A REGULARITY RESULT FOR A LOCUS OF BRILL TYPE

ABDELMALEK ABDESSELAM AND JAYDEEP CHIPALKATTI

Abstract. Let \(n, d\) be a positive integers, with \(d\) even (say \(d = 2e\)). Write \(N = \binom{n+d}{d} - 1\), and let \(X_{(n,d)} \subseteq \mathbb{P}^N\) denote the locus of degree \(d\) hypersurfaces in \(\mathbb{P}^n\) which consist of two \(e\)-fold hyperplanes. We calculate a bound on the Castelnuovo-regularity of its defining ideal, moreover we show that this variety is \(r\)-normal for \(r \geq 2\). The latter part is proved by reducing the question to a combinatorial calculation involving Feynman diagrams and hypergeometric series. As such, it is a result of a tripartite collaboration of algebraic geometry, classical invariant theory, and modern theoretical physics.

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1. Introduction

The set of hypersurfaces of degree \(d\) in \(\mathbb{P}^n\) is parametrized by the projective space \(\mathbb{P}^N\), where \(N = \binom{n+d}{d} - 1\). Assume that \(d\) is even (say \(d = 2e\)), and consider the subset of hypersurfaces which consist of two (possibly coincident) \(e\)-fold hyperplanes. In algebraic terms, this is the set of \((n+1)\)-ary degree \(d\) forms \(F\) which factor as \(F = L_1^e L_2^e \ldots L_d^e\) for some linear forms \(L_i\). This forms a subvariety of \(\mathbb{P}^N\), which we will denote by \(X_{(n,d)}\). It may be called of Brill type, in analogy with the following problem considered (and solved) by Brill: Find necessary and sufficient conditions for \(F\) to factor into a product of linear forms, \(F = L_1 L_2 \ldots L_d\).

Let us write \(X\) for \(X_{(n,d)}\) if no confusion is likely. (We exclude the trivial case \(n = 1, d = 2\) throughout.) Define
\[m_0 = \left\lfloor 2n + 1 - \frac{n}{e} \right\rfloor.\]

Our main result is the following:
Theorem 1.1. \quad (i) The ideal $I_X$ is $m_0$-regular. A fortiori, it is minimally generated in degrees $\leq m_0$.

(ii) The linear series on $X$ cut out by degree $r$ hypersurfaces in $\mathbb{P}^N$ is complete for $r \geq 2$.

The natural action of the group $SL_{n+1}$ on the imbedding $X \subseteq \mathbb{P}^N$ will be essential to the proof. As a byproduct we will get a formula for the image of each graded piece $(I_X)_r$ in the Grothendieck ring of $SL_{n+1}$-modules.

Part (ii) of the theorem is the more delicate one. We reduce it to a question about transvectants of binary forms, and then resolve the issue using an explicit computation with Feynman diagrams. We hope that this technique should find a wider application.

In section §3 we give classical invariant theoretic descriptions of the generators of $I_X$ for the cases $(n, d) = (1, 8), (2, 4)$.

Remark 1.2. It will be apparent that this is an instance of a problem which can be formulated rather generally. Given any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of $d$, one can define a subvariety of forms which factor as $\prod L_i^{\lambda_i}$. It is a natural problem to find $SL_{n+1}$-invariant equations for this variety; it is not completely settled even for binary forms.

The case $\lambda = (d)$ corresponds to the Veronese imbedding (see [22]), and $\lambda = (1^d)$ to the Chow variety of degree $d$ zero cycles on $\mathbb{P}^d$ (see [19, Ch. 4]). The case $\lambda = (e, e)$ is perhaps the next in order of complexity. A result for the case $(\lambda_1, \lambda_2)$ (with $\lambda_1 \neq \lambda_2$) is under preparation.

1.1. Preliminaries. The base field will be $\mathbb{C}$. Let $V$ denote a complex vector space of dimension $n+1$, and write $W = V^*$. All subsequent constructions will be $SL(V)$-equivariant; see [18, Ch. 6 and 15] for the relevant representation theory. We will abbreviate $\text{Sym}^d V, \text{Sym}^r(\text{Sym}^d V)$ as $S_d, S_r(S_d)$ etc. If $\lambda$ is a partition, then $S_{\lambda}(-)$ will denote the associated Schur functor. All terminology from algebraic geometry follows [23].

Fix a positive integer $d = 2e$, and let $N = \binom{n+d}{d} - 1$. Given the symmetric algebra

$$R = \bigoplus_{r \geq 0} S_r(S_d V),$$

the space of degree $d$ hypersurfaces in $\mathbb{P}V$ is identified with

$$\mathbb{P}^N = \mathbb{P} S_d W = \text{Proj } R.$$
Now define
\[ X_{(n,d)} = \{ [F] \in \mathbb{P}^N : F = (L_1L_2)^e \text{ for some } L_1, L_2 \in W \}. \] (1)
This is an irreducible 2n-dimensional projective subvariety of \( \mathbb{P}^N \).

Recall the definition of regularity according to Mumford [29, Ch. 6].

**Definition 1.3.** Let \( F \) be a coherent \( \mathcal{O}_{\mathbb{P}^N} \)-module, and \( m \) an integer. Then \( F \) is said to be \( m \)-regular if \( H^q(\mathbb{P}^N, F(m-q)) = 0 \) for \( q \geq 1 \).

It is known that \( m \)-regularity implies \( m' \)-regularity for all \( m' \geq m \). Let \( M \) be a graded \( R \)-module containing no submodules of finite length. Then (for the present purpose) we will say that \( M \) is \( m \)-regular if its sheafification \( \tilde{M} \) is. In our case, \( M = I_X \) (the saturated ideal of \( X \)), and \( \tilde{I}_X = \mathcal{I}_X \).

We have the usual short exact sequence
\[ 0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0. \] (2)

The map
\[ \mathbb{P}W \times \mathbb{P}W \xrightarrow{f} \mathbb{P}S_d W, \quad (L_1, L_2) \mapsto (L_1L_2)^e \] (3)
induces a natural isomorphism of \( X \) with the quotient \((\mathbb{P}W \times \mathbb{P}W)/\mathbb{Z}_2\), and of the structure sheaf \( \mathcal{O}_X \) with \((f_\ast \mathcal{O}_{\mathbb{P}W \times \mathbb{P}W})^\mathbb{Z}_2 \).

Using the Leray spectral sequence and the Künneth formula,
\[ H^q(\mathbb{P}^N, f_\ast \mathcal{O}_{\mathbb{P}W \times \mathbb{P}W}(r)) = \bigoplus_{i+j=q} H^i(\mathbb{P}W, \mathcal{O}_{\mathbb{P}^n}(re)) \otimes H^j(\mathbb{P}W, \mathcal{O}_{\mathbb{P}^n}(re)). \]

This group can be nonzero only in two cases: \( i, j \) are either both 0 or both \( n \) (see [23, Ch. III, §5]).

**Corollary 1.4.** We have an isomorphism \( H^0(\mathcal{O}_X(r)) = S_2(S_{re}) \) for \( r \geq 0 \). Moreover \( H^{2n}(\mathcal{O}_X(r)) = 0 \) for \( re \geq -n \).

2. **The Proof of Theorem 1.1**

Define the predicate
\[ \mathcal{R}(q) : H^q(\mathbb{P}^N, \mathcal{I}_X(m_0 - q)) = 0. \]
We would like to show $\mathcal{R}(q)$ for $q \geq 1$. Tensor the short exact sequence (2) by $O_{P^N}(m_0 - q)$ and consider the long exact sequence in cohomology. If $q \neq 1, 2n + 1$, then $\mathcal{R}(q)$ is immediate. By the choice of $m_0$, we have

$$e(m_0 - 2n - 1) \geq -n.$$ 

Hence $H^{2n}(O_X(m_0 - 2n - 1)) = 0$, which implies $\mathcal{R}(2n + 1)$. Now $\mathcal{R}(1)$ is the case $r = m_0 - 1$ of the following result (which is part (ii) of the main theorem).

