The Maximum Principle in Time-Inconsistent LQ Optimal Control Problem for Jump Diffusions

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Abstract

In this paper, we consider a general time-inconsistent optimal control problem for a non homogeneous linear system, in which its state evolves according to a stochastic differential equation with deterministic coefficients, when the noise is driven by a Brownian motion and an independent Poisson point process. The running and the terminal costs in the objective functional, are explicitly dependent on some general discounting coefficients which cover the non-exponential and the hyperbolic discounting situations. Furthermore, the presence of some quadratic terms of the conditional expectation of the state process as well as a state-dependent term in the objective functional makes the problem time-inconsistent. Open-loop Nash equilibrium controls are constructed instead of optimal controls, by using a version of the stochastic maximum principle approach. This approach involves a stochastic system that consists of a flow of forward-backward stochastic differential equations and an equilibrium condition. As an application, we study some concrete examples.

Keywords: Stochastic maximum principle, Time inconsistency, Linear quadratic control problem, Equilibrium control, Variational inequality.

MSC 2010 subject classifications, 93E20, 60H30, 93E99, 60H10.

1 Introduction

Time-inconsistent stochastic control problems have received remarkable attention in the recent years. The risk aversion attitude of a mean-variance investor [2], [3] and [9], such as the portfolio optimization with non-exponential discount function [6] and [7], provide two well-known examples of time-inconsistency in mathematical finance. Motivated by these practical examples, this paper studies optimality conditions for time-inconsistent linear quadratic stochastic control problem, where the state is described by an n-dimensional non homogeneous controlled SDE with jump processes, defined on a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\)

\[
\begin{aligned}
&dX(s) = \{A(s)X(s) + B(s)u(s) + b(s)\} \, ds + \sum_{j=1}^{d} \{C_j(s)X(s) + D_j(s)u(s) + \sigma_j(s)\} \, dW^j(s) \\
&+ \int \{E(s,z)X(s-)+F(s,z)u(s)+c(s,z)\} \, \tilde{N} (ds,dz), \; s \in [0,T], \\
X(0) &= x_0 (\in \mathbb{R}^n) .
\end{aligned}
\]

(1.1)

The coefficients \(A(\cdot), B(\cdot), b(\cdot), C_j(\cdot), D_j(\cdot), \sigma_j(\cdot), E(\cdot), F(\cdot), c(\cdot)\) are deterministic matrix-valued functions of suitable sizes. As time evolves, it is natural to consider the linear controlled stochastic

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differential equation starting from the situation $(t, \xi) \in [0, T] \times \mathbb{L}^2 (\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$

\[
\begin{cases}
    dX (s) = \{ A (s) X (s) + B (s) u (s) + b (s) \} ds + \sum_{j=1}^{d} \{ C_j (s) X (s) + D_j (t) u (s) + \sigma_j (s) \} dW^j (s) \\
    + \int \{ E (s, z) X (s-) + F (s, z) u (s) + c (s, z) \} \bar{N} (ds, dz), \ s \in [t, T], \\
    X (t) = \xi.
\end{cases}
\]  

(1.2)

Under some conditions, for any initial situation $(t, \xi)$ and any admissible control $u (.)$ the state equation is uniquely solvable, we denote by $X (.) = X^{t, \xi} (.; u (.) )$ its solution, for $s \in [t, T]$. Different controls $u (.)$ will lead to different solutions $X (.)$. To measure the performance of $u (.)$, we introduce the following cost functional

\[
J (t, \xi, u (.) ) = \mathbb{E}^t \left[ \int_t^T \frac{1}{2} \left( \langle Q (t, s) X (s), X (s) \rangle + \langle \tilde{Q} (t, s) \mathbb{E}^t [X (s)], \mathbb{E}^t [X (s)] \rangle + \langle R (t, s) u (s), u (s) \rangle \right) ds \\
+ \langle \mu_1 (t) \xi + \mu_2 (t), X (T) \rangle \\
+ \frac{1}{2} \left( \langle G (t) X (T), X (T) \rangle + \langle G (t) \mathbb{E}^t [X (T)], \mathbb{E}^t [X (T)] \rangle \right) \right].
\]  

(1.3)

The coefficients $Q (..), \tilde{Q} (..), R (..), G (..), \mu_1 (..)$ and $\mu_2 (..)$ are deterministic matrix-valued functions of suitable sizes, which explicitly depend on the initial time $t$ in some general way. Our objective in this paper, is to investigate a general discounting linear quadratic optimal control problem for jump diffusions, which is time-inconsistent in the sense that, it does not satisfy the Bellman optimality principle, since a restriction of an optimal control for a specific initial pair on a later time interval might not be optimal for that corresponding initial pair. The novelty of this work lies in the fact that, our calculations are not limited to the exponential discounting framework, the time-inconsistency of the LQ optimal controls that we are going to consider, is due to the presence of some general discounting coefficients, involving the so-called hyperbolic discounting situations. In addition, the presence of some quadratic terms of the expected controlled state process, in either the running cost or the terminal cost, make the problem time-inconsistent, this can be motivated by the reward term in the mean-variance portfolio choice model. The term $\mu_1 (t) \xi + \mu_2 (t)$ stems from a state-dependent utility function in economics [9]. Each of these terms introduces time-inconsistency of the underlying model, in somewhat different ways.

The main difficulty when facing a time-inconsistent optimal control problem is that, we cannot use the dynamic programming and the standard HJB techniques, in general. However, the main approach to handle the time-inconsistent optimal control problems, is by viewing them within a game theoretic framework. Nash equilibriums are therefore considered instead of optimal solutions, see e.g. [2], [4], [5], [6], [7], [9], [10], [14], [15], [16], [19], [20] and [21]. The fundamental idea is that the control action that the controller makes at every instant of time, is considered as a game against all the control actions that the future incarnations of the controller are going to make. Strotz [19], was the first who used this game perspective to handle the dynamic time-inconsistent decision problem on the deterministic Ramsey problem [16]. Then by capturing the idea of non-commitment, by letting the commitment period being infinitesimally small, he characterized a Nash equilibrium strategy. Further work which extend [16], are [10], [16] and [15] and [8]. Ekland and Lazrak [8] and Ekland and Pirvu [7] apply this game perspective to investigate the optimal investment-consumption problem under general discount functions, in both, deterministic and stochastic framework. Then, by means of the so-called "local spike variation" they provide a formal definition of feedback Nash equilibrium controls in continuous time. The work [2] extends the idea to the stochastic framework where the controlled process is Markovian. In addition, an extended HJB equation is derived, along with a verification argument that characterizes a Markov subgame perfect Nash equilibrium. In [19], Yong studied a time-inconsistent deterministic linear quadratic model, and he derive a closed-loop equilibrium strategy, via a forward ordinary differential equation coupled with a backward Riccati–Volterra integral equation. Hu et al [9] studied a time-inconsistent stochastic linear–quadratic control model, which is originated from the mean-variance portfolio selection problem with state-dependent risk aversion, and by means of variational method they derive a general sufficient condition for equilibria, through a new class of forward-backward stochastic differential equation (FBSDE in short) along with some equilibrium conditions. In [21] Yong investigate a time-inconsistent stochastic problem for stochastic differential equation. By introducing a family of N-person non-cooperative differential games he characterize a closed-loop equilibrium strategy.
The purpose of this paper is to characterize Nash equilibrium controls for a general time-inconsistent stochastic linear quadratic optimal control problem. The objective functional includes the cases of hyperbolic discounting, as well as, the continuous-time Markowitz’s mean-variance portfolio selection problem, with state-dependent risk aversion. We accentuate that, our model covers some class of time-inconsistent stochastic LQ optimal control problem studied by [9], and some relevant cases appeared in [20]. Note that, in [9] the weighting matrices do not depend on current time $t$ and in [20] the terminal cost do not depend on current state $\xi$. Moreover, we have defined the equilibrium controls in open-loop sense (in a manner similar to [9]), which is different from the feedback form (see [2], [5], [11], [16], [17], [30], [21] and [13]).

The rest of the paper is organized as follows. In Section 2, we describe the model and formulate the objective. In Section 3 we present the first main result of this work (Theorem 3.2), which characterizes the equilibrium control via a stochastic system, which involves a flow of forward-backward stochastic differential equation with jumps (FBSDEJ in short), along with some equilibrium conditions. In Section 4, by decoupling the flow of the FBSDEJ, we investigate a feedback representation of the equilibrium control, via some class of ordinary differential equations, which do not have a symmetry structure. Section 5 is devoted to some applications, we solve a continuous time mean–variance portfolio selection model and some one-dimensional general discounting LQ problems. The paper ends with Appendix containing some proofs.

2 Problem setting

Let $\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P$ be a filtered probability space such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t \in [0,T]}$ satisfies the usual conditions. We assume that $(\mathcal{F}_t)_{t \in [0,T]}$ is generated by a $d$-dimensional standard Brownian motion $(W(t))_{t \in [0,T]}$ and an independent Poisson measure $\mathcal{N}$ on $[0,T] \times \mathbb{Z}$ where $\mathcal{N} \subset \mathbb{R} - \{0\}$. We assume that the compensator of $\mathcal{N}$ has the form $\mu(dt, dz) = \theta (dz) dt$ for some positive and $\sigma$–finite Levy measure on $Z$, endowed with its Borel $\sigma$–field $\mathcal{B}(Z)$. We suppose that $\int 1 \wedge |z|^2 \theta (dz) < \infty$ and write $\tilde{\mathcal{N}} (dt, dz) = N (dt, dz) - \theta (dz) dt$ for the compensated jump martingal random measure of $\mathcal{N}$. Obviously, we have

$$\mathcal{F}_t = \sigma \left[ \int_{\alpha(x; \theta)} N(dr, de); s \leq t, A \in \mathcal{B}(Z) \right] \cup \sigma [B_s; s \leq t] \cup \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\nu$–null sets, and $\sigma_1 \cup \sigma_2$ denotes the $\sigma$–field generated by $\sigma_1 \cup \sigma_2$.

2.1 Notations

Throughout this paper, we use the following notations:

- $\mathbb{S}^n$ : the set of $n \times n$ symmetric real matrices.
- $C^\top$ : the transpose of the vector (or matrix) $C$.
- $\langle ., . \rangle$ : the inner product in some Euclidean space.

For any Euclidean space $H = \mathbb{R}^n$, $\mathbb{R}^{n \times m}$ or $\mathbb{S}^n$ with Frobenius norm $|.|$ we let for any $t \in [0,T]$

- $L^p (\Omega, \mathcal{F}_t, \mathbb{P}; H) := \{ \xi : \Omega \to H \mid \xi \text{ is } \mathcal{F}_t \text{ – measurable}, \text{ with } E[|\xi|^p] < \infty \}$, for any $p \geq 1$.
- $L^2 (Z, \mathcal{B}(Z), \theta; H) := \{ r(.) : Z \to H \mid r(.) \text{ is } \mathcal{B}(Z) \text{ – measurable}, \text{ with } \int_{Z} |r(z)|^2 \theta (dz) < \infty \}$.
- $\mathcal{S}_{\mathcal{F}}^2 (t, T; H) := \{ X(.) : [t, T] \times \Omega \to H \mid X(.) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} \text{ – adapted}, s \mapsto X(s) \text{ is càdlàg}, \text{ with } E \sup_{s \in [t,T]} |X(s)|^2 ds < \infty \}$.
- $L^2_{\mathcal{F}} (t, T; H) := \{ X(.) : [t, T] \times \Omega \to H \mid X(.) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} \text{ – adapted}, \text{ with } E \left[ \int_t^T |X(s)|^2 ds \right] < \infty \}$.
- $L^2_{\mathcal{F}} ([t, T] \times Z; H) := \{ R(.,.) : [t, T] \times \Omega \times Z \to H \mid R(.,.) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} \text{ – adapted measurable process on } [t,T] \times \Omega \times Z, \text{ with } E \left[ \int_t^T \int_Z |R(s, z)|^2 \theta (dz) ds \right] < \infty \}$.  

