Quasi-Frobenius algebras and their integrable $N$-parametric deformations generated by compatible $(N \times N)$-metrics of constant Riemannian curvature\footnote{This work was supported by the Alexander von Humboldt Foundation (Germany), the Russian Foundation for Basic Research (Grant No. 02–01–00803), and INTAS (Grant No. 99–1782).}

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1 Introduction

In this paper, we prove that the description of pencils of compatible $(N \times N)$-metrics of constant Riemannian curvature is equivalent to a special class of integrable $N$-parametric deformations of quasi-Frobenius (in general, noncommutative) algebras.

A finite-dimensional algebra $Q$ is said to be quasi-Frobenius if the identity
\[(ab)c = (ac)b, \quad a, b, c \in Q\] (1.1)
is fulfilled and also an invariant nondegenerate symmetric bilinear form $<a, b>$ is given,
\[<ab, c> = <a, cb>, \quad <a, b> = <b, a>, \quad a, b, c \in Q.\] (1.2)

Recall that a finite-dimensional commutative associative algebra equipped with an invariant nondegenerate symmetric bilinear form is called a Frobenius algebra (here, we do not require an existence of a unit in Frobenius algebra). Any commutative quasi-Frobenius algebra is always Frobenius, i.e., if the identity $ab = ba$ (commutativity) is fulfilled in a quasi-Frobenius algebra, then the identities
\[(ab)c = a(bc) \quad (associativity),\] (1.3)
\[<ab, c> = <a, bc> \quad (invariance \ of \ bilinear \ form)\] (1.4)
are also always fulfilled in the algebra.

Identity (1.1), meaning commutativity of the operators of right-sided multiplication in the algebra: $R_a R_b = R_b R_a$, where $R_a b = ba$, naturally arises as one of identities in algebras describing some special classes of first-order Poisson brackets linear in field variables (see [1]–[3]). The theory of left-symmetric algebras with the additional identity (1.1), which correspond to linear one-dimensional Poisson brackets of hydrodynamic type and are called Novikov algebras, was developed in [4]–[6] (left-symmetric algebras or Vinberg algebras, i.e., algebras with the identity $a(bc) - (ab)c = b(ac) - (ba)c$ or, equivalently, $L_a L_b - L_b L_a = L_{ab - ba}$, where $L_a b = ab$, were considered in [6]). Identity (1.1) naturally...
arises also in algebras describing all multidimensional Poisson brackets of hydrodynamic type (the algebras were studied by the present author in [8]).

Introducing the considered in this paper notion of quasi-Frobenius algebras is motivated by Dubrovin’s theory of Frobenius manifolds [9] and by a natural generalization of Frobenius manifolds (quasi-Frobenius manifolds) that is related to arbitrary flat pencils of metrics [10], see also [11], [12]. Dubrovin proved that two-dimensional topological field theories (Frobenius manifolds) correspond to a special class of integrable deformations of Frobenius algebras, and also to a special class of quasihomogeneous flat pencils of metrics (see [8], [13]). General flat pencils of metrics locally correspond to quasi-Frobenius manifolds (manifolds with quasi-Frobenius structure on tangent spaces, see [10]) and, respectively, to special deformations of quasi-Frobenius algebras. Integrability of nonlinear equations describing these deformations of quasi-Frobenius algebras was proved by the present author in [14], [15] by the method of the inverse scattering problem, see also [11], [16], [17] (in [18], a Lax pair for these equations was also indicated). In the present work, we prove that pencils of compatible metrics of constant Riemannian curvature also generate special integrable deformations of quasi-Frobenius algebras. We very hope that integrable deformations of quasi-Frobenius algebras, constructed in the paper, will also prove to be useful in two-dimensional topological field theories. In particular, we conjecture that the deformations of noncommutative associative algebras, recently found by S.M. Natanzon in [19] (see also [20]) and corresponding to open-closed two-dimensional topological field theories, are integrable by the method of the inverse scattering problem and are special reductions of the considered class of integrable deformations of noncommutative quasi-Frobenius algebras. In this connection, the study of deformations of noncommutative quasi-Frobenius algebras and, especially, the extraction of associative deformations among them is of a great interest.

