The Quantum Stochastic Differential Equation Is Unitarily Equivalent to a Symmetric Boundary Value Problem for the Schrödinger Equation

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Abstract. We prove that the solution of the Hudson–Parthasarathy quantum stochastic differential equation in the Fock space coincides with the solution of a symmetric boundary value problem for the Schrödinger equation in the interaction representation generated by the energy operator of the environment. The boundary conditions describe the jumps in the phase and the amplitude of the Fourier transforms of the Fock vector components as any of its arguments changes the sign. The corresponding Markov evolution equation (the Lindblad equation or the “master equation”) is derived from the boundary value problem for the Schrödinger equation.

Key words: singular perturbation, quantum noises, boundary value problem in the Fock space, open quantum system, master equation, quantum dynamical semigroup.

§1. Introduction

Our interest in the theory of quantum stochastic differential equations is primarily due to the fact that it can be used to represent solutions of the Markov evolution equation, which generalizes the Heisenberg, Kolmogorov–Feller, and heat equations [1–4].

Solutions of Markov evolution equations are studied in the theory of open quantum systems [5–8]; they are called quantum dynamical semigroups. One of the consequences of the stochastic representation of quantum dynamical semigroups in von Neumann algebras is a simple relationship between the conservativity of a quantum dynamical semigroup and the isometric property of a solution of the quantum stochastic differential equation. The conservativity (or the unital property) of a quantum dynamical semigroup means that the unit of the operator algebra is preserved; this corresponds to the preservation of the total probability by the solutions of the equations for the transition probabilities of the stochastic processes in the classical and quantum cases. Violations of this property are related to critical phenomena such as explosions, escape of the solution to infinity, accumulation of infinitely many discontinuities, nonunique solvability of the Cauchy problem, the existence of growing solutions to formally dissipative equations, etc. [9–10]. The study of conditions necessary and sufficient for the existence or nonexistence of such phenomena is currently of considerable interest [11–18].

Sufficient criteria for solutions of quantum stochastic differential equations to be isometric or unitary were obtained in recent years mainly by perturbation methods [19–21]. However, symmetric operators responsible for these properties of solutions were not found. The difficulties encountered in this approach include the violation of the group property by solutions of quantum stochastic differential equations [22] and the fact that their formal generators fail to be symmetric on domains consisting of smooth functions. More precisely [23], the formal generators for the Schrödinger equations unitarily equivalent to quantum stochastic differential equations have the form of dissipative operators perturbed by singular quadratic forms in the sense of [24–25]. Since the dissipative part of the generator has an unbounded anti-Hermitian part, the general theory of self-adjoint extensions of symmetric operators bounded below and perturbed by singular quadratic forms [24–26] does not apply directly to this case.

In the present paper, we consider quantum stochastic differential equations whose solutions in the interaction representation are limits of solutions of a family of Schrödinger equations in the Fock space.
[23]. The fundamental property of functions belonging to the range of the resolvent of the limit unitary group is that they experience *amplitude* and *phase jumps* at the points where the quadratic form of
the generator has singularities. Evaluating the action of the generator of the limit group on functions with amplitude and phase jumps, we obtain a symmetric operator; its self-adjointness can be verified independently under assumptions less restrictive than those used to construct the resolvent explicitly. Thus, we prove the equivalence of the quantum stochastic differential equation and the boundary value problem for the Schrödinger equation in the Fock space. Straightforward computations permit one to derive the Markov evolution equation directly from the boundary value problem for the Schrödinger equation.

We start from a simple example showing the distinction between the formal weak limit, the strong resolvent limit of a sequence of generators of unitary groups, and the action of the generator of the limit unitary group on the range of its resolvent.

