Poincaré gauge theory from higher derivative matter Lagrangians

Pradip Mukherjee

Department of Physics, Presidency College, 86/1 College Street, Kolkata - 700 073, India
E-mail: mukhpradip@gmail.com

Received 7 May 2010, in final form 26 July 2010
Published 28 September 2010
Online at stacks.iop.org/CQG/27/215008

Abstract
Starting from matter Lagrangians containing a higher order derivative than the first order, we construct the Poincaré gauge theory by localizing the Poincaré symmetry of the matter theory. The construction is shown to follow the usual geometric procedure of a gravitational coupling, thereby buttressing the geometric interpretation of the Poincaré gauge theory.

PACS numbers: 04.50.Kd, 04.40.-b

1. Introduction
Poincaré gauge theory (PGT) is a celebrated framework where the gauge principle, so successful in the other branches of theoretical physics, is applied in the context of the theories of gravitation [1–3]. The edifice of the PGT is constructed by localizing the Poincaré symmetry in Minkowski space. One starts with a matter theory invariant under global Poincaré transformations. Naturally, this does not remain invariant when the parameters of the Poincaré transformations are the functions of spacetime. The PGT emerges from attempts to modify the matter theory so that it becomes invariant under the local Poincaré transformations. Compensating potentials are introduced in the process, the dynamics of which is provided by invariant densities constructed out of the field strengths obtained from the usual gauge theoretic procedure. The theory has been ubiquitous in classical gravity [3] as well as in its extension to noncommutative spacetime [4–8].

A very important aspect of the PGT is the correspondence of the gauge fields introduced here with the tetrad and spin-connection in Riemann–Cartan spacetime. The geometric interpretation of the PGT, based on local Lorentz transformations (LLT) and general coordinate transformations or diffeomorphisms (diff), is a well-known fact [3]. This has recently been explicitly demonstrated by comparing the transformations of the geometrical structures such as the metric and the connection under LLT plus diff with those under the Poincaré gauge transformations [9]. The usefulness of the PGT stems from this geometric interpretation, i.e.
the fact that theories invariant under local Poincaré transformations can be viewed as invariant theories in curved spacetime. We can thus view the construction of the PGT as a means of coupling the initial particle theory with gravity.

The geometric interpretation of the PGT is a cherished feature which is however more of an observed fact rather than a product of some underlying fundamental principle. Primarily, the PGT originates from the localization of the global Poincaré symmetry of matter theories. It is a gauge theory in the Minkowski space and there is no \textit{a priori} reason that the transformations of the gauge fields should mimic the transformations of the geometric structures in the curved spacetime. This correspondence is found \textit{en route} \cite{9} which enables one to identify the theory as carrying the symmetry of the curved spacetime. It is thus notable in this context that this procedure has been developed in the literature for Lagrangians containing first-order derivatives only \cite{3}. The questions like whether the geometric interpretation will persist if we apply the gauge principle to a Poincaré invariant theory containing higher derivative Lagrangians are nontrivial and demands thorough analysis.

The construction of the PGT starting from the matter theory with higher derivative Lagrangians has not, so far, attracted much attention in the literature. On the other hand, this problem has a direct bearing on the geometrical interpretation of the PGT. If we look from the point of view of extending the gauge principle to theories with higher derivative Lagrangians we find that it is a complicated task which in general requires the introduction of new tensor gauge fields \cite{10}. However, from the geometric point of view the coupling of Lagrangian field theories with gravity is straightforward. Employing the principle of general covariance, such a theory formulated in the Minkowski space is coupled to gravity by replacing the Minkowski metric by the metric of the curved spacetime and substituting ordinary derivatives by covariant derivatives \cite{11}. The localization of the Poincaré invariance of higher order matter theory, i.e. the construction of the PGT from such theories is thus worth investigating. The point is to verify whether the PGT construction corresponds to the geometric coupling when matter theories with higher derivative Lagrangians are taken as the starting point. This will indeed be an alternative independent check of the geometric interpretation of the PGT. Moreover, Lagrangian theories with higher order derivatives are interesting in their own right. These have been discussed over a long period of time \cite{12–21}. Such theories have appeared in different contexts: higher derivative terms naturally occur as quantum corrections to the lower order theories; various stringy models are shown to be equivalent to higher derivative theories; the literature is rich in possible higher derivative theories of gravity which find application in quantum gravity \cite{22}. It will thus be instructive to pursue the PGT construction directly by localizing the global Poincaré symmetry of matter Lagrangians, containing higher derivatives of the fields than the first. We propose to address this question in the present paper.

The status of the PGT in the realm of the gauge theory is indeed very special. The transformation of the potentials in the PGT obtained from canonical analysis is different from what is obtained from the localization of the Poincaré symmetry. The two can be mapped only modulo the equations of motion \cite{9, 23}. The construction of the PGT from more general Lagrangians thus becomes more interesting in this perspective. Our results indeed vividly illustrate the unique role of the PGT. At first terms up to the second derivatives are retained in the matter Lagrangian. The question arises how to modify the second derivative terms so that the theory has the local Poincaré invariance. Taking the cue from the geometric interpretation of the usual theory, we substitute the second derivatives $\partial_k \partial_m$ by $\nabla_k \nabla_m$, where $\nabla$ is the covariant derivative. We find that the transformation of $\nabla_k \nabla_m \phi$ under local Poincaré transformations \textit{is} of the same form as that of $\partial_k \partial_m \phi$ under global Poincaré transformations. No new gauge fields are required—a remarkable fact from the point of view of the extension of the gauge principle to the higher derivative theories since it is known that in general such
a procedure involves the introduction of new (tensor) gauge fields [10]. The observation is in correspondence with the geometric interpretation of the PGT [3, 9] and thus manifests the geometric significance of the PGT in a more general context.

