Helicity formalism revisited for polarised particle decays

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Abstract

We discuss a revision of the helicity formalism in light of its application to multi-body decays involving polarised particles, notably fermions. The issue of the phase definition for helicity states is clarified and solved by proposing a slight modification of the helicity formalism. We present a method to write helicity amplitudes with a consistent matching of final particle spin states valid for fermions as well. The need for the proposed revision of the helicity formalism is stressed by considering the consequences of an improper phase definition of spin states on the decay distributions of polarised particles.

1 Introduction

The helicity formalism, originally proposed by Jacob and Wick to treat relativistic processes involving particles with spin, is still one of the mostly used tools for performing amplitude analyses of particle decays. Nowadays, hadron colliders provide unprecedented samples of baryons, allowing for detailed amplitude analyses of their multi-body decays. The helicity formalism is in principle able to deal with initial and final particles with spin, their polarisation introduced in the decay distributions by means of their associated density operators.

In practice, a consistent definition of final particle spin states for decays involving different intermediate states turned out to be an issue, since the helicity formalism holds for helicity states which have a different definition for each intermediate state. This issue has been addressed in different ways, but none properly considered the underlying problem of a consistent phase definition of spin states in full generality for multi-body decays.

In this article we first consider the issue of the phase definition for spin states in Section 2. The phase of spin states depends on the choice of the spin operators describing spin components orthogonal to the spin quantisation axis, so that a rotation around this axis change the phase definition of spin states. We show that the phase of helicity states is undefined by construction and must be specified. Moreover, spin states transform under rotations differently from coordinate systems, under representations of the SU(2) group. For fermions, this introduces a sign difference between spin states reached by rotations differing by an overall $2\pi$ rotation.

In Section 3 we show that an inconsistent definition of helicity state phases for two-body decays is present in the helicity formalism, which can be settled by choosing the same spin coordinate system for the two daughter particles.

In Section 4 we present a method to write helicity amplitudes with a consistent definition of final particle spin states for different intermediate states, applicable to any multi-body decay topology. This way, the final particle spin coordinate systems are matched for different intermediate states, without introducing unphysical phases between their decay amplitudes. Helicity amplitudes have been explicitly derived for three-body decays.

The need for a consistent definition of final particle spin state phases is showed by considering the consequences of an improper phase definition of spin states on the decay distributions of polarised particles. The differential decay rate for particles with polarisation, which is described by spin density operators, is briefly introduced in Section 5. Section 6 deals with the effects produced by an unphysical phase which can be...
introduced among decay amplitudes if final particle spin state are not correctly matched. Such a phase produces unphysical interference effects in the decay distributions, and may be absorbed into helicity coupling values in certain cases. For generic particle polarisations, we demonstrate that an unphysical phase has observable consequences on the decay distributions and prevents the measurement of particle polarisations from experimental data.

2 Phase definition for spin states

In quantum mechanics, the spin of a particle is described by a vector of spin operators $\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$, which defines a right-handed spin coordinate system $(x, y, z)$. The spin states $|s, m\rangle$ are defined as the simultaneous eigenstates of the spin squared modulus $\hat{S}^2$ and $\hat{S}_z$, with eigenvalues $s(s + 1)$ and $m$, respectively. The $z$ axis is called quantisation axis, while $x$ and $y$ axes will be referred to as orthogonal axes.

The choice of the quantisation axis specifies spin states up to a phase, which depends on the orthogonal axes of the spin coordinate system. Indeed, rotations around the $z$ quantisation axis change the phase definition of the spin states,

$$R_z(\alpha) |s, m\rangle = e^{-i\alpha \hat{S}_z} |s, m\rangle = e^{-i\alpha m} |s, m\rangle ,$$

and it is in this sense that the choice of $x$ and $y$ axes enters the $|s, m\rangle$ state definition. Therefore, orthogonal axes must be consistently defined to avoid introducing unphysical phases. This is especially important when considering interference effects, which make phase differences observable quantities.

Let’s consider the phase definition for helicity spin states $|s, \lambda\rangle$, eigenstates of the helicity operator $\lambda$ defined as the spin projection along the particle momentum $p$,

$$\lambda = \hat{S}_z \cdot \frac{p}{p} .$$

Given a reference frame in which the particle has momentum $p$, the helicity coordinate system can be obtained by applying an Euler rotation $R(\phi, \theta, \psi)$, defined in [Appendix A] aligning the $z$ axis with $p$. Here, $\theta$ and $\phi$ are the polar and azimuthal angles of $p$ in the $(x, y, z)$ reference frame coordinate system.

$$\cos \theta = \frac{p_z}{|p|} ,$$

$$\phi = \text{atan2} (p_y, p_x) .$$

while the third angle $\psi$, associated to a rotation around $p$, determines the choice of the orthogonal spin coordinate axes. The orthogonal axes can be chosen arbitrarily, but once defined must be consistently specified to avoid introducing unphysical phase differences. In the following, the simplest choice $\psi = 0$ will be employed for the definition of helicity states.

