Absolute continuity of stable foliations for mappings of Banach spaces

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Abstract

We prove the absolute continuity of stable foliations for mappings of Banach spaces satisfying conditions consistent with time-t maps of certain classes of dissipative PDEs. This property is crucial for passing information from submanifolds transversal to the stable foliation to the rest of the phase space; it is also used in proofs of ergodicity. Absolute continuity of stable foliations is well known in finite dimensional hyperbolic theory. On Banach spaces, the absence of nice geometric properties poses some additional difficulties.

1 Introduction and Setting

In finite dimensional dynamical systems theory, positive Lebesgue or Riemannian measure sets have often been equated with observable events, and the absolute continuity of stable foliations has been a very useful tool for connecting positive measure sets on unstable manifolds to positive measure sets in the phase space. Here we have assumed that the phase space supports a meaningful notion of volume, e.g., it is a Riemannian manifold, and the measures in question are associated with volumes or induced volumes on unstable manifolds. The connection above has made it possible for dissipative systems with chaotic attractors to have a natural notion of physically relevant invariant measures. Indeed one of the most important advances in finite dimensional hyperbolic theory in the last half century is the idea of SRB measures, which govern the distributions of positive Lebesgue measure sets of initial conditions thanks to the absolute continuity of stable foliations (see e.g. [4, 16, 20]). An equally important use of this property is in proofs of ergodicity, via the well known argument of Hopf [6]. This argument has been used many times: we mention applications to geodesic flows on manifolds of negative curvature (see, e.g., [1]) and to dispersing billiards (e.g., [18, 13]); see also [15, 16].

In infinite dimensional dynamical systems, such as those on Banach spaces, there is no natural notion of volume, hence no obvious concept of “observable events”; yet the idea of what constitutes a “typical solution” for a PDE seems no less important. It is in the context of attempting to offer an answer to these questions that the idea of absolute continuity of stable foliations appears. Using Haar measure to define a notion of “positivity of measure” on finite dimensional subspaces or submanifolds of Banach spaces, it has been shown that for a Banach space system with a center manifold, there is a strong stable foliation that is absolutely continuous [11]. Via this strong stable foliation, properties that are determined by asymptotic

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future orbit distributions are passed from the center manifold to the rest of the phase space, and the absolute continuity of this foliation enables us to define a notion of “typical initial condition”, a notion of “almost everywhere” in Banach spaces, that is dynamically connected to volumes on center manifolds.

In this paper, we extend the idea of absolute continuity of stable foliations to dissipative dynamical systems with quasi-compact derivative operators on Banach spaces without assuming the existence of center manifolds. We state and carry out in detail a complete proof of this result for strong stable manifolds of nonuniformly hyperbolic dynamical systems. As a corollary, we show that the basins of SRB measures with nonzero Lyapunov exponents are “visible”, in the sense that for many families of initial conditions smoothly parametrized by \([0,1]^k\), orbits starting from a positive Lebesgue measure subset are described by SRB measures.

There are several proofs of absolute continuity in finite dimensions, a testimony to the centrality of this result in the subject. Our proof follows in outline the one sketched in [20], and is different than [16, 15, 9]. We mention that [9], as well as the very recent paper [10], both prove a similar result for mappings of Hilbert spaces. An important difference between Hilbert and Banach spaces is that the latter need not have good geometry. Any proof of absolute continuity hinges on (i) the action of holonomy maps (defined by sliding along stable manifolds) on balls or objects with nice geometric shapes, and (ii) covering lemmas on transversals by objects of the same kind. In this paper, we have had to devise ways to overcome the difficulty that Banach space balls are not necessarily nice. We believe our proof is to-the-point and concise, perhaps one of the most direct even among finite dimensional proofs. We have also included a complete proof of the formula for the Radon-Nikodým derivatives of holonomy maps, a fact often claimed without proof in papers in finite dimensions.

The setting of this paper is as follows: Let \(B\) be a Banach space with norm \(|\cdot|\). We consider \((f,\mu)\), where \(f : B \to B\) is a map and \(\mu\) is an \(f\)-invariant Borel probability measure. We assume:

\[(H1)\] (i) \(f\) is injective and \(C^2\) Fréchet differentiable;
(ii) the derivative of \(f\) at \(x \in B\), denoted \(df_x\), is also injective.

\[(H2)\] (i) \(f\) leaves invariant a compact set \(\mathcal{A} \subset B\), with \(f(\mathcal{A}) = \mathcal{A}\);
(ii) \(\mu\) is supported on \(\mathcal{A}\).

\[(H3)\] We assume
\[
l_\alpha(x) := \lim_{n \to \infty} \frac{1}{n} \log |df^n_x|_\alpha < 0 \quad \text{for } \mu - \text{a.e. } x.
\]
Here \(|df^n_x|_\alpha\) is the Kuratowski measure of noncompactness of the set \(df^n_x(B)\), where \(B\) is the unit ball in \(B\) (see, e.g., [14] for properties of \(|\cdot|_\alpha\)).

Conditions (H1), (H2)(i) and (H3) are known to hold for systems defined by large classes of dissipative PDEs (see [5]); the compact set \(\mathcal{A}\) is often an attractor. The existence of invariant measures on \(\mathcal{A}\) is not an additional assumption; such measures always exist.

To motivate the material in Sections 2 and 3, we first state a rough version of one of our main results, containing yet-to-be-defined terms.

**Provisional Theorem** Let \(W^{ss}\) be a strong stable foliation of \(f\). We assume \(W^{ss}\) has codimension \(k \in \mathbb{Z}^+\), and let \(\Sigma^1, \Sigma^2\) be two embedded \(k\)-dimensional disks in \(B\), close to one another and roughly parallel, both transversal to \(W^{ss}\). We assume that the holonomy map
\[
p : \Sigma^1 \to \Sigma^2
\]
from \(\Sigma^1\) to \(\Sigma^2\) along \(W^{ss}\) is defined on \(\tilde{\Sigma}^1 \subset \Sigma^1\), i.e. for \(x \in \tilde{\Sigma}^1\), \(p(x)\) is the unique point in \(W^{ss}_{loc,x} \cap \Sigma^2\), where \(W^{ss}_{loc,x}\) is a local \(W^{ss}\)-manifold at \(x\). Then \(p\) is absolutely continuous, in the
sense that if $B \subset \Sigma^2$ is a Borel set such that $\nu_{\Sigma^2}(B) = 0$, then $\nu_{\Sigma^1}(p^{-1}B) = 0$. Here $\nu_\Sigma$ is the induced volume on an embedded disk $\Sigma$.

Induced volumes and other preliminaries are given in Section 2. More technical preparation, including the strong stable foliation and transversals, are discussed in Section 3. The Provisional Theorem above is formulated precisely as Theorem A and proved in Section 4. Theorem B, which gives precise Radon-Nikodym derivatives of holonomy maps, is stated and proved in Section 5. Section 6 contains some consequences of these results for SRB measures with no zero Lyapunov exponents, including Theorem C, on ergodic decomposition, and Theorem D, on the “visibility” of SRB measures.

2 Preliminaries

2.1 Banach space geometry

First we explain what is meant by induced volume in the statement of the Provisional Theorem.

**Definition 2.1.** Let $E \subset B$ be a finite-dimensional subspace. We define the induced volume $m_E$ on $E$ to be the unique Haar measure on $E$ for which

$$m_E\{u \in E \mid |u| \leq 1\} = \omega_k$$

where $k = \dim E$ and $\omega_k$ is the Lebesgue measure of the Euclidean unit ball in $\mathbb{R}^k$.

Once volumes are defined, the notion of determinant follows naturally: Let $A : B \to B$ be a bounded operator, and let $E \subset B$ be a subspace of finite dimension. Let $B_E$ denote the closed unit ball in $E$. Then

$$\det(A|E) = \begin{cases} m_{AE}(AB_E) & A|E \text{ injects} \\ m_E(B_E) & \text{else.} \end{cases}$$

The notion of induced volume above is defined for one subspace at a time. For it to be useful, it is necessary to ensure some regularity as subspaces are varied. The Hausdorff distance between two closed subspaces $E, E' \subset B$ is defined to be

$$d_H(E, E') = \max\{\sup\{d(e, S_{E'}) : e \in S_E\}, \sup\{d(e', S_E) : e' \in S_{E'}\}\}$$

where $S_E = \{v \in E \mid |v| = 1\}$.

**Proposition 2.2 ([3], Proposition 2.15).** For any $k \geq 1$ and any $M > 1$ there exist $L, \epsilon > 0$ with the following properties. If $A_1, A_2 : B \to B$ are bounded linear operators and $E_1, E_2 \subset B$ are $k$-dimensional subspaces for which

$$|A_j|, \ |(A_j|_{E_j})^{-1}| \leq M \quad j = 1, 2,$$

$$|A_1 - A_2|, \ d_H(E_1, E_2) \leq \epsilon,$$

then we have the estimate

$$\left| \log \frac{\det(A_1|E_1)}{\det(A_2|E_2)} \right| \leq L(|A_1 - A_2| + d_H(E_1, E_2)) \ .$$  \hspace{1cm} (1)
Remark 2.3 (see [3]). Later, when we apply Proposition 2.2 to distortion estimates, we will need to use the dependence of the constants $\epsilon, L$ on the parameters $k, M$. One can show that there exists a constant $C_k \geq 1$, depending only on the dimension $k \in \mathbb{N}$, such that we may take $\epsilon = (C_k M^{10k})^{-1}$ and $L = C_k M^{10k}$ in the conclusion to Proposition 2.2.

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Treating induced volumes on finite dimensional linear subspaces as volume elements, one obtains by the usual construction a notion of induced volume $\nu_W$ on a finite dimensional submanifold $W$ (see, e.g., Sect. 5.3 in [3]). This is the measure on transversals used in the statement of the Provisional Theorem in Section 1.

For computations, it is often convenient to work with the gap $\text{Gap}(E, E')$, defined by

$$\text{Gap}(E, E') = \sup_{v \in S_E} d(v, E').$$

The quantities Gap and $d_H$ are related as follows:

**Lemma 2.4 ([8]).** For all closed subspaces $E, E'$, we have

$$d_H(E, E') \leq \max\{\text{Gap}(E, E'), \text{Gap}(E', E)\} \leq 2d_H(E, E').$$

If additionally $E, E' \subset B$ are closed subspaces with the same finite codimension $q$, then

$$\text{Gap}(E', E) \leq \frac{q \text{Gap}(E, E')} {1 - q \text{Gap}(E, E')} ,$$

so long as the denominator in the above expression is $> 0$.

For a more complete discussion of results on Banach space geometry, induced volumes and determinants etc. that are relevant for the extension of finite dimensional ergodic theory to Banach space maps, see [3], Section 2.

### 2.2 Multiplicative Ergodic Theorem (MET)

To fix notation, we recall the following version of the MET, which is adequate for our purposes: Let $X$ be a compact metric space, and let $f : X \to X$ be a homeomorphism preserving a Borel probability measure $\mu$ on $X$. We consider a continuous map $T : X \to B(B)$ where $B(B)$ denotes the space of bounded linear operators on $B$, the topology on $B(B)$ being the operator norm topology. We assume additionally that $T_x := T(x)$ is injective for every $x \in X$, and write $T^n_x = T_{f^n-1} x \circ \cdots \circ T_x$. Define

$$l_\alpha(x) = \lim_{n \to \infty} \frac{1}{n} \log |T^n_x|_\alpha$$

for $\mu$-almost every $x \in X$ (as in (H3) in Section 1).

**Theorem 2.5 (Multiplicative ergodic theorem [19]).** Under the hypotheses above, for any measurable function $\lambda_\alpha : X \to \mathbb{R}$ for which $\lambda_\alpha > l_\alpha$ $\mu$-almost surely, there is a measurable, $f$-invariant set $\Gamma \subset X$ with $\mu(\Gamma) = 1$, a measurable function $r : \Gamma \to \mathbb{Z}_{\geq 0}$, and on the level sets of $r$ a collection of measurable functions $\lambda_1, \cdots, \lambda_r(x) : X \to \mathbb{R}$ such that

$$\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_r(x) > \lambda_\alpha(x),$$

for which the following properties hold. For any $x \in \Gamma$, there is a splitting

$$B = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x) \oplus F(x)$$

such that
(a) for each \( i = 1, 2, \ldots, r(x) \), \( \dim E_i(x) = m_i(x) \) is finite, \( T_xE_i(x) = E_i(fx) \), and for any \( v \in E_i(x) \setminus \{0\} \), we have
\[
\lambda_i(x) = \lim_{n \to \infty} \frac{1}{n} \log |T^n_x v| = - \lim_{n \to \infty} \frac{1}{n} \log |(T^n_{f^{-n}x})^{-1}v| ;
\]

(b) the distribution \( F \) is closed and finite-codimensional, satisfies \( T_xF(x) \subset F(fx) \) and
\[
\lambda_\alpha(x) \geq \limsup_{n \to \infty} \frac{1}{n} \log |T^n_x|F(x)| ;
\]

(c) the mappings \( x \mapsto E_i(x) \), \( x \mapsto F(x) \) are \( \mu \)-continuous along the level sets of \( r \) (see Definition 2.6 below), and

(d) writing \( \pi_i(x) \) for the projection of \( B \) onto \( E_i(x) \) via the splitting at \( x \), we have
\[
\lim_{n \to \pm \infty} \frac{1}{n} \log |\pi_i(f^n x)| = 0 \quad a.s.
\]

**Definition 2.6.** Let \( X \) be a compact metric space and \( \mu \) a Borel probability on \( X \), and let \( Z \) be a metric space. We say that a mapping \( \Psi : X \to Z \) is \( \mu \)-continuous if there is an increasing sequence of compact subsets \( \bar{K}_1 \subset \bar{K}_2 \subset \cdots \subset X \) with the properties that (i) \( \Psi|_{\bar{K}_n} \) is a continuous mapping for all \( n \) and (ii) \( \mu(\cup_n \bar{K}_n) = 1 \).

For related facts on \( \mu \)-continuity, see [3], Section 3.

