A Proof On Weinstein Conjecture On Cotangent Bundles *

Renyi Ma
Department of Mathematics
Tsinghua University
Beijing, 100084
People’s Republic of China
rma@math.tsinghua.edu.cn

Abstract

In this article, we prove that there exists at least one closed characteristics of Reeb vector field in a connected contact manifolds of induced type in the cotangent bundles of any open smooth manifolds which confirms completely the Weinstein conjecture in cotangent bundles of open manifold.

Keywords J-holomorphic curves, Cotangent bundles, Closed characteristics.

2000 MR Subject Classification 32Q65, 53D35, 53D12

1 Introduction and results

A contact structure on a manifold is a field of a tangent hyperplanes (contact hyperplanes) that is nondegenerate at any point. Locally such a field is defined as the field of zeros of a 1–form λ, called a contact form. The nondegeneracy condition is that dλ is nondegenerate on the hyperplanes on which λ vanishes; equivalently, in (2n − 1)–space:

*Project 19871044 Supported by NSF
The important example of contact manifold is the well-known projective cotangent bundles defined as follows:

Let $N = T^*M$ be the cotangent bundle of a smooth connected compact manifold $M$. $N$ carries a canonical symplectic structure $\omega = -d\lambda$ where $\lambda = \sum_{i=1}^{n} y_i dx_i$ is the Liouville form on $N$, see [2, 14]. Let $P = PT^*M$ be the oriented projective cotangent bundle of $M$, i.e. $P = \cup_{x \in M} PT^*_x M$. It is well known that $P$ carries a canonical contact structure induced by the Liouville form and the projection $\pi : T^*M \mapsto PT^*M$.

Let $(\Sigma, \lambda)$ be a smooth closed oriented manifold of dimension $2n - 1$ with a contact form $\lambda$. Associated to $\lambda$ there are two important structures. First of all the so-called Reed vectorfield $X_\lambda$ defined by

$$i_{X_\lambda} \lambda \equiv 1, \quad i_{X_\lambda} d\lambda \equiv 0$$

and secondly the contact structure $\xi = \xi_\lambda \mapsto \Sigma$ given by

$$\xi_\lambda = \ker(\lambda) \subset T\Sigma$$

by a result of Gray, [8], the contact structure is very stable. In fact, if $(\lambda_t)_{t \in [0,1]}$ is a smooth arc of contact forms inducing the arc of contact structures $(\xi_t)_{t \in [0,1]}$, there exists a smooth arc $(\psi_t)_{t \in [0,1]}$ of diffeomorphisms with $\psi_0 = Id$, such that

$$T\Psi_t(\xi_0) = \xi_t \quad (1.1)$$

here it is important that $\Sigma$ is compact. From (1.1) and the fact that $\Psi_0 = Id$ it follows immediately that there exists a smooth family of maps $[0,1] \times \Sigma \mapsto (0, \infty) : (t, m) \mapsto f_t(m)$ such that

$$\Psi_t^* \lambda_t = f_t \lambda_0 \quad (1.2)$$

In contrast to the contact structure the dynamics of the Reeb vectorfield changes drastically under small perturbation and in general the flows associated to $X_t$ and $X_s$ for $t \neq s$ will not be conjugated, see[2, 5].

Let $M$ be a Riemann manifold with Riemann metric, then it is well known that there exists a canonical contact structure in the unit sphere of its tangent bundle and the motion of geodesic line lifts to a geodesic flow on the unit sphere bundles. Therefore the closed orbit of geodesic flow or Reeb flow on the sphere bundle projects to a closed geodesics in the Riemann manifolds, conversely the closed geodesic orbit lifts to a closed Reeb
orbit. The classical work of Ljusternik and Fet states that every simply connected Riemannian manifold has at least one closed geodesics, this with the Cartan and Hadamard’s results on non-simply closed Riemann manifold implies that any closed Riemann manifolds has a closed geodesics, i.e., the sphere bundle of a closed Riemann manifold with standard contact form carries at least one closed Reeb orbits which is a lift of closed geodesics of base manifold. Its proof depends on the classical minimax principle of Ljusternik and Schnirelman or minimalization of Hadamard and Cartan,[14], an \( J \)–holomorphic curve’s proof can be found in [19]. In sympletic geometry, Gromov [9] introduces the global methods to proves the existences of symplectic fixed points or periodic orbits which depends on the nonlinear Fredholm alternative of \( J \)–holomorphic curves in the symplectic manifolds. In this paper we use the \( J \)–holomorphic curve’s method to prove

**Theorem 1.1** Let \((\Sigma, \lambda)\) be a contact manifold with contact form \(\lambda\) of induced type or Weinstein type in the cotangent bundles of any open smooth manifold with symplectic form \(\sum_{i=1}^{n} dp_i \wedge dq_i\) induced by Liouville form \(\alpha = \sum_{i=1}^{n} p_i dq_i\), i.e., there exists a transversal vector field \(Z\) to \(\Sigma\) such that \(L_{Z}\omega = \omega, \lambda = i_{Z}\omega\). Let \(X_{\lambda}\) be its Reeb vector field. Then, there exists at least one closed characteristic for \(X_{\lambda}\).

This gives a complete solution on the well-known Weinstein conjecture in cotangent bundles of smooth open manifold. Note that Viterbo [25] first proved the above result for any contact manifolds \(\Sigma\) of induced type in \(R^{2n} = T^*R^n\) after Rabinowitz [21] and Weinstein [27, 28]. After Viterbo’s work many results were obtained in [6, 10, 11, 12, 16, 17] etc by using variational method or Gromov’s \(J\)–holomorphic curves via nonlinear Fredholm alternative, see survey paper [4].

**Corollary 1.1** ([17])If \(M = N \times R\), \(N\) is any smooth manifold, then Theorem 1.1 holds true.

Through the variational method by Hofer and Viterbo[11], especially, Viterbo finally in [26] proved the following result.

**Corollary 1.2** (Viterbo[26])If \(M\) is an open simply connected manifolds and \([\lambda - p_i dq_i] = 0\), then Theorem 1.1 holds true.

