A variety of vortex state solutions of Ginzburg-Landau equation on superconducting mesoscopic plates

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Abstract We report numerical solutions of the Ginzburg-Landau equation for superconducting mesoscopic square plate under a gradient magnetic field that changes linearly in strength in one direction. All the obtained vortex configurations have mirror symmetry. We found two different solutions containing the same number of the total flux quantum under the same field. Our calculation results imply that the obtained metastable states can be actually observed at almost the same frequency as the most stable state.

1. Introduction

In mesoscopic superconductors, effects on vortex states caused by shapes of the samples cannot be negligible. It is observed experimentally that flux quantum configurations have the same symmetry as the symmetry of the shape of the sample[1][2]. Especially, in the case of square mesoscopic sample in a homogeneous magnetic field, it has been studied experimentally in detail [1]. Near and below the transition temperature, theoretical calculations show that anti-vortex and giant-vortex can appear in order to maintain the symmetric vortex configuration[3][4]. This is because, near and below the transition temperature, the Ginzburg-Landau (GL) equation can be linearized, since the order parameter $\psi(r)$ becomes very small and higher order terms of $\psi(r)$ can be neglected. In general quantum mechanics, symmetry of the Hamiltonian determines symmetry of eingen functions of the Schrödinger equation. The linearized GL (LGL) equation has a same form as the Schrödinger equation. Therefore, the LGL equation has vortex states solutions of having symmetry reflected by the symmetry of the sample shape. At low temperatures, the vortex configuration does not have to match the symmetry of the system because higher order terms of $\psi(r)$ in GL equation cannot be negligible. Here we discuss the vortex state of a mesoscopic square superconducting plate under a gradient magnetic field that changes linearly in strength in the direction of a side of the square. The gradient magnetic field changes the symmetry of the system from the four-fold symmetry into the mirror symmetry. If the square sample is in a homogeneous field, breaking of the four-fold symmetry of the vortex configuration occurs with lowering the temperature, but the sample is in the gradient field, we consider that mirror symmetry of the vortex configuration maintains in a wide temperature range. And we expect to find peculiar vortex state that cannot be found in the homogeneous field case.
2. Formalism

Here we consider the superconducting 2D square plate is in an external field \( \mathbf{H}(r) \). The whole area that we will simulate is denoted by \( \Omega \). The modified Ginzburg-Landau free energy functional[5] \( F(\psi, A) \) of the superconducting order parameter \( \psi \) and vector potential \( A \) with constraints \( \nabla \cdot A = 0 \)
is,

\[
F(\psi, A) = \int_{\Omega} \left[ \frac{1}{2} \sqrt{\beta} |\psi(r)|^2 + \frac{\alpha(T)}{\sqrt{\beta}} \right] \left[ 1 + \frac{1}{2m_\phi} \left( -i \nabla - e \frac{A(r)}{\epsilon} \right) \psi(r) \right]^2 + \frac{|h(r) - H(r)|}{8\pi} + \left( \nabla \cdot A(r) \right)^2 \right] d\Omega \, , \tag{1}
\]

where \( \alpha(T) \) is the temperature dependent coefficient as

\[
\alpha(T) = a(T - T_c) \quad (a > 0) , \tag{2}
\]

\( \beta \) is a positive constant, \( e \) and \( m_\phi \) charge and mass of the Cooper pair respectively, and the microscopic magnetic field is denoted by \( h(r) \). Rewriting (1) by

\[
\bar{F}(\bar{\psi}, \bar{A}) = \frac{F(\psi, A)}{\alpha(T)\xi(T)^2 / \beta} \, , \tag{3}
\]

\[
\bar{\psi}(r) = \frac{\psi(r)}{\sqrt{\alpha(T) / \beta}} \quad \text{and} \quad \bar{A}(r) = \frac{2\pi A(r)}{\Phi_0} , \tag{4}
\]

we obtain,

\[
\bar{F}(\bar{\psi}, \bar{A}) = \int_{\Omega} \left[ \frac{1}{\xi(T)^2} \left( \frac{1}{2} |\psi(r)|^2 - 1 \right) + \left( -i \nabla - \bar{A}(r) \right) |\psi(r)|^2 d\Omega + \kappa \xi(T)^2 \int_{\Omega} \nabla \times \bar{A}(r) - \frac{2\pi H(r)}{\Phi_0} \right] \left| \nabla \cdot \bar{A}(r) \right| d\Omega \, , \tag{5}
\]

where \( \kappa \) is the GL parameter and \( \xi(T) \) is the temperature dependence GL coherence length which obeys \( \xi(T) = \xi_0 / \sqrt{1 - T / T_c} \). Here the transition temperature at zero field is denoted by \( T_c \).

