ON THE CONSTRUCTION OF ASYMMETRIC ORBIFOLD MODELS

Kenichiro Aoki\textsuperscript{a}, Eric D’Hoker\textsuperscript{b} and D.H. Phong\textsuperscript{c}

\textsuperscript{a} Hiyoshi Department of Physics
Keio University, Yokohama 223-8521, Japan
\textsuperscript{b} Department of Physics and Astronomy
University of California, Los Angeles, CA 90095, USA
\textsuperscript{c} Department of Mathematics
Columbia University, New York, NY 10027, USA

Abstract

Various asymmetric orbifold models based on chiral shifts and chiral reflections are investigated. Special attention is devoted to the consistency of the models with two fundamental principles for asymmetric orbifolds: modular invariance and the existence of a proper Hilbert space formulation for states and operators. The interplay between these two principles is non-trivial. It is shown, for example, that their simultaneous requirement forces the order of a chiral reflection to be 4, instead of the naive 2. A careful explicit construction is given of the associated one-loop partition functions. At higher loops, the partition functions of asymmetric orbifolds are built from the chiral blocks of associated symmetric orbifolds, whose pairings are determined by degenerations to one-loop.

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1 Introduction

String theories are built of independent left and right movers on the worldsheet, and both Type II [1] and Heterotic [2] superstring theories are inherently chiral on the worldsheet. As such, they admit asymmetric compactifications for left and right degrees of freedom. This fact is basic to the construction of the Heterotic string [2] and to other models such as the Narain compactification [3]. Compactifications by orbifolds of flat space-time ([4, 5]; for reviews see [6, 7, 8]), are particularly important because their conformal field theories are explicitly solvable from free field theories via group theoretic methods, and yet their space-time properties are non-trivial.

For symmetric orbifolds, the orbifold group acts identically on left and right movers. The associated worldsheet conformal field theory admits a consistent construction equivalently via functional integral or operator methods [9, 10, 11], and the functional integral formulation naturally guarantees modular invariance.

For asymmetric orbifolds, the orbifold group acts differently on left and right movers. This circumstance renders functional integral formulations problematic. Therefore, modular invariance and the validity of a Hilbert space interpretation cannot be taken for granted. Some general principles for compactifications on asymmetric orbifolds have been formulated in [12], but relatively few examples have been studied explicitly (see for example [13]. (At special compactification radii, asymmetric orbifold conformal field theories may be reformulated in terms of free fermionic degrees of freedom alone. The action of the orbifold group on chiral fermions is well understood and systematic studies of asymmetric free fermion models are available in [14, 15].)

Asymmetric orbifolds are clearly needed when orbifolding the Heterotic string (for example, see [16]). More recently, asymmetric orbifold models for Type II strings generated by chiral reflections and shifts were also proposed [17, 18, 19, 20]. These “Kachru-Kumar-Silverstein” (KKS) models are of particular interest, since they exhibit space-time supersymmetry breaking and vanishing cosmological constant at one-loop order. Initially, there were hopes that the cosmological constant would continue to vanish to two loops [17, 18] (see also [21]), via an independent cancellation of the sums over spin structures for left and right movers, but this is now known not to be the case [22].

In the present paper, we take the opportunity of the study of the KKS models to examine the principles of the construction of asymmetric orbifold models based on chiral shifts and twists in greater detail, and to add to the hitherto relatively short list of examples worked out explicitly. In view of the methods of chiral splitting [23, 24, 25] (see also [26, 27]) and two-loop superstring perturbation theory developed over the past few years in [28, 29, 30, 31, 32] (see also [33, 34]), it is natural to postulate that the partition
function for an asymmetric orbifold is obtained by suitably pairing the chiral blocks of symmetric orbifold theories. Thus, the main problem of asymmetry is the determination of the corresponding pairing matrix (see [22] and §6). It can be determined in principle by degenerations to one-loop. Thus, the key problem reduces to finding all the one-loop traces \( Z_{gh} = \text{Tr}_{\mathcal{H}_h}(gq^{L_0}q^{\bar{L}_0}) \), where \( g, h \) runs over all the elements of the orbifold group, and \( \mathcal{H}_h \) is the Hilbert space of the sector twisted by \( h \).

Even for the relatively simple asymmetric orbifold models based on chiral shifts and twists, there are important subtleties in the construction: (1) chiral operators \( g, h \) are defined only up to phases, which can affect their orders and change the whole structure of the theory; (2) it is unclear how to incorporate in the construction of the Hilbert space \( \mathcal{H}_h \) for the asymmetric theory the ground state degeneracies of the corresponding symmetric orbifold theories; (3) one has to extend to \( \mathcal{H}_h \) the operator \( g \) which was originally defined only on the untwisted Hilbert space.

We find that for the asymmetric orbifold models based on chiral shifts and twists, the combined requirements of modular covariance and Hilbert space interpretation constrain the orders, and hence the phases, of the elements of the orbifold group. The degeneracies in the Hilbert spaces for the twisted asymmetric theory can be handled by suitable selection rules on the larger set of ground states coming from the symmetric theory. And upon this construction, the operators of the theory admit consistent extensions to the twisted sectors. It can be hoped that similar considerations will apply to more general asymmetric orbifold models.

### 1.1 Principles of Orbifold Constructions

Let \( G \) be an orbifold group acting on a flat torus \( \mathbb{T}^n \). The orbifold group \( G \) will be taken to be finite and Abelian for simplicity.\(^1\) The following principles will be taken as the starting point for orbifold constructions on the worldsheet of a torus with modulus \( \tau \);

- **The existence of a consistent Hilbert space formulation**

  The partition function \( Z_G \) of the \( G \)-orbifold theory is given by a summation over partition traces \( Z_{gh} \),

  \[
  Z_G(\tau) = \frac{1}{|G|} \sum_{g,h \in G} Z_{gh}(\tau), \quad Z_{gh}(\tau) \equiv \text{Tr}_{\mathcal{H}_h}(gq^{L_0}q^{\bar{L}_0}), \quad (1.1)
  \]

\(^1\)For a more detailed summary of symmetric and asymmetric orbifold constructions, see [22]. The orbifold construction may be carried out either by coseting flat Euclidean space \( \mathbb{R}^n \) by a full orbifold group including the action of translations, or by coseting the flat torus \( \mathbb{T}^n \) by the point group \( P_G \). For symmetric orbifolds, these procedures are equivalent, as is demonstrated in some simple cases in Appendix B. For asymmetric orbifolds, the coseting procedure starting from the torus is taken as a definition.
where \( q \equiv \exp\{2\pi i \tau \} \) and \( |G| \) is the order of \( G \). The key assumption in this principle is that a consistent Hilbert space formulation exists in which \( \mathcal{H}_h \) is the Hilbert space of the sector twisted by \( h \), and the group elements \( g \in G \) have a consistent operator realization in each of these twisted Hilbert spaces.

- **Modular Covariance of partition traces**

Modular invariance of the partition function \( Z_G \) is guaranteed by the stronger condition of modular covariance of each of the partition traces,

\[
Z^g_h(\tau + 1) = Z^{g^{-1}}_h(\tau) \\
Z^g_h(-1/\tau) = Z^{h^{-1}}_g(\tau)
\]

or more generally,

\[
Z^g_h(a\tau + b/c\tau + d) = Z^{g^{-1}h^{-1}}_{g^{-1}}(\tau) \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})
\]  

(1.2)

Modular covariance of the partition traces implies that \( Z^g_h(\tau + n_h) = Z^g_h(\tau) \) for any element \( h \) of order \( n_h \). This relation, in turn, is equivalent to the familiar requirement of level matching,

\[
L_0 - \bar{L}_0 \in \mathbb{Z}
\]

(1.4)

when acting on the Hilbert space \( \mathcal{H}_h \) of states twisted by an element \( h \) of order \( n_h \).

Finally, the transformation in (1.3) given by \( a = d = -1 \) and \( b = c = 0 \) belongs to the center of \( SL(2, \mathbb{Z}) \), reverses the orientation of both \( A \) and \( B \) cycles, and corresponds to charge conjugation symmetry.

When the orbifold group \( G \) acts on genuine well-defined fields, and a functional integral formulation is available, the above principles may be deduced from the functional integral formulation using standard quantum field theory methods. In particular, a simple change of variables in the functional integral will imply the the modular covariance relations for the partition traces expressed above.

For general asymmetric orbifolds, a proper functional integral formulation may be lacking and it may not be possible to derive the above relations from first principles. In such theories, the above principles will simply be postulated as necessary and sufficient conditions for the existence of physically viable asymmetric orbifold models. It has been argued in [9] that level matching suffices to insure the full modular covariance (1.2) of the theory. This condition is not always sufficient to guarantee the full modular covariance of the partition traces. In this paper some examples will be presented where this lack of modular covariance conflicts with the existence of a proper Hilbert space interpretation.
1.2 Algorithm for the construction of asymmetric orbifolds

In this paper, asymmetric orbifold models based on chiral shifts and reflections (such as arise in the constructions of [17]) are examined in detail. Special attention is devoted to a proper construction of states and operators in the untwisted and twisted sectors of Hilbert space and to the modular covariance of the partition traces. Models with only chiral shifts are constructed first and shown to satisfy modular covariance, at least in some critical dimensions. In models involving chiral reflections, difficulties are found with modular covariance, even though the level matching condition (1.4) is satisfied. These problems are traced to the precise definition of the chiral reflection operators in Hilbert space. In particular, it is shown that a chiral reflection of order 2 leads to conflicts with the modular transformation $\tau \rightarrow -1/\tau$, but that a chiral reflection of order 4 is consistent with modular invariance.

Generally, in models satisfying level matching, it had been expected that the modular orbit method would yield all partition traces $Z_{gh}^g(\tau)$ from applying the modular covariance condition (1.2) to the partition traces in the untwisted sector $Z_{g1}^g(\tau)$. The models with chiral reflections examined here show that the method must be applied with some care: modular transformations may not generate all partition traces from the untwisted sector, and even when they do, the objects they generate may not have a proper interpretation as partition traces in a twisted sector.

While our discussions later in the paper will be in terms of specific examples based on shifts and twists, the methods developed there may be recast in terms of a recursive algorithm, which we shall now summarize.

1. It is assumed that all the group elements $g \in G$ in the orbifold group have well-defined operator realizations on the untwisted Hilbert space $H_1$. The partition traces $Z_{g1}^g$ in the untwisted sector for all $g \in G$ are then well-defined and may be calculated.

2. Modular transformations applied to the partition traces $Z_{h1}^h$ yield $Z_{1h}^h$. The Hilbert space representation of $Z_{h1}^h(\tau) = \text{Tr}_{H_h} q^{L_0} \bar{q}^{L_0}$ may be viewed as a spectral density function for the Hilbert space $H_h$, which yields the conformal spectrum, including the multiplicities for all states in $H_h$. This essentially determines $H_h$.

3. The further application of modular transformations to $Z_{h1}^h$ yields $Z_{hn}^h$, from which the action of the operators $h^n$ on $H_h$ may be deduced.

4. Combining the knowledge of the action of operators $h^n$ on $H_h$ with the properties of the symmetric operators in the associated symmetric orbifold theories has allowed us, in all cases considered here, to construct also the action of elements $g$ on $H_h$,
even when \( g \) is not a power of \( h \). This last step permits us to calculate the remaining partition traces \( Z^g_h \).

The twisted Hilbert space \( \mathcal{H}_h \) which emerges from the above construction is of the following form. Let \( G \) be the asymmetric orbifold group whose elements consist of pairs \( h = (h_L; h_R) \), where \( h_L \) and \( h_R \) act respectively on the left and the right sectors. The left and right elements themselves span groups \( G_L \) and \( G_R \) and \( G \) may be viewed as a subgroup of the product \( G_L \times G_R \). The starting point is the two symmetric orbifold theories with symmetric orbifold groups \( G_L \) and \( G_R \) consisting of pairs \((h_L; h_L)\) and \((h_R; h_R)\) respectively. Let \( \mathcal{H}_{h_L}(p_L) \) and \( \mathcal{H}_{h_R}(p_R) \) be the Hilbert spaces of the sectors twisted by \((h_L, h_L)\) and \((h_R, h_R)\) in the symmetric theories. Here, we denote by \( p_L \in M_L \) and \( p_R \in M_R \) any additional labels of the blocks, such as for example the internal loop momenta. \( M_L \) and \( M_R \) may be viewed as the degeneracies of the ground states in the symmetric theories. Then the Hilbert space of the \( h \)-twisted sector in the asymmetric theory is of the form

\[
\mathcal{H}_h = \bigoplus_{(p_L, p_R) \in \mathcal{I}} \left( \mathcal{H}_{h_L}(p_L) \otimes \mathcal{H}_{h_R}(p_R) \right) \tag{1.5}
\]

Here, \( \mathcal{I} \) is a pairing or selection rule, and we keep only chiral oscillators in \( \mathcal{H}_{h_L}(p_L) \) and \( \mathcal{H}_{h_R}(p_R) \). The set \( \mathcal{I} \) is usually strictly smaller than the set of all \( M_L \times M_R \) possible choices for \((p_L, p_R)\). It is determined by modular transformations from the asymmetric traces in the untwisted sector. In [12], it was proposed to obtain the partition function of the asymmetric theory by taking square roots of suitably extended symmetric theories. Our prescription is a significant departure from this, since it avoids square roots altogether and replaces them by selection rules after a suitable chiral splitting.

The remainder of this paper is organized as follows. In section 2, a brief summary is given of the circle theory. In section 3, orbifolds generated by chiral shifts only are constructed. In section 4, orbifolds with chiral reflections only are investigated and subtle issues of modular invariance are addressed and solved. In section 5, the results of \( \S 3 \) and \( \S 4 \) are combined and orbifolds with both chiral shifts and reflections are solved, and the effects of adding worldsheet fermions are included. In section 6, general rules for the partition function in higher genus are formulated in terms of chiral splitting and a summation over pairings of chiral blocks. Finally, in appendix \( \S A \), the construction of symmetric orbifolds is briefly reviewed; in \( \S B \), the equivalence is demonstrated for symmetric orbifolds between orbifolding the full line by the full orbifold group and orbifolding the circle by the point group; and in \( \S C \), useful \( \vartheta \)-function identities are collected.
2 The circle theory

We begin by recalling some basic facts about the $S^1$ theory, mainly for normalizations and conventions. The torus $\Sigma$ with modulus $\tau = \tau_1 + i\tau_2$, $\tau_2 > 0$, is parametrized by $z$, with the identifications $z \sim z + 1$ and $z \sim z + \tau$. It is equipped with the metric $ds^2 = \tau_2^{-1} dzd\bar{z}$.

