Abstract. Noncommutative spectral geometry offers a purely geometric explanation for the standard model of strong and electroweak interactions, including a geometric explanation for the origin of the Higgs field. Within this framework, the gravitational, the electroweak and the strong forces are all described as purely gravitational forces on a unified noncommutative space-time. In this study, we infer a constraint on one of the three free parameters of the model, namely the one characterising the coupling constants at unification, by linearising the field equations in the limit of weak gravitational fields generated by a rotating gravitational source, and by making use of recent experimental data. In particular, using data obtained by Gravity Probe B, we set a lower bound on the Weyl term appearing in the noncommutative spectral action, namely $\beta \gtrsim 10^{-6} \text{m}^{-1}$. This constraint becomes stronger once we use results from torsion balance experiments, leading to $\beta \gtrsim 10^4 \text{m}^{-1}$. The latter is much stronger than any constraint imposed so far to curvature squared terms.
1 Introduction: Elements of Noncommutative Spectral Geometry

One of the main quests in theoretical particle physics is the unification of all interactions, including gravity. While at low energy scales one can consider an effective theory with physics being described by the sum of the Einstein-Hilbert action — based on diffeomorphism invariance — and the Standard Model (SM) action — based upon internal symmetries of a gauge group — this is no longer valid at high energy scales. As one approaches the Planck energy scale, the quantum nature of space-time reveals itself and the correct formulation of geometry should be within a quantum framework. In constructing a quantum theory of gravity coupled to matter, we will adopt the philosophy that the interaction between gravity and matter is the most important aspect of the whole dynamics. One may speculate that at very high energy scales quantum gravity will enforce a wildly noncommutative space-time, while at intermediate scales — close but below the Planck scale — the algebra of coordinates may be assumed as moderately noncommutative, and if appropriately chosen it can lead to a purely geometric explanation of the SM coupled to gravity [1].

NonCommutative Spectral Geometry (NCSG) [2, 3] proposes that the SM fields and gravity are packaged into geometry and matter on a Kaluza-Klein noncommutative space. Its main goal is to unfold the small-scale structure of space-time from our knowledge at the electroweak scale; in that sense NCSG follows a bottom-up approach complementary to the top-down approach of string theory. The Standard Model of strong and electroweak interactions is considered, within the NCSG framework, as a phenomenological model which will dictate the structure of space-time. According to this proposal, a few orders of magnitude below the Planck energy scale, geometry is composed by a two-sheeted space, made from the product of a four-dimensional compact Riemannian manifold $M$ (with a fixed spin structure) — describing the geometry of space-time — and a discrete noncommutative space $F$ — describing the internal space of the particle physics model. Hence, gravity and the SM fields are put together into matter and geometry on a noncommutative space made from the product $M \times F$. Such a product space, seen as a four-dimensional internal Kaluza-Klein space attached to each point with the fifth dimension being a discrete zero-dimensional space, leads to an almost commutative geometry. Its physical interpretation is that left- and right-handed fermions are placed on two different sheets with the Higgs fields being the gauge fields in the discrete dimensions; the Higgs can be seen as the difference (thickness) between the two sheets.

The choice of a two-sheet geometry — an almost commutative manifold — has a deep physical meaning. As it has been highlighted in Ref. [4], this structure is essential in order to accommodate the gauge symmetries of the SM, while in addition it incorporates the seeds...
of quantisation [4]. More recently, it has been also shown [5] that this structure can account for neutrino mixing [1, 6].

The noncommutative nature of $F$ is encoded in the spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$. The algebra $\mathcal{A}_F = C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}_F$ of Euclidean fermions; it plays the rôle of the algebra of coordinates. The operator $D_F$ is the Dirac operator $\partial_M = \sqrt{-\gamma^s \nabla_\mu}$ on the spin manifold $\mathcal{M}$; it corresponds to the inverse of the Euclidean propagator of fermions and is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi–Maskawa mixing parameters. The operator $D_F$ is such that $JD_F = \epsilon' D_F J$, where $J$ is an anti-linear isometry of the finite dimensional Hilbert space with $J^2 = \epsilon$, $J\gamma = \epsilon'' \gamma J$;

$\gamma$ is the chirality operator and $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$. The internal space $F$ has dimension 6 to allow fermions to be simultaneously Weyl and chiral, while it is discrete to avoid the infinite tower of massive particles that are produced in string theory.

