On Tightness of Tsaknakis-Spirakis Descent Methods for Approximate Nash Equilibria

Zhaohua Chen\textsuperscript{1}, Xiaotie Deng\textsuperscript{1}, Wenhan Huang\textsuperscript{2}, Hanyu Li\textsuperscript{1}, and Yuhao Li\textsuperscript{3}

\textsuperscript{1}Peking University
\textsuperscript{2}Shanghai Jiao Tong University
\textsuperscript{3}Columbia University

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Abstract

This article explores the minimum approximation ratio for Nash equilibrium in bi-matrix games, focusing on the Tsaknakis and Spirakis (TS) methods. The previous SOTA, TS algorithm, achieved an approximation ratio of 0.3393, but efforts to improve the analysis of the TS algorithm have been unsuccessful. This work demonstrates that the bound of 0.3393 is tight for the TS algorithm and presents a theoretical worst-case analysis. A condition for identifying tight instances is provided, along with a generator. While most generated instances are unstable, indicating potential improvements, stable instances exist where perturbations cannot enhance the 0.3393 bound. Other approximate algorithms, such as regret-matching and fictitious play, achieve better ratios on these instances. The generated instances can serve as benchmarks for approximate Nash equilibrium algorithms. The article also mentions progress in the TS algorithm, achieving an approximation ratio of 1/3, which can be further studied using the presented techniques.

1 Introduction

Nash equilibria characterize a static status of players in which each player cannot gain more utilities by changing her current strategy. Ever since Nash proved the existence of such an equilibrium [22], it has become the fundamental concept in non-cooperative game theory and economics. In view of computer science, computing Nash equilibria rises in great importance from computational complexity theory [5, 9, 28], algorithmic game theory [20, 24, 27], and learning theory [1, 4, 17]. It has been shown that Nash equilibrium computing lies in the complexity class PPAD introduced by Papadimitriou [26]. The PPAD-completeness results have been established...
for 3NASH (finding an approximate solution of Nash equilibria for at-least-three-player games) by Daskalakis, Goldberg, and Papadimitriou [9], and for 2NASH (finding an exact/approximate solution of Nash equilibria for two-player games) by Chen, Deng, and Teng [5]. It is well believed that PPAD-complete problems can hardly have a polynomial-time algorithm. These completeness results thus lead to a great many efforts to find an $\epsilon$-approximate Nash equilibrium (see the formal definition in Section 2) in polynomial time for some small constant $\epsilon > 0$.

Early works by Kontogiannis et al. [18] and Daskalakis et al. [11] introduced polynomial-time algorithms to reach an approximation ratio of $\epsilon = 3/4$ and $\epsilon = 1/2$, respectively. Their algorithms are based on searching strategies with small supports. Conitzer [7] also showed that the well-known fictitious play algorithm [3] gives a $1/2$-approximate Nash equilibrium within constant rounds, matching Feder et al.’s lower-bound result [15]. Subsequently, Daskalakis et al. [10] gave an algorithm with an approximation ratio of 0.38 by enumerating arbitrarily large supports. The same result was achieved by Czumaj et al. [8] with a totally different approach by finding the Nash equilibria of two zero-sum games and further making a convex combination between the solution and the corresponding best responses. The result was achieved by Czumaj et al. [8] with a totally different approach by finding the Nash equilibria of two zero-sum games and further making a convex combination between the solution and the corresponding best responses. Bosse et al. [2] provided another algorithm based on the previous work of Kontogiannis and Spirakis [19], and their algorithm reaches a 0.36-approximation ratio. Concurrent with these works, Tsaknakis and Spirakis [29] established the best previously known approximation ratio of 0.3393. The basic idea of the Tsaknakis and Spirakis (TS) algorithm is to directly optimize the approximation ratio itself by a descent procedure, after which a further adjustment is conducted by making a convex combination between the solution produced by the optimization and the corresponding best responses.

The original paper of Tsaknakis and Spirakis [29] proved that the approximation ratio of the algorithm is at most 0.3393. However, it leaves open whether 0.3393 is tight for the algorithm. In the literature, the experimental performance of the algorithm is far better than 0.3393 [30]. The worst approximation ratio in experiments reported ahead of our paper is provided in Fearnley et al. [14], where the TS algorithm on a game finds a 0.3385-approximate Nash equilibrium.

In this work, exploring the descent procedure in the TS algorithm, we present a delicate analysis of the lower bound of the TS algorithm, which is illustrated with several images of the worst cases. It provides a full understanding of the worst cases of the TS algorithm. The analysis allows us to prove that 0.3393 is indeed the tight bound for the TS algorithm by providing a bimatrix game instance. Subsequently, we solve the open problem regarding the well-followed TS algorithm. Furthermore, we characterize all game instances that are able to attain the tight bound. Based on this characterization, we identify a region of payoff matrices that the game instances generated are precisely tight instances. This identification allows us to propose a generator of tight instances and conduct various experiments on the generated instances.

Despite the tightness of 0.3393 for the TS algorithm, our extensive experiments show that it is rather difficult to reach a 0.3393 bound on generated instances in practice by brute-force enumerations. Such results imply that the experimental bound is usually inconsistent with the theoretical bound. To figure out the reasons for such a gap, we further explore the stability of generated instances. Our empirical studies show that most generated instances of large sizes are unstable. The word “unstable” means that a small perturbation near a 0.3393 solution would make the TS algorithm find another faraway solution with a much better approximation ratio. With the game size growing large, the probability of finding a stable tight instance plunges and even vanishes. These results help to understand the gap: Even if a tight instance is met, the TS algorithm usually escapes the 0.3393 solution and reaches a far better approximation ratio. Based
on these results, we give a time-saving and effective suggestion on the practical usage of the TS algorithm in Section 7.

We also use the generated game instances to measure the performances of the Czumaj et al.’s algorithm [8], the regret-matching algorithm in online learning [16], and the fictitious play algorithm [3]. The regret-matching algorithm and the fictitious play algorithm perform well on these instances. Interestingly, the algorithm of Czumaj et al. always reaches an empirical approximation ratio of 0.3393 on generated game instances, implying that the tight instance generator for the TS algorithm also makes a totally different algorithm perform poorly. Such results show that our instances generated against the TS algorithm serve as a necessary benchmark in the design and analysis of approximate Nash equilibrium algorithms.

Subsequent Work. After the conference version [6] of the presented paper, very recently, the work by Deligkas, Fasoulakis, and Markakis [12] provides a polynomial-time algorithm computing a 1/3-approximate Nash equilibrium. Their algorithm is also based on the same descent procedure but makes a more delicate adjustment. We show the generality of our study to provide a proof using similar techniques that the approximation ratio 1/3 of their algorithm is tight.

This paper is organized as follows. In Section 2, we introduce the basic definitions and notations that we use throughout the paper. In Section 3, we restate the TS algorithm [29] and propose two other auxiliary methods which help to analyze the original algorithm. In Section 4, we dive into the worst-case analysis, show what a tight instance looks like, and then prove by a game instance that 0.3393 is indeed the tight bound for the TS algorithm. Further, we characterize all tight instances and present a generator that outputs tight game instances in Section 5. With similar techniques, we show that the upper bound 1/3 of the DFM algorithm is indeed tight by presenting matching-bound instances in Section 6. We conduct extensive experiments to reveal the properties of the stationary points and compare the descent methods with other approximate Nash equilibrium algorithms in Section 7. At last, we present several open problems raised from our study in Section 8.

2 Definitions and Notations

We focus on finding an approximate Nash equilibrium in normal-form games with two players. We use $R_{m \times n}$ and $C_{m \times n}$ to denote the payoff matrices of the row player and column player, where the row player and the column player have $m$ and $n$ strategies, respectively. Furthermore, we suppose that both $R$ and $C$ are normalized so that all their entries belong to $[0, 1]$. In fact, concerning Nash equilibria, any game is equivalent to a normalized game with appropriate shifting and scaling of both payoff matrices.

For two vectors $u$ and $v$ with the same length, we say $u \geq v$ if each entry of $u$ is greater than or equal to the corresponding entry of $v$. Meanwhile, let us denote by $e_k$ a $k$-dimension vector with all entries equal to 1. We use a probability vector to define either player’s behavior, which describes the probability that a player chooses any pure strategy to play. Specifically, the row player’s strategy and the column player’s strategy lie in $\Delta_m$ and $\Delta_n$, respectively, where

\[
\Delta_m = \{ x \in \mathbb{R}^m : x \geq 0, x^T e_m = 1 \},
\]

\[
\Delta_n = \{ y \in \mathbb{R}^n : y \geq 0, y^T e_n = 1 \}.
\]

\footnote{We call it DFM algorithm for short.}
For a strategy pair \((x, y) \in \Delta_m \times \Delta_n\), we call it an \(\epsilon\)-approximate Nash equilibrium, if for any \(x' \in \Delta_m, y' \in \Delta_n\), the following inequalities hold:
\[
    x'^T R y \leq x^T R y + \epsilon, \\
    x^T C y' \leq x^T C y + \epsilon.
\]

Therefore, a Nash equilibrium is an \(\epsilon\)-approximate Nash equilibrium with \(\epsilon = 0\).

To simplify our further discussion, for any probability vector \(u\), we use
\[
    \text{supp}(u) = \{i : u_i > 0\},
\]

to denote the support of \(u\), and
\[
    \text{suppmax}(u) = \{i : \forall j, u_i \geq u_j\}, \\
    \text{suppmin}(u) = \{i : \forall j, u_i \leq u_j\},
\]
to denote the index set of all entries equal to the maximum/minimum entry of vector \(u\).

At last, we use \(\max(u)\) to denote the value of the maximal entry of vector \(u\), and \(\max_S(u)\) to denote the value of the maximal entry of vector \(u\) confined in the index set \(S\).

### 3 Algorithms

In this section, we first restate the TS algorithm [29], and then propose two auxiliary adjusting methods, which help to analyze the bound of the TS algorithm.

The TS algorithm formulates the approximate Nash equilibrium problem into an optimization problem. Specifically, we define the following functions:
\[
    f_R(x, y) := \max(Ry) - x^T Ry, \\
    f_C(x, y) := \max(C^T x) - x^T Cy, \\
    f(x, y) := \max \{f_R(x, y), f_C(x, y)\}.
\]

The goal is to minimize \(f(x, y)\) over \(\Delta_m \times \Delta_n\).

The relationship between the above function \(f\) and approximate Nash equilibrium is as follows. Given strategy pair \((x, y) \in \Delta_m \times \Delta_n\), \(f_R(x, y)\) and \(f_C(x, y)\) are the respective maximum deviations of row player and column player. By definition, \((x, y)\) is an \(\epsilon\)-approximate Nash equilibrium if and only if \(f(x, y) \leq \epsilon\). In other words, as long as we obtain a point with \(f\) value no greater than \(\epsilon\), an \(\epsilon\)-approximate Nash equilibrium is reached.

The idea of the TS algorithm is to find a stationary point (to be defined in Definition 2) of the objective function \(f\) by a descent procedure and make a further adjustment from the stationary point\(^2\). To give the formal definition of stationary points, we need to define the Dini directional derivative of \(f\) as follows:

\(^2\)We will see in Remark 2 that finding a stationary point is not enough to reach a good approximation ratio; therefore the adjustment step is necessary.
Definition 1. Given \((x, y), (x', y') \in \Delta_m \times \Delta_n\), the Dini directional derivative [13] of \((x, y)\) in direction \((x' - x, y' - y)\) is

\[
Df(x, y, x', y') := \lim_{\theta \to 0^+} \frac{1}{\theta} \left( f(x + \theta (x' - x), y + \theta (y' - y)) - f(x, y) \right).
\]

\(Df_R(x, y, x', y')\) and \(Df_C(x, y, x', y')\) are defined similarly with respect to \(f_R\) and \(f_C\).

Remark 1. Note that the notion of \(Df(x, y, x', y')\) in Definition 1 is not the directional derivative we usually consider. The latter should be defined as

\[
\lim_{\theta \to 0^+} \frac{1}{\theta} \left( f(x + \theta \frac{x' - x}{\|x' - x\|}, y + \theta \frac{y' - y}{\|y' - y\|}) - f(x, y) \right).
\]

We give an example to show the difference. Let \((x'', y'') = (x, y) + 1/2(x' - x, y' - y)\). It is clear that \((x'' - x, y'' - y)\) represents the same direction as \((x' - x, y' - y)\). However,

\[
Df(x, y, x'', y'') = \lim_{\theta \to 0^+} \frac{1}{\theta} \left( f(x + \theta (x'' - x), y + \theta (y'' - y)) - f(x, y) \right) = \frac{1}{2} \lim_{\theta \to 0^+} \frac{1}{\theta/2} \left( f(x + \theta/2(x' - x), y + \theta/2(y' - y)) - f(x, y) \right) = \frac{1}{2} Df(x, y, x', y').
\]

So even when we consider points in the same direction, their \(Df\) values will be different by a multiple. Why do we not normalize the direction vectors but keep using Dini derivatives? We will see soon that this definition shares good properties: It is easier to analyze and compute the steepest direction.

Now we present the definition of stationary points.

**Definition 2.** \((x, y) \in \Delta_m \times \Delta_n\) is a stationary point if and only if for any \((x', y') \in \Delta_m \times \Delta_n\),

\[
Df(x, y, x', y') \geq 0.
\]

We use a descent procedure to find a stationary point. The descent procedure is presented in Appendix B. Due to time and precision limit, we cannot expect the descent procedure always finds an exact stationary point. Instead, we seek a \(\delta\)-stationary point as in the following definition.

**Definition 3.** Given \(\delta \geq 0\), \((x, y) \in \Delta_m \times \Delta_n\) is a \(\delta\)-stationary point if and only if

\[
f_R(x, y) = f_C(x, y)
\]

and for any \((x', y') \in \Delta_m \times \Delta_n\),

\[
Df(x, y, x', y') \geq -\delta.
\]

It has already been proved that the procedure runs in the polynomial-time of precision \(\delta\) to find a \(\delta\)-stationary point [30].

