Asymptotic Behavior of Blowup Solutions for Elliptic Equations with Exponential Nonlinearity and Singular Data

LEI ZHANG

UNIVERSITY OF ALABAMA AT BIRMINGHAM
DEPARTMENT OF MATHEMATICS,
452 CAMPBELL HALL
1300 UNIVERSITY BOULEVARD
BIRMINGHAM, AL 35294-1170
leizhang@math.uab.edu*

Abstract

We consider a sequence of blowup solutions of a two dimensional, second order elliptic equation with exponential nonlinearity and singular data. This equation has a rich background in physics and geometry. In a work of Bartolucci-Chen-Lin-Tarantello it is proved that the profile of the solutions differs from global solutions of a Liouville type equation only by a uniformly bounded term. The present paper improves their result and establishes an expansion of the solutions near the blowup points with a sharp error estimate.

Mathematics Subject Classification (2007): 35J60, 35B45, 53C21
Keywords: Liouville equation, Blowup analysis.

1 Introduction

Two dimensional semilinear elliptic equations with exponential nonlinearities arise naturally in conformal geometry and physics. The study of these equations is always

*Supported by National Science Foundation Grant 0600275. Running title: Asymptotic expansion for blowup solutions
related to their blowup phenomena. When a sequence of solutions tends to infinity near a blowup point, the asymptotic behavior of the solutions near the blowup point carries important information. In some applications it is crucial to completely understand the asymptotic behavior of blowup solutions. In this article we study the following equation:

$$\Delta u + |x|^{2\alpha} H(x)e^u = 0, \quad \text{in } B_1 \subset \mathbb{R}^2,$$

(1.1)

where $B_1$ is the unit ball in $\mathbb{R}^2$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, ($\mathbb{N}$ is the set of natural numbers) and $H \in C^3(B_1)$ is a positive function. If a sequence of solutions $\{u_i\}$ tends to infinity near a point other than 0, the coefficient function $|x|^{2\alpha} H(x)$ is bounded above and below near the blowup point. This situation has been extensively studied and the asymptotic behavior of the blowup solutions is well understood (see, for example [3], [5], [13], [24], [25], [10], [36]). In this article we mainly consider a sequence of solutions $\{u_i\}$ of (1.1) such that

$$u_i(z_i) = \max_{\overline{B}_1} u_i \to \infty, \quad z_i \to 0, \quad 0 \text{ is the only blowup point in } \overline{B}_1.$$

(1.2)

We shall describe the asymptotic profile of $\{u_i\}$ near 0 under natural assumptions on $H$ and the oscillation of $\{u_i\}$ on $\partial B_1$.

The blowup analysis for (1.1) near 0 reflects the bubbling feature of a few important equations or systems of equations in physics or geometry. For example, the following mean field equation is defined on Riemann surfaces:

$$\Delta_g w + \rho \left( \frac{1}{h(x)e^w} - \frac{1}{|M|} \right) = 4\pi \sum_{j=1}^{m} \alpha_j (\delta_{p_j} - \frac{1}{|M|})$$

(1.3)

where $M$ is a compact smooth Riemann surface without boundary, $h$ is a positive smooth function on $M$, $\rho$ is a positive constant, $|M|$ is the volume of $M$, $\Delta_g$ is the Laplace-Beltrami operator and $\alpha_j \delta_{p_j}$ are Dirac sources. For (1.3), the profile of blowup solutions near each $p_j$ is exactly described by (1.1) and $\alpha_j$ in (1.3) plays the same role as $\alpha$ in (1.1). From the physical point of view, it is important to consider the case $\alpha_j > 0$ in (1.3) as it is closely related to the self-dual equations in the Abelian Chern-Simons-Higgs theory (see [16], [18], [19], etc) and the Electroweak theory (see [11], [23], etc). There is a considerable literature on the mean field equation (1.3) and closely related topics, we list the following as a partial list: [3], [8], [10], [11], [12], [15], [17], [26], [27], [28], [29], [30], [31].

Another application comes from the 2-dimensional open Toda system for $SU(N+
1). The simplest example is

\[
\begin{cases}
-\Delta u_k^1 = 2h_k^1 e^{u_k^1} - h_k^2 e^{u_k^2} & \text{in } B_1 \\
-\Delta u_k^2 = 2h_k^2 e^{u_k^2} - h_k^1 e^{u_k^1}
\end{cases}
\]

where \( u_k^1, u_k^2 \) are sequences of blowup solutions and \( h_k^1, h_k^2 \) are positive, smooth functions very close to 1. Even for this simplest example, the blowup analysis is challenging because \( u_k^1 \) and \( u_k^2 \) may have common blowup points and the asymptotic behavior of them near their common blowup points is not yet well understood (see [20], [21], [22], [33], [30] and the references therein for recent development). The analysis for (1.1) is closely related to the Toda system and the result we prove in this work (Theorem 1.2) helps to understand the system.

In addition to the background in physics, (1.1) has a well known interpretation in geometry. Let \( g_0 \) be the Euclidean metric on \( B_1 \), then \( |x|^{2\alpha} H_i(x) e^{u_i} \) is the Gauss curvature under metric \( e^{u_i} g_0 \). In this sense (1.1) is related to the Nirenberg problem or more generally the Kazdan-Warner problem.

When \( \alpha \) is equal to 0, the behavior of a sequence of blowup solutions \( \{u_i\} \) to

\[
\Delta u_i + H_i(x) e^{u_i} = 0 \quad \text{in } B_1
\]

has been extensively studied through the works of Brezis-Merle [5], Li-Shafrir [25], Li [24], Chen-Lin [10] and the references therein. In [24] Li proved that if a sequence of blowup solutions \( \{u_i\} \) of (1.4) has a bounded oscillation near their blowup point, then in a neighborhood of this point \( \{u_i\} \) is only \( O(1) \) different from a sequence of standard bubbles appropriately scaled. Later Chen-Lin [10] and the author [36] improved Li’s estimate to the sharp form by different approaches. Since the case \( \alpha > 0 \) is more meaningful in physics, it is important to obtain similar results for equation (1.1) when \( \alpha > 0 \).

