Relativistic extension of the dynamical friction in stellar systems I: General formalism

Caterina Chiari,1 Pierfrancesco Di Cintio,1,2,3,4*

1Dipartimento di Fisica e Astronomia, Università di Firenze, via G. Sansone 1, I-50019 Sesto Fiorentino, Italy
2INAF - Sezione di Firenze, via G. Sansone 1, I-50019 Sesto Fiorentino, Italy
3INAF - Osservatorio Astrofisico di Arcetri, Largo Enrico Fermi 5, I-50125 Firenze, Italy
4CNR - ISC, Via Madonna del piano 10, I-50019 Sesto Fiorentino, Italy

Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT

Aiming at investigating the collisional dynamics of massive stars in dense stellar systems hosting a black hole, two relativistic generalizations of the Chandrasekhar dynamical friction formula are derived and discussed. We extend the original formalism to the case of relativistic velocities and distributions, and we account for strong encounters with massive objects by including the first post-Newtonian correction (1PN), so that the effects of general relativity, at the lowest order, are accounted.

Key words: stars: kinematics and dynamics – galaxies: kinematics and dynamics – stars: black holes – methods: analytical

1 INTRODUCTION

Dynamical friction (hereafter DF) is an important physical phenomenon, with several consequences in stellar dynamics (and in plasma physics). It can be qualitatively thought of as the slowing-down of a test particle of mass $M$, moving at $v_T$ in a background of field particles of mass $m$, mean number density $n$ and velocity distribution $f(v_e)$, due to the cumulative effect of their long-range gravitational (or Coulomb) interactions.

An analytical estimation of the DF was evaluated for the first time for stellar systems by Chandrasekhar (1943), who found that $M$ must experience a slowing down along its initial direction of propagation as

$$\frac{d v_T}{dt} = -4\pi G^2 n m (M + m) \log \Lambda \frac{\Xi(v_T)}{v_T^2} v_T.$$  \hspace{1cm} (1)

In the equation above, $G$ is the gravitational constant, $\log \Lambda$ is the so-called Coulomb logarithm (in analogy with the analogous quantity in plasma physics, Spitzer 1965) of the ratio $\Lambda$ of the maximum and minimum impact parameters $b_{\text{max}}$ and $b_{\text{min}}$ and

$$\Xi(v_T) \equiv 4\pi \int_0^{v_T} f(v_e)v_e^2 dv_e$$ \hspace{1cm} (2)

is the fractional velocity volume function.

The process of DF appears to be crucial for the evolution of collisional systems (see e.g. Binney & Tremaine 1987) from the "large scales", of galaxies clusters (Ostriker & Tremaine 1975; Richstone 1976; Gunn & Tinsley 1976; Adhikari et al. 2016), to the "smaller scales" for its consequences on the motion of supermassive black holes (SMBHs) in galactic cores (Antonini & Merritt 2012; Tremmel et al. 2018; Di Cintio et al. 2020; Ricarte et al. 2021; Chen et al. 2022), globular clusters (GCs) orbiting their host galaxies (Weinberg 1989; Colpi & Pallavicini 1998; Bertin et al. 2003; Arena et al. 2006; Arena & Bertin 2007), or exotic stellar objects such as for example blue straggler stars (BSS, Ferraro et al. 1995) in GCs (see Paresce et al. 1992; Ferraro et al. 1995; Proctor Sills et al. 1995; Ferraro et al. 2001, 2009; Ransom et al. 2005; Pooley & Hut 2006; Alessandri et al. 2014, 2016; Miocchi et al. 2015; Pasquato et al. 2018; Pasquato & Di Cintio 2020).

Since the pioneering work of Chandrasekhar, the DF formalism, initially conceived for a point-like particle in an infinitely extended background of scatterers, has been extended to the case of finite-sized objects (e.g. see Mulder 1983; Zel’nikov & Kuskov 2016) sinking in the host stellar system, flattened or spherical models with self-gravity (e.g. see Kalnajs 1971; Tremaine & Weinberg 1984) or spheroids with anisotropic velocity distribution (Binney 1977).

More recently, Ciotti & Binney (2004) and Nipoti et al. (2008) derived an expression for the DF formula in the case of modified Newtonian dynamics (hereafter MOND, Milgrom 1983; Bekenstein & Milgrom 1984), and performed $N$-body simulations of sinking satellites in MOND, while Ciotti (2010) considered the effect of a mass spectrum for the field particles and Silva et al. (2016) that of a non-thermal,
power-law-like velocity distribution.

The main conclusion of these works is that, in general, using the original formulation of the DF for idealized infinite systems (cfr. Eq. 1), leads to a substantial underestimation (even of a factor 10) of its effectiveness when applied to more realistic models.

Prompted by the detection of gravitational waves (GWs) from binary compact objects announced by the LIGO and VIRGO collaborations (Abbott et al. 2016a,b), a renewed interest in relativistic stellar dynamics (Shapiro & Teukolsky 1985; Hamers et al. 2014) has recently widespread, in particular with respect to the formation and migration processes of single and binary black holes (BHs) in GCs (Torniamenti et al. 2022; Ellis et al. 1983; Bhat et al. 2022), Supermassive Black Holes in galactic cores (Kandrup 1984; Correia; and references therein), on the other hand much less has been done in the case of a relativistic test particle crossing an isotropic distribution. In this work we explore this matter further aiming at formulating a general treatment of DF that can be applied in different regimes dominated by collisional effects, involving the precession effect induced by the contribution of the field particles as a hyperbolic two body problem in the frame centered on the field particle. Let \((x_T, v_T)\) and \((x_r, v_r)\) be the positions and velocities of \(M\) and \(m\), respectively, and let

\[
\mathbf{r} = x_T - x_r; \quad \mathbf{v} = v_T - v_r
\]

be their relative position and velocity. We recall that the equation of motion for a fictitious particle of reduced mass \(\mu = mM/(M + m)\) moving in the Keplerian potential of the fixed body of mass \(M + m\), is

\[
\frac{mM}{m + M} \mathbf{F} = -\frac{GMm}{r^2} \mathbf{e}_r. \tag{4}
\]

The energy conservation along the orbit of \(\mu\) for a given encounter with impact parameter \(b\), implies that the relative velocity vector \(\mathbf{V}\) is deflected by an angle \(\pi - 2\psi\), in the orbital plane defined by

\[
\cos \psi = \frac{1}{\sqrt{1 + \frac{b^2}{c^2(M + m)^2}}}. \tag{5}
\]

Using finite differences, we can always express the relative velocity change as

\[
\Delta \mathbf{V} = \Delta \mathbf{V}_F - \Delta \mathbf{V}_T, \tag{6}
\]

where \(\Delta \mathbf{V}_F\) and \(\Delta \mathbf{V}_T\) are the velocity variations of \(m\) and \(M\) during the encounter. Since the velocity of the center of mass is constant (by definition) during the encounter, we have that \(m\Delta \mathbf{V}_F + M\Delta \mathbf{V}_T = 0\).

Eliminating \(\Delta \mathbf{V}_F\) in the two equations above yields

\[
\Delta \mathbf{V}_T = -\left(\frac{m}{m + M}\right) \Delta \mathbf{V}. \tag{7}
\]

We must now evaluate \(\Delta \mathbf{V}\) in order to find \(\Delta \mathbf{V}_T\). The conserved angular momentum per unit mass of the reduced particle is \(L = b \mathbf{V}\). Let us now label with \(\theta_{\text{defl}}\) the deflection angle. The relation between the radius and azimuthal angle of a particle on a Keplerian orbit becomes

\[
\frac{1}{r} = C \cos(\psi - \psi_0) + \frac{G(M + m)}{b^2 V^2}, \tag{9}
\]

where the constant \(C\) and the phase angle \(\psi_0 = \psi(t = 0)\) are determined by the initial conditions. Deriving (9) with respect to the time we obtain

\[
\frac{dr}{dt} = C r^2 \dot{\psi} \sin(\psi - \psi_0) = C b V \sin(\psi - \psi_0), \tag{10}
\]

where the last term arises from \(L = r^2 \dot{\psi}\). If we impose that \(\psi = 0\) when \(t \to -\infty\) we obtain from (10)

\[
-V = C b \sin(-\psi_0). \tag{11}
\]

Evaluating equation (9) we then have

\[
0 = C \cos(\psi_0) + \frac{G(M + m)}{b^2 V^2}, \tag{12}
\]

and eliminating \(C\) from the equations above we obtain

\[
\tan \psi_0 = -\frac{b V^2}{G(M + m)}. \tag{13}
\]
From Equations (9) and (10) we can appreciate that the point of closest approach is reached when $\psi = \psi_0$ and, since the orbit is symmetrical about this point, the deflection angle is $\theta_{\text{def}} = 2\psi_0 - \pi$. Thanks to the conservation of energy, after the encounter, the modulus of the relative velocity, equals the modulus of the initial relative velocity and therefore, the components of $\Delta V$, parallel and perpendicular to the initial relative velocity vector $V$, $\Delta V_\parallel$ and $\Delta V_\perp$ become

$$
||\Delta V_\perp|| = \frac{2bV^3}{G(M+m)} \left[ 1 + \frac{b^2V^4}{G^2(M+m)^2} \right]^{-1} ;
$$

(14)

and

$$
||\Delta V_\parallel|| = 2V \left[ 1 + \frac{b^2V^4}{G^2(M+m)^2} \right]^{-1}.
$$

(15)

