The dynamics of bright matter wave solitons in a quasi one-dimensional Bose–Einstein condensate with a rapidly varying trap

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Abstract

The dynamics of a bright matter wave soliton in a quasi one-dimensional Bose–Einstein condensate (BEC) with a periodically rapidly varying time trap is considered. The governing equation is based on averaging the fast modulations of the Gross–Pitaevskii (GP) equation. This equation has the form of a GP equation with an effective potential of a more complicated structure than an unperturbed trap. In the case of an inverted (expulsive) quadratic trap corresponding to an unstable GP equation, the effective potential can be stable. For the bounded space trap potential it is showed that bifurcation exists, i.e. the single-well potential bifurcates to the triple-well effective potential. The stabilization of a BEC cloud on-site state in the temporary modulated optical lattice is found. This phenomenon is analogous to the Kapitza stabilization of an inverted pendulum. The analytical predictions of the averaged GP equation are confirmed by numerical simulations of the full GP equation with rapid perturbations.

1. Introduction

Bright matter wave solitons have recently been observed in a Bose–Einstein condensate (BEC) [1, 2]. In the experiment described in [1] propagation in an anisotropic BEC of cigar-type geometry was considered. In [2] the soliton was monitored by projecting the bound state of about 5000 atoms into an expulsive harmonic potential. The soliton was observed to propagate without changing its form for distances of the order of \( \sim 1 \text{ mm} \). The expulsive harmonic trap used in the experiment corresponds to an unstable potential for the Gross–Pitaevskii (GP) equation, and a soliton can exist only as a metastable state in BEC [3, 4]. It is of interest to investigate the dynamics of a bright matter wave soliton in inhomogeneous time-dependent systems, in particular, the possible stabilization of unstable dynamics or the complicated dynamics of solitons in a BEC.
An interesting example of such a trap is an optical trap which is rapidly varied in time. This is realized, for instance, with a condensate in a dipole trap formed by a strong off-resonant laser field [5]. A typical model is a trap formed by a harmonic potential, with different frequencies, which is cut at an energy, $V_c$. Numerical simulations of the condensate dynamics in a one-dimensional BEC with a positive scattering length under a periodically shaken trap performed in [5] show the existence of splittings in the condensate. The same form of wave equation governs the propagation of spatial solitons in periodically modulated parabolic waveguides. This problem appears with the investigation of intense light beams in a nonlinear waveguide with an inhomogeneous distribution of a transverse refractive index. A typical distribution can be approximated by a quadratic profile. The rapid variation in this profile along the longitudinal direction leads to a rapidly varying quadratic potential in the nonlinear Schrödinger (NLS) equation.

The general mathematical problem is to investigate the localized states for a nonlinear wave equation with cubic nonlinearity and with a rapidly varying (not small) potential of the form $V(x,t)/\epsilon$, $\epsilon \ll 1$. It is also of interest to investigate the possibilities of stabilizing a soliton using rapid perturbations. For a multi-dimensional NLS equation with a rapidly varying in space periodic potential such a problem is discussed in [6].

The paper is organized as follows. In section 2 we describe the procedure of obtaining the averaged equation for the GP equation with an external potential, which is rapidly varying in time and inhomogeneous in space and has the form $f(x)\cos(\Omega t)$, where $\Omega = 1/\epsilon$. In section 3 we consider some important potentials for applications which have the forms given by $f(x) = x^2$, $1/cosh(bx)$ and $cos(\kappa x)$.

2. The averaged Gross–Pitaevskii equation

The dynamics of a BEC is described by the time-dependent GP equation

$$i\psi_t = -\frac{\hbar^2}{2m}\Delta \psi + V_x(r,t)\psi + g|\psi|^2\psi,$$

where $m$ is the atom mass, $g = 4\pi\hbar^2a_s/m$ and $a_s$ is the atomic scattering length. $a_s > 0$ corresponds to the BEC with a repulsive interaction between the atoms and $a_s < 0$ to an attractive interaction. The trap potential is given by $V_x = m\omega_x^2(y^2 + z^2)/2 + \alpha(t)(m\omega_y^2x^2/2 + V_1(x,t))$, where $V_1(x,t)$ is a bounded potential or the optical lattice potential and $\alpha(t)$ describes the time dependence of the potential. We will specify the form of $V_1$ later. We will now consider the condensate of a cigar type with $\omega_x^2 \gg \omega_y^2$. Within such restrictions we can look for the solution of equation (1) of the form $\psi(x,y,z,t) = R(y,z)\phi(x,t)$, where $R$ satisfies the equation

