Abstract. An explicit categorical equivalence is defined between a proper subvariety of the class of PMV-algebras, as defined by Di Nola and Dvurečenskij, to be called PMV\(_{f}\)-algebras, and the category of semi-low \(f_u\)-rings. This categorical representation is done using the prime spectrum of the MV-algebras, through the equivalence between MV-algebras and \(l_u\)-groups established by Mundici, from the perspective of the Dubuc–Poveda approach, that extends the construction defined by Chang on chains. As a particular case, semi-low \(f_u\)-rings associated to Boolean algebras are characterized.

Keywords: PMV-algebra, PMV\(_{f}\)-algebra, \(l_u\)-ring, Prime ideal, Spectrum.

1. Introduction

In this paper the categorical equivalence between the variety of PMV\(_{f}\)-algebras and the category of semi-low \(f_u\)-rings is described. The variety of PMV\(_{f}\)-algebras is defined as a subvariety of the MVW-rigs defined by Estrada [7]. The class of PMV\(_{f}\)-algebras is a variety of classical universal algebra, and is a proper subvariety of the PMV-algebras defined by Di Nola and Dvurečenskij [4]. On the other hand, the variety of commutative unitary PMV-algebras studied by Montagna [8], to be called in this paper PMV\(_1\)-algebras, is a proper subvariety of the PMV\(_{f}\)-algebras. The MVW-rigs contain strictly the variety of PMV-algebras, because every MV-algebra with the infimum as product, is an MVW-rig (Proposition 4.8), and it can happen that it is not a PMV-algebra; for example, the Łukasiewicz MV-algebras or the MV-algebra [0, 1].

The equivalence between the category of PMV\(_{f}\)-algebras and the category of semi-low \(f_u\)-rings is established based on the equivalence proved by Mundici [10], but applying the construction introduced by Dubuc–Poveda.
[5,6], since it does not require the *good sequences*, and relies in the representation of any MV-algebra as a subdirect product of totally ordered MV-algebras, that will be called from here on chain MV-algebras or MV-chains. This representation only requires the prime spectrum of an MV-algebra and the equivalence between chain MV-algebras and the chain l_u-groups, established by Chang [2].

It is proved that for the representation established in this paper, it is enough with the prime spectrum of the subjacent MV-algebra, since every PMV_f-algebra A is a PMV-algebra that satisfies that \( xy \leq x \land y \), and \( x(y \ominus z) = xy \ominus xz \), for every \( x, y, z \in A \), and every prime ideal of the subjacent MV-algebra is an ideal of the PMV_f-algebra A.

This construction finds explicit representations for the rings associated to notable examples of PMV_f-algebras. For example, the MV-algebra \([0, 1]\) with the usual product, the MV-algebra of the functions from \([0, 1]\) to \([0, 1]\) with the usual product, or the PMV_f-algebra of boolean algebras with product defined by the infimum. In this representation, the semi-low \( f_u \)-ring associated to the boolean algebra \( 2^n \) is precisely the ring \( \mathbb{Z}^n \).

In Section 2, the preliminary concepts about MV-algebras are presented. In Section 3, the MV-algebras with product are defined, and in that context, the varieties of PMV_1, PMV_f, PMV-algebras and MVW-rigs. Some properties of the MVW-rigs are presented, with examples that illustrate the independence of the axioms chosen. Besides, it is shown that the inclusions between the categories are strict. In Section 6 the semi-low \( l_u \)-rings are presented, (Definition 6.9). In Section 7 we find the key results of this paper, Theorem 7.8, where the distributive property of the product for PMV_f-chains is proven, and as well as the main result of this paper, the construction of the equivalence is extended to the category of PMV_f-algebras with product and the category of semi-low \( f_u \)-rings. Finally, in Section 9, some consequences of the equivalence are presented, and in particular the construction of the ring associated to the boolean algebras is sketched.

2. MV-Algebras

Some properties of the theory of MV-algebras are presented, that are relevant to this work. The reader can find more complete information in [3].

**Definition 2.1. (MV-algebra)** An MV-algebra is a structure \((A, \oplus, \neg, 0)\) such that \((A, \oplus, 0)\) is a commutative monoid and the operation \(\neg\) satisfies:
(i) \(-(-x) = x\),
(ii) \(x \oplus -0 = -0\),
(iii) \(-(-x \oplus y) \oplus y = -(y \oplus x) \oplus x\).

Because of properties of MV-algebras, \(0 \leq a \leq u\) for all \(a \in A\), with \(u = -0\). The operation \(-\) is called **negation**, while the operation \(\oplus\) is called **sum**.

**Affirmation 2.2.** (Order) Every MV-algebra \(A\) is ordered by the relation,

\[ x \leq y \text{ if and only if } x \ominus y = 0, \text{ for all } x, y \in A. \]

**Definition 2.3.** (Homomorphism) Given two MV-algebras \(A\) and \(B\), a function \(f : A \rightarrow B\) is a homomorphism of MV-algebras if for every \(x, y\) in \(A\):

(i) \(f(0) = 0\),
(ii) \(f(x \oplus y) = f(x) \oplus f(y)\),
(iii) \(f(-x) = -(f(x))\).

**Definition 2.4.** (Ideal of an MV-algebra) A non-empty subset \(I\) of an MV-algebra \(A\) is an ideal if and only if:

(i) If \(a \leq b\) and \(b \in I\), then \(a \in I\).
(ii) If \(a, b \in I\), then \(a \oplus b \in I\).

The set of all ideals of the MV-algebra \(A\) will be denoted by \(Id(A)\).

**Definition 2.5.** (Prime ideal of an MV-algebra) An ideal \(P\) of an MV-algebra \(A\), is prime if for all \(a, b \in A\), \(a \land b \in P\) implies \(a \in P\) or \(b \in P\).

The set of all prime ideals of the MV-algebra \(A\) will be called \(Spec(A)\), the spectrum of \(A\).

**Theorem 2.6.** (Chang Representation Theorem [2]) Every non trivial MV-algebra is isomorphic to a subdirect product of MV-chains.

### 3. MV-Algebras with Product

**Definition 3.1.** An MV-algebra with product is a structure \((A, \oplus, \cdot, -, 0)\) such that \((A, \oplus, -, 0)\) is an MV-algebra, and \((A, \cdot)\) is a semigroup.

The operation \(\cdot\) is called **product**, and the notation used is:

\[ a \cdot a \cdot \ldots \cdot a = a^n. \]

\(n\)-times
Next, four varieties of $MV$-algebras with product are defined, namely the $MVW$-rigs, the $PMV$-algebras, the $PMV_f$-algebras and the unitary $PMV_1$-algebras. Some of their properties are proved and in particular, we show that there is a order relation among them.

From here on, all products are supposed to be commutative.

**Definition 3.2. ($MVW$-rig [7, 2.4])** An $MVW$-rig $(A, \oplus, \cdot, \neg, 0)$ is an $MV$-algebra with product such that

(i) $a0 = 0a = 0$,
(ii) $(a(b \oplus c)) \ominus (ab \oplus ac) = 0$,
(iii) $(ab \ominus ac) \ominus (a(b \ominus c)) = 0$.

**Observation 3.3.** For every $a, b, c \in A$, axiom (ii) is equivalent to

$$a(b \oplus c) \leq ab \oplus ac$$

and axiom (iii) is equivalent to

$$ab \ominus ac \leq a(b \ominus c).$$

**Definition 3.4.** An $MVW$-rig $A$ is called unitary if there exists an element $s$ with the property that for every $x$ in $A$ $sx = xs = x$. It is follows that $s$ is unique.

**Definition 3.5. ($PMV$ [4])** A $PMV$-algebra $A$ is an $MV$-algebra with product such that for every $a, b, c \in A$: (i) $a \circ b = 0$ implies $ac \circ bc = 0$; (ii) $a \circ b = 0$ implies $c(a \oplus b) = ca \oplus cb$.

In [4, Theorem 3.1], it is shown that the class $PMV$ is equationally definable.

**Definition 3.6. ($PMV_f$)** A $PMV_f$-algebra is an $MVW$-rig such that for every $a, b, c \in A$, $ab \leq a \land b$, and $a(b \ominus c) = ab \ominus ac$.

**Definition 3.7. ($PMV$-unitary algebra [9])** A $PMV$-unitary algebra is an $MV$-algebra $A$ with product such that for every $a, b, c \in A$, $au = a$, and $a(b \ominus c) = ab \ominus ac$.

