ADVECTION-DOMINATED ACCRETION FLOWS AROUND KERR BLACK HOLES

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ABSTRACT

We derive all relevant equations needed for constructing a global general relativistic model of advection-cooled, very hot, optically thin accretion disks around black holes and present solutions that describe advection-dominated flows in the gravitational field of a Kerr black hole.

Subject headings: accretion, accretion disks — black hole physics — hydrodynamics — relativity

1. ADVECTION-DOMINATED BLACK HOLE ACCRETION DISKS

Recently, solutions corresponding to a new type of hot, optically thin accretion disk cooled mostly by advection have been discovered (see Abramowicz & Lasota 1996, Abramowicz 1996, and Narayan 1996 for reviews). The luminosity, \( L \), of these disks is much lower than that of the standard disks with the same rate of mass supply \( M \). The radiative cooling efficiency, defined as \( \eta = L/Mc^2 \), is typically less than 0.01 in the newly discovered disks, while in the case of the standard disks it is typically \( \eta \gtrsim 0.1 \). Thus, the advection-dominated accretion flows (ADAFs) hide the evidence that they are consuming a lot of matter—for this reason, Abramowicz & Lasota (1996) called them “secret guzzlers.” A typical secret guzzler appears to be very hot and underluminous.

Lasota, Narayan, and collaborators have discussed properties of the secret guzzler models and have compared them with observed properties of sources that are believed to contain accreting black holes and appear both underluminous and very hot, like Sgr A* (Narayan, Yi, & Mahadevan 1995), soft X-ray transients (Narayan, McClintock, & Yi 1996b; Lasota et al. 1996a), and NGC 4258 (Lasota, Narayan, & Yi 1996b). These detailed discussions, based on calculations of electromagnetic spectra, convincingly demonstrated that secret guzzlers may explain these particular types of sources, with no need for ad hoc hypothesis about additional optically thin components in the model or some unusual properties of the viscosity.

Models of secret guzzlers are described by similar equations as the well-known slim accretion disks (Abramowicz et al. 1988). The only difference is in equations for the radiative processes: secret guzzlers are hot and optically thin, and the ion temperature is, most probably, much higher than that of the electrons (see, e.g., Narayan 1996b) and slim disks are cool and optically thick. All the previously calculated models of slim disks and secret guzzlers are not fully satisfactory at least in one respect: they assume an approximate model for the gravity of the black hole. It is either just the plain Newton’s potential of a point mass \( M \), \( \Phi = -GM/r \), or the Paczyński’s pseudopotential \( \Phi = -GM/(r - r_g) \), \( r_g = 2GM/c^2 \) (Paczyński & Wiita 1980, hereafter PW). No general relativistic solution has ever been obtained.

The PW paradigm is an excellent approximation for the calculations of the hydrodynamical properties of accretion disks around nonrotating black holes. It is both very simple and very accurate. However, it has two shortcomings: (1) it does not describe the relativistic effects in the light propagation, and (2) it does not include the effect of the inertial frame dragging, an important property of rotating black holes. For this reason, at the TAD2 meeting at Garching (see Duschl et al. 1994), one of us suggested that the time has come to construct solutions for slim disks that would be fully consistent with general relativity. The first step should be a derivation of the relevant equations. Ramesh Narayan argued that it should be the job of the proposer, so this was done in Lasota (1994, hereafter L94).

We repeat these calculations here, correcting some computational errors of L94 and using a slightly more practical notation. The general assumption and definition are presented in § 2. In § 3 we give the set of equations describing a general relativistic model of a hot accretion flow. The detailed microphysics that describes the radiative cooling is briefly in §§ 4 and 5. After a discussion of the transonic character of the flow in § 6, numerical solutions of advection-dominated flows in the gravitational field of a Kerr black hole are presented in § 7. Section 8 contains the conclusions.

2. ASSUMPTIONS AND DEFINITIONS

We use geometrical units here that are linked to the physical units for length, time and mass by

\[
\text{length in physical units} = \text{length in geometrical units} \times \frac{r}{\text{length in geometrical units}},
\]

\[
\text{time in physical units} = \frac{1}{c} \text{(time in geometrical units)},
\]

\[
\text{mass in physical units} = \frac{c^2}{G} \text{(mass in geometrical units)}.
\]

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We use the Boyer-Lindquist spherical coordinates \( t, r, \theta, \varphi \) to describe the Kerr black hole metric. The cylindrical vertical coordinate \( z = \cos \theta \) is defined very close to the equatorial plane, \( z = 0 \). The metric of the Kerr black hole on the equatorial plane, accurate up to the \((z/r)^0\) terms, takes the form given by Novikov & Thorne (1973, hereafter NT),

\[
ds^2 = -\frac{r^2 \Delta}{A} \, dt^2 + \frac{A}{r^2} \left( d\varphi - \omega \, dt \right)^2 + \frac{r^2}{\Delta} \, dr^2 + dz^2 ,
\]

\[
\Delta = r^2 - 2Mr + a^2 , \quad A = r^4 + r^2 a^2 + 2Mr a , \quad \omega = \frac{2Mar}{A} .
\]

Here \( M \) is the mass and \( a \) the total specific angular of the Kerr black hole. The metric (eq. [4]) was also used in L94. Note, that the determinant of the metric tensor corresponding to equation (4) is \( g = -r^2 \).