The above discussions apply to Lagrangian theories with second derivative terms. The purpose of the paper would remain unfulfilled if we do not demonstrate the validity of our construction for the general case. We prove the validity of our construction in the general setting by induction. First, we demonstrate that if our construction is valid for a theory containing the $n$th order derivatives then it is true for the $(n + 1)$th order derivatives. But we have proved that the construction goes through for $n = 2$. By this inductive procedure the same geometric connection derived from a limited class of first-order theories [1–3] is shown to be valid for the most general Lagrangian theory.

The organization of the paper is as follows. In section 2 the PGT construction from the usual Lagrangian theories where the Lagrangian contains only the first derivatives of the fields, is reviewed. While this material is standard, it sets the stage for the extension of the construction for more general theory. In the following section, the construction of the PGT from higher derivative matter theory is presented. The paper ends with a few concluding remarks in section 4.

2. Poincaré gauge theory—a review

We begin with the algorithm of constructing the PGT from a matter theory when the Lagrangian does not contain higher derivatives [1–3]. The starting point of the PGT is the global Poincaré transformations in the Minkowski space:

$$x^\mu \rightarrow x^\mu + \xi^\mu,$$

(1)

where

$$\xi^\mu = \theta^\mu_\nu x^\nu + \epsilon^\mu,$$

with both $\theta_{\mu\nu}$ and $\epsilon^\mu$ being infinitesimal constants and $\theta^{\mu\nu}$ being the antisymmetric. We construct a local basis $e_i$ at each spacetime point which are related to the coordinate basis $e_\mu$ by

$$e_i = \delta^{\mu}_i e_\mu.$$

The fields will be assumed to refer to the local basis. The Latin indices refer to the local Lorentz frame and the Greek indices refer to the coordinate frame. The differentiation between the local and the coordinate bases appears to be a formal one here, but it will be a necessity in curved spacetime.

Now consider a matter Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi, \partial_k \phi).$$

The corresponding action is

$$I = \int d^4x \mathcal{L}(\phi, \partial_k \phi).$$

(2)

Under (1) the action transforms as

$$I \rightarrow I + \Delta I,$$

where

$$\Delta I = \int d^4x \Delta \mathcal{L}(\phi, \partial_k \phi).$$

(3)
The variation $\Delta L$ is given by
\[ \Delta L = \delta L + \xi^\mu \partial_\mu L + \partial_\mu \xi^\mu L, \] (4)
where $\delta L$ is the form variation of the Lagrangian
\[ \delta L = L'(x) - L(x), \] (5)
with $L'$ being the transformed Lagrangian. The condition for the invariance of the theory is
\[ \Delta L = 0. \] (6)

It is useful to scrutinize the above invariance condition with care. For global Poincaré transformations,
\[ \partial_\mu \xi^\mu = 0. \] (7)

Also, the field and its derivative transform as
\[ \delta \phi = \left( \frac{1}{2} \theta^{ij} \Sigma_{ij} - \xi^\mu \partial_\mu \right) \phi = \mathcal{P} \phi \]
\[ \delta \partial_k \phi = \left( \frac{1}{2} \theta^{ij} \Sigma_{ij} - \xi^\mu \partial_\mu \right) \partial_k \phi + \theta_{k}^{\ i} \partial_j \phi \]
\[ = \mathcal{P} \partial_k \phi + \theta_{k}^{\ i} \partial_j \phi, \] (9)

where $\Sigma_{ij}$ are the Lorentz spin matrices, the form of which depends on the particular representation to which $\phi$ belong. These are matrices with constant elements that satisfy the Lorentz algebra
\[ [\Sigma_{ij}, \Sigma_{kl}] = \eta_{il} \Sigma_{jk} - \eta_{ik} \Sigma_{jl} + \eta_{ij} \Sigma_{lk}. \] (10)

Equations (7)–(9) are instrumental for the invariance condition (6) to be satisfied. The form variation $\delta L$ is explicitly given by
\[ \delta L = \frac{\delta L}{\delta \phi} \delta \phi + \frac{\delta L}{\delta \partial_k \phi} \delta \partial_k \phi, \] (11)
and it is the form of variations given by (8), (9) and condition (7) which lead to the invariance of the theory.

