Optimal quantum measurements for spin-1 and spin-$\frac{3}{2}$ particles

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Abstract

Positive operator valued measures (POVMs) are presented that allow an unknown pure state of a spin-1 particle to be determined with optimal fidelity when 2 to 5 copies of that state are available. Optimal POVMs are also presented for a spin-$\frac{3}{2}$ particle when 2 or 3 copies of the unknown state are available. Although these POVMs are optimal they are not always minimal, indicating that there is room for improvement.

1 Introduction

The problem of determining an unknown quantum state of which only a finite number of copies is available is one that has attracted much attention in recent years. A fruitful method of attacking the problem was suggested by Peres and Wootters [1], who pointed out that a judicious joint measurement on all the copies can sometimes yield more information than any measurements on the individual copies. This suggestion, whose formal basis is embodied in the theory of Positive Operator Valued Measurements (POVMs) [2], has served as the springboard for almost all subsequent work on the problem. Initial work focussed on determining an unknown (pure) state of a qubit of which $N$ copies were available [3-5], but this work was later extended to mixed states of qubits [6] and to pure states of qudits (i.e. $D$-state quantum systems) as well [7,8]. The relationship of quantum state estimation to quantum cloning machines was discussed in [9]. Other aspects of the state estimation problem have also been discussed recently [10].

The purpose of this paper is to present some new POVMs that allow optimal determinations of unknown qudit states in certain cases. Section II presents POVMs that allow an unknown pure state of a spin-1 particle to be determined with optimal fidelity when 2 to 5 copies of that state are available. Section III presents optimal POVMs for a spin-$\frac{3}{2}$ particle when 2 or 3 copies of the unknown (pure) state are available. The POVMs for spin-1 are based on the geometry of the 24-cell and the 600-cell, two four-dimensional regular polytopes, while the
POVMs for spin-$\frac{3}{2}$ are based on the 40 states of the "Penrose dodecahedron"[11] as well as another configuration of 60 states. The geometric structures underlying the various POVMs presented here have been exploited earlier to provide proofs of the Bell-Kochen-Specker[12] and Bell theorems[13,14], and it is interesting that they should prove of use in the state estimation problem as well.

2 POVMs for spin-1

Given $N$ identical copies of a $D$-state quantum system in an unknown pure state, how can one determine that state as reliably as possible? We first address this problem for general $N$ and $D$, introducing our terminology and notation in the process, before specializing to the case $D = 3$ in this section and $D = 4$ in the next. The (now) standard method of attacking this problem is to implement a POVM (or generalized measurement) on the $N$ copies and to use the outcome of this measurement to make a judicious guess about the input state. The success of a guess is gauged by a "fidelity", which is generally taken to be the squared modulus of the overlap between the input state and the guess made for it. The problem of state estimation consists of devising a POVM for which the average fidelity (i.e. the fidelity averaged over all possible occurences of the input state) is as large as possible. If the input state is distributed with uniform probability over the entire projective Hilbert space of the qudit, it has been shown[7,8] that the average fidelity is bounded from above by the quantity $\frac{N+D}{N+1}$. The task of constructing a POVM that achieves this upper bound for any $N$ and $D$ was addressed in fairly general terms in ref.[8], with the analysis for the cases $N = 2$ and 3 (but arbitrary $D$) being carried somewhat further. However only for the case $D = 3$ and $N = 2$ was a POVM explicitly constructed. This section will present optimal POVMs for $D = 3$ and $N = 2-5$.

An optimal POVM[2] for arbitrary $N$ and $D$ is characterized by a set of positive numbers $c_r$ and system states $|\Psi_r\rangle$ ($r = 1,\ldots,k$) such that

$$\sum_{r=1}^{k} c_r |\Psi_r\rangle^{\otimes N} \langle \Psi_r| = I,$$

where $|\Psi_r\rangle^{\otimes N}$ denotes the tensor product of the state $|\Psi_r\rangle$ with itself $N$ times and $I$ is the identity operator in the maximally symmetric subspace of the space of $N$ qudits. The number of states in the POVM ($= k$) can be larger than the dimensionality of the maximally symmetric subspace ($= \frac{(N+D-1)!}{N!(D-1)!}$) in which the POVM acts. A standard von Neumann measurement can be regarded as a POVM in which all the $c_r$ are unity and the operators that effect the resolution of the identity are projectors onto non-overlapping subspaces. A POVM can be implemented in practice by coupling the system of interest to an auxiliary system (the "ancilla") and carrying out a von Neumann measurement on the...
enlarged system; this measurement then appears in the space of the system alone as a POVM[2].

