ASYMPTOTIC BEHAVIOR OF MULTIDIMENSIONAL SCALAR RELAXATION SHOCKS

BONGSUK KWON AND KEVIN ZUMBRUN

Abstract. We establish pointwise bounds for the Green function and consequent linearized stability for multidimensional planar relaxation shocks of general relaxation systems whose equilibrium model is scalar, under the necessary assumption of spectral stability. Moreover, we obtain nonlinear $L^2$ asymptotic behavior/sharp decay rate of perturbed weak shocks of general simultaneously symmetrizable relaxation systems, under small $L^1 \cap H^{[d/2]+3}$ perturbations with first moment in the normal direction to the front.

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1. Introduction

In this paper, we investigate the time-asymptotic stability of multidimensional planar shocks of general relaxation systems whose equilibrium model is scalar.

Consider hyperbolic relaxation systems of general form

\[
\begin{pmatrix}
u
\end{pmatrix}	 + \sum_{j=1}^{d}
\begin{pmatrix}
f_{j}(u,v) \\
g_{j}(u,v)
\end{pmatrix}_{x_j} = \begin{pmatrix}0 \\
-\tau^{-1}q(u,v)
\end{pmatrix}
\]

\(u, f_j, g_j, q \in \mathbb{R}^1, v, q \in \mathbb{R}^r,\) with the condition

\[
\text{Re } \sigma(q_v(u, v^*(u))) < 0
\]

along a smooth equilibrium manifold defined by

\[
E := \{(u, v)|q(u, v) = 0\}
\]

and \(\tau\) determines relaxation time. The first equation and the second \(r\) equations represent a conservation law for \(u\) and relaxation rate equations for \(v\), respectively. The condition \((1.2)\) implies that a perturbed solution eventually relaxes to the equilibrium state.

The first equation for \(u\) can be approximated by a hyperbolic conservation law and a parabolic conservation law to “zeroth order” and “first order”, respectively. Here, the corresponding order is with respect to parameter \(\tau\) determining the relaxation time. To “zeroth order”, the corresponding “relaxed” scalar equation is

\[
u_t + \sum_{j=1}^{d} f_{j}^{*}(u)_{x_j} = 0,
\]
where \( f_j(u) := f_j(u, v_u(u)) \), and to “first order”, the corresponding parabolic conservation laws is

\[
(1.5) \quad u_t + \sum_{j=1}^{d} f_j^*(u)_{xj} = (b_j^k(u) u_{xk})_{xj},
\]

where

\[
(1.6) \quad b_j^k = \begin{cases} 
-f_j q_v^{-1} (g_j - g_j q^{-1}_v q_u) - (f_j - f_j q^{-1}_v q_u) q^{-1}_v q_u), & \text{if } j = k \\
-\frac{1}{2} \left( f_j q_v^{-1} (g_j - g_j q^{-1}_v q_u) + (f_k - f_k q^{-1}_v q_u) q^{-1}_v q_u) \right), & \text{if } j \neq k
\end{cases}
\]

is determined by the expansion of Fourier symbol (Chapmann-Enskog expansion) as in Appendix A.

These approximation equations suggest us to investigate the existence of shock wave solutions. A planar relaxation shock wave is a traveling wave solution of (1.1) satisfying

\[
(1.7) \quad (u, v)(x, t) = (\bar{u}, \bar{v})(x_1 - st),
\]

where the end states \((u_\pm, v_\pm)\) satisfy \(v^*(u_\pm) = v_\pm\) and \(u_\pm\) is a shock solution of (1.4).

Such traveling wave solutions are known to exist for small amplitude profiles, see for example, [Liu, YoZ, MZ1]. However, profiles of large amplitude may develop “subshocks” or jump discontinuities. We restrict here to the smooth and small-amplitude case.

Stability of such multidimensional planar shock wave solutions has been studied for specific models. Nonlinear stability of planar shock fronts for the 3 × 3 Jin-Xin model in two spatial dimension has been proved in [Li]. For a two-dimensional shallow river model, Ha and Yu proved nonlinear stability of small amplitude shocks in [HY]. Both of these analyses follow the approach introduced by Goodman in [Go] to treat the related scalar viscous case, based on energy estimates and conservation of mass, yielding sup-norm convergence to the unperturbed front with no rate. This method has since been greatly sharpened in the scalar viscous case, using shock-tracking and spectral (inverse Laplace-transform) methods and pointwise estimates on the resolvent to obtain asymptotic behavior and sharp rates
of decay for general scalar models; see e.g., [GM, HoZ1, HoZ2]. However, up to now, no comparable result has been carried out for the relaxation case.

In the present paper, generalizing the results of [HoZ1, HoZ2] in the viscous case, we prove stability, with asymptotic behavior and sharp rates of decay, of small-amplitude multidimensional planar relaxation shocks of $N \times N$ general systems (1.1) whose equilibrium model is scalar, under the following assumptions.

**Assumptions 1.1.**

$(H0)$ \( f^j, g^j, q \in C^{m+1}, m \geq [d/2] + 2. \)

$(H1)$ (i) \( \sigma(\sum_j \xi_j (df^j, dg^j))^l(u, v) \) real, semi-silultiplicity, for all \( \xi \in \mathbb{R}^d \), and (ii) \( (\sigma(df^1, dg^1))^l(u, v) \) different from \( s. \)

$(H2)$ \( \sigma(\sum_j \xi_j df^j_\delta(u_\pm)) \) real, distinct and different from \( s. \)

$(H3)$ \( \Re(\sum_{j=1}^d i \xi_j (df^j, dg^j))^l(u_\pm, v_\pm) - (0, dq)^l(u_\pm, v_\pm) \leq -\theta|\xi|^2/(1 + |\xi|^2) \) for all \( \xi \in \mathbb{R}^d \), \( \theta > 0. \)

$(H4)$ The set of solutions of (1.1) forms a smooth manifold \((\bar{u}_\delta, \bar{v}_\delta), \delta \in \mathcal{U} \in \mathbb{R}^1.\)

Let \( D(\lambda, \tilde{\xi}) \) as in Definition 2.6, Section 2.2 denote the Evans function associated with Fourier transform \( L_{\tilde{\xi}} \) of the linearized operator about the wave, a function that is analytic in \( \lambda \) for \( \Re \lambda \geq -\theta, \theta > 0, \) with zeros corresponding with eigenvalue of \( L_{\tilde{\xi}}. \) (For history and further discussion of the Evans function, see [AGJ, GZ, PZ] and references therein.)

**Assumptions 1.2. (Strong spectral stability conditions)**

$(D1)$ \( D(\cdot, \tilde{\xi}) \) has no zeroes in \( \{\Re \lambda \geq 0\} \) except at \( \tilde{\xi} = \lambda = 0. \)

$(D2)$ \( (d/d\lambda)D(0, 0) \neq 0. \)

$(D3)$ A zero \( \lambda_*(\tilde{\xi}) \) of \( D(\cdot, \tilde{\xi}) \) satisfies \( \lambda_*(0) = 0 \) and \( \Re \lambda_*(\tilde{\xi}) \leq -\theta|\tilde{\xi}|^2 \) for \( |\tilde{\xi}| \) sufficiently small. (Existence and local uniqueness of \( \lambda_*(\tilde{\xi}) \) are guaranteed by the Implicit Function Theorem and (D2).)

\(^1\) Though the analysis of [Go] also involves an approximate front location, this is determined by a zero residual mass condition convenient for energy estimates rather than considerations of asymptotic behavior, and involves errors of the same magnitude as the perturbation itself; see the discussions in [GM, HoZ1, HoZ2].
Sometimes it is more convenient to write (1.1) in the abbreviated form

\[(1.8) \quad U_t + \sum_{j=1}^{d} A^j(U) U_{x_j} = \tau^{-1} Q(U)\]

where \(U = (u, v)^t\), \(A^j(U) = (df^j, dg^j)^t(u, v)\) and \(Q(U) = (0, q(u, v))^t\).

**Assumptions 1.3.**

(A1) (1.8) is symmetrizable in the sense that there exists \(A^0\) symmetric, positive definite such that \(A^0 A^j\) are symmetric for all \(j = 1, 2, ..., d\) and \(A^0 dQ\) is symmetric, negative semidefinite.

(A2) (Kawashima condition) There exists the operator \(K(\partial_x)\) such that

\[(1.9) \quad \hat{K}(\partial_x) f(\xi) = i\hat{K}(\xi) \hat{f}(\xi)\]

where \(\hat{K}(\xi)\) is a skew-symmetric operator which is smooth and homogeneous degree one in \(\xi\) satisfying

\[(1.10) \quad \Re(\xi^2 A^0 dQ - \sum_{j=1}^{d} \xi_j \hat{K}(\xi) A^j)_\pm \leq -\theta |\xi|^2 \text{ for all } \xi \text{ in } \mathbb{R}^d\]

**Remark 1.4.** If (1.8) satisfies (A1) and the Genuine-Coupling condition that no eigenvector of \(\sigma \left( i \sum_{j=1}^{d} \xi_j (df^j, dg^j)^t(u_\pm, v_\pm) \right)\) lies in the kernel of \(dQ(u_\pm, v_\pm)\), then (A2) holds \([K, SK, MZ5, Z4, Z5]\). Moreover, conditions (A1)–(A2) imply (H3) \([K, SK, Ze]\).

Conditions (H0)–(H4), (A1)–(A2) are the standard set of hypotheses proposed by W.A. Yong for relaxation systems \([Y0]\), as adapted to the shock case by Mascia and Zumbrun \([MZ1]\). As described in \([MZ1, MZ5]\), (A1)–(A2) are satisfied for a wide variety of relaxation systems, in which case all of (H0)–(H4) are satisfied for sufficiently small-amplitude profiles under the single condition (H1)(ii).

Before we state our main theorem, we briefly go over the idea used by Goodman and Miller \([GM]\) to give a formal qualitative description of the behavior of the linear perturbation
\[ U(x, t) := \tilde{U}(x, t) - \bar{U}(x_1), \]
where \( \tilde{U}(x, t) \) is a solution of (1.1) and \( \bar{U}(x_1) \) is a shock wave solution. Linearizing (1.1) about \( \bar{U}(x_1) \), we obtain the linearized perturbation equations

\[ (1.11) \quad U_t = LU := -\sum_{j=1}^{d} (A^jU)_{x_j} + (dQ)U \]

where

\[ (1.12) \quad A^j := dF^j(\bar{U}(x_1)), \quad dQ := dQ(\bar{U}(x_1)) \]

depend only on the normal direction \( x_1 \).

We can give a heuristic approach to describe the behavior of the perturbation \( U = \tilde{U} - \bar{U} \). First, we approximate the operator \( e^{L_t} \) by its formal spectral projection

\[ (1.13) \quad e^{\lambda^* t} \varphi(\tilde{x}, \tilde{U}_0), \]

onto the top eigenfunction of \( L_{\tilde{\xi}} \), assuming a perturbation expansion

\[ (1.14) \quad \lambda^*(\tilde{\xi}) = \tilde{\gamma}^1 \cdot \tilde{\xi} + \tilde{\xi}^2 \tilde{\beta} + \ldots \]

\[ = i\tilde{\alpha} \cdot \tilde{\xi} - \tilde{\xi}^2 \tilde{\beta} + \ldots \]

of the corresponding eigenvalue \( \lambda^* \). Next we apply the method of stationary phase to the inverse Fourier transform to obtain the approximation

\[ (1.15) \quad U(x, t) = e^{L_t} U_0 = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i\tilde{\xi} \cdot \tilde{x}} e^{i\tilde{\xi} \cdot \tilde{U}_0(t)} d\tilde{\xi} \]

\[ \approx \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{\lambda^* t} \varphi(\tilde{\psi}, \tilde{U}_0) e^{i\tilde{\xi} \cdot \tilde{x}} d\tilde{\xi} \]

\[ \approx -\frac{\tilde{U}'(x_1)}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i\tilde{\xi} \cdot \tilde{x}} e^{(i\tilde{\alpha} \cdot \tilde{\xi} - \tilde{\xi}^2 \tilde{\beta}) t} \delta_0(\tilde{\xi}) d\tilde{\xi} \]

\[ \approx -\bar{U}'(x_1) \delta(\tilde{x}, t) \]

where \( \delta(\tilde{x}, t) \) satisfies the transverse convection-diffusion equation

\[ (1.16) \quad \delta_t + \tilde{\alpha} \cdot \nabla_{\tilde{x}} \delta = \text{div}_{\tilde{x}}(\tilde{\beta} \nabla_{\tilde{x}} \delta) \]
with initial data

\( \delta_0(\tilde{x}) = \langle \tilde{\psi}, U_0(x) \rangle_{L^2(x_1)} = -([u]^{-1}, 0) \int U_0(x)dx_1. \)

The following theorem shows that the formal linear approximation \( U(x, t) \sim -\delta(\tilde{x}, t)\bar{U}'(x_1) \) is in fact valid at the nonlinear level.

**Theorem 1.5.** For fixed \( U_- \), let \( \bar{U} \) be a relaxation shock profile (1.7) satisfying (H0)–(H5), (A1)–(A2), (D1)–(D3), with amplitude \( |U_+ - U_-| \) sufficiently small. If \( |\tilde{U}_0 - \bar{U}|_{L^1}, |\tilde{U}_0 - \bar{U}|_{L^2}, |x_1(\tilde{U}_0 - \bar{U})|_{L^1}, |\tilde{U}_0 - \bar{U}|_{H^{d/2}+3} \leq \zeta_0 \) sufficiently small, then for arbitrary small \( \sigma > 0 \), there holds

\[ |\tilde{U}(x, t) - \bar{U}(x_1 - \delta(\tilde{x}, t))|_{L^2(x)} \leq C\zeta_0(1 + t)^{-\frac{1}{2(d-1)+1/2+\sigma}} \]

for dimensions \( d \geq 2 \), with \( \delta \) as defined in (1.16)–(1.17) and \( \tilde{\alpha}, \tilde{\beta} \) as in (1.14). Moreover, the above result holds with \( \sigma = 0 \) for dimensions \( d \geq 3 \).

1.1. Discussion and open problems. Previous results [Li, HY] on multidimensional scalar relaxation fronts rely on the structure of specific models and give only stability without rates or behavior. Theorem 1.5 by contrast applies to general equations, giving sharp decay rates and a detailed picture of asymptotic behavior. On the other hand, it relies on the assumption of spectral stability, which must be verified.

We conjecture that conditions (D1)–(D3) might be verified by a singular perturbation argument like that of [PZ] for the one-dimensional (system) case. Verification by this or other means is an important open problem. Another interesting open problem would be to remove the restrictive hypothesis (H1)(ii) as discussed in [MZ5, MZ6].

The main obstruction to the application to the relaxation problem of the spectral techniques of [GM, HoZ1, HoZ2] is to treat the more singular high-frequency behavior associated with the hyperbolic nature of the equations. In the viscous case, the linearized operator about the wave is sectorial, generating an analytic semigroup, and high-frequency contributions are essentially negligible. In the relaxation case, the linearized operator generates a \( C^0 \) semigroup, and there is substantial high-frequency contribution. We conjecture that it is this difficulty that has so far prevented application of these techniques, despite their advantages of generality and detailed information on asymptotic behavior.
This difficulty was overcome in the one-dimensional analysis of [MZ1] by direct calculation/detailed asymptotic expansion. However, this would appear quite complicated to carry out in the multi-dimensional case. Here, we follow instead a simplified version of an approach suggested in [Z4] in the context of hyperbolic–parabolic systems, based on high-frequency energy estimates. This is quite general, and should find application to other problems with delicate high-frequency behavior; in particular, it applies with little modification to the case of relaxation equations whose equilibrium models are systems, preparing the way for a treatment of multidimensional shock fronts in this case, another important open problem. We regard this method of treating high frequencies as perhaps the main new contribution of this paper.

A point that should be mentioned is that the analysis of [HoZ1, HoZ2] was for arbitrary amplitude shock waves, whereas the present analysis is limited to the small-amplitude case. The reasons for this are two. First, there is a limitation already at the level of the existence problem, since subshocks may form for too-large amplitudes in general [Lm]. However, supposing existence of a sufficiently smooth shock profile, we face a technical problem in carrying out the energy estimates of Section 1 in the presence of characteristics moving both to the left and to the right. This was overcome with great difficulty in the one-dimensional case in [MZ5]; we do not know how or whether this is possible in multi-dimensions.

As a final open problem, we mention the treatment for relaxation systems of existence and stability of relaxation shock layers in the small-relaxation time limit, analogous to the small-viscosity analysis of [GMWZ] in the hyperbolic–parabolic case.

2. Preliminaries

We begin with a series of preparatory steps, loosely following [MZ1, Z4].

2.1. Spectral resolution formulae. We derive the spectral resolution formula by proving that the linearized operator about the wave generates $C^0$ semigroup.

Linearizing (1.1) about the wave $\bar{U}(x_1)$, we have the linearized equations

\[ U_t = LU := - \sum_j (\bar{A}_j U)_{x_j} + dQU, \quad U(0) = U_0, \]

2 (1)
where \( dQ = dQ(\bar{U}(x_1)) \), \( \bar{A}^j = dF^j(\bar{U}(x_1)) \). Taking the Fourier transform in the transverse directions \( \tilde{x} := (x_2, \ldots, x_d) \), we reduce to a family of partial differential equations (PDE)

\[
\dot{\hat{U}}_t = L_{\bar{\xi}} \hat{U} := -(\bar{A}^1 \hat{U})' - i \sum_{j=2}^d \xi_j \bar{A}^j \hat{U} + dQ \hat{U}, \quad \hat{U}(0) = \hat{U}_0
\]

in \((x_1, t)\) indexed by frequency \( \tilde{\xi} \in \mathbb{R}^{d-1} \), where \( \hat{U} = \hat{U}(x_1, \tilde{\xi}, t) \) denotes the Fourier transform of \( U = U(x, t) \) in \( \tilde{x} \) and \( "'" \) denotes \( d/dx_1 \). Taking the Laplace transform in \( t \), we reduce to the resolvent equation

\[
(\lambda - L_{\bar{\xi}}) \hat{U} = \hat{U}_0
\]

where \( \hat{U}(x_1, \tilde{\xi}, \lambda) \) denotes the Laplace-Fourier transform of \( U = U(x, t) \).

**Definition 2.1.** (a) The Green function \( G(x, t; y) \) associated with the linearized equations (2.1) is defined by

(i) \( (\partial_t - L_{\bar{\xi}})G = 0 \) in the distributional sense, for all \( t > 0 \) and,

(ii) \( G(x, t; y) \to \delta(x - y) \) as \( t \to 0 \).