**Remark 2.7.** The function \( \lambda_\alpha \) appearing in Theorem 2.5 should be thought of as mitigating a cutoff, prescribed in advance, for the Lyapunov spectrum of \( (f, \mu; T) \). In the case where \((f, \mu, T)\) is ergodic, \( l_\alpha \) is constant almost surely, and so for all purposes it suffices to apply the MET with \( \lambda_\alpha \) equal to any constant strictly greater than \( l_\alpha \). When \((f, \mu, T)\) is not ergodic, \( l_\alpha \) is a measurable function taking values in \( (-\infty, 0) \) (see (H3)), and it may be natural, even necessary, for \( \lambda_\alpha \) to be nonconstant. Given \( l_\alpha \), an example of \( \lambda_\alpha \) may be as follows: For arbitrarily fixed constants \( \gamma \in (0, 1) \) and \( \lambda_\alpha \in (-\infty, 0) \), define
\[
\lambda_\alpha(x) = \begin{cases} 
(1 - \gamma)l_\alpha(x) & l_\alpha(x) > -\infty \\
\lambda_\alpha & l_\alpha(x) = -\infty 
\end{cases}.
\]

Observe that the function \( \lambda_\alpha \) so defined has the property that \( l_\alpha < \lambda_\alpha < 0 \); it converges to \( l_\alpha \) as \( \gamma \to 0 \) and \( \lambda_\alpha \to -\infty \), and importantly, it is an \( f \)-invariant function.

**Invariant sets defined by splitting of the Lyapunov spectrum**

It is convenient to represent \( \bar{\Gamma} \) as a countable union of positive \( \mu \)-measure invariant subsets on which certain quantities in the MET have uniform bounds. Here is one way to systematically enumerate such a collection of invariant sets:

For \( \lambda^* \in \mathbb{R}, m, p \in \mathbb{Z}_{>0} \), define
\[
\Gamma(\lambda^*; m, p) = \{ x \in \bar{\Gamma} : \lambda_\alpha(x) < \lambda^* - \frac{1}{p}, \ min_i |\lambda_i(x) - \lambda^*| > \frac{1}{p}, \ \dim E^+_x = m \} .
\]

When \( \lambda_\alpha \) is \( f \)-invariant, each \( \Gamma(\lambda^*; m, p) \) is invariant under \( f \), and that the countable union
\[
\bigcup_{m, p \in \mathbb{Z}_{>0}} \bigcup_{\lambda^* \in \mathbb{Q}} \Gamma(\lambda^*; m, p)
\]
is a full \(\mu\)-measure set. On \(\Gamma(\lambda^*; m, p)\), we have the following splitting: Let \(\mathcal{B}_x\) denote the tangent space at \(x \in \Gamma(\lambda^*; m, p)\). Then \(\mathcal{B}_x = E^+_x \oplus E^-_x\), where \(E^+_x\) and \(E^-_x\) are defined by

\[
E^+_x = \bigoplus_{i: \lambda_i(x) > \lambda^*} E_i(x) \quad \text{and} \quad E^-_x = F(x) \oplus \bigoplus_{i: \lambda_i(x) < \lambda^*} E_i(x),
\]

Thus \(\dim(E^+) = m\) and \(df_x(E^+_x) = F^+\), while \(\dim(E^-) = \infty\) and \(df_x(E^-_x) \subseteq E^-_{fx}\).

From here on: the setting in the Introduction is assumed. Let \(l_a\) be the function in (H3). We fix an \(f\)-invariant cutoff function \(\lambda_\alpha\) with \(l_\alpha < \lambda_\alpha < 0\), and apply the MET to the derivative cocycle \((f, \mu; df)\). All notation is as in the MET. Paring off sets of zero measure, we may assume that there exists an increasing sequence of Borel sets \(K_1 \subset K_2 \subset \cdots \subset \mathcal{A}\) for which (i) \(\mathcal{T} = \bigcup_n K_n\) and (ii) the Oseledets subspaces \(E_i, F\) are continuous on each \(K_n\) (see Section 3.1 in [3]).

As we will see, in most of our arguments it will suffice to consider one \(\Gamma(\lambda^*; m, p)\) at a time. Specifically, from here to the end of Section 3, we fix \(\lambda^*, m, p\), and write

\[\Gamma = \Gamma(\lambda^*; m, p)\]  

As we are interested only in splittings in which \(df^n|E^-\) is strictly contracting, we may further assume \(\lambda^* < 1/2p\).

### 2.3 Adapted norms

We recall here without proof some standard results on adapted norms, modifying results from Section 4 of [3] as follows. Instead of decomposing the tangent space at \(x \in \Gamma\) into \(\mathcal{B}_x = E^+_x \oplus E^-_x\) or \(E^u_x \oplus E^s_x\) as is done in [3], here we have \(\mathcal{B}_x = E^+_x \oplus E^-_x\) where \(E^+_x\) and \(E^-_x\) are as defined in (4) above; we will sometimes refer to \(E^-_x\) as the “strong stable” direction.

Letting

\[\lambda^+ = \lambda^* + \frac{1}{2p}\]  
\[\lambda^- = \lambda^* - \frac{1}{2p}\]

we have that \(\lambda^- < 0\), \(\lambda^+ > \lambda^-\), and \(\lambda^+\) can be positive or negative. Analogous to the construction in [3], we define the adapted norms \(|\cdot|'_x\) as follows:

\[
|u|^'_x = \sum_{n=-\infty}^{0} \frac{|df^n u|}{e^{n\lambda^+}} \text{ for } u \in E^+_x,
\]
\[
|w|^'_x = \sum_{n=0}^{\infty} \frac{|df^n w|}{e^{n\lambda^-}} \text{ for } w \in E^-_x,
\]

and for \(v = u + w \in \mathcal{B}_x, u \in E^+_x, w \in E^-_x\), we define \(|v|^'_x = \max\{|u|^'_x, |w|^'_x\}\).

For \(x \in \Gamma\) and \(r > 0\), we will sometimes refer to the domain \(\mathcal{B}_x(r) = \{v \in \mathcal{B}_x : |v|^'_x \leq r\}\) equipped with the adapted norm \(|\cdot|'_x\) as a “chart”, or a “Lyapunov chart”, a term borrowed from finite dimensional nonuniform hyperbolic theory. Accordingly, we define chart maps \(f_x : \mathcal{B}_x(r) \to \mathcal{B}_{fx}\) by \(f_x = \exp_{fx}^{-1} \circ f \circ \exp_x\). The proofs of the following results are identical to those in [3].

**Lemma 2.8.**

(a) **(One-step hyperbolicity)** For any \(u \in E^+_x, w \in E^-_x\), we have

\[
|df_x u|^'_{fx} \geq e^{\lambda^+}|u|^'_x
\]
\[
|df_x w|^'_{fx} \leq e^{\lambda^-}|w|^'_x
\]
There exists $\delta_1 > 0$ for which the following hold: Given any $\delta_2 > 0$, there is a Borel measurable function $l : \Gamma \to \mathbb{R}^+$, with
\begin{equation}
l(f^\pm x) \leq e^{\delta_2} l(x) \quad \text{for } \mu - \text{a.e. } x, \tag{6}\end{equation}
such that for all $x \in \Gamma$,

(b) the norms $| \cdot |_x'$ and $| \cdot |$ are related by
\[ \frac{1}{2}|v| \leq |v|_x' \leq l(x)|v|; \]

c) for any $\delta \leq \delta_1$, the following hold for $\tilde{f}_x$ restricted to $\tilde{B}_x(\delta l(x)^{-1})$:

(i) $\text{Lip}'(\tilde{f}_x - (d\tilde{f}_x)_{0}) \leq \delta$;

(ii) the mapping $z \mapsto (d\tilde{f}_x)_z$ satisfies $\text{Lip}'(d\tilde{f}_x) \leq l(x)$.

Here, $\text{Lip}'$ refers to the Lipschitz constant taken with respect to the $| \cdot |'$ norm.

Throughout, the parameters $\delta_1$ and $\delta_2 > 0$ are fixed with $\delta_2 \ll \lambda^+ - \lambda^-$, while $\delta \leq \delta_1$ may be shrunk a finite number of times. The function $l : \Gamma \to [1, \infty)$ is as in Lemma 2.8. Paring off a set of zero measure, we may assume that (6) holds for pointwise $x \in \Gamma$.

It follows from (a) and (b) above that for all $n \in \mathbb{Z}^+$,
\[ |df_x^n w| \leq 2l(x)e^{n\lambda^-}|w| \quad \text{for all } w \in E_x^-, \]
\[ |df_x^{-n} u| \leq 2l(x)e^{-n\lambda^+}|u| \quad \text{for all } u \in E_x^+. \]

Hereafter, we write $\Gamma_{l_0} = \{ x \in \Gamma : l(x) \leq l_0 \}$ for $l_0 > 1$, and refer to these as uniformity sets.

3 Preparation: $W^{ss}$-manifolds and transversals

Notation: for $x \in \Gamma$, $r > 0$ we write $\tilde{B}_x^\pm(r) = \{ v \in E_x^\pm : |v|_x' \leq r \}$, so that $\tilde{B}_x(r) = B_x^+(r) + B_x^-(r)$. We write $\pi_x^+$, $\pi_x^-$ for the projection operators corresponding to the splitting $B_x = E_x^+ \oplus E_x^-$, and for notational simplicity, we write $\tilde{f}_x^n$ instead of $\tilde{f}_{f_x^n \circ \cdots \circ f_x}$. We will sometimes omit mention of the point $x \in \Gamma$ at which the adapted norm is taken when it is clear from context.

3.1 Local strong stable manifolds

We state the following result, the proof of which is identical to that of the usual local stable manifolds theorem; see, e.g., [12].

**Theorem 3.1.** There is a constant $\delta'_1 \leq \delta_1$ with the property that for all $\delta \leq \delta'_1$, there is a family of functions $\{ h_x : \tilde{B}_x^-(\delta l(x)^{-1}) \to \tilde{B}_x^+(\delta l(x)^{-1}) \}_{x \in \Gamma}$ such that
\[ h_x(0) = 0 \quad \text{and} \quad \tilde{f}_x(\text{graph } h_x) \subset \text{graph } h_{f_x} \quad \text{for all } x \in \Gamma. \]

With respect to the norms $| \cdot |_x'$, the family $\{ h_x \}_{x \in \Gamma}$ has the following additional properties.

(a) $h_x$ is $C^{1+\text{Lip}}$, Fréchet differentiable, with $(dh_x)_0 = 0$.

(b) $\text{Lip}'h_x \leq \frac{1}{10}$ and $\text{Lip}'dh_x \leq Cl(x)$, where $C > 0$ is independent of $x$.

(c) For any $z_1, z_2 \in \text{graph } h_x$, we have the estimate
\[ |\tilde{f}_x z_1 - \tilde{f}_x z_2|'_x \leq (e^{\lambda^-} + \delta)|z_1 - z_2|'_x. \]
(d) The set $\text{graph}(h_x)$ is characterized by
\[
\text{graph}(h_x) = \{ y \in \bar{B}_x(\delta l(x)^{-1}) : \bar{f}_x^n y \in \bar{B}_{f^n x}(\delta(l(x)e^{\delta l x})^{-1}) \text{ and } |\bar{f}_x^n y|_{f^n x} \leq (e^{\lambda^-} + \delta)^n |y|_x \forall n \geq 1 \} .
\]

The local strong stable manifold at $x$, written $W_{x,loc}^{ss}$, is defined to be $\exp_x \text{graph } h_x$.

Theorem 3.1 is obtained via the backwards graph transform, i.e., the graph transform taken with respect to $f^{-1}$. More precisely, we define
\[
W_0^{ss}(x) = \{ h : \bar{B}_x(\delta l(x)^{-1}) \to \bar{E}_x : h(0) = 0 \text{ and } \text{Lip } h \leq \frac{1}{10} \} .
\]

For $h \in W_0^{ss}(x)$, we say the backwards graph transform of $h$ is well defined and is equal to $G_x h$ if there exists a unique mapping $G_x h : \bar{B}_{f^{-1} x}(\delta l(f^{-1} x)^{-1}) \to E_{f^{-1} x}$ such that
\[
\bar{f}_x(\text{graph } G_x h) \subset \text{graph } h .
\]

We state without proof the following basic lemma which implies the existence and Lipschitzness of the family $\{h_x\}$ above and which will be used again later on.

Lemma 3.2. For all $\delta \leq \delta_1$ and $x \in \Gamma$, $G_x : W_0^{ss}(x) \to W_0^{ss}(f^{-1} x)$ is well defined and is a contraction mapping, i.e.,
\[
\|G_x h_1 - G_x h_2\|_{f^{-1} x, ss} \leq q \|h_1 - h_2\|_{x, ss}
\]
for $h_1, h_2 \in W_0^{ss}(x)$, where the norm $\| \cdot \|_{x, ss}$ on $W_0^{ss}(x)$ is defined by
\[
\|h\|_{x, ss} = \sup_{v \in \bar{B}_x(\delta l(x)^{-1})} \frac{|h(v)|_x}{|v|_x} ,
\]
and $q \in (0, 1)$ is a constant independent of $x$.

### 3.2 Iterated transversals

In the local version of our result, the transversals $\Sigma^1$ and $\Sigma^2$ (see Theorem A) will be pieces of manifolds contained inside the domains of a chart at some $x \in \Gamma$, and they will be of the form $\exp_x \text{graph } (g_i)$ for some $g_i : E_i^+ \to E_i^-$, $i = 1, 2$. For reasons to become clear momentarily, it will be necessary to consider shrinking charts. Let $\lambda_c := \frac{1}{2}\lambda^- < 0$ when $\lambda^+ > 0$, and $\lambda_c := \frac{1}{2}(\lambda^+ + \lambda^-)$ when $\lambda^+ < 0$: the exponent $\lambda_c < 0$ will be the rate at which our charts shrink.

Lemma 3.3. For $\delta > 0$ sufficiently small, the following holds. Fix $l_0 \geq 1$ and $c_0 \geq l_0$. Let $x \in \Gamma_{l_0}$, and let $g_0 : \bar{B}_x(\delta l_0^{-1}) \to \bar{E}_x(\delta l_0^{-1})$ be a $C^{1+\text{Lip}}$-Fréchet differentiable map for which $\text{Lip}'(g_0) \leq \frac{1}{10}$. Then, writing $c_n = e^{-n\lambda_c}c_0$, there exists a sequence of $C^{1+\text{Lip}}$ maps
\[
g_n : \bar{B}_{f^n x}(\delta c_n^{-1}) \to \bar{E}_{f^n x}(\delta c_n^{-1}) , \quad n \geq 1 ,
\]
with the following properties:

(a) $\{g_n\}_{n \geq 0}$ is a (forward) graph transform sequence along the charts $\{\bar{B}_{f^n x}(\delta c_n^{-1})\}_{n \geq 0}$, i.e., for all $n \geq 1$ we have
\[
\text{graph } g_{n+1} = \bar{B}_{f^{n+1} x}(\delta c_{n+1}^{-1}) \cap \bar{f}_n x(\text{graph } g_n) ,
\]
(b) For all \( n \geq 1 \), we have that:
- \( \text{Lip}'(g_n) \leq \frac{1}{10} \), and
- \( \text{Lip}'(d_{gn}) \leq 2e^{n\lambda_2} (l_0 + \text{Lip}'(dg_0)) \).

