Circle-valued Morse theory, Reidemeister torsion, and Seiberg-Witten invariants of 3-manifolds

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Abstract

Let $X$ be a compact oriented Riemannian manifold and let $\phi : X \to S^1$ be a circle-valued Morse function. Under some mild assumptions on $\phi$, we prove a formula relating:

(a) the number of closed orbits of the gradient flow of $\phi$ of any given degree;
(b) the torsion of a “Morse complex”, which counts gradient flow lines between critical points of $\phi$; and
(c) a kind of Reidemeister torsion of $X$ determined by the homotopy class of $\phi$.

When $\dim(X) = 3$ and $b_1(X) > 0$, we state a conjecture analogous to Taubes’s “$SW=Gromov$” theorem, and we use it to deduce (for closed manifolds, modulo signs) the Meng-Taubes relation between the Seiberg-Witten invariants and the “Milnor torsion” of $X$.

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# Introduction and statement of results

It has long been known that one can obtain information about the homology of a manifold from the structure of the critical points of a Morse function defined on it. A sharp statement of this relationship is that there is an isomorphism between the homology of the manifold and the homology of a “Morse complex” whose chains are critical points and whose differential counts gradient flow lines between critical points.

Novikov [14] generalized this relationship to circle-valued (and other multiply-valued) Morse functions. One can still define a Morse complex in terms of gradient flow lines between critical points, which is now a module over the ring of integer Laurent series in one variable. Novikov obtained bounds on the betti numbers of the manifold in terms of algebraic invariants of the homology of this complex.

The novelty of this paper is that we also consider the closed orbits of the gradient flow of a circle-valued Morse function. These turn out to be related not to homology, but rather to Reidemeister torsion. (For nonsingular flows, relations between closed orbits and Reidemeister torsion have been investigated by D. Fried [2] [3].)
1.1 Counting closed orbits

Let $X^n$ be a compact oriented Riemannian manifold and let $\phi : X \to S^1 = \mathbb{R}/\mathbb{Z}$ be a generic Morse function. (See §2.2 for the definition of “generic”.) Assume $0 \in \mathbb{R}/\mathbb{Z}$ is a regular value of $\phi$, and let $\Sigma = \phi^{-1}(0)$. If $p \in \Sigma$, we can flow upwards from $p$ along the gradient vector field of $\phi$. As $\phi$ goes once around $S^1$, we will return to $\Sigma$, if we do not get sucked into a critical point first. Let $f(p)$ denote this point of $\Sigma$, if it exists. Thus $f$ is a function from a subset of $\Sigma$ to a subset of $\Sigma$. Also $f^k$ is a function defined on a smaller subset of $\Sigma$. Let $\text{Fix}(f^k)$ denote the signed number of fixed points of $f^k$. (A fixed point of a function corresponds to an intersection point of the graph with the diagonal, and the sign of this intersection number determines the sign of the fixed point.)

The Morse complex

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} \cdots \xrightarrow{d} M^n$$

is defined as follows. Let $L_{\mathbb{Z}}$ be the ring of Laurent series in one variable $t$ with integer coefficients, i.e. formal sums $\sum_{k=k_0}^{\infty} a_k t^k$ with $a_k \in \mathbb{Z}$. Let $M^i$ be the free $L_{\mathbb{Z}}$-module generated by $\text{Crit}^i$, the set of critical points of index $i$. If $x \in \text{Crit}^i$, define

$$dx := \sum_{y \in \text{Crit}^{i+1}} (dx, y)y,$$

where $(dx, y)$ is a Taylor series whose $n^{th}$ coefficient is the signed number of gradient flow lines from $x$ to $y$ that cross $\Sigma$ $n$ times. (The sign conventions, and other details, are explained in §2.2.)

Let us recall the definition of Reidemeister torsion. Suppose $C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^m$ is an acyclic complex of finite dimensional vector spaces over a field $F$, and suppose that each vector space $C^i$ has a volume form chosen on it. Choose $\omega_i \in \wedge^* C^i$, $i = 0, \ldots, m-1$, so that $d\omega_{i-1} \wedge \omega_i \in \wedge^{\text{top}} C^i$. Then the Reidemeister torsion $\tau(C)$ is defined to be

$$\tau(C) := \prod_{i=0}^{m} \text{vol}(d\omega_{i-1} \wedge \omega_i)^{(-1)^i}$$

(1)

(where we interpret $d\omega_{-1} = 1$). One can check, using the fact that $d^2 = 0$, that $\tau(C)$ does not depend on the choice of $\omega_i$’s. If the differential has degree -1 instead of 1, the definition is analogous, but we adopt the following sign convention: if we have a complex $C_m \xrightarrow{\partial} C_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0$, then we choose $\omega_i \in \wedge^* C_i$, $i = 1, \ldots, m$, so that $\partial \omega_{i+1} \wedge \omega_i \in \wedge^{\text{top}} C_i$ and define

$$\tau(C) := \prod_{i=0}^{m} \text{vol}(\partial \omega_{i+1} \wedge \omega_i)^{(-1)^{m-i}}.$$
There is a natural symmetric bilinear form $\langle \ , \ \rangle$ on the Morse complex, in which the critical points are orthonormal. This defines a volume form on the $L_Q$-vector space $M^* \otimes L_Q$. (Here $L_Q$ is the field of rational Laurent series.) If the complex $M^* \otimes L_Q$ is acyclic, then the Reidemeister torsion of $M^* \otimes L_Q$ is defined, up to sign. We denote this simply by $\tau(M)$. (One can define this in the non-acyclic case as well, but we will not need to.)

Let $\tilde{X}$ be the infinite cyclic cover of $X$ induced by $\phi$, i.e. the fiber product of $X$ and $\mathbb{R}$ over $S^1$:

$$
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \\
X & \phi \longrightarrow & \mathbb{R}/\mathbb{Z}.
\end{array}
$$

We think of $\tilde{X}$ as a subset of $X \times \mathbb{R}$. There is a covering transformation of $\tilde{X}$ which shifts points “down”, i.e. which sends $(p, \lambda) \mapsto (p, \lambda - 1)$ for $p \in X$ and $\lambda \in \mathbb{R}$. Let $A : H_*(\tilde{X}; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ be the map in rational homology induced by this covering transformation.

We can now state the formula for counting closed orbits.

**Theorem 1.1** Assume $H^*(M^* \otimes L_Q) = 0$. On $\partial X$, assume that $\text{grad}(\phi)$ is parallel to the boundary and has no zeroes there. Then

$$
\sum_{k=1}^{\infty} t^k \text{Fix}(f^k) - (-1)^n t \frac{d}{dt} \log \tau(M) = \text{Tr}(tA(1 - tA)^{-1}) + m,
$$

where $m \in \mathbb{Z}$.

**Notes.** Here Tr denotes the graded trace. We will see in Corollary 2.6 that $H_*(\tilde{X}; \mathbb{Q})$ is finite dimensional (thanks to our assumption that $H^*(M^* \otimes L_Q) = 0$), so this trace is well defined. Also $\frac{d}{dt} \log \tau$ means $\tau^{-1} \frac{d}{dt} \tau$, which is well defined even though $\tau$ has a sign ambiguity.

**Example.** If there are no critical points, then $\tau(M) = 1$, $f$ is a diffeomorphism of $\Sigma$, and our theorem reduces to the Lefschetz fixed point formula for $f$. Thus we can think of the theorem as Lefschetz fixed point formula for certain partially defined functions, in which the $R$-torsion of the Morse complex appears as a correction term.

### 1.2 A refinement

This theorem is the logarithmic derivative of a slightly sharper formula, which we will also prove. To state it we need two more definitions. First, define a

...
\[
\zeta(f) := \exp \left( \sum_{k \geq 1} \text{Fix}(f^k) \frac{t^k}{k} \right).
\]

(2)

Note that
\[
\zeta(f) = \prod_{\gamma \in \mathcal{O}} (1 - t^{k(\gamma)})^{-\varepsilon(\gamma)},
\]

where \( \mathcal{O} \) is the set of irreducible, connected closed orbits, and for \( \gamma \in \mathcal{O}, k(\gamma) \) is the degree of \( \phi : \gamma \to S^1 \) and \( \varepsilon(\gamma) \) is the sign of each of the corresponding \( k \) fixed points of \( f^k \).

Second, let \( C_\ast(\tilde{X}) \) be the chain complex associated to a cell decomposition of \( \tilde{X} \) lifted from a cell decomposition of \( X \). This is a \( \mathbb{Z}[t, t^{-1}] \) module, where \( t \) acts via the upward covering transformation. Let \( Q(\mathbb{Z}[t, t^{-1}]) \) be the field of fractions of \( \mathbb{Z}[t, t^{-1}] \). Our assumption on the acyclicity of the rational Morse complex implies that \( C_\ast(\tilde{X}) \otimes_{\mathbb{Z}[t, t^{-1}]} Q(\mathbb{Z}[t, t^{-1}]) \) is acyclic. (This follows from Corollary 2.5.) Furthermore this complex has a volume form, well defined up to multiplication by \( \pm t^k \), consisting of a wedge product of cells, one for each cell in \( X \). Thus the torsion of this complex, which we denote by \( \tau(X, \phi) \), is a well defined element of \( Q(\mathbb{Z}[t, t^{-1}])/\pm t^k \). (Of course the homology of this complex is isomorphic to \( H_\ast(\tilde{X}) \), viewed as a \( \mathbb{Z}[t, t^{-1}] \) module. We will see in Lemma 2.7 that this homology alone, and not the choice of cell decomposition, determines the torsion.)

**Example.** Let \( K \subset S^3 \) be a knot, and let \( X \) be the 3-manifold obtained from \((0, 1)\) surgery on \( K \). Let \( \phi : X \to S^1 \) be a Morse function whose homotopy class in \( H^1(X; \mathbb{Z}) \) is the Alexander dual of \( K \), i.e. which sends a loop \( \gamma \) in \( X \) to the linking number of \( \gamma \) with \( K \). Then
\[
\tau(X, \phi) = \frac{\Delta_K(t)}{(1 - t)^2},
\]

where \( \Delta_K(t) \) is the Alexander polynomial of \( K \).