**Theorem 3.8.** The following inclusions hold:

$$PMV_1 \subset PMV_f \subset PMV \subset MVW$$

**Proof.** The first inclusion, $PMV_1 \subset PMV_f$, follows from [9, Lemma 2.9, iii] and Example 5.2.

For the second inclusion, given $a, b, c \in PMV_f$, if $a \circ b = 0$, since $ac \leq a$ and $bc \leq b$ then $ac \circ bc \leq a \circ b = 0$. On the other hand, $a \circ b = 0$
implies \( a \leq u \odot b \) and therefore \( ca \leq c(u \odot b) \leq cu \), and this last inequality implies [9, 2.7–vii] \( c(b \oplus a) = c(u \ominus ((u \ominus b) \oplus a)) = cu \ominus (c(u \ominus b) \ominus ca) = (cu \ominus c(u \ominus b)) \ominus ca = cb \ominus ca \). To see that it is strict inclusion, it is enough to consider [4, Example 3.6, 2], where every finite MV-algebra \( M \), with two unique atoms \( a, b \) with orders \( n > 1 \) and \( n^2 \) respectively admits a product, such that \( aa = b \) and \( ab = ba = bb = 0 \). Thus \( M \) is a PMV-algebra and it is not a \( PMV^f \)-algebra, because \( aa = b \nleq a \).

The inclusion \( PMV \subset MVW \)-rig is proven in Proposition 9.6. To see that it is a strict inclusion, see Example 4.6.

4. Examples and Properties of the \( MVW \)-Rigs

Example 4.1. Every MV-algebra with the product defined by \( ab = 0 \), for all \( a, b \in A \), is an \( MVW \)-rig.

Example 4.2. The MV-algebra \([0,1]\) with the usual multiplication inherited from \( \mathbb{R} \) is a commutative \( MVW \)-rig with unitary element \( u = 1 \).

Example 4.3. The MV-algebra \([0, u]\) of real numbers with \( 0 \leq u < 1 \) is a commutative \( MVW \)-rig, but it is not unitary.

Example 4.4. The MV-algebra of the continuous functions from \([0,1]^n\) to \([0,1]\) with the usual product of functions is an \( MVW \)-rig with the property: \( xy \leq x \land y \).

Example 4.5. [7, 2.10] The algebra \( \widetilde{L}_n = \{ \frac{m}{n^k} \in \mathbb{Q} \cap [0,1] | k, m \in \mathbb{N} \} \) obtained by closing the Lukasiewicz algebra \( L_n \) under products, where \( L_n = \langle \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}, \oplus, \neg \rangle \) with the usual product, is an \( MVW \)-rig.

Example 4.6. [7, 2.11] \( \mathbb{Z}_n = \{0, 1, \ldots, n\} \) with \( n \in \mathbb{N} \), \( x \oplus y = \min \{n, x+y\}, \neg x = n - x \) and \( xy = \min \{n, x \cdot y\} \), is an \( MVW \)-rig where sum and product are the usual operations on the natural numbers.

\( \mathbb{Z}_n \) is unitary and \( \neg 0 \neq 1 \). The cancellation law does not hold, because the product of two elements can be larger than their supremum. In some cases, the strict inequality in (2) holds, even though the equality (1) is always true.

For example, in \( \mathbb{Z}_{10} \), \( 2(7 \odot 6) = 2[-(7 \oplus 6)] = 2[-(3 \oplus 6)] = 2[-9] = 2(1) = 2 \succ (2)(7) \oplus (2)(6) = 10 \oplus 10 = 0 \).

\( \mathbb{Z}_n \) is not always a \( PMV \) algebra either, because for example in \( \mathbb{Z}_{10} \), \( (3)(2) \odot (3)(2) = 6 \odot 6 = 2 \), even though \( 2 \odot 2 = 0 \).
Example 4.7. \( \hat{L}_{n+1} = \langle \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}, \oplus, \cdot, \neg, 0, 1 \rangle \), with product defined by \( \frac{x \cdot y}{n} = \min\{n, x \cdot y\} \), for \( x, y \in \{0, 1, \ldots, n\} \), is an MVW-rig isomorphic to \( \mathbb{Z}_n \), and the isomorphism is defined by \( \varphi : \mathbb{L}_{n+1} \longrightarrow \mathbb{Z}_n, \frac{x}{n} \mapsto x \).

Proposition 4.8. Every MV-algebra \( A \) with product defined by the infimum \( x \cdot y = x \land y \) is an MVW-rig.

Proof. Because the product is defined in terms of the infimum, by the Chang representation theorem, it is enough to show that the result holds for every MV-chain. Since the infimum is associative and commutative, it is enough to prove the inequalities (1) and (2). Consider \( a, b, c \in A \) an MV-chain. If \( b \oplus c \leq a \) then \( b \oplus c = a \land (b \oplus c) \leq b \oplus c = (a \land b) \oplus (a \land c) \). If on the contrary \( a \leq b \oplus c \), and \( a \leq b, c \), then \( a = a \land (b \oplus c) \leq a \land b = (a \land b) \oplus (a \land c) \). If \( a \leq b \oplus c \) and \( b \leq a \leq c \), then \( a = a \land (b \oplus c) \leq a \land a = (a \land b) \oplus (a \land c) \). Similarly it can be proved that \( a \land (b \ominus c) \geq (a \land b) \ominus (a \land c) \).

Observation 4.9. Even though every MV-algebra is an MVW-rig with the product given by the infimum, in general it is not a PMV-algebra, as is shown in the next example.

The Lukasiewics MV-algebra \( \mathbb{L}_4 \) with the product defined by the infimum is not a PMV-algebra because \( \frac{1}{3} \ominus \frac{1}{3} = 0 \) and \( \frac{1}{3} = \frac{1}{3} \land \left( \frac{1}{3} \oplus \frac{1}{3} \right) < \frac{1}{3} \oplus \frac{1}{3} = \frac{2}{3} \).

Proposition 4.10. Every MV-algebra \( A \) with product defined by the supremum for non-zero elements, namely, \( ab = a \lor b \), if \( a \neq 0 \) and \( b \neq 0 \) and zero otherwise, is an MVW-rig.

Proof. It is enough to show (1) and (2) for totally ordered MVW-rigs.

Example 4.11. An interesting and relevant particular case of the Proposition is when \( A \) is a boolean algebra. A boolean algebra \( A \) can be considered as an MV-algebra, where the sum is given by the supremum and negation is the complement. If the product is defined as in the Propositions 4.8 or 4.10, every boolean algebra is naturally an MVW-rig.

Proposition 4.12. Axiom (iii) in Definition 3.2, is independent of the other axioms for MVW-rig. Similarly, axiom (i) is independent of the others.

Proof. Consider the Lukasiewicz MV-algebra \( \mathbb{L}_4 \) with product defined by:

\[
a \cdot b = \begin{cases} 
0 & \text{if } a = 0 \text{ or } b = 0, \\
 a \oplus b & \text{if } a \ominus b = 0, \\
 a \ominus b & \text{if } a \ominus b \neq 0.
\end{cases}
\]
In this structure axiom (iii) does not hold, but the others do. The product is equivalent to the sum on the integers mod 3, \( \mathbb{Z}_3 \) for the elements of \( L_4 - \{0\} \). Therefore, this product is associative and commutative.

On the other hand, every MV-algebra with the supremum as product is a model for all the axioms of MVW-rigs, except for axiom (i). The proof is similar to the one given in Proposition 4.10.

**Proposition 4.13.** [7, 2.5] For every \( a, b, c \in A \) a commutative MVW-rig the following properties hold:

(i) If \( a \leq b \) then \( ac \leq bc \),

(ii) \( u^2 \leq u \),

(iii) \( a \leq b \) and \( c \leq d \) then \( ac \leq bd \).

**Proof.** Property (iii) follows from (i); in fact, \( a \leq b \) and \( c \leq d \) imply that \( ac \leq bc \) and \( cb \leq db \).

**Ideals and Homomorphisms of MVW-Rigs**

**Definition 4.14.** Given MVW-rigs \( A \) and \( B \), a function \( f : A \rightarrow B \) is a homomorphism of MVW-rigs in and only if

(i) \( f \) is a homomorphism of MV-algebras and

(ii) \( f(ab) = f(a)f(b) \).

**Definition 4.15.** The kernel of a homomorphism \( \varphi : A \rightarrow B \) of MVW-rigs is

\[
ker(\varphi) := \varphi^{-1}(0) = \{ x \in A | \varphi(x) = 0 \}.
\]

**Definition 4.16.** An ideal of an MVW-rig \( A \) is a subset \( I \) of \( A \) that has the following properties:

(i) \( I \) is an ideal of the subjacent MV-algebra \( A \).

(ii) Given \( a \in I \), and \( b \in A \), \( ab \in I \) (Absorbent Property).

\( Id_w(A) \) denotes the set of all ideals of the MVW-rig \( A \).