**Proposition 2.1.** Let $r \geq 2$. Then the morphism

$$\alpha_r : H^0(O_{P^N}(r)) \longrightarrow H^0(O_X(r))$$

is surjective.

**Proof.** The map $f$ can be factored as

$$\mathbb{P}W \times \mathbb{P}W \longrightarrow \mathbb{P}S_e W \times \mathbb{P}S_e W \longrightarrow \mathbb{P}S_d W.$$ 

Tracing this backwards, we see that $\alpha_r$ is the composite

$$S_r(S_d) \xrightarrow{1} S_r(S_e \otimes S_e) \xrightarrow{2} S_r(S_e) \otimes S_r(S_e) \xrightarrow{3} S_{re} \otimes S_{re} \xrightarrow{4} S_2(S_{re}),$$

where 1 is given by applying $S_r(-)$ to the coproduct map, 2 is the projection coming from the ‘Cauchy decomposition’ (see [3]), 3 is the multiplication map, and 4 is the symmetrisation. Now we have a plethysm decomposition

$$H^0(O_X(r)) = S_2(S_{re}) = \bigoplus_p S_{(rd - 2p, 2p)},$$

where the direct sum is quantified over $0 \leq p \leq \lfloor \frac{r}{2} \rfloor$. Let $\pi_p$ denote the projection onto the $p$-th summand. Since any finite dimensional $SL(V)$-module is completely reducible, the cokernel of $\alpha_r$ is a direct summand of $H^0(O_X(r))$. We will show that $\pi_p \circ \alpha_r \neq 0$ for any $p$, then Schur’s lemma will imply that the cokernel is zero.

The entire construction is functorial in $V$, hence if $U \subseteq V$ is any subspace, then the diagram

$$\begin{array}{ccc}
S_r(S_d U) & \longrightarrow & S_{(rd - 2p, 2p)} U \\
\downarrow & & \downarrow \\
S_r(S_d V) & \longrightarrow & S_{(rd - 2p, 2p)} V
\end{array}$$
is commutative. If we further assume that \( \dim U \geq 2 \), then both vertical maps are injective. (Recall that \( S_\lambda(V) \) vanishes if and only if the number of parts in \( \lambda \) exceeds \( \dim V \).) Hence we may as well assume that \( \dim V = 2 \). Thus we are reduced to the following statement:

**Proposition 2.2.** Assume \( \dim V = 2 \). Then the morphism \( \pi_p \circ \alpha_r \) is nonzero for any \( 0 \leq p \leq \left\lfloor \frac{r^2}{4} \right\rfloor \).

The proof will be given in the sections 4 and 5. Tentatively we will take the main theorem as proved, and interpose some examples. The following is a simple corollary to the theorem.

**Corollary 2.3.** In the Grothendieck ring of finite-dimensional \( SL(V) \)-modules, we have the equality

\[
[(I_X)_r] = [S_r(S_d)] - \sum_{0 \leq p \leq \left\lfloor \frac{r^2}{4} \right\rfloor} [S(rd-2p,2p)]
\]

Here \([-\] denotes the formal character of a representation.

**Proof.** This follows because \( (I_X)_r = \ker \alpha_r \).  

Decomposing the plethysm \( S_r(S_d) \) is in general a difficult problem. Explicit formulae are known only in very special cases – see [10, 28] and the references therein. In particular the decomposition of \( S_3(S_d) \) is given by Thrall’s formula (see [33]), and then \( (I_X)_3 \) can be calculated in any specific case.

**Remark 2.4.** Note that \( (I_X)_2 = 0 \), i.e., the ideal has no quadratic generators. If \( n = 1 \), then \( I_X \) is generated in degree 3 and has regularity 3, so its minimal resolution is linear.

3. Examples

3.1. **Binary octavics.** We will write down a complete set of invariant theoretic conditions necessary and sufficient for a degree eight binary form to lie in \( X(1,8) \). By what we have proved, the ideal has all of its generators in degree 3. By Corollary 2.3 and Thrall’s formula,

\[
[(I_X)_3] = [S_{18} \oplus S_{14} \oplus S_{12} \oplus S_{10} \oplus S_8 \oplus S_6].
\]

Now (for instance) \( S_{12} \) corresponds to a covariant of binary octavics of degree 3 and order 12. (This formalism is explained in [11]).
module $S_3(S_8)$ contains two copies of $S_{12}$, so there is a two dimensional space of such covariants. An inspection reveals that

$$A = (F^2, F)_6, \quad B = ((F, F)_2, F)_4$$

can be taken as a basis for this space. (The fact that $A$ and $B$ have the right degree and order is clear from their definitions, so we only need to show that they are linearly independent for general $F$. This can be done by specializing $F$ to $x_0^6 x_1^2 + x_0 x_1^7$ and calculating directly.)

Hence the required covariant must be $c_1 A + c_2 B$ for some constants $c_i$. Now specialize $F$ to $x_0^4 x_1^4$, then by hypothesis the covariant vanishes. This gives a system of linear equations for the $c_i$, it has the solution $c_1 : c_2 = 13 : -63$. This determines the covariant (of course up to a scalar). By the same procedure, we can identify all the summands in (6) as follows:

$$(F^2, F)_3, \quad (F^2, F)_5, \quad 13 (F^2, F)_6 - 63 (F^2, F)_4,$$

$$(F^2, F)_7, \quad ((F, F)_6, F)_3, \quad 195 (F^2, F)_8 - 2744 ((F, F)_2, F)_6.$$  (7)

We conclude that a form $F \in \mathbb{P}^8$ belongs to $X_{(1,8)}$, iff all the covariants in (7) vanish on $F$.

**Remark 3.1.** If $n \geq 2$, then $X_{(n,2)}$ is the variety of quadrics of rank at most 2. It is a symmetric determinantal variety in the sense of [25], and its entire minimal resolution is deduced there. The ideal is generated in degree 3 by the piece

$$(I_X)_3 = S_{(2,2,2)} \subseteq S_3(S_2).$$

3.2. Ternary quartics. Assume $n = 2, d = 4$. By the main theorem, we know that the generators of $I_X$ lie in degrees $\leq 4$. We will find them using an elimination theoretic computation. Define

$$L_1 = a_0 x_0 + a_1 x_1 + a_2 x_2,$$

$$L_2 = b_0 x_0 + b_1 x_1 + b_2 x_2,$$

$$F = c_0 x_0^4 + c_1 x_0^3 x_1 + \cdots + c_{14} x_2^4;$$

where the $a, b, c$ are indeterminates. Write $F = (L_1 L_2)^2$ and then equate the coefficients of the monomials in $x_0, x_1, x_2$. This expresses each $c_i$ as a function of $a_0, \ldots, b_2$, and hence defines a ring map

$$\mathbb{C} [c_0, \ldots, c_{14}] \rightarrow \mathbb{C} [a_0, \ldots, b_2].$$

The kernel of this map is $I_X$. We calculated this in Macaulay-2, it turned out that in fact all the minimal generators are in degree 3. By
Corollary 2.3 and Thrall’s formula, 

\[(I_X)_3 = S_{(9,3)} \oplus S_{(6,0)} \oplus S_{(6,3)} \oplus S_{(4,2)} \oplus S_{(0,0)}\].

Now each summand corresponds to a concomitant of ternary quartics, e.g., \(S_{(9,3)}\) corresponds to one of degree 3, order 6 and class 3. It is not difficult to identify the concomitants symbolically (see [12] for the procedure), they are

\[
\alpha_x^2 \beta_x^2 \gamma_x (\alpha \gamma u)^2 (\beta \gamma u), \quad \alpha_x^2 \beta_x^2 \gamma_x^2 (\alpha \beta \gamma)^2,
\]
\[
\alpha_x^2 \beta_x (\beta \gamma u)^2 (\alpha \gamma u)(\alpha \beta \gamma), \quad \alpha_x \beta_x (\alpha \gamma u)(\beta \gamma u)(\alpha \beta \gamma)^2,
\]
\[(\alpha \beta \gamma)^4.\]

This is a rephrasing of the calculation in geometric terms:

**Theorem 3.2.** Let \(F\) be a ternary quartic with zero scheme \(C \subseteq \mathbb{P}^2\). Then \(C\) consists of two (possibly coincident) double lines iff all the concomitants in (8) vanish on \(F\).