The action $S[x]$ for a theory of a single bosonic scalar field $x$ is normalized to be

$$S[x] = \frac{1}{4\pi\ell^2} \int_\Sigma d^2z \partial x \partial \bar{x},$$  \hspace{1cm} (2.1)$$

where $\ell$ is the string scale.\footnote{The parameter $\ell$ is related to the Regge slope parameter $\alpha'$ by $2\ell^2 = \alpha'$. Customary conventions used in the literature are as follows. In [16], the Regge slope parameter $\alpha'$ is exhibited explicitly; in [24] and [7], one sets $\ell = 1$; in [6], one sets $\ell = 1/2$ instead.}

In the $S^1$ theory, the field $x(z)$ takes values in a circle of radius $R$, and we have the identification $x \sim x + 2\pi R$. The evaluation of the functional integral over all instanton sectors defined by the boundary conditions $x(z + 1) = x(z) + 2\pi m_1 R$ and $x(z + \tau) = x(z) + 2\pi m_2 R$ gives the well-known partition function

$$Z_{S^1_R}(\tau) = \frac{R}{\ell\sqrt{2\tau_2} |\eta(\tau)|^2} \sum_{m_1,m_2 \in \mathbb{Z}} \exp\left\{ -\frac{\pi R^2}{2\ell^2 \tau_2} |m_1 \tau - m_2|^2 \right\}$$ \hspace{1cm} (2.2)$$

Here, the argument of the exponential is the action of the corresponding instanton solution, and the prefactor combines the contributions of the quantum fluctuations and the zero mode integration. The partition function $Z_{S^1_R}$ can be recast in Hamiltonian language by a Poisson resummation in $m_2$

$$Z_{S^1_R}(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{m_1,m_2 \in \mathbb{Z}} q^{\frac{1}{2}p_L^2} q^{\frac{1}{2}p_R^2}, \hspace{1cm} q \equiv e^{2\pi i \tau},$$ \hspace{1cm} (2.3)$$

where the left and the right (dimensionless) momenta $p_L$ and $p_R$ are given by

$$p_L = \frac{\ell}{R} m_2 - \frac{R}{2\ell} m_1, \hspace{1cm} p_R = \frac{\ell}{R} m_2 + \frac{R}{2\ell} m_1, \hspace{1cm} m_1, m_2 \in \mathbb{Z}$$ \hspace{1cm} (2.4)$$

In particular $(p_L, p_R)$ belongs to an even Lorentzian lattice $p_L^2 - p_R^2 = -2m_1 m_2 \in 2\mathbb{Z}$.

Henceforth, we consider the $S^1$ theory at the self-dual radius $R^2 = 2\ell^2$. The left and right momenta simplify to

$$\sqrt{2} p_L = m_2 - m_1 = n_L, \hspace{1cm} \sqrt{2} p_R = m_2 + m_1 = n_R,$$ \hspace{1cm} (2.5)$$
with \( n_L, n_R \in \mathbb{Z} \) and \( n_L + n_R \in 2\mathbb{Z} \). The momentum summations may be carried out in terms of \( \vartheta \)-functions. At the self-dual radius, it is convenient to cast the \( \vartheta \)-functions results in terms of the \( \zeta \)-function, defined by

\[
\zeta[\alpha|\beta](\tau) \equiv \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(n+\alpha)^2} e^{2\pi i n \beta} = \frac{\vartheta[\alpha|\beta](0, 2\tau)}{\eta(\tau)} e^{-2\pi i \alpha \beta} \tag{2.6}
\]

Various transformation properties of its arguments are listed in Appendix C. One finds,

\[
Z_{S^1_R}^\text{st}(\tau) = |\zeta[0|0](\tau)|^2 + |\zeta[1|2](\tau)|^2 \tag{2.7}
\]

where the first term arises from the contributions with \( n_L, n_R \in 2\mathbb{Z} \) and the second term from contributions with \( n_L, n_R \in 2\mathbb{Z} + 1 \).

The Hilbert space \( \mathcal{H} \) of the circle theory can be built from chiral sectors as follows

\[
\mathcal{H} = \bigoplus_{n_L + n_R \in 2\mathbb{Z}} \left( \bigotimes_{n_j = 1}^\infty x_{-n_1} \cdots x_{-n_k}|n_L \rangle_L \bigotimes \bigotimes_{n_j = 1}^\infty \bar{x}_{-n_1} \cdots \bar{x}_{-n_k}|n_R \rangle_R \right) \tag{2.8}
\]

Here, the states \( |n_L \rangle_L, |n_R \rangle_R \) are momentum eigenstates, characterized by their quantum numbers \( n_L, n_R \) as defined in (2.5), and \( x_{n_j}, \bar{x}_{n_j} \) are oscillators for each chiral sector. Then \( Z_{S^1_R} \) becomes

\[
Z_{S^1_R}^\text{st}(\tau) = \text{Tr}_\mathcal{H}(q^L q^\dagger L_0) \tag{2.9}
\]

with the following definitions of the Virasoro generators,

\[
L_0 = -\frac{1}{24} + \frac{1}{2} p_L^2 + \sum_{n=1}^{\infty} n x_{-n} x_n \quad \bar{L}_0 = -\frac{1}{24} + \frac{1}{2} p_R^2 + \sum_{n=1}^{\infty} n \bar{x}_{-n} \bar{x}_n \tag{2.10}
\]

Finally, at any radius \( R \), the circle theory has a chiral \( \hat{U}(1)_L \times \hat{U}(1)_R \) symmetry, generated by the chiral currents \( J_L^3 = \partial x \) and \( J_R^3 = \partial \bar{x} \). At the self-dual radius \( R^2 = 2\ell^2 \), the symmetry is enhanced to \( SU(2)_L \times SU(2)_R \), which arises due to the existence of the extra currents \( J_L^\pm = \exp\{\pm i\sqrt{2} x_+\} \) and \( J_R^\pm = \exp\{\pm i\sqrt{2} x_-\} \), where \( x_\pm \) denote the chiral parts of the field \( x \). The latter symmetry will be exploited in section 4.2.

## 3 Orbifolds defined by a chiral shift

The operator \( s \) realizing a shift \( x \to x + \pi R \) by a half-circumference commutes with the oscillators \( x_n \) and \( \bar{x}_n \) and acts on the momentum ground states by \( s |n_L \rangle_L \otimes |n_R \rangle_R = e^{i\pi (p_L + p_R) R / (2\ell)} |n_L \rangle_L \otimes |n_R \rangle_R \). At the self-dual radius \( R^2 = 2\ell^2 \), this becomes simply

\[
s |n_L \rangle_L \otimes |n_R \rangle_R = e^{i\pi (n_L + n_R) / 2} |n_L \rangle_L \otimes |n_R \rangle_R \tag{3.1}
\]
This form naturally permits chiral splitting into actions of left and right chiral shift operators $s_L$ and $s_R$, which also commute with the oscillators $x_n$ and $\tilde{x}_n$. Their action on the ground states is defined by

$$s_L |n_L\rangle_L \otimes |n_R\rangle_R = e^{\frac{i\pi n_L}{2}} |n_L\rangle_L \otimes |n_R\rangle_R$$
$$s_R |n_L\rangle_L \otimes |n_R\rangle_R = e^{\frac{i\pi n_R}{2}} |n_L\rangle_L \otimes |n_R\rangle_R$$

(3.2)

Clearly these definitions reproduce $s = s_L s_R$. Note that each operator $s_L$ and $s_R$ has order 4 while their product $s$ has order 2 in view of the fact that $n_L + n_R \in 2\mathbb{Z}$. In this section, orbifolds generated by either $s_R$ or $s_R^2$ will be considered.

### 3.1 The $s_R$ models

Recall that the partition function of the orbifold theory defined by a finite abelian group $G$ is given (1.1), where $|G|$ is the order of $G$, and $\mathcal{H}_h$ is the Hilbert space of the twisted sector defined by $h$. As is reviewed in Appendix A, in the case of the symmetric shift $s = s_L s_R$, the Hilbert space in the twisted sector corresponding to an element $s^{-a}$ is generated by oscillators from the ground states $|n_L + \frac{a}{2}\rangle \otimes |n_R - \frac{3a}{2}\rangle$.

In the asymmetric orbifold theory generated by $s_R$, the Hilbert space of the sector twisted by the element $(s_R)^{-a}$ is defined to be the space generated by applying the oscillators $x_n$ and $\tilde{x}_n$ to the ground states of the form

$$|n_L\rangle_L \otimes |n_R - \frac{a}{2}\rangle_R$$

$n_L + n_R \in 2\mathbb{Z}$

(3.3)

The untwisted sector may simply be viewed as the sector twisted by $(s_R)^{-a}$ for $a = 0$, an observation that permits us to treat all sectors at once. This construction is consistent with the fact that $s_R$ is of order 4, since a shift $a \rightarrow a + 4$ may be compensated by relabeling the states by $n_R \rightarrow n_R + 2$, a transformation that preserves the condition $n_L + n_R \in 2\mathbb{Z}$.

The action of $(s_R)^b$ on a state in the sector twisted by $(s_R)^{-a}$ is given by

$$(s_R)^b |n_L\rangle_L \otimes |n_R - \frac{a}{2}\rangle_R = \exp \left\{ \frac{i\pi b}{2} (n_R - \frac{a}{2}) \right\} |n_L\rangle_L \otimes |n_R - \frac{a}{2}\rangle_R$$

(3.4)

with, as always, the constraint that $n_L + n_R \in 2\mathbb{Z}$.

The partition traces for a single scalar field are easily calculated. The oscillator contributions are all equal to those of the untwisted sector and result in the familiar $\eta$-function factors. The momentum dependence may be read off from the momentum assignments of the states. Abbreviating $\mathcal{Z}^{g_h} = \mathcal{Z}^{b_a}$ when $h = (s_R)^{-a}$ and $g = (s_R)^b$, the partition traces become,

$$\mathcal{Z}^{b_a} (\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{n_L + n_R \in 2\mathbb{Z}} e^{b \frac{i\pi}{2} (n_R - \frac{a}{2})} q^{\frac{1}{4} n_L^2} q^{\frac{1}{4} (n_R - \frac{a}{2})^2}$$

(3.5)
The sums over \( n_L \) and \( n_R \) are conveniently expressed in terms of the \( \zeta \)-function defined in (2.6). The constraint \( n_L + n_R \in 2\mathbb{Z} \) in (3.4) may be solved explicitly and gives rise to two parts: the first in which \( n_L \) and \( n_R \) are independent even integers (first term on rhs) and the second in which they are independent odd integers (second term on rhs),

\[
Z_{b,a}^d(\tau) = e^{-\frac{i\pi ab}{4}} \left( \zeta[0|0](\tau) \frac{a}{4} \frac{b}{2} (0, 2\tau) + \zeta[1|0](\tau) \zeta[1|0](\tau) \right) (3.6)
\]

From the partition traces of the \( s_R \)-twisted theory of a single boson it is straightforward to obtain the partition traces of a theory of \( d \) bosons taking values in a square torus of all self-dual radii and where \( s_R \) acts as a right chiral shift simultaneously on all directions. As the \( d \) bosons in this model are independent of one another, the corresponding partition traces are simply given by \( (Z_{b,a}^d)^d \).

### 3.1.1 Interpretation in terms of chiral blocks constructions

With the extension to higher loops in mind, it is useful to interpret the previous formulas in the following manner. We would like to view the traces \( Z_{b,a}^d(\tau) \) of the \( s_L \) theory as built from the chiral blocks of the symmetric \( s = (s_L, s_R) \) theory and the untwisted theory, with a suitable pairing. Now the partition function for the symmetric \( s \) theory at the self-radius \( R^2 = 2\ell^2 \) is simply given by the partition function for the circle theory at \( R^2 = \ell^2 / 2 \) and hence given by [10]

\[
Z_{R^2=\ell^2/2}^\text{circle}(\tau) = \frac{1}{2} \sum_{\epsilon,\delta} Z_{\epsilon,\delta}^\text{circle}(\tau)
\]

(3.7)

where \([\epsilon|\delta]\) runs over all half-characteristics, and the “trace” \( Z_{\epsilon,\delta}^\text{circle}(\tau) \) is given by

\[
Z_{\epsilon,\delta}^\text{circle}(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{\gamma \in \{0, \frac{1}{2}\}} |\vartheta(\gamma + \frac{1}{2} \epsilon|\delta)(0, 2\tau)|^2
\]

(3.8)

On the other hand, the partition function of the self-dual circle model in arbitrary genus \( h \) is given by

\[
Z_{R^2=2\ell^2}^\text{circle}(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{\gamma \in \{0, \frac{1}{2}\}^h} |\vartheta(\gamma|0)(0, 2\tau)|^2
\]

(3.9)

where \( \tau \) is now viewed as the period matrix and \( \eta(\tau) \) as the chiral boson partition function at genus \( h \). The formula (3.6) suggests that, for fixed \( b, a \in \mathbb{Z}_4 = \{0, 1, 2, 3\} \), the chiral
blocks of the $s$ theory and the untwisted theory can be defined respectively as

$$Z_{R^2=\ell^2/2}^{s-\text{theory}} = \frac{1}{\eta(\tau)} \theta[\gamma + \frac{1}{4} a | \beta |](0, 2\tau)$$

$$Z_{R^2=\ell^2/2}^{\text{circle}} = \frac{1}{\eta(\tau)} \theta[\gamma | 0](0, 2\tau)$$

(3.10)

where $\gamma \in \{0, \frac{1}{2}\}$ and we have expressed $b$ as

$$b = 2(\beta + c), \quad \beta \in \{0, \frac{1}{2}\}, \quad c \in \{0, 1\}.$$  

(3.11)

With this choice of phases for the chiral blocks arising from the symmetric theories, the blocks $Z^b_a(\tau)$ of the asymmetric $s_L$ theory arise by pairing them with the matrix

$$K_{\gamma \bar{\gamma}} = \delta_{\gamma \bar{\gamma}} e^{2\pi i c \gamma}$$

(3.12)

so that we indeed recover the earlier partition traces,

$$Z^b_a(\tau + 1) = e^{i\pi a^2/8} Z^a_a(\tau)$$

$$Z^b_a(-1/\tau) = e^{-i\pi ab/4} Z^a_{-b}(\tau)$$

(3.15)

Note that the exponent $a^2/16$ is as expected in the $\tau \rightarrow \tau + 1$ transformation law because the ground state in the sector twisted by $(s_R)^{\pm a}$ has conformal dimension $a^2/16 \mod 1$. As a result, the operator $s_R$ in the twisted sector actually has order 16 with $s_R^8 = -1$. To properly restore the order of $s_R$ to be 4 as it was in the untwisted sector, the dimension of the torus (see the last paragraph of the preceding section) must be divisible by 4.

More critically, the orbifold model for a single boson does not satisfy the modular covariance requirement of (1.2). Partition traces with full modular invariance are obtained only when the dimension of the torus satisfies $d \in 16\mathbb{N}$. 