There are two Killing vectors in the geometry (eq. [4]), connected to the time symmetry (metric is independent of \( t \)) and axial symmetry (metric is independent of \( \varphi \))

\[
\eta^i = \delta^{i(t)} , \quad \xi^i = \delta^{i(\varphi)} ,
\]

where \( \delta^{i(j)} \) is the Kronecker delta. Using Killing vectors (eq. [6]) one defines some useful scalar functions: angular velocity of the dragging of inertial frames \( \omega \), gravitational potential \( \Phi \), and radius of gyration \( \bar{R} \),

\[
\omega = -\frac{(\eta \xi)}{(\xi \xi)} , \quad e^{-2\Phi} = (\eta \eta) - \omega^2 (\xi \xi) , \quad \bar{R}^2 = (\xi \xi) e^{2\Phi} .
\]

In the Boyer-Lindquist coordinates the scalar products of the Killing vectors (6) are given by the components of the metric,

\[
(\eta \eta) = g_{tt} , \quad (\eta \xi) = g_{t\varphi} , \quad (\xi \xi) = g_{\varphi\varphi} ,
\]

and therefore quantities defined by equation (7) can by explicitly written down in terms of the Boyer-Lindquist coordinates:

\[
\bar{R}^2 = \frac{A^2}{r^4 \Delta} , \quad e^{-2\Phi} = \frac{r^2 \Delta}{A} .
\]

The black hole surface (event horizon) is at

\[
r_+ = M + (M^2 - a^2)^{1/2} .
\]

The unit timelike vector

\[
n^i = e^{\Phi}(\eta^i + \omega \xi^i) ,
\]

is orthogonal to the spacelike surfaces \( t = \text{const} \). It corresponds to the four-velocity of the local inertial observer or ZAMO, i.e., zero angular momentum observers (Bardeen 1973).

The four-velocity of matter \( u^i \) has components \( u^t, u^\varphi, u^r \),

\[
u^i = u^t \delta^{i(t)} + u^\varphi \delta^{i(\varphi)} + u^r \delta^{i(r)} .
\]

One defines the angular velocity \( \Omega \) with respect to the stationary observer, and the angular velocity \( \bar{\Omega} \) with respect to the local inertial observer by

\[
\Omega = \frac{u^\varphi}{u^t} , \quad \bar{\Omega} = \Omega - \omega .
\]

The angular frequencies of the corotating (+) and counterrotating (−) Keplerian orbits are

\[
\Omega^\pm = \pm \frac{M^{1/2}}{r^{3/2} \pm a M^{1/2}} ,
\]

and the Keplerian specific angular momentum is given by

\[
\mathcal{L}^\pm = \pm \frac{M^{1/2}(r^2 \mp 2a M^{1/2} r^{1/2} + a^2)}{r^{3/2} (r^{3/2} - 3M r^{1/2} + 2a M^{1/2})^{1/2}} .
\]

The Keplerian angular momentum has a minimum at the marginally stable orbit

\[
r_m^s = M \left\{ 3 + Z_2 \mp \left[ (3 - Z_1) (3 + Z_1 + 2Z_2) \right]^{1/2} \right\} ,
\]

\[
Z_1 = 1 + (1 - a^2/M^2)^{1/3} [(1 + a/M)^{1/3} + (1 - a/M)^{1/3}] ,
\]

\[
Z_2 = (3a^2/M^2 + Z_1)^{1/2} .
\]

In the reference frame of the local inertial observer, the four-velocity takes the form

\[
u^i = \gamma (n^i + v^{\varphi i} \xi^{i(\varphi)} + v^{r i} \xi^{i(r)}) .
\]
The vectors $\mathbf{\epsilon}^{(\phi)}$ and $\mathbf{\epsilon}^{(r)}$ are the unit vectors in the coordinate directions $\phi$ and $r$. The Lorentz gamma factor $\gamma$ equals

$$\gamma = \frac{1}{\sqrt{1 - (v^{(\phi)})^2 - (v^{(r)})^2}}. \quad (18)$$

The relation between the Boyer-Lindquist and the physical velocity component in the azimuthal direction is

$$v^{(\phi)} = \tilde{R} \Omega. \quad (19)$$

It will be convenient to use the (rescaled) radial velocity component $V$ defined by the formula

$$V = \frac{v^{(r)}}{\sqrt{1 - V^2}} = \frac{v^{(r)}}{\sqrt{1 - \Omega^2 R^2}}. \quad (20)$$

The Lorentz gamma factor may be written as

$$\gamma^2 = \left( \frac{1}{1 - \Omega^2 R^2} \right) \left( \frac{1}{1 - V^2} \right), \quad (21)$$

and from this follows a simple expression for $V$ in terms of the velocity components measured in the frame of the local inertial observer,

$$V = \frac{v^{(r)}}{\sqrt{1 - (v^{(\phi)})^2}} = \frac{v^{(r)}}{\sqrt{1 - \Omega^2 R^2}}. \quad (22)$$

Thus, $V$ is the radial velocity of the fluid as measured by an observer at fixed $r$ who corotates with the fluid. In the notation used in L94 (cf. his eq. (17)),

$$v^{(r)} = [v^r]^1_{94}. \quad (23)$$

Although a different quantity could have been chosen as the definition of the "radial velocity," only $V$ has the three very convenient properties, all guaranteed by its definition: (1) everywhere in the flow $|V| \leq 1$, (2) on the horizon $|V| = 1$, and (3) at the sonic point $|V| \approx c_s$, with $c_s$ being the local sound speed. To see that the first property holds, let us define $\mathbf{v}^2 = u^i u_i = u^r g_{rr} \geq 0$. Then, one has $V^2 = (\mathbf{v}^2)/(1 + \mathbf{v}^2) \leq 1$. Writing $V^2 = 1/[1 + \Delta/(r^2 u^i u^j)]^{1/2}$ demonstrates property (2). Property (3) of $V$ will be proved in § 6. Other possible choices of the "radial velocity" are not that convenient. For example, the radial velocity $u = |u^r|$, chosen by Shapiro & Teukolsky (1983), has none of these properties and would be less convenient in the present context.