4. Transvectants

In this section we will break down Proposition 2.2 into two separate questions about transvectants of binary forms. A general account of transvectants may be found in [20] and [30].

We begin by describing the map \(\alpha_r\) from (4) in coordinates. (It is as yet unnecessary to assume \(\dim V = 2\).) Let

\[
x^{(i)} = (x_0^{(i)}, \ldots, x_n^{(i)}), \quad 1 \leq i \leq r,
\]

be \(r\) sets of \(n + 1\) variables, with their ‘copies’

\[
y^{(i)} = (y_0^{(i)}, \ldots, y_n^{(i)}), \quad 1 \leq i \leq r.
\]

Let \(F_i(x^{(i)}), 1 \leq i \leq r\) be degree \(d\) forms, then the image \(\alpha_r(\bigotimes_{i=1}^r F_i)\) is calculated as follows:

- For each \(F_i\), apply the polarization operator

\[
\sum_{\ell=0}^n y^{(i)}_\ell \frac{\partial}{\partial x^{(i)}_\ell}
\]

altogether \(e\) times, denote the result by \(F_i(x^{(i)}, y^{(i)})\).
• Take the product $\prod_{i} F_i(x^{(i)}, y^{(i)})$, and make substitutions

$$x^{(i)}_{\ell} = x_\ell, \quad y^{(i)}_{\ell} = y_\ell,$$

for all $i, \ell$. (This is tantamount to ‘erasing’ the upper indices.)

This gives a form having degree $re$ each in $x, y$, which is the image of $\otimes F_i$ via $\alpha_r$. Since it is symmetric in the sets $x, y$, it can be thought of as an element of $S_2(S_{re})$.

Suppose now that $\dim V = 2$. We will show by induction on $r$ that $\pi_p \circ \alpha_r$ is not identically zero.

4.1. Case $r = 2$. We now specialize the $F_i$. Let $F_i = l_i^d$, where $l_i(x_0, x_1)$ are linear forms. Then $\alpha_r(\otimes F_i) = Q(x)^e Q(y)^e$, where $Q = \prod l_i$.

Introduce the Omega operator

$$\Omega = \frac{\partial^2}{\partial x_0 \partial y_1} - \frac{\partial^2}{\partial y_0 \partial x_1}.$$

The projection $\pi_p$ corresponds to applying $\Omega^{2p}$ and substituting $y := x$, which is the same as taking the $2p$-th transvectant of $Q^e$ with itself. Hence we are reduced to showing the following statement:

**Lemma 4.1.** If $Q$ is the generic binary quadratic, then

$$(Q^e, Q^e)_{2p} \neq 0, \quad \text{for } 0 \leq p \leq e.$$

**Proof.** See Section 5. □

4.2. The induction step. For the transition from $r$ to $r + 1$, consider the commutative diagram

$$\begin{array}{ccc}
S_r(S_d) \otimes S_d & \xrightarrow{\alpha_r \otimes 1} & H^0(\mathcal{O}_X(r)) \otimes S_d \\
\downarrow & & \downarrow u_r \\
S_{r+1}(S_d) & \xrightarrow{\alpha_{r+1}} & H^0(\mathcal{O}_X(r + 1))
\end{array}$$

Assume that $\alpha_r$ (and hence $\alpha_r \otimes 1$) is surjective. If we show that $u_r$ is surjective, then it will follow that $\alpha_{r+1}$ is surjective. We need to
understand the action of $u_r$ on the summands of the decomposition (5). The map

$$u_r^{(p,p')} : S_{rd-4p} \otimes S_d \rightarrow S_{(r+1)d-4p'}$$

is defined as the composite

$$S_{rd-4p} \otimes S_d \rightarrow (S_{re} \otimes S_{re}) \otimes S_d \rightarrow (S_{re} \otimes S_{re}) \otimes (S_e \otimes S_e) \rightarrow S_{(r+1)e} \otimes S_{(r+1)e} \rightarrow S_{(r+1)d-4p'}.$$

Let $C \in S_{rd-4p}, D \in S_d$. Following the component maps, we will get a recipe for calculating the image of $C \otimes D$ via $u_r^{(p,p')}$. Let

$$\Gamma_a = \sum_{i=0}^{re} \binom{re}{i} a_i x_0^{re-i} x_1^i, \quad \Gamma_b = \sum_{i=0}^{re} \binom{re}{i} b_i y_0^{re-i} y_1^i,$$

be two generic forms of degree $re$. (That is to say, the $a_i, b_i$ are thought of as independent indeterminates.)

- Let $T_1 = (\Gamma_a, \Gamma_b)_{2p}$ and $T_2 = (C, T_1)_{rd-4p}$. Then $T_2$ does not involve $x_i$.
- Obtain $T_3$ by making the substitutions

$$a_i = x_1^{re-i}(-x_0)^i, \quad b_i = y_1^{re-i}(-y_0)^i,$$

in $T_2$.
- Let

$$T_4 = (y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1})^e D,$$

and $T_5 = T_3 T_4$.
- Let $T_6 = \Omega^{2p'} T_5$. Finally $u_r^{(p,p')} (C \otimes D)$ is obtained by substituting $x_0, x_1$ for $y_0, y_1$ in $T_6$.

Hence it is enough to show the following:

For $p'$ in the range $0 \leq p' \leq \frac{(r+1)e}{2}$, there exists a $p$ such that $u_r^{(p,p')} (C \otimes D)$ is nonzero for some forms $C, D$ of degrees $rd - 4p, d$ respectively.

We translate this statement into the symbolic calculus of classical invariant theory (see [20]). Introduce symbolic letters $c, d$, and let $c_x$ stand for $c_0 x_0 + c_1 x_1$ etc. Write $\omega = x_0 y_1 - x_1 y_0$. Then the claim becomes

**Lemma 4.2.** Given $r \geq 2$ and $0 \leq p' \leq \frac{(r+1)e}{2}$, there exists a $p$ in the range $0 \leq p \leq \frac{re}{2}$, such that the expression

$$\{\Omega^{2p'}(\omega^{2p} c_x^{re-2p} c_y^{re-2p} d_x^e d_y^e)\}|_{y=x}$$
is nonzero.

**Proof.** See Section 5. □

At this point, modulo Lemmata 4.1 and 4.2, the proof of the main theorem is complete.

**Remark 4.3.** It would be unnecessary to make an inductive argument if we could prove the following statement:

Assume \( r \geq 2 \), and let \( R \) denote the generic binary form of degree \( r \). Then \( (R^e, R^e)_{2p} \neq 0 \) for \( 0 \leq p \leq \left\lfloor \frac{r}{2} \right\rfloor \).

However we do not see how to do this.

## 5. The combinatorics of Feynman diagrams

In this section we will complete the proof of the key Proposition 2.2 by proving Lemmata 4.1 and 4.2. Although there might be a simple geometric or representation theoretic argument allowing the derivation of these lemmas, we were unable to find one, and relied instead on explicit combinatorial computation. This is an instance of what one might call the combinatorics of invariants of binary forms which were at the heart of classical invariant theory. Although neglected for the last century, it is a fascinating subject with ramifications in many fields of current mathematical and physical interest like the theory of angular momentum [5, 6], classical hypergeometric series [21], the spin network approach to quantum gravity [32, 35], as well as knot and 3-manifold invariants [8].