When the Poincaré symmetry (1) is assumed to be a local symmetry, the parameters $\theta$ and $\epsilon$ are no longer constants but become the functions of the spacetime coordinates. Here, it is advantageous to take the parameters $\xi^\mu = \theta^{ij} x^j + \epsilon^\mu$ and $\theta^{ij}$ as the independent parameters. It is natural that the action (2) which was invariant under (1) with constant parameters (global Poincaré transformations) will cease to remain invariant when the parameters become the functions of spacetime (local Poincaré transformations). The reasons are:

(i) equation (7) is no longer true and
(ii) though the fields transform as equation (8):
\[ \delta \phi = \left( \frac{1}{2} \theta^{ij} \Sigma_{ij} - \xi^\mu \partial_\mu \right) \phi = \mathcal{P} \phi, \]
their derivatives $\partial_k \phi$ transform as
\[ \delta \partial_k \phi = \left( \frac{1}{2} \theta^{ij} \Sigma_{ij} - \xi^\mu \partial_\mu \right) \partial_k \phi - \partial_k \xi^\nu \partial_\nu \phi + \frac{1}{2} \partial_k \theta^{ij} \Sigma_{ij} \phi \]
\[ = \mathcal{P} \partial_k \phi - \partial_k \xi^\nu \partial_\nu \phi + \frac{1}{2} \partial_k \theta^{ij} \Sigma_{ij} \phi, \] (12)

which is different from their counterpart in equation (9).

1 The form variation of a quantity will always be denoted by the precedent $\delta$.

2 In the most general case, a total divergence may be added to the right-hand side. However, condition (6) is sufficiently general in (3+1) dimensions.
To modify the matter action so as to be invariant under the local Poincaré transformations, one has to remedy the above departures. The first thing is to replace the ordinary derivative $\partial_k \phi$ by some covariant derivative $\nabla_k \phi$ which will transform as in (9). This is done in two steps as follows.

(i) In the first step, the $\theta$-covariant derivative $\nabla_\mu$ is introduced which eliminates the $\partial_\mu \theta_{ij}$ term from (12). We define $\nabla_\mu$ as

$$\nabla_\mu = \partial_\mu + \frac{1}{2} \omega^{ij}_\mu \Sigma_{ij}, \quad (13)$$

where $\omega^{ij}_\mu$ are the ‘gauge potentials’. The required transformation of $\nabla_\mu \phi$ is

$$\delta \nabla_\mu \phi = P \nabla_\mu \phi - \partial_\mu \xi^\nu \nabla_\nu \phi. \quad (14)$$

The transformation of the ‘gauge field’ $\omega^{ij}_\mu$ is determined from this requirement.

(ii) In the next step, the covariant derivative in the local frame is constructed as

$$\nabla_k = b^\mu_k \nabla_\mu, \quad (15)$$

where $b^\mu_k$ is another compensating field. For later convenience, we define $b^\mu_k$ as the inverse to $b_k^\mu$. If the fields $b^\mu_k$ and $\omega^{ij}_\mu$ transform as

$$\delta b^\mu_k = \theta^i_k b^\mu_k - \partial_\mu \xi^\rho b^\rho_k - \xi^\rho \partial_\rho b^\mu_k$$

$$\delta \omega^{ij}_\mu = \theta^i_k \omega^{kj}_\mu + \theta^j_k \omega^{ik}_\mu - \partial_\mu \theta^{ij} - \partial_\mu \xi^\rho \omega^{ij}_\rho - \xi^\rho \partial_\rho \omega^{ij}_\mu, \quad (16)$$

then $\nabla_k \phi$ transforms as

$$\delta \nabla_k \phi = P \nabla_k \phi + \theta^j_k \nabla_j \phi. \quad (17)$$

This transformation rule is formally identical with (9). So we get the required transformation. The matter Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_k \phi)$ which was invariant under global Poincaré transformations is converted to an invariant density $\tilde{\mathcal{L}}$ under local Poincaré transformations by replacing the ordinary derivative $\partial_k$ by the covariant derivative $\nabla_k$, i.e.

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\phi, \nabla_k \phi).$$

The departure from equation (7) can be accounted for by altering the measure of spacetime integration suitably. An invariant action is now constructed as

$$I = \int d^4 x \, b \tilde{\mathcal{L}}(\phi, \nabla_k \phi),$$

where $b = \det b^\mu_k$. The invariance of this action is ensured by the transformations (16) of the ‘potentials’.

At this point, a very interesting property of the above construction should be noted. The transformations (16) comprise the Poincaré gauge transformations. Their structure suggests a geometric interpretation. The basic fields $b^\mu_k$ and $\omega^{ij}_\mu$ mimic the tetrad and the spin connection in curved spacetime. The most general invariance group in curved spacetime consists of the LLT plus diff. Observe in (16) that the Latin indices transform as under LLT with parameters $\theta^{ij}$, and the Greek indices transform as under diff with parameters $\xi^\mu$. This suggests a correspondence between the Poincaré gauge transformations and the geometric transformations of the curved spacetime. That this correspondence is an equivalence, has recently been demonstrated explicitly [9]. The geometric interpretation of the PGT is a crucial step. It emerges from the application of the gauge principle. At this point, it is notable that the construction is limited to Lagrangians which contain only the first derivative of the fields. The question of what happens for higher derivative theories begs to be addressed. One wonders
whether the geometric interpretation will still hold? Otherwise the geometric connection will be episodic. The question of higher derivative theories will thus provide an independent check of the geometric connection.