3.1.2 Modular invariance of the $s_R$ model for $d \in 16\mathbb{N}$ bosons

The modular covariance properties (1.2) of the partition traces follow from the modular transformation laws (C.11) for the functions $\zeta[a | \beta](\tau)$ and the relations (C.9) and (C.12). One finds,

$$Z^b_a(\tau + 1) = e^{i\pi a^2/8} Z^a_a(\tau)$$

$$Z^b_a(-1/\tau) = e^{-i\pi ab/4} Z^a_{-b}(\tau)$$

(3.15)

Note that the exponent $a^2/16$ is as expected in the $\tau \rightarrow \tau + 1$ transformation law because the ground state in the sector twisted by $(s_R)^{\pm a}$ has conformal dimension $a^2/16 \mod 1$. As a result, the operator $s_R$ in the twisted sector actually has order 16 with $s_R^8 = -1$. To properly restore the order of $s_R$ to be 4 as it was in the untwisted sector, the dimension of the torus (see the last paragraph of the preceding section) must be divisible by 4.

More critically, the orbifold model for a single boson does not satisfy the modular covariance requirement of (1.2). Partition traces with full modular invariance are obtained only when the dimension of the torus satisfies $d \in 16\mathbb{N}$. 

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3.1.3 Asymmetry of the $s_R$ partition function for $d = 16$ bosons

Often, an orbifold constructed from the action of an asymmetric orbifold group still possesses a symmetric partition function, i.e. $Z(\tau) = \overline{Z(\tau)}$. It will be verified below that the orbifold theory constructed from 16 bosons and the asymmetric group action of $s_R$ is in fact asymmetric and satisfies $Z(\tau) \neq \overline{Z(\tau)}$. The starting point is the partition function defined by the partition traces of (3.6),

$$Z(\tau) = \frac{1}{4} \sum_{a,b=0,1,2,3} (Z_{b_a(\tau)})^{16} \tag{3.16}$$

The most convenient study of the asymmetry issue is in terms of $\vartheta$-functions of modulus $\tau$. To this end, $Z$ is first expressed in terms of $\vartheta$-functions of modulus $2\tau$, which are then converted into $\vartheta$-functions with modulus $\tau$ by using the doubling formulas. To simplify this calculation, note that a shift $b \rightarrow b + 2$ produces a relative sign between the two terms in the parenthesis of (3.6). It is convenient to restrict the range of $b = 0,1$ and isolate the summation over the shifts $b \rightarrow b + 2$ using a new variable $c = 0,1$. One obtains,

$$\left(Z^{b+2c_a(\tau)}\right)^2 = \frac{1}{|\eta(\tau)|^4} \left(\vartheta(0|0) \vartheta\left[\frac{a+b}{4}\right] + (-)^c \vartheta\left[\frac{1}{2}\right] \vartheta\left[\frac{a+b}{4}\right]\right)^2$$

(0, 2\tau) \tag{3.17}

the partition function may be expressed as follows, To compute the above partition traces in terms of $\vartheta$-functions with argument $\tau$, $\vartheta_i(0,\tau)$, with $i = 2,3,4$, we use the doubling formula for $\vartheta$-functions given in Appendix C. One finds (here $a,c = 0,1$)

$$\left(Z^{2c_2a(\tau)}\right)^2 = \frac{1}{2|\eta(\tau)|^4}\left(|\vartheta_3|^4 + (-)^a |\vartheta_4|^4 + (-)^c |\vartheta_2|^4\right)$$

$$\left(Z^{1+2c_2a(\tau)}\right)^2 = \frac{1}{2|\eta(\tau)|^4}\left(|\vartheta_3|^2 + (-)^a |\vartheta_4|^2\right) \bar{\vartheta}_3 \bar{\vartheta}_4$$

$$\left(Z^{2c_1(\tau)}\right)^2 = \left(Z^{2c_3(\tau)}\right)^2 = \frac{1}{2|\eta(\tau)|^4}\left(-i\vartheta_4^2 + (-)^c \vartheta_2^2\right) \bar{\vartheta}_2 \bar{\vartheta}_4$$

$$\left(Z^{1+2c_1(\tau)}\right)^2 = \left(Z^{1+2c_3(\tau)}\right)^2 = \frac{1}{2|\eta(\tau)|^4}\left(-i\vartheta_4^2 + (-)^c \vartheta_2^2\right) \bar{\vartheta}_2 \bar{\vartheta}_4$$

(3.18)

Combining all contributions, one obtains

$$Z = \frac{1}{2\eta(\tau)^{16}} \left\{(|\vartheta_3|^4 + |\vartheta_4|^4 + |\vartheta_2|^4)^8 + (|\vartheta_3|^4 + |\vartheta_4|^4 - |\vartheta_2|^4)^8 + (|\vartheta_3|^4 - |\vartheta_4|^4 + |\vartheta_2|^4)^8 + (|\vartheta_3|^4 - |\vartheta_4|^4 - |\vartheta_2|^4)^8 + 2(|\vartheta_3^2 + |\vartheta_4^2|)^8 + (|\vartheta_3^2 - |\vartheta_4^2|)^8 (\vartheta_3 \vartheta_4)^8 + 2(|\vartheta_3^2 - |\vartheta_4^2|)^8 + (|\vartheta_3^2 + |\vartheta_4^2|)^8 (\vartheta_3 \bar{\vartheta}_2)^8 + 2(|\vartheta_3^2 - |\vartheta_4^2|)^8 + (|\vartheta_3^2 + |\vartheta_4^2|)^8 (\bar{\vartheta}_3 \vartheta_2)^8 + 2(|\vartheta_3^2 - |\vartheta_4^2|)^8 + (|\vartheta_3^2 + |\vartheta_4^2|)^8 (\bar{\vartheta}_3 \bar{\vartheta}_2)^8 \right\}$$

(3.19)
Using (C.4), it is straightforward to verify that $Z$ is modular invariant.

The partition function $Z$ for the 16-dimensional orbifold by $(1, s_R)$ is asymmetric. To see this, it suffices to show that $Z - \bar{Z} \neq 0$. The contribution of the first 4 terms in $Z$ is manifestly symmetric and cancels out of $Z - \bar{Z}$. The remaining terms may be simplified using the Jacobi identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$, and expressed as a function of a single homogeneous variable $t \equiv \vartheta_4^4/\vartheta_3^4$,

$$Z - \bar{Z} = \frac{15|\vartheta_3|^3}{2^7|\eta(\tau)|^{32}}(t - \bar{t})(1 - t\bar{t})(1 - t - \bar{t})(t + \bar{t} - t\bar{t})$$  \hspace{1cm} (3.20)

Manifestly, this is not zero and the model has asymmetric conformal weights.

### 3.2 The $s^2_R$ models

The $s^2_R$ can be easily derived as a subsector of the $s_R$ models, consisting of the blocks $Z^b_a(\tau)$ with $a, b = 0, 2$. The Hilbert space interpretation of all the blocks is a direct consequence of the Hilbert space interpretation provided for the $s_R$ theory. Because fewer blocks appear in the $s^2_R$ theory, the requirement of modular invariance block by block is less restrictive. Explicitly,

$$Z^0_0(\tau) = |\zeta[0|0]\zeta[1^20]\bar{0}|^2 + |\zeta[1^20]\zeta[0|0]\bar{1}|^2$$

$$Z^2_0(\tau) = |\zeta[0|0]\zeta[1^20]\bar{0}|^2 - |\zeta[1^20]\zeta[0|0]\bar{1}|^2$$

$$Z^0_2(\tau) = \zeta[0|0]\zeta[2^20]\bar{0} + \zeta[2^20]\zeta[0|0]\bar{2}$$

$$Z^2_2(\tau) = e^{-i\pi}(\zeta[0|0]\zeta[1^20]\bar{1} - \zeta[1^20]\zeta[0|0]\bar{1})$$  \hspace{1cm} (3.21)

Level matching requires the number $d$ of bosonic fields to be a multiple of 4, in which case we have modular covariance of the partition traces. The resulting partition function is

$$Z(\tau) = \frac{1}{2} \sum_{\sigma = \pm 1} (|\zeta[0|0]\zeta[1^20]|^2 + \sigma|\zeta[2^20]|^2) + \frac{1}{2} \sum_{\sigma = \pm 1} (\zeta[0|0]\zeta[1^20] + \sigma\zeta[1^20]\zeta[0|0])^d$$  \hspace{1cm} (3.22)

for $d \in 4\mathbb{N}$. We observe that, although the orbifold group consists of asymmetric elements, the resulting partition function $Z$ turns out to be symmetric in this case. For the minimal dimension of $d = 4$, one may alternatively express the partition function in terms of $\vartheta$-function with modulus $\tau$. The result is

$$Z(\tau) = \frac{1}{2|\eta|^8} \left\{ |\vartheta_2|^8 + |\vartheta_3|^8 + |\vartheta_4|^8 \right\}$$  \hspace{1cm} (3.23)

It is easy to see that this partition function equals the partition function for 8 decoupled Majorana fermions with $SO(8)$ invariant action.
4 Orbifolds defined by a chiral reflection

The symmetric reflection $r : x(z) \to -x(z)$ is realized as a unitary operator on the untwisted Hilbert space $\mathcal{H}$ by $r|n_L\rangle_L \otimes |n_R\rangle_R = \pm| - n_L\rangle_L \otimes | - n_R\rangle_R$, and on the oscillators by $r x_n = -x_n r$, $r \tilde{x}_n = -\tilde{x}_n r$. A chiral reflection $r_L$ should reflect $n_L$ while preserving $n_R$, while a chiral reflection $r_R$ should reflect $n_R$ while preserving $n_L$. Although these actions of the chiral operators $r_L$ and $r_R$ are natural and unique, the splitting $r = r_L r_R$ generally leaves some phases undetermined. Truly chiral operators will be obtained only if the phases associated with $r_L$ only depend upon $n_L$ (and the phases associated to $r_R$ only depend on $n_R$). It may not always be possible to achieve this, in which case the splitting of the operator $r = r_L r_R$ may be viewed as anomalous. Here, models will be sought which are anomaly free. Thus, the following action of $r_L$ and $r_R$ will be postulated,

$$
\begin{align*}
    r_L|n_L\rangle_L \otimes |n_R\rangle_R &= \rho_L(n_L)|-n_L\rangle_L \otimes |n_R\rangle_R \\
    r_R|n_L\rangle_L \otimes |n_R\rangle_R &= \rho_R(n_R)|n_L\rangle_L \otimes |-n_R\rangle_R \\
    r_L x_n &= -x_n r_L \\
    r_L \tilde{x}_n &= \tilde{x}_n r_L \\
    r_R x_n &= x_n r_R \\
    r_R \tilde{x}_n &= -\tilde{x}_n r_R
\end{align*}
$$

(4.1)

where $\rho_L$ and $\rho_R$ are phases which may depend on $n_L$ and $n_R$ respectively. The requirement that $r = r_L r_R$ is a further constraint on the phases : $\rho_L(n_L) \rho_R(n_R) = \pm 1$ whenever $n_L + n_R \in 2\mathbb{Z}$. Furthermore, note that the choice of $\rho_L(n_L)$ and $\rho_R(n_R)$ dictates the order of the chiral operators $r_L$ and $r_R$. It will be shown next that – contrary to naive expectation – the order of the operators $r_L$ and $r_R$ cannot be 2, but must be larger.

4.1 The order of the chiral operator $r_L$

In order for the expression [11] to correspond to a trace of projections onto invariant subspaces, the operators $r_L$ and $r_R$ should have the same order in all twisted sectors. Therefore, it suffices to rule out the order 2 in the untwisted sector. In this sector, the Hilbert space is well-known, and the partition traces for the $r_L$ theory easily computed. The partition trace $Z^1$ is, of course, the same as $Z^0$ in (3.6), and $Z^{r_L}$ may be computed using the definition of $r_L$ given above,

\begin{align*}
    Z^1_1(\tau) &= |\zeta[0|0]^2(\tau) + |\zeta[1\tau/2|0]^2(\tau) \\
    Z^{r_L}_1(\tau) &= \rho_L(0) \zeta[0|1\tau/2|^2(\tau) \zeta[0|0](\tau)
\end{align*}

(4.2)

In the second line above, the insertion of $r_L$ in the trace causes only the states generated from $n_L = 0$ to contribute, and produces a factor $(-)^k$ when acting on the state
\[ x_{-n_1} \cdots x_{-n_k}|n_L\rangle_L. \] Note that the traces in the untwisted sector only involve \( \rho_L(0) \). Under \( \tau \to \tau + 1 \), the partition trace \( Z^{r_L} \) is invariant, as expected.

It will be assumed that the partition traces satisfy modular covariance (possibly when raised to some critical power \( d \)), and it will be shown that the assumption \( r_L^2 = 1 \) leads to a contradiction. Indeed, if \( r_L^2 = 1 \), only the traces \( Z^1_{r_L} \) and \( Z^{r_L}_{r_L} \) in the sector twisted by \( r_L \) remain to be determined. Modular covariance gives,

\[
Z^1_{r_L}(\tau) = Z^{r_L}(1/\tau) = \rho_L(0) \zeta \left[ \frac{1}{4} \right] \left( \zeta[0|0] + \zeta[1/2|0] \right)(\tau)
\]

\[
Z^{r_L}_{r_L}(\tau) = Z^1_{r_L}(\tau + 1) = \rho_L(0) e^{i\pi/8} \zeta \left[ \frac{1}{4} \right] \left( \zeta[0|0] + e^{-i\pi/2} \zeta[1/2|0] \right)(\tau)
\]

Assuming that \( r_L \) is of order two, \( (r_L^2 = 1) \) implies that, \( Z^1_{r_L}(\tau + 2) = \rho_L(0) e^{i\pi/4} \zeta \left[ \frac{1}{4} \right] (\zeta[0|0] - \zeta[1/2|0]) \)(\tau). But this result is clearly distinct from \( Z^1_{r_L}(\tau) \), and hence violates modular invariance and in particular the level matching condition of (4.3), irrespective of the dimension \( d \). Therefore, the order of \( r_L \) cannot be 2. On the other hand, an analogous calculation yields \( Z^1_{r_L}(\tau + 4) = e^{i\pi/2} Z^1_{r_L}(\tau) \). Therefore, \( r_L \) is of order 4 provided that additionally the dimension \( d \) is a multiple of 4 and that \( \rho_L(0)^4 = 1 \).