The operator identity (1) can be turned into a scalar equation by taking its expectation value in $|\Psi\rangle^\otimes N$, the $N$-fold tensor product of the arbitrary system state $|\Psi\rangle$. One then finds that

$$
\sum_{r=1}^{k} c_r N^\otimes (|\Psi_r\rangle^\otimes N N^\otimes (|\Psi_r\rangle^\otimes N N = \sum_{r=1}^{k} c_r \left(|(\Psi|\Psi_r\rangle)^2\right)^N = 1. \quad (2)
$$

We now specialize the discussion to a spin-1 system for which the spin value, $J$, is equal to 1 and the Hilbert space dimension is $D = 2J + 1 = 3$. The projective Hilbert space of a spin-1 system has dimension $2D - 2 = 4$ and an arbitrary state, $|\Psi\rangle$, in this space can be parametrized as[15]

$$
|\Psi\rangle = (e^{i\chi_1}\sin \theta \cos \phi, e^{i\chi_2}\sin \theta \sin \phi, \cos \theta) \quad \text{where} \quad 0 \leq \chi_1, \chi_2 \leq 2\pi, \quad 0 \leq \theta, \phi \leq \pi/2. \quad (3)
$$

If one introduces the real variables $x_1 = \cos \phi \cos \chi_1, x_2 = \cos \phi \sin \chi_1, x_3 = \sin \phi \cos \chi_2$ and $x_4 = \sin \phi \sin \chi_2$ that define a point on the surface of the four-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, the state $|\Psi\rangle$ can be expressed as

$$
|\Psi\rangle = (\sin \theta (x_1 + ix_2), \sin \theta (x_3 + ix_4), \cos \theta), \quad 0 \leq \theta \leq \pi/2, \quad -1 \leq x_1, x_2, x_3, x_4 \leq 1. \quad (4)
$$

Our strategy for constructing an optimal POVM for $D = 3$ is as follows. For each $c_r$ and $\theta_r$, we choose a set of points $x_{1i}, x_{2i}, x_{3i}, x_{4i}$ ($i = 1, \ldots, n$) on the surface of the four-dimensional unit sphere (the same set turns out to suffice for every $r$) in such a way that when the sum in (2) is carried out over all $i$ for a fixed $r$, the dependence on the angular variables $\phi, \chi_1$, and $\chi_2$ cancels out, leaving a function of $\theta$ alone. The sum on the left side of (2) then reduces to a polynomial of degree $k - 1$ in $\cos \theta$ that depends parametrically on the as yet undetermined elements $c_r$ and $\theta_r$ ($r = 1, \ldots, k$) of the POVM. We finally nail down the POVM by choosing $k$ arbitrary angles $\theta_r$ and fixing the constants $c_r$ in such a way that the polynomial on the left side of (2) reduces identically to the unity.

It remains for us to specify how the points $x_{1i}, x_{2i}, x_{3i}, x_{4i}$ ($i = 1, \ldots, n$) should be chosen so that the cancellation over the angular variables $\phi, \chi_1$, and $\chi_2$ can be accomplished on the left side of (2). For $N = 2$ or 3 it turns out that choosing the 24 vertices of a 24-cell does the trick, while for $N = 2 - 5$ choosing the 120 vertices of a 600-cell does the trick (the 24-cell and 600-cell are two of the four-dimensional regular polytopes[16]). The unit vectors to the vertices of a 24-cell are the 16 vectors $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and the 8 vectors $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0)$ and $(0, 0, 0, \pm 1)$ (i.e. the vertices of a hypercube plus those of a hyperoctahedron), while the unit vectors to the vertices of a 600-cell

$$
\text{3}
$$
are the 24 vectors just mentioned plus the 96 vectors obtained by taking all even permutations of $1\frac{1}{2}(\pm \tau, \pm 1, \pm \tau^{-1}, 0)$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ is the golden mean.

