On a parametrized difference equation connecting chaotic and integrable mappings

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Abstract

We present a new difference equation with two parameters $c \in [0, 1]$ and $A \in [1, 4]$. This equation is equivalent to the logistic mapping if $c = 1$ and the Morishita mapping if $c = 0$, which are the well-known chaotic and integrable mappings, respectively. We first consider the case $A = 4$ and investigate the time evolution by changing the parameter $c \in [0, 1]$. We next change both two parameters $A \in [3, 4]$ and $c \in [0, 1]$ and present the corresponding 3D bifurcation diagram.

1 Introduction

The logistic equation,

\[
\frac{du}{dt} = au(1 - u) \quad u(0) = u_0, \tag{1}
\]

where $a$ is a positive constant and $u = u(t)$ is an unknown function, is a model equation describing population dynamics and possesses a solution

\[
u(t) = \frac{u_0e^{at}}{u_0e^{at} + 1 - u_0} \tag{2}
\]

Concerning its discrete version, the following two difference equations are well-known:

\[
u_{n+1} = Au_n(1 - u_n), \tag{3}
\]

\[
u_{n+1} = Au_n(1 - \frac{u_{n+1}}{u_n}) \Leftrightarrow u_{n+1} = \frac{Au_n}{1 + Au_n}, \tag{4}
\]

where $A \in (1, 4]$ is a given constant.

Eq. (3) is the well-known logistic mapping and exhibits a chaotic behavior if $A$ exceeds the value $A = 3.5699456 \cdots$ (See \cite{1}, for example.)

On the other hand, eq. (4), which is called the Morishita mapping, is an integrable mapping \cite{2, 3}. In other words, eq. (4) is linearized by taking its reciprocal and putting $v_n = 1/u_n$ as

\[
u_{n+1} = \frac{1}{A}v_n + 1.
\]

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This is solved as

\[ u_n = \frac{A}{A - 1} + \frac{1}{A^n} \left( \frac{1}{u_0} - \frac{A}{A - 1} \right) \]

and therefore \( u_n \) is given by

\[ u_n = \frac{A^n(A - 1)u_0}{A(A^n - 1)u_0 + A - 1}, \]  

which converges to \( \frac{A - 1}{A} \) as \( n \) tends to \( \infty \).

The purpose of this paper is to present a new difference equation connecting the above two different mappings. We also investigate the time evolution of this new equation and calculate the Lyapunov exponent. The bifurcation diagram is also presented.

## 2 A new difference equation connecting the logistic and Morishita mappings

In this section, we put \( A = 4 \) and consider a new mapping including a variable parameter \( c \in [0, 1] \),

\[ u_{n+1} = 4u_n(1 - cu_n - (1 - c)u_{n+1}) \]

\[ \Leftrightarrow u_{n+1} = \frac{4u_n(1 - cu_n)}{1 + 4(1 - c)u_n} = f(c, u_n), \]  

where \( f(c, x) = \frac{4x(1 - cx)}{1 + 4(1 - cx)x} \). It is easy to observe that the mapping (6) is equivalent to the Morishita mapping if \( c = 0 \) and to the logistic mapping if \( c = 1 \).

Concerning the properties of \( f(c, x) \), we have the following theorem.

**Theorem 1** The function \( f(c, x) \) \((0 \leq x \leq 1, 0 \leq c \leq 1)\) satisfies the following properties.

\[ f(c, \frac{3}{4}) = \frac{3}{4} \]  

\[ 0 \leq f(c, x) \leq 1 \] \hspace{1cm} (7)

\[ 0 \leq f(0, x) = 4x(1 - 4x) < 1, \]  

\[ 0 \leq f(1, x) = 4x(1 - x) \leq 1, \] \hspace{1cm} (8)

The relation (7) means that \( u_n = \frac{3}{4} \) is an equilibrium point of the mapping (6) and (8) means that if \( u_0 \in [0, 1] \) we have \( u_n \in [0, 1] \) for any \( n = 1, 2, \ldots \).

**Proof:** The relation (7) is easily shown through direct calculation. In order to prove (8), we take a derivative of \( f(c, x) \) with respect to \( c \), which is given by

\[ \frac{\partial}{\partial c}f(c, x) = \frac{4x^2(3 - 4x)}{(1 + 4x - 4cx)^2}. \]

Hence we have

\[ f(0, x) \leq f(c, x) \leq f(1, x) \] \hspace{1cm} (0 \leq x \leq \frac{3}{4}),

\[ f(1, x) \leq f(c, x) \leq f(0, x) \] \hspace{1cm} (\frac{3}{4} \leq x \leq 1).

Together with the facts,

\[ 0 \leq f(0, x) = \frac{4x}{1 + 4x} < 1, \]  

\[ 0 \leq f(1, x) = 4x(1 - x) \leq 1, \]

the relation (8) follows.

By changing \( c \) in an interval \([0, 1]\), we calculate the time evolutions of \( \{u_n\} \), which are given in Fig. 1.