**Theorem 1.2** Under the assumptions of Theorem 1.1, we have
\[
\zeta(f)^{(-1)^{n-1}} \tau(M) = \tau(X, \phi),
\]

up to multiplication by \( \pm t^k \).
Remark. The right side of this formula clearly depends only on the homotopy class of $\phi$ in $H^1(X;\mathbb{Z})$. It is not too hard to show directly that the left hand side is also invariant, but this requires our assumption that the rational Morse complex is acyclic. To see this, note that the zeta function is determined by the intersection number of the graph of $f^k$ and the diagonal in $\Sigma \times \Sigma$, for $k = 1, 2, \ldots$. If we deform $\phi$, this intersection number will change exactly when the boundary of the graph crosses the diagonal. But we will see that the boundary of the graph consists of points of the form $(p, q)$ where $p$ is in the descending manifold of a critical point $x$, and $q$ is in the ascending manifold of $x$. When the ascending and descending manifolds cross each other this way, the effect is to replace $dx$ by $(1 - t^k)^{\pm 1}dx$ and $d^*x$ by $(1 - t^k)^{\mp 1}d^*x$. In the acyclic case, this multiplies or divides $\tau(M)$ by $(1 - t^k)$, which exactly cancels the change in the zeta function. But in the nonacyclic case, if say $dx = d^*x = 0$, then $\tau(M)$ does not change, even though $\zeta(f)$ changes.

Of course this formula might generalize to the nonacyclic case if a suitable correction term is included. However several of the steps of our proof do not appear to have natural generalizations to the nonacyclic case.

Example. Suppose $X = S^1$ and $\phi : S^1 \to S^1$ has degree $k$. Then $\tilde{X}$ has $k$ components, and the upward covering transformation permutes them in a $k$-cycle, so we find that $\tau(X, \phi) = (1 - t^k)^{-1}$. If $\phi$ has no critical points then $\zeta(f) = (1 - t^k)^{-1}$ and $\tau(M) = 1$. If there are $c > 0$ critical points then $\zeta(f) = 1$, while the differential $d$ of the Morse complex has the form

\[
\begin{pmatrix}
t^{a_1} & -t^{b_1} & 0 & \cdots & 0 \\
0 & t^{a_2} & -t^{b_2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-t^{b_c} & 0 & \cdots & 0 & t^{a_c}
\end{pmatrix}
\]

(for a certain choice of orientations). Since $\phi$ has degree $k$, we have $\sum a_i - \sum b_i = k$, so up to signs and powers of $t$, $\tau(M) = (1 - t^k)^{-1}$.

1.3 Relation to Seiberg-Witten theory

Suppose now that $X$ is a closed oriented 3-manifold with $b^1(X) > 0$. Let $S$ denote the set of $Spin^c$ structures on $X$. A $Spin^c$ structure is a $U(2)$ bundle $S \to X$ such that the fiber over each point is an irreducible Clifford module over the tangent space. The set $S$ is an $H^2(X;\mathbb{Z})$-torsor; $\alpha \in H^2(X;\mathbb{Z})$ sends $S$ to $S \otimes \mathcal{L}$, where $\mathcal{L}$ is the complex line bundle with $c_1(\mathcal{L}) = \alpha$. 

6
Suppose an orientation is chosen on the rational homology of $X$. Then the Seiberg-Witten invariant

$$SW : S \to \mathbb{Z}$$

is defined. (See e.g. [6], [13], [8], or [24], [12] for the four dimensional case.) (When $b^1(X) = 1$, the definition of $SW$ requires some care; see §4.1.)

Taubes [18] has shown that for a symplectic 4-manifold with a metric compatible with the symplectic form, the Seiberg-Witten invariant is equal to the Gromov invariant (see Taubes [19] for a precise definition), which counts pseudoholomorphic curves. He has also made some progress on generalizing this result to nonsymplectic 4-manifolds [20]. Here one considers 2-forms which are symplectic except on a set of circles, and pseudoholomorphic curves bounded by this set of circles.

In three dimensions, a possible analogue of a symplectic form is a harmonic 1-form, and the analogue of pseudoholomorphic curves is flow lines of the dual vector field. Actually we do not need to assume that the 1-form is harmonic, but only that it has no index 0 or 3 critical points. In §4.1 we define an analogue of the Gromov invariant out of such a 1-form. This turns out to be a map

$$I : S \to \mathbb{Z}.$$ It is similar to the left hand side of Theorem 1.2, but it is sharper, because it keeps track of the relative homology classes of closed orbits and gradient flow lines (and not just the intersection numbers with $\Sigma$). A priori $I$ depends on the choice of an integral cohomology class, but we conjecture (Conjecture 4.3) that it does not, and in fact

$$SW = \pm I.$$ (The basic idea of this was suggested to us by Taubes.)

In §4.2, we apply Theorem 1.2 to prove:

**Theorem 1.3 (assuming Conjecture 4.3)** Let $X$ be a closed oriented 3-manifold with $b^1 > 0$ and $0 \neq \alpha \in H^1(X; \mathbb{Z})$. Then

$$\sum_{S \in S} SW(S)t^{\alpha(c_1(det S))/2} = \begin{cases} \tau(X, \phi) & \text{if } M^* \otimes L_\mathbb{Q} \text{ acyclic} \\ 0 & \text{otherwise} \end{cases}$$

modulo multiplication by $\pm t^k$, where $\phi : X \to S^1$ is in the homotopy class determined by $\alpha$.

When $b^1(X) > 1$, the $t^k$ ambiguity in the right hand side of this theorem can be resolved by applying the “charge conjugation invariance” of the
Seiberg-Witten equations, which tells us that the left hand side is invariant under $t \mapsto t^{-1}$. (See Witten [24] or Morgan [12] for the 4-dimensional case. The 3-dimensional case is also easy to deduce from Conjecture 4.3, by replacing $\eta$ in §4.1 with $-\eta$.)

Note that even if we apply this theorem to every $\alpha \in H^1(X;\mathbb{Z})$, we cannot recover all of the Seiberg-Witten invariants of $X$, because the theorem does not distinguish between $Spin^c$ structures that differ by the action of a torsion element of $H^2(X;\mathbb{Z})$. However we can recover the theorem of Meng and Taubes [8] relating $SW$ to “Milnor torsion” (modulo signs, in the case of closed manifolds). We explain this in §4.2. (Actually our methods might also be applicable to the Meng-Taubes formula for manifolds with boundary, because the boundary conditions in Theorem 1.1 and in [8] are similar.)

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2 Preliminaries

2.1 Notation and conventions

The symbol · always denotes intersection number. We take $\alpha \cdot \beta$ to be zero if $\alpha$ and $\beta$ do not have complementary dimension.

The differential in the Morse complex is $d$, its adjoint with respect to the natural inner product $\langle \cdot, \cdot \rangle$ is $d^*$, and $\Delta := dd^* + d^*d$.

The symbol ‘Tr’ always denotes graded trace.

If $R$ is an integral domain, $Q(R)$ is its field of fractions, and $L_R$ is the ring of Laurent series with coefficients in $R$, i.e. functions $\mathbb{Z} \to R$ supported away from $-\infty$.

Regarding orientations: if $Y$ is an oriented manifold, $\psi$ is a Morse function on $Y$, and $\lambda \in \mathbb{R}$ is a regular value of $\psi$, we orient the level set $W = \psi^{-1}(\lambda)$ by declaring $TY|W = \mathbb{R} \cdot \text{grad}(\psi) \oplus TW$ to be an isomorphism of oriented vector bundles. If $Y$ is an oriented manifold with boundary, we orient the boundary via the convention $TY|\partial Y = \mathbb{R} \cdot \nu \oplus T(\partial Y)$.
2.2 Morse homology

We will now present a direct approach to the relation between Morse homology and ordinary homology. We will then deduce some simple lemmas concerning circle-valued Morse homology.

The approach here is very useful for understanding Theorem 1.1, but there is one technical difficulty, which is that we have to compute ordinary homology using special singular chains. These chains must have well defined intersection numbers with the descending manifolds of the critical points, and the set of such chains must be preserved by the gradient flow. For example, we can use smooth chains with conical singularities, in which the smooth set and the singular sets intersect the descending manifolds of the critical points transversely. (See Laudenbach [7] for a proof that these are preserved by gradient flow.) One can use standard techniques to show that an arbitrary chain may be approximated by these special chains, so that the homology of the complex of special chains is isomorphic to ordinary homology. We will not go into further details about this.

Let \( Y^n \) be an oriented Riemannian manifold and let \( \psi : Y \to (-\infty, 0] \) be a generic Morse function with \( \partial Y = \psi^{-1}(0) \). (“Generic” means that the gradient flow is Morse-Smale, i.e. the ascending and descending manifolds of different critical points intersect transversely.) Let \( \text{Crit} \) denote the set of critical points. For \( x \in \text{Crit} \), we define the ascending manifold of \( x \), \( \mathcal{A}(x) \), to be the closure of the set of all \( p \in Y \) such that downward gradient flow from \( p \) converges to \( x \). We define the descending manifold \( \mathcal{D}(x) \) analogously, using upward gradient flow. Choose orientations on \( \mathcal{A}(x) \) and \( \mathcal{D}(x) \) such that the intersection number \( \mathcal{A}(x) \cdot \mathcal{D}(x) \) is +1 in \( Y \).

We define the Morse complex as follows. Let \( M^i \) be the free \( \mathbb{Z} \)-module generated by \( \text{Crit}^i \), the set of critical points of index \( i \). Define \( d : M^i \to M^{i+1} \) by

\[
dx := \sum_{y \in \text{Crit}^{i+1}} \langle dx, y \rangle y,
\]

where \( \langle dx, y \rangle \) is the signed number of gradient flow lines from \( x \) to \( y \). A flow line counts with positive sign when the intersection number \( (\mathcal{D}(x) \cap \psi^{-1}(\lambda)) \cdot (\mathcal{A}(y) \cap \psi^{-1}(\lambda)) \) is +1 in \( \psi^{-1}(\lambda) \).

If \( \alpha \in C_*(Y) \) is a generic chain, define \( \mathcal{F}(\alpha) \) to be the closure of the union, over all \( s \in [0, \infty) \), of the time \( s \) upward gradient flow applied to \( \alpha \), oriented so that the orientations on \( \partial \mathcal{F}(\alpha) \) and \( \alpha \) disagree. We need just one geometric observation:
Lemma 2.1 If $\alpha$ is a generic chain, then
$$\partial \mathcal{F}(\alpha) = \mathcal{F}(\alpha) \cap \partial Y - \alpha - \mathcal{F}(\partial \alpha) + \sum_{x \in \text{Crit}} (D(x) \cdot \alpha) \mathcal{A}(x).$$

Proof Sketch. This is straightforward, using the Morse lemma to model the behavior near the critical points. (It is a little easier to first assume all the critical points are at the same height, and then use induction.) The idea is illustrated in Figure 1. □

Lemma 2.2 If $x$ is a critical point then
$$\partial \mathcal{A}(x) = \mathcal{A}(x) \cap \partial Y + \mathcal{A}(dx).$$
(Here we are extending $\mathcal{A}$ to a map from $M^*$ to $C_{n-*}(Y)$.)

