On the Hyperbolicity of the Complements of Curves in Algebraic Surfaces: The Three Component Case

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1 Introduction

In complex analysis hyperbolic manifolds have been studied extensively, with close relationships to other areas (cf. eg. [20]). Hyperbolic manifolds are generalizations of hyperbolic Riemann surfaces to higher dimensions. Despite the fact that the general theory of hyperbolic manifolds is well-developed, only very few classes of hyperbolic manifolds are known. But one could hope that ‘most’ of the pseudoconvex quasi-projective varieties are in fact hyperbolic. In particular it is believed that e.g. the complements of most hypersurfaces in $\mathbb{P}_n$ are hyperbolic, if only their degree is at least $2n+1$. More precisely according to Kobayashi [18], and later Zaidenberg [30] one has the following:

Conjecture: Let $C(d_1,\ldots,d_k)$ be the space of $k$ tupels of hypersurfaces $C = (C_1,\ldots,C_k)$ in $\mathbb{P}_n$, where $\deg(C_i) = d_i$. Then for all $(d_1,\ldots,d_k)$ with $\sum_{i=1}^k d_i =: d \geq 2n+1$ the set $\mathcal{H}(d_1,\ldots,d_k) = \{ C \in C(d_1,\ldots,d_k) : \mathbb{P}_n \setminus \bigcup_{i=1}^k C_i \}$
is complete hyperbolic and hyperbolically embedded\} contains the complement of a proper algebraic subset of \(C(d_1, \ldots, d_k)\).

In this paper we shall restrict ourselves to the two dimensional case. However we consider also more general quasi-projective complex surfaces than the complements of curves in the projective plane.

Concerning the above conjecture, the following was known: It seems that the conjecture is the more difficult the smaller \(k\) is. Other than in the case of 5 lines (\(C(1, 1, 1, 1, 1)\)), the conjecture was previously proved by M. Green in \([14]\) in the case of a curve \(C\) consisting of one quadric and three lines (\(C(2, 1, 1, 1)\)). Furthermore, it was shown for \(C(d_1, \ldots, d_k)\), whenever \(k \geq 5\), by Babets in \([3]\). A result which went much further was given by Eremenko and Sodin in \([9]\), where they proved a Second Main Theorem of value distribution theory in the situation \(k \geq 5\). Green proved in \([14]\) that for any hypersurface \(C\) consisting of at least four components in \(\mathbb{P}_2\) any entire curve \(f : \mathbb{C} \to \mathbb{P}_2 \setminus C\) is algebraically degenerate. Knowing this, it follows immediately that for generic configurations, any such algebraically degenerate map is constant, hence the conjecture is true for any family \(C(d_1, \ldots, d_k)\) with \(k \geq 4\) (cf. \([8]\)). (The degeneracy locus of the Kobayashi pseudometric was studied by Adachi and Suzuki in \([1\), \([2]\).)

In our paper \([8]\) we gave a proof of the conjecture for 3 quadrics (\(C(2, 2, 2)\)), based on methods from value distribution theory. The three quadric case had been previously studied by Grauert in \([11]\) who used differential geometry. However, certain technical problems still exist with this approach. For \(C(2, 2, 1)\), i.e. two quadrics and a line, we proved with similar methods the existence of an open set in the space of all such configurations, which contains a quasi-projective set of codimension one, where the conjecture is true.

The paper contains two main results. The first is Theorem 6.1. It states that the conjecture is true for almost all three component cases, namely for \(C(d_1, d_2, d_3)\) with \(d_1, d_2, d_3 \geq 2\) and at least one \(d_i \geq 3\). Together with our result for three quadrics (which, by the way, occur on the borderline of the method used in this paper) this means that the conjecture is true for three components whenever none of them is a line. We finally remark that we get a weaker conclusion also for \(C(d_1, d_2, d_3)\) where, up to enumeration, \(d_1 = 1, d_2 \geq 3, d_3 \geq 4\): Namely we show that any holomorphic map \(f : \mathbb{C} \to X\) is algebraically degenerated, i.e. \(f(\mathbb{C})\) is contained in a proper algebraic subset of \(X\).

The other main result is Theorem 6.2. We consider a smooth surface \(\bar{X}\) in \(\mathbb{P}_3\) of degree at least five for which every curve on \(\bar{X}\) is the complete intersection
with another hypersurface. Surfaces of this kind are much more general than $\mathbb{P}_2$ – by the Noether-Lefschetz theorem (cf. [8]) the ‘generic’ surface in $\mathbb{P}_3$ of any given degree at least four has this property (‘generic’ here indicates the complement of a countable union of proper varieties). Let $C$ be a curve on $\bar{X}$ consisting of three smooth components intersecting transversally. From our assumptions we know that $C$ is a complete intersection of $\bar{X}$ and a hypersurface $B$. We assume that the degree of $B$ is at least five. Now Theorem 6.2 states the hyperbolicity of any such $X = \bar{X} \setminus C$. Moreover $X$ is complete hyperbolic and hyperbolically embedded.

Our method of proof is the following: We heavily use a theorem due to S. Lu (cf. [22]). It states that for a certain class of differentials $\sigma$, which may have logarithmic poles along the curve $C$, and any holomorphic map $f : \mathbb{C} \to \mathbb{P}_2 \setminus C$ the pull-back $f^*(\sigma)$ vanishes identically. This can be interpreted as algebraic degeneracy of the tangential map corresponding to $f : \mathbb{C} \to X$. Our aim is to show algebraic degeneracy of the map $f$ itself.

The paper is organized as follows: In section 2 we collect, for the convenience of the reader, some basics from value distribution theory. (Readers who are familiar with these may skip this section). In section 3 we fix the notation and quote some theorems which are needed in the following proof, especially Lu’s theorem. Furthermore we examine more closely the spaces of sections which are used in Lu’s theorem and get sections with special zero sets. The essential step of our paper is the proof of Theorem 4.2 in section 4. It states the algebraic degeneracy of holomorphic maps $f : \mathbb{C} \to X$, if $\text{Pic}(\bar{X}) = \mathbb{Z}$, and under assumptions on the determinant bundle and the Chern numbers of the logarithmic cotangent bundle on $\bar{X}$ with respect to $C$. The proof uses value distribution theory and the existence of the special sections which were constructed in section 3. In section 5 we compute the Chern numbers and the determinant bundle in the situation where $\bar{X}$ is a complete intersection (Theorem 5.3). We apply this to $\bar{X} = \mathbb{P}_2$, and to hypersurfaces in $\mathbb{P}_3$ using the Noether Lefschetz theorem (Theorem 5.4). Finally in section 6 we apply Theorem 5.4 and get Theorem 6.1, using an argument like in our paper [8] to prove the nonexistence of algebraic entire curves in generic complements. Furthermore we apply Theorem 5.4 using results of Xu [29] and Clemens [7] to get Theorem 6.2.

