Self-organized critical and synchronized states in a nonequilibrium percolation model

Siegfried Clar¹, Barbara Drossel², and Franz Schwabl¹

¹Institut für Theoretische Physik,
Physik-Department der Technischen Universität München,
James-Franck-Str., D-85747 Garching, Germany

²Massachusetts Institute of Technology,
Physics Department 12-104,
Cambridge, MA 02139, USA

(December 31, 2021)

Abstract

We introduce a nonequilibrium percolation model which shows a self-organized critical (SOC) state and several periodic states. In the SOC state, the correlation length diverges slower than the system size, and the corresponding exponent depends non universally on the parameter of the model. The periodic states contain an infinite cluster covering only part of the system. PACS numbers: 05.40.+j, 05.70.Jk, 05.70.Ln
During the past years, systems which exhibit self-organized criticality (SOC) have attracted much attention, since they might explain part of the abundance of $1/f$-noise and fractal structures in nature [1]. Their common features are slow driving or energy input (e.g. dropping of sand grains, increase of strain, growing of trees) and rare dissipation events which are instantaneous on the time scale of driving (e.g. avalanches, earthquakes, fires). In the stationary state, the size distribution of dissipation events obeys a power law, irrespective of initial conditions and without the need to fine-tune parameters (see [2] for examples). In general, essentially three different types of avalanche size distributions are possible in systems with slow energy input and instantaneous relaxation events: Either relaxation events occur quite frequently but release only little energy (localized avalanches), or they occur very seldom and release a finite portion of the systems energy (avalanches which cover the whole system), or the system is at the borderline between both regimes and exhibits a power-law size distribution of avalanches. In SOC systems, some mechanism pushes the system naturally to the critical point, e.g. the local conservation of sand in the sandpile model, or separation of time scales in the forest-fire model. A model which has the two noncritical regimes, separated by a critical point, has been introduced e.g. in [3].

In this paper, we present a model that belongs to the mentioned class of systems, but has surprising features not known so far: The region of small avalanches and the region of infinite avalanches are separated by a finite region of critical behavior, where the correlation length diverges slower than the system size. Since criticality is not restricted to a single point, this model can be counted among the SOC systems. In the region of infinite avalanches, the system shows synchronization with a period that depends on the value of the control parameter.

Our model can be obtained from the SOC forest-fire model [4] by taking the tree density $\rho$ as the control parameter instead of the ratio between lightning probability and tree growth probability $f/p$. It is defined as follows: Initially, the fraction $\rho$ of all sites in a square lattice with $L^2$ sites are randomly chosen to be occupied. We call an occupied site “tree”. The remaining sites are empty. Then we repeat the following rules: (i) Lightning strikes
an arbitrary site in the system. If the site is occupied, the whole cluster of $s$ trees which is connected to this site (by nearest-neighbor coupling) burns down, i.e. the trees of that cluster turn to empty sites. (ii) We grow $s$ new trees at randomly chosen empty sites.

With these rules, the tree density $\rho$ is a globally conserved quantity. The model therefore does not apply to any kind of burning or reaction process. It would be more realistic to speak of “explosions” instead of fires: At a randomly chosen site, an explosion takes place, and the whole cluster connected to the explosion site is blown into the air and settles down somewhere else in the system. Or one might think of colonies of animals which are dispersed into all directions by some enemy or other event. Or, from a purely mathematical point of view, one has a nonequilibrium percolation problem. But since the model originally was derived from a forest-fire model, we continue to describe it in terms of trees and fires.

In the following, we discuss the properties of the stationary state as function of the tree density. Let $\bar{s}$ be the average number of trees destroyed by a lightning stroke. For small tree densities, there exist only small forest clusters, and consequently only few trees are destroyed by a lightning stroke, i.e. $\bar{s}$ is small. With increasing tree density, the size of the largest forest cluster increases, and $\bar{s}$ increases, too. There exists a critical density $\rho^1_c \approx 0.41$, where the size of the largest cluster and $\bar{s}$ both diverge. The critical behavior close to $\rho^1_c$ can be described by exponents which are defined as in percolation theory [5]. The size distribution of forest clusters is

$$n(s) \propto s^{-\tau}C(s/s_{\text{max}}), \quad s_{\text{max}} \propto (\rho^1_c - \rho)^{-1/\sigma}$$

with a cutoff function $C$. The fractal dimension of the clusters is defined via

$$R(s) \propto s^{1/\mu},$$

where $R$ is the radius of gyration of a cluster. The correlation length is given by $\xi \propto (\rho^1_c - \rho)^{-\nu}$. The quantity $\bar{s}$ can be considered as susceptibility and diverges as $\bar{s} \propto (\rho^1_c - \rho)^{-\gamma}$. The exponents are related via the scaling relations $1/\sigma = \gamma/(3-\tau) = \mu\nu$. In our simulations, we found $\tau = 2.15(3), \mu = 1.96(2), \nu = 1.20(5)$, and $\gamma = 2.09(5)$. 
These exponents are identical to those of the SOC forest-fire model for \( f/p = (1 - \rho)/\bar{s}\rho \), if they are appropriately redefined (see [6–9]). In a system which is much larger than the correlation length, the difference between a globally conserved tree density and a tree density which is only conserved on an average cannot be seen on length scales comparable to the correlation length.