To better deal with \(Df(x, y, x', y')\), we give an explicit form of \(Df\). Detailed calculations are presented in Appendix A. In [29], they have provided alternative characterizations of \(Df\). Below we provide a similar development, but emphasize its min-max structure, which is utilized in a series
of proofs later. For now we only care about cases when \( f_R(x, y) = f_C(x, y) \) (which is a necessary condition for a stationary point to be proved in Proposition 3). Let \( S_C(x) := \text{suppmax}(C^T x) \), \( S_R(y) := \text{suppmax}(R y) \). Then the formula is

\[
D f(x, y, x', y') = \max\{ D f_R(x, y, x', y'), D f_C(x, y, x', y') \}
\]

\[
= \max\{ T_1(x, y, x', y'), T_2(x, y, x', y') \} - f(x, y),
\]

where

\[
T_1(x, y, x', y') = \max_{S_R(y)}(R y') - x'^T R y - x'^T R y' + x'^T R y,
\]

\[
T_2(x, y, x', y') = \max_{S_C(x)}(C^T x') - x'^T C y - x'^T C y' + x'^T C y.
\]

A key component of \( D f \) is \( \max\{T_1, T_2\} \), for which several maximum operators are applied. To smoothen these maximum operations, we introduce linear convex combinations via \( \rho, w \) and \( z \):

\[
T(x, y, x', y', \rho, w, z) := \rho(w^T R y' - x'^T R y - x'^T R y' + x'^T R y) + (1 - \rho)(x'^T C z - x'^T C y' - x'^T C y + x'^T C y),
\]

where \( \rho \in [0,1] \), \( w \in \Delta_m \), \( \text{supp}(w) \subseteq S_R(y) \), \( z \in \Delta_n \), \( \text{supp}(z) \subseteq S_C(x) \). When \( f_R(x, y) = f_C(x, y) \), we have the following identities

\[
\max_{\rho, w, z} T(x, y, x', y', \rho, w, z)
\]

\[
= \max_{\rho} \max_{w, z} T(x, y, x', y', \rho, w, z)
\]

\[
= \max_{\rho} \left( \rho \max_{w} (w^T R y' - x'^T R y - x'^T R y' + x'^T R y) + (1 - \rho) \max_{z} (x'^T C z - x'^T C y' - x'^T C y + x'^T C y) \right)
\]

\[
= \max_{\rho} \left( \rho T_1(x, y, x', y') + (1 - \rho) T_2(x, y, x', y') \right)
\]

\[
= \max \{ T_1(x, y, x', y'), T_2(x, y, x', y') \}.
\]

Thus

\[
D f(x, y, x', y') = \max_{\rho, w, z} T(x, y, x', y', \rho, w, z) - f(x, y).
\]

Now define

\[
V(x, y) := \min_{x', y'} \max_{\rho, w, z} T(x, y, x', y', \rho, w, z).
\]

By Definition 2, \((x, y)\) is a stationary point if and only if \( V(x, y) \geq f(x, y) \). Further, by substituting \( x', y' \) with \( x, y \), we have \( V(x, y) \leq \max_{\rho, w, z} T(x, y, x, y, \rho, w, z) \), which is the same as \( f(x, y) \) because \( T_1(x, y, x, y) = f_R(x, y) \) and \( T_2(x, y, x, y) = f_C(x, y) \) by their definitions.

Therefore, we have the following proposition.

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3Throughout the paper, we require that \((x, y), (x', y') \in \Delta_m \times \Delta_n\), and \( \rho \in [0,1] \), \( w \in \Delta_m \), \( \text{supp}(w) \subseteq S_R(y) \), \( z \in \Delta_n \), \( \text{supp}(z) \subseteq S_C(x) \). These restrictions are omitted afterward for fluency of presentation.
**Proposition 1.** \((x, y)\) is a stationary point if and only if

\[ V(x, y) = f_R(x, y) = f_C(x, y). \]

By making some variations on \(T\) (see Appendix A for detail calculations), we can show that \(T\) is a bilinear form in \((\rho w, (1 - \rho)z)\) and \((x', y')\), i.e., \(T\) is equal to

\[ (\rho w^T, (1 - \rho)z^T) G(x, y) \begin{pmatrix} y' \\ x' \end{pmatrix} \]

for some \((m + n) \times (m + n)\) matrix \(G(x, y)\). Thus by applying von Neumann’s minimax theorem [23], we have

**Proposition 2.**

\[ V(x, y) = \max_{\rho, w, z} \min_{x, y'} T(x, y, x', y', \rho, w, z), \]

and there exist \(\rho_0, w_0, z_0\) such that

\[ V(x, y) = \min_{x', y'} T(x, y, x', y', \rho_0, w_0, z_0). \]

We call the tuple \((\rho_0, w_0, z_0)\) a dual solution as it can be calculated by dual linear programming. See Appendix A for the calculations in detail.

In the following context, we fix \((x^*, y^*)\) to denote a stationary point and use \((\rho^*, w^*, z^*)\) to denote the corresponding dual solution about \(G(x^*, y^*)\).

As we will see in Remark 2, a stationary point may only achieve an approximation ratio of \(1/2\) in the worst case. To find a better solution, we adjust the stationary point to another point lying in the following square:

\[ \Lambda := \{ (\alpha w^* + (1 - \alpha)x^*, \beta z^* + (1 - \beta)y^*) : \alpha, \beta \in [0, 1] \}. \]

Different adjustments on \(\Lambda\) derive different algorithms to find an approximate Nash equilibrium. We present three of these methods below, of which the first one is the solution by the TS algorithm, and the other two are for the sake of analysis in Section 4. For simplicity of the presentation, we define the following two subsets of the boundary of \(\Lambda\):

\[ \Gamma_1 := \{ (ax^* + (1 - a)w^*, y^*) : a \in [0, 1] \} \cup \{ (x^*, \beta y^* + (1 - \beta)z^*) : \beta \in [0, 1] \}, \]

\[ \Gamma_2 := \{ (ax^* + (1 - a)w^*, z^*) : a \in [0, 1] \} \cup \{ (w^*, \beta y^* + (1 - \beta)z^*) : \beta \in [0, 1] \}. \]

**Method 1. Method in the original TS algorithm [29].** The first method is the original adjustment given by [29] (known as the TS algorithm in literature). Define the quantities

\[ \lambda := \min_{y' : \text{supp}(y') \subseteq S_C(x^*)} \{(w^* - x^*)^T R y'\}, \]

\[ \mu := \min_{x' : \text{supp}(x') \subseteq S_R(y^*)} \{x'^T C (z^* - y^*)\}. \]

The adjusted strategy pair is

\[ (x_{TS}, y_{TS}) := \begin{cases} \left( \frac{1}{1+\lambda-\mu}w^* + \frac{\lambda-\mu}{1+\lambda-\mu}x^*, z^* \right), & \lambda \geq \mu, \\ \left( w^*, \frac{1}{1+\lambda-\mu}z^* + \frac{\mu-\lambda}{1+\lambda-\mu}y^* \right), & \lambda < \mu. \end{cases} \]
Method 2. **Minimum point on** $\Gamma_2$. For the second method, define

$$\alpha^* := \arg\min_{\alpha\in[0,1]} f(\alpha w^* + (1 - \alpha)x^*, z^*),$$

$$\beta^* := \arg\min_{\beta\in[0,1]} f(w^*, \beta z^* + (1 - \beta)y^*).$$

From a geometric view, our goal is to find the minimum point of $f$ on $\Gamma_2$. The strategy pair given by the second method is

$$(x_{MB}, y_{MB}) := \begin{cases} 
(\alpha^* w^* + (1 - \alpha^*)x^*, z^*), & f_C(w^*, z^*) \geq f_R(w^*, z^*), \\
(w^*, \beta^* z^* + (1 - \beta^*)y^*), & f_C(w^*, z^*) < f_R(w^*, z^*). 
\end{cases}$$

When we write the strategy into such two different cases, it is not trivial to see $(x_{MB}, y_{MB})$ is indeed the minimum point of $f$ on $\Gamma_2$. We will prove this fact in **Lemma 3**.

Method 3. **Intersection point of linear bound of** $f_R$ and $f_C$ on $\Gamma_2$. As we will see later, $(x_{MB}, y_{MB})$ always behaves no worse than $(x_{TS}, y_{TS})$ theoretically. However, it is rather hard to quantitatively analyze the exact approximation ratio of $(x_{MB}, y_{MB})$ given in the second method. Therefore, we propose a third adjustment method. It is not hard to see directly from definitions that $f_R(x, y)$, $f_C(x, y)$ and $f(x, y)$ are all convex and linear-piecewise functions with either $x$ or $y$ fixed. Therefore, on the boundary of $\Lambda$, they can be bounded by linear functions. Formally, for $0 \leq p, q \leq 1$, we have

$$f_R(pw^* + (1 - p)x^*, z^*) = (f_R(w^*, z^*) - f_R(x^*, z^*))p + f_R(x^*, z^*), \quad (1)$$

$$f_C(pw^* + (1 - p)x^*, z^*) \leq f_C(w^*, z^*)p; \quad (2)$$

$$f_C(w^*, qz^* + (1 - q)y^*) = (f_C(w^*, x^*) - f_C(w^*, y^*))q + f_C(w^*, y^*), \quad (3)$$

$$f_R(w^*, qz^* + (1 - q)y^*) \leq f_R(w^*, z^*)q. \quad (4)$$

Taking the minimum of terms on the right hand sides of Eq. (1) and Eq. (2), Eq. (3) and Eq. (4) respectively, i.e.,

$$p^* \in \arg\min_{p\in[0,1]} \{(f_R(w^*, z^*) - f_R(x^*, z^*))p + f_R(x^*, z^*), f_C(w^*, z^*)p\},$$

$$q^* \in \arg\min_{q\in[0,1]} \{(f_C(w^*, x^*) - f_C(w^*, y^*))q + f_C(w^*, y^*), f_R(w^*, z^*)q\},$$

we derive the following quantities\(^4\)

$$p^* := \frac{f_R(x^*, z^*)}{f_R(x^*, z^*) + f_C(w^*, z^*) - f_R(w^*, z^*)},$$

$$q^* := \frac{f_C(w^*, y^*)}{f_C(w^*, y^*) + f_R(w^*, z^*) - f_C(w^*, z^*)}.$$

\(^4\)The denominator of $p^*$ or $q^*$ may be zero. In this case, we simply define $p^*$ or $q^*$ to be 0.
The adjusted strategy pair is now defined as
\[(x_{IL}, y_{IL}) := \begin{cases} (p^* w^* + (1 - p^*) x^*, z^*), & f_C(w^*, z^*) \geq f_R(w^*, z^*), \\ (w^*, q^* z^* + (1 - q^*) y^*), & f_C(w^*, z^*) < f_R(w^*, z^*). \end{cases} \]

In Section 4, we will see that it is easy to analyze this strategy pair quantitatively, and it is a key auxiliary structure that brings about a thorough worst-case analysis of the TS algorithm.

We remark that the outcome of all these three methods can be calculated in polynomial-time of \(m\) and \(n\).

4 A Tight Instance for All Three Methods

We now show the tight bound of the TS algorithm that we presented in the previous section, with the help of two auxiliary adjustment methods proposed in Section 3. [29] has shown that the TS algorithm gives an approximation ratio of no greater than \(b \approx 0.3393\). In this section, we construct a game on which the TS algorithm attains the tight bound \(b \approx 0.3393\). In detail, the payoff matrices of the game are presented in Eq. (5), where \(b \approx 0.3393\) is the tight bound, \(\lambda_0 \approx 0.582523\) and \(\mu_0 \approx 0.812815\) are the real numbers to be derived in Lemma 6. The game attains the tight bound \(b \approx 0.3393\) at the stationary point \(x^* = y^* = (1, 0, 0)^T\) with dual solution \(\rho^* = \mu_0 / (\lambda_0 + \mu_0), w^* = z^* = (0, 0, 1)^T\). Additionally, the bound stays \(b \approx 0.3393\) for this game even when we try to find the minimum point of \(f\) on the entire space of square \(\Lambda\).

\[
R = \begin{pmatrix} 0.1 & 0 & 0 \\ 0.1 + b & 1 & 1 \\ 0.1 + b & \lambda_0 & \lambda_0 \end{pmatrix}, \quad C = \begin{pmatrix} 0.1 & 0.1 + b & 0.1 + b \\ 0 & 1 & \mu_0 \\ 0 & 1 & \mu_0 \end{pmatrix}. \tag{5}
\]

The formal statement of this result is presented in the following Theorem 1.

**Theorem 1** (Tightness of the generalized TS algorithm). There exists a game such that for some stationary point \((x^*, y^*)\) with dual solution \((\rho^*, w^*, z^*)\),

\[
b = f(x^*, y^*) = f(x_{IL}, y_{IL}) = f(x_{MB}, y_{MB}) \leq f(\alpha w^* + (1 - \alpha) x^*, \beta z^* + (1 - \beta) y^*)
\]

holds for any \(\alpha, \beta \in [0, 1]\).

The proof of Theorem 1 is done by verifying the tight instance Eq. (5) above. However, it is not direct to verify this fact. More importantly, it is far from triviality how this tight instance comes about. Below, we present the thread of our idea by a series of lemmas and propositions that help us find the tight instance Eq. (5).

We sketch our preparation into three steps. First, we give an equivalent condition of the stationary point in Proposition 3, which makes it easier to construct payoff matrices with a given stationary point and its corresponding dual solution. Second, we draw figures of functions \(f_R\) and \(f_C\) on \(\Lambda\) and subsequently reveal the relationship among the three adjusting strategy pairs presented in Section 3. Finally, we present some estimations over \(f\) and show when these estimations are exactly tight.
During the preparations (or more precisely, attempts), we have found more accurate constraints for tight instances. A parameterization method arises naturally, and thus the tight instances were found by trying very few cases.\(^5\)

We have seen that stationary points are closely linked to von Neumann minimax theorem. We here utilize it again but in a more delicate way. Specifically, the following proposition shows how to construct payoff matrices with a given stationary point \((x^*,y^*)\) and its dual solution \((\rho^*,w^*,z^*)\).

**Proposition 3.** Let

\[
A(\rho,y,z) := -\rho Ry + (1-\rho)C(z-y),
\]
\[
B(\rho,x,w) := \rho R^T (w-x) - (1-\rho)C^T x.
\]

Then \((x^*,y^*)\) is a stationary point if and only if \(f_R(x^*,y^*) = f_C(x^*,y^*)\) and there exist \(\rho^*,w^*,z^*\) such that

\[
\text{supp}(x^*) \subset \text{suppmin}(A(\rho^*,y^*,z^*)),
\]
\[
\text{supp}(y^*) \subset \text{suppmin}(B(\rho^*,x^*,w^*)).
\]

**Proof.** First, we show that \(f_R(x^*,y^*) = f_C(x^*,y^*)\) is the necessary condition for \((x^*,y^*)\) to be a stationary point. We prove the contraposition. Suppose that \(f_R(x^*,y^*) > f_C(x^*,y^*)\), then we have \(f_R(x^*,y^*) > 0\), which implies that \(\max(Ry^*) > x^T Ry^*\). Therefore \(\text{supp}(x^*) \not\subset \text{suppmax}(Ry^*)\).

Suppose without loss of generality that \(1 \in \text{suppmax}(Ry^*), 2 \not\in \text{suppmax}(Ry^*)\) and \(2 \in \text{supp}(x^*)\). Let \(E := (1,-1,0,\cdots,0)^T \in \mathbb{R}^m\). For sufficiently small \(\theta_0 > 0\), we have \((x^* + \theta_0 E, y^*) \in \Delta_m \times \Delta_n\) and \(f_R(x^* + \theta_0 E, y^*) > f_C(x^* + \theta_0 E, y^*)\). One can verify that

\[
Df(x^*,y^*,x^* + \theta_0 E, y^*) = Df_R(x^*,y^*,x^* + \theta_0 E, y^*)
\]
\[
= -\theta_0 E^T Ry^* < 0.
\]

Therefore \((x^*,y^*)\) is not a stationary point. The case of \(f_C(x^*,y^*) > f_R(x^*,y^*)\) is similar.