**Example 4.17.** (Boolean Algebras) Every boolean algebra is an MVW-rig, taking the supremum as the sum and the infimum as the product. The ideals of this MVW-rig are the ideals of the MV-algebra, that are at the same time ideals for the lattice.

**Observation 4.18.** Note that in Proposition 4.10, the MVW-rig has no proper non trivial ideals. Its only ideals are zero and the MVW-rig.
Definition 4.19. (Prime ideal of an MVW-rig) An ideal $P$ of an MVW-rig, is called prime if for every $a, b \in A$, $ab \in P$ implies $a \in P$ or $b \in P$.

The set of all prime ideals of the MVW-rig $A$ is denoted $\text{Spec}_w(A)$.

Proposition 4.20. [3, 7] There is a bijective correspondence between the set of all ideals of an MVW-rig $A$ and the set of its congruences. Namely, given an ideal $I$ of the MVW-rig $A$, the binary relation defined by $x \equiv_I y$ if and only if $(x \oplus y) \oplus (y \ominus x) \in I$ is a congruence relation, and given any congruence relation $\equiv$ on $A$, the set $\{x \in A \mid x \equiv 0\}$ is an ideal of $A$.

Because it is relevant, the proof of the compatibility of the product is reproduced. The full proof can be found on [7, 2.29].

Proof. Since $A$ is an MV-algebra and $I$ is an MV-ideal, $a \equiv_I b$ and $c \equiv_I d$ imply $a \oplus c \equiv_I b \oplus d$ and $\neg a \equiv_I \neg b$, [3, 1.2.6]. It is left then to prove that $a \equiv_I b$ and $c \equiv_I d$ imply $ac \equiv_I bd$.

Given $a \equiv_I b$ and $c \equiv_I d$ then $a \oplus b \in I$ and $c \ominus d \in I$ respectively, so $ac \leq (a \oplus b)(c \ominus d) = ((a \oplus b) \oplus b)((c \ominus d) \oplus d) \leq (a \oplus b)((c \ominus d) \oplus d) \oplus b((c \ominus d) \oplus d) \leq (a \oplus b)(c \ominus d) \ominus (a \oplus b)d \ominus b(c \ominus d) \ominus bd$. Equivalently

$$ac \ominus bd \leq (a \oplus b)(c \ominus d) \ominus (a \oplus b)d \ominus b(c \ominus d) \in I,$$

because $(a \oplus b)$ and $(c \ominus d) \in I$ and $I$ is absorbent, $ac \ominus bd \in I$. Similarly $bd \ominus ac \in I$, then $(ac \ominus bd) \ominus (bd \ominus ac) \in I$, and therefore $ac \equiv_I bd$. \hfill \blacksquare

Observation 4.21. For $a \in A$, the equivalence class of a respect to $\equiv_I$ will be denoted by $[a]_I$ and the quotient set $A/\equiv_I$ by $A/I$.

Since $\equiv_I$ is a congruence, the operations $\neg [a]_I = [-a]_I$, $[a]_I \oplus [b]_I = [a \oplus b]_I$ and $[a]_I[b]_I = [ab]_I$ are well defined over $A/I$.

Proposition 4.22. [7, 2.31] If $I \in \text{Id}_w(A)$, then $A/I$ is an MVW-rig.

Corollary 4.23. Consider $I \in \text{Spec}(A)$ and $A$ an MVW-rig. If $I$ is absorbent, then $A/I$ is a totally ordered MVW-rig.

5. Examples and Properties of the $PMV_f$-Algebras

Example 5.1. The MV-algebra $[0, 1]$ with the usual multiplication inherited from $\mathbb{R}$ is a $PMV_f$-algebra.

Example 5.2. The MV-algebra $[0, u]$ with the usual multiplication , and $0 < u < 1$, is a non-unitary $PMV_f$-algebra.
Example 5.3. The set of all continuous functions from \([0, 1]^n\) to \([0, 1]\) with truncated sum and the usual multiplication is a \(PMV_f\)-algebra.

Example 5.4. \(F[x_1, \ldots, x_n] \) the set of all continuous functions from \([0, 1]^n\) to \([0, 1]\), that are constituted by finite polynomials in \(\mathbb{Z}[x_1, \ldots, x_n]\), namely, \(f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n] \iff \exists p_1, \ldots, p_k \in \mathbb{Z}[x_1, \ldots, x_n]\) such that for all \(z \in [0, 1]^n\), \(f(z) = p_i(z)\), for some \(i \in \{1, \ldots, n\}\), is a \(PMV_f\)-algebra.

Example 5.5. Every boolean algebra, as in the Example 4.17, is a \(PMV_f\)-algebra.

Example 5.6. Every \(MV\)-algebra with multiplication defined by the infimum is an \(MVW\)-rig, not necessarily \(PMV_f\)-algebra, as was established on Example 4.8.

Proposition 5.7. Given a \(PMV_f\)-algebra \(A\), \(Id_W(A) = Id(A)\) and furthermore \(Spec_W(A) \subseteq Spec(A)\).

Proof. By definition, \(Id_W(A) \subseteq Id(A)\). Additionally, given \(I \in Id(A)\) and \(a \in I\), for all \(c \in A\), \(ac \leq a \land c \in I\), so \(I \in Id_W(A)\). On the other hand, given \(I \in Spec_W(A)\) and \(a, b \in A\) such that \(a \land b \in I\), then \(ab \in I\) because \(ab \leq a \land b\). Consequently \(a \in I\) or \(b \in I\), therefore \(I \in Spec(A)\).

Proposition 5.8. If \(I \in Id_W(A)\) and \(A\) is a \(PMV_f\)-algebra, then \(A/I\) is a \(PMV_f\)-algebra.

Proof. It follows directly from Proposition 4.20 and Proposition 4.22.

6. \(l_u\)-Rings

Definition 6.1. \((l\text{-group } [1,3])\) An \(l\)-group \(G\) is a lattice abelian group \((G, +, -, 0)\), such that, the order \(<\) is compatible with the sum.

Definition 6.2. For each \(x\) in an \(l\)-group \(G\), its absolute value is defined by \(|x| = x^+ + x^-\), where \(x^+ = x \lor 0\) is the positive part of \(x\) and \(x^- = -x \lor 0\) is the negative part.

Definition 6.3. A strong unit \(u\) of an \(l\)-group \(G\) is an element \(u\) such that \(0 \leq u \in G\) and for all \(x \in G\) there exists an integer \(n \geq 0\) with \(|x| \leq nu\).

An \(l\)-group with strong unit \(u\) will be called an \(l_u\)-group.

Definition 6.4. \((l\text{-ideal})\) An \(l\)-ideal of an \(l\)-group \(G\) is a subgroup \(J\) of \(G\) that satisfies: if \(x \in J\) and \(|y| \leq |x|\) then \(y \in J\).
Definition 6.5. \textit{(l-prime ideal)} An \textit{l-ideal} $P$ of an \textit{l_u-group} $G$, is prime if and only if $G/P$ is a chain.

The set of all \textit{l-prime ideals} of $G$ is called the spectrum of $G$ and denoted by $\text{Spec}_l(G)$.

Definition 6.6. \cite[XXVII.1]{1} An \textit{l_u-ring} is a ring $R = (\|R|, +, \cdot, \leq, u)$ such that $(R, +, \leq, u)$ is an \textit{l_u-group} and, $0 \leq x, 0 \leq y$ implies $0 \leq xy$, where $\|R|$ denotes the subjacent set.

From this point on, all rings will be assumed to be commutative.

Definition 6.7. \textit{(L-ideal [1, XVII.3])} An \textit{L-ideal} $I$ from an \textit{l_u-ring} $R$ is an \textit{l_u-ideal} such that for every $y \in I$ and $x \in R$, $xy \in I$. $I$ is called irreducible if and only if $R/I$ is totally ordered.

The set of all \textit{L-ideals} of $R$ is called $\text{Id}(R)$ and the set of all \textit{l_u-ideals} of the subjacent group is called $\text{Id}_g(R)$.

Definition 6.8. \textit{(Low l-ring [8])} An \textit{l_u-ring} is called low if and only if, for all $x, y \geq 0 \in R$ we have that $xy \leq x \wedge y$.

Definition 6.9. \textit{(Semi-low l_u-ring)} A \textit{l_u-ring} $R$ is semi-low if and only if, for all $a, b \in [0, u]$, $ab \leq a \wedge b$.

Theorem 6.10. Given an \textit{l-ring} $R$ and $u \in R$, $u > 0$, and the segment $[0, u] = \{a \in R \mid 0 \leq a \leq u\}$ with $[0, u]^s \subset R$, the subring generated by $[0, u]$, then:

(a) For every $A \subset [0, u]$, a \textit{PMV}_f-algebra, $A^s \subset [0, u]^s$, the subring generated by $A$, $A^s$ is a semi-low \textit{l_u-ring} with strong unit $u$ and $A = \Gamma(A^s, u) := \{x \in A^s \mid 0 \leq x \leq u\}$.