Proof. Let $s \in \mathbb{R}$ be slightly larger than $\psi(x)$, and let $\alpha = \mathcal{A}(x) \cap \psi^{-1}(s)$. Apply Lemma 2.1 to obtain
$$\partial(\mathcal{A}(x) \cap \psi^{-1}[s, 0]) = \mathcal{A}(x) \cap \partial Y - \alpha + \mathcal{A}(dx).$$
Since $\partial(\mathcal{A}(x) \cap \psi^{-1}(-\infty, s]) = \alpha$, we are done. □

It follows from this lemma that
$$\mathcal{A} : M^* \to C_{n-*}(Y, \partial Y)$$
is a chain map. It then follows from $\partial^2 = 0$ that
$$d^2 = 0,$$
since the ascending manifolds of different critical points are disjoint.

Proposition 2.3 The chain map $\mathcal{A}$ induces an isomorphism
$$H^*(M^*) \simeq H_{n-*}(Y, \partial Y).$$
Under this isomorphism, the connecting homomorphism $\delta$ in the relative homology exact sequence
$$H_* (Y) \longrightarrow H_* (Y, \partial Y) \longrightarrow H_{*+1}(\partial Y)$$
is given by
$$\delta(x) = \mathcal{A}(x) \cap \partial Y.$$
(Classical references for this are Thom [21], Smale [17], and Milnor [9]. A more general statement is proved in Floer [1]. For more novel proofs see Witten [23], Helffer-Sjöstrand [4], and Schwarz [16].)

Proof. Define $G : C_*(Y) \to M^{n-*}$ by

$$G(\alpha) = \sum_{x \in \text{Crit}} (\mathcal{D}(x) \cdot \alpha)x.$$ 

Clearly $G$ annihilates $C_*(\partial Y)$ and therefore defines a map $C_*(Y, \partial Y) \to M^{n-*}$. Applying $\partial$ to Lemma 2.1 and using Lemma 2.2, we obtain

$$0 = -A(G(\partial \alpha)) + A(dG(\alpha))$$

in $C_*(Y, \partial Y)$. It follows that $G$ is a chain map. We claim that the induced map on homology is the inverse of the map induced by $A$. By definition, $G \circ A$ is equal to the identity on $N_*$. On the other hand, Lemma 2.1 asserts that $F$ is a chain homotopy between $A \circ G$ and the identity on $C_*(Y, \partial Y)$.

The assertion about the connecting homomorphism is true more or less by definition. □

We now consider the generalization to circle-valued Morse functions. Let the notation be as in the introduction. Let $C_*(\tilde{X}, +\infty)$ be the complex of locally finite chains in $\tilde{X}$ that are supported away from the lower end of $\tilde{X}$, i.e. for each chain there exists $R \in \mathbb{R}$ such that the chain is supported in $\{(x, \lambda) \in \tilde{X} \mid \lambda \geq R\}$. If $x \in \text{Crit}$, let $A(t^kx) \subset \tilde{X}$ denote the ascending manifold of a lift $(x, \lambda) \in \tilde{X}$ of $x$ with $k < \lambda < k + 1$.

The following (without $A$) was observed by Novikov [14]:

**Proposition 2.4** $A$ is a chain map and induces an isomorphism of $L_\mathbb{Z}$ modules

$$H^*(M^*) \simeq H_{n-*}(\tilde{X}, +\infty).$$

Proof. Same idea as the proof of Proposition 2.3. □

This proposition has two corollaries which we will also need.

**Corollary 2.5** Let $C_*(\tilde{X})$ be the chain complex associated to an equivariant cell decomposition of $\tilde{X}$, as in the introduction. Then we have an isomorphism of $L_\mathbb{Z}$-modules

$$H_*(C_*(\tilde{X}) \bigotimes_{\mathbb{Z}[t,t^{-1}]} L_\mathbb{Z}) \simeq H^{n-*}(M^*).$$
Proof. There is an obvious chain map

\[ C_\ast(X) \bigotimes_{\mathbb{Z}[t,t^{-1}]} L_Z \rightarrow C_\ast(X, +\infty). \]

This induces an isomorphism in homology, thanks to the equivalence between cellular and singular homology. Now compose this isomorphism with the isomorphism of Proposition 2.4. □

Corollary 2.6 Suppose \( H^\ast(M^\ast \otimes L_\mathbb{Q}) = 0 \). Then the map \( H_\ast(\Sigma; \mathbb{Q}) \rightarrow H_\ast(\tilde{X}; \mathbb{Q}) \) induced by the inclusion \( p \mapsto (p, 0) \) is surjective.

Proof. Let \( \alpha \in C_\ast(\tilde{X}; \mathbb{Q}) \) be a cycle. By Proposition 2.4 (tensored with \( L_\mathbb{Q} \)) we can write \( \alpha = \partial \beta \) for some \( \beta \in C_\ast(\tilde{X}, +\infty; \mathbb{Q}) \). Observe that

\[ \partial(\beta \cap \phi^{-1}(-\infty, 0]) = \alpha \cap \phi^{-1}(-\infty, 0] + \beta \cap (\Sigma \times \{0\}). \] (3)

Now we can turn this reasoning upside down and apply Proposition 2.4 to the Morse function \( -\phi \) to find \( \gamma \in C_\ast(\tilde{X}, -\infty; \mathbb{Q}) \) with \( \alpha = \partial \gamma \). (The Morse complex for \( -\phi \) is acyclic because it is just the dual of the Morse complex for \( \phi \).) We have

\[ \partial(\gamma \cap \phi^{-1}[0, \infty)) = \alpha \cap \phi^{-1}[0, \infty] - \gamma \cap (\Sigma \times \{0\}). \] (4)

Subtracting (4) from (3), we see that \( (\gamma - \beta) \cap (\Sigma \times \{0\}) \) is a cycle in \( \Sigma \times \{0\} \) homologous to \( \alpha \). □

2.3 Torsion

Let \( R \) be an integral domain and let

\[ C^0 \overset{d}{\longrightarrow} C^1 \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} C^m \]

be a complex of finitely generated free \( R \)-modules. Suppose that \( C^\ast \otimes Q(R) \) is acyclic. Using a volume form on \( C^i \otimes Q(R) \) coming from a free basis for \( C^i \), we can define the Reidemeister torsion

\[ \tau(C \otimes Q(R)) \in \frac{Q(R)}{\{\text{units of } R\}} \]

as in (1). (Choosing a different free basis multiplies \( \tau \) by a unit.)

Note that if \( h : Q(R) \rightarrow F \) is an inclusion into a larger field, then \( C \otimes F \) is acyclic if and only if \( C \otimes Q(R) \) is, and

\[ \tau(C \otimes F) = h(\tau(C \otimes Q(R))). \]
We will denote $\tau(C \otimes Q(R))$ simply by $\tau(C)$.

To compute $\tau(C)$, we will use the following result of Turaev [22, §2.1], generalizing a theorem of Milnor [11]. If $E$ is a finitely generated module over a ring $R$, let $\text{Fitt}_1(E)$ be the first Fitting ideal of the module, which is generated by the determinants of the $n \times n$ minors of the matrix of relations for a presentation of $E$ with $n$ generators. This does not depend on the presentation. (See e.g. Rolfsen [15].) If greatest common divisors exist in $R$, let $\text{ord}(E)$ denote the greatest common divisor of the elements in $\text{Fitt}_1(E)$, which is well-defined up to units in $R$.

**Lemma 2.7** Suppose $R$ is a Noetherian UFD, and $C$ is as above. Then

$$\tau(C) = \prod_{i=0}^{m} (\text{ord} H^i(C))^{(-1)^i}$$

up to units of $R$.

In particular, the rings $\mathbb{Z}[t, t^{-1}]$ and $L_\mathbb{Z}$ satisfy the hypothesis on $R$ above. This lemma is an easy exercise in the special case when $\text{Ker}(d)$ is a free $R$-module with a free complement.

### 2.4 Equivalence of Theorems 1.1 and 1.2

This follows from:

**Lemma 2.8** (a) The leading coefficients of the left and right sides of Theorem 1.2 are equal, up to sign.

(b) Theorem 1.1 is the logarithmic derivative of Theorem 1.2.

**Proof.** (a) By definition, the leading coefficient of $\zeta(f)$ is 1, so we need to check that the leading coefficients of $\tau(M)$ and $\tau(X, \phi)$ agree. We defined

$$\tau(X, \phi) = \tau \left( C_*(\tilde{X}) \bigotimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}[t, t^{-1}] \right),$$

where $C_*(\tilde{X})$ is the complex associated to an equivariant cell decomposition of $\tilde{X}$. We can tensor $C_*(\tilde{X}) \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}[t, t^{-1}]$ with $L_\mathbb{Z}$, and this will not affect the torsion (or more precisely will include the torsion into $L_\mathbb{Q}$). Thus

$$\tau(X, \phi) = \tau \left( C_*(\tilde{X}) \bigotimes_{\mathbb{Z}[t, t^{-1}]} L_\mathbb{Z} \right).$$
By Corollary 2.5,

$$H_* \left( C_*(\tilde{X}) \bigotimes_{\mathbb{Z}[t, t^{-1}]} L_{\mathbb{Z}} \right) \simeq H^{n-*}(M_*) .$$

From Lemma 2.7, we see that the torsion of a complex of $L_{\mathbb{Z}}$-modules depends only on the homology, up to units in $L_{\mathbb{Z}}$. Thus

$$\tau(M) = \tau(X, \phi)$$

up to units in $L_{\mathbb{Z}}$. But a unit in $L_{\mathbb{Z}}$ must have leading coefficient $\pm 1$, so we are done.