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• $C([0,T];H) := \{ f : [0,T] \to H | f(\cdot) \text{ is continuous} \}$.

• $D[0,T] := \{(t,s) \in [0,T] \times [0,T], \text{ such that } s \geq t \}$.

• $C(D[0,T];H) := \{ f(\cdot) : D[0,T] \to H | f(\cdot) \text{ is continuous} \}$.

• $C^{0,1}(D[0,T];H) := \{ f(\cdot) : D[0,T] \to H | f(\cdot) \text{ and } \frac{\partial f}{\partial s}(\cdot) \text{ are continuous} \}$.

### 2.2 Problem statement

We consider a $n$-dimensional non homogeneous linear controlled jump diffusion system

\[
\begin{aligned}
    dX(s) &= \{ A(s) X(s) + B(s) u(s) + b(s) \} \, ds + \sum_{j=1}^{d} \{ C_j(s) X(s) + D_j(t) u(s) + \sigma_j(s) \} \, dW^j(s) \\
    &\quad + \int_{Z} \{ E(s,z) X(s-)+ F(s,z) u(s) + c(s,z) \} \, \tilde{N}(ds,dz), \quad s \in [t,T],
\end{aligned}
\]

where $(t,\xi,u(\cdot)) \in [0,T] \times \mathbb{L}^2(\Omega,F_t,P;\mathbb{R}^n) \times \mathcal{L}^2_F(t,T;\mathbb{R}^m)$. Note that $\mathcal{L}^2_F(t,T;\mathbb{R}^m)$ is the space of all admissible strategies. Our aim is to minimize the following expected discounted cost functional

\[
J(t,\xi,u(\cdot)) = \mathbb{E}^t \left[ \int_t^T \frac{1}{2} \left( \langle Q(t,s) X(s), X(s) \rangle + \langle \tilde{Q}(t,s) \mathbb{E}^t [X(s)], \mathbb{E}^t [X(s)] \rangle + \langle R(t,s) u(s), u(s) \rangle \right) \, ds \right. \\
&\quad \left. + \langle \mu_1(t) \xi + \mu_2(t), X(T) \rangle + \frac{1}{2} \left( \langle G(t) X(T), X(T) \rangle + \langle \tilde{G}(t) \mathbb{E}^t [X(T)], \mathbb{E}^t [X(T)] \rangle \right) \right],
\]

over $u(\cdot) \in \mathcal{L}^2_F(t,T;\mathbb{R}^m)$, where $X(\cdot) = X^{t,\xi}(\cdot;u(\cdot))$ and $\mathbb{E}^t[\cdot] = \mathbb{E}[\cdot | F_t]$.

We need to impose the following assumptions about the coefficients

(H1) The functions $A(\cdot), C_j(\cdot) : [0,T] \to \mathbb{R}^{n \times n}, B(\cdot), D_j(\cdot) : [0,T] \to \mathbb{R}^{n \times m}, b(\cdot), \sigma_j(\cdot) : [0,T] \to \mathbb{R}^n, E(\cdot) : [0,T] \times Z \to \mathbb{R}^{n \times n}, F(\cdot) : [0,T] \times Z \to \mathbb{R}^{n \times m}$, and $c(\cdot) : [0,T] \times Z \to \mathbb{R}^n$ are continuous, and the coefficients on the cost functional satisfy

\[
\begin{aligned}
    Q(\cdot), \tilde{Q}(\cdot) &\in C(D[0,T];S^m), \\
    R(\cdot) &\in C(D[0,T];S^m), \\
    G(\cdot), \tilde{G}(\cdot) &\in C([0,T];S^m), \\
    \mu_1(\cdot) &\in C([0,T];\mathbb{R}^{n \times n}), \\
    \mu_2(\cdot) &\in C([0,T];\mathbb{R}^n).
\end{aligned}
\]

(H2) The functions $R(\cdot), Q(\cdot)$ and $G(\cdot)$ satisfy $R(t,t) \geq 0, G(t) \geq 0, \forall t \in [0,T], and Q(t,s) \geq 0, \forall (t,s) \in D[0,T]$.

Under (H1) for any $(t,\xi,u(\cdot)) \in [0,T] \times \mathbb{L}^2(\Omega,F_t,P;\mathbb{R}^n) \times \mathcal{L}^2_F(t,T;\mathbb{R}^m)$, the state equation (2.1) has a unique solution $X(\cdot) \in S^2_F(t,T;\mathbb{R}^n)$, see for example [11]. Moreover, we have the following estimate

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X(s)|^2 \right] \leq K \left( 1 + \mathbb{E} \left[ |\xi|^2 \right] \right),
\]

for some positive constant $K$. The optimal control problem can be formulated as follows.

**Problem (LQJ).** For any given initial pair $(t,\xi) \in [0,T] \times \mathbb{L}^2(\Omega,F_T,P;\mathbb{R}^n)$, find a control $\hat{u}(\cdot) \in \mathcal{L}^2_F(t,T;\mathbb{R}^m)$ such that

\[
J(t,\xi,\hat{u}(\cdot)) = \min_{u(\cdot) \in \mathcal{L}^2_F(t,T;\mathbb{R}^m)} J(t,\xi,u(\cdot))
\]
Remark 2.1. 1) The dependence of the weighting matrices of the current time \( t \), the dependence of the terminal cost on the current state \( \xi \) and the presence of quadratic terms of the expected controlled state process in the cost functional make the Problem (LQ) time-inconsistent.

2) One way to get around the time-inconsistency issue is to consider only precommitted controls (i.e., the controls are optimal only when viewed at the initial time).

2.3 An example of time-inconsistent optimal control problem

We present a simple illustration of stochastic optimal control problem which is time-inconsistent. Our aim is to show that the classical SMP approach is not efficient in the study of this problem if it’s viewed as time-consistent. For \( n = d = 1 \), consider the following controlled SDE starting from \( (t, x) \in [0, T] \times \mathbb{R} \)

\[
\begin{cases}
    dX^{t,x}(s) = bu(s) \, ds + \sigma dW(s), & s \in [t, T], \\
    X^{t,x}(t) = x, 
\end{cases}
\]

(2.3)

where \( b \) and \( \sigma \) are real constants. The cost functional given by

\[
J(t, x, u(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_t^T |u(s)|^2 \, ds + h(t) \left( X^{t,x}(T) - x \right)^2 \right],
\]

(2.4)

where \( h(\cdot) : [0, T] \to (0, \infty) \), is a general deterministic non-exponential discount function satisfying \( h(0) = 1 \), \( h(s) \geq 0 \) and \( \int_0^T h(t) \, dt < \infty \). We want to address the following stochastic control problem.

**Problem (E).** For any given initial pair \( (t, x) \in [0, T] \times \mathbb{R} \), find a control \( \bar{u}(\cdot) \in \mathcal{L}^2_T(t, T; \mathbb{R}) \) such that

\[
J(t, x, \bar{u}(\cdot)) = \min_{u(\cdot) \in \mathcal{L}^2_T(t, T; \mathbb{R})} J(t, x, u(\cdot)),
\]

At a first stage, we consider Problem (E) as a standard time consistent stochastic linear quadratic problem. Since \( J(t, x, \cdot) \) is convex and coercive, there exists then a unique optimal control for this problem for each fixed initial pair \( (t, x) \in [0, T] \times \mathbb{R} \). Notice that the usual Hamiltonian associated to this problem is closed \( \mathbb{H} : [0, T] \times \mathbb{R}^4 \to \mathbb{R} \) such that for every \( (s, y, v, p, q) \in [0, T] \times \mathbb{R}^4 \) we have

\[
\mathbb{H}(s, y, v, p, q) = pv + \sigma q - \frac{1}{2}v^2.
\]

Let \( u^{t,x}(\cdot) \) be an admissible control for \( (t, x) \in [0, T] \times \mathbb{R} \). Then the corresponding first order and second order adjoint equations are given respectively by

\[
\begin{cases}
    dp^{t,x}(s) = q^{t,x}(s) \, dW(s), & s \in [t, T], \\
    p^{t,x}(T) = -h(t) \left( X^{t,x}(T) - x \right),
\end{cases}
\]

and

\[
\begin{cases}
    dP^{t,x}(s) = Q^{t,x}(s) \, dW(s), & s \in [t, T], \\
    P^{t,x}(T) = -h(t),
\end{cases}
\]

the last equation has only the solution \( (P^{t,x}(s), Q^{t,x}(s)) = (-h(t), 0) \), \( \forall s \in [t, T] \).

Note that, the corresponding \( \mathbb{H} \)-function is given by

\[
\mathbb{H}(s, y, v) = \mathbb{H}(s, y, v, p^{t,x}(s), q^{t,x}(s)) = p^{t,x}(s) y + \sigma q^{t,x}(s) - \frac{1}{2}v^2,
\]

which is a concave function of \( v \). Then according to the sufficient condition of optimality, see e.g. Theorem 5.2 pp 138 in [13], for any fixed initial pair \( (t, x) \in [0, T] \times \mathbb{R} \), Problem (E) is uniquely solvable with an optimal control \( \bar{u}^{t,x}(\cdot) \) having the representation

\[
\bar{u}^{t,x}(s) = b p^{t,x}(s), \text{ } \forall s \in [t, T],
\]

such that the process \((p^{t,x}(\cdot), q^{t,x}(\cdot))\) is the unique adapted solution to the BSDE

\[
\begin{cases}
    dp^{t,x}(s) = q^{t,x}(s) \, dW(s), & s \in [t, T], \\
    p^{t,x}(T) = -h(t) \left( X^{t,x}(s) - x \right).
\end{cases}
\]
By standard arguments we can show that the processes \( \bar{p}^{t,x}(\cdot), \bar{q}^{t,x}(\cdot) \) are explicitly given by

\[
\begin{align*}
\bar{p}^{t,x}(s) &= -M^t(s)(\bar{X}^{t,x}(s) - x), \quad s \in [t, T], \\
\bar{q}^{t,x}(s) &= -\sigma M^t(s), \quad s \in [t, T],
\end{align*}
\]

where \( \bar{X}^{t,x}(\cdot) \) is the solution of the state equation corresponding to \( \bar{u}^{t,x}(\cdot) \), given by

\[
\begin{align*}
d\bar{X}^{t,x}(s) &= b^2 \bar{p}^{t,x}(s) \, ds + \sigma dW(s), \quad s \in [t, T], \\
\bar{X}^{t,x}(t) &= x.
\end{align*}
\]

and

\[
M^t(s) = \frac{h(t)}{b^2 h(t)(T-s) + 1}, \quad \forall s \in [t, T].
\]

A simple computation show that

\[
\bar{\pi}^{t,x}(s) = -\frac{bh(t)}{b^2 h(t)(T-s) + 1} \left( \bar{X}^{t,x}(s) - x \right), \quad \forall s \in [t, T],
\]

clearly we have

\[
\bar{\pi}^{t,x}(s) \neq 0, \quad \forall s \in (t, T]. \quad (2.5)
\]

In the next stage, we will see that Problem (E) is time-inconsistent, for this we first fix the initial data \( (t,x) \in [0,T] \times \mathbb{R} \). Note that, if we assume that the Problem (E) is time-consistent, in the sense that for any \( r \in [t,T] \) the restriction of \( \bar{u}^{t,x}(\cdot) \) on \( [r,T] \) is optimal for Problem (E) with initial pair \( (r, \bar{X}^{t,x}(r)) \), however as Problem (E) is uniquely solvable for any initial pair, we should have then \( \forall r \in (t,T] \)

\[
\bar{u}^{t,x}(s) = \bar{u}^{r,X^{t,x}(r)}(s) = -\frac{bh(r)}{b^2 h(r)(T-s) + 1} \left( \bar{X}^{r,X^{t,x}(r)}(s) - \bar{X}^{t,x}(r) \right), \quad \forall s \in [r, T],
\]

where \( \bar{X}^{r,X^{t,x}(r)}(\cdot) \) solves the SDE

\[
\begin{align*}
d\bar{X}^{r,X^{t,x}(r)}(s) &= b^2 \frac{h(r)}{b^2 h(r)(T-s) + 1} \left( \bar{X}^{r,X^{t,x}(r)}(s) - \bar{X}^{t,x}(r) \right) \, ds + \sigma dW(s), \quad \forall s \in [r, T], \\
\bar{X}^{r,X^{t,x}(r)}(r) &= \bar{X}^{t,x}(r).
\end{align*}
\]

In particular by the uniqueness of solution to the state SDE we should have

\[
\bar{u}^{t,x}(r) = -\frac{bh(r)}{b^2 h(r)(T-r) + 1} \left( \bar{X}^{r,X^{t,x}(r)}(r) - \bar{X}^{t,x}(r) \right) = 0,
\]

is the only optimal solution of the Problem (E), this contradict (2.5). Therefore, Problem (E) is not time-consistent, and more precisely, the solution obtained by the classical SMP is wrong and the problem is rather trivial since the only optimal solution equal to zero.

