THE MOMENT MAP ON SYMPLECTIC VECTOR SPACE
AND OSCILLATOR REPRESENTATION

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ABSTRACT. The aim of this paper is to show that the canonical quantization of the moment maps on symplectic vector spaces naturally gives rise to the oscillator representations. More precisely, let \((W, \omega)\) denote a real symplectic vector space, on which a Lie group \(G\) acts symplectically on the left, where \(G\) denotes a real reductive Lie group \(\text{Sp}(n, \mathbb{R}), \text{U}(p, q)\) or \(\text{O}^*(2n)\) in this paper. Then we quantize the moment map \(\mu : W \to g_0^*\) with \(g_0^*\) the dual space of the Lie algebra \(g_0\) of \(G\). Namely, after taking a complex Lagrangian subspace \(V\) of the complexification of \(W\), we assign an element of the Weyl algebra for \(V\) to \(\langle \mu, X \rangle\), which we denote by \(\langle \hat{\mu}, X \rangle\), for each \(X \in g_0\). It is shown that the map \(X \mapsto i \langle \hat{\mu}, X \rangle\) gives a representation of \(g_0\) which extends to the one of \(g\), the complexification of \(g_0\), by linearity. With a suitable choice of the complex Lagrangian subspace \(V\) in each case, the representation coincides with the oscillator representation of \(g\). Taking the direct sum of \(k\) copies of \(W\) produces the Howe duality in the cases of the reductive dual pairs \((\text{sp}(n, \mathbb{R}), \text{O}(k)), (\text{u}(p, q), \text{U}(k))\) and \((\text{o}^*(2n), \text{Sp}(k))\) respectively.

1. INTRODUCTION

Let \((W, \omega)\) be a symplectic vector space and \(\text{Sp}(W)\) the group of linear symplectic isomorphisms of \(W\). Then it is well known that each component of the moment map, i.e., the Hamiltonian function \(H_X\) on \(W\), is quadratic in the coordinates for any \(X \in \text{sp}(W)\) (see e.g. [CG97]). Therefore, taking account of the fact that the commutators among the quantized operators corresponding to the coordinate functions are central, one can see that the canonical quantization gives a representation of \(\text{sp}(W)\) since the map \(X \mapsto H_X\) is a Lie algebra homomorphism from \(\text{sp}(W)\) into \(C^\infty(W)\), where the latter is regarded as a Lie algebra of infinite dimension with respect to the Poisson bracket.

The main aim of this paper is to show that for real reductive Lie groups \(G = \text{Sp}(n, \mathbb{R}), \text{U}(p, q)\) and \(\text{O}^*(2n)\), the canonical quantization of the moment map on real symplectic \(G\)-vector space \((W, \omega)\) gives rise to the oscillator (or Segal-Shale-Weil) representation of the complexified Lie algebra \(\mathfrak{g}\) of \(g_0 := \text{Lie}(G)\) in a natural way. Here, we understand that the canonical quantization is to construct a mapping from the space of smooth functions on \(W\) into the ring of polynomial coefficient differential operators on a complex Lagrangian subspace \(V\) of the complexification \(W_C\) of \(W\), the so called Weyl algebra for \(V\), that induces a Lie algebra homomorphism from \(\mathfrak{g}\) into the Weyl...
algebra. We remark that a different choice of a Lagrangian subspace results in a different quantization, and hence a different representation of the Lie algebra. In fact, when $G = U(p, q)$ and $O^*(2n)$, we will find in §§3-5 that one choice of a Lagrangian subspace produces finite-dimensional irreducible representations of $g$, while another produces infinite-dimensional ones (i.e. the oscillator representations).

Note that each Lie group $G$ we consider in this paper is a counterpart of the Howe’s reductive dual pair $(G, G')$ with $G'$ compact, i.e., $G$ and $G'$ are commutant to each other in a symplectic group $Sp(N, \mathbb{R})$ for some $N$. One can also obtain the oscillator representations by embedding $G$ into $Sp(N, \mathbb{R})$ (see [NOT01, Yam01]).

It was shown in [Has11] that for the classical Hermitian symmetric pairs $(G, K) = (SU(p, q), S(U(p) \times U(q))), (Sp(n, \mathbb{R}), U(n))$, and $(SO^*(2n), U(n))$, one obtains generating functions of the principal symbols of $K_C$-invariant differential operators on $G/K$ in terms of determinant or Pfaffian of a certain $g$-valued matrix whose entries are the total symbols of the differential operators corresponding to the holomorphic discrete series representations realized via Borel-Weil theory, where $K_C$ denotes a complexification of $K$. We note that the $K_C$-invariant differential operators play a prominent rôle in the Capelli identity (see [HU91]). It was also clarified in [Has11] that the $g$-valued matrix mentioned above can be regarded as the twisted moment map $\mu_\lambda$ on the cotangent bundle of $G/K$ which reduces to the moment map $\mu$ on the cotangent bundle when $\lambda \to 0$, where $\lambda$ is an element of the dual space of the Lie algebra of $G$ that parametrizes the representations. In summary, one can say that the moment map relates non-commutative objects (representation operators which are realized as differential operators) to commutative ones (symbols of the differential operators). Now in this paper, we will proceed in the reverse direction: from non-commutative objects to commutative ones.

The oscillator representations have been extensively studied in relation to the Howe duality (see e.g. [KV78]) and to the minimal representations. As for another approach to a construction of the oscillator representations, we should mention [HKMØ12], in which they construct the oscillator representations via Jordan algebras when $G$ is an arbitrary Hermitian Lie group of tube type.

In the remainder of this section, we review a few relevant notions from symplectic geometry briefly, and state our main result.

Let $(M, \omega)$ be a real symplectic manifold. For $f \in C^\infty(M)$, the space of smooth $\mathbb{R}$-valued functions on $M$, let $\xi_f$ denote the vector field on $M$ satisfying $\iota(\xi_f)\omega = df$, where $\iota$ stands for the contraction. Then we define the Poisson bracket by

$$\{f, g\} := \omega(\xi_g, \xi_f) \quad (f, g \in C^\infty(M)), \quad (1.1)$$

which we extend to the space of smooth $\mathbb{C}$-valued functions by linearity. If we denote the quantum observable corresponding to a classical observable $f \in C^\infty(M)$ by $\hat{f}$, then the quantization principles in particular require that

$$\text{if } \{f_1, f_2\} = f_3 \text{ then } [\hat{f}_1, \hat{f}_2] = -i \hbar \hat{f}_3, \quad (1.2)$$

where $\hbar$ is the Planck constant (see e.g. [Woo91]); we set $\hbar = 1$ for simplicity in what follows.
Suppose that a Lie group $G$ acts on $M$ symplectically, i.e., $g^\ast \omega = \omega$ for all $g \in G$. A smooth map $\mu : M \to g_0^\ast$ is called the moment map if the following conditions hold: $\mu$ is $G$-equivariant, and satisfies

$$d(\mu, X) = \iota(X_M)\omega \quad \text{for all } X \in g_0,$$

(1.3)

where $g_0^\ast$ is the dual space of $g_0$ and $X_M$ denotes the vector field on $M$ defined by

$$X_M(p) = \frac{d}{dt} \bigg|_{t=0} \exp(-tX).p \quad (p \in M).$$

(1.4)

We often identify $g_0^\ast$ with $g_0$ via the nondegenerate symmetric invariant bilinear form $B$ defined by

$$B(X, Y) = \begin{cases} \frac{1}{2} \operatorname{tr}(XY) & \text{if } g_0 = \mathfrak{sp}(n, \mathbb{R}) \text{ or } \mathfrak{o}^\ast(2n); \\ \operatorname{tr}(XY) & \text{if } g_0 = \mathfrak{u}(p, q), \end{cases}$$

(1.5)

which extends to the one on $\mathfrak{g} = \mathfrak{sp}_n$, $\mathfrak{o}_{2n}$, or $\mathfrak{gl}_{p+q}$, the complexification of $g_0 = \mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{o}^\ast(2n)$ or $\mathfrak{u}(p, q)$. If there is no risk of confusion, we denote the composition of $\mu$ and the isomorphism $g_0^\ast \simeq g_0$ also by $\mu$. Our symplectic $G$-manifold $(M, \omega)$ will be a real symplectic vector space.

The main result of this paper is the following, which we prove case by case:

**Theorem.** Let $G = \operatorname{Sp}(n, \mathbb{R}), \operatorname{U}(p, q)$ and $\operatorname{O}^\ast(2n)$, and let $(W, \omega)$ be the real symplectic $G$-vector spaces with $W = \mathbb{R}^{2n}$, $(\mathbb{C}^{p+q})_\mathbb{R}$ and $(\mathbb{C}^{2n})_\mathbb{R}$ and $\omega$ given by

$$\omega(v, w) = \begin{cases} i v J_n w & \text{if } W = \mathbb{R}^{2n}, \\ \operatorname{Im}(v I_{p,q} w) & \text{if } W = (\mathbb{C}^{p+q})_\mathbb{R}, \\ \operatorname{Im}(v I_{n,n} w) & \text{if } W = (\mathbb{C}^{2n})_\mathbb{R}. \end{cases}$$

for $v, w \in W$, respectively. Then, with certain choice of complex Lagrangian subspace of the complexification $W_\mathbb{C}$ of $W$, the canonical quantization of the moment map $\mu : W \to g_0^\ast$ provides the oscillator representations.

The rest of this paper is organized as follows. In §2, we consider the case where $G = \operatorname{Sp}(n, \mathbb{R})$, which is the most fundamental in this paper in the sense that a choice of a complex Lagrangian subspace is the key to obtain the oscillator representation. The original motivation of this project lies in this case with $n = 1$. In §3, we turn to the case where $G = \operatorname{U}(p, q)$ and show that the canonical quantization of the moment map with a certain choice of a complex Lagrangian subspace yields irreducible finite-dimensional representations of $\mathfrak{gl}_{p+q}$, and postpone showing that another choice leads to the oscillator representations of $\mathfrak{gl}_{p+q}$ until §5. In §4, we treat the case $G = \operatorname{O}^\ast(2n)$, in which the moment map can be expressed in two ways due to the fact that the quaternionic vector space $\mathbb{H}^n$ is $\mathbb{C}$-isomorphic to $\mathbb{C}^{2n}$ and to $\operatorname{Mat}_{n \times 2} (\mathbb{C})$. In §5, we take complex Lagrangian subspaces different from the ones considered in §§3 and 4 in the cases of $\operatorname{U}(p, q)$ and $\operatorname{O}^\ast(2n)$: one leading to finite-dimensional irreducible representations when $\mathfrak{g} = \mathfrak{o}_{2n}$, and one leading to the oscillator representation when $\mathfrak{g} = \mathfrak{gl}_{p+q}$.

**Notation:** (i) Let $g_0 = t_0 + p_0$ denote the Cartan decomposition for $g_0$ with respect to the Cartan involution $\theta$ given by $\theta X := -X^\ast$, $X \in g_0$, and let $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ denote the corresponding complexified Cartan decomposition for $\mathfrak{g} = g_0 \otimes \mathbb{C}$. 

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2.1. Let
\[ B(X_\alpha, X_\beta^\vee) = \delta_{\alpha, \beta}, \]
where \( \delta_{\alpha, \beta} \) is the Kronecker’s delta, i.e., is equal to 1 if \( \alpha = \beta \) and 0 otherwise.

(ii) For a positive integer \( i \), we set
\[ \bar{t} := \begin{cases} n + i & \text{if } g = sp_n \text{ or } o_{2n}; \\ p + i & \text{if } g = gl_{p+q}, \end{cases} \]
where \( sp_n \), \( o_{2n} \) and \( gl_{p+q} \) denote the complexified Lie algebras of \( sp(n, \mathbb{R}) \), \( o'(2n) \) and \( u(p, q) \) respectively.

