amounts to a functional shift and set $X^\mu = X^\mu_{\text{cl}} + Y^\mu$ where $X^\mu_{\text{cl}} = R(e^\mu_+ \tau^+ + e^\mu_- \tau^-)$ which, as in (2.4), is a solution of the equation of motion of the free part of the action. Since all additional terms in the effective action involve higher derivatives, $X^\mu_{\text{cl}}$ continues to be a solution of the full EOM, and $Y^\mu$ continues to have the interpretation of a fluctuation around a classical background.

While carrying out field redefinitions of the $Y$-field, care should be taken not to upset the $(0,0)$ nature of the $X$-field. As this property is not explicit in the $Y$-formulation, it is very easy to upset.

Since the fluctuation field $Y$ derives from the $X$-field, one could consider the following principle of “X-uniformity”: All expressions involving $Y^\mu$ and its derivatives, $e^\mu_\pm, R$ must be such they are derivable from expressions involving $X^\mu$ and its derivatives only.

Many field redefinitions of the $Y$-field will not be permissible if this principle is adopted. In fact the field redefinitions found by Drummond involving $R^{-2}$ and $R^{-3}$ terms of the Polchinski-Strominger action are of a type that do not satisfy this principle.

If this principle of $X$-uniformity is applied, the $(0,0)$ property of $Y^\mu$ is never upset. In fact, without this principle, it seems unclear how to define precisely the $(0,0)$ property for the fluctuation field. Still, this does not necessarily mean that such a definition is impossible in case $X$-uniformity is not imposed.

Further motivation for the imposition of $X$-uniformity is as follows. When it is broken, the fluctuation field $Y$ and derived spectrum thereof no longer have a direct interpretation in terms of excitations about some background, since of course it is not possible in such a case to write down the background in question. Without such an interpretation, the physical significance or meaning of the fluctuations may be difficult to appreciate.

On the other hand, if one is given an effective theory describing fluctuations, there is a priori no clear reason why one should not define the field differently, and indeed one could set out with such an effective theory without even having seen the original $X$ field.

6. Equivalence of field theories

As much of the discussion below will centre around the issue of equivalence of field theories under a change of variables (what is called field redefinition) we review here known facts and establish some new points in the context of effective string theories.

As far as effective string theories are concerned two distinct possibilities exist as to their interpretation as quantum theories. One is to treat them as a tool to calculate the so-called tree diagrams only. In the early days of effective chiral symmetric field theories
Now, it is well known in chiral perturbation theory that such a limited approach would have missed many essential features like universal logarithmic behaviour and would have made a discussion of analytic properties of scattering amplitudes beyond perview. The more modern approach is to make the calculation of even loop amplitudes meaningful and the lack of renormalisability is handled through a larger set of arbitrary parameters. If we interpret the effective string theories this way, all features of a full quantum field theory are to be considered.

6.1 Classical field theories

Equivalence in classical field theories is easily proved; see for example [9] for an explicit demonstration for the case of point transformations, namely, cases where the redefinition does not involve time derivatives. At a classical level this can be extended also to more general redefinitions. This classical equivalence means that for such things as the currents and energy-momentum tensor, either using the lagrangian formalism on the original lagrangian and then transforming the fields to a new set gives the same results as first transforming the fields and then applying the formalism. The point is that the change of variables and the lagrangian formalism commute, and a field redefinition can thus be done at will to simplify the procedure without further thought.

When the redefinition in question is an infinitesimal one it is possible, classically, to eliminate terms in the action that are proportional to the leading order (in an appropriate sense) EOM. Equivalently, it can be stated that two classical actions differing from each other by EOM terms in this sense are physically equivalent. Therefore the PS prescription [1] of dropping all EOM terms as irrelevant is certainly justified at a classical level. The important question is to what extent this procedure is also valid quantum-mechanically. We are able to make some progress on this issue at least within the path-integral formulation. We argue, somewhat formally, that up to order $R^{-3}$ classical equivalence also implies quantum equivalence, but beyond that things are uncertain. This means that our original proof of the absence of order-$R^{-3}$ corrections to the spectrum is valid and complete even when considering quantum equivalence. It also means that the claim of absence of order-$R^{-4}$ and $-R^{-5}$ corrections in the action made by Drummond in [5] is certainly true classically but its validity quantum-mechanically is not certain.