We showed how spin states can be defined from spin coordinate systems, however, the two transform differently under rotations. Spin coordinate systems transform under vector (spin 1) representations of the $SO(3)$ group, while spin states transform under spin $s$ representations of the $SU(2)$ group. This implies that the relation between spin coordinate systems and spin states is not bijective. For fermion states the mismatch is particularly visible since the spin coordinate system does not determine the sign of spin states. Indeed, for a $2\pi$ rotation around any axis, leaving the spin coordinate system $(x, y, z)$ unchanged, fermion states change sign. For instance,

$$R_z(2\pi) |s, m\rangle = e^{-2i\pi \hat{S}_z} |s, m\rangle = (-1)^s |s, m\rangle .$$

Therefore, particular care must be taken when defining spin coordinate systems for fermions, since unphysical sign differences can be introduced if the same coordinate system is reached by a rotation sequence differing by an overall $2\pi$ rotation. In mathematical terminology, an unphysical minus sign is introduced if the same spin coordinate system is reached by topologically inequivalent trajectories in the $SU(2)$ group space.

3 Helicity formalism revisited

Relativistic processes involving particles with spin can be described by applying the spin-orbit formalism of non-relativistic quantum mechanics to helicity states, a method known as helicity formalism, first proposed by Jacob and Wick [1]. The key point underlying the helicity formalism is the invariance of helicity states under rotations, which is exploited to construct twoparticle states with definite value of total angular momentum. Following the notation of [3], one can express two-particle helicity states having definite momentum $|p, \phi, \lambda_1, \lambda_2\rangle$ in terms of two-particle helicity states $|p, J, M, \lambda_1, \lambda_2\rangle$ with definite value of total angular momentum. Here, $p, \phi, \lambda$ are the spherical coordinates of particle 1 momentum $p_1$ in the two particle centre-of-mass rest frame, with particle 2 momentum determined by $p_2 = -p_1$; while $J(J+1)$ and $M$ are the eigenvalues of the total angular momentum operators $\hat{J}^2$ and $\hat{J}_z$, respectively. The relation between the two states is

$|^p, \phi, \lambda_1, \lambda_2\rangle = (2\pi)^{J(J+1)/2} \delta^{(2)} (p - p_1) \delta^{(2)} (\phi - \phi) \delta^{(2)} (\lambda_1 - \lambda_2) |p, J, M, \lambda_1, \lambda_2\rangle .$$

$^1$The function $\text{atan2} (y, x) \in [-\pi, \pi]$ computes the signed angle between the $x$ axis and the vector having components $(x, y)$. 

$^2$The function $|p, \phi, \lambda_1, \lambda_2\rangle$ is normalized so that $\langle p', \phi', \lambda', \lambda' | p, \phi, \lambda_1, \lambda_2 \rangle = \delta^{(4)} (p - p') \delta^{(2)} (\phi - \phi') \delta^{(2)} (\lambda_1 - \lambda') \delta^{(2)} (\lambda_2 - \lambda') .
worked out in \[1,6\],
\[
|p, \theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2J + 1}{4\pi}} D_{M,\lambda_1 \rightarrow \lambda_2}^J(\phi, \theta, 0) \\
\times |p, J, M, \lambda_1, \lambda_2\rangle ,
\]  
(5)
in which \(D\) is a Wigner \(D\)-matrix, defined in Appendix A.

For a two-body decay \(A \rightarrow 1, 2\), Eq. (5) allows to write the decay amplitude as
\[
A_{m_A, \lambda_1, \lambda_2}(\theta_1, \phi_1) = \langle \theta_1, \phi_1, \lambda_1, \lambda_2 | \hat{T} | s_A, m_A \rangle = H_{\lambda_1, \lambda_2} D_{m_A, \lambda_1 \rightarrow \lambda_2}^s(\phi_1, \theta_1, 0),
\]  
(6)
in which \(\hat{T}\) is the transition operator and
\[
H_{\lambda_1, \lambda_2} \equiv \langle J = s_A, M = m_A, \lambda_1, \lambda_2 | \hat{T} | s_A, m_A \rangle ,
\]  
(7)
are complex numbers called helicity couplings, describing the decay dynamics. Here, the Wigner \(D\)-matrix describes the rotation from the spin coordinate system of the \(A\) particle to the helicity coordinate system of particle 1; the angles \(\theta_1, \phi_1\) are the spherical coordinates of \(p_1\) in the \(A\) coordinate system, \(\lambda_1\) is the particle 1 spin projection along \(p_1\) and \(\lambda_2\) is the particle 2 spin projection along \(p_2 = -p_1\).