To get the SM, the choice of the algebra should be such that it can account for massive neutrinos and neutrino oscillations — thus it cannot be left-right symmetric — while non-commutative geometry imposes constraints on the algebras of operators in Hilbert space; in addition one should avoid fermion doubling. These considerations lead to the algebra [7]

$$\mathcal{A}_F = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) ,$$

with $k = 2a$; $\mathbb{H}$ is the algebra of quaternions, which encodes the noncommutativity of the manifold. The first possible value for $k$ is 2, corresponding to a Hilbert space of four fermions. This choice is however ruled out from the existence of quarks. The next possible value is $k = 4$ leading to the correct number of $k^2 = 16$ fermions in each of the three generations. Note that the number of generations is a physical input in the theory. Let us emphasise that the choice of $\mathcal{A}_F$ is the underlying input which determines the physical implications of the model, in particular the particle content of the theory. In Ref. [1] it has been chosen so that it leads to the Standard Model of particle physics.

The spectral geometry in the product $\mathcal{M} \times F$ is given by the product rules:

$$\mathcal{A} = C^\infty(\mathcal{M}) \oplus \mathcal{A}_F ,$$

$$\mathcal{H} = L^2(\mathcal{M}, S) \oplus \mathcal{H}_F ,$$

$$D = D_M \oplus 1 + \gamma_5 \oplus D_F ,$$

where $L^2(\mathcal{M}, S)$ is the Hilbert space of $L^2$ spinors and $D_M$ is the Dirac operator of the Levi-Civita spin connection on $\mathcal{M}$.

To obtain the NCSG action one applies the spectral action principle to the product geometry $\mathcal{M} \times F$. The bare bosonic Euclidean action is

$$\text{Tr}(f(D_A/\Lambda)) ,$$

where $D_A = D + A + \epsilon' JAJ^{-1}$ (with $A$ a self-adjoint operator $A = A^*$ of the form $A = \sum_j a_j[D, b_j]$, $a_j, b_j \in \mathcal{A}$) are uni-modular inner fluctuations, $f$ is a cutoff function and $\Lambda$ fixes the energy scale; we include the eigenvalues of the Dirac operator that are smaller than the
cutoff scale $\Lambda$. This action can be seen à la Wilson as the bare action at the mass scale $\Lambda$. To obtain the full action functional one has to include the fermionic part
\[
\frac{1}{2} \langle J\psi, D\psi \rangle ,
\]
where $J$ is the real structure on the spectral triple and $\psi$ is a spinor in the Hilbert space $\mathcal{H}$ of the quarks and leptons.

We will concentrate on the bosonic part of the action. Using heat kernel methods, the trace $\text{Tr}(f(D_A/\Lambda))$ can be written in terms of the geometrical Seeley-De Witt coefficients $a_n$ — known for any second order elliptic differential operator — as $\sum_{n=0}^{\infty} F_{4-n} \Lambda^{4-n} a_n$; the function $F$ is defined such that $F(D^2_A) = f(D_A)$. Hence, the bosonic part of the spectral action expanded in powers of $\Lambda$ reads [8]
\[
\text{Tr}(f(D_A/\Lambda)) \sim \sum_{k \in \text{DimSp}} f_k \Lambda^k \int |D_A|^{-k} + f(0) \zeta_{D_A(0)} + \mathcal{O}(1) ,
\]
with $f_k$ the momenta of the smooth even test (cutoff) function which decays fast at infinity:
\[
\begin{align*}
f_0 &\equiv f(0) , \\
f_k &\equiv \int_0^\infty f(u) u^{k-1} du , \quad \text{for } k > 0 , \\
f_{-2k} &= (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0) .
\end{align*}
\]
In Eq. (1.4) above, the noncommutative integration is defined in terms of residues of zeta functions $\zeta_{D_A}(s) = \text{Tr}(|D_A|^{-s})$ at poles of the zeta function and the sum is over points in the dimension spectrum of the spectral triple.