With the question of higher derivative theories pending for the time being, we come back to the construction of the PGT in the usual context. Corresponding to the basic fields $b^i_\mu$ and $\omega^{ij}_\mu$, the Lorentz field strength $R^{ij}_{\mu\nu}$ and the translation field strength $T^{i}_{\mu\nu}$ are obtained following the usual procedure in the gauge theory. The commutator of two $\theta$-covariant derivatives gives $R^{ij}_{\mu\nu}$:

$$[\nabla_{\mu}, \nabla_{\nu}]\phi = \frac{1}{2} R^{ij}_{\mu\nu} \Sigma_{ij} \phi,$$

whereas the commutator of two $\nabla_k$ derivatives furnishes the additional fields $T^{i}_{\mu\nu}$ as

$$[\nabla_k, \nabla_l]\phi = \frac{1}{2} b^{i}_\mu b^{j}_\nu R^{ij}_{\mu\nu}/\Sigma_{1ij} \phi,$$

where

$$\phi = \frac{1}{2} b^{i}_\mu b^{j}_\nu R^{ij}_{\mu\nu} \Sigma_{ij} \phi - b^{i}_\mu b^{j}_\nu T^{i}_{\mu\nu} \nabla_i \phi$$

These defining equations give the following expressions for the field strengths:

$$T^{i}_{\mu\nu} = \partial \mu b^{i}_\nu + \omega^{i}_\mu b^{j}_\nu - \partial \nu b^{i}_\mu - \omega^{j}_\nu b^{i}_\mu,$$

$$R^{ij}_{\mu\nu} = \partial \mu \omega^{ij}_\nu - \partial \nu \omega^{ij}_\mu + \omega^{i}_\mu \omega^{jk}_\nu - \omega^{j}_\nu \omega^{ik}_\mu.$$

We have already observed the connection of Poincaré gauge symmetry with LLT plus diff invariances in curved spacetime. This connection may further be pursued at the level of the field strengths. From the point of view of the PGT the transformations of the Lorentz field strength $R^{ij}_{\mu\nu}$ and the translation field strength $T^{i}_{\mu\nu}$, can easily be obtained from (16) and (19),

$$\delta R^{ij}_{\mu\nu} = \theta^i_k R^{kj}_{\mu\nu} + \theta^j_k R^{ik}_{\mu\nu} - \partial \mu \xi^i \ R^{ij}_{\mu\nu} - \partial \nu \xi^j \ R^{ij}_{\mu\nu} - \xi^i \partial \mu R^{ij}_{\mu\nu}$$

and

$$\delta T^{i}_{\mu\nu} = \theta^i_k T^{k}_{\mu\nu} - \partial \mu \xi^i \ T^{i}_{\mu\nu} - \partial \nu \xi^i \ T^{i}_{\mu\nu} - \xi^i \partial \mu T^{i}_{\mu\nu}.$$

As noted earlier in the context of the transformations (16), the Latin indices transform as under LLT with parameters $\theta^{ij}$ and the Greek indices transform as under diff with parameters $\xi^\mu$. We get the expected transformations under LLT and diff.

The locally Poincaré invariant theory is constructed in the Minkowski space and has been developed as a gauge theory. Using the geometric interpretation, the Lorentz field strength $R^{ij}_{\mu\nu}$, and the translation field strength $T^{i}_{\mu\nu}$ may be identified with the Riemann tensor and the torsion, respectively. Using these basic structures, gravity can be formulated in the framework of the PGT. The gravitational dynamics follows in general from the combination of the Riemann tensor and the torsion which is invariant under (16). The procedure of obtaining the most general gravitational dynamics has been elaborately studied in the literature [24], but details of these are not required in the present discussion. We only note the crucial role of the geometric correspondence in casting PGT as the theory of gravity.

3. Localization of Poincaré symmetry of theories with a higher order derivative

So far, we have been reviewing the methodology of coupling Lagrangian field theories in the framework of the PGT when the Lagrangian of the theory contains only first-order derivatives. While this material is standard, it paves the way to further generalization. We will attempt to extend the same basic mechanism for theories with a higher order derivative. For simplicity, we will begin with theories inclusive of a second-order derivative. Note that the extension of the gauge principle to higher derivative theories is known to be associated with the introduction of new gauge fields [10]. It is thus remarkable that we do not introduce any new gauge potential.
We consider theories whose action is given by
\[ I = \int d^4 x \, L(\phi, \partial_k \phi, \partial_k \partial_l \phi). \] (22)
It is assumed that the theory is invariant under global Poincaré transformations. This means that under the transformation (1)
\[ \delta I = \int d^4 x \, \Delta L = 0, \] (23)
where
\[ \Delta L = \delta L + \xi^\mu \partial_\mu L + \partial_\mu \xi^\mu L \] (24)
with
\[ \delta L = \delta L_{\phi \phi} + \delta L_{\partial_k \phi \partial_\mu \phi} + \delta L_{\partial_\mu \partial_l \phi \partial_\nu \phi}. \] (25)
Our concern is to modify the action (22) in such a way so that it is invariant under local Poincaré transformation. The catch is in (25). Note the change in (25) compared with (11). Due to the presence of the second derivative in the Lagrangian an extra term has appeared in (25) as compared with (11). Let us calculate the form variation of \( \partial_k \partial_l \phi \) under global Poincaré transformation (1). This is
\[ \delta \partial_k \partial_l \phi = \mathcal{P} \partial_k \partial_l \phi + \theta^m_{kl} \partial_m \partial_l \phi + \theta^m_{kl} \partial_k \partial_m \phi. \] (26)
To get a theory of the form (22) invariant under the local Poincaré gauge transformation one has to look for pieces that transform as (26). A possible candidate is \( \nabla_k \nabla_l \phi \), but one has to be sure about its transformation properties.