Let’s consider the definition of the orthogonal axes for the daughter particles 1,2. For particle 1, \(x\) and \(y\) axes are well defined by the Euler rotation \(R(\phi_1, \theta_1, 0)\). For particle 2, the coordinate system is defined from the relation \(p_2 = -p_1\) by mirroring the \(z\) axis defined for particle 1. But the mirror operation is not valid, since it changes right-handed coordinate systems to left-handed ones! Therefore, the spin coordinate system employed for particle 2 in the helicity formalism is wrong, and the phase of particle 2 spin states is undefined.

We propose a straightforward modification of the helicity formalism to ensure a correct definition of particle 2 spin states: the particle 2 spin should be referred to the helicity coordinate system of particle 1, so that particle 2 states \(|s_2, \lambda_2\rangle\) represent spin projection eigenstates in the direction opposite to \(p_2\). The quantity \(\hat{\lambda}\) is the opposite of the helicity,
\[
\hat{\lambda} = -\hat{S} \cdot \frac{p}{p},
\]  
(8)
and obviously share the same properties of the helicity, so that Eq. (5) holds with the substitution \(-\lambda_2 \rightarrow \lambda_2\),
\[
|p, \theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2J + 1}{4\pi}} D_{M,\lambda_1 \rightarrow \lambda_2}^J(\phi, \theta, 0) \\
\times |p, J, M, \lambda_1, \lambda_2\rangle .
\]  
(9)
The two-body decay amplitude becomes
\[
A_{m_A, \lambda_1, \lambda_2}(\theta_1, \phi_1) = \langle \theta_1, \phi_1, \lambda_1, \lambda_2 | \hat{T} | s_A, m_A \rangle = H_{\lambda_1, \lambda_2} D_{m_A, \lambda_1 \rightarrow \lambda_2}^s(\phi_1, \theta_1, 0),
\]  
(10)
and the possible helicity values are determined by angular momentum conservation, requiring
\[
|\lambda_1| \leq s_1, \quad |\lambda_2| \leq s_2, \quad |\lambda_1 + \lambda_2| \leq s_A .
\]  
(11)

Note that the choice of which of the particle 1 or 2 spin is expressed in terms of helicity or opposite helicity states is arbitrary, and it does not assign a preferential role to one of the two particles. For instance, if both particles have subsequent decays, \(1 \rightarrow 3, 4\) and \(2 \rightarrow 5, 6\), the same spin coordinate system can be used as a starting point to define the helicity-opposite helicity states for 3,4 and 5,6 pairs, via Eq. (10). Instead, in the previous definition of helicity states there was an asymmetry between particle 1, with a well defined spin coordinate system, and particle 2, whose spin states are referred to a spin coordinate system with undefined orthogonal spin components. This badly defined system introduces an unphysical phase ambiguity in the decay amplitudes, and it cannot be used to define the helicity states of the particle 2 daughter particles 5,6.

### 4 Helicity amplitudes with matching of final particle spin states

Decay amplitudes for multi-body particle decays can be obtained in the helicity formalism by breaking the decay chain in sequential two-body decays mediated by intermediate particles states. For the sake of clarity, we consider a three-body decay \(A \rightarrow 1, 2, 3\), but the way the revised helicity formalism is applied in order to match final particle spin state definitions is applicable to any decay topology.

A three body decay is treated in the helicity formalism by breaking it into two binary decays involving an intermediate state. Three decay chains, involving three kind of intermediate states, are possible: \(A \rightarrow R(\rightarrow 1, 2), 3\), \(A \rightarrow S(\rightarrow 1, 3), 2\) and \(A \rightarrow U(\rightarrow 2, 3), 1\). We first consider the \(A \rightarrow R(\rightarrow 1, 2), 3\) decay chain. The \(A \rightarrow R, 3\) decay can be expressed by Eq. (11),
\[
A_{m_A, \lambda_R, \lambda_3}^{A \rightarrow R, 3}(\theta_R, \phi_R) = \langle \theta_R, \phi_R, \lambda_R, \lambda_3 | \hat{T} | s_A, m_A \rangle
\]  
(12)
and the \(R \rightarrow 1, 2\) decay can be written in the same form by applying Eq. (11) to the \(R\) state \(|s_R, \lambda_R\rangle\) as decaying...
particle,

$$A_{\lambda R, \lambda R_1}^{1,2}(\theta_1, \phi_1^R) = \langle \theta_1, \phi_1^R | R \bar{\lambda}_1, \lambda_R \rangle | s_R, \lambda_R \rangle = \mathcal{H}_{\lambda R, \lambda R_1}^{1,2} D^{s_{R_1}}_{m_R, \lambda R_1 + \lambda_R} (\phi_1^R, \theta_1^R, 0),$$

