Semiclassical string computation of strong-coupling corrections to dimensions of operators in Konishi multiplet

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Abstract

Following our earlier work in arXiv:0906.4294 we show how to use semiclassical string quantization approach to compute the leading corrections to the energy of $AdS_5 \times S^5$ string states on the first excited string level that should correspond to operators in the Konishi multiplet of $\mathcal{N} = 4$ SYM theory. Compared to examples in our previous paper the string solutions we consider here carry an extra component of $S^5$ angular momentum $J$. This facilitates their identification with operators in the Konishi multiplet. We show that for such string states with $J = 2$ the coefficient of the subleading $\lambda^{-1/4}$ term in the large string tension expansion of the energy is twice the one found in arXiv:0906.4294. The resulting value matches the one found for the Konishi state in the $sl(2)$ sector from the Y-system/TBA approach, resolving an apparent disagreement claimed earlier.

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1 Introduction

This paper is a direct sequel to our earlier work [1] where we discussed how a semiclassical quantization of “short” strings in AdS$_5 \times S^5$ may be used to determine the structure of large string tension ($T = \sqrt{\lambda} \gg 1$) expansion of the AdS energy of quantum string states on the first excited string level. The analysis of the structure of the “short” string expansion [2, 1] or of the marginality condition for string vertex operators [1, 3] implies that the energy of such states for large $\sqrt{\lambda}$ may be written as

$$E = 2\sqrt{\lambda} + b_0 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{\sqrt{\lambda}} + O\left(\frac{1}{(\sqrt{\lambda})^3}\right).$$

The coefficients $b_1, b_2, ...$ should be the same for all states in one supermultiplet with $b_0$ being an integer related to a position of a given state in the supermultiplet. As was argued in [1], for the Konishi multiplet $b_0 + 4 = \Delta_0$ should be equal to bare ($\lambda = 0$) dimension of the corresponding gauge-theory operator, while $b_1$ should receive contributions only from the classical string energy and the string 1-loop correction to it.

Generic string solutions [4, 5, 6] carry global $SO(2,4) \times SO(6)$ charges ($E, S_1, S_2; J_1, J_2, J_3$), i.e. the energy and components of spins/orbital momenta in 2+3 orthogonal planes of AdS$_5 \times S^5$. While in the standard semiclassical expansion the values of these charges are assumed to be of order of the string tension which is taken to be large, one may formally interpolate the semiclassical results to fixed values of $S_r$ and $J_i$ by considering “short” string limit [2, 1] when $S_r = \frac{S_r}{\sqrt{\lambda}} \ll 1$ and $J_i = \frac{J_i}{\sqrt{\lambda}} \ll 1$.

Establishing a correspondence between semiclassical string states and the dual gauge-theory operators is, in general, a highly non-trivial enterprise. The values of the global charges may be used as a guide in constructing such a map. In particular, quantum string states and the dual gauge theory operators should be highest-weight states with Dynkin labels$^1$

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_1, s_2, s_3)}$. For example, the singlet operator $\text{Tr}(\Phi^4 \Phi_4)$ is which is the “ground-state” of the Konishi multiplet [7, 8]$^2$ corresponds to $[0, 0, 0]_{(0,0)}$ with $\Delta_0 = 2$ while its supersymmetry descendents in the $sl(2)$ and $su(2)$ sectors are, respectively, $\text{Tr}(\Phi_1 D^2 \Phi_1)$ with $S = J_1 = 2$ or $[0, 2, 0]_{(1,1)}$, and $\text{Tr}((\Phi_1, \Phi_2)^2)$ with $J_1 = J_2 = 2$ or $[2, 0, 2]_{(0,0)}$, both with bare dimension $\Delta_0 = 4$.

The semiclassical description of string states is very crude (unless the values of all hidden charges are specified, as it is the case in the integrability-based description). In particular, one may not be able to distinguish between zero and finite values of spins at large string tension, so that it is important to have additional cross-checks. In particular, one should verify that the string theory predicts universal (i.e. equal) values of anomalous dimensions for states that are expected to belong to the same supermultiplet. Such a universal behavior was observed in [1]. The states considered there formally fit (under some particular assignment of their global charges) into some representations present in the Konishi multiplet table. There was, however,

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$^1$We follow the standard notation used in [1]: $SU(2) \times SU(2)$ labels $(s_L, s_R)$ for $SO(4)$ and the Dynkin labels $[p_1, q, p_2]$ for $SU(4)$ are given by $s_L, s_R = \frac{1}{2} (S_1 \pm S_2)$, and $p_{1,2} = J_2 \mp J_3$, $q = J_1 - J_2$.

$^2$We copy the table of its states (in the form taken from [8]) at the end of this paper.
only a circumstantial evidence\(^3\) that these states correspond indeed to members of the Konishi multiplet, implying that one needs some additional input or data.

An additional assumption used in [1] was that in identifying semiclassical spinning string states with quantum string states one may use analogy with the bosonic or NSR string spectrum in flat space \(E^2 = 2\sqrt{\lambda}(N - 2)\) where \(N = 2, 4, \ldots\) is the maximal value of spin at a given string level (i.e. \(N = 2\) at the massless level, \(N = 4\) at the first excited string level, etc.). Following such pattern would suggest that one should shift the value of the total spin of a semiclassical string by -2 when comparing to quantum string states. However, in describing the light-cone GS superstring spectrum one identifies the massless string level with the ground state not carrying any genuine string oscillators: all massless (supergravity) string modes are obtained by quantizing the superparticle representing the point-like string limit (with the spin of the graviton carried by the fermion 0-modes building the vacuum supermultiplet, etc.; the same picture applies in the case of the superstring in a pp-wave background [9, 10]). Then the spin of an extended string solution should be associated only with “extra” oscillators acting on the ground state.