**Sketch of proofs:** We work in the framework as in [9, 18]. In Section 2, we study the linear Cauchy-Riemann operator and sketch some basic properties. In section 3, first we construct a Lagrangian submanifold \(W\) under the
assumption that there does not exist a closed Reeb orbit in $(\Sigma, \lambda)$; second, we study the space $D(V, W)$ of contractible disks in manifold $V$ with boundary in Lagrangian submanifold $W$ and construct a Fredholm section of tangent bundle of $D(V, W)$. In section 4, following [9, 18], we prove that the Fredholm section is not proper by using a special anti-holomorphic section as in [9, 18]. In section 5, we transform the non-homogeneous Cauchy-Riemann equation as $J$-holomorphic curves. In the final section, we use nonlinear Fredholm trick in [9, 18] to complete our proof.

2 Linear Fredholm theory

For $100 < k < \infty$ consider the Hilbert space $V_k$ consisting of all maps $u \in H^{k,2}(D, C^n)$, such that $u(z) \in R^n \subset C^n$ for almost all $z \in \partial D$. $L_{k-1}$ denotes the usual Hilbert $L^{k-1}$ space $H^{k-1}(D, C^n)$. We define an operator $\bar{\partial} : V_p \mapsto L_p$ by

$$\bar{\partial} u = u_s + iu_t \quad (2.1)$$

where the coordinates on $D$ are $(s, t) = s + it$, $D = \{z \mid |z| \leq 1\}$. The following result is well known (see [3, 29]).

Proposition 2.1 $\bar{\partial} : V_p \mapsto L_p$ is a surjective real linear Fredholm operator of index $n$. The kernel consists of the constant real valued maps.

Let $(C^n, \sigma = -Im(\cdot, \cdot))$ be the standard symplectic space. We consider a real $n$-dimensional plane $R^n \subset C^n$. It is called Lagrangian if the skew-scalar product of any two vectors of $R^n$ equals zero. For example, the plane $p = 0$ and $q = 0$ are Lagrangian subspaces. The manifold of all (nonoriented) Lagrangian subspaces of $R^{2n}$ is called the Lagrangian-Grassmanian $\Lambda(n)$. One can prove that the fundamental group of $\Lambda(n)$ is free cyclic, i.e. $\pi_1(\Lambda(n)) = Z$. Next assume $(\Gamma(z))_{z \in \partial D}$ is a smooth map associating to a point $z \in \partial D$ a Lagrangian subspace $\Gamma(z)$ of $C^n$, i.e. $(\Gamma(z))_{z \in \partial D}$ defines a smooth curve $\alpha$ in the Lagrangian-Grassmanian manifold $\Lambda(n)$. Since $\pi_1(\Lambda(n)) = Z$, one have $[\alpha] = ke$, we call integer $k$ the Maslov index of curve $\alpha$ and denote it by $m(\Gamma)$, see ([2]).

Now let $z : S^1 \mapsto R^n \subset C^n$ be a smooth curve. Then it defines a constant loop $\alpha$ in Lagrangian-Grassmanian manifold $\Lambda(n)$. This loop defines the Maslov index $m(\alpha)$ of the map $z$ which is easily seen to be zero.

Now Let $(V, \omega)$ be a symplectic manifold and $W \subset V$ a closed Lagrangian submanifold. Let $u : D^2 \mapsto V$ be a smooth map homotopic to constant map with boundary $\partial D \subset W$. Then $u^*TV$ is a symplectic vector bundle and
$(u|_{\partial D})^*TW$ be a Lagrangian subbundle in $(u|_{\partial D})^*TV$. Since $u$ is contractible, we can take a trivialization of $u^*TV$ as

$$\Phi(u^*TV) = D \times C^n$$

and

$$\Phi((u|_{\partial D})^*TW) \subset S^1 \times C^n$$

Let

$$\pi_2 : D \times C^n \to C^n$$

then

$$\bar{u} : z \in S^1 \to \{\pi_2\Phi((u|_{\partial D})^*TW(z))\} \in \Lambda(n).$$

Write $\bar{u} = u|_{\partial D}$.

**Lemma 2.1** Let $u : (D^2, \partial D^2) \to (V,W)$ be a $C^k$-map ($k \geq 1$) as above. Then,

$$m(u|_{\partial D}) = 0$$

**Proof.** Since $u$ is contractible in $V$ relative to $W$, we have a homotopy $\Phi_s$ of trivializations such that

$$\Phi_s(u^*TV) = D \times C^n$$

and

$$\Phi_s((u|_{\partial D})^*TW) \subset S^1 \times C^n$$

Moreover

$$\Phi_0((u|_{\partial D})^*TW) = S^1 \times R^n$$

So, the homotopy induces a homotopy $\bar{h}$ in Lagrangian-Grassmannian manifold. Note that $m(\bar{h}(0,\cdot)) = 0$. By the homotopy invariance of Maslov index, we know that $m(u|_{\partial D}) = 0$.

Consider the partial differential equation

$$\bar{\partial}u + A(z)u = 0 \text{ on } D \quad (2.2)$$

$$u(z) \in \Gamma(z)R^n \text{ for } z \in \partial D \quad (2.3)$$

$$\Gamma(z) \in GL(2n,R) \cap Sp(2n) \quad (2.4)$$

$$m(\Gamma) = 0 \quad (2.5)$$

For $100 < k < \infty$ consider the Banach space $\bar{V}_k$ consisting of all maps $u \in H^{k,2}(D,C^n)$ such that $u(z) \in \Gamma(z)$ for almost all $z \in \partial D$. Let $L_{k-1}$ the usual $L_{k-1}$-space $H_{k-1}(D,C^n)$ and
\[ L_{k-1}(S^1) = \{ u \in H^{k-1}(S^1) \mid u(z) \in \Gamma(z) R^n \text{ for } z \in \partial D \} \]

We define an operator \( P: \bar{V}_k \to L_{k-1} \times L_{k-1}(S^1) \) by

\[ P(u) = (\bar{\partial}u + Au, u|_{\partial D}) \] (2.6)

where \( D \) as in (2.1).

