Rationality of the zeta function for Ruelle-expanding maps

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Abstract

We will prove that the zeta function for Ruelle-expanding maps is rational.

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1 Introduction

For any map \( f \) with a finite number of periodic points for each period we can associate its zeta function defined as

\[
\zeta_f(t) = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n \right)
\]

where \( N_n(f) \) is the number of periodic points with period \( n \). In some cases, it is known that \( \zeta_f(t) \) is a rational function. Those cases include the Markov subshifts of finite type (unilateral and bilateral) and Axiom A diffeomorphisms. Besides, in the case of the subshifts, an explicit formula relates the topological entropy and the radius of convergence of the zeta function. Another class of maps with this property is the Ruelle-expanding maps. This concept, created by Ruelle, generalizes the notion of expanding maps defined on manifolds, freeing its essence from the derivative’s constraints. Our main result will be the following

Theorem 1.1 If \( f \) is Ruelle-expanding, then its zeta function is rational.

Its proof will emulate the classical argument used to ensure the rationality of the zeta function for a \( C^1 \) diffeomorphism defined on a hyperbolic set with local product structure, which profits by the existence of Markov partitions with arbitrarily small diameter. Within the Ruelle-expanding setting, we will prove the existence of a finite cover with analogous properties, which will play the same role the Markov partition did. Moreover, we will see that there is also a relation between the topological entropy and the radius of convergence of the zeta function in this case.

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2 The zeta function

**Definition 2.1** If \( f \) is a continuous map of a topological space \( X \), let \( N_n(f) \) denote the number of periodic points with period \( n \), that is, the points \( x \) for which \( f^n(x) = x \). If \( N_n(f) < \infty, \forall n \in \mathbb{N} \), we define the **zeta function** of \( f \) as

\[
\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n\right)
\]

As the exponential is an entire function, the radius of convergence of \( \zeta_f(t) \) is given by

\[
\rho = \frac{1}{\limsup \sqrt[n]{N_n(f)}} = \frac{1}{\limsup \sqrt[n]{N_n(f)}}
\]

(since \( \lim \sqrt[n]{n} = 1 \)).

Let \( L = -\log \rho \), so that \( \rho = e^{-L} \). Then, we have

\[
L = -\log \frac{1}{\limsup \sqrt[n]{N_n(f)}} = \limsup(1/n) \log N_n(f)
\]

### 2.1 Examples

#### 2.1.1 Markov subshifts of finite type

Let \( k \) be a natural number and \([k]\) the set \( \{1, 2, \ldots, k\} \) with the discrete topology. Consider \( \Sigma(k) \) the product space \( [k]^\mathbb{Z} \), whose elements are the sequences \( a = (\ldots, a_{-1}, a_0, a_1, \ldots) \), with \( a_n \in [k], \forall n \in \mathbb{Z} \).

This space has a product topology, which can be generated by the metric given by

\[
d(a, b) = \sum_{n=-\infty}^{\infty} \frac{\delta_n(a, b)}{2^{2|n|}}
\]

where \( \delta_n(a, b) \) is 0 when \( a_n = b_n \) and 1 otherwise. Notice that

\[
0 \leq d(a, b) \leq \sum_{n \in \mathbb{Z}} \frac{1}{2^{2|n|}} = 1 + 2 \sum_{n \in \mathbb{N}} \frac{1}{2^{2n}} = \frac{5}{3}
\]

and that \( d(a, b) \geq 1 \iff a_0 \neq b_0 \), since

\[
\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{1}{2^{2|n|}} = 2 \sum_{n \in \mathbb{N}} \frac{1}{2^{2n}} = \frac{2}{3} < 1
\]

On \( \Sigma(k) \) we have defined a homeomorphism, called *shift*, by

\[
(\sigma(a))_i = a_{i+1}, i \in \mathbb{Z}
\]

This way, \( \sigma \) has a special class of closed invariant sets. Let \( M_k \) be the set of \( k \times k \) matrices with entries 0 or 1. For each \( A \in M_k \), we define \( \Sigma_A = \{a \in \Sigma(k) : A_{a_0, a_{i+1}} = 1\} \), which is a closed invariant subspace of \( \Sigma(k) \). The pair \((\Sigma_A, \sigma_A)\), where \( \sigma_A = \sigma|\Sigma_A \), is called a **subshift of finite type**.

A matrix \( A \in M_k \) is said to be **irreducible** if \( \forall i, j \in [k], \exists n \in \mathbb{N} : (A^n)_{ij} > 0 \). In this case, by the **Perron-Frobenius Theorem**, we know that it has a non-negative simple eigenvalue \( \lambda \) which is greater than
the absolute value of all the others eigenvalues, that is, such that \( \max_{i \in [k]} |\lambda_i| = \lambda \), where \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are all the eigenvalues of \( A \). Besides, its entropy is \( \log \lambda \). In particular, the entropy of the full shift \( \sigma : \Sigma(k) \to \Sigma(k) \) is \( \log k \). (See [2]). For such a \( \sigma_A \), we can actually compute the zeta function: it is a rational function and \( L \) is precisely the entropy of \( f \). Let us recall why.

We say that a finite sequence \( a_0a_1\ldots a_n \) of elements in \([k]\) is admissible if \( A_{a_i}a_{i+1} = 1 \). Let \( N_n(p, q, A) \) denote the number of admissible sequences of length \( n + 1 \) which start at \( p \) and end at \( q \).

**Proposition 3** \( N_n(p, q, A) = (A^n)_{pq} \)

**Proof:** We use induction over \( n \). For \( n = 1 \), this is true by definition of \( A \). Suppose this is true for \( n = m - 1 \). Then, for \( n = m \) we have

\[
N_m(p, q, A) = \sum_{r=1}^{k} N_{m-1}(p, r, A)r_{rq} = \sum_{r=1}^{k} (A^{m-1})_{pr}r_{rq} = (A^m)_{pq}
\]

and the number of admissible sequences of length \( n + 1 \) which start and end with the same element of \([k]\) is

\[
\sum_{p=1}^{k} N_n(p, p, A) = \sum_{p=1}^{k} (A^n)_{pp} = \text{tr}(A^n)
\]

Notice that \( p \in \Sigma_A \) is a fixed point of \( \sigma^n_A \) if and only if \( a_i = a_{i+n}, \forall i \in \mathbb{Z} \). Then, for each fixed point of \( \sigma^n_A \) given by

\[
p = (\ldots, a_0, a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots) = (\ldots, a_0, a_1, a_2, \ldots, a_0, a_1, a_2, \ldots)
\]

we can associate a unique admissible sequence of length \( n + 1 \) given by \( a_0a_1a_2\ldots a_{n-1}a_0 \). Therefore, the number of fixed points of \( \sigma^n_A \) is \( N_n(\sigma_A) = \text{tr}(A^n) \).

**Theorem 3.1** \( \zeta_{\sigma_A}(t) = 1/\det(I - tA) \)

**Proof:** Let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be the eigenvalues of \( A \), so that

\[
\det(tI - A) = (t - \lambda_1)(t - \lambda_2)\ldots(t - \lambda_k)
\]

Replacing \( t \) by \( t^{-1} \), we get

\[
\det(t^{-1}I - A) = (t^{-1} - \lambda_1)(t^{-1} - \lambda_2)\ldots(t^{-1} - \lambda_k)
\]

and, multiplying both sides by \( t^k \), we get

\[
t^k \det(t^{-1}I - A) = t^k(t^{-1} - \lambda_1)(t^{-1} - \lambda_2)\ldots(t^{-1} - \lambda_k)
\]

\[
\det(I - tA) = (1 - \lambda_1t)(1 - \lambda_2t)\ldots(1 - \lambda_kt)
\]

Besides, we have

\[
\zeta_{\sigma_A}(t) = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(\sigma_A)}{n} t^n \right) = \exp \left( \sum_{n=1}^{\infty} \frac{\text{tr}(A^n)}{n} t^n \right)
\]
Since the eigenvalues of $A^n$ are $\lambda_1^n, \lambda_2^n, ..., \lambda_k^n$, we get $\text{tr}(A^n) = \sum_{m=1}^{k} \lambda_m^n$. So,

$$\zeta_{\sigma_A}(t) = \exp \left( \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{k} \lambda_m^n}{n} t^n \right) = \exp \left( \sum_{m=1}^{k} \left( \sum_{n=1}^{\infty} \frac{(\lambda_m t)^n}{n} \right) \right)$$

Moreover, since $\sum_{n=1}^{\infty} \frac{t^n}{n} = \log \left( \frac{1}{1-t} \right)$, we have

$$\zeta_{\sigma_A}(t) = \exp \left( \sum_{m=1}^{k} \log \left( \frac{1}{1 - \lambda_m t} \right) \right) = \exp \left( \log \left( \prod_{m=1}^{k} \left( \frac{1}{1 - \lambda_m t} \right) \right) \right) = \frac{1}{\prod_{m=1}^{k} (1 - \lambda_m t)} = \frac{1}{\det(I - tA)}$$

Remark: However, there are closed invariant subsets of $\Sigma(k)$ for which the zeta function for the restriction of $\sigma$ to those sets is not rational. In fact:

- The set of rational functions defined in a neighborhood of zero of the form $\exp \left( \sum_{n=1}^{\infty} \frac{N_n}{n} t^n \right)$, with $N_n \in \mathbb{Z}, \forall n \in \mathbb{N}$, is countable. In particular, the set of rational functions which are zeta functions for some restriction of $\sigma$ is countable.

- There is a noncountable collection of closed invariant subsets of $\Sigma(k)$ such that the zeta function for the restriction of $\sigma$ to those sets is distinct from each other.

Proposition 4 Let $A$ be an irreducible matrix with entries 0 or 1. Then the topological entropy of $\sigma_A$ is $-\log \rho$, where $\rho$ is the radius of convergence of $\zeta_{\sigma_A}$.