(b) Observe that

$$\frac{d}{dt} \log \det(1 - tA_i) = -\text{Tr}(A_i(1 - tA_i)^{-1}).$$

Combining this with (2), we see that we need

$$\tau(X, \phi) = c \prod_{i=0}^{n} \det(1 - tA_i)^{(-1)^{n-i}}$$

for some $c \in \mathbb{R}$, up to units in $\mathbb{Z}[t, t^{-1}]$. (Note that the integer $m$ on the right side of Theorem 1.1 absorbs this ambiguity in $\tau(X, \phi)$.) Now $\tau(X, \phi)$ is the torsion of the complex $C_*(\tilde{X}) \bigotimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z}[t, t^{-1}]$, and the homology of this complex is isomorphic to $H_*(\tilde{X})$ as a $\mathbb{Z}[t, t^{-1}]$-module. So we will show that

$$\text{ord}(H_i(\tilde{X})) = c_i \det(1 - tA_i)$$

for some $c_i \in \mathbb{Z}$, and then we will be done by Lemma 2.7.

We can choose a set $S \subset H_i(\tilde{X})$ which projects to a basis for $H_i(\tilde{X})/\text{Torsion}$ over $\mathbb{Z}$. ($S$ will be finite thanks to Corollary 2.6.) We can then choose a finite set $T$ that generates the torsion part of $H_i(\tilde{X})$ as a $\mathbb{Z}[t, t^{-1}]$-module. The matrix of relations for $H_i(\tilde{X})$ is the following:

$$
\begin{pmatrix}
S & T & T \\
S & 1 - tA_i & 0 & 0 \\
T & ? & D & ?
\end{pmatrix}
$$

Here the columns represent relations. The only relations on $H_i(\tilde{X}; \mathbb{Q})$ are $1 - tA_i$, and when we lift these to $H_i(\tilde{X})$ via our choice of $S$, there may be an additional component in $T$, which is the lower left block of the matrix. $D$ is a diagonal matrix of integers asserting that the elements of $T$ are torsion.
The lower right block of the matrix expresses whatever additional relations
the elements of $T$ may satisfy amongst themselves.

Now every minor of this matrix is divisible by $\det(1-tA_i)$, so $\det(1-tA_i)$
divides $\text{ord}(H_i(X))$. On the other hand one of the minors is $\det(D)\det(1-
tA_i)$, so $\text{ord}(H_i(X))$ divides $\det(D)\det(1-tA_i)$. Our claim follows.  \qed

3 Proof of Theorem 1.1

3.1 Outline of the proof

One of the classical proofs of the Lefschetz fixed point formula on a manifold
$\Sigma$ goes as follows. We wish to calculate the intersection number of the graph
with the diagonal in $\Sigma \times \Sigma$. We can replace the diagonal with a homologous
cycle in $C_*(\Sigma) \otimes C_*(\Sigma)$, and then we are reduced to intersection theory in $\Sigma$.
We will attempt to extend this reasoning to our situation, where the graph
is no longer a cycle. Since intersection number with a chain with boundary
does not descend to homology, more care is required.

First of all let us assume that $\partial X = \emptyset$; it is easy to remove this restriction
at the end, in §3.7.

Define a chain $\Gamma$ in $\Sigma \times \Sigma$ with Taylor series coefficients by
\[
\Gamma := \sum_{k=1}^{\infty} t^k (\text{graph of } f^k).
\]

Let $\text{diag} \subset \Sigma \times \Sigma$ be the diagonal. Then the first term in Theorem 1.1 is the
intersection number $\Gamma \cdot \text{diag}$.

We wish to replace $\text{diag}$ with a homologous cycle in $C_*(\Sigma; L_Q)^{\otimes 2}$. However
$\Gamma$ is not a cycle, as its closure has nontrivial boundary involving the ascending
and descending manifolds of critical points. (We will not distinguish $\Gamma$ from
its closure in the notation.) So we will first find $Z \in C_*(\Sigma; L_Q)^{\otimes 2}$ such that
\[
\partial(\Gamma - Z) = 0.
\]

Such a $Z$ exists by the Eilenberg-Zilber theorem, but is not canonical. However
we will see in §3.4 that when the rational Morse complex is acyclic, there
is a canonical choice of $Z$, constructed directly out of the gradient flow.

Next, let $\{e_i\}$ be a set of cycles in $\Sigma$ that represent a basis for $H_*(\Sigma; \mathbb{Q})$, and let $\{e^*_i\}$ be cycles representing the (Poincaré) dual basis, i.e. $e_i \cdot e^*_j = \delta_{ij}$.
Then $\text{diag} - \sum_i e_i \times e^*_i$ is homologous to zero. It follows that the intersection number
\[
(\Gamma - Z) \cdot (\text{diag} - \sum_i e_i \times e^*_i) = 0.
\]
Direct calculations in §3.5 will show that for the natural choice of $Z$ mentioned above,

**Lemma 3.1**  

(a) $Z \cdot \text{diag} = (-1)^n t \frac{d}{dt} \log \tau(M)$.  

(b) $(\Gamma - Z) \cdot \sum_i e_i \times e_i^* = \text{Tr}(B : H_*(\Sigma; L_Q) \to H_*(\Sigma; L_Q)))$.

Here $B \in \text{End}(C_*(\Sigma; L_Q))$ is a natural chain map constructed in §3.3 out of the gradient flow. Roughly speaking, $B$ is $\sum_k t^k f^k$, plus a correction term that makes it a chain map.

In §3.6 we prove:

**Lemma 3.2** The diagram

$$
\begin{array}{ccc}
H_*(\Sigma) & \xrightarrow{B} & H_*(\Sigma) \\
\downarrow \iota_* & & \downarrow \iota_* \\
H_*(\tilde{X}) & \xrightarrow{t^A(1-t^A)^{-1}} & H_*(\tilde{X})
\end{array}
$$

commutes. (Here all homology is with $L_Q$ coefficients, and $\iota : \Sigma \to \tilde{X}$ sends $p \mapsto (p, 0)$.)

The proof of this lemma is quite natural. Let $\gamma \in C_*(\Sigma; Q)$ be a cycle, and suppose the upward gradient flow takes $\gamma$ around $X k$ times without hitting any critical points. Then $\gamma \times \{-k\}$ and $f^k(\gamma) \times \{0\}$ are homologous in $\tilde{X}$, because their difference is the boundary of the entire gradient flow between them. But $\gamma \times \{-k\}$ is the $k^{th}$ downward deck transformation of $\gamma \times \{0\}$, so the $k^{th}$ downward deck transformation of $\gamma \times \{0\}$ is homologous in $\tilde{X}$ to $f^k(\gamma) \times \{0\}$. This means that $\iota_* f^k(\gamma) = A^k(\iota_* \gamma)$. More generally, if the upward gradient flow of $\gamma$ hits some critical points, then the gradient flow no longer gives a homology between $\gamma \times \{-k\}$ and $f^k(\gamma) \times \{0\}$, because it has additional boundary components arising from the critical points. But the extra term in $B$ is exactly what is needed to cancel these.

By Corollary 2.6, $\iota_*$ is surjective, so

$$
\text{Tr}(B) = \text{Tr}(t^A(1-t^A)^{-1}) + \text{Tr}(B|\text{Ker}(\iota_*))
$$

So if we can show $\text{Tr}(B|\text{Ker}(\iota_*)) \in \mathbb{Z}$, we are done. We can understand $\text{Ker}(\iota_*)$ in terms of the Morse theory using Proposition 2.3. We then compute the restriction of $B$ to this kernel, and we use a cheap trick to show that its trace is an integer: we argue that all the nonconstant terms in $\text{Tr}(B|\text{Ker}(\iota_*))$ vanish a priori, basically because they have the wrong degree.

To carry out the above computations, we need to develop some formalism. §3.2 proves some simple geometrical facts we need, along the lines of Lemma 2.1, and §3.3 encodes these geometrical facts into algebraic formalism.
3.2 Geometric observations

Let $Y$ be an oriented manifold and let $\psi : Y \to [0, r]$ be a generic Morse function. Let $Y_0 = \psi^{-1}(0)$, $Y_1 = \psi^{-1}(r)$, and assume $\partial Y = Y_0 \cup Y_1$. If $x$ is a critical point, define $A(x)$ and $D(x)$, and orient them, as in §2.2. The gradient flow of $\psi$ defines a diffeomorphism

$$g : Y_0 \setminus \bigcup_{x \in \text{Crit}} D(x) \rightarrow Y_1 \setminus \bigcup_{x \in \text{Crit}} A(x).$$

Lemma 3.3  
(a) $\partial(\text{graph of } g) = \sum_i (-1)^i \sum_{x \in \text{Crit}^i} (D(x) \cap Y_0) \times (A(x) \cap Y_1)$.

(b) If $\alpha$ is a chain in $Y_0$, then

$$\partial g(\alpha) = g(\partial \alpha) - \sum_{x \in \text{Crit}} (D(x) \cdot \alpha)(A(x) \cap Y_1).$$

(c) If $\alpha$ is a chain in $Y_1$, then

$$\partial g^{-1}(\alpha) = g^{-1}(\partial \alpha) + \sum_{x \in \text{Crit}} (\alpha \cdot A(x))(D(x) \cap Y_0).$$

Proof Sketch. (a) is straightforward. (As in the proof of Lemma 2.1, it is easiest to first assume all the critical points are at the same height and then use induction.) (b) follows by applying $\partial$ to Lemma 2.1. (c) is analogous to (b). □

3.3 Some formalism

If $x \in \text{Crit}^i$, let $A(t^k x) \subset \tilde{X}$ (resp. $D(t^k x)$) be the ascending (resp. descending) manifold of a lift $(x, \lambda) \in \tilde{X}$ of $x$ with $k < \lambda < k + 1$. Define a map

$$\pi_+ : M^* \otimes L_Z \rightarrow C_{n-1-*}(\Sigma; L_Z)$$

by setting

$$\pi_+(x) := \sum_{j=-\infty}^{\infty} t^j A(x) \cap (\Sigma \times \{j\}).$$

for all $x \in M^*$. (Here we are identifying $\Sigma \times \{j\}$ with $\Sigma$.)

In English, $\pi_+$ of a critical point is the set of all points in $\Sigma$ such that downward gradient flow converges to the critical point, multiplied by $t^k$, where $k$ is the number of times the gradient flow crosses through $\Sigma$ before reaching the critical point.

We define

$$\pi_- : M^* \otimes L_Z \rightarrow C_{*-1}(\Sigma; L_Z)$$
similarly; for \( x \in M^* \), let
\[
\pi_-(x) := \sum_j t^j D(x) \cap (\Sigma \times \{-j\}).
\]

**Lemma 3.4** \( \partial \Gamma = t \sum_i (-1)^i \sum_{y \in \text{Crit}} \pi_-(y) \times \pi_+(y). \)

**Proof.** We use Lemma 3.3(a). To calculate the \( t^k \) term, let
\[
Y = \{(p, \lambda) \in \tilde{X} \mid 0 \leq \lambda \leq k \} \subset \tilde{X}.
\]
So in the notation of §3.2, \( Y_0 = Y_1 = \Sigma \) and \( g = f^k \).