Next, we prove that under the condition that \(f_R(x^*,y^*) = f_C(x^*,y^*)\), \(V(x^*,y^*) = f(x^*,y^*)\) if and only if Eq. (6) and Eq. (7) hold for some \(\rho^*,w^*,z^*\).

Suppose \(f(x^*,y^*) = V(x^*,y^*)\), by Proposition 2, there exist \(\rho^*,w^*,z^*\) such that

\[
f(x^*,y^*) = \min_{x',y'} T(x^*,y^*,x',y',\rho^*,w^*,z^*).
\]

Rewrite \(T\) as

\[
T(x^*,y^*,x',y',\rho^*,w^*,z^*) = x'^T A(\rho^*,y^*,z^*) + B(\rho^*,x^*,w^*)^T y' + \rho^* x^T Ry^* + (1-\rho^*) x^T C y^*.
\]

Notice that

\[
T(x^*,y^*,x^*,y^*,\rho^*,w^*,z^*) = f(x^*,y^*)
\]
\[
= \min_{x',y'} T(x^*,y^*,x',y',\rho^*,w^*,z^*).
\]

---

\(^5\)Such a procedure can be written into an algorithm, as we will show in Section 5.
Therefore,

\[
\text{supp}(x^*) \subseteq \text{suppmin} \ A(\rho^*, y^*, z^*), \\
\text{supp}(y^*) \subseteq \text{suppmin} \ B(\rho^*, x^*, w^*)
\]

must hold.

Now suppose Eq. (6) and Eq. (7) hold. Similarly, we have

\[
V(x^*, y^*) = \min_{x', y'} T(x^*, y^*, x', y', \rho^*, w^*, z^*) \\
= T(x^*, y^*, y^*, \rho^*, w^*, z^*) = f(x^*, y^*).
\]

\[\square\]

Note that given stationary point \((x^*, y^*)\) and dual solution \((\rho^*, w^*, z^*)\), we can restrict \(R\) and \(C\) by simple linear constraints: Finding \((R, C)\) becomes a problem of solving linear equations and linear inequalities.

Now we turn to the second step, i.e., plotting the figure of \(f_R\) and \(f_C\) on the rectangle \(\Lambda\) in general cases. To avoid burden notations, we define

\[
F_I(\alpha, \beta) := f_I(\alpha w^* + (1 - \alpha)x^*, \beta z^* + (1 - \beta)y^*), I \in \{R, C\}, \alpha, \beta \in [0, 1].
\]

Alternatively, we will show the figures of \(F_R(\alpha, \beta)\) and \(F_C(\alpha, \beta)\). An instance is presented in Figure 1 and Figure 2.

To understand why \(f_R\) and \(f_C\) have such a shape, we first define quantities \(\lambda^*\) and \(\mu^*\) as follows, which have both geometric and algebraic meanings. Let

\[
\lambda^* := (w^* - x^*)^T R z^* = f_R(x^*, z^*) - f_R(w^*, z^*) = F_R(0, 1) - F_R(1, 1), \\
\mu^* := w^*^T C(z^* - y^*) = f_C(w^*, y^*) - f_C(w^*, z^*) = F_C(1, 0) - F_C(1, 1).
\]

The vertical dashed colored lines in Figure 1 show the geometric meaning of these quantities: they are height differences. The following lemma shows that \(\lambda^*\) and \(\mu^*\) are always nonnegative height differences shown in Figure 1.

**Lemma 1.** If \(\rho^* \in (0, 1)\), then \(\lambda^*, \mu^* \in [0, 1]\). And if \(\rho^* \in \{0, 1\}\), stationary point \((x^*, y^*)\) is a Nash equilibrium.

**Proof.** We have \(\lambda^*, \mu^* \leq 1\) as all entries of \(R\) and \(C\) belong to \([0, 1]\). Suppose \(\rho^* \in (0, 1)\). By Proposition 1, \(0 \leq f(x^*, y^*) \leq T(x^*, y^*, x^*, z^*, \rho^*, w^*, z^*) = \rho^* \lambda^*\), therefore \(\lambda^* \geq 0\). Similarly, \(0 \leq f(x^*, y^*) \leq T(x^*, y^*, y^*, \rho^*, w^*, z^*) = (1 - \rho^*) \mu^*\), therefore \(\mu^* \geq 0\).

Suppose \(\rho^* \in \{0, 1\}\). By the previous inequalities, \(0 \leq f(x^*, y^*) \leq \min\{\rho^* \lambda^*, (1 - \rho^*) \mu^*\} = 0\). Therefore \((x^*, y^*)\) is a Nash equilibrium. \(\square\)

Below we always assume \(\rho^* \in (0, 1)\), since otherwise a Nash equilibrium \((x^*, y^*)\) is found, which does not match our goal of finding a tight instance.

The following lemma shows how \(F_R\) and \(F_C\) look like in the section when either \(\alpha\) or \(\beta\) is fixed. The colored solid lines in Figure 1 present the image of this lemma.

**Lemma 2.** The following two statements hold:
Figure 1: \( f_R \) and \( f_C \) on rectangle \( \Lambda \). We also tag the critical points \((x^*, y^*)\) and \((x_{MB}, y_{MB})\) and the height differences \(\lambda^*\) and \(\mu^*\). Text \((x^*, y^*)\) in the figure means \(\alpha = 0, \beta = 0\). Similarly, \((x^*, z^*)\) means \(\alpha = 0, \beta = 1\); \((w^*, y^*)\) means \(\alpha = 1, \beta = 0\); \((w^*, z^*)\) means \(\alpha = 1, \beta = 1\).

1. Given \(\beta\), \(F_C(\alpha, \beta)\) is an increasing, convex and piecewise-linear function of \(\alpha\); \(F_R(\alpha, \beta)\) is a decreasing and linear function of \(\alpha\).

2. Given \(\alpha\), \(F_R(\alpha, \beta)\) is an increasing and convex, piecewise-linear function of \(\beta\); \(F_C(\alpha, \beta)\) is a decreasing and linear function of \(\beta\).

Proof. We only prove the first statement here and the second one is symmetric. Let \(x_\alpha := \alpha w^* + (1 - \alpha)x^*, \ y_\beta := \beta z^* + (1 - \beta)y^*\).

Notice that 
\[
F_C(\alpha, \beta) = \max(C^T x_\alpha) - x_\alpha^T C y_\beta,
\]
therefore is convex and piecewise-linear in \(\alpha\) with fixed \(\beta\). A similar argument holds for \(F_R(\alpha, \beta)\). We then show the increasing property for \(F_R\). In fact,
\[
F_R(\alpha, \beta) = \max(R y_\beta) - x_\alpha^T R y_\beta,
\]
therefore is linear in \(\alpha\) with fixed \(\beta\). Further by \(\text{supp}(w^*) \subseteq S_R(y^*)\) and Lemma 1, we have
\[
F_R(0, \beta) - F_R(1, \beta) = (1 - \beta)(w^* - x^*)^T R y^* + \beta (w^* - x^*)^T R z^*
\geq \beta \lambda^* \geq 0,
\]
which shows that \(F_R(\alpha, \beta)\) is decreasing with fixed \(\beta\).
At last, to prove that $F_C(a, \beta)$ is increasing in $a$, by convexity, it suffices to show that $D f_C(x^*, y^*, w^*, y^*) \geq 0$. Note that $D f_R(x^*, y^*, w^*, y^*) \leq 0$. By the definition of stationary point, we must have

$$0 \leq D f(x^*, y^*, w^*, y^*) = D f_C(x^*, y^*, w^*, y^*) = \max_{S_C(x^*)} (C^T w^*) - \max_{S_C(x^*)} (C^T x^*) + (x^* - w^*)^T C y^*. \tag{8}$$

Notice that $f_C(x^*, z^*) = 0$ and $f_C(x, z^*) \geq 0$ for all valid $x$, so

$$0 \leq D f_C(x^*, z^*, w^*, z^*) = \max_{S_C(x^*)} (C^T w^*) - \max_{S_C(x^*)} (C^T x^*) + (x^* - w^*)^T C z^*. \tag{9}$$

Combining Eq. (8) and Eq. (9), we have

$$D f_C(x^*, y^*, w^*, y^*) = \max_{S_C(x^*)} (C^T w^*) - \max_{S_C(x^*)} (C^T x^*) + (x^* - w^*)^T C y^* = \beta D f_C(x^*, z^*, w^*, z^*) + (1 - \beta) D f_C(x^*, y^*, w^*, z^*) \geq 0.\qedhere$$

Recall that the second adjustment method yields the strategy pair $(x_{MB}, y_{MB})$. We have the following lemma indicating that $(x^*, y^*)$ and $(x_{MB}, y_{MB})$ are the minimum points on the boundary of $\Lambda$. They are the colored dots in Figure 1.

**Lemma 3.** The following two statements hold:
1. \((x^*, y^*)\) is the minimum point of \(f\) on \(\Gamma_1 = \{(ax^* + (1 - \alpha)w^*, y^*) : \alpha \in [0, 1]\} \cup \{(x^*, \beta y^* + (1 - \beta)z^*) : \beta \in [0, 1]\}\).

2. \((x_{MB}, y_{MB})\) is the minimum point of \(f\) on \(\Gamma_2 = \{(ax^* + (1 - \alpha)w^*, z^*) : \alpha \in [0, 1]\} \cup \{(w^*, \beta y^* + (1 - \beta)z^*) : \beta \in [0, 1]\}\).

Proof. Let \(x_\alpha := \alpha w^* + (1 - \alpha)x^*, \ y_\beta := \beta z^* + (1 - \beta)y^*\). For the first part, by Proposition 1, \(f_R(x^*, y^*) = f_C(x^*, y^*) = f(x^*, y^*)\). Meanwhile, Lemma 2 shows that \(f_C(x_\alpha, y^*)\) is an increasing function of \(\alpha\), and \(f_R(x_\alpha, y^*)\) is a decreasing function of \(\alpha\), therefore \(f(x_\alpha, y^*) = f_C(x_\alpha, y^*) \geq f_C(x^*, y^*) = f(x^*, y^*)\). Similarly, \(f(x^*, y_\beta) = f_R(x^*, y_\beta) \geq f(x^*, y^*)\). As a result, \((x^*, y^*)\) is the minimum point of \(f\) on \(\Gamma_1\).

For the second part, suppose \(f_C(w^*, z^*) \geq f_R(w^*, z^*)\). Again, by Lemma 2 and a similar argument, \(f(w^*, y_\beta) = f_C(w^*, y_\beta) \geq f_C(w^*, z^*) = f(w^*, z^*) \geq f(x_\alpha, z^*)\). Therefore \((x_{MB}, y_{MB}) = (x_\alpha, z^*)\) is the minimum point on \(\Gamma_2\). A similar argument holds for the case \(f_R(w^*, z^*) > f_C(w^*, z^*)\).

From the above lemmas, it is clear how Figure 1 comes about. Now we turn to find tight instances. To take a further step, it is not enough only to plot a sketch. We need a quantitative analysis, in other words, calculating the exact heights of \(F_R\) and \(F_C\) in Figure 1. We then try to make the lowest point of the figure as high as possible, which may lead to a tight instance. We first do such a process on the boundary of \(\Lambda\), denoted by \(\partial\Lambda\). Then we show that it naturally leads to the worst-case analysis on the whole square \(\Lambda\) as well.

As Lemma 3 suggests, either \((x^*, y^*)\) or \((x_{MB}, y_{MB})\) is a minimum point on \(\partial\Lambda\), depending on their \(f\) values. \((x^*, y^*)\) is a stationary point; thus it owns many properties owing to its dual LP structure, which can be utilized to estimate \(f(x^*, y^*)\). The main barrier, however, is to analyze \((x_{MB}, y_{MB})\), whose position in Figure 1 seems random, let alone its \(f\) value. We give an estimable upper bound of \(f(x_{MB}, y_{MB})\) by developing a shifted point along the boundary of square \(\Lambda\) as follows.

Recall that Lemma 2 shows when \(\alpha\) or \(\beta\) is fixed, the figure of \(F_C\) and \(F_R\) is convex. For every section, the figure of \(F_R\) and \(F_C\) becomes a convex curve. Fixing the two endpoints of the curve, we stretch the convex curve into a line, which gives an upper bound of \(F_R\) or \(F_C\). After stretching every section, the figure is stretched to a smooth surface with every section linear. Since we only care about \((x_{MB}, y_{MB})\), by the definition of \((x_{MB}, y_{MB})\), we only need to stretch \(F_R\) or \(F_C\) so that \((x_{MB}, y_{MB})\) is lifted. Such a procedure is shown in Figure 3 and the result is shown in Figure 4.

Such an upper bound can be expressed in inequalities as well (we have presented them in Section 3 to define \((x_{IL}, y_{IL})\)):

\[
\begin{align*}
f_C(pw^* + (1 - p)x^*, z^*) & \leq f_C(w^*, z^*)p, \\
f_R(w^*, qz^* + (1 - q)y^*) & \leq f_R(w^*, z^*)q.
\end{align*}
\]

After stretching, the original intersection point \((x_{MB}, y_{MB})\) shifts to a new point, which is exactly the definition of \((x_{IL}, y_{IL})\) in Section 3. \((x_{IL}, y_{IL})\) appears easier to calculate and estimate.

Let us leave the process of estimations for a while. Remember that we are doing worst-case analysis, so an upper bound is not enough if it is never tight. Luckily, we have the following lemma and proposition that point out when such an upper bound becomes tight. From a geometric view, they present the equivalent condition that stretched figures and original figures are identical and that \((x_{MB}, y_{MB})\) coincides with \((x_{IL}, y_{IL})\).
Figure 3: The procedure of stretching. Every section of $F_C$ when $\beta$ is fixed is stretched to a segment. Such a procedure lifts $(x_{MB}, y_{MB})$ to $(x_{IL}, y_{IL})$.

