An AGM Approach to Revising Preferences

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Abstract

We look at preference change arising out of an interaction between two elements: the first is an initial preference ranking encoding a pre-existing attitude; the second element is new preference information signaling input from an authoritative source, which may come into conflict with the initial preference. The aim is to adjust the initial preference and bring it in line with the new preference, without having to give up more information than necessary. We model this process using the formal machinery of belief change, along the lines of the well-known AGM approach. We propose a set of fundamental rationality postulates, and derive the main results of the paper: a set of representation theorems showing that preference change according to these postulates can be rationalized as a choice function guided by a ranking on the comparisons in the initial preference order. We conclude by presenting operators satisfying our proposed postulates. Our approach thus allows us to situate preference revision within the larger family of belief change operators.

1 Introduction

Preferences play a central role in theories of decision making, as part of the mechanism underlying intentional behavior and rational choice, both in economic models of rational agency as well as in formal models of artificial agents supposed to interact with the world and each other (Boutilier et al. 2004; Domshlak et al. 2011; Rossi, Venable, and Walsh 2011; Pigozzi, Tsoukiás, and Viappiani 2016). Since such interactions take place in dynamic environments, it can be expected that preferences change in response to new developments.

In this paper we are interested in preference change occurring when new preference information, denoted by $o$, becomes available and has to be taken at face value, thereby prompting a change in prior preference information, denoted by $\pi$. The change, we require, should preserve as much useful information from $\pi$ as can be afforded.

Preference change thus described is a pervasive phenomenon, arising in many contexts spanning the realms of both human and artificial agency. Thus, there is a distinguished tradition in Economics and Philosophy that looks at examples of conflict between an agent’s subjective preference (what we call here the initial, or prior preference $\pi$) and a second-order preference, often standing for a commitment or moral rule (what we call here the new preference information $o$): subjective versus ‘ethical’ preferences (Harsanyi 1955), lack of will, or akrasia (Jeffrey 1974), moral commitments (Sen 1977), second-order volitions (Frankfurt 1988) and second-order preferences (Nozick 1994) all fall under this heading.

The same challenge can occur in technological applications, from updating CP-nets (Cadilhac et al. 2015) to changing the order in which search results are displayed (Russell 2019): an artificial agent dealing with humans will have to learn their preferences, but as it cannot do so instantaneously, it must presumably acquire the relevant information in intermediate steps, revising along the way.

Thus, whether it is the internal conflict between an agent’s private leanings and the better angels of its nature, or a content provider wanting to tailor its products for a better user experience, many cases of preference change involve a conflict between two types of preferences, one of which is perceived as having priority over the other. However, even though the need to reconcile conflicting preferences in favor of one of them is widely acknowledged, a concrete mechanism for resolving preference conflicts, that works for general preference orders, is often overlooked.

In keeping with prominent approaches to belief change, which model rational change using a plausibility relation over the states of affairs undergoing revision, and echoing a suggestion of Amartya Sen to the effect that conflicts among preferences can be understood using rankings over the preferences themselves (Sen 1977), we propose formalizing preference change using preferences over the basic elements of a preference order, as illustrated in the following example.

Example 1. The initial preference $\pi$ is such that, as a result of explicit assertion, item 1 is ranked better than 2 and 2 is ranked better than 3; by virtue of transitivity, it is also inferred that 1 is considered better than 3. We want to revise $\pi$ by a preference $o$, according to which 3 is better than 1 (see Figure 1). The simplest solution is to add $o$ to $\pi$ (i.e., include the comparisons contained in both), but the transitivity requirement leads to a cycle between 1, 2 and 3, which we
Figure 1: Revising preference order \( \pi \) by \( o \): simply adding \( o \) to \( \pi \) leads to a cycle, so if \( o \) is accepted then a choice needs to be made regarding which of the initial comparisons of \( \pi \) to keep; potential candidates for the revised order are \( \pi_1, \pi_2 \) or \( \pi_3 \). A direct comparison ranking \( i \) better than \( j \) is depicted by a solid arrow from \( i \) to \( j \), with comparisons inferred by transitivity depicted by dotted arrows.

would like to avoid. We are thus in a situation where \( \pi \) and \( o \) cannot be jointly accepted, but since \( o \), we stipulate, must be accepted, something must be given up from \( \pi \) (though, we ask, no more than strictly necessary). How is the decision to be made? We suggest that an implicit preference over the comparisons of \( \pi \) that were explicitly provided can provide an answer: if the comparison of 1-vs-2 (the edge from 1 to 2 in Figure 1) is preferred to the one of 2-vs-3 then the result is \( \pi_1 \), which holds on to 1-vs-2 from \( \pi \) and together with \( o \) infers, by transitivity, that 3 is better than 2; alternatively, a preference for 2-vs-3 over 1-vs-2 leads to \( \pi_2 \) as the result, while indifference between the two comparisons means that both are given up, resulting in \( \pi_3 \). Thus, preference over comparisons in \( \pi \) translates as choice over how to go about revising \( \pi \). Interestingly, we may also reason in the opposite direction: observing choice behavior across different instances of revision allows us to infer preferences over comparisons in \( \pi \), e.g., revising to \( \pi_1 \), rather than to \( \pi_2 \) or \( \pi_3 \), can be rationalized as saying that the comparison of 1-vs-2 is considered better than 2-vs-3.

Our purpose here is to formalize the type of reasoning illustrated in Example 1 by rationalizing preference change as a type of choice function on what we will call the direct comparisons of \( \pi \), i.e., the explicit preferences assumed to be given in \( \pi \). Since a conflict between \( \pi \) and \( o \) forces some of the direct comparisons of \( \pi \) to be renounced, additional information in the form of a preference order over the direct comparisons of \( \pi \) will serve as guide to the choice function. The aim, in this, is not legislate on what is the right choice to make; rather, it is to make sure that whatever the choice is, it is made in a coherent way.

Contributions. We present a mechanism for revising a preference order \( \pi \) that is based on an underlying preference relation over the basic, atomic comparisons of \( \pi \). This mechanism proceeds sequentially, by working its way through the underlying preference relation and adding as many of the direct comparisons of \( \pi \) as possible, while avoiding a conflict with \( o \). We present a set of conditions under which the preference order on direct comparisons of \( \pi \) exists and has desired properties, and characterize the revision mechanism using a set of intuitive normative principles, i.e., rationality postulates in the AGM mould (Alchourrón, Gärdenfors, and Makinson 1985). The significance of our approach lies in laying bare the theoretical requirements and basic assumptions for mechanisms intended to revise preferences.

Related work. Our work complements existing research, but manages to occupy a distinct niche in a broader landscape. Some previous work labeled as preference revision (Bradley 2007; Lang and van der Torre 2008; Liu 2011), looks at changes in preferences prompted by a change in beliefs. Here we abstract away from the source of the new information, choosing to focus exclusively on a mechanism that can be used for resolving conflicts: the rational thing to do when knowing that, for some reason or other, one’s preference has to change. Other work (Cadilhac et al. 2015) describes preference change when preferences are represented using CP-nets (Boutilier et al. 2004), or dynamic epistemic logic (Benthem and Liu 2014), in the context of declarative debugging (Dell’Acqua and Pereira 2005), or databases (Chomicki 2003), and therefore comes with additional structural constraints. In contrast, we have opted to represent preferences as strict partial orders over a set of items: we believe this straightforward formulation allows the basic issue signaled by Amartya Sen (Sen 1977), to be visible and tackled head on.