If \( y \in \text{Crit} \), let \( \phi(y)_0 \in (0,1) \) be a representative of \( \phi(y) \in \mathbb{R}/\mathbb{Z} \). Then the critical points of \( g \) are of the form \((y, \phi(y)_0 + j)\) for \( y \in \text{Crit} \) and \( j = 0, \ldots, k - 1 \). The intersection of the descending manifold of such a critical point with \( \phi^{-1}(0) \) is the \( t^j \) term of \( \pi_-(y) \). The intersection of the ascending manifold with \( \phi^{-1}(k) \) is the \( t^{k-j-1} \) term of \( \pi_+(y) \). So by Lemma 3.3(a),
\[
\partial(\text{graph of } f^k) = \sum_i (-1)^i \sum_{y \in \text{Crit}} \sum_{j=0}^{k-1} (t^j \text{ term of } \pi_-(y)) \times (t^{k-j-1} \text{ term of } \pi_+(y)).
\]
This proves the \( t^k \) term of the lemma. \( \square \)

In the sequel, we will omit the details when applying Lemma 3.3 in a straightforward way as above.

**Lemma 3.5** If \( d \) is the differential in the Morse complex and \( d^* \) is its adjoint with respect to the natural inner product \( \langle , \rangle \), then
(a) \( \partial \pi_+ = -\pi_+ d \),
(b) \( \partial \pi_- = (-1)^{i-1} \pi_- d^* \) on \( M^* \).

**Proof.** Part (a) follows from Lemma 3.3(b). Part (b) follows similarly from Lemma 3.3(c). The sign here arises when we switch from intersections in \( X \) (as in Lemma 3.3) to intersections in \( \Sigma \) (in the definition of \( d \) in §2.2). \( \square \)

The following will be used in §3.5. Let \( f \) be the partially defined endomorphism of \( \Sigma \) from the introduction. Define an endomorphism \( f^+ \) of \( C_*(\Sigma) \otimes L_\mathbb{Z} \) by
\[
f^+ := (1 - tf)^{-1}.
\]
It turns out that \( f^+ \) is not a chain map. To understand this, define a map \( \xi : C_*(\Sigma) \to M^{n-*} \) by requiring that
\[
\langle x, \xi(\alpha) \rangle := \pi_-(x) \cdot \alpha
\]
for all \( \alpha \in C_*(\Sigma) \) and \( x \in M^{n-*} \). (Note that \( \xi \) is an analogue of the map \( G \) in the proof of Proposition 2.3.)
Lemma 3.6 If $\alpha \subset \Sigma$ is a generic chain, then
\[
\partial f^+(\alpha) = f^+(\partial \alpha) - t\pi_+\xi(\alpha).
\]

Proof. This follows from Lemma 3.3(b).

We can add a correction term to $f^+$, using $\xi$, to make it a chain map. (This will arise naturally in §3.5.) Namely, define an endomorphism $B$ of $C_*(\Sigma; L_{\mathbb{Q}})$ by
\[
B := f^+ - 1 - t\pi_+ d^* \Delta^{-1} \xi.
\]
($\Delta$ is invertible thanks to our assumption that the rational Morse complex is acyclic.)

Lemma 3.7 (a) $d\xi = \xi \partial$.

(b) $\partial B = B \partial$.

Proof. (a) Let $\alpha \in C_i(\Sigma)$ and $x \in M^{n-i+1}$. Then
\[
\langle d\xi(\alpha), x \rangle = \pi_- d^* x \cdot \alpha
= (1)^{n-i} \partial \pi_- x \cdot \alpha
= \pi_- x \cdot \partial \alpha
= \langle \xi(\partial \alpha), x \rangle.
\]
(The sign in the third line is tricky, because $\partial$ is not quite a signed derivation with respect to intersections. Also, an alternate proof of (a) can be given by mimicking the proof that $G$ is a chain map in §2.2.)

(b) By Lemma 3.6,
\[
\partial B = f^+ \partial - t\pi_+ \xi - \partial - t\partial \pi_+ d^* \Delta^{-1} \xi.
\]
We can write
\[
t\pi_+ \xi = t\pi_+ d^* \Delta^{-1} \xi + t\pi_+ d^* \Delta^{-1} d\xi.
\]
By Lemma 3.5(a),
\[
t\pi_+ d^* \Delta^{-1} \xi = - t\partial \pi_+ d^* \Delta^{-1} \xi.
\]
By part (a),
\[
t\pi_+ d^* \Delta^{-1} d\xi = t\pi_+ d^* \Delta^{-1} \xi \partial.
\]
Substitute the above two equations into (6), and put the result into (5).
3.4 Closing off the boundary of the graph

As explained in the outline, we now want to find $Z \in C_*(\Sigma; L_Q) \otimes^2$ with $\partial Z = \partial \Gamma$. Let $P$ be the composition

$$\text{Hom}(M^i, M^j) \xrightarrow{\rho} M^j \otimes M^i \xrightarrow{\pi_- \otimes \pi_+} C_*(\Sigma; L_Q) \otimes^2.$$ 

Here $\rho$ is the canonical isomorphism given by the inner product $\langle \cdot, \cdot \rangle$. Our ansatz will be $Z = P(W)$ for some $W \in \text{Hom}(M^*, M^{i+1})$.

**Lemma 3.8** Let $W = \sum_{i=0}^{n-1} W^i$ with $W^i \in \text{Hom}(M^i, M^{i+1})$. Then

$$\partial P(W) = \sum_{i=0}^{n-1} (-1)^i P(d^*W_i + W_{i-1}d^*).$$

(Here we interpret $W_{-1} = 0$.)

**Proof.** By Lemma 3.5,

$$\partial P(W) = \sum_i ((-1)^i \pi_- \otimes \pi_+ - (-1)^i \pi_- \otimes \pi_+ d) \rho(W_i).$$

(The factor of $(-1)^i$ on the right arises because $\partial (a \times b) = \partial a \times b + (-1)^{\dim(a)} a \times \partial b$, and $\pi_- M^{i+1}$ has dimension $i$.) Now use the facts $(d^* \otimes 1)\rho(W_i) = \rho(d^* W_i)$ and $(1 \otimes d)\rho(W_i) = \rho(W_id^*)$. □

In this notation, Lemma 3.4 says that

$$\partial \Gamma = t \sum_{i=0}^{n-1} (-1)^i P(1|M^i).$$

So by Lemma 3.8, $\partial Z = \partial \Gamma$ if and only if

$$d^*W + Wd^* = t.$$ 

Such a $W$ exists if and only if the rational Morse complex is acyclic, which we assumed to be true. The natural choice, which we will adopt, is

$$W := t \Delta^{-1} d.$$
3.5 Calculating intersection numbers

We will now prove Lemma 3.1. In these calculations, it is convenient to choose a basis \( \{ x^i_j \} \) for Ker\((d^*|M^i)\) with \( \| x^i_j \| = 1 \) and \( \Delta x^i_j = \lambda^i_j x^i_j \). (To find such a basis we may have to extend to coefficients in the algebraically closed field \( L_C \), which causes no problems.)

**Lemma 3.9** \( \tau(M) = \prod_{i=0}^{n-1} \left( \prod_j \sqrt{\lambda^i_j} \right)^{(1)^{i-1}} \).

**Proof.** Take \( \omega_i = \bigwedge_j x^i_j \) in (1).

**Lemma 3.10** Let \( x \in M^* \) and \( y \in M^{*+1} \). Then
\[
\pi_- y \cdot \pi_+ x = \frac{d}{dt} \langle dx, y \rangle - \left\langle d \left( \frac{d}{dt} x \right), y \right\rangle - \left\langle dx, \frac{d}{dt} y \right\rangle.
\]

**Proof.** If \( x, y \in \text{Crit} \), then the two rightmost terms are zero, and the formula follows directly from the definitions. The general case follows by expanding \( x \) and \( y \) in powers of \( t \).

**Proof of Lemma 3.1.** (a) We have
\[
Z = t \sum_{i,j} (\lambda^i_j)^{-1} \pi_- (dx^i_j) \times \pi_+ (x^i_j).
\]

If \( \alpha, \beta \) are two chains of complementary dimension then \( (\alpha \times \beta) \cdot \text{diag} = (-1)^{\dim(\beta)} \alpha \cdot \beta \). Thus
\[
Z \cdot \text{diag} = t \sum_{i,j} (-1)^{n-i-1} (\lambda^i_j)^{-1} \pi_- (dx^i_j) \cdot \pi_+ (x^i_j).
\]

Writing \( x = x^i_j \), Lemma 3.10 gives
\[
\pi_- (dx) \cdot \pi_+ (x) = \frac{d}{dt} \| dx \|^2 - \left\langle d \left( \frac{d}{dt} x \right), dx \right\rangle - \left\langle dx, \frac{d}{dt} dx \right\rangle.
\]

Thanks to the special properties of \( x = x^i_j \), the middle term on the right vanishes:
\[
\left\langle d \left( \frac{d}{dt} x^i_j \right), dx^i_j \right\rangle = \left\langle \frac{d}{dt} x^i_j, d^i dx^i_j \right\rangle = \lambda^i_j \left\langle \frac{d}{dt} x^i_j, x^i_j \right\rangle = \frac{\lambda^i_j}{2} \frac{d}{dt} \| x^i_j \|^2 = 0.
\]
Thus
\[ \pi_-(dx^i_j) \cdot \pi_+(x^i_j) = \frac{1}{2} \frac{d}{dt} \|dx^i_j\|^2 = \frac{1}{2} \frac{d}{dt} \lambda^i_j. \]
Substituting this into (7) and comparing with Lemma 3.9 proves (a).

