A remark on representations of infinite symmetric groups

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We simplify construction of Thoma representations of an infinite symmetric group.

1. Spherical representations. Let $G$ be a group, $K$ a subgroup. The pair $(G, K)$ is called spherical if for any irreducible unitary representation $\rho$ of $G$ the dimension of the subspace of $K$-fixed vectors is $\leq 1$. A unit vector of this subspace is called a spherical vector. Recall that the spherical function of an irreducible spherical representation $\rho$ of $G$ is given by

$$\Phi(g) := \langle \rho(g)v, v \rangle,$$

where $v$ is the unit $K$-fixed vector.

2. The double of symmetric group. Let $\Omega$ be a countable set. A permutation of $\Omega$ is called finite if it fixes all but finite number elements of $\Omega$. Denote by $S(\Omega)$ the group of finite permutations of a countable set $\Omega$. We also use an alternative notation $S_\infty$ for such groups.

Now let $G = S_\infty \times S_\infty$ be the product of two copies of $S_\infty$, let $K \simeq S_\infty$ be the diagonal subgroup. By [5], the pair $(G, K)$ is spherical.

3. Thoma formula. Let $G, K$ be the same as in the previous subsection. Representations of $G$ spherical with respect to $K$ are parametrized by collection of positive numbers

$$\alpha_1 \geq \alpha_2 \geq \ldots, \quad \beta_1 \geq \beta_2 \geq \ldots, \quad \sum \alpha_i + \sum \beta_j \leq 1,$$

finite and empty collections are admissible. Spherical functions are given by the formula

$$\Phi(\alpha, \beta, \sigma, \tau) = \prod_{k=2}^{\text{number of cycles of } \sigma \tau^{-1}\text{ of length } k} \left( \sum_{i} \alpha_i^k + (-1)^{k-1} \sum_{j} \beta_j^k \right)$$

(1) (the product is finite), Thoma (1964) formulated this statement on another language (see [6], [5]).

Explicit constructions of Thoma representations were obtained by Vershik and Kerov (see [7], another version was done by Vershik [8]). Olshanski [5] proposed a transparent construction (which also is a modification of [7]), but it does not cover all possible values of parameters, we add a missing element to his construction.

4. Olshanski construction. Let

$$\sum \alpha_i + \sum \beta_j = 1$$

(2)

Consider a Hilbert space $H = H_\Omega \oplus H_\Gamma$. Fix an orthonormal basis $e_i$ in $H_\Omega$ and $f_j \in H_\Gamma$. Consider a unit vector $\xi \in H \otimes H$ given by

$$\xi := \sum_i \alpha_i^{1/2} e_i \otimes e_i + \sum_j \beta_j^{1/2} f_j \otimes f_j.$$
We consider the infinite super-tensor product\(^2\)
\[
\mathcal{H} := (H \otimes H, \xi) \otimes (H \otimes H, \xi) \otimes (H \otimes H, \xi) \otimes \ldots
\]
(recall that the definition of an infinite tensor product requires distinguished unit vectors). The subgroup \(S_\infty \times e \subset G\) acts in \(\mathcal{H}\) by permutations of first factors in brackets \((H \otimes H, \xi)\), the group \(e \times S_\infty\) acts by permutations of the second factors. The diagonal subgroup \(K\) acts by permutations of the whole brackets \((H \otimes H, \xi)\) and \(\xi^{\otimes \infty}\) is the \(K\)-spherical vector.

However, the condition (2) is essential\(^3\), the same difficulty arises for other spherical pairs considered by Olshanski \(\[5\]\) and in a more general setting discussed in \(\[3\], \[4\]\).

5. Products of spherical functions. Consider two irreducible spherical representations of \(G\) corresponding to parameters \(\alpha, \beta\) and \(\alpha', \beta'\). It can be easily shown that their tensor product contains only one \(K\)-fixed vector\(^4\), and the spherical function is the product of spherical functions, it has the form (1) with parameters
\[
\tilde{\alpha} = \{\alpha_i \alpha'_i\}, \{\beta_j \beta'_j\}, \quad \tilde{\beta} = \{\alpha_i \beta'_j\}, \{\beta_j \alpha'_i\}.
\]
Therefore to construct all spherical representations it suffices to construct a representation\(^5\) corresponding to a single-element collection \(\alpha\) and empty collection \(\beta\), the spherical function of this representation is
\[
\Psi(\sigma, \tau) = \alpha^{\text{number of } i \text{ such that } \sigma_i = \tau_i}.
\]

6. Group of affine isometries and Araki scheme. For details, see, e.g., [2], V.1.6-7, X.1. Denote by \(F_n\) the Hilbert space of holomorphic functions on \(\mathbb{C}^n\) with the inner product
\[
\langle f, g \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z) g(z) e^{-\langle z, z \rangle} \, dz \, d\overline{z}.
\]
Consider the natural embeddings \(J_n : F_n \to F_{n+1}\) given by
\[
J_n f(z_1, \ldots, z_n, z_{n+1}) = f(z_1, \ldots, z_n).
\]
Evidently, \(J_n\) is an isometric embedding. We consider the union of the chain
\[
\ldots \to F_n \to F_{n+1} \to \ldots
\]
and its completion \(F_\infty\) (the boson Fock space), see, e.g., [2], VI.1.

