Descriptive Complexity of Computable Sequences Revisited

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February 5, 2019

Abstract

The purpose of this paper is to answer two questions left open in [B. Durand, A. Shen, and N. Vereshchagin, Descriptive Complexity of Computable Sequences, Theoretical Computer Science 171 (2001), pp. 47–58]. Namely, we consider the following two complexities of an infinite computable 0-1-sequence $\alpha$: $C^{0'}(\alpha)$, defined as the minimal length of a program with oracle $0'$ that prints $\alpha$, and $M_\infty(\alpha)$, defined as $\liminf C(\alpha_{1:n}|n)$, where $\alpha_{1:n}$ denotes the length-$n$ prefix of $\alpha$ and $C(x|y)$ stands for conditional Kolmogorov complexity. We show that $C^{0'}(\alpha) \leq M_\infty(\alpha) + O(1)$ and $M_\infty(\alpha)$ is not bounded by any computable function of $C^{0'}(\alpha)$, even on the domain of computable sequences.

1 Introduction

The notion of Kolmogorov complexity for finite binary strings was introduced in the 60ies independently by Solomonoff, Kolmogorov and Chaitin [8, 4, 1]. There are different versions (plain Kolmogorov complexity, prefix complexity, etc. see [9] for the details) that differ from each other not more than by an additive term logarithmic in the length of the argument. In the sequel we are using plain Kolmogorov complexity $C(x)$ as defined in [4], but similar results can be obtained for prefix complexity.

When an infinite 0-1-sequence is given, we may study the complexity of its finite prefixes. If prefixes have high complexity, the sequence is random (see [5, 7] for details and references); if prefixes have low complexity, the sequence is computable. In the sequel, we study the latter type.

Let $C(x)$, $C(x|y)$ denote the plain Kolmogorov complexity of a binary string $x$ and the conditional Kolmogorov complexity of $x$ when $y$ (some other binary

*The article was prepared within the framework of the HSE University Basic Research Program and funded by the Russian Academic Excellence Project '5-100'. The author was in part funded by RFBR according to the research project 19-01-00563.
string) is known. Let \( \alpha_{1:n} \) denote first \( n \) bits (= length-\( n \) prefix) of the sequence \( \alpha \). Let us recall the following criteria of computability of \( \alpha \) in terms of complexity of its finite prefixes.

(a) \( \alpha \) is computable if and only if \( C(\alpha_{1:n}|n) = O(1) \). This result is attributed in [6] to A.R. Meyer (see also [5] [7]).

(b) \( \alpha \) is computable if and only if \( C(\alpha_{1:n}) < C(n) + O(1) \) [2].

(c) \( \alpha \) is computable if and only if \( C(\alpha_{1:n}) < \log_2 n + O(1) \) [2].

These results provide criteria of the computability of infinite sequences. For example, (a) can be reformulated as follows: a sequence \( \alpha \) is computable if and only if \( M(\alpha) \) is finite, where

\[
M(\alpha) = \max_n C(\alpha_{1:n}|n) = \max_n \min_p \{ l(p) | p(n) = \alpha_{1:n} \}.
\]

Here \( l(p) \) stands for the length of program \( p \); \( p(n) \) denotes its output on \( n \). As usual in Kolmogorov complexity theory, we assume that some optimal programming language \( U \) is fixed. That is, \( (p, n) \mapsto U(p, n) \) is a computable function such that for any other computable function \( V(p, n) \) there is a constant \( c \) such that for all \( p \) there is \( p' \) with \( l(p') \leq l(p) + c \) and \( U(p', n) = V(p, n) \) for all \( n \). By \( p(n) \) we then denote \( U(p, n) \); conditional Kolmogorov complexity is defined as \( C(x|n) = \min l(p) | p(n) = x \} \) and unconditional Kolmogorov complexity is defined as \( C(x) = C(x|0) \). (For more details see [5] [7].)

Therefore, \( M(\alpha) \) can be considered as a complexity measure of computable sequences. Another straightforward approach is to define complexity of a sequence \( \alpha \) as the length of the shortest program computing \( \alpha \): \( C(\alpha) = \min \{ l(p) | \forall n \ p(n) = \alpha_{1:n} \} \),

(and by definition \( C(\alpha) = \infty \) if \( \alpha \) is not computable.)