For a four-dimensional Riemannian geometry, the $\text{Tr}(f(D_A/\Lambda))$ can be expressed perturbatively as [9, 10]
\[
\text{Tr}(f(D_A/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots .
\]
Since the Taylor expansion of the $f$ function vanishes at zero, the asymptotic expansion of the spectral action, in terms of the geometrical Seeley-De Witt coefficients $a_n$, reduces to
\[
\text{Tr}(f(D_A/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 .
\]
Hence, the cutoff function $f$ plays a rôle only through its momenta $f_0, f_2, f_4$, three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively. This action has to be considered as the bare action at unification scale; to make extrapolations to lower energy scales one has to use renormalisation group equations and consider nonperturbative effects in the NCSG action.

The noncommutative spectral geometry model offers [1] a purely geometric approach to the SM of particle physics, where the fermions provide the Hilbert space of a spectral triple for the algebra and the bosons are obtained through inner fluctuations of the Dirac operator of the product $\mathcal{M} \times \mathcal{F}$ geometry. The model is in agreement with particle physics data, such as the top quark mass [1] and, as recent studies have shown [11, 12], it is also consistent with the Higgs mass. It is worth noting that in the original model [1], the Higgs mass in
the zeroth order approximation was found to be 170 GeV, inconsistent with recent particle physics experiments. However, in the original approach, the real scalar singlet, associated with the Majorana mass of the right-handed neutrino, was integrated out and replaced by its vacuum expectation value. It was then shown [11] that this singlet, whose presence has been also argued in Ref. [12], is non-trivially mixed with the Higgs doublet. This results to their masses being shifted, rendering the model consistent with a 125 GeV Higgs and a 170 GeV top quark. Finally, let us note that extensions to the Pati-Salam model have been considered more recently [13].

The NCSG model lives by construction at the Grand Unified Theories (GUTs) scale — the cutoff scale \( \Lambda \) is set at the unification scale — offering a natural framework to study early universe cosmology [14]-[21]. The gravitational part of the asymptotic expression for the bosonic sector of the NCSG action, including the coupling between the Higgs field \( \phi \) and the Ricci curvature scalar \( R \), in Lorentzian signature — obtained through Wick rotation in imaginary time — reads [1]

\[
S^L_{\text{grav}} = \int \left( \frac{R^2}{2\kappa^2} + \alpha_0 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tau_0 R^* R^* - \xi_0 |H|^2 \right) \sqrt{-g} d^4x ;
\]

where \( \kappa^2 \equiv 8\pi G \) and

\[
H = (\sqrt{af_0/\pi})\phi, \quad \text{with } a \text{ a parameter related to fermion and lepton masses and lepton mixing.}
\]

At unification scale (set up by \( \Lambda \)), \( \alpha_0 = -3f_0/(10\pi^2) \).

At this point one may wonder whether the quadratic curvature terms in the action functional indicate the emergence of negative energy massive graviton modes [22]. We will briefly highlight that this is not the case. The higher derivative terms that are quadratic in curvature lead to [23]

\[
\int \left( \frac{1}{2\eta} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{\omega}{3\eta} R^2 + \frac{\theta}{\eta} E \right) \sqrt{-g} d^4x ;
\]

\( E = R^* R^* \) denotes the topological term which is the integrand in the Euler characteristic

\[
\int E \sqrt{-g} d^4x = \int R^* R^* \sqrt{-g} d^4x .
\]

The running of the coefficients \( \eta, \omega, \theta \) of the higher derivative terms is determined by the renormalisation group equations [23]. The coefficient \( \eta \) goes slowly to zero in the infrared limit, so that \( 1/\eta = \mathcal{O}(1) \) up to scales of the order of the size of the universe. Note that \( \eta(t) \) varies by at most one order of magnitude between the Planck scale and infrared energies. All three coefficients \( \eta(t), \omega(t), \theta(t) \) run to a singularity at a very high energy scale \( \mathcal{O}(10^{23}) \text{GeV} \) (i.e., above the Planck scale). To avoid low energy constraints, the coefficients of the quadratic curvature terms \( R_{\mu\nu} R^{\mu\nu} \) and \( R^2 \) should not exceed \( 10^{74} \) [23], which is indeed the case for the running of these coefficients.