2 Compatible metrics of constant Riemannian curvature

Recall that two pseudo-Riemannian contravariant metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ are said to be compatible if for any linear combination $g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u)$ of these metrics, where $\lambda_1$ and $\lambda_2$ are arbitrary constants such that $\det(g^{ij}(u)) \neq 0$, the coefficients of the corresponding Levi-Civita connections and the components of the corresponding Riemannian curvature tensors are related by the same linear formula: $\Gamma^{ij}_k(u) = \lambda_1 \Gamma^{ij}_{1,k}(u) + \lambda_2 \Gamma^{ij}_{2,k}(u)$ and $R^{ij}_{kl}(u) = \lambda_1 R^{ij}_{1,kl}(u) + \lambda_2 R^{ij}_{2,kl}(u)$ (in this case, we also say that the metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ form a pencil of compatible metrics) [17], [21]. Flat pencils of metrics, which are none other than compatible nondegenerate local Poisson brackets of hydrodynamic type (compatible Dubrovin–Novikov brackets [22]), were introduced in [9]. Two pseudo-Riemannian contravariant metrics $g^{ij}_1(u)$ and $g^{ij}_2(u)$ of constant Riemannian curvatures $K_1$ and $K_2$, respectively, are said to be compatible if any linear combination $g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u)$ of these metrics, where $\lambda_1$ and $\lambda_2$ are arbitrary constants such that $\det(g^{ij}(u)) \neq 0$, is a metric of
constant Riemannian curvature $\lambda_1 K_1 + \lambda_2 K_2$, and the coefficients of the corresponding Levi-Civita connections are related by the same linear formula: $\Gamma^j_{ik}(u) = \lambda_1 \Gamma^j_{i1k}(u) + \lambda_2 \Gamma^j_{i2k}(u)$ \cite{17, 21}. In this case, we also say that the metrics $g^{ij}(u)$ and $g^{ij}_2(u)$ form a pencil of compatible metrics of constant Riemannian curvature \cite{17, 21}. It is obvious that all these definitions are consistent with one another: if compatible metrics are metrics of constant Riemannian curvature, then they form a pencil of compatible metrics of constant Riemannian curvature, and if compatible metrics are flat, then they form a flat pencil of metrics.

In \cite{23} nonlocal Poisson brackets of hydrodynamic type that have the following form (the Mokhov–Ferapontov brackets):

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b_{ij}^k(u(x)) u^k_x + Ku^i_x \left( \frac{d}{dx} \right)^{-1} u^i_x \right) \frac{\delta J}{\delta u^j(x)} dx, \quad (2.1)$$

where $I[u]$ and $J[u]$ are arbitrary functionals on the space of functions (fields) $u^i(x)$, $1 \leq i \leq N$, of one independent variable $x$, $u = (u^1, ..., u^N)$ are local coordinates on a certain given smooth $N$-dimensional manifold $M$, the coefficients $g^{ij}(u)$ and $b_{ij}^k(u)$ of bracket (2.1) are smooth functions of local coordinates, $K$ is an arbitrary constant, were introduced and studied. These nonlocal Poisson brackets play an important role in the theory of systems of hydrodynamic type. The form of bracket (2.1) is invariant with respect to local changes of coordinates. A bracket of form (2.1) is said to be nondegenerate if $\det(g^{ij}(u)) \neq 0$. If $\det(g^{ij}(u)) \neq 0$, then bracket (2.1) is a Poisson bracket if and only if $g^{ij}(u)$ is an arbitrary pseudo-Riemannian contravariant metric of constant Riemannian curvature $K$, $b_{ij}^k(u) = -g^{is}(u)\Gamma_{sk}^j(u)$, where $\Gamma_{sk}^j(u)$ is the Riemannian connection generated by the metric $g^{ij}(u)$ (the Levi-Civita connection) \cite{23} (note that the coefficients $g^{ij}(u)$ and $b_{ij}^k(u)$ of bracket (2.1) are transformed as corresponding differential-geometric objects under local changes of coordinates: a contravariant metric $g^{ij}(u)$ and a connection $b_{ij}^k(u) = -g^{is}(u)\Gamma_{sk}^j(u)$, respectively, $K$ is an invariant). For $K = 0$ we have the local Poisson brackets of hydrodynamic type (the Dubrovin–Novikov brackets \cite{22}).