The stress-energy tensor $T^{ik}$ of the matter in the disk is given by

$$T^{ik} = (\epsilon + p) u^i u^k + p g^{ik} + S^{ik} + u^i q^k + u^k q^i, \quad (24)$$

where $\epsilon$ is the total energy density, $p$ is the pressure,

$$S_{ik} = \nu \rho \sigma_{ik}, \quad (25)$$

is the viscous stress tensor, $\rho$ is the rest mass density, and $q^i$ is the radiative energy flux. In the last equation $\nu$ is the kinematic viscosity coefficient, and $\sigma_{ik}$ is the shear tensor of the velocity field. From the first law of thermodynamics, it follows that

$$d \epsilon = \frac{\epsilon + p}{\rho} \, d \rho + \rho T \, d S, \quad (26)$$

where $T$ is the temperature and $S$ is the entropy per unit mass. Note that in the physical units that $\epsilon = \rho c^2 + \Pi$, where $\Pi$ is the internal energy. For nonrelativistic fluids, $\Pi \ll \rho c^2$ and $p \ll \rho c^2$, and therefore

$$\epsilon + p \approx c^2 \rho. \quad (27)$$

We shall use this approximation (in geometrical units $\epsilon + p \approx \rho$) in all our calculations. This approximation does not automatically ensure that the sound speed is below $c$, and one should check this a posteriori when models are constructed. We write the first law of thermodynamics in the form

$$d U = -\Pi \left( \frac{1}{\rho} \right)^2 + T \, d S, \quad (28)$$

where $U = \Pi/\rho$.

3. SLIM DISK EQUATIONS IN KERR GEOMETRY

It is convenient to write the final form of all the slim disk equations at the equatorial plane, $z = 0$. Only these equations that do not refer to the vertical structure could be derived directly from the quantities at the equatorial plane with no further
approximations. All other equations are approximated—either by expansion in terms of the relative disk thickness \( H/r \) or by vertical averaging. The thickness of the disk \( H \) is defined as in equation (44).

3.1. **Mass Conservation Equation**

From the general equation of mass conservation,

\[
V (\rho u) = 0 ,
\]

and the definition of the surface density \( \Sigma \),

\[
\Sigma = \int_{-H(r)}^{+H(r)} \rho(r, z) dz \approx 2H \rho ,
\]

we derive the mass conservation equation,

\[
\dot{M} = -2\pi A^{1/2} \frac{V}{\sqrt{1 - V^2}} .
\]

It is identical to that derived by NT. In L94 the numerical factor on the right-hand side is \(-4\pi\).

3.2. **Equation of Angular Momentum Conservation**

From the general form of the angular momentum conservation,

\[
\nabla_i (T^{ik} \xi_i) = 0 ,
\]

we derive, after some algebra,

\[
\frac{\dot{M}}{2\pi} \frac{d\mathcal{L}}{dr} + \frac{1}{r} \frac{d}{dr} \left( \Sigma v A^{3/2} \frac{\Lambda^{1/2} \gamma^3}{r^4} \frac{d\Omega}{dr} \right) - F^- \mathcal{L} = 0 ,
\]

where \( F^- = 2q_z \) is the vertical flux of radiation and

\[
\mathcal{L} \equiv -(u \xi) = -u_\rho = \frac{1}{r^3} A \frac{\Lambda^{1/2}}{3} \Omega
\]

is the specific (per unit mass) angular momentum. Equation (33) differs from that derived in L94. In addition to a trivial misprint in the L94 equation, there is also a more significant difference—we keep the term \( F^- \mathcal{L} \) that was rejected from the final form of the L94 equation. This term represents angular momentum losses through radiation. Although it was always fully recognized that angular momentum may be lost this way, it has been argued that this term must be very small. Rejection of this term would enormously simplify numerical calculations because it is obvious that with \( F^- \mathcal{L} = 0 \), equation (33) can be trivially integrated:

\[
\frac{\dot{M}}{2\pi} (\mathcal{L} - \mathcal{L}_0) = -\Sigma v A^{3/2} \frac{\Lambda^{1/2} \gamma^3}{r^4} \frac{d\Omega}{dr} ,
\]

where \( \mathcal{L}_0 \) is the specific angular momentum of matter at the horizon (\( \Lambda = 0 \)). In the numerical scheme for integrating the slim Kerr equations (with \( F^- \mathcal{L} \) assumed to be zero), the quantity \( \mathcal{L}_0 \) plays an important role: it is the eigenvalue of the solutions that passes regularly through the sonic point.

Several authors have pointed out that in some astrophysically relevant situations, the \( F^- \mathcal{L} \) cannot be put to zero, because it could be a significant part (up to 0.4) of the angular momentum balance (see Lamb 1996 for discussion and most recent references).

3.3. **Equation of Momentum Conservation**

From the \( r \)-component of the equation \( \nabla_i T^{ik} = 0 \) one derives

\[
\frac{V}{1 - \frac{V^2}{r^2}} \frac{dV}{dr} = \mathcal{A} + \frac{1}{r} \frac{dP}{dr} ,
\]

where \( P = 2Hp \) is the vertically integrated pressure and

\[
\mathcal{A} = -\frac{MA \left( \Omega - \Omega_k^+ \right) \left( \Omega - \Omega_k^- \right)}{r^3 \Omega_k \Omega_k^+ \Omega_k^- \left( 1 - \frac{\Lambda^2 R^2}{\Delta} \right)} .
\]

The L94 version of the momentum conservation equation contains few trivial misprints.

Using equation (20) one shows that equation (36) has the correct limit corresponding to the spherical accretion (\( \Omega = 0 \)) on the nonrotating (\( a = 0 \)) black hole: in this limit, it becomes identical with equation (G.7) in Shapiro & Teukolsky (1983, p. 569).
3.4. Equation of Energy Conservation

From the general form of the energy conservation
\[ \nabla_i (T^{ik} \eta_k) = 0 \, , \]
and the first law of thermodynamics,
\[ T = \frac{1}{\rho} \left( \frac{\partial e}{\partial s} \right)_p \, , \quad p = \rho \left( \frac{\partial e}{\partial \rho} \right)_s - \epsilon \, , \]
the energy equation can be written in general as
\[ F^{adv} = F^+ - F^- \, , \] (40)
where
\[ F^+ = \Sigma V A^2 r^5 \gamma^4 \left( \frac{d\Omega}{dr} \right)^2 \] (41)
is the surface heat generation rate, \( F^- \) is the radiative cooling flux (both surface) which is discussed in § 5, and \( F^{adv} \) is the advective cooling rate due to the radial motion of the gas. It is expressed as
\[ F^{adv} = \frac{\Sigma V}{\sqrt{1 - V^2}} \frac{\Delta s^{1/2}}{r} T \frac{dS}{dr} \implies - \frac{\dot{M}}{2\pi r} \frac{dS}{dr} \] (42)

