Revisiting the classical electron model in general relativity

Farook Rahaman\textsuperscript{1} \textbullet{} Mubasher Jamil\textsuperscript{2} \textbullet{} Kaushik Chakraborty\textsuperscript{3}

Abstract Motivated by earlier studies [Tiwari et al. 1984, Herrera & Varela 1994], we model electron as a spherically symmetric charged perfect fluid distribution of matter. The existing model is extended assuming a matter source that is characterized by quadratic equation of state in the context of general theory of relativity. For the suitable choices of the parameters, our charged fluid models almost satisfy the physical properties of electron.

Keywords Electromagnetic mass; General relativity; Einstein-Maxwell field equations.

1 Introduction

Attempts to give a classical model of charged particles, in particular, electrons, have been going on since the time of Lorentz (Lorentz 1904). The model proposed by Lorentz in which electrons have only ‘electromagnetic mass’ and no ‘true’ or ‘mechanical mass’ is commonly known as the Electromagnetic Mass Model. He assumed electron to be an extended object consisting of pure charge and no matter. This model was proved to be unstable but later on with the arrival of special relativity, the model was subsequently improved. The model was motivated from the hypothesis that gravitational mass and other physical quantities emerge from the electromagnetic field alone. In the past, this model has been investigated in the classical, quantum and relativistic regimes, and we are here re-analyzing the same model in the last scheme. The earliest developments were made by Lorentz (Lorentz 1892) by describing the properties of particles by their interaction with the ether. Further, Thomson (Thomson 1881) found that the kinetic energy of the charged sphere increases by its motion through a medium of finite specific inductive capacity, and he concluded that this increase in kinetic energy results in the increase of mass of the charged sphere, a phenomenon later on termed as the ‘electromagnetic mass’. Later on, the improvement was made by Mie (Mie 1912) through constructing a model of electrons having their origin from the electromagnetic fields alone. The drawbacks of Mie’s model were later removed by the Einstein’s general theory of relativity, with the inclusion of gravitational mass in the model to stabilize the electron (Einstein 1919) (see also [Herrera & Varela 1994, Rahaman 2009] for the history of the electromagnetic mass model).

Tiwari and co-workers [Tiwari et al. 1984] gave an extensive model of electron in the context of general relativity. They assumed electron as charged sphere of perfect fluid and obtained the pressure inside the charged sphere negative. It vanishes at $r = a$, where $a$ is the radius of the sphere. This result was later explained by Gron (Gron 1985) in the context of vacuum polarization. Tiwari et al showed that in their model the gravitational mass, pressure and mass density of electron vanishes when charge is zero [Tiwari & Ray 1991]. In a pioneering work, Bonnor and Cooperstock (Bonnor & Cooperstock 1989) discussed the model of electron as a static charged sphere that obeys Einstein-Maxwell theory of relativity and found that it must contain some negative rest mass density. Moreover, they found that the electron has negative active gravitational mass within the sphere of radius $a = 10^{-16}$ cm. Here-
rera and Varela (Herrera & Varela 1994) proposed an electromagnetic mass model satisfying the pure charge condition, \( p + \rho = 0 \), \( p \) and \( \rho \) being the pressure and mass density respectively. They obtained energy density to be negative for the radius 10^{-16} \text{ cm} which is experimentally verified upper limit of the radius of electron. Some authors have also discussed the Electromagnetic mass models in different context (Radinschi 2008; Capozziello et al. 2006). This quadratic EoS, \( p = p_0 + \alpha \rho + \beta \rho^2 \), where \( p_0, \alpha \) and \( \beta \) are parameters, is nothing but the Taylor expansion of arbitrary barotropic EoS, \( p(\rho) \). According to (Ananda & Bruni 2004), the quadratic EOS may describe dark energy or unified dark matter. In the string theory, the gravity is a truly higher dimensional interaction, which becomes effective 4D at low enough energies. In brane world models, inspired by string theory, the physical fields in our four dimensional Universe are confined to the three brane, while gravity can access the extra dimension. In brane world scenario, the gravity on the brane can give the following form for the electric field 

\[
e^{-\lambda} \left[ \frac{\lambda'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} = 8\pi \rho + E^2,
\]

\[
e^{-\lambda} \left[ \frac{1}{r^2} + \frac{\nu'}{r} \right] - \frac{1}{r^2} = 8\pi \rho - E^2,
\]

\[\frac{1}{2} e^{-\lambda} \left[ \frac{1}{2}(\nu')^2 + \nu'' - \frac{1}{2} \lambda' \nu' + \frac{1}{2} \frac{1}{\tau} (\nu' - \lambda') \right] = 8\pi \rho + E^2.\]