The description problem for compatible metrics of constant Riemannian curvature is equivalent to that for compatible nonlocal Poisson brackets of hydrodynamic type generated by metrics of constant Riemannian curvature (compatible Mokhov–Ferapontov brackets). Recall that Poisson brackets are said to be compatible if their linear combination is also a Poisson bracket (Magri, \cite{24}). As was shown in \cite{25, 26} (see also \cite{17, 27, 28}), compatible Mokhov–Ferapontov brackets are described by a consistent nonlinear system integrable by the method of the inverse scattering problem (the case of compatible Dubrovin–Novikov brackets, i.e., compatible local Poisson brackets of hydrodynamic type, was integrated by the author earlier in \cite{14, 15}, see also \cite{17, 18}).
3 Compatible nonlocal Poisson brackets of hydrodynamic type

Lemma 3.1 In the classification problem for an arbitrary pair of compatible nonlocal Poisson brackets of form (2.1), one can always consider one of these two Poisson brackets as local without loss of generality.

Actually, if two compatible nonlocal Poisson brackets of form (2.1)
\[
\{ I, J \}_{0} \quad \text{(with a corresponding constant } K_{0} \text{ in the nonlocal term)}
\]
and
\[
\{ I, J \}_{1} \quad \text{(with a constant } K_{1})
\]
are linear independent, then in the pencil of these Poisson brackets, i.e., among the Poisson brackets
\[
\lambda_{0}\{ I, J \}_{0} + \lambda_{1}\{ I, J \}_{1},
\]
where \( \lambda_{0} \) and \( \lambda_{1} \) are arbitrary constants, there is necessarily a nonzero local Poisson bracket
\[
\{ I, J \} = \lambda_{0}^{'}\{ I, J \}_{0} + \lambda_{1}^{'}\{ I, J \}_{1},
\]
where \( \lambda_{0}^{'} \) and \( \lambda_{1}^{'} \) are arbitrary constants satisfying the relation
\[
\lambda_{0}^{'}K_{0} + \lambda_{1}^{'}K_{1} = 0,
\]
which can be taken as one of the generators for all the considered pencil of compatible Poisson brackets (it is obvious that if this local Poisson bracket \( \{ I, J \}_{0} \) is identically zero, then the Poisson brackets \( \{ I, J \}_{0} \) and \( \{ I, J \}_{1} \) are linearly dependent: \( \lambda_{0}^{'}\{ I, J \}_{0} + \lambda_{1}^{'}\{ I, J \}_{1} \equiv 0 \)).

Consider the problem of compatibility for a pair of nonlocal and local Poisson brackets of hydrodynamic type
\[
\{ I, J \}_{1} = \int \frac{\delta I}{\delta u^{i}(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_{k}(u(x)) u_{x}^{k} + K_{1}u_{x}^{i} \right) \frac{\delta J}{\delta u^{j}} \, dx \tag{3.1}
\]
and
\[
\{ I, J \}_{2} = \int \frac{\delta I}{\delta u^{i}(x)} \left( g^{ij}_{2}(u(x)) \frac{d}{dx} + b^{ij}_{2,k}(u(x)) u_{x}^{k} \right) \frac{\delta J}{\delta u^{j}} \, dx, \tag{3.2}
\]
i.e., the condition that for any constant \( \lambda \) the bracket
\[
\{ I, J \} = \{ I, J \}_{1} + \lambda\{ I, J \}_{2} \tag{3.3}
\]
is a Poisson bracket (thus, formula (3.3) defines a pencil of compatible Poisson brackets).

We assume further that the local bracket \( \{ I, J \}_{2} \) is nondegenerate, i.e., \( \det(g^{ij}_{2}(u)) \neq 0 \), but we do not impose any additional conditions on the bracket \( \{ I, J \}_{1} \), i.e., generally speaking, this bracket may be degenerate. Bracket (3.3) can be degenerate, therefore here we indicate the general relations for the coefficients of a bracket of form (2.1) which are equivalent to the condition that bracket (2.1) is a Poisson bracket. These general relations (without the assumption of nondegeneracy) were obtained in the present author’s work \( [29] \) (see also \( [30], [31] \)):
\[
g^{ij}(u) = g^{ji}(u), \tag{3.4}
\]
\[
\frac{\partial g^{ij}}{\partial u^{k}} = b^{ij}_{k}(u) + b^{ij}_{k}(u), \tag{3.5}
\]
\[
g^{is}(u)b^{js}_{s}(u) = g^{js}(u)b^{is}_{s}(u), \tag{3.6}
\]
\[ g^{is}(u) \left( \frac{\partial b^{jr}}{\partial u^s} - \frac{\partial b^{jr}}{\partial u^k} \right) + b^j_s(u) b^k_s(u) - b^s_r(u) b^j_k(u) = K\left( g^{ir}(u) \delta^j_k - g^{ij}(u) \delta^r_s \right), \quad (3.7) \]