3.5. Equation of Vertical Balance of Forces

One should calculate the condition for the hydrostatic vertical equilibrium in the frame that comoves with matter. NT have approximated this frame by the one that moves along circular Keplerian orbits at the equatorial plane. It is known that in the innermost part of the disk, the motion of matter differs considerably from the circular Keplerian motion. For this reason, one needs to repeat the calculations in the comoving frame of matter.\(^5\)

Like NT, we demand that the vertical pressure force should be equal to the vertical component of the tidal gravitational force,
\[ \frac{1}{\rho} \left( \frac{\partial p}{\partial z} \right)_H = HR^{ij}_{\text{ijkl}} t^{m}_{i} g_{m_{i}} t^{n}_{(z)} w_{i} u_{j} d \approx H \gamma^2 R^{ij}_{\text{ijkl}} t^{m}_{i} g_{m_{i}} t^{n}_{(z)} w_{i} u_{j} = H \gamma^2 R^{ij}_{\text{ijkl}} \] (43)

however, we demand that this happen in the comoving frame of the fluid (note the fluid’s four-velocity \( u^i \) multiplying the Riemann tensor \( R^{ijkl}_{\text{ijkl}} \)) while NT demand that equality of forces should occur in the frame rotating with the Keplerian velocity (NT have \( u^i \) \( w^l \) multiplying the Riemann tensor). In the formula, \( t^{(z)} \) is the unit vector in the \( z \)-direction. From this we derive the final formula,
\[ \frac{p}{\rho H^2} = \gamma^2 \frac{M}{r^3} \left[ \frac{(r^2 + a^2)^2 + 2 \Delta a_z^2}{(r^2 + a^2)^2 - \Delta a_z^2} \right] = \gamma^2 \gamma' \] (44)
that should be compared with NT formula (5.7.2). The difference is that in our formula the redshift factor \( \gamma \) is given by equation (21), which corresponds to the motion of matter and not to a fictitious fluid with strictly Keplerian rotation. The NT formula is singular at the location of the photon circular trajectory, \( r = r_{\text{ph}} \), and our formula at the horizon. L94 gives the same formula for the vertical equilibrium.

The singularity of equation (44) on the horizon is an artifact of approximations used in equation (43). We assumed that the flow is vertical hydrostatic equilibrium, in the ZAMO frame, down to the black hole surface. This requirement implies an infinite boost at the horizon. In reality, the flow near the horizon is supersonic and therefore practically in a free fall. Equation (43) should be completed by a term \( v' (\partial v^0 / \partial r) \) that, in the supersonic flow, will dominate over the pressure gradient term. As a result equation (43) will become the (nonsingular) geodesic deviation equation. However, because we are adopting here a one-dimensional approach in which “vertical” quantities are averaged, it is not possible to include \( v \) terms in the equations. Since the “singularity” appears only in the supersonic part of the transonic flow and since the reason for its appearance is clear, it has no influence on the physically important properties of the accretion flow (see also discussion in § 6).

3.6. Viscosity Prescription

The standard assumption for the viscosity coefficient is
\[ v = (2/3) \pi c_s H \, , \] (45)
where \( c_s = (\rho / \rho)^{1/2} \) is the isothermal sound speed.

\(^5\) This reason differs from that discussed recently by Riffert & Herold (1995) who have also redervied some of the NT formulae. They have argued that for some of these derivations, the Kerr metric must be accurate to second order in \( z/r \). Although they have kept the \( (z/r)^2 \) terms in geometry, they have rejected physical quantities of this order: radial pressure gradient, radial advective cooling, etc., which is not a self-consistent scheme.
4. THERMODYNAMICAL RELATIONS

The equation of state can be expressed in the form
\[ p = p_r + \frac{\mathcal{R}}{\mu_i} \rho T_i + \frac{\mathcal{R}}{\mu_e} \rho T_e + \frac{B^2}{24\pi} , \]  
(46)
where \( p_r \) is the radiation pressure, \( \mathcal{R} \) is the gas constant, \( \mu_i \) and \( \mu_e \) are the mean molecular weights of ions and electrons respectively, \( T_i \) and \( T_e \) are ion and electron temperatures, \( a \) is the radiation constant, and \( B \) is the intensity of an isotropically tangled magnetic field, including the radiation, gas, and magnetic pressures. The radiation pressure \( p_r \), the gas pressure \( p_g \), and the magnetic pressure \( p_m \) correspond, respectively, to the first term, the second and third terms, and the last term in equation (46).

The mean molecular weights of ions and electrons can be well approximated by
\[ \mu_i \approx \frac{4}{4X + Y} , \quad \mu_e \approx \frac{2}{1 + X} , \]  
(47)
where \( X \) is the relative mass abundance of hydrogen and \( Y \) is that of helium. We may define a temperature as
\[ T = \mu \left( \frac{T_i + T_e}{\mu_i + \mu_e} \right) , \]  
(48)
where
\[ \mu = \left( \frac{1}{\mu_i} + \frac{1}{\mu_e} \right)^{-1} \approx \frac{2}{1 + 3X + 1/2Y} \]  
(49)
is the mean molecular weight in the standard approximation (Cox & Giuli 1968). In the case of a one-temperature gas \( T_i = T_e \), one has \( T = T_i = T_e \). For an optically thick gas, \( p_r = \frac{1}{2} a T_i^4 \).