We will now obtain the transformation of \( \nabla_k \nabla_l \phi \) under the local Poincaré transformation. Substituting the expression of \( \nabla_k \) from (15) we can write
\[ \nabla_k \nabla_l \phi = b^\nu_k \nabla_\mu (b^\nu_l \nabla_\nu \phi). \] (27)
The covariant derivative operator \( \nabla_\mu \) operates everything on the right of it. The action on \( b^\nu_k \) is computed from the same general expression (13), where \( \Sigma_{ij} \) is now given by the appropriate vector representation
\[ [\Sigma_{ij}]^m = (\eta_{jm} \delta_i^j - \eta_{im} \delta_j^j). \] (28)
Using this we get from (27)
\[ \nabla_k \nabla_l \phi = b_k^\nu (\partial_\mu b_\nu^\mu - \omega^m_{\mu \nu} b^\nu_\mu) \nabla_\nu \phi + b_k^m \nabla_\mu \nabla_\nu \phi. \] (29)
The form variation of \( \nabla_k \nabla_l \phi \) is to be worked out from (29). A straightforward way is to expand (29) using (13) and then take the variation. After a long calculation, we get the variation of \( \nabla_k \nabla_l \phi \) as
\[ \delta \nabla_k \nabla_l \phi = \mathcal{P} \nabla_k \nabla_l \phi + \theta^m_{kl} \nabla_m \nabla_l \phi + \theta^m_{kl} \nabla_k \nabla_m \phi. \] (30)
The transformation (30) is quite of the required form (26).

The transformation relations obtained in (30) allow us to check the internal consistency of the PGT in the following way. Using (30) we can calculate \( \delta [\nabla_k, \nabla_l] \phi \) as
\[ \delta [\nabla_k, \nabla_l] \phi = \mathcal{P} [\nabla_k, \nabla_l] \phi + \theta^m_{kl} [\nabla_m, \nabla_l] \phi + \theta^m_{kl} [\nabla_k, \nabla_m] \phi. \] (31)
We have calculated the variation of \( [\nabla_k, \nabla_l] \phi \) using our expression (30) for \( \delta \nabla_k \nabla_l \phi \). Now this can also be computed in an alternative way. \( [\nabla_k, \nabla_l] \phi \) is given by (18) where it is expressed in terms of the field strengths which have known transformation properties (equations (20) and (21)). Using (18) we can write
\[ [\nabla_k, \nabla_l] \phi = \frac{1}{2} R^j_{kl} \Sigma_{ij} \phi - T^j_{kl} \nabla_j \phi. \] (32)
where we have used the definitions
\[ R_{ij}^{kl} = b^j_\mu b^i_\nu R_{ij}^{\mu\nu} \]
\[ T_{ij}^{kl} = b^j_\mu b^i_\nu T_{ij}^{\mu\nu}. \]  

The form variation of these fields may be worked out using equations (16), (20) and (21). First we express
\[ \delta \left[ \nabla_k, \nabla_l \right] \phi = \frac{1}{2} \delta R_{ij}^{kl} \Sigma_{ij} \phi + \frac{1}{2} R_{ij}^{kl} \Sigma_{ij} \delta \phi - \delta T_{ij}^{kl} \nabla_s \phi - T_{ij}^{kl} \delta \nabla_s \phi. \]  

Substituting \( \delta R_{ij}^{kl} \) and \( \delta T_{ij}^{kl} \) working from (33) in (34) we get
\[ \delta \left[ \nabla_k, \nabla_l \right] \phi = P \left[ \nabla_k, \nabla_l \right] \phi + \theta_{\mu k} \nabla_{\mu} \nabla_l \phi + \theta_{\mu l} \nabla_{\mu} \nabla_k \phi + P \left[ \nabla_l, \nabla_k \right] \phi. \]  

The variation (30) is thus found to fit in the general scheme. From the geometric point of view the variation (30) is what it should be under diff and LLT. We thus continue to note the correspondence between the geometric and the gauge structures.

Coming back to the variation of \( \nabla_k \nabla_l \phi \) obtained in (30) we note that it is formally identical with the transformation of \( \delta \partial_k \partial_l \phi \) given by equation (26). The situation is then similar to what we observed for the usual theory. We have a theory (22) which is invariant under global Poincaré transformation. The invariance is ensured by the specific forms of the variations of \( \phi, \partial_k \phi \) and \( \partial_k \partial_l \phi \) given by equations (8), (9) and (26), respectively. When the global Poincaré transformations are made local, all these pieces naturally do not satisfy the same forms of variations, but we can construct a covariant derivative \( \nabla_k \phi \) so that \( \nabla_k \phi \) and \( \nabla_k \nabla_l \phi \) transform, respectively, as (9) and (26).