2. REDUCTIVE DUAL PAIR \((sp(n, \mathbb{R}), O_k)\)

In this section, let \( G \) denote the symplectic group \( Sp(n, \mathbb{R}) \) of rank \( n \) over \( \mathbb{R} \) which we realize as
\[ Sp(n, \mathbb{R}) = \{ g \in SL_{2n}(\mathbb{R}); \; 'gJ_n g = J_n \} \]
with \( J_n = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} \). Set \( g_0 = sp(n, \mathbb{R}) \), the Lie algebra of \( G \), and take a basis for \( g_0 \) as follows:
\[
\begin{align*}
X_{i,j}^0 &= E_{i,j} - E_{j,i} & (1 \leq i, j \leq n), \\
X_{i,j}^+ &= E_{i,j} + E_{j,i} & (1 \leq i < j \leq n), \\
X_{i,j}^- &= E_{i,j} - E_{j,i} & (1 \leq i < j \leq n),
\end{align*}
\]
(2.1)
where \( E_{i,j} \) denotes the matrix unit of size \( 2n \times 2n \), i.e., its \((i, j)\)-th entry is 1 and all other entries are 0. Note that they also form a basis for \( g = sp_n \).

2.1. Let \( W = \mathbb{R}^{2n} \) which is equipped with the canonical symplectic form \( \omega \) given by
\[ \omega(v, w) = 'vJ_n w \quad (v, w \in W). \]
(2.2)
Obviously, the natural left action of \( G \) on \( W \) defined by \( v \mapsto gv \) (matrix multiplication) for \( v \in W \) and \( g \in G \) is symplectic, i.e., \( g^* \omega = \omega \) for all \( g \in G \). If we identify the canonical base vectors \( e_i := (0, \ldots, 0, 1_i, 0, \ldots, 0) \) with \( \partial_x \) for \( i = 1, 2, \ldots, n \) and with \( \partial_{y_{i+n}} \) for \( i = 1, 2, \ldots, n \), then it is written as
\[ \omega = \sum_{i=1}^n dx_i \wedge dy_i \]
(2.3)
at \( v = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in W \).

**Lemma 2.1.** The vector fields on \( W \) generated by the basis \((2.1)\) for \( g_0 = sp(n, \mathbb{R}) \) in the sense of \((1.4)\) are given by
\[
\begin{align*}
(X_{i,j}^0)_W &= -x_i \partial_{x_j} + y_i \partial_{y_j} & (1 \leq i, j \leq n), \\
(X_{i,j}^+)_W &= -(y_j \partial_{x_i} + x_i \partial_{y_j}) & (1 \leq i < j \leq n), \\
(X_{i,j}^-)_W &= -(x_j \partial_{y_i} + x_i \partial_{y_j}) & (1 \leq i < j \leq n),
\end{align*}
\]
(2.4)

**Proof.** It is an easy exercise to show these formulæ. 
\[ \square \]
Therefore, it follows from Lemma 2.1 that

\[ \mu(v) = v'vJ_n = \begin{bmatrix} -x'y' & x'x \\ -y'y & y'y \end{bmatrix} \]  \hspace{1cm} (2.5)\]

for \( v = (v_1, \ldots, v_n) \in W \) with \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \). In particular, \( \mu \) is \( G \)-equivariant and is \( O(1) \)-invariant.

**Proof.** In order to make this paper self-contained, we include the proof (see, however, e.g. [CG97] Proposition 1.4.6). It follows from Lemma 2.1 that

\[
d(\mu, X^0_{i,j}) = \iota((X^0_{i,j})_W) \omega 
= \iota(-x_i \partial_{x_i} + y_i \partial_{y_i}) \sum_{k=1}^n dx_k \wedge dy_k 
= -x_j dy_i - y_i dx_j = -d(y_i x_j).
\]

Hence one obtains that

\[ \langle \mu, X^0_{i,j} \rangle = -y_i x_j. \]

Similar calculations yield

\[ \langle \mu, X^+_{i,j} \rangle = -y_i y_j \quad \text{and} \quad \langle \mu, X^-_{i,j} \rangle = x_i x_j. \]

Therefore,

\[
\mu(v) = \sum_{i,j} \langle \mu, X^0_{i,j} \rangle (X^0_{i,j})^\dagger + \sum_{i \neq j} \langle \mu, X^+_{i,j} \rangle (X^+_{i,j})^\dagger + \sum_{i \neq j} \langle \mu, X^-_{i,j} \rangle (X^-_{i,j})^\dagger 
= \sum_{i,j} (-y_i x_j (E_{i,j} - E_{j,i}) + \sum_{i \neq j} (-y_i y_j) 2^{-\delta_{ij}} (E_{i,j} + E_{j,i}) + \sum x_i x_j 2^{-\delta_{ij}} (E_{i,j} + E_{j,i}) 
= \sum_{i,j} (-x_i y_j E_{i,j} + x_i x_j E_{i,j} - y_i y_j E_{i,j} + y_i x_j E_{i,j}) 
= \begin{bmatrix} -x'y' & x'x \\ -y'y & y'y \end{bmatrix} = v'vJ_n
\]

for \( v = (v_1, \ldots, v_n) \) with \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \).

Now the \( O(1) \)-invariance of \( \mu \) is trivial, and the \( G \)-equivariance can be verified as follows:

\[ \mu(gv) = gv'(gv)J_n = gv'v'gJ_n = gv'vJ_n g^{-1} = \text{Ad}(g) \mu(v) \]

since \( 'gJ_n = J_n g^{-1} \) for \( g \in G \). This completes the proof. \( \square \)

It follows from the definitions of the Poisson bracket (1.1) and the symplectic form (2.3) that

\[ \{x_i, y_j\} = -\delta_{i,j}, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0 \]  \hspace{1cm} (2.6)\]

for \( i, j = 1, \ldots, n \). In view of (2.6), we quantize the classical observables by assigning

\[ \widehat{x}_i = \text{multiplication by } x_i, \quad \widehat{y}_i = -i \partial_{x_i}. \]  \hspace{1cm} (2.7)\]
so that $[\hat{\pi}_i, \hat{\pi}_j] = i \delta_{i,j}$, as required. In what follows, we simply denote the multiplication operator by a function $f$ by the same letter $f$ if there is no risk of confusion.

Note that the quantization (2.7) corresponds to taking a Lagrangian subspace of $W$ spanned by $e_1, \ldots, e_n$. However, in order to obtain a representation of the complex Lie algebra $\mathfrak{g} = \text{sp}_n$, we will take a complex Lagrangian subspace of the complexification $W_\mathbb{C}$ defined by

$$V := \langle e_1, \ldots, e_n \rangle_\mathbb{C}. \quad (2.8)$$

Therefore, the classical observables $x_j$, $j = 1, \ldots, n$, are now the complex coordinates on $V$ with respect to this basis.

Now, we quantize the moment map $\mu$ according to (2.7) as follows and denote the quantized moment map by $\hat{\mu}$:

$$\hat{\mu} :=\begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{y}_n \end{bmatrix}$$

$$\{\hat{x}_1, \ldots, \hat{y}_n\} J_n = \begin{bmatrix} x \\ -i \partial_x \end{bmatrix} \begin{bmatrix} \hat{x}_1, \ldots, \hat{y}_n \end{bmatrix}$$

$$= \begin{bmatrix} i x \partial_x & x \partial_x \\ \partial_x \partial_x & -i \partial_x \partial_x \end{bmatrix}$$

with $x = \langle x_1, \ldots, x_n \rangle$ and $\partial_x = \langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle$.

Let $\mathcal{P}(V)$ denote the space of complex coefficient polynomial functions on $V$, i.e., $\mathcal{P}(V) = \mathbb{C}[x_1, \ldots, x_n]$, and $\mathcal{P}D(V)$ the ring of polynomial coefficient differential operators on $V$. Thus each entry of $\hat{\mu}$ is an element of $\mathcal{P}D(V)$.

**Theorem 2.3.** For $X \in \mathfrak{g} = \text{sp}_n$, set $\pi(X) = i \langle \hat{\mu}, X \rangle$. Then

$$\pi : \mathfrak{g} \to \mathcal{P}D(V)$$

is a Lie algebra homomorphism. In terms of the basis (2.1), it is given by

$$\pi(X) = \begin{cases} \frac{i}{2}(x_i \partial_{x_j} + \partial_{x_j} x_i) & \text{if } X = X^0_{i,j}; \\ i \partial_{x_i} \partial_{x_j} & \text{if } X = X^+_{i,j}; \\ i x_i x_j & \text{if } X = X^-_{i,j}. \end{cases} \quad (2.10)$$

**Proof.** Of course, one can verify the commutation relations among the explicit form (2.10) which can be easily deduced from (2.9), coincides with those of the basis $\{X^*_i\}$ for $\mathfrak{g}$. However, we will give another proof in the following.

The moment map $\mu$ induces a Lie algebra homomorphism from $\mathfrak{g}_0$ to $C^\infty(W)$, i.e., if we write $H_X := \langle \mu, X \rangle$ for $X \in \mathfrak{g}_0$, then we have

$$[H_X, H_Y] = H_{[X,Y]} \quad (X, Y \in \mathfrak{g}_0). \quad (2.11)$$

Taking account of the fact that both Poisson bracket and commutator are derivations and $\pi(X) = i \hat{H}_X$, we see that the relation (2.11) implies that

$$[\pi(X), \pi(Y)] = \pi([X,Y]) \quad (X, Y \in \mathfrak{g}_0)$$

since each function $H_X$ is quadratic in the coordinate functions $x_i, y_j$ for any $X \in \mathfrak{g}_0$ (see [CG97]) and the commutators among $\hat{x}_i$ and $\hat{y}_j$ are in the center of $\mathcal{P}D(V)$ for $i, j = 1, \ldots, n$. Now, extend the result to the complexification by linearity. \(\square\)
Remark 2.4. By (2.9), one can rewrite \( \pi(X) = i \langle \mu, X \rangle, \) \( X \in \mathfrak{g}, \) as follows:

\[
\pi(X) = \frac{i}{2} \text{tr} (\tilde{\mu} X) = \frac{i}{2} \text{tr} \left( \begin{bmatrix} x \\ -i \partial_x \end{bmatrix} (\langle x, -i \partial_x \rangle J_n X) \right) \\
= \frac{i}{2} (i \langle \partial_x, \langle x \rangle \rangle x \begin{bmatrix} x \\ -i \partial_x \end{bmatrix},
\]

where the last equality follows from the fact that \( X \) is a member of \( \mathfrak{g}. \) Namely, our quantized moment map \( \tilde{\mu} \) is essentially identical to the map \( \varphi \) given in Example of [KV95] Chap. I, §6]. This observation was the original motivation of the present work.

It is well known that the irreducible decomposition of the representation \((\pi, P(V))\) of \( \mathfrak{g} \) is given by \( P(V) = P(V)^+ \oplus P(V)^- \) where \( P(V)^+ \) and \( P(V)^- \) are the subspaces consisting of even polynomials \( f(x) \) satisfying \( f(-x) = f(x) \) and of odd polynomials \( f(x) \) satisfying \( f(-x) = -f(x) \) respectively. It is also well known that this phenomena can be explained by the type of representations of \( O(1) \) which acts on \( V \) on the right.

