MAXIMAL SURFACE AREA OF A CONVEX SET IN $\mathbb{R}^n$ WITH RESPECT TO EXPONENTIAL ROTATION INVARIANT MEASURES.

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Abstract. Let $p$ be a positive number. Consider probability measure $\gamma_p$ with density $\varphi_p(y) = c_{n,p} e^{-\|y\|^p}$. We show that the maximal surface area of a convex body in $\mathbb{R}^n$ with respect to $\gamma_p$ is asymptotically equal to $C_p n^\frac{1}{4}$, where constant $C_p$ depends on $p$ only. This is a generalization of Ball’s [Ba] and Nazarov’s [N] bounds, which were given for the case of the standard Gaussian measure $\gamma_2$.

1. Introduction

As usual, $|\cdot|$ denotes the norm in Euclidean $n$-space $\mathbb{R}^n$, and $|A|$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. We will write $B^n_2 = \{x \in \mathbb{R}^n : |x| \leq 1\}$ for the unit ball in $\mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ for the unit $n$-dimensional sphere. We will denote by $\nu_n = |B^n_2| = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}$.

In this paper we will study the geometric properties of measures $\gamma_p$ on $\mathbb{R}^n$ with density

$$\varphi_p(y) = c_{n,p} e^{-\|y\|^p},$$

where $p \in (0, \infty)$ and $c_{n,p}$ is the normalizing constant.

Many interesting results are known for the case $p = 2$ (standard Gaussian measure). One must mention the Gaussian isoperimetric inequality of Borell [B] and Sudakov, Tsirelson [ST]: fix some $a \in (0, 1)$ and $\varepsilon > 0$, then among all measurable sets $A \subset \mathbb{R}^n$, with $\gamma_2(A) = a$ the set for which $\gamma_2(A + \varepsilon B^n_2)$ has the smallest Gaussian measure is half-space. We refer to books [Ba] and [LT] for more properties of Gaussian measure and inequalities of this type.

Mushtari and Kwapien asked the reverse version of isoperimetric inequality, i.e. how large the Gaussian surface area of a convex set $A \subset \mathbb{R}^n$ can be. In [Ba] it was shown, that Gaussian surface area of a convex body in $\mathbb{R}^n$ is asymptotically bounded by $C n^{1/4}$, where $C$ is an absolute constant. Nazarov in [N] gave the complete solution to this problem by proving the sharpness of Ball’s result:

$$0.28 n^{\frac{1}{4}} \leq \max \gamma_2(\partial Q) \leq 0.64 n^{\frac{1}{4}},$$

where maximum is taken over all convex bodies. Further estimates for $\gamma_2(\partial Q)$ were provided in [K].

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Isoperimetric inequalities for rotation invariant measures were studied by Sudakov, Tsirelson [ST], who proved that for a measure \( \gamma \) with density \( e^{-h(\log|x|)} \), where \( h(t) \) is a positive convex function, there exist derivative of a function \( M_Q(a) = \gamma(aQ) \) (where \( Q \) is a convex body), and minimum of \( M_Q(1) \) among all convex bodies is attained on half spaces. Thus the result can be applied to measures \( \gamma_p \) by setting \( h(t) = e^{pt} \). Some interesting results for manifolds with density were also provided by Bray and Morgan [BM] and further generalized by Maurmann and Morgan [MM].

The main goal of this paper is to compliment the study of isoperimetric problem for rotation invariant measures and to prove an inverse isoperimetric inequality for \( \gamma_p \), which is done using the generalization of Nazarov’s method from [N].

We remind that the surface area of a convex body \( Q \) with respect to the measure \( \gamma_p \) is defined to be

\[
\gamma_p(\partial Q) = \lim_{\epsilon \to 0^+} \frac{\gamma_p((Q + \epsilon B_n^p) \setminus Q)}{\epsilon}
\]

One can also provide an integral formula for \( \gamma_p(\partial Q) \):

\[
\gamma_p(\partial Q) = \int_{\partial Q} \varphi_p(y) d\sigma(y) = c_{n,p} \int_{\partial Q} e^{\frac{|y|^p}{p}} d\sigma(y),
\]

where \( d\sigma(y) \) stands for Lebesgue surface measure. We refer to [K] for the proof in the case \( p = 2 \).