(b) The resolvent kernel \( G_{\lambda, \bar{\xi}}(x_1, y_1) \) associated with the resolvent equation (2.3) is defined as a distributional solution of

\[
(\lambda - L_{\bar{\xi}})G_{\lambda, \bar{\xi}}(x_1, y_1) = \delta(x - y).
\]

Formally, one can write

\[
G(x, t; y) := e^{Lt} \delta(x - y)
\]

and

\[
G_{\lambda, \bar{\xi}}(x_1, y_1) := (\lambda - L_{\bar{\xi}})^{-1} \delta(x_1 - y_1)
\]

**Proposition 2.2 ([MZ1, Z4]).** Under assumptions \((H0)-(H4), (A1)-(A2)\), \( L \) generates a \( C^0 \) semigroup \( |e^{Lt}| \leq C e^{|\eta|t} \) on \( L^2 \) with domain \( \mathcal{D}(L) := \{ U : U, LU \in L^2 \} \), satisfying the generalized spectral resolution formula, for some \( \eta > \eta_0 \),

\[
G(x, t; y) = \frac{1}{(2\pi i)^d} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} \int_{\mathbb{R}^{d-1}} e^{i\tilde{\xi} \cdot \tilde{x} + \lambda t} G_{\lambda, \bar{\xi}}(x_1, y_1) d\tilde{\xi} d\lambda,
\]
Proof. Performing an elementary energy estimate, under the assumption of symmetrizability of \( (1.8) \), we establish that

\[
|A^0 U|_{L^2} \leq |\lambda - \lambda_*|^{-1} |A^0 (L - \lambda) U|_{L^2}
\]

for a symmetrizer \( A^0 \) and all \( U \in D(L) \) and real \( \lambda \) greater than some value \( \lambda_* \). If, in addition, \( A^j \) and \( Q \) are asymptotically constant as \( x_1 \to \pm \infty \), then it is shown that \( L \) generates a \( C^0 \) semigroup \( e^{Lt} \) on \( L^2 \), satisfying \( |e^{Lt}|_{L^2} \leq C e^{\omega t} \) for some real \( \omega \) [MZ2]. This is done by \((2.8)\) and a standard result of Henry. Therefore, by [Pa], p.1., the inverse Laplace-Fourier Transform formula holds for \( L, e^{Lt} \):

\[
e^{Lt} f = \frac{1}{(2\pi i)^d} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} \int_{\mathbb{R}^{d-1}} e^{\tilde{\xi} \cdot \tilde{x} + \lambda (L_{\tilde{\xi}} - \lambda)^{-1} d\tilde{\xi} d\lambda,}
\]

As a consequence, \((2.7)\) holds in the distribution sense. \( \square \)

2.2. The Evans function. Consider the homogeneous eigenvalue equation

\[
(\lambda - L_{\tilde{\xi}}) W = (\lambda + i \sum_{j=2}^{d} \xi_j A^j - dQ) W + (A^1 W)' = 0
\]

and its constant-coefficient limits as \( x_1 \to \pm \infty \),

\[
(L_{\pm,\tilde{\xi}} - \lambda) W = (dQ_{\pm} - A_{\pm,\tilde{\xi}} - \lambda) W - (A^1_{\pm} W)' = 0
\]

or, alternatively,

\[
W' = (A^1_{\pm})^{-1} \left( dQ_{\pm} - \sum_{j=2}^{d} i \xi_j A_{j,\pm} - \lambda \right) W.
\]

Definition 2.3. The domain of consistent splitting \( \Lambda \) is defined as the connected component of \( (\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} \) containing \( \tilde{\xi} = 0 \) and \( \lambda \) going to real \(+\infty\), for which the coefficients

\[
(A^1_{\pm})^{-1} \left( dQ_{\pm} - \sum_{j=2}^{d} i \xi_j A_{j,\pm} - \lambda \right)
\]

in \((2.12)\) have \( k \) eigenvalues of negative real part and \( N - k \) eigenvalues of positive real part, with no pure imaginary eigenvalues.
Lemma 2.4. Under assumptions (H0)-(H1), (H3),

\begin{equation}
\Lambda \subset \{ (\lambda, \tilde{\xi}) : \Re \lambda > -\theta |\tilde{\xi}|^2/(1 + |\tilde{\xi}|^2) \}.
\end{equation}

In particular, for \(|(\lambda, \tilde{\xi})| \geq r > 0\), arbitrary, \(\Lambda \subset \{ \lambda : \Re \lambda \geq -\eta \} \times \mathbb{R}^{d-1}\), where \(\eta(r) := \theta r^2 > 0\).

Proof. Noting that eigenvalues \(\mu(\lambda, \tilde{\xi})\) of coefficient (2.13) relate to solutions of the dispersion-relation

\[ \lambda(\xi) \in \sigma \left( dQ_{\pm} - \sum_{j=1}^{d} i\xi_j A_{j,\pm} - \lambda \right) \]

by the relation \(\mu = i\xi_1\), we find by Assumption (H3) that the coefficient has no pure imaginary eigenvalues when \(\Re \lambda > -\theta |\tilde{\xi}|^2/(1 + |\tilde{\xi}|^2)\) for \(\xi = (\xi_1, \tilde{\xi})\), all \(\xi_1 \in \mathbb{R}\), or equivalently \(\Re \lambda > -\theta |\tilde{\xi}|^2/(1 + |\tilde{\xi}|^2)\). A straightforward homotopy argument taking \(\lambda\) to real plus infinity then gives the result; see [MZ1, Z3, Z4]. □

Proposition 2.5. Under assumptions (H0)-(H1), (H3), for \((\lambda, \tilde{\xi})\) in the domain of consistent splitting \(\Lambda\), there are solutions of (2.10)

\begin{equation}
\{ \varphi^{+}_{1}(x_1; \lambda, \tilde{\xi}), \ldots, \varphi^{+}_{k}(x_1; \lambda, \tilde{\xi}) \}
\end{equation}

and

\begin{equation}
\{ \varphi^{-}_{k+1}(x_1; \lambda, \tilde{\xi}), \ldots, \varphi^{-}_{N}(x_1; \lambda, \tilde{\xi}) \},
\end{equation}

\(N = r+1\), which are locally analytic (in \((\lambda, \tilde{\xi})\)) bases for the stable and unstable manifolds as \(x_1 \to +\infty\) and \(x_1 \to -\infty\), respectively, that is, the (unique) manifolds of solutions decaying exponentially as \(x_1 \to \pm \infty\). There are also solutions of (2.11)

\begin{equation}
\{ \psi^{-}_{1}(x_1; \lambda, \tilde{\xi}), \ldots, \psi^{-}_{k}(x_1; \lambda, \tilde{\xi}) \}
\end{equation}

and

\begin{equation}
\{ \psi^{+}_{k+1}(x_1; \lambda, \tilde{\xi}), \ldots, \psi^{+}_{N}(x_1; \lambda, \tilde{\xi}) \}
\end{equation}

which are locally analytic (in \((\lambda, \tilde{\xi})\)) bases for stable and unstable manifolds as \(x_1 \to -\infty\) and \(x_1 \to +\infty\), respectively, that is, manifolds of solutions blowing up exponentially as \(x_1 \to \pm \infty\) (not unique).
Proof. This standard result holds for general variable-coefficient systems whose coefficients converge exponentially as \( x_1 \to \pm \infty \) (a consequence of the gap and conjugation lemmas of Appendix B; see [MZ1, MZ3, Z3, Z4]. □

Definition 2.6. (Evans function) For \((\lambda, \tilde{\xi})\) in the domain of consistent splitting, we define the Evans function as

\[
D(\lambda, \tilde{\xi}) := \det (\varphi_1^+, \ldots, \varphi_k^+, \varphi_{k+1}^-, \ldots, \varphi_N^-)|_{x_1=0}.
\]

Evidently, the Evans function is locally analytic in \((\lambda, \tilde{\xi})\) in the domain of consistent splitting, with zeros of \(D(\cdot, \tilde{\xi})\) corresponding to eigenvalues of \(L_{\tilde{\xi}}\). We shall show in Section 2.4 that \(\varphi_k^\pm\), hence \(D\) as well, extend analytically to a domain

\[
(\lambda, \tilde{\xi}) \in \{\lambda : \Re \lambda \geq -\eta\} \times \mathbb{R}^{d-1}, \quad \eta > 0.
\]

2.3. Construction of the resolvent kernel. We next derive explicit representation formulae for the resolvent kernel \(G_{\lambda, \tilde{\xi}}\). We seek a solution of form

\[
G_{\lambda, \tilde{\xi}}(x_1, y_1) = \begin{cases} 
\Phi^+(x_1; \lambda, \tilde{\xi})N^+(y_1; \lambda, \tilde{\xi}), & x_1 > y_1 \\
\Phi^-(x_1; \lambda, \tilde{\xi})N^-(y_1; \lambda, \tilde{\xi}), & x_1 < y_1
\end{cases}
\]

where

\[
\Phi^+(x_1; \lambda, \tilde{\xi}) = (\varphi_1^+(x_1; \tilde{\xi}, \lambda), \ldots, \varphi_k^+(x_1; \tilde{\xi}, \lambda)) \in \mathbb{R}^{N \times k}
\]

and

\[
\Phi^-(x_1; \lambda, \tilde{\xi}) = (\varphi_{k+1}^-(x_1; \tilde{\xi}, \lambda), \ldots, \varphi_N^-(x_1; \tilde{\xi}, \lambda)) \in \mathbb{R}^{N \times (N-k)}.
\]

Imposing the jump condition

\[
\begin{pmatrix} 
\Phi^+(y_1; \lambda, \tilde{\xi}) & \Phi^-(y_1; \lambda, \tilde{\xi}) 
\end{pmatrix} 
\begin{pmatrix} 
N^+(y_1; \lambda, \tilde{\xi}) \\
N^-(y_1; \lambda, \tilde{\xi}) 
\end{pmatrix} = -(A^1)^{-1}(y_1),
\]

and inverting (2.23), we express for the resolvent kernel \(G_{\lambda, \tilde{\xi}}\):
\begin{equation}
G_{\lambda,\tilde{\xi}}(x_1, y_1) = \begin{cases} 
-\left( \Phi^+(x_1; \lambda, \tilde{\xi}) 0 \right) \left( \Phi^+ \Phi^- \right)^{-1}(y_1; \lambda, \xi)(A^1)^{-1}(y_1), & x_1 > y_1 \\
(0 \Phi^-(x_1; \lambda, \tilde{\xi})) \left( \Phi^+ \Phi^- \right)^{-1}(y_1; \lambda, \xi)(A^1)^{-1}(y_1), & x_1 < y_1 
\end{cases}
\end{equation}

Now, consider the dual equation of (2.10).

\begin{equation}
(L^*_{\tilde{\xi}} - \lambda^*) \tilde{W} = 0
\end{equation}

where

\[ L^*_{\tilde{\xi}} \tilde{W} := (A^1)^* \tilde{W}' + \left( dQ^* - A^*_{\tilde{\xi}} \right) \tilde{W} \]

\textbf{Lemma 2.7.} For any \(W, \tilde{W}\) such that \((L_{\tilde{\xi}} - \lambda) W = 0\) and \((L^*_{\tilde{\xi}} - \lambda^*) \tilde{W} = 0\), there holds

\begin{equation}
\langle \tilde{W}, A^1 W \rangle \equiv \text{constant}
\end{equation}

\textbf{Proof.}

\[ \langle \tilde{W}, A^1 W \rangle' = \langle (A^1)^* \tilde{W}', W \rangle + \langle \tilde{W}, (A^1 W)' \rangle \]
\[ = \langle (\lambda^* I - dQ^* + A^*_{\tilde{\xi}}) \tilde{W}, W \rangle + \langle \tilde{W}, (-\lambda I + dQ - A_{\tilde{\xi}}) W \rangle \]
\[ = 0 \]

From (2.26), it follows that if there are \(k\) independent solutions \(\varphi^+_1, \ldots, \varphi^+_k\) of \((L_{\tilde{\xi}} - \lambda I) W = 0\) decaying at \(+\infty\) and \(N - k\) independent solutions \(\varphi^-_{k+1}, \ldots, \varphi^-_N\) of the same equation decaying at \(-\infty\), then there exist \(N - k\) independent solutions \(\tilde{\psi}^+_1, \ldots, \tilde{\psi}^+_N\) of \((L^*_{\tilde{\xi}} - \lambda^* I) \tilde{W} = 0\) decaying at \(+\infty\) and \(k\) independent solutions \(\tilde{\psi}^-_1, \ldots, \tilde{\psi}^-_k\) of the same equation decaying at \(-\infty\). More precisely, setting

\begin{equation}
\Psi^+(x_1; \lambda, \tilde{\xi}) = \left( \psi^+_1(x_1; \lambda, \tilde{\xi}) \cdots \psi^+_N(x_1; \lambda, \tilde{\xi}) \right) \in \mathbb{R}^{N \times (N-k)}
\end{equation}

\begin{equation}
\Psi^-(x_1; \lambda, \tilde{\xi}) = \left( \psi^-_1(x_1; \lambda, \tilde{\xi}) \cdots \psi^-_k(x_1; \lambda, \tilde{\xi}) \right) \in \mathbb{R}^{N \times k}
\end{equation}
where \( \psi_j^\pm \) are the exponentially growing solutions at \( \pm \infty \) respectively, of \((L_\tilde{\xi} - \lambda I)W = 0\) as described above. We may define dual exponentially decaying and growing solutions \( \tilde{\psi}_j^\pm \) and \( \tilde{\phi}_j^\pm \) via

\[
(2.30) \quad (\tilde{\psi} \quad \tilde{\phi})^*_\pm A_1^\dagger (\psi \quad \Phi)^\dagger_\pm = I
\]

We seek the Green function \( G_{\lambda, \tilde{\xi}} \) in the form

\[
(2.31) \quad G_{\lambda, \tilde{\xi}}(x_1, y_1) = \begin{cases} 
\Phi^+(x_1; \lambda, \tilde{\xi}) M^+(\lambda, \tilde{\xi}) \tilde{\Psi}^-*(y_1; \lambda, \tilde{\xi}) , & x_1 > y_1 \\
\Phi^-(x_1; \lambda, \tilde{\xi}) M^-(\lambda, \tilde{\xi}) \tilde{\Psi}^+*(y_1; \lambda, \tilde{\xi}) , & x_1 < y_1
\end{cases}
\]

where

\[
(2.32) \quad M(\lambda, \tilde{\xi}) := \begin{pmatrix} -M^+(\lambda, \tilde{\xi}) & 0 \\ 0 & M^-(\lambda, \tilde{\xi}) \end{pmatrix} = \Phi^{-1}(z; \lambda, \tilde{\xi}) (A^1)^{-1}(z) \tilde{\Psi}^{-1*}(z; \lambda, \tilde{\xi})
\]

and

\[
(2.33) \quad \tilde{\Psi} := (\tilde{\Psi}^- \quad \tilde{\Psi}^+).
\]

Note that the independence of the righthand side with respect to \( z \) is a consequence of the previous lemma. Thus,

\[
(2.34) \quad G_{\lambda, \tilde{\xi}}(x_1, y_1) = \begin{cases} 
-\left( \Phi^+(x_1; \lambda, \tilde{\xi}) \ 0 \right) M(\lambda, \tilde{\xi}) \left( \tilde{\Psi}^- (y_1; \lambda, \tilde{\xi}) \ 0 \right)^* , & x_1 > y_1 \\
\left( 0 \ \Phi^-(x_1; \lambda, \tilde{\xi}) \right) M(\lambda, \tilde{\xi}) \left( 0 \ \tilde{\Psi}^+ (y_1; \lambda, \tilde{\xi}) \right)^* , & x_1 < y_1
\end{cases}
\]
Proposition 2.8. For \((\lambda, \tilde{\xi}) \in \Lambda\), there hold

\[
G_{\lambda, \tilde{\xi}}(x_1, y_1) = \begin{cases} 
\sum_{k,j} M_{jk}^+(\lambda, \tilde{\xi}) \varphi_j^+(x_1; \lambda, \tilde{\xi}) \tilde{\psi}_k^-(y_1; \lambda, \tilde{\xi})^* & \text{for } y_1 \leq 0 \leq x_1, \\
\sum_{k,j} d_{jk}^+(\lambda, \tilde{\xi}) \varphi_j^-(x_1; \lambda, \tilde{\xi}) \tilde{\psi}_k^-(y_1; \lambda, \tilde{\xi})^* & \text{for } y_1 \leq x_1 \leq 0, \\
\sum_{k,j} d_{jk}^-(\lambda, \tilde{\xi}) \varphi_j^-(x_1; \lambda, \tilde{\xi}) \tilde{\psi}_k^-(y_1; \lambda, \tilde{\xi})^* & \text{for } x_1 \leq y_1 \leq 0, \\
+ \sum_k \varphi_k^-(x_1; \lambda, \tilde{\xi}) \tilde{\varphi}_k^-(y_1; \lambda, \tilde{\xi})^* & \text{for } x_1 \leq y_1 \leq 0, 
\end{cases}
\]

with

\[
M^+ = (-I, 0) \left( \Phi^+ \Phi^- \right)^{-1} \Psi^- 
\]

and

\[
d^+ = (0, I) \left( \Phi^+ \Phi^- \right)^{-1} \Psi^- .
\]

A symmetric representation holds for \(y_1 \geq 0\).

Proof. This follows exactly as in the one-dimensional case [MZ1], by (2.32) together with Kramer’s rule. □

Remark 2.9. Representation (2.34) together with uniform exponential decay of \(\Phi^\pm\), \(\tilde{\Psi}^\pm\), Proposition 2.5 and the fact that \(d^\pm\) are bounded when the Evans function \(D := \det(\Phi^+, \Phi^-)\) does not vanish yields uniform bounds

\[
|G_{\lambda, \tilde{\xi}}(x, y)| \leq Ce^{-\theta |x-y|},
\]

\(\theta > 0\), on the resolvent set \(\rho(L_{\tilde{\xi}})\), in particular (by assumption \((D1)\)) for \(\Re \lambda \geq -\eta, \eta > 0\) on intermediate frequencies \(1/R \leq |(\lambda, \tilde{\xi})| \leq R, R > 0\) arbitrary. However, we shall not use this in our analysis, carrying out instead energy-based resolvent estimates for intermediate and high frequencies. We shall use (2.34) only in the low frequency regime \(|(\lambda, \tilde{\xi})| \ll 1\).
2.4. **Low frequency bounds.** We now examine in further detail behavior for small frequencies, carrying out at the same time the analytic extension of normal modes and Evans function beyond the region of consistent splitting. As in [HoZ2], for our later arguments it will be important to enlarge the domain of $\tilde{\xi}$ to complex values $\tilde{\xi} \in \mathbb{C}^{d-1}$, and so we will do this at the same time.