We now expand upon the above explanation to show how our optimal POVMs for $N = 2 - 5$ are constructed. The POVM for $N = 2$ is based on three numbers (or "weights") $c_1, c_2$ and $c_3$ and three angles $\theta_1, \theta_2$ and $\theta_3$, with each angle giving rise to 24 states based on the geometry of a 24-cell as indicated in eqn.(4). On using this POVM in the left side of (2) and taking $\theta_1 = \pi/4, \theta_2 = \pi/2$ and $\theta_3 = 0$, we find that the summation can be carried out analytically to yield the polynomial expression

$$2c_1 + 8c_2 + (8c_1 - 16c_2) \cos^2 \theta + (-4c_1 + 8c_2 + c_3) \cos^4 \theta. \quad (5)$$

For this to be identically equal to 1, it is necessary that the constant term be equal to 1 and that the coefficients of $\cos^2 \theta$ and $\cos^4 \theta$ both vanish. This is achieved by the choice of weights $c_1 = \frac{1}{6}, c_2 = \frac{1}{12}$ and $c_3 = 0$. We have therefore constructed an optimal POVM for $N = 2$ which is characterized by two weights $c_1 = \frac{1}{6}$ and $c_2 = \frac{1}{12}$ and a total of 48 states (24 arising from each of the angles $\theta_1 = \pi/4$ and $\theta_2 = \pi/2$). An optimal POVM for $N = 3$ can be constructed in a similar manner. This POVM involves four weights $c_1, c_2, c_3, c_4$ and four angles $\theta_1, \theta_2, \theta_3, \theta_4$ and contains $4 \times 24 = 96$ states obtained by combining each $\theta$ value with the vertices of a 24-cell in the manner indicated in (4). On using this POVM with $\theta_1 = \pi/6, \theta_2 = \pi/4, \theta_3 = \pi/3$ and $\theta_4 = \pi/2$ in the left side of (2) we obtain a third-degree polynomial in $\cos^2 \theta$ which reduces identically to 1 if one chooses the weights as $c_1 = \frac{2}{27}, c_2 = \frac{1}{18}, c_3 = \frac{2}{9}$ and $c_4 = \frac{7}{108}$. Our optimal POVM for $N = 3$ is therefore characterized by 4 weights and 96 states.

The above construction cannot be used to obtain POVMs for $N \geq 4$, because the use of a 24-cell no longer allows the angular dependence on $\phi$, $\chi_1$ and $\chi_2$ to be cancelled out on the left side of (2). However, as mentioned earlier, replacing the 24-cell by a 600-cell allows this cancellation to be accomplished for all $N$ values from 2 to 5. For $N = 2$ and 3 the 600-cell yields POVMs that are not as economical as the ones found earlier, so we skip over these cases and pass to $N = 4$ and 5. The construction of the POVMs for $N = 4, 5$ proceeds in the same manner as explained earlier for $N = 2$ and 3, but with the difference that the role of the 24-cell is now taken over by the 600-cell. Each $\theta$ value in the POVM gives rise to 120 states, and the total number of states in the POVM is a multiple of 120. The details of our POVMs for $N = 4$ and 5 (i.e. the weights $c_i$ and angles $\theta_i$) are summarized in the bottom half of Table I, while the top half summarizes our results for $N = 2$ and 3. For $N = 2$ ref.[8] presents a POVM with just 8 states, which is more economical than the one found here. However for $N \geq 3$ no optimal POVMs of any kind have been reported earlier.

3 POVMs for spin-3/2

Before presenting our POVMs for spin-$\frac{3}{2}$, we return to eqn.(2) and recast it in an alternative form based on the Bloch vector description of qudit systems.
In this description[15], the state vector $|\Psi\rangle$ of a $D$-state quantum system is represented by a unit vector – the generalized "Bloch" vector $\vec{n}$ – in a real $(D^2 - 1)$-dimensional space. The inner product of two state vectors is related to the scalar product of the corresponding Bloch vectors by the equation

$$|\langle \Psi_i | \Psi_j \rangle|^2 = \frac{1}{2J+1} (1 + 2J \vec{n}_i \cdot \vec{n}_j),$$

(6)

where the spin value $J$ is related to $D$ by $D = 2J + 1$. On substituting (6) into (2) and performing some manipulations, (2) can be recast in the form of a hierarchy of equations that must be satisfied by the POVM elements $(c_r, \vec{n}_r)$ for $r = 1, ..., k$. We now turn to these equations[8].

