How to build Hamiltonians that transport noncommuting charges in quantum thermodynamics

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Noncommuting conserved quantities have recently launched a subfield of quantum thermodynamics. In conventional thermodynamics, a system of interest and an environment exchange quantities—energy, particles, electric charge, etc.—that are globally conserved and are represented by Hermitian operators. These operators were implicitly assumed to commute with each other, until a few years ago. Freeing the operators to fail to commute has enabled many theoretical discoveries—about reference frames, entropy production, resource-theory models, etc. Little work has bridged these results from abstract theory to experimental reality. This paper provides a methodology for building this bridge systematically: We present a prescription for constructing Hamiltonians that conserve noncommuting quantities globally while transporting the quantities locally. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. Our Hamiltonians may be realized physically with superconducting qudits, with ultracold atoms, and with trapped ions.

One of thermodynamics’ most fundamental and ubiquitous interactions is the exchange of quantities between a system of interest and an environment. Example quantities include energy, particles, and electric charge. As the quantities are conserved globally, we call them ‘charges.’ (We call even the local quantities ‘charges’ for convenience, even though the quantities are not conserved locally.) Such exchanges happen, for example, in electrochemical batteries, in a cooling cup of coffee, and when a few spins flip to align with a magnetic field. Given such exchanges’ pervasiveness, studying their quantum facets is essential for (i) developing the field of quantum thermodynamics\textsuperscript{1,2} and (ii) discovering nonclassical features of quantum many-body thermalization in condensed matter; atomic, molecular and optical (AMO) physics; high-energy physics; and chemistry. One important quantum phenomenon is operators’ failure to commute with each other: Noncommutation underlies uncertainty relations, measurement disturbance, and more. Therefore, studying exchanges of noncommuting charges is crucial for understanding quantum thermodynamics. As a result, noncommuting charges have been enjoying a heyday\textsuperscript{3–29} in quantum-information-theoretic (QIT) thermodynamics.

Lifting the assumption that exchanged charges commute\textsuperscript{3–29, 30, 51} has led to discoveries of truly quantum thermodynamics. Example discoveries include a generalization of the microcanonical state\textsuperscript{5}, resource theories\textsuperscript{6, 7, 11, 32}, a generalization of the majorization preorder\textsuperscript{12}, a reduction of entropy production by charges’ noncommutation\textsuperscript{13}, and reference-frame designs\textsuperscript{14, 15}. These discoveries and others have turned noncommuting thermodynamic charges into a growing subfield.

Most of the discoveries have, until recently, belonged in QIT thermodynamics. However, given their fundamental and nonclassical nature, exchanges of thermodynamic noncommuting charges call for bridges to experiments and to many-body physics. Building these bridges requires Hamiltonians that transport noncommuting observables locally while conserving them globally: As stated in the quantum-thermodynamics review\textsuperscript{1}, ‘an abstract view of dynamics, minimal in the details of Hamiltonians, is often employed in quantum information’ and so in QIT thermodynamics. In contrast, experiments, simulations, and many-body theory require microscopic Hamiltonians.

Before the present work, it was unknown (i) whether Hamiltonians that transport noncommuting observables locally, while conserving them globally, exist; (ii) how such Hamiltonians look, if they exist; (iii) how to construct such Hamiltonians for given noncommuting charges; and (iv) for which charges such Hamiltonians can be constructed. We answer these questions, enabling the system-and-environment exchange of noncommuting charges to progress from its QIT-thermodynamic birthplace to many-body physics and experiments. Example predictions that merit experimental exploration

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include (i) the emergence of the quantum equilibrium state in [8, 9], (ii) the decrease in entropy production by noncommuting charges [13], (iii) applications of the entropy decrease to quantum engines [33], (iv) the conjecture that noncommuting charges hinder thermalization [7], and (v) the conjecture’s application to quantum memories. We open the door to experiments by prescribing how to construct the needed Hamiltonians. Our construction also enables the generalization, to noncommuting charges, of many-body–thermalization tools in condensed matter, AMO physics, and high-energy. Examples include the eigenstate thermalization hypothesis, out-of-time-ordered correlators, and random unitary circuits (e.g., [44, 42]).

This paper introduces a prescription for constructing Hamiltonians that overtly move noncommuting charges between subsystems while conserving the charges globally. The charges form a finite-dimensional semisimple complex Lie algebra. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. The prescription also produces a convenient basis for the algebra—a basis of charges explicitly transported locally, and conserved globally, by the Hamiltonian. The prescription is general, being independent of any physical platforms. Consequently, the Hamiltonians can be realized with diverse physical systems, such as superconducting circuits, ultracold atoms, and trapped ions.

In a special case, the charges form the Lie algebra $su(D)$, $N$ identical subsystems form the global system, and each subsystem corresponds to the Hilbert space $\mathbb{C}^D$. In this example the Schur-Weyl duality describes the Hamiltonians’ forms $|j\rangle$. Let the global system (formed from the system of interest and the environment) be many copies of the system of interest. The Hamiltonians are the linear combinations of the permutations of the copies. (Hamiltonians have also been engineered to have SU($D$) symmetry without regard to whether noncommuting charges are transported [48, 49].) Our results are more general than the Schur-Weyl duality and elucidate the dynamics’ physical interpretation. First, our prescription governs a much wider class of algebras: all finite-dimensional, semisimple Lie algebras in which the Killing form induces a metric. Many physically significant algebras satisfy these assumptions—for example, the simple Lie algebras, which include $su(D)$. Second, our results are not restricted to systems whose Hilbert spaces are $\mathbb{C}^D$. Finally, the Hamiltonian form specified by the Schur-Weyl duality—a linear combination of permutations—is an abstract construct. How to implement an arbitrary linear combination of permutations is not obvious. In contrast, our Hamiltonians have a clear physical interpretation, manifestly transporting noncommuting charges between subsystems. To our knowledge, no other class of Hamiltonians that transport charges locally and conserve them globally, comparably general to our class, is known.

This paper begins with our setup, detailed in Sec. I. Section II introduces the Hamiltonian-construction prescription pedagogically. We also review mathematical background and illustrate the prescription with an example familiar in quantum information, the Lie algebra $su(2)$. Section III synthesizes the prescription, crystallizing the main result, and presents two properties of the prescription. A richer example provides intuition in Sec. IV Hamiltonians that transport and conserve charges in the Lie algebra $su(3)$. Section V concludes with potential realizations of our Hamiltonians in condensed matter, AMO, and high-energy and nuclear physics.

I. SETUP

Consider a global closed quantum many-body system, as in recent thermalization experiments [47–58]. As in conventional statistical mechanics, the global system is an ensemble of $N$ identical subsystems. (We use the term ‘ensemble’ in the traditional sense of statistical physics: a collection of many identical copies of a system of interest. Such ensembles are often invoked to determine equilibrium probability distributions [59, p. 62].) A few of the subsystems form the system of interest; and the rest, an effective environment. Each subsystem corresponds to a Hilbert space $\mathcal{H}$ of finite dimensionality $d$.

We will construct global Hamiltonians, $H^{\text{tot}}$, that conserve extensive charges defined as follows. Let $Q_\alpha$ denote a Hermitian operator defined on $\mathcal{H}$. We denote by $Q^{(j)}_\alpha$ the observable defined on the $j^\text{th}$ subsystem’s $\mathcal{H}$. Each global observable

$$Q^{\text{tot}}_\alpha := \sum_{j=1}^{N} Q^{(j)}_\alpha = \sum_{j=1}^{N} 1^\otimes(j-1) \otimes Q^{(j)}_\alpha \otimes 1^\otimes(N-j) \tag{1}$$

will be conserved by design:

$$[H^{\text{tot}}, Q^{\text{tot}}_\alpha] = 0. \tag{2}$$

Although the local $Q^{(j)}_\alpha$’s are not conserved, we will sometimes call them, and the $Q_\alpha$, ‘charges’ for convenience. One might know, initially, of only $\ell$ charges’ existence.

These $\ell$ $Q_\alpha$’s generate a complex Lie algebra $\mathcal{A}$, which we assume to be finite-dimensional. $\mathcal{A}$ consists of all the charges (as well as non-Hermitian operators, which we ignore). Lie algebras describe many conserved physical quantities: particle number, angular momentum, electric charge, color charge, weak isospin, and our space-time’s metric [44, 60, 61]. We focus on non-Abelian Lie algebras, motivated by quantum thermodynamics that highlights noncommutation: The commutator exemplifies the Lie bracket, $[Q_\alpha, Q_\beta]$.

We assume four more properties of the algebra, to facilitate our proofs. $\mathcal{A}$ is finite-dimensional and semisimple. Representing an observable, $\mathcal{A}$ is over the complex numbers. Also, on $\mathcal{A}$ is defined a Killing form (reviewed
below) that induces a metric. Many physically significant algebras satisfy these assumptions—for example, the simple Lie algebras (see the Supplementary Note 1 and [44, 60, 61]).

II. PEDAGOGICAL EXPLANATION

This section describes the prescription for constructing Hamiltonians $H^{\text{tot}}$ that conserve noncommuting charges globally [Eq. (2)] while transporting them locally:

$$ [H^{\text{tot}}, Q^{(j)}_{\alpha}] \neq 0 $$

for some site $j$. (In every such commutator throughout this paper, one argument implicitly contains tensor factors of $I$, so that both arguments operate on the same Hilbert space.) We construct two-body interaction terms, then combine them into many-body terms. This explanation provides a pedagogical introduction; the presentation is synopsized in Sec. III Here, we illustrate each step with an algebra familiar in quantum information, $su(2)$, which describes spin-1/2 angular momentum.

Table I lists the simple Lie algebras. Every Cartan-Weyl basis has a significance that we will encounter shortly.

A representation of $A$ is a Lie-bracket-preserving map from $A$ to a set of linear transformations. The adjoint representation maps from $A$ to linear transformations defined on $A$. If $x \in A$, the adjoint representation $\text{ad}(x)$ acts on $y \in A$ as $\text{ad}(x)(y) := [x, y]$. The adjoint representation features in the Killing form, which we review now. The definition of $A$ involves a vector space $V$ defined over a field $F$. A map $V \times V \rightarrow F$ is a form. The Killing form is the symmetric bilinear form

$$(x, y) := \text{Tr}(\text{ad}(x) \text{ad}(y)).$$

We say that $x$ and $y$ are Killing-orthogonal if $(x, y) = 0$. We say that subalgebras $A_1$ and $A_2$ are Killing-orthogonal if, for all $x \in A_1$ and $y \in A_2$, $(x, y) = 0$. We will use the Killing form to construct the preferred basis of charges for $A$.