We can now form a prescription for the localization of the Poincaré invariance of higher order theories. It is simple: replace \( \partial_k \phi \) by \( \nabla_k \phi \) and \( \partial_k \partial_l \phi \) by \( \nabla_k \nabla_l \phi \) in a theory which was invariant under global Poincaré transformations and a theory invariant under local Poincaré transformations will be obtained. No new gauge field is required. This is again equivalent to the formulation of the theory in curved spacetime. This illustrates the correspondence of the PGT with theories in curved spacetime from a more general perspective than was considered earlier [3, 9]. Giving independent dynamics for the gravitational fields completes the formulation of the theory under gravity.

Before passing to the next part of the analysis a particular feature is worth mentioning. Since \( \nabla_k \) does not commute with \( \nabla_l \) there is an ordering ambiguity. Thus, for instance, we could replace \( \partial_k \partial_l \phi \) by \( D_{kl} \phi \) given by
\[ D_{kl} \phi = \frac{1}{2} \left( \nabla_k \nabla_l \phi + \nabla_l \nabla_k \phi \right). \]  

Indeed, because of (30)
\[ \delta D_{kl} \phi = \frac{1}{2} \left[ P \nabla_k \nabla_l \phi + \theta_{\mu k} \nabla_\mu \nabla_l \phi + \theta_{\mu l} \nabla_\mu \nabla_k \phi + P \nabla_l \nabla_k \phi + \theta_{\mu k} \nabla_\mu \nabla_l \phi + \theta_{\mu l} \nabla_\mu \nabla_k \phi \right]. \]  

Simplifying, we get
\[ \delta D_{kl} \phi = P D_{kl} \phi + \theta_{\mu k} D_{\mu l} \phi + \theta_{\mu l} D_{\mu k} \phi. \]  

We find that the form variation of \( D_{kl} \phi \) is again the same as (26). This shows that the prescription of the replacement of \( \partial_k \partial_l \phi \) is not unique. This nonuniqueness is indeed due to the noncommutativity of the covariant derivative \( \nabla_k \).

We have so far considered up to the second-order derivatives of the fields in the Lagrangian. The generalization to an arbitrary order may be obtained by induction. To build the induction process it will be advantageous to redo the calculation of \( \delta \nabla_k \nabla_l \phi \) in another way where the variation of \( \nabla_k \phi \), given by (17), will be directly used. Using (29) we can write
\[ \nabla_k \nabla_l \phi = b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega_{\mu j} \Sigma_{ij} \right) \nabla_l \phi - b_k^\mu \omega_{\mu j} \nabla_l \phi. \]  

8
In the above expression the matrix $\Sigma_{ij}$ is in the representation of $\phi$ and acts on it. Taking the form variation we get

$$\delta (\nabla_k \nabla_j \phi) = \delta \left( b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \right) \nabla_i \phi \right) - \delta \left( b_k^\mu \omega^j_{\mu} \nabla_i \phi \right).$$  \hspace{1cm} (40)

Now

$$\delta (b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right)) = \left( \delta b_k^\mu \right) \partial_\mu \nabla_i \phi + b_k^\mu \delta \left( \partial_\mu \Sigma_{ij} \nabla_i \phi \right) + \left( \delta b_k^\mu \right) \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi.$$ \hspace{1cm} (41)

To compute rhs of (41) we substitute $\delta b_k^\mu$ and $\delta \omega^i_{\mu}$ from (16). Also note that

$$\delta (\partial_\mu \nabla_i \phi) = \partial_\mu (P \nabla_i \phi + \theta^m_i \nabla_m \phi),$$ \hspace{1cm} (42)

which is obtained from (17). Substituting all these in (41) we get

$$\delta (b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right)) = P (b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right)) + \theta^m_i (b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right)) + b_k^\mu \delta (\partial_\mu \Sigma_{ij} \nabla_i \phi) + b_k^\mu \partial_\mu \theta^m_i \nabla_m \phi.$$ \hspace{1cm} (43)

Similarly, we find

$$\delta (b_k^\mu \omega^i_{\mu} \nabla_i \phi) = P (b_k^\mu \omega^i_{\mu} \nabla_i \phi) + \theta^m_i (b_k^\mu \omega^i_{\mu} \nabla_i \phi) + \theta^m_i (b_k^\mu \omega^i_{\mu} \nabla_i \phi) + b_k^\mu \delta (\partial_\mu \Sigma_{ij} \nabla_i \phi) + b_k^\mu \partial_\mu \theta^m_i \nabla_m \phi.$$ \hspace{1cm} (44)

Substituting (43) and (44) into (40) we get the form of $\delta (\nabla_k \nabla_j \phi)$ which of course agrees with (30). The merit of the expansion (39) is that it allows us to find $\delta (\nabla_k \nabla_j \phi)$ from (16). Substituting all these in (41) we get

$$\delta b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right) = \left( \partial_\mu \Sigma_{ij} \nabla_i \phi \right) \partial_\mu \phi - b_k^\mu \omega^i_{\mu} \nabla_i \phi - b_k^\mu \omega^i_{\mu} \nabla_i \phi$$

$$- b_k^\mu \omega^i_{\mu} \nabla_i \phi - \cdots - b_k^\mu \omega^i_{\mu} \nabla_i \phi.$$ \hspace{1cm} (45)

We will use (45) in the following for the demonstration of our construction for Lagrangians containing the derivatives of an arbitrary order.