Proof: In fact, since $\zeta_{\sigma_A}(t) = 1/\det(I - tA)$ and

$$\det(I - tA) = 0 \iff \prod_{m=1}^{k} (1 - \lambda_m t) = 0 \iff \exists m \in [k]: t = 1/\lambda_m \land \lambda_m \neq 0$$

the radius of convergence of $\zeta_{\lambda_A}$ is

$$\rho = \min \{|1/\lambda_i|: i \in [k] \land \lambda_i \neq 0\} = 1/\max \{|\lambda_i|: i \in [k] \land \lambda_i \neq 0\} = 1/\lambda$$

Then $L = -\log \rho = \log \lambda$, and this value is precisely the topological entropy of $\sigma_A$. □

Remark: However, there are closed invariant subsets of $\Sigma(k)$ for which the zeta function for the restriction of $\sigma$ to those sets is not rational. In fact:

- The set of rational functions defined in a neighborhood of zero of the form $\exp \left( \sum_{n=1}^{\infty} \frac{N_n}{n} t^n \right)$, with $N_n \in \mathbb{Z}, \forall n \in \mathbb{N}$, is countable. In particular, the set of rational functions which are zeta functions for some restriction of $\sigma$ is countable.

- There is a noncountable collection of closed invariant subsets of $\Sigma(k)$ such that the zeta function for the restriction of $\sigma$ to those sets is distinct from each other.
Therefore, there is a noncountable collection of closed invariant subsets of $\Sigma(k)$ such that the zeta function for the restriction of $\sigma$ to those sets is not rational. For example, let $k = 2$ and $S \subseteq \Sigma(2)$ be the set whose elements are the sequences with only one '1' and the periodic sequences with at most one '1' in a minimal period. Then, $S$ is a closed invariant subset of $\Sigma(2)$. Also, the number of periodic points of period $n$ in $S$ is equal to the sum of the divisors of $n$, $\sigma(n)$, plus one, that is, $N_n(\sigma|_S) = \sigma(n) + 1$ and hence,

$$
\zeta_{\sigma|_S}(t) = \exp \left( \sum_{n=1}^{\infty} \frac{\sigma(n) + 1}{n} t^n \right) = \exp \left( \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} t^n + \sum_{n=1}^{\infty} \frac{t^n}{n} \right) = \exp (-\log(s(t)) - \log(1-t)) = \frac{1}{(1-t)s(t)}
$$

where $s(t) = 1 - t - t^2 + t^5 + t^7 - t^{12} + t^{15} - \ldots$ is a power series with arbitrarily long sequences of coefficients equal to zero. Since $s(t)$ isn’t rational, $\zeta_{\sigma|_S}$ is not rational as well.

2.1.2 Expansive maps

Definition 4.1 Let $(X, d)$ be a metric space and $f : X \to X$ a continuous map. We say that $\varepsilon$ is an expansive constant for $f$ if

$$
d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \in \mathbb{N}_0 \implies x = y
$$

The map $f$ is called expansive if it has an expansive constant. If $f : X \to X$ is a homeomorphism, we say that $\varepsilon$ is an expansive constant for $f$ (and $f$ is expansive) if

$$
d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \in \mathbb{Z} \implies x = y
$$

This property ensures that the periodic points of $f$ of period $n$ are isolated and the sets $N_n(f)$ are finite, $\forall n \in \mathbb{N}$ (see [2]). Moreover

Proposition 5 If $(X, d)$ is a compact metric space and $f : X \to X$ is expansive, then $\zeta_f$ has a positive radius of convergence.

Proof: Suppose that $f$ is a continuous map with expansive constant $\varepsilon$. Let $U_1, \ldots, U_r$ be a cover of $X$ with $\text{diam}(U_i) \leq \varepsilon, \forall i \in [r]$. For each $x \in X$, let $\phi(x) = (a_0, a_1, a_2, \ldots)$, with $a_n = \min \{ i \in [r] : f^n(x) \in U_i \}$. We can see that $\phi(x) = \phi(y) \Rightarrow d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \in \mathbb{N}_0 \Rightarrow x = y$, so $\phi$ is injective. Also, if $x$ is periodic with period $n$, then so is $\phi(x)$. Since the number of periodic points in $[r]^\mathbb{N}_0$ with period $n$ is $r^n$, we have $N_n(f) \leq r^n$ and

$$
L = \limsup (1/n) \log N_n(f) \leq \log r \implies \rho \geq 1/r > 0
$$

If $f$ is a homeomorphism with expansive constant $\varepsilon$, then the proof is similar (we associate to each point of $X$ an unique sequence in $[r]^\mathbb{Z}$, which is periodic if the point is periodic).
Since, for each expansive map, there is some \( r \in \mathbb{N} \) such that \( N_n(f) \leq r^n, \forall n \in \mathbb{N} \), we may also deduce that

**Corollary 5.1**

\[
1 - r \, |t| \leq |\zeta_f(t)| \leq \frac{1}{1 - r \, |t|}
\]

**Proof:**

\[
|\zeta_f(t)| = \left| \exp \left( \sum_{n=1}^{\infty} \frac{N_n(f)}{n} \, t^n \right) \right| = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(f)}{n} \, R(t^n) \right) \leq \exp \left( \sum_{n=1}^{\infty} \frac{r^n}{n} \right) = \exp \left( \sum_{n=1}^{\infty} \frac{(r \, |t|)^n}{n} \right) = \exp \left( \log \left( \frac{1}{1 - r \, |t|} \right) \right) = \frac{1}{1 - r \, |t|}
\]

and, similarly,

\[
|\zeta_f(t)| = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(f)}{n} \, R(t^n) \right) \geq \exp \left( \sum_{n=1}^{\infty} \frac{r^n}{n} (- |t^n|) \right) = 1 - r \, |t|
\]

for all \( t \) with \( |t| < 1/r \) (recall that \( \rho \geq 1/r \)).

\[\square\]

### 2.1.3 Axiom A Diffeomorphisms

**Definition 5.1** Let \( f \) be a \( C^1 \) diffeomorphism defined on a manifold \( M \). A subset \( \Lambda \subseteq M \) is **hyperbolic** if it is compact, \( f \)-invariant \( (f(\Lambda) = \Lambda) \) and there is a decomposition \( T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda \) such that

\[
\begin{align*}
D_x f(E^s_x) &= E^s_{f(x)}, \forall x \in \Lambda \\
D_x f(E^u_x) &= E^u_{f(x)}, \forall x \in \Lambda \\
\exists c > 0, \lambda \in ]0, 1[ : \forall x \in \Lambda, \forall n \geq 0, \\
\|D_x f^n(v)\| &\leq c \lambda^n \|v\|, \forall v \in E^s_x \text{ and } \|D_x f^{-n}(v)\| \leq c \lambda^n \|v\|, \forall v \in E^u_x
\end{align*}
\]

For each \( x \in \Lambda \), these expanding and contracting subbundles are tangent to the stable and unstable submanifolds,

\[
W^s(x) = \{ y \in M : d(f^n(x), f^n(y)) \to 0 \}
\]

\[
W^u(x) = \{ y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0 \}
\]

Besides, for small \( \varepsilon \), the local submanifolds

\[
W^s_\varepsilon(x) = \{ y \in M : d(f^n(x), f^n(y)) < \varepsilon, \forall n \geq 0 \}
\]

\[
W^u_\varepsilon(x) = \{ y \in M : d(f^{-n}(x), f^{-n}(y)) < \varepsilon, \forall n \geq 0 \}
\]

are \( C^1 \) disks embedded in \( M \) and there is \( \delta > 0 \) such that, if the distance between two points \( x \) and \( y \) in \( \Lambda \) is less then \( \delta \), then \( W^s_\varepsilon(x) \) and \( W^u_\varepsilon(y) \) intersect transversely at an unique point, denoted by \([x, y]\).
In particular, if $y = x$ then $W^s_*(x) \cap W^u_*(x) = \{x\}$, which means that $\varepsilon$ is an expansive constant for $f$ (see [1]). We say that $\Lambda$ has a local product structure if $[x, y] \in \Lambda, \forall x, y \in \Lambda$.

If $f$ is a $C^1$ diffeomorphism defined on a hyperbolic set with local product structure, then $f$ is expansive, so $N_n(f) < \infty, \forall n \in \mathbb{N}$ and we can define the zeta function for $f$. And moreover, as proved in [1],

**Theorem 5.1** The zeta function of a $C^1$ diffeomorphism on a hyperbolic set with local product structure is rational.

As a consequence, if $f$ is a $C^1$ diffeomorphism such that $\overline{\text{Per}(f)}$ is hyperbolic, then $\zeta_f(t)$ is a rational function: in fact, it is known that, if $\overline{\text{Per}(f)}$ is hyperbolic, then it has a local product structure; and $\zeta_f(t) = \zeta_{\overline{\text{Per}(f)}}(t)$. In particular, if $f$ is Axiom A (\(\Omega(f)\) is hyperbolic and $\overline{\text{Per}(f)} = \Omega(f)$), then $\zeta_f(t)$ is rational.

The main ingredient of the classical argument to prove this theorem is the existence of a Markov partition of arbitrarily small diameter, which establishes a codification of most of the orbits of $f$ through a subshift of finite type (for which we already know how to count the periodic points), and a sharp way to translate the properties of the zeta function from the subshift to the diffeomorphism setting.

### 6 Ruelle-expanding maps

Here, we will explain the nature of another class of maps, called Ruelle-expanding, whose zeta function is rational.

**Definition 6.1** Let $(K, d)$ be a compact metric space and $f : K \to K$ a continuous map. We say that $f$ is Ruelle-expanding if there are $r > 0, 0 < \lambda < 1$ and $c > 0$ such that:

- $\forall x, y \in K, x \neq y \land f(x) = f(y) \implies d(x, y) > c$
- $\forall x \in K, \forall a \in f^{-1}\{\{x\}\}, \exists \phi : B_r(x) \to K$ with
  - $\phi(x) = a$
  - $(f \circ \phi)(y) = y, \forall y \in B_r(x)$
  - $d(\phi(y), \phi(z)) \leq \lambda d(y, z), \forall y, z \in B_r(x)$

**Examples**

- Let $M$ be a compact manifold and $f : M \to M$ a $C^1$ map. We say that $f$ is expanding if $\exists \lambda \in \{0, 1\} : \forall x \in M, ||D_x f(v)|| \geq 1/\lambda ||v||$. It can be proved (see [3]) that, in this particular case, this condition is equivalent to the previous two from the last definition. So, $f$ is expanding if and only if it is Ruelle-expanding.

