FIEDLER LINEARIZATIONS OF MULTIVARIABLE STATE-SPACE SYSTEM AND ITS ASSOCIATED SYSTEM MATRIX

NAMITA BEHERA ∗ AND AVISEK BIST †

Abstract. Linearization is a standard method in the computation of eigenvalues and eigenvectors of matrix polynomials. In the last decade a variety of linearization methods have been developed in order to deal with algebraic structures and in order to construct efficient numerical methods. An important source of linearizations for matrix polynomials are the so called Fiedler pencils, which are generalizations of the Frobenius companion form and these linearizations have been extended to regular rational matrix function which is the transfer function of LTI State-space system in [6]. We consider a multivariable state-space system and its associated system matrix S(λ). We introduce Fiedler pencils of S(λ) and describe an algorithm for their construction. We show that Fiedler pencils are linearizations of the system matrix S(λ).

Key words. rational matrix valued function, matrix polynomial, linearization, linearization, Rosenbrock system matrix.

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1. Introduction. We denote by C[λ] the polynomial ring over the complex field C. Further, we denote by C^{m×n} and C[λ]^{m×n}, respectively, the vector spaces of m×n matrices and matrix polynomials over C.

Consider a matrix polynomial P(λ) = ∑_{j=0}^{m} λ^j A_j, A_j ∈ C^{n×n}. Then a matrix polynomial P(λ) is said to be regular if det(P(λ)) ≠ 0 for some λ ∈ C. Linearization is a standard method for solving polynomial eigenvalue problems P(λ)x = 0. Let P(λ) be an n × n matrix polynomial (regular or singular) of degree m. Then an mn × mn matrix pencil L(λ) := X + λY is said to be a linearization [12] of P(λ) if there are mn × mn unimodular matrix polynomials U(λ) (the determinant of U(λ), is a nonzero constant for all λ ∈ C.) and V(λ) such that

U(λ)L(λ)V(λ) = diag(I_{(m−1)n}, P(λ))

for all λ ∈ C, where I_k denotes the k × k identity matrix. Linearizations of matrix polynomials have been studied extensively over the years, see [12, 13] and references therein. Recently, a new family of linearizations of matrix polynomials referred to as Fiedler linearizations (or Fiedler pencils) has been introduced and is an active area of research, see [5, 11] and references therein. One of the distinctive features of a Fiedler pencil L(λ) of the matrix polynomial P(λ) is that its construction is operation free, that is, block entries of L(λ) are either 0 or I_n or the coefficient matrices of P(λ), and that L(λ) allows an easy (operation free) recovery of eigenvectors of P(λ) from the eigenvectors of L(λ) [11, 5].

In this paper we extend the concept of Fiedler linearization from LTI state-space system to general multivariable state-space system and associated system matrix. In particular, in this paper we discuss the solution (finding eigenvalues λ ∈ C and
eigenvectors } v \in \mathbb{C}^n \text{ of multivariable state-space system } \Sigma \\
A \left( \frac{d}{dt} \right) x(t) = B u(t), \\
y(t) = C x(t) + D \left( \frac{d}{dt} \right) u(t) \quad t \geq 0, 
\tag{1.1}

\text{such that } S(\lambda)v = 0, \text{ where } A(\lambda) = \sum_{j=0}^{d_A} \lambda^j A_j \in \mathbb{C}[\lambda]^{n \times n} \text{ is a regular matrix polynomial of degree } d_A, \text{ } D(\lambda) = \sum_{j=0}^{d_D} \lambda^j D_j \in \mathbb{C}[\lambda]^{m \times m} \text{ is a matrix polynomial of degree } d_D, \text{ and } C \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m}, \text{ and its associate Rosenbrock system matrix } S(\lambda), \\
S(\lambda) = \begin{bmatrix} A(\lambda) & -B \\ C & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{n+m,n+m} 
\tag{1.2}

\text{and the associated transfer function } \\
R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m,m}. \tag{1.3}

\text{Next, consider a more general linear multivariable time invariant state-space system } \Sigma_1 \text{ on the positive half line } \mathbb{R}_+ \text{ in the representation} \\
0 = A \left( \frac{d}{dt} \right) x(t) + B \left( \frac{d}{dt} \right) u(t), \\
y(t) = C \left( \frac{d}{dt} \right) x(t) + D \left( \frac{d}{dt} \right) u(t). \tag{1.4}

\text{The function } u : \mathbb{R}_+ \to \mathbb{R}^m \text{ is the input vector, } x : \mathbb{R}_+ \to \mathbb{R}^r \text{ is the state vector, } \\
y : \mathbb{R}_+ \to \mathbb{R}^p \text{ is the output vector, and for } M(\lambda) = \sum_{i=0}^{d} M_i \lambda^i \in \mathbb{C}[\lambda]^{p,p} \text{ we use } M \left( \frac{d}{dt} \right) \text{ to denote the differential operator } \sum_{i=0}^{d} M_i \frac{d^i}{dt^i}, \text{ where } \frac{d}{dt} \text{ denotes time-differentiation.} \\
\text{The associated matrix polynomial is} \\
S(\lambda) := \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(p+m),(p+m)}. \tag{1.5}

\text{The associate transfer function is defined by} \\
R(\lambda) := D(\lambda) - C(\lambda)A(\lambda)^{-1}B(\lambda) \in \mathbb{C}(\lambda)^{p,p}, \tag{1.6}

\text{where, denoting by } \mathbb{C}[\lambda]^{p,p} \text{ the vector space of } p \times p \text{ matrix polynomials, we assume that } A(\lambda) \in \mathbb{C}[\lambda]^{n,m}, B(\lambda) \in \mathbb{C}[\lambda]^{m,p}, C(\lambda) \in \mathbb{C}[\lambda]^{p,m}, D(\lambda) \in \mathbb{C}[\lambda]^{p,p}. 

\text{Notice that in } (1.2) \text{ we consider } B \text{ and } C \text{ are constant matrices.}

\text{Rational eigenvalue problems arise in many applications, see e.g. } \text{[10, 18, 19, 22]} \text{ and the references therein. Rational matrix value functions of this form arise e.g. in linear system theory, see e.g. } \text{[21].} \\
\text{If } A(\lambda) \text{ is regular, i.e., } \det A(\lambda) \text{ does not vanish identically, then performing a Schur complement, one obtains the rational matrix function } (1.6) \text{ which, in frequency domain, describes the transfer function from the Laplace transformed input to the Laplace transformed output of the system. In this case } S(\lambda) \text{ is called a Rosenbrock system matrix, see } \text{[20].} \\
\text{Conversely, if one has a given rational matrix function of the form } (1.6), \text{ then one can always interpret it as originating from a Rosenbrock system matrix of the form}
Such rational matrix valued functions arise from realizations of input-output data, see e.g. [16], or in model order reduction, see e.g. [4, 13].

We consider the general square polynomial eigenvalue problem

\[
S(\lambda_0) \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} := \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0
\] (1.7)

If \(A\) and \(D\) are square and regular, then one can form the rational function \(R(\lambda)\) as in (1.6) and, since \(\det S(\lambda) = \det A(\lambda) \det R(\lambda)\), it is clear that the eigenvalues of \(S(\lambda)\) are the eigenvalues of \(A\) and \(R\) combined and the eigenvalues of \(A(\lambda)\) are the poles of \(R\). We restrict ourselves to rational functions of the form (1.6) with regular \(A(\lambda)\) and we assume for simplicity that \(B, C\) are constant matrices in \(\lambda\). All the results can be extended (with a lot of technicalities) to the case that \(B, C\) depend on \(\lambda\).

The system \(\Sigma_1\) given in (1.4) is said to be in state-space form if it is given by

\[
\begin{aligned}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + P(\lambda)u(t),
\end{aligned}
\] (1.8)

where \(P(\lambda) \in \mathbb{C}[\lambda]^{n \times n}\) is a matrix polynomial and \(A, E \in \mathbb{C}^{m \times m}\) with \(E\) being nonsingular, \(B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{m \times n}\) are constant matrices, see [21]. For linear time invariant (LTI) state-space system given in (1.8), there is a state-space framework developed in [1] to study the zeros of LTI system in state space form.

For computing zeros of a linear time-invariant system \(\Sigma\) in state-space form, it has been introduced Fiedler-like pencils and Rosenbrock linearization of the Rosenbrock system polynomial \(S(\lambda)\) associated with \(\Sigma\). Also, it has shown that the Fiedler-like pencils are Rosenbrock linearizations of the system polynomial, see [1, 2, 3, 7].

Next, for the higher order linear time invariant (LTI) state-space system \(\Sigma_1\) given by

\[
\begin{aligned}
P \left( \frac{d}{dt} \right) x(t) &= Bu(t) \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
\] (1.9)

where \(P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j \in \mathbb{C}[\lambda]^{n \times n}\) is regular matrix polynomial of degree \(m\) and \(D \in \mathbb{C}^{r \times r}, C \in \mathbb{C}^{r \times n}, B \in \mathbb{C}^{n \times r}\), there is a state-space framework developed in [8] to study the zeros of \(\Sigma_1\). To study the eigenvalues and eigenvectors of the system matrix associated with \(\Sigma_1\) recently, it has been introduced Fiedler-like pencils and Rosenbrock linearization system polynomial associated with \(\Sigma_1\). Also, it has shown that the Fiedler-like pencils are Rosenbrock linearizations of the system polynomial, see [8].