Here \( \rho(r), \lambda(r) \) and \( E(r) \) represent the fluid pressure, energy density and electric field, respectively, for a charged fluid sphere. In our model, we use the quadratic equation of state

\[ p(r) = \alpha \rho(r) + \beta [\rho(r)]^2. \]

The electric field is expressed as

\[ (r^2 E)' = 4\pi r^2 \sigma \mathbf{e} \mathbf{z}. \]

where \( \sigma(r) \) is the charge density on the sphere. Eq. (9) gives the following form for the electric field

\[ E(r) = \frac{1}{r^2} \int_0^r 4\pi r'^2 \sigma \mathbf{e} \mathbf{z} \, dr' = \frac{q(r)}{r^2}, \]

with \( q(r) \) is the total charge of the sphere under consideration. Also, the hydrodynamical equilibrium condition \( (T^\mu_\nu)_{\nu} = 0 \) is given by

\[ \frac{dp}{dr} + (\rho + p) \frac{\nu'}{2} = \frac{1}{8\pi r^4} \frac{d}{dr} q(r). \]
The above expression represents the Tolman-Oppenheimer-Volkoff equation for a charged sphere. In the next section, we are going to solve the equations (5)–(10), to get the metric coefficients. These solutions describe different type of charged fluid spheres for given effective densities. For a sphere of radius $r_0$, the pressure must vanish at $r_0$ and consequently the pressure gradient must be decreasing function of $r$. Since $\rho + p = 0$ (due to the fact $\nu + \lambda = 0$), it yields that density to vanish at the surface too. To avoid this inconsistency, it is necessary that the sphere must be charged.

3 Solutions of Einstein-Maxwell field equations

One can note that the term $\sigma e^{\frac{2A}{r}}$ occurring inside the integral sign in equation (10), is equivalent to the volume charge density. We will discuss two models assuming the volume charge density being polynomial function of $r$. Hence we use the condition

$$\sigma e^{\frac{2A}{r}} = \sigma_0 r^s,$$

where $s$ is arbitrary constant and the constant $\sigma_0$ is the charge density at $r = 0$, the center of the charged matter [Tiwari & Ray 1991; Rahaman 2009].

3.1 Electron model with constant effective radial pressure

In this specialization, we assume the effective radial pressure is constant as

$$p_{eff}^e = 8\pi p - E^2 = p_0,$$

where $p_0$ is an arbitrary constant. Here, we assume effective radial pressure is minimum at the center, $r = 0$ and equal to constant, say $p_0$. Equation (9) together with (12) yields

$$E^2(r) = \frac{16\pi^2\sigma_o^2}{(s + 3)^2}r^{2s+2},$$

while Eqs. (10) and (14) give

$$q^2(r) = \frac{16\pi^2\sigma_o^2}{(s + 3)^2}r^{2s+6}.$$  

Using equations (8), (13) and (14), one gets the following expressions of pressure and energy density as

$$p = \frac{p_0}{8\pi} + \frac{2\pi\sigma_o^2}{(s + 3)^2}r^{2s+2},$$

$$\rho = -\alpha + \sqrt{\alpha^2 - 4\beta^2 \left[\frac{p_0}{8\pi} + \frac{2\pi\sigma_o^2}{(s + 3)^2}r^{2s+2}\right]} \frac{1}{2\beta}.$$  

Using the field equations and the above obtained parameters, we get the first metric coefficient as

$$e^\nu = \left(\frac{\alpha + 1 + \sqrt{\alpha^2 - 4\beta^2 \left[\frac{p_0}{8\pi} + \frac{2\pi\sigma_o^2}{(s + 3)^2}r^{2s+2}\right]} }{2\beta} \right)^{\frac{1}{\alpha + 1}}.$$  