### Table I: Simple Lie algebras

| Algebra  | Dimension (c) | Rank (r) | c/r |
|----------|---------------|----------|-----|
| $so(2D)$ | $D(2D - 1)$   | $D$      | $2D - 1$ |
| $sl(D + 1)$ | $(D + 1)^2 - 1$ | $D + 2$ |         |
| $so(2D + 1)$ | $D(2D + 1)$   | $D$      | $2D + 1$ |
| $sp(2D)$ | $D(2D + 1)$   | $D$      | $2D + 1$ |

Our construction begins with another basis: Every finite-dimensional semisimple complex Lie algebra $A$ has a Cartan-Weyl basis. In fact, $A$ has infinitely many. Convention may distinguish one Cartan-Weyl basis. We use the conventional $su(2)$ basis for concreteness. We use this basis, in our example, for concreteness. In general, one selects an arbitrary Cartan-Weyl basis. The basis contains generators of two types: Hermitian operators and ladder operators.

The number of Hermitian operators is the algebra’s rank, $r$. These operators commute with each other. If $r > 1$, we rescale the operators to endow them with unit Hilbert-Schmidt norms:

$$\text{Tr}(Q^\dagger_\alpha Q_\alpha) = 1.$$

We include these operators, $Q_{\alpha=1,2,...,r}$, in our preferred basis. In the $su(2)$ example, $r = 1$; and $Q_1 = \sigma_z$, whose eigenstates $|\pm\rangle$ correspond to the eigenvalues $\pm 1$. The $Q_\alpha$’s generate a subalgebra, a Cartan subalgebra.

The Cartan-Weyl basis contains, as well as Hermitian operators, ladder operators. They form pairs $L_{\pm\beta}$, for $\beta = 1, 2, \ldots, g/2$. Since the Cartan-Weyl basis has $c$ elements, and $r$ of them are Hermitian, there are $c - r$ ladder operators. Each $\beta$ corresponds to two ladder operators, one raising $(+\beta)$ and one lowering $(-\beta)$. Hence $\beta$ runs from 1 to $g/2$. Each $L_{\pm\beta}$ raises or lowers at least one $Q_\alpha$. In the $su(2)$ example, the ladder operators $\sigma_{\pm z} = \frac{1}{2}(\sigma_x \pm i \sigma_y)$ raise and lower $\sigma_z$: $L_{\pm 1} |\pm\rangle = |\pm\rangle$. In other algebras, an $L_{\pm\beta}$ can raise and/or lower multiple $Q_\alpha$’s. Examples include $su(3)$ (Sec. IV).

From each ladder-operator pair, we construct an interaction that couples subsystems $j$ and $j'$. Let $J^{(j,j')}_\beta$ denote a hopping frequency. An interaction that transports all the charges between $j$ and $j'$, while conserving each charge globally, has the form

$$H^{(j,j')} \propto \sum_{\beta=1}^{(c-r)/2} J^{(j,j')}_\beta \left( L^{(j)}_{+\beta} L^{(j')}_{-\beta} + L^{(j)}_{-\beta} L^{(j')}_{+\beta} \right).$$

We assemble the other terms in $H^{(j,j')}$ from other
Cartan-Weyl bases, constructed as follows. Let $U$ denote a general element of our first Cartan-Weyl basis: For $\alpha = 1, 2, \ldots, r$ and $\beta = 1, 2, \ldots, \frac{r-r}{2}$,

$$Q_\alpha \mapsto U^\dagger Q_\alpha U = Q_{\alpha+r}, \quad \text{and}$$

$$L_{+\beta} \mapsto U^\dagger L_{+\beta} U = L_{+}(\beta, \frac{r}{2}).$$

We include the new $Q_\alpha$’s (for which $\alpha = r+1, r+2, \ldots, 2r$) in our preferred basis for the algebra.

We constrain $U$ such that each new $Q_\alpha$ is Killing-orthogonal to (i) each other new charge $Q_\beta$ and (ii) each original charge $Q_\gamma$:

$$(Q_\alpha, Q_\beta) = (Q_\alpha, Q_\gamma) = 0$$

for all $\alpha, \beta = r+1, r+2, \ldots, 2r$ and all $\gamma = 1, 2, \ldots, r$. This orthogonality restricts $U$, though not completely. The new $Q_\alpha$’s generate a Cartan subalgebra Killing-orthogonal to the original Cartan subalgebra. The new ladder operators contribute to the interaction:

$$H^{(j,j')} \propto \sum_{\beta=1}^{r} J_{\beta}^{(j,j')} \left( L_{+\beta}^{(j)} L_{-\beta}^{(j')} + \text{h.c.} \right).$$

In the $su(2)$ example, $U$ can be represented by

$$\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix},$$

wherein $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. The prescription restricts $U$ only via the Killing-orthogonality of $U^\dagger \sigma_{z} U$ to $U$. We enforce only this restriction in the Supplementary Note 2. Here, we choose a $U$ for pedagogical simplicity: $U = (1 + i\sigma_y)/\sqrt{2}$, such that $Q_{a+r} = Q_2 = \sigma_z$. The new ladder operators, $\sigma_{\pm z} := (1 + i\sigma_y)/\sqrt{2}$, create and annihilate quanta of the $z$-component of the angular momentum. The interaction becomes

$$H^{(j,j')} \propto \sum_{\beta=x,y,z} J_{\beta}^{(j,j')} \left( \sigma_{+\beta}^{(j)} \sigma_{-\beta}^{(j')} + \text{h.c.} \right).$$

We repeat the foregoing steps: Write out the form of a general $U \in \mathcal{G}$. Conjugate each element of the original Cartan-Weyl basis with $U$. Constrain $U$ such that the new $Q_\alpha$’s are orthogonal to each other and to the older $Q_\alpha$’s. Include the new $Q_\alpha$’s in our preferred basis for the algebra. Form a term, in $H^{(j,j')}$, from the new ladder operators $L_{+\beta}$.

Each Cartan-Weyl basis contributes $r$ elements $Q_\alpha$ to the preferred basis. The basis contains $c$ elements, so we form $c/r$ mutually orthogonal Cartan-Weyl bases. $c/r$ equals an integer for the finite-dimensional semisimple complex Lie algebras, according to Proposition 1 in Sec. IIII Table 1 confirms the claim for the simple Lie algebras. Our algebra’s finite dimensionality ensures that our prescription halts. The two-body interaction is now

$$H^{(j,j')} = \sum_{\beta=1}^{c/r} J_{\beta}^{(j,j')} \left( L_{+\beta}^{(j)} L_{-\beta}^{(j')} + \text{h.c.} \right).$$

Why is the preferred basis $\{Q_\alpha\}$ preferable? First, the basis endows the Hamiltonian with a simple physical interpretation: $H^{(j,j')}$ transports all these charges locally while conserving them globally. Second, the basis is (Killing-)orthogonal.

In the $su(2)$ example, $c/r = 3/1 = 3$. Hence we construct three Cartan-Weyl bases, using two SU(2) elements. If the first unitary was $(1 + i\sigma_y)/\sqrt{2}$, the second unitary is $(1 - i\sigma_x + i\sigma_y + i\sigma_z)/2$, to within a global phase. Consequently, $Q_3 = \sigma_y$, the preferred basis for $A$ is $\{\sigma_z, \sigma_x, \sigma_y\}$, and

$$H^{(j,j')} = \sum_{\beta=x,y,z} J_{\beta}^{(j,j')} \left( \sigma_{+\beta}^{(j)} \sigma_{-\beta}^{(j')} + \text{h.c.} \right).$$

Next, we constrain the interaction to conserve every global charge:

$$[H^{(j,j')}, Q_\alpha^{\text{tot}}] = 0 \quad \forall \alpha = 1, 2, \ldots, c.$$
\[ H^{(j',j'',...j^{(t)})}, Q^\text{tot}_\alpha = 0, \] and subtracting off any few-
body terms that appear in the product. Section III details the formalism. In the \( \text{su}(2) \) example, a three-body interaction has the form (see Supplementary Note 2)
\[ H^{(j',j'',...j^{(t)})} \propto H^{(j,j')} H^{(j'',...j^{(t)})} H^{(j,j''')} \]
\[ \propto J^{(j',j'',...j^{(t)})} (\sigma_x \sigma_y \sigma_z + \sigma_y \sigma_z \sigma_x + \sigma_z \sigma_x \sigma_y) \]
\[ - (\sigma_z \sigma_y \sigma_x + \sigma_x \sigma_z \sigma_y + \sigma_y \sigma_x \sigma_z). \]
wherein \( J^{(j',j'',...j^{(t)})} \in \mathbb{R} \).

The Hamiltonian we constructed may be integrable. For example, the one-dimensional (1D) nearest-neighbor Heisenberg model is integrable \[^2\]. Integrable Hamiltonians have featured in studies of noncommuting charges in thermodynamics \[^22\]. But one might wish for the system to thermalize as much as possible, as is promoted by nonintegrability \[^37\]. Geometrically nonlocal couplings, many-body interactions, and multidimensional lattices tend to break integrability. Hence one can add terms \( H^{(j,j')} \) and \( H^{(j',j'',...j^{(t)})} \) to the global Hamiltonian \( H^\text{tot} \), and keep growing the lattice’s dimensionality, until \( H^\text{tot} \) becomes nonintegrable. Nonintegrability may be diagnosed with, e.g., energy-gap statistics \[^37\]. In the \( \text{su}(2) \) example, one can break integrability by creating next-nearest-neighbor couplings or by making the global system two-dimensional \[^20\].