For arbitrary $D = 2J + 1$ and $N = 2$, the equations to be satisfied by the POVM elements are[8]

$$\sum_{r=1}^{k} c_r = (2J + 1)(J + 1),$$

(7)

$$\sum_{r=1}^{k} c_r (\vec{n}_r \cdot \vec{n}_s) = 0,$$

(8)

and

$$\sum_{r=1}^{k} c_r (\vec{n}_r \cdot \vec{n}_s)^2 = \frac{2J+1}{4J}.$$  

(9)

The two last equations must be satisfied for each value of $s$, so the above requirements amount to a total of $(2k + 1)$ equations.

For arbitrary $D = 2J + 1$ and $N = 3$, the equations to be satisfied by the POVM elements are[8]

$$\sum_{r=1}^{k} c_r = \frac{(2J + 3)(2J + 1)(J + 1)}{3},$$

(10)

$$\sum_{r=1}^{k} c_r (\vec{n}_r \cdot \vec{n}_s) = 0,$$

(11)

and

$$\sum_{r=1}^{k} c_r (\vec{n}_r \cdot \vec{n}_s)^2 = \frac{(2J + 3)(2J + 1)}{12J}.$$  

(12)
and \[ \sum_{r=1}^{k} c_r (\vec{n}_r \cdot \vec{n}_s)^3 = \frac{(2J+1)(2J-1)}{12J^2}. \] (13)

Again the last three equations have to be satisfied for each \( s \), leading to a total of \((3k + 1)\) equations altogether. It is interesting to note, from a comparison of (7)-(9) with (10)-(13), that any POVM for three copies can be turned into one for two copies simply by reducing each of the constants \( c_r \) by the factor \( (2J + 3)/3 \). This observation will prove to be of use below.

We now introduce the 40 states of the "Penrose dodecahedron"[11] and show how they can be used to construct POVMs for a spin-\( \frac{3}{2} \) particle. Twenty of these states (termed "explicit rays" in [11]) are the spin +\( \frac{1}{2} \) projections of a spin-\( \frac{3}{2} \) particle along the twenty directions from the center of a regular dodecahedron to its vertices. The remaining twenty states (termed "implicit" rays in [11]) are also associated with the vertices of the dodecahedron and have the property that the implicit ray associated with any vertex, together with the explicit rays associated with the three neighboring vertices, constitute a mutually orthogonal set of states. Explicit expressions for all 40 Penrose states are given in Table II, in the basis afforded by four of these states. From this table it is easily verified that each state is orthogonal to exactly 12 others and makes the same, constant angle with the 27 states it is not orthogonal to. This implies that the scalar products of Bloch vectors for pairs of Penrose rays have only the two values

\[ \vec{n}_r \cdot \vec{n}_s = -\frac{1}{3} \] for orthogonal rays, \( \quad \) (14)

\[ \text{or} \quad \vec{n}_r \cdot \vec{n}_s = \frac{1}{9} \] for non-orthogonal rays. (15)

Using (14) and (15) it is readily verified that a POVM satisfying (10)-(13) with \( J = \frac{3}{2} \) is obtained by taking all forty Penrose rays with a common weighting factor of \( c_r = \frac{1}{6} \) for each. Using the remark just below (13), it follows that a POVM satisfying (7)-(9) is obtained by taking all forty Penrose rays with a common weighting factor of \( c_r = \frac{4}{9} \) for each. The correctness of these POVMs can also be checked directly, but much more tediously, by verifying that they make the left side of (2) reduce identically to the unity for an arbitrary choice of system state \(|\Psi\rangle\).

We have discovered yet another POVM for \( J = \frac{3}{2} \) and \( N = 2 \) or 3, consisting of the 60 states in Table III. For \( N = 2 \) these states are to be taken with the common weighting factor of \( c_r = \frac{1}{6} \), while for \( N = 3 \) they are to be taken with the common weighting factor of \( c_r = \frac{1}{3} \). The correctness of these POVMs can be verified in two distinct ways. The first is to note that each of these 60 states is orthogonal to 15 others, makes a constant angle with 32 others, and a second constant angle with the remaining 12 others. In other words, if \( \vec{n}_1 \) denotes the
Bloch vector of any one of these states and $\vec{n}_r$ ranges over the Bloch vectors of all the others, the scalar product of $\vec{n}_1$ with $\vec{n}_r$ can assume only one of the following three values:

$$\vec{n}_1 \cdot \vec{n}_r = -\frac{1}{3} \quad \text{for 15 orthogonal rays,} \quad (16)$$

$$\vec{n}_1 \cdot \vec{n}_r = 0 \quad \text{for 32 non-orthogonal rays,} \quad (17)$$

and

$$\vec{n}_1 \cdot \vec{n}_r = \frac{1}{3} \quad \text{for 12 non-orthogonal rays.} \quad (18)$$