2.2. Let us consider the vector space \( W_k := W \oplus \cdots \oplus W, \) the direct sum of \( k \) copies of \( W = \mathbb{R}^{2n}, \) which can be identified with \( \text{Mat}_{2n \times k}(\mathbb{R}). \) It is a symplectic vector space equipped with symplectic form \( \omega_k \) given by

\[
\omega_k(v, w) = \text{tr} \left( \langle v J_n, w \rangle \right) \quad (v, w \in W^k).
\]

Let \( e_{i,a} \) denote the matrix unit of size \( 2n \times k \) for \( i = 1, \ldots, n \) and \( a = 1, \ldots, k. \) Under the identification \( e_{i,a} \leftrightarrow \partial_{x_i,a} \) and \( e_{i,a} \leftrightarrow \partial_{y_i,a} \), we write \( v = [x_1, \ldots, x_n, y_1, \ldots, y_n] \in W^k \) with \( x_i = (x_{i,1}, \ldots, x_{i,k}) \) and \( y_i = (y_{i,1}, \ldots, y_{i,k}) \) being row vectors of size \( k \) for \( i = 1, \ldots, n. \) Then \( \omega_k \) is given by

\[
\omega_k = \sum_{1 \leq i \leq n, 1 \leq a \leq k} \text{d}x_{i,a} \wedge \text{d}y_{i,a} \quad (2.12)
\]

at \( v = [x_1, \ldots, y_n]. \) Note that \( G = \text{Sp}(n, \mathbb{R}) \) acts on \( W^k = \text{Mat}_{2n \times k}(\mathbb{R}) \) on the left, while the real orthogonal group \( O(k) \) acts on the right. Both actions are symplectic.

For brevity, let us write \( x_* \cdot y_* = \sum_{a=1}^k x_{*,a} y_{*,a}, \) the standard inner product, for row vectors \( x_* = (x_{*,1}, \ldots, x_{*,k}) \) and \( y_* = (y_{*,1}, \ldots, y_{*,k}) \) of size \( k \) in what follows.

Proposition 2.5. Let \( (W^k, \omega_k) \) be the symplectic \( G \)-vector space. Then the moment map \( \mu : W^k \rightarrow \mathfrak{g}_0^* \cong \mathfrak{g}_0 \) is given by

\[
\mu(v) = v^t \cdot J_n 
\]

\[\text{Remark}\] More precisely, one should write an element \( v \in W^k = \text{Mat}_{2n \times k}(\mathbb{R}) \) as \( v = [x_1, \ldots, x_n, y_1, \ldots, y_n]; \) however, we will adopt this abbreviated notation in what follows.
respectively.

Laé as in Lemma 2.1, and thus the same argument given in the proof of Proposition 2.2 and all other brackets vanish. Therefore, we quantize them by assigning

\[
\mu = \{x_{i,a}, y_{i,b}\} = \delta_{i,j} \delta_{a,b}
\]

and all other brackets vanish. Therefore, we quantize them by assigning

\[
\hat{x}_{i,a} = x_{i,a} \quad \text{and} \quad \hat{y}_{i,a} = -i \partial_{x_{i,a}}
\]

for \( i = 1, \ldots, n \) and \( a = 1, \ldots, k \).

Let \( V^k \) denote the direct sum \( V \oplus \cdots \oplus V \) \((k \text{ copies})\) with \( V \) given in (2.8). Since \( V^k \) can be identified with \( \text{Mat}_{n \times k}(\mathbb{C}) \), the upper half of \( W^k = \text{Mat}_{2n \times k}(\mathbb{C}) \), we write an element of \( V^k \) as \( x = (x_{i,a})_{i=1,\ldots,n; a=1,\ldots,k} \). Let \( \mathcal{D}(V^k) = \mathbb{C}[x_{i,a}; i = 1, \ldots, n, a = 1, \ldots, k] \) be the algebra of complex polynomial functions on \( V^k \), and \( \mathcal{P}(V^k) \) the ring of polynomial coefficient differential operators on \( V^k \). Note that \( x_{i,a} \)'s are now complex variables and that the complex general linear group \( \text{GL}_k \) acts on \( V^k \) by matrix multiplication on the right, and thus on \( \mathcal{P}(V^k) \) by right translation:

\[
\rho(g)f(x) := f(xg) \quad (g \in \text{GL}_k, f \in \mathcal{P}(V^k)).
\]  

The right-action of \( \text{GL}_k \) on \( V^k \) is the restriction of the one on \( W^k \).

The quantized moment map \( \hat{\mu} \) in this case is also given by the same formula as (2.9):

\[
\hat{\mu} = \begin{bmatrix}
ix \partial_x & x'x \\
\partial_x \partial_x & -i \partial_x x
\end{bmatrix}.
\]

In this case, however, \( x \) and \( \partial_x \) are \( n \times k \)-matrices whose \((i,a)\)-th entries are the multiplication operator \( x_{i,a} \) and the differential operator \( \partial_{x_{i,a}} \) for \( i = 1, \ldots, n \) and \( a = 1, \ldots, k \), respectively.
Lemma 2.6. For \( x = (x_{i,a})_{i=1,...,n,a=1,...,k} \in V^k \) and \( g \in \text{GL}_k \), the following relations hold in \( \text{End}(\mathcal{P}(V^k)) \):

\[
\rho(g)^{-1} \partial_{s_{i,a}} \rho(g) = \sum_{b} g_{ab} \partial_{x_{i,b}}, \tag{2.15}
\]

\[
\rho(g)^{-1} x_{i,a} \rho(g) = \sum_{b} g^{ba} x_{i,b}, \tag{2.16}
\]

where \( g = (g_{ab}) \) and \( g^{-1} = (g^{ab}) \).

Proof. Since \( \partial_{s_{i,a}} \) is identified with \( e_{i,a} \in \text{Mat}_{n \times k}(\mathbb{C}) \), one sees that

\[
(\partial_{e_{i,a}}(\rho(g)f))(x) = \frac{d}{dt} \bigg|_{t=0} f(xg + te_{i,a}g) = \sum_{b=0}^{k} g_{ab} \frac{\partial f}{\partial x_{i,b}}(xg),
\]

and hence

\[
\left( \rho(g)^{-1} \partial_{s_{i,a}} \rho(g) \right) f = \sum_{b=1}^{k} g_{ab} \frac{\partial f}{\partial x_{i,b}}
\]

for \( f \in \mathcal{P}(V^k) \). Thus one obtains (2.15).

On the other hand, since

\[
\left( \rho(g)^{-1} (x_{i,a}f) \right)(x) = \left( \sum_{b=1}^{k} x_{i,b} g^{ba} \right) f(xg^{-1})
\]

one has

\[
\left( \rho(g)^{-1} x_{i,a} \rho(g) \right) f = \left( \sum_{b=1}^{k} g^{ba} x_{i,b} \right) f
\]

and (2.16). \qed

Let us abbreviate as \( \rho(g) a \rho(g)^{-1} =: \text{Ad}_{\rho(g)} a \) for \( a \in \mathcal{P}(V^k) \) and \( g \in \text{GL}_k \). Moreover, for a given matrix \( A = (a_{ij}) \) with \( a_{ij} \in \mathcal{P}(V^k) \), let us denote by \( \text{Ad}_{\rho(g)} A = (\text{Ad}_{\rho(g)} a_{ij}) \), the matrix whose \((i, j)\)-th entries are equal to \( \text{Ad}_{\rho(g)} a_{ij} \).

Corollary 2.7. For \( X \in \mathfrak{g} = \mathfrak{sp}_n \), set \( \pi(X) = i \left( \hat{\mu}, X \right) \). Then

\[\pi : \mathfrak{g} \to \mathcal{P}(V^k)\]

is a Lie algebra homomorphism. In terms of the basis (2.1), it is given by

\[
\pi(X) = \begin{cases} 
-\frac{1}{2} \sum_{a=1}^{k} x_{i,a} \partial_{x_{j,a}} + \partial_{x_{j,a}} x_{i,a} & \text{if } X = X_{i,j}^0; \\
i \sum_{a=1}^{k} \partial_{s_{i,a}} \partial_{x_{j,a}} & \text{if } X = X_{i,j}^+; \\
i \sum_{a=1}^{k} x_{i,a} \partial_{x_{j,a}} & \text{if } X = X_{i,j}^-; 
\end{cases} \tag{2.17}
\]

Moreover, \( \pi(X) \) commutes with the action of the complex orthogonal group\(^3\) \( O_k \), i.e., \( \pi(X) \in \mathcal{P}(V^k)^{O_k} \) for all \( X \in \mathfrak{sp}_n \).

\(^3\)We realize the complex orthogonal group as \( O_k = \{ g \in \text{GL}_k ; \ g^2 = 1 \} \) in this section.
Proof. The same argument as in the proof of Theorem 2.3 shows that $\pi : g \to \mathcal{P}(V^k)$ is a Lie algebra homomorphism and that (2.17) holds.

For the last statement, it follows from Lemma 2.6 that

$$\text{Ad}_{\rho(g)^{-1}} x_i = x_i g^{-1} \quad \text{and} \quad \text{Ad}_{\rho(g)^{-1}} \partial_x = \partial_x 'g$$

with $x_i = (x_{i,1}, \ldots, x_{i,k})$ and $\partial x_i = (\partial x_{i,1}, \ldots, \partial x_{i,k})$ for $g \in \text{GL}_k$. Hence, if $g \in O_k$ then one has

$$\left[ \begin{array}{c} \text{Ad}_{\rho(g)^{-1}} x \\ -i \text{Ad}_{\rho(g)^{-1}} \partial \end{array} \right] = \left[ \begin{array}{c} x \\ -i \partial \end{array} \right] 'g$$

with $x = \{x_1, \ldots, x_n\}$ and $\partial = \{\partial x_1, \ldots, \partial x_n\}$, since $'g = g^{-1}$. Therefore,

$$\text{Ad}_{\rho(g)^{-1}} \hat{\mu} = \left[ i \text{Ad}_{\rho(g)^{-1}} (x'y_x) \quad \text{Ad}_{\rho(g)^{-1}} (x'y_x) \right]$$

$$= \left[ i \text{Ad}_{\rho(g)^{-1}} x ' \left( \text{Ad}_{\rho(g)^{-1}} \partial_x \right) \quad \text{Ad}_{\rho(g)^{-1}} x ' \left( \text{Ad}_{\rho(g)^{-1}} \partial_x \right) \right]$$

$$= \left[ \text{Ad}_{\rho(g)^{-1}} x \\ -i \text{Ad}_{\rho(g)^{-1}} \partial_x \right] \left[ x, \quad -i '\partial \right] J_n = \hat{\mu}.$$

This completes the proof. \qed

It is well known that the irreducible decomposition of $\mathcal{P}(V^k)$ under the joint action of $(\mathfrak{sp}_n, O_k)$ is given by

$$\mathcal{P}(V^k) \cong \sum_{\sigma \in \widehat{O}_k, L(\sigma) \neq [0]} L(\sigma) \otimes V_\sigma,$$

where $V_\sigma$ is a representative of the class $\sigma \in \widehat{O}_k$, the set of equivalence classes of the finite-dimensional irreducible representation of $O_k$, and $L(\sigma) := \text{Hom}_{O_k}(V_\sigma, \mathcal{P}(V^k))$ which is an infinite-dimensional irreducible representation of $\mathfrak{sp}_n$. Moreover, it is also known that the action $\pi$ restricted to $1$ lifts to the double cover $\tilde{K}$ of the maximal compact subgroup $K$ of $G = \text{Sp}(n, \mathbb{R})$, which implies that $L(\sigma)$ is an irreducible $(\mathfrak{g}, \tilde{K})$-module (see [KV78] and [Yam01]).

3. Reductive Dual Pair $(u(p, q), \text{GL}_k)$

Let $G$ denote the indefinite unitary group defined by

$$U(p, q) = \{ g \in \text{GL}_n(\mathbb{C}) ; g^t I_{p,q} g = I_{p,q} \}$$

with $I_{p,q} = \left[ \begin{array}{cc} 1_p & 0 \\ 0 & -1_q \end{array} \right]$, and put $n = p + q$ only in this section for brevity. Set $\mathfrak{g}_0 = u(p, q)$, the Lie algebra of $G$, and take a basis for $\mathfrak{g}_0$ as follows:

$$X_{i,j} = E_{i,j} - E_{j,i} \quad (1 \leq i < j \leq p, \text{ or } p + 1 \leq i < j \leq n),$$

$$Y_{i,j} = i (E_{i,j} + E_{j,i}) \quad (1 \leq i < j \leq p, \text{ or } p + 1 \leq i < j \leq n),$$

$$X_{i,j}'' = E_{i,j} + E_{j,i} \quad (1 \leq i < p, 1 \leq j \leq q),$$

$$Y_{i,j}'' = i (E_{i,j} - E_{j,i}) \quad (1 \leq i < p, 1 \leq j \leq q),$$

$$Y_{i,j}''' = i (E_{i,j} + E_{j,i}) \quad (1 \leq i < p, 1 \leq j \leq q).$$

(3.1)
where $E_{i,j}$ denotes the matrix unit of size $n \times n$. Note that $E_{i,j}$, $i, j = 1, \ldots, n$, form a basis for $g = gl_n$, the complexified Lie algebra of $g_0 = u(p, q)$.