As for lemma 4.1, we will prove that

\[
\Omega_{xy}^{2p} Q(x)^e Q(y)^e \bigg|_{y=x} = \mathcal{N}_{e,p}^t Q(x)^{2e-2p} (-\Delta)^p
\]

where \( \Delta \) is the discriminant of the quadratic form \( Q \) and \( \mathcal{N}_{e,p}^t \) is a strictly positive numerical constant. We will give two proofs of this result. The first is a combinatorially explicit calculation with Feynman diagrams which explains why \( \mathcal{N}_{e,p}^t > 0 \). The second is perhaps less transparent, but it allows the exact computation of the constant \( \mathcal{N}_{e,p}^t \). Apart from a harmless normalisation factor, it is a special value of Wigner’s 3-j-symbol (or Clebsch-Gordan coefficient, see [21]) which can be computed using Dixon’s summation theorem for the \( _3F_2 \) hypergeometric series. The comparison of both methods yields an interesting formula for the weighted enumeration of a class of bipartite
graphs which have vertex degree at most two. Note that Proposition 2.2 can also be considered as a statement concerning a sum over bipartite graphs with vertex degree bounded by \( r \), the degree of the binary form \( Q \). However a direct combinatorial approach seems very difficult at this point. The representation theoretic arguments involved in our inductive proof of Proposition 2.2 and its reduction to Lemmata 4.1 and 4.2 can be credited for taming a significant part of the combinatorial complexity of such sums over graphs.

The proof of Lemma 4.2 uses Feynman diagrammatic generating function techniques which are implicit in the work of J.Schwinger [36] and its reformulation by V. Bargmann [4]. This allows us to prove that

\[
\Omega_{x^y c^r d^e}^{2p' r^{e-p} c^{e-2p} d^{x^y}}\bigg|_{y=x} = \mathcal{N}_{r,e,p',p}^{II} \times (cd)^{p'-p} c^{2(Re-p'-p)} d^{2(e-p'+p)},
\]

(10)

where \( \mathcal{N}_{r,e,p',p}^{II} \) is a numerical constant that we compute explicitly. An easy and tempting shortcut at this point would have been to use analysis, akin to what Bargmann did in [4]. This would however obscure the fact that what is at play is purely combinatorial algebra, with no real need for transcendental methods.

5.1. First proof of Lemma 4.1. The following presentation is semiformal yet completely rigorous. The reader who needs a strictly formal exposition of Feynman diagrams and their rigorous mathematical use should consult [11] (see also [17]). For the present purposes, let us simply say that a Feynman diagram is essentially the combinatorial data needed to encode a complex tensorial expression built from a predefined collection of elementary tensors, exclusively using contraction of tensor indices. The word “tensor” here is used as meaning the multidimensional analog of a matrix which is therefore basis dependent. Coordinates are needed in order to state the necessary definitions, but are almost never actually used in the computations. Here the basic tensors are

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]

(11)
made of formal indeterminates, and the two matrices $Q$ and $\epsilon$ in $M_2(\mathbb{C})$. $Q$ is symmetric and gives the quadratic form $Q(x) = x^T Q x$,

$$
\epsilon \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

(12)

is antisymmetric and defines the symbolic brackets as well as Cayley’s Omega operator. We also need the vectors of differential operators

$$
\partial_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right), \quad \partial_y = \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right)
$$

(13)

We now introduce a graphical notation for the entries of these elementary tensors (indices belong to the set $\{1, 2\}$),

Now to any diagram obtained by assembling any number of these elementary pieces by gluing pairs of index-bearing lines, one associates an expression, called the amplitude of the diagram. For example

$$
Q_{\alpha\beta} \overset{\text{def}}{=} \sum_{\alpha, \beta = 1}^{2} x_\alpha Q_{\alpha\beta} x_\beta = Q(x)
$$

(14)

the quadratic form itself. also

$$
Q_{\alpha\beta} \overset{\text{def}}{=} \sum_{\alpha, \beta, \gamma, \delta = 1}^{2} Q_{\alpha\beta} \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} Q_{\gamma\delta}
$$

(15)

$$
= 2(Q_{11} Q_{22} - Q_{12} Q_{21})
$$

(16)

$$
= 2 \det(Q)
$$

(17)

Whenever we write a diagram inside an equation what is meant is the amplitude of the diagram. Now we will use the fact $Q_{\alpha\beta}$ has an inner structure. This is related to the notion of combinatorial plethysm (see \[24\]). Indeed, since $\mathbb{C}$ is algebraically closed, one can factor $Q$ as
$Q(x) = R_1(x)R_2(x)$ where

$$R_1 = \begin{pmatrix} R_{1,1} \\ R_{1,2} \end{pmatrix}, \quad R_2 = \begin{pmatrix} R_{2,1} \\ R_{2,2} \end{pmatrix}$$  \hfill (18)

in $\mathbb{C}^2$, are dual to the homogenous roots of $Q$. Now for any indices $\alpha$ and $\beta$

$$Q_{\alpha\beta} = \frac{1}{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} Q(x)$$  \hfill (19)

$$= \frac{1}{2} (R_{1,\alpha} R_{2,\beta} + R_{1,\beta} R_{2,\alpha})$$  \hfill (20)

which we write more suggestively as

$$Q_{\alpha \beta} = R_2 R_1 R_1 R_2$$  \hfill (21)

$$= \frac{1}{2} R_2 R_1 R_1 R_2 + \frac{1}{2} R_2 R_1 R_2 R_1$$  \hfill (22)

This implies, for instance that

$$Q \circ Q = \frac{1}{4} R_1 R_2 + \frac{1}{4} R_2 R_1$$  \hfill (23)

since reversing the direction of an $\epsilon$ arrow produces a minus sign, and therefore

$$R_1 R_1 = R_2 R_2 = 0$$  \hfill (24)

As a result

$$Q \circ Q = \frac{1}{2} \Delta$$  \hfill (25)

where

$$\Delta \overset{\text{def}}{=} \left( \begin{pmatrix} R_2 \\ R_1 \end{pmatrix} \right)^2$$  \hfill (26)

is the discriminant of $Q$.

Now the quantity we are interested in is

$$\Omega_{xy}^{2p} Q(x)^c Q(y)^c \bigg|_{y=x} = F(x, x)$$  \hfill (27)
where
\[ F(x, y) = \left( \begin{array}{c} Q \\uparrow \\ x \quad \downarrow \\ Q \end{array} \right)^{2p} \left( \begin{array}{c} Q \\uparrow \\ x \quad \downarrow \\ Q \end{array} \right)^e \left( \begin{array}{c} Q \\uparrow \\ y \quad \downarrow \\ Q \end{array} \right)^e \] (28)

Summing over where exactly the derivatives act, via Leibnitz’s rule, generates a sum over Feynman diagrams which, once we let \( y = x \), condenses into the following sum over vertex-labelled bipartite multi-

graphs.

\[ F(x, y) = \sum_G w_G A_G \] (29)

Here, the graph \( G \) can be seen as a matrix \((m_{ij})\) in \( \mathbb{N}^{L \times R} \) where \( L \) and \( R \) are fixed sets of cardinality \( e \) labelling the \( Q(x) \) and \( Q(y) \) factors in (28) respectively. \( w_G \) is a combinatorial weight and \( A_G \) is the amplitude of a Feynman diagram encoded by the graph \( G \). The latter has to satisfy the following conditions

\[ \sum_{i \in L, j \in R} m_{ij} = 2p \]

\[ \forall i \in L, \; l_i \overset{\text{def}}{=} \sum_{j \in R} m_{ij} \leq 2 \]

and

\[ \forall j \in R, \; c_j \overset{\text{def}}{=} \sum_{i \in L} m_{ij} \leq 2 \] (30)

The combinatorial weight is easily seen to be

\[ w_G = \frac{(2p)!2^{2e}}{\prod_{ij}(m_{ij})! \times \prod_i (2 - l_i)! \times \prod_j (2 - c_j)!} \] (31)

