A new result on the uniqueness of the CMC foliation in asymptotically flat manifold

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Abstract

In this article we consider the uniqueness of stable constant mean curvature spheres in asymptotically flat 3-manifolds with more general asymptotics. We remove the radius condition in my original article [7].

1 Introduction

In Generality Relativity we usually study the asymptotically flat 3-manifolds. It can be considered as the initial data set of the Einstein Equation. To study the geometry of such manifolds is also interesting and useful. In 1996 Huisken and Yau proved in [5] that in the asymptotically Schwarzschild manifold with positive mass, there exists a foliation by strictly stable constant mean curvature(CMC) spheres. They also use this foliation to defined the center of mass. The uniqueness of such foliation is a harder problem. Huisken and Yau proved that for $1/2 < q \leq 1$, stable CMC sphere outside $B_H - r(0)$ is unique, where $H$ is the constant mean curvature of the surface. In 2002, Jie Qing and Gang Tian removed this radius condition and proved a sharper uniqueness theorem in [8]. They found a scaling invariant integral to detect the positive mass. To calculate this integral they blow down the constant mean curvature spheres in three differential scales and use some technique from harmonic maps to deal with the intermediate part. Then Lan-hsuan Huang considered in [4] the general asymptotically flat manifolds with Regge-Teitelboim condition. She proved a similar result as Huisken and Yan. Her uniqueness result also needs radius condition of the form $r_1 \leq C_1 r_0^{\frac{q}{2}}$ for some $a$ satisfying $\frac{5-a}{2(2+q)} < a \leq 1$. In recent papers [2, 3], Eichmair and Metzger considered the existence and uniqueness of isoperimetric surfaces in a kind of asymptotically flat manifolds which is $C^0$ asymptotic to Schwarzschild(for uniqueness they require more smoothness). In [7] I studied the uniqueness problem in $(m, k, \varepsilon)$-AF-RT manifold which requires the manifold to be close to asymptotically Schwarzschild manifold in some weak sense and under the weaker radius condition $\log(r_1) \leq C r_0^{1/4}$ I proved the uniqueness of the stable CMC spheres outside some sufficiently large compact set. In this article, we remove both the radius condition and the
condition that the manifold being close to Schwarzschild. First we give some main definitions.

A three-manifold \( M \) with a Riemannian metric \( g \) and a two-tensor \( K \) is called an initial data set \((M, g, K)\) if \( g \) and \( K \) satisfy the constraint equations

\[
R_g - |K|^2_g + (tr_g(K))^2 = 16\pi\rho \\
\text{div}_g(K) - d(tr_g(K)) = 8\pi J
\]

where \( R_g \) is the scalar curvature of the metric \( g \), \( tr_g(K) \) denotes \( g^{ij}K_{ij} \), \( \rho \) is the observed energy density, and \( J \) is the observed momentum density.

In this paper we consider asymptotically flat manifold of the following kind:

**Definition 1.** We say \((M, g, K)\) is asymptotically flat (AF) if it is an initial data set, and there is a compact subset \( \tilde{K} \subset M \) such that \( M \setminus \tilde{K} \) is diffeomorphic to \( \mathbb{R}^3 \setminus B_1(0) \) and there exists coordinate \( \{x^i\} \) such that

\[
g_{ij}(x) = \delta_{ij} + h_{ij}(x)
\]

\[
h_{ij}(x) = O_5(|x|^{-1}) \quad K_{ij}(x) = O_1(|x|^{-2})
\]

Also, \( \rho \) and \( J \) satisfy

\[
\rho(x) = O(|x|^{-4}) \quad J(x) = O(|x|^{-4})
\]

Here, \( f = O_k(|x|^{-q}) \) means \( \partial^l f = O(|x|^{-l-q}) \) for \( l = 0, \cdots, k \).

\( M \setminus \tilde{K} \) is called an end of this asymptotically flat manifold. Here we only consider the asymptotically flat manifolds with one end.

We can define mass for this end as:

\[
m = \lim_{r \to \infty} \frac{1}{16\pi} \int_{|x|=r} (h_{ij,j} - h_{jj,i}) v^i_g d\mu_g,
\]

where \( v_g \) and \( d\mu_g \) are the unit normal vector and volume form with respect to the metric \( g \). From [1] we know the mass is well defined if the scalar curvature \( R_g \) is \( L^1 \) integrable. From the constraint equation we have \( R_g \) decays like \( r^{-4} \) which is in \( L^1 \) so the mass is well defined.

For the definition of center of mass we introduce the Regge-Teitelboim(RT) condition [9]:

**Definition 2.** We say \((M, g, K)\) is asymptotically flat satisfying the Regge-Teitelboim condition (AF-RT) if it is AF, and \( g, K \) satisfy these asymptotically even/odd conditions

\[
h_{ij}^{\text{odd}}(x) = O_2(|x|^{-1}) \quad K_{ij}^{\text{even}}(x) = O_1(|x|^{-3})
\]
Also, $\rho$ and $J$ satisfy

$$
\rho^{\text{odd}}(x) = O(|x|^{-5}) \quad J^{\text{odd}}(x) = O(|x|^{-5})
$$

where $f^{\text{odd}}(x) = f(x) - f(-x)$ and $f^{\text{even}}(x) = f(x) + f(-x)$.

For (AF-RT) manifolds, the center of mass $C$ is defined as

$$
C^{\alpha} = \frac{1}{16\pi m} \lim_{r \to \infty} \left( \int_{|x|=r} x^\alpha (h_{ij,i} - h_{ii,j}) v_j^\alpha d\mu_g - \int_{|x|=r} (h_{i\alpha} v_i^\alpha - h_{i\alpha} v_{\alpha}^i) d\mu_g \right). \quad (7)
$$

From [4], we know it is well defined.

Let $\Sigma$ be a constant mean curvature (CMC for short) surface. We say it is stable if the second variation operator has only non-negative eigen values when restricted to the functions with 0 mean value, i.e.

$$
\int_{\Sigma} (|A|^2 + \text{Ric}(v_g, v_g)) f^2 d\mu \leq \int_{\Sigma} |\nabla f|^2 d\mu \quad (8)
$$

for function $f$ with $\int_{\Sigma} f d\mu = 0$, where $A$ is the second fundamental form, and $\text{Ric}(v_g, v_g)$ is the Ricci curvature in the normal direction with respect to the metric $g$.

