Dynamical characterization in different slow quenching processes

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Recently, dynamical characterization through quench dynamics has attracted growing interests. Previous studies have shown that the dynamical characterization of bulk topology of \textit{the final phase} can be realized in sudden quench and further extended into slow quench. In this paper, take the two-dimensional Chern insulator as an example, we investigate the quenching processes not only between nontrivial regime and trivial regime, but also between the regimes with different topological invariants under slow quench dynamics. Three typical vanishing polarizations, whose position are defined as initial spin inversion surface, final spin inversion surface, and final band inversion surface, are found in the time-averaged spin polarization. Each quenching process shows its unique features to the initial spin inversion surface, final spin inversion surface, and final band inversion surface, on which both the bulk topology of \textit{initial phase and final phase} can be characterized. Importantly, compared with the sudden quench, the initial phase and final phase can be distinguished by the difference between initial spin inversion surface, final spin inversion surface, and final band inversion surface. Finally, we show our findings are robust to an arbitrary initial state. All the dynamical characterization schemes are entirely based on the experimentally measurable quantity time-averaged spin polarization, and thus one can expect our findings may provide reference for future experiments.

\section{I. INTRODUCTION}

Topological quantum phases have been extensively studied in last two decades \cite{1–9}. Under the equilibrium theory, the topological quantum phases can be classified and characterized by its topological invariants defined with Bloch functions. The tightness of the topological invariants and robust boundary modes immune to moderate disorder or defects, is called the famous bulk-boundary correspondence. According to the bulk-boundary correspondence, one can identify the topological phases by resolving the boundary modes with angle-resolved photoelectron spectroscopy and transport measurements experimentally \cite{10–13}. In addition to the equilibrium theory, the notion of characterizing topological phases can also be extended to the nonequilibrium theory.

Nonequilibrium theory can be considered as a probe in characterizing topological properties of equilibrium topological phases such as (non-) Hermitian quantum systems, (non-) correlated systems, higher-order (lower-order) topological insulators, etc \cite{14–24}. Recently, a dynamical bulk-surface correspondence, which states a generic $dD$ topological phase can be characterized by the $(d-1)$ D invariant defined on the so-called band inversion surface (BIS), is established by Liu and his coworkers in Hermitian systems \cite{25, 26}. Due to the platforms of well-designed optical lattice in ultracold atomic systems, the measurement of the time-averaged spin polarization (TASP) by spin-resolved time-of-flight absorption imaging becomes possible. Thus, this dynamical bulk-surface correspondence is further verified experimentally \cite{27–31}. In momentum-space, after suddenly quenching the system from a trivial phase to a topological phase, the bulk topology of a $dD$ equilibrium phase of the postquench Hamiltonian can be easily determined with high-precision by the winding of dynamical field on BIS in the TASP. In addition, a dynamical topological invariant after a sudden quench is proposed by Chen and his co-workers \cite{32}, which shows the intrinsic relation to the difference in the topological invariant of the initial and final static Hamiltonian. The dynamical topological invariant is zero if the initial Hamiltonian and the final Hamiltonian lie in the same phase. On the contrary, the dynamical topological invariant is not zero if the initial Hamiltonian and the final Hamiltonian lie in the different phases including different topological phases.

Compared with the above sudden quench, a general dynamical characterization scheme based on the slow quench protocol is provided in our previous works \cite{33, 34}. It has been found that, nonadiabatic slow quench dynamics can indeed provide an alternative position named spin inversion surface (SIS) in characterizing the topological phases. However, the previous studies only consider the dynamical characterization in one type of quenching process, and the TASP only capture the topological invariant of the postquench Hamiltonian. One may wonder whether it is critical for dynamical characterization to quench from a topologically trivial phase. What knowledge could we obtain by reversing the quenching processes (i.e. from “topological set of parameters” to “non-topological ones”) or by quenching between regimes with different topological invariants? Moreover, the initial state of the previous studies was only set as the ground state of the initial phase, and the influence of the initial state has not been considered yet. Under an arbitrary initial state with the system being slowly quenched with some finite quenching rate, one would expect more rich physical phenomenons emergent in different types of quenching processes.