6.2 Canonical formulation of quantum field theories

Proofs of equivalence in the canonical, or operator, formalism of quantum field theories are not completely straightforward, the main obstacle being the non-commutativity of various operator expressions; this has been very clearly analysed in [9, 10]. Although in that paper they do manage to prove the quantum equivalence for point transformations, such a proof for field redefinitions that go beyond point transformations does not seem to exist. In the context of effective string theories the field redefinitions considered are certainly not of the point-transformation type.
We shall therefore analyse this issue within the path-integral formalism. There is of course an intimate connection (and usually assumed equivalence) between the operator and path-integral formulations of quantum field theories. Though there are very important structural and conceptual differences, the final results for physically relevant issues are expected to be the same. This equivalence between canonical and path-integral formulations is subtle and delicate as shown long ago by Lee and Yang [11].

6.3 Path-integral formulation

In the path-integral formulation, given a particular field definition, there are three important issues: the action; the transformation laws leaving the action invariant; the invariance of the measure under said transformation laws. For non-linear transformation laws the transformation of the measure can of course be quite involved.

In the case of the PS transformation law, it is easy to see that the naive measure \( DX \) is indeed not invariant. We calculate now the change in the naive measure under the PS transformation law. Consider the relation between the untransformed field \( Y^\mu \) and the (infinitesimally) transformed field \( Y'^\mu \):

\[
Y'^\mu = Y^\mu + R e^+ e^\mu_+ + e^- \partial_- Y^\mu + \frac{\beta a^2}{R^2} e^\nu_+ \partial^\nu_+ e^- Y^\mu + \frac{2\beta a^2}{R^2} \partial^\nu_+ e^\mu_+ (e^- \partial_+ Y^\mu + e_+ \cdot \partial_+ Y) + \cdots
\]

(6.1)

Denoting the naive measure by \( DY \), its change under the above transformation is \( DY' = J DY \), where \( J \) is the determinant of \( J^\nu_\mu (\xi, \xi') = \frac{\delta Y'^\mu (\xi')}{\delta Y^\nu (\xi)} \) given by

\[
J^\nu_\mu (\xi, \xi') = \delta^\nu_\mu \delta^2 (\xi - \xi') + \delta^\nu_\mu e^- \partial_- \delta^2 (\xi - \xi') + \frac{\beta a^2}{R^2} \delta^\nu_\mu \partial^2 e^- e^- \partial_+ \delta^2 (\xi - \xi')
\]

\[
+ \frac{2\beta a^2}{R^2} \partial^\nu_+ e^\mu_+ (e_+ \partial_+ \delta^2 + e_- \partial_- \delta^2) + \frac{2\beta a^2}{R^3} \partial^2 e^- \partial_+ Y^\mu (e_+ \partial_- \delta^2 + e_- \partial_+ \delta^2) + \frac{2\beta a^2}{R^3} \partial^\nu_+ \partial^\nu_- e^- (e_+ \cdot \partial_+ Y + e_+ \cdot \partial_- Y)
\]

\[
+ \frac{2\beta a^2}{R^3} \partial^2 e^- e^\mu_+ (4(e_+ \partial_- \delta^2 + e_- \partial_+ \delta^2) (e_+ \cdot \partial_+ Y + e_+ \cdot \partial_- Y)
\]

\[
+ (\partial_+ Y^\nu \partial_+ \delta^2 + \partial_+ Y^\nu \partial_- \delta^2)) + \cdots
\]

(6.2)

The appearance of \( \delta^2 (\xi - \xi') \) and its derivatives makes the evaluation of \( J \) very delicate and a proper regularisation scheme, like for example some form of \( \zeta \)-function regularisation, is needed to do this carefully. There are two possible fates for these singular terms.

One possibility is that the singular terms from the measure could cancel against singular non-covariant expressions arising on a careful use of Feynman rules in the apparently covariant path integral. This is what was shown in [13]. The same situation shows up in perturbative quantum gravity also [12].

The other possibility is that there are singular terms in the measure which do not vanish or cancel in the above fashion, leaving non-trivial factors to be taken into account.
The only relevant changes are contained in the $Y$-dependent part of $J$ and these come from those parts of the transformation law having at least quadratic terms in the fluctuation field $Y^\mu$.

We conclude that at least formally, by which we mean that of course a careful regularised definition of the path integral in principle should be taken into account, one can say that the variation of the measure under the PS transformation law is order-$R^{-3}$ and hence does not affect our work in sections 4 where only variations that are of order $R^{-2}$ are relevant. Nevertheless, the important point evident from the above expression is that when any analysis is carried out for order-$R^{-4}$ terms and beyond, this could become an important issue and must be handled with care.