**Lemma 2.10.** Under the assumptions of Theorem 1.1, for $|{(\tilde{\xi}, \lambda)}|$ sufficiently small, $\tilde{\xi}$ now taken in $\mathbb{C}^{d-1}$, the eigenvalue equation $(L_{\pm \xi} - \lambda)W = 0$ associated with the limiting, constant-coefficient operator $L_{\pm \xi}$ has a basis of $(1 + r)$ solutions, for $m = 1, \ldots, r$,

\begin{equation}
W_m^{\pm} = e^{\mu_m^{\pm}(\lambda, \tilde{\xi})}V_m^{\pm}(\lambda, \tilde{\xi}),
\end{equation}

$\mu_m^{\pm}, V_m^{\pm}$, analytic in $\lambda$ and $\tilde{\xi}$, consisting of $r$ “fast” modes

\begin{equation}
\mu_m^{\pm} = \gamma_m^{\pm} + \mathcal{O}(\lambda, \tilde{\xi}), \quad V_m^{\pm} = (A_1^{\pm})^{-1}S_m^{\pm} + \mathcal{O}(\lambda, \tilde{\xi}),
\end{equation}

$S_m^{\pm} = (0, s_m^{\pm})^t$ where $\gamma_m^{\pm}, s_m^{\pm}$ are eigenvalues and associated right eigenvectors of $dq_{\pm}(A_1^{\pm})^{-1}(0, 1)^t$ (equivalently, $\gamma_m^{\pm}, S_m^{\pm}$ are nonzero eigenvalues and associated right eigenvectors of $(Q_{\pm} - i \sum_{j \neq 1} \xi_j A_1^{\pm})(A_1^{\pm})^{-1}$), and 1 “slow” mode

\begin{equation}
\mu_{r+1}^{\pm}(\lambda, \tilde{\xi}) = (-\frac{1}{a_1^{\pm}})(\lambda + i\tilde{\xi} \cdot \tilde{a}^{\pm}) + \frac{b_1^{\pm}}{(a_1^{\pm})^2}(\lambda + i\tilde{\xi} \cdot \tilde{a}^{\pm})^2
\end{equation}

\begin{equation}
- \frac{1}{a_1^{\pm}} \tilde{\xi}^2 B_1^{\pm} + \mathcal{O}(|\lambda + |\tilde{\xi}||^3)
\end{equation}

and

\begin{equation}
V_{r+1}^{\pm}(\lambda, \tilde{\xi}) := R_1^{\pm} + \mathcal{O}(\lambda, \tilde{\xi}),
\end{equation}

where $a_1^{\pm} = a_1^*(\pm \infty)$, $\tilde{a}^{\pm} = (a_2^*, \ldots, a_d^*)^*(\pm \infty)$, $R_1^{\pm} = V_1^*(\pm \infty)$, and $b_1^{\pm} = b_1^*(\pm \infty)$, with $a_j^*, V_1^*$, and $b_j^*$ as defined in (A.22), (A.25), and (A.24), Appendix A. Likewise, the adjoint eigenvalue equation $(L_{\pm \xi} - \lambda)^*Z = 0$ has a basis of solutions

\begin{equation}
\tilde{W}_m^{\pm} = e^{-\mu_m^{\pm}(\lambda, \tilde{\xi})}x_1 \tilde{V}_m^{\pm}(\lambda, \tilde{\xi}),
\end{equation}

where

\begin{equation}
\tilde{V}_m^{\pm}(\lambda, \tilde{\xi}) = \tilde{T}_m^{\pm} + \mathcal{O}(\lambda, \tilde{\xi})
\end{equation}
and

\begin{equation}
\tilde{V}_{r+1}^\pm (\lambda, \tilde{\xi}) = \tilde{L}_1^\pm \ast + \mathcal{O}(\lambda, \tilde{\xi}),
\end{equation}

where \( \tilde{V} \) is analytic in \((\lambda, \tilde{\xi})\), \( \tilde{T}_m \) are the left eigenvectors of

\[ (Q_\pm - i \sum_{j \neq 1} \xi_j A_j^\pm) (A_1^\pm)^{-1} \]

associated with the nonzero eigenvalues \(-\mu_\pm^\pm\), and \( L_1^\pm = (1, 0)^T \).

**Proof.** By the inversion of the expressions

\begin{equation}
\lambda(\xi) = -i \xi \cdot a^\ast - \xi^\ast B^\ast \xi + \cdots
\end{equation}

\[ = -i \xi_1 a_1 - (b_{11}^\ast \xi_1^2 + \sum_{j \neq 1} b_{j1}^\ast \xi_j \xi_j + \sum_{k \neq 1} b_{1k}^\ast \xi_k \xi_1) - i \xi^\ast \tilde{a} - \xi^\ast B_{11}^\ast \tilde{\xi} + \cdots \]

\[ = -i \xi_1 (a_1 - i \sum_{j \neq 1} b_{j1}^\ast \xi_j + \sum_{k \neq 1} b_{j1}^\ast \xi_k) - b_{11}^\ast \xi_1^2 - i \xi^\ast \tilde{a} - \xi^\ast B_{11}^\ast \tilde{\xi} + \cdots \]

\[ = -i \xi_1 (a_1 - 2i \sum_{j \neq 1} b_{j1}^\ast \xi_j) - b_{11}^\ast \xi_1^2 - i \xi^\ast \tilde{a} - \xi^\ast B_{11}^\ast \tilde{\xi} + \cdots \]

carried out in Appendix [A] for the dispersion curves near \( \xi = 0 \), together with the fundamental relation \( \mu = i \xi_1 \), we have \( R_1^\pm = V_1^\pm (\pm \infty) \) and

\[
\mu_{r+1}^\pm (\lambda, \tilde{\xi}) := \frac{\lambda}{a_1^\pm - 2i \sum_{j \neq 1} b_{j1}^\pm \xi_j} + \frac{b_{11}^\pm}{(a_1^\pm - 2i \sum_{j \neq 1} b_{j1}^\pm \xi_j)^3} \lambda^2 + \mathcal{O}(\lambda^3)
\]

\[ = \frac{-1}{a_1^T}(\lambda + i \tilde{\xi} \cdot \tilde{a}^\pm + \tilde{\xi}^\ast B_{11}^\ast \tilde{\xi}) + \frac{b_{11}^\pm}{(a_1^T)^3}(\lambda + i \tilde{\xi} \cdot \tilde{a}^\pm + \tilde{\xi}^\ast B_{11}^\ast \tilde{\xi})^2
\]

\[ + \mathcal{O}(|\lambda + |\tilde{\xi}||^3) \]

\[ = \left( -\frac{1}{a_1^T} + \frac{2 b_{11}^\pm}{(a_1^T)^3} \tilde{\xi}^\ast B_{11}^\ast \tilde{\xi} \right) \lambda + \frac{b_{11}^\pm}{(a_1^T)^3} \lambda^2 - \frac{1}{a_1^T} \tilde{\xi}^\ast B_{11}^\ast \tilde{\xi}
\]

\[ - \frac{b_{11}^\pm}{(a_1^T)^3}(\tilde{\xi} \cdot \tilde{a}^\pm)^2 + i(\tilde{\xi} \cdot \tilde{a}^\pm) \left( -\frac{1}{a_1^T} + \frac{2 b_{11}^\pm}{(a_1^T)^3} \right) + \mathcal{O}(|\lambda + |\tilde{\xi}||^3) \]

\[ = \left( -\frac{1}{a_1^T}(\lambda + i \tilde{\xi} \cdot \tilde{a}^\pm) + \frac{b_{11}^\pm}{(a_1^T)^3}(\lambda + i \tilde{\xi} \cdot \tilde{a}^\pm)^2
\]

\[- \frac{1}{a_1^T} \tilde{\xi}^\ast B_{11}^\ast \tilde{\xi} + \mathcal{O}(|\lambda + |\tilde{\xi}||^3). \]
The corresponding adjoint computation yields $L^{\pm}_1 = (1, 0)^T$. (See, for example, the one-dimensional computation of [MZ1], which is sufficient to determine $L^{\pm}_1$.)

□

Normal modes. Consider again the variable-coefficient eigenvalue equations

\begin{equation}
(L_\xi - \lambda)W = (dQ - A_\xi - \lambda)W - (A^1 W)' = 0
\end{equation}

and the limiting constant-coefficient equations

\begin{equation}
(L_{\pm, \xi} - \lambda)W = (dQ_{\pm} - A_{\xi, \pm} - \lambda)W - (A_{1 \pm} W)' = 0.
\end{equation}

We now relate the normal modes of (2.46) to those of (2.47).

Lemma 2.11. (normal modes) Under the assumptions of Theorem 1.1, for $(\lambda, \tilde{\xi}) \in B(0, r)$, $r$ sufficiently small ($\tilde{\xi}$ now complex), there exist solutions $W^\pm_m(x_1; \lambda, \tilde{\xi})$ of (2.46), $C^2$ in $x_1$ and analytic in $\lambda$ and $\tilde{\xi}$, satisfying

\begin{equation}
W^\pm_m(x_1; \lambda, \tilde{\xi}) = e^{\mu^\pm_m x_1} V^\pm_m(\lambda, \tilde{\xi})
\end{equation}

\begin{equation}
\left(\frac{\partial}{\partial \lambda}\right)^k \left(\frac{\partial}{\partial \tilde{\xi}}\right)^l V^\pm_m(x_1; \lambda, \tilde{\xi}) = \left(\frac{\partial}{\partial \lambda}\right)^k \left(\frac{\partial}{\partial \tilde{\xi}}\right)^l V^\pm_m(\lambda, \tilde{\xi}) + O(e^{-\tilde{\theta}|x_1|} |V^\pm_m(\lambda, \tilde{\xi})|),
\end{equation}

for any $k \geq 0$ and $0 < \tilde{\theta} < \theta$, where $\mu^\pm_m(\lambda, \tilde{\xi})$, $V^\pm_m(\lambda, \tilde{\xi})$ are as in the previous lemma.

Proof. This is a direct consequence of the previous lemma and the gap lemma, Lemma B.2

□

Proposition 2.12. There is a neighborhood of $(0, 0)$ in $(\lambda, \tilde{\xi})$ space ($\tilde{\xi}$ now complex) in which, for $y_1 \geq 0$,

\begin{equation}
G_{\lambda, \tilde{\xi}}(x_1, y_1) = G^1_{\lambda, \tilde{\xi}}(x_1, y_1) + G^2_{\lambda, \tilde{\xi}}(x_1, y_1),
\end{equation}
where

\begin{equation}
G_{\lambda,\xi}^1(x_1, y_1) = \begin{cases} 
0 & , y_1 \leq 0 \leq x_1 \\
- \sum_j \psi_j^-(x_1; \lambda, \xi) \tilde{\psi}_j^-(y_1; \lambda, \xi) & , y_1 \leq x_1 \leq 0 \\
\sum_j \varphi_j^-(x_1; \lambda, \xi) \tilde{\varphi}_j^-(y_1; \lambda, \xi) & , x_1 \leq y_1 \leq 0 
\end{cases}
\end{equation}

\begin{equation}
G_{\lambda,\xi}^2(x_1, y_1) = \begin{cases} 
\sum_{k,j} M_{jk}^+(\lambda, \xi) \varphi_j^+(x_1; \lambda, \xi) \tilde{\varphi}_k^-(y_1; \lambda, \xi) & , y_1 \leq 0 \leq x_1 \\
\sum_{k,j} d_{jk}^+ \varphi_j^+(x_1; \lambda, \xi) \tilde{\varphi}_k^-(y_1; \lambda, \xi) & , y_1 \leq x_1 \leq 0 \\
\sum_{k,j} d_{jk}^- \varphi_j^-(x_1; \lambda, \xi) \tilde{\varphi}_k^-(y_1; \lambda, \xi) & , x_1 \leq y_1 \leq 0 
\end{cases}
\end{equation}

where

\begin{equation}
|M_{jk}^+|, |d_{jk}^-| \leq C_1 |D^{-1}|
\end{equation}

and \(D(\lambda, \tilde{\xi}) = \det \Phi\) is the Evans function.

Moreover,

\begin{equation}
G_{\lambda,\xi}^1(x_1, y_1) = \begin{cases} 
0 & , y_1 \leq 0 \leq x_1 \\
e^{\mu_{r+1}(x_1-y_1)} \begin{pmatrix} c_1(x_1) \\ c_2(x_1) \end{pmatrix} & , y_1 \leq x_1 \leq 0 \\
+ e^{\mu_{r+1}(x_1-y_1)} & , x_1 \leq y_1 \leq 0 
\end{cases}
\end{equation}
\[
\left( \frac{\partial}{\partial y_1} \right) G_{\lambda, \tilde{\xi}}^1(x_1, y_1) = \begin{cases}
0, & y_1 \leq 0 \leq x_1 \\
\left[ e^{\mu_{r+1}(x_1-y_1)} \right. \\
\times O(|\lambda| + |	ilde{\xi}|) + O(e^{-\theta|x_1-y_1|})
\end{cases},
\]

\[
(2.55)
\]

\[
G_{\lambda, \tilde{\xi}}^2(x_1, y_1) = CD^{-1}(\lambda, \tilde{\xi}) e^{-\mu_{r+1}y_1} \bar{U}'(x_1)(1, 0) + O\left(|D^{-1}(\lambda, \tilde{\xi})|(|\lambda| + |	ilde{\xi}|) e^{-\theta|x_1-y_1|} e^{Re \mu_{r+1}(\lambda, \tilde{\xi})(x_1-y_1)}\right)
\]

\[
+ O(e^{-\theta(x_1-y_1)})
\]

\[
(2.56)
\]

\[
\left( \frac{\partial}{\partial y_1} \right) G_{\lambda, \tilde{\xi}}^2(x_1, y_1) = O\left(|D^{-1}(\lambda, \tilde{\xi})|(|\lambda| + |	ilde{\xi}|) e^{-\theta|x_1-y_1|} e^{Re \mu_{r+1}(\lambda, \tilde{\xi})(x_1-y_1)}\right)
\]

\[
(2.57)
\]

**Proof.** By proposition 2.6, we decompose \( G \) into two parts \( G^1 \) and \( G^2 \) where \( G^1 \) has no poles and \( G^2 \) has all terms with poles, respectively. Thus, we have \(2.51\) and \(2.52\). Without loss of generality, we can set

\[
\varphi_1 = \bar{U}'(x_1) + \Lambda(x_1, \lambda, \tilde{\xi}) e^{-c|x_1|} \text{ for } c > 0
\]

where \( \Lambda(x_1, \lambda, \tilde{\xi}) = O(|\lambda| + |	ilde{\xi}|) \) is differentiable and exponentially decaying in \( x_1 \). This is possible since it solves \(2.10\), and is only bounded solution at zero frequency. On the other hand, we can check by inspection that

\[
\tilde{\psi}_{r+1} = [(1, 0) + \Theta(y_1, \lambda, \tilde{\xi})] e^{-\mu_{r+1}|y_1|},
\]

\[
(2.59)
\]
where \( \Theta(y_1, \lambda, \tilde{\xi}) = \mathcal{O}(|\lambda| + |\tilde{\xi}|) \) is differentiable and exponentially decaying in \( y_1 \). By lemma 3.2, all fast modes can be written as

\[
\psi_j^- = e^{\mu_j^+ x_1} V_j^-(x_1, \lambda, \tilde{\xi})
= e^{\mu_j^+ x_1} V_j^-(x_1, 0, 0) + e^{\mu_j^+ x_1} \Psi(x_1, \lambda, \tilde{\xi}) V_j^-(x_1, 0, 0),
\]

where \( \Psi(x_1, \lambda, \tilde{\xi}) = \mathcal{O}(|\lambda| + |\tilde{\xi}|) \) is differentiable and exponentially decaying in \( x_1 \). Since all modes are fast except for one slow mode \( \tilde{\psi}_{r+1} \), for \( y_1 \leq x_1 \leq 0 \), we have

\[
(2.60) \quad \sum_j \psi_j^- \tilde{\psi}_j^- = \psi_{r+1}^- \tilde{\psi}_{r+1}^- + \sum_{j \neq 1} \psi_j^- \tilde{\psi}_j^- = e^{\mu_{r+1}^- x_1} (V_j^-(x_1, 0, 0) + \Psi(x_1, \lambda, \tilde{\xi}) V_j^-(x_1, 0, 0))
\]

\[
\times e^{-\mu_{r+1}^- y_1} [(1, 0) + \Theta(y_1, \lambda, \tilde{\xi})] + \sum_{j \neq 1} \psi_j^- \tilde{\psi}_j^- - O(e^{-\theta |x_1 - y_1|}).
\]

Using (2.59) and (2.60). For \( x_1 \leq y_1 \leq 0 \), since all \( \varphi_j^- \) are fast modes, we have,

\[
(2.62) \quad \sum_j \varphi_j^-(x_1; \lambda, \tilde{\xi}) \tilde{\varphi}_j^-(y_1; \lambda, \tilde{\xi})^* = \mathcal{O}(e^{-\theta |x_1 - y_1|}).
\]

Thus, we have (2.54).

It is easy to check (2.55) by differentiating (2.54) with respect to \( y_1 \).
For $G_{\lambda,\xi}^2(x_1, y_1)$, we have, for $y_1 \leq 0 \leq x_1$,

$$G_{\lambda,\xi}^2(x_1, y_1) = \sum_{j,k} M_{j,k}^+ \varphi_j^+(x_1; \lambda, \xi) \tilde{\psi}_k^- (y_1; \lambda, \xi)^*$$

$$= CD^{-1} (e^{-\mu_{r+1} y_1} \tilde{U}'(x_1)(1,0)$$

$$+ \Psi(x_1, \lambda, \tilde{\xi})(1,0) e^{-c x_1} e^{-\mu_{r+1} y_1} + \tilde{U}'(y_1, \lambda, \tilde{\xi}) e^{-\mu_{r+1} y_1}$$

$$+ \Psi(x_1, \lambda, \tilde{\xi}) \Theta(y_1, \lambda, \tilde{\xi}) e^{-c x_1} e^{-\mu_{r+1} y_1})$$

$$+ \sum_{(j,k) \neq (1, x+1)} M_{j,k}^+ \varphi_j^+(x_1; \lambda, \tilde{\xi}) \tilde{\psi}_k^- (y_1; \lambda, \tilde{\xi})^*$$

$$= CD^{-1}(\lambda, \tilde{\xi}) e^{-\mu_{r+1} y_1} \tilde{U}'(x_1)(1,0)$$

$$+ O(|\lambda| + |\tilde{\xi}|) e^{-\frac{|x_1|}{c} e^{-Re \mu_{r+1}(\lambda, \tilde{\xi})(x_1-y_1)} D^{-1}(\lambda, \tilde{\xi})}$$

$$+ O(e^{-\theta |x_1-y_1|}).$$

Similarly, for $x_1 \leq y_1 \leq 0$ and for $y_1 \leq x_1 \leq 0$, respectively, we have the same bounds. It is also easy to check (2.57) by differentiating (2.56) with respect to $y_1$.

\[ \square \]

**Corollary 2.13.** There is a neighborhood of $(0, 0)$ in $(\lambda, \tilde{\xi})$ space ($\tilde{\xi}$ now complex) in which, for $y_1 \geq 0$,

$$G_{\lambda,\xi}^1(x_1, y_1) \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{cases} 0, & y_1 \leq 0 \leq x_1 \\
[ e^{-Re \mu_{r+1}(x_1-y_1)} \\
\times O(|\lambda| + |\tilde{\xi}|) \\
+ O(e^{-\theta |x_1-y_1|}) \end{cases}, & y_1 \leq x_1 \leq 0 \\
O(e^{-\theta |x_1-y_1|}) \end{cases}, & x_1 \leq y_1 \leq 0$$

(2.63)

$$G_{\lambda,\xi}^2(x_1, y_1)(0, I)^t = O((D^{-1}(\lambda, \tilde{\xi})(|\lambda| + |\tilde{\xi}|) e^{-\frac{|x_1|}{c} e^{-Re \mu_{r+1}(\lambda, \tilde{\xi})(x_1-y_1)})}$$

(2.64)

**Proof.** If we multiply (2.54) by $(0, I)^t$, then we have the result. \[ \square \]
Remark 2.14. Note that \( G^j_{\lambda,\xi}(x_1, y_1)(0, I)^t \) has almost the same bounds as \( \frac{\partial}{\partial y_1} \tilde{G}^j_{\lambda,\xi}(x_1, y_1) \) for \( j = 1, 2 \). This will be important in our later derivation of pointwise Green function bounds, yielding that differentiation by \( y \) is roughly comparable to right-multiplication by \((0, I)^t\).