Similarly, from Eq. (6) we obtain the second metric coefficient as

$$e^{-\lambda} = 1 - \frac{2M(r)}{r} = 1 - \frac{16\pi^2\sigma_o^2}{(s + 3)^2(2s + 5)}r^{2s+4}$$

$$- \frac{8\pi}{r} \int \left[ -\alpha + \sqrt{\alpha^2 - 4\beta^2 \left[\frac{p_0}{8\pi} + \frac{2\pi\sigma_o^2}{(s + 3)^2}r^{2s+2}\right]} \frac{1}{2\beta} \right] r^2 dr.$$  

where $M(r) \equiv \int \frac{4\pi r^2T^0_0 dr}{r} = 4\pi \int r^2(\rho + E^2/8\pi)dr$, is the effective gravitational mass. One can find the exact analytical forms of (19) for different values of the parameter $s$.

3.1.1 $s = 0$

$$e^{-\lambda} = 1 - \frac{16\pi^2\sigma_o^2}{45}r^4 + \frac{4\pi\alpha}{3\beta}r^2 - \left[ \frac{128\pi^2\sigma_o^2}{9\beta} \right]^\frac{1}{2}$$

$$\times \left[ \frac{(A^2 + r^2)^{3/2}}{4} - A^2\sqrt{A^2 + r^2} \right]$$

$$\times \ln(r + \sqrt{A^2 + r^2}).$$  

Here

$$A^2 = \frac{9(2\pi\alpha^2 - p_0\beta)}{-16\pi^2\sigma_o^2\beta}.$$  

3.1.2 $s = \frac{1}{2}$

$$e^{-\lambda} = 1 - \frac{32\pi^2\sigma_o^2}{147}r^5 + \frac{4\pi\alpha}{3\beta}r^2 - \frac{8\pi\alpha}{9\beta}$$

$$\left[ \frac{(\eta' + r^3)^{3/2}}{r} \right] - \frac{32\pi^2\sigma_o^2}{49\alpha}.$$  

(22)
Here
\[ \eta' = 1 + \frac{p_0\beta}{2\pi\alpha}. \]

3.1.3 \( s = 2 \)
\[ e^{-\lambda} = 1 - \frac{16\pi^2\sigma_a^2}{225} r^2 + \frac{4\pi\alpha}{3\beta} r^2 - \frac{4\pi\alpha}{3\beta} \left( \frac{8\pi\beta\sigma_a^2}{25\alpha} \right)^2 r^2 \]
\[ \times \left[ \frac{r^2}{2} \sqrt{\rho^2 + k^2} + \frac{k^2}{2r} \ln(r^3 + \sqrt{r^6 + k^2}) \right], \]
(24)
where
\[ k^2 = \frac{1 + \frac{p_0\beta}{2\pi\alpha}}{2s\sigma_a^2}. \]

3.1.4 \( s = -\frac{1}{2} \)
\[ e^{-\lambda} = 1 - \frac{16\pi^2\sigma_a^2}{25} r^2 + \frac{4\pi\alpha}{3\beta} r^2 \]
\[ - \frac{4\pi\alpha}{3\beta} \frac{2(15a^2r^2 - 12abr + 8b^2)}{105a^3r} (ar + b)^{3/2}. \]
(26)
where
\[ a = \frac{32\pi\beta\sigma_a^2}{25\alpha}, \quad b = 1 + \frac{p_0\beta}{2\pi\alpha}. \]

3.1.5 \( s = -2 \)
\[ e^{-\lambda} = 1 - 16\pi^2\sigma_a^2 + \frac{4\pi\alpha}{3\beta} r^2 + \frac{4\pi\alpha}{3\beta} \left( \frac{ar^2 + b}{r} \right)^{3/2}, \]
(28)
where
\[ a = 1 + \frac{p_0\beta}{2\pi\alpha}, \quad b = \frac{8\pi\beta\sigma_a^2}{\alpha}. \]