For simplicity and since it will not influence our results, in what follows we neglect the nonminimal coupling between the Higgs field and the Ricci curvature. The NCSG equations of motion are [14]

\[
G^{\mu\nu}_{(\text{NCSG})} = \kappa^2 T^{\mu\nu}_{(\text{matter})} ,
\]

where \( \kappa^2 = 8\pi G \) and

\[
G^{\mu\nu}_{(\text{NCSG})} \equiv G^{\mu\nu} + \frac{1}{\beta^2} [2\nabla_{\lambda} \nabla_{\kappa} C^{\mu\nu\lambda\kappa} + C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] ;
\]
\( G^{\mu\nu} \) is the (zeroth order) Einstein tensor, \( T^{\mu\nu}_{\text{matter}} \) the energy-momentum tensor of matter and \( \beta^2 = \frac{5\pi^2}{(6\kappa^2 f_0)} \).

Using the Bianchi identity \( \nabla^\sigma R_{\mu\lambda\sigma} = -\nabla_\lambda R_{\mu\nu} + \nabla_\mu R_{\lambda\nu} \) and \( 2\nabla^\sigma R_{\lambda\nu} = \nabla_\lambda R \), the second term above reads

\[
2\nabla^\lambda \nabla^\sigma C_{\mu\lambda\sigma} + C_{\lambda\mu\nu\sigma} R^{\lambda\sigma} = -\nabla_\lambda \left( R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R \right) + \frac{1}{3} \nabla_\mu \nabla_\nu R
\]  

where \( \Box \equiv \nabla_\mu \nabla^\mu \).

The aim of this paper is to constrain the parameter \( \beta \), which corresponds to a restriction on the particle physics at unification by making use of recent results obtained from Gravity Probe B satellite, and then to improve this constrain by using results from torsion balance experiments. We will thus extend previous studies [18, 19] of one of us and collaborators, where by using recent observations of pulsar timing, we were able to set \( \beta \geq 7.55 \times 10^{-13} m^{-1} \).

It is worth noting that one cannot constrain the other two free parameters, namely \( f_2, f_4 \) unless one makes a (unjustified to our opinion) ansatz on how the coefficients of the terms appearing in the action functional run with energy in the renormalisation group equations.

2 Gravitational Waves in Noncommutative Spectral Geometry

Neglecting the nonminimal coupling between the Higgs field and the Ricci curvature, NCSG does not lead to corrections for homogeneous and isotropic cosmologies [14]. We will therefore consider linear perturbations

\[
g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu},
\]

around a Minkowski background metric \( \eta_{\mu\nu} \), so that to first order \( g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} \). Considering the weak field approximation we will be able to get analytically a lower bound on \( f_0 \).

Defining

\[
\Box_\eta \equiv \partial_\mu \partial^\mu, \quad \tilde{\gamma}_{\mu\nu} \equiv \gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \gamma,
\]

with

\[
\gamma = \gamma^\mu_\mu = \eta^{\mu\nu} \gamma_{\mu\nu},
\]

the \( G^{\mu\nu}_{\text{(NCSG)}} \) is corrected by higher order contributions.

In the Lorentz (synchronous) gauge \( \partial_\mu \tilde{\gamma}^\mu_{\nu} = 0 \), it hence assumes the form [18]

\[
G^{\mu\nu}_{\text{(NCSG)}} = -\frac{1}{2} \Box_\eta \tilde{\gamma}^{\mu\nu} + \frac{1}{2\beta^2} \left[ \Box_\eta \tilde{\gamma}^{\mu\nu} + \frac{1}{3} \left( \eta^{\mu\nu} \Box_\eta - \partial_\mu \partial^\nu \right) \gamma \right].
\]

Introducing the tensor [18]

\[
\tilde{h}_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{3\beta^2} Q^{-1} (\eta_{\mu\nu} \Box_\eta - \partial_\mu \partial_\nu) \gamma,
\]

with \(^1\)

\[
Q \equiv 1 - \frac{1}{\beta^2} \Box_\eta,
\]

\(^1\)We assume that the operator \( Q \) can be inverted since the terms in the r.h.s. of Eq. (2.4) are well-defined and we are considering linear perturbations in the weak field approximation.
the trace of $\bar{h}_{\mu\nu}$ reads [18]

$$\bar{h} = -\left(1 + \frac{1}{\beta^2} Q^{-1} \Box_{\eta}\right) \gamma = -Q^{-1} \gamma.$$  (2.6)