The difference between \( C(\alpha) \) and \( M(\alpha) \) can be explained as follows: \( M(\alpha) \leq m \) means that for every \( n \) there is a program \( p_n \) of size at most \( m \) that computes \( \alpha_{1:n} \) given \( n \); this program may depend on \( n \). On the other hand, \( C(\alpha) \leq m \) means that there is a one such program that works for all \( n \). Thus, \( M(\alpha) \leq C(\alpha) \) for all \( \alpha \), and one can expect that \( M(\alpha) \) may be significantly less than \( C(\alpha) \). (Note that the known proofs of (a) give no bounds of \( C(\alpha) \) in terms of \( M(\alpha) \).)

Indeed, Theorem 3 from [3] shows that there is no computable bound for \( C(\alpha) \) in terms of \( M(\alpha) \): for any computable function \( f(m) \) there exist computable infinite sequences \( \alpha^0, \alpha^1, \alpha^2 \ldots \) such that \( M(\alpha^m) \leq m + O(1) \) and \( C(\alpha^m) \geq \alpha(m) \).

The situation changes surprisingly when we compare “almost all” versions of \( C(\alpha) \) and \( M(\alpha) \) defined in the following way:

\[
C_{\infty}(\alpha) = \min \{ l(p) | \forall^\infty n \ p(n) = \alpha_{1:n} \}
\]

\[
M_{\infty}(\alpha) = \limsup_n C(\alpha_{1:n}|n) = \min \{ m | \forall^\infty n \exists p \ (l(p) \leq m \text{ and } p(n) = \alpha_{1:n}) \},
\]
(∀n \in \mathbb{N} \text{ stands for “for all but finitely many } n\). It is easy to see that \(M_\infty(\alpha)\) is finite only for computable sequences. Indeed, if \(M_\infty(\alpha)\) is finite, then \(M(\alpha)\) is also finite, and the computability of \(\alpha\) is implied by Meyer’s theorem. All the four complexity measures mentioned above are “well calibrated” in the following sense: there are \(\Theta(2^m)\) sequences whose complexity does not exceed \(m\).

Surprisingly, it turns out that \(C_\infty(\alpha) \leq 2M_\infty(\alpha) + O(1)\) so the difference between \(C_\infty\) and \(M_\infty\) is not so large as between \(C\) and \(M\). As this bound is tight: Theorem 6 from \([3]\) proves that for every \(m\) there is a sequence \(\alpha\) with \(C_\infty(\alpha) \geq 2m\) and \(M(\alpha) \leq m + O(1)\) (and hence \(M_\infty(\alpha) \leq m + O(1)\)).

Finally, by Theorem 2 from \([3]\), \(M(\alpha)\) cannot be bounded by any computable function of \(C_\infty(\alpha)\) (and hence of \(M_\infty(\alpha)\)).

It is interesting also to compare \(C_\infty\) and \(M_\infty\) with relativized versions of \(C\). For any oracle \(A\) one may consider a relativized Kolmogorov complexity \(C^A\) allowing programs to access the oracle. Then \(C^A(\alpha)\) is defined in a natural way. The results of this comparison are shown by a diagram (Fig. 1).

![Diagram](image-url)

**Figure 1:** Relations between different complexity measures for infinite sequences. Arrows go from the bigger quantity to the smaller one (up to \(O(1)\)-term, as usual). Bold arrows indicate inequalities that are immediate consequences of the definitions. Other arrows are provided by \([3]\) Theorems 1 and 4).

On this diagram no arrow could be inverted. We have mentioned this for the rightmost four arrows. For the remaining three arrows this is obvious. Indeed, \(C^{0''}(\alpha)\) is finite while \(C^{0'}(\alpha)\) is infinite for a sequence \(\alpha\) that is \(0''\) computable but not \(0'\)-computable. Therefore the leftmost downward arrow cannot be inverted. The leftmost leftward arrows cannot be inverted for similar reasons: \(C^{0'}(\alpha)\) and \(C^{0''}(\alpha)\) are finite while \(C_\infty(\alpha)\) and \(M_\infty(\alpha)\) are infinite for a sequence that is \(0'\)-computable but not computable.