$$-\frac{\hbar^2}{2m}\Delta R + \frac{m\omega^2}{2}(y^2 + z^2)R = \lambda R,$$

Averaging over the transverse mode $R$ (i.e. multiplying by $R^*$,

$$|R_0|^2 = \frac{m\omega}{\pi\hbar} \exp\left(\frac{m\omega}{\hbar}\rho^2\right),$$

and integrating over $\rho$) we obtain the quasi one-dimensional GP for $\phi$ [7]

$$i\hbar\phi_t = -\frac{\hbar^2}{2m}\phi_{xx} + \alpha(t)\left(\frac{m\omega_y^2x^2}{2} + V_1(x,t)\right)\phi + G|\phi|^2\phi,$$

where $G = g \int |R|^4 dx dy / \int |R|^2 dx dy = (2\hbar|a_s|\omega)$. In the dimensionless variables

$$t = \omega\tau/2, \quad x = x/l, \quad \lambda = \frac{\hbar}{m\omega}, \quad u = \sqrt{2|a_s|\phi},$$
we have the governing equation
\[ iu_t + \beta u_{xx} + 2\sigma |u|^2 u = \alpha(t) f(x)u \]  
where we will suppose a periodic modulation of the trap
\[ \alpha(t) = \alpha_0 + \alpha_1 \sin(\Omega t). \]  
For example, for \( V_f = n_0 \omega^2 x^2 / 2 \) we have \( \alpha_0 f(x) = (\alpha_1 / \omega) x^2 \). We introduced the parameter \( \beta = \pm 1 \) in order to be able to use the results obtained for optical beam propagation. For the BEC system \( \beta = 1 \). \( \sigma = \pm 1 \) corresponds to the attractive and repulsive two-body interactions, respectively.

The field \( u(x, t) \) can be represented in the form of the sum of slowly and rapidly varying parts \( U(x, t) \) and \( \xi(x, t) \)
\[ u(x, t) = U(x, t) + \xi(x, t). \]  
For obtaining the equation for an averaged field we will apply the asymptotic procedure suggested in [6], namely we will present the rapidly varying part of the field a Fourier series expansion
\[ \xi = A \sin(\Omega t) + B \cos(\Omega t) + C \sin(2\Omega t) + D \cos(2\Omega t) + \cdots \]  
where \( A, B, C, D \) are functions of \((x, t)\) that are slowly varying in the scale of \( O(1) \) functions. By substituting equations (5) and (6) into (3) we obtain the next set of equations for the slowly varying field and the coefficients of the expansion for the rapidly varying component
\[ iU_t + \beta U_{xx} + 2|U|^2 U + U^*(A^2 + B^2 + C^2 + D^2) \]
\[ + 2U(|A|^2 + |B|^2 + |C|^2 + |D|^2) + \left( BCA^* - |A|^2D + ACB^* + |B|^2D + \alpha \right) + \frac{A^2D^*}{2} + \frac{B^2D^*}{2} = \alpha f(x)U \]
\[ iA_t - i\Omega B + \beta A_{xx} + 4|U|^2 A + 2U^*(BC - AD) \]
\[ + 2U^2A^* + 2U(-DA^* + CB^* + BC^* - AD^*) \]
\[ + \left( \frac{3}{2} |A|^2 A + \cdots \right) = \alpha_1 f(x)U - \frac{\alpha_1}{2} f(x)A, \]
\[ i\Omega A + iB_t + \beta B_{xx} + 4|U|^2 B + 2U^2 B^* + 2U^*(AC + BD) + \cdots = \frac{\alpha_1}{2} f(x)C, \]
\[ iC_t - 2i\Omega D + \beta C_{xx} + 4|U|^2 C + 2U^* AB + \cdots = \frac{\alpha_1}{2} f(x)B, \]
\[ 2i\Omega C + iD_t + \beta D_{xx} + 4|U|^2 D + U^*(B^2 - A^2) + \cdots = -\frac{\alpha_1}{2} f(x)A. \]