The amplitude \( A_G \) factors over the connected components of \( G \). These components are of four possible types: cycles containing an even number of \( \epsilon \) arrows of alternating direction, chains with both endpoints in \( L \), chains with both endpoints in \( R \) and finally chains with one endpoint in \( L \) and another in \( R \). However, the last type of connected component gives a zero contribution. Indeed, such a chain contains an odd number of \( \epsilon \) arrows and therefore its amplitude changes sign if one reverses all the orientations of the arrows, an operation which,
followed by a 180 degrees rotation, puts the chain back in its original form; because we have already set $y = x$. For example

\[
\begin{align*}
L & \xrightarrow{x} R \\
\begin{array}{c}
Q \\
Q \\
Q
\end{array} & = \begin{array}{c}
\begin{array}{c}
Q \\
Q \\
Q
\end{array}
\end{array} \\
\begin{array}{c}
Q \\
Q \\
Q
\end{array} & = - \begin{array}{c}
\begin{array}{c}
Q \\
Q \\
Q
\end{array}
\end{array} \\
\begin{array}{c}
Q \\
Q \\
Q
\end{array} & = - \begin{array}{c}
\begin{array}{c}
Q \\
Q \\
Q
\end{array}
\end{array} \\
\begin{array}{c}
Q \\
Q \\
Q
\end{array} & = 0
\end{align*}
\] (32)

Now to calculate the amplitudes of the other three kinds of components, one needs to use the inner structure of $Q$. Namely for each cycle of even length $2m$, incorporating the decomposition (22) at each $Q$ vertex, produces a sum of $2^{2m}$ terms all of which vanish except for two of them. Indeed, once one chooses the precise connections between the “inner” and “outer” part of what was a particular $Q$ vertex, the connections for the remaining vertices are forced, if one wants to avoid the appearance of the vanishing factors

\[
\begin{array}{c}
\begin{array}{c}
R_1 \xrightarrow{R_2}
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
R_2 \xrightarrow{R_1}
\end{array}
\end{array}
\]

Besides, the alternating pattern for the orientations of the $\epsilon$ arrows makes it so that we collect an equal number $m$ of

\[
\begin{array}{c}
\begin{array}{c}
R_1 \xrightarrow{R_2}
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
R_2 \xrightarrow{R_1}
\end{array}
\end{array}
\]

factors. As a result, the amplitude of the cycle is exactly $2^{1-2m}(-\Delta)^m$.

Likewise, a chain with both endpoints in $L$, or both endpoints in $R$, and with a necessarily even number $2m$ of $\epsilon$ arrows (and thus $2m + 1$ $Q$ vertices) gives as an amplitude

\[
\frac{2}{2^{2m+1}} \xrightarrow{x} R \left( \begin{array}{c}
\begin{array}{c}
R_2 \xrightarrow{R_1}
\end{array}
\end{array} \right)^m \xrightarrow{x}
\]
\[ = 2^{-2m}(-\Delta)^m Q(x) \quad (36) \]

Therefore an easy count shows that the amplitude of a bipartite graph \( G \) in (29) is
\[
A_G = 2^{C(G) - 2p} Q(x)^{2e - 2p} (-\Delta)^p \quad (37)
\]
where \( C(G) \) is the number of cycles in \( G \). Finally,
\[
F(x, x) = N_{e,p}^1 Q(x)^{2e - 2p} (-\Delta)^p \quad (38)
\]
where
\[
N_{e,p}^1 \overset{\text{def}}{=} \sum_G \frac{(2p)!2^{2e-2p+C(G)}}{\prod_i (2 - l_i)! \prod_j (2 - c_j)!} \quad (39)
\]
and the last sum is over all graphs \( G = (m_{ij}) \) satisfying the three constraints (30) and the additional condition of not having any connected component which is a chain starting in \( R \) and ending in \( L \). It is easy to see that given \( e \geq 1 \) and \( p, 0 \leq p \leq e \), there always exists such graphs \( G \), i.e. \( N_{e,p}^1 > 0 \) which proves Lemma 4.1.

5.2. Second proof of Lemma 4.1 We specialize the quadratic form \( Q(x) = x_1 x_2 \), for which \( \Delta = 1 \). We use
\[
\Omega_{xy}^{2p} = \sum_{i=0}^{2p} (-1)^i \left( \begin{array}{c} 2p \\ i \end{array} \right) \frac{\partial^{2p}}{\partial x_1^{2p-i} \partial x_2^i} \frac{\partial^{2p}}{\partial y_1^i \partial y_2^{2p-i}} \quad (40)
\]
to obtain
\[
\Omega_{xy}^{2p} Q(x) Q(y) \bigg|_{y=x} = \sum_{i=\max(0,2p-e)}^{\min(2p,e)} (-1)^i \left( \begin{array}{c} 2p \\ i \end{array} \right) \times 
\frac{e!^4}{(e-2p+i)!^2(e-i)!^2} \times x_1^{2e-2p} x_2^{2e-2p} \quad (41)
\]
\[
= N_{e,p}^1 Q(x)^{2e - 2p} (-\Delta)^p \quad (42)
\]
with
\[
N_{e,p}^1 = \sum_{i=\max(0,2p-e)}^{\min(2p,e)} (-1)^{p+i} \left( \begin{array}{c} 2p \\ i \end{array} \right) \frac{e!^4}{(e-2p+i)!^2(e-i)!^2} \quad (43)
\]
By \( GL_2(\mathbb{C}) \) change of coordinate and density of quadratic forms with nonzero discriminants, (42) is valid for any quadratic form \( Q \).
Let us suppose \( e \geq 2p \geq 0 \). It is then easy to rewrite \( \mathcal{N}_{e,p}^1 \), using Pochhammer’s symbol \((a)_n \overset{\text{def}}{=} a(a+1) \cdots (a+n-1)\), as

\[
\mathcal{N}_{e,p}^1 = \frac{(-1)^p e!^2}{(e-2p)!^2} \sum_{i \geq 0} \frac{(-2p)i(-e)_i(-e)_i}{i!(e-2p+1)_i(e-2p+1)_i}
\]

(44)

\[
= \frac{(-1)^p e!^2}{(e-2p)!^2} \left[ \begin{array}{c}
-2p, -e, -e \\
-2p+1, e-2p+1 ; 1
\end{array} \right]
\]

(45)

The terminating classical hypergeometric series that appears in the last formula, is of the form

\[
\left. \begin{array}{c}
3F_2 \\
1 + a - b, 1 + a - c ; 1
\end{array} \right| a, b, c
\]

and can therefore be evaluated thanks to Dixon’s summation theorem (see [37]):

\[
3F_2 \left[ \begin{array}{c}
1 + a - b, 1 + a - c ; 1
\end{array} \right] = \frac{\Gamma(1 + \frac{1}{2}a)\Gamma(1 + \frac{1}{2}a - b - c)\Gamma(1 + a - b)\Gamma(1 + a - c)}{\Gamma(1 + a)\Gamma(1 + a - b - c)\Gamma(1 + 1\frac{1}{2}a - b)\Gamma(1 + 1\frac{1}{2}a - c)}
\]

(46)

which is valid in the domain of analyticity \( \Re(1 + \frac{1}{2}a - b - c) > 0 \). Here we want to take \( a = -2p \) and \( b = c = -e \); one therefore has to be careful with the \( \frac{\Gamma(1 + \frac{1}{2}a)}{\Gamma(1 + a)} \) factor and rewrite it as

\[
\frac{\pi}{\Gamma(-\frac{a}{2})\sin(-\frac{\pi a}{2})} \times \frac{\Gamma(-a)\Gamma(-a + 1)}{\pi} = \cos\left(\frac{\pi a}{2}\right)\frac{\Gamma(-a + 1)}{\Gamma(-\frac{a}{2} + 1)}
\]

(47)

The end result is

\[
\mathcal{N}_{e,p}^1 = \frac{(2p)!(2e - p)!e!^2}{p!(2e - 2p)!(e - p)!^2}
\]

(48)

if \( 0 \leq p \leq \frac{e}{2} \).