  One example of such a map is the application

  $f : S^1 \to S^1$, $z \mapsto z^k$, with $k \in \mathbb{Z}$ and $k > 1$

  (it is easy to see that $f$ is expanding, with $\lambda = 1/k$). Notice that, for this map, we have $N_n(f) = k^n - 1$. So,
\[\zeta_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{k^n-1}{n} t^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(kt)^n}{n} - \sum_{n=1}^{\infty} \frac{t^n}{n}\right) = \exp\left(\log \left(\frac{1}{1-kt}\right) - \log \left(\frac{1}{1-t}\right)\right) = \exp\left(\frac{1-t}{1-kt}\right)\]

which is a rational function (with a pole at \(\frac{1}{k}\), so \(\rho = \frac{1}{k} = \exp(-\log(k)) = \exp(-h(f))\)).

- Let \(\Sigma(k)^+\) be the product space \([k]^{\mathbb{N}_0}\), whose elements are the sequences \(a = (a_0, a_1, \ldots)\), with \(a_n \in [k], \forall n \in \mathbb{N}_0\). As its bilateral version, this space has a product topology which can be generated by the metric given by

\[d(a, b) = \sum_{n=0}^{\infty} \frac{\delta_n(a, b)}{2^n}\]

where \(\delta_n(a, b)\) is 0 when \(a_n = b_n\) and 1 otherwise. The **unilateral shift** is the map of \(\Sigma(k)^+\) given by

\[(\sigma^+(a))_i = a_{i+1}, i \in \mathbb{N}_0\]

For each \(A \in M_k\), we define \(\Sigma^+_A = \{a \in \Sigma(k)^+ : A_{a_0a_1a_2\ldots} = 1\}\). The pair \((\Sigma^+_A, \sigma^+_A)\), where \(\sigma^+_A = \sigma^+_1|_{\Sigma^+_A}\), is called a **unilateral subshift of finite type**. If \(A\) is irreducible, then it is easy to see that \(\sigma^+_A\) is Ruelle-expanding, with \(r = 1\) and \(\lambda = c = 1/2\), since:

- If \(a \neq b\) and \(\sigma^+_A(a) = \sigma^+_A(b)\), then \(a_0 \neq b_0\), so \(d(a, b) \geq 1 > c\).

- If \(r = 1\), then, for any \(a \in \Sigma^+_A\) we have \(B_r(a) = \{b \in \Sigma^+_A : b_0 = a_0\}\) since, as we have seen, \(b_0 \neq a_0 \Rightarrow d(a, b) \geq 1 = r\). Also, the pre-images of \(a = (a_0, a_1, a_2, \ldots)\) are of the form \((x, a_0, a_1, \ldots)\), where \(A_{xan} = 1\) (there is at least one \(x \in [k]\) such that \(A_{xan} = 1\) because \(A\) is irreducible). If we define \(\phi(b) = (x, b_0, b_1, b_2, \ldots)\) for \(b = (b_0, b_1, b_2, \ldots) \in B_r(a)\) (that is, with \(a_0 = b_0\)), then \(\sigma^+_A(\phi(b)) = b\) and

\[d(\phi(b), \phi(c)) = \sum_{n=1}^{\infty} \frac{\delta_{n-1}(b, c)}{2^n} = \sum_{n=0}^{\infty} \frac{\delta_n(b, c)}{2^{n+1}} = \frac{d(b, c)}{2} = \lambda d(b, c), \forall b, c \in B_r(a)\]

If, to simplify the notation, we denote by \(\sigma\) the map \(\sigma^+_A\), then \(a \in \Sigma^+_A\) is a fixed point of \(\sigma^n\) if and only if \(a_i = a_{i+n}, \forall i \in \mathbb{N}_0\). For each fixed point of \(\sigma^n\) given by

\[a = (a_0, a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots) = (a_0, a_1, a_2, \ldots, a_0, a_1, a_2, \ldots)\]

we can associate a unique admissible sequence of length \(n+1\) given by \(a_0a_1a_2\ldots a_{n-1}a_0\). So, the number of fixed points of \(\sigma^n\) is \(N_n(\sigma) = \text{tr}(A^n)\) and \(\zeta_\sigma(t) = 1/\det(I - tA)\), also a rational function (with poles at the inverses of the eigenvalues of \(A\)).

**Definition 6.2** Let \(f : K \to K\) be Ruelle-expanding and \(S \subseteq K\). Given \(n \in \mathbb{N}\), we say that \(g : S \to K\) is a **contractive branch** of \(f^{-n}\) if
\( (f^n \circ g)(x) = x, \forall x \in S \)

\( d((f^j \circ g)(x), (f^j \circ g)(y)) \leq \lambda^{n-j} d(x, y), \forall x, y \in S, j \in \{0, 1, \ldots, n\} \)

It is easy to see (details in \[3\]) that, given \( x \in K \) and \( a \in f^{-n}(\{x\}) \) for some \( n \in \mathbb{N} \), there is always a contractive branch \( g : B_r(x) \to K \) of \( f^{-n} \) with \( g(x) = a \). Moreover,

**Proposition 7** Let \( B(n, \varepsilon, x) = \{y \in K : d(f^j(x), f^j(y)) < \varepsilon, \forall j \in \{0, \ldots, n\}\} \). There is some \( \varepsilon_0 < r \) such that, for every \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon_0 \), we have

- \( \forall n \in \mathbb{N}, B(n, \varepsilon, x) = g(B_{\varepsilon}(f^n(x))), \) where \( g : B_{\varepsilon}(f^n(x)) \to K \) is a contractive branch of \( f^{-n} \) with \( g(f^n(x)) = x \)
- \( \varepsilon \) is an expansive constant for \( f \)

**Proof:** See \[3\]. \( \square \)

**Proposition 8** \( K = \bigcup_{n \geq 0} f^{-n}(\text{Per}(f)), \) where \( \text{Per}(f) \) is the set of periodic points for \( f \). In particular, \( \text{Per}(f) \neq \emptyset \).

**Proof:** See \[3\]. \( \square \)

Notice that, since \( f \) is expansive, we can consider the zeta function for \( f \) and, as \( \text{Per}(f) \neq \emptyset \), given \( x \in K \) with \( f^k(x) = x \) for some \( k \in \mathbb{N} \), we have \( f^{nk}(x) = x \) and \( N_{nk}(f) \geq 1, \forall n \in \mathbb{N} \), which implies that \( L \geq \limsup \frac{1}{nk} \log N_{nk}(f) \geq 0 \) and \( \rho \leq 1 \).

Is there any relation between \( L \) and \( h(f) \) if \( f \) is Ruelle-expanding? In fact, we have that \( L \leq h(f) \) but, to prove it, we need to simplify the calculus of \( h(f) \). Let us first recall briefly how to evaluate, in general, this number.

Given a metric space \((X, d)\) and a uniformly continuous map \( f : X \to X \), for every \( n \in \mathbb{N} \), we define a dynamical metric \( d_n \) on \( X \) by

\[
d_n(x, y) = \max\{d(f^i(x), f^i(y)), i \in \{0, 1, \ldots, n-1\}\}
\]

and the corresponding open dynamical ball, with center \( x \) and radius \( r \),

\[
B(n-1, r, x) = \{y \in K : d(f^j(x), f^j(y)) < r, \forall j \in \{0, \ldots, n-1\}\} = \bigcap_{i=0}^{n-1} f^{-i}(B_r(f^i(x)))
\]

and closed dynamical ball

\[
\overline{B}(n-1, r, x) = \{y \in K : d(f^j(x), f^j(y)) \leq r, \forall j \in \{0, \ldots, n-1\}\} = \bigcap_{i=0}^{n-1} f^{-i}(\overline{B}_r(f^i(x)))
\]

Accordingly,

**Definition 8.1** Let \( n \in \mathbb{N}, \varepsilon > 0 \) and \( K \) be a compact subset of \( X \). Given a subset \( F \) of \( X \), we say that \( F \) \((n, \varepsilon)\)– spans \( K \) with respect to \( f \) if

\[
\forall x \in K, \exists y \in F : d_n(x, y) \leq \varepsilon
\]
or, equivalently,

\[ K \subseteq \bigcup_{y \in F} \overline{B}(n - 1, \varepsilon, y) \]

**Definition 8.2** Let \( n \in \mathbb{N} \), \( \varepsilon > 0 \) and \( K \) be a compact subset of \( X \). We define \( r_n(\varepsilon, K) \) as the smallest cardinality of any \((n, \varepsilon)\) spanning set for \( K \) with respect to \( f \).

Notice that, since \( K \) is compact, we have \( r_n(\varepsilon, K) < \infty \); and \( \varepsilon_1 < \varepsilon_2 \implies r_n(\varepsilon_1, K) \geq r_n(\varepsilon_2, K) \).

**Definition 8.3** Let \( \varepsilon > 0 \) and \( K \) be a compact subset of \( X \). Then

\[ r(\varepsilon, K) = r(\varepsilon, K, f) = \limsup_{n \to \infty} (1/n) \log r_n(\varepsilon, K) \]

**Definition 8.4** If for each compact subset \( K \) of \( X \) we denote by \( h(f, K) \) the limit \( \lim_{\varepsilon \to 0} r(\varepsilon, K, f) \), then the *topological entropy* of \( f \) is \( h(f) = \sup\{ h(f, K), K \text{ compact subset of } X \} \).

Sometimes it is useful to use an equivalent way of defining topological entropy which uses separated sets instead of spanning ones.

**Definition 8.5** Let \( n \in \mathbb{N} \), \( \varepsilon > 0 \) and \( K \) be a compact subset of \( X \). Given a subset \( E \) of \( K \), we say that \( E \) is \((n, \varepsilon)\) separated with respect to \( f \) if

\[ \forall x, y \in E, d_n(x, y) \leq \varepsilon \implies x = y \]

or, equivalently,

\[ \forall x \in E, \overline{B}(n - 1, \varepsilon, x) = \{x\} \]

**Definition 8.6** Let \( n \in \mathbb{N} \), \( \varepsilon > 0 \) and \( K \) be a compact subset of \( X \). We define \( s_n(\varepsilon, K) \) as the largest cardinality of any \((n, \varepsilon)\) separated set for \( K \) with respect to \( f \).

