A new measure of nonclassical distance

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Abstract. In the present paper we shall propose a new measure of the nonclassical
distance \[.\] The proposed modification is based on the following considerations. If
\(\rho_1\) and \(\rho_2\) are density operators, and \(F(\rho_1, \rho_2)\) is the corresponding fidelity, then
from the inequalities \[.\] it is evident that the quantity
\[
\phi(\rho) = \sup_{\rho_{cl}} F(\rho_{cl}, \rho)
\]
can be used in the same extent as a measure of the distance of the state \(\rho\) to the set
of classical states \(\rho_{cl}\) as the Hillery measure \(\delta(\rho) = \sup_{\rho_{cl}} \|\rho_1 - \rho_2\|\) \[.\] \(\phi(\rho_{cl}) = 1\)
for any classical state and \(\phi(\Gamma(\rho)) \geq \phi(\rho)\) if the map \(\Gamma\) is the Gaussian noise map \[.\]

Short title: A new measure of nonclassical distance

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1. Introduction

In [1], Hillery gave a definition of the nonclassical distance of radiation in terms of the trace norm as

\[
\delta(\rho) = \inf_{\rho_{cl}} ||\rho - \rho_{cl}||,
\]

where \(\rho\) is the density matrix of the nonclassical radiation and \(\rho_{cl}\) is that of an arbitrary classical field, while \(||A||_1\) is the trace norm of the operator \(A\). The Hillery nonclassical distance has the following properties:

(i) \(\delta(V(u)\rho V(u)^\dagger) = \delta(\rho)\);
(ii) \(0 \leq \delta(\rho) \leq 2\);
(iii) \(\delta(\theta\rho_1 + (1 - \theta)\rho_2) \leq \theta\delta(\rho_1) + (1 - \theta)\delta(\rho_2)\).

where the unitary operators \(V(u)\) are the well known Weyl operators giving a projective unitary representation of the vector group \(R^{2n}\) (see the next section). This measure seems to be quite universal. Unfortunately, it is not easy to use this measure in actual calculations. In practice, one can only provide the upper bound and lower bound of the measure.

It is possible to define the distance between two quantum states described by density operators in many ways [5, 6, 7, 8]. If the distance is small these two density operators can be considered very similar to each other. On the other hand a large distance means very different density operators.

Another, more physical point of view is that presented in [2]. According to this point of view "the only physical means available with which to distinguish two quantum states is that specified by the general notion of quantum mechanical measurement". Instead of a metrical point of view a statistical point of view is taken into account. A measurement being necessarily indeterministic and statistic the more physical measures of distance between two quantum states are those which are based on the statistical-hypothesis testing procedures.

In the present paper we shall propose a new measure of the nonclassical distance [1]. The proposed modification is based on the following considerations. If \(\rho_1\) and \(\rho_2\) are density operators, and \(F(\rho_1, \rho_2)\) is the corresponding fidelity, then from the inequalities [2]

\[
2(1 - \sqrt{F(\rho_1, \rho_2)}) \leq ||\rho_1 - \rho_2||_1 \leq 2[1 - (F(\rho_1, \rho_2))]^{\frac{1}{2}}
\]

it is evident that the quantity

\[
\phi(\rho) = \sup_{\rho_{cl}} F(\rho_{cl}, \rho)
\]

can be used in the same extent as a measure of the distance of the state \(\rho\) to the set of classical states \(\rho_{cl}\) as Hillery’s measure \(\delta(\rho)\).

Let \(\rho_1\) and \(\rho_2\) be two density operators which describe two mixed states. The transition probability \(P(\rho_1, \rho_2)\) has to satisfy the following natural axioms:

(i) \(P(\rho_1, \rho_2) \leq 1\) and \(P(\rho_1, \rho_2) = 1\) if and only if \(\rho_1 = \rho_2\);
(ii) \(P(\rho_1, \rho_2) = P(\rho_2, \rho_1)\);
(iii) If $\rho_1$ is a pure state, $\rho_1 = |\psi_1><\psi_1|$ then

$$P(\rho_1, \rho_2) = <\psi_1|\rho_2|\psi_1>$$

(iv) $P(\rho_1, \rho_2)$ is invariant under unitary transformations on the state space;

(v) $P(\rho_1|_A, \rho_2|_A) \geq P(\rho_1, \rho_2)$ for any complete subalgebra of observables $A$;

(vi) $P(\rho_1 \otimes \sigma_1, \rho_2 \otimes \sigma_2) = P(\rho_1, \rho_2)P(\sigma_1, \sigma_2)$.

(vii) $P(\mu_1 \rho_1 + \mu_2 \rho_2, \sigma)$ $\geq$ $\mu_1 P(\rho_1, \sigma) + \mu_2 P(\rho_2, \sigma)$ when $0 \leq \mu_1, \mu_2 \leq 1, \mu_1 + \mu_2 = 1$.