\section{Characterization of equilibrium strategies}

The purpose of this paper is to characterize open-loop Nash equilibriums instead of optimal controls. We use the game theoretic approach to handle the time inconsistency in the same perspective as Ekeland and Lazrak \[6\], Bjork and Murgoci \[2\]. Let us briefly describe the game perspective that we will consider, as follows.

\begin{itemize}
  \item We consider a game with one player at each point \( t \) in \([0,T]\). This player represents the incarnation of the controller at time \( t \) and is referred to as “player \( t \)”.
  \item The \( t-th \) player can control the system only at time \( t \) by taking his/her strategy \( u(t,\cdot) : \Omega \to \mathbb{R}^m \).
  \item A control process \( u(\cdot) \) is then viewed as a complete description of the chosen strategies of all players in the game.
  \item The reward to the player \( t \) is given by the functional \( J (t,\xi,u(\cdot)) \). Note that \( J (t,\xi,u(\cdot)) \) depends only on the restriction of the control \( u(\cdot) \) to the time interval \([t,T]\).
\end{itemize}
In the above description, we have presented the concept of a "Nash equilibrium point" of the game. This is an admissible control process \( \hat{u}(\cdot) \) satisfying the following condition: Suppose that every player \( s \), such that \( s > t \), will use the strategy \( \hat{u}(s) \). Then the optimal choice for player \( t \) is that, he/she also uses the strategy \( \hat{u}(t) \).

Nevertheless, the problem with this "definition", is that the individual player \( t \) does not really influence the outcome of the game at all. He/she only chooses the control at the single point \( t \), and since this is a time set of Lebesgue measure zero, the control dynamics will not be influenced. Therefore, to characterize open-loop Nash equilibrium control, we define an equilibrium by local spike variation, given an admissible control \( \hat{u}(\cdot) \) for Problem (LQJ) if

\[
\text{Definition 3.1} \quad \text{Nash equilibrium control for Problem (LQJ) if}
\]

\[
\text{In addition, sometimes we simply call } \hat{u}(\cdot) \text{ an equilibrium control instead of open-loop Nash equilibrium control when there is no ambiguity.}
\]

We define an equilibrium by local spike variation, given an admissible control \( \hat{u}(\cdot) \in \mathcal{L}_F^2(0,T;\mathbb{R}^m) \). For any \( t \in [0,T] \), \( v \in \mathcal{L}^2(\Omega,F_t,P;\mathbb{R}^m) \) and for any \( \varepsilon > 0 \), define

\[
u^\varepsilon(s) = \begin{cases} \hat{u}(s) + v, & \text{for } s \in [t,t+\varepsilon), \\ \hat{u}(s), & \text{for } s \in [t+\varepsilon,T), \end{cases}
\]

(3.1)

we have the following definition.

**Definition 3.1** (Open-loop Nash equilibrium). An admissible strategy \( \hat{u}(\cdot) \in \mathcal{L}_F^2(0,T;\mathbb{R}^m) \) is an open-loop Nash equilibrium control for Problem (LQJ) if

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ J(t,\hat{X}(t),u^\varepsilon(\cdot)) - J(t,\hat{X}(t),\hat{u}(\cdot)) \right\} \geq 0,
\]

(3.2)

for any \( t \in [0,T] \), and \( v \in \mathcal{L}^2(\Omega,F_t,P;\mathbb{R}^m) \). The corresponding equilibrium dynamics solves the following SDE with jumps

\[
\begin{aligned}
d\hat{X}(s) &= \left\{ A\hat{X}(s) + B\hat{u}(s) + b \right\} ds + \sum_{j=1}^{d} \left\{ C_j\hat{X}(s) + D_j\hat{u}(s) + \sigma_j \right\} dW^j(s) \\
&\quad + \int_Z \left\{ E(z)\hat{X}(s-) + F(z)\hat{u}(s) + c(z) \right\} \tilde{N}(ds,dz), s \in [0,T],
\end{aligned}
\]

\( \hat{X}_0 = x_0. \)

### 3.1 The flow of adjoint equations

First, we introduce the adjoint equations involved in the stochastic maximum principle which characterize the open-loop Nash equilibrium controls of Problem (LQJ). Define the Hamiltonian \( \mathcal{H} : D[0,T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R} \) by

\[
\mathcal{H}(t,s,X,u,p,q,r(\cdot)) = \langle p, AX + Bu + b \rangle + \sum_{j=1}^{d} (q_j, D_j X + C_j u + \sigma_j) - \frac{1}{2} \langle R(t,s)u,u \rangle \\
+ \int_Z \langle r(z), E(z)X + F(z)u + c(z) \rangle \theta(dz) - \frac{1}{2} \left( \langle Q(t,s)X,X \rangle + \langle \tilde{Q}(t,s)E^t[X],E^t[X] \rangle \right).
\]

(3.3)

Let \( \hat{u}(\cdot) \in \mathcal{L}_F^2(0,T;\mathbb{R}^m) \) and denote by \( \hat{X}(\cdot) \) the corresponding controlled state process. For each \( t \in [0,T] \), we introduce the first order adjoint equation defined on the time interval \([t,T]\), and satisfied by the triple of processes \( (p(\cdot,t),q(\cdot,t),r(\cdot,t)) \) as follows

\[
\begin{aligned}
dp(s;t) &= -\left\{ A^t p(s;t) + \sum_{j=1}^{d} C_{j}^t q_{j}(s;t) + \int_Z E(z)^t r(s,z;t) \theta(dz) - Q(t,s)\hat{X}(s) \right\} \\
&\quad - \bar{Q}(t,s)E^t\left[\hat{X}(s)\right] ds + \sum_{j=1}^{d} q_{j}(s;t) dW^j(s) + \int_Z r(s-,z;t) \tilde{N}(ds,dz), s \in [t,T],
\end{aligned}
\]

\[
p(T;t) = -G(t)\hat{X}(T) - G(t)E^t\left[\hat{X}(T)\right] - \mu_1(t)\hat{X}(t) - \mu_2(t),
\]

(3.4)
where \( q(.; t) = (q_1(.; t), ..., q_d(.; t)) \). Similarly, we introduce the second order adjoint equation defined on the time interval \([t, T]\), and satisfied by the triple of processes \((P(.; t), \Lambda (.; t), \Gamma (.; ; t))\) as follows

\[
\begin{aligned}
\left\{ \begin{array}{l}
    dP(s; t) = - \left\{ A^T P(s; t) + P(s; t) A + \sum_{j=1}^d C_j^T P(s; t) C_j \\
    + C_j^T \Lambda (s; t)) + \int_Z E(z)^T (\Gamma (s, z; t) + P(s; t)) E(z) \theta (dz) \\
    + \int_Z E(z)^T \Gamma (s, z; t) \theta (dz) - Q(t, s) \right\} ds \\
    + \sum_{j=1}^d \Lambda (s; t) dW^j_s + \int_Z \Gamma (s, z; t) \tilde{N}(ds, dz), \ s \in [t, T],
\end{array} \right.
\end{aligned}
\]

(3.5)

where \( \Lambda (.; t) = (\Lambda_1 (.; t), ..., \Lambda_d (.; t)) \). Under \( (H1) \) the BSDE (3.4) is uniquely solvable in \( \mathcal{S}^2 (t, T; \mathbb{R}^n) \times \mathcal{L}^2 (t, T; \mathbb{R}^{n \times d}) \times \mathcal{L}^{d, 2} ([t, T] \times Z; \mathbb{R}^n) \), see e.g. \[11\]. Moreover there exists a constant \( K > 0 \) such that

\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |p(s; t)|^2 \right] + \mathbb{E} \left[ \int_t^T |q(s; t)|^2 ds \right] + \mathbb{E} \left[ \int_t^T \int_Z |r(s, z; t)|^2 \theta (dz) ds \right] \leq K \left( 1 + |x_0|^2 \right).
\]

(3.6)

In another hand, noting that the final data of the equation (3.5) is deterministic, it is straightforward to look at a deterministic solution. In addition we have the following representation

\[
\left\{ \begin{array}{l}
    dP(s; t) = - \left\{ A^T P(s; t) + P(s; t) A + \sum_{j=1}^d C_j^T P(s; t) C_j \\
    + \int_Z E(z)^T P(s; t) E(z) \theta (dz) - Q(t, s) \right\} ds, \ s \in [t, T],
\end{array} \right.
\]

(3.7)

which is a uniquely solvable matrix-valued ordinary differential equation. Indeed, if we define the function \( \Phi (s, .) \) for each \( s \in [0, T] \), as the fundamental solution of the following linear SDE

\[
\left\{ \begin{array}{l}
    d\Phi (s, r) = A (r) \Phi (s, r) dr + \sum_{j=1}^d C_j (r) \Phi (s, r) dW^j (r) + \int_Z E (r, z) \Phi (s, r) \tilde{N}(dr, dz), \ r \in [s, T],
\end{array} \right.
\]

(3.8)

\( \Phi (s, s) = I \).

Then, by standard arguments based on the Ito’s formula we can prove that the triple \((P (.; t), \Lambda (.; t), \Gamma (.; ; t))\) solution to (3.7) is explicitly given by

\[
\left\{ \begin{array}{l}
    P(s; t) = \mathbb{E}^s \left[ - \Phi (s, T)^T G(t) \Phi (s, T) - \int_s^T \Phi (s, r)^T Q(r, \Phi (s, r)) ds \right], \ s \in [t, T], \\
    \Lambda_j (s; t) = 0, \ s \in [t, T], \text{ for } j = 1, 2, ..., d, \\
    \Gamma (s, z; t) = 0, \ (s, z) \in [t, T] \times Z.
\end{array} \right.
\]

(3.9)

Next, for each \( t \in [0, T] \), associated with the 6-tuple \((\hat{u}(.), \hat{X}(.), p(.; t), q(.; t), r (.; ; t), P (.; ; t))\) we define the \( \mathcal{H}_t \)-function as follows

\[
\mathcal{H}_t (s, X, u) = \mathbb{H} (t, s, X, \hat{u} (s) + u, p(s; t), q(s; t), r (s; ; t)) + \frac{1}{2} \sum_{j=1}^d u^T D_j^T P(s; t) D_j u \\
+ \frac{1}{2} \int_Z u^T F(z)^T P(s; t) F(z) u \theta (dz),
\]

(3.10)

where \((s, X, u) \in [t, T] \times L^1 (\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \times \mathbb{R}^m \).
3.2 A stochastic maximum principle for equilibrium controls

In this section, we present a version of Pontryagin’s stochastic maximum principle which characterizes the equilibrium controls of Problem (LQJ). We derive the result by using the second order Taylor expansion in the special form spike variation (3.1). Here, we don’t assume the non-negativity condition about the matrices $Q, G$ and $R$ as in [9] and [20].

