Robust solutions of uncertain mixed-integer linear programs using structural-reliability and decomposition techniques

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Abstract

Structural-reliability based techniques has been an area of active research in structural design during the last decades, and different methods have been developed, such as First Order Second Moment (FOSM) or First Order Reliability (FORM) methods. The same has occurred with robust optimization, which is a framework for modeling optimization problems involving data uncertainty. If we focus on linear programming (LP) problems with uncertain data and hard constraints within an ellipsoidal uncertainty set, this paper provides a different interpretation of their robust counterpart (RC) inspired from structural-reliability methods. This new interpretation allows the proposal of an ad-hoc decomposition technique to solve the RC problem with the following advantages: i) it improves tractability, specially for large-scale problems and those including binary decisions, and ii) it provides exact bounds for the probability of constraint violation in case the probability distribution of uncertain parameters are completely defined by using first and second probability moments. An attractive aspect of our method is that it decomposes the initial linear mathematical programming problem in a deterministic linear master problem of the same size of the original problem and different quadratically constraint problems (QCP) of considerable lower size. The optimal solution is achieved through the solution of these master and subproblems within an iterative scheme.

Keywords: Stochastic programming, Conic programming and interior point methods, Decision analysis under uncertainty, Reliability analysis,

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1. Introduction

The concept of robust optimization was developed to drop the classical assumption in mathematical programming that the input data is precisely known and equal to given nominal values. It is well known that in practice, most of the data involved in optimization problems is uncertain, and optimal solutions using nominal values might no longer be optimal or even feasible. Robust optimization techniques deal with the problem of designing solutions that are immune to data uncertainty [1, 2, 3, 4, 5, 6, 7] by solving equivalent deterministic problems. The main advantage of these techniques is that it is not required to know the probability density function (PDF) of the uncertain data. The decision-maker searches for the optimal solution of all-possible realizations of uncertain data within the uncertainty set, and in addition, probabilistic bounds of constraint violation valid for different probability density functions are available.

Stochastic programming is also a framework for modelling problems that involve uncertainty [8]. In this particular case, uncertain data is assumed to follow a given probability distribution and are usually dealt with by using scenario models or finite sampling from the PDFs [9, 10]. However, the number of scenarios needed to represent the actual stochastic processes can be very large, which may result in intractable problems. That is the most important reason why robust optimization is gaining widespread among practitioners with respect to stochastic programming, not only in the operational research community but also for design engineers [11, 12].

Stochastic programming in the context of engineering design and optimization, i.e. reliability-based structural optimization [13, 14, 15, 16, 17], has also broadened using as risk measure the failure probability. In this context, it is required to know: i) the joint probability density function of all random variables involved and ii) a method for calculating the probabilities of failure for a given design. Since the evaluation of failure probabilities is not an easy task, different methods have been developed, such as First Order Second Moment (FOSM, [18]) or First Order Reliability (FORM) methods [19, 20].

Despite the analogies among the problems treated within these different frameworks, i.e. stochastic programming, robust optimization, and reliability-based structural optimization, research trends and solution techniques have
followed different paths. To the best of our knowledge, a few works have taken advantage of methods from one field to be applied on any other. For instance, [21] proposes a new method to solve certain classes of stochastic programming problems based on FORM and mathematical programming decomposition techniques. Their method focus on an specific type of problems where: i) the joint probability distribution of the random variables involved is given or can be estimated parametrically, ii) distributions do not depend on the decision variables, and iii) the random variables only affect the objective function. Recently, [22] proposed a new risk measure, the buffered failure probability, which allows the generalization of the CVaR concept from stochastic programming [9, 10] to reliability-based structural optimization problems using finite sampling. One of the aims of the present work is to give a new perspective and apply concepts originating from structural reliability to robust optimization, we attempt to shed new light on existing problems and as such stir innovative thinking.

In particular, we focus on the type of problems dealt with on work [5], i.e. linear mathematical programming problems with hard constraints that must be satisfied for any possible realization of the uncertain data. [5] proposes to obtain robust solutions of an uncertain LP problem with ellipsoidal uncertainty sets, whose RC results is a conic quadratic problem, i.e. a convex and tractable problem that can be solved in polynomial time by interior point algorithms. This paper proposes an alternative and decomposable solution technique inspired on FOSM, which allows reaching the optimal solution of the RC problem by solving two kind of problems within an iterative scheme: one master problem of the same size as the original RC problem, and one subproblem of considerable lower size for each hard constraint. This strategy of decomposing a problem into smaller pieces has proved to be effective to improve tractability in many different applications [23].

The proposed method has the following feature which might make it attractive for practical use: it improves tractability with respect to interior point algorithms, specially for large-scale problems and those including binary decision variables, because i) the master problem remains linear and ii) the subproblems are QCP with just one quadratic constraint, which under certain circumstances have analytical solutions and otherwise can be solved efficiently by using linear programming or specific solvers [24]. In addition and due to its relationship with reliability-based structural techniques, it allows to calculate exact bounds for the probability of constraint violation in case the probability distributions of uncertain parameters are completely
defined by using first and second moments (mean and variance-covariance). This feature could encourage engineers used to work with failure probabilities to take advantage of robust optimization techniques, even without using the proposed iterative method. An additional feature of the method proposed in this paper is that it constitutes an alternative formulation, specially for problems including binary variables, to that proposed by [7] using cardinality constrained uncertainty sets. Both approaches solve linear programming problems, the main difference is that the method proposed by [7] requires the definition of the maximum number of parameters allowed to reach its worst possible values, while the approach proposed in this paper requires the definition of the protection level. Besides, the proposed method would not require the definition of bounds for the uncertain parameters, which might be of interest for certain applications.

The rest of the paper is organized as follows. Section 2 introduces robust formulations of linear programming problems, while in section 3 FOSM technique is explained in detail, paying special attention to its relationship with ellipsoidal uncertainty sets. In Section 4 a detailed description of the decomposition method proposed in this work is given. In Sections 5 and 6 an illustrative example and a realistic case study are respectively described, solved, analyzed, and compared with existing approaches. Finally, in Section 7 some relevant conclusions are duly drawn.

2. Robust counterpart of an uncertain linear programming problem

Consider the following problem:

\[
\text{Maximize } \quad c^T x, \quad (1)
\]

subject to

\[
A x \leq 0 \quad (2)
\]

\[
l \leq x \leq u. \quad (3)
\]

where \(x(n \times 1)\) is the decision variable vector which might include binary variables, \(c(n \times 1)\) and \(A(m \times n)\) are data coefficients, and \(l(n \times 1)\) and \(u(n \times 1)\) are lower and upper decision variable bounds. We assume without loss of generality that the only uncertain coefficients are those belonging to
matrix $A(m \times n)$. For those cases where vector $c(n \times 1)$ is uncertain, or even the right hand side of equation (2) is uncertain and equal to $b(m \times 1)$, it is possible to rewrite the original problem as (1)-(3) (see [25]).

The RC of problem (1)-(3) is the same problem but replacing constraint set (2) by:

$$Ax \leq 0; \forall A \in U$$

(4)

where $U$ is an uncertainty set. According to [25] (check reference [11]), an LP with a certain objective is a constraint-wise problem and its solution does not change if the uncertainty set is extended to the product of its projections on the subspaces of the constrains, i.e. constraint set (4) is equivalent to:

$$a_{(i)^T}x \leq 0; \forall a_{(i)} \in U_i, i = 1,\ldots,m,$$

(5)

where $a_{(i)^T}; i = 1,\ldots,m$ are the rows of matrix $A$ and $U_i$ is the projection of $U$ on the subspace of the data of $a_{(i)}$.