In this article we address the case \( \alpha \notin \mathbb{N} \). Our work is based on a result by Bartolucci-Chen-Lin-Tarantello [2] who studied (1.3) and established a result similar to Li’s result for \( \alpha = 0 \). More specifically, let \( \{u_i\} \) be a sequence of functions solving

\[
\Delta u_i + |x|^{2\alpha} H_i(x) e^{u_i} = 0, \quad \text{in } B_1, \tag{1.5}
\]

such that (1.2) holds. Suppose \( \{u_i\} \) has bounded oscillation on \( \partial B_1 \) and a bound on the energy:

\[
\begin{cases}
|u_i(x) - u_i(x')| \leq C_0 & \forall x, x' \in \partial B_1, \\
\int_{B_1} |x|^{2\alpha} H_i(x) e^{u_i} \leq C_0.
\end{cases}
\]

Then the following result is proved in [2]:
Theorem 1.1. (Bartolucci-Chen-Lin-Tarantello) Let \( \{u_i\} \) satisfy (1.5), (1.2), (1.6) and let \( H_i \) satisfy
\[
0 < \frac{1}{C_1} \leq H_i(x) \leq C_1, \quad \|\nabla H_i\|_{L^\infty(B_1)} \leq C_1.
\]
Then for \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N} \), there exists \( C > 0 \) such that
\[
\left| u_i(x) - \log \frac{e^{u_i(0)}}{1 + \frac{H_i(0)}{8(1 + \alpha)^2} e^{u_i(0)} |x|^{2\alpha + 2}} \right| \leq C \quad \text{in } B_1.
\]

Theorem 1.1 is a refinement of a result in Bartolucci-Tarantello [3], concerning the quantization phenomena for blowup solutions of (1.1) by using an improved argument based on the Pohozaev Identity. The idea of using Pohozaev’s type arguments was first introduced by Bartolucci-Tarantello [3] in the analysis of the bubbling phenomena for (1.1) with \( \alpha > 0 \).

In our main result below we assume the following on \( H_i \):
\[
0 < \frac{1}{C_1} \leq H_i(x) \leq C_1, \quad \|H_i\|_{C^3(B_1)} \leq C_1,
\] (1.7)
and we consider the harmonic function \( \psi_i \) solving
\[
\begin{cases}
\Delta \psi_i = 0 & \text{in } B_1, \\
\psi_i = u_i - \frac{1}{2\pi} \int_{\partial B_1} u_i dS & \text{on } \partial B_1.
\end{cases}
\] (1.8)
Clearly \( \psi_i \) is a bounded function on \( B_1 \) and \( \psi_i(0) = 0 \). For simplicity, we define
\[
V_i(x) = H_i(x)e^{\psi_i(x)},
\]
and introduce the following two constants:
\[
\Lambda_1 = -\frac{\pi}{V_i(0) \sin \left( \frac{\pi}{1 + \alpha}(1 + \alpha) \right)} \left( \frac{8(1 + \alpha)^2}{V_i(0)} \right)^{\frac{1}{1 + \alpha}}, \quad \Lambda_2 = \frac{\pi}{V_i^2(0) \sin \left( \frac{\pi}{1 + \alpha}(1 + \alpha) \right)} \left( \frac{8(1 + \alpha)^2}{V_i(0)} \right)^{\frac{1}{1 + \alpha}}.
\] (1.9) (1.10)
Note that \( \psi_i(0) = 0 \) implies the following:
\[
V_i(0) = H_i(0), \quad \nabla V_i(0) = \nabla H_i(0) + H_i(0) \nabla \psi_i(0)
\]
\[
\Delta V_i(0) = \Delta H_i(0) + 2\nabla H_i(0) \cdot \nabla \psi_i(0) + H_i(0) |\nabla \psi_i(0)|^2.
\]
Our main theorem improves Theorem 1.1 as follows:
Theorem 1.2. Let \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N} \). Suppose \( u_i \) satisfies (1.5), (1.2), (1.6) and (1.7). Then in \( B_1 \) there holds
\[
\begin{align*}
  u_i(x) &= \log \left( \frac{e^{u_i(0)}}{1 + \frac{V_i(0)}{8(1 + \alpha)} e^{u_i(0)} |x|^{2\alpha + 2}} \right) + \psi_i(x) \\
  &\quad - \frac{2(1 + \alpha)}{\alpha V_i(0)} \frac{\nabla V_i(0) \cdot x}{1 + \frac{V_i(0)}{8(1 + \alpha)} e^{u_i(0)} |x|^{2\alpha + 2}} \\
  &\quad + \left( \Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2 \right) \log \left( 2 + e^{\frac{u_i(0)}{2(1 + \alpha)} |x|} \right) e^{-\frac{u_i(0)}{1 + \alpha}} + O(e^{-\frac{u_i(0)}{1 + \alpha}}),
\end{align*}
\]
where \( \psi_i, \Lambda_1, \Lambda_2 \) are defined by (1.8), (1.9) and (1.10), respectively.

Note that we use \( O(e^{-\frac{u_i(0)}{1 + \alpha}}) \) to denote a smooth function in \( B_1 \) whose \( C^3 \) norm is bounded by \( C(\alpha, C_0, C_1) e^{-\frac{u_i(0)}{1 + \alpha}} \).

Theorem 1.2 corresponds to the results of Chen-Lin [10] and the author [36] for the case \( \alpha = 0 \). One important application of getting sharp blowup estimates is to derive a degree counting formula for the mean field equation (1.3). When the right hand side of (1.3) is 0, Chen-Lin [11] established a degree counting formula in terms of the genus of the Riemann surface using the sharp estimate in [10]. Another way of counting the degree has been proved by Malchiodi [32]. We expect to use Theorem 1.2 to derive a degree counting formula for the general mean field equation (1.3) in a forthcoming paper.

The assumption \( \alpha \not\in \mathbb{N} \) is essential to Theorem 1.2. When (1.5) is compared with the case \( \alpha = 0 \), many important features are different. For the latter case, the method of moving spheres and the Pohozaev’s type arguments are very effective. However, these well known methods do not seem to be useful for the former case. Instead we mainly use the potential theory iteratively to obtain the sharp estimate. Heuristically, the main difference between the two cases is that when \( \alpha > 0 (\alpha \not\in \mathbb{N}) \), the linearized operator of (1.5) along a standard bubble is "invertible”. This invertibility forces the maximum points of blowup solutions to be very close to 0 (see Corollary 2.1). While for the \( \alpha = 0 \) case, this invertibility is lost and it is important to obtain the sharp vanishing rate of the gradient of some coefficient functions, see [36] for details.

We also note that the assumption \( \alpha \not\in \mathbb{N} \) cannot be removed in general. In fact even Theorem 1.1 may not hold for \( \alpha \in \mathbb{N} \) (see [2] [35]). However the case \( \alpha \in \mathbb{N} \) has important applications (see [34]) and needs to be better understood. A theorem similar to Theorem 1.2 in this case, even with more assumptions, should still be of much interest.
The structure of this paper is as follows: In section 2 we prove Theorem 1.2. Our proof is based on Theorem 1.1. Then in the appendix we include some detailed estimates.

Acknowledgement The author would like to thank the referee for scrutinizing the paper and for his/her many insightful remarks. He is also grateful to M. Lucia for stimulating discussions.

2 Proof of Theorem 1.2

We shall always use C to denote a constant depending on $\alpha$, $C_0$, $C_1$ only, unless we specify otherwise.

2.1 A uniqueness lemma

Lemma 2.1. Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, $\phi$ be a $C^2$ function that verifies

\[
\begin{cases}
\Delta \phi + |x|^{2\alpha} e^{U_\alpha} \phi = 0 & \text{in } \mathbb{R}^2, \\
\phi(0) = 0, \quad |\phi(x)| \leq C(1 + |x|)^\tau & x \in \mathbb{R}^2.
\end{cases}
\]

where

\[U_\alpha(x) = \log \frac{8(\alpha + 1)^2}{(1 + |x|^{2\alpha+2})^2}\]

and $\tau \in [0, 1)$.