The following theorem is the first main result of this work, it’s providing a necessary and sufficient condition to characterize the open-loop Nash equilibrium controls for time-inconsistent Problem (LQJ).

**Theorem 3.2 (Stochastic Maximum Principle For Equilibriums).** Let \( (H1) \) holds. Then an admissible control \( \bar{u}(.) \in L^2_F(0,T;\mathbb{R}^m) \) is an open-loop Nash equilibrium, if and only if, for any \( t \in [0,T] \), there exist a unique triple of adapted processes \( (p(.;t), q(.;t), r(.;.;t)) \) which satisfy the BSDE (3.4) and a deterministic matrix-valued function \( P(.;t) \) which satisfies the ODE (3.7), such that the following condition holds, for all \( u \in \mathbb{R}^m \),

\[
\begin{align*}
\mathbb{H} \left( t, t, \bar{X}(t), \bar{u}(t) + u, p(t; t), q(t; t), r(t; ..; t) \right) & - \mathbb{H} \left( t, t, \bar{X}(t), \bar{u}(t), p(t; t), q(t; t), r(t; ..; t) \right) \\
+ \frac{1}{2} \sum_{j=1}^{d} u^T D_j (t)^T P(t; t) D_j (t) u + \frac{1}{2} \int_{Z} u^T F(t, z)^T P(t; t) F(t, z) u \theta (dz) \leq 0, \quad \mathbb{P} - \text{a.s.}
\end{align*}
\]

or equivalently, we have the following two conditions, the first order equilibrium condition

\[
R(t,t) \bar{u}(t) - B(t) p(t; t) - \sum_{j=1}^{d} D_j (t)^T q_j (t; t) - \int_{Z} F(t, z)^T r(t, z; t) \theta (dz) = 0, \quad \mathbb{P} - \text{a.s},
\]

and the second order equilibrium condition

\[
R(t,t) - \sum_{j=1}^{d} D_j (t)^T P(t; t) D_j (t) - \int_{Z} F(t, z)^T P(t; t) F(t, z) \theta (dz) \geq 0.
\]

We point that the above result provides a characterisation of open-loop Nash equilibrium controls via a stochastic maximum principle which is not in the same setting that the classical stochastic maximum principle for optimal controls [22] in the sense that, the above result involves the existence of solutions \( \left( \bar{X}(.), (p(.;t), q(.;t), r(.;.;t))_{t \in [0,T]} \right) \) to a "flow" of forward-backward stochastic differential equations parameterized by \( t \in [0,T] \), while the Pontryagin’s stochastic maximum principle for optimal controls involve only one system of forward-backward stochastic differential equation. Note that for each \( t \in [0,T] \), (3.4) and (3.5) are backward stochastic differential equations. So, as we consider all \( t \in [0,T] \), all their corresponding adjoint equations form essentially a "flow" of BSDEs. Moreover, there is an additional constraint (3.11) which is equivalent to the conditions (3.12) and (3.13) that acts on the flow only when \( s = t \).

Our goal now, is to give a proof of the Theorem 3.2. The main idea is still based on the variational techniques in the same spirit of proving the stochastic Pontryagin’s maximum principle [17].

Let \( \bar{u}(.) \in L^2_F(0,T;\mathbb{R}^m) \) be an admissible control and \( \bar{X}(.) \) the corresponding controlled process solution to the state equation. Consider the perturbed control \( u^\varepsilon(.) \) defined by the spike variation (3.1) for some fixed arbitrary \( t \in [0,T] \), \( v \in L^2 _{\mathcal{F}_t} (\Omega,\mathbb{F},\mathbb{P};\mathbb{R}^m) \) and \( \varepsilon \in [0,T-t] \). Denote by \( \bar{X}^\varepsilon(.) \) the solution of the state equation corresponding to \( u^\varepsilon(.) \). Since the coefficients of the controlled state equation are linear, then by the standard perturbation approach, see e.g. [17], we have

\[
\bar{X}^\varepsilon (s) - \bar{X} (s) = y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s), \quad s \in [t,T],
\]

where \( y^{\varepsilon,v}(.) \) and \( z^{\varepsilon,v}(.) \) solve the following linear stochastic differential equations, respectively

\[
\begin{align*}
\begin{cases}
\,dy^{\varepsilon,v} (s) = Ay^{\varepsilon,v} (s) ds + \sum_{j=1}^{d} \{ C_j y^{\varepsilon,v} (s) + D_j v_{1[t,t+\varepsilon]} (s) \} dW^j (s) \\
+ \int_{Z} \left\{ E (z) y^{\varepsilon,v} (s-) + F (z) v_{1[t,t+\varepsilon]} (s) \right\} \tilde{N} (ds,dz), \quad s \in [t,T],
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
    d z^{\varepsilon,v}(s) &= \left\{ A z^{\varepsilon,v}(s) + B v 1_{[t,t+\varepsilon)}(s) \right\} ds + \sum_{j=1}^{d} C_j z^{\varepsilon,v}(s) dW^j(s) \\
    &\quad + \int_{Z} E(z) z^{\varepsilon,v}(s) \tilde{N}(ds,dz), \ s \in [t,T],
\end{align*}
\] (3.16)

Now, we present the following technical lemma needed later in this study, see the Appendix A.1. for its proof.

**Lemma 3.3.** Under assumption (H1), the following estimates hold
\[
    \mathbb{E}^t \left[ |y^\varepsilon(s)|^2 \right] = 0, \ a.e. \ s \in [t,T] \ and \ \sup_{s \in [t,T]} \mathbb{E}^t \left[ |z^\varepsilon(s)|^2 \right] = O(\varepsilon^2),
\] (3.17)
\[
    \mathbb{E}^t \left[ \sup_{s \in [t,T]} |y^\varepsilon(s)|^2 \right] = O(\varepsilon) \ and \ \mathbb{E}^t \left[ \sup_{s \in [t,T]} |z^\varepsilon(s)|^2 \right] = O(\varepsilon^2). \] (3.18)

Moreover, we have the equality
\[
    J \left( t, \hat{X}(t), u^\varepsilon(.) \right) - J \left( t, \hat{X}(t), \hat{u}(.) \right) = -\mathbb{E}^t \left\{ \mathbb{H} \left( t, s, \hat{X}(s), \hat{u}(s) + v, p(s;t), q(s;t), r(s,.;t) \right) \\
    \quad \quad \quad - \mathbb{H} \left( t, s, \hat{X}(s), \hat{u}(s), p(s;t), q(s;t), r(s,.;t) \right) \right\} 1_{(t,t+\varepsilon]}(s) ds \\
    + \frac{1}{2} \sum_{j=1}^{d} v^\top D_j^T P(s;t) D_j v + \frac{1}{2} \int_{Z} v^\top F(z)^\top P(s;t) F(z) v\theta(dz) \right\} 1_{(t,t+\varepsilon]}(s) ds + o(\varepsilon).
\] (3.19)

Now, we are ready to give the proof of the Theorem 3.2.

**Proof of Theorem 3.2.** Given an open-loop Nash equilibrium \( \hat{u}(.) \), then for any \( t \in [0,T] \) and \( v \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m) \), we have clearly
\[
    \lim_{\varepsilon \downarrow 0} \mathbb{E}^t \left\{ J \left( t, \hat{X}(t), \hat{u}(.) \right) - J \left( t, \hat{X}(t), u^\varepsilon(.) \right) \right\} \leq 0,
\]
from which we deduce
\[
    \lim_{\varepsilon \downarrow 0} \frac{1}{2} \mathbb{E}^t \left\{ \int_{t}^{T} \mathbb{H} \left( t, s, \hat{X}(s), \hat{u}(s) + v, p(s;t), q(s;t), r(s,.;t) \right) \\
    \quad \quad \quad - \mathbb{H} \left( t, s, \hat{X}(s), \hat{u}(s), p(s;t), q(s;t), r(s,.;t) \right) \right\} 1_{(t,t+\varepsilon]}(s) ds \leq 0,
\]
which leads to
\[
    \mathbb{H} \left( t, t, \hat{X}(t), \hat{u}(t) + v, p(t;t), q(t;t), r(t,.;t) \right) - \mathbb{H} \left( t, t, \hat{X}(t), \hat{u}(t), p(t;t), q(t;t), r(t,.;t) \right) \\
    + \frac{1}{2} \sum_{j=1}^{d} v^\top D_j^T (p(t) D_j v + \frac{1}{2} \int_{Z} v^\top F(z)^\top P(t) F(z) v\theta(dz) \right\} 1_{[t,T]}(s) ds \leq 0, \ \mathbb{P} - a.s.
\]

Therefore, by setting \( v \equiv u \) for an arbitrarily \( u \in \mathbb{R}^m \) we obtain (3.11).

Conversely, given an admissible control \( \hat{u}(.) \in L^2_{\mathcal{P}}(0,T;\mathbb{R}^m) \). Suppose that for any \( t \in [0,T] \), the variational inequality (3.11) holds. Then for any \( v \in L^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m) \) it yields
\[
    \mathbb{H} \left( t, t, \hat{X}(t), \hat{u}(t) + v, p(t;t), q(t;t), r(t,.;t) \right) - \mathbb{H} \left( t, t, \hat{X}(t), \hat{u}(t), p(t;t), q(t;t), r(t,.;t) \right) \\
    + \frac{1}{2} \sum_{j=1}^{d} v^\top D_j^T (p(t) D_j v + \frac{1}{2} \int_{Z} v^\top F(t,z)^\top P(t) F(t,z) v\theta(dz) \right\} 1_{[t,T]}(s) ds \leq 0, \ \mathbb{P} - a.s.
\]
which satisfies the BSDE
\[ (3) \]
control, if and only if, for any \( t \)

Corollary 3.4.
is obviously satisfied. Therefore, we summarize the main theorem into the following Corollary.

