Abstract

We characterize the virtually soluble profinite groups of finite rank that are finitely axiomatizable in the class of all profinite groups.
1 Introduction

We give a group-theoretic criterion of finite axiomatizability for certain profinite groups, namely the virtually soluble groups of finite rank; this solves Problem 1 of [NST]. To articulate it precisely we need to recall some definitions.

- Throughout, \( L \) is a first-order language: either the language of groups \( L_{gp} \), or the language \( L_\pi \) for some set of primes \( \pi \): this is \( L_{gp} \) augmented with unary function symbols \( P_\lambda \), one for each \( \lambda \in \mathbb{Z}_\pi = \prod_{p \in \pi} \mathbb{Z}_p \); these are interpreted in a profinite group as profinite powers, \( P_\lambda(g) = g^\lambda \) (cf. [FJ], Chap. 1, Ex. 9).

- A profinite group \( G \) has an \( L \)-presentation in a class \( C \) of profinite groups if \( G \) has a finite generating set \( \{g_1, \ldots, g_d\} \) and there is an \( L \)-formula \( \psi(x_1, \ldots, x_d) \) such that \( G \models \psi(g) \) and for any profinite group \( H \in C \) and \( h_1, \ldots, h_d \in H \), if \( H \models \psi(h) \) then the map sending \( g_i \) to \( h_i \) for each \( i \) extends to an epimorphism \( G \to H \) (see [NST] §5.3).

- A profinite group \( G \) has rank \( r = \text{rk}(G) \) if every closed (equivalently, open) subgroup can be (topologically) generated by \( r \) elements, and \( r \) is the least such integer (see [DDMS], §3.2).

- A profinite group \( G \) is finitely axiomatizable (FA) in \( C \) if there is a sentence \( \sigma_G \) in \( L \) such that \( G \models \sigma_G \), and for any profinite group \( H \in C \), if \( H \models \sigma_G \) then \( H \) is isomorphic to \( G \). When \( C \) is the class of all profinite groups we say that \( G \) is FA.

- A profinite group \( G \) satisfies the OS condition (for Oger-Sabbagh) if the image of \( Z(G) \) in the abelianization of \( G \) is periodic.

Here \( Z(G) \) denotes the centre of \( G \). The set of primes \( p \) such that \( G \) has a nontrivial Sylow pro-\( p \) subgroup will be denoted \( \pi(G) \). A profinite group is said to be virtually \( C \) for some class \( C \) if it has an open normal \( C \)-subgroup.

- \( C_\pi \) denotes the class of all pronilpotent groups \( G \) with \( \pi(G) \subseteq \pi \).
A profinite group $G$ is in $C^q$ if $G$ has an open normal subgroup $H$ such that $H \in C_\pi$ and $G^q \leq H$. The notation assumes that $q$ is a $\pi$-number, so $\pi(G) \subseteq \pi$.

Note that $G \in C_\pi$ if and only if $G$ is a Cartesian product of pro-$p$ groups with $p \in \pi$.

Before stating the main result we need the following observation, proved in the next section:

**Lemma 1.1** Let $G$ be a virtually prosoluble group of finite rank with $\pi := \pi(G)$ finite. Then $G \in C_\pi^q$ for some $\pi$-number $q$.

Now we can state

**Theorem 1.2** Let $G$ be a virtually soluble profinite group of finite rank, and assume (1) $\pi(G)$ is finite and (2) $G$ has an $L$-presentation in $C_\pi^q$, where $G \in C_\pi^q$. Then $G$ is finitely axiomatizable if and only if every open subgroup of $G$ satisfies the OS condition.

**Remarks**

1. The hypothesis of an $L$-presentation is automatically fulfilled when $L$ is $L_\pi$, $\pi = \pi(G)$; see \[2\] below. When $L$ is $L_{gp}$, a sufficient (though by no means necessary) condition is that $G$ be the $C_\pi^q$ completion of a finitely presented abstract group $\Gamma$, for example a polycyclic group (in view of Lemma [1.] this is in fact the same as the pro-$\pi$ completion of $\Gamma$ when $\Gamma$ is virtually soluble of finite rank); cf. [NST], Prop. 5.13(i), which is the case where $q = 1$.

The unavoidability of some such hypothesis is discussed in the introduction of [NST]; in fact it is an (obvious) consequence of ‘finite axiomatizability’ if the latter is defined to include a generating tuple, as in Theorem 5.15 of [NST]. It will be clear from the proof - which is an application of that result - that Theorem 1.2 holds as well with such an amended definition.

2. Suppose that instead of (1) we assume that $G$ is pronilpotent. Then $G$ is FA if and only (a) every open subgroup of $G$ satisfies the OS condition and (b) $\pi(G)$ is finite. Indeed, Proposition 1.3 of [NST] shows that if $G$ is pronilpotent and FA then $\pi(G)$ is finite.

3. The theorem generalises Theorem 5.16 of [NST] which deals with the nilpotent case; in that case, the OS condition for $G$ is automatically inherited by all open subgroups (a simple exercise).

4. The title of this paper refers to C. Lasserre [L], who in a similar way characterizes finite axiomatizability for virtually polycyclic groups in the class of finitely generated abstract groups. Note that a profinite group is soluble of finite rank if and only if it is poly-procyclic.
5. **Problem**: Is the pronilpotency hypothesis required in Remark 2. above? In other words, does $G$ being FA imply that $\pi(G)$ is finite, for a virtually soluble profinite group $G$ of finite rank? This would yield a more elegant characterization; but it seems hard to either prove or disprove.