Traditionally, parameter uncertainty $a_{ij} \in A$ within robust optimization is modeled as a symmetric and bounded random variable $\tilde{a}_{ij}$ [5] that takes values in the interval $[a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]$ following an unknown probability distribution. Elements $a_{ij} \in A$ represent nominal values and $\delta_{ij}; i = 1,\ldots,m; j = 1,\ldots,n$ are the maximum absolute value deviations from the nominal values. This interval characterization of parameter uncertainty is required if worst-case oriented methods, or box uncertainty sets are used. This is the solution proposed by [1], where each uncertain parameter $\tilde{a}_{ij}; i = 1,\ldots,m; j = 1,\ldots,n$ takes its worst possible value within the given interval. This strategy leads to an excessively conservative solution.

To address this excessive conservatism, [5] proposes an alternative uncertainty set. Let us assume that uncertain parameter vectors $\tilde{a}_{(i)}; i = 1,\ldots,m$ have nominal or expected values $a_{(i)}; i = 1,\ldots,m$ and variance-covariance matrix $\Sigma^{(i)}; i = 1,\ldots,m$, respectively. According to [5], the ellipsoidal uncertainty set can be written using the Mahalanobis distance as follows:

$$U_i(\beta_i) = \left\{ (\tilde{a}^{(i)} - a^{(i)})^T \left(\Sigma^{(i)}\right)^{-1} (\tilde{a}^{(i)} - a^{(i)}) \leq \beta_i^2 \right\}; i = 1,\ldots,m,$$

(6)

so that the RC of problem (1)-(3) is the same problem but replacing constraint set (2) by:

$$\left(\begin{array}{c}
\text{Maximum} \\
\tilde{a}^{(i)^T}x
\end{array}\right) \leq 0; \tilde{a}^{(i)} \in U_i(\beta_i), i = 1,\ldots,m,$$

(7)
where parameters $\beta_i; \ i = 1, \ldots, m$ control the size and protection level of the ellipsoidal sets for each constraint. According to [5], for given values of the decision variables $\mathbf{x}$, the probability that each uncertain constraint is violated is given by the following expression:

$$\text{Prob} \left( \mathbf{a}^{(i)\top} \mathbf{x} \right) \leq e^{-\beta_i^2/2}, \ i = 1, \ldots, m.$$ \hspace{1cm} (8)

Note that contrary to the worst case approach proposed by [1], only first and second order moments of the random parameters without lower and upper bounds are initially considered in this paper for the ellipsoidal uncertainty set. There are two reasons, firstly, in case the uncertain parameters follow a multivariate normal distribution, the probability bound (8) can be replaced by the exact probability, which might be of interest for practitioners, and secondly, it is straightforward to include also the interval limitations $[a_{ij} - \delta_{ij}, a_{ij} + \delta_{ij}]$ of the random parameters within the proposed solution method, although tighter bounds with respect to those in (8) would required a more sophisticated development. We will come back to these matters in the next sections.

In order to solve constraints (7), the uncertainty sets (6) are transformed using an affine mapping into balls of radius $\beta_i; \ i = 1, \ldots, m$, respectively, resulting in the following set of alternative constraints:

$$\left( \begin{array}{c}
\text{Maximum} \\
z^{(i)}
\end{array} \right) \left( \begin{array}{c}
\mathbf{a}^{(i)} + \mathbf{L}^{(i)}z^{(i)}
\end{array} \right) \mathbf{x} \leq 0; \quad \|z^{(i)}\| \leq \beta_i, \ i = 1, \ldots, m, \hspace{1cm} (9)$$

where $\mathbf{L}^{(i)}$ is the mapping matrix which can be obtained from Cholesky decomposition of variance-covariance matrix $\Sigma^{(i)} = \mathbf{L}^{(i)}\mathbf{L}^{(i)\top}$, $z^{(i)}$ represents a perturbation vector and $\| \cdot \|$ stands for Euclidean norm. The analytical solution of constraints (9) in terms of the objective function is (see [5]):

$$\mathbf{a}^{(i)\top} \mathbf{x} + \beta_i \|\mathbf{L}^{(i)\top} \mathbf{x}\| \leq 0; \ i = 1, \ldots, m.$$ \hspace{1cm} (10)

Finally, the RC of problem (1)-(3) using ellipsoidal uncertainty sets becomes:

$$\text{Maximize} \quad \sum_{j=1}^{n} c_j x_i; \ i = 1, \ldots, n$$ \hspace{1cm} (11)
subject to

\[
\sum_{j=1}^{n} a_{ij} x_j + \beta_i \sqrt{\sum_{k=1}^{n} \left( \sum_{j=k}^{n} L_{jk}^{(i)} x_j \right)^2} \leq 0; \ i = 1, \ldots, m, \quad (12)
\]

\[
l_j \leq x_j \leq u_j; \ j = 1, \ldots, n. \quad (13)
\]

Problem (11)-(13) is a conic quadratic problem, if decision variable vector \( \mathbf{x} \) does not contain binary variables, it is still a convex tractable formulation which could be solved efficiently using interior point algorithms. However, as pointed out by [7] it is not particularly attractive for solving robust linear discrete optimization models. Note that [7] proposed an alternative linear RC problem to avoid (11)-(13) formulation if binary variables are involved, which instead of allowing all random parameters to take their worst possible value within the given interval such as [1], only allows a pre-established number of parameters \( \Gamma \). This alternative formulation remains linear, and it also provides a robust solution in terms of probability of infeasibility.

3. First-Order-Second-Moment (FOSM) methods

Structural reliability has been an area of active research during the last decades [see 26, 18, 27, 28, 29, 14, 30]. It primarily focuses on the quantification and evaluation of the safety of structures considering that the variables involved are uncertain. Assuming that constraints (5) represent limit-state equations, the random variables involved in each constraint \( \tilde{a}^{(i)} \) belong to an \( n \)-dimensional space, which for given values of the decision variables \( \mathbf{x} \) can be divided into two regions with respect to the limit-state equation \( g_i(\mathbf{x}, \tilde{a}^{(i)}) \), the feasible (\( \mathcal{F} \)) and the unfeasible (\( \mathcal{U} \)) regions. That is,

\[
\mathcal{F}_i \equiv \{(\tilde{a}_{i,1}, \tilde{a}_{i,2}, \ldots, \tilde{a}_{i,n}) \mid g_i(\mathbf{x}, \tilde{a}^{(i)}) = a^{(i)^T} \mathbf{x} \leq 0\}; \ i = 1, \ldots, m,
\]

\[
\mathcal{U}_i \equiv \{(\tilde{a}_{i,1}, \tilde{a}_{i,2}, \ldots, \tilde{a}_{i,n}) \mid g_i(\mathbf{x}, \tilde{a}^{(i)}) = a^{(i)^T} \mathbf{x} > 0\}; \ i = 1, \ldots, m, \quad (14)
\]

where the feasible regions \( \mathcal{F}_i \) corresponds to feasible realizations of the uncertain parameters associated with constraint \( i \), and the unfeasible regions \( \mathcal{U}_i \) to unfeasible realizations of the uncertain parameters related to constraint \( i \). Structural reliability methods focus on the quantification of the probability of random variables associated with the limit-state equations \( g_i; \forall i \) to be in the unfeasible region, also called the failure probability. The final aim is
the selection of the appropriate decision variables so that this probability of failure is below a given threshold $\alpha$.