**Proposition 2.2** \( \bar{\partial}: \bar{V}_p \to L_p \) is a real linear Fredholm operator of index \( n \).

**Proof:** see [3, 9, 29].

### 3 Nonlinear Fredholm theory

#### 3.1 Construction of Lagrangian submanifold

Let \( M \) be an open manifold and \((T^*M, p, dq_i)\) be the cotangent bundle of open manifold with the Liouville form \( p_i dq_i \). Since \( M \) is open, there exists a function \( g : M \to R \) without critical point. The translation by \( tTdg \) along the fibre give a hamilton isotopy of \( T^*M \):

\[ h^T_t (q, p) = (q, p + tTdg(q)) \] (3.1)

\[ h^T_t (p_i dq_i) = p_i dq_i + tTdg. \] (3.2)

**Lemma 3.1** For any given compact set \( K \subset T^*M \), there exists \( T = T_K \) such that \( h^T_1 (K) \cap K = \emptyset \).

**Proof.** Similar to [9, 15]

Let \( \Sigma \subset T^*M \) be a closed hypersurface, if there exists a vector field \( V \) defined in the neighbourhood \( U \) of \( \Sigma \) transversal to \( \Sigma \) such that \( L_V \omega = \omega \), here \( \omega = dp_i \wedge dq_i \) is a standard symplectic form on \( T^*M \) induced by the Liouville form \( p_i dq_i \), we call \( \Sigma \) the contact manifold of induced type in \( T^*M \) with the induced contact form \( \lambda = i_V \omega \).

Let \((\Sigma, \lambda)\) be a contact manifold of induced type or Weinstein’s type in \( T^*M \) with contact form \( \lambda \) and \( X \) its Reeb vector field, then \( X \) integrates to a Reeb flow \( \eta_s \) for \( s \in R^1 \).

By using the transversal vector field \( V \), one can identify the neighbourhood \( U \) of \( \Sigma \) foliated by flow \( f_t \) of \( V \) and \( \Sigma \), i.e., \( U = \cup_t f_t(\Sigma) \) with the
neighbourhood of \( \{0\} \times \Sigma \) in the symplectization \( R \times \Sigma \) by the exact symplectic transformation (see [17, 25]).

Consider the form \( d(e^a\lambda) \) at the point \((a, x)\) on the manifold \((R \times \Sigma)\), then one can check that \( d(e^a\lambda) \) is a symplectic form on \( R \times \Sigma \). Moreover one can check that

\[
i_X(e^a\lambda) = e^a \quad (3.3)
\]
\[
i_X(d(e^a\lambda)) = -de^a \quad (3.4)
\]

So, the symplectization of Reeb vector field \( X \) is the Hamilton vector field of \( e^a \) with respect to the symplectic form \( d(e^a\lambda) \). Therefore the Reeb flow lifts to the Hamilton flow \( h_s \) on \( R \times \Sigma \) (see [2, 5]).

Let

\[
(V', \omega') = (T^*M \times T^*M, d(p_1^1dq_1^1 \ominus p_2^2dq_2^2))
\]

be the anti-product of cotangent bundles and

\[
\mathcal{L} = \{(\sigma, \sigma)| \sigma \in \Sigma \subset T^*M\}
\]

be a closed isotropic submanifold contained in \((\Sigma', \lambda') = (\Sigma \times \Sigma, \lambda \ominus \lambda)\), i.e., there exists a smooth diagonal embedding \( Q : \mathcal{L} \to \Sigma' \) such that \( Q^*\lambda|_{\mathcal{L}} = 0 \).

Let

\[
W' = \mathcal{L} \times R, \quad W'_s = \mathcal{L} \times \{s\} \quad (3.5)
\]

define

\[
G' : W' \to V' \\
G'(w') = G'(l, s) = (\sigma, \eta_s(\sigma)) \quad (3.6)
\]

we also denote \( W' = G'(W') \)

**Lemma 3.2** There does not exist any Reeb closed orbit in \((\Sigma, \lambda)\) if and only if \( G'(W'(s)) \cap G'(W'(s')) \) is empty for \( s \neq s' \).

Proof. Obvious.

**Lemma 3.3** If there does not exist any Reeb closed orbit for \( X_\lambda \) in \((\Sigma, \lambda)\) then there exists a smooth embedding \( G' : W' \to V' \) with \( G'(l, s) = (\sigma, \eta_s(\sigma)) \) such that

\[
G'_K : \mathcal{L} \times (-K, K) \to V' \quad (3.7)
\]

is a regular open Lagrangian embedding for any finite positive \( K \).
Proof. One first checks

\[ G^*(e^a \lambda') = \lambda_1 - \eta(\cdot, \cdot) \lambda_2 = \lambda_1 - (\eta^*_s \lambda_2 + i_X \lambda ds) = -ds \quad (3.8) \]

Recall that \( \Sigma \) is a contact manifold of induced type in \( T^*M \), let \( \lambda = i_Z(dp_i \wedge dq_i) \). Since \( d\lambda = dp_i \wedge dq_i \), we know that \( \theta = \lambda - p_i dq_i \) is a close form which determines a cohomology \([\theta] \in H^1(\Sigma)\). Let \( \theta_1 = \lambda_1 - p_1 dq_1 \) and \( \theta_2 = \lambda_2 - p_2 dq_2 \). Since \([\theta_1] = [\theta_2]\), we have

\[ (\lambda_1 - p_1 dq_1) - (\lambda_2 - p_2 dq_2) = df \quad (3.9) \]

So,

\[ G^*(p_1 dq_1^1 - p_2 dq_2^2)) = -ds - df. \quad (3.10) \]

This shows that \( W' \) is an exact Lagrangian submanifold in \((T^*M \times T^*M, dp_1 \wedge dq_1^1 \ominus dp_2 \wedge dq_2^2)\).

Now we modify the above construction as follows[20]:

\[ F'_0 : \mathcal{L} \times R \times R \rightarrow (R \times \Sigma) \times (R \times \Sigma) \]

\[ F'_0(((0, \sigma), (0, \sigma)), s, b) = ((0, \sigma), (b, \eta_s(\sigma))) \]

Now we embed a elliptic curve \( E \) long along \( s - axis \) and thin along \( b - axis \) such that \( E \subset [-s_1, s_2] \times [0, \varepsilon] \). We parametrize the \( E \) by \( t' \).

