Entire convex curvature flow in Minkowski space

Zhizhang Wang · Ling Xiao

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Abstract
In this paper, we study fully nonlinear curvature flows of noncompact spacelike hypersurfaces in Minkowski space. We prove that if the initial hypersurface satisfies certain conditions, then the flow exists for all time. Moreover, we show that after rescaling the flow converges to the future timelike hyperboloid, which is a self-expander.

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1 Introduction

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$  

In this paper, we study fully nonlinear curvature flows of noncompact spacelike hypersurfaces in Minkowski space. Spacelike hypersurfaces $\mathcal{M} \subset \mathbb{R}^{n,1}$ have an everywhere timelike normal field, which we assume to be future directed and to satisfy the condition $\langle \nu, \nu \rangle = -1$. Such hypersurfaces can be locally expressed as the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying $|Du(x)| < 1$ for all $x \in \mathbb{R}^n$.

We consider a family of spacelike embeddings

$$X(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^{n,1}.$$  

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with corresponding hypersurfaces \( \mathcal{M}(t) = \{ (x, u(x, t)) \mid x \in \mathbb{R}^n \} \) satisfying the evolution equation

\[
\frac{\partial X}{\partial t} = F^\alpha u.
\]

Here \( \alpha > 0 \), \( F = f(\kappa_1, \ldots, \kappa_n) = \frac{1}{f_s(\kappa_1^{-1}, \ldots, \kappa_n^{-1})} \), and \( f_s \) satisfies the following conditions:

\[
f_s \text{ is a concave function in } \Gamma_n^+, \quad (1.2)
\]

where \( \Gamma_n^+ := \{ \kappa \in \mathbb{R}^n : \text{each component } \kappa_i > 0 \} \),

\[
f_s > 0 \text{ in } \Gamma_n^+, \quad f_s = 0 \text{ on } \partial \Gamma_n^+, \quad (1.3)
\]

and

\[
f^i_s := \frac{\partial f_s}{\partial \kappa_i} > 0 \text{ in } \Gamma_n^+, \quad 1 \leq i \leq n. \quad (1.4)
\]

In addition, we shall assume \( f_s \) satisfies the following more technical assumptions. These include

\[
f_s \text{ is homogeneous of degree one,} \quad (1.5)
\]

there exists \( \epsilon_0 > 0 \) such that \( \sum f^i_s \kappa_i^2 \geq \epsilon_0 f_s \sum \kappa_i \), \( (1.6) \)

and for every \( C > 0 \) and every compact set \( E \) in \( \Gamma_n^+ \), there exists \( R = R(E, C) > 0 \) such that

\[
f_s(\kappa_1, \ldots, \kappa_{n-1}, \kappa_n + R) > C, \forall \kappa \in E. \quad (1.7)
\]

Finally, for our convenience, we also suppose \( f_s \) is normalized

\[
f_s(1, \ldots, 1) = 1. \quad (1.8)
\]

All of these assumptions are satisfied by \( f_s = (s_n)^{\beta} (s_k)^{\frac{1-\beta}{n}} \), where \( 0 \leq k \leq n, \beta \in (0, 1] \), and

\[
s_k = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \kappa_{i_1} \ldots \kappa_{i_k}
\]

is the normalized \( k \)-th elementary symmetric polynomial \( (s_0 = 1) \). One can find more examples of \( f_s \) in [10].

We also assume the initial hypersurface \( \mathcal{M}_0 = \{ (x, u_0(x)) \mid x \in \mathbb{R}^n \} \) satisfies:

**Conditions A:** (1) strictly convex, (2) spacelike, (3) the curvatures of \( \mathcal{M}_0 \), denoted by \( \kappa[\mathcal{M}_0] = (\kappa_1, \ldots, \kappa_n) \), and their derivatives are uniformly bounded. We also need the following condition:

\[
\sqrt{|x|^2 + C_0} < u_0(x) < \sqrt{|x|^2 + C_1}, \quad (4)
\]

for some \( C_1 > C_0 > 0 \).

Note that condition (4) is needed to obtain the local \( C^1 \) estimates (see Subsection 4.1).

**Remark 1** It is clear that condition (4) implies

\[
u^*_0(x) \rightarrow |x|, \text{ as } |x| \rightarrow +\infty.
\]

In view of Lemma 14 of [15], we obtain the Legendre transform of \( u_0 \) which we denote by \( u^*_0 \), satisfies \( u^*_0 = 0 \) on \( \partial B_1 \). By the strict convexity of \( u^*_0 \) we have \( u^*_0 < 0 \) in \( B_1 \).
There is a great deal of literature concerning the entire graphical curvature flow. In Minkowski space, Ecker [7] proved the long time existence for entire graphical solutions of the mean curvature flow. In Aarons [2], Aarons studied the forced mean curvature flow of entire graphs in Minkowski space. With initial data $M_0$ being a smooth strictly spacelike hypersurface, Aarons proved long time existence of solutions to the flow where the forcing term is smooth. Furthermore, Aarons also obtained some convergence results when the forcing term is a constant and $M_0$ has bounded curvatures. In Bayard and Schnürer [3] constructed solutions to the logarithmic Gauss curvature flow for convex spacelike hypersurfaces with prescribed Gauss map images. They showed that the solutions converge to the constant Gauss curvature hypersurfaces. In Euclidean space, entire graphical solutions of the mean curvature flow have been studied by Ecker and Huisken in [8]. Entire graphical solutions of fully-nonlinear flows were recently studied by Daskalopoulos and her collaborators in [4, 9].

The flow (1.1) is well understood when the initial entire graphical hypersurface is co-compact. It was shown in [1] that for any co-compact, spacelike, uniformly locally convex initial embedding $M_0$, there exists a unique co-compact solution $M_t$ satisfying for any fixed $t \in (0, \infty)$, $u(x, t) - |x| \to 0$ as $|x| \to \infty$. The rescaled embeddings $\tilde{X} = \frac{X}{\left[\left(1+\alpha\right)F(I)\right]^{\frac{1}{1+\alpha}}}$ converge in $C^\infty$ to an embedding with image equal to the future timelike hyperboloid $H^n$. As it was proved in Section 12 of [1], this result implies a convergence result for a class of special solution of the cross-curvature flow (see [5]) of negatively curved Riemannian metric on three-manifolds.

This work is concerned with the long time existence and convergence of (1.1) for initial data $M_0$ which is an entire graph satisfying Conditions A. In particular, we prove the following theorem.

**Theorem 2** Let $M_0$ be an entire spacelike hypersurface satisfying Conditions A, and the dual of function $f$, i.e., $f_*$, satisfies (1.4)–(1.7). Then, given $M_0$, there exists a unique $C^\infty$ smooth, entire, spacelike, strictly convex solution $M_t := \{ (x, u(x, t)) | (x, t) \in \mathbb{R}^n \times (0, \infty) \}$ of the Eq. (1.1) satisfying for any fixed $t \in (0, \infty)$, $u(x, t) - |x| \to 0$ as $|x| \to \infty$. The rescaled embeddings given by $\tilde{X} = \frac{X}{\left[\left(1+\alpha\right)F(I)\right]^{\frac{1}{1+\alpha}}}$ converge in $C^\infty$ to an embedding with image equal to the future timelike hyperboloid $H^n$, where $\tilde{t} = (1 + \alpha)^{-1} + t$ and $I$ is the identity matrix.

**Remark 3** Notice that here we study the complete flow instead of the co-compact flow, we need to deal with the behavior of the flow at infinity. Therefore, our techniques here are completely different from those in [1]. Moreover, in order to obtain that this flow converges, we need to develop new methods to obtain interior estimates. The methods we developed here are completely different from those used in studying curvature flows with forcing terms (see [2, 3] for examples).

**Remark 4** In Sect. 5 we will see that the rescaled embeddings $\tilde{X}$ converge to a self-expander satisfying

$$F_\alpha = -\langle X, v \rangle,$$

which happens to be the future timelike hyperboloid. On the other hand, for any hypersurface $M$ satisfies (1.9), we have $[(1 + \alpha)t]^{\frac{1}{1+\alpha}} X$ satisfies flow Eq. (1.1). By the result in [16], we know that there exist infinitely many hypersurfaces that satisfying (1.9). Therefore, in order for the rescaled flow to converge to the hyperboloid, the condition (4) in Conditions A is necessary.

The organization of the paper is as follows. In Sect. 2 we give a special solution to Eq. (1.1). This solution inspires us to formulate the approximate problem and construct barrier
functions in later sections. In Sect. 3 we prove the long time existence of the approximate problem. Local $C^1$ and $C^2$ estimates are established in Sect. 4. Combining with Sect. 3, we prove the existence of solution to (1.1) for all time. In Sect. 5 we show that after rescaling the solution of (1.1) converges to the unit future hyperboloid as $t \to \infty$.

2 Special solution of equation (1.1)

In this section, we will introduce a special solution for Eq. (1.1). We will also discuss the Legendre Transform of this special solution. In later sections, we will use this special solution to construct various barrier functions.

Let’s first note that since $\mathcal{M}$ is spacelike, the position vector of $\mathcal{M}$ can be written as $X = (x, u(x))$. After reparametrization, we can rewrite (1.1) as following equation

$$
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} = F^\alpha \left( \frac{1}{w} \gamma^{ik} u_{kl} \gamma_{lj} \right) w \\
u(\cdot, 0) = u_0(x),
\end{cases}
\end{aligned}
$$

where $w = \sqrt{1 - |Du|^2}$, $\gamma^{ik} = \delta_{ik} + \frac{u_i u_k}{w(1+w)}$ is the square root of the inverse metric, i.e., $\sum_k \gamma^{ik} \gamma_{kj} = g_{ij}$, and $u_{kl} = D^2_{x_k, x_l} u$ is the ordinary Hessian of $u$. Let $z = \sqrt{|x|^2 + [(1+\alpha)t]^{\frac{2}{1+\alpha}}}$, then a straightforward calculation yields

$$
\begin{aligned}
z_t &= \left( |x|^2 + [(1+\alpha)t]^{\frac{2}{1+\alpha}} \right)^{-\frac{1}{2}} [(1+\alpha)t]^{\frac{1}{1+\alpha}} , \\
k[\mathcal{M}_t] &= [(1+\alpha)t]^{\frac{1}{1+\alpha}} (1, 1, \cdots, 1) , \\
F^\alpha (k[\mathcal{M}_t]) &= [(1+\alpha)t]^{\frac{1}{1+\alpha}} ,
\end{aligned}
$$

and

$$
w = \sqrt{1 - \frac{|x|^2}{|x|^2 + [(1+\alpha)t]^{\frac{2}{1+\alpha}}}} = \frac{[(1+\alpha)t]^{\frac{1}{1+\alpha}}}{z} .
$$

Therefore, $z = \sqrt{|x|^2 + [(1+\alpha)t]^{\frac{2}{1+\alpha}}}$ satisfies (2.1) for $t \in (0, \infty)$. We can see that, for $\mathcal{M}_t = \{(x, z(x, t)) \mid (x, t) \in \mathbb{R}^n \times \mathbb{R}_+\}$, $k[\mathcal{M}_t] \to 0$ as $t \to \infty$. Moreover, let $\tilde{X} = X_{[1+(\alpha)t]^{\frac{1}{1+\alpha}}}$ where $X$ is the position vector of $\mathcal{M}_t$. Then, for any fixed $t > 0$, the rescaled graph $\tilde{\mathcal{M}}(t) = \{(x, \sqrt{|x|^2 + 1}) \mid x \in \mathbb{R}^n\}$ is the standard hyperboloid satisfying $F^\alpha (k[\mathcal{M}]) = -\langle X, v \rangle$.