(b) Every semi-low \textit{l_u-ring} is generated by its segments,

$$[0, u]^s = \{x \in R \mid \exists n \in \mathbb{N}, |x| \leq nu, \}.$$ 

Proof. (a) Given a \textit{PMV}_f-algebra $A = \langle |A|, \oplus, \cdot, \neg, 0 \rangle$, then $A = \langle |A|, \oplus, \neg, 0 \rangle$ is an \textit{MV}-algebra; call $A^s$ the associated \textit{l_u-group}. Then the subjacent sets are equal $|A^s| = |A^s|$, because for every $a \in A^s$, $a = \sum \epsilon_i b_i c_i + \sum \delta_j d_j$, with $b_i, c_i, b_i c_i, d_j \in A$ and $\epsilon_i, \delta_j \in \{1, -1\}$, is a sum of elements of $A$. Therefore $A = \Gamma(A^s, u)$ because of \cite[Theorem 1.2, a]{5}. On the other hand, $x, y \in A^s \cap [0, u]$ implies $x, y \in A y xy \leq x \wedge y$.

(b) From \cite[Theorem 1.2, b]{5}, it follows that

$$[0, u]^s = \{x \in R \mid \exists n \in \mathbb{N}, |x| \leq nu\}$$
is an $l_u$-group with strong unit $u$, and for the reasons exposed above, the subjacent sets are $|R| = |[0, u]^*| = |[0, u]^2|$, so $R = [0, u]^2$.

7. Categorical Equivalence Between $\mathcal{CPMV}_f$ and $\mathcal{CLR}_u$

**Definition 7.1.** $\mathcal{PMV}_f$ and $\mathcal{CPMV}_f$ are the categories whose objects are $PMV_f$-algebras and $PMV_f$-chains and morphisms are homomorphisms between them respectively.

**Definition 7.2.** $\mathcal{LR}_u$ and $\mathcal{CLR}_u$ are the categories whose objects are semi-$l_u$-rings and chain semi-$l_u$-rings, and morphisms are homomorphisms between them respectively.

**Theorem 7.3.** (Chang’s construction of the $l_u$-group $A^*$ [2, Lemma 5]) Given an $MV$-chain $A$, $A^* = \langle \mathbb{Z} \times A, +, \leq \rangle$ together with the operations $(m + 1, 0) = (m, u)$, $(-m - 1, -a) = -(m, a)$, and $(m, a) + (n, b) = (m + n, a \oplus b)$ if $a \oplus b \neq u$, or $(m, a) + (n, b) = (m + n + 1, a \circ b)$ if $a \oplus b = u$, is a chain $l_u$-group with strong unit $u = (1, 0) = (0, u)$, where the order $\leq$ is given by $(m, a) \leq (n, b)$ if and only if $m < n$ or $m = n$ and $a \leq b$.

**Observation 7.4.** Denote $|A^*| = \mathbb{Z} \times A$.

The functor $(-)^\sharp: \mathcal{CPMV}_f \to \mathcal{CLR}_u$

**Definition 7.5.** Given a $PMV_f$-chain $A$, the structure $A^\sharp$ is defined following the Chang’s construction [2] as follows. $A^\sharp = \langle |A^*|, +, u, \leq \rangle$, with $\langle |A^*|, +, u, \leq \rangle$, its associated $l_u$-group where the product is defined by $(m, a) \cdot (n, b) := mn(0, u^2) + m(0, bu) + n(0, au) + (0, ab)$.

with $n(0, x) = \underbrace{(0, x) + \ldots + (0, x)}_{n \text{-times}}$ for $n \geq 0$ and $n(0, x) = -\underbrace{(0, x) - \ldots - (0, x)}_{n \text{-times}}$ for $n < 0$.

**Proposition 7.6.** The product defined above is well defined.

**Proof.** Note that in $A^*$, $(m + 1, 0) = (m, u)$; it is enough then to observe directly from the definition of the product that $(m, u) \cdot (n, b) = (m + 1, 0) \cdot (n, b)$.

**Affirmation 7.7.** For every $x, y, z \in A$, $(0, x)(0, y \circ z) = (0, xy) + (0, xz) - (0, xu)$.
PROOF. The equality follows directly from Theorem 3.8 and Definition 7.5, if \( y \odot z = 0 \). On the other hand, \( y \odot z \neq 0 \iff -y \oplus -z \neq u, \iff -y \odot -z = 0 \), implies

\[
(0, x) (0, y \odot z) = (0, x) (0, -(y \oplus z)) \\
= (0, x) [-(1, -(y \oplus z))] \\
= -(0, x) (1, -(y \oplus z)) \\
= -(0, x) [(0, -y) + (1, -z)] \\
= -(0, x) (0, -y) - (0, x) (1, -z) \\
= (0, x) [-(0, -y)] + (0, x) [-(1, -z)] \\
= (0, x) (1, -y) + (0, x) (0, z) \\
= (0, xy) - (0, xu) + (0, xz) .
\]

\[\square\]

THEOREM 7.8. Given a PMV\(_f\)-chain \( A \), \( A^z \) is a chain semi-low \( l_u \)-ring.

PROOF. It is clear that \( A^z \) with the sum operation and the associated order is a chain \( l_u \)-group. It is enough to show that with the product given by Definition 7.5, it is a semi-low ring.

For every \((m, x), (n, y), (s, z) \in A^z\), the following properties hold:

Distributivity

\[
(m, x) [(n, y) + (s, z)] = (m, x) (n, y) + (m, x) (s, z).
\]

Because of Theorems 3.8 and 7.3, \( z \odot y = 0 \) implies \( xz \odot xy = 0 \), and \( x(z \oplus y) = xz \oplus xy \), so, \((m, z) + (n, y) = (m + n, z \oplus y) \) and \((m, xz) + (n, xy) = (m + n, xz \oplus xy) \).

This affirmation will be proved dividing the proof in two cases.

**Case 1.** \( y \odot z = 0 \).

\[
(m, x) [(n, y) + (s, z)] = (m, x) [(n + s, y \oplus z)] \\
= m(n + s)(0, u^2) + m(0, (y \oplus z)u) + (n + s)(0, xu) \\
+ (0, x(y \oplus z)) \\
= mn(0, u^2) + ms(0, u^2) + m(0, yu \oplus zu) + n(0, xu) \\
+ s(0, xu) + (0, xy \oplus xz) \\
= mn(0, u^2) + ms(0, u^2) + m(0, yu) + m(0, zu) \\
+ n(0, xu) + s(0, xu) + (0, xy) + (0, xz) \\
= [mn(0, u^2) + m(0, yu) + n(0, xu) + (0, xy)]
\]
\[ + [ms(0, u^2) + m(0, zu) + s(0, xu) + (0, xz)] \]
\[ = (m, x)(n, y) + (m, x)(s, z). \]

From Theorem 7.3, \( y \odot z \neq 0 \) implies, \( (n, y) + (s, z) = (n + s + 1, y \odot z) \), and form Affirmation 7.7, \( (0, x)(0, y \odot z) = (0, x(y \odot z)) = (0, xy) + (0, xz) - (0, xu) \).

**Case 2.** \( y \odot z \neq 0 \)

\[
(m, x)[(n, y) + (s, z)] = (m, x)[(n + s + 1, y \odot z)] \\
= m(n + s + 1)(0, u^2) + m(0, (y \odot z)u) \\
+ (n + s + 1)(0, xu) + (0, x(y \odot z)) \\
= m(n + s + 1)(0, u^2) + m[0, yu + (0, zu) - (0, u^2)] \\
+ (n + s + 1)(0, xu) + [(0, xy) + (0, xz) - (0, xu)] \\
= mn(0, u^2) + ms(0, u^2) + m(0, u^2) + m(0, yu) \\
+ m(0, zu) - m(0, u^2) + n(0, xu) + s(0, xu) \\
+ (0, xu) + (0, xy) + (0, xz) - (0, xu) \\
= [mn(0, u^2) + m(0, yu) + n(0, xu) + (0, xy)] \\
+ [ms(0, u^2) + m(0, zu) + s(0, xu) + (0, xz)] \\
= (m, x)(n, y) + (m, x)(s, z).
\]