Using these scalar products in (7)-(9) and (10)-(13) allows us to confirm that the 60 rays, taken with the appropriate weighting factors, are indeed POVMs for $N = 2, 3$. The second way of checking the correctness of these POVMs is to substitute them into the left side of (2) and show that it reduces identically to the unity for an arbitrary choice of system state $|\Psi\rangle$. We say a word about the origin of this POVM: the first 24 states in Table III were introduced by Peres[17] and used by him to prove the Bell-Kochen-Specker theorem; we added the remaining 36 states to obtain a set satisfying all the conditions for a POVM.

This completes our presentation of optimal POVMs for $D = 4 \ (or \ J = \frac{3}{2})$ and $N = 2, 3$. We have not succeeded in finding any optimal POVMs for $D = 4$ and $N \geq 4$.

For $N = 2$ the Penrose rays provide an optimal, but perhaps not minimal, POVM since Acin et.al.[8] constructed a smaller POVM involving only 15 Bloch vectors equally inclined to each other; however Acin et.al. did not demonstrate that physical states corresponding to these Bloch vectors exist, so their solution cannot be regarded as complete. For $N = 3$ the Penrose rays definitely provide a minimal POVM because it was shown by Acin et.al.[8] that a minimal POVM in this case cannot consist of fewer than 40 states. Our 60-state POVM, though not minimal, is nevertheless of interest because of the neat solution it provides to the same problem.

4 Some open questions

The optimal POVMs for spin-1 presented in this paper all involve a large number of states. For $N = 3$, for example, our POVM involves 96 states whereas the minimal number is expected[8] to be a little above 18. We suspect, therefore, that there is considerable room for improvement in our POVMs, as far as their economy is concerned. Realizing this improvement poses an interesting mathematical challenge, but also one that is not without physical interest since more economical POVMs would likely lead to more streamlined experiments.
The POVMs presented in this paper are "special purpose" ones, tailored to specific spin values and small numbers of copies. It is clearly desirable to generalize the algorithms presented here and devise POVMs for any spin value and any finite number of copies. In the case of qubits such a general algorithm was proposed in ref.[4], where it was shown how a POVM for $N$ copies can always be constructed out of a suitable set of $(N+1)^2$ states. The technique underlying this construction is to distribute unit vectors over the Bloch sphere in such a way that a cancellation over the azimuthal angle $\phi$ is first achieved, following which a cancellation over the polar angle $\theta$ is achieved by suitably adjusting the weights, $c_r$, in the POVM. A very similar technique of angular cancellation (first over the "azimuthal" angles $\phi, \chi_1$ and $\chi_2$, followed by a cancellation over the "polar" angle $\theta$) was employed in the construction of our special purpose POVMs for spin-1. We suspect that this technique can be generalized to yield POVMs for any spin and any number of copies, but the specific way to do this has so far eluded us.

The problem of realizing the optimal POVMs proposed here as von Neumann measurements on an enlarged space of the system and an ancilla is an interesting one worth addressing. This would bring the scheme of optimal measurements one step closer to experimental realization and also help highlight any problems connected with its practical implementation.

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TABLE I. Optimal POVMs for estimating an unknown (pure) state of a spin-1 particle of which \(N = 2\) − 5 copies are available. The POVMs for \(N = 2, 3\) are based on the geometry of a 24-cell, while those for \(N = 4, 5\) are based on the geometry of a 600-cell. The states of the POVMs are constructed by combining the \(\theta\) values in column 2 with the vertices of a 24- or 600-cell in the manner indicated in eqn.(4), the total number of states in the POVM being either 24 or 120 times the number of \(\theta\) values in column 2 (this total is indicated in column 4). The "weights" associated with all states sharing a common \(\theta\) value are indicated in column 3.
TABLE II. The 40 states of the ”Penrose dodecahedron”. The ket vectors corresponding to the states are shown as row (rather than column) vectors for convenience. The first 20 states, without primes, are the ”explicit rays” while the last 20 states, with primes, are the ”implicit rays”. Each state bears a subscript indicating the vertex of the dodecahedron with which it is associated (see the papers in ref.11 for a picture of a dodecahedron with its vertices labelled with the letters used here). The states $|\Psi_F\rangle, |\Psi_E\rangle, |\Psi_B\rangle$ and $|\Psi_A\rangle$, consisting of an implicit ray and the explicit rays at the three neighboring dodecahedron vertices, form a mutually orthogonal set and are used as a basis in which the components of all the states are expressed. The symbol $p$ stands for $\exp(i\pi/5)$ and a normalization factor of $1/\sqrt{3}$ is omitted from many of the states. Note that each state is orthogonal to exactly 12 others and makes a constant angle with the remaining 27.