Apart from the issues raised in the Economics literature about second-order desires (Harsanyi 1955; Jeffrey 1974; Sen 1977; Frankfurt 1988; Nozick 1994), the basic phenomenon of preference change has also been raised in explicit connection to belief change (Hansson 1995; Grüne-Yanoff and Hansson 2009b; Grüne-Yanoff 2013), but a representation in terms of preferences on the comparisons present in the preference orders, along the lines suggested here, has, to the best of our knowledge, not yet been given. Much existing work proceeds by putting forward some concrete preference revision mechanism, possibly by shifting some elements of the original preference around, and occasionally with a remark on the similarity between this operation and a belief revision operation (Freund 2004; Chomicki and Song 2005; Liu 2011; Ma, Benferhat, and Liu 2012). What our work adds to these models is an analysis in terms of postulates and representation results.

The postulates we put forward for preference revision bear a distinct resemblance to the AGM postulates employed for belief revision (Alchourrón, Gärdenfors, and Makinson 1985; Katsuno and Mendelzon 1992; Fermé and Hansson 2018): given that changing one’s mind involves choosing some parts of a belief to keep and some to remove, this is no coincidence. Indeed, the two problems are similar, though the structural particularities of preferences (in particular, the requirement that they are transitive) mean that transfer of insights from belief revision to preference revision is by no means straightforward.
2 Preliminaries

We assume a finite set $V$ of items, standing for the objects an agent can have preferences over. If $\pi$ is a binary relation on a set $V$ of items, then $\pi$ is a strict partial order (spo) on $V$ if $\pi$ is transitive and irreflexive, and we write $\mathcal{O}_V$ for the set of strict partial orders on $V$. If $\pi$ is an spo on a set $V$ of items, then $\pi$ is a strict linear order on $V$ if $\pi$ is also total, in addition to being transitive and irreflexive. A chain on $V$ is a strict linear order on a subset of $V$. We write $\mathcal{C}_V$ for the set of chains on $V$. Finally, $\pi$ is a total preorder on $V$ if $\pi$ is transitive and total, with $\mathcal{T}_V$ being the set of total preorders on $V$. Note that in the following we will typically be interested in total preorders on $V \times V$, i.e., total preorders on the set of comparisons of items in $V$.

If $\pi$ is an spo on a set of items $V$, then a comparison $(i, j)$ of $\pi$ is an element $(i, j) \in \pi$, for some items $i, j \in V$, interpreted as saying that, in the context of $\pi$, $i$ is considered strictly better than $j$. To simplify notation, we sometimes also refer to comparisons with the letter $c$. We often have to consider the union $\pi_1 \cup \pi_2$ of two spo, which is not guaranteed to be an spo, since transitivity is not preserved under unions. If this is the case, we typically have to substitute $\pi_1 \cup \pi_2$ for its transitive closure, denoted by $(\pi_1 \cup \pi_2)^\pi$. Since preferences are required to be transitive, we write a sequence of comparisons $\{(1, 2), (2, 3) \ldots, (m-1, m)\}$ as $(1, \ldots, m)$.

If $\pi = (1, \ldots, m)$ is a chain on $V$, a direct comparison of $\pi$ is a comparison $(i_k, i_{k+1}) \in \pi$, i.e., a comparison between $i_k$ and its direct successor in $\pi$, with $\delta_\pi$ being the set of direct comparisons of $\pi$. The assumption is that direct comparisons are the result of explicit information, and are basic in the sense that they cannot be inferred by transitivity using other comparisons in $\pi$. Given preference orders $\pi \in \mathcal{C}_V$ and $o \in \mathcal{O}_V$, we want to carve out the possible options for the revision of $\pi$ by $o$. For this we use the set $[o]_\pi$ of $\pi$-completions of $o$, defined as:

$$[o]_\pi = \{(o \cup \delta)^\pi \in \mathcal{O}_V \mid \delta \subseteq \delta_\pi\}.$$  

The intuition is that a $\pi$-completion of $o$ is a preference order constructed from $o$ using some, and only, direct comparisons in $\pi$, i.e., information originating exclusively from the two sources given as input. We will expect that a preference revision operator selects one element of this set as the revision result.

Though taking $(\pi \cup o)^\pi$ as the result of revising $\pi$ by $o$ is not, in general, feasible, we still want to identify parts of $(\pi \cup o)^\pi$ that are uncontroversial. To that end, the cycle-free part $\alpha^\pi_o$ of $(\pi \cup o)^\pi$ is defined as:

$$\alpha^\pi_o = \{(i, i+1) \in (\pi \cup o)^\pi \mid (i+1, i) \notin (\pi \cup o)^\pi\},$$

i.e., the set of comparisons of $(\pi \cup o)^\pi$ not involved in a cycle with the comparisons of $o$. The cyclic part $\kappa^\pi_o$ of $\pi$ with respect to $o$ is defined as:

$$\kappa^\pi_o = \{(i, i+1) \in \delta_\pi \mid (i+1, i) \in (\pi \cup o)^\pi\},$$

i.e., the set of direct comparisons of $\pi$ involved in a cycle with $o$.

Example 2. For $\pi$ and $o$ as in Example 1, we have that $\delta_\pi = \{(1, 2), (2, 3)\}$, while the $\pi$-completions of $o$ are $[o]_\pi = \{(3, 1, 2), (2, 3, 1), (3, 1)\}$, i.e., the spo $\pi$-completions of $\pi$ obtained by adding to $o$ either of the elements of $\delta_\pi$, or none (corresponding to $\pi_1, \pi_2$ and $\pi_3$). The cyclic part of $\pi$ with respect to $o$ is $\kappa^\pi_o = \{(1, 2), (2, 3)\}$ and the cycle-free part of $\pi$ with respect to $o$ is $\alpha^\pi_o = \emptyset$.

3 A General Method for Revising Preferences

A preference revision operator $\triangleright$ is a function $\triangleright : \mathcal{C}_V \times \mathcal{O}_V \rightarrow \mathcal{O}_V$ taking a chain $\pi$ and an spo $o$ as input, and returning an spo $\pi \triangleright o$ as output.

The choice of input and output can be motivated by imagining that $\pi$ stands for an existing priority ranking, e.g., the ordering of items on a webpage, whereas the new information $o$ is provided by a user and is more likely to be incomplete.

In addition, we may look at this in light of the material that is to come: since we will be rationalizing preference revision operators using preferences (i.e., preorders) on comparisons, an spo as output reflects the fact that certain comparisons are considered equally good, and must be given up together. The unfortunate effect of this is that the input and output formats do not match, which makes it unclear, at this point, whether we can iterate the revision operation. That being said, the output can (and will) be tightened to a chain: provided that the preferences guiding revision are a linear order (i.e., there are no ties). We touch on this aspect at the end of Section 6.