### III. PRESCRIPTION FOR CONSTRUCTING THE HAMILTONIANS

Here, we synopsise the prescription elaborated on in Sec. III. Then, we present two results pertinent to the prescription. We construct, as follows, Hamiltonians that transport noncommuting charges locally and conserve the charges globally:

1. Identify an arbitrary Cartan-Weyl basis for the algebra, \( \mathcal{A} \).
2. The Cartan-Weyl basis contains \( r \) Hermitian operators that commute with each other. Scale each such operator such that it has a unit Hilbert-Schmidt norm [Eq. (5)]. Label the results \( Q_{\alpha=1,2,...,r} \). Include them in the preferred basis for the algebra.
3. The other Cartan-Weyl-basis elements are ladder operators that form raising-and-lowering pairs: \( L_{\pm \beta} \), for \( \beta = 1, 2, \ldots c-r \). From each pair, form one term in the two-body interaction, \( H^{(j,j')} \) [Eq. (6)].
4. Write out the form of the most general element \( U \in \mathcal{G} \) of the Lie group \( \mathcal{G} \) generated by \( \mathcal{A} \). Conjugate each charge \( Q_{\alpha} \) and each ladder operator \( L_{\pm \beta} \) with \( U \) [Eq. (7)]. The new charges and new ladder operators, together, form another Cartan-Weyl basis.
5. Constrain \( U \) such that every new charge \( Q_{\alpha} \) is Killing-orthogonal to (i) each other new charge and (ii) each charge already in the basis [Eq. (9)].
6. Include each new \( Q_{\alpha} \) in the basis for \( \mathcal{A} \).
7. From each new pair \( L_{\pm \beta} \) of ladder operators, form a term in the two-body interaction \( H^{(j,j')} \) [Eq. (10)].
8. Repeat steps 4 until having identified \( c/r \) Cartan-Weyl bases, wherein \( c \) denotes the algebra’s dimension. Each Cartan-Weyl basis contributes \( r \) elements \( Q_{\alpha} \) to the preferred basis for \( \mathcal{A} \). The basis is complete, containing \( r \cdot \frac{c}{r} = c \) elements.
9. Constrain the two-body interaction to conserve each global charge [Eq. (11)], for all \( \alpha = 1,2,\ldots,c \). Solve for the frequencies \( J^{(j',j'')} \) that satisfy this constraint.
10. If a \( k \)-body interaction is desired, for any \( k > 2 \): Perform the following substeps for \( \ell = 3,4,\ldots,k \): Multiply together \( \ell \) unconstrained two-body interactions [12] cyclically:
\[ H^{(j',j'',...j^{(t)})} = H^{(j,j')} H^{(j'',...j^{(t)})} \ldots H^{(j^{(t-1)},j^{(t)})} \]
\[ \times H^{(j^{(t)},j)}. \]
Constrain the couplings so that \( [H^{(j',j'',...j^{(t)})}, Q^\text{tot}_\alpha] = 0 \) for all \( \alpha \). If \( H^{(j',j'',...j^{(t)})} \) contains fewer-body terms that conserve all the \( Q^\text{tot}_\alpha \), subtract those terms off.
11. Sum the accumulated interactions \( H^{(j,j',...j^{(k)})} \) over the subsystems \( j,j',\ldots \) to form \( H^\text{tot} \).
12. If \( H^\text{tot} \) is to be nonintegrable, add longer-range interactions and/or large-\( k \) \( k \)-body interactions until breaking integrability, as signaled by, e.g., energy-gap statistics.

Having synopsized our prescription, we present two properties of it. The first property ensures that the prescription runs for an integer number of iterations (step 5).

**Proposition 1.** Consider any finite-dimensional semisimple complex Lie algebra. The algebra’s dimension, \( c \), and rank, \( r \), form an integer ratio: \( c/r \in \mathbb{Z}_{>0} \).

We prove this proposition in the Supplementary Note 4. The second property characterizes the prescription’s output.

**Theorem 1.** The charges \( Q_1, Q_2,\ldots,Q_c \) produced by the prescription form a basis for the algebra \( \mathcal{A} \).

**Proof.** The charges are Killing-orthogonal by construction: \( [Q_\alpha, Q_\beta] = 0 \) for all \( \alpha, \beta \). The Killing form induces a metric on \( \mathcal{A} \) by assumption. Therefore, the \( Q_\alpha \) are linearly independent according to this metric.
The prescription produces \( c \) charges (step 3). \( c \) denotes the algebra’s dimension, the number of elements in each basis for \( \mathcal{A} \). Hence every linearly independent set of \( c \) \( \mathcal{A} \) elements forms a basis for \( \mathcal{A} \). Hence the \( Q_{\alpha} \) form a basis.

**IV. \( \mathfrak{su}(3) \) Example**

Section 3 illustrated the Hamiltonian-construction prescription with the algebra \( \mathfrak{su}(2) \). The \( \mathfrak{su}(2) \) example offered simplicity but lacks other algebras’ richness: In other algebras, each Cartan-Weyl basis contains multiple Hermitian operators and multiple ladder-operator pairs. We demonstrate how our prescription accommodates this richness, by constructing a two-body Hamiltonian that transports \( \mathfrak{su}(3) \) elements locally while conserving them globally. Such Hamiltonians may be engineered for superconducting qutrits, as sketched in Sec. V. However, this \( \mathfrak{su}(3) \) example only illustrates our more general prescription, which works for all finite-dimensional semisimple complex Lie algebras on which the Killing form induces a metric.

Each basis for \( \mathfrak{su}(3) \) contains \( c = 8 \) elements. The most famous basis consists of the Gell-mann matrices, \( \lambda_k \equiv 1, 2, \ldots, 8 \). The \( \lambda_k \) generalize the Pauli matrices in certain ways, being traceless and Killing-orthogonal. From the Gellmann matrices is constructed the conventional Cartan-Weyl basis, reviewed in the Supplementary Note 5. The \( r = 2 \) Hermitian elements are Gellmann matrices:

\[
Q_1 = \lambda_3, \quad \text{and} \quad Q_2 = \lambda_8.
\]

\( Q_1 \) and \( Q_2 \) belong in the preferred basis of charges for \( \mathfrak{su}(3) \). For pedagogical clarity, we will identify all the charges before addressing the ladder operators.

A general element \( U \in \text{SU}(3) \) contains eight real parameters. In the Euler parameterization \([62]\):

\[
U = e^{i\lambda_3\phi_1/2}e^{i\lambda_2\phi_2/2}e^{i\lambda_5\phi_3/2}e^{i\lambda_6\phi_4/2} \\
\times e^{i\lambda_7\phi_5/2}e^{i\lambda_8\phi_6/2}e^{i\lambda_9\phi_7/2}e^{i\lambda_{10}\phi_8/2}.
\]

The parameters \( \phi_1, \phi_3, \phi_5, \phi_7 \in [0, 2\pi] \); \( \phi_2, \phi_4, \phi_6, \phi_8 \in [0, 2\sqrt{3}\pi] \); and \( \phi_9 \) are free. We now constrain \( U \), identifying the instances \( U_i \) that map the first charges to \( Q_3 = U_i^\dagger Q_1 U_i \) and \( Q_4 = U_i^\dagger Q_2 U_i \) that are Killing-orthogonal to each other and to the original charges. Supplementary Note 5 contains the details. We label with a superscript (i) the parameters used to fix \( U_i \): \( \phi_1^{(i)}, \phi_3^{(i)}, \phi_5^{(i)}, \phi_7^{(i)}, \phi_8^{(i)} \), and \( n^{(i)} \). For convenience, we package several parameters together: \( a^{(i)} := \frac{1}{2} \left( \phi_3^{(i)} - \phi_7^{(i)} - \sqrt{3}\phi_8^{(i)} + \pi n^{(i)} + \frac{\pi}{2} \right) \), and \( b^{(i)} := a^{(i)} + \phi_5^{(i)} \). In terms of these parameters, the new charges have the forms (see Supplementary Note 5)

\[
Q_3 = \frac{1}{\sqrt{3}} \left[ (-1)^{n^{(i)}+1} \sin \left( a^{(i)} - b^{(i)} \right) \lambda_1 \right] + \lambda_4.
\]

\( Q_4 \) has the same form as \( Q_5 \) and \( Q_7 \), which satisfy the same Killing-orthogonality conditions. Similarly, \( Q_4 \) has the same form as \( Q_6 \) and \( Q_8 \). The later charges’ parameters \( a^{(i)} \) and \( b^{(i)} \) are more restricted, however (see Supplementary Note 5). We have identified our preferred basis of charges.

Let us construct the ladder operators and Hamiltonian. Each Cartan-Weyl basis contains \( c - r = 8 - 2 = 6 \) ladder operators. The conventional Cartan-Weyl basis contains ladder operators formed from Gell-mann matrices:

\[
L_{\pm 1} := \frac{1}{2}(\lambda_1 \pm i\lambda_2), \quad L_{\pm 2} := \frac{1}{2}(\lambda_4 \pm i\lambda_5), \quad \text{and} \quad L_{\pm 3} := \frac{1}{2}(\lambda_6 \pm i\lambda_7).
\]

Transforming these operators with unitaries \( U_{ii,iii,iv} \) yields \( L_{\pm 4} \) through \( L_{\pm 12} \), whose forms appear in the Supplementary Note 5. From each ladder operator, we form one term in the two-body Hamiltonian \([62]\).

Finally, we determine the hopping frequencies \( J^{(j,j')}_{\alpha} \), demanding that \( [H^{(j,j')}, Q^{(j')}_{\alpha}] = 0 \) for all \( \alpha \). For all possible values of the \( a^{(i)}, b^{(i)} \), and \( n^{(i)} \), if all the frequencies are nonzero, then all the frequencies equal each other. We set \( J^{(j,j')}_{\alpha} := \frac{4}{3} J^{(j,j')} \), such that

\[
H^{(j,j')} = J^{(j,j')} \sum_{a=1}^{2} \lambda^{(j)}_{\alpha} \lambda^{(j')}_{\alpha} \propto \sum_{a=1}^{2} Q^{(j)}_{\alpha} Q^{(j')}_{\alpha}.
\]

The Hamiltonian collapses to a simple form analogous to the \( \mathfrak{su}(2) \) example’s Eq. \([10]\) (see Supplementary Note 3).