(13)

The $R$ superscript is put on helicity values and angles of particles 1,2 to stress that their definition is specific to the $A \to R(\to 1,2), 3$ decay chain.

The total amplitude of the $A \to R(\to 1,2), 3$ decay is written introducing $R$ as intermediate state, and summing the amplitudes over the helicity values $\lambda_R$ satisfying the angular momentum conservation requirements Eq. (11),

$$A_{m_A, \lambda A_1}^{2, 3, 1} (\Theta_1, \Theta_2, \Theta_3) = \sum_{\lambda_R} \langle \Theta_1, \Theta_2, \Theta_3 | R \bar{\lambda}_1, \lambda_R \rangle | s_A, m_A \rangle \times \langle \Theta_1, \Theta_2, \Theta_3 | R \bar{\lambda}_1, \lambda_R \rangle | s_A, m_A \rangle \times \langle \Theta_1, \Theta_2, \Theta_3 | R \bar{\lambda}_1, \lambda_R \rangle | s_A, m_A \rangle \times \mathcal{H}_{\lambda R, \lambda_R_1}^{1,2} D^{s_{R_1}}_{m_A, \lambda R_1 + \lambda_R} (\phi_1^R, \theta_1^R, 0).$$

(14)

Note that the angles entering the decay amplitude are in general not independent from each other, but depend on the phase variables describing the decay, denoted collectively as $\Omega$.

Now, let’s consider the $A \to S(\to 1,3), 2$ decay chain. Its associated amplitude is, following Eq. (13),

$$A_{m_A, \lambda A_1}^{2, 3, 1} (\Theta_1, \Theta_2, \Theta_3) = \sum_{\lambda_R} \langle \Theta_1, \Theta_2, \Theta_3 | R \bar{\lambda}_1, \lambda_R \rangle | s_A, m_A \rangle \times \langle \Theta_1, \Theta_2, \Theta_3 | R \bar{\lambda}_1, \lambda_R \rangle | s_A, m_A \rangle \times \langle \Theta_1, \Theta_2, \Theta_3 | R \bar{\lambda}_1, \lambda_R \rangle | s_A, m_A \rangle \times \mathcal{H}_{\lambda R, \lambda_R_1}^{1,2} D^{s_{R_1}}_{m_A, \lambda R_1 + \lambda_R} (\phi_1^R, \theta_1^R, 0).$$

(15)

Helicity values and angles denoted by the $\bar{\lambda}$ superscripts are defined specifically for the $A \to S(\to 1,3), 2$ decay chain: indeed, the spin coordinate systems of final particles are different from those employed to express the amplitude for the $R$ intermediate state.

To write the total amplitude of the $A \to 1,2,3$ decay, amplitudes associated to different intermediate states must be summed coherently to properly include interference effects. The sum can be performed only if the final particle spin coordinate systems match among different decay chains. Since helicity coordinate systems are different for different decay chains, they must be rotated to a common reference system, for each final particle. The matching of the final particle spin coordinate systems has already been addressed in diverse ways [3,5], but none considered it in light of the underlying problem of a consistent phase definition of the spin states, described in Section 2, and its consequences, which will be shown in Section 6. In the following, a method for the matching of the final particle spin coordinate systems with a consistent phase definition of spin states applicable to any multi-body decay topology is presented.

In principle, any final particle spin coordinate system can be selected to express amplitudes, provided that the rotations needed to reach such systems from the different helicity systems introduce no $2\pi$ rotation differences among the rotation sequences from initial to final particle systems of different decay chains. Otherwise, see Eq. (4), an unphysical minus sign is introduced between different amplitudes for fermion states. A simple way to bypass this problem is to rotate the spin coordinate systems to the mother particle one unwinding the rotation sequence for each decay chain, i.e. inverting the rotation sequence step-by-step in order to avoid topologically inequivalent trajectories in the $SU(2)$ group space. The final particle spin coordinate systems therefore coincide with the mother particle one, which is the canonical choice of spin states [4].