(b) First of all,
\[ \Gamma \cdot \sum_k e_k \times e^*_k = \sum_{n=1}^{\infty} t^n \sum_i (-1)^{\dim(e_k)} f^n(e_k) \cdot e^*_k. \]  \hfill (8)

Second,
\[ Z \cdot \sum_k e_k \times e^*_k = t \sum_{i,j,k} (-1)^{\dim(e_k)} ((\lambda^i_j)^{-1} \pi_- dx^i_j \cdot e_k)(\pi_+(x^i_j) \cdot e^*_k) \]
\[ = t \sum_{i,j,k} (-1)^{\dim(e_k)} (\lambda^i_j)^{-1} \langle \xi(e_k), dx^i_j \rangle (\pi_+ (x^i_j) \cdot e^*_k). \]  \hfill (9)

By Lemma 3.7(a), \( d\xi(e_k) = 0 \), so we can write
\[ \xi(e_k) = dd^* \Delta^{-1} \xi(e_k). \]
Then
\[ \langle \xi(e_k), dx^i_j \rangle = \langle d^* \Delta^{-1} \xi(e_k), d^* dx^i_j \rangle \]
\[ = \lambda^i_j \langle d^* \Delta^{-1} \xi(e_k), x^i_j \rangle. \]

Putting this into (9) gives
\[ Z \cdot \sum_k e_k \times e^*_k = t \sum_k (-1)^{\dim(e_k)} (\pi_+ d^* \Delta^{-1} \xi(e_k)) \cdot e^*_k. \]
Subtracting this from (8) gives
\[ Z \cdot (\Gamma - \sum_k e_k \times e^*_k) = \sum_k (-1)^{\dim(e_k)} B(e_k) \cdot e^*_k. \]

3.6 Understanding \( B \)

We will now prove Lemma 3.2. To do this we need a bit more formalism. If \( \alpha \) is a chain in \( \Sigma \), define
\[ \mathcal{F}(t^k \alpha) := \mathcal{F}(\alpha \times \{k\}) \subset \check{X}. \]
If \( \alpha \in C_*(\Sigma; L) \) and \( k \in \mathbb{Z} \), define
\[ \mathcal{F}_k(\alpha) := \{(x, \lambda) \in \mathcal{F}(\alpha) \mid \lambda \leq k \}. \]
If \( x \in M^* \), define
\[ \mathcal{A}_k(x) := \{(x, \lambda) \in \mathcal{A}(x) \mid \lambda \leq k \}. \]
Lemma 3.11  

(a) If $\gamma \in C_\ast(\Sigma; \mathbb{Q})$ is a cycle and $k > 0$, then

$$\partial F_k(\gamma) = f^k(\gamma) \times \{k\} - \gamma \times \{0\} + A_k(\xi(\gamma)).$$

(b) If $x \in M^\ast$ then

$$\partial A_k(x) = A_k(dx) + (\pi_+ x)^{k-1} \times k,$$

where $(\pi_+ x)^{k-1}$ is the coefficient of $t^{k-1}$ in $\pi_+ x$.

Proof. (a) follows directly from Lemma 2.1. (b) follows from Lemma 2.2. □

Proof of Lemma 3.2. Let $\gamma \in C_\ast(\Sigma; \mathbb{Q})$ be a cycle. We need to show:

(a) If $k > 0$, then $(B\gamma)^k \times \{k\}$ is homologous to $\gamma \times \{0\}$ in $\tilde{X}$ (where $(B\gamma)^k$ is the $t^k$ coefficient in $B(\gamma)$).

(b) If $k \leq 0$, then $(B\gamma)^k \times \{k\}$ is nullhomologous in $\tilde{X}$.

Suppose $k > 0$. By Lemma 3.7(a), $d\xi(\gamma) = 0$, so we can write

$$\xi(\gamma) = d(d^* \Delta^{-1} \xi \gamma).$$

Putting $x = d^* \Delta^{-1} \xi \gamma$ into Lemma 3.11(b) gives

$$\partial A_k(d^* \Delta^{-1} \xi \gamma) = A_k(\xi \gamma) + (\pi_+ d^* \Delta^{-1} \xi \gamma)^{k-1} \times \{k\}.$$ 

Subtracting this from Lemma 3.11(a) gives

$$\partial(\text{something}) = f^k(\gamma) \times \{k\} - (\pi_+ d^* \Delta^{-1} \xi \gamma)^{k-1} \times \{k\} - \gamma \times \{0\}$$

$$= (B\gamma)^k \times \{k\} - \gamma \times \{0\}.$$ 

This proves (a).

If $k \leq 0$, then $F_k(\xi(\gamma)) = 0$, since

$$\xi(\gamma) = \sum_{y \in \text{Crit}} (\pi_-(y) \cdot \gamma) y$$

is a Taylor series. Then Lemma 3.11(b) gives

$$\partial A_k(d^* \Delta^{-1} \xi \gamma) = (\pi_+ d^* \Delta^{-1} \xi \gamma)^{k-1} \times \{k\}.$$ 

This implies (b). □

Let $V_+ \subset H_\ast(\Sigma; \mathbb{Q})$ be the subspace generated by cycles of the form $(\pi_+ x)^0$, where $x \in M^\ast$ and $(dx)^{\leq 0} = 0$. (Here $(\pi_+ x)^0$ denotes the constant coefficient of $\pi_+ x$, and $(dx)^{\leq 0}$ the portion of $dx$ containing nonpositive powers of $t$.) Similarly, let $V_- \subset H_\ast(\Sigma; \mathbb{Q})$ be the subspace generated by cycles of the form $(\pi_- y)^0$, where $y \in M^\ast$ and $(d^* y)^{\leq 0} = 0$. 

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Lemma 3.12  (a) $\text{Ker}(i_* : H_*(\Sigma; \mathbb{Q}) \to H_*(\tilde{X}; \mathbb{Q})) = V_+ \oplus V_-.$

(b) Our assumption that $H^*(M^* \otimes L\mathbb{Q}) = 0$ implies $V_+ \cap V_- = \{0\}.$

Proof. Define

$$\tilde{X}^+ := \{(x, \lambda) \in \tilde{X} \mid \lambda \geq 0\},$$

$$\tilde{X}^- := \{(x, \lambda) \in \tilde{X} \mid \lambda \leq 0\}.$$

The relative homology exact sequence

$$H_{k+1}(\tilde{X}^-, \Sigma) \xrightarrow{\delta} H_k(\Sigma) \to H_k(\tilde{X}^-).$$

and Proposition 2.3 imply that the kernel of the map $H_*(\Sigma) \to H_*(\tilde{X}^-)$ is $V_+.$ (Here we are identifying $\Sigma$ with $\Sigma \times \{0\} \subset \tilde{X},$ and all homology is with rational coefficients.) Also $H_*(\tilde{X}, \tilde{X}^-) \cong H_*(\tilde{X}^+, \Sigma)$ by excision, and the connecting homomorphism $\delta$ in the exact sequence

$$H_{k+1}(\tilde{X}, \tilde{X}^-) \xrightarrow{\delta} H_k(\tilde{X}^-) \to H_k(\tilde{X}).$$

sends this to $V_-.$ This proves (a).

To prove (b), suppose $u \in V_+ \cap V_-.$ Write $u = (\pi_+ x)^0 = (\pi_- y)^0.$ Let $v \in H_{k+1}(\tilde{X})$ be the cycle obtained by gluing together the upward gradient flow of $x$ (up to $\Sigma$) and the downward gradient flow of $y$ (down to $\Sigma$). Note that $u$ is the image of $v$ under the connecting homomorphism $\delta$ in the Mayer-Vietoris sequence

$$H_{k+1}(\tilde{X}^-, \Sigma) \oplus H_{k+1}(\tilde{X}^+) \to H_{k+1}(\tilde{X}) \xrightarrow{\delta} H_k(\Sigma).$$

By Corollary 2.6, $v$ is in the image of $i_* : H_{k+1}(\Sigma) \to H_{k+1}(\tilde{X}).$ But $\delta i_* = 0,$ so $u = 0.$ $\square$

We will now compute $B|\text{Ker}(i : H_*(\Sigma; \mathbb{Q}) \to H_*(\tilde{X}; \mathbb{Q})).$ Let $R$ be an operator that sends $t^n$ to $t^{-n}.$

Lemma 3.13  (a) If $(dx)^{\leq 0} = 0$ then $B((\pi_+ x)^0) = -(\pi_+ x)^{\leq 0}.$

(b) If $(d^* y)^{\leq 0} = 0$ then $B((\pi_- y)^0) = R((\pi_- y)^{< 0}).$

Proof. We might as well assume that $x^{> 0} = 0.$ Then

$$f^+((\pi_+ x)^0) = (\pi_+ x)^{\geq 0}.$$
From the definitions of $\xi$ and $d$, we have
\[ \xi((\pi x)^0) = t^{-1}(dx)^0 = t^{-1}dx. \]

Then
\[ t\pi d^* \Delta^{-1} \xi((\pi x)^0) = \pi x - \pi dd^* \Delta^{-1} x \]
\[ = \pi x + \text{(nullhomologous cycle)} \]
by Lemma 3.5. Putting this into the definition of $B$ proves (a).

To prove (b), let $\gamma$ be a perturbation of $(\pi - y)^0$. (We need to do this because $(\pi - y)^0$ does not intersect the descending manifolds of the critical points in $y$ transversely, so $(\pi - y)^0$ is not a generic chain on which $f^+$ is defined.) We create $\gamma$ by replacing $(\pi - y)^0$ with the intersection of $\Sigma \times \{0\}$ and the descending flow of a sum of small spheres linking the ascending manifolds of the critical points in $y$ at generic points. For any positive integer $k$, we can choose these spheres to be small enough that
\[ f^+(\gamma) = R((\pi - y)^{\leq 0}) + \text{(nullhomologous cycle)} + O(t^k). \]

Similarly
\[ \xi(\gamma) = \pm R((d^*y)^{\leq 0}) + O(t^k) \]
\[ = O(t^k). \]

We can then complete the proof as in part (a).

\[ \square \]

**Lemma 3.14** \( \text{Tr}(B|\text{Ker}(\iota_*)) \in \mathbb{Z}. \)

**Proof.** By Lemma 3.12,
\[ \text{Tr}(B|\text{Ker}(\iota_*)) = \text{Tr}(B|V_+) + \text{Tr}(B|V_-). \]

Now here is the cheap trick. Since
\[ (\Gamma - Z) \cdot (\text{diag} - \sum_i e_i \times e_i^*) = 0, \]
the combination of Lemmas 3.1, Lemma 3.2, and Corollary 2.6 implies that
\[ \text{Tr}(B|\text{Ker}(\iota_*)) = \Gamma \cdot \text{diag} - (-1)^n t \frac{d}{dt} \log \tau(M) - \text{Tr}(tA(1 - tA)^{-1}). \]

In particular, we see that \( \text{Tr}(B|\text{Ker}(\iota_*)) \) is a Taylor series. For \( \Gamma \cdot \text{diag} \) and \( \text{Tr}(tA(1 - tA)^{-1}) \) contain only positive powers of \( t \) by definition, and it is easy to see that \( t \frac{d}{dt} \log \) of a Laurent series is a Taylor series.