In this paper, in the framework of slow nonadiabatic quench, we show how the dynamical characterization is in different types of quenching processes of the two-dimensional Chern insulator. By analyzing the precession behavior of the spin polarization under the slow quench dynamics, we find that there exist three typical vanishing polarizations in TASP, whose positions are defined as initial spin inversion surface (ISIS), final
spin inversion surface (FSIS), and final band inversion surface (FBIS), respectively. Then after obtaining the TASP in different types of quenching processes, we show the winding of the dynamical field on the FSIS (FBIS), and ISIS exactly reflects the final topological invariant and initial topological invariant of the system, respectively. Each type of quenching process shows its unique features to these vanishing polarizations. Specifically, if both FBIS and FSIS emerge in the TASP, the final phase of the system will be characterized. On the contrary, if only ISIS emerges in the TASP, the initial phase of the system will be characterized. In addition, if the FBIS, FSIS, and ISIS all exist in the TASP, both initial and final phases can be characterized. That is to say, no matter what phases is experienced during quenching, the TASP only captures the topological invariants of the initial phase and final phase. Compared with the sudden quench, the initial phase and final phase can be distinguished by the difference between FBIS, FSIS, and ISIS. Finally, the robustness of the above results to an arbitrary initial state is also verified. All the dynamical characterization schemes are entirely based on the TASP, which can be observed directly in experiment. Therefore, one can expect our findings may provide reference for future experiments.

The rest of this paper is organized as follows. In Sec. II, we introduce the generic method of characterizing bulk topology under slow non-adiabatic quench dynamics. Then in Sec. III, we present the results of dynamical characterization in different types of quenching processes. In addition, we discuss the influence of an initial state to the TASP in Sec. IV. Finally, we provide a brief discussion and summary to the main results of the paper in Sec. V and Sec. VI, respectively.

II. NONADIABATIC QUENCH DYNAMICS TO CHARACTERIZE THE TOPOLOGICAL PHASES

A. Nonadiabatic slow quench protocol

We first give a general introduction to the slow quench protocol that can be utilized in the non-adiabatic dynamical characterization of topological phases. In general, the Hamiltonian in momentum space is described in the following form

\[ \mathcal{H}(k, t) = \mathbf{h}(k, t) \cdot \mathbf{\gamma} = h_0(k, t)\mathbf{\gamma}_0 + \sum_{i=1}^{d} h_i(k)\mathbf{\gamma}_i. \]  

(1)

The matrices \( \mathbf{\gamma} \) satisfy the anticommutation relations \( \{ \mathbf{\gamma}_i, \mathbf{\gamma}_j \} = 2\delta_{ij} \) and are of dimensionality \( n_d = 2^{d/2} \) (or \( 2^{(d+1)/2} \)) when \( d \) is even (or odd). In 1D and 2D systems, \( \mathbf{\gamma} \) are the Pauli matrices. In higher dimensional systems, \( \mathbf{\gamma} \) take the Dirac form.

Here, we consider a specific protocol like \( h_0(k, t) = g/t + h_0(k) \) with \( g \) determines the quenching rate. The parameter \( g \) varies from 0 to \( \infty \), corresponding to a continuous crossover from the sudden quench limit \( (g = 0) \) to the adiabatic limit \( (g \rightarrow \infty) \). In such a protocol, the form of \( g/t \) enables us to quench the system from an initial Hamiltonian at \( t = t_i \) to a final (postquench) Hamiltonian at \( t = t_f \) by keeping other parameters unchanged. During the quenching process, the evolution of the state vector is fully determined by the Schrödinger equation \( i\hbar \frac{\partial}{\partial t} |\psi(k, t)\rangle = \mathcal{H}|\psi(k, t)\rangle \) after preparing an initial state \( |\psi(k, t_i)\rangle \). Then the pseudospin defined as

\[ \langle \gamma(k, t) \rangle = \langle \psi(k, t)|\gamma|\psi(k, t)\rangle \]  

(2)

will precession about the effective field \( \mathbf{h}(k, t) \). After a certain time \( \Delta t \), one can observe an approximately stable oscillation of the pseudospin \( \langle \gamma(k, t) \rangle \). Thus, the final TASP over a period \( T \) at each \( k \) point can be obtained as:

\[ \langle \gamma(k) \rangle = \frac{1}{T} \int_{t_i + \Delta t}^{t_i + \Delta t + T} \langle \gamma(k, t) \rangle dt. \]  

(3)

Here, the time point \( t_i \) can be obtained directly in experiment.

B. The dynamical topological characterization under slow nonadiabatic quench dynamics

The dynamical topological characterization depends on the winding of the dynamical field on the position of the vanishing polarizations in the component of TASP \( \langle \gamma_0(k) \rangle \). As described before, the quench dynamics leads to a precession of the pseudospin \( \langle \gamma(k, t) \rangle \) about the effective field \( \mathbf{h}(k, t) \), and thus there may be two typical cases for the formation of the vanishing polarizations in \( \langle \gamma_0(k) \rangle \): (i) the pseudospin \( \langle \gamma(k, t) \rangle \) is perpendicular to the effective field \( \mathbf{h}(k, t) \); (ii) the pseudospin \( \langle \gamma(k, t) \rangle \) is not perpendicular to the effective field \( \mathbf{h}(k, t) \), however, the component of pseudospin \( \langle \gamma_0(k) \rangle \) oscillates between positive and negative value.