For the three-body decay under consideration, the amplitude for $R$ intermediate states must be rotated back to the $R$ helicity coordinate system and then to the $A$ particle spin coordinate system. For particle 1, the helicity states $|s_1, \lambda_1^R \rangle$ are rotated to $|s_1, m_1' \rangle$ states in the $R$ spin coordinate system by $\mathcal{R}(0, -\theta_1^R, -\phi_1^R)$, which are then rotated to the canonical basis $|s_1, m_1 \rangle$ by $\mathcal{R}(0, -\theta_R, -\phi_R)$; see Appendix A for the definition of inverse rotations. In terms of Wigner $D$-matrices,

$$|s_1, m_1 \rangle = \sum_{m_1'} D_{m_1, m_1'}^{s_1} (0, -\theta_R, -\phi_R) |s_1, m_1' \rangle = \sum_{m_1'} D_{m_1, m_1'}^{s_1} (0, -\theta_R, -\phi_R) \times \sum_{\lambda_1^R} D_{\lambda_1^R, m_1'}^{s_1} (0, -\theta_1^R, -\phi_1^R) |s_1, \lambda_1^R \rangle. \quad (16)$$

The rotation sequence for particle 2 is exactly identical to particle 1, since their spin states are expressed in the same coordinate system, while for particle 3, whose spin states are already expressed in the $R$ spin coordinate system, only the second rotation is needed. The
amplitude for \( R \) intermediate states is therefore
\[
A_{m_A,m_1,m_2,m_3}^{A \rightarrow R,3 \rightarrow 1,2,3} (\Omega) = \sum_{m'_1, \lambda'_1} D_{\lambda'_1}^{m'_1,m_1} (0, -\theta_R, -\phi_R) \times D_{m_1,m_2}^{m'_1, m_1} (0, -\theta_R, -\phi_R) \times D_{m_2,m_3}^{m'_1, m_1} (0, -\theta_R, -\phi_R) \times A_{m_A,m_1,m_2,m_3}^{A \rightarrow R,3 \rightarrow 1,2,3} (\Omega).
\]

The amplitude for \( S \) intermediate states can be written analogously, taking into account the different decay topology
\[
A_{m_A,m_1,m_2,m_3}^{A \rightarrow S,2 \rightarrow 1,2,3} (\Omega) = \sum_{m'_1, \lambda'_1} D_{\lambda'_1}^{m'_1,m_1} (0, -\theta_S, -\phi_S) \times D_{m_1,m_2}^{m'_1, m_1} (0, -\theta_S, -\phi_S) \times D_{m_2,m_3}^{m'_1, m_1} (0, -\theta_S, -\phi_S) \times A_{m_A,m_1,m_2,m_3}^{A \rightarrow S,2 \rightarrow 1,2,3} (\Omega),
\]
in which now the \( m'_1 \) spin values are referred to the \( S \) helicity coordinate system.

Finally, the total decay amplitude is obtained summing the amplitudes associated to each intermediate state,
\[
A_{m_A,m_1,m_2,m_3}^{A \rightarrow 1,2,3} (\Omega) = \sum_i A_{m_A,m_1,m_2,m_3}^{A \rightarrow R_i,3 \rightarrow 1,2,3} (\Omega) + \sum_j A_{m_A,m_1,m_2,m_3}^{A \rightarrow S_j,2 \rightarrow 1,2,3} (\Omega) + \sum_k A_{m_A,m_1,m_2,m_3}^{A \rightarrow U_k,1 \rightarrow 1,2,3} (\Omega),
\]
in which the amplitude for \( U \) intermediate states can be written similarly as for \( R \) and \( S \) states.

5 Polarised differential decay rate

The generic spin state of a statistical ensemble of particles is defined by the associated density operator \( \hat{\rho} \) [2]:
given an ensemble of spin states \( |\psi_i\rangle \) occurring with probability \( p_i \), the density operator is
\[
\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i|,
\]
so that the expectation value of any operator \( \hat{X} \) can be expressed as
\[
\langle \hat{X} \rangle = \sum_i p_i \langle \psi_i | \hat{X} | \psi_i \rangle = \text{Tr} [\hat{\rho} \hat{X}].
\]

From the definition Eq. (20) follows that the density operator is hermitian with unit trace.