In this paper we prove the following uniqueness theorem:

**Theorem 1.1.** Suppose $(M, g, K)$ is AF-RT manifold with positive mass. Then there exists a compact set $\tilde{K}$, such that for any $H > 0$ sufficiently small, there is only one stable sphere with constant mean curvature $H$ that separates infinity from $\tilde{K}$.

This theorem follows form the key lemma below and Huang’s uniqueness theorem:

**Lemma 1.2.** If $(M, g)$ is asymptotically flat in the following sense

$$
g_{ij} = \delta_{ij} + h_{ij}(x),
$$

where $h_{ij}(x) = O_3(|x|^{-1})$ and $h^{\text{odd}}_{ij}(x) = O_2(|x|^{-2})$. And the scalar curvature $R_g$ is $L^1$ integrable. Then if the mass is positive. For any sequence of stable CMC spheres $\Sigma_n$ which separate infinity from the compact part, if

$$
\lim_{n \to \infty} r_0(\Sigma_n) = \infty
$$

then there exist some constant $C$ and some sufficiently large compact set $T$ such that for any $\Sigma_n$ outside $T$ we have $r_1(\Sigma_n)/r_0(\Sigma_n) \leq C$. Where

$$
r_0(\Sigma_n) = \inf\{|x| : x \in \Sigma_n\} \\
r_1(\Sigma_n) = \sup\{|x| : x \in \Sigma_n\}
$$
Now we state the main idea of this article. To detect the positive mass, we follow the idea of Qing and Tian [8], using an integral:

\[ \int_{\Sigma} (H - H_e) < v_e, b > d\mu_e, \]

where \( H \) is the constant mean curvature of the sphere \( \Sigma \), \( H_e \) is the mean curvature in Euclidean metric, \( v_e \) is the out-point unit normal vector of \( \Sigma \) and \( b \) is some constant vector to be chosen later. In [7] I used harmonic coordinates to calculate this integral which needs the original metric to be close to Schwarzschild manifold. Under the radius condition \( \log(r_1) \leq Cr_0^{1/4} \), I proved that the center of the sphere can not go very far away after suitably blowing-down which is sufficient to prove the uniqueness. In this article, I find a direct way to calculate this integral in Chapter 5 which does not need the manifold to be close to Schwarzschild. Also we find a better estimate on the second fundamental form, Lemma 4.8 which removes the radius condition.

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2 Curvature estimate

All the curvature estimates in my original paper [7] are valid in this case if we require \( q = 1 \) in that paper. We state the results directly.

Lemma 2.1. Suppose \( \Sigma \) is a stable constant mean curvature sphere in the asymptotically flat manifold. We have for \( r_0 \) sufficiently large \( \hat{A} \)

\[ \int_{\Sigma} |\hat{A}|^2 d\mu \leq Cr_0^{-1} \]

\[ H^2|\Sigma| \leq C \]

(9)

\[ \int_{\Sigma} H^2 d\mu = 16\pi + O(r_0^{-1}) \]

(10)

Lemma 2.2. Suppose \( \Sigma \) is a CMC sphere in an asymptotically flat end \( (R^3 \setminus B_1(0)) \), then we have:

\[ \int_{\Sigma} H_e^2 d\mu = 16\pi + O(r_0^{-1}), \]

where \( H_e \) denotes the mean curvature with respect to the background Euclidean metric.
Proof. From the following explicitly expression

\[
H - H_e = -f^{ik}h_{kl}f^{lj}A_{ij} + \frac{1}{2}H v^i v^j h_{ij} - f^{ij} v^l \nabla_i h_{jl} + \frac{1}{2} f^{ij} v^l \nabla_i h_{ij} \pm C |h| |\nabla h| \pm C |h|^2 |A| \tag{11}
\]

and the lemma above we can deduce the result.

\[
\int \int f^2 \, d\mu \leq C \left( \int |\nabla f| \, d\mu + \int H |f| \, d\mu \right)
\]

\textbf{Lemma 2.3.} Suppose \( \Sigma \) is a CMC sphere in the asymptotically flat end with \( r_0 \) sufficiently large and that \( \int_{\Sigma} H^2 \leq C \), then:

\[
\left( \int f^2 \, d\mu \right)^{1/2} \leq C \left( \int |\nabla f| \, d\mu + \int H |f| \, d\mu \right).
\]

\textbf{Lemma 2.4.} Suppose \( \Sigma \) is a CMC sphere in an asymptotically flat end with \( r_0(\Sigma) \) sufficiently large, then:

\[
C_1 H^{-1} \leq \text{diam}(\Sigma) \leq C_2 H^{-1},
\]

where the \( \text{diam}(\Sigma) \) denotes the diameter of \( \Sigma \) in the Euclidean space \( \mathbb{R}^3 \).

In particular, if the surface \( \Sigma \) separates the infinity from the compact part, then:

\[
C_1 H^{-1} \leq r_1(\Sigma) \leq C_2 H^{-1}.
\]

Then from Simons identity and Sobolev inequality Lemma 2.3 we have the following basic curvature estimate:

\textbf{Theorem 2.5.} Suppose that \( (\mathbb{R}^3 \setminus B_1(0), g) \) is an asymptotically flat end. Then there exist positive numbers \( \sigma_0, \delta_0 \) such that for any CMC surface in the end, which separates the infinity from the compact part, we have:

\[
|\hat{A}|^2(x) \leq C |x|^{-2} \int_{B_{\hat{r}_0}(x)} |\hat{A}|^2 \, d\mu + C |x|^{-4} \leq C |x|^{-2} r_0^{-1}
\]

\[
|\nabla \hat{A}|^2(x) \leq C |x|^{-2} \int_{B_{\hat{r}_0}(x)} |\nabla \hat{A}|^2 \, d\mu + C |x|^{-6} \leq C |x|^{-4} r_0^{-1/2}
\]