In a homogeneous background of particles equal masses $m$, all $\Delta V_{T,\perp}$ sum to zero by symmetry, (using the so-called "Jeans swindle", see e.g. Binney & Tremaine 1987), while the parallel velocity changes add up, and thus the mass $M$ will experience a deceleration (cfr Eq. (8)) as a result of the DF. Therefore, it is sufficient to evaluate $\Delta V_{T,\perp}$ as

$$
||\Delta V_{T,\perp}|| = \frac{2mV}{M+m} \left[ 1 + \frac{b^2V^4}{G^2(M+m)^2} \right]^{-1}.
$$

(16)

In a system defined by the phase-space distribution function $F = nf(\nu F)$, where $n$ is a constant number density and $f(\nu F)$ be the velocity distribution, the rate at which the mass $M$ encounters stars with impact parameter between $b$ and $b+db$, and velocities between $\nu S$ and $\nu F + d\nu F$, is

$$
n_{\text{enc}} = 2\pi bdbV nf(\nu F) d^3 \nu F,
$$

(17)

where $d^3 \nu F$ is the velocity-space element. The total change in velocity suffered by $M$ is found by adding all the contributions of $||\Delta V_{T,\perp}||$ due to particles with impact parameters from $b_{\text{min}}$ to $b_{\text{max}}$ and then summing over all velocities of stars as

$$
\left. \frac{d\nu F}{dt} \right|_{\nu F} = V nf(\nu F) d^3 \nu F \int_{b_{\text{min}}}^{b_{\text{max}}} ||\Delta V_{T,\perp}|| 2\pi bdb =
$$

$$
= V nf(\nu F) d^3 \nu F \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{2mV}{M+m} \left[ 1 + \frac{b^2V^4}{G^2(M+m)^2} \right]^{-1} 2\pi bdb.
$$

(18)

Let us first perform the integral over $b$

$$
\int_{b_{\text{min}}}^{b_{\text{max}}} \frac{2mV}{M+m} \left[ 1 + \frac{b^2V^4}{G^2(M+m)^2} \right]^{-1} 2\pi bdb =
$$

$$
= 2\pi G^2(M+m) \frac{V^3}{V^3} \log \left[ 1 + \frac{b_{\text{max}}^2V^4}{G^2(M+m)^2} \right] \frac{1}{1 + \frac{b_{\text{min}}^2V^4}{G^2(M+m)^2}}.
$$

(19)

We must note that, choosing the minimum and maximum impact parameters is a rather delicate step. When using the impulsive approximation (i.e. $\mu ||\Delta V_\perp|| = 2GMm/V^2$; see e.g. Ciotti 2010), the so-called "ultraviolet divergence" occurs when performing the integral over $b$ in (18) and setting $b_{\text{min}} = 0$. The latter divergence is actually artificial as it disappears when the full solution of the hyperbolic two body problem is taken into account as in this discussion. Conversely, the "infrared divergence" appearing for $b \to \infty$ cannot be eliminated in an infinite system and therefore one must put a upper cutoff with a suitable choice of $b_{\text{max}}$.

Not surprisingly, this point is still source of debate (see the discussion in Van Albada & Szomoru 2020 and references therein). The problem lies in what should dominate between few strong encounters with nearby stars (see Chandrasekhar 1941, 1942, 1943, see also Kandrup 1983), or many weak encounters with distant stars (see Spitzer 1987; Binney & Tremaine 1987). In the first interpretation $b_{\text{max}}$ should be of the order of the average inter-particle distance, while in the second, it should be of the order of size of the system. Both views are in principle plausible, the former as it is more intuitive to think that the largest contribution must be due to nearest stars and the latter because there is no screening in gravitational systems, at variance with (quasi-)neutral Plasmas where charges of opposite signs are present. In this work we follow the Spitzer approach.

We note that, under most conditions of practical interest in astrophysics, the quantity $\Delta^2 = b^2V^4/G^2(M+m)^2$ is typically much greater than unity. For this reason, we will now on replace $\log 1 + \Delta^2$ with $2 \log \Delta$, recovering the widely used definition of the Coulomb logarithm as $\log \left[ \frac{b_{\text{max}}^2V^4}{G^2(M+m)^2} \right]$.1

In this approximation, the right hand side of Eq. (19) becomes

$$
2\pi G^2(M+m) \frac{V^3}{V^3} \log \left[ 1 + \frac{b_{\text{max}}^2V^4}{G^2(M+m)^2} \right] \approx 4\pi G^2(M+m) \frac{V^3}{V^3} \log \Delta.
$$

(20)

Combining the expression above with Eq. (18) yields

$$
\frac{d\nu F}{dt} = -4\pi G^2 nm(M+m) \log \Delta \int f(\nu F) \frac{\nu F - \nu F}{||\nu F - \nu F||^3} d^3 \nu F,
$$

(21)

where we have replaced $V$ with its definition (cfr. Eq. 3) and assumed $\log \Delta$ as the velocity averaged Coulomb logarithm (e.g. see the discussion in Ciotti 2021). The velocity integral in Eq. (21) is often referred to as the first Rosenbluth potential (see e.g. Rosenbluth et al. 1957).

Remarkably, the problem of computing the acceleration $d\nu F/dt$ integrating over all field star velocities, is formally equivalent to that of evaluating the gravitational field at $\nu F$ generated by the "mass density" $\rho(\nu F) = 4\pi \log M(Gm(M+m)f(\nu F)$. Assuming an isotropic (spherically symmetric) velocity distribution, in virtue of the second Newton’s theorem (e.g. see Chandrasekhar 1995) we have that only the stars such that $\nu F < \nu F$ contribute to the slowing down of $M$, hence

$$
\frac{d\nu F}{dt} = -16\pi^2 G^2 nm(M+m) \frac{\nu F}{V^2} \int_{0}^{\nu F} f(\nu F) e^{\nu F} d\nu F.
$$

(22)

In the special case where $f(\nu F)$ is a Maxwellian with dispersion $\sigma$,

$$
f(\nu F) = \frac{1}{(2\pi \sigma^2)^{3/2}} \exp \left( -\frac{\nu F}{2\sigma^2} \right).
$$

(23)

1 In most systems of astrophysical interest $\log \Delta$ is a number of order 10. For example, in a globular cluster of mass $M_{\text{GC}} = 10^6 M_\odot$ sinking at $V \sim 100 \text{km/s}$ through a galaxy of radius $b_{\text{max}} \approx 2 \text{ kpc}$ in which the stars have mass of the order of one Solar mass, we have that $\log \left[ 1 + \frac{b_{\text{max}}^2V^4}{G^2(M+m)^2} \right] \approx 17$. 

MNRAS 000, 1–13 (2022)
evaluating the velocity volume function integral in Eq. (22) yields
\[
\frac{d\mathbf{v}_T}{dt} = -4\pi G^2 n m (M+m) \log \Lambda \left[ \text{Erf} \left( \frac{v_T}{\sqrt{2} \sigma} \right) - \frac{2 v_T e^{-v_T^2/2 \sigma^2}}{\sqrt{2 \pi \sigma}} \right] \mathbf{v}_T, \\
\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.
\]
where Erf(x) is the standard error function defined (see Arfken et al. 2012) as
\[
\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.
\]

3 DYNAMICAL FRICTION: THE RELATIVISTIC GENERALIZATION

3.1 Relativistic velocity composition

As mentioned above, the Chandrasekhar DF formula was extended to relativistic velocities by Syer (1994), but only in the weak scattering limit (i.e. \( b \gg r_o \approx GM/c^2 \) and small deflection angle \( \theta_{\text{defl}} \)). We will now derive a more general expression for a general \( \theta_{\text{defl}} \), therefore also accounting for the case of strong scattering; (i.e. \( \theta_{\text{defl}} \to \pi/2 \)).

In this derivation we will keep the classical 1/r² Newtonian force and replace velocity composition with its relativistic counterpart. For this purpose, we define an inertial frame \( S' \), in which the test star of mass \( M \) is stationary at the beginning of the encounter (i.e. the field star \( m \) is at infinity). In said frame, \( \theta_{\text{defl}} = \pi - 2\psi_i \), (with \( \psi_i \) given by Eq (5)), is the deflection angle according to the classical unbound two body problem, where now \( V \) is the relativistic relative velocity given by (e.g. see Landau & Lifshitz 1976)
\[
V^2 = \frac{||\mathbf{v}_T - \mathbf{v}_F||^2 - \frac{1}{2\sigma^2} ||\mathbf{v}_T \wedge \mathbf{v}_F||^2}{(1 - \frac{v_T^2}{c^2})},
\]
for arbitrary choices of \( \mathbf{v}_T \) and \( \mathbf{v}_F \). We stress the fact that, in the relativistic case, the velocity \( \mathbf{v}_{\text{AB}} \) of a body A respect to another B is not equal to \( -\mathbf{v}_{\text{BA}} \) of B with respect to A. This loss of symmetry is related to the Thomas precession and the fact that two subsequent Lorentz transformations rotate the coordinate system (cfr. Weinberg 1972). This rotation however, has no effect on the magnitude of a vector and hence, the modulus of the relative velocity is symmetric.

Let \( v_{F,\mu} = \gamma_F (c, v_F) \) and \( v_{T,\mu} = \gamma_T (c, v_T) \) be the fourvelocities of the field and test stars, respectively and where \( \gamma_{F,T} = \left(1 - v_{F,T}^2/c^2\right)^{-1/2} \) are the Lorentz factors of \( M \) and \( m \). Let us consider a single encounter in the "laboratory frame" \( S \). This process can be expressed as a product of a Lorentz boost \( \Gamma \) in \( S' \), a rotation \( \mathcal{R}(\theta_{\text{defl}}) \) in the 3-space and, finally, an inverse boost \( \Gamma^{-1} \), reverting back to \( S \).

