HODGE STRUCTURE OF A COMPLETE INTERSECTION OF QUADRICS IN A PROJECTIVE SPACE

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Let $V$ be a smooth projective variety of dimension $n$. One may say that $V$ has motivic dimension less than $d + 1$ if the cohomology of $V$ comes from varieties of dimensions less than $d + 1$ in some geometric way. More precisely, we say that $V$ has motivic dimension less than $d + 1$ if there exists a (nonconnected) smooth projective variety $W$ of dimension less than $d + 1$ and an algebraic correspondence $\Gamma$ on $W \times X$ such that $\Gamma$ induces a surjection $H^\ast(W) \to H^\ast(V)$.

Suppose the Generalized Hodge conjecture (GHC) ([G], [L]) holds for $V$ and suppose the level of $V$ is less than $l$ (i.e. level($H^i(V)$) < $l$ for all $i$). This implies that $V$ has a motivic dimension less than $l$. Conversely, if $V$ has motivic dimension less than $d + 1$, i.e. there is a smooth projective variety $W$ of dimension less than $d + 1$ and a surjection $\phi : H^\ast(W) \to H^\ast_n(V)$ induced by a correspondence $\Gamma$ on $W \times X$, then the level of the Hodge structure $H^\ast_n(V)$ is less than $d + 1$ because a morphism of Hodge structures preserves the level. The existence of $W$ does not imply that the GHC holds for $V$ of course, because the dimension of the variety $W$ is not small enough to conclude the GHC for $V$. However, this gives a way to reduce the GHC for $V$ to the GHC for $W$, the variety with the smaller dimension. And we think that this can be a practical intermediate step for checking GHC for a smooth projective variety.

For a smooth complete intersection of $k$ quadrics $V = Q_1 \cap \cdots \cap Q_k$ in $\mathbb{P}^{n+k}$, it can be checked that the level of the Hodge structure $H^\ast_n(V)$ is less than $k$ (for example by checking Hodge number of it). In this paper, we show that a smooth complete intersection of $k$ quadrics in $\mathbb{P}^{n+k}$ has motivic dimension less than $k$ (Theorem 1.2). As a corollary of this, we get the Hodge conjecture for $V$ holds if $k < 4$.

A brief outline of the proof of main theorem is as follows. By using the construction in [O], we form a family of quadrics parametrized by $\mathbb{P}^{k-1}$ with the base locus $V$. In case when $n+k$ is odd, this is a family of even dimensional quadrics. Then we use the fact that an even dimensional quadric contains two irreducible families of linear spaces of the expected dimension, and we choose $W$ to be a double covering of $\mathbb{P}^{n+k}$. In the case when $n+k$ is even, then quadrics in the original family are odd dimensional. So in order to find $W$, we pass to the family of singular fibers over the discriminant variety which can be understood as a family of quadrics of even dimension. Then we form a double covering of the discriminant variety to find $W$.

All varieties in this paper will be defined over $\mathbb{C}$ and the cohomology without coefficient would be the singular cohomology with rational coefficient.

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1. Main Theorem

Let $V = Q_1 \cap \cdots \cap Q_k$ be a smooth complete intersection of $k$ quadrics in a projective space $\mathbb{P}^{n+k}$. Then $V$ is a smooth subvariety of $\mathbb{P}^{n+k}$ of dimension $n$. Consider the family of quadrics $Q_t$ $(t \in \mathbb{P}^{k-1})$ with the base locus $V$. We can give more precise description of this family. Let $Q_t = \{F_t = 0\}$ for $t = (t_0, \ldots, t_{k-1}) \in \mathbb{P}^{k-1}$, where $F_t(x_0, x_1, \ldots, x_{n+k}) = \sum_{i,j=0}^{n+k} c_{ij}^t x_i x_j$, $[c_{ij}^t]_{0 \leq i,j \leq n+k}$ is a symmetric $(n+k+1) \times (n+k+1)$ matrix. Then for a general $t = (t_0, \ldots, t_{k-1}) \in \mathbb{P}^{k-1}$, the fiber $Q_t$ is given by the equation $\sum_{l=0}^{k-1} t_l F_{l+1}(x_0, \ldots, x_{n+k}) = 0$. Let $\Delta \subset \mathbb{P}^{k-1}$ be the subvariety of $\mathbb{P}^{k-1}$ parametrizing all singular fibers in the family. Then $\Delta$ is a hypersurface in $\mathbb{P}^{k-1}$ of degree $n + k + 1$. We assume that $\Delta$ is smooth. Set

$$X = \{(t, x) \in \mathbb{P}^{k-1} \times \mathbb{P}^{n+k} \mid x \in Q_t \subset \mathbb{P}^{n+k}\} \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n+k}$$

and let $p_1 : X \to \mathbb{P}^{k-1}$ and $p_2 : X \to \mathbb{P}^{n+k}$ be projections. Note that $X$ is a smooth projective variety of dimension $n + 2k - 2$.

(1)

Let $i_X : X \hookrightarrow \mathbb{P}^{k-1} \times \mathbb{P}^{n+k}$ and $i_V : V \hookrightarrow \mathbb{P}^{n+k}$ be inclusions. Then by Lefschetz theorem, the restriction maps $i_X^* : H^{n+2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) \to H^{n+2k-2}(X)$ and $i_V^* : H^n(\mathbb{P}^{n+k}) \to H^n(V)$ are injections. Set

$$H^{n+2k-2}_0(X) = H^{n+2k-2}(X)/\text{im } i_X^*$$

and

$$H^n_0(V) = H^n(V)/\text{im } i_V^*$$

Remark 1.1. In fact,

$$H^{n+2k-2}_0(X) \cong \begin{cases} H^{n+2k-1}_{c}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) - X & \text{if } n \text{ is even} \\ H^{n+2k-2}(X) & \text{if } n \text{ is odd} \end{cases}$$

and

$$H^n_0(V) \cong \begin{cases} H^{n+1}_{c}(\mathbb{P}^{n+k} - V) & \text{if } n \text{ is even} \\ H^n(V) & \text{if } n \text{ is odd} \end{cases}$$

as Hodge structures.

Now we can state our main theorem precisely:

Theorem 1.2. Let $V$ be a smooth complete intersection of $k$ quadrics in $\mathbb{P}^{n+k}$. Then $V$ has motivic dimension less than $k$. More precisely, there is a smooth projective variety $W$ of dimension $k - 1$ (resp. $k - 2$) and surjection of rational Hodge structures

$$\Theta : H^{k-1}(\tilde{W})(-q) \to H^n(V) \quad \text{if } n \text{ is odd and } k \text{ is even}$$

(resp. $\Theta : H^{k-2}(\tilde{W})(-q) \to H^n(V)$ if $n$ is odd and $k$ is odd)

where $q = \frac{n-k+1}{2}$ (resp. $\frac{n-k+2}{2}$) and $\tilde{W}$ is a disjoint union of finitely many copies of $W$,

$$\Theta : \bigoplus_r H^{2r}(W)(-q_r) \to H^n_0(V) \quad \text{if } n \text{ is even}$$
where \( q_r = \frac{n-2r}{2} \) and \( l_r \) are positive integers given by explicit formula.

We will prove this theorem by considering two cases depending on the parity of \( n + k \) in the last two sections. One immediate corollary of this theorem is

**Corollary 1.3.** If \( k \leq 4 \), then Hodge conjecture holds for \( V \).

2. Intermediate Step

Consider the projection \( p_2 : X \rightarrow \mathbb{P}^{n+k} \). Note that for any \( q = (q_0, ..., q_{n+k}) \in \mathbb{P}^{n+k} - V \),

\[
p_2^{-1}(q) = \{(t, q) \in \mathbb{P}^{k-1} \times \mathbb{P}^{n+k} \mid q \in Q_t\}
\]

\[
= \left\{ t = (t_0, \ldots, t_{k-1}) \in \mathbb{P}^{k-1} \mid \sum_{l=0}^{k-1} \sum_{i,j=0}^{n+k} E_{ij} t_l q_j = 0 \right\} \cong \mathbb{P}^{k-2}
\]

and for any \( q \in V \),

\[
p_2^{-1}(q) = \{(t, q) \in \mathbb{P}^{k-1} \times \mathbb{P}^{n+k} \mid q \in Q_t\} = \mathbb{P}^{k-1} \times \{q\}
\]

since \( V \) is the base locus of the family. Hence we have the following diagram, which we will use throughout this section:

\[
\begin{array}{c}
\mathbb{P}^{n+k} - V \\
p_2 \downarrow \\
\mathbb{P}^{n+k} \\
\end{array}
\quad \xrightarrow{i_E} \quad
\begin{array}{c}
\mathbb{P}^{k-1} \times \mathbb{P}^{n+k} \\
p_2 \downarrow \\
\mathbb{P}^{k-1} \\
\end{array}
\quad \xrightarrow{p_2} \quad
\begin{array}{c}
\mathbb{P}^{n+k} - V \\
p_2 \downarrow \\
\mathbb{P}^{n+k} \\
\end{array}
\]

**Lemma 2.1.**

\[ \dim H^0_0(V) = \dim H^{n+2k-2}_0(X) \]

**Proof.** From the diagram (2), we get

\[ \chi(X) = (k-1)(n+k+1) + \chi(V) \]

Since \( X \) is a very ample divisor in \( \mathbb{P}^{k-1} \times \mathbb{P}^{n+k} \) and \( V \) is a smooth complete intersection in \( \mathbb{P}^{n+k} \), by Lefschetz Theorem we have

\[ H^i(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) \cong H^i(X), \quad H^j(\mathbb{P}^{n+k}) \cong H^j(V) \]

for any \( i < \dim X = n + 2k - 2 \) and \( j < \dim V = n \).