Until the end of this section we will assume that \( p_r = \frac{1}{2} a T_i^4 \). As noted by Narayan & Yi (1995b, hereafter NY), the frozen-in magnetic field pressure \( p_m \sim B^2 \sim \rho^{4/3} \); therefore, we may write the internal energy as
\[ U = \frac{a T_i^4}{\rho} + \frac{\mathcal{R} T}{\mu \gamma - 1} + e_0 \rho^{1/3} , \]  
(50)
where \( e_0 \) is a constant \( (p_m = 1/3 e_0 \rho^{4/3}) \) and \( \gamma \) is the ratio of the specific heats of the gas. We define
\[ \beta = \frac{p_g}{\rho} , \quad \beta_m = \frac{p_m}{p_g + p_m} , \quad \beta^* = \frac{4 - \beta_m}{3\beta_m} \beta . \]  
(51)
From equations (46) and (50) one obtains the following formulae (see, e.g., Cox & Giuli 1968) for the specific heat at constant volume:
\[ c_v = \frac{\mathcal{R}}{\mu \gamma - 1} \left( \frac{12(1 - \beta/\beta_m)(\gamma_0 - 1) + \beta^*}{\beta} \right) = \frac{4 - 3\beta^*}{\Gamma_3 - 1} \frac{p}{\rho T} , \]  
(52)
and the adiabatic indices:
\[ \Gamma_3 - 1 = \frac{(4 - 3\beta^*)(\gamma_0 - 1)}{12(1 - \beta/\beta_m)(\gamma_0 - 1) + \beta} , \]  
(53)
\[ \Gamma_1 = \beta^* + (4 - 3\beta^*)(\Gamma_3 - 1) . \]  
(54)
The ratio of specific heats is \( \gamma = c_v/c_r = \Gamma_1/\beta \). For \( \beta = \beta_m \), we have \( \Gamma_3 = \gamma_0 \) and \( \Gamma_1 = (4 - \beta)/3 + \beta(\gamma_0 - 1) \). For an equipartition magnetic field \( (\beta = 0.5) \), one gets \( \Gamma_1 = 1.5 \), and for \( \beta = 0.95, \Gamma_1 = 1.65 \) (here we have used \( \gamma_0 = 5/3 \)). Our formula for \( \Gamma_1 \) is different from that used by NY, but the numerical values differ by less than 5%. One expects \( \beta_m \sim 0.5 \). Since
\[ T \frac{dS}{dr} = c_r T \left[ \frac{d\ln T}{dr} - (\Gamma_3 - 1) \left( \frac{d\ln \Sigma}{dr} - \frac{d\ln H^*}{dr} \right) \right] , \]  
(55)
in the numerical scheme based on Chen & Taam (1993) (see also Abramowicz et al. 1995), the advective flux is written in the form
\[ F_{\text{adv}} = \frac{M}{2\pi r^2 \rho} \frac{p}{\rho} \xi_u , \]  
(56)
where
\[ \xi_u = \left[ \frac{4 - 3\beta^*}{\Gamma_3 - 1} \frac{d\ln T}{d\ln r} - (4 - 3\beta^*) \frac{d\ln \Sigma}{d\ln r} \right] . \]  
(57)
Here the term $\propto d \ln H/d \ln r$ is neglected. Since no rigorous vertical averaging procedure exists, the presence or absence of the $d \ln H/d \ln r$-type terms in this (and other) equations may be decided only by comparison with two-dimensional calculations (see, e.g., Narayan & Yi 1995a and also Chen et al. 1995 for a discussion of this point).

For $\beta_m = 1$, we recover the formulae used in Abramowicz et al. (1995). The formulae derived in this section are valid for the optically thin case $\tau = 0$ if one assumes $\beta = \beta_m$.

5. RADIATIVE COOLING

We have to complete the set of equations by specifying the physical processes that will be involved in the radiative cooling. In our opinion the most convenient and general description of cooling processes has been presented by NY. They consider a two-temperature plasma cooled by synchrotron radiation, inverse-Compton process, and bremsstrahlung emission. They neglect electron-positron pair creation and annihilation, but, as shown by Björnsson et al. (1996) and Kusunose & Mineshige (1996), this is justified in most cases of interest. Below, for the reader’s convenience we will recall the formulae used by NY.

5.1. Heating of Electrons by Ions

NY use the formula of Stepney & Guilbert (1983) that has been modified in order to account for an $X = 0.75$ and $Y = 0.25$ composition (the effective molecular weight of ions $\mu_i = 1.23$ and of electrons $\mu_e = 1.14$). Here we give the general formula in which the factor $n_j$ in the Stepney & Guilbert (1983) formula is replaced by

$$\bar{n} = \sum Z_j^2 n_j,$$

(58)

where $Z_j$ and $n_j$ are, respectively, the charge and the number density of $j$th species. The Coulomb collisions transfer energy from (hotter) ions to electrons at a volume transfer rate

$$f^{ie} = \frac{3}{2} n_e \bar{n} \sigma_T c \frac{k T_i - k T_e}{K_3(1/\theta_i) K_3(1/\theta_e)} \ln \Lambda \left[ \frac{2(\theta_e + \theta_i)^2 + 1}{\theta_e + \theta_i} + K_0 \left( \frac{\theta_e + \theta_i}{\theta_{ei}} \right) + 2K_0 \left( \frac{\theta_e + \theta_i}{\theta_{ei}} \right) \right] \text{ergs cm}^{-3} \text{s}^{-1},$$

(59)

where the $K$s are modified Bessel functions, $\ln \Lambda \approx 20$ is the Coulomb logarithm, and the dimensionless electron and ion temperatures are defined by

$$\theta_e = k T_e/m_e c^2, \quad \theta_i = k T_i/m_e c^2.$$

(60)

5.2. Bremsstrahlung Cooling

The bremsstrahlung includes emission from both ion-electron and electron-electron collisions (Stepney & Guilbert 1983; Svensson 1982; NY):

$$f_{be} = f_{ei} + f_{ee}. $$

(61)

The ion-electron bremsstrahlung cooling is given by

$$f_{ei} = n_e \bar{n} \sigma_T c \alpha_f m_e c^2 F_e(\theta_e),$$

(62)

where $\alpha_f = 1/137$ is the fine-structure constant and

$$F_e(\theta_e) = \frac{4}{(2 \theta_e)^{1/2}} \left( 1 + 1.781 \theta_e^{3/4} \right), \quad \theta_e < 1,$$

$$F_e(\theta_e) = \frac{90}{2\pi} [\ln (1.123 \theta_e + 0.48) + 1.5], \quad \theta_e > 1.$$

(63)

In the original formula quoted by Stepney & Guilbert (1983), there is a number 0.42 instead of 0.48 (see NY).