§2. The weak limit and the resolvent limit of a family of generators of one-parameter unitary groups

Consider the one-parameter family of unitary groups \( \exp\{it\hat{H}_\alpha\} = U_t(\alpha) \) in \( L_2(\mathbb{R}) \) given by

\[
U_t(\alpha) \psi(x) = \psi(x-t) \exp \left\{ i\lambda \int_0^t d\tau V_\alpha(x - t + \tau) \right\}, \quad x, \lambda \in \mathbb{R},
\]

where \( V_\alpha(x) = (2\pi\alpha)^{-1/2} \exp\{-x^2/2\alpha\}, \alpha \in \mathbb{R}_+ \). Obviously, \( V_\alpha(x) \rightarrow \delta(x) \) as \( \alpha \rightarrow +0 \). Therefore, the weak limit of the family of essentially self-adjoint operators \( \hat{H}_\alpha = i\partial_x + \lambda V_\alpha(x) \) is described by the bilinear form

\[
\hat{H}_\alpha[\varphi, \psi] = (\varphi, \hat{H}_\alpha \psi) = i(\varphi, \psi') + \lambda \varphi(0) \psi(0), \tag{2.1}
\]

which is well defined on \( W^1_2(\mathbb{R}) \). On the other hand,

\[
\int_0^\infty d\tau V_\alpha(x - t + \tau) \rightarrow I_{(0,t)}(x),
\]

where \( I_T(x) \) is the characteristic function of a Borel set \( T \subseteq \mathbb{R} \). Hence, the strong limit of the family \( U_t(\alpha) \) is

\[
\lim_{\alpha \rightarrow +0} U_t(\alpha) \psi(x) = U_t \psi(x) = e^{it\hat{H}_\alpha} \psi(x) = \psi(x-t)e^{i\lambda I_{(0,t)}(x)}. \tag{2.2}
\]

Note the identity \( e^{i\lambda I_{(0,t)}(x)} = (e^{i\lambda} - 1)I_{(0,t)}(x) + 1 \). Therefore, the bilinear form of the resolvent limit

\[
\hat{H}_R = r\lim  \hat{H}_\alpha \text{ is well defined on } W^1_2(\mathbb{R}) \text{ and is given by}
\]

\[
\hat{H}_R[\varphi, \psi] = i^{-1} \lim_{t \rightarrow 0} \left. \frac{d}{dt} (\varphi, U_t \psi) \right|_{t=0} = i(\varphi, \psi') + i(e^{i\lambda} - 1)\varphi(0) \psi(0). \tag{2.2}
\]

Comparing (2.1) with (2.2), we conclude that \( \hat{H}_R = r\lim  \hat{H}_\alpha \neq w\lim  \hat{H}_\alpha = \hat{H}_w \).

The range of the resolvent of the limit unitary group \( U_t \) is the natural domain of the generator \( \hat{H}_R \) and can be described explicitly in our example:

\[
R_\mu \psi(x) = \int_0^\infty dt e^{-\mu t} \psi(x-t) + \theta(x)(e^{i\lambda} - 1)e^{-\mu x} \int_0^\infty dt e^{-\mu t} \psi(-t),
\]

where \( \theta(x) \) is the characteristic function of the semi-axis \( \mathbb{R}_+ \). This structure of the resolvent means that the functions belonging to the domain of \( \hat{H}_R \) have a phase jump at the origin \( x = 0 \):

\[
\lim_{x \rightarrow +0} R_\mu \psi(x) = e^{i\lambda} \lim_{x \rightarrow -0} R_\mu \psi(x).
\]

