Crumpling of Curved Sheets: Generalizing Föppl-von Kármán

J. Leo van Hemmen¹ and Mark A. Peterson²

¹Physik Department, Technische Universität München ,
85747 Garching bei München, Germany, and

²Physics Department, Mount Holyoke College, South Hadley, MA 01075-6420

Abstract

We generalize the Föppl-von Kármán equations to an initially precurved sheet and present the underlying derivation. A geometrically computed moment of strain replaces the notion of bending moment and results in a geometric formulation of the theory of shells. As the curvature approaches zero, i.e., the sheet becomes flat, the new equations reduce to the classic Föppl-von Kármán ones. The present theory solves the long-standing problem of formulating these equations for an a priori curved shell and applies, for instance, both to shell theory and to strongly curved biomembranes of cells as closed surfaces, exhibiting crumpling as the membrane thickness goes to zero.

PACS numbers: 46.25.-y, 68.60.Bs, 87.16.Dg, 02.40.Hw
Crumpling of thin membranes or sheets and the existence of auxetic (negative Poisson ratio) materials have challenged physical imagination, and explanation, for quite a while [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Figure 1 shows a striking example stemming from the biophysics of membranes [15, 16, 17]. At present it appears that all these phenomena occurring in thin sheets are realizations of the Föppl-von Kármán (FvK) equations [18, 19, 20, 21], an important case of the equilibrium equations of elasticity theory, incorporating nonlinearity, and aiming at describing strongly curved sheets of finite thickness $h > 0$. The FvK equations only apply, however, to a thin sheet that is planar to start with and is then bent strongly. Their derivation could not handle a sheet being precurved, although this often occurs in practice – as in Fig. 1, for instance. The latter problem has been tantalizing the physics of membranes ever since Föppl [18] and von Kármán [19] proposed their equations a century ago.

The FvK derivation includes the assumption that the sheet in question is thin, but it allows for the possibility that deviations from the initial shape may be large: it goes to quadratic order in the normal deformation parameter. Because the sheet is thin, by assumption, it is still consistent to assume a linear stress-strain relationship within the sheet. Several studies have shown [21] that this description makes physical sense for large deformation, up to and including sharp creasing – a regime well beyond what Föppl and von Kármán originally had in mind.

There is yet another assumption in FvK, which greatly limits its applicability to physical systems of interest, namely, that the initial shape of the sheet to be strongly bent or even crumpled is planar. How would deformation of an initially curved, anisotropic surface interact with the pre-existing anisotropy? This and other fundamental questions of strong bending and crumpling of surfaces cannot even be asked in the context of the original FvK theory. We present here the generalization necessary to handle such questions.

If one looks to shell theory, a notoriously difficult part of elasticity theory that deals with thin elastic surfaces [22], one finds that existing formulations are inadequate for the purpose of handling a priori curved sheets. Shell theory was developed in the context of mechanical engineering with practical applications in mind. The vocabulary of shell theory is entirely physical. One would not know from reading shell theory that there is already a highly developed mathematical language for dealing with surfaces and their deformations, namely differential geometry. In this paper we show how to use methods of differential geometry to
FIG. 1: Scanning electron micrograph of a ‘ghost’, a red blood cell (erythrocyte) voided of its cytoplasm through a small hole created by osmotic rupture of the cell (perhaps the dark area in the membrane). It consists of a closed cell membrane (plus reticulum) of thickness $h \approx 10$ nm; the cell’s diameter is about $10 \, \mu m$. The ridges give the membrane a crumpled structure. As was suggested by Lobkovsky, one can understand this “chaotic” structure through a system of two coupled nonlinear partial differential equations due to Föppl and von Kármán (FvK). One of the two differential operators of the form $\Delta^2$ has a small prefactor scaling as $h^2$ where $h$ is the membrane thickness; cf. (14) and (24) below. In the limit $h \rightarrow 0$ it cannot be neglected but gives rise to crumpling – analogous to $R_e^{-1}\Delta$ in hydrodynamics where it generates turbulence as the Reynolds number $R_e \rightarrow \infty$. The FvK equations have been derived as a local deviation from the flat state. The cell membrane, however, was never flat and clearly shows that in biological physics this assumption often has no justification. Picture courtesy of T.L. Steck. The original has been contrast enhanced.
formulate, and physically interpret, the equations of crumpling for a general sheet. It will be evident that this generality is necessary in describing deformations in which the change in curvature is comparable to pre-existing curvature. By way of motivation we call attention to a typical, biological, example in Fig. an initially curved surface that has subsequently been wrinkled.