**Lemma 3.4** If there does not exist any closed Reeb orbit in \((\Sigma, \lambda)\), then

\[ F_0 : \mathcal{L} \times S^1 \rightarrow (R \times \Sigma) \times (R \times \Sigma) \]

\[ F_0(((0, \sigma), (0, \sigma)), t') = ((0, \sigma), (b(t'), \eta_{s(t')})(\sigma)) \]

is a compact Lagrangian submanifold. Moreover

\[ l(V', F_0(\mathcal{L} \times S^1, d(e^a \lambda - e^b \lambda)) = area(E) \]

Proof. We check that

\[ F_0^*(e^a \lambda \ominus e^b \lambda) = -e^{b(t')}ds(t') \]

So, \( F_0 \) is a Lagrangian embedding.

If the circle \( C \) homotopic to \( C_1 \subset \mathcal{L} \times s_0 \) then we compute

\[ \int_C F_0^*(e^b \lambda) = \int_{C_1} F_0^*(e^b \lambda) = 0. \]
since $\lambda|C_1 = 0$ due to $C_1 \subset \mathcal{L}$ and $\mathcal{L}$ is Legendre submanifold.

If the circle $C$ homotopic to $C_1 \subset l_0 \times S^1$ then we compute

$$\int_C F_0^*(e^b \lambda) = \int_{C_1} F_0^*(e^b \lambda) = n(area(E)).$$

This proves the Lemma.

Now we modify the above construction as follows:

$$F' : \mathcal{L} \times R \times R \to (0 \times \Sigma) \times ([0, \varepsilon] \times \Sigma) \subset T^*M \times T^*M$$
$$F'(((0, \sigma), (0, \sigma)), s, b) = ((0, \sigma), (b, \eta_s(\sigma)))$$
(3.17)

Now we embed a elliptic curve $E$ long along $s-axis$ and thin along $b-axis$ such that $E \subset [-s_1, s_2] \times [0, \varepsilon]$. We parametrize the $E$ by $t'$.

**Lemma 3.5** If there does not exist any closed Reeb orbit in $(\Sigma, \lambda)$, then

$$F : \mathcal{L} \times S^1 \to (R \times \Sigma) \times (R \times \Sigma)$$
$$F(((0, \sigma), (0, \sigma)), t') = ((0, \sigma), (b(t'), \eta_{s(t')}(\sigma)))$$
(3.18)

is a compact Lagrangian submanifold. Moreover

$$l(V', F(\mathcal{L} \times S^1), d(p^1_i dq^1_i - p^2_i dq^2_i)) = area(E)$$
(3.19)

Proof. We check that

$$F^*(p^1_i dq^1_i - p^2_i dq^2_i) = F^*(e^a \lambda \Theta e^b \lambda + df) = -e^{b(t')} ds(t') + df(l, t')$$
(3.20)

So, $F$ is a Lagrangian embedding.

If the circle $C$ homotopic to $C_1 \subset \mathcal{L} \times s_0$ then we compute

$$\int_C F^*(p^1_i dq^1_i - p^2_i dq^2_i) = \int_{C_1} F^*(e^b \lambda + df) = 0.$$ 
(3.21)

since $\lambda|C_1 = 0$ due to $C_1 \subset \mathcal{L}$ and $\mathcal{L}$ is “Legendre” submanifold.

If the circle $C$ homotopic to $C_1 \subset l_0 \times S^1$ then we compute

$$\int_C F^*(p^1_i dq^1_i - p^2_i dq^2_i) = \int_{C_1} F^*(e^b \lambda + df) = n(area(E)).$$
(3.22)

This proves the Lemma.

Now we construct an isotopy of Lagrangian embeddings as follows:

$$F' : \mathcal{L} \times S^1 \times [0, 1] \to V'$$
$$F'(l, t', t) = (\sigma, h^1_t(b(t'), \eta_{h(t')}(\sigma)))$$
$$F'_t(l, t') = F'(l, t', t), \ l = (\sigma, \sigma).$$
(3.23)
Lemma 3.6 If there does not exist any Reeb closed orbit for $X_\lambda$ in $(\Sigma, \lambda)$ then $F'$ is an weakly exact isotopy of Lagrangian embeddings. Moreover for the choice of $T = T_\Sigma$ satisfying $\Sigma \cap h_1'(\Sigma) = \emptyset$, then $F'_0(\mathcal{L} \times S^1) \cap F'_1(\mathcal{L} \times S^1) = \emptyset$.

Proof. By Lemma 3.1-3.5 and below.

Let $(V', \omega') = (T^*M \times T^*M, dp_i^1 \wedge dq_i^1 \oplus dp_i^2 \wedge dq_i^2), W' = \mathcal{F}(\mathcal{L} \times S^1)$, and $(V, \omega) = (V' \times C, \omega' \oplus \omega_0)$. As in [9], we use symplectic figure eight trick invented by Gromov to construct a Lagrangian submanifold in $V$ through the Lagrange isotopy $F'$ in $V'$. Fix a positive $\delta < 1$ and take a $C^\infty$-map $\rho : S^1 \to [0, 1]$, where the circle $S^1$ is parametrized by $\Theta \in [-1, 1]$, such that the $\delta$–neighborhood $I_0$ of $0 \in S^1$ goes to $0 \in [0, 1]$ and $\delta$–neighbourhood $I_1$ of $\pm 1 \in S^1$ goes $1 \in [0, 1]$. Let

\[
\tilde{l} = h^{T^*}_{\rho}(p_i^1 dq_i^1 \ominus p_i^2 dq_i^2) = p_i^1 dq_i^1 - p_i^2 dq_i^2 - \rho(\Theta) T d\rho
\]

\[
= -(e^{b(t')} ds(t') + d\beta) - \rho T d\rho = (-e^{b(t')} ds(t') + d\beta + d\rho T g) + T g d\rho
\]

\[
= (-e^{b(t')} ds(t') + d\beta + d\rho T g) - T g \rho'(\Theta) d\Theta
\]

\[
= (-e^{b(t')} ds(t') + d\beta + d\rho T g) - \Phi d\Theta
\]

(3.24)

be the pull-back of the form $\tilde{l}' = (-e^{b(t')} ds(t') + d\beta + d\rho T g) - \psi(s, t) dt$ to $W' \times S^1$ under the map $(w', \Theta) \to (w', \rho(\Theta))$ and assume without loss of generality $\Phi$ vanishes on $W' \times (I_0 \cup I_1)$. Since $[\tilde{l}']W' \times \{t\} = [(-e^{b(t')} ds(t'))$ is independent of $t$, so $F'$ is weakly exact. It is crucial here $| - \psi(s, t)| \leq M_0$ and $M_0$ is independent of $area(E)$.