For further background and motivation, see the introduction to [NST]. We recall that a pro-$p$ group has finite rank if and only if it is $p$-adic analytic; for this and more information about these groups see [DDMS].

I would like to thank Andre Nies for introducing me to finite axiomatizability in general, and to Lasserre’s paper in particular.

2 Initial observations

We briefly recall some material from [NST], Section 2. All formulae are $L$-formulae. A subset $S$ of a group $G$ is definable if

$$S = \phi(\mathbf{a}; G) := \{ g \in G \mid G \models \phi(\mathbf{a}; g) \}$$

where $\phi(t_1, \ldots, t_r, x)$ is a formula and $\mathbf{a} = (a_1, \ldots, a_r) \in G^r$ (here $r$ may be zero). $S$ is definably closed if in addition, for every profinite group $H$ and $b \in H^r$ the subset $\phi(b; H)$ is closed in $H$. If $S$ is a definably closed (normal) subgroup of $G$, we can (and will) assume that

$$\phi(t; x) \land \phi(t; y) \to \phi(t; x^{-1}y) \quad \text{(subgroup)}$$

$$\phi(t; x) \to \phi(t; x^y) \quad \text{(normal)}.$$  

Then for $H$ and $b$ as above the subset $\phi(b; H)$ is a closed (normal) subgroup of $H$.

First-order formulae equivalent to various other useful group-theoretic assertions can be found in [NST], Section 2.

If $G$ is a finitely generated profinite group then every subgroup of finite index is open and definably closed, and the derived group $G'$ is definably closed (see [NST], Thm. 2.1). More specifically, If $G$ is an $r$-generator pronilpotent group then every element of $G'$ is equal to a product of $r$ commutators (cf. [DDMS], proof of Prop. 1.19). In general, if $G$ has finite rank then every centralizer is definably closed, because every subgroup is finitely generated.

If $H$ is a definable subset and $N$ is a definable normal subgroup of a group $G$, a definable subset of $H$ is definable in $G$, and $S$ is a definable subset of $G/N$ iff $\pi^{-1}(S)$ is definable in $G$ where $\pi : G \to G/N$ is the quotient mapping.

We will use these observations without special mention.

A formula constructed to axiomatize a group $G$ will often take the parametric form $\chi(a_1, \ldots, a_r)$ where the $a_i$ are elements of $G$; to avoid repeating the obvious, it should then be understood that the corresponding axiom will be $\exists x_1 \ldots \exists x_r. \chi(x_1, \ldots, x_r)$.

I will write

$$H <_o G$$
to mean that \( H \) is an open normal subgroup of \( G \).

**Lemma 2.1** Let \( G \) be a virtually prosoluble group of finite rank with \( \pi := \pi(G) \) finite. Then \( G \in C_\pi^q \) for some \( \pi \)-number \( q \).

**Proof.** Put \( r = \text{rk}(G) \). Let \( K \) be the intersection of the kernels of all homomorphisms \( G \to \text{GL}_r(F_p), p \in \pi \), and let \( H < G \) be prosoluble. Then \( H_0 := K \cap H < G \).

Now suppose \( N < H_0 \). Then \( N \geq N_0 \) for some \( N_0 < G \), and we have a chain

\[
N_0 < N_1 < \ldots < N_k = H_0
\]

with each \( N_i \) normal in \( G \) and \( N_i/N_{i-1} \cong \mathbb{F}_p^s \) for some \( p \in \pi \) and \( s \leq r \). Now \( K \) centralizes each such factor, so \( H_0/N_0 \) is nilpotent. It follows that \( H_0 \) is pronilpotent, hence \( H_0 \in C_\pi \). Set \( q = |G : H_0| \). \( \blacksquare \)

**Proposition 2.2** If \( \pi \) is finite, \( G \in C_\pi^q \) and \( G \) has finite rank then \( G \) has an \( L_\pi \)-presentation in \( C_\pi^q \).

**Proof.** [NST], Proposition 5.13(ii) is the case where \( G \in C_\pi \); but the proof only uses the fact that \( G \) is virtually a product of uniform pro-\( p \) groups, \( p \in \pi \), and this still holds if \( G \) is virtually \( C_\pi \) (of finite rank). (In the proof of [NST], Lemma 5.14, one should add the first-order condition saying that each of the words \( w_i \) is a product of \( m \) \( q \)-th powers, for some suitable \( m \)).

It also uses a formula

\[
\beta_d(y_1, \ldots, y_d)
\]

which asserts for a \( C_\pi \) group \( H \) that \( \{y_1, \ldots, y_d\} \) generates \( H \); in effect this asserts that \( \{y_1, \ldots, y_d\} \) generates \( H \) modulo its Frattini subgroup \( \Phi(H) \), which is open in \( H \). But \( \Phi(H) \) is open in \( G \), and a similar formula may be constructed to assert that \( \{y_1, \ldots, y_d\} \) generates \( G \) modulo \( \Phi(H) \); this does the job since \( \Phi(H) \subseteq \Phi(G) \). \( \blacksquare \)

### 3 ‘Only if’

The analogue for finitely generated abstract groups was established by Francis Oger in Theorem 3 of [O]. The following proof is essentially his argument, adapted to deal with profinite groups in place of finitely generated groups.