In this paper we shall use this superstring pattern when comparing semiclassical strings to quantum string states, i.e. we shall assume that in the flat-space limit \(E^2 = 2\sqrt{\lambda}N\), \(N = 0, 2, \ldots\), with the maximal spin carried by extended string motion being \(N\). States on the first excited string level \(N = 2\) will have just two oscillators, and the resulting values of spins they may carry are \(S_1 = 2\), \(S_2 = J_i = 0\), or \(S_1 = S_2 = 1\), \(J_i = 0\), or \(S_1 = S_2 = 0\), \(J_1 = 2\), or \(S_1 = J_1 = 1\), \(S_2 = J_i = 0\), or \(J_1 = J_2 = 1\), \(S_i = J_3 = 0\). States with such small values of spins will not correspond to non-trivial states in the “minimal” closed \(sl(2)\) or \(su(2)\) sectors on the gauge theory side. Also, identification to members of the Konishi multiplet table appears to be ambiguous.

There is, however, a natural resolution: instead of considering classical solutions that carry no orbital angular momentum \(J = J_3\) in \(S^5\) (as was the case in [1]) one should switch on a minimal non-zero value of \(J\). While having \(J \neq 0\) will not influence the value of the string level it will allow for a natural identification of the corresponding semiclassical string solutions with operators in the \(sl(2)\) or \(su(2)\) sectors and with members in the Konishi multiplet (for \(J = 2\)).

Thus the main point of the present paper is that one should focus on the simplest extended string states built “on top” of the BPS vacuum state with charge \(J = 2\) (i.e. the chiral primary operator \(Tr(\Phi^2)\) on the gauge theory side or a superparticle state on the string side). For this reason, to describe string states dual to the Konishi multiplet states here we shall consider simplest string solutions that carry a (minimal) non-zero value of the \(S^5\) orbital momentum, i.e. their center of mass will be orbiting a circle in \(S^5\).\(^4\)

As we shall see below, the direct generalizations of the semiclassical string states discussed in [1] to states that carry an additional momentum \(J = 2 \ll \sqrt{\lambda}\) have their natural places in the Konishi multiplet table of states.\(^5\) The corresponding value of \(b_1\) in (1.1) is shifted by the

\(^3\)For example, one may check that in the flat space limit the small circular string states considered in [1] apparently belong to the same supersymmetry multiplet.

\(^4\)As we shall discuss below, the simplest “short” strings without such \(S^5\) momentum like folded spinning strings or pulsating strings with just one quantum number (for which \(b_1\) in (1.1) happens to be transcendental [2, 14, 15]) presumably do not to correspond to members of the Konishi multiplet.

\(^5\)The state with \(J = 2\) in the supergravity multiplet, dual to a chiral primary operator, may be described
“classical” contribution proportional to $J^2$ as

$$b_1(J) = b_1(0) + \frac{1}{4} J^2.$$  \hfill (1.2)

As the universal value of the coefficient $b_1(0)$ found in [1] for different solutions was 1, this implies then that $b_1(2) = 2$. This is precisely the value for the Konishi multiplet state in the $sl(2)$ sector (having $S = J = 2$) found in [12], and recently also in [13], from the Y-system/TBA equations (starting from the weak coupling region and interpolating numerically to strong coupling).

The $J^2$ term in (1.2) has a simple classical origin.\(^6\) Consider, e.g., a “short” string with some charge $N$ (e.g. spin or oscillation number) orbiting also in $S^5$. Then for small $\frac{N}{\sqrt{\lambda}}$ we have for the classical string energy $E_0^2 = 2\sqrt{\lambda}N + aN^2 + J^2 + ...$ ($a$ is a state-dependent constant). Assuming that $Q, J \ll \sqrt{\lambda}$ we then find the following expansion

$$E_0 = \sqrt{2\sqrt{\lambda}}N \left[ 1 + \frac{1}{4\sqrt{\lambda}} \left( aN + \frac{J^2}{N} \right) + ... \right].$$  \hfill (1.3)

For $N = 2$ we then get a state on the first excited string level. Switching on non-zero $J$ thus shifts the value of $b_1$ coefficient in (1.1) by the $J^2$ term given in (1.2).

Below we shall illustrate and verify this argument on several explicit examples. We shall consider the same solutions as in [1] and generalize them to the case of non-zero orbital momentum $J$ in $S^5$. In particular, we will show that for rigid circular strings switching on a non-zero value of $J \ll \sqrt{\lambda}$ does not alter the value of the 1-loop string correction to $b_1$ found in [1]. The resulting states with $N = 2$ (to be on the first excited string level, cf. (1.3)) and $J = 2$ will transform in the same representations as the corresponding members of the Konishi multiplet and for all of them we will find that $b_1 = 2$, matching the Y-system result of [12].

2 Examples of semiclassical string states dual to

$\Delta_0 = 6$ states in Konishi multiplet

We shall start with examples of simple circular string solutions which have three non-vanishing spins and generalize the 2-spin solutions considered in [1]. For each such solution (with values

\(6\)This term was already mentioned in footnote 34 in [1]. There we considered the example of a short $(S, J)$ folded string [5, 16] but we were wrongly assuming that $S = 4$ (instead of $S = 2$) in which case there is no corresponding state in the Konishi multiplet. Similar $J^2$ term in $b_1$ was mentioned also in [17, 18].
of spins representing a state on the first excited string level) there will be a state in the corresponding representation in the Konishi multiplet table. Members of the same supermultiplet are expected to have the same anomalous dimension and we will see the evidence of that in the universality of the predicted value of $b_1 = 2$ in (1.1).