Next, consider a map $\alpha$ of the annulus $S^1 \times [\Phi_, \Phi_+]$ into $R^2$, where $\Phi_-$ and $\Phi_+$ are the lower and the upper bound of the function $\Phi$ correspondingly, such that

(i) The pull-back under $\alpha$ of the form $dx \wedge dy$ on $R^2$ equals $-d\Phi \wedge d\Theta$.

(ii) The map $\alpha$ is bijective on $I \times [\Phi_-, \Phi_+]$ where $I \subset S^1$ is some closed subset, such that $I \cup I_0 \cup I_1 = S^1$; furthermore, the origin $0 \in R^2$ is a unique double point of the map $\alpha$ on $S^1 \times 0$, that is

$0 = \alpha(0, 0) = \alpha(\pm 1, 0),$

and $\alpha$ is injective on $S^1 = S^1 \times 0$ minus $\{0, \pm 1\}$.

(iii) The curve $S_0 = \alpha(S^1 \times 0) \subset R^2$ “bounds” zero area in $R^2$, that is $\int_{S_0} xdy = 0$, for the 1-form $xdy$ on $R^2$.

Proposition 3.1 Let $V'$, $W'$ and $F'$ as above. Then there exists an exact Lagrangian embedding $F : W' \times S^1 \to V' \times R^2$ given by $F(w', \Theta) = (F'(w', \rho(\Theta))), \alpha(\Theta, \Phi))$. Denote $W = F(W' \times S^1)$. Then $W$ is contained in $T^*M \times T^*M \times B_R(0)$, here $4\pi R^2 = 8M_0$.

Proof. Similar to [9, 2.3B_3].
3.2 Formulation of Hilbert manifolds

Let $(\Sigma, \lambda)$ be a closed $(2n-1)$-dimensional manifold with a contact form $\lambda$ of induced type in $T^*M$, it is well-known that $T^*M$ is a Stein manifold, so it is exhausted by a proper pluri-subharmonic function, in fact if $M$ is closed one can take $f = \frac{1}{2} |p|^2$, if $M$ is an open manifold one can take a proper Morse function $g$ to modify $f$, i.e., $f_1 = f + \pi^* g$. Since $\Sigma$ is compact and $W' = G'(\mathcal{L} \times R)$ is contained in $\Sigma$, by our construction we have $W$ is contained in a compact set $V_c$, $V_c \subset T^*M \times T^*M \times R^2$ for $M$ is an open manifold. If $M$ is closed and $\pi(\Sigma) = M$, we know that $W_K = F_K(W'_K \times S^1)$ is a bounded set in $T^*M \times T^*M \times R^2$.

We choose an almost complex structure $J_1$ on $T^*M$ tamed by $\omega_1 = dp_i \wedge dq_i$ and the metric $g_1 = \omega_1(\cdot, J_1 \cdot)$ (see[9]). Let $(V', \omega') = (T^*M \times T^*M, p_1^* dq_1 \oplus p_2^* dq_2^2)$ By above discussion we know that $W'$ and $\Sigma \times \Sigma$ contained in $\{f_1 \leq c\} \times \{f_1 \leq c\}$ for $c$ large enough, i.e., contained in a compact set $V'_c$ in $T^*M \times T^*M$. Then we expanding near $f_1^{-1}(c)$ to get a complete exact symplectic manifold with a complete Riemann metric with injective radius $r_0 > 0$ (see[17]).

In the following we denote by $(V, \omega) = (V' \times R^2, \omega' \oplus dx \wedge dy)$ with the metric $g = g' \oplus g_0$ induced by $\omega(\cdot, J \cdot)(J = J' \oplus i)$ and $W \subset V$ a Lagrangian submanifold which was constructed in section 3.1.

Let

$$D^k(V, W, p) = \{ u \in H^k(D, V)| u(x) \in W \ a.e \ for \ x \in \partial D and u(1) = p \}$$

for $k \geq 100$.

**Lemma 3.7** Let $W$ be a closed Lagrangian submanifold in $V$. Then,

$$D^k(V, W, p) = \{ u \in H^k(D, V)| u(x) \in W \ a.e \ for \ x \in \partial D and u(1) = p \}$$

is a pseudo-Hilbert manifold with the tangent bundle

$$T^k(D^k(V, W, p)) = \bigcup_{u \in D^k(V, W, p)} \Lambda^{k-1}(u^* TV, u|_{\partial D}^* TW, p) \quad (3.25)$$

here

$$\Lambda^{k-1}(u^* TV, u|_{\partial D}^* TW, p) = \{ H^{k-1} - sections of (u^*(TV), (u|_{\partial D})^* TL) which \ vanishes \ at \ 1 \}$$
Proof: See [3, 14].

Now we consider a section from $D^k(V, W, p)$ to $TD^k(V, W, p)$ follows as in [3, 9], i.e., let $\bar{\partial} : D^k(V, W, p) \rightarrow TD^k(V, W, p)$ be the Cauchy-Riemmann section

$$\bar{\partial}u = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t}$$  \hspace{1cm} (3.26)

for $u \in D^k(V, W, p)$.