In this section, we consider only the case where the Brownian motion is one-dimensional \((d = 1)\) for simplicity of presentation. There is no essential difficulty with the multidimensional Brownian motions. All the indices \( j \) will then be dropped. Our goal is to obtain a state feedback representation of an equilibrium control for Problem (LQJ) via some class of ordinary differential equations. Suppose that \( \hat{u} (\cdot) \) is an equilibrium control and denote by \( \hat{X} (\cdot) \) the corresponding controlled process. Then in view of Theorem 3.2, there exists a flow of triple of adapted processes \((p (\cdot; t), q (\cdot; t), r (\cdot, \cdot; t))_{t \in [0, T]}\) for which the 3-tuple \( (\hat{u} (\cdot), \hat{X} (\cdot), (p (\cdot; t), q (\cdot; t), r (\cdot, \cdot; t))_{t \in [0, T]}) \) solves the following flow of forward-backward SDE with jumps,
parametrized by \( t \in [0, T] \)

\[
\begin{align*}
    d\tilde{X}(s) &= \left\{ A\tilde{X}(s) + B\tilde{u}(s) + b \right\} ds + \left\{ C\tilde{X}(s) + D\tilde{u}(s) + \sigma \right\} dW(s) \\
    &\quad + \int_Z \left\{ E(z) \tilde{X}(s-) + F(z) \tilde{u}(s) + c(z) \right\} \tilde{N}(ds, dz), \quad s \in [0, T], \\
    dp(s; t) &= -\left\{ A^T p(s; t) + C^T q(s; t) + \int_Z E(z)^T r(s, z; t) \theta(dz) - Q(t, s) \tilde{X}(s) \right\} ds + q(s; t) dW(s) + \int_Z r(s-, z; t) \tilde{N}(ds, dz), \quad 0 \leq t \leq s \leq T, \\
    \tilde{X}_0 &= x_0, \quad p(T; t) = -G(t) \tilde{X}(T) - G(t) \mathbb{E}^f \left[ \tilde{X}(T) \right] - \mu_1(t) \tilde{X}(t) - \mu_2(t), \quad t \in [0, T],
\end{align*}
\]

with the condition

\[
R(t, t) \tilde{u}(t) - B^T p(t; t) - D(t)^T q(t; t) - \int_Z F(t, z)^T r(t, z; t) \theta(dz) = 0, \quad P - a.s, \quad \forall t \in [0, T].
\]

Now, to solve the above stochastic system, we conjecture that \( \tilde{X}(\cdot) \) and \( p(\cdot; t) \) for \( t \in [0, T] \) are related by the following relation

\[
    p(s; t) = -M(t, s) \tilde{X}(s) - \tilde{M}(t, s) \mathbb{E}^f \left[ \tilde{X}(s) \right] - \mathbb{Y}(t, s) \tilde{X}(t) - \varphi(t, s), \quad (t, s) \in D[0, T],
\]

for some deterministic functions \( M(\cdot, \cdot), \tilde{M}(\cdot, \cdot), \mathbb{Y}(\cdot, \cdot) \in C^{0,1}(D[0, T], \mathbb{R}^{n \times n}) \) and \( \varphi(\cdot, \cdot) \in C^{0,1}(D[0, T], \mathbb{R}^n) \) such that

\[
    M(t, T) = G(t), \quad \tilde{M}(t, T) = \tilde{G}(t), \quad \mathbb{Y}(t, T) = \mu_1(t), \quad \varphi(t, T) = \mu_2(t), \quad t \in [0, T].
\]

Applying Itô’s formula to (4.3) and using (4.1), it yields

\[
\begin{align*}
    dp(s; t) &= \left\{ -\frac{\partial M}{\partial s}(t, s) \tilde{X}(s) - \frac{\partial \tilde{M}}{\partial s}(t, s) \tilde{X}(s) - \frac{\partial \mathbb{Y}}{\partial s}(t, s) \tilde{X}(t) - \frac{\partial \varphi}{\partial s}(t, s) \right\} ds \\
    &\quad - M(t, s) \left( A\tilde{X}(s) + Bu(s) + b \right) - \tilde{M}(t, s) \left( A\mathbb{E}^f \left[ \tilde{X}(s) \right] + B\mathbb{E}^f [u(s)] + b \right) ds \\
    &\quad - \int_Z M(t, s) \left( E(z) \tilde{X}(s-) + F(z) \tilde{u}(s) + c(z) \right) \tilde{N}(ds, dz), \\
    &= \left\{ A^T p(s; t) + C^T q(s; t) + \int_Z E(z)^T r(s, z; t) \theta(dz) - Q(t, s) \tilde{X}(s) \right\} ds + q(s; t) dW(s) + \int_Z r(s-, z; t) \tilde{N}(ds, dz), \quad s \in [t, T],
\end{align*}
\]

and we obtain

\[
\begin{align*}
    q(s; t) &= -M(t, s) \left( C\tilde{X}(s) + D\tilde{u}(s) + \sigma \right), \quad \text{for } t \in [0, T], \\
    r(s, z; t) &= -M(t, s) \left( E(z) \tilde{X}(s) + F(z) \tilde{u}(s) + c(z) \right), \quad \text{for } t \in [0, T].
\end{align*}
\]
By taking (4.6) and (4.7) in (4.2) we get

\[
0 = R(t,t) \dot{u}(t) + B(t)^	op \left( (M(t,t) + \ddot{M}(t,t) + \Upsilon(t,t)) \dot{X}(t) + \varphi(t,t) \right) \\
+ D(t)^	op M(t,t) \left( C(t) \dot{X}(t) + D(t) \dot{u}(t) + \sigma(t) \right) \\
+ \int_Z F(t,z)^	op M(t,t) \left( E(t,z) \dot{X}(t) + F(t,z) \dot{u}(t) + c(t,z) \right) \theta(dz),
\]

\[
= \left\{ R(t,t) + D(t)^	op M(t,t) D(t) + \int_Z F(t,z)^	op M(t,t) F(t,z) \theta(dz) \right\} \dot{u}(t) \\
+ \left\{ B(t)^	op \left( (M(t,t) + \ddot{M}(t,t) + \Upsilon(t,t)) + D(t)^	op M(t,t) C(t) \right) \\
+ \int_Z F(t,z)^	op M(t,t) E(t,z) \theta(dz) \right\} \dot{X}(t) \\
+ B(t)^	op \varphi(t,t) + D(t)^	op M(t,t) \sigma(t) + \int_Z F(t,z)^	op M(t,t) c(t,z) \theta(dz).
\]

Thus if we assume that \( \Theta(t) = \left( R(t,t) + D(t)^	op M(t,t) D(t) + \int_Z F(t,z)^	op M(t,t) F(t,z) \nu(dz) \right)^{-1} \) exists, then we deduce that \( \dot{u}(\cdot) \) admits the following feedback representation

\[
\dot{u}(t) = -\Psi(t) \dot{X}(t) - \psi(t),
\]

where \( \Psi(t) \) and \( \psi(t) \) are given by

\[
\left\{ \begin{array}{l}
\Psi(t) = \Theta(t) \left[ B(t)^	op (M(t,t) + \ddot{M}(t,t) + \Upsilon(t,t)) + D(t)^	op M(t,t) \right. \\
+ \int_Z F(t,z)^	op M(t,t) E(t,z) \theta(dz),
\end{array} \right.
\]

\[
\psi(t) = \Theta(t) \left[ B(t)^	op \varphi(t,t) + D(t)^	op M(t,t) \sigma(t) + \int_Z F(t,z)^	op M(t,t) c(t,z) \theta(dz) \right].
\]

Therefore, for any \((t,s) \in D[0,T]\), we have

\[
\mathbb{E}^t [\dot{u}(s)] = \Psi(s) \mathbb{E}^t [\dot{X}(s)] + \psi(s).
\]

Next, comparing the \( ds \) term in (4.5) by the one in (4.1), then by using the expressions (4.8) and (4.10), we obtain

\[
0 = \left\{ \frac{\partial M}{\partial s}(t,s) + M(t,s) A + A^\top M(t,s) + C^\top M(t,s) C + \int_Z E(z)^	op M(t,s) E(z) \theta(dz) \right. \\
- \left( M(t,s) B + C^\top M(t,s) D + \int_Z E(z)^	op M(t,s) F(z) \theta(dz) \right) \Psi(s) + Q(t,s) \right\} \dot{X}(s) \\
+ \left\{ \frac{\partial \dot{M}}{\partial s}(t,s) + \ddot{M}(t,s) A + A^\top \ddot{M}(t,s) - \ddot{M}(t,s) B\Psi(s) + \dot{Q}(t,s) \right\} \mathbb{E}^t [\dot{X}(s)] \\
+ \left\{ \frac{\partial \Upsilon}{\partial s}(t,s) + A^\top \Upsilon(t,s) \right\} \dot{X}(t) \\
+ \frac{\partial \varphi}{\partial s}(t,s) + (M(t,s) + \ddot{M}(t,s)) (b - B\psi(s)) + A^\top \varphi(t,s) \\
+ C^\top M(t,s) (\sigma - D\psi(s)) + \int_Z E(z)^	op M(t,s) (c(z) - F(z) \psi(s)) \theta(dz)
\]

This suggests that the functions \( M(\cdot,\cdot) \), \( \ddot{M}(\cdot,\cdot) \), \( \Upsilon(\cdot,\cdot) \) and \( \varphi(\cdot,\cdot) \) solve the following system of ordinary
differential equations, for \((t, s) \in D[0, T]\)

\[
\begin{aligned}
0 &= \frac{\partial M}{\partial s}(t, s) + M(t, s) A + A^T M(t, s) C + \int_Z E(z)^T M(t, s) E(z) \theta(dz) \\
&\quad - \left(\int_M(t, s) B + C^T M(t, s) D + \int_Z E(z)^T M(t, s) F(z) \theta(dz)\right) \Psi(s) + Q(t, s), \\
0 &= \frac{\partial M}{\partial t}(t, s) + \dot{M}(t, s) A + A^T \dot{M}(t, s) - M(t, s) B \Psi(s) + \dot{Q}(t, s), \\
0 &= \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial s}(t, s) + A^T \frac{\partial}{\partial s}(t, s), \\
0 &= \frac{\partial}{\partial s}(t, s) + (M(t, s) + \dot{M}(t, s))(b - B \psi(s)) + A^T \varphi(t, s) + C^T M(t, s)(\sigma - D \psi(s)) \\
&\quad + \int_Z (E(z)^T M(t, s) c(z) - F(z) \psi(s)) \theta(dz), \\
M(t, T) &= G(t), \quad \dot{M}(t, T) = \dot{G}(t), \quad \Upsilon(t, t) = \mu_2(t), \quad \varphi(t, T) = \mu_2(t), \quad t \in [0, T],
\end{aligned}
\]

where

\[
\det \left[ R(t, t) + D(t)^T M(t, t) D(t) + \int_Z F(t, z)^T M(t, t) F(t, z) \theta(dz) \right] \neq 0, \quad t \in [0, T],
\]

and \(\Psi(s), \psi(s)\) are given by (4.9).

### 4.1 Verification Theorem

The following theorem provides a verification argument.

**Theorem 4.1.** let (H1)-(H2) hold. Let \(M(\ldots), \dot{M}(\ldots), \Upsilon(\ldots)\) and \(\varphi(\ldots)\) be the solution of the system (4.11). Then \(\hat{u}(.\hspace{1em})\) given by (4.8) is an equilibrium control.

**Proof.** First, we can check that \(\Psi(\ldots)\) and \(\psi(\ldots)\) in (3.15) are both uniformly bounded. Then the following linear SDE

\[
\begin{aligned}
d\hat{X}(s) &= \left\{(A - B \Psi(s)) \hat{X}(s) + b - B \psi(s)\right\} ds \\
&\quad + \left\{(C - D \Psi(s)) \hat{X}(s) + \sigma - D \psi(s)\right\} dW(s) \\
&\quad + \int_Z \left\{ (E(z)^T - F(z) \Psi(s)) \hat{X}(s) - c(z) - F(z) \psi(s) \right\} \tilde{N}(ds, dz), \quad s \in [0, T], \\
\hat{X}(0) &= x_0,
\end{aligned}
\]

is uniquely solvable, and the following estimate holds

\[
E \left[ \sup_{s \in [0, T]} \left| \hat{X}(s) \right|^2 \right] \leq K (1 + x_0^2).
\]

So the control \(\hat{u}(.\hspace{1em})\) defined by (4.8) is admissible. Moreover, by definition of \((p(s; t), q(s; t), r(s, z; t))\) via (4.3), (4.6) and (4.7), respectively, and by applying the Ito’s formula, we can easily check that, for each \(t \in [0, T]\) the processes \((p(.; t), q(.; t), r(.; t))\) satisfy (3.4).