The following theorem is the main result of this paper:

**Theorem 1.** For any positive \( p \)

\[
e^{-\frac{2}{p}n^{\frac{3}{4}-\frac{1}{p}}} \leq \max \gamma_p(\partial Q) \leq C(p)n^{\frac{3}{4}-\frac{1}{p}},
\]

where \( C(p) \approx 2\sqrt{2\pi}c_1e^{-\left(\frac{2}{p} + c_2p\right)p^2} \).

In Theorem 1 and further we will denote by "\( \approx \)" an asymptotic equality while \( p \) tends to infinity and by \( c_1, c_2, \ldots \) different absolute constants. We shall also use notation \( \lesssim \) for an asymptotic inequality.

Using the trick from [Ba] one can find an easy estimate from above for the surface area by \( e^{\frac{1}{p} - \frac{1}{2}n^{1 - \frac{1}{p}}} \). The calculation is given in the Section 2, as well as some other important preliminary facts. The upper bound from Theorem 1 is obtained in the Section 3, and the lower bound is shown in the Section 4.

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### 2. Preliminary lemmas.

We remind that \( \gamma_p \) is a probability measure on \( \mathbb{R}^n \) with density \( \varphi_p(y) = c_{n,p}e^{\frac{|y|^p}{p}} \), where \( p \in (0, \infty) \). The normalizing constant \( c_{n,p} \) equals to \( \nu_nJ_{n-1,p}^{-1} \), where

\[
J_{a,p} = \int_0^\infty t^a e^{-\frac{t^p}{p}} dt.
\]
We need to give an asymptotic estimate for $J_{a,p}$. Our main tool is the Laplace method, which can be found, for example, in [15]. For the sake of completeness, we shall present it here:

**Lemma 2.** Let $h(x)$ be a function on an interval $(a, b) \ni 0$ having at least two continuous derivatives (here $a$ and $b$ may be infinities). Let 0 be the global maxima point for $h(x)$ and assume for convinience that $h(0) = 0$. Assume also that $h''(0) < 0$ and that the integral $\int_a^b e^{th(x)}dx < \infty$. Then

$$\int_a^b e^{th(x)}dx \approx \sqrt{-\frac{2\pi}{h''(0)t}}, \quad t \to \infty.$$ 

**Proof.** First, using conditions of the lemma and Taylor formula, for a sufficiently small $h''(0) >> \epsilon > 0$ there exist positive $\delta = \delta(\epsilon)$, such that for any $x \in (-\delta, \delta)$ it holds that $|h(x) - \frac{h''(0)x^2}{2}| \leq \epsilon x^2$. Thus the integral

$$\int_{-\delta}^\delta e^{th(x)}dx \leq \frac{1}{\sqrt{-h''(0) + \epsilon}} \int_{-\delta}^\delta \sqrt{-h''(0) + \epsilon} e^{\frac{\epsilon x^2}{2}} dy \leq \sqrt{\frac{2\pi}{(h''(0) + \epsilon)t}}.$$ 

Note that for any constant $C > 0$,

$$\int_C^\infty e^{-\frac{y^2}{2}} dy \geq e^{-\frac{(t-1)c^2}{2}} \int_C^\infty e^{-\frac{y^2}{2}} dy = C' e^{-C''t},$$

thus (4) is asymptotically equivalent to $\sqrt{\frac{2\pi}{(h''(0) + \epsilon)t}}$. It remains to prove that the whole integral is coming from the small interval about zero under the lemma conditions on $h(x)$. Indeed, for an arbitrary $\epsilon$ we choose $\delta(\epsilon)$, and then by condition of the lemma, we pick $\eta(\delta) = \eta(\epsilon)$, so that

$$\int_{(a,-\delta)\cup(\delta,b)} e^{th(x)} \leq e^{-(t-1)\eta(\delta)} \int_a^b e^{th(x)}dx = C' e^{-C''t}.$$ 

Thus,

$$\int_a^b e^{th(x)}dx \approx \sqrt{-\frac{2\pi}{h''(0) + \epsilon}t}.$$ 

Similarly to (4) and by (5), the reverse inequality holds:

$$\sqrt{-\frac{2\pi}{(h''(0) - \epsilon)t}} \approx \int_a^b e^{th(x)}dx, \quad t \to \infty.$$ 

Taking $\epsilon$ small enough we finish the proof. \[\square\]

We will now apply the Laplace’s method to deduce the asymptotic estimate for $J_{a,p}$.

