REGULARITY FOR A QUASILINEAR CONTINUOUS CASTING PROBLEM

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Abstract. In this paper we study the regularity of weak solutions to the continuous casting problem

\(|\nabla u|^{p-2}\nabla u - v\beta(u)| = 0 \tag{♯}

for prescribed constant velocity \(v\) and enthalpy \(\beta(u)\) with jump discontinuity at \(u = 0\). We establish the following estimates: local log-Lipschitz \(p > 2\) for \(u\) and \(BMO\) for \(\nabla u\) for two phase, \(Lipschitz p > 1\) for one phase and linear growth up-to boundary near the contact points. We also prove that the free boundary is continuous curve in the direction of \(v\) in two spatial dimensions.

The proof is based on a delicate argument exploiting Sard’s theorem for \(W^{2,2+}\), \(\eta > 0\) functions and circumventing the lack of comparison principle for the solutions of (♯).

1. Introduction

Let \(\Omega \subset \mathbb{R}^{N-1}\) be a bounded domain with \(C^{1,\alpha}\) boundary. Denote \(C_L = \Omega \times (0, L) \subset \mathbb{R}^N\) where \(L > 0\) is given. In what follows we denote the points in \(C_L\) by \(X = (x, z)\) where \(x \in \Omega, z \in (0, L)\).

Consider the steady state continuous casting problem in \(C_L\) with constant convection in the direction of the \(z\)-axis

\[
\begin{aligned}
\Delta_p u &= \partial_z \beta(u) \quad \text{in} \quad C_L, \\
u &= m^+ \quad \text{on} \quad \Omega \times \{L\}, \\
u &= -m^- \quad \text{on} \quad \Omega \times \{0\}, \\
u &= g \quad \text{on} \quad \partial \Omega \times (0, L),
\end{aligned}
\tag{1}
\]

where \(m^+, m^-\) are two positive constants. The quasilinear degenerate elliptic operator

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty
\]

is called the \(p\)-Laplacian. The boundary data \(g\) on the lateral boundary of \(C_L\) is \(C^{1,\alpha}\) regular for some \(\alpha \in (0, 1)\) and satisfies the compatibility conditions

\[
g(x, 0) = -m^-, \quad g(x, L) = m^+ \quad \forall x \in \partial \Omega. \tag{2}
\]

The enthalpy \(\beta = \beta(u)\) is defined as follows

\[
\beta(s) = \begin{cases}
\text{as} & \text{if} \ s < 0, \\
\in [0, \ell] & \text{if} \ s = 0, \\
\text{as} + \ell & \text{if} \ s > 0.
\end{cases} \tag{3}
\]

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Here $a > 0$, $\ell > 0$ are given constants. An equivalent definition of $\beta$, which will useful in the analysis of the equation $\Delta_p u = \partial_z \beta(u)$, is $\beta(s) = as + H(s)$, with $H(s)$ being the Heaviside function

$$H(s) = \begin{cases} 
    s, & \text{if } s > 0, \\
    0, & \text{if } s \leq 0.
\end{cases}$$

The equations of the form

$$\Delta_p u = \text{div}[v\beta(u)] + f$$

have a number of physical applications [2]. One may interpret $u(X)$ as the normalized temperature at a point $X \in C_L$ whereas $f$ accounts for sources and $v(X)$ is the velocity of convection. (3) manifests the heat conservation of thermodynamical system with enthalpy $\beta(u)$ when the liquid phase has velocity $v$. The case of constant convection $v = e_N$ models the solidification of molten steel extracted at constant speed and is used intensively in steel production. We shall mainly focus on this case.

In order to study the problem mathematically we first formulate it in weak sense. Let $f$ be a given continuous function and $v$ a Lipschitz continuous vectorfield defined in the cylinder $C_L$. In what follows $W^{1,p}(C_L), p > 1$ denotes the Sobolev space of weakly differentiable functions $v \in L^p$ such that the weak derivatives of $v$ are in $L^p(C_L)$. The subspace of $W^{1,p}$ of functions with vanishing trace is denoted by $W_0^{1,p}$.

**Definition 1.** Let $v \in C^{0,1}(\overline{C_L})$ and $f$ continuous. Then $u \in W^{1,p}(C_L)$ is said to be a weak solution of (3) in $C_L$ if

$$-\int_{C_L} \beta(u)v \cdot \nabla \phi + \int_{C_L} |\nabla u|^{p-2} \nabla u \nabla \phi = -\int_{C_L} f \phi, \quad \forall \phi \in C_0^\infty(C_L).$$

is satisfied. Here $\beta$ is the maximal monotone graph given by (3).

For given function $g \in C^{1,\alpha}(\partial C_L), \alpha \in (0, 1)$ we consider the weak solutions to Dirichlet problem

$$\left\{ \begin{array}{ll}
\Delta_p u = \text{div}[v\beta(u)] + f & \text{in } C_L, \\
u(x, 0) = -m^- & x \in \Omega, \\
u(x, L) = m^+ & x \in \Omega, \\
u = g(X) & \text{on } \Sigma = \partial \Omega \times (0, L).
\end{array} \right.$$

(DP)

**Definition 2.** Let $v \in C^{0,1}(\overline{C_L})$. A pair $(u, \eta)$ is said to be a weak solution to (DP) if $u \in W^{1,p}(C_L), \eta \in \beta(u)$, $u = g$ on $\Sigma := \partial \Omega \times (0, L)$ (in the trace sense), $u(x, 0) = -m^-, u(x, L) = m^+, x \in \Omega$ and for any $\phi \in W_0^{1,p}(C_L)$

$$-\int_{C_L} \eta v \cdot \nabla \phi + \int_{C_L} |\nabla u|^{p-2} \nabla u \nabla \phi = -\int_{C_L} f \phi.$$

The condition $v \in C^{0,1}(\overline{C_L})$ on the convection $v$ is of technical nature and later will be replaced by a stronger one, namely $\mathbf{v} = e_N$ which corresponds to the continuous casting problem. Let $u^+ = \max(u, 0), u^- = -\min(u, 0)$ so that $u = u^+ - u^-$. If $\partial \{u > 0\}$ is $C^1$ smooth then the following free boundary condition is satisfied

$$|\nabla u^+|^{p-2} \nabla u^+ \cdot \nu^+ - |\nabla u^-|^{p-2} \nabla u^- \cdot \nu^- = \ell \nu \cdot \nu^+$$
where $v^+, v^-$ are the outer normals of $\{u > 0\} \cap C_L$ and $\{u < 0\} \cap C_L$, respectively, see [21] equation (5).

**Remark 3.** It is known that there is a unique weak solution of the problem such that $\|u\|_\infty \leq M < \infty$, see Theorems 2 and 3 in [21]. For the classical case $p = 2$ we refer to [18] where it is shown that $u \in C^\alpha(C_L)$ provided that $g \in C^{\alpha}(\partial C_L)$.

2. Main results

In this section we formulate our main results.

**Theorem 1.** Let $2 < p < \infty$ and $u$ be a bounded weak solution to the equation $\Delta_p u = \partial_z(\beta(u))$. Then $u$ is locally in $BMO$ and consequently it is locally log-Lipschitz continuous in $C_L$.

Notice that in Theorem 1 the weak solution $u$ may change sign. The condition $p > 2$ is dictated by the non-variational structure of this equation. Indeed, as we shall see below (see Remark 8) for $1 < p < 2$ our technique gives only Hölder continuity of $u$.

It is worthwhile to point out that for one phase problem, $p > 2$, the BMO estimate above implies a linear growth from free boundary, see Lemma 13. However, the same conclusion holds for any $p > 1$ as the next theorem shows.

**Theorem 2.** Let $u$ be a non-negative bounded weak solution to (1) and $1 < p < \infty$.

1° Then $u$ grows linearly away from the free boundary $\partial\{u > 0\} \cap C_L$, provided that $v \in L^\infty(C_L, \mathbb{R}^N)$ and $f \in C(C_L)$. This means that for every subdomain $D \subset \subset C_L$ there is a constant $C$ depending only on $N, p, a, L, \ell, \text{dist}(D, \partial C_L), \|v\|_\infty, \|f\|_C$ such that

$$u(x) \leq C|x - x_0|, \quad x \in D, x_0 \in D \cap \partial\{u > 0\}.$$  

2° Furthermore if $v = c_N$ then $u$ is locally Lipschitz continuous in $C_L$.