Therefore, the domain \( \mathcal{D}_\lambda \) of the generator \( \hat{H}_R \) consists of the functions

\[
\psi: \quad \psi \in W^1_2(\mathbb{R} \setminus \{0\}), \quad \lim_{x \rightarrow +0} \psi(x) = e^{i\lambda} \lim_{x \rightarrow -0} \psi(x). \tag{2.3}
\]
The operator $\widehat{H}_R$ acts as $i\partial_x$ for $x \neq 0$. For functions $\psi \in \mathcal{D}_\lambda$, the left and the right limits at the origin exist by virtue of the embedding $W^1_2(\mathbb{R} \setminus \{0\}) \subset C(\mathbb{R} \setminus \{0\})$. The symmetry property of $\widehat{H}_R$ follows from integration by parts and from the identity
\[
\varphi(x)\psi(x)|_{x=0}^{-0} = 0,
\]
which holds for $\varphi, \psi \in \mathcal{D}_\lambda$; the self-adjointness follows [27] from the fact that for $\mu = \pm 1$ the equation $(\widehat{H}_R + i\mu)\psi(x) = f(x)$, $x \neq 0$, with the boundary condition (2.3) is solvable in $\mathcal{D}_\lambda$ for every right-hand side $f \in L^2_2(\mathbb{R})$.

§3. The weak and the strong resolvent limit of solutions of the Schrödinger equation in the Fock space

Suppose that $\mathcal{H}$ is a Hilbert space, $\Gamma^S(L^2_2(\mathbb{R}))$ is the symmetric Fock space and $\mathfrak{h} = \mathcal{H} \otimes \Gamma^S(L^2_2(\mathbb{R}))$ is their tensor product. For $v, g \in L^2_2(\mathbb{R})$, by $A^+(v)$ and $A(g)$ we denote the standard creation and annihilation operators in $\Gamma^S(L^2_2(\mathbb{R}))$. Consider the family of Schrödinger equations $\partial_t \psi_t = i\mathcal{H}\psi_t$ with self-adjoint Hamiltonian $\mathcal{H} = H_0 \otimes I + I \otimes \widehat{E} + H_{\text{int}}$ in $\mathfrak{h}$:
\[
H_{\text{int}} = K \otimes A^+(g)A(g) + R \otimes A^+(f) + R^* \otimes A(f), \quad \widehat{E} = \int \omega a^+(\omega) a(\omega) \, d\omega.
\]
From now on, it is assumed for simplicity that the operators $H_0$, $K$, and $R \in C(\mathcal{H})$ commute and have a joint spectral family $E_\lambda$. More precisely, let
\[
H_0 = \int \nu(\lambda) \, dE_\lambda, \quad K = \int \lambda \, dE_\lambda, \quad R = \int \rho(\lambda) e^{i\Phi(\lambda)} \, dE_\lambda,
\]
where $\nu, \rho$, and $\Phi$ are measurable real functions corresponding to the operators $H_0$, $K$, and $R$, so that the operators $H_0$ and $K$ are self-adjoint and $R$ is normal.

We denote by $\widehat{P}_t(\lambda)$ the one-parameter group of unitary operators in $L^2_2(\mathbb{R})$ with the generator $\widehat{N}(\lambda) = \omega + \lambda g(\langle g \rangle)$. The unitary group $U_t = \exp\{i\mathcal{H}t\}$ can be constructed by transforming the Hamiltonian to the canonical form [28], or by using the interaction representation generated by the operator $I \otimes \widehat{E} + K \otimes A^+(g)A(g)$. Theorem 3.1 describes the action of $U_t$ on coherent vectors.

**Theorem 3.1.** The unitary one-parameter group $U_t = \exp\{i t \mathfrak{h}\}$, where $\mathfrak{h} = H_0 \otimes I + I \otimes \widehat{E} + H_{\text{int}}$ and $H_{\text{int}}$ is defined by Eq. (3.1), acts as follows:
\[
U_t h \otimes \psi(v) = \int e^{i\nu(\lambda)t} \, dE_\lambda h \otimes \psi(v_\lambda) \exp\left\{i\rho(\lambda)e^{-i\Phi(\lambda)} \int_0^t \left(f, v_\lambda(\lambda)\right) \, ds\right\},
\]
where
\[
v_\lambda(\lambda) = \widehat{P}_t(\lambda)v + i\rho(\lambda)e^{i\Phi(\lambda)} \int_0^t \widehat{P}_s(\lambda)f \, ds.
\]
Let $L^+_{2,1}(\mathbb{R}) \subset \overline{W}^1_2(\mathbb{R})$ be the set of functions with positive absolutely integrable Fourier transform; $\overline{W}^1_2(\mathbb{R})$ is the Fourier transform of the Sobolev space $W^1_2(\mathbb{R})$. Let $f, g \in L^+_{2,1}(\mathbb{R})$ be real functions such that $f(0) = g(0) = 1/\sqrt{2\pi}$, and let $f^{(\alpha)}(\omega) = f(\alpha \omega)$, $g^{(\alpha)}(\omega) = g(\alpha \omega)$.