We consider the equilibrium state of an elastic body occupying a region \( D \subset \mathbb{R}^3 \) and subject to a conservative force density \( f \) derivable from a potential \( U \). The body is described by an elastic energy density \( E_{el} \) quadratic in strain \( u_{ij} \). Thus its equilibrium is characterized by

\[
0 = \delta \int_D dV \left( E_{el} + U \right) = \int_D dV \left( \sigma^{ij} \delta u_{ij} - f^i \delta u_i \right) \tag{1}
\]

where the stress \( \sigma^{ij} \) is symmetric and linear in strain, and \( \delta u_i \) is an arbitrary small deformation, with \( \delta u_{ij} \) a function of \( \delta u_i \). Now we ask what this means in case the body is a thin sheet, meaning that the region \( D \) is a smooth surface \( M \) thickened along the normal direction by a small amount \( h \), with \( M \) as the midsection. (Generalizations to such cases as \( h \) variable rather than constant, elastic constants dependent on position, etc., are straightforward, and will not be emphasized in what follows.) Let the Euclidean metric of three dimensional space be \( g_{ij} \), and let the first, second, and third fundamental forms of \( M \) be \( g_{\alpha\beta}, h_{\alpha\beta}, k_{\alpha\beta} \), where Greek indices take values 1, 2. In particular the first fundamental form \( g_{\alpha\beta} \) is just \( g_{ij} \) restricted to \( M \). We assume \( M \) is oriented by choice of a unit normal \( n \).

As is well known, there is an orthogonal coordinate system \((q^1, q^2)\) on \( M \) such that the level lines of the coordinates are tangent to the principal curvature directions at every point (umbilic points may be coordinate singularities). Furthermore, in a thickened three-dimensional neighborhood of \( M \) this coordinate system may be extended to an orthogonal system \((q^1, q^2, z)\), where the third coordinate \( z \) is along the normal to \( M \) through \((q^1, q^2)\), positive in the direction of \( n \). Methods for computing geometrical objects within this framework have been given in. In particular, in these coordinates the metric \( g_{ij} \) in \( D_M \) takes the form

\[
g_{ij} = \text{diag} \left( g_{11}(1 - \kappa_1 z)^2, g_{22}(1 - \kappa_2 z)^2, 1 \right). \tag{2}
\]

Here \( \text{diag}(g_{11}, g_{22}) \) is the first fundamental form of \( M \) and \((\kappa_1, \kappa_2)\) are the principal curvatures of \( M \), functions of \((q^1, q^2)\) but not \( z \). The minus signs define the sign conventions relating \((\kappa_1, \kappa_2)\) and \( n \). For example, if \( n \) is the inner normal on the sphere, then the princi-
pal curvatures are positive. There is still the freedom to choose \((q^1, q^2)\) such that at a given point \(P \in \mathcal{M}\) the first fundamental form obeys \(g_{11,1} = g_{22,2} = 0\) (at \(P\)). Throughout what follows, \(\alpha\) denotes a partial derivative \(\partial/\partial x_\alpha\) with respect to the coordinate \(x_\alpha\) following the comma. The only non-vanishing Christoffel symbols at \(P\) are now
\[
\Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{2} g^{11} g_{11,2}
\]
\[
\Gamma^1_{22} = \frac{1}{2} g^{11} g_{22,1}
\]
and the corresponding ones with \(1 \leftrightarrow 2\).