For $p = 2$ the local regularity for two phase problem is discussed in [10], and [12]. The regularity of free boundary is more delicate, our main result here states that if $N = 2$ and $u$ is a Lipschitz continuous solution of (DP) and $\partial_z u \geq 0$, then the free boundary is a continuous graph in $z-$direction. In order to prove this result we first show that for suitable boundary data $g$ we have $\partial_z u \geq 0$.

**Proposition 4.** Let $u \geq 0$ be a weak solution of (DP) in the sense of Definition 2 $N = 2 < p < \infty, m^- = 0$ and assume further that

$$\liminf_{z \to z_0} \frac{g(x, z) - g(x, z_0)}{z - z_0} \geq 0, \quad \forall x \in \partial\Omega, z_0 \in [0, L], \quad \partial_z g(X) = 0, \quad X \in \partial\Sigma,$$

where $g \in W^{2,2+m}(\Sigma), \eta_0 > 0$, and $\Sigma$ is the lateral boundary of $C_L$. Then $u$ is monotone nondecreasing in $z$ direction.

Finally we formulate our main result concerning the regularity of free boundary in two spatial dimensions.
Theorem 3. Let \( u \) be a nonnegative weak solution to \((\text{DP})\) in \( C_{L}, N = 2 < p, m^{-} = 0 \) such that \( u \) is nondecreasing in \( z \)-direction. Let \( g \in W^{2,2+\eta}(\Sigma), \eta_{0} > 0 \), where \( \Sigma \) is the lateral boundary of \( C_{L} \). Then for any subdomain \( D \subset C_{L}, \Gamma(u) = \partial\{u > 0\} \cap D \) is locally a continuous graph in \( e_{2} \)-direction.

The main difficulty in the proof is the lack of the ellipticity of the operator \( \Delta_{p} \). We circumvent this difficulty by a delicate argument based on an approximation of \( u \) and Sard’s theorem for \( W^{2,2+\eta} \) functions.

Remark 5. Notice that if \( g \geq 0 \), i.e. we consider the one phase problem, then for \( g \in C^{1,\alpha}, \alpha > 0 \) we cannot have strict monotone (i.e. strict inequality in (8)) boundary condition (8) because at the free boundary points on the lateral boundary \( \Sigma = \partial \Omega \times (0, L) \) one has \( \partial_{z} g = 0 \) as \( g = 0 \) is a minimal value.

Remark 6. One can take more general boundary data and consider the following problem

\[
\begin{aligned}
\Delta_{p} u &= \text{div}[\mathbf{v} \beta(u)] + f & \text{in } C_{L}, \\
u(x, 0) &= h_{0}(x) & x \in \Omega, \\
u(x, L) &= h_{L}(x) & x \in \Omega, \\
u(u) &= g(X) & \text{on } \Sigma = \partial \Omega \times (0, L).
\end{aligned}
\]

One can extend all of the results to this general case under suitable conditions on \( \mathbf{v} \) and \( f \) and the boundary data \( h_{0}, h_{1} \). For instance if \( f = 0 \) and \( (\mathbf{v} \cdot e_{N}) \geq 0 \) with \( \mathbf{v} \in C^{0,1}(C_{L}) \) then the free boundary is a continuous curve in the \( z \) direction in two spatial dimensions.

The paper is organised as follows: In Section 4 we prove some BMO estimates by testing \( u \) against its \( p \)-harmonic replacement in small balls. Theorem 1 will follow as a consequence of Lemma 7. In Section 5 we prove Theorem 2. The argument is based on a dyadic scaling method. The regularity of the free boundary in two spatial dimensions is discussed in Section 6. In Section 7 we prove that nonnegative solutions \( u \) have at most linear growth at the contact points where the free boundary touches the fixed boundary.

We shall also sketch how one can extend the results to uniformly elliptic quasilinear equations in Section 8. The paper also contains the proofs of a version of Caccioppoli type estimate and Hopf’s lemma included in the Appendix.

3. Notations

- \( C_{0}, C_{1}, C_{D} \ldots \)
- \( C_{0}, C_{1}, C_{D} \ldots \) generic constants
- \( \chi_{D} \)
- \( \chi_{D} \) the characteristic function of a set \( D \subset \mathbb{R}^{N}, N \geq 2 \)
- \( \Omega \)
- \( \Omega \) the closure of \( \Omega \)
- \( \partial \Omega \)
- \( \partial \Omega \) the boundary of \( \Omega \)
- \( \nu \)
- \( \nu \) outer unit normal
- \( X = (x, z) \in \mathbb{R}^{N} \)
- \( X = (x, z) \in \mathbb{R}^{N} \) \( x = (x_{1}, \ldots, x_{N-1}, 0) \)
- \( \nabla u \)
- \( \nabla u = (\partial_{x_{1}} u, \partial_{x_{2}} u, \ldots, \partial_{z} u) \)
- \( \partial_{X_{i}} = \frac{\partial}{\partial X_{i}}, 1 \leq i \leq N - 1, \partial_{z} = \frac{\partial}{\partial z} \)
\[(\nabla u)_{x_0,\rho} := \int_{B_{\rho}(x_0)} \nabla u\]

$C_L$ the cylinder $C_L = \Omega \times (0, L), L > 0$ for some $\Omega \subset \mathbb{R}^{N-1}$

$\Sigma$ lateral boundary of $C_L, \partial \Omega \times (0, L)$

$B_r(X) = \{Y \in \mathbb{R}^N : |Y - X| < r\}$

$B_r(0)$

4. BMO estimate

**Lemma 7** (Continuity of weak solutions). Let $u \in W^{1,p}(C_L)$ be a solution of \([\text{I}]\). Then for $p > 2$, there exist $c > 0$ and $B > 0$ depending only on $a, \ell, p, N$ and $\sup_{C_L} |u|$ such that

$$\phi(r) \leq cr^N \left( \frac{\phi(R)}{R^2} + B \right),$$

for all $0 < r \leq R \leq \text{dist}(X_0, \partial C_L)$, where

$$\phi(r) := \sup_{t \leq r} \|\nabla u - (\nabla u)_{x_0,t}\|_{L^2(B_r(x_0))}.$$ 

and $X_0 \in \partial \{u > 0\}$.

In particular, we have that $\nabla u \in \text{BMO}(D)$, for any bounded subdomain $D \subset C_L$, and thus $u$ is locally log-Lipschitz continuous. Furthermore, $\nabla u \in L^q(D)$ for any $1 < q < \infty$.

**Proof.** Fix $R \geq r > 0$ and $x_0 \in D$ such that $B_{2R}(x_0) \subset D$. Let $v$ be the solution of

$$\begin{cases}
\Delta_p v = 0 & \text{in } B_{2R}(X_0), \\
v = u & \text{on } \partial B_{2R}(X_0).
\end{cases}$$

From Definition [\text{I}] we have

$$\int_{B_{2R}(X_0)} |\nabla u|^{p-2} \nabla u (\nabla u - \nabla v) = \int_{B_{2R}(X_0)} \beta(u)(u_z - v_z),$$

$$\int_{B_{2R}(X_0)} |\nabla v|^{p-2} \nabla v (\nabla u - \nabla v) = 0.$$ 

After subtracting the second equation from the first one we obtain

\[(10) \int_{B_{2R}(X_0)} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) (\nabla u - \nabla v) = \int_{B_{2R}(X_0)} \beta(u)(u_z - v_z).\]

Recall that by Lemma 5.7 [16] there is a generic constant $\mu > 0$ depending only on $p$ and $N$ such that

\[(11) (|\xi|^{p-2}\xi - |\eta|^{p-2}\xi)(\xi - \eta) \geq \mu \left\{ \begin{array}{ll}
|\xi - \eta|^p & \text{if } p > 2, \\
|\xi - \eta|^2(|\xi| + |\eta|)^{p-2} & \text{if } 1 < p \leq 2,
\end{array} \right.\]
for all $\xi, \eta \in \mathbb{R}^d$. Hence

$$
\mu \int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p \leq \frac{\varepsilon p}{p} \int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p + \frac{1}{\varepsilon p'} \int_{B_{2R}(X_0)} |\beta(u)|^{p'}
$$

$$
\leq \frac{\varepsilon}{p} \int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p + \frac{1}{\varepsilon p'} (\ell + aM)\varepsilon R^N \omega_N
$$

where $M = \sup_{\mathbb{C}_L} |u|$. Consequently, we get that

$$
\int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p \leq CR^N,
$$

(13)

where

$$
C = \frac{(\ell + aM)\varepsilon R^N \omega_N}{(\mu - \varepsilon p/p)\varepsilon p'}.
$$

We infer the estimate

$$
\int_{B_{2R}(X_0)} |\nabla u - \nabla v|^{2p} \leq CR^N,
$$

(14)

with some tame constant $C > 0$.

