The Blow-up Rate Estimates for a System of Heat Equations with Nonlinear Boundary Conditions

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Abstract

This paper deals with the blow-up properties of positive solutions to a system of two heat equations

\[ u_t = \Delta u, \quad v_t = \Delta v \quad \text{in} \quad B_R \times (0, T), \]

with Neumann boundary conditions

\[ \frac{\partial u}{\partial \eta} = e^{v^p}, \quad \frac{\partial v}{\partial \eta} = e^{u^q} \quad \text{on} \quad \partial B_R \times (0, T), \]

where \( p, q > 1 \), \( B_R \) is a ball in \( \mathbb{R}^n \), \( \eta \) is the outward normal. The upper bounds of blow-up rate estimates were obtained. It is also proved that the blow-up occurs only on the boundary.

1 Introduction

In this paper, we consider the system of two heat equations with coupled nonlinear Neumann boundary conditions, namely

\[
\begin{align*}
    u_t &= \Delta u, & v_t &= \Delta v, & (x, t) \in B_R \times (0, T), \\
    \frac{\partial u}{\partial \eta} &= e^{v^p}, & \frac{\partial v}{\partial \eta} &= e^{u^q}, & (x, t) \in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in B_R,
\end{align*}
\]

(1.1)

where \( p, q > 1 \), \( B_R \) is a ball in \( \mathbb{R}^n \), \( \eta \) is the outward normal, \( u_0, v_0 \) are smooth, radially symmetric, nonzero, nonnegative functions satisfy the condition

\[
\Delta u_0, \Delta v_0 \geq 0, \quad u_0(|x|), v_0(|x|) \geq 0, \quad x \in \overline{B_R}.
\]

(1.2)

The problem of system of two heat equations with nonlinear Neumann boundary conditions defined in a ball,

\[
\begin{align*}
    u_t &= \Delta u, & v_t &= \Delta v, & (x, t) \in B_R \times (0, T), \\
    \frac{\partial u}{\partial \eta} &= f(v), & \frac{\partial v}{\partial \eta} &= g(u), & (x, t) \in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in B_R,
\end{align*}
\]

(1.3)
was introduced in \[1, 2, 3, 6\], for instance, in \[1\] it was studied the blow-up solutions to the system (1.3), where

\[ f(v) = v^p, \quad g(u) = u^q, \quad p, q > 1. \] (1.4)

It was proved that for any nonzero, nonnegative initial data \((u_0, v_0)\), the finite time blow-up can only occur on the boundary, moreover, it was shown in \[5\] that, the blow-up rate estimates take the following form

\[ c \leq \max_{x \in \Omega} u(x, t)(T - t)^{\frac{p+1}{pq} - 1} \leq C, \quad t \in (0, T), \]

\[ c \leq \max_{x \in \Omega} v(x, t)(T - t)^{\frac{q+1}{pq} - 1} \leq C, \quad t \in (0, T). \]

In \[2, 6\], it was considered the solutions of the system (1.3) with exponential Neumann boundary conditions model, namely

\[ f(v) = e^{pv}, \quad g(u) = e^{qu}, \quad p, q > 0. \] (1.5)

It was proved that for any nonzero, nonnegative initial data, \((u_0, v_0)\), the solution blows up in finite time and the blow-up occurs only on the boundary, moreover, the blow-up rate estimates take the following forms

\[ C_1 \leq e^{qu(R, t)}(T - t)^{1/2} \leq C_2, \quad C_3 \leq e^{pu(R, t)}(T - t)^{1/2} \leq C_4. \]

In this paper, we prove that the upper blow-up rate estimates for problem (1.1) take the following form

\[ \max_{B_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), \quad 0 < t < T, \]

\[ \max_{B_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), \quad 0 < t < T, \]

where \(\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}\). Moreover, the blow-up occurs only on the boundary.

2 Preliminaries

The local existence and uniqueness of classical solutions to problem (1.1) is well known by \[8\]. On the other hand, every nontrivial solution blows up simultaneously in finite time, and that due to the known blow-up results of problem (1.3) with (1.4) and the comparison principle \[8\].

In the following lemma we study some properties of the classical solutions of problem (1.1). We denote for simplicity \(u(r, t) = u(x, t)\).

Lemma 2.1. Let \((u, v)\) be a classical unique solution of (1.1). Then

(i) \(u, v\) are positive, radial. Moreover, \(u_r, v_r \geq 0\) in \([0, R] \times (0, T)\).

(ii) \(u_t, v_t > 0\) in \(B_R \times (0, T)\).
3 Rate Estimates

In order to study the upper blow-up rate estimates for problem (1.1), we need to recall some results from [3, 5].