Finally, in view of Corollary 3.4, it’s remains to check that the condition (3.11) holds. To this end, we substitute \((p(t; t), q(t; t), r(t, z; t), \hat{u}(t))\) taken from (4.3), (4.6), (4.7) and (4.8), respectively, in the expression

\[
K(t) = R(t, t) \hat{u}(t) - B(t)^T p(t, t) - D(t)^T q(t, t) - \int_Z F(t, z)^T r(t, z; t) \theta(dz),
\]

we get

\[
K(t) = R(t, t) \hat{u}(t) + B(t)^T \left\{ (M(t, t) + \dot{M}(t, t) + \Upsilon(t, t)) \hat{X}(t) + \varphi(t, t) \right\} + D(t)^T M(t, t) \left(C(t) \hat{X}(t) + \dot{M}(t, t) + \Upsilon(t, t)\right) + \int_Z F(t, z)^T M(t, t) \left(E(t, z) \hat{X}(t) + F(t, z) \hat{u}(t) + c(t, z)\right) \theta(dz)
\]

\[
= \Theta(t)^{-1} \left( \hat{u}(t) + \Psi(t) \hat{X}(t) + \psi(t) \right),
\]

the representation (4.8) shows that \(K(t) = 0\). Hence, by Corollary 3.4, \(\hat{u}(.\hspace{1em})\) is an open-loop Nash equilibrium control. \(\blacksquare\)
Note that, the verification theorem (Theorem 4.1) assumes the existence of a solution to the system (4.11). However, since the ODEs which should be solved by $M(\cdot,\cdot)$ and $\tilde{M}(\cdot,\cdot)$ do not have a symmetry structure. The general solvability for this type of ODEs when $(n>1)$ remains an outstanding open problem. We will see in the next section two examples in the case when $n=1$, this case is important, especially in financial applications as the mean–variance portfolio selection model. Also, we remark that a special feature of the case when $n=1$ is that the state $X(\cdot)$ is one-dimensional, so are the unknown $M(\cdot,\cdot), \tilde{M}(\cdot,\cdot), Y(\cdot,\cdot)$ and $\phi(\cdot,\cdot)$ of the system (4.11). This makes it easier to solve (4.11).

5 Some applications

5.1 Mean-variance portfolio selection problem

In this subsection, we discuss the continuous-time Markowitz’s mean-variance portfolio selection problem. We apply Theorem 4.1 to obtain a state feedback representation of an equilibrium control for the Problem. In the absence of Poisson random jumps this problem is discussed in [9].

The problem is formulated as follows: We consider a financial market, in which two securities are traded continuously. One of them is a bond, with price $S^0(\cdot)$ at time $s \in [0,T]$ governed by

$$dS^0(s) = S^0(s) r(s) \, ds, \quad S^0(0) = s_0 > 0.$$  \hspace{1cm} (5.1)

There is also a stock with unit price $S^1(\cdot)$ at time $s \in [0,T]$ governed by

$$dS^1(s) = S^1(s) \left( \alpha(s) \, ds + \beta(s) \, dW(s) + \int_{\mathbb{R}^+} \gamma(s,z) \, \tilde{N}(ds,dz) \right), \quad S^1(0) = s^1 > 0.$$  \hspace{1cm} (5.2)

where $r : [0,T] \to (0,\infty)$, $\alpha, \beta : [0,T] \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ are assumed to be deterministic and continuous, we also assume a uniform ellipticity condition as follow $\sigma(t)^2 + \int_{\mathbb{R}^+} \gamma(t,z)^2 \, \theta(dz) \geq \delta, \ a.e.$ for some $\delta > 0$. For an investor, a portfolio $\pi(\cdot)$ is a process represents the amount of money invested in the stock. The wealth process $X^{x_0,\pi(\cdot)}$ corresponding to initial capital $x_0 > 0$, and portfolio $\pi(\cdot)$, satisfies then the equation

$$
\begin{cases}
    dX(s) = (r(s) X(s) + \pi(s)(\alpha(s) - r(s))) \, ds + \pi(s) \beta(s) \, dW(s) \\
        + \pi(s) \int_{\mathbb{R}^+} \gamma(s,z) \, \tilde{N}(ds,dz), \quad \text{for } t \in [0,T], \\
    X(0) = x_0.
\end{cases}
$$  \hspace{1cm} (5.3)

As time evolves, we need to consider the controlled stochastic differential equation parametrized by $(t,\xi) \in [0,T] \times L^2(\Omega,\mathcal{F}_t,P;\mathbb{R})$ and satisfied by $X(\cdot)$

$$
\begin{cases}
    dX(s) = (r(s) X(s) + \pi(s)(\alpha(s) - r(s))) \, ds + \pi(s) \beta(s) \, dW(s) \\
        + \pi(s) \int_{\mathbb{R}^+} \gamma(s,z) \, \tilde{N}(ds,dz), \quad \text{for } s \in [t,T], \\
    X(t) = \xi.
\end{cases}
$$  \hspace{1cm} (5.4)

The objective is to maximize the conditional expectation of terminal wealth $\mathbb{E}^\xi[X(T)]$, and at the same time to minimize the conditional variance of the terminal wealth $\text{Var}^\xi[X(T)]$, over controls $\pi(\cdot)$ valued in $\mathbb{R}$. Then, the mean-variance portfolio optimization problem is denoted as: minimizing the cost $J(t,\xi,\cdot)$, given by

$$J(t,\xi,\pi(\cdot)) = \frac{1}{2} \text{Var}^\xi[X(T)] - (\mu_1(t) \, \xi + \mu_2(t)) \, \mathbb{E}^\xi[X(T)],$$  \hspace{1cm} (5.5)

subject to (5.4). Here $\mu_1, \mu_2 : [0,T] \to (0,\infty)$ are some deterministic non constant, continuous and bounded functions. The above model cover the one in [3], since, in our case, the weight between the conditional variance and the conditional expectation depends on the current wealth level, as well as, the current time, while in [9] the weight depends on the current wealth level only. Hence, in the above model, there are three different sources of time-inconsistency. Moreover, the above model is mathematically a special case of the general LQ problem formulated earlier in this paper, with $n = d = m = 1$. Then we can apply Theorem 4.1 to obtain a Nash equilibrium control. We recall that, the definition of equilibrium controls is in the sense of open-loop, which is different from the feedback one in [3], [4] and [22].
The optimal control problem associated with (5.4) and (5.5) is equivalent to minimize

\[ J(t, \xi, u(\cdot)) = \frac{1}{2} \left( \mathbb{E}^t \left[ X(T)^2 \right] - \mathbb{E}^t \left[ X(T) \right]^2 \right) - (\mu_1(t) \xi + \mu_2(t)) \mathbb{E}^t [X(T)] \]

subject to (5.4). Denote

\[ \rho(t) = \beta(t)^2 + \int_{R} \gamma(t, z)^2 \theta(dz) . \]

Thus, the system (4.11) reduces to

\[
\begin{aligned}
\frac{\partial M}{\partial s}(t, s) + \left\{ 2r(s) - \frac{(\alpha(s) - r(s))^2}{M(s, s) \rho(s)} \left( M(s, s) + \bar{M}(s, s) + \Upsilon(s, s) \right) \right\} M(t, s) &= 0, \quad \forall s \in [0, T], \\
\frac{\partial \bar{M}}{\partial s}(t, s) + \left\{ 2r(s) - \frac{(\alpha(s) - r(s))^2}{M(s, s) \rho(s)} \left( M(s, s) + \bar{M}(s, s) + \Upsilon(s, s) \right) \right\} \bar{M}(t, s) &= 0, \quad \forall s \in [0, T], \\
\frac{\partial \Upsilon}{\partial s}(t, s) + r(s) \Upsilon(t, s) &= 0, \quad \forall (t, s) \in D [0, T], \\
\frac{\partial \varphi}{\partial s}(t, s) + r(s) \varphi(t, s) &= 0, \quad \forall (t, s) \in D [0, T], \\
\bar{M}(t, T) &= 1, \quad \bar{M}(t, T) = -1, \quad \Upsilon(t, T) = \mu_1(t), \quad \varphi(t, T) = \mu_2(t), \quad \forall t \in [0, T].
\end{aligned}
\]

(5.6)

Clearly, if \( M(\cdot, \cdot) \) and \( \bar{M}(\cdot, \cdot) \) are solutions to the first and the second equations, respectively, in (5.6), then \( \bar{M}(\cdot, \cdot) = (M + M)(\cdot, \cdot) \) solves the following ODE

\[
\begin{aligned}
\frac{\partial \bar{M}}{\partial s}(t, s) + \left\{ 2r(s) - \frac{(\alpha(s) - r(s))^2}{M(s, s) \rho(s)} \left( M(s, s) + \bar{M}(s, s) + \Upsilon(s, s) \right) \right\} \bar{M}(t, s) &= 0, \quad \forall (t, s) \in [0, T], \\
\bar{M}(t, T) &= 0, \quad t \in [0, T],
\end{aligned}
\]

which is equivalent to

\[
\bar{M}(t, s) = \bar{M}(t, T) e^{\int_s^T \left( 2r(\tau) - \frac{(\alpha(\tau) - r(\tau))^2}{M(\tau, \tau) \rho(\tau)} \left( \bar{M}(\tau, \tau) + \Upsilon(\tau, \tau) \right) \right) d\tau},
\]

hence

\[
\bar{M}(t, s) + M(t, s) = \bar{M}(t, s) = 0, \quad \forall (t, s) \in D [0, T].
\]

Moreover, we remark that all data of the ODEs which should be solved by \( M(\cdot, \cdot) \) and \( \bar{M}(\cdot, \cdot) \) are not influenced by \( t \), thus (5.6) reduces to

\[
\begin{aligned}
\frac{dM}{ds}(s) + 2r(s) M(s) - M(s) \frac{(\alpha(s) - r(s))^2}{M(s) \rho(s)} \Upsilon(s, s) &= 0, \quad \forall s \in [0, T], \\
M(s) &= -M(s), \quad \forall s \in [0, T], \\
\frac{\partial \Upsilon}{\partial s}(t, s) + r(s) \Upsilon(t, s) &= 0, \quad \forall (t, s) \in D [0, T], \\
\frac{\partial \varphi}{\partial s}(t, s) + r(s) \varphi(t, s) &= 0, \quad \forall (t, s) \in D [0, T], \\
\Upsilon(T) &= 1, \quad \Upsilon(t, T) = \mu_1(t), \quad \varphi(t, T) = \mu_2(t), \quad \forall t \in [0, T].
\end{aligned}
\]

(5.8)

which is explicitly solved by

\[
\begin{aligned}
M(s) &= e^{\int_s^T r(\tau) d\tau} \left\{ 1 + \int_s^T e^{-\int_s^\tau r(\tau') d\tau'} \mu_1(\tau) \frac{(\alpha(\tau) - r(\tau))^2}{\rho(\tau)} d\tau \right\}, \quad \forall s \in [0, T], \\
\bar{M}(s) &= -e^{\int_s^T r(\tau) d\tau} \left\{ 1 + \int_s^T e^{-\int_s^\tau r(\tau') d\tau'} \mu_1(\tau) \frac{(\alpha(\tau) - r(\tau))^2}{\rho(\tau)} d\tau \right\}, \quad \forall s \in [0, T], \\
\Upsilon(t, s) &= \mu_1(t) e^{\int_s^T r(\tau) d\tau}, \quad \forall (t, s) \in D [0, T], \\
\varphi(t, s) &= \mu_2(t) e^{\int_s^T r(\tau) d\tau}, \quad \forall (t, s) \in D [0, T].
\end{aligned}
\]

(5.9)
In view of Theorem 4.1, the representation of the Nash equilibrium control (4.8) then gives
\[ \hat{u} (s) = \Psi (s) \hat{X} (s) + \psi (s), \forall s \in [0, T], \]
where, \( \forall s \in [0, T] \)
\[ \Psi (s) = \frac{(\alpha (s) - r (s))}{M (s) \rho (s)} \Upsilon (s, s) \] and \( \psi (s) = \frac{(\alpha (s) - r (s))}{M (s) \rho (s)} \varphi (s, s). \]

The corresponding equilibrium dynamics solves the SDEJ
\[
\begin{cases}
    d\hat{X} (s) = \left\{ (r (s) - \Psi (s) (\alpha (s) - r (s))) \hat{X} (s) - \psi (s) (\alpha (s) - r (s)) \right\} ds \\
    - \left\{ \Psi (s) \hat{X} (s) + \psi (s) \right\} \left\{ \beta (s) dW (s) + \int_{\mathbb{R}^*} \gamma (s, z) \tilde{N} (ds, dz) \right\}, \text{ for } s \in [0, T], \\
    \hat{X} (0) = x_0.
\end{cases}
\]

**Remark 5.1.** In the absence of Poisson random jumps, we have the following items

1. In the case where \( \mu_1 (t) = \mu_2 > 0 \) and \( \mu_2 (t) = \mu_2 > 0 \), the solution obtained, for the mean–variance problem, coincides with that obtained by [9].

