D-dimensional Klein-Gordon equation with spatially q-deformed of radial momentum for Kratzer potential analyzed by using hypergeometric method

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Abstract. Hypergeometric method can be used to solve D-dimensional Klein-Gordon equation with q-deformed of radial momentum for Kratzer potential. By substituting q-deformed of radial momentum variable, the D-dimensional and the potential parameters to obtain the second order of D-dimensional Klein-Gordon equation which is used to determine the energy and wave function equations. The analytical result of energy can be founded with D-dimension and q-deformed of radial momentum variations. As a result, the rise of D-dimension values cause the rise of energy values, while the rice of q deformation values causes the decline of energy values.

1. Introduction
In quantum mechanics, the motion of a particle in strong potential field is showing relativistic effect [1]. The characteristic of the zero spin particles, mesons, are explained by using Klein-Gordon equation [2,3]. Non-trivial nonrelativistic limit appears when $S(r) = V(r)$ [3,4,5] and potential function become $2V(r)$, not $V(r)$, then by changing potential scale in radial part $V(r)$ as $2V(r)$ [12].

The Klein-Gordon equation can be solved by a lot of methods, such as Supersymmetry Quantum Mechanics (SUSY) [2], Asymptotic Iteration Method (AIM) [1], Nikiforov-Uvarov (NU) [6], Fabrication method [7] and Hypergeometric. The Hypergeometric method has solution that the second-order differential equation of Hypergeometric function can be obtained with substitution new variable [8].

Some potentials are applied by using Hypergeometric method, specialty for shape invariance potentials, like Kratzer, Gendenhstein, Morse, Pöschl-Teller, Rosen Morse, Scarf, Manning Rosen, Eckart, Wood-Saxon and Symmetrical Top potentials [9,10]. Kratzer potential is used to explain the molecule structure and molecule interaction with each other [11].

The aim of this research are to obtain and to calculate energy and wave function for D-dimensional Klein-Gordon equation with q-deformed of radial momentum for Kratzer potential using Matlab R2013 Software.

This paper covers several sections, they are D-dimensional Klein-Gordon equation in section 1, Hypergeometric method in section 2, result and discussion in section 3, and conclusion in section 4.
2. Numerical Methods

2.1. D-dimensional Klein-Gordon Equation

D-dimensional Klein-Gordon equation with \( M \) is mass of particle, scalar \( S(r) \) and vector \( V(r) \) potentials can be expressed by [12]:

\[
\left\{ \nabla_2^2 + \frac{1}{\hbar^2 c^2} \left[ (E - V(r))^2 - (Mc^2 + S(r))^2 \right] \right\} \psi_{\ell_{1}, \ell_{2}, \ldots, \ell_{d-2}}^{(\ell_{1}, \ell_{2}, \ldots, \ell_{d-2})}(x) = 0
\]

where \( \nabla_2^2 \) is Laplacian operator, \( \hbar \) is Planck's constant, \( c \) is speed of light and \( E \) is relativistic energy.

D-dimensional Klein-Gordon in natural unit \( (\hbar = 1, c = 1) \) is [13,14]

\[
\left\{ \nabla_2^2 + (V(r) - E)^2 - (S(r) + M)^2 \right\} \psi_{\ell_{1}, \ell_{2}, \ldots, \ell_{d-2}}^{(\ell_{1}, \ell_{2}, \ldots, \ell_{d-2})}(x) = 0
\]

D-dimensional Laplacian is [14]

\[
\nabla_2^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \sum_{j=1}^{D} \sin^2 \theta_j \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{d-1} \times \left\{ \frac{1}{\sin \theta_j} \left( \frac{\partial}{\partial \theta_j} \sin \theta_j \frac{\partial}{\partial \theta_j} \right) \right\}
\]

for central potential, equation (3) is rewritten as

\[
\nabla_2^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) - \frac{L_{D-1}^2}{r^2}
\]

with \( L_{D-1}^2 \) is angular momentum.