Defining \( p_{\mu} = m \gamma_F (c, v_F) \) as the 4-momentum of \( m \) before the encounter, we have, that \( p^\mu = \Lambda^\mu_\nu p^\nu \), where \( \Lambda = \Gamma^{-1} \mathcal{R}(\theta_{\text{defl}}) \Gamma \) and thus formally obtain
\[
\Delta p^\mu = (\Gamma^{-1} \mathcal{R}(\theta_{\text{defl}}) \Gamma - 1) p^\mu.
\]
Since the motion is planar, as we are still dealing with a classical two body problem, we can simplify the notation involving 4-vectors by using 3-vectors instead, where only two of space dimensions are maintained; one parallel and one perpendicular to \( \mathbf{v}_T \). However, as argued before, by reasons of symmetry, only the parallel component of \( V \) contributes to the DF. Denoting with \( \varphi \), the angle between \( \mathbf{v}_T \) and \( \mathbf{v}_F \), we can now write
\[
p^\mu = m \gamma_F \left( v_F \cos \varphi \frac{c}{v_T} \right).
\]
With such choice \( \Gamma \), \( \mathcal{R} \) and \( \Gamma^{-1} \) read
\[
\Gamma = \begin{pmatrix}
\frac{\gamma_T}{c} & -\gamma_T & 0 \\
\gamma_T & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\mathcal{R}(\theta_{\text{defl}}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{\text{defl}} & -\sin \theta_{\text{defl}} \\
0 & \sin \theta_{\text{defl}} & \cos \theta_{\text{defl}}
\end{pmatrix},
\Gamma^{-1} = \begin{pmatrix}
\gamma_T & -\gamma_T & 0 \\
\gamma_T & \gamma_T & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

After the encounter the 4-momentum of the field star \( p^\mu \) changes by \( \Delta p^\mu \) defined as
\[
\Delta p^\mu = m \gamma_F \begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
\]
where, respectively,
\[
A = c^2 \gamma_T^2 - \gamma_T^2 \frac{v_T}{c} \gamma_F \cos \varphi + \frac{v_T}{c} \gamma_T [(-v_T \gamma_T + \gamma_T v_F \cos \varphi) \times \cos \theta_{\text{defl}} - v_F \sin \varphi \sin \theta_{\text{defl}}] - c
\]
\[
B = \frac{v_T}{c} \gamma_T^2 (c - \frac{v_T}{c} \gamma_F \cos \varphi) + \gamma_T [(-v_T \gamma_T + \gamma_T v_F \cos \varphi) \times \cos \theta_{\text{defl}} - v_F \sin \varphi \sin \theta_{\text{defl}}] - \gamma_T v_F \cos \varphi
\]
\[
C = (-v_T \gamma_T + \gamma_T v_F \cos \varphi) \sin \theta_{\text{defl}} + v_F \sin \varphi \cos \theta_{\text{defl}} - v_F \sin \varphi,
\]
and where
\[
\theta_{\text{defl}} = \pi - 2 \cos^{-1} \left( \frac{1}{\sqrt{1 + \frac{1}{4 \gamma_T^4 V^2}}} \right)
\]
is the deflection angle².
We now have to multiply \( \Delta p^\mu \) for the differential number of encounters \( d\nu_{\text{enc}} = 2\pi n V \theta_{\text{defl}} dv dv dB dt \) in the laboratory frame. The latter, at variance with the one given by Eq. (17), is a Lorentz-invariant quantity³, where \( V = ||\mathbf{v}_T - \mathbf{v}_F||^2 - 1/c^2 ||\mathbf{v}_T \wedge \mathbf{v}_F||^2 \)^{1/2}.

In integral form, the momentum variation of the field particle \( m \) is now given by
\[
\frac{dp^\mu}{dt} = \int 2\pi n \theta_{\text{defl}} \mathcal{D} \mathcal{P} \mathcal{F}_f (\mathbf{v}_F) d^3 \mathbf{v}_F.
\]

The momentum of the test particle \( m \), \( P^\mu = M \gamma_T (c, v_T) \), will suffer the opposite change
\[
\frac{dP^\mu}{dt} = - \frac{dp^\mu}{dt}.
\]

² We recall that in the derivation carried out by Syer (1994) the expression for the deflection angle \( \theta_{\text{defl}} \) is taken in the limit for a massless particle (i.e. photon) \( \theta_{\text{defl}} = 2(1 + c^2/V^2)GMc^2/b \), see Misner et al. (2017).
³ Notably, due to the relativistic length contraction, the number density \( n \) along the direction of \( M \) would increase in the rest frame of \( m \), cfr. Landau & Lifshitz (1976).
We now substitute the expression for the deflection angle (32) in Eqs. (29-31). Using the standard trigonometric identities, after some trivial but tedious algebra we obtain
\[
\cos \theta_{\text{def}} = \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}, \quad \sin \theta_{\text{def}} = \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}.
\]

(35)

and
\[
A = c \gamma T - \frac{\gamma T}{c} \frac{1}{c} v F \cos \varphi + \frac{\gamma T}{c} \gamma T \left[ (-v T \gamma T + \gamma T v F \cos \varphi) \times \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}} {1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \sin \varphi \frac{2 \lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - c \right]
\]
\[
B = \frac{\gamma T}{c} (c - \frac{v T}{c} v F \cos \varphi) + \gamma T \left[ (-v T \gamma T + \gamma T v F \cos \varphi) \times \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}} {1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \sin \varphi \frac{2 \lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \cos \varphi \right]
\]
\[
C = \gamma T \left[ (-v T \gamma T + \gamma T v F \cos \varphi) \times \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}} {1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \sin \varphi \frac{2 \lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \cos \varphi \right]
\]
\[
\cdot \frac{2 \lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \sin \varphi.
\]

(36)

It is important to stress that for the derivation the DF formula, only the parallel component of \(v_F\) contributes, so it is sufficient to evaluate
\[
\frac{dp}{dt} = \int 2\pi n b m_\gamma v F f(v_F) \left\{ \frac{v T}{c} \gamma T (c - \frac{v T}{c} v F \cos \varphi) + \gamma T \left[ (-v T \gamma T + \gamma T v F \cos \varphi) \times \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}} {1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \sin \varphi \frac{2 \lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - c \right] \right\} dB d^3 v_F.
\]

(37)

Following the classical derivation discussed in Sect. 2, we perform first the integral over the impact parameter \(b\) obtaining
\[
\frac{dp}{dt} = \frac{\left( b_{\text{max}}^2 - b_{\text{min}}^2 \right)}{2} \int 2\pi n b m_\gamma v F f(v_F) v_F \cos \varphi d^3 v_F + -2G^2(M+m)^2 \log \Lambda \int 2\pi n b m_\gamma v F f(v_F) \gamma T \times \left[ (-v T \gamma T + \gamma T v F \cos \varphi) \times \frac{-1 + \frac{\lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}}} {1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \sin \varphi \frac{2 \lambda^4}{1 + \frac{\lambda^4 G^2(M+m)^2}{c^4}} - v F \cos \varphi \right] dB d^3 v_F.
\]

(39)

We note that, as we are considering a relativistic set-up, a convenient choice for minimum impact parameter in Eq. (38) could be the Schwarzschild radius of the reduced mass \(b_{\text{min}} = r_{s,\mu} \equiv 2G\mu/c^2\).

In the limits of \(c \to +\infty\), and \(V \to ||v_T - v_F||\) in the equation above, Equation (21) is recovered with \((M+m)^2\) in lieu of \(m(M+m)\). The reason of this discrepancy with the classical case is due to the fact that, in the relativistic treatment we used the total relativistic momentum conservation Equation (34), where classically one uses Eqs. (6-8). The application of the latter would in principle commute with the limit \(c \to +\infty\) (and the term \((M+m)\) at the denominator in 8 would elide with another identical factor in 38), while Eq. (34) loses meaning in such limit, as the time component would diverge.

We note that Karhuj (1972) also encounters a similar disagreement between the Chandrasekhar expression and the particle limit of his fluid formalism, i.e. the frictional force on the test mass is proportional to \(M^2 m\) rather than \(M(M+m)\). However, the reason of this difference is ascribed to the fluid picture where the force on \(M\) is due to a continuous mass density distribution (see e.g. Kandrup 1983).

We note also that, in the fully kinetic approach on DF based on the Fluctuation-Dissipation theorem, pioneered among the others by Bekeinstein & Mao (1992), one recovers the original expression formulated by Chandrasekhar with the term \(mM(M+m)\).