The parameters \( \alpha_1 \) and \( \Omega \) are assumed to be \( \gg 1 \). With this system the coefficients can be represented in an expanded form as follows:
\[ A = \frac{a_1}{\Omega^2} + \frac{a_2}{\Omega^4}, \quad B = \frac{b_1}{\Omega} + \frac{b_2}{\Omega^3}, \]
\[ C = \frac{c_1}{\Omega^3} + \frac{c_2}{\Omega^5}, \quad D = \frac{d_1}{\Omega^2} + \frac{d_2}{\Omega^4}. \]  
For the expansion coefficients we have the expressions
\[ a_1 = -i\alpha_1 f(x)U - \alpha_1 f^2(x)U_{xx} - 2\alpha_1 f(x)|U|^2U + \alpha_0 \alpha_1 f^2(x)U, \]
\[ b_1 = i\alpha_1 f(x)U, \quad d_1 = \frac{a_1^2 f(x)^2}{4}U, \quad c_1 = \cdots. \]
From (7), (12) and (13) we obtain the averaged equation for $U$

$$iU_t + \beta U_{xx} + 2|U|^2U = a_0 f(x)U - \frac{\epsilon^2}{2} f^2(x)U_t$$

$$- \beta \frac{\epsilon^2}{2} f(x)(f(x)U)_{xx} - 2\epsilon^2 f^2(x)|U|^2U + \frac{\epsilon^2 a_0}{2} f^2(x)U$$

(14)

where $\epsilon = a_1/\Omega$.

This equation has the conserved quantity

$$\int_{-\infty}^{\infty} dx \left( 1 + \frac{\epsilon^2 f^2(x)}{2} \right) |U|^2 = \text{constant.}$$

(15)

It is, therefore, useful to introduce the new field $V$ by

$$V = \left( 1 + \frac{\epsilon^2 f^2(x)}{2} \right)^{1/2} U.$$  

(16)

Substituting (16) into (14) and keeping the terms of order $\epsilon^2$ we obtain the equation

$$iV_t + \beta V_{xx} + 2|V|^2V = \left( a_0 f(x) + \frac{\epsilon^2}{2} \beta [f_x(x)]^2 \right) V + O(\epsilon^4).$$

(17)

The averaged equation has the form of a modified NLS equation with a slowly varying potential

$$W(x) = a_0 f(x) + \frac{\epsilon^2}{2} \beta [f_x(x)]^2.$$  

(18)

This result shows that the soliton dynamics can have a more complicated character than in the case of a BEC with slowly varying parameters. We can expect here the stabilization of the unstable dynamics of a soliton by rapidly varying perturbation. It is a direct analogy with the stabilization of systems with a few degrees of freedom under rapidly varying time parameters.

### 3. The dynamics of the different models of trap potentials

#### 3.1. A quadratic potential

As an example, we will consider an attractive BEC with a quadratic trap potential. In this case we have a perturbed trap potential $[-a_0 + a_1 \sin(\Omega t)]f(x)$ with $f(x) = x^2$ and $\beta = 1$. An averaged effective potential is given by

$$W = (-a_0 + 2\epsilon^2)x^2,$$

(19)

where $\epsilon = a_1/\Omega$. The case of $a_0 > 0$ corresponds to the inverted (repulsive) harmonic potential case which is known to be unstable [4]. As seen, when $2\epsilon^2 > |a_0|$ the sign of the effective potential is reversed and the solitary wave dynamics become stable. The frequency of oscillations of the soliton centre is now

$$\Omega_0 = 2\sqrt{(2\epsilon^2 - a_0)}.$$  

(20)

This result can also be obtained using the moments method. It is useful to introduce two new variables: a position of a solitary solution of the governing equation (3)

$$X(t) = \int_{-\infty}^{\infty} dx \, |u(x, t)|^2$$  

(21)

and a field momentum

$$P(t) = \int_{-\infty}^{\infty} dx \, (u^*(x, t)u_x(x, t) - u(x, t)u^*_x(x, t)).$$  