**Lemma 3.1** (about four limits). Let $\widehat{P}^{(\alpha)}_t(\lambda)$ be the one-parameter unitary group in $L^2_2(\mathbb{R})$ with generator $\widehat{N}_0(\lambda) = \omega + \lambda |g^{(\alpha)}(\langle g^{(\alpha)} \rangle)|$, and let $\pi(t) = F_{t \rightarrow \omega} F_{\omega}^{-1} t$ be a family of projections in $\mathcal{H}(L^2_2(\mathbb{R}))$. Then the following limits exist as $\alpha \to 0$:

1) $\lim_{\alpha \to 0} \int_0^t ds \left(g^{(\alpha)}, \widehat{P}^{(\alpha)}_s(\lambda)f^{(\alpha)}\right) = (2 - i\lambda)^{-1}$;
2) \( w\text{-}\lim \int_0^t ds \hat{P}_s(\lambda)f^{(\alpha)}(\omega) = e^{i\omega t} \hat{I}_{[0,t]}(\omega)(1-i\lambda/2)^{-1}; \)
3) \( w\text{-}\lim \\langle g^{(\alpha)}, \hat{P}_t^{(\alpha)}(\lambda)v \rangle = (1-i\lambda/2)^{-1} F^{*}_{\omega \to v}; \)
4) \( s\text{-}\lim \hat{P}_t^{(\alpha)}(\lambda) = \exp\{iZ(\lambda)\hat{\pi}_{(0,t)}\} = \hat{P}_t(\lambda), \exp\{iZ(\lambda)\} = (2+i\lambda)/(2-i\lambda). \)

Let us consider the family \( H_\alpha \) of self-adjoint Hamiltonians (3.1) parametrized by the functional arguments \( f^{(\alpha)}(\omega), g^{(\alpha)}(\omega) \) of the creation and annihilation operators. Lemma 3.1 in [23] justifies the passage to the limit in Eq. (3.2) as \( \alpha \to +0 \). The limits 1)–4) correspond to the passage to the limit in the four components of the solution (3.2) depending on \( \alpha \). By substituting 1)–4) into Eq. (3.2), we obtain the limit unitary group \( U_t = \exp\{iHt\} = s\text{-}\lim_{\alpha \to 0} \exp\{iH_\alpha t\}: \)

\[
U_t h \otimes \psi(v) = \int e^{i \hat{H}(\lambda)t} dE \chi \left( e^{iZ(\lambda)\hat{\pi}_{(0,t)}} e^{i\omega t} v + \frac{2i}{2-i\lambda} \rho(\lambda) e^{i\Phi(\lambda) \hat{I}_{[0,t]}} \right) \\
\times \exp\left\{ \rho(\lambda) e^{-i\Phi(\lambda)} \right\}^{\frac{2i}{2-i\lambda} \hat{I}_{[0,t]}, e^{i\omega t} v}, \quad (3.3)
\]