**Associativity**

\[
(m, x)[(n, y)(s, z)] = (m, x)[ns(0, u^2) + n(0, zu) + s(0, yu) + (0, yz)] \\
= mn(0, u^3) + ns(0, xu^2) + mn(0, zu^2) + n(0, xzu) \\
+ ms(0, yu^2) + s(0, xyu) + m(0, yzu) + (0, xyz) \\
= mn(0, u^3) + mn(0, zu^2) + ms(0, yu^2) + m(0, zyu) \\
+ ns(0, xu^2) + n(0, xzu) + s(0, xyu) + (0, xyz) \\
= [mn(0, u^2) + m(0, yu) + n(0, xu) + (0, xy)](s, z) \\
= [(m, x)(n, y)](s, z).
\]

Given \( (0, 0) \leq (m, x) \) and \( (0, 0) \leq (n, y) \) it is clear that \( 0 \leq m \) and \( 0 \leq n \), so

\[
(0, 0) \leq mn(0, u^2) + m(0, yu) + n(0, xu) + (0, xy) = (m, x)(n, y).
\]

Now it can be proved that \( A^2 \) is semi-low.
Given $(0,0) \leq (m,x), (n,y) \leq (0,u)$ it must be that $m = n = 0$, so
\[(0,x)(0,y) = (0,xy) \leq (0,x \land y) = (0,x) \land (0,y).\]

\[\text{Corollary 7.9.} \text{ For every } PMV_f\text{-chain } A, \]
\[\left(\sum_{i=1}^{n}(0,x_i)\right) \left(\sum_{i=1}^{n}(0,y_i)\right) = \sum_{i=1}^{n}(0,x_iy_i),\]
in $A^\sharp$.

\[\text{Proposition 7.10.} (--)^\sharp \text{ is functorial.} \]

For $h : A \to B$ in $\mathcal{CPMV}_f$, define $h^\sharp : A^\sharp \to B^\sharp$ en $\mathcal{CLR}_u$ as follows: $h^\sharp(m,a) := (m,h(a))$. By construction $[5, 2.2]$, $h^\sharp$ is a homomorphism of $l_u$-groups, so it is enough to prove that $h^\sharp$ is a homomorphism of $l_u$-rings. This follows directly from Definition 7.5. Namely, $h^\sharp[(m,a)(n,b)] = h^\sharp(m,a)h^\sharp(n,b)$.

Besides, given $(m,a) \in A^\sharp$, $(gh)^\sharp(m,a) = (m,gh(a)) = g^\sharp(m,h(a)) = g^\sharp \circ h^\sharp(m,a)$.

\[\text{The Functor } \Gamma : \mathcal{CLR}_u \to \mathcal{CPMV}_f \]

\[\text{Definition 7.11.} \text{ For } (R,u) \text{ a chain semi-low } l_u\text{-ring, define } \Gamma(R,u) = \{x \in R \mid 0 \leq x \leq u\} \text{ together with the operations } x \oplus y = (x + y) \land u, \neg x = u - x \text{ and, } x \cdot y = xy. \text{ The multiplication is well defined because } xy \leq x \land y \leq u. \]

\[\text{Proposition 7.12.} [4, 3.2] \text{ Given an } l_u\text{-ring } (R,u) \text{ that satisfies } u^2 \leq u, \Gamma(R,u) \text{ is a } PMV\text{-algebra.} \]

\[\text{Observation 7.13.} \text{ If } R \text{ is a chain semi-low } l_u\text{-ring, then } x(y \lor z) = xy \lor xz. \text{ In fact, it can be assumed without loss of generality that } y \leq z. \text{ Then } x(y \lor z) = xz = xy \lor xz. \text{ A similar statement for the infimum is true. Consequently, in this case } x(y \land z) = xy \land xz. \]

\[\text{Corollary 7.14.} \text{ For every } R \text{ a chain semi-low } l_u\text{-ring, } \Gamma(R,u) \text{ is an } PMV_f. \]

\[\text{Affirmation 7.15.} \Gamma \text{ is functorial.} \]

Given $\alpha : (R,u) \to (H,v)$ in $\mathcal{CLR}_u$, define $\Gamma(\alpha) : \Gamma(R,u) \to \Gamma(H,v)$ in $\mathcal{CPMV}_f$ as follows: $\Gamma(\alpha) := \alpha[0,u]$. By construction $\Gamma(\alpha)$ is a homomorphism of totally ordered $MV$-algebras. Then it is enough to see that it respects products, that is,
\[\Gamma(\alpha)(a)\Gamma(\alpha)(b) = \alpha(a)\alpha(b) = \alpha(ab) = \Gamma(\alpha)(ab).\]
Therefore, $\Gamma(\alpha)$ is a morphism in $\mathcal{CLR}_u$, such that for all $x \in \Gamma(R,u)$, it holds that $\Gamma(\beta)\Gamma(\alpha)(x) = \Gamma(\beta)(\Gamma(\alpha)(x)) = \Gamma(\beta)(\alpha(x)) = \beta(\alpha(x)) = (\beta\alpha)(x) = \Gamma(\beta\alpha)(x)$.

**Theorem 7.16.** For every PMV$_f$-chain $A$ and every chain semi-low $l_u$-ring $(R,u)$, the following are isomorphisms:

$$A \cong \Gamma(A^2, u) \text{ and } R \cong (\Gamma(R,u))^2$$

**Proof.** The correspondences $i$ and $v$ :

$$i : A \rightarrow \Gamma(A^2, u) \quad v : (\Gamma(R,u))^2 \rightarrow R$$

$$a \mapsto (0, a) \quad (m, x) \mapsto mu + x$$

are isomorphisms of MV-algebras and $l_u$-groups respectively [2, Lemma 6]. It is then enough to prove that they respect the product. For $a, b \in A$,

$$i(ab) = (0, ab) = (0, a)(0, b) = i(a)i(b).$$

On the other hand, given $(m, a), (n, b) \in A^2$, it is true that:

$$v[(m, a)(n, b)] = v[mn(0, u^2) + m(0, bu) + n(0, au) + (0, ab)]$$

$$= mn[v(0, u^2)] + m[v(0, bu)] + n[v(0, bu)] + v(0, ab)$$

$$= mnu^2 + mbu + nau + ab = mu(nu + b) + a(nu + b)$$

$$= (mu + a)(nu + b) = v(m, a)v(n, b).$$

It is now easy to prove that the isomorphisms defined based on Chang’s construction given in Theorem 7.3, $i$ and $v$, determine a categorical equivalence.

**Theorem 7.17.** The isomorphisms $i$ and $v$ defined above are natural transformations associated to the functors $\Gamma(-)^2$ and $(-)^2\Gamma$ respectively and establish an equivalence of categories

$$\mathcal{CPMV}_f \xrightarrow{(-)^2} \mathcal{CLR}_u \quad \mathcal{CLR}_u \xrightarrow{\Gamma} \mathcal{CPMV}_f$$

**Proof.** The proof is analogous to [5, Theorem 2.2]. Given $A \xrightarrow{h} B$ in $\mathcal{CPMV}_f$, and with $(R,u) \xrightarrow{\varphi} (H,w)$ in $\mathcal{CLR}_u$, the naturality of $i$ and $v$ follows from the commutativity of the following diagrams:

$$A \xrightarrow{i} \Gamma(A^2, u) \xrightarrow{(\Gamma^2)} A^2 \quad \Gamma(R,u)^2 \xrightarrow{v} (R,u)$$

$$\xrightarrow{h} \quad \xrightarrow{(\Gamma^2)} \quad \xrightarrow{(\Gamma^2)}$$

$$B \xrightarrow{i} \Gamma(B^2, u) \xrightarrow{(\Gamma^2)} B^2 \quad \Gamma(H,w)^2 \xrightarrow{v} (H,w)$$
Given \( a \in A \), it holds that \( \Gamma(h^\sharp)i(a) = \Gamma(h^\sharp)(0, a) = h^\sharp(0, h(a)) = ih(a) \), and for \((n, x) \in \Gamma(R, w)^\sharp\), then \( \varphi v(n, x) = \varphi(nu + x) = nw + \varphi(x) = v(n, \varphi(x)) = v(n, (\Gamma\varphi)(x)) = v(\Gamma\varphi)^\sharp(n, x) \).

8. Categorical Equivalence Between the Categories \( \mathcal{PMV}_f \) and \( \mathcal{LR}_u \)

Subdirect Representation of \( \mathcal{PMV}_f \)-Algebras by Chains. Recall the partial order isomorphism between the ideals of an \( l_u \)-group \( G \) and the ideals of its \( MV \)-algebra \( \Gamma(G, u) \), established in [3, Theorem 7.2.2].

**Theorem 8.1.** Given an \( l_u \)-group \( G \) and \( A = \Gamma(G, u) \), the correspondence

\[
\phi : \mathcal{I}(A) \longrightarrow \mathcal{I}(G)
\]

\[
J \longmapsto \phi(J) = \{ x \in G \mid |x| \land u \in J \}
\]

is a partial order isomorphism between the ideals of the \( MV \)-algebra \( \Gamma(G, u) \) and the \( l \)-ideals of the \( l_u \)-group, and its inverse is given by \( H \longmapsto \psi(H) = H \cap [0, u] \).