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2 Some tools from Value Distribution Theory

In this section we fix some notations and quote some facts from Value Distribution Theory. We give references but do not trace these facts back to the original papers.

We define the characteristic function and the counting function, and give some formulas for these.

Let \( ||z||^2 = \sum_{j=0}^{n} |z_j|^2 \), where \((z_0, \ldots, z_n) \in \mathbb{C}^{n+1}\), let \( \Delta_t = \{ \xi \in \mathbb{C} : ||\xi|| < t \} \), and let \( dc^t = (i/4\pi)(\partial - \overline{\partial}) \). Let \( r_0 \) be a fixed positive number and let \( r \geq r_0 \). Let \( f : \mathbb{C} \to \mathbb{P}_n \) be entire, i.e. \( f \) can be written as \( f = [f_0 : \ldots : f_n] \) with holomorphic functions \( f_j : \mathbb{C} \to \mathbb{C}, j = 0, \ldots, n \) without common zeroes. Then the characteristic function \( T(f, r) \) is defined as

\[
T(f, r) = \int_{r_0}^{r} \int_{\Delta_t} d\xi \log ||f||^2
\]

Let furthermore \( D = V(P) \) be a divisor in \( \mathbb{P}_n \), given by a homogeneous polynomial \( P \). Assume \( f(\mathbb{C}) \not\subset \text{support}(D) \). Let \( n_f(D, t) \) denote the number of zeroes of \( P \circ f \) inside \( \Delta_t \) (counted with multiplicities). Then we define the counting function as

\[
N_f(D, r) = \int_{r_0}^{r} n_f(D, t) \frac{dt}{t}
\]

Stokes Theorem and transformation to polar coordinates imply (cf. [28]):

\[
T(f, r) = \frac{1}{4\pi} \int_{0}^{2\pi} \log ||f||^2(re^{i\vartheta})d\vartheta + O(1).
\]

The characteristic function as defined by Nevanlinna for a holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) is

\[
T_0(f, r) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\vartheta})|d\vartheta.
\]

For the associated map \( [f : 1] : \mathbb{C} \to \mathbb{P}_1 \) one has

\[
T_0(f, r) = T([1 : f], r) + O(1)
\]

(cf. [16]).

By abuse of notation we will, from now on, for a function \( f : \mathbb{C} \to \mathbb{C} \), write \( T(f, r) \) instead of \( T_0(f, r) \). Furthermore we sometimes use \( N(f, r) \) instead of \( N_f(z_0 = 0, r) \).

We state some elementary properties of the characteristic function:
Lemma 2.1 Let $f, g, f_j : \mathbb{C} \to \mathbb{C}$ be entire holomorphic functions for $j = 0, \ldots, n$. Then

a) $T(f \cdot g, r) \leq T(f, r) + T(g, r) + O(1)$

b) $T([f_0 : \ldots : f_n], r) \leq \sum_{j=0}^{n} T(f_j, r) + O(1)$

c) $T(f + g, r) \leq T(f, r) + T(g, r) + O(1)$

Proof: Property a) is obvious for $T_0$ and generalizes to $T$ because of (2). Property b) is a consequence of

$$\log \sum_{j=0}^{n} |f_j|^2 \leq \sum_{j=0}^{n} \log(1 + |f_j|^2).$$

Property c) is a consequence of

$$\log^+ |f + g| \leq \log(1 + |f| + |g|) \leq \log(1 + |f|) + \log(1 + |g|) \leq \log^+ |f| + \log^+ |g| + 2$$

Later on we will use the concept of finite order.

Definition 2.2 Let $s(r)$ be a positive, monotonically increasing function defined for $r \geq r_0$. If

$$\lim_{r \to \infty} \frac{\log s(r)}{\log r} = \lambda$$

then $s(r)$ is said to be of order $\lambda$. For entire $f : \mathbb{C} \to \mathbb{P}_n$ or $f : \mathbb{C} \to \mathbb{C}$ we say that $f$ is of order $\lambda$, if $T(f, r)$ is.

Remark 2.3 Let $f = [f_0 : \ldots : f_n] : \mathbb{C} \to \mathbb{P}_n$ be a holomorphic map of finite order $\lambda$. Then $\log T(f, r) = O(\log r)$.

For holomorphic maps to $\mathbb{P}^1$ whose characteristic function only grows like $\log r$ we have the following characterization (cf. [16]):
Lemma 2.4 Let $f = [f_0 : f_1] : \mathbb{C} \to \mathbb{P}^1$ be entire. Then $T(f, r) = O(\log r)$ if and only if the meromorphic function $f_0/f_1$ is equal to a quotient of two polynomials.

We need the following:

Lemma 2.5 Assume that $f : \mathbb{C} \to \mathbb{P}_n$ is an entire map and misses the divisors \{\(z_j = 0\)\} for \(j = 0, \ldots, n\) (i.e. the coordinate hyperplanes of \(\mathbb{P}_n\)). Assume that $f$ has order at most $\lambda$. Then $f$ can be written as $f = [1 : f_1 : \ldots : f_n]$ with $f_j(\xi) = e^{P_j(\xi)}$, where the $P_j(\xi)$ are polynomials in $\xi$ of degree $d_j \leq \lambda$.

Proof: We write $f = [1 : f_1 : \ldots : f_n]$ with holomorphic $f_j : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Now we get with equations (1) and (2) for $j = 1, \ldots, n$:

$$T(f_j, r) = T([1 : f_j], r) + O(1) \leq T(f, r) + O(1),$$

hence the $f_j$ are nonvanishing holomorphic functions of order at most $\lambda$. This means that

$$\limsup_{r \to \infty} \frac{T(f_j, r)}{r^{\lambda+\epsilon}} = 0$$

for any $\epsilon > 0$. From this equation our assertion follows with the Weierstraß theorem as it is stated in [10].

We state the First and the Second Main Theorem of Value Distribution Theory which relate the characteristic function and the counting function (cf. [22]):

Let $f : \mathbb{C} \to \mathbb{P}_n$ be entire, and let $D$ be a divisor in $\mathbb{P}_n$ of degree $d$, such that $f(\mathbb{C}) \not\subset \text{support}(D)$. Then:

**First Main Theorem**

$$N_f(D, r) \leq d \cdot T(f, r) + O(1)$$

Assume now that $f(\mathbb{C})$ is not contained in any hyperplane in $\mathbb{P}_n$, and let $H_1, \ldots, H_q$ be distinct hyperplanes in general position. Then

**Second Main Theorem**

$$(q - n - 1)T(f, r) \leq \sum_{j=0}^{q} N_f(H_j, r) + S(r)$$

where $S(r) \leq O(\log(rT(f, r)))$ for all $r \geq r_0$ except for a set of finite Lebesque measure. If $f$ is of finite order, then $S(r) \leq O(\log r)$ for all $r \geq r_0$. 