The probabilities of failure or infeasibility can be calculated using the joint probability distribution functions of all random variables involved, by means of the following multidimensional integral over the unfeasible regions $\mathcal{U}_i$:

$$
\text{Prob}(g_i(x, \tilde{a}^{(i)}) > 0) = \int_{g_i(x, \tilde{a}^{(i)}) > 0} f(\tilde{a}^{(i)}) \, da_{i1} \cdots da_{in}; \ i = 1, \ldots, m, \quad (15)
$$

where $f(\tilde{a}^{(i)})$ is the joint probability density function of the random parameters involved, which assuming that only expectations $a^{(i)}; \forall i$ and variance-covariances $\Sigma^{(i)}; \forall i$ are being used, it corresponds to a multivariate normal distribution. This is a requirement to use FOSM methods, although alternative multivariate probability distribution models could be used instead (FORM).

To perform the integration required in (15), the dependent normally distributed random variables $\tilde{a}^{(i)}$ can be transformed into independent standard normal random variables $z^{(i)}$ that can be integrated straightforwardly. Using the linear Hasofer and Lind transformation [18], the relationships between both set of variables are:

$$
\tilde{a}^{(i)} = a^{(i)} + L^{(i)} z^{(i)}; \ i = 1, \ldots, m, \quad (16)
$$

where $L^{(i)}$ is the Cholesky factorization of the variance-covariance matrix, i.e. $\Sigma^{(i)} = L^{(i)} L^{(i)T}$. This factorization is always possible for positive definite matrices, and variance-covariance matrices must be positive definite by definition because their eigenvalues represent variances which must be always positive. Transformation (16) is equivalent to the mapping transformation in (9). Finally, (15) can be evaluated by solving the following optimization problem for each constraint:

$$
\begin{align*}
\beta_i &= \text{Minimize} \quad \sqrt{\sum_{j=1}^{n} z_j^{(i)2}}, \\
\text{subject to} \quad &\tilde{a}^{(i)} = a^{(i)} + L^{(i)} z^{(i)}, \\
&g_i(x, \tilde{a}^{(i)}) = \tilde{a}^{(i)^T} x = 0
\end{align*}
\quad \iota = 1, \ldots, m, \quad (17)
$$
where the optimal solution $\hat{z}^{(i)}$ corresponds to the closest point to the origin located on the limit-state equation in the standard and independent normal random space, $\beta_i$ is the minimum distance so-called reliability index in the structural reliability scientific community, and $\tilde{a}^{(i)*}$ is the point of maximum likelihood, i.e. the actual values of the uncertain parameters located on the limit of the feasibility region where the probability is higher, and it represents the most likely values of the random parameters that produce infeasibility. Note that the reliability index $\beta_i$ is an algebraic quantity with a negative value for probabilities of failure larger than 0.5, however, solution of problems (17) always results in a positive reliability index value. The appropriate sign can be obtained checking the sign of the limit state equation particularized for the expected or nominal values of the uncertain parameters:

$$\begin{align*}
a^{(i)T}x \leq 0 & \implies \beta_i \geq 0, \\
a^{(i)T}x > 0 & \implies \beta_i < 0,
\end{align*}$$

which means that the sign of $\beta_i$ is positive if constraint is feasible for the nominal values of the uncertain parameters, otherwise, the sign is negative.

The final probability of infeasibility (15) is related to the reliability index by the relation

$$\text{Prob}(g_i(x, \tilde{a}^{(i)}) > 0) = \Phi(-\beta_i); \forall i = 1, \ldots, m,$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variables. This method provides the exact probability if the limit-state equation is linear in the standard normal random space, i.e., if the resulting limit-state distribution is normally distributed, which is the case if uncertain parameters are normally distributed.

In Figure 1, a 2-D graphical interpretation of the FOSM method is shown. In the independent standard normal random space $z^{(i)}$, the joint probability distribution function contours are $n$-spheres centered at the origin, which in the figure becomes concentric circumferences at the origin such that the further the point $(z_1^{(i)}, z_2^{(i)})$ is away from the origin, the lower the value of the joint probability distribution function. The solution of problem (17) provides the closest point to the origin in the limit-state equation. This limit-state equation is linear whenever the random variables are normally distributed.

An important aspect with respect to problems in (17) is that, at the
optimum $\beta_i$, $\hat{z}^{(i)}$ and $\hat{a}^{(i)}$, they are equivalent to the following problems:

\[
\begin{aligned}
\underset{z^{(i)}}{\text{Maximize}} & \quad \hat{a}^{(i)^T} x, \\
\text{subject to} & \quad \hat{a}^{(i)} = a^{(i)} + L^{(i)} z^{(i)} ; \ \lambda^{(i)}, \\
& \quad \sqrt{\sum_{j=1}^{n} z_j^{(i)^2}} \leq \beta_i ; \mu_i
\end{aligned}
\]

where $\lambda^{(i)}$ and $\mu_i$ are the dual variables associated with constraints. The reason is that they have equivalent KKT conditions [31]. Proof of this statement can be found in [21]. Besides, these problems (20) are the same as problems in (9). As we mentioned previously, this problem has an analytical solution, whose optimal objective function value is contained in constraint (10), i.e. $\hat{a}^{(i)^T} x = a^{(i)^T} x + \beta_i \| L^{(i)^T} x \|$ where $\hat{a}^{(i)^T}$ is the optimal solution of the random model parameters. However, no explicit expressions for these variables $\hat{a}^{(i)}$, $\hat{\lambda}^{(i)}$ and $\hat{\mu}_i$ were available because it is not required in the formulation given by [5]. Nevertheless and for convenience, we have used findings on duality for convex quadratic programming with a quadratic constraint [32] to obtain the analytical expressions for the optimal primal and dual variable solutions, i.e.:

\[
\begin{aligned}
\hat{\lambda}^{(i)} & = L^{(i)^T} x ; \ i = 1, \ldots, m \\
\hat{\mu}_i & = \sqrt{x^T \Sigma^{(i)} x} \ ; \ i = 1, \ldots, m \\
\hat{a}^{(i)} & = a^{(i)} + \frac{\Sigma^{(i)} x}{2\mu_i} ; \ i = 1, \ldots, m.
\end{aligned}
\]

Replacing $\hat{\mu}_i$ from (22) into expression (23), $\hat{a}^{(i)}$ simplifies as follows:

\[
\hat{a}^{(i)} = a^{(i)} + \beta_i \frac{\Sigma^{(i)} x}{\sqrt{x^T \Sigma^{(i)} x}} ; \ i = 1, \ldots, m.
\]

It is easy to prove that this is the optimal solution of problem (20) by just evaluating its objective function at this point (24), i.e.:

\[
\begin{aligned}
\hat{a}^{(i)^T} x & = a^{(i)^T} x + \beta_i \frac{x^T \Sigma^{(i)^T} x}{\sqrt{x^T \Sigma^{(i)^T} x}} ; \ i = 1, \ldots, m, \\
& = a^{(i)^T} x + \beta_i \sqrt{x^T \Sigma^{(i)} x} ; \ i = 1, \ldots, m,
\end{aligned}
\]

\[10\]
which is exactly the same optimal objective function as expression (10). This completes the proof.

These expressions (21)-(24) are very important for the proposed decomposition method. Note that the proper sign of the dual variable $\hat{\mu}_i$ in (22) is the same as the one from the reliability index in (18).