Observe that \( r_n(\varepsilon, K) \leq s_n(\varepsilon, K) \leq r_n(\varepsilon/2, K) \) and so, since \( r_n(\varepsilon/2, K) < \infty \), we have \( s_n(\varepsilon, K) < \infty \); besides, \( \varepsilon_1 < \varepsilon_2 \implies s_n(\varepsilon_1, K) \geq s_n(\varepsilon_2, K) \).

**Definition 8.7** Let \( \varepsilon > 0 \) and \( K \) be a compact subset of \( X \). We define

\[ s(\varepsilon, K) = s(\varepsilon, K, f) = \limsup_{n \to \infty} (1/n) \log s_n(\varepsilon, K) \]

As a consequence of the previous inequalities, we get \( r(\varepsilon, K) \leq s(\varepsilon, K) \leq r(\varepsilon/2, K) \) and so (see [2])

**Proposition 9**

(a) *For any compact subset \( K \) of \( X \), we have \( h(f, K) = \lim_{\varepsilon \to 0} s(\varepsilon, K) \).*

(b) \( h(f) = \sup_{K} h(f, K) = \sup_{K} \lim_{\varepsilon \to 0} s(\varepsilon, K, f) \).

(c) *In case \( X \) is compact, then*

\[ h(f) = h(f, X) = \lim_{\varepsilon \to 0} \sup(1/n) \log r_n(\varepsilon, X) = \lim_{\varepsilon \to 0} \sup(1/n) \log s_n(\varepsilon, X) \]

Let us now go back to Ruelle-expanding maps.

**Proposition 10** If \( f : X \to X \) is a Ruelle-expanding map of a compact metric space \((X, d)\), then

\[ h(f) = r(\varepsilon_0, X) = s(\varepsilon_0, X) \text{ for all } \varepsilon_0 < \varepsilon/4, \text{ where } \varepsilon \text{ is an expansive constant for } f. \]
**Proof:** See [2]. (Although the proof is for expansive homeomorphisms, it can be easily adapted for expansive maps.) □

**Corollary 10.1** For any Ruelle-expanding map we have $L \leq h(f)$, that is, the radius of convergence of the zeta function is $\rho \geq \exp(-h(f))$.

**Proof:** Let $p$ and $q$ be periodic points of $f$, with $f^n(p) = p$ and $f^n(q) = q$ for some $n \in \mathbb{N}$. Then, we have

$$d_n(p, q) \leq \varepsilon_0 \implies d_n(p, q) \leq \varepsilon \implies d(f^n(p), f^n(q)) \leq \varepsilon, \forall i \in \{0, 1, \ldots, n - 1\} \implies d(f^i(p), f^i(q)) \leq \varepsilon, \forall i \in \mathbb{N}_0 \implies p = q$$

So, the set $P_n$ of periodic points $p$ with $f^n(p) = p$ is a $(n, \varepsilon_0)$ separated set for $X$ and $s_n(\varepsilon_0, X) \geq \text{card}(P_n) = N_n(f)$. Consequently,

$$L = \limsup(1/n) \log N_n(f) \leq \limsup(1/n) \log s_n(\varepsilon_0, X) = s(\varepsilon_0, X) = h(f)$$

□

This yields a link between $h(f)$ and the number of pre-images of the points in $X$ by $f$.

**Lemma 10.1** If $(X, d)$ is a compact metric space and $f : X \to X$ is a Ruelle-expanding map, then there is a $k \in \mathbb{N}$ such that $\text{card}(f^{-1}(\{x\})) \leq k, \forall x \in X$.

**Proof:** If we set $E = f^{-1}(\{x\})$ then we have $f(u) = f(v) = x, \forall u, v \in E, u \neq v$, so $d_1(u, v) = d(u, v) > c$ and $E$ is a $(1, c)$ separated set. Since $\text{card}(E) \leq s_1(c, X) < \infty$, we can take $k = s_1(c, X)$.

**Proposition 11** $h(f) \leq \log(k)$, with equality if $\text{card}(f^{-1}(\{x\})) = k, \forall x \in X$.

**Proof:** Let $\varepsilon_0 < \min\{\varepsilon/4, c, r\}$. Since $X$ is compact, there is a finite set $F$ for which we can write

$$X = \bigcup_{y \in F} \mathcal{P}_{\varepsilon_0}(y)$$

Given $x \in X$ and $n \in \mathbb{N}$, let $y \in F$ be such that $d(f^n(x), y) \leq \varepsilon_0$ and let $g : B_r(f^n(x)) \to X$ be a contractive branch of $f^{-n}$ with $g(f^n(x)) = x$. If we take $z = g(y)$, we have

- $f^n(z) = f^n(g(y)) = y \implies z \in f^{-n}(F)$
- $d(f^n(x), f^n(z)) = d(f^n(g(f^n(x))), f^n(g(y))) \leq \lambda^{n-1}d(f^n(x), y) \leq \lambda^{n-1}\varepsilon_0 \leq \varepsilon_0, \forall i \in \{0, 1, \ldots, n - 1\} \implies d_n(x, z) \leq \varepsilon_0$

So, $f^{-n}(F)$ is a $(n, \varepsilon_0)$ spanning set for $X$. Therefore, $r_n(\varepsilon_0, X) \leq \text{card}(f^{-n}(F)) \leq k^n \text{card}(F), \forall n \in \mathbb{N}$ and we get

$$h(f) = r(\varepsilon_0, X) = \limsup(1/n) \log r_n(\varepsilon_0, X) \leq \limsup(1/n) \log(k^n \text{card}(F)) = \limsup(\log k + (1/n) \log(\text{card}(F))) = \log k$$
As a consequence, we have $0 \leq L \leq \log k$ and $1/k \leq \rho \leq 1$.

Suppose now that there is some $k \in \mathbb{N}$ such that $\text{card}(f^{-1}(\{x\})) = k, \forall x \in X$. Take a point $x \in X$. If we consider $E_n = f^{-n}(\{x\})$, then we have $f^n(u) = f^n(v) = x, \forall u, v \in E_n, u \neq v$. If $f(u) = f(v)$, then $d_n(u, v) \geq d(u, v) > c$, otherwise, we have $f(u) \neq f(v)$. Admitting the last case, if $f^2(u) = f^2(v)$, then $d_n(u, v) \geq d(f(u), f(v)) > c$, otherwise, we have $f^2(u) \neq f^2(v)$. Proceeding, and since we have $f^n(u) = f^n(v)$, there must be some $j \in \{1, \ldots, n\}$ for which $f^j(u) = f^j(v)$ and $f^{j-1}(u) \neq f^{j-1}(v)$, so $d_n(u, v) \geq d(f^{j-1}(u), f^{j-1}(v)) > c$ and $E_n$ is a $(n, c)$ separated set. Since $\text{card}(E_n) = k^n$, we have $k^n \leq s_n(c, X) \leq s_n(\varepsilon_0, X)$ and we get

$$h(f) = s(\varepsilon_0, X) = \limsup(1/n) \log s_n(\varepsilon_0, X) \geq \limsup(1/n) \log(k^n) = \log k$$

which allow us to conclude that, in this particular case, $h(f) = \log k$.

Now, our goal will be to prove the rationality of the zeta function for Ruelle-expanding maps. Recall that the existence of a Markov partition was an essential ingredient in the proof of the rationality of the zeta function for $C^1$ diffeomorphisms defined on a hyperbolic set with local product structure. In the case of Ruelle-expanding maps, we will prove the existence of a finite cover with analogous properties, which will play the same role the Markov partition did.

**Proposition 12** Let $f$ be a Ruelle-expanding map defined on a compact set $K$. Let $\varepsilon$ denote an expansive constant for $f$. Then, $K$ has a finite cover $\{R_1, \ldots, R_n\}$ with the following properties:

- Each $R_i$ has a diameter less than $\min\{\varepsilon, c/2\}$ and is proper, that is, equal to the closure of its interior.
- $R_i \cap R_j = \emptyset, \forall i, j \in [n], i \neq j$.
- $f(R_i) \cap R_j \neq \emptyset \implies R_j \subseteq f(R_i)$.

**Remark:** If $R_i \subseteq f(R_i)$, then $R_j = \overline{R_j} \subseteq f(R_i) \subseteq f(\overline{R_i}) = f(R_i)$ and the last condition means that $f(R_i) \cap R_j \neq \emptyset \implies R_j \subseteq f(R_i)$.

To prove this proposition, we will begin by a shadowing lemma.

**Lemma 12.1** Let $f : K \to K$ be Ruelle-expanding. For any $\beta \in [0, r]$ there is some $\alpha > 0$ such that, if $(x_n)_{n \in \mathbb{N}_0}$ is a $\alpha$-pseudo orbit in $K$ (that is, if $d(f(x_n), x_{n+1}) < \alpha, \forall n \in \mathbb{N}_0$), then it admits a $\beta$-shadow (that is, a point $x \in K$ such that $d(f^n(x), x_n) < \beta, \forall n \in \mathbb{N}_0$). Besides, the $\beta$-shadow is unique if $\beta < \varepsilon/2$, where $\varepsilon$ is an expansive constant for $f$.