Then $\phi \equiv 0$.

Proof of Lemma 2.1: Let $k \geq 1$ be an integer. We define

\[\phi_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi(r \cos \theta, r \sin \theta) \cos(k\theta) d\theta.\]

Then $\phi_k$ satisfies

\[
\begin{cases}
\phi_k''(r) + \frac{1}{r} \phi_k'(r) + (r^{2\alpha} e^{U_\alpha} - \frac{k^2}{r^2}) \phi_k(r) = 0, & 0 < r < \infty, \\
\lim_{r \to 0} \phi_k(r) = 0, \quad |\phi_k(r)| \leq C(1 + r)^\tau, & r > 0.
\end{cases}
\]

Putting $r = e^t$ and using Theorem 8.1 of [14, Theorem 8.1, p.92] one can see there are two fundamental solutions of (2.1) that behave like those of

\[\tilde{\phi}''(r) + \frac{1}{r} \tilde{\phi}'(r) - \frac{k^2}{r^2} \tilde{\phi}(r) = 0\]


Let us denote by \( \phi_{k,1} \) and \( \phi_{k,2} \) the two fundamental solutions of (2.1) so that \( \phi_{k,1} \sim r^k, \phi_{k,2} \sim r^{-k} \) as \( r \to \infty \). Since \( \phi_k = c_1 \phi_{k,1} + c_2 \phi_{k,2} \) and \( |\phi_k| \) grows no faster than \( r^\alpha \) as \( r \to \infty \), we conclude that \( c_1 = 0 \). Hence \( |\phi_k| \sim r^{-k} \) for \( r \) large. On the other hand, when \( r \) is close to 0, the term \( r^{2\alpha}e^{U_0} \) is a perturbation again, which means there are two fundamental solutions, say, \( \tilde{\phi}_{k,1} \) and \( \tilde{\phi}_{k,2} \), comparable to \( r^k \) and \( r^{-k} \) as \( r \to 0^+ \), respectively. Since \( \phi_k(0) = 0 \), we know that \( |\phi_k(r)| \sim r^k \) as \( r \to 0^+ \).

We claim that \( \phi_k \equiv 0 \) for all \( k \geq 1 \). To see this, let \( f_k(s) = \phi_k(s^{1+\alpha}) \), then we have

\[
f''_k(s) + \frac{1}{s} f'_k(s) + \left( \frac{8}{(1+s^2)^2} - \frac{k^2}{(1+\alpha)^2 s^2} \right) f_k(s) = 0, \quad 0 < s < \infty. \tag{2.2}
\]

Using \( \phi_k = O(r^{-k}) \) at infinity and \( \phi_k = O(r^k) \) at 0 we conclude that

\[
f_k(s) = O(s^{-\frac{1}{1+\alpha}}) \quad \text{at} \; \infty, \quad f_k(s) = O(s^{\frac{1}{1+\alpha}}) \quad \text{at} \; 0. \tag{2.3}
\]

Let \( \delta_1 = \frac{k}{1+\alpha} \). Since \( \alpha \notin \mathbb{N} \) we see clearly that \( \delta_1 \neq 1 \). Then by direct computation we verify that the following two functions are two fundamental solutions of (2.2):

\[
f_{11}(s) = \frac{(\delta_1 + 1)s^{\delta_1} + (\delta_1 - 1)s^{\delta_1+2}}{1 + s^2},
\]

\[
f_{12}(s) = \frac{(\delta_1 + 1)s^{-\delta_1} + (\delta_1 - 1)s^{-\delta_1}}{1 + s^2}. \tag{2.4}
\]

Consequently \( f_k = c_1 f_{11} + c_2 f_{12} \) where \( c_1 \) and \( c_2 \) are two constants. From (2.3) we see that \( c_1 = 0 \) because otherwise \( |f_k| \sim s^{\delta_1} \) at \( \infty \). Similarly by observing the behavior of \( f_k \) at 0 we have \( c_2 = 0 \) because otherwise \( |f_k| \sim s^{-\delta_1} \) near 0, which is a contradiction to (2.3) again. So \( f_k \equiv 0 \) for all \( k \geq 1 \). This is equivalent to \( \phi_k \equiv 0 \) for all \( k \geq 1 \). The same argument also shows that the projection of \( \phi \) along \( \sin k \theta \) (\( \forall k \geq 1 \)) is 0. Finally, from \( \phi(0) = 0 \) we conclude \( \phi(x) \equiv 0. \square \)

### 2.2 Initial refinement

In this subsection, we give the first improvement of Theorem 1.1. The main result in this subsection is Proposition 2.1.

Let \( \delta_i = e^{-\frac{u_i(0)}{2\alpha}} \) and we denote by \( z_i \) the maximum point of \( u_i \). It is proved in [2] that \( \delta_i^{-1} z_i \to 0 \). Note that we use \( u_i(0) \) to define \( \delta_i \) because \( u_i(0)/u_i(z_i) = 1 + o(1) \). From the definition of \( \psi_i \), \( u_i - \psi_i \) has no oscillation on \( \partial B_1 \) and it satisfies

\[
\Delta (u_i - \psi_i) + |x|^{2\alpha} V_i(x)e^{(u_i-\psi_i)(x)} = 0, \quad \text{in} \; B_1
\]
where \( V_i(x) = H_i(x)e^{\psi_i(x)} \) is defined as before. Without loss of generality we assume 
\[ V_i(0) = H_i(0) \to 8(\alpha + 1)^2. \]

Let \( x_i \) be the maximum point of \( u_i - \psi_i \), then we still have \( \delta_i^{-1}x_i \to 0 \) by the argument in [2]. Now we define \( v_i \) as 
\[ v_i(y) = u_i(\delta_i y) - \psi_i(\delta_i y) - u_i(0), \quad y \in \Omega_i := B(0, \delta_i^{-1}). \]

We list some properties of \( v_i \) implied by its definition:

\[
\begin{align*}
\Delta v_i(y) + |y|^{2\alpha}V_i(0)e^{v_i(y)} &= 0, \quad y \in \Omega_i := B(0, \delta_i^{-1}), \\
v_i(0) &= 0, \quad y_i := \delta_i^{-1}x_i \to 0. \\
v_i(y) &\to (-2) \log(1 + |y|^{2\alpha+2}) \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^2). \\
v_i(y_1) &= v_i(y_2), \quad \forall y_1, y_2 \in \partial\Omega_i := \partial B(0, \delta_i^{-1}).
\end{align*}
\]

Note that in the second equation above we use \( y_i \) to denote the maximum point of \( v_i \). Let 
\[ U_i(y) = (-2) \log \left( 1 + \frac{V_i(0)}{8(\alpha + 1)^2}|y|^{2\alpha+2} \right) \]
be a standard bubble which satisfies 
\[ \Delta U_i + |y|^{2\alpha}V_i(0)e^{U_i(y)} = 0, \quad \text{in} \quad \mathbb{R}^2. \quad (2.5) \]

Then the conclusion of Theorem [1.1] can be written as:
\[ |v_i(y) - U_i(y)| \leq C, \quad |y| \leq \delta_i^{-1}. \]

Let \( w_i(y) = v_i(y) - U_i(y) \), then the following proposition is the first improvement of Theorem [1.1].