Next, let’s consider the Legendre transform of $z$. By definition we have

$$
\begin{aligned}
z^* &= x \cdot Dz - z = -\frac{[(1+\alpha)t]^{\frac{2}{1+\alpha}}}{\sqrt{|x|^2 + [(1+\alpha)t]^{\frac{2}{1+\alpha}}}} \\
&= -[(1+\alpha)t]^{\frac{1}{1+\alpha}} \sqrt{1 - |Dz|^2} = -[(1+\alpha)t]^{\frac{1}{1+\alpha}} \sqrt{1 - |\xi|^2} .
\end{aligned}
$$
It’s easy to verify that $z^*$ is a solution of
\[
\begin{cases}
  z^*_a = -F^{-a}_*(w^*y^*_ikz^*_klY^*_lj)w^* & \text{in } B_1 \times (0, \infty) \\
  z^*(\cdot, t) = 0 & \text{on } \partial B_1 \times [0, \infty) \\
  z^*(\cdot, 0) = 0 & \text{in } B_1 \times [0, 
\end{cases}
\]
where $w^* = \sqrt{1-|\xi|^2}$, $Y^*_ik = \delta_{ik} - \frac{\xi_i\xi_k}{1+w^*}$, $u^*_kl = \frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j}$, and
\[
F^{-a}_*(w^*y^*_iku^*_klY^*_lj) = f^{-a}_*(\kappa^*[w^*y^*_iku^*_klY^*_lj]).
\]
Here $\kappa^*[w^*y^*_iku^*_klY^*_lj] = (\kappa^*_1, \cdots, \kappa^*_n)$ are the eigenvalues of the matrix $(w^*y^*_iku^*_klY^*_lj)$. Since when $\mathcal{M}_t$ satisfies Eq. (1.1), its support function $v = \langle X, v \rangle$ satisfies $v_t = -F^g(\kappa[\mathcal{M}_t])$. In this paper we will consider the solvability of the following problem:
\[
\begin{cases}
  u^*_r = -F^{-a}_*(w^*y^*_iku^*_klY^*_lj)w^* & \text{in } B_1 \times (0, \infty) \\
  u^*(\cdot, t) = 0 & \text{on } \partial B_1 \times [0, \infty) \\
  u^*(\cdot, 0) = u^*_0 & \text{on } B_1 \times [0, 
\end{cases}
\]

### 3 Solvability of the approximate problem

Notice that Eq. (2.3) is degenerate. Inspired by the special solution $z^*$ of (2.2), in this section we will consider the solvability of the following approximate problem:
\[
\begin{cases}
  (u^*_r)_r = -F^{-a}_*(w^*y^*_iku^*_klY^*_lj)w^* & \text{in } B_r \times (0, T) \\
  u^*_r(\cdot, t) = \hat{u}^*_0 - \left[(1+\alpha)\hat{t} \right]^{\frac{1}{1+\alpha}} \sqrt{1-r^2} & \text{on } \partial B_r \times [0, T] \\
  u^*_r(\cdot, 0) = \hat{u}^*_0 - \sqrt{1-|\xi|^2} = u^*_0 & \text{on } B_r \times [0, 
\end{cases}
\]
where $\hat{t} = t + (1+\alpha)^{-1}$ and $T > 0$ is an arbitrary positive number.

**Remark 5** Note that for our convenience, in the rest of this paper, we always assume $\hat{u}^*_0 = u^*_0 + \sqrt{1-|\xi|^2}$ to be strictly convex. For if it’s not, we can consider the following equation instead
\[
\begin{cases}
  (u^*_r)_r = -F^{-a}_*(w^*y^*_iku^*_klY^*_lj)w^* & \text{in } B_r \times (0, T) \\
  u^*_r(\cdot, t) = \hat{u}^*_0 - \epsilon \left[(1+\alpha)\hat{t} \right]^{\frac{1}{1+\alpha}} \sqrt{1-r^2} & \text{on } \partial B_r \times [0, T] \\
  u^*_r(\cdot, 0) = \hat{u}^*_0 - \epsilon \sqrt{1-|\xi|^2} = u^*_0 & \text{on } B_r \times [0, 
\end{cases}
\]
where $\epsilon > 0$ is a small constant such that $\hat{u}^*_0 = u^*_0 + \epsilon \sqrt{1-|\xi|^2}$ is strictly convex.

#### 3.1 $C^0$ estimates for $u^*_r$ – solution of (3.1)

In this subsection, we will establish the uniform upper and lower bounds for the solution $u^*_r$ of (3.1).

Let’s denote $u^c = \sqrt{|x|^2 + a^c_0}$ and $\tilde{u}^c = \sqrt{|x|^2 + a^c_1}$, where $a^c_0 < \min\{C_0, 1\}$, $a^c_1 > \max\{C_1, 1\}$, and the constants $C_0, C_1$ are the same constants as in (4) of **Conditions A**. We
will denote the Legendre transform of \( \tilde{u}^s \), \( u^s \) by \( \tilde{u}^{s*} \), \( u^{s*} \) respectively. Applying Lemma 13 of [15] we know \( \tilde{u}^{s*} < u^s_0 < u^{s*} \). Moreover, a straightforward calculation yields
\[
F^\alpha(\kappa[\mathcal{M}_{\tilde{u}^s}]) = a_1^{-\alpha} < -(X_{\tilde{u}^s}, v_{\tilde{u}^s}) = a_1, \text{ for } a_1 > 1,
\]
where \( X_{\tilde{u}^s}, v_{\tilde{u}^s} \) are position vector and normal vector of \( \mathcal{M}_{\tilde{u}^s} \). We get \( \tilde{u}^{s*} \) satisfies
\[
-F^{-\alpha}_*(w^* \gamma^*_{ik} \tilde{u}^{s*}_{kl} \gamma^*_l) \sqrt{1 - |\xi|^2} > \tilde{u}^{s*}.
\]

Consider \( \tilde{u}^s = [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \tilde{u}^{s*} \), we have
\[
\tilde{u}^*_t = [(1 + \alpha)\tilde{t}]^{-\alpha} \tilde{u}^{s*} < -[(1 + \alpha)\tilde{t}]^{-\alpha} F^{-\alpha}_*(w^* \gamma^*_{ik} \tilde{u}^{s*}_{kl} \gamma^*_l) w^*
\]
\[
= -F^{-\alpha}_*(w^* \gamma^*_{ik} \tilde{u}^{s*}_{kl} \gamma^*_l) w^*. \tag{3.3}
\]

**Lemma 6** Denote \( u^{s*}_b = \tilde{u}^s - [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \tilde{u}^{s*} \), then \( u^{s*}_b \) is a subsolution of (3.1) on \( \tilde{B}_t \times [0, T] \) for any \( T > 0 \).

**Proof** On \( \tilde{B}_t \times \{0\} \) we have
\[
u^{s*}_b(\cdot, 0) = \tilde{u}^{s*} - \sqrt{1 - |\xi|^2} < u^s_0 \text{ in } \tilde{B}_t,
\]
where \( \tilde{u}^{s*} = -a_1 \sqrt{1 - |\xi|^2} \).

On \( \partial \tilde{B}_t \times [0, \infty) \),
\[
u^{s*}_b(\cdot, t) = [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \tilde{u}^{s*} - [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \sqrt{1 - r^2} < \tilde{u}^{s*}_0 - [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \sqrt{1 - r^2}.
\]
Moreover, by earlier discussion it’s not hard to see that
\[
\frac{\partial u^{s*}_b}{\partial t} + F^{-\alpha}_*(w^* \gamma^*_{ik} (u^{s*}_b)_{kl} \gamma^*_l) w^* < 0.
\]
Therefore, Lemma 6 is proved. \( \square \)

Similarly, we can show

**Lemma 7** Denote \( u^{r*}_s = [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} u^{s*} \), then \( u^{r*}_s \) is a supersolution of (3.1) on \( \tilde{B}_r \times [0, T] \) for any \( T > 0 \).

**Proof** On \( \tilde{B}_r \times \{0\} \) we have
\[
u^{r*}_s(\cdot, 0) = \tilde{u}^{r*} > u^s_0.
\]
On \( \partial \tilde{B}_r \times (0, T] \) we want to show
\[
[(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} u^{r*} > \tilde{u}^s_0 - [(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \sqrt{1 - r^2}.
\]
This is equivalent to show
\[
[(1 + \alpha)\tilde{t}] \frac{1}{1 + \alpha} \left( -a_0 \sqrt{1 - r^2} + \sqrt{1 - r^2} \right) > \tilde{u}^s_0
\]
we can always choose \( 0 < a_0 < 1 \) small such that the l.h.s > 0. By our assumption on the initial hypersurface, we have \( \tilde{u}^s_0 \) is a strictly convex function with \( \tilde{u}^s_0 = 0 \) on \( \partial B_1 \), this
implies \textbf{r.h.s} < 0. We have shown \( u_{s}^{r*} > u_{r}^{s} \) on the parabolic boundary. Moreover, it’s easy to see that \( u_{s}^{r*} \) satisfies
\[
(u_{s}^{r*})_{s} > -F_{s}^{-\alpha} \left( w_{s}^{n} (u_{s}^{r*})_{k} (y_{ij}^{s}) \right) w^{*}.
\]
This completes the proof of the Lemma.

Combining Lemma 6 and 7, when we apply the maximum principle we obtain that

\[ \textbf{Lemma 8} \text{ Let } u_r^* \text{ be the solution of Eq. (3.1) on } \tilde{B}_r \times [0, T] \text{ then } u_{b}^{r*} < u_{r}^{s} < u_{s}^{r*}. \]

### 3.2 C^1 estimates for \( u_r^* \) – solution of (3.1)

In this subsection, we will apply Anmin Li’s idea [12] to construct barrier functions and establish the gradient estimates for the solution \( u_r^* \) of (3.1).

\[ \textbf{Lemma 9} \text{ Let } \varphi(\xi) \text{ be a strictly convex, smooth function defined on } B_1, \text{ we will show for any } \hat{\xi} \in \partial B_r, \text{ } r \in (0, 1), \text{ there exists } \bar{v} = b_1 \xi_1 + \ldots + b_n \xi_n + d \text{ such that} \]

(1) \( \bar{v}(\xi) = \varphi(\hat{\xi}), \)

(2) \( \bar{v}(\xi) < \varphi(\xi) \) for any \( \xi \in \tilde{B}_r \setminus \{ \hat{\xi} \}. \)

Here \( b_i = b_i(|\varphi|_{C^1}) \) for \( 1 \leq i \leq n \), and \( d = d(|\varphi|_{C^1}). \)

\[ \textbf{Proof} \text{ Without loss of generality we assume } \hat{\xi} = (r, 0, \ldots, 0). \text{ Since } \varphi \text{ is strictly convex we obtain} \]

\[
\varphi(\hat{\xi}) > \varphi(r, 0, \ldots, 0) + \varphi_1(r, 0, \ldots, 0)(\xi_1 - r) + \sum_{i=2}^{n} \varphi_i(r, 0, \ldots, 0)\xi_n \text{ for any } \xi \in \tilde{B}_r \setminus \{ \hat{\xi} \}. \]

Let \( \bar{v} = \varphi(r, 0, \ldots, 0) + \varphi_1(r, 0, \ldots, 0)(\xi_1 - r) + \sum_{i=2}^{n} \varphi_i(r, 0, \ldots, 0)\xi_n, \) then we are done.

We also note that \( b_i \) depends on \( |\varphi|_{C^1} \) for \( 1 \leq i \leq n \) and \( d \) depends on \( |\varphi|_{C^1}. \)

\[ \textbf{Lemma 10} \text{ Let } \varphi(\xi) \text{ be a strictly convex, smooth function defined on } B_1, \text{ we will show for any } \hat{\xi} \in \partial B_r, \text{ } r \in (0, 1), \text{ there exists } \bar{v} = b_1 \xi_1 + \ldots + b_n \xi_n + d \text{ such that} \]

(1) \( \bar{v}(\xi) = \varphi(\hat{\xi}), \)

(2) \( \bar{v}(\xi) > \varphi(\xi) \) for any \( \xi \in \tilde{B}_r \setminus \{ \hat{\xi} \}. \)

Here \( b_1 \) and \( d \) depend on \( |\varphi|_{C^2} \), while \( b_i = b_i(|\varphi|_{C^1}) \) for \( 2 \leq i \leq n. \)

\[ \textbf{Proof} \text{ Following the idea of [12], we consider an arbitrary great circle } c(t) \text{ on } \partial B_r \text{ passing through } \hat{\xi}. \text{ Without loss of generality, we assume } \hat{\xi} = (r, 0, \ldots, 0) \text{ and the great circle is} \]

\[
\xi_1 = r \cos s, \xi_2 = r \sin s, \xi_3 = \ldots = \xi_n = 0, \quad -\pi \leq s \leq \pi.
\]

Consider \( \hat{\varphi}(s) = b_1 r \cos s + b_2 r \sin s + d - \varphi(r \cos s, r \sin s, 0, \ldots, 0), \) then
\[
\hat{\varphi}(0) = b_1 r + d - \varphi(r, 0, \ldots, 0).
\]
We let \( b_1 = \frac{\varphi(r, 0, \ldots, 0) - d}{r} \) so that \( \hat{F}(0) = 0 \). Now differentiating \( \hat{F} \) with respect to \( s \) we get
\[
\frac{d\hat{F}}{ds} = -b_1 r \sin s + b_2 r \cos s + \varphi_1 r \sin s - \varphi_2 r \cos s.
\]

Let \( b_2 = \varphi_2(r, 0, \ldots, 0) \) then we obtain \( \frac{d}{ds} \hat{F}(0) = 0 \) and
\[
\frac{d^2\hat{F}}{ds^2} = (d - \varphi(r, 0, \ldots, 0)) \cos s - b_2 r \sin s + \frac{d^2\varphi}{ds^2}.
\] (3.4)

When \( s \in [-\frac{\pi}{4}, \frac{\pi}{4}] \), i.e., \( \cos s \geq \frac{\sqrt{2}}{2} \), choosing \( d > 0 \) sufficiently large we have
\[
\frac{d^2\hat{F}}{ds^2} \geq \frac{\sqrt{2}}{2} (d - \varphi(r, 0, \ldots, 0)) - C(\varphi_{C^2}) > 0.
\]

When \( s \in [-\pi, -\frac{\pi}{4}) \cup (\frac{\pi}{4}, \pi] \) and \( d > 0 \) sufficiently large, we have
\[
\hat{F}(s) = (d - \varphi(r, 0, \ldots, 0)) \cos s + b_2 r \sin s + d - \varphi(r \cos s, r \sin s, 0, \ldots, 0)
\]
\[
> d \left( 1 - \frac{\sqrt{2}}{2} \right) - C(\varphi_{C^1}) > 0.
\] (3.5)

Therefore, we have \( \tilde{u}(\xi) > \varphi(\xi) \) on \( \partial B_r \setminus \{ \hat{\xi} \} \) and \( \tilde{v}(\hat{\xi}) = \varphi(\hat{\xi}) \). In view of the strict convexity of \( \varphi \), it’s easy to see that \( \triangle(\tilde{u} - \varphi) < 0 \) thus \( \tilde{u}(\xi) > \varphi(\xi) \) in \( B_r \).