Assume that the construction is valid for a theory which contains up to $n$-th order derivative. This means that we replaced $\partial_\mu \cdots \partial_\mu \phi$ by $\nabla_k \cdots \nabla_k \phi$ such that

$$\delta \nabla_k \cdots \nabla_k \phi = P \nabla_k \cdots \nabla_k \phi + \theta^m_k \nabla_m \cdots \nabla_m \phi + \cdots \theta^m_k \nabla_k \cdots \nabla_m \phi$$ \hspace{1cm} (46)

which ensures the local Poincaré invariance. We assume that equation (46) holds for some $n$. Assume now that we have a theory containing the $(n+1)$th derivative. We will show that

$$\delta \nabla_k \cdots \nabla_k \phi$$

also satisfies equation (46) for $n = n + 1$. Indeed, using (45) we get

$$\delta \nabla_k \cdots \nabla_k \phi = \delta \left( b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right) \right) \nabla_k \cdots \nabla_k \phi - \delta \left( b_k^\mu \omega^i_{\mu} \nabla_i \phi \right)$$

$$- \delta \left( b_k^\mu \omega^i_{\mu} \nabla_i \phi \right) - \cdots - \delta \left( b_k^\mu \omega^i_{\mu} \nabla_i \phi \right).$$ \hspace{1cm} (47)

Compare this expression with that given by (40). There we found the transformations of the first term generate such pieces which are either canceled by the appropriate terms coming from the second or couple with them so that the left-hand side conforms to the covariant transformation given by equation (30). Similar things happen with equation (47). We can expand the first term as

$$\delta \left( b_k^\mu \left( \partial_\mu + \frac{1}{2} \omega^i_{\mu} \Sigma_{ij} \nabla_i \phi \right) \right) \nabla_k \cdots \nabla_k \phi = \left( \delta b_k^\mu \right) \partial_\mu \left( \nabla_k \cdots \nabla_k \phi \right)$$

$$+ b_k^\mu \delta \left( \partial_\mu \nabla_k \cdots \nabla_k \phi \right) + \frac{1}{2} \left( \delta b_k^\mu \right) \omega^i_{\mu} \Sigma_{ij} \nabla_k \cdots \nabla_k \phi$$

$$+ \frac{1}{2} b_k^\mu \delta \left( \omega^i_{\mu} \right) \Sigma_{ij} \nabla_k \cdots \nabla_k \phi + \frac{1}{2} b_k^\mu \omega^i_{\mu} \Sigma_{ij} \delta \left( \nabla_k \cdots \nabla_k \phi \right).$$ \hspace{1cm} (48)
Note that the operators $\delta$ and $\partial$ commute. The rhs of equation (48) may be computed using equations (16) and (46). The first term of the rhs of (47) thus comes out to be
\[
\left(\frac{1}{2} \theta^{ab} \Sigma_{ab} - \xi^a \partial_a \right) \left[ b_{k_1}^\mu \left( \tilde{\partial}_\mu + \frac{1}{2} \omega_{\mu j}^{ij} \Sigma_{ij} \right) \nabla_{k_2} \cdots \nabla_{k_{n+1}} \phi \right]
\]
\[\quad + \theta_{k_1}^m \left[ b_{m}^\mu \left( \tilde{\partial}_\mu + \frac{1}{2} \omega_{\mu j}^{ij} \Sigma_{ij} \right) \nabla_{k_2} \cdots \nabla_{k_{n+1}} \phi \right]
\]
\[\quad + \cdots + \theta_{k_{n+1}}^m \left[ b_{k_1}^\mu \left( \tilde{\partial}_\mu + \frac{1}{2} \omega_{\mu j}^{ij} \Sigma_{ij} \right) \nabla_{k_2} \cdots \nabla_{m} \phi \right]
\]
\[\quad + \tilde{\partial}_\mu \theta_{m}^{k_2} \nabla_{m} \cdots \nabla_{k_{n+1}} \phi + \cdots + b_{k_1}^\mu \tilde{\partial}_\mu \theta_{m}^{k_2} \nabla_{m} \cdots \nabla_{k_{n+1}} \phi.
\] (49)

Similarly, other terms on the rhs of (47) may be calculated. We find the second term as
\[
\left(\frac{1}{2} \theta^{ab} \Sigma_{ab} - \xi^a \partial_a \right) \left( b_{k_1}^\mu \omega_{k_2}^{ij} \nabla_{i} \cdots \nabla_{k_{n+1}} \phi \right) + \theta_{k_1}^m b_{m}^\mu \omega_{k_2}^{ij} \nabla_{i} \cdots \nabla_{k_{n+1}} \phi + \cdots
\]
\[\quad + \theta_{k_{n+1}}^m \left[ b_{k_1}^\mu \omega_{k_2}^{ij} \nabla_{i} \cdots \nabla_{m} \phi + b_{k_1}^\mu \tilde{\partial}_\mu \theta_{m}^{k_2} \nabla_{m} \cdots \nabla_{k_{n+1}} \phi \right].
\] (50)

All terms of (50) except the last one fit in the covariant structure of (49), while the last term cancels one noncovariant piece of (49). Similar results follow from the other terms of (47).