**Theorem 3.1** Let \( G \) be a finitely generated profinite group that is virtually \( C_\pi \), where \( \pi \) is a finite set of primes. Suppose that \( G \) has an open subgroup that fails to satisfy the OS condition. Let \( \sigma \) be a sentence satisfied by \( G \). Then for almost all primes \( q \) there exists a profinite group \( G_q \) that satisfies \( \sigma \) and contains elements of order \( q \).
In particular, such a group $G_q$ cannot be isomorphic to $G$ when $q \notin \pi(G)$; in view of Lemma 1.1 this suffices to establish the ‘only if’ direction of Theorem 1.2.

The first step is the following lemma, which is proved just like Prop. 1 of [O], replacing $\mathbb{Z}$-modules by $\mathbb{Z}_p$-modules where appropriate:

**Lemma 3.2** Suppose that the f.g. profinite group $\Gamma$ is virtually pro-$p$ and that some open subgroup of $\Gamma$ fails to satisfy the OS condition. Then $\Gamma$ has closed normal subgroups $A, N$ such that $A \cap N = 1$, $AN$ is open in $G$, and $A \cong \mathbb{Z}_p^r$ for some finite $r \geq 1$.

Now let $G$ be as in Theorem 3.1. Thus $G$ has an open normal subgroup $Q = Q_1 \times \cdots \times Q_k$ where $Q_i$ is a pro-$p_i$ group for each $i$ and $\pi = \{p_1, \ldots, p_k\}$; and $G$ has an open subgroup $L$ that fails to satisfy the OS condition. Set $L_i = L \cap Q_i$. Say $z \in Z(L)$ has infinite order modulo $L'$. Then $z^m = y_1 \cdots y_k$ with $y_i \in Z(L_i)$, where $m = |G : Q|$, and for at least one value of $i$ the element $y_i$ has infinite order modulo $L_i'$. Let's assume that $i = 1$, and put $p = p_1$. Thus $Q_1$ is a pro-$p$ group and its open subgroup $L_1$ fails to satisfy the OS condition.

Put $M = Q_2 \times \cdots \times Q_k$. Applying Lemma 3.2 to $\Gamma = G/M$ we find closed normal subgroups $A^* / M, N / M$ of $G / M$ such that $A^* \cap N = M, A^* N$ is open in $G$, and $A^*/M \cong \mathbb{Z}_p^r$ for some finite $r \geq 1$. Put $A = A^* \cap Q_1$. Then $A^* \cap Q = A \times M$, so $A \cong (A^* \cap Q) / M$ which is open in $A^* / M$ and so $A \cong \mathbb{Z}_p^r$. Also

$$A \cap N = A_1 \cap Q_1 \cap N = M \cap Q_1 = 1$$
$$AN = AMN = (A^* \cap Q)N \leq_o A_1N \leq_o G.$$  

Thus $G$ has closed normal subgroups $A, N$ such that $K := A \times N$ is open in $G$ and $A \cong \mathbb{Z}_p^r$ for some finite $r \geq 1$.

Write $F = G / K$, so the action of $G$ makes $A$ into a $\mathbb{Z}_p F$-module, which we write additively. $F$ is a finite group. The matrices representing $G$ relative to a $\mathbb{Z}_p$-basis $\{e_1, \ldots, e_r\}$ of $A$ have entries in a finitely generated subring of $\mathbb{Z}_p$; then $E = \bigoplus e_i R$ is an $R G$-module, and as $\mathbb{Z}_p G$-modules

$$A \cong E \otimes_R \mathbb{Z}_p.$$  

By a standard argument (cf. paragraph 5 in the proof of [O], Theorem 2), we can embed $A$ in the $\mathbb{Q}_p (G / A)$-module $\tilde{A} = \mathbb{Q}_p A \cong \mathbb{Q}_p^r$ and $G$ in a group $\tilde{G}$ such that

$$\tilde{G} \cap \tilde{A} = A$$
$$\tilde{G} = \tilde{A} G = \tilde{A} \rtimes T;$$

here $T / N$ is a complement to $\tilde{K} / N := (A \times N) / N$ in the extension $\tilde{G} / N$ of $\tilde{K} / N \cong \mathbb{Q}_p^r$ by the finite group $\tilde{G} / \tilde{K} \cong F$.

As $T / N \cong G / K$ is finite, we can set $A_1 = p^{-e} A$ for some finite $e$ to obtain

$$H := A_1 G = A_1 \rtimes T;$$

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and $G$ has finite index $s$, say, in $H$.

In particular $H$ is again a finitely generated profinite group, and so $G = \kappa(H)$ for some formula $\kappa$ (with parameters) that defines $G$ as a closed subgroup. Then $H$ satisfies the sentence

$$
\psi := \text{res}(\kappa; \sigma) \land s(\kappa) \land \text{ind}^*(\kappa; s)
$$

asserting that $\kappa$ defines a closed subgroup that satisfies $\sigma$ and has index exactly $s$ (see [NST], Section 2).

Now suppose that $H_q$ is a profinite group that satisfies $\psi$ and contains an element $y$ of order $q$, where $q$ is a prime not dividing $s$. Then $G_q := \kappa(H_q)$ is a closed subgroup that satisfies $\sigma$, and $G_q$ has index $s$ so $y \in G_q$. To complete the proof it will therefore suffice to construct groups like $H_q$ for almost all primes $q$.