(Case 1) If \( n \) is even : then we have

\[ \chi(V) = 2 \sum_{i=0}^{n-1} (-1)^i b_i(V) + b_n(V) = n + b_n(V) \]

\[ \chi(X) = 2 \sum_{i=0}^{n+2k-3} (-1)^i b_i(X) + b_{n+2k-2}(X) \]

\[ = 2 \left( \frac{k(k+1)}{2} + k \cdot \frac{n-2}{2} \right) + b_{n+2k-2}(X) \]

\[ = nk + k^2 - k + b_{n+2k-2}(X) \]

Therefore by (3) we get

\[ nk + k^2 - k + b_{n+2k-2}(X) = (k-1)(n+k+1) + n + b_n(V) \]
is well-defined. In the case when \( i \)
\( b_{n+2k-2}(X) = b_n(V) + k - 1 \)

Note that
\[
\dim H^{n+2k-2}_0(X) = b_{n+2k-2}(X) - \dim H^{n+2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) = b_{n+2k-2}(X) - k
\]
and
\[
\dim H^n_0(V) = b_n(V) - \dim H^n(\mathbb{P}^{n+k}) = b_n(V) - 1
\]
therefore we get
\[
\dim H^{n+2k-2}_0(X) = \dim H^n_0(V)
\]
in this case.

(Case 2) If \( n \) is odd : then by the similar calculation, we get
\[
\chi(X) = nk + k^2 - b_{n+2k-2}(X), \quad \chi(V) = (n + 1) - b_n(V)
\]
Then, again by (3)
\[
k + k^2 - b_{n+2k-2}(X) = (k - 1)(n + k + 1) + (n + 1) - b_n(V) = nk + k^2 - b_n(V)
\]
i.e.
\[
b_{n+2k-2}(X) = b_n(V)
\]
Hence, we get
\[
\dim H^{n+2k-2}_0(X) = \dim H^{n+2k-2}_0(X) = \dim H^n(V) = \dim H^n_0(V)
\]
in this case also, which finishes the proof of the Lemma. \( \Box \)

Set \( E = \mathbb{P}^{k-1} \times V \subset X \). Then we have a \( \mathbb{P}^{k-1} \)-bundle \( p_2 : E \to V \). Let \( i_E : E \hookrightarrow X \) be an inclusion. Then we have the following of morphism of Hodge structures
\[
\phi : H^n(V) \xrightarrow{p_2^*} H^n(E) \xrightarrow{i_E_*} H^{n+2k-2}(X)
\]
where \( i_{E_*} : H^n(E) \to H^{n+2k-2}(X) \) is the Gysin map. (Note that \( \text{codim}(E, X) = k - 1 \))

**Theorem 2.2.** \( \phi : H^n(V) \to H^{n+2k-2}(X) \) induces a morphism
\[
\tilde{\phi} : H^n_0(V)(-k + 1) \to H^{n+2k-2}_0(X)
\]
which is an isomorphism of rational Hodge structures.

**Proof.** First we prove that the induced morphism
\[
\tilde{\phi} : H^n_0(V) \to H^{n+2k-2}_0(X)
\]
is well-defined. In the case when \( n \) is odd, then \( \tilde{\phi} = \phi \), hence the morphism is well-defined. In case when \( n \) is even, it is enough to show that \( \phi(\text{im } i_E^* V) \subseteq \text{im } i_X^* \). Let \([E] \in H^{2k-2}(X)\) be the fundamental class of \( E \). Note that \( i_X^* : H^{2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) \to H^{2k-2}(X) \) is an isomorphism by Lefschetz theorem since \( 2k - 2 < \dim X = n + 2k - 2 \). Hence there is \( \gamma \in H^{2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) \) such that
\[
i_X^*(\gamma) = [E]. \text{ In fact, we can write } \gamma = [\mathbb{P}^{k-1} \times S] \text{ where } [S] \in H^{2k-2}(\mathbb{P}^{n+k}) \text{ such that}
\]
\[
i_X^*(\gamma) = [X \cap (\mathbb{P}^{k-1} \times S)] = [\mathbb{P}^{k-1} \times V]
\]
To show this, we consider the following commutative diagram:

\[
\begin{array}{ccc}
H^0(E) & \xrightarrow{(i_X \circ i_E)} & H^{2k-2}(X) \\
\downarrow_{i^*_X} & & \downarrow_{i^*_X} \\
H^{2k-2}(P^{k-1} \times \mathbb{P}^{n+k}) & \cong & \mathbb{P}^{n+k}
\end{array}
\]

Since \( \gamma \in H^{2k-2}(P^{k-1} \times \mathbb{P}^{n+k}) \), we may write \( \gamma = \sum_{i=0}^{k-1} [a_i H^i \times b_i L^{k-1-i}] \), where \( H^i \) (resp. \( L^j \)) is a linear space in \( \mathbb{P}^{k-1} \) (resp. \( \mathbb{P}^{n+k} \)) of codimension \( i \) (resp. \( j \)). By commutativity of the diagram, we have

\[
[\mathbb{P}^{k-1} \times V] = i_X [E] = i_X i^*_X (\gamma) = \gamma \cup [X] = \sum_{i=0}^{k-1} [a_i H^i \times b_i L^{k-1-i}] \cap [X]
\]

Hence \( a_i = 0 \) for \( i \neq 0 \), since \( H^i \not\subseteq \mathbb{P}^{k-1} \) for \( i \neq 0 \) and then this forces \( b_i = 0 \) for \( i \neq 0 \) since \( b_i L^{k-1-i} \notin H^{2k-2}(P^{k-1} \times \mathbb{P}^{n+k}) \) for \( i > 0 \). Hence

\[
\gamma = [\mathbb{P}^{k-1} \times b_0 L^{k-1}]
\]

We take \( S \) to be \( b_0 L^{k-1} \) and we will use this \( S \) later.

Now consider the following diagram:

\[
\begin{array}{ccccc}
H^n(V) & \xrightarrow{i_E} & H^n(E) & \xrightarrow{\phi} & H^{n+2k-2}(X) \\
\downarrow_{i^*_E} & \downarrow_{i^*_E} & \downarrow_{i^*_E} & \downarrow_{i^*_E} & \\
H^n(P^{n+k}) & \xrightarrow{pr_2} & H^n(P^{k-1} \times \mathbb{P}^{n+k}) & \xrightarrow{\Phi} & H^{n+2k-2}(P^{k-1} \times \mathbb{P}^{n+k})
\end{array}
\]

where

1. \( j_E : E \to P^{k-1} \times \mathbb{P}^{n+k} \), \( i_E : E \to X \) and \( i_X : X \to P^{k-1} \times \mathbb{P}^{n+k} \) are inclusions,
2. \( pr_2 : P^{k-1} \times \mathbb{P}^{n+k} \to \mathbb{P}^{n+k} \) is the projection to the second factor,
3. \( \Phi : H^n(P^{k-1} \times \mathbb{P}^{n+k}) \to H^{n+2k-2}(P^{k-1} \times \mathbb{P}^{n+k}) \) is defined by \( \Phi(\alpha) = \alpha \cup \gamma \).

Commutativity of (II) and (III) are clear and commutativity of (I) follows from the following commutative diagram:

\[
\begin{array}{ccc}
E = V \times \mathbb{P}^{n+k} & \xrightarrow{j_E} & P^{k-1} \times \mathbb{P}^{n+k} \\
\downarrow_{p_2} & \downarrow_{p_2} & \\
V & \xrightarrow{i_V} & \mathbb{P}^{n+k}
\end{array}
\]

We show the commutativity of (IV). Let \( \alpha \in H^n(P^{k-1} \times \mathbb{P}^{n+k}) \). Then

\[
i^*_X \circ \Phi(\alpha) = i^*_X (\alpha \cup \gamma) = i^*_X (\alpha) \cup i^*_X (\gamma) = i^*_X (\alpha) \cup [E]
\]

by the definition of \( \gamma \). Hence (IV) commutes. Therefore we have

\[
\phi \circ i^*_V = i^*_X \circ \Phi \circ pr_2^*
\]

and hence \( \phi(\text{im } i^*_V) \subseteq \text{im } i^*_X \) and \( \phi \) induces a well-defined morphism \( \tilde{\phi} : H^n_0(V) \to H^{n+2k-2}_0(X) \) in this case also.
To show that $\bar{\phi}$ is an isomorphism, note that we have

\[(6) \quad H^n(E) \cong (H^n(V) \otimes H^0(\mathbb{P}^{k-1})) \oplus \left(\bigoplus_{i=1}^{k-1} H^{n-2i}(\mathbb{P}^{n+k}) \otimes H^2(\mathbb{P}^{k-1})\right)\]

by the Künneth formula and Lefschetz theorem.