3 Blow down analysis

Now we have the three blow-downs as usual. First we consider

\[
\hat{\Sigma} = \frac{1}{2} H \Sigma = \{ \frac{1}{2} H x : x \in \Sigma \}
\]
Suppose that there is a sequence of constant mean curvature surfaces \( \{ \Sigma_i \} \) such that
\[
\lim_{i \to \infty} r_0(\Sigma_i) = \infty,
\]
we have known that
\[
\lim_{i \to \infty} \int_{\Sigma_i} H^2 d\mu = 16\pi.
\]

Then from L. Simon [10] Theorem 3.1, we have

**Lemma 3.1.** Suppose that \( \{ \Sigma_i \} \) is a sequence of constant mean curvature surfaces in a given asymptotically flat end \((\mathbb{R}^3 \setminus B_1(0), g)\) and that
\[
\lim_{i \to \infty} r_0(\Sigma_i) = \infty.
\]
And suppose that \( \Sigma_i \) separates the infinity from the compact part. Then, there is a subsequence of \( \{ \Sigma_i \} \) which converges in Gromov-Hausdorff distance to a round sphere \( S_1^2(a) \) of radius 1 and centered at \( a \in \mathbb{R}^3 \). Moreover, the convergence is in \( C^{2,\alpha} \) sense away from the origin.

Then, we use a smaller scale \( r_0 \) to blow down the surface
\[
\widetilde{\Sigma} = r_0(\Sigma)^{-1} = \{ r_0^{-1} x : x \in \Sigma \}.
\]

**Lemma 3.2.** Suppose that \( \{ \Sigma_i \} \) is a sequence of constant mean curvature surfaces in a given asymptotically flat end \((\mathbb{R}^3 \setminus B_1(0), g)\) and that
\[
\lim_{i \to \infty} r_0(\Sigma_i) = \infty.
\]
And suppose that
\[
\lim_{i \to \infty} r_0(\Sigma_i)H(\Sigma_i) = 0.
\]
Then there is a subsequence of \( \{ \Sigma_i \} \) converges to a 2-plane at distance 1 from the origin. Moreover the convergence is in \( C^{2,\alpha} \) in any compact set of \( \mathbb{R}^3 \).

We must understand the behavior of the surfaces \( \Sigma_i \) in the scales between \( r_0(\Sigma_i) \) and \( H^{-1}(\Sigma_i) \). We consider the scale \( r_i \) such that
\[
\lim_{i \to \infty} \frac{r_0(\Sigma_i)}{r_i} = 0 \quad \lim_{i \to \infty} r_i H(\Sigma_i) = 0
\]
and blow down the surfaces
\[
\Sigma_i = r_i^{-1} \Sigma = \{ r_i^{-1} x : x \in \Sigma \}.
\]
Lemma 3.3. Suppose that $\{\Sigma_i\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ and that

$$\lim_{i \to \infty} r_0(\Sigma_i) = \infty$$

And suppose that $r_i$ are such that

$$\lim_{i \to \infty} \frac{r_0(\Sigma_i)}{r_i} = 0 \quad \lim_{i \to \infty} r_i H(\Sigma_i) = 0$$

Then there is a subsequence of $\{\Sigma_i\}$ converges to a 2-plane at the origin in Gromov-Hausdorff distance. Moreover the convergence is $C^{2,\alpha}$ in any compact subset away from the origin.

4 Asymptotically analysis

In this chapter, we mainly follow the same idea as [8] or [7]. However in the end we will derive a new estimate on the second fundamental form Lemma 4.8 which makes the uniqueness possible. First let us revise the properties of harmonic function on a column. Denote

$$\|u\|_{2,1} = \int_{[(i-1)L_i L] \times S^1} |u|^2 + |\nabla u|^2 dt d\theta,$$

where $(t, \theta)$ is the standard column coordinate.

Lemma 4.1. Suppose $u \in W^{1,2}(N, R^k)$ satisfies

$$\Delta u + A \cdot \nabla u + B \cdot u = h$$

in $N$, where $N = [0, 3L] \times S^1$. And suppose that $L$ is given and large. Then there exists a positive number $\delta_0$ such that if

$$\|h\|_{L^2(N)} \leq \delta_0 \max_{1 \leq i \leq 3} \|u\|_{1,i}$$

and

$$\|A\|_{L^\infty(N)} \leq \delta_0 \quad \|B\|_{L^\infty(N)} \leq \delta_0$$

then,

(a) $\|u\|_{1,3} \leq e^{-\frac{1}{2}L} \|u\|_{1,2}$ implies $\|u\|_{1,2} < e^{-\frac{1}{2}L} \|u\|_{1,1}$

(b) $\|u\|_{1,1} \leq e^{-\frac{1}{2}L} \|u\|_{1,2}$ implies $\|u\|_{1,2} < e^{-\frac{1}{2}L} \|u\|_{1,3}$

(c) If both $\int_{L \times S^1} u d\theta$ and $\int_{L \times S^1} u d\theta \leq \delta_0 \max_{1 \leq i \leq 3} \|u\|_{1,i}$, then either $\|u\|_{1,2} < e^{-\frac{1}{2}L} \|u\|_{1,1}$ or $\|u\|_{1,2} < e^{-\frac{1}{2}L} \|u\|_{1,3}$
Given a surface \( \Sigma \) in \( \mathbb{R}^3 \). Recall
\[
\Delta_{\varepsilon} v + |\nabla_{\varepsilon} v|^2 v = \nabla_{\varepsilon} H_{\varepsilon},
\]
where \( v \) is the Gauss map from \( \Sigma \to S^2 \). For the constant mean curvature spheres in the asymptotically flat end \( (\mathbb{R}^3 \setminus B_1(0), g) \), we have

**Lemma 4.2.**
\[
|\nabla_{\varepsilon} H_{\varepsilon}|(x) \leq C|x|^{-3}
\]

**Proof.** Because of the uniform equivalence of the metric \( g \) and the euclidean metric, we can prove:
\[
|\nabla H_{\varepsilon}|(x) \leq C|x|^{-3}
\]
instead. From the expression of \( H - H_{\varepsilon} \), we have
\[
|\nabla H_{\varepsilon}| \leq |\nabla h_{ij}| |A| + |h_{ij}| |A|^2 + |h_{ij}| |\nabla A_{ij}| + H |A| |h_{ij}| + H |\nabla h_{ij}|
\]
\[
+ |A| |\nabla h_{ij}| + |\nabla^2 h|
\]
\[
\leq C|x|^{-3}
\]
(24)

Suppose \( \Sigma \) is a CMC surface in the asymptotically flat end. Set
\[
A_{r_1,r_2} = \{ x \in \Sigma : r_1 \leq |x| \leq r_2 \}
\]
and \( A_{r_1,r_2}^0 \) stand for the standard annulus in \( \mathbb{R}^2 \). Consider the behavior of \( v \) on \( A_{Kr_0(\Sigma), sH^{-1}(\Sigma)} \) of \( \Sigma \) where \( K \) will be fixed large and \( s \) will be fixed small. The lemma below gives us a good coordinate on the surface.