2.5. Decomposition of the Green function. For fixed small \( \delta_1, \theta_1, r > 0 \) to be chosen later, define a “low-frequency” part \( G^I \) and a “high-frequency” part \( G^{II} \) of \( G \), respectively by

\[
G^I(x,t; y) := \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq \delta_1} \oint_{\Gamma_0} e^{i\tilde{\xi} \cdot (\tilde{x} - \tilde{y}) + \lambda t} G_{\lambda,\xi}(x_1, y_1) d\lambda d\tilde{\xi},
\]

where \( \Gamma_0 = [-\eta - ir, \eta - ir] \cup [\eta - ir, \eta + ir] \cup [2\eta + ir, -\eta - ir] \), and

\[
G^{II}(x,t; y) := \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \int_{|\tilde{\xi}| \geq \delta_1 \text{ or } |\Im \lambda| \geq r} e^{i\tilde{\xi} \cdot (\tilde{x} - \tilde{y}) + \lambda t} G_{\lambda,\xi}(x_1, y_1) d\tilde{\xi} d\lambda.
\]

Then, by the spectral resolution formula (2.7) together with Cauchy’s Theorem, we have a decomposition formula for \( G(x,t; y) \) of

\[
G(x,t; y) = G^I(x,t; y) + G^{II}(x,t; y).
\]

3. Low-frequency estimates

3.1. Pointwise bounds. The stationary phase approximation

\[
U \sim \bar{U}'(x_1)\delta(\bar{x}, t)
\]

of (1.15) can be expressed alternatively (recalling that the low-frequency part of \( G \) generally determines large-time behavior) as

\[
G^I(x,t; y) \sim \bar{U}'(x_1)([u]^{-1}, 0)g(\bar{x} - \bar{y}, t),
\]
where \( g(\tilde{x}, t) \) denotes the Green function for the constant-coefficient \((d - 1)\)-dimensional equation \((1.16)\) approximately governing normal deformation of the front. Using the analysis of \([\text{HoZ1}]\) together with the low-frequency description of the resolvent kernel in Proposition 3.1, we now establish the following pointwise description of the low-frequency part of the Green function \( G^I \), sharpening the formal prediction of \((3.1)\).

**Proposition 3.1.** Under the assumptions of Theorem 1.1, for \(y_1 \geq 0, |\alpha| \leq 1\), and some \(\eta, M > 0\),

\[
D_y^\alpha G^I = \chi_{\{|x_1 - y_1| \leq |a_1 + t|\}} \chi_{\{y \geq y_1 + a_1 + t\}} D_y^\alpha \overline{U}'(x_1)(|u|^{-1}, 0) \overline{g}^+(\tilde{x}, t; y) + \chi_{\{x_1 \geq 0\}} D_y^\alpha K^+(x, t; y) + R^+_\alpha,
\]

where

\[
\overline{g}^+(\tilde{x}, t; y) := c_{\beta} t^{-\frac{d}{2}} e^{-\frac{(\tilde{x} - y - \alpha^+ (y_1 + t))^{\beta^{-1}}}{\beta}}
\]

and

\[
K^+(x, t; y) := c_{B^*} t^{-\frac{d}{2}} e^{-\frac{(x - y - a^+ + t)^{B^*}}{B^*}}
\]

are \((d - 1)\)- and \(d\)-dimensional convected heat kernels, respectively, with

\[
\alpha^+ := (1 - \frac{|y_1|}{|\alpha_1 + t|}) \tilde{\alpha} + \frac{|y_1|}{|\alpha_1 + t|} \alpha^+,
\]

\[
\beta^+ := (1 - \frac{|y_1|}{|\alpha_1 + t|}) \tilde{\beta} + \frac{|y_1|}{|\alpha_1 + t|} (b^*_1 + \tilde{B}^* + \frac{b^*_1}{|\alpha_1 + t|^2} (\tilde{a}^+ - \tilde{a} - a^+ b^*) (\tilde{a}^+ - \tilde{a} - a^+ b^*)^{t})
\]

and

\[
\tilde{B}^* := B^* - b^* b^t,
\]
where \( \bar{a}^+ \) is given by \( a^+ = (a_1^+, \bar{a}^+) = (df_1^+(u_1), df_2^+(u_1), \ldots df_N^+(u_1)), b^*, B^*, \tilde{a}, \tilde{b} \) are as in \([A, 26]\) and \([C.5]\), and the (faster decaying) residual terms \( R_{\alpha}^+ \) and \( \Theta_{\alpha}^+ \) satisfy

\[
|R_{\alpha}^+| \leq C \chi_{\{x_1 - y_1 \leq |a_1^+|t\}} t^{-d + \frac{1}{2}} |x|^{-\frac{1}{2}} e^{-\eta|y_1|} e^{\alpha_1 e^{-\eta|x_1|} - \frac{|x_1 - \bar{a}^+|^2}{m t}} + Ce^{-\frac{|x-y-a^+|^2}{m t}} t^{-d - |a|} e^{-\eta|x_1|}
\]

\[
+ Ce^{-\frac{|x-y-a^+|^2}{m t}} t^{-d - \frac{|a|}{2}} ((1 + t)^{-\frac{1}{2}} + \alpha_1 e^{-\eta|y_1|}) e^{-\eta|x_1|} e^{-\frac{|x_1 - \bar{a}^+|^2}{m t}}
\]

\[
+ C \chi_{\{|x_1 - y_1| \geq M(t + |x_1 - y_1|)\}} \prod_{j=2}^{d} (1 + |x_j - y_j|)
\]

A symmetric description holds for \( y_1 \leq 0 \).

Before proving Proposition 3.1, we first establish the following lemma, a simplified version of Proposition 2.8 in \([HoZ2]\), allowing us to vary \( \tilde{\xi} \) in the complex plane. Following \([HoZ2]\), denote \( \tilde{\xi} \in \mathbb{C} \) by

\[
\tilde{\xi} = \xi_1 + i \xi_2.
\]

**Lemma 3.2.** Under the assumptions of Theorem 1.1, for some \( \delta_2, \eta_1 > 0 \) sufficiently small, \( |\bar{x} - \bar{y}| \leq M(t + |x_1 - y_1|) \), and any \( |\tilde{\xi}|^2 \leq \delta_2 \),

\[
G^I(x, t; y) = \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}| \leq \delta_1, |\tilde{\xi}| \leq \delta_2} \oint_{C(t)} e^{i\tilde{\xi}(\bar{x} - \bar{y}) + \lambda t} G_{\lambda, \tilde{\xi}}(x_1, y_1) d\lambda d\tilde{\xi}
\]

\[
+ O(e^{-\eta_1(|x-y|+t)}).
\]

**Proof.** By Cauchy’s Theorem, it is equivalent to show

\[
\int_{|\tilde{\xi}| = \delta_1, |\tilde{\xi}| \leq \delta_2} \oint_{C(t)} e^{i\tilde{\xi}(\bar{x} - \bar{y}) + \lambda t} G_{\lambda, \tilde{\xi}}(x_1, y_1) d\lambda d\tilde{\xi} = O(e^{-\eta_1(|x-y|+t)}).
\]

By assumption (D1) and continuity, there are no zeros of the Evans function for \( |\tilde{\xi}| = \delta_1 \) and \( \Re \lambda \geq -\theta \), some \( \theta > 0 \). Thus, by Cauchy’s Theorem again, (3.11) is equivalent in turn to

\[
\int_{|\tilde{\xi}| = \delta_1, |\tilde{\xi}| \leq \delta_2} \oint_{C(t)} e^{i\tilde{\xi}(\bar{x} - \bar{y}) + \lambda t} G_{\lambda, \tilde{\xi}}(x_1, y_1) d\lambda d\tilde{\xi} = O(e^{-\eta_1(|x-y|+t)}).
\]

Taking \( \eta \ll \delta_1 \), we have on the domain of integration by our previous bounds (see for example Remark 2.9) that

\[
|G_{\lambda, \tilde{\xi}}(x_1, y_1)| \leq Ce^{-\theta_2|x_1 - y_1|}
\]
for \( \theta_2 > 0 \). Taking \( \delta_2 \ll \theta_2, \eta \), we thus have

\[
|e^{i\xi(x - \tilde{y}) + \lambda t} G_{\lambda, \xi}(x_1, y_1)| \leq Ce^{-\theta_2|x_1 - y_1|} \leq Ce^{-\eta t/2} e^{-\theta_2|x_1 - y_1|/2},
\]

yielding the result for \( \eta_1 := \min\{\theta_2, \eta\} \). \( \square \)

We have also the following weakened version of Proposition 2.7, [HoZ2].

**Lemma 3.3.** Under the assumptions of Theorem 1.1, for \( M(t + |x_1 - y_1|) \),

\[
|G^I(x, t; y)| \leq C \frac{e^{-\eta t(|x_1 - y_1| + t)}}{\prod_{j=2}^d (1 + |x_j - y_j|)}.
\]

**Proof.** We first consider the simplest case of dimension \( d = 2 \). Moving \( \tilde{\xi}_2 \) from 0 to \( c := \frac{\delta_2(x - \tilde{y})}{|x - \tilde{y}|} \) in

\[
\int_{|\xi_1| \leq \delta_1, \tilde{\xi}_2 = \tilde{\xi}_2} \int_{\Gamma_0} e^{i\xi(x - \tilde{y}) + \lambda t} G_{\lambda, \xi}(x_1, y_1) d\lambda d\xi,
\]

\( \delta_2 > 0 \) fixed, by the argument of Lemma 3.2 yields a change of order

\[
e^{-\eta(t + |x_1 - y_1|)} \int_0^{\delta_2} e^{-z|x - \tilde{y}|} dz \leq \frac{Ce^{-\eta(t + |x_1 - y_1|)}}{1 + |x - \tilde{y}|}.
\]

On the other hand,

\[
\int_{|\xi_1| \leq \delta_1, \tilde{\xi}_2 = c} \int_{\Gamma_0} e^{i\xi(x - \tilde{y}) + \lambda t} G_{\lambda, \xi}(x_1, y_1) d\lambda d\xi \leq Ce^{-\delta_2|x - \tilde{y}|} e^{C_1(t + |x_1 - y_1|)}
\]

\[
\leq Ce^{-(\delta_2/2)(t + |x - y|)},
\]

since by assumption \( |\tilde{x} - \tilde{y}| >> (t + |x_1 - y_1|) \). This completes the proof for dimension \( d = 2 \).

For dimensions \( d > 2 \), we proceed by induction, moving one component of \( \tilde{\xi}_2 \) at a time, starting with the component for which \( |x_j - y_j| \) is largest, without loss of generality, \( j = 2 \). At the first step, then, this yields a change of order

\[
e^{-\eta(t + |x_1 - y_1|)} \int_0^{\delta_2} e^{-z|x_2 - y_2|} dz \leq \frac{Ce^{-\eta(t + |x_1 - y_1|)}}{1 + |x_2 - y_2|}
\]

times the maximum of a family of \( (d - 2) \)-dimensional integrals in \( (\xi_3, \ldots, \xi_d) \) of similar form to the original one, plus a new integral of negligible order \( O(e^{-(\delta_2/2)(t + |x - y|)}) \). Moving
the next component similarly, in each of the family of \((d - 2)\)-dimensional integrals, yields
a factor
\[
\frac{C}{1 + |x_3 - y_3|}
\]
times the maximum of a family of \((d - 3)\)-dimensional integrals of similar form, plus a new
integral of smaller order \(O(e^{-\eta|x_3 - y_3|})\) times another \((d - 3)\)-dimensional integral of similar
form. Continuing this process, we obtain the result. \(\square\)

\textbf{Remark 3.4.} For \(|\tilde{x} - \tilde{y}| \geq Mt\), \(M\) sufficiently large, we have \(G \equiv 0\) by finite speed
of propagation. Thus, (3.13) reflects a certain inefficiency in our splitting scheme. Note
that the righthand side is time-exponentially decaying in \(L^p\), \(p > 1\), whereas usual error
terms \(O(e^{-\eta|t+|x-y|)})\) are time-exponentially decaying in all \(L^p\) norms; thus, for practical
purposes it is almost but not quite optimal. For the present analysis, in which we consider
only \(2 \leq p \leq \infty\), it is harmless.

\textit{Proof of Proposition 3.1.} With our preparations, this follows now by exactly the argument
used in [HoZ2] to establish the corresponding bounds on the full solution operator for
\(|x - y| \geq Mt\), \(M \gg 1\). (Note: what is actually estimated for this regime in [HoZ2] is the
low-frequency part, with the rest shown to be negligible.)

Specifically, we note that the description of the resolvent kernel in Proposition 2.31 agrees
with that for the viscous case in Proposition 2.5, [HoZ2] in the sense that the principal
terms are identical, with the rest consisting of fast-decaying \(O(e^{-\theta|x_1 - y_1|})\) terms leading
to negligible errors. The only difference in the relaxation case is that there are more (at
most \(r\)) of the latter terms than in the viscous case (at most 1).

Then, the rest of the proof goes word for word as in the arguments of [HoZ2], Sections
3, 4, and 5, based on careful stationary phase estimates of the various terms. For this
(complicated) argument, we refer to [HoZ2]. \(\square\)

\subsection{3.2. \(L^1 \to L^p\) bounds.} From Proposition 3.1 we readily obtain the following bounds on
the solution operator.

\textbf{Lemma 3.5.} For \(t \geq 1\) and \(\alpha\) is a multi-index with \(|\alpha| \leq 1\), there holds
\begin{equation}
|D_x^\alpha G^I(x, t; y)|_{L^p(x)} \leq C(p) t^{((d-1)/2)(1-1/p) - |\alpha|/2}
\end{equation}
for all \( p > 1 \), with \( C \to \infty \) as \( p \to 1 \). Moreover, for \( f \in L^1 \) and \( p > 1 \), there holds

\[
(3.15) \quad \left| \int G^I(x, t; y)f(y)dy \right|_{L^p(x)} \leq C(p)(1 + t)^{-(d-1)/2(1-1/p)}|f|_{L^1}
\]

and

\[
(3.16) \quad \left| \int G^I_{y_j}(x, t; y)f(y)dy \right|_{L^p(x)} \leq C(p)(1 + t)^{-(d-1)/2(1-1/p)-1/2}|f|_{L^1}.
\]

**Proof.** By calculating the \( L^p \) norm of the \( (d-1) \)-dimensional heat kernel, together with the expression of \( (3.2) \), we have

\[
(3.17) \quad \left| D^0_x G^I \right|_{L^p(x)} \leq C \left( \left| \tilde{U}'(x_1) D^0_x \tilde{g}^+ \right|_{L^p} + \left| D^0_x \tilde{K}^+ \right|_{L^p} + \left| R^+_{\alpha} \right|_{L^p} \right)
\]

\[
\leq Ct^{(d-1)/2(1-1/p)-|\alpha|/2} + C(p)e^{-\eta t},
\]

\( \eta > 0 \). The inequalities \((3.15)\) and \((3.16)\) follows from \((3.14)\) and the triangle inequality. \( \square \)

**Lemma 3.6.** For \( p > 1 \) and \( C(p) \) as in Lemma 3.5

\[
(3.18) \quad \left| \int G^I(x, t, y)(0, I_x)^{t} f(y)dy \right|_{L^p} \leq C(p)(1 + t)^{-(d-1)/2(1-1/p)-1/2}|f|_{L^1}.
\]

**Proof.** By \((2.63)\) and \((2.64)\), we know that \( G^I_{\lambda \tilde{\xi}}(x_1, y_1)(0, I)^{t} \) has the same bound as \( \frac{\partial}{\partial \lambda} G^I_{\lambda \tilde{\xi}}(x_1, y_1) \).