Let $\varpi$ denote the set of primes $q$ such that $qR \neq R$. The complement of $\varpi$ is finite ($R$ is ‘generically free’ over $\mathbb{Z}$, [E], Thm. 14.4). For each $q \in \varpi$ we choose a maximal ideal $m_q$ containing $q$; then $R/m_q = \Phi_q \cong \mathbb{F}_{q^f(q)}$ for some finite $f(q)$ ([AM], Cor. 5.24).

Now as $\mathbb{Z}_p T$-modules, $A_1 \cong A \cong E \otimes_R \mathbb{Z}_p$. For each $q \in \varpi$ let $B_q = E/Em_q \cong E \otimes_R \Phi_q$, and put

$$
H_q = (B_q \oplus A_1) \rtimes T.
$$

This is a profinite group having elements of order $q$, and it remains to show that $H_q$ satisfies $\psi$ for almost all $q$. Put

$$
\varpi^* := \{ q \in \varpi \mid H_q \models \neg \psi \}.
$$

**Lemma 3.3** Suppose that $\varpi^*$ is infinite. Let $\mathcal{U}$ be a non-principal ultrafilter on the set $\varpi^*$. Then

$$
H^\mathcal{U} \cong \left( \prod_{q \in \varpi^*} H_q \right) / \mathcal{U},
$$

i.e. the two ultraproducts are isomorphic as groups.

As $H \models \psi$, it follows by Lós’s Theorem that $\varpi^*$ must be finite, and the proof is complete, modulo the

**Proof of Lemma 3.3.** This is essentially contained in the proof of [O], Theorem 3, with $R$ replacing $\mathbb{Z}$. For clarity, I sketch the argument here.

Observe that

$$
B_q \oplus A_1 \cong E \otimes (\Phi_q \oplus \mathbb{Z}_p)
$$

as $RT$-modules, where $\otimes = \otimes_R$. Let $K$ denote the field of fractions of $R$. 




Consider the $R$-modules

$$Q := \left( \prod_{q \in \pi^*} \Phi_q \right) / \mathcal{U},$$

$$P := \prod_{q \in \pi^*} \Phi_q,$$

$$V := \bigcap_{0 \neq r \in R} Pr.$$ 

Here $Q$ is a vector space of dimension $2^{\aleph_0}$ over $K$, while $V$ is a divisible submodule of the torsion-free $R$-module $P$. This implies both that $V$ is a $K$-vector space (of dimension bigger than $2^{\aleph_0}$), and that $V$ is a direct summand of $P$. Thus $V \cong Q \oplus V$ and $P = V \oplus S$ for some $R$-submodule $S$, whence

$$P \cong Q \oplus V \oplus S = Q \oplus P.$$ 

It follows that

$$A_1^U \cong E \otimes P \cong E \otimes (Q \oplus P) \cong \prod_{q \in \pi^*} (B_q \oplus A_1) / \mathcal{U}. \tag{1}$$

These are $R$-module isomorphisms (the tensor and ultraproduct operations commute because $E \cong R^d$), and also $T$-module automorphisms, since $T$ acts trivially on the right-hand factors.

Recall now that $H = A_1 \rtimes T$ is an extension of $A_1 \times N$ by the finite group $F = T/N$, that splits over $A_1$ (so a corresponding 2-cocycle maps $F \times F$ into $N$). It follows that $H^U$ is similarly an extension of $(A_1 \times N)^U$ by $F$ that splits over $A_1^U$:

$$H^U \cong A_1^U \rtimes \tilde{T}$$

where $\tilde{T} = N^U \rtimes T \, \text{ (with } T \text{ diagonally embedded in } T^d)$. Similarly

$$\left( \prod_{q \in \pi^*} H_q \right) / \mathcal{U} \cong \prod_{q \in \pi^*} (B_q \oplus A_1) / \mathcal{U} \times \tilde{T}.$$

As the action of $\tilde{T}$ on the respective modules factors through $T$, the lemma now follows from (1). ■

4 Proof of Theorem 1.2, ‘if’

Suppose now that $G \in C^q_\pi$ is soluble of finite rank, where $\pi = \pi(G)$ is finite. Assume that $G$ has an $L$-presentation in $C^q_\pi$. If $q = 1$, Theorem 5.15 of [NST] asserts that $G$ is finitely axiomatizable in $C_\pi$. However the proof works equally well in the more general case. Thus we may suppose that $G$ is finitely axiomatizable in $C^q_\pi$.

To complete the proof of Theorem 1.2 it will therefore suffice to establish
Theorem 4.1 Let $G \in C_2^\pi$ be virtually soluble of finite rank. Assume that every open subgroup of $G$ satisfies the OS condition. Then $G$ satisfies a sentence $\chi_G$ such that every profinite group satisfying $\chi_G$ is in $C_2^\pi$.

The first step reduces to the case where $G \in C_\pi$. Let us call a sentence $\chi$ such that every profinite group satisfying $\chi$ is in $C_2^\pi$ a $C_2^\pi$-sentence.

Lemma 4.2 Suppose that the f.g. profinite group $G$ has an open normal subgroup $H \in C_\pi$ with $G^\prime \leq H$, and that $H$ satisfies a $C_\pi$-sentence. Then

(1) $G$ satisfies a $C_2^\pi$-sentence;

(2) if also $G \in C_\pi$ then $G$ satisfies a $C_\pi$-sentence.