**Lemma 4.3.** Suppose \( \Sigma \) is a constant mean curvature surface in a given asymptotically flat end \( (\mathbb{R}^3 \setminus B_1(0), g) \). Then, for any \( \varepsilon > 0 \) and \( L \) fixed and large, there are \( M, s \) and \( K \) such that, if \( r_0 \geq M \) and \( Kr_0(\Sigma) < r < sH^{-1}(\Sigma) \), then \( (r^{-1} A_{r, r+L}, r^{-2} g_{\varepsilon}) \) may be represented as \( (A_{r_1,r_2}^0, \tilde{\gamma}) \) and
\[
\|\tilde{\gamma} - |dx|^2\|_{C^1(A_{r_1,r_2}^0)} \leq \varepsilon.
\]
(25)

In other words, in the cylindrical coordinates \( (S^1 \times [\log r, L + \log r], \tilde{\gamma}) \)
\[
\|\tilde{\gamma} - (dt^2 + d\theta^2)\|_{C^1(S^1 \times [\log r, L + \log r])} \leq \varepsilon
\]
(26)

Now consider the cylindrical coordinates \((t, \theta)\) on \( (S^1 \times [\log Kr_0, \log sH^{-1}]) \), then the tension field
\[
|\tau(v)| = r^2 |
\]
\[
\nabla_{\varepsilon} H_{\varepsilon} | \leq C r^{-1}
\]
(27)
for \( t \in [\log Kr_0, \log sH^{-1}] \). Thus,
\[
\int_{S^1 \times [t, t+L]} |\tau(v)|^2 dtd\theta \leq C r^{-2}
\]
(28)
Let $I_i$ stand for $S^1 \times \{ \log K r_0 + (i-1)L, \log K r_0 + iL \}$, and $N_i$ stand for $I_{i-1} \cup I_i \cup I_{i+1}$. On $\Sigma_n$ we assume $\log(sH^{-1}) - \log(K r_0) = l_0 L$.

Now we get the energy decay by an argument which is a little different from that of [8]. Suppose there is only a map $\tilde{L}$ uniformly bounded in $v$ field $\tilde{R}$ which is close to the unit ball of $I$. Now the Gauss map $v_n : \Sigma_n \to S^2$ induces a map $\tilde{v}_n : B_1(0) \to S^2$. Note that the energy of $\tilde{v}$ will concentrate at the origin of $B_1(0)$ and the tension field $\tilde{\tau}$ of the map $\tilde{v}$ satisfies

$$|\tilde{\tau}| \leq C|x|^{-3}|x|^4(K r_0)^{-2} = C|x|(K r_0)^{-2} \leq \frac{(K r_0)^{-1}}{\sqrt{4s^2 e^{-2l_0 L} + r^2}}$$

where $r$ denotes the radius function of the unit ball. We notice that the tension field $\tilde{\tau}$ is not uniformly bounded in $L^p(B_1(0))$. But for any $p \in (1,2)$, it is uniformly bounded in $L^p(B_1(0))$. To use the $L^p$ theory of harmonic maps, we first find the weak limit of the map $\tilde{v}_n$ as $n$ tend to infinity. By Lemma 3.2 we can find a subsequence of $\tilde{v}_n$ (also denoted by $\tilde{v}_n$ ) which converges weakly in $W^{1,2}(B_1(0))$ to a constant map $\tilde{v}_0$ which is the unit normal vector of the limit plane of Lemma 3.2. Now we introduce Theorem 6.5.1 of [8].

**Theorem 4.4. (Theorem 6.5.1 of [8])** Let $M$ be a Riemannian surface without boundary. For any $p > 1$, assume that $\{u_i\} \subset W^{1,2}(M, S^{L-1})$ are such that the tension fields:

$$\tau(u_i) \equiv \Delta u_i + |\nabla u_i|^2 u_i$$

are bounded in $L^p(M)$. If $\{u_i\}$ converges to $u$ weakly in $W^{1,2}(M, S^{L-1})$, then there exist finitely many harmonic $S^2$'s, $\{\omega_j\}_{j=1}^{l_1}, \{a_i^j\}_{j=1}^{l_1} \subset M$, $\{\lambda_i^j\}_{j=1}^{l_1} \subset R_+$ such that

$$\lim_{i \to \infty} \|u_i - u - \sum_{j=1}^{l_1} \omega_j^i\|_{L^\infty(M)} = 0$$

and hence

$$\lim_{i \to \infty} \|u_i - u - \sum_{j=1}^{l_1} \omega_j^i\|_{W^{1,2}(M)} = 0$$

where

$$\omega_j^i(\cdot) = \omega_j(\cdot - \frac{a_j^i}{\lambda_j^i}) - \omega_j(\infty)$$