6
3 Setup and Basic Methods

We denote by $\bar{X}$ a non-singular projective surface and by $C$ a curve in $\bar{X}$ whose irreducible components are smooth and intersect each other only in normal crossings. Let $X = \bar{X} \setminus C$.

We denote by $E$ the dual of the bundle $\Omega^1_{\bar{X}}(\log C)$ of holomorphic one forms of $\bar{X}$ with logarithmic poles along $C$. Then we define the projectivized logarithmic tangent bundle $p : \mathbb{P}(E) \to \bar{X}$ over $\bar{X}$ to be the projectivized bundle whose fibers correspond to the one dimensional subspaces of the fibers of $E$. Furthermore let $\mathcal{O}_{\mathbb{P}(E)}(-1)$ be the sheaf associated to the tautological line bundle on $\mathbb{P}(E)$, for which we have the canonical isomorphism between the total space of $\mathcal{O}_{\mathbb{P}(E)}(-1) \setminus \{\text{zerosection}\}$ and the total space of $E \setminus \{\text{zerosection}\}$.

Let $D$ be a divisor on $\bar{X}$. According to a Theorem of Kobayashi-Ochiai (cf. [19]), the cohomology, in particular the holomorphic sections, of a symmetric power of $E^*$ tensorized with the bundle $[-D]$, corresponds to the cohomology of the $m$-th power of the dual of the tautological line bundle on $\mathbb{P}(E)$, tensorized with the pull-back of $[-D]$:

$$H^0(\bar{X}, S^m(E^*) \otimes [-D]) \simeq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*[-D]),$$

and in particular

$$H^0(\bar{X}, S^m(E^*)) \simeq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)).$$

Let $f : \mathbb{C} \to X$ be a holomorphic map. Denote by

$$(f, f') : T(\mathbb{C}) \to T(X)$$

the induced map from $T(\mathbb{C})$ to the holomorphic tangent bundle $T(X)$, which gives rise to a meromorphic map

$$F : \mathbb{C} \to \mathbb{P}(T(X))$$

from $\mathbb{C}$ to $\mathbb{P}(T(X))$. Since the domain is of dimension one, points of indeterminacy can be eliminated, more precisely the map $F$ extends holomorphically into the points $\xi \in \mathbb{C}$ where $f'(\xi) = 0$. We denote the extended holomorphic map on $\mathbb{C}$ again by $F$. Since the restriction of $E$ to $X$ is isomorphic to the holomorphic tangent bundle of $X$, any map

$$f : \mathbb{C} \to X$$
has a unique holomorphic lift $F : \mathbb{C} \to \mathbb{P}(E)$.

The following theorem, which is a special case of Theorem 2 of Lu in [22] (it actually follows already from Proposition 4.1 there) imposes restrictions to such lifts $F$.

**Theorem 3.1 (Lu)** Assume that the divisor $D$ is ample and that there exist a non-trivial holomorphic section

$$0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*[\mathcal{O}(D)])$$

Then for any non-constant holomorphic map $f : \mathbb{C} \to X$ the holomorphic lift $F : \mathbb{C} \to \mathbb{P}(E)$ has values in the zero-set of $\sigma$.

In order to apply Lu’s theorem in a given situation it is important to guarantee the existence of suitable sections.

Let $(\bar{X}, C)$ be given as above. The logarithmic Chern classes $\bar{c}_j(X)$ are by definition the Chern classes of the logarithmic tangent bundle $E$:

$$\bar{c}_j(X) = c_j(E) = c_j(X, (\Omega^1_{\bar{X}}(\log C))^*)$$

Now the existence of suitable sections is guaranteed by the following theorem of Bogomolov (cf. [5] and also Lu [22] (Proof of Proposition 3.1 and localization to the divisor $D$).

**Theorem 3.2 (Bogomolov)** Let $D$ be a divisor on $\bar{X}$, $D$ effective (i.e. $D \geq 0$). Assume that

$$\bar{c}_1^2(X) - \bar{c}_2(X) > 0,$$

and that

$$\det(E^*)$$

is effective. Then there exist positive constants $A, B$ and $m_0, n_0 \in \mathbb{N}$, such that

$$A \cdot m^3 \leq h^0(\bar{X}, S^{mn_0}(E^*) \otimes [-D]) \leq B \cdot m^3$$

for all $m \geq m_0$.

We have the following nonexistence statement, which is a consequence of the logarithmic version of the Bogomolov’s lemma due to Sakai (cf. [26]). It will also become important for the following proofs.
Lemma 3.3 Assume that the divisor \(D\) is ample. Then the following group vanishes:

\[ H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^*[−D]) = \{0\}. \]

In particular, there is no logarithmic 1-form on \(X\) which vanishes on \(D\).

**Proof:** The existence of a non-trivial section \(s \in H^0(\overline{X}, \Omega^1_{\overline{X}}(\log C) \otimes [−D])\) implies that the invertible sheaf \(L := [D]\) can be realized as a subsheaf of \(\Omega^1_{\overline{X}}(\log C)\). According to a result of Sakai (7.5), this implies that the \(L\)-dimension of \(\overline{X}\) equals one, which is clearly impossible since \([D]\) is ample. □

Next we deal with divisors in \(\mathbb{P}(E)\) which project down to all of \(\overline{X}\):

**Definition 3.4** Consider the projection \(p : \mathbb{P}(E) \rightarrow \overline{X}\). We call a divisor \(Z \subset \mathbb{P}(E)\) horizontal, if \(p(Z) = \overline{X}\).

Those horizontal divisors which occur as parts of the zero sets \(V(\sigma)\) of sections \(0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes p^*[−D])\) will play an important role in the sequel. We study this relationship somewhat closer.

**Lemma 3.5** Given

\[ 0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[−D]) \]

there exist divisors \(E_j\), \(j = 1, \ldots, l\) on \(\overline{X}\), and numbers \(a_j, n_j \in \mathbb{N}\) such that \([\sum a_j \cdot E_j − D] \geq 0\) and sections \(s_j \in H^0(\mathbb{P}(E), \mathcal{O}(n_j) \otimes p^*[−E_j])\), \(\tau \in H^0(\mathbb{P}(E), p^*[\sum a_j \cdot E_j − D])\) such that \(\sigma = \tau \otimes_{1 \leq j \leq l} s_j^{a_j}\) with the following property: The zero-sets of \(s_j\) are precisely the irreducible horizontal components of \(V(\sigma)\).