**Proposition 2.1.** For any \( \epsilon \in (0, \frac{1}{2}) \), there exists \( C(\epsilon) > 0 \) such that for all large \( i \),
\[ |w_i(y)| \leq C(1 + |y|)^\epsilon, \quad \text{in} \quad \Omega_i. \]

**Proof of Proposition 2.1.** We write the equation for \( w_i \) as
\[
\begin{align*}
\Delta w_i + r^{2\alpha}V_i(0)e^{\xi_i}w_i &= O(\delta_i)(1 + r)^{-3-2\alpha}, \\
w_i(0) &= 0, \quad |w_i(y)| \leq C, \quad y \in \Omega_i, \quad w_i|_{\partial\Omega_i} = \tilde{a}_i,
\end{align*}
\]
where $\xi_i$ is obtained by the mean value theorem. From Theorem 1.1 we immediately have $\tilde{a}_i = O(1)$. Let

$$\tilde{\Lambda}_i = \max \frac{|w_i(y)|}{\delta_i(1 + |y|)^c} \quad y \in \bar{\Omega}_i.$$ 

Our goal is to show $\tilde{\Lambda}_i = O(1)$. We prove this by a contradiction. Suppose $\tilde{\Lambda}_i \to \infty$, then we use $\tilde{y}_i$ to denote a point where $\tilde{\Lambda}_i$ is assumed. Let

$$\tilde{\Lambda}_i = \max \frac{|w_i(y)|}{\delta_i(1 + |y|)^c}.$$ 

It readily follows from the definition of $\tilde{\Lambda}_i$ that $|\tilde{w}_i(y)| \leq \frac{(1 + |y|)^c}{(1 + |\tilde{y}_i|)^c}$, which means $\tilde{w}_i$ is uniformly bounded over any fixed compact subset of $\mathbb{R}^2$. Therefore we conclude that, along a subsequence, $\tilde{w}_i$ converges in $C^2_{loc}(\mathbb{R}^2)$ to a solution $w$ of

$$\begin{cases}
\Delta w + r^{2\alpha} e^{U_\alpha} w = 0, & \mathbb{R}^2, \\
w(0) = 0, & |w(y)| \leq C(1 + |y|)^c.
\end{cases}$$

If $\tilde{y}_i$ converges to $y_0 \in \mathbb{R}^2$, we have $|w(y_0)| = 1$ by continuity. However this is impossible because an application of Lemma 2.1 shows that $w \equiv 0$. Therefore the only case to consider is $\tilde{y}_i \to \infty$.

It follows from $|\tilde{w}_i(\tilde{y}_i)| = 1$ and the Green’s representation formula that

$$\pm 1 = \tilde{w}_i(\tilde{y}_i) = \int_{\Omega_i} G(\tilde{y}_i, \eta) \left\{ |\eta|^{2\alpha} V_i(0) e^{\xi_i(\eta)} \frac{w_i(\eta)}{\tilde{\Lambda}_i \delta_i(1 + |\eta|)^c} \frac{(1 + |\eta|)^c}{(1 + |\tilde{y}_i|)^c} + O(1)(1 + |\eta|^{-3-2\alpha}) \right\} d\eta - \int_{\partial \Omega_i} \frac{\partial G}{\partial \nu}(\tilde{y}_i, \eta) \frac{\tilde{a}_i}{\tilde{\Lambda}_i \delta_i(1 + |\tilde{y}_i|)^c} dS$$

(2.6)

where $G$ is the Green’s function over $\Omega_i$ with respect to the Dirichlet boundary condition. Recall that the Green’s function over $\Omega_i$ is

$$G(y, \eta) = -\frac{1}{2\pi} \log |y - \eta| + \frac{1}{2\pi} \log \left( \frac{|y|}{\delta_i} \frac{\delta_i^2 y}{|y|^2} - \eta \right).$$

Since $\tilde{w}_i(0) = 0$, the Green’s representation formula gives

$$0 = \int_{\Omega_i} G(0, \eta) \left\{ |\eta|^{2\alpha} V_i(0) e^{\xi_i(\eta)} \frac{w_i(\eta)}{\tilde{\Lambda}_i \delta_i(1 + |\eta|)^c} \frac{(1 + |\eta|)^c}{(1 + |\tilde{y}_i|)^c} + O(1)(1 + |\eta|)^{-3-2\alpha} \right\} d\eta - \int_{\partial \Omega_i} \frac{\partial G}{\partial \nu}(0, \eta) \frac{\tilde{a}_i}{\tilde{\Lambda}_i \delta_i(1 + |\tilde{y}_i|)^c} dS.$$

(2.7)
To deal with the two boundary integral terms in (2.6) and (2.7), we observe that

\[
\int_{\partial \Omega_i} \left( \frac{\partial G(\tilde{y}_i, \eta)}{\partial \nu} - \frac{\partial G(0, \eta)}{\partial \nu} \right) \tilde{a}_i \tilde{\Lambda}_i \delta_i (1 + |\tilde{y}_i|)^\epsilon dS = 0.
\]

This equality follows by using the well known identity

\[
\int_{\partial \Omega} \partial_\nu G(\xi, \eta) dS = -1, \quad \forall \xi \in \Omega_i.
\]

From (2.6) and (2.7) we have

\[
1 \leq \int_{\Omega_i} |G(\tilde{y}_i, \eta) - G(0, \eta)| \left( \frac{(1 + |\eta|)^{-4 - 2\alpha + \epsilon}}{(1 + |\tilde{y}_i|)^\epsilon} + \circ(1) \frac{(1 + |\eta|)^{-3 - 2\alpha}}{(1 + |\tilde{y}_i|)^\epsilon} \right) d\eta. \quad (2.8)
\]

Note that in the above we used

\[
\left| \frac{w_i(\eta)}{\bar{\Lambda}_i \delta_i (1 + |\eta|)^\epsilon} \right| \leq 1, \quad e^{\xi_i(\eta)} \leq C(1 + |\eta|)^{1 - 4\alpha}, \quad \bar{\Lambda}_i \to \infty.
\]

To get a contradiction to (2.8) we only need to show the following:

\[
\int_{\Omega_i} |G(\tilde{y}_i, \eta) - G(0, \eta)| \left( \frac{(1 + |\eta|)^{-4 - 2\alpha + \epsilon}}{(1 + |\tilde{y}_i|)^\epsilon} + \circ(1) \frac{(1 + |\eta|)^{-3 - 2\alpha}}{(1 + |\tilde{y}_i|)^\epsilon} \right) d\eta = \circ(1). \quad (2.9)
\]

We consider two cases: If $|\tilde{y}_i| = \circ(1) \delta_i^{-1}$, $G(\tilde{y}_i, \eta)$ can be written as

\[
G(\tilde{y}_i, \eta) = -\frac{1}{2\pi} \log |\tilde{y}_i - \eta| + \frac{1}{2\pi} \log \delta_i^{-1} + \circ(1).
\]

In this case it is enough to show

\[
\int_{\Omega_i} \left| \frac{\log |\tilde{y}_i - \eta|}{|\eta|} \right| \left( \frac{(1 + |\eta|)^{-4 - 2\alpha + \epsilon}}{(1 + |\tilde{y}_i|)^\epsilon} + \circ(1) \frac{(1 + |\eta|)^{-3 - 2\alpha}}{(1 + |\tilde{y}_i|)^\epsilon} \right) d\eta = \circ(1),
\]

which follows from standard elementary estimates.