In addition, another way of obtaining the TASP is to consider the famous Landau-Zener problem. Specifically, for a two-level system, if one prepares an initial state \( |\psi(k, t_i)\rangle \), the system will undergo a non-adiabatic transition during the evolution, and finally at time \( t = t_f \), the system will not only stay on the final ground state \( |\rangle \) with probability \( P_g(k) \), but also on the excited state \( |\langle \rangle \rangle \) with probability \( P_e(k) \). One can only find the common vanishing polarizations in all components of TASP on the momentum point with \( P_u = P_d = 0 \), i.e., the points where the transition probability of initial ground state to the final excited state and the final ground state is equal. Here, we define the position with \( P_u = P_d = 0 \) as SIS. Therefore, the vanishing polarization on the SIS is actually the one as the case (i) described. We then define the position of the vanishing polarization in the case of (ii) as FBIS, on which only one component of TASP \( \langle \gamma_0(k) \rangle \) is zero. Surprisingly, we find the SIS here can be subdivided into two types: one is FSIS, and its position is close to the FBIS; the other is ISIS, and its position is far away from the FBIS. Therefore, all the positions of these vanishing polarizations will be identified after obtaining the TASP, which can be directly measurable in experiment.

After identifying the vanishing polarizations in TASP, one can then obtain the dynamical field on the ISIS, FSIS and FBIS. On the ISIS and FSIS, all the components of TASP vanish, one needs to define a new field \( \overline{\gamma}_{0, i} \) to characterize the topology, while on the FBIS, one can characterize the
topology by simply measuring the value of TASP \( \langle \gamma_{so,i}(k) \rangle \). Both the dynamical field \( \langle \gamma_{so,i}(k) \rangle \) and \( g_{so,i} \) can be shown to be proportional to the spin-orbit field \( h_{so,i} \). Here, the spin-orbit field \( h_{so,i} \) means one of the remaining components of \( h(k,t) \) except \( h_0(k,t) \), and \( \langle \gamma_{so,i}(k) \rangle \) means one of the remaining components of TASP except \( \langle \gamma_0(k) \rangle \). Finally, one can easily capture the topological information of the system from the winding of \( h_{so} \) on the ISIS and FSIS (FBIS), corresponding to the bulk topological invariants of the initial Hamiltonian \( \mathcal{H}(t_i) \) and the final Hamiltonian \( \mathcal{H}(t_f) \), respectively. For clarity, we summarize all the features about FBIS, FSIS and ISIS in Table. I. The information related to the FBIS and SIS is always marked in purple and green in the following description, respectively.

### III. DIFFERENT TYPES OF QUENCHING PROCESSES FOR THE 2D TOPOLOGICAL INSULATOR

Now we apply different types of quenching processes to the 2D Chern insulator, which is generically described by a two-band Hamiltonian: \( \mathcal{H}(k,t) = h(k,t) \cdot \sigma \), with the vector field given by:

\[
\begin{align*}
    h_0(k,t) &= h_z(k,t) = g \frac{t}{t_o} + m_z - t_0 \cos k_x - t_0 \cos k_y, \\
    h_1 &= h_x = t_{so} \sin k_x, \\
    h_2 &= h_y = t_{so} \sin k_y.
\end{align*}
\]

This Hamiltonian without time-dependent term \( \frac{g}{t} \) has been realized in recent experiment of quantum anomalous Hall effect [35]. Here, we slowly quench the \( z \)-component of vector field from \( t = t_i \) to \( t_f \) (a large number compared with \( g \)), and study the emergent topological characterization after quenching. For \( 0 < \frac{g}{t} + m_z < 2t_0 \), the Hamiltonian gives a topological phase with Chern number \( C = -1 \). In addition, for \( -2t_0 < \frac{g}{t} + m_z < 0 \), the Hamiltonian describes a topologically nontrivial phase with Chern number \( C = +1 \).

#### A. quench the system from a trivial phase to a topological phase

The exact solutions of the Landau-Zener problem are difficult to find and only a few special classes can be solved exactly. In the previous papers [33, 34], we have given an exact solutions in a special case. Briefly speaking, we slowly quench the system from a deep trivial phase (far from the phase boundary) to a topological phase with time \( t \) varies from \( t_i \rightarrow 0 \) to \( t_f \rightarrow \infty \), and the initial state \( |\psi(0)\rangle \) is always the ground state of the initial Hamiltonian \( \mathcal{H}(t_i) \). Under these conditions, the TASP can be given by

\[
\langle \sigma_i \rangle = (P_u - P_d) \frac{h_i}{\varepsilon} = \frac{(e^{-2\pi g} \sinh 2\pi g) h_i}{\varepsilon}.
\]

with \( \varepsilon \) is the energy of final Hamiltonian \( H(t_f) \). Here, we still slowly quench the system from a trivial phase to a topological phase, however, the initial trivial phase is close to the phase boundary with \( t_i \neq 0 \).