(Case 1) If $n$ is odd: in this case $\phi = \bar{\phi}$ and we show that $\phi = i_{E*} \circ p^*_2$ is surjective. Since $n$ is odd, $H^{n-2i}(\mathbb{P}^{k-1}) = 0$ for all $i$. Hence (6) gives an isomorphism

\[p^*_2 : H^n(V) \xrightarrow{\cong} H^n(E)\]

Now for the morphism $i_{E*} : H^n(E) \rightarrow H^{n+2k-2}(X)$, consider the following Gysin exact sequence

\[(7) \quad \cdots \rightarrow H^{n+2k-3}(X-E) \rightarrow H^n(E) \xrightarrow{i_{E*}} H^{n+2k-2}(X) \rightarrow H^{n+2k-2}(X-E) \rightarrow \cdots\]

Since $X - E \cong \mathbb{P}^{k-2} \times (\mathbb{P}^{n+k} - V)$,

\[H^{n+2k-2}(X-E) = H^{n+2k-2}(\mathbb{P}^{k-2} \times (\mathbb{P}^{n+k} - V)) = \bigoplus_{i=0}^{k-2} H^{(n+2k-3)-2i}(\mathbb{P}^{n+k} - V)(-i) = 0\]

Hence, $i_{E*} : H^n(E) \rightarrow H^{n+2k-2}(X)$ is surjective and hence we get the surjection $\phi = i_{E*} \circ p^*_2$. Now lemma 2.7 implies that $\phi = \phi$ is an isomorphism in this case.

(Case 2) If $n = 2l$ is even: then (6) gives that $p^*_2 : H^n(V) \rightarrow H^n(E)$ is an injection. Since $X - E \cong \mathbb{P}^{k-2} \times (\mathbb{P}^{n+k} - V)$, we get

\[H^{n+2k-3}(E) = H^{n+2k-3}(\mathbb{P}^{k-2} \times (\mathbb{P}^{n+k} - V)) = \bigoplus_{i=0}^{k-2} H^{(n+2k-3)-2i}(\mathbb{P}^{n+k} - V)(-i) = 0\]

Then Gysin exact sequence (6) implies that $i_{E*} : H^n(E) \rightarrow H^{n+2k-2}(X)$ is injective. Since $\phi = i_{E*} \circ p^*_2$, we get the injection $\phi = i_{E*} \circ p^*_2 : H^n(V) \rightarrow H^{n+2k-2}(X)$. We show that the induced map $\phi$ is also an injection in this case: Suppose $\bar{\phi}(\bar{\alpha}) = 0$. Then

\[\phi(\alpha) \in \text{im}[i_X : H^{n+2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) \rightarrow H^{n+2k-2}(X)]\]

where $\alpha \in H^n(V)$ which maps to $\bar{\alpha} \in H^n(E)$. Let $\beta \in H^{n+2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k})$ such that $i_X(\beta) = \phi(\alpha)$. We claim that $\beta$ is a cycle in $H^0(\mathbb{P}^{k-1}) \otimes H^{n+2k-2}(\mathbb{P}^{n+k})$.

By applying $i_{E*}$ to $\phi(\alpha) = i_X(\beta) \in H^{n+2k-2}(X)$, we get

\[(8) \quad i_{E*}(\phi(\alpha)) = i_{E*}i_{E*}p^*_2(\alpha) = p^*_2(\alpha) \cup c_{k-1}(N_{E/X}) = i_{E*}(i_X(\beta)) = (i_X \circ i_{E*})(\beta)\]

where the second equality comes from the self-intersection formula (\ref{1}, p103).

From the inclusions $E \xrightarrow{i_E} X \xrightarrow{i_X} \mathbb{P}^{k-1} \times \mathbb{P}^{n+k}$, we have

\[0 \rightarrow i_{E*}(N_{X/\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}}) \rightarrow N_{E/\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}} \rightarrow N_{E/X} \rightarrow 0\]
Hence
\[ c(N_{E/P^{k-1}X^+P^{n+k}}) = c(i_E^*(N_{X/P^{k-1}X^+P^{n+k}})) \cdot c(N_{E/X}) \]

In particular,
\[ c_k(N_{E/P^{k-1}X^+P^{n+k}}) = c_1(i_E^*(N_{X/P^{k-1}X^+P^{n+k}})) \cdot c_{k-1}(N_{E/X}) = i_E^*(\{X\}|_X) \cup E|_E \]

since \( X \) is a divisor in \( P^{k-1} \times P^{n+k} \), where \( \cup_E \) is the cup product on \( E \). Since the inclusion \( i_X \circ i_E : E \rightarrow P^{k-1} \times P^{n+k} \) is actually \( (id_{P^{k-1}}, i_V) : P^{k-1} \times V \rightarrow P^{k-1} \times P^{n+k} \), we have \( N_{E/P^{k-1}X^+P^{n+k}} = p_2^s(N_{V/P^{n+k}}) \) and hence
\[ c_k(N_{E/P^{k-1}X^+P^{n+k}}) = p_2^s(c_k(N_{V/P^{n+k}})) \in H^0(P^{k-1}) \otimes H^{2k}(V) \subset H^{2k}(E) \]

Now since \( X \) is an ample divisor in \( P^{k-1} \times P^{n+k} \), \([X] = [aH+bL] \in H^2(P^{k-1} \times P^{n+k}) \) where \( H \) (resp. \( L \)) is a hyperplane in \( P^{k-1} \) (resp. \( P^{n+k} \)) and \( a, b \geq 0 \) such that \( ab \neq 0 \). Now by using \([4]\),
\[ i_E([X]|_X) \cup E|_E = i_E^*(([X]|x) \cup X \cup X) = i_E^*([X]|_X \cup X \cup X) \]

Then by \([9, 10]\), we have \( a = 0 \) and hence \([X] = [bL] \in H^2(P^{k-1} \times P^{n+k}) \). Set \( P \) be a hypersurface of degree \( b \) in \( P^{n+k} \) such that \([X] = [P^{k-1} \times P]\). Since \( X \) contains \( E = P^{k-1} \times V \), we may assume that \( V \subset P \). Then \( N_{E/X} = p_2^s(N_{V/P}) \), and hence
\[ c_k(N_{E/X}) = p_2^s(c_{k-1}(N_{V/P})) \in H^0(P^{k-1}) \otimes H^{2k-2}(V) \]

Now, since \( p_2^s(\alpha) \in H^0(P^{k-1}) \otimes H^n(V) \), we have
\[ i_E^*(\alpha) = p_2^s(\alpha) \cup c_{k-1}(N_{E/X}) \in H^0(P^{k-1}) \otimes H^{n+2k-2}(V) \]

Then by \([8]\), we have
\[ i_E^*(\alpha) = (i_E \circ i_X)^*(\beta) = (id_{P^{k-1}}, i_V^*)(\beta) \in H^0(P^{k-1}) \otimes H^{n+2k-2}(V) \]

Hence \( \beta \in H^0(P^{k-1}) \otimes H^{n+2k-2}(P^{n+k}) \).

Next note that by cupping with \( \gamma = [P^{k-1} \times S] \), \( \Phi : H^n(P^{k-1} \times P^{n+k}) \rightarrow H^{n+2k-2}(P^{k-1} \times P^{n+k}) \) maps the Künneth component \( H^i(P^{k-1}) \otimes H^{n-i}(P^{n+k}) \) of \( H^n(P^{k-1} \times P^{n+k}) \) to the Künneth component \( H^{i+2k-2}(P^{k-1}) \otimes H^{n+2k-2-i}(P^{n+k}) \) of \( H^{n+2k-2}(P^{k-1} \times P^{n+k}) \) isomorphically. Hence we have \( \eta \in H^0(P^{k-1}) \otimes H^n(P^{n+k}) \) such that \( \eta \cup \gamma = \Phi(\eta) = \beta \). Now \( \eta \in \text{im}(p_2^s) : H^n(P^{n+k}) \rightarrow H^n(P^{k-1} \times P^{n+k}) \) and we may consider \( \eta \in H^n(P^{n+k}) \) since \( p_2^s \) is injective. Then by commutativity of diagram \([5]\), we have
\[ \phi \circ i_E^*(\eta) = i_X^* \circ \Phi \circ p_2^s(\eta) = i_X^*(\eta \cup \gamma) = i_X^*(\beta) = \phi(\alpha) \]

Since \( \phi \) is injective, we have \( \alpha = i_E^*(\eta) \). Hence \( \tilde{\phi} \) is also an injection. Now by lemma \([24]\) we can conclude that \( \phi : H^n(V) \rightarrow H^{n+2k-2}(X) \) is an isomorphism. \( \square \)
3. Proof of Theorem 1.2 when $n + k$ is odd

Throughout this section, we assume that $n + k = 2m + 1$.

In order to prove theorem 1.2 we use the construction of O’Grady [O]. We give a brief outline of his construction here. For detailed construction, see [O].