Lemma 3.1. [5] Let $A(t)$ and $B(t)$ be positive $C^1$ functions in $[0, T)$ and satisfy

$$A'(t) \geq c \frac{B^p(t)}{\sqrt{T - t}}, \quad B'(t) \geq c \frac{A^q(t)}{\sqrt{T - t}}$$

for $t \in [0, T)$,

where $p, q > 0, c > 0$ and $pq > 1$. Then there exists $C > 0$ such that

$$A(t) \leq C(T - t)^{-\alpha/2}, \quad B(t) \leq C(T - t)^{-\beta/2}, \quad t \in [0, T),$$

where $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$.

Lemma 3.2. [3] Let $x \in B_R$. If $0 \leq a < n - 1$. Then there exist $C > 0$ such that

$$\int_{S_R} \frac{ds_y}{|x - y|^a} \leq C.$$

Theorem 3.3. (Jump relation. [3]) Let $\Gamma(x, t)$ be the fundamental solution of heat equation, namely

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{(n/2)}} \exp\left[-\frac{|x|^2}{4t}\right]. \quad (3.1)$$

Let $\varphi$ be a continuous function on $S_R \times [0, T]$. Then for any $x \in B_R, x^0 \in S_R, 0 < t_1 < t_2 \leq T$, for some $T > 0$, the function

$$U(x, t) = \int_{t_1}^{t_2} \int_{S_R} \Gamma(x - y, t - z)\varphi(y, z) d\sigma_y d\tau$$

satisfies the jump relation

$$\frac{\partial}{\partial \eta} U(x, t) \to -\frac{1}{2} \varphi(x^0, t) + \frac{\partial}{\partial \eta} U(x^0, t), \quad \text{as} \quad x \to x^0.$$

Theorem 3.4. Let $(u, v)$ be a solution of (1.1), which blows up in finite time $T$. Then there exist positive constants $C_1, C_2$ such that

$$\max_{B_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), \quad 0 < t < T,$$

$$\max_{B_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), \quad 0 < t < T.$$
Proof. We follow the idea of [5], define the functions $M$ and $M_b$ as follows

\[ M(t) = \max_{B_R} u(x, t), \quad \text{and} \quad M_b(t) = \max_{S_R} u(x, t). \]

Similarly,

\[ N(t) = \max_{B_R} v(x, t), \quad \text{and} \quad N_b(t) = \max_{S_R} v(x, t). \]

Depending on Lemma 2.1, both of $M, M_b$ are monotone increasing functions, and since $u$ is a solution of heat equation, it cannot attain interior maximum without being constant, therefore,

\[ M(t) = M_b(t). \quad \text{Similarly} \quad N(t) = N_b(t). \]

Moreover, since $u, v$ blow up simultaneously, therefore, we have

\[ M(t) \to +\infty, \quad N(t) \to +\infty \quad \text{as} \quad t \to T^-. \quad (3.2) \]

As in [4, 5], for $0 < z_1 < t < T$ and $x \in B_R$, depending on the second Green’s identity with assuming the Green function:

\[ G(x, y; z_1, t) = \Gamma(x - y, t - z_1), \]

where $\Gamma$ is defined in (3.1), the integral equation to problem (1.1) with respect to $u$, can be written as follows

\[
\begin{align*}
\int_{B_R} & \Gamma(x - y, t - z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{S_R} e^{\nu(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau \\ & - \int_{z_1}^{t} \int_{S_R} u(y, \tau) \frac{\partial \Gamma}{\partial \eta_y} (x - y, t - \tau) ds_y d\tau,
\end{align*}
\]

As in [4], letting $x \to S_R$ and using the jump relation (Theorem 3.3) for the third term on the right hand side of the last equation, it follows that

\[
\begin{align*}
\frac{1}{2} u(x, t) & = \int_{B_R} \Gamma(x - y, t - z_1) u(y, z_1) dy + \int_{z_1}^{t} \int_{S_R} e^{\nu(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau \\ & - \int_{z_1}^{t} \int_{S_R} u(y, \tau) \frac{\partial \Gamma}{\partial \eta_y} (x - y, t - \tau) ds_y d\tau,
\end{align*}
\]

for $x \in S_R, 0 < z_1 < t < T$.