In the case of \( S(r) = V(r) \), the D-dimensional Klein-Gordon in equation (2) can be written as [5,12]

\[
\left\{ \nabla_2^2 + (E^2 - M^2 - (E + M)V(r)) \right\} \psi_{\ell_{1}, \ell_{2}, \ldots, \ell_{d-2}}^{(\ell_{1}, \ell_{2}, \ldots, \ell_{d-2})}(x) = 0
\]

By setting the wave function equation which is given as

\[
\psi_{\ell_{1}, \ell_{2}, \ldots, \ell_{d-2}}^{(\ell_{1}, \ell_{2}, \ldots, \ell_{d-2})}(x) = R_l(r) Y_{\ell_{1}, \ell_{2}, \ldots, \ell_{d-2}}^{(\ell_{1}, \ell_{2}, \ldots, \ell_{d-2})}(\theta_1, \theta_2, \ldots, \theta_{D-1})
\]

By substituting equations (4) and (6) into equation (5) with \( R_l(r) = \frac{E}{r^{D-2}} \), we get

\[
\left\{ \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) - \frac{L_{D-1}^2}{r^2} \right\} + \left( E^2 - M^2 - (E + M)V(r) \right) \frac{F}{r^2} \psi_{\ell_{1}, \ell_{2}, \ldots, \ell_{d-2}}^{(\ell_{1}, \ell_{2}, \ldots, \ell_{d-2})}(\theta_1, \theta_2, \ldots, \theta_{D-1}) = 0
\]

Equation (7) is multiplied by \( \frac{r^{D-1}}{r^{D-2}} \), the radial part of wave function from equation (7) is given as

\[
\frac{\partial^2 F}{\partial r^2} - \frac{\left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right)}{r^2} F + \left( E^2 - M^2 - (E + M)V(r) \right) F = \frac{\lambda_{D-1}}{r} F
\]

with \( V(r) \) is the central potential only.

2.2. Spatially Deformed Momentum

In this section, we will discuss canonical algebra to determine energy values and wave functions for q-deformed of radial momentum. The spatially deformed momentum [17] can be written

\[
\frac{\partial^2}{\partial r^2} = D_r^q
\]

where the q deformed of radial momentum \( D_r^q \) is expressed by

\[
D_r^q = (1 + qr) \frac{\partial}{\partial r} \rightarrow D_r^q = (1 + qr)^2 \frac{\partial^2}{\partial r^2} + q(1 + qr) \frac{\partial}{\partial r}
\]
By changing new variable [17]
\[ x = \ln(1 + qr) \rightarrow e^{qx} = (1 + qr) \rightarrow r = \frac{e^{qx} - 1}{q} \]  
(11)

By substituting equation (12) into (10), equation (9) becomes
\[ D_r^2 = e^{2qx} \left( e^{-2qx} \frac{\partial^2}{\partial x^2} - qe^{-2qx} \frac{\partial}{\partial x} \right) + qe^{qx} \left( e^{-qx} \frac{\partial}{\partial x} \right) = \frac{\partial^2}{\partial x^2} \]  
(12)

By using variable substitution equation (10), then the q-deformed of radial momentum reduces into usual square of momentum as shown in equation (12).

2.3. Hypergeometric Methods
Hypergeometric method suggested by Gauß. This method from the differential of Hypergeometric function that the equation is [15]
\[ z(1-z) \frac{\partial^2 \phi}{\partial z^2} + \left( c' - (a' + b' + 1)z \right) \frac{\partial \phi}{\partial z} - a' b' \phi = 0 \]  
(13)

By using regular singular point approximation at \( z = 0 \), we get the series equation as
\[ \phi = z^n \sum a_n z^n \]  
(14)

The solution of equation (13) is [2,15]
\[ z F_i(a', b', c'; z) = \Phi_i(z) = \sum_{n=0}^{\infty} \left( \frac{(a')_n (b')_n}{n! (c')_n} \right) z^n \]  
(15)

and by setting \( a' = -n \) or \( b' = -n \). The equation (15) have solution of by polynomial series rank \( n \).