In terms of $\bar{h}^{\mu\nu}$ the linearised NCSG equation of motion is [18]

$$\left(1 - \frac{1}{\beta^2} \Box_{\eta}\right) \Box_{\eta}\bar{h}^{\mu\nu} = -2\kappa^2 T^{\mu\nu}_{\text{matter}};$$  (2.7)

$T^{\mu\nu}_{\text{matter}}$ is taken to lowest order in $\gamma^{\mu\nu}$, so that it is independent of $\gamma^{\mu\nu}$ and satisfies the conservation equation $\partial^{\nu} T^{\mu\nu}_{\text{matter}} = 0$. We restrict to $\alpha_0 < 0$ for Minkowski to be a stable vacuum of the theory [18], implying $\beta^2 > 0$.

Let us define the tensor

$$\chi^{\mu\nu} \equiv -\frac{1}{\beta^2} \Box_{\eta}\bar{h}^{\mu\nu},$$  (2.8)

with $\chi^{\mu\nu}$ satisfying the

$$(\Box_{\eta} - \beta^2)\chi^{\mu\nu} = -2\kappa^2 T^{\mu\nu}_{\text{matter}}.$$  (2.9)

We denote by $\gamma^{\mu\nu}_{(GR)}$ the Einstein’s theory of General Relativity (GR) metric. It fulfills the equation

$$\Box_{\eta}\gamma^{\mu\nu}_{(GR)} = -2\kappa^2 \left(T^{\mu\nu}_{\text{matter}} - \frac{1}{2} \eta^{\mu\nu} T_{\text{matter}}\right),$$  (2.10)

where $T_{\text{matter}}$ is the trace of the energy-momentum tensor, with solution

$$\gamma^{\mu\nu}_{(GR)}(r) = -2\kappa^2 \int d^3 r' T^{\mu\nu}_{\text{matter}}(r'),$$  (2.11)

where

$$T^{\mu\nu}(r') \equiv T^{\mu\nu}_{\text{matter}}(r') - \frac{1}{2} \eta^{\mu\nu} T_{\text{matter}}(r').$$  (2.12)

The linearised equation of motion, Eq. (2.7), reads

$$\Box_{\eta}(\bar{h}^{\mu\nu} + \chi^{\mu\nu}) = -2\kappa^2 T^{\mu\nu}_{\text{matter}},$$  (2.13)

which has the same form as the linearised equation for Einstein’s theory of GR in the synchronous gauge. Hence its solution can be written as

$$\bar{h}^{\mu\nu} = \bar{\gamma}^{\mu\nu}_{(GR)} - \chi^{\mu\nu}.$$  (2.14)

From Eqs. (2.4), (2.14) and writing $\gamma = -Q(\bar{\gamma}_{(GR)} - \chi)$, with $\bar{\gamma}_{(GR)} = -\gamma_{(GR)}$ and $\chi = \chi^{\mu\nu}$, we get

$$\gamma^{\mu\nu} = \gamma^{\mu\nu}_{(GR)} - \bar{\chi}^{\mu\nu} + \phi^{\mu\nu},$$  (2.15)

where

$$\bar{\chi}^{\mu\nu} = \chi^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \chi,$$

$$\phi^{\mu\nu} = \left[\frac{1}{2} \eta^{\mu\nu}(1 - Q) - \Omega^{\mu\nu}\right](\gamma_{(GR)} - \chi),$$

$$= \frac{1}{3\beta^2} \Pi^{\mu\nu}(\gamma_{(GR)} - \chi),$$  (2.16)
with

\[
\Pi_{\mu\nu} \equiv \frac{1}{2} \eta_{\mu\nu} \Box \eta + \partial_\mu \partial_\nu, \\
\Omega_{\mu\nu} \equiv \frac{1}{3 \beta r^2} (\eta_{\mu\nu} \Box \eta - \partial_\mu \partial_\nu).
\] (2.17)

In what follows we shall compute the terms in the right-hand-side of Eq. (2.15).