2.1 Small circular spinning string with $J_1 = J_2$ and $J_3 \neq 0$

The “small” circular spinning string with $J_1 = J_2 = J_3$ in $S^5$ has its classical energy given simply by its flat-space expression [1], i.e. $E_0 = \sqrt{2\lambda(J_1 + J_2)} = 2\sqrt{\lambda J'}$. Its generalization to include a non-zero orbital momentum $J = J_3$ in a third plane is given by [6, 19]  

$$
X_1 = a e^{i(w\tau + \sigma)} , \quad X_2 = a e^{i(w\tau - \sigma)} , \quad X_3 = \sqrt{1 - 2a^2} e^{i\nu \tau} ,
$$

(2.1)

$$
\mathcal{E}_0^2 = \kappa^2 = 4a^2 + \nu^2 = \nu^2 + \frac{4J'}{\sqrt{1 + \nu^2}} , \quad w^2 = 1 + \nu^2 ,
$$

(2.2)

$$
J' \equiv J_1 = J_2 = a^2 w , \quad J \equiv J_3 = (1 - 2a^2) \nu , \quad \nu = \frac{J}{1 - \frac{\nu}{\sqrt{1 + \nu^2}}} .
$$

(2.3)

Expanding the classical string energy for $J' = J_3' \ll 1, \ J = \frac{J}{\sqrt{\lambda}} \ll 1$ we get (cf. (1.3))

$$
E_0 = 2\sqrt{\lambda J'} \left[ 1 + \frac{1}{\sqrt{\lambda} 8J'} \frac{J^2}{128J'^2} \left( \frac{J^4}{128J'^4} - \frac{J^2}{4} \right) + \ldots \right] .
$$

(2.4)

One can give a general argument (see sec. 2.4 below) that the leading term in the 1-loop correction to string energy expanded in $J = \frac{J}{\sqrt{\lambda}} \ll 1$ should not depend on $J$, i.e. should have the same value as in the $J = 0$ case considered in [1]. For this particular solution we explicitly prove this in the Appendix (using fluctuation frequencies found in [20]). As a result, the 1-loop corrected expression for the energy can be written as

$$
E = E_0 + E_1 = 2\sqrt{\lambda J'} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{J^2}{8J'} + \frac{1}{2} \right) + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2}) \right] + b_0 .
$$

(2.5)

The value of the constant term $b_0$ is sensitive to the 0-mode contributions and appears to be discontinuous with $J$: equal to 0 for $J = 0$ but equal to 2 for $J = 2$ (cf. [1]).

To get a state on the first excited string level we should choose $J' = 1$, i.e. $J_1 = J_2 = 1$. Choosing the minimal non-trivial value of $J = J_3 = 2$ and relabeling the momenta (as

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7We follow closely the notation of [1].

8Here we are assuming that $J = \mathcal{O}(1) \ll \sqrt{\lambda}$; for large $J \sim \sqrt{\lambda}$ a linear term in $J$ will appear already in the classical string energy. In general, to systematically reproduce such contributions to $b_0$ one would need to perform 0-mode superparticle quantization.

9As already discussed in the introduction, this identification of charges is different from the one in [1] where it was assumed that $N = 2J'$ should be renormalized as $2J' \rightarrow 2J'_{eff} = 2J' - 2$. In general, fixing constant quantum integer shifts of charges (like energy and spins) is a subtle issue in the semiclassical approach. Its resolution requires to start with the correct definition of the charge operators, see also [1]. The identification in [1] was based on an NSR-type interpretation of states, where the first excited string level has two left-moving and two right-moving oscillators. The first excited level of the GS string has only one left-moving and one right-moving oscillator, which is what we are assuming here.
$J_1 = 2, J_2 = J_3 = 1$) to compare to states in the Konishi multiplet table, we conclude that, remarkably, there is a unique corresponding state there – $[0, 1, 2]_{(0,0)}$ (at level $\Delta_0 = 6$). The resulting value of $b_1$ in (1.1) that follows from (2.5) is then consistent with (1.2),

$$b_1 = 2 \left( \frac{J^2}{8J'} + \frac{1}{2} \right)_{J=2,J'=1} = 2$$.

(2.6)

### 2.2 Small circular spinning string with $S_1 = S_2$ and $J \neq 0$

The same discussion can be repeated for a rigid circular string with two equal spins in $AdS^5$ and orbital momentum $J = J_1$ in $S^5$. The solution is $[6, 19]$

$$Y_0 + iY_5 = \sqrt{1 + 2r^2} e^{i\kappa \tau}, \quad Y_1 + iY_2 = r e^{i(w\tau + \sigma)}, \quad Y_3 + iY_4 = r e^{i(w\tau - \sigma)},$$

(2.7)

$$X_1 + iX_2 = e^{i\nu \tau}, \quad w^2 = \kappa^2 + 1, \quad \kappa^2 = 4r^2 + \nu^2,$$

(2.8)

$$\mathcal{E}_0 = (1 + 2r^2)\kappa = \kappa + \frac{2\kappa S}{\sqrt{1 + \kappa^2}}, \quad S = S_1 = S_2 = r^2 w, \quad J = \nu.$$  

(2.9)

The parameter $\kappa$ is determined from the conformal gauge condition

$$\kappa^2 = \frac{4}{\sqrt{1 + \kappa^2}} S + J^2,$$

(2.10)

and leads to the following “short” string expansion of the classical energy ($E_0 = \sqrt{\lambda} \mathcal{E}_0$):

$$E_0 = 2\sqrt{S} \left( 1 + S + \frac{J^2}{8S} + ... \right).$$

(2.11)

The leading 1-loop correction to the energy for $J = 0$ was $-\sqrt{S}$ [1, 21] and, as in the previous section, it should be the same for small $J$ as well. The constant shift $b_0$ was zero for $J = 0 [1]$ but now we expect to find $b_0 = 2$ for $J = 2$. We thus end up with

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda} S} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( S + \frac{J^2}{8S} - \frac{1}{2} \right) + O \left( \frac{1}{(\sqrt{\lambda})^2} \right) \right] + b_0.$$  

(2.12)

As for the small circular string in $S^5$, in the flat space limit the state on the first excited level associated to the string with two equal spins in $AdS_5$ appears to have only two excited oscillators, i.e. the corresponding state should have $S = S_1 = S_2 = 1$. Then, for $J = 2$, the dual state should be in representation $[0, 2, 0]_{(1,0)}$. As in the previous example, there is just one such state in the Konishi multiplet table, at level $\Delta_0 = 6$. The resulting value of $b_1$ is the same as in (2.6)

$$b_1 = 2 \left( S + \frac{J^2}{8S} - \frac{1}{2} \right)_{S=1,J=2} = 2.$$

(2.13)