A central ingredient in this approach is the observation that the two dimensional strain tensor on \(\mathcal{M}\) is related to the metric tensor by
\[
\delta u^\mathcal{M}_{\alpha\beta} = \frac{1}{2} \delta g_{\alpha\beta}
\]
where \(\delta g_{\alpha\beta}\) is the change in the metric tensor under a deformation \(V = (\delta u^1, \delta u^2, \psi)\), a vector field. Here the notation for the components of \(V\) has been chosen to suggest that tangential displacements \(\delta u^\alpha\) are small, but that the normal displacement \(\psi\) is not necessarily small, reflecting the anisotropy of e.g. a biomembrane. The change in the metric \(g_{\alpha\beta}\) under \(V\), in turn, can be computed as a Lie-Taylor series, using the Lie derivative \(\mathcal{L}_V\)[27, 28]
\[
\delta g_{\alpha\beta} = (\mathcal{L}_V g)(\partial_\alpha, \partial_\beta) + \frac{1}{2} (\mathcal{L}_V (\mathcal{L}_V g))(\partial_\alpha, \partial_\beta) + \ldots
\]
Here \(\mathcal{L}_V g\) is the Lie derivative of the three-dimensional metric, with the result restricted to \(\mathcal{M}\) by evaluating on tangent vectors \(\partial_\alpha\) to \(\mathcal{M}\). The result depends only on the restriction of \(V\) to \(\mathcal{M}\), and if \(V\) is given only on \(\mathcal{M}\), \(V\) may be extended in any smooth way for the purpose of the computation, for example the components of \(V\) could be independent of \(z\). We therefore interpret \(\delta u^\mathcal{M}_{\alpha\beta}\) as the strain in \(D_{\mathcal{M}}\), averaged across the thickness \(h\). The result is, keeping terms linear in the deformation \(\delta u^\alpha\), and going to quadratic terms in \(\psi\),
\[
\delta u^\mathcal{M}_{\alpha\beta} = (\nabla_\alpha \delta u_\beta + \nabla_\beta \delta u_\alpha + \nabla_\alpha \psi \nabla_\beta \psi + \psi^2 k_{\alpha\beta})/2
\]
\[-\psi h_{\alpha\beta}
\]
where \(\nabla_\alpha\) is the covariant derivative on \(\mathcal{M}\). (First derivatives of the function \(\psi\) are, of course, just ordinary partial derivatives.) We illustrate the method by computing one component of the first Lie derivative. The vector field \(V\) is the first order differential operator
\[
V = \delta u^1 \partial_1 + \delta u^2 \partial_2 + \psi \partial_z
\]
Then we have, evaluating at $P$ for simplicity, and noting that $z = 0$ there,

$$(L_V g)(\partial_1, \partial_1) = V g(\partial_1, \partial_1) + 2g([V, \partial_1], \partial_1)$$

$$= \delta u^2 g_{11,2} - 2\psi \kappa_{11} g_{11} + 2g_{11}\delta u^1_{11}$$

$$= 2(\delta u^1_{11} + \delta u^2_{2\Gamma_{11}^2}) - 2\psi g_{11}\kappa_{11}$$

and similarly for other components. Since the result is a tensor and coincides with $\nabla_\alpha \delta u_\beta + \nabla_\beta \delta u_\alpha - 2\psi h_{\alpha\beta}$ in this coordinate system, it must be this tensor. The second derivative computation is similar.

Since the three-dimensional body $D_M$ is thin, we assume the strain is at most linear in $z$. But when $D_M$ is bent, it is relatively compressed on one side and extended on the other, so it is clear that there is a physically important strain that is linear in $z$. From the geometrical meaning of the second fundamental form $h_{\alpha\beta}$ as the rate of rotation of the unit normal $n$, it is clear that this contribution is $-z\delta h_{\alpha\beta}$, where $\delta h_{\alpha\beta}$ is the change in the second fundamental form of $M$ under the deformation.

Unlike the first fundamental form, the second fundamental form is not the restriction of a tensor field in three dimensions to $M$, and so its change under deformation must be computed in a different way. For example, the second fundamental form of the varied surface $M'$ is, up to a factor $-2$, the Lie derivative of $g$ with respect to the unit normal on $M'$, according to (7), and there are also more classical ways to compute it [24]. The result is

$$\delta u^B_{\alpha\beta} = -z\delta h_{\alpha\beta} = -z(\nabla_\alpha \nabla_\beta \psi - \psi k_{\alpha\beta}) .$$

Here we have only gone to linear order in $\psi$, as the bending term is usually small in any case, and the second order terms in $\psi$ are complicated. Again we illustrate the method. Let the equation of the varied surface $M'$ be $z = \psi$, the result of deforming $M$ by the vector field $U = \psi \partial_z$. Tangent vectors to $M'$ are

$$X_\mu = \partial_\mu + [\partial_\mu, U] = \partial_\mu + \psi_\mu \partial_z .$$