Now if \( \frac{e}{2} \leq p \leq e \), then by setting \( i = 2p - e + j \) in (43) and again writing the resulting sum over \( j \) as a terminating hypergeometric series one gets

\[
\mathcal{N}_{e,p}^1 = \frac{(-1)^{p+e}(2p)!e!^3}{(2p - e)!(2e - 2p)!^2} \left[ \begin{array}{c}
-2e + 2p, -2e + 2p, -e \\
1, 2p - e + 1
\end{array} \right]
\]

(49)

The same method using Dixon’s theorem gives

\[
\mathcal{N}_{e,p}^1 = \frac{(2p)!(3e - 3p)!e!^3}{(2p - e)!(2e - 2p)!^2(e - p)!^3}
\]

(50)
if $\frac{r}{e} \leq p \leq e$.

This again shows that, in either case, $N_{e,p}^1 > 0$ and completes our second proof of Lemma 4.1. This also gives a closed form evaluation of the sum in (39).

### 5.3. Proof of Lemma 4.2

Let $r$, $e$, $p'$ and $p$ be integers satisfying $r \geq 2$, $e \geq 1$, $0 \leq 2p' \leq (r + 1)e$ and $0 \leq 2p \leq re$. Let

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

be two elements of $\mathbb{C}^2$ and

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

be two vectors of indeterminates. The quantity we would like to compute is

$$G(x) \overset{\text{def}}{=} \Omega_{2y}^2 (xy)^{2p} c_x c_y^{re - 2p} d_x^e d_y^e \bigg|_{y=x}$$

or, in matrix notation,

$$G(x) = \left[ \partial_x^T \epsilon \partial_y \right]^{2p'} (x^T e y)^{2p} (x^T c c^T y)^{re - 2p} (x^T d d^T y)^e \bigg|_{y=x}$$

We now introduce two new vectors of indeterminates

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \overline{\phi} = \begin{pmatrix} \overline{\phi}_1 \\ \overline{\phi}_2 \end{pmatrix}$$

It is then easy to see that

$$G(x) = (2p')!(2p)!(re - 2p)! e! \left[ \frac{\partial_x^T \epsilon \partial_y}{{(2p')!} (2p)!} \left[ (\phi + x)^T e (\overline{\phi} + x) \right]^{2p} \right]$$

$$\times \left[ (\overline{\phi} + x)^T c c^T (\phi + x) \right]^{re - 2p} \bigg|_{x=x} \times \left[ (\overline{\phi} + x)^T d d^T (\phi + x) \right]^{e} \bigg|_{y=x}$$

i.e.

$$G(x) = (2p')!(2p)!(re - 2p)! e! [h^{2p'} u^{2p} v^{re - 2p} w^e] \mathcal{Z}$$

where $[h^{2p'} u^{2p} v^{re - 2p} w^e] \mathcal{Z} \in \mathbb{C}[[x_1, x_2]]$ denotes the coefficient of the monomial $h^{2p'} u^{2p} v^{re - 2p} w^e$ in $\mathcal{Z} \in \mathbb{C}[[x_1, x_2, h, u, v, w]]$, the generating function defined by

$$\mathcal{Z} \overset{\text{def}}{=} \sum_{n \geq 0} \frac{h^n}{n!} \left[ \partial_x^T \epsilon \partial_y \right]^{2n} e^S \bigg|_{x=0, \phi=0}$$
where $S \in \mathbb{C}[[\phi_1, \phi_2, \phi_1, \phi_2, x_1, x_2, h, u, v, w]]$ is given by

$$S \overset{\text{def}}{=} (\phi + x)^T(-u\epsilon + M)(\phi + x)$$

(59)

and

$$M \overset{\text{def}}{=} vcc^T + wdd^T \in M_2(\mathbb{C}[[v, w]]).$$

(60)

Note that there is no problem of convergence since we work over rings of formal power series with their usual topology. With obvious notations, one can rewrite $Z$ as

$$Z = \exp(h\partial^T_{\phi} \epsilon \partial_{\phi}) \exp(\phi^T A \phi + J^T \phi + \phi^T K + S_0) \bigg|_{\phi=0} \bigg|_{\phi=0}$$

(61)

with $A \overset{\text{def}}{=} -u\epsilon + M$, $J^T \overset{\text{def}}{=} -ux^T \epsilon + x^T M$, $K = -u\epsilon x + Mx$ and $S_0 \overset{\text{def}}{=} v(x^T c)^2 + w(x^T d)^2$. Therefore $Z = e^{S_0} \tilde{Z}$ with

$$\tilde{Z} \overset{\text{def}}{=} \exp(h\partial^T_{\phi} \epsilon \partial_{\phi}) \exp(\phi^T A \phi + J^T \phi + \phi^T K) \bigg|_{\phi=0} \bigg|_{\phi=0}$$

(62)

$\tilde{Z}$ can now be expressed as a sum over Feynman diagrams built, like in Section 5.1, from the following pieces

by plugging the $\partial_{\phi}$'s onto the $\phi$'s and the $\partial_{\phi}$ onto the $\phi$'s, in all possible ways. More precisely, given any finite set $E$, we define a Feynman diagram on $E$ as any sextuple $F = (E_{\phi}, E_{\phi}, \pi_A, \pi_J, \pi_K, C)$ where $E_{\phi}$, $E_{\phi}$ are subsets of $E$, and $\pi_A$, $\pi_J$, $\pi_K$ are, each, sets of subsets of $E$ and $C$ is a map $E_{\phi} \to E_{\phi}$, satisfying the following axioms

- $E_{\phi}$ and $E_{\phi}$ have equal cardinality and they form a two set partition of $E$.
- The union of the elements in $\pi_A$, that of the elements of $\pi_J$, and likewise for $\pi_K$ form a three set partition of $E$.
- $C$ is bijective.
- Every element of $\pi_A$ has two elements, one in $E_{\phi}$ and one in $E_{\phi}$.
- Every element of $\pi_J$ has only one element which is in $E_{\phi}$.
- Every element of $\pi_K$ has only one element which is in $E_{\phi}$.

The set of Feynman diagrams on $E$ is denoted by $\text{Fey}(E)$. The point is that the association $E \to \text{Fey}(E)$ defines an endofunctor of the groupoid category of finite sets with bijections $\mathbb{24}$ $\mathbb{11}$ $\mathbb{17}$, since, given
a Feynman diagram $\mathcal{F}$ on $E$ and a bijective map $\sigma : E \to E'$, there is a natural way to transport $\mathcal{F}$ along $\sigma$ to obtain a Feynman diagram $\mathcal{F}' = \text{Fey}(\sigma)(\mathcal{F})$ on $E'$. Let us take for example $E = \{1, 2, \ldots, 8\}$, $E_\phi = \{1, 2, 3, 4\}$, $E_\sigma = \{5, 6, 7, 8\}$, $\pi_A = \{\{2, 6\}, \{3, 7\}, \{4, 8\}\}$, $\pi_J = \{\{1\}\}$, $\pi_K = \{\{5\}\}$, and $C$ given by $C(5) = 1$, $C(6) = 3$, $C(7) = 4$ and $C(8) = 2$. This corresponds to the diagram

![Diagram](image)

where we put the elements of $E$ next to the corresponding half-line. The amplitude of such a pair $(E, \mathcal{F})$ is in this example

$$A(E, \mathcal{F}) = (J^T(h\epsilon)K) \times tr([h\epsilon A]^3)$$  \hspace{1cm} (63)