Recall the diagram (1) and consider the projection $p_1 : X \to \mathbb{P}^{k-1}$ and recall that $\Delta$ is the discriminant variety, which is a smooth hypersurface in $\mathbb{P}^{k-1}$ by our assumption. In case when $n + k$ is odd, for a general $t \in \mathbb{P}^{k-1}$, the fiber $p_1^{-1}(t) = Q_t$ is a smooth quadric of dimension $n + k - 1 = 2m$ in $\mathbb{P}^{n+k}$. Hence it contains two irreducible families of $m$–planes parametrized by $F_i^1$ and $F_i^2$. Note that $F_i^1 \cong F_i^2$ and $\dim F_i^1 = \frac{m(m+1)}{2}$ [GH]. Let $F$ be the abstract variety to which $F_i^1$ is isomorphic for $i = 1, 2$ and for any $t$. Let $W$ be a double covering of $\mathbb{P}^{k-1}$ branched over $\Delta$ and let $\sigma : W \to \mathbb{P}^{k-1}$ be the covering map. Set

$$P = \{ M \subset X \mid p_1(M) = t \in \mathbb{P}^{k-1} \text{ a point, } p_2(M) \cong \mathbb{P}^m \subset \mathbb{P}^{n+k} \}$$

Then there is a natural map $\psi : P \to \mathbb{P}^{k-1}$ defined by $\psi(M) = p_1(M) \in \mathbb{P}^{k-1}$. Then the Stein factorization of $\psi$ is factored through $W$ and get a composition

$$\psi : P \xrightarrow{f} W \xrightarrow{\sigma} \mathbb{P}^{k-1}$$

and set

$$\Gamma = \{ (M, x) \in P \times X \mid x \in M \subset X \} = \{ (M, x) \in P \times X \mid p_1(x) = p_1(M) \in \mathbb{P}^{k-1}, \ p_2(x) = p_2(M) \cong \mathbb{P}^m \}$$

Let $pr_1 : \Gamma \to P$ and $pr_2 : \Gamma \to X$ be the projections. We summarize the construction in the following diagram:

$$\xymatrix{ & \Gamma \ar[dl]_{pr_1} \ar[dr]^{pr_2} & \\
P \ar[dr]_{f} & X \ar[dl]^{p_1} & \\
W \ar[rr]_{\sigma} & & \mathbb{P}^{k-1}}$$

Note that for any $w \in W$,

$$f^{-1}(w) = \{ M \in X \mid p_1(M) = \sigma(w), p_2(M) \cong \mathbb{P}^m \subset \mathbb{P}^{n+k} \} \cong \{ \mathbb{P}^m \subset Q_{\sigma(w)} \} \cong F$$

and for any $M \in P$,

$$pr_1^{-1}(M) = \{ (M, x) \in P \times X \mid x \in M \subset X \} \cong \{ x \in X \mid p_1(x) = p_1(M), \ p_2(x) \cong \mathbb{P}^m \} = \mathbb{P}^m$$

Hence $f : P \to W$ is $F$–bundle and $pr_1 : \Gamma \to P$ is $\mathbb{P}^m$–bundle.

Now we give a proof of theorem 1.2 in case of $n + k$ odd.

Proof of theorem 1.2 when $n + k$ is odd. Consider the morphism of Hodge structures

$$pr_{2*} : H_{n+2k-2}(\Gamma) \to H_{n+2k-2}(X)$$

We show that $pr_{2*}$ is a surjection. First note that $pr_2 : \Gamma \to X$ is a surjection. Hence we can take an iterated hyperplane section $\Gamma_1$ of $\Gamma$ such that $\dim \Gamma_1 = \dim X = n + 2k - 2$ and $\Gamma_1$ surjects onto $X$. Let $g = pr_2|_{\Gamma_1} : \Gamma_1 \to X$. Then $g$ is a generically finite map. Let $j : \Gamma_1 \to \Gamma$ be an inclusion. Then we have the following commutative diagram:
where $P_{\Gamma_1}$ and $P_X$ are isomorphisms from Poincaré duality. Since $g_* = pr_{2*} \circ j_*$, it is enough to show that $g_*$ is surjective, but it is clear since $g_* \circ g^* = \deg g \cdot \operatorname{id}$.

Now since $pr_1 : \Gamma \to P$ is $\mathbb{P}^m$–bundle, by Künneth formula we have

$$H_{n+2k-2}(\Gamma) = \bigoplus_{j=0}^{m} (H_{n+2k-2-2j}(P) \otimes H_{2j}(\mathbb{P}^m)) = \bigoplus_{j=0}^{m} H_{n+2k-2-2j}(P)(j)$$

We claim that for each $j$, there is an isomorphism

$$H_{n+2k-2-2j}(P) = \bigoplus_{r+2s=n+2k-2-2j} (H_r(W) \otimes H_{2s}(F))$$

To show this, first note that we can use cohomology instead of homology since all our varieties considered are smooth. Let $U_W \subset W$ be a Zariski open set in $W$ such that $f^{-1}(U_W) \cong U_W \times F$. Since $F$ has a cellular decomposition \[E\], $H_*(F)$ are generated by algebraic cycles. Let $\alpha_1, \ldots, \alpha_l$ be the algebraic cycles generating $H_*(F)$, i.e. there are algebraic subvarieties $Z_1, \ldots, Z_l$ of $F$ such that the fundamental classes of them are $\alpha_1, \ldots, \alpha_l$. Let $p : U_W \times F \to F$ be the projection to the second factor and consider the algebraic cycles $p^*(\alpha_i)$ in $H^*(U_W \times F)$ which are supported on $U_W \times Z_i$ for $i = 1, 2, \ldots, l$. Let $\beta_i$ be the closures of $p^*(\alpha_i)$ in $P$ for each $i = 1, \ldots, l$.

Then this gives the splitting of the restriction map $H^r(P) \to H^r(F)$. So we may apply the Leray–Hirsch theorem \[S\]. Therefore we have

$$H^s(P) = \bigoplus_{r+2s=q} (H^r(W) \otimes H^{2s}(F))$$

for any $q$. In particular,

$$(11) \quad H^{2d-(n+2k-2-2j)}(P) = \bigoplus_{*} (H^{2d-(n+2k-2-2j)-2s}(W) \otimes H^{2s}(F))$$

where $d = \dim P$.

(Case 1) If $n$ is odd (and hence $k$ is even) : in this case, from (11) we have

$$H^{2d-(n+2k-2-2j)}(P) \cong H^{k-1}(W) \otimes H^{2d-(n+3k-3-2j)}(F)$$

for each $j$, since $W$ is simply connected, $H^r(W) = 0$ for all odd $r$ such that $r \neq \dim W = k - 1$, or equivalently

$$H_{n+2k-2-2j}(P) \cong H_{k-1}(W) \otimes H_{n+k-1-2j}(F) = H_{k-1}(W)(m-j)^{\otimes l},$$
where \( l_j = \dim H_{n+k-1-2j}(F) \). Therefore we have
\[
H_{n+2k-2}(\Gamma) = \bigoplus_{j=0}^{m} H_{n+2k-2-2j}(P)(j) \\
\cong \bigoplus_{j=0}^{m} \left( H_{k-1}(W)(m-j)^{\oplus l_j} \right)(j) = H_{k-1}(W)(m)^{\oplus N}
\]
where \( N = \sum_{j=0}^{m} l_j \). Thus, we have a surjection
\[
pr_{2*} : H_{k-1}(W)(m)^{\oplus N} \to H_{n+2k-2}(X)
\]
and by Poincaré duality, we get a surjection
\[
H^{k-1}(W)(-m)^{\oplus N} \to H^{n+2k-2}(X)
\]
By choosing \( \tilde{W} \) to be the disjoint union of \( N \) copies of \( W \) and by composing with \( \phi^{-1} \) in theorem 2.2 we get a surjection
\[
\Theta : H^{k-1}(\tilde{W})(-m + k - 1) \to H^n(V)
\]
as we claimed.

(Case 2) If \( n \) is even (and hence \( k \) is odd) : in this case, (11) gives
\[
H^{2d-(n+2k-2-2j)}(P) \cong \bigoplus_{s} \left( H^{2r}(W) \otimes H^{2d-(n+2k-2-2j)-2r}(F) \right)
\]
for each \( j \), or equivalently
\[
H_{n+2k-2-2j}(P) \cong \bigoplus_{s} \left( H_{2s}(W) \otimes H_{n+2k-2-2j-2s}(F) \right)
\]
Therefore we have
\[
H_{n+2k-2}(\Gamma) = \bigoplus_{j=0}^{m} H_{n+2k-2-2j}(P)(j) \\
\cong \bigoplus_{j=0}^{m} \bigoplus_{s} \left( H_{2s}(W) \otimes H_{n+2k-2-2j-2s}(F) \right)(j) \\
= \bigoplus_{s} H_{2s}(W) \otimes \left( \bigoplus_{j=0}^{m} H_{n+2k-2-2j-2s}(F) \right)(j) \\
= \bigoplus_{s} H_{2s}(W)(q_s)^{\oplus l_s}
\]
where \( q_s = \frac{n+2(k-1)-2s}{2} \) and \( l_s = \sum_{j=0}^{m} \dim H_{n+2k-2-2j-2s}(F) \). Thus, we have a surjection
\[
pr_{2*} : \bigoplus_{s} H_{2s}(W)(q_s)^{\oplus l_s} \to H_{n+2k-2}(X)
\]
Recall that \( H^{n+2k-2}(X) = H^{n+2k-2}(X)/\im i^*_X \). Hence we get a composition of surjections
\[
\bigoplus_{s} H^{2s}(W)(-q_s)^{\oplus l_s} \to H^{n+2k-2}(X) \to H^{n+2k-2}(X)
\]
Therefore, by composing with $\tilde{\phi}^{-1}$ from theorem 2.2 we get a surjection of Hodge structures

$$\Theta : \bigoplus_s H^{2s}(W)(-q_s + k - 1)^{\oplus s} \twoheadrightarrow H^+_0(V)$$

(Note that $q_s - k + 1 = \frac{n-2s}{2}$.) This completes the proof of theorem 1.2 in case when $n + k$ is odd.

4. Proof of Theorem 1.2 when $n + k$ is even

Throughout this section, we assume that $n + k = 2m$ is even.