Indeed, as $p > 2$ we have by Hölder’s inequality

$$
\left(\int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p \right)^{\frac{1}{p}} \geq \left(\int_{B_{2R}(X_0)} |\nabla u - \nabla v|^2 \right)^{\frac{1}{2}}
$$

and (14) follows.

Furthermore, for any $\rho > 0$, we set

$$
(\nabla u)_{X_0, \rho} := \int_{B_{\rho}(X_0)} \nabla u.
$$

Then, from Hölder’s inequality we have

$$
|((\nabla v)_{X_0, r} - (\nabla u)_{X_0, r})|^2 \leq \left(\int_{B_r(X_0)} |\nabla v - \nabla u| \right)^2
$$

$$
\leq \int_{B_r(X_0)} |\nabla v - \nabla u|^2.
$$

(15)

We would also need the following estimate for a $p-$harmonic function $v$: there is $\alpha > 0$ such that for all balls $B_{2R}(X_0) \subset D$, with $R \geq r > 0$, there exists a universal constant $c > 0$ such that the following Companato type estimate is valid

$$
\int_{B_r(X_0)} |\nabla v - (\nabla v)_{X_0, r}|^2 \leq c \left(\frac{r}{R}\right)^{\alpha} \int_{B_R(X_0)} |\nabla v - (\nabla v)_{X_0, R}|^2.
$$

(16)

See [5] Theorem 5.1.
Denote \( \| \cdot \|_{L^2(B_r(X_0))} = \| \cdot \|_{2,r} \), then, using (15), we obtain
\[
\| \nabla u - (\nabla u)_{X_0,r} \|_{2,r} \leq \| \nabla u - \nabla v \|_{2,r} + \| \nabla u - (\nabla v)_{X_0,r} \|_{2,r} \\
+ \| (\nabla v)_{X_0,r} - (\nabla u)_{X_0,r} \|_{2,r} \\
\leq 2 \| \nabla u - \nabla v \|_{2,r} + \| \nabla v - (\nabla v)_{X_0,r} \|_{2,r} \\
\leq 2 \| \nabla u - \nabla v \|_{2,r} + C \left( \frac{r}{R} \right)^{\frac{N+n}{2}} \| \nabla v - (\nabla v)_{X_0,R} \|_{2,R},
\]
(17)
where, in order to get (17), we used Campanato type estimate (16).

From the triangle inequality for \( L^2 \) norm we have
\[
\| \nabla v - (\nabla v)_{X_0,r} \|_{2,R} \leq 2 \| \nabla u - \nabla v \|_{2,R} + \| \nabla u - (\nabla u)_{X_0,R} \|_{2,R},
\]
and so, combining this with (14), we obtain
\[
\| \nabla u - (\nabla u)_{X_0,r} \|_{2,r} \leq 2 \| \nabla u - \nabla v \|_{2,r} \\
+ C \left( \frac{r}{R} \right)^{\frac{N+n}{2}} \left[ 2 \| \nabla u - \nabla v \|_{2,R} + \| \nabla u - (\nabla u)_{X_0,R} \|_{2,R} \right] \\
\leq C \left\{ \| \nabla u - \nabla v \|_{2,R} + \left( \frac{r}{R} \right)^{\frac{N+n}{2}} \| \nabla u - (\nabla u)_{X_0,R} \|_{2,R} \right\} \\
\leq A \left( \frac{r}{R} \right)^{\frac{N+n}{2}} \| \nabla u - (\nabla u)_{X_0,R} \|_{2,R} + BR^{\frac{N}{2}},
\]
for some tame positive constants \( A \) and \( B \).

Introduce
\[
\phi(r) := \sup_{t \leq r} \| \nabla u - (\nabla u)_{X_0,t} \|_{2,t},
\]
then the former inequality can be rewritten as
\[
\phi(r) \leq A \left( \frac{r}{R} \right)^{\frac{N+n}{2}} \phi(R) + BR^{\frac{N}{2}},
\]
with some positive constants \( A, B, \alpha. \) Applying Lemma 2.1 from \( \textit{K} \) Chapter 3, we conclude that there exist \( R_0 > 0 \) and \( c > 0 \) such that
\[
\phi(r) \leq cr^{\frac{N}{2}} \left( \frac{\phi(R)}{R^{\frac{N}{2}}} + B \right),
\]
for all \( r \leq R \leq R_0 \), and hence
\[
\int_{B_r(X_0)} |\nabla u - (\nabla u)_{X_0,r}|^2 \leq Cr^N,
\]
for some tame constant \( C > 0 \). This shows that \( \nabla u \) is locally BMO. The log-Lipschitz estimate for \( p > 2 \) now follows from \( \textit{K} \) Theorem 3.

\[\square\]

**Remark 8.** If \( 1 < p < 2 \) then using the equation (10) and inequality (11) in conjunction with the comparison of \( u \) with its \( p \)-harmonic replacement in a small ball \( B_{2R}(X_0) \) centred at a free boundary...
point, renders the following inequality
\[
\int_{B_{2R}(X_0)} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \lesssim \int_{B_{2R}(X_0)} |u_z - v_z| \leq \left( \int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p \right)^{\frac{1}{p}} |B_{2R}|^{1-\frac{1}{p}}.
\]
Consequently, setting \(\sigma = \frac{p(2-p)}{2} > 0\) and applying Hölder's inequality we infer the estimate
\[
\int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p = \left( \int_{B_{2R}(X_0)} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right)^{\frac{p}{2}} \left( \int_{B_{2R}(X_0)} (|\nabla u| + |\nabla v|)^{2p} \right)^{\frac{1-p}{2}} \lesssim \left( \int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p \right)^{\frac{1}{2}} |B_{2R}|^{1-\frac{1}{2}} \left( \int_{B_{2R}(X_0)} |\nabla u|^p \right)^{\frac{1}{2}}
\]
which yields the estimate
\[
\int_{B_{2R}(X_0)} |\nabla u - \nabla v|^p \lesssim |B_{2R}|^{p-1} \left( \int_{B_{2R}(X_0)} |\nabla u|^p \right)^{2-p}.
\]
Then the Caccioppoli type inequality from \([14]\) and the technique above give that \(u\) is Hölder continuous.