**Lemma 3.** Let $p > 0$. Then

$$J_{a,p} \approx \sqrt{\frac{2\pi}{a^p}} a^{-\frac{a^p}{p}} e^{-\frac{a^p}{p}}, \quad a \to \infty.$$
Thus where the expression for \( \gamma \):

\[
\gamma = \frac{\partial Q}{\partial C} \leq C'(p)e^{-C''(p)n} \text{ for any convex body } Q \subset \mathbb{R}^n.
\]

Proof. First, assume that \(|y| < (1 - \Delta_p)(n - 1)^{\frac{1}{p}} \) for any \( y \in \partial Q' \). Then

\[
\gamma_p(\partial Q') \leq \frac{1}{n\nu_nJ_{n-1,p}} \int_{\partial Q'} e^{-\frac{|y|^p}{p}} d\sigma(y) \leq \frac{1}{n\nu_nJ_{n-1,p}} |\partial Q'|.
\]

Since \( Q' \subset (1 - \Delta_p)(n - 1)^{\frac{1}{p}} B_2^n \), it holds that \(|\partial Q'| \leq (1 - \Delta_p)^{n-1}(1 - 1)^{\frac{n-1}{p}} n\nu_n \). By the choice of \( \Delta_p \), \( \square \) is exponentially small.

Assume now that for any \( y \in \partial Q'' \) it holds that \(|y| > (1 + \Delta_p)(n - 1)^{\frac{1}{p}} \). We can rewrite the expression for \( \gamma_p(\partial Q'') \) using a trick from \( \square \). Notice, that

\[
e^{-\frac{|y|^p}{p}} = \int_{|y|}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} dt = \int_{0}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} \chi_{[-t,t]}(|y|) dt.
\]

Under this assumptions on \( y \), for any \( t \leq (1 + \Delta_p)(n - 1)^{\frac{1}{p}} \) it holds that \( \chi_{[-t,t]}(|y|) = 0 \) and

\[
e^{-\frac{|y|^p}{p}} = \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{\frac{-t^p}{p}} \chi_{[-t,t]}(|y|) dt.
\]

Thus

\[
\gamma_p(\partial Q'') = \frac{1}{n\nu_nJ_{n-1,p}} \int_{\partial Q''} e^{-\frac{|y|^p}{p}} d\sigma(y)
\]

\[
= \frac{1}{n\nu_nJ_{n-1,p}} \int_{\partial Q''} \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} \chi_{[-t,t]}(|y|) dt d\sigma(y)
\]

\[
= \frac{1}{n\nu_nJ_{n-1,p}} \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} |\partial Q'' \cap tB_2^n| dt
\]

\[
\leq \frac{1}{J_{n-1,p}} \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} dt.
\]

From the previous lemmas it is clear that for any constant \( \delta > 0 \), we get

\[
\int_{(1+\delta)(n-1)^{\frac{1}{p}}}^{\infty} t^{n+p-2} e^{-\frac{t^p}{p}} dt \leq C'(p)e^{-C''(p)n},
\]
for some positive $C'(p)$ and $C''(p)$. Thus
\[
\gamma_p(\partial Q'') \leq \frac{C'(p)e^{-C''(p)n}}{n^{\frac{1}{2}}n^\frac{1}{p}e^{-\frac{n}{p}}},
\]
which is exponentially small as well.

Note, that using same trick from [Ba], one can obtain a rough bound for $\gamma_p$-surface area of a convex body. Namely,
\[
\gamma_p(\partial Q') = \frac{1}{n\nu_n J_{n-1,p}} \int_{\partial Q} e^{-\frac{|y|^p}{2}} dx = \frac{1}{n\nu_n J_{n-1,p}} \int_0^\infty t^{p-1}e^{-\frac{t^p}{2}} |\partial Q \cap tB^n_2| dt \leq \frac{J_{n+p-2,p}}{J_{n-1,p}} \approx n^{1-\frac{1}{p}}, \quad n \to \infty.
\]
This bound is not best possible. The next section is dedicated to the best possible asymptotic upper bound.