(c) Let \( 0 \leq k < n \) and let \( u_n^i \in \tilde{B}_{f_n}^+ (\delta c_n^{-1}) \), \( i = 1, 2 \). Then \( u_n^i + g_n(u_n^i) = \tilde{f}_{n,k}^{-n-k} (u_k^i + g_k(u_k^i)) \) for some \( u_k^i \in \tilde{B}_{f_k}^+ (\delta c_k^{-1}) \), and

\[
|u_n^1 - u_n^2|_{f_n,x} \geq (e^{\lambda^+} - \delta)^{n-k} |u_k^1 - u_k^2|_{f_k,x}.
\]

As \( E^+ \) is finite-dimensional, the proof of Lemma 3.3 follows from standard graph transform arguments which we summarize here:

1. Even though \( \mathcal{df}_x|_{E^+} \) may be contracting, the presumptive domain \( \pi^+_{f,x} \circ f_x(\text{graph}(g_0)) \) of the graph transform \( g_1 \) contains \( \tilde{B}^+ (\delta c_1^{-1}) \) because \( \lambda_c < \lambda^+ \); the same comment applies to all subsequent steps.

2. Since graph \( g_0 \) need not pass through 0, it must be checked that each graph \( g_n \) sits inside the diminished chart at \( f^n(x) \); this is ensured because there is a point \( z \in \text{graph} \cap \text{graph} h_x \), where \( W^{ss \text{loc,}_x} = \exp_x \text{graph} h_x \), and \( f_n(z) \) tends to 0 much more quickly than the rate at which chart sizes shrink, i.e., \( \lambda_c > \lambda^- \).

Though not yet justified at this point, we will refer to the manifolds \( \exp_{f^n,x}(\text{graph}(g_n)) \) where the \( g_n \) are as in Lemma 3.3 as transversals to the \( W^{ss} \)-foliation.

We record below two properties of the sequence of transversals defined by \( \{g_n\} \). The first says that they become increasingly “flat” in a sense to be made precise, and the second gives a distortion estimate on the dynamics restricted to these transversals. The setting and notation are as in Lemma 3.3.

**Lemma 3.4.** The sequence of functions \( g_n \) has the property that

\[
\sup_{u \in \tilde{B}_{f_n}^+ (\delta c_n^{-1})} |(dg_n)u|_{f_n,x} \to 0 \quad \text{exponentially fast as } n \to \infty,
\]

with uniform bounds (independent of \( x \)) depending only on \( l_0, \lambda^+, \lambda^- \) and \( \text{Lip}'(dg_0) \).

**Proof.** Let \( z_0 \in \text{graph}(g_0) \cap \text{graph}(h_x) \), \( z_n = \tilde{f}_x^n(z_0) \), and write \( z_n = u_n + g_n(u_n) \). From \( z_n \in \text{graph} h_x \), it follows from contraction along \( W^{ss} \)-leaves and a standard graph transform argument that

\[
|dg_n(u_n)|' \lesssim e^{n(\lambda^- - \lambda^+ - \delta)},
\]

where \( \lesssim \) refers to inequality up to a multiplicative constant depending only on \( l_0 \). The lemma follows from this, together with the fact that \( \text{Lip}'(dg_n) \leq 2e^{n\lambda_2} (l_0 + \text{Lip}'(dg_0)) \) and the domain of \( g_n \) has diameter \( \delta c_0^{-1}e^{\lambda_-n} \), which shrinks faster than \( \text{Lip}'(dg_n) \) can grow. \( \square \)

**Lemma 3.5.** For any \( l, L \geq 1 \) there exists a constant \( D_{l,L} > 0 \) with the following property. Let \( x \in \Gamma, c_0 \geq l(x) \), and let \( g_0 \) be as in Lemma 3.3; set \( L_0 = \text{Lip}'(dg_0) \). Then, for any \( n \geq 1 \) and \( y^1, y^2 \in f^n(\exp_{f^n,x} \text{graph} g_n) \), we have the estimate

\[
\left| \log \frac{\det(df^n_{y^1}|T_{y^1}(\exp_x \text{graph} g_0))}{\det(df^n_{y^2}|T_{y^2}(\exp_x \text{graph} g_0))} \right| \leq D_{l(x),L_0} \cdot \max \{ (e^{-\delta_2} (e^{\lambda^+} - \delta))^{-n}, 1 \} \cdot |f^n y^1 - f^n y^2|,
\]

where \( T_y W \) denotes the tangent space to the manifold \( W \) at \( y \).
Proof. The case \( \lambda^+ > 0 \) follows verbatim from the proof of Proposition 5.8 in [3]. In the case \( \lambda^+ < 0 \), Lemma 3.5 follows from similar arguments to those in [3]. The only substantive difference is that an expansion estimate along unstable manifolds is replaced with the following ‘weak contraction’ estimate along transversals (c.f. Lemma 3.3, item (c)):

\[
|z_k^1 - z_k^2|^f_{k,x} \leq (e^{\lambda^+} - \delta)^{(n-k)}|z_n^1 - z_n^2|^f_{n,x}
\]

for \( 0 \leq k < n \); here we have written \( z_i^j = \exp^{-1}_x f^k y^i \) for \( i = 1, 2 \) and \( 0 \leq k \leq n \).

Another difference is that the constant \( D \) appearing in the distortion estimate depends now on the Lipschitz constant \( L_0 \) of \( dg_0 \); this, however, does not substantially change the arguments in [3]. \( \square \)

3.3 Continuity of holonomy maps along the \( W^{ss} \) “foliation”

In preparation for the proof of absolute continuity of holonomy maps, we first establish their continuity, which we carry out in some detail, following the outline below:

Step 1. continuity of \( x \mapsto E_x^- \) (Lemma 3.6)
Step 2. continuity of \( k \)-step backward graph transforms (Lemma 3.7)
Step 3. continuity of \( x \mapsto W^{ss}_{loc,x} \) (Lemma 3.8), and finally
Step 4. continuity of holonomy maps along \( W^{ss}_{loc} \)-leaves (Lemma 3.9)

We begin with the continuity of the distribution \( E^- \). For \( l_0 > 1 \) we write

\[
\Gamma_{l_0}^+ = \{ x \in \Gamma : |df^n_x|_{E_x^-} \leq l_0 e^{n\lambda^-} \text{ for all } n \geq 0 \};
\]

the sets \( \Gamma_{l_0}^+ \) are referred to as forward uniformity sets, as they only detect information along forward trajectories.

**Lemma 3.6.** Let \( l_0 > 1 \) be fixed. Then, \( x \mapsto E_x^- \) varies continuously in the Hausdorff metric \( d_H \) as \( x \) varies in \( \Gamma_{l_0}^+ \).

Observe by Lemma 2.8 that \( \Gamma_{l_0} \subset \Gamma_{l_0}^+ \) for any \( l_0 > 1 \), and so Lemma 3.6 implies the continuity of \( x \mapsto E_x^- \) across the uniformity sets \( \Gamma_{l_0} \) as well.

**Proof of Lemma 3.6.** Let \( x^n \to x \) be a convergent sequence in \( \Gamma_{l_0}^+ \). To show \( d_H(E_{x^n}^-, E_x^-) \to 0 \), we will prove \( \text{Gap}(E_{x^n}^-, E_x^-) \to 0 \) as \( n \to \infty \), where \( \text{Gap} \) is as defined in Sect. 2.1 (see Lemma 2.4). Assume the contrary. Then there exists a sequence of unit vectors \( v^n \in E_{x^n}^- \) such that, writing

\[
v^n = w^{n,+} + w^{n,-}
\]

according to the splitting \( B = E^+ \oplus E^x_- \), we have \( |w^{n,+}| \geq c \) for some constant \( c > 0 \).

We use the shorthand \( x_k = f^k x, x^n_k = f^k x^n \). Then for arbitrary \( k, n \),

\[
(*) := |df^k_x v^n| \geq |df^k_x w^{n,+}| - |df^k_x w^{n,-}|
\]

\[
\geq \frac{1}{2} l(x)^{-1} e^{k(\lambda^- - \delta_2)} - l_0 e^{k\lambda^-} \| \pi_x^- \|
\]

after carrying out the change of norms and using \( x \in \Gamma_{l_0}^+ \). On the other hand, since \( x^n \in \Gamma_{l_0}^+ \) and \( v_n \in E_{x^n}^- \),

\[
(*) = |df^k_x v^n| \leq |(df^k_x - df^k_{x^n}) v^n| + |df^k_{x^n} v^n| \leq \| df^k_x - df^k_{x^n} \| + l_0 e^{k\lambda^-}.
\]

Taking the limit as \( n \to \infty \), we have shown that for all \( k \),

\[
(1 + \| \pi_x^- \|) l_0 e^{k\lambda^-} \geq \frac{1}{2} cl(x)^{-1} e^{k(\lambda^- - \delta_2)}.
\]

For \( k \) large enough, this is a contradiction. \( \square \)
Next we treat the continuity of backward graph transforms. Let $\mathcal{G}_x$ be as defined in Sect. 3.1. It is easy to see that this transform can be extended to the set of functions

$$\mathcal{W}_{\text{ss}}^{3.3}(x) := \{ h \in \widetilde{B}_x^-(\delta l(x^{-1}) \rightarrow \widetilde{E}_x^+ : (i) \text{ Lip}(h) < \frac{1}{10} \text{ and } (ii) \exists \tilde{z}, \text{ with } |\tilde{z}|_{f^{-1}-x}, |\tilde{z}|_{x} < \frac{1}{10}\delta l(x^{-1}) \text{ s.t. } \tilde{f}_{f^{-1}-x} \tilde{z} = z \in \text{ graph } h \}.$$  

The following notation will be used for ‘chart switching’: Let $x, y \in \Gamma$ and let $\phi_y : \text{ Dom}(\phi_y) \rightarrow E_y^+$ be a Lipschitz map, where $\text{ Dom}(\phi_y) \subset E_y^-$. We write $\phi_y^x : \text{ Dom}(\phi_y^x) \rightarrow E_x^+$ for the map, if it exists, for which

$$\exp_y \text{ graph } \phi_y = \exp_x \text{ graph } \phi_y^x.$$

Lemma 3.7. Let $x, y^n \in \Gamma_{l_0}$ be such that $y^n \rightarrow x$ as $n \rightarrow \infty$, and fix arbitrary $k \in \mathbb{Z}^+$. Writing $x_k = f^k x$ and $y^n_k = f^k y^n$, we let $O_{x_k} : \mathcal{E}_{x_k}^- \rightarrow \mathcal{E}_{x_k}^+$ and $O_{y^n_k}^k : \mathcal{E}_{y^n_k}^- \rightarrow \mathcal{E}_{y^n_k}^+$ be the functions that are identically equal to zero. Then for all large enough $n$, $O_{y^n_k}^k : \mathcal{E}_{y^n_k}^- \rightarrow \mathcal{E}_{y^n_k}^+$ are defined, as are the backward graph transforms

$$\mathcal{G}_x^k(O_{y^n_k}^k) := \mathcal{G}_x \circ \cdots \circ \mathcal{G}_{x_{k-1}} \circ \mathcal{G}_{x_k}(O_{y^n_k}^k),$$

as mappings $\widetilde{B}_x^-(\delta l_0^{-1}) \rightarrow \mathcal{E}_{x_k}^+$; moreover, $\|\mathcal{G}_x^k(O_{y^n_k}^k) - \mathcal{G}_x^k(O_{y^n_k}^k)\|_{C^0} \rightarrow 0$ as $n \rightarrow \infty$, where $\| \cdot \|_{C^0}$ is taken with respect to the adapted norm $\| \cdot \|_x$.

Proof. As $E^-$ is continuous on $\Gamma_{l_0 e^{\delta l_2}}$, the well-definedness of $O_{y^n_k}^k$ and $\mathcal{G}_x^k(O_{y^n_k}^k)$ for large enough $n$ is clear, and the only statement that requires a proof is the last statement on $C^0$ convergence. For this it suffices to prove the continuity of the backward graph transform for one step. For definiteness, let us work with $\Gamma_{l_1}$. We will show that for $h_1, h_2 \in \mathcal{W}_{\text{ss}}^{3.3}(x_1)$, we have

$$\|\mathcal{G}_{x_1}(h_1) - \mathcal{G}_{x_1}(h_2)\|_{C^0} \leq C\|h_1 - h_2\|_{C^0} \text{ for } C = 2e^{-\lambda^+}.$$

Fix arbitrary $v \in \mathcal{E}_{x_1}^-(\delta l_1^{-1})$, and let $z_i = (\mathcal{W}_{x_1}(h_i))(v + v, i = 1, 2$. We omit the subscripts in $\tilde{f}_x, \pi_x^+$, $| \cdot |_x$ etc. when they are obvious from context. By Lemma 3.3(b),(c),

$$|\pi^-(\tilde{f}_2 - \tilde{f}_1)| \leq \frac{1}{10}|\pi^+(\tilde{f}_2 - \tilde{f}_1)| \quad \text{ and } \quad |\pi^+(\tilde{f}_2 - \tilde{f}_1)| \geq (e^{\lambda^+} - \delta)|z_1 - z_2| \quad .$$

Let $w = \pi^+\tilde{f}_1$. Then

$$|h_2(w) - h_1(w)| \geq |\pi^+(\tilde{f}_2 - \tilde{f}_1)| - |h_2(w) - \pi^+(\tilde{f}_2)|$$

$$\geq |\pi^+(\tilde{f}_2 - \tilde{f}_1)| - \frac{1}{10}|\pi^-(\tilde{f}_2 - \tilde{f}_1)| \quad \text{ since Lip}(h_2) < \frac{1}{10}$$

$$\geq \frac{99}{100}|\pi^+(\tilde{f}_2 - \tilde{f}_1)| \quad \text{ by (9)} .$$

Using (9) again, we conclude from this that

$$|z_1 - z_2| \leq 2e^{-\lambda^+}|h_2(w) - h_1(w)| \leq 2e^{-\lambda^+}\|h_1 - h_2\|_{C^0} .$$

The next lemma defines what we will refer to as a stack of strong stable leaves. Below, we write $B^\pm_x(r) = \{ v \in E^\pm_x : |v| \leq r \}$ (note the difference between $\tilde{B}$ and $B$).