Again we start by considering the projection $p_1 : X \to \mathbb{P}^{k-1}$ in diagram (1). Recall that the discriminant variety $\Delta$ is a smooth hypersurface of degree $n + k + 1$ in $\mathbb{P}^{k-1}$ by our assumption. Set $U = \mathbb{P}^{k-1} - \Delta$ and let $X_\Delta = p_1^{-1}(\Delta)$ and $X_U = p_1^{-1}(U)$.

$$\begin{array}{cccc}
X_\Delta & \hookrightarrow & X & \twoheadrightarrow & X_U \\
p_1 & \downarrow & p_1 & \downarrow & p_1 \\
\Delta & \longrightarrow & \mathbb{P}^{k-1} & \longrightarrow & U = \mathbb{P}^{k-1} - \Delta
\end{array}$$

(12)

Note that for any $t \in \Delta$, $p_1^{-1}(t)$ is a singular quadric in the family. Since we have assumed that $\Delta$ is smooth, a singular fiber is a cone through a 0-plane (i.e. a point) over a quadric of rank $n + k$ in $\mathbb{P}^{n+k-1}$, i.e. all singular fibers are cones over a smooth quadric $\tilde{Q}_t$ of dimension $n + k - 2$ in $\mathbb{P}^{n+k-1}$. For any $t \in \Delta$, we denote $p_1^{-1}(t) = C_t$ a cone with a vertex $0_t$. We can form a family of quadrics of dimension $n + k - 2$ over $\Delta$ as follows: Let $s : \Delta \to X_\Delta$ be a section of $p_1$ defined by $s(t) = 0_t$ for any $t \in \Delta$. Let $Y_0$ be a general hyperplane section of $X_\Delta - s(\Delta)$ and let $p_1|_{Y_0} : Y_0 \to \Delta$ be the obvious map. Let $Y$ be a smooth compactification of $Y_0$. Then by using theorem by Hironaka, we may assume that the rational map $Y \to \Delta$ is actually a morphism. We denote this morphism by $\pi : Y \to \Delta$. Then for a general $t \in \Delta$, $\pi^{-1}(t) = \tilde{Q}_t$ a smooth quadric of dimension $n + k - 2$ in $\mathbb{P}^{n+k-1}$.

$$\begin{array}{cccc}
Y & \longleftarrow & Y_0 & \longrightarrow & X_\Delta \\
\pi & \downarrow & p_1|_{Y_0} & \downarrow & p_1 \\
\Delta & \longrightarrow & \Delta & \longrightarrow & \mathbb{P}^{k-1}
\end{array}$$

Lemma 4.1. For any $p$

$$H^p(X_\Delta) \cong H^{p-2}(Y)(-1) \quad \text{as Hodge structures}$$

Proof. Consider the Leray spectral sequence associated to the map $p_1 : X_\Delta \to \Delta$

$$'E_2^{pq} = H^p(\Delta, R^q p_{1*} \mathbb{Q}) \Rightarrow H^{p+q}(X_\Delta, \mathbb{Q})$$

and one associated to the map $\pi : Y \to \Delta$

$$''E_2^{pq} = H^p(\Delta, R^q \pi_{*} \mathbb{Q}) \Rightarrow H^{p+q}(Y, \mathbb{Q})$$

Since for any $t \in \Delta$, $C_t$ is a cone through a point $0_t$ over a smooth quadric $\tilde{Q}_t$ in $\mathbb{P}^{n+k-1}$, we have

$$R^q p_{1*} \mathbb{Q} = R^q(p_1|_{Y_0})_{*} \mathbb{Q} = (R^{2(n+k-1)-q}(p_1|_{Y_0})_{*} \mathbb{Q})^*(-n-k+1)$$

$$= (R^{2(n+k-1)-q} \pi_{*} \mathbb{Q})^*(-n-k+1) = R^{q-2} \pi_{*} \mathbb{Q}(-1)$$
Hence we have
\[ E_2^{pq} = E_2^{p-q-2}(-1) \]
and so
\[ H^{p+q}(X_\Delta) \cong H^{p+q-2}(Y)(-1) \]
as Hodge structures. \( \square \)

In particular,
\[ H^{n+2k-2}(X_\Delta) \cong H^{n+2k-4}(Y)(-1) \]

**Lemma 4.2.** There is an injection of Hodge structures
\[ 0 \to H_0^{n+2k-2}(X) \to H^{n+2k-4}(Y)(-1) \]

*Proof.* From the top row of the diagram (12), we have an exact sequence of mixed Hodge structures
\[ \cdots \to H_c^{n+2k-2}(X_U) \to H^{n+2k-2}(X) \to H^{n+2k-2}(X_\Delta) \to \cdots \]

By using (13) and by taking the exact functor \( \text{Gr}^W_{n+2k-2} \), we get
\[ 0 \to \text{Gr}^W_{n+2k-2}H_c^{n+2k-2}(X_U) \to H^{n+2k-2}(X) \to H^{n+2k-4}(Y)(-1) \to \cdots \]

Consider the morphism \( p_1 : X_U \to U \) and the Leray spectral sequence associated to it
\[ E_2^{pq} = H_c^p(U, R^q p_1)_* \Rightarrow H_c^{p+q}(X_U, \mathbb{Q}) \]

Note that
\[ E_2^{pq} = \begin{cases} 0 & \text{if } q \text{ is odd} \\ H_c^p(U, \mathbb{Q}) & \text{if } q \text{ is even} \end{cases} \]
since \( (R^q p_1)_* = H^q(Q_t, \mathbb{Q}) \) and \( Q_t \) is a smooth quadric of dimension \( n+k-1 \text{(odd)} \). Also, we have
\[ 0 \to H_0^0(U) \to H^0(\mathbb{P}^{k-1}) \to H^0(\Delta) \to H_1^1(U) \to 0 \]
\[ 0 \to H^{2i-1}(\Delta) \to H^{2i}(U) \to H^{2i}(\mathbb{P}^{k-1}) \to H^{2i}(\Delta) \to H^{2i+1}(U) \to 0 \]
for \( 1 \leq i \leq k-2 \) and
\[ H^{2k-2}(U) \cong H^{2k-2}(\mathbb{P}^{k-1}) \]

Now, since \( \Delta \) is a smooth hypersurface in \( \mathbb{P}^{k-1} \), by Lefschetz theorem we get
\[ H^j(\mathbb{P}^{k-1}) \cong H^j(\Delta), \quad \text{for } j < \dim \Delta = k-2 \]

Hence \( H^{2i-1}(\Delta) = 0 \) and hence \( H^{2i}(U) = H^{2i+1}(U) = 0 \) for \( i \) such that \( 2i < \dim \Delta = k-2 \). By applying duality on \( H^{2i-1}(\Delta) \), we get \( H^j(U) = 0 \) unless \( j = k-1 \) or \( j = 2k-2 \). Hence the Leray spectral sequence degenerates at \( E_2 \) and we have a short exact sequence
\[ 0 \to E^{2k-2,n}_\infty \to H_c^{n+2k-2}(X_U) \to E_{\infty}^{k-1,n+k-1} \to 0 \]

Note that \( E_{\infty}^{k-1,n+k-1} = E_2^{k-1,n+k-1} = 0 \) since \( n+k \) is even. Hence we have
\[ E_{\infty}^{2k-2,n} \cong H_c^{n+2k-2}(X_U) \]

(Case 1) If \( n \) is odd: Then \( E^{2k-2,n}_{\infty} = 0 \) also and hence we have \( H_c^{n+2k-2}(X_U) = 0 \). So \( \text{(13)} \) gives an injection
\[ 0 \to H^{n+2k-2}(X) \to H^{n+2k-4}(Y)(-1) \]
Recall that Lemma 4.3. There are surjections of Hodge structures by Lefschetz theorem. Hence, we can rewrite the exact sequence (15) as follows:

\[ 0 \to H^{2k-2}(\mathbb{P}^{k-1}) \otimes H^n(Q_i) \cong H^{2k-2}(\mathbb{P}^{k-1}) \otimes H^n(\mathbb{P}^{n+k}) \to \cdots \]

Recall that \( H^{n+2k-2}(X) \cong H^{n+2k-2}(\mathcal{X})/\text{im} \imath_X \) and note that \( \text{im} h_\ast \cap H^{n+2k-2}(X) = \emptyset \). In fact, \( h_\ast \) can be factorized as

\[ H^{2k-2}(\mathbb{P}^{k-1}) \otimes H^n(\mathbb{P}^{n-k}) \xrightarrow{h_\ast} H^{n+2k-2}(\mathbb{P}^{k-1} \times \mathbb{P}^{n+k}) \xrightarrow{\imath_X^\ast} H^{n+2k-2}(\mathcal{X}) \]

Hence \( \text{im} h_\ast \subseteq \text{im} \imath_X^\ast \), and we get an injection

\[ 0 \to H^{n+2k-2}(X) \to H^{n+2k-4}(\mathcal{X})(-1) \]

