ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF RIORDAN GROUPS OVER FINITE FIELDS

GI-SANG CHEON, NHAN-PHU CHUNG, AND MINH-NHAT PHUNG

Abstract. In this paper, we investigate algebraic and topological properties of the Riordan groups over finite fields. These groups provide a new class of topologically finitely generated, abstract, pro-p groups with finite width. We also introduce, characterize index-subgroups of our Riordan groups, and finally we show exactly the range of Hausdorff dimensions of these groups. The latter results are analogous to the work of Barnea and Klopsch for the Nottingham groups.

1. Introduction

The Riordan group over the field of real or complex numbers was introduced in 1991 by Shapiro and his collaborators [27] in the framework of enumerative combinatorics. After that, it has been investigated by many authors with applications in combinatorics, computer science, group theory, matrix theory, number theory, Lie groups and Lie algebras, orthogonal polynomials, graph theory, Heisenberg-Weyl algebra. A non-exhaustive list of works related to Riordan groups is given by [7, 14, 21–24, 28, 29] and references there in. Recently, in 2017 via the inverse limit approaches of Riordan groups initiated in [20], the authors established an infinite-dimensional Fréchet Lie group and the corresponding Lie algebra on the Riordan group [8].

In this article, we study Riordan groups $R(\mathbb{K})$ over general unital commutative rings $\mathbb{K}$ and more specially for $\mathbb{K} = \mathbb{F}_q$ finite fields. To our knowledge, this is the first paper investigating properties of these Riordan groups. Instead of using the representation of Riordan groups via Riordan arrays which are infinite lower triangular matrices, we construct them as semi-direct products of $\mathcal{H}(\mathbb{K})$, the group of formal series with the free coefficient is 1, and $\mathcal{N}(\mathbb{K})$, the substitution group of formal power series. The group $\mathcal{N}(\mathbb{K})$ was first introduced in 1954 by Jennings [15], and was brought attention later to the group theory community by Johnson [16] and his Ph.D student York [31, 32]. It is known under the name of the Nottingham group now. Since then, the group $\mathcal{N}(\mathbb{F}_p)$, where $\mathbb{F}_p$ is the finite field with a prime number $p$, has been extensively studied in the literature. It has many remarkable properties and plays an important role in the theory of profinite groups [2, 4–6, 9–11, 18, 30].
As our Riordan group $\mathcal{R}(\mathbb{F}_p)$ contains the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ as a subgroup, we would like to investigate which properties of $\mathcal{N}(\mathbb{F}_p)$ still hold for $\mathcal{R}(\mathbb{F}_p)$.

It was proved that the Nottingham groups $\mathcal{N}(\mathbb{F}_p)$ and $\mathcal{N}(\mathbb{Z})$ are topologically finitely generated [3, 6]. On the other hand, the group $\mathcal{N}(\mathbb{F}_p)$ has finite width [19].

Our first main result is to show that $\mathcal{R}(\mathbb{F}_p)$ and $\mathcal{R}(\mathbb{Z})$ are topologically finitely generated, and $\mathcal{R}(\mathbb{F}_p)$ has finite width.

**Theorem 1.1.** Let $\mathbb{K}$ be a finite commutative unital ring, $p \geq 2$ be a prime number and $\mathbb{F}_q$ be a finite field with characteristic $p$. Then

1. the Riordan group $\mathcal{R}(\mathbb{K})$ is profinite;
2. the Riordan group $\mathcal{R}(\mathbb{F}_q)$ is a pro-$p$-group with finite width;
3. the Riordan groups $\mathcal{R}(\mathbb{F}_p)$ and $\mathcal{R}(\mathbb{Z})$ are topologically finitely generated. Furthermore, for every topologically generating set $S_N$ of the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ (respectively $\mathcal{N}(\mathbb{Z})$), the group $\mathcal{R}(\mathbb{F}_p)$ (respectively $\mathcal{R}(\mathbb{Z})$) is generated by
   $$\{(1+x,x), (1,g) : g \in S_N\}.$$

The definitions of $\mathcal{R}(\mathbb{K})$, topologically finitely generating sets and groups with finite width will be presented in Section 2.

On the other hand, the study of Hausdorff dimension in profinite groups has been initiated by Abercrombie [1]. Given a filtration $\{G_n\}_n$ of an infinite profinite group $G$, we can define an induced invariant metric on $G$ and then we can compute the Hausdorff dimension of a closed subgroup $H$ of $G$ with respect to this metric via the following formula

$$\dim_H(H) = \liminf_{n \to \infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}.$$

After that, in [5], given an infinite profinite group $G$, Barnea and Shalev investigated the dimension spectrum of $G$

$$\text{hspec}(G) := \{\dim_H(H) : H \text{ is a closed subgroup of } G\}.$$  

Using Lie algebra ideas they showed that if $p \geq 5$ is a prime number then

$$\left\{\frac{1}{s} : s \in \mathbb{N}\right\} \subset \text{hspec}(\mathcal{N}(\mathbb{F}_p)) \subset [0, \frac{3}{p}] \cup \left\{\frac{1}{s} : s \in \mathbb{N}\right\}.$$  

Later, Barnea and Klopsch introduced index-subgroups, a large class of closed subgroups of $\mathcal{N}(\mathbb{F}_p)$ [4]. Then they partially extended the above result of [5] as they proved that for $p > 2$

$$\text{inspec}(\mathcal{N}(\mathbb{F}_p)) = [0, \frac{1}{p}] \cup \left\{\frac{1}{p} + \frac{1}{p^r} : r \in \mathbb{N}\right\} \cup \left\{\frac{1}{s} : s \in \mathbb{N}\right\},$$

where

$$\text{inspec}(\mathcal{N}(\mathbb{F}_p)) := \{\dim_H(H) : H \text{ is an index-subgroup of } \mathcal{N}(\mathbb{F}_p)\} \subset \text{hspec}(\mathcal{N}(\mathbb{F}_p)).$$
Inspired by the work of [4], we introduce index-subgroups of the Riordan group \( \mathcal{R}(\mathbb{F}_p) \). In our second main result of the article, we characterize these index-subgroups and given a standard filtration of \( \mathcal{R}(\mathbb{F}_p) \) we describe exactly the set

\[
\text{inspec}(\mathcal{R}(\mathbb{F}_p)) := \{ \dim_H(H) : H \text{ is an index-subgroup of } \mathcal{R}(\mathbb{F}_p) \} \subset \text{hspec}(\mathcal{R}(\mathbb{F}_p)).
\]

**Theorem 1.2.** Let \( p > 2 \) be a prime number and given the filtration \( \{ \mathcal{H}^n(\mathbb{F}_p) \times \mathcal{N}^n(\mathbb{F}_p) \}_n \) of \( \mathcal{R}(\mathbb{F}_p) \), we have

\[
\text{inspec}(\mathcal{R}(\mathbb{F}_p)) = \left[ 0, \frac{1}{p} \right] \cup \left\{ \frac{1}{2p^r} + \frac{1}{3p^r} : r \in \mathbb{N} \right\} \cup \left\{ \frac{1}{2} + \frac{1}{2p^r} + \frac{1}{2p^r} : r \in \mathbb{N} \right\} \\
\cup \bigcup_{s < p} \left\{ \frac{1}{2^s}, \frac{1}{2^s} + \frac{1}{p} \right\} \cup \left\{ \frac{1}{2sp^r} + \frac{1}{2sp^r} : s, u \in \mathbb{N}, r \in \mathbb{N} \cup \{0\} \right\}.
\]

In contrast to the Nottingham group \( \mathcal{N}(\mathbb{F}_p) \), Hausdorff dimensions of the Riordan group \( \mathcal{R}(\mathbb{F}_p) \) do depend on the choice of the filtrations.

Our paper is organized as follows. In Section 2 we define our Riordan group \( \mathcal{R}(\mathbb{K}) \) as a semi-direct product of \( \mathcal{H}(\mathbb{K}) \) and \( \mathcal{N}(\mathbb{K}) \), and then we present it as an inverse limit of topological groups. After that we show that both \( \mathcal{R}(\mathbb{F}_p) \) and \( \mathcal{R}(\mathbb{Z}) \) are topologically finitely generated, calculate the lower central series of \( \mathcal{R}(\mathbb{F}_p) \) and then prove Theorem 1.1. Finally, in Section 3 we first review the Hausdorff dimension on metric spaces arising from profinite groups, introduce and characterize the index-subgroups of \( \mathcal{R}(\mathbb{F}_p) \), and finally we present a proof of Theorem 1.2.

**Acknowledgements:** This work was partially supported by Science Research Center Program through the National Research Foundation of Korea (NRF) Grant funded by the Korean Government (MSIP) (NRF-2016R1A5A1008055). G.-S. Cheon was partially supported by the NRF-2019R1A2C1007518. N.-P. Chung and M.-N. Phung were partially supported by the NRF-2019R1C1C1007107. We thank Minho Song for pointing out [13, Proposition 3.3] to us.

### 2. Riordan groups and their properties

In this note, we put \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( p \) is always a prime number.

For every unital commutative ring \( \mathbb{K} \), we denote by \( \mathbb{K}[[x]] \) the set of all formal series with coefficients in \( \mathbb{K} \), i.e.

\[
\mathbb{K}[[x]] := \left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{K}, \text{ for every } i \in \mathbb{N}_0 \right\}.
\]

Under the usual multiplication, we know that an element \( g = \sum_{i=0}^{\infty} a_i x^i \in \mathbb{K}[[x]] \) is invertible if and only if \( a_0 \) is invertible in \( \mathbb{K} \).

We denote \( \mathcal{H}(\mathbb{K}) := \{ 1 + a_1 x + \cdots \in \mathbb{K}[[x]] \} \) and endow it with the usual multiplication operation \( \cdot : \mathcal{H}(\mathbb{K}) \times \mathcal{H}(\mathbb{K}) \to \mathcal{H}(\mathbb{K}) \) then \( (\mathcal{H}(\mathbb{K}), \cdot) \) is a group. We consider \( \mathcal{N}(\mathbb{K}) \) the set of all \( g = x + a_2 x^2 + \cdots + a_n x^n + \cdots \in \mathbb{K}[[x]] \), and define the
substitution operation \( \circ : \mathcal{N}(\mathbb{K}) \times \mathcal{N}(\mathbb{K}) \rightarrow \mathcal{N}(\mathbb{K}) \) by

\[
 h \circ g := g(h) = h + \sum_{i=2}^{\infty} a_i h^i,
\]

for every \( g = x + \sum_{i=2}^{\infty} a_i x^i, h = x + \sum_{i=2}^{\infty} b_i x^i \in \mathcal{N}(\mathbb{K}) \). It is well known that \((\mathcal{N}(\mathbb{K}), \circ)\) is a group [15, Theorem 1.11]. It is called the Nottingham group which has been studied intensively.