Lemma 3.8. Let $l_0$ and $n_0 > 1$ be fixed, and fix $x_0 \in \Gamma_{l_0} \cap K_{n_0}$ ($K_n$ as in the end of Section 2.2). For $\epsilon > 0$ and $x \in \Gamma$, we let

$$U(x, \epsilon) := \Gamma_{l_0} \cap K_{n_0} \cap \{ y : |x - y| < \epsilon \} ,$$

and let $\{ h_x \}_{x \in \Gamma}$ be as in Theorem 3.1. Then, for any $\delta \leq \frac{1}{4}\delta_1'$, there exists $\epsilon_0 > 0$ sufficiently small so that the following hold:
(a) For any \( y \in U(x_0, \epsilon_0) \), the map \( h^{x_0}_y \) is defined on \( B^{-}_{x_0}(\delta l^{-3}_0) \) with \( \operatorname{Lip}(h^{x_0}_y|_{B^{-}_{x_0}(\delta l^{-3}_0)}) \leq \frac{1}{10} \).

(b) The mapping \( \Theta : U(x_0, \epsilon_0) \rightarrow C^0(B^{-}_{x_0}(\delta l^{-3}_0), B^{+}_{x_0}(\delta l^{-3}_0)) \) defined by setting \( \Theta(y) = h^{x_0}_y|_{B^{-}_{x_0}(\delta l^{-3}_0)} \) is continuous in the uniform norm.

**Proof.** (a) follows from the continuity of \( h^{x_0}_y \) and Theorem 3.1; the extra copies of \( l^{-1}_0 \) come from norm changes and the reduction of domain size to keep the graphs “flat”. For more detail see the proof of Lemma 5.5 in [3].

For (b), we fix \( x, y^n \in U(x_0, \epsilon_0) \) with \( y^n \rightarrow x \) as \( n \rightarrow \infty \). To prove the continuity of \( \Theta \) at \( x \), it suffices to show that given any \( \gamma > 0 \), when restricted to \( B^{-}_{x}(2\delta l^{-3}_0) \) we have \( \|h^{x_0}_y - h^{x}_z\| < \gamma \) for all large enough \( n \) (here \( \| \cdot \| \) refers to the uniform \( (C^0) \) norm taken using the standard norm \( | \cdot | \) on \( \mathcal{B} \)). For \( k \in \mathbb{Z}^+ \), write \( x_k = f^k z \) and \( y_k^n = f^k y_n \). Since \( x_k, y_k^n \in \Gamma_{\log^k\mathcal{S}_2} \), we have, by Lemma 3.6, \( E^{-}_{y_k^n} \rightarrow E^{-}_{x_k} \) as \( n \rightarrow \infty \). (We could not have concluded this from the continuity of \( E^{-} \) on \( K_{n_0} \) alone because we may have \( df_{x}^{-}(E^{-}_{x}) \nsubseteq E_{f^{-}x}^{-} \).) Using the notation in Lemma 3.7, we have that

\[
\|h^{x_0}_y - h^{x}_z\| \leq \|h^{x_0}_y - \mathcal{G}^k_{x_k}(0_{x_k})\| + \|\mathcal{G}^k_{x_k}(0_{x_k}) - \mathcal{G}^k_{x_k}(0_{x_k})\| + \|\mathcal{G}^k_{x_k}(0_{x_k}) - h^{x}_z\|.
\]

From Lemma 3.2 and the uniform equivalence of \( | \cdot | \) and \( | \cdot |' \) norms on uniformity sets, we have that the first and third terms above are \( < \gamma/3 \) for \( k \) sufficiently large. Fixing one such \( k \), Lemma 3.7 tells us that the middle term is \( < \gamma/3 \) for \( n \) large enough, completing the estimate. \( \square \)

Letting \( \bar{U} \subset U(x_0, \epsilon_0) \) be any compact subset, we refer to

\[
\Sigma = \bigcup_{x \in \bar{U}} \exp x_0 \text{graph} \Theta(x)
\]

as a stack of strong stable leaves. We remark that for \( x, y \in \bar{U} \), either \( \Theta(x) = \Theta(y) \) or \( \Theta(x) \cap \Theta(y) = \emptyset \). This follows easily from Theorem 3.1(d).

We finish with a lemma on the continuity of holonomy maps.

**Lemma 3.9.** Let \( \Sigma = \bigcup_{x \in \bar{U}} \exp x_0 \text{graph} \Theta(x) \) be as above. For \( i = 1, 2 \), we let

\[
\Sigma^i = \exp x_0 \text{graph} \sigma^i, \quad \text{where} \quad \sigma^i : B^{-}_{x_0}(\delta l^{-3}_0) \rightarrow B^{-}_{x_0}(\delta l^{-3}_0)
\]

has \( \operatorname{Lip}(\sigma^i) \leq \frac{1}{10} \), and let

\[
\Sigma^i = \Sigma \cap \Sigma^i.
\]

Then the holonomy map \( p : \Sigma^1 \rightarrow \Sigma^2 \) defined by letting \( p(z) \) be the unique point in \( \exp x_0 (\text{graph} \Theta(x)) \cap \Sigma^2 \) for \( z \in \exp x_0 (\text{graph} \Theta(x)) \cap \Sigma^1 \) is a homeomorphism.

**Proof.** Define \( \psi_1 : \bar{U} \rightarrow \Sigma^1 \) by \( \{\psi_1(x)\} = \exp x_0 \text{graph} \Theta(x) \cap \Sigma^1 \). First we observe that \( \psi_1 \) is continuous by Lemma 3.8(b). In more detail, if \( x, y^n \in \bar{U} \) and \( y^n \rightarrow x \),

\[
|\psi_1(y^n) - \psi_1(x)| \leq |\pi_{x_0}^+(\psi_1(y^n) - \psi_1(x))| + |\pi_{x_0}^-(\psi_1(y^n) - \psi_1(x))| := A^+ + A^-,
\]

with \( A^- \leq \frac{1}{10} A^+ \) by the condition on \( \operatorname{Lip}(\sigma^i) \). Letting \( w = \pi_{x_0}^\pm \psi_1(x) \), we also have, by the condition on \( \operatorname{Lip}(\Theta(y^n)) \) in Lemma 3.8(a),

\[
|\Theta(y^n)w - \Theta(x)w| \geq A^+ - \frac{1}{10} A^-.
\]

Thus \( |\Theta(y^n)w - \Theta(x)w| \rightarrow 0 \) as \( n \rightarrow \infty \) implies \( |\psi_1(y^n) - \psi_1(x)| \rightarrow 0 \). Introducing the equivalence relation \( \sim \) on \( \bar{U} \) where \( x \sim y \) iff \( \Theta(x) = \Theta(y) \), \( \psi_1 \) gives rise to a continuous map \( \tilde{\psi}_1 : (\bar{U} / \sim) \rightarrow \Sigma^1 \), which is injective and therefore a homeomorphism by the compactness of \( \bar{U} \). Defining \( \tilde{\psi}_2 : (\bar{U} / \sim) \rightarrow \Sigma^2 \) analogously, we have that \( p = \tilde{\psi}_2 \circ \tilde{\psi}_1^{-1} \) is a homeomorphism. \( \square \)
4 Local version of absolute continuity result

In Sect. 4.1, we formulate the precise statement of Theorem A and give an outline of the proof. Details are given in Sects. 4.2 and 4.3.

4.1 Theorem A: precise formulation and outline of proof

Setting. Let \((f, \mu)\) be as in Section 1, satisfying (H1)–(H3). We fix \(\Gamma = \Gamma(\lambda^*; m, p)\) with \(\lambda^* < 1/2p\) and \(\mu(\Gamma) > 0\). The setup consists of a stack \(S\) of strong stable manifolds and a pair of transversals \(\Sigma^1\) and \(\Sigma^2\) to the leaves of \(S\). More precisely, we apply the constructions of Section 2.3. Fix \(l_0 > 1\) and \(n_0 \in \mathbb{N}\). Let \(x_0 \in \Gamma_{l_0} \cap K_n\) (as in the end of Section 2.2). We fix \(\delta \leq \frac{1}{4}\delta_1\) small enough for the results in Section 3 to apply. Let \(\epsilon_0 > 0\) is as in Lemma 3.8, and let \(S\) be the stack of strong stable leaves defined as in (10) through points in a compact set \(\bar{U} \subset U(x_0, \epsilon_0) \subset \Gamma_{l_0} \cap K_n\). For the transversals, we let \(\Sigma^i = \exp_{x_0}(\text{graph } \sigma_i), i = 1, 2, \) where \(\sigma_i : B^+_{x_0}(2\delta_1^{-3}) \to B^+_{x_0}(\frac{1}{4}\delta_1^{-4})\) are \(C^{1+\text{Lip}}\) maps satisfying \(\text{Lip}(\sigma_i) \leq \frac{1}{\mu_{l_0}}\). These conditions ensure that for all \(x \in \bar{U}\), \(g_i^0 := \sigma_i^*|_{\Sigma^i} \exp_{x_0}(\text{graph } \sigma_i)\) satisfies the assumptions in Lemma 3.3 (see Lemma 4.3). As in Lemma 3.9, we define \(\Sigma^i := \Sigma^i \cap S\), and let \(p : \tilde{\Sigma}^1 \to \tilde{\Sigma}^2\) be the holonomy map.

Theorem A In the setting above, assume that \(\nu_{\Sigma^1}(\tilde{\Sigma}^1) > 0\). Then the holonomy map \(p\) is absolutely continuous with respect to the induced volumes \(\nu_{\Sigma^1}\) and \(\nu_{\Sigma^2}\) restricted to \(\tilde{\Sigma}^1\) and \(\tilde{\Sigma}^2\), respectively. Moreover, \(p\) has uniformly bounded Jacobian, i.e., there exists a constant \(C > 0\) with the property that for any Borel set \(A \subset \tilde{\Sigma}^1\),

\[ C^{-1}\nu_{\Sigma^1}(A) \leq \nu_{\Sigma^2}(p(A)) \leq C\nu_{\Sigma^1}(A). \]

The goal of Section 4 is to prove this result. Section 5 proves an explicit formula for the Radon-Nikodym derivative of \(p\).

Remark 4.1. Theorem A has been proved a number of times for diffeomorphisms of finite dimensional Riemannian manifolds [16, 15, 9]. We remark on some of the more significant issues in passing from these settings to Banach space maps. The first is that a priori there is no notion of volume on transversals in Banach spaces. There is, however, a well defined measure class, namely that generated by Haar measure on finite dimensional subspaces, and that is adequate for the definition of absolute continuity for \(W^{ss}\)-foliations, but not for statements on Radon-Nikodym derivatives of holonomy maps. A second issue is that in the proofs, one needs to compare Jacobians of high iterates of the map, at different phase points and restricted to different subspaces. This requires not only the introduction of volume elements on finite dimensional subspaces (which we have done in [3]) but proofs of regularity of volume elements and determinants as subspaces are varied; Proposition 2.2 is in this spirit. Finally, as we will see, absolute continuity of the \(W^{ss}\)-foliation ultimately boils down to one’s ability to pass “round balls”, or sets with nice geometries, between nearby, roughly parallel transversals (see, e.g., [16, 9, 20]). Banach-space geometry is not always nice; indeed in some Banach spaces, \(x \mapsto |x|\) is not even differentiable. Finite dimensional techniques such as overcovering by round balls with controlled intersections have no obvious analogs in Banach spaces.

We have found that it is technically simpler to work with the following surrogate for balls on embedded submanifolds.

Definition 4.2. Let \(W \subset \mathcal{B}\) be an embedded submanifold. For \(x \in W\) and \(r > 0\) we define the \(\Omega\)-ball of radius \(r\) in \(W\) centered at \(x\) to be

\[ \Omega_W(x, r) = \text{the connected component of } W \cap \{y \in \mathcal{B} : |x - y| \leq r\} \text{ containing } x. \]
We are primarily interested in the case where $W$ is a finite dimensional embedded $C^1$ submanifold and $r > 0$ is very small.

**Notation:** Below and throughout Section 4 we use the shorthand $\Sigma^i_n = f^i \Sigma^i$, $\Sigma^i_{n+1} = f^i \Sigma^i_n$, and write $p_n : \Sigma^i_n \to \Sigma^i_{n+1}$ for the conjugated holonomy $p_n := f^{-n} \circ p \circ f^n$. The symbols $\lesssim$, $\gtrsim$ denote $\leq$, $\geq$, respectively, up to a multiplicative constant independent of $n$ (but perhaps depending on $l_0$); the symbol $\approx$ means that both of $\lesssim$ and $\gtrsim$ hold.

**Outline of proof:** It suffices to show there exists a constant $C > 0$ such that for every compact set $A \subset \bar{\Sigma}^1$, we have $\nu_{\Sigma^1}(p(A)) \leq C \nu_{\Sigma^1}(A)$. This is because all bounded Borel sets can be approximated from the inside by compact sets, and the other inequality can be obtained by reversing the roles of $\Sigma^1$ and $\Sigma^2$. Let $A$ be given, and let $\mathcal{O} \supset A$ be an open neighborhood for which $\nu_{\Sigma^1}(\mathcal{O}) \leq 2 \nu_{\Sigma^1}(A)$. We will show that for some large $n$, there is a collection of open $\Omega$-balls $\{\Omega_1, \Omega_2, \ldots, \Omega_M\}$ of $\Sigma^1_n$ for which $\{f^{-n} \Omega_i\}$ has the following properties:

(a) $A \subset \bigcup_i f^{-n} \Omega_i \subset \mathcal{O}$;

(b) $\sum_i \nu_{\Sigma^1}(f^{-n} \Omega_i) \lesssim \nu_{\Sigma^1}(\bigcup f^{-n} \Omega_i)$;

(c) $\nu_{\Sigma^1}(p(\Sigma^1_n \cap f^{-n} \Omega_i)) \lesssim \nu_{\Sigma^1}(f^{-n} \Omega_i)$ for each $i$.