3.2 Relativistic velocity distribution

As we are accounting for relativistic velocities, when performing the velocity integral in (38), with \(V\) given by Eq. (26), we need to use a distribution function in covariant form. In this work we adopt the Maxwell-Jüttner distribution (see Jüttner 1911, see also Fackerell 1968)
\[
f(v_F) = \frac{\gamma_5 v_F^5}{c^3 \Theta K_2(\Theta^{-1})} \exp \left( -\frac{\gamma_5 v_F^5}{c^3 \Theta K_2(\Theta^{-1})} \right)
\]

(40)

In the expression above \(\Theta = \sigma^2/3c^2\) and \(K_2(\mu)\) is the so-called Neumann function, often also dubbed modified Bessel function of the second kind (see e.g. Arfenk et al. 2012). With such choice Eq. (38) becomes
\[
\mathcal{V} = V (1 - v_T \cdot v_F/c^2).
\]
The four integrals appearing in Equation (41), can be separated independently in a similar fashion as

\[
I_i \equiv \int_0^{v_{\text{v}}} \left( \ldots, \ldots, \ldots \right) d\nu \gamma + \int_{v_{\text{v}}}^c \left( \ldots, \ldots, \ldots \right) d\nu \gamma + \int_{v_{\text{v}}}^{c_{\text{finite}}} \left( \ldots, \ldots, \ldots \right) d\nu \gamma
\]

where the last term is negligible due to the rapid decay of the factors \(\exp(-3\epsilon^2 \gamma^2 / \sigma^2)\) in the limit \(\gamma \to 0\). It is therefore sufficient to evaluate the remaining terms. Let us now recover the strong scattering case restarting from Eq. (38). When \(\theta_{\text{def}} \to \pi/2\) in Eq. (32), we have that

\[
\cos^{-1}\left( \frac{1}{\sqrt{1 + \frac{\nu^2}{\sigma^2(M+m)}}} \right) = \frac{\pi}{4}.
\]

In the limit \(b^2 V^4 / G^2 (M+m)^2 \to 1\) the DF formula (38) can be re-written as

\[
\frac{d\nu}{d\eta} \bigg|_{\eta=1} = \int 2\pi b \, d\mu \gamma \nu f(\nu) \times \left[ \frac{\nu^2}{c^2} \left( \frac{c - \nu \nu \gamma \sin \phi - \nu \nu \gamma \cos \phi}{c^2} \right) \right] d\nu \gamma
\]

in virtue of the limits \(\lim_{\theta_{\text{def}} \to \pi/2} \cos \theta_{\text{def}} = 0\) and \(\lim_{\theta_{\text{def}} \to \pi/2} \sin \theta_{\text{def}} = 1\).

Since \(V^2 = (\sqrt{1 - v_T^2 \nu \gamma} \cos \phi / c^2)^2\), with the aid of (43), we may replace \(\sqrt{V}\) with \(\sqrt{G(M+m)/b(1 - v_T^2 \nu \gamma \cos \phi / c^2)}\), so that Eq. (44), with the assumption of an isotropic velocity distribution, becomes

\[
\frac{d\nu}{d\eta} \bigg|_{\eta=1} = \int 2\pi b \gamma \nu \sqrt{G(M+m)/b} \left( 1 - \frac{v_T \nu \gamma \cos \phi}{c^2} \right) 4\pi \nu^2 \nu f(\nu) \times \left[ \frac{\nu^2}{c^2} \left( \frac{c - \nu \nu \gamma \sin \phi - \nu \nu \gamma \cos \phi}{c^2} \right) \right] d\mu d\nu \gamma.
\]

In analogy with the classical case we perform first the integral over the impact parameter, that after some algebraic manipulation yields

\[
\frac{d\nu}{d\eta} \bigg|_{\eta=1} = \frac{16\pi^2 m \sqrt{G(M+m)}(\theta_{\text{max}} - \theta_{\text{min}})}{3} \times \left( \nu_T^2 F(\nu_T) \left( 1 - \frac{v_T \nu \gamma \cos \phi}{c^2} \right) \times \left( \nu_T \nu \gamma \sin \phi - \nu_T \nu \gamma \cos \phi \right) \right) d\nu_T.
\]

We notice that, while in Eq. (41) the dependence on \(\phi\) is only virtual, as it integrates out in all members by the assumption of isotropy of the velocity (and density) distributions, in Eq. (46), due to the choice of \(\theta_{\text{def}}\) angle, which contains \(\phi\) in its definition (cfr. Eqs. 43 and 26) such dependence cannot be neglected. This amounts to an additional integration over such angular variable when evaluating (numerically) Equation (46) for a given choice of \(f(\nu_T)\).

4 POST-NEWTONIAN APPROXIMATION

So far, we have derived a formal generalization of the dynamical friction formula in the limit of large test particle velocities or relativistic velocity distributions assuming classical Newtonian forces. We now carry out an alternative derivation involving strong gravitational scattering in the post-Newtonian regime.

Introduced by Einstein (1915) (see also Weinberg 1972; Blanchet 2010 and references therein) to study the precision of the perihelion of Mercury, the post-Newtonian approximation consists in an expansion in orders of the parameter \(v/c\), such that at the zero-th order it reduces to Newtonian gravity, while at higher orders (nPN) the acceleration on the mass \(m\) due to the mass \(M\) is augmented by corrections of order \((v/c)^{2n}
\)

\[
\sim (GM/rc)^n.
\]

4.1 Non relativistic velocities

The cores of dense star clusters are often dominated by massive objects due to dynamical mass segregation, it is therefore interesting to evaluate the DF on a test particle in such an environment where strong scattering by large masses are likely to happen, even though the velocity distribution might not be relativistic. To do so, we begin with a naive derivation of \(d\nu/d\eta\) in impulsive approximation keeping the Galilean transformations of velocities, but using the 1PN-acceleration

\[
a_{1\text{PN}} = -\frac{G(M+m)}{r^2} + \frac{G(M+m)}{c^2 V^2} \left\{ (4+2\eta) \frac{G(M+m)}{r} + (1+3\eta) V^2 + \frac{3}{2} \frac{\hat{r}}{r} \right\} \cos^2 \phi.
\]

in the frame centered on the field particle \(m\) (see Mora & Will 2004; Cashen et al. 2017). In the equation above \(\mu\), \(V\) and \(\hat{r}\) have the same meaning as in Sect. 2 and \(\eta = \mu/(M+m)\). In impulsive approximation we can express the velocity change of the test particles in a discrete time interval \(\Delta t = 2b/V\) as \(||\Delta V\parallel \sim a\Delta t = 2ab/V\), so that, see e.g. Ciotti (2021), its parallel component becomes

\[
\Delta V_{||} \sim \frac{\mu}{M} \frac{||\Delta V_{\perp}\parallel}{2V^2} V.
\]

Defining \(r = \hat{r} \sim b \hat{r}\) and \(V = V \hat{V}\), we obtain the perpendicular relative velocity change as

\[
\Delta V_{\perp} \sim a_{1\text{PN}} \frac{2b}{V} \sim -\frac{2G(M+m)}{V b} \hat{r} + \frac{2G(M+m)}{c^2 V b} \left\{ (4+2\eta) \frac{G(M+m)}{b} + (1+3\eta) V^2 + \frac{3}{2} \frac{\hat{r}}{r} \right\} \cos^2 \phi
\]

and its square as

\[
||\Delta V_{\perp}\parallel^2 \sim \frac{4G^2 (M+m)^2}{V^2 b^2} - \frac{8G^2 (M+m)^2}{c^2 V^2 b^2} \times \left( (4+2\eta) \frac{G(M+m)}{b} + 3V^2 - \frac{7}{2} \frac{\hat{r}}{r} \right) \cos^2 \phi.
\]

where we have assumed that \(\hat{r} \cdot \hat{V} \sim 1\) and dropped the terms proportional to \(1/c^4\). The parallel velocity change of the test
 particle becomes

\[ \Delta v_{\parallel} \sim - \frac{\mu}{M} \frac{||\Delta V||^2}{2V^2} V \sim \]

\[ \sim \left[ - \frac{2mG^2(M + m)}{b^2V^4} + \frac{4G^2m(M + m)^2(4 + 2\eta)}{c^2b^2V^2} \right] V, \]  

so that, assuming again as in Section 2 that the number of encounters is given by (17), the finite differences velocity change of particle \( M \) is expressed as

\[ \frac{\Delta v_T}{\Delta t} = 2\pi bdbV n f(v_T) d^3v_T \left[ - \frac{2mG^2(M + m)}{b^2V^4} V + \frac{4G^2m(M + m)^2(4 + 2\eta)}{c^2b^2V^2} \right]. \]  

With the standard integration over the impact parameter \( b \), we easily obtain

\[ \frac{dv_{T\parallel}}{dt} = -4\pi mnG^2(M + m) \log \Lambda \int \frac{f(v_T)}{V^3} d^3v_T \]

\[ + \frac{16\pi mnG^2(M + m)^2(2 + \eta)}{c^2} \left( \frac{1}{b_{\min}} - \frac{1}{b_{\max}} \right) \int \frac{f(v_T) V}{V^3} d^3v_T \]

\[ + \frac{8\pi mnG^2(M + m)(3 - \frac{7}{2}\eta)}{c^2} \log \Lambda \int \frac{f(v_T)}{V} d^3v_T. \]  

In practice, a naive 1PN extension of the classical case, independently on the specific choice of \( f(v_T) \) augments the Chandrasekhar expression of the DF of two additional pieces. The first one has a different dependence on the impact parameter, but the same integral on \( V \), while the second contains the classical Coulomb logarithm, but a different integral on \( V \).

Surprisingly, such extra terms, have the net effect of reducing the DF drag force, being both positively defined.

### 4.2 Relativistic velocities

We now extend the DF formula to the case where the particles velocities are relativistic and the density is such that the force is to be evaluated at 1PN order during close encounters.

Following Lee (1969), we consider a mass \( M \) moving at \( v_T \) in a uniform medium of field stars, of constant number density \( n \), with isotropic velocity distribution \( f(w_T) \), in an inertial frame \( S \). As usual, in a time interval \( \Delta t \), its velocity will change by an amount \( \sum \Delta w_T \) after \( n_{enc} \) encounters.

As the effect of General Relativity will be accounted only during the deflection of the test particle \( M \), its velocity will be transformed according to the Lorentz transformations of Special Relativity.