Now we define our Riordan group as a semidirect product of \( \mathcal{H}(\mathbb{K}) \) and \( \mathcal{N}(\mathbb{K}) \) as follows. We denote by \( \text{Aut}(\mathcal{H}(\mathbb{K})) \) the group of all automorphisms of \( \mathcal{H}(\mathbb{K}) \). For every \( h = 1 + \sum_{i=1}^{\infty} b_i x^i \in \mathcal{H}(\mathbb{K}) \) and \( g \in \mathcal{N}(\mathbb{K}) \), the substitution \( g \circ h = 1 + \sum_{i=1}^{\infty} b_i g^i \) is still in \( \mathcal{H}(\mathbb{K}) \). Let \( \varphi : \mathcal{N}(\mathbb{K}) \rightarrow \text{Aut}(\mathcal{H}(\mathbb{K})) \) be the map defined by \( \varphi(g)(h) = g \circ h \) for every \( h \in \mathcal{H}(\mathbb{K}), g \in \mathcal{N}(\mathbb{K}) \) then it is clear that \( \varphi \) is well defined and indeed it is a homomorphism. Let \( \mathcal{R}(\mathbb{K}) \) be the semi-direct product \( \mathcal{H}(\mathbb{K}) \rtimes_{\varphi} \mathcal{N}(\mathbb{K}) \). The multiplication operation in \( \mathcal{H}(\mathbb{K}) \rtimes_{\varphi} \mathcal{N}(\mathbb{K}) \) is defined by

\[
 (h_1, g_1) \cdot (h_2, g_2) := (h_1 \varphi(g_1) h_2, g_1 \circ g_2) = (h_1 h_2(g_1), g_1 \circ g_2),
\]

for every \((h_1, g_1), (h_2, g_2) \in \mathcal{H}(\mathbb{K}) \times \mathcal{N}(\mathbb{K}) \). For every \((h, g) \in \mathcal{H}(\mathbb{K}) \rtimes_{\varphi} \mathcal{N}(\mathbb{K}) \), its inverse is \((h, g)^{-1} = (\varphi(\bar{g})(h^{-1}), \bar{g}) = (h^{-1}(\bar{g}), \bar{g})\), where \( hh^{-1} = 1 \) and \( g \circ \bar{g} = \bar{g} \circ g = x \).

We start with the following elementary lemma which will be used several times later.

**Lemma 2.1.** Let \( h = 1 + \sum_{i=n}^{\infty} a_i x^i \in \mathcal{H}(\mathbb{K}) \) and \( g = x + \sum_{j=m}^{\infty} b_j x^j \in \mathcal{N}(\mathbb{K}) \). Then

1. the invertible element \( h^{-1} \) in \( \mathcal{H}(\mathbb{K}) \) is \( 1 + \sum_{i=n}^{\infty} (p_i - a_i) x^i \), where \( p_n = 0 \) and \( p_k \) is a polynomial depending only on \( a_n, \ldots, a_{k-1} \) for every \( k \geq n \);
2. For every \( k \geq n \), the coefficient at degree \( k \) of \( h(g) \) is:
   (i) \( a_k \) if \( n \leq k < m + n - 1 \),
   (ii) \( na_n b_m + a_{m+n-1} \) if \( k = m + n - 1 \),
   (iii) \( na_n b_{k-n+1} + q_k \), where \( q_k \) is a polynomial depending only on \( a_n, \ldots, a_k \) and \( b_m, \ldots, b_{k-n} \) if \( k > m + n - 1 \);
3. For every \( k \geq n \), the coefficient at degree \( k \) of \( h(g)h^{-1} \) is:
   (i) \( 0 \) if \( n \leq k < m + n - 1 \),
   (ii) \( na_n b_m \) if \( k = m + n - 1 \),
   (iii) \( na_n b_{k-n+1} + r_k \), where \( r_k \) is a polynomial depending only on \( a_n, \ldots, a_k \) and \( b_m, \ldots, b_{k-n} \) if \( k > m + n - 1 \).

**Proof.** (1) We write \( h^{-1} \) in the form \( 1 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + \sum_{i=n}^{\infty} (p_i - a_i) x^i \). Because \( hh^{-1} = 1 \) we get that \( c_1 = \cdots = c_{n-1} = 0, p_n = 0 \) and for every \( k \geq n + 1 \), \( p_n \) must satisfy the recurrence

\[
 a_k + p_k - a_k + \sum_{i=n}^{k-1} a_i (p_{k-i} - a_{k-i}) = 0.
\]
(2) Reminding that \( \left( \sum_{j=1}^{\infty} x_j \right)^n = \sum t c_t \prod_{j=1}^{\infty} x_j^{t_j} \), where \( t = (t_1, t_2, \ldots) \subset \mathbb{N} \cup \{0\} \) satisfies \( \sum_{j=1}^{\infty} t_j = n \), and the coefficient \( c_t = \prod_{j=1}^{n} t_j! \). Using the formula for \( n = i, x_1 = 1 \) and \( x_{j-m+2} = b_j x_j^{j-1}, j \geq m \) to expand \( h(g) \) we get that

\[
h(g) = 1 + \sum_{i=n}^{\infty} a_i x^i (1 + \sum_{j=m}^{\infty} b_j x_j^{j-1})^i
\]

\[
= 1 + \sum_{i=n}^{\infty} a_i \left( \sum_{(t_j)_{j \geq m}} \frac{i!}{(i-S_1((t_j)^{t_j}_{j \geq m})!) \prod_j b_j^{t_j}} \right) x^{i+S_2((t_j)^{t_j}_{j \geq m})},
\]

where \( (t_j)^{t_j}_{j \geq m} \) is taken over all sequences \( (t_j)_{j \geq m} \subset \mathbb{N} \cup \{0\} \) such that \( S_1 := \sum_{j=m}^{\infty} t_j \leq i \), and where \( S_2((t_j)^{t_j}_{j \geq m}) := \sum_{j=m}^{\infty} (j-1) t_j \). In order to find coefficient for \( x^i \), we consider \( i + S_2((t_j)^{t_j}_{j \geq m}) = k \).

(i) For \( k < m + n - 1 \) we only have the case \( i = k, S_2((t_j)^{t_j}_{j \geq m}) = 0 \) so \( S_1 = 0 \) and \( \prod_j b_j^{t_j} = 1 \). Thus, the coefficient is \( a_k \).

(ii) For \( k = m + n - 1 \) we have \( i = n, S_2((t_j)^{t_j}_{j \geq m}) = m - 1 \) or \( i = m + n - 1, S_2((t_j)^{t_j}_{j \geq m}) = 0 \). If \( i = n, S_2((t_j)^{t_j}_{j \geq m}) = m - 1 \) we have \( na_m b_m \) and if \( i = m + n - 1, S_2((t_j)^{t_j}_{j \geq m}) = 0 \) we have \( a_{m+n-1} \).

(iii) If \( t_j > 0 \) for \( j > k - n + 1 \) then \( S_2((t_j)^{t_j}_{j \geq m}) > k - n \). So, we only consider \( t_j = 0 \) for \( j > k - n + 1 \). For \( j = k - n + 1 \) if \( t_j > 1 \) then \( S_2((t_j)^{t_j}_{j \geq m}) \geq 2(k-n) > k-n \). So \( t_{k-n+1} = 0 \) or \( t_{k-n+1} = 1 \). The case \( t_{k-n+1} = 1 \) leads to \( S_2((t_j)^{t_j}_{j \geq m}) = k - n \), this means that \( t_{k-n+1} \) is the only non-zero number in the sequence and \( i = n \). Thus, \( S_1 = 1 \) and \( \prod_j b_j^{t_j} = b_{k-n+1} \). Hence, we get the part \( na_m b_{k-n+1} \) in the coefficient. For \( q_k \), we have \( t_j = 0 \) for \( j \geq k - n + 1 \) so we get the desired result.

(3) From (1) we get that the coefficient at degree \( k \) of \( h(g) h^{-1} \) is \( \sum_{i=0}^{k} c_i (p_{k-i} - a_{k-i}) \), where \( c_i \) is the coefficient at degree \( i \) of \( h(g) \).

(i) For \( n \leq k < m + n - 1 \) we have \( c_i = 0 \) if \( 0 < i < n \), and \( c_i = a_i \) if \( n \leq i \leq k \).

\[
\sum_{i=0}^{k} c_i (p_{k-i} - a_{k-i}) = a_k + (p_k - a_k) + \sum_{n}^{k-1} a_i (p_{k-i} - a_{k-i}) = 0.
\]

(ii) Because \( c_{m+n-1} = na_n b_m + a_{m+n-1} \) so

\[
\sum_{i=0}^{k} c_i (p_{k-i} - a_{k-i}) = na_n b_m + a_k + (p_k - a_k) + \sum_{n}^{k-1} a_i (p_{k-i} - a_{k-i}) = na_n b_m.
\]
(iii) For \( k > m + n - 1 \), we have
\[
\sum_{i=0}^{k} c_i(p_{k-i} - a_{k-i}) = na_n b_{k-n+1} + q_k + (p_k - a_k) + \sum_{n}^{k-1} c_i(p_{k-i} - a_{k-i}).
\]
We see that the property \( r_k = q_k + (p_k - a_k) + \sum_{n}^{k-1} c_i(p_{k-i} - a_{k-i}) \) is a polynomial depending only on \( a_n, \ldots, a_k \) and \( b_m, \ldots, b_{k-n} \) follows from (1) and (2).

\[\square\]

2.1. Profiniteness of \( \mathcal{R}(\mathbb{K}) \). Before proving properties of the Riordan group \( \mathcal{R}(\mathbb{K}) \), let us review the definitions of inverse limit groups and profinite groups. For more details of profinite groups, see \([25,33]\). A directed set is a partially order set \( (I, \leq) \) such that for all \( a, b \in I \), there exists an element \( c \in I \) such that \( a \leq c \) and \( b \leq c \). Let \( I \) be a directed set. An inverse system \( (X_i, \pi_{ij})_{i,j \in I, i \leq j} \) of topological spaces consists of a family \( (X_i)_{i \in I} \) of topological spaces and a family of continuous maps \( (\pi_{ij} : X_j \to X_i)_{i,j \in I, i \leq j} \) such that \( \pi_{ii} = Id_{X_i} \), the identity map of \( X_i \) for every \( i \in I \), and \( \pi_{ik} = \pi_{ij} \pi_{jk} \) for every \( i \leq j \leq k \). If each \( X_i \) is a topological group and each \( \pi_{ij} \) is a continuous homomorphism then \( (X_i, \pi_{ij}) \) is called an inverse system of topological groups. An inverse limit of an inverse system \( (X_i, \pi_{ij}) \) of topological groups is a topological group \( X \) with a continuous homomorphisms \( \pi_i : X \to X_i \) satisfying \( \pi_{ij} \circ \pi_j = \pi_i \) for every \( i \leq j \) in \( I \) and the following universal property: whenever \( (\varphi_i : Y \to X_i) \) is a family of continuous homomorphisms such that \( \pi_{ij} \varphi_j = \varphi_i \) for every \( i \leq j \) in \( I \) there is a unique continuous homomorphism \( \varphi : Y \to X \) such that \( \pi_i \varphi = \varphi_i \) for every \( i \in I \). By the universal property, two inverse limit group of \( (X_i, \pi_{ij}) \) will be isomorphic. Let \( P = \prod_{i \in I} X_i \) denote the direct product of groups \( X_i \) then
\[
X := \{(x_i) \in P : \pi_{ij}(x_j) = x_i \text{ for all } i, j \in I \text{ such that } i \leq j \}
\]
is a subgroup of \( P \). Then the group \( X \) is the inverse limit group of \( (X_i) \) and we denote \( X = \lim_{\leftarrow I} X_i \). We endow \( X \) with the topology induced from the product topology of \( \prod_{i \in I} X_i \).

Let \( \mathcal{C} \) be a class of groups. A group \( H \in \mathcal{C} \) is called a \( \mathcal{C} \)-group and an inverse limit group \( G \) of \( \mathcal{C} \)-groups is called a pro-\( \mathcal{C} \)-group. In the special case where \( \mathcal{C} \) is the class of all finite groups endowed with the discrete topology, we call an inverse limit group \( G \) of \( \mathcal{C} \)-groups is profinite. Furthermore, if \( p \) is a prime and \( G = \lim_{\rightarrow} G_i \), where \( G_i \) is a \( p \)-group for every \( i \), we say that \( G \) is a pro-\( p \)-group. Recall that a finite group is called a \( p \)-group, where \( p \) is a prime, if its order is a power of \( p \).

Now we are ready to prove our first result in this section that the Riordan group \( \mathcal{R}(\mathbb{K}) \) is profinite for every finite commutative ring \( \mathbb{K} \).