| $|\Psi_A\rangle$ | $|\Psi_F\rangle$ | $|\Psi_B\rangle$ | $|\Psi_E\rangle$ |
|-----------------|-----------------|-----------------|-----------------|
| $(1, p, p^2, 0)$ | $(1, 0, 0, 0)$   | $(0, 1, 0, 0)$   | $(0, 0, 1, 0)$   |
| $|\Psi_L\rangle$ | $|\Psi_G\rangle$ | $|\Psi_C\rangle$ | $|\Psi_D\rangle$ |
| $(-1, 0, p^2, 1)$| $(0, -1, p, 1)$ | $(p^2, 1, 0, 1)$| $(p, 0, 1, 1)$   |
| $|\Psi_J\rangle$ | $|\Psi_K\rangle$ | $|\Psi_R\rangle$ | $|\Psi_M\rangle$ |
| $(0, p^2, 1, -1)$| $(1, p^{-2}, 0, 1)$| $(0, p, -1, 1)$| $(p^{-1}, 0, 1, 1)$|
| $|\Psi_H\rangle$ | $|\Psi_I\rangle$ | $|\Psi_F\rangle$ | $|\Psi_Q\rangle$ |
| $(1, 0, p, -1)$   | $(1, p^2, 0, 1)$| $(p^{-2}, 1, 0, 1)$| $(0, 1, p^2, -1)$|
| $|\Psi_S\rangle$ | $|\Psi_N\rangle$ | $|\Psi_U\rangle$ | $|\Psi_T\rangle$ |
| $(1, 1, 0, 1)$    | $(0, 1, 1, -1)$ | $(-1, 0, 1, 1)$ | $(p^2, p, 1, 0)$ |
| $|\Psi'_A\rangle$ | $|\Psi'_F\rangle$ | $|\Psi'_B\rangle$ | $|\Psi'_E\rangle$ |
| $(0, 0, 0, 1)$    | $(0, p^2, 1, p)$| $(p, 0, 1, p^2)$| $(p^{-2}, p^2, 0, 1)$|
| $|\Psi'_L\rangle$ | $|\Psi'_G\rangle$ | $|\Psi'_C\rangle$ | $|\Psi'_D\rangle$ |
| $(0, 1, 1, p)$    | $(1, 0, -1, p^{-1})$| $(1, 0, -1, p)$| $(1, 1, 0, p^2)$ |
| $|\Psi'_J\rangle$ | $|\Psi'_K\rangle$ | $|\Psi'_R\rangle$ | $|\Psi'_M\rangle$ |
| $(1, 1, 0, p^{-2})$| $(0, 1, 1, p^{-1})$| $(-1, 1, p^{-1}, 0)$| $(-1, 1, p, 0)$ |
| $|\Psi'_H\rangle$ | $|\Psi'_I\rangle$ | $|\Psi'_P\rangle$ | $|\Psi'_Q\rangle$ |
| $(p^2, -1, 1, 0)$| $(p, 1, -1, 0)$| $(1, p^{-1}, 1, 0)$| $(1, p, 1, 0)$ |
| $|\Psi'_S\rangle$ | $|\Psi'_N\rangle$ | $|\Psi'_U\rangle$ | $|\Psi'_T\rangle$ |
| $(1, p^2, 0, p^{-2})$| $(0, 1, p^2, p)$| $(p, 0, p^2, 1)$| $(1, -1, 1, 0)$ |
| $|\Psi_1\rangle = (1,0,0,0)$ | $|\Psi_2\rangle = (0,1,0,0)$ | $|\Psi_3\rangle = (0,0,1,0)$ | $|\Psi_4\rangle = (0,0,0,1)$ |
| $|\Psi_5\rangle = (1,1,1,1)$ | $|\Psi_6\rangle = (-1,1,-1,1)$ | $|\Psi_7\rangle = (-1,-1,1,1)$ | $|\Psi_8\rangle = (1,-1,-1,1)$ |
| $|\Psi_9\rangle = (1,1,1,-1)$ | $|\Psi_{10}\rangle = (1,-1,-1,1)$ | $|\Psi_{11}\rangle = (1,-1,1,1)$ | $|\Psi_{12}\rangle = (1,1,-1,1)$ |