### 4.2 Chiral reflections of order 4

Henceforth, it will be assumed that \( r_L \) is of order 4, and that the dimension \( d \) is a multiple of 4. In the preceding subsection, it was shown that these conditions are necessary in order to have modular invariance and a consistent Hilbert space formulation. It was also necessary for the factor \( \rho_L(0) \) to be a 4-th root of unity. Since all effects of \( \rho_L(0) \) will disappears from all the partition traces in all dimensions \( d \) which are multiples of 4, we may simply set \( \rho_L(0) = 1 \) without loss of generality. The traces \( Z^1_1 \) and \( Z^{r_L} \) are unchanged from (4.2) but are now considered with \( \rho_L(0) = 1 \). Using modular covariance and charge conjugation symmetry, one has the following traces,

\[
Z^{r_L}(\tau) = Z^1_{r_L}(\tau) = \zeta \left[ \frac{1}{2} \right] \zeta[0|0]|\rangle_L(\tau)
\]

\[
Z^1_{r_L}(\tau) = Z^1_{r_L}(1/\tau) = \zeta \left[ \frac{1}{4} \right] \left( \zeta[0|0] + \zeta[1/2|0] \right)(\tau)
\]

\[
Z^{r_L}_{r_L}(\tau) = Z^1_{r_L}(\tau + 1) = \rho_L(0) e^{i\pi/8} \zeta \left[ \frac{1}{4} \right] \left( \zeta[0|0] + e^{-i\pi/2} \zeta[1/2|0] \right)(\tau)
\]

\[
Z^{r_L}_{r_L}(\tau) = Z^1_{r_L}(\tau + 1) = \rho_L(0) e^{i\pi/8} \zeta \left[ \frac{1}{4} \right] \left( \zeta[0|0] + e^{-i\pi/2} \zeta[1/2|0] \right)(\tau)
\]

Note that the partition traces in the untwisted sector and in the sector twisted by \( r_L^2 \) indeed yield different results.
4.2.1 The sector twisted by \( r_L \)

The next key step is to construct a Hilbert space \( \mathcal{H}_{r_L} \) for the sector twisted by \( r_L \) which can reproduce the traces \( Z^1_{r_L} \) and \( Z^{r_L}_{r_L} \). Here, we encounter the key difficulty of how to determine the pairing between left and right movers in the twisted sector. Specifically, the chiral operator \( r_L \) produces two twisted ground states, which may both be taken to be eigenstates, \( r_L|\pm\rangle_L \sim |\pm\rangle_L \). Therefore, the Hilbert space in the sector twisted by \( r_L \) should be a subspace of the full direct product space of left and right movers,

\[
\mathcal{H}_{r_L} = \{ x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} |+\rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} |n_R\rangle_R, \, n_R \in \mathbb{Z} \} \\
\oplus \{ x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} |-\rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} |n_R\rangle_R, \, n_R \in \mathbb{Z} \}
\]

To determine which subspace \( \mathcal{H}_{r_L} \) is the appropriate choice and how \( r_L \) should act on this space, use will be made of the partition traces \( Z^1_{r_L} \) and \( Z^{r_L}_{r_L} \), which were obtained earlier by modular covariance.

The partition trace \( Z^1_{r_L} \) plays the role of a spectral density, as each state contributes a factor \( q^{L_0} \bar{q}^{\bar{L}_0} \) times its multiplicity factor. The spectral density (for a theory of bosons only) suffers no accidental cancellations and may be used as an accurate guide for how left and right chiralities are to be paired against one another. The expansion of \( Z^1_{r_L} \) in powers of \( q \) and \( \bar{q} \) suggests defining \( \mathcal{H}_{r_L} \) by the following pairing,

\[
\mathcal{H}_{r_L} = \{ x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} |+\rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} |n_R\rangle_R, \, n_R \in 2\mathbb{Z} \} \\
\oplus \{ x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} |-\rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} |n_R\rangle_R, \, n_R \in 2\mathbb{Z} + 1 \}
\]

The operator theoretic trace \( \text{Tr}_{\mathcal{H}_{r_L}}(q^{L_0} \bar{q}^{\bar{L}_0}) \) agrees then with \( Z^1_{r_L}(\tau) \), up to a factor of \( e^{i\pi/8} \). Taking the dimension \( d \) to be a multiple of 16, complete agreement is obtained.

More generally, the partition traces \( Z^b_{r_L} \), with \( b \in \mathbb{Z} \) play the role of a spectral density weighed by the eigenvalues of the operator \( r^b_L \) on the subspace \( \mathcal{H}_{r_L} \), as each state contributes a factor \( r^b_L q^{L_0} \bar{q}^{\bar{L}_0} \) times its multiplicity factor. Clearly, when \( b \) is not a multiple of 4, this quantity may suffer from accidental cancellations between states with identical conformal weight but different \( r^b_L \) eigenvalue. Therefore, this partition trace will not, in general, determine the action of \( r^b_L \) in a unique manner, and one will have to restrict to searching for internally consistent choices. Taking the operator \( r_L \) to have order 4 on \( \mathcal{H}_{r_L} \), one has the following natural assignment,

\[
r_L |+\rangle_L \otimes |n_R\rangle_R = |+\rangle_L \otimes |n_R\rangle_R, \quad n_R \in 2\mathbb{Z} \\
r_L |-\rangle_L \otimes |n_R\rangle_R = e^{-i\frac{1}{2}\pi} |-\rangle_L \otimes |n_R\rangle_R, \quad n_R \in 2\mathbb{Z} + 1
\]
and the usual commutation relations with the oscillators. With this, the evaluation of all traces in the sector twisted by $r_L$ may be completed, and one finds for $b \in \mathbb{Z}$,

$$Z_{r_L}^{r_L}(\tau) = \zeta^{\frac{1}{4}b^2} \left( \zeta[0|0] + e^{i\pi b/2} \zeta[1|0] \right) (\tau) \quad (4.8)$$

For $b = 0, 1$, this formula indeed reproduces the result of (4.3), (which was obtained by modular transformations from partition traces in the untwisted sector) up to an overall factor of $e^{-i\pi b/8}$. Therefore, the Hilbert space formulation of the operator $r_L$ in the sector twisted by $r_L$, given above, will be consistent with modular covariance only if the dimension $d$ is a multiple of 16.

### 4.2.2 The sector twisted by $r^3_L$

Charge conjugation symmetry, represented by the operator $C = S^2$, ensures that the partition traces behave naturally under the reversal of orientation of the homology cycles, and implies the following relation on the partition traces, $Z_{g,h}^g(\tau) = \pm Z_{g^{-1},h^{-1}}^g(\tau)$. As the dimension is always assumed to be even, the $\pm$ factor is immaterial and will be omitted.

The charge conjugation relation may be used to investigate the partition traces and the Hilbert space in the sector twisted by $r^3_L = r_L^{-1}$. Applying charge conjugation to (4.8), it is readily derived that for $b \in \mathbb{Z}$,

$$Z_{r^3_L}^{r^3_L}(\tau) = \zeta^{\frac{1}{4}b^2} \left( \zeta[0|0] + e^{-i\pi b/2} \zeta[1|0] \right) (\tau) \quad (4.9)$$

Notice the difference in the relative phase factor between the two terms in the parenthesis above when $b$ is odd. As a result of this phase difference the structure of the Hilbert space $H_{r^3_L}$ is very close to that of $H_{r_L}$ but not identical, as the operator $r_L$ has a different action on both spaces. Therefore, extra care is needed in constructing also this space correctly. The two degenerate ground states in the sector twisted by $r^3_L$ will be denoted by $|\pm \rangle\rangle_{r^3_L}$, where the superscript $C$ stands for charge conjugation. The Hilbert space is

$$H_{r^3_L} = \{ x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} |+\rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} |n_R\rangle, n_R \in 2\mathbb{Z} \}$$

$$\oplus \{ x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} |-\rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} |n_R\rangle, n_R \in 2\mathbb{Z} + 1 \} \quad (4.10)$$

and the analysis of the partition traces in (4.9) leads to the following definition of the action of the operator $r_L$ on $H_{r^3_L}$,

$$r_L |+\rangle_L \otimes |n_R\rangle_R = |+\rangle_L \otimes |n_R\rangle_R, \quad n_R \in 2\mathbb{Z}$$

$$r_L |-\rangle_L \otimes |n_R\rangle_R = e^{\frac{1}{2}i\pi} |-\rangle_L \otimes |n_R\rangle_R, \quad n_R \in 2\mathbb{Z} + 1 \quad (4.11)$$
and the usual commutation relations with the oscillators. With these assignments, all partition traces in $\mathcal{H}_{r_L^3}$ are consistent with modular covariance and the values of the remaining partition traces $Z_{r_L^3}^{r_L^1} = Z_{r_L^1}^{r_L^3}$.

### 4.2.3 The sector twisted by $r_L^2$

It remains to analyze the sector twisted by $r_L^2$. Since $r_L$ is not of order 2 but rather 4, the Hilbert space $\mathcal{H}_{r_L^2}$ in the sector twisted by $r_L^2$ is expected to be distinct from the untwisted Hilbert space. The starting point is the partition trace with $r_L^2$ inserted in the untwisted sector. The eigenvalues of $r_L^2$ in the untwisted sector are deduced from (4.1),

$$r_L^2|n_L\rangle_L \otimes |n_R\rangle_R = \rho_L(-n_L)\rho_L(n_L)|n_L\rangle_L \otimes |n_R\rangle_R$$  \hspace{1cm} (4.12)

Since $r_L^4 = 1$, the combination $\rho_L(-n_L)\rho_L(n_L)$ can take eigenvalues $\pm 1$ only. There are two natural assignments,

$$\begin{cases}
\text{I} & \rho_L(-n_L)\rho_L(n_L) = 1 \quad \text{for all } n_L \in \mathbb{Z} \\
\text{II} & \rho_L(-n_L)\rho_L(n_L) = (-1)^n_L \quad \text{for all } n_L \in \mathbb{Z}
\end{cases}$$ \hspace{1cm} (4.13)

In the next subsection, it will be shown that these are not only natural assignments, but in fact the only ones that are compatible with the $\hat{SU}(2)_L \times \hat{SU}(2)_R$ symmetry of the theory.

In case I, $r_L^2$ acts as the identity in the untwisted sector and modular covariance requires $Z^1 = Z_{r_L^2}^{r_L^1} = Z_{r_L^1}^{r_L^2} = Z_{r_L^3}^{r_L^1}$. The full partition function is therefore

$$Z_1 = \left( |\zeta[0][0]|^2 + |\zeta[1][0]|^2 \right)^d + \frac{1}{2} |\zeta[0][0]|^d |\zeta[0][0]|^d + \frac{1}{2} |\zeta[0][0]|^d |\zeta[0][0]|^d + \frac{1}{2} \sum_{b=0}^{3} \frac{1}{4} \frac{b}{2} \left( \zeta[0][0] + e^{\frac{1}{2}i\pi b} \zeta[1][0] \right)^d$$ \hspace{1cm} (4.14)

This theory appears to have the undesirable property that the order of $r_L$ is 4 in the sector twisted by $r_L$ but that the order is only 2 in the untwisted sector. The counting of the low lying states suggests that the theory is inconsistent.

In case II, we have the following partition trace in the untwisted sector,

$$Z_{r_L^2}^{r_L^1} = |\zeta[0][0]|^2 - |\zeta[1][0]|^2$$ \hspace{1cm} (4.15)

Modular covariance may be used to derive the remaining partition traces,

$$Z^1_{r_L^2} = |\zeta[0][0]|^2 + |\zeta[1][0]|^2$$

$$Z^{r_L^3}_{r_L^2} = |\zeta[0][0]^2| + |\zeta[1][0]|^2$$

$$Z^{r_L^1}_{r_L^2} = |\zeta[0][0] + |\zeta[1][0]|^2$$

$$Z^{r_L^2}_{r_L^2} = |\zeta[0][0]^2 - |\zeta[0][0]|^2$$ \hspace{1cm} (4.16)
Proceeding as in the other twisted sectors, the Hilbert space $\mathcal{H}_{r_L^2}$ is obtained from $Z_{1, r_L^2}^1$,
\[
\mathcal{H}_{r_L^2} = \{ x_{-n_1} \cdots x_{-n_k} \tilde{x}_{-n_1} \cdots \tilde{x}_{-n_k} | n_L \otimes | n_R - 1 \rangle_R, \ n_L + n_R \in 2\mathbb{Z} \} \quad (4.17)
\]
The trace over this Hilbert space of the operator $q^L \bar{q}^L$ indeed reproduces $Z_{1, r_L^2}^1$. The action of $r_L$ on $\mathcal{H}_{r_L^2}$ is inferred from $Z_{1, r_L^2}^b$. The following phase assignment will be assumed, consistently with the fact that the chiral parts of $\mathcal{H}_{r_L^2}$ are identical to the chiral parts of the untwisted sector,
\[
r_L|n_L \rangle_L \otimes | n_R - 1 \rangle_R = \rho_L(n_L)| n_L \rangle_L \otimes | n_R - 1 \rangle_R \quad (4.18)
\]
We continue to assume that $\rho_L(0) = 1$ without loss of generality. As only states with $n_L = 0$ contribute to $Z_{r_L^2}^{r_L^2} = Z_{r_L^2}^{r_L^2}$, these partition traces are readily reproduced by the above assignments. From the expression for $Z_{r_L^2}^{r_L^2}$, it is clear that $r_L^2$ has eigenvalue +1 when $n_L$ is even, and −1 when $n_L$ is odd. Therefore, $\rho(n_L)$ must satisfy
\[
\rho_L(-n_L)\rho(n_L) = (-1)^{n_L} \quad (4.19)
\]
consistently with the fact that $r_L^4 = 1$. There is no canonical choice $\rho_L$ satisfying this relation; a convenient choice is $\rho_L(n_L) = \exp\{i\pi|n_L|\}$, which has the additional property that $\rho_L(n_L)^4 = 1$ for all $n_L \in \mathbb{Z}$.

Altogether, the $r_L$ theory with $d \in 16\mathbb{N}$ fields is modular covariant, possesses a consistent Hilbert space interpretation and has the following full partition function
\[
Z_{11} = \frac{1}{4} \sum_{\sigma = \pm 1} \left( |\zeta[0][0]|^2 + \sigma |\zeta[1/2][0]|^2 \right)^{d/2} + \frac{1}{4} \sum_{\sigma = \pm 1} \left( |\zeta[0][0]|^2 + \sigma |\zeta[0][0]|^2 \right)^{d/2} \quad (4.20)
\]
\[
+ \frac{1}{2} \zeta[0][0] \frac{1}{2} \zeta[0][0] \frac{1}{2} \zeta[1/2][0] \frac{1}{2} \zeta[1/2][0] + \frac{1}{2} \zeta[1/4][b] \frac{1}{2} \zeta[1/4][b] + \frac{1}{2} \sum_{b=0}^3 \zeta[1/4][b] \frac{1}{2} \zeta[1/4][b] + e^{\frac{1}{2} i \pi b} \zeta[1/2][0]^2 \right)^{d/2}
\]
Term by term comparison reveals that this partition function is identical to (the complex conjugate of) the partition function of the model twisted by $s_R$, given by combining (3.6) with (3.16) for dimension $d = 16$. The significance of this identification will be elucidated in the next section.