(22)
By using the governing equation (3) we come to the following set of integro-differential equations:

\[-iP_t + 2 \int_{-\infty}^{\infty} dx' \frac{\partial W(x')}{\partial x'} |u(x', t)|^2 = 0, \quad iX_t - \beta P = 0. \tag{23}\]

In the case of a quadratic effective potential \(W(x) = \alpha x^2\) the set of equations (23) is closed and we get the equation of motion for the soliton solution centre, \(X(t)\), in the form

\[X_{tt} + 4\beta \alpha X = 0. \tag{24}\]

We would like to point out that the set (23) is exact. No additional assumptions were made in obtaining this set of equations, and it is valid for any solitary solution of the governing equation (3). In addition, the following two remarkable facts connected with the dynamics of a solitary solution of the NLS equation in the presence of a quadratic potential (or dynamics of BEC in a quadratic trap) should be noted:

(i) the equations of motion (23) for the soliton centre are exact, and
(ii) after averaging, the initially quadratic potential remains quadratic but its sign may be reversed.

The results of the numerical simulation of the full GP equation with the modulated over time explosive (inverted) harmonic potential are shown in figure 1. Two cases with \(\alpha_1 = 4.7124\) and 3.1415 were considered. For both cases \(\alpha_0 = 0.0493\) and \(\Omega = 10\). The results of the computation are as follows:

(a) \(\alpha_1 = 4.7124, \Omega_0 = 1.309, (T = 4.8)\). Calculation using equation (24) gives \(\Omega_{cal} = 1.257, (T_{cal} = 5.0)\).

(b) \(\alpha_1 = 3.1415, \Omega_0 = 0.7705, (T = 8.155)\). Calculation using equation (24) gives \(\Omega_{cal} = 0.7695, (T_{cal} = 8.165)\).

In both cases we observe the emergence of stable oscillations of the soliton position. We obtain good agreement between the full simulations and the averaged equation. As can be seen, when the value of \(\epsilon\) increases, the greater the discrepancy is between the results of the calculations using equation (24) and those of the full numerical simulations using the NLS equation. We can expect this since for these parameter values the ratio \(\alpha_1/\Omega\) is not small. Let us estimate the values of parameters for the experimental situation. In the experiment in [1] the longitudinal frequency was \(\omega_l \approx 2\pi \times 70\) Hz and in [2] it was \(\omega_l \approx 2\pi \times 50\) Hz, so with the temporal modulations of this potential at frequencies of \(\Omega = 2\pi \times 700\) and \(2\pi \times 500\) Hz we should observe the stable bright matter wave soliton.

3.2. The case of a bounded potential

Let us consider a bounded space trap potential for the atomic BEC [5] under fast temporal perturbations \((-\alpha_0 + \alpha_1 \sin(\Omega t)) f(x)\) with \(f(x) = 1/\cosh(bx)\) and \(\beta = 1\). For the BEC case we should add the constant \(\alpha_0\) to the potential, i.e. \(f(x) = \alpha_0 (1 - 1/\cosh(bx))\) to give the asymptotic, \(W_0 \rightarrow \alpha_0, |x| \rightarrow \infty\).

After averaging over the period of the fast perturbations, the effective potential \(W(x)\) has the form

\[W(x) = -\frac{\alpha_0}{\cosh(bx)} + \frac{\epsilon^2 b^2}{2} \frac{\sinh(bx)^2}{\cosh(bx)^2}, \tag{25}\]

provided that

\[\epsilon^2 b^2 > \sqrt{\frac{27}{2}} \alpha_0. \tag{26}\]
Figure 1. Numerical simulations of the full GP equation for $f(x) = x^2$ with $\Omega = 10$, $\alpha_0 = 0.0493$ and values of the perturbation parameter $\alpha_1$ reversing the initially inverted harmonic potential. Two cases are presented: (a) $\alpha_1 = 4.7124$ and (b) $\alpha_1 = 3.1415$.