| $|\Psi_{13}\rangle = (1,0,1,0)$ | $|\Psi_{14}\rangle = (0,1,0,1)$ | $|\Psi_{15}\rangle = (1,0,-1,0)$ | $|\Psi_{16}\rangle = (0,1,0,-1)$ |
| $|\Psi_{17}\rangle = (1,1,0,0)$ | $|\Psi_{18}\rangle = (1,-1,0,0)$ | $|\Psi_{19}\rangle = (0,0,1,1)$ | $|\Psi_{20}\rangle = (0,0,1,-1)$ |
| $|\Psi_{21}\rangle = (-1,0,0,-1)$ | $|\Psi_{22}\rangle = (0,-1,-1,0)$ | $|\Psi_{23}\rangle = (-1,0,0,1)$ | $|\Psi_{24}\rangle = (0,-1,1,0)$ |
| $|\Psi_{25}\rangle = (1,i,i,1)$ | $|\Psi_{26}\rangle = (1,-i,-i,1)$ | $|\Psi_{27}\rangle = (1,-i,i,-1)$ | $|\Psi_{28}\rangle = (1,i,-i,-1)$ |
| $|\Psi_{29}\rangle = (-1,1,-i,-i)$ | $|\Psi_{30}\rangle = (-1,-1,i,i)$ | $|\Psi_{31}\rangle = (-1,1,-i,i)$ | $|\Psi_{32}\rangle = (-1,1,i,i)$ |
| $|\Psi_{33}\rangle = (-1,-i,1,-i)$ | $|\Psi_{34}\rangle = (-1,1,-i,-i)$ | $|\Psi_{35}\rangle = (-1,-i,-1,i)$ | $|\Psi_{36}\rangle = (-1,1,1,1)$ |
| $|\Psi_{37}\rangle = (1,0,1,i)$ | $|\Psi_{38}\rangle = (1,0,0,-i)$ | $|\Psi_{39}\rangle = (0,1,i,0)$ | $|\Psi_{40}\rangle = (0,1,-i,0)$ |
| $|\Psi_{41}\rangle = (1,0,i,0)$ | $|\Psi_{42}\rangle = (1,0,-i,0)$ | $|\Psi_{43}\rangle = (0,1,0,i)$ | $|\Psi_{44}\rangle = (0,1,0,-i)$ |
| $|\Psi_{45}\rangle = (1,i,0,-i)$ | $|\Psi_{46}\rangle = (1,-i,i,-i)$ | $|\Psi_{47}\rangle = (1,i,-i,1)$ | $|\Psi_{48}\rangle = (1,-i,i,1)$ |
| $|\Psi_{49}\rangle = (1,i,0,0)$ | $|\Psi_{50}\rangle = (1,-i,0,0)$ | $|\Psi_{51}\rangle = (0,0,1,i)$ | $|\Psi_{52}\rangle = (0,0,1,-i)$ |
| $|\Psi_{53}\rangle = (1,i,1,i)$ | $|\Psi_{54}\rangle = (1,-i,1,-i)$ | $|\Psi_{55}\rangle = (1,i,-1,i)$ | $|\Psi_{56}\rangle = (1,-i,-1,i)$ |
| $|\Psi_{57}\rangle = (1,i,1,i)$ | $|\Psi_{58}\rangle = (1,-i,i,-i)$ | $|\Psi_{59}\rangle = (1,1,-i,-i)$ | $|\Psi_{60}\rangle = (1,1,1,i)$ |

Table III. A set of 60 states yielding optimal POVMs for $J = \frac{3}{2}$ and $N = 2, 3$ (note that most of these states are unnormalized). The POVM for $N = 2$ is obtained by taking these states with an equal weighting factor of $c_r = \frac{1}{3}$, while the POVM for $N = 3$ is obtained by taking these states with an equal weighting factor of $c_r = \frac{1}{5}$. 