In this paper we study the relationship between the eigenvalues and eigenvectors of a rational eigenvalue problem given in the form of a transfer function (1.6), its polynomial representation as a Rosenbrock matrix and associated linearizations. We introduce Fiedler linearizations of the system matrix \(S(\lambda)\) given in (1.2) which is also helpful to study zeros of the system \(\Sigma\) given in (1.1). This problem has recently has been studied for higher order state-space system in [8] and we will extend these results to the Multivariable state-space case.

The paper is organized as follows. In section 2 we recall the definition and some properties of matrix which we need throughout this paper. In section 3 we extend
the results of Fiedler pencils for Rosenbrock system from [11] to multivariable state space system. That is, we show that the state-space framework so developed in [11] could be gainfully used to linearize (Fiedler linearizations) a multivariable state-space system. In particular, we define Fiedler pencils for $S(\lambda)$ given in [12] and describe an algorithm for their construction. In Section 4 we prove that these are linearizations for associate system matrix $S(\lambda)$.

**Notation.** An $m \times n$ rational matrix function $R(\lambda)$ is an $m \times n$ matrix whose entries are rational functions of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are scalar polynomials in $\mathbb{C}[\lambda]$. We denote the $j$-th column of the $n \times n$ identity matrix $I_n$ by $e_j$ and the transpose of a matrix $A$ by $A^T$.

**2. Basic Concepts.**

**Definition 2.1.** [11] Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q}$. Then the Kronecker product (tensor product) of $A$ and $B$ is defined by

$$A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix} \in \mathbb{C}^{mp \times nq}.$$  

One of the properties of Kronecker product is as follows: Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q}, C \in \mathbb{C}^{n \times q}, D \in \mathbb{C}^{m \times r}$. Then $(A \otimes B)(C \otimes D) = (AC \otimes BD) \in \mathbb{C}^{mr \times pt}$.

In order to systematically generate the Fiedler linearizations for Rosenbrock system matrices, we need a few concepts introduced in [11].

**Definition 2.2.** Let $\sigma : \{0, 1, \ldots, p - 1\} \rightarrow \{1, 2, \ldots, p\}$ be a bijection.

1. For $j = 0, \ldots, p - 2$, the bijection is said to have a consecution at $j$ if $\sigma(j) < \sigma(j + 1)$ and $\sigma$ has an inversion at $j$ if $\sigma(j) > \sigma(j + 1)$.
2. The tuple $\text{CIS} \sigma(\sigma) := (c_1, i_1, c_2, i_2, \ldots, c_l, i_l)$ is called the consecutive-inversion structure sequence of $\sigma$, where $\sigma$ has $c_1$ consecutive consecutions at $0, 1, \ldots, c_1 - 1$; $i_1$ consecutive inversions at $1, c_1 + 1, \ldots, c_1 + i_1 - 1$ and so on, up to $i_l$ inversions at $p - 1 - i_l, \ldots, p - 2$.
3. The total number of consecutions and inversions in $\sigma$ is denoted by $c(\sigma)$ and $i(\sigma)$, respectively, i.e., $c(\sigma) = \sum_{j=1}^{l} c_j$, $i(\sigma) = \sum_{j=1}^{l} i_j$, and $c(\sigma) + i(\sigma) = p - 1$.

**Definition 2.3.** [11] Let $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^m A_m$ be a matrix polynomial of degree $m$. For $k = 0, \ldots, m$, the degree $k$ Horner shift of $P(\lambda)$ is the matrix polynomial $P_k(\lambda) := A_{m-k} + \lambda A_{m-k+1} + \cdots + \lambda^k A_m$. These Horner shifts satisfy the following:

$$P_0(\lambda) = A_m, \quad P_{k+1}(\lambda) = \lambda P_k(\lambda) + A_{m-k-1}, \text{ for } 0 \leq k \leq m - 1, \quad P_m(\lambda) = P(\lambda).$$

**Definition 2.4.** [12] Matrix polynomial $U(\lambda)$ is said to be unimodular if det $U(\lambda)$ is a nonzero constant, independent of $\lambda$. Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be equivalent if there exists unimodular matrices $U(\lambda)$ and $V(\lambda)$, such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$. If $U(\lambda), V(\lambda)$ are constant matrices, then $P(\lambda)$ and $Q(\lambda)$ are said to be strictly equivalent.

Let $X(\lambda) \in \mathbb{C}(\lambda)^{m \times n}$ be a rational matrix function. The normal rank [20] of $X(\lambda)$ is denoted by $\operatorname{rank}(X)$ and is given by $\operatorname{rank}(X) := \max_{\lambda} \operatorname{rank}(X(\lambda))$ where the maximum is taken over all $\lambda \in \mathbb{C}$ which are not poles of the entries of $X(\lambda)$. If $\operatorname{rank}(X) = n = m$ then $X(\lambda)$ is said to be regular, otherwise $X(\lambda)$ is said to be singular.
A complex number $\lambda$ is said to be an eigenvalue of the system matrix $S(\lambda)$ if $\text{rank}(S(\lambda)) < \text{rank}(S)$. An eigenvalue $\lambda$ of $S(\lambda)$ is called an invariant zero of the system $\Sigma$. We denote the set of eigenvalues of $S(\lambda)$ by $\text{Sp}(S)$.

Let $G(\lambda) \in \mathbb{C}(\lambda)^{m \times n}$ be a rational matrix function and let

$$\text{SM}(G(\lambda)) = \text{diag}\left(\frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0_{m-k,n-k}\right)$$

be the Smith-McMillan form \[14, 20\] of $G(\lambda)$, where the scalar polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are monic, are pairwise coprime and, $\phi_i(\lambda)$ divides $\phi_{i+1}(\lambda)$ and $\psi_i(\lambda)$ divides $\psi_{i+1}(\lambda)$, for $i = 1, 2, \ldots, k - 1$. The polynomials $\phi_1(\lambda), \ldots, \phi_k(\lambda)$ and $\psi_1(\lambda), \ldots, \psi_k(\lambda)$ are called invariant zero polynomials and invariant pole polynomials of $G(\lambda)$, respectively. Define

$$\phi_G(\lambda) := \prod_{j=1}^{k} \phi_j(\lambda) \quad \text{and} \quad \psi_G(\lambda) := \prod_{j=1}^{k} \psi_j(\lambda).$$

A complex number $\lambda$ is said to be a zero of $G(\lambda)$ if $\phi_G(\lambda) = 0$ and a complex number $\lambda$ is said to be a pole of $G(\lambda)$ if $\psi_G(\lambda) = 0$. The spectrum $\text{Sp}(G)$ of $G(\lambda)$ is given by $\text{Sp}(G) := \{\lambda \in \mathbb{C} : \phi_G(\lambda) = 0\}$. That is $\text{Sp}(G)$ is the set of zeros of $G(\lambda)$, see \[1\].

### 3. Fiedler pencils for Rosenbrock system matrix.

In this section we define Fiedler pencils for system polynomial $S(\lambda)$ and describe an algorithm for their construction. Let us consider a Rosenbrock system of the form \[1.2\] with $B, C$ constant in $\lambda$,

$$S(\lambda) = \begin{bmatrix} A(\lambda) & -B \\ C & D(\lambda) \end{bmatrix} \in \mathbb{C}^{n+m,n+m}$$

and the associated transfer function

$$R(\lambda) = D(\lambda) + CA(\lambda)^{-1}B \in \mathbb{C}(\lambda)^{m,m}$$

where for $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i \in \mathbb{C}[\lambda]^{n,n}$ is regular and $D(\lambda) = \sum_{j=0}^{d_D} \lambda^j D_j \in \mathbb{C}[\lambda]^{m,m}$. Our aim is to study linearizations of $R(\lambda)$ and its relation to linearizations of $S(\lambda)$.

The most simple way to perform a direct linearization is to consider a first companion form

$$C_1(\lambda)w := (\lambda X + Y)w = 0,$$

where

$$X := \begin{bmatrix} A_{d_A} & I_n & \cdots & I_n \\ & D_{d_D} & I_m & \cdots & I_m \\ & & 5 & & \end{bmatrix}$$
An important class of linearizations (which include the first companion form \((3.2)\) as special case) that has recently received a lot of attention are the Fiedler pencils, \([5, 9, 11]\). Introducing Fiedler matrices \([5, 9, 11]\).\n
Based on the Fiedler matrices, then for given \(A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}\) of degree \(d_A\), defined by

\[
M_{d_A} := \begin{bmatrix}
A_{d_A-1} & I_{(d_A-1)n} \\
I_{(d_A-1)n} & -A_n \\
I_{(d_A-1)n} & I_n \\
I_{(d_A-1)n} & I_n \\
\end{bmatrix},
\]

\[
M_i := \begin{bmatrix}
I_{(d_A-i-1)n} & -A_{i-1} & I_n \\
I_{(d_A-i-1)n} & I_n \\
\end{bmatrix},
\]

\(i = 1, \ldots, d_A - 1\), \((3.3)\)

see e.g. \([11]\).

If \(\sigma : \{0, 1, \ldots, m-1\} \to \{1, 2, \ldots, m\}\) is a bijection, then one furthermore defines the products \(M_\sigma := M_{\sigma(1)}M_{\sigma(2)} \cdots M_{\sigma(m)}\). Note that \(\sigma(i)\) describes the position of the factor \(M_i\) in the product \(M_\sigma\); i.e., \(\sigma(i) = j\) means that \(M_i\) is the \(j\)th factor in the product.