\(\square\)

**Lemma 11** Let \( u^*_r \) be the solution of Eq. (3.1) on \( B_r \times [0, T] \), then \( |Du^*_r| \leq C(|u^*_0|_{C^2}, t) \) in \( B_r \times [0, T] \).

**Proof** We only need to show \( |Du^*_r| \) is bounded on \( \partial B_r \). For any \( \hat{\xi} \in \partial B_r \), let
\[
u^*_r = -[(1 + \alpha)\hat{\xi}]^{\frac{1}{1-\alpha}} \sqrt{1 - |\hat{\xi}|^2} + \nu
\]
and
\[
u^*_r = -[(1 + \alpha)\hat{\xi}]^{\frac{1}{1-\alpha}} \sqrt{1 - |\hat{\xi}|^2} + \tilde{u}.
\]

Here \( \nu \) and \( \tilde{u} \) are linear functions constructed in Lemmas 9 and 10 by letting \( \varphi(\xi) = \nu^*_0(\xi) \). Notice that \( u^*_r, \tilde{u}^*_r, \) and \( u^*_r \) all satisfies the flow Eq. (3.1). Moreover, at the parabolic boundary \( (B_r \times \{ 0 \}) \cup \{(\partial B_r \times [0, T]) \) we have \( u^*_r \leq u^*_r \leq \tilde{u}^*_r \) with equalities hold at \( \hat{\xi} \times [0, T] \). By the maximum principle we obtain
\[
u^*_r < u^*_r < \tilde{u}^*_r \text{ in } B_r \times (0, T).
\]

Therefore,
\[
|Du^*_r(\hat{\xi}, t)| < \max \left\{ |Du^*_r(\hat{\xi}, t)|, |\tilde{Du}^*_r(\hat{\xi}, t)| \right\}.
\]

This completes the proof of Lemma 11. \(\square\)

Let \( S^{n-1}(r) = \{ \xi \in \mathbb{R}^n \mid \sum \xi_k^2 = r^2 \} \), and we denote the angular derivative \( \xi_k \frac{\partial}{\partial \xi_l} - \xi_l \frac{\partial}{\partial \xi_k} \) on \( S^{n-1}(r) \) by \( \partial_{k,l} \) or simply by \( \partial \) when no confusion arises.

**Lemma 12** Let \( u^*_0 \) be the Legendre transform of the initial hypersurface \( u_0 \), then \( |\partial u^*_0| \) is bounded by a constant depending on \( |u^*_0|_{C^3} \).

\(\square\)
Proof Since \( \det(D^2 u_0^*) = \frac{1}{K(\xi)} (1 - |\xi|^2)^{-\frac{n+2}{2}} \) we have
\[
(u_0^*)_{ij} = \frac{\partial}{\partial \xi_k} \left[ \log(1 - |\xi|^2)^{-\frac{n+2}{2}} \right] - \frac{\partial \log K}{\partial \xi_k}.
\]
By the assumption that \( u_0^* \) satisfies Conditions A we get
\[
(u_0^*)_{ij} (\partial u_0^*)_{ij} = -\partial \log K \geq -nC_1
\]
for some positive \( C_1 \) depending only on \( |u_0^*|_{C^3} \). Note that in the above calculation we used \( \partial |\xi|^2 = 0 \). The inequality (3.6) yields
\[
(u_0^*)_{ij} (\partial u_0^* + C_1 u_0^*)_{ij} \geq 0.
\]
Thus, we obtain \( \partial u_0^* \leq -C_1 u_0^* \). Similarly, we can show \( \partial u_0^* \geq C_2 u_0^* \).

Lemma 13 Let \( u_0^* \) be the Legendre transform of the initial hypersurface \( u_0 \), then \( \partial^2 u_0^* \) is bounded from above by a constant depending on \( |u_0^*|_{C^4} \).

Proof Differentiating \( (u_0^*)_{ij} (\partial u_0^*)_{ij} = -\partial \log K \) we get
\[
(u_0^*)_{ij} \partial[(\partial u_0^*)_{ij}] + \partial(u_0^*)_{ij}(\partial u_0^*)_{ij} = \partial^2 \log K.
\]
Following the proof of Lemma 5.2 of \([12]\) we obtain,
\[
(u_0^*)_{ij} [(\partial^2 u_0^*)_{ij}] \geq -nC_3,
\]
for some positive \( C_3 \) depending only on \( |u_0^*|_{C^4} \). This implies
\[
\partial^2 u_0^* \leq -C_3 u_0^*.
\]

3.3 Upper and lower bounds for \( F_* \)

In this subsection we will show that along the flow, \( F_* \) is bounded from above and below.

We take the hyperplane \( \mathbb{P} := \{ X = (x_1, \ldots, x_n, x_{n+1}) | x_{n+1} = 1 \} \) and consider the projection of \( \mathbb{H}^n(-1) \) from the origin into \( \mathbb{P} \). Then \( \mathbb{H}^n(-1) \) is mapped in a one-to-one fashion onto an open unit ball \( B_1 := \{ \xi \in \mathbb{R}^n \mid \sum \xi_k^2 < 1 \} \). The map \( P \) is given by
\[
P : \mathbb{H}^n(-1) \to B_1; \ (x_1, \ldots, x_{n+1}) \mapsto (\xi_1, \ldots, \xi_n),
\]
where \( x_{n+1} = \sqrt{1 + x_1^2 + \ldots + x_n^2} \), \( \xi_i = \frac{x_i}{x_{n+1}} \). Recall that when \( u_r^* \) satisfies (3.1), let \( v_r = \frac{u_r^*}{u_r^{\prime \prime}} \) then \( v_r \) satisfies
\[
\begin{cases}
(\alpha)_{ij} = \tilde{F}^{-1}(\Lambda_{ij}) := \tilde{G}(\Lambda_{ij}) & \text{in } P^{-1}(B_r) \times (0, T] := U_r \times (0, T], \\
v_r = \frac{\tilde{u}_0^*}{\sqrt{1 - r^2}} - \frac{1}{(1 + \alpha \tilde{r})^{\frac{1}{1+\alpha}}} & \text{on } \partial U_r \times [0, T], \quad (3.7)
\end{cases}
\]
\[
v_r(\cdot, 0) = \frac{\tilde{u}_0^*}{\sqrt{1 - |\xi|^2}} - 1 := v_0 \quad \text{on } U_r \times \{0\}.
\]
Here, \( \tilde{F} = F_*^\prime, \Lambda_{ij} = \tilde{\nabla}_{ij} v_r - v_r \delta_{ij}, \) and \( \tilde{\nabla}_{ij} v \) denote covariant derivatives with respect to the hyperbolic metric. In the following, when there is no confusion, we will drop the subscription on \( v_r \). Moreover, we also write \( v_{ij} := \tilde{\nabla}_{ij} v \).
Lemma 14 Assume (3.7) is satisfied by $v$ on $\bar{U}_r \times [0, T]$, then there exists $c_0 > 0$ depending only on the initial hypersurface $M_0$ such that $\bar{F} > \frac{1}{c_0}$ in $\bar{U}_r \times [0, T]$.

Proof From (3.7) and $\bar{G} = -\bar{F}^{-1}$, we derive
\[
\bar{G}_t = \bar{G}^{ij}(\dot{v}_{ij} - \dot{v}\delta_{ij}) = \bar{G}^{ij}\tilde{\nabla}_{ij}\bar{G} - \bar{G} \sum \bar{G}^{ii}.
\]

By the maximum principle we know that $\bar{G}$ achieves its minimum at the parabolic boundary $(\partial U_r \times (0, T)) \cup (\bar{U}_r \times \{0\})$. Since on $\partial U_r \times (0, T]$ we have
\[
\bar{F} = [(1 + \alpha)\bar{t}]^\frac{a}{1+a} > 1,
\]
the Lemma is proved. \hfill $\square$

Lemma 15 Assume (3.7) is satisfied by $v$ on $\bar{U}_r \times [0, T]$, then there exists $c_1(t) > 0$ depending only on the initial hypersurface $M_0$, $T$, and time $t$ such that $\bar{F}(\cdot, t) < \frac{1}{c_1(t)}$ in $\bar{U}_r \times [0, T]$.

Proof Let $\hat{G} = \bar{F}^{-1} = -\bar{G}$ and $\mathcal{L} := \frac{\partial}{\partial t} - \bar{G}^{ij}\tilde{\nabla}_{ij}$. Then we have
\[
\mathcal{L}\hat{G} = \frac{\partial}{\partial t}\hat{G} - \bar{G}^{ij}\tilde{\nabla}_{ij}\hat{G} = -\hat{G} \sum \bar{G}^{ii}.
\]

Denote $\hat{v} = -v$, and by our assumption on the initial data we know there exist $a_1 > a_0 > 0$ such that
\[
-a_1\sqrt{1 - |\xi|^2} < u_0^* < -a_0\sqrt{1 - |\xi|^2},
\]
which yields $\max \hat{v}(\cdot, 0) < a_1$. Moreover, by the Sect. 3.1 we get
\[
-a_1[(1 + \alpha)\bar{t}]^\frac{1}{1+a} \sqrt{1 - |\xi|^2} - [(1 + \alpha)\bar{t}]^\frac{1}{1+a} \sqrt{1 - |\xi|^2} < u_r^*
\]
\[
< -a_0[(1 + \alpha)\bar{t}]^\frac{1}{1+a} \sqrt{1 - |\xi|^2}.
\]

This implies
\[
a_0[(1 + \alpha)\bar{t}]^\frac{1}{1+a} < \hat{v}(\cdot, t) < a_1[(1 + \alpha)\bar{t}]^\frac{1}{1+a} + [(1 + \alpha)\bar{t}]^\frac{1}{1+a}
\]
and
\[
a_0 + [(1 + \alpha)\bar{t}]^\frac{1}{1+a} - 1 < \hat{v}(\partial U_r, t) < a_1 + [(1 + \alpha)\bar{t}]^\frac{1}{1+a} - 1.
\]

In view of (3.7) we obtain
\[
\mathcal{L}\hat{v} = (1 + \alpha)\hat{G} - \hat{v} \sum \bar{G}^{ii}.
\]

Now consider the test function $\varphi = \hat{G}/\hat{v}$. If $\varphi$ achieves its minimum at an interior point, then at this point we obtain
\[
\mathcal{L}\varphi = \frac{\mathcal{L}\hat{G}}{\hat{v}} - \hat{G}/\hat{v}^2 \mathcal{L}\hat{v}
\]
\[
= -\varphi \sum \bar{G}^{ii} \frac{\hat{G}}{\hat{v}^2} \left((1 + \alpha)\hat{G} - \hat{v} \sum \bar{G}^{ii}\right)
\]
\[
= -(1 + \alpha)\varphi^2.
\]

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Without loss of generality let’s assume $\varphi_{\text{min}} \leq 1$, then consider $\tilde{\varphi} = e^{(1+\alpha)t} \varphi$. We know that $\tilde{\varphi}$ doesn’t achieve interior minimum that is less than or equal to 1. Let $c = \min F_0^*(\cdot, 0)$, then
\[
e^{(1+\alpha)t} \varphi \geq \min_{0 \leq t \leq T} \left\{ \frac{c}{\max \hat{v}(\cdot, 0)}, \frac{[(1+\alpha)\hat{t}]^{-\frac{q}{1-q}}}{\hat{v}(\partial U_r, t)}, 1 \right\}.
\]
Here, the three values on the right hand side are corresponding to when the minimum is achieved at $t = 0$, when the minimum is achieved at the boundary, and when the minimum is achieved at an interior point respectively. Therefore, we have $\tilde{G} \geq c_1(t)$ for some $c_1(t)$ depends on $M_0$, $T$, and $t$. This completes the proof of Lemma 15.

\[\Box\]

### 3.4 $C^2$ boundary estimates for $u^\tau_*$ – solution of (3.1)

First, we will need to construct a subsolution to (3.1). Consider
\[
u^* = \hat{u}_0^* - [(1+\alpha)\hat{t}]^{\frac{1}{1-q}} \sqrt{1 - |\xi|^2},
\]
then $u^*_r = -[(1+\alpha)\hat{t}]^{-\frac{q}{1-q}} \sqrt{1 - |\xi|^2}$. Moreover, we have
\[
\kappa^*[w^*_i \gamma^*_k, u^*_l \gamma^*_j] > [(1+\alpha)\hat{t}]^{-\frac{q}{1-q}} \kappa^* \left[ w^*_i \gamma^*_k (-\sqrt{1 - |\xi|^2})_k \gamma^*_j \right] = [(1+\alpha)\hat{t}]^{-\frac{q}{1-q}} (1, \cdots, 1).
\]
Here, we have used the convexity of $\hat{u}_0^*$. Therefore, we get
\[
u^*_r + F_0^*(w^*_i \gamma^*_k, u^*_l \gamma^*_j)w^*_r < 0,
\]
which implies $\nu^*$ is a subsolution of (3.1).