where \( \hat{H}(\lambda) = \nu(\lambda) + i\rho^2/(2-i\lambda) \). Let \( W \) and \( L \) be the operators with spectral densities \((2+i\lambda)/(2-i\lambda)\) and \(2i/(2-i\lambda) \rho(\lambda) e^{i\Phi(\lambda)}\), respectively. Then \( 2i\rho(\lambda) e^{-i\Phi(\lambda)}/(2-i\lambda) \) is the spectral density of the operator \(-L^*W\), where \( W = (2+iK)/(2-iK) \) is the Cayley transformation of the self-adjoint operator \( 2K \), and \( L = 2i/(2-iK) \) is a densely defined operator such that \( \text{dom} L \supseteq \text{dom} R \), and \( \hat{H}(\lambda) \) is the spectral function of the operator \( iG = H_0 - \frac{1}{4} L^*KL + \frac{i}{2} L^*L \).

In the following, we suppose that the operator \(-G\) is the generator of a strongly continuous one-parameter contraction semigroup \( W_t = \exp\{-Gt\} \) in \( \mathcal{H} \), and moreover,

\[
D = \text{dom} H \cap \text{dom} L^*L \subseteq \text{dom} G \subseteq \text{dom} L, \quad G^*\varphi + G\varphi = L^*L\varphi \quad \forall \varphi \in D,
\]

where the operator \( H = -H_0 + \frac{1}{4} L^*KL \) is symmetric on \( D \). In this notation, the bilinear form

\[
\mathbf{H}[g \otimes \psi(f), h \otimes \psi(v)] = \lim_{t \to +0} \frac{1}{i} \frac{d}{dt} \langle g \otimes \psi(f), U_t h \otimes \psi(v) \rangle,
\]

where \( g, h \in D \subseteq \mathcal{H} \) and \( f, v \in \tilde{W}_2^1(\mathbb{R}) \), acts as follows:

\[
\mathbf{H}[g \otimes \psi(f), h \otimes \psi(v)] = e^{i(f,v)_L} \left( i(g, Gh)_{\mathcal{H}} - i(g, Lh)_{\mathcal{H}} f(0) + i(Lg, Wh)_{\mathcal{H}} \tilde{v}(0) \right) + (g, h)_{\mathcal{H}} \int d\omega \overline{f(\omega)g(\omega)} + i(g, (I - W)h)_{\mathcal{H}} \overline{f(0)\tilde{v}(0)} \right). \quad (3.4)
\]

The bilinear form (3.4) has the regular dissipative component \( \frac{1}{2} L^*L \otimes I \) and the singular component [24–25]

\[
-iL \otimes A^+ \left( \frac{1}{\sqrt{2\pi}} \right) + iL^*W \otimes A \left( \frac{1}{\sqrt{2\pi}} \right) + i(I - W) \otimes A^+ \left( \frac{1}{\sqrt{2\pi}} \right) A \left( \frac{1}{\sqrt{2\pi}} \right),
\]

vanishing on the subset \( D_0 \)

\[
D_0 = \{ \Phi : \Phi = h \otimes \psi(v), V \in \tilde{W}_2^1(\mathbb{R}), \tilde{v}(0) = 0, h \in D \},
\]

which is total in \( \mathfrak{h} \). At the same time, the weak limit of the sequence \( H_\alpha \) is described by a bilinear form with different regular and singular components:

\[
\lim_{\alpha \to 0} H_\alpha[g \otimes \psi(f), h \otimes \psi(v)] = e^{i(f,v)_L} \left( (g, H_0h)_{\mathcal{H}} + (g, h)_{\mathcal{H}} \int d\omega \overline{f(\omega)g(\omega)} \right.

\[
+ (g, Kh)_{\mathcal{H}} \overline{f(0)\tilde{v}(0)} + (g, Rh)_{\mathcal{H}} \overline{f(0)\tilde{v}(0)} - (Rg, h)_{\mathcal{H}} \overline{\tilde{v}(0)} \right).
\]

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Just as in the example in §2, here we have

\[ r - \lim_{\alpha \to 0} H_{\alpha} \neq w - \lim_{\alpha \to 0} H_{\alpha}. \]