**Theorem 3.1** The Cauchy-Riemann section $\bar{\partial}$ defined in (3.26) is a Fredholm section of Index zero.

Proof. According to the definition of the Fredholm section, we need to prove that $u \in D^k(V, W, p)$, the linearization $D\bar{\partial}(u)$ of $\bar{\partial}$ at $u$ is a linear Fredholm operator. Note that

$$D\bar{\partial}(u) = D\bar{\partial}_{[u]}$$  \hspace{1cm} (3.27)

where

$$(D\bar{\partial}_{[u]})v = \frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v$$  \hspace{1cm} (3.28)

with

$$v|_{\partial D} \in (u|_{\partial D})^*TW$$

here $A(u)$ is $2n \times 2n$ matrix induced by the torsion of almost complex structure, see [3, 9] for the computation.

Observe that the linearization $D\bar{\partial}(u)$ of $\bar{\partial}$ at $u$ is equivalent to the following Lagrangian boundary value problem

$$\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} + A(u)v = f, \ v \in \Lambda^k(u^*TW)$$  \hspace{1cm} (3.29)

$v(t) \in T_{u(t)}W, \ t \in \partial D$

One can check that (3.29) defines a linear Fredholm operator. In fact, by proposition 2.2 and Lemma 2.1, since the operator $A(u)$ is a compact, we know that the operator $\bar{\partial}$ is a nonlinear Fredholm operator of the index zero.

**Definition 3.1** Let $X$ be a Banach manifold and $P : Y \rightarrow X$ the Banach vector bundle. A Fredholm section $F : X \rightarrow Y$ is proper if $F^{-1}(0)$ is a compact set and is called generic if $F$ intersects the zero section transversally, see [3, 9].

**Definition 3.2** $\text{deg}(F, y) = \frac{\sharp F^{-1}(0)}{2}$ is called the Fredholm degree of a Fredholm section (see[3, 9]).
Theorem 3.2 Assum that the Fredholm section $F = \bar{\partial} : D^k(V, W, p) \to T^* D^k(V, W, p)$ constructed in (3.26) is proper. Then,

$\deg(F, 0) = 1$

Proof: We assume that $u : D \mapsto V$ be a $J$–holomorphic disk with boundary $u(\partial D) \subset W$ and by the assumption that $u$ is homotopic to the constant map $u_0(D) = p$. Since almost complex structure $J$ tamed by the symplectic form $\omega$, by stokes formula, we conclude $u : D \to V$ is a constant map. Because $u(1) = p$, We know that $F^{-1}(0) = p$. Next we show that the linearization of $DF(p)$ of $F$ at $p$ is an isomorphism from $T_p D(V, W, p)$ to $E$. This is equivalent to solve the equations

$$\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} + Av = f$$

$$(3.30)$$

$v|_{\partial D} \subset T_p W$  $$(3.31)$$

here $J = J(p) = i$ and $A$ a constant matrix. By Lemma 2.1, we know that $DF(p)$ is an isomorphism. Therefore $\deg(F, 0) = 1$.

4 Non-properness of a Fredholm section

In this section we shall construct a non-proper Fredholm section $F_1 : D \to E$ by perturbing the Cauchy-Riemann section as in [3, 9].

4.1 Anti-holomorphic section

Let $(V', \omega')$ and $(V, \omega) = (V' \times C, \omega' \oplus \omega_0)$, and $W$ as in section 3 and $J = J' \oplus i$, $g = g' \oplus g_0$, $g_0$ the standard metric on $C$.

Now let $c \in C$ be a non-zero vector or nonzero constant vector field on $C$. We consider the equations

$$v = (v', f) : D \to V' \times C$$

$$\bar{\partial}_v v' = 0, \bar{\partial} f = c$$

$$v|_{\partial D} : \partial D \to W$$

$$(4.1)$$

here $v$ homotopic to constant map $\{p\}$ relative to $W$. Note that $W \subset V \times B_R(0)$ for a positive number $R$ large enough.
Lemma 4.1 Let $v$ be the solutions of (4.1), then one has the following estimates

$$E(v) = \int_D \left( g'(x) \frac{\partial v'}{\partial x} J' \frac{\partial v'}{\partial y} + g'(y) \frac{\partial v'}{\partial y} J' \frac{\partial v'}{\partial x} \right) + g_0 \left( i \frac{\partial f}{\partial x}, i \frac{\partial f}{\partial y} \right) d\sigma \leq 4\pi R^2.$$ (4.2)

Proof: Since $v(z) = (v'(z), f(z))$ satisfy (4.1) and $v(z) = (v'(z), f(z)) \in V' \times C$ is homotopic to constant map $v_0 : D \to \{p\} \subset W$ in $(V, W)$, by the Stokes formula

$$\int_D v^*(\omega' \oplus \omega_0) = 0$$ (4.3)

Note that the metric $g$ is adapted to the symplectic form $\omega$ and $J$, i.e.,

$$g = \omega(\cdot, J \cdot)$$ (4.4)

By the simple algebraic computation, we have

$$\int_D v^* \omega = \frac{1}{4} \int_D (|\partial v|^2 - |\bar{\partial} v|^2) = 0$$ (4.5)

and

$$|\nabla v| = \frac{1}{2} (|\partial v|^2 + |\bar{\partial} v|^2)$$ (4.6)

Then

$$E(v) = \int_D |\nabla v| = \int_D \left\{ \frac{1}{2} (|\partial v|^2 + |\bar{\partial} v|^2) \right\} d\sigma = \pi |c|^2 g_0$$ (4.7)

By the equations (4.1), one get

$$\bar{\partial} f = c \text{ on } D$$ (4.8)

We have

$$f(z) = \frac{1}{2} c \bar{z} + h(z)$$ (4.9)

here $h(z)$ is a holomorphic function on $D$. Note that $f(z)$ is smooth up to the boundary $\partial D$, then, by Cauchy integral formula

$$\int_{\partial D} f(z) dz = \frac{1}{2} c \int_{\partial D} \bar{z} dz + \int_{\partial D} h(z) dz$$
So, we have
\[ |c| = \frac{1}{\pi} \left| \int_{\partial D} f(z) dz \right| \]  \hspace{1cm} (4.11)

Therefore,
\[
E(v) \leq \pi |c|^2 \leq \frac{1}{\pi} \left| \int_{\partial D} f(z) dz \right|^2 \\
\leq \frac{1}{\pi} \int_{\partial D} |f(z)||dz|^2 \\
\leq 4\pi |\text{diam}(pr_2(W))|^2 \\
\leq 4\pi R^2. \hspace{1cm} (4.12)
\]

This finishes the proof of Lemma.