Uhlmann’s transition probability for mixed states $[9, 10, 11, 12]$

$$P(\rho_1, \rho_2) = \left[ \text{Tr} \left( \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right)^{1/2} \right]^2$$

(4)

satisfies properties 1–7. The fidelity is defined by $F(\rho_1, \rho_2) = P(\rho_1, \rho_2)$. A detailed analysis for the structure of the transition probability was hampered by the factors containing square roots. Due to technical difficulties in the computation of fidelities, few concrete examples of analytic calculations are known. The first results in an infinite-dimensional Hilbert space were recently obtained by Twamley [13] for the fidelity of two thermal squeezed states and by Paraoanu and Scutaru [14] for the case of two displaced thermal states. In [15] Scutaru has developed another calculation method which allowed getting the result for the case of two displaced thermal squeezed states in a coordinate-independent form. A general formula for the fidelity of any two mixed Gaussian states (i.e. multimode displaced thermal states $[16, 17, 18, 19, 20, 21]$, from which the previous results can be obtained as particular cases, has been obtained recently $[22]$.

Another modification is based on the following considerations $[23]$. If $\rho_1$ and $\rho_2$ are density operators, then it is easy to establish $[23]$ the inequalities

$$2(1 - \text{Tr} \sqrt{\rho_1} \sqrt{\rho_2}) \leq ||\rho_1 - \rho_2||_1 \leq 2[1 - (\text{Tr} \sqrt{\rho_1} \sqrt{\rho_2})^2]^2$$

(5)

from which it is evident that the quantity

$$\chi(\rho) = \sup_{\rho_{cl}} \text{Tr} \sqrt{\rho_{cl} \sqrt{\rho}}$$

(6)

can be used in the same extent as a measure of the distance of the state $\rho$ to the set of classical states $\rho_{cl}$ as the Hillery measure $\delta(\rho)$. In the same time it is more easy to calculate this quantity than Hillery’s measure. When $\rho_1$ and $\rho_2$ are the density operators of multimode displaced squeezed thermal states then the quantity $\text{Tr} \sqrt{\rho_1} \sqrt{\rho_2}$ can be computed in an explicit way $[23]$.

In the following we shall take as a fundamental test for a good nonclassical distance the fact that it must increase under the action of a Gaussian (or thermal) noise $[3, 4, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]$. A drawback of the measure of the nonclassical distance which is based on the Holevo overlap is given by the fact that in this case the condition is not fulfilled.

2. Multimode thermal squeezed states

Let $(E, \sigma)$ be a phase space i.e. a vector space with a symplectic structure $\sigma$. Then the commutation relations on $(E, \sigma)$ acting in a Hilbert space $\mathcal{H}$ are defined by a
A continuous family of unitary operators \( \{V(u), u \in E\} \) on \( \mathcal{H} \) which satisfy the Weyl relations \([10,21]\):

\[
V(u)V(v) = \exp \frac{i}{2} \sigma(u,v) V(u + v).
\]

(7)

Hence the family \( \{V(tu), -\infty < t < \infty\} \) for a fixed \( u \in E \) is a group of unitary operators.

Then by the Stone theorem

\[
V(u) = \exp i R(u),
\]

(8)

where \( R(u) \) is a selfadjoint operator. From the Weyl relations we have

\[
\exp it R(u) \exp is R(v) = \exp its \sigma(u,v) \exp is R(v) \exp it R(u).
\]

By differentiation and taking \( t = s = 0 \) one obtains

\[
[R(u), R(v)] = -i \sigma(u,v) I.
\]

(9)

The operators \( \{R(u), u \in E\} \) are called canonical observables.

The phase space \( E \) is of even real dimension 2n and there exist in \( E \) symplectic bases of vectors \( \{e_j, f_j\}_{j=1,...,n} \), i.e., reference systems such that \( \sigma(e_j, e_k) = \sigma(f_j, f_k) = 0 \) and \( \sigma(e_j, f_k) = -\sigma(f_j, e_j) = \delta_{jk}, j,k = 1,...,n \). The coordinates \( (\xi^j, \eta^j) \) of a vector \( u \in E \) in a symplectic basis \( u = \sum_{j=1}^{n} (\xi^j e_j + \eta^j f_j) \) are called symplectic coordinates. The well known coordinate and momentum operators are defined by \( Q_k = R(f_k) \) and \( P_k = R(e_k) \) for \( k = 1,2,...,n \). Then the canonical observables \( R(u) \) are linear combinations of the above defined coordinate and momentum operators:

\[
R(u) = \sum_{j=1}^{n} (\xi^j P_j + \eta^j Q_j).
\]