3. Result and Discussion
Kratzer potential is defined by [12,16]
\[ V(r) = -\frac{V_1}{r} + \frac{V_2}{r^2} \quad \text{and} \quad S(r) = -\frac{S_1}{r} + \frac{S_2}{r^2} \]  
(16)

with \( V_1, V_2, S_1, \) dan \( S_2 \) are parameters of potential. If \( V_1 = S_1 \) and \( V_2 = S_2 \), and by substituting equations (16), (11) and (12) into equation (8), we get
\[ D_r^2 F - \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 \left( e^{qx} - 1 \right)^2 F + \left[ E^2 - M^2 - \lambda_{\alpha,1} - (E + M) \left( \frac{V_2 q}{e^{qX} - 1} \right) \right] F = 0 \]  
(17)

The exponential equation in equation (17) can be changed by using cosines hyperbolic rule, we obtain
\[ \frac{\partial^2}{\partial x} F - \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 \left( E + M \right) V_2 q^2 \left( e^{qx} \right) F + \left[ \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 + \left( E + M \right) V_2 q^2 \right] F = 0 \]  
(18)
By setting
\[ A_s = -\left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 + \frac{(E + M) V \, q^2}{2} \right) \]
\[ B_s = -\left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 + \frac{(E + M) V \, q^2}{2} - \frac{(E + M) V \, q^2}{2} \right) \]
\[ E'_s = -\left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 + \frac{(E + M) V \, q^2}{2} + \frac{(E + M) V \, q^2}{2} - \frac{(E^2 - M^2)}{2} + \lambda_{p-1} \right) \]

Equation (18) can be written by
\[ \frac{\partial^2}{\partial x^2} F + A_s \, cosh^2 \frac{q x}{2} F - B_s \, coth \frac{q x}{2} F = E_s \, F \]

By using the approximation of variable, we have
\[ v(v - 1) = A_s \rightarrow v = 1 + \sqrt{1 + 4z} \]
\[ 2\mu = B_s \rightarrow \mu = \frac{B_s}{2} \]

Equation (22) can be solved by using variable and parameter approximations.

\[ \coth \frac{q x}{2} = 1 - 2z \quad \text{and} \quad cosech^2 \frac{q x}{2} = 4z(z - 1) \]

\[ F = z^\alpha (1 - z)^\beta f(z) \]

By substituting equations (24) and (25) into equation (22), we get the second order of Hypergeometric function like equation (13)
\[ z(1 - z) f'' + \left( (2\alpha + 1) - (2\alpha + 2\beta + 2)z \right) f' + \left( \nu (\nu - 1) - \left( \alpha^2 + \beta^2 \right)(\alpha + \beta + 1) \right) f = 0 \]

with \[ \alpha = \frac{\nu - 1 - n}{\nu - 1 - n} \] \[ \beta = \frac{\nu - 1 - n + \mu}{\nu - 1 - n} \]

By setting (23) with \[ E'_n = E'_s \] and by substituting equation (21) into equation (28), we obtain
\[ (E^2 - M^2) = \left( \frac{D-1}{2} \right) \left( \frac{D-3}{2} \right) q^2 + \frac{(E + M) V \, q^2}{2} + \frac{(E + M) V \, q^2}{2} \right) - \lambda_{p-1} \]

with \[ A_s \] and \[ B_s \] in equations (19) and (20), respectively. Equation (29) is the energy spectra equation that can be numerically calculated by using Matlab R2013 software with diatomic molecules variations \[ I_2 \] and \[ D_2 \] that showed in Table 1. which is adopted from reference [11]. Besides that, the result of energy spectra for D-dimensional Klein-Gordon Equation is showed in Table 2.
Table 1. The characteristic of spectroscopic and reduces mass for diatomic molecules in ground state

| Parameter | \( I_2 \) (eV) | \( O_2 \) (eV) |
|-----------|----------------|----------------|
| \( D_e(eV) \) | 1.581791863    | 5.156658828    |
| \( a(\text{Å}) \) | 2.662          | 1.208          |
| \( M(\text{amu}) \) | 63.45223502    | 7.997457504    |