The unit normal $n'$ on $M'$ is orthogonal to these and normalized, hence to first order in $\psi$

$$n' = -g^{11}\psi_1 \partial_1 - g^{22}\psi_2 \partial_2 + \partial_z .$$
Then computing one component, at a point over $P$ for simplicity, we have

$$(\mathcal{L}_{n'}g)(X_1, X_1) = n'g(X_1, X_1) + 2g([X_1, n'], X_1)$$

$=-\psi g_{22}g_{11,2} + 2g(\psi, X_1 - \kappa_1) - \psi_{11}.$

We evaluate at $z = \psi$, finding

$$(\mathcal{L}_{n'}g)(X_1, X_1) = -2g_{11}\kappa_1 + 2\psi g_{11}\kappa_1^2 - 2(\psi_{,11} + \psi_2\Gamma^2_{11})$$

and similarly for other components, giving the result above. Going to higher order in $\psi$ would only require solving to higher order in (11) as was done, for example, in [26].

As stress is linear in strain, it too will be linear in $z$:

$$\sigma^{\alpha\beta} = \sigma^{\alpha\beta}_M + z\sigma^{\alpha\beta}_B.$$  \hspace{1cm} \text{(12)}

Putting $\delta u_{\alpha\beta} = \delta u^{\alpha}_{\alpha\beta} + \delta u^{B}_{\alpha\beta}$ from (7), (9), and $\sigma^{\alpha\beta}$ from (12) into (1), and integrating $z$ from $-\h/2$ to $\h/2$ we obtain

$$0 = \int_{\mathcal{M}} dA \sqrt{g} \left[ h\sigma^{\alpha\beta}_M \delta u^{\alpha}_{\alpha\beta} + \frac{h^3}{12} \sigma^{\alpha\beta}_B \delta u^{B}_{\alpha\beta} - h(f\delta u_{\alpha} + f^2 \psi) \right]$$

where $g := \text{det}(g_{\alpha\beta})$. Finally, integrating by parts and ignoring boundary terms (if $\mathcal{M}$ is a closed surface, for example), and recognizing that the variations are arbitrary, we obtain the equations of equilibrium

$$0 = \nabla^{\beta} \sigma^{\alpha\beta}_M + f^{\alpha},$$  \hspace{1cm} \text{(13)}

$$0 = \frac{h^3}{12} (\nabla_{\alpha} \nabla_{\beta} \sigma^{\alpha\beta}_B - k_{\alpha\beta} \sigma^{\alpha\beta}_B)$$

$$+ h\sigma^{\alpha\beta}_M (h_{\alpha\beta} + \nabla_{\alpha} \nabla_{\beta} \psi - k_{\alpha\beta} \psi) + h (f_z - \nabla_{\alpha} \psi f^{\alpha}) ,$$  \hspace{1cm} \text{(14)}

a nonlinear system of equations for deformation $(u_{\alpha}, \psi)$ in response to the force field $f$.

It is interesting to notice that one effect of keeping second-order terms in $\psi$ in (7) is to correct the second fundamental form $h_{\alpha\beta}$ in the second parenthesis of (14) for the change in curvature due to the normal displacement $\psi$ — compare (9), which contains this same expression, arrived at differently. Thus one effect of the nonlinearity is to replace the original $h_{\alpha\beta}$ on $\mathcal{M}$ by $h_{\alpha\beta}(\psi)$ in the balance of normal stress. This can be a large effect, because the tangential stress $\sigma^{\alpha\beta}_M$ can be large, and it is frequently the $\sigma^{\alpha\beta}_M h_{\alpha\beta}$ term which is most
important in balancing applied normal stress. If \( \sigma_{\mathcal{M}}^{\alpha\beta} = \Sigma g^{\alpha\beta} \), for example, corresponding to surface tension \( \Sigma \) in \( D_M \), then that term is the normal stress \( \Sigma H \), where \( H = (\kappa_1 + \kappa_2) \) is the mean curvature.

The insight of Föppl and von Kármán can be appreciated by taking the special case of an initially flat \( M \), so that \( h_{\alpha\beta} = k_{\alpha\beta} = 0 \). Then, taking Cartesian coordinates, the covariant derivatives are ordinary derivatives, and the equilibrium equations (13) and (14) reduce to

\[
0 = \partial_\beta \sigma_{\mathcal{M}}^{\alpha\beta} + f^\alpha, \tag{15}
\]

\[
0 = \frac{h^3}{12} (\partial_\alpha \partial_\beta \sigma_{\mathcal{M}}^{\alpha\beta}) + h \sigma_{\mathcal{M}}^{\alpha\beta} (\partial_\alpha \partial_\beta \psi) + hf_z. \tag{16}
\]

Equation (16) is essentially the second Föppl-von Kármán equation. Its middle term, representing the normal stress due to \( \sigma_{\mathcal{M}}^{\alpha\beta} \), is absent without the nonlinear term in (7), leaving just bending stress to balance normal stress. The corresponding linear theory, which this theory was designed to correct, greatly underestimates the strength of the membrane to resist normal stress. As we see in (14), the contribution of the nonlinear term persists in the case of a curved \( M \), but its effect is less dramatic.