From the proof of this theorem we find the theorem holds also for $M = B_1(0)$ which is a Riemann surface with boundary. In the case we consider, there is only one bubble $\omega$ blown up at the origin. So from the theorem above we have:

$$\lim_{n \to \infty} \|\tilde{v}_n - \tilde{v}_0 - (\omega(\frac{\cdot}{s \cdot e^{-l_0 L}}) - \omega(\infty))\|_{L^\infty(B_1(0))} = 0.$$
So from Lemma 3.1, for $s$ sufficiently small we have:

$$\text{OSC}_{B_1(0) \setminus B_{e^{-i-1}L}(0)} \tilde{v}_n \leq \text{OSC}_{B_1(0) \setminus B_{e^{-i-1}L}(0)} \omega \left( \frac{\cdot}{s \cdot e^{-i-1}L} \right) + o(1)$$

can be arbitrarily small. So we have

**Lemma 4.5.** For any $\varepsilon > 0$, there is some $\delta > 0$ and $M > 0$ such that if $0 < s < \delta$ and $n > M$ we have

$$\text{OSC}_{\Sigma_n \cap B_{sH^{-1}}(0)} \leq \varepsilon.$$  

Now in the cylindrical coordinates $(t, \theta)$ on $(S^1 \times [\log(Kr_0, \log sH^{-1})])$, we consider the equation satisfied by $v_n - v_0$, where $v_0 = \tilde{v}_0$. If we denote the Laplacian and gradient in this coordinate as $\hat{\Delta} = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2}$ and $\hat{\nabla} = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right)$, then

$$\hat{\Delta}(v_n - v_0) + v_n \hat{\nabla}v_n \cdot \hat{\nabla}(v_n - v_0) = \tau$$

where $|\tau| \leq Ce^{-t}$. And

$$|v_n \hat{\nabla}v_n| \leq |\hat{\nabla}v_n| \leq C(s + r_0^{-1/2})$$

which can be very small. At last from the lemma above for any $1 \leq i \leq l_n$, we have

$$\int_{iL \times S^1} (v_n - v_0) d\theta \leq \varepsilon,$$

so we can use Lemma 4.1 to get the energy decay:

**Lemma 4.6.** For each $i \in [3, l_n - 2]$, there exists a geodesic $\gamma$ such that

$$\int_{I_i} |\hat{\nabla}v_n|^2 dt d\theta \leq C(e^{-iL} + e^{-(i-1)L})(s^2 + r_0^{-1}). \quad (29)$$

**Lemma 4.7.** Suppose that $\{\Sigma_n\}$ is a sequence of constant mean curvature surfaces in a given asymptotically flat end $(\mathbb{R}^3 \setminus B_1(0), g)$ and that

$$\lim_{i \to \infty} r_0(\Sigma_n) = \infty \quad (30)$$

And suppose that

$$\lim_{n \to \infty} r_0(\Sigma_n) H(\Sigma_n) = 0 \quad (31)$$

Then there exist a large number $K$, a small number $s$ and $n_0$ such that, when $n \geq n_0$,

$$\max_{I_i} |\hat{\nabla}v| \leq C(e^{-\frac{s}{2}L} + e^{-\frac{(n-1)}{2}L})(s + r_0^{-\frac{1}{2}}) \quad (32)$$

where

$$I_i = S^1 \times [\log(Kr_0(\Sigma_n)) + (i-1)L, \log(Kr_0(\Sigma_n)) + iL] \quad (33)$$

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and

\[ i \in [0, l_n] \quad \log(Kr_0(S_n)) + l_n L = \log(sH^{-1}(S_n)) \]  

(34)

From the Lemma above, we get the new estimate of the second fundamental form:

**Lemma 4.8.** If \( \Sigma \) is a stable CMC sphere in the asymptotically flat end, then the second fundamental form of \( \Sigma \) has the following estimate: For a point \( x \in (B_{Kr_0e^{(i+1)L}} \setminus B_{Kr_0e^{iL}}) \cap \Sigma \),

\[ |A(x)| \leq C|x|^{-1}(e^{-\frac{1}{2}L} + e^{-\frac{(l_n-i)L}{2}})(s + r_0^{-\frac{1}{2}}) \]

where \( sH^{-1} = Kr_0 \cdot e^{iL}. \)

**Proof.** Note that

\[ |A(x)| \leq C|\nabla v(x)| \leq C|x|^{-1} \sup_{I_i} |\nabla v| \leq C|x|^{-1}(e^{-\frac{1}{2}L} + e^{-\frac{(l_n-i)L}{2}})(s + r_0^{-\frac{1}{2}}) \]

\[ \square \]

**Corollary 4.9.** Assume the same condition as Proposition 4.7. Let \( v_n = v(p_n) \) for some \( p_n \in I_{\frac{1}{2}L} \). Then

\[ \sup_{I_i} |v - v_n| \leq C(e^{-\frac{1}{2}iL} + e^{-\frac{1}{2}(l_n-i)L})(s + r_0^{-\frac{1}{2}}) \]  

(35)

for \( i \in [0, \frac{1}{2}l_n] \)

\[ \sup_{I_i} |v - v_n| \leq C(e^{-\frac{1}{2}l_nL} + e^{-\frac{1}{2}(l_n-i)L})(s + r_0^{-\frac{1}{2}}) \]  

(36)

for \( i \in [\frac{1}{2}l_n, l_n] \)

5 Mass integral

In this section we consider the integral in a very different way compared with [7]. In that paper, we use the harmonic coordinates to reduce the integral to explicit form. But here we calculate it directly. We have a new estimate on the second fundamental form Lemma 4.8 so we can deal with the bad term in the integral.

Now by contradiction we assume that Lemma 4.7 were false. Then we could find a subsequence of stable CMC spheres \( \{S_n\} \) such that : \( S_n = \frac{1}{2}H \Sigma = \{\frac{1}{2}Hx : x \in S_n\} \) converges to some sphere \( S_1(a) \), for some unit vector \( a \). Then the origin lies on \( S_1(a) \). For \( b = -a \), we consider the integral:

\[ \int_{\Sigma} (H - H_e) < v_e \cdot b >_e d\mu_e = \int_{\Sigma} (-f^{ik}h_{kl}f^{ij}A_{ij} + \frac{1}{2}Hv_i^jv^jh_{ij} - f^{ij}v_i^j\nabla h_{ij} + \frac{1}{2}f^{ij}v_i^j\nabla h_{ij} \pm C|h|\nabla h \pm C|h|^2|A|) < v_e \cdot b >_e d\mu_e + O(r_0^{-1}), \]  