Based on the Fiedler matrices, then for given \(A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}\) of degree \(d_A\) and a bijection \(\sigma\), in \([11]\) the associated Fiedler pencil is defined as the \(d_A n \times d_A n\) matrix pencil

\[
L_\sigma(\lambda) := \lambda M_{d_A} - M_{\sigma-1} \cdots M_{\sigma-1(d_A)} = \lambda M_{d_A} - M_\sigma.
\]

This concept was extended in \([11, 2, 3, 7, 6, 8]\) for square Rosenbrock systems of the state-space form \([1, 8]\) and \([1, 9]\). In \([1]\) also a multiplication-free algorithm is presented to construct Fiedler pencils for square system polynomials and it is shown...
that these Fiedler pencils are linearizations of the system polynomial and as well as
of the associated transfer functions under some appropriate conditions.

Extending the definition of [1], based on the idea of the companion like form (3.2),
we define \( nd_A \times nd_A \) Fiedler matrices associated with \( A(\lambda) \in \mathbb{F}[\lambda]^{n,n} \) as in (3.3), and
Fiedler matrices associated with the matrix polynomial \( D(\lambda) \in \mathbb{F}[\lambda]^{m,m} \) by

\[
N_d := \begin{bmatrix} D_d & I_{(d_D-1)m} \\ -D_i & I_m \\ I_n & 0 \\ I_{(i-1)m} \end{bmatrix}, \quad N_0 := \begin{bmatrix} I_{(d_D-1)m} \\ -D_0 \end{bmatrix},
\]

\[
N_i := \begin{bmatrix} -D_i & I_p \\ I_m & 0 \\ I_{(i-1)m} \end{bmatrix}, \quad i = 1, \ldots, d_D - 1. \quad (3.5)
\]

Based on the Fiedler matrices, then for given \( D(\lambda) \in \mathbb{F}[\lambda]^{m,m} \) of degree \( d_D \) and a
bijection \( \sigma \), in [1] the associated Fiedler pencil is defined as the \( d_Dm \times d_Dm \) matrix pencil

\[
T_\sigma(\lambda) := \lambda N_d - N_{\sigma^{-1}(1)} \cdots N_{\sigma^{-1}(d_D)} = \lambda N_d - N_\sigma. \quad (3.6)
\]

Note that \( M_i, M_j = M_j M_i, N_i N_j = N_j N_i \) for \( |i-j| > 1 \) and except for the terms with
index 0, \( d_A \) and \( d_D \), respectively, each \( M_i \) and \( N_i \) is invertible. We then have the
following definition of Fiedler matrices for Rosenbrock matrices \( S(\lambda) \in \mathbb{C}[\lambda]^{n+m,n+m} \)
given in (1.2).

**Definition 3.1.** Consider a system polynomial \( S(\lambda) \) as in (1.2) and let \( d = \max\{d_A, d_D\}, r = \min\{d_A, d_D\} \). Define \((d_A n + d_D m) \times (d_A n + d_D m)\) matrices
\( M_0, \ldots, M_d \) by

\[
M_0 := \begin{bmatrix} I_{(d_A-1)n} & -A_0 & (e_{d_A} e_{d_D}^T) \otimes B \\ -e_{d_A} e_{d_D}^T & I_{(d_D-1)m} & -D_0 \end{bmatrix},
\]

\[
M_d := \begin{bmatrix} A_d & I_{(d_A-1)n} \\ D_d & I_{(d_D-1)m} \end{bmatrix},
\]

\[
M_i := \begin{bmatrix} M_i \\ N_i \end{bmatrix}, \quad i = 1, \ldots, r - 1,
\]

\[7\]
and

\[
M_i := \begin{bmatrix}
I_{(d_A-i-1)n} & -A_i & I_n \\
-A_i & I_n & 0 \\
I_n & 0 & I_{(i-1)n}
\end{bmatrix}_n
\]

\[
= \begin{bmatrix}
M_i \\
I_{d_Dm}
\end{bmatrix}
\]

\[
, \quad i = r, r + 1, \ldots, d_A - 1, \text{ if } d_D < d_A,
\]

\[
M_i := \begin{bmatrix}
I_{d_A n} & I_{(d_D-i-1)m} \\
-I_{d_A n} & I_{m} & 0 \\
0 & I_{(i-1)m}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I_{d_A n} \\
N_i
\end{bmatrix}
\]

\[
, \quad i = d_A, d_A + 1, \ldots, d_D - 1, \text{ if } d_D > d_A.
\]

Observe that as in [1] one has \(M_i M_j = M_j M_i\) for \(|i - j| > 1\) and all \(M_i\) (except possibly \(M_0, M_d\)) are invertible.

The associated Fiedler pencils are then defined as follows.

**Definition 3.2.** Consider a system polynomial \(S(\lambda)\) as in (1.2) and let \(d = \max\{d_A, d_D\}, r = \min\{d_A, d_D\}\). Let \(M_0, \ldots, M_d\) be Fiedler matrices associated with \(S(\lambda)\) as in Definition 3.1. Given any bijection \(\sigma : \{0, 1, \ldots, d - 1\} \rightarrow \{1, 2, \ldots, d\}\), the matrix pencil

\[
L_{\sigma}(\lambda) := \lambda M_d - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(m)} =: \lambda M_d - M_{\sigma}.
\]  

(3.7)

is called the Fiedler pencil of \(S(\lambda)\) associated with \(\sigma\). We also refer to \(L_{\sigma}(\lambda)\) as a Fiedler pencil of \(R(\lambda)\).

The companion like form given in (3.2), then is

\[
C_1(\lambda) = \lambda M_d - M_{d-1} \cdots M_1 M_0
\]
and the associated second companion form of $S(\lambda)$ is

$$C_2(\lambda) = \lambda \mathbb{M}_d + \mathbb{M}_0 \mathbb{M}_1 \cdots \mathbb{M}_{d-2} \mathbb{M}_{d-1}$$

(3.8)

$$= \lambda 
\begin{bmatrix}
A_{dA} & I_n & & \\
& & \ddots & \\
& & & I_n \\
& & & & D_{dD} \\
& & & & & I_m \\
& & & & & & \ddots \\
& & & & & & & I_m \\
\end{bmatrix}$$

(3.9)

$$+ 
\begin{bmatrix}
A_{dA-1} & -I_n & & \\
& & \ddots & \\
& & & -I_n \\
& & & & 0 \\
& & & & & \ddots \\
& & & & & & \ddots \\
& & & & & & & -I_m \\
0 & & & & & & & 0 \\
& & & & & & & \ddots \\
& & & & & & & D_{dD-1} \\
& & & & & & & & D_{dD} \\
\end{bmatrix}.$$
By using the commutativity relation it is easy to check that $\mathbb{L}_{\sigma_1}(\lambda) = \mathbb{L}_{\sigma_2}(\lambda)$.

**Example 3.2.** Let $G(\lambda) = A(\lambda) + CD(\lambda)^{-1}B \in \mathbb{C}^{n \times n}$ with $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$, $A_i \in \mathbb{C}^{n \times n}$ and $D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \lambda^3 D_3 + \lambda^4 D_4$, $D_i \in \mathbb{C}^{m \times m}$. Here, $d_A = 3$, $d_D = 4$, $d_A < d_D$ and $r = 3$, $d = 4$. Consider $\mathbb{L}_{\sigma}(\lambda) = \lambda M_4 - M_2 M_0 M_1 M_3$. Then the Fiedler matrices for $G(\lambda)$ are given by

\[
M_0 = \begin{bmatrix}
  i_n & 0 & 0 & 0 \\
  0 & i_m & 0 & 0 \\
  0 & 0 & -A_0 & 0 \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & i_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -C & 0
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
  i_n & 0 & 0 & 0 \\
  0 & i_m & 0 & 0 \\
  0 & 0 & -A_0 & i_m \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & i_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -C & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
  -A_0 & i_n & 0 & 0 \\
  0 & i_n & 0 & 0 \\
  0 & 0 & 0 & J_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -D_2 & 0 \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}, \quad M_3 = \begin{bmatrix}
  i_n & 0 & 0 & 0 \\
  0 & i_m & 0 & 0 \\
  0 & 0 & -A_0 & i_m \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & i_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -C & 0
\end{bmatrix}, \quad M_4 = \begin{bmatrix}
  A_3 & 0 & 0 & 0 \\
  -A_1 & 0 & 0 & 0 \\
  0 & i_n & 0 & 0 \\
  0 & 0 & 0 & J_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -D_2 & 0 \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}.

Then

\[
M_{\sigma} = \begin{bmatrix}
  -A_2 & -A_1 & i_n & 0 & 0 & 0 & 0 \\
  i_n & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -A_0 & 0 & 0 & 0 & B & 0 \\
  0 & 0 & -D_3 & i_m & 0 & 0 & 0 \\
  0 & 0 & 0 & -D_2 & 0 & -D_1 & I_m \\
  0 & 0 & 0 & I_m & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -D_0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.