**Proof:** Let \(0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[−D])\) be a non-trivial section and \(V(\sigma)\) its zero divisor. We denote by \(S_j; j = 1, \ldots, l\) the irreducible horizontal components of \(V(\sigma)\). Since \(\text{Pic}(\mathbb{P}(E)) = \text{Pic}(X) \oplus \mathbb{Z}\), we get \([S_j] = \mathcal{O}_{\mathbb{P}(E)}(n_j) \otimes p^*[−E_j]\) for certain divisors \(E_j \subset \overline{X}\) with \(n_j \geq 1\). This fact follows by restricting the bundles \([S_j]\) to a generic fiber of \(p\). Let \(a_j\) be the multiplicities of \(\sigma\) with respect to \(S_j\), then in particular \(a_1 n_1 + \ldots + a_l n_l = m\). (This fact follows again by restricting bundles and sections to a generic fiber of \(p\).) Canonical sections of \([S_j]\) give rise to non-trivial sections \(s_j \in H^0(\mathbb{P}(E), \mathcal{O}(n_j) \otimes p^*[−E_j])\) which vanish exactly on \(S_j\). Thus \(\tau := \sigma/(s_1^{a_1} \cdot \ldots \cdot s_l^{a_l})\) is a (holomorphic) section of \(H^0(\mathbb{P}(E), p^*[\sum a_j \cdot E_j − D])\). In particular \([\sum a_j \cdot E_j − D] \geq 0\). □
In order to control the horizontal divisors of a section of \( H^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[-D]) \), the number \( m \) will be chosen minimal in the following sense.

For any \( k \in \mathbb{N} \) we set
\[
\mu_k := \inf \{m; h^0(\mathbb{P}(E), \mathcal{O}(m) \otimes p^*[-kD]) \neq 0\}.
\]
and
\[
\mu := \inf_{k \in \mathbb{N}} \{\mu_k\}.
\]

**Lemma 3.6** Assume that \( \text{Pic}(\bar{X}) = \mathbb{Z} \) and that \([D]\) is the ample generator of \( \text{Pic}(\bar{X}) \).

Then we have
\[
2 \leq \mu < \infty
\]
and if \( k_0 = \min\{k \in \mathbb{N} : \mu_k = \mu\} \), there exists a non-trivial section
\[
0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0D])
\]
such that exactly one horizontal component of \( V(\sigma) \) exists and has multiplicity one.

**Proof:** Since some multiple of \( D \) is a very ample and hence linear equivalent to an effective divisor, we have \( \mu < \infty \) from Theorem 3.2. From Lemma 3.3 we then get \( \mu \geq 2 \).

Now take any section \( 0 \neq \sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0D]) \). We use Lemma 3.3. Since \([D]\) is a generator of \( \text{Pic}(\bar{X}) \), there exist \( b_j \in \mathbb{Z} \) such that \([E_j] = b_j \cdot [D]\).

Since \( \sum a_j \cdot E_j - k_0D \geq 0 \) we have \( \sum a_j b_j \geq k_0 \). Since all \( a_j \geq 0 \), there must be at least one \( b_j > 0 \), say \( b_1 > 0 \). Now \( s_1 \in H^0(\mathbb{P}(E), \mathcal{O}(n_1) \otimes p^*[-b_1 \cdot D]) \) is a non-trivial section. By definition of \( \mu \) we have \( n_1 \geq \mu \) which means \( n_1 = \mu \), since \( \sum a_j n_j = \mu \). So in terms of the notion of Lemma 3.5 \( \sigma = \tau \cdot s_1 \), i.e. \( S \) contains only one horizontal component. This component has multiplicity one. \( \square \)

## 4 Algebraic Degeneracy Of Entire Curves

Let \( \bar{X} \) be a non-singular (connected) projective surface.

**Definition 4.1** Let \( f : \mathbb{C} \to \bar{X} \) be a holomorphic map. We call \( f \) algebraically degenerate, if there exists an algebraic curve \( A \subset \bar{X} \) such that \( f(\mathbb{C}) \) is contained in \( A \).
Our main result on algebraic degeneration is:

**Theorem 4.2** Let $C \subset \bar{X}$ be a curve consisting of three smooth components with normal crossings. Assume that:

1. $\text{Pic}(\bar{X}) = \mathbb{Z}$
2. The logarithmic Chern numbers of $X = \bar{X} \setminus C$ satisfy the inequality
   \[ \bar{c}_1^2(X) - \bar{c}_2(X) > 0 \]
3. The line bundle $\text{det}(E^*)$ is effective, where $E^* = \Omega^1_X(\log C)$ is the logarithmic cotangent bundle.

Then any holomorphic map $f : \mathbb{C} \to \bar{X} \setminus C$ of order at most two is algebraically degenerate.

Remark: The theorem also holds without the assumption on the order of the map $f$, but since we are mostly interested in the hyperbolicity of the complement, we include this assumption, because it slightly simplifies the proof.

The rest of this section is devoted to the proof of this Theorem.

Let again $[D]$ be an ample generator of $\text{Pic}(\bar{X})$. Let $k \in \mathbb{N}$ be a natural number such that $[kD]$ is very ample. Then by Theorem 3.2 there exists a symmetric differential $\omega \in H^0(\bar{X}, S^m(E^*) \otimes [-kD])$ which is not identically zero. By Theorem 3.1 we know that $f^*\omega \equiv 0$.

The proof now will work as follows: The three components of the curve $C$ give rise to a morphism $\Phi : \bar{X} \to \mathbb{P}_2$ which maps $C$ to the union of the three coordinate axis. In the first step of the proof we show that we can ‘push down’ the symmetric differential $\omega$ by this morphism to some symmetric rational differential $\Omega$ on $\mathbb{P}_2$ and that we still have $(\Phi \circ f)^*(\Omega) \equiv 0$. Since $\Phi \circ f$ maps the complex plane to the complement of the three coordinate hyperplanes in $\mathbb{P}_2$, we will be able to interpret this, in the second step of the proof, as an equation for nonvanishing functions with coefficients which may have zeroes, but which grow of smaller order, only. In such a situation we then can apply Value Distribution Theory.

**First step:** We first remark that the intersection number of any two curves $D_1$ and $D_2$ is positive (including self intersection numbers). Let $[D]$ be the ample generator of $\text{Pic}(\bar{X}) = \mathbb{Z}$. Now $[D_j] = a_j[D]$; $a_j \in \mathbb{Z}$, and $0 < D_j \cdot D = a_jD^2$ (cf. the easy implication of the Nakai criterion). Hence all $a_j$ are positive, and

\[ D_1 \cdot D_2 = a_1a_2D^2 > 0. \]
We can find $a_j \in \mathbb{N}; j = 1, 2, 3$ such that $[a_1C_1] = [a_2C_2] = [a_3C_3]$, since the divisors $C_j j = 1, 2, 3$ are effective. Let $\sigma_j \in H^0(\bar{X}, L)$ be holomorphic sections which vanish exactly on $C_j$. Then

$$\Phi = [\sigma_1 : \sigma_2 : \sigma_3] : \bar{X} \to \mathbb{P}_2$$

defines a rational map, which is a morphism, since the three components do not pass through any point of $\bar{X}$.