To the orders relevant for our original proof, the invariant measure is therefore just the naive measure. At higher orders this is no longer true and the issue of the correct measure has to be properly dealt with.

6.4 Jacobian for field redefinitions

When a generic field redefinition (change of variables in the path integral) is made, the new action is obtained from the old by simple substitution. The new transformation laws are what are induced by the redefinition. Unless the field redefinition is invertible it is not possible to work out the induced transformation laws. For infinitesimal changes in the fields, this can always be done.

Of course, this is not the whole picture; the change in the measure for path integration consequent to the change in variables should also be taken into account. Simple counting arguments of the type developed above in section 3 and improved upon in 6 show that the most general field redefinitions have the structure

$$\Delta Y^\mu = c_1(R)[Y] + c_2(R)[YY] + c_3(R)[YYY] + \cdots$$

(6.3)

where $[Y], [YY], \ldots$ symbolically denote terms that are linear, quadratic, etc. in the $Y$-field. Further it can be shown that the coefficients $c_n(R)$ can at most be of order $R^{-(n+1)}$. The dominant field redefinition in fact has the form

$$Y'^\mu = Y^\mu + \frac{\alpha}{R^2} \partial_+ Y^\mu + \frac{\beta}{R^3} \partial_+ Y^\mu (e_+ \cdot \partial_- Y) + \cdots$$

$$+ \frac{\gamma}{R^4} \partial_+ Y^\mu (\partial_+ Y \cdot \partial_- Y) + \cdots$$

(6.4)

The jacobian for this transformation can again be formally computed as before. Again we encounter singular expressions and careful regularisation is required to make further progress. The non-trivial part of the jacobian, by which we mean the $Y$-dependent terms, can only come from terms that are cubic in the $Y$-field in the field redefinition and those are of order $R^{-4}$. These are not relevant for the proof we have presented in section 3 and in 4.

We conclude this section with two points. Firstly, as far as the proof of the absence of $R^{-3}$ corrections to the Nambu-Goto spectrum is concerned, quantum equivalence is the same as classical equivalence.
Secondly, if a careful evaluation of these jacobians leads to local terms, one may still be able to use Drummond’s results [6] for absence of terms to order $R^{-5}$ to show that classical equivalence implies quantum equivalence up to this higher order. A rough argument in support of this could be constructed by making field redefinitions and performing partial integration as done by Drummond to eliminate all terms up to order $R^{-6}$. An inspection of the field redefinitions carried out in [6] reveals that at least formally the resultant jacobian also does not get any contributions to order $R^{-5}$. Nevertheless, this is only a sketch and we can have confidence in it only after it is carefully established. Until then, quantum equivalence to orders $R^{-4}$ and beyond should be taken as unproven.

7. Field Definitions and Redefinitions

Our analysis of the spectrum, as we mentioned in the introduction, does not involve a field redefinition. Of course, one can make the casual statement that irrelevant terms can be removed by a field redefinition, and thus be led to think that such a redefinition has been made when such terms are discarded. In fact, it is more precise to say that we have through this procedure made a certain choice of field definition, as opposed to a field redefinition. This is an important distinction which we now explain fully.

7.1 Choice of field definition

The PS procedure can be stated algorithmically as follows: Firstly, write down all possible $(1,1)$ terms (in the sense used by PS); Secondly, discard all terms proportional to the leading order constraints and their derivatives; Finally, use integration by parts to relate equivalent terms.

At this point one will have terms with and without ‘mixed derivatives’, terms sporting mixed derivatives being what we have called irrelevant. The PS prescription then is to discard all irrelevant terms and find transformation laws that leave the relevant terms in the action invariant.

This is still not uniquely specified as not all the relevant terms are independent and some can be related to others through integration by parts and additional irrelevant terms. The unambiguous method is to first express all the relevant terms in terms of a particular choice of a minimal set (this choice in itself being arbitrary) and additional irrelevant terms. Next, one chooses a subset of the irrelevant terms and simply drops all the rest. Finally, one finds transformation laws that leave this combination of relevant and irrelevant terms invariant (modulo issues of quantum equivalence discussed earlier).