**V. Outlook**

We have presented a prescription for constructing Hamiltonians that transport noncommuting charges locally while conserving the charges globally. The Hamiltonians can couple arbitrarily many subsystems together and can be integrable or nonintegrable. The prescription produces, as well as Hamiltonians, preferred bases of charges that are (i) overtly transported locally and conserved globally and (ii) Killing-form-orthogonal. This construction works whenever the charges form a finite-dimensional semisimple complex Lie algebra on which the Killing form induces a metric. Whether there exists any Hamiltonians that transport charges locally, while conserving the charges globally, outside of those found...
by our prescription, is an interesting open question for theoretical exploration.

This work provides a systematic means of bridging noncommuting thermodynamic charges from abstract quantum information theory to condensed matter, AMO physics, and high-energy and nuclear physics. The mathematical results that have accrued can now be tested experimentally, via our construction. This paper’s introduction highlights example results that merit testing. Such experiments’ benefits include the simulation of quantum systems larger than what classical computers can simulate, the uncovereding of behaviors not predicted by theory, and the grounding of abstract QIT thermodynamics in physical reality.

In addition to harnessing controlled platforms to study noncommuting charges’ quantum thermodynamics, one may leverage that quantum thermodynamics to illuminate high-energy and nuclear physics. Such physics includes non-Abelian gauge theories, such as quantum chromodynamics. How to define and measure such theories’ thermalization is unclear. One might gain insights by using our dynamics as a bridge from quantum thermodynamics to non-Abelian field theories.

As mentioned above, the Heisenberg model can be implemented with ultracold atoms and trapped ions. Reference details how to harness these setups to study noncommuting thermodynamic charges. We introduce a more complex example here: We illustrate, with superconducting qubits, how today’s experimental platforms can implement the $su(3)$ instance of our general prescription.

Superconducting circuits can serve as qudits with Hilbert-space dimensionalities $d \geq 2$. Qutrits have been realized with transmons, slightly anharmonic oscillators. The lowest two energy levels often serve as a qubit, but the second energy gap nearly equals the first. Hence the third level can be addressed relatively easily. Superconducting qutrits offer a tabletop platform for transporting and conserving $su(3)$ charges as in Sec. IV.

Experiments with $\leq 5$ qutrits have been run. Furthermore, many of the tools used to control and measure superconducting qubits can be applied to qutrits. A noncommuting-charges-in-thermodynamics experiment may begin with preparing the qutrits in an approximate microcanonical subspace, a generalization of the microcanonical subspace that accommodates noncommuting charges. Such a state preparation may be achieved with weak measurements, which have been performed on superconducting qudits through cavity quantum electrodynamics.

$T_2^*$ relaxation times of $\sim 39 \mu s$, for the lowest energy gap, and $\sim 14 \mu s$, for the second-lowest gap, have been achieved. Meanwhile, two-qutrit gates can be realized in $\sim 10 - 10^2$ ns. Some constant number of such gates may implement one three-level gate that simulates a term in our Hamiltonian. If the number is order-10, information should be able to traverse an 8-qutrit system $\sim 10$ times before the qutrits decohere detrimentally. According to numerics in [1], a small subsystems such as qutrit pairs, can be read out via quantum state tomography. Hence superconducting qutrits, and other platforms, can import noncommuting charges from quantum thermodynamics to many-body physics, by simulating the Hamiltonians constructed here.

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**AUTHOR CONTRIBUTIONS**

NYH developed the prescription, managed the project, and led the paper writing. SM worked out the $su(3)$ example, proofs, Supplementary Notes, and superconducting-qutrit details, in addition to leading the referee revisions.

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**Appendix A** **THE KILLING FORM INDUCES A METRIC ON EVERY SIMPLE LIE ALGEBRA.**

Here, we prove a claim made in Sec. II.A of the main text: The Killing form induces a metric on every simple Lie algebra. The proof relies on background material reviewed in Sec. II.B of the main text.

Every inner product defines a metric. Therefore, proving that the Killing form induces an inner product suffices. On a simple Lie algebra, all symmetric bilinear forms equal each other to within a multiplicative constant. The Killing form is one symmetric bilinear form; another is $\text{Tr}(Q_\alpha Q_\beta)$. Hence $(Q_\alpha, Q_\beta) \propto \text{Tr}(Q_\alpha Q_\beta) = \text{Tr}(Q_\beta^\dagger Q_\alpha)$. The final equality follows from the charges’ Hermiticity. The final expression is the Hilbert-Schmidt inner product. Hence...
the Killing form induces an inner product.

Appendix B GENERAL HAMILTONIAN THAT TRANSPORTS $\mathfrak{su}(2)$ ELEMENTS LOCALLY WHILE CONSERVING THEM GLOBALLY

Section II.B of the main text illustrated how to construct Hamiltonians that transport $\mathfrak{su}(2)$ elements locally while conserving them globally. The illustration was not maximally general; we restricted a unitary $U$ more than required, for pedagogy. We generalize the construction here. For clarity of presentation, we derive the charges’ forms first (Supplementary Note [B 1]), and the ladder operators’ forms second (Supplementary Note [B 2]). We then construct the two-body Hamiltonian $H^{(i,j)}$ and a three-body Hamiltonian (Supplementary Note [B 3]).

B 1 Preferred basis of charges for $\mathfrak{su}(2)$

The conventional Cartan-Weyl basis contains the Hermitian operator

$$Q_1 = \sigma_z.$$  (B1)

To identify the next Cartan-Weyl basis, we invoke a general unitary $U \in \mathfrak{SU}(2)$. In the Euler parameterization,

$$U = e^{i\sigma_z\phi_1/2}e^{i\sigma_y\phi_2/2}e^{i\sigma_z\phi_3/2},$$  (B2)

wherein $\phi_1 \in [0,2\pi)$, $\phi_2 \in [0,\pi]$, and $\phi_3 \in [0,2\pi)$. We restrict this general unitary to a $U_i$ that maps $Q_2$ to a Killing-orthogonal charge $Q_2 = U_i^\dagger Q_1 U_i$. For $X,Y \in \mathfrak{su}(D)$, the Killing form evaluates to $(X,Y) = \text{Tr}(XY)$ [92]. Hence the Killing form between the charges is

$$0 = (U_i^\dagger Q_1 U_i, Q_1) = \text{Tr} \left( U_i^\dagger Q_1 U_i Q_1 \right) = 2 \cos \left( \phi_2^{(i)} \right).$$  (B3)

The superscript $(i)$, here and below, labels a parameter as belonging to $U_i$. The equation, with $\phi_2^{(i)} \in [0,\pi]$, implies that $\phi_2^{(i)} = \pi/2$. The unitary and charge assume the forms

$$U_i = e^{i\sigma_z\phi_1^{(i)}/2}e^{i\sigma_y\pi/4}e^{i\sigma_z\phi_3^{(i)}/2} \quad \text{and} \quad Q_2 = \cos \left( \phi_3^{(i)} \right) \sigma_x + \sin \left( \phi_3^{(i)} \right) \sigma_y.$$  (B4)

Having identified the second charge, we identify the final one. We transform $Q_1$ with a unitary $U_{ii} \in \mathfrak{SU}(2)$ such that $Q_3 = U_{ii}^\dagger Q_1 U_{ii}$ is Killing-orthogonal to the first two charges. The first orthogonality constraint has the form of Eq. (B3), except that a $(ii)$ replaces the superscript $(i)$. The second orthogonality constraint is

$$0 = \text{Tr} \left( U_{ii}^\dagger Q_1 U_{ii}, Q_2 \right) = \text{Tr} \left( U_{ii}^\dagger Q_1 U_{ii} Q_2 \right) = 2 \cos \left( \phi_3^{(ii)} - \phi_3^{(i)} \right).$$  (B5)

Hence $\phi_3^{(ii)} = \phi_3^{(i)} + \pi \left( n^{(ii)} - \frac{1}{2} \right)$, wherein $n^{(ii)} \in \mathbb{Z}$. Hence $U_{ii}$ and $Q_3$ have the forms

$$U_{ii} = e^{i\sigma_z\phi_1^{(ii)}/2}e^{i\sigma_y\pi/4}e^{i\sigma_z\phi_3^{(ii)}/2} + \pi(n^{(ii)} - \frac{1}{2})/2 \quad \text{and} \quad Q_3 = (-1)^{n^{(ii)}} \left[ \sin \left( \phi_3^{(i)} \right) \sigma_x - \cos \left( \phi_3^{(i)} \right) \sigma_y \right].$$  (B6)

Equations (B7), (B4), and (B1) specify the preferred basis of charges for $\mathfrak{su}(2)$.

B 2 General ladder operators for $\mathfrak{su}(2)$

The conventional Cartan-Weyl basis contains operators that raise and lower $\sigma_z$:

$$L_{\pm 1} = \sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y).$$  (B8)

Conjugation with $U_i$ yields the ladder operators for $Q_2$, and conjugation with $U_{ii}$ yields the ladder operators for $Q_3$:

$$L_{\pm 2} = U_i^\dagger L_{\pm 1} U_i = -\frac{e^{\mp \phi_3^{(i)}}}{2} \left[ \sigma_z \pm i(\sin \left( \phi_3^{(i)} \right) \sigma_x - \cos \left( \phi_3^{(i)} \right) \sigma_y) \right], \quad \text{and} \quad$$  (B9)

$$L_{\pm 3} = U_{ii}^\dagger L_{\pm 1} U_{ii} = -\frac{e^{\mp \phi_3^{(ii)}}}{2} \left[ \sigma_z \mp i(-1)^{n^{(ii)}} \left[ \cos \left( \phi_3^{(i)} \right) \sigma_x + \sin \left( \phi_3^{(i)} \right) \sigma_y \right] \right].$$  (B10)
B.3 Two-body and three-body Hamiltonians for $\mathfrak{su}(2)$

To form $H^{(j,j')}$, we substitute for the ladder operators from Eqs. (B8) and (B9) into Eq. (12). We require that $H^{(j,j')}$ conserve each global charge, imposing Eq. (14). This equation holds, algebra reveals, if and only if the hopping frequencies $J^{(j,j')}_\alpha$ equal each other. The Hamiltonian simplifies to Eq. (15). The final expression does not depend on our choice of $\phi_k^{(i)}, \phi_k^{(ii)}$, or $n^{(i)}$.