(37)
here $i, j$ ran from 1 to 3, and $f_{ij}$ is the restriction of $g_{ij}$. From

$$
\int_{\Sigma_n} -f_{ij} v^l (\nabla_i h_{jl}) v^m b^m d\mu_e
= \frac{1}{2} \int_{\Sigma_n} (f_{ij} h_{kji} - H v^i v^j h_{jl}) v^m b^m d\mu_e + \frac{1}{2} \int_{\Sigma_n} f_{ij} v^l h_{jl} A_{ik} f_{km} b^m d\mu_e
- \frac{1}{2} \int_{\Sigma_n} f_{ij} v^l (\nabla_i h_{jl}) v^m b^m d\mu_e,
\tag{38}
$$

we change the integral into:

$$
\int_{\Sigma_n} (H - H_e) < v_e \cdot b > e d\mu_e = \int_{\Sigma_n} -\frac{1}{2} f^{ik} h_{kl} f^{ij} A_{ij} v^m b^m + \frac{1}{2} f^{ij} v^l h_{jl} A_{ik} f_{km} b^m
- \frac{1}{2} f^{ij} v^l (\nabla_i h_{jl}) v^m b^m + \frac{1}{2} f^{ij} v^l h_{lj} v^m b^m d\mu_e + O(r_0^{-1})
\tag{39}
$$

Now for fixed $s$ sufficiently small and $K$ sufficiently large, we divide the integral into three parts: $\int_{\Sigma_n \cap B_{K\rho}^c}, \int_{\Sigma_n \cap (B_{sH}^c \setminus B_{K\rho}^c)}$, $\int_{\Sigma_n \cap (B_{sH}^c \setminus B_{K\rho}^c)}$.

For $\int_{\Sigma_n \cap B_{sH}^c \setminus B_{K\rho}^c}$ if we blow down $\Sigma_n$ by $H/2$, and denote $\tilde{\Sigma}_n = H\Sigma_n/2$, we have

$$
\int_{\Sigma_n \cap B_{sH}^c \setminus B_{K\rho}^c} -\frac{1}{2} h^r_{ij} A_{ij} v^m b^m + \frac{1}{2} h_{ij} A_{im} v^i v^m b^m - \frac{1}{2} v^l \partial_\alpha h_{ij} v^m b^m + \frac{1}{2} v^l \partial_\alpha h_{ij} v^m b^m d\mu_e
= \int_{\Sigma_n \cap B_{sH}^c/2} -\frac{1}{2} h^r_{ij} (\tilde{\Sigma}_n) v^m b^m + \frac{1}{2} h_{ij} (\tilde{\Sigma}_n) v^i v^m b^m
- \frac{1}{2} v^l \partial_\alpha h^r_{ij} (\tilde{\Sigma}_n) v^m b^m + \frac{1}{2} v^l \partial_\alpha h^r_{ij} (\tilde{\Sigma}_n) v^m b^m d\mu_e(\tilde{\Sigma}_n),
$$

where $h^r_{ij}(x) = r \cdot h_{ij}(rx)$. Now from Theorem 3.1 and the estimate of the second fundamental form we have for fixed $s$ small, as $n \to \infty$, $\tilde{\Sigma}_n \cap B_{sH}^c/2$ will converge in $C^{2,\alpha}$ sense to $S_1(a) \cap B_{sH/2}^c$. So we have $A_{ij}(\tilde{\Sigma}_n) \to f_{ij}(\tilde{\Sigma}_n)$ and $v^i \to x^i - a^i$. We know there is some constant $C$ such that

$$
|h^r_{ij}(x)|_{C^0} \leq C|x|^{-1}, |h^r_{ij,k}(x)|_{C^0} \leq C|x|^{-2}.
$$
Now for $\varepsilon > 0$ we can choose $n$ sufficiently large such that:

$$
\left| \int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} h_{ij}^{2/H} A_{ij}(\tilde{\Sigma}_n)v^m b^m + \frac{1}{2} h_{il}^{2/H} A_{im}(\tilde{\Sigma}_n)v^l b^m \\
- \frac{1}{2} v^l \partial_a h_{al}^{2/H} v^m b^m + \frac{1}{2} v^l \partial h_{al}^{2/H} v^m b^m \, d\mu_e(\tilde{\Sigma}_n) \\
- \int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} h_{aa}^{2/H} v^m b^m + \frac{1}{2} h_{al}^{2/H} v^l b^a \\
- \frac{1}{2} v^l \partial_a h_{al}^{2/H} (x^m - a^m)b^m + \frac{1}{2} v^l \partial h_{al}^{2/H} (x^m - a^m)b^m \, d\mu_e(\tilde{\Sigma}_n) \right| 
\leq \varepsilon/8,
$$

where the Greek indices ran from 1 to 2. By a simple argument we have:

$$
\int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} h_{ii}^{2/H} v^m b^m + \frac{1}{2} h_{il}^{2/H} v^l b^i \\
+ \frac{1}{2} v^l \partial h_{ii}^{2/H} (x^m - a^m)b^m \, d\mu_e(\tilde{\Sigma}_n) \\
= \int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} h_{ii}^{2/H} v^m b^m + \frac{1}{2} h_{il}^{2/H} v^l b^i - \frac{1}{2} v^l \partial h_{il}^{2/H} (x^m - a^m)b^m \\
+ \frac{1}{2} v^l \partial h_{il}^{2/H} (x^m - a^m)b^m \, d\mu_e(\tilde{\Sigma}_n) \\
= \int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} h_{ii}^{2/H} v^m b^m + \frac{1}{2} h_{il}^{2/H} v^l b^i - \frac{1}{2} v^l \partial h_{il}^{2/H} x^m b^m + \frac{1}{2} v^l \partial h_{ii}^{2/H} x^m b^m \, d\mu_e(\tilde{\Sigma}_n) \\
+ \int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} v^l \partial h_{ii}^{2/H} + \frac{1}{2} v^l \partial h_{il}^{2/H} \, d\mu_e(\tilde{\Sigma}_n).
$$