It is worth examining other aspects of the flat \( M \) theory to see what else persists in the more general case. The usual linear relation between stress and strain, which we have not yet invoked, should still hold in general,

\[
\sigma^{\alpha\beta} = \frac{E}{1 - \sigma^2} (g^{\alpha\mu} g^{\beta\nu} \delta u_{\mu\nu} + \sigma \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \delta u_{\mu\nu}) \tag{17}
\]

where \( E \) is Young’s modulus, \( \sigma \) is Poisson’s ratio, and \( \epsilon^{\alpha\beta} \) is the antisymmetric tensor with \( \epsilon^{12} = 1/\sqrt{g} \) and \( g = \det(g_{\alpha\beta}) \). In particular,

\[
\sigma_B^{\alpha\beta} = \frac{E}{1 - \sigma^2} (g^{\alpha\mu} g^{\beta\nu} \delta u_B^{\mu\nu} + \sigma \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \delta u_B^{\mu\nu}) \tag{18}
\]

with \( \delta u_B^{\mu\nu} \) given in (9). Equivalently, strain and stress are related by

\[
\delta u_{\alpha\beta} = \frac{1}{E} (g_{\alpha\mu} g_{\beta\nu} \sigma^{\mu\nu} - \sigma \epsilon_{\alpha\mu} \epsilon_{\beta\nu} \sigma^{\mu\nu}) \tag{19}
\]

and in particular

\[
\delta u_{\alpha\beta}^M = \frac{1}{E} (g_{\alpha\mu} g_{\beta\nu} \sigma_{\mathcal{M}}^{\mu\nu} - \sigma \epsilon_{\alpha\mu} \epsilon_{\beta\nu} \sigma_{\mathcal{M}}^{\mu\nu}) \tag{20}
\]

In the flat case

\[
\partial_\beta \sigma_{\mathcal{M}}^{\alpha\beta} = 0 \tag{21}
\]
implies $\sigma^{\alpha\beta}_M$ is derivable from an Airy potential $\chi$,

$$\sigma^{\alpha\beta}_M = \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \partial_\mu \partial_\nu \chi$$

(22) eliminating one variable. In this case, putting (22) into the right side of (20) and $u^\alpha_M$ from (7) into the left side of (20), and applying the operator $\epsilon_{\alpha\kappa} \epsilon_{\beta\lambda} \partial_\kappa \partial_\lambda$ to both sides, we find the first Föppl-von Kármán equation

$$-\det(\partial_\alpha \partial_\beta \psi) = \frac{1}{E} \Delta^2 \chi$$

(23) and putting (22) and (18) into (16) we obtain the second Föppl-von Kármán equation

$$0 = -\kappa_c \Delta^2 \psi + h \epsilon^{\alpha\mu} \epsilon^{\beta\nu} (\partial_\mu \partial_\nu \chi)(\partial_\alpha \partial_\beta \psi) + hf_z .$$

(24) Here $\kappa_c = Eh^3/[12(1 - \sigma^2)]$ is the bending rigidity, with $\kappa_c/h \propto h^2$. Tangential displacements $\delta u^\alpha$ being small, $\det(\partial_\alpha \partial_\beta \psi)$ in (23) equals the Gaussian curvature $K = \kappa_1 \kappa_2$ of the membrane surface $M$ to fair approximation. Bending in two orthogonal directions, which is what bending in general boils down to, implies $\kappa \neq 0$.

The general equations (13) and (14) solve the long-standing problem of describing dynamic equilibrium of a precurved sheet under strong deformations. They can be widely used in shell theory and, for instance, to analyze crumpling of a naturally precurved cell membrane, such as the one in Fig. 1. Unfortunately the above program for an originally flat surface cannot get started in the general, curved case. The non-commutativity of covariant derivatives means there is no Airy representation to eliminate (13) even if $f^\alpha = 0$. The equilibrium equations, using all the tricks known for less general situations, remain Eqs. (13) and (14), with $\sigma^{\alpha\beta}$ given by (17). Through the second and third fundamental form ($h_{\alpha\beta}$) and ($k_{\alpha\beta}$) and the covariant derivatives they explicitly show how curvature must be taken into account. Evaluating the consequences of these equations, especially that of taking the limit $h \to 0$, will be a true challenge to membrane physics for some time to come.