There is a one-to-one correspondence between the symplectic bases and the linear operators \( J \) on \( E \) defined by \( Je_k = -f_k \) and \( Jf_k = e_k \), \( k = 1,...,n \). The essential properties of these operators are:

\[
\sigma(Ju,u) \geq 0, \quad \sigma(Ju,v) + \sigma(u,Jv) = 0 \quad (u,v \in E \text{ and } J^2 = -I, I \text{ denotes the identity operator on } E).
\]

Such operators are called complex structures. In the following we shall use the matricial notations with \( u \in E \) as column vectors. Then \( \sigma(u,v) = u^T J v \) and the scalar product is given by \( \sigma(Ju,v) = u^T v, u,v \in E \). A linear operator \( S \) on \( E \) is called a symplectic operator if \( S^T JS = J \). When \( S \) is a symplectic operator then \( S^T \) and \( S^{-1} \) are also symplectic operators. The group of all symplectic operators \( Sp(E,\sigma) \) is called the symplectic group of \( (E,\sigma) \). The Lie algebra of \( Sp(E,\sigma) \) is denoted by \( sp(E,\sigma) \) and its elements are operators \( R \) on \( E \) with the property:

\[
(JR)^T = JR.
\]

Hence an operator \( R \) on \( E \) belongs to \( Sp(E,\sigma) \) if \( R^2 = -I \). If \( J \) and \( K \) are two complex structures, there exists a symplectic transformation \( S \) such that \( J = S^{-1} KS \). For any symplectic operator \( S \) we can define a new system of Weyl operators \( \{V(Su); u \in E\} \). Then from a well known result on the unicity of the the systems of Weyl operator up to a unitary equivalence it follows that there exists a unitary operator \( U(S) \) on \( \mathcal{H} \) such that

\[
V(Su) = U(S)^TV(u)U(S).
\]

For any nuclear operator \( O \) on \( \mathcal{H} \) one defines the characteristic function

\[
CF(u) = TrOV(u), \quad u \in E.
\]

(10)

We give the properties of the characteristic function which are important in the following \([19]\).
(i) \( CF_0(O) = TrO; \)
(ii) \( CF_u \left[ V(v)^\dagger OV(v) \right] = CF_u \left[ O \exp i\sigma(v,u) \right]; \)
(iii) \( CF_u(O_1O_2) = \frac{1}{(2\pi)^n} \int CF_v(O_1)CF_{u-v}(O_2) \exp \frac{i}{2}\sigma(v,u)dv; \)
(iv) \( CF_{Su}(O) = CF_u(U(S)OU(S)^\dagger). \)

The multimode thermal squeezed states are defined by the density operators \( \rho \) whose characteristic functions are Gaussians \[15, 19, 21\]

\[
CF_u(\rho) = \exp \left\{ -\frac{1}{4}u^T Au \right\}. 
\]  
(11)

where \( A \) is a \( 2n \times 2n \) positive definite matrix, called correlation matrix. From the last property of the characteristic function, enumerated above, it follows that:

\[
A_{U(S)\rho U(S)^\dagger} = S^T A \rho S \]

(12)

Because the correlation matrix \( A \) is positive definite it follows \[21, 36\] that there exists \( S \in Sp(E,\sigma) \) such that

\[
A = S^T D S
\]

(13)

where \( D = \left( \begin{array}{cc} D & 0 \\ 0 & D \end{array} \right) \) and \( D \geq I \) is a diagonal \( n \times n \) matrix. The most general real symplectic transformation \( S \in Sp(E,\sigma) \) has \[21, 37\] the following structure:

\[
S = O M O' \]

(14)

where

\[
M = \left( \begin{array}{cc} M & 0 \\ 0 & M^{-1} \end{array} \right)
\]

(15)

and \( O, O' \) are symplectic and orthogonal \( (O^T O = I) \) operators, and where \( M \) is a diagonal \( n \times n \) matrix. As a consequence the most general form of a correlation matrix \( A \) is given by:

\[
A = O'^T M O^T D O M O' \]

(16)

Various particular kinds of such matrices are obtained taking \( O, O', D \) or \( M \) to be equal or proportional to the corresponding identity operator. A pure squeezed state is obtained when \( D = I \). If this condition is not satisfied, the state is a mixed state called thermal squeezed state \[38\]. When \( M = I \) there is no squeezing and the corresponding states are pure coherent states or thermal coherent states. All these states have correlations between the different modes produced by the orthogonal symplectic operators \( O \) and \( O' \).