The statements we cited do not tell us whether the inequality \(C^{0'}(\alpha) \leq M_\infty(\alpha) + O(1)\) is true or not. Another question left open in \([3]\) is the following: are the inequalities

\[
C_\infty(\alpha) \leq C^{0'}(\alpha) + O(1), \quad M_\infty(\alpha) \leq C^{0''}(\alpha) + O(1)
\]

true on the domain of computable sequences? In this paper we answer the first question in positive and the remaining two questions in negative (Theorems \([4]\) and \([2]\) below). Thus we get the following diagram for complexities of computable sequences:
The sign 2 near the arrow means that the larger quantity is at most 2 times the smaller quantity (up to an additive constant), and the sign ∞ means that the larger quantity cannot be bounded by any computable function of the smaller quantity even for computable sequences.

It is instructive to compare these results with similar results for finite sequences (i.e. strings). For $x \in \{0, 1\}^*$ let $M_\infty(x) = \liminf C(x|n)$ and $C_\infty(x) = \min\{l(p) \mid p(n) = x \text{ for almost all } n\}$. From definitions it is straightforward that $M_\infty(x) \leq C_\infty(x) \leq C_0'(x)$ for all $x$ (up to an additive constant). And by [10] we have $C_0'(x) \leq M_\infty(x) + O(1)$, hence all the three quantities coincide up to an additive constant. Similar inequality holds for infinite sequences as well (Theorem 1 from the present paper). However, the analog of the straightforward inequality $M_\infty(x) \leq C_0'(x) + O(1)$ is not true for infinite sequences, even on the domain of computable sequences.

2 Theorems and proofs

Theorem 1. $C_0'(\alpha) \leq M_\infty(\alpha) + O(1)$.

Proof. Fix $k$ and consider the set $S$ of all binary strings $x$ with $C(x|l(x)) \leq k$. This set is computably enumerable uniformly on $k$. The width of $S$ is less than $2^{k+1}$ (this means that for all $n$ the set contains less than $2^{k+1}$ strings of length $n$).

We will view the set $\{0, 1\}^*$ of all binary strings as a rooted tree. Its root is the empty string $\Lambda$ and each edge connects a vertex $x$ with its children $x0$ and $x1$.

An infinite path in $S$ is an infinite sequence of vertices $x_0, x_1, x_2, \ldots$ from $S$ such that $x_i$ is a child of $x_{i-1}$ for all $i > 0$. Let us stress that we do not require infinite paths start in the root, that is, $x_0$ may be non-empty.

If $M_\infty(\alpha) \leq k$, then for some $n$ prefixes of $\alpha$ of length at least $n$ form an infinite path in $S$. We have to show that in this case $C_0'(\alpha) \leq k + O(1)$.

The proof will follow from two lemmas. To state the lemmas we need yet another definition. A set $T$ of strings is called leafless, if for all $x \in T$ at least one child of $x$ is in $T$.

Lemma 1 (on trimming leaves). For every computably enumerable set $S \subset \{0, 1\}^*$ of width at most $w$ there is a computably $0'$-decidable set $T \subset \{0, 1\}^*$ such that

\[C_\infty(\alpha) \xrightarrow{\infty} C(\alpha) \xrightarrow{2} \infty\]

\[C_0'(\alpha) \xrightarrow{\infty} M_\infty(\alpha) \xrightarrow{\infty} M(\alpha)\]

\[\text{If, moreover, } M(\alpha) \leq k, \text{ then that path starts in the root.}\]
(1) $T$ is leafless,

(2) the width of $T$ is at most $w$,

(3) $T$ includes all infinite paths in $S$.

The program of the algorithm that $0'$-recognizes $T$ can be found from $w$ and the program enumerating $S$.

**Lemma 2.** Let $T$ be a leafless set of width at most $w$. The for any infinite 0-1-sequence $\alpha$ whose sufficiently large prefixes form and infinite path in $T$ we have $C^T(\alpha) \leq \log w + O(1)$. The constant $O(1)$ does not depend on $T, \alpha, w$.