We denote the inside of $\tilde{\Sigma}_n$ by $int(\tilde{\Sigma}_n)$. Then by divergence formula we have

$$
\int_{\tilde{\Sigma}_n \cap B_{\varepsilon}/2} -\frac{1}{2} h_{ii}^{2/H} v^m b^m + \frac{1}{2} h_{il}^{2/H} v^l b^i \\
- \frac{1}{2} v^l \partial h_{ii}^{2/H} x^m b^m + \frac{1}{2} v^l \partial h_{il}^{2/H} x^m b^m \, d\mu_e(\tilde{\Sigma}_n) \\
= \int_{int(\tilde{\Sigma}_n) \cap \partial B_{\varepsilon}/2(0)} -\frac{1}{2} h_{ii}^{2/H} v^m b^m + \frac{1}{2} h_{il}^{2/H} v^l b^i \\
- \frac{1}{2} v^l \partial h_{ii}^{2/H} x^m b^m + \frac{1}{2} v^l \partial h_{il}^{2/H} x^m b^m \, d\mu_e \\
- \frac{1}{2} \int_{int(\tilde{\Sigma}_n) \cap B_{\varepsilon}/2(0)} (h_{ii}^{2/H} - h_{ii,il}^{2/H})(x^m b^m) \, dv.
$$

Note that the scalar curvature $R_g$ is $L^1$ integrable and $R_g = h_{ij,ij} - h_{ii,ij} + O(|x|^{-4})$. So $h_{ij,ij} - h_{ii,ij}$ is $L^1$ integrable. Define

$$
F(r) = \int_{M \cap B_{\varepsilon}/2(0)} |h_{ij,ij} - h_{ii,ij}| \, d\mu_e.
$$
We have
\[ \lim_{r \to \infty} F(r) = 0. \]

So we have
\[
\left| \int_{\text{int}(\Sigma_n) \cap B_{s/2}(0)} (h_{ii,il}^{2/H} - h_{ii,il}^{2/H})(x^m b^m) dv \right| \leq C \int_{\text{int}(\Sigma_n) \cap B_{s/2}(0)} |h_{ii,il}^{2/H} - h_{ii,il}^{2/H}| dv
\]
\[
= C \int_{\text{int}(\Sigma_n) \cap B_{s/2}(0)^{1+1}} |h_{ii,il} - h_{ii,il}| dv
\]
\[
\leq CF(sH^{-1}).
\]

So
\[
\lim_{n \to \infty} \int_{\text{int}(\Sigma_n) \cap B_{s/2}(0)} (h_{ii,il}^{2/H} - h_{ii,il}^{2/H})(x^m b^m) dv = 0.
\]

And
\[
\left| \int_{\text{int}(\Sigma_n) \cap \partial B_{s/2}(0)} - \frac{1}{2} \nu^l \partial_i h_{il}^{2/H} x^m b^m + \frac{1}{2} \nu^l \partial_i h_{ii,il}^{2/H} x^m b^m d\mu_e \right| \leq Cs
\]

which is small when \( s \) is small.

For the integral
\[
\int_{\tilde{\Sigma}_n \cap B_{s/2}(0)} \frac{1}{2} \nu^l \partial_i h_{il}^{2/H} + \frac{1}{2} \nu^l \partial_i h_{ii,il}^{2/H} d\mu_e(\tilde{\Sigma}_n),
\]
by divergence formula
\[
\int_{\tilde{\Sigma}_n \cap B_{s/2}(0)} - \frac{1}{2} \nu^l \partial_i h_{il}^{2/H} + \frac{1}{2} \nu^l \partial_i h_{ii,il}^{2/H} d\mu_e(\tilde{\Sigma}_n)
\]
\[
= \int_{\text{int}(\Sigma_n) \cap \partial B_{s/2}(0)} - \frac{1}{2} \nu^l \partial_i h_{il}^{2/H} + \frac{1}{2} \nu^l \partial_i h_{ii,il}^{2/H} d\mu_e(\tilde{\Sigma}_n)
\]
\[
- \frac{1}{2} \int_{\text{int}(\Sigma_n) \cap B_{s/2}(0)} (h_{ii,il}^{2/H} - h_{ii,il}^{2/H}) dv.
\]

The second integral on the right hand side converges to \( 0 \) as before. The first integral on the right hand side is close to the mass integral on half sphere when \( s \) is small. From RT condition it is half of the mass integral, which converges to \( 4\pi m \), where \( m \) is the mass of the end. (Note the limit is taken first as \( n \to \infty \), then \( s \to 0 \).)
Now we deal with the integral on $\Sigma_n \cap B_{K_0}$.

$$\int_{\Sigma_n \cap B_{K_0}} \frac{1}{2} h_{ij} A_{ij} v^m b^m + \frac{1}{2} h_{il} A_{im} v^i b^m - \frac{1}{2} v^l \partial_i h_{il} v^m b^m + \frac{1}{2} v^l \partial_i h_{il} v^m b^m \, d\mu_c$$

$$= \int_{\Sigma_n \cap B_K} -\frac{1}{2} h_{ij}^0 A_{ij} (\hat{\Sigma}_n) v^m b^m + \frac{1}{2} h_{il}^0 A_{im} (\hat{\Sigma}_n) v^i b^m$$

$$- \frac{1}{2} v^l \partial_i h_{il}^0 v^m b^m + \frac{1}{2} v^l \partial_i h_{il}^0 v^m b^m \, d\mu_c(\hat{\Sigma}_n),$$

where $\hat{\Sigma}_n = r_0^{-1} \Sigma_n$. From Theorem 3.2 and Lemma 4.3, $A_{ij}(\hat{\Sigma}_n) \to 0$ and $v^m \to b^m$. $h_{ij}^0$ and $h_{ij}^0$ is bounded. So the integral above converges to

$$\int_{\hat{\Sigma}_n} -\frac{1}{2} v^l \partial_i h_{il}^0 + \frac{1}{2} v^l \partial_i h_{il}^0 \, d\mu_c(\hat{\Sigma}_n).$$

Again by divergence formula

$$\int_{\Sigma_n \cap B_K} -\frac{1}{2} v^l \partial_i h_{il}^0 + \frac{1}{2} v^l \partial_i h_{il}^0 \, d\mu_c(\hat{\Sigma}_n)$$

$$= \int_{\partial B_K \setminus int(\hat{\Sigma}_n)} -\frac{1}{2} v^l \partial_i h_{il}^0 + \frac{1}{2} v^l \partial_i h_{il}^0 \, d\mu_c - \frac{1}{2} \int_{B_K \setminus int(\hat{\Sigma}_n)} (h_{il}^0 - h_{il}^0) \, d\mu_c.$$

The second integral on the right hand side converges to 0. The first integral on the right hand side is close to the mass integral on half sphere when $K$ is large. So the integral converges to $-4\pi m$. (Note the limit is taken first as $n \to \infty$, then $K \to \infty$.)