From the property 3 of the characteristic function we have for two density operators \( \rho_1 \) an \( \rho_2 \)

\[
CF_u(\rho_1\rho_2) = \left[ det \left( \frac{A_1 + A_2}{2} \right) \right]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{4}u^T \left[ A_2 - (A_2 - iJ)(A_1 + A_2)^{-1}(A_2 + iJ) \right] u \right\}. 
\]
When $\rho_1 = \rho_2$ we have

$$CF_u(\rho^2) = \left(\det A\right)^{-\frac{1}{4}} \exp \left\{ -\frac{1}{4} u^T \left( A - JA^{-1}J \right) u \right\}.$$  \hspace{1cm} (17)

A state $\rho$ is pure iff $\rho^2 = \rho$. Then from the equality $CF_u(\rho^2) = CF_u(\rho)$ it follows that a Gaussian state is pure iff

$$A = -JA^{-1}J,$$  \hspace{1cm} (18)

i.e. a Gaussian state is pure iff $JA \in Sp(E, \sigma)$. Analogously, for a mixed state $\rho^2 < \rho$. Then $CF_u(\rho^2) < CF_u(\rho)$ and as a consequence $\frac{A - JA^{-1}J}{2} > A$. Hence for any Gaussian state the correlation matrix $A$ must satisfy the following restriction \[21\]

$$A \leq -JA^{-1}J.$$  \hspace{1cm} (19)

3. The classical Gaussian states

A multimode squeezed thermal state is a classical state when it has a $P$-representation. The $P$-distribution on the phase space which describes such a state is the symplectic Fourier transform of the normal ordered characteristic function

$$CF_u^N(\rho) = \exp \left\{ -\frac{1}{4} u^T (A - I) u \right\}.$$  \hspace{1cm} (20)

The necessary and sufficient condition for the existence of the symplectic Fourier transform is the positive definiteness of the matrix $A - I$. Then one has

$$P(v) = \pi^{-n} \left( \sqrt{\det (A - I)} \right)^{-1} \exp \left\{ \frac{1}{4} v^T J (A - I)^{-1} J v \right\}.$$  \hspace{1cm} (21)

4. The Gaussian noise

One form of noise which has been extensively studied is the thermal noise. The admixture of the thermal noise is described by the semigroup mapping of a fiducial state $\rho$ into a state $\Gamma(\rho)$ with a number of thermal photons “added”. Generalizing an idea from \[3\] we shall define a Gaussian noise map $\Gamma : \rho \rightarrow \Gamma(\rho)$ for any density operator $\rho$ in the following way:

$$\Gamma(\rho) = \int p_G(v) V(v) \rho V(-v) dv$$  \hspace{1cm} (22)

where $p_G(v)$ is a probability distribution on the phase space of Gaussian form:

$$p_G(v) = \pi^{-n} \sqrt{\det G} \exp \left\{ -v^T G v \right\}.$$  \hspace{1cm} (23)

($G$ is a positive definite $2n \times 2n$ matrix) and $V(u)$ are the Weyl operators. It is easy to see that in the case when $\rho$ is a quasifree state with the characteristic function

$$CF_u(\rho) = \exp \left\{ -\frac{u^T Au}{4} \right\}$$  \hspace{1cm} (24)
the characteristic function of the state $\Gamma(\rho)$ is given by

$$CF_u(\Gamma(\rho)) = \exp\left\{-\frac{u^T(A - JG^{-1}J)u}{4}\right\}$$

(25)

In the following we shall use the notation $\Gamma(A) = A - JG^{-1}J$. In the general case $G$ must be of the following form $G = O_T G \left( \begin{array}{cc} D_G & 0 \\ 0 & D_G^{-1} \end{array} \right) O_G$ where $O_G$ is an orthogonal symplectic matrix. Hence

$$\Gamma(A) = A + O_T G \left( \begin{array}{cc} D_G^{-1} & 0 \\ 0 & D_G^{-1} \end{array} \right) O_G.$$  

(26)

4.1. The Gaussian noise in the one mode case

Let us concentrate on the one mode case when $A = dS^T S$ where $S = OMO'$ with $O$ and $O'$ orthogonal matrices and $M = \left( \begin{array}{cc} m & 0 \\ 0 & \frac{1}{m} \end{array} \right)$. Then it is evident that $TrA = d(m^2 + \frac{1}{m^2})$ and $detA = d^2$. Hence the squeezing parameter $m$ is determined by the invariants $detA$ and $TrA$ of the correlation matrix $A$ from the equation

$$(m^2 + \frac{1}{m^2}) = \frac{TrA}{\sqrt{detA}}$$

(27)

Evidently, to the correlation matrix $\Gamma(A)$ there corresponds a new squeezing parameter $\Gamma(m)$ given by the analogous equation with $A$ replaced with $\Gamma(A)$. From the formula (26) it follows that

$$det\Gamma(A) = det(g^{-1}I + d(O_T G)^{-1}S^T SO_G^{-1})$$

(28)

and

$$Tr\Gamma(A) = TrG^{-1} + TrA$$

(29)

From the formula (28) we obtain that

$$det\Gamma(A) = detG^{-1} + \frac{TrG^{-1}TrA}{2} + detA$$

(30)