We first finish the proof of the theorem assuming the lemmas. By applying Lemma 1 to the set $S = \{ x \mid C(x|l(x)) \leq k \}$ we obtain a leafless set $T$ of width less than $2^{k+1}$ that includes all infinite paths in $S$ and is $0'$-decidable uniformly on $k$. If $M_\infty(\alpha) \leq k$, then by the second lemma $C^T(\alpha) \leq k + O(1)$. Since $T$ is $0'$-decidable uniformly on $k$ and we can retrieve $k$ from the length of the program witnessing the inequality $C^T(\alpha) \leq k + O(1)$, we can conclude that $C^{0'}(\alpha) \leq k + O(1)$.

It remains to prove the lemmas. We start with the proof of the simpler Lemma 2.

**Proof of Lemma 2.** Basically we have to number infinite paths in $T$ in such a way that given the number of a path we can find all its vertices. We will imagine that we have tokens with numbers from 1 to $w$, and move those tokens along infinite paths in $T$. The number of an infinite path in $T$ will be the number of the token that moves along that path.

More specifically, we start an enumeration of the set $T$. Observing string enumerated in $T$ we will place tokens on some of them; vertices baring tokens will be called distinguished. We will do that so that the following be true:

(1) distinguished vertices are pair wise inconsistent (neither of them is a prefix of another one),

(2) every string enumerated so far in $T$ is a prefix of some distinguished vertex,

(3) tokens move only from a vertex to its descendant (=extension).

At the start no strings are enumerated so far and all tokens are not used. When a new string $x$ is enumerated into $T$, we first look whether it is a prefix of a distinguished vertex. In that case we do nothing, since property (2) remains true.

Otherwise property (2) has been violated. If $x$ is an extension of a distinguished string $y$ (such a vertex $y$ is unique by property (1)), then we move the token from $y$ to $x$ keeping (1) and (3) true and restoring (2).

Finally, if $x$ is inconsistent with all distinguished nodes, we take a new token and place it on $x$ restoring (2) and keeping (1).
Since $T$ is leafless and its width is at most $w$, the set $T$ cannot have more than $w$ pairwise inconsistent strings (for all large enough $n$ each of those strings has a length-$n$ extension in $T$ and those extensions are pairwise different). Therefore we do not need more than $w$ tokens.

By construction for every infinite path in $T$ a token is at certain time placed on a vertex of the path and moves along the path infinitely long.

To every natural number $i$ from 1 to $w$ we assign a program $p_i$ that for input $n$ waits until the $i$th token is placed on a string $x$ of length at least $n$, then it prints the first $n$ bits of $x$. \qed

It remains to prove the first lemma.

Proof of Lemma. It seems natural to let $T$ be the union of all infinite paths in $S$. In this case the conditions (1)–(3) hold automatically. However, this set is only $\Pi_2$, since

$$T = \{ x | \forall i \text{ there is an extension of } x \text{ of length } l(x) + i \text{ in } S \}.$$ 

It is not hard to find an example of a c.e. set $S$ for which this set $T$ is $\Pi_2$ complete (and hence is not $0'$-decidable). The set $T$ we construct will be larger in general case than the union of all infinite paths in $S$.

We will be using Cantor topology on the set of subsets of $\{0,1\}^*$. Its base consists of sets of the form:

$$\{ X \subset \{0,1\}^* | A \subset X, B \cap X = \emptyset \},$$

where $A, B$ are any finite subsets of $\{0,1\}^*$. Open sets in Cantor topology are arbitrary unions of these sets. It is well known that this topological space is compact.

We will consider leafless sets $T$ such that the width of the set $T \cup S$ does not exceed $w$. Such sets will be called acceptable. For instance, the empty set is acceptable. The key observation is the following: the family of acceptable sets is closed in Cantor topology.

The set $T$ is defined as the largest acceptable set with respect to some linear order. More specifically, consider the lexicographical order on binary strings (for strings of different length, the shorter string is less than the longer one). Then we define $X < Y$ for different sets $X, Y \subset \{0,1\}^*$ if the lex first string in the symmetric difference of $X, Y$ belongs to $Y \setminus X$ (in other words, we compare sets according to the lexicographical order on their characteristic sequences). Not every non-empty family of subsets of $\{0,1\}^*$ has the largest set with respect to this order. However, this holds for closed families. Hence there exists the largest acceptable set $T$.

