Wavefunction collapse induced by gravity in a relativistic Schrödinger-Newton model

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Abstract

A relativistic version of the Schrödinger-Newton equation is analyzed within the recently proposed Grave de Peralta approach [L. Grave de Peralta, Results Phys. 18 (2020) 103318], which include relativistic effects by a parametrization of the non-relativistic hamiltonian, so as to impose that the average kinetic energy of the system coincide with its relativistic kinetic energy. The reliability of this method is tested for the particle in a box. By applying this method to the Schrödinger-Newton equation we shows that the characteristic length of the model [L. Diósi, Phys. Lett. 105A (1984) 199] goes to zero for a mass of the order of the Planck mass, suggesting a collapse of the wavefuncton, induced by gravity.

1 Introduction

The Schrödinger-Newton equation (SNE) [1][2][3][4][5][6] is a model which describe the time evolution of a Schrödinger quantum field coupled to a Newtonian gravitational field, aimed to elucidate the role of gravity on quantum state reduction [7][8][9]. For a
single-point particle the equation is written as

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 - m\Phi(x, t) \right] \Psi(x, t), \quad (1)$$

where $\Phi(x, t)$ is the gravitational potential determined by the mass distribution $m|\Psi(x, t)|^2$.

Eqs. (1) involves three parameters: $\hbar$, the action quantum constant, $G$, the gravitational constant, and $m$ a free parameter that couples both fields. Therefore, the solutions of the SNE are length scaled by

$$l_D = \frac{\hbar^2}{Gm^3}, \quad (2)$$

here called the Diósi length, as it was initially proposed by Diósi [1], as a measure of the quantum uncertainty of the stationary soliton-like solutions of the SNE. As observed by Diósi [1], for a distance of the order of $l_D$ the gravitational force balances the dispersion of the quantum field $\Psi$ by the action of the $p^2/2m$ operator, leading to a stationary state. The $l_D \sim m^{-3}$ dependence point out to the suppression of the macroscopic quantum behavior due to gravitational collapse for a enough massive particle.

Note that when $m$ is equal to the Planck mass ($m_P = \sqrt{\hbar c/G}$), $l_D$ match the Planck distance ($l_P = \sqrt{\hbar G/c^3}$), and the reduced Compton wavelength ($\lambda_C = \hbar/mc$) of the particle. Considering that $\lambda_C$ is the cut-off for a relativistic quantum field description and that $l_P$ is a fundamental distance, below which gravity is no longer a classical field, the SNE should break down when the mass approach the Planck mass.

As suggested elsewhere [6], the Eqs. (1) is the Newtonian limit of the semi-classical Einstein equations [10, 11]. Hence, a relativistic treatment require to deal with the Einstein field equations, couple to a Dirac or Klein-Gordon field. Rather, in the present work we use the Grave de Peralta approach [12, 13, 14, 15, 16] which include relativistic effects to the non-relativistic hamiltonian by introducing a parameter which value is fixed by the condition that the average kinetic energy of the system should coincide with its relativistic kinetic energy.

The paper is organized as follow. In section 2 the Grave de Peralta approach to relativity is briefly summarized; then, the method is apply to well known quantum problem of a particle an infinite well in section 3 after that, a relativistic version of the Schrödinger-Newton equation is given and the relativistic Diósi length is derived in section 4 finally, some conclusions are given in section 5.

# 2 The Grave de Peralta equation

Consider the hamiltonian

$$\hat{H} = \hat{K}^{\text{(GP)}} + \hat{V}, \quad (3)$$

where

$$\hat{K}^{\text{(GP)}} = -\frac{\hbar^2}{(1 + \gamma) m} \nabla^2, \quad (4)$$
is the Grave de Peralta (GP) kinetic energy operator \[12, 13, 14, 15, 16\] obtained by
first quantization \( p \to \hat{p} \equiv \hbar \nabla \) of the relativistic kinetic energy \[17\]
\[
\frac{p^2}{(1 + \gamma)m}
\] (5)
where \( \gamma = \left(1 + \frac{v^2}{c^2}\right)^{-1/2} \) is the Lorentz’s factor.

Note that
\[
\hat{K}^{(GP)} = \frac{2}{(1 + \gamma)} \hat{K}^{(S)},
\] (6)
where \( \hat{K}^{(S)} \) is the well known Schrödinger (S) kinetic energy operator, which result
from the quantization of the non-relativistic version of Eq. (5).

In Eqs. (4) and (6), \( \gamma \) is not an operator, instead is a parameter which value is
chosen by the condition
\[
\langle \hat{K}^{(GP)} \rangle = (\gamma - 1)mc^2
\] (7)
where \( \langle \cdot \rangle \) is the average in an appropriated quantum state.

Substituting Eq. (6) in (7) and solving for the parameter \( \gamma \) we obtain,
\[
\gamma = \sqrt{1 + \frac{2\langle \hat{K}^{(S)} \rangle}{mc^2}}
\] (8)
and note that in general, \( \gamma \) has a coordinate dependence.