### 4.3 The symmetry $\text{SU}(2)_L \times \text{SU}(2)_R$

At the self-dual radius, $R^2 = 2\ell^2$, the conformal field theory of a single boson possesses the symmetry $\text{SU}(2)_L \times \text{SU}(2)_R$. The generators of $\text{SU}(2)_L$ are the holomorphic currents
\[
J_L^3 = \partial x_+ \quad \left\{ \begin{array}{l}
J_L^+ = \frac{1}{\sqrt{2}}(J_L^1 + iJ_L^2) = e^{+i\sqrt{2}x_+} \\
J_L^- = \frac{1}{\sqrt{2}}(J_L^1 - iJ_L^2) = e^{-i\sqrt{2}x_+}
\end{array} \right. \quad (4.21)
\]
where $x_+$ denotes the left chiral part of the field $x$. Using the fact that under $s_L$, one has the transformation $x_+ \to x_+ + \pi/\sqrt{2}$, we are in a position to obtain the transformation laws of these generators under the chiral operators, (see e.g. [2]),

$$
\begin{align*}
[J^a_L, r_R] &= 0 & r_L J^1_L r_L^\dagger &= +J^1_L & r_L J^{2,3}_L r_L^\dagger &= -J^{2,3}_L \\
[J^a_L, s_R] &= 0 & s_L J^3_L s_L^\dagger &= +J^3_L & s_L J^{1,2}_L s_L^\dagger &= -J^{1,2}_L
\end{align*}
$$

Thus, $r_L$ and $s_L$ both correspond to $SU(2)_L$ rotations by 180 degrees, $r_L$ about the 1-axis and $s_L$ about the 3-axis. This observation explains (1) why the orders of $r_L$ and $s_L$ must be equal – to 4; (2) why the critical dimension of the asymmetric orbifold models constructed from them coincide – and are multiples of 16; (3) why their partition functions coincide.

The presence of this enhanced symmetry also has consequences for the phase assignments $\rho_L(n_L)$ associated with $r_L$. The operator $r^2_L$ corresponds to a 360 degree rotations and, as expected, commutes with the currents $J^a_L$ and must thus assume constant eigenvalue throughout any irreducible representation of $SU(2)_L$. In the untwisted sector, two representations occur, one of spin 0 corresponding to $n_L$ even, and one of spin 1/2 corresponding to $n_L$ odd. The first representation of $r^2_L$ would be the trivial (identity) representation, but this was ruled out for being inconsistent with modular invariance. The only other non-trivial representation is then,

$$
\begin{align*}
r^2_L |n_L\rangle &= (-)^{n_L} |n_L\rangle & r_L |n_R\rangle &= |n_R\rangle \\
r^2_R |n_R\rangle &= (-)^{n_R} |n_R\rangle & r_R |n_L\rangle &= |n_L\rangle
\end{align*}
$$

Notice that these assignments are in complete analogy with the action of $s^2_{L,R}$ in the untwisted sector. For $s_{L,R}$, however, the phase assignment of their squares dictated their own action. For $r_{L,R}$, the condition is weaker as the invariant generator $J^1_L$ is not diagonal in the basis $|n_L\rangle_L$. It simply amounts to the relation $\rho_L(-n_L)\rho(n_L) = (-1)^{n_L}$, already derived in (4.19).

### 4.3.1 Interpretation in terms of chiral blocks constructions

As we did earlier for the asymmetric $s_L$ theory, we can give an interpretation for the asymmetric $r_L$ theory in terms of chiral blocks of the symmetric $r = (r_L, r_R)$ theory. Now the chiral operator $r_L$ is of higher order than the symmetric operator $r = (r_L, r_R)$, and at first sight, there seems to be no way of differentiating between the conformal blocks of, say, $r^b_L$ and $r^{b+2}_L$. But the one-loop case shows that the blocks come with a specific assignment of phases, and it is these phases which distinguish between the blocks of $r^b_L$ and $r^{b+2}_L$. More precisely, let $b, a \in \mathbb{Z}_4$ as before, with $b$ expressed uniquely in terms of $\beta \in \{0, \frac{1}{2}\}$ and $c \in \{0, 1\}$ as in (3.11). In [10], it was shown that the orbifold of the circle theory at
self dual radius by \( r = (r_L, r_R) \) coincides with the circle theory at half the self-dual radius. Thus the chiral blocks of the \( r = (r_L, r_R) \) theory and the chiral blocks of the untwisted theory are given as before by (3.10). This time, our formulas obtained earlier by Hilbert space methods show that they are paired differently in order to give the partial traces of the asymmetric \( r_L \) theory. In fact, the pairing matrix is \( K_{\gamma \bar{\gamma}} = \delta_{\gamma \bar{\gamma}} e^{2\pi i (c+2\beta)\gamma}, \) and the partial traces given by

\[
Z^a_b(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{\gamma \in \{0, \frac{1}{2}\}} e^{2\pi i (c+2\beta)\gamma} |a| |\bar{\beta}| (0, 2\tau) \overline{\vartheta[\gamma][0]}(0, 2\tau),
\]

(4.24)

Hence the partition function of the \( r_L \) theory for \( d \) bosons is given by

\[
Z(\tau) = \frac{1}{4} \sum_{a,b \in \mathbb{Z}_4} \left\{ \frac{1}{|\eta(\tau)|^2} \sum_{\gamma \in \frac{1}{2}\mathbb{Z}} e^{2\pi i (c+2\beta)\gamma} |\gamma| + \frac{1}{4} |a| |\bar{\beta}| (0, 2\tau) \overline{\vartheta[\gamma][0]}(0, 2\tau) \right\}^d.
\]

(4.25)

As noted before, the partition functions of the \( s_L \) and the \( r_L \) theories are the same, although the blocks for the assignment \( s_L^b \) and \( s_R^l \) on the \( B \) and \( A \) cycles differ by phases from the blocks for the assignment of \( r_L^b \) and \( r_R^l \) on the same cycles.

5 Orbifolds defined by chiral shifts and reflections

The next step towards analyzing the models of Kachru-Kumar-Silverstein is the consideration of asymmetric orbifold models which involve both chiral shifts \( s_{L,R} \) and chiral reflections \( r_{L,R} \). First, a detailed study is presented of a purely bosonic model with only a single chiral generator \( f \equiv (r_L, s_R) \); second, the model is extended to include two chiral generators \( f \) and \( g \equiv (s_L, r_R) \); third, fermions are included as well and the full KKS model is analyzed.

5.1 The \( f = (r_L, s_R) \) model

The proper definition of the chiral operator \( f \) involves some of the same delicate issues that arose when defining \( r_L \). First, in the untwisted sector, \( f \) will involve the phases \( \rho_L(n_L), \)

\[
f|n_L\rangle_L \otimes |n_R\rangle_R = \rho_L(n_L) e^{i\pi n_R/2} |n_L\rangle_L \otimes |n_R\rangle_R
f^2|n_L\rangle_L \otimes |n_R\rangle_R = \rho_L(-n_L) \rho_L(n_L) (-1)^n_R |n_L\rangle_L \otimes |n_R\rangle_R
\]

(5.1)

Depending on whether \( \rho_L(-n_L) \rho_L(n_L) \) equals 1 or \((-1)^n_L\), the order of \( r_L \) is 2 or 4, while the order of \( f \) will be 4 or 2 respectively (the latter since \( n_L + n_R \in 2\mathbb{Z} \)). Which of these choice (if any) leads to a consistent \( f \)-orbifold is analyzed below.
5.1.1 Ruling out the case of trivial phases \( \rho_L(n_L) \)

It is first shown that the naive definition of \( r_L \), for which \( \rho_L(−n_L)\rho_L(n_L) = 1 \) for all \( n_L \in \mathbb{Z} \), and \( f \) is of order 4, is incompatible with the dual requirements of modular covariance and a proper Hilbert space interpretation. With this naive choice, one has \( f^2 = (1, s^2_R) \neq 1 \).

The 4 partition traces in the untwisted sector are readily evaluated to be

\[
\begin{align*}
Z^1_{f^2} &= |\zeta[0]^{1/2}[0]|^2 + |\zeta[1/2]^{1/2}[0]|^2 \\
Z^2_{f^2} &= |\zeta[0]^{1/2}[0]|^2 - |\zeta[1/2]^{1/2}[0]|^2 \\
Z^1_f &= |\zeta[0]^{1/2}[1]|^2 \\
Z^2_f &= |\zeta[0]^{1/2}[1]|^2 \\
\end{align*}
\]

in the sectors twisted by \( f \) and \( f^3 \), and

\[
\begin{align*}
Z^1_{f^2} &= \zeta[0][1/2][0][1/2] + \zeta[0][1/2][1/2][1/2] \\
Z^2_{f^2} &= e^{-i\pi/2}(\zeta[0][1/2][0] - \zeta[0][1/2][1/2]) \\
Z^1_f &= |\zeta[0]^{1/2}[1]|^2 \\
Z^2_f &= |\zeta[0]^{1/2}[1]|^2 \\
\end{align*}
\]

This allows an independent evaluation of \( Z^f_{f^2}(\tau) \) as an operator trace; one finds,

\[
\text{Tr}_{H_{f^2}} \left( f q^{L_0} \bar{q}^{\bar{L}_0} \right) = - \frac{1}{q^{1/24} \prod_{n=1}^{\infty} (1 + q^n)} \frac{\vartheta[1/2, 1/2](0, \tau)}{\eta(\tau)} = 0
\]

This contradicts the earlier, non-vanishing, formula for \( Z^f_{f^2}(\tau) \) obtained from modular covariance

\[
0 \neq Z^f_f \rightarrow Z^f_f \rightarrow Z^f_{f^2} \rightarrow Z^f_{f^2} \neq 0
\]

Therefore, the assumption that the phase assignment of \( r_L \) is trivial cannot lead to a consistent orbifold theory.
5.1.2 Consistency of the case of non-trivial phases $\rho_L(n_L)$

For the non-trivial phase assignment, $\rho_L(-n_L)\rho_L(n_L) = (-1)^n_L$, one has $f^2 = 1$ in the untwisted sector and the theory truncates. The traces $Z^1_f$ and $Z^f_f$ in the untwisted sector do not depend on the choice of $\rho_L(n_L)$ (having set $\rho_L(0) = 1$ without loss of generality), and are still given by their expressions in (5.2). Since the traces $Z^1_f$ and $Z^f_f$ follow by modular covariance, they are also given by their expressions in (5.3).

The only issue that remains to be verified is that the traces in the sector twisted by $f$ can indeed be realized as partition traces in a twisted Hilbert space $H_f$, where $f$ is realized as an operator of order 2. It is well-known $[6, 7]$ that the chiral sector twisted by $r_L$ contains two degenerate ground states, which we shall denote by $|\pm\rangle_L$. Also, the chiral sector twisted by $s_R$ contains two sectors $|n_R - 1/2\rangle_R$ with $n_R$ either even or odd. Following the method of §4, the Hilbert space is found to be

$$H_f = \{ (x_{-n_1 + \frac{1}{2}} \cdots x_{-n_k + \frac{1}{2}} | + \rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} | n_R - \frac{1}{2} \rangle_R, \ n_R \in 2\mathbb{Z} \}$$

$$\oplus \{ (x_{-n_1 + \frac{1}{2}} \cdots x_{-n_k + \frac{1}{2}} | - \rangle_L \otimes \bar{x}_{-n_1} \cdots \bar{x}_{-n_k} | n_R - \frac{1}{2} \rangle_R, \ n_R \in 2\mathbb{Z} + 1 \}$$

Note that the pairing between the $|\pm\rangle_L$ and $|n_R - 1/2\rangle_R$ provides a highly non-trivial piece of information on how the left and right chiral Hilbert spaces need to be combined. The pairing indeed implies that out of the 4 ground states arising from a full tensor product of left and right, only 2 are to be retained in the asymmetric orbifold theory. The same pairing issue arose already in the theory twisted by the symmetric operator $r = (r_L, r_R)$, where also only 2 states out of a total of 4 should be retained.$^3$ It is readily verified that,

$$Z^1_f = \text{Tr}_{H_f}(q^L_0 \bar{q}^L_0) = 2 |\zeta[\frac{1}{4}]_0|^2$$

(5.9)

as desired. To obtain $Z^f_f$ from $H_f$ as well, it suffices to define $r_L$ on the momenta ground states in $H_f$ as

$$r_L |+\rangle_L \otimes |n_R - \frac{1}{2}\rangle_R = |+\rangle_L \otimes |n_R - \frac{1}{2}\rangle_R, \ n_R \in 2\mathbb{Z}$$

$$r_L |-\rangle_L \otimes |n_R - \frac{1}{2}\rangle_R = -i |-\rangle_L \otimes |n_R - \frac{1}{2}\rangle_R, \ n_R \in 2\mathbb{Z} + 1$$

(5.10)

and similarly for excited states. With this construction, $\text{Tr}_{H_f}(q^L_0 \bar{q}^L_0)$ is indeed found to agree with the expression found earlier for $Z^f_f$ by modular covariance.

$^3$Although the issue is well-known in the symmetric theory $[6]$, it has nonetheless given rise to confusion when the proper pairing was not taken into account.

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Altogether, the theory obtained this way is modular covariant and admits a Hilbert space realization in terms of twisted sectors. Its partition function in dimension \(d \in 4\mathbb{N}\) is given by
\[
Z = \frac{1}{2} \left\{ (|\zeta[0]|^2 + |\zeta[1]|^2)^d + |\zeta[0]|^2 + 2^d |\zeta[1]|^2 + 2^d |\zeta[1]|^2 \right\}
\] (5.11)
which coincides with that of the \(d\)-dimensional circle theory orbifolded by \(r = (r_L, r_R)\).

5.2 Orbifolds generated by \(f\) and \(g\)

Next, we consider the theory of \(d\) bosonic scalar fields moded out by the group \(G\) generated by the two elements \(f = (r_L, s_R)\) and \(g = (r_R, s_L)\). This group is Abelian; indeed, in view of (5.1), one has in the untwisted sector
\[
\langle n_L \rangle_L \otimes \langle n_R \rangle_R = \rho_L(n_L) e^{i mn_R/2} \langle -n_L \rangle_L \otimes \langle n_R \rangle_R
\]
and therefore
\[
gf \langle n_L \rangle_L \otimes \langle n_R \rangle_R = e^{i \pi (n_R - n_L)} fg \langle n_L \rangle_L \otimes \langle n_R \rangle_R
\] (5.12)
In view of the fact that in the untwisted sector \(n_L + n_R \in 2\mathbb{Z}\), the phase factor on the rhs is just 1, and one has \(fg = gf\). Since also \(f^2 = g^2 = 1\), the orbifold group generated by \(f, g\) is just the group \(\mathbb{Z}_2 \times \mathbb{Z}_2\). As usual, it taken for granted that the group composition laws remain the same in all sectors.

5.2.1 The sectors twisted by \(f\) and \(g\)

The partition traces \(Z^f_{\mathbb{I}^1}\), (and by interchanging left and right movers also \(Z^g_{\mathbb{I}^1}\)), have already been determined in (5.2) and we have e.g. \(Z^f_{\mathbb{I}^1} = Z^g_{\mathbb{I}^1} = |\zeta[0]|^2\). By modular transformation, the partition traces are found in the twisted sectors, \(Z^1 f = Z^1 g = 2|\zeta[1]|^2\).

The Hilbert space \(\mathcal{H}_f\) in the sector twisted by \(f\), (and by interchanging left and right movers also the Hilbert space \(\mathcal{H}_g\) in the sector twisted by \(g\)) was given in (5.8).