Depending on Lemma 2.1 we notice that $u, v$ are positive and radial. Thus

\[
\begin{align*}
\int_{B_R} & \Gamma(x - y, t - z_1) u(y, z_1) dy > 0, \\
\int_{z_1}^{t} & \int_{S_R} e^{\nu(y, \tau)} \Gamma(x - y, t - \tau) ds_y d\tau = \int_{z_1}^{t} e^{\nu(R, \tau)} \left[ \int_{S_R} \Gamma(x - y, t - \tau) ds_y \right] d\tau.
\end{align*}
\]
This leads to
\[
\frac{1}{2} M(t) \geq \int_{z_1}^t e^{N_p(\tau)} \left[ \int_{S_R} \Gamma(x - y, t - \tau) ds_y \right] d\tau \\
- \int_{z_1}^t M(\tau) \int_{S_R} \frac{\partial \Gamma}{\partial y} (x - y, t - \tau) |ds_y| d\tau, \quad x \in S_R, 0 < z_1 < t < T.
\]

It is known that (see [3]) there exist \( C_0 > 0 \), such that \( \Gamma \) satisfies
\[
\left| \frac{\partial \Gamma}{\partial y} (x - y, t - \tau) \right| \leq \frac{C_0}{(t - \tau)^\mu} \cdot \frac{1}{|x - y|^{(n+1-2\mu-\sigma)}}, \quad x, y \in S_R, \sigma \in (0, 1).
\]

Choose \( 1 - \sigma^2 < \mu < 1 \), from Lemma 3.2, there exist \( C^* > 0 \) such that
\[
\int_{S_R} \frac{ds_y}{|x - y|^{(n+1-2\mu-\sigma)}} < C^*.
\]

Moreover, for \( 0 < t_1 < t_2 \) and \( t_1 \) is closed to \( t_2 \), there exists \( c > 0 \), such that
\[
\int_{S_R} \Gamma(x - y, t_2 - t_1) ds_y \geq \frac{c}{\sqrt{t_2 - t_1}}.
\]

Thus
\[
\frac{1}{2} M(t) \geq c \int_{z_1}^t e^{N_p(\tau)} \frac{1}{\sqrt{T - \tau}} d\tau - C \int_{z_1}^t \frac{M(\tau)}{|t - \tau|^{\mu}} d\tau.
\]

Since for \( 0 < z_1 < t_0 < t < T \), it follows that \( M(t_0) \leq M(t) \), thus the last equation becomes
\[
\frac{1}{2} M(t) \geq c \int_{z_1}^t e^{N_p(\tau)} \frac{1}{\sqrt{T - \tau}} d\tau - C^*_1 M(t) |T - z_1|^{1-\mu}.
\]

Similarly, for \( 0 < z_2 < t < T \), we have
\[
\frac{1}{2} N(t) \geq c \int_{z_2}^t e^{M_q(\tau)} \frac{1}{\sqrt{T - \tau}} d\tau - C^*_2 N(t) |T - z_2|^{1-\mu}.
\]

Taking \( z_1, z_2 \) so that
\[
C^*_1 |T - z_1|^{1-\mu} \leq 1/2, \quad C^*_2 |T - z_2|^{1-\mu} \leq 1/2,
\]

it follows
\[
M(t) \geq c \int_{z_1}^t e^{N_p(\tau)} \frac{1}{\sqrt{T - \tau}} d\tau, \quad N(t) \geq c \int_{z_2}^t e^{M_q(\tau)} \frac{1}{\sqrt{T - \tau}} d\tau.
\]

Since both of \( M, N \) increasing functions and from (3.2), we can find \( T^* \) in \((0, T)\) such that
\[
M(t) \geq q^{\frac{1}{(p-1)}} \cdot p^{\frac{1}{(p-1)}}, \quad N(t) \geq p^{\frac{1}{(p-1)}}, \quad \text{for } T^* \leq t < T.
\]
Thus
\[ e^{M(t)} \geq e^{qM(t)}, \quad e^{N(t)} \geq e^{pN(t)}, \quad T^* \leq t < T. \]

Therefore, if we choose \( z_1, z_2 \) in \( (T^*, T) \), then (3.3) becomes
\[ e^{M(t)} \geq c \int_{z_1}^{t} \frac{e^{pN(\tau)}}{\sqrt{T - \tau}} d\tau \equiv I_1(t), \quad e^{N(t)} \geq c \int_{z_2}^{t} \frac{e^{qM(\tau)}}{\sqrt{T - \tau}} d\tau \equiv I_2(t). \]

Clearly,
\[ I'_1(t) = c \frac{e^{pN(t)}}{\sqrt{T - t}} \geq c I_{p}^{\frac{q}{2}}, \quad I'_2(t) = c \frac{e^{qM(t)}}{\sqrt{T - t}} \geq c I_{q}^{\frac{q}{2}}. \]

By Lemma 3.1, it follows that
\[ I_1(t) \leq \frac{C}{(T - t)^{\frac{q}{2}}}, \quad I_2(t) \leq \frac{C}{(T - t)^{\frac{q}{2}}}, \quad t \in (\max \{z_1, z_2\}, T). \quad (3.4) \]

