The Rise and Fall of a Networked Society

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We propose a simple model of the evolution of a social network which involves local search and volatility (random decay of links). The model captures the crucial role the network plays for information diffusion. This is responsible for a feedback loop which results in a first-order phase transition between a very sparse network regime and a highly-connected phase. Phase coexistence and hysteresis take place for intermediate value of parameters. We derive a mean-field theory which correctly reproduces this behavior, including the distribution of degree connectivity and the non-trivial clustering properties.

Recent phenomenological studies on complex networks in social sciences have uncovered ubiquitous nontrivial statistical properties, such as scale free distribution of connectivity or small world phenomena. These properties have striking consequences on the processes which take place on such networks, such as percolation, diffusion, phase transitions and epidemic spreading. The research on complex networks raises questions of a new type as it addresses phenomena where the topology of interactions is part of the dynamic process. This contrasts with traditional statistical mechanics, where the topology of the interaction is fixed a priori by the topology of the embedding space.

Phenomena of this type are quite common in social sciences where agents purposefully establish cooperative links. Links between individuals in a social network support not only the socioeconomic interactions that determine their payoffs, but it also carries information about the state of the network. This aspect has important consequences in the long run if the underlying environment is volatile. In this case, former choices tend to become obsolete and individuals must swiftly search for new opportunities to offset negative events. The role of the network for information diffusion is particularly apparent, for example, pertaining to the way in which individuals find new job opportunities. It has been consistently shown by sociologists and economists alike that personal acquaintances or neighborhood effects play a prominent role in job search. This, in turn, leads to significant correlation in employment across friends, relatives, or neighbors. The common thesis proposed to explain this evidence is that, in the presence of environmental volatility, the quantity and quality of one’s social links – sometimes referred to as her social capital – is a key basis for search and adaptability to change.

The study of complex networks has been mainly concerned with simple phenomenological models reproducing the main stylized facts. In contrast, the socioeconomic literature has studied game-theoretic models of network formation in a fixed environment. Here, our objective is to integrate and enrich both approaches, proposing a stylized model of a society that embodies the following three features: (i) agent interaction, (ii) search cum adjustment, and (iii) volatility (i.e. random link removal). Individuals are involved in bilateral interaction, as reflected by the prevailing network. Through occasional update, some of the existing links have their value deteriorate and are therefore lost. In contrast, the individuals also receive opportunities to search that, when successful, allow the establishment of fresh new links. Over time, this leads to an evolving social network that is always adapting to changing conditions. The model aims to capture the continuous struggle of search against volatility, which is at the heart of network’s dynamics. In the long run, the so-called Red Queen Principle applies: “...it takes all the running you can do, to keep in the same place.” Agents’ continuous search must be strong enough to offset volatility if a dense and effective social network is to be preserved. On the other hand, search can be effective only in a densely networked society. So information diffusion and a dense network of interactions are two elements of a feedback self-reinforcing loop. As a result, the system displays a discontinuous phase transition and hysteresis, enjoying some resistance to a moderate deterioration of the underlying environmental conditions. Such a resilience can be interpreted as consequence of the buffer effects and enhanced flexibility enjoyed by a society that has accumulated high levels of social capital.

These features are captured by a mean field theory which is in good agreement with numerical simulation results. This theory highlights the particular role that clustering plays in the dynamics of the model. Indeed search is particularly effective when clustering is low whereas it is suppressed in a high clustered society.

The model introduced here is a simplification of a more
elaborate model proposed by one of us in [15] to understand how the network dynamics impinges on strategic behavior. The model is also similar to that proposed in Ref. [16] to explain the emergence of the small-world property [3] in social networks. There are, however, important differences between [3] and our approach, as we shall discuss later at the end of the paper. Related issues, within the vast recent literature on network dynamics, have also been addressed in Ref. [17], that studied the evolution of network among agents involved in an iterated Prisoner’s Dilemma, and in Ref. [18], that found a topological phase transition in networks that minimize a suitably chosen cost function.

Formally, the network is given by a set of nodes $N$ and the corresponding adjacency matrix $A$ with elements $A_{ij} \in \{0, 1\}$, for $i,j \in \{1, 2, \ldots, N\}$. The value $A_{ij} = 1$ means that there is a link connecting nodes $i$ and $j$, while $A_{ij} = 0$ holds otherwise. We will require $A_{ii} = 0$ (no on-site loops) and $A_{ij} = A_{ji}$ (unoriented links). The matrix stochastic process $A(t)$ in continuous time $t$ represents the evolution of the network.