Now we have a family \( \pi : Y \to \Delta \) of quadrics of dimension \( n + k - 2 \) which is even, so we can form a double covering of \( \Delta \) as in the construction of O’Grady ([O]). For a general \( t \in \Delta \), the fiber \( Q_t \), which is a smooth quadric of dimension \( n + k - 2 = 2(m - 1) \), contains two \( \frac{m(m-1)}{2} \)-dimensional irreducible families \( F_1^t, F_2^t \) of \( (m-1) \)-planes of \( \mathbb{P}^{n+k} \). As in the case of \( n + k \) odd, we can form the following diagram

\[
\begin{array}{ccc}
  & & \Gamma \downarrow \quad & & \\
 & P \xleftarrow{\psi} Y \xrightarrow{\pi} \mathbb{P}^{n+k-1} \quad & & \pi \\
 W \xrightarrow{\sigma} \Delta & & & & \\
  & & \downarrow \quad & & \\
 & & \end{array}
\]

where

1. \( P = \{ M \subset Y \mid \pi_1(M) = t \in \Delta \text{ a point, } \pi_2(M) \cong \mathbb{P}^{m-1} \subset \mathbb{P}^{n+k-1} \} \)
2. \( \Gamma = \{(M, y) \in P \times Y \mid \pi(y) = \pi(M) \in \Delta, \pi(y) \in \pi_2(M) \cong \mathbb{P}^{m-1} \subset \tilde{Q}_{\pi(M)} \} \), \( pr_1 \) and \( pr_2 \) are projections.
3. \( \psi : P \to \Delta \) is a natural map defined by \( \psi(M) = \pi(M) = t \in \Delta \)
4. \( f : P \to W \) is \( F \)-bundle and \( pr_1 : \Gamma \to P \) is \( \mathbb{P}^{m-1} \)-bundle, where \( F \) is the abstract variety such that \( F_i \) is isomorphic to it for \( i = 1, 2 \) and \( t \in \Delta \)
5. \( \sigma : W \to \Delta \) is a double covering branched over the discriminant variety \( \Delta_1 \) of the family \( \pi : Y \to \Delta \).

**Lemma 4.3.** There are surjections of Hodge structures

1. \( H^{k-2}(\tilde{W})(-m+1) \to H^{n+2k-4}(\mathcal{X}) \) if \( n \) is odd
   where \( \tilde{W} \) is a disjoint union of finitely many copies of \( W \);
2. \( \bigoplus_r H^{2r}(W)(-q_r)^{\oplus l_r} \to H^{n+2k-4}(\mathcal{X}) \) if \( n \) is even
   where \( q_r = \frac{n+2k-4-2r}{2} \) and \( l_r = \sum_j \dim H_{n+2k-4-2j-2r}(F) \).
Proof. First note that by using the same arguments in the proof of theorem [1,2] in the case of \( n + k \) odd, we can show that

\[ pr_{2*} : H_{n+2k-4}(\Gamma) \to H_{n+2k-4}(Y) \]

is surjective and

\[
H_{n+2k-4}(\Gamma) = \bigoplus_{j=0}^{m-1} (H_{n+2k-4-2j}(P) \otimes H_{2j}(\mathbb{P}^{m-1}))
\]

\[
= \bigoplus_{j=0}^{m-1} H_{n+2k-4-2j}(P)(j)
\]

\[
= \bigoplus_{j=0}^{m-1} \bigoplus_{s} (H_{n+2k-4-2j-2s}(W) \otimes H_{2s}(F))(j)
\]

(Case 1) If \( n \) is odd: In this case, \( n + 2k - 4 - 2j - 2s \) is odd. But \( H_{r}(W) \) is zero for an odd number \( r \) unless \( r = k - 2 = \dim W \). Hence we have

\[
H_{n+2k-4-2j}(P) = H_{k-2}(W) \otimes H_{n+k-2-2j}(F) = H_{k-2}(W)(m - 1 - j)^{\oplus l_j}
\]

where \( l_j = \dim H_{n+k-2-2j}(F) \). Thus we have

\[
H_{n+2k-4}(\Gamma) = \bigoplus_{j=0}^{m-1} H_{k-2}(W)(m - 1 - j)^{\oplus l_j}(j)
\]

\[
= \bigoplus_{j=0}^{m-1} H_{k-2}(W)(m - 1)^{\oplus N}
\]

where \( N = \sum_{j=0}^{m} l_j \). Hence by (17) and using Poincaré duality, we get a surjection

\[ pr_{2*} : H^{k-2}(W)(-m + 1)^{\oplus N} \to H^{n+k-4}(Y) \]

By setting \( \tilde{W} \) to be a disjoint union of \( N \) copies of \( W \), we get the desired surjection in this case.

(Case 2) If \( n \) is even: In this case, \( n + 2k - 4 - 2j \) is even. Hence we have

\[
H_{n+2k-4-2j}(P) = \bigoplus_{s} (H_{2s}(W) \otimes H_{n+2k-4-2j-2s}(F))
\]

Thus we have

\[
H_{n+2k-4}(\Gamma) = \bigoplus_{j=0}^{m-1} H_{n+2k-4-2j}(P)(j)
\]

\[
= \bigoplus_{j=0}^{m-1} \left( \bigoplus_{s} (H_{2s}(W) \otimes H_{n+2k-4-2j-2s}(F)) \right)(j)
\]

\[
= \bigoplus_{s} H_{2s}(W) \otimes \left( \bigoplus_{j=0}^{m-1} H_{n+2k-4-2j-2s}(F) \right)(j)
\]

\[
= \bigoplus_{s} H_{2s}(W)(q_s)^{\oplus l_s}
\]
where \( q_s = \frac{n+2k-1-2s}{2} \) and \( l_s = \sum_{j=0}^{m-1} \dim H_{n+2k-4-2j-2s}(F) \). Hence by (17) and using Poincaré duality, we get a surjection
\[
p_{2*} : \bigoplus_r H^{2r}(W)(-q_r) \rightarrow H^{n+2k-4}(Y)
\]
in this case.

We can summarize lemmas 4.2 and 4.3 in the following diagram:
\[
\begin{array}{ccc}
H^*(W) & \xrightarrow{\psi_1} & H^{n+2k-4}(Y)(-1) \\
\downarrow{\theta} & & \downarrow{\psi_2} \\
0 \rightarrow H_0^{n+2k-2}(X) & \xrightarrow{\psi_2} & Gr_{n+2k-2}H_{n+2k-1}(X_U) \rightarrow 0
\end{array}
\]
where with the notation in theorem 4.3
\[
H^*(W) = \begin{cases} 
H^{k-2}(\Delta, R^{n+k-2} \pi_s \mathbb{Q}) & \text{if } n \text{ is odd} \\
H^{k-2}(\Delta, R^{n+k-2} \pi_s \mathbb{Q}) \oplus \bigoplus_{p=0, p \neq \frac{k}{2}}^{n-2} (\alpha_p \otimes \beta_p) \mathbb{Q} & \text{if } n \text{ is even}
\end{cases}
\]
and \( \theta : H^*(W) \rightarrow H^{n+2k-4}(Y)(-1) \) is the surjection in lemma 4.3.

To finish the proof of theorem 1.2 in this case, we need to lift the surjection \( \theta \) to a surjection onto \( H_0^{n+2k-2}(X) \). In order to do that, we construct a section \( s_U \) of \( \psi_2 \) in geometric way. Following two lemmas will lead us the desired section \( s_U \).

**Lemma 4.4.** There is a decomposition
\[
H^{n+2k-4}(Y) \cong \begin{cases} 
H^{k-2}(\Delta, R^{n+k-2} \pi_s \mathbb{Q}) & \text{if } n \text{ is odd} \\
H^{k-2}(\Delta, R^{n+k-2} \pi_s \mathbb{Q}) \oplus \bigoplus_{p=0, p \neq \frac{k}{2}}^{n-2} (\alpha_p \otimes \beta_p) \mathbb{Q} & \text{if } n \text{ is even}
\end{cases}
\]
where \( \alpha_p \) (resp. \( \beta_p \)) is an algebraic cycle generating \( H^{2p}(\Delta) \) (resp. \( H^{n+2k-4-2p}(\tilde{\pi}_t) \)).

**Proof.** Consider the Leray spectral sequence associated to the map \( \pi : Y \rightarrow \Delta \).
\[
E_2^{pq} = H^p(\Delta, R^q \pi_* \mathbb{Q}) \Rightarrow H^{p+q}(Y, \mathbb{Q})
\]
Note that
\[
E_2^{pq} = \begin{cases} 
0 & \text{if either } q \text{ is odd or } p \text{ is odd, } p = k - 2 \\
H^p(\Delta, \mathbb{Q}) & \text{if } p \text{ is even and } q \text{ is odd, } q \neq n + k - 2 \\
H^p(\Delta, R^{n+k-2} \pi_s \mathbb{Q}) & \text{if } p \text{ is even and } q = n + k - 2
\end{cases}
\]
where \( (R^{n+k} \pi_s \mathbb{Q})_t = \mathbb{Q}^2 \) for \( t \in \Delta \). Then the spectral sequence degenerates at \( E_2 \).