3.1. Let $W = (\mathbb{C}^n)_{\mathbb{R}}$, the underlying real vector space of the complex vector space $\mathbb{C}^n$, and $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ the indefinite Hermitian form given by

$$H(z, w) := z^* I_{p,q} w \quad (z, w \in \mathbb{C}^n).$$

We regard $W$ as symplectic manifold with symplectic form $\omega = \text{Im} H$, where $\text{Im} H$ stands for the imaginary part of $H$. Under the identification $e_j \leftrightarrow \partial_{x_j}$ and $i e_j \leftrightarrow \partial_{y_j}$ for $j = 1, \ldots, n$, it is explicitly given by

$$\omega = \sum_{j=1}^n \epsilon_j \, dx_j \wedge dy_j \quad (3.2)$$

at $z = x + iy \in W$ with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, where

$$\epsilon_j := \begin{cases} 1 & (j = 1, \ldots, p) \\ -1 & (j = p + 1, \ldots, n). \end{cases} \quad (3.3)$$

Then $(W, \omega)$ is a symplectic $G$-manifold since the natural action of $G$ on $\mathbb{C}^n$ preserves the Hermitian form $H$.

**Lemma 3.1.** The vector fields on $W$ generated by the basis (3.1) for $g_0 = u(p, q)$ in the sense of (1.4) are given by

$$(X_{i,j}^c)_W = -x_i \partial_{x_j} - y_j \partial_{y_i} + x_j \partial_{x_i} + y_i \partial_{y_j},$$

$$(Y_{i,j}^c)_W = y_j \partial_{x_i} - x_i \partial_{y_j} + y_i \partial_{x_j} - x_j \partial_{y_i},$$

$$(X_{i,j}^n)_W = -x_j \partial_{x_i} - x_i \partial_{x_j} + y_i \partial_{y_j} - y_j \partial_{y_i},$$

$$(Y_{i,j}^n)_W = y_i \partial_{x_j} - x_j \partial_{x_i} - y_j \partial_{y_i} + x_i \partial_{y_j}. \quad (3.4)$$

Note that the unitary group $U(1)$ also acts on $W$ symplectically on the right.

**Proposition 3.2.** Let $(W, \omega)$ be as above and $G = U(p, q)$. Then the moment map $\mu : W \to g_0^* \cong g_0$ is given by

$$\mu(z) = -\frac{i}{2} z z^* I_{p,q} \quad (3.5)$$

for $z = x + iy \in W$ with $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. In particular, $\mu$ is $G$-equivariant and is $U(1)$-invariant.

**Proof.** It follows from Lemma 3.1 that

$$\langle \mu, X \rangle = \begin{cases} \epsilon_i (x_i y_j - x_j y_i) & \text{if } X = X_{i,j}^c, \\ \epsilon_j (x_j x_i + y_i y_j) & \text{if } X = Y_{i,j}^c, \\ x_i y_j - x_j y_i & \text{if } X = X_{i,j}^n, \\ x_i x_j + y_i y_j & \text{if } X = Y_{i,j}^n. \end{cases} \quad (3.6)$$
which can be rewritten in terms of the complex coordinates defined by \( z_j = x_j + i y_j \) \((j = 1, \ldots, n)\) and their complex conjugates as

\[
\langle \mu, X \rangle = \begin{cases}
\frac{i}{2} \epsilon_j(z_j \bar{z}_j - z_j \bar{z}_j) & \text{if } X = X^c_{i,j}; \\
\frac{i}{2} \epsilon_j(z_j \bar{z}_j + z_j \bar{z}_j) & \text{if } X = Y^c_{i,j}; \\
\frac{i}{2} (z_j \bar{z}_j - z_j \bar{z}_j) & \text{if } X = X^n_{i,j}; \\
\frac{i}{2} (z_j \bar{z}_j + z_j \bar{z}_j) & \text{if } X = Y^n_{i,j}.
\end{cases}
\]  

(3.7)

Hence,

\[
\mu(z) = \sum_{i<j} \langle \mu, X^c_{i,j} \rangle (X^c_{i,j})^\vee + \sum_{i<j} \langle \mu, Y^c_{i,j} \rangle (Y^c_{i,j})^\vee + \sum_{>} \langle \mu, X^n_{i,j} \rangle (X^n_{i,j})^\vee + \sum_{i<j} \langle \mu, Y^n_{i,j} \rangle (Y^n_{i,j})^\vee
\]

\[
= -\frac{i}{2} \sum_{1 \leq i < j \leq p} z_i \bar{z}_j E_{i,j} + \frac{i}{2} \sum_{1 \leq i < j \leq q} z_i \bar{z}_j E_{i,j} + \frac{i}{2} \sum_{1 \leq i < j \leq p, 1 \leq i < q} z_i \bar{z}_j E_{i,j} - \frac{i}{2} \sum_{1 \leq i < j \leq p, 1 \leq j < q} z_i \bar{z}_j E_{i,j}
\]

\[
= -\frac{i}{2} zz^\ast I_{p,q},
\]

with \( z = (z_1, \ldots, z_n) \).

The U(1)-invariance of \( \mu \) is obvious, and the \( G \)-equivariance can be verified as follows:

\[
\mu(gz) = -\frac{i}{2} (gz)(gz)^\ast I_{p,q} = -\frac{i}{2} gzz^\ast g^\ast I_{p,q} = \text{Ad}(g)\mu(z)
\]

since \( g^\ast I_{p,q} = I_{p,q}g^{-1} \) for \( g \in U(p, q) \). \( \square \)

It follows from (3.3) that the Poisson brackets among the real coordinate functions \( x_i, y_i, i = 1, \ldots, n, \) are given by

\[
\{x_i, y_j\} = -\epsilon_{i,j} \quad (i, j = 1, 2, \ldots, n),
\]  

(3.8)

and all other brackets vanish. In terms of the complex coordinates \( z_j = x_j + i y_j, j = 1, 2, \ldots, n, \) and their conjugates, it follows from (3.8) that the Poisson brackets among \( z_j \) and \( \bar{z}_j \) are given by

\[
\{z_i, \bar{z}_j\} = 2i \epsilon_{i,j}, \quad \{z_i, z_j\} = \{\bar{z}_i, \bar{z}_j\} = 0
\]  

(3.9)

for \( i, j = 1, 2, \ldots, n. \) In view of (3.9) we quantize \( z_i \) and \( \bar{z}_i \) by assigning

\[
\hat{z}_i = z_i, \quad \hat{\bar{z}}_i = -2i \epsilon_{i,j} \nabla z_i,
\]  

(3.10)

so that they satisfy

\[
[\hat{z}_i, \hat{\bar{z}}_j] = 2i \epsilon_{i,j}, \quad [\hat{z}_i, \hat{z}_j] = [\hat{\bar{z}}_i, \hat{\bar{z}}_j] = 0
\]  

(3.11)

for \( i, j = 1, 2, \ldots, n. \) Therefore, we quantize the moment map \( \mu \) as follows and denote the quantized moment map by \( \hat{\mu} \):

\[
\hat{\mu} = -\frac{i}{2} \begin{bmatrix} \hat{z}_1 \\ \vdots \\ \hat{z}_n \end{bmatrix} \begin{bmatrix} \hat{z}_1, \ldots, \hat{z}_n \end{bmatrix} I_{p,q} = i \begin{bmatrix} \hat{z}_1 \\ \vdots \\ \hat{z}_n \end{bmatrix} (\partial_{z_1}, \ldots, \partial_{z_n}) = iz^\ast \partial z
\]  

(3.12)
with $\partial_z = (\partial_{z_1}, \ldots, \partial_{z_n})$. Note that the quantization (3.10) corresponds to taking a complex Lagrangian subspace $V$ given by

$$V := \left\{ \frac{1}{2}(e_1 - i I e_1), \ldots, \frac{1}{2}(e_n - i I e_n) \right\} \subseteq \mathbb{C},$$

where $I$ denotes the complex structure on $V$ defined by $e_j \mapsto ie_j$, $ie_j \mapsto -e_j$ for $j = 1, \ldots, n$. The classical observables $z_j = x_j + iy_j$ can be regarded as the coordinates on $V$ with respect to this basis under the identification $e_j \leftrightarrow \partial_{x_j}$ and $ie_j \leftrightarrow \partial_{y_j}$, $j = 1, \ldots, n$, and $V$ is naturally identified with $\mathbb{C}^n$. Let $\mathcal{P}(V)$ denote the algebra of complex coefficient polynomial functions on $V$, i.e., $\mathcal{P}(V) = \mathbb{C}[z_1, \ldots, z_n]$, and $\mathcal{P}\mathcal{D}(V)$ the ring of polynomial coefficient differential operators on $V$.

**Theorem 3.3.** For $X \in \mathfrak{g} = \mathfrak{gl}_n$, set $\pi(X) = i \langle \mu, X \rangle$. Then

$$\pi : \mathfrak{g} \rightarrow \mathcal{P}\mathcal{D}(V)$$

is a Lie algebra homomorphism. In terms of the basis $\{E_{i,j}\}$ for $\mathfrak{g}$, it is given by

$$\pi(E_{i,j}) = -z_j \partial_{z_i}$$

for $i, j = 1, \ldots, n$.

**Proof.** The same argument as in Theorem 2.3 shows that $\pi$ is a Lie algebra homomorphism, and (3.14) follows immediately from (3.12). $\square$

It is clear from (3.14) that $\pi(X) \in \mathcal{P}\mathcal{D}(V)^{GL_1}$ for all $X \in \mathfrak{g}$, where $GL_1$ acts on $V$ on the right.

3.2. Now let us consider $W^k$, the direct sum of $k$ copies of $W = (\mathbb{C}^n)_{\mathbb{R}}$, which is identified with the underlying real vector space of $\text{Mat}_{n \times k}(\mathbb{C})$. It is equipped with a symplectic form $\omega_k$ given by

$$\omega_k(z, w) = \text{Im tr} \left( z^* I_{p,q} w \right) \quad (z, w \in W^k),$$

and is still acted on by $G = U(p, q)$ symplectically by matrix multiplication on the left. Under the identification of $e_{i,a} \leftrightarrow \partial_{x_{a,i}}$ and $ie_{i,a} \leftrightarrow \partial_{y_{a,i}}$, we write an element of $W^k$ as $z = [z_1, \ldots, z_n]$, where $z_i = x_i + iy_i$ are complex row vectors with $x_i = (x_{i,1}, \ldots, x_{i,n})$ and $y_i = (y_{i,1}, \ldots, y_{i,k})$ being real row vectors of size $k$ for $i = 1, \ldots, n$. Then $\omega_k$ is given by

$$\omega_k = \sum_{1 \leq i \leq n, 1 \leq a \leq k} e_i \, dx_{i,a} \wedge dy_{i,a}$$

at $z = [z_1, \ldots, z_n] \in W^k$. Note that $U(p, q)$ acts on $W$ on the left, while $U(k)$ acts on it on the right and that both actions are symplectic.