Proof. The subgroup $H$ is definably closed: that is, $H = \kappa(G)$ where $\kappa$ always defines a closed normal subgroup in any profinite group.

(1) Take $\chi$ to assert, for a group $G$, that the index of $\kappa(G)$ in $G$ is equal to $|G : H|$ and that $\kappa(G)$ satisfies $\chi_1$, where $\chi_1$ is the $C_\pi$-sentence satisfied by $H$. Then $\chi$ is a $C_2^\pi$-sentence.

(2) The Frattini subgroup $\Phi(H) = H'H^m$ is open in $G$; here $m = \prod_{p \in \pi} p$. Since now $G$ is pro-nilpotent, we have $\gamma_n(G) \leq H'H^m$ for some $n$, and this is expressible by a first-order sentence $\psi$, say, since $H$ and $\Phi(H)$ are definable in $G$. The conjunction $\chi \land \psi$ is then a $C_\pi$-sentence satisfied by $G$. For if $G \models \chi \land \psi$ and $\bar{H} = \kappa(G)$ then $\bar{H}$ is a $C_\pi$ group and $|\bar{G} : \bar{H}| = |G : H|$, so $G$ is a pro-$\pi$ group; and $\bar{G}$ is pronilpotent because $\bar{G}/\Phi(\bar{H})$ is nilpotent, which implies that $\bar{G}/N$ is nilpotent for every open normal subgroup $N$ of $\bar{G}$ contained in $\bar{H}$. □

Replacing $G$ by a suitable open normal subgroup, we may henceforth assume that $G \in C_\pi$, and have to prove that $G$ satisfies a $C_\pi$-sentence (at this point we are only using claim (1) of the lemma). Note that then $G$ is soluble.

We will often use the fact that a $C_\pi$ group of finite rank satisfies the maximal condition for closed subgroups (cf. [DDMS] Ex. 1.14). In particular, for such a group $G$ the Fitting subgroup $\text{Fit}(G)$ of $G$ is the unique maximal nilpotent closed normal subgroup.

Proposition 4.3 Let $G$ be a soluble $C_\pi$ group of finite rank and set $F := \text{Fit}(G)$. Then $G/F$ is virtually abelian, $C_G(F) = Z(F)$, and $F$ is definably closed, by a formula $\phi_1(S; -)$ ($S$ a finite set of parameters).

Proof. The first two claims are well known: $G$ is a linear group by [DDMS], Thm. 7.19, hence virtually nilpotent-by-abelian by the Lie-Kolchin Theorem; the second claim holds for every soluble group $G$. The first one implies that $G$ has a (definable) open normal subgroup $G^\dagger$ such that $G^\dagger \leq F \leq G^\dagger$. Then $\text{Fit}(G^\dagger) = F$, and if $F$ is definably closed in $G^\dagger$ then it is definably closed in $G$. So for the final claim we may replace $G$ by $G^\dagger$ and assume that $G/F$ is abelian.

Say $F$ is generated by the finite set $T$, and is nilpotent of class $c$. Then $x \in F$ iff $\phi_1(T; x)$ holds where

\[ \phi_1(T; x) \iff \{t, x\} = 1 \text{ for each } t \in T; \]
to see this, note that $\phi_1(T; x)$ implies that $F \langle x \rangle / F'$ is nilpotent, whence $F \langle x \rangle$ is nilpotent as well as normal in $G$. It is easy to see that $\phi_1(S; -)$ defines a closed set in any profinite group with a given subset $S$.

Next we prove a special case of Theorem 4.1. Recall that the FC-centre of a group $G$ is the set $Z_f(G)$ of all elements whose conjugacy class is finite. If $G$ is a profinite group of finite rank then $Z_f(G)$ is the unique maximal member of the family of subgroups whose centralizer is open.

**Proposition 4.4** Let $G$ be a torsion-free soluble pro-$p$ group of finite rank. Assume that $G/\text{Fit}(G)$ is infinite and abelian, and that $Z_f(G) = 1$. Then $G$ satisfies a $C\{p\}$-sentence.

This depends on the next few lemmas.

**Lemma 4.5** Let $X$ be a profinite group and $A$ a profinite $X$-module such that

$$\text{for } a \in A, \ x \in X, \ \ ax = a \implies (a = 0 \lor x = 1),$$

$$pA + A(X - 1) < A.$$  \hspace{1cm} (3)

Then $X$ is a pro-$p$ group.

**Proof.** Let $q \neq p$ be a prime and $Y = \langle y \rangle$ a pro-$q$ subgroup of $X$. Assuming that $y \neq 1$ we derive a contradiction. Let $a \in A \setminus (pA + A(X - 1))$.

Now $y = z^p$ for some $z \in Y$. Set $u = z - 1$. Then $au \in A(Y - 1)$, so $au = b(y - 1)$ for some $b \in A$, and then

$$a(y - 1) = a((u + 1)^p - 1)$$

$$= au(u^{p-1} + pw)$$

$$= b(y - 1)(u^{p-1} + pw) = b(u^{p-1} + pw)(y - 1)$$

where $w = u^{p-2} + \cdots + 1$.

Since $y \neq 1$ this implies that $a = b(u^{p-1} + pw) \in pA + A(X - 1)$, contradicting hypothesis. \hspace{1cm} ■

**Lemma 4.6** Let $X$ be a profinite group and $A$ a profinite $X$-module such that

$$\bigcap_{1 \neq x \in X} A(x - 1) = 0.$$  \hspace{1cm} (4)

If (2) and (3) hold then $A$ is a pro-$p$ group.