Finally we consider the case $|\tilde{y}_i| \sim \delta_i^{-1}$. For the Green’s function we use

\[
|G(\tilde{y}_i, \eta) - G(0, \eta)| \leq C(\log(1 + |\eta|) + \log \delta_i^{-1}), \quad C \text{ universal}.
\]

Then it is easy to obtain (2.9) by elementary estimates. Proposition 2.1 is established. □

From Proposition 2.1 we obtain an estimate on $x_i$ more precise than $|x_i| = \circ(\delta_i)$ (Recall that $x_i$ is the place where the maximum of $u_i - \psi_i$ occurs, $y_i$ is the point where the maximum of $v_i$ is attained.)
Corollary 2.1.

|y_i| = | \delta_i^{-1} x_i| = O(\delta_i^{1/(2\alpha + 1)}). \quad (2.10)

Proof of Corollary 2.1. Using Proposition 2.1 and standard elliptic estimates we have

|v_i(y) - U_i(y)| \leq C\delta_i |y|, \quad |y| \leq 10

and

|\nabla v_i(y) - \nabla U_i(y)| \leq C\delta_i, \quad |y| \leq 10.

Therefore |\nabla U_i(y_i)| = O(\delta_i) because \nabla v_i(y_i) = 0. Then (2.10) follows from the definition of U_i. □

Remark 2.1. Recall that z_i is the place where the maximum of u_i is attained. For z_i we also have z_i = O(\delta_i^{2\alpha +2}) by the same argument.

2.3 Further refinement of the expansion

U_i is the first term in the expansion of v_i. To determine the second term in the expansion we need a radial function g_i that satisfies the following:

\begin{align*}
\left\{ \begin{array}{l}
g''_i(r) + \frac{1}{r}g'_i(r) + (r^{2\alpha} V_i(0)e^{U_i(r)} - \frac{1}{r^2})g_i(r) = -r^{2\alpha +1}e^{U_i(r)}, \quad 0 < r < \infty, \\
\lim_{r \to 0^+} g_i(r) = \lim_{r \to \infty} g_i(r) = 0, \quad |g_i(r)| \leq C \frac{r}{1 + r^2}.
\end{array} \right.
\end{align*}

(2.11)

By direct computation one checks that the following expression verifies the above:

\[ g_i(r) = -\frac{2(1 + \alpha)}{\alpha V_i(0)} \frac{r}{1 + \frac{V_i(0)}{8(1 + \alpha)^\alpha} r^{2\alpha +2}}. \quad (2.12) \]

Let

\[ \phi_i(y) = g_i(r) \delta_i \sum_{j=1}^2 \partial_j V_i(0) \theta_j, \quad \theta_j = y_j / r. \quad (2.13) \]

Then \phi_i satisfies

\[ \Delta \phi_i + |y|^{2\alpha} V_i(0)e^{U_i(y)} \phi_i = -\sum_{j=1}^2 \delta_i \partial_j V_i(0) y_j |y|^{2\alpha} e^{U_i(y)}, \quad \text{in } \Omega_i. \quad (2.14) \]

We claim that \phi_i is the second term in the expansion of v_i. This is verified in the following
Proposition 2.2. For any $\epsilon \in (0, \frac{1}{2})$, there is $C > 0$ depending only on $C_0, C_1, \alpha, \epsilon$ such that

$$|v_i(y) - (U_i + \phi_i)(y)| \leq C\delta_i^2(1 + |y|)^\epsilon,$$

$$|\nabla v_i(y) - \nabla(U_i + \phi_i)(y)| \leq C\delta_i^2(1 + |y|)^{\epsilon-1},$$

in $\Omega_i$.

Proof of Proposition 2.2 Let $b_i = v_i - U_i - \phi_i$. Then $b_i$ satisfies $b_i(0) = 0$ and

$$\Delta b_i + r^{2\alpha} V_i(\delta_i y)e^{\xi} b_i = r^{2\alpha} (V_i(0)e^{U_i} - V_i(\delta_i y)e^{\xi}) \phi_i + O(\delta_i^2) r^{2\alpha+2} e^{U_i}.$$

By using Proposition 2.1 and (2.11) in the estimate of the right hand side of the above, we have

$$\Delta b_i + r^{2\alpha} V_i(\delta_i y)e^{\xi} b_i = O(\delta_i^2)(1 + r)^{2-2\alpha}.$$

Proposition 2.2 follows from the same argument as in Proposition 2.1 $\Box$.

Remark 2.2. The essential difference between the $\alpha \not\in \mathbb{N}$ case and the $\alpha = 0$ case lies in the fundamental solutions of the equation in (2.11). The two sets of fundamental solutions behave very differently. As a result, for $\alpha \not\in \mathbb{N}$, we have the presence of $\phi_i$ and the maximum point of $v_i$ very close to the singularity 0 (see Corollary 2.1). On the other hand, for the $\alpha = 0$ case, it is crucial to obtain the vanishing rate of $\nabla V_i(0)$ from the Pohozaev Identity (see [36]).