Now a particular choice of mixed derivative terms amounts to a choice of field definition. A different choice of the irrelevant terms amounts to yet another choice of field parametrisation. As long as the conditions for equivalence under field redefinitions hold, one can go from one parametrisation to another with the help of a field redefinition. The transformation laws in the new definition can be worked out as induced by the field redefinition (this is somewhat more than mere substitution).
Let us illustrate this with an example. Take the partial set of terms at order $R^{-2}$ to be
\[
\alpha \frac{\partial^2_x X \cdot \partial^2_x X}{L} + \beta \frac{\partial^2_x X \cdot \partial_x X \partial^2_x X \cdot \partial_x X}{L^2} \\
+ \delta \frac{\partial_x \partial^2_x X \cdot \partial^2_x X}{L^2} + \eta \frac{\partial^2_x X \cdot \partial_x X \partial^2_x X \cdot \partial_x X}{L^2}
\]
(7.1)

In [5] it is claimed that the two relevant actions here are equivalent modulo total derivatives. This is not true and this has some bearing on the issues discussed; using the identity
\[
\frac{\partial^2_x X \cdot \partial^2_x X}{L} = \frac{\partial^2_x X \cdot \partial_x X \partial^2_x X \cdot \partial_x X}{L^2} \\
+ \frac{\partial_x \partial^2_x X \cdot \partial^2_x X}{L^2} - \frac{\partial^2_x X \cdot \partial_x X \partial^2_x X \cdot \partial_x X}{L^2} \\
+ \partial_x \left( \frac{\partial^2_x X \cdot \partial^2_x X}{L} \right) - \partial_x \left( \frac{\partial^2_x X \cdot \partial^2_x X}{L} \right)
\]
(7.2)

(7.1) can be rewritten as
\[
\beta \frac{\partial^2_x X \cdot \partial_x X \partial^2_x X \cdot \partial_x X}{L^2} + \delta \frac{\partial_x X \cdot \partial^2_x X}{L} + \eta \frac{\partial_x \partial^2_x X \cdot \partial_x X}{L^2} + \beta' \frac{\partial^2_x \partial^2_x X \cdot \partial^2_x X \cdot \partial_x X}{L^2}
\]
(7.3)

with $\beta' = \beta + \alpha, \delta' = \delta + \alpha$ and $\eta' = \eta - \alpha$. Now the choice $\delta' = 0, \eta' = 0$ yields one field definition, say, $X'$ with the effective action
\[
\beta' \frac{\partial^2_x X' \cdot \partial_x X' \partial^2_x X' \cdot \partial_x X'}{L^2}
\]
(7.4)

while the choice $\beta' = \alpha, \delta' = \alpha$ and $\eta' = -\alpha$ gives another field definition, say, $X^*$ with the effective action
\[
\alpha \frac{\partial^2_x X^* \cdot \partial^2_x X^*}{L^*}
\]
(7.5)

Thus specific choices for coefficients of the mixed derivative terms merely pick out specific parametrisations and have nothing to do with redefinitions. Having chosen a particular parametrisation, one has to work out the transformation laws leaving the action invariant, the consequent stress tensors, and so on.

In this particular example, though the two forms of relevant terms are related by a non-trivial field redefinition, the transformation laws for the two cases are identical. In fact there are families of field redefinitions that do not induce any additional terms in the transformation laws; for examples see section 8.3. Of course, in generic cases field redefinitions induce additional terms in the transformation laws.

We conclude this section with two somewhat subtle points about such redefinitions. Firstly, care has to be taken to carry through this procedure consistently to all orders. In particular, it should be noticed that with field redefinitions of order $\epsilon$ there is a residual error of at least order $\epsilon^2$ which must be included in the analysis of the most general effective action to be carried out to the next higher order. The other subtle but important point is that a redefinition eliminating one set of irrelevant terms may bring another set through the transformation of the measure.
8. Field redefinitions and irrelevant terms

With regard to field redefinitions, we have examined issues of measures and jacobians in section 6.4 above. Whereas clearly such matters cannot simply be asserted not to affect the calculations, in light of our analysis, it appears that indeed these may not be issues of consequence to the orders required in proving the absence of corrections to the spectra in the present work and in [5].