As shown in Figs. 1(a), 1(b), and 1(c), we plot the three components of TASP after quenching the \( h_z \) axis from a trivial phase to a topological phase with topological invariant \(-1\). The purple ring is identified as FBIS, on which only one component of TASP have vanishing polarization. The green ring in \( \langle \sigma_z \rangle \) is identified as FSIS, on which all three components of TASP have vanishing polarization. As the dynamical field plotted in Fig. 1(e), both the winding of the dynamical field on FBIS and FSIS can characterize the bulk topology of final phase.

In Figs. 2(a), 2(b), and 2(c), three components of TASP are shown after quenching the \( h_x \) axis from a trivial phase to a topological phase with topological invariant \(+1\). The purple

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### TABLE I. The comparison of FBIS, FSIS, and ISIS in the two-level system.

| Name | Bulk topology | Effective field | Position in TASP | Transition Probability |
|------|---------------|-----------------|------------------|------------------------|
| ISIS | Initial       | \( h(k,t) \cdot \langle \gamma(k,t) \rangle \) | Far away from FBIS | \( P_u - P_d = 0 \) |
| FSIS | Final         | \( h(k,t) \cdot \langle \gamma(k,t) \rangle \) | Close to FBIS     | \( P_u - P_d = 0 \) |
| FBIS | Final         | \( h(k,t) \cdot \langle \gamma(k,t) \rangle \) | Only \( \langle \gamma(k) \rangle = 0 \) | \( P_u - P_d \neq 0 \) |

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**FIG. 1.** (a) The specific quenching processes of slow quench. "0" represents the trivial phase, and other numbers represent the topological phase with different topological invariant. (b)-(d) The TASP after quenching \( h_z \) axis from a trivial phase to a topological phase with \( C = 1 \). The system is quenched from \( t_i = \frac{1}{2} \) to \( t_f = 5000 \) with \( 5t_{so} = t_0 = g = m_z = 1 \). The quench protocol is \( \frac{g}{t} \) and the initial state is the ground state of \( \mathcal{H}(t_i) \). \( \langle \sigma_z \rangle \) represents the \( z \) component of TASP after quenching \( h_z \) axis, and so on. The green dashed line (inner ring) and purple dashed line (outer ring) represent the positions of FSIS and FBIS, respectively. (e) The dynamical field of the system plotted by the normalized spin-orbit field \( h_{so} \).
FIG. 2. (a) The specific quenching processes of slow quench. "0" represents the trivial phase, and other numbers represent the topological phases with different topological invariants. (b)-(d) The TASP after quenching $h_z$ axis from a trivial phase to a topological phase with $C = 1$. The system is quenched from $t_i = \frac{1}{2}$ to $t_f = 5000$ with $5t_{so} = t_0 = g = -m_z = 1$. The quench protocol is $\frac{g}{t}$ and the initial state is the ground state of $\mathcal{H}(t_i)$. (e) The dynamical field of the system plotted by the normalized spin-orbit field. The purple dashed lines and the green lines in $\langle \sigma_z \rangle$ are also identified as FBIS and FSIS, respectively. Compared with the dynamical field shown in Fig. 1(e), the dynamical field plotted in Fig. 2(c) converges at a central point inside FBIS and FSIS rather than spread out to the outside of FBIS and FSIS. Thus, the opposite Chern number $C = +1$ is given by the dynamical field both on FBIS and FSIS.

What we mean here is that, no matter what phases the quenching process undergoes, the TASP only records the topological invariant of the final phase. In fact, the topological invariant of initial phase should also be recorded in the TASP. However, the initial trivial phase here cannot bring any topological information on TASP. Thus, only the information related to final topological phase is recorded in the TASP. In experiment, it is no obstacle to obtain bulk topological invariant of the final phase based on the winding of the dynamical field on the FBIS or FSIS, and even no need to distinguish them. To avoid redundancy, we no longer show the dynamical field hereafter.

FIG. 3. (a) The specific quenching processes in (b), (c) and (d). "0" represents the trivial phase, and other numbers represent the topological phases with different topological invariants. (b) The TASP after quenching $h_z$ axis from a topological phase with $C = -1$ to a topological phase with $C = 1$. The system is quenched from $t_i = 1/2$ to $t_f = 5000$ with $5t_{so} = t_0 = g = m_z = -1$. The solid green line represents the position of IS. (c) The TASP after quenching $h_z$ axis from a topological phase with $C = 1$ to a topological phase with $C = -1$. The system is quenched from $t_i = 1/2$ to $t_f = 5000$ with $5t_{so} = t_0 = g = m_z = 1$. (d) The TASP after quenching $h_z$ axis from a topological phase with $C = 1$ to a trivial phase. The system is quenched from $t_i = 1/3$ to $t_f = 5000$ with $5t_{so} = t_0 = g = 1$ and $m_z = -4$.