Note that there is a natural equivalence relation between pairs of finite sets equipped with a Feynman diagram. It is given by letting $(E, \mathcal{F}) \sim (E', \mathcal{F}')$ if and only if there exists a bijection $\sigma : E \to E'$ such that $\mathcal{F}' = \text{Fey}(\sigma)(\mathcal{F})$. One also has the notion of automorphism group $\text{Aut}(E, \mathcal{F})$ of a pair $(E, \mathcal{F})$ which is the set of bijections $\sigma : E \to E$ such that $\text{Fey}(\sigma)(\mathcal{F}) = \mathcal{F}$. Now one can check that

$$\tilde{Z} = \sum_{[E, \mathcal{F}]} \frac{A(E, \mathcal{F})}{|\text{Aut}(E, \mathcal{F})|}$$  \hspace{1cm} (64)

where the sum is over equivalence classes of pairs $(E, \mathcal{F})$, $A(E, \mathcal{F})$ is the amplitude, and $|\text{Aut}(E, \mathcal{F})|$ the cardinality of the automorphism group of any representative in the class $[E, \mathcal{F}]$. We leave it to the reader to check (otherwise see [11, 17]) that

$$\log \tilde{Z} = \sum_{[E, \mathcal{F}] \text{ connected}} \frac{A(E, \mathcal{F})}{|\text{Aut}(E, \mathcal{F})|}$$  \hspace{1cm} (65)
A REGULARITY RESULT FOR A LOCUS OF BRILL TYPE

\[ \sum_{n \geq 1} \frac{1}{n} tr((h \epsilon A)^n) + \sum_{n \geq 0} J^T(h \epsilon A)^n(h \epsilon)K \]

(66)
since the only connected diagrams are pure \( A \) cycles or \( A \) chains joining a \( J \) to a \( K \) vertex. As a result

\[ \tilde{Z} = \frac{1}{\det(I - h \epsilon A)} \exp(J^T(I - h \epsilon A)^{-1}(h \epsilon)K) \]

(67)
After straightforward but tedious computations with 2 by 2 matrices, which we spare the reader, this implies

\[ Z = \frac{1}{(1 - hu)^2 + h^2 vw(c^t ed)^2} \exp\left(\frac{v(x^T c)^2 + w(x^T d)^2}{(1 - hu)^2 + h^2 vw(c^t ed)^2}\right) \]

(68)
or in classical notation

\[ Z = \frac{1}{(1 - hu)^2 + h^2 vw(cd)^2} \exp\left(\frac{vc_x^2 + wd_x^2}{(1 - hu)^2 + h^2 vw(cd)^2}\right) \]

(69)
We now expand

\[ Z = \sum_{\mu \geq 0} \frac{1}{\mu!} (vc_x^2 + wd_x^2)^\mu \left(1 - 2hu + h^2 u^2 + h^2 vw(cd)^2\right)^{-(\mu + 1)} \]

(70)
\[ = \sum_{\mu, \nu \geq 0} \frac{(-1)^\nu (\mu + \nu)!}{\mu!^2 \nu!} (vc_x^2 + wd_x^2)^\mu (-2hu + h^2 u^2 + h^2 vw(cd)^2)^\nu \]

(71)
\[ = \sum_{m, n, \alpha, \beta, \gamma \geq 0} \frac{(-1)^{\alpha + \beta + \gamma} (m + n + \alpha + \beta + \gamma)!}{(m + n)!m!n!\alpha!\beta!\gamma!} \times \]

\[ (vc_x^2)^m (wd_x^2)^n (-2hu)^\alpha (h^2 u^2)^\beta (h^2 vw(cd)^2)^\gamma \]

(72)
\[ = \sum_{m, n, \alpha, \beta, \gamma \geq 0} \frac{(-1)^{\beta + \gamma} (m + n + \alpha + \beta + \gamma)!2^\alpha}{(m + n)!m!n!\alpha!\beta!\gamma!} \times \]

\[ h^{\alpha + 2\beta + \gamma} u^{\alpha + 2\beta} v^{m + \gamma} w^{n + \gamma} \times \]

(73)
\[ c_{x}^{2m} d_{x}^{2n} (cd)^{2\gamma} \]

(74)
The coefficient of \( h^{2p'} u^{2p} v^{e-2p} w^e \) is a sum over the single index \( \beta \), \( 0 \leq \beta \leq p \), as a result of solving for \( \alpha = 2p - 2\beta \), \( \gamma = p' - p \), \( m = re - p' - p \), \( n = e - p' + p \). Therefore

\[ G(x) = \Lambda_{r,e,p',p''}^{\Pi} c_{x}^{2(re-p'-p)} d_{x}^{2(e-p'+p)} (cd)^{2(p'-p)} \]

(75)
where

\[
\mathcal{N}_{r,e,p',p}^{II} = \mathbb{1}_{\left\{ p' - p \geq 0, e - p' + p \geq 0 \right\}} \times (-1)^{p' - p}(2p'!(2p')!e! \times J_{s,p}) (77)
\]

\[
(p' - p)!(e - p' + p)!(re - p' - p)!((r + 1)e - 2p'!)\times J_{s,p} (78)
\]

where \( \mathbb{1}_{\{\cdots\}} \) denotes the characteristic function of the condition between braces and

\[
J_{s,p} \overset{\text{def}}{=} \sum_{\beta = 0}^{p} \frac{(-1)^{\beta}2^{2p-2\beta}(s + 2p - \beta)!}{(2p - 2\beta)!\beta!} (79)
\]

with \( s = (r + 1)e - p' - p \geq e \) when the characteristic function is nonzero. Note that \( J_{s,p} \) can be rewritten as a Gauss hypergeometric series and can be summed by the Chu-Vandermonde theorem (see [37] for instance)

\[
J_{s,p} = \frac{(s + p)!(s + \frac{3}{2})_{p}}{p!(\frac{1}{2})_{p}} (80)
\]

As a result, the characteristic function alone dictates whether \( \mathcal{N}_{r,e,p',p}^{II} \) vanishes or not. Now for \( r \geq 2, e \geq 1 \) and \( 0 \leq p' \leq \frac{(r+1)e}{2} \) it is easy to see that one can always find an integer \( p \) with \( 0 \leq p \leq \frac{re}{2}, p' - p \geq 0, e - p' + p \geq 0 \) and \( re - p' - p \geq 0 \). Indeed take \( p = p' \) if \( 0 \leq p' \leq \frac{re}{2} \), and otherwise take \( p = p' - e \) if \( \frac{re}{2} \leq p' \leq \frac{(r+1)e}{2} \). This completes the proof of Lemma 4.2.

6. A note on terminology and history

The simultaneous relevance to our approach of the literature from many fields of mathematics and physics requires the following *mise au point*. Firstly, our choice of terminology was based on a simple majority rule: we adopted the denomination, “Feynman diagrams”, of the largest community, that of theoretical physics, which uses the corresponding concept. Secondly, the historical roots of this notion, especially in the context of invariant theory, go much further back in time than Feynman’s work. There is a good account of the history of the diagrammatic notation in physics and group theory in Chapter 4 of [15] to which we refer the reader. This needs, however, to be complemented by the following pieces of information.

Feynman diagrams, as known to physicists, seem to have first appeared *in print* in [16], with due credit to the previously unpublished
work of R. P. Feynman. However, the idea of using discrete combinatorial structures (e.g., graphs) in order to describe the outcome of repeated applications of differential operators with polynomial coefficients (called “operandators” most probably by Sylvester) such as the polarization, the Omega, and the Aronhold processes of invariant theory goes back to A. Cayley [9]. Our diagrammatic approach is a presentation, guided by modern physical wisdom, of the original work of Sylvester [39] and Clifford [14] (see also [26]). It is remarkable that Clifford used what would now be called Fermionic or Berezin integration to explain the translation from graphs to actual covariants. The diagrams we used are a direct visualization of the classical symbolic notation: arrows correspond to bracket factors, and each vertex corresponds to a symbolic letter to be repeated a number of times equal to the degree of the vertex. There is however one extremely powerful extra feature of the 19th century methods, in comparison to the physicists’ “diagrammar”. It is the realization, by Aronhold and Clebsch [2, 13], that one can do all the calculations while pretending that the ground forms are powers of linear forms. This is the main obstacle lying before a modern who would like to understand the classics. Nevertheless, this obstacle can be easily overcome, for instance by using the umbral methods developed by Rota and his school [27]. Here is another way to rigorously justify this simplification, which we believe offers more flexibility, like for instance the possibility of iteration, that of mixed interpretation of some variables as true and others as symbolic within the same computation, as well as that of using, in intermediate steps, the same symbolic letter a number of times exceeding the degree of the form. It goes as follows: do all the needed calculations as stated with symbolic letters $a$, $b$, $c$... considered as formal indeterminates, and afterwards, act on the resulting expression with the product

$$Q\left(\frac{\partial}{\partial a}\right)Q\left(\frac{\partial}{\partial b}\right)Q\left(\frac{\partial}{\partial c}\right)\ldots$$

where $Q$ is the ground form under consideration. This will insert the coefficients of the form at the right place in the symbolical expression, i.e., at the right vertex of the diagram.