$^{10}$The present solution $(E, S_1, S_2; J)$ is related to the $(E; J_1, J_2, J_3)$ circular solution by an analytic continuation so the fluctuation spectra should be similar.
2.3 Small circular spinning string with $S = J_1$ and $J_2 \neq 0$

Next, let us consider a $J = J_2 \neq 0$ generalization of the rigid circular solution with one spin in $AdS_5$ and one angular momentum in $S^5$ discussed in [1]

$$Y_0 + iY_5 = \sqrt{1 + r^2} e^{i\nu \tau}, \quad Y_1 + iY_2 = r e^{i(w\tau - \sigma)}, \quad w^2 = \kappa^2 + 1,$$

$$X_1 + iX_2 = a e^{i(w'\tau - \sigma)}, \quad X_3 + iX_4 = \sqrt{1 - a^2} e^{i\nu \tau}, \quad w'^2 = \nu^2 + 1,$$

$$\kappa^2 - \nu^2 = 2r^2 + 2a^2, \quad r^2 w = a^2 w', \quad (2.14)$$

$$X_1 + iX_2 = a e^{i(w'\tau - \sigma)}, \quad X_3 + iX_4 = \sqrt{1 - a^2} e^{i\nu \tau}, \quad w'^2 = \nu^2 + 1,$$

$$\kappa^2 - \nu^2 = 2r^2 + 2a^2, \quad r^2 w = a^2 w', \quad (2.15)$$

The parameters $\kappa$ and $\nu$ may be found by solving the equations

$$\kappa^2 - \nu^2 = \frac{2S}{\sqrt{1 + \kappa^2}} + \frac{2S}{\sqrt{1 + \nu^2}}, \quad J_2 = \nu - \frac{\nu S}{\sqrt{1 + \nu^2}}. \quad (2.16)$$

Clearly, $\kappa^2$ is a series in $\nu^2 \ll 1$; since, moreover, $\kappa \neq 0$ at $\nu = 0$, it follows that $\kappa$ is also a series in $\nu^2 = J_2^2 + ...$. One finds that the classical energy is

$$E_0 = \kappa + \frac{\kappa S}{\sqrt{1 + \kappa^2}}, \quad S = r^2 w = a^2 w' = J_1, \quad J_2 = (1 - a^2)\nu. \quad (2.17)$$

The leading 1-loop correction to the energy for $J_2 = 0$ was vanishing in [1], and, as we shall argue in sec.2.4 the same should be true also for non-zero $J_2 \ll \sqrt{\lambda}$. Therefore, we find that

$$E = E_0 + E_1 = 2\sqrt{\lambda S} \left[1 + \frac{1}{\sqrt{\lambda}} \left(\frac{1}{2} S + \frac{J_2^2}{8S}\right) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right)\right] + b_0. \quad (2.18)$$

Similarly to the other two circular string solutions, to get a state on the first excited level we should choose $S = J_1 = 1$. Then for $J_2 = 2$ the corresponding representation is (relabeling $J_1$ and $J_2$) [1, 1, 1]. Unlike the two previous examples, there exist several states in the Konishi multiplet table which transform in this representation (at levels $\Delta_0 = 6$ and $\Delta_0 = 4, 8$). Since in the flat space limit this state happens to combine with the previous two circular string states into a single Lorentz-invariant multiplet, we expect that it corresponds to a level $\Delta_0 = 6$ state. The resulting $b_1$ takes again the same value:

$$b_1 = 2\left(\frac{1}{2} S + \frac{J_2^2}{8S}\right)_{s=1, j_2=2} = 2. \quad (2.21)$$

2.4 On the independence of the leading 1-loop correction to string energy on $J \ll \sqrt{\lambda}$

One may give a general argument suggesting that if a semiclassical angular momentum $J = \sqrt{\lambda} J \ll \sqrt{\lambda}$ is added to a classical solution carrying other charges (of the same order or larger) in the quantum corrections to the string energy its presence will first show up at order $J^2 = (\frac{J^2}{(\sqrt{\lambda})^2})$. The leading $\frac{1}{\sqrt{\lambda}}$ order (for fixed $J$) correction is then not affected by $J$ as it happened in the examples discussed above.
The $\mathcal{J} \to -\mathcal{J}$ symmetry of the string Lagrangian expanded around a solution of the type described above implies that it depends on $\mathcal{J}$ only through $\mathcal{J}^2$. This need not, however, immediately apply to quantum corrections to the energy or other charges. For example, for special fluctuation modes the corresponding solutions of the characteristic equations may depend, in fact, on $|\mathcal{J}|$ suggesting that such terms may potentially appear in the 1-loop correction to the energy.

One may make an assumption that if $\mathcal{J}$ is added to a non-trivial classical solution, then observables should be continuous functions of $\mathcal{J}$, i.e. that the limit $\mathcal{J} \to 0$ should give the same answer whether one approaches the $\mathcal{J} = 0$ point from $\mathcal{J} > 0$ or from $\mathcal{J} < 0$.\footnote{We do not assume, however, that this $\mathcal{J} \to 0$ limit reproduces the value of the observable computed directly at $\mathcal{J} = 0$. The energies of folded spinning string solutions discussed in the next section are examples of cases when these two quantities are different.}

Suppose we consider a string partition function as function of parameters of the classical solution and evaluate the derivative of it with respect to $\mathcal{J}$. Potential terms containing $|\mathcal{J}|$ would make this derivative discontinuous at $\mathcal{J} = 0$ as $\frac{d\mathcal{J}}{d\mathcal{J}} = \text{sgn}(\mathcal{J})$. As discussed in [22], the semiclassical parameter $\mathcal{J}$ may be given the interpretation of the chemical potential for the angular momentum $J$. The conformal gauge constraint relates this chemical potential to chemical potentials for other charges and for the energy, i.e. $\kappa = \kappa(\mathcal{J}^2, ...)$.