The decay rate of multi-body decays \( A \rightarrow \{i = 1, \ldots, n\} \) for definite spin eigenstates is the squared modulus of the transition amplitude between the \( A \) particle initial state \( |s_A, m_A\rangle \) and the final particle product state \( \{|s_i\}, \{m_i\} \rangle = \otimes_i |s_i, m_i\rangle \),
\[
p_{m_A, \{m_i\}} (\Omega) = |\langle s_A, m_A |\hat{T}| \{s_i\}, \{m_i\} \rangle|^2 = |A_{m_A, \{m_i\}} (\Omega)|^2.
\]
Generic polarisation states are described by introducing the density operators for the initial particle state \( \hat{\rho}^A \) and the final particle states \( \hat{\rho}^{(i)} \), which are included in the decay rate Eq. (22) by inserting suitable identity resolutions, obtaining
\[
p(\hat{\rho}^A, \hat{\rho}^{(i)}; \Omega) = \text{Tr} [\hat{\rho}^A \hat{T} \hat{\rho}^{(i)} \hat{T}^\dagger]
\]
\[
= \sum_{m_A, m'_A} \sum_{\{m_i\}, \{m'_i\}} \hat{\rho}_{m_A, m'_A}^{A, \{m_i\}, \{m'_i\}} \times A_{m_A, \{m_i\}} (\Omega) A_{m'_A, \{m'_i\}}^\dagger (\Omega)
\]

6 Effects of an unphysical phase in spin state definition

We stress the need for the proposed matching of final particle spin states by considering the effects that an unphysical phase arising from an inconsistent definition of the final particle spin state phase has on the decay distributions involving polarised particles.

We consider the general case of a particle \( A \) decay to \( \{i = 1, \ldots, n\} \) final state particles with spin, passing through two intermediate states \( R \) and \( S \). The \( A \) spin state \( |s_A, m_A\rangle \) is defined with respect to a given, well-defined spin coordinate system, while the final particle spin states \( |s_i, m_i\rangle \) are defined with respect to a spin coordinate system derived from the decay kinematics, e.g. helicity systems. The associated amplitudes are denoted as \( A_{m_A, \{m_i\}}^R \) and \( A_{m_A, \{m_i\}}^S \), which in the helicity formalism can be expressed in the general form
\[
A_{m_A, \{m_i\}}^R = \sum_{\lambda_R} \mathcal{H}_{m_A, \{m_i\}}^{R} D_{m_A, \lambda_R; \{m_i\}}^R (\Omega),
\]
in which \( \mathcal{H}_{m_A, \{m_i\}}^{R} \) is the set of complex couplings describing the decay dynamics for the combinations of \( \{m_i\} \) and \( \lambda_R \) (helicity of the \( R \) states) allowed by angular momentum conservation, and \( D_{m_A, \lambda_R; \{m_i\}}^R (\Omega) \) are
functions collecting the phase space dependence of the amplitude. For a three-body decay the amplitudes have been explicitly obtained in Eq. (17) and (18), respectively.

Let’s suppose that, due to an inconsistent definition of the final particle spin state phase, an unphysical phase difference $\exp(i\psi)$ is introduced between the two amplitudes. The unpolarised decay rate is

$$p(\Omega) = \sum_{m_A, \{m_i\}} |A^R_{m_A, \{m_i\}} + \exp(i\psi)A^S_{m_A, \{m_i\}}|^2$$

$$= \sum_{m_A, \{m_i\}} \left\{ |A^R_{m_A, \{m_i\}}|^2 + |A^S_{m_A, \{m_i\}}|^2 \right\} + 2\text{Re} \left[ \exp(-i\psi)A^R_{m_A, \{m_i\}}A^S_{m_A, \{m_i\}}^\ast \right],$$

(25)

and it is affected by the unphysical phase via the interference term

$$2\text{Re} \left[ \exp(-i\psi) \times \sum_{m_A, \{m_i\}} \sum_{\lambda_R, \lambda_S} \mathcal{H}^R_{\lambda_R, \{m_i\}} \mathcal{H}^S_{\lambda_S, \{m_i\}} \right] \times D^R_{m_A, \lambda_R, \{m_i\}}(\Omega)D^S_{m_A, \lambda_S, \{m_i\}}(\Omega).$$

(26)

To understand the effect of the unphysical phase in an amplitude fit, we must consider that the fit is able to discriminate among products of helicity couplings which enter the decay rate via different dependences $\mathcal{D}^R_{m_A, \lambda_R, \{m_i\}}(\Omega)D^S_{m_A, \lambda_S, \{m_i\}}(\Omega)$ on the phase space variables. If the amplitude model consists of a sum of independent combinations of helicity couplings, each one having a different phase space dependence, then the fit is not able to distinguish helicity couplings from the exp(i\psi) phase, and the unphysical phase is absorbed into the fitted values of the couplings. On the contrary, if there is at least one combination of helicity couplings which enters multiple phase space dependencies with a different relative phase, the unphysical phase has observable effects on the amplitude fit, since in this case it is not possible to redefine independently the helicity couplings to absorb it. Whether an unphysical phase has been absorbed into the helicity couplings or it affects physical results depends on the specific decay under consideration.