Next, we will show $|D^2 \nu^*|$ is bounded on $\partial B_r \times (0, T]$. For our convenience, we will the second covariant derivatives of $v$ instead, i.e., we will consider the following equation
\[
\begin{align*}
\nu_t &= -F_0^*(\Lambda_{ij}) & \text{in } U_r \times (0, T], \\
v &= \frac{\hat{u}_0^*}{\sqrt{1 - r^2}} - [(1+\alpha)\hat{t}]^{\frac{1}{1-q}} & \text{on } \partial U_r \times [0, T], \\
v(\cdot, 0) &= \frac{\hat{u}_0^*}{\sqrt{1 - |\xi|^2}} - 1 := v_0 & \text{on } U_r \times \{0\}.
\end{align*}
\]

**Lemma 16** Let $v$ be the solution of (3.12), then the second tangential derivatives on the boundary satisfy $|\tilde{\nabla}_{\alpha\beta} v| \leq C$ on $\partial U_r \times (0, T]$ for $\alpha, \beta < n$.

**Proof** Let $\nu = \frac{\hat{u}_0^*}{\sqrt{1 - |\xi|^2}} - [(1+\alpha)\hat{t}]^{\frac{1}{1-q}}$ and $\tau_\alpha$, $\tau_\beta$ be some tangential vector fields on $\partial U_r$. Since $\nu = v_0 \equiv 0$ on $\partial U_r$ we have
\[
\tilde{\nabla}_{\alpha\beta}(v - \nu) = -\tilde{\nabla}_{\alpha\beta}(v - \nu) I I (\tau_\alpha, \tau_\beta) \text{ on } \partial U_r.
\]
Here $\tau_\alpha$ is the interior unit normal vector field to $\partial U_r$ and $II$ denotes the second fundamental form of $\partial U_r$. Therefore Lemma 16 follows from Lemma 11

\[\Box\]
Next, we will show that $|\tilde{\nabla}_{\alpha n} v|$ is bounded. In the following, we will denote
\[\mathcal{L} \phi := \phi_t - F_v^{-2} \tilde{F}_v^{ij} \tilde{\nabla}_i \phi + \phi \tilde{F}_v^{-2} \sum_i \tilde{F}_v^{ii}\]
for any smooth function $\phi$. Here $\tilde{F}_v(\Lambda_{ij}) = F^t_\alpha(\Lambda_{ij})$, $\tilde{F}_v^{ij} = \frac{\delta \tilde{F}_v}{\delta x_i}$, and $\Lambda_{ij} = \tilde{\nabla}_i v - v \delta_{ij}$.

**Lemma 17** Let $v$ be a solution of (3.12), $v$ be the subsolution of (3.12) which is defined in the proof of Lemma 16, and $h = (v - \bar{v}) + A \left( \frac{1}{\sqrt{1 - r^2}} - x_{n+1} \right)$. Then for any $A_1 > 0$, there exists $A > 0$ such that
\[\mathcal{L}h > \frac{A_1}{\tilde{F}_v^2} \sum \tilde{F}_v^{ii}.\]

**Proof** By equation (7.5) in [13], it’s straightforward to see that $\mathcal{L} \left( \frac{1}{\sqrt{1 - r^2}} - x_{n+1} \right) \geq \tilde{F}_v^{-2} \tilde{F}_v^{ii}$. To prove Lemma 17 we only need to show there exists $A_2 > 0$ such that
\[\mathcal{L}(v - \bar{v}) > - \left( \bar{v} + \tilde{F}^{-1}(\Delta_{ij}) \right) - \frac{A_2}{\tilde{F}_v^2} \sum \tilde{F}_v^{ii},\]
where $\Delta_{ij} = \tilde{\nabla}_iv - v \delta_{ij}$. This is equivalent to show
\[v_t - \tilde{F}_v^{-2} \tilde{F}_v^{ii} \Lambda_{ii} - v_t + \tilde{F}_v^{-2} \tilde{F}_v^{ii} \Lambda_{ij} + \frac{A_2}{\tilde{F}_v^2} \sum \tilde{F}_v^{ii} \geq -v_t - \tilde{F}^{-1}(\Delta_{ij}).\]
This implies
\[- \frac{\alpha + 1}{\tilde{F}_v} + \tilde{F}_v^{-2} \tilde{F}_v^{ii} \Lambda_{ii} + \frac{A_2}{\tilde{F}_v^2} \sum \tilde{F}_v^{ii} \geq -\tilde{F}^{-1}(\Delta_{ij}). \tag{3.13}\]

By condition (1.2) we know $\tilde{F}^{\frac{1}{\alpha}}$ is concave, which gives
\[\frac{1}{\alpha} \tilde{F}_{v}^{\frac{1}{\alpha} - 1} \tilde{F}_v^{ij} \Lambda_{ij} \geq \tilde{F}^{\frac{1}{\alpha}}(\Lambda_{ij}).\]
Moreover, by condition (1.5) we get
\[\frac{1}{\alpha} \tilde{F}_{v}^{\frac{1}{\alpha} - 1} \sum \tilde{F}_v^{ii} \geq 1.\]
Therefore,
\[\text{l.h.s. of (3.13)} \geq - \frac{1 + \alpha}{\tilde{F}_v} + \tilde{F}_v^{-2} \left( \frac{\alpha \tilde{F}_{v}^{\frac{1}{\alpha}}(\Lambda_{ij})}{\tilde{F}_{v}^{\frac{1}{\alpha} - 1}} \right) + \frac{A_2}{\tilde{F}_v^2} \cdot \frac{\alpha}{\tilde{F}_{v}^{\frac{1}{\alpha} - 1}}.\]

**Claim:** $\alpha \tilde{F}_{v}^{\frac{1}{\alpha}}(\Lambda_{ij}) + A_2 \alpha + \frac{1}{\tilde{F}(\Lambda_{ij})} \tilde{F}_v^{1+\frac{1}{\alpha}} \geq (1 + \alpha) \tilde{F}_{v}^{-2}$.

Proof of the claim: A straightforward calculation yields $C_1 < \tilde{F}(\Lambda_{ij}) < C_2(T)$, where $C_1$ only depends on the initial hypersurface $\mathcal{M}_0$ while $C_2$ also depends on the time $T$. When $\tilde{F}_v > C_3(T) := (\alpha + 1)C_2(T)$ we have
\[\frac{1}{\tilde{F}(\Lambda_{ij})} \tilde{F}_v^{1+\frac{1}{\alpha}} \geq (1 + \alpha) \tilde{F}_{v}^{-2};\]
when $\bar{F}_v \leq C_3(T)$ we can choose $A_2 = A_2(C_3, \alpha) > 0$ such that $A_2 \alpha > (1 + \alpha)C_3^{\frac{1}{2}}$. Thus the claim holds. This completes the proof of Lemma 17.

**Lemma 18** Let $v$ be a solution of (3.12) and suppose $\tau_n$ is the interior unit normal vector field of $\partial U_r$. We have $|\nabla_{\alpha n} v| \leq C$ on $\partial U_r \times (0, T]$.

**Proof** Consider any fixed point $p$ on $\partial B_r$, we may assume it to be $(0, \cdots, 0, -r)$. Choose a new coordinates $\{\hat{\xi}_1, \ldots, \hat{\xi}_n\}$ around $p$ such that $p$ is the origin and $\hat{\xi}_n$ axis is the interior normal to $\partial B_r$ at $p$. Denote $T := \partial_\alpha + \frac{1}{r} (\hat{\xi}_\alpha \partial_n - \hat{\xi}_n \partial_\alpha)$, where $\partial_i := \frac{\partial}{\partial \hat{\xi}_i}$ for $1 \leq i \leq n$ and $\alpha < n$. Let $\phi(\cdot, t) := u_\alpha(\cdot, t) - u^*(\cdot, t)$, where $u_\alpha^*$ is the solution of (3.1) and $u^*$ is given in (3.10). Then at $t = 0$ we have

$$\phi(\cdot, 0) = u_\alpha^*(\cdot, 0) - u^*(\cdot, 0) = 0, \quad (3.14)$$

and

$$T \phi(\cdot, 0) = 0. \quad (3.15)$$

Now let’s denote $\Omega_\epsilon := B_r \cap B_\epsilon(p)$, where $B_\epsilon(p)$ is a ball centered $p$ with radius $\epsilon$, then for any $t \in (0, T]$, on $\partial B_r \cap B_\epsilon(p)$ we have $\phi(\cdot, t) \equiv 0$. This implies for any $t \in (0, T]$, $|T \phi(\cdot, t)| \leq C|\hat{\xi}|^2$ on $\partial B_r \cap B_\epsilon(p)$. Consider $\psi := T \phi - C|\hat{\xi}|^2 = T u_\alpha^* - T u^* - C|\hat{\xi}|^2$. In the following we denote

$$\bar{G}(w^*_i \gamma^*_{ik}(u^*_j)_{kl}\gamma^*_{lj}) = -F_{-\alpha}^{-\alpha}(\kappa^*[w^*_i \gamma^*_{ik}(u^*_j)_{kl}\gamma^*_{lj}]).$$

Then

$$\frac{\partial u^*_i}{\partial t} = \bar{G}(w^*_i \gamma^*_{ik}(u^*_j)_{kl}\gamma^*_{lj}) w^* = 0,$$

and

$$T = \frac{1}{r} \left( \xi_\alpha \frac{\partial}{\partial \xi_n} - \xi_n \frac{\partial}{\partial \xi_\alpha} \right)$$

is an angular derivative vector. By Lemma 29 of [13] we get

$$\frac{\partial (Tu^*_i)}{\partial t} - \left( \bar{G}^ij(w^*_i \gamma^*_{ik}(Tu^*_j)_{kl}\gamma^*_{lj}) \right) w^* = 0, \quad (3.16)$$

where if we denote $a_{ij} = w^*_i \gamma^*_{ik}(u^*_j)_{kl}\gamma^*_{lj}$ then $\bar{G}^ij = \frac{\partial \bar{G}}{\partial a_{ij}}$. Recall that $u^*_i = \tilde{u}^*_0 - [(1 + \alpha)\mathbf{I} + \frac{1}{1 + \alpha} \sqrt{1 + |\hat{\xi}|^2}]^{-1} \bar{G}^ij \frac{u}{w^*} \delta_{ij} = w^*_i \gamma^*_{ik} u_{kl} \gamma^*_{lj}$. Applying Lemma 15 of [15] we know

$$\nabla_{ij} \left( \frac{u}{w^*} \right) - \frac{u}{w^*} \delta_{ij} = w^*_i \gamma^*_{ik} u_{kl} \gamma^*_{lj}. \quad (3.17)$$

Combining with (3.16) we obtain

$$(Tv)_{ij} - \bar{G}^ij \tilde{\nabla}_{ij}(Tv) + (Tv) \sum \bar{G}^{lij} = 0, \quad (3.18)$$

where $Tv = T \left( \frac{u^*_i}{w^*} \right) = \frac{T u^*_i}{w^*}$, and we also note that here $\bar{G}^ij = \bar{F}_v^{-2} \bar{F}^{ij}_v$. Moreover, it’s easy to see that $|\nabla Tv| \leq C_1 \sum \bar{G}^{lij}$. Now, since $|\hat{\xi}|^2 = \sum_{\alpha < n} \xi_{\alpha}^2 + (\xi_n + r)^2$ we have

$$\left| \frac{\nabla |\hat{\xi}|^2}{w^*} \right| \leq C_2 \sum \bar{G}^{lij}.$$
Therefore, choosing $C > 0$ large the function $\bar{\Psi} := Tv - T_0v - \frac{C}{w^\alpha}$ satisfies
\[ \bar{\Psi}(\cdot, 0) \leq 0 \text{ in } P^{-1}(\Omega_c) \times \{0\}, \]
\[ \bar{\Psi}(\cdot, t) \leq 0 \text{ on } \partial P^{-1}(\Omega_c) \times (0, T), \]
and $|\mathcal{L}\bar{\Psi}| \leq C_3 \sum \bar{G}^{ij}$. When $A > 0$ very large we have
\[ h \geq \bar{\Psi} \text{ in } P^{-1}(\Omega_c) \times \{0\} \text{ and on } \partial P^{-1}(\Omega_c) \times (0, T), \]
\[ \mathcal{L}(\bar{\Psi} - h) \leq 0 \text{ in } \Omega_c \times (0, T). \]
Therefore, at any point $(p, t) \in \partial U_r \times (0, T)$ we get $h_n > \bar{\Psi}_n$. This implies $\bar{\nabla}_{\alpha n} v(p, t) \leq C_4$. Similarly, by considering $Tv - T_0v - C \frac{1}{w^\alpha}$ we obtain $\bar{\nabla}_{\alpha n} v \geq C_5$.