Two local parameters will be of central importance for our discussion namely the node degree $c_i(t) = \sum_j A_{ij}(t)$ and the local clustering coefficient

$$q_i(t) = \frac{\sum_{j \neq k} A_{ik} A_{ij} A_{jk}}{\sum_{j \neq k} A_{ik} A_{ij}}. \quad (1)$$

The latter measures the fraction of pairs of neighbors of $i$ who are also neighbors among themselves. The averages of these two quantities over sites will be simply denoted $c$ and $q$. While random networks have $q \sim 1/N$, social networks typically have a clustering coefficient $q$ bounded above zero.

Denote by $F_i = \{j | A_{ij} = 1\}$ the set of neighbors (“friends”) of the node $i$. The network evolves due to the following three processes.

1. **Long distance search**: At rate $\eta$, each node $i$ gets the opportunity to make a link to a node $j$ randomly selected (if the link is already there nothing happens).

2. **Short distance search**: At rate $\xi$, each node $i$ picks at random one of its neighbors $j \in F_i$ and $j$ then randomly selects (i.e. “refers to”) one of its other neighbors $k \in F_j \setminus \{i\}$. If $k \notin F_i$ then the link between $i$ and $k$ is formed. If $F_i = \emptyset$ or $F_j = \{i\}$ or $k \in F_i$ nothing happens.

3. **Decay**: At rate $\lambda$, each existing link decays and it is randomly deleted.

For $\xi = 0$, the dynamics is very simple and the stationary network is a random graph with average degree $c = \eta/\lambda$. For $\eta \ll \lambda$ the network is composed of many disconnected parts. Fig. 1 reports what happens when the local search rate $\xi$ is turned on. For small $\xi$, network growth is limited by the global search process that proceeds at rate $\eta$. Clusters of more than 2 nodes are rare and when they form local search quickly saturates the possibilities of forming new links. Suddenly, at a critical value $\xi_2$, a giant component connecting a finite fraction of the nodes emerges. The average degree $c$ indeed jumps abruptly at $\xi_2$. The distribution $p(c)$ of $c_i$ is peaked with an exponential decrease for large $c$ and a power law $p(c) \sim c^{-\alpha}$ for $c$ small. The network becomes more and more densely connected as $\xi$ increases further. But when $\xi$ decreases, we observe that the giant component remains stable also beyond the transition point ($\xi < \xi_2$). Only at a second point $\xi_1$ does the network lose stability and the population gets back to an unconnected state. There is a whole interval $[\xi_1, \xi_2]$ where both a dense-network phase and one with a nearly empty network coexist. This behavior is typical of first-order phase transitions. The coexistence region $[\xi_1, \xi_2]$ shrinks as $\eta$ increases and it disappears for $\eta > 0.05\lambda$.

![FIG. 1: Average degree $c$ (top) and clustering coefficient $q$ (bottom) from numerical simulations with $\eta/\lambda = 0.01$ for populations of size $n = 1000$ and 2000. Runs were equilibrated for a time $t_{eq} = 3000/\lambda$ before taking averages.](image)

The average clustering coefficient $q$ shows a non-trivial behavior. In the unconnected phase, $q$ increases with $\xi$ as expected. In this phase, $q$ is close to one because the expansion of the network is mostly carried out through global search, and local search quickly saturates all possibilities of new connections. On the other hand, in the dense-network phase, $q$ takes relatively small values. This makes local search very effective. Remarkably we find that $q$ decreases with $\xi$ in this phase, which is rather counterintuitive: increasing the rate $\xi$ at which bonds between neighbors form through local search, the density $q$ of these bonds decreases. In fact, similar behavior is found if, fixing $\xi$ and $\eta$, the volatility rate $\lambda$ decreases.