Therefore the equation for the squeezing parameter $\Gamma(m)$ is

$$\Gamma(m)^2 + 1 = \frac{TrG^{-1} + TrA}{detG^{-1} + \frac{TrG^{-1}TrA}{2} + detA}$$

(31)

With the above parametrization we have

$$\Gamma(m)^2 = \sqrt{\frac{1}{g} + \frac{dm^2}{m^2}}$$

(32)

Also we have

$$\Gamma(d) = \sqrt{\frac{1}{g} + dm^2}\left(\frac{1}{g} + \frac{d}{m^2}\right)$$

(33)
and
g^{-1} = \frac{d(m^2 + \frac{1}{m^2}) + \sqrt{(m^2 - \frac{1}{m^2})^2 + 4\Gamma(d)^2}}{2}\tag{34}

A direct relation between \(\Gamma(m)\) and \(\Gamma(d)\) is the following:

\[
\Gamma(m) = \sqrt{\frac{m^2 - \frac{1}{m^2}}{2\Gamma(d)^2}} + \sqrt{1 + \frac{(m^2 - \frac{1}{m^2})^2}{2\Gamma(d)^2}}\tag{35}
\]

4.2. Comparison with the previous results

In order to compare the above results with those from the papers \cite{4, 34} the following identifications are made:

\[
g = \frac{1}{2}, \quad m = \Gamma(m), \quad \Gamma(d) = 1 = 2\bar{n} + 1\text{ and } m = \exp(r).
\]

Then the first two equations (A6) from the Appendix A of the paper \cite{4} become:

\[
m^2 - \frac{1}{m^2} = -\Gamma(1)(\Gamma(m)^2 - \frac{1}{\Gamma(m)^2})\tag{36}
\]

and

\[
\frac{1}{2g} + \frac{1}{4}(m^2 + \frac{1}{m^2}) = \frac{\Gamma(1)}{4}(\Gamma(m)^2 + \frac{1}{\Gamma(m)^2})\tag{37}
\]

The minus sign from the equation (36) is not correct. Without this sign the equation (36) becomes:

\[
\sqrt{\frac{1 + gm^2}{1 + \frac{g}{m^2}}} = \frac{g(m^2 - \frac{1}{m^2})}{\sqrt{(1 + gm^2)(1 + \frac{g}{m^2})}}\tag{38}
\]

and is evidently fulfilled. The equation (37) becomes:

\[
\frac{(\frac{1}{g} + m^2) + (\frac{1}{g} + \frac{1}{m^2})}{\sqrt{(\frac{1}{g} + m^2)(\frac{1}{g} + \frac{1}{m^2})}} = \sqrt{\frac{\frac{1}{g} + m^2}{\frac{1}{g} + \frac{1}{m^2}}} + \sqrt{\frac{\frac{1}{g} + \frac{1}{m^2}}{\frac{1}{g} + m^2}}\tag{39}
\]

and is also evidently fulfilled.

In the general case \(1 < \Gamma(m) \leq m\), i.e. the Gaussian noise map reduces the squeezing but it does not suppress it. Also \(\Gamma(d) \geq d\) i.e. the Gaussian noise map increases the number of thermal photons. A nonclassical state \(\rho\) becomes a classical one under the map \(\Gamma\) when \(\Gamma(d) > \Gamma(m)^2\), i.e. when \(\frac{1}{g} + \frac{d}{m^2} > 1\). This inequality is valid for any values of \(d\) and \(m\) when \(g \leq 1\), which is evidently fulfilled.

5. The Holevo distance

The characteristic function of a multimode squeezed thermal state is given by:

\[
CF_u(\rho) = \exp\{-\frac{1}{4}u^TAu\}\tag{40}
\]

where \(A = 2\Sigma\) and \(\Sigma\) is the correlation matrix of the state. We have for any two density operators \(\rho_1\) and \(\rho_2\):

\[
CF_u(\rho_1, \rho_2) = (2\pi)^{-n} \int CF_{\nu^+\nu}(\rho_1)CF_{\nu^2-\nu}(\rho_2)\exp\{-\frac{iv^TJu}{2}\}dv\tag{41}
\]
where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). This becomes in the particular case when \( \rho_1 = \rho_2 = \sqrt{\rho} \):

\[
CF_u(\rho) = (2\pi)^{-n} \int CF_{\frac{u}{\sqrt{\rho}} + v} CF_{\frac{u}{\sqrt{\rho}} - v} \exp\left\{ \frac{iv^TJu}{2} \right\} dv
\]

(42)

If we suppose that

\[
CF_u(\sqrt{\rho}) = K \exp\left\{ -\frac{1}{4} u^T \phi(A) u \right\}
\]