Care must be taken in any case when reproducing an amplitude model from the results of an amplitude fit: if the model differs from that used for fitting the data by a phase, the resonance pattern in the decay distributions is distorted following Eq. (25). If interference effects are large, significant discrepancies between model and experimental data decay distributions will appear.

Now, let’s consider the case of a polarised decay in which both initial and final particles are polarised along their quantisation axes. The associated density operators are diagonal in the spin states with real matrix elements,

$$\hat{\rho} = \sum_{i} p_i \langle s, m_z \rangle_i \langle s, m_z \rangle,$$

(27)

so that the polarised decay rate Eq. (26) is

$$p(\hat{\rho}, \hat{\rho}^i; \Omega) = \sum_{m_A, \{m_i\}} \rho^A_{m_A, m_A} \rho^i_{\{m_i\}, \{m_i\}} \times |A^R_{m_A, \{m_i\}} + \exp(i\psi)A^S_{m_A, \{m_i\}}|^2,$$

(28)

whose interference term is

$$\sum_{m_A, \{m_i\}} \rho^A_{m_A, m_A} \rho^i_{\{m_i\}, \{m_i\}}$$

$$\times 2\text{Re} \left[ \exp(-i\psi) \sum_{m_A, \{m_i\}} \rho^A_{m_A, m_A} \rho^i_{\{m_i\}, \{m_i\}} \times \mathcal{D}^R_{m_A, \lambda_R, \{m_i\}}(\Omega)D^S_{m_A, \lambda_S, \{m_i\}}(\Omega) \right].$$

(29)

The effect of the unphysical phase is analogous to the case of unpolarised decays: non-zero z-axis polarisation components just introduce real combinations of products of helicity couplings not present in the unpolarised rate, all multiplied by the same exp(i\psi) phase. Whether the unphysical phase can be absorbed into the helicity couplings or not depends on the presence of independent combinations of helicity coupling constrained by different phase space distributions, as for unpolarised decays. Note that the presence of non-zero polarisation introduces both new helicity coupling combinations and new phase space dependences. If the polarised decay rate is used to measure the particle polarisations, by fitting the density matrix elements from experimental data, the unphysical phase can affect the measured polarisation values if not absorbed in the helicity couplings.

In the case of generic polarisations, the density operators can have non-diagonal complex matrix elements, representing the orthogonal polarisation components.
The decay rate following from Eq. (28) is

\[ p(\hat{\rho}^A, \hat{\rho}^{(i)}; \Omega) = \sum_{m_A, m_A'} \sum_{m_i, m_i'} \hat{\rho}^A_{m_A, m_A'} \hat{\rho}^{(i)}_{m_i, m_i'} \times \left( A_{m_A, m_i}^R + \exp(i\psi) A_{m_A, m_i}^S \right) \times \left( A_{m_A', m_i}^R + \exp(-i\psi) A_{m_A', m_i}^S \right), \]

and the part depending on the unphysical phase is

\[ \sum_{m_A, m_A'} \sum_{m_i, m_i'} \hat{\rho}^A_{m_A, m_A'} \hat{\rho}^{(i)}_{m_i, m_i'} \times \left( A_{m_A, m_i}^R \exp(-i\psi) A_{m_A, m_i}^S \right) + \left( A_{m_A', m_i}^R \exp(i\psi) A_{m_A', m_i}^S \right). \]

(30)

The part of the decay rate sum involving the diagonal elements of density matrices \((m_A = m_A', m_i = m_i')\) has already been considered in Eq. (28). Since density matrices are hermitian, the remaining part of the rate can be written as

\[ 2Re \left[ \exp(-i\psi) \sum_{m_A \neq m_A'} \sum_{m_i \neq m_i'} \hat{\rho}^A_{m_A, m_A'} \hat{\rho}^{(i)}_{m_i, m_i'} \times \sum_{\lambda_R, \lambda_S} H_{\lambda_R, m_i}^R H_{\lambda_S, m_i}^S \times D_{m_A, \lambda_R, m_i}^R(\Omega) D_{m_A, \lambda_S, m_i}^S(\Omega) \right]. \]

(31)

(32)

The complex density matrix elements now introduce complex combinations of the helicity couplings, and their value can not be separated from the couplings or the unphysical phase as before, since the \(Re\) operation is not linear on complex density matrix elements. Therefore, an unphysical phase has observable consequences on the decay distributions, and it changes the value of the fitted polarisation components. The direct impact of the unphysical phase on the orthogonal polarisation components is a consequence of the mixing of the orthogonal spin eigenstates caused by the \(\exp(i\psi)\) phase.