Let us consider the set function \( u(s, t) = J_s U_{t-s} J_t^* \), where \( J_t \) is the unitary one-parameter group generated by the operator \( \hat{E} = \int d\omega \omega \phi(\omega) a(\omega) \) [7], which acts in \( \Gamma^S \) as \( J_t \psi(v) = \psi(e^{i\omega t} v) \). Taking into account the fact that \( J_t^* \psi(v) = \psi(e^{-i\omega t} v) \), from Eq. (3.3) we obtain

\[
\begin{align*}
  u(T)h \otimes \psi(v) &= \int e^{iH(\lambda) t} dE \{ h \otimes \psi \left( e^{iZ(\lambda)\hat{T} v} + i\rho(\lambda) e^{i\Phi(\lambda)} \frac{2}{2 - i\lambda} \hat{T} \right) \}
  &\times \exp \left\{ i\rho(\lambda) e^{-i\Phi(\lambda)} \frac{2}{2 - i\lambda} (\hat{T}, v) \right\}. 
\end{align*}
\]

(3.5)

For disjoint sets, the family of commuting operators \( u(T) \) satisfies the composition law \( u(T_1 \cup T_2) = u(T_1)u(T_2) \), and the differential of the bilinear form \((g \otimes \psi(f), u(0, t)h \otimes \psi(v))\) satisfies the weak quantum stochastic differential equation [2]

\[
d(h \otimes \psi(v), u(0, t)h \otimes \psi(v)) = i(h \otimes \psi(v), u(0, t)H(dt_{\pi})h \otimes \psi(v)),
\]

(3.6)

where

\[
\begin{align*}
  iH(T) &= M(T) = \int_t dt J_t(\hat{H} - \hat{E} \otimes I) J_t^*, \\
  M(T) &= -G \otimes \text{mes} T + L \otimes A^+(T) - L^* W \otimes A(T) + (W - I) \otimes \Lambda(T).
\end{align*}
\]

Thus, the following statement holds.

**Theorem 3.2.** The family of solutions of the Schrödinger equation with Hamiltonian (3.1) in \( \mathfrak{h} \) strongly converges, up to the unitary transformation \( J_t^* \), to the solution \( u(0, t) \) of the stochastic equation (3.6), \( u(0, t) = s \cdot \lim_{\alpha \to 0} U_{\alpha}^{(\alpha)} J_t^* \).

In what follows, we shall see that, quite unexpectedly, the natural domain of the generator of the group \( U_t \) (the range of the resolvent) does not contain functions on which the bilinear form (3.4) is well defined: the Fourier transforms of the Fock components of the resolvent experience amplitude and phase jumps at the points where their arguments change sign.

**§4. Surprising properties of the resolvent**

Consider the following vector \( \Phi \in \mathfrak{h} = \mathcal{H} \otimes \Gamma^S(L_2(\mathbb{R})) \):

\[ \Phi = R_{\mu} h \otimes \psi(v) = \int_0^{\infty} dt e^{-\mu t} U_t h \otimes \psi(v) = \{ \Phi_n(\omega) \}, \quad \Phi_n(\cdot): \mathbb{R}^n \to \mathcal{H}, \quad \omega = \{ \omega_1, \ldots, \omega_n \}, \]

with components (3.3)

\[
\begin{align*}
  \Phi_n(\omega) &= \int_0^{\infty} dt \exp \left\{ -\mu (G + t - L^* W) \int_0^t \varphi(-\tau) d\tau \right\} \phi_{n,t}(\omega), \\
  \varphi_{\tau \cdot t}(\omega) &= \prod_1^n \left( (W - 1) \pi_{1,t}(\omega_k) e^{i\omega_k t} v(\omega_k) + e^{i\omega_k t} v(\omega_k) + L \hat{T}_{1,t}(\omega_k) \right) h,
\end{align*}
\]

where \( L, W, \) and \( G \) are the above-described commuting operators with spectral densities

\[
\begin{align*}
  L(\lambda) &= 2i\rho(\lambda) e^{-i\Phi(\lambda)(2 - i\lambda)^{-1}}, \\
  W(\lambda) &= e^{iZ(\lambda)}, \\
  G(\lambda) &= -i\nu(\lambda) + \frac{\rho(\lambda)^2}{(2 - i\lambda)^2},
\end{align*}
\]