**Lemma 19** Let $v$ be a solution of (3.12) and suppose $\tau_n$ is the interior unit normal field of $\partial U_r$. We have $|\bar{\nabla}_{\alpha n} v| \leq C$ on $\partial U_r \times (0, T)$.

**Proof** For any fixed point $(p, t) \in \partial U_r \times (0, T)$ we may assume $(\bar{\nabla}_{\alpha \beta} v(p, t))$, $1 \leq \alpha, \beta < n$ to be diagonal. Then at this point
\[
\Lambda_{ij} = \begin{bmatrix}
v_{11} - v & 0 & \cdots & v_{1n} \\
0 & v_{22} - v & \cdots & v_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
v_{1n} & v_{2n} & \cdots & v_{nn} - v
\end{bmatrix}.
\]
By Lemma 1.2 in [6], we know if $v_{nn}$ is very large, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\Lambda_{ij}$ are
\[
\lambda_\alpha = v_{\alpha \alpha} - v + o(1) \\
\lambda_n = v_{nn} - v + O(1).
\]
Since $F_*$ is bounded we know $v_{nn}$ is bounded from above. $v_{nn}$ is bounded from below comes from the strict convexity of the flow. $\square$

### 3.5 $C^2$ global estimates for $u^*_r$ – solution of (3.1)

In this subsection, we will still use the hyperbolic model and study the Eq. (3.12). We will estimate $|\bar{\nabla}^2 v|$ on $\bar{U}_r \times [0, T]$. Keep in mind that a bound on $|\bar{\nabla}^2 v|$ yields a bound on $|D^2 u^*_r|$.

**Lemma 20** Let $v$ be the solution of (3.12). Denote the eigenvalues of $(\bar{\nabla}_{ij} v - v \delta_{ij})$ by $\lambda[\bar{\nabla}_{ij} v - v \delta_{ij}] = (\lambda_1, \ldots, \lambda_n)$. Then, $\lambda[\bar{\nabla}_{ij} v - v \delta_{ij}]$ is bounded from above.

**Proof** Differentiating (3.12) twice we get
\[
(v_t)_i = \bar{F}^{-2} \bar{F}^{kl} \Lambda_{kli},
\]  
\[
(v_t)_{ij} = \bar{F}^{-2} \left\{ \bar{F}^{kl} \Lambda_{kij} + \bar{F}^{pq, rs} \Lambda_{pqi} \Lambda_{rsj} \right\} - 2 \bar{F}^{-3} \left( \bar{F}^{kl} \Lambda_{kli} \right) \left( \bar{F}^{kl} \Lambda_{kij} \right).
\]
Set $M = \max_{(p, t) \in \bar{U}_r \times [0, T]} \max_{|x| = 1, \tilde{x} \in T_p \mathbb{H}^n} (\log \Lambda_{\tilde{x} \tilde{x}} + N x_{n+1})$, where $N$ is a constant to be determined later and $x_{n+1}$ is the coordinate function. By the discussion in Sect. 3.4 we already know that $|\lambda|$ is bounded on $\partial U_r \times [0, T]$. Therefore, in the following, we may assume $M$ is

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achieved at an interior point \((p_0, t_0)\) for some direction \(\xi_0\). Choosing an orthonormal frame \(\{\tau_1, \ldots, \tau_n\}\) around \(p_0\) such that \(\tau_1(p_0) = \xi_0\) and \(\Lambda_{ij}(p_0, t_0) = \lambda_i \delta_{ij}\).

Now, let’s consider the test function \(\phi = \log \Lambda_{11} + N x_{n+1}\). At its maximum point \((p_0, t_0)\), we have

\[
0 = \frac{\Lambda_{11}}{\Lambda_{11}} + N (x_{n+1})_i
\]

which implies

\[
0 \geq \frac{\Lambda_{11}}{\Lambda_{11}} - \frac{\Lambda_{11}^2}{\Lambda_{11}^2} + N (x_{n+1})_{ii}.
\]

Therefore,

\[
\phi_t - \tilde{F}^{-2} \tilde{F}^{ii} \phi_{ii} = \frac{(\Lambda_{11})_t}{\Lambda_{11}} - \tilde{F}^{-2} \tilde{F}^{ii} \left\{ \frac{\Lambda_{11}^2}{\Lambda_{11}} - \frac{\Lambda_{11}^2}{\Lambda_{11}^2} + N (x_{n+1})_{ii} \right\}
\]

\[
= \frac{(v_t)_{11} - v_t}{\Lambda_{11}} - \tilde{F}^{-2} \tilde{F}^{ii} \left\{ \frac{\Lambda_{ii11} + \Lambda_{ii} - \Lambda_{11}}{\Lambda_{11}} - \frac{\Lambda_{11}}{\Lambda_{11}^2} + N x_{n+1} \delta_{ii} \right\}
\]

\[
= \frac{1}{\Lambda_{11}} \tilde{F}^{-2} \left\{ \tilde{F}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} - 2 \tilde{F}^{-1} (\nabla_1 \tilde{F})^2 \right\}
\]

\[
+ \frac{1}{\tilde{F} \Lambda_{11}} - \tilde{F}^{-2} \tilde{F}^{ii} \left\{ \frac{\Lambda_{ii} - \Lambda_{11}}{\Lambda_{11}} - \frac{\Lambda_{11}}{\Lambda_{11}^2} + N x_{n+1} \delta_{ii} \right\}.
\]

By our assumption (1.2) we know that \(\tilde{F}^{\frac{1}{\alpha}}\) is concave, which implies

\[
\tilde{F}^{pp,qq} \Lambda_{pp1} \Lambda_{qq1} + \left(1 - \frac{1}{\alpha}\right) \tilde{F}^{-1} (\nabla_1 \tilde{F})^2 \leq 0.
\]

Since

\[
\tilde{F}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} = \tilde{F}^{pp,qq} \Lambda_{pp1} \Lambda_{qq1} + \sum_{p \neq q} \tilde{F}^{pp} - \tilde{F}^{qq} \Lambda_{pq1},
\]

we have

\[
\phi_t - \tilde{F}^{-2} \tilde{F}^{ii} \phi_{ii} \leq \frac{2}{\Lambda_{11}} \tilde{F}^{-2} \sum_{p > 1} \frac{\tilde{F}^{pp} - \tilde{F}^{11}}{\lambda_p - \lambda_1} \Lambda_{11p}^2 + \frac{1 - \alpha}{\tilde{F} \Lambda_{11}}
\]

\[
-(N x_{n+1} - 1) \tilde{F}^{-2} \sum \tilde{F}^{ii} + \tilde{F}^{-2} \tilde{F}^{ii} \frac{\Lambda_{11}}{\Lambda_{11}^2}.
\]

Now let

\[
I := \{j : \tilde{F}^{jj} \leq 4 \tilde{F}^{11}\}, \quad J := \{j : \tilde{F}^{jj} > 4 \tilde{F}^{11}\},
\]

Eq. (3.24) becomes

\[
\phi_t - \tilde{F}^{-2} \tilde{F}^{ii} \phi_{ii}
\]
\[ \frac{2}{\Lambda_{11}} \tilde{F}^{-2} \sum_{j \in I} \frac{-3 \tilde{F}^{jj}}{4 \Lambda_{11}} \Lambda_{11j}^2 + \frac{1 - \alpha}{\tilde{F} \Lambda_{11}} \\ -(N\bar{\gamma}_{n+1} - 1) \tilde{F}^{-2} \sum_{i \in I} 4 \tilde{F}_{111}^2 \Lambda_{i1}^2 + \tilde{F}^{-2} \sum_{j \in I} \tilde{F}^{jj} \Lambda_{11j}^2. \]

Combining with (3.22), we get

\[ \phi_t - \tilde{F}^{-2} \tilde{F}^{ii} \phi_{ij} \]
\[ \leq \frac{1 - \alpha}{\tilde{F} \Lambda_{11}} - (N\bar{\gamma}_{n+1} - 1) \tilde{F}^{-2} 4\alpha \tilde{F}^{-2} |I| \frac{\tilde{F} N^2 (\bar{\gamma}_{n+1})^2}{\Lambda_{11}} \]

notice that here we used \( \tilde{F}^{11} \leq \frac{\alpha \tilde{F}}{\Lambda_{11}} \). Moreover, since \( F_\ast \) is bounded, concave and satisfies (1.3), we have \( \sum \tilde{F}^{ii} \geq \alpha F_\ast^{-1} > c_0 \). Choosing \( N = 2 \) we can see that if \( \phi \) achieves its maximum at an interior point \( (p_0, t_0) \), then at this point \( |\tilde{V}^2 v| \) is bounded from above. Otherwise, the maximum is achieved at \( (\partial U_r \times (0, T)) \bigcup (U_r \times \{0\}) \). Therefore, Lemma 20 is proved. \( \square \)

### 4 Local estimates

In this section we want to show that there exists a subsequence of \( \{w_r^\ast\} \) that converges to the desired solution \( u^\ast \) of (2.3).

#### 4.1 Local C^1 estimates

In this subsection we will prove the local C^1 estimates for \( v_r \). We will study (3.1) and consider the local C^1 estimates for \( u_r^\ast \) instead. For readers’ convenience, we rewrite Eq. (3.1) here:

\[
\begin{align*}
(u_r^\ast)_r &= -F_\ast^{-\alpha} (w^\ast \gamma_{k}^\ast u_{ik}^\ast \gamma_{ij}^\ast) w^\ast & \text{in } B_r \times (0, T) \\
u_r^\ast (\cdot, t) &= \hat{u}_0^\ast - [(1 + \alpha)\tilde{t}] \frac{1}{1 - r^2} \sqrt{1 - |\bar{\xi}|^2} & \text{on } \partial B_r \times [0, T] \\
u_r^\ast (\cdot, 0) &= \hat{u}_0^\ast & \text{on } B_r \times \{0\},
\end{align*}
\]

where \( \tilde{t} = \tilde{t} + (1 + \alpha)^{-1} \). In the following, denote \( \tilde{F} = F_\ast^\alpha \) and when there is no confusion, we will omit the superscript \( r \).

**Lemma 21** Let \( u_r^\ast \) be the solution of (4.1) and suppose \( \partial \) is some angular derivative. Then we have \( |\partial u_r^\ast| \) is bounded on \( B_r \times [0, T] \).

**Proof** We assume that \( \partial = \xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2} \). Since \( u_r^\ast = -\tilde{F}^{-1} (\gamma_{k}^\ast u_{ik}^\ast \gamma_{ij}^\ast) (w^\ast)^{1-\alpha} \), by Lemma 29 of [13] we get

\[ (\partial u^\ast)_r - (w^\ast)^{1-\alpha} \tilde{F}^{-2} \tilde{F}^{kli} (\partial u^\ast)_{ik} \gamma_{ij}^\ast = 0. \]

Using the maximum principle we get that \( \partial u^\ast \) achieves its maximum and minimum on the parabolic boundary. Lemma 21 follows directly from Lemma 12. \( \square \)

**Lemma 22** Let \( u_r^\ast \) be the solution of (4.1) and suppose \( \partial \) is some angular derivative. Then we have \( \partial^2 u_r^\ast \) is bounded from above on \( B_r \times [0, T] \).
Applying the convexity of $u$, straightforward calculation yields, there is a positive constant $b$ such that here the inequality is due to $\tilde{h}$. Combining with (4.2), we get where $\tilde{a}_{kl} = \gamma_{ki}^* u_{ij}^* \gamma_{jl}^*$. By Lemma 30 of [13] we get

$$\left( \partial^2 u^* \right)_t - (w^*)^{1-\sigma} \tilde{G}^{kl} \left( \gamma_{ki}^* (\partial^2 u^*)_{ij} \gamma_{jl}^* \right) \leq 0,$$

here the inequality is due to $\tilde{G}$ is concave. Therefore, $\partial^2 u^*$ achieves its maximum at the parabolic boundary and Lemma 22 follows directly from Lemma 13. □

**Lemma 23** Let $u^*_r$ be the solution of (4.1) and suppose $\partial$ is some angular derivative. Then there is a positive constant $b$ such that

$$\frac{(r - |\xi|)\partial^2 u^*_r}{\sqrt{1 - |\xi|^2}} > -b[(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}}$$

on $\tilde{B}_r \times [0, T]$. Here, $b$ only depends on $u^*_0$.