After simplification of (47) in this way we get
\[
\delta \nabla_{k_1} \cdots \nabla_{k_{n+1}} \phi = \mathcal{P} \nabla_{k_1} \cdots \nabla_{k_{n+1}} \phi + \theta_{k_1}^m \nabla_{m} \cdots \nabla_{k_{n+1}} \phi + \cdots \theta_{k_{n+1}}^m \nabla_{m} \cdots \nabla_{k_{n+1}} \phi.
\] (51)

This shows that our assertion (46), if true for $n$, holds good for $n + 1$ also. So the PGT construction demonstrated for $n = 2$ is valid for $n = 3$ and so on.

4. Conclusion

We have discussed the extension of the PGT framework to matter theories containing higher order derivatives in the Lagrangian. The PGT [1, 2] is constructed by localizing the Poincaré invariance of a matter theory applying the well-known gauge principle. The gauge fields introduced in the process are observed to have one-to-one correspondence with the geometric structures such as the tetrad and spin-connection in Riemann–Cartan spacetime [3, 9]. This correspondence enables one to cast gravity in the form of the PGT. However, the usual analysis matter theories limited to Lagrangians containing the first derivative only have been considered. The question of Poincaré gauge theoretic construction from higher derivative matter theory is not discussed much in the literature. Looking from the point of view of the gauge theory, such construction appears to be complicated because it has been found that in general in the extension of the gauge principle to higher derivative theories, new tensor gauge fields are required to be introduced in the process [10]. On the other hand, the gravitational coupling to the particle theories is usually a straightforward process. The principle of general covariance allows one to couple a Poincaré invariant theory to gravity by substituting the Minkowski metric by the metric of curved spacetime and replacing ordinary derivatives by the covariant derivative. We have shown by localizing the Poincaré invariance of a higher derivative matter theory that the resulting PGT follows this geometric procedure. Thus, our results buttresses the geometric interpretation of the PGT observed in the literature from a different perspective. The following ambiguity, following from the noncommutativity of the covariant derivative, is manifest in our construction.

Acknowledgments

The author would like to thank Rabin Banerjee, Saurav Samanta and Debraj Roy for useful discussions. He also acknowledges the facilities extended to him during his visit to the IUCAA, Pune and later to S N Bose National Centre for Basic Sciences, Kolkata as visiting associate.
References

[1] Utiyama R 1956 Phys. Rev. 101 1597
[2] Kibble T W B 1961 J. Math. Phys. 2 212
[3] Blagojevic M 2002 Gravitation and Gauge Symmetries (Bristol: Institute of Physics Publishing) 522 p
[4] Chamseddine A H 2001 Phys. Lett. B 504 33 (arXiv:hep-th/0009153)
[5] Calmet X and Kobakhidze A 2005 Phys. Rev. D 72 045010 (arXiv:hep-th/0506157)
[6] Mukherjee P and Saha A 2006 Phys. Rev. D 74 027702 (arXiv:hep-th/0605287)
[7] Banerjee R, Mukherjee P and Samanta S 2007 Phys. Rev. D 75 125020 (arXiv:hep-th/0703128)
[8] For a review, see Banerjee R, Chakraaborty B, Ghosh S, Mukherjee P and Samanta S 2009 Found. Phys. 39 1297–345 (arXiv:0809.1000 [hep-th])
[9] Banerjee R, Gangopadhyay S, Mukherjee P and Roy D 2010 J. High Energy Phys. JHEP02(2010)075 (arXiv:0912.1472 [gr-qc])
[10] Hamamoto S 1995 Prog. Theor. Phys. 94 105 (arXiv:hep-th/9502158)
[11] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: Wiley) 657 p
[12] Lee T D and Wick G C 1969 Nucl. Phys. B 9 209
Lee T D and Wick G C 1970 Phys. Rev. D 2 1033
[13] Gitman D M and Tyutin I V 1983 Sov. Phys. J. 26 730
[14] Gitman D M and Tyutin I V 1990 Quantumization of Fields with Constraints (Berlin: Springer) 291 p
[15] de Souza Dutra A and Natividade C P 1996 Mod. Phys. Lett. A 11 775–83
[16] Hawking S W and Hertog T 2002 Phys. Rev. D 65 103515 (arXiv:hep-th/0107088)
[17] Rivelles V O 2003 Phys. Lett. B 577 137 (arXiv:hep-th/0304073)
[18] Smilga A V 2005 Nucl. Phys. B 706 598 (arXiv:hep-th/0407231)
[19] Kruglov S I 2006 Ann. Fond. Broglie 31 343–56 (arXiv:hep-th/0606128)
[20] Carone C D and Lebed R F 2009 J. High Energy Phys. JHEP01(2009)043 (arXiv:0811.4150 [hep-ph])
[21] Andrzejewski K, Gonera J, Machalski P and Maslanka P 2010 arXiv:1005.3941
[22] Buchbinder I L, Odintsov S D and Shapiro I L 1992 Effective Action in Quantum Gravity (Bristol: Institute of Physics Publishing) 413 p
[23] Blagojevic M and Cvetkovic B 2009 J. High Energy Phys. JHEP05(2009)073 (arXiv:0812.4742)
[24] Hayashi K and Shirafuji T 1980 Prog. Theor. Phys. 64 866