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Let $\tilde{\varphi}_{n,t}(\tau)$ be the Fourier transform of $\varphi_{n,t}(\omega)$,

$$\tilde{\varphi}_{n,t}(\tau) = \prod_{k=1}^{n}((W - I)\tilde{\upsilon}(\tau_k - t) + \tilde{\upsilon}(\tau_k - t) + LI_{[0, t]}(\tau_k))h,$$

where $\tau = \{\tau_1, \ldots, \tau_n\}$. Let $\mathcal{K} \subset \{1, \ldots, n\}$, and let $\mathcal{K}^c$ be the complement of $\mathcal{K}$. Set

$$P^{(n)}_{\mathcal{K}, i}(\tau) = \prod_{k \in \mathcal{K}} ((W - I)\tilde{\upsilon}(\tau_k - t) + LI_{[0, t]}(\tau_k)) \in \mathcal{B}(\mathcal{H}).$$

Then

$$\tilde{\varphi}_{n,t}(\tau) = \sum_{\mathcal{K}} \left( \prod_{m \in \mathcal{K}^c} \tilde{\upsilon}(\tau_m - t) \right) h. \quad (4.1)$$

Obviously, the functions $P^{(n)}_{\mathcal{K}, i}(\tau)$ have discontinuities of the first kind at the points where the variables $\tau_k$ change the sign:

$$\lim_{\tau_k \to 0} \frac{P^{(n)}_{\mathcal{K}, i}(\tau)}{P^{(n)}_{\mathcal{K}, i}(\tau)} = I_{\mathcal{K}^c}(k)P^{(n)}_{\mathcal{K}, i}(\tau),$$

$$\lim_{\tau_k \to +0} \frac{P^{(n)}_{\mathcal{K}, i}(\tau)}{P^{(n)}_{\mathcal{K}, i}(\tau)} = I_{\mathcal{K}^c}(k)P^{(n)}_{\mathcal{K}, i}(\tau) + ((W - I)\tilde{\upsilon}(-t) + L)P^{(n-1)}_{\mathcal{K} \setminus \{k\}, t}(\tau). \quad (4.2)$$

Therefore, Eq. (4.2) implies

$$P^{(n)}_{\mathcal{K}, i}(\tau)|_{\tau_k = 0} = -(W - I)\tilde{\upsilon}(-t) + L)P^{(n-1)}_{\mathcal{K} \setminus \{k\}, t}(\tau)I_{\mathcal{K}}(k). \quad (4.3)$$

Let us calculate the jump of $\tilde{\varphi}_{n,t}(\tau)$ at the points where $\tau_k$ changes sign. Note that

$$\lim_{\tau_k \to 0} \tilde{\varphi}_{n,t}(\tau) = \tilde{\upsilon}(-t)\tilde{\varphi}_{n-1,t}(\tau_1, \ldots, \tau_k-1, \tau_k+1, \ldots, \tau_n).$$

Taking into account Eqs. (4.1) and (4.3), we find the amplitude and phase jumps of the functions from the domain of the infinitesimal operator of $U_t$:

$$\lim_{\tau_k \to +0} \tilde{\varphi}_{n,t}(\tau) = W \lim_{\tau_k \to 0} \tilde{\varphi}_{n,t}(\tau) + L\tilde{\varphi}_{n-1,t}(\tau_1, \ldots, \tau_k-1, \tau_k+1, \ldots, \tau_n). \quad (4.4)$$