The Grave de Peralta equation (GPE) is the Schrödinger-like equation,
\[
\frac{i\hbar}{\partial t} \psi(x, t) = \left[ -\frac{\hbar^2}{(1 + \gamma)m} \nabla^2 + V(x) \right] \psi(x, t)
\] (9)
with \( \gamma \) given by Eq. (8). The corresponding stationary solutions fulfills the eigenvalue
equation
\[
\left[ -\frac{\hbar^2}{(1 + \gamma)m} \nabla^2 + V(x) \right] \psi(x) = E\psi(x),
\] (10)

In the next section we show that the GdPE include relativistic corrections in a re-
liable way, for the elementary system of a particle in a one-dimensional box of size \( L \).[13]

3  Relativistic particle in a box

In this case, \( V(x) = 0 \) for \( 0 \leq x \leq L \), and the non-relativistic energy eigenvalues are
given by \[18\],
\[
E_n^{(S)} = \frac{\hbar^2 n^2 \pi^2}{2mL^2}
\] (11)
Considering that \( \langle \hat{K}^{(S)} \rangle = \langle \hat{K}^{(S)} \rangle \) inside the box, and taking the average in a Schrödinger eigen-
state, Eq. (8) adopt the form,
\[
\gamma = \sqrt{1 + \frac{\hbar^2 n^2 \pi^2}{m^2 c^2 L^2}}
\] (12)
Finally, the energy of the $n$-th level, within the relativistic Grave de Peralta approach will be

$$E_n^{(GP)} = \frac{\hbar^2 n^2 \pi^2}{2mL^2} - \frac{\hbar^4 n^4 \pi^4}{8m^3 c^2 L^4} + O(L^{-6})$$  \hspace{1cm} (13)$$

where the first order coincides with the non-relativistic energy, Eq. (11), and the second order gives the mass-velocity relativistic correction to the kinetic energy.

A comparison of the energies computed with Eq. (13) and those reported in reference [19], is given in Figure 11 for different sizes of the well. Apart from disagreements for lower $n$, the present model reliably predict, with very good agreement, most of the reported data and describe the tendency when the energy increase and the size of the well reduce.

In general for a particle confined in a region of size $L$, the relativistic parameter $\gamma$ of Eq. (8) can be evaluated as,

$$\gamma = \sqrt{1 + \left(\frac{\lambda_c}{L}\right)^2}$$  \hspace{1cm} (15)$$
where $\lambda_C = \hbar/mc$ is the reduced Compton wavelength of the particle. Figure 2 shows the behavior of $\gamma$ as a function of the particle size in units of $\lambda_C$. From the figure, the parameter appreciably deviates from 1 when the localization region of the particle approaches the Compton wavelength. Moreover, when the particle extends to a spatial region several orders larger than its Compton wavelength, the relativistic effects become negligible. It is worth noting that for a particle with a mass close to the Planck mass, the relativistic effects start to become relevant when the particle is localized in a region of the order of the Planck length, which coincides with $\lambda_C$ for $m = m_P$.

4 The relativistic Diósi length

Now consider the eigenvalue equation,

$$
\left[ \frac{\hbar^2}{(1 + \gamma) m} \nabla^2 - \frac{Gm^2}{|x - x'|} \right] \psi = E \psi,
$$

which is the Schrödinger-Newton model with the Grave de Peralta modification to include relativistic effects.

Using Eq. (15) we can qualitatively evaluate the energy of a stationary solution of Eq. (16), for a particle localized in a region of size $l$, in the form,

$$
E \approx \frac{\hbar^2}{\left[ 1 + \sqrt{1 + \left( \frac{4}{\lambda_C^2} \right)^2} \right] m^2 l^2} - \frac{Gm^2}{l}. \tag{17}
$$

This equation is the relativistic version of those previously obtained by Diósi [1].
Figure 3: Length scales in units of the Planck length as a function of the mass in units of the Planck mass. See text for details.

namely

\[ E \approx \frac{\hbar^2}{2ml^2} - \frac{Gm^2}{l}. \]  

(18)

By minimizing this expression with respect to \( l \), Diósi arrived to the distance \( l_D \) given by Eq. (2), which represent the characteristic length of a stationary solution of the SNE. Indeed, this length give a measure of the size of the quantum field when its expansion, due to the dispersive \( p^2/2m \) energy term, is stopped by the gravitational self-force.

Following Diósi, it is straightforward to shows that the value of \( l \) for which Eq. (17) has a minimum is,

\[ l_D(r) \approx l_D \sqrt{1 - \left(\frac{\lambda_C}{l_D}\right)^2}, \]  

(19)

where the supra-index \( r \) in \( l_D(r) \) stand for "relativistic".

In the same way, \( l_D(r) \) represent the length scale for the solutions of Eq. (16), and clearly indicate that this characteristic length goes to zero when the Diósi length approach the Compton wavelength of the particle. This happens for a mass of the particle equal to the Planck mass, when it is verified that \( l_D = \lambda_C = l_P \). Figure 3 shows the behavior of the different length scales in units of the Planck scale, where it is evident that close to the Planck mass the relativistic corrected Diósi length start to depart from the non-relativistic one and sharply drop to zero, then becoming undefined for larger values. This sort of collapse behavior observed for the characteristic length, may be given a glimpse of a truly collapse behavior of a quantum field induced by gravity, in a full relativistic approach.
The Eq. (19) may be written in the following appealing form

\[ l_D = l_D \sqrt{1 - \left(\frac{m_0}{m_P}\right)^4} \]  \hspace{1cm} (20)

5 Conclusions

The Grave de Peralta proposal to account for relativistic effects by the Eqs. (8), (9) and (10), shows a good agreement with the values obtained by integration of the Dirac equation for a particle in a box. Although some appreciable disagreements are observed above the \( 2m_0c^2 \) threshold, our values reproduce with good agreement most of the reported data. The major advantage of this method rely on the possibility of include relativistic corrections by solving just an equation beyond a Schrödinger-like equation.

After extend the Schrödinger-Newton model to the relativistic domain using the Grave de Peralta approach, we obtained that length scale of the model is bounded from above at a mass of the particle of the order of the Planck mass. This indicates that the coupling between a relativistic quantum field and a classical gravitational field may lead to a collapse of the wavefunction induced by gravity.

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