We start, then, by presenting a general procedure for revising preferences that, as advertised, utilizes total preorders on the set $\delta_\pi$ of direct comparisons of $\pi$: thus, a preference assignment $a$ is a function $a : \mathcal{C}_V \rightarrow \mathcal{T}_V$ mapping every preference $\pi \in \mathcal{C}_V$ to a total preorder $\preceq_\pi$ on elements of $V \times V$, i.e., on pairwise comparisons on the items of $V$, of which we are interested only in the preorder on $\delta_\pi$. In typical AGM manner, a comparison $c_i \preceq_\pi c_j$ in the context of a preorder $\preceq_\pi$ on $\delta_\pi$ means that $c_i$ is better than $c_j$.

If $\pi \in \mathcal{C}_V$, $o \in \mathcal{O}_V$ and $\preceq_\pi$ is a total preorder on $\delta_\pi$, then, for $i \geq 1$, the $\preceq_\pi$-level $i$ of $\delta_\pi$, denoted $\text{lvl}^i_{\preceq_\pi}(\delta_\pi)$, contains the $i$th best elements of $\delta_\pi$ according to $\preceq_\pi$, i.e., $\text{lvl}^i_{\preceq_\pi}(\delta_\pi) = \min_{\preceq_\pi}(\delta_\pi \setminus \text{lvl}^{i-1}_{\preceq_\pi}(\delta_\pi)) = \min_{\preceq_\pi}(\delta_\pi \setminus \text{lvl}^i_{\preceq_\pi}(\delta_\pi))$. Note that the $\preceq_\pi$-levels of $\delta_\pi$ partition $\delta_\pi$ and, since $\delta_\pi$ is finite, there exists a $j > 0$ such that $\text{lvl}^{j}_{\preceq_\pi}(\delta_\pi) = \emptyset$, for all...
Intuitively, the addition operator starts by adding to $\pi \cup \{o\}$ comparisons from $o$ (and is added at the beginning anyway).

$i \geq j$. The addition operator $\text{add}_{\preceq \pi}^i(o)$ is defined, for any $o \in O_V$ and $i \geq 0$, as follows:

$$\text{add}_{\preceq \pi}^0(o) = (\pi \cup \{o\})^+,$$

$$\text{add}_{\preceq \pi}^i(o) = \left\{ \begin{array}{ll}
\text{add}_{\preceq \pi}^{i-1}(o) \cup (\{\delta \mid \delta \preceq \pi) \cap \alpha^o_{\pi}\}^+) & \text{if in } O_V, \\
\text{add}_{\preceq \pi}^{i-1}(o) & \text{otherwise}.
\end{array} \right.$$ 

Intuitively, the addition operator starts by adding to $\pi$ all the direct comparisons of $\pi$ that are not involved in a cycle with it, i.e., which are not under contention by the accrual of new preference information. Then, at every further step $i > 0$, the addition operator tries to add all comparisons on level $i$ of $\pi$, i.e., those that are involved in a cycle with $o$, if the resulting set of comparisons can be construed as a spo (by taking its transitive closure) the operation is successful, and the new comparisons are added; if not, the addition operator does nothing. Since the addition of new comparison follows the order $\preceq \pi$, this ensures that better quality comparisons are considered before lower quality ones.

Note that this procedure guarantees that there are always some comparisons in $\pi \triangleright o$, i.e., we have that $o \subseteq \pi \triangleright o$, regardless of anything else. Note, also, that the number of non-empty levels in $\delta_\pi$ is finite and the addition operation eventually reaches a fixed point, i.e., there exists $j \geq 0$ such that $\text{add}_{\preceq \pi}^j(o) = \text{add}_{\preceq \pi}^i(o)$, for any $i \geq j$. We denote by $\text{add}_{\preceq \pi}^j(o)$ the fixed point of this operator and take it as the defining expression of a preference revision operator: if $\alpha$ is a preference assignment, then the $a$-induced preference revision operator $\triangleright a$ is defined, for any $\pi \in C_V$ and $o \in O_V$, as:

$$\pi \triangleright a o = \text{add}_{\preceq \pi}^*(o).$$

Note that, by design, $\text{add}_{\preceq \pi}^* \in O_V$, i.e., the operator $\triangleright a$ is well defined.

**Example 3.** Consider initial preference order $\pi = (1, 2, 3, 4)$ and new information $o = (3, 1)$. We obtain that the direct comparisons of $\pi$ are $\delta_\pi = \{(1, 2), (2, 3), (3, 4)\}$. Suppose, now, that there is a total preorder $\preceq_\pi$ on $\delta_\pi$ according to which $(1, 2) \preceq_\pi (2, 3) \preceq_\pi (3, 4)$, as depicted in Figure 2. To construct $\pi \triangleright o$, the addition operator starts from $\text{add}_{\preceq \pi}^0(o) = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$, i.e., $o$ itself together with $\alpha^o_\pi$, the cycle-free part of $\pi$ with respect to $o$. At the next step the addition operator tries to add $(1, 2)$, which it can do successfully; at the next step it attempts to add $(2, 3)$, which creates a conflict with $(3, 1)$ and $(1, 2)$, added previously. After this there are no more comparisons to add.

4 Postulates for Preference Revision

We show now that the procedure described in Section 3 can be characterized with a set of AGM-like postulates that do not reference any concrete revision procedure and are, by themselves, intuitive enough to provide reasonable constraints on any preference revision operator.

The first two postulates we consider apply to any chain $\pi \in C_V$, spo $o \in O_V$ and preference revision operator $\triangleright : C_V \times O_V \rightarrow O_V$, and are as follows:

$$\newline (P_1) \ \pi \triangleright o \in [o]_\pi, \newline (P_2) \ \alpha^o_\pi \subseteq \pi \triangleright o.$$

Postulates $P_1 - 2$ require the result to be obtained by adding elements from $\pi$ to the new information $o$, and to be of a certain admissible type, i.e., a spo. They are meant to capture preference revision in its most uncontroversial aspects, yet they still require some careful unpacking.

Postulate $P_1$ states that $\pi \triangleright o$ is a $\pi$-completion of o, i.e., a preference order constructed only by adding direct comparisons from $\pi$ to $o$. Unfolding its consequences, postulate $P_1$ ensures that:

(i) $\pi \triangleright o \in O_V$, i.e., $\pi \triangleright o$ is a chain on $V$,
(ii) $o \subseteq \pi \triangleright o$, i.e., $\pi \triangleright o$ contains all the information present in $o$, and
(iii) $\pi \triangleright o \subseteq (\pi \cup o)^+$, i.e., $\pi \triangleright o$ is contained in the binary relation obtained by simply adding $o$ to $\pi$, and adding all the comparisons inferred by transitivity.