Table 2. The Energy Spectra for D-Dimensional Klein-Gordon Equation with q-deformed of radial momentum for Kratzer potential in natural unit

| \( D \) | \( q \) | \( E_{I_2} \) | \( E_{O_2} \) | \( D \) | \( q \) | \( E_{I_2} \) | \( E_{O_2} \) |
|-------|-------|-------------|-------------|-------|-------|-------------|-------------|
| 3     | 0     | -7.31413290554 | -7.17610898467 | 3     | 0     | -7.31413290554 | -7.17610898467 |
| 3     | 0.01  | -7.31413290554 | -7.17610898467 | 3     | 0.05  | -7.31413442363 | -7.17680838122 |
| 3     | 0.1   | -7.31413594171 | -7.17889450984 | 3     | 0.5   | -7.31414808677 | -7.239949242962 |
| 3     | 1     | -7.31416326897 | -7.44151169460 | 4     | 0     | -7.24544961751 | -7.10609176411 |
| 4     | 0.01  | -7.24546043060 | -7.10614532350 | 4     | 0.05  | -7.24571064060 | -7.1074330859 |
| 4     | 0.1   | -7.24648828450 | -7.11134897628 | 4     | 0.5   | -7.27051114790 | 7.225859720785 |
| 4     | 1     | -7.339402422904 | 7.651919981986 |

Table 2. shows that the energy values depend on q-deformed of radial momentum and D-dimension parameters. The rise of D-dimension values cause the rise of energy values, while the rise of q deformation values causes the decline of energy values. The energy values are negative because of presence the interaction of particles with potential and the bound energy to core of the particle.

Furthermore, the wave function equation for D-dimensional Klein-Gordon equation with q-deformed of radial momentum for Kratzer potential has solution with substitution equation (26) into (25), which is

\[
R = \left( \frac{1 - \coth \frac{qx}{2}}{2} \right)^{\alpha} \left( \frac{1 + \coth \frac{qx}{2}}{2} \right)^{\beta} \; _2F_1(a', b', c', z)
\]

where \( \alpha \) and \( \beta \) parameters in equations (27). The equation (30) becomes
\[ R = \frac{1 - \coth \frac{qx}{2}}{2} \left(1 + \coth \frac{qx}{2}\right) \]

with \( \nu \) and \( \mu \) in equation (31). Then, the wave function of D-dimensional Klein-Gordon equation influence Kratzer potential that is showed in Table 3.

**Table 3.** The Wave Function for D-Dimensional Klein-Gordon Equation with q-deformed of radial momentum for Kratzer potential with quantum number variations

| \( n \) | \( R_n (r) \) |
|-------|----------------|
| 0     | \[ \frac{1 - \coth \frac{qx}{2}}{2} \left(1 + \coth \frac{qx}{2}\right) \]
| 1     | \[ \frac{1 - \coth \frac{qx}{2}}{2} \left(1 + \coth \frac{qx}{2}\right) \]
| 2     | \[ \frac{1 - \coth \frac{qx}{2}}{2} \left(1 + \coth \frac{qx}{2}\right) \]

4. Conclusions
Hypergeometric method can be used to solve D-dimensional Klein-Gordon equation with q-deformed of radial momentum for Kratzer potential. The rise of D-dimension values cause the rise of energy values, then the rise of q-deformed parameter causes the decline of energy values. Energy values are negative and positive because of presence the interaction of particles with potential and the bound energy to core of the particle, respectively.

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6. References

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