As in the classical case, the isotropy of \( f(w_T) \) allows us to write

\[ \sum \Delta w_T \wedge w_T = 0, \]  

so it is sufficient to perform

\[ \sum \Delta w_{T\parallel} = \sum \frac{\Delta w_T \cdot w_T}{w_T}. \]  

Let us consider a single encounter of a test star with a field star: the velocity \( w_T \) will become \( w'_T \), so we have

\[ \Delta w_{T\parallel} = (w'_T - w_T) \cdot w_T. \]  

It is now useful to introduce a second reference frame, \( S' \), in which the test star is, initially, at rest. Let \( v_T' \) be the velocity of the test star after the encounter, in the frame \( S' \). With our assumptions (the motion of all the stars will be approximately described by straight line) we can write, without losing generality, that

\[ w_T = \left( \frac{v_T + w_T}{c^2} \right), \]  

and therefore

\[ w_T \cdot w_T = \left( \frac{v_T + w_T}{c} \right) \cdot w_T \rightarrow \frac{1 + \frac{1}{c^2} v_T w_T}{1 + \frac{1}{c^2} v_T w_T}. \]  

Noting that \( \Delta v_T = v'_T, \) (i.e., the test star is initially at rest in \( S' \)) and keeping only terms of order \( 1/c^3 \), Eq. (56) becomes after some algebra

\[ \Delta w_{T\parallel} = \frac{1}{w_T} \left[ \frac{\Delta v_T}{w_T} \right] \rightarrow \frac{1}{w'_T} \left[ \frac{\Delta v_T}{w_T} (1 - \frac{w_T^2}{c^2}) \right]. \]  

We must now sum Eq. (59) over all possible values of \( \Delta v_T \). We recall that \( w_T \) and \( v_T \) are the initial velocities of field stars in the frames \( S \) and \( S' \) and \( b \) is, as usual, the impact parameter of the encounter, defined in \( S' \). We also have that \( v_T, b = v_T b \cos \phi \) while \( \Delta v_T, \) decomposed into its components parallel and perpendicular to \( v_T, \) becomes

\[ \Delta v_T = \Delta v_{T\parallel} \frac{v_T}{v_T} + \Delta v_{T\perp} \frac{b}{b}. \]  

The velocity distribution of field particles in \( S' \) (e.g. see Landau & Lifshitz 1976)

\[ f'(v_T) = \frac{f(w_T)}{\sqrt{1 - \frac{v_T^2}{c^2}}} \left( 1 - \frac{w_T^2}{c^2} \right)^2, \]  

where \( w_T = w_T(v_T, w_T) \). Let us denote with

\[ \Delta t' = \sqrt{1 - \frac{w_T^2}{c^2}} \Delta t \]  

the transformed time interval during which the encounters with the field particles are summed.

It is important to note that \( f'(v_T) \) in the frame \( S' \) is only homogeneous, but no more isotropic. As a consequence of this, the impact parameter \( b \) and the deflection angle \( \varphi \) are randomly distributed for a given value of \( v_T \). Integrating Eq. (59) over all values of \( \Delta v_T \) we obtain the formal result

\[ \int \frac{df(w_T)}{w_T} \int \frac{f(w_T)}{V} d^3v_T \int_{b_{\min}}^{b_{\max}} \pibdb \times \]

\[ \Delta v_T \cdot w_T \left( 1 - \frac{w_T^2}{c^2} \right) \frac{\Delta v_T \cdot w_T^2}{c^2} \right] d\phi . \]  

The quantity \( \Delta v_T(v_T, b) \) appearing in the equation above must be expressed as a function of \( v_T \) and \( b \) and can be computed with the help of the two body problem 1PN-Lagrangian. For this purpose, it is convenient to introduce a third reference frame \( S'' \), in which the center of mass (c.o.m.)
of the encounter is at rest at the origin of coordinates\(^4\), where we denote the particles velocities by \(u\).

At 1PN order the equations of motion for the two-body encounter are usually derived from the Einstein-Infeld-Hoffmann Lagrangian (see Einstein et al. 1938, see also Eddington & Clark 1938)

\[
\mathcal{L}_{\text{EIH}} = \frac{1}{2} m u^2 + \frac{1}{2} M u_T^2 + \frac{1}{8c^2} (m u^4 + M u_T^4) + \frac{GmM}{r} \times \left\{ 1 - \frac{1}{2c^2} \left[ 7(u_T \cdot u_T) + (u_T \cdot n)(u_T \cdot n) - 3(u_T \cdot u_T)^2 \right] \right\}.
\]

Remarkably, qualitatively similar (but slightly simpler) equations of motion can be derived using the gravitational analog of the Darwin Lagrangian of electrodynamics (see e.g. Jackson 1975, see also Essén 2007), often referred as Fokker Lagrangian (see Deruelle & Uzan 2018)

\[
\mathcal{L}_{\text{Darwin}} = \frac{1}{2} m u^2 + \frac{1}{2} M u_T^2 + \frac{1}{8c^2} (m u^4 + M u_T^4) + \frac{GmM}{r} \times \left\{ 1 - \frac{1}{2c^2} \left[ u_T \cdot u_T + (u_T \cdot n)(u_T \cdot n) \right] \right\}.
\]

In Eqs. (64) and (65), \(r = r_F - r_T\) is the (instantaneous) relative position vector and \(n = r / ||r||\), (see Fagundes et al. 1976; Damour & Deruelle 1985; Zürcher 2017, for details). In both cases, the c.o.m. coordinates at the 1PN order are

\[
r_{\text{c.m.}} = \frac{\delta_T r_T + \delta_F r_F}{\delta_T + \delta_F}.
\]

where

\[
\delta_T = M c^2 + \frac{1}{2} M u_T^2 - \frac{1}{2} \frac{GmM}{||r_T - r_F||},
\]

\[
\delta_F = m c^2 + \frac{1}{2} m u_F^2 - \frac{1}{2} \frac{GmM}{||r_T - r_F||}.
\]

Since the c.o.m. is defined at rest at the origin of \(S'\), (i.e. \(r_{\text{c.m.}} = 0\)), in terms of the relative velocity \(u\) we can always write

\[
u_T = \frac{m}{m + M} u + O(1/c^2); \quad u_F = -\frac{M}{m + M} u + O(1/c^2),
\]

so that

\[
\frac{1}{2} (M u_T^2 + m u_F^2) = \frac{1}{2} \left( \frac{mM}{M + m} u^2 \right) + O(1/c^3).
\]

At variance with Lee (1969), we will assume hereafter the Lagrangian (65), that in terms of the relative velocity \(u\), once transformed in polar coordinates, becomes

\[
\mathcal{L}_D = \frac{1}{2} \mu \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 + \Phi + \frac{1}{8c^2} \mu \left( \frac{\dot{\mu}}{\mu_3} \right)^3 \right] \times \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 + \Phi + \frac{1}{2c^2} \frac{\mu}{M} \left[ 2 \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right] \right].
\]

In the equation above, \(\mu\) is again the reduced mass, \(M = M + m\), \(\mu_3 = [m^3 M^3/(m^3 + M^3)]^{1/3}\), and \(\Phi = G\mu M/r\) is the Newtonian gravitational potential energy.

Instead of deriving the equations of motion by using the principle of least action, it is easier to find their first integrals:

\[
p_\theta = \frac{\partial \mathcal{L}_D}{\partial (d\theta/dt)}; \quad E = p_r \frac{dr}{dt} + p_\theta \frac{d\theta}{dt} - \mathcal{L}_D \equiv \mathcal{H}_D,
\]

where \(\mathcal{H}_D\) is the Darwin Hamiltonian, in analogy with the electromagnetic case. At the lowest order, (i.e. when \(\mathcal{L}_{\text{Darwin}} \equiv \mathcal{L}_{\text{Newton}}\), we have

\[
p_\theta = \mu r^2 \frac{d\theta}{dt}; \quad \mathcal{H}_D = \frac{1}{2} \mu \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right] - \Phi.
\]

These expressions will be used to eliminate \(dr/dt\) and \(d\theta/dt\) from \(p_\theta\) and \(\mathcal{H}_D\) in all terms of order \(1/c^2\). This is possible because \(\mathcal{L}_{\text{Darwin}} = \mathcal{L}_{\text{Newton}} + 1/c^2(\ldots)\) and all terms in parentheses, already containing a factor of order \(1/c^2\), are multiplied by another \(1/c^2\) factor, out of parentheses, therefore adding up to the terms in \(1/c^4\), that neglected in the 1PN approximation, (a more sophisticated proof of this argument is given, for example, in Damour & Deruelle 1985).