5. PROOF OF THEOREM \([2]\)

1° We show that for any compact set \(K \subset \subset C_L\) there exists a tame constant \(C\), depending on \(\text{dist}(K, \partial C_L)\) such that
\[
\sup_{B_{2^{-k-1}}(X)} u \leq \max \left( C2^{-k}, \sup_{B_{2^{-k}}(X)} u \right), \quad \forall X \in K \cap \partial \{u > 0\}.
\]
If this inequality is false then there exist a sequence of weak solution \(u_j\) such that \(0 \leq u_j \leq M\) for some fixed constant \(M > 0\), a sequence \(\{k_j\} \subset \mathbb{N}, X_j \in K \cap \partial \{u_j > 0\}\) such that
\[
(18) \quad \sup_{B_{2^{-k_j-1}}(X)} u_j > \max \left( j2^{-k_j}, \frac{1}{2} \sup_{B_{2^{-k_j}}(X_j)} u_j \right).
\]
Consider the scaled functions
\[
v_j(X) = \frac{u_j(X_j + 2^{-k_j}X)}{S_j},
\]
where \(S_j = \sup_{B_{2^{-j+1}}(X_j)} u_j\). It is obvious that
\[
(19) \quad v_j(0) = 0,
\]
Moreover, it follows from (18) that
\[
(20) \quad \frac{2^{-k_j}}{S_j} < \frac{1}{j}, \quad \sup_{B_{2^{-k_j}}} v_j \geq \frac{1}{2}, \quad 0 \leq v_j(X) \leq 2, \quad X \in B_1.
\]
Since, by assumption, the weak solutions \( u_j \) are bounded it follows from \( 13 \) that \( M > j^{2^{-k_j}} \), implying that \( k_j \to \infty \).

If \( u_j \) solves \( 11 \), then from the scale invariance properties of \( \Delta_p \) it follows that \( v_j \) solves the following equation

\[
\text{div}(|\nabla v_j|^{p-2}\nabla v_j) = \frac{2^{-pk_j}}{S_j^{p-1}}(\Delta_p u_j)(X_j + 2^{-k_j}X) = \left[ \frac{2^{-k_j}}{S_j} \right]^{p-1} \text{div}[\beta(v_j) \nabla (X_j + 2^{-k_j}X)] + f_j
\]

where

\[
F_j = \left[ \frac{2^{-k_j}}{S_j} \right]^{p-1} \beta(v_j) \nabla (X_j + 2^{-k_j}X),
\]

\[
f_j = \frac{2^{-pk_j}}{S_j^{p-1}} f(X_j + 2^{-k_j}X) = S_j \left[ \frac{2^{-k_j}}{S_j} \right]^{p} f(X_j + 2^{-k_j}X).
\]

From \( v \in L^\infty(C_L, \mathbb{R}^N) \) we obtain, using \( 20 \), definition of \( S_j \) and \( 3 \), the inequality

\[
|F_j| \leq \left[ \frac{2^{-k_j}}{S_j} \right]^{p-1} \beta(2) \sup_{B_1} |v| \leq \left[ \frac{1}{j} \right]^{p-1} \beta(2) \sup_{B_1} |v| \to 0.
\]

Similarly we obtain \( \sup_{B_1} |f_j(X)| \to 0 \).

From the Caccioppoli inequality (see Appendix) it follows that \( \{v_j\} \) is bounded in \( W^{1,p}(B_{\frac{3}{4}}) \). Furthermore, utilizing \( 21 \) and \( 20 \) and Serrin’s theorem for quasilinear divergence form elliptic operators \( 19 \), we infer that the sequence \( \{v_j\} \) is uniformly Hölder continuous in \( B_{\frac{3}{4}} \). Now employing a customary compactness argument and the estimates for \( \{F_j\} \) and \( \{f_j\} \), we can extract subsequences \( j_m \) such that \( X_{j_m} \to X_0 \), \( \{v_{j_m}\} \subset \{v_j\} \) which uniformly converges to some \( v_0 \) in \( B_{\frac{3}{4}} \) and weakly in \( W^{1,p}(B_{\frac{3}{4}}) \). Moreover, we can check that

\[
-\int \nabla v_0 \cdot \nabla \phi + \int |\nabla v_{j_m}|^{p-2} \nabla v_{j_m} \nabla \phi = \int f_{j_m} \cdot \nabla \phi \to 0, \quad \forall \phi \in C_0^\infty(B_{\frac{3}{4}}).
\]

To see this we first prove

**Claim 9.** For every \( q \geq p \) there is a tame constant \( \gamma \) independent of \( j_m \) such that

\[
\|\nabla v_{j_m}\|_{L^{q}(B_{\frac{3}{4}})} \leq \gamma, \quad \|\nabla v_0 - \nabla v_{j_m}\|_{L^{q}(B_{\frac{3}{4}})} \to 0.
\]

**Proof.** Indeed, let us define \( h_j \in W^{1,2}_0(B_1) \) as the solution of the following homogeneous Dirichlet problem

\[
\Delta h_j = f_j \quad \text{in } B_1, \quad h_j = 0 \quad \text{on } \partial B_1.
\]

From the a priori bound \( 3.12 \) in \( 7 \) it follows that

\[
\sup_{B_1} |h_j| \leq C \sup_{B_1} |f_j|
\]
with some tame constant $C > 0$. Observe that $h_j$ is the convolution of $f_j$ and the Green function of $B_1$. Using the estimates for the Green potentials and the fact that $\sup_{B_1} |f_j(X)| \to 0$ it follows that

$$\|h_j\|_{C^{1,\sigma}(B_1)} \to 0 \text{ as } j \to \infty$$

for any $\sigma \in (0, 1)$, see estimates (4.45) and (4.46) in [7]. Let $H_j = F_j + \nabla h_j$ and define $F_j = |H_j|^{\frac{1}{p-1}}H_j$ then

$$\Delta_p v_j = \text{div} F_j + f_j = \text{div} H_j = \text{div}(|F_j|^{p-2}F_j)$$

and $F_j \in L^\infty(B_1)$. In fact,

$$\|F_j\|_{L^\infty(B_1)} \leq \left(\|H_j\|_{L^\infty(B_1)}\right)^{\frac{1}{p-1}} \leq \left(\|H_j\|_{L^\infty(B_1)} + \|\nabla h_j\|_{L^\infty(B_1)}\right)^{\frac{1}{p-1}} \to 0 \text{ as } j \to \infty.$$ 

Now we are in position to apply Theorem 1.2 [5] to infer that for any $q \geq p$ the following estimate is true

$$\|\nabla(\zeta v_j)\|_{L^q(\mathbb{R}^N)} \leq \gamma(\|\zeta F_j\|_{L^q(\mathbb{R}^N)} + \|\zeta v_j\|_{L^p(\mathbb{R}^N)})$$

where $\zeta \in C_0^\infty(B_1)$ is a cut-off function and $\gamma$ is a tame constant. This in particular yields

$$\|\nabla v_{j_m}\|_{L^q(B_{\frac{3}{4}})} \leq \gamma, \quad \|\nabla v_0 - \nabla v_{j_m}\|_{L^q(B_{\frac{3}{4}})} \to 0$$

for any $q \geq p$ and a suitable subsequence $j_m$ with some tame constant $\gamma$ independent of $j_m$, because the weak convergence of the gradients in $L^{q'}$ implies strong convergence in $L^q$ if $q' > q$. This finishes the proof of the claim. $\square$

It remains to note that

$$\left|\int_{B_{\frac{3}{4}}} \left(|\nabla v_0|^{p-2}\nabla v_0 - |\nabla v_{j_m}|^{p-2}\nabla v_{j_m}\right) \nabla \phi \right| \leq \sup |\nabla \phi| \int_{B_{\frac{3}{4}}} \left|\nabla v_0|^{p-2}\nabla v_0 - |\nabla v_{j_m}|^{p-2}\nabla v_{j_m}\right|$$

$$\leq \gamma(p) \sup |\nabla \phi| \int_{B_{\frac{3}{4}}} |\nabla v_0 - \nabla v_{j_m}|^{p-1}$$

provided that $1 < p \leq 2$, where the last estimate follows from [13] page 43. Note that

$$\int_{B_{\frac{3}{4}}} |\nabla v_0 - \nabla v_{j_m}|^{p-1} \leq \left(\int_{B_{\frac{3}{4}}} |\nabla v_0 - \nabla v_{j_m}|^p\right)^{\frac{p-1}{p}} \to 0.$$ 

As for the case $p > 2$ again from [13] page 43 by choosing $q > 2(p-2)$ we have

$$\int_{B_{\frac{3}{4}}} \left|\nabla v_0|^{p-2}\nabla v_0 - |\nabla v_{j_m}|^{p-2}\nabla v_{j_m}\right| \leq (p-1) \int_{B_{\frac{3}{4}}} |\nabla v_0 - \nabla v_{j_m}|(|\nabla v_0|^{p-2} + |\nabla v_{j_m}|^{p-2})$$

$$\leq \left(\int_{B_{\frac{3}{4}}} |\nabla v_0 - \nabla v_{j_m}|^2 \int_{B_{\frac{3}{4}}} |\nabla v_0|^{2(p-2)} + |\nabla v_{j_m}|^{2(p-2)}\right)^{\frac{1}{2}} \to 0 \text{ as } j_m \to \infty.$$ 

Thus $v_0 \in W^{1,p}(B_{\frac{3}{4}})$ is a nonnegative continuous solution of $\text{div}(\nabla v_0|^{p-2}\nabla v_0) = 0$ in $B_{\frac{3}{4}}$. On the other hand, it follows from uniform convergence $v_{j_m} \to v_0$ that [19] translates to $v_0$ and we have
$v_0(0) = 0$ and also $\sup_{B^+_L} v_0 = \frac{1}{2}$ thanks to \([20]\). However this is in contradiction with the strong maximum principle and the proof of linear growth from the free boundary follows.