**Proposition 8.2.** For a semi-low \( l_u \)-ring \( R \), an ideal \( J \) of the \( \mathcal{PMV}_f \)-algebra \( \Gamma(R, u) \) and \( \phi(J) \) the ideal of the \( l_u \)-group \( (R, +, u) \) as in Theorem 8.1, it holds that \( \phi(J) = J^\sharp \) with

\[
J^\sharp = \left\{ x \in R \mid x = \sum_{i=1}^{m} \epsilon_i c_i, \ c_i \in J, \epsilon_i \in \{-1, 1\} \right\}.
\]

**Proof.** \( J^\sharp \) is an \( l \)-ideal of the \( l_u \)-group \( (R, +, u) \). In fact, \( J^\sharp \) is a subgroup of \( R \) by construction. Next it must be proven that given \( x \in J^\sharp \) and \( y \in R \) such that \( |y| \leq |x|, y \in J^\sharp \). Suppose without loss of generality that \( |x| = x^+ = x \) and \( |y| = y^+ = y \).

Since \( x = \sum_{i=1}^{m} \epsilon_i c_i \) with \( c_i \in J \),

\[
x \land u = \left( \sum_{i=1}^{m} \epsilon_i c_i \right) \land u = \left| \sum_{i=1}^{m} \epsilon_i c_i \right| \land u \leq \left( \sum_{i=1}^{m} c_i \right) \land u = \bigoplus_{i=1}^{n} c_i \in J, \tag{3}
\]

therefore, \( x \land u \in J \).

By [5, Theorem 1.5, c], it is enough to consider \( x = \sum_{k=0}^{n} a_k \), with \( a_k = (x - ku) \land u \lor 0 \), where \( 0 < x < nu \) for some \( n \in \mathbb{N} \), since the elements are in an \( l_u \)-group.

If \( x - ku > 0 \), \( a_k \in J \) because

\[
(x - ku) \land u \lor 0 = (x - ku) \land u \leq x \land u \in J,
\]
by inequality (3). If \((x - ku) < 0\) then \(a_k = 0 \in J\).

Since \(0 \leq y \leq x \leq nu\) implies \(b_k = (y - ku) \wedge u \vee 0 \leq (x - ku) \wedge u \vee 0 = a_k \in J\), then \(y = \sum_{k=0}^{n} b_k \in J^\sharp\).

The same proof can be used if \(x = x^-\) and \(y = y^-\). Because \(|x| = x^+ + x^-\) and \(|y| = y^+ + y^-\), both are sums of positive elements and \(J^\sharp\) is a subgroup of \(R\), \(|y| \leq |x|\) and \(x \in J^\sharp\) imply \(y \in J^\sharp\).

By construction \(J \subseteq J^\sharp\), and for inequality (3), \(J^\sharp \cap [0, u] \subseteq J\) so

\[J^\sharp \cap [0, u] = J = \phi(J) \cap [0, u],\]

thus, by the isomorphism given in Theorem 8.1, \(J^\sharp = \phi(J)\). □

**Corollary 8.3.** \(\phi(J) = J^\sharp\) is an ideal of the \(l_u\)-ring.

**Proof.** It is enough to show that \(J^\sharp\) is absorbent. For any \(r \in R\), \(r = \sum_{j=1}^{m} \alpha_j d_j\) with \(d_j \in [0, u]\) and \(\alpha_j \in \{-1, 1\}\), because of Theorem 6.10, and given \(x \in J^\sharp\), \(x = \sum_{i=1}^{n} \epsilon_i c_i\), with \(c_i \in J\), then

\[rx = \sum_{j=1}^{m} \alpha_j d_j \sum_{i=1}^{n} \epsilon_i c_i = \sum_{i,j=1}^{mn} \alpha_j \epsilon_i d_j c_i,\]

where \(d_j c_i \in J\), since this is absorbent, therefore \(rx \in J^\sharp\). □

**Corollary 8.4.** In a semi-low \(l_u\)-ring every \(l\)-ideal is an \(L\)-ideal.

\[Id_g(R) = Id(R).\]

**Proof.** For any \(J \in Id_g(R)\), because of Theorem 8.1 it holds that \(J \cap [0, u] \in Id(\Gamma(R, u))\). Because \(\Gamma(R, u) \in PMV_f\), \(J \cap [0, u]\) absorbs, then \(J \cap [0, u] \in Id_w(\Gamma(R, u))\) and consequently by Proposition 8.2, \(J = (J \cap [0, u])^\sharp \in Id(R)\). In particular, \(Spec_g(R) \subseteq Id(R)\). □

**Corollary 8.5.** For any \(J \in Id_g(R)\) where \(R\) is a semi-low \(l_u\)-ring, \(R/J\) is a semi-low \(l_u\)-ring.

**Theorem 8.6.** For any \(J \in Id_g(R)\) where \(R\) is a semi-low \(l_u\)-ring,

\[\Theta: \Gamma(R/J, uJ) \to \Gamma(R, u)/(J \cap [0, u]); [x]_J \mapsto [x]_{J \cap [0, u]}\]

is an isomorphism of \(PMV_f\)-algebras.

**Proof.** Because the \(MV\)-algebras are isomorphic, due to [3, Theorem 7.2.4], it is enough to see that the isomorphism respects products. Using Corollary 8.5, Proposition 7.12 and the definition of \(\Theta\) it follows that

\[\Theta([a]_J [b]_J) = \Theta([ab]_J) = [ab]_{J \cap [0, u]} = ([a]_{J \cap [0, u]}) ([b]_{J \cap [0, u]}).\]
Corollary 8.7. If \( J \in \text{Spec}_g(R) \) then \( \Theta \) is an isomorphism of PMV\(_f\)-chains.

Proof. It follows from the last theorem and Corollary 7.14.

Theorem 8.8. Every PMV\(_f\)-algebra is isomorphic to a subdirect product of PMV\(_f\)-chains.

Proof. For any PMV\(_f\)-algebra \( A \) there is an injective homomorphism of \( MV \)-algebras,

\[
\hat{()}: A \to \prod_{P \in \text{Spec}(A)} A/P,
\]

mapping \( a \mapsto \hat{a} \) where \( \hat{a}: \text{Spec} A \to \bigcup_{P \in \text{Spec} A} A/P \) with \( \hat{a}(P) = [a]_P \). It is a homomorphism of PMV\(_f\)-algebras, and \( \pi_P \circ \hat{()} : A \to A/P \) is a surjective homomorphism for each prime ideal \( P \in \text{Spec}(A) \). In fact, every prime ideal \( P \) in the \( MV \)-algebra is an ideal in the PMV\(_f\)-algebra, as proven in Proposition 5.7, where \( \hat{ab} = \hat{a} \cdot \hat{b} \), due to the correspondence between ideals and congruences in any PMV\(_f\)-algebra.

Corollary 8.9. Every PMV\(_f\)-equation [3, Section 1.4] that holds in any PMV\(_f\)-chain holds in every PMV\(_f\)-algebra.

Corollary 8.10. In every PMV\(_f\)-algebra it holds that \( a(b \land c) = ab \land ac \), \( a(b \lor c) = ab \lor ac \).

Corollary 8.11. If \( (R, u) \) is a semi-low \( l_u\)-ring, \( \Gamma((R, u)) \) is a PMV\(_f\)-algebra.

Proof. From the Corollary 7.14 it follows that every \( \Gamma((R, u))/P \) is a PMV\(_f\)-chain for every \( P \), and the other hand, \( \Gamma((R, u)) \) is isomorphic to a subdirect product of \( \prod_{P \in \text{Spec}(\Gamma((R, u)))} \Gamma((R, u))/P \) and therefore \( \Gamma((R, u)) \) is a PMV\(_f\)-algebra.

Theorem 8.12. Every semi-low \( l_u\)-ring \( R \) is isomorphic to a subdirect product of chains.