**Proof.** Lemma 4.5 shows that $X$ is pro-$p$. Let $B$ be the pro-$p'$ component of $A$. Then for $1 \neq x \in X$ we have $B(x - 1) = B(x - 1)^2$ (coprime action), and it follows from (2) that $B = B(x - 1)$. Now (1) implies that $B = 0$. (‘Coprime action’ refers to the fact that if a finite $p$-group acts nilpotently on an abelian $p'$ group, it acts trivially; this transfers to the profinite case.) \hspace{1cm} ■
Note that (4) holds automatically if $X$ acts faithfully on $A$ and is infinite: for the open normal subgroups $U$ of $A \times X$ intersect in \{1\}, and for each such $U$ we may choose $x \in U \cap X \setminus \{1\}$ giving $A(x - 1) \subseteq U$.

Let $G$ be a group, $A$ a non-zero $G$-module, and set $X := G/C_G(A)$.

- $A$ is nice if $a \in A \setminus \{0\} \implies |G : C_G(a)|$ is infinite,
- $A$ is very nice if both (4) and (2) hold.

Note that if $A$ is a definable abelian normal subgroup of $G$, then for $A$ to be very nice as a $G$-module is a first-order property of $G$.

**Lemma 4.7** Let $A \neq 0$ be a nice $G$-module, where $G/C_G(A)$ is abelian. Let $a \in A \setminus \{0\}$ and suppose that $C_G(a)$ is maximal among centralizers of nonzero elements. Put $B = C_A(C_G(a))$. Then $B$ is very nice, and if $B \neq A$ then $A/B$ is nice.

**Proof.** Put $Y = C_G(a)$. Suppose $0 \neq b \in B$. Then $C_G(b) \supseteq Y$ so $C_G(b) = Y = C_G(B)$; thus $B$ satisfies (2). That $B$ satisfies (4) follows from the fact that $|G : C_G(B)| = |G : C_G(a)|$ is infinite.

Now let $c \in A \setminus B$ and let $Y$ be the centralizer of $c$ mod $B$. Then for $x \in X$,

$$y \in Y \implies c(x - 1)(y - 1) = c(y - 1)(x - 1) = 0,$$

so $Y \subseteq C_G(c(x - 1))$. If for some $x \in X$ we have $b := c(x - 1) \neq 0$ then $|G : C_G(b)|$ is infinite, hence so is $|G : Y|$. Otherwise, $c \in C_A(X) = B$. Thus $A/B$ is nice. 

**Corollary 4.8** Let $G$ be a pro-$p$ group of finite rank such that $Z_f(G) = 1$. Let $A \neq 1$ be a definable torsion-free abelian closed normal subgroup, with $G/C_G(A)$ abelian. Then $A$ has a chain of $G$-submodules (of length $k \geq 1$)

$$0 = Z_0 < Z_1 < \ldots < Z_k = A$$

such that each factor $A_i := Z_i/Z_{i-1}$ is a very nice $G$-module. Moreover each $Z_i$ is definably closed in $G$.

**Proof.** The hypothesis $Z_f(G) = 1$ implies that $A$ is a nice $G$-module. Let $C_1$ be maximal among centralizers in $G$ of non-zero elements of $A$ and set $Z_1 = C_A(C_1)$. Lemma 4.7 shows that $Z_1$ is very nice, and that if $Z_1 < A$ then $A/Z_1$ is nice. Note that $Z_1$ is definably closed (with parameters) because $C_1$ is finitely generated.

Now $A/Z_1$ is torsion-free, and if $Z_1 < A$ we can iterate.

**Lemma 4.9** Let $G$ be a soluble profinite group and $F$ a nilpotent closed normal subgroup with $C_G(F) = Z(F)$. Assume that $F/F'F''$ is finite. If $Z(F)$ is a pro-$p$ group then so is $C_G(F/F'F'')$.  

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**Proof.** The pro-$p'$ component of $F$ is normal but intersects $Z(F)$ trivially, so it is trivial. Thus $F$ is a pro-$p$ group. It follows that the Frattini subgroup of $F$ is $F'/F''$, so $F$ is finitely generated and $F'/F''$ is closed ([DDMS], Cor. 1.20). Now [DDMS], Prop. 5.5 shows that $C_G(F/F''p)/C_G(F)$ is a pro-$p$ group, and the result follows. 

Now we can complete the

**Proof of Proposition 4.4** For simplicity in the following discussion, I will omit various parameters; it should be clear where these are needed.

$G$ is a torsion-free soluble pro-$p$ group of finite rank, $Z_f(G) = 1$, and $G/F$ is infinite and abelian where

$$F = \text{Fit}(G) = \phi_1(G)$$

is definably closed (Proposition 4.3). Set $A = Z(F)$; then $A$ is definable, and $A \neq 1$ (easy exercise). The condition $Z_f(G) = 1$ implies that $A$ is nice as a $G$ module.

Say $F$ is nilpotent of class $c$. Note that $|F/F''p| = p^d$ for some $d \leq \text{rk}(G)$, so also $G/C_G(F/F''p)$ is finite, of order $p^f$ say.