For every \( n \in \mathbb{N} \), we define
\[
\mathcal{H}^0(\mathbb{K}) := \{ h(x) = 1 + a_n x^n + a_{n+1} x^{n+1} + \cdots \in \mathcal{H}(\mathbb{K}) \}, \quad n \geq 2,
\]
\[
\mathcal{H}^1(\mathbb{K}) := \mathcal{H}(\mathbb{K}),
\]
RIORDAN GROUPS OVER FINITE FIELDS

\[ \mathcal{N}^n(\mathbb{K}) := \{ g(x) = x + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots \in \mathcal{N}(\mathbb{K}) \}, \quad n \geq 2, \]

\[ \mathcal{N}^1(\mathbb{K}) := \mathcal{N}(\mathbb{K}). \]

For every \( m, n \in \mathbb{N} \), we consider the subsets \( \mathcal{R}^{m,n}(\mathbb{K}) := \mathcal{H}^m(\mathbb{K}) \times_{\varphi_n|_{\mathcal{H}^m(\mathbb{K})}} \mathcal{N}^n(\mathbb{K}) \) of \( \mathcal{R}(\mathbb{K}) \), where \( \varphi_n \) is the restriction of \( \varphi : \mathcal{N}(\mathbb{K}) \to \text{Aut}(\mathcal{H}(\mathbb{K})) \) to \( \mathcal{N}^n(\mathbb{K}) \) and \( \varphi_n|_{\mathcal{H}^m(\mathbb{K})} \) is defined by the restriction of \( \varphi_n(g) \) to \( \mathcal{H}^m(\mathbb{K}) \) for each \( g \in \mathcal{N}^n(\mathbb{K}) \). We use the notation \( \mathcal{R}^n(\mathbb{K}) \) when \( m = n \).

**Lemma 2.2.**

1. For every \( m, n \in \mathbb{N} \), the map \( \varphi_n|_{\mathcal{H}^m(\mathbb{K})} : \mathcal{N}^n(\mathbb{K}) \to \text{Aut}(\mathcal{H}^m(\mathbb{K})) \) is well defined and therefore so is \( \mathcal{R}^{m,n}(\mathbb{K}) \).

2. \( \mathcal{R}^{m_1,n_1}(\mathbb{K}) \) is a normal subgroup of \( \mathcal{R}^{m_2,n_2}(\mathbb{K}) \) for \( m_2 + n_1 \geq m_1 \geq m_2 \) and \( n_1 \geq n_2 \).

**Proof.**

(1) We need to show that for any \( g \in \mathcal{N}^n(\mathbb{K}) \) and \( h \in \mathcal{H}^m(\mathbb{K}) \) there exists a unique \( h' \in \mathcal{H}^m(\mathbb{K}) \) such that \( h' \circ g = h \).

Since we already had \( \varphi(g) \in \text{Aut}(\mathcal{H}(\mathbb{K})) \) so there exists a unique \( h' \in \mathcal{H}(\mathbb{K}) \) such that \( h' \circ g = h \). We want to show \( h' \in \mathcal{H}^m(\mathbb{K}) \).

Suppose \( h' \in \mathcal{H}(\mathbb{K}) \setminus \mathcal{H}^m(\mathbb{K}) \) then there exist \( m' < m \) and \( a_{m'} \neq 0 \) such that \( h' = 1 + a_m x^m + O(x^{m+1}) \). Because \( g = x + O(x^{n+1}) \) so \( h = h' \circ g = 1 + a_{m'} x^{m'} + O(x^{m'+1}) \notin \mathcal{H}^m(\mathbb{K}) \), a contradiction.

(2) For every \( h_i \in \mathcal{H}^{m_i}, g_j \in \mathcal{N}^{n_j} \), we have

\[
(h_2, g_2)^{-1}(h_1, g_1)(h_2, g_2) = (\varphi(g_2)(h_2^{-1}), \varphi(g_1)(h_2, g_2)) = (\varphi(g_2)(h_2^{-1})\varphi(g_2)(h_1), g_2 \circ g_1)(h_2, g_2) = (\varphi(g_2)[h_2^{-1}h_1\varphi(g_1)(h_2)], g_2 \circ g_1 \circ g_2).
\]

Let \( h = 1 + ax^m + O(x^{m+1}) \) and \( g = x + bx^n + O(x^{n+2}) \) then applying Lemma 2.1 we have

\[
h(g)h^{-1} = (h + abmx^{m+n} + O(x^{m+n+1}))h^{-1} \in \mathcal{H}^{m+n}(\mathbb{K}).
\]

Hence, \( \varphi(g_2)[h_2^{-1}h_1\varphi(g_1)(h_2)] \in \mathcal{H}^{m_1}(\mathbb{K}) \). From [2, Lemma 2.3], we also have \( g_2 \circ g_1 \circ g_2 \in \mathcal{N}^{m_1}(\mathbb{K}) \). The proof is complete. \( \square \)

If \( \mathbb{K} \) is a finite unital commutative ring then the group \( \mathcal{R}_n(\mathbb{K}) := \mathcal{R}(\mathbb{K})/\mathcal{R}^n(\mathbb{K}) \) is finite for every \( n \in \mathbb{N} \). Furthermore, if \( p \) is a prime and \( \mathbb{K} = \mathbb{F}_q \) a finite field with characteristic \( p \) then \( \mathcal{R}_n(\mathbb{K}) \) is a \( p \)-group for every \( n \in \mathbb{N} \). Therefore, we get the following corollary.

**Corollary 2.3.** For every unital commutative ring \( \mathbb{K} \), \( \mathcal{R}(\mathbb{K}) = \varprojlim \mathcal{R}_n(\mathbb{K}) \). If \( \mathbb{K} \) is finite then \( \mathcal{R}(\mathbb{K}) \) is profinite. Furthermore, if \( \mathbb{K} = \mathbb{F}_q \) a finite field with characteristic \( p \), where \( p \) is a prime then \( \mathcal{R}(\mathbb{K}) \) is a \( p \)-group.

**Remark 2.4.** We have many other representations of \( \mathcal{R}(\mathbb{K}) \) as inverse limits of topological groups, for example \( \mathcal{R}(\mathbb{K}) = \varprojlim \mathcal{P}_n(\mathbb{K}) \), where \( \mathcal{P}_n(\mathbb{K}) := \mathcal{R}(\mathbb{K})/\mathcal{H}(\mathbb{K}) \times \mathcal{N}^n(\mathbb{K}) \) for every \( n \in \mathbb{N} \). However, we choose our representation \( \mathcal{R}(\mathbb{K}) = \varprojlim \mathcal{R}_n(\mathbb{K}) \) and its induced inverse limit topology for the remaining of the paper.
From now on, if there is no other statement we assume the action is induced from substitutions. We just denote the semi-direct products by $\rtimes$ instead of $\rtimes_{\varphi}$ or $\rtimes_{\varphi|_{\mathcal{H}_n}}$.

For every element $h = \sum_{n=0}^{\infty} h_n x^n \in \mathbb{K}[x]$ and every $m \in \mathbb{N}_0$, we denote $[x^m]h := h_m \in \mathbb{K}$. For each $(h, g) \in \mathcal{R}(\mathbb{K})$ we associate the matrix $A(h, g) \in M_{\mathcal{N}_0}(\mathbb{K})$ with entries $a_{ij} \in \mathbb{K}$ defined by \( a_{ij} := [x^i]h(x)g^j(x) \) for every $i, j \in \mathbb{N}_0$. Then $A(h, g)$ is a lower triangle matrix, i.e., $a_{ij} = 0$ for every $j > i$, and furthermore, $a_{ii} = 1$ for all $i \in \mathbb{N}_0$. The map $A : \mathcal{R}(\mathbb{K}) \to GL_{\mathcal{N}_0}(\mathbb{K}), (h, g) \mapsto A(h, g)$ is a homomorphism and indeed it is a monomorphism. We denote the map $\tilde{\mathcal{R}}(\mathbb{K})$ as the image $A(\mathcal{R}(\mathbb{K}))$ and denote $\tilde{\mathcal{N}}(\mathbb{K})$ is the subset of $\tilde{\mathcal{R}}(\mathbb{K})$ such that $a_{ij} = 0$ for $j < i < n$. We have the following proposition.

**Proposition 2.5.** For every $n \in \mathbb{N}$, $\tilde{\mathcal{N}}(\mathbb{K})$ is a normal subgroup of $\tilde{\mathcal{R}}(\mathbb{K})$ and is isomorphic to $\mathcal{H}^n(\mathbb{K}) \rtimes \mathcal{N}^n(\mathbb{K})$ by the restriction on $A$. Then, naturally, $\tilde{\mathcal{R}}(\mathbb{K}) / \tilde{\mathcal{N}}(\mathbb{K})$ is isomorphic to $\mathcal{R}_n(\mathbb{K})$.

**Proof.** First, we show that $A(\mathcal{H}^n(\mathbb{K}) \rtimes \mathcal{N}^n(\mathbb{K})) \subseteq \tilde{\mathcal{R}}(\mathbb{K})$. Indeed, let $h = 1 + h_n x^n + \ldots \in \mathcal{H}^n(\mathbb{K})$ and $g = x + g_n x^{n+1} + \ldots \in \mathcal{N}^n(\mathbb{K})$ then we see that

\[
A(h, g)_{ij} = [x^i]h(x)g^j(x) = [x^{i-j}](1 + h_n x^n + \ldots)(1 + g_n x^{n+} \ldots)^j = [x^{i-j}](1 + (h_n^j + g_n^j) x^n + (h_{n+1}^j + \ldots) x^{n+1} + \ldots)
\]

for $j < i < n$. On the other hand, let $A(h, g) \in \tilde{\mathcal{R}}(\mathbb{K})$. We write $h = 1 + h_1 x + \ldots$ and $g = x + g_1 x^2 + \ldots$ and see that $0 = [A(h, g)]_{i0} = h_i$ for $0 < i < n$ so $h \in \mathcal{H}^n(\mathbb{K})$. We also see that $0 = [A(h, g)]_{i1} = \sum_{k=0}^{i-1} h_k g_{i-1-k} (h_0 = g_0 = 1)$ for $1 < i < n$. Using induction on $i$ we get that $g \in \mathcal{N}^n(\mathbb{K})$. Hence, $A(\mathcal{H}^n(\mathbb{K}) \rtimes \mathcal{N}^n(\mathbb{K})) = \tilde{\mathcal{R}}(\mathbb{K})$.

As every Riordan matrix is a lower triangular matrix, for every $m \in \mathbb{N}$, we have a natural homomorphism $\pi_m : \tilde{\mathcal{R}}(\mathbb{K}) \to GL_m(\mathbb{K})$ defined by $\pi_m((d_{i,j})_{i,j\in \mathbb{N}}) := (d_{i,j})_{i,j=0,\ldots,m-1}$ for every $(d_{i,j})_{i,j\in \mathbb{N}} \in \tilde{\mathcal{R}}(\mathbb{K})$. For every $m \in \mathbb{N}$, put $\tilde{\mathcal{R}}_m := \pi_m(\tilde{\mathcal{R}}(\mathbb{K}))$ then we have $\mathcal{R}_m \cong \tilde{\mathcal{R}}(\mathbb{K}) / \tilde{\mathcal{N}}(\mathbb{K})$ and the natural map $\varphi_{m+1} : \tilde{\mathcal{R}}_{m+1} \to \tilde{\mathcal{R}}_m$ given by $\varphi_{m+1}((d_{i,j})_{i,j=0,\ldots,m+1}) := (d_{i,j})_{i,j=0,\ldots,m}$ is a homomorphism. Then the Riordan group is isomorphic to the inverse limit group $\lim_{\leftarrow m} \{ (\tilde{\mathcal{R}}_m, \varphi_m)_{m \in \mathbb{N}} \}$.

**Lemma 2.6.** We have that

\[
\mathcal{H}^m(\mathbb{K}) = \langle \varphi(g)(h)h^{-1} \mid g \in \mathcal{N}^{m-1}(\mathbb{K}), h \in \mathcal{H}(\mathbb{K}) \rangle = \langle \varphi(g)(h)h^{-1} \mid g \in \mathcal{N}^{m-1}(\mathbb{K}), h = 1 + x \rangle,
\]

for any $m \geq 2$.