At last we deal with the intermediate part

$$\int_{\Sigma_n \cap (B_{sH-1} \setminus B_{K_0})} -\frac{1}{2} h_{ij} A_{ij} v^m b^m + \frac{1}{2} h_{il} A_{im} v^i b^m - \frac{1}{2} v^l \partial_i h_{il} v^m b^m + \frac{1}{2} v^l \partial_i h_{il} v^m b^m \, d\mu_c.$$

First from the new estimate for the second fundamental form Lemma 4.8 we have

$$|\int_{\Sigma_n \cap (B_{sH-1} \setminus B_{K_0})} -\frac{1}{2} h_{ij} A_{ij} v^m b^m + \frac{1}{2} h_{il} A_{im} v^i b^m|$$

$$\leq \sum_{i=1}^{l_n} \int_{\Sigma_n \cap (B_{K_{\infty}^{(i-1)L}} \setminus B_{K_{\infty}^{(i+1)L}})} C|x|^{-2} (e^{-\frac{l}{2L}} + e^{-\frac{l}{2L}} (r_0^{-\frac{l}{2L}} + s)) \, d\mu_c$$

$$\leq C(r_0^{-\frac{l}{2L}} + s).$$

For the second part

$$\int_{\Sigma_n \cap (B_{sH-1} \setminus B_{K_0})} -\frac{1}{2} v^l \partial_i h_{il} v^m b^m + \frac{1}{2} v^l \partial_i h_{il} v^m b^m \, d\mu_c.$$

For each $n$ we can choose $p_n \in \Sigma_n \cap B_{K_{\infty}^{(i+1)L}} \setminus B_{K_{\infty}^{(i-1)L}}$ such that for
$v_n = v(p_n)$ Corollary 4.9 holds. So we have

$$\int_{\Sigma_n \cap (B_{rH^{-1}} \setminus B_{K\rho_0})} - \frac{1}{2} v^l \partial_l h_{il} v^m b^m + \frac{1}{2} v^l \partial_l h_{ii} v^m b^m d\mu_e$$

$$= \int_{\Sigma_n \cap (B_{rH^{-1}} \setminus B_{K\rho_0})} - \frac{1}{2} v^l \partial_l h_{il} (v^m - v^m_n) b^m + \frac{1}{2} v^l \partial_l h_{ii} (v^m - v^m_n) b^m d\mu_e$$

$$+ (v^m_n b^m) \int_{\Sigma_n \cap (B_{rH^{-1}} \setminus B_{K\rho_0})} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e.$$

For the first term on the right hand side, we have:

$$\left| \int_{\Sigma_n \cap (B_{rH^{-1}} \setminus B_{K\rho_0})} - \frac{1}{2} v^l \partial_l h_{il} (v^m - v^m_n) b^m + \frac{1}{2} v^l \partial_l h_{ii} (v^m - v^m_n) b^m d\mu_e \right|$$

$$\leq \sum_{i=1}^{l_n} \int_{\Sigma_n \cap (B_{K\rho \cdot e^{-i(1)} L} \setminus B_{K\rho \cdot e^{-i(1)} L})} - \frac{1}{2} v^l \partial_l h_{il} (v^m - v^m_n) b^m + \frac{1}{2} v^l \partial_l h_{ii} (v^m - v^m_n) b^m d\mu_e$$

$$\leq \sum_{i=1}^{l_n/2} C(e^{-\frac{i}{2}L} + e^{-\frac{i}{4}L})(s + r_0^{-\frac{i}{2}}) + \sum_{i=l_n/2+1}^{l_n} C(e^{-\frac{i}{4}L} + e^{-\frac{i}{4}(l_n-i)L})(s + r_0^{-\frac{i}{2}})$$

$$\leq C(s + r_0^{-\frac{i}{2}})$$

At last we prove

$$\lim_{s \to 0} \lim_{K \to \infty} \lim_{n \to \infty} \int_{\Sigma_n \cap (B_{rH^{-1}} \setminus B_{K\rho_0})} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e = 0.$$

First for sufficiently large $r$

$$\int_{\Sigma_n} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e - \frac{1}{2} \int_{B_r(0) \setminus int(\Sigma_n)} \left( h_{il,il} - h_{ii,il}\right) d\mu_e$$

$$= \int_{\partial B_r(0)} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e$$

by divergence formula. So

$$\left| -8\pi m - \int_{\Sigma_n} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e \right| \leq F(r_0).$$

And from RT condition

$$\lim_{s \to 0} \lim_{n \to \infty} \int_{\Sigma_n \cap (B_{rH^{-1}}^c \setminus B_{\rho_0})} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e = -4\pi m,$$

$$\lim_{K \to \infty} \lim_{n \to \infty} \int_{\Sigma_n \cap (B_{K\rho_0})} \left(- \frac{1}{2} v^l \partial_l h_{il} + \frac{1}{2} v^l \partial_l h_{ii}\right) d\mu_e = -4\pi m.$$
So we know
\[
\lim_{s \to 0,K \to \infty} \lim_{n \to \infty} \int_{\Sigma_n \cap (B_{s,H-1} \setminus B_{K,0})} \left( -\frac{1}{2} v^i \partial_i h_{il} + \frac{1}{2} v^i \partial_i h_{il} \right) d\mu_e = 0.
\]

Now we combine all the terms we get
\[
\lim_{n \to \infty} \int_{\Sigma_n} (H - H_e) < v \cdot b > d\mu_e = -4\pi m - 4\pi m = -8\pi m
\]
where \( m > 0 \) is the mass of the manifold. This is a contradiction. So we prove Lemma 1.2 and the main theorem.

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