In terms of AGM propositional belief revision, postulate $P_1$ does the same duty as the Closure, Success, Inclusion and Consistency postulates (Hansson 2017; Fermé and Hansson 2018). These postulate mandate that the revision result should be a propositional theory (i.e., have a required format), that the new information should be accepted, and that, unless the new information is inconsistent, the revision result should be consistent.

Given this observation, a question emerges as to why not use conditions (i)-(iii) as postulates instead of the proposed $P_1$. The reason is that $P_1$ contains an element that lacks from conditions (i)-(iii): what $P_1$ adds is the requirement that $\pi \triangleright o$ is to be constructed using only direct comparisons of $\pi$ (in addition to $o$), and the reason why such a condition is desirable is to prevent $\pi \triangleright o$ from having opinions on items over which no opinion had been expressed before revision. The issue is illustrated in Example 4.

**Example 4.** Consider preferences $\pi$ and $o$ as in Example 1, and an additional spo $\pi_4 = \{(3, 1), (3, 2)\}$. Note that $\pi_4$ is such that $o \subseteq \pi_4 \subseteq (\pi \cup o)^+$ and therefore satisfies conditions (i)-(iii) expressed above, so that according to conditions (i)-(iii) preference $\pi_4$ is a viable revision result.

At the same time, we do not want to consider $\pi_4$ as a potential candidate for the revision result: the comparison $(3, 2)$ occurs neither in $\pi$ nor in $o$ as a direct comparison,
and there is reason to think that adding it would be unjustified: a rational preference revision operator should not be allowed to return $\pi_3$ when revising $\pi$ by $o$. By contrast, when the comparison $(3, 2)$ does occur, e.g., in the desirable preference order $\pi_1 = (3, 1, 2)$, it occurs as the result of inference from $(3, 1)$, which is added from $o$, and $(1, 2)$, which is preserved from $\pi$.

Postulate $P_2$ says that the cycle-free part of $\pi$ with respect to $o$ is to be preserved in $\pi \bowtie o$, and is meant to preserve the parts of $(\pi \cup o)^+$ that are not up for dispute. Note that in the case when $(\pi \cup o)^+$ does not contain a cycle then $\alpha^+_o = (\pi \cup o)^+$, and $P_2$ together with $P_1$ imply that $\pi \bowtie o = (\pi \cup o)^+$: this is the case when revision is easy, and nothing special needs to be done. Throughout all this, postulate $P_2$ serves the same function as the Vacuity postulate in propositional revision (Hansson 2017; Farme and Hansson 2018): in the ideal case, when $o$ can simply be added to $\pi$, applying postulate $P_2$ results in the union of the two structures.

So far we have established that, if there is no conflict between $\pi$ and $o$, i.e., no cycle arises by adding $o$ to $\pi$, then we can simply add $o$ to $\pi$; and if there is a conflict, then $\bowtie$ must choose between the direct comparisons of $\pi$ involved in the cycle. This choice, however, must be coherent in a precise sense: we expect the choices to be indicative of an underlying preference over direct comparisons, which remains stable across different instances of revision. This sense of coherence is illustrated by Example 5.

**Example 5.** Consider revising $\pi = (1, 2, 3, 4)$, depicted in Figure 2, by $o_1 = (4, 1)$. Since adding $(\pi \cup o)^+$ contains a cycle, revision requires a choice between comparisons $(1, 2)$, $(2, 3)$ and $(3, 4)$: assume $(1, 2)$ is chosen, suggesting $(1, 2)$ is better than $(2, 3)$ and $(3, 4)$. Suppose, now, that we add $o_2 = \{(3, 4)\}$ and revise by $(o_1 \cup o_2)^+=\{(3, 4), (4, 1), (3, 1)\}$: another cycle is formed, and a choice is necessary, this time only between $(1, 2)$ and $(2, 3)$. In accordance with the previous decision, $(1, 2)$ should be chosen here as well.

The choice behavior of a revision operator has to reflect an implicit preference order over the direct comparisons of $\pi$, and this is handled by the following postulates, meant to apply to any chain $\pi \in C_V$, spos $o_1, o_2 \in O_V$ such that $(o_1 \cup o_2)^+ \in O_V$, and a preference revision operator $\bowtie$:

\[
(P_3) \quad \pi \bowtie (o_1 \cup o_2)^+ \subseteq ((\pi \bowtie o_1) \cup o_2)^+.
\]

\[
(P_4) \quad \text{If } ((\pi \bowtie o_1) \cup o_2)^+ \subseteq O_V, \text{ then } ((\pi \bowtie o_1) \cup o_2)^+ \subseteq \pi \bowtie (o_1 \cup o_2)^+.
\]

There is a similarity between postulates $P_3$ and $P_4$ and the Superexpansion and Subexpansion postulates, respectively, from propositional belief revision (Hansson 2017; Farme and Hansson 2018), which ensure that the choice between two options is stable and independent of alternatives not directly involved. Postulates $P_3$–$P_4$ are meant to ensure the same here. However, it turns out that in the context of preference revision this happens only under a specific set of conditions, which we elaborate on in the following section.

\[
\text{Figure 3: Postulates } P_{3-4} \text{ are satisfied only if } o_1 \text{ and } o_2 \text{ are coordinated with respect to } \pi.
\]

## 5 Coordination

In this section we identify the precise conditions under which it makes sense to apply postulates $P_{3-4}$, presented in Section 4. Before doing so, we introduce some additional notation.

If $o_1$ and $o_2$ are spos, we say that $o_1$ and $o_2$ are coordinated with respect to $\pi$ if for any set $\delta \subseteq \kappa^+_\pi$ such that for every direct comparison $(i, i+1) \in \delta$, neither $(i, i+1)$ nor $(i+1, i)$ is in $(o_1 \cup o_2)^+$, it holds that if $(o_1 \cup \delta)^+ \in O_V$, then $((o_1 \cup o_2)^+ \cup \delta)^+ \in O_V$. In other words, if $\pi$ and $o_1$ form a cycle and we want to add $o_2$ as well, then we direct our attention to the direct comparisons in $\pi$ that are not directly ruled out by $(o_1 \cup o_2)^+$, i.e., such that neither of these comparisons nor their inverses are contained in $(o_1 \cup o_2)^+$. The property of coordination says that if we can consistently add some of these comparisons to $o_1$, then it must be the case that we can also add them to $(o_1 \cup o_2)^+$. Intuitively, coordination means that adding extra information $o_2$ does not step on $o_1$’s toes, by rendering unviable any comparisons that were previously viable. The following example makes this clearer.

**Example 6.** Take $\pi = (1, 2, 3, 4)$ and $o_1 = (4, 1), o_2 = (3, 1)$. The direct comparisons of $\pi$ that are involved in a cycle with $o_1$ are $\kappa^+_{o_1} = \{(1, 2), (2, 3), (3, 4)\}$, so that revision by $o_1$ requires making a choice between these comparisons. This choice, we expect, is done on the basis of some implicit preference over the comparisons, which guides revision even when we add additional information in the form of $o_2$. Notice, now, that neither of $(1, 2), (2, 3)$ and $(3, 4)$ is individually ruled out by $(o_1 \cup o_2)^+$: we have, for instance, that $(1, 2) \notin (o_1 \cup o_2)^+$ and $(2, 1) \notin (o_1 \cup o_2)^+$; the same holds for $(2, 3)$ and $(3, 4)$. The significance of this is that adding $o_2$ to $o_1$ does not alter the menu: the choice is still one over comparisons $(1, 2), (2, 3)$ and $(3, 4)$.