**Proof.** Let $h = 1 + a_1 x + a_2 x^2 + \cdots \in \mathcal{H}(\mathbb{K})$ and $g = x + b_m x^m + \cdots \in \mathcal{N}^{m-1}(\mathbb{K})$. From Lemma 2.1 we get that $\varphi(g)(h)h^{-1} = 1 + a_1 b_1 x + (a_1 b_{m+1} + r_{m+1}) x^{m+1} + \ldots$, where $r_n (n > m)$ is a polynomial which depends on $a_1, \ldots, a_{n-1}, b_m, \ldots, b_{n-1}$, for every $n \geq m + 1$. Hence
Let $\langle \varphi(g)(h)h^{-1}, g \in \mathcal{H}^{m-1}(\mathbb{K}), h \in \mathcal{H}(\mathbb{K}) \rangle \subset \mathcal{H}^{m}(\mathbb{K})$.

If $h = 1 + x$ then $h^{-1} = 1 + \sum_{i=1}^{\infty}(-1)^{i}x^{i}$. Therefore, for every $g = x + \sum_{j=m}^{\infty}b_{j}x^{j} \in \mathcal{H}^{m-1}(\mathbb{K})$, we have

$$h(g)h^{-1} = [(1 + x) + \sum_{j=m}^{\infty}b_{j}x^{j}](1 + \sum_{i=1}^{\infty}(-1)^{i}x^{i})$$

$$= 1 + b_{m}x^{m} + \sum_{j=m+1}^{j-1}(\sum_{i=0}^{j-1}(-1)^{i}b_{j-i})x^{j}.$$  

By the freedom of choice of $b_{j} \in \mathbb{K}$ for every $j \geq m$ we get that

$$\mathcal{H}^{m}(\mathbb{K}) \subset \langle \varphi(g)(h)h^{-1}, g \in \mathcal{H}^{m-1}(\mathbb{K}), h = 1 + x \rangle.$$  

Next we will present topological finite generating sets of $\mathcal{R}(\mathbb{F}_{p})$ and $\mathcal{R}(\mathbb{Z})$. Let us recall the definition of (topological) generating sets of a profinite group.

**Definition 2.7.** Let $S$ be a subset of a profinite group $G$. We call that $S$ (topologically) generates $G$ or $S$ is a (topologically) generating set of $G$ if the subgroup $\langle S \rangle$ of $G$ generated by $S$ is dense in $G$. A profinite group $G$ is called finitely generated if it has a finitely generating set $S$.

**Proposition 2.8.** The Riordan groups $\mathcal{R}(\mathbb{F}_{p})$ and $\mathcal{R}(\mathbb{Z})$ are finitely generated, for every prime number $p \geq 2$. Furthermore, for every generating set $S_{N}$ of the Nottingham group $\mathcal{N}(\mathbb{F}_{p})$ (respectively $\mathcal{N}(\mathbb{Z})$), the group $\mathcal{R}(\mathbb{F}_{p})$ (respectively $\mathcal{R}(\mathbb{Z})$) is generated by

$$\{(1 + x, x), (1, g) : g \in S_{N}\}.$$  

**Proof.** We know that $\mathcal{N}(\mathbb{F}_{p}) = \lim N(\mathbb{F}_{p})/N^{m}(\mathbb{F}_{p})$, $\mathcal{N}(\mathbb{Z}) = \lim N(\mathbb{Z})/N^{m}(\mathbb{Z})$, and both $\mathcal{N}(\mathbb{F}_{p})$ and $\mathcal{N}(\mathbb{Z})$ are topologically finitely generated, see for example [2, Theorem 4.5] and [3, Theorem 1.1]. Let $\mathbb{K} = \mathbb{F}_{p}$ or $\mathbb{Z}$ and let $S_{N}$ be a finite generator set of $\mathcal{N}(\mathbb{K})$. From Lemma 2.6, the group $\mathcal{H}^{2}(\mathbb{K}) \times \{x\}$ is generated by $(1 + x, x)$ and $\{1\} \times N(\mathbb{K})$ so for every $m \geq 2$ the group $\mathcal{H}^{2}(\mathbb{K}) \times N(\mathbb{K})/\mathcal{R}^{m}(\mathbb{K})$ is generated by the left cosets of $\mathcal{R}^{m}(\mathbb{K})$: $(1 + x, x)\mathcal{R}^{m}(\mathbb{K})$ and $\{1\} \times N(\mathbb{K})/\mathcal{R}^{m}(\mathbb{K})$. Therefore the group $\mathcal{H}^{2}(\mathbb{K}) \times N(\mathbb{K})/\mathcal{R}^{m}(\mathbb{K})$ is generated by $(1 + x, x)\mathcal{R}^{m}(\mathbb{K})$ and $\{(1, s)\mathcal{R}^{m}(\mathbb{K}) : s \in S_{N}\}$ for every $m \geq 2$.

On the other hand, for every $a_{i} \in \mathbb{K}$, $1 + a_{1}x$ and $\mathcal{H}^{2}(\mathbb{K})$ generate the set $\{1 \pm a_{1}x + a_{2}x^{2} + a_{3}x^{3} + \ldots | a_{i} \in \mathbb{K}, i \geq 2\}$ because of the two following equations. The first one is

$$1 + a_{1}x + a_{2}x^{2} + a_{3}x^{3} + \ldots = (1 + a_{1}x)(1 + p_{2}x^{2} + p_{3}x^{3} \ldots),$$

where $p_{2} = a_{2}, p_{n+1} = a_{n+1} - a_{1}p_{n}$; and the second one is

$$1 - a_{1}x = (1 + a_{1}x + a_{1}^{2}x^{2} + a_{1}^{3}x^{3} + \ldots)^{-1}. $$

□
Using induction, we claim that $1 + x$ and $\mathcal{H}^2(K)$ generate $1 + a_1 x$ for every $a_1 \in K$. In fact, we have the following equation:

$$1 + (a_1 + 1)x = (1 + x)(1 + a_1 x - a_1 x^2 + a_1 x^3 - a_1 x^4 \ldots).$$

Hence if $K = F_p$ we get the claim. If $K = \mathbb{Z}$ then we get that $1 + x$ and $\mathcal{H}^2(K)$ generate $1 + a_1 x$ for every $a_1 \geq 0$. On the other hand, from above, we also have that $1 + a_1 x$ and $\mathcal{H}^2(K)$ generate $1 - a_1 x$ for every $a_1 \in K$. Therefore, we also obtain the claim for the case $K = \mathbb{Z}$.

Therefore, $1 + x$ and $\mathcal{H}^2(K)$ generate the whole $\mathcal{H}(K)$. This leads that $(1 + x, x)\mathcal{R}^m(K)$ and $\mathcal{H}^2(K) \rtimes N(K)/\mathcal{R}^m(K)$ generate the whole group $\mathcal{R}(K)/\mathcal{R}^m(K)$ for every $m \geq 2$. Therefore, $(1 + x, x)\mathcal{R}^m(K)$ and $\{(1, s)\mathcal{R}^m(K) | s \in S_N\}$ can generate the whole $\mathcal{R}(K)/\mathcal{R}^m(K)$ for every $m \geq 2$. Therefore, we get that $(1 + x, x)$ and $S_N$ topologically generate $\mathcal{R}(K)$ when $K = F_p$ or $K = \mathbb{Z}$.

2.2. **Lower central series of** $\mathcal{R}(K)$. Let $G$ be a group and let $G_1, G_2$ be subgroups of $G$. For elements $g$ and $h$ of $G$, the **commutator** of $g$ and $h$ is $[g, h] := ghg^{-1}h^{-1}$. The **commutator subgroup** of $G_1$ and $G_2$ is the group generated by

$$\{[g_1, g_2] : g_1 \in G_1, g_2 \in G_2\}.$$ 

We denote by $\gamma_2(G) = [G, G]$, the commutator subgroup of $G$. The **lower central series** of $G$ is defined inductively as $\gamma_n(G) := [G, \gamma_{n-1}(G)]$ for every $n \geq 3$.

In this subsection we investigate the lower central series of $\mathcal{R}(K)$.

The following lemma is a slight generalization of [13, Proposition 3.3] when the group $H$ is commutative.

**Lemma 2.9.** Let $G$ be a group and $G_1, G_2$ be normal subgroups of $G$. Let $H$ be a commutative group and $H_1, H_2$ be normal subgroups of $H$. Suppose that there is an action $\varphi : G \to \text{Aut}(H)$ such that we have $\varphi(g_i)|_{H_i} \in \text{Aut}(H_i)$ for all $i = 1, 2$ and $g_i \in G_i$. Let $L$ be the subgroup of $H$ generated by

$$\{\varphi(g_1)(h_2)^{-1}, \varphi(g_2)(h_1)^{-1} : g_i \in G_i, h_i \in H_i, i = 1, 2\}.$$ 

Then $\varphi(g)|_{L} \in \text{Aut}(L)$ for every $g \in [G_1, G_2]$ and

$$(1) \quad [H_1 \rtimes \varphi G_1, H_2 \rtimes \varphi G_2] = L \rtimes \varphi [G_1, G_2].$$

**Proof.** For any $g, g' \in G$ and $h \in H$ we have

$$\varphi(g)(\varphi(g')(h)h^{-1}) = \varphi(gg')(h)h^{-1} \cdot (\varphi(g)(h)h^{-1})^{-1}.$$ 

As $G_1$ is a normal subgroup of $G$ we get that $gg' \in G_1$ for every $g \in [G_1, G_2], g' \in G_1$. Therefore, $\varphi(g)(\varphi(g')(h)h^{-1}) = \varphi(gg')(h)h^{-1} \cdot (\varphi(g)(h)h^{-1})^{-1} \in L$ for every $g \in [G_1, G_2], g' \in G_1, h \in H_2$. Similarly, $\varphi(g)(\varphi(g')(h)h^{-1}) \in L$ for every $g \in [G_1, G_2], g' \in G_2, h \in H_1$. Hence $\varphi(g)(L) \subset L$ for every $g \in [G_1, G_2]$.

Let $g \in [G_1, G_2]$. It is clear that $\varphi(g)$ is injective. The surjectivity of $\varphi(g)|_{L}$ follows from the following equation.

$$\varphi(g)(\varphi(g^{-1}g')(h)h^{-1} \cdot (\varphi(g^{-1})(h^{-1})h)) = \varphi(g')(h)h^{-1}, \text{ for every } g' \in G, h, h' \in H.$$
Hence \( \varphi(g)|_L \in \operatorname{Aut}(L) \) for every \( g \in [G_1, G_2] \).

Now we are ready to prove (1). For every \((h_i, g_i) \in H_i \rtimes \varphi G_i\) we have

\[
[(h_1, g_1), (h_2, g_2)] = (\varphi(g_1)(h_2)h_2^{-1} \varphi(g_1g_2g_2^{-1})(h_1^{-1})h_1 \varphi([g_1, g_2] )(h_2^{-1}h_2, [g_1, g_2]).
\]

Therefore \([H_1 \rtimes \varphi G_1, H_2 \rtimes \varphi G_2] \subset L \rtimes \varphi [G_1, G_2].\) On the other hand for every \(g_i \in G_i, h_i \in H_i, i = 1, 2,\) we have

\[
(\varphi(g_1)(h_2)h_2^{-1}, 1) = [(1, g_1), (h_2, 1)] \quad \text{and} \quad (\varphi(g_2)(h_1)h_1^{-1}, 1) = [(h_1, 1), (1, g_2)]^{-1}.
\]

Hence, \(L \rtimes \varphi [G_1, G_2] \subset [H_1 \rtimes \varphi G_1, H_2 \rtimes \varphi G_2].\) \qed

Now we are ready to calculate the lower central series of the Riordan group \(\mathcal{R}(\mathbb{F}_q)\).