2° Let us take $X_0 \in C_L$ such that $u(X_0) > 0$ and set $r = \text{dist}(X_0, \partial \{ u > 0 \})$. Then by linear growth $u(X_0) \leq C r$ with some tame constant $C > 0$. Consider the scaled function $v(X) = \frac{u(X_0 + r X)}{r}$, $X \in B_1$. Then $v \geq 0$ solves the equation

$$\text{div}(\| \nabla v(X) \|^{p-2} \nabla v(X)) = \text{div}(F(X_0 + r X)) + r f(X_0 + r X)$$

in $B_1$ and $v(0) \leq C$. Here $F(X) = au(X)e_N \in C^\alpha(B_1)$ by Lemma \([7]\). From the weak Harnack inequality, \([19]\) Theorem 7, we have that

$$\sup_{B^+_{1/2}} v \leq c_0(v(0) + \| F \|_{L^\infty(B_1)}^{\frac{1}{p-1}} + (r \| f \|_{L^\infty(B_1)})^{\frac{1}{p-1}})$$

$$\leq c_0(C + \| F \|_{L^\infty(B_1)}^{\frac{1}{p-1}} + (r \| f \|_{L^\infty(B_1)})^{\frac{1}{p-1}}).$$

From the local gradient estimates \([13]\) we infer that

$$\sup_{B_{1/4}} |\nabla v| \leq c_0(C + \| F \|_{L^\infty(B_1)}^{\frac{1}{p-1}} + (\| F \|_{C^\alpha(B_1)})^{\frac{1}{p-1}} + (r \| f \|_{L^\infty(B_1)})^{\frac{1}{p-1}})$$

where the star above means the classical weighted Hölder norm using the radius of the ball. In particular $|\nabla v(0)| \leq \tilde{C}$ for some tame constant $\tilde{C}$ and rescaling back we infer $|\nabla u(X_0)| \leq \tilde{C}$. This completes the proof of Theorem 2. \( \square \)

6. Regularity of free boundary, Proof of Theorem \([3]\)

6.1. Proof of Proposition \([4]\). We recast Proposition \([4]\) here.

**Lemma 10.** Let $u \geq 0$ be a weak solution of \([\text{DP}]\) in the sense of Definition \([2]\) $N = 2 < p < \infty$, $m^- = 0$ and assume further that

\[
(22) \quad \liminf_{z \to z_0} \frac{g(x,z) - g(x,z_0)}{z - z_0} \geq 0, \quad \forall x \in \partial \Omega, z_0 \in [0,L], \quad \partial_z g(X) = 0, X \in \partial \Sigma
\]

where $g \in W^{2,2+n}(\Sigma), \eta_0 > 0$, and $\Sigma$ is the lateral boundary of $C_L$. Then $u$ is monotone nondecreasing in $z$ direction.

**Proof.** For $\varepsilon > 0$ small let us consider the mollified problem

\[
(23) \quad \text{div} \left( (\varepsilon^2 + |\nabla u^\varepsilon(X)|^2)^{\frac{p-2}{2}} \nabla u^\varepsilon(X) \right) = \partial_z(\beta(u^\varepsilon(X))), \quad X \in C_L,
\]

with boundary condition $u^\varepsilon(X) = g(X)$ on $\partial C_L$, where $g \in W^{2,2+n}(\Sigma), \eta_0 > 0$ is satisfying \([4]\). The existence of $u^\varepsilon$ for each $\varepsilon > 0$ follows from standard penalisation argument for uniformly elliptic equations.

**Claim 11.** Let $\delta > 0$ be small. Then $u^\varepsilon \in W^{2,2+n}(\{u^\varepsilon > \delta\} \cap C_L)$ for some $\eta > 0$ which depends on $\varepsilon$ and $\| g \|_{W^{2,2+n}}$. 
Proof. In order to prove this claim we first extend $u^\varepsilon$ to the cylinder $(0, \frac{3L}{2}) \times \Omega := \tilde{C}_L$ such that the extended function $\tilde{u}^\varepsilon$ solves the equation [23]. Introduce the upper extension of $g$ as follows

$$\tilde{g}(x, z) = \begin{cases} g(x, z) & \text{if } x \in \partial\Omega, z \in (0, L), \\ g(x, 2L - z) & \text{if } x \in \partial\Omega, z \in (L, \frac{3L}{2}). \end{cases}$$

Since $g(x, L) = m^+ = \text{const}, x \in \partial\Omega$ and by assumption [8] $\partial_2 g(x, L) = 0, x \in \partial\Omega$ it follows that $\tilde{g} \in W^{2,2+\eta_0}(\tilde{C}_L) \cap C^{1,\alpha_0}(\tilde{C}_L)$ where $\alpha_0 = \frac{\eta_0}{2+\eta_0}$. This can be seen from the embedding of the Sobolev space $W^{2,2+\eta_0}(\Sigma)$ where $\Sigma$ is the lateral boundary of $\tilde{C}_L$ (recall that $N = 2$). The upper extension of $u^\varepsilon$ are defined accordingly

$$\tilde{u}^\varepsilon(x, z) = \begin{cases} u^\varepsilon(x, z) & \text{if } x \in \Omega, z \in (0, L), \\ u^\varepsilon(x, 2L - z) & \text{if } x \in \Omega, z \in (L, \frac{3L}{2}). \end{cases}$$

Let us check that $\tilde{u}^\varepsilon$ is a solution of (23) across $\Omega \times \{L\}$. Note that by the continuity of $u^\varepsilon$, which follows from Theorem [11] and Remark [8] near $\Omega \times \{L\}$ $v = u^\varepsilon - m^+$ solves the equation

$$\text{div} \left( \left( \varepsilon^2 + |\nabla v|^2 \right)^{\frac{p-2}{2}} \nabla v \right) = a \partial_z v \text{ and } v = 0 \text{ on } \Omega \times \{L\}.$$ 

By Proposition 1 [20] it follows that $u^\varepsilon \in W_{loc}^{2,2}(C_L)$.