### 3. Upper bound

We will use the approach developed by Nazarov in [N]. Let us consider "polar" coordinate system $x = X(y, t)$ in $\mathbb{R}^n$ with $y \in \partial Q$, $t > 0$. Then
\[
\int_{\mathbb{R}^n} \varphi_{p}(y) d\sigma(y) = \int_0^\infty \int_{\partial Q} D(y, t) \varphi_{p}(X(y, t)) d\sigma(y) dt,
\]
where $D(y, t)$ is a Jacobian of $x \to X(y, t)$. Define
\[
(7) \quad \xi(y) = \varphi_{p}^{-1}(y) \int_0^\infty D(y, t) \varphi_{p}(X(y, t)) dt.
\]
Then
\[
1 = \int_{\partial Q} \varphi_{p}(y) \xi(y) dy,
\]
and thus
\[
\int_{\partial Q} \varphi_{p}(y) dy \leq \frac{1}{\min_{y \in \partial Q} \xi(y)}.
\]
Following [N], we shall consider two such systems.

#### 3.1. First coordinate system.
Consider "radial" polar coordinate system $X_1(y, t) = yt$. The Jacobian $D_1(y, t) = t^{n-1}|y|\alpha$, where $\alpha = \alpha(y)$, denotes the absolute value of cosine of an angle between $y$ and $\nu_y$. Here $\nu_y$ stands for a normal vector at $y$. From (7),
\[
(8) \quad \xi_1(y) = e^{\frac{|y|^p}{p}} \alpha |y|^{1-n} J_{n-1} \approx \sqrt{\frac{2\pi}{p}} e^{\frac{|y|^p}{p}} \alpha |y|^{1-n} n^{\frac{1}{2}} \frac{1}{e} F((n-1)\frac{1}{2}), \quad n \to \infty,
\]
where $F(t) = (n - 1) \log t - \frac{n}{p}$. Since $(n - 1)\frac{1}{2}$ is the maxima point for $F(t)$, for all $y \in \mathbb{R}^n$, $F((n - 1)\frac{1}{2}) \geq F(|y|)$. So we can estimate (8) from below by
\[
(9) \quad \xi_1(y) \gtrsim \sqrt{\frac{2\pi}{p}} n^{\frac{1}{2}} \frac{1}{e} \alpha.
\]
3.2. Second coordinate system. Now consider "normal" polar coordinate system $X_2(y, t) = y + tv_y$. Then $D_2(y, t) \geq 1$ for all $y \notin Q$. Thus, by cosine rule, namely, $|x + y|^2 = x^2 + y^2 - 2xy \cos \beta$, where $\beta$ is an angle between vectors $x$ and $y$, we get:

\[
\xi_2(y) \geq e^{-\frac{|y|^p}{p}} \int_0^\infty e^{-\frac{(|y|^2 + x^2 + 2t|y|\alpha)^2}{p}} \frac{e^{\frac{t\nu}{y}}}{y} \, dt.
\]

Note, that for any positive function $f(x)$ defined on the interval $I$,

\[
\int_I e^{-f(t)} \, dt \geq e^{-f(t_0)} |\{t : f(t) < f(t_0)\} \cap I|.
\]

Consider

\[
f(t) = \frac{(|y|^2 + t^2 + 2t|y|\alpha)^{\frac{p}{2}}}{p}.
\]

By intermediate value theorem there is $t_1$ such that

\[
(|y|^2 + t_1^2 + 2t_1|y|\alpha)^\frac{p}{2} = |y|^p + 1.
\]

Since $f(t)$ is increasing, from (11) and (12) we get

\[
\xi_2(y) \geq e^{-\frac{1}{p}} t_1.
\]

Now we need to estimate $t_1$ from below. Using (12) and taking $y \in A_p$, we apply Mean Value Theorem and get

\[
t_1 = \sqrt{\alpha^2|y|^2 - |y|^2 + (|y|^p + 1)^\frac{p}{2} - \alpha|y|} \approx \sqrt{\alpha^2|y|^2 + \frac{2}{p}|y|^{2-p} - \alpha|y|}.
\]

Multiplying the last expression by a conjugate and applying the inequality $\sqrt{a + b} \leq \sqrt{a + \sqrt{b}}$,

we get:

\[
\xi_2(y) \geq e^{-\frac{1}{p}} \sqrt{\frac{2}{p} |y|^{1-\frac{2}{p}}} \frac{1}{1 + \sqrt{2p\alpha}|y|^{\frac{p}{2}}}
\]