Consider a real Hilbert space \(H\). Denote by \(O(H)\) the group of orthogonal (i.e., real unitary) operators. Consider the group \(\text{Isom}(H)\) generated by \(O(H)\) and translations, we get a semi-direct product \(\text{Isom}(H) = O(H) \ltimes H\), this group acts in \(H\) by affine transformations
\[
h \mapsto Ah + v, \quad \text{where } A \in O(H), v \in H.
\]

\(\[2\]\) Consider a \(\mathbb{Z}_2\)-graded space \(V = \mathbb{V}_\mathbb{R} \oplus \mathbb{V}_{-\mathbb{R}}\). Then \(V \otimes V\) is the usual tensor product, but the operator of transposition of factors is another: for homogeneous elements \(v, w\), we have \(v \otimes w \to (-1)^{\varepsilon} w \otimes v\), where \(\varepsilon = 1\) if \(v, w \in \mathbb{V}_+\) and \(\varepsilon = 0\) if at least one of the vectors \(v, w\) is contained in \(\mathbb{V}_{-\mathbb{R}}\). On \(n\)-factor product \(V \otimes \ldots \otimes V\) we have an action of the symmetric group \(S_n\), any permutation is a product of transpositions, and action of a transposition was described above.

\(\[3\]\) Otherwise the length of \(\xi\) is not 1 and \(\[3\]\) is not well-defined.

\(\[4\]\) but this representation can be reducible

\(\[5\]\) Another construction of this representation can be found in [1].
Next, we construct the linear representation $\text{Exp} (\cdot)$ of $\text{Isom}(\ell_2)$ in the Fock space $\mathcal{F}_\infty$. Orthogonal transformations act in the Fock space by

$$\text{Exp} (A) f(z) = f(zA), \quad A \in \text{O}(\ell_2),$$

translations by

$$\text{Exp} (v) f(z) = f(z + v) e^{-(z,v) - \frac{i}{2} (v,v)}, \quad v \in \ell_2.$$

Thus we get an irreducible unitary representation of $\text{Isom}(\ell_2)$. The function $f(z) = 1$ is $\text{O}(\ell_2)$-invariant, the spherical function is $e^{-\frac{i}{2} (h,h)}$.

One of the most common ways$^6$ (Araki scheme) to construct representations of infinite-dimensional groups $G$ is embeddings of $G$ to the group of isometries of Hilbert space and restrictions of the representation $\text{Exp}(\cdot)$ to $G$.

Let we have a unitary representation $U$ of a group $G$ in a Hilbert space $H$. Let

$$\Xi : G \to H$$

be a function satisfying

$$\Xi(g_1 g_2) = T(g_1) \Xi(g_2) + \Xi(g_1).$$

Then affine isometric transformations

$$\Xi(g_1) h = U(g) h + \Xi(g)$$

satisfy

$$\tilde{U}(g_1) \tilde{U}(g_2) = \tilde{U}(g_1 g_2)$$

and we get embedding $G \to \text{Isom}(H)$ (this is straightforward). For a fixed vector $\eta \in H$ the function

$$\Xi(g) := U(g) \eta - \eta$$

satisfies the equation $[3]$. This solution of (5) is not interesting because such correction $\Xi(\cdot)$ is equivalent to a change of origin of coordinates.

Now let $G$ acts by linear transformations of a larger linear space $\hat{H} \supset H$, $\eta \in \hat{H} \setminus H$. If $U(g) \eta - \eta \in H$ for all $g$, then we get a nontrivial affine isometric action of $G$.