This effective potential takes a triple-well structure with a central minimum and two lateral minima. In figure 2, a typical form of the potential is presented, where $\alpha_0 = 0.05$, $\alpha_1 = 9$, $\Omega = 10$ and $b = 0.6$. (As can be seen, condition (26) makes one take a large ratio of the parameters $\alpha_1$ and $\alpha_0$.) The two lateral minima are positioned at $x = \pm 3.96$ and the central one at $x = 0$.

The results of the numerical simulations of the GP equation are shown in figure 3. To describe small oscillations of the soliton solution centre and to obtain the equations of motion in the general case, we will have to make some approximations. As the ansatz, we suppose that the solution takes the form $u(x, t) = u(x - X(t), t)$. From here on, for the sake of simplicity, we will also suppose that the minimum is at $X_0 = 0$ and that the soliton solution (the dynamics of which we are investigating) is symmetrical, i.e.

$$|u(x, t)|^2 = |u(-x, t)|^2. \tag{27}$$

Now we can consider the second term of equation (23). We initially rewrite the effective potential as a sum of the antisymmetric and symmetric parts $W(x) = W^A(x) + W^S(x)$, where $W^A(x) = \frac{1}{2}(W(x) - W(-x))$ and $W^S(x) = \frac{1}{2}(W(x) + W(-x))$. Then, substituting $W(x)$ into the second term of equation (23) and expanding $|u(x - X(t), t)|^2$ into terms of $X(t)$ and by holding the first two terms

$$|u(x' + X, t)|^2 = |u(x, t)|^2 + \frac{\partial |u(x', t)|^2}{\partial x'}, \tag{28}$$
we have
\[
2 \int_{-\infty}^{\infty} dx' \frac{\partial W(x')}{\partial x'} |\mu(x' + X, t)|^2 = 2 \int_{-\infty}^{\infty} dx' \frac{\partial W^A(x')}{\partial x'} |\mu(x', t)|^2
+ 2X \int_{-\infty}^{\infty} dx' \frac{\partial W^S(x')}{\partial x'} \frac{\partial |\mu(x', t)|^2}{\partial x'}.
\]
(29)
By introducing the new parameters
\[
\omega^2 = 2\beta \int_{-\infty}^{\infty} dx' \frac{\partial W^S(x')}{\partial x'} \frac{\partial |\mu(x', t)|^2}{\partial x'}, \quad \Delta X = \frac{\int_{-\infty}^{\infty} dx' \frac{\partial W^A(x')}{\partial x'} |\mu(x', t)|^2}{\omega^2},
\]
(30)
we get the following equation of motion for the centre of the soliton
\[
X_{tt} + \omega^2 (X + \Delta X) = 0.
\]
(31)
The calculation of the parameters \(\Delta X\) and \(\omega^2\) for this case (\(\alpha_0 = 0.05, \alpha_1 = 9, \Omega = 10\) and \(b = 0.6\) with \(f(x) = 1/\cosh(bx)\)) gives the following.

1. For the central minimum \(X^W_0 = 0, \omega = 0.2539\) (\(T = 24.74\)), \(\Delta X = 0, X'' = 0\), the results of the numerical simulation are \(\omega = 0.251\) (or \(T = 25.0\)), \(\Delta X = 0\).
2. For the lateral minimum \(X^W_0 = 3.96, \omega = 0.00599\) (\(T = 104.73\)), \(\Delta X = 0.321, X'' = X^W_0 + \Delta X = 4.18\), the results of the numerical simulation are \(\omega = 0.0539\) (or \(T = 116.8\)), \(X'' = 4.3\).

Here \(X^W_0\) is the position of a given minimum of the potential \(W(x)\), \(X''_0\) is the stationary point (about which the oscillations occur), \(\Delta X\) is the shift of a stationary point and \(\omega\) and \(T\) are the frequency and period of the oscillations.