As an example of the sort of issue which can arise, let us consider how a particular field redefinition is used in [5]. In that paper, a field redefinition is used to bring the more complicated Lagrangian for the fluctuation field \( Y \) of the PS action to a simple form. In fact, the redefinition of \( Y \) reduces the PS action modulo the terms of order higher than \( R^{-3} \) to the particular free action

\[
\mathcal{L}^{(0)} = \frac{1}{4\pi a^2} \partial_+ \tilde{Y} \cdot \partial_- \tilde{Y}. \tag{8.1}
\]

Now, looking at the action (8.1), the casual reader would have expected the standard stress tensor

\[
-\frac{1}{2a^2} \partial_- \tilde{Y} \cdot \partial_- \tilde{Y}, \tag{8.2}
\]

but instead one is confronted with a different and non-trivial energy momentum tensor in [5]. That is, the energy-momentum tensor, if obtained from the old energy-momentum tensor via substitution of the redefined field \( \tilde{Y} \), is non-trivial, and of course does not coincide with (8.2).

The issue at hand is how to make the correct ‘choice’ of energy-momentum tensor, between eqn.(8.2) and the non-trivial energy momentum tensor obtained in [5] by substitution of the field redefinition. Clearly this question is not spurious, as it involves a redefinition of the field, and we in fact resolve this issue in section 8.1 below.

It is indeed not enough, as mentioned in [6], for the theory to remain conformal with critical central charge and non-trivial energy momentum tensor. Under field redefinitions central charge certainly cannot change, and neither can the conformal nature of a theory; this is somewhat beside the point, as the issue is really whether or not there are nontrivial effects on the spectrum.

Although it may appear [6] that our procedure of iteratively solving field equations is equivalent to the procedure of [5] involving field redefinition, the following comparisons can be made, showing that in fact there are significant differences.

In our approach, based on the iterative solution to the EOM, we have no need to rework either the action or the transformation laws. In the approach of [5], both of these have to be done 4. In addition to computation of the action to obtain the two-point function relevant for the OPE calculation, the method also requires the computation of the transformation law to know whether the energy-momentum tensor obtained by substitution is the canonical one or not.

\(^3\text{eqns.}(2.17-2.19)\)

\(^4\text{In [5], the induced transformation law was not computed. We compute it in section 8.3.}\)
In our method, as we explain in section 7.1, we make use of no field redefinition. In consequence, there are no issues of quantum equivalence. In this respect, differences between the two procedures would be more pronounced at higher order. Indeed, iterative solutions to field equations are possible (in principle) to any order, but as is mentioned in [3], certain subtleties of the method involving field redefinition (at least in the present context of effective strings) seem to be restricted to order $R^{-3}$.

In general solving the full EOM iteratively involves non-locality. It is a fortuitous circumstance here (perhaps because of 2-d) that it is quasi-local. Field redefinitions are typically local (not necessarily point transformations) by contrast, with nonlocal field redefinitions being a largely uninvestigated and difficult subject.

All one could have concluded had one designed a field redefinition based on the iterative solution to the full equations of motion is that $\tilde{Y}$ would be solution of the EOM of eqn.(8.1), but of course that does not guarantee that the action is given by eqn.(8.1). There are many actions (including the quadratic part of the terms in PS) whose EOM is satisfied by such a redefined field. Significantly, they generically have different two-point functions and consequently different OPEs. Such a procedure also does not guarantee that the transformation law is the standard one associated with free actions. One would have had to recompute both these, though the latter was not done in [5, 6]. In the case of [3], as we mentioned above at the beginning of section 8 while the induced action turns out to be free, the induced transformation law turns out to be non-trivial. Our method here does not require the consideration of these issues.

Before further examining the nature of the irrelevant terms, we pause here to note that in our analysis of possible terms in section 3 it took some care to decide whether a given term vanishes (by partial integration), or alternately can be reduced to an irrelevant (proportional to the EOM) term. In fact, this makes no difference for our calculations, given the way we have handled the irrelevant terms. Again making a comparison, using the method of [3, 4] this would make a difference, as what might have been thought to be an irrelevant term, necessitating a field redefinition and the attendant induced transformation law, could actually be shown to vanish if it turns out to be a total derivative. In our opinion, this motivates the use of the method in the present paper where the field definition, as explained in section 7, is made clear from the outset and the inclusion of various irrelevant terms is merely a choice of parametrisation.

