Disintegration of cylindrical measures

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Abstract

We show that the existence of disintegration for cylindrical measures follows from a general disintegration theorem for countably additive measures.

1 Notation

A probability space \((X, \mathcal{A}, P)\) is a nonempty set \(X\) together with a sigma-algebra \(\mathcal{A}\) on \(X\) and a probability \(P\) on \(\mathcal{A}\) (that is, a nonnegative countably additive measure with \(PX = 1\)). When \(C\) is a set of subsets of a given set, \(\sigma(C)\) is the smallest sigma-algebra that contains \(C\). When \(\mathcal{A}\) and \(\mathcal{B}\) are sigma-algebras on \(X\) and \(Y\), respectively, \(\sigma(\mathcal{A} \otimes \mathcal{B})\) is the smallest sigma-algebra making the canonical projections \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\) measurable.

When \((X, \mathcal{A}, P)\) and \((Y, \mathcal{B}, Q)\) are two probability spaces, a probability \(S\) on \(\sigma(\mathcal{A} \otimes \mathcal{B})\) is a joint probability if \(\pi_X[S] = P\) and \(\pi_Y[S] = Q\).

A lattice on \(X\) is a class of subsets of \(X\) that is closed under finite unions and finite intersections.

A class \(\mathcal{K}\) of sets is semicompact if every countable class \(\mathcal{K}_0 \subseteq \mathcal{K}\) such that \(\bigcap \mathcal{K}_0 = \emptyset\) contains a finite class \(\mathcal{K}_{00} \subseteq \mathcal{K}_0\) such that \(\bigcap \mathcal{K}_{00} = \emptyset\).

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Let \((X, \mathcal{A}, P)\) be a probability space and \(\mathcal{K} \subseteq \mathcal{A}\). We say that \(\mathcal{K}\) approximates \(P\) if for every \(E \in \mathcal{A}\) and \(\varepsilon > 0\) there is a \(K \in \mathcal{K}\) such that \(K \subseteq E\) and \(P(E \setminus K) < \varepsilon\).

All linear spaces are assumed to be over the field \(\mathbb{R}\) of reals. When \(Y\) and \(Z\) are locally convex spaces, \(\mathcal{L}(Y, Z)\) is the set of continuous linear mappings from \(Y\) to \(Z\).

When \(I\) and \(J\) are two index sets such that \(J \subseteq I\), the canonical projection from \(\mathbb{R}^I\) onto \(\mathbb{R}^J\) is denoted \(p^I_J\).

When \(I\) is an infinite index set, let \(\mathcal{C}(I)\) be the set of all subsets of \(\mathbb{R}^I\) of the form \(p^I_J(C)\) where \(J\) is finite, \(\emptyset \neq J \subseteq I\) and \(C\) is a compact subset of \(\mathbb{R}^J\). Then \(\mathcal{C}(I)_\delta\) is the set of all countable intersections of sets in \(\mathcal{C}(I)\).

## 2 Disintegration theorem

The following theorem is an immediate consequence of Theorem 3.5 in [4].

**Theorem 1** Let \((X, \mathcal{A}, P)\) and \((Y, \mathcal{B}, Q)\) be two probability spaces, and let \(S\) be a joint probability on \(\sigma(\mathcal{A} \otimes \mathcal{B})\). Let \(Q\) be complete, and let \(\mathcal{K}\) be a semicompact lattice closed under countable intersections and such that \(\mathcal{A} = \sigma(\mathcal{K})\) and \(\mathcal{K}\) approximates \(P\).

Then there exists a family of probabilities \(\{P_y\}_{y \in Y}\) on \(A\) such that
(a) for every \(E \in \mathcal{A}\), the function \(y \mapsto P_yE\) is \(\mathcal{B}\)-measurable;
(b) for every \(E \in \mathcal{A}\) and \(F \in \mathcal{B}\) we have
\[
S(E \times F) = \int_F P_yE \, dQ(y).
\]

## 3 Application to cylindrical measures

In his study of non-linear images of cylindrical measures, Krée [1] raised the problem of the disintegration of cylindrical measures. Here we show that the
disintegration in the sense of 4.A.b in [1] always exists. However, the disintegration constructed here need not be continuous or measurable in the sense of 4.A.c-e in [1]. In fact, the remark on page 36 in [2] seems to imply that there is an example, due to Schwartz, of a cylindrical measure with no measurable disintegration.