(43)

then it follows that

\[
CF_u(\rho) = K^2 (\det \phi(A))^{-\frac{1}{4}} \exp\left\{ -\frac{1}{8} u^T (\phi(A) - J\phi(A)^{-1}J) u \right\}
\]

(44)

In order that this equality be valid for all values of \( u \) we must have \( K = (\det \phi(A))^{\frac{1}{4}} \) and

\[
\phi(A) - J\phi(A)^{-1}J = 2A
\]

(45)

We can put this equation in the following form

\[
J\phi(A)J\phi(A) - I = 2J\phi(A)JA
\]

(46)

Now we shall prove that this equation has a solution which is given by

\[
\phi(A) = A(I + \sqrt{I + (JA)^{-2}})
\]

(47)

Indeed, we have

\[
A = S^T D S
\]

(48)

with \( S^T J S = J \) (i.e. \( S \) is a symplectic matrix) and \( D \geq I \) is a diagonal matrix. It is well known that if \( S \) is a symplectic matrix then \( S^T \) and \( S^{-1} \) are also symplectic matrices. From this fact we obtain that

\[
(JA)^{-2} = -S^{-1}D^{-2}S
\]

(49)

Also we have

\[
J\phi(A)JA = -S^{-1}D(D + \sqrt{D^2 - I})S
\]

(50)

and

\[
J\phi(A)J\phi(A) = -S^{-1}(D + \sqrt{D^2 - I})^2S
\]

(51)

Because \((D + \sqrt{D^2 - I})^2 + I = 2D(D + \sqrt{D^2 - I})\) the desired result follows. The following form of the function \( \phi(A) \) is also useful:

\[
\phi(A) = -S^{-1}(D + \sqrt{D^2 - I})S
\]

(52)

It is interesting to point out that in the case of a pure state \( D = I \) and \( \phi(A) = A \).

Now we can calculate \( Tr\sqrt{\rho_1\rho_2} \). From the general formula:

\[
TrB_1B_2 = (2\pi)^{-n} \int CF_u(B_1)CF_{-u}(B_2)du
\]

(53)
it follows that
\[
Tr\sqrt{\rho_1}\sqrt{\rho_2} = (2\pi)^{-n} \int CF_u(\sqrt{\rho_1})CF_{-u}(\sqrt{\rho_2})du
\]  
(54)

It is easy to compute this integral. The result is
\[
Tr\sqrt{\rho_1}\sqrt{\rho_2} = \sqrt{\frac{\det\phi(A_1)\det\phi(A_2)}{\det(\frac{\phi(A_1)+\phi(A_2)}{2})}}
\]  
(55)

When \(\rho_2\) is a squeezed state (i.e. when \(A_2 = O_T^2 \left( \begin{array}{cc} M^2 & 0 \\ 0 & M^{-2} \end{array} \right) O_2 \)) it is plausible to suppose that the maximum value of this quantity is obtained for \(D_1 = I\), and \(O_1 = O_2\). In this case
\[
\chi(\rho) = \frac{1}{\det(M+M^{-1})}
\]  
(56)

(where we have denoted the density matrix \(\rho_2\) by \(\rho\)). It is clear from this formula that the nonclassicity of the state \(\rho\) is entirely due to the squeezing. When \(M = I\) we have no squeezing and \(\chi(\rho) = 1\).

5.1. The one-mode case

In the one-mode case the most general classical state is given by a characteristic function with \(A_1 = O_T^1 \left( \begin{array}{cc} d_1 m_1^2 & 0 \\ 0 & \frac{d_1}{m_1^2} \end{array} \right) O_1\), where \(O_T O = I\) and the positivity of \(A_1 - I\) requires the validity of the inequalities \(d_1 \geq m_1^2\) and \(d_1 \geq m_1^{-2}\) which are not independent. When one of them is satisfied then the other is also satisfied. Because for any symplectic matrix \(S\) we have \(\det S = 1\) it follows that \(\det\phi(A_k) = (d_k + \sqrt{d_k^2 - 1})^2\) for \(k = 1, 2\). If \(\rho_2\) is also a classical state then
\[
\text{sup}_{\rho_1} Tr\sqrt{\rho_1}\sqrt{\rho_2} = 1
\]  
(57)

is obtained for \(\rho_1 = \rho_2\).