Now apply Lemma 4.8 to obtain a chain (5) with each $Z_i$ definable by a formula $\eta_i$, and each factor $A_i := Z_i/Z_{i-1}$ a very nice $G$-module. Since $G$ is pro-$p$, the following holds for each $i$:

$$pA_i + A_i(G - 1) \neq A_i.$$  

Thus $G$ satisfies a sentence $\alpha_i$ asserting, for any group $\widetilde{G}$, that $\widetilde{A}_i := \eta_i(\widetilde{G})/\eta_{i-1}(\widetilde{G})$ is a very nice $\widetilde{G}$-module and that (3) holds with $\widetilde{A}_i$ for $A$ and $\widetilde{G}$ for $X$.

Now let $\chi$ be the conjunction of $\alpha_1 \land \ldots \land \alpha_k$ with sentences asserting the following for a group $\widetilde{G}$, with $\widetilde{F} = \phi_1(\widetilde{G})$, $\widetilde{Z} = Z(\widetilde{F})$, $\widetilde{Z}_i = \eta_i(\widetilde{G})$ :

- $\widetilde{F} \triangleleft \widetilde{G}$ and $\widetilde{F}$ is nilpotent of class at most $c$
- $\widetilde{G}/\widetilde{F}$ is abelian and $C_{\widetilde{G}}(\widetilde{F}) = \widetilde{Z}$
- $|\widetilde{F}/\widetilde{F}'\widetilde{p}| = p^d$ and $|\widetilde{G}/C_{\widetilde{G}}(\widetilde{F}/\widetilde{F}'\widetilde{p})| = p^f$
- $0 < \widetilde{Z}_1 < \ldots < \widetilde{Z}_k = \widetilde{Z}$.

Suppose that $\widetilde{G}$ is a profinite group satisfying $\chi$, and $\widetilde{F}$, $\widetilde{Z}$, $\widetilde{Z}_i$ are as defined above. Then $\widetilde{F}$ is a closed nilpotent normal subgroup of $\widetilde{G}$ with $\widetilde{G}/\widetilde{F}$ is abelian and $C_{\widetilde{G}}(\widetilde{F}) = Z(\widetilde{F})$. Lemma 4.6 with $\alpha_i$ shows that $\widetilde{A}_i := \widetilde{Z}_i/\widetilde{Z}_{i-1}$ is pro-$p$, for each $i$. It follows that $\widetilde{Z}$ is a pro-$p$ group. Then Lemma 4.9 shows that $C_{\widetilde{G}}(\widetilde{F}/\widetilde{F}'\widetilde{p})$ is pro-$p$, and it follows that $G$ is a pro-$p$ group. 

**Corollary 4.10** Let $G$ be a torsion-free soluble $C_{\pi}$ group of finite rank, where $\pi$ is finite. Assume that $G/\text{Fit}(G)$ is infinite and abelian, and that $Z_f(G) = 1$. Then $G$ satisfies a $C_{\pi}$-sentence.

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Proof. We have $G = G_1 \times \cdots \times G_k$ where $G_i \neq 1$ is the Sylow pro-$p_i$ subgroup of $G$ and $\pi(G) = \{p_1, \ldots, p_k\} \subseteq \pi$; evidently $\text{Fit}(G) = \text{Fit}(G_1) \times \cdots \times \text{Fit}(G_k)$ and $Z_f(G_i) = 1$ for each $i$. Suppose that $G_i/\text{Fit}(G_i)$ is finite, for some $i$. Then $Z(\text{Fit}(G_i)) \leq Z_f(G_i) = 1$; this implies that $\text{Fit}(G_i) = 1$ and hence that $G_i = 1$, a contradiction.

Thus each $G_i/\text{Fit}(G_i)$ is infinite and abelian. Applying Proposition 4.4 we find for each $i$ a $C_{[p, 1]}$-sentence $\chi$, satisfied by $G_i$. Since $Z(G) \leq Z_f(G) = 1$ the subgroup $G_i$ is definable in $G$ as the centralizer of $\prod_{j \neq i} G_j$ (since $G$ has finite rank, every centralizer is definable). Say $G_i = \kappa_i(G)$. Thus $G$ satisfies a sentence $\chi$ which asserts (a) $G$ is the direct product of the $\kappa_i(G)$ and (b) for each $i$ the group $\kappa_i(G)$ satisfies $\chi_i$. Any profinite group satisfying $\chi$ is then in $C_\pi$.

To prove Theorem 5.1 in full generality we make some more reductions.

From now on we assume that $G$ is a soluble $C_\pi$ group of finite rank, and that every open subgroup of $G$ satisfies the OS condition. We shall prove that $G$ satisfies some $C_\pi$-sentence.

In view of Lemma 4.2 (2), we may reduce $G$ by any open normal subgroup. Since $G$ is virtually torsion-free ([DDMS], Cor. 4.3) and virtually nilpotent-by-abelian, we may assume henceforth that $G$ is torsion-free and nilpotent-by-abelian.

Suppose now that $F := \text{Fit}(G)$ is open in $G$. Then $F$ is nilpotent and satisfies the OS condition. The proof of Theorem 5.16 of [NST] now shows that $F$ satisfies a $C_\pi$-sentence $\chi_1$, and again we are done by Lemma 4.2 (2). (The theorem in question also assumes that $F$ has an $L$-presentation, and asserts that then $F$ is FA in profinite groups; but the $L$-presentation is not used for the weaker assertion just quoted.)