To shed light on these numerical results, we study the dynamics of the distribution $P(c_i, q_i, t)$ of the degrees $c_i$ and clustering coefficients $q_i$. Specifically, we study a mean field approximation that assumes $q_i = q$ for all $i = 1, \ldots, N$ and

$$P(c_i, q, t) = \prod_{i=1}^N p(c_i, t) \delta(q_i - q). \quad (2)$$
It is convenient to set $\lambda = 1$, by an appropriate time rescaling. Then, the transition rates that enter into the master equation for $p$ have the form:

$$
w(c \to c + 1) = 2\eta + \beta\theta(c) + \gamma c \quad (3)$$
$$
w(c \to c - 1) = c \quad (4)
$$

where $\theta(k) = 0$ for $k \leq 0$ and $\theta(k) = 1$ for $k > 0$. In Eq. (3), the term $2\eta$ accounts for long distance search. The factor 2 counts both the processes when the search opportunity is given to site $i$ and when it is given to another site, which selects $i$ as end point. The second term in Eq. (3) arises from local search and it requires that $c_i > 1$. Here $\beta = \xi(1 - q)P\{c_j > 1 | j \in F_i\}$ accounts for the fact that the selected friend $j \in F_i$ must have at least one more friend $k \neq i$ and that $k \notin F_i$, which occurs with probability $1 - q$.

Finally the last term accounts for indirect local search opportunities given to a friend $k$ of a friend $j \in F_i$ of $i$. This process is proportional to $c_j$ and $\gamma = \xi(1 - q)c_j^{-1}$ accounts for the probability that $i \notin F_k$ and $k$ selects $j$. Note that the probability that $j \in F_k$ selects $i$ is $1/(c_j - 1)$. This, combined with the multiplicity $c_j - 1$ of second neighbors $k \in F_j$ of $i$, contributes a factor $(c_j - 1)/(c_j - 1) = 1$ to the rate.

Both $\beta$ and $\gamma$ will be determined self-consistently. The master equation for $p(c)$, in the stationary state, can be solved using the generating function $\pi(s) = \langle s^c \rangle$:

$$
\pi(s) = \frac{\beta + 2\eta(1 - \gamma s)^{-\mu}}{\beta + 2\eta(1 - \gamma)^{-\mu}}, \quad \mu = \frac{2\eta + \beta}{\gamma}. \quad (5)
$$

Note that $p(c) \sim c^d$ for small $c$ and $p(c) \sim c^{-\ln \gamma|c|}$ for large $c$ which perfectly matches the behavior observed in numerical simulations.

Eq. (5), allows us to compute the distribution $P\{c_j = k | j \in F_i\} = \tilde{p}(k)$ for the degree $c_j$ of $j \in F_i$. The larger $c_j$ the more likely is $j$ a neighbor of $i$. Thus, $\tilde{p}(k) \propto k\hat{p}(k)$ which, in terms of the generating function, implies $\tilde{\pi}(s) = s\pi'(s)/\pi'(1)$. Using this to compute averages over $c_j$ and $c_k$ we arrive at the self-consistent equations:

$$
\beta = \xi(1 - q) \left[1 - \frac{\pi'(0)}{\pi'(1)}\right] \quad (6)
$$
$$
\gamma = \xi(1 - q) \frac{1 - \pi(0)}{\pi'(1)} \quad (7)
$$

As we should, in the limit $\xi \to 0$ we find $\beta, \gamma \to 0$ and we recover a pure Poisson distribution with mean $2\eta$. But with constant $q$ Eqs. (6,7) are not able to reproduce the observed behavior. It just predicts a smooth crossover and no phase coexistence. This means that, in order to shed light on our observations, it is essential to allow for $q$ to depend on the parameters of the model and the distribution $p(k)$.

In order to derive an equation for $q$, we consider the evolution of clustering for a node $i$ with $c_i$ friends. Let $Q_i = \sum_{j<k} A_{ij} A_{jk} = q_i c_i (c_i - 1)/2$ be the number of pairs of friends of $i$ which are also friends among themselves. Only local search processes contributes to an increase in $Q_i$ through two different routes.

The first is when a local search opportunity is given to site $i$ itself and has already been discussed above. Its rate is $W_i(Q_i \to Q_i + 1) = \xi(1 - q)P\{c_i > 1\}$. The second occurs when a local search opportunity is given to some $j \in F_i$, who then asks to $i$ about his other friends $k \in F_i$ ($k \neq j$). This may lead to the formation of the link between $j$ and $k$, thus increasing $Q_i$ by one. The rate of this process is given by $W_2(Q_i \to Q_i + 1) = \xi(c_i \theta(c_i - 1))\langle c_j^{-1}\rangle(1 - q)$. Here, $1/c_j$ is the probability that $j$ picks $i$ from his neighbors and $1 - q$ is the probability that $k \notin F_j$. This rate should be multiplied by the number $c_i$ of neighbors of $i$, but is zero unless $c_i \geq 2$. Finally, we must account for the link-decay process that, contrary to the former two, decreases $Q_i$. The rate at which this happens is $W_3(Q_i \to Q_i - 1) = Q_i = \langle c_i (c_i - 1)\rangle q/2$.