**Lemma 4.** The following two statements hold:

1. $F_C(\alpha, \beta) = f_C(\alpha w^* + (1 - \alpha)x^*, \beta z^* + (1 - \beta)y^*)$ is a linear function of $\alpha$ if and only if
   \[ S_C(x^*) \cap S_C(w^*) \neq \emptyset. \tag{10} \]

2. $F_R(\alpha, \beta) = f_R(\alpha w^* + (1 - \alpha)x^*, \beta z^* + (1 - \beta)y^*)$ is a linear function of $\beta$ if and only if
   \[ S_R(y^*) \cap S_R(z^*) \neq \emptyset. \tag{11} \]

**Proof.** We only prove the first statement, and the second one is similar. Let $y_\beta = \beta z^* + (1 - \beta)y^*$. Since $F_C(\alpha, \beta)$ is a convex function of $\alpha$, it suffices to prove that $Df_C(w^*, y_\beta, x^*, y_\beta) = -Df_C(x^*, y_\beta, w^*, y_\beta)$ if and only if Eq. (10) holds. One can verify that
   \[ Df_C(w^*, y_\beta, x^*, y_\beta) = \max_{S_C(w^*)} (C^T x^*) - x^*^T R y_\beta - \max_{S_C(w^*)} w^*^T C y_\beta, \]
   \[ Df_C(x^*, y_\beta, w^*, y_\beta) = \max_{S_C(x^*)} (C^T w^*) - w^*^T R y_\beta - \max_{S_C(x^*)} x^*^T C y_\beta. \]

Sum up these two equations and we have
   \[ Df_C(w^*, y_\beta, x^*, y_\beta) + Df_C(x^*, y_\beta, w^*, y_\beta) = \max_{S_C(x^*)} (C^T w^*) - \max_{S_C(w^*)} (C^T w^*) + \max_{S_C(x^*)} (C^T x^*) - \max_{S_C(x^*)} (C^T x^*) \leq 0, \]

and the equality holds if and only if $S_C(x^*) \cap S_C(w^*) \neq \emptyset.$
Figure 4: The result of stretching. Note that $(x_{MB}, y_{MB})$ is now lifted to $(x_{IL}, y_{IL})$.

**Proposition 4.** $f(x_{TS}, y_{TS}) \geq f(x_{MB}, y_{MB})$ and $f(x_{IL}, y_{IL}) \geq f(x_{MB}, y_{MB})$ always hold. Meanwhile, $f(x_{MB}, y_{MB}) = f(x_{IL}, y_{IL})$ holds if and only if

$$\begin{cases}
S_C(x^*) \cap S_C(w^*) \neq \emptyset, & \text{if } f_C(w^*, z^*) > f_R(w^*, z^*), \\
S_R(y^*) \cap S_R(z^*) \neq \emptyset, & \text{if } f_C(w^*, z^*) < f_R(w^*, z^*), \\
f_R(w^*, z^*) = f_C(w^*, z^*). & \\end{cases}$$

**Proof.** $f(x_{TS}, y_{TS}) \geq f(x_{MB}, y_{MB})$ and $f(x_{IL}, y_{IL}) \geq f(x_{MB}, y_{MB})$ are directly deducted by Lemma 3. We now prove the second part.

If $f_R(w^*, z^*) = f_C(w^*, z^*)$, then by Lemma 2 and Lemma 3, we obtain that $(x_{MB}, y_{MB}) = (x_{IL}, y_{IL}) = (w^*, z^*)$.

Suppose now $f_C(w^*, z^*) > f_R(w^*, z^*)$. Let

$$x_\alpha := \alpha w^* + (1 - \alpha) x^*,$$

so $(x_{MB}, y_{MB}) = (x_\alpha, z^*)$. Notice that $f_R(x^*, z^*) \geq f_C(x^*, z^*) = 0$, by intermediate value theorem and Lemma 2, the unique minimum point of $f$ on $\Gamma_2$, $(x_{MB}, y_{MB})$, lying on $\Phi = \{(x_\alpha, z^*) : \alpha \in [0, 1]\}$, is the intersection of $f_C$ and $f_R$. Again, by Lemma 2, $f_R$ is linear on $\Phi$ and $f_C$ is piecewise-linear on $\Phi$, therefore $(x_{MB}, y_{MB})$ coincides with $(x_{IL}, y_{IL})$ if and only if both $f_C$ is also linear on $\Phi$. By Lemma 4, $f_C(x_\alpha, z^*)$ is linear on $\Phi$ if and only if $S_C(x^*) \cap S_C(w^*) \neq \emptyset$, which completes the proof of the case.

The case $f_R(w^*, z^*) > f_C(w^*, z^*)$ is symmetric, which we omit.

We note that $(x_{TS}, y_{TS})$ is automatically included in Proposition 4. Thus our analysis involves the adjustment in the original TS algorithm as well.

Now we turn back to estimations. We present the following estimations and inequalities for $f(x^*, y^*)$ and $f(x_{MB}, y_{MB})$ and show when the equality holds.
Lemma 5. The following two estimations hold:

1. If $f_C(w^*, z^*) > f_R(w^*, z^*)$, then
   \[ f(x^*_IL, y^*_IL) = \frac{f_R(x^*, z^*)(f_C(w^*, y^*) - \mu^*)}{f_C(w^*, y^*) + \lambda^* - \mu^*} \leq \frac{1 - \mu^*}{1 + \lambda^* - \mu^*}. \]
   And symmetrically, when $f_R(w^*, z^*) > f_C(w^*, z^*)$, we have
   \[ f(x^*_IL, y^*_IL) = \frac{f_C(w^*, y^*)(f_R(x^*, z^*) - \lambda^*)}{f_R(x^*, z^*) + \mu^* - \lambda^*} \leq \frac{1 - \lambda^*}{1 + \mu^* - \lambda^*}. \]
   Furthermore, if $(x^*, y^*)$ is not a Nash equilibrium, the equality holds if and only if $f_C(w^*, y^*) = f_R(x^*, z^*) = 1$.

2. $f(x^*, y^*) \leq \min\{\rho^*\lambda^*, (1 - \rho^*)\mu^*\} \leq \frac{\lambda^*\mu^*}{\lambda^* + \mu^*}$.

Proof. The value of $f(x^*_IL, y^*_IL)$ is obtained immediately by definition. We now show the inequality holds. We only prove the case when $f_C(w^*, z^*) > f_R(w^*, z^*)$ and the other case is symmetric. Notice that
   \[ f(x^*_IL, y^*_IL) = \frac{f_R(x^*, z^*)(f_C(w^*, y^*) - \mu^*)}{f_C(w^*, y^*) + \lambda^* - \mu^*} \leq \frac{(f_C(w^*, y^*) - \mu^*)}{f_C(w^*, y^*) + \lambda^* - \mu^*} \leq \frac{1 - \mu^*}{1 + \lambda^* - \mu^*}. \]
The second line holds as $f(x^*_IL, y^*_IL) \geq 0$ and $f_R(x^*, z^*) \leq 1$, and the third line holds as
   \[ G(t) = \frac{t}{t + \lambda^*} \]
is increasing on $(0, 1 - \mu^*]$. Moreover, by the proof of Lemma 1, $\lambda^* > 0$ as $f(x^*, y^*) > 0$. As a result, the equality holds if and only if $f_C(w^*, y^*) = f_R(x^*, z^*) = 1$.

For the second part, notice that
   \[ f(x^*, y^*) = \min_{x', y'} T(x^*, y^*, x', y', \rho^*, w^*, z^*). \]
Therefore,
   \[ f(x^*, y^*) \leq T(x^*, y^*, x^*, z^*, \rho^*, w^*, z^*) = \rho^*\lambda^*, \]
   \[ f(x^*, y^*) \leq T(x^*, y^*, w^*, y^*, \rho^*, w^*, z^*) = (1 - \rho^*)\mu^*, \]
which immediately derives that $f(x^*, y^*) \leq \min\{\rho^*\lambda^*, (1 - \rho^*)\mu^*\} \leq \lambda^*\mu^*/(\lambda^* + \mu^*)$. \hfill \Box

Remark 2. Lemma 5 tells us that at worst a stationary point could reach an approximation ratio of $1/2$. In fact, by the average value inequality, $f(x^*, y^*) \leq \lambda^*\mu^*/(\lambda^* + \mu^*) \leq (\lambda^* + \mu^*)/4 \leq 1/2$. We now give the following game to demonstrate this. Consider the payoff matrices:

\[
R = \begin{pmatrix}
0.5 & 0 \\
1 & 1
\end{pmatrix}, \quad
C = \begin{pmatrix}
0.5 & 1 \\
0 & 1
\end{pmatrix}.
\]
One can verify by Proposition 3 that \( ((1,0)^T, (1,0)^T) \) is a stationary point with dual solution \( \rho^* = 1/2, w^* = z^* = (0,1)^T \) and \( f(x^*, y^*) = 1/2 \). Therefore, merely a stationary point itself cannot beat a straightforward algorithm given by [11], which always finds a solution with an approximation ratio no greater than 1/2.

The following lemma gives a numerical bound of the estimations in Lemma 6 and the equivalent condition that the quality holds.

**Lemma 6 ([29]).** Let

\[
  b = \max_{s,t \in [0,1]} \min \left\{ \frac{st}{s+t}, \frac{1-s}{1+t-s} \right\},
\]

Then \( b \approx 0.339321 \), which is attained exactly at \( s = \mu_0 \approx 0.582523 \) and \( t = \lambda_0 \approx 0.812815 \).

For now, all the preparations are finished. All conditions that lead to a tight instance are given.\(^6\) After several trials, one can find a tight instance.

At last, we prove Theorem 1 by verifying the tight instance Eq. (5) with stationary point \( x^* = y^* = (1,0,0)^T \) and dual solution \( \rho^* = \mu_0/(\lambda_0 + \mu_0), w^* = z^* = (0,0,1)^T \). Note that the theorem also guarantees the bound 0.3393 when we try to adjust on the rectangle \( \Lambda \), not only on the boundary \( \partial \Lambda \).

**Proof of Theorem 1.** We prove the theorem by verifying game Eq. (5) with stationary point \( x^* = y^* = (1,0,0)^T \) and dual solution \( \rho^* = \mu_0/(\lambda_0 + \mu_0), w^* = z^* = (0,0,1)^T \). Let \( x_a = \alpha w^* + (1-\alpha)x^*, y_\beta = \beta z^* + (1-\beta)y^* \).

**Step 1.** Verify that \( (x^*, y^*) \) is a stationary point. We have \( A(\rho^*, y^*, z^*) = f_C(x^*, y^*) = b \).

Therefore \( \{1\} = \text{supp}(x^*) \subset \{1,2,3\} = \text{suppmin}(A(\rho^*, y^*, z^*)) \). Condition Eq. (6) holds. Similarly, condition Eq. (7) holds, and the former statement is proved Proposition 3.

**Step 2.** Verify that \( S_C(x^*) \cap S_C(w^*) \neq \emptyset \) and \( f_C(w^*, z^*) > f_R(w^*, z^*) \). The latter can be checked by direction calculation. One can calculate that \( S_C(x^*) = \{2,3\} \) and \( S_C(w^*) = \{2\} \), therefore their intersection is \( \{2\} \neq \emptyset \). Consequently, by Proposition 4, \( f(x_{IL}, y_{IL}) = f(x_{MB}, y_{MB}) \) and by Lemma 4, \( f_C(x_a, y_\beta) \) is a linear function of \( \alpha \).

**Step 3.** Verify that \( \lambda^* = \lambda_0, \mu^* = \mu_0 , f_R(x^*, z^*) = f_C(w^*, y^*) = 1 \), and \( f(x^*, y^*) = b \). One can check these claims by calculation. Then by Lemma 5 and Lemma 6, \( f(x_{IL}, y_{IL}) = f(x_{MB}, y_{MB}) = b \).

**Step 4.** Verify that \( b \leq f(\alpha w^* + (1-\alpha)x^*, \beta z^* + (1-\beta)y^*) \) for any \( \alpha, \beta \in [0,1] \).

First, we do a verification similar to step 2: \( S_R(y^*) = \{2,3\} \) and \( S_R(w^*) = \{2\} \), therefore \( S_R(y^*) \cap S_R(z^*) = \{2\} \neq \emptyset \), and \( f_R(x_a, y_\beta) \) is a linear function of \( \beta \).

\(^6\)Precisely, these conditions guarantee that if we only make adjustments on boundary \( \partial \Lambda \), we will attain a tight bound 0.3393. But it suffices for the original TS algorithm.
Since \( f_1(x, y) \) \((I \in \{R, C\})\) is a linear function of \( \alpha \) or \( \beta \), we can calculate the minimum point \((x^\alpha(\beta), y^\alpha)\) of \( f \) given specific \( \beta \).

\[
\begin{align*}
  f_R(x, y) &= b + (1 - b)\beta - (b + (\lambda_0 - b)\beta)\alpha, \\
  f_C(x, y) &= b - b\beta + (1 - b + (b - \mu_0)\beta)\alpha.
\end{align*}
\]

and \( x^\alpha(\beta) \) satisfies

\[
 f_R(x^\alpha(\beta), y) = f_C(x^\alpha(\beta), y) \iff x^\alpha(\beta) = \frac{\beta}{1 + (\lambda_0 - \mu_0)\beta}w^* + \left(1 - \frac{\beta}{1 + (\lambda_0 - \mu_0)\beta}\right)y^*.
\]

Now let

\[
g(\beta) := f(x^\alpha(\beta), y) = b + (1 - b)\beta - \frac{(b + (\lambda_0 - b)\beta)\beta}{1 + (\lambda_0 - \mu_0)\beta}.
\]

As \( \lambda_0 > \mu_0 \), to prove that \( \min_\beta g(\beta) = b \), it is sufficient to show

\[
(1 - b)\beta(1 + (\lambda_0 - \mu_0)\beta) - (b + (\lambda_0 - b)\beta)\beta \geq 0.
\]

Or equivalently,

\[
h(\beta) := (1 - 2b)\beta + (b(1 + \mu_0 - \lambda) - \mu_0)\beta^2 \geq 0.
\]

Notice that \( h(\beta) \) has a negative coefficient on the square term, therefore \( h(\beta) \) is a concave function. Further, we have \( h(0) = 0 \) and \( h(1) = 1 - 2b + b(1 + \mu_0 - \lambda) - \mu_0 > 0 \). By concavity, \( h(\beta) \geq \beta h(1) + (1 - \beta)h(0) \geq 0 \).

Now we complete the proof.

From the proof of Theorem 1, we obtain the following useful corollaries.

**Corollary 1.** Suppose \( f(x, y^*) = f(x_{1L}, y_{1L}) = b \). If either of the following two statements holds:

1. \( S_C(x^*) \cap S_C(w^*) \neq \emptyset \) and \( f_C(w^*, z^*) > f_R(w^*, z^*) \),
2. \( S_R(y^*) \cap S_R(z^*) \neq \emptyset \) and \( f_R(w^*, z^*) > f_C(w^*, z^*) \),

then for any \((x, y)\) on the boundary of \( \Lambda \), \( f(x, y) \geq b \).

**Corollary 2.** Suppose \( f(x^*, y^*) = f(x_{1L}, y_{1L}) = b \), \( S_C(x^*) \cap S_C(w^*) \neq \emptyset \) and \( S_R(y^*) \cap S_R(z^*) \neq \emptyset \). Then for any \( \alpha, \beta \in [0, 1] \), \( f(\alpha w^* + (1 - \alpha)x^*, \beta z^* + (1 - \beta)y^*) \geq b \).