In the nonrelativistic approximation, one gets from equation (62)

$$f_{ei} = 1.57 \times 10^{-27} n_e \bar{\theta}_e T_e^{1/2} \text{ergs cm}^{-3} \text{s}^{-1},$$

(64)

which for Population I abundances ($X = 0.7$, $Y = 0.28$) gives

$$f_{ei} = 5.63 \times 10^{20} \rho^2 T_e^{1/2} \text{ergs cm}^{-3} \text{s}^{-1}. $$

(65)

For the electron-electron bremsstrahlung, Svensson (1982) gives the following formula for $\theta_e < 1$:

$$f_{ee} = n_e^2 c^2 r_e^2 m_e c^2 \alpha_f \frac{20}{9 \pi^{1/3}} (44 - 3 \pi^2) \theta_e^{3/2}(1 + 1.1 \theta_e^2 - 1.25 \theta_e^{5/2}) \text{ergs cm}^{-3} \text{s}^{-1},$$

(66)

and for $\theta_e > 1$:

$$f_{ee} = n_e^2 c^2 r_e^2 m_e c^2 \alpha_f 24 \theta_e \left[ \ln (\eta \theta_e) + 1.28 \right] \text{ergs cm}^{-3} \text{s}^{-1},$$

(67)

where $r_e = e^2/m_e c^2$ is the classical radius of electron and $\eta = \exp (-\gamma_e) = 0.5616$. Again, in the original formula there is a 5/4 instead of 1.28 (see NY).
5.3. Synchrotron Cooling

Following NY, we give formulae for the synchrotron emission of a relativistic Maxwellian distribution of electrons. These formulae are valid only for $\theta > 1$, but they are sufficient for applications to ADAFs (see NY):

$$f_{\text{synch}}^- = \frac{2\pi}{3c^2} kT_e(r) \frac{dv_e(r)}{dr},$$

(68)

where

$$v_e = \frac{3}{2} \frac{eB}{2\pi m_e c} \theta_e^2 x_M,$$

(69)

and $x_M$ is the solution of the transcendental equation

$$\exp(1.8899x_M^{3/2}) = 2.49 \times 10^{-10} \frac{4\pi m_e r}{B} \frac{1}{\theta_e^3 K_a(1/\theta_e)} \left( \frac{1}{x_M^{7/6}} + \frac{0.40}{x_M^{17/12}} + \frac{0.5316}{x_M^{5/3}} \right),$$

(70)

where radius $r$ is in physical unit of cm.

5.4. Compton Cooling

NY use the Dermer, Liang, & Canfield (1991) prescription for the Compton energy enhancement factor $\eta$:

$$\eta = \frac{1 + p(A - 1)}{1 - PA} \left[ 1 - \left( \frac{x}{3\theta_e} \right)^{-1 - \ln P/\ln A} \right]^{-1} + \eta_1 + \eta_2 \left( \frac{x}{\theta_e} \right)^{\eta_3},$$

(71)

where

$$x = \frac{\hbar v_e}{m_e c^2}, \quad P = 1 - \exp(-\tau_{es}), \quad A = 1 + 4\theta_e + 16\theta_e^2,$$

$$\eta_1 = \frac{p(A - 1)}{1 - PA}, \quad \eta_2 = -3^{-\eta_1} \eta_1, \quad \eta_3 = -1 - \ln P/\ln A,$$

(72)

and $\tau_{es}$ is the electron scattering optical depth. Note that $\eta$, $P$, and $A$ here are different from the ones defined in previous sections.

The Comptonization of the bremsstrahlung radiation is given by

$$f_{br,c}^- = 3\eta_1 q^{br} \left( 1 - \frac{x_e}{3\theta_e} \right) - \frac{1}{\eta_3 + 1} \left( \frac{1}{3\theta_e} \right)^{\eta_1 + 1} \left( 1 + \frac{1}{3\theta_e} \right)^{\eta_1 + 1},$$

(73)

and the Comptonized synchrotron emission is

$$f_{\text{synch},c}^- = f_{\text{synch}}^- \left[ \eta_1 - \eta_2 (x_e/\theta_e)^{\eta_3} \right],$$

(74)

where $x_e = v_e/m_e c^2$.

5.5. Total Radiative Cooling

In the optically thin case, the total radiative cooling will be given by

$$f^- = \frac{F^-}{2H} = f_{br}^{-} + f_{\text{synch}}^{-} + f_{br,c}^{-} + f_{\text{synch},c}^{-},$$

(75)

whereas in the optically thick case, one can use

$$f^- = \frac{8\sigma T_e^4}{3H\tau},$$

(76)

where $\tau = \tau_{abs} + \tau_{es} = 0.5(\kappa_{abs} + \kappa_{es})\Sigma$ is the total optical depth. In the intermediate case, one should solve the transfer equation to get reliable results. In the present context, one can use the solution of the gray problem obtained by Hubeny (1990). NY give the following formula:

$$f^- = \frac{4\sigma T_e^4}{H} \left[ \frac{3\tau}{2} + \sqrt{3} + \frac{4\sigma T_e^4}{H} \times (f_{br}^- + f_{\text{synch}}^- + f_{br,c}^- + f_{\text{synch},c}^-)^{-1} \right]^{-1},$$

(77)

where the relation

$$\tau_{abs} = \frac{H}{4\sigma T_e^3} (f_{br}^- + f_{\text{synch}}^- + f_{br,c}^- + f_{\text{synch},c}^-)$$

(78)

has been used. The opacity appearing in equation (77) is the Planck mean opacity.
The radiation pressure can be expressed as
\[ p_r = \frac{f^*H}{2c} \left( \tau + \frac{2}{\sqrt{3}} \right). \]  
(79)

5.6. Thermal Balance Equations

Since the viscous heating concerns the ions, we shall write equation (40) in the form of two equations (NY):
\[ f^+ = f^{adv} + f^{ie}, \quad f^{ie} = f^-, \]  
(80)
where \( f^+ = F^+/(2H), f^{adv} = F^{adv}/(2H), f^{ie}, \) and \( f^- \) are given by equations (59) and (76) respectively.