From (a)–(c), it follows immediately that

$$
\nu_{\Sigma^1}(p(A)) \leq \sum_i \nu_{\Sigma^1}(p(\Sigma^1_n \cap f^{-n} \Omega_i)) \\
\lesssim \sum_i \nu_{\Sigma^1}(f^{-n} \Omega_i) \quad \text{by (c)}
\lesssim \nu_{\Sigma^1}(\bigcup f^{-n} \Omega_i) \quad \text{by (b)}
\lesssim \nu_{\Sigma^1}(\mathcal{O}) \leq 2 \nu_{\Sigma^1}(A) \quad \text{by (a) and the choice of } \mathcal{O},
$$

giving the desired result.

To complete the proof, then, it suffices to produce $\{\Omega_1, \ldots, \Omega_M\}$ with properties (a)–(c) above, and to be sure that the constants in “$\lesssim$” are independent of $A$. In the proof to follow, $\Omega_i$ will be chosen to be $\Omega$-balls (in the sense of Definition 4.2), and they will be of the form $\Omega_{\Sigma^1_i}(f^n y_i, e^{n \lambda_b})$ for suitable choices of $y_i \in \bar{\Sigma}^1$. Here $\lambda_b < 0$ is a new lengthscale satisfying

$$
\lambda^- < \lambda_b < \lambda_c < \lambda^+.
$$

We assume $\lambda_b$ is fixed and bounded away from $\lambda^-$ and $\lambda_c$ by small numbers to be specified.

### 4.2 Holonomies of “large” $\Omega$-balls

To prove Theorem A, we need to show $\nu_{\Sigma^1}(p(A)) \approx \nu_{\Sigma^1}(A)$ for all Borel subsets $A \subset \bar{\Sigma}^1$. We consider in this subsection a situation where $A$ is an $\Omega$-ball the radius of which is much larger than the distance between the two transversals, and explain how that is relevant to the original problem.

**Lemma 4.3.** The following hold with uniform bounds for all $x \in \bar{U}$. For $i = 1, 2$ let $g^i_0 = \sigma_i^1 |_{\bar{B}^+(\delta l_0^{-3})}$ (using the notation just before Lemma 3.7). Then, $g^i_0$ has range contained in $\bar{B}^+(\delta l_0^{-3})$, and we have the estimates $\text{Lip}^i(g^i_0) \leq 1/10$ and $\text{Lip}(dg^i_0) \leq 5 l_0 \text{Lip}(d\sigma^i)$. Here, $\text{Lip}^i$ refers to the adapted norm $|\cdot|_x^i$ at $x$. 



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The proof of Lemma 4.3 follows from the considerations in Section 5.2 in [3], which we do not repeat here. Lemma 4.3 permits us to apply forward graph transforms as in Lemma 3.3, with \( c_0 = l_0, c_k = e^{-k\lambda)c_0} \), to obtain the graph transform sequence \( \{g^i_k : \tilde{B}^+_{f^i_k x}(\delta c^{-1}_k) \rightarrow \tilde{B}^+_{f^i_k x}(\delta c^{-1}_k)\}_{k \geq 0} \) in the charts system along the trajectory \( \{f^n x\}_{n \geq 0} \) for any \( x \in \bar{U} \).

In what follows we will use the notation \( \Sigma^i_{n,x} := \exp_{f^n x} \text{graph } g^i_0 \), and will be comparing \( \Omega \)-balls in \( \Sigma^i_{n,x}, i = 1, 2 \) for some large \( n \). In addition to other quantities to be specified, it should be assumed throughout that the choice of \( n \) will depend implicitly on \( \delta, l_0 \) and \( \text{Lip}(d\sigma) \) (but it must not depend on \( x \in \bar{U} \)).

In the rest of Sect. 4.2, we fix attention on an arbitrary \( y^1 \in \Sigma^1 \). We write \( p(y^1) = y^2 \) and \( y^n_i = f^n y^i \), and let \( x \in \bar{U} \) be the corresponding point for which \( y^i \in W^s_{\text{loc},x} \) for \( i = 1, 2 \). We first establish that sets of the form \( \Omega_{\Sigma^i_{n}}(y^n_i, 10e^{n\lambda_b}) \) lie well inside the domains of the charts system along \( \{f^n x\} \).

**Lemma 4.4.** For all \( n \) large enough, we have

(i) \( \Omega_{\Sigma^1_{n}}(y^n_1, 10e^{n\lambda_b}) \subset \Sigma^i_{n,x} \)

(ii) \( \text{diam}(f^{-n}\Omega_{\Sigma^1_{n}}(y^n_1, e^{n\lambda_b})) \rightarrow 0 \) as \( n \rightarrow \infty \).

**Proof.** (i) Let \( \hat{y}^i \in \Omega_{\Sigma^1_{n}}(y^n_1, 10e^{n\lambda_b}) \). We estimate:

\[
|\hat{y}^i - f^n x|_{f^n x} \leq |\hat{y}^i - y^n_1|_{f^n x} + |y^n_1 - f^n x|_{f^n x} \leq l(f^n x)|\hat{y}^i - y^n_1| + (e^{n\lambda^-} + \delta)^n|y^i - x|_x \\
\leq 10l_0 e^{n(\delta_2 + \lambda_b)} + (e^{n\lambda^-} + \delta)^n \cdot \delta c_0^{-1},
\]

which is \( \leq \frac{1}{2} \delta c_0^{-1} \) assuming \( e^{\lambda^-} + \delta < e^{\lambda_b} \) and \( \delta_2 + \lambda_b < \lambda_c \).

As for (ii), notice that for \( k = 1, 2, \ldots, n \), \( f^{-k}\Omega_{\Sigma^1_{n}}(y^n_1, 10e^{n\lambda_b}) \subset \Sigma^i_{n-k,x} \), so that \( \text{diam}(f^{-n}\Omega_{\Sigma^1_{n}}(y^n_1, 10e^{n\lambda_b})) \sim e^{n\lambda_b}(e^{-\lambda^+} + \delta)^n \), which tends to 0 as \( n \rightarrow \infty \). \( \Box \)

**Lemma 4.5.** For any \( \epsilon > 0 \), there exists \( n \) sufficiently large (depending on \( \epsilon \)) for which we have

\[
p_n(\Sigma^1_n \cap \Omega_{\Sigma^1_{n}}(y^n_1, e^{n\lambda_b})) \subset \Omega_{\Sigma^2_{n}}(y^n_2, (1 + \epsilon)e^{n\lambda_b})
\]

**Proof.** Let \( \hat{y}_n \in \Sigma^1_n \), and let \( \hat{x} \in \bar{U} \) be such that \( f^{-n}\hat{y}_n \in W^s_{\text{loc},\hat{x}} \). Then

\[
|\hat{y}_n - p_n(\hat{y}_n)| \leq 2|\hat{y}_n - p_n(\hat{y}_n)|_{f^n\hat{x}} \leq 2(\epsilon^{\lambda^-} + \delta)^n \cdot \delta c_0^{-1}.
\]

If, additionally, \( \hat{y}_n \in \Omega_{\Sigma^1_{n}}(y^n_1, e^{n\lambda_b}) \), then

\[
|p_n(\hat{y}_n) - y^n_2| \leq |p_n(\hat{y}_n) - \hat{y}_n| + |\hat{y}_n - y^n_1| + |y^n_1 - y^n_2| \leq e^{n\lambda_b} + 4\delta c_0^{-1}(e^{\lambda^-} + \delta)^n \leq (1 + \epsilon)e^{n\lambda_b}
\]

proving the containment. \( \Box \)

We consider next the relation between \( \Omega \)-balls in \( \Sigma^i_{n,x} \) and in \( E_{f^i_n x}^+ \). Define \( \Psi^i = \Psi^i_{n,x} : \Sigma^i_{n,x} \rightarrow E_{f^i_n x}^+ \) by \( \Psi^i := \pi^+_{f^i_n x} \circ \exp_{f^i_n x}^{-1} \).

**Lemma 4.6.** Fix \( \epsilon > 0 \). Then for all \( n \) sufficiently large (depending on \( \epsilon \)), we have

(i) \( \frac{1}{1 + \epsilon}|y - y'| \leq |\Psi^i(y) - \Psi^i(y')| \leq (1 + \epsilon)|y - y'| \) for all \( y, y' \in \Sigma^i_{n,x} \),

(ii) \( \frac{1}{1 + \epsilon} \leq \det(d\Psi^i) \leq 1 + \epsilon \).

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Proof. For (i), observe that \( y = f^n x + \Psi^i(y) + g^i_n \circ \Psi^i(y) \) and similarly for \( y' \), so that
\[
|y - y'| \leq (1 + \text{Lip } g^i_n) | \Psi^i(y) - \Psi^i(y') | \leq (1 + \epsilon) | \Psi^i(y) - \Psi^i(y') |
\]
on taking \( n \) sufficiently large so that \(|(d g^i_n)|_u| \leq \epsilon \) for \( u \in B^*_f n x (\delta c)^{-1} \) by Lemma 3.4 (note that we have passed from the adapted norm \(| \cdot | f n x \) in the conclusions of Lemma 3.4 to the standard norm \(| \cdot | \)); the lower bound works similarly.

For (ii), note that \( \Psi^i = (\text{exp}_{f n x} \circ (\text{Id} + g^i_n))^{-1} \). The desired estimate follows on applying Lemma 3.4 to make \(|d g^i_n| \) sufficiently small and on applying the simple bound \((1 - |V|)\dim E \leq \det(\text{Id} + V | E) \leq (1 + |V|)\dim E \) to \( V = d g^i_n, E = E^+_{f n x} \).

We summarize the results thus far a vis a vis the Outline of proof in Sect. 4.1: For given small \( \epsilon \) and \( n \) large enough depending on \( l_0, \text{Lip}(d \sigma^1) \) and \( \epsilon \), we have shown that \( \Omega^1 := \Omega_{\Sigma^1 n}(y^1_n, e^n \lambda^k) \) has the properties
\[
\begin{align*}
(i) & \quad f^{-n}(\Omega^1) \subset \mathcal{O} \quad \text{Lemma 4.3(ii)} \\
(ii) & \quad p_n(\Omega^1 \cap \Sigma^1) \subset \Omega_{\Sigma^1 n}(y^1_n, (1 + \epsilon)e^n \lambda^k) := \Omega^2 \quad \text{Lemma 4.4} \\
(iii) & \quad \Psi^i_n(\Omega^1) \supset \Omega_{E^+_n} \left( \Psi^i_n y_n, (1 + \epsilon)^{-1} e^n \lambda^k \right) \\
& \quad \text{and} \quad \Psi^2_n(\Omega^2) \subset \Omega_{E^+_n} \left( \Psi^2_n y_n, (1 + \epsilon)^2 e^n \lambda^k \right) \quad \text{Lemma 4.5(i)}
\end{align*}
\]

Notice that \( E^+_n \) is a linear subspace, and \( \Omega \)-balls in \( E^+_n \) are usual Banach space balls. Combining the above and using Lemma 4.5(ii), we obtain
\[
\nu_{\Sigma^1 n}(\Omega^2) \leq (1 + \epsilon) \cdot m_{E^+_n} \left( \Psi^2_n y_n, (1 + \epsilon)^2 e^n \lambda^k \right) = (1 + \epsilon) \cdot (1 + \epsilon)^{\dim E^+_n} \cdot m_{E^+_n} \left( \Psi^1_n y_n, (1 + \epsilon)^{-1} e^n \lambda^k \right) \leq (1 + \epsilon)^{2 + \dim E^+_n} \cdot \nu_{\Sigma^1 n}(\Omega^1).
\]

Here we have used the translation invariance and scaling properties of the induced volumes \( m_{E^+_n} \) on the linear subspaces \( E^+_n \) (see Sect. 2.1). The discussion above suggests that we take \( \{\Omega_i\} \) in the Outline to consist of sets of the form \( \Omega^1 \).

### 4.3 A cover by \( \Omega \)-balls

We construct here the cover \( \{\Omega_1, \cdots, \Omega_M\} \) of \( f^n(A) \) in the Outline in Sect. 4.1. We continue to use the notation from Sect. 4.2, but as we will be working exclusively with iterates of \( \Sigma^1 \), we will drop the superscript 1 in \( \Sigma^1_n \). We say a cover has multiplicity \( \leq C \) if no point is contained in more than \( C \) elements of the cover.

**Proposition 4.7.** Let \( A \subset \Sigma \) be compact, and let \( n \) be large enough that Lemma 4.5 holds with \( \epsilon = 1 \). Then there is a finite set \( S = S_{n,A} = \{y_i\}_{i=1}^M \subset f^n(A) \) with the property that \( \{\Omega_{\Sigma^1 n}(y_i, e^n \lambda^k)\}_{i=1}^M \) is a cover of \( f^n(A) \) with multiplicity \( \leq C_{m^+} \), where the constant \( C_{m^+} \) depends only on \( m^+ := \dim E^+ \).

**Proof.** Writing \( r = e^n \lambda^k \), we take \( S = \{y_i\}_{i=1}^M \) to be a \((\frac{\epsilon}{2})\)-maximal separated set in \( f^n(A) \), i.e.,
\[
\begin{align*}
(a) & \quad \text{for any } 1 \leq i < j \leq M, \quad \Omega_{\Sigma^1 n}(y_i, \frac{\epsilon}{2}) \cap \Omega_{\Sigma^1 n}(y_j, \frac{\epsilon}{2}) = \emptyset \\
(b) & \quad \text{for any } y \in f^n(A), \quad \Omega_{\Sigma^1 n}(y, \frac{\epsilon}{2}) \cap \Omega_{\Sigma^1 n}(y_i, \frac{\epsilon}{2}) \neq \emptyset \text{ for some } i \in \{1, \cdots, M\}.
\end{align*}
\]
That such a set exists and is finite follows from the compactness of \( f^n A \) and of \( \Sigma_n \) for all \( n > 0 \); details are left to the reader.

To complete the proof, we will show that (i) \( \{ \Omega_{\Sigma_n}(y_i,r) \}_{i=1}^M \) is a cover of \( f^n A \) and (ii) the multiplicity of this cover is bounded by a constant depending only on \( m^+ \).

That (i) holds follows from the following: Given \( y \in f^n A \), let \( y_i \) be given by property (b), and let \( z \in \Omega_{\Sigma_n}(y_i,\frac{r}{2}) \cap \Omega_{\Sigma_n}(y_i,\frac{r}{2}) \). Since \( y \) and \( z \) both lie in a connected component of \( \Sigma_n \cap \{ w \in B : |w - y| \leq \frac{r}{2} \} \), there is a continuous path in \( \Sigma_n \cap \{ w : |w - y| \leq \frac{r}{2} \} \) connecting \( y \) and \( z \). Likewise, there is a continuous path in \( \Sigma_n \cap \{ |w - y| \leq \frac{r}{2} \} \) connecting \( z \) and \( y_i \).