\[ \sum_{(i,j,r)} \left[ b^{si}_p(u) \left( \frac{\partial b^{jr}}{\partial u^s} - \frac{\partial b^{jr}}{\partial u^p} \right) + K\left( b^j_p(u) - b^i_p(u) \right) \delta^r_p \right] = 0, \quad (3.8) \]

where \( \sum_{(i,j,r)} \) means summation over all cyclic permutations of the indices \( i, j, r \).

4 Canonical form for compatible pairs of brackets

According to the Dubrovin–Novikov theorem \[22\], for any nondegenerate local Poisson bracket of hydrodynamic type \( \{ I, J \}_2 \), there always exist local coordinates \( u^1, ..., u^N \) (flat coordinates of the metric \( g^{ij}_2(u) \)) in which this bracket is constant, i.e., \( g^{ij}_2(u) = \eta^{ij} = \text{const} \), \( b^{ij}_2(u) = \Gamma^{ij}_2(u) = 0 \). Thus we can choose flat coordinates of the metric \( g^{ij}_2(u) \) and further we consider that the Poisson bracket \( \{ I, J \}_2 \) is constant and has the form

\[ \{ I, J \}_2 = \int \frac{\delta I}{\delta u^i(x)} \eta^{ij} \frac{d}{dx} \frac{\delta J}{\delta u^j(x)} dx, \quad (4.1) \]

where \( \eta^{ij} = \eta^{ji}, \eta^{ij} = \text{const}, \det(\eta^{ij}) \neq 0 \). In the sequel, in the considered flat coordinates, we use also the covariant metric \( \eta^{ij}_s \), which is inverse to the contravariant metric \( \eta^{ij} \): \( \eta^{is} \eta_{sj} = \delta^{ij} \).

**Theorem 4.1** ([27]) An arbitrary nonlocal Poisson bracket \( \{ I, J \}_1 \) of form (3.1) (may be degenerate) is compatible with the constant Poisson bracket (4.1) if and only if it has the form

\[ \{ I, J \}_1 = \int \frac{\delta I}{\delta u^i(x)} \left[ \eta^{is} \frac{\partial H^j}{\partial u^s} + \eta^{js} \frac{\partial H^i}{\partial u^s} - K_1 u^i u^j \right] \frac{d}{dx} + \frac{\delta J}{\delta u^j(x)} \right] dx, \quad (4.2) \]

where \( H^i(u), 1 \leq i \leq N, \) are smooth functions defined in a certain domain of local coordinates.

In the flat case of compatible Dubrovin–Novikov brackets \( (K_1 = 0) \), the corresponding statement was formulated and proved by the present author in \[32\]–\[34\] (see also the conditions on flat pencils of metrics in \[1\]).
5 Integrable equations for canonical compatible pairs of brackets

Theorem 5.1 ([27]) An arbitrary nonlocal bracket of form (4.2) (may be degenerate) is a Poisson bracket if and only if the following equations are satisfied:

\[
\frac{\partial^2 H_i}{\partial u^k \partial u^s} \eta^{kp} \frac{\partial^2 H_j}{\partial u^p \partial u^l} = \frac{\partial^2 H_j}{\partial u^k \partial u^s} \eta^{lp} \frac{\partial^2 H_i}{\partial u^p \partial u^l},
\]

(5.1)

\[
\left( \eta^{br} \frac{\partial H^s}{\partial u^r} + \eta^{sr} \frac{\partial H^i}{\partial u^r} - K_1 u^i u^s \right) \eta^{jp} \frac{\partial^2 H_k}{\partial u^p \partial u^s} = \left( \eta^{jr} \frac{\partial H^s}{\partial u^r} + \eta^{sr} \frac{\partial H^j}{\partial u^r} - K_1 u^j u^s \right) \eta^{ip} \frac{\partial^2 H_k}{\partial u^p \partial u^s},
\]

(5.2)

In the flat case \((K_1 = 0)\), the corresponding theorem was obtained by the present author in [33], where was also stated the conjecture on the integrability of system (5.1), (5.2) for \(K_1 = 0\) by the method of the inverse scattering problem. This conjecture was proved by the author in the works [14], [15], [17] (see also [18], where a Lax pair for system (5.1), (5.2) was indicated). The corresponding general conditions on flat pencils of metrics were indicated in [9].