At 1PN we then find

\[
p_\theta = \mu r^2 \frac{d\theta}{dt} \left( 1 + \frac{1}{\mu c^2} \left[ \frac{\mu_3^3}{\mu_3^2} + \Phi \left( \frac{\mu_3^2}{\mu_3} + \frac{\mu}{M} \right) \right] \right),
\]

from which

\[
\frac{d\theta}{dt} = \frac{p_\theta}{\mu r^2} \left( 1 + \ldots \right) \approx \frac{p_\theta}{\mu r^2} \left( 1 - \frac{1}{\mu c^2} \left[ \frac{\mu_3^3}{\mu_3^2} + \Phi \left( \frac{\mu_3^2}{\mu_3} + \frac{\mu}{M} \right) \right] \right),
\]

and

\[
p_r = \mu c \frac{dr}{dt} \left( 1 + \frac{1}{\mu c^2} \left[ \frac{\mu_3^3}{\mu_3^2} + \Phi \left( \frac{\mu_3^2}{\mu_3} + \frac{\mu}{M} \right) \right] \right).
\]

In order to find \(\mathcal{H}_D\) as a function only of \(r\) and \(dr/d\theta\), we have to replace

\[
\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt}
\]

in \(p_r\) and \(\mathcal{L}_D\), while \(d\theta/dt\) is given by (74). Therefore, keeping only terms of order \(1/c^2\), we obtain

\[
\mathcal{L}_D = \frac{1}{2} \mu p_\theta^2 \left( \frac{dr}{d\theta} \right)^2 + \frac{1}{2} \mu p_r^2 + \Phi + \frac{\Phi}{Mc^2} \left( \frac{dr}{d\theta} \right)^2 \frac{p_\theta^2}{\mu r^2} - \frac{1}{\mu c^2} \times \left\{ \frac{3 \mu_3^3}{\mu_3^2} + \frac{3 \mu_3^2}{\mu_3} + \frac{3 \mu_3}{\mu} + 3 \frac{\mu_3}{\mu} \frac{\Phi}{\mu_3^2} + \Phi \right\},
\]

and finally

\[
\mathcal{H}_D = p_\theta \frac{dr}{d\theta} + p_r \frac{d\theta}{dt} - \mathcal{L}_D = \frac{1}{2} \mu p_\theta^2 \left( \frac{dr}{d\theta} \right)^2 \left( 1 - \frac{2\Phi}{Mc^2} \right) + \frac{1}{2} \mu p_r^2 \left( \frac{dr}{d\theta} \right)^2 \frac{2\Phi}{Mc^2} - \Phi - \frac{1}{\mu c^2} \left( \frac{\mu_3^3}{\mu_3^2} + \frac{\Phi}{\mu_3^2} \right) \times \left( \frac{\mu_3^3}{\mu_3^2} - \frac{\mu_3}{\mu} \right) + \frac{1}{2} \frac{\mu_3}{\mu} \left( \frac{\mu_3^3}{\mu_3^2} - \frac{2\mu}{\mu} \right) = E.
\]
At this point, we need to express the azimuthal angle, $\vartheta$, in $S'$. In order to do so, let us manipulate the Equation (78), by isolating the terms in $d\vartheta$ and $dr/r$ obtaining

$$\left[ \frac{1}{2} \frac{p^2}{\mu^2 r^2} \left( \frac{dr}{d\vartheta} \right)^2 + \frac{1}{2} \frac{p^2}{\mu^2 r^2} \right] \left( 1 - \frac{2\Phi}{\mu c^2} \right) = E + \Phi + \frac{1}{\mu c^2} \left[ \frac{1}{2} \delta'' \frac{\mu^3}{\mu^3} + \delta' \Phi \left( \frac{\mu^3}{\mu^3} - \frac{\mu}{M} \right) + \frac{1}{2} \Phi^2 \left( \frac{\mu^3}{\mu^3} - 2 \frac{\mu}{M} \right) \right],$$

from which, with a little rearrangement of terms we then get

$$\left[ \frac{1}{2} \frac{p^2}{\mu^2 r^2} \left( \frac{dr}{d\vartheta} \right)^2 \right] = \left\{ \frac{E + \Phi}{\left( 1 - \frac{2\Phi}{\mu c^2} \right)} - \frac{1}{2} \frac{p^2}{\mu^2 r^2} + \frac{1}{\mu c^2} \left[ 1 - \frac{2\Phi}{\mu c^2} \right] \right\} \times \left[ \frac{1}{2} \delta'' \frac{\mu^3}{\mu^3} + \delta' \Phi \left( \frac{\mu^3}{\mu^3} - \frac{\mu}{M} \right) + \frac{1}{2} \Phi^2 \left( \frac{\mu^3}{\mu^3} - 2 \frac{\mu}{M} \right) \right].$$

The angle $\vartheta$ is obtained in integral form as

$$\vartheta = 2 \int_{r_c}^{r_e} \frac{dx}{x} \left\{ \frac{2\mu^2}{p_0^2 x^2} E + \Phi + \frac{1}{2} \frac{p_0^2}{\mu^2 x^2} + \frac{1}{\mu c^2} \left[ 1 - \frac{2\Phi}{\mu c^2} \right] \times \left[ \frac{1}{2} \delta'' \frac{\mu^3}{\mu^3} + \delta' \Phi \left( \frac{\mu^3}{\mu^3} - \frac{\mu}{M} \right) + \frac{1}{2} \Phi^2 \left( \frac{\mu^3}{\mu^3} - 2 \frac{\mu}{M} \right) \right] \right\}^{1/2} dr,$$

(80)

where $r_c$ is the distance of closest approach, that can be found by setting $dr/d\vartheta = 0$ in Eq. (80). It is now useful to make a change of variable, by introducing $x = r/r_c$ so that Equation (81) becomes

$$\vartheta = 2 \int_{0}^{1} \frac{dx}{x} \left\{ \frac{2\mu^2}{p_0^2 x^2} E + \Phi + \frac{1}{2} \frac{p_0^2}{\mu^2 x^2} + \frac{1}{\mu c^2} \left[ 1 - \frac{2\Phi}{\mu c^2} \right] \times \left[ \frac{1}{2} \delta'' \frac{\mu^3}{\mu^3} + \delta' \Phi \left( \frac{\mu^3}{\mu^3} - \frac{\mu}{M} \right) + \frac{1}{2} \Phi^2 \left( \frac{\mu^3}{\mu^3} - 2 \frac{\mu}{M} \right) \right] \right\}^{1/2}.$$

(82)

Furthermore, let $G\mu/r_c c^2 = \delta/r_c \ll 1$, so that we can perform the expansion $(1 - 2\delta x/r_c)^{-1} \approx 1 + 2\delta x/r_c$ and neglect the terms proportional to $1/c^2$ obtaining

$$\vartheta = 2 \int_{0}^{1} \left[ - a_1 x^2 + a_2 x + a_3 x^3 \right]^{-1/2} dx,$$

(83)

where we have defined the following quantities to simplify the notation,

$$a_1 = 1 - \frac{G\mu^2 \delta M}{p_0^2 c^2} \left( \frac{\mu^3}{\mu^3} + 2 \frac{\mu}{M} \right),$$

$$a_2 = \frac{2G\mu^2 \delta M}{p_0^2} \left( 1 + \frac{\delta}{c^2} \left[ \frac{1}{\mu} \frac{\mu^3}{\mu^3} + \frac{\mu}{M} \right] \right),$$

$$a_3 = \frac{2\mu E}{p_0^2} \left( 1 + \frac{\delta}{2\mu c^2} \frac{\mu^3}{\mu^3} \right).$$

(84)

Solving the elementary integral in Eq. (83), yields

$$\vartheta = \frac{2}{\sqrt{a_1}} \sin^{-1} \left( \frac{a_2 r_c}{\sqrt{a^2 r_c^2 + 4a_3 a_1 r_c^2}} \right) - \sin^{-1} \left( \frac{a_2 x_c - 2 a_1}{\sqrt{a^2 x_c^2 + 4a_3 a_1 x_c^2}} \right).$$

To eliminate the variable $r_c$ from the expression above, we solve \([-a_1 x_c^2 + a_2 r_c x_c + a_3 x_c^2]_{x_c=1} = 0\) and substitute its positive root in Eq. (85), getting

$$\vartheta = \frac{2}{\sqrt{a_1}} \left( \sin^{-1} \left( \frac{4 a_2 a_1}{a^2} \right)^{1/2} + \frac{\pi}{2} \right).$$

(86)

The latter is the azimuthal angle of the collision in the frame $S'$. In this form $\vartheta$ is given as a function of the first integrals $p_0$ and $\delta'$. We now proceed to express it in terms of the impact parameter $b$ and the asymptotic relative velocity of the stars in the frame $S'$. Let $u, u_T, u_F$ be the asymptotic values of $u, u_T, u_F$ in $S'$ after the encounter. In this limit (i.e. $r \rightarrow +\infty$) we have that $r^2 d\vartheta/dr \rightarrow b u\Phi \rightarrow 0$ and $E \rightarrow \mu u^2/2$, so Equations (73,78) become

$$p_0 \approx \mu b u \left( 1 + \frac{\mu^2}{2c^2} \frac{\mu^3}{\mu^3} \right),$$

(87)

Introducing the effective impact parameter for sharp deflections $\delta' = G\mu M/u^2$, the factors $a_1, a_2, a_3$ in Eq. (83) become

$$a_1 = 1 - \frac{4a_2^2}{b^2 c^2} \left( \frac{\mu^3}{\mu^3} + 2 \frac{\mu}{M} \right),$$

$$a_2 = \frac{2G\mu^2 \delta M}{b^2 c^2} \left( 1 + \frac{\delta}{c^2} \left( \frac{\mu^3}{\mu^3} + 2 \frac{\mu}{M} \right) \right),$$

$$a_3 = \frac{2 \mu E}{b^2 c^2} \left( 1 + \frac{\delta}{2\mu c^2} \frac{\mu^3}{\mu^3} \right).$$

(88)

The azimuthal angle $\vartheta$ is finally given as a function of $b$ and $u$ as

$$\vartheta = \frac{2}{\sqrt{a_1}} \sin^{-1} \left( \frac{4 a_2 a_1}{a^2} \right) \approx \frac{2}{\sqrt{a_1}} \sin^{-1} \left( 1 + \frac{\mu^2}{2c^2} \frac{\mu^3}{\mu^3} \right),$$

(89)

from which one may recover the net deflection angle as $\theta_{\text{def}} = \vartheta - \pi$.