The novel partition traces to be calculated in \(\mathcal{H}_f\) are \(Z^g_{\mathbb{I}^f}\) and \(Z^{fg}_{\mathbb{I}^f}\). They belong to a modular orbit that does not include any partition trace evaluated in the untwisted sector, and is thus not derivable from preceding results. To evaluate \(Z^g_{\mathbb{I}^f}\), it suffices to find \(g\) on the ground states of \(\mathcal{H}_f\),
\[
g\langle \pm \rangle_L \otimes \langle n_R - \frac{1}{2} \rangle_R = (s_L\langle \pm \rangle_L) \otimes (r_R\langle n_R - \frac{1}{2} \rangle_R)
\]
\[
\sim (s_L\langle \pm \rangle_L) \otimes (| - n_R + \frac{1}{2} \rangle_R)
\] (5.14)
Here, the ∼ sign has been used, because a phase factor may arise when applying \( r_R \) to \( \langle \rangle_R \) states. The half-integer moding of the \( \langle \rangle_R \) states alone guarantees that the operator \( g \) has no diagonal matrix elements in \( \mathcal{H}_f \), and must therefore have vanishing trace, \( \mathcal{Z}^g_f = 0 \). Modular invariance then implies that all partition traces in the same orbit must vanish,

\[
\mathcal{Z}^g_f = \mathcal{Z}^{f g}_f = \mathcal{Z}^{f g}_{fg} = \mathcal{Z}^{f g}_g = \mathcal{Z}^{g f}_g = 0 \tag{5.15}
\]

In the sector \( \mathcal{H}_f \), the vanishing of the partition trace \( \mathcal{Z}^{f g}_f \) holds for the same reasons as the vanishing of \( \mathcal{Z}^g_f \) did, namely that \( fg \) has vanishing diagonal matrix elements in \( \mathcal{H}_f \). The verification of the traces \( \mathcal{Z}^{f g}_f \) and \( \mathcal{Z}^{g f}_g \) will require the construction of the Hilbert space \( \mathcal{H}_{fg} \) in the sector twisted by \( fg \), to be given below.

Finally, it is worth noting that although \( fg = gf \) when \( f, g \) are viewed as elements of the point group, they do not commute when they are viewed as elements of the full orbifold group, since \( fg^{-1}g^{-1} = (s_L^{-2}, s_R^2) \). For asymmetric orbifold groups, we are aware of no arguments based on first principles that would guarantee the vanishing of the torus contribution for twists \( f, g \) that do not commute in the full orbifold group (but do commute in the point group). Nonetheless, this vanishing actually does take place in the model considered here.

### 5.2.2 The action of \( s_{L,R} \) in the sector twisted by \( r_{L,R} \)

For the sake of completeness, as well as for later use, we briefly discuss the action of \( s_L \) on the twisted states \( \langle \pm \rangle_L \), which enters (5.14) but is not actually needed for the evaluation of \( \mathcal{Z}^g_f \). In the left-right symmetric conformal field theory of a single scalar field, twisted by the symmetric reflection operator \( r \), the symmetric ground states \( \langle \pm \rangle \) are associated with the fixed points of \( r \). Applying \( r \) to the circle of radius \( R \), there are two solutions to the fixed point equation \( r(x) \equiv -x \mod 2\pi R \), namely \( x = 0 \) and \( x = \pi R \). Under \( r \), each fixed point is mapped into itself, which is tantamount to \( r \langle + \rangle \sim \langle + \rangle \) and \( r \langle - \rangle \sim \langle - \rangle \), and the state \( \langle + \rangle \) may be associated with the fixed point 0 while \( \langle - \rangle \) is associated with the fixed point \( \pi R \). On the other hand, the operator \( s \) acts by shifts \( s(x) = x + \pi R \) and thus interchanges the two twisted ground states, \( s \langle + \rangle \sim \langle - \rangle \) and \( s \langle - \rangle \sim \langle + \rangle \). It is natural and internally consistent to induce analogous actions on the chiral twisted states, namely \( r_L \langle \pm \rangle_L \sim \langle \pm \rangle_L \) and \( s_L \langle \pm \rangle_L \sim \langle \mp \rangle_L \), and \( r_R \langle \pm \rangle_R \sim \langle \pm \rangle_R \) and \( s_R \langle \pm \rangle_R \sim \langle \mp \rangle_R \).

### 5.2.3 The sector twisted by \( fg \)

The only partition trace in the untwisted sector which has not yet been determined is \( \mathcal{Z}^{fg}_1(\tau) \). Since the product \( fg \) involves reflection operators on both left and right movers, only states with \( n_L = n_R = 0 \) will contribute in the untwisted sector and the partition
trace manifestly reduces to that of the symmetric operator $r = r_{LR}$; it is readily computed and one finds,

$$Z^{fg}_{1} = |\zeta[0][\frac{1}{2}]|^2$$

(5.16)

Modular covariance implies

$$Z^{1}_{fg} = 2|\zeta[\frac{1}{4}[0]]|^2 \quad Z^{fg}_{fg} = 2|\zeta[\frac{1}{4}[\frac{1}{2}]]|^2$$

(5.17)

The first result yields the Hilbert space of the sector twisted by $fg$

$$\mathcal{H}_{fg} = \oplus\{x_{-n_1+\frac{1}{2}} \cdots x_{-n_k+\frac{1}{2}} \bar{x}_{-n_1+\frac{1}{2}} \cdots \bar{x}_{-n_k+\frac{1}{2}} |\pm\}'\}$$

(5.18)

Here, $|\pm\>'$ are two ground states in the sector twisted by $fg = (r_{LS}, s_{LR})$, both of which are of conformal weight $1/16$.

If the presence of the translation operators $s_L$ and $s_R$ were ignored, $fg$ would coincide with the non-chiral reflection $r$, and $|\pm\>' = |\pm\>$ would just be the ground states in the sector twisted by $r$. The problem with this simplified picture is that the vanishing of $Z^{fg}_{fg}$ does not allow for a chiral formulation of the states $|\pm\>$ and the operator $fg$ consistent with the relations $r_L|+\> = |+\>_L$ and $r_L|-\> = -i|-\>_L$, used successfully in other sectors, such as (4.7), (4.11), (5.10).

If the presence of the translation operators $s_L$ and $s_R$ is carefully taken into account, a consistent chiral formulation of the twisted states $|\pm\>'$ and the action thereupon by the chiral operators $r_{LR}$ and $s_{LR}$ does exist and reproduces the corresponding partition traces predicted from modular covariance. The key observation is that the states twisted by $(r_L, r_R)$ and by $fg = (r_{LS}, s_{LR})$ are different but isomorphic to one another. Following the spirit of section 5.2.2, the twisted states may be associated with geometrical fixed points in the symmetric theory. While the fixed points of $(r_L, r_R)$ on the circle are 0 and $\pi R$, those of $fg = (r_{LS}, s_{LR})$ are shifted by $\pi R/2$, i.e. they are $+\pi R/2$ and $-\pi R/2$. Denoting associated twisted states by $|+\>'$ and $|\sim 1\>'$ respectively, it is now clear that one should expect to have $r|+\>' \sim |-\>'$, $r|\sim 1\>' \sim |+\>'$, as well as $s|+\>' \sim |-\>', s|\sim 1\>' \sim |+\>'$. On the other hand, the states $|\pm\>'$ are eigenstates of the operators $rs$.

The above geometry-inspired picture may be realized concretely in terms of chiral states and operators. We start with the chiral ground states $|\pm\>_{L,R}$ of the sectors twisted by $r_L$ and $r_R$ and the action of the chiral reflection operators on these states,

$$r_L|+\>_L = |+\>_L \quad r_L|-\>_L = -i|-\>_L$$

$$r_R|+\>_R = |+\>_R \quad r_R|-\>_R = +i|-\>_R$$

(5.19)

Here, $|\pm\>_{L,R}$ are two ground states in the sector twisted by $fg = (r_{LS}, s_{LR})$, both of which are of conformal weight $1/16$. If the presence of the translation operators $s_L$ and $s_R$ were ignored, $fg$ would coincide with the non-chiral reflection $r$, and $|\pm\>_{L,R}$ would just be the ground states in the sector twisted by $r$. The problem with this simplified picture is that the vanishing of $Z^{fg}_{fg}$ does not allow for a chiral formulation of the states $|\pm\>$ and the operator $fg$ consistent with the relations $r_L|+\> = |+\>_L$ and $r_L|-\> = -i|-\>_L$, used successfully in other sectors, such as (4.7), (4.11), (5.10).
Using the standard relations \( s_L^a r_L s_L^a = r_L \) and \( s_R^a r_R s_R^a = r_R \) for any \( a \in \mathbb{R} \), one deduces
\[
(r_L s_L)|+\rangle_L' = |+\rangle_L' \\
(r_L s_L)|-\rangle_L' = -i|\rangle_L' \\
(s_R r_R)|+\rangle_R' = |+\rangle_R' \\
(s_R r_R)|-\rangle_R' = +i|\rangle_R'
\]
Thus, the action of the operator \( r_L s_L \) on the twisted states \(|\pm\rangle_L'\) is isomorphic to the action of \( r_L \) on the twisted states \(|\pm\rangle_L\). The action of the chiral operators \( r_L, r_R \) on \(|\pm\rangle_L, |\pm\rangle_R\), however, is now calculable from the above definitions, from \( s_L^a r_L s_L^a = r_L \), and from the fact that \( s_L, s_R |\pm\rangle_{L,R} \sim |\mp\rangle_{L,R} \), and we find,
\[
r_L|\pm\rangle_L' \sim |\mp\rangle_L' \\
r_R|\pm\rangle_R' \sim |\mp\rangle_R' \tag{5.21}
\]
This relation readily guarantees that \( f \) and \( g \) have vanishing diagonal matrix elements in \( \mathcal{H}_{fg} \), so that \( \mathcal{Z}^{\prime}_{fg} = \mathcal{Z}^{\prime g}_{fg} = 0 \).

The final relation to be implemented is the partition trace \( \mathcal{Z}^{fg}_{fg} = 2|\zeta[\frac{1}{4} |\frac{1}{2}]|^2 \), which effectively requires that \( fg \) be the identity on both states \(|\pm\rangle'\). In view of \( fg = (r_L s_L, s_R r_R) \) and (5.20), this provides with a unique correspondence between the chiral and non-chiral twisted states and we have
\[
|+\rangle' = |+\rangle_L' \otimes |+\rangle_R' \\
|-\rangle' = |-\rangle_L' \otimes |-\rangle_R'
\]
With this construction, all the partition traces of (5.2.3) and (5.15) are indeed reproduced in a chiral fashion. The partition function for the orbifold generated by \( f \) and \( g \) is, in dimension \( d \in 4\mathbb{N} \) is therefore given by
\[
Z = \frac{1}{4} \left\{ \langle |\zeta[0|0]|^2 + |\zeta[\frac{1}{2}|0|^2 \rangle d + 3|\zeta[0|^2 \rangle d + 3 \cdot 2^d |\zeta[\frac{1}{4}|0|^2 \rangle d + 3 \cdot 2^d |\zeta[\frac{1}{2}|0|^2 \rangle d \right\} \tag{5.23}
\]
Notice that this partition function involves the same blocks as that of the orbifold by \( f \) alone, but the relative proportions of the three non-vanishing modular orbits is different.

## 5.3 Including worldsheet fermions

It is straightforward to include worldsheet fermions \( \psi_{\pm}^{\mu} \) in models twisted by shifts \( s \) and reflections \( r \). Given a canonical homology basis of 1-cycles \( A, B \), the worldsheet fermions are defined with a spin structure \( \delta = (\delta'|\delta'') \). The models of greatest interest are those with worldsheet supersymmetry \([35]\). Invariance of the matter supercurrent \( S_m = -1/2|\psi_{\pm}^{\mu} \partial_{\pm} x^{\mu} \) thus forces \( \psi_{\pm}^{\mu} \) to undergo the same transformation as the bosonic field \( \partial_{\pm} x^{\mu} \). Shifts do not act on \( \psi_{\pm}^{\mu} \). Reflections may be parametrized by a half-characteristic \( \varepsilon = (\varepsilon'|\varepsilon'') \), with
\[ \varepsilon', \varepsilon'' = 0, \frac{1}{2}, \text{such that } \psi_{\pm}^{\mu} \text{ is double-valued around the cycle } D_{\varepsilon} = \varepsilon'A + \varepsilon''B. \] The combined boundary conditions due to the spin structure \( \delta \) and the reflection \( \varepsilon \) are,

\[
\psi_{+}^{\mu}(z + 1) = -(-)^{2\delta' + 2\varepsilon'} \psi_{+}^{\mu}(z) \\
\psi_{-}^{\mu}(z + \tau) = -(-)^{2\delta'' + 2\varepsilon''} \psi_{-}^{\mu}(z)
\] (5.24)

The chiral partition function is given by

\[
\mathcal{F}_{\psi}[\delta; \varepsilon] = \frac{\vartheta[\delta + \varepsilon](0, \tau)}{\eta(\tau)}
\] (5.25)

When \( \delta + \varepsilon \) equals \( \frac{1}{2}[\frac{1}{2}] \), the chiral partition function vanishes, \( \mathcal{F}[\delta; \varepsilon] = 0 \).

Notice that while the bosonic models twisted by \( r \) and by \( s \) are identical, once fermions are included, they will differ, since the fermions are twisted by \( r \) but not by \( s \).

### 6 Asymmetric Orbifolds via chiral splitting

The operator and Hilbert space methods used earlier in this paper to construct the partition traces and full partition functions for asymmetric orbifolds on a torus worldsheet do not easily generalize to higher loop order. The construction of symmetric orbifolds is, of course, well-understood, both in the functional integral and the operator formulations (see, for example [5, 6, 7, 10, 11]). The construction of the chiral blocks has also been extensively investigated using operator methods, the chiral OPE, and the implications of modular invariance [10, 11]. Furthermore, in recent work [22], the full \( \mathbb{Z}_2 \)-twisted chiral blocks were calculated in the presence of non-trivial supermoduli and simple expressions in terms of \( \vartheta \)-functions were derived. It is the construction of the full partition functions for asymmetric orbifolds that remains much less well-understood.

By chiral splitting, it is clear that the correct blocks of an asymmetric orbifold theory should be defined as the products of the blocks of the left and right chiral halves. To be concrete, the asymmetric orbifold group \( G \) is viewed as a subgroup of the product \( G_L \times G_R \) of left and right groups \( G_L \) and \( G_R \), so that elements of \( G \) may be labeled as pairs \( (g_L; g_R) \), with \( g_L \in G_L \) and \( g_R \in G_R \). The starting point of the proposed construction is two symmetric orbifolds, one with group \( G_L \), the other with group \( G_R \). For simplicity, the discussion will be carried out here for the torus, with boundary conditions around a single \( A \) and a single \( B \) cycle; the generalization to higher genus are analogous.