**Proposition 4.1** For \(|c| \geq 3R\), then the equations (4.1) has no solutions.

Proof. By (4.11), we have
\[
|c| \leq \frac{1}{\pi} \int_{\partial D} |f(z)||dz| \\
\leq \frac{1}{\pi} \int_{\partial D} \text{diam}(pr_2(W))||dz| \\
\leq 2R. \hspace{1cm} (4.13)
\]

It follows that \(c = 3R\) can not be obtained by any solutions.

### 4.2 Modification of section \(c\)

Note that the section \(c\) is not a section of the Hilbert bundle in section 3 since \(c\) is not tangent to the Lagrangian submanifold \(W\), we must modify it as follows:

Let \(c\) as in section 4.1, we define
\[
c_{\chi, \delta}(z, v) = \begin{cases} 
s \quad \text{if } |z| \leq 1 - 2\delta, \\
0 \quad \text{otherwise}
\end{cases} \hspace{1cm} (4.14)
\]

Then by using the cut off function \(\varphi_h(z)\) and its convolution with section \(c_{\chi, \delta}\), we obtain a smooth section \(c_\delta\) satisfying
\[
c_\delta(z, v) = \begin{cases} 
s \quad \text{if } |z| \leq 1 - 3\delta, \\
0 \quad \text{if } |z| \geq 1 - \delta.
\end{cases} \hspace{1cm} (4.15)
\]
for $h$ small enough, for the convolution theory see [13].

Now let $c \in C$ be a non-zero vector and $c_\delta$ the induced anti-holomorphic section. We consider the equations

$$v = (v', f) : D \to V' \times C$$
$$\bar{\partial}v' = 0, \bar{\partial}f = c_\delta$$
$$v|_{\partial D} : \partial D \to W$$

(4.16)

which is a slight modification of (4.1) Note that $W \subset V \times B_R(0)$. Then by repeating the same argument as section 4.1., we obtain

**Lemma 4.2** Let $v$ be the solutions of (4.16) and $\delta$ small enough, then one has the following estimates

$$E(v) \leq 4\pi R^2.$$  

(4.17)

and

**Proposition 4.2** For $|c| \geq 3R$, then the equations (4.16) has no solutions.

### 4.3 Modification of $J \oplus i$

Let $(\Sigma, \lambda)$ be a closed contact manifold with a contact form $\lambda$ of induced type in $T^*M$. Let $J_M$ be an almost complex structure on $T^*M$ and $J_1 = J_M \oplus J_M \oplus i$ the almost complex structure on $T^*M \times T^*M \times R^2$ tamed by $\omega' \oplus \omega_0$. Let $J_2$ be any almost complex structure on $T^*M \times T^*M \times R^2$.

Now we consider the almost complex structure on the symplectic fibration $D \times V \to D$ which will be discussed in detail in section 5.1., see also [9].

$$J_{x,\delta}(z, v) = \begin{cases} 
  i \oplus J_1 & \text{if } |z| \leq 1 - 2\delta, \\
  i \oplus J_2 & \text{otherwise}
\end{cases}$$

(4.18)

Then by using the cut off function $\varphi_h(z)$ and its convolution with section $J_{x,\delta}$, we obtain a smooth section $J_\delta$ satisfying

$$J_\delta(z, v) = \begin{cases} 
  i \oplus J_1 & \text{if } |z| \leq 1 - 3\delta, \\
  i \oplus J_2 & \text{if } |z| \geq 1 - \delta.
\end{cases}$$

(4.19)

as in section 4.2.

Then as in section 4.2, one can also reformulation of the equations (4.16) and get similar estimates of Cauchy-Riemann equations, we leave it as exercises to reader.
Theorem 4.1 The Fredholm sections $F_1 = \bar{\partial} + c_\delta : D^k(V, W, p) \to T(D^k(V, W, p))$ is not proper for $|c|$ large enough.

Proof. See [3, 9].

5 J–holomorphic section

Recall that $W \subset V = T^*M \times T^*M \times B_R(0)$ as in section 3. The Riemann metric $g$ on $V' \times R^2$ induces a metric $g|W$.

Now let $c \in C$ be a non-zero vector and $c_\delta$ the induced anti-holomorphic section. We consider the nonlinear inhomogeneous equations (4.16) and transform it into $\bar{\partial}$–holomorphic map by considering its graph as in [3, 9].

Denote by $Y(1) \to D \times V$ the bundle of homomorphisms $T_s(D) \to T_v(V)$. If $D$ and $V$ are given the disk and the almost Kähler manifold, then we distinguish the subbundle $X(1) \subset Y(1)$ which consists of complex linear homomorphisms and we denote $\tilde{X}(1) \to D \times V$ the quotient bundle $Y(1)/X(1)$. Now, we assign to each $C^1$-map $v : D \to V$ the section $\bar{\partial}v$ of the bundle $\tilde{X}(1)$ over the graph $\Gamma_v \subset D \times V$ by composing the differential of $v$ with the quotient homomorphism $Y(1) \to \tilde{X}(1)$. If $c_\delta : D \times V \to \tilde{X}$ is a $H^k$–section we write $\bar{\partial}v = c_\delta$ for the equation $\bar{\partial}v = c_\delta|\Gamma_v$.

Lemma 5.1 (Gromov[9]) There exists a unique almost complex structure $J_g$ on $D \times V$ (which also depends on the given structures in $D$ and in $V$), such that the (germs of) $J_\delta$–holomorphic sections $v : D \to D \times V$ are exactly and only the solutions of the equations $\bar{\partial}v = c_\delta$. Furthermore, the fibres $z \times V \subset D \times V$ are $J_\delta$–holomorphic (i.e. the subbundles $T(z \times V) \subset T(D \times V)$ are $J_\delta$–complex) and the structure $J_\delta|z \times V$ equals the original structure on $V = z \times V$. Moreover $J_\delta$ is tamed by $k\omega_0 \oplus \omega$ for $k$ large enough which is independent of $\delta$.