In other words, one can define $T$ recursively: enumerate all binary strings $x_1, x_2, \ldots$ according to the lexicographical order, then put $x_i$ in $T$ if there is an acceptable set $R$ which includes the set $T \cap \{x_1, \ldots, x_{i-1}\}$, or, equivalently,

$$R \cap \{x_1, \ldots, x_{i-1}\} = T \cap \{x_1, \ldots, x_{i-1}\}.$$
This definition guarantees that for all \( i \) there is an acceptable set \( R \) with \( R \cap \{x_1, \ldots, x_{i-1}\} = T \cap \{x_1, \ldots, x_{i-1}\} \). Since the family of acceptable sets is closed, this implies acceptability of \( T \). And by construction this \( T \) is larger than or equal to every acceptable set.

Properties (1) and (2) hold automatically for \( T \). Let us verify the property (3). Let \( \alpha \) be an infinite path in \( S \). Consider the set \( T' = T \cup \alpha \). It is leafless (since every vertex from \( \alpha \) has a child in \( \alpha \)). Besides, \( T' \cup S = T \cup S \) (as \( \alpha \subset S \)), hence \( T' \) is acceptable. The definition of \( T \) implies that it is a maximal w.r.t. inclusion acceptable set. Therefore \( T' = T \), or, in other words, \( \alpha \subset T \).

It remains to show that \( T \) is \( 0' \)-decidable. Assume that we already know for every string among \( x_1, \ldots, x_{i-1} \) whether it belongs to \( T \) or not. We have to decide whether \( x_i \in T \). By construction \( x_i \) is in \( T \) if and only if there is an acceptable set including the set \( T \cap \{x_1, \ldots, x_{i-1}\} \) and \( x_i \). Thus it suffices to prove that for any finite \( E \subset \{0,1\}^* \) we can decide with the help of \( 0' \) whether there is an acceptable set including \( E \) or not. To this end we reformulate this property of \( E \). Fix a computable enumeration of \( S \) and denote by \( S_j \) the subset of \( S \) consisting of all strings enumerated in \( j \) steps.

Call a set \( R \) acceptable at time \( j \) if it is leafless and the width of \( R \cup S_j \) is at most \( w \). We claim that

there is an acceptable set including \( E \) if and only if for all \( j \) there is a set \( R_j \) including \( E \) that is acceptable at time \( j \).

Since acceptability implies acceptability at time \( j \) for all \( j \), one direction is straightforward. In the other direction: assume that for every \( j \) there is a set \( R_j \supset E \) which is acceptable at time \( j \). We have to construct an acceptable set \( R \supset E \).

By compactness arguments, the sequence \( R_1, R_2, \ldots \) has an accumulation point \( R \). Since both properties “to include \( E \)” and “be leafless” are closed, the set \( R \) possesses these properties. It remains to show that the width of the set \( R \cup S \) is at most \( w \). For the sake of contradiction assume that there are \( w + 1 \) strings of the same length \( n \) that belong to \( R \cup S \). Then consider the (open) family that consists of all sets \( R' \) such that the set \( R' \cup S \) includes all those strings. Since \( R \) is in this family, for infinitely many \( j \) the set \( R_j \) is in this family. Choose such a \( j \) for which \( S_j \) includes all those strings. We obtain a contradiction, as the width of the set \( R_j \cup S_j \) is at most \( w \).

It remains to show decidability of the following property of the pair \( E, j \): there is a set \( R \) including \( E \) that is acceptable at time \( j \) (indeed, in this case the oracle \( 0' \) is able to decide whether this property holds for all \( j \)). Indeed, the sets \( S_j \) and \( E \) are finite. Let \( n \) be the maximal length of strings from these sets. Without loss of generality we may assume that each string \( x \in R \) of length \( n \) or larger has exactly one child in \( R \), namely, \( x0 \), and all strings from \( R \) of length larger than \( n \) are obtained from strings of length \( n \) from \( R \) by appending zeros.