4. Decomposition method for the RC

This paper provides a methodology to solve problem (11)-(13) using mathematical programming decomposition techniques. The mentioned problem can be rewritten as follows:

\[
\begin{align*}
\text{Maximize} \quad & \sum_{i=1}^{n} c_i x_i, \\
\text{subject to} \quad & \sum_{j=1}^{n} \hat{a}_{ij} x_j \leq 0; \ i = 1, \ldots, m, \\
& l_j \leq x_j \leq u_j; \ j = 1, \ldots, n,
\end{align*}
\]

where $\hat{a}_{ij}; i = 1, \ldots, m; j = 1, \ldots, n$ are the solutions of the following optimization problems:

\[
\begin{align*}
\hat{a}^{(i)} &= \text{Maximum} \quad \hat{a}^{(i)T} x, \\
\text{subject to} \quad & \hat{a}^{(i)} = a^{(i)} + L^{(i)} z^{(i)}, \\
& \sqrt{\sum_{j=1}^{n} z_j^{(i)2}} \leq \beta_i
\end{align*}
\]

which is the same problem as problem (20).

Note that although both problems (11)-(13) and (26)-(29) are equivalents, in this second alternative, the points of maximum likelihood for each constraint $\hat{a}^{(i)}$ are used explicitly in (27). This allows decomposing the problem in two procedures:

1. **Decision making**: For given values of the points of maximum likelihood $\hat{a}^{(i)}; i = 1, \ldots, m$, the decision variables $\hat{x}$ which maximize the objective function are obtained. This is considered the master problem.
2. **Point of maximum likelihood updating:** For the decisions made in the previous step, FOSM is used to update the values of the random variables required to achieve the target security criterium. These are considered the subproblems.

Although, problem (1)-(2) can be solved directly according to [5] formulation, in the procedure presented in this paper the solution is achieved by means of an iterative scheme, which is repeated until the points $\hat{a}^{(i)}; i = 1, \ldots, m$ associated with constraints (27) truly corresponds to maximum likelihood points. The main reasons for proposing this decomposition are:

1. Problem (26)-(28) for given values of the parameter values $\hat{a}^{(i)}$ is a linear mathematical programming problem of the same size as the original problem, which can be solved very efficiently even if decision vector $a$ contains discrete variables.
2. Problems (29) for given values of the decision variables $a$ can be solved for each constraint independently. Each of these problems is a QCP with just one quadratic constraint, which assuming only the first two moments of the probability distributions are used, it has an analytical solution (equations (21)-(23)).

The proposed iterative scheme is described step by step on the following algorithm.

**Algorithm 4.1. (Iterative method).**

**Input:** Selection of protection levels associated with constraints $\beta_i; \forall i$, objective function, constraints, the statistical description of the involved random parameters $\tilde{a}^{(i)}; \forall i$ and the tolerance of the process $\varepsilon$. These data are selected by the decision maker.

**Step 1: Initialization.** Initialize the iteration counter to $\nu = 1$, and the initial values of the random parameters to their expected or nominal values $\tilde{a}_\nu^{(i)} = a^{(i)}; \forall i$. We select this initial point because we know in advance that $\beta_i = 0; \forall i$, i.e. the probability of infeasibility from the resulting design is $\Phi(-\beta_i) = 0.5$.

**Step 2: Solving the decision making problem at iteration $\nu$.** This is considered the master problem. For given values of the random parameters $\tilde{a}_\nu^{(i)}$, obtain new values of the decision variables for the actual iteration by solving problem (26)-(28). The result provides values of...
the decision variables $x_\nu$. Note that at the first iteration ($\nu = 1$) this problem corresponds to the traditional expected value analysis.

**Step 3: Convergence checking.** If the iteration counter is equal to 1 ($\nu = 1$) go to **Step 4**, otherwise, proceed to check convergence, i.e., if $||x_\nu - x_{\nu-1}|| \leq \varepsilon$ go to **Step 5**, else continue in **Step 4**.

**Step 4: Evaluating the new points of maximum likelihood.** For given values of the decision variables $x(\nu)$, we update the points of maximum likelihood associated with each constraint so that the protection levels $\beta_i; \forall i$ hold. This is achieved by solving problem (29) for each constraint, i.e. the subproblems. Note that the optimal solution is given in (24). Thus, updated representative values of the random parameters for the next iteration $\hat{a}^{(i)}_{\nu+1}; i = 1, \ldots, m$ are obtained. The iteration counter must then be updated $\nu \rightarrow \nu + 1$ before proceeding to **Step 2**.

**Step 5: Output.** The solution for a given tolerance corresponds to $\hat{x} = x(\nu)$, and $\hat{a}^{(i)} = \hat{a}^{(i)}_{\nu}$.

We examine the size of formulations to solve problem (1)-(3) (initial problem) using the RC formulation (11)-(13) proposed by [5], and the master (26)-(28) and subproblems (29) proposed in this paper. Assuming that all nominal values in matrix $A$ are subject to uncertainty, and given that the original problem has $n$ variables and $m$ constraints (not counting decision variable bounds), (11)-(13) is a second-order cone mathematical programming problem (QCP) with $n$ variables and $m$ quadratic constrains, the master problem (26)-(28) is a linear mathematical programming problem with $n$ variables and $m$ constrains, and each of the $m$ subproblems is a QCP with $n$ variables, and one quadratic constraint. The non-linearities in formulation (11)-(13) do not make this problem particularly attractive for solving robust discrete optimization models, however, the proposed decomposition remove those non-linearities from the master problem, which can be solved efficiently using standard mixed-integer linear programming algorithms.

4.1. Bounded random variables

So far, the method developed in this paper deals with uncertainty coefficients by using the first and second probability moments associated with
their probability distributions, and without assuming any particular type of distribution. This feature makes the method to be an approximation unless the random parameters truly follows a normal distribution. However, most of robust uncertainty sets proposed in the literature deal with random variables within a given interval. In our opinion, since we are already using an ellipsoidal set, we rather not to constraint the parameters to be inside a given interval because there are variables, specially associated with environmental conditions such as wind speed (necessary to calculate energy production), whose probability distributions are well-known unbounded distributions (Weibull for the case of wind speed) and it is difficult to define a fixed interval. Nevertheless, this information could be easily incorporated in the proposed method by including the following constraint set into the \( i \)-th-subproblem (29):

\[
-\delta^{(i)} \leq \tilde{a}^{(i)} - a^{(i)} \leq \delta^{(i)}.
\]

These constraints constitute bounded convex polytopes both in the original space associated with random model parameters \( a^{(i)}; \forall i \) and also in the transformed space \( z^{(i)}; \forall i \). To solve this modified subproblem, the best strategy is to use the analytical solution (24) and check afterwards whether constraints (30) hold, which means that the optimal solution has been achieved, or do not, which means that the target reliability index \( \beta_i \) is not achievable and the optimal solution is found by solving the following alternative problem:

\[
\begin{align*}
\hat{a}^{(i)} &= \text{Maximum} \quad \tilde{a}^{(i)T} x, \\
\text{subject to} & \\
\tilde{a}^{(i)} &= a^{(i)} + L^{(i)} z^{(i)}, \\
-\delta^{(i)} &\leq \hat{a}^{(i)} - a^{(i)} \leq \delta^{(i)},
\end{align*}
\]

The latter case is computationally more involved but it is still a linear mathematical programming problem. Given an equal reliability index \( \beta \), a 2-D graphical interpretation of the subproblems in the \( z \)-space including constrains (30) is shown in Figure 2. Figure 2 (a) shows the case where the parameter bounds are inactive, so that the analytical solution remains valid, and Figure 2 (b) shows the case where bounds are active. In this latter case the optimal solution can be obtained solving the linear formulation that remains from removing the quadratic constraint.
4.2. Convergence

Discussion about the convergence of the method is given in Appendix A.