It is obvious that any set of \(N\) linear functions \(H^i(u) = a^i_k u^k + a^i\), where \(a^i_k\) is an arbitrary constant matrix, \(a^i = const\), \(a^i = const\), is always a solution of the nonlinear system (5.1), (5.2) and, consequently, always generates a corresponding canonical pair of compatible Poisson brackets (4.1), (4.2) (see [27]).

In the flat case \((K_1 = 0)\), a set of \(N\) quadratic functions \(H^i(u) = c^i_{jk} u^j u^k\), where \(c^i_{jk} = c^i_{kj}\), \(c^i_{jk} = const\), is a solution of nonlinear system (5.1), (5.2) if and only if the structural constants \(a^i_{jk} = \eta^{is} c^j_{sk}\) satisfy the relations (see [10])

\[
a^i_{jk} a^i_{sk} = a^i_{sk} u^j, \quad (a^i_{ks} + a^i_{sk}) a^i_{jk} = (a^i_{js} + a^i_{sj}) a^i_{sk},
\]

(5.3)

(5.4)

which are equivalent to the condition that \(N\)-dimensional algebra with basis \(e^1, \ldots, e^N\) and multiplication

\[
e^i \cdot e^j = a^i_{jk} e^k
\]

is a Novikov algebra, i.e., the identity

\[
(e^i \cdot e^j) \cdot e^k = (e^i \cdot e^k) \cdot e^j, \quad e^i \cdot (e^j \cdot e^k) - (e^i \cdot e^j) \cdot e^k = e^i \cdot (e^j \cdot e^k) - (e^j \cdot e^i) \cdot e^k
\]

(5.5)

(5.6)
are fulfilled (see [10]). Moreover, the invariant nondegenerate symmetric bilinear form

\[ (e^i, e^j) = \eta^{ij} \]

is defined on this Novikov algebra, i.e.,

\[ (e^i \cdot e^j, e^k) = (e^i, e^k \cdot e^j), \quad (e^i, e^j) = (e^j, e^i). \]

Thus, in this case, we get a special class of quasi-Frobenius algebras, namely, exactly the class of left-symmetric quasi-Frobenius algebras describing linear one-dimensional local Poisson brackets of hydrodynamic type (see [3]).

**Lemma 5.1** The condition of left-symmetry (5.4) in an arbitrary algebra \( \mathcal{F} \) with multiplication \( e^i \cdot e^j = f_k^{ij} e^k \) and identity (5.4) (and consequently in any quasi-Frobenius algebra) is equivalent to the condition that the symmetric bilinear form

\[ <e^i, e^j> = (f_k^{ij} + f_k^{ji})u^k \]  

(5.7)

is invariant for any \( u = (u^1, ..., u^N) \), i.e.,

\[ <e^i, e^j>, <e^i, e^k> = <e^j, e^k \cdot e^i>. \]  

(5.8)

The parameters \( u^1, ..., u^N \) realize a deformation of the invariant form.

In the general case of compatible metrics of constant Riemannian curvature, it is interesting to study algebraic structures related to sets of \( N \) cubic functions \( H^i(u) = c_{jkl}^i u^j u^k u^l \), where \( c_{jkl}^i = c_{kjl}^i = c_{jlk}^i \) = const. Such a set of cubic functions is a solution of nonlinear system (5.1), (5.2) if and only if for the structural constants \( a_{jkl}^{ij} = \eta^{is} c_{sij}^k, a_{jkl}^{ij} = a_{jkl}^{ij} \), the following relations are satisfied:

\[ a_{ms}^{ki} a_{nl}^{sj} - a_{ms}^{kj} a_{nl}^{si} + a_{ms}^{ki} a_{nl}^{sj} - a_{ms}^{kj} a_{nl}^{si} = 0, \]  

(5.9)

\[ \sum_{[l,m,n]} [(3a_{mn}^{is} + 3a_{nm}^{si} - K_1 \delta_{mn} \delta_{is})a_{ls}^{jk} - (3a_{mn}^{js} + 3a_{nm}^{sj}) - K_1 \delta_{mn} \delta_{js})a_{ls}^{jk}] = 0, \]  