From now on, we may therefore assume that $G$ is torsion-free and that $G/\text{Fit}(G)$ is abelian and infinite.

The FC-centre $Z_f(G)$ of $G$ was defined above. Put

$$G_1 = \lambda(G) := C_G(Z_f(G)),$$

$$Z_1 = Z(G_1).$$

Since $G$ has finite rank, $G_1$ is open in $G$ and so $Z_1 = Z(Z_f(G)).$

For $i > 0$ set

$$\frac{G_i}{Z_{i-1}} := \lambda\left(\frac{G_{i-1}}{Z_{i-1}}\right),$$

$$\frac{Z_i}{Z_{i-1}} := Z\left(\frac{G_i}{Z_{i-1}}\right).$$

For some finite $n$ we have $Z_{n+1} = Z_n$; set $Z^*(G) = Z_n$ and $G^* = G_n$. Then $G^* \lhd_0 G$ and

$$Z_f(G^*/Z^*(G)) = 1,$$

$$Z^*(G) = \zeta_n(G^*)$$

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This implies that for each \( x \in \text{r} \) where \( Z \) is nilpotent profinite group \( G \)

\[
\text{OS condition for } H \implies \text{centralizes } Q \leq \pi \text{ is a pro-group.}
\]

\[
\text{Remark As open subgroups of } G \text{ are definable, each } G_i \text{ and each } Z_i \text{ is definable, in particular } Z^*(G) \text{ and } G^* \text{ are definable, indeed definably closed. (Not uniformly: the definition depends on } G)\]

**Lemma 4.11 ([NST], Lemma 5.17)** There is a sentence \( \psi_\pi \) such that for any nilpotent profinite group \( N \),

\[
Z(N) \in C_\pi \implies N \models \psi_\pi \implies N/Z(N) \in C_\pi.
\]

Now we are ready to complete the proof of Theorem 4.1 In view of Lemma 4.2, it will suffice to show that \( G^* \) satisfies some \( C_\pi \)-sentence \( \chi \).

Set \( H = G^* \) and \( Y = Z^*(G) \); then \( Z_\pi(H/Y) = 1 \). Set \( F = \text{Fit}(H) \); then \( F = H \cap \text{Fit}(G) \geq Y \) and \( F/Y = \text{Fit}(H/Y) \), because \( Y = \zeta_n(H) \). It is easy to see that \( H/Y \) then satisfies the hypotheses of Corollary 4.10 consequently \( H/Y \models \beta \) for some \( C_\pi \)-sentence \( \beta \).

The OS condition for \( H \) implies that \( Z(H)^m \subseteq H' \) for some \( \pi \)-number \( m \).
This implies that for each \( x \in Z(H) \), \( x^m \) is a product of \( r \) commutators in \( H \), where \( r = \text{rk}(G) \).

Now \( F = \phi_1(H) \) and \( Y = \eta(H) \) are definably closed normal subgroups of \( H \).

The group \( H \) satisfies a sentence \( \chi \) which asserts the following for a group \( \tilde{H} \):

- \( x \in Z(\tilde{H}) \implies x^m = \prod_{i=1}^c \left[u_i, v_i\right] \) for some \( u_i, v_i \in \tilde{H} \)
- \( \tilde{F} = \phi_1(\tilde{H}) \triangleleft \tilde{H} \) and \( \tilde{F} \) is nilpotent of class \( c \)
- \( \tilde{F} \models \psi_\pi \)
- \( \tilde{Y} = \eta(\tilde{H}) \triangleleft \tilde{H} \), \( \tilde{Y} \subseteq \tilde{F} \) and \( [\tilde{H}, \tilde{Y}] = 1 \)
- \( \tilde{H}/\tilde{Y} \models \beta \).

Now suppose that \( \tilde{H} \) is a profinite group satisfying \( \chi \). We show that \( \tilde{H} \in C_\pi \).

Define \( \tilde{F} \) and \( \tilde{Y} \) as above. These are both closed normal subgroups; also \( \tilde{H}/\tilde{Y} \in C_\pi \) because of \( \beta \), and \( \tilde{F}/Z(\tilde{F}) \in C_\pi \) because of \( \psi_\pi \). Thus \( \tilde{H}/(\tilde{Y} \cap Z(\tilde{F})) \) is a pro-\( \pi \) group.

I claim that \( \tilde{H} \) is pronilpotent. Suppose \( N \triangleleft_o \tilde{H} \). Then \( \tilde{H}/N\tilde{Y} \) is nilpotent and \( N\tilde{Y}/N \leq \zeta_n(\tilde{H}/N) \), so \( \tilde{H}/N \) is nilpotent, and the claim follows.

Let \( Q \) be a Sylow pro-\( q \) subgroup of \( \tilde{H} \) where \( q \notin \pi \) is a prime. Then \( Q \leq \tilde{Y} \cap Z(\tilde{F}) \). The \( C_\pi \) group \( \tilde{H}/F \) acts nilpotently on \( Q \), hence by coprime action it centralizes \( Q \). Hence \( Q \leq Z(\tilde{H}) \), and as \( Q = Q^m \) it follows that \( Q \leq \tilde{H}' \). As \( \tilde{H} \) is pronilpotent it follows that \( Q \cap \tilde{H}' = Q' \), so \( Q = Q' \).

It follows that \( Q = 1 \). Thus \( \tilde{H} \in C_\pi \).
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