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It follows from Lemma 3.13 that Tr(B|V_+) ∈ Z, since all the negative degree terms must vanish. We also see from Lemma 3.13 that the coefficients of Tr(B|V_-) are exactly minus what the nonconstant coefficients of Tr(B|V_+) would be if we inverted the Morse function (with appropriate new orientations on the ascending and descending manifolds). So Tr(B|V_-) = 0. □

This completes the proof of Theorems 1.1 and 1.2 in the case ∂X = ∅, as explained in §3.1.

3.7 Extension to manifolds with boundary

We will now deduce Theorem 1.2 for manifolds with boundary. Let L(X, φ) denote the left hand side of Theorem 1.2.

Lemma 3.15 Let −X be X with the opposite orientation. Then

\[ L(X, \phi) = L(-X, \phi), \]
\[ \tau(X, \phi) = \tau(-X, \phi), \]

modulo the sign ambiguity in L and the ±t^k ambiguity in τ.

Proof. Suppose we change the orientation of X. The zeta function is not affected. We can switch the orientation of the descending manifold of each critical point to restore the condition that the descending and ascending manifolds of a critical point have intersection number +1 at the critical point. After we do this, the differential in the Morse complex is exactly the same as it was before, so τ(M) is not affected. The complex out of which τ(X, φ) is defined is not changed either. □

Lemma 3.16 Let X be closed, and let Y ⊂ X be a hypersurface separating X into X_1 and X_2. Let φ : X → S^1 be a Morse function such that grad(φ) is parallel to Y and nonzero on Y. Assume that the rational Morse complexes for X_1 and X_2 are acyclic. Then the rational Morse complex for X is acyclic, and

(a) \[ L(X, \phi) = \frac{L(X_1, \phi)L(X_2, \phi)}{L(Y, \phi)}, \]
(b) \[ \tau(X, \phi) = \frac{\tau(X_1, \phi)\tau(X_2, \phi)}{\tau(Y, \phi)}. \]

Proof. We have

\[ M^*(X) = M^*(X_1) ⊕ M^*(X_2), \]
where $M^*(X)$ is the Morse complex for $X$, since our assumption on $\phi$ assures that gradient flow lines can never cross $Y$, and there are no critical points on $Y$. We then trivially obtain
\[
\tau(M(X)) = \tau(M(X_1))\tau(M(X_2)).
\]
Since $\tau(M(Y)) = 1$, we can rewrite this as
\[
\tau(M(X)) = \frac{\tau(M(X_1))\tau(M(X_2))}{\tau(M(Y))}.
\]
Also every closed orbit lies in either $X_1$ or $X_2$, so we have
\[
\zeta(X) = \frac{\zeta(X_1)\zeta(X_2)}{\zeta(Y)}.
\]
(We have to divide by $\zeta(Y)$ to avoid counting the closed orbits in $Y$ twice.) This proves (a).

Next, observe that
\[
\tilde{X} = \tilde{X}_1 \bigcup_{\tilde{Y}} \tilde{X}_2.
\]
Thus we have a short exact sequence of complexes of $\mathbb{Z}[t,t^{-1}]$-modules
\[
0 \to C_*(\tilde{Y}) \to C_*(\tilde{X}_1) \oplus C_*(\tilde{X}_2) \to C_*(\tilde{X}) \to 0.
\]
Now (b) follows from the product formula for torsion [10, Thm. 3.1].

Proof of Theorem 1.2 when $\partial X \neq \emptyset$. Given an oriented manifold $X$ with boundary, we can form the double $2X = X \cup_{\partial X} (-X)$. Since $\operatorname{grad}(\phi)$ is parallel to $\partial X$, we can extend $\phi$ in the obvious way to $2X$, as a $C^1$ function. Using Lemmas 3.15 and 3.16, and applying Theorem 1.2 to the closed manifolds $\partial X$ and $2X$, we have
\[
L(X,\phi)^2 = L(\partial X,\phi)L(2X,\phi)
= \tau(\partial X,\phi)\tau(2X,\phi)
= \tau(X,\phi)^2.
\]
Since Theorem 1.2 is not sensitive to signs, we are done.

4 Seiberg-Witten invariants of 3-manifolds

From now on we assume that $X$ is a closed oriented 3-manifold with $b^1 > 0$. In §4.1, we define a possible analogue of the Gromov invariant for $X$, and we conjecture that this is equal to the Seiberg-Witten invariant. In §4.2 we use Theorem 1.2 to show that this conjecture implies the Meng-Taubes formula (for closed manifolds, up to signs) relating the Seiberg-Witten invariant to Milnor torsion.
4.1 An analogue of the Gromov invariant

Let $\phi : X \to S^1$ be a generic Morse function, and let $\Sigma = \phi^{-1}(0)$ as usual. Let $\eta = d\phi$. Assume that $\phi$ has no index 0 or 3 critical points. In particular this implies that the homology class of $\eta$ is nontrivial.

Let $X' = X \setminus \text{Crit}$. Let $H(\eta)$ denote the set of $\alpha \in H_1(X',\partial X')$ whose boundary is the sum of the index 2 critical points minus the sum of the index 1 critical points. The Gromov invariant counts unions of flow lines and closed orbits of the gradient flow of $\phi$ whose homology class is in $H(\eta)$. We then need some way of identifying $H(\eta)$ with the set $S$ of $\text{Spin}^c$-structures.

Let us take care of this last point first. Given a $\text{Spin}^c$ structure $S$ on $X$, let $E \subset S|_{X'}$ be the $-i$ eigenspace of Clifford multiplication by $\eta/|\eta|$. 

**Lemma 4.1** The map that sends a $\text{Spin}^c$ structure $S$ to the Poincaré-Lefschetz dual of $c_1(E)$ in $H_1(X',\partial X')$ defines an isomorphism of $H^2(X;\mathbb{Z})$-torsors 

$$\Psi_\eta : S \to H(\eta).$$

**Proof.** Given $S$, the Poincaré dual of $c_1(E)$ lives in $H(\eta)$ because $S$ can be trivialized in a neighborhood of a critical point, but on a sphere around the critical point, $\eta/|\eta|$ defines a map $S^2 \to S^2$ of degree $\pm 1$, so $E|S^2$ is the Hopf line bundle or its inverse.

Given $E$, we can recover $S$ as follows. Let $K^{-1}$ denote the kernel of $\eta : TX' \to \mathbb{R}$. This inherits a complex structure from the metric and orientation on $X$. Define 

$$S := E \oplus (K^{-1} \otimes E).$$

The Clifford action is as follows. If $v \in TX$ is dual to $\eta$, then 

$$c(v) := |v| \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$ 

If $v \in TX$ is annihilated by $\eta$, i.e. $v \in K^{-1}$, then for $e \in E$,

$$c(v)e := v \otimes e, \quad c(v)(v \otimes e) := -|v|^2 e.$$ 

Every $\text{Spin}^c$ structure $S$ must be of this form, because if $E_{\pm}$ is the $\pm i$ eigenspace of Clifford multiplication by $\eta/|\eta|$, then Clifford multiplication by $K^{-1}$ defines an isomorphism 

$$K^{-1} = \text{Hom}(E_-, E_+),$$

so $E_+ = K^{-1} \otimes E_-$, etc.
The $H^2(X;\mathbb{Z})$-torsor structure on $H(\eta)$ is as follows: any two elements of $H(\eta)$ differ by an element of $H_1(X',\partial X')$ which is annihilated by $\delta : H_1(X',\partial X') \to H_0(\partial X')$, and hence extends to an element of $H_1(X) = H^2(X;\mathbb{Z})$. This clearly corresponds to the $H^2(X;\mathbb{Z})$ action on $\mathcal{S}$. \hfill $\Box$

Now choose orientations on the ascending and descending manifolds of the critical points, and choose orientations on the vector spaces $\mathbb{Q}^{\text{Crit}^1}$ and $\mathbb{Q}^{\text{Crit}^2}$. Note that $\text{Crit}^1$ and $\text{Crit}^2$ have the same cardinality, because $\chi(X) = 0$.

Let $\Lambda$ be the Novikov ring of $H_1(X',\partial X')$ with respect to intersection with $\Sigma$. This is the ring of functions $\lambda : H_1(X',\partial X') \to \mathbb{Z}$ such that for any $k \in \mathbb{Z}$, the set
\[
\{ \alpha \in H_1(X',\partial X') \mid \lambda(\alpha) \neq 0, \alpha \cdot \Sigma < k \}
\]
is finite. The multiplication is given by the convolution product
\[
(\lambda_1 \lambda_2)(\alpha) := \sum_{\beta} \lambda_1(\beta) \lambda_2(\alpha - \beta).
\]
This is a generalization of the ring of Laurent series. (See [5] for more discussion.) We will write elements of $\Lambda$ like Laurent series, in the form $\sum_i f(\alpha_i) \alpha_i$.

Define a map
\[
P : \mathbb{Q}^{\text{Crit}^1} \to \mathbb{Q}^{\text{Crit}^2} \otimes \Lambda
\]
as follows. If $x \in \text{Crit}^1$ and $y \in \text{Crit}^2$, let $\mathcal{P}(x,y)$ be the set of flow lines from $x$ to $y$ (of the flow dual to $\eta$), with the orientation induced by $\eta$. Given $\gamma \in \mathcal{P}(x,y)$, let $\epsilon(\gamma)$ be the sign of the intersection of the descending manifold of $y$ and the ascending manifold of $x$ (in a local slice orthogonal to $\eta$). For $x \in \text{Crit}^1$, define
\[
P(x) := \sum_{y \in \text{Crit}^2} y \otimes \sum_{\gamma \in \mathcal{P}(x,y)} \epsilon(\gamma)[\gamma].
\]
Here $[\gamma]$ denotes the homology class of $\gamma$.

Let $\mathcal{O}$ be the set of closed orbits of the flow dual to $\eta$. For $\gamma \in \mathcal{O}$, let $\epsilon(\gamma)$ be the sign of $\det(df - 1)$, as in the introduction. Now define
\[
I_\eta := \prod_{\gamma \in \mathcal{O}} (1 - [\gamma])^{-\epsilon(\gamma)} \det(P) \in \Lambda.
\]
(10)

By the definition of determinant, the function $I_\eta : H_1(X',\partial X') \to \mathbb{Z}$ is nonzero only on elements of $H(\eta)$. We now wish to define
\[
I := I_\eta \circ \Psi_\eta : \mathcal{S} \to \mathbb{Z}.
\]
If we change the orientation choices above, this will multiply $I$ by $\pm 1$. 29
Proposition 4.2  Modulo the above sign ambiguity, $I$ depends only on the cohomology class of $\eta$.