5. Illustrative example

In order to illustrate the proposed method and the graphical interpretation of the iterative process, a simple example with only two decision variables is presented below.

Let consider the following problem:

\[
\text{Maximize } c_1 x_1 + c_2 x_2, \quad (32)
\]

subject to

\[
a_{11} x_1 + a_{12} x_2 \leq b_1, \quad (33) \\
a_{21} x_1 + a_{22} x_2 \leq b_2, \quad (34)
\]

where \( c = (3 \ 3)^T \), \( A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \), and \( b = (4 \ 4)^T \). The only uncertain parameters are those of matrix \( A \) so that their expected values are equal to their nominal values and the variance-covariance matrix associated with each constraint (33) and (34) are, respectively:

\[
\Sigma^{(1)} = \begin{pmatrix} 0.1^2 & 0.016 \\ 0.016 & 0.2^2 \end{pmatrix}; \quad \Sigma^{(2)} = \begin{pmatrix} 0.2^2 & -0.01 \\ -0.01 & 0.1^2 \end{pmatrix}.
\]

Figure 29 (a) shows the graphical illustration of the problem (32)-(34). The feasible region of the nominal problem is defined by the two gray constraints, while the black lines are contours associated with different values of the objective function. Note that its value is higher inside the infeasible region. The gray and light gray shadows are indeed 1000 different realizations of the uncertain constraints, and it can be observed that assuming as optimal solution the values given by the nominal parameters (light gray circle \( (\hat{x}_1, \hat{x}_2) \)), this point is infeasible for many realizations of the uncertain constraints.

The RC of problem (32)-(34) using [5] formulation is:

\[
\text{Maximize } c_1 x_1 + c_2 x_2, \quad (35)
\]
subject to

\[ a_{11}x_1 + a_{12}x_2 + \beta_1 \sqrt{\sum_{k=1}^{2} \left( \sum_{j=k}^{2} L_{jk}^{(1)} x_j \right)^2} \leq b_1, \quad (36) \]

\[ a_{21}x_1 + a_{22}x_2 + \beta_2 \sqrt{\sum_{k=1}^{2} \left( \sum_{j=k}^{2} L_{jk}^{(2)} x_j \right)^2} \leq b_2, \quad (37) \]

which assuming \( \beta_1 = \beta_2 = 3 \), results in the following optimal solution:

\( f = 6.47959, \hat{x}_1 = 1.20784, \hat{x}_2 = 0.95202 \). Once the optimal values of

the decision variables are obtained we check the true values of the reliability

indexes by solving problem (17), which confirms that values of the reliability

indexes truly corresponds to \( \beta_1 = \beta_2 = 3 \), and allows obtaining the maximum

likelihood points for each constraint, i.e. \( (\hat{a}_{11}, \hat{a}_{12}) = (1.27676, 2.58174) \) and

\( (\hat{a}_{21}, \hat{a}_{22}) = (2.55219, 0.96358) \).

In contrast, if the iterative method proposed in this paper is used, the

convergence to the optimal solution within an pre-specified tolerance to \( \varepsilon = 10^{-8} \)

is achieved in 4 iterations. The evolution of variables for the master and

subproblems at every iteration is given in Table 1. Note that the algorithm

converges to the same optimal values from problem (35)-(37). The graphical

illustration of the iterative process is also shown in Figure 2. For the first

iteration we select as maximum likelihood points of the uncertain parameters

their nominal values. The solution of the first master problem corresponds to

\( (\hat{x}_1, \hat{x}_2) = (1.333, 1.333) \), as shown in Figure 2(a). Once the actual values of

the decision variables are known, we update the maximum likelihood points

(M.L.P.) using the analytical solutions (21)-(23) associated with subproblems

(29) for each constraint. These process is shown in Figures 2(b) and (c),

respectively. Note that the new estimates are in the circumference of radius

\( \beta_1 = \beta_2 = 3 \) and their values are given in the second row of Table 1. Using

these M.L.P., the master problem is solved again. The solution corresponds

to \( (\hat{x}_1, \hat{x}_2) = (1.21004, 0.95116) \) and it is shown in Figure 2(d). Note that it

corresponds to the intersection given by the constrains associated with the

M.L.Ps. and it is inside the feasible region defined by the nominal values of

uncertain parameters. With these new values of the decision variables, the

M.L.Ps are updated as shown in Figures 2(e) and (f) which remain in the

circumference radius \( \beta_1 = \beta_2 = 3 \). Note that these values are very close from

those in the previous iteration. The process is repeated and shown in the

last three panels of Figure 2.
$\nu \quad \hat{a}_{11} \quad \hat{a}_{12} \quad \hat{a}_{21} \quad \hat{a}_{22} \quad \hat{x}_1 \quad \hat{x}_2 \quad ||x_\nu - x_{\nu-1}||$

| $\nu$ | 1.00000 | 2.00000 | 2.00000 | 1.00000 | 1.33333 | 1.33333 | -- |
|------|--------|--------|--------|--------|--------|--------|----|
| 1    | 1.27239 | 2.58668 | 2.51962 | 1.00000 | 1.21004 | 0.95116 | 0.16125419 |
| 2    | 1.27681 | 2.58168 | 2.55247 | 0.96322 | 1.20785 | 0.95202 | 0.00000554 |
| 3    | 1.27676 | 2.58174 | 2.55219 | 0.96359 | 1.20785 | 0.95202 | 0.00000000 |

Table 1: Evolution of the iterative algorithm for the illustrative example.

Note that in terms of computational time, results achieved using the QCP approach are slightly better. Nevertheless, we implemented all the problems using GAMS [33] and did not make any special efforts to implement the individual steps of our algorithm efficiently, for instance, by taking advantage that the problems differ slightly on the values of the parameters, specially at the latest iterations. Note also that GAMS takes some time to build the models and this is done at every iteration. Such savings could potentially improve the running time of the algorithm, but not change the number of iterations required. To compare computational performance in a more meaningful way we present the following case study.

6. Case study: Optimal Truss Design

This section considers an adapted example about optimal truss design previously used in different works [22, 34, 35], a simple supported truss with 7 elements (bars) as shown in the upper part of Figure 4. Yield stress of all members $\hat{a}_i; i = 1, \ldots, 7$ are random variables with the following mean and variances: $E[\hat{a}] = (-100, -100, -200, -200, -200, -200, -200)^T; N/mm^2$ and $E[(\hat{a} - E[\hat{a}])^2] = (15^2, 15^2, 40^2, 40^2, 40^2, 40^2, 40^2)^T; N^2/mm^4$. Note that we use negative values for yield stress because it is more convenient from the formulation perspective. There is a vertical load applied on the structure which is also normally distributed with mean $p = 100kN$ and standard deviation $\sigma_p = 40kN$.