Then according to [22] $\frac{d}{d\mathcal{J}} \ln Z = \frac{d}{d\mathcal{J}} \langle E \rangle - \langle J \rangle$. Since $\kappa$ depends on $\mathcal{J}^2$ and $\langle E \rangle$ and $\langle J \rangle$ are assumed to be continuous functions at $\mathcal{J} = 0$, it then follows that $\frac{d}{d\mathcal{J}} \ln Z$ must also be continuous, in particular, $\ln Z$ cannot contain terms proportional to $|\mathcal{J}|$. Higher odd powers of $|\mathcal{J}|$ are not forbidden by this argument, but they would not contribute at leading order ($|\mathcal{J}|^k = |\mathcal{J}|^{k\sqrt{\lambda}}$).

The assumption of continuity of observables as $\mathcal{J} \to 0$ is potentially subtle, and it is possible that the limit $\mathcal{J} \to 0$ may or may not be analytic, depending on a solution in question. Since, however, $\mathcal{J}$ is turned on in the presence of other classical charges, one may expect this non-analyticity to be weaker than, e.g., in the example discussed in Appendix A in [1]. There we found different results when taking $\mathcal{J} \to 0$ before or after doing summation over infinite number of quantum modes. Unlike the setup discussed here, in that case the classical solution was becoming trivial in the limit $\mathcal{J} \to 0$.

3 Examples of semiclassical string states dual to
$\Delta_0 = 4$ states in Konishi multiplet

Similar observations regarding the generalization to nonvanishing orbital momentum $J$ in $S^5$ go through in the case of more complicated folded spinning string solutions.

3.1 Folded string with spin $S$ and orbital momentum $J$

Let us start with the case of folded spinning string with $J = 0$. The “short”-string (i.e. $S = \frac{S}{\sqrt{\lambda}} \ll 1$) expansion of the 1-loop corrected energy of this solution was considered in [2] (using near-flat space expansion) and in [14] (by directly diagonalizing the corresponding
fluctuation problem for any $S$). The resulting expression for the energy was found to be \[ E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}S} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{3}{8}S + a_{01}^{(0)} \right) + ... \right] + b_0^{(0)} , \] \[ a_{01}^{(0)} = 3 - 4\ln 2, \quad b_0^{(0)} = 1 . \] (3.1)

The generalization of (3.1) to the case of the folded spinning string with non-zero angular momentum $J \ll \sqrt{\lambda}$ in $S^5$ [5] is given by \[ E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}S} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{3}{8}S + \frac{J^2}{4S} + a_{01}^{(0)} \right) + ... \right] + b_0 . \] (3.2)

A direct calculation of the 1-loop coefficients $a_{01}, b_0$ starting with the string fluctuation Lagrangian appears to be non-trivial due to mixing of the fluctuation modes [5, 16].

The computation of $a_{01}, b_0$ in (3.3) was first performed in [24]. It started with the folded spinning string solution with $J = \frac{S}{\sqrt{\lambda}} \neq 0$ [5, 23] and used the algebraic curve approach [25] to extract the fluctuation spectrum. The resulting 1-loop correction to the energy was then expanded first in $J \ll 1$ and then in $S \ll 1$ with the result being (3.1) with coefficients different from (3.2) \[ a_{01} = -\frac{1}{4} , \quad b_0 = 0 . \] (3.4)

The difference between (3.2) and (3.4) implies that in this case the limit $J \to 0$ is non-analytic.\(^{13}\)

Starting with (3.3),(3.4) we then get a state on the first excited string level if we set $S = 2$. Assuming also $J = J_1 = 2$ we can then associate to this state a state $[0,2,0]_{(1,1)}$ in the Konishi multiplet with $\Delta_0 = 4$ which is the familiar Konishi descendant in the $sl(2)$ sector. The value of $b_0 = 0$ is consistent with $\Delta_0 = 4$. Then the value of $b_1$ in (1.1) is found to be the same as in the examples in the previous section and in agreement with [12, 13]: \[ b_1 = 2 \left( \frac{3}{8}S + \frac{J^2}{4S} - \frac{1}{4} \right)_{S=J=2} = 2 . \] (3.5)

### 3.2 Folded string with spin $J_1$ and orbital momentum $J_2$

One may repeat the above discussion for the folded 2-spin string solution in $S^5$ [27]. For the string with c.o.m. at rest ($J_2 = 0$) one finds [15] (cf. (3.1),(3.2)) \[ E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}J_1} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{1}{8}J_1 + a_{01}^{(0)} \right) + ... \right] + b_0^{(0)} , \] \[ a_{01}^{(0)} = 2 - 4\ln 2 , \quad b_0^{(0)} = 2 . \] (3.6)

The results of [24] where numeric, implying that $a_{01} = -0.25...$. In [1] it was assumed that $a_{01}$ is actually equal to $-\frac{1}{4}$, as that led to a consistency between values of 1-loop corrections for other string states that may appear on the first excited level. Very recently the expressions in (3.4) were derived analytically [26] using the same algebraic curve method as in [24]. We thank N. Gromov for informing us about this result prior to its publication.

\(^{13}\)Possible reasons for this non-analyticity are the non-trivial mixing of fluctuation modes which was absent in the $J = 0$ case and the difference in the number of massless fluctuation modes between the $J = 0$ and $J \neq 0$ cases.
Switching on non-zero $J_2$ we get (cf. (3.3))

\[ E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}J_1} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{1}{8} J_1^{\frac{3}{2}} J_1 + \frac{J_2^2}{4J_1} + a_{01} \right) + \ldots \right] + b_0 \ . \] (3.8)

Since the two folded string solutions \((E, S; J)\) and \((E; J_1, J_2)\) are closely related when all spins are non-zero (cf. [23]) the corresponding results for the 1-loop corrections in the “short”-string (near-flat space) limit should be similar. We may thus conjecture, following [1], that $a_{01}$ here should be the same as (3.4) up to the opposite sign (reflecting the opposite sign of the curvature of $S^3$ as compared to $AdS_3$)\(^{14}\)

\[ a_{01} = \frac{1}{4} , \quad b_0 = 0 \ . \] (3.9)

Here the limit $J_2 \to 0$ in the 1-loop correction to energy should again be non-analytic.