From the present analysis we conclude that averaging the GP equation over rapidly varying perturbations works well in the case of a bounded perturbation potential even when the expansion parameter \(\epsilon\) is large (in our case \(\epsilon = 0.9\)). The effective trap potential bifurcates from the single-well form to the triple-well structure. This can lead to the splitting of a single attractive BEC into three parts.
3.3. A periodic potential

Let \( f(x) = \cos(kx) \), then the averaged equation coincides not with the unperturbed one but with the effective potential

\[
F(x) = -\alpha_0 \cos(kx) + \epsilon^2 k^2 \left( 1 - \cos(2kx) \right). \quad (32)
\]

This potential has a more complicated structure and, as a result, the motion of the soliton centre has new properties. Let us consider the motion of one soliton in such a potential. The single soliton solution is

\[
V(x, t) = 2i\eta \text{sech}[2\eta(x - \zeta)] \exp \left[ \frac{\xi z}{\eta} + 4i(\eta^2 + \xi^2) + i\delta \right]. \quad (33)
\]

where \( \varepsilon = 2\eta(x - \zeta) \) and \( \zeta = 4\xi t \) for the unperturbed problem. The effective potential acting on the soliton mass centre is

\[
W_{sol} = \frac{\pi \alpha k}{2\eta \sinh \left( \frac{\xi}{2\eta} \right)} \cos(k\xi) - \frac{\pi \epsilon^2 k^3}{2\eta \sinh \left( \frac{\xi}{2\eta} \right)} \cos(2k\xi). \quad (34)
\]
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The equation of motion is obtained via
\[ \ddot{\zeta} = -\frac{\partial W_s}{\partial \zeta}. \] (35)

Thus, for the soliton centre \( \zeta \), we have the following equation of motion:
\[ \ddot{\zeta} = -\frac{\pi \epsilon^2 k^4}{2 \eta \sinh(k)} \sin(2k\zeta) + \frac{\pi k^2 \alpha}{2 \eta \sinh(k)} \sin(k\zeta). \] (36)

As can be seen the motion of the soliton centre occurs under the action of a *two-harmonic* effective perturbation. It represents a close analogy with the stabilization of a pendulum with the rapidly oscillating pivot point (the Kapitza pendulum). As is well known [8], the oscillations of the pendulum at the point \( k\zeta = \pi \) are unstable. The rapid perturbations stabilize this fixed point. The stabilization condition follows from the effective potential for the soliton in the periodic field (34) and is given by
\[ \epsilon^2 k^2 > \alpha \cosh \left( \frac{k\pi}{4\eta} \right). \] (37)

The results of the numerical simulation of the full GP equation are presented in figure 4. The parameters \( \alpha_0 \) and \( \omega \) are the same as for the bounded potential considered earlier. Figure 4(a) depicts the dynamics of the soliton centre in the periodic potential \(-\alpha_0 \cos(kx)\) with its initial position \( \zeta = \pi / k \) corresponding to the unstable point when \( \alpha_1 = 0 \) (the fast perturbation is turned off). It can be seen that the motion of the soliton is infinite. Figure 4(b) shows the case when \( \alpha_1 = 9 \) (the fast perturbation is turned on). In this case the motion of the soliton becomes finite and the soliton centre oscillates with the frequency \( \Omega_1 = 0.303 \) whereas the predicted value given by equation (36) is 0.308. One can see that the agreement with the theory is remarkable.

Figure 4. Numerical simulations of the full GP equation for the spatial periodic potential \( f(x) = \cos(bx) \) with the initial soliton positioned at an unstable point \( X_0 = \pi / k \). Case (a) (dashed curve) corresponds to the infinite motion of the soliton centre when \( \alpha_1 = 0 \) and the fast perturbation is turned off. Case (b) corresponds to \( \alpha_1 = 9 \) when the fast perturbation stabilizes the motion of the soliton centre. The parameters of perturbation are \( \Omega_1 = 10, \alpha_0 = 0.05 \), as in the case of a bounded potential.
4. Conclusion

We have investigated the propagation of a bright matter wave soliton in a BEC with a trap potential that varies rapidly over time.

The cases of periodically modulated quadratic, bounded and periodic trap potentials have been analysed. For the repulsive (unstable) trap potential it is shown that there exists critical value of the modulation parameter which occurs when the matter wave soliton is stabilized. The analogous phenomenon of the stabilization of unstable fixed points is found for the motion under spatial periodic modulations (optical lattices). For the bounded trap potential it is shown that the effective trap potential can bifurcate from the single well to the triple-well structure; and so may give rise to the splitting of a single, attractive BEC into three parts.

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