2. In the case where \( \mu_1 (t) = \mu_2 > 0 \) and \( \mu_2 (t) = 0 \), the solution obtained, coincides with that obtained by [3].

3. In the case where \( \mu_1 (t) = 0 \) and \( \mu_2 (t) = \mu_2 > 0 \), the solution, obtained coincides with that obtained by [1].

### 5.2 General discounting LQ regulator

In this subsection, we consider an example of a general discounting time-inconsistent LQ model. The objective is to minimize the expected cost functional, that is earned during a finite time horizon
\[
J (t, \xi, u (\cdot)) = \frac{1}{2} \mathbb{E}^\mathbb{P} \left[ \int_t^T |u (s)|^2 ds + h (t) |X (T) - \xi|^2 \right] 
\] (5.11)
where \( h (\cdot) : [0, T] \to (0, \infty) \), is a general deterministic non-exponential discount function satisfying \( h (0) = 1 \), \( h (s) \geq 0 \) and \( \int_0^T h (t) dt < \infty \). Subject to a controlled one dimensional SDE parametrized by \( (t, \xi) \in [0, T] \times L^2 (\Omega, \mathcal{F}_t, P; \mathbb{R}) \)
\[
\begin{cases}
    dX (s) = (aX (s) + bu (s)) ds + \sigma dW (s) + c \int_{\mathbb{R}^*} \tilde{N} (ds, dz), \ s \in [0, T], \\
    X (t) = \xi,
\end{cases}
\] (5.12)
where \( a, b, c \) are real constant. As mentioned in [2], this is atime-inconsistent version of the classical linear quadratic regulator, we want control the system so that the final sate \( X (T) \) is close to \( \xi \) while at the same time we keep the control energy (formalized by the integral term) small. Note that, here the time-inconsistency is due to the fact that the terminal cost depends explicitly on the current state \( \xi \) as well as the current time \( t \). Hence there are two different sources of time-inconsistency. For this example, the system (4.11) reduces to
\[
\begin{cases}
    \frac{\partial M (t, s)}{\partial s} + 2aM (t, s) - b^2 M (t, s) \left\{ M (s, s) + \bar{M} (s, s) + \Upsilon (s, s) \right\} = 0, \ \forall (t, s) \in D [0, T], \\
    \frac{\partial \bar{M} (t, s)}{\partial s} + 2a\bar{M} (t, s) - b^2 \bar{M} (t, s) \left\{ M (s, s) + \bar{M} (s, s) + \Upsilon (s, s) \right\} = 0, \ \forall (t, s) \in D [0, T], \\
    \frac{\partial \Upsilon (t, s)}{\partial s} + a\Upsilon (t, s) = 0, \ \forall (t, s) \in D [0, T], \\
    \frac{\partial \varphi (t, s)}{\partial s} + a\varphi (t, s) - b^2 \left\{ M (t, s) + \bar{M} (t, s) \right\} \varphi (s, s) = 0, \ \forall (t, s) \in D [0, T], \\
    \bar{M} (t, T) = h (t), \quad \bar{M} (t, T) = 0, \quad \Upsilon (t, T) = h (t), \quad \varphi (t, T) = 0, \ \forall t \in [0, T], \\
\end{cases}
\] (5.13)
and the final data problem where the general discounting LQ regulator model, formulated earlier in this paragraph, in the case when

\begin{equation}
\text{Remark 5.3}
\end{equation}

The first equation in Theorem 5.2.

\begin{proof}
See the proof in Appendix A.2.
\end{proof}

Moreover, we can check that \( M (.,.) \) and \( \varphi (.,.) \) solve (5.13), if and only if, they solve the following system of integral equations

\begin{equation}
\begin{aligned}
M (t, s) = M (t, T) e^{\int_s^T \{ 2a - b^2 (M(r, r) + \hat{\theta}(r)) \} dr}, \forall (t, s) \in D [0, T]. \\
M (t, s) = M (t, T) e^{\int_s^T \{ 2a - b^2 (M(r, r) + \hat{\theta}(r)) \} dr}, \forall (t, s) \in D [0, T]. \\
\varphi (t, s) = \varphi (t, T) e^{a(T - s)} - b^2 \int_s^T e^{a(r - s)} (M (t, r) + \hat{M} (t, r)) \varphi (r, r) dr, \forall (t, s) \in D [0, T],
\end{aligned}
\end{equation}

on the other hand, we have \( M (t, T) = \varphi (t, T) = 0 \), then (5.15) reduces to

\begin{equation}
\begin{aligned}
M (t, s) = M (t, T) e^{\int_s^T \{ 2a - b^2 (M(r, r) + \hat{\theta}(r)) \} dr}, \forall (t, s) \in D [0, T]. \\
M (t, s) = 0, \forall (t, s) \in D [0, T]. \\
\varphi (t, s) = -b^2 \int_s^T e^{a(r - s)} M (t, r) \varphi (r, r) dr, \forall (t, s) \in D [0, T].
\end{aligned}
\end{equation}

It is clear that if \( M (.,.) \) is the solution of the first equation in (5.16), then

\begin{equation}
\varphi (s, s) = -b^2 \int_s^T e^{a(r - s)} M (s, r) \varphi (r, r) dr, \forall s \in [0, T],
\end{equation}

thus, there exists some constant \( L > 0 \) such that \( |\varphi (s, s)| \leq L \int_s^T |\varphi (r, r)| dr \), then by Gronwall Lemma, we conclude that \( \varphi (s, s) = 0, \forall s \in [0, T] \). Therefore

\begin{equation}
\varphi (t, s) = 0, \forall (t, s) \in D [0, T].
\end{equation}

is the unique solution to the last equation in the system (5.16).

Now, it’s remains to solve the first equation in the system (5.16). We have the following Theorem.

**Theorem 5.2.** The first equation in (5.16) has a unique solution in \( C (D [0, T], \mathbb{R}^+) \).

**Proof.** See the proof in Appendix A.2.

In view of Theorem 4.1, the representation (4.8) of the Nash equilibrium control, then gives

\begin{equation}
\hat{u} (s) = b \{ \Upsilon (s, s) - M (s, s) \}, \forall s \in [0, T],
\end{equation}

and the corresponding equilibrium dynamics solves the SDE

\begin{equation}
\begin{aligned}
dX (s) = \{ a + b^2 (\Upsilon (s, s) - M (s, s)) \} X (s) ds + \sigma dW (s) + c \int_{\mathbb{R}^2} z \tilde{N} (ds, dz), \ s \in [0, T], \\
X (0) = x_0.
\end{aligned}
\end{equation}

To conclude this section let us present the following remark.

**Remark 5.3.** The Problem (E) given by the subsection 2.3, is in fact shown to be a particular case of the general discounting LQ regulator model, formulated earlier in this paragraph, in the case when \( a = c = 0 \), and the final data \( \xi = x \), this leads to the following representation of the Nash equilibrium control of this problem

\(\begin{equation}
\hat{u} (s) = b (h (s) - M (s, s)) X (s), \forall s \in [0, T],
\end{equation}\)

where \( M (t, s) \) solves

\(\begin{equation}
M (t, s) = h (t) e^{\int_s^T -b^2 (M(r, r) + h(r)) dr}, \text{ for } (t, s) \in D [0, T],
\end{equation}\)

and the corresponding equilibrium dynamics solves the SDE

\(\begin{equation}
\begin{aligned}
dX (s) = b^2 \{ h(s) - M (s, s) \} X (s) ds + \sigma dW (s), \ s \in [0, T], \\
X (0) = x_0.
\end{aligned}
\end{equation}\)

This in fact, the correct solution of Problem (E).
Conclusion and future work. In this paper, we have studied a class of dynamic decision problems of a general time-inconsistent type. We have used the game theoretic approach to handle the time inconsistency. During this study open-loop Nash equilibrium controls are constructed as an alternative of optimal controls. This has been accomplished through stochastic maximum principle that includes a flow of forward-backward stochastic differential equations under maximum condition. The inclusion of concrete examples confirms the validity of our proposed study. The work can be extended in several ways. For example, this approach can be extended to a mean field game to construct decentralized strategies and obtain an estimate of their performance. The research on this topic is in progress and will appear in our forthcoming paper.

6 Appendix: Additional proofs

A.1. Proof of Lemma 3.3. Let $t \in [0, T], v \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^m)$ and $\varepsilon \in [0, T-t]$. Since $\mathbb{E}^t[y^\varepsilon,v(.)]$ and $\mathbb{E}^t[z^\varepsilon,v(.)]$ solve the following ODEs, respectively

\[
\begin{align*}
\frac{d\mathbb{E}^t[y^\varepsilon,v(s)]}{ds} &= A\mathbb{E}^t[y^\varepsilon,v(s)] + \mathbb{E}^t[y^\varepsilon,v(t)], \\
\frac{d\mathbb{E}^t[z^\varepsilon,v(s)]}{ds} &= \mathbb{E}^t[z^\varepsilon,v(s)], \\
\frac{d\mathbb{E}^t[y^\varepsilon,v(t)]}{ds} &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\frac{d\mathbb{E}^t[z^\varepsilon,v(s)]}{ds} &= \mathbb{E}^t[z^\varepsilon,v(s)] + B\mathbb{E}^t[v]_{[t,t+\varepsilon]}(s), \\
\frac{d\mathbb{E}^t[y^\varepsilon,v(t)]}{ds} &= 0.
\end{align*}
\]

Thus, it is clear that $\mathbb{E}^t[y^\varepsilon,v(s)] = 0$, a.e. $s \in [t, T]$. According to Gronwall’s inequality there exists a positive constant $K$ such that $\sup_{s \in [t, T]} |\mathbb{E}^t[z^\varepsilon,v(s)]|^2 \leq K\varepsilon^2$. Moreover, by Lemma 2.1. in [17], we obtain (3.18).