B. quench the system from a topological phase to a topological phase

Now, we discuss other processes, which quench the system from a topological phase to another topological phase. We first quench the system from a topological phase with topological invariant $C = -1$ to a topological phase with topological invariant $C = +1$. As shown in Fig. 3(b), three vanishing polarizations appear in $\langle \sigma_z \rangle$. The positions of the adjacent vanishing polarizations in $\langle \sigma_z \rangle$ are identified as FBIS and FSIS. Then one can easily distinguish the FBIS and FSIS by the vanishing polarizations in other components of TASP $\langle \sigma_x, y \rangle$. The positions of the adjacent vanishing polarizations in $\langle \sigma_x, y \rangle$ are identified as FBIS and FSIS. For FBIS, there is no more vanishing polarizations in other components of TASP, and thus we identify the purple dashed line as FBIS. For FSIS, the vanishing polarizations can be
found in other components of TASP, and thus we identify the green dashed line as FSIS. Naturally, the position of the remaining vanishing polarization in $\langle \sigma_z \rangle$ is identified as ISIS, which is also denoted by the green solid line. Similar to the quenching process described before, the topology of the initial Hamiltonian $\mathcal{H}(t_i)$ and the final Hamiltonian $\mathcal{H}(t_f)$ can still be characterized from the winding of the dynamical field on the ISIS and FSIS(FBIS), respectively.

The above characterization schemes remain valid for the quenching process from a topological phase with topological invariant $C = +1$ to a topological phase with topological invariant $C = -1$. In order to reverse the quenching process, we change the quench protocol $g/t$ as $-g/t$. As shown in Fig. 3(c), the adjacent vanishing polarizations are also emergent in the TASP, on which the bulk topology of the final phase with topological invariant $C = -1$ can be characterized. As long as the initial and final phases of one quenching process are in the same phases as another quenching process, the results of TASP of two quenching processes will be similar.

Based on the identification of FSIS, FBIS and ISIS, the initial phase and final phase can be easily distinguished in slow quench. One can imagine how awkward the situation would be if these vanishing polarizations could not be identified. Compared with the sudden quench (shown in Discussion V), one can really feel the advantages of our characterization schemes.

C. quench the system from a topological phase to a trivial phase

All the final phases in above processes are topological, we further explore how the dynamical characterization in other types of quenching processes, which quench the system from a topological phase with topological invariant $C = +1$ to a trivial phase. As shown in Fig. 3(c), only one polarization (green solid line) is observed in the TASP, however, we can still obtain the bulk invariant of initial phase $C = +1$ by the winding of the field on the position of this vanishing polarization. The position of this vanishing polarization is actually the ISIS described before due to the fact that the FBIS and FSIS is always appeared at the same time. In other words, there is actually no FBIS exists, and thus no FSIS. Evidently, the final trivial phase cannot bring any topological information on the TASP. Thus, the TASP only record the topological information related to initial phase.

Overall, each type of quenching process shows its unique features to the ISIS, FSIS and FBIS in the TASP, and the winding of the field on them exactly reflects the topological invariants of the system. No matter what phases the quenching process undergoes, the TASP only records the topological invariants of the initial phase and final phase. In particular, the initial phase and final phase can be distinguished by the difference between ISIS, FSIS and FBIS. The similar results can be found in the 2D topological insulator with high integer invariant [33]. For the three-dimensional (3D) chiral insulator, we also give a discussion in Appendix C.

D. the relation between the Hamiltonian and the position of the vanishing polarizations in TASP

The reason why the TASP can capture the bulk topology of initial phase and final phase of the system is that it actually records all the information about the Hamiltonian before and after quenching. For a two-level system, all the information about the Hamiltonian are $h_z(t_i), h_z(t_f), h_x,$ and $h_y.$ Thus, in general, one can find the zero points of these axes correspond to the positions of the vanishing polarizations in the TASP. For example, one can always see the zero points of $h_x = \sin x$ and $h_x = \sin y$ correspond to the vanishing polarizations on the lines $k_x = 0$ and $k_y = 0$ in $\langle \sigma_x, g \rangle$, respectively. However, only the initial phase or the final phase lies in topological regime, the TASP records the information of $h_z(t_i)$ on ISIS or $h_z(t_f)$ on FBIS (FSIS). We further uncover the inherent relation of FBIS, FSIS and ISIS to the Hamiltonian.

As shown in Fig. 4, we extract the TASP on the lines $k_x = h_y$ from the TASP in Fig. 3(b). The solid lines and dashed lines denote the zero points of $h_z(t_i)$ and $h_z(t_f)$, respectively. When $g$ changes, the FBIS is always at zero points of $h_z(t_f)$. Thus, the FBIS here is actually the BIS described in previous papers.