7 Conclusions

In this article, we consider the issues related to the phase definition of spin states and their consequences on the helicity formalism. We show the helicity formalism has an inconsistent definition of helicity state phases for two-body decays, which can be solved by choosing the same spin coordinate system for the two daughter particles.

We present a method to write helicity amplitudes with a consistent definition of final particle spin states for different intermediate states, applicable to any multi-body decay topology. Helicity amplitudes have been explicitly derived for three-body decays.

We demonstrate the need for a consistent definition of final particle spin state phases by evaluating the impact of an improper phase definition on the decay distributions of polarised particles. Such unphysical phase affects the decay distributions via the interference terms associated to different intermediate states. The decay distributions described by a fixed amplitude model are thus modified if interference effects are significant. In amplitude fits in which the helicity couplings describing the decay dynamics are extracted from experimental data, the unphysical phase may be absorbed into the fitted couplings according to the specific decay under consideration, but only for unpolarised particles or particles polarised along their quantisation axes. The unphysical phase affects in any case the decay distributions for particles polarised orthogonally to their quantisation axes, this preventing the measurement of particle polarisations from experimental data.

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Appendix A: Euler rotations and their representation on spin states

The rotation of a coordinate system \((x, y, z)\) to a new one \((X, Y, Z)\) can be described by means of Euler rotations, which, in the \(z-y-z\) convention for the rotation axes, are

\[ R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma) = e^{-i\alpha S_z} e^{-i\beta S_y} e^{-i\gamma S_z}, \]

(A.1)

in which rotations are expressed with respect to the original axes \((x, y, z)\).

The action of an Euler rotation on spin states \(|s, m)\) associated to the spin coordinate system \((x, y, z)\) is

\[ R(\alpha, \beta, \gamma) |s, m) = \sum_{m'} D^s_{m', m}(\alpha, \beta, \gamma) |s, m'), \]

(A.2)

in which the Wigner D-matrices \(D^s_{m', m}(\alpha, \beta, \gamma)\) are

\[ D^s_{m', m}(\alpha, \beta, \gamma) = \langle s, m' | R(\alpha, \beta, \gamma) | s, m \rangle. \]

(A.3)
Following from Eq. (A.1), the Wigner $D$-matrices can be factorised as
\[
D_{s',m,s}^m(\alpha,\beta,\gamma) = \langle s,m'|\hat{e}^{-i\alpha}\hat{S}_z\hat{e}^{-i\beta}\hat{S}_y\hat{e}^{-i\gamma}\hat{S}_z|s,m\rangle = e^{-im\alpha}d_{m',m}^S(\beta)e^{-im'\gamma},
\]  
(A.4)
in which the Wigner $d$-matrix elements are real combinations of trigonometric functions of $\beta$.

The inverse rotation from the final coordinate system $(X, Y, Z)$ to the initial one $(x, y, z)$ follows from Eq. (A.1),
\[
R^{-1}(\alpha,\beta,\gamma) = R_z^{-1}(\gamma)R_y^{-1}(\beta)R_z^{-1}(\alpha) = e^{i\gamma\hat{S}_z}e^{i\beta\hat{S}_y}e^{i\alpha\hat{S}_z} = R(-\gamma,-\beta,-\alpha),
\]  
(A.5)
and the Wigner $D$-matrix associated to $|s,m\rangle$ states is $D_{s',m,s}^m(-\gamma,-\beta,-\alpha)$.

References

1. M. Jacob and G. C. Wick, *On the general theory of collisions for particles with spin*, Annals Phys. 7 (1959) 404.
2. E. Leader, *Spin in particle physics*, vol. 15, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol., 2011.
3. LHCb collaboration, R. Aaij et al., *Observation of $J/\psi p$ resonances consistent with pentaquark states in $\Lambda_b \to J/\psi p K^-$ decays*, Phys. Rev. Lett. 115 (2015) 072001, arXiv:1507.03414.
4. H. Chen and R.-G. Ping, *Coherent helicity amplitude for sequential decays*, Phys. Rev. D95 (2017), no. 7 076010, arXiv:1704.05184.
5. M. Mikhasenko et al., *Dalitz-plot decomposition for three-body decays*, arXiv:1910.04566.
6. J. D. Richman, *An experimenters guide to the helicity formalism*, CALT-68-1148.