By these estimates, we can calculate the difference

\[
J\left(t, \hat{X}(t), u^\varepsilon(.)\right) - J\left(t, \bar{X}(t), \hat{u}(.)\right)
\]

\[
= \mathbb{E}^t \left[ \int_t^T \left\{ \left( Q(t,s) \hat{X}(s) + Q(t,s) \mathbb{E}^t \left[ \hat{X}(s), y^\varepsilon,v(s) + z^\varepsilon,v(s) \right] + \frac{1}{2} \left( Q(t,s) (y^\varepsilon,v(s) + z^\varepsilon,v(s)), y^\varepsilon,v(s) + z^\varepsilon,v(s) \right) \langle \varepsilon, v \rangle \right) + \left( \langle R(t,s) \hat{u}(s), v \rangle 1_{[t,t+\varepsilon]}(s) \right) \right\} ds \right]
\]

\[
= \mathbb{E}^t \left[ \int_t^T \left\{ \left( Q(t,s) \hat{X}(s) + Q(t,s) \mathbb{E}^t \left[ \hat{X}(s), y^\varepsilon,v(s) + z^\varepsilon,v(s) \right] \right) \right\} ds \right]
\]

In an another hand, from (H1) and (3.17) – (3.18) the following estimate holds

\[
\mathbb{E}^t \left[ \int_t^T \frac{1}{2} \left( Q(t,s) \mathbb{E}^t \left[ y^\varepsilon,v(s) + z^\varepsilon,v(s) \right] \right) ds \right] \]

\[
+ \frac{1}{2} \left( \langle G(t) \mathbb{E}^t \left[ y^\varepsilon,v(T) + z^\varepsilon,v(T) \right] \rangle, y^\varepsilon,v(T) + z^\varepsilon,v(T) \right)
\]

\[
= o(\varepsilon).
\]

Then, from the terminal conditions in the adjoint equations, it follows that

\[
J\left(t, \hat{X}(t), u^\varepsilon(.)\right) - J\left(t, \bar{X}(t), \hat{u}(.)\right)
\]

\[
= \mathbb{E}^t \left[ \int_t^T \left\{ \left( Q(t,s) \hat{X}(s) + Q(t,s) \mathbb{E}^t \left[ \hat{X}(s), y^\varepsilon,v(s) + z^\varepsilon,v(s) \right] \right) \right\} ds \right]
\]

\[
+ \frac{1}{2} \left( \langle R(t,s) \hat{u}(s), v \rangle 1_{[t,t+\varepsilon]}(s) \right) \]

\[
+ \frac{1}{2} \left( \langle R(t,s) \hat{u}(s), v \rangle 1_{[t,t+\varepsilon]}(s) \right) \]

\[
= o(\varepsilon).
\]
Now, by applying Ito’s formula to \( s \mapsto \langle p (s; t), y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \rangle \) on \([t, T]\), we get

\[
\langle p (T; t), y^{\varepsilon,v} (T) + z^{\varepsilon,v} (T) \rangle = \int_t^T \left\{ (Bv)^T p (s; t) 1_{[t,T]} (s) + (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T \left( Q (t,s) X (s) + Q (t,s) \mathbb{E} \left[ X (s) \right] \right) \right. \\
+ \sum_{j=1}^d (D_j v)^T q_j (s; t) 1_{[t,T]} (s) + \int_Z \left( F (z) v^T r (s,z) 1_{[t,T]} (s) \theta (dz) \right) \bigg\} ds \\
+ \int_t^T \int_Z \left\{ \left( C_j (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s)) + D_j v 1_{[t,T]} (s) \right)^T p (s; t) + (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T q_j (s; t) \right\} dW^j (s) \\
+ \int_t^T \int_Z \left\{ \left( E (z) (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s)) + F (z) v 1_{[t,T]} (s) \right)^T p (s; t) \\
+ (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T r (s,z) \right\} \tilde{N} (ds, dz).
\]

By applying Ito’s formula to \( s \mapsto \langle p (s; t), y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \rangle, y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \rangle \) on \([t, T]\), we get

\[
\langle p (T; t), y^{\varepsilon,v} (T) + z^{\varepsilon,v} (T) \rangle, y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \rangle \rangle = \int_t^T \left\{ 2 (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T P (s; t) B v 1_{[t,T]} (s) + (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T Q (t,s) (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s)) \right. \\
+ \sum_{j=1}^d \left( 2 (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T C_j^T P (s; t) D_j v 1_{[t,T]} (s) \right) \\
+ \left. \int_Z \left\{ 2 (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T E (z)^T P (s; t) F (z) v 1_{[t,T]} (s) \right. \\
+ \left. v^T F (z)^T P (s; t) F (z) v 1_{[t,T]} (s) \theta (dz) \right) \bigg\} ds \\
+ 2 \int_t^T \int_Z \left\{ \left( y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \right)^T P (s; t) \left( E (z) (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s)) + F (z) v 1_{[t,T]} (s) \right) \right\} \tilde{N} (ds, dz). \tag{6.3}
\]

Moreover, we conclude from \( (H1) \) together with (3.17) – (3.18) that

\[
\mathbb{E}^t \left[ \int_t^T \left( y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \right)^T P (s; t) B v 1_{[t,T]} (s) \right] = o (\varepsilon), \\
\mathbb{E}^t \left[ \int_t^T \left( y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s) \right)^T C_j^T P (s; t) D_j v 1_{[t,T]} (s) \right] = o (\varepsilon), \tag{6.5}
\]

By taking the conditional expectation in (6.3) and (6.4), then by invoking (6.5) it hold that

\[
\mathbb{E}^t \left[ \langle p (T; t), y^{\varepsilon,v} (T) + z^{\varepsilon,v} (T) \rangle \right] = \mathbb{E}^t \left[ \int_t^T \left\{ u^T B^T p (s; t) 1_{[t,T]} (s) + (y^{\varepsilon,v} (s) + z^{\varepsilon,v} (s))^T \left( Q (t,s) X (s) + Q (t,s) \mathbb{E} \left[ X (s) \right] \right) \right. \\
+ \sum_{j=1}^d u^T D_j^T q_j (s; t) 1_{[t,T]} (s) + \left. \int_Z u^T F (z)^T r (s,z) 1_{[t,T]} (s) \theta (dz) \bigg\} ds. \tag{6.6} \right.
\]
and
\[
\frac{1}{2} E^T \left[ \langle P(T; t) (y^{w,v}(T) + z^{w,v}(T)) \rangle , y^{w,v}(T) + z^{w,v}(T) \rangle \right] \\
= \frac{1}{2} E^T \left[ \int_t^T \left\{ \left( y^{w,v}(s) + z^{w,v}(s) \right) Q(t, s) \left( y^{w,v}(s) + z^{w,v}(s) \right) + \sum_{j=1}^d v^T D_j P(s; t) D_j v_{1(t,t+\varepsilon)}(s) \right. \right. \\
\left. + \left. \int_Z \left. v^T F(z) P(s; t) F(z) v_{1(t,t+\varepsilon)}(s) \theta(dz) \right] ds \right] + o(\varepsilon). \tag{6.7}
\]
By taking (6.6) and (6.7) in (6.2), it follows that
\[
J \left( t, \hat{X}(t), u^*(\cdot) \right) - J \left( t, \hat{X}(t), \tilde{u}(\cdot) \right) \\
= -E^T \left[ \int_t^T \left\{ v^T B^T p(s; t) + \sum_{j=1}^d v^T D_j q_j(s, t) + \frac{1}{2} \sum_{j=1}^d v^T D_j P(s; t) D_j v \right. \\
\left. - v^T R(t, s) \tilde{u}(s) - \frac{1}{2} v^T R(t, s) v \right. \right. \\
\left. + \left. \int_Z \left( r(s, z; t)^T F(z) v + \frac{1}{2} v^T F(z) P(s; t) F(z) v \right) \theta(dz) \right\} 1_{(t,t+\varepsilon)}(s) ds \right] + o(\varepsilon),
\]
which is equivalent to (3.19).

**A.2. Proof of Theorem 5.2.** For a constant \( \beta > 0 \), to be fixed later, we introduce the following norm, for \( f(\cdot, \cdot) \in C(D[0,T], \mathbb{R}) \)
\[
\left\| f \right\|_{\infty, \beta} = \sup_{(t,s) \in D[0,T]} \left| e^{-\beta(T-s)} f(t, s) \right|,
\]
it is easy to check that \( e^{-\beta T} \left\| f \right\|_{\infty} \leq \left\| f \right\|_{\infty, \beta} \leq \left\| f \right\|_{\infty} \), for every \( f \in C(D[0,T], \mathbb{R}) \), hence the norm \( \left\| \cdot \right\|_{\infty, \beta} \) is equivalent to \( \left\| \cdot \right\|_{\infty} \) on the Banach space \( C(D[0,T], \mathbb{R}) \). We introduce the following nonlinear operator, \( \tilde{L} \) : \( C(D[0,T], \mathbb{R}^+) \rightarrow C(D[0,T], \mathbb{R}^+) \), such that for all \( f(\cdot, \cdot) \in C(D[0,T], \mathbb{R}^+) \), we have
\[
\tilde{L} [f(t, s)] = \tilde{L} [f(t, s)] = h(t) e^{\int_s^T (2a-b^2 T(r,r)-b^2 f(r,r)) dr}.
\]
Then our problem is equivalent to a fixed point problem for the operator \( \tilde{L} \) in the closed subset \( C(D[0,T], \mathbb{R}^+) \) of the Banach space \( C(D[0,T], \mathbb{R}), \left\| \cdot \right\|_{\infty, \beta} \).

1) **Existence of solution.** It is clear that \( \tilde{L} \) is well defined. Now, consider \( f_1, f_2 \in C([0,T], \mathbb{R}^+) \), then
\[
\tilde{L} [f_1](t, s) - \tilde{L} [f_2](t, s) = h(t) e^{\int_s^T (2a-b^2 T(r,r)-b^2 f(r,r)) dr} \left( e^{-b^2 T(f_1(r,r) dr - e^{-b^2 T f_2(r,r) dr} \right), \tag{6.8}
\]
we put \( \lambda(t, s) = h(t) e^{\int_s^T (2a-b^2 T(r,r)-b^2 f(r,r)) dr} \), \( \forall (t, s) \in D[0,T] \), obviously \( \lambda(\cdot, \cdot) \) is uniformly bounded. Then there exists some constant \( K > 0 \), such that
\[
\left| \tilde{L} [f_1](t, s) - \tilde{L} [f_2](t, s) \right| \leq K \left| e^{-b^2 T f_1(r,r) dr} - e^{-b^2 T f_2(r,r) dr} \right|,
\]
moreover, since \( e^{-x} - e^{-y} \leq |x - y|, \forall x, y \in [0, \infty) \), then
\[
\left| \tilde{L} [f_1](t, s) - \tilde{L} [f_2](t, s) \right| \leq K b^2 \int_s^T |f_1(t, \tau) - f_2(t, \tau)| d\tau, \tag{6.9}
\]
thus
\[
e^{-\beta(T-s)} \left| \tilde{L} [f_1](t, s) - \tilde{L} [f_2](t, s) \right| \leq e^{-\beta(T-s)} K b^2 \int_s^T |f_1(t, \tau) - f_2(t, \tau)| d\tau, \\
= e^{-\beta(T-s)} K b^2 \int_s^T e^{\beta(T-\tau)} e^{-\beta(T-\tau)} |f_1(t, \tau) - f_2(t, \tau)| d\tau, \\
\leq K b^2 \left( 1 - e^{-\beta(T-s)} \right) \left\| f_1 - f_2 \right\|_{\infty, \beta},
\]

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\[
\| \tilde{L} [f_1] - \tilde{L} [f_2] \|_{\infty, \beta} \leq \frac{Kb^2 (1 - e^{-\beta T})}{\beta} \| f_1 - f_2 \|_{\infty, \beta}.
\]

Therefore \( \tilde{L} [\cdot] \) is a contraction mapping for \( \beta \) large enough.

2) **Uniqueness of solution.** Let \( f_1, f_2 \in C (D [0, T], \mathbb{R}^+) \) be two solutions, then
\[
f_1 (t, s) = \tilde{L} [f_1] (t, s) \quad \text{and} \quad f_2 (t, s) = \tilde{L} [f_2] (t, s), \quad \forall (t, s) \in D [0, T].
\]
From (6.9) we have
\[
| f_1 (s, s) - f_2 (s, s) | \leq Kb^2 \int_s^T | f_1 (\tau, \tau) - f_2 (\tau, \tau) | d\tau, \forall s \in [0, T],
\]
therefore, by Gronwall Lemma, we conclude that \( | f_1 (s, s) - f_2 (s, s) | = 0, \forall s \in [0, T], \) hence
\[
f_1 (t, s) = \tilde{L} [f_1] (t, s)
= h (t) e^{\int_s^T (2a - b^2 \Upsilon (\tau, \tau) - b^2 f_1 (\tau, \tau)) d\tau}
= h (t) e^{\int_s^T (2a - b^2 \Upsilon (\tau, \tau) - b^2 f_2 (\tau, \tau)) d\tau}
= f_2 (t, s), \quad \forall (t, s) \in D [0, T]
\]
This completes the proof.

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