**Proposition 3.4.** Let $(W^k, \omega_k)$ be the symplectic $G$-vector space as above. Then the moment map $\mu : W^k \rightarrow g_0^* \cong g_0$ is given by the same formula as (3.5)

$$\mu = -\frac{i}{2} z z^* I_{p,q}$$

with $z \in W^k = \text{Mat}_{n \times k}(\mathbb{C})$. In particular, $\mu$ is $G$-equivariant and is $U(k)$-invariant.
equivariance is verified as in Proposition 3.2. □

It follows from (3.15) that the Poisson brackets among the real coordinate functions $x_{i,a}, y_{i,a}, i = 1, \ldots, n; a = 1, \ldots, k$, are given by
\begin{equation}
\{x_{i,a}, y_{j,a}\} = -\varepsilon \delta_{i,j} \delta_{a,b} \quad (i, j = 1, \ldots, n; a, b = 1, \ldots, k),
\end{equation}
and all other brackets vanish. It follows from (3.16) that the Poisson brackets among the complex coordinates $z_{i,a} = x_{i,a} + i y_{i,a}$ and their conjugates are given by
\begin{equation}
\{z_{i,a}, \bar{z}_{j,a}\} = 2i \varepsilon \delta_{i,j} \delta_{a,b}
\end{equation}
for $i, j = 1, \ldots, n; a, b = 1, \ldots, k$, and all other brackets vanish. Therefore, we quantize $z_{i,a}$ and $\bar{z}_{i,a}$ by assigning
\begin{equation}
\tilde{z}_{i,a} = z_{i,a}, \quad \tilde{\bar{z}}_{i,a} = -2\varepsilon \delta_{i,a},
\end{equation}
so that the nontrivial commutators are given by
\begin{equation}
[\tilde{z}_{i,a}, \tilde{\bar{z}}_{j,a}] = 2\varepsilon \delta_{i,j} \delta_{a,b}.
\end{equation}

Proof. As in the proof of Proposition 2.5, if we regard $x_i, y_i, \partial_{x_i}$ and $\partial_{y_i}$ as row vectors and the products as the inner product on the space of row vectors, then similar argument to Proposition 3.2 shows that the moment map $\mu : W^k \to \mathfrak{g}_0$ is given by (3.5), with the understanding that $z \in \text{Mat}_{nk}(\mathbb{C})$. The $U(k)$-invariance is obvious, and the $G$-equivariance is verified as in Proposition 3.2.

Let $V^k$ denote the direct sum of $k$ copies of $V$, with $V$ as in (3.13). Since $V^k$ can be identified with $\text{Mat}_{nk}(\mathbb{C})$, we write an element of $V^k$ as $z = (z_{i,a})_{i=1,\ldots,n,a=1,\ldots,k}$. Note then that $GL_k$ acts on $V^k$ by matrix multiplication on the right, and hence on $\mathcal{P}(V^k)$ by right regular representation, which we denote also by $\rho$ as in (2.14). Let $\mathcal{P}(V^k) = \mathbb{C}[z_{i,a}; i = 1, \ldots, n, a = 1, \ldots, k]$ be the algebra of complex polynomial functions on $V^k$, and let $\mathcal{P} \mathcal{D}(V^k)$ the ring of polynomial coefficient differential operators on $V^k$.

The quantized moment map $\hat{\mu}$ is also given by the same formula as (3.12):
\begin{equation}
\hat{\mu} = i z \partial_z.
\end{equation}
In this case, however, $z$ and $\partial_z$ are $n \times k$-matrices whose $(i, a)$-th entries are the multiplication operator $z_{i,a}$ and the differential operator $\partial_{z_{i,a}}$ for $i = 1, \ldots, n$ and $a = 1, \ldots, k$, respectively.

Corollary 3.5. For $X \in \mathfrak{g} = \text{gl}_n$, set $\pi(X) = i \langle \hat{\mu}, X \rangle$. Then
\begin{equation}
\pi : \mathfrak{g} \to \mathcal{P} \mathcal{D}(V)
\end{equation}
is a Lie algebra homomorphism. In terms of the basis $\{E_{i,j}\}$ for $\mathfrak{g}$, it is given by
\begin{equation}
\pi(E_{i,j}) = -\sum_{a=1}^k z_{i,a} \partial_{z_{i,a}} \quad (3.20)
\end{equation}
for $i, j = 1, \ldots, n$. Moreover, $\pi(X)$ commutes with the action of the complex general linear group $GL_k$, i.e., $\pi(X) \in \mathcal{P} \mathcal{D}(V)^{GL_k}$ for all $X \in \mathfrak{g}$.

Proof. The first statement that $\pi$ is a Lie algebra homomorphism can be shown as in the proof of Theorem 2.3. It remains to show that $\hat{\mu}$ commutes with the action of $GL_k$, which can be done in the following way. By Lemma 2.6 one obtains that
\begin{equation}
\text{Ad}_{\rho(g) \cdot \hat{z}} \cdot z = zg^{-1} \quad \text{and} \quad \text{Ad}_{\rho(g) \cdot \hat{z}} \cdot \partial_z = \partial_z' g,
\end{equation}
from which it follows that
\[ \text{Ad}_{\rho(g)}(z \partial_z) = (\text{Ad}_{\rho(g)} z)' (\text{Ad}_{\rho(g)}^{-1} \partial_z) = z g^{-1} g' \partial_z. \]

This completes the proof. \( \square \)

Similarly to the case of \( \text{Sp}(n, \mathbb{R}) \), it is well known that the irreducible decomposition of \( \mathcal{P}(V^k) \) under the joint action of \((\mathfrak{gl}_n, \text{GL}_k)\) is given by
\[ \mathcal{P}(V^k) \simeq \sum_{\sigma \in \mathbb{GL}_k, L(\sigma) \neq 0} L(\sigma) \otimes V_{\sigma}, \]
where \( V_{\sigma} \) is a representative of the class \( \sigma \in \mathbb{GL}_k \), the set of equivalence classes of the finite-dimensional irreducible representation of \( \text{GL}_k \), and \( L(\sigma) := \text{Hom}_{\text{GL}_k}(V_{\sigma}, \mathcal{P}(V^k)) \) which is a finite-dimensional irreducible representation of \( \mathfrak{gl}_n \). It is also well known that the action \( \pi \) restricted to \( \mathfrak{l} \) lifts to the maximal compact subgroup \( K \) of \( G = \text{U}(p, q) \), which implies that \( L(\sigma) \) is an irreducible \((\mathfrak{g}, K)\)-module (see [GW10]).

4. Reductive Dual Pair \((\mathfrak{o}^*(2n), \text{Sp}_k)\)

In this section, let \( G \) denote the linear Lie group defined by
\[ \mathfrak{o}^*(2n) = \{ g \in \text{U}(n, n); 'g S g = S \} \]
\[ = \{ g \in \text{GL}_{2n}(\mathbb{C}); g^* I_{n, n} g = I_{n, n}, 'g S g = S \}, \]
where \( S \) denotes the nondegenerate symmetric matrix \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) of size \( 2n \times 2n \). Set \( \mathfrak{g}_0 = \mathfrak{o}^*(2n) \), the Lie algebra of \( G \), and take a basis for \( \mathfrak{g}_0 \) as follows:
\[
\begin{align*}
X^i_{-j} &= E_{i,j} - E_{j,i} + E_{i,j} - E_{j,i} & (1 \leq i < j \leq n), \\
Y^i_{-j} &= i(E_{i,j} + E_{j,i} - E_{i,j} - E_{j,i}) & (1 \leq i < j \leq n), \\
X^i_{+i} &= E_{i,j} - E_{j,i} + E_{i,j} + E_{j,i} & (1 \leq i < j \leq n), \\
Y^i_{+i} &= i(E_{i,j} - E_{j,i} + E_{i,j} - E_{j,i}) & (1 \leq i < j \leq n),
\end{align*}
\]

(4.1)

where \( E_{i,j} \) denotes the matrix unit of size \( 2n \times 2n \). The complexified Lie algebra \( \mathfrak{o}_{2n} \) of \( \mathfrak{g}_0 = \mathfrak{o}^*(2n) \) is realized as
\[ \mathfrak{o}_{2n} = \{ X \in \text{Mat}_{2n}(\mathbb{C}); 'XS + SX = O \} \]
(4.2)
in this section, which we will denote by \( \mathfrak{g} \) below. It has a basis
\[
\begin{align*}
X^i_{-j} &= E_{i,j} - E_{j,i} & (1 \leq i, j \leq n), \\
X^i_{+i} &= E_{i,j} - E_{j,i} & (1 \leq i < j \leq n), \\
X^i_{-i} &= E_{j,i} - E_{i,j} & (1 \leq i < j \leq n).
\end{align*}
\]
(4.3)

4.1. Let \( W = (\mathbb{C}^{2n})_\mathbb{R} \) and \( \omega = \text{Im } H \), where \( H : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C} \) is the Hermitian form given by
\[ H(u, v) = u^* I_{n, n} v \quad (u, v \in \mathbb{C}^{2n}). \]
Namely, we consider the case we have discussed in §3 with \( p = q = n \). Note in particular that \( \omega \) can be written as
\[ \omega = \sum_{j=1}^{n} (dx_j \wedge dy_j - dx_j \wedge dy_j) \]
(4.4)
at \( v = x + iy \in W \) with \( x = (x_1, \ldots, x_{2n}), \ y = (y_1, \ldots, y_{2n}) \in \mathbb{R}^{2n} \). Then \((W, \omega)\) is a symplectic \( G \)-vector space, as above.

**Remarks 4.1.** (i) There is another realization of the Lie group \( O^*(2n) \) as a group consisting of the complex orthogonal matrices. Namely,

\[
O^*(2n) = \{ g \in GL_{2n}; \ 'gg = 1, \ 'gJ_gJ = J \};
\]

we temporarily denote this realization of \( O^*(2n) \) by \( G^* \), because the former realization \( G \) is isomorphic to \( G^* \) by \( G \). The vector fields on \( W \) generated by the basis

\[
\text{sense of } G
\]

we regard as a left \( \mathbb{C} \)-linear transformations on \( \mathbb{C}^{2n} \); one sees that

\[
\phi_1 : \mathbb{H}^n \to \mathbb{C}^{2n}, \quad v = v' + jv'' \mapsto \begin{bmatrix} v' \\ v'' \end{bmatrix} \quad (v', v'' \in \mathbb{C}^n),
\]

which is in fact a \( \mathbb{C} \)-isomorphism. Then \( G^* \) is characterized as the group consisting of \( \mathbb{C} \)-linear transformations on \( \mathbb{H}^n \) that preserve the quaternionic skew-Hermitian form \( C \) given by

\[
C(u, v) := u^*jv \quad (u, v \in \mathbb{H}^n)
\]

(see [GW10] for details).

(ii) There is another identification of \( \mathbb{H}^n \) with a \( \mathbb{C} \)-vector space. Namely, there is an isomorphism of \( \mathbb{H}^n \) onto \( \text{Mat}_{n \times 2}(\mathbb{C}) \) given by

\[
\phi_2 : \mathbb{H}^n \to \text{Mat}_{n \times 2}(\mathbb{C}), \quad v = v' + jv'' \mapsto \begin{bmatrix} v' \\ v'' \end{bmatrix}.
\]

In this case, however, \( \mathbb{H}^n \) is regarded as a left \( \mathbb{H} \)-vector space, and the map \( \phi_2 \) is a \( \mathbb{C} \)-isomorphism in this sense. Since \( jv'' = \bar{v}' \) for \( v'' \in \mathbb{C}^n \), one sees that

\[
(\phi_2 \circ \phi_1^{-1})(\begin{bmatrix} v' \\ v'' \end{bmatrix}) = [v', \bar{v}'].
\]

Note that \( \phi_2 \circ \phi_1^{-1} \) is an \( \mathbb{R} \)-isomorphism from \( \mathbb{C}^{2n} \) onto \( \text{Mat}_{n \times 2}(\mathbb{C}) \).