Take $\phi \in C_0^\infty(\tilde{C}_L)$ such that for some ball $B$ we have supp $\phi \subset B \subset \tilde{C}_L$. Denote $D^- = B \cap \tilde{C}_L$ and $D^+ = B \cap (\Omega \times (\frac{L}{2}, L))$ and fix $t > 0$ small. Then from the divergence theorem we get

$$I^\varepsilon_+ := \int_{D^- \cap \{z < L - t\}} \left( \varepsilon^2 + |\nabla \tilde{u}^\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla \tilde{u}^\varepsilon \cdot \nabla \phi = \int_{D^- \cap \{z < L - t\}} \phi \text{div} \left( \left( \varepsilon^2 + |\nabla \tilde{u}^\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla \tilde{u}^\varepsilon \right)$$

$$= \int_{D^- \cap \{z < L - t\}} \phi \varepsilon^2 |\partial_z \tilde{u}^\varepsilon|^2 \partial_z \tilde{u}^\varepsilon \phi \text{div} \left( \left( \varepsilon^2 + |\nabla \tilde{u}^\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla \tilde{u}^\varepsilon \right)$$

where the last equality follows from the $W_{loc}^{2,2}(C_L)$ estimates mentioned above. Similarly we have that

$$I^\varepsilon_- := \int_{D^+ \cap \{z > L + t\}} \left( \varepsilon^2 + |\nabla \tilde{u}^\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla \tilde{u}^\varepsilon \cdot \nabla \phi = -\int_{D^+ \cap \{z > L + t\}} \phi \varepsilon^2 |\partial_z \tilde{u}^\varepsilon|^2 \partial_z \tilde{u}^\varepsilon - \int_{D^- \cap \{z > L + t\}} \phi a \partial_z \tilde{u}^\varepsilon.$$

From the gradient estimates near the flat portions of the boundary [13] it follows that $\nabla u^\varepsilon$ is Hölder continuous near $\Omega \times \{L\}$, therefore

$$\lim_{t \to 0} \left( \int_{D^- \cap \{z < L - t\}} \phi \varepsilon^2 |\partial_z \tilde{u}^\varepsilon|^2 \partial_z \tilde{u}^\varepsilon \cdot \nabla \phi = \int_{D^+ \cap \{z > L + t\}} \phi a \partial_z \tilde{u}^\varepsilon. \right)$$

Consequently

$$\int \left( \left( \varepsilon^2 + |\nabla \tilde{u}^\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla \tilde{u}^\varepsilon \right) \cdot \nabla \phi = \int \phi a \partial_z \tilde{u}^\varepsilon.$$

Near the lateral boundary of $(\Omega \times (0, \frac{3L}{2})) \cap \{\tilde{u}^\varepsilon > \delta\}$ we can take $\phi = (\tilde{u}^\varepsilon - g) \zeta, \zeta \in C_0^\infty$ such that $\phi \in W_0^{1,p}(\tilde{C}_L)$. The finite differences of $\nabla \tilde{u}^\varepsilon$ in the $z$ variable (for sufficiently small step size compared to $\delta$) satisfy uniform $W^{2,2}$ estimates, see the proof of Lemma 8.12 in [17] (recall that by assumption $N = 2$ and hence the lateral boundary is flat). Thus $\partial_z \tilde{u}^\varepsilon, \partial_{zz} \tilde{u}^\varepsilon \in W^{2,2}(\{\tilde{u}^\varepsilon > \delta\} \cap \tilde{C}_L)$ and $\partial_{zz} \tilde{u}^\varepsilon \in W^{2,2}(\{\tilde{u}^\varepsilon > \delta\} \cap \tilde{C}_L)$ follows directly from the equation.
Differentiating the equation \( \text{div} \left( (\varepsilon^2 + |\nabla u^\varepsilon|^2)^{\frac{p-2}{2}} \nabla u^\varepsilon \right) = a \partial_x u^\varepsilon \) in \( \{ u^\varepsilon > 0 \} \) we get

\[
(24) \quad \text{div} \left( (\varepsilon^2 + |\nabla u^\varepsilon|^2)^{\frac{p-2}{2}} \left[ id + \frac{(p-2)}{\varepsilon^2 + |\nabla u^\varepsilon|^2} \nabla \partial_x u^\varepsilon \right] \right) = a \partial_x (\partial_x u^\varepsilon).
\]

For each \( \varepsilon > 0 \) the matrix

\[
A(x) = (\varepsilon^2 + |\nabla u^\varepsilon|^2)^{\frac{p-2}{2}} \left[ id + \frac{(p-2)}{\varepsilon^2 + |\nabla u^\varepsilon|^2} \nabla \partial_x u^\varepsilon \right]
\]

is strictly elliptic. Hence \( u^\varepsilon \in W^{2,2+\eta}(\{ u^\varepsilon > \delta \} \cap \overline{C_L}) \) follows from the strict ellipticity of \( A(x) \) and a standard application of Gehring’s Lemma, see [8] Theorem 2.1, page 136. This finished the proof of the claim.

In view of (24) the function \( w = \partial_x u^\varepsilon \) solves the equation \( \text{div}(A(x)\nabla w) = a \partial_x w \) with strictly elliptic matrix \( A \). Hence from minimum principle and it follows that

\[
\min_{\partial \{ u^\varepsilon > 0 \} \cap \overline{C_L}} \partial_x u^\varepsilon = \min_{\{ u^\varepsilon > 0 \} \cap \overline{C_L}} \partial_x u^\varepsilon.
\]

Take an arbitrary \( \delta > 0 \) small and let us show that \( \min_{\partial \{ u^\varepsilon > \delta \} \cap \overline{C_L}} \partial_x u^\varepsilon \geq 0 \). Applying Claim 11 we have that \( u^\varepsilon \in W^{2,2+\eta}(\{ u^\varepsilon > \delta \} \cap \overline{C_L}) \) for some \( \eta > 0 \) which depends on \( \varepsilon \) and \( \|g\|_{W^{2,2+\eta}} \).

From Sard’s theorem [6] it follows that the one dimensional Lebesgue measure of the critical values of \( u^\varepsilon \) is zero. Consequently, \( \partial \{ u^\varepsilon > \delta \} \) is a regular curve for a.e. \( \delta > 0 \) and the trace of \( u^\varepsilon \) is well defined on it.

That said, let us consider the following cases:

- \( X_0 = (x_0, z_0) \in \partial \{ u^\varepsilon > \delta \} \cap \overline{C_L} \) then for \( (x, z_0) \in \partial \{ u > 0 \} \cap \overline{C_L} \) the lower bound follows immediately

\[
\liminf_{z \to z_0} \frac{u(x_0, z) - u(X_0)}{z - z_0} = \liminf_{z \to z_0} \frac{u(x_0, z) - \delta}{z - z_0} \geq 0.
\]

- \( X_0 \in \partial \{ u^\varepsilon > \delta \} \cap \partial (\Omega \times \{ 0, L \}) \) then on the lateral boundary the tangential derivative agrees with that of \( g \).

- \( X_0 \in \overline{\Omega} \times \{ L \} \) it follows from Hopf’s lemma, because by the maximum principle \( m^+ = \max u^\varepsilon \) and \( \nabla u^\varepsilon \) is Hölder continuous near \( \Omega \times \{ L \} \) (see the reflection argument in the proof of Claim 11 and the application of boundary gradient estimates from [13]) hence \( \partial_x u^\varepsilon > 0 \).

Consequently we conclude that \( \min_{\partial \{ u^\varepsilon > 0 \} \cap \overline{C_L}} \partial_x u^\varepsilon \geq 0 \) because \( \delta > 0 \) was arbitrary small number.

Since the solution \( u \) is unique, in view of Remark 3 then \( u^\varepsilon \to u \) weakly in \( W^{1,p}(\overline{C_L}) \) and thus

\[
0 \leq \partial_x u(X), \quad X \in \{ u > 0 \} \cap \overline{C_L}.
\]

\[\square\]

**Proof of Theorem 3** Let \( N = 2 \) and \( \overline{\Omega} \) holds, we show that the free boundary is a continuous curve over \( \Omega \).
For \( x \in \Omega \) introduce the following height functions
\[
\begin{align*}
    h^+(x) &= \sup \{ z \text{ s.t. } u(x, z) = 0 \}, \\
    h^-(x) &= \inf \{ z \text{ s.t. } u(x, z) = 0 \}.
\end{align*}
\] (25)

If \( h^+(x_0) > h^-(x_0) \) for some \( x_0 \in \Omega \) then it follows from \( \partial_z u \geq 0 \) (see Lemma 10) that the free boundary contains a vertical segment of the form \( I_0 = \{ x_0 \} \times (a, b) \) for some \( a < b \). On \( I_0 \) we have that \( \partial_z u = 0 \). On the other hand the free boundary condition (7) is satisfied in the classical sense on \( I_0 \). Hence \( |u_x|^{p-2}u_x = 0 \) implying
\[
\begin{align*}
    \partial_z u = 0 & \quad \text{on } I_0.
\end{align*}
\] (26)

However, since there is a touching ball from \( \{ u > 0 \} \) at the points on \( \{ x_0 \} \times (a + \varepsilon, b - \varepsilon) \) for small \( \varepsilon > 0 \) then it follows from Hopf’s lemma (see the Appendix) that \( |\nabla u| \neq 0 \) which is in contradiction with (26).