**Lemma 4.3** The morphism $\Phi$ is a branched covering.

*Proof:* Since $C_2 \cdot C_3 > 0$, the fiber $\Phi^{-1}(1 : 0 : 0) = C_2 \cap C_3$ is non-empty. By assumption $C_2 \cap C_3$ consists of at most finitely many points. Hence $\Phi$ is surjective and has discrete generic fibers. Finally $\Phi$ has no positive dimensional fibers at all: Applying Stein factorization we would get a bimeromorphic map. Since there are no curves of negative self-intersection, no exceptional curves exist on $\bar{X}$ (cf. [4]). Hence there exist no positive dimensional fibers of $\Phi$. $\Box$

Hence the morphism $\Phi$ is a finite branched covering of $\bar{X}$ over $\mathbb{P}_2$ with, let us say $N$ sheets. Let $R$ be the ramification divisor of $\Phi$, $B = \Phi(R)$ the branching locus and $R' = \Phi^{-1}(B)$. Then

$$\Phi : \bar{X} \setminus R' \to \mathbb{P}_2 \setminus B$$

is an unbranched covering with $N$ sheets.

We now want to construct a meromorphic symmetric $mN$-form $\Omega$ defined on $\mathbb{P}_2 \setminus B$ from the meromorphic symmetric $m$-form $\omega$ on $\bar{X}$: For any point $w^0 \in \mathbb{P}_2 \setminus B$, there exists a neighborhood $U = U(w^0)$ of $w^0$ and $N$ holomorphic maps $a_i(w), a_i : U \to \bar{X} \setminus R'$; $i = 1, \ldots, N$ such that $\Phi \circ a_i = \text{id}_U$. By pulling back the symmetric $m$-form $\omega$ by means of these maps we get $N$ meromorphic symmetric $m$-forms $(a_i)^*(\omega)(w)$ on $U$. Taking now the symmetric product of these $m$-forms, we get the symmetric $(Nm)$-form $\Omega$ on $U$:

$$\Omega(w) = \prod_{i=1}^{N} a_i^*(\omega(w)).$$

Let $M = Nm$. Defining $g = \Phi \circ f$, we then have:

**Lemma 4.4** The form $\Omega$ extends to a rational symmetric $M$-form on $\mathbb{P}_2$, which we again denote by $\Omega$. We have $\Omega \neq 0$, but $g^*\Omega = 0$. 

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The first statement of this lemma is probably well known, and the second statement is considered to be obvious. But since we did not find a reference, we will include a proof of this Lemma at the end of this section.

We proceed with the proof of Theorem 4.2.

Denote the homogeneous coordinates of \( \mathbb{P}^2 \) by \( w_0, w_1, w_2 \). On \( \mathbb{P}^2 \setminus V(w_0) \) we have inhomogeneous coordinates \( \xi_1 = w_1/w_0, \xi_2 = w_2/w_0 \). Hence on \( \mathbb{P}^2 \setminus V(w_0) \) the symmetric \( M \)-form \( \Omega \) can be written as

\[
\Omega = \sum_{i=1}^{M} R_i(\xi_1, \xi_2)(d\xi_1)^i(d\xi_2)^{M-i}
\]

where multiplication means the symmetric tensor product here, and the coefficients \( R_i(\xi_1, \xi_2) \) are rational functions in \( \xi_1 \) and \( \xi_2 \).

Now \( g : \mathbb{C} \rightarrow \mathbb{P}^2 \) has values in the complement \( \mathbb{P}^2 \setminus V(w_0w_1w_2) \) of the three coordinate axis, hence the functions \( g_j = \xi_j \circ g \) are holomorphic and without zeroes. Since \( g^*\omega \equiv 0 \) on \( \mathbb{C} \), equation (3) implies

\[
\sum_{i=1}^{M} R_i(g_1(\eta), g_2(\eta))(g_1'(\eta))^i(g_2'(\eta))^{M-i} \equiv 0
\]

for all \( \eta \in \mathbb{C} \). This equation still holds if we clear the denominators of the \( R_i(\xi_1, \xi_2) \) simultaneously, so without loss of generality we may assume from now on that in equation (3) the \( R_i(g_1(\eta), g_2(\eta)) \) are polynomials in \( g_1(\eta) \) and \( g_2(\eta) \), i.e. we have

\[
R_i(g_1(\eta), g_2(\eta)) = \sum_{j,k} a_{ijk}(g_1(\eta))^j(g_2(\eta))^k
\]

Under our assumptions we are able to say more about the functions \( g_i; i = 1, 2 \). Since the holomorphic map \( f : \mathbb{C} \rightarrow \tilde{X} \setminus C \) was of finite order at most two, this is also true for \( g = \Phi \circ f \) by Lemma 2.1, since the components of \( g \) are polynomials in the components of \( f \).

Hence by Lemma 2.1, we have

\[
g_i(\eta) = \exp(p_i(\eta))
\]

where the \( p_i(\eta); i = 1, 2 \) are polynomials in \( \eta \) of degree at most two. Furthermore we may assume that both polynomials are non-constant, otherwise \( g \) would be linearly degenerate and so \( f \) would be algebraically degenerate, and we were done.
Replacing equation (3) and equation (4) in equation (4) we get
\[ \sum_{i=1}^{M} \sum_{j,k} a_{ijk} \exp\{(i+j)p_1(\eta) + (M-i+k)p_2(\eta)\}(p'_1(\eta))^i(p'_2(\eta))^{M-i} \equiv 0. \] (7)

If we still allow linear combinations of the above summands with constant coefficients \( c_{ijk} \) in equation (3) we can pass to a subset \( S \) of indices which occur in this equation and get a relation
\[ \sum_{(i,j,k) \in S} c_{ijk} a_{ijk} \exp\{(i+j)p_1(\eta) + (M-i+k)p_2(\eta)\}(p'_1(\eta))^i(p'_2(\eta))^{M-i} \equiv 0 \] (8)
but now with the additional property that \( S \) is minimal with equation (8). Let \( S \) have \( L \) elements.

Since we may assume that the polynomials \( p_i(\eta) \) are nonconstant, we know that the \( p'_i(\eta) \) are not identically zero and hence that \( L \geq 2 \).

For the rest of this proof we will distinguish between two cases:

**Case 1:** There exist two summands in equation (8) the quotient of which is not a rational function in the variable \( \eta \).

**Case 2:** The quotient of any two summands in equation (8) is a rational function in the variable \( \eta \).

We shall show that the first case is impossible whereas in the second case algebraic degeneracy is shown.

**Case 1:** We could immediately finish up the proof under the assumptions of case 1 by using a Second Main Theorem for moving targets to equation (8), as to be found e.g. in the paper of Ru and Stoll [25]. Another approach is to treat equation (8) directly with a generalized Borel’s theorem (we can regard this equation as a sum of nonvanishing holomorphic functions with coefficients which may vanish, but which grow of a smaller order than the nonvanishing functions, only). We present here a more elementary argument based on the Second Main Theorem which might also be considered somewhat simpler.