By chiral splitting, the blocks are given by \( |\mathcal{F}_{g_L, h_L}(p_L; \tau)|^2 \) and \( |\mathcal{F}_{g_R, h_R}(p_R; \tau)|^2 \) respectively, i.e. the left and right chiral blocks are complex conjugates of one another. Here, \( p_L, p_R \) stand for any extra labels of the chiral blocks, such as, for example, dependence on
the internal loop momentum. A block $Z^g_h$ of the asymmetric orbifold with $h = (h_L; h_R)$ and $g = (g_L; g_R)$ is then defined as the product of the corresponding chiral blocks; as illustrated by the diagram below,

\[
\begin{array}{c}
\text{Symm } |F^{g_L}_{h_L}|^2 \\
(F^{g_L}_{h_L})^* \rightarrow \downarrow \rightarrow \rightarrow \\
(F^{g_L}_{h_L}) \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \\
\text{Asymm } F^{g_L}_{h_L} \times (F^{g_R}_{h_R})^*
\end{array}
\]

and may be expressed by the following formula,

\[
Z^{(g_L; g_R)}_{(h_L; h_R)}(p_L, p_R; \tau, \bar{\tau}) = F^{g_L}_{h_L}(p_L; \tau) F^{g_R}_{h_R}(p_R; \tau) \quad (6.1)
\]

The key difficulty with asymmetric orbifolds resides in the precise pairing between the left and right chiral blocks, i.e. in the superposition of the blocks $Z^{(g_L; g_R)}_{(h_L; h_R)}$.

Following \[22\], the superposition of the blocks may be expressed in terms of a pairing matrix on left and right chiral blocks, which is denoted $K$. The full partition function is then given by

\[
Z_G(\tau, \bar{\tau}) = \frac{1}{|G|} \sum_{g, h \in G} \sum_{p_L, p_R} K(p_L, g_L; p_R, g_R, h) F^{h_L}_{g_L}(p_L; \tau) \left(F^{h_R}_{g_R}(p_R; \tau)\right)^* \quad (6.2)
\]

Here the summations are over $g = (g_L; g_R)$ and $h = (h_L; h_R)$. Clearly, the matrix $K$ cannot depend upon moduli $\tau, \bar{\tau}$ as this would violate chiral splitting. Many models admit the same symmetrized theories, and hence the same chiral blocks. It is then the pairing matrix $K$ which differentiates between different models, and the key issue is its determination.

Modular invariance places strong constraints on $K$, which may be derived as follows. The modular transformations of the blocks of the symmetric theory are as in (1.2). Using chiral splitting, it follows that the transformations of the chiral blocks themselves are known up to phases $\varphi$ and mixing coefficients $S$, which are independent of moduli,

\[
\begin{align*}
F^g_h(p_L; \tau + 1) &= e^{i\varphi(g, h)} F^{g h^{-1}}_h(p_L; \tau) \\
F^g_h(p_L; -1/\tau) &= \sum_{p'_L} S(p_L, p'_L) F^{h^{-1}}_g(p'_L; \tau)
\end{align*}
\]  

(6.3)

\^4It is useful to note that, when considering the compactification and orbifolding of several dimensions $d > 1$, the group elements $g_L, h_L, g_R, h_R$ represent the action of the orbifold group on all dimensions and the chiral blocks $F$ represent the chiral blocks for $d$ dimensions. When the $d$ dimensions are orthogonal, as has been the case in this paper, the blocks $F$ themselves are the product of $d$ one-dimensional blocks and the action of a group element $g$ accordingly decomposes into $d$ group elements each acting on a one-dimensional block.
A sufficient condition for modular invariance of the full partition function is the requirement that $K$ satisfy the following relations,

$$K \left( p_L, g_L h_L^{-1}, h_L; p_R, g_R h_R^{-1}, h_R \right) = e^{i\varphi(g_L, h_L) - i\varphi(g_R, h_R)} K \left( p_L, g_L, h_L; p_R, g_R, h_R \right)$$

(6.4)

$$K \left( p_L; h_L^{-1}, g_L; p_R, h_R^{-1}, g_R \right) = \sum_{p'_L, p'_R} S(p_L, p'_L) S(p_R, p'_R)^* K \left( p_L, g_L, h_L; p_R, g_R, h_R \right)$$

These conditions are also necessary when all the chiral block-functions $\mathcal{F}_{gh}$ are linearly independent. If the block-functions exhibit non-trivial linear dependences, the matrix $K$ can be reduced to pair only the linearly independent block-functions; modular invariance may then be expressed on this reduced $K$ matrix just as in (6.4).

Modular invariance cannot, in general, determine $K$ completely, even when all the block-functions are linearly independent. The action of modular transformations $\tau \to (a\tau + b)(c\tau + d)^{-1}$ (with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$) induces the following transformations on pairs of group elements

$$(g_L, h_L) \to (g_L^a h_L^b, g_L^c h_L^d) \quad (g_R, h_R) \to (g_R^a h_R^b, g_R^c h_R^d)$$

(6.5)

Generally, this action will decompose into a several disjoint modular orbits. Clearly, the modular transformation laws of $K$ in (6.4) cannot serve to relate $K$ on different orbits; this information must come from physical input, derived from the Hilbert space formulation and the proper actions of the chiral group elements $g_L, h_L$.

Since $K$ is independent of moduli, its value may be determined by taking various degeneration limits of moduli space in which the corresponding chiral blocks do not all vanish. For example, in the case of genus two, the separating degeneration will produce two tori with prescribed twist sectors. On each of these tori, the matrix $K$ is known. Therefore, the genus two matrix $K$ will be known for the subset of twists for which the genus one limits of the chiral blocks are non-vanishing (and linearly independent). We expect that the matching of the Hilbert space construction with the expression of (6.2) produces a unique pairing matrix $K$ to higher genus.

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A Orbifolding by symmetric orbifold groups

In this appendix, some concrete orbifold partition functions for symmetric orbifolds are constructed. The general prescription in terms of conjugacy classes is reviewed. Finally, simple examples of symmetric orbifolds based on shifts and reflections are worked out.

A.1 The functional integral and conjugacy class prescription

Consider a scalar field \( x \) with classical action \( S(x) \) on a torus worldsheet with modulus \( \tau \) subject to the following twisted boundary conditions \( \mathcal{B} \) on the \( A \)- and \( B \)-cycles, corresponding to \( z \to z + 1 \) and \( z \to z + \tau \) respectively,

\[
\mathcal{B} \left\{ \begin{array}{l}
x(z + 1) = h x(z) \\
x(z + \tau) = g x(z)
\end{array} \right.
\]

(A.1)

The functional integral with boundary conditions \( \mathcal{B} \) is denoted by \( Z_{gh}^g(\tau) \) and given by

\[
Z_{gh}^g(\tau) \equiv \int_{\mathcal{B}} Dx e^{-S(x)}
\]

(A.2)

It is well-known that when \( gh \neq hg \), the above boundary conditions have no solutions, and such sectors do not contribute to the functional integral. Since the boundary conditions have also the following two equivalent representations,

\[
\mathcal{B} \iff \left\{ \begin{array}{l}
x(z + 1) = h x(z) \\
x(z + \tau) = g h x(z)
\end{array} \right. \iff \left\{ \begin{array}{l}
x(z + \tau) = g x(z) \\
x(z - 1) = h^{-1} x(z)
\end{array} \right.
\]

(A.3)

the following modular identities immediately result from the functional integral representation,

\[
Z_{gh}^g(\tau) = Z_{gh}^{gh}(\tau + 1) = Z_{gh}^{h^{-1}}(-1/\tau)
\]

(A.4)

which, in turn, are equivalent to the modular transformation laws of (1.2).

The general prescription for the one-loop orbifold partition function \( Z_G \) defined by the group \( G \) is given in terms of a summation over all possible twisted boundary conditions \( g, h \in G \). The following two formulas are equivalent,

\[
Z_G = \frac{1}{|G|} \sum_{g,h \in G} \ Z_{gh}^g = \sum_i \frac{1}{|N_i|} \sum_{h \in N_i} \ Z_{C_i \ h}
\]

(A.5)

The second expression above is in terms of a summation over all conjugacy classes of \( G \), which are indexed by \( i \) and any representative is denoted by \( C_i \) in the above formula. Also, \( N_i \) is the stabilizer of \( C_i \), i.e. the subgroup of all \( h \in G \) such that \( g \) commutes with every element of \( C_i \). The equivalence of both expressions is readily established by using the fact that \( Z_{gh}^g = Z_{hgh}^{u^{-1}} \) for all \( g, h, u \in G \), and by breaking up the sum over all \( g \in G \) into a sum over \( g' \) and \( u \) with \( g = ugu^{-1} \).
A.2 The Poisson resummation formula

The reformulation of momentum and winding mode summations is carried out using the Poisson resummation formula. Let $A$ be an invertible $n \times n$ symmetric matrix and $B, C$ two $n$-column vectors,

$$\sum_{m^1, \ldots, m^n \in \mathbb{Z}} \exp \left\{ -\pi A_{ij} (m^i + C^i) (m^j + C^j) + 2\pi i B_i (m^i + C^i) \right\} = \frac{1}{(\det A)^{\frac{1}{2}}} \sum_{m^1, \ldots, m^n \in \mathbb{Z}} \exp \left\{ -\pi (A^{-1})_{ij} (m^i - B_i) (m^j - B_j) + 2\pi i C^i (m^i - B_i) \right\}$$

(A.6)

The derivation of this formula is standard [24].

A.3 Symmetric Abelian orbifolds defined by $\mathbb{Z}_N$ shifts

Consider the symmetric orbifold generated by $\mathbb{Z}_N$ shifts, $sx = x + 2\pi R/N$. In this subsection, the theory $S^1_{R/s}$ is worked out using the conjugacy class prescription and it is verified to coincide with the theory $S^1_{R/N}$, as should be expected on geometric grounds. The prescription of the preceding subsection for this case gives,

$$Z_{S^1_{R/s}} = \frac{1}{N} \sum_{g,h=1}^{s^{N-1}} Z^g_h$$

(A.7)

By definition, $Z^1_1 = Z^s_h$ is the untwisted partition function.

General shifted boundary conditions may be specified by the characteristics $\delta = (\delta', \delta'')$ where $\delta', \delta'' = 0, 1/N, 2/N, \ldots (N-1)/N$ and the following correspondence with the group elements, $h = s^{N\delta'}$ and $g = s^{N\delta''}$,

$$x(\sigma^1 + 1, \sigma^2) = x(\sigma^1, \sigma^2) + 2\pi R \delta' \quad (\text{mod } 2\pi R)$$
$$x(\sigma^1, \sigma^2 + 1) = x(\sigma^1, \sigma^2) + 2\pi R \delta'' \quad (\text{mod } 2\pi R)$$

(A.8)

The associated instanton solutions and action are given by

$$x^\delta_{m_1, m_2}(\sigma) = 2\pi R \sigma^1 (m_1 + \delta') + 2\pi R \sigma^2 (m_2 + \delta'')$$
$$S[x^\delta_{m_1, m_2}] = \frac{\pi R^2}{2\ell^2 \tau_2} \left| \tau (m_1 + \delta') - (m_2 + \delta'') \right|^2$$

(A.9)

The field $y(\sigma) \equiv x(\sigma) - x^\delta_{m_1, m_2}(\sigma)$ is now a doubly periodic scalar function. Its functional integral produces the well-known factor produces a factor $\text{Det}' \Delta = \tau_2 |\eta(\tau)|^4$ from the non-zero modes and a factor of $(8\pi^2)^{-\frac{1}{4}} 2\pi R/\ell = R/\sqrt{2}\ell$ from the zero mode of $y$. Assembling
all contributions, one finds, (still with $h = s^{N\delta'}$ and $g = s^{N\delta''}$),

$$Z_h^g(\tau) = \frac{R}{\sqrt{2\pi}2^\ell |\eta(\tau)|^2} \sum_{m_1,m_2 \in \mathbb{Z}} \exp \left\{ -\frac{\pi R^2}{2\ell^2 \tau_2} |\tau(m_1 + \delta') - (m_2 + \delta'')|^2 \right\}$$  \hspace{1cm} (A.10)

Under the modular transformations $\tau \to \tau + 1$, one has $\delta' \to \delta'$ and $\delta'' \to \delta'' - \delta'$, i.e. $h \to h$ and $g \to gh^{-1}$ in accord with (A.10). Under the modular transformations $\tau \to -1/\tau$, one has $\delta' \to \delta''$ and $\delta'' \to -\delta'$, i.e. $h \to g$ and $g \to h^{-1}$, also in accord with (A.10).

The formula (A.10) obtained above has manifest modular transformation properties but does not exhibit the chiral block structure of the theory. To make the chiral block structure manifest, Poisson resummation in $m_2$ is carried out, using (A.6) for $n = 1$, $A = R^2/2\ell^2 \tau_2$, $B = 0$ and $C = \delta'' - \tau_1(m_1 + \delta')$,

$$Z_h^g(\tau) = \frac{1}{|\eta(\tau)|^2} \sum_{(p_L,p_R) \in \Gamma_R^\ell} e^{2\pi i \delta''(p_L + p_R)\bar{\ell}/2} q^{2p_L^2} q^{2p_R^2}$$  \hspace{1cm} (A.11)

Here, the momenta $(p_L, p_R)$ span the lattice $\Gamma_{R/\ell}$, defined by,

$$\Gamma_{R/\ell} \equiv \left\{ (p_L, p_R) = \left( \frac{\ell}{R} m_2 - \frac{R}{2\ell} (m_1 + \delta'), \frac{\ell}{R} m_2 + \frac{R}{2\ell} (m_1 + \delta') \right) \mid m_1, m_2 \in \mathbb{Z} \right\}$$  \hspace{1cm} (A.12)

The modular transformation properties of (A.11) of course follow from those of (A.10) which were discussed after (A.10). But they may also be derived directly from (A.11). This is manifest for $\tau \to \tau + 1$, while for $\tau \to \bar{\tau} = -1/\tau$, it is achieved by applying the Poisson resummation formula (A.6) in both $m_1$ and $m_2$ for $n = 2$ and

$$A = \begin{pmatrix} \frac{R^2}{2\ell^2 \tau_2} & -i\bar{\tau} \\ -i\bar{\tau} & 2\frac{R^2}{2\ell^2 \tau_2} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \delta'' \end{pmatrix} \quad C = \begin{pmatrix} \delta' \\ 0 \end{pmatrix}$$  \hspace{1cm} (A.13)

and one recovers results in accord with (A.10).