It is worth noting that the game with payoff matrices Eq. (5) has a pure Nash equilibrium with \( x = y = (0, 1, 0)^T \), and the stationary point

\[
(x^*, y^*) = ((1, 0, 0)^T, (1, 0, 0)^T)
\]

is a strictly-dominated strategy pair. However, the supports of strategies forming a Nash equilibrium never include strictly-dominated pure strategies! We can also construct lots of games that are able to attain the tight bound but own distinct characteristics. For instance, we can give a game with no dominant strategies but attain the tight bound. Some examples are listed in Appendix D. Such results suggest that stationary points may not be an optimal concept (in theory) for a better calculation of approximate Nash equilibrium.
5 Generating Tight Instances

In Section 4, we prove the existence of tight game instances. Furthermore, as our preparations suggest, we can mathematically profile all games that are able to attain the tight bound. In this section, we gather properties in the previous sections and present a generator for such games. Using the generator, we can dig into the previous three approximate Nash equilibrium algorithms and reveal the behavior of these algorithms and also the features of stationary points. Algorithm 1 is the generator of tight instances, in which the inputs are arbitrary \((x^*, y^*), (w^*, z^*) \in \Delta_m \times \Delta_n\). The algorithm outputs games such that \((x^*, y^*)\) is a stationary point and \((\rho^* = \lambda_0 / (\lambda_0 + \mu_0), w^*, z^*)\) is a corresponding dual solution, or outputs “NO” if there is no such game.

The main idea of the algorithm is as follows. Proposition 3 shows an easy-to-verify equivalent condition of the stationary point; and all additional equivalence conditions required by a tight instance are stated in Proposition 4, Lemma 5 and Lemma 6. All of these conditions form convex linear restrictions over \((R, C)\). Therefore, if we enumerate all pairs of possible pure strategies in \(S_R(z^*)\) and \(S_C(w^*)\) respectively, whether there exists a tight instance solution becomes a linear programming problem.

\textbf{Algorithm 1}  
**Tight Instance Generator**

\begin{itemize}
  \item \textbf{Input} \((x^*, y^*), (w^*, z^*) \in \Delta_m \times \Delta_n\).
  \item 1: if \(\text{supp}(x^*) = \{1, 2, \ldots, m\}\) or \(\text{supp}(y^*) = \{1, 2, \ldots, n\}\) then
    \item 2: Output “NO”
    \item 3: end if
  \item 4: \(\rho^* \leftarrow \mu_0 / (\lambda_0 + \mu_0)\).
  \item 5: // Enumerate \(k \in S_R(z^*)\) and \(l \in S_C(w^*)\).
  \item 6: for \(k \in \{1, \ldots, m\}\ \text{\& supp}(x^*), l \in \{1, \ldots, n\}\ \text{\& supp}(y^*)\) do
  \item 7: Solve a feasible \(R = (r_{ij})_{m \times n}, C = (c_{ij})_{m \times n}\) from the following LP with no objective function:
    \begin{itemize}
      \item 8: // basic requirements.
      \item 9: \(0 \leq r_{ij}, c_{ij} \leq 1\) for \(i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\),
      \item 10: \(\text{supp}(w^*) \subseteq S_R(y^*), \text{supp}(z^*) \subseteq S_C(x^*)\),
      \item 11: \(k \in S_R(z^*), l \in S_C(w^*)\),
      \item 12: // ensure \((x^*, y^*)\) is a stationary point.
      \item 13: \(\text{supp}(x^*) \subseteq \text{suppmin}(-\rho^* R y^* + (1 - \rho^*) (C z^* - C y^*))\),
      \item 14: \(\text{supp}(y^*) \subseteq \text{suppmin}(\rho^* (R^T w^* - R^T x^*) - (1 - \rho^*) R^T x^*)\),
      \item 15: // ensure \(f(x^*, y^*) = b\).
      \item 16: \((w^* - x^*)^T R y^* = x^T C (z^* - y^*) = b\),
      \item 17: // ensure \(f(x_{IL}, y_{IL}) = b\).
      \item 18: \(x^T R z^* = w^T C y^* = 0\),
      \item 19: \(r_{kj} = 1\) for \(j \in \text{supp}(z^*), c_{ij} = 1\) for \(i \in \text{supp}(w^*)\),
      \item 20: \(w^T R z^* = \lambda_0, w^T C z^* = \mu_0\),
      \item 21: // ensure \(f(x_{MB}, y_{MB}) = f(x_{IL}, y_{IL})\).
    \end{itemize}
\end{itemize}
\begin{align*}
l \in S_C(x^*). \\
\text{if LP is feasible then} \\
\quad \text{Output feasible solutions} \\
\text{end if} \\
\text{end for} \\
\text{if LP is infeasible in any round then} \\
\quad \text{Output “No”} \\
\text{end if}
\end{align*}

**Proposition 5.** Given \((x^*, y^*), (w^*, z^*) \in \Delta_m \times \Delta_n\), all the feasible solutions of the LP in Algorithm 1 are all the games \((R, C)\) satisfying

1. \((x^*, y^*)\) is a stationary point,
2. tuple \((\rho^* = \mu_0 / (\lambda_0 + \mu_0), w^*, z^*)\) is the dual solution\(^7\),
3. \(f_C(w^*, z^*) > f_R(w^*, z^*)\), and
4. \(f(x, y) \geq b\) for all \((x, y)\) on the boundary of \(\Lambda\).

If such a game exists, and the output is “NO” if no such game exists.

\textbf{Proof.} By Proposition 3, line 13 and line 14 together form an equivalent condition of the first two statements that \((x^*, y^*)\) is a stationary point and \((\rho^*, w^*, z^*)\) is the corresponding dual solution.

Now we prove the last two statements. By Lemma 3, it suffices to prove that the algorithm outputs all games such that \(f_C(w^*, z^*) > f_R(w^*, z^*), f(x^*, y^*) \geq b\) and \(f(x_{MB}, y_{MB}) \geq b\). By Lemma 5 and Lemma 6, we already have

\[
\min\{f(x^*, y^*), f(x_{IL}, y_{IL})\} \leq b,
\]

and the equality holds if and only if \(f(x^*, y^*) = f(x_{IL}, y_{IL}) = b\). By Proposition 4, \(f(x_{IL}, y_{IL}) \geq f(x_{MB}, y_{MB})\), so it suffices to show that \(f(x^*, y^*) = f(x_{MB}, y_{MB}) = f(x_{IL}, y_{IL}) = b\) and \(f_C(w^*, z^*) > f_R(w^*, z^*)\).

Line 16 ensures that \(f(x^*, y^*) = b\). Line 18 and line 19 together ensure that

\[
f_R(x^*, z^*) = f_C(w^*, y^*) = 1.
\]

Line 18 and line 20 together ensure that \(\lambda^* = \lambda_0\) and \(\mu^* = \mu_0\). By Lemma 5 and Lemma 6, these are equivalent conditions such that \(f(x_{IL}, y_{IL}) = b\), and it naturally leads to \(f_C(w^*, z^*) > f_R(w^*, z^*)\) by Lemma 5.

At last, by Proposition 4, line 22 is the equivalent condition such that

\[
f(x_{IL}, y_{IL}) = f(x_{MB}, y_{MB}).
\]

\(\Box\)

\(^7\)One can verify that the value of \(\rho^*\) in the dual solution of any tight stationary point has to be \(\mu_0 / (\lambda_0 + \mu_0)\), by the second part of Lemma 5.
For the sake of experiments, there are three main concerns of the generator we take into account.

First, sometimes we want to generate games such that the minimum value of $f$ on the entire $\Lambda$ is also $b \approx 0.3393$. By Corollary 2, it suffices to add a constraint $S_R(y^*) \cap S_R(z^*) \neq \emptyset$ to the LP in Algorithm 1. This is not a necessary condition though.

Second, the dual solution of the LP is usually not unique, and we cannot expect which dual solution the LP algorithm yields. [21] gives some methods to guarantee that the dual solution is unique. In practice, we simply make sure that $w^*$ and $z^*$ are pure strategies. The reason is that even if the dual solution is not unique, the simplex algorithm will end up with some optimal dual solution on a vertex, in which cases, $w^*$ and $z^*$ are often both pure strategies.

Third, all feasible LP solutions form a convex polyhedron, which indicates that the cardinality of solutions is a continuum. Hence we need a sampling method to generate tight instances. A simple approach is to set a random object function, and the LP algorithm will find different vertices of the convex polyhedron. Make convex combinations of these vertices, and the results are samples of tight instances.

### 6 Tightness of the Deligkas-Fasoulakis-Markakis Algorithm

Very recently, the work by Deligkas, Fasoulakis, and Markakis [12] provides a polynomial-time algorithm computing a $1/3$-approximate Nash equilibrium. The DFM algorithm is also based on the same descent procedure but equipped with a more complicated adjustment method by using convex combinations with additional best response strategies beyond square $\Lambda$. They prove that such an adjustment method yields an upper bound approximation ratio of $1/3$. In this section, based on techniques developed in Section 3 and Section 4, we show that $1/3$ is also the lower bound of the DFM algorithm.

We first introduce the adjustment method of the DFM algorithm. Suppose that $(x^*, y^*)$ is a stationary point and the corresponding dual solution is $(\rho^*, w^*, z^*)$. Recall that in Section 4, we define $\lambda^* = (w^* - x^*)^T R z^*$ and $\mu^* = w^T C (z^* - y^*)$. The adjustment presented in Algorithm 2 is divided into four cases. In the case that $1/2 < \lambda^* \leq 2/3 < \mu^*$ and its symmetric case, the adjustment is delicate.

**Algorithm 2 Adjustment Method in the DFM algorithm.**

**Input** $(x^*, y^*), (w^*, z^*) \in \Delta_m \times \Delta_n, \lambda^*, \mu^* \in [0, 1]$.

1. if min{$\lambda^*, \mu^*$} $\leq 1/2$ or max{$\lambda^*, \mu^*$} $\leq 2/3$ then
2. Output $(x^*, y^*)$
3. end if
4. if min{$\lambda^*, \mu^*$} $\geq 2/3$ then
5. Output $(w^*, z^*)$
6. end if
7. if $1/2 < \lambda^* \leq 2/3 < \mu^*$ then
8. $\tilde{y} \leftarrow (y^* + z^*)/2$.
9. Pick $\tilde{w} \in \Delta_m$ such that supp($\tilde{w}$) $\subseteq$ suppmax($R\tilde{y}$).
10. $t_r \leftarrow \tilde{w}^T R\tilde{y} - w^R T R\tilde{y}, v_r \leftarrow w^T R y^* - \tilde{w}^T R y^*, \hat{\mu} \leftarrow \hat{\mu} C z^* - \hat{\omega} C y^*$.
11. if $v_r + t_r \geq (\mu^* - \lambda^*)/2$ and $\hat{\mu} \geq \mu^* - v_r - t_r$ then
12: \[ \alpha \leftarrow \frac{2(v_r + t_r) - (\mu^* - \lambda^*)}{2(v_r + t_r)}. \]
13: Take \((x', y')\) between \((x^*, y^*)\) and \((\alpha w^* + (1 - \alpha) \hat{w}, z^*)\) minimizing \(f(x', y')\).
14: Output \((x', y')\)
15: else
16: \[ \beta \leftarrow \frac{1 - \mu^*/2 - t_r}{1 + \mu^*/2 - \lambda^* - t_r}; \]
17: Take \((x', y')\) between \((x^*, y^*)\) and \((w^*, (1 - \beta) \hat{y} + \beta z^*)\) minimizing \(f(x', y')\).
18: Output \((x', y')\)
19: end if
20: else
21: Do the procedure symmetric to the previous “if” case.
22: end if

We then show tight instances of the DFM algorithm matching the upper bound of 1/3. Notice that for the first two cases (line 1-6), one can verify that the 1/3 bound is achieved with the following game, modified from game Equation (5):

\[
R = \begin{pmatrix} 0 & 0 & 0 \\ 1/3 & 1 & 1 \\ 1/3 & 1/2 & 1/2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1/3 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{12}
\]

Game Eq. (12) attains the tight bound 1/3 of the DFM algorithm at the stationary point \(x^* = y^* = (1, 0, 0)^T\) with dual solution \(\rho^* = 2/3, w^* = z^* = (0, 0, 1)^T\).

In the rest of this section, we focus on the last two cases (line 7-22), which are more sophisticated. We prove that for arbitrarily small \(\epsilon > 0\), an approximation ratio of \(1/3 - \epsilon\) can be reached by some instances. Such instance family is presented in Equation (13). Again, it is a modification of game Equation (5).

\[
R = \begin{pmatrix} 0 & 0 & 0 \\ 1/3 & 1 & 1 \\ 1/3 & 2/3 - \epsilon/2 & 2/3 - \epsilon/2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1/3 - \epsilon & 1/3 - \epsilon \\ 0 & 1 & 2/3 + \epsilon \\ 0 & 1 & 2/3 + \epsilon \end{pmatrix}. \tag{13}
\]

The DFM algorithm reaches an approximation ratio of \(1/3 - \gamma(\epsilon)\) with stationary point \(x^* = y^* = (1, 0, 0)^T\) and dual solution \(\rho^* = 1/2, w^* = z^* = (0, 0, 1)^T\), where \(\gamma(\epsilon) > 0\) and \(\gamma(\epsilon) \to 0\) as \(\epsilon \to 0\).

We verify that for this instance, Algorithm 2 terminates in case 3 (line 7) and outputs a strategy profile with the claimed approximation ratio.

First, we use Proposition 3 to check that \((x^*, y^*)\) is indeed a stationary point with dual solution \((\rho^*, w^*, z^*)\). The direct calculation shows that

\[
A(\rho^*, y^*, z^*) = \left( \frac{1 - 3\epsilon}{6}, \frac{1 + 3\epsilon}{6}, \frac{1 + 3\epsilon}{6} \right)^T
\]

and

---

\(\gamma(\epsilon)\) is a positive function that approaches zero as \(\epsilon\) approaches zero. The proof of \(\gamma(\epsilon)\) is similar to the proof of \(\beta(\epsilon)\), and both functions are used to show that the DFM algorithm reaches the claimed approximation ratio. The proof involves showing that the algorithm terminates in case 3 and outputs a strategy profile with the claimed approximation ratio.

---

The analysis on the proof of the upper bound in [12] suggests that if some instance reaches an approximation ratio of 1/3 in the last two cases, then it must hold that \(\lambda^* = \mu^* = 2/3\). However, due to the boundary condition of line 1, such instance should terminate in the first case and never fall into the last two cases. Such contradiction implies that 1/3 is not attainable in the last two cases.
Thus
\[ \text{supp}(x^*) = \{1\} = \text{suppmin}(A(\rho^*, y^*, z^*)) \]
and
\[ \text{supp}(y^*) = \{1\} = \text{suppmin}(B(\rho^*, x^*, w^*)). \]

Second, it can be shown that \( \lambda^* = 2/3 - \epsilon/2 \) and \( \mu^* = 2/3 + \epsilon \), thus the input of Algorithm 2 is valid and it falls exactly into case 3 (line 7).

Third, we calculate values of variables in case 3. \( \hat{y} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}^T \), \( \hat{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T \), \( t_r = 1/6 + \epsilon/4, v_r = 0, \hat{\mu} = 2/3 + \epsilon \). Thus when \( \epsilon \leq 1/3 \), \( v_r + t_r \geq (\mu^* - \lambda^*)/2 \) and \( \hat{\mu} \geq \mu^* - v_r - t_r \).