Proof Sketch. Given two cohomologous $\eta$'s, one can show that they are homotopic through closed forms in the same cohomology class with no index 0 or 3 critical points. If we generically deform $\eta$ and/or the metric over time, during a time interval when $\phi$ remains generic, none of the terms in the definition of $I_\eta$ change. At certain times, $\eta$ may fail to be generic in one of the following two ways:

(a) Two critical points of index 1 and 2 with a single short flow line $\gamma$ between them annihilate each other (or are created).

(b) There is a flow line $\gamma$ connecting two critical points with the same index.

In case (a), if $x$ is an index 1 point with a flow line to the index 2 point being annihilated, and if $y$ is an index 2 point with a flow line to the index 1 point being annihilated, then after the annihilation, these two flow lines fuse into a single flow line from $x$ to $y$. In other words, the path matrix $P$ changes as follows:

$$\left( \pm [\gamma] \begin{array}{c} v \\ w \end{array} P^t \right) \mapsto (P^t + wv).$$

(Here $v$ and $w$ are the row and the column associated to the index 2 point and the index 1 point, respectively.) This divides the determinant, and hence $I_\eta$, by $\pm [\gamma]$. However the isomorphisms $\Psi_\eta$ before and after the annihilation also differ by a factor of $\pm [\gamma]$. Thus $I$ is changed only by $\pm 1$.

In case (b), suppose without loss that the two critical points $x, y$ have index one. Let the flow line $\gamma$ start at $x$ and end at $y$. Suppose first that $x \neq y$. The effect of this catastrophe is to replace $P(x)$ by $P(x) \pm [\gamma]P(y)$. This does not change $I_\eta$ for the same reason that the left side of Theorem 1.2 is invariant, which was remarked upon in the introduction.

When $x = y$, $P(x)$ is multiplied by $(1 - [\gamma])^{\pm 1}$, but a closed orbit is created or destroyed to cancel this. This will not change $I_\eta$ for the same reason that the left side of Theorem 1.2 is invariant, which was remarked upon in the introduction.

Note that the set of times at which changes of type (b) occur is in general not discrete. However, for any integer $k$, the set of times at which terms $[\gamma]$ in $I_\eta$ with $|\gamma \cdot \Sigma| \leq k$ change is discrete. So if we discard terms $[\gamma]$ in $I_\eta$ with $|\gamma \cdot \Sigma| > k$, the resulting expression is invariant. Taking $k \to \infty$, it follows that $I_\eta$ is invariant. □

Conjecture 4.3 let $X$ be a closed oriented 3-manifold with $b^1(X) > 0$. Then $I$ does not depend on the choice of $\eta \in H^1(X; \mathbb{Z})$, and

$$SW = \pm I.$$
Note that when $b^1(X) = 1$, we define $SW$ to be the limit of the number of solutions to the equations perturbed by $-ir \ast \eta$, where $r$ is a large real number. (See Meng-Taubes [8].)

This conjecture is analogous to Taubes’ results relating the Seiberg-Witten and the Gromov invariants in 4 dimensions. The funny way that closed orbits are counted in $I$ is analogous to more intricate results in [19]. The idea of (part of) the proof is that if we have a nonzero Seiberg-Witten invariant and take the limit as $r \to \infty$, we get a sequence of Seiberg-Witten solutions such that the zero set of the $E$ component of the spinor converges to one of the submanifolds that $I$ counts.

4.2 The Meng-Taubes formula

Proof of Theorem 1.3. Let $\eta$ be the harmonic 1-form representing $\alpha$, and perturb it slightly so that $\eta$ is $d$ of a generic Morse function $\phi : X \to S^1$. Since we started with a harmonic form, there can be no index 0 or 3 critical points. Let 0 be a regular value of $\phi$, and let $\Sigma = \phi^{-1}(0)$.

Let $S$ be a $\text{Spin}^c$ structure. We have

$$\alpha(c_1(\det S)) = \int_{\Sigma} c_1(\det S).$$

On $\Sigma$, we have the decomposition

$$S = E \oplus K^{-1}E$$

from §4.1. Clearly $K^{-1}|\Sigma = T\Sigma$, so

$$\alpha(c_1(\det S)) = \chi(\Sigma) + 2\Sigma \cdot \Psi_\eta(S).$$

By Conjecture 4.3,

$$\sum_{S \in \mathcal{S}} SW(S) t^{\Sigma \cdot \Psi_\eta(S)} = \pm \rho(I_\eta),$$

where $\rho : \Lambda \to L_\mathbb{Z}$ sends $\gamma \mapsto t^{\Sigma \cdot \gamma}$. Thus

$$\sum_{S \in \mathcal{S}} SW(S) t^{\alpha(c_1(\det S))/2} = \pm t^{\chi(\Sigma)/2} \rho(I_\eta).$$

To compute $\rho(I_\eta)$, observe that in the notation of §1,

$$\rho \left( \prod_{\gamma \in \mathcal{O}} (1 - [\gamma])^{-\tau(\gamma)} \right) = \zeta(f),$$

$$\rho(\det(P)) = \det(d : M^1 \otimes L_\mathbb{Q} \to M^2 \otimes L_\mathbb{Q}).$$
Since \( M^1 \) and \( M^2 \) are the only nontrivial terms in the Morse complex, \( \det(d : M^1 \otimes L_Q \to M^2 \otimes L_Q) \) is \( \tau(M) \) if \( M^* \otimes L_Q \) is acyclic, and zero otherwise. So by \( (10) \), \( \rho(I_\eta) \) equals the left side of Theorem 1.2 when \( M^* \otimes L_Q \) is acyclic, and zero otherwise. We are done by Theorem 1.2.

**Milnor torsion.** Let \( H = H_1(X) / \text{Torsion} = H^2(X; \mathbb{Z}) / \text{Torsion} \), and let \( \hat{X} \) be the covering of \( X \) whose monodromy is the projection \( \pi_1(X) \to H \). The “Milnor torsion” \( MT \) of \( X \) is the torsion of the complex \( C_*(\hat{X}) \otimes Q(\mathbb{Z}[H]) \), where \( C_*(\hat{X}) \) is the cellular complex coming from an equivariant cell decomposition, and \( \mathbb{Z}[H] \) is the group ring of \( H \) (where \( H \) is written multiplicatively). This is a well defined element of \( Q(\mathbb{Z}[H])/H \), up to sign. In fact the sign can be specified, by the same data needed to specify the sign of \( SW \) (see Meng-Taubes \[8\], Turaev \[22\]). The Milnor torsion is defined to be zero if the complex is not acyclic.

When \( b_1(X) > 1 \), it turns out that \( MT \in \mathbb{Z}[H]/H \). (See Turaev \[22, Thm. 1.1.2\].) Furthermore there is a unique element in this equivalence class invariant under the map that sends \( h \to h^{-1} \) for \( h \in H \) \[22, §1.11.5\]. When \( b_1(X) > 1 \) we will identify \( MT \) with this element of \( \mathbb{Z}[H] \).

Following Meng-Taubes \[8\], define

\[
SW := \sum_{S \in S} SW(S) \frac{c_1(\det S)}{2} \in \mathbb{Z}[[H]].
\]

Here \( \mathbb{Z}[[H]] \) is the set of functions \( H \to \mathbb{Z} \) that do not necessarily have finite support. We can now deduce part of the Meng-Taubes formula:

**Theorem 4.4 (assuming Conjecture 4.3)** Let \( X \) be a closed oriented 3-manifold with \( b_1(X) > 0 \). Then

\[
SW = \pm MT.
\]

**Lemma 4.5** Let \( G \) be a free abelian group on \( m \) generators and let \( f, g \in \mathbb{Z}[G] \). Suppose that for every homomorphism \( \alpha : G \to \mathbb{Z} \), \( \alpha(f) = \alpha(g) \) in \( \mathbb{Z}[\mathbb{Z}] \), up to sign. Then \( f = \pm g \).

**Proof.** Let \( \{e_i\} \) be a free basis for \( G \). Choose an integer \( N \) such that \( f \) and \( g \) are supported in the set \( \{ \sum a_i e_i \mid |a_i| < N \} \). Let \( \alpha \) send \( e_i \) to \( (2N)^i \). Then for any integer \( k \), the hyperplane \( \{ x \in G \mid \alpha(x) = k \} \) contains at most one point of union of the supports of \( f \) and \( g \). Apply the hypothesis to this \( \alpha \). \( \square \)

**Proof of Theorem 4.4.** If \( b_1(X) = 1 \) then this is just Theorem 1.3. Assume \( b_1(X) > 1 \). We have already remarked that \( MT \in \mathbb{Z}[H] \). We also have
SW ∈ Z[H], by the well known a priori bounds on the for the Seiberg-Witten equations (see Witten [24]). (Note that we do not necessarily have SW ∈ Z[H] when b¹ = 1, because here we are making a large perturbation to the equations which destroys the a priori bounds, and the Seiberg-Witten invariants are not invariant under perturbation when b¹(X) = 1.)

If α ∈ H¹(X;Z) then α extends to a function Z[H] → Z[Z] = Z[t, t⁻¹], and the left hand side of Theorem 1.3 is α(SW), modulo signs. On the other hand the right side of Theorem 1.3 is α(MT). (This is easy when both the complexes involved are acyclic, and the general case follows from Turaev [22, Thm. 1.1.3] and Corollary 2.5.) So Theorem 1.3 says that α(SW) = α(MT), modulo signs and powers of t. Since both SW and MT are symmetric, α(SW) = α(MT) modulo signs. We are done by Lemma 4.5.

Final remark. Under favorable circumstances one can define the torsion of larger (i.e. not free) abelian coverings of X. (See e.g. Fried [2].) If X is a fibration over S¹ (of any dimension), then Fried [2] shows that the torsion of the universal abelian cover can be identified with the zeta function, which is our I in the 3-dimensional case. This is slightly stronger than our result, since a 3-manifold fibered over S¹ may have torsion in H¹. We can use an equivariant version of circle-valued Morse theory to extract more information about the Seiberg-Witten invariants for other 3-manifolds, and we intend to discuss this in a future paper.

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