Let $\theta_j := \frac{r}{y_j}$, $j = 1, 2$, we list the following well known identities for convenience.

$$-\frac{d^2}{d\theta^2}(\theta_1 \theta_2) = 4\theta_1 \theta_2, \quad -\frac{d^2}{d\theta^2}(\theta_j^2 - \frac{1}{2}) = 4(\theta_j^2 - \frac{1}{2}), \quad j = 1, 2. \quad (2.15)$$

Now we rewrite the equation for $v_i$ using Proposition 2.2. Let $b_i = v_i - U_i - \phi_i$, then we have

$$\Delta(U_i + \phi_i + b_i) + r^{2\alpha}(V_i(0) + \delta_i \nabla V_i(0) \cdot y + F_1 + O(\delta_i^3 r^{3})) e^{U_i + \phi_i + b_i} = 0 \quad (2.16)$$

where

$$F_1 = \delta_i^2 \left( \frac{1}{2} \partial_{11} V_i(0)y_1^2 + \frac{1}{2} \partial_{22} V_i(0)y_2^2 + \partial_{12} V_i(0)y_1 y_2 \right)$$

$$= \delta_i^2 \left( \frac{1}{2} \partial_{11} V_i(0)(y_1^2 - \frac{r^2}{2}) + \frac{1}{2} \partial_{22} V_i(0)(y_2^2 - \frac{r^2}{2}) + \partial_{12} V_i(0)y_1 y_2 \right)$$

$$+ \frac{1}{4} \delta_i^2 r^2 \Delta V_i(0) = F_{11} + F_{12} \quad (2.17)$$
\[ F_{11} = \delta^2 r^2 \left( \frac{1}{2} \partial_{11} V_i(0)(\theta^2 - \frac{1}{2}) + \frac{1}{2} \partial_{22} V_i(0)((\theta^2 - \frac{1}{2}) + \partial_{12} V_i(0)\theta_1 \theta_2) \right) \]

\[ F_{12} = \frac{1}{4} \delta^2 r^2 \Delta V_i(0) \]  \hspace{1cm} (2.18)

By using Proposition 2.2 we have \( |b_i| = O(\delta^2 r^\epsilon) \) and \( b_i + \phi_i = O(\delta_i (1 + r)^{-\frac{3}{2}}) \). Since \( e^{U_i} = O((1 + r)^{-\frac{1}{4} - 4\alpha}) \) we conclude that

\[ e^{U_i + \phi_i + b_i} = e^{U_i} (1 + \phi_i + b_i + \frac{1}{2} (\phi_i + b_i)^2 + O(\delta^3 r^{-\frac{3}{2}})) \]

\[ = e^{U_i} + e^{U_i} \phi_i + e^{U_i} b_i + \frac{1}{2} e^{U_i} \phi_i^2 + O(\delta^3_i (1 + r)^{-5 - 4\alpha + 2\epsilon}) \]

With this expression (2.16) is reduced to

\[ \Delta (U_i + \phi_i + b_i) + r^{2\alpha} \left( V_i(0) + \delta_i \nabla V_i(0) \cdot y + F_1 + O(\delta^3 r^3) \right) \]

\[ \cdot \left( e^{U_i} + e^{U_i} \phi_i + e^{U_i} b_i + \frac{1}{2} e^{U_i} \phi_i^2 + O(\delta^3_i (1 + r)^{-5 - 4\alpha + 2\epsilon}) \right) = 0. \]

The last term on the left hand side of the above is

\[ r^{2\alpha} \left( V_i(0) e^{U_i} + V_i(0) e^{U_i} \phi_i + V_i(0) e^{U_i} b_i + \frac{1}{2} V_i(0) e^{U_i} \phi_i^2 + \delta_i \nabla V_i(0) \cdot ye^{U_i} \right. \]

\[ + \delta_i \nabla V_i(0) \cdot ye^{U_i} \phi_i + F_1 e^{U_i} + O(\delta^3_i (1 + r)^{-1 - 4\alpha}) \right). \]

By combining (2.14), (2.5), (2.17) we can write the equation for \( b_i \) as

\[ \Delta b_i + r^{2\alpha} V_i(0) e^{U_i} b_i + \frac{V_i(0)}{2} r^{2\alpha} e^{U_i} \phi_i^2 + \delta_i \nabla V_i(0) \cdot ye^{U_i} \phi_i + r^{2+2\alpha} + \frac{r^{2+2\alpha}}{4} \Delta V_i(0) \delta^2 \nabla e^{U_i} + r^{2\alpha} F_{11} e^{U_i} + O(\delta^3_i (1 + r)^{-1 - 2\alpha}) = 0. \]  \hspace{1cm} (2.19)

Without loss of generality we assume \( \nabla V_i(0) = |\nabla V_i(0)| e_1 \). Then the sum of the
third and fourth term on the left hand side of (2.19) is

\[
\frac{V_i(0)}{2} r^{2\alpha} e^{U_i} \phi_i^2 + \delta_i r^{2\alpha} \nabla V_i(0) \cdot ye^{U_i} \phi_i
\]

\[
= \frac{V_i(0)}{2} r^{2\alpha} e^{U_i} g_i^2(r) \delta_i^2 |\nabla V_i(0)|^2 \theta_i^2 + \delta_i^2 g_i(r) |\nabla V_i(0)|^2 r^{1+2\alpha} \theta_i^2 e^{U_i}
\]

\[
= \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \theta_i^2 \left( \frac{V_i(0)}{2} g_i^2(r) + g_i(r) r \right)
\]

\[
= \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \left( \frac{V_i(0)}{2} g_i^2(r) + g_i(r) r \right)
\]

\[
+ \frac{1}{2} \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \left( \frac{V_i(0)}{2} g_i^2(r) + g_i(r) r \right)
\]

\[= C_{11} + C_{12}.
\]

For \(C_{11}\) and \(r^{2\alpha} F_{11} e^{U_i}\) we can find \(c_i\) that satisfies

\[
\begin{cases}
\Delta c_i + r^{2\alpha} V_i(0) e^{U_i} c_i + C_{11} + r^{2\alpha} F_{11} e^{U_i} = 0, & 0 < r < \delta_i^{-1} \\
|c_i(x)| \leq C \delta_i^2 \frac{r^2}{(1+r)^2}, & 0 < r < \delta_i^{-1}.
\end{cases}
\]

The existence of \(c_i\) and its estimate are established in the appendix. Let \(d_i = b_i - c_i\). Then the equation for \(d_i\) is

\[
\Delta d_i + r^{2\alpha} V_i(0) e^{U_i} d_i + E_i + O(\delta_i^3 (1+r)^{-1-2\alpha}) = 0.
\]

where

\[
E_i = \frac{r^{2+2\alpha}}{4} \Delta V_i(0) \delta_i^2 e^{U_i} + \frac{1}{2} \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 \left( \frac{V_i(0)}{2} g_i^2(r) + g_i(r) r \right).
\]

It follows from the definition of \(d_i\), Proposition 2.2 and (2.20), that

\[
d_i(0) = 0, \quad |d_i(y)| \leq C \delta_i^2 (1+r)^c, \quad r \leq \delta_i^{-1}.
\]

Also, it is implied by (2.21) and (2.22) that

\[
\int_{\Omega_i} \Delta d_i = O(\delta_i^2).
\]

In the following proposition we evaluate the value of \(d_i\) on \(\partial \Omega_i\):

**Proposition 2.3.**

\[
d_i = (\Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2) \delta_i^2 \log \delta_i^{-1} + O(\delta_i^2) \quad \text{on} \quad \partial \Omega_i.
\]
Proof of Proposition [2.3]: Let
\[ f_i(y) = \frac{1 - a_i r^{2\alpha + 2}}{1 + a_i r^{2\alpha + 2}}, \quad a_i = \frac{V_i(0)}{8(1 + \alpha)^2}. \]

Direct computation shows
\[ \Delta f_i(y) + r^{2\alpha} V_i(0) e^{U_i} f_i(y) = 0, \quad \text{in } \mathbb{R}^2. \tag{2.24} \]