At last, it can be calculated that \( f(x^*, y^*) = 1/3 \) and
\[
 f(\alpha w^* + (1 - \alpha)\hat{w}, z^*) = \max \left\{ \left(1 - \frac{9\epsilon}{2 + 3\epsilon}\right) \left(1 + \frac{\epsilon}{2}\right), 1 - \frac{\epsilon}{3} \right\} = \frac{1}{3} - \gamma(\epsilon).
\]

As \( \epsilon \to 0 \), \( f(\alpha w^* + (1 - \alpha)\hat{w}, z^*) \) is arbitrarily close to 1/3, as desired.

7 Experimental Analysis

In this section, we further explore the characteristics of the algorithms presented in Section 3 with the help of numerical experiments. Such empirical results may provide us with a deep understanding of the behavior of these algorithms, specifically, the behavior of stationary points and the descent procedure. Furthermore, we are interested in the tight instance generator itself presented in Section 5, particularly, on the probability that the generator outputs an instance given random inputs. At last, we compare the algorithms with other approximate Nash equilibrium algorithms, additionally showing the potentially implicit relationships among these different algorithms.

Readers can refer to Appendix C for the details of the experiments. We here list the key results and insights we gain from these experiments.

1. Our studies on the behavior of algorithms presented in Section 3 show that even in a uniformly sampled tight game instance, it is almost impossible for a uniformly-picked initial strategy pair to fall into the tight stationary point at the termination. Such results suggest that uniform initialization leads to the dramatic inconsistency of tight instances of stationary point algorithms between theory and practice.

2. We then study the stability of tight stationary points. A stationary point \((x^*, y^*)\) is stable if that, when we arbitrarily make a slight perturbation on \((x^*, y^*)\) and run the TS algorithm again, the algorithm generally terminates near \((x^*, y^*)\). We explore the stability on randomly generated tight instances with different sizes. In experiments, most tight instances of large sizes are not stable. Moreover, with the game size growing larger, the probability to find a stable tight instance becomes smaller and even vanishes. Thus it is really hard to meet an empirical approximation ratio of 0.3393 in large-size games. Based on this result and further empirical studies, we give a time-saving and effective suggestion about the practical usage of the TS algorithm: If the algorithm terminates with a bad approximation ratio, slightly perturb the
solution, and continue the algorithm. If the algorithm still terminates near the bad solution, randomly
pick an initial point outside a small neighborhood of the solution, and rerun the algorithm.

3. Next, we turn to the tight instance generator described in Section 5. Given two arbitrary
strategy pairs \((x^*, y^*)\) and \((w^*, z^*)\) in \(\Delta_m \times \Delta_n\), we are interested in whether the generator
outputs a tight game instance. The result shows that the intersecting proportion of \((x^*, y^*)\)
and \((w^*, z^*)\) plays a vital role in whether a tight game instance can be successfully generated
from these two pairs. It suggests that neither \(x^*\) and \(w^*\) share support, nor \(y^*\) and \(z^*\).

4. At last, we measure how other algorithms behave on these tight game instances. Surprisingly,
Czumaj et al.’s algorithm [8] terminates at an approximation ratio \(b \approx 0.3393\) for all cases and
all trials. Meanwhile, regret-matching algorithms [16] always find a pure Nash equilibrium
of a 2-player game if there exists, which is the case for all generated tight instances. Finally,
fictitious play algorithm [3] behaves well on these instances, with a median approximation
ratio of approximately \(1 \times 10^{-3}\) to \(1.2 \times 10^{-3}\) for games with different sizes.

8 Discussion

We present three problems that are expected to elicit a further understanding of stationary points
and the underlying structure of Nash equilibria.

1. Analyze the dynamics of the descent procedure of the TS algorithm and provide stability
analysis and smoothed analysis for worst cases.

On both theoretical and experimental sides, we have yet to determine which kinds of stationary
points are easier to reach and which are not. It is also noticeable that a minor perturbation on the
initial point leads to a significant difference in the convergence. All these phenomena are owing
to the atypical behavior of the descent procedure. When we take these into consideration, the
bound analysis becomes stability analysis and smoothed analysis.

2. Propose a benchmark for approximate Nash equilibrium computing such that most existing
polynomial-time algorithms have few advantages on the generated games.

There is a natural extension to our tight instance generator: Find a class of games rendering the
performances of most existing polynomial-time algorithms unsatisfying. It is worth noting that the
classic game generator GAMUT [25] is conquered by the TS algorithm [14]: On games generated
by GAMUT, the TS algorithm always finds a solution with an approximation ratio far better than
0.3393. Therefore, a new benchmark is required, which is of great significance to understanding
the hardness of Nash equilibrium computing.

3. Propose a novel solution concept that calculates an \(\epsilon\)-approximate Nash equilibrium directly
without any further adjustment.

The ultimate goal is to improve the approximation ratio. We summarize that all non-trivial
polynomial-time approximation algorithms presented up till now involve two steps: first, to find
a polynomial-time-solvable concept (usually by linear programming), and second, to make an
adjustment step if the concept has an unsatisfying approximation ratio [2, 8, 10, 12, 29]. The
real challenge here is to propose a novel concept that characterizes the \(\epsilon\)-approximate Nash equilibria directly without any adjustment, which could show some insightful unknown structures of approximate Nash equilibria.

References

[1] Avrim Blum and Yishay Mansour. From external to internal regret. *J. Mach. Learn. Res.*, 8:1307–1324, 2007.

[2] Hartwig Bosse, Jaroslav Byrka, and Evangelos Markakis. New algorithms for approximate Nash equilibria in bimatrix games. *Theor. Comput. Sci.*, 411(1):164–173, 2010.

[3] George W. Brown. Iterative solution of games by fictitious play. *Activity analysis of production and allocation*, 13(1):374–376, 1951.

[4] Nicolò Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge University Press, 2006.

[5] Xi Chen, Xiaotie Deng, and Shang-Hua Teng. Settling the complexity of computing two-player Nash equilibria. *J. ACM*, 56(3):14:1–14:57, 2009.

[6] Zhaohua Chen, Xiaotie Deng, Wenhan Huang, Hanyu Li, and Yuhao Li. On tightness of the tsaknakis-spirakis algorithm for approximate Nash equilibrium. In Ioannis Caragiannis and Kristoffer Arnsfelt Hansen, editors, *Algorithmic Game Theory - 14th International Symposium, SAGT 2021, Aarhus, Denmark, September 21-24, 2021, Proceedings*, volume 12885 of *Lecture Notes in Computer Science*, pages 97–111. Springer, 2021.

[7] Vincent Conitzer. Approximation guarantees for fictitious play. In *2009 47th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 636–643. IEEE, 2009.

[8] Artur Czumaj, Argyrios Deligkas, Michail Fasoulakis, John Fearnley, Marcin Jurdzinski, and Rahul Savani. Distributed methods for computing approximate equilibria. In Yang Cai and Adrian Vetta, editors, *Web and Internet Economics - 12th International Conference, WINE 2016, Montreal, Canada, December 11-14, 2016, Proceedings*, volume 10123 of *Lecture Notes in Computer Science*, pages 15–28. Springer, 2016.

[9] Constantinos Daskalakis, Paul W. Goldberg, and Christos H. Papadimitriou. The complexity of computing a Nash equilibrium. *SIAM J. Comput.*, 39(1):195–259, 2009.

[10] Constantinos Daskalakis, Aranyak Mehta, and Christos H. Papadimitriou. Progress in approximate Nash equilibria. In *Proceedings 8th ACM Conference on Electronic Commerce (EC-2007), San Diego, California, USA, June 11-15, 2007*, pages 355–358. ACM, 2007.

[11] Constantinos Daskalakis, Aranyak Mehta, and Christos H. Papadimitriou. A note on approximate Nash equilibria. *Theor. Comput. Sci.*, 410(17):1581–1588, 2009.

[12] Argyrios Deligkas, Michail Fasoulakis, and Evangelos Markakis. A polynomial-time algorithm for 1/3-approximate Nash equilibria in bimatrix games. In Shiri Chechik, Gonzalo Navarro, Eva Rotenberg, and Grzegorz Herman, editors, *30th Annual European Symposium*
[13] Vladimir F. Demyanov. *Nonsmooth Optimization*, pages 55–163. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[14] John Fearnley, Tobenna Peter Igwe, and Rahul Savani. An empirical study of finding approximate equilibria in bimatrix games. In *Experimental Algorithms - 14th International Symposium, SEA 2015, Paris, France, June 29 - July 1, 2015, Proceedings*, volume 9125 of *Lecture Notes in Computer Science*, pages 339–351. Springer, 2015.

[15] Tomás Feder, Hamid Nazerzadeh, and Amin Saberi. Approximating Nash equilibria using small-support strategies. In *Proceedings 8th ACM Conference on Electronic Commerce (EC-2007), San Diego, California, USA, June 11-15, 2007*, pages 352–354. ACM, 2007.

[16] Amy Greenwald, Zheng Li, and Casey Marks. Bounds for regret-matching algorithms. In *International Symposium on Artificial Intelligence and Mathematics, ISAIM 2006, Fort Lauderdale, Florida, USA, January 4-6, 2006*, 2006.

[17] Junling Hu and Michael P. Wellman. Multiagent reinforcement learning: Theoretical framework and an algorithm. In Jude W. Shavlik, editor, *Proceedings of the Fifteenth International Conference on Machine Learning (ICML 1998), Madison, Wisconsin, USA, July 24-27, 1998*, pages 242–250. Morgan Kaufmann, 1998.

[18] Spyros C. Kontogiannis, Panagiota N. Panagopoulou, and Paul G. Spirakis. Polynomial algorithms for approximating Nash equilibria of bimatrix games. *Theor. Comput. Sci.*, 410(17):1599–1606, 2009.

[19] Spyros C. Kontogiannis and Paul G. Spirakis. Efficient algorithms for constant well supported approximate equilibria in bimatrix games. In *Automata, Languages and Programming, 34th International Colloquium, ICALP 2007, Wroclaw, Poland, July 9-13, 2007, Proceedings*, volume 4596 of *Lecture Notes in Computer Science*, pages 595–606. Springer, 2007.

[20] Elias Koutsoupias and Christos H. Papadimitriou. Worst-case equilibria. *Comput. Sci. Rev.*, 3(2):65–69, 2009.

[21] Olvi Mangasarian. Uniqueness of solution in linear programming. Technical report, University of Wisconsin-Madison Department of Computer Sciences, 1978.

[22] John Nash. Non-Cooperative Games. *Annals of Mathematics*, 54(2):286–295, 1951.

[23] John von Neumann. Zur theorie der gesellschaftsspiele. *Mathematische annalen*, 100(1):295–320, 1928.

[24] Noam Nisan and Amir Ronen. Algorithmic mechanism design. *Games Econ. Behav.*, 35(1-2):166–196, 2001.

[25] Eugene Nudelman, Jennifer Wortman, Yoav Shoham, and Kevin Leyton-Brown. Run the GAMUT: A comprehensive approach to evaluating game-theoretic algorithms. In *3rd International Joint Conference on Autonomous Agents and Multiagent Systems (AAMAS 2004), 19-23 August 2004, New York, NY, USA*, pages 880–887. IEEE Computer Society, 2004.
A Missing Calculations in the Main Body

A.1 Calculating Derivatives

We used explicit forms of $Df$, $Df_R$ and $Df_C$ in Section 3 but omitted the calculations of them. Since these calculations are rather complex, we present the detailed calculations here for completeness.

We first calculate the explicit form of $Df_R(x, y, x', y')$. The main calculation is the directional derivative of $\max (Ry)$ with respect to $y$. Let $h := y' - y$. Notice that all entries of $\max (Ry)$ are continuous in $y$. Therefore, for sufficient small $\theta > 0$,

$$\max (R(y + \theta h)) = \max_{S_R(y)} (R(y + \theta h)).$$

Since all entries of $Ry$ over $S_R(y)$ are equal (called property ($*$)), for sufficient small $\theta > 0$, we have

$$\max (R(y + \theta h)) - \max (Ry) = \max_{S_R(y)} (R(y + \theta h) - \max (Ry)) = \max_{S_R(y)} (R(y + \theta h) - Ry) \quad \text{(by property (*))}$$

$$= \max_{S_R(y)} (\theta Rh) = \theta \max_{S_R(y)} (R(y' - y)) = \theta \max_{S_R(y)} (Ry' - Ry) = \theta \left( \max_{S_R(y)} (Ry') - \max_{S_R(y)} (Ry) \right) \quad \text{(by property (*))}$$

$$= \theta \left( \max_{S_R(y)} (Ry') - \max_{S_R(y)} (Ry) \right).$$
As a result, we have
\[
\lim_{\theta \to 0+} \frac{1}{\theta} \left( \max(R(y + \theta(y' - y))) - \max(Ry) \right) = \max(Ry') - \max(Ry).
\]

By basic calculus, we have
\[
\lim_{\theta \to 0+} \frac{1}{\theta} \left( (x + \theta(x' - x))^T R(y + \theta(y' - y)) - x^T R y \right)
= \lim_{\theta \to 0+} \frac{1}{\theta} \left( (\theta(x' - x))^T R y + x^T R (\theta(y' - y) + (\theta(x' - x))^T R (\theta(y' - y))) \right)
= (x' - x)^T R y + x^T R (y' - y).
\]

Combining these results, we get the following formula
\[
Df_R(x, y, x', y') = \max(Ry') - \max(Ry) - (x' - x)^T R y - x^T R (y' - y)
= \max(Ry') - x^T R y - x^T R y' + x^T R y - f_R(x, y).
\]

Similarly, we have
\[
Df_C(x, y, x', y') = \max(C^T x') - \max(C^T y) - (x' - x)^T C y - x^T C (y' - y)
= \max(C^T x') - x^T C y - x^T C y' + x^T C y - f_C(x, y).
\]

Now we calculate the derivative \(Df(x, y, x', y')\). First, we consider the case that \(f_R(x, y) \neq f_C(x, y)\). By continuity of \(f_R\) and \(f_C\), if \(f_R(x_0, y_0) > f_C(x_0, y_0)\), then \(f(x, y) = f_R(x, y)\) in some neighborhood of \((x_0, y_0)\); and if \(f_R(x_0, y_0) < f_C(x_0, y_0)\), then \(f(x, y) = f_C(x, y)\) in some neighborhood of \((x_0, y_0)\). Consequently,
\[
Df(x, y, x', y') = \begin{cases} 
Df_R(x, y, x', y'), & f_R(x, y) > f_C(x, y), \\
Df_C(x, y, x', y'), & f_R(x, y) < f_C(x, y).
\end{cases}
\]

Finally, we calculate the derivative \(Df(x, y, x', y')\) under the constraint that \(f_R(x, y) = f_C(x, y)\), which is rather difficult by direct calculations. We develop the following lemma to handle it.