Proof. It is enough to show that the injective homomorphism of \( l_u \)-groups given by

\[
\hat{(-)^g}: R \to \prod_{P \in \text{Spec}_g(R)} R/P; \quad x \mapsto [x]_P,
\]

is an \( l_u \)-ring homomorphism. In fact, from [3, Theorem 7.2.2] and Corollary 8.3 it follows directly that \( R/P \) is a semi-low \( l_u \)-ring for every \( P \in \text{Spec}_g(R) \).
Extension of the Functors \((-\cdot)^{\sharp}\) and \(\Gamma\)

The diagram on the left will be completed to extend the construction of Chang to a functor \(\mathcal{PMV}_f \xrightarrow{(-\cdot)^{\sharp}} \mathcal{LR}_u\), and then it will be proven that this extends the equivalence from the first row to an equivalence in the second.

\[
\begin{array}{ccc}
\mathcal{CPMV}_f & \xrightarrow{(-\cdot)^{\sharp}} & \mathcal{CLR}_u \\
\downarrow & & \downarrow \\
\mathcal{PMV}_f & \xrightarrow{-} & \mathcal{LR}_u \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{CLR}_u & \xrightarrow{\Gamma} & \mathcal{CPMV}_f \\
\downarrow & & \downarrow \\
\mathcal{LR}_u & \xrightarrow{\Gamma} & \mathcal{PMV}_f \\
\end{array}
\]

**Definition 8.13.** For any \(PMV_f\)-algebra \(A\), we define

\[A^0 = \{(0, \hat{a}) : a \in A\} \subseteq \prod_{P \in \text{Spec} A} (A/P)^{\sharp}.
\]

**Definition 8.14.** \(\text{(Associate } l_u\text{-ring)}\) For any \(PMV_f\)-algebra \(A\) we define \(A^2 = \text{gen}(A^0)\) as the \(l\)-ring generated in the \(l\)-ring \(\prod_{P \in \text{Spec} A} (A/P)^{\sharp}\).

**Notation.** \(|A^*| = \left\{ x \in \prod_{P \in \text{Spec} A} (A/P)^* | x = \sum_{i=1}^n \epsilon_i(0, \hat{a}_i), a_i \in A, n \in \mathbb{N} \right\} \).  

**Affirmation 8.15.** \(A^2\) is a semi-low \(l_u\)-ring and \(A^2 = \langle |A^*|, +, \cdot, u, \leq \rangle\) where \(A^* = \langle |A^*|, +, u, \leq \rangle\) is the \(l_u\text{-group associated to the subjacent MV-algebra } A\), and the product is defined as follows:

\[
\varphi : |A^*|^2 \longrightarrow |A^*| \\
(x, y) \longmapsto \varphi(x, y) := x \cdot y.
\]

with \(x = \sum_{i=1}^n \epsilon_i(0, \hat{a}_i), y = \sum_{j=1}^m \delta_j(0, \hat{b}_j), \quad x \cdot y = \sum_{i,j=1}^{nm} \epsilon_i \delta_k(0, \hat{a}_i \hat{b}_j), \quad \epsilon_i, \delta_j \in \{-1, 1\}\) and, \(a_i, b_j \in A\).

**Proof.** \(\varphi\) is well defined because for each \(P \in \text{Spec}(A)\) the product \((x \cdot y)(P) = x(P) \cdot y(P)\) coincides with the product given in Definition 7.5 and described in Corollary 7.9. From Theorem 7.8 it follows that the operation is associative and distributive.

On the other hand, \(\langle |A^*|, +, \cdot, \leq \rangle\) is a semi-low \(l_u\)-ring because for every \(0 \leq x, y \leq u\),

\[
x = \sum_{i=1}^n (0, \hat{a}_i) = (0, \oplus_{i=1}^n \hat{a}_i) \\
y = \sum_{j=1}^m (0, \hat{b}_j) = (0, \oplus_{j=1}^m \hat{b}_j),
\]

\[
(x \cdot y)(P) = x(P) \cdot y(P) = (0, [\oplus a_i]_P) \cdot (0, [\oplus b_j]_P) = (0, [\oplus a_i]_P \cdot [\oplus b_j]_P) \\
\leq (0, [\oplus a_i]_P \wedge [\oplus b_j]_P) = (0, [\oplus a_i]_P) \wedge (0, [\oplus b_j]_P) = x(P) \wedge y(P).
\]
Since $A^o \subseteq |A^*|$, and every $l_u$-ring $H$ that contains $A^o$, must contain all finite sums and products of elements of $A^o$, $\langle |A^*|, +, \cdot, u, \leq \rangle \subseteq H$.

**Definition 8.16.** For any $h : A \to B$ in the category $\mathcal{PMV}_f$, we define $h^\sharp : A^\sharp \to B^\sharp$ in $\mathcal{LR}_u$ by $h^\sharp \left( \sum_{i=1}^n \epsilon_i(0, \hat{a}_i) \right) := \sum_{i=1}^n \epsilon_i \left(0, \hat{h}(a_i)\right)$.

**Theorem 8.17.** The application $(-)^\sharp : \mathcal{PMV}_f \to \mathcal{LR}_u$ that assigns to each $PMV_f$-algebra $A$ the $l_u$-ring $A^\sharp$, is functorial.

**Proof.** For every $h : A \to B$ in the category $\mathcal{PMV}_f$, $h^\sharp$ is a homomorphism of $l_u$-rings and $h^\sharp$ is a homomorphism of $l$-rings such that the following diagram is commutative.

\[
\begin{array}{ccccccccc}
A & \xrightarrow{\sim} & A^\circ \subset \Gamma(A^\sharp, u) & \xrightarrow{\sim} & A^\sharp & \xrightarrow{\prod_{P \in \text{Spec} A}} & \prod_{P \in \text{Spec} A} (A/P)^\sharp \\
\downarrow h & & \downarrow h^\sharp & & \downarrow h^\sharp & & \downarrow h^\sharp \\
B & \xrightarrow{\sim} & B^\circ \subset \Gamma(B^\sharp, u) & \xrightarrow{\sim} & B^\sharp & \xrightarrow{\prod_{Q \in \text{Spec} B}} & \prod_{Q \in \text{Spec} B} (B/Q)^\sharp
\end{array}
\]

According to [5, Theorem 3.3], $h^\sharp$ is a homomorphism of $l_u$-groups and $h^\sharp$ is a homomorphism of $l$-groups. Recall that on the proof of [5, Theorem 3.3], for any $Q \in \text{Spec}(B)$ the well defined morphism $h|Q : A/h^{-1}Q \to B/Q$; $h|Q \left( [a]_{h^{-1}Q} \right) = [h(a)]_Q$, makes the following diagram commute

\[
\begin{array}{ccc}
A/h^{-1}Q & \xrightarrow{i} & (A/h^{-1}Q)^\sharp \\
\downarrow h|Q & & \downarrow (h|Q)^\sharp \\
B/Q & \xrightarrow{i} & (B/Q)^\sharp
\end{array}
\]

and therefore the group homomorphisms $h^\sharp$ can be defined as follows: given $\sigma \in \prod_{P \in \text{Spec} A} (A/P)^\sharp$, $h^\sharp(\sigma)(Q) = (h|Q)^\sharp(\sigma(h^{-1}Q))$, $(h_1 h_2)^\sharp = h_1^\sharp h_2^\sharp$, and $h^\sharp|_{A^\sharp} = h^\sharp$.

To finish the proof, it is enough to show that $h^\sharp$ respects products in the generators of $A^\circ$.

Given $P \in \text{Spec} A$, it follows from Proposition 7.10 that:

\[
h^\sharp \left( (0, \hat{a}) \left(0, \hat{b}\right) \right) (P) = h^\sharp \left( (0, [a]_P) (0, [b]_P) \right) = h^\sharp \left( (0, [a]_P) \right) h^\sharp \left( (0, [b]_P) \right).
\]

As seen in the Affirmation 8.15, $A^\sharp$ is a semi-low $l_u$-ring, and if $h : A \to B$ is a homomorphism of $PMV_f$-algebras, $h^\sharp : A^\sharp \to B^\sharp$ is a homomorphism of semi-low $l_u$-rings.

On the other hand, given $A \xrightarrow{h} B \xrightarrow{g} C \in \mathcal{PMV}_f$, $(gh)^\sharp = g^\sharp h^\sharp$ follows directly from Definition 8.16.
Theorem 8.18. \( \Gamma : \mathcal{LR}_u \to \mathcal{PMV}_f \) is a functor, where \( \Gamma(R, u) = [0, u] \) and \( \Gamma(h) = h|_{[0, u]} \) for every homomorphism of \( l_u \)-rings \( h : R \to R' \).

Proof. It is follows directly from Corollary 8.11 and Affirmation 7.15.