Diagrams, in the context of classical invariant theory, were reintroduced in the work of Olver and Shakiban [31] which is a slightly different formalism because of a normal ordering procedure explained in
Chapter 6 of [30]. Finally, although somewhat atypical, the interesting pedagogical work of computer graphics pioneer J. F. Blinn [7], who was inspired by the book [38], deserves to be mentioned.

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References

[1] A. Abdesselam.
Feynman diagrams in algebraic combinatorics.
Sém. Lothar. Combin., vol. 49, Article B49c, 2003.

[2] S. Aronhold.
Theorie der homogenen Functionen dritten Grades von drei Veränderlichen.
J. Reine Angew. Math., vol. 55, pp. 97–191, 1858.

[3] K. Akin, D. Buchsbaum, and J. Weyman.
Schur functors and Schur complexes.
Adv. in Math., vol. 44, pp. 207–278, 1982.

[4] V. Bargmann.
On the representations of the rotation group.
Rev. Modern Phys., vol. 34, pp. 829–845, 1962.

[5] L. C. Biedenharn, and J. D. Louck.
Angular Momentum in Quantum Physics.
Encyclopedia of Mathematics and its Applications, 8.
Addison-Wesley Publishing Co., Reading, Mass. , 1981.

[6] L. C. Biedenharn, and J. D. Louck.
The Racah-Wigner Algebra in Quantum Theory.
Encyclopedia of Mathematics and its Applications, 9.
Addison-Wesley Publishing Co., Reading, Mass. , 1981.

[7] J. F. Blinn.
Quartic discriminants and tensor invariants.
IEEE Computer Graphics and Applications, vol. 22 (2), pp.86–91, 2002.

[8] J. S. Carter, D. E. Flath, and S. Masahico.
The Classical and Quantum 6j-symbols.
Mathematical Notes, 43. Princeton University Press, 1995.

[9] A. Cayley.
On the theory of the analytical forms called trees.
Philos. Mag., vol. 13, pp. 19-30, 1857.

[10] Y. Chen, A. Garsia, and J. Remmel.
A REGULARITY RESULT FOR A LOCUS OF BRILL TYPE

Algorithms for plethysm.
In Combinatorics and Algebra, volume 34 of Contemp. Math., pages pp. 109–153. Amer. Math. Soc., 1984.

[11] J. Chipalkatti.
On equations defining coincident root loci.
J. Algebra, vol. 267, no. 1, pp. 246–271, 2003.

[12] J. Chipalkatti.
Decomposable ternary cubics.
Exper. Math., vol. 11, no. 1, pp. 69–80, 2002.

[13] A. Clebsch.
Ueber symbolische Darstellung algebraischer Formen.
J. Reine Angew. Math., vol. 59, pp. 1–62, 1861.

[14] W. Clifford.
Extract of a letter to Mr. Sylvester from Prof. Clifford of University College, London.
Amer. J. Math., vol. 1, pp. 126–128, 1878.

[15] P. Cvitanović.
Group Theory.
“webbook” available at http://www.nbi.dk/GroupTheory, 2002.

[16] F. J. Dyson.
The radiation theories of Tomonaga, Schwinger, and Feynman.
Phys. Rev., vol. 75, pp. 486–502, 1949.

[17] D. Fiorenza.
Sums over graphs and integration over discrete groupoids.
math.CT/0211389, preprint, 2002.

[18] W. Fulton and J. Harris.
Representation Theory, A First Course.
Graduate Texts in Mathematics. Springer–Verlag, New York, 1991.

[19] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky.
Discriminants, Resultants and Multidimensional Determinants.
Birkhäuser, Boston., 1994.

[20] J. H. Grace and A. Young.
The Algebra of Invariants, 1903.
Reprinted by Chelsea Publishing Co., New York, 1965.

[21] R. A. Gustafson.
Invariant theory and special functions.
In Invariant theory, volume 88 of Contemp. Math., pp. 125–144. Amer. Math. Soc., 1989.

[22] J. Harris.
Algebraic Geometry, A First Course.
Graduate Texts in Mathematics. Springer–Verlag, New York, 1992.

[23] R. Hartshorne.
Algebraic Geometry.
Graduate Texts in Mathematics. Springer–Verlag, New York, 1977.

[24] A. Joyal.
Une théorie combinatoire des séries formelles.
*Adv. in Math.*, vol. 42, pp. 1–82, 1981.

[25] T. Józefiak, P. Pragacz, and J. Weyman.
Resolutions of determinantal varieties.
In *Young Tableaux and Schur functors in algebra and geometry*,
Astérisque vol. 87-88, 1980.

[26] A. B. Kempe.
On the application of Clifford’s graphs to ordinary
binary quantics.
*Proc. London Math. Soc.*, vol. 17, pp. 107–121, 1885.

[27] J. P. S. Kung, and G.-C. Rota.
The invariant theory of binary forms.
*Bull. Amer. Math. Soc.*, vol. 10, pp. 27–85, 1984.

[28] I. G. MacDonald.
*Symmetric Functions and Hall Polynomials*.
Oxford University Press, 2nd edition, 1995.

[29] D. Mumford.
*Lectures on Curves on an Algebraic Surface*.
Ann. of Math. Studies, No 59. Princeton University Press, 1966.

[30] P. J. Olver.
*Classical Invariant Theory*.
London Mathematical Society Student Texts. Cambridge University Press, 1999.

[31] P. J. Olver, and C. Shakiban.
Graph theory and classical invariant theory.
*Adv. in Math.*, vol. 75, pp. 212–245, 1989.

[32] R. Penrose.
Angular momentum; an approach to combinatorial space time.
In *Quantum theory and beyond*, ed. T. Bastin.
Cambridge University Press, 1971.

[33] S. P. O. Plunkett.
On the plethysm of S-functions.
*Canad. J. of Math.*, vol. 24, pp. 541–552, 1972.

[34] O. Porras.
Rank varieties and their resolutions.
*J. of Algebra*, vol. 186, No. 3, pp. 677–723, 1996.

[35] C. Rovelli, and L. Smolin.
Spin networks and quantum gravity.
*Phys. Rev. D*, vol. 52, pp. 5743–5759, 1995.

[36] J. Schwinger.
On angular momentum.
U. S. Atomic Energy Comm. NYO-3071, 1952.
Published in Quantum Theory of Angular Momentum (Compiled by L. C. Biedenharn and H. van Dam), pp. 229–279, Academic Press, New York, 1965.

[37] L. J. Slater.
Generalized Hypergeometric Functions.
Cambridge University Press, Cambridge, 1966.

[38] G. E. Stedman.
A Diagram Technique in Group Theory.
Cambridge University Press, Cambridge, 1990.

[39] J. J. Sylvester.
On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, with three appendices.
Amer. J. Math., vol. 1, pp. 64–125, 1878.

Abdelmalek Abdesselam
Department of Mathematics
University of British Columbia,
1984 Mathematics Road,
Vancouver, B. C. , V6T 1Z2, CANADA.

and
LAGA, Institut Galilée, CNRS UMR 7539,
Université Paris XIII,
99 Avenue J.B. Clément
F93430 Villetaneuse, FRANCE.

abdessel@math.ubc.ca
JAYDEEP CHIPALKATTI
Department of Mathematics
University of British Columbia,
1984 Mathematics Road,
Vancouver, B.C., V6T 1Z2, CANADA.

jaydeep@math.ubc.ca