The problem, however, is that whereas with $o_1$ the choice is relatively unconstrained, meaning we can choose any proper subset of $\{(1, 2), (2, 3), (3, 4)\}$ to add to $(4, 1)$, adding the additional comparison $(3, 1)$ complicates things. To see how, consider the set of comparisons $\delta = \{(1, 2), (2, 3)\}$. These comparisons can be consistently added to $o_1$, i.e., $(o_1 \cup \delta)^+ \in O_V$, but not to $(o_1 \cup o_2)^+$, i.e., $((o_1 \cup o_2)^+ \cup \delta)^+ \notin O_V$. According to our definition, this implies that $o_1$ and $o_2$ are not coordinated with respect to $\pi$. Thus, whereas with $o_1$ can be augmented with both $(1, 2)$ and $(2, 3)$, $o_1$ and $o_2$ do not allow adding both comparisons.
together. This, then, has a knock-down effect in that it makes it possible to add comparison (3, 4), irrespective of where it is in the preorder on comparisons.

In such a situation, then, the specific details of how the choice problem is constructed makes the position of (3, 4) in the overall preference order over comparisons irrelevant. Consequently, expecting our axioms to take the preference order into account will land us into trouble. To see this, consider preorder \( \leq_\pi \) in Figure 3, where (3, 4) is the least preferred comparison, and the revision operator \( \triangleright \) induced by it. We have that (3, 4) \( \in (3, 4) \in \pi \triangleright (o_1 \cup o_2)^+ \), but (3, 4) \( \not\in ((\pi \triangleright o_1) \cup o_2)^+ \), i.e., postulate \( P_3 \) is not satisfied.

This fact is related with the lack of coordination between \( o_1 \) and \( o_2 \), as the addition of \( o_2 \) tampers with the choice problem: though we can still add either one of the three comparisons, as mentioned above, we cannot add (1, 2) and (2, 3) together anymore, which in turn means that (3, 4) can be added regardless of its position in \( \leq_\pi \) the preorder.

Example 6 is a case in which lack of coordination creates a situation where postulate \( P_3 \) is not satisfied. We do not mean to imply, however, that there is anything wrong with postulate \( P_3 \), or with uncoordinated preference information. Rather, we take the moral to be that we need postulates tailored to cases that do not look like the one in Example 6, in which preference information over the direct comparisons is rendered unusable by the overriding structural constraints of working with preference orders.

In other words, we want the behavior of a revision operator to reflect the preference information over the direct comparisons: however, the requirement of transitivity means that, in the interest of consistency, we sometimes have to add comparisons that were not explicitly chosen, and this can interfere with the preference information over the comparisons of \( \pi \). Thus, the significance of coordination, as the following theorem shows, is that it is needed in order for postulates \( P_{3-4} \) to be effective at ensuring that choice across different types of incoming preferences is coherent.

**Theorem 1.** If \( \alpha : C \rightarrow \mathcal{T}_V \) is a preference assignment and \( \triangleright^\alpha \) is the \( \alpha \)-induced revision operator, then \( \triangleright^\alpha \) satisfies postulates \( P_{3-4} \) if and only if, for any chain \( \pi \in C \) and sops \( o_1, o_2 \in \mathcal{O}_V \), it holds that \( o_1 \) and \( o_2 \) are coordinated with respect to \( \pi \).

**Proof.** (\( \Leftarrow \)) Take \( o_1, o_2 \in \mathcal{O}_V \) that are coordinated with respect to \( \pi \). We will show that, for any preorder \( \leq_\pi \) on \( \delta_\pi \), the \( \alpha \)-induced revision operator \( \triangleright^\alpha \) satisfies postulates \( P_{3-4} \). Since \( \triangleright^\alpha \) satisfies postulates \( P_{3-4} \) trivially if \( (\pi \cup o_1)^+ \in \mathcal{O}_V \), we look at the case when \( \delta_\pi \neq \emptyset \), i.e., when \( (\pi \cup o_1)^+ \) contains a cycle.

For postulate \( P_3 \), assume there is a comparison \( c^* \in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2) \) such that \( c^* \not\in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2) \). If \( c^* \in (o_1 \cup o_2)^+ \), then a contradiction follows immediately. We thus have to look at the case when \( c^* \not\in (o_1 \cup o_2)^+ \), which contains two subcases of its own.

**Case 1.** If \( c^* \in \delta_\pi \), then by our assumption we have that \( c^* \in \kappa_\pi \), i.e., \( c^* \) is involved in some cycle with \( o_1 \). From \( c^* \not\in \text{add}_{\leq_\pi}^+ (o_1) \) we infer that there must be a set \( \delta \subseteq \delta_\pi \) of direct comparisons of \( \pi \) that precede \( c^* \) in \( \leq_\pi \), are added to \( o_1 \) before it, and prevent \( c^* \) itself from being added. In particular, this means that \( (o_1 \cup \delta)^+ \in \mathcal{O}_V \), but \( ((o_1 \cup \delta)^+ \cup \{c^*\})^+ \not\in \mathcal{O}_V \). At the same time, we know that \( c^* \in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \), i.e., \( c^* \) can be consistently added to \( (o_1 \cup o_2)^+ \). Note that this happens after all the comparisons in \( \delta \), which precede it in \( \leq_\pi \), have been considered as well. This implies that not all of the comparisons in \( \delta \) can be added to \( (o_1 \cup o_2)^+ \), since if they could, then the cycle formed with \( o_1 \), \( \delta \) and \( c^* \) would be reproduced here as well. If not all of the comparisons in \( \delta \) can be added to \( (o_1 \cup o_2)^+ \), this must be because \( ((o_1 \cup o_2)^+ \cup \delta)^+ \not\in \mathcal{O}_V \). This now contradicts the fact that \( o_1 \) and \( o_2 \) are coordinated with respect to \( \pi \).

**Case 2.** If \( c^* \) is not a direct comparison of \( \pi \), then it is inferred by transitivity using at least one direct comparison of \( \pi \) added previously. We apply the reasoning in Case 1 to these direct comparisons to show that they are in \( \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \), which implies that \( c^* \in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \) as well.