**Proof** Without loss of generality we assume

$$\xi = (\xi_1, 0, \cdots , 0), \quad |\xi| = \xi_1, \quad \text{and} \quad \partial = \xi_1 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_1}.$$

Then at $\xi$ we have

$$\partial^2 u^* = \xi_1^2 u^*_{22} - \xi_1 u^*_1. \quad (4.2)$$

Notice that by the convexity of $u^*$ we have $u^*_r > 0$. Now at the point $\xi$, applying Lemma 8 we know

$$u^* > -a_1 \sqrt{1 - |\xi|^2} [(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}} - \sqrt{1 - r^2}[(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}}.$$

Applying the convexity of $u^*_r$ again we get

$$-a_1 \sqrt{1 - |\xi|^2} [(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}} - \sqrt{1 - r^2}[(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}} + (r - |\xi|)u^*_1 < -\sqrt{1 - r^2}[(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}}.$$

Therefore,

$$u^*_1 < \frac{a_1 \sqrt{1 - |\xi|^2} [(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}}}{r - |\xi|}. \quad (4.3)$$

Combining with (4.2), we get

$$\partial^2 u^* > -|\xi| \frac{a_1 \sqrt{1 - |\xi|^2} [(1 + \alpha)\hat{r}]^{\frac{1}{1+\sigma}}}{r - |\xi|}.$$

This completes the proof of Lemma 23. □

**Lemma 24** Suppose $u^*_r$ is the solution of (4.1). Let $\frac{1}{2} < \hat{r} < r < 1$, $S^{n-1}(\hat{r}) = \{ \xi \in \mathbb{R}^n \mid \sum \xi_i^2 = \hat{r}^2 \}$. For any $\hat{\xi} \in S^{n-1}(\hat{r})$, there exists a function

$$h = -a\sqrt{1 - |\xi|^2} + b_1 \xi_1 + \cdots + b_n \xi_n + d \quad (4.3)$$

such that $h(\hat{\xi}) = u^*_r(\hat{\xi}, t)$ and $h(\xi) > u^*_r(\xi, t)$ for any $(\xi, t) \in \left( S^{n-1}(\hat{r}) \setminus \{ \hat{\xi} \} \right) \times (0, T]$. □
Proof The proof of this Lemma is a small modification of the proof of Lemma 10. For readers’ convenience, we include it here. Since $\kappa^*[w^*\gamma_h^{ik} h_{kl}^{ij}] = a\delta_{ij}$, we can choose $a > 0$ such that $a^2 \leq \frac{1}{c_0}$, where $c_0 > 0$ is the same as the one in Lemma 14. This gives

$$
\tilde{F}(\kappa^*[u_r^* \gamma_h^{ik} h_{kl}^{ij}]) < \tilde{F}(\kappa^*[u_r^* (., t))h_{kl}^{ij}]).
$$

By rotating the coordinate we may assume $\tilde{\xi} = (\tilde{r}, 0, \cdots, 0)$. We choose

$$
b_k = \frac{\partial u_r^* (\tilde{r}, 0, \cdots, 0, t)}{\partial \xi_k}, \quad k = 2, 3, \cdots, n
$$

and choose $b_1$ such that

$$
u_r^*(\tilde{r}, 0, \cdots, 0, t) = -a\sqrt{1 - \tilde{r}^2} + b_1\tilde{r} + d.
$$

To choose $d$ we consider an arbitrary great circle $c(s)$ on $\mathbb{S}^n(\tilde{r})$ passing through $\tilde{\xi}$. For example, the circle

$$
\xi_1 = \tilde{r}\cos{s}, \xi_2 = \tilde{r}\sin{s}, \xi_3 = \cdots = \xi_n = 0, \quad \text{where} -\pi \leq s \leq \pi.
$$

Let

$$
\tilde{F}(s) = (h - u_r^*)|c(s) = [u_r^*(\tilde{r}, 0, \cdots, 0, t) + a\sqrt{1 - \tilde{r}^2} - d] \cos{s} + b_2\tilde{r}\sin{s} + d - u_r^*(s, t)
$$

$$
- a\sqrt{1 - \tilde{r}^2},
$$

where $u_r^*(s, t) = u_r^*(\tilde{r}\cos{s}, \tilde{r}\sin{s}, 0, \cdots, 0, t)$. We have $\tilde{F}(0) = 0, \frac{d\tilde{F}}{ds}(0) = 0$, and

$$
\frac{d^2\tilde{F}(s)}{ds^2} = [d - u_r^*(\tilde{r}, 0, \cdots, 0, t) - a\sqrt{1 - \tilde{r}^2}] \cos{s} - b_2\tilde{r}\sin{s} - \frac{d^2u_r^*}{ds^2}.
$$

When $-\frac{\pi}{4} \leq s \leq \frac{\pi}{4}$ we have

$$
\frac{d^2\tilde{F}(s)}{ds^2} \geq [d - u_r^*(\tilde{r}, 0, \cdots, 0, t) - a\sqrt{1 - \tilde{r}^2}] \frac{1}{\sqrt{2}} - c_1,
$$

where $c_1 > 0$ is a constant determined by Lemma 21 and 22. When $s \in [-\pi, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, \pi]$ we have

$$
\tilde{F}(s) = d(1 - \cos{s}) + [u_r^*(\tilde{r}, 0, 0, \cdots, 0, t) + a\sqrt{1 - \tilde{r}^2}] \cos{s} + b_2\tilde{r}\sin{s} - u_r^*(s, t)
$$

$$
- a\sqrt{1 - \tilde{r}^2}
$$

$$
\geq \left(1 - \frac{\sqrt{2}}{2}\right)d - 2a\sqrt{1 - \tilde{r}^2} - 2 \max_{-\pi \leq s \leq \pi} |u_r^*(s, t)| - \frac{|du_r^*(s, t)|}{ds}.
$$

Choosing $d > 0$ large we are done.

Lemma 25 Suppose $v_r$ is the solution of (3.12). Let $\frac{1}{2} < \tilde{r} < r < 1, \mathbb{S}^n(\tilde{r}) = [\xi \in \mathbb{R}^n | \sum \xi_i^2 = \tilde{r}^2]$. For any $(\tilde{\xi}, t) \in \mathbb{S}^n(\tilde{r}) \times (0, T)$ we have $\tilde{\nabla}_i v_r(\tilde{\xi}, t) > c_3, \ 1 \leq i \leq n$, for some constant $c_3$ that is independent of $\tilde{r}$ and $r$. 

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Proof Without loss of generality we consider the derivative of \( v_r \) at the point \( \hat{r} = (\hat{r}, 0, \cdots, 0) \). Without causing confusion, in the following we will drop the subscript \( r \).

A straightforward calculation yields (for details see the proof of Lemma 15 in [15])

\[
\tilde{v}_i v = \gamma_{ik}^* u_k^* + v \xi_i \\
= (\delta_{ik} - \frac{\xi_i \xi_k}{1 + w^*}) u_k^* + v \xi_i \\
= u_i^* - \frac{\xi_i \xi_k}{1 + w^*} u_k^* + v \xi_i.
\]

(4.8)

Then at \( \hat{r} \), we have

\[
\tilde{v}_i v = \sqrt{1 - \hat{r}^2} u_i^* + v \hat{r}
\]

and \( \tilde{v}_i v = u_i^* \) for \( i \geq 2 \). By Lemma 24 we obtain

\[
u_i^* > h_i^* = \frac{a \hat{r}}{\sqrt{1 - \hat{r}^2}} + b_1.
\]

Therefore,

\[
\tilde{v}_i v > a \hat{r} + b_1 \sqrt{1 - \hat{r}^2} + v \hat{r}.
\]

Note that, by the Subsection 3.1 we get

\[
-(a_1 + 1)[(1 + \alpha)\hat{r}]^{\frac{1}{1 + \alpha}} \sqrt{1 - |\hat{r}|^2} < u_i^* < -a_0[(1 + \alpha)\hat{r}]^{\frac{1}{1 + \alpha}} \sqrt{1 - |\hat{r}|^2}.
\]

This implies

\[
-(a_1 + 1)[(1 + \alpha)\hat{r}]^{\frac{1}{1 + \alpha}} < v(\cdot, t) < -a_0[(1 + \alpha)\hat{r}]^{\frac{1}{1 + \alpha}},
\]

which in turn gives an uniform lower bound for \( \tilde{v}_i v \). When \( i = 2, \ldots, n \), by Lemma 21, we conclude that \( \tilde{v}_i v = u_i^* \) is bounded at \((\hat{r}, t)\) for any \( t \in [0, T] \) directly. Therefore, Lemma 25 is proved. \( \square \)

Following the steps of Lemma 24 we can show

Lemma 26 Suppose \( u_i^* \) is the solution of (4.1). Let \( \frac{1}{2} < \hat{r} < r < 1, \mathbb{S}_{n-1}^{n-1}(\hat{r}) = \{ \xi \in \mathbb{R}^n \mid \sum \xi_i^2 = \hat{r}^2 \} \). For any \( \hat{\xi} \in \mathbb{S}_{n-1}^{n-1}(\hat{r}) \), there exists a function

\[
h = -a\sqrt{1 - |\hat{r}|^2} + a_1 \xi_1 + \ldots + a_n \xi_n - a
\]

(4.9)

such that \( h(\hat{\xi}) = u_i^*(\hat{\xi}, t) \) and \( h(\xi) < u_i^*(\xi, t) \) for any \( (\xi, t) \in (\mathbb{S}_{n-1}^{n-1}(\hat{r}) \setminus \{\hat{\xi}\}) \times (0, T] \).

Here \( a \) is determined by \( c_1(t) \) of Lemma 15, \( a_1, \ldots, a_n \) are constants, and \( a > 0 \) is chosen to be

\[
a = 10 \left[ \frac{b\sqrt{1 - \hat{r}^2}}{r - \hat{r}} [(1 + \alpha)\hat{r}]^{\frac{1}{1 + \alpha}} + C(|u_0^*|) [(1 + \alpha)\hat{r}]^{\frac{1}{1 + \alpha}} + a\sqrt{1 - \hat{r}^2}} \right],
\]

where \( b \) is from Lemma 23.

Lemma 27 Suppose \( v_r \) is the solution of (3.12). Let \( \frac{1}{2} < \hat{r} < r < 1, \mathbb{S}_{n-1}^{n-1}(\hat{r}) = \{ \xi \in \mathbb{R}^n \mid \sum \xi_i^2 = \hat{r}^2 \} \). For any \( (\hat{\xi}, t) \in \mathbb{S}_{n-1}^{n-1}(\hat{r}) \times (0, T] \), we have \( \tilde{v}_i v_r < 2a + 2a\sqrt{1 - \hat{r}^2} + c, \ 1 \leq i \leq n \), where \( c \) is from Lemma 21 and \( a, a \) are from Lemma 26.
Therefore, we can use this yields \( u \). In the following, we denote Moreover, on \( \partial \kappa \) for \( t > 0 \). From Eq. (4.1) and Lemma 14 we know \( (u^*_r)_t < 0 \), this yields \( l(\xi_0) - l_0^*(\xi_0, t) > l(\xi_0) - l_0^*(\xi_0) \) for \( t > 0 \). Moreover, on \( \partial B_r \), when \( t \leq T \) and \( r > r_0(T) \) we have \( u^*_r(\xi, t) - l(\xi) > u_0^*(\xi) + \sqrt{1 - r^2} - [(1 + \alpha)t]^{1/\alpha} \sqrt{1 - r^2} - l(\xi) > 0 \). Therefore, we can use \( l(\xi) \) to construct cutoff function and obtain the following local \( C^2 \) estimates.

**Lemma 28** Let \( u^*_r \) be the solution of (4.1) For any \( \xi_0 \in B_1 \) and \( T > 0 \), we have

\[
\left| \kappa^*[w^*\gamma^*_i(u^*_r)_{kl}\gamma^*_j]^{\xi_0, t} \right| < C
\]

for \( t \in [0, T] \) and \( 1 > r > r_0(T) \). Here \( C = C(T) \) is independent of \( r \).