7. Behaviour of \( \partial \{ u > 0 \} \) near the fixed boundary

In this section we show that at the contact points \( \partial \{ u > 0 \} \cap \Sigma \) u grows at most linearly and this is contained in Lemma 14. We begin with a simple observation

**Lemma 12.** If \( u \) is a weak solution of (1) with \( u_z \geq 0 \) then \( u \) is a weak solution of the following differential inequality
\[
\begin{align*}
    \Delta_p u - au_z &\geq 0 \quad \text{in } C_L.
\end{align*}
\] (27)

**Proof.** Indeed, for \( \zeta \geq 0, \zeta \in C_0^\infty(B) \), with some ball \( B \subset C_L \) it follows that
\[
\begin{align*}
    0 &= \int_{C_L} |\nabla u|^{p-2} \nabla u \nabla \zeta - \int_{C_L} \beta(u) \zeta \\
    &= \int_{C_L} |\nabla u|^{p-2} \nabla u \nabla \zeta - \int_{C_L} au \zeta - \int_{C_L} \ell \chi_{\{u>0\}} \zeta.
\end{align*}
\] (28)

Thus it is enough to show that \( \int_{C_L} \chi_{\{u>0\}} \zeta \leq 0 \). Let us choose a sequence \( \gamma_k(t), t \in \mathbb{R} \) such that \( \gamma'_k(t) \geq 0, r \in \mathbb{R} \) and \( \gamma_k \to \chi_{\{t>0\}} \) weak star. It follows that
\[
\begin{align*}
    \int_{C_L} \chi_{\{u>0\}} \zeta &= \lim_{k \to \infty} \int \gamma_k(u) \zeta \\
    &= - \lim_{k \to \infty} \int_B \gamma'_k(u)u_z \zeta \\
    &= - \lim_{k \to \infty} \int_B \gamma'_k(u)(u_x^+ - u_x^-) \zeta \\
    &= - \lim_{k \to \infty} \int_B \gamma'_k(u)u_x^+ \zeta \\
    &\leq 0
\end{align*}
\] (29)

where we used the notation \( v^+ = \max(v, 0) \), \( v^- = -\min(v, 0) \) and the last line follows from Lemma 10. \( \Box \)
The next lemma is true in all dimensions $N \geq 2$.

**Lemma 13.** Let $p > 2$ and $u \geq 0$ be a weak solution to $\Delta_p u = \partial_z \beta(u)$ in $C_L$. Then there is a constant $C_0 > 0$ depending only on $\ell, a, N, p$ and $\|u\|_{L^\infty(C_L)}$ such that

$$|\nabla u(X)| \leq C_0 \quad \forall X \in \partial\{u > 0\} \cap C_L.$$ 

**Proof.** Let $p > 2$ then if $X \in \partial\{u > 0\} \cap C_L$ then $d(X) = \text{dist}(X, \partial C_L) > 0$. For $r < \frac{d(X)}{2}$

$$\int_{B_r(X)} \left| \nabla u - \int_{B_r(X)} \nabla u \right|^2 \leq Cr^N$$

where $C$ depends on $a, \ell, p, \|u\|_\infty$ and $N$, see Lemma 7.

Let $\varepsilon > 0$ be small, for $\rho > \varepsilon$ and $X_0 \in \partial\{u > 0\}, B_{2\rho}(X_0) \subset C_L$ we have

$$\frac{1}{\rho^N} \int_{B_{\rho}(X_0)} u = \frac{1}{\varepsilon^N} \int_{B_{\rho}(X_0)} u + \int_{\varepsilon}^{\rho} \frac{d}{dt} \left( \frac{1}{t^N} \int_{B_t(X_0)} u \right) dt.$$ 

From the log-Lipschitz continuity of $u$ it follows that the Lebesgue Differentiation Theorem holds everywhere. Thus, it follows that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^N} \int_{B_{\varepsilon}(X_0)} u = 0$$

and consequently

$$\frac{1}{\rho^N} \int_{B_{\rho}(X_0)} u = \int_0^\rho \frac{d}{dt} \left( \frac{1}{t^N} \int_{B_t(X_0)} u \right) dt$$

$$= \int_0^\rho \frac{d}{dt} \left( \int_{B_t} u(X_0 + tX) dX \right) dt$$

$$= \int_0^\rho \frac{1}{t^N} \int_{B_t} \nabla u(X_0 + tX) \frac{X}{t} dX dt$$

$$= \int_0^\rho \frac{1}{t^N} \int_{B_t} \nabla u(X_0 + Y) \frac{Y}{t} dY dt$$

$$= \int_0^\rho \left( \frac{1}{t^N} \int_{B_t} \nabla u(X_0 + Y) - \int_{B_t(X_0)} \nabla u \right) \frac{X - X_0}{t} dX dt$$

$$\leq \int_0^\rho \left( \frac{1}{t^N} \int_{B_t(X_0)} \nabla u(X) - \int_{B_t(X_0)} \nabla u \right) dX dt$$

$$\leq C \rho$$

where the last line follows from Lemma 7. It remains to apply Harnack’s inequality in order to finish the proof. Let $v(X) = \frac{u(X_0 + \rho X)}{\rho}$, $X \in B_1$ then $\Delta_p v(X) = \partial_z (\beta(u(X_0 + \rho X)))$. By Lemma 12

$$\text{div}(|\nabla v|^{p-2} \nabla v) \geq a \rho v_z$$
Denote $A(v, \xi) = |\xi|^{p-2}\xi, B(\xi) = a\rho N$. Thus $v$ solves an inequality of the following form \( \text{div} A(v, \nabla v) \geq B(\nabla v) \) and

\[
|A(v, \xi)| \leq |\xi|^{p-1},
\]

\[
|B(\xi)| \leq \frac{|\xi|^p}{p} + \frac{(a\rho)^p}{p'}
\]

\[
A(v, \xi) \cdot \xi \geq \frac{p-1}{p} |\xi|^p
\]

where \( 1/p + 1/p' = 1 \). Thus $A$ and $B$ satisfy the structural conditions (3.5) in [16]. From the weak Harnack inequality, Corollary 3.10 [16] we infer that

\[
\sup_{B_2} v \leq C \left[ \sup_{\partial B_2} v + \kappa \right]
\]

where $C$ depends on $p, N$ and $\kappa$ depends on $p, \ell, N$ and $a$. Now the desired estimate follows from (32).

Observe that Lemma 13 is stronger than Theorem 2 since the constant $C_0$ does not depend on the distance of the point $X_0$ from $\Sigma$.

**Lemma 14.** Let $u \geq 0$ be as in Theorem 3, $N = 2 < p$. There is a constant $\mu > 0$ such that

\[
u(X) \leq \mu |X - X_0|
\]

for any $X_0 \in \partial\{u > 0\} \cap \Sigma$.