**Example 3.3.** Let $G(\lambda) = A(\lambda) + CD(\lambda)^{-1}B \in \mathbb{C}^{n \times n}$ with $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3$ where $A_i \in \mathbb{C}^{n \times n}$ and $D(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \lambda^3 D_3$ where $D_i \in \mathbb{C}^{m \times m}$. Here, $d_A = 3$, $d_D = 3$, $r = 3$ and $d = 3$. Consider $\mathbb{L}_{\sigma}(\lambda) = \lambda M_3 - M_{\sigma} = \lambda M_3 - M_2 M_0 M_1 M_3$. Then the Fiedler matrices for $G(\lambda)$ are given by

\[
M_0 = \begin{bmatrix}
  i_n & 0 & 0 & 0 \\
  0 & i_m & 0 & 0 \\
  0 & 0 & -A_0 & 0 \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & i_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -C & 0
\end{bmatrix}, \quad M_1 = \begin{bmatrix}
  i_n & 0 & 0 & 0 \\
  0 & i_m & 0 & 0 \\
  0 & 0 & -A_0 & i_m \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & i_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -C & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
  -A_0 & i_n & 0 & 0 \\
  0 & i_n & 0 & 0 \\
  0 & 0 & 0 & J_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -D_2 & 0 \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}, \quad M_3 = \begin{bmatrix}
  i_n & 0 & 0 & 0 \\
  0 & i_m & 0 & 0 \\
  0 & 0 & -A_0 & i_m \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & i_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -C & 0
\end{bmatrix}, \quad M_4 = \begin{bmatrix}
  A_3 & 0 & 0 & 0 \\
  -A_1 & 0 & 0 & 0 \\
  0 & i_n & 0 & 0 \\
  0 & 0 & 0 & J_m \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -D_2 & 0 \\
  0 & 0 & J_m & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}.

Then,

\[
M_{\sigma} = \begin{bmatrix}
  -A_2 & -A_1 & i_n & 0 & 0 & 0 \\
  i_n & 0 & 0 & 0 & 0 & 0 \\
  0 & -A_0 & 0 & 0 & B & 0 \\
  0 & 0 & -D_3 & i_m & 0 & 0 \\
  0 & 0 & 0 & -D_2 & 0 & -D_1 \\
  0 & 0 & 0 & I_m & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -D_0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
Having introduced the basic idea of generating Fiedler pencils for Rosénbrock system polynomials given in \([12]\), now we will analyze these constructed pencils.

**Theorem 3.3.** Let \(S(\lambda)\) be given in \([13]\). Let \(d = \max(d_A, d_D)\) and \(\sigma : \{0, 1, \ldots, d - 1\} \to \{1, 2, \ldots, d\}\) be a bijection. Let \(L_\sigma(\lambda)\), \(T_\sigma(\lambda)\) and \(L_\sigma(\lambda)\) be the Fiedler pencils of \(A(\lambda)\) of degree \(d_A\), \(D(\lambda)\) of degree \(d_D\) and \(S(\lambda)\), respectively, associated with \(\sigma\), that is, \(L_\sigma(\lambda) := \lambda M_{d_A} - M_\sigma\), \(T_\sigma(\lambda) := \lambda N_{d_D} - N_\sigma\) and \(L_\sigma(\lambda) := \lambda M_d - M_\sigma\). If \(\sigma^{-1} = (\sigma_1^{-1}, 0, \sigma_2^{-1})\) for some bijections \(\sigma_1\) and \(\sigma_2\), then

\[
L_\sigma(\lambda) = \frac{L_\sigma(\lambda)}{N_{\sigma_1}(e_{d_D}e_{d_A}^T C)M_{\sigma_2}} = \frac{-M_{\sigma_1}(e_{d_A}e_{d_D}^T B)N_\sigma}{T_\sigma(\lambda)}. 
\]

Further, if \(\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_1, i_1)\) then

\[
L_\sigma(\lambda) = \left[ \begin{array}{c|c}
L_\sigma(\lambda) & -c_{d_A}e_{d_A}^T B \\
\hline
0 & T_\sigma(\lambda)
\end{array} \right] , \quad \text{if } c_1 > 0
\]

and

\[
L_\sigma(\lambda) = \left[ \begin{array}{c|c}
L_\sigma(\lambda) & -c_{d_A}e_{d_A}^T B \\
\hline
0 & T_\sigma(\lambda)
\end{array} \right] , \quad \text{if } c_1 = 0.
\]

Thus the map \((\text{Fiedler}(A), \text{Fiedler}(D)) \to \text{Fiedler}(S)) \lambda M_{d_A} - M_\sigma, \lambda N_{d_D} - N_\sigma \to \lambda M_m - M_\sigma\) is a bijection, where Fiedler(A), Fiedler(D) and Fiedler(S), respectively, denote the set of Fiedler pencils of \(A(\lambda), D(\lambda)\) and \(S(\lambda)\).

**Proof.** We have \(L_\sigma(\lambda) = \lambda M_m - M_\sigma = \lambda M_m - M_\sigma, M_0 M_\sigma\)

\[
\begin{align*}
= & \lambda \left[ \begin{array}{c|c}
M_{d_A} & 0 \\
\hline
0 & N_{d_D}
\end{array} \right] - \left[ \begin{array}{c|c}
M_\sigma & 0 \\
\hline
0 & N_{d_A}
\end{array} \right] \left[ \begin{array}{c|c}
M_0 & (e_{d_A}e_{d_A}^T C)B \\
\hline
N_0 & T_\sigma(\lambda)
\end{array} \right] \left[ \begin{array}{c|c}
M_{\sigma_1} & 0 \\
\hline
0 & N_{\sigma_2}
\end{array} \right] \\
= & \lambda \left[ \begin{array}{c|c}
M_{d_A} & 0 \\
\hline
0 & N_{d_D}
\end{array} \right] - \left[ \begin{array}{c|c}
M_{d_A} & 0 \\
\hline
0 & N_{d_A}
\end{array} \right] \left[ \begin{array}{c|c}
M_{\sigma_1} & 0 \\
\hline
0 & N_{\sigma_2}
\end{array} \right] \\
= & \left[ \begin{array}{c|c}
L_\sigma(\lambda) & -M_{\sigma_1}(e_{d_A}e_{d_A}^T C)M_{\sigma_2} \\
\hline
N_{\sigma_1}(e_{d_A}e_{d_A}^T C)M_{\sigma_2} & T_\sigma(\lambda)
\end{array} \right] .
\end{align*}
\]

Now suppose that \(\text{CISS}(\sigma) = (c_1, i_1, \ldots, c_1, i_1)\).

Case I : Suppose that \(c_1 > 0\). Then by commutativity relation we have \(M_\sigma = M_{\sigma_1}M_0 M_{c_1} \cdots M_{c_1}\) with \(c_1 + 1 \in \sigma_1\). Thus \(M_\sigma = M_{\sigma_1}M_0 M_{\sigma_2}\), where \(M_{\sigma_2} = M_{c_1} \cdots M_{c_1}\). Hence

\[
M_\sigma = \left[ \begin{array}{c|c}
M_{\sigma_1} & 0 \\
\hline
- (e_{d_A}e_{d_A}^T C) & N_{\sigma_1}
\end{array} \right] \left[ \begin{array}{c|c}
M_0 & (e_{d_A}e_{d_A}^T C)B \\
\hline
N_0 & T_\sigma(\lambda)
\end{array} \right] \left[ \begin{array}{c|c}
M_{\sigma_1} & 0 \\
\hline
0 & N_{\sigma_2}
\end{array} \right] \\
= \left[ \begin{array}{c|c}
M_{\sigma_1}M_0 M_{\sigma_2} & 0 \\
\hline
M_{\sigma_1}M_0 M_{\sigma_2} & T_\sigma(\lambda)
\end{array} \right] .
\]

Since \(j \in \sigma\) implies that \(j \geq c_1 + 1\), we have \(M_{\sigma_1} = \left[ \begin{array}{c|c}
\circ & I_{c_1, n} \\
\hline
I_{c_1, m} & \circ
\end{array} \right] \) and \(N_{\sigma_1} = \left[ \begin{array}{c|c}
\circ & I_{c_1, m} \\
\hline
I_{c_1, n} & \circ
\end{array} \right] \). This shows that \(M_{\sigma_1}(e_{d_A} I_n) = e_{d_A} I_n\) and \(N_{\sigma_1}(e_{d_A} I_m) = e_{d_A} I_m\). So, we have \(M_{\sigma_1}(e_{d_A} B) = e_{d_A} B\). Next, we have \(N_{\sigma_1}(e_{d_A} e_{d_A}^T C)M_{\sigma_2} = \).
Now, we have

\[ (e_{d_A}^T \otimes I_n) M_1 = (e_{d_A}^T \otimes I_n) \begin{pmatrix} I_{(d_A-2)n} & -A_1 & I_n \\ I_n & 0 & \end{pmatrix} = (e_{d_A-1}^T \otimes I_n), \]

\[ (e_{d_A}^T \otimes I_n) M_1 M_2 = (e_{d_A}^T \otimes I_n) \begin{pmatrix} I_{(d_A-3)n} & -A_2 & I_n \\ I_n & 0 & \end{pmatrix} = (e_{d_A-2}^T \otimes I_n), \]

and so on. Thus \((e_{d_A}^T \otimes I_n) M_1 M_2 \cdots M_{c_1} = (e_{d_A-c_1}^T \otimes I_n)\). Hence \((e_{d_A}^T \otimes I_n) M_{\sigma_2} = (e_{d_A-c_1}^T \otimes I_n)\) and \((-e_{d_A-c_1}^T \otimes C) M_{\sigma_2} = -(e_{d_A-c_1}^T \otimes C)\). Similarly, we have

\[ (e_{d_D}^T \otimes I_m) N_1 = (e_{d_D}^T \otimes I_m) \begin{pmatrix} I_{(d_D-2)m} & -D_1 & I_m \\ I_m & 0 & \end{pmatrix} = (e_{d_D-1}^T \otimes I_m), \]

\[ (e_{d_D}^T \otimes I_m) N_1 N_2 = (e_{d_D}^T \otimes I_m) \begin{pmatrix} I_{(d_D-3)m} & -D_2 & I_m \\ I_m & 0 & \end{pmatrix} = (e_{d_D-2}^T \otimes I_m), \]