In addition, the positon of FSIS is close to the FBIS, and the spin polarizations on it are all zero. As shown in Fig. 4(b), we denote the change of FSIS by black arrows. The position of FSIS depends on the quenching rate $g$, and totally overlaps with FBIS when $g = 0.$ Thus, in a certain range of $g$ ($0 < g < 10$), there must be a FSIS in TASP similar to the shape of FBIS.

Finally, the position of ISIS is far from the FBIS, and all the spin polarizations on it are vanished. One can see the position of ISIS is still dependent on $g$, however, compared with FSIS, the influence of the change of $g$ to ISIS is very insensitive. When $g$ tends to zero, the ISIS is at $h_z(t_i) = 0$.

From the perspective of phase transition, in the process of slow quench, $h_z(k, t)$ varies with time, and the bulk gap closes while $h_z(k, t) = h_x, y = 0$, which leads to a topological phase transition with the topological invariant changed. In particular, $h_{x,y} = 0$ is the position of topological charge, i.e., the singularities enclosed in the FBIS, FSIS, and ISIS. Analogous to the Gaussian theorem [36, 37], the topological charge is actually a monopole, whose quantized flux through the FBIS (FSIS) and ISIS are viewed as the initial topological invariant and final topological invariant, respectively. In other words, the topological charge is dual to dynamical field on the FBIS, FSIS and ISIS. Thus, one can also determine the topological invariant of the system by its topological charge.

IV. ROBUSTESS OF THE RESULTS TO THE INITIAL STATE

All the initial states of the above quenching processes are the ground states of the initial Hamiltonian. It is necessary
to explore whether the above results are robust to an arbitrary initial state. As shown below, we plot the TASP in different quenching processes for an arbitrary initial state [superposition state of the ground state and excited state of $H(t_i)$].

For the quench shown in Fig. 5(b), we can still identify the purple ring and the green ring as FBIS, FSIS, respectively. The change of initial state only influences the shape of FSIS. The FBIS and FSIS still share the same topology, which is ensured by the common topological charge contained in the FBIS and FSIS at $k_x = k_y = 0$.

For the quenches shown in Fig. 5(c) and Fig. 5(d), all the types of the vanishing polarizations (ISIS, FSIS, FBIS) can be identified according to the features described earlier. Also, we can still know the topological invariants of the initial phase and final phase. The change of initial state only produces another vanishing polarization similar to ISIS. In addition, the different configuration of the superposition state only influences the shape of this new emergent vanishing polarization.

V. DISCUSSION

The dynamical characterization schemes we discussed above are only based on the TASP, and thus one can easily obtain the topological information of the system from our schemes in experiment. The key point of this paper is: in the framework of slow quench, identifiable vanishing polarizations in TASP give more clear information of the phases we are characterizing. One does not need to know the prior knowledge that the quenches have undergone even how the initial state is before quenching.

To prove this, we also present the results of different types of quenching processes in sudden quench. One can obviously see from Fig. 6(b) and Fig. 6(c), there is no difference for vanishing polarizations between two reversed quenching processes.

In fact, all the vanishing polarizations in sudden quench have same features in TASP. Therefore, one cannot distinguish the initial phase and final phase even though their topological invariants have already recorded in TASP. As shown in Fig. 6(d), although the topological invariants of the initial phase and final phase are captured on purple dashed line and purple solid line, respectively, they cannot be distinguished.

VI. CONCLUSIONS

In summary, take the 2D Chern insulator as an example, we uncover how the dynamical characterization is in different types of quenching processes. We show that each type of quenching process shows its own features for three typical vanishing polarizations in the TASP. One can obtain the topological invariants of initial phase and final phase from three typical vanishing polarizations, whose positions are defined...
as ISIS, FSIS and FBIS. Compared with the sudden quench, the initial phase and final phase can be distinguished by the difference between ISIS, FSIS and FBIS. In a word, no the prior knowledge of the specific quenching process and even the initial state, one can obtain all the topological information from the dynamical winding on the FBIS, FSIS and ISIS in the TASP. It is worthwhile that the dynamical characterization schemes here is only based on the TASP, and thus one can expect our characterization schemes may provide more reference for future experiments. After all, all the Hamiltonian of this paper are based on Gamma matrices, we will soon discuss the characterization schemes about the Hamiltonian based on the Gell-Mann matrices in our next paper.

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Appendix A: The TASP and dynamical field in nonadiabatic quench dynamics

In sudden quench, a two-level Hamiltonian can be written as:

\[
\mathcal{H}_f = \mathbf{h} \cdot \sigma = \left( \frac{\varepsilon \cos \theta_f}{\sin \phi_f \varepsilon} - \frac{\varepsilon \sin \theta_f e^{-i\phi_f}}{-\varepsilon \cos \theta_f} \right). \tag{A1}
\]

Here, \(\varepsilon\) is the energy of the final Hamiltonian. The parameters \(\theta_f\) and \(\phi_f\) are space angles that describe the direction of the effective field \(\mathbf{h}\).