Concatenating these two paths, we obtain that \( y \in \Omega_{\Sigma_n}(y_i,r) \).

To prove (ii), for each \( i \) we let \( S_i = \{ j \in \{ 1, \ldots, M \} \setminus \{ i \} : \Omega_{\Sigma_n}(y_i,r) \cap \Omega_{\Sigma_n}(y_j,r) \neq \emptyset \} \).

Then the multiplicity of the cover \( \{ \Omega_{\Sigma_n}(y_i,r) \}_{i=1}^M \) is no worse than
\[
\max_{1 \leq i \leq M} \# S_i + 1 ,
\]
so it suffices to bound \( \# S_i \) by a constant depending only on \( m^+ = \dim E^+ \).

For fixed \( i \), we let \( x_i \in f^n \bar{U} \) be such that \( f^{-n} y_i \in W_{\text{loc},f^{-n}x_i} \), and recall that \( \Omega_{\Sigma_n}(y_i,10r) \subset \tilde{B}_{x_i}(\delta c^{-1}) \) (Lemma 4.4). Letting \( \Psi = \pi^+_x \circ \exp_{x_i}^{-1} \), we now pass from \( \Omega \)-balls in \( \Sigma_n \) to balls in \( E_{x_i}^+ \) via Corollary 4.6: For \( j \in S_i \), since \( \Omega_{\Sigma_n}(y_j,\frac{r}{2}) \subset \Omega_{\Sigma_n}(y_i,3r) \), we have \( \Omega_{E_{x_i}^+}(\Psi y_j,\frac{r}{4}) \subset \Omega_{E_{x_i}^+}(\Psi y_i,6r) \).

As the sets \( \Omega_{\Sigma_n}(y_j,\frac{r}{2}) \) are pairwise disjoint by property (a) above, so are the sets \( \Omega_{E_{x_i}^+}(\Psi y_j,\frac{r}{4}) \).

By volume count, the maximum number of such sets that can fit inside \( \Omega_{E_{x_i}^+}(\Psi y_i,6r) \) is no more than \( 24^{\dim E^+} \). We have thus shown that \( \# S_i \leq 24^{\dim E^+} \), completing the proof.

Summarizing what we have proved via a vis the Outline in Sect. 4.1: In addition to the requirement in Proposition 4.7, let \( n \) be chosen large enough that \( f^{-n}(\Omega_{\Sigma_n}(f^n z,e^{n\lambda_b})) \subset O \) for all \( z \in A \), and let \( \{ \Omega_i \} \) be the cover \( \{ \Omega_{\Sigma_n}(y_i,e^{n\lambda_b}) \}_{i=1}^M \) in Proposition 4.7. Then (a) and (b) in the Outline hold, the constant in “\( \lesssim \)” in (b) being the multiplicity of this cover.

### 4.4 Completing the proof

We continue to use the notation in Sects. 4.1 and 4.2. To prove the remaining item in the Outline, item (c), it suffices to prove the following.

**Proposition 4.8.** There exists \( D > 0 \) such that for all \( n \) sufficiently large and for all \( y \in \bar{\Sigma}^1 \),
\[
\nu_{\Sigma^2}(p(\bar{\Sigma}^1 \cap f^{-n}\Omega_{\Sigma_n}^1(f^n y,e^{n\lambda_b}))) \leq D \nu_{\Sigma^1}(f^{-n}\Omega_{\Sigma_n^1}(f^n y,e^{n\lambda_b})).
\]

**Proof.** By Lemma 4.5, it suffices to bound from above the ratio
\[
(*) \quad \frac{\nu_{\Sigma^2}(f^{-n}\Omega_{\Sigma_n^2}(f^n p(y),(1 + \epsilon) e^{n\lambda_b}))}{\nu_{\Sigma^1}(f^{-n}\Omega_{\Sigma_n^1}(f^n y,e^{n\lambda_b}))}.
\]

By the change of variables formula, \( \nu_{\Sigma_n^2}(f^{-n}\Omega_{\Sigma_n^2}(f^n y,e^{n\lambda_b})) \) is related to \( \nu_{\Sigma_n^1}(\Omega_{\Sigma_n^1}(f^n y,e^{n\lambda_b})) \) by the Jacobian of \( f^n |(f^{-n}\Omega_{\Sigma_n^1}(f^n y,e^{n\lambda_b})) \), and this in turn is related to the corresponding Jacobian at the point \( y \) by the distortion estimate in Lemma 3.5. Examining the estimate in Lemma 3.5, note that the \( \Omega \)-balls we consider have radius \( e^{n\lambda_b} \), which contracts faster than the contraction rate \( e^{n\lambda^+} \) along transversals. Thus for any fixed \( \epsilon > 0 \), we may take \( n \) sufficiently large (depending on \( \epsilon, l_0 \) and the Lipschitz constant \( L_0 = \text{Lip}'(dg_0) \)) so that the the right-hand side in Lemma 3.5 is \( \leq \log(1 + \epsilon) \).
Applying this estimate to both the numerator and denominator of (*)& and invoking Lemma 4.5, we obtain

$$(*) \leq (1 + \epsilon)^2 \cdot \frac{\nu_{\Sigma_2} \Omega_{\Sigma_2} (f^n p(y), (1 + \epsilon)e^{n\lambda_0})}{\nu_{\Sigma_1} \Omega_{\Sigma_1} (f^n y, e^{n\lambda_0})}.$$  

As Term II has been bounded at the end of Sect. 4.2, it remains to bound Term I. For $N \in \mathbb{N}$, we introduce the function $\Delta_N : \Sigma^1 \to [0, \infty)$ by

$$\Delta_N(y) := \frac{\det(df_{N,y}^n | T_{y} \Sigma^1)}{\det(df_{p,y}^n | T_{p(y)} \Sigma^2)} = \prod_{n=0}^{N-1} \frac{\det(df_{N,y}^n | T_{f^n y} \Sigma_1^n)}{\det(df_{p,y}^n | T_{f^n p(y)} \Sigma_1^n)}.$$  

To complete the proof of Proposition 4.8, it suffices to show that there exists $\hat{D}$ (depending only on $l_0$) such that $\hat{D}^{-1} \leq \Delta_N(y) \leq \hat{D}$ for all $y \in \hat{\Sigma}$ and for all $N \in \mathbb{Z}^+$. This follows from Lemma 4.9 below. \hfill \Box

We prove a stronger result than needed here, namely the existence of the $N \to \infty$ limit, which is needed in Section 5. Observe that $y \mapsto \Delta_N(y)$ is continuous in $y \in \hat{\Sigma}$ for any fixed $N > 0$ by the continuity of $p$ (Lemma 3.9) and the regularity of det (Proposition 2.2).

**Lemma 4.9.** For any $y \in \hat{\Sigma}$, the limit $\Delta(y) := \lim_{N \to \infty} \Delta_N(y)$ exists. The convergence $\Delta_N \to \Delta$ is uniform, so $y \mapsto \Delta(y)$ is therefore continuous. Moreover, there is a constant $\hat{D} = \hat{D}_{l_0} > 0$ (depending only on $l_0$) such that $\hat{D}^{-1} \leq \Delta(y) \leq \hat{D}$ for any $y \in \hat{\Sigma}$.

**Proof of Lemma 4.9.** Given $N$ and $y$, we define for $k \in \mathbb{Z}^+$

$$\Delta_{N,k}(y) = \frac{\det(df_{N,y}^k | T_{f^k y} \Sigma_1^k)}{\det(df_{p,y}^k | T_{p(y)} \Sigma_2^k)} = \prod_{n=0}^{k-1} \frac{\det(df_{N,y}^{n+1} | T_{f^{n+1} y} \Sigma_1^{n+1})}{\det(df_{p,y}^{n+1} | T_{f^{n+1} p(y)} \Sigma_2^{n+1})}.$$  

We will show there exists $D_1$ (depending on $l_0$ but not on $y$ or $N$) such that for all $k \geq 1$:

$$\Delta_{N,k}(y) \leq D_1 \cdot e^{N\left(\frac{1}{2}(\lambda^+ - \lambda^-) + 2\lambda_2\right)}.$$  

The proof of (12) relies on regularity properties of the determinant function, which should not be taken granted as our notion of volume on finite dimensional subspaces was defined one subspace at a time. We state formally the estimate used:

**Claim 4.10.** Let $\hat{x} \in \Gamma$, $\hat{y}^1, \hat{y}^2 \in \mathcal{D}_\hat{x}(\delta l(\hat{x})^{-1})$, and let $L_1, L_2 : E^+_{\hat{x}} \to E^-_{\hat{x}}$ be linear maps for which $|L_i|_{\hat{x}} \leq 1/10$ for $i = 1, 2$. Write $E_i = (\text{Id} + L_i)E^+_{\hat{x}}$. Then, we have the estimate

$$\left| \log \frac{\det(df_{\hat{y}^1}^q | E_1)}{\det(df_{\hat{y}^2}^q | E_2)} \right| \leq \text{Const.} \cdot l(\hat{x})^q \left( |\hat{y}^1 - \hat{y}^2| + |L_1 - L_2| \right),$$  

where $q \in \mathbb{N}$ depends only on $\dim E^+$.

To deduce this inequality from Proposition 2.2, observe that $M$ as in Proposition 2.2 is determined by $|\langle df_{\hat{y}^1}^q | E_1 \rangle|^{-1} \leq 2e^{-\lambda^+}l(\hat{x})$, and $\epsilon$ can be taken as small as need be by introducing intermediate points of the form $\check{y}^{1,\ell} := \hat{y}^1 + \ell\gamma(\hat{y}^2 - \hat{y}^1)$ and linear maps $L_{1,\ell} := L_1 + \ell\gamma(L_2 - L_1)$ for $\ell = 1, 2, \ldots, \gamma^{-1}, \gamma \ll 1$, and applying Proposition 2.2 to $\check{y}^{1,\ell}$ and $\check{y}^{1,\ell+1}$, $L_{1,\ell}$ and $L_{1,\ell+1}$. The constant on the right side of (13) follows from Remark 2.3 after Proposition 2.2.
Letting $x \in \bar{U}$ be such that $y \in W^{ss}_{loc,x}$ and using Claim 4.10 we estimate
\[
|\log \frac{\det(df_{f_N y}^k | T_{f_N y}^k \Sigma_N)}{\det(df_{f_N y}^k | T_{f_N y}^k \Sigma_N)}| \leq \sum_{n=0}^{k-1} \left| \log \frac{\det(df_{f_N y}^{n+y} | T_{f_N y}^{n+y} \Sigma_N)}{\det(df_{f_N y}^{n+y} | T_{f_N y}^{n+y} \Sigma_N)} \right|
\]
where $u_n^1 = \pi_{J_{N+n}}^+ \circ \exp_{J_{N+n}}^{-1}(f^{N+n}y)$ and $u_n^2 = \pi_{J_{N+n}}^+ \circ \exp_{J_{N+n}}^{-1}(f^{N+n+p}(y))$. To bound the RHS of (14), recall the estimates $l(f^{N+n}x) \leq e^{(N+n)\delta_2} l_0$ and $|f^{N+n}y - f^{N+n+p}(y)| \lesssim (e^{\lambda^+} + \delta)^{N+n}$. For the last term, the estimate (8) in the proof of Lemma 3.4 gives the bound
\[
|\langle df_{N+n} \rangle u_{N+n}^1 - \langle df_{N+n} \rangle u_{N+n}^2| \lesssim e^{(N+n)}(\lambda^- - \lambda^+) + \delta_2.
\]

Assuming, as we may, that $\delta_2 \ll \frac{1}{4} (\lambda^- - \lambda^+)$, the desired result follows. \hfill \Box

The proof of Theorem A is now complete.

5 Derivative computation

The setting is as in the beginning of Section 4. We now compute explicitly the Radon-Nikodym derivative of the holonomy map $p$.

**Theorem B** For all $y \in \Sigma^1$,
\[
\frac{dp^{-1} \nu_{\Sigma^1}}{dp} \bigg|_{\Sigma^1} (y) = \Delta(y)
\]
where $\Delta(y)$ is given by Lemma 4.9.

5.1 Outline of proof

As $y \mapsto \Delta(y)$ is continuous on $\Sigma^1$, by considering small enough sets on which $\Delta(y)$ is nearly constant, one deduces Theorem B from

**Proposition 5.1.** Let $A \subset \Sigma^1$ be compact. Then,
\[
\nu_{\Sigma^1}(p(A)) \leq \sup_{y \in A} \Delta(y) \cdot \nu_{\Sigma^1}(A).
\]

Below, we fix $\epsilon > 0$, to be regarded as acceptable error in our pursuit of the inequality in Proposition 5.1. As the main source of the overestimate in the proof of Theorem A comes from the overcovering by $\Omega$-balls of $f^N A$, we now replace this over-cover by a collection of pairwise disjoint sets. An important requirement for this new cover is that the volumes of its elements must be transformed nicely by holonomy maps, a property we have, up until now, proved only for $\Omega$-balls that are large in radius compared to the distance between transversals (Sect. 4.2).

**Construction of a special cover.** For $n$ sufficiently large, we apply Proposition 4.7 to obtain a cover $\{\Omega_i\}_{i=1}^M$ of $f^n A$ by balls of the form $\Omega_i = \Omega_{\Sigma^1}(y_i, e^{n\lambda_0})$, where $\{y_i\} \subset f^n A$. Writing $\alpha \Omega_i = \Omega_{\Sigma^1}(y_i, e^{n\lambda})$ for $\alpha > 0$, we define the collection $\{V_i\}_{i=1}^M$ of pairwise disjoint measurable sets that will comprise this special cover as follows:
\[
V_i = \Omega_i \setminus \left( \bigcup_{j \neq i} \Omega_j \cup \bigcup_{i < j} \frac{1}{2} \Omega_j \right).
\]

The following are immediate:
(i) \( \frac{1}{2} \Omega_i \subset V_i \subset \Omega_i \) for any \( 1 \leq i \leq M \), and
(ii) \( f^{-n}(\cup_i V_i) \supset A \).

Let \( S_i \) be as in the proof of Proposition 4.7, that is to say, \( S_i \) consists of those indices \( j \neq i \) such that \( \Omega_j \cap \Omega_i \neq \emptyset \). Observe that only those \( \Omega_j \) with \( j \in S_i \) are involved in the construction of \( V_i \), and that as shown in Proposition 4.7, the cardinality of \( S_i \) is bounded by a constant that depends only on \( m^+ = \dim E^+ \). In particular, it is independent of \( M \), which can grow exponentially with \( n \). As we will see, our control on the “geometry” of the sets \( V_i \) will depend crucially on this uniform bound on the cardinality of \( S_i \).