When \(\rho_2\) is a squeezed state (i.e. when \(A_2 = O_T^2 \left( \begin{array}{cc} d_2 m_2^2 & 0 \\ 0 & \frac{d_2}{m_2^2} \end{array} \right) O_2\)) then we have:
\[
Tr\sqrt{\rho_{\text{class}}} = \frac{2}{\sqrt{(\phi(d_1) - \phi(d_2))^2 + F(\Delta\theta, m_1, m_2)}}
\]  
(58)

where we have denoted with \(F\) the following function
\[
F(\Delta\theta, m_1, m_2) = 2 + \sin(\Delta\theta)^2(m_1^2m_2^2 + \frac{1}{m_1^2m_2^2}) + \cos(\Delta\theta)^2((\frac{m_1}{m_2})^2 + (\frac{m_2}{m_1})^2)
\]  
(59)

The maximum value of this function corresponds to the minimum value of the function under the square-root:
\[
H(d_1, d_2, \Delta\theta, m_1, m_2) = \frac{(\phi(d_1) - \phi(d_2))^2}{\phi(d_1)\phi(d_2)} + F(\Delta\theta, m_1, m_2)
\]  
(60)
It is evident that the minimum value of $H$ is attained for $d_1 = d_2$ and for those values of $\Delta \theta$, $m_1$ and $m_2$ which minimize the function $F$. The minimum value of the function $F$ is equal with 2 and is attained either for $\Delta \theta = 0$, and $m_1 = m_2$ or for $\Delta \theta = \frac{\pi}{2}$, and $m_1 = m_2^{-1}$. But when $\rho_2$ is not a classical state, then $d_1 = d_2 < m_2^2 = m_1^2$ or $d_1 = d_2 > m_2^{-2} = m_1^{-2}$ in the first case and $d_1 = d_2 < m_2^2 = m_1^{-2}$ or $d_1 = d_2 > m_2^{-2} = m_1^2$ in the second case. In all these situations the conditions for the classicality of the state $\rho_1$ are not satisfied. Because the function $F$ is monotonically decreasing for $m_1 \leq \sqrt{d_2} < m_2$ or for $m_1 \geq \frac{1}{\sqrt{d_2}} > m_2$ it follows that the minimum value of $F$ is in both cases equal with $(\sqrt{\frac{m_2}{m_1}} + \frac{m_2}{d})^2$. Hence we have obtained that the nonclassical distance for a thermal squeezed state $\rho$:

$$\chi(\rho) = \frac{2}{\sqrt{d} + \frac{m}{\sqrt{d}}} \quad (61)$$

### 5.2. The increase of nonclassical distance under Gaussian noise

The nonclassical distance $\chi(\Gamma(\rho))$ in the case of a one mode thermal squeezed state $\rho$ is then given by

$$\chi(\Gamma(\rho)) = \frac{2}{\sqrt{\Gamma(m)} + \frac{\Gamma(m)}{\sqrt{\Gamma(d)}}} \quad (62)$$

when $\Gamma(d) > \Gamma(m)^2$ or by $\chi(\rho) = 1$ when $\Gamma(d) < \Gamma(m)^2$. The intuitive fact according to which $\Gamma(\rho)$ is closer to a classical state than $\rho$ is reflected quantitatively in the inequality

$$\chi(\Gamma(\rho)) \geq \chi(\rho). \quad (63)$$

Hence we must prove that

$$\frac{\Gamma(m)\sqrt{\Gamma(d)}}{m\sqrt{d}} \geq \frac{\Gamma(d) + \Gamma(m)^2}{d + m^2} \quad (64)$$

or in a more convenient form

$$\frac{\Gamma(m)^2\Gamma(d)}{(\Gamma(d) + \Gamma(m)^2)^2} \geq \frac{d}{(m^2 + 1)^2} \quad (65)$$

which becomes

$$\frac{(\frac{d}{m^2} + \frac{1}{g})^2}{(\frac{d}{m^2} + \frac{1}{g} + 1)^2} \geq \frac{d}{(m^2 + 1)^2} \quad (66)$$

From this it follows that the inequality $\chi(\Gamma(\rho)) \geq \chi(\rho)$ is valid only for $\frac{1}{g} \leq \frac{m^2}{d} - \frac{d}{m^2}$. This limitation on the number of thermal photons $2\bar{n} = \frac{1}{g}$ introduced by the Gaussian noise map is unacceptable.

### 6. The fidelity distance

The fidelity $F(\rho_1, \rho_2)$ for two density operators $\rho_1$ and $\rho_2$ is defined by

$$F(\rho_1, \rho_2) = Tr\left(\sqrt{\rho_1}\rho_2\sqrt{\rho_1}\right)^2 \quad (67)$$
Since the characteristic function of a product of operators whose characteristic functions are Gaussians is also a Gaussian and the characteristic function of the square root of a Gaussian density operator is a Gaussian we can find a simple formula for the characteristic function of the operator $\sqrt{\rho_1 \rho_2 \sqrt{\rho_1}}$:

$$CF_z(\sqrt{\rho_1 \rho_2 \sqrt{\rho_1}}) = \sqrt{L} \exp \left\{ -\frac{1}{4} z^T O z \right\},$$