6 Proof of Theorem 1.1

Theorem 6.1 There exists a non-constant $J$–holomorphic map $u : (D, \partial D) \to (V' \times C, W)$ with $E(u) \leq 4\pi R^2$.

Proof. The results in section 4 shows the solutions of equations (4.16) must denegerate to a cusp curves, i.e., we obtain a Sacks-Uhlenbeck’s bubble, i.e., $J$–holomorphic sphere or disk with boundary in $W$, the exactness of $T^*M \times$
$T^*M \times R^2$ rules out the possibility of $J$–holomorphic sphere. So, we get a holomorphic disk. For the more detail, see the proof of Theorem 2.3.B in [9].

**Proof of Theorem 1.1.** By the assumption of Theorem 1.1, we know that the Lagrangian submanifold $W$ in $T^*M \times T^*M \times R^2$ is embedded. Moreover $l = l(T^*M \times T^*M \times R^2, W, \omega) = \inf \{ \int_D f^*\omega > 0 | f : (D, \partial D) \to (T^*M \times T^*M \times R^2, W) \} = area(E)$. By Theorem 6.1, $l \leq 4\pi R^2$. If $area(E)$ is large enough, this is a contradiction. This implies the assumption that $L$ has no self-intersection point under Reeb flow does not hold.

**References**

[1] Arnold, V. I., First steps in symplectic topology, Russian Math. Surveys 41(1986),1-21.

[2] Arnold, V.& Givental, A., Symplectic Geometry, in: Dynamical Systems IV, edited by V. I. Arnold and S. P. Novikov, Springer-Verlag, 1985.

[3] Audin, M& Lafontaine, J., eds.: Holomorphic Curves in Symplectic Geometry. Progr. Math. 117, (1994) Birkhaüser, Boston.

[4] Eliashberg, Y., Symplectic topology in the nineties, Differential geometry and its applications 9(1998)59-88.

[5] Eliashberg,Y.& Gromov, M., Lagrangian Intersection Theory: Finite-Dimensional Approach, Amer. Math. Soc. Transl. 186(1998): 27-118.

[6] Floer, A., Hofer, H.& Viterbo, C., The Weinstein conjecture in $P \times C^l$, Math.Z. 203(1990)469-482.

[7] Givental, A. B., Nonlinear generalization of the Maslov index, Adv. in Sov. Math., V.1, AMS, Providence, RI, 1990.

[8] Gray, J.W., Some global properties of contact structures. Ann. of Math., 2(69): 421-450, 1959.

[9] Gromov, M., Pseudoholomorphic Curves in Symplectic manifolds. Inv. Math. 82(1985), 307-347.

[10] Hofer, H., Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjection in dimension three. Inventions Math., 114(1993), 515-563.
[11] Hofer, H& Viterbo, C., The Weinstein conjecture in cotangent bundle and related results, Ann.Scuola. Norm.sup.Pisa. Serie 4,15 (1988), 411-415.

[12] Hofer, H.& Zehnder, E., Periodic solutions on hypersurfaces and a result by C. Viterbo, Invent. Math. 90(1987)1-9.

[13] Hörmander, L., The Analysis of Linear Partial Differential Operators I, Springer-Verlag, 1983.

[14] Klingenberg, K., Lectures on closed Geodesics, Grundlehren der Math. Wissenschaften, vol 230, Springer-Verlag, 1978.

[15] Lalonde, F & Sikorav, J.C., Sous-Vari`et`es Lagrangiennes et lagrangiennes exactes des fibrès cotangents, Comment. Math. Helvetici 66(1991) 18-33.

[16] Ma, R., A remark on the Weinstein conjecture in $M \times R^{2n}$. Nonlinear Analysis and Microlocal Analysis, edited by K. C. Chang, Y. M. Huang and T. T. Li, World Scientific Publishing, 176-184.

[17] Ma, R., Symplectic Capacity and Weinstein Conjecture in Certain Cotangent bundles and Stein manifolds. NoDEA.2(1995):341-356.

[18] Ma, R., Legendrian submanifolds and A Proof on Chord Conjecture, Boundary Value Problems, Integral Equations and Related Problems, edited by J K Lu & G C Wen, World Scientific, 135-142.

[19] Ma, R., The existence of $J$–holomorphic curves and applications to the Weinstein conjecture. Chin. Ann. of Math. 20B:4(1999), 425-434.

[20] Mohunke, K.: Holomorphic Disks and the Chord Conjecture, Annals of Math., (2001), 154:219-222.

[21] Rabinowitz, P., Periodic solutions of Hamiltonian systems, Comm. Pure. Appl. Math 31, 157-184, 1978.

[22] Sacks, J. and Uhlenbeck,K., The existence of minimal 2-spheres. Ann. Math., 113:1-24, 1983.

[23] Smale, S., An infinite dimensional version of Sard's theorem, Amer. J. Math. 87: 861-866, 1965.

[24] Thurston, W., The theory of foliations in codimension greater than one, Comm. Math. Helv. 214-231, 49(1974).
[25] Viterbo, C., A proof of the Weinstein conjecture in $R^{2n}$, Ann. Inst. Henri. Poincaré, Analyse nonlinéaire, 4: 337-357, 1987.

[26] Viterbo, C., Exact Lagrange submanifolds, Periodic orbits and the cohomology of free loop spaces, J.Diff.Geom., 47(1997), 420-468.

[27] Weinstein, A., Periodic orbits for convex Hamiltonian systems. Ann. Math. 108(1978),507-518.

[28] Weinstein, A., On the hypothesis of Rabinowitz’s periodic orbit theorems, J. Diff. Eq.33, 353-358, 1979.

[29] Wendland, W., Elliptic systems in the plane, Monographs and studies in Mathematics 3, Pitman, London-San Francisco, 1979.