For postulate \( P_4 \), take \( c^* \in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \) and assume \( c^* \not\in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \). As before, the non-obvious case is when \( c^* \not\in (o_1 \cup o_2)^+ \). If \( c^* \in \delta_\pi \), then from the assumption that \( c^* \not\in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \), we conclude that there is a set \( \delta \subseteq \kappa_\pi \) of comparisons that precede \( c^* \) in \( \leq_\pi \), are added to \( (o_1 \cup o_2)^+ \) before it and, in concert with \( (o_1 \cup o_2)^+ \), block \( c^* \) from being added, i.e., such that:

\[( (o_1 \cup o_2)^+ \cup \delta)^+ \in \mathcal{O}_V , \]

but \( ((o_1 \cup o_2)^+ \cup \delta)^+ \not\in \mathcal{O}_V \), where \( \delta = \delta + \{c^*\} \). From the second to last result we infer that \( \delta \) can be added consistently to \( (o_1 \cup o_2)^+ \) and, since we have that \( c^* \in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \) as well, we obtain that \( o_1 \) and \( c^* \) can be added consistently to \( o_1 \). In other words, it holds that \( (o_1 \cup \delta)^+ \in \mathcal{O}_V \). Together with the previous result this contradicts the fact that \( o_1 \) and \( o_2 \) are coordinated with respect to \( \pi \).

The case when \( c^* \not\in (o_1 \cup o_2)^+ \) is treated analogously as for postulate \( P_3 \).

\((\Rightarrow)\) Assume that there are \( o_1, o_2 \in \mathcal{O}_V \) not coordinated with respect to \( \pi \), i.e., there exists a set \( \delta \subseteq \kappa_\pi \) of direct comparisons of \( \pi \) that are involved in a cycle with \( o_1 \) and are such that \( (o_1 \cup \delta)^+ \in \mathcal{O}_V \) and \( (o_1 \cup o_2)^+ \cup \delta)^+ \not\in \mathcal{O}_V \). Additionally, we have that neither of the comparisons in \( \delta \), or their inverses, are in \( (o_1 \cup o_2)^+ \). We infer that there must exist a comparison \( c^* \in (\kappa_\pi \setminus \delta) \) that completes the cycle. We will show that there exists a preorder \( \leq_\pi \) such that the revision operator induced by it does not satisfy \( P_3 \). Take a preorder \( \leq_\pi \) on \( \delta_\pi \) that arranges the elements of \( \delta \) in a linear order at the bottom of \( \leq_\pi \), i.e., such that \( c_j <_\pi c_i \), for any \( c_j < c_i \) and \( c_j \not\in \delta \), and \( c^* \) the maximal element in \( \leq_\pi \), i.e., \( c_j < \pi c^* \), for any \( c_j < \pi c^* \). This implies, in particular, that \( c_j < \pi c^* \), for any \( c_j < c^* \).

Note, now, that \( c^* \in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \) is because, by assumption, not all of the comparisons in \( \delta \) can be added to \( (o_1 \cup o_2)^+ \), and this makes it possible for \( c^* \) to be added. On the other hand, \( c^* \not\in \text{add}_{\leq_\pi}^+ (o_1 \cup o_2)^+ \) is because here we can, again by assumption, consistently add \( o_1 \) and, since \( c^* \) is the last in line to be added, the inevitability of creating a cycle with \( \delta \) and the rest of the comparisons of
o₁ makes it impossible to do so consistently. We obtain that \( c^* \in \text{add}_{≤}^*(o₁ \cup o₂) \) but \( c^* \notin (\text{add}_{≤}^*(o₁) \cup \text{add}_{≤}^*(o₂)) \), i.e., postulate P₃ is not satisfied. Concurrently, there will be a comparison in \( δ \) that occurs in \((\text{add}_{≤}^*(o₁) \cup \text{add}_{≤}^*(o₂)) \) that does not make it into \((\text{add}_{≤}^*(o₁ \cup o₂)) \), showing that P₄ is not satisfied either.

Theorem 1 shows that coordination is needed in order to make sure that postulates P₃₋₄ work, and we will henceforth assume that o₁ and o₂ are coordinated with respect to \( π \) whenever we apply these postulates.

6 Characterizing Preference Revision as Choice Over Comparisons

We show now that the procedure described in Section 3 is characterized by the postulates introduced in Section 4, under the restrictions established through Theorem 1. Theorem 2 shows that the procedure in Section 3 yields preference revision operators that satisfy postulates P₁₋₄.

Theorem 2. If \( α : ov \rightarrow ov \times ov \) is a preference assignment, then the revision operator \( ⊲ \) induced by it satisfies postulates P₁₋₄, for any \( π \in ov \) and o₁, o₂ \( ∈ ov \) such that o₁, o₂ are coordinated with respect to \( π \).

Proof. Satisfaction of postulates P₁₋₂ is straightforward.

For P₃, since at every step \( \text{add}_{≤}^*(o₁ \cup o₂) \) selects some direct comparisons in π to add to o, the end result satisfies the condition for being in \( |o|_π \).

For P₄, note that \((π∪o) \) selects \( \text{add}_{≤}^*(o₁) \cup o₂ \) of o₁ and o₂ are assumed to be coordinated with respect to π, satisfaction of postulates P₃₋₄ is guaranteed by Theorem 1.

For the converse, we want to show that any preference revision operator satisfying P₁₋₄ can be rationalized using a preference assignment.

To that end, we will construct the preorder \( ≤ \) from binary comparisons, but we must first figure out how to compare two direct comparisons \((k, k+1) \) and \((l, l+1) \). This is done by creating a situation where we cannot add both and hence one has to be given up. We will use a special type of preference order to induce a choice between these comparisons. If \( π \in ov \) is a chain and \((k, k+1), (l, l+1) \) is a direct comparison of π, the choice inducing preference \( oκ,l \) for \((k, k+1) \) and \((l, l+1) \) is defined as \( oκ,l = \{(k+1, l), (l+1, k)\} \). The following example illustrates this notion.

Example 7. To induce a choice between direct comparisons \((1, 2) \) and \((3, 4) \) in Figure 4, revise by \( o₁,₃ = \{(2, 3), (4, 1)\} \). Note that effectiveness of this maneuver hinges on the choice being confined to the direct comparisons of π: if inferred comparisons were allowed to be part of the choice, o₁,₃ loses its power to discriminate between \((1, 2) \) and \((3, 4) \): if, for instance, \((1, 3) \) and \((2, 4) \) are chosen, then \((2, 1) \) and \((4, 3) \) have to be inferred, leaving no space for a choice between \((1, 2) \) and \((3, 4) \), i.e., o₁,₃ would tell us nothing about the implicit preference between \((1, 2) \) and \((3, 4) \). We can also see that comparison of \((1, 2) \) and \((2, 3) \) is done by revising by \((3, 1) \).

Conversely, if \((k, k+1) \), \((l, l+1) \) is considered less important than \( π \), o₁,₃ardinferences \( π ⊲ o \). Bullets indicate other potential items in \( π \), faded arrows indicate comparisons that may not be in \( π \), but can be consistently added to it.

Lemma 1. If \( ⊲ \) satisfies postulates P₁₋₄, then the revealed preorder \( ≤ ⊲ \) is transitive.