Taking a variation of \( \delta \bar{F} = 0 \), we obtain a couple of equations,

\[
\int_{\Omega} \left( \left( i \nabla \psi(r) - \bar{A}(r) \bar{\psi}(r) \right) \left[ -i \nabla \bar{\psi}^*(r) - \bar{A}(r) \psi(r) \right] + \left( i \nabla \bar{\psi}(r) - \bar{A}(r) \bar{\psi}(r) \right) \left[ -i \nabla \psi^*(r) - \bar{A}(r) \psi(r) \right] \right) d\Omega \quad (6)
\]

\[
+ \int_{\Omega} \left( \frac{1}{\xi(T)^2} \right) \left( |\psi(r)|^2 - 1 \right) \left( \psi(r) \delta \psi^*(r) + \psi^*(r) \delta \psi(r) \right) d\Omega = 0
\]

\[
\int_{\Omega} \kappa \xi(T)^2 \left[ \nabla \cdot \bar{A}(r) \right] \left( \nabla \cdot \bar{\psi}(r) \right) + \left( \nabla \times \bar{A}(r) \right) \left( \nabla \times \bar{\psi}(r) \right) d\Omega 
\]

\[
\quad + |\bar{\psi}(r)|^2 \bar{A} \cdot \delta \bar{A} - \frac{i}{2} \left\{ \bar{\psi}^*(r) \nabla \bar{\psi}(r) - \bar{\psi}(r) \nabla \bar{\psi}^*(r) \right\} \delta \bar{A}(r) \right) d\Omega \quad (7)
\]

To solve these equations numerically, we divide the area \( \Omega \) into small triangular finite elements as Fig. 1. Using the Galerkin’s method of the first order, a type of the finite element method (FEM), the functions \( \bar{\psi}(r) \) and \( \bar{A}(r) \) in the \( j \)-th element are expressed by a linear combination of their values at the vertices \( i \) of the \( j \)-th element \( \bar{\psi}_i^{(j)}, \bar{A}_i^{(j)} \) as follows,

\[
\bar{\psi}(x, y) = \sum_{i=1}^3 N_i^{(j)}(x, y) \bar{\psi}_i^{(j)}, \quad \bar{A}(x, y) = \sum_{i=1}^3 N_i^{(j)}(x, y) \bar{A}_i^{(j)} \, . \tag{8}
\]
Here we supposed $\Omega$ is in $x$-$y$ plane. The area coordinates $N_j(x, y)^{(i)} (j = 1, 2, 3)$ are equal to the area of three small triangles, which are made by drawing three line segments from the point $(x, y)$ to the three vertices of the $j$-th finite element, and the definition of the area coordinate is shown in figure 2. By iterative solving the set of equations (6) and (7) self-consistently, we have to set the initial values of $\vec{A}(r)$ and $\vec{\psi}(r)$. The initial value of $\vec{A}(r)$ was determined by the external magnetic field $H(r)$, and the initial value of $\vec{\psi}(r)$ was given by using a random number. Since the iterative calculation starts from various initial values, the same converged solution cannot always be obtained each time, however they are solutions of the most stable state or at least metastable states. Therefore, we calculate over 20 times in each set of physical parameters.

3. Results and discussions

We consider a mesoscopic square plate with a side length of $L = 29\xi_0$ placed on the $x$-$y$ plane. The region of the whole area $\Omega$ is $30\xi_0 \leq x \leq 30\xi_0$, $30\xi_0 \leq y \leq 30\xi_0$. The superconducting plate occupies the region of $0.5\xi_0 \leq x \leq 29.5\xi_0$, $0.5\xi_0 \leq y \leq 29.5\xi_0$, and the rest is the vacuum area. A magnetic field is applied along the $z$-axis. In following discussion, field strength is denoted by renormalized field strength that is defined by $f = HL^2/\Phi_0$. The field strength has a linear gradient toward the $x$-axis direction as

$$f = f_{\text{min}} + (f_{\text{max}} - f_{\text{min}})x / 30\xi_0.$$  \hspace{1cm} (9)

Figure 3 shows vortex distributions of the plate in various field strengths. Here, the GL parameter of the plate is $\kappa = 10$ and the temperature is $T = 0.80T_c$. We set the gradient field parameter as $f_{\text{min}} = 0$ and $f_{\text{max}} = 8.03, 10.07, 13.38, 16.6$. There is more than one solution for each $f_{\text{max}}$. All the obtained vortex distributions have mirror symmetry respect to the straight line of $y = 15\xi_0$. In the
most stable state, as the $f_{\text{max}}$ increases, the number of flux quanta passing through the plate also monotonically increases. It is noteworthy here that there are more than one state that have equal numbers of total flux that passing through the plate in the same field ($f_{\text{max}} = 16.06$ case shown in figure 3), which cannot be seen in the case of uniform magnetic field.

In solving the equations (5) and (6), by changing the initial value of order parameter, resulting solution also may change, and it is not necessary to converge to the most stable solution. In some cases, a specific metastable solution may be obtained more frequently than the most stable solution. Therefore, it is incorrect to abandon the metastable solutions obtained here, and we consider that the metastable states can be realized as well as the most stable state. However, in order to obtain strong evidence of the "stability" of metastable state, numerical studies using the TDGL theory would also be necessary.

4. Conclusions
We obtained vortex states of the mesoscopic square superconducting plate under the gradient magnetic field by solving the GL equations make use of the FEM. When a gradient magnetic field is applied, a combined system of the square plate and the gradient magnetic field has the mirror symmetry. Then the obtained vortex configurations including metastable states of them have the mirror symmetry. We also found two different solutions containing the same number of the total flux quantum under the same gradient field, which cannot be seen under the homogeneous field. When solving the GL equation self-consistently, we obtained the metastable state solutions at almost the same frequency as the stable solution, which implies that the metastable states obtained here can be observed as the most stable state.

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