The subelliptic heat kernel on SU(2): representations, asymptotics and gradient bounds

Fabrice Baudoin · Michel Bonnefont

Received: 8 April 2008 / Accepted: 22 September 2008 / Published online: 22 October 2008
© Springer-Verlag 2008

Abstract The Lie group SU(2) endowed with its canonical subriemannian structure appears as a three-dimensional model of a positively curved subelliptic space. The goal of this work is to study the subelliptic heat kernel on it and some related functional inequalities.

Keywords Gradient estimate · Heat kernel · Log-Sobolev inequality · Poincaré inequality · SU(2) · Sublaplacian

Contents

1 Introduction .............................................. 648
2 Preliminaries on SU(2) ........................................ 649
3 The subelliptic heat kernel on SU(2) ................................. 650
   3.1 Spectral decomposition of the heat kernel ............................ 650
   3.2 Integral representation of the heat kernel ............................. 652
   3.3 Asymptotics of the heat kernel in small times .......................... 656
4 Gradient bounds for the heat kernel measure ............................. 661
   4.1 A first gradient bound ....................................... 662
   4.2 Li-Yau type inequality ........................................ 663
   4.3 The reverse spectral gap inequality ................................ 665
   4.4 $L^p$ gradient bounds ........................................ 668
      4.4.1 Long-time behavior ..................................... 668
      4.4.2 Short-time behavior ..................................... 670

F. Baudoin (✉)
Department of Mathematics, Purdue University, West Lafayette, IN, USA
e-mail: fbaudoin@math.purdue.edu

M. Bonnefont
Institut de Mathématiques de Toulouse, Université de Toulouse, Toulouse, France
e-mail: bonnefon@math.ups-tlse.fr
1 Introduction

The goal of this work is to study in details the heat kernel and some related functional inequalities in one of the simplest sub-elliptic models after the Heisenberg group: the Lie group \( SU(2) \) endowed with its canonical subriemannian structure (coming from the Hopf fibration \( S^2 \to S^3 \), see [25]). In the classification of three-dimensional homogeneous subriemannian structures (see page 22 in [15]) the role played by this group could be compared to the role played by the sphere in Riemannian geometry: It should be a three-dimensional model of a compact positively curved subriemannian space.

In the flat three-dimensional subelliptic model, that is the Heisenberg group, the subelliptic heat kernel is quite well understood. In his celebrated paper [14], Gaveau gave a useful integral representation and deduced from it small times asymptotics. Since, numerous papers have been devoted to the study of this kernel (see for instance [9, 22] and the references therein). In the case of \( SU(2) \), we will see that a quite similar study can be made: we will obtain an integral representation of the heat kernel and will deduce from it the small times asymptotics. These asymptotics give, in particular, a way to compute explicitly the Carnot–Carathéodory distance associated to the subriemannian structure of \( SU(2) \).

On the other hand, recent works have started to study gradient estimates for subelliptic semigroups (see for instance [5, 13, 22, 24]). From the point of view of partial differential equations (see [4, 21]), gradient estimates had proved to be a very efficient tool for the control of the rate of convergence to equilibrium, quantitative estimates on the regularization properties of heat kernels, functional inequalities such as Poincaré, logarithmic Sobolev, Gaussian isoperimetric inequalities for heat kernel measures, etc. When dealing with linear heat equations, those gradient estimates often rely on the control of the intrinsic Ricci curvature associated to the generator of the heat equation (Bakry–Emery criterion, see [2]). Those methods basically require some form of ellipticity of the generator and fail in typical subelliptic situations, like for instance in the Heisenberg group (see [5]). From the point of view of geometry, these gradient estimates are interesting, because they should contain information on the curvature of the space. For instance, in Riemannian geometry (see [3, 31]), the functional inequality \( \| \nabla e^{\Delta_1} f \|^2 \leq e^{-2\rho t} e^{\Delta_1} (\| \nabla f \|^2) \) is equivalent to the lower bound \( \text{Ric} \geq \rho \), where \( \text{Ric} \) denotes the Ricci curvature. In subriemannian geometry there is no real analogue of Ricci curvature; for instance, in Lott–Villani–Sturm sense (see [23, 28, 29]), the Ricci curvature of the simplest subelliptic model, the Heisenberg group, is \(-\infty\) (see [17]). However, we will show in this paper that we obtain exponential decays for the long-time behaviour of gradient estimates of the subelliptic semigroup on the model space \( SU(2) \) and controls on the small-time behaviour. Nevertheless, as it appears from our methods, the exponential decays we obtain are optimal but are mainly consequences of spectral properties, so that we do not really rely on any notion of intrinsic Ricci curvature excepted in the Li-Yau type estimate that we obtain. In the future, we hope to extend those methods to cover more general situations and to make the link with more geometrically oriented works like for instance [27], where a Bonnet-Myers type theorem is obtained in a hypoelliptic situation.

So, finally, this work is mainly divided into two parts. In a first part (Sect. 3), we will study the subelliptic heat kernel on \( SU(2) \). We provide its spectral decomposition, prove an integral representation of it and compute its small times asymptotics. In the second part (Sect. 4) we will focus on gradient estimates, using the previous results.