Also it is straightforward to verify that
\[ f_i(y) = -1 + O(\delta_i^{2\alpha + 2}), \quad \frac{\partial f_i}{\partial r} = O(\delta_i^{3+2\alpha}) \quad \text{on } \partial \Omega_i. \tag{2.25} \]

A direct consequence of (2.21), (2.24) and the Green’s formula is
\[ \int_{\partial \Omega_i} \frac{\partial f_i}{\partial \nu} d_i - \frac{\partial d_i}{\partial \nu} f_i = \int_{\Omega_i} E_i f_i + O(\delta_i^{2+2\alpha}). \tag{2.26} \]

From (2.25) we have
\[ \int_{\partial \Omega_i} \frac{\partial f_i}{\partial \nu} d_i = O(\delta_i^{2+2\alpha}). \tag{2.27} \]

Since \( f_i \) is a radial function, we have, by (2.23) and (2.25),
\[ \int_{\partial \Omega_i} \frac{\partial d_i}{\partial \nu} f_i = (\int_{\Omega_i} \Delta d_i) f_i|_{\partial \Omega_i} = -\int_{\Omega_i} \Delta d_i + O(\delta_i^{2+2\alpha}). \tag{2.28} \]

So we conclude from (2.26), (2.27) and (2.28) that
\[ \int_{\Omega_i} \Delta d_i = \int_{\Omega_i} E_i f_i + O(\delta_i^{2+2\alpha}). \tag{2.29} \]

On the other hand, from \( d_i(0) = 0 \) and (2.21) we have
\[ 0 = \int_{\Omega_i} G(0, \eta)(|\eta|^{2\alpha} V_i(0) e^{U_i(\eta)} d_i(\eta) + E_i(\eta)) + d_i|_{\partial \Omega_i} + O(\delta_i^2). \]

Since
\[ G(0, \eta) = -\frac{1}{2\pi} \log |\eta| + \frac{1}{2\pi} \log \delta_i^{-1} \]

and
\[ \int_{\Omega_i} \log |\eta|(|\eta|^{2\alpha} V_i(0) e^{U_i(\eta)} d_i(\eta) + E_i(\eta)) = O(\delta_i^2), \]
we obtain from elementary estimates
\[
\begin{align*}
\left. d_i \right|_{\partial \Omega_i} &= \frac{1}{2\pi} \log(\delta_i^{-1}) \int_{\Omega_i} E_i f_i + O(\delta_i^2) \\
&= (\log \delta_i^{-1}) \delta_i^2 (\Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2) + O(\delta_i^2).
\end{align*}
\]

Proposition 2.3 is established. □

We now finish the proof of Theorem 1.2 by a standard application of the maximum principle. Let
\[
M_i(y) = (\Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2) \delta_i^2 \log |y| + \delta_i^2 M (1 - r^{-\alpha})
\]
where \( M \) is a large number to be determined. Thanks to (2.21),(2.22) and the estimate of \( E_i \), the equation for \( d_i \) can be written as
\[
\Delta d_i = O(\delta_i^2)(1 + r)^{-2\alpha}, \quad \text{in } \Omega_i.
\]
By choosing \( M \) large enough we have \( M_i(y) > d_i(y) \) on \( \partial B_2 \) and \( \partial \Omega_i \). On the other hand
\[
\Delta M_i(y) = -M \alpha^2 \delta_i^2 r^{-2-\alpha} < \Delta d_i(y) \quad 2 < r < \delta_i^{-1}.
\]
Therefore by the maximum principle
\[
d_i(y) \leq M_i(y) \quad 2 < r < \delta_i^{-1}.
\]
Similarly we can also show
\[
d_i(y) \geq (\Lambda_1 \Delta V_i(0) + \Lambda_2 |\nabla V_i(0)|^2) \delta_i^2 \log |y| - \delta_i^2 M (1 - r^{-\alpha}) \quad 2 < r < \delta_i^{-1}
\]
for \( M \) large. Theorem 1.2 is established. □

3 Appendix

In this section we prove the existence of \( c_i \) that satisfies (2.20). Recall that
\[
F_{11} = \delta_i^2 r^2 \left( \frac{1}{2} \partial_{11} V_i(0)(\theta_2^2 - \frac{1}{2}) + \frac{1}{2} \partial_{22} V_i(0)(\theta_2^2 - \frac{1}{2}) + \partial_{12} V_i(0)\theta_1\theta_2 \right)
\]
\[
C_{11} = \delta_i^2 r^{2\alpha} e^{U_i} |\nabla V_i(0)|^2 (\theta_1^2 - \frac{1}{2}) \left( \frac{V_i(0)}{2} g_i^2(r) + g_i(r)r \right).
\]
The three terms in $r^{2\alpha}F_{11}e^{V_i}$ and $C_{11}$ can all be written in the form $\delta_i^2 f(\theta)Q_i(r)$ where $f(\theta)$ is one of the spherical harmonics ( $\theta_1^2 - \frac{1}{2}, \theta_2^2 - \frac{1}{2}, \theta_1 \theta_2$ ) and $Q_i(r)$ is a radial function that satisfies

$$|Q_i(r)| \leq C \frac{r^{2+2\alpha}}{(1 + a_i r^{2+2\alpha})^2}.$$ 

If we can find a function $h_1(r)$ that solves

$$\begin{cases}
h''_1(r) + \frac{1}{r}h'_1(r) + \left( r^{2\alpha} V_i(0) e^{U_i} - \frac{4}{r^2} \right) h_1(r) = -Q_i(r), & 0 < r < \delta_i^{-1} \\
|h_1(r)| \leq C \frac{r^2}{(1+r)^2}, & 0 < r < \delta_i^{-1},
\end{cases}$$

then by (2.15) $h_1(r) f(\theta)$ solves

$$(\Delta + r^{2\alpha} V_i(0) e^{U_i})(f(\theta) h_1(r)) + f(\theta) Q_i(r) = 0, \quad \Omega_i.$$ 

Consequently $c_i$ is the sum of four such functions. So all we need to establish is (3.1). Let

$$r = \left( \sqrt{\frac{8(\alpha + 1)^2}{V_i(0)}} s \right)^{1+\alpha} \quad \text{and} \quad f_1(s) = h_1 \left( \sqrt{\frac{8(\alpha + 1)^2}{V_i(0)}} s \right)^{1+\alpha}.$$ 

Then the equation for $f_1(s)$ is

$$f''_1(s) + \frac{1}{s} f'_1(s) + \left( \frac{8}{(1 + s^2)^2} - \frac{4\delta^2}{s^2} \right) f_1(s) = l_1(s), \quad 0 < s < \infty,$$

where $\delta = \frac{1}{1+\alpha}$,

$$|l_1(s)| \leq C \frac{s^{2\delta}}{(1 + s^2)^2}.$$ 

Since $\alpha$ is not an integer, $2\delta - 1 \neq 0$, two fundamental solutions of the homogeneous equation ($f_{21}$ and $f_{22}$) can be found in explicit form:

$$f_{21}(s) = \frac{(2\delta + 1)s^{2\delta} + (2\delta - 1)s^{2\delta+2}}{1 + s^2},$$

$$f_{22}(s) = \frac{(2\delta + 1)s^{2-2\delta} + (2\delta - 1)s^{-2\delta}}{1 + s^2}.$$