Proof. Take \( π \in ov \) and \((i, i+1), (j, j+1), (k, k+1) ∈ δπ \) such that \((i, i+1) \) is a \( (j, j+1) \) \( ∈ (k, k+1) \) (we can assume that \( i < j < k \)). To show that \((i, i+1) \) is a \( (j, j+1) \) \( ∈ (k, k+1) \), take o ∈ ov such that contains all direct comparisons in π up to k, except \((i, i+1), (j, j+1) \), and \((k, k+1) \), plus the comparison \((k+1, i) \). In other words, o is such that \((i, i+1), (j, j+1) \) and \((k, k+1) \) were added to it, a cycle would form. The first step involves showing that \((k, k+1) \) is a \( π \). To see why this is the case, note first that, by design, not all of \((i, i+1), (j, j+1) \) and \((k, k+1) \) can be in \( π \), i.e., at least one of them must be left out. We now do a case analysis to show that, either way, \((k, k+1) \) ends up being left out.

Case 1. If \((k, k+1) \) is a \( π \), the conclusion is immediate.

Case 2. If \((j, j+1) \) is a \( π \), then we can safely add \((i, i+1) \) to \( π \); this is because the inference of the opposite comparison, i.e., \((i, i+1) \), can be done only by adding all comparisons on the path from \( i+1 \) to \( i \), and the absence of \((j, j+1) \) means this inference is blocked. Using P₄₋₅ we can now conclude that \((π ⊲ o) \cup \{(i, i+1)\} \) is a preorder (see Figure 5). Note, we can separate \( o \cup \{(i, i+1)\} \) into \( o₅,k = \{(k, k+1), (j, j+1) \}) \) and all the
comparisons on the path from \( k+1 \) to \( j \), plus the comparisons on the path from \( j+1 \) to \( k \). Call this latter preference \( o' \). We thus have that \( (o \cup \{(i,i+1)\})^+ = (o_j,k \cup o')^+ \) and, applying \( P_3 \), we obtain that:

\[
\pi \sqsupset (o \cup \{(i,i+1)\})^+ = \pi \sqsupset (o_{j,k} \cup o')^+ \subseteq ((\pi \sqsupset o_{j,k}) \cup o')^+.
\]

Since, by definition, \( (k,k+1) \notin \pi \sqsupset o_{j,k} \) and \( (k,k+1) \notin o' \), it follows that:

\[
(k,k+1) \notin \pi \sqsupset (o \cup \{(i,i+1)\})^+,
\]

then:

\[
(k,k+1) \notin ((\pi \sqsupset o) \cup \{(i,i+1)\})^+,
\]

and, finally, that:

\[
(k,k+1) \notin \pi \sqsupset o.
\]

Case 3. If \( (i,i+1) \notin \pi \sqsupset o \), then we can safely add \( (k,k+1) \) to \( \pi \sqsupset o \) and, by reasoning similar to above, show that \( (j,j+1) \notin \pi \sqsupset o \). Here we invoke Case 2.

With the fact that \( (k,k+1) \notin \pi \sqsupset o \) in hand, we can add \( (j,j+1) \) to \( \pi \sqsupset o \) (by reasoning similar to above), because the path from \( j+1 \) to \( j \) in \( \pi \sqsupset o \) is blocked by the absence of \( (k,k+1) \). Using postulates \( P_3-4 \), we conclude that:

\[
(\pi \sqsupset o) \cup \{(j,j+1)\})^+ = \pi \sqsupset (o \cup \{(j,j+1)\})^+ = \pi \sqsupset ((i+1, \ldots, k), (k+1, j))}^+ = \pi \sqsupset (o_{i,k}) \cup \{(i+1, \ldots, k)\}^+.
\]

Since \( (k,k+1) \notin ((\pi \sqsupset o) \cup \{(j,j+1)\})^+ \), we conclude that \( (k,k+1) \notin \pi \sqsupset o_{i,k} \), which implies that \( (i,i+1) \leq_{\pi} (k,k+1) \).

Lemma 1 is crucial for the following representation result, as it indicates that we can identify the revealed preference relation with the underlying preference over direct comparisons of \( \pi \) driving revision.

**Theorem 3.** If \( \triangleright \) is a revision operator satisfying postulates \( P_{1-4} \), for any \( \pi \in C_V \) and \( o, o_1, o_2 \in O_V \) such that \( o_1, o_2 \) are coordinated with respect to \( \pi \), then there exists a preference assignment \( a \) such that \( \triangleright \) is the \( a \)-induced revision operator.

**Proof.** For any \( \pi \in C_V \), take \( \leq_{\pi} \) to be the revealed preference relation \( \leq_{\pi} \). By Lemma 1, we know that \( \leq_{\pi} \) is transitive, so the only thing left to show is that \( (\pi \triangleright o) = add_{\leq_{\pi}}(o) \). We do this in two steps.

(“\( \subseteq \)” For one direction. Take \( (j,k) \in (\pi \triangleright o) \) and suppose \( (j,k) \notin add_{\leq_{\pi}}(o) \). Clearly, it cannot be the case that \( (j,k) \in o \), so we conclude that \( (j,k) \) is either a direct comparison of \( o \), or is inferred by transitivity using direct comparisons in \( \pi \) and \( o \).

Case 1. If \( (j,k) \in \delta_{\pi} \), then we can write \( (j,k) \) as \( (j,j+1) \). Suppose that \( (j,j+1) \) is on level \( i \) of \( \delta_{\pi} \); this means that if \( (j,j+1) \) does not get added to \( add_{\leq_{\pi}}(o) \) at step \( i \), then, since it cannot be inferred by transitivity, it does not get added at all. The fact that \( (j,j+1) \notin add_{\leq_{\pi}}(o) \) thus means that \( (j,j+1) \) forms a cycle with some comparisons in \( o \) and comparisons in \( \pi \) on levels \( l \leq i \). First, note that \( (j,j+1) \) cannot form a cycle with elements of \( o \) only, since the above would imply that \( (j+1, j) \in o \) and that would exclude the possibility that \( (j,j+1) \in \pi \triangleright o \). Thus, at least one other comparison in the cycle must come from \( \pi \). We can state, now, that, since \( (j+1, j) \in \pi \triangleright o \), then at least one of these comparisons must be absent in \( \pi \triangleright o \), i.e., there exists a direct comparison \( (k,k+1) \in \delta_{\pi} \) such that \( (k,k+1) \in \{k,j\} \triangleright o \) for some \( j \leq i \). \( (k,k+1) \notin \pi \triangleright o \) and \( (j,j+1) \), \( (k,k+1) \), \( (j,j+1) \), plus some other comparisons in \( o \) and \( \pi \) form a cycle. This means that \( o \) is safe to add \( \triangleright \) to \( \pi \triangleright o \), where \( o \) contains all comparisons on the path from \( k+1 \) to \( j \), plus the comparison on the path from \( j+1 \) to \( k \). We can rewrite \( o' \) by separating out \( (k,j+1) \) and \( (j,k+1) \), i.e., \( o' = (o_{j,k} \cup o')^+ \). Applying postulates \( P_{3-4} \), we now get that:

\[
((\pi \triangleright o) \cup o')^+ = \pi \triangleright (o \cup o')^+ \subseteq ((\pi \triangleright o_{j,k}) \cup o')^+.
\]

Using the assumption that \( (j,j+1) \in (\pi \triangleright o) \) and the fact that \( (j,j+1) \notin o' \), we can thus infer that \( (j,j+1) \in \pi \triangleright o_{j,k} \).