and so on. Thus \((e_{d_D}^T \otimes I_m) N_1 N_2 \cdots N_{c_1} = (e_{d_D-c_1}^T \otimes I_m)\). Hence \((e_{d_D}^T \otimes I_m) N_{\sigma_2} = (e_{d_D-c_1}^T \otimes I_m)\). Now, we have \(N_{\sigma_1} (e_{d_D} e_{d_A}^T \otimes I_n) M_{\sigma_2} = N_{\sigma_1} (e_{d_D} \otimes I_m) (e_{d_A}^T \otimes C) M_{\sigma_2} = (e_{d_D} e_{d_A}^T \otimes I_n)\) and \(-N_{\sigma_1} e_{d_D} e_{d_A}^T \otimes C) M_{\sigma_2} = -(e_{d_D} e_{d_A}^T \otimes C)\). Similarly, \(M_{\sigma_1} (e_{d_A} e_{d_D}^T \otimes B) N_{\sigma_2} = (e_{d_A} e_{d_D}^T \otimes C)\). Consequently, we have

\[ \mathbb{L}_{\sigma}(\lambda) = \lambda M_{\pi_m} - M_{\sigma} = \begin{pmatrix} N_{\sigma} & M_{\sigma} \\ 0 & \end{pmatrix} \begin{pmatrix} N_{\sigma} & M_{\sigma} \\ 0 & \end{pmatrix} = \begin{pmatrix} \mathbb{L}_{\sigma}(\lambda) & -e_{d_D} e_{d_A}^T \otimes C \\ e_{d_A} e_{d_D}^T \otimes B & \mathbb{L}_{\sigma}(\lambda) \end{pmatrix}. \]

Case II: Suppose that \(c_1 = 0\). Then \(\sigma\) has \(i_1\) inversions at 0. Hence by commutativity relations we have \(M_{\sigma} = M_{i_1} \cdots M_{i_1} M_0 M_{\sigma_2} = \lambda M_{\pi_m} M_{\sigma_2} \) with \(i_1 + 1 \in \sigma_2\). Hence

\[ M_{\sigma} = \begin{pmatrix} M_{\sigma_1} & M_0 \\ M_0 & \end{pmatrix} \begin{pmatrix} (e_{d_A} e_{d_D}^T \otimes B) N_{\sigma_2} \\ 0 & \end{pmatrix} = \begin{pmatrix} M_{\sigma_1} M_0 M_{\sigma_2} & M_{\sigma_1} (e_{d_A} e_{d_D}^T \otimes B) N_{\sigma_2} \\ -N_{\sigma_1} (e_{d_D} e_{d_A} \otimes C) M_{\sigma_2} & N_{\sigma_1} M_0 N_{\sigma_2} \end{pmatrix}. \]

Since \(j \in \sigma_2\) implies that \(j \geq i_1 + 1\), we have \(M_{\sigma_2} = \begin{pmatrix} * & 0 \\ 0 & \end{pmatrix} \) and \(N_{\sigma_2} = \begin{pmatrix} * & 0 \\ 0 & \end{pmatrix} \). This shows that \((e_{d_A} \otimes I_n) M_{\sigma_2} = e_{d_A} \otimes I_n\) and \((e_{d_D} \otimes I_m) N_{\sigma_2} = e_{d_D} \otimes I_m\).
Hence \((-e^T_{d_A} \otimes C)M_{\sigma_2} = -e^T_{d_A} \otimes C\). Next, we have

\[
M_1(e_{d_A} \otimes I_n) = \begin{bmatrix}
I_{(m-2)n} & -A_1 & I_n \\
I_n & 0 & 0 \\
0 & I_n & 0
\end{bmatrix} (e_{d_A} \otimes I_n) = (e_{d_A-1} \otimes I_n),
\]

\[
M_2M_1(e_{d_A} \otimes I_n) = \begin{bmatrix}
I_{(d_A-3)n} & -A_2 & I_n \\
I_n & 0 & 0 \\
0 & I_n & 0
\end{bmatrix} (e_{d_A-1} \otimes I_n) = (e_{d_A-2} \otimes I_n).
\]

Thus \(M_1 \cdots M_2M_1(e_{d_A} \otimes I_n) = (e_{d_A-1} \otimes I_n)\). Hence \(M_{\sigma_1}(e_{d_A} \otimes I_n) = (e_{d_A-i_i} \otimes I_n)\) and \(M_{\sigma_1}(e_{d_A} \otimes B) = (e_{d_A-i_i} \otimes B)\). Similarly, we have

\[
N_1(e_{d_D} \otimes I_m) = \begin{bmatrix}
I_{(d_D-2)m} & -D_1 & I_m \\
I_m & 0 & 0 \\
0 & I_m & 0
\end{bmatrix} (e_{d_D} \otimes I_m) = (e_{d_D-1} \otimes I_m),
\]

\[
N_2N_1(e_{d_D} \otimes I_m) = \begin{bmatrix}
I_{(d_D-3)m} & -D_2 & I_m \\
I_m & 0 & 0 \\
0 & I_m & 0
\end{bmatrix} (e_{d_D-1} \otimes I_m) = (e_{d_D-2} \otimes I_m).
\]

Thus \(N_1 \cdots N_2N_1(e_{d_D} \otimes I_m) = (e_{d_D-1} \otimes I_m)\). Hence \(N_{\sigma_1}(e_{d_D} \otimes I_m) = (e_{d_D-i_i} \otimes I_m)\). Now, we have \(N_{\sigma_1}(e_{d_D} e^T_{d_A} \otimes I_n)M_{\sigma_2} = N_{\sigma_1}(e_{d_D} \otimes I_m)(e^T_{d_A} \otimes C)M_{\sigma_2} = (e_{d_D-1} \otimes e^T_{d_A} \otimes I_n)\) and \(-(N_{\sigma_1}e_{d_D} e^T_{d_A} \otimes I_n)M_{\sigma_2} = -(e_{d_D-1} \otimes e^T_{d_A} \otimes C)\). Similarly, \(M_{\sigma_1}(e_{d_A} e^T_{d_D} \otimes B)N_{\sigma_2} = (e_{d_A-i_i} e^T_{d_D} \otimes B)\). Consequently, we have

\[
\mathbb{L}_\sigma(\lambda) = \lambda M_m - M_\sigma = \begin{bmatrix}
L_\sigma(\lambda) & -e_{(d_A-i_i)} e^T_{d_A} \otimes B \\
-e_{(d_D-i_i)} e^T_{d_D} \otimes C & T_\sigma(\lambda)
\end{bmatrix}.
\]

Note that for each \(i, j \in \sigma\), we have \(M_i M_j = M_j M_i, N_i N_j = N_j N_i \Leftrightarrow M_i M_j = M_j M_i\). Hence it follows that \#(Fiedler(A), Fiedler(D)) = #(Fiedler(S)). This completes the proof. \(\square\)

**Theorem 3.4.** Let \(S(\lambda)\) be in \((L2)\) with \(A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i, A_i \in \mathbb{C}^{n \times n}, \sum_{i=0}^{d_D} \lambda^i D_i, D_i \in \mathbb{C}^{m \times m}\). Suppose that \(d_A > d_D\). Let \(\sigma : \{0, 1, \ldots, d_A - 1\} \rightarrow \{1, 2, \ldots, d_A\}\) be a bijection. The following algorithm constructs a sequence of matrices \(\{W_0, W_1, \ldots, W_{d_A-2}\}\), where each matrix \(W_i\) for \(i = 1, 2, \ldots, d_A - 2\) is partitioned into blocks in such a way that the blocks of \(W_{i-1}\) are blocks of \(W_i\).
Algorithm 1 Construction of $M_\sigma$ for $I_\sigma(\lambda) := \lambda M_{d_A} - M_\sigma$.

Input: $S(\lambda) = \begin{bmatrix} \sum_{i=0}^{d_A} \lambda^i A_i & -B \\ \sum_{i=0}^{d_B} \lambda^i D_i & C \end{bmatrix}$ and a bijection $\sigma : \{0, 1, \ldots, d_A - 1\} \to \{1, 2, \ldots, d_A\}$.

Output: $M_\sigma$

if $\sigma$ has a consecution at 0 then
    $w_0 := \begin{bmatrix} -A_0 & 0 & 0 \\ 0 & B & 0 \\ -C & 0 & -D_0 \end{bmatrix}$
else
    $w_0 := \begin{bmatrix} -A_1 & 0 & 0 \\ 0 & -A_0 & 0 \\ 0 & -C & B \\ 0 & 0 & -D_1 \end{bmatrix}$
end if

If $d_A > d_D$

for $i = 1 : d_D - 2$ do
    if $\sigma$ has a consecution at $i$ then
        $w_i := \begin{bmatrix} W_{i-1}(1 : i + 1, 1) & 0 & 0 & W_{i-1}(1 : i + 1, 2 : i + 1) \\ W_{i-1}(3 : i + 3, 1) & 0 & 0 & W_{i-1}(3 : i + 3, 2 : i + 1) \\ 0 & 0 & 0 & W_{i-1}(3 : i + 3, 2 : i + 2) \\ 0 & 0 & 0 & W_{i-1}(3 : i + 3, i + 3 : 2i + 2) \end{bmatrix}$, where
        