If we now take the averages on $c_i$ and $c_j$ with probability distributions $p(k)$ and $\tilde{p}(k)$ respectively, we can impose stationarity on $Q_i$, i.e. $\langle \Delta Q_i \rangle = W_i + W_2 - W_3 = 0$. After some algebra, this condition becomes the desired equation for $q$:

$$
\frac{q}{2} = \xi(1 - q) [2 - \pi(0)] \left[1 - \frac{\pi'(0)}{\pi'(1)}\right] \quad (8)
$$

Using Eq. (5), we arrive at a set of three self-consistent equations for $\beta$, $\gamma$ and $q$.

![Mean degree $c$ (top) and clustering coefficient $q$ (middle) as a function of $\xi/\lambda$ for $\eta/\lambda = 0.19$. Bottom: phase diagram in the mean field approximation. The behaviour of $c$ and $q$ along the dashed line is reported above.](image-url)
values of $\xi$ and $\eta$ we find either a unique solution with small $c$ and large $q$ (corresponding to a dilute network) or a unique solution with large $c$ and small $q$ (dense network), or both solutions simultaneously. In particular the solution correctly reproduces the behavior of $q$ in the two phases: $q$ increases with $\xi$ in the dilute network phase whereas it decreases with $\xi$ when a giant component forms. Our approach shows that this is not just a by-product of our analysis but rather an essential ingredient for understanding the network’s dynamics. Fig. 2 also depicts the phase diagram predicted by the mean field. In the shaded region, Eqs. 10 11 12 have three solutions, of which two are stable and correspond to the two coexisting phases.

The coexistence interval $[\xi_1, \xi_2]$ predicted by the mean field theory is much larger than that observed in numerical simulations, as can be seen by comparing Fig. 1 and Fig. 2. For example, the critical point $\eta_c/\lambda \approx 0.226 \ldots$ above which there is a smooth crossover in the mean field is an order of magnitude larger than that suggested by numerical simulations. We believe this is due to the fact that mean field theory underestimates fluctuations and neglects correlations.

It is straightforward to repeat our approach to obtain the mean field equations for the model of Ref. 10. The network growth process of Ref. 10 mixes local and global search in a different way. (For example, search is always effective since when it is unsuccessful locally, the agent nevertheless creates a link through global search.) In addition volatility affects sites instead of removing bonds, i.e. Eq. 10 is replaced with $w(c_i \to 1) = p$. This changes considerably the stationary state distribution, since for $c > 1$ the stationarity in the master equation implies $p(c) = \frac{c-1}{c+c^{c-1}} p(c-1)$, i.e. a power law behavior $p(c) \sim c^{-\nu/\gamma-1}$, as observed in Ref. 13. The solution of the self-consistent equations is always unique, implying that there no phase transition in this model. These conclusions illustrate that the present approach is a rather powerful tool in the analysis of network dynamics in a wide range of different setups.

To sum up, our aim in this paper has been to study a simple model of network formation whose implications shed some light on the evolution (rise and fall) of a networked society. The induced network dynamics displays a first-order transition that, as the environmental conditions improve, lead from a sparse phase to a qualitatively different regime where the rich potential of a network society is realized. Thus, in particular, social interaction becomes dense (i.e. average connectivity is high) and individual search turns effective (i.e. redundant search is avoided by low clustering).

These findings explains the apparently paradoxical observation that a networked society does not necessarily materialize even under favorable conditions while, by contrast, it displays a significant resilience to deteriorating conditions. This may help understand the origin of the “miracles” and “anti-miracles” in economic development 21, which are still an unresolved puzzle for modern economic theory.

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[19] There is a third process that may give $\Delta Q_i = +1$. This occurs when $j \in F_i$ is given a local search opportunity, then a site $\ell \neq i$ is picked as intermediate node and $k \in F_i \cap F_i$ is selected to form a link to $j$. It is hard to determine the joint probability of these events precisely because fourth-order correlations (i.e. four sites) are involved. In the spirit of the mean field approach, we may neglect it. In any case, it is easy to check that $W_2(Q_i \to Q_i+1)$ is a safe upper bound to the rate of this process. We could therefore account for this fact by intro-
ducing a factor \( \alpha \in [1, 2] \) in front of \( W_2 \) above. However, since the final results depend very weakly on \( \alpha \), we choose to set \( \alpha = 1 \) and ignore this last process altogether.

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