In order to switch back to $S'$ and evaluate $\Delta v_T$, let $u_T^i$ and $u_T^f$ be the initial and final velocity of the test star in $S'$. By decomposing them into the components parallel and perpendicular to $u_T^f$, we have

$$u_T^i = \frac{u_T^i \cdot u_T^f}{||u_T^f||}$$

(90)

$$u_T^\perp = \frac{||u_T^i \wedge u_T^f||}{||u_T^f||} = 0$$

(91)

$$u_T^i = \frac{u_T^i \cdot u_T^f}{||u_T^f||} = \mu r \cos(\vartheta - \pi)$$

(92)

$$u_T^\perp = \frac{||u_T^i \wedge u_T^f||}{||u_T^f||} = \mu r \sin(\vartheta - \pi).$$

(93)

In the $S'$ frame, $\Delta v_T \equiv v_T^i$, since $v_T^f = 0$. Moreover, as the test star moves with velocity $-u_T$ in $S''$, we can write in components
\[ \Delta v_{\perp} = \frac{u_T \sin(\vartheta - \pi) \sqrt{1 - \frac{u_T^2}{c^2}}}{1 - \frac{u_T^2 \cos(\vartheta - \pi)}{c^2}} \approx \left[ u_T \sin(\vartheta - \pi) \right] \left( 1 - \frac{u_T^2}{2c^2} \right) \times \left( 1 + \frac{u_T^2}{c^2} \right) \approx \left[ u_T \sin(\vartheta - \pi) \right] \left[ 1 + \frac{u_T^2 \cos(\vartheta - \pi)}{2c^2} \right]. \]

The two equations above should then be expressed in terms of \( v_F \), instead of \( u_T \) and \( u \). Let us remind that \( u \) is the relative velocity in \( S' \), as given by Eq. (26), where \( u_T, v_T \) are defined in Eq. (68).

Since we only need the relative velocity at IPN approximation, expanding Eq.(26) an keeping the terms in \( 1/c^2 \) yields

\[ \frac{dw_{T||}}{dt} = \int n f'(v_F) \sqrt{1 - \frac{u_T^2}{c^2}} v_F d^3v_F \int_{\min}^{b_{\max}} 2\pi b db \left( \frac{w_T \cdot b}{v_F} \right)^2 \left( \Delta v_{T\perp} \right)^2 - 2 \left( \frac{w_T \cdot b}{v_F} \right) \left( \Delta v_{T\parallel} \right) \left( \Delta v_{T\perp} \right) + \left( \frac{w_T \cdot b}{v_F} \right)^2 \left( \Delta v_{T\perp} \right)^2 \right) d\phi. \]

We note that the all linear terms in the equation above containing \( \frac{w_T \cdot b}{v_F} \) vanish when they are integrated in \( d\phi \), while the quadratic terms yield

\[ \int_0^{2\pi} \frac{1}{2\pi} \left( \frac{w_T \cdot b}{v_F} \right)^2 d\phi = \frac{1}{2} \left( \frac{w_T v_F}{v_F} \sin \varphi \right)^2 = \frac{1}{2} \left( \frac{w_T \cdot v_F}{v_F} \right)^2. \]

In the equation above, we have used the fact that \( v_F \cdot b = 0 \), (i.e. \( v_F \) is perpendicular to \( b \) and then \( \sin \varphi = 1 \)). We then rewrite Eq. (99) as

\[ \frac{dw_{T\parallel}}{dt} = \int n f'(v_F) \sqrt{1 - \frac{u_T^2}{c^2}} v_F d^3v_F \int_{\min}^{b_{\max}} 2\pi b \times \left( \frac{w_T \cdot v_F}{v_F} \right) \left( \Delta v_{T\parallel} \right) - \frac{1}{2} \left( \frac{w_T \cdot v_F}{v_F} \right)^2 \left( \Delta v_{T\parallel} \right)^2 + \left( \frac{w_T \cdot v_F}{v_F} \right)^2 \left( \Delta v_{T\perp} \right)^2 \right) \]

This is established, we now have to evaluate \( \Delta v_{T\parallel} \) and \( \left( \Delta v_{T\perp} \right)^2 \), using Eqs. (89), (94) and (95). To do so, we should first recover \( \cos(\vartheta - \pi) \) and \( \sin(\vartheta - \pi) \) in IPN approximation. In such limit we have that \( \cos(1/c^2, ...) \approx 1 \) and \( \sin(1/c^2, ...) \approx 1/(c^2, ...) \). Using the standard trigonometry and some further algebraic manipulation we define the angle \( \varphi \) by its sine and cosine as

\[ \cos(\vartheta - \pi) \approx 1 - \frac{2}{1 + \mu^2/\mu'^2} - \frac{2\mu^2}{\mu'^2} - \frac{2\mu'^2}{\mu^2} \left( \frac{\mu^2}{\mu'^2} - \frac{\mu'^2}{\mu^2} \right) \left( \frac{\mu^2}{\mu'^2} + \frac{\mu'^2}{\mu^2} \right) \left( \sin^{-1} \left( 1 + b^2/\mu'^2 \right)^{-1/2} + \pi/2 \right) \]

\[ \sin(\vartheta - \pi) \approx \frac{2b}{\mu'^2} + \frac{\mu'^2}{\mu^2} \left( \frac{\mu^2}{\mu'^2} - \frac{\mu^2}{\mu'^2} \right) + \left( \frac{\mu^2}{\mu'^2} + \frac{\mu'^2}{\mu^2} \right) \left( \sin^{-1} \left( 1 + b^2/\mu'^2 \right)^{-1/2} + \pi/2 \right). \]
We may now express the (finite) parallel and perpendicular velocity changes as
\[
\Delta v_{||} \approx -2u_T \left\{ \frac{1}{1 + b^2/\mathscr{A}^2} + \frac{u^2}{c^2} \left( \frac{1}{1 + b^2/\mathscr{A}^2} - \frac{1}{1 + b^2/\mathscr{A}^2} \right) \right\},
\]
and substitute them in Eq. (101). Performing the integral over \(b\) keeping only the terms yielding the Coulomb logarithm in
\[
\int_{b_{\text{min}}}^{b_{\text{max}}} 2\pi b \Delta v_{||} \, db = -2\pi u_T \int_{b_{\text{min}}}^{b_{\text{max}}} 2b \left\{ \frac{1}{1 + b^2/\mathscr{A}^2} + \frac{u^2}{c^2} \left( \frac{1}{1 + b^2/\mathscr{A}^2} - \frac{1}{1 + b^2/\mathscr{A}^2} \right) \right\} \, db \approx
\]
\[
\approx -2\pi u_T \mathscr{A}^2 \log \frac{1 + b_{\text{max}}^2/\mathscr{A}^2}{1 + b_{\text{min}}^2/\mathscr{A}^2} \left[ 1 + \frac{u^2}{c^2} \left( \frac{1}{1 + b^2/\mathscr{A}^2} - \frac{1}{1 + b^2/\mathscr{A}^2} \right) \right],
\]
(106)
gives
\[
\int_{b_{\text{min}}}^{b_{\text{max}}} 2\pi b (\Delta v_{||})^2 \, db \approx 4u_T^2 \pi \int_{b_{\text{min}}}^{b_{\text{max}}} \frac{2b^3 \, db}{\mathscr{A}^2 (1 + b^2/\mathscr{A}^2)^2} \approx
\]
\[
\approx 4u_T^2 \pi \mathscr{A}^2 \log \frac{1 + b_{\text{max}}^2/\mathscr{A}^2}{1 + b_{\text{min}}^2/\mathscr{A}^2}.
\]
(107)
After some further algebraic manipulation, we finally obtain the 1PN dynamical friction formula in compact form as
\[
\frac{dw_{||}}{dt} = -4\pi G^2 (M + m)^2 \log \Lambda \int f'(v_F) \left\{ 1 - \frac{v^2}{c^2} \right\} \frac{v_F}{v_T} \times
\]
\[
\times \left\{ \frac{1}{u_T^2} \left( 1 - \frac{u_T^2}{c^2} \right) \frac{w_T \cdot v_F}{v_T} \times
\]
\[
\times \left[ 1 + \frac{u^2}{c^2} + \frac{u^2}{c^2} \left( \frac{1}{1 + b^2/\mathscr{A}^2} - \frac{1}{1 + b^2/\mathscr{A}^2} \right) \right] \right\} \, d^3v_F,
\]
(108)
where again we have assumed the velocity averaged Coulomb logarithm. In analogy with Equation (41), it is always possible to evaluate numerically the integral in Eq. (108) for a given choice of \(f'(v_F)\).

It is possible, however, to infer that the deflection of a test star (in the encounters center of mass frame \(S''\)) is higher in 1PN approximation than in Newtonian regime. This fact, evident from the first term in brackets of Eq. (108), always increases the drag effect on \(M\) with respect to the purely classical treatment. An additional increase of the dynamical friction force is caused by the term \((w_T \wedge v_F)^2\), arising due to the transformation from the inertial frame \(S\) to that of the test mass, \(S''\). This confirms what found by previous work (Lee 1969; Syer 1994) when considering relativistic velocities, although it seems to be in apparent contradiction with the somewhat counter intuitive finding when using the simpler approach carried out in Sect. 4.1 (cf. Eq. 53). In the latter case however, particle velocities were assumed to be strictly classical.

Equation (108), although highly simplified by the assumptions made above, represents the most general relativistic derivation of the dynamical friction formula. It is now worth to discuss its behaviour in its two fundamental limits; i.e. \(c \to \infty\) and \(M \gg \mu\). In the first case, Equation (108) yields back exactly its classical counterpart (cf. Eq. 21) in scalar form, at variance with the classical limit of Eq. (38) where the dynamical friction formula turns out to be proportional to \((M + m)^2\).