**Lemma 7.** Let \(g, h\) be functions from \([0, 1]\) to \(\mathbb{R}\). \(g, h\) are differentiable in the positive direction at 0, and \(g(0) = h(0)\). Then \(\phi(x) = \max\{g(x), h(x)\}\) is differentiable in the positive direction at 0, and
\[
\phi_+'(0) := \lim_{\theta \to 0+} \frac{\phi(\theta) - \phi(0)}{\theta} = \max\{g_+(0), h_+(0)\}.
\]

**Proof.** Without loss of generality, suppose that \(f(0) = g(0) = 0\) and \(g'(0) \geq h'(0)\). By the result from analysis,
\[
\begin{align*}
g(x) &= g_+(0)x + a_1(x), \\
h(x) &= h_+(0)x + a_2(x),
\end{align*}
\]

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where \( a_1(x) = o(x), a_2(x) = o(x) \) when \( x \to 0^+ \). It suffices to prove that
\[
\phi(x) = g'_+(0)x + o(x).
\]

If \( g'_+(0) = h'_+(0), \max\{g(x), h(x)\} = g'_+(0)x + \max\{a_1(x), a_2(x)\} \). Since both \( a_1(x) \) and \( a_2(x) \) are \( o(x) \), clearly \( \max\{a_1(x), a_2(x)\} = o(x) \).

If \( g'_+(0) > h'_+(0) \), then for sufficient small \( x_0 > 0, (g'_+(0) - h'_+(0))x + a_1(x) - a_2(x) > 0 \) holds for all \( x \in (0, x_0) \), therefore \( \phi(x) = g(x) = g'_+(0)x + o(x) \) when \( x \to 0^+ \). \( \square \)

By Lemma 7, we obtain that
\[
Df(x, y, x', y') = \max\{Df_R(x, y, x', y'), Df_C(x, y, x', y')\}

= \max\{T_1(x, y, x', y'), T_2(x, y, x', y')\} - f(x, y),
\]
where
\[
T_1(x, y, x', y') = \max_{S(y)}(Ry' - x^TRy - x^TRy' + x^TRy),
\]
\[
T_2(x, y, x', y') = \max_{S_C(x)}(C^Tx' - x^TCy - x^TCy' + x^TCy).
\]

## A.2 Calculating the Bilinear Form of \( T \) and Dual Solution \((\rho_0, w_0, z_0)\) from Dual LP

In this part, we give the detailed variations on \( T \) that make \( T \) a bilinear form and convert the problem \( \min_{x', y'} \max_{\rho, w, z} T \) to the dual LP form. In Section 2, we denote an \( n \)-dimensional column vector with all entries equal to 1 by \( e_n \). We will use this notation in the calculations below. Recall the definition of \( T \) in Section 3 that
\[
T(x, y, x', y', \rho, w, z) = \rho(w^TRy' - x^TRy' - x^TRy + x^TRy)

+ (1 - \rho)(x^TCy - x^TCy' - x^TCy + x^TCy).
\]
We have the following identities.
\[
\rho(w^TRy' - x^TRy' - x^TRy + x^TRy)
\]
\[
= \rho w^TRy' - \rho (w^T e_m) x^TRy' - \rho (w^T e_m) y^T R x' + \rho (w^T e_m) (e_m^T x') x^TRy
\]
\[
= (\rho w^T) (R - e_m x^T R) y' + (\rho w^T) (-e_m y^T R + e_m^T x^TRy)x'.
\]

Similarly,
\[
(1 - \rho)(x^TCy - x^TCy' - x^TCy + x^TCy)
\]
\[
= ((1 - \rho)z^T)(-e_n x^T C + e_n e_n^T x^TCy)y' + ((1 - \rho)z^T)(C - e_n y^T C) x'.
\]

Combine these and we get
\[
T(x, y, x', y', \rho, w, z)
\]
\[
= (\rho w^T, (1 - \rho)z^T) \begin{pmatrix} R - e_m x^T R & -e_m y^T R + e_m^T x^TRy \\ -e_n x^T C + e_n e_n^T x^TCy & C - e_n y^T C \end{pmatrix} \begin{pmatrix} y' \\ x' \end{pmatrix}
\]
\[
= (\rho w^T, (1 - \rho)z^T) G(x, y) \begin{pmatrix} y' \\ x' \end{pmatrix}.
\]

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That is the bilinear form in the main body. To show that minimaxing this bilinear form produces a dual LP, we need the following calculations. Let \( u_{i,n} \) denote an \( n \)-dimensional column vector whose \( i \)th entry is 1 and the other entries are 0. Let \( 0_n \) denote a \( n \)-dimensional zero column vector. Consider primal problem about variable \( x', y' \):

\[
\minimize \max_{\rho \in [0,1], \supp(w) \subseteq S_R(y), \supp(z) \subseteq S_C(x), \quad (w, z) \in \Delta_m \times \Delta_n} T(x, y, x', y', \rho, w, z)
\]

s.t. \( (x', y') \in \Delta_m \times \Delta_n \).

It is equivalent to the standard LP about variable \( x', y', \delta_{1+}, \delta_{1-} \) below:

\[
\minimize \delta_{1+} - \delta_{1-}
\]

s.t. \( -(u_{i,m}^T u_{j,n}^T) G(x, y) \left( y' \right) \left( x' \right) + \delta_{1+} - \delta_{1-} \geq 0, \quad i \in S_R(y), j \in S_C(x), \)

\[
\sum_{i=1}^{n} y'_i \geq 1,
\]

\[
\sum_{i=1}^{n} -y'_i \geq -1,
\]

\[
\sum_{i=1}^{m} x'_i \geq 1,
\]

\[
\sum_{i=1}^{m} -x'_i \geq -1,
\]

\[
x'_i \geq 0, \quad i = 1, \ldots, m,
\]

\[
y'_i \geq 0, \quad i = 1, \ldots, n,
\]

\[
\delta_{1+} \geq 0,
\]

\[
\delta_{1-} \geq 0.
\]

The dual problem about variable \( \rho, w, z \) is

\[
\maximize \min_{(x', y') \in \Delta_m \times \Delta_n} T(x, y, x', y', \rho, w, z)
\]

s.t. \( (w, z) \in \Delta_m \times \Delta_n, \)

\[
\rho \in [0,1],
\]

\[
\supp(w) \subseteq S_R(y),
\]

\[
\supp(z) \subseteq S_C(x).
\]

It is equivalent to the standard dual LP about variable \( w', z', \delta_{21+}, \delta_{21-}, \delta_{22+}, \delta_{22-} \) below:

\[
\maximize (\delta_{21+} - \delta_{21-}) + (\delta_{22+} - \delta_{22-})
\]
s.t. \(- (u_i, n^T \ 0_m^T) G(x, y)^T \begin{bmatrix} w_r' \\ z_r' \end{bmatrix} + \delta_{21+} - \delta_{21-} \leq 0, \ i = 1, \ldots, n,\)
\(- (0_n^T \ u_i, m^T) G(x, y)^T \begin{bmatrix} w_r' \\ z_r' \end{bmatrix} + \delta_{22+} - \delta_{22-} \leq 0, \ i = 1, \ldots, m,\)
\[w'_i \leq 0, \ i \in \{1, \ldots, m\} \setminus S_R(y),\]
\[z'_i \leq 0, \ i \in \{1, \ldots, n\} \setminus S_C(x),\]
\[\sum_{i=1}^m w'_i + \sum_{i=1}^n z'_i \leq 1,\]
\[- \sum_{i=1}^m w'_i - \sum_{i=1}^n z'_i \leq -1,\]
\[w'_i \geq 0, \ i = 1, \ldots, m,\]
\[z'_i \geq 0, \ i = 1, \ldots, n,\]
\[\delta_{21+} \geq 0,\]
\[\delta_{21-} \geq 0,\]
\[\delta_{22+} \geq 0,\]
\[\delta_{22-} \geq 0.\]

Then the correspondence between dual solution \((\rho_0, w_0, z_0)\) and optimal solution \((w'_0, z'_0)\) of the dual LP above is \(\rho_0 = \sum_i w'_0 i, \ w' = \rho_0 w_0,\) and \(z' = (1 - \rho_0) z_0.\) It is clear that these two LPs are the corresponding primal and dual LPs in the standard form.
B Descent Procedure

In this section, we show how to find a $\delta$-stationary point in polynomial time of precision $\delta$. Recall that $V(x, y)$ is the min-max value of $T$ and $V(x, y) \leq f(x, y)$ always holds. Then immediately, we have

**Lemma 8.** A strategy pair $(x, y)$ is a $\delta$—stationary point, if and only if

$$ f_R(x, y) = f_C(x, y) $$

and

$$ V(x, y) - f(x, y) \geq -\delta. $$

When $\delta = 0$, we have $V(x, y) = f_R(x, y) = f_C(x, y)$, and by Proposition 1, $(x, y)$ is a stationary point.

Now we state the descent procedure given by [30]. The descent procedure is partitioned into 2 steps: seeking descent direction and line search.

Seeking descent direction, as the name suggests, is to find the steepest direction of $f$ in the sense of Dini directional derivative. We first fix one of $x, y$ and adjust the other to make $f_R = f_C$. Then we calculate the min-max value $V$ of $T$ and the steepest direction $(x' - x, y' - y)$ that attains $V$.

Having obtained the steepest direction $(x' - x, y' - y)$, we now search for a proper step size in that direction. This step is called line search in literature.

The pseudo-code of the descent procedure is presented in **Algorithm 3**.

**Algorithm 3** Searching for a $\delta$—stationary point

**Input** Precision $\delta$, payoff matrices $R_{m \times n}, C_{m \times n}$ and initial strategy $(x, y) \in \Delta_m \times \Delta_n$.

1: // seeking descent direction.
2: if $f_R(x, y) \neq f_C(x, y)$ then
3: if $f_R(x, y) > f_C(x, y)$ then
4: Fix $y$ and solve the following LP with respect to $x$:
   $$ \min_x \{ \max(Ry) - x^T Ry \}, $$
   s.t. $f_R(x, y) \geq f_C(x, y),$
   $$ x \in \Delta_m. $$
5: end if
6: if $f_R(x, y) < f_C(x, y)$ then
7: Fix $x$ and solve the following LP with respect to $y$:
   $$ \min_y \{ \max(C^T x) - x^T Cy \}, $$
   s.t. $f_C(x, y) \geq f_R(x, y),$
   $$ y \in \Delta_n. $$
8: end if
9: end if
10: Solve the following minimax problem with respect to \((x', y')\):
\[
\begin{align*}
\min_{x', y'} \max_{\rho, w, z} T(x, y, x', y', \rho, w, z), \\
\text{s.t. } (x', y') &\in \Delta_m \times \Delta_n, \\
\rho &\in [0, 1], \supp(w) \subset S_R(y) \text{ and } \supp(z) \subset S_C(x).
\end{align*}
\]

11: Let the optimal value of LP be \(V\) and the corresponding solution be \((x', y')\) and \((\rho, w, z)\).
12: if \(V - f(x, y) \geq -\delta\) then
13: Output \((x, y)\)
14: return
15: end if
16: // line search.
17: \(\hat{S}_R(y) \leftarrow \{1, 2, \ldots, m\} \setminus S_R(y), \hat{S}_C(x) \leftarrow \{1, 2, \ldots, n\} \setminus S_C(x)\).
18: \[
\begin{align*}
\epsilon^*_1 &\leftarrow \min_{i \in \hat{S}_R(y)} \left( \frac{\max(\mathbf{R}y) - (\mathbf{R}y)_i}{\max(\mathbf{R}y) - (\mathbf{R}y)_i + (\mathbf{R}y')_i - \max_{\hat{S}_R(y)}(\mathbf{R}y')} \right), \\
\epsilon^*_2 &\leftarrow \min_{j \in \hat{S}_C(x)} \left( \frac{\max(\mathbf{C}^T\mathbf{x}) - (\mathbf{C}^T\mathbf{x})_j}{\max(\mathbf{C}^T\mathbf{x}) - (\mathbf{C}^T\mathbf{x})_j + (\mathbf{C}^T\mathbf{x}')_j - \max_{\hat{S}_C(x)}(\mathbf{C}^T\mathbf{x}')} \right), \\
\epsilon^* &\leftarrow \min\{\epsilon^*_1, \epsilon^*_2, 1\}.
\end{align*}
\]
19: \(H \leftarrow \min\{(x' - x)^T(Ry' - y), (x' - x)^T(Cy' - y)\}\).
20: if \(H < 0\) then
21: \(\epsilon^* \leftarrow \min\{\epsilon^*, |V - f(x, y)|/(2|H|)\}\).
22: end if
23: \(x \leftarrow x + \epsilon^*(x' - x), y \leftarrow y + \epsilon^*(y' - y)\).
24: goto line 2

The minimax problem at line 10 can be solved by using the dual LP in Appendix A. We have the following convergence result.

**Theorem 2 ([29])**. Algorithm 3 terminates with a \(\delta\)-stationary point in \(O(\delta^{-2})\) steps for any \(\delta > 0\). Thus Algorithm 3 finds a \(\delta\)-stationary point in time \(O(\delta^{-2} \text{poly}(m, n))\).
C Details of Experiments

Throughout this section, we consider the distance induced by $L^\infty$ norm in $\Delta_m \times \Delta_n$.

C.1 Behavior of the Stationary Point Algorithms

In the very first experiment, we are quite interested in the behavior of algorithms we present in Section 3. Specifically, given a tight game instance, we care much on the probability that these algorithms reach the tight bound $b \approx 0.3393$ with respect to the random choice of initial strategies. By the convexity of function $f(x, y)$, we can obtain the optimal adjustment by a ternary search algorithm. Therefore, this adjustment gives a lower bound for all convex combinations in square $\Lambda$.

We generate 20 games of size $3 \times 3$, 15 games of size $3 \times 4$, 10 games of size $4 \times 4$, 3 games of size $5 \times 5$ and one game of size $6 \times 6$ by Algorithm 1, with respect to a random choice of $(x^*, y^*)$ and $(w^*, z^*)$ in $\Delta_m \times \Delta_n$. For $3 \times 3, 3 \times 4, 4 \times 4, 5 \times 5$ and $6 \times 6$ games, we partition the total space $\Delta_m \times \Delta_n$ into lattices of side length $1/10, 1/10, 1/8, 1/6$ and $1/5$ respectively, and uniformly sample an initial strategy pair from each of them. Table 1 shows the behavior of the algorithm with the optimal adjustment on the approximation ratio, given the sampled initial strategies.

Table 1: Statistics on approximation ratios on games with different sizes. The statistics is obtained by the algorithm with the optimal adjustment in Section 3.

| Game Size | \#[Sampled Points] | \#[f > 0.01] | \#[f > 0.339] | Pr[f > 0.339] |
|-----------|--------------------|--------------|----------------|----------------|
| $3 \times 3$ | 200,000            | 0            | 0              | 0              |
| $3 \times 4$ | 1,500,000          | 2            | 2              | $1.3 \times 10^{-6}$ |
| $4 \times 4$ | 2,621,440          | 0            | 0              | 0              |
| $5 \times 5$ | 5,038,848          | 0            | 0              | 0              |
| $6 \times 6$ | 9,765,625          | 0            | 0              | 0              |

The result provides us with the following insight: even though we can promise the existence of a tight stationary point by Algorithm 1, we cannot promise a high probability to actually find them in practice! This result roughly implies the inconsistency of tight instances of stationary point algorithms between theory and practice.