The authors thank Ted Steck for providing Fig. 1. J.L.v.H. gratefully acknowledges constructive discussions during a stimulating workshop at the Aspen Center for Physics, where it all started. M.A.P. thanks the SFB 563 for support while this work was done at the TU Munich.

[1] D.R. Nelson and L. Peliti, J. Phys. France 48, 1085 (1987).
[2] H.S. Seung, D.R. Nelson, Phys. Rev. A 38, 1005 (1988).
[3] A.E. Lobkovsky, S. Gentges, H. Li, D. Morse, and T.A. Witten, Science 270, 1482 (1995).
[4] A. Lobkovsky, Phys. Rev. E 53, 3750 (1996).
[5] A.E. Lobkovsky and T.A. Witten, Phys. Rev. E 55, 1577 (1997).
[6] M. Ben Amar and Y. Pomeau, Proc. R. Soc. Lond. A 453, 729 (1997) and Philos. Mag. B 78, 235 (1998).
[7] E.M. Kramer and T.A. Witten, Phys. Rev. Lett. 78, 1303 (1997); E.M. Kramer, J. Math. Phys. 38, 830 (1997).
[8] E. Cerda, S. Chaieb, F. Melo, and L. Mahadevan, Nature 401, 46 (1999).
[9] B. Audoly, Phys. Rev. Lett. 83, 4124 (1999).
[10] A. Boudaoud, P. Patricio, Y. Couder, and M. Ben Amar, Nature 407, 718 (2000).
[11] K. Matan, R.B. Williams, T.A. Witten, and S.R. Nagel, Phys. Rev. Lett. 88, 076101 (2002).
[12] B.A. DiDonna, T.A. Witten, S.C. Venkataramani, and E.M. Kramer Phys. Rev. E 65, 016603 (2002).
[13] M. Bowick, A. Cacciuto, G. Thorleifsson, and A. Travesset, Phys. Rev. Lett. 87, 148103 (2001).
[14] J. Lidmar, L. Mirny, and D.R. Nelson, Phys. Rev. E 68, 051910 (2003).
[15] J.F. Hainfeld and T.L. Steck, J. Supramol. Struct. 6, 301 (1977).
[16] E. Sackmann, FEBS Lett. 346, 3 (1994).
[17] D.R. Nelson, T. Piran, and S. Weinberg, Eds., Statistical Mechanics of Membranes and Surfaces, 2nd ed., (World Scientific, Singapore, 2004).
[18] A. Föppl, Vorlesungen über Technische Mechanik V (Teubner, Leipzig, 1907), pp. 132–144; esp. p. 139.
[19] Th. von Kármán, in: Encyklopädie der Mathematischen Wissenschaften, Vol. IV:4 (Teubner, Leipzig, 1910), 311–385; esp. pp. 348–350.
[20] L.D. Landau and E.M. Lifshitz, Theory of Elasticity (Pergamon, London, 1959) §14.
[21] E.H. Mansfield, The Bending and Stretching of Plates, 2nd ed. (Cambridge University Press, Cambridge, 1989).
[22] W. Flügge, Stresses in Shells (Springer, New York, 1960).
[23] R.D. Kamien, Rev. Mod. Phys., 74, 953 (2002).
[24] J.L. van Hemmen and C. Leibold, TU Munich Preprint.
[25] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (Chelsea, New York, 1952), p. 187.

[26] M.A. Peterson, J. Math. Phys. 26, 711 (1985).

[27] B. Schutz, *Geometrical Methods of Mathematical Physics*, (Cambridge University Press, 1980), p. 79.

[28] T. Frankel, *The Geometry of Physics*, (Cambridge University Press, 1997), p. 625.

[29] G.B. Airy, Philos. Trans. R. Soc. London 153, 49 (1863).

[30] R.S. Millman and G.D. Parker, *Elements of Differential Geometry* (Prentice-Hall, Englewood Cliffs, NJ, 1977), Sects. 4-8 & 4-9.