The following theorem improves Proposition (34) in [1]. The terminology and notation are as in [1] and [3].

**Theorem 2** Let $Y$ and $Z$ be two locally convex spaces and let $m$ be a cylindrical probability on $Y \times Z$ whose canonical image $m_1$ on $Y$ is countably additive. Extend $m_1$ to the complete probability $Q$ defined on a sigma-algebra $\mathcal{B}$.

Then there exists a family $\{m_y\}_{y \in Y}$ of cylindrical probabilities on $Z$ such that for every positive integer $n$, every $b \in L(Z, R^n)$ and every Borel set $E \subseteq R^n$,

(a) the function $y \mapsto b[m_y]E$ is $\mathcal{B}$-measurable on $Y$;

(b) for every $F \in \mathcal{B}$ we have

$$\int_Y (b \times b)[m](F \times E) = \int_F b[m_y]E \ dQ(y).$$

The proof of the theorem uses the following result about projective limits of measures. The original version of this result is due to Bochner. For a proof of a more general result, see e.g. [3].

**Theorem 3** Let $I$ be an infinite index set. Then

(a) $C(I)_δ$ is a semicompact class of subsets of $R^I$;

(b) if $m$ is a nonnegative finitely additive measure on the algebra generated by $C(I)$ and the image $p_I[m]$ in $R^I$ is countably additive for each finite $J \subseteq I$ then $m$ is countably additive and its unique extension to a countably additive measure on the sigma-algebra $\sigma(C(I))$ is approximated by $C(I)_δ$.

**Proof of Theorem 2**. Schwartz ([3], pp. 177-180) points out that cylindrical measures on $Z$ are in one-to-one correspondence with cylindrical measures on $R^I$ for some index set $I$. Namely, $Z$ is embedded in the algebraic dual $Z'$. 

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of its topological dual $Z'$, and $Z'^*$ is isomorphic to $\mathbb{R}^I$ when $I$ is chosen of the same cardinality as an algebraic basis of $Z'$. This yields an injective continuous linear mapping $h : Z \rightarrow \mathbb{R}^I$ such that $\mu \mapsto h[\mu]$ is a one-to-one correspondence between cylindrical probabilities $\mu$ on $Z$ and cylindrical probabilities on $\mathbb{R}^I$. In the following we choose one such $h$ and keep it fixed.

By Theorem 3, every cylindrical probability on $\mathbb{R}^I$ is countably additive, and we have a one-to-one correspondence between cylindrical probabilities on $Z$ and countably additive probabilities on the sigma-algebra $\sigma(\mathcal{C}(I))$ in $\mathbb{R}^I$.

Consider the cylindrical probability $m_0 = (I_Y \times h)[m]$ on the space $Y \times \mathbb{R}^I$ and its projections on $Y$ and $\mathbb{R}^I$. The projection $\pi_Y[m_0] = m_1$ is countably additive by the assumption. The projection $\pi_{\mathbb{R}^I}$ is countably additive as shown above; let $P$ be its unique countably additive extension to the sigma-algebra $\mathcal{A} = \sigma(\mathcal{C}(I))$.

By a result of Marczewski [3], $m_0$ is countably additive. Thus $m_0$ has a unique extension to a countably additive probability $S$ on the sigma-algebra $\sigma(\mathcal{A} \otimes \mathcal{B})$.

Hence we can apply Theorem 1 with $X = \mathbb{R}^I$ and $\mathcal{A}$, $\mathcal{P}$, $\mathcal{Y}$, $\mathcal{B}$, $Q$ and $S$ as defined. For each probability $P_y$ on $\mathcal{A}$ obtained from Theorem 1 let $m_y$ be the unique cylindrical probability on $Z$ such that $h[m_y] = P_y$.

Since $\mathbb{R}^I$ identifies with $Z'^*$, every $b \in \mathcal{L}(Z, \mathbb{R}^n)$ factors through $h$, and it is now straightforward to verify the properties of $m_y$ stated in the theorem.

**Remark.** Note that the proof does not use the fact that $Y$ is a vector space. Thus the same result holds true, with the same proof, in the more general case of $Y$-cylindrical probabilities ([2], p. 34), for a topological or measurable space $Y$.

**References**

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