C.2 Stability of Tight Stationary Points

A tight stationary point is hard to find in practice, implying that even if we start the descent procedure near the tight stationary point, the TS algorithm may terminate at a faraway solution with a better approximation ratio. We call a stationary point stable if, under most slight perturbations, the TS algorithm will ultimately fall back to the same stationary point; Otherwise, we call it unstable.

9We count the cases without distinguishing the specific games of the same size, as the result is similar for every game of the same size.
The stability in experiments interprets as follows. Let \((x^*, y^*)\) be a tight stationary point. Choose a ball centered at \((x^*, y^*)\) with radius \(r\) (called perturbation ball). Randomly pick \(s\) points in the ball and run the TS algorithm with every picked point as the initial point. If the algorithm terminates with a solution whose distance to \((x^*, y^*)\) is less than \(r\), we call event “fall-back” occurs. We say \((x^*, y^*)\) is stable if “fall-back” always occurs for any picked point.

Specifically, randomly generate 1,000 games of size 3 × 3, 600 games of size 5 × 5, 300 games of size 7 × 7, 150 games of size 10 × 10, 80 games of size 15 × 15, and 50 games of size 20 × 20. For a game of size \(m \times n\), we randomly pick \(s = 16(m - 1)(n - 1)\) points in the perturbation ball with radius \(r = 0.01\). To sample games and perturbed points, we use different random methods as in Appendix C.6. In the decent procedure of the TS algorithm, we choose precision parameter \(\delta = 0.001\). For each trial, we do the above perturbation experiment and count the number of “fall-back”s. We count the number of tight stationary points that are stable for each size of the games. The result is presented in Table 2.

Table 2: Statistics of stable tight stationary points of different game sizes. We also calculate the probability that a randomly generated tight instance has a stable tight stationary point. By viewing the trials as independent Bernoulli trials, we calculate a 95% confidence interval of the probability to be stable using Wilson’s score method.

| Game Size | #[trials] | #[stable] | Pr[stable] (95% CI) |
|-----------|-----------|-----------|---------------------|
| 3 × 3     | 1,000     | 752       | 0.752 (0.724-0.778) |
| 5 × 5     | 600       | 101       | 0.168 (0.141-0.200) |
| 7 × 7     | 300       | 32        | 0.107 (0.077-0.147) |
| 10 × 10   | 150       | 0         | 0 (0-0.025)         |
| 15 × 15   | 80        | 0         | 0 (0-0.046)         |
| 20 × 20   | 50        | 0         | 0 (0-0.071)         |

As the table shows, in large games, most tight instances are not stable. Moreover, with the game size growing larger, the probability to find a tight instance with a stable tight stationary point becomes smaller and even vanishes. Thus it is really hard to meet an empirical approximation ratio of 0.3393.

C.3 “Outside-the-Ball” Strategy

As the previous part suggests, the TS algorithm performs quite well in practice. But what can we do if we, very unluckily, meet stable tight instances? We propose a strategy called “outside-the-ball”, i.e., randomly select an initial point outside the perturbation ball. Then we expect the algorithm would find better solutions.

We use the stable instances in the previous experiment. For each instance, randomly select \(s'\) initial points outside the perturbation ball and run the TS algorithm with each initial point. To sample initial points, we use the random method as in Appendix C.6. We say the “outside-the-ball” strategy is effective for one trial of one game if the TS algorithm terminates with a solution outside the perturbation ball and whose approximation ratio is below 0.339. If in 95% of the trials of a game, the “outside-the-ball” strategy is effective, we also say the “outside-the-ball” strategy is
effective for this game. The parameters to do the experiment are as follows. The perturbation ball has radius \( r = 0.01 \). For a game of size \( m \times n \), we have \( s' = 16(m - 1)(n - 1) \) initial points to try. We choose precision parameter \( \delta = 0.01 \) in the decent procedure. The statistics of the experiment are shown in Table 3.

Table 3: Results of “outside-the-ball” strategy. In fact, for any game, there are at most two trials in which “outside-the-ball” strategy is not effective.

| Game Size | # [stable] | # [“outside-the-ball” effective] |
|-----------|------------|---------------------------------|
| 3 × 3     | 752        | 752                             |
| 5 × 5     | 101        | 101                             |
| 7 × 7     | 32         | 32                              |

All trials of the “outside-the-ball” strategy succeed! Thus it is a good strategy if we have bad luck to meet stable tight instances.

C.4 Behavior of the Tight Instance Generator

In this part, we turn to the tight instance generator we described in Section 5. We focus on the “efficiency” of the tight instance generator, or formally, given two random strategy pairs \((x^*, y^*)\) and \((w^*, z^*)\) in \( \Delta_m \times \Delta_n \), the probability that the generator outputs a tight game instance. As we have already shown in Proposition 5, our generator can generally provide all tight game instances. Therefore, the results may bring us with further understanding on the distribution of tight stationary points in a general sense.

For the experiment setting, we consider games with 5 different sizes from 3 × 3 to 7 × 7. Further, for the restriction on the stationary point and its dual solution, we give four different conditions: (1) no restriction as the control group, (2) \( \text{supp}(x^*) \cap \text{supp}(w^*) = \text{supp}(y^*) \cap \text{supp}(z^*) = \emptyset \), (3) \( \text{supp}(x^*) \cap \text{supp}(w^*) \neq \emptyset \) and \( \text{supp}(y^*) \cap \text{supp}(z^*) \neq \emptyset \), and (4) \( \text{supp}(w^*) \subseteq \text{supp}(x^*) \) and \( \text{supp}(z^*) \subseteq \text{supp}(y^*) \). Under each of the 5 × 4 = 20 possible combinations of a game size and a restriction, we uniformly sample 1,000 pairs of \((x^*, y^*)\) and \((w^*, z^*)\) and count the success rate that a tight game instance can be generated. The result is shown in Table 4.

Table 4 shows that when the supports of \( x^*, w^* \) and those of \( y^*, z^* \) do not intersect, respectively, the successful generating rate increases with the game size. Meanwhile, when their support always intersect (Condition (3)), the successful generating rates are all small under experimented game sizes. At last, surprisingly, we discover that when \( \text{supp}(w^*) \subseteq \text{supp}(x^*) \) and \( \text{supp}(z^*) \subseteq \text{supp}(y^*) \), the success rate remains zero whatever the game size is!

C.5 Comparison with Other Algorithms

In the very last, we make an experiment on how other approximate Nash equilibrium algorithms behave on those tight instances that stationary point algorithms are expected not to perform well, and therefore compare stationary point algorithms with these algorithms. Specifically, we consider three algorithms: Czumaj et al.’s algorithm [8] with an approximation ratio of 0.38, regret-matching algorithms [16] in online learning, and fictitious play algorithm [3] with an approximation ratio of 1/2 within constant rounds [7, 15].
We generate 1,860 tight game instances with different sizes by Algorithm 1 for the test: 1,000 $3 \times 3$ games, 500 $5 \times 5$ games, 200 $7 \times 7$ games, 100 $10 \times 10$ games, 50 $15 \times 15$ games and 10 $20 \times 20$ games. For every game instance, we run each of the three algorithms 20 times provided the randomness of these algorithms and count the approximation ratio.

The results turn out to be surprising. First, Czumaj et al.’s algorithm terminates at an approximation ratio of 0.3393 for all cases and all trials. The reason for such a consequence is that the Nash equilibrium of the zero-sum game specified by the algorithm already satisfies the required approximation ratio of 0.38; therefore, the further adjustment step never happens.

Meanwhile, regret-matching algorithms always find a pure Nash equilibrium of a 2-player game if there exists one, which is the case for all generated tight instances. However, we believe that there still exists some tight game instances with no pure Nash equilibrium.

At last, Figure 5 shows the performance of fictitious play algorithm on these tight instances. For games of all sizes, fictitious play algorithm shows a bell-shaped distribution on the approximation ratios, with median value approximately $1.2 \times 10^{-3}$ on games with size no smaller than $5 \times 5$. For $3 \times 3$ games, fictitious play algorithm behaves even better, with lots of instances finding a Nash equilibrium, and median value decreasing to approximately $1.0 \times 10^{-3}$. We explain the good performance of fictitious play algorithm as the set of tight instances of fictitious play algorithm are of zero measure over the set of tight instances of algorithms presented in Section 3. In other words, there are some special latent relationships between fictitious play algorithm and stationary points.

C.6 Random Methods for Sampling

In this part, we introduce the omitted details of random samplings in previous experiments. We employ different random methods for three objects: games in most experiments, perturbed points in Appendix C.2, and initial points of “outside-the-ball” in Appendix C.3.

\[\text{Table 4: Success rate of generating a tight game instance } (R, C) \text{ with different game size under different restrictions on } (x^*, y^*) \text{ and } (w^*, z^*).\]

| Restrictions                  | $3 \times 3$ | $4 \times 4$ | $5 \times 5$ | $6 \times 6$ | $7 \times 7$ |
|-------------------------------|--------------|--------------|--------------|--------------|--------------|
| No Restriction                | 3.0%         | 5.0%         | 5.7%         | 6.8%         | 6.3%         |
| $\text{supp}(x^*) \cap \text{supp}(w^*) = \emptyset$, $\text{supp}(y^*) \cap \text{supp}(z^*) = \emptyset$ | 56.4%        | 85.6%        | 94.7%        | 98.0%        | 99.3%        |
| $\text{supp}(x^*) \cap \text{supp}(w^*) \neq \emptyset$, $\text{supp}(y^*) \cap \text{supp}(z^*) \neq \emptyset$ | 0.0%         | 0.0%         | 0.4%         | 0.3%         | 1.1%         |
| $\text{supp}(w^*) \subseteq \text{supp}(x^*)$, $\text{supp}(z^*) \subseteq \text{supp}(y^*)$ | 0.0%         | 0.0%         | 0.0%         | 0.0%         | 0.0%         |

\[\text{Similar to the first experiment, we count the cases without distinguishing the specific games of the same size, since the result is similar for every game of the same size.}\]
Games. To sample $t$ games of size $m \times n$, we need 10 groups of $x^*, y^*, w^*, z^*$ that Algorithm 1 can use to efficiently generate games (see Appendix C.4), and then sample $t/10$ tight instances for each $x^*, y^*, w^*, z^*$.

Step 1. Uniformly pick nonempty sets $\text{supp}(x^*)$, $\text{supp}(y^*)$, $\text{supp}(w^*)$ and $\text{supp}(z^*)$ satisfying

$$\text{supp}(x^*) \neq \{1, 2, \ldots, m\},$$

$$\text{supp}(y^*) \neq \{1, 2, \ldots, n\},$$

$$\text{supp}(x^*) \cap \text{supp}(w^*) = \emptyset,$$

and

$$\text{supp}(y^*) \cap \text{supp}(z^*) = \emptyset.$$

For each vector $x^*, y^*, w^*, z^*$, uniformly and independently pick real numbers from $[0, 1]$ for the support indices. Normalize these vectors to $\Delta_m$ or $\Delta_n$. By this method, we obtain valid $x^*, y^*, w^*, z^*$. Input it to Algorithm 1.

Step 2. For the LP in each enumeration of $k \in S_R(z^*)$ and $l \in S_C(w^*)$ in Algorithm 1, uniformly and independently pick $m$ vectors in $[0, 1]^{2mn}$ as the coefficients of the object function for the LP. And then for each function, choose to maximize or minimize it with equal probabilities. Let the optimal solutions for $m$ object functions be $(R_i, C_i)_{i=1}^m$. Uniformly and independently pick $t/10$ vectors from $[0, 1]^{mn} \times [0, 1]^{mn}$ and normalize them. Make $t/10$ convex combinations of $(R_i, C_i)$ using these vectors as weights and output the combination results.

Perturbed points. Suppose the radius of the perturbation ball is $r$. Uniformly and independently pick $m + n$ real numbers from $[-r, r]$. Add $m$ of them to $x^*$ and $n$ of them to $y^*$ index by index. Make the negative entries in the added vectors be 0. Normalize the vectors and output them as a perturbed point.
Initial points of “outside-the-ball”. Uniformly and independently pick two vectors from $[0, 1]^m$ and $[0, 1]^n$, respectively. Normalize these two vectors. If the normalization result lies in the perturbation ball, redo the previous procedure. Otherwise, output the result as an initial point of the strategy “outside-the-ball”.
D More Tight Instances

We present two more tight instances without proof. One can check their correctness by similar steps in Theorem 1.

Example 1 (Tight instances of every size). Suppose \( m, n > 2 \). The game with payoff matrices Eq. (14) attains the tight bound \( b \) at stationary point \( x^* = (1, 0, \ldots, 0)_m^T, y^* = (1, 0, \ldots, 0)_n^T \) and dual solution \( \rho^* = \mu_0 / (\lambda_0 + \mu_0), w^* = (0, 1, 0, \ldots, 0)_m^T \) and \( z^* = (0, 1, 0, \ldots, 0)_n^T \).

\[
R = \begin{pmatrix}
0.1 & 0 & \ldots & 0 \\
0.1 + b & \lambda_0 & \ldots & \lambda_0 \\
0.1 + b & 1 & \ldots & 1 \\
0.1 + b & 1 & \ldots & 1
\end{pmatrix}_{m \times n},
\]

\[
C = \begin{pmatrix}
0.1 & 0.1 + b & \ldots & 0.1 + b \\
0 & \mu_0 & \ldots & 1 \\
0 & \mu_0 & \ldots & 1
\end{pmatrix}_{m \times n}.
\] (14)

Example 2 (Tight instance with no dominated strategy). The game with payoff matrices Eq. (15) attains the tight bound \( b \) at stationary point \( x^* = y^* = (1/2, 1/2, 0, 0)^T \), and dual solution \( \rho^* = \mu_0 / (\lambda_0 + \mu_0), w^* = z^* = (0, 0, 1/2, 1/2)^T \). One can verify that there is no dominated strategy for either player in this game.

\[
R = \begin{pmatrix}
2b + 0.2 & 0 & 0 & 0 \\
0 & 2b + 0.2 & 0 & 0 \\
2b + 0.17 & 2b + 0.03 & 1 & 1 \\
2b + 0.03 & 2b + 0.17 & 2\lambda_0 - 1 & 2\lambda_0 - 1
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
2b + 0.2 & 0 & 2b + 0.17 & 2b + 0.03 \\
0 & 2b + 0.2 & 2b + 0.03 & 2b + 0.17 \\
0 & 0 & 1 & 2\mu_0 - 1 \\
0 & 0 & 1 & 2\mu_0 - 1
\end{pmatrix}.
\] (15)