More generally, let us consider \((\mathbb{H}^n)^k\), the direct sum of \( k \) copies of \( \mathbb{H}^n \), which we regard as a left \( \mathbb{H} \)-vector space as above. Then the multiplication on \((\mathbb{H}^n)^k\) on the right by an element of \( \text{Mat}_k(\mathbb{H}) \) gives a \( \mathbb{H} \)-module, say, \( a + bj \) with \( a, b \in \text{Mat}_k(\mathbb{C}) \), corresponds to the multiplication on \( \text{Mat}_{n \times 2k}(\mathbb{C}) \) on the right by the complex \( 2k \times 2k \)-matrix \( \begin{bmatrix} a & b \\ -b & \bar{a} \end{bmatrix} \).

**Lemma 4.2.** The vector fields on \( W \) generated by the basis \( \{4.1\} \) for \( g_0 = o^*(2n) \) in the sense of \( [1,4] \) are given by

\[
\begin{align*}
(X^c_{j,i})_W &= -x_i \partial_{x_j} - y_j \partial_{x_i} + x_i \partial_{x_j} + y_j \partial_{x_i}, \\
(Y^c_{j,i})_W &= y_j \partial_{x_i} - x_i \partial_{y_j} - y_j \partial_{y_i} + x_i \partial_{y_j} - x_i \partial_{y_j} - y_j \partial_{x_i} + x_i \partial_{y_j}, \\
(X^s_{j,i})_W &= -x_i \partial_{x_j} - x_i \partial_{x_j} + x_i \partial_{x_j} - y_j \partial_{x_i} - y_j \partial_{x_i} + x_i \partial_{y_j} - y_j \partial_{y_i} + y_j \partial_{y_i} - y_j \partial_{y_i}, \\
(Y^s_{j,i})_W &= y_j \partial_{x_i} - y_j \partial_{x_i} + y_j \partial_{x_i} - x_i \partial_{y_j} + x_i \partial_{y_j} - x_i \partial_{y_j} - y_j \partial_{x_i} + x_i \partial_{y_j}.
\end{align*}
\]
For a given $v = \left[\begin{array}{l}v' \\ v'' \end{array}\right] \in \mathbb{C}^{2n}$ with $v', v'' \in \mathbb{C}^n$, we set $v_+ := (\phi_2 \circ \phi_1^{-1})(v) = [v', \bar{v}''] \in \text{Mat}_{n \times 2}(\mathbb{C})$ for brevity. By Remark 4.1 (ii), $\text{Sp}(1)$ acts on $W$ on the right via the $\mathbb{R}$-isomorphism $\phi_2 \circ \phi_1^{-1}$.

**Proposition 4.3.** Let $(W, \omega)$ be as above and $G = \mathcal{O}^*(2n)$. Then the moment map $\mu : W \to g_0^* \cong g_0$ is given by

$$
\mu(v) = -\frac{i}{2} (v^t I_{n,n} - S (v^t I_{n,n}) S) \quad (4.10a)
$$

$$
= -\frac{i}{2} \begin{bmatrix} v_+ v^*_+ & -v^*_+ J_1 v_+ \\ -v_+ J_1^t v^*_+ & -v^*_+J_1 v_+ \end{bmatrix} \quad (4.10b)
$$

for $v = x + i y \in W$ with $x = \langle x_1, \ldots, x_{2n} \rangle, y = \langle y_1, \ldots, y_{2n} \rangle \in \mathbb{R}^{2n}$. In particular, $\mu$ is $G$-equivariant and is $\text{Sp}(1)$-invariant.

**Proof.** It follows from Lemma 4.2 that

$$
\langle \mu, X \rangle = \begin{cases} 
-x_i y_j - x_j y_i - x_i y_j + x_j y_i & \text{if } X = X^t_{i,j} ; \\
-x_i x_j + x_j x_i + y_i y_j + y_j y_i & \text{if } X = Y^t_{i,j} ; \\
x_i y_j - x_j y_i - x_i y_j + x_j y_i & \text{if } X = X_i^t_{i,j} ; \\
x_i x_j - x_j x_i - y_i y_j + y_j y_i & \text{if } X = Y_i^t_{i,j} ,
\end{cases}
$$

which can be rewritten in terms of complex coordinates defined by $z_i := x_i + i y_i, i = 1, \ldots, 2n$, and their complex conjugates as

$$
\langle \mu, X \rangle = \begin{cases} 
-\frac{1}{2} (\bar{z}_i z_j - \bar{z}_j z_i - \bar{z}_i z_j + \bar{z}_j z_i) & \text{if } X = X^t_{i,j} ; \\
\frac{1}{2} (z_i z_j + \bar{z}_i z_j + \bar{z}_j z_i) & \text{if } X = Y^t_{i,j} ; \\
-\frac{1}{2} (\bar{z}_i z_j - \bar{z}_j z_i + \bar{z}_i z_j - \bar{z}_j z_i) & \text{if } X = X_i^t_{i,j} ; \\
\frac{1}{2} (z_i z_j - \bar{z}_i z_j + \bar{z}_j z_i) & \text{if } X = Y_i^t_{i,j} .
\end{cases}
$$

Thus, setting $v' := \langle z_1, \ldots, z_n \rangle$ and $v'' := \langle \bar{z}_1, \ldots, \bar{z}_n \rangle$, one obtains that

$$
\mu(v) = \sum_{i < j} \langle \mu, X^t_{i,j} \rangle (X^t_{i,j})^\vee + \sum_{i < j} \langle \mu, Y^t_{i,j} \rangle (Y^t_{i,j})^\vee + \sum_{i < j} \langle \mu, X_i^t_{i,j} \rangle (X_i^t_{i,j})^\vee + \sum_{i < j} \langle \mu, Y_i^t_{i,j} \rangle (Y_i^t_{i,j})^\vee
$$

$$
= -\frac{i}{2} \sum_{i,j} \left( (z_i \bar{z}_j + \bar{z}_i z_j) E_{i,j} - (\bar{z}_i z_j + \bar{z}_i z_j) E_{i,j} - (z_i \bar{z}_j - \bar{z}_i z_j) E_{i,j} - (z_i \bar{z}_j - \bar{z}_i z_j) E_{i,j} \right)
$$

$$
= -\frac{i}{2} \left[ v' \bar{v}' + \bar{v}'' v'' - v'' \bar{v}' + \bar{v}' v'' \right] = -\frac{i}{2} \left[ \begin{bmatrix} v' \\ v'' \end{bmatrix} \left( \bar{v}', -\bar{v}'' \right) + \left[ \bar{v}'' \bar{v}' \right] \left(v'', -v' \right) \right]
$$

$$
= -\frac{i}{2} (v^t I_{n,n} - S (v^t I_{n,n}) S).
$$

Rewriting (4.10a) one obtains the second expression (4.10b).

The $\text{Sp}(1)$-invariance of $\mu$ immediately follows from (4.10b), and the $G$-equivariance can be verified in the following way. If $g \in G$, then

$$
\mu(gv) = \frac{i}{2} (gv^t g^* I_{n,n} - S (gv^t g^* I_{n,n}) S) = \frac{i}{2} (gv^t I_{n,n} g^{-1} - S (gv^t I_{n,n} g^{-1}) S)
$$
since \(g^* I_{n,n} = I_{n,n} g^{-1}\). The second term in the brace of the right-hand side equals

\[
S' g^{-1} (vv^* I_{n,n})' g S = g S' (vv^* I_{n,n}) S g^{-1}
\]

since \(g' S = S g^{-1}\). Thus,

\[
\mu(gv) = -\frac{i}{2} \left( gvv^* I_{n,n} g^{-1} - g S' (vv^* I_{n,n}) S g^{-1} \right) = \text{Ad}(g) \mu(v).
\]

This completes the proof. \(\square\)

It follows from (4.4) that the Poisson brackets among \(x_i, y_j, i = 1, \ldots, 2n\), are given by

\[
\{x_i, y_j\} = -\delta_{i,j}, \quad \{x_i, y_j\} = \delta_{i,j} \tag{4.13}
\]

for \(i, j = 1, \ldots, n\), and all other brackets vanish. In terms of complex coordinates \(z_j = x_j + i y_j\) for \(j = 1, \ldots, 2n\) and their conjugates, it follows from (4.13) that the Poisson brackets among them are given by

\[
\{z_i, \bar{z}_j\} = \{z_i, z_j\} = 2i \delta_{i,j} \tag{4.14}
\]

for \(i, j = 1, \ldots, n\) and all other brackets vanish, as in (3.9). In view of (4.14), we quantize them by assigning

\[
\hat{z}_i = z_i, \quad \hat{\bar{z}}_i = -2 \partial_{z_i},
\]

\[
\hat{\bar{z}}_i = \bar{z}_i, \quad \hat{z}_i = -2 \partial_{\bar{z}_i} \tag{4.15}
\]

for \(i = 1, \ldots, n\) so that the nontrivial commutators among the quantized operators are given by

\[
[\hat{z}_i, \hat{\bar{z}}_j] = [\hat{\bar{z}}_i, \hat{z}_j] = 2 \delta_{i,j} \tag{4.16}
\]

for \(i, j = 1, \ldots, n\).

Let \(I\) denote a complex structure on \(W\) defined by \(e_j \mapsto i e_j\) and \(i e_j \mapsto -e_j\) for \(j = 1, \ldots, 2n\). Under the identification \(e_j \leftrightarrow \partial_{z_j}\) and \(i e_j \leftrightarrow \partial_{\bar{z}_j}\), the classical observables \(z_j\) and \(\bar{z}_j\) introduced above can be regarded as the coordinate functions on \(W_c\) with respect to the basis \(\frac{1}{2}(e_j - i e_j)\) and \(\frac{1}{2}(e_j + i e_j)\) respectively for \(j = 1, \ldots, 2n\). Note that \(\bar{z}_j\) is no longer the complex conjugate of \(z_j\) since \(x_i\) and \(y_i\) are now complex functions. Then the quantization (4.15) corresponds to taking a complex Lagrangian subspace \(V\) given by

\[
V = \left\{\frac{1}{2}(e_j - i e_j), \frac{1}{2}(e_j + i e_j); j = 1, \ldots, n\right\}_c. \tag{4.17}
\]

For simplicity, we set \(w_j := \bar{z}_j, j = 1, \ldots, n\), and write an element of \(V = \text{Mat}_{n \times 2}(\mathbb{C})\) as \([z, w]\) with \(z = (z_1, \ldots, z_n)\) and \(w = (w_1, \ldots, w_n)\) in what follows.

Now, we quantize the moment map \(\mu\) according to (4.15) using its first expression (4.10a) as follows and denote the quantized moment map by \(\hat{\mu}\):

\[
\hat{\mu} = -\frac{i}{2} \left[ \begin{array}{c}
\hat{z}_1 \\
\vdots \\
\hat{z}_n
\end{array} \right] (\bar{z}_1, \ldots, \bar{z}_n) I_{n,n} - S I_{n,n} \left[ \begin{array}{c}
\hat{z}_1 \\
\vdots \\
\hat{z}_n
\end{array} \right] (\bar{z}_1, \ldots, \bar{z}_n) S \tag{4.18a}
\]

\[
= -\frac{i}{2} \left[ \begin{array}{c}
z \\
-2 \partial_w
\end{array} \right] (-2 ' \partial_z, 'w) I_{n,n} - S I_{n,n} \left[ \begin{array}{c}
z \\
-2 \partial_w
\end{array} \right] (\bar{z}, -2 ' \partial_w) S
\]

\[ i \left[ z'\partial_z + w'\partial_w, \frac{1}{2}(z'w - w'z) \right] \] (4.18b)

where \( z' = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n), \partial_z = (\partial_{z_1}, \ldots, \partial_{z_n}) \) and \( \partial_w = (\partial_{w_1}, \ldots, \partial_{w_n}) \).