Thus, \( \left| \int G^I(x, t, y)(0, I_x)^{t} f(y)dy \right|_{L^p} \) has the same bound as \( \left| \int G^I_{y_j}(x, t; y)f(y)dy \right|_{L^p(x)} \). By \((3.16)\), we have the result. \( \square \)

**Lemma 3.7.** The low frequency part of Green function \( G^I(x, t; y) \) associated with \((\partial_t - L)\) satisfies

\[
(3.19) \quad \left| \int (G^I(x, t; y) - \tilde{U}'(x_1)\Pi'(x_1)g(\tilde{x} - \tilde{y}, t)) f(y)dy \right|_{L^p(x)} \leq Ct^{-(d-1)/2(1-1/p)-1/2} \left( \left| x_1 f(x) \right|_{L^1(x)} + \left| f(x) \right|_{L^1(x)} \right)
\]

for all \( t \geq 0 \), where \( \Pi(x_1) \) is the left zero eigenvector dual to the right zero eigenvector \( \tilde{U}'(x_1) \) at \( \tilde{\xi} = 0 \)
Proof. By (3.2), we have

\begin{align*}
|G^I(x,t;0,\tilde{y}) - \bar{U}'(x_1)\Pi^I(x_1)g(\bar{x} - \tilde{y},t)f(y)dy|_{L^p(x)} \\
\leq Ct^{-(d-1)/2}(1-1/p-1/2).
\end{align*}

Then using (3.20), we have

\begin{align*}
|\int G^I(x,t;0,\tilde{y}) - \bar{U}'(x_1)\Pi^I(x_1)g(\bar{x} - \tilde{y},t)f(y)dy|_{L^p(x)} \\
\leq \int |G^I(x,t;0,\tilde{y}) - \bar{U}'(x_1)\Pi^I(x_1)g(\bar{x} - \tilde{y},t)f(y)dy|_{L^p(x)}|f|_{L^1(x)} \\
\leq Ct^{-(d-1)/2}(1-1/p-1/2).
\end{align*}

On the other hand, we have

\begin{align*}
|\int G^I(x,t;y_1,\tilde{y}) - G^I(x,t;0,\tilde{y})f(y)dy|_{L^p(x)} \\
= |\int \int_0^1 \partial_{y_1} G^I(x,t;\theta y_1,\tilde{y})y_1 f(y)d\theta dy|_{L^p(x)} \\
\leq \sup_y |\partial_{y_1} G^I(x,t;y)|_{L^p(x)}|x_1 f(x)|_{L^p(x)} \\
\leq t^{-(d-1)/2}(1-1/p-1/2)|x_1 f(x)|_{L^p(x)}.
\end{align*}

Combining (3.21) and (3.22) together with the triangle inequality, we have the result. \(\Box\)

4. Damping and high frequency estimates

We now carry out the main new work of the paper, establishing high-derivative and high-frequency bounds by energy estimates following the approach introduced in [Z4] for the hyperbolic-parabolic case.

We denote \(U = (u,v)^t\) for our convenience. Let \(\bar{U}\) be a solution of (1.1) and \(\tilde{U}\) be a traveling wave solution of (1.1). Define the nonlinear perturbation \(U(x,t) := \bar{U}(x,t) - \bar{\bar{u}}(x,t)\) where \(\bar{\bar{u}}(x,t) = \bar{U}(x_1 - \delta(\bar{x},t))\). We also denote \(\bar{A} = A(\bar{U}), \bar{\bar{A}} = A(\bar{\bar{U}}), d\bar{Q} = dQ(\bar{U})\) and \(d\bar{\bar{Q}} = dQ(\bar{\bar{U}})\).
Lemma 4.1. Multi-dimensional scalar relaxation equations (1.1) can be put in a quasilinear form as follows:

\[ U_t + \sum_j \tilde{A}_j U_{x_j} - d\tilde{Q}U = (\partial_t - L)\delta(\tilde{x}, t)\tilde{U}'(x_1) - R_{x_1} + (0, I_r)^t N(U, U) - M =: f \]

where

\[ M = \sum_{j=1}^d (\tilde{A}_j - \bar{A}_j) \bar{U}_{x_j} = O(|U||\tilde{U}'||\nabla \tilde{x} \delta|), \]

\[ R_{x_1} = O(|\delta_t||\tilde{U}'| |\delta| + |\nabla \tilde{x} \delta||\tilde{U}'| |\delta|) x_1 = O(|\delta_t|(|\delta_t| + |\nabla \tilde{x} \delta|)|\tilde{U}''|) \]

and

\[ N(U, U) = O(|U|^2). \]

Proof. Consider the multi-dimensional scalar relaxation equations.

\[ \begin{pmatrix} u \\ v \end{pmatrix}_t + \sum_{j=1}^d \begin{pmatrix} f^j(u, v) \\ g^j(u, v) \end{pmatrix}_{x_j} = \begin{pmatrix} 0 \\ q(u, v) \end{pmatrix} \]

For \( \tilde{U} \), there holds

\[ \tilde{U}_t + \sum_j \begin{pmatrix} f^j(\tilde{U}) \\ g^j(\tilde{U}) \end{pmatrix}_{x_j} - \begin{pmatrix} 0 \\ q(\tilde{U}) \end{pmatrix} = -\delta_t \tilde{U}'|_{x_1 - \delta} + \sum_{j \neq 1} (-\delta_{x_j}) \begin{pmatrix} df^j(\tilde{U}) \\ dg^j(\tilde{U}) \end{pmatrix}_{x_1 - \delta} \tilde{U}'|_{x_1 - \delta} + \begin{pmatrix} f^1(\tilde{U}) \\ g^1(\tilde{U}) \end{pmatrix}_{x_1} - \begin{pmatrix} 0 \\ q(\tilde{U}) \end{pmatrix} = 0 \]

\[ = -\delta_t \tilde{U}'(x_1) + \sum_{j \neq 1} (-\delta_{x_j}) \begin{pmatrix} df^j(\tilde{U}(x_1)) \\ dg^j(\tilde{U}(x_1)) \end{pmatrix} \tilde{U}'(x_1) + \begin{pmatrix} f^1(\tilde{U}) \\ g^1(\tilde{U}) \end{pmatrix}_{x_1} - \begin{pmatrix} 0 \\ q(\tilde{U}) \end{pmatrix} - R_{x_1} \]

\[ = -(\partial_t - L)\delta(\tilde{x}, t)\tilde{U}'(x_1) - R_{x_1} \]
where

\begin{equation}
R_{x_1} = -\delta_t \bar{U}'|_{x_1 - \delta} + \sum_{j \neq 1} (-\delta_{x_j}) \begin{pmatrix}
\frac{df^j(\bar{U})}{dx_j} \\
\frac{dg^j(\bar{U})}{dx_j}
\end{pmatrix} \bar{U}'|_{x_1 - \delta}.
\end{equation}

The last equality is true since the stationary shock wave solution \(\bar{U}'(x_1)\) satisfies

\begin{equation}
0 = \begin{pmatrix}
    f^1(\bar{U}) \\
    g^1(\bar{U})
\end{pmatrix}_{x_1} - \begin{pmatrix}
    0 \\
    q(\bar{U})
\end{pmatrix}_{x_1} = \begin{pmatrix}
    d\frac{f^1(\bar{U}(x_1))}{dx_1} \\
    d\frac{g^1(\bar{U}(x_1))}{dx_1}
\end{pmatrix}_{x_1} \bar{U}'(x_1) - \begin{pmatrix}
    0 \\
    d\frac{q(\bar{U}(x_1))}{dx_1}
\end{pmatrix} \bar{U}'(x_1).
\end{equation}

The last equation is obtained by multiplying the shock wave equation by \(\delta\) and adding to the second last equation.

Let \(\tilde{U}\) be a solution of

\begin{equation}
\tilde{U}_t + \sum_j \begin{pmatrix}
    f^j(\tilde{U}) \\
    g^j(\tilde{U})
\end{pmatrix}_{x_j} = \begin{pmatrix}
    0 \\
    q(\tilde{U})
\end{pmatrix}_{x_1}.
\end{equation}

After subtracting (4.6) from (4.9), we have the perturbation equation for \(U\)

\begin{equation}
U_t + \sum_{j=1}^d \left( A_j^i(\bar{U}) \tilde{U}_{x_j} - A_i^j(\bar{U}) \bar{U}_{x_j} \right) - (Q(\bar{U}) - Q(\bar{U}))
= U_t + \sum_{j=1}^d \left( F^j(\bar{U}) - F^j(\tilde{U}) \right)_{x_j} - (Q(\bar{U}) - Q(\tilde{U}))
= (\partial_t - L)\delta \tilde{U}' - R_{x_1}.
\end{equation}

Keeping a quasilinear form on the left hand side, we have

\begin{equation}
U_t + \sum_j A_j^i U_{x_j} - d\tilde{Q} U
= (\partial_t - L)\delta(\tilde{x},t)\tilde{U}'(x_1) - R_{x_1} + (0, I_r)^t N(U,U) - \sum_{j=1}^d (\tilde{A}_j^i - \bar{A}_j^i) \bar{U}_{x_j}
= (\partial_t - L)\delta(\tilde{x},t)\tilde{U}'(x_1) - R_{x_1} + (0, I_r)^t N(U,U) - M =: f
\end{equation}
where

\[(4.12)\]
\[M = \sum_{j=1}^{d} (\tilde{A}^j - \bar{A}^j) \tilde{U}_{x_j} = O(|U||\tilde{U}'||\nabla_x \delta|),\]

\[(4.13)\]
\[R_{x_1} = O(|\delta_t||\tilde{U}'||\delta| + |\nabla_x \delta||\tilde{U}'||\delta|)_{x_1} = O(|\delta|(|\delta_t| + |\nabla_x \delta|)|\tilde{U}''|)\]

and

\[(4.14)\]
\[N(U, U) = O(|U|^2)\]

since \(\tilde{A}^j - \bar{A}^j = A^j(\tilde{U}) - A^j(\bar{U}) = \int_0^1 dA^j(\tilde{U} + \theta(\bar{U} - \tilde{U})) U d\theta = O(U)\) and \(\tilde{U}_{x_j} = -\bar{U}'\delta_{x_j}\). □

**Assumption 4.2.** The operator \(\hat{K}(\partial_x)\) is defined by

\[(4.15)\]
\[\hat{K}(\partial_x)f(\xi) = i\hat{K}(\xi)\tilde{f}(\xi)\]

where \(\hat{K}(\xi)\) is a skew-symmetric operator which are smooth and homogeneous degree one in \(\xi\) satisfying

\[(4.16)\]
\[\Re\sigma(|\xi|^2 A^0 dQ - \sum_{j=1}^{d} \xi_j \hat{K}(\xi)A^j)_{\pm} \leq -\theta|\xi|^2\]

for all \(\xi\) in \(\mathbb{R}^d\).

**Remark 4.3.** This is a standard assumption of Kawashima which is satisfied when \(1.8\) is simultaneously symmetrizable and satisfies genuine-coupling condition. (See [Yo] and [Ze].)

**Proposition 4.4 (Damping estimate).** If \(|U|_{H^s}(t) \leq \varepsilon\) sufficiently small for \(0 \leq t \leq T\) where \(s \geq \left[\frac{d}{2}\right] + 2\), then, for some \(\tilde{\theta} > 0\), there holds

\[(4.17)\]
\[|U|^2_{H^s}(t) \leq e^{-\tilde{\theta}t}|U|^2_{H^s}(0) + C \int_0^t e^{-\tilde{\theta}(t-s)} \left(|U|^2_{L^2}(s) + \epsilon(s)\right) ds\]

for \(0 \leq t \leq T\), where

\[(4.18)\]
\[\epsilon(t) \leq C|\nabla_{t, x}\delta|^2_{L^2} = C\zeta_0^2(1 + t)^{-\frac{d+1}{4}}.\]

**Proof.** Let \(\alpha\) be a multi-index with \(|\alpha| = r \geq 1\). Taking a differential operator \(\partial^\alpha_x\) on the equation \((4.11)\), we have

\[(4.19)\]
\[\partial^\alpha_x U_t + \sum_{j=1}^{d} \partial^\alpha_x (\tilde{A}^j U_{x_j}) - \partial^\alpha_x (d\tilde{Q} U) = \partial^\alpha_x f\]
where \( f := (\partial_t - L)\delta(x, t)\tilde{U}'(x_1) - R_{x_1} + (0, I_r)'N(U, U) - M. \)

Taking the \( L^2 \) inner product of \( A^0\partial_x^\alpha U \) against \( \partial_x^\alpha U \), we have the energy estimate:

\[
\frac{1}{2} \frac{d}{dt} \langle A^0\partial_x^\alpha U, \partial_x^\alpha U \rangle = \frac{1}{2} \langle A^0\partial_x^\alpha U, \partial_x^\alpha U \rangle + \langle A^0\partial_x^\alpha U_t, \partial_x^\alpha U \rangle \\
\leq \frac{1}{2} \langle (A^0_t + \sum_j (A^0\tilde{A}^j)_{x_j})\partial_x^\alpha U, \partial_x^\alpha U \rangle + \langle (A^0d\tilde{Q})\partial_x^\alpha U, \partial_x^\alpha U \rangle \\
- C\langle \sum_{|\alpha|} \sum_j A^0(\partial_x^k\tilde{A}^j)(\partial_x^{|\alpha|-k}U_{x_j}), \partial_x^\alpha U \rangle \\
+ \sum_{k=1} \langle A^0(\partial_x^k\tilde{Q}d\tilde{Q})(\partial_x^{|\alpha|-k}U), \partial_x^\alpha U \rangle + \langle A^0\partial_x^\alpha f, \partial_x^\alpha U \rangle \\
\leq C(|U|_{W^{1,\infty}} + |U'_{W^{1,\infty}}|U|_{H^r} + \varepsilon|U|^2_{H^r} + C|U|^2_{L^2} \\
+ \langle A^0d\tilde{Q}\partial_x^\alpha U, \partial_x^\alpha U \rangle + \varepsilon(t), \\
\leq C(|U|_{W^{1,\infty}} + |U'_{W^{1,\infty}}|U|_{H^r} + \varepsilon|U|^2_{H^r} + C|U|^2_{L^2} \\
+ \langle (A^0d\tilde{Q})\partial_x^\alpha U, \partial_x^\alpha U \rangle + \varepsilon(t),
\]

where

\[
\varepsilon(t) = O\left(\|\nabla_{t, x}\delta^2_{L^2}\| = C(1 + t)^{-\frac{d+1}{2}} - 1. \right)
\]

The second last inequality is true by Moser’s inequalities and Sobolev inequalities. For each \( \alpha \) with \( |\alpha| = r \geq 1 \), we define \( \tilde{\alpha} := \alpha - e_j \) where \( j = \min \{ i : \alpha_i \text{ is maximal} \} \). Then, \( |\tilde{\alpha}| = r - 1 \). Let \( S_r := \{ (\alpha, \tilde{\alpha}) : |\alpha| = r \} \). Similarly, taking the \( L^2 \) inner product of \( K(\partial_x^{|\alpha|-\tilde{\alpha}})\partial_x^\alpha U \) against \( \partial_x^\alpha U \), we have the auxiliary energy estimate:

\[
\frac{1}{2} \frac{d}{dt} \langle K(\partial_x^{|\alpha|-\tilde{\alpha}})\partial_x^\alpha U, \partial_x^\alpha U \rangle = \frac{1}{2} \frac{d}{dt} \langle i\tilde{K}(\xi^{|\alpha|-\tilde{\alpha}})(i\xi)^{\tilde{\alpha}}\hat{U}, (i\xi)^{\tilde{\alpha}}\hat{U} \rangle \\
= \langle i\tilde{K}(\xi^{\tilde{\alpha}})(i\xi)^{\tilde{\alpha}}\hat{U}, (i\xi)^{\tilde{\alpha}}\hat{U}_t \rangle \\
= \langle (i\xi)^{2\tilde{\alpha}}\hat{U}, -\sum_j \tilde{K}(\xi^{\tilde{\alpha}}) \xi_j A^\dagger \hat{U} \rangle \\
+ \langle i\tilde{K}(\xi^{\tilde{\alpha}})(i\xi)^{\tilde{\alpha}}\hat{U}, (i\xi)^{\tilde{\alpha}}\hat{H} \rangle.
\]

using Plancherel’s inequality together with the equation

\[
\hat{U}_t = -\sum_j i\xi_j A^\dagger \hat{U} + \hat{H},
\]
where

\begin{equation}
(4.23) \quad H := \sum_j (A_j^2 - \tilde{A}_j^2) U + (d\tilde{Q}) U + f.
\end{equation}

By a direct calculation with the Moser inequality and the assumption of smallness of \( \tilde{U} \), we have

\begin{equation}
(4.24) \quad |(i\xi)^\alpha \tilde{H}|_{L^2(\xi)} = \left| \partial_x^\alpha H \right|_{L^2(x)} \leq C |U|_{L^\infty} |U|_{H^{r-1}} + \varepsilon |U|_{H^{r-1}}.
\end{equation}

Thus, we have

\begin{equation}
(4.25) \quad \frac{1}{2} \frac{d}{dt} \sum_{|\alpha| = r} \left( \langle A^0_\alpha U, \partial_x^{\alpha} U \rangle + \langle K(\partial_x^{\alpha-\tilde{\alpha}})\partial_x^{\tilde{\alpha}} U, \partial_x^{\tilde{\alpha}} U \rangle \right)
\end{equation}

\begin{align*}
&\leq \sum_{(\alpha,\tilde{\alpha}) \in S_r} \xi^{2\tilde{\alpha}} \left( (\xi^{2(\alpha-\tilde{\alpha})} (A^0 dQ)_- - \sum_{j=1}^d \xi_j \tilde{K}(\xi^{\alpha-\tilde{\alpha}}) A^j) \tilde{U}, \tilde{U} \right) \\
&\quad + \varepsilon |U|^2_{H^{r}} + C |U|^2_{L^2} + \epsilon(t) \\
&\leq -\theta \sum_{|\alpha|=r} \langle \xi^{2\alpha} \tilde{U}, \tilde{U} \rangle + \varepsilon |U|^2_{H^{r}} + C |U|^2_{L^2} + \epsilon(t) \\
&\leq -\theta |U|^2_{H^{r}} + \varepsilon |U|^2_{H^{r}} + C |U|^2_{L^2} + \epsilon(t)
\end{align*}

The second last inequality is true by (4.16). By (4.25) and choosing \( \varepsilon \leq \theta/2 \), we obtain

\begin{equation}
(4.26) \quad \frac{1}{2} \frac{d}{dt} \left( \langle A^0_\alpha U, U \rangle + \sum_{r=1}^s \sum_{|\alpha| = r} c^{-r} \left( \langle A^0_\alpha U, \partial_x^{\alpha} U \rangle + \langle K(\partial_x^{\alpha-\tilde{\alpha}})\partial_x^{\tilde{\alpha}} U, \partial_x^{\tilde{\alpha}} U \rangle \right) \right)
\end{equation}

\begin{align*}
&\leq \sum_{r=1}^s \sum_{(\alpha,\tilde{\alpha}) \in S_r} c^{-r} \left( \xi^{2\tilde{\alpha}} \left( (\xi^{2(\alpha-\tilde{\alpha})} (A^0 dQ)_- - \sum_{j=1}^d \xi_j \tilde{K}(\xi^{\alpha-\tilde{\alpha}}) A^j) \tilde{U}, \tilde{U} \right) \\
&\quad + \sum_{r=1}^s \frac{\theta}{2c^r} |U|^2_{H^{r}} + C |U|^2_{L^2} + \epsilon(t) \\
&\leq -\frac{\theta}{2c^s} |U|^2_{H^{r}} + C |U|^2_{L^2} + \epsilon(t)
\end{align*}

so long as \( |U|_{W^{1,\infty}} \) is small. We define

\begin{equation}
(4.27) \quad \mathcal{E}(t) := \langle A^0_\alpha U, U \rangle + \sum_{r=1}^s \sum_{|\alpha| = r} c^{-r} \left( \langle A^0_\alpha U, \partial_x^{\alpha} U \rangle + \langle K(\partial_x^{\alpha-\tilde{\alpha}})\partial_x^{\tilde{\alpha}} U, \partial_x^{\tilde{\alpha}} U \rangle \right)
\end{equation}

\begin{align*}
&\sim |U|^2_{H^{r}}(t)
\end{align*}
It is easy to check that $\mathcal{E}(t)$ is equivalent to $|U|_{H^s}^2(t)$. Using (4.26) and (4.27), we have the Gronwall-type inequality

$$\frac{d}{dt}\mathcal{E}(t) \leq -\bar{\theta}\mathcal{E}(t) + C\left(|U|_{L^2}^2(t) + \epsilon(t)\right).$$

Therefore, we have the result. \qed

**Lemma 4.5.** (High-Frequency Operator Estimate) Let $G^H$ be the high-frequency part of Green function associated with $(\partial_t - L)$. For any $f \in H^3$, there holds

$$\left| \int G^H(x, t; y)f(y)dy \right|_{L^2} \leq C e^{-\theta t}|f|_{H^3}.$$  

for some $\theta > 0$.