Such sets \( R \) are essentially finite objects and there are finitely many of them. For any such set we can decide whether it includes \( E \) and is acceptable at time \( j \). The lemma is proved.
Remark 1. The set $T$ constructed in the proof of Lemma 2 can be defined in several ways. In the original proof, it was defined as the limit of the sequence $R_1, R_2, \ldots$ where $R_j$ is the largest set that is acceptable at time $j$. One can show that this sequence has a limit indeed. So defined, $T$ is obviously $0'$-decidable. B. Bauwens suggested another way to define (the same) set $T$: include $x_i$ in $T$ if for all $j$ there is a set that is acceptable at time $j$ and includes $T \cap \{x_1, \ldots, x_{i-1}\}$ and $x_i$. Again, so defined $T$ is obviously $0'$-decidable. In the above proof, we defined $T$ in a way that is independent on the chosen enumeration of the set $S$. This construction of $T$ simplifies the verification of properties (1)-(3), but proving $0'$-decidability of $T$ becomes harder.

Now we know that $C^0'(\alpha) \leq M_\infty(\alpha) + O(1)$. How large can be the gap between $C^0'(\alpha)$ and $M_\infty(\alpha)$? For $\alpha$ equal to the characteristic sequence of $0'$ the gap is infinite, since $C^0'(0')$ is finite while $M_\infty(0')$ is infinite. However, we are mostly interested in computable sequences, thus we refine the question: How large can be the gap between $C^0'(\alpha)$ and $M_\infty(\alpha)$ for computable sequences $\alpha$?

It turns out that this such gap can be arbitrary large: $M_\infty(\alpha)$ cannot be bounded by any computable function of $C^0'(\alpha)$. More specifically, the following holds:

**Theorem 2.** For any computable function $f : \mathbb{N} \to \mathbb{N}$ for all $m$ there is a computable sequence $\alpha$ for which $C^0'(\alpha) \leq m + O(1)$ while $M_\infty(\alpha) \geq f(m)$. The constant $O(1)$ depends on the function $f$.

**Proof.** A natural approach to construct such sequence $\alpha$ is to take a sufficiently long prefix of $0'$ and extend it by zeros. More specifically, let $x$ stand for a prefix encoding of the first $m$ bits of $0'$, say $x = 0^m10'^{1:m}$, and let $\alpha = x000\ldots$. This approach fails, as whatever $m$ we choose the complexities $C^0'$ and $M_\infty$ of this sequence coincide up to an additive constant. Indeed, $C^0'(\alpha) \geq C^0'(m) - O(1)$, since from $\alpha$ we can find $m$. On the other hand, $M_\infty(\alpha) \leq C^0'(m) + O(1)$: pick a program $p$ with oracle $0'$ whose length is $C^0'(m)$ and that prints $m$. Assume that $n$ is larger than the number of steps needed to enumerate all numbers at most $m$ into $0'$ and is larger than all queries by $p$ to its oracle. Then we can find $x$ from $n$ and $p$: first make $n$ steps of enumerating $0'$ and run $p$ with the subset $A$ of $0'$ we have obtained instead of the full oracle $0'$. The program $p$ will print $m$. Then we find $x$, as the length-$m$ prefix of the characteristic sequence of $A$ and output $\alpha_{1:n}$.

To prove the theorem we will use the Game Approach. Assume that a natural parameter $w$ is fixed. Consider the following game between two players, Alice and Bob. Players turn to move alternate. On each move each player can paint any string or do nothing. We will imagine that Alice uses green color and Bob uses red color (each string can be painted in both colors). For every $n$ Alice may paint at most $w$ strings of length $n$. The player make infinitely many moves and then the game ends. Alice wins if (1) for some $n$ there are $w$ strings of length $n$ who all have been painted by both players ($w$ red-green strings of the same length), or (2) there is an infinite 0-1-sequence $\alpha$ such that $\alpha_{1:n}$ is the
lex first green string of length \( n \) for all \( n \) and \( \alpha_{1:n} \) has not been painted by Bob (is not red) for infinitely many \( n \).

From the rules of the game it is clear that it does not matter who starts the game (postponing a move does not hurt).

**Lemma 3.** For every \( w \) Alice has a winning strategy in this game and this strategy is computable uniformly on \( w \).