Let us construct a Hamiltonian $H^{(j,j',j'')}$ that transfers $\mathfrak{su}(2)$ charges between three sites—$j$, $j'$, and $j''$—while conserving the charges globally. We multiply three two-body Hamiltonians together cyclically:

$$H^{(j,j',j'')} \propto H^{(j,j')} H^{(j',j'')} H^{(j'',j)}$$

We substitute in from Eq. (13), the $H^{(j,j')}$ expression in which the hopping frequencies have not yet been restricted. The frequencies can assume different values, when $[H^{(j,j')}, Q^{\text{tot}}]_\alpha = 0$, than when $[H^{(j,j')}, Q^{\text{tot}}]_\alpha = 0$. Imposing the first commutator equation yields four sets of solutions for the $J^{(j)}_\alpha$’s, when $J^{(j)}_\alpha \neq 0$ for all $\alpha$:

1. $J_1 = J_2 = J_3 = J_4 = J_5 = J_6$, and $J_7 = J_8 = J_9$.
2. $J_1 = J_2 = -J_3$, $J_4 = J_5 = -J_6$, and $J_7 = J_8 = -J_9$.
3. $J_1 = J_2 = -\frac{J_3}{2}$, $J_4 = J_5 = -\frac{J_6}{2}$, and $J_7 = J_8 = -\frac{J_9}{2}$.
4. $J_1 = J_2 = \frac{J_3}{2}$, $J_1 + J_2 = -J_3$, $J_4 = J_5 = -J_6$, and $J_7 + J_8 = -J_9$.

We have omitted superscripts for conciseness. The four solutions lead to distinct Hamiltonians[^1].

For concreteness, we detail the first set of solutions, item (1). We collect three of the frequencies to simplify notation: $J^{(j,j',j'')} = J_j^{(j)}, J_j^{(j')}, J_j^{(j'')}$. Substituting the $J^{(j,j',j'')}_\alpha$’s into the Hamiltonian (B11) yields

$$H^{(j,j',j'')} \propto J^{(j,j',j'')} \left\{ 3 \sum_\alpha \left[ H^{(j,j')} - H^{(j'',j')} + H^{(j',j'')} \right] + i \left\{ \sigma_x \sigma_y \sigma_z + \sigma_y \sigma_z \sigma_x + \sigma_z \sigma_x \sigma_y - (\sigma_z \sigma_y \sigma_x + \sigma_x \sigma_z \sigma_y + \sigma_y \sigma_x \sigma_z) \right\} \right\}.$$  (B12)

We have omitted some superscripts to simplify notation. The first term is trivial, terms 2-4 are two-body, and each of terms 1-4 conserves each $Q^{\text{tot}}_\alpha$. Subtracting these terms off yields the solely three-body Hamiltonian, Eq. (18). We have absorbed the $i$ into the coefficient such that $J^{(j,j',j'')} \in \mathbb{R}$.

Appendix C  SIMPLE FORM TO WHICH A TWO-BODY HAMILTONIAN MAY COLLAPSE

In the $\mathfrak{su}(2)$ example, $H^{(j,j')}$ collapsed to the simple form (16). The $\mathfrak{su}(3)$ $H^{(j,j')}$ collapses to an analogous form, we shown in Sec. II.D. This form generalizes to

$$\sum_{\alpha=1}^\nu Q^{(j)}_\alpha Q^{(j')}_{\alpha'}.$$  (C1)

This expression generally conserves noncommuting charges globally, and transport the charges locally, as proved below. However, the expression’s equality with a two-body Hamiltonian that clearly, overtly transports local charges from site to site is proved only in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ examples.

**Proposition 2.** Consider any Lie algebra whose structure constants have the antisymmetry property

$$f^{\gamma}_{\alpha\beta} = -f^{\gamma}_{\beta\alpha}.$$  (C2)

A two-body Hamiltonian of the form (C1) conserves the algebra’s elements globally.

[^1]: However, each solution contains a little redundancy: Consider picking one of the four solutions, then cycling the indices in (1, 2, 3) identically to the indices in (4, 5, 6) and to the indices in (7, 8, 9). The resulting $J^{(j,j',j'')}_\alpha$’s specify a Hamiltonian identical to the original.
Every compact semisimple Lie algebra has such structure constants \[ f^\gamma_{\alpha\beta} \].

**Proof.** First, we substitute from Eq. (C1) into the conservation law. Then, we invoke the commutator’s linearity and the arguments’ tensor-product forms:

\[
0 = \left[ H^{(j,j')}, Q_{\alpha}^{\text{tot}} \right] = \sum_{\beta=1}^{c} Q^{(j)}_{\beta} Q^{(j')}_{\beta}, \quad Q^{(j)}_{\alpha} \otimes 1^{(j')} + 1^{(j)} \otimes Q^{(j')}_{\alpha} \]  
(C3)

\[
= \sum_{\beta=1}^{c} \left( Q^{(j)}_{\beta}, Q^{(j')}_{\beta}, 1^{(j')} + 1^{(j)} \otimes Q^{(j')}_{\beta} \right) \]  
(C4)

\[
= \sum_{\beta=1}^{c} \left( Q^{(j)}_{\beta}, Q^{(j)}_{\beta} \right) Q^{(j')}_{\beta} + Q^{(j)}_{\beta} \left[ Q^{(j')}, Q^{(j')}_{\alpha} \right]. \quad \text{(C5)}
\]

Let \( f^\gamma_{\alpha\beta} \) denote the Lie algebra’s structure constants. The \( f \)’s dictate how a Lie bracket decomposes as a linear combination of the algebra’s elements:

\[
[Q_{\alpha}, Q_{\beta}] = \sum_{\gamma=1}^{c} f^\gamma_{\alpha\beta} Q_{\gamma}. \quad \text{(C6)}
\]

We substitute into Eq. (C6), then pull the sums and constants out front:

\[
0 = \sum_{\beta=1}^{c} \left( \sum_{\gamma=1}^{c} f^\gamma_{\beta\alpha} Q^{(j)}_{\gamma} \right) Q^{(j')}_{\beta} + \left( \sum_{\gamma=1}^{c} f^\gamma_{\beta\alpha} Q^{(j)}_{\gamma} \right) \]  
(C7)

The final equation holds if \( f^\gamma_{\alpha\beta} = - f^\gamma_{\beta\alpha} \). Consider relabeling the index \( \alpha \) as \( \beta \) and vice versa. Equation (C2) results. \( \square \)

Having proved that the simple operator (C1) conserves noncommuting charges globally, we prove that it transports charges locally.

**Proposition 3.** The simple two-body Hamiltonian (C1) transports the charges \( Q_{\alpha} \) locally.

*Proof.** Charge \( Q_{\alpha} \) is transported locally if it satisfies Eq. (3), having a nonzero commutator

\[
\left[ H^{(j,j')}, Q_{\alpha}^{(j)} \right] = \sum_{\beta=1}^{c} Q^{(j)}_{\beta} Q^{(j')}_{\beta}, \quad Q^{(j)}_{\alpha}, Q^{(j')}_{\alpha} \]  
(C8)

The final expression vanishes if \( Q_{\alpha} \) commutes with all the other charges \( Q_{\gamma} \) in the preferred basis. If a Lie algebra has a basis of which one element commutes with the others, the algebra is Abelian, by definition (92). We assume that the algebra \( \mathcal{A} \) is non-Abelian (Sec. II.A of the main text). Therefore, the right-hand side of (C8) is nonzero, and the Hamiltonian transports the charges locally. \( \square \)

**Appendix D**  **PROOF OF PROPOSITION 1**

Proposition 1 states that the algebra \( \mathcal{A} \) has an integer ratio \( c/r \), wherein \( c \) denotes the algebra’s dimension and \( r \) denotes the rank.

*Proof.** For every finite-dimensional complex Lie algebra, there exists a corresponding connected Lie group that is unique to within finite coverings. The Lie algebra has the same dimension and rank as each of the corresponding Lie groups. Thus, if Proposition 1 holds for all semisimple Lie groups, it holds for all semisimple Lie algebras. We prove the group claim.

Every Lie group has a maximal torus \( \mathbb{T}^r \), which is the group generated by a Cartan subalgebra of the Lie algebra. The torus’ dimensionality equals the group’s rank, \( r \). A torus is an \( r \)-fold Cartesian product of \( S^1 \) manifolds [equivalently, of the group \( U(1) \)]. Quotienting out the torus’ action from the Lie group yields a finite-dimensional coset space. Every finite-dimensional coset space’s dimensionality is a positive integer \( n \in \mathbb{Z}_{>0} \). Thus, the semisimple Lie group’s dimension is \( c = rn \).  \( \square \)
II.D illustrated the Hamiltonian-construction prescription with \( \mathfrak{su}(3) \). We flesh out the explanation here. Appendix E.1 reviews the conventional Cartan-Weyl basis for \( \mathfrak{su}(3) \). Appendix E.2 identifies the preferred basis of charges for \( \mathfrak{su}(3) \). Appendix E.3 presents the ladder operators from which we construct a Hamiltonian.

### E 1 Conventional Cartan-Weyl basis for \( \mathfrak{su}(3) \)

\( \mathfrak{su}(3) \) has dimension \( c = 8 \) and rank \( r = 2 \). The conventional Cartan-subalgebra generators are denoted by \( t_z = \lambda_3/2 \) and \( y = \lambda_8/\sqrt{3} \), wherein \( \lambda_3 \) and \( \lambda_8 \) denote Gell-mann matrices \( \mathbb{1} \). These generators, in the three-dimensional representation of \( \mathfrak{su}(3) \), manifest as

\[
T_z = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{E1}
\]

\( t_z \) and \( y \) are orthogonal relative to the Killing form. They (more precisely, rescaled versions of them) belong in our preferred basis of charges: \( Q_1 \propto t_z \), and \( Q_2 \propto y \).