        $W_{i2} := \begin{bmatrix} W_{i-1}(1 ; i + 1) & 0 & 0 & W_{i-1}(1 ; i + 1, 2 : i + 1) \\ W_{i-1}(2 ; i + 1, 1 ; i + 1) & 0 & 0 & W_{i-1}(2 ; i + 1, 3 ; i + 3) \\ 0 & 0 & 0 & W_{i-1}(2 ; i + 2, 2 ; i + 2) \\ 0 & 0 & 0 & W_{i-1}(2 ; i + 2, 3 ; i + 3) \end{bmatrix}$
    else
        $w_i := \begin{bmatrix} -A_{i+1} & 0 & 0 & W_{i-1}(1 ; i + 1) \\ 0 & 0 & W_{i-1}(2 ; i + 1, 1 ; i + 1) \\ 0 & 0 & 0 & W_{i-1}(3 ; i + 3, i + 3 : 2i + 2) \\ 0 & 0 & 0 & W_{i-1}(3 ; i + 3, i + 3 : 2i + 2) \end{bmatrix}$
    end if
end for

for $i = d_D - 1 : d_A - 2$ do
    if $\sigma$ has a consecution at $i$ then
        $w_i := \begin{bmatrix} -A_{i+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & W_{i-1}(1 ; i + 1) \\ 0 & 0 & W_{i-1}(2 ; i + 1, 1 ; i + 1) \\ 0 & 0 & 0 & W_{i-1}(2 ; i + 2, 2 ; i + 2) \end{bmatrix}$
    else
        $w_i := \begin{bmatrix} -A_{i+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & W_{i-1}(1 ; i + 1) \\ 0 & 0 & W_{i-1}(2 ; i + 1, 1 ; i + 1) \\ 0 & 0 & 0 & W_{i-1}(2 ; i + 2, 2 ; i + 2) \end{bmatrix}$
    end if
end for

$f_{d_A} := W_{d_A - 2}$

Proof. We prove the result by induction on the degree $d_A = \max(d_A, d_D)$. Then the rest of the proof follows directly from proof of Theorem 3.11 in [1]. \[\square\]

**Theorem 3.5.** Let $S(\lambda)$ be in \((122)\) with $A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i, A_i \in \mathbb{C}^{n \times n}, \sum_{i=0}^{d_B} \lambda^i D_i, D_i \in \mathbb{C}^{m \times n}$. Suppose that $d_A < d_D$. Let $\sigma : \{0, 1, \ldots, d_D - 1\} \to \{1, 2, \ldots, d_D\}$ be a bijection. The following algorithm constructs a sequence of matrices \(\{W_0, W_1, \ldots, W_{d_D - 2}\}\), where each matrix $W_i$ for $i = 1, 2, \ldots, d_D - 2$ is partitioned into blocks in such a way
that the blocks of $W_{i-1}$ are blocks of $W_i$.

Algorithm 2 Construction of $M_\sigma$ for $L_\sigma(\lambda) := \lambda M_{d_D} - M_\sigma$.

Input: $S(\lambda) = \begin{bmatrix} \sum_{i=0}^{d_A} \lambda A_i \\ \sum_{i=0}^{d_D} \lambda D_i \end{bmatrix}$ and a bijection $\sigma: \{0, 1, \ldots, d_D - 1\} \rightarrow \{1, 2, \ldots, d_A\}$

Output: $M_\sigma$

if $\sigma$ has a consecution at 0 then

$W_0 = \begin{bmatrix} -A_1 & I_n \\ -A_0 & 0 \end{bmatrix}$

else

$W_0 = \begin{bmatrix} -A_1 & A_0 \\ I_n & -D_1 \end{bmatrix}$

end if

if $d_D > d_A$ then

for $i = 1 : d_A - 2$ do

if $\sigma$ has a consecution at $i$ then

$W_i := \begin{bmatrix} -A_{i+1} & I_n \\ \hat{W}_{i-1}(1 : i+1, 1, 1) & -D_{i+1} \end{bmatrix}$, where

$\hat{W}_{i-1}(3 : i+3, 1) 0 \quad 0 \quad \hat{W}_{i-1}(3 : i+3, 2 : i+1)$

$\hat{W}_{i-1}(1 : i+1, i+2) 0 \quad 0 \quad \hat{W}_{i-1}(1 : i+1, i+3 : 2i+2)$

$\hat{W}_{i-1}(3 : i+3, i+2) 0 \quad \hat{W}_{i-1}(3 : i+3, i+3 : 2i+2)$

else

$W_i = \begin{bmatrix} -A_{i+1} & \hat{W}_{i-1}(1 : i+1) \\ 0 & I_n \end{bmatrix}$

end if

end for

for $i = d_A - 1 : d_D - 2$ do

if $\sigma$ has a consecution at $i$ then

$W_i := \begin{bmatrix} -A_{i+1} & \hat{W}_{i-1}(1 : i+1, 1, 1) \\ 0 & I_n \end{bmatrix}$

else

$W_i = \begin{bmatrix} -A_{i+1} & \hat{W}_{i-1}(1 : i+1, 1, 1) \\ 0 & I_n \end{bmatrix}$

end if

end for

$M_\sigma := W_{d_D-2}$

Proof. We prove the result by induction on the degree $d_D = \max(d_A, d_D)$. Then the rest of the proof follows directly from proof of Theorem 3.11 in [1].

4. Fiedler linearizations of Rosenbrock system matrix. In this section we show that the constructed Fiedler pencils associated with Rosenbrock systems are indeed linearizations. To do this we have to recall a few basic facts.

Definition 4.1 (System equivalence). Let $S_1(\lambda)$ and $S_2(\lambda)$ be $(n+m) \times (n+m)$ Rosenbrock system polynomials of the form $[A, B]$, partitioned conformably. Then $S_1(\lambda)$ is said to be system equivalent to $S_2(\lambda)$ (denoted as $S_1(\lambda) \sim_{SE} S_2(\lambda)$), if there exist unimodular matrix polynomials $U(\lambda), V(\lambda) \in \mathbb{P}^{n,n}, \tilde{U}(\lambda) \in \mathbb{P}^{m,m}$, and $\tilde{V}(\lambda) \in \mathbb{P}^{m,m}$.
\[ F^{m,m} \text{ such that for all } \lambda \in \mathbb{C} \text{ we have} \]
\[
\begin{bmatrix}
U(\lambda) & 0 \\
0 & U(\lambda)
\end{bmatrix}
S_1(\lambda)
\begin{bmatrix}
V(\lambda) & 0 \\
0 & V(\lambda)
\end{bmatrix} = S_2(\lambda). \tag{4.1}
\]

**Definition 4.2 (Rosenbrock linearization).** Let \( S(\lambda) \) be an \((n+m) \times (n+m)\) system polynomial of the form \((1.3)\) with degree \(d = \max\{d_A, d_D\}\). A linear matrix polynomial \( L(\lambda) \) is called a Rosenbrock linearization of \( S(\lambda) \), if it has the form
\[
L(\lambda) := \begin{bmatrix}
A(\lambda) & B \\
C & D(\lambda)
\end{bmatrix},
\]
with matrix polynomials \( A(\lambda), D(\lambda) \) of degree less than or equal to 1, (constant in \( \lambda \)) matrices \( B, C, \) and \( S(\lambda) \) is system equivalent to
\[
S(\lambda) := \begin{bmatrix}
U(\lambda) & 0 \\
0 & U(\lambda)
\end{bmatrix} L(\lambda)
\begin{bmatrix}
V(\lambda) & 0 \\
0 & V(\lambda)
\end{bmatrix} = \begin{bmatrix}
I_{(d-1)n} & 0 \\
0 & S(\lambda)
\end{bmatrix}. \tag{4.2}
\]

If, in addition, \( U(\lambda) \) and \( V(\lambda) \) in \((4.2)\) are constant matrices, then \( S(\lambda) \) is said to be a strict Rosenbrock linearization of \( S(\lambda) \).

Let \( E := (E_{ij}) \) be a block \( m \times n \) matrix with \( p \times q \) blocks \( E_{ij} \). The block transpose of \( E \), see \([11]\), denoted by \( E^B \), is the block \( n \times m \) matrix with \( p \times q \) blocks defined by \((E^B)_{ij} := E_{ji} \). We slightly modify this definition for the special structure of Rosenbrock linearizations.

**Definition 4.3 (Rosenbrock block transpose).** Consider a Rosenbrock system matrix of the form \((1.3)\) and let \( S \) be an \((d_A n + m) \times (d_A n + m)\) Rosenbrock linearization of the form \((4.3)\), where \( B := -(e_i e_j^T) \otimes B \) and \( C := (e_k e_l^T) \otimes C \) with \( B \in F^{m,m} \), \( C \in F^{p,n} \), where \( A := [A_{ij}] \) is an \( d_A \times d_A \) block matrix with \( A_{ij} \in F^{n \times n} \), and where \( D \) is a \( d_D \times d_D \) block matrix with \( D_{ij} \in F^{m \times m} \), or \( F^{p \times p} \). The Rosenbrock block transpose of \( S \), denoted by \( S^B \), is defined by
\[
S^B := \begin{bmatrix}
A(\lambda)^B & -(e_i e_j^T) \otimes B \\
(e_k e_l^T) \otimes C & D(\lambda)^B
\end{bmatrix}.
\]

For \( C_1(\lambda) \) and \( C_2(\lambda) \) given in \((3.2)\) and \((3.3)\), respectively, we have \( C_2(\lambda) = C_1(\lambda)^B \).

Extending \([11],\) Definition 4.2 we define auxiliary matrix polynomials associated with Horner shifts for system polynomials.