(5.10)

where \( \sum_{[l,m,n]} \) means summation over all permutations of the indices \( l, m, n \). In this case, the identity

\[ (a \ast b) \ast c = (a \ast c) \ast b \]  

(5.11)

is fulfilled in \( N \)-dimensional algebra \( \mathcal{C}(u) \) with basis \( e^1, ..., e^N \) and multiplication \( e^i \ast e^j = a_{jkl}^{ij} u^l e^k \) for any \( u = (u^1, ..., u^N) \), and, moreover, this algebra \( \mathcal{C}(u) \) possesses two invariant symmetric bilinear forms \( (e^i, e^j) = \eta^{ij} \) and

\[ <e^i, e^j> = 3(a_{kkl}^{ij} + a_{klj}^{ij})u^k u^l - K_1 u^i u^j, \]  

(5.12)

i.e.,

\[ (e^i \ast e^j, e^k) = (e^i, e^k \ast e^j), \quad <e^i, e^j, e^k> = <e^i, e^k \ast e^j>. \]

Thus, in this case, we get \( N \)-parametric deformations of quasi-Frobenius algebras \( \mathcal{C}(u), (\cdot, \cdot) \) and \( \mathcal{C}(u), <\cdot, \cdot> \).
Theorem 5.2 ([25], [26]) The system of nonlinear equations (5.1), (5.2) is consistent and integrable by the method of the inverse scattering problem.

Note that the system of nonlinear equations that was found and integrated in [25], [26] is equivalent to system (5.1), (5.2) and, consequently, also describes compatible nonlocal Poisson brackets of form (2.1), but in different special local coordinates, which are much more convenient for the integration (the metrics of both compatible brackets are diagonal in these coordinates). In the flat case ($K_1 = 0$), the corresponding system in special “diagonal” local coordinates was integrated by the author in [14], [15].

6 Quasi-Frobenius algebras and their integrable deformations

Consider the Poisson bracket (4.2) defining the canonical pair of compatible brackets and for any $u = (u^1, ..., u^N)$ define an algebra $A(u)$ in $N$-dimensional vector space with basis $e^1, ..., e^N$ and multiplication

$$e^i \circ e^j = \eta^{js} \frac{\partial^2 H^j}{\partial u^s \partial u^k} e^k. \quad (6.1)$$

Moreover, define a symmetric bilinear form (“a metric of constant Riemannian curvature $K_1$”)

$$< e^i, e^j > = g^{ij}(u) \quad (6.2)$$
on the algebra $A(u)$, i.e.,

$$< e^i, e^j > = \eta^{is} \frac{\partial H^j}{\partial u^s} + \eta^{js} \frac{\partial H^i}{\partial u^s} - K_1 u^i u^j. \quad (6.3)$$

Then equations (5.1) mean that the identity

$$(e^i \circ e^j) \circ e^k = (e^k \circ e^i) \circ e^j \quad (6.4)$$
is fulfilled in the algebra $A(u)$, and equations (5.2) are equivalent to the identity

$$< e^i \circ e^j, e^k > = < e^i, e^k \circ e^j >. \quad (6.5)$$

Thus, any solution of the integrable nonlinear system (5.1), (5.2) defines a quasi-Frobenius algebra $A(u)$ with the multiplication $a \circ b$ and the invariant symmetric bilinear form $< a, b >$ for any fixed $u = (u^1, ..., u^N)$, and system (5.1), (5.2) defines an $N$-parametric deformation of this quasi-Frobenius algebra. Note that also there is always a nondeformed invariant nondegenerate symmetric bilinear form (“a flat metric”)

$$(e^i, e^j) = \eta^{ij}, \quad (e^i \circ e^j, e^k) = (e^i, e^k \circ e^j). \quad (6.6)$$
on the constructed quasi-Frobenius algebra $\mathcal{A}(u)$. Thus, the compatible metrics of constant Riemannian curvature $\eta^{ij}$ and $\eta^{js} \frac{\partial H^s}{\partial u^p} + \eta^{ps} \frac{\partial H^s}{\partial u^p} - K_1 u^j u^j$ define two "compatible" quasi-Frobenius algebras $(\mathcal{A}(u), \langle \cdot, \cdot \rangle)$ and $(\mathcal{A}(u), \langle \cdot, \cdot \rangle)$ for any $u = (u^1, \ldots, u^N)$. Quasi-Frobenius structures and quasi-Frobenius manifolds arising in the flat case for $K_1 = 0$ were considered in [10].