**Proof** From the discussion in Sect.2 of [15] we know that \( \lambda(\tilde{\nabla}_{ij}v - v\delta_{ij}) = \kappa^*[w^*\gamma^*_i(u^*_r)_{kl}\gamma^*_j] \) for \( v = \frac{u^*_r}{w^r} \). Therefore, in this proof we will consider Eq. (3.12) and obtain an upper bound for \( |\lambda| \) instead. Let \( l(\xi) \) be a linear function described as above and let \( \tilde{s} = \frac{l(\xi)}{w^r} \), then by Lemma 15 of [15] we obtain

\[
\tilde{\nabla}_{ij}\tilde{s} = \tilde{s}\delta_{ij}.
\]

In the following, we denote

\[
\eta = \tilde{s} - v \quad \text{and} \quad \vartheta = v^2 - |\tilde{\nabla}v|^2.
\]

We also denote \( \tilde{G} = -\tilde{F}^{-1} \), where \( \tilde{F} = F_x^\alpha \). Following [14], let’s consider

\[
W(p, t) = \eta^\vartheta \exp \Phi(\vartheta) \Lambda_{\xi\xi},
\]
where $\xi \in T_{p}^{\mathbb{H}^n}$ is a unit vector. We may assume $W$ achieves its maximum at $(p_0, t_0)$ for some direction $\xi_0$. Choosing an orthonormal frame $\{\tau_1, \ldots, \tau_n\}$ around $p_0$ such that $\tau_1(p_0) = \xi_0$ and $\Lambda_{ij}(p_0, t_0) = \delta_{ij}$. Differentiating $W$ at $(p_0, t_0)$ we obtain

$$0 = \hat{W}_i = \frac{\beta \eta_i}{\eta} + \Phi' \partial_i + \frac{\Lambda_{1i}}{\Lambda_{11}}, \quad (4.10)$$

and

$$0 \geq \frac{\beta \eta_{ii}}{\eta} - \frac{\beta \eta_i^2}{\eta^2} + \Phi' \partial_{ii} + \Phi''(\partial_i)^2 + \frac{\Lambda_{1i}}{\Lambda_{11}} + \frac{\Lambda_{ii}}{\Lambda_{11}} - 1 - \left( \frac{\Lambda_{1i}}{\Lambda_{11}} \right)^2, \quad (4.11)$$

where we have used the equality $\Lambda_{1i} = \Lambda_{i11} + \Lambda_{ii} - \Lambda_{11}$. Moreover, at $(p_0, t_0)$ we have

$$0 \leq \frac{\beta}{\eta} \mathcal{L} \eta + \Phi' \mathcal{L} \partial + \frac{1}{\Lambda_{11}} \mathcal{L} \Lambda_{11} + \beta \tilde{G}^{ii} \left( \frac{\eta_i}{\eta} \right)^2 - \Phi'' \tilde{G}^{ii}(\partial_i)^2 + \tilde{G}^{ii} \left( \frac{\Lambda_{1i}}{\Lambda_{11}} \right)^2. \quad (4.12)$$

A direct calculation gives

$$\mathcal{L} \eta = \eta_t - \tilde{G}^{ii} \eta_{ii} = -v_t - \tilde{G}^{ii} (\tilde{s}_{ii} - v_{ii}) = -\tilde{G} - \tilde{s} \sum \tilde{G}^{ii} + \tilde{G}^{ii} (\Lambda_{ii} + v \delta_{ii}) = -\eta \tilde{G}^{ii} - (1 + \alpha) \tilde{G}, \quad (4.13)$$

where we have used $\tilde{G}^{ii} = -\alpha \tilde{G}$. Note that differentiating Eq. (3.12) twice we get

$$v_{11} = \tilde{G}^{ii} \Lambda_{ii1} + \tilde{G}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1}. \quad (4.14)$$

Moreover, a straightforward calculation yields

$$G(\partial)_t = 2uv_t - 2v_k (v_k)_t = 2u \tilde{G} - 2v_k \tilde{G}^{ii} \Lambda_{ii}, \quad (4.15)$$

$$\partial_i = 2uv_i - 2v_k v_{ki} = -2v_i \Lambda_{ii}. \quad (4.16)$$

and

$$\partial_{ii} = 2v_i^2 + 2uv_{ii} - 2v_{ki}^2 - 2v_k v_{kii} = -2 \Lambda_{ii}^2 - 2v \Lambda_{ii} - 2v_k \Lambda_{iik}. \quad (4.17)$$
Thus we get
\[
(\partial_t - \tilde{G}^{ii} \partial_{ii}) = 2v\tilde{G}(1 - \alpha) + 2\tilde{G}^{ii} \Lambda_{ii}^2. \tag{4.18}
\]
Therefore, (4.12) becomes
\[
0 \leq \frac{\beta}{\eta} (-\eta \sum \tilde{G}^{ii} - (1 + \alpha)\tilde{G})
+ \Phi' \left(2v\tilde{G}(1 - \alpha) + 2\tilde{G}^{ii} \Lambda_{ii}^2 \right)
+ \frac{1}{\Lambda_{11}} \left[ \tilde{G}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} + (\alpha - 1)\tilde{G} + \Lambda_{11} \sum \tilde{G}^{ii} \right]
+ \beta \tilde{G}^{ii} \left( \frac{\eta}{\eta} \right)^2 - \Phi'' \tilde{G}^{ii} (\partial_t) - \tilde{G}^{ii} \left( \frac{\Lambda_{11i}}{\Lambda_{11}} \right)^2.
\tag{4.19}
\]

Denote \( I = \{ j : \tilde{g}_j \leq 4\tilde{g}_1 \} \) and \( J = \{ j : \tilde{g}_j > 4\tilde{g}_1 \} \). When \( j \in I \) we have,
\[
\tilde{g}_j \left( \frac{\Lambda_{11j}}{\Lambda_{11}} \right)^2 = \tilde{g}_j \left( \frac{\beta \eta_j}{\eta} + \Phi' \partial_j \right)^2
\leq (1 + \epsilon)(\Phi')^2 \tilde{g}_j (\partial_j)^2 + \left( 1 + \frac{1}{\epsilon} \right) \beta^2 \tilde{g}_j \left( \frac{\eta_j}{\eta} \right)^2.
\tag{4.20}
\]
When \( j \in J \) we have,
\[
\beta \tilde{g}_j \left( \frac{\eta_j}{\eta} \right)^2 = \frac{1}{\beta} \tilde{g}_j \left( \Phi' \partial_j + \frac{\Lambda_{11j}}{\Lambda_{11}} \right)^2
\leq \frac{1 + \epsilon}{\beta} (\Phi')^2 \tilde{g}_j (\partial_j)^2 + \left( 1 + \frac{1 - \epsilon}{\beta} \right) \tilde{g}_j \left( \frac{\Lambda_{11j}}{\Lambda_{11}} \right)^2.
\tag{4.21}
\]
Therefore,
\[
\beta \sum_{j=1}^{\infty} \tilde{g}_j \left( \frac{\eta_j}{\eta} \right)^2 + \sum_{j=1}^{\infty} \tilde{g}_j \left( \frac{\Lambda_{11j}}{\Lambda_{11}} \right)^2
\leq 4n[\beta + (1 + \epsilon^{-1})\beta^2] \tilde{g}_1 \left[ \frac{|\nabla \eta|^2}{\eta^2} \right]
+(1 + \epsilon)(1 + \beta^{-1})(\Phi')^2 \sum_{j=1}^{\infty} \tilde{g}_j (\partial_j)^2
+[1 + (1 + \epsilon^{-1})\beta^{-1}] \sum_{j \in J} \tilde{g}_j \left( \frac{\Lambda_{11j}}{\Lambda_{11}} \right)^2.
\tag{4.22}
\]
Combining (4.22) with (4.19) we get
\[
0 \leq (-\beta + 1) \sum \tilde{G}^{ii} - \frac{(1 + \alpha)\tilde{G} \beta}{\eta} + 2v\tilde{G}(1 - \alpha) \Phi' + \frac{(\alpha - 1)\tilde{G}}{\Lambda_{11}}
+ 2\Phi' \sum \tilde{g}_i \Lambda_{ii}^2 + \frac{1}{\Lambda_{11}} \tilde{G}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1}
+ 4n[\beta + (1 + \epsilon^{-1})\beta^2] \tilde{g}_1 \left[ \frac{|\nabla \eta|^2}{\eta^2} \right]
\]
Therefore, we can find a subsequence of \( u^\sim \). Since \( \tilde{\nabla} \eta = \gamma_i^{\alpha} \tilde{d}(\xi) + \tilde{s} \xi_i \), which yields \( |\tilde{\nabla} \eta| \) is bounded. Let \( \Phi(\vartheta) = -\log(\vartheta + A) \), where \( A = A(T, \mathcal{M}_0) > 0 \) large such that \( \vartheta + A > 1 \). Note that \( \tilde{g}_j \vartheta_j^2 = 4\tilde{g}_j v_j^2 \Lambda_{jj}^1 \), we want

\[
[(1 + \epsilon)(1 + \beta^{-1})(\Phi')^2 - \Phi''] \sum_{j=1}^n \tilde{g}_j (\vartheta_j)^2 + [1 + (1 + \epsilon^{-1})\beta^{-1}] \sum_{j \in J} \tilde{g}_j \left( \frac{\Lambda_{jj}}{\Lambda_{11}} \right)^2. 
\]  

(4.23)

Here we point out that \( \tilde{\nabla}_i \tilde{\nabla}_j = \gamma_i^{\alpha} \tilde{d}(\xi) + \tilde{s} \xi_i \), which yields \( |\tilde{\nabla} \eta| \) is bounded. Let \( \Phi(\vartheta) = -\log(\vartheta + A) \), where \( A = A(T, \mathcal{M}_0) > 0 \) large such that \( \vartheta + A > 1 \). Note that \( \tilde{g}_j \vartheta_j^2 = 4\tilde{g}_j v_j^2 \Lambda_{jj}^1 \), we want

\[
[(1 + \epsilon)(1 + \beta^{-1})(\Phi')^2 - \Phi''] \sum_{j=1}^n \tilde{g}_j (\vartheta_j)^2 + [1 + (1 + \epsilon^{-1})\beta^{-1}] \sum_{j \in J} \tilde{g}_j \left( \frac{\Lambda_{jj}}{\Lambda_{11}} \right)^2. 
\]  

(4.23)

This is equivalent to

\[
(\epsilon + \beta^{-1} + \epsilon \beta^{-1})4|\nabla v|^2 - (\vartheta + A) < 0. 
\]  

(4.24)

By choosing \( \epsilon > 0 \) small and \( \beta > 0 \) large inequality (4.24) can be satisfied. Then (4.23) can be reduced to

\[
0 \leq \frac{C_1}{\eta} + C_2 - \frac{1}{\vartheta + A} \sum \tilde{g}_i \Lambda_{ii}^2 + \frac{1}{\Lambda_{11}} \tilde{G}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} + \frac{C_3(\beta + \beta^2)\tilde{g}_1}{\eta^2} \sum \tilde{g}_j \left( \frac{\Lambda_{jj}}{\Lambda_{11}} \right)^2. 
\]  

(4.25)

Since \( \tilde{g} \) is concave we have

\[
\frac{1}{\Lambda_{11}} \tilde{G}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} \leq \frac{2}{\Lambda_{11}} \sum \frac{\tilde{g}_1 - \tilde{g}_j}{\Lambda_{11} - \Lambda_{jj}} \lambda_{11} \lambda_{11}^j \leq -\frac{3}{2\lambda_{11}} \sum \tilde{g}_j \Lambda_{11}^2. 
\]

When \( 1 + \frac{1}{\beta} + \frac{1}{\beta \epsilon} < \frac{3}{2} \) we have

\[
0 \leq \frac{C_1}{\eta} + C_2 - \frac{1}{\vartheta + A} \sum \tilde{g}_i \Lambda_{ii}^2 + \frac{C_3(\beta + \beta^2)\tilde{g}_1}{\eta^2} \sum \tilde{g}_j \Lambda_{j1}^2. 
\]  

(4.26)

Since \( \tilde{g}_i = \alpha F_{-}^{\alpha} F_{ii}^* \), we get \( \tilde{g}_1 < \frac{\alpha(\tilde{G})}{\lambda_{11}} \). By (4.26) and (1.7) we conclude

\[
\frac{1}{\vartheta + A} |\tilde{G}| \lambda_1 \leq \frac{C_1}{\eta} + C_2 + \frac{C_3(\beta + \beta^2)|\tilde{G}|}{\lambda_{11} \eta^2}. 
\]

Therefore, \( \lambda_{11} \eta \leq C_4 \) and the Lemma is proved. \( \square \)

In Sect. 3 we proved the existence of the solution \( u^*_r \) to Eq. (3.1). In Sect. 4 we obtained the local estimates of \( u^*_r \) for any compact subset \( K \subset B_1 \) and \( T > 0 \) Therefore, we can find a subsequence of \( u^*_r \) such that \( u^*_r \rightarrow u^* \), and \( u^* \) satisfies (2.3) in \([0, T]\). Since \( T > 0 \) is arbitrary, we obtain the existence of a solution to Eq. (2.3) in \([0, \infty)\). Applying the standard maximum principle, we also know that the solution to (2.3) is unique. The Legendre transform of \( u^* \), which we denote by \( u \), is the desired entire solution of (2.1).
5 Convergence

In this section, we will show after rescaling, the entire solution to Eq. (2.1) converges to the unit future hyperboloid as \( t \to \infty \).