Continuing to allow dependence on \( \delta, l_0 \) and \( \text{Lip}(\sigma^i) \), our main estimate is the following:

**Lemma 5.2.** Assume that \( n \) is sufficiently large depending on \( \epsilon > 0 \). Then, for any \( 1 \leq i \leq M \) we have that
\[
\nu_{\Sigma^2_n}(p_n(V_i \cap \Sigma^1_n)) \leq (1 + \epsilon)\nu_{\Sigma^1_n}(V_i).
\]

The proof of Lemma 5.2 is deferred to the next subsection.

**Proof of Proposition 5.1 assuming Lemma 5.2.** Let \( \epsilon > 0 \) be given. We fix an open set \( \mathcal{O} \supset A \) with the property that \( \nu_{\Sigma^1_n}(\mathcal{O}\setminus A) \leq \epsilon \nu_{\Sigma^1_n}(A) \). The value of \( n \) will be increased a finite number of times as we go along. First we assume it is large enough that \( f^{-n} V_i \subset \mathcal{O} \) where \( \{V_i\} \) is as constructed above. We then bound \( \nu_{\Sigma^2_n}(p(A)) \) by
\[
\nu_{\Sigma^2_n}(p(A)) \leq \sum_{i=1}^{M} \nu_{\Sigma^2_n}(f^{-n}p_n(V_i \cap \Sigma^1_n)). \tag{15}
\]

As before, we have, from Lemma 3.5,
\[
\frac{\nu_{\Sigma^2_n}(f^{-n}p_n(V_i \cap \Sigma^1_n))}{\nu_{\Sigma^1_n}(f^{-n}V_i)} \leq (1 + \epsilon)^2 \Delta_n(f^{-n}y_i) \cdot \frac{\nu_{\Sigma^2_n}(p_n(V_i \cap \Sigma^1_n))}{\nu_{\Sigma^1_n}(V_i)},
\]
and assume \( n \) is large enough that \( \Delta_n \leq (1 + \epsilon)\Delta \) on \( \Sigma^1 \) (Lemma 4.9). Applying these inequalities together with Lemma 5.2 to the right side of (15) and summing, we obtain
\[
\nu_{\Sigma^2_n}(p(A)) \leq (1 + \epsilon)^3 \sup_{y \in A} \Delta(y) \cdot \nu_{\Sigma^2_n}(\mathcal{O}) \leq (1 + \epsilon)^4 \sup_{y \in A} \Delta(y) \cdot \nu_{\Sigma^1_n}(A).
\]
Taking \( \epsilon \to 0 \) completes the proof. \( \square \)

### 5.2 Proof of Lemma 5.2

For fixed \( n \) and \( i \), we let \( V^1_i = V_i \) be as defined in the last subsection, and extend this notation in the following ways:

(i) Let \( y^1_j = y_j, y^2_j = p_\nu(y^1_j) \), and define \( V^2_i \) analogously, with \( y^2_j \) in the place of \( y^1_j \).

(ii) For \( k = 1, 2 \) and \( \alpha > 1 \), we define
\[
V^k_i(\alpha) := \alpha \Omega^k_i \setminus \left( \bigcup_{j < i} \alpha^{-1} \Omega^k_j \cup \bigcup_{j > i} \frac{\alpha^{-1}}{2} \Omega^k_j \right). \tag{16}
\]
Notice that \( V^k_i(\alpha) \subset V^k_i(\alpha') \) for \( \alpha < \alpha' \), and for \( \alpha > 1 \), the sets \( V^k_i(\alpha) \) and \( V^k_j(\alpha) \) are not necessarily pairwise disjoint.
(iii) We consider next analogous constructions on $E^+$. Let $\Psi_i = \pi_{x_i} \circ (\exp_{x_i})^{-1}$ be projection to $E^+_x$ where $x_i$ is such that $f^{-n}x_i \in U$ and $y_j^k \in W_{\text{loc},x_i}$. Let $S'_i := \{ j : y_j^k \in \Omega_{\Sigma^1_i}(y_1^i, 3e^{\lambda n}) \}$. For $j \in S'_i$, we let $\Omega_j = \Omega_{E^+_x}(\Psi_i(y_j^k), e^{\lambda n_0})$, and define, for $\alpha \approx 1$,

$$\tilde{V}_i^k(\alpha) := \alpha \Omega_i^k \setminus \left( \bigcup_{j < i, j \in S'_i} \alpha^{-1} \tilde{\Omega}_j \cup \bigcup_{j > i, j \in S'_i} \frac{\alpha^{-1}}{2} \tilde{\Omega}_j \right).$$

Reasoning similar to those in Section 4 shows that for $n$ large enough, $j \in S'_i$ are the only indices involved in the definition of $\tilde{V}_i^k(\alpha)$, and that $\# S'_i \leq C''_{m+}$ for all $i$, where $C''_{m+}$ depends on $m^+ = \dim E^+$ alone. It is important to note that $\tilde{V}_i^k(\alpha)$ is not the $\Psi_i$-image of $V_i^k$, and that it is constructed using real balls in $E^+_x$, the centers of which are projections of those used in the construction of $V_i^k$.

**Proof of Lemma 5.2.** Let $\epsilon > 0$ be given. We first choose $\epsilon' = \epsilon(\epsilon) > 0$ and $\alpha = \alpha(\epsilon, \epsilon') > 1$ with $\epsilon', |\alpha - 1|$ sufficiently small, and then $n = n(\epsilon, \epsilon', \alpha)$ sufficiently large; exact dependences will become clear in the course of the proof. Let $\{V_i^1\}$ be a special cover of $f^n(A)$ as defined in Sect. 5.1. We assume $\{V_i^1\}$ is constructed from $\{\Omega_i^1\}$, where each $\Omega_i^1 = \Omega_{\Sigma^1_i}(y_1^i, e^{\lambda n})$, and let $i$ be fixed throughout. We will show that the assertion in Lemma 5.2 follows from the following sequence of approximations:

1. $p_n(V_i^1 \cap \Sigma^n_i) \subset V_i^2(\alpha)$,
2. $\Psi_i(V_i^1) \supset \tilde{V}_i^1(\alpha^{-1})$ and $\Psi_i(V_i^2(\alpha)) \subset \tilde{V}_i^2(\alpha^2)$ ,
3. $\tilde{V}_i^2(\alpha^2) \subset \tilde{V}_i^1(\alpha^3)$ ,
4. $m_{E^+_x}(\tilde{V}_i^1(\alpha^3)) \leq (1 + \epsilon') m_{E^+_x}(\tilde{V}_i^1(\alpha^{-1}))$.

Applying (1)-(4) in the order stated together with Lemma 4.6(ii), we obtain

$$\nu_{\Sigma^1_i}(p_n(V_i \cap \Sigma^n_i)) \leq \nu_{\Sigma^1_i}(V_i^2(\alpha)) \leq (1 + \epsilon') m_{E^+_x}(\tilde{V}_i^1(\alpha^2)) \leq (1 + \epsilon') m_{E^+_x}(\tilde{V}_i^1(\alpha^3)) \leq (1 + \epsilon')^2 m_{E^+_x}(\tilde{V}_i^1(\alpha^{-1})) \leq (1 + \epsilon')^3 \nu_{\Sigma^1_i}(V_i^1).$$

It remains to prove (1)-(4).

**Proof of (1).** Lemma 4.5 asserts that for $n$ large enough,

$$p_n(\Omega_i^1 \cap \Sigma^n_i) \subset \alpha \Omega_i^2.$$ 

A similar proof applied to $p_n^{-1}$ gives, for $j \in S_i$,

$$p_n^{-1}(\alpha^{-1} \Omega_j^1 \cap \Sigma^n_i) \subset \Omega_j^1 \quad \text{and} \quad \nu_{\Sigma^2_i}(p_n^{-1}(\alpha^{-1} \Omega_j^1 \cap \Sigma^n_i)) \subset \frac{1}{2} \Omega_j^1.$$ 

Combining these relations give the desired result.

**Proof of (2).** This follows from the bi-Lipschitz property of $\Psi_i$ wth Lipschitz constant $\approx 1$ (Lemma 4.6(i)). It implies in particular $\Psi_i(\Omega_i^1) \supset \alpha^{-1} \Omega_i^1$ and $\Psi_i(\Omega_j^1) \subset \alpha \Omega_j^1$ for $j \in S_i$, the latter being valid because $\Omega_i^1 \subset \Omega_{\Sigma^2_i}(y_1^i, 10e^{\lambda n})$. The second containment is proved similarly.

**Proof of (3).** It suffices to estimate $|\Psi_i(y_j^1) - \Psi_i(y_j^2)|$ where $j \in S'_i \cup \{ i \}$; the rest of the containments are as before. This quantity is equal to

$$|\pi_{x_i}(p_n(y_j^1) - y_j^1)| \leq |\pi_{x_i}||p_n(y_j^1) - y_j^1| \leq 2l_0 \epsilon^{\alpha \delta^2} \cdot 2\delta^2 c_0^{-1}(e^{\lambda n} - \delta)n,$$
which can be made arbitrarily small relative to $e^{\lambda n}$ by taking $n$ large.

**Proof of (4).** We will show

$$m_{E^+_i}(V_1^1(\alpha^3)) \setminus V_i^1(\alpha^{-1})) \leq e' m_{E^+_i}(V_i^1(\alpha^{-1})) . \quad (17)$$

Let $\omega$ denote the volume of the unit ball in $\mathbb{R}^{m^+}$ where $m^+ = \dim E^+$. Then the left side of (17) is bounded from above by

$$e^{\lambda n m^+} \omega[(\alpha^{3m^+} - \alpha^{-m^+}) + (#S_i^l)(\alpha^{m^+} - \alpha^{-3m^+})] .$$

As for the right side of (17), recall that we have made sure $V_i^1(\alpha') \supset \frac{1}{2} \bar{\Omega}_i^1$ for any $\alpha' > 1, \alpha' - 1 \ll 1$. Thus

$$m_{E^+_i}(V_i^1(\alpha^{-1})) \geq \left( \frac{1}{2} e^{\lambda n} \right)^{m^+} \omega ,$$

proving (17) provided $\alpha$ is sufficiently close to 1.

**Remark 5.3.** In the proof of (4) above, we have used the implicitly the fact that norm balls $B$ in $E^+$ are star convex, i.e., they contain a point (the origin 0) with the property that any other point $q$ of $B$ is connected to $B$ by a segment $\ell$ connecting 0 and $q$. This is the geometric property that enables us to estimate boundaries of norm balls by scaling, as we have done.

## 6 SRB measures and phase-space observability

In this section we discuss some consequences of **Theorem A** when applied to SRB measures with no zero Lyapunov exponents. **Theorem C**, which asserts that every such SRB measure can be decomposed into at most a countable number of ergodic SRB measures, is proved in Sect. 6.1. **Theorem D**, which asserts, in a sense to be clarified, the “visibility” of SRB measures as a subset of the phase space, is proved in Sect. 6.2.

### 6.1 Ergodic components of SRB measures

In addition to the hypotheses (H1)-(H3) at the beginning of this paper, we introduce

(H4) The Lyapunov exponents of $(f, \mu)$ are nonzero $\mu$-a.e.

The aim of this subsection is to prove

**Theorem C.** Assume (H1)-(H4), and that $\mu$ is an SRB measure. Then

$$\mu = \sum_{i=1}^{\infty} c_i \mu_i \mod 0$$

where $c_i \geq 0$ and each $\mu_i$ is an ergodic SRB measure.

To define SRB measures, we first recall the idea of stacks of local unstable manifolds from [3]. As there are no zero Lyapunov exponents and we are interested only in the splitting $E^u = E^+ \text{ and } E^s = E^-$, it suffices to consider $\Gamma = \Gamma(0; m, p)$. Let $\Gamma_{l_0}$ and $K \subset \Gamma$ be as before. For $\epsilon > 0$, $x_0 \in B$, we write $U(x_0, \epsilon) = \{ x \in B : |x - x_0| < \epsilon \}$.

**Lemma 6.1** (Lemma 5.5 in [3]). Let $l_0 \geq 1, n_0 \in \mathbb{N}$, and let $x_0 \in \Gamma_{l_0} \cap K_{n_0}$. Then, there exist $c_0 > 0$ such that for each $x \in U(x_0, c_0) \cap \Gamma_{l_0} \cap K_{n_0}$, there is a $C^{1+\text{Lip}}$ mapping $\Theta^u(x) : B_{x_0}^+(\delta l_0^{-3}) \rightarrow B_{x_0}^-(\delta l_0^{-3})$ such that $\exp_{x_0} \text{graph } \Theta^u(x) \subset W_{x_0}^u$, $\text{Lip}(\Theta^u(x)) \leq \frac{1}{n_0}$ and $\text{Lip}(d\Theta^u(x)) \leq C_u l_0^2$, where $C_u > 0$ is a constant independent of $\delta$. Moreover, the assignment $x \mapsto \Theta^u(x)$ varies continuously in the uniform norm on $C^0(B_{x_0}^+(\delta l_0^{-3}), B_{x_0}^-(\delta l_0^{-3}))$. 

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An unstable stack $S^u$ is a set of the form

$$S^u = \bigcup_{x \in U} \exp_{x_0} \text{graph} \Theta^u(x)$$

for some fixed compact $\bar{U} \subset U(x_0, \epsilon_0) \cap \Gamma_0 \cap K_{n_0}$. Given $S^u$ with $\mu(S^u) > 0$, let $\eta$ denote the (measurable) partition of $S^u$ into unstable leaves. We consider the canonical disintegration \( \{\mu_W\}_{W \in \eta} \) of $\mu|_{S^u}$ with respect to $\eta$, i.e., for Borel $K \subset S^u$, we have

$$\mu(K) = \int_{S^u/\eta} (\mu_W(K \cap W)) d\mu^T(W).$$

Here, $\mu^T$ is the quotient measure on $S^u/\eta$; for details, see [17].