First, it is easy to see that \( L \geq 3 \), since for \( L = 2 \) we would get, by dividing in equation (8) through one of the exponential terms, that the exponential of a nonconstant polynomial is equal to a quotient of two other polynomials, which is absurd.

Let \( \psi_1, \ldots, \psi_L \) be some enumeration of the summands which occur in equation (8). Then, after factoring out possible common zeroes of the entire holomorphic functions \( \psi_1, \ldots, \psi_L \) we get an entire holomorphic curve
\[ \Psi : \mathbb{C} \to \mathbb{P}^{L-1}; \eta \to [\psi_1(\eta) : \ldots : \psi_L(\eta)]. \]
If we denote the homogenous coordinates of this $\mathbb{P}^{L-1}$ by $[z_1 : \ldots : z_L]$, the image of $\Psi$ is contained in the hyperplane $H = \{z_1 + \ldots + z_L = 0\}$ and does not hit any of the coordinate hyperplanes $H_i = \{z_i = 0\}$. So we can regard $\Psi$ also as an entire holomorphic mapping with values in the hyperplane $H$ (which is isomorphic to $\mathbb{P}^{L-2}$) which does not intersect the $L$ different hyperplanes $H \cap H_i$ in $H$. It is now an important fact that these hyperplanes are in general position in $H$, and that the entire curve $\Psi$ is not mapping $\mathbb{C}$ entirely into any hyperplane in $H$ (the latter follows from the minimality condition in equation (9)), because under these conditions we can apply the Second Main Theorem (cf. section 2), which yields:

$$(L - (L - 2) - 1)T(\Psi, r) \leq \sum_{i=1}^{L} N_{\Psi}(H \cap H_i, r) + O(\log r) \quad (9)$$

because the entire curve $\Psi$ is of finite order at most two by Lemma 2.1. Now we have

$$N_{\Psi}(H \cap H_i, r) = N(\{ \psi_i = 0 \}, r) \leq M(N(\{ p_1' = 0 \}, r) + N(\{ p_2' = 0 \}, r)).$$

The First Main Theorem (cf. section 2) and Lemma 2.4 imply that the right hand side grows at most of order $O(\log r)$ only, so equation (9) yields that

$$T(\Psi, r) = O(\log r). \quad (10)$$

We know by the assumption of case 1 that there exist indices $i, j$ such that $\psi_i(\eta)/\psi_j(\eta)$ is not a rational function in $\eta$. Then $[\psi_i(\eta) : \psi_j(\eta)] : \mathbb{C} \to \mathbb{P}^1$ is an entire curve for which by Lemma 2.4 the characteristic function $T([\psi_i : \psi_j])$ grows faster than $\log r$, so (by the formula for the characteristic function given in equation (9)) this is also true for $T(\Psi, r)$ contradicting equation (10). So we have shown that under the assumptions of case 1 we get a contradiction.

**Case 2:** We want to show first that there exist nonvanishing complex numbers $\gamma$ and $\lambda$ such that

$$\lambda p_1'(\eta) = \gamma p_2'(\eta). \quad (11)$$

We only need to show that $p_1'(\eta)$ and $p_2'(\eta)$ are linearly dependent, because if one of them is the zero polynomial, we have algebraic degeneracy of $g$ and hence of $f$. So assume that $p_1'(\eta)$ and $p_2'(\eta)$ are linearly independent. Then no linear combination of $p_1(\eta)$ and $p_2(\eta)$ is a constant polynomial. So under
the assumptions of case 2 get that for all \((i, j, k) \in S\) the terms \(i + j\) in the summands
\[
c_{ijk}a_{ijk}\exp((i + j)p_1(\eta)) + (M - i + k)p_2(\eta))(p'_1(\eta))^i(p'_2(\eta))^{M-i}
\]
are equal, and also the terms \(k + (M - i)\) are the same as well. But then for a given \(i_0\) there can be at most one \((i_0, j, k) \in S\). So by factoring out the exponential function in equation (8) we get a nontrivial homogenous equation of degree \(M\) in \(p'_1(\eta)\) and \(p'_2(\eta)\), which then can be factored in linear factors. Since then one of the linear factors has to vanish identically we get the linear dependency of \(p'_1(\eta)\) and \(p'_2(\eta)\) again, so the assumption of linear independency was wrong.

We now want to construct a special symmetric form with at most logarithmic poles as singularities along the curve \(C\) which is annihilated by \(f\).

Let us simply state equation (11) in terms of the original entire curve \(f\). We have
\[
p'_i(\eta) = \frac{dg_i(\eta)}{g_i(\eta)} = (\Phi \circ f)^* \frac{d\xi_i}{\xi_i} = f^*\omega_i
\]
where \(\omega_i; i = 1, 2\) is a differential one form on \(\bar{X}\) with at most logarithmic poles along \(C\). Define \(\omega_0 = \lambda\omega_1 - \gamma\omega_2\). Then \(\omega_0 \in H^0(\bar{X}, E^*)\), and since
\[
\omega_0 = \Phi^*(\lambda \frac{d\xi_1}{\xi_1} - \gamma \frac{d\xi_2}{\xi_2})
\]
and the map \(\Phi\) is a local isomorphism outside the branching, we have
\[
\omega_0 \neq 0
\]
Furthermore by equations (12) and (11) we have
\[
f^*\omega_0 \equiv 0
\]
Now the proof of the fact that \(f\) is algebraically degenerate is almost finished:

Let \(\sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0D])\) be the section constructed in Lemma 3.4 and \(\tilde{\sigma} \in H^0(\mathbb{P}(E), \mathcal{O}(1))\) the section which corresponds to \(\omega_0\). We recall that both sections are nontrivial, that \(\mu \geq 2\), and that \(V(\sigma)\) contains only one horizontal component, which we will denote by \(S_\sigma\), with multiplicity one. We also recall that by Theorem 3.1, the lift of \(f\) to \(\mathbb{P}(E)\), which we denoted by \(F\), maps entirely into \(V(\sigma)\). We may assume that it maps into \(S_\sigma\),
otherwise by projecting down to $\bar{X}$ we get that $f$ is algebraically degenerate and we are done.

If $\bar{\sigma}$ does not vanish identically on $S_\sigma$, $F$ maps into the zero set of $\bar{\sigma}$ in $S_\sigma$, which has codimension at least two. So projecting down to $\bar{X}$ again yields algebraic degeneracy of $f$.