**Proof:** We will start proving this assertion for finite $\alpha$-pseudo orbits. Let $\beta \in [0, r]$ and $(x_0, x_1, \ldots, x_n)$ be such that $d(f(x_{k-1}), x_k) < \alpha, \forall k \in [n]$ for some $\alpha > 0$. If $y_n = x_n$, then $d(y_n, x_n) = 0 < \beta$. Now, suppose that $d(y_k, x_k) < \beta$ for $k \in [n]$. Since $d(f(x_{k-1}), x_k) < \alpha$, we have $d(y_k, f(x_{k-1})) < \alpha + \beta < r$ if we assume $\alpha < r - \beta$. Then, we can take $y_k = g(y_k)$, where $g : B_r(f(x_{k-1})) \to K$ is a contractive branch of $f^{-1}$ with $g(f(x_{k-1})) = x_k$, and we have $d(y_k, x_k) \leq \lambda d(y_k, f(x_{k-1})) < \lambda(\alpha + \beta) < \beta$ if we assume $\alpha < \frac{1}{\lambda + \beta}$. Also, notice that $y_k = f(y_{k-1}), \forall k \in [n]$, so that $y_k = f^k(x), \forall k \in [n]$ for $x = y_0$. Hence, it suffices to take $\alpha < \min\{r - \beta, \frac{1}{\lambda + \beta}\}$.
Now, take $\beta \in [0, r]$ and let $(x_n)_{n \in \mathbb{N}}$ be a $\alpha$-pseudo orbit, with $\alpha < \min\{r - \beta/2, \frac{\beta}{2}\}$. Let $z_n$ be a $\beta/2$-shadow of $(x_0, x_1, \ldots, x_n)$; since $K$ is compact, there is some subsequence $(z_{n_k})_k$ converging to some point $z \in K$. We have $d(f^i(z_{n_k}), x_i) < \beta/2, \forall i \in \{0, 1, \ldots, n_k\}$, so, for $i \in \mathbb{N}_0$ fixed we get $d(f^i(z), x_i) = \lim d(f^i(z_{n_k}), x_i) \leq \beta/2 < \beta$ and we conclude that $z$ is a $\beta$-shadow of $(x_n)_{n \in \mathbb{N}}$.

For the uniqueness of the $\beta$-shadow when $\beta < \varepsilon/2$, suppose that $z$ and $z'$ are both $\beta$-shadows of $(x_n)_{n \in \mathbb{N}}$. Then, we have $d(f^i(z), f^i(z')) \leq d(f^i(z), x_i) + d(f^i(z'), x_i) < 2\beta < \varepsilon, \forall i \in \mathbb{N}_0$, so $z = z'$.

Let $\varepsilon$ be an expansive constant for $f$ with $\varepsilon < r$ and fix some $\beta < \min\{\varepsilon/2, c/4\}$. Let $\alpha$ be given by the previous lemma and $\gamma \in [0, \alpha/2]$ be such that $d(x, y) < \gamma \Rightarrow d(f(x), f(y)) < \alpha/2, \forall x, y \in K$. Since $K$ is compact, we can take $\{p_1, \ldots, p_k\}$ such that $K = \bigcup_{i=1}^k B_i(p_i)$. We define a matrix $A \in M_k$ by

$$A_{ij} = 1 \text{ if } d(f(p_i), p_j) < \alpha \text{ and } A_{ij} = 0 \text{ otherwise.}$$

For every $\mathbf{a} \in \Sigma^+_A$ the sequence $(p_{a_i})_{i \in \mathbb{N}_0}$ is a $\alpha$-pseudo orbit, so it admits an unique $\beta$-shadow which we will denote by $\theta(\mathbf{a})$. Therefore, we have defined a map $\theta : \Sigma^+_A \to K$.

**Lemma 12.2** $\theta$ is a semiconjugacy of $\sigma_A^+$ and $f$, that is, $\theta$ is surjective, continuous and verifies $f \circ \theta = \theta \circ \sigma_A^+$.  

**Proof:** Given $x \in K$, we can take $a_i \in [k]$ so that $d(f^i(x), p_{a_i}) < r$ for any $i \in \mathbb{N}_0$; then, $d(f(p_{a_i}), p_{a_{i+1}}) \leq d(f(p_{a_i}), f(f^i(x))) + d(f^{i+1}(x), p_{a_{i+1}}) < \alpha/2 + \gamma < \alpha$ and $(p_{a_i})_{i \in \mathbb{N}_0}$ is a $\alpha$-pseudo orbit. So, $x = \theta(\mathbf{a})$ and $\theta$ is surjective.

For the continuity, since $K$ is compact it suffices to see that, for any two sequences $(\mathbf{a}^n)_{n \in \mathbb{N}}$ and $(\mathbf{b}^n)_{n \in \mathbb{N}}$ converging to the same limit $l$ in $\Sigma^+_A$ whose images under $\theta$ converge respectively to $s$ and $t$ in $K$, we have $s = t$. Fix some $i \in \mathbb{N}_0$; for any $n \in \mathbb{N}$, we have $d(f^i(\theta(\mathbf{a}^n)), p_{a^n}) < \beta$ and $d(f^i(\theta(\mathbf{b}^n)), p_{b^n}) < \beta$. So, taking limits we have $d(f^i(s), p_i) \leq \beta$ and $d(f^i(t), p_i) \leq \beta$. Hence, $d(f^i(s), f^i(t)) \leq 2\beta < \varepsilon$ and, since $\varepsilon$ is an expansive constant for $f$, we get $s = t$.

Finally, the relation $f \circ \theta = \theta \circ \sigma_A^+$ is a consequence of the unicity of the $\beta$-shadow and the fact that, if $x$ is a $\beta$-shadow for $(p_{a_i})$, then $f(x)$ is a $\beta$-shadow for $(p_{a_{i+1}}) = (p_{\sigma_A^+(a_i)})$.

Let $T_i = \{\theta(a) : a_0 = i\}$ for $i \in [k]$. Then, $T_i = \theta(C_i)$ where $C_i = \{\mathbf{a} \in \Sigma^+_A : a_0 = i\}$ and, since $\Sigma^+_A = \bigcup_{i=1}^k C_i$, we have $K = \bigcup_{i=1}^k T_i$ because $\theta$ is surjective. Hence, $\{T_i, i \in [k]\}$ is a finite closed cover of $K$ ($T_i$ is closed since $C_i$ is compact and $\theta$ is continuous).

**Lemma 12.3** If $A_{ij} = 1$, then $T_j \subseteq f(T_i)$ and $T_j \subseteq f(T_i)$. Also, given $x \in T_i$ with $f(x) \in T_j$, if $g : B_{\varepsilon}(f(x)) \to K$ is a contractive branch of $f^{-1}$ with $g(f(x)) = x$, then $g(T_j) \subseteq T_i$ and $g(T_j) \subseteq T_i$.

**Proof:** Given any $y \in T_j$, we have $y = \theta(\mathbf{b})$ for some $\mathbf{b} \in \Sigma^+_A$ with $b_0 = j$. Since $A_{ij} = 1$, we can take $z = (i, b_0, b_1, b_2, \ldots) \in \Sigma^+_A$, and so $y = \theta(\mathbf{b}) = \theta(\sigma_A^+(z)) = f(\theta(z)) \in f(\theta(C_i)) = f(T_i)$. Then, $T_j \subseteq f(T_i)$

Notice that $T_j \subseteq B_{\beta}(p_j)$. Since $d(f(x), p_j) < \beta$, we have $T_j \subseteq B_{\beta}(f(x)) \subseteq B_{\beta}(f(x))$. Let $g : B_{\varepsilon}(f(x)) \to K$ be a contractive branch of $f^{-1}$ with $g(f(x)) = x$. Given $y \in T_j$, we have $y = f(z)$ for some $z \in T_i$. Then,
Given Lemma 12.4

Proof: Let \( y \) and \( g \) satisfy the following properties:

- \( x \in R(x) \) (because \( x \in T_i^*(x) \), \( \forall i \in [k] \))
- \( R(x) \subseteq T_i^* \) for some \( i \in [k] \)
  (since \( \bigcap_{i=1}^{k} K \setminus T_i = K \setminus \bigcup_{i=1}^{k} T_i = \emptyset \), we must have \( x \in T_i^* \) for some \( i \in [k] \))
- If \( R(x) \cap R(y) \neq \emptyset \), then \( R(x) = R(y) \)
  (in fact, we have \( R(x) \cap R(y) \neq \emptyset \) \( \Rightarrow \forall i \in [k], T_i^*(x) \cap T_i^*(y) \neq \emptyset \) \( \Rightarrow \forall i \in [k], T_i^*(x) = T_i^*(y) \) \( \Rightarrow R(x) = R(y) \))

Moreover,

**Lemma 12.4** Given \( x \in Z \cap f^{-1}(Z) \), we have \( g(R(f(x))) \subseteq R(x) \), where \( g : B_r(f(x)) \to K \) is a contractive branch of \( f^{-1} \) with \( g(f(x)) = x \).

**Proof:** Let \( y \in R(f(x)) \). Notice that \( y \in Z \) and \( f(x) \in R(y) \).

For \( i \in [k] \), if \( x \in T_i \) then \( x = \theta(\mathbf{a}) \) for some \( \mathbf{a} \in \Sigma_A^+ \) with \( a_0 = i \). Let \( j = a_1 \). Then, \( f(x) = \theta(\sigma(\mathbf{a})) \) and \( f(x) \in T_j \), so that \( y \in R(f(x)) \subseteq T_j \Rightarrow g(y) \in g(T_j) \). Since \( A_{ij} = 1 \), by the previous lemma we get \( g(T_j) \subseteq T_i \) and, hence, \( g(y) \in T_i \).

On the other hand, if \( g(y) \in T_i \) then \( g(y) = \theta(\mathbf{b}) \) for some \( \mathbf{b} \in \Sigma_A^+ \) with \( b_0 = i \). Let \( j = b_1 \). Then, \( y = f(g(y)) = \theta(\sigma(\mathbf{b})) \) and \( y \in T_j \), so that \( f(x) \in R(y) \subseteq T_j \Rightarrow x = g(f(x)) \in g(T_j) \). Since \( A_{ij} = 1 \), by the previous lemma we get \( g(T_j) \subseteq T_i \) and, hence, \( x \in T_i \). So, \( x \in T_i \Leftrightarrow g(y) \in T_i, \forall i \in [k] \)
Similarly, using the previous lemma we get \( x \in \overset{\circ}{T}_i \Leftrightarrow g(y) \in \overset{\circ}{T}_i, \forall i \in [k] \). This way, we conclude that 
\[ \text{g}(y) \in \text{R}(x). \]

Let \( R = \{ \text{R}(x), x \in \text{Z} \} \). Since \( R \) is obviously a finite set, we can write \( R = \{ R_1, \ldots, R_n \} \) with \( R_i \neq R_j \) if \( i \neq j \). Also, since \( Z \) is dense in \( K \), we have \( K = \bigcup_{x \in Z} \text{R}(x) = \bigcup_{x \in Z} \text{R}(x) = \bigcup_{i=1}^n R_i \), that is, \( R \) is a finite closed cover of \( K \). Let us see that \( R \) satisfies the required properties.