8.1 Treating the irrelevant terms

In investigating the possible $R^{-3}$ corrections to the spectrum, we have dropped any order-$R^{-4}$ terms in the action. It has been suggested by Drummond [5] that this is inconsistent, and the following explanation is offered. It is shown in [3] that, in the field definition chosen in that paper, the PS action is of order $R^{-4}$ and subsequently claimed that neglecting these $R^{-4}$ terms can only be done after taking due account of the changes in the transformation law induced by the field redefinition that allowed the reduction of the PS action to $O(R^{-4})$. From this point of view, our treatment of the parity violating terms where we simply drop the $R^{-2}$ and $R^{-3}$ terms as being reducible to mixed derivative terms must appear to be incorrect.
In fact, this explanation is overly complex, and the consistency and correctness of our procedure is easily seen as follows. Summarising once more, what we have done is drop all the irrelevant terms, and find the symmetry variations for what is retained. This means that, to the order we were originally interested in, the only relevant action terms are the PS term and the free action; the PS transformation law leaves them invariant, actually to order $R^{-3}$. Noting this, nothing more need be done.

As we have explained, our treatment merely amounts to a specific choice of field parametrisation and not to a redefinition. We feel that the distinction we made between these in section 7.1 is an important one. In contrast to this straightforward procedure, the alternative proposed in [6] is that one should first have determined the transformation laws that would leave invariant the free and PS terms along with the irrelevant terms in the parity violation action, then carried out the field redefinition that would remove the irrelevant terms, and finally computed the modifications to the transformation laws. We now show through explicit calculation that these extra ingredients are unnecessary.

For this purpose consider a generic irrelevant term

$$\mathcal{L}_{\text{irr}} = F^\mu \partial^+ X_\mu$$

where $F^\mu$ can be any general expression constructed out of derivatives of $X^\mu$. In particular it can also contain additional mixed derivative terms. Now we wish to modify the transformation law so that it leaves $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{irr}}$ invariant. That this can always be done follows from the fact that the variation of the EOM is proportional to the EOM (in a functional sense) which is just another way of saying the EOM is covariant under symmetry variations. Now we evaluate the variation of $\mathcal{L}_{\text{irr}}$ under

$$\delta^{(0)} X^\mu = \epsilon^- \partial_- X^\mu,$$

yielding

$$\delta^{(0)} \mathcal{L}_{\text{irr}} = \delta^{(0)} F^\mu \partial^+_X X_\mu + F^\mu \delta^{(0)} (\partial^+_X X_\mu)$$

$$= \delta^{(0)} F^\mu \partial^+_X X_\mu - \epsilon^- \partial_- F^\mu \partial^+_X X_\mu),$$

(8.5)

where we have dropped total derivative terms as usual. Therefore the modification to the transformation law under which $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{irr}}$ will be invariant is

$$\delta X^\mu = 2\pi a^2 (\delta^{(0)} F^\mu - \epsilon^- \partial_- F^\mu).$$

(8.6)

On the other hand the field redefinition needed to remove the irrelevant term is

$$\Delta X^\mu = -2\pi a^2 F^\mu.$$

(8.7)

This induces the additional terms

$$\delta X^\mu = \delta^{(0)} \Delta X^\mu - \epsilon^- \partial_- (\Delta X^\mu)$$

$$= -2\pi a^2 (\delta^{(0)} F^\mu - \epsilon^- \partial_- F^\mu)$$

(8.8)
and this contribution exactly cancels the modification demanded by the irrelevant term.

Generalisation of these arguments to include an arbitrary number of relevant terms in the action is straightforward as long as one keeps track of the order of terms carefully.

We conclude that the procedure suggested in \( \text{[6]} \) is equivalent to simply dropping all irrelevant terms and working with the transformation laws that leave only the relevant terms invariant, which is exactly what we have done in the first sections of the present work. As we have described, this is in fact a particular field definition. Thereafter dropping the irrelevant parity-violating terms one is left with just the free action and PS terms as relevant; as the PS transformation laws expanded to the next higher order left them invariant, this exercise serves as a check on our methodology in sections \( \text{[3]} \) and \( \text{[4]} \).

At this stage it may occur to the reader that following such a procedure, the entire PS action itself might be dropped. Although it has been suggested in \( \text{[6]} \) that this indicates there is something amiss with our methodology, it is in fact a valid possibility which we instead take seriously in the following section.

\section*{8.2 Relevant part of the PS action}

In the previous section we discussed our dropping of \( \mathcal{O}(R^{-4}) \) terms. Naturally, one is tempted to see how far this procedure can be carried, and to determine which terms must be retained and which may be discarded. Indeed, it did not go unnoticed \( \text{[6]} \) that using our methods, one could be led to discard the PS term in the action in its entirety.