Let us consider a rotating source producing a static gravitational field, we then have

\[
T_{(\text{matter})}^{\mu\nu}(r) = -\rho(r) u^\mu u^\nu,
\] (2.18)

where \( u^\mu \) is the four velocity \( (u^\mu u^\mu = -1) \) and \( \rho = T_{(\text{matter})} \) is the time independent matter density referred to the frame rotating with the source. Setting the origin of coordinates at the centre of mass, we get for large \( r \):

\[
\frac{1}{|r - r'|} = \frac{1}{r} + \frac{1}{r^3} \sum_{i=1}^{3} x^i x'^i + \ldots
\] (2.19)

where \( r = |r| \) and \( r = (x^1, x^2, x^3) \).

Hence, we obtain the standard GR result (dipole approximation):

\[
\gamma_{00}(\text{GR}) &= \frac{2GM}{r}, \quad \gamma_{ij}(\text{GR}) = \frac{2GM}{r} \delta_{ij}, \\
\gamma_{0i}(\text{GR}) &= \gamma_{i0}(\text{GR}) = -\frac{4G}{r^3} (r \wedge J)_i,
\] (2.20)

where

\[
M = \int \rho(r') d^3r', \quad J = \int \rho(r') [r' \wedge v] d^3r'.
\] (2.21)

Note that we have neglected the kinetic energy term being of the second order in the spatial velocity.

Using the series expansion

\[
e^{-\beta |r - r'|} \approx e^{-\beta r} \left[ \frac{1}{r} + \frac{1 + \beta r}{r^3} \sum_{i=1}^{3} x^i x'^i + \ldots \right],
\] (2.22)

the solution

\[
\chi_{\mu\nu}(r) = 2\kappa^2 \int d^3r' T_{(\text{matter})}^{\mu\nu}(r') \frac{e^{-\beta |r - r'|}}{|r - r'|} e^{-\beta |r - r'|},
\] (2.23)

with \( \beta > 0 \), of Eq. (2.9) can be explicitly written as

\[
\chi_{00} = 4GM \frac{e^{-\beta r}}{r}, \\
\chi_{ij} \sim \mathcal{O}(u^i u^j) \sim 0, \\
\chi_{0i} = \chi_{i0} = -4G \frac{(1 + \beta r) e^{-\beta r}}{r^3} (r \wedge J)_i.
\] (2.24)
Hence, in the dipole approximation, we get

\[ \tilde{\chi}_{00} = 2GM e^{-\beta r}, \]

\[ \tilde{\chi}_{ij} = 2GM e^{-\beta r} \delta_{ij}, \]

\[ \tilde{\chi}_{0i} = \tilde{\chi}_{i0} = -4G \left( 1 + \beta r \right) e^{-\beta r} \left( r \wedge \mathbf{J} \right)_i. \]  

(2.25)

For a static gravitational field, the non-vanishing components of \( \phi_{\mu\nu} \) are \( \phi_{00} \) and \( \phi_{ij} \) given by:

\[ \phi_{00} = -\frac{2GM}{3} e^{-\beta r}, \]

\[ \phi_{ij} = -\frac{4GM}{3\beta^2 r^3} \left\{ \left[ 1 + \left( 1 + \beta r - \frac{\beta^2 r^2}{2} \right) e^{-\beta r} \right] \delta_{ij} - \frac{3x^i x^j}{r^2} \left[ 1 + \left( 1 + \beta r + \frac{\beta^2 r^2}{3} \right) e^{-\beta r} \right] \right\}, \]

(2.26)

where we have used that

\[ \gamma_{(GR)} - \chi = 4GM \left( \frac{1 + e^{-\beta r}}{r} \right). \]

(2.27)

Introducing the metric potentials \( \Phi, \Psi \) and the vector potential \( \mathbf{A} \), the metric reads

\[ ds^2 = -(1 + 2\Phi)dt^2 + 2\mathbf{A} \cdot d\mathbf{x} dt + (1 + 2\Psi)d\mathbf{x}^2. \]

(2.28)

Assuming that for satellite orbits, the relation

\[ x^i x^j = \frac{r^2}{3} \delta_{ij} \]

(2.29)

holds on the average, Eq. (2.26) simplifies to

\[ \phi_{ij} = \frac{10GM}{9} e^{-\beta r} \frac{1}{3} \delta_{ij}. \]