**Proof.** Let $X_0 \in \partial\Omega \times (0, L)$ and take $r$ small, say $r = \text{diam}\Omega/100$ such that $B_r(X_0) \cap \mathcal{C}_L$ is a half ball entirely in $\mathcal{C}_L$. Recall that $N = 2$ and therefore the lateral boundary of $\mathcal{C}_L$ is flat. Let $w$ be the solution to the following Dirichlet problem

\[
\begin{aligned}
\Delta_p w - aw_z &= 0 & & \text{in} & \mathcal{C}_L \cap B_r(X_0), \\
w &= u & & \text{on} & \partial(\mathcal{C}_L \cap B_r(X_0)).
\end{aligned}
\]

Since $g \in W^{2,2+w}(\Sigma)$ then we can apply the interior gradient estimates from [13] in the half ball $B_{r/2}^+ = \mathcal{C}_L \cap B_r(X_0)$ to infer the following estimate

\[
\sup_{B_{r/2}^+} |\nabla w| \leq C \frac{\sup_{B^+} w}{r} = C \frac{\sup_{\partial B^+} w}{r} \leq C(r)
\]

where $C(r)$ also depends on $\|g\|_{W^{2,2+w}}$. By (27) $u$ is a subsolution and hence we can apply the comparison principle Theorem 3.5.1 [17] and the boundary gradient estimate (34) in order to obtain $u \leq w \leq C|X - X_0|$ in $B_2^+(X_0)$ with tame constant $C > 0$ depending only on $a, \ell, \|g\|_{W^{2,2+w}}, r, p$. \( \square \)

8. CONCLUDING REMARKS

When the governing quasilinear equation is uniformly elliptic then the arguments can be considerably simplified. As an example let us consider the following operator [1]: Let $F(t)$ be a function in $C^{2,1}[0, \infty)$ satisfying

\[
F(0) = 0, \quad c_0 \leq F'(t) \leq C_0, \quad 0 \leq F''(t) \frac{C}{1 + t},
\]
for some positive constants $c_0, C_0$. From here we find that $f(\xi) = F(|\xi|^2)$ is convex and the following holds

\[(36) \quad \gamma |\xi|^2 \leq f(\xi) \leq \frac{1}{\gamma} |\xi|^2,\]

\[f_\xi(\xi) \cdot \xi \geq \gamma |\xi|^2,\]

\[\gamma |\eta|^2 \leq \sum_{ij} \frac{\partial^2 f(\xi)}{\partial \xi_i \partial \xi_j} \eta_i \eta_j \leq \frac{1}{\gamma} |\eta|^2,\]

where $f_\xi = \nabla_\xi f$ and $\gamma > 0$. The quasilinear operator $Lv = \text{div}(f_\xi(\nabla v))$ is now uniformly elliptic and Theorems 1-2 can be extended to the solutions of $L^z u = \partial_z (\beta(u))$ with less efforts. Moreover, by differentiating $Lu = au$ in $z$-direction we can see that $u_z$ solves a strictly elliptic operator and hence applying the comparison principle (which can be proved by a standard method discussed in [3]) one can infer that $u_z \geq 0$ provided that (8) and $u \geq 0$ hold.

Notwithstanding its simple form the equation $\Delta_p u = \partial_z (\beta(u))$ differs drastically from its strictly elliptic counterpart. This in particular includes:

- The strong comparison principle is not known for the $p-$Laplace structure as opposed to the case $p = 2$, see [3] Lemma 2.1. The reason is that one cannot define the corresponding strictly elliptic adjoint problem and hence Kamin’s argument cannot be generalised directly.

- When $p = 2$ then one can deduce that the solution $u \geq 0$ is non-degenerate at the free boundary points by using Biacchi’s transformation. This allows to transform the continuous casting problem to an obstacle like problem and apply the techniques developed for the latter for a class of divergence form elliptic equations [11]. This argument fails to work when $p \neq 2$ due to the nonlinear structure of the operator $\Delta_p u$.

It would be interesting to find out whether any of these difficulties can be circumvented which will lead to stronger free boundary regularity.

**Appendix**

8.1. The Caccioppoli inequality.

**Lemma 15.** Let $u$ be a weak solution of the equation $\Delta_p u = \partial_z (\beta(u)) + f$ in $B_1$, $f \in C(\overline{B_1})$. Then there is a constant $\Gamma$ depending only on $\text{sup}_{B_1} |u|, \text{sup}_{C_4} |f|, a, \ell, p$ and $N$ such that

\[(37) \quad \int_{B_\frac{4}{p}} |\nabla u|^p \leq \gamma.\]

**Proof.**

\[(38) \quad \int |\nabla u|^{p-2} \nabla u \nabla \zeta = \int \beta(u) \partial_z \zeta - \int f \zeta.\]

Let $\zeta = u \phi^p$ where $\phi \geq 0$ in $B_\frac{1}{4}, \phi = 0$ in $\mathbb{R}^N \setminus B_1, \phi = 1$ in $B_\frac{1}{4}$, and $|\nabla \phi| \leq C$ with some tame constant $C > 0$. Then we have

\[(39) \quad \int |\nabla u|^p \phi^p + |\nabla u|^{p-2} \nabla u p u \nabla \phi \phi^{p-1} = \int \beta(u) (\partial_z u \phi^p + p u \phi^{p-1} \partial_z \phi) - \int f u \phi^p.\]
Rearranging the order of integrals and after applying the Hölder inequality we get
\begin{equation}
\int |\nabla u|^p \phi^p \leq \int |\nabla u|^{p-1} |u| |\nabla \phi|^{p-1} + \int \beta(u)(|\partial_x u| \phi^p + p|u| \phi^{p-1} |\partial_x \phi|) + \int |f| |u| \phi^p \\
\leq \frac{\varepsilon^p}{p} \int |\nabla u|^p \phi^p + \frac{1}{\varepsilon^p} \int |u|^p |\nabla \phi|^p + \\
+ \beta(\sup_{B_{t}} |u|) \left[ \frac{\varepsilon^p}{p} \int |\nabla u|^p \phi^p + \frac{1}{\varepsilon^p} \int \phi^p + p \int |u| \phi^{p-1} |\nabla \phi|^p \right] + \int |f| |u| \phi^p.
\end{equation}
Here $p'$ is the conjugate of $p$. Choosing $\varepsilon$ small enough we infer the inequality
\begin{equation}
\int |\nabla u|^p \phi^p \leq \gamma \left[ \int |u|^p |\nabla \phi|^p + \int \phi^p + \int |u| \phi^{p-1} |\nabla \phi|^p \right] + \int |f| |u| \phi^p.
\end{equation}
where $\gamma$ depends on $\sup |u|, \sup_{\partial B} |f|, p, a, \ell$ and $N$. Now the required estimate follows from the fact that $\phi = 1$ in $B_{\frac{r}{2}}$. 

8.2. Hopf’s lemma.

**Lemma 16.** Let $u$ be a weak solution of $\Delta_p u - a \partial_x u = 0$ in $B_r$, $u \in C(\overline{B_r})$ such that $u \geq 0$ in $B_r$ and $u(X_0) = 0$ for some $X_0 \in \partial B_r$. Then
\[ \frac{\partial u(X_0)}{\partial \nu} < 0 \]
where $\nu$ is the unit outer normal at $X_0$. If the normal derivative does not exist then
\[ \limsup_{X \to X_0} \frac{u(X_0) - u(X)}{|X_0 - X|} < 0. \]

**Proof.** Let $b = \gamma(e^{-\lambda |X|^2} - e^{-\lambda r^2})$ then (see for instance [21])
\[ \Delta_p b = \gamma e^{-\lambda |X|^2} \left( 2 \gamma e^{-\lambda |X|^2} |X| \right)^{p-2} \left[ 4 \lambda (p-1) |X|^2 - 2(N + p - 2) \right] \]
Hence we have that
\[ \Delta_p b - a \partial_x b = \gamma e^{-\lambda |X|^2} \left( 2 \gamma e^{-\lambda |X|^2} |X| \right)^{p-2} \left[ 2 \lambda (p-1) |X|^2 - (N + p - 2) \right] + az \]
if we choose $\lambda$ sufficiently large, say $\lambda \geq \frac{2(N+p-2)}{p-1}$ and $\gamma = \frac{\inf_{B_{r/2}} u}{e^{-\lambda r^2/4} - e^{-\lambda r^2}}$ then we infer that $\Delta_p b - a \partial_x b \geq 0$. By comparison principle Theorem 3.5.1 [17] we have $b \leq u$ and consequently
\[ 0 > \limsup_{X \to X_0} \frac{b(X_0) - b(X)}{|X_0 - X|} \geq \limsup_{X \to X_0} \frac{u(X_0) - u(X)}{|X_0 - X|} \]
If the normal derivative exists then this becomes
\[ 0 > \frac{\partial b}{\partial \nu} \geq \frac{\partial u}{\partial \nu}. \]

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