**Definition 4.4.** Let \( A(\lambda) = \sum_{i=0}^{d_A} \lambda^i A_i \in \mathbb{C}[\lambda]^{n,n} \) be of degree \( d_A \) and let \( P_i(\lambda) \) be the degree \( i \) Horner shift of \( A(\lambda) \). For \( 1 \leq i \leq d_A - 1 \), define the matrix polynomials
\[
Q_i(\lambda) := \begin{bmatrix}
I_{(i-1)n} & I_n & \lambda I_n & 0_n \\
I_n & I_n & I_{(d_A-i-1)n}
\end{bmatrix},
\]
\[
R_i(\lambda) := \begin{bmatrix}
I_{(i-1)n} & 0_n & I_n & \lambda I_n \\
0_n & -I_n & P_i(\lambda) & I_{(d_A-i-1)n}
\end{bmatrix},
\]

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Let $Q_i, R_i, T_i, D_i$ be as in Definition 4.4 and $M_i$’s be Fiedler matrices associated with $A(\lambda)$. Then the following relations hold for $i = 1, \ldots, d_A - 1$.

(a) $Q_i^B(\lambda D_i)R_i = \lambda D_{i+1} + T_i$, and $Q_i^B(M_{d_A-(i+1)}D_{d_A-i})R_i = M_{d_A-(i+1)} + T_i$.
(b) $R_i^B(\lambda D_i)Q_i = \lambda D_{i+1} + T_i^B$, and $R_i^B(M_{d_A-(i+1)}D_{d_A-i})Q_i = M_{d_A-(i+1)} + T_i^B$.
(c) $T_iM_j = M_jT_i = T_i$ and $T_i^B M_j = M_j T_i^B = T_i^B$ for all $j \leq d_A - i - 2$.

Definition 4.6. Let $D(\lambda) = \sum_{i=0}^{d_D} \lambda^i D_i$ be an $m \times m$ matrix polynomial, and let $P_i(\lambda)$ be the degree $i$ Horner shift of $D(\lambda)$. For $1 \leq i \leq d_D - 1$, define the following $md_D \times md_D$ matrix polynomials:

\[
Z_i(\lambda) := \begin{bmatrix}
I_{(i-1)m} & I_m & \lambda I_m \\
0_m & I_m & I_{(d_D-i-1)m}
\end{bmatrix},
\]

\[
J_i(\lambda) := \begin{bmatrix}
I_{(i-1)m} & 0_m & \lambda I_m \\
0_m & I_m & P_i(\lambda) \\
0_m & 0_m & I_{(d_D-i-1)m}
\end{bmatrix},
\]

\[
H_i(\lambda) := \begin{bmatrix}
0_{(i-1)m} & 0_m & \lambda P_{i-1}(\lambda) \\
0_m & \lambda I_m & \lambda^2 P_{i-1}(\lambda) \\
0_{(d_D-i-1)m} & 0_m & I_{(d_D-i-1)m}
\end{bmatrix},
\]

\[
E_i(\lambda) := \begin{bmatrix}
0_{(i-1)m} & P_{i-1}(\lambda) & 0_m \\
0_m & 0_m & I_m \\
0_{(d_D-i-1)m} & 0_m & I_{(d_D-i-1)m}
\end{bmatrix},
\]

and $E_{d_D}(\lambda) := \text{diag} \left[ 0_{(d_D-1)m}, P_{d_D-1}(\lambda) \right]$. For simplicity, we often write $Z_i, J_i, H_i, E_i$ in place of $Z_i(\lambda), J_i(\lambda), H_i(\lambda), E_i(\lambda)$. Note that $E_1(\lambda) = M_{d_D}$, and $Z_i(\lambda), J_i(\lambda)$ are unimodular for all $i = 1, \ldots, d_D - 1$. Also note that $J_i^B(\lambda) = J_i(\lambda)$.

Remark 4.7. Consider the auxiliary matrices $Z_i(\lambda), J_i(\lambda), H_i(\lambda), E_i(\lambda)$ given in Definition 4.6. Then the Lemma 4.5 also holds for $Z_i(\lambda), J_i(\lambda), H_i(\lambda), E_i(\lambda)$.

Definition 4.8 (Auxiliary system polynomials). Let $Q_i(\lambda), R_i(\lambda), T_i(\lambda), D_i(\lambda)$ be as in Definition 4.4. Let $Z_i(\lambda), J_i(\lambda), H_i(\lambda), E_i(\lambda)$ be as in Definition 4.6.
4.6 Let \( d = \max\{d_A, d_D\} \), and \( r = \min\{d_A, d_D\} \). For \( i = 1, \ldots, d - 1 \), define \((nd_A + md_D) \times (nd_A + md_D)\) system polynomials:

\[
Q_i(\lambda) = \begin{cases} 
Q_i(\lambda) & \text{if } 1 \leq i \leq r - 1 \\
0 & \text{if } r \leq i \leq d - 1 \text{ and } d_A > d_D \\
I_{d_{\text{dom}}} & \text{if } r \leq i \leq d - 1 \text{ and } d_A < d_D 
\end{cases}
\]

\[
R_i(\lambda) = \begin{cases} 
R_i(\lambda) & \text{if } 1 \leq i \leq r - 1 \\
0 & \text{if } r \leq i \leq d - 1 \text{ and } d_A > d_D \\
I_{d_{\text{dom}}} & \text{if } r \leq i \leq d - 1 \text{ and } d_A < d_D 
\end{cases}
\]

\[
T_i(\lambda) = \begin{cases} 
T_i(\lambda) & \text{if } 1 \leq i \leq r - 1 \\
0 & \text{if } r \leq i \leq d - 1 \text{ and } d_A > d_D \\
I_{d_{\text{dom}}} & \text{if } r \leq i \leq d - 1 \text{ and } d_A < d_D 
\end{cases}
\]

\[
D_i(\lambda) = \begin{cases} 
D_i(\lambda) & \text{if } 1 \leq i \leq r - 1 \\
0 & \text{if } r \leq i \leq d - 1 \text{ and } d_A > d_D \\
I_{d_{\text{dom}}} & \text{if } r \leq i \leq d - 1 \text{ and } d_A < d_D 
\end{cases}
\]

and \( D_d := \begin{bmatrix} D_d(\lambda) & 0 \\ 0 & E_d(\lambda) \end{bmatrix} \), where \( d = \max\{d_A, d_D\} \).

Note that \( D_1(\lambda) = \begin{bmatrix} D_1(\lambda) & 0 \\ 0 & E_1(\lambda) \end{bmatrix} = \begin{bmatrix} M_{d_A} & 0 \\ 0 & N_{d_D} \end{bmatrix} = M_{d_A} \) and that \( Q_i(\lambda) \) and \( R_i(\lambda) \) are unimodular matrix polynomials for \( i = 1, \ldots, d - 1 \). Also, note that \( R_{\text{fin}}(\lambda) = R_i(\lambda) \) for \( i = 1, \ldots, d - 1 \).

The auxiliary system polynomials satisfy the following relations.

**Lemma 4.9.** Let \( Q_i, R_i, T_i, D_i \) be the system polynomials given in Definition 4.8 and \( M_i \)'s be Fiedler matrices associated with \( S(\lambda) \). Then the following system equivalence relations hold for \( i = 1, \ldots, d - 1 \).
(a) \( Q_i^B(\lambda D_i) R_i = \lambda D_{i+1} + T_i \), and \( Q_i^B(M_{d-(i+1)} M_{d-i}) R_i = M_{d-(i+1)} + T_i \).

(b) \( R_i^B(\lambda D_i) Q_i = \lambda D_{i+1} + T_i^B \), and \( R_i^B(M_{d-(i+1)} M_{d-i}) Q_i = M_{d-(i+1)} + T_i^B \).

(c) \( T_i M_{d_j} = M_{d_j} T_i = T_i \) and \( T_i^B M_{d_j} = M_{d_j} T_i^B = T_i^B \) for all \( j \leq d-i-2 \).

Proof.  
(a) We have

\[
Q_i^B(\lambda D_i) R_i = \begin{bmatrix} Q_i^B \lambda D_i \lambda E_i \\ 0 \end{bmatrix} \begin{bmatrix} R_i \end{bmatrix} = \begin{bmatrix} Q_i^B(\lambda D_i) R_i \\ Z_i^B(\lambda E_i ) J_i \end{bmatrix} = \begin{bmatrix} \lambda D_{i+1} + T_i \end{bmatrix} \lambda E_{i+1} + H_i (By \ Lemma 4.5(a) \ and \ Remark 4.7)
\]

\[
= \begin{bmatrix} \lambda D_{i+1} J_i \\ \lambda E_{i+1} \end{bmatrix} \lambda E_{i+1} + H_i = \lambda D_{i+1} + T_i \hspace{1cm} \text{and}
\]

\[
Q_i^B M_{d-(i+1)} M_{d-i} R_i = \begin{bmatrix} Q_i^B M_{d-(i+1)} M_{d-i} R_i \\ Z_i^B D_{d-(i+1)} D_{d-i} J_i \end{bmatrix} = \begin{bmatrix} \lambda D_{i+1} + T_i \end{bmatrix} \begin{bmatrix} D_{d-(i+1)} + H_i \end{bmatrix} (Lemma 4.5(a) \ and \ Remark 4.7)
\]

\[
= \begin{bmatrix} M_{d-(i+1)} + T_i D_{d-(i+1)} + H_i \end{bmatrix} = M_{d-(i+1)} + T_i.
\]