In the region \( \rho > \rho^1_c \), which was inaccessible in the SOC forest–fire model, one might naively expect an infinite forest cluster which spans the whole system, as in percolation theory. However, the values of the critical exponents for \( \rho \lesssim \rho^1_c \) show that this cannot be the case here. The hyperscaling relation \( d = \mu(\tau - 1) \) is violated [6–9], which means that not every part of the system contains a spanning cluster [6]. This can be seen in Fig. 1(a), where in addition to large forest clusters also large clusters of empty sites exist. In contrast to ordinary percolation, there is no homogeneously distributed set of large clusters that could join at \( \rho = \rho^1_c \) to form the infinite cluster.

Nevertheless, the largest forest cluster has to be infinitely large for \( \rho > \rho^1_c \). For \( \rho^1_c < \rho < \rho^2_c \approx 0.435 \), we observe the following behavior: The size of the largest cluster \( s_{\text{max}} \) diverges, but slower than \( L^2 \), and the correlation length diverges slower than \( L \). We find \( s_{\text{max}} \propto L^{\phi_1}, \xi \propto L^{\phi_2}, \) and \( \bar{s} \propto L^{\phi_3} \) with \( \rho \)-dependent exponents \( \phi_{1,2,3} \), while \( \tau \) and \( \mu \) remain unchanged. Tab. 1 shows the values of the exponents for different densities. Fig. 1(b) shows a snapshot of the system for \( \rho = 0.43 \), and Fig. 2 shows the size distribution of fires \( s_n(s) \) for different system sizes at fixed \( \rho \) (a) before and (b) after rescaling. With Eqs. (1)–(2) one can derive the scaling relations \( \phi_1 = \mu \phi_2 \) and \( \phi_3 = (3 - \tau)\phi_1 \), which are confirmed by the simulations. For \( \rho \to \rho^2_c \), \( \phi_{1,2,3} \) approach the values \( \mu, 1, \) and \( (3 - \tau)\mu \).

At \( \rho = \rho^2_c \), the correlation length becomes proportional to the system size, and the critical state becomes unstable. We observe a discontinuous phase transition to a state where the largest cluster contains a finite portion of all trees in the system. Fig. 1(c) shows a snapshot of the system for \( \rho = 0.45 \). The system has five homogeneous and equally large subphases with different densities. The subphase with highest density contains an infinite cluster. When lightning strikes one of the small clusters in this state, only few trees burn
down, and the state of the system essentially remains unchanged. When lightning strikes
the infinite cluster, a large portion of all trees in the subphase with the highest density burn
down and are regrown at empty sites all over the system. The subphase which used to have
the highest density now has the lowest density, while the density of the other subphases has
increased. The values of the five densities are the same as before, except that they are now
associated with different subphases. If we measure time in units of large fires, the state of
the system is periodic with a period 5. Increasing $\rho$ further, one finds four, three (Fig. 1(d)),
two (Fig. 1(e)), and finally one (sub)phase (Fig. 1(f)), where the infinite cluster spans the
whole system. The shape of the subphases depends on the boundary conditions.

To understand the occurrence of subphases with different densities, we consider first a
system with a density far above the percolation threshold 0.59 for random site percolation
[5], and we start with a random initial state. In the first iteration step, lightning strikes
either the infinite cluster, which consists of nearly all trees in the system, or it strikes one of
the finite clusters, which are very small. In the first case, only a few small clusters survive
the fire, and most of the trees are redistributed randomly in the system. In the second case,
only a small number of trees are redistributed. In both cases, however, the state of the
system changes little and the new state is close to a completely random state. This remains
true even after many iterations, since the clusters which survived the first large fire will burn
down during one of the following large fires, so that correlations cannot increase with time.

If we decrease $\rho$, the clusters which survive the first large fire become larger. When the
burnt trees are regrown, part of them grow in or near these surviving clusters, where the
density then will be larger than the mean density. In the space between these clusters the
density consequently is lower than the mean density. If $\rho$ is below a threshold $\rho^*$, this density
between the surviving clusters becomes smaller than the percolation threshold 0.59. Then
there exists no infinite cluster in the system after the first time step, and the state with one
phase becomes unstable. Evidently $\rho^* > 0.59$. In our simulations, we find $\rho^* \simeq 0.625$.