**Lemma 2.10.** Let \(\mathbb{F}_q\) be finite field with characteristic \(p > 2.\) We have for \(n \geq 2,\)

\[(2) \quad \gamma_n(\mathcal{R}(\mathbb{F}_q)) = \mathcal{H}^n + [(n-2)/(p-1)](\mathbb{F}_q) \rtimes \mathcal{N}^n + 1 + [(n-2)/(p-1)](\mathbb{F}_q).\]

**Proof.** For convenience, we denote \( n + [(n-2)/(p-1)] \) just by \(n\).

Applying Lemmas 2.9, 2.6 for \(m = 2,\) and \([6, \text{Remark 1 ii}]\) (or \([19, \text{Proposition 12.4.24}]\), formula (2) is true for the case \(n = 2.\)

Suppose that formula (2) is true for \(n \geq 2,\) applying Lemma 2.9 and \([6, \text{Remark 1 ii}]\) (or \([19, \text{Proposition 12.4.24}]\) we get

\[
\gamma_{n+1}(\mathcal{R}(\mathbb{F}_q)) = [\mathcal{H}(\mathbb{F}_q) \rtimes \mathcal{N}(\mathbb{F}_q), \mathcal{H}^n(\mathbb{F}_q) \rtimes \mathcal{N}^{n+1}(\mathbb{F}_q)] = L \rtimes \mathcal{N}^{n+1+1},
\]

where \(L\) is the subgroup of \(\mathcal{H}(\mathbb{F}_q)\) generated by

\[
\{h'(g)h'^{-1}, h(g')h'^{-1} : g \in \mathcal{N}(\mathbb{F}_q), g' \in \mathcal{N}^{n+1}(\mathbb{F}_q), h \in \mathcal{H}(\mathbb{F}_q), h' \in \mathcal{H}^n(\mathbb{F}_q)\}.
\]

We need to check that \(L = \mathcal{H}^{n+1}(\mathbb{F}_q).\) Let \(g \in \mathcal{N}(\mathbb{F}_q), g' \in \mathcal{N}^{n+1}(\mathbb{F}_q), h \in \mathcal{H}(\mathbb{F}_q), h' \in \mathcal{H}^n(\mathbb{F}_q)\) then write

\[
h' = 1 + a_{\tau_n} x^{\tau_n} + a_{\tau_n+1} x^{\tau_n+1} + \ldots,
\]

\[
g = x + b_1 x^2 + b_3 x^3 + \ldots.
\]

Applying Lemma 2.1, the smallest positive degree for \(h'(g)h'^{-1}\) is \(\tau_n + 1\) and the coefficient of \(\tau_n + i - 1\)-degree \((i \geq 2)\) term is of the form

\[
\tau_n a_{\tau_n} b_i + \tau_{\tau_n+i-1},
\]

where \(\tau_{\tau_n+i-1}\) is a polynomial depends on \(b_2, \ldots, b_{\tau_n-1}, a_{\tau_n}, \ldots, a_{\tau_n+i-1}\). Applying Lemma 2.6 for \(m = \tau_n + 2, \mathcal{H}^{\tau_n+2}(\mathbb{F}_q) = \langle h(g')h'^{-1} \rangle = \mathcal{H}^{\tau_n+2}(\mathbb{F}_q) \subseteq L \subseteq \mathcal{H}^{\tau_n+1}(\mathbb{F}_q).\)

If \(\tau_n\) is not divisible by \(p\) then \(\tau_n + 1 = \tau_{n+1}\), because \([(n - 1)/(p - 1)] = [(n - 2)/(p - 1)]\). Also, because \(\tau_n\) is invertible in \(\mathbb{F}_q\), by choosing \(a'_{\tau_n} = 1\) we are done.

If \(\tau_n\) is divisible by \(p\) then \(\tau_n a_{\tau_n} b_i = 0\) in \(\mathbb{F}_q\) so by Lemma 2.1 the coefficient of \(\tau_n + 1\)-degree is zero. The smallest positive degree of \(h'(g)h'^{-1}\) is \(\tau_n + 2 = \tau_{n+1}\). Hence, \(L = \mathcal{H}^{\tau_n+2}(\mathbb{F}_q) = \mathcal{H}^{\tau_n+1}(\mathbb{F}_q).\) \qed
**Definition 2.11.** ([17] or [19, Definition 12.1.5]) Let $G$ be an infinite pro-$p$-group. We say that $G$ has finite width if
\[
\sup_n \log_p(\frac{\gamma_n(G)}{\gamma_{n+1}(G)}) < \infty.
\]

The concept of coclass is very useful in approaching a classification of finite $p$-groups. The notion finite width arises naturally as a generalization of coclass to classify pro-$p$-groups [17], [19, Chapter 12].

**Corollary 2.12.** Let $\mathbb{F}_q$ be a finite field with characteristic $p$. Then $\mathcal{R}(\mathbb{F}_q)$ is a pro-$p$-group with finite width.

**Proof.** We see that when we increase $n$ by 1 then the smallest positive degree in $H_n + \lfloor (n-2)/(p-1) \rfloor$ and the smallest degree which is greater than 1 in $\mathcal{N}_{n+1} + \lfloor (n-2)/(p-1) \rfloor$ are increasing by 1 or 2. So there are maximum $q^4$ elements that are in $\gamma_n(\mathcal{R}(\mathbb{F}_q))$ but not in $\gamma_{n+1}(\mathcal{R}(\mathbb{F}_q))$. Hence, $|\gamma_n(\mathcal{R}(\mathbb{F}_q))/\gamma_{n+1}(\mathcal{R}(\mathbb{F}_q))| \leq q^4$, this means $\mathcal{R}(\mathbb{F}_q)$ has finite width. \(\square\)

**Proof of Theorem 1.1.** Theorem 1.1 follows from Corollaries 2.3, 2.12 and Proposition 2.8. \(\square\)

3. INDEX-SUBGROUPS AND HAUSDORFF DIMENSIONS

We begin this section by recalling the definitions of Hausdorff dimension, and lower and upper box dimensions in metric spaces. For more details, see [12].

Let $(X, d)$ be a metric space, $S$ be a subset of $X$, and $\varepsilon, \alpha > 0$. We define

\[
\dim_{H}^{\alpha}(S) := \inf \sum_k (\text{diam}S_k)^{\alpha},
\]

where the infimum is taken over all covers $\{S_k\}_{k \in \mathbb{N}}$ of $S$ with diam$S_k \leq \varepsilon$ for every $k \in \mathbb{N}$. Here diam$S_k := \sup_{x, y \in S_k} d(x, y)$, the diameter of $S_k$. For every $\alpha > 0$, $\dim_{H}^{\alpha}(S)$ is non-increasing with $\varepsilon$ and hence the limit

\[
\dim_{H}(S) := \lim_{\varepsilon \to 0} \dim_{H}^{\alpha}(S)
\]

exists.

The *Hausdorff dimension* of $S$ is defined as follows.

\[
\dim_{H}(S) := \sup\{\alpha > 0 : \dim_{H}^{\alpha}(S) = \infty\} = \inf\{\alpha > 0 : \dim_{H}^{\alpha}(S) = 0\}.
\]

It is clear that if $S \subset S'$ then $\dim_{H}(S) \leq \dim_{H}(S')$.

Now we assume further that the metric space $(X, d)$ is compact. As $X$ is compact, for every $\varepsilon > 0$ and for every cover $\{B(x, \varepsilon)\}_{x \in I}$ of $S$ we can choose a finite subcover. Here, $B(x, \varepsilon) := \{y \in X : d(y, x) < \varepsilon\}$ is the open ball in $X$ with the center at $x$ and the radius $\varepsilon$. We define $N_{\varepsilon}(X)$ the minimal number of open balls of radius $\varepsilon$ required to cover $S$. The *lower and upper box dimensions* of $S$ are defined as follows.

\[
\dim_{B}(S) := \liminf_{\varepsilon \to 0} \frac{\log N_{\varepsilon}(S)}{-\log \varepsilon} \quad \text{and} \quad \overline{\dim}_{B}(S) := \limsup_{\varepsilon \to 0} \frac{\log N_{\varepsilon}(S)}{-\log \varepsilon}.
\]
For every bounded subset $S$ of a metric space $X$, we always have $\dim_H(S) \leq \dim_B(S)$ [12, Proposition 3.4]. Note that although [12, Proposition 3.4] only stated the result for bounded subsets of $X = \mathbb{R}^n$, it still holds for general metric spaces.

Now let $G$ be a profinite group. A filtration is a descending chain of open normal subgroups $G_1 = G \supseteq G_2 \supseteq \cdots$ which form a base for the neighborhoods of the identity of $G$. If $\{G_n\}$ is a filtration of $G$ then $\bigcap_{n \in \mathbb{N}} G_n = \{1_G\}$.

Given a filtration $\{G_n\}$ of $G$, we define an invariant metric $d$ on $G$ by

$$d(g, h) := \inf \left\{ \frac{1}{|G : G_n|} : gh^{-1} \in G_n \right\}.$$ 

Then every set of diameter $\frac{1}{|G : G_n|}$ is contained in some coset of $G_n$. Let $H$ be a subgroup of $G$. If $\rho = \frac{1}{2|G : G_n|}$ then $N_\rho(H) = |HG_n : G_n| = |H : H \cap G_n|$ and hence

$$\dim_B(H) = \liminf_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G/G_n|} \quad \text{and} \quad \dim_B(H) = \limsup_{n \to \infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}.$$ 

In [1, Proposition 2.6], Abercrombie showed that $\dim_B(H) \geq \liminf_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G/G_n|}$ for every closed subgroup $H$ of $G$. Therefore, for every closed subgroup $H$ of $G$ and every number $p \geq 2$, we have that [5, Theorem 2.4]

$$\dim_H(H) = \dim_B(H) = \liminf_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G/G_n|} = \liminf_{n \to \infty} \frac{\log p |HG_n : G_n|}{\log p |G/G_n|}.$$ 

The Hausdorff dimension does depend on the choice of the filtration $\{G_n\}$ [5, Example 2.5].

The Hausdorff spectrum of the profinite group $G$ is the set

$$\text{hspec}(G) := \{ \dim_H(H) : H \text{ is a closed subgroup of } G \}.$$ 

The study of Hausdorff spectrum of profinite groups has been initiated by Barnea and Shalev [5]. In [5, Theorem 1.6] we know that for Nottingham group $\mathcal{N}(\mathbb{F}_p)$ with prime $p \geq 5$ it holds that for every closed subgroup $H$ of $\mathcal{N}(\mathbb{F}_p)$, $\dim_H(H)$ does not depend on the choice of the filtration and

$$\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \text{hspec}(\mathcal{N}(\mathbb{F}_p)) \subset [0, \frac{3}{p}] \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$ 

Furthermore, later via introducing index-subgroups of $\mathcal{N}(\mathbb{F}_p)$, Barnea and Klopsch showed that [4, Theorem 1.8] for every $p > 2$,

$$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup [0, \frac{1}{p}] \cup \left\{ \frac{1}{p} + \frac{1}{r} : r \in \mathbb{N} \right\} \subset \text{hspec}(\mathcal{N}(\mathbb{F}_p)).$$ 

Inspired by [4], in this section we will introduce index-subgroups of $\mathcal{R}(\mathbb{F}_p)$ and study $\text{hspec}(\mathcal{R}(\mathbb{F}_p))$. We start with constructions of some filtrations of $\mathcal{R}(\mathbb{K})$.

**Lemma 3.1.** Let $\mathbb{K}$ be a unital commutative ring. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a function such that $\sigma(1) = 1, \sigma(m+n) \leq \sigma(m) + \sigma(n)$, non-decreasing and $\lim_{n \to \infty} \sigma(n) = \infty$. 