Let

$$w_1(s) := f_{21}(s) f'_{22}(s) - f'_{21}(s) f_{22}(s) = 4\delta(1 - 4\delta^2)s^{-1}.$$
and
\[ f_1(s) = -\int_s^\infty \frac{f_{22}(\tau)l_1(\tau)}{w_1(\tau)} d\tau f_2(\tau) + \int_0^s \frac{f_{21}(\tau)l_1(\tau)}{w_1(\tau)} d\tau f_2(s), \]
then it is straightforward to verify
\[ |f_1(s)| \leq C \frac{s^{2\delta}}{(1+s)^3}, \quad 0 < s < \infty. \]
(3.1) is established. \(\square\)

References

[1] J. Ambjorn, P. Oleson, A magnetic condensate solution of the classical electroweak theory, Physics Letters B Volume 218, Issue 1, 9 February 1989, Pages 67-71.

[2] D. Bartolucci, C.C. Chen, C. S. Lin, G. Tarantello, Profile of blow-up solutions to mean field equations with singular data. Comm. Partial Differential Equations 29 (2004), no. 7-8, 1241–1265.

[3] D. Bartolucci, G. Tarantello, Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory, Comm. Math. Phys. 229 (2002), 3-47.

[4] H. Brezis, Y.Y. Li, I. Shafrir, A sup + inf inequality for some nonlinear elliptic equations involving exponential nonlinearities, J. Functional Analysis 115 (1993), 344-358.

[5] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of \(-\Delta u = v(x)e^u\) in two dimensions. Comm. Partial Differential Equations, 16 (1991) 1223-1253.

[6] L. Caffarelli, Y. Yang, Vortex condensation in the Chern-Simons Higgs model: an existence theorem. Comm. Math. Phys. 168 (1995), no. 2, 321–336.

[7] D. Chae, O.Y. Imanuvilov The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory Comm. Math. Phys. 215 (2000), no. 1, 119–142.
[8] S. A. Chang, C. C. Chen, C. S. Lin, Extremal functions for a mean field equation in two dimension. Lectures on partial differential equations, 61–93, New Stud. Adv. Math., 2, Int. Press, Somerville, MA, 2003.

[9] S. Chanillo, M. Kiessling, Surfaces with prescribed Gauss curvature. Duke Math. J. 105 (2000), no. 2, 309–353.

[10] C. C. Chen, C. S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. Comm. Pure Appl. Math. 55 (2002), no. 6, 728–771.

[11] C. C. Chen, C. S. Lin, Topological degree for a mean field equation on Riemann surfaces. Comm. Pure Appl. Math. 56 (2003), no. 12, 1667–1727.

[12] C. C. Chen, C. S. Lin, On the symmetry of blowup solutions to a mean field equation. (English, French summaries) Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001), no. 3, 271296.

[13] X. Chen, Remarks on the existence of branch bubbles on the blowup analysis of equation $-\Delta u = e^{2u}$ in dimension two. Comm. Anal. Geom. 7 (1999), no. 2, 295–302.

[14] E. A. Coddington; N. Levinson, Theory of ordinary differential equations. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[15] W. Ding, J. Jost, J. Li, G. Wang, Existence results for mean field equations. Ann. Inst. H. Poincaré Anal. Non Linéaire 16 (1999), no. 5, 653–666.

[16] G. Dunne, Self-dual Chern-Simons Theories. Lecture Notes in Physics, m36. Berlin Hidelberg: Spinger Verlag.

[17] P. Esposito, M. Grossi, A. Pistoia, On the existence of blowing-up solutions for a mean field equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), no. 2, 227–257.

[18] J. Hong, Y. Kim, P.Y. Pac, Multivortex solutions of the abelian Chern-Simons-Higgs theory. Phys. Rev. Lett. 64 (1990), no. 19, 2230–2233.

[19] R. Jackiw, E.J. Weinberg, Self-dual Chern-Simons vortices. Phys. Rev. Lett. 64 (1990), no. 19, 2234–2237.

[20] J. Jost, G. Wang, Analytic aspects of the Toda system. I. A Moser-Trudinger inequality. Comm. Pure Appl. Math. 54 (2001), no. 11, 1289–1319.
[21] J. Jost, G. Wang, Classification of solutions of a Toda system in $\mathbb{R}^2$. Int. Math. Res. Not. 2002, no. 6, 277–290.

[22] J. Jost, C. S. Lin, G. Wang, Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. Comm. Pure Appl. Math. 59 (2006), no. 4, 526–558.

[23] C. H. Lai, Selected papers on Gauge Theory of Weak and Electromagnetic Interactions. Singapore World Scientific, 1981.

[24] Y. Y. Li, A Harnack Type Inequality: the Method of Moving Planes, Comm. Math. Phys. 200 (1999), 421-444.

[25] Y. Y. Li, I. Shafrir, Blow up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, Indiana Univ. Math. J. 43 (1994), 1255-1270.

[26] C. S. Lin, Topological degree for mean field equations on $S^2$. Duke Math. J. 104 (2000), no. 3, 501–536.

[27] C. S. Lin, An expository survey on the recent development of mean field equations. Discrete Contin. Dyn. Syst. 19 (2007), no. 2, 387–410.

[28] C. S. Lin, M. Lucia, One-dimensional symmetry of periodic minimizers for a mean field equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (2007), no. 2, 269–290.

[29] M. Lucia, A blowing-up branch of solutions for a mean field equation. (English summary) Calc. Var. Partial Differential Equations 26 (2006), no. 3, 313330.

[30] M. Lucia, M. Nolasco, SU(3) Chern-Simons vortex theory and Toda systems. J. Differential Equations 184 (2002), no. 2, 443–474.

[31] M. Lucia, L. Zhang, A priori estimates and uniqueness for some mean field equations. J. Differential Equations 217 (2005), no. 1, 154–178.

[32] A. Malchiodi, Morse theory and a scalar field equation on compact surfaces, preprint.

[33] H. Ohtsuka, T. Suzuki, Blow-up analysis for SU(3) Toda system. J. Differential Equations 232 (2007), no. 2, 419–440.
[34] J. Prajapet, G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results. Proc. Roy. Soc. Edinburgh Sect. A 131 (2001), no 4, 967-985.

[35] G. Tarantello, Analytical aspects of Liouville type equations with singular sources, Handbook Diff. Eqs., North Holland, Amsterdam, Stationary partial differential equations, I (2004), 491-592.

[36] L. Zhang, Blowup solutions for some nonlinear elliptic equations involving exponential nonlinearities, Comm. Math. Phys, 268, (2006) no 1: 105-133.