3.2 Electron model with constant effective energy density

In this specialization, we assume the effective energy density is constant as
\[ \rho_{\text{eff}} = 8\pi\rho + E^2 = \rho_o \]
(30)
where \( \rho_o \) is an arbitrary constant. We assume here the effective energy density has just opposite behavior to the previous case i.e. it is maximum at the center, say \( \rho_0 \). In this case the forms of \( E \) and \( q \) are same as
(14) and (15). Following the procedure in the previous section, the other parameters can be found as
\[ \rho = \frac{\rho_0}{8\pi} - \frac{2\pi\sigma_a^2}{(s + 3)^2} r^{2s + 2}, \]
(31)
\[ p = \alpha \left[ \frac{\rho_0}{8\pi} - \frac{2\pi\sigma_a^2}{(s + 3)^2} r^{2s + 2} \right] \]
\[ + \beta \left[ \frac{\rho_0}{8\pi} - \frac{2\pi\sigma_a^2}{(s + 3)^2} r^{2s + 2} \right]^2, \]
(32)
\[ e^{-\lambda} = 1 - \frac{\rho_0 r^2}{3}. \]
(33)
\[ \nu = \int re^{-\lambda}(8\pi - E^2 + 1) dr - \int \frac{dr}{r}, \]
(34)
which gives
\[ \nu = \alpha\rho_o \int \frac{1}{1 - \frac{\rho_o r^2}{3}} dr + \frac{\beta\rho_o^2}{8\pi} \int \frac{1}{1 - \frac{\rho_o r^2}{3}} dr \]
\[ - \frac{8\pi\sigma_a^2(2\pi\alpha + \beta s)}{\rho_o} \int \frac{r^{s+3}}{1 - \frac{\rho_o r^2}{3}} dr \]
\[ + \frac{32\pi^3 s^3 \sigma_a^4 \beta}{9} \int \frac{r^{s+5}}{1 - \frac{\rho_o r^2}{3}} dr - \frac{16\pi^2\sigma_a^2}{(s + 3)^2} \int \frac{r^{s+3}}{1 - \frac{\rho_o r^2}{3}} dr \]
\[ - \int \frac{dr}{r(1 - \frac{\rho_o r^2}{3})} - \int \frac{dr}{r}. \]
(35)

On simplification, we obtain
\[ \nu = (\alpha\rho_o + \frac{\beta\rho_o^2}{8\pi}) \sqrt{3} \rho_o \frac{\text{Tanh}^{-1}(r\sqrt{\rho_o/3})}{\rho_o} \]
\[ - \frac{8\pi\sigma_a^2(2\pi\alpha + \beta s)}{\rho_o} \frac{r^{4+2s}}{2(2 + s)} \]
\[ + \frac{32\pi^3 s^3 \sigma_a^4 \beta}{9} \frac{3r^{6+4s}}{18 + 12s} \]
\[ + \frac{32\pi^3 s^3 \sigma_a^4 \beta}{9} \frac{r^{4+2s}}{(s + 3)^2} \frac{2F_1(2 + s, 1, 3 + s, \frac{r^{2}\rho_o}{3})}{2} \]
\[ - 2\ln r + \frac{1}{2} \ln(r^2 \rho_o^2 - 3). \]
(36)

Here we are interested in the the case for \( s = 0 \), which yields
\[ \nu = (\alpha\rho_o + \frac{\beta\rho_o^2}{8\pi}) \sqrt{3} \rho_o \frac{\text{Tanh}^{-1}(r\sqrt{\rho_o/3})}{\rho_o} \]
\[ - \frac{8\pi\sigma_a^2(2\pi\alpha + \beta s)}{9} \frac{r^4}{4} \frac{2F_1(2, 1, 3, \frac{r^2\rho_o}{3})}{2} \]
\[ + \frac{32\pi^3 s^3 \sigma_a^4 \beta}{9} \frac{3r^6}{18} \frac{2F_1(3, 1, 4, \frac{r^2\rho_o}{3})}{2} \]
\[ - \frac{16\pi^2\sigma_a^2}{9} \frac{r^4}{4} \frac{2F_1(2, 1, 3, \frac{r^2\rho_o}{3})}{2} \]
\[ - 2\ln r + \frac{1}{2} \ln(r^2 \rho_o^2 - 3). \]
(37)
4 Discussions

We have provided two new toy electromagnetic mass models which suggest that mass of electron has the ultimate origin from the electromagnetic field alone. For this, we have considered a static spherically symmetric charged perfect fluid with the quadratic EoS $p = \alpha \rho + \beta \rho^2$. In the first model, the total effective pressure is taken to be a combination of the barotropic pressure and the one generated due to the electric field. While in the second model, the effective energy density is taken to be sum of barotropic energy density and due to the electric field. The parameter in the EoS, namely $\alpha$ is related to sound speed for the fluid as follows: The squared of sound velocity, $v_s^2 = \partial p / \partial \rho = \alpha + 2 \beta \rho$, is always positive irrespective of matter density. In literature, more generally employed EoS parameter $\omega = p / \rho = \alpha + \beta \rho \omega$. Hence, $\alpha$ is related to sound speed and EoS parameter as $\alpha = 2 \omega - v_s^2$.