Considering (8) and (13) with $|y| \in A_p$, we get

\[
(\xi_2(y) := \xi_1(y) + \xi_2(y) \geq n^\frac{1}{p} \left( \sqrt{\frac{2p}{\alpha}} + \frac{C}{C_1\alpha} \sqrt{n + 1} \right),
\]

where $C_1 = \sqrt{2p(2 - e^{-\frac{1}{p}})^\frac{p}{2}}$; for $0 < p \leq 2 C = \sqrt{\frac{2}{p} e^{\frac{1}{2} - \frac{p}{2}}}$, and for $p \geq 2$

\[
C = \sqrt{\frac{2}{p} e^{-\frac{1}{p}} (2 - e^{-\frac{1}{p}})^{\frac{1}{2}}}
\]

Note that (14) is minimized whenever $\alpha = \sqrt{\frac{2p}{\alpha}} \sqrt{\frac{C}{C_1}} n^{-\frac{1}{2}}$. The minimal value of (14) is

\[
C(p)^{-1} n^\frac{1}{p} \sqrt{n^{-\frac{1}{2}}}, \quad \text{where } C(p) = 2^{\frac{1}{2p}} \sqrt{\frac{C}{C_1}}. \quad \text{This implies, that}
\]

\[
\gamma_p(\partial Q \cap A_p) \leq C(p) n^\frac{3}{4} \frac{1}{p}.
\]

One can note that $C(p)$ tends to infinity while $p$ tends to infinity or to zero. Applying Lemma 4 we finish the proof of the upper bound from the Theorem 1.
Remark 5. It was noticed by Nazarov, that his construction in \( N \) also implies that any polytope \( P_K \) with \( K \) faces has Gaussian surface area bounded by \( C \sqrt{\log K} \). The same way, in the general case \( \gamma_p(\partial P_K) \leq C(p)n^{\frac{1}{2}-\frac{1}{p}}\sqrt{\log K} \). Indeed, let \( H(\rho) \) be a hyperplane distanced at \( \rho \) from the origin. By the Mean Value Theorem, the surface area of \( H(\rho) \) is bounded from above by

\[
\frac{1}{\sqrt{2\pi}} n^{\frac{1}{2}-\frac{1}{p}} e^{-\frac{\rho^2}{2}}. \tag{15}
\]

By (13), and since \( \alpha|y| = \rho \) for \( y \in H(\rho) \), we note that

\[
\gamma_p(\partial P_K) \leq \sum_{\rho \geq \sqrt{2\log K} n^{\frac{1}{p}}} \gamma_p(H(\rho)) + \left( e^{-\frac{\rho^2}{2}p} \frac{|y|^{2-p}}{\sqrt{2\pi} |y|^{1-\frac{2}{p}} + 2\sqrt{2\log K} n^{\frac{1}{p}-\frac{1}{2}}} \right)^{-1}. \tag{16}
\]

The first summand is about a constant times \( n^{\frac{1}{2}-\frac{1}{p}} \) (by (15)). The second summand is bounded by \( C(p)n^{\frac{1}{2}-\frac{1}{p}}\sqrt{\log K} \).

4. Lower bound

Let’s consider \( N \) uniformly distributed random vectors \( x_i \in S^{n-1} \). Let \( \rho = n^{\frac{1}{p}-\frac{1}{4}} \) and \( r = r_w = n^{\frac{1}{p}} + w \), where \( w \in [-W, W] \), and \( W = n^{\frac{1}{p}-\frac{1}{2}} \). Consider random polytope \( Q \) in \( \mathbb{R}^n \), defined as follows:

\[ Q = \{ x \in \mathbb{R}^n : < x, x_i > \leq \rho, \ \forall i = 1, \ldots, N \}. \]

The expectation of \( \gamma_p(\partial Q) \) is

\[
\frac{1}{n^\nu_n J_{n-1}} N \int_{\mathbb{R}^{n-1}} \exp(- \frac{(|y|^2 + \rho^2)^{\frac{p}{2}}}{p})(1 - p(|y|))^{N-1} dy, \tag{17}
\]

where \( p(t) \) is the probability that the fixed point on the sphere of radius \( \sqrt{t^2 + \rho^2} \) is separated from the origin by hyperplane \( < x, x_i > = \rho \).