Then $G^n_i = \mathcal{H}(\sigma^n(\mathbb{K})) \times \mathcal{N}^n(\mathbb{K})$ is a filtration of $\mathcal{R}(\mathbb{K})$ satisfying that $[G^n_i, G^n_j] \subseteq G^n_{i+j}$ for every $i, j \in \mathbb{N}$.

**Proof.** From Lemma 2.2, we see that $G^n_i$ form a chain of descending chain of normal subgroups of $\mathcal{R}(\mathbb{K})$. Remember that for $h = 1 + ax^n + O(x^{n+1}) \in \mathcal{H}(\mathbb{K})$ and $g = x + bx^{m+1} + O(x^{m+2}) \in \mathcal{N}(\mathbb{K})$, we have

$$h(g)h^{-1} = (h + abnx^{n+m} + O(x^{n+m+1}))h^{-1} \in \mathcal{H}^{m+n}(\mathbb{K}).$$

Combining with the following equation

$$[(h_1, g_1), (h_2, g_2)] = (\varphi(g_1)(h_2)h_2^{-1} \cdot \varphi(g_1g_2g_1)(h_1^{-1})h_1 \cdot \varphi([g_1, g_2])(h_2^{-1})h_2, [g_1, g_2]),$$

we get that

$$[\mathcal{H}(\sigma(m)\mathbb{K}) \times \mathcal{N}^m(\mathbb{K}), \mathcal{H}(\sigma(n)\mathbb{K}) \times \mathcal{N}^n(\mathbb{K})] \subseteq \mathcal{H}^{\min\{\sigma(m)+n, m+\sigma(n)\}}(\mathbb{K}) \times \mathcal{N}^{m+n}(\mathbb{K}).$$

By the properties of $\sigma$, we have $\sigma(m) + n \geq \sigma(m) + \sigma(n) \geq \sigma(m + n)$ and similarly for $m + \sigma(n) \geq \sigma(m + n)$. Hence, $G^n_i$ is a filtration with the property we are looking for. \qed

Now we introduce the important notion of this section, the index-subgroups of $\mathcal{R}(\mathbb{F}_p)$. For every $I, J \subseteq \mathbb{N}$ we define

$$\mathcal{H}(I) : = \{1 + \sum_{i \in I} a_i x^i | a_i \in \mathbb{F}_p\},$$

$$\mathcal{N}(J) : = \{x + \sum_{j \in J} b_j x^{j+1} | b_j \in \mathbb{F}_p\},$$

$$\mathcal{R}(I, J) : = \{(1 + \sum_{i \in I} a_i x^i, x + \sum_{j \in J} b_j x^{j+1}) | a_i, b_j \in \mathbb{F}_p\}.$$  

It is clear the $\mathcal{R}(I, J)$ is a closed subset of $\mathcal{R}(\mathbb{F}_p)$ for every $I, J \subseteq \mathbb{N}$. We call $(I, J)$ an admissible index-pair if $\mathcal{R}(I, J)$ is a subgroup of $\mathcal{R}(\mathbb{F}_p)$. If $(I, J)$ is an admissible index-pair then we say that $\mathcal{R}(I, J)$ is an index-subgroup of $\mathcal{R}(\mathbb{F}_p)$.

**Lemma 3.2.** The set $\mathcal{H}(I)$ is a subgroup of $\mathcal{H}(\mathbb{F}_p)$ if and only if $I$ is closed under addition.

**Proof.** From the follow equation

$$(1 + \sum_{i \in I} a_i x^i)(1 + \sum_{i' \in I} a_{i'} x^{i'}) = 1 + \sum_{i, i' \in I} a_i a_{i'} x^{i+i'},$$

$\mathcal{H}(I)$ is closed under multiplication if and only if $I$ is closed under addition. We only need to prove that $\mathcal{H}(I)$ is closed under inversion if $I$ is closed under addition. Suppose there exists $h \in \mathcal{H}(I)$ such that $h^{-1} \notin \mathcal{H}(I)$. Let $i_0$ be the smallest positive degree of $h^{-1}$ with non-zero coefficient such that $i_0 \notin I$. The coefficient of $i_0$-degree in $hh^{-1}$ is $c = \sum_{i=0}^{i_0} a_i a_{i_0-i}^*$, where $a_i, a_j^*$ are coefficients at degree $i, j$ of $h, h^{-1}$ respectively. As $i_0 \notin I$, we see that $a_i a_{i_0-i}^* \neq 0$ only if $i = 0$ or $i, i_0 - i \in I$. But
$i, i_0 - i$ cannot both be inside $I$ when $I$ is closed under addition so the only case $a_i a_{i_0 - i} \neq 0$ is when $i = 0$. Hence, $c = a_{i_0}' \neq 0$, which is a contradiction. \[\bigtriangleup\]

**Lemma 3.3.** Let $I, J \subset \mathbb{N}$. Then $(I, J)$ is an admissible index-pair if and only if all the followings happen

1. For $j \in J$ and $n \in \{1, \ldots, j + 1\}$ with $\binom{j + 1}{n}$ is not divisible by $p$ we have $\{j + n j' | j' \in J\} \subseteq J$;

2. $I$ is closed under addition;

3. For $i \in I$ and $n \in \{1, \ldots, i\}$ with $\binom{i}{n}$ is not divisible by $p$ we have $\{i + n j | j \in J\} \subseteq I$.

**Proof.** First, we show the “only if” part. Assume that $(I, J)$ is an admissible index-pair. As $\mathcal{N}(J)$ is a subgroup of $\mathcal{N}(\mathbb{F}_p)$ applying [4, Theorem 1.1], we get condition (1). On the other hand, $\mathcal{H}(I)$ is a subgroup of $\mathcal{H}(\mathbb{F}_p)$, applying Lemma 3.2 we get (2).

For the last condition, we observe that for every $h \in \mathcal{H}(I)$ and $g \in \mathcal{N}(J)$ we have $(1, g)(h, g) = (h(g), x)$ and hence $h(g) \in \mathcal{H}(I)$. Take $h = 1 + x^i$ and $g = x + x^{j+1}$ with $i \in I$ and $j \in J$, we have

$$h(g) = 1 + (x + x^{j+1})^i = 1 + \sum_{n=0}^{i} \binom{i}{n} x^{nj+i}.$$

Therefore, if $\binom{i}{n}$ is not divisible by $p$ we have $\{i + n j | j \in J\} \subseteq I$.

Now, we show the “if” part. From the definition of the operation on Riordan group

$$(h', g)(h, g') = (h'h(g), g'(g)) \quad \text{and} \quad (h, g)^{-1} = (h^{-1}(g), \bar{g}).$$

To make $\mathcal{R}(I, J)$ a group we need $h'h(g), h^{-1}(\bar{g}) \in \mathcal{H}(I)$ and $g'(g), \bar{g} \in \mathcal{N}(J)$ for every $h, h' \in \mathcal{H}(I)$ and $g, g' \in \mathcal{N}(J)$. From (1) and [4, Theorem 1.1] we get that $g'(g), \bar{g} \in \mathcal{N}(J)$ for every $g, g' \in \mathcal{N}(J)$. Lemma 3.2 and condition (2) give us that $\mathcal{H}(I)$ is a subgroup of $\mathcal{H}(\mathbb{F}_p)$. Thus, we now only need to show $h(g) \in \mathcal{H}(I)$ for every $h \in \mathcal{H}(I), g \in \mathcal{N}(J)$. Put

$$\mathcal{S} := \{x + ax^{j+1} : j \in J, a \in \mathbb{F}_p\}.$$

Let $\mathcal{F}$ be the sub-semigroup generated by $\mathcal{S}$, i.e.

$$\mathcal{F} := \{g_1 \circ g_2 \circ \cdots \circ g_s | s \in \mathbb{N}, g_i \in \mathcal{S} \text{ for every } 1 \leq s\}.$$

We endow $\mathcal{N}(\mathbb{F}_p) = \varprojlim \mathcal{N}(\mathbb{F}_p)/\mathcal{N}^{n+1}(\mathbb{F}_p)$ with its inverse limit topology. Then $\text{cl}(h(\mathcal{F}))$, the topological closure of $h(\mathcal{F})$ in $\mathcal{N}(\mathbb{F}_p)$ is also a semigroup and hence
from [4, Lemma 2.2] it is a subgroup of $N(F_p)$. For $h = 1 + \sum_{i \in I} a_i x^i \in \mathcal{H}(I)$ and $g = x + bx^{j+1} \in \mathcal{S}$, we have

$$h(g) = 1 + \sum_{i \in I} \sum_{n=0}^{i} a_i \left( \begin{array}{c} i \\ n \end{array} \right) b^n x^{nj+i} \in \mathcal{H}(I).$$

So $h(g) \in \mathcal{H}(I)$ for any $h \in \mathcal{H}(I)$ and $g \in \mathcal{S}$. From continuity, $h(\text{cl}(\mathcal{S})) \subseteq \text{cl}(h(\mathcal{S})) \subseteq \mathcal{H}(I)$. The result follows. □

From now on, for simplicity, we denote $N = N(F_p), N^n = N^n(F_p), \mathcal{H}^n = \mathcal{H}^n(F_p)$ and $\mathcal{R} = \mathcal{R}(F_p)$ for every $n \in \mathbb{N}$. We also fix the filtration $\{\mathcal{H}^n \times N^n\}_n$. Before studying the Hausdorff dimensions of index-subgroups of $\mathcal{R}$, let us recall the lower and upper density. Let $I \subset \mathbb{N}$. The lower and upper density of $I$ are defined as follows.

$$\text{ldense}(I) : = \lim_{m \to \infty} \inf \frac{|\{i \in I : i \leq m\}|}{m},$$

$$\text{udense}(I) : = \lim_{m \to \infty} \sup \frac{|\{i \in I : i \leq m\}|}{m}.$$ 

If $\text{ldense}(I) = \text{udense}(I)$ then the density of $I$, $\text{dense}(I)$, is defined as this common value.

We define

$$\text{inspec}(\mathcal{R}) := \{\text{dim}_H(\mathcal{R}(I,J)) | (I,J) \text{ is an admissible index-pair} \} \subset \text{hspec}(\mathcal{R}).$$

Before investigating further $\text{dim}_H(\mathcal{R}(I,J))$, where $(I,J)$ is an admissible index-pair, let us define functions $W : \mathbb{N} \to [0,1]$ and $w : (p\mathbb{N} - 1) \to [0, \frac{1}{p}]$, which have been used in [4], as follows. For $m \in \mathbb{N}$ with $m = \sum_{n=0}^{\infty} m_n p^n$, we put

$$W(m) := \sum_{n=0}^{\infty} m_n p^{-n-1}, \quad m_n \in \{0, 1, \ldots, p-1\}.$$ 

The function $w : p\mathbb{N} - 1 \to [0, \frac{1}{p}]$ is defined by $w(j) := W(j+1)$ for every $j \in p\mathbb{N} - 1$. For every $\xi \geq 0$, we define

$$J(\xi) := \{j \in (p\mathbb{N} - 1) | w(j) < \xi \}.$$ 

The following lemma is an adaptation of [4, Lemma 4.1].

**Lemma 3.4.** Let $\xi \in [0, \frac{1}{p}]$ then $J(\xi)$ satisfies the condition 1) of Lemma 3.3 and $\text{dense}(J(\xi) \cap s\mathbb{N}) = s^{-1} \xi$ for any $s$ which is not divisible by $p$.