According to previous models [Herrera & Varela 1994; Bonnor & Cooperstock 1989], an electron (experimentally obtained radius $r \sim 10^{-16}$ cm, the inertial mass $m \sim 10^{-56}$ cm, charge $Q \sim 10^{-34}$ cm in relativistic units) is modeled as a charge fluid sphere obeying Einstein-Maxwell theory. So, the interior metric should be matched with the exterior Reissner-Nordström metric. If $M$ be the effective gravitational mass within the fluid sphere, then matching at the junction interface implies $M = m - Q^2 / 2s$. Putting the above values of inertial mass, charge and radius of the electron, one gets the value of the effective gravitational mass within the fluid sphere is $\sim -10^{-52}$ cm.

Since, we have studied two models assuming the volume charge density being polynomial function of $r$, so for both the models the electric field and total charge of the sphere under consideration are the same. If we assume the charge density $\sigma_0$ at $r = 0$, the center of the charged matter is $\sigma_0 \sim 10^{12}$, then charge of the fluid is nearly equal to $q \sim 10^{-34}$ cm in relativistic units (see the fig 1) which is equivalent to the charge of electron. In a similar way, if one assumes the suitable values of the parameters, then all physical parameters like, pressure, density and effective gravitational mass of the charged fluid that represents the electron can be obtained (see figs. 3, 4 and 5).

In model 1, one can note that for the following values of the parameters: $\sigma_0 \sim 10^{12}, s = 0, \alpha = -10^{-4}, \beta = -0.1$, the term $4 \pi \rho \sigma_0 r^2$ is the dominating term in equation (20). In this case the effective gravitational mass $\sim -10^{-52}$ (see the figure 5). Note that this result of negative mass is consistent with that of [12], if the size of electron is taken to be less then $10^{-16}$ cm. This negative mass and gravitational repulsion is a signature of the strain of the vacuum due to vacuum polarization [8]. For model 2, if one chooses the values of the parameters $\rho_0 = -10^{-4}$ and $s = 0$, then the effective gravitational mass $\sim -10^{-52}$ (see the figure 8). In figs. (4) and (7), we see that the energy density $\rho$ has decreasing behavior against the radial parameter and it takes negative values. This behavior arises since $\rho = -p$ for $p > 0$ (vacuum fluid). Thus figures (3) and (6) manifest the rising behavior of pressure.

In the present work, we have considered charged fluid of radius $\sim 10^{-16}$ (i.e. electron) obeying quadratic equation of state. We hope other people would be motivated by our approach and in future will try to extrapolate the present investigation to the astrophysical bodies, specially quark or strange stars.

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Fig. 1 The diagram of the Electric charge $q$ with respect to radial coordinate $r$ for $s = 0$.

Fig. 2 The diagram of the Electric field strength $E$ with respect to radial coordinate $r$ for $s = 0$. 
Fig. 3  The diagram of the radial pressure \( p \) with respect to radial coordinate \( r \) for \( s = 0 \).

Fig. 4  The diagram of the energy density with respect to radial coordinate \( r \) for \( s = 0 \).

Fig. 5  The diagram of the effective gravitational mass function \( M \) with respect to radial coordinate \( r \) for suitable values of the parameters.

Fig. 6  The diagram of the radial pressure \( p \) with respect to radial coordinate \( r \) for \( s = 0 \) for model 2.

Fig. 7  The diagram of the energy density \( \rho \) with respect to radial coordinate \( r \) for \( s = 0 \) for model 2.

Fig. 8  The diagram of the effective gravitational mass function \( M \) with respect to radial coordinate \( r \) for the model 2 for suitable value of the parameter, \( \rho_0 = -10^{-3} \).