(b) We have

\[
R_i^B(\lambda D_i) Q_i = \begin{bmatrix} R_i^B \lambda D_i \lambda E_i \\ 0 \end{bmatrix} \begin{bmatrix} Q_i \end{bmatrix} = \begin{bmatrix} R_i^B(\lambda D_i) Q_i \\ J_i^B(\lambda E_i) Z_i \end{bmatrix} = \begin{bmatrix} \lambda D_{i+1} + T_i^B \end{bmatrix} \lambda E_{i+1} + H_i^B (From \ Lemma 4.5(b) \ and \ Remark 4.7)
\]

\[
= \begin{bmatrix} \lambda D_{i+1} J_i \end{bmatrix} \lambda E_{i+1} + H_i^B = \lambda D_{i+1} + T_i^B \hspace{1cm} \text{and}
\]

\[
R_i^B M_{d-(i+1)} M_{d-i} Q_i = \begin{bmatrix} R_i^B M_{d-(i+1)} M_{d-i} Q_i \\ J_i^B D_{d-(i+1)} D_{d-i} Z_i \end{bmatrix} = \begin{bmatrix} M_{d-(i+1)} + T_i^B \end{bmatrix} \begin{bmatrix} D_{d-(i+1)} + H_i^B \end{bmatrix} (By \ Lemma 4.5(b) \ and \ Remark 4.7)
\]

\[
= \begin{bmatrix} M_{d-(i+1)} \end{bmatrix} D_{d-(i+1)} + H_i^B = M_{d-(i+1)} + T_i^B.
\]
(c) We have
\[
\begin{align*}
T_iM_j &= \begin{bmatrix} T_i & M_j \\ H_i & D_j \end{bmatrix} = \begin{bmatrix} T_iM_j & H_iD_j \\ 0 & -E_{dD} \end{bmatrix} \\
\quad &= \begin{bmatrix} M_jT_i & D_jH_i \end{bmatrix} (\text{by Lemma 4.5(c) and Remark 4.7}) \\
\quad &= \begin{bmatrix} M_j & D_j \end{bmatrix} \begin{bmatrix} T_i & H_i \end{bmatrix} = M_jT_i
\end{align*}
\]
\[
\begin{align*}
T_i^B M_j &= \begin{bmatrix} T_i^B & M_j \\ H_i^B & D_j \end{bmatrix} = \begin{bmatrix} T_i^B M_j & H_i^B D_j \\ 0 & -E_{dD} \end{bmatrix} \\
\quad &= \begin{bmatrix} M_jT_i^B & D_jH_i^B \end{bmatrix} (\text{by Lemma 4.5(c) and Remark 4.7}) \\
\quad &= \begin{bmatrix} M_j & D_j \end{bmatrix} \begin{bmatrix} T_i^B & H_i^B \end{bmatrix} = M_jT_i^B.
\end{align*}
\]

\[\Box\]

**Definition 4.10.** Let \( L_\sigma(\lambda) = \lambda M_d - M_\sigma \) be the Fiedler pencil of \( S(\lambda) \) given in (1.2) associated with a bijection \( \sigma \). For \( j = 1, 2, \ldots, d \), define
\[
M^{(j)}_\sigma := \prod_{\sigma^{-1}(i) \leq d-j} M_{\sigma^{-1}(i)},
\]
where the factors \( M_{\sigma^{-1}(i)} \) are in the same relative order as they are in \( M_\sigma \). Note that \( M^{(1)}_\sigma = \prod_{\sigma^{-1}(i) \leq d-1} M_{\sigma^{-1}(i)} = M_\sigma \) and that \( M^{(d)}_\sigma = M_0 \). Also for \( j = 1, 2, \ldots, d \) define the \((nd_d + md_D) \times (nd_d + md_D)\) system pencils \( L^{(j)}_\sigma(\lambda) := \lambda D_j(\lambda) - M^{(j)}_\sigma \). Observe that \( L^{(1)}_\sigma(\lambda) = \lambda D - M^{(1)}_\sigma = \lambda M_m - M_\sigma = L_\sigma \) and that
\[
\begin{align*}
L^{(d)}_\sigma(\lambda) &= \lambda D - M^{(d)}_\sigma = \lambda \begin{bmatrix} \frac{D_{dA} - M_d}{0} & 0 \\ 0 & -E_{dD} \end{bmatrix} - M_0 \\
\quad &\quad = \begin{bmatrix} -I_{(d_A-1)n} & -(e_{dA}^T e_{dD}) \otimes B \\ (e_{dD} e_{dA}) \otimes C & A(\lambda) \end{bmatrix} \begin{bmatrix} A(\lambda) & -(e_{dA}^T e_{dD}) \otimes B \\ -I_{(d_D-1)m} & D(\lambda) \end{bmatrix}.
\end{align*}
\]

The next result shows that \( L^{(i)}_\sigma(\lambda) \sim_se L^{(i+1)}_\sigma(\lambda) \) for \( i = 1, 2, \ldots, d-1 \).

**Lemma 4.11.** We have \( L^{(i)}_\sigma(\lambda) \sim_se L^{(i+1)}_\sigma(\lambda) \) for \( i = 1, 2, \ldots, d-1 \). More precisely, if \( Q_1 \) and \( R_1 \) are the system polynomials given in Definition 4.8, then
\[
L^{(i+1)}_\sigma(\lambda) = \begin{cases} Q^{B}_1 L^{(i)}_\sigma(\lambda) R_1, & \text{if } \sigma \text{ has a consecution at } d - i - 1 \\
R^{B}_1 L^{(i)}_\sigma Q_1, & \text{if } \sigma \text{ has an inversion at } d - i - 1.
\end{cases}
\]

**Proof.** The proof is exactly the same as that of Lemma 4.5 in [11]. \( \Box \)

It is now immediate that a Fiedler pencil is a Rosenbrock linearization of \( S(\lambda) \).

**Theorem 4.12.** (Rosenbrock linearization). Let \( S(\lambda) \) be an \((n + m) \times (n + m)\) system polynomial (regular or singular) given in (1.2). Then a Fiedler pencil \( L_\sigma(\lambda) \) of the system polynomial \( S(\lambda) \) is a Rosenbrock linearization of \( S(\lambda) \).
Proof. By Lemma 4.11 we have $d - 1$ system equivalences

$$L_\sigma(\lambda) = L_\sigma^{(1)}(\lambda) \sim_{sc} L_\sigma^{(2)}(\lambda) \sim_{sc} \cdots \sim_{sc} L_\sigma^{(d)}(\lambda)$$

$$= \begin{bmatrix}
-I_{(d_A-1)n} & -(e_{d_A}^T e_{d_D}^T) \otimes B \\
(e_{d_D} e_{d_A}^T) \otimes C & -I_{(d_D-1)m} \\
A(\lambda) & D(\lambda)
\end{bmatrix},$$

where $L_\sigma^{(i)}(\lambda)$ is as in Lemma 4.11. This shows that $L_\sigma(\lambda) \sim_{sc} I_{(d_A-1)n} \oplus S(\lambda) \oplus I_{(d_D-1)m}$. □

Corollary 4.13. Let $L_\sigma(\lambda)$ be the Fiedler pencil of $S(\lambda)$ given in (1.2) associated with a bijection $\sigma$, and $Q_i, R_i$ for $i = 1, 2, \ldots, d - 1$, be as in Definition 4.8. Then

$$U(\lambda) L_\sigma(\lambda) V(\lambda) = \begin{bmatrix}
-I_{(d_A-1)n} & -(e_{d_A}^T e_{d_D}^T) \otimes B \\
(e_{d_D} e_{d_A}^T) \otimes C & -I_{(d_D-1)m} \\
A(\lambda) & D(\lambda)
\end{bmatrix}$$

$$\sim_{sc} I_{(d_A-1)n} \oplus S(\lambda) \oplus I_{(d_D-1)m},$$

where $U(\lambda)$ and $V(\lambda)$ are $(nd_A + md_D) \times (nd_A + md_D)$ unimodular system polynomials given by

$$U(\lambda) := U_0 U_1 \cdots U_{d-3} U_{d-2}, \text{ with } U_i = \begin{cases}
Q_{d-i+1}, & \text{if } \sigma \text{ has a consecution at } i, \\
R_{d-i+1}, & \text{if } \sigma \text{ has an inversion at } i
\end{cases}$$

$$V(\lambda) := V_{d-2} V_{d-3} \cdots V_1 V_0, \text{ with } V_i = \begin{cases}
R_{d-i+1}, & \text{if } \sigma \text{ has a consecution at } i, \\
Q_{d-i+1}, & \text{if } \sigma \text{ has an inversion at } i
\end{cases}$$

The indexing of $U_i$ and $V_i$ factors in $U(\lambda)$ and $V(\lambda)$, respectively, in Corollary 4.13 has been chosen for simplification of notation and has no other special significance.

Remark 4.14. If we consider $D(\lambda)$ is a matrix polynomial of degree 1 then the Fiedler pencils $L_\sigma(\lambda)$ are linearizations of the system matrix of LTI state-space system, see [7].

Remark 4.15. Consider the system matrix $S(\lambda)$ and associated transfer function $R(\lambda)$ given in (1.2) and (1.3), respectively. Given an eigenvector $x$ of $L_\sigma(\lambda)$ one can determine an eigenvector of $S(\lambda)$ from $x$. That is, one can recover eigenvectors of $R(\lambda)$ and $S(\lambda)$ from those of the Fiedler pencils of $R(\lambda)$. It is directly follows from the Theorem 4.10 and Theorem 4.11 in [S].

5. Conclusions. We have considered a multivariable state-space system and its associated system matrix $S(\lambda)$. We have introduced Fiedler pencils of $S(\lambda)$ and described an algorithm for their construction. Finally, we have shown that Fiedler pencils are linearizations of $S(\lambda)$.

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