**Proof.** First, we establish the “high-frequency” resolvent bound by Kawashima-type energy estimate. To this end, consider the eigenvalue equation

$$(\lambda - L\tilde{\xi})U = f$$

Set

$$\langle A^0(\lambda - L\tilde{\xi})(1 + |\tilde{\xi}| + \partial_{x_1})U, (1 + |\tilde{\xi}| + \partial_{x_1})U \rangle = \langle A^0(1 + |\tilde{\xi}| + \partial_{x_1})f, (1 + |\tilde{\xi}| + \partial_{x_1})U \rangle.$$

We obtain by the simpler linear version of Kawashima-type energy estimate as in Proposition 4.4 (the damping estimate), that

$$\Re \lambda \langle (1 + |\tilde{\xi}| + \partial_{x_1})U \rangle_{L^2(x_1)}^2 \leq -\theta \langle (1 + |\tilde{\xi}| + \partial_{x_1})U \rangle_{L^2(x_1)}^2$$
$$+ C^* \langle (1 + |\tilde{\xi}| + \partial_{x_1})f \rangle_{L^2(x_1)}^2 + C^* |U|_{L^2(x_1)}^2.$$

Taking the imaginary part of the $L^2$ inner product of $U$ against $\lambda U = L\tilde{\xi}U + f$, we have

$$\Im \lambda |U|_{L^2(x_1)}^2 \leq$$
$$C_2 |f|_{L^2(x_1)}^2 + (2\epsilon)^{-1} |U|_{L^2(x_1)}^2 + \epsilon \langle (1 + |\tilde{\xi}| + \partial_{x_1})U \rangle_{L^2(x_1)}^2.$$

By (4.32) and (4.33), we establish the high-frequency resolvent bound

$$(\lambda - L\tilde{\xi})^{-1} \leq C$$
for $|(\tilde{\xi}, \lambda)| \geq R$ and $\Re \lambda \geq -\theta$. 

for some $R, C > 0$ sufficiently large and $\theta > 0$ sufficiently small. Moreover, we have the intermediate frequency bound,

\begin{equation}
(\lambda - L_\xi)^{-1}|_{H^1(x)} \leq C \quad \text{for} \quad R^{-1} \leq |(\xi, \lambda)| \leq R \quad \text{and} \quad \Re \lambda \geq -\theta,
\end{equation}

for any $R > 0$ and $C = C(R) > 0$ sufficiently large and $\theta > 0$ sufficiently small. This follows by compactness of the set of frequencies under consideration together with the fact that the resolvent $(\lambda - L_\xi)^{-1}$ is analytic with respect to $H^1$ in $(\xi, \lambda)$.

On the other hand, it is easy to check the following resolvent identity using analyticity on the resolvent set $\rho(L)$ of the resolvent $(\lambda - L_\xi)^{-1}$, for all $f \in \mathcal{D}(L_\xi)$,

\begin{equation}
(\lambda - L_\xi)^{-1}f = \lambda^{-2}(\lambda - L_\xi)^{-1}L_\xi^2f + \lambda^{-2}L_\xi f + \lambda^{-1}f
\end{equation}

Using the resolvent identity (4.34), the high-frequency solution operator $S^{HI}$ can be written as

\[
S^{HI}f = \text{P.V.} \int_{-\theta + i\infty}^{\theta - i\infty} \int_{\mathbb{R}^d} \chi_{|\xi|^2 + |3\lambda|^2 \geq \theta_1 + \theta_2} e^{i\xi \cdot (\tilde{x} - \tilde{y}) + \lambda t} (\lambda - L_\xi)^{-1} \hat{f}(x, \xi) d\xi d\lambda
\]

\[
= \text{P.V.} \int_{-\theta + i\infty}^{\theta - i\infty} \int_{\mathbb{R}^d} \chi_{|\xi|^2 + |3\lambda|^2 \geq \theta_1 + \theta_2} e^{i\xi \cdot (\tilde{x} - \tilde{y}) + \lambda t} \lambda^{-2}(\lambda - L_\xi)^{-1}L_\xi^2 \hat{f}(x, \xi) d\xi d\lambda
\]

\[
+ \text{P.V.} \int_{-\theta + i\infty}^{\theta - i\infty} \int_{\mathbb{R}^d} \chi_{|\xi|^2 + |3\lambda|^2 \geq \theta_1 + \theta_2} e^{i\xi \cdot (\tilde{x} - \tilde{y}) + \lambda t} L_\xi \hat{f}(x, \xi) d\xi d\lambda
\]

\[
+ \text{P.V.} \int_{-\theta + i\infty}^{\theta - i\infty} \int_{\mathbb{R}^d} \chi_{|\xi|^2 + |3\lambda|^2 \geq \theta_1 + \theta_2} e^{i\xi \cdot (\tilde{x} - \tilde{y}) + \lambda t} \lambda^{-1} \hat{f}(x, \xi) d\xi d\lambda
\]

\[
=: A + B + C.
\]

For $f \in H^3$, there holds

\begin{equation}
|A|_{H^1} \leq Ce^{-\theta_1 t} \sup \lambda |(\lambda - L_\xi)^{-1}|_{H^1} |f|_{H^3}
\end{equation}

\[
\leq C^* e^{-\theta_1 t} |f|_{H^3}.
\]
Similarly, we have

\[(4.38) \quad |B|_{H^2} \leq Ce^{-\theta_1 t}|f|_{H^2}.\]

Using the triangle inequality, we have

\[(4.39) \quad |C|_{H^1} \leq \left| \text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \lambda^{-1} e^{\lambda t} \sum_{x, \xi} e^{i\xi \cdot \tilde{x}} \hat{f}(x, \xi) d\lambda \right|_{H^1} + \left| \int_{-\theta_1-ir}^{-\theta_1+ir} \lambda^{-1} e^{\lambda t} \sum_{x, \xi} e^{i\xi \cdot \tilde{x}} \hat{f}(x, \xi) d\lambda \right|_{H^1} \leq 2r/\theta e^{-\theta_1 t}|f|_{H^1}.\]

The last inequality is true since the first term is zero, by the inverse Laplace transform identity

\[\text{P.V.} \int_{-\theta_1-i\infty}^{-\theta_1+i\infty} \lambda^{-1} e^{\lambda t} d\lambda = \begin{cases} 0 & \text{if } \theta_1 t > 0 \\ \theta_1 t & \text{if } \theta_1 t < 0. \end{cases}\]

By the expression of $S^{II}$, together with (4.37), (4.38) and (4.39), we have

\[(4.40) \quad |S^{II}|_{H^{3} \to L^2(t)} \leq Ce^{-\theta t}.\]

Therefore, we have the result. \[\square\]

**Remark 4.6.** The argument of Lemma 4.5 based on the initial-value problem with homogeneous forcing, greatly simplifies the treatment in [Z4] based on the initial-value problem with homogeneous data and inhomogeneous forcing. The argument is quite general, in particular extending without modification to the case that the equilibrium model is a system.

5. **Nonlinear $L^2$ decay and asymptotic behavior.**

We now carry out the nonlinear analysis by an argument combining the approach of [HoZ2] in the low-frequency domain with that of [Z4] in the high-frequency domain. Let the approximate shock deformation $\delta(\tilde{x}, t)$ be a solution of the constant coefficient equation

\[(5.1) \quad \delta_t + \vec{a} \cdot \nabla \delta = \text{div}_x(\tilde{\beta} \nabla \tilde{z} \delta)\]

with initial data

\[(5.2) \quad \delta_0(\tilde{x}) = -([u]^{-1}, 0) \int_{-\infty}^{\infty} \tilde{U}(x_1, \tilde{x}, 0) - \tilde{U}(x_1) dx_1.\]
Here, $\bar{a}$ and $\bar{\beta}$ are as in Appendix A.

We define a smooth approximation of $\delta$, denoted by $\delta^\varepsilon$, by

$$(5.3) \quad \delta^\varepsilon(\cdot,t) := \eta^\varepsilon * \delta(\cdot,t)$$

where $\eta^\varepsilon$ is a smooth mollifier supported in $B(0,\varepsilon)$. Note that $\delta^\varepsilon$ satisfies the same convected heat equation as $\delta$ does with $C^\infty$ initial data $\delta^\varepsilon_0 = \eta^\varepsilon * \delta_0$. Define

$$(5.4) \quad U(x,t) := \tilde{U}(x,t) - \bar{U}(x_1 - \delta^\varepsilon(\tilde{x},t)).$$

**Lemma 5.1.** For $|\tilde{U}_0 - \bar{U}|_{L^1} \leq \zeta_0$, a multi-index $|\alpha| \leq K$, there holds

$$(5.5) \quad |\partial^\alpha_x \delta^\varepsilon(\cdot,t)|_{L^2} \leq C\zeta_0(1 + t)^{-(d-1)/4 - |\alpha|/2}$$

and

$$(5.6) \quad |\delta^\varepsilon(\cdot,t) - \delta(\cdot,t)|_{L^2} \leq C\zeta_0 t^{-(d-1)/4 - 1/2}$$

where $C = C(\varepsilon, K)$ is a constant.

**Proof.** Letting $g(x,t)$ be a Green function for (5.2), we have

$$(5.7) \quad \delta^\varepsilon(\cdot,t) = g(\cdot,t) * \delta^\varepsilon(\cdot,0)$$

and

$$(5.8) \quad \delta^\varepsilon(\cdot,t) - \delta(\cdot,t) = g(\cdot,t) * (\delta^\varepsilon(\cdot,0) - \delta(\cdot,0)).$$

Thus, we have

$$(5.9) \quad |\partial^\alpha_x \delta^\varepsilon(\cdot,t)|_{L^2} \leq |\partial^\alpha_x g(\cdot,t)|_{L^2} |\delta^\varepsilon(\cdot,t)|_{L^1}$$

$$\leq |\partial^\alpha_x g(\cdot,t)|_{L^2} |\eta^\varepsilon|_{L^1} |\delta^\varepsilon_0|_{L^1}$$

$$\leq C\zeta_0 t^{-(d-1)/4 - |\alpha|/2}$$

by the standard fact that a heat kernel $g$ decays as

$$(5.10) \quad |\partial^\alpha_x g(\cdot,t)|_{L^2} \leq C t^{-(d-1)/4 - |\alpha|/2}$$
and observing that $|\delta_0|_{L^1} \leq C|\tilde{U}_0 - \bar{U}|_{L^1} \leq C\zeta_0$ by definition of $\delta_0$. On the other hand, we have also

\begin{equation}
|\partial_x^\alpha \delta^\varepsilon(\cdot, t)|_{L^2} \leq |g(\cdot, t)|_{L^1}|\partial_x^\alpha \eta^\varepsilon(\cdot, t)|_{L^2} \leq C\zeta_0|\alpha|.
\end{equation}

By (5.9) and (5.11), we have the first claim. Expressing

\begin{equation}
\delta^\varepsilon(\cdot, t) - \delta(\cdot, t) = (g^\varepsilon(\cdot, t) - g(\cdot, t)) \ast \delta(\cdot, 0),
\end{equation}

and noting that

\begin{equation}
|(g^\varepsilon(\tilde{y}, t) - g(\tilde{y}, t))|_{L^2(\tilde{y})} = \left| \int (g(\tilde{y} - \tilde{z}, t) - g(\tilde{y}, t))\eta^\varepsilon(\tilde{z})d\tilde{z} \right|_{L^2(\tilde{y})}
= \left| \int \int_0^1 (\nabla \tilde{z} g(\tilde{y} - \theta \tilde{z}, t) \cdot \tilde{z})\eta^\varepsilon(\tilde{z})d\theta d\tilde{z} \right|_{L^2(\tilde{y})}
\leq \int_0^1 |\nabla \tilde{z} g(\tilde{y} - \theta \tilde{z}, t)|_{L^2(\tilde{y})} |\tilde{z} \eta^\varepsilon(\tilde{z})|_{L^1(\tilde{z})}d\theta
\leq \varepsilon t^{-(d-1)/4-1/2}.
\end{equation}

In the last inequality, we have used the fact that $|\tilde{z} \eta^\varepsilon(\tilde{z})|_{L^1(\tilde{z})} \leq C\varepsilon$. Thus, we obtain

\begin{equation}
|\delta^\varepsilon(\cdot, t) - \delta(\cdot, t)|_{L^2} \leq |g^\varepsilon(\cdot, t) - g(\cdot, t)|_{L^2} |\delta(\cdot, 0)|_{L^1},
\end{equation}

\begin{equation}
\leq C\varepsilon\zeta_0 t^{-(d-1)/4-1/2}.
\end{equation}

\[\square\]

**Lemma 5.2.** If $|\tilde{U}_0 - \bar{U}|_{L^1}$, $|\tilde{U}_0 - \bar{U}|_{L^\infty}$, $|x_1(\tilde{U}_0 - \bar{U})|_{L^1} \leq \zeta_0$. Then,

\begin{equation}
|U_0|_{L^1}, |U_0|_{L^\infty}, |x_1 U_0|_{L^1} \leq C(\varepsilon)\zeta_0.
\end{equation}

**Proof.** We have

\begin{equation}
\bar{U}(x_1) - \tilde{U}(x_1 - \delta_0) = \int_0^1 \bar{U}'(x_1 - \theta \delta_0)\delta_0 d\theta.
\end{equation}
Then we have

\begin{align}
|\bar{U}(x_1) - \bar{U}(x_1 - \delta_0^\varepsilon)|_{L^p(x)} & \leq \int_0^1 |\bar{U}'(x_1 - \theta \delta_0^\varepsilon)|_{L^p(x)} d\theta \\
& \leq \int_0^1 |\bar{U}'|_{L^p(x)} |\delta_0^\varepsilon|_{L^p(x)} d\theta \leq C \zeta_0
\end{align}

By (5.17) together with the triangle inequality, we have

\begin{align}
|U_0|_{L^p(x)} & \leq |\tilde{U}_0 - \bar{U}|_{L^p(x)} + |\bar{U}(x_1) - \tilde{U}(x_1 - \delta_0^\varepsilon)|_{L^p(x)} \leq (C + 1) \zeta_0.
\end{align}

Similarly, we can prove the other inequality. \hfill \Box

**Lemma 5.3.** The nonlinear residual \( U(x,t) \) defined in (5.4) satisfies

\begin{align}
U_t + \sum_{j=1}^d (A_j(\bar{U})U)_{x_j} - dQ(\bar{U})U & = (\partial_t - L)\delta^\varepsilon \bar{U}' - R_{x_1} + \sum_{j=1}^d N_{x_j}^j + (0, I_r)^t N^0 + \sum_{j=1}^d S_{x_j}^j + (0, I_r)^t S^0
\end{align}

where

\begin{align}
R_{x_1} & = O(|\delta_t^\varepsilon||\bar{U}'|)|\delta^\varepsilon| + |\nabla_x \delta^\varepsilon||\bar{U}'||\delta^\varepsilon|)_{x_1} = O(|\delta_t^\varepsilon|(|\delta_t^\varepsilon| + |\nabla_x \delta^\varepsilon||\bar{U}'|))
\end{align}

and

\begin{align}
N^j & = O(|U|^2) \text{ and } S^j = O(|\delta^\varepsilon||\bar{U}'||U|) \text{ for } j = 0, 1, ..., d.
\end{align}

**Proof.** Let \( \bar{U} \) be a solution of

\begin{align}
\bar{U}_t + \sum_{j=1}^d A_j(\bar{U})\bar{U}_{x_j} - Q(\bar{U}) = 0.
\end{align}
For $\bar{U}(x,t) = \bar{U}(x_1 - \delta^\varepsilon(\bar{x},t))$, there holds

\begin{equation}
\bar{U}_t + \sum_{j=1}^{d} A^j(\bar{U})\bar{U}_{x_j} - Q(\bar{U})
\end{equation}

\begin{equation}
= -\delta^\varepsilon \bar{U}' + \sum_{j=2}^{d} A^j(\bar{U})(-\delta^\varepsilon_{x_j})\bar{U}'
\end{equation}

\begin{equation}
= -\delta^\varepsilon \bar{U}'(x_1) + \sum_{j=2}^{d} A^j(\bar{U}(x_1))(-\delta^\varepsilon_{x_j})\bar{U}'(x_1) + R_{x_1}
\end{equation}

\begin{equation}
= - (\partial_t - L)\delta^\varepsilon \bar{U}' + R_{x_1}.
\end{equation}

If we subtract (5.23) from (5.22), we have

\begin{equation}
U_t + \sum_{j=1}^{d} (A^j(\bar{U})\bar{U}_{x_j} - A^j(\bar{U})\bar{U}_{x_j}) - (Q(\bar{U}) - Q(\bar{U}))
\end{equation}

\begin{equation}
= U_t + \sum_{j=1}^{d} (F^j(\bar{U}) - F^j(\bar{U}))_{x_j} - (Q(\bar{U}) - Q(\bar{U}))
\end{equation}

\begin{equation}
= (\partial_t - L)\delta\bar{U}' - R_{x_1}
\end{equation}

By Taylor expansion of $F^j$ about $\bar{U}$ and (5.24), we have

\begin{equation}
U_t + \sum_{j=1}^{d} (A^j(\bar{U})U)_{x_j} - dQ(\bar{U})U
\end{equation}

\begin{equation}
= (\partial_t - L)\delta\bar{U}' - R_{x_1} + \sum_{j=1}^{d} N^j_{x_j} + (0, I_r)^{t} N^0(U, U)
\end{equation}
By Taylor expansion of $A^j$ about $\bar{U}$ and (5.25), we have

\begin{equation}
U_t + \sum_{j=1}^{d} (A^j(\bar{U})x_j - dQ(\bar{U})U
\end{equation}

\begin{equation}
= (\partial_t - L)\delta^\epsilon \bar{U}' - R_{x_1} + \sum_{j=1}^{d} N^j_{x_j} + (0, I_r)^{j}N^0
\end{equation}

\begin{equation}
+ \sum_{j=1}^{d} \left( (A^j(\bar{U}) - A^j(\bar{U}))x_j - (dQ(\bar{U}) - dQ(\bar{U}))U
\right)
\end{equation}

\begin{equation}
= (\partial_t - L)\delta^\epsilon \bar{U}' - R_{x_1} + \sum_{j=1}^{d} N^j_{x_j} + (0, I_r)^{j}N^0 + \sum_{j=1}^{d} S^j_{x_j} + (0, I^r)S^0
\end{equation}

where

\begin{equation}
R_{x_1} = O(|\delta^\epsilon||\bar{U}'||\delta^\epsilon| + |\nabla_x \delta^\epsilon||\bar{U}'||\delta^\epsilon|)_{x_1} = O(|\delta^\epsilon||\nabla_x \delta^\epsilon||\bar{U}'||\delta^\epsilon|),
\end{equation}

\begin{equation}
N^j = O(|U|^2) \text{ for } j = 0, 1, ..., d
\end{equation}

and

\begin{equation}
S^j = O(|\delta^\epsilon||\bar{U}'||U|) \text{ for } j = 0, 1, ..., d.
\end{equation}

\begin{lemma}
For $f \in C^2 \cap L^1$, there holds

\begin{equation}
\int_{0}^{t} \int G(x, t - s; y)(\partial_s - L_y)f(y, s)dyds = f(x, t) - \int G(x, t; y)f(y, 0)dy
\end{equation}

\end{lemma}

\begin{proof}
Integrating by parts, we have

\begin{equation}
\int_{0}^{t} \int G(x, t - s; y)(\partial_s - L_y)f(y, s)dyds
\end{equation}

\begin{equation}
= \int G(x, \varepsilon; y)f(y, t - \varepsilon)dy - \int G(x, t; y)f(y, 0)dy
\end{equation}

\begin{equation}
+ \int_{0}^{t-\varepsilon} (\partial_s - L_y)^*G(x, t - s; y)f(y, s)dyds.
\end{equation}

By duality and letting $\varepsilon \to 0$, we obtain the result.