In the limit of very large test particle mass \(M\) we obtain
\[
\frac{dw_{||}}{dt} = -4\pi G^2 M^2 \rho \log \Lambda \frac{\gamma w_T^{-3}}{v_F^2} \int f'(v_F) \frac{M c^2 + m v_F^2}{M c^2 + 4m v_F^2} \, d^3v_F,
\]
(109)
where we have used the limits of Eqs. (97-98) for \(M \gg m\), \(\rho = \mu m\) is the mass density of the field particles, and \(\gamma w_T = 1/\sqrt{1 - u_T^2/c^2}\) is the Lorentz factor of \(M\). Curiously, Equation (109) still bears an explicit dependence on the rest energy of \(M\) and the mass of the field particles \(m\), differently from what one would obtain in the same limit for the classical expression, (see the discussion in Binney & Tremaine 2008 and Ciotti 2010). In other words, in a relativistic set-up the dynamical friction acting on a massive particle is different in two systems with the same mass density \(\rho\) but different field particle mass \(m\). We notice that, Eq. (109) has a 1/\(\gamma w_T^3\) dependence on the Lorentz factor, instead of 1/\(\gamma w_T\) as in the weak scattering limit for a large test mass in a background of particles of vanishingly small masses \(m\), (cf. Eq. 2.26 of Syer 1994).

5 SUMMARY AND PERSPECTIVES

In this preparatory work on the the dynamical friction in relativistic systems, we have explored two relevant cases, the one involving relativistic velocity distribution and classical forces between particles, and the other with strong scattering with and without relativistic velocities.

We find that, independently on the specific force law under consideration (Newtonian or first order PN), a particle moving through a medium associated with a relativistic distribution suffers a slightly larger drag with respect to its counterpart evaluated with the standard classical Chandrasekhar approach.

In addition we also derived a more general DF formula for a relativistic massive particle in a system of lighter scatterers, accounting for both large and small scattering angles, extending the asymptotic expression of Syer (1994), only valid for small angles.

Remarkably, we also found out that a naive reconstruction of the DF formula in the case of strong scattering in post Newtonian approximation and non relativistic velocity distribution underestimates the drag on the test particle with respect to the purely classical calculation. This puzzling result

\[\text{To be precise, even in the } c \to \infty \text{ limit, the integral in Eq. (108) would still contain a multiplicative factor } \cos \theta, \text{ with } \theta \text{ the angle between } w_T \text{ and } v_F \text{ arising from the dot product between the two. Such angle between two vectors in } S \text{ and } S'' \text{ zeroes out in the limit of } c \to \infty \text{ and thus } \cos \theta \to 1. \text{ Therefore, in this case, the angular dependence is not dropped by symmetry arguments.}\]
appears to be confirmed by preliminary direct $N$–body simulations (to be published elsewhere, see Chiari & Di Cintio 2022), where we have studied the orbital decay of a $10^5 M_\odot$ BH placed initially on circular orbit in a star cluster with and without the post Newtonian correction to the force law, finding a slightly (of a factor 1.1) larger in-spiraling time in the runs with the post Newtonian corrections.

So far, the results discussed in the present paper could be relevant for the dynamics of massive objects at the centre of dense star clusters where both strong deflections and large velocity dispersion may occur, as well as models of hot Dark Matter with relativistic velocities. It is worth to mention that the formalism used in Sect. 4.2 could be also extended to treat relativistic charged particles (in the limit of negligible radiation loss) using the "real" Darwin Lagrangian.

This established, it remains to be determined whether the inclusion of explicitly dissipative terms (i.e. the effect of momentum loss due to the emission of gravitational waves) would alter significantly the relativistic dynamical friction drag. To do so, in principle one should include in Eq. (47) all terms up to the order 2.5. Moreover, an additional generalization that seems to be feasible could be in the direction of systems with a mass spectrum with mass-dependent average relativistic factor, in other words, where for example only the low mass particles have relativistic velocities.

In this paper we have explored only infinitely extended systems in the spirit of the original treatment of the dynamical friction. As mentioned above star cluster simulations are in the works. In the next paper of this series, we will investigate by means of post Newtonian $N$–body simulations the collisional dynamics of compact objects kicked by gravitational waves emission in dense stellar systems and the relativistic corrections on their dynamical friction induced retention.

ACKNOWLEDGMENTS

We thank Lapo Casetti, for the useful discussions at an early stage of this work. One of us (PFDC) wishes to acknowledge partial financial support from the MIUR-PRIN2017 project Coarse-grained description for non-equilibrium systems and transport phenomena (CO-NEST) n. 201798CZL.

DATA AVAILABILITY

This paper does not contain any data analysis nor numerical simulation.

REFERENCES

Abbott B. P., et al., 2016a, Phys. Rev. Lett., 116, 061102
Abbott B. P., et al., 2016b, p1, 116, 221101
Adhikari S., Dalal N., Clampitt J., 2016, J. Cosmology Astropart. Phys., 2016, 022
Alessandrini E., Lanzoni B., Miocchi P., Ciotti L., Ferraro F. R., 2014, ApJ, 795, 169
Alessandrini E., Lanzoni B., Ferraro F. R., Miocchi P., Vesperini E., 2016, ApJ, 833, 252
Antonini F., Merritt D., 2012, ApJ, 745, 83
Antonini F., Gieles M., Gualandris A., 2019, MNRAS, 486, 5008
Arena S. E., Bertin G., 2007, A&A, 463, 921

Arena S. E., Bertin G., Liseikina T., Pegoraro F., 2006, A&A, 453, 9
Arfken G. B., Weber H. J., Harris F. E., 2012, Mathematical Methods for Physicists: A Comprehensive Guide. Elsevier Science Publishing Co Inc
Barausse E., 2007, MNRAS, 382, 826
Bekenstein J. D., Maoz E., 1992, ApJ, 390, 79
Bekenstein J., Milgrom M., 1984, ApJ, 286, 7
Bertin G., Liseikina T., Pegoraro 2003, A&A, 405, 73
Bhat A., Irigang A., Heber U., 2022, arXiv e-prints, p. arXiv:2204.01594
Binney J., 1977, mnras, 181, 735
Binney J., Tremaine S., 1987, Galactic dynamics. Princeton Series in Astrophysics
Binney J., Tremaine S., 2008, Galactic Dynamics: Second Edition. Princeton Series in Astrophysics
Blanchet L., 2010, Post-Newtonian theory and the two-body problem (arXiv:0907.3596)
Cashen B., Aker A., Kesden M., 2017, Phys. Rev. D, 95, 064014
Chandrasekhar S., 1941, ApJ, 94, 511
Chandrasekhar S., 1942, Principles of Stellar Dynamics. Dover Publications, Inc.
Chandrasekhar S., 1943, ApJ, 97, 255
Chandrasekhar S., 1995, Newton’s Principia for the Common Reader. Clarendon Press
Chavanis P.-H., 2020a, European Physical Journal Plus, 135, 290
Chavanis P.-H., 2020b, European Physical Journal Plus, 135, 310
Chen N., Ni Y., Tremmel M., Di Matteo T., Bird S., DeGraf C., Feng Y., 2022, MNRAS, 510, 531
Chiari C., Di Cintio P., 2022, in preparation
Ciotti L., in Bertin G., de Luca F., Lodato G., Pozzoli R., Romé M., eds, American Institute of Physics Conference Series Vol. 1242, Plasmas in the Laboratory and the Universe: Interactions, Patterns, and Turbulence. pp 117–128 (arXiv:1001.3531), doi:10.1063/1.4060114
Ciotti L., 2021, Introduction to Stellar Dynamics. Cambridge University Press, doi:10.1017/97805117360117
Ciotti L., Binney J., 2004, mnras, 351, 285
Colpi M., Pallavicini A., 1998, ApJ, 502, 150
Correia M., 2022, Phys. Rev. D, 105, 084041
Costa L. F. O., Natário J., 2014, General Relativity and Gravitation, 46, 1792
Damour T., Deruelle N., 1985, Annales de l’I.H.P. Physique théorique, 43, 107
Deruelle N., Uzan J.-P., 2018, Relativity in Modern Physics. Oxford University press
Di Cintio P., Ciotti L., Nipoti C., 2020, in Bragaglia A., Davies M., Sills A., Vesperini E., eds, Vol. 351, Star Clusters: From the Milky Way to the Early Universe. pp 93–96 (arXiv:1908.04283), doi:10.1017/S1743921319007221
Eddington A., Clark G. L., 1938, Proceedings of the Royal Society of London Series A, 166, 465
Einstein A., 1915, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin, pp 831–839
Einstein A., Infeld L., Hoffmann B., 1938, Annals Phys. (Leipzig), 5, 777
Ellis G. F. R., Matravers D. R., Treciokas R., 1983, Annals of Physics, 150, 455
Essén H., 2007, EPL (Europhysics Letters), 79, 60002
Fackerell E. D., 1968, ApJ, 153, 643
Fagundes F. H., Zimerman H. A., Ragusa S., 1976, Derivation of Bazinski’s Lagrangian in a Lorentz Covariant Theory of Gravitation, http://sbfisica.org.br/bjp/download/v06/v06a20.pdf
Fang Y., Huang Q.-G., 2020, Phys. Rev. D, 102, 104002
Frias J. P., Tan J. C., Eyre L., 2020, ApJ, 900, 14
Ferraro F., Fusi Pecci F., Bellazzini M., 1995, aap, 294, 80

MNRAS 000, 1–13 (2022)