At first this seems surprising, and an argument against it could begin with the assertion that the PS action has left its trace in terms of a non-trivial energy momentum tensor and a non-trivial transformation law, somehow remembering the parameters (i.e. \( \beta \)) of the irrelevant terms. Also, one could argue, it was the PS term that rendered the effective theories valid in all dimensions.

Following this putative argument, our irrelevant terms also could in principle have left their nontrivial traces, but these are simply not there in our way of handling things. Consider, though, that we have explicitly and quite generally proved in the previous section that the field redefinitions exactly compensate any modifications to the transformation laws brought forth by the irrelevant terms: How can these two apparently contradicting situations be reconciled?

Before we go on we wish to categorically state that the \( R^{-4} \)-order terms are totally irrelevant for our analysis in sections \( \text{[3]} \) and \( \text{[4]} \). This is also true for Drummond’s analysis in \( \text{[5]} \). The simple reason for this is that the variation of such terms under the PS transformation law can at most be of order \( R^{-3} \) and consequently their contribution to the Virasoro generators can also be at most \( R^{-3} \). The above-mentioned imprints in the transformation law must all be from \( R^{-2} \) or \( R^{-3} \) terms.

Now, let us go on and examine the case of PS action more carefully. If we had simply dropped the irrelevant terms from the PS action we would have ended up only with (modulo \( R^{-4} \) and higher order terms)

\[
\frac{1}{2\pi a^2} \int \partial^+ Y \cdot \partial^- Y \quad (8.9)
\]
Now according to our prescription we should just work with this relevant action and the transformation laws leaving it invariant. Somewhat surprisingly, it turns out that not just the standard
\[ \delta^{(0)}_{-} Y^\mu = \epsilon^- \partial_- Y^\mu. \] (8.10)
but at least the entire two-parameter family
\[ \delta^{(0)}_{-} Y^\mu = \epsilon^- \partial_- Y^\mu + \epsilon^{-} R e^\mu \]
\[ + \frac{\beta' a^2}{R} \partial^2_+ \epsilon_+^\mu + \frac{2 \beta' a^2}{R^2} \partial^2_+ \epsilon_+^\mu e_+ \cdot \partial_- Y \] (8.11)
leaves the above action invariant (exactly). Also, they constitute the conformal group (for fixed \( R \) and \( \beta' \)) as can be verified by
\[
[\delta_- (\epsilon_1^-), \delta_- (\epsilon_2^-)] = \delta_- (\epsilon_{12}^-); \quad \epsilon_{12}^- = \epsilon_1^- \partial_- \epsilon_2^- - \epsilon_2^- \partial_- \epsilon_1^- \] (8.12)
It should be emphasised that at this stage the parameter \( \beta' \) has nothing to do with \( \beta \).

The resulting canonical energy-momentum tensor is indeed the same with \( \beta \) replaced by \( \beta' \) as eqn(19) of [4] or eqn(2.20) of [5] (after correcting a sign error). The central charge depends on the free parameter \( \beta' \) and can be adjusted for consistency in all dimensions.

On the other hand, if we had included the irrelevant terms used in [5], the transformation law that would have left the above combination of relevant and irrelevant terms invariant would have been modified to
\[ \delta^{\text{tot}}_- = \delta^{(0)}_- Y^\mu + \frac{\beta a^2}{R} \partial^2_+ \epsilon_+^\mu + \frac{2 \beta a^2}{R^2} \partial^2_+ \epsilon_+^\mu e_+ \cdot \partial_- Y \] (8.13)
If now one had done a field redefinition to remove the irrelevant terms there would be additional terms induced in the above equation. As shown above their effect would be to cancel the \( \beta \)-dependent terms and just leave eqn.(8.11).

If one computes the transformation law to which the PS-transformation law would have been modified by the field redefinition, one finds it to be just eqn.(8.11) with \( \beta' \) replaced by \( \beta \), modulo some harmless terms arising from the ambiguities in field redefinitions.

The question now is: How did \( \beta' \) get to be replaced by \( \beta \) which is a parameter belonging entirely to the irrelevant terms? The resolution is the following. According to the above construction what would leave the combination of relevant and irrelevant terms of this example would be eqn.(8.13), but this is clearly not the PS-transformation law expanded to suitable orders. Nevertheless in the scheme used in [5], this would have been taken to be the PS-transformation suitably expanded. This is consistent with eqn.(8.13) if and only if \( \beta' \) had been equated to \( \beta \).