Numerical calculations are performed by Python 3.9.
Figure 1. Time evolutions of the mapping \( f \) for \( c = 0.7 \), \( c = 0.9 \) (period 2), \( c = 0.93 \) (period 4), \( c = 0.94 \) (period 8), \( c = 0.97 \) (chaotic), \( c = 0.98 \) (period 3)
We here investigate the Fig. 1 in a detailed manner. If \(0 \leq c < \frac{5}{6}\), we can observe that \(u_n\) converges to \(\frac{3}{4}\). This is confirmed as follows. We put \(u_n = \frac{3}{4} + \varepsilon\) in eq. (6) and have the Taylor expansion of \(u_{n+1} = f(c,u_n)\) around \(\varepsilon = 0\) as follows.

\[
\begin{align*}
    u_{n+1} &= (3 + 4\varepsilon)(1 - c(\frac{3}{4} + \varepsilon))
                = \frac{3}{4} + \frac{1}{1 + (1-c)(3+4\varepsilon)} + \cdots.
\end{align*}
\]

If \(0 \leq c < \frac{5}{6}\), we have \(|\frac{1}{1 + (1-c)(3+4\varepsilon)}| < 1\) and therefore the equilibrium point \(u_n = \frac{3}{4}\) is stable.

If \(c\) exceeds \(\frac{5}{6}\), we have \(|\frac{1}{1 + (1-c)(3+4\varepsilon)}| > 1\) and therefore \(u_n = \frac{3}{4}\) is unstable. It should be noted that \(u_n = 0\) is another equilibrium point, which is, however, unstable for \(c \in [0,1]\).

Through straightforward calculations, if \(\frac{5}{6} < c\), \(u_n\) converges to a periodic orbit with period 2. That is, \(u_n\) takes alternately two values \(u_2, \pm\) which are solutions of the equation \(f(c,f(c,x)) = x\) except \(x = 0, \frac{3}{4}\), as \(n\) tends to \(\infty\). In other words, we have

\[
    f(c,u_{2,\pm}) = u_{2,\mp}.
\]

The Taylor expansion of \(f(c,f(c,u_{2,\pm} + \varepsilon))\) around \(\varepsilon = 0\) is given by

\[
    f(c,f(c,u_{2,\pm} + \varepsilon)) = u_{2,\pm} + \frac{63c^2 - 84c + 25}{3c^2 - 4c} \varepsilon + \cdots.
\]

Hence if \(c\) exceeds the value \(\frac{44 + \sqrt{256}}{66} = 0.922902\cdots\), which is a solution to \(|\frac{63c^2 - 84c + 25}{3c^2 - 4c}| = 1\), the period of the sequence \(\{u_n\}\) becomes 4.

If we further increase the parameter \(c\), the period of the sequence \(\{u_n\}\) is doubled as \(2^3, 2^4, 2^5\cdots\) and finally a chaotic behavior appears at \(c = 0.942\cdots\). We next calculate the value \(c_n\) where \(u_n\) is \(2^n\)-periodic if \(c_n < c < c_{n+1}\). We further calculate

\[
    F_n = \frac{c_{n+1} - c_n}{c_{n+2} - c_{n+1}},
\]

which is expected to converge to the Feigenbaum constant 4.669201\cdots as is shown in Table 1.
Figure 2. Lyapunov exponent

Figure 3. Bifurcation diagram of the mapping \(c\) (0.8 \(\leq c \leq 1\))

Figure 4. Enlarged bifurcation diagram. (Left : 0.92 \(\leq c \leq 0.95\), Right : 0.938 \(\leq c \leq 0.945\))
Table 2. $F_n$ in Eq. (11)

| $n$ | 1    | 2    | 3    | 4    |
|-----|------|------|------|------|
| $A$ | 3.80 | 5.89888 | 4.82824 | 4.70431 | 4.67570 |
|     | 3.85 | 5.90474 | 4.82648 | 4.70576 | 4.67548 |
|     | 3.90 | 5.90988 | 4.82597 | 4.70410 | 4.67736 |
|     | 3.95 | 5.91525 | 4.82493 | 4.70450 | 4.67634 |

We can observe that $\lambda$ takes negative value for $c < 0.942 \cdots$ and then takes positive value.

The corresponding bifurcation diagrams, where $(c, u_n)$ ($n = 200, \cdots, 400$) are plotted, are given in Fig. 3. If we expand a certain region of the diagram, we can find a self-similar diagram, as is observed in Fig. 4.

3 The mapping with two parameters and 3D bifurcation diagram

Next we extend Eq. (6) to a mapping with two parameters $A \in [3, 4]$ and $c \in [0, 1]$, given by

\[
\begin{align*}
    u_{n+1} &= Au_n(1 - cu_n - (1 - c)u_{n+1}) \\
    \Leftrightarrow u_{n+1} &= \frac{Au_n(1 - cu_n)}{1 + A(1 - c)u_n} = f(A, c, u_n),
\end{align*}
\]

where $f(A, c, x) = \frac{Ax(1 - cx)}{1 + A(1 - c)x}$.

Changing the two parameters $A$ and $c$, we obtain the corresponding bifurcation diagrams shown in Fig. 5 in which the upper and lower figures stand for the same diagram looked from different viewpoints. It should be noted that if $A \leq 3$ $u_n$ converges to the value $1 - \frac{1}{A}$ so we omitted.

We next calculate $F_n = F_n(A)$ of Eq. (9), which is given as Table 2.

4 Concluding Remarks

We have investigated a mapping with two parameters connecting the logistic mapping and the Morishita mapping, which are famous chaotic and integrable difference equations, respectively. It is very interesting to note that a period orbit of period 3 appears and a self-similarity of the bifurcation diagram is observed. These phenomena are also observed in the logistic mapping $u_{n+1} = Au_n(1 - u_n)$ ($0 < A \leq 4$). From the obtained results, we may conclude that the difference equation (11) is a chaotic mapping.

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Figure 5. 3D bifurcation diagram corresponding to Eq. (11) from different viewpoints.
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