\end{proof}
We are now ready to prove our main result.

**Proof of Theorem 5.3.** By Lemma 5.1, it is sufficient to show

\[ |\tilde{U}(x, t) - \tilde{U}(x_1 - \delta^\varepsilon(\hat{x}, t))|_{L^2(x)} \leq C\zeta_0(1 + t)^{-(d-1)/4-1/2+\sigma}. \]  

Define

\[ \zeta(t) := \sup_{0 \leq s \leq t} |U(\cdot, s)|_{L^2}(1 + s)^{(d-1)/4+1/2-\sigma}. \]

**Claim 5.5.** For all \( t \geq 0 \), there holds

\[ \zeta(t) \leq C_1(\zeta^2(t) + \zeta_0 \zeta(t) + \zeta_0^2) \leq C_2(\zeta_0 + \zeta^2(t)). \]

From this result, it follows by continuous induction that

\[ \zeta(t) \leq 2C_2\zeta_0 \text{ for all } t \geq 0, \]

provided \( \zeta_0 < \frac{1}{4C_2} \), i.e., \( \zeta(t) \) remains small for all \( t \geq 0 \).

**Proof of claim.** Applying Duhamel’s principle, we can express

\[ U(x, t) = \left( \int G(x, t; y)U_0(y)dy - \int_0^t \int G(x, t-s; y)(\partial_x - L)\delta^\varepsilon\tilde{U}'(y, s)dyds \right) \]

\[ + \int_0^t \int G^I(x, t-s; y)(-R_{y_1} + \sum_{j=1}^d N_{y_j}^j + \sum_{j=1}^d S_{y_j}^j)(y, s)dyds \]

\[ + \int_0^t \int G^{II}(x, t-s; y)(-R_{y_1} + \sum_{j=1}^d N_{y_j}^j + \sum_{j=1}^d S_{y_j}^j)(y, s)dyds \]

\[ + \int_0^t \int G^I(x, t-s; y)((0, I_r)^tN^0 + (0, I_r)^tS^0)(y, s)dyds \]

\[ + \int_0^t \int G^{II}(x, t-s; y)((0, I_r)^tN^0 + (0, I_r)^tS^0)(y, s)dyds \]

\[ = I + II + III + IV + V. \]

First, we establish that

\[ |I|_{L^2} \leq C\zeta_0(1 + t)^{-(d-1)/4-1/2}. \]
Using (5.19), we have

\[
\begin{align*}
|\int G^I(x, t; y)U_0(y)dy|_{L^2} & \\
& \leq \left|\bar{U}'(x_1) \int g(\bar{x} - \bar{y}, t)\left(\Pi_1^I \int U(y_1, \bar{y}, 0)dy_1\right)dy\right|_{L^2(x)} + C\zeta_0 t^{-(d-1)/4-1/2} \\
& = \left|\bar{U}'(x_1)(\delta^\varepsilon(\bar{x}, t) - \delta(\bar{x}, t))\right|_{L^2(x)} + C\zeta_0 t^{-(d-1)/4-1/2} \\
& \leq C\zeta_0 t^{-(d-1)/4-1/2}.
\end{align*}
\]

By standard \(C^0\) semigroup theory, we obtain the short-time bound

\[
|\int G(x, t; y)U_0(y)dy|_{L^2(x)} \leq C|U_0|_{L^2} \leq C\zeta_0 \text{ for } t \leq 1.
\]

By (5.38) and (5.39), we have

\[
|\int G^I(x, t; y)U_0(y)dy|_{L^2(x)} \leq C\zeta_0(1 + t)^{-(d-1)/4-1/2}.
\]

By (4.29),

\[
|\int G^{II}(x, t; y)U_0(y)dy|_{L^2} \leq e^{-\theta t}|U_0|_{H^3} \leq C\zeta_0(1 + t)^{-(d-1)/4-1/2}.
\]

On the other hand, by (5.30), we have

\[
\begin{align*}
|\int_0^t \int G(x, t - s; y)(\partial_s - L)\delta^\varepsilon \bar{U}'(y, s)dyds|_{L^2} & \\
& = \left|\delta^\varepsilon \bar{U}'(x, t) - \int G(x, t; y)(\delta^\varepsilon \bar{U}')(y, 0)dy\right|_{L^2} \\
& \leq \left|\delta^\varepsilon \bar{U}'(x, t) - \bar{U}'(x_1) \int g(\bar{x} - \bar{y}, t)\delta^\varepsilon(\bar{y}, 0)dy\right|_{L^2} + C\zeta_0 t^{-(d-1)/4-1/2} \\
& = C\zeta_0 t^{-(d-1)/4-1/2}.
\end{align*}
\]

Combining this with the short time bound, we obtain

\[
|\int_0^t \int G(x, t - s; y)(\partial_s - L)\delta^\varepsilon \bar{U}'(y, s)dyds|_{L^2} \leq C\zeta_0(1 + t)^{-(d-1)/4-1/2}.
\]
We now establish that

\[(5.44)\quad |II|_{L^p} \leq C\zeta_0 (1 + t)^{-(d-1)/2(1-1/p)-1/2}.\]

Using (3.16), (5.5), and (5.21), together with the definition of \(\zeta(t)\), we have

\[(5.45)\quad |II_a|_{L^2} = \left| \int_0^t \int \partial_{x_j} G^I(x, t-s; y) N^j(y, s) dy ds \right|_{L^2}
\leq C \int_0^t (1 + t-s)^{-(d-1)/4-1/2} |U|^2_{L^2}(s) ds
\leq C\zeta^2(t) \int_0^t (1 + t-s)^{-(d-1)/4-1/2} (1 + s)^{-(d-1)/2-1/2+\sigma} ds
\leq C\zeta^2(t)(1 + t)^{-(d-1)/4-1/2+\sigma}
\]

and

\[(5.46)\quad |II_b|_{L^2} = \left| \int_0^t \int \partial_{x_j} G^I(x, t-s; y) S^j(y, s) dy ds \right|_{L^2}
\leq C \int_0^t (1 + t-s)^{-(d-1)/4-1/2} |S^j|_{L^1} ds
\leq C\zeta_0 \int_0^t (1 + t-s)^{-(d-1)/4-1/2} |\delta_\varepsilon|^2_{L^2}(s) |U|_{L^2}(s) ds
\leq C\zeta_0 \zeta(t)(1 + t)^{-(d-1)/4-1/2+\sigma}.
\]

The last inequality is true due to the following calculations;

\[(5.47)\quad \int_0^{t/2} (1 + t-s)^{-(d-1)/4-1/2} (1 + s)^{-(d-1)/4} (1 + s)^{-(d-1)/4-1/2+\sigma} ds
\leq (1 + t/2)^{-(d-1)/4-1/2} \int_0^{t/2} (1 + s)^{-(d-1)/2-1/2+\sigma} ds
\leq (1 + t/2)^{-(d-1)/4-1/2} \left| (1 + t/2)^{-(d-1)/2+1/2+\sigma} - 1 \right|
\leq C(1 + t)^{-(d-1)/4-1/2+\sigma} \left| (1 + t)^{-(d-1)/2+1/2} - (1 + t)^{-\sigma} \right|
\leq C(1 + t)^{-(d-1)/4-1/2+\sigma}
\]

It is true with arbitrary small \(\sigma > 0\) for \(d = 2\) and \(\sigma = 0\) for \(d \geq 3\).
\begin{align}
\int_{t/2}^{t} (1 + t - s)^{-\frac{(d-1)}{4}-1/2}(1 + s)^{-\frac{(d-1)}{4}-1/2+\sigma} ds &
\leq (1 + t)^{-\frac{(d-1)}{4}-1/2}(1 + s)^{-\frac{(d-1)}{4}-1/2+\sigma} ds \\
&\leq (1 + t)^{-\frac{(d-1)}{4}-1/2}(1 + t - s)^{-\frac{(d-1)}{4}-1/2+\sigma} ds \\
&\leq (1 + t)^{-\frac{(d-1)}{4}-1/2+(d-1)/4-1/2+\sigma} ds \\
&\leq (1 + t)^{-\frac{(d-1)}{4}-1/2+\sigma}.
\end{align}

The last inequality is true if we choose arbitrary small $\epsilon > 0$ for $d = 3$ so that $-(d-1)/4 - 1/2 + \epsilon \neq -1$ and choose $\epsilon = 0$ otherwise.

Similarly, using (3.16), (5.5), and (5.20),

\begin{align}
|II|_{L^2} &= \left| \int_0^t \int_0^t \partial_{x_1} G^I(x, t - s; y) R(y, s) dy ds \right|_{L^2(x)} \\
&\leq C \int_0^t (1 + t - s)^{-\frac{(d-1)}{4}-1/2} |R|_{L^1(s)} ds \\
&\leq C \int_0^t (1 + t - s)^{-\frac{(d-1)}{4}-1/2} |\delta_{x_1}^\epsilon|_{L^2(s)} |\delta_{x_j}^\epsilon|_{L^2(s)} ds \\
&\leq C_0^2 \int_0^t (1 + t - s)^{-\frac{(d-1)}{4}-1/2}(1 + s)^{-\frac{(d-1)}{2}-1/2} ds \\
&\leq C_0^2 (1 + t)^{-\frac{(d-1)}{4}-1/2}.
\end{align}

By (5.45), (5.46) and (5.49), we have

\begin{align}
|II|_{L^2} &\leq C(\zeta^2(t) + \zeta_0 \zeta(t) + \zeta_0^2)(1 + t)^{-\frac{(d-1)}{4}-1/2+\sigma}.
\end{align}

We now establish that

\begin{align}
|III|_{L^2} &\leq C(\zeta_0^2 + \zeta_0 \zeta(t) + \zeta^2(t))(1 + t)^{-\frac{(d-1)}{4}-1/2+\sigma}.
\end{align}
By (4.29), (4.17), and (5.20)–(5.21), we have

\[
\left| \int_0^t \int G^{II}(x, t - s; y)(N^j_{y_j} + S^j_{y_j} + R_{y_1})dyds \right|_{L^2} 
\leq \int_0^t e^{-\theta(t-s)}|N^j_{x_j} + S^j_{x_j} + R_{x_1}|_{H^3}ds 
\leq \int_0^t e^{-\theta(t-s)}|N^j + S^j + R|_{H^4}ds 
\leq C \int_0^t e^{-\theta(t-s)}(|U|_{L^\infty} |U|_{H^4} + |\delta^\varepsilon|_{L^\infty} |U|_{H^4} + |\delta^\varepsilon|_{L^\infty} |\nabla x \delta^\varepsilon|_{H^4})ds 
\leq C(\zeta_0^2 + \zeta_0 \zeta(t) + \zeta^2(t))(1 + t)^{-\frac{d-1}{4}-\frac{1}{2}+\sigma}
\]

as long as $|U|_{H^s}$ and thus $|U|_{W^{1,\infty}}$ remains sufficiently small. We shall verify in a moment that it indeed remains small.

Similarly, we can establish the estimates for $IV$ and $V$. By the same proof as the one for $II$, together with (3.18), we have

\[
|IV|_{L^2} \leq C(\zeta^2(t) + \zeta_0 \zeta(t) + \zeta_0^2)(1 + t)^{-\frac{d-1}{4}-\frac{1}{2}+\sigma}.
\]

By an identical calculation as for the $III$ term except for the fact that $V$ has one less derivatives, we can get

\[
|V|_{L^2}(t) \leq C(\zeta^2(t) + \zeta_0 \zeta(t) + \zeta_0^2)(1 + t)^{-\frac{d-1}{4}-\frac{1}{2}+\sigma}.
\]

Therefore,

\[
|U|_{L^2}(t) \leq C_1(\zeta^2(t) + \zeta_0 \zeta(t) + \zeta_0^2)(1 + t)^{-\frac{d-1}{4}-\frac{1}{2}+\sigma}.
\]

We have the desired inequality,

\[
\zeta(t) \leq C_1(\zeta^2(t) + \zeta_0 \zeta(t) + \zeta_0^2) \leq C_2(\zeta_0 + \zeta^2(t))
\]

so long as $|U|_{H^s} \leq \varepsilon$ sufficiently small.

To complete the proof, we show that $|U|_{H^s}(t) \leq \varepsilon$ remains small for all $t \geq 0$ if we choose $|U|_{H^s}(0) \leq \zeta_0$ small enough, by the continuation argument. By local well-posedness, for sufficiently small $\varepsilon > 0$, we can define

\[
T := \sup \{\tau > 0 : |U|_{H^s}(\tau) < \varepsilon \} > 0.
\]
As shown above, we have

\[(5.58) \quad \zeta(t) \leq C_2(\zeta_0 + \zeta^2(t)) \text{ for } t \in [0, T),\]

which implies

\[(5.59) \quad \zeta(t) \leq 2C_2\zeta_0 \text{ for } t \in [0, T).\]

Using (4.17) again, we have, by choosing \(\zeta_0\) so small that there holds

\[(5.60) \quad |U|_{H^s}(T) \leq e^{-\theta T}\zeta_0 + \frac{2C_2^2\zeta_0}{\theta} < \varepsilon.\]

By continuity of \(H^s\)-norm, there exists \(h > 0\) such that \(|U|_{H^s}(t) < \varepsilon\) for \(t \in [0, T + h)\), which contradicts to the definition of \(T\). So, \(T = \infty\), i.e., \(|U|_{H^s}(t) < \varepsilon\) for \(t \in [0, \infty)\). Thus, the desired inequality (5.56) is true for all \(t \geq 0\), which implies that

\[(5.61) \quad \zeta(t) \leq 2C_2\zeta_0 \text{ for all } t \in [0, \infty).\]

This completes the proof. \(\Box\)

**Theorem 5.6.** If \(\tilde{U}_0 - \overline{U}|_{H^{[d/2]+5}} \leq \zeta_0\) sufficiently small, then there holds

\[(5.62) \quad |\tilde{U}(x, t) - \overline{U}(x_1 - \delta^\varepsilon(\tilde{x}, t))|_{L^\infty(x)} \leq C\zeta_0(1 + t)^{-(d-1)/2-1/2}.\]

**Proof.** Using the expression (5.36) similarly as in the previous theorem, we get the \(L^\infty\) bounds for each term.

\[(5.63) \quad |I|_{L^\infty} \leq C\zeta_0(1 + t)^{-(d-1)/2-1/2}.\]

Using (3.16), (5.3) and (1.18), we have

\[
|II_n|_{L^\infty} \leq \int_0^t |G^n_{x_j}|_{L^\infty(t-s)}|U|^2_{L^2}(s)ds \\
\leq C\zeta_0^2 \int_0^t (1 + t - s)^{-(d-1)/2-1/2}(1 + s)^{-(d-1)/2-1+2\sigma}ds \\
\leq C\zeta_0^2(1 + t)^{-(d-1)/2-1/2},
\]
\[|II_b|_{L^\infty} \leq \int_0^t |G^t_{x_j}|_{L^\infty}(t - s)|U|_{L^2}(s)|\delta^\varepsilon|_{L^2}(s) \, ds \leq C\zeta_0 \int_0^t (1 + t - s)^{-(d-1)/2-1/2}(1 + s)^{-(d-1)/4-1/2+\sigma}(1 + s)^{-(d-1)/4} \, ds \leq C\zeta_0(1 + t)^{-(d-1)/2-1/2+\sigma}
\]

and

\[|II_c|_{L^\infty} \leq \int_0^t |G^t_{x_j}|_{L^\infty}(t - s)|\delta^\varepsilon|_{L^2}(s)|\delta^\varepsilon|_{L^2}(s) \, ds \leq C\zeta_0^2 \int_0^t (1 + t - s)^{-(d-1)/2-1/2}(1 + s)^{-(d-1)/4-1/2}(1 + s)^{-(d-1)/4} \, ds \leq C\zeta_0^2(1 + t)^{-(d-1)/2-1/2}.
\]

We establish that

\[(5.64) \quad |III|_{L^\infty} \leq C\zeta_0(1 + t)^{-(d-1)/2-1/2+\sigma}.
\]