(68)

where

$$L^{-1} = \det \Phi(A_1)^{-1} \det \left( \frac{\Phi(A_1) + A_2}{2} \right)$$

$$\det \left( \frac{A_2 + \Phi(A_1) - \mathcal{U}}{2} \right),$$

(69)

where $\mathcal{U} = (A_2 - iJ)\Phi(A_1) + A_2)^{-1}(A_2 + iJ)$, and

$$O = \Phi(A_1) - (\Phi(A_1) - iJ[A_2 + \Phi(A_1) - (A_2 - iJ)(\Phi(A_1) + A_2)^{-1}(A_2 + iJ)]^{-1})(\Phi(A_1) + iJ).$$

Then applying the result of the preceding section we can obtain the characteristic function of $\sqrt{\rho_1 \rho_2 \sqrt{\rho_1}}$,

$$CF_z \left( \sqrt{\rho_1 \rho_2 \sqrt{\rho_1}} \right) =$$

$$[L \det \Phi(O)]^\frac{1}{2} \exp \left\{ -\frac{1}{4} z^T \Phi(O) z \right\}.$$

(70)

From this formula and the property 1 of the characteristic function we obtain

$$F(\rho_1, \rho_2) = \sqrt{L \det \Phi(O)}.$$  

(71)

We remark that

$$\det \Phi(O) = \det \mathcal{O} \det \left[ I + \sqrt{I + (J\mathcal{O})^{-2}} \right].$$

(72)

In order to simplify the formula for the fidelity we observe that

$$t_{ik} = \text{Tr} \rho_i \rho_j \rho_k = \det \left( \frac{A_i + A_j}{2} \right)$$

$$\det \left[ A_j + A_k - (A_j - iJ)(A_i + A_j)^{-1}(A_j + iJ) \right],$$

and that $t_{123} = t_{231} = t_{312}$. If we take in this last identity $\Phi(A_1)$ instead of $A_1$ we obtain

$$\det \left[ \frac{\Phi(A_1) + A_2}{2} \right]$$

$$\det \left[ \frac{A_2 + \Phi(A_1) - \mathcal{U}}{2} \right]$$

$$= \det \left( \frac{A_1 + A_2}{2} \right) \det \Phi(A_1).$$
Hence we get
\[ L = \left[ \det \left( \frac{A_1 + A_2}{2} \right) \right]^{-1}. \] (73)

6.1. The one mode case

In [15] we have obtained an expression for the fidelity in the one mode case. This formula can be reobtained as a consequence of the above general formula. In the one mode case all matrices are $2 \times 2$ matrices. For a $2 \times 2$ matrix $O$ we have
\[ \Phi(O) = \epsilon O, \] (74)
where $\epsilon = 1 + \sqrt{1 - \frac{1}{\det O}}$ and $\det \Phi(O) = (\sqrt{\det O} + \sqrt{\det O - 1})^2$. From these considerations it follows that
\[ F(\rho_1, \rho_2) = \frac{2}{\sqrt{\det(A_1 + A_2)}(\sqrt{\det O} - \sqrt{\det O - 1})}. \] (75)
Thus it is sufficient to compute $\det O$. We shall denote by $P$ the product $(\det A_1 - 1)(\det A_2 - 1)$. After simple but long computations we obtain
\[ \det O = 1 + \frac{P}{\det(A_1 + A_2)}, \] (76)
which gives the result of [15]
\[ F(\rho_1, \rho_2) = \frac{2}{\sqrt{\det(A_1 + A_2) + P - \sqrt{P}}}. \] (77)

With the parametrization taken in the subsection 4.1. we have:
\[ F(\rho_1, \rho_2) = \frac{2}{\sqrt{d_1^2 d_2^2 + 1 + d_1 d_2 |F(\Delta \theta, m_1, m_2) - 2| - \sqrt{(d_1^2 - 1)(d_2^2 - 1)}}}. \] (78)
Then
\[ \phi(\rho) = \sup_{\rho \in \mathcal{L}} F(\rho_1, \rho) = \frac{2}{\sqrt{(d^2 - 1)^2 + d^2 [\frac{\sqrt{d}}{m} + \frac{m}{\sqrt{d}}]^2 - (d^2 - 1)}}. \] (79)

We stress the fact that for nonclassical Gaussian states we have $\sqrt{d} < m$. The fact that $\phi(\Gamma(\rho)) \geq \phi(\rho)$ is a direct consequence of the definition of the Gaussian noise map and of the property 7 of the fidelity (transition probability) given in the introduction.

7. Conclusions

We have considered the problem of nonclassical distance from the point of view of distinguishability between quantum states. In the case of Gaussian states, using an explicit formula for the fidelity, we have obtained an explicit formula for the nonclassical distance. The Gaussian noise was used to eliminate an attractive candidate for the definition of nonclassical distance. In the particular case of pure
Gaussian states our results are comparable with the upper bounds obtained for Hillery’s nonclassical distance, which is defined using the trace norm, because these upper bounds contain the overlaps between the squeezed states and the coherent states.

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