6. THE SONIC POINT AND THE BOUNDARY CONDITIONS

Our system of equations, like its Newtonian and PW versions, possesses singular points. In this section, we will use the vertically averaged quantities \( P \) and \( \Sigma \). From equation (31), we have
\[ \frac{d\ln P}{d\ln r} = \frac{1}{V(1 - V^2)} \frac{dV}{d\ln r} - B, \]  
(81)
where
\[ B = \frac{d\ln(\Delta^{1/2})}{d\ln r} = r(r - M). \]  
(82)
The radial gradient of \( P \) in equation (36) can be derived from equation (44) and (46) as
\[ \frac{d\ln P}{d\ln r} = 24(3 - 3\beta^*) \frac{d\ln T}{d\ln r} + 3\beta^* - 1 \frac{d\ln \Sigma}{d\ln r} + \frac{1 - \beta^*}{1 + \beta^*} \frac{d\ln \gamma^2 \beta}{d\ln r}. \]  
(83)
Thus, by combining the radial motion equation (36) and the energy equation (40), one gets
\[ \frac{d\ln V}{d\ln r} = \frac{N}{D} (1 - V^2), \]  
(84)
where
\[ D = \mathcal{A} c_s^2 - \left[ 1 + \frac{2(1 - \beta^*)}{1 + \beta^*} \right] V^2, \]  
(85)
\[ N = -\mathcal{A} - \mathcal{B} c_s^2 \left[ \mathcal{B} - \frac{2(1 - \beta^*)}{1 + \beta^*} \right] - (\Gamma_3 - 1)4\pi r^2 (F^+ - F^-) - \frac{(1 - \beta^*)}{(1 + \beta^*)} c_s^2 \left[ d\ln \mathcal{B} + \frac{2\nu(\nu/d\nu)/d\nu}{1 - (\nu/d\nu)^2} + 2\beta \right]. \]  
(86)
\( \mathcal{A} \) is defined by equation (37) and
\[ c_s^2 = \frac{P}{\Sigma}, \quad \mathcal{B} = 1 + 2 \frac{\Gamma_1 - 1}{1 + \beta^*}. \]  
(87)
The equation for the temperature is then
\[ \frac{d\ln T}{d\ln r} = (1 - \Gamma_3) \frac{N}{D} + (1 - \Gamma_3) \frac{2\pi r^2 (F^+ - F^-)}{Mc_T} + \frac{2\pi r^2 (F^+ - F^-)}{Mc_T}. \]  
(88)
There are two singular points of these equations: horizon \( r = r_h \), defined by \( \Delta(r_h) = 0 \), and sonic point \( r = r_s \), defined by the condition
\[ V^2(r_s) = \frac{\mathcal{B}(r_s)c_s^2(r_s)}{1 + 2c_s^2(1 - \beta^*)/(1 + \beta^*)}. \]  
(89)

1. The horizon.—At the horizon \( \Delta = 0 \), and therefore \( B = \infty, \mathcal{A} = \infty \). However, one may demonstrate that \( \mathcal{A} = 1 \) and \( \mathcal{B}/\mathcal{A} = -1 \). Since on the horizon \( V^2(r_h) = 1 \), it follows that at the horizon \( dV/dr \) is nonsingular. For \( \beta^* \neq 1 \), the gradients \( d\Sigma/dr, dP/dr \) and \( dT/dr \) are singular only through the infinite boost required by the hydrostatic equilibrium equation (44). Note, that from equation (34), it follows that at the horizon \( \Omega = 0 \). There are no extra regularity conditions at the horizon.

2. The sonic point.—For large radii, the flow is subsonic, \( V^2 < c_s^2 \), and as we have just seen, it crosses the horizon with the light speed, \( 1 = V^2 > c_s^2 \). Thus, somewhere in the flow it must be \( V^2 = c_s^2 \), i.e., the flow must pass through the sonic point. Derivatives \( dV/dr \) and \( dT/dr \) are nonsingular at the sonic point only if
\[ N = D = 0. \]  
(90)
This gives an extra algebraic condition that must be satisfied by an acceptable, regular at the sonic point, global solution of the slim disk equations. The regularity condition at the sonic point was the main source of numerical difficulties in solving the PW slim disk equations: only for one particular choice of the eigenvalue \( \mathcal{L}_0 \), could the condition (89) be satisfied.
Fig. 1.—Radial structure of the Mach number ($M$), the pressure ($p$ in cgs unit), the angular momentum ($L$ in units of $M$), and the sound speed ($c_s$ in units of the speed of light). Here the mass of the black hole is $10 M_\odot$, $z = 0.1$ and $M/M_\odot = 10^{-5}$. The solid, dotted, and dashed lines represent the cases of $a/M = 0, 0.5, \text{ and } 0.99$, respectively. The heavy dots represent solutions obtained with the pseudo-Newtonian potential. These solutions are excellent approximation to the solutions representing the Schwarzschild black hole flows (in the case $a = 0$). The corresponding Keplerian angular momenta of test particles around Kerr black holes are also shown for comparison (thin lines).

One may demonstrate that in the limit of spherical accretion ($\Omega = 0$, $N = 1$) onto a nonrotating black hole ($a = 0$), the condition (eq. [89]) becomes (for $\beta^* = 1$),

$$V^2(r_S) = \frac{M}{2[r_S - (3/2)M]},$$

which in the notation of Shapiro & Teukolsky (1983) takes the form

$$u^2(r_S) = \frac{M}{2r_S},$$

which is identical with their formula (G.17). On the other hand, using equation (44) in the form

$$H \gamma^2 \beta = c_s^2$$

and including into equation (80) the $dH/dr$ term

$$\beta = -\frac{d}{dr} \left( \log \Delta^{1/2} H \right) = -\frac{r - M}{\Delta} - \frac{1}{r} N,$$
where \( \mathcal{N} = d \log H / d \log r \), one obtains from equations (84) and (85) (for \( \beta^* = 1 \)) the following condition for the velocity at the sonic point:

\[
V(r_0) = \frac{2c_s}{1 + \Gamma_1},
\]

which is the same condition as the one obtained by Narayan et al. (1996a).