1. \( R_i \) has a diameter less than \( \min\{\varepsilon, c/2\} \) and is proper.

Take \( x \in Z \) such that \( R_i = \text{R}(x) \) and \( j \in [k] \) such that \( R(x) \subseteq \overset{\circ}{T}_j \). Then, \( R_i = \text{R}(x) \subseteq \overset{\circ}{T}_j \subseteq T_j \) and \( \text{diam}(R_i) \leq \text{diam}(T_j) \leq 2\beta < \min\{\varepsilon, c/2\} \). Also, using the fact that the closure of the interior of a set is just the closure of the interior of that set, we have 
\[
\overset{\circ}{R}_i = \bigcup_{x \in Z} \text{R}(x) = \bigcup_{x \in Z} \text{R}(x) = \bigcup_{x \in Z} \text{R}(x) = R_i \text{ because } R_i \text{ is open.}
\]

2. \( \overset{\circ}{R}_i \cap \overset{\circ}{R}_j = \emptyset, \forall i, j \in [n], i \neq j \)

Take \( x, y \in Z \) such that \( R_i = \text{R}(x) \) and \( R_j = \text{R}(y) \). Suppose that \( \overset{\circ}{R}_i \cap \overset{\circ}{R}_j \neq \emptyset \); using the fact that any open set that intersects the closure of a set also intersects the set itself, we get 
\[
\overset{\circ}{R}(x) \cap \overset{\circ}{R}(y) \neq \emptyset \Rightarrow \overset{\circ}{R}(x) \cap \overset{\circ}{R}(y) \neq \emptyset \Rightarrow \overset{\circ}{R}(x) \cap \overset{\circ}{R}(y) \neq \emptyset \Rightarrow \overset{\circ}{R}(x) \cap R(y) \neq \emptyset \Rightarrow R(x) \cap R(y) \neq \emptyset \Rightarrow R_i = R_j \Rightarrow i = j
\]

3. \( f(\overset{\circ}{R}_i) \cap \overset{\circ}{R}_j \neq \emptyset \Rightarrow \overset{\circ}{R}_j \subseteq f(\overset{\circ}{R}_i) \)

Since \( f \) takes open sets into open sets and \( Z \) is dense in \( K \), \( f^{-1}(Z) \) is also dense in \( K \). Also, \( Z \) is a nonempty open set, so \( Z \cap f^{-1}(Z) \) is dense in \( Z \), and, hence, \( Z \cap f^{-1}(Z) \) is dense in \( K \). Since \( \overset{\circ}{R}_i \cap f^{-1}(\overset{\circ}{R}_j) \) is a nonempty open set, we have \( Z \cap f^{-1}(Z) \cap \overset{\circ}{R}_i \cap f^{-1}(\overset{\circ}{R}_j) \neq \emptyset \), so we can take \( x \in Z \cap \overset{\circ}{R}_i \) with \( f(x) \in Z \cap \overset{\circ}{R}_j \). Notice that \( x \in R(x) \subseteq R(x) \Rightarrow \overset{\circ}{\text{R}}_i \cap \overset{\circ}{\text{R}}(x) \neq \emptyset \Rightarrow R_i = \text{R}(x) \) and, similarly, \( R_j = \text{R}(f(x)) \). Using the previous lemma and the fact that \( g \) is continuous, we get 
\[
g(R_i) = g(\overset{\circ}{\text{R}}(f(x))) \subseteq g(\overset{\circ}{\text{R}}(f(x))) \subseteq \overset{\circ}{\text{R}}(x) = R_i \Rightarrow R_j = f(g(R_i)) \subseteq f(R_i).
\]

Now we will see that there is a semiconjugacy between \( f \) and a unilateral subshift of finite type. Let \( \{ R_1, \ldots, R_k \} \) be a partition of \( K \) as above. We can define a matrix \( A \in M_k \) by
\[
A_{ij} = 1 \text{ if } f(\overset{\circ}{R}_i) \cap \overset{\circ}{R}_j \neq \emptyset, \quad A_{ij} = 0 \text{ otherwise.}
\]

**Lemma 12.5** Let \( (a_0, \ldots, a_n) \) be an admissible sequence for \( A \). Then, \( \bigcap_{i=0}^n f^{-i}(\overset{\circ}{R}_{a_i}) \neq \emptyset \).
Besides, if \( x \) is continuous, it is a semiconjugacy of \( f \).

**Proof:** The lemma is trivial for sequences with just one element. Suppose now that the lemma is valid for the admissible sequence \((a_1, \ldots, a_n)\), so that \( \bigcap_{i=0}^{n-1} f^{-1}(R_{a_{i+1}^{-1}}) \neq \emptyset \). Let \( y \in \bigcap_{i=0}^{n-1} f^{-1}(R_{a_{i+1}^{-1}}) \). Since \( A_{a_{n}a_{1}} = 1 \), we have \( \hat{R}_1 \subseteq f(\hat{R}_0) \). So, \( y = f(x) \) for some \( x \in \hat{R}_0 \) and it is easy to see that \( x \in \bigcap_{i=0}^{n} f^{-i}(\hat{R}_{a_{i+1}}) \). \( \square \)

As a consequence of this lemma, we can see that, for each sequence \( a = (a_n)_{n \in \mathbb{N}_0} \in \Sigma_A^+ \), if \( F_n = \bigcap_{i=0}^{n-1} f^{-i}(R_{a_i}) \) then \( (F_n)_n \) is a decreasing sequence of nonempty compact sets, so its limit is nonempty. Besides, if \( x \) and \( y \) are two points in this intersection, then \( \forall i \in \mathbb{N}_0, d(f^i(x), f^i(y)) \leq \text{diam}(R_{a_i}) < \varepsilon \), so \( x = y \). Therefore, we can define a map \( \Pi : \Sigma_A^+ \to K \) by

\[
\{\Pi(a)\} = \lim_{n \to \infty} F_n = \bigcap_{n=0}^{\infty} f^{-n}(R_{a_n})
\]

Let \( a \in \Sigma_A^+ \). Notice that, since \( f \) is surjective, \( f(f^{-1}(L)) = L \) for any \( L \subseteq K \). Then, we have

\[
\{\Pi(\Pi(a))\} = f \left( \bigcap_{n=0}^{\infty} f^{-n}(R_{a_n}) \right) \subseteq f(R_{a_n}) \cap \bigcap_{n=1}^{\infty} f^{-n}(R_{a_n}) = f(R_{a_0}) \cap \bigcap_{n=0}^{\infty} f^{-n}(R_{a_{n+1}}) = \bigcap_{n=0}^{\infty} f^{-n}(R_{a_{n+1}}) = \{\Pi(\sigma_A^+(a))\}
\]

(recall that \( A_{a_{n}a_{1}} = 1 \) implies \( f(R_{a_0}) \supseteq R_{a_1} \)). So, \( \{\Pi(a)\} = \Pi(\sigma_A^+(a)) \) and, since \( \Pi \) is surjective and continuous, it is a semiconjugacy of \( \sigma_A^+ \) and \( f \). A point in \( K \) can have more than one preimage under \( \Pi \), but we will show that it can not have more than \( k \) preimages.

**Lemma 12.6** Let \((a_0, \ldots, a_n)\) and \((b_0, \ldots, b_n)\) be two admissible sequences for \( A \) with \( a_n = b_n \). If \( \forall i \in \{0, \ldots, n\}, R_{a_i} \cap R_{b_i} \neq \emptyset \), then the sequences are equal.

**Proof:** We have seen in the previous lemma that \( \bigcap_{i=0}^{n} f^{-i}(R_{a_i}) \neq \emptyset \), so there is some \( x \in K \) with \( f^i(x) \in \hat{R}_{a_i} \). By hypothesis, \( R_{a_n} = R_{b_n} \). Suppose now that, for \( i \in [n] \), we have \( R_{a_i} = R_{b_i} \). Since \( A_{a_{n-1}a_0} = A_{b_{n-1}b_0} = 1 \), we get \( \hat{R}_{a_0} \subseteq f(R_{a_{n-1}}) \) and \( \hat{R}_{b_0} \subseteq f(R_{b_{n-1}}) \). Then, since \( f^i(x) \in \hat{R}_{a_i} \subseteq R_{b_i} \), there are \( y \in R_{a_{n-1}} \) and \( z \in R_{b_{n-1}} \) such that \( f^i(x) = f(y) = f(z) \). Also, \( d(y, z) \leq \text{diam}(R_{a_{n-1}}) + \text{diam}(R_{b_{n-1}}) \leq \varepsilon \) because \( R_{a_{n-1}} \cap R_{b_{n-1}} \neq \emptyset \). So, \( y = z \) and \( R_{a_{n-1}} \cap R_{b_{n-1}} \neq \emptyset \). Since different elements of the partition must have disjoint interior, we conclude that \( R_{a_{n-1}} = R_{b_{n-1}} \). \( \square \)

Therefore,

**Proposition 13** Any point of \( K \) has no more than \( k \) preimages under \( \Pi \), where \( k \) is the number of rectangles of the partition.

**Proof:** Suppose, by contradiction, that there is a point in \( x \in K \) with \( k+1 \) distinct preimages. Call these preimages \( x^1, x^2, \ldots, x^{k+1} \). Then, for \( n \) big enough, the admissible sequences \((x^i_0, \ldots, x^i_n)\) must be different from each other. But, since we have \( k+1 \) sequences, at least two of them must have the same last element of the sequence, so they should be equal by the previous lemma (recall that, by definition of \( \Pi \), \( f^m(x) \in R_{a_{m+1}^{-1}} \) for every \( m \in \{0, \ldots, n\} \) and \( i \in [k+1] \)). \( \square \)
Proposition 14 The preimages of periodic points of $f$ are periodic points of $\sigma = \sigma_\mathcal{A}^r$.