Let \( F^* \) be the filtration on \( H^{n+2k-4}(Y) \) obtained from the spectral sequence. Then for each \( p = 0, 1, ..., 2(k-2) \) we have a short exact sequence
\[
0 \rightarrow F^{p+1} \rightarrow F^p \rightarrow Gr_F^p H^{n+2k-4}(Y) \rightarrow 0
\]
where
\[
Gr_F^p H^{n+2k-4}(Y) = E_\infty^{p,n+2k-4-p} = H^p(\Delta, R^{n+2k-4-p} \pi_s \mathbb{Q})
\]
(Case 1) If \( n \) is odd (and hence \( k \) is odd): then \( n + 2k - 4 \) is odd. Hence \( E_2^{pq} = 0 \) unless \( (p, q) = (k - 2, n + k - 2) \). Hence we have
\[
H^{n+2k-4}(Y) \cong H^{k-2}(\Delta, R^{n+k-2} \pi_s \mathbb{Q})
\]
(Case 2) If $n$ is even (and hence $k$ is even): then $n + 2k - 4$ is also even. Note $F^{2p-1} = F^{2p}$ from \([19]\). Hence we may write above sequence as

\[
0 \to F^{2p+2} \to F^{2p} \xrightarrow{t_2} \text{Gr}_F^{2p} H^{n+2k-4}(Y) \to 0
\]

First by using descending induction on $p$, we show that for $p$ such that $k - 2 < 2p \leq 2(k - 2)$ there is a natural splitting of exact sequences \([22]\) such that

\[
F^{2p} \cong F^{2p+2} \oplus \text{Gr}_F^{2p} H^{n+2k-4}(Y) \cong \bigoplus_{l=p+1}^{k-2} (\alpha_l \otimes \beta_l) \mathbb{Q}
\]

where $\alpha_l$ is an algebraic cycle generating $H^{2l}(\Delta) \cong \mathbb{Q}$ and $\beta_l$ is an algebraic cycle generating $H^{n+2k-4-2l}(\tilde{Q}_l)(-1) \cong \mathbb{Q}$.

If $2p = 2(k - 2)$, then

\[
F^{2(k-2)} \cong \text{Gr}_F^{2(k-2)} H^{n+2k-4}(Y) = E_{2(k-2),n}^2 = H^{2(k-2)}(\Delta) \otimes H^n(\tilde{Q}_t) \cong (\alpha_{k-2} \otimes \beta_{k-2}) \mathbb{Q}
\]

where $\alpha_{k-2}$ (resp. $\beta_{k-2}$) is an algebraic cycle generating $H^{2(k-2)}(\Delta) = \mathbb{Q}$ (resp. $H^n(\tilde{Q}_t) = \mathbb{Q}$). Now suppose $k - 2 < 2p < 2(k - 2)$ and consider the short exact sequence \([22]\). Since $2p > k - 2$, $\dim H^{2p}(\Delta) = \dim H^{n+2k-4-2p}(\tilde{Q}_t) = 1$, hence we can choose an algebraic cycle $\alpha_p$ (resp. $\beta_p$) which generates $H^{2p}(\Delta)$ (resp. $H^{n+2k-4-2p}(\tilde{Q}_t)$) and hence

\[
\text{Gr}_F^{2p} H^{n+2k-4}(Y) = H^{2p}(\Delta, R^{n+2k-4-2p} \pi_+ \mathbb{Q}) = H^{2p}(\Delta) \otimes H^{n+2k-4-2p}(\tilde{Q}_t) = (\alpha_p \otimes \beta_p) \mathbb{Q}
\]

and by induction hypothesis $F^{2p+2}$ has a decomposition

\[
F^{2p+2} \cong \bigoplus_{l=p+1}^{k-2} (\alpha_l \otimes \beta_l) \mathbb{Q}
\]

with a basis $\mathcal{B}_{p+1} = \{\alpha_{p+1} \otimes \beta_{p+1}, \ldots, \alpha_{k-2} \otimes \beta_{k-2}\}$, where $\alpha_l$ and $\beta_l$ are algebraic cycles. Let $\mathcal{B}_p = \{v_p, \alpha_p \otimes \beta_p, \ldots, \alpha_{k-2} \otimes \beta_{k-2}\}$ be a basis of $F^{2p}$ extending the basis $\mathcal{B}_{p+1}$. Then $t_2(v_p) = q_p(\alpha_p \otimes \beta_p)$ for some nonzero $q_p \in \mathbb{Q}$. Define

\[
s_p : \text{Gr}_F^{2p} H^{n+2k-4}(Y) \to F^{2p}
\]

by $s_p(\alpha_p \otimes \beta_p) = \frac{1}{q_p}v_p$. Then $s_p$ is a section of $t_p$ and hence we have a decomposition

\[
F^{2p} \cong F^{2p+2} \oplus \text{Gr}_F^{2p} H^{n+2k-4}(Y) = \bigoplus_{l=p+1}^{k-2} (\alpha_l \otimes \beta_l) \mathbb{Q} \oplus s_p(\alpha_p \otimes \beta_p) \mathbb{Q}
\]

with a basis $\mathcal{B}_p = \{\alpha_p \otimes \beta_p, \alpha_{p+1} \otimes \beta_{p+1}, \ldots, \alpha_{k-2} \otimes \beta_{k-2}\}$ of $F^{2p}$ by identifying $\alpha_p \otimes \beta_p$ with its image under $s_p$.

Now for splitting for $2p < k - 2$ in the exact sequence \([22]\), the argument is same as the case when $2p > k - 2$, since $\text{Gr}_F^{2p} H^{n+2k-4}(Y) = H^{2p}(\Delta) \otimes H^{n+2k-4-2p}(\tilde{Q}_t) = \mathbb{Q}$. Hence we can choose a natural splitting of each exact sequence \([22]\) and get a decomposition

\[
H^{n+2k-4}(Y) \cong H^{k-2}(\Delta, R^{n+k-2} \pi_+ \mathbb{Q}) \oplus \bigoplus_{p=0, p \neq \frac{k-2}{2}}^{k-2} (\alpha_p \otimes \beta_p) \mathbb{Q}
\]

\[\square\]
Lemma 4.5. Let $L$ be the subspace of $H^{n+2k-4}(Y)$ generated by $\{\alpha_p \otimes \beta_p \mid p = 0, \ldots, \frac{k-2}{2}, \ldots, k-2\}$. Then,

$$H^{k-2}(\Delta, R^{n+k-2}p_*\mathbb{Q}) \cong L^\perp$$

where $\perp$ is the orthogonal complement with respect to the cup product on $H^{n+2k-4}(Y)$.

Proof. First we show that

$$(\alpha_i \otimes \beta_i) \cup (\alpha_j \otimes \beta_j) = \begin{cases} 0 & \text{if } i + j \neq k-2 \\ \deg \Delta & \text{if } i + j = k-2 \end{cases}$$

Since

$$H^{2i}(\Delta, R^{n+2k-4-2i}p_*\mathbb{Q}) \otimes H^{2j}(\Delta, R^{n+2k-4-2j}p_*\mathbb{Q}) \xrightarrow{\cup} H^{2(i+j)}(\Delta, R^{2n+4k-8-2(i+j)}p_*\mathbb{Q})$$

$$(\alpha_i \otimes \beta_i) \cup (\alpha_j \otimes \beta_j) = 0 \text{ for } i + j \neq k-2. \text{ For } i + j = k-2, \text{ we observe } \alpha_i, \beta_i \text{ closely. Note that for any } i \neq \frac{k-2}{2}, \alpha_i = h^i_\Delta \text{ where } h^i_\Delta \in H^{2i}(\Delta) \text{ is a class corresponding to an iterated hyperplane section of } \Delta \text{ of codimension } i. \text{ For } \beta_i, \text{ let } \Pi_i \in H^{n+k-2}(\tilde{Q}_t) \text{ be a class corresponding to } \mathbb{P}^{n+k-2-\frac{4}{2}} \subset \tilde{Q}_t \text{ and } H^r(\tilde{Q}_t) \text{ a class corresponding to an iterated hyperplane section of } \tilde{Q}_t \text{ of codimension } r. \text{ Note that } \Pi \text{ can be chosen in either families } F^i_t \text{ or } F^r_t \text{ of } \mathbb{P}^{n+k-2-\frac{4}{2}} \text{ contained in } \tilde{Q}_t(\mathbb{R}). \text{ Then,}$$

$$\beta_i = \begin{cases} H^{\frac{n+k-4-2i}{2}} & \text{if } i > \frac{k-2}{2} \\ \Pi \cup H^{k-2-i} & \text{if } i < \frac{k-2}{2} \end{cases}$$

Then for $i + j = k + 2$,

$$(\alpha_i \otimes \beta_i) \cup (\alpha_j \otimes \beta_j) = \deg \Delta$$

Now we show $L \cap L^\perp = 0$. Let $\eta = \sum_{i=0, i \neq k-2}^{k-2} c_i(\alpha_i \otimes \beta_i) \in L \cap L^\perp$. Then, we have $c_i \cdot \deg \Delta = 0$ for any $i \neq \frac{k-2}{2}$. Hence $\eta = 0$. Hence

$$H^{n+2k-4}(Y) = L \oplus H^{k-2}(\Delta, R^{n+k-2}p_*\mathbb{Q}) = L \oplus L^\perp$$

Hence we get

$$H^{k-2}(\Delta, R^{n+k-2}p_*\mathbb{Q}) = L^\perp$$

\[\blacksquare\]

From the above lemmas, we have an injection

$$h : H^{k-2}(\Delta, R^{n+k}p_*\mathbb{Q}) \hookrightarrow H^{n+2k-2}(X_\Delta)$$

We refer the diagram (18) for the following lemma.

Lemma 4.6. There is a section defined in geometric way

$$s_U : Gr^W_{n+2k-2}H^{n+2k-1}_c(X_U) \to H^{n+2k-4}(Y)(-1)$$

of $\psi_2$.