**Proof.** From [4, Example 3.5] we know that $J(\xi)$ satisfies the condition 1) of Lemma 3.3.
We write $\xi$ in the form $\sum_{k=1}^{\infty} \xi_k p^{-k-1}$, where $\xi_k \in \{0, 1, \ldots, p-1\}$. For every $m \in \mathbb{N}$, we define

$$J_m := \bigcup_{0 \leq t < \xi_m} \{ i \in s\mathbb{N} | i \equiv_{p^{m+1}} 1 + \sum_{n=1}^{m-1} \xi_n p^n + tp^m \},$$

and

$$S_m := \bigcup_{\xi_m < t \leq p-1} \{ i \in s\mathbb{N} | i \equiv_{p^{m+1}} 1 + \sum_{n=1}^{m-1} \xi_n p^n + tp^m \}.$$

For every $m \in \mathbb{N}$ and $0 \leq t < \xi_m$, because $s$ is coprime with $p^{m+1}$, there exists a unique element $k \in \{1, \ldots, p^{m+1} - 1\}$ such that $ks \equiv_{p^{m+1}} 1 + \sum_{n=1}^{m-1} \xi_n p^n + tp^m$. Thus, for every $m \in \mathbb{N}$, $J_m$ is a union of disjoint arithmetic progressions with increment $sp^{m+1}$. Note that the density of an arithmetic progression with increment $t$ is $1/t$. Hence $\text{dense}(J_m) = \frac{\xi_m}{s} p^{-m-1}$ for every $m \in \mathbb{N}$. Therefore, for every $m \in \mathbb{N}$, we have

$$\text{ldense}(J(\xi)) \geq \text{ldense}(\bigcup_{i=1}^{m} J_i) = \sum_{i=1}^{m} \text{dense}(J_i) = \frac{1}{s} \sum_{i=1}^{m} \xi_i p^{-i-1}. $$

Letting $m \to \infty$ we get that $\text{ldense}(J(\xi)) \geq s^{-1} \xi$.

Similarly, for every $m \in \mathbb{N}$ we have

$$\text{ldense}(S) \geq \text{ldense}(\bigcup_{i=1}^{m} S_i) = \sum_{i=1}^{m} \text{dense}(S_i) = \frac{1}{s} \sum_{i=1}^{m} (p - 1 - \xi_i) p^{-i-1}. $$

Letting $m \to \infty$ we get that $\text{ldense}(S) \geq s^{-1} (1/p - \xi)$. On the other hand, we have $J(\xi)$ and $S$ are disjoint, $J(\xi) \cup S = s\mathbb{N} \cap (p\mathbb{N} - 1) \setminus w^{-1}(\xi)$, and the set $w^{-1}(\xi)$ has at most one element. Hence, as $s$ is coprime with $p$ we get that

$$\text{udense}(J(\xi)) = \frac{1}{s} \left[ 1/p - (1/p - \xi) \right] = \frac{\xi}{s}. $$

$\square$
Inspired by [4, Proposition 4.3], we characterize the index-subgroups of $\mathcal{R}(\mathbb{F}_p)$ as follows.

**Lemma 3.5.** Let $(I, J)$ be an admissible pair. Then one of the following holds:

1. $I$ is empty and $J$ satisfies the condition 1) of Lemma 3.3;
2. $I$ is a cofinite subset of $sp^r\mathbb{N}$, where $s \in \mathbb{N}$ is not divisible by $p$ and $r \in \mathbb{N}\cup\{0\}$.

In this case, $J \subseteq s\mathbb{N}$ and furthermore $J$ must satisfies one of the following properties:

(i) $J \subseteq (p\mathbb{N} - 1) \cap s\mathbb{N}$ and so $\text{ldense}(J) \in \left[0, \frac{1}{sp}\right]$;
(ii) there exists $s_0 \in \mathbb{N}$ such that $J$ is a cofinite set of $s_0\mathbb{N}$ and so $\text{dense}(J)$ has the form of $\frac{1}{su}$, for some $u \in \mathbb{N}$;
(iii) There exist $s_1, v \in \mathbb{N}$ with $s_1$ is not divisible by $p$ such that $J \subseteq s_1\mathbb{N} \cap (p^v\mathbb{N} \cup (p^v\mathbb{N} - 1))$ and $I \cap (p^v\mathbb{N} \cup (p^v\mathbb{N} - 1))$ is cofinite subset of $s_1\mathbb{N} \cap (p^v\mathbb{N} \cup (p^v\mathbb{N} - 1))$. Moreover, there exists $t \in \{1, \ldots, p^{v-1}\}$, $u \in \mathbb{N}$ such that $\text{dense}(J) = \frac{1+t}{su}$.

**Proof.** As $(I, J)$ is an admissible pair, applying Lemma 3.3 we get that $I$ is closed under addition and hence $I$ is either empty or there exists $t \in \mathbb{N}$ such that $I$ is an cofinite subset of $t\mathbb{N}$. In fact, take $t = \text{gcd}(I)$ when $I$ is not empty, we prove that for large enough $N \in \mathbb{N}$ we have $\{sn|n \geq N, n \in \mathbb{N}\} \subseteq I$. Because $I$ is non-empty, take two elements $a_0, a_1 \in I$ which are not necessarily different. Let $s_1 := \text{gcd}(a_0, a_1)$, then there exist $u, v \in \mathbb{N}_0$ such that $1 = ub_0 - vb_1$, where $a_i = b_is_1$. Then $(kb_1 + l)s_1 = lua_0 + (k - lv)a_1 \in I$ for any $k \geq b_1v, 0 \leq l \leq b_1 - 1$. In other word, $\{s_1n|n \geq b_1^2v, n \in \mathbb{N}\} \subseteq I$. If $s_1 = t$ then we are done. If $s_1 > t$ then there exists $a_2 \in I$ such that $t \leq s_2 := \text{gcd}(s_1, a_2) < s_1$. There exists $a'_1 \in \{s_1n|n \geq b_0^2v, n \in \mathbb{N}\}$ such that $s_2 = \text{gcd}(a'_1, a_2)$ (take $n$ is coprime with $s_2$). With the same argument as $s_1$ there exists $N_2$ such that $\{s_2n|n \geq N_2, n \in \mathbb{N}\} \subseteq I$. We make the same process until we get $s_m = t$.

If $I$ is empty then we get the case 1).

Now we consider that $I$ is a cofinite subset of $q\mathbb{N}$ for some $q \in \mathbb{N}$. We can write $q$ as $sp^r$, where $s \in \mathbb{N}$ is not divisible by $p$ and $r \in \mathbb{N}\cup\{0\}$. As $s$ is not divisible by $p$, we get that $\left(\frac{sp^r}{p^r}\right)$ is not divisible by $p$ and hence $I + p^rJ \subseteq I$. This leads to $J \subseteq s\mathbb{N}$.

If $J \subseteq (p\mathbb{N} - 1) \cap s\mathbb{N}$ we get the case i) and $\text{ldense}(J) \in \left[0, \frac{1}{sp}\right]$.

Now assume that $J \not\subseteq (p\mathbb{N} - 1)$. We define $s_0 := \text{gcd}(S)$, where $S := J \setminus (p\mathbb{N} - 1)$.

For every $j \in S$ we have $\left(\frac{j + 1}{1}\right)$ is not divisible by $p$ and hence from Lemma 3.3 we get that $J + S \subseteq J$. Hence there exists $N \in \mathbb{N}$ such that $J + \{s_0n : n \in \mathbb{N} \cap \mathbb{N}_0\} \subseteq J$.
Lemma that $J$ is finite and dense $(J) = \frac{1}{s_0^v}$. We get the case ii). Since $J \subseteq s\mathbb{N}$ that only happens if $s|s_0$. Then dense $(J)$ has the form of $\frac{1}{su}$ for some $u \in \mathbb{N}$.

Next, let us assume that $J \not\subseteq s_0\mathbb{N}$. Let $j \in J \setminus s_0\mathbb{N}$ then by the definition of $s_0$ we get that $j \in (p\mathbb{N} - 1)$ and hence

$$J \subseteq s_0\mathbb{N} \cup (p\mathbb{N} - 1).$$

On the other hand, from (3), we get $j + \{s_0n|n \geq M\} \subset J \setminus s_0\mathbb{N} \subset (p\mathbb{N} - 1)$. Hence $s_0$ is divisible by $p$. Therefore, $s_0 = p^v s_1$ for some $v, s_1 \in \mathbb{N}$ with $s_1$ is not divisible by $p$.

Now we will prove that $J \subset s_1\mathbb{N}$. As $S \subset s_0\mathbb{N} \subset s_1\mathbb{N}$, it is sufficient to show that $J \cap (p\mathbb{N} - 1) \subset s_1\mathbb{N}$. Let $j \in J \cap (p\mathbb{N} - 1)$. Let $p^k$ be the highest power of $p$ dividing $M + 1$. Applying [4, Lemma 2.1] we have that $(M + 1)p^v s_1 = (M + 1)s_0 \in J$ satisfies \( (M + 1)s_0 + 1 \) is not divisible by $p$. Therefore applying Lemma 3.3, we get $\ell := (M + 1)p^v s_1 + p^{v+k} j \in J$. As $\ell + 1$ is not divisible by $p$, we have $\ell \in s_1\mathbb{N}$ and hence $p^{v+k} j = \ell - (M + 1)p^v s_1 \in s_1\mathbb{N}$. Thus, $j \in s_1\mathbb{N}$ and so we obtain that $J \subset s_1\mathbb{N}$.

By (3), we have $J \subset s_1\mathbb{N} \cap (p^v\mathbb{N} \cup (p\mathbb{N} - 1))$ and hence

$$2 \leq |T| \leq 1 + |\{n|0 < n < s_0\text{ with } s_1|n, p|(n + 1)\}| = 1 + p^{k-1}.$$ 

From (2) and $s|s_1$, we get that dense $(J) = \frac{|T|}{s_0} = \frac{1 + t}{sup v}$ for some $u \in \mathbb{N}, t \in \{1, 2, \ldots, p^{k-1}\}$.

Now we show that $J \cap (p^v\mathbb{N} \cup (p^k\mathbb{N} - 1))$ is a cofinite subset of $s_1\mathbb{N} \cap (p^v\mathbb{N} \cup (p^k\mathbb{N} - 1))$. Since $p^v$ and $s_1$ are coprime, from (3), it is sufficient to prove that $J \cap (p^v\mathbb{N} - 1) \neq \emptyset$. We will prove this by induction. By assumption, we have $J \cap (p\mathbb{N} - 1) \neq \emptyset$. Suppose that $j = np^a - 1 \in J$ for some $a, n \in \mathbb{N}$ with $n$ is not divisible by $p$. Then from Lemma 3.3, $n^2 p^{2a} - 1 = j + (j + 1) j \in J$ and therefore $J \cap (p^{a+1}\mathbb{N} - 1) \neq \emptyset$. Hence by induction, $J \cap (p^v\mathbb{N} - 1) \neq \emptyset$. $\square$
Proof of Theorem 1.2. Let \((I, J)\) be an admissible pair then

\[
\dim_H(\mathcal{R}(I, J)) = \liminf_{n \in \mathbb{N}} \frac{\log |\mathcal{H}(I)\mathcal{H}^{n+1}/\mathcal{H}^{n+1}| + \log |N(J)\mathcal{N}^{n+1}/\mathcal{N}^{n+1}|}{\log |\mathcal{H}/\mathcal{H}^{n+1}| + \log |\mathcal{N}/\mathcal{N}^{n+1}|}
\]

\[
= \liminf_{n \in \mathbb{N}} \frac{|\{i \in I | i \leq n\}| + |\{j \in J | j \leq n\}|}{2n}
\]

\[
= \frac{1}{2} (\text{ldense}(I) + \text{ldense}(J)).
\]

If \((I, J)\) satisfies the condition 1) in Lemma 3.5 then combining Lemma 3.3 (1) and [4, Theorems 1.1 and 1.8] we get that

\[
[0, \frac{1}{2p}] \cup \left\{ \frac{1}{2p} + \frac{1}{2p^r} | r \in \mathbb{N} \right\} \cup \left\{ \frac{1}{2s} | s \in \mathbb{N} \right\} = \{ \dim_H(\mathcal{R}(\emptyset, J)) | \text{J satisfies Lemma 3.3 (1)} \}. \tag{5}
\]

If \((I, J)\) satisfies the condition 2)ii) in Lemma 3.5 then for this case, if \(r \geq 1\)

\[
\dim_H(\mathcal{R}(I, J)) \leq \frac{1}{2sp^r} + \frac{1}{2sp} \leq \frac{1}{p}. \quad \text{If } r = 0, \dim_H(\mathcal{R}(I, J)) \leq \frac{1}{2s} + \frac{1}{2sp} \quad \text{and if } s > p
\]

we still have \(\dim_H(\mathcal{R}(I, J)) \leq \frac{1}{p}\). If \((I, J)\) satisfies the condition 2)ii) in Lemma 3.5 then

\[
\dim_H(\mathcal{R}(I, J)) = \frac{1}{2sp^r} + \frac{1}{2su}.
\]

If \((I, J)\) satisfies the condition 2)ii) in Lemma 3.5 then \(\dim_H(\mathcal{R}(I, J)) = \frac{1}{2sp^r} + \frac{1+t}{2supv}\). If \(r > 1\) or \(s > 1\) or \(u > 1\) or \(t < p^{-1}\) then \(\dim_H(\mathcal{R}(I, J)) \leq \frac{1}{p}\). If \(r \leq 1, s = 1, u = 1\) and \(t = p^{-1}\) then \(\dim_H(\mathcal{R}(I, J)) = \frac{1}{2p^r} + \frac{1}{2p} + \frac{1}{2p^r}\).