(3) The boundary conditions.—The set of equations describing the flow is of the fifth order with four independent radial first derivatives (say, \( d\Omega/dr, dV/dr, dE/dr, \) and \( d^2T/dr^2 \)) and one second derivative (\( d^2\Omega/dr^2 \)). Five integration constants are required. The equation of mass conservation is integrated analytically, and this fixes one integration constant, \( M \), which is a free parameter. Furthermore, since the angular momentum equation has been analytically integrated from the horizon, an (unknown) value of the specific angular momentum at the horizon, \( \mathcal{L}_0 \), appears in the angular momentum equation. This constant, however, is not a free parameter and is determined such that the flow is transonic. The two regularity conditions \( (N = 0, D = 0) \) at the sonic point provide one condition for the sonic point location and one extra condition for the five integration constants. Three boundary conditions are assumed at the outer boundary. There are no other boundary assumed conditions, in particular no other boundary conditions at the horizon. The solution is determined by the regularity conditions (\( N = 0 \), \( D = 0 \)) as the outer boundary conditions, and by varying \( \mathcal{L}_0 \), we find its particular value (the eigenvalue of the problem) that fulfills the regularity conditions (\( N = 0 \), \( D = 0 \)) (for details, see Chen & Taam 1993).

7. NUMERICAL SOLUTIONS

The set of Kerr black hole disk equations was solved by using the method described in Chen & Taam (1993). This method has been modified to take into account the terms introduced by general relativistic effects. Here we focus only on the case of advection-dominated accretion flows (ADAFs) and study the effects of the rotation of the black hole. Specifically, we assume the gas is optically thin (\( \tau = 0 \)) and consider the \( \beta^* = 1 \) case. The local radiative cooling is only through nonrelativistic bremsstrahlung emission (eq. [65]). The corresponding solution in pseudo-Newtonian potential approximation was obtained by Chen, Abramowicz, & Lasota (1996) and Narayan et al. (1996a).

The mass accretion rate is measured in the Eddington limit, which is defined as

\[
\dot{M}_E = 4\pi GM/(c\kappa_{es}), \quad \kappa_{es} = 0.34,
\]

and the units for \( r \) and angular momentum (such as \( \mathcal{L}, \alpha \)) are \( r_0 \) and \( M \), respectively.

We present in Figure 1 examples of solutions with three different rotation rates of the black hole, \( a/M = 0 \) (solid line), 0.5 (dotted line), and 0.99 (dashed line). The black hole mass is \( M = 10 M_\odot \), the viscosity parameter is \( \alpha = 0.1 \), and the accretion rate is \( \dot{M}/\dot{M}_E = 10^{-5} \). The pseudo-Newtonian solution is also plotted for comparison (heavy dots). One can see that the pseudo-Newtonian solution is an excellent approximation of the solution for a Schwarzschild black hole (\( a = 0 \)). As the black hole rotates faster, the sonic point moves inward, mainly because of the increase of the sound speed. This is due to the deepening of the potential well depth with increasing black hole rotation since the ADAF temperature is close to the virial one. The specific angular momentum lost to the black hole is smaller for a faster rotating black hole, which is also the effect of the hole becoming smaller with increasing rotation. At the black hole surface, the pressure goes to infinity because of the infinite boost required by equation (44).

8. CONCLUSION

We have derived a complete set of equations describing accretion flows with nonzero angular momentum around a rotating black hole (non-Keplerian accretion disks). We used the Novikov and Thorne form of the Kerr metric, which is accurate up to the \( (H/r)^3 \) terms. Since ADAFs can be heated up to the virial temperature and therefore \( H/r \leq 1 \), the slim disk approximation may appear to be not adequate. However, it is known from the study of ADAFs in the Newtonian case (Narayan & Yi 1995a) that the quasi-spherical structure of ADAFs can be well represented by suitably interpreted vertically averaged quantities. We do not expect general relativistic effects to modify this conclusion.

We obtained several test solutions that show that ADAFs in a Kerr spacetime have the same basic properties as the pseudo-Newtonian ADAFs. The rotation of the hole allows, however, for higher temperatures and pressures which should modify the observed emission from ADAFs. This property, as well as the influence of the general relativistic effects on light propagation, will be studied in a future work.

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Note added in proof.—Recently, M. Abramowica, A. Lanza, and M. J. Percival (ApJ, submitted [1996]) derived two equations that describe the vertical equilibrium. Both equations contain the vertical component of velocity which is convenient to express, in the spherical coordinates that they use, by the quantity

\[ u \equiv \left( \frac{u_\theta}{\cos \theta} \right)_{\theta = \theta_0}. \]

Here \( \theta = \theta(r) \) is the location of the surface of the disk, \( \cos^2 \Theta = (H/r)^2 \). The first of the two equations follows from \([V, T_\phi = 0]_{\theta = \theta_0}\), and it is a nonsingular version of equation (44):

\[ -2 \frac{P}{\rho} + \left( \frac{H}{r} \right)^2 \left( \frac{u_\phi^2}{\rho} - u_z^2 + a^2 \right) + a^2 - \Delta u \frac{du}{dr} = 0. \]  

(44a)

The second equation is a surface boundary condition. It demands that there should be no velocity component orthogonal to the surface of the disk

\[ \frac{u}{v_r} = \frac{1}{r} \frac{d \ln (\pi/2 - \Theta)}{d \ln r}. \]  

(44b)

Note that this condition must be imposed for the mathematical completeness. It determines what in the body of the present paper is denoted as \( d \ln H/d \ln r \). In the standard accretion disk model, this quantity is not self-consistently calculated, but assumed ad hoc. Equations (44a) and (44b) that should replace equation (44) were derived in the Kerr geometry according to a general discussion given by J.-P. Lasota and M. A. Abramowicz (Classical Quantum Gravity, in press [1996]. In this paper the Schwarzschild version of equation (44a) was derived.