Considering the case with $J_1 = J_2 = 2$ we then get a state on the first excited string level with the corresponding representation $[2, 0, 2]_{(0,0)}$ present in the Konishi multiplet at level $\Delta_0 = 4$. The dual operator should be the $su(2)$ sector descendant of the Konishi operator. The corresponding value of $b_1$ in (1.1) that follows from (3.8) is once again equal to 2:

\[ b_1 = 2 \left( \frac{1}{8} J_1 + \frac{J_2^2}{4J_1} + \frac{1}{4} \right)_{J_1=J_2=2} = 2 \ . \] (3.10)

### 3.3 Comments on other states

As in [1] one may also consider a string folded and spinning (with spins $S$ and $J_1$) in both $AdS_5$ and $S^5$. It was conjectured in [1] that in this case the leading 1-loop contributions to $a_{01}$ cancel out (see (3.4) and (3.9); cf. also (2.20)) so that the leading correction to the energy is given just by the classical term. Adding extra orbital momentum in $S^5$, i.e. generalizing the $S^5$ part to $(J_1, J_2)$ folded spinning string we then get as in [1]

\[ E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}(S + J_1)} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{3}{8} S + \frac{1}{8} J_1 + \frac{J_2^2}{4(S + J_1)} \right) + \ldots \right] + b_0 \] (3.11)

for $S + J_1 = 2$ we then get a state on the first excited level; with $J_2 = 2$ the corresponding representation is (interchanging $J_1$ and $J_2$) $[1, 1, 1]_{(\frac{3}{2}, \frac{3}{2})}$, i.e. the same as in the case of the circular $(S, J_1; J_2)$ solution in sec. 2.3. There is no contradiction as there are several such $[1, 1, 1]_{(\frac{3}{2}, \frac{3}{2})}$ states in the Konishi multiplet table (e.g. four at level $\Delta_0 = 6$). Once again,

\[ b_1 = 2 \left[ \frac{3}{8} S + \frac{1}{8} J_1 + \frac{J_2^2}{4(S + J_1)} \right]_{S=J_1=1, J_2=2} = 2 \ . \] (3.12)

The transcendental values of the 1-loop coefficients in (3.2),(3.7) for the single-spin folded string solutions imply that corresponding states should not belong to the Konishi multiplet. One may wonder if such semiclassical states cannot actually be interpolated to true quantum string states.

\(^{14}\)Note a similar opposite sign of 1-loop corrections in (2.5) and in (2.12).
Another open question is the identification of the string states associated to singlet (spinless) states in the Konishi multiplet. A natural guess could be that they are small pulsating strings in $AdS_5$ or $R \times S^2$ [28, 15]. However, the corresponding 1-loop coefficient $a_{01}$ contains again $\ln 2$ terms [15]. By analogy with the folded string case one may conjecture that first adding a non-zero orbital momentum $J$ and then expanding in $J \ll \sqrt{\lambda}$ will make $a_{01}$ rational. For example, for a string pulsating in $AdS_5$ with the oscillation number $N \ll \sqrt{\lambda}$ and orbital momentum $J \ll \sqrt{\lambda}$ in $S^5$ one finds as in [15] and above (cf. (1.3))

$$E = E_0 + E_1 = \sqrt{2\sqrt{\lambda}N} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{5}{8}N + \frac{J^2}{4N} + a_{01} \right) + \ldots \right] + b_0.$$  \hspace{1cm} (3.13)

The choice of $N = 2$ and $J = 2$ then gives a state on the first excited string level. A candidate dual state in the Konishi multiplet is $[0, 2, 0]_{(0,0)}$ at level $\Delta_0 = 4$. To reproduce the same universal value $b_1 = 2$ as above we would then need $a_{01} = -\frac{3}{4}$ (instead of $a_{01}^{(0)} = \frac{5}{2} - 4 \ln 2$ found in [15] by starting directly with the solution having $J = 0$).

The value of the string center-of-mass orbital $S^5$ momentum $J$ does not influence the value of the string excitation level which is determined by spins and oscillation numbers related to extended nature of the string (cf. (1.3)). The states with $N = 2$ in (1.3) but $J > 2$ will still be on the first excited level. In general, they will not belong to the Konishi multiplet but rather to its Kaluza-Klein descendants (see [8]).

A non-vanishing value of the orbital angular momentum $J$ is crucial for relating some of the classical solutions we discussed to members of the Konishi multiplet. For example, for $J = 0$, the circular string solution with one spin in $AdS_5$ and one in $S^5$ discussed in section 2.3 should have $S = 1 = J_1$ (to represent a state on the first excited string level). It should also correspond to a gauge-theory operator in the $sl(2)$ sector, but the minimal R-charge in the $sl(2)$ sector is $J_1 = 2$, so such $S = 1 = J_1$ state should be excluded.

Solutions with vanishing orbital angular momentum $J$ have a natural place on higher string levels, with level determined by values of the $AdS_5$ and $S^5$ spins. Choosing the spins so that the solutions correspond to the second excited string level ($N = 4$ in (1.3)) we find from the eqs. (2.5), (2.12), (2.20) the following values of $b_1$ in the analog of (1.1) ($E = \sqrt{2N \sqrt{\lambda} + b_0} + \frac{b_1}{\sqrt{\lambda}} + \ldots$): $b_1(J_1 = J_2 = 2) = \sqrt{2}$; $b_1(S_1 = S_2 = 2) = 3\sqrt{2}$; $b_1(S = J_1 = 2) = 2\sqrt{2}$. From the eqs. (3.3), (3.8), (3.11) we get:\footnote{The Konishi multiplet also contains a state in the representation $[0, 0, 0]_{(2,2)}$, which has $S = 4$. It should correspond to a string state on the first excited level with 2 units of spin carried by the two string oscillators and the other two units of spin carried by the graviton vacuum state.} $b_1(S = 4) = \frac{5}{2} \sqrt{2}$; $b_1(J = 4) = \frac{3}{2} \sqrt{2}$; $b_1(S = J_1 = 2) = 2\sqrt{2}$. These different values of the coefficient $b_1$ for states on the same string level are not in contradiction with supersymmetry. Indeed, the corresponding level in the flat space string spectrum contains 81 physical long multiplets, each of them generating a KK tower in $AdS_5 \times S^5$ [8]. While the leading order terms in the energy of all such states are equal to $2\sqrt{2\sqrt{\lambda}}$, the order $\frac{1}{\sqrt{\lambda}}$ corrections can be different.