The aim of the design problem is to determine the cross-sectional areas of the bars $x_i; i = 1, \ldots, 7$, so that the probability of failure of each bar due to the uncertainty on yield stress and load is at most 0.001. Instead of working with failure probabilities, and since random parameters are normally distributed, we use relationship (19) to define the protection level of each bar $\beta_i = 3.09; i = 1, \ldots, 7$. In case alternative distributions were used, the bound
Given by [5] could be used instead. The advantage of this example is that the problem can be easily augmented in size by simply replicating the same block structure as shown in the lower part of Figure 4. Assuming that there are \( n_b \) blocks, the robust formulation of the design problem can be written as follows:

\[
\text{Minimize} \quad \sum_{k=1}^{n_b} \sum_{i=1}^{7} c_i x_{ik},
\]

subject to

\[
p_k/\tau_i + a_{ik} x_{ik} \leq 0; \quad (a_{ik}, p_k) \in U_{ik}; \quad i = 1, \ldots, 7; \quad k = 1, \ldots, n_b \quad (39)
\]

\[
0.5 \leq x_{ik} \leq 2; \quad i = 1, \ldots, 7; \quad k = 1, \ldots, n_b, \quad (40)
\]

where \( c_i = 1 \) are the cost coefficients, and \( \tau_i \) are factors that depend on geometry and the load which are equal to \( \tau_i = 1/(2\sqrt{3}) \) for \( i = 1, 2 \), and \( \tau_i = 1/(\sqrt{3}) \) for \( i = 3, 4, 5, 6, 7 \). The left hand side of constraints (39) correspond to the difference between the actual stresses induced by the vertical load and the actual strength of the bars, note that the negative sign is implicitly included in the yield stress parameter. The optimal solution of one block in terms of decision variables is the same for all blocks, independently of the number of blocks \( n_b \) selected, for this reason we can use this example to compare computational performance between the traditional QCP and the proposed decomposition methods on problems of different size, and checking afterwards if the optimal solution is attained. Besides, we compare results assuming that cross sectional areas might take any possible value between bounds (40) and also that they can only take specific values from a given catalogue, i.e. \( x_{ik} \in \{0.5, 0.6, 0.7, \ldots, 1.8, 1.9, 2\} \).

The traditional formulation (11)-(13) proposed by [5] for this example becomes:

\[
\text{Minimize} \quad \sum_{k=1}^{n_b} \sum_{i=1}^{7} c_i x_{ik},
\]

subject to

\[
p/\tau_i + a_i x_i + \beta_i \sqrt{(\sigma_p/\tau_i)^2 + \sigma_i^2 x_{ik}^2} \leq 0; \quad i = 1, \ldots, 7; \quad k = 1, \ldots, n_b \quad (42)
\]

\[
0.5 \leq x_{ik} \leq 2; \quad i = 1, \ldots, 7; \quad k = 1, \ldots, n_b \quad (43)
\]

where \( p \) is the nominal value of all loads, \( a_i; \quad i = 1, \ldots, 7 \) and \( \sigma_i; \quad i = 1, \ldots, 7 \) are, respectively, the nominal and standard deviation of yield stresses associated with bars. Problem (41)-(43) corresponds to the case where \( x_{ij} \in \mathbb{R}; \forall ij \)
and it is a conic quadratic problem. In case cross sectional areas can only take specific values from catalogue, we have to include the following constraint:

\[ x_{ik} \in \{0.5, 0.6, 0.7, \ldots, 1.8, 1.9, 2\}; \ i = 1, \ldots, 7; \ k = 1, \ldots, n_b, \] (44)

and problem (41)-(44) is a mixed-integer conic quadratic problem.

In contrast, the master and subproblems are defined as follows:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{k=1}^{n_b} \sum_{i=1}^{7} c_i x_{ik}, \\
\text{subject to} & \quad \hat{p}_{ik}/\tau_i + \hat{a}_{ik} x_{ik} \leq 0; \ i = 1, \ldots, 7; \ k = 1, \ldots, n_b \\
& \quad 0.5 \leq x_{ik} \leq 2; \ i = 1, \ldots, 7; \ k = 1, \ldots, n_b, 
\end{align*}
\] (45)

and

\[
\begin{align*}
\text{Maximize} & \quad \tilde{p}_k/\tau_i + \tilde{a}_{ik} x_{ik} \\
\text{subject to} & \quad z_{pk} = \frac{\tilde{p}_k - p}{\sigma_p} \\
& \quad z_{ik} = \frac{\tilde{a}_{ik} - a_i}{\sigma_i} \\
& \quad z_{pk}^2 + z_{ik}^2 \leq \frac{\sigma_i^2}{\beta_i^2}, 
\end{align*}
\] (48)

The optimal solution of subproblem (48) according to (21)-(23) is:

\[
\mu_{ik} = \sqrt{\left( \frac{1}{\tau_i} x_{ik} \right) \left( \frac{\sigma_p^2}{\sigma_i^2} \frac{0}{\sigma_i^2} \frac{1}{\tau_i} \frac{\sigma_i^2}{\sigma_i^2} x_{ik} \right) / (4 \beta_i^2); \ \forall i, k} \] (49)

\[
\begin{pmatrix} \hat{p}_k \\ \hat{a}_{ik} \end{pmatrix} = \begin{pmatrix} p \\ a_i \end{pmatrix} + \begin{pmatrix} \sigma_p^2 \\ 0 \\ 0 \\ \sigma_i^2 \end{pmatrix} \left( \frac{1}{\tau_i} \frac{1}{x_{ik}} \right) / (2 \mu_{ik}); \ \forall i, k. \] (50)

The optimal solution associated with the RC and QCP problem (41)-(43) depends on the number of blocks considered \( n_b \) and it is equal to:

\[
\hat{f} = 6.63435 n_b \\
\hat{x}^{(k)} = (0.805, 0.805, 1.005, 1.005, 1.005, 1.005, 1.005)^T; \ \forall k = 1, \ldots, n_b. \] (51)
The corresponding problem has $7 \times n_b$ quadratic constraints, and $7 \times n_b + 1$ bounded and continuous variables. Analogously, the optimal solution associated with the RC and MIQCP problem (41)-(44) is equal to:

$$\hat{f} = 7.3n_b$$
$$\hat{x}^{(k)} = (0.9, 0.9, 1.1, 1.1, 1.1, 1.1, 1.1)^T; \forall k = 1, \ldots, n_b. \tag{52}$$

The associated problem has $7 \times n_b$ quadratic constraints, and $7 \times n_b$ integer variables with 15 different possible values and one continuous variable.

We also set the robust formulation proposed by [7] particularized for this example:

$$\text{Minimize} \quad x_{ik}; i = 1, \ldots, 7; k = 1, \ldots, n_b \sum_{k=1}^{n_b} \sum_{i=1}^{7} c_i x_{ik}, \tag{53}$$

subject to

$$p_{ik}/\tau_i + a_{ik} x_{ik} + \Gamma_i z_{ik} + \gamma_{ik}^p/\tau_i + \gamma_{ik}^a \leq 0; \, i = 1, \ldots, 7; k = 1, \ldots, n_b \tag{54}$$
$$0.5 \leq x_{ik} \leq 2; \, i = 1, \ldots, 7; k = 1, \ldots, n_b, \tag{55}$$
$$z_{ik} + \gamma_{ik}^p \geq \delta_i^p; \, i = 1, \ldots, 7; k = 1, \ldots, n_b \tag{56}$$
$$z_{ik} + \gamma_{ik}^a \geq \delta_i^a y_{ik}; \, i = 1, \ldots, 7; k = 1, \ldots, n_b \tag{57}$$
$$-y_{ik} \leq x_{ik} \leq y_{ik}; \, i = 1, \ldots, 7; k = 1, \ldots, n_b \tag{58}$$
$$z_{ik} \geq 0, y_{ik} \geq 0, \gamma_{ik}^p \geq 0, \gamma_{ik}^a \geq 0; \, i = 1, \ldots, 7; k = 1, \ldots, n_b \tag{59}$$

where auxiliary variables $z_{ik}, y_{ik}, \gamma_{ik}^p, \gamma_{ik}^a$ are positive variables required by the formulation, and $\delta_i^p = 100, \delta_i^a = 37.5; i = 1, 2$ and $\delta_i^a = 100; i = 3, \ldots, 7$ are the maximum absolute value deviations of the loads and bar strengths from their respective nominal values. In this particular case we have selected those values in such a way that they correspond to 2.5 times the standard deviation associated with each random parameter.