The evolution of state vector can be written as: \(|\psi(t)\rangle = e^{-i\mathcal{H}_f t/\hbar} |\psi(0)\rangle\). In addition, an arbitrary initial state can be expressed as the superposition state of eigenstates of the postquench Hamiltonian: \(|\psi(0)\rangle = C_1 |+\rangle + C_2 |−\rangle\). After some algebras, we show

\[
\langle \sigma_i \rangle = \frac{(C_1^2 - C_2^2) h_i}{\varepsilon}. \tag{A2}
\]

If one choose the initial state as the ground state of the initial Hamiltonian \(\mathcal{H}_{\theta_{i\text{nt}}, \varphi_{i\text{nt}}}\), the TASP \(\langle \sigma_i \rangle\) will become

\[
[-\cos(\theta_f - \theta_{i\text{nt}}) - \sin \theta_f \sin \theta_{i\text{nt}} (\cos \varphi_f - \varphi_{i\text{nt}}) - 1] h_i. \tag{A3}
\]

Especially, if the initial ground state is a completely polarized state, that is to say, the \(h_i\) is infinite, and thus the initial phase is a trivial phase. In this case, the parameter \(\theta_{i\text{nt}}\) will equal to 0 or \(\pi\), and thus the TASP becomes

\[
\langle \sigma_i \rangle = -\cos \theta_f \frac{h_i}{\varepsilon} = -\frac{h_i^f h_i}{\varepsilon^2}. \tag{A3}
\]

Similarly, if the final phase is a trivial phase, the parameter \(\theta_f\) will equal to 0 or \(\pi\). Then the TASP becomes

\[
\langle \sigma_i \rangle = -\cos \theta_{i\text{nt}} \frac{h_i}{\varepsilon} = -\frac{h_{i\text{nt}} h_i}{\varepsilon^2}. \tag{A4}
\]

The corresponding TASP of the expression Eq. (A3) and Eq. (A4) is shown in Fig. 6(b) and Fig. 6(c), respectively.

Overall, whatever the initial state we choose, the vanishing spin polarizations in each component of TASP can be two cases: one is the common spin polarizations \(C_1^2 - C_2^2 = 0\), the other is unique spin polarizations \(h_i = 0\). Here, we define the direction \(k_{\perp}\) to be perpendicular to the common spin polarizations \((C_1^2 - C_2^2 = 0)\). Thus, the dynamical field on the \((C_1^2 - C_2^2 = 0)\) can be shown to be proportional to the spin-orbit field:

\[
\partial_{k_{\perp}} \langle \sigma_{so,i} \rangle \propto \lim_{k_{\perp} \rightarrow 0} \frac{1}{\varepsilon} \frac{h_{so,i} + O(k_{\perp})}{k_{\perp}} \frac{1}{\varepsilon} k_{\perp} = \frac{h_{so,i}}{\varepsilon}. \tag{A5}
\]

In slow quench, if we prepare the system in its initial ground state \(|\psi(t_i)\rangle\), the system will undergo a non-adiabatic transition, and finally at infinite time, the system will not only stay on the final ground state \(|−\rangle\) with LZ probability \(P_d\), but also on the excited state \(|+\rangle\) with LZ probability \(P_u\).

Up to a total phase factor \(\phi\), the state vector at the long-time limit is given:

\[
|\psi(t)\rangle = \sqrt{P_u} e^{-i\mathbf{h} t/\hbar} |+\rangle + \sqrt{P_d} e^{i\mathbf{h} t/\hbar} |−\rangle. \tag{A6}
\]

Here, \(\phi\) is 0 in the case of sudden quench. \(|\pm\rangle\) satisfy the eigen equation \(H_f |\pm\rangle = \pm \varepsilon |\pm\rangle\) with \(H_f = \mathbf{h} \cdot \sigma\) and \(\varepsilon = \sqrt{\sum_{i=0}^{n} h_i^2}\) being the final Hamiltonian and the eigen energy of final Hamiltonian, respectively.

After some algebras, the TASP can be obtained by using the relation \(|\pm H_f |\pm\rangle = \pm \varepsilon |\pm\rangle\) and ignoring the terms dependent on time,

\[
\langle \sigma_i \rangle = (P_u - P_d) \frac{\partial \varepsilon}{\partial h_i} = (P_u - P_d) \frac{h_i}{\varepsilon}. \tag{A7}
\]

Thus, the vanishing polarizations in \(\langle \sigma_0 \rangle\) can be \(h_0(t_f) = 0\), \(P_u - P_d = 0\), which correspond to the FBIS and SIS, respectively.