Hence we now may assume that $\bar{\sigma}$ vanishes identically on $S_\sigma$. Since $\bar{\sigma} \in H^0(\mathbb{P}(E), \mathcal{O}(1))$ the degree of $V(\bar{\sigma})$ with respect to a generic fiber of the map $p : \mathbb{P}(E) \to \bar{X}$ is one (cf. the argument in the proof of Lemma 3.5). However since $S_\sigma$ is the only horizontal component of the zero set of $\sigma \in H^0(\mathbb{P}(E), \mathcal{O}(\mu) \otimes p^*[-k_0D])$ with $\mu \geq 2$ and has multiplicity one, and since $\bar{\sigma}$ vanishes on $S_\sigma$, the degree of $V(\bar{\sigma})$ with respect to such a generic fiber must be at least two, which is a contradiction. So this case cannot occur and the proof of Theorem 4.2 is complete.

**Proof of Lemma 4.4:** The assertion $g^*\Omega \equiv 0$ is clear from $f^*\omega \equiv 0$ and the construction of $\Omega$.

In order to prove the assertion $\Omega \not\equiv 0$, we choose a point $\xi^0 \in \mathbb{P}_2 \setminus (B \cup \{w_0 = 0\})$. In a small neighborhood $U(\xi^0)$ we have the $N$ biholomorphic functions $a_i(\xi), i = 1, ..., N$ which invert the map $\Phi$ on $U(\xi^0)$. Then we have

$$((a_i)^*(\omega))(\xi) = \sum_{j=0}^m b_{ij}(\xi)(d\xi_1)^j(d\xi_2)^{m-j}$$

After possibly moving the point $\xi^0$ in $U(\xi^0)$ we may assume that the meromorphic functions $b_{ij}(\xi)$ either vanish identically on $U(\xi^0)$ or have no zero or singularity in $\xi^0$. Let now for each $i = 1, ..., N$ the index $j(i)$ be the maximal $j \in \{0, ..., m\}$ such that $b_{ij}(\xi^0) \neq 0$. Let $k = \sum_{i=1}^N j(i)$. Then the $(d\xi_1)^k(d\xi_2)^{M-k}$-monomial of $\Omega$ in the point $\xi_0$ is equal to $\prod_{i=1}^N b_{ij(i)}(\xi^0)$, which is not equal to zero by construction.

Last we have to show that $\Omega$ extends to a rational symmetric $M$-form on $\mathbb{P}_2$.

We only have to show how $\Omega$ can be extended over smooth points of the branching locus, because by Levi’s extension theorem (cf. [13]) we then can extend it over the singular locus which is of codimension two. Then, by Chow’s Theorem it is rational. So assume $P \in B$ is a smooth point of $B$. Then (cf. [12]) there exists a neighborhood of $P$ over which $\Phi$ is an analytically branched covering of a very special form: For every point $Q$ over $P$ one can introduce local coordinates $\xi_1, \xi_2$ around $P$ and $z_1, z_2$ around $Q$ such that $\xi_1(P) = \xi_2(P) = z_1(Q) = z_2(Q) = 0$, and neighborhoods $U = \{|\xi_1| < 1, |\xi_2| < 1\}$, $V = \{|z_1| < 1, |z_2| < 1\}$ such that, for some $b \in \{1, ..., N\}$, we have

$$\Phi : V \to U; (z_1, z_2) \to (z_1^b, z_2)$$

In order to prove our assertion in a neighborhood of $P \in B$, it is sufficient to prove it for the analytically branched covering in equation (14).
For $k = 0, \ldots, b - 1$ let

$$g_k : V \to V; (z_1, z_2) \to (\exp \left( \frac{2\pi ik}{b} \right) z_1, z_2).$$

Then $G = \{g_0, \ldots, g_{b-1}\}$ is just the group of deck transformations, i.e. automorphisms which respect $\Phi$. For the meromorphic symmetric $m$-form $\omega$ on $V$, let $\hat{\Omega}$ be the symmetric product of the $b$ meromorphic symmetric $m$-forms $(g_i)^{*}(\omega)$ on $V$. We are done, if we show that by projecting down with $\Phi$ this form gives rise to a meromorphic symmetric $M$-form on $U$, because in $U \setminus B$ this is just the form $\Omega$. The symmetric $M$-form $\hat{\Omega}$ can be uniquely written in the form

$$\hat{\Omega}(z_1, z_2) = \sum_{i=0}^{M} r_{ij}(z_1, z_2) \left( \frac{dz_1}{z_1} \right)^{i} \left( \frac{dz_2}{z_2} \right)^{M-i}$$

with meromorphic functions $r_{ij}$ in the variables $(z_1, z_2)$. Now $\hat{\Omega}(z_1, z_2)$ is invariant under the action of $G$, $(\frac{dz_1}{z_1})^i$ and $(\frac{dz_2}{z_2})^{M-i}$ are also $G$-invariant. Moreover $b \frac{dz_1}{z_1} = \frac{d\xi_1}{\xi_1}$ and $dz_2 = d\xi_2$. Hence the $r_{ij}$ are $G$-invariant functions, i.e. these are pull-backs of meromorphic functions on $U$. 

\[ \square \]

5 Application to the Projective Plane and Complete Intersections

We shall apply Theorem 4.2. Throughout this section, we make the following assumptions:

Let the smooth complex surface $\bar{X}$ be a complete intersection

$$\bar{X} = V_{2}^{(a_1, \ldots, a_r)} \subset \mathbb{P}_{r+2}, \; r \geq 0$$

of hypersurfaces of degrees $a_j$, $j = 1, \ldots, r$ in $\mathbb{P}_{r+2}$. Set $A = \prod_{i=1}^{r} a_i$ and $a = \sum_{i=1}^{r} a_i$. Let smooth curves $C_j$, $j = 1, 2, 3$ be given in $\bar{X}$ which intersect in normal crossings. We assume that these curves are transversal intersections of $\bar{X}$ with hypersurfaces of degrees $b_j$. We set $b = b_1 + b_2 + b_3$.

Lemma 5.1 The Euler numbers of $\bar{X}$ and $C_j$ are:

$$e(\bar{X}) = A \left( 2 + (a - r - 1)^2 \right)$$

and

$$e(C_j) = Ab_j \left( 3 + r - a - b_j \right).$$
The Proof is a direct consequence of the Riemann-Roch Theorem. According to [17], Theorem 22.1.1, the $\chi_y$-characteristic of a complete intersection can be computed from a generating function. Its value at $y = -1$ yields the Euler number. □

In order to determine, when the assumptions of Theorem 4.2 are satisfied, we first compute $\bar{c}_1^2(X) - \bar{c}_2(X)$.

**Proposition 5.2** In the above situation
\[ \bar{c}_1^2(X) - \bar{c}_2(X) = A((a - r - 3)(b - 4) - 6 + \sum_{i<j} b_i \cdot b_j), \]
and
\[ \det(E^*) = (a + b - 3 - r)\bar{H}, \]
where $\bar{H}$ is a hyperplane section.