This, in turn, means that \((j,j+1) \notin \pi \triangleright o \), which implies that \( (i,i+1) \leq_{\pi} (k,k+1) \). We can now apply the reasoning from Case 1 to the direct comparisons of \( \pi \) that go into inferring \( (j,k) \), to show that they must be in \( add_{\leq_{\pi}}(o) \).

This, in turn, implies that \( (j,k) \) will be in \( add_{\leq_{\pi}}(o) \) as well. The reasoning for the other direction is similar.

Theorems 2 and 3 describe preference revision operators that rely on total preorders \( \leq_{\pi} \) on \( \delta_{\pi} \), where a tie between two direct comparisons means that if they cannot both be added, then they are both passed over. We can eliminate this indecision by using linear orders on \( \delta_{\pi} \) instead of preorders: this ensures that any two direct comparisons of \( \pi \) can be clearly ranked with respect to each other, and that a revision operator is always in a position to choose among them. On the postulate site, linear orders can be characterized by tightening the notions of a \( \pi \)-completion and, with it, postulate \( P_1 \). Thus, a decisive \( \pi \)-completion of \( o \) is defined as:

\[
[o]_{\pi}^D = \{ (o \cup \delta)^+ \in O_V \mid \emptyset \subseteq \delta \subseteq \delta_{\pi} \}.
\]

Changing the format of the revision output requires changing the postulate that speaks about this format as well. The decisive version of \( P_1 \) is then written, for any \( \pi \in C_V \) and \( o \in O_V \), as:

\[
(P_D) \ (\pi \triangleright o \in [o]_{\pi}^D).
\]

A decisive preference assignment \( a \) is a function \( a \colon C_V \to C_V \times V \) mapping every \( \pi \in C_V \) to a linear preorder \( <_{\pi} \) on \( \delta_{\pi} \). We can now show the following result.

**Theorem 4.** A revision operator \( \triangleright \) satisfies postulates \( P_D \) and \( P_{2-4} \) if and only if there exists a decisive preference assignment \( a \) such that, for any \( \pi \in C_V \) and \( o, o_1, o_2 \in O_V \) such that \( o_1, o_2 \) are coordinated with respect to \( \pi \), \( \triangleright \) is the \( a \)-induced preference revision operator.
Proof. The proofs for Theorems 2 and 3 work here with minimal adjustments. Note that when choosing between two direct comparisons, postulate $P_2$ does not allow $\triangleright$ to be indifferent anymore. This means that the revealed preference relation on $\delta_\pi$ ends up being linear.

We can see, thus, that what seems like a weakness in the original formulation of the problem, i.e., the mismatch in type between the input (a chain) and the output (an spn) of a revision operator, can be resolved by requiring the ranking on comparisons to be strict. However, in the present setup this amounts to a less general result, which is why we presented our work in this manner.

7 Concrete Preference Revision Operators

Theorems 2, 3 and 4 articulate an important lesson: preference revision performed in a principled manner, i.e., in accordance with $P_1 - 4$ or $P_1^D$ and $P_2 - 4$, involves having preferences over comparisons. Thus, to obtain concrete operators one must look at ways of ranking the comparisons in a preference $\pi$. We sketch here two simple solutions, as proof of concept.

The trivial assignment $a^t$ is defined by taking:

$$(i, j+1) \simeq^t_j (j, j+1),$$

while the lexicographic assignment $a^{\text{lex}}$ is defined by taking:

$$(i, i+1) \lessapprox^{\text{lex}}_\pi (j, j+1),$$

if $i < j$, for any $\pi \in \mathcal{C}_V$ and $(i, i+1), (j, j+1) \in \pi$. Intuitively, the trivial assignment makes all direct comparisons of $\pi$ equally desirable, while the lexicographic assignment orders them in lexicographic order.

These assignments induce the trivial and lexicographic operators $\triangleright^t$ and $\triangleright^{\text{lex}}$, respectively. It is straightforward to see that $\lessapprox^t_\pi$ is a preorder and $\lessapprox^{\text{lex}}_\pi$ is a linear order, prompting the following result.

Proposition 1. The operator $\triangleright^t$ satisfies postulates $P_1 - 4$.

The operator $\triangleright^{\text{lex}}$ satisfies postulates $P_1^D$ and $P_2 - 4$.

The following example illustrates that the two operators can give different results on the same input.

Example 8. For $\pi$ and $o$ as in Example 1, the trivial operator ranks all direct comparisons of $\pi$, i.e., $(1, 2)$ and $(2, 3)$, equally, and hence either adds all or none of them to $o$. Since adding both leads to a cycle, it ends up adding none and hence $\pi \triangleright^t o = (3, 1)$.

The lexicographic assignment ranks $(1, 2)$ as better than $(2, 3)$, and hence adds $(1, 2)$ after which it runs out of options, i.e., $\pi \triangleright^{\text{lex}} o = (3, 1, 2)$.

8 Conclusion

We have presented a model of preference change according to which revising a preference $\pi$ goes hand in hand with having preferences over the comparisons of $\pi$, thereby providing a rigorous formal treatment to intuitions found elsewhere in the literature (Sen 1977; Grüne-Yanoff and Hansson 2009a). Interestingly, the postulates describing preference revision are analogous to existing postulates offered for propositional enforcement (Haret, Wallner, and Woltran 2018), an operation used to model changes in Abstract Argumentation Frameworks (AFs) (Dung 1995).

Our treatment unearthed interesting aspects of preference revision, such as the issue of coordination between successive instances of new preference information (Section 4) and the non-obvious solution to the question of how to rank two comparisons relative to each other (Section 6). These aspects are taken for granted in regular propositional revision, but prove key to successful application of revision to the more specialized context of transitive relations on a set of items, i.e., preference orders. In this respect, preference revision is akin to revision for fragments of propositional logic (Delgrande, Peppas, and Woltran 2018; Creignou et al. 2018), and raises the possibility of exporting this approach to other formalisms in this family. The addition procedure in particular, lends itself to application in other formalisms by slight tweaking of the acceptance condition, and could thus supply some interesting lessons for revision in general, in particular to revision-like operators for specialized formalisms, such as that of AFs, mentioned above.

There is also ample space for future work with respect to the present framework itself. To facilitate exposition of the main ideas we imposed certain restrictions on the primary notions. Lifting these restrictions would yield broader results that would potentially cover more ground and apply to a more diverse set of inputs. We can consider, for instance, revising strict partial orders in general (not just linear orders), and using rankings that involve all comparisons of the initial preference order (not just the direct ones). As the space of possibilities becomes larger, the choice problems on this space become increasingly more complex as well. Finding the right conditions under which the choice mechanism corresponds to a set of appealing postulates requires a delicate balance of many elements, and holds the promise for interesting results.
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