Combining all these cases, we have

\[
\text{inspec}(\mathcal{R}) \subseteq [0, \frac{1}{p}] \cup \left\{ \frac{1}{p} + \frac{1}{2p^r} | r \in \mathbb{N} \right\} \cup \left\{ \frac{1}{2} + \frac{1}{2p} + \frac{1}{2p^r} | r \in \mathbb{N} \right\} \cup
\]

\[
\bigcup_{s \leq p} \left[ \frac{1}{2s}, \frac{1}{2s}(1 + \frac{1}{p}) \right] \cup \left\{ \frac{1}{2sp^r} + \frac{1}{2su} | s, u \in \mathbb{N}, r \in \mathbb{N} \cup \{0\} \right\}.
\]

Now we will prove the “\(\supseteq\)” inclusion.

Let \(I = p\mathbb{N}, J = I(\xi)\) for \(\xi \in [0, 1/p]\). Then by Lemmas 3.3 and 3.4 we get that

\((I, J)\) is an admissible pair and \(\dim_H(\mathcal{R}(I, J)) = \frac{1}{2p} + \frac{\xi}{2}\). So \(\left[ \frac{1}{2p}, \frac{1}{p} \right] \subseteq \text{inspec}(\mathcal{R})\).
Also with $I = p\mathbb{N}$ we can take $J = p^r\mathbb{N} \cup (p\mathbb{N} - 1)$ and so $\{ \frac{1}{p} + \frac{1}{2p^r} | r \in \mathbb{N} \} \subseteq \text{inspec}(\mathcal{R})$. Take $I = \mathbb{N}, J = p^r\mathbb{N} \cup (p\mathbb{N} - 1)$ we have $\left\{ \frac{1}{2} + \frac{1}{2p} + \frac{1}{2p^r} | r \in \mathbb{N} \right\} \subseteq \text{inspec}(\mathcal{R})$.

Take $I = s\mathbb{N}, J = I(\xi) \cap s\mathbb{N}$ for $s < p, \xi \in [0, 1/p]$. By Lemmas 3.3 and 3.4 we have $\dim_H(\mathcal{R}(I, J)) = \frac{1}{2s} + \frac{1}{2s}(1 + \frac{1}{p}) \subseteq \text{inspec}(\mathcal{R})$.

For $\left\{ \frac{1}{2sp^r} + \frac{1}{2su} | s, u \in \mathbb{N}, r \in \mathbb{N} \cup \{0\} \right\} \subseteq \text{inspec}(\mathcal{R})$, we take $I = sp^r\mathbb{N}$ and $J = su\mathbb{N}$. \qed

Remark 3.6. (a) In contrast to the Nottingham group $N(\mathbb{F}_p)$, Hausdorff dimensions of the Riordan group $\mathcal{R}(\mathbb{F}_p)$ do depend on the choice of the filtrations.

(b) If $\mathcal{R}(I, J)$ is just infinite then either $I$ or $J$ is an empty set. Recall that an infinite topological group is just infinite if every non-trivial closed normal subgroup has finite index.

Proof. (a) Look back to Lemma 3.1, when we choose $\sigma(n) = n$ we have

$$\dim_H(\mathcal{R}(I, J)) = \frac{1}{2}(\text{ldense}(I) + \text{ldense}(J)).$$

But if we choose $\sigma(n) = \lceil n/2 \rceil$ we have

$$\dim_H(\mathcal{R}(I, J)) = \frac{1}{3}\text{ldense}(I) + \frac{2}{3}\text{ldense}(J).$$

(b) Suppose that $I$ and $J$ are not empty. The set $\mathcal{H}(I) \times \{x\}$ is a non-trivial closed normal subgroup of $\mathcal{R}(I, J)$ with the index equal to $p^{|J|}$. Because $\mathcal{R}(I, J)$ is just infinite so $|J|$ must be finite. Let $j = \max J = sp^r - 1$ then $j + p^r j \in J$, which is a contradiction. \qed

Remark 3.7. From [5, Theorem 1.6] we know that $\text{hspec}(N(\mathbb{F}_p)) \cap (\frac{3}{5}, 1) = \emptyset$ if $p > 5$ and $\text{hspec}(N(\mathbb{F}_p)) \cap (\frac{3}{5}, 1) = \emptyset$ if $p = 5$. In particular, 1 is an isolated point of $\text{hspec}(N(\mathbb{F}_p))$ for $p \geq 5$. It is interesting to investigate whether 1 is an isolated point of $\text{hspec}(\mathcal{R}(\mathbb{F}_p))$.

References

[1] A. G. Abercrombie, Subgroups and subrings of profinite rings, Math. Proc. Cambridge Philos. Soc. 116 (1994), no. 2, 209–222.

[2] I. K. Babenko, Algebra, geometry and topology of the substitution group of formal power series, Uspekhi Mat. Nauk 68 (2013), no. 1(409), 3–76 (Russian, with Russian summary); English transl., Russian Math. Surveys 68 (2013), no. 1, 1–68.
[3] I. K. Babenko and S. A. Bogaty˘ı, On the substitution group of formal integer series, Izv. Ross. Akad. Nauk Ser. Mat. 72 (2008), no. 2, 39–64 (Russian, with Russian summary); English transl., Izv. Math. 72 (2008), no. 2, 241–264.

[4] Y. Barnea and B. Klopsch, Index-subgroups of the Nottingham group, Advances in Mathematics 180 (2003), no. 1, 187 - 221.

[5] Y. Barnea and A. Shalev, Hausdorff dimension, pro-p groups, and Kac-Moody algebras, Trans. Amer. Math. Soc. 349 (1997), no. 12, 5073–5091.

[6] R. Camina, The Nottingham group, New horizons in pro-p groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 205–221.

[7] G.S. Cheon, J.H. Jung, B. Kang, H. Kim, S.R. Kim, S. Kitaev, and S.A. Mojallal, Counting independent sets in Riordan graphs, Discrete Math. 343 (2020), no. 11, 112043, 10.

[8] G.S. Cheon, A. Luzón, M.A. Morón, L. F. Prieto-Martinez, and M. Song, Finite and infinite dimensional Lie group structures on Riordan groups, Adv. Math. 319 (2017), 522–566.

[9] M. du Sautoy and I. Fesenko, Where the wild things are: ramification groups and the Nottingham group, New horizons in pro-p groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 287–328.

[10] M. Ershov, The Nottingham group is finitely presented, J. London Math. Soc. (2) 71 (2005), no. 2, 362–378.

[11] ———, On the commensurator of the Nottingham group, Trans. Amer. Math. Soc. 362 (2010), no. 12, 6663–6678.

[12] K. Falconer, Fractal geometry, 3rd ed., John Wiley & Sons, Ltd., Chichester, 2014. Mathematical foundations and applications.

[13] D. L. Gonçalves and J. Guaschi, The lower central and derived series of the braid groups of the sphere, Trans. Amer. Math. Soc. 361 (2009), no. 7, 3375–3399.

[14] T.X. He, Shift operators defined in the Riordan group and their applications, Linear Algebra Appl. 496 (2016), 331–350.

[15] S. A. Jennings, Substitution groups of formal power series, Canadian J. Math. 6 (1954), 325–340.

[16] D. L. Johnson, The group of formal power series under substitution, J. Austral. Math. Soc. Ser. A 45 (1988), no. 3, 296–302.

[17] G. Klaas, C. R. Leedham-Green, and W. Plesken, Linear pro-p-groups of finite width, Lecture Notes in Mathematics, vol. 1674, Springer-Verlag, Berlin, 1997.

[18] B. Klopsch, Automorphisms of the Nottingham group, J. Algebra 223 (2000), no. 1, 37–56.

[19] C. R. Leedham-Green and S. McKay, The structure of groups of prime power order, London Mathematical Society Monographs. New Series, vol. 27, Oxford University Press, Oxford, 2002. Oxford Science Publications.

[20] A. Luzón, D. Merlini, M. A. Morón, L. F. Prieto-Martinez, and R. Sprugnoli, Some inverse limit approaches to the Riordan group, Linear Algebra Appl. 491 (2016), 239–262.

[21] A. Luzón and M. A. Morón, Ultrametrics, Banach’s fixed point theorem and the Riordan group, Discrete Appl. Math. 156 (2008), no. 14, 2620–2635.

[22] D. Merlini and M. Nocentini, Algebraic generating functions for languages avoiding Riordan patterns, J. Integer Seq. 21 (2018), no. 1, Art. 18.1.3, 25.

[23] D. Merlini and R. Sprugnoli, Algebraic aspects of some Riordan arrays related to binary words avoiding a pattern, Theoret. Comput. Sci. 412 (2011), no. 27, 2988–3001.

[24] D. Merlini, R. Sprugnoli, and M. C. Verri, Combinatorial sums and implicit Riordan arrays, Discrete Math. 309 (2009), no. 2, 475–486.

[25] L. Ribes and P. Zalesskii, Profinite groups, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics
and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, Springer-Verlag, Berlin, 2010.

[26] A. Shalev, Some problems and results in the theory of pro-p groups, Groups ’93 Galway/St. Andrews, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 212, Cambridge Univ. Press, Cambridge, 1995, pp. 528–542.

[27] L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991), no. 1-3, 229–239. Combinatorics and theoretical computer science (Washington, DC, 1989).

[28] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1994), no. 1-3, 267–290.

[29] ______, Combinatorial sums through Riordan arrays, J. Geom. 101 (2011), no. 1-2, 195–210.

[30] B. Szegedy, Almost all finitely generated subgroups of the Nottingham group are free, Bull. London Math. Soc. 37 (2005), no. 1, 75–79.

[31] I. O. York, The exponent of certain finite p-groups, Proc. Edinburgh Math. Soc. (2) 33 (1990), no. 3, 483–490.

[32] ______, The Group of Formal Power Series under Substitution, Ph.D. Thesis, Nottingham (1990).

[33] J. S. Wilson, Profinite groups, London Mathematical Society Monographs. New Series, vol. 19, The Clarendon Press, Oxford University Press, New York, 1998.

department of mathematics, sungkyunkwan university, 2066 seobu-ro, jangan-gu, suwon-si, gyeonggi-do 16419, korea. tel: +82 031-299-4819

applied algebra and optimization research center, sungkyunkwan university, korea.

Institute of applied mathematics, university of economics Ho Chi Minh city, Vietnam.

Email address: Gi-Sang Cheon, gscheon@skku.edu
Email address: Nhan-Phu Chung, phucn@ueh.edu.vn; phuchung82@gmail.com
Email address: Minh-Nhat Phung, pmnt1114@gmail.com