(2.30)

In terms of \( \Phi, \Psi, A_i \), the components of \( \gamma_{\mu\nu} \) are

\[ \gamma_{00} = -2\Phi = \frac{2GM}{r} \left( 1 - \frac{4}{3} e^{-\beta r} \right), \]

\[ \gamma_{0i} = \gamma_{i0} = A_i = \frac{-4G}{r^3} \left[ 1 - (1 + \beta r) e^{-\beta r} \right] (r \wedge \mathbf{J})_i, \]

\[ \gamma_{ij} = 2\Psi \delta_{ij} = \frac{2GM}{r} \left[ 1 + \frac{5}{9} e^{-\beta r} \right] \delta_{ij}, \]

(2.31)

and the non-vanishing Christoffel symbols read

\[ \Gamma^0_{0i} = \Gamma^i_{00} = -\partial_i \Phi, \]

\[ \Gamma^0_{0j} = \frac{1}{2} \left( \partial_i A_j - \partial_j A_i \right), \]

\[ \Gamma^i_{jk} = \delta_{jk} \partial_i \Psi - \delta_{ij} \partial_k \Psi - \delta_{ik} \partial_j \Psi. \]

(2.32)
Notice that the modifications induced by the NCSG action to the Newtonian potentials \( \Phi \) and \( \Psi \) as appear in Eq. (2.31) are similar to those induced by a fifth-force through a potential [24]
\[
V(r) = -\frac{GMm}{r} \left(1 + \frac{\alpha r}{\lambda} \right),
\]
where \( \alpha \) is a dimensionless strength parameter and \( \lambda \) a length scale. In the following, we will put a lower bound on \( \beta \), or equivalently an upper bound on \( \lambda \). We will then see that by using current experimental data that constrain \( \lambda \) we can set a stronger constraint on the \( \beta \) parameter of our model.

3 Constraints from Gravity Probe B and from Torsion Balance

The Gravity Probe B satellite contains a set of gyroscopes (in low circular polar orbit with altitude \( h = 650 \text{ km} \)) that, according to general relativity, will undergo a geodesic precession in the orbital plane, as well as a Lense-Thirring — frame-dragging — precession in the plane of the Earth equator. The Lense-Thirring precession is related to the off-diagonal components of the metric tensor of a rotating gravitational source, so its experimental verification will test the Einstein theory for gravitation. The values (in units of milliarcsec/year) of the geodesic precession and the Lense-Thirring precession measured by the Gravity Probe B satellite and those predicted by General Relativity are [25]

| Effect                        | Measured  | Predicted |
|-------------------------------|-----------|-----------|
| Geodesic precession           | 6602 ± 18 | 6606      |
| Lense-Thirring precession     | 37.2 ± 7.2| 39.2      |

Splitting the rate of an orbiting gyroscope precession into a part generated by the metric potentials and one generated by the vector potential, we get the following spin equation of motion for the gyro spin three-vector \( \mathbf{S} \) [26, 27]:
\[
\frac{d\mathbf{S}}{dt} = \left[ \Omega_G \wedge \mathbf{S} \right]_{G} + \left[ \Omega_{LT} \wedge \mathbf{S} \right]_{LT},
\]
where the instantaneous geodesic precession is
\[
\left[ \frac{d\mathbf{S}}{dt} \right]_{G} = \Omega_G \wedge \mathbf{S} \quad \text{with} \quad \Omega_G = \frac{1}{2} [\nabla(\Phi - 2\Psi)] \wedge \mathbf{v}
\]
and the instantaneous Lense-Thirring precession is
\[
\left[ \frac{d\mathbf{S}}{dt} \right]_{LT} = \Omega_{LT} \wedge \mathbf{S} \quad \text{with} \quad \Omega_{LT} = \frac{1}{2} \nabla \wedge \mathbf{A}.
\]
The geodesic and Lense-Thirring precession, \( \Omega_G \) and \( \Omega_{LT} \), respectively, can be written as the sum of two terms, one obtained within GR and the other being the NCSG contribution. Thus,
\[
\Omega_G = \Omega_{G(GR)} + \Omega_{G(NCG)},
\]
with
\[
\begin{align*}
\Omega_{G(GR)} &= \frac{3GM}{2r^3} (r \wedge \mathbf{v}), \\
\Omega_{G(NCG)} &= -\frac{20}{27}(1 + \beta r) e^{-\beta r} \Omega_{G(GR)}.
\end{align*}
\]
Similarly,
\[ \Omega_{LT} = \Omega_{LT(GR)} + \Omega_{LT(NCSG)}, \]  
with
\[ \Omega_{LT(GR)} = \frac{-2G}{r^3} \mathbf{J}, \]
\[ \Omega_{LT(NCSG)} = -e^{-\beta r}(1 + \beta r + \beta^2 r^2)\Omega_{LT(GR)}, \]
where we have assumed that on the average \( \langle (\mathbf{J} \cdot \mathbf{r}) \rangle = 0. \)