Let \( \bar{X} = \frac{X}{[(1+\alpha)\bar{t}]^{1/\alpha}} \) and \( \bar{t} = \int_0^t [(1+\alpha)\bar{t}]^{-1} dt \). Then \( \bar{X} \) satisfies

\[
(\bar{X})_{\bar{t}} = (\bar{X})_{t} \cdot \frac{\partial t}{\partial \bar{t}} = F^\alpha(\kappa)[(1+\alpha)\bar{t}]^{\alpha/\alpha} v - \frac{X}{[(1+\alpha)\bar{t}]^{1/\alpha}}.
\]

(5.1)

Since \( \bar{\kappa} = [(1+\alpha)\bar{t}]^{1/\alpha} \kappa \), (5.1) becomes

\[
(\bar{X})_{\bar{t}} = F^\alpha(\bar{\kappa}) v - \bar{X}.
\]

(5.2)

Denote \( \bar{v} = \langle \bar{X}, v \rangle \), then under this rescaling we get

\[
\bar{v}_{\bar{t}} = -F_{*}^{-\alpha}(\bar{\Lambda}_{ij}) + \bar{v},
\]

(5.3)

where \( \bar{\Lambda}_{ij} = \bar{\nabla}_{ij} \bar{v} - \bar{v} \delta_{ij} \). A straightforward calculation gives

\[
(\bar{\Lambda}_{ii})_{\bar{t}} = (\bar{\nabla}_{ii} \bar{v})_{\bar{t}} - \bar{v}_{\bar{t}}
= -\bar{\nabla}_{ii} F_{*}^{-\alpha} - \bar{\nabla}_{ii} \bar{v} + F_{*}^{-\alpha}(\bar{\Lambda}) + \bar{v}
= -\bar{\nabla}_{ii} F_{*}^{-\alpha} + F_{*}^{-\alpha}(\bar{\Lambda}) - \bar{\Lambda}_{ii}.
\]

(5.4)

This yields

\[
(F_{*}^{-\alpha})_{\bar{t}} = -\alpha F_{*}^{-\alpha-1} F_{kk}^{\alpha}(\bar{\Lambda}_{kk})_{\bar{t}}
= \alpha F_{*}^{-\alpha-1} F_{kk} \bar{\nabla}_{kk} F_{*}^{-\alpha} - \alpha F_{*}^{-2\alpha-1} \sum F_{kk} + \alpha F_{*}^{-\alpha}.
\]

(5.5)

In the following, we denote \( L := \frac{\partial}{\partial \bar{t}} - \alpha F_{*}^{-\alpha-1} F_{kk}^{\alpha} \bar{\nabla}_{kk} \) and \( \Phi = F_{*}^{-\alpha}(\bar{\Lambda}) + \bar{v} \). Then we have

\[
L\Phi = -\left(1 + \alpha F_{*}^{-\alpha-1} \sum F_{kk}^{\alpha}\right) \Phi.
\]

Consider \( \Psi = e^{2\bar{t}} \Phi^2 \), then \( \Psi \) satisfies

\[
L\Psi \leq -2\alpha F_{*}^{-\alpha-1} \sum F_{kk}^{\alpha} \Psi \leq 0.
\]

(5.6)

Therefore, by the maximum principle we know \( \Phi^2 \leq e^{-2\bar{t}} c_0 \). This implies \( \Phi \to 0 \) as \( \bar{t} \to \infty \). Since \( \bar{v} = \frac{(X,v)}{[(1+\alpha)\bar{t}]^{1/\alpha}} = \frac{v}{[(1+\alpha)\bar{t}]^{1/\alpha}} \), by Lemma 8 we have \(-a_1 + 1 \leq \bar{v} \leq -a_0 \).

From earlier estimates for \( F_{*}^{-\alpha}(\bar{\Lambda}) + \bar{v} \), we get when \( \bar{t} > t_0 > 0 \),

\[
c_1 \leq F_{*}^{-\alpha}(\bar{\Lambda}) \leq c_2.
\]

(5.7)

Now suppose \( \bar{u}^*_{\infty} = -\sqrt{1 - |\xi|^2} \), it’s easy to see that \( \bar{u}^*_{\infty} \) satisfies

\[
\begin{aligned}
F_{*}^{-\alpha}(w^*_{ijkl}(u^*_{\infty})_{ki} \gamma_{ij}^*) w^* &= -u^*_{\infty} & \text{in } B_1 \\
u^*_{\infty} &= 0 & \text{on } \partial B_1.
\end{aligned}
\]

(5.8)
Moreover, the graph of the Legendre transform of $\tilde{u}^*_\infty$, denoted by $\tilde{M}_\infty = \{(x, \tilde{u}_\infty(x)) \mid x \in \mathbb{R}^n\}$, is a hyperboloid.

Let $u^*$ be the solution of (2.3) and $\tilde{u}^*$ be the rescaling of $u^*$. Then $\tilde{u}^*$ satisfies

$$
\left\{
\begin{array}{ll}
(\tilde{u}^*)_t = -F_*^{-\alpha}(w^*\gamma^*_{ij}(\tilde{u}^*)_{kl}\gamma^*_i j)w^* - \tilde{u}^* & \text{in } B_1 \times (0, \infty) \\
\tilde{u}^*(\cdot, \tilde{\tau}) = 0 & \text{on } \partial B_1 \times (0, \infty) \\
\tilde{u}^*(\cdot, 0) = u_0^* & \text{on } B_1 \times \{0\}.
\end{array}
\right.
$$

(5.9)

**Lemma 29** Let $\tilde{u}^*$ be the solution of (5.9). Then for each sequence $\tilde{\tau}_j \rightarrow \infty$ there is a subsequence $\tilde{\tau}_{j_k} \rightarrow \infty$ such that $\tilde{u}^*(\cdot, \tilde{\tau}_{j_k}) \rightarrow \tilde{u}^*_\infty(\cdot)$ locally smoothly on $K$, where $K \subset B_1$ is a compact set.

**Proof** Without loss of generality we assume (5.7) holds for $\tilde{\tau} \geq 0$. In view of (5.7) we obtain

$$
c_1[(1 + \alpha)\tilde{\tau}]^{-\frac{\alpha}{1+\alpha}} \leq F_*^{-\alpha}(\Lambda) \leq c_2[(1 + \alpha)\tilde{\tau}]^{-\frac{\alpha}{1+\alpha}}.
$$

(5.10)

In Lemma 24 we can choose $a = c_3[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}}$, where $0 < c_3 < \left(\frac{1}{c_2}\right)^\frac{1}{\alpha}$, then we obtain

$$
\tilde{\nabla}_i v > c_4[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}},
$$

(5.11)

for any $1 \leq i \leq n$ and $\xi \in S^{n-1}(\tilde{\tau})$, $\frac{1}{2} < \tilde{\tau} < 1$. Similarly by choosing $a = c_5[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}}$ where $c_5 > \left(\frac{1}{c_1}\right)^\frac{1}{\alpha}$ in Lemma 26, we get

$$
\tilde{\nabla}_i v > c_6[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}}.
$$

(5.12)

Now in Lemma 28 we will choose $\tilde{s} = -\frac{c}{w^*}[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}}$, where $c > 0$ is a small number. Moreover, we let $\Phi(\vartheta) = -\log(\vartheta + A[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}})$, here $A > 0$ such that $\vartheta + A[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}} > [(1 + \alpha)\tilde{\tau}]^2$. Then by Lemma 28 we get in $K \times [0, \infty)$ we have

$$
\Lambda_{\text{max}} \leq c_7[(1 + \alpha)\tilde{\tau}]^{\frac{1}{1+\alpha}}.
$$

This combining with (5.7) and condition (1.8) yields in $K \times [0, \infty)$

$$
0 < c_8 \leq \Lambda_{\text{min}} \leq \Lambda_{\text{max}} \leq c_7.
$$

Applying standard regularity and convergence theorems, we prove Lemma 29. \qed

**Remark 30** The solutions of the following equation

$$
F^*\theta(k) = -\langle X, \nu \rangle
$$

is called the self-expander. According to Husiken [11], the singularity of the mean curvature flow of a strictly closed convex hypersurface is a self-shrinker in Euclidean space. Here, since the unit normal is time-like, we have the opposite sign.

**Remark 31** We want to point out that **Conditions A** is invariant under Lorentz transformation. It’s clear that the conditions (1),(2),(3) of **Conditions A** are invariant. We only need to check condition (4). Let $u_0$ satisfy

$$
\sqrt{|x|^2 + C_0} < u_0(x) < \sqrt{|x|^2 + C_1},
$$
for some suitable positive constant $C_0, C_1$. Recall that the Lorentz transformation is

$$
\begin{align*}
  x'_1 &= \frac{x_1 - \alpha x_{n+1}}{\sqrt{1 - \alpha^2}} \\
  x'_i &= x_i \\
  x'_{n+1} &= \frac{x_{n+1} - \alpha x_1}{\sqrt{1 - \alpha^2}}.
\end{align*}
$$

(5.13)

It's easy to check that

$$(x'_{n+1})^2 < (x'_1)^2 + \cdots + (x'_n)^2 + C_1$$

provided that $(x_{n+1})^2 < (x_1)^2 + \cdots + (x_n)^2 + C_1$, and

$$(x'_{n+1})^2 > (x'_1)^2 + \cdots + (x'_n)^2 + C_0$$

provided that $(x_{n+1})^2 > (x_1)^2 + \cdots + (x_n)^2 + C_0$.

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**References**

1. Andrews, Ben, Chen, Xuzhong, Fang, Hanlong: McCoy, James expansion of co-compact convex spacelike hypersurfaces in Minkowski space by their curvature. Indiana Univ. Math. J. **64**(2), 635–662 (2015)
2. Aarons, Mark AS.: Mean curvature flow with a forcing term in Minkowski space. Calc. Var. Partial Differ. Equ. **25**(2), 205–246 (2006)
3. Bayard, Pierre, Schnürer, Oliver C.: Entire spacelike hypersurfaces of constant Gauß curvature in Minkowski space. J. Reine Angew. Math. **627**, 1–29 (2009)
4. Choi, Kyeongsu, Daskalopoulos, Panagiota, Kim, Lami, Lee, Ki-Ahm.: The evolution of complete non-compact graphs by powers of Gauss curvature. J. Reine Angew. Math. **757**, 131–158 (2019)
5. Chow, Bennett, Hamilton, Richard S.: The cross curvature flow of 3-manifolds with negative sectional curvature. Turkish J. Math. **28**(1), 1–10 (2004)
6. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. Acta Math. **155**(3–4), 261–301 (1985)
7. Ecker, Klaus: Interior estimates and longtime solutions for mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space. J. Differential Geom. **46**(3), 481–498 (1997)
8. Ecker, Klaus: Huisken, Gerhard interior estimates for hypersurfaces moving by mean curvature. Invent. Math. **105**(3), 547–569 (1991)
9. Daskalopoulos, P., Huisken, G.: Inverse mean curvature evolution of entire graphs. arXiv:1709.06665
10. Guan, Bo.: Spruck, Joel Locally convex hypersurfaces of constant curvature with prescribed boundary. Comm. Pure Appl. Math. **57**(10), 1311–1331 (2004)
11. Huisken, Gerhard: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**(1), 285–299 (1990)
12. Li, Anmin: Spacelike hypersurfaces with constant Gauß–Kronecker curvature in the Minkowski space. Arch. Math. (Basel) **64**(6), 534–551 (1995)
13. Ren, C., Wang, Z., Xiao, L.: Entire spacelike hypersurfaces with constant $\sigma_{n-1}$ curvature in Minkowski space. arXiv:2005.06109
14. Sheng, Weinming, Urbas, John, Wang, Xu-Jia.: Interior curvature bounds for a class of curvature equations. Duke Math. J. **123**, 235–264 (2004)
15. Wang, Zhi Zhang, Xiao, Ling: Entire spacelike hypersurfaces with constant $\sigma_k$ curvature in Minkowski space. Math. Ann. **382**(3–4), 1279–1322 (2022)
16. Wang, Z., Xiao, L.: Entire convex self-expanders of power of $\sigma_k$ curvature flow in Minkowski space. J. Funct. Anal. **284**(8), 109866 (2023)

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