On the other hand, for \( t^* = 2t - T \) (Assuming that \( t^* \) is close to \( T \)).
\[ I_1(t) \geq c \int_{t^*}^{t} \frac{e^{pN(\tau)}}{\sqrt{T - \tau}} d\tau \geq ce^{pN(t^*)} \int_{2t - T}^{t} \frac{1}{\sqrt{T - \tau}} d\tau = 2c(\sqrt{2} - 1)\sqrt{T - t}e^{pN(t^*)}. \]

Combining the last inequality with (3.3) yields
\[ e^{N(t^*)} \leq \frac{C}{2c(\sqrt{2} - 1)(T - t) \frac{q+1}{2p(pq-1)}} \leq \frac{2^{\frac{q+1}{p}} C}{2c(\sqrt{2} - 1)(T - t^*) \frac{q+1}{2p(pq-1)}}. \]

Thus, there exists a constant \( c_1 > 0 \) such that
\[ e^{N(t^*)}(T - t^*)^{\frac{q+1}{2p(pq-1)}} \leq c_1. \]

In the same way we can show
\[ e^{M(t^*)}(T - t^*)^{\frac{q+1}{2p(pq-1)}} \leq c_2. \]

This leads to, there exists \( C_1, C_2 > 0 \) such that
\[ \max_{\overline{B}_R} u(x, t) \leq \log C_1 - \frac{\alpha}{2} \log(T - t), \quad 0 < t < T, \quad (3.5) \]
\[ \max_{\overline{B}_R} v(x, t) \leq \log C_2 - \frac{\beta}{2} \log(T - t), \quad 0 < t < T. \quad (3.6) \]
4 Blow-up Set

In order to show that the blow-up to problem (1.1) occurs only on the boundary, we need to recall the following lemma from [6].

Lemma 4.1. Let \( w \) is a continuous function on the domain \( \overline{B}_R \times [0,T) \) and satisfies
\[
\begin{align*}
    w_t &= \Delta w, & (x,t) \in B_R \times (0,T), \\
    w(x,t) &\leq \frac{C}{(T-t)^m}, & (x,t) \in S_R \times (0,T), & m > 0.
\end{align*}
\]

Then for any \( 0 < a < R \)
\[
\sup\{w(x,t) : 0 \leq |x| \leq a, 0 \leq t < T\} < \infty.
\]

Proof. Set
\[
h(x) = (R^2 - r^2)^2, \quad r = |x|,
\]
\[
z(x,t) = \frac{C_1}{[h(x) + C_2(T-t)]^m}.
\]

We can show that:
\[
\Delta h - \frac{(m+1)|\nabla h|^2}{h} = 8r^2 - 4n(R^2 - r^2) - (m+1)16r^2 \\
\geq -4nR^2 - 16R^2(m+1),
\]
\[
z_t - \Delta z = \frac{C_1 m}{[h(x) + C_2(T-t)]^{m+1}}\left(C_2 + \frac{(m+1)|\nabla h|^2}{h + C_2(T-t)}\right) \\
\geq \frac{C_1 m}{[h(x) + C_2(T-t)]^{m+1}}\left(C_2 - 4nR^2 - 16R^2(m+1)\right).
\]

Let
\[
C_2 = 4nR^2 + 16R^2(m+1) + 1
\]
and take \( C_1 \) to be large such that
\[
z(x,0) \geq w(x,0), \quad x \in B_R.
\]

Let \( C_1 \geq C(C_2)^m \), which implies that
\[
z(x,t) \geq w(x,t) \quad \text{on} \quad S_R \times [0,T).
\]

Then from the maximum principle [7], it follows that
\[
z(x,t) \geq w(x,t), \quad (x,t) \in \overline{B}_R \times (0,T)
\]
and hence
\[
\sup\{w(x,t) : 0 \leq |x| \leq a, 0 \leq t < T\} \leq C_1(R^2 - a^2)^{-2m} < \infty, \quad 0 \leq a < R.
\]

\[\square\]
**Theorem 4.2.** Let the assumptions of Theorem 3.4 be in force. Then \((u,v)\) blows up only on the boundary.

**Proof.** Using equations (3.5), (3.6)

\[
\begin{align*}
  u(R,t) &\leq \frac{c_1}{(T-t)^\alpha}, & v(R,t) &\leq \frac{c_2}{(T-t)^\beta}, & t \in (0,T).
\end{align*}
\]

From Lemma 4.1 it follows that

\[
\sup\{u(x,t) : (x,t) \in B_a \times [0,T)\} \leq C_1(R^2 - a^2)^{-\alpha} < \infty,
\]

\[
\sup\{v(x,t) : (x,t) \in B_a \times [0,T)\} \leq C_1(R^2 - a^2)^{-\beta} < \infty,
\]

for \(a < R\).

Therefore, \(u,v\) blow up simultaneously and the blow-up occurs only on the boundary. \(\Box\)

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