Finally, it goes without saying that the construction of a systematic quantization of $AdS_5 \times S^5$ superstring in the perturbative large tension expansion remains an important open problem. Some related recent work this direction appeared in [29, 30]. The consistent results found in the semiclassical approach should provide an important guidance.
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We would like to thank N. Gromov for informing us about a forthcoming paper [26] in which eq. (3.4) is derived analytically and the conclusion that the semiclassical approach applied to the case of the folded spinning string with $S = J = 2$ leads to the same value $b_1 = 2$ as in [12] is reached independently. We also thank B. Vallilo for sending us a draft of [31] which also claims, using a different method, that $b_1 = 2$ for a different state (with zero $S^5$ momentum) on the first excited string level.

Appendix: 1-loop correction to energy of 3-spin solution of sec. 2.1

Here we present some details of the calculation of the 1-loop correction to the energy of the 3-spin solution [6, 20] in the small-string regime discussed in section 2.1. In this case $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}' \ll 1$. The equations of motion and the conformal gauge constraints may be easily solved and the resulting expansions in small $\mathcal{J}$ for fixed $\mathcal{J}'$ are

$$\nu = \frac{\mathcal{J}}{1 - 2 \mathcal{J}'} \left[ 1 - \frac{\mathcal{J}'}{(1 - 2 \mathcal{J}')^3} \mathcal{J}^2 + \frac{3 \mathcal{J}'(1 + 2 \mathcal{J}')}{4(1 - 2 \mathcal{J}')^6} \mathcal{J}^4 + \mathcal{O}(\mathcal{J}^6) \right], \quad (A.1)$$

$$\mathcal{E}_0 = \kappa = 2 \sqrt{\mathcal{J}'} \left[ 1 + \frac{\mathcal{J}^2}{8(1 - 2 \mathcal{J}') \mathcal{J}'} - \frac{(12 \mathcal{J}'^2 - 4 \mathcal{J}' + 1) \mathcal{J}^4}{128(1 - 2 \mathcal{J}')^4 \mathcal{J}'^2} + \mathcal{O}(\mathcal{J}^6) \right], \quad (A.2)$$

leading to the expansion of the classical energy in (2.4).

The 1-loop calculation may proceed either by evaluating the frequencies of the physical fluctuations around the solution and constructing $E_1 = \frac{1}{2 \kappa} \sum_{n = -\infty}^{+\infty} (-)^{F_n} \omega_n$, or by directly evaluating the determinant of the quadratic fluctuation operator. We will use the fluctuation frequencies found in [20] and expand them in small $\mathcal{J}$. Then $E_1$ has the following expansion

$$E_1 = \frac{1}{2 \kappa} \sum_{n = -\infty}^{+\infty} (-)^{F_n} \omega_n = \frac{1}{\kappa} \left[ f_0(\mathcal{J}') + f_1(\mathcal{J}') \mathcal{J} + f_2(\mathcal{J}') \mathcal{J}^2 + \mathcal{O}(\mathcal{J}^3) \right]. \quad (A.3)$$

The analyticity properties of the functions $f_i(\mathcal{J}')$ determine the order in the $\frac{1}{\sqrt{\lambda}}$ expansion (for fixed $J, J'$) to which they contribute. In general, these functions appear to be analytic, or at most have simple poles in $\sqrt{\mathcal{J}'}$. In particular, the third term, proportional to $\mathcal{J}^2$, is of too high an order in the large $\sqrt{\lambda}$ expansion to be relevant for the leading $\frac{1}{\sqrt{\lambda}}$ correction that we are concerned with here. The argument outlined in section 2.4 suggests that $f_1(\mathcal{J}') = 0$. We will see that this is indeed the case.

\textsuperscript{16}Let us mention that starting with $\frac{1}{(\sqrt{\lambda})^2}$ order, the energy depends not just on $\mathcal{J}'^2$, but independently on $J$ and $\mathcal{J}'$ (e.g. one finds terms like $\mathcal{J}_1^2 \mathcal{J}_2^2$).
As found in [20], apart from two massless modes and four bosonic modes with \( \omega_n = \sqrt{n^2 + \kappa^2} \), the characteristic frequencies are the roots of the “bosonic” and “fermionic” polynomials (\( \Omega \equiv \omega^2 \) and \( q \equiv 2J'/\sqrt{1+\nu^2} \))