We will use Eqs. (3.5) and (3.7) to constrain the parameter \( \beta. \) Here \( r \) the sum of the Earth radius \( R_\oplus \) and the altitude \( h \) of the satellite. Setting the geodesic precession \( |\Omega_{G(GR)}| = 6606 \text{ mas/y} \) and requiring that \( |\Omega_{G(NCSG)}| \lesssim |\delta \Omega_G|, \) where \( |\delta \Omega_G| = 18 \text{ mas/y}, \) we get
\[ \beta \gtrsim 10^{-6} \text{ m}^{-1}. \]  

Note that we get the same lower bound for \( \beta \) from the Lense-Thirring precession, where \( |\Omega_{LT(GR)}| = 39.2 \text{ mas/y} \) and \( |\Omega_{LT(NCSG)}| \lesssim |\delta \Omega_{LT}| \) where \( |\delta \Omega_{LT}| = 7.2 \text{ mas/y}. \)

The constraint on \( \beta, \) Eq. (3.8) above, provides an upper bound on \( \lambda = \beta^{-1}, \) namely \( \lambda < 10^6 \text{m}. \) It has been shown [28] that the inverse-square law holds down to a length scale \( \lambda = 56 \mu \text{m} \) for \( |\alpha| \leq 1. \) Note that in our case \( \alpha \sim O(1), \) as one can easily see from Eqs. (2.31) and (2.33).

A more stringent constraint on \( \beta \) can be obtained once we use results from laboratory experiments design to test the fifth force. Hence, by constraining \( \lambda \) through torsion balance experiments we will subsequently obtain a stronger lower bound to \( \beta, \) or equivalently an upper bound to the momentum \( f_0 \) of the cutoff function \( f. \)

The test masses have a typical size of \( \sim 10 \text{mm} \) and their separation is smaller than their size. As we have already mentioned above, for our study \( |\alpha| \sim O(1), \) so that the tightest constraint on \( \lambda \) provided by Eöt-Wash [29] and Irvine [30] experiments is [28]
\[ \lambda \lesssim 10^{-4} \text{m}, \]  
implying
\[ \beta \gtrsim 10^4 \text{m}^{-1}. \]

4 Conclusions

In the context of NCSG we have studied the linearised field equations in the limit of weak gravitational fields generated by a rotating gravitational source. Then making use of recent experimental data, we were able to constrain one of the free parameters of the model, namely the moment of the cutoff function that is related to the coupling constants at unification. First, we have studied the precession of spin of a gyroscope orbiting about a rotating gravitational source. Such a gravitational field gives rise, according to General Relativity predictions, to the geodesic and the Lense-Thirring precessions, the latter being strictly related to the off-diagonal terms of the metric tensor generated by the rotation of the source. We have focused in particular on the gravitational field generated by the Earth, and on the recent experimental results obtained from the Gravity Probe B satellite, which has tested the geodesic and Lense-Thirring spin precession with high precision. We have calculated the corrections of the precession induced by NCSG corrections. Requiring that the corrections are below the experimental errors, we have inferred a lower bound on \( \beta, \) namely that
We then used laboratory fifth force experiments to impose a more stringent constraint on the parameter $\beta$; we thus obtained $\beta \gtrsim 10^4 \text{m}^{-1}$. Note that this is a stronger constraint than the one imposed [19] by studying the energy lost from binary systems via emission of gravitational waves, and much stronger than any constraint imposed so far to curvature squared terms (as for instance in Ref. [22]).

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