**Proof:** According to a result of Sakai [26] we have $\det(E^*) = [\Gamma]$, where $\Gamma = K_{\bar{X}} + C$. Then the second claim follows from the Adjunction Formula. Furthermore (cf. [26]),
\[ c_1^2(E) - c_2(E) = c_1^2(E^*) - c_2(E^*) = \Gamma^2 - e(\bar{X}\setminus C) = \Gamma^2 - e(\bar{X}) + e(C), \]
where $\Gamma^2$ denotes the self intersection. It equals
\[ \Gamma^2 = A(a + b - r - 3)^2. \]
For the Euler number of $\bar{X}$ we use Proposition 5.1. The Euler number of $C$ is evaluated in terms of the Euler numbers $e(C_j)$ of the components and the respective intersection numbers
\[ C_i \cdot C_j = Ab_i b_j \]
to be
\[ e(C) = \sum_{j=1}^3 e(C_j) - \sum_{i<j} C_i \cdot C_j. \]
From these equalities we get immediately the above formula for $c_1^2 - c_2$. □

Now Theorem 4.2 yields

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Theorem 5.3 Let $X = \bar{X} \setminus C$ as above. Then any entire holomorphic curve $f : \mathbb{C} \to X$ of order at most two is algebraically degenerate, if

i) $\text{Pic}(\bar{X}) = \mathbb{Z}$

ii) $(a - r - 3)(b - 4) + \sum_{i<j} b_i b_j > 6$

iii) $a + b \geq r + 3$

The proof follows from Theorem 4.2 and Proposition 5.2.

Theorem 5.4 Let $X = \bar{X} \setminus C$ as above. Then any entire holomorphic curve of order at most two $f : \mathbb{C} \to X$ is algebraically degenerate in any of the following cases:

a) $\text{Pic}(\bar{X}) = \mathbb{Z}$, and $a \geq r + 3$, $b \geq 5$.

b) $\bar{X} \subset \mathbb{P}_3$ is a ‘generic’ hypersurface of degree at least four, and $b \geq 5$.

c) Let $\bar{X} = \mathbb{P}_2$ (i.e. $a_1 = \ldots a_r = 1$, $r \geq 0$). Let $b_1, b_2, b_3 \geq 2$ and at least one $b_j \geq 3$, or up to enumeration $b_1 = 1, b_2 \geq 3, b_3 \geq 4$.

Remark: ‘Generic’ indicates the complement of a countable union of proper varieties in space of all hypersurfaces.

Proof: Case a) is obvious. Case b) is an application of the Noether-Lefschetz theorem [6] and case a). For case c) we set e.g. $r = a_1 = 1$. Then

$$\bar{c}_1^2(X) - \bar{c}_2(X) = -3(b - 4) - 6 + \sum_{i<j} b_i \cdot b_j$$

is equal to

$$(b_1 - 2)(b_2 - 2) + (b_1 - 2)(b_3 - 2) + (b_2 - 2)(b_3 - 2) + b - 6$$

or to

$$(b_1 - 1)(b_2 - 1) + (b_1 - 1)(b_3 - 2) + (b_2 - 3)(b_3 - 4) + (2b_2 + b_3) - 9$$

From these facts the assertion of case c) follows immediately.
6 Algebraic Degeneracy of Entire Curves Versus Hyperbolicity

Theorem 6.1 Let $C$ be the union of three smooth curves $C_j$ $j = 1, 2, 3$ in $\mathbb{P}_2$ of degree $d_j$ with
\[ d_1, d_2, d_3 \geq 2 \text{ and at least one } d_j \geq 3. \]

Then for generic such configurations $\mathbb{P}^2 \setminus C$ is complete hyperbolic and hyperbolically embedded in $\mathbb{P}_2$.

More precisely this is the case, if the curves intersect only in normal crossings, and if one curve is a quadric there must not exist a line which intersects the two other curves only in one point each and which intersects the quadric just in these two points.

Proof: In order to prove that $\mathbb{P}_2 \setminus C$ is hyperbolic and hyperbolically embedded in $\mathbb{P}_2$, we only will have to prove, by an easy Corollary of a Theorem of M.Green (cf. [15]), that there does not exist a non-constant entire curve $f : \mathbb{C} \to \mathbb{P}_2 \setminus C$ of order at most two.

We know from Theorem 5.4 that the entire curve $f : \mathbb{C} \to \mathbb{P}_2 \setminus C$ of order at most two is contained in an algebraic curve $A \subset \mathbb{P}_2$ of degree $d_0$ say.

Now the proof is almost the same as in [3]. Assume that there exists an irreducible algebraic curve $A \subset \mathbb{P}_2$ such that $A \setminus C$ is not hyperbolic. We know that $A \cap C$ consists of at least 2 points $P$ and $Q$. Moreover, $A$ cannot have a singularity at $P$ or $Q$ with different tangents, because $A$ had to be reducible in such a point, and $A \setminus C$ could be identified with an irreducible curve with at least three punctures. (This follows from blowing up such a point or considering the normalization).

So $A \cap C$ consists of exactly 2 points $P$ and $Q$ with simple tangents. We denote the multiplicities of $A$ in $P$ and $Q$ by $m_P$ and $m_Q$. Then the inequality (cf. [10])
\[ m_P(m_P - 1) + m_Q(m_Q - 1) \leq (d_0 - 1)(d_0 - 2) \]
implies
\[ m_P, m_Q < d_0 \text{ or } d_0 = m_P = m_Q = 1. \quad (16) \]

After a suitable enumeration of its components we may assume that $P \in C_1 \cap C_2$ and $Q \in C_3$. If $Q \notin C_2 \cup C_1$ we are done, since then we may assume
that $A$ is not tangential to $C_2$, and then, computing intersection multiplicities according to [16], we have

$$m_P = I(P, A \cap C_2) = d_2d_0$$

which contradicts equation (16). So we may assume that $Q \in C_2 \cap C_3$. Now $A$ has to be tangential to $C_1$ in $P$ and to $C_3$ in $Q$, otherwise we again get $m_P = d_1d_0$ or $m_Q = d_3d_0$ what contradicts equation (16). But then $C_2$ is not tangential to $A$ in $P$ or $Q$, so we have

$$m_P + m_Q = I(P, A \cap C_2) + I(Q, A \cap C_2) = d_2d_0$$

Again by equation (16) this is only possible if $d_2 = 2$ and $m_P = m_Q = d_0 = 1$, but then we are in a situation which we excluded in Theorem 6.1, which is a contradiction.

We make the same assumptions as in section 5.

**Theorem 6.2** Let $\bar{X} \subset \mathbb{P}^3$ be a ‘generic’ smooth hypersurface of degree $d \geq 5$ and $b \geq 5$. Then $X = \bar{X} \setminus C$ is hyperbolic and hyperbolically embedded in $\bar{X}$.

**Proof:** According to Xu [23] and Clemens [7] $\bar{X}$ does not contain any rational or elliptic curves. Hence Theorem 5.4 yields the claim. \qed

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