The Equivalence

Theorem 8.19. Given any PMV\(_f\)-algebra \( A \) and any semi-low \( l_u \)-ring \((R, u)\), the following homomorphisms are isomorphisms of PMV\(_f\)-algebras and semi-low \( l_u \)-rings.

\[ A \cong \Gamma(A^2, u) \text{ and } (R, u) \cong (\Gamma(R, u))^2. \]

Proof. From [6, Proposition 3.2], it follows that \( A \cong A^o \) as PMV\(_f\)-algebras, with \( A^o \subset A^2 \) as shown in the following commutative diagram

\[
\begin{array}{c}
A \xrightarrow{(\cdot)_A} \prod_{P \in \text{Spec } A} (A/P) \xrightarrow{i} \prod_{P \in \text{Spec } A} (A/P)^2 \\
\cong \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{with } i_P \text{ the application defined for each } P \text{ as:}
\]

\[ i_P : A/P \longrightarrow (A/P)^2 \]

\[ [a]_P \longmapsto (0, [a]_P) \]

Because of Theorem 6.10 a), \( A^o = \Gamma(A^2, u) \), and so \( A \cong \Gamma(A^2, u) \).
τ_{R} is well defined because the homomorphism \((\widehat{(-)})^g\) is injective. The fact that \(\tau_{R}\) is injective follows from the fact that for every \(x, y \in \Gamma(R, u)\), \(\widehat{x}^g = \widehat{y}^g\) implies \(x = y\), and so \(\widehat{x} = \widehat{y}\), since \((\widehat{(-)})\) is a homomorphism. \(\tau_{R}\) is surjective because for every \(x \in (R, u)\) it holds that \(x = \sum_{i=1}^{n} \epsilon_i x_i\) for some \(x_i \in [0, u]\) by Theorem 6.10, b). Consequently \(\tau_{R}(\sum_{i=1}^{n} \epsilon_i (0, \widehat{x}_i)) = \sum_{i=1}^{n} \epsilon_i \tau_{R}(0, \widehat{x}_i) = x\).

**Theorem 8.20.** For every \(A \in \mathcal{P}MV_{f}\) and \(R \in L\mathcal{R}_u\) the isomorphisms

\[
A \xymatrix{\widehat{(-)}_A} \Gamma(A^\sharp, u) \quad \quad \quad \Gamma(R, u)^\sharp \xymatrix{\tau_R} (R, u)
\]

are natural transformations.

**Proof.** It follows directly from [5, Theorem 3.3].

### 9. \(PMV_{f}\) Versus \(f\)-Rings

**Definition 9.1.** (\(f\)-rings [1, XVII.5]) A function ring or \(f\)-ring is an \(l\)-ring that satisfies

\[a \land b = 0 \text{ and } c \geq 0 \text{ implies } ac \land b = a \land cb = 0.\]

**Proposition 9.2.** [1, XVII.5] In every \(f\)-ring it holds that

\[a \land b = 0 \implies ab = 0.\]

**Theorem 9.3.** (Fuchs [1, XVII.5]) An \(l\)-ring is an \(f\)-ring if and only if all its \(l\)-closed ideals are \(L\)-ideals.

**Proposition 9.4.** For any \(PMV_{f}\) \(A\), \(A^\sharp\) is a semi-low \(f_u\)-ring.

**Proof.** From Affirmation 8.15, \(A^\sharp\) is a semi-low \(l_u\)-ring. From Corollary 8.4 and Theorem 9.3, it is an \(f\)-ring, since all its \(l\)-ideals are \(L\)-ideals.

**Example 9.5.** \([0,1]^\sharp = \mathbb{R}\).

**Proposition 9.6.** \(PMV \subset MVW\)-rig.

**Proof.** From [4, Theorem 4.2], it follows that for any \(PMV\)-algebra \(A\) there exists an \(l_u\)-ring \(R\) such that \(\Gamma(R, u) \cong A\) and because of Proposition 7.12, \(A\) is an \(MVW\)-rig. The inclusion is strict because of remark 4.9.

**Affirmation 9.7.** For any set \(X\) the semi-low \(f_u\)-ring associated to the \(PMV_{f}\)-boolean algebra \(2^X\) is isomorphic to the ring of bounded functions of \(\mathbb{Z}^X\), \(B(\mathbb{Z}^X)\).
Categorical Equivalence Between PMV\textsubscript{f}-Product Algebras

**Proof.** It is enough to see that $(2^X)^\sharp \cong B(\mathbb{Z}^X)$. The application $\Theta$ defined on the generators, for all $f \in 2^X$,

$$(2^X)^\sharp \xrightarrow{\Theta} B(\mathbb{Z}^X)$$

$$i \hat{f} \longmapsto \hat{f}: X \longrightarrow \mathbb{Z}$$

$x \longmapsto f(x)$

with $\Theta \left( \sum_{j=1}^{k} i \hat{f}_j \right) = \sum_{j=1}^{k} \Theta(i \hat{f}_j)$ and $\Theta(\hat{f} \hat{g}) = \Theta((i \hat{f})) \Theta((i \hat{g}))$, is a ring isomorphism.

Because $2^X$ is a hyper-archimedean MV-algebra, every prime ideal is maximal and of the form $P_x = \{ f \in 2^X : f(x) = 0 \}$ and $[f]_{P_x} = f(x)$. Then, if $x \in X$, $\hat{f}(x) \neq \hat{g}(x) \Leftrightarrow f(x) \neq g(x) \Leftrightarrow f \neq g \Leftrightarrow [f]_{P_x} \neq [g]_{P_x} \Leftrightarrow \hat{f} \neq \hat{g}$, implies that $\Theta$ is well defined and injective, with $P_x \in \text{Spec}(2^X)$.

On the other hand, for $h \in B(\mathbb{Z}^X)$ it holds that $h = \sum_{k=-n}^{n} k \lambda_k$ with $|h| \leq n$, $\lambda_k \in 2^X$ such that $\lambda_k(x) = 1$ if $h(x) = k$ and zero elsewhere. Therefore $\Theta$ is surjective. By construction $\Theta$ is a homomorphism of $l$-rings.

**Example 9.8.** The semi-low $f_u$-ring $(2^n)^\sharp$ is isomorphic to the ring $\mathbb{Z}^n$.

**Corollary 9.9.** Every boolean algebra seen as a PMV\textsubscript{f}-algebra is a subalgebra of $2^X$ for some set $X$. Since the functor $(-)^\sharp$ preserves subalgebras, the semi-low $f_u$-ring associated to a boolean algebra is a subring of the semi-low $f_u$-ring $B(\mathbb{Z}^X)$.

**Example 9.10.** $F[x] \subset C([0,1][0,1])$, the semi-low $f_u$-ring of continuous functions defined as follows:

$f \in F[x] \Leftrightarrow \exists P_1, \ldots, P_k \in \mathbb{Z}[x]$, such that $\forall x \in [0,1] f(x) = P_i(x)$,

for some $1 \leq i \leq k$, is the semi-low $f_u$-ring associated to the PMV\textsubscript{f}-algebra $\Gamma(F[x])$. This algebra is the minimum PMV\textsubscript{f}-algebra that contain the MV-algebra $Free_1$.

10. Conclusions

The construction of Dubuc–Poveda [5] lets you visualize the associate ring of each PMV\textsubscript{f}-algebra, because it does not use the good sequences, and use the easy construction by Chang [2] for chains. The explicit construction of this equivalence permits us to study some properties of commutative algebra for the class of semi-low $f_u$-rings, in relationship with PMV\textsubscript{f}-algebras. Although
the problem of studying the free algebras of $f_u$-rings, is known its relationship with the $PMV_f$-algebras let us see it from a new perspective [11].

References

[1] Birkhoff, G., *Lattice Theory*. 3rd ed., Colloquium Publications No. 25, American Mathematical Society, Providence, 1967.
[2] Chang, C. C., A new proof of the completeness of the Lukasiewicz axioms. *Transactions of the American Mathematical Society* 93:74–90, 1959.
[3] Cignoli, R. L. I. M. L. D’Ottaviano, and D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.
[4] Di Nola, A., and A. Dvurečenskij, Product MV-algebras. *Multiple-Valued Logics* 193–215, 2001.
[5] Dubuc, E. J., and Y. A. Poveda, On the Equivalence Between MV-Algebras and l-Groups whit Strong Unit. *Studia Logica* 103(4):807–814, 2015.
[6] Dubuc, E. J., and Y. A. Poveda, Representation Theory of MV-Algebras, *Annals of Pure and Applied Logic*, 161, 2010.
[7] Estrada, A., *MVW-rigs*. Master’s Thesis, Universidad Tecnológica de Pereira, 2016.
[8] Montagna, F., An algebraic approach to propositional fuzzy logic, *Journal of Logic, Language and Information* 9:91–124, 2000.
[9] Montagna, F., Subreducts of MV-algebras with product and product residuation. *Algebra Universalis* 53(1):109–137, 2005.
[10] Mundici, D., Interpretation of AF C*-algebras in Lukasiewicz sentential calculus, *Journal of Functional Analysis* 65:15–63, 1986.
[11] Zuluaga, S., *Los MVW-rigs provenientes de las MV-álgebras libres*. Master’s Thesis, Universidad Tecnológica de Pereira, 2017.