The \(P_u - P_d\) only have one zero point (ISIS or FSIS) in the quench including a trivial phase. However, in the quench between different topological regimes, the \(P_u - P_d\) have two zero points in the TASP, and thus both the ISIS and IFIS exist. A direct verification can be made by comparing the results with the TASP of sudden quench. As discussed before, the positions of FSIS and ISIS \((P_u - P_d = 0)\) are dependent on the quenching rate \(q\). When \(q\) tends to zero, the positions of FSIS and ISIS overlap with FBIS \([h_0(t_f) = 0]\) and \([h_0(t_i) = 0]\), respectively.

Of course, one can also show the dynamical field is proportional to spin-orbit field as sudden quench.
Appendix B: Different types of quenching processes in sudden quench

![Diagram of specific quenching processes](image)

FIG. 6. (a) The specific quenching processes in (b), (c) and (d). "0" represents the trivial phase, and other numbers represent the topological phases with different topological invariants. (b) The TASP after suddenly quenching $h_z$ axis from a trivial phase to a topological phase with $C = +1$. The system is quenched from $m_z = 20$ to $m_z = -1$ with $5\nu_3 = 0 = 1$. The initial state is the ground state of $H(m_z = 20)$. (c) The TASP after suddenly quenching $h_z$ axis from a topological phase with $C = +1$ to a trivial phase. The system is quenched from $m_z = -1$ to $m_z = -10$ with $5\nu_3 = 0 = 1$. The initial state is the ground state of $H(m_z = -1)$. (d) The TASP after suddenly quenching $h_z$ axis from a topological phase with $C = -1$ to a topological phase with $C = +1$. The system is quenched from $m_z = 1$ to $m_z = -1$ with $5\nu_3 = 0 = 1$. The initial state is the ground state of $H(m_z = 1)$.

Appendix C: 3D chiral topological insulator

Here, we apply the above characterization schemes to the 3D chiral topological insulator. The Hamiltonian of 3D topological phase can be written as

$$h_0(k, t) = \frac{g}{t} + m - t_0 \sum_{j=x,y,z} \cos k_j,$$

$$h_{1,2,3} = t_{so} \sin k_{x,y,z},$$

$$h_4 = 0.$$  \hfill (C1)

The system will lie in different phases depending on the value of $\frac{g}{t} + m$: for $|\frac{g}{t} + m| > 3t_0$, the trivial phase; for $t_0 < \frac{g}{t} + m < 3t_0$, the topological phase with winding number $\nu_3 = -1$; for $-t_0 < \frac{g}{t} + m < t_0$ with $\nu_3 = 2$, and for $-3t_0 < \frac{g}{t} + m < -t_0$, with $\nu_3 = -1$.

We quench this system from a topological phase with invariant $-1$ to a topological phase with invariant $2$. The corresponding 3D configuration of FSIS and FBIS with a small $g$ are shown in Figs. 7(e) and 7(f), respectively. In addition, the 3D configuration of ISIS is an enclosed sphere surface, which gives the winding number $-1$. At this time, both the FBIS and FSIS are unenclosed, and thus the bulk invariant $2$ of final Hamiltonian cannot be well obtained from winding of the dynamical field on it. However, the distinct configurations of ISIS and FBIS (FSIS) still correspond to initial topological phase and final topological phase with the winding number $-1$ and $2$, respectively. That is to say, the TASP of 3D chiral insulator here can also capture the different information belong to the initial phase and the final phase, respectively.

When $g$ is greater than 1, the FSIS will transform to the configuration in Figs. 7(g), and the configuration of ISIS and FBIS are almost unchanged. As shown in Figs. 7(a), 7(b), and 7(c), we extract the TASP on the plane $k_z = 0$ from the 3D configuration of TASP. At this time, the FSIS no longer surrounds the FBIS, and thus the initial phase and the final phase cannot be distinguished easily. However, compared with the sudden quench, the unique advantage of slow quench here is the topological phase with bulk invariant $2$ can be characterized. To be specific, the topological numbers defined on a pair of hemisphere surfaces and the central surface cancel out with each other, and then the remaining two pairs of hemisphere surfaces give the bulk invariant $2$. Therefore, in other quenching processes including a trivial phase, the initial topological phase and the final topological phase with bulk invariant $2$ can still be characterized.

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FIG. 7. (a) The specific quenching processes of slow quench for 3D topological insulator. "0" represents the trivial phase, and other numbers represent the topological phases with different topological invariants. (b)-(d) The TASP on the plane $k_z = 0$ after slowly quenching $h_0$ axis from a topological phase to another topological phase. The system is quenched from $t_i = 1/2$ to $t_f = 5000$ with $\tau_{so} = \frac{\tau}{2} = 1$, $g = 1$. The quench protocol is $\frac{g}{t}$ and the initial state is the ground state of $H(t_i)$. (e) The configuration of BIS. (f) The configuration of SIS when $g$ is less than 1. (g) The configuration of SIS when $g$ is greater than 1.