These charges are raised and lowered by \( c - r = 8 - 2 = 6 \) ladder operators, \( t_z = (\lambda_1 \pm i\lambda_2)/2 \), \( v_\pm = (\lambda_4 \pm i\lambda_5)/2 \), and \( u_\pm = (\lambda_6 \pm i\lambda_7)/2 \). In the three-dimensional representation of \( \mathfrak{su}(3) \), the ladder operators manifest as

\[
T_+ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_- = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_+ = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_- = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{E2}
\]

\[
U_+ = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad U_- = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{E3}
\]

The ladder operators participate in the following commutation relations with the charges:

\[
[t_z, t_\pm] = \pm t_\pm, \quad [y, t_\pm] = 0, \tag{E4}
\]

\[
[t_z, v_\pm] = \pm \frac{1}{2} v_\pm, \quad [y, v_\pm] = \pm v_\pm \tag{E5}
\]

\[
[t_z, u_\pm] = \mp \frac{1}{2} u_\pm, \quad \text{and} \quad [y, u_\pm] = \pm u_\pm. \tag{E6}
\]

These relations imply that (i) \( t_\pm \) raises and lowers \( t_z \), whereas (ii) \( v_\pm \) raises or lowers both \( t_z \) and \( y \), as does \( u_\pm \). We can prove this physical significance easily: Let \( L_\pm \) denote a ladder operator (a \( t_\pm \), a \( v_\pm \), or a \( u_\pm \)) that raises/lowers a charge \( Q \). Let \( |\psi\rangle \) denote a \( Q \) eigenstate associated with the eigenvalue \( q: Q |\psi\rangle = q |\psi\rangle \). Consider operating on the state with the ladder operator: \( L_\pm |\psi\rangle \). Suppose, for notational convenience, that, (i) if \( L_+ \) operates, \( q \) is not the greatest \( Q \) eigenvalue and (ii) if \( L_- \) operates, \( q \) is not the least \( Q \) eigenvalue. The resulting state is a \( Q \) eigenstate associated with the eigenvalue \( q \pm a \), wherein \( a = 1 \) or 1/2. To prove this claim, we operate on the new state with the charge: \( Q(L_\pm |\psi\rangle) \). Invoking the appropriate commutation relation (Eqs. (E4)-(E6)) yields

\[
QL_\pm |\psi\rangle = (L_\pm Q \pm L_\pm) |\psi\rangle = L_\pm (Q \pm a 1) |\psi\rangle = L_\pm (q \pm a) |\psi\rangle = (q \pm a) L_\pm |\psi\rangle. \tag{E7}
\]

By Eqs. (E4)-(E7), \( t_\pm \) raises/lowers the \( t_z \) charge by one quantum and preserves \( y \). \( u_\pm \) lowers/raises \( t_z \) by half a quantum and raises/lowers \( y \) by one quantum. \( v_\pm \) raises/lowers each of \( t_z \) and \( y \) by one quantum.

Having reviewed the conventional Cartan-Weyl basis for \( \mathfrak{su}(3) \), we dispense with the conventional notation \( (t_z, t_\pm, \text{etc.}) \). We revert to the notation introduced in the main text \( (Q_\alpha \text{ and } L_\pm \alpha) \).

### E 2 Preferred basis of charges for \( \mathfrak{su}(3) \)

The first two charges appear in Eqs. (20). We construct two new charges from \( Q_1, Q_2 \), and a unitary \( U \in \text{SU}(2) \). The general form of such a \( U \), appears, in the Euler parameterization, in Eq. (21). We constrain \( U \) with the Killing-orthogonality conditions (9), obtaining a unitary \( U_1 \). The transformed charges have the forms \( Q_3 = U_1^\dagger Q_1 U_1 \) and
$Q_1 = U_i^t Q_2 U_i$. The new charges are Killing-orthogonal to each other by unitarity: $0 = \text{Tr} \left( \left[ U_i^t Q_1 U_i \right] \left[ U_i^t Q_2 U_i \right] \right) = \text{Tr}(Q_1 Q_2) = 0$. Killing-orthogonality to the old charges, Eq. (20), with the form of the $su(D)$ Killing form [12], implies

$$0 = \text{Tr} \left( \left[ U_i^t Q_1 U_i \right] \right) Q_2 = -\cos(\phi_2)/3, \quad 0 = \text{Tr} \left( \left[ U_i^t Q_1 U_i \right] \right) Q_1 = -\frac{1}{2\sqrt{3}} \cos(\phi_3 + \phi_5), \quad (E8)$$

$$0 = \text{Tr} \left( \left[ U_i^t Q_2 U_i \right] \right) Q_1 = \frac{1}{2} \left( \cos(\phi_4 + \phi_5) \right), \quad \text{and} \quad 0 = \text{Tr} \left( \left[ U_i^t Q_2 U_i \right] \right) Q_2 = -\cos(\phi_6)/3. \quad (E9)$$

Since $\phi_2, \phi_4, \phi_6 \in [0, \pi]$ and $\phi_3, \phi_5 \in [0, 2\pi)$, $\phi_2 = \frac{\pi}{2}$, $\phi_4 = \cos(-1/3)$, $\phi_6 = \frac{\pi}{2}$ and $\phi_5 = \pi(n - 1/2) - \phi_3$, for $n \in \{1, 2, 3, 4\}$. Transforming $Q_1$ and $Q_2$ with a $U_{ii} \in SU(3)$ yields the charges $Q_5$ and $Q_6$, and transforming $Q_1$ and $Q_2$ with a $U_{ii} \in SU(3)$ yields $Q_7$ and $Q_8$. These last four charges are Killing-orthogonal to $Q_1$ and $Q_2$, like $Q_3$ and $Q_4$. So $U_{ii}$ and $U_{ii}$ share the form of $U_i$. However, parameters $a^{(ii)}$ and $b^{(ii)}$, or $a^{(iii)}$ and $b^{(iii)}$, replace the $a^{(i)}$ and $b^{(i)}$. The later unitaries’ parameters are more constrained than the $U_i$ parameters. Similarly, $Q_5$ through $Q_8$ share the forms of $Q_3$ and $Q_4$, apart from their more-constrained parameters.

Evaluating the restrictions on all the charges simultaneously will prove useful. First, the conditions for $Q_5$ to be orthogonal to $Q_3$ and $Q_4$ are

$$0 = \text{Tr}(Q_5 Q_3) \propto (-1)^{n^{(ii)} + a^{(ii)}} \cos(a^{(i)} - a^{(iii)} - b^{(i)} + b^{(ii)}) + \cos(a^{(i)} - a^{(iii)}) + \cos(b^{(i)} - b^{(ii)}), \quad (E10)$$

$$0 = \text{Tr}(Q_5 Q_4) \propto (-1)^{n^{(ii)} + a^{(ii)}} \sin(a^{(i)} - a^{(iii)} - b^{(i)} + b^{(ii)}) - \sin(a^{(i)} - a^{(iii)}) + \sin(b^{(i)} - b^{(ii)}). \quad (E11)$$

The orthogonality conditions for $Q_6$ impose the same constraints, since $\text{Tr}(Q_6 Q_3) \propto \text{Tr}(Q_5 Q_4)$ and $\text{Tr}(Q_6 Q_4) \propto \text{Tr}(Q_5 Q_3)$ (as can be checked explicitly). Similarly, the orthogonality conditions on $Q_7$ evaluate to

$$0 = \text{Tr}(Q_7 Q_3) \propto (-1)^{n^{(ii)} + a^{(ii)}} \cos(a^{(i)} - a^{(iii)} + b^{(i)} + b^{(ii)}) + \cos(a^{(i)} - a^{(iii)}) + \cos(b^{(i)} - b^{(ii)}), \quad (E12)$$

$$0 = \text{Tr}(Q_7 Q_4) \propto (-1)^{n^{(ii)} + a^{(ii)}} \sin(a^{(i)} - a^{(iii)} + b^{(i)} + b^{(ii)}) - \sin(a^{(i)} - a^{(iii)}) + \sin(b^{(i)} - b^{(ii)}), \quad (E13)$$

$$0 = \text{Tr}(Q_7 Q_5) \propto (-1)^{n^{(ii)} + a^{(ii)}} \cos(a^{(i)} - a^{(iii)} - b^{(i)} + b^{(ii)}) + \cos(a^{(i)} - a^{(iii)}) + \cos(b^{(i)} - b^{(ii)}), \quad (E14)$$

$$0 = \text{Tr}(Q_7 Q_6) \propto (-1)^{n^{(ii)} + a^{(ii)}} \sin(a^{(i)} - a^{(iii)} - b^{(i)} + b^{(ii)}) - \sin(a^{(i)} - a^{(iii)}) + \sin(b^{(i)} - b^{(ii)}). \quad (E15)$$

The orthogonality conditions for $Q_8$ impose the same constraints [Eqs. (E12)-(E15)].

We now identify sets of $a^{(ii)}, b^{(ii)}$, and $n^{(ii)}$ that are solutions for all six constraints, Eqs. (E10)-(E15). First, we define $x_{lm} := a^{(i)} - a^{(m)}$ and $y_{lm} := b^{(i)} - b^{(m)}$ for $(l, m) = (2, 3), (2, 4), (3, 4)$. By these definitions, $x_{23} = x_{24} = x_{34} + x_{34}$, and $y_{24} = y_{23} + y_{34}$. Second, the values of the $n^{(ii)}$ themselves are irrelevant. Only whether $n^{(ii)} + n^{(m)}$ is even or odd matters. Only four unique possibilities for the $n^{(ii)}$ exist: All the $n^{(ii)} + n^{(m)}$ are even; or one $n^{(ii)} + n^{(m)}$ is even, while the other two sums are odd. A solution can therefore be expressed in terms of just four quantities: $x_{23}, x_{34}, y_{23}$, and $y_{34}$. Each solution is periodic:

$$x_{23}, x_{34}, y_{23}, y_{34} \equiv (x_{23}, x_{34}, y_{23}, y_{34}) + (2\pi n, 2\pi n, 2\pi n, 2\pi n), \quad (E16)$$

wherein $n \in \mathbb{Z}$. Therefore, we omit the $2\pi n$ when listing the solutions below.

First, suppose that all the $n^{(ii)} + n^{(m)}$ are even. The constraints (E10)-(E16) admit of 18 solutions. The first ten are

$$(x_{23}, x_{34}, y_{23}, y_{34}) = \left\{ \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3} \right\}, \left\{ 0, 0, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3} \right\}, \left\{ 0, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3}, 0 \right\}, \left\{ \pm \frac{2\pi}{3}, 0, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3} \right\},$$

$$\left\{ \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{2\pi}{3} \right\}. \quad (E17)$$

The next eight solutions are identical to the first eight, except that each $x_{lm}$ is swapped with the corresponding $y_{lm}$.