For $\rho < \rho^*$, a stationary state with two subphases occurs, the density of one subphase
being above, the density of the other subphase being far below the threshold $\rho^*$. Decreasing
Further, eventually the density of the high-density subphase drops below $\rho^*$, and a state with three subphases is formed, and so on.

Let $\rho_1, \ldots, \rho_n$ be the densities in a state with $n$ subphases, starting with the highest density. Additionally we define the density $\rho_{n+1}$ of the subphase which contained the infinite cluster, immediately after the infinite cluster has been removed from the system, and before the removed trees are regrown. As consequence of a large fire, the different subphases just exchange their densities, i.e.

$$
\rho_{i-1} = \rho_i + (\rho_1 - \rho_{n+1}) \cdot \frac{(1 - \rho_i)}{(n (1 - \rho) + \rho_1 - \rho_{n+1})}
$$

for $i = 2, \ldots, n + 1$. The last factor on the r.h.s. represents the fraction of trees of the infinite cluster that are regrown in the subphase with density $\rho_i$. We finally obtain

$$
\frac{1 - \rho_1}{1 - \rho_2} = \frac{1 - \rho_2}{1 - \rho_3} = \ldots = \frac{1 - \rho_n}{1 - \rho_{n+1}}.
$$

(3)

Together with $\rho = \frac{1}{n} \sum_{i=1}^{n} \rho_i$ we have $n$ equations for $n + 1$ densities.

In our simulations, we observed a maximum number of $n = 5$ subphases. In the following, we argue that there exists an upper limit for the number of subphases even in an infinitely large system: A state with $n$ subphases is only stable if the subphase with the second highest density has no infinite cluster. In the limit $n \to \infty$, we would have $\rho_1 - \rho_2 \to 0$ and $\rho_1 \to \rho^*$. In such a state, the density of a given subphase would increase continuously in time, until it reaches $\rho^*$. Then the infinite cluster in this subphase would be destroyed, and the whole cycle would restart. We simulated this dynamics, and measured the percentage of trees in the largest cluster as function of the density (which itself is a function of time). This percentage became finite at $\rho \simeq 0.595 < \rho^*$, indicating that subphases with densities above 0.595 contain an infinite cluster. Consequently states with $\rho_2 > 0.595$ cannot exist, which gives with Eq. (3) a maximum possible number of $n = 11(\pm 2)$ subphases, and a corresponding minimum mean density $\rho = 42(\pm 0.3)\%$.

Since the transition between states with different number of subphases is discontinuous, there are hysteresis effects. When the density $\rho$ is decreased, a state becomes unstable
when $\rho_1$ falls below $\rho^* \simeq 0.625$, and its number of subphases increases by one. When $\rho$ is increased, however, the subphase with the highest density never becomes unstable, and the transition from an $(n+1)$-subphase state to an $n$-subphase state takes place when the subphase with second highest density $\rho_2$ starts to contain an infinite cluster, i.e. when $\rho_2$ approaches $\simeq 0.595$.

To conclude, we have described a nonequilibrium percolation model which shows several new phenomena which are unknown in equilibrium percolation. Besides the earthquake model [10], this is the first model with a SOC state where the correlation length diverges slower than the system size and the corresponding exponent depends continuously on the parameter. In contrast to the earthquake model, the exponent which characterizes the size distribution of avalanches remains constant. In addition to the SOC state, the system shows synchronization leading to periodic states with 1 to 5 subphases.

This work was supported by the Deutsche Forschungsgemeinschaft (DFG) under Contracts No Dr 300/1-1 and No Schw 348/7-1.
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FIGURES

FIG. 1. Stationary states for different densities and boundary conditions (trees are black, empty sites are white): (a) $\rho = 0.41$, $L = 4096$, periodic b.c. (b) $\rho = 0.43$, $L = 4096$, p.b.c. (c) $\rho = 0.45$, $L = 4096$, p.b.c. (d) $\rho = 0.50$, $L = 2048$, absorbing b.c. (e) $\rho = 0.55$, $L = 1024$, a.b.c. (f) $\rho = 0.63$, $L = 1024$, a.b.c.

FIG. 2. Normalized size distribution of fires for $\rho = 0.43$ and $L = 512, 1024, 2048, 4096$, (a) before and (b) after rescaling.
| $\rho$ | 0.41   | 0.42   | 0.43   | 0.435  |
|-------|--------|--------|--------|--------|
| $\phi_1$ | 1.46(8) | 1.72(6) | 1.86(8) | 1.92(8) |
| $\phi_2$ | 0.79(2) | 0.92(3) | 0.97(3) | 0.99(2) |
| $\phi_3$ | 1.23(3) | 1.50(3) | 1.62(3) | 1.69(5) |

**TABLE I.** The exponents $\phi_{1,2,3}$ of $s_{\text{max}}$, $\xi$, and $\bar{s}$ for various densities $\rho_c^1 < \rho < \rho_c^2$ ($L = 512, 1024, 2048, 4096$)