7. The construction for the double of the symmetric group. Consider the space $\ell_2$ with basis $e_j$. The group $G = S_\infty \times S_\infty$ acts in $\ell_2 \otimes \ell_2$ by linear transformations

$$U(\sigma, \tau) e_i \otimes e_j = e_{\sigma i} \otimes e_{\tau j}$$

We fix $s > 0$, set

$$\eta := s \sum_{j=1}^{\infty} e_j \otimes e_j$$

and define $\Xi(\sigma, \tau)$ by (7). Note that $\eta \notin \ell_2 \otimes \ell_2$ but $\Xi(\sigma, \tau) \in \ell_2 \otimes \ell_2$. Also, $\Xi(\sigma, \sigma) = 0$. We define affine isometric action of $G = S_\infty \times S_\infty$ by (3) and restrict the representation of $\text{Isom}(\ell_2 \otimes \ell_2)$ to $G$. Then the function $f = 1$ is a unique $K$-fixed vector, the spherical function is (4) with $\alpha = e^{-s^2}$.

8. Symmetric group and hyper-octahedral subgroup. Consider two copies of $\mathbb{N}$, say $\mathbb{N}_+$ and $\mathbb{N}_-$. Denote their points by $1_+, 2_+, \ldots, 1_-, 2_-, \ldots$. Let $G_1 = S(\mathbb{N}_+ \cup \mathbb{N}_-) =: S_{2\infty}$. The \textit{hyperoctahedral group} $K_1$ is the subgroup in $G_1$ consisting of permutations such that for any $j \in \mathbb{N}$ the pair $(\sigma j_+, \sigma j_-)$ has the form $(m_+, m_-)$ or $(m_-, m_+)$. Evidently, $K_1$ is a semidirect product, $K_1 = S_\infty \rtimes \mathbb{Z}_2^\infty$.

$^6$See numerous examples in [2], Sections VIII.6.8, IX.1.6, IX.2.5, IX.5.4, IX.1.4, IX.1.5, IX.3.12, IX.4.6, F.4.
The pair \((G_1, K_1)\) is spherical, see \[5\].

Now we consider the Hilbert spaces \(\ell_2(N_\pm)\) with bases \(e_j^\pm\). Consider the Hilbert space

\[ H = (\ell_2(N_+) \oplus \ell_2(N_-)) \otimes (\ell_2(N_+) \oplus \ell_2(N_-)) \]

The group \(G_1 = S(N_+ \sqcup N_-)\) acts in this space in a natural way (on each tensor factor). Next, we define the vector

\[ \eta := s \cdot \sum_j (e_j^+ \otimes e_j^- + e_j^- \otimes e_j^+) \]

and construct an embedding of \(G_1 = S(N_+ \sqcup N_-)\) to \(\text{Isom}(H)\) as above.

This gives \(K_1\)-spherical representations of \(G_1\), which were not covered by explicit construction of \[5\].

9. **One more example.** Consider the pair \(G_2 \supset K_2\), where \(G_2 = G_1\) and \(K_2 \subset K_1\) is the group of permutations \(\sigma\) sending any ordered pair \((j_+, j_-)\) to a pair \((m_+, m_-)\). In fact,

\[ K_2 = S_\infty \subset S_\infty \times \mathbb{Z}_2^\infty = K_1 \]

This pair is spherical (see \[4\], note that this fact has not counterpart for finite symmetric groups). Now we fix real parameters \(s, t\), set

\[ \eta = s \sum_j e_j^+ \otimes e_j^- + t \sum_j e_j^- \otimes e_j^+ \]

and repeat the same arguments.

10. **Triple products.** Now let \(G_3 = S_\infty \times S_\infty \times S_\infty\), \(K_3 \simeq S_\infty\) be the diagonal. This pair is spherical, see \[3\], \[4\]. We set

\[ H := \ell_2 \otimes \ell_2 \otimes \ell_2 \]

and

\[ \eta = s \sum_j e_j \otimes e_j \otimes e_j. \]

**References**

[1] A.V. Dudko, N.I. Nessonov Invariant states on the wreath product. Preprint [arXiv:0903.4987]
[2] Neretin, Yu. A. Categories of symmetries and infinite-dimensional groups. Oxford University Press, New York, 1996;
[3] Neretin, Yu. A. Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories. Int. Math. Res. Notices (2011), Vol. 2012 501-523
[4] Neretin, Yu. A. Infinite symmetric group and combinatorial descriptions of semigroups of double cosets. Preprint. [arXiv:1106.1161] (2011)
[5] Olshanski, G.I., Unitary representations of \((G, K)\)-pairs connected with the infinite symmetric group \(S(\infty)\). Leningr. Math. J. 1, No.4, 983–1014 (1990).
[6] Thoma, E. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. Math. Z. 85, 40–61 (1964).
[7] Vershik, A. M.; Kerov, S. V. Characters and factor representations of the infinite symmetric group. Soviet Math. Dokl. 23 (1981), no. 2, 389–392.
[8] A. M. Vershik Totally nonfree actions and the infinite symmetric group. Mosc. Math. J., 12:1 (2012), 193–212

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