The summation over $\delta''$ forces $m_2 = (p_L + p_R) \frac{2\ell}{N_{\ell}}$ to be an integer multiple of $N$, so that

$$m_2 = Nn_2 \text{ with } n_2 \in \mathbb{Z},$$

and eliminates the overall factor of $\frac{1}{N}$. Next, the summation over $\delta'$ is reproduced by replacing $m_1$ with $\frac{1}{N_{\ell}}n_1$, where $n_1 \in \mathbb{Z}$. Combining all, one finds,

$$Z_{S_{R/\ell}/s} = \frac{1}{|\eta(\tau)|^2} \sum_{(p_L,p_R) \in \Gamma_{R/\ell}} q^{2p_L^2} q^{2p_R^2}$$  \hspace{1cm} (A.14)

This expression clearly coincides with $Z_{S_{R/\ell}/s}$. The above formula can be understood in the Hamiltonian picture as a trace over a Hilbert space $\mathcal{H}_{s^a}$ twisted by $s^a$ with the insertion of the $b$-th power of the translation operator $s$ of shifts by $2\pi R/N$,

$$Z_h^g = \text{Tr}_{s^{a^b}} \left( s^{b} q^{p_L^2/2} q^{p_R^2/2} \right) \quad s \equiv \exp \left( 2\pi i (p_L + p_R) \frac{R}{2\ell N} \right)$$  \hspace{1cm} (A.15)
B The point group versus by the full orbifold group

In the construction of an orbifold theory of flat space $\mathbb{R}^n$, the orbifold group $G$ (or more generally the space group of $G$) is a discrete subgroup of the Euclidean group of $\mathbb{R}^n$, consisting of elements $g = (R_g, v_g)$, where $R_g \in O(n)$ is a rotation and $v_g \in \mathbb{R}^n$ a translation. The maximal subgroup of pure translations is denoted $\Lambda_G$ while the subgroup of all elements $R_g$ is the point group $P_G$. The normal subgroup $\bar{P}_G \equiv G/\Lambda_G$ is isomorphic to $P_G$, while the coset $\mathbb{R}^n/\Lambda_G = T^n_G$ is a torus. Symmetric orbifolds may be constructed either coseting $\mathbb{R}^n$ by the full $G$ or coseting $T^n$ by $\bar{P}_G$,

$$\mathbb{R}^n/G = T^n_G/\bar{P}_G \quad (B.1)$$

In this appendix, the simplest non-trivial such case when $n = 1$ and $G = \mathbb{Z}_2 \times \mathbb{Z}$ will be shown to yield the same partition function when treated either way.

B.1 The functional integral from $S^1/\mathbb{Z}_2$

Here, the field $x$ takes values in the circle $S^1_R$. The point group $\bar{P} = \mathbb{Z}_2 = \{(1, 0), (-1, 0)\}$ is Abelian, and hence each element forms a conjugacy class by itself. The centralizer of each element is always the full $\bar{P}$, so that the cardinality of the centralizer is always 2, producing a factor $\frac{1}{2}$. In the twisted sectors, consider all solutions to the twisted boundary conditions. For example, when the twist is placed on the $B$-cycle,

$$x(\sigma^1 + 1, \sigma^2) = +x(\sigma^1, \sigma^2) \mod (2\pi R)$$
$$x(\sigma^1, \sigma^2 + 1) = -x(\sigma^1, \sigma^2) \mod (2\pi R) \quad (B.2)$$

the general solution is a combination of oscillating solutions, plus a constant,

$$x_{m_1, m_2}(\sigma^1, \sigma^2) = x_0 + \exp \left\{ 2\pi i \left[ m_1 \sigma^1 + (m_2 + \frac{1}{2}) \sigma^2 \right] \right\} \quad (B.3)$$

The constant may be interpreted as the center of mass of the string, as usual. It must satisfy $x_0 = -x_0 \mod (2\pi R)$, which produces two solutions or fixed points $x_0 = 0$ and $x_0 = \pi R$. Both fixed points produce equal contributions to the functional integral, whence a factor of 2. Putting all together, one finds

$$\int Dx \ e^{-S[x]} = \frac{1}{2} \ Z_{S^1_R} + \sum_{i=2,3,4} \left| \frac{\eta(\tau)}{\vartheta_i(0, \tau)} \right| \quad (B.4)$$

This formula agrees with [6].
B.2 The functional integral from $\mathbb{R}/(\mathbb{Z}_2 \times \mathbb{Z})$

Quotienting by the full orbifold group, the conjugacy classes are

\[
\begin{align*}
C_1 &= \{(1, 0)\} & N_1 &= G \\
C_2 &= \{(1, m), \ m > 0\} & N_2 &= \{(1, n), \ n \in \mathbb{Z}\} \\
C_3 &= \{(-1, 0)\} & N_3 &= \{(1, 0), (-1, 0)\} \\
C_4 &= \{(-1, 0)\} & N_4 &= \{(1, 0), (-1, 1)\}
\end{align*}
\]

(B.5)

For the classes $C_3$ and $C_4$, the centralizers have cardinality 2, resulting in an overall factor of 1/2 in the partition function. On the other hand, the contributions from $C_3$ and $C_4$ are identical, therefore cancelling the factor of 1/2. Their contributions result in the twisted functional integrals with $\vartheta_i$ for $i = 3, 4$. For the classes $S_1$ and $S_2$, the cardinality of the centralizers are

\[
\#(N_1) = 2\#(\Lambda_R) \quad \#(N_2) = \#(\Lambda_R)
\]

(B.6)

Recall that $\Lambda_R = \{2\pi nR, \ n \in \mathbb{Z}\}$. The factor of 2 arises because $N_1 = \mathbb{Z}_2 \times \Lambda_R$.

The elements in $N_1$ of the type $(-1, n)$ produce $\#(\Lambda_R)$ identical copies of the twisting $(-1, 0)$, which yields the contribution with $\vartheta_2$. To derive the precise weight of this contribution is a little tricky. For a given element $(-1, n)$, the boundary conditions are

\[
\begin{align*}
x(\sigma^1 + 1, \sigma^2) &= +x(\sigma^1, \sigma^2) \\
x(\sigma^1, \sigma^2 + 1) &= -x(\sigma^1, \sigma^2) + 2\pi nR
\end{align*}
\]

(B.7)

The general solution of these equations is

\[
x_{m_1, m_2}(\sigma^1, \sigma^2) = \pi nR + e^{2\pi i((m_1 + \frac{1}{2})\sigma^1 + m_2\sigma^2)}
\]

(B.8)

Here, $n \in \mathbb{Z}$ specifies the center of mass of the string at $\pi nR$. These positions may be viewed as translates of one another by the lattice $\Lambda_{\frac{1}{2}R}$. The original shifts in the boundary conditions were $2\pi nR$, which may be viewed as translates in the lattice $\Lambda_R$. Although both quantities are infinite, their ratio is well-defined, and given by $\#(\Lambda_{\frac{1}{2}R}) = 2\#(\Lambda_R)$. As a result, the factor of $\#(\Lambda_{\frac{1}{2}R})$ cancels the factor of $1/\#(N_1)$ and the contribution involving $\vartheta_2$ is recovered with coefficient 1. The elements in $N_1$ of the type $(1, n)$ combine with all of $N_2$ to produce half of the untwisted partition function.
C \ \vartheta\text{-function identities}

Throughout, the following correspondence of \vartheta\text{-function notations is being used, (see [36]),}

\[
\begin{align*}
\vartheta_1(z, \tau) &= \vartheta_{\frac{1}{2} \frac{1}{2}}(z, \tau) & \vartheta_3(z, \tau) &= \vartheta_{00}(z, \tau) \\
\vartheta_2(z, \tau) &= \vartheta_{\frac{1}{4} 0}(z, \tau) & \vartheta_4(z, \tau) &= \vartheta_{0 \frac{1}{2}}(z, \tau)
\end{align*}
\] (C.1)

where the \vartheta\text{-functions with half-characteristics are defined by,}

\[
\vartheta[\alpha|\beta](z, \tau) \equiv \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n+\alpha)^2+2\pi i(n+\alpha)(z+\beta)}
\] (C.2)

The \vartheta\text{-constants are defined by setting } z = 0. The Dedekind \eta\text{-function is defined by}

\[
\eta(\tau) \equiv e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})
\] (C.3)

and satisfies \(2\eta(\tau)^3 = \vartheta_2(0, \tau)\vartheta_3(0, \tau)\vartheta_4(0, \tau)\). The modular transformations of the \vartheta\text{-constants and \eta\text{-function are given by}

\[
\begin{align*}
\vartheta_2(0, \tau + 1) &= e^{i\pi/4}\vartheta_2(0, \tau) & \vartheta_2(0, -1/\tau) &= \sqrt{-i} \vartheta_4(0, \tau) \\
\vartheta_3(0, \tau + 1) &= \vartheta_4(0, \tau) & \vartheta_3(0, -1/\tau) &= \sqrt{-i} \vartheta_3(0, \tau) \\
\vartheta_4(0, \tau + 1) &= \vartheta_3(0, \tau) & \vartheta_4(0, -1/\tau) &= \sqrt{-i} \vartheta_2(0, \tau) \\
\eta(\tau + 1) &= e^{i\pi/12}\eta(\tau) & \eta(-1/\tau) &= \sqrt{-i} \eta(\tau)
\end{align*}
\] (C.4)

The Jacobi \vartheta\text{-identity is } \vartheta_2(0, \tau)^4 + \vartheta_4(0, \tau)^4 = \vartheta_3(0, \tau)^4. As a result, the following combinations of \vartheta^4 may be expressed in terms of 8-th powers of \vartheta,

\[
\begin{align*}
2\vartheta_2^2\vartheta_3^2 &= \vartheta_2^8 + \vartheta_3^8 - \vartheta_4^8 \\
2\vartheta_3^4\vartheta_4^4 &= \vartheta_3^8 + \vartheta_4^8 - \vartheta_2^8 \\
-2\vartheta_4^2\vartheta_2^2 &= \vartheta_2^8 + \vartheta_4^8 - \vartheta_3^8
\end{align*}
\] (C.5)

The following doubling identities, which relate \vartheta\text{-constants with modulus } 2\tau \mathrm{ to those of modulus } \tau, \text{ will be needed. For general characteristics,}

\[
\vartheta[\alpha|\beta](0, 2\tau)^2 = \frac{1}{2} \left( \vartheta[0|0] \vartheta[2\alpha|\beta] + e^{-2\pi i \alpha} \vartheta[0|\frac{1}{2}] \vartheta[2\alpha|\beta + \frac{1}{2}] \right)(0, \tau)
\] (C.6)

For some of the characteristics needed here, for example,

\[
\begin{align*}
\vartheta_{\frac{1}{4} \frac{1}{2}}(0, 2\tau)^2 &= \frac{1}{2} \vartheta_2\vartheta_3(0, \tau) & \vartheta_{\frac{3}{4} \frac{1}{2}}(0, 2\tau)^2 &= \frac{1}{2} \vartheta_2\vartheta_3(0, \tau) \\
\vartheta_{\frac{1}{4} \frac{1}{2}}(0, 2\tau)^2 &= +\frac{i}{2} \vartheta_2\vartheta_4(0, \tau) & \vartheta_{\frac{3}{4} \frac{1}{2}}(0, 2\tau)^2 &= -\frac{i}{2} \vartheta_2\vartheta_4(0, \tau)
\end{align*}
\] (C.7)
C.1 The function $\zeta[\alpha|\beta](\tau)$

The definition of the function $\zeta[\alpha|\beta](\tau)$ may be given either in terms of $\vartheta$-functions for modulus $2\tau$, or directly in terms of momentum summations, (with $q \equiv e^{2\pi i \tau}$),

$$\zeta[\alpha|\beta](\tau) \equiv e^{-2\pi i \alpha \beta} \frac{\vartheta[\alpha|\beta](0, 2\tau)}{\eta(\tau)} = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(n+\alpha)^2} e^{\pi i \alpha \beta} \quad (C.8)$$

The following periodicity properties are readily derived,

$$\zeta[-\alpha| -\beta](\tau) = \zeta[\alpha|\beta](\tau)$$
$$\zeta[\alpha + 1|\beta](\tau) = e^{-2\pi i \beta} \zeta[\alpha|\beta](\tau)$$
$$\zeta[\alpha|\beta + 1](\tau) = \zeta[\alpha|\beta](\tau) \quad (C.9)$$

Modular transformations act as follows

$$\zeta[\alpha|\beta](\tau + 1) = \omega e^{2\pi i \alpha \beta} \zeta[\alpha|\beta + 2\alpha](\tau)$$
$$\zeta[\alpha|\beta]\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{2}} e^{-2\pi i \alpha \beta} \left(\zeta\left[-\frac{\beta}{2}\right][2\alpha](\tau) + e^{2\pi i \alpha} \zeta\left[\frac{1}{2} - \frac{\beta}{2}\right][2\alpha](\tau)\right) \quad (C.10)$$

where $\omega = \exp\{-i\pi/12\}$. Actually, it is useful to spell out the modular transformations on the characteristics that are needed in the orbifold constructions in this paper,

$$\zeta[0|0](\tau + 1) = \omega \zeta[0|0](\tau) \quad \zeta[0|0]\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{2}} \left(\zeta[0|0] + \zeta[\frac{1}{2}|0]\right)(\tau)$$
$$\zeta[0|\frac{1}{2}](\tau + 1) = \omega \zeta[0|\frac{1}{2}](\tau) \quad \zeta[0|\frac{1}{2}\left(-\frac{1}{\tau}\right) = \sqrt{2} \zeta[\frac{1}{4}|0](\tau)$$
$$\zeta[\frac{1}{2}|0]\left(\tau + 1\right) = \omega e^{i\pi/2} \zeta[\frac{1}{2}|0](\tau) \quad \zeta[\frac{1}{2}|0]\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{2}} \left(\zeta[0|0] - \zeta[\frac{1}{2}|0]\right)(\tau)$$
$$\zeta[\frac{1}{4}|0]\left(\tau + 1\right) = \omega e^{i\pi/8} \zeta\left[\frac{1}{4} \frac{1}{2}\right](\tau) \quad \zeta[\frac{1}{4}|0]\left(-\frac{1}{\tau}\right) = \frac{1}{\sqrt{2}} \zeta[0|\frac{1}{2}](\tau)$$
$$\zeta[\frac{1}{4}|\frac{1}{2}](\tau + 1) = \omega e^{i\pi/8} \zeta\left[\frac{1}{4} \frac{1}{2}\right](\tau) \quad \zeta[\frac{1}{4}|\frac{1}{2}\left(-\frac{1}{\tau}\right) = \zeta[\frac{1}{4} \frac{1}{2}](\tau) \quad (C.11)$$

Another useful fact is that for $a \in 2\mathbb{Z} + 1$,

$$\zeta[\frac{a}{4} + \frac{1}{2} | \frac{b}{2}] = e^{i\pi(a+1)b/2} \zeta[\frac{a}{4} | \frac{b}{2}] \quad (C.12)$$

both of which are proportional (with a $\pm$ factor) to $\zeta[\frac{1}{4} | \frac{b}{4}]$. Other useful identities are,

$$\frac{q^{-1/24}}{\prod_{n=1}^{\infty} (1 + q^n)} = \zeta[0|\frac{1}{2}](\tau)$$
$$\frac{q^{1/48}}{\prod_{n=1}^{\infty} (1 - (-) b q^n)} = \zeta[\frac{1}{4} | \frac{b}{2}](\tau) \quad (C.13)$$
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