\[
B_8(\Omega) = \Omega^4 + \Omega^3 \left( -8 - 4n^2 + 20q - 8\kappa^2 \right) \\
+ \Omega^2 \left( 16 + 8n^2 + 6n^4 - 80q - 36n^2q + 96q^2 + 32\kappa^2 + 16n^2\kappa^2 - 80q\kappa^2 + 16\kappa^4 \right) \\
+ \Omega \left( -32n^2 + 8n^4 - 4n^6 + 96n^2q + 12n^4q - 96n^2q^2 - 32n^2\kappa^2 - 8n^4\kappa^2 + 48n^2\kappa^2 \right) \\
+ 16n^4 - 8n^6 + n^8 - 16n^4q + 4n^6q \\
F_8(\Omega) = 2\Omega^4 + \Omega^3 \left( -8 - 12\kappa^2 - 8n^2 + 20q \right) \\
+ \Omega^2 \left( 12 + 28\kappa^2 + 18\kappa^4 + 8n^2 + 28\kappa^2n^2 + 12n^4 - 52q - 64\kappa^2q - 36n^2q + 59q^2 \right) \\
+ \Omega \left( -8 - 20\kappa^2 - 20\kappa^4 - 8\kappa^6 + 8n^2 + 8n^2\kappa^2 - 20\kappa^4n^2 + 8n^4 - 20\kappa^4n^4 \right) \\
- 8n^6 + 44q + 80\kappa^2q + 44\kappa^4q - 24n^2q + 32\kappa^2n^2q + 12n^4q - 78q^2 \\
- 79\kappa^2q^2 + 2n^2q^2 + 45q^3 \right) \\
+ 2 + 4\kappa^2 + 2\kappa^4 - 8n^2 - 4\kappa^2n^2 - 4\kappa^4n^2 + 12n^4 - 4\kappa^2n^4 + 2\kappa^4n^4 \\
- 8n^6 + 4\kappa^2n^6 + 2n^8 - 12q - 16\kappa^2q - 4\kappa^4q + 28qn^2 + 16\kappa^2n^2q \\
+ 4\kappa^4n^2q - 20n^4q + 4n^6q + 27q^2 + 21\kappa^2q^2 + 2\kappa^4q^2 - 30n^2q^2 \\
- 11\kappa^2n^2q^2 + 11n^4q^2 - 27q^3 - 9\kappa^2q^3 + 9n^2q^3 + \frac{81}{8}q^4 \right). \quad (A.4)
\]

A closed-form expression for the characteristic frequencies does not seem to exist but one may readily find their expansion in small \( J \). It turns out that it is more convenient to use \( \nu \propto J \) (cf. eq. (A.1)) as an expansion parameter. We will also limit ourselves to terms of order \( \nu \), as higher order terms scale at least as \( (J/\nu)^J \) for fixed \( J \).

The expansion of the squares of the four nontrivial bosonic frequencies is

\[
\Omega_1^b = n^2 - \frac{2\sqrt{J'n^2}}{(n^2 + 2J' - n^2J' - 1)^{1/2}}\nu + \mathcal{O}(\nu^2), \\
\Omega_2^b = n^2 + \frac{2\sqrt{J'n^2}}{(n^2 + 2J' - n^2J' - 1)^{1/2}}\nu + \mathcal{O}(\nu^2), \\
\Omega_3^b = 4 + n^2 - 4J' - 4\sqrt{J'^2 - n^2J' + n^2} + 0 \times \nu + \mathcal{O}(\nu^2), \\
\Omega_4^b = 4 + n^2 - 4J' + 4\sqrt{J'^2 - n^2J' + n^2} + 0 \times \nu + \mathcal{O}(\nu^2)
\]

and the expansion of the squares of the four nontrivial fermionic frequencies is

\[
\Omega_1^f = 1 + J' + n^2 - 2\sqrt{n^2(1 - J')} + \frac{\sqrt{J'} \left( n^2 - \sqrt{n^2(1 - J') + J' + 1} \right)}{\sqrt{n^2(1 - J') + J'}} \nu + \mathcal{O}(\nu^2) \\
\Omega_2^f = 1 + J' + n^2 + 2\sqrt{n^2(1 - J')} + \frac{\sqrt{J'} \left( n^2 + \sqrt{n^2(1 - J') + J' + 1} \right)}{\sqrt{n^2(1 - J') + J'}} \nu + \mathcal{O}(\nu^2) \\
\Omega_3^f = 1 + J' + n^2 - 2\sqrt{n^2(1 - J')} + \frac{\sqrt{J'} \left( n^2 - \sqrt{n^2(1 - J') + J' + 1} \right)}{\sqrt{n^2(1 - J') + J'}} \nu + \mathcal{O}(\nu^2) \\
\Omega_4^f = 1 + J' + n^2 - 2\sqrt{n^2(1 - J')} + \frac{\sqrt{J'} \left( n^2 + \sqrt{n^2(1 - J') + J' + 1} \right)}{\sqrt{n^2(1 - J') + J'}} \nu + \mathcal{O}(\nu^2)
\]
\[ \Omega'_4 = 1 + J' + n^2 + 2\sqrt{n^2(1-J')} + J' + \frac{\sqrt{J'} \left( n^2 + \sqrt{n^2(1-J') + J' + 1} \right)}{\sqrt{n^2(1-J') + J'}} \nu + O(\nu^2) \]

As discussed in [1], when carrying out the frequency sum it is important to isolate the modes that have a non-analytic dependence in \( J' \). This is the case for the \( n = 0, \pm 1 \) modes, for which some of the \( \Omega = \omega^2 \) or some of the frequencies \( \omega \) acquire \( \sqrt{J'} \)-dependent terms. We will also isolate the \( n = \pm 2 \) mode. Carrying out the sum and expanding at small \( J' \) leads to the following results for the functions in (A.3)

\[
\begin{align*}
  f_0(J') &= \left( \frac{7}{3} - \frac{1}{3} \right) J' + \left( -\frac{3445}{432} + \frac{1}{432} \left[ 3121 - 2592 \zeta(3) \right] \right) J^2 + O(J'^3) \\
  f_1(J') &= 2J' - \frac{3}{4} \left[ 1 + 8\zeta(3) \right] J^2 + O(J'^3) \\
  f_2(J') &= 0, \quad f_2(J') = O(J') .
\end{align*}
\] (A.5)

In each parenthesis on the first line of (A.5) the first number is the contribution of the \( n = 0, \pm 1, \pm 2 \) modes and the second number is the contribution of the other modes. The vanishing of the function \( f_1(J') \) implies the absence of terms linear in \( J \), in agreement with the general argument in sec. 2.4.

Extracting the leading correction to the energy of the small 3-spin circular string in the limit \( J \ll 1 \) we find that it is independent of \( J \)

\[ E_1 = \sqrt{\lambda J'} \left[ \frac{1}{\sqrt{\lambda}} + O\left( \frac{1}{(\sqrt{\lambda})^2} \right) \right] . \] (A.7)

Combining this with the tree-level energy in (A.2) we find that the total energy is given by (2.5).

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Table 1: Long Konishi multiplet