**Conjecture.** The deformations of noncommutative associative algebras, recently found by S.M. Natanzon in [19] (see also [20]) and corresponding to open-closed two-dimensional topological field theories, belong to an integrable class of deformations of noncommutative quasi-Frobenius algebras and are integrable by the method of the inverse scattering problem.

The construction of the corresponding integrable "associative" reductions is a very interesting and important problem.

### 7 Deformations of Frobenius algebras

Assume that multiplication in the algebra $\mathcal{A}(u)$ is commutative: $e^i \circ e^j = e^j \circ e^i$, i.e.,

$$\eta^{is} \frac{\partial^2 H^j}{\partial u^s \partial u^k} = \eta^{js} \frac{\partial^2 H^i}{\partial u^s \partial u^k}. \tag{7.1}$$

Then there exist $c_{pl} = \text{const}$ such that

$$\eta^{ps} \frac{\partial H^s}{\partial u^p} = \eta^{ls} \frac{\partial H^s}{\partial u^p} + c_{lp} - c_{pl}, \tag{7.2}$$

and, consequently,

$$\frac{\partial (\eta^{ps} H^s + c_{ps} u^s)}{\partial u^l} = \frac{\partial (\eta^{ls} H^s + c_{ls} u^s)}{\partial u^p}. \tag{7.3}$$

Thus, there exists a function $\Phi(u)$ ("a potential") such that

$$H^i(u) = \eta^{is} \left( \frac{\partial \Phi}{\partial u^s} - \frac{1}{2} (c_{sk} - c_{ks}) u^k \right), \tag{7.4}$$

where we use that the symmetric part of the matrix $(c_{ks})$ can easily be included in "the potential" $\Phi(u)$:

$$\frac{1}{2} (c_{sk} + c_{ks}) u^k = \frac{1}{2} \frac{\partial (c_{pr} u^p u^r)}{\partial u^s}. \tag{7.5}$$

Thus, for the commutative algebra $\mathcal{A}(u)$, we have:

$$e^i \circ e^j = \eta^{is} \eta^{jp} \frac{\partial^3 \Phi}{\partial u^p \partial u^r \partial u^k} e^k, \tag{7.6}$$

$$\langle e^i, e^j \rangle = 2 \eta^{is} \eta^{jp} \frac{\partial^2 \Phi}{\partial u^p \partial u^r} - K_1 u^j u^j. \tag{7.7}$$
It follows from identities (6.4) and (6.5) that the commutative algebra $A(u)$ is associative:

$$(e^i \circ e^j) \circ e^k = e^i \circ (e^j \circ e^k), \quad (7.7)$$

and is equipped with an invariant symmetric bilinear form:

$$< e^i \circ e^j, e^k > = < e^i, e^j \circ e^k >, \quad (7.8)$$

i.e., for any $u = (u^1, ..., u^N)$ the algebra $A(u)$ is Frobenius.

As a result of the reduction (7.4), equations (5.1) assume the form of the Witten–Dijkgraaf–Verlinde–Verlinde–Dubrovin equations of associativity (see [9], [35]–[38]):

$$\frac{\partial^3 \Phi}{\partial u^k \partial u^i \partial u^s} \eta^{sp} \frac{\partial^3 \Phi}{\partial u^p \partial u^j \partial u^l} = \frac{\partial^3 \Phi}{\partial u^k \partial u^j \partial u^s} \eta^{sp} \frac{\partial^3 \Phi}{\partial u^p \partial u^i \partial u^l}, \quad (7.9)$$

and equations (5.2) have the following form in this case:

$$\left( \frac{\partial^2 \Phi}{\partial u^s \partial u^s} - \frac{K_1}{2} \eta_{ir} \eta_{as} u^r u^l \right) \eta^{sp} \frac{\partial^3 \Phi}{\partial u^p \partial u^j \partial u^k} = \left( \frac{\partial^2 \Phi}{\partial u^s \partial u^s} - \frac{K_1}{2} \eta_{ir} \eta_{as} u^r u^l \right) \eta^{sp} \frac{\partial^3 \Phi}{\partial u^p \partial u^i \partial u^k}. \quad (7.10)$$

Note that in the flat case (for $K_1 = 0$) all Dubrovin’s potentials $\Phi(u)$ corresponding to two-dimensional topological field theories are always solutions for both corresponding equations (7.9) and (7.10) with $K_1 = 0$ (see [8]).

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