Second, $n^{(ii)} + n^{(iii)}$ can be even while $n^{(ii)} + n^{(i)}$ and $n^{(ii)} + n^{(iii)}$ are odd. The constraints (E10)-(E15) admit of another 18 solutions. The first ten are

$$(x_{23}, x_{34}, y_{23}, y_{34}) = \left\{ \pi, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right\}, \left\{ \pi, \pi, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right\}, \left\{ \pi, \pi, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right\}, \left\{ \pi, \pi, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right\},$$

$$\left\{ \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3} \right\}. \quad (E18)$$

wherein $n \in \mathbb{Z}$. Therefore, we omit the $2\pi n$ when listing the solutions below.
The next eight solutions are identical to the first eight, except that each \( x_{\ell m} \) is swapped with the corresponding \( y_{\ell m} \). Third, \( n^{(i)} + n^{(ii)} \) can be even while \( n^{(i)} + n^{(iii)} \) and \( n^{(ii)} + n^{(iii)} \) are odd. The constraints (E10)–(E13) admit of another 18 solutions. The first ten are

\[
(x_{23}, x_{34}, y_{23}, y_{34}) = \left(0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}\right), \left(0, \pi, \pm \frac{2\pi}{3}, \mp \frac{\pi}{3}\right), \left(0, \mp \pi, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}\right), \left(\pm \frac{2\pi}{3}, \pi, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}\right), \\
\left(\pm \frac{2\pi}{3}, \mp \frac{\pi}{3}, \mp \frac{2\pi}{3}, \mp \frac{\pi}{3}\right).
\]  

(E19)

The next eight solutions are identical to the first eight, except that each \( x_{\ell m} \) is swapped with the corresponding \( y_{\ell m} \). Fourth, suppose that \( n^{(i)} + n^{(iii)} \) is even while \( n^{(i)} + n^{(ii)} \) and \( n^{(i)} + n^{(iii)} \) are odd. The constraints (E10)–(E15) admit of another 18 solutions. The first ten are

\[
(x_{23}, x_{34}, y_{23}, y_{34}) = \left(\pi, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}\right), \left(\pi, 0, \pm \frac{\pi}{3}, \mp \frac{2\pi}{3}\right), \left(\pi, \mp \frac{2\pi}{3}, \mp \frac{\pi}{3}, 0\right), \left(\pm \frac{\pi}{3}, 0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}\right), \\
\left(\pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}, \mp \frac{2\pi}{3}\right).
\]  

(E20)

The next eight solutions are identical to the first eight, except that each \( x_{\ell m} \) is swapped with the corresponding \( y_{\ell m} \). One can check explicitly that the tuple \((x_{23} + y_{23}, x_{34} + y_{34})\) has three possible values: \((x_{23} + y_{23}, x_{34} + y_{34}) = (\pm 2\pi/3, \pm 2\pi/3), (\pm 4\pi/3, \pm 4\pi/3), (\pm \pi/3, \mp \pi/3)\). Three sets of solutions follow. For example, the first set of solutions is \((x_{23} + y_{23}, x_{34} + y_{34}) = (\pm 2\pi/3, \pm 2\pi/3)\). Hence

\[
a^{(i)} - a^{(ii)} + b^{(i)} - b^{(ii)} = \pm \frac{2\pi}{3}, \quad a^{(i)} - a^{(iii)} + b^{(i)} - b^{(iii)} = \pm \frac{2\pi}{3},
\]

(E21)

\[
a^{(\ell)} - a^{(m)} \in \left\{0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}\right\}, \quad \text{and} \quad b^{(\ell)} - b^{(m)} \in \left\{0, \pm \frac{\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{\pi}{3}\right\},
\]

(E22)

for \((\ell, m) = (2, 3)\) and \((3, 4)\). All the solutions lead to the same Hamiltonian, Eq. (25).

### E 3 Ladder operators for \( su(3) \)

The conventional Cartan-Weyl basis contains six ladder operators [Eqs. (24)]. We transform \( L_{\pm 1,2,3} \) with the unitaries \( U_1, U_3, \) and \( U_{iii} \) of Sec. [E.2] to construct the rest of the ladder operators: \( L_{\pm 4} = U_1^\dagger L_{\pm 1} U_1, L_{\pm 5} = U_3^\dagger L_{\pm 2} U_3, \) and \( L_{\pm 6} = U_{iii}^\dagger L_{\pm 3} U_{iii} \). Substituting in for \( L_{\pm 1,2,3} \) from Eq. (24) yields

\[
L_{\pm 4} = \frac{i e^{\pm i \phi_{1}^{(i)}}}{6} \left(2i \cos(a^{(i)} - b^{(i)}) \lambda_1 - 2i \sin(a^{(i)} - b^{(i)}) \lambda_2 \mp \sqrt{3} \mp i(-1)^{n^{(i)}} \cos(a^{(i)} - b^{(i)}) \lambda_4 - \sin(a^{(i)}) \lambda_5\right)
\]

(E23)

\[
L_{\pm 5} = \frac{i e^{\pm i \phi_{1}^{(i)}}}{6} \left(i \cos(a^{(i)} - b^{(i)}) - \mp \sqrt{3}(-1)^{n^{(i)}} \sin(a^{(i)} - b^{(i)})\right) \lambda_1 - i \left[\cos(a^{(i)} - b^{(i)}) + \sqrt{3}(-1)^{n^{(i)}} \cos(a^{(i)} - b^{(i)})\right] \lambda_2 \pm \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \sin(a^{(i)}) \pm i \cos(a^{(i)})\right] + \sqrt{3} e^{\pm i a^{(i)}}\right\} \lambda_4
\]

\[
\pm \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \cos(a^{(i)}) \pm i \sin(a^{(i)})\right] + \sqrt{3} e^{\pm i a^{(i)}}\right\} \lambda_5
\]

\[
\pm \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \sin(b^{(i)}) \pm i \cos(b^{(i)})\right] - \sqrt{3} e^{\mp i b^{(i)}}\right\} \lambda_6
\]

\[
\pm \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \cos(b^{(i)}) \mp i \sin(b^{(i)})\right] \mp i \sqrt{3} e^{\pm i b^{(i)}}\right\} \lambda_7 \mp \sqrt{3}(-1)^{n^{(i)}} \lambda_3 + \sqrt{3} i \lambda_8, \quad \text{and}
\]

(E24)

\[
L_{\pm 6} = \frac{i e^{\pm i \phi_{2}^{(i)}}}{6} \left(-i \left[\cos(a^{(i)} - b^{(i)}) + \sqrt{3}(-1)^{n^{(i)}} \sin(a^{(i)} - b^{(i)})\right] \lambda_1
\]

\[
+i \left[\cos(a^{(i)} - b^{(i)}) + \sqrt{3}(-1)^{n^{(i)}} \sin(a^{(i)} - b^{(i)})\right] \lambda_2 \mp \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \sin(a^{(i)}) \pm i \cos(a^{(i)})\right] - \sqrt{3} e^{i a^{(i)}}\right\} \lambda_4
\]

\[
\pm \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \cos(a^{(i)}) \mp i \sin(a^{(i)})\right] \pm i \sqrt{3} e^{i a^{(i)}}\right\} \lambda_5
\]

\[
\pm \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \sin(b^{(i)}) \mp i \cos(b^{(i)})\right] \pm i \sqrt{3} e^{i b^{(i)}}\right\} \lambda_6
\]

\[
\mp \frac{1}{2} \left\{(-1)^{n^{(i)}} \left[3 \cos(b^{(i)}) \pm i \sin(b^{(i)})\right] \pm i \sqrt{3} e^{i b^{(i)}}\right\} \lambda_7
\]

\[
\mp i \sqrt{3}(-1)^{n^{(i)}} \lambda_3 + \sqrt{3} i \lambda_8, \quad \text{and}
\]

(E25)
\[ \pm \frac{1}{2} \left\{ (-1)^{n(i)} \left[ 3 \cos \left( \theta^{(i)} \right) \mp i \sin \left( \theta^{(i)} \right) \right] \pm i \sqrt{3} \epsilon^{(i)} \right\} \lambda_7 \mp \sqrt{3} (-1)^{n(i)} \lambda_3 - \sqrt{3} i \lambda_8 \right\}. \]

(E25)

\( L_{\pm7} \), \( L_{\pm8} \), and \( L_{\pm9} \) have the same forms. However, (ii)’s replace the superscripts (i)’s. \( L_{\pm10} \), \( L_{\pm11} \), and \( L_{\pm12} \) likewise have the same form, except that (iii)’s replace the (i)’s.