The optimal solution associated with the RC problem (53)-(59) depends on the number of blocks considered $n_b$ and it is equal to:

$$\hat{f} = 7.23427n_b$$
$$\hat{x}^{(k)} = (0.924, 0.924, 1.077, 1.077, 1.077, 1.077, 1.077)^T; \forall k = 1, \ldots, n_b. \tag{60}$$

The corresponding problem has $35 \times n_b$ linear constraints, $7 \times n_b + 1$ bounded and continuous variables and $28 \times n_b$ positive and continuous variables. Analogously, the optimal solution associated with the RC and MIP
problem (53)-(59) including constraint (44) is equal to:

\[
\hat{f} = 7.5n_b \\
\hat{x}^{(k)} = (1, 1, 1.1, 1.1, 1.1, 1.1)^T; \quad \forall k = 1, \ldots, n_b.
\]  

(61)

In order to compare the above procedures, the following problems have been solved using different current mathematical modelling solvers and different number of blocks \(n_b \in \{10, 100, 1000, 10000\}\):

1. QCP problem (41)-(43) using CONOPT [36] and COUENNE (http://www.coin-or.org/) solvers.

2. LP\(^{(1)}\) problem (45)-(50) using the decomposition procedure proposed in this paper. Note that master LP problems are solved using CBC (https://projects.coin-or.org/Cbc), CONOPT, CPLEX (http://www.ilog.com) and MINOS [37] solvers.

3. LP\(^{(2)}\) problem (53)-(59) using the robust formulation proposed by [7]. Note that master LP problems are solved using CBC (https://projects.coin-or.org/Cbc), CONOPT, CPLEX (http://www.ilog.com) and MINOS [37] solvers.

4. MIQCP problem (41)-(44) with different number of blocks \(n_b \in \{10, 100, 1000, 10000\}\), and using BONMIN[38], COUENNE, DICOPT[39] and SBB[39] solvers.

5. MIP\(^{(1)}\) problem (45)-(50) including constraint (44) and using the decomposition procedure proposed in this paper. Note that the master MIP problems are solved using BONMIN and CPLEX solvers.

6. MIP\(^{(2)}\) problem (53)-(59) including constraint (44) and using the robust formulation proposed by [7]. Note that the master MIP problems are solved using BONMIN and CPLEX solvers.

All computations have been performed on a desktop PC with four processors clocking at 2.39GHz and 3.25GB of RAM. It is worth mentioning that all results associated with the decomposition procedure (LP and MIP) are obtained after four or five iterations of the proposed method, using a tolerance of \(\epsilon = 10^{-8}\). We imposed a time limit of 9600 seconds so that if the solver does not find an optimal solution within that time the process is stopped.

Table 2 provides the computational times in seconds taken for each solver to reach the optimal solution for the different cases and problems considered. According these results, the following observations are pertinent:
| Problem       | Solver | $n_b = 10$ | $n_b = 10^2$ | $n_b = 10^3$ | $n_b = 10^4$ |
|--------------|--------|------------|---------------|---------------|---------------|
| (41)-(43) QCP| CONOPT | 0.719      | 0.703         | 51.203        | 9054.015      |
|              | COUENNE| 14.032     | 1832.890      | 9542.219      | 9600*         |
|              | CBC    | 0.469      | 0.500         | 1.828         | 68.312        |
| (45)-(50) LP(1) | CONOPT | 0.578      | 0.609         | 1.719         | 67.656        |
|              | CPLEX  | 0.500      | 0.484         | 1.703         | 67.313        |
|              | MINOS  | 0.531      | 0.500         | 2.703         | 195.141       |
|              | CBC    | 0.110      | 0.187         | 2.718         | 134.313       |
| (53)-(59) LP(2) | CONOPT | 0.125      | 2.922         | 286.828       | 9600*         |
|              | CPLEX  | 0.125      | 0.265         | 1.297         | 31.313        |
|              | MINOS  | 0.110      | 0.453         | 46.468        | 7888.547      |
|              | CBC    | 0.110      | 0.187         | 2.718         | 134.313       |
| (41)-(44) MIQCP | CONOPT | 0.218      | 1.812         | 80.813        | 9600*         |
|              | DICOPT | 1.078      | 1318.094$\S$ | 834.562$\S$  | 9600*         |
|              | SBB    | 2.156      | 42.875        | 6161.563      | 9600*         |
| (44)-(50) MIP(1) | BONMIN | 11.407     | 595.938       | 9600*         | 9600*         |
|              | CPLEX  | 0.453      | 0.500         | 1.5320        | 60.046        |
| (53)-(59),(44) MIP(2) | BONMIN | 15.625     | 1548.390      | 9600*         | 9600*         |
|              | CPLEX  | 0.234      | 0.344         | 4.172         | 363.500       |

(1) $\Rightarrow$ Proposed iterative method.
(2) $\Rightarrow$ Bertsimas and Sim (2004) method [7].
* : Maximum time limit reached and no optimal solution found.
$\S$ : No optimal solution found within maximum time limit.

Table 2: Computational results of the case study using different methods, solvers and problem types and sizes.
1. Computational time for QCP formulation increases faster with respect to the size of the problem than LP\(^{(1)}\) formulation proposed in this paper. Note in the first two rows of Table 2 that both solvers almost reached the time limit of 9600 seconds, in fact, solver COUENNE reached the time limit of 9600 without achieving the optimal solution of the problem.

2. Computational time for LP\(^{(1)}\) formulation associated with the proposed iterative method is considerably faster than the equivalent QCP formulation, as shown in rows 3 to 6 of Table 2. Note that all solvers, even those for non-linear programming, reached the solution in a reasonable amount of time.

3. Computational time for LP\(^{(2)}\) formulation associated with the method proposed by [7] is analogous to that obtained for LP\(^{(1)}\) if linear programming solvers CBC and CPLEX are used. For non-linear programming solvers the computational time grows rapidly, in fact solver CONOPT reached the time limit of 9600 without achieving the optimal solution of the problem.

4. When the number of block structures is relatively low, i.e. below 100, computational times related to the proposed decomposition procedure might be higher than the alternative quadratic formulations. This effect is produced by the deficiencies in the implementation of the algorithm, which includes model generation times and other times that could be avoided if an ad-hoc solver implementing the proposed procedure is programmed.

5. Computational time for MIQCP formulation, analogously to the QCP formulation, increases rapidly with respect to the size of the problem. For 10000 blocks none of the solvers succeed on finding the optimal solution within the maximum time frame of 9600 seconds considered in this work, this result confirms conclusion by [7] that robust optimization using ellipsoidal uncertainty sets is not particularly attractive if integer variables are involved. Note that solver DICOPT is not appropriate even for small size problems because it was no capable of reaching the optimal solution for 100 and 1000 blocks.