Proof: Suppose that $x \in K$ is such that $f^p(x) = x$ for some $p \in \mathbb{N}$. Let $x^1, x^2, \ldots, x^n$ be the preimages of $x$, distinct from each other by hypothesis. Then, for every $i \in \{r\}$, we have $\Pi(\sigma^p(x^i)) = f^p(\Pi(x^i)) = f^p(x) = x$, so that $\sigma^p(x^1), \sigma^p(x^2), \ldots, \sigma^p(x^n)$ are also preimages of $x$.

Assume that there are $i, j \in \{r\}$, $i \neq j$, with $\sigma^p(x^i) = \sigma^p(x^j)$; in particular, we have $x^i_p = x^j_p$. Then, the admissible sequences $(x^i_0, \ldots, x^i_p)$ and $(x^j_0, \ldots, x^j_p)$ verify the hypothesis of the previous lemma, therefore they must be equal. So, $x^i = (x^i_0, x^i_1, \ldots, x^i_p, x^i_{p+1}, \ldots) = (x^j_0, x^i_1, \ldots, x^i_p, x^i_{p+1}, \ldots) = x^j$, which contradicts the assumption that $x^1, x^2, \ldots, x^n$ are distinct from each other.

So, $\sigma^p(x^1), \sigma^p(x^2), \ldots, \sigma^p(x^n)$ are also distinct from each other and, therefore, they are precisely the preimages of $x$, so that there is some permutation $\mu \in S$, such that $\sigma^p(x^i) = \bar{x}^\mu(i)$ for every $i \in \{r\}$. So, $\sigma^{\text{ord}(\mu)p}(x^i) = \bar{x}^{\text{ord}(\mu)}(i) = x^i$ for every $i \in \{r\}$.

Proposition 15 If $\bar{s}$ and $\bar{t}$ are two preimages of a periodic point $x \in K$ with $s_i = t_i$ for some $i \in \mathbb{N}_0$, then $\bar{s} = \bar{t}$.

Proof: In fact, since $\bar{s}$ and $\bar{t}$ are both periodic points, there is some common period $n$ such that $\sigma^n(\bar{s}) = \bar{s}$ and $\sigma^n(\bar{t}) = \bar{t}$. Then, the sequences $(s_i, s_{i+1}, \ldots, s_{i+n})$ and $(t_i, t_{i+1}, \ldots, t_{i+n})$ verify the hypothesis of the previous lemma: they end with the same element ($s_{i+n} = s_i = t_i = t_{i+n}$) and, by definition of $\Pi$, $f^m(x) \in R_{s_m}$ and $f^m(x) \in R_{t_m}$ for every $m \in \{i, \ldots, i+n\}$.

For each $r \in \{k\}$, define:

Definition 15.1 $I_r = \{\{s_1, \ldots, s_r\} \subset [k] : \bigcap_{i=1}^r R_{s_i} \neq \emptyset\}$ where we assume that $s_1 < s_2 < \ldots < s_r$.

Definition 15.2 $A^{(r)}$ and $B^{(r)}$ as matrices with coefficients indexed in the set $I_r$ given, satisfying the following conditions: fixing $s, t \in I_r$ with $s = \{s_1, \ldots, s_r\}$ and $t = \{t_1, \ldots, t_r\}$,

1. if there is an unique permutation $\mu \in S_r$ such that $A_{s_\mu t_{\text{ord}(\mu)}}(i) = 1$ for every $i \in \{r\}$, then $A_{st}^{(r)} = 1$ and $B_{st}^{(r)} = \text{sgn}(\mu)$, where $\text{sgn}(\mu)$ denotes the signature of the permutation $\mu$ (1 if the permutation is even and -1 if it is odd);

2. otherwise, $A_{st}^{(r)} = B_{st}^{(r)} = 0$.

Let $\Sigma^+_r = I_r^{\mathbb{N}_0}$ be the set of sequences indexed by $\mathbb{N}_0$ whose elements belong to $I_r$ and $\Sigma(A^{(r)})^+ \subseteq \Sigma^+_r$ be the subset of admissible sequences according to the matrix $A^{(r)}$. Also, let $\sigma^{(r)}_+$ denote the unilateral shift defined on these sets. Now, we will see how to define a codification map $\Pi_r : \Sigma(A^{(r)})^+ \rightarrow K$.

Given a sequence $\tilde{a} = (\tilde{a}_n)_{n} \in \Sigma(A^{(r)})^+$, with $\tilde{a}_n = \{a^1_n, \ldots, a^n_n\} \in I_r$, for every $n \in \mathbb{N}_0$, there is, by definition of $\Sigma(A^{(r)})^+$, an unique permutation $\mu_n$ such that $A_{\sigma^n_{\mu_{n+1}} a^n_{\mu_{n+1}}}(i) = 1, \forall i \in \{r\}$. Consider the permutations defined by $\nu_0 = \text{id}$ $\nu_n = \mu_{n-1} \circ \ldots \circ \mu_1 \circ \mu_0$.

Notice that $\mu_n \circ \nu_n = \nu_{n+1}, \forall n \in \mathbb{N}_0$. 

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For each $i \in [r]$ and $m \in \mathbb{N}_0$, let $\alpha^i_m = a^i_m$. Then, $\alpha^i = \langle \alpha^i_m \rangle_m$ belongs to $\Sigma^+_r$, for every $i \in [r]$. In fact, we have:

$$A_{\alpha^i_m \alpha^i_{m+1}} = A_{\alpha^i_m \alpha^i_{m+1}}(a^i_m) = A_{\alpha^i_m \alpha^i_{m+1}}(a^i_m) = 1, \forall m \in \mathbb{N}_0$$

For each $m \in \mathbb{N}_0$, since $\tilde{a}_m \in I_r$, we know that there is some $y_m \in \bigcap_{i=1}^r R_{\alpha^i_m}$. So, for all $i, j \in [r]$ we have:

$$d(f^m(\Pi(\alpha^i)), f^m(\Pi(\alpha^j))) \leq d(f^m(\Pi(\alpha^i)), y_m) + d(y_m, f^m(\Pi(\alpha^j))) \leq$$

$$\leq 2 \max_{n \in [k]} \{diam(R_n)\} < \delta < \varepsilon/2$$

which implies that $\Pi(\alpha^i) = \Pi(\alpha^j)$.

Hence, for each $r \in [k]$, we can define a map $\hat{\Pi}_r : \Sigma(A^r)^+ \rightarrow K$ by setting $\hat{\Pi}_r(\hat{\alpha}) = \Pi(\alpha^r)$, which does not depend on the choice of the index $i \in [r]$.

Let us verify that $\hat{\Pi}_r(Per_p(\Sigma(A^r)^+)) \subseteq Per_p(f)$. Given $\hat{\alpha} \in Per_p(\Sigma(A^r)^+)$, we have

$$\hat{\Pi}_r(\hat{\alpha}) = \{\Pi(\alpha^r)\} = \bigcap_{n \in \mathbb{N}_0} f^{-n}(R_{\alpha^r_n})$$

for any $i \in [r]$. So,

$$\hat{\Pi}_r(\hat{\alpha}) = \bigcap_{i \in [r]} \bigcap_{n \in \mathbb{N}_0} f^{-n}(R_{\alpha^r_n}) = \bigcap_{n \in \mathbb{N}_0} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_n} \right) = \bigcap_{n \in \mathbb{N}_0} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_n} \right)$$

and

$$\{f^p(\hat{\Pi}_r(\hat{\alpha}))\} = f^p \left( \bigcap_{n \in \mathbb{N}_0} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_n} \right) \right) \subseteq \bigcap_{n \in \mathbb{N}_0} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_n} \right) \subseteq \bigcap_{n \in \mathbb{N}_0, n \geq p} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_n} \right) =$$

$$= \bigcap_{n \in \mathbb{N}_0} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_{n+p}} \right) = \bigcap_{n \in \mathbb{N}_0} f^{-n} \left( \bigcap_{i \in [r]} R_{\alpha^r_n} \right) = \{\hat{\Pi}_r(\hat{\alpha})\}$$

because $f$ is surjective and $\tilde{a}_m = \tilde{a}_{m+p}, \forall n \in \mathbb{N}_0$. So, $f^p(\hat{\Pi}_r(\hat{\alpha})) = \hat{\Pi}_r(\hat{\alpha})$.

On the other hand, if $x \in Per_p(f)$, let $\tilde{\alpha}^1, \ldots, \tilde{\alpha}^r$ be the preimages of $x$ under the map $\Pi$ (notice that $r \leq k$, by a previous proposition). For each $m \in \mathbb{N}_0$ and $i \in [r]$, we have $f^m(x) \in R_{\alpha^i_m}$, so $\bigcap_{i \in [r]} R_{\alpha^i_m} \neq \emptyset$ and, since $\alpha^i_m \neq \alpha^j_m$ for $i \neq j$ (by the previous proposition), we can define an element $\tilde{a}_m \in I_r$ and, therefore, build a sequence $\tilde{\alpha} = (\tilde{a}_m)_{m \in \mathbb{N}_0} \in \Sigma^+_r$.

Let us see that $\mu = id$ is the only permutation in $S_r$ such that $A_{\alpha^i_m \alpha^i_{m+1}}(a^i_m) = 1, \forall i \in [r]$. Take a permutation $\mu \in S_r$, with order $r$, such that $A_{\alpha^i_m \alpha^i_{m+1}}(a^i_m) = 1, \forall i \in [r]$. Given any $j \in [r]$, consider the two admissible sequences
\[ \alpha_n^j \alpha_{n+1}^{\mu(j)} \cdots \alpha_{n+q}^{\mu(j)} \alpha_{n+q+1}^{\mu^2(j)} \cdots \alpha_{n+(r-1)q}^{\mu^r-1(j)} \alpha_n^j \]

and

\[ \alpha_n^j \alpha_{n+1}^{j} \cdots \alpha_{n+q}^{j} \alpha_{n+q+1}^{j} \cdots \alpha_{n+(r-1)q}^{j} \]

where \( q \) is a common period of the preimages of \( x \). By a previous lemma, they must be equal; in particular, \( \alpha_{n+1}^j = \alpha_n^j \). Then, the last proposition tells us that \( \mu(j) = j \) and, therefore, \( \mu = \text{id} \).