Passing to polar coordinates, we shall estimate (17) from below by

\[
\frac{\nu_{n-1}}{\nu_n J_{n-1,p}} N \int_W f(n^{\frac{1}{p}} + w)(1 - p(r_w))^{N-1} dy, \tag{18}
\]

where \( f(t) = t^{n-2} e^{-\frac{(t^2 + \rho^2)^{\frac{p}{2}}}{p}} \). Note, that \( \frac{\nu_{n-1}}{\nu_n} \approx \frac{\sqrt{\pi}}{\sqrt{2\pi}} \). Thus we estimate (18) from below by

\[
\frac{1}{\sqrt{2\pi}} n^{n-\frac{2}{p}} e^{\frac{n}{p}} f(n^{\frac{1}{p}} + W) N \int_W (1 - p(r_w))^{N-1} dy. \tag{19}
\]

Next,

\[
f(n^{\frac{1}{p}} + W) \geq n^{\frac{n-2}{p}} (1 + n^{-\frac{1}{2}}) n^{-2} e^{-\frac{n}{p}} e^{-\frac{3}{2} \sqrt{\pi}} \approx n^{\frac{1}{p}} e^{-\frac{n}{p}} n^{-\frac{2}{p}} e^{-\frac{\sqrt{\pi}}{2}}. \]

Thus (19) is greater than

\[
\frac{1}{\sqrt{2\pi}} n^{n-\frac{2}{p}} e^{-\frac{\sqrt{\pi}}{2}} N \int_W (1 - p(r_w))^{N-1} dy. \tag{20}
\]
Next, we estimate the probability $p(r)$. The same way, as in [N], by Fubini Theorem,

\[ p(r) = \left( \int_{-\sqrt{r^2+\rho^2}}^{\sqrt{r^2+\rho^2}} \left( 1 - \frac{t^2}{r^2+\rho^2} \right)^{\frac{n-3}{2}} dt \right)^{-1} \int_{\rho}^{\sqrt{r^2+\rho^2}} \left( 1 - \frac{t^2}{r^2+\rho^2} \right)^{\frac{n-3}{2}} dt. \]

Directly by Laplace method (or due to the fact that it represents the sphere surface area) the first integral is approximately equal to $\sqrt{2\pi n} \frac{1}{r}\, e^{-\frac{1}{2}}$.

Using an elementary inequality that $1 - a \leq e^{-\frac{a^2}{2}} e^{-a}$, for all $a > 0$, one can estimate the second integral in (21) by

\[
\int_{\rho}^{\infty} \exp\left(-\frac{n-3}{4(r^2+\rho^2)^2} t^4\right) \cdot \exp\left(-\frac{n-3}{r^2+\rho^2} t^2/2 \right) dt \\
\leq \exp\left(-\frac{n-3}{4(r^2+\rho^2)^2} \rho^2\right) \int_{\rho}^{\infty} \exp\left(-\frac{n-3}{r^2+\rho^2} t^2/2 \right) dt.
\]

The first multiple is of order $e^{-\frac{1}{4}}$ under these assumptions on $r$ and $\rho$. The second integral can be estimated with usage of inequality

\[
\int_{\rho}^{\infty} e^{-a^2} \leq \frac{1}{a} e^{-\frac{a^2}{2}}.
\]

We note that $a\rho^2$ is of order $\frac{n-3}{\rho^2 r^2} \sim n^{\frac{1}{2}}(1 - 3n^{-\frac{1}{4}})$ up to an additive error $\sim n^{-\frac{1}{2}}$. Hence one can write that

\[ p(r) \leq \frac{e^{\frac{1}{4}}}{\sqrt{2\pi}} n^{-\frac{1}{4}} e^{-\frac{\sqrt{n}}{2}}. \]

Now, one can choose $N = \sqrt{2\pi} n^{\frac{1}{4}} e^{\sqrt{n}/2}$. From (20) and (22) it now follows that the expectation of a $\gamma_p$-surface area is greater than

\[ e^{-\frac{1}{4}} n^{\frac{3}{4}}. \]

which finishes the proof of the Theorem 1. □

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