The quantization of the second expression (4.18b), i.e.,

\[ \bar{\mu} = -\frac{i}{2} \begin{bmatrix} \bar{\nu}_+ \bar{\nu}_+ & -\bar{\nu}_+ J_1 \bar{\nu}_+ \\ -\bar{\nu}_+ J_1 \bar{\nu}_+ & -\bar{\nu}_+ \bar{\nu}_+ \end{bmatrix} \] (4.18c)

produces the same result as (4.18b), where \( \bar{\nu}_+ = [z, w] \) and \( \hat{\nu}_+ = [-2\partial_z, -2\partial_w] \).

Let \( \mathcal{P}(V) \) denote the algebra of complex coefficient polynomials on \( V \), i.e., \( \mathcal{P}(V) = \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n] \), and \( \mathcal{P}\mathcal{D}(V) \) the ring of polynomial coefficient differential operators on \( V \). Note that the complex symplectic group of rank one

\[ \text{Sp}_1 = \{ g \in \text{GL}_2; \; \hat{\nu}_1 g = J_1 \} \]

acts on \( V \) by matrix multiplication on the right, and hence on \( \mathcal{P}(V) \) by right regular representation, which we denote by \( \rho \), as in (2.14). The right-action of \( \text{Sp}_1 \) on \( V \) coincides with the one on \( \text{Mat}_{n \times 2}(\mathbb{C}) \) mentioned in Remark 4.1 (ii).

**Theorem 4.4.** For \( X \in \mathfrak{g} = \mathfrak{o}_{2n} \), set \( \pi(X) = i \bar{\mu}, X \). Then

\[ \pi : \mathfrak{g} \to \mathcal{P}\mathcal{D}(V) \]

is a Lie algebra homomorphism. In terms of the basis (4.3) for \( \mathfrak{g} \), it is given by

\[ \pi(X) = \begin{cases} -(z_j \partial_z + w_j \partial_w + \delta_{ij}) & \text{if } X = X^0_{i,j} \\ 2(\partial_z w_j - \partial_w z_j) & \text{if } X = X^1_{i,j} \end{cases} \] (4.19)

Moreover, \( \pi(X) \) commutes with the action of \( \text{Sp}_1 \) i.e., \( \pi(X) \in \mathcal{P}\mathcal{D}(V)^{\text{Sp}_1} \) for all \( X \in \mathfrak{g} \).

**Proof.** It suffices to prove that \( \pi(X) \) commutes with the right-action of \( \text{Sp}_1 \). For this, we use the second expression (4.18c) of \( \bar{\mu} \). It follows from Lemma 2.6 that

\[ \text{Ad}_{\rho(g)} \bar{\nu}_+ = \bar{\nu}_+ g^{-1} \quad \text{and} \quad \text{Ad}_{\rho(g)} \hat{\nu}_+ = \hat{\nu}_+ \bar{g} \]

for \( g \in \text{GL}_2 \). Therefore, if \( g \in \text{Sp}_1 \) then one obtains

\[ \text{Ad}_{\rho(g)} \bar{\mu} = -\frac{i}{2} \begin{bmatrix} \bar{\nu}_+ g^{-1} \bar{\nu}_+ & -\bar{\nu}_+ g^{-1} J_1 \bar{\nu}_+ \\ -\bar{\nu}_+ J_1 \bar{\nu}_+ & -\bar{\nu}_+ \bar{\nu}_+ \end{bmatrix} \]

\[ = -\frac{i}{2} \begin{bmatrix} \bar{\nu}_+ \bar{\nu}_+ & -\bar{\nu}_+ J_1 \bar{\nu}_+ \\ -\bar{\nu}_+ J_1 \bar{\nu}_+ & -\bar{\nu}_+ \bar{\nu}_+ \end{bmatrix} = \bar{\mu} \]

since \( \hat{\nu}_1 g = J_1 \). This completes the proof. \( \square \)

4.2. Now let us consider \( W^k \), the direct sum of \( k \) copies of \( W = (\mathbb{C}^2)^n \), which we identify with \( \text{Mat}_{2n \times 2}(\mathbb{C}) \). It is equipped with a symplectic form given by

\[ \omega_k(u, v) = \text{Im} \; \text{tr} \; (u^* \mathcal{I}_{2n} v) \quad (u, v \in W^k), \]

and is still acted on by \( G = O^*(2n) \) symplectically by matrix multiplication on the left. Under the identification of \( e_{i,a} \leftrightarrow \partial_{x_{ia}} \) and \( i e_{i,a} \leftrightarrow \hat{\partial}_{x_{ia}} \), we write an element of \( W^k \) as \( v = [v_1, \ldots, v_{2n}] \), where \( v_i = x_i + iy_i \) are complex row vectors with \( x_i = (x_{i,1}, \ldots, x_{i,k}) \)
and \( y_i = (y_{i,1}, \ldots, y_{i,k}) \) being real row vectors of size \( k \) for \( i = 1, \ldots, 2n \). Then \( \omega_k \) is given by

\[
\omega_k = \sum_{1 \leq i \leq n, 1 \leq a \leq k} (dx_{i,a} \wedge dy_{i,a} - dx_{i,a} \wedge dy_{i,a})
\]  
(4.20)

at \( v = [v_1, \ldots, v_{2n}] \in \text{Mat}_{2n \times k}(\mathbb{C}) \). Moreover, the isomorphisms \( \phi_1 \) and \( \phi_2 \) defined by (4.5) and (4.7) respectively naturally extend to the one between \( (\mathbb{H}^n)^k \) and \( \text{Mat}_{2n \times k}(\mathbb{C}) \) and the one between \( (\mathbb{H}^n)^k \) and \( \text{Mat}_{n \times 2k}(\mathbb{C}) \) respectively, which we denote by the same symbols. Then \( \text{Sp}(k) \) acts on \( W^k \) on the right via the \( \mathbb{R} \)-isomorphism \( \phi_2 \circ \phi_1^{-1} \), as above.

**Proposition 4.5.** Let \( (W^k, \omega_k) \) be the symplectic \( G \)-vector space as above. Then the moment map \( \mu : W^k \to \mathfrak{g}_0^* \cong \mathfrak{g}_0 \) is given by the same formulae as (4.10). Namely, for \( v = [v', v''] \in W^k \) with \( v', v'' \in \text{Mat}_{2k}(\mathbb{C}) \),

\[
\mu(v) = -\frac{i}{2} (v'v^n I_{n,n} - S' (vv'' I_{n,n}) S')
\]

\[
= -\frac{i}{2} \begin{pmatrix} v_+ v'_+ & -v_+ J_k v'_+ \\ -\bar{v}_+ J_k v'_+ & -\bar{v}_+ v'_+ \end{pmatrix},
\]

(4.21)

where \( v_+ = (\phi_2 \circ \phi_1^{-1})(v) \in \text{Mat}_{n \times 2k}(\mathbb{C}) \). In particular, \( \mu \) is \( G \)-equivariant and is \( \text{Sp}(k) \)-invariant.

**Proof.** The vector fields on \( W^k \) generated by the basis for \( \mathfrak{g}_0 \) are given by the same formulae as (4.9) in Lemma 4.2, with the understanding that \( x_i, y_i, \partial_i, \bar{\partial}_i \) are row vectors and the products stand for the inner product of row vectors. Now, exactly the same argument as in Proposition 4.3 implies the proposition. \( \square \)

It follows from (4.20) that the Poisson brackets among the real coordinate functions \( x_{i,a}, y_{i,a} \) are given by

\[
\{x_{i,a}, y_{j,b}\} = -\delta_{i,j} \delta_{a,b}, \quad \{x_{i,a}, y_{j,b}\} = \delta_{i,j} \delta_{a,b}
\]

(4.22)

and all other brackets vanish, and hence the nontrivial ones among the complex coordinate functions are given by

\[
\{z_{i,a}, \bar{z}_{j,b}\} = \{\bar{z}_{i,a}, z_{j,b}\} = 2i \delta_{i,j} \delta_{a,b}
\]

(4.23)

for \( i, j = 1, \ldots, n \) and \( a, b = 1, \ldots, k \). Therefore, we quantize \( z_{i,a} \) and \( \bar{z}_{i,a} \) by assigning

\[
\hat{z}_{i,a} = z_{i,a}, \quad \hat{\bar{z}}_{i,a} = -2\partial_{i,a},
\]

(4.24)

so that the nontrivial commutators among the quantized operators are given by

\[
[\hat{z}_{i,a}, \hat{z}_{j,b}] = [\hat{\bar{z}}_{i,a}, \hat{\bar{z}}_{j,b}] = 2\delta_{i,j} \delta_{a,b}
\]

(4.25)

for \( i, j = 1, \ldots, n \) and \( a, b = 1, \ldots, k \).

Let \( V^k \) denote the direct sum of \( k \) copies of \( V \), with \( V \) as in (4.17). Since \( V^k \) can be identified with \( \text{Mat}_{n \times 2k}(\mathbb{C}) \), we write an element of \( V^k \) as \( [z, w] \), where \( z = (z_{i,a}) \) and \( w = (w_{i,a}) \) are elements of \( \text{Mat}_{n \times 2k}(\mathbb{C}) \), and we set \( w_{i,a} = \bar{z}_{i,a} \) for \( i = 1, \ldots, n \) and \( a = 1, \ldots, k \) for simplicity, as above. Let \( \mathcal{P}(V^k) = \mathbb{C}[z_{i,a}, w_{i,a}; i = 1, \ldots, n, a = 1, \ldots, k] \) be the algebra of complex polynomial functions on \( V^k \), and \( \mathcal{P}D(V^k) \) the ring of polynomial coefficient differential operators on \( V^k \). Then the complex symplectic group \( \text{Sp}_k \)
acts on \( V^k \) by matrix multiplication on the right, and hence on \( \mathcal{P}(V^k) \) by right regular representation, which we denote by \( \rho \), as usual.

The quantized moment map \( \tilde{\mu} \) is given by the same formula as (4.18):

\[
\tilde{\mu} = \begin{bmatrix}
z' \partial_z + w' \partial_w & \frac{1}{2}(z' w - w' z) \\
2(\partial_z' w - \partial_w' w) & -(\partial_w' w + \partial_z' z)
\end{bmatrix}.
\]

Here, \( z \) (resp. \( w \)) and \( \partial_z \) (resp. \( \partial_w \)) now denote \( n \times k \)-matrices whose \((i, a)\)-th entries are the multiplication operator \( z_{i,a} \) (resp. \( w_{i,a} \)) and the differential operator \( \partial_{z_{i,a}} \) (resp. \( \partial_{w_{i,a}} \)) for \( i = 1, \ldots, n \) and \( a = 1, \ldots, k \).

Corollary 4.6. For \( X \in \mathfrak{g} = \mathfrak{o}_{2n} \), set \( \pi(X) = i(\tilde{\mu}, X) \). Then

\[
\pi : \mathfrak{g} \to \mathcal{P}(\mathbb{D}(V^k))
\]

is a Lie algebra homomorphism. In terms of the basis \((4.3)\) for \( \mathfrak{g} \), it is given by

\[
\pi(X) = \begin{cases}
-\sum_{i=1}^{k} (z_{i,a} \partial_{z_{i,a}} + w_{i,a} \partial_{w_{i,a}} + k \partial_{l_{i,j}}) & \text{if } X = X_{i,j}^0,
2 \sum_{i=1}^{k} (\partial_{z_{i,a}} w_{i,a} - \partial_{w_{i,a}} z_{i,a}) & \text{if } X = X_{i,j}^1,
\frac{1}{2} \sum_{i=1}^{k} (z_{i,a} w_{i,a} - w_{i,a} z_{i,a}) & \text{if } X = X_{i,j}^2.
\end{cases}
\]

Moreover, \( \pi(X) \) commutes with the action of the complex symplectic group \( \text{Sp}_k \), i.e., \( \pi(X) \in \mathcal{P}(\mathbb{D}(V^k))^{\text{Sp}_k} \) for all \( X \in \mathfrak{g} \).