So, we have \( \hat{\alpha} \in \Sigma(A^{(r)})^+ \). Also, as we have seen before, the set of preimages of \( x \) is invariant by \( \sigma^p \). Then, for each \( m \in \mathbb{N}_0 \), the element \( \hat{a}_{m+p} \in I_r \), whose elements are \( \alpha_m^q, \ldots, \alpha_{m+p}^q \), is the same as the element \( \hat{a}_m \in I_r \), because its elements, \( \alpha_m^q, \ldots, \alpha_{m+p}^q \), are the same (although not necessarily in the same order). Therefore \( \hat{a}_{m+p} = \hat{a}_m \), that is, \( \hat{\alpha} \in \text{Per}_p(\Sigma(A^{(r)})^+) \).

The next proposition provides a formula for the number of periodic points of \( f \).

**Proposition 16** For all \( p \in \mathbb{N} \),

\[ N_p(f) = \sum_{r=1}^{L} (-1)^{r-1} \text{tr}((B^{(r)^p})) \]

where \( L \) is the largest value of \( r \in [L] \) for which \( I_r \neq \emptyset \) (notice that, if \( I_r \neq \emptyset \), then \( I_{r'} \neq \emptyset \) for \( r' < r \)).

**Proof:** Given \( x \in \text{Per}_p(f) \), consider the function given by

\[ \Phi(x) = \sum_{t=1}^{L} \left( \sum_{\hat{\alpha} \in \hat{\Pi}^{-1}(x) \cap \text{Per}_p(\Sigma(A^{(t)})^+)} (-1)^{t-1} \text{sgn} (\nu) \right) \]

where \( \nu \) is the unique permutation in \( S_t \) such that \( \alpha_{\nu(i)}^\mu = \alpha_i^\mu, \forall i \in [t] \), with \( \alpha_i^\mu, i \in [t] \) the elements of \( \Sigma_A^\mu \) constructed as before. We want to show that \( \Phi(x) = 1 \). Let \( \Pi^{-1}(x) = \{ \alpha_1^\mu, \ldots, \alpha_r^\mu \} \) and \( \mu \) be the permutation such that \( \sigma^p(\alpha_i^\mu) = \alpha_{\nu(i)}^\mu, \forall i \in [r] \), that is, the permutation induced by the action of \( \sigma^p \) on \( \Pi^{-1}(x) \). We can write \( \mu \) as the product of disjoint cycles \( \mu_1, \ldots, \mu_s \) (eventually with length 1) which act on the sets \( K_1, \ldots, K_s \), respectively, and these sets form a partition of \( [r] \).

Given \( \hat{\alpha} \in \hat{\Pi}^{-1}(x) \), we can build \( t \) distinct preimages of \( x \) under \( \Pi \), with \( t \leq r \). Let \( J \subseteq [r] \) be such that these preimages are \( (\alpha_i^\mu)_{j \in J} \). If we suppose additionally that \( \hat{\alpha} \in \text{Per}_p(\Sigma(A^{(t)})^+) \), then \( J \) is invariant under \( \nu \), so we can write \( J = \bigcup_{m \in B} K_m \) for some \( \emptyset \neq B \subseteq [s] \). On the other hand, for each nonempty subset \( B \) of \( [s] \), we can take \( J = \bigcup_{m \in B} K_m \) and associate to it a sequence \( \hat{\alpha} \) given by the set of distinct preimages \( (\alpha_i^\mu)_{j \in J} \).

So, for each \( t \in [L] \) and \( \hat{\alpha} \in \hat{\Pi}^{-1}(x) \cap \text{Per}_p(\Sigma(A^{(t)})^+) \), we can associate an unique nonempty subset \( B \) of \( [s] \), and we have

\[ t = \text{card}(J) = \text{card} \left( \bigcup_{m \in B} K_m \right) = \sum_{m \in B} \text{card}(K_m) \]

Since \( \mu_m \) is a cycle of length \( \text{card}(K_m) \), we have
Suppose now this is true for \( n \) at \( m \) for \( A \). Since \( a \in \{ \ldots, 0, 1, \ldots, n \} \), \( \nu \in \text{Per}_n(\Sigma(A^{(i)})) \text{ and end at } \hat{a}_0, \hat{a}_1 \in I_t \) we have \( \nu_1 = \mu_0 \), so

\[
\text{sgn}(\nu_1) = \text{sgn}(\mu_0) = (B^{(i)})^{\hat{a}_0\hat{a}_1}
\]

Suppose now this is true for \( n = m - 1 \). Then, for \( n = m \) we have

\[
\sum_{S(\hat{a}_0, \hat{a}_m, m)} \text{sgn}(\nu_m) = \sum_{S(\hat{a}_0, \hat{a}_m, m)} \text{sgn}(\mu_{m-1}) \text{sgn}(\nu_{m-1}) = \sum_{\{\hat{a}_m-1 \in I_t : A^{(i)}_{\hat{a}_m-1\hat{a}_m} = 1\}} \left( \sum_{S(\hat{a}_0, \hat{a}_{m-1}, m-1)} \text{sgn}(\nu_{m-1}) \right) \text{sgn}(\mu_{m-1}) = \ldots
\]

Hence,

\[
(-1)^{t-1} \text{sgn}(\nu) = (-1)^{2^{t-1} + \text{card}(B)} = -(-1)^{\text{card}(B)}
\]

\[
\Phi(x) = \sum_{t=1}^{L} \left( \sum_{\hat{a} \in \Pi_t^{-1}(x) \cap \text{Per}_p(\Sigma(A^{(i)}))} (-1)^{t-1} \text{sgn}(\nu) \right) = -\sum_{\emptyset \neq B \subseteq [s]} (-1)^{\text{card}(B)} - \sum_{q=1}^{s} \sum_{B \subseteq [s], \text{card}(B) = q} (-1)^{\text{card}(B)} = -\sum_{q=1}^{s} \binom{s}{q} (-1)^{q} = \binom{s}{0} (-1)^{0} - \sum_{q=0}^{s} \binom{s}{q} (-1)^{q} = 1 - (1 - 1)^{s} = 1
\]

Since \( \text{Per}_p(\Sigma(A^{(i)})) \subseteq \Pi_t^{-1}(\text{Per}_p(f)) \), we have

\[
N_p(f) = \sum_{x \in \text{Per}_p(f)} \Phi(x) = \sum_{x \in \text{Per}_p(f)} \sum_{t=1}^{L} \left( \sum_{\hat{a} \in \Pi_t^{-1}(x) \cap \text{Per}_p(\Sigma(A^{(i)}))} (-1)^{t-1} \text{sgn}(\nu) \right) = \sum_{t=1}^{L} \left( \sum_{\hat{a} \in \text{Per}_p(\Sigma(A^{(i)}))} (-1)^{t-1} \text{sgn}(\nu) \right) = \sum_{t=1}^{L} (-1)^{t-1} \left( \sum_{\hat{a} \in \text{Per}_p(\Sigma(A^{(i)}))} \text{sgn}(\nu) \right)
\]
\[
\sum_{\hat{a}_{m-1} \in I, \hat{a}_{m-1} \hat{a}_m = 1} \left( (B^{(t)})^{m-1} \hat{a}_0 \hat{a}_{m-1} B^{(t)}_{\hat{a}_{m-1} \hat{a}_m} = (B^{(t)})^m \hat{a}_0 \hat{a}_m \right)
\]

In particular, we get

\[
\sum_{S(\hat{a}_0, \hat{a}_0, n)} sgn(\nu_n) = (B^{(t)})^n \hat{a}_0 \hat{a}_0
\]

As for each sequence \( \hat{a} \in Per_p(\Sigma(A^{(t)})^+) \) we can associate an unique element of \( S(\hat{a}_0, \hat{a}_0, p) \) which verifies \( \nu_p = \nu \), we conclude that

\[
\sum_{\hat{a} \in Per_p(\Sigma(A^{(t)})^+)} sgn(\nu) = \sum_{\hat{a}_0 \in I_t} ((B^{(t)})^p) \hat{a}_0 \hat{a}_0 = \text{tr}((B^{(t)})^p)
\]

and therefore

\[
N_p(f) = \sum_{t=1}^{L} (-1)^{t-1} \text{tr}((B^{(t)})^p)
\]

\[\square\]

**Theorem 16.1** If \( f \) is Ruelle-expanding, then its zeta function is rational.

**Proof:** As

\[
N_n(f) = \sum_{r=1}^{L} (-1)^{r-1} \text{tr}((B^{(r)})^n) = \sum_{r \in [L], r \text{ odd}} \text{tr}((B^{(r)})^n) - \sum_{r \in [L], r \text{ even}} \text{tr}((B^{(r)})^n)
\]

we have

\[
\zeta_f(t) = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(f)}{n} t^n \right) = \exp \left( \sum_{n=1}^{\infty} \sum_{r \in [L], r \text{ odd}} \frac{\text{tr}((B^{(r)})^n)}{n} t^n - \sum_{r \in [L], r \text{ even}} \frac{\text{tr}((B^{(r)})^n)}{n} t^n \right)
\]

\[
= \exp \left( \sum_{n=1}^{\infty} \sum_{r \in [L], r \text{ odd}} \frac{\text{tr}((B^{(r)})^n)}{n} t^n \right) - \exp \left( \sum_{n=1}^{\infty} \sum_{r \in [L], r \text{ even}} \frac{\text{tr}((B^{(r)})^n)}{n} t^n \right)
\]

\[
= \prod_{r \in [L], r \text{ odd}} \exp \left( \sum_{n=1}^{\infty} \frac{\text{tr}((B^{(r)})^n)}{n} t^n \right) - \prod_{r \in [L], r \text{ even}} \exp \left( \sum_{n=1}^{\infty} \frac{\text{tr}((B^{(r)})^n)}{n} t^n \right)
\]

\[
= \prod_{r \in [L], r \text{ odd}} \det(I - tB^{(r)}) \frac{1}{\prod_{r \in [L], r \text{ even}} \det(I - tB^{(r)})}
\]

which is clearly a rational function.

\[\square\]
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