REFERENCES

[1] Vinjanampathy, S. & Anders, J. Quantum thermodynamics. *Contemp. Phys.* 57, 545–579 (2016).
[2] Goold, J., Huber, M., Riera, A., del Rio, L. & Skrzypczyk, P. The role of quantum information in thermodynamics — a topical review. *J. Phys. A: Math. Appl.* 49, 143001 (2016). URL [http://stacks.iop.org/1751-8121/49/i=14/a=143001](http://stacks.iop.org/1751-8121/49/i=14/a=143001)
[3] Lostaglio, M., Jennings, D. & Rudolph, T. Quantum dynamic resource theories, non-commutativity and maximum entropy principles. *New J. Phys.* 19, 043008 (2017). URL [http://stacks.iop.org/1367-2630/19/i=4/a=043008](http://stacks.iop.org/1367-2630/19/i=4/a=043008)
[4] Guryanova, Y., Popescu, S., Short, A. J., Silva, R. & Skrzypczyk, P. Thermodynamics of quantum systems with multiple conserved quantities. *Nat. Commun.* 7, 12049 (2016). URL [http://dx.doi.org/10.1038/ncomms12049](http://dx.doi.org/10.1038/ncomms12049)
[5] Yung Halpern, N. Beyond heat baths ii: framework for generalized thermodynamic resource theories. *J. Phys. A: Math. Anal.* 51, 094001 (2018). URL [http://stacks.iop.org/1751-8121/51/i=9/a=094001](http://stacks.iop.org/1751-8121/51/i=9/a=094001)
[6] Lostaglio, M. The resource theory of quantum thermodynamics. Master’s thesis, Imperial College London (2014).
[7] Yung Halpern, N., Faist, P., Oppenheim, J. & Winter, A. Microcanonical and resource-theoretic derivations of the thermal state of a quantum system with noncommuting charges. *Nat. Commun.* 7, 12051 (2016). URL [http://www.ncbi.nlm.nih.gov/pmc/articles/PMC4941045/](http://www.ncbi.nlm.nih.gov/pmc/articles/PMC4941045/)
[8] Vaccaro, J. A. & Barnett, S. M. Information erasure without an energy cost. *Proc. Math. Phys. Eng. Sci.* 467, 1770–1778 (2011).
[9] Sparaciari, C., Del Rio, L., Scandolo, C. M., Faist, P. & Oppenheim, J. The first law of general quantum resource theories. *Quantum* 4, 259 (2020).
[10] Khanian, Z. B. From quantum source compression to quantum thermodynamics. *Preprint at https://arxiv.org/abs/2012.14143* (2020).
[11] Khanian, Z. B., Bera, M. N., Riera, A., Lewenstein, M. & Winter, A. Resource theory of heat and work with non-commuting charges: yet another new foundation of thermodynamics. *Preprint at https://arxiv.org/abs/2011.08020* (2020).
[12] Gour, G., Jennings, D., Buscemi, F., Duan, R. & Marvian, I. Quantum majorization and a complete set of entropic conditions for quantum thermodynamics. *Nat. Commun.* 9, 5532 (2018). URL [https://doi.org/10.1038/s41467-018-06261-7](https://doi.org/10.1038/s41467-018-06261-7)
[13] Manzano, G., Parrondo, J. M. & Landi, G. Non-abelian quantum transport and thermosqueezing effects. *Preprint at https://arxiv.org/abs/2011.04560* (2020).
[14] Popescu, S., Sainz, A. B., Short, A. J. & Winter, A. Quantum reference frames and their applications to thermodynamics. *Philos. Trans. Royal Soc. A* 376, 20180111 (2018).
[15] Popescu, S., Sainz, A. B., Short, A. J. & Winter, A. Reference frames which separately store noncommuting conserved quantities. *Phys. Rev. Lett.* 125, 090601 (2020).
[16] Ito, K. & Hayashi, M. Optimal performance of generalized heat engines with finite-size baths of arbitrary multiple conserved quantities beyond independent-and-identical-distribution scaling. *Phys. Rev. E* 97, 012129 (2018). URL [https://link.aps.org/doi/10.1103/PhysRevE.97.012129](https://link.aps.org/doi/10.1103/PhysRevE.97.012129)
[17] Bera, M. N., Riera, A., Lewenstein, M., Khanian, Z. B. & Winter, A. Thermodynamics as a consequence of information conservation. *Quantum* 3, 121 (2019).
[18] Mur-Petit, J., Relaño, A., Molina, R. A. & Jaksh, D. Revealing missing charges with generalised quantum fluctuation relations. *Nat. Commun.* 9, 2006 (2018). URL [https://doi.org/10.1038/s41467-018-04407-1](https://doi.org/10.1038/s41467-018-04407-1)
[19] Manzano, G. Sneezed thermal reservoir as a generalized equilibrium reservoir. *Phys. Rev. E* 98, 042123 (2018).
[20] Yung Halpern, N., Beverland, M. E. & Kalev, A. Noncommuting conserved charges in quantum many-body thermalization. *Phys. Rev. E* 101, 042117 (2020). URL [https://link.aps.org/doi/10.1103/PhysRevE.101.042117](https://link.aps.org/doi/10.1103/PhysRevE.101.042117)
[21] Manzano, G. et al. Hybrid thermal machines: Generalized thermodynamic resources for multitasking. *Phys. Rev. Res.* 2, 043302 (2020).
[22] Fukai, K., Nozawa, Y., Kawahara, K. & Ikeda, T. N. Noncommutative generalized gibbs ensemble in isolated integrable quantum systems. *Phys. Rev. Res.* 2, 023403 (2020).
[23] Mur-Petit, J., Relaño, A., Molina, R. A. & Jaksh, D. Fluctuations of work in realistic equilibrium states of quantum systems with conserved quantities. *SciPost Phys. Proc.* 3 (2020).
[24] Scandi, M. & Perarnau-Llobet, M. Thermodynamic length in open quantum systems. *Quantum* 3, 197 (2019).
[25] Boes, P., Wilming, H., Eisert, J. & Gallego, R. Statistical ensembles without typicality. *Nat. Commun.* 9, 1–9 (2018).
[26] Mitsuhashi, Y., Kaneko, K. & Sagawa, T. Characterizing symmetry-protected thermal equilibrium by work extraction. *Preprint at https://arxiv.org/abs/2103.06080* (2021).
[27] Croucher, T., Wright, J., Carvalho, A. R. R., Barnett, S. M. & Vaccaro, J. A. Information Erasure, 713–730 (Springer International Publishing, Cham, 2018). URL [https://doi.org/10.1007/978-3-319-99046-0_29](https://doi.org/10.1007/978-3-319-99046-0_29)
[28] Wright, J. S., Gould, T., Carvalho, A. R., Bedkhal, S. & Vaccaro, J. A. Quantum heat engine operating between thermal and spin reservoirs. *Phys. Rev. A* 97, 052104 (2018).
Cahn, R. N. *Semi-Simple Lie Algebras and Their Representations* (Dover, 2006).

Byrd, M. Differential geometry on su (3) with applications to three state systems. *J. Math. Phys.* **39**, 6125–6136 (1998).

Mueller, N., Zache, T. V. & Ott, R. Thermalization of gauge theories from their entanglement spectrum. *Preprint at arxiv.org/abs/2107.11416* (2021).

Jané, E., Vidal, G., Dürr, W., Zoller, P. & Cirac, J. I. Simulation of quantum dynamics with quantum optical systems. *Quantum Inf. Comput.* **3**, 15–37 (2003).

Barredo, D., de Léséleuc, S., Lienhard, V., Lahaye, T. & Browaeys, A. An atom-by-atom assember of defect-free arbitrary two-dimensional atomic arrays. *Science* **354**, 1021–1023 (2016). URL [https://science.sciencemag.org/content/354/6315/1021](https://science.sciencemag.org/content/354/6315/1021)

de Léséleuc, S. *et al.* Observation of a symmetry-protected topological phase of interacting bosons with rydberg atoms. *Science* **365**, 775–780 (2019).

Zhang, J. *et al.* Observation of a many-body dynamical phase transition with a 53-qubit quantum simulator. *Nature* **551**, 601–604 (2017).

Fukuhara, T. *et al.* Microscopic observation of magnon bound states and their dynamics. *Nature* **502**, 76 EP – (2013). URL [https://doi.org/10.1038/nature12541](https://doi.org/10.1038/nature12541)

You, J. Q. & Nori, F. Atomic physics and quantum optics using superconducting circuits. *Nature* **474**, 589–597 (2011). URL [https://doi.org/10.1038/nature10122](https://doi.org/10.1038/nature10122)

Koch, J. *et al.* Charge-insensitive qubit design derived from the cooper pair box. *Phys. Rev. A* **76**, 042319 (2007).

Bianchetti, R. *et al.* Control and tomography of a three level superconducting artificial atom. *Phys. Rev. Lett.* **105**, 223601 (2010).

Morvan, A. *et al.* Qutrit randomized benchmarking. *Phys. Rev. Lett.* **126**, 210504 (2021).

Blek, M. S. *et al.* Quantum information scrambling on a superconducting qutrit processor. *Phys. Rev. X* **11**, 021010 (2021).

Xu, H. *et al.* Coherent population transfer between uncoupled or weakly coupled states in ladder-type superconducting qutrits. *Nat. Commun.* **7**, 1–6 (2016).

Kumar, K., Vepsäläinen, A., Danilin, S. & Paraoanu, G. Stimulated raman adiabatic passage in a three-level superconducting circuit. *Nat. Commun.* **7**, 1–6 (2016).

Tan, X. *et al.* Topological maxwell metal bands in a superconducting qutrit. *Phys. Rev. Lett.* **120**, 130503 (2018).

Vepsäläinen, A., Danilin, S. & Paraoanu, G. S. Superadiabatic population transfer in a three-level superconducting circuit. *Sci. Adv.* **5**, eaau5999 (2019).

Lu, X.-J. *et al.* Nonleaky and accelerated population transfer in a transmon qutrit. *Phys. Rev. A* **96**, 023843 (2017).

Vepsäläinen, A., Danilin, S., Paladino, E., Falci, G. & Paraoanu, G. S. Quantum control in qutrit systems using hybrid rabi-stirap pulses. In *Photonic*, vol. 3, 62 (Multidisciplinary Digital Publishing Institute, 2016).

Yang, C.-P., Su, Q.-P. & Han, S. Generation of greenberger-horne-zeilinger entangled states of photons in multiple cavities via a superconducting qutrit or an atom through resonant interaction. *Phys. Rev. A* **86**, 022329 (2012).

Shilyakov, A. *et al.* Quantum metrology with a transmon qutrit. *Phys. Rev. A* **97**, 022115 (2018).

Danilin, S., Vepsäläinen, A. & Paraoanu, G. S. Experimental state control by fast non-abelian holonomic gates with a superconducting qutrit. *Phys. Scr.* **93**, 055101 (2018).

Shnyrkov, V., Soroka, A. & Turutanov, O. Quantum superposition of three macroscopic states and superconducting qutrit detector. *Phys. Rev. B* **85**, 224512 (2012).

Naghiloo, M. Introduction to experimental quantum measurement with superconducting qubits. *Preprint at arxiv.org/abs/1904.09291* (2019).

Huang, H.-L., Wu, D., Fan, D. & Zhu, X. Superconducting quantum computing: a review. *Sci. China Inf. Sci.* **63**, 1–32 (2020).

Humphreys, J. E. *Introduction to Lie algebras and representation theory*, vol. 9 (Springer Science & Business Media, 2012).

Metha, M., Normand, J. & Gupta, V. A property of the structure constants of finite dimensional compact simple lie algebras. *Commun. Math. Phys.* **90**, 69–78 (1983).