**Definition 6.2.** Let $(f, \mu)$ satisfy (H1)–(H4), and assume that $\lambda_1 > 0$ $\mu$-a.e. We say $\mu$ is an SRB measure if for any $\Gamma = \Gamma(0; m, p)$ and any unstable stack $S^u$ of positive $\mu$-measure consisting of leaves through $x \in \bar{U} \subset \Gamma_0 \cap K_{n_0}$, the disintegration $\{\mu_W\}_{W \in \eta}$ has the property that for $\mu^T$-almost every $W \in \eta$, $\mu_W$ is equivalent to $\nu_W$, the measure induced on $W$ from volume elements in $B$.\(^1\)

The following terminology will be useful: Consider a homeomorphism $T$ of a compact metric space $X$ preserving an invariant probability $\nu$. We say a point $x \in X$ is future-generic with respect to $(T, \nu)$ if for every continuous $\phi : X \to \mathbb{R}$, we have that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ T^i(y) = \int \phi \, d\nu.$$  \(\text{(18)}\)

Past genericity is defined similarly with $T$ replaced by $T^{-1}$, and we say $T$ is generic if it is both future and past generic. It follows from the Birkhoff Ergodic Theorem that $\nu$-a.e. $x \in X$ is generic with respect to $(T, \nu)$ if and only if $(T, \nu)$ is ergodic. Furthermore, by the Ergodic Decomposition Theorem, for any invariant probability $\nu$, $\nu$-a.e. $x$ is generic with respect to some ergodic measure $\nu_x^*$, and $\nu = \int \nu_x^* d\nu(x)$.

**Proof of Theorem C.** We will show that $\mu$ is locally ergodic in the following sense: For arbitrary $\Gamma = \Gamma(0; m, p)$ and $l_0, n_0$ for which $\mu(\Gamma_0 \cap K_{n_0}) > 0$, it is easy to see that $\Gamma_0 \cap K_{n_0}$ is the union of a countable number of positive $\mu$-measure sets $U$, each one of which is small enough that it can be used to define both a stack of stable manifolds $S^u_\mathcal{U}$ (Lemma 3.8) and a stack of unstable leaves $S^u_\mathcal{U} = \cup W$ (see above). We will show that for each such $U$, there is an ergodic measure $\mu_x^* = \mu_x^\mathcal{U}$ with respect to which $\mu$-a.e. $x \in S^u_\mathcal{U}$ is generic.

Let $\bar{U}$ be fixed. Since $\mu$ is an SRB measure, it follows that for $\mu^T$-a.e. $W$ and $\nu_W$-a.e. $x \in W$, there is an ergodic measure $\mu_x^W$ with respect to which $x$ is generic. First we note that if $x, y$ lie in the same $W$, then $\mu_x^W = \mu_y^W$ because orbits through $x$ and $y$ are backward asymptotic. Thus for $\mu^T$-a.e. $W$ in $S^u$, there is an ergodic measure $\mu_x^W$ with respect to which $x$ is generic for $\mu_W$-a.e. $x \in W$. To connect the $\mu_x^W$ for different $W$, observe that by Theorem A, either (a) $\nu_W(W \cap S^a) > 0$ for every $W$, or (b) $\nu_W(W \cap S^s) = 0$ for every $W$. Since $\mu(\bar{U}) > 0$ and $\nu_W$ is absolutely continuous with respect to $\mu_W$, it follows that (a) must hold. Furthermore, by the equivalence of $\mu_W$ and $\nu_W$ on $\mu^T$-a.e. $W$, we have that $\mu_x^W$ is defined for $\nu_W$-a.e. $x \in W \cap S^s$. This together with $\mu_x^* = \mu_x^W$ for $y \in W_x^s$ implies that $\mu_x^W = \mu_y^W$ for $\mu^T$-a.e. $W, W'$. This

\(^1\)We remark that Definition 6.2 is slightly stronger than the definition of SRB measures given in [3]: here we assume not only that $\mu_W$ is absolutely continuous with respect to $\nu_W$ but that the densities are strictly positive $\nu_W$-a.e. This definition is more convenient for us; the results in [3] hold also under this definition.
common measure \( \mu_*^W \) is \( \mu_{\bar{U}} \). By the ergodic decomposition argument above, we have that \( \mu_{\bar{U}} \) and \( \mu \) coincide on \( S^u_\bar{U} \) mod zero.

Since a countable union of sets of the form \( \bar{U} \) has full \( \mu \)-measure, it follows that \( \mu \) has at most a countable number of ergodic components \( \mu_i \), each given by \( \mu_{\bar{U}_i} \) for some \( \bar{U}_i \).

It remains to show that each \( \mu_i \) is an SRB measure. That is, we need to verify Definition 6.2 for the stack \( S^u_\bar{U} \) for every small compact set \( \bar{U} \) with \( \mu(\bar{U}) > 0 \) for which the stable/unstable stacks \( S^{s/u}_\bar{U} \) are defined as in Lemma 3.8/Lemma 6.1. We will do so by checking that \( \mu_i \) and \( \mu \) coincide mod zero on \( S^u_\bar{U} \). Repeating the above arguments, we have that \( \mu \)-almost every \( x \in S^u_\bar{U} \) is generic to an ergodic measure \( \mu_{\bar{U}_i} \), i.e. \( \mu|S^u_\bar{U} = \mu_{\bar{U}_i}|S^u_\bar{U} \). That \( \mu_{\bar{U}_i} \) follows from the fact that \( \mu_i \) is an ergodic component of \( \mu \), and \( \mu_i(\bar{U}) > 0 \).

6.2 Global holonomy and “visibility” of SRB measures

Consider the setting in Sect. 2.3 – Section 3, with the notation and chart systems defined there. For \( x \in \Gamma \), define

\[
W^ss_x := \{y \in \mathcal{B} : \limsup_{n \to \infty} \frac{1}{n} \log d(f^n x, f^n y) \leq \lambda^- \}.
\]

Because chart sizes shrink more slowly than \( \lambda^- \), it is easy to see that

\[
W^ss_x = \bigcup_{n=0}^{\infty} f^{-n}(W^ss_{f^n x, \text{loc}}),
\]

where \( h_x : \bar{B}_x^-(\delta'_1 l(x)^{-1}) \to \bar{B}_x^+(\delta'_1 l(x)^{-1}) \) and \( W^ss_{x, \text{loc}} = \exp_x \text{graph} h_x \) are as in Theorem 3.1. The sets \( W^ss_x \) are global strong stable sets associated with points \( x \in \Gamma \). In the setting under consideration, they are not guaranteed to be immersed submanifolds, as \( df_x \) is generally not onto and therefore not invertible. The manifold structure of \( W^ss_x \) can be proved under the following assumption, which holds for the time-\( t \) solution mappings of a broad class of dissipative parabolic PDEs [5].

(D) For any \( x \in \mathcal{B} \), the operator \( df_x \) has dense range in \( \mathcal{B} \).

**Proposition 6.3** ([5]). Assume that \( f \) satisfies assumption (D) in addition to (H1) - (H3), and let \( W \) be an embedded submanifold of codimension \( k \). Then, \( f^{-1} W \) is an embedded submanifold of codimension \( k \).

We include the proof of Proposition 6.3 for completeness.

**Proof of Proposition 6.3.** Fix arbitrary \( p \in f^{-1} W \), and let \( g : U \to \mathbb{R}^k \) be a \( C^1 \) submersion (i.e. \( dg \) has full rank) on an open set \( U \subset \mathcal{B} \) with \( f(p) \in U \) and for which \( g^{-1}(0) = W \cap U \). Observe that \( f^{-1}(W \cap U) = (g \circ f)^{-1}(0) \), so it suffices to check that \( d(g \circ f)_q = df_q \circ dg_q : \mathcal{B} \to \mathbb{R}^k \) has full rank for \( q \) in a neighborhood of \( p \). Now there exists a \( k \)-dimensional complement \( E \) to \( \ker(dg_p) \) such that \( E \subset (df_p)\mathcal{B} \), by the dense range assumption. Let \( E' \subset \mathcal{B} \) be the \( k \)-dimensional subspace for which \( df_p E' = E \). Let \( V \subset \mathcal{B} \) be a small enough neighborhood of \( p \) such that \( f(V) \subset U \) and the following hold for all \( q \in V \): \( df_q|E' \) is injective and \( d\mu(E, df_q(E')) \) is sufficiently small that \( df_q|E' \) is injective. This implies that \( d(g \circ f) \) has full rank on \( V \) as desired.

From Proposition 6.3 we immediately obtain the following.

**Corollary 6.4** (Global Strongly Stable Manifold Theorem). Assume that \( f \) satisfies (H1)–(H3) and (D). For any \( x \in \Gamma \), \( W^ss_x \) is an immersed \( C^1 \) submanifold of \( \mathcal{B} \) having the same finite codimension as \( W^ss_{x, \text{loc}} \).
Under Assumption (D) then, we may refer to $W_x^{ss}$ as the global strong stable manifold at $x$ associated with the rate of convergence $\lambda^-$.

Corollary 6.4 makes possible the extension of local results on absolute continuity of $W^{ss}$-foliations such as those in Theorem A to holonomy maps along global strong stable manifolds. There are many ways to formulate results of this kind, all of which boil down to their reduction to local holonomy maps. Here we present one version that has a strong implication on the “visibility” of SRB measures.

For an ergodic measure $\mu$ of $f$, we define the basin of $\mu$ to be the set
\[ \mathcal{N}(\mu) := \{ x \in \mathcal{B} : \lim_{n \to \infty} d(f^n x, \mathcal{A}) = 0 , \text{ and} \] \[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ f^i x = \int \phi \, d\mu \quad \text{for any } \phi \in C_b^0(\mathcal{B}) \}, \]
where $C_b^0(\mathcal{B})$ denotes the set of bounded continuous functions on $\mathcal{B}$. The set $\mathcal{N}(\mu)$ so defined is a Borel subset of $\mathcal{B}$ by an elementary analysis lemma, a proof of which we have included in the Appendix.

We wish to state next that the basin of an SRB measure occupies a significant subset of the phase space. In the absence of a reference measure on $\mathcal{B}$ that plays the role of Lebesgue measure on $\mathbb{R}^n$, we resort to the use of finite-dimensional “probes”. For a finite dimensional manifold $W \subset \mathcal{B}$, the measure $\nu_W$ on $W$ induced from volume elements on finite dimensional subspaces of $\mathcal{B}$ is a natural reference measure. Theorem D expresses the fact that the basins of ergodic SRB measures are “visible” with respect to these reference measures on suitably placed finite-dimensional probes.

**Theorem D.** In addition to (H1)-(H4) and (D), we assume $\mu$ is an ergodic SRB measure. Let $W$ be a $C^2$-embedded disk of dimension $k \geq \dim E^u$. If $W$ meets $W_x^{ss}$ transversally at one point for some density point $x_0 \in \Gamma$ of $\mu$, then $\nu_W(\mathcal{N}(\mu)) > 0$.

**Proof.** Assume first that $k = \dim E^u$. Then by iterating forward, there is an $N \in \mathbb{Z}^+$ such that a component of $f^n(W)$ in the chart at $f^N x_0$ satisfies the condition for $\Sigma^1$ at the beginning of Section 4 with $E^+ = E^u$ and $E^- = E^s$. (This involves proving the analog of what is sometimes referred to as an “inclination lemma” in finite dimensions; the proof follows from techniques similar to those used in Section 3 and is omitted.) By the fact that $f^N x_0$ is also a density point of $\mu$, it follows from Lemma 6.1 that it lies in a stack of unstable leaves $S^u$ with $\mu(S^u) > 0$. Theorem A together with the SRB property of $\mu$ then implies that $\nu_{f^N W}(\mathcal{N}(\mu)) > 0$. Since $f^N|_W$ is a diffeomorphism with a $C^1$ inverse, we conclude that $\nu_W(\mathcal{N}(\mu)) > 0$ as well.

If $\dim W > \dim E^u$, it is easy to decompose $W$ into a smooth family $W = \cup D_\alpha$ where each $D_\alpha$ is a disk having dimension $\dim E^u$ and transversal to $W_x^{ss}$. The argument above applies to each $D_\alpha$; we then integrate the result. \(\square\)

**Remark 6.5.** There are many extensions of the notion of “Lebesgue measure zero” to the setting of infinite-dimensional spaces; for a survey, see Chapter 6 in [2]. The property possessed by the basin $\mathcal{N}(\mu)$ of SRB measures as shown in Theorem D is stronger than many of these notions. For example, it implies that $\mathcal{N}(\mu)$ is not of ‘measure zero’ in the framework of prevalence/shyness [7] (shyness is called Haar null in [2]). The proof is similar to that showing that in $\mathbb{R}^n$, positive Lebesgue measure sets are not shy (see [2] or [7]); modifications are left to the reader.
Lemma 6.6. Let $A \subset Y$ be a compact subset of a metric space $Y$. Let $h : Y \to Y$ be a continuous map with $h^{-1}A = A$, and let $\nu$ be a Borel probability measure on $A$. Then the basin of $\nu$ (as defined in Sect.6.2) is a Borel subset of $Y$.

Proof. The concern is that $C^0_b(Y)$ can be large. As $C^0(A)$ has a countable dense subset $D$, it suffices to show that trajectory averages for $\phi \in C^0(A)$ can be approximated by those for functions in $D$. Let $\phi \in C^0_b(Y)$ be given. Fix $\epsilon > 0$ and let $\psi \in D$ be such that $\|\phi|_A - \psi\|_{C^0(A)} < \epsilon$. By the Tietze Extension Theorem, $\psi$ has a bounded continuous extension $\tilde{\psi}$ to all of $Y$. For each $i \geq 0$, let $y_i \in A$ be such that $d(h^i x, A) = d(h^i x, y_i)$. Then

$$
\left| \sum_{i=0}^{n-1} \phi \circ h^i x - \sum_{i=0}^{n-1} \tilde{\psi} \circ h^i x \right| 
\leq \left| \sum_{i=0}^{n-1} \phi \circ h^i x - \sum_{i=0}^{n-1} \phi(y_i) \right| + \left| \sum_{i=0}^{n-1} \phi(y_i) - \sum_{i=0}^{n-1} \tilde{\psi}(y_i) \right| + \left| \sum_{i=0}^{n-1} \tilde{\psi}(y_i) - \sum_{i=0}^{n-1} \tilde{\psi} \circ h^i x \right|
$$

The middle term is $\leq \epsilon n$. For the first term we use the fact that there exists $\hat{\delta} > 0$ (depending on $\epsilon$ and $\phi$) such that for all $x \in A$ and $y \in Y$ with $d(x, y) < \hat{\delta}$, we have $|\phi(x) - \phi(y)| < \epsilon$. The third term is disposed of similarly.

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