**Proof.** Alice's strategy is recursive. If \( w = 1 \), then Alice just paints the strings \( \Lambda,0,00,000,\ldots \) (on her \( i \)th move she paints the string \( 0^i \)).

Assume now that we have already defined Alice's winning strategy in the \( w \)-game, we will call it the \( w \)-strategy. Then Alice can win \((w + 1)\)-game as follows: she paints first the empty string and then runs the \( w \)-strategy in the subtree with the root \( 1^\infty \). If \( w \)-strategy wins in the first way (that is, for some \( n \) there are \( w \) red-green strings of length \( n \) that start with 1), then Alice stops \( w \)-strategy. She then paints the strings \( 0,00,000,\ldots,0^m \) where \( m \) is larger than the length of all strings painted by \( w \)-strategy. Then Alice runs \( w \)-strategy for the second time, but this time in the subtree with the root \( 0^m1 \). Again, if the second run of \( w \)-strategy wins in the first way, then Alice stops it and paints the strings \( 0^{m+1},0^{m+2},\ldots,0^l \) where \( l \) is larger than the length of all strings painted green so far. And so on.

The \( w + 1 \)-strategy is constructed. Let us show that it obeys the rules, that is, for all \( n \) it paints at most \( w + 1 \) strings of length \( n \). Indeed, for each \( n \) at most \( w \) strings of length \( n \) were painted by a run of \( w \)-strategy (different runs of \( w \)-strategy paint strings of different lengths) and besides the string \( 0^n \) might be painted.

Let us show that \( w + 1 \)-strategy wins the game. We will distinguish two cases.

**Case 1.** We have run \( w \)-strategy infinitely many times. Then each its run has won in the first way. Hence for infinitely many \( n \) there exist \( w \) red-green strings of length \( n \) and all those strings have a 1 (indeed, we have run \( w \)-strategies only in subtrees with roots of the form 000...01). Consider now the strings \( 0^n \) for those \( n \)'s. If at least one of them has been painted by Bob, then we have won in the first way. Otherwise the nodes \( \Lambda,0,00,\ldots \) are lex first green strings of lengths 0, 1, 2, \ldots and infinitely many of them are not red. This means that we have won in the second way.

**Case 2.** A run of \( w \)-strategy, say in the subtree with root \( 0^l1 \), has not been stopped and hence it won in the second way. Then the string \( 0^l1 \) has been extended by an infinite green path \( P \) (including the string \( 0^l1 \) itself) that contains infinitely many non-red nodes. Since, all the nodes 0, 00, 000, \ldots, \( 0^l \) are also green, the path \( 0^l1P \) consists entirely of green nodes, starts in the root, contains infinitely many non-red nodes and all its nodes are lex first green nodes (recall that all strings with prefix 000 have not been painted by Alice). \( \square \)

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2Formally, that means that Alice adds prefix 1 to every move made by the \( w \)-strategy, postpones Bob's moves that do not start with 1, and for every Bob's move of the form 1x tells the \( w \)-strategy that Bob has made the move x.
To prove the theorem we apply $2^{f(m)}$-strategy against the following “blind” Bob’s strategy: Bob paints a string $x$ of length $n$ when he finds a program $p$ of length less than $f(m)$ with $p(n) = x$ (he runs all programs of length less than $f(m)$ on all inputs in a dovetailing style). This strategy is computable and for all $n$ it paints less than $2^{f(m)}$ strings of length $n$. Hence Alice wins in the second way: there is an infinite green path whose infinitely many nodes are not red. Call this path $\alpha$. By construction $M_\infty(\alpha) \geq f(m)$.

On the other hand, the set of all green nodes is computably enumerable and its width is at most $f(m)$. Hence $M(\alpha) < f(m) + O(1)$ and by Meyer’s theorem $\alpha$ is computable.

Finally, the path $\alpha$ can be computed from $m$ with oracle $0'$: for every $n$ we can find the lex first green string for length $n$ Hence $C_0'(\alpha) < \log m + O(1) < m + O(1)$. The theorem is proved.

**Acknowledgments**

The author is sincerely grateful to Bruno Bauwens for providing an alternative proof of Lemma [1]

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