By (4.29), we have

\[
\left| \int_0^t \int G^{II}(x, t - s; y)N^j_{y_j} \, dy ds \right|_{L^\infty} \leq \left| \int_0^t \int G^{II}(x, t - s; y)N^j_{y_j} \, dy ds \right|_{H^{[d/2]+1}} \leq \int_0^t e^{-\theta(t-s)}|N^j_{x_j}|_{H^{[d/2]+4}} \, ds \leq C \int_0^t e^{-\theta(t-s)}|U|_{L^\infty}|U|_{H^{[d/2]+5}} \, ds \leq C\zeta_0^2(1 + t)^{-(d-1)/2-1+2\sigma},
\]

\[
\left| \int_0^t \int G^{II}(x, t - s; y)S^j_{y_j} \, dy ds \right|_{L^\infty} \leq \left| \int_0^t \int G^{II}(x, t - s; y)S^j_{y_j} \, dy ds \right|_{H^{[d/2]+1}} \leq \int_0^t e^{-\theta(t-s)}|S^j_{x_j}|_{H^{[d/2]+4}} \, ds \leq C \int_0^t e^{-\theta(t-s)}|\delta^\varepsilon|_{L^\infty}|U|_{H^{[d/2]+5}} \, ds \leq C\zeta_0^2(1 + t)^{-3(d-1)/4-1/2+\sigma},
\]
\[ \left\| \int_0^t \int G^{II}(x, t - s; y) R_{y_1} dy ds \right\|_{L^\infty} \leq \left\| \int_0^t \int G^{II}(x, t - s; y) R_{y_1} dy ds \right\|_{H^{d/2}+1} \]
\[ \leq \int_0^t e^{-\theta(t-s)} |R_{x_1}|_{H^{d/2}+1} ds \]
\[ \leq C \zeta^2_0 (1 + t)^{-\frac{d}{2}-\frac{1}{2}+\sigma}. \]

Thus we have established (5.64). The identical calculation as in the estimation of III gives the desired bound for \( V \), which is

\[ |V|_{L^\infty} \leq C \zeta^2_0 (1 + t)^{-\frac{d}{2}-\frac{1}{2}+\sigma}. \]

Therefore, we have

\[ |U|_{L^\infty} \leq C \zeta^2_0 (1 + t)^{-\frac{d}{2}-\frac{1}{2}+\sigma}. \]

\[ \square \]

**Appendix A. Expansion of the Fourier Symbol**

*Low frequency expansion.* We carry out the expansion of \( P(\xi) \) in \( \xi \) about zero.

\[ P(\xi) = dQ - i \sum_{j=1}^d \xi_j A^j = \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix} - i \sum_{j=1}^d \xi_j \begin{pmatrix} f^j_u & f^j_v \\ g^j_u & g^j_v \end{pmatrix} \]

**Claim A.1.** To the second order, dispersion relations

\[ \lambda(\xi) = \sigma(dQ - i \sum_{j=1}^d \xi_j A^j), \quad \lambda(0) = 0 \]

are given by

\[ \lambda(\xi) = -i \xi \cdot a^* - \xi^t B^* \xi + \cdots \]

and

\[ V(\xi) = V^0 + \sum_{j=1}^d \xi_j V^1_j + \sum_{j,k=1}^d \xi_j \xi_k V^2_{jk} + \cdots \]
with \( a = (a_1, a_2, ..., a_d) \) and \( B^* = \left[ b_{jk}^* \right]_{j,k=1}^d \), where
\[
A.5 \quad a_j^* = f_u^j - f_v^j q_u^{-1} q_u,
\]
\[
A.6 \quad b_{jk}^* = \begin{cases} 
- f_u^j q_v^{-1} (g_u^j - g_v^j q_v^{-1} q_u - (f_u^j - f_v^j q_v^{-1} q_u) q_v^{-1} q_u) & \text{if } j = k \\
- \frac{1}{2} \left( f_u^j q_v^{-1} (g_u^j - g_v^j q_v^{-1} q_u + (f_u^j - f_v^j q_v^{-1} q_u) q_v^{-1} q_u) + f_v^k q_v^{-1} (g_u^k - g_v^k q_v^{-1} q_u + (f_u^j - f_v^j q_v^{-1} q_u) q_v^{-1} q_u) \right) & \text{if } j \neq k
\end{cases}
\]
\[
A.7 \quad V^0 = \begin{pmatrix} 1 \\ - q_v^{-1} q_u \end{pmatrix},
\]
and
\[
A.8 \quad V_j^1 = \begin{pmatrix} 1 \\ s_j^1 \end{pmatrix} = \begin{pmatrix} 1 \\ - q_v^{-1} q_u + i q_v^{-1} (g_u^j - g_v^j q_v^{-1} q_u + a_j^* q_v^{-1} q_u) \end{pmatrix}.
\]

Proof. Set
\[
A.9 \quad 0 = (dQ - i \sum_{j=1}^d \xi_j A^j - \lambda(\xi)) V(\xi) = (dQ - i \sum_{j=1}^d \xi_j A^j + i \sum_{j=1}^d \xi_j a_j^* I + \sum_{j,k=1}^d \xi_j \xi_k b_{jk}^* I + \cdots) \times (V^0 + \sum_{j=1}^d \xi_j V_j^1 + \sum_{j,k=1}^d \xi_j \xi_k V_{jk}^2 + \cdots).
\]
and let \( V_j^m = \begin{pmatrix} r_j^m \\ s_j^m \end{pmatrix} \) for \( m = 0, 1, 2 \). Collecting the 0th order term, we have
\[
A.10 \quad dQV^0 = 0.
\]
which yields
\[
A.11 \quad V^0 = \begin{pmatrix} 1 \\ - q_v^{-1} q_u r^0 \end{pmatrix}
\]
Collecting the 1st order terms, we have
\[(A.12) \quad dQV_j^1 - i(A^j - a_j^* I)V^0 = 0 \text{ for } j = 1, 2, \ldots, d.\]

Examining the first coordinate, we have
\[(A.13) \quad 0 = i(f_j^u - f_v^j q_v^{-1} q_u - a_j^* I)r^0\]

So, \(r^0\) is the right eigenvector of \(f_u^j - f_v^j q_v^{-1} q_u\) corresponding to the eigenvalue \(a_j^*\). Let \(l^0\) be the counterpart left-eigenvector. Then, the eigenvalue \(a_j^*\) is given by
\[(A.14) \quad a_j^* = l^0(f_j^u - f_v^j q_v^{-1} q_u) r^0.\]

Examining the second coordinate equation, we have
\[(A.15) \quad q_u r_j^1 + q_v s_j^1 = i(g_u^j - g_v^j q_v^{-1} q_u + a_j^* q_v^{-1} q_u)r^0,\]

which yields
\[(A.16) \quad s_j^1 = -q_v^{-1} q_u r_j^1 + i q_v^{-1}(g_u^i - g_v^i q_v^{-1} q_u + a_j^* q_v^{-1} q_u)r^0.\]

So, \(V_j^1 = \begin{pmatrix} r_j^1 \\ s_j^1 \end{pmatrix} = \begin{pmatrix} r_j^1 \\ -q_v^{-1} q_u r_j^1 + i q_v^{-1}(g_u^i - g_v^i q_v^{-1} q_u + a_j^* q_v^{-1} q_u)r^0 \end{pmatrix}.

Collecting 2nd order terms (\(\xi_j \xi_k\) term),
\[(A.17) \quad b_{jk}^* V^0 + i(a_j^* I - A^j) V_k^1 + dQV_{jk}^2 = 0\]

For \(j = k\), the first coordinate equation yields
\[(A.18) \quad b_{jj}^* r^0 + i \left( (a_j^* - f_j^u) r_j^1 + f_v^j (q_u^{-1} q_u r_j^1 - i q_v^{-1}(g_u^i - g_v^i q_v^{-1} q_u + a_j^* q_v^{-1} q_u)r^0) \right) = b_{jj}^* r^0 + i(a_j^* - f_j^u + f_v^j q_v^{-1} q_u)r_j^1 + f_v^j q_v^{-1}(g_u^i - g_v^i q_v^{-1} q_u + a_j^* q_v^{-1} q_u)r^0 = 0\]

For \(j \neq k\), we have
\[(A.19) \quad b_{jk}^* V^0 + i(a_j^* I - A^j) V_k^1 + dQV_{jk}^2 + \beta_{jk}^* V^0 + i(a_k^* I - A^k) V_j^1 + dQV_{kj}^2 = 0.\]
The first coordinate equation gives
\[
\begin{align*}
\text{(A.20)} & \quad b^*_{jk} r^0 + i(a^*_j - f^j_u + f^j_v q_v^{-1} q_u) r^1_k + f^j_v q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u + a^*_k q_v^{-1} q_u) r^0 \\
&\quad + b^*_{kj} r^0 + i(a^*_k - f^k_u + f^k_v q_v^{-1} q_u) r^1_j + f^k_v q_v^{-1} (g^k_u - g^k_v q_v^{-1} q_u + a^*_j q_v^{-1} q_u) r^0 \\
&\quad = 2b^*_{jk} r^0 + i(a^*_j - f^j_u + f^j_v q_v^{-1} q_u) r^1_k + f^j_v q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u + a^*_k q_v^{-1} q_u) r^0 \\
&\quad + i(a^*_k - f^k_u + f^k_v q_v^{-1} q_u) r^1_j + f^k_v q_v^{-1} (g^k_u - g^k_v q_v^{-1} q_u + a^*_j q_v^{-1} q_u) r^0 = 0
\end{align*}
\]

For the scalar case, i.e., \( n = 1 \), we will denote, for simplicity, \( a^*_i = (a^*_{i1}, \ldots, a^*_{id}) \) by \( a^* = (a^*_{1}, \ldots, a^*_{d}) \). Then, we have
\[
\begin{align*}
\text{(A.21)} & \quad V^0 = \begin{pmatrix} 1 \\ -q_v^{-1} q_u \end{pmatrix}, \\
\text{(A.22)} & \quad a^*_j = f^j_u - f^j_v q_v^{-1} q_u, \\
\text{(A.23)} & \quad b^*_{jj} = -f^j_v q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u - a^*_j q_v^{-1} q_u) \\
&\quad = -f^j_v q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u) - (f^j_u - f^j_v q_v^{-1} q_u) q_v^{-1} q_u), \\
\text{(A.24)} & \quad b^*_{jk} = -\frac{1}{2} \left( f^j_v q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u + a^*_k q_v^{-1} q_u) + f^k_v q_v^{-1} (g^k_u - g^k_v q_v^{-1} q_u + a^*_j q_v^{-1} q_u) \right) \\
&\quad = -\frac{1}{2} \left( f^j_v q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u + (f^k_u - f^k_v q_v^{-1} q_u) q_v^{-1} q_u) \\
&\quad + f^k_v q_v^{-1} (g^k_u - g^k_v q_v^{-1} q_u + (f^j_u - f^j_v q_v^{-1} q_u) q_v^{-1} q_u) \right)
\end{align*}
\]

and
\[
\begin{align*}
\text{(A.25)} & \quad V^1_j = \begin{pmatrix} 1 \\ s^1_j \end{pmatrix} = \begin{pmatrix} 1 \\ -q_v^{-1} q_u + i q_v^{-1} (g^j_u - g^j_v q_v^{-1} q_u + a^*_j q_v^{-1} q_u) \end{pmatrix}.
\end{align*}
\]

Moreover, we let \( B^* \) be the viscosity matrix \([b^*_{jk}]\) as in (1.6) and write
\[
\begin{align*}
\text{(A.26)} & \quad B^* = b^*_{11} \begin{pmatrix} 1 & -b^* \\ b^* & B^* \end{pmatrix}
\end{align*}
\]
where $b^* \in \mathbb{R}^{d-1}$ and $B^* \in \mathbb{R}^{(d-1) \times (d-1)}$.

**Appendix B. Asymptotic ODE: gap and conjugation lemmas**

Consider a general family of first-order ODE

(B.1) \[ W' - A(x_1, \Lambda)W = F \]

indexed by a spectral parameter $\Lambda \in \Omega \subset \mathbb{C}^m$, where $W \in \mathbb{C}^N$, $x_1 \in \mathbb{R}$ and $'n$ denotes $d/dx_1$.

**Assumption B.1.**

(h0) Coefficient $A(\cdot, \Lambda)$, considered as a function from $\Omega$ into $C^0(x_1)$ is analytic in $\Lambda$. Moreover, $A(\cdot, \Lambda)$ approaches exponentially to limits $A_{\pm}$ as $x_1 \to \pm\infty$, with uniform exponential decay estimates

(B.2) \[ |(\partial/\partial x_1)^k (A - A_{\pm})| \leq C_1 e^{-\theta|x_1|/C_2}, \quad \text{for } x_1 \gtrless 0, 0 \leq k \leq K, \]

$C_j, \theta > 0$, on compact subsets of $\Omega$.

**Lemma B.2** (The gap lemma [KS, GZ, ZH]). Consider the homogeneous version $F \equiv 0$ of (B.1), under assumption (h0). If $V^-(\Lambda)$ is an eigenvector of $A_-$ with eigenvalue $\mu(\Lambda)$, both analytic in $\Lambda$, then there exists a solution of (B.1) of form

(B.3) \[ W(x_1, \Lambda) = V(x_1, \Lambda)e^{\mu(\Lambda)x_1}, \]

where $V$ is $C^1$ in $x_1$ and locally analytic in $\Lambda$ and, for any fixed $\bar{\theta} < \theta$, satisfies

(B.4) \[ V(x_1, \Lambda) = V^-(\Lambda) + O(e^{-\bar{\theta}|x_1||V^-(\Lambda)|}), \quad x_1 < 0. \]

**Lemma B.3.** (The conjugation lemma). Given (h0), there exist locally to any given $\Gamma_0 \in \Omega$ invertible linear transformations $P_+(x_1, \Gamma) = I + \Theta_+(x_1, \Gamma)$ and $P_-(x_1, \Gamma) = I + \Theta_-(x_1, \Gamma)$ defined on $x_1 \geq 0$ and $x_1 \leq 0$, respectively, $\Phi_\pm$ analytic in $\Gamma$ as functions from $\Omega$ to $C^0[0, \pm\infty)$, such that:
(i) For any fixed $0 < \tilde{\theta} < \theta$ and $0 \leq k \leq K + 1$, $j \geq 0$,

$$
|\frac{\partial}{\partial \Lambda}^j (\partial / \partial x_1)^k \Theta_\pm| \leq C(j) C_1 C_2 e^{-\theta |x_1| / C_2} \quad \text{for } x_1 \gtrsim 0.
$$

(ii) The change of coordinates $\mathbb{W} = P_\pm Z$, $F = P_\pm G$ reduces (B.1) to

$$
Z' - A_\pm Z = G \quad \text{for } x_1 \gtrsim 0.
$$

Equivalently, solutions of (B.1) may be factored as

$$
\mathbb{W} = (I + \Theta_\pm) Z_\pm,
$$

where $Z_\pm$ satisfy the limiting, constant-coefficient equations (B.6) and $\Theta_\pm$ satisfy bounds (B.5).

**Example B.4.** Consider the linearized equations:

$$
U_t = LU := -\sum_{j=1}^{d} (A^j U)_{x_j} + QU
$$

$$
\hat{U}_t = L_\xi \hat{U} := - (A^1 \hat{U})' - \sum_{j=2}^{d} i \xi_j A^j \hat{U} + Q \hat{U}
$$

Consider a non-homogeneous eigenvalue problem:

$$
(L_\xi - \lambda) W = f
$$

Eq. (B.10) can be expressed in the form:

$$
W' = -(A^1)^{-1} (A^1)' + i \sum_{j=2}^{d} \xi_j A^j - Q + \lambda I) W - (A^1)^{-1} f
$$

$$
= \mathbb{A}(x_1, \Lambda) W - F
$$
**Appendix C. Series expansion of the top eigenvalue of $L_{\tilde{\xi}}$**

Consider

(C.1) \[ L_{\tilde{\xi}} U = -(\bar{A}^1 U)' - i \sum_{j \neq 1} \xi_j \bar{A}^j U + QU. \]

It can be shown that there exists a unique, analytic eigenvalue

(C.2) \[ \lambda_0(\tilde{\xi}) = 0 + \tilde{\gamma}_1 \cdot \tilde{\xi} + \tilde{\xi}^t \tilde{\gamma}_2 \tilde{\xi} + O(|\tilde{\xi}|^3), \]

of $L_{\tilde{\xi}}$ perturbing from the top eigenvalue $\lambda = 0$ of the operator $L_0$, with associated analytic right and left eigenfunctions

(C.3) \[ \varphi(\tilde{\xi}) = \varphi^0 + \varphi^1 \cdot \tilde{\xi} + \tilde{\xi}^t \varphi^2 \tilde{\xi} + O(|\tilde{\xi}|^3) \]

and

(C.4) \[ \pi(\tilde{\xi}) = \pi^0 + \pi^1 \cdot \tilde{\xi} + \tilde{\xi}^t \pi^2 \tilde{\xi} + O(|\tilde{\xi}|^3). \]

with $\varphi^0 = \bar{U}'$ and $\pi^0 = ([u]^{-1}, 0)$.

**Lemma C.1.** The expansions (C.2), (C.3) and (C.4) hold with $i\tilde{\gamma}_1 = \tilde{a}, \tilde{a} \in \mathbb{R}^{d-1}$, and $\tilde{\beta} \in \mathbb{R}^{(d-1)\times(d-1)}$, where $-\tilde{\gamma}_2 = \tilde{\beta}$ is positive definite. Here

(C.5) \[ \tilde{a} = (\bar{a}_2, ..., \bar{a}_d) = -([f_2^*][u]^{-1}, ..., [f_d^*][u]^{-1}) \]

**Proof.** Let $\varphi(\tilde{\xi}) = \varphi_0 + \sum_{j \neq 1} \Phi_j^\delta \xi_j + \sum_{j,k \neq 1} \Psi_{jk}^\delta \xi_j \xi_k + O(|\tilde{\xi}|^3)$.

The eigenvalue equation $L_{\tilde{\xi}} \varphi = \lambda_0 \varphi$ leads to

(C.6) \[ -(\bar{A}^1 \Phi_j)' - iF^j(\bar{U})' + \bar{Q}\Phi^j = \gamma_j \bar{U}' \text{ for } j = 2, ..., d. \]

Integrating from $-\infty$ to $\infty$ both sides, we have

(C.7) \[ \int_{-\infty}^{\infty} -(\bar{A}^1 \Phi_j)' - iF(\bar{U})' + (0 \, dq)^t \Phi^j \, dx_1 = \int_{-\infty}^{\infty} \gamma_j \bar{U}' \, dx_1 \]

which yields, by looking at $u$ coordinate,

(C.8) \[ -i[F_j^j] = \gamma_j [U_j] \text{ for } j = 2, ..., d. \]
For $u$ scalar case, we have

(C.9) \[ \tilde{a}_j = i \gamma_j = - \frac{[f^j_i]}{u} \text{ for } j = 2, \ldots, d. \]

On the other hand, by (D3) in Assumptions 1.2 and the analytic eigenvalue expansion (C.2), $-\tilde{\gamma}^2 = \tilde{\beta}$ is positive definite.

\[ \square \]

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Department of Mathematics, Indiana University, Bloomington, IN 47402

E-mail address: bkwon@indiana.edu

Department of Mathematics, Indiana University, Bloomington, IN 47402

E-mail address: kzumbrun@indiana.edu