It seems then that, contrary to folklore, a free bosonic string theory with an adjustable central charge can be made consistent provided the conformal transformation law is chosen appropriately. This can be viewed as an alternative approach to the spectrum of free strings that appears consistent in all dimensions. How far such an approach to string theory can be extended further is currently under investigation [13].
8.3 Ambiguities in field redefinitions

All field redefinitions are ambiguous up to terms of the type

$$\Delta X^\mu = N^\mu, \quad N \cdot E = 0 \quad (8.14)$$

where $E^\mu$ is the EOM.

Not all field redefinitions induce changes in transformation laws. Some examples are:

$$\Delta_1 X^\mu = \frac{\partial_{+-} X^\mu}{L}$$
$$\Delta_2 X^\mu = \frac{(\partial_+ X^\mu \partial_- X + \partial_- X^\mu \partial_+ X \cdot \partial_- X)}{L^2}$$
$$\Delta_3 X^\mu = \frac{(\partial_+ X^\mu \partial_- X - \partial_- X^\mu \partial_+ X \cdot \partial_- X)}{L^2}$$

It should be noted that $\Delta_3 X^\mu$ is of the form of an ambiguity. It is so even when $L^{-2}$ is replaced by any power of $L$, but the transformation law is unaffected only for $L^{-2}$. Only the choice $L^{-2}$ maintains the $(0,0)$ character of $X^\mu$.

9. OPE and Virasoro algebra to higher order

In the first part of this paper, sections 3 and 4, we consistently expanded the PS action to order $R^{-3}$, and the PS stress tensor to order $R^{-2}$. This analysis also implies that it is enough to expand the transformation law given by PS to include order-$R^{-2}$ terms. Incidentally, the transformation law is closed (satisfying the Virasoro algebra) to order $R^{-3}$; only at order $R^{-4}$ do the PS transformations fail to close.

Here we also point out that the variation of the entire PS action, eqn (1) of [4], without truncating to any order, under the entire PS variation, eqn(2) of [4], again without any truncations, has only $\beta^2$ terms and these are of order $R^{-4}$. Furthermore, the untruncated PS transformation has the closure property of Virasoro algebra also to order $R^{-4}$ as is very easily verified. The terms that spoil closure are again $\beta^2$ terms. Also, the form of the higher corrections do not obviously affect the central charge. Taken together these guarantee that there are no issues with either the OPE of stress tensors or of the validity of the Virasoro algebra generated by their moments to the order to which we extended the PS results. The fact that our higher order energy-momentum tensor is conserved is another consistency check.

10. Conclusions and comments

Not only is the Polchinski-Strominger action [1] the unique effective first-order action for a consistent conformal theory of long strings, but as we have carefully shown in sections 3 and 4 it is essentially unique up to and including terms of third order in the inverse string length. The only remaining freedom in the action is the use of irrelevant terms, as for example the other ‘equivalent’ form of the PS term, what we have called the other relevant term, which we mentioned in section 3 and which does not alter our results for the
spectrum. As we have discussed in section 7, this is an example of choice of field definition, and the possibilities become more numerous as one considers higher-order actions.

Furthermore, the spectrum is found to coincide with that of the Nambu-Goto theory, including third order terms. This universality explains why comparisons between potentials and excited state energies in lattice computations [14, 15, 16, 17, 18] and Nambu-Goto theory have been favourable in the past even beyond the universal Lüscher term [19, 20] (in the case of the ground state energy), despite the inconsistency of the Nambu-Goto string outside the critical dimension.

Again with regard to the above statements about subsequent higher-order terms in the action, it has now been proven carefully [6] that the action does not contain relevant terms all the way up to and including order-$R^{-5}$ terms, when consideration is restricted to the parity-conserving sector. This is significant, and one would like to ask what consequence this has for corrections to the spectrum. An obvious path for further work is to determine when corrections to the Nambu-Goto spectrum do in fact occur. In the parity-violating sector, we have shown in section 3.2 that correction terms begin at order $R^{-5}$ and it would also be interesting to consider the consequences of this fact on the spectrum.

In the context of effective string theory, we have also reached some conclusions about field redefinitions. We have shown that our results on the spectrum in section 4 can indeed be justified in a quantum-mechanical sense, and have motivated our position that it is wise to avoid field-redefinitions without a careful consideration of the transformation of the measure. We introduced the idea of X-uniformity in section 5 and given some motivation for its application; still, this principle also seems somewhat restrictive and we subsequently abandoned it for a time in section 8.2 in order to consider dropping the PS term itself.

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