6. Computational time for MIP\(^{(1)}\) formulation associated with the proposed iterative method allows solving robust optimization problems using ellipsoidal uncertainty sets provided that the appropriate mixed integer solver, such as CPLEX, is used. Note that computational times are considerably lower than those related to MIQCP formulation.
7. Computational time for MIP\(^{(2)}\) formulation associated with the method proposed by [7] is analogous to that obtained for MIP\(^{(1)}\) provided that the appropriate mixed integer solver, such as CPLEX, is used. Note that when the size of the problem grows, MIP\(^{(2)}\) formulation seems to be a bit slower than MIP\(^{(1)}\), although this result is not statistically significant. However, we can conclude that both approaches MIP\(^{(1)}\) and MIP\(^{(2)}\) are alternative formulations to solve robust optimization problems involving mixed integer variables.

To further reinforce our conclusions, specially the last one in terms of tractability, we run an example using \(n_b = 30000\) blocks and solved the robust problems associated with MIP\(^{(1)}\) and MIP\(^{(2)}\) formulations. Problem MIP\(^{(1)}\) has 210001 equations, 210001 continuous variables and 210,000 discrete variables, and it was solved in 2421.110 seconds of CPU time. In contrast, problem MIP\(^{(2)}\) has 1050001 equations, 1050001 continuous variables and 210000 discrete variables, and it was solved in 1886.015 seconds of CPU time. Note that although the second alternative is faster for this particular case, both computational times are of the same order of magnitude.

Finally, we run an additional example using \(n_b = 10^5\) blocks and solved the robust problems associated with MIP\(^{(1)}\) and MIP\(^{(2)}\) formulations. Problem MIP\(^{(1)}\) has 700,001 equations, 700,001 continuous variables and 700000 discrete variables, and it was solved in 35714.485 seconds of CPU time. However, problem MIP\(^{(2)}\), which has 3500001 equations, 3500001 continuous variables and 700000 discrete variables, could not be solved due to memory problems. CPLEX solver produced an error 1001: Out of memory.

7. Conclusions

Based on first-order second-moment methods from structural reliability and decomposition techniques, this paper proposes an iterative method for solving RC of uncertain mixed-integer linear programs with ellipsoidal uncertainty. The method is specially suitable for problems where first and second order moments of the probability distributions of the uncertain parameters involved are available.

Additional advantages of the proposed method are:

1. If the uncertain parameters follow a jointly normal distribution, then the exact probability of constraint violation can be calculated through the reliability index.
2. The decoupling between decision variable and maximum likelihood updating problems allows a richer interpretation of the solution.

3. The decomposition procedure presents good convergence properties and computational behavior, making it suitable to solve large scale linear problems including integer variables.

4. The proposed method constitutes a plausible alternative for solving mixed integer linear robust problems with respect to that proposed by [7]. Both methods present analogous computational time performance, with the only advantage that the method proposed in this paper requires less memory resources.

In summary, the method proposed in this paper allows the applicability of the RC developed by [5] for a wider range of problems, even making it attractive for solving robust discrete optimization models. This decomposition procedure makes RC with ellipsoidal uncertainty to be an alternative method in terms of tractability with respect to the one proposed by [7], which uses cardinality constrained uncertainty sets.

Finally, this procedure might set the basis for alternative decomposition algorithms to solve conic programming problems besides the robust linear formulation presented in this paper. This is a subject for further research.

Appendix A. Analysis and discussion of the convergence of the proposed iterative method

In order to discuss the convergence characteristics of the proposed method and without loss of generality, the illustrative example given in Section 5 is used.

According to [5] the robust counterpart of the linear problem described in the illustrative example is a conic problem defined by the alternative constraints (10). Note that the initial linear constraints are shown as gray continues lines in Figure A.5, while the corresponding robust conic constraints:

\[ \mathbf{a}^{(i)^T} \mathbf{x} + \beta_i \sqrt{\mathbf{x}^T \Sigma^{(i)} \mathbf{x}} = b_i; \quad i = 1, 2, \]

are shown in red color continues lines. The optimal solution of the robust counterpart problem corresponds to the intersection of these conic constraints.

The proposed method recommends to use as initial values of the random parameters the nominal values \( \hat{\mathbf{a}}^{(i)}_1 = \mathbf{a}^{(i)}; i = 1, 2, \) so that the optimal
solution at the first iteration $x(1)$ corresponds to the intersection of gray line constraints (point with a grey circle specifier). This solution is far from the feasible region defined by the conic constraints, and thus according to the method proposed in this paper the next step consists of updating the point of maximum likelihood by solving the problem (29) for each constraint, that has an analytical solution. The optimal solutions are $\hat{a}_2(i); i = 1, 2$, and the resulting linear equations $\hat{a}_2(i)^T x = b_i; i = 1, 2$ correspond to tangent hyperplanes with respect to the conic constraints, as shown by the dashed gray lines in Figure A.5. Let remind the reader that according to equations (24)-(25) the optimal solutions $\hat{a}_2(i); i = 1, 2$ are the derivatives of each conic constraint with respect to decision variables $x$. Note also that hyperplanes $\hat{a}_2(i)^T x = \hat{a}_2(i)^T x(1); i = 1, 2$ are parallel to those tangent hyperplanes but going through the solution point $x(1)$ and it holds that $\hat{a}_2(i)^T x(1) > b_i; i = 1, 2$, as shown by the green dashed lines in Figure A.5, because they are deeper inside the unfeasibility region with respect the tangent hyperplanes. The next solution of the master problem corresponds to the intersection of both tangent hyperplanes. This new point is closer to the final optimal solution than the solution related to the previous iteration, and new tangent hyperplanes to the conic constraints are calculated. This process is repeated until the final solution is achieved within the required tolerance.

According to this graphical interpretation, the iterative process proposed in this paper solves a linear programming problem based on first order Taylor series approximations of the conic constraints. Conceptually, this process is a successive linear programming approach, which has shown its ease and robustness in implementation for large scale problems provided that an efficient and stable linear programming solver is available [40]. If the optimum is a vertex of the linearized feasible region, which is the case for the robust counterpart problems dealt with in this paper, then a rapid convergence is obtained. In fact, according to [40], once the algorithm enters a relatively close neighborhood of such a solution and under suitable regularity assumptions, which are always satisfied for this specific application due to the convexity character of the robust problem, it behaves like Newton’s algorithm applied to the binding constraints and a quadratic rate of convergence is obtained. That is the reason why all illustrative examples and case studies were solved in 4-5 iterations within a tolerance of $\epsilon = 10^{-8}$. 


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Joint probability distribution function contours

Limit state equation:

\[ g_i(x, \tilde{a}^{(i)}) = 0 \]

\[ g_i(x, \tilde{a}^{(i)}) < 0 \]

\[ g_i(x, \tilde{a}^{(i)}) > 0 \]

Feasible region
Unfeasible region

a) Initial random space \((\tilde{a}^{(i)})\)

b) Transformed random space \((z^{(i)})\)

Figure 1: Joint probability distribution function contours, limit-state equation, and design points in: a) the initial random space \(\tilde{a}^{(i)}\), and b) the unit standard normal random space \(z^{(i)}\).
Figure 2: Bi-dimensional graphical interpretation of the subproblems in the $z$-space including upper and lower bounds of the uncertain parameters: a) inactive bounds and b) active bounds.
Figure 3: Graphical illustration of the master and subproblem solutions associated with the iterative solution for the illustrative example. Panels on the left side correspond to master problem solutions, panels on the middle are related to subproblems for the first constraint and panels on the right side correspond to subproblems for the second constraint.
Figure 4: Truss design example.
Figure A.5: Graphical representation of the robust counterpart associated with the illustrative example and convergence interpretation.