Proof. The proof is essentially the same as that of Theorem 4.4. \( \square \)

Similarly to the cases discussed above, it is well known that the irreducible decomposition of \( \mathcal{P}(V^k) \) under the joint action of \( (\mathfrak{o}_{2n}, \text{Sp}_k) \) is given by

\[
\mathcal{P}(V^k) \cong \bigoplus_{\sigma \in \text{Sp}_k, L(\sigma) \neq 0} L(\sigma) \otimes V_{\sigma},
\]

where \( V_{\sigma} \) is a representative of the class \( \sigma \in \text{Sp}_k \), the set of equivalence classes of the finite-dimensional irreducible representation of \( \text{Sp}_k \), and \( L(\sigma) := \text{Hom}_{\text{Sp}_k}(V_{\sigma}, \mathcal{P}(V^k)) \) which is an infinite-dimensional irreducible representation of \( \mathfrak{o}_{2n} \). It is also well known that the action \( \pi \) restricted to \( \mathfrak{l} \) lifts to the maximal compact subgroup \( K \) of \( G = O^*(2n) \), which implies that \( L(\sigma) \) is an irreducible \((\mathfrak{g}, K)\)-module (see [Yam01]).

5. Lagrangian Subspace

In this section, we take complex Lagrangian subspaces of \( W_C \) different from the ones considered in the previous sections in the cases where \( G = O^*(2n) \) and \( U(p, q) \). Finally, we make an observations that the image of the Lagrangian subspace coincides with the associated variety of the corresponding irreducible \( \mathfrak{g} \)-modules occurring in the irreducible decomposition of the space consisting of polynomial functions on the Lagrangian subspace under the joint action of \((\mathfrak{g}, G')\).
5.1. Let $G = O^*(2n)$ and let $(W, \omega)$ be the symplectic $G$-vector space we discussed in §3, i.e., $W = (\mathbb{C}^{2n})_\mathbb{R}$ and $\omega$ is given by $(4.4)$. Let us now consider another complex Lagrangian subspace $V' \subset W_\mathbb{C}$ defined by

$$V' := \left\{ \frac{1}{2}(e_1 - i e_1), \ldots, \frac{1}{2}(e_{2n} - i e_{2n}) \right\}_\mathbb{C} \tag{5.1}$$

and the corresponding quantization

$$\tilde{z}_i = z_i, \quad \tilde{z}_j = -2e_i \partial_{z_i} \tag{5.2}$$

for $i = 1, \ldots, 2n$ as in §3 which also satisfy $(4.25)$. Here $I$ denotes the complex structure on $W$ mentioned in §3. Then the quantized moment map, which we denote by the same symbol $\tilde{\mu}$, is given by

$$\tilde{\mu} = -\frac{i}{2} \left[ \begin{array}{c} \tilde{z}_1 \\ \vdots \\ \tilde{z}_n \end{array} \right] (\tilde{z}_1, \ldots, \tilde{z}_n) I_{n,n} - S I_{n,n} \left[ \begin{array}{c} \tilde{z}_1 \\ \vdots \\ \tilde{z}_n \end{array} \right] S \right)$$

$$= -i \left[ -\tilde{z}' \partial_{\tilde{z}} - \partial_{\tilde{z}'} \tilde{z}' - \tilde{z}'' \partial_{\tilde{z}} + \partial_{\tilde{z}'} \tilde{z}' \right]$$

with $\tilde{z}' = \langle z_1, \ldots, z_n \rangle, \tilde{z}'' = \langle \tilde{z}_1, \ldots, \tilde{z}_n \rangle, \partial_{\tilde{z}} = \langle \partial_{z_1}, \ldots, \partial_{z_n} \rangle$ and $\partial_{\tilde{z}'} = \langle \partial_{\tilde{z}_1}, \ldots, \partial_{\tilde{z}_n} \rangle$. Therefore, in terms of the basis $(4.3)$ for $\mathfrak{g} = \mathfrak{so}_{2n}$, $\pi(X) := i(\tilde{\mu}, X)$ is given by

$$\pi(X) = \begin{cases} z_j \partial_{z_i} - z_i \partial_{z_j} & \text{if } X = X_{i,j}^0; \\ z_j \partial_{z_i} - z_i \partial_{z_j} & \text{if } X = X_{i,j}^j; \\ z_i \partial_{z_j} - z_j \partial_{z_i} & \text{if } X = X_{i,j}^j. \end{cases} \tag{5.4}$$

Since each $\pi(X)$ preserves the degree of a homogeneous polynomial $f \in \mathcal{P}(V') = \mathbb{C}[z_1, \ldots, z_{2n}]$ for $X \in \mathfrak{g}$, any irreducible representation appearing in the irreducible decomposition of $\mathcal{P}(V')$ is finite-dimensional.

5.2. On the contrary, we will apply the quantization procedure introduced in §4 to the case discussed in §3. Namely, let $G = U(p, q)$ and let $(W, \omega)$ be the symplectic $G$-vector space, i.e., $W = (\mathbb{C}^{p+q})_\mathbb{R}$ and $\omega$ is given by $(3.2)$. Now we quantize the complex coordinate functions $z_j = x_j + iy_j$ and $\tilde{z}_j = x_j - iy_j$ in the following way (cf. $(4.15)$):

$$\tilde{z}_i = z_i, \quad \tilde{z}_i = -2 \partial_{z_i}, \quad (i = 1, \ldots, p);$$
$$\tilde{z}_j = z_j, \quad \tilde{z}_j = -2 \partial_{z_j}, \quad (j = 1, \ldots, q), \tag{5.5}$$

which also satisfy $(3.11)$. This quantization corresponds to taking a complex Lagrangian subspace $V' \subset W_\mathbb{C}$ defined by

$$V' = \left\{ \frac{1}{2}(e_i - i e_i), \frac{1}{2}(e_j + i e_j); i = 1, \ldots, p, j = 1, \ldots, q \right\}_\mathbb{C}, \tag{5.6}$$

where $I$ denotes the complex structure on $W$ mentioned in §3. For simplicity, we will write $w_j := \tilde{z}_j$, $j = 1, \ldots, q$, as in the previous section, and write an element of $V$ as $[\begin{smallmatrix} \cdot \\ w \end{smallmatrix}]$ with $z \in \mathbb{C}^p$ and $w \in \mathbb{C}^q$. Then the quantized moment map, which we denote by the
same symbol \( \hat{\mu} \), is given by

\[
\hat{\mu} = -\frac{i}{2} \begin{bmatrix}
\bar{z}_1 \\
\vdots \\
\bar{z}_n
\end{bmatrix} I_{p,q} = -\frac{i}{2} \begin{bmatrix}
z \\
-2\partial_w \end{bmatrix} (\partial_z - \partial_w')
\]

with \( z = (z_1, \ldots, z_p), \partial_z = \partial_{z_1}, \ldots, \partial_{z_p}, \) \( w = (w_1, \ldots, w_q) \) and \( \partial_w = \partial_{w_1}, \ldots, \partial_{w_q} \).

Let us now consider the \( k \) direct sum \( W^k \), and its subspace \( V^k \) given in (5.7) which is identified with \( \text{Mat}_{p\times k}(\mathbb{C}) \). Then \( GL_k \) acts on \( V^k \) on the right by

\[
\pi(X) = \begin{cases}
-z_j \partial_{z_i} & \text{if } X = E_{i,j} \ (i, j = 1, \ldots, p); \\
2\partial_{z_i} \partial_{w_j} & \text{if } X = E_{i,j} \ (i = 1, \ldots, p; \ j = 1, \ldots, q); \\
-\frac{1}{2}z_j w_i & \text{if } X = E_{i,j} \ (i = 1, \ldots, q; \ j = 1, \ldots, p); \\
\partial_{w_j} w_i & \text{if } X = E_{i,j} \ (i, j = 1, \ldots, q).
\end{cases}
\]

for \( g \in GL_k \) with \( z \in \text{Mat}_{p\times k}(\mathbb{C}) \) and \( w \in \text{Mat}_{q\times k}(\mathbb{C}) \), and hence on \( \mathcal{P}(V^k) \) by right regular representation, which we denote by \( \rho \), as usual. Note that (5.9) is the holomorphic extension of the standard right-action of \( U(k) \) on \( \text{Mat}_{p\times k}(\mathbb{C}) \) given by \( Z \mapsto Z g \) for \( Z \in \text{Mat}_{p\times k}(\mathbb{C}) \) and \( g \in U(k) \). Then, understanding that \( z \) and \( \partial_z \) (resp. \( w \) and \( \partial_w \)) in (5.7) stand for \( p \times k \)-matrices (resp. \( q \times k \)-matrices) as in the previous sections, one obtains the following.

**Theorem 5.1.** For \( X \in \mathfrak{g} = \mathfrak{gl}_n \), set \( \pi(X) := i \langle \hat{\mu}, X \rangle \). Then

\[
\pi : \mathfrak{g} \to \mathcal{P}(\mathcal{D}(V^k))
\]

is a Lie algebra homomorphism. Moreover, \( \pi(X) \) commutes with the action of \( GL_k \) on \( V^k \), i.e., \( \pi(X) \in \mathcal{P}(\mathcal{D}(V^k))^{GL_k} \) for all \( X \in \mathfrak{g} \).

**Proof.** We only show that \( \pi(X) \in \mathcal{P}(\mathcal{D}(V^k)) \) for \( X \in \mathfrak{g} \). It follows from Lemma 2.6 that

\[
\text{Ad}_{\rho(g)^{-1}} z = z g^{-1}, \quad \text{Ad}_{\rho(g)^{-1}} w = w' g,
\]

\[
\text{Ad}_{\rho(g)^{-1}} \partial_z = \partial_z' g, \quad \text{Ad}_{\rho(g)^{-1}} \partial_w = \partial_w g^{-1}
\]

for \( g \in GL_k \). Hence one obtains that

\[
\text{Ad}_{\rho(g)^{-1}} \hat{\mu} = -\frac{i}{2} \begin{bmatrix}
\text{Ad}_{\rho(g)^{-1}} z \\
-2\text{Ad}_{\rho(g)^{-1}} \partial_w \end{bmatrix} \begin{bmatrix}
-2(\text{Ad}_{\rho(g)^{-1}} \partial_z), \ (\text{Ad}_{\rho(g)^{-1}} w)
\end{bmatrix}
\]

\[
= -\frac{i}{2} \begin{bmatrix}
z g^{-1} \\
-2 \partial_w g^{-1}
\end{bmatrix} \begin{bmatrix}
-2 g' \partial_z, \ g' w
\end{bmatrix}
\]

\[
= -\frac{i}{2} \begin{bmatrix}
z \\
-2 \partial_w
\end{bmatrix} \begin{bmatrix}
g^{-1} \begin{bmatrix}
-2 g' \partial_z, \ g' w
\end{bmatrix}
\end{bmatrix} = \hat{\mu}.
\]

This completes the proof. \( \square \)
Therefore, the irreducible decomposition of \( P(V^k) \) is given by
\[
P(V^k) \simeq \sum_{\sigma \in \hat{\text{GL}}_k, L(\sigma) \neq \{0\}} L(\sigma) \otimes V_\sigma,
\]
where \( V_\sigma \) is a representative of the class \( \sigma \in \hat{\text{GL}}_k \), the set of equivalence classes of the finite-dimensional irreducible representation of \( \text{GL}_k \), and \( L(\sigma) := \text{Hom}_{\text{GL}_k}(V_\sigma, P(V^k)) \).

It is well known that \( L(\sigma) \) is an irreducible \((\mathfrak{g}, K)\)-module of infinite dimension for any \( \sigma \in \hat{\text{GL}}_k \) such that \( L(\sigma) \neq \{0\} \), where \( \mathfrak{g} = \mathfrak{gl}_{p+q} \) and \( K \) is the maximal compact subgroup of \( G = \text{U}(p, q) \) (see [KV78] and [Yam01]).

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