Proof. First recall $H^{n+2k-4}(Y)(-1) \cong H^{n+2k-2}(X_\Delta)$ and $\psi_2$ is the connecting homomorphism in the exact sequence of mixed Hodge structures (14)

$$\cdots \to H^{n+2k-2}(X) \to H^{n+2k-2}(X_\Delta) \xrightarrow{\psi_2} Gr^W_{n+2k-2}H^{n+2k-1}_c(X_U) \to 0$$
As in the proof of lemma 4.2, we can show that the the Leray spectral sequence associated to \( p_1 : X_U \to U \) gives rise to an exact sequence

\[
0 \to E^{2k-2,n+1}_\infty \to H_c^{n+2k-1}(X_U) \to E^{2k-1,n+k}_\infty \to 0
\]

By taking the exact functor \( \text{Gr}^W_{n+2k-2} \), we get an isomorphism

\[
l : \text{Gr}^W_{n+2k-2} H_c^{n+2k-1}(X_U) \xrightarrow{\cong} \text{Gr}^W_{k-2} H_c^{k-1}(U)
\]

since \( E^{2k-2,n+1}_\infty = H_c^{2k-2}(U)(-\frac{n+1}{2}) \cong H_c^{2k-2}(\mathbb{P}^{k-1})(-\frac{n+1}{2}) \) is a pure Hodge structure of weight \( n + 2k - 2 \), if \( n \) is odd and 0 if \( n \) is even. Hence we have a following commutative diagram:

\[
\begin{array}{ccc}
H^{n+2k-2}(X_\Delta) & \xrightarrow{\psi_2} & \text{Gr}^W_{n+2k-2} H^{n+2k-1}(X_U) \\
\downarrow h & & \downarrow l \cong \\
H^{k-2}(\Delta, R^{n+k}p_1_* \mathbb{Q}) & \xrightarrow{\delta} & \text{Gr}^W_{k-2} H^{k-1}_c(U)(-\frac{n+k}{2})
\end{array}
\]

where \( h \) is the injection of (2.5) and \( \delta : H^{k-2}(\Delta, R^{n+k}p_1_* \mathbb{Q}) \to \text{Gr}^W_{k-2} H^{k-1}_c(U)(-\frac{n+k}{2}) \) is the connecting homomorphism of long exact sequence induced by a short exact sequence of sheaves on \( \mathbb{P}^{k-1} \)

\[
0 \to j(R^{n+k}p_1 \mathbb{Q})|_U \to R^{n+k}p_1 \mathbb{Q} \to i_\Delta_* (R^{n+k}p_1 \mathbb{Q})|_\Delta \to 0
\]

where \( j : U \to \mathbb{P}^{k-1} \) and \( i_\Delta : \Delta \to \mathbb{P}^{k-1} \) are inclusions. Hence in order to choose a section of \( \psi_2 \), it is enough to construct a section of \( \delta \).

Note that \( (R^{n+k}p_1 \mathbb{Q})_t = \mathbb{Q}^2 \) for \( t \in \Delta \) and \( R^{n+k}p_1 \mathbb{Q}|_U = \mathbb{Q}_U \) is the constant sheaf. Let \( G \) be the monodromy group of \( R^{n+k}p_1 \mathbb{Q}|_\Delta \). Then \( R^{n+k}p_1 \mathbb{Q}|_\Delta = \mathbb{Q} \) is a subsheaf of \( R^{n+k}p_1 \mathbb{Q}|_\Delta \) and let \( R^{n+k}p_1 \mathbb{Q}|_\Delta \hookrightarrow R^{n+k}p_1 \mathbb{Q}|_\Delta \) be an injection. Let

\[
s : i_\Delta_* (R^{n+k}p_1 \mathbb{Q}|_\Delta) \to i_\Delta_* (R^{n+k}p_1 \mathbb{Q}|_\Delta)
\]

be the induced map. Then there is a sheaf \( s^*(R^{n+k}p_1 \mathbb{Q}) \) on \( \mathbb{P}^{k-1} \) which fits into the following commutative diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{j} & j(R^{n+k}p_1 \mathbb{Q}|_U) \\
& & \downarrow s^* \\
0 & \xrightarrow{j} & R^{n+k}p_1 \mathbb{Q} \\
& & \downarrow i_\Delta_*
\end{array}
\]

From this, we get a commutative diagram

\[
\begin{array}{ccc}
\cdots H^{k-2}(\mathbb{P}^{k-1}, s^*(R^{n+k}p_1 \mathbb{Q})) & \xrightarrow{\delta} & H^{k-2}(\Delta, \mathbb{Q}) \\
& \downarrow s & \downarrow \delta_1 \cong \text{Gr}^W_{k-2} H^{k-1}_c(U, \mathbb{Q}) \to 0 \\
\cdots H^{k-2}(\mathbb{P}^{k-1}, R^{n+k}p_1 \mathbb{Q}) & \xrightarrow{\delta} & H^{k-2}(\Delta, R^{n+k}(p_1|_\Delta)_* \mathbb{Q}) \\
& \downarrow s & \downarrow \delta_1 \cong \text{Gr}^W_{k-2} H^{k-1}_c(U) \to 0
\end{array}
\]

i.e.

\[
\delta_1 = \delta \circ s
\]

Now from the exact sequence

\[
\cdots \to H^{k-2}(\mathbb{P}^{k-1}, s^*) \xrightarrow{\delta} H^{k-2}(\Delta) \xrightarrow{\delta_1} \text{Gr}^W_{k-2} H^{k-1}_c(U) \to 0
\]
we have an isomorphism
\[
\text{Gr}^W_{k-2} H^{k-1}_c(U) \xrightarrow{\cong} H^{k-2}(\Delta) / \text{im } i_{\Delta}^*
\]
hence we can choose a natural section \( s_1 : \text{Gr}^W_{k-2} H^{k-1}_c(U) \to H^{k-2}(\Delta) \) of \( \delta_1 : H^{k-2}(\Delta) \to \text{Gr}^W_{k-2} H^{k-1}_c(U) \), i.e. \( \delta_1 \circ s_1 = \text{id} \). Then from (27) we get a section \( s \circ s_1 \) of \( \delta \). By combining all these, we get a commutative diagram:

\[
\begin{array}{ccc}
H^{n+2k-2}(X_\Delta) & \xrightarrow{\psi_2} & \text{Gr}^W_{n+2k-2} H^{n+2k-1}(X_U) \\
\downarrow h & & \downarrow \cong \\
H^{k-2}(\Delta, R^{n+k} p_{1*} \mathbb{Q}) & \xrightarrow{\delta} & \text{Gr}^W_{k-2} H^{k-1}_c(U)(-\frac{n+k}{2}) \\
\downarrow s_1 & & \downarrow s_1 \\
H^{k-2}(\Delta) & \xrightarrow{\delta_1} & \text{Gr}^W_{k-2} H^{k-1}_c(U)(-\frac{n+k}{2})
\end{array}
\]

Set
\( s_U = h \circ s \circ s_1 \circ l \)
then
\( \psi_2 \circ s_U = \psi_2 \circ (h \circ s \circ s_1 \circ l) = l^{-1} \circ (\delta \circ s \circ s_1) \circ l = l^{-1} \circ l = \text{id} \)

Hence \( s_U \) is a section of \( \psi_2 \).

\( \square \)

**Proof of theorem 1.2 when \( n + k \) is even.** Now we finish the proof of the Theorem 1.2 in case when \( n + k \) is even. Recall the diagram (18)

\[
\begin{array}{ccc}
0 \to H^{n+2k-2}_0(X) & \xrightarrow{\psi_1} & H^{n+2k-4}(Y)(-1) & \xrightarrow{\psi_2} & \text{Gr}^W_{n+2k-2} H^{n+2k-1}(X_U) \to \cdots \\
\downarrow \theta & & \downarrow \psi_2 \circ \theta \\
H^*(W) & \xrightarrow{\psi_2 \circ \theta} & \cdots
\end{array}
\]

where \( s_U \) is the section of \( \psi_2 \) constructed in lemma 1.6

Let
\( \Theta = \theta - s_U \circ \psi_2 \circ \theta : H^*(W) \to H^{n+2k-4}(Y)(-1) \)

First note that \( \psi_2 \circ \Theta = 0 \): In fact,
\[
\begin{align*}
\psi_2 \circ \Theta &= \psi_2 \circ (\theta - s_U \circ \psi_2 \circ \theta) \\
&= \psi_2 \circ \theta - \psi_2 \circ s_U \circ \psi_2 \circ \theta = \psi_2 \circ \theta - \psi_2 \circ \theta = 0
\end{align*}
\]

Hence \( \text{im } \Theta \subseteq \ker \psi_2 \) and we may consider that \( \Theta \) is mapped into \( H^{n+2k-2}_0(X) \) since \( \psi_1 \) is an injection.
Now we show this map is in fact a surjection. Let $\alpha \in H^{n+2k-2}_0(X)$. Since $\psi_1$ is an injection we may identify $\alpha$ with $\psi_1(\alpha)$. Then there is $\beta \in H^*(W)$ such that 
\[
\theta(\beta) = \psi_1(\alpha).
\]
Then
\[
\Theta(\beta) = (\theta - s_U \circ \psi_2 \circ \theta)(\beta) = \theta(\beta) - s_U \circ \psi_2 \circ \theta(\beta) = \psi_1(\alpha) - s_U \circ \psi_2 \circ \psi_1(\alpha) = \psi_1(\alpha) = \alpha
\]
Hence we have a surjection
\[
\Theta : H^*(W) \longrightarrow H^{n+2k-2}_0(X)
\]
as we claimed. This finishes the proof of theorem 1.2.

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