Let $D_{W \otimes L} = D \otimes \Gamma^S(\mathbb{W}^1(\mathbb{R} \setminus \{0\}) \subset h$ be the vector subspace of elements satisfying condition (4.4), and let $A(\delta_\pm)$, $\Lambda(\delta_\pm)$, and $\tilde{N}$ be the operators acting on the Fock vectors by the rule

$$\langle \Phi, \Lambda(\delta_\pm)\Psi \rangle = \lim_{\epsilon \to +0} \frac{1}{n!} \sum_{k=1}^{n} \sum_{m \neq k}^{n} \int_{\mathbb{R} \setminus \{0\}} |d\tau_m(\tilde{\Phi}_n, \tilde{\Psi}_n)_{\mathcal{H}}|_{\tau_k = \pm \epsilon},$$

$$\mathcal{F}_{\omega \to \tau} \left( A(\delta_\pm)\Psi \right)_n(\tau) = \lim_{\epsilon \to +0} \frac{1}{n!} \sum_{k=1}^{n} \tilde{\Psi}_{n+1}(\tau)|_{\tau_k = \pm \epsilon}, \quad \tilde{N}\Psi_n(\omega) = n\Psi_n(\omega).$$

In this notation, the boundary condition (4.4) in $\Gamma^S$ acquires the form

$$\left( \tilde{N} + 1^{-1}(I \otimes A(\delta_+) - W \otimes A(\delta_-)) \right)\Psi = L \otimes I\Psi. \quad (4.5)$$

This condition is imposed on the Fock vector components $\Psi_1, \Psi_2, \ldots$ and extends condition (2.2) to the Fock space.
Let us prove that the operator
\[ \hat{H} = iG \otimes I + I \otimes \hat{E} + iL^*W \otimes A(\delta_-), \quad \hat{E} = \int_{\mathbb{R}\setminus\{0\}} d\tau a^+(\tau)a(\tau)i\partial_\tau, \]
(4.6)
is symmetric in \( D_{W,L} \). Let \( \Phi, \Psi \in D_{W,L} \), and let \( B \) be a Hermitian operator such that \( \text{dom } B \otimes I \supset D_{W,L} \). Integration by parts yields the following identity, where the integrated terms are expressed via the operators \( \Lambda(\delta_\pm) \):
\[
\langle \Phi, B \otimes \hat{E}\Psi \rangle - \langle B \otimes \hat{E}\Phi, \Psi \rangle = i\langle \Phi, B \otimes (\Lambda(\delta_-) - \Lambda(\delta_+))\Psi \rangle
\]
\[= i \sum_{n} \frac{1}{n!} \sum_{k=1}^{n} \int_{(\mathbb{R}\setminus\{0\})^{n-1}} \prod_{m \neq k} d\tau_m (\hat{\Phi}_n(\tau), B\hat{\Psi}_n(\tau)) \bigg|_{\tau_k = -0}^{\tau_k = +0}.
\]
(4.7)

Using the boundary condition (4.5) for \( \tilde{\varphi}_n \) and \( \tilde{\psi}_n \), we find the integrated terms in (4.7):
\[
i\langle \Phi, B \otimes (\Lambda(\delta_+) - \Lambda(\delta_-))\Psi \rangle = i\langle \Phi, (W^*BW - B)\Lambda(\delta) - \Psi \rangle
\]
\[+ i\langle L\Phi, BL\Psi \rangle + i\langle W(A(\delta_-)\Phi, BL\Psi) + i\langle L\Phi, BW A(\delta_-)\Psi \rangle.
\]
(4.8)

In particular, Eq. (4.8) is simplified for \( B = I \) as follows:
\[
I(\Phi, I \otimes (\Lambda(\delta_+) - \Lambda(\delta_-))\Psi) = i\langle \Phi, L^*L\Psi \rangle - i\langle L^*W \otimes A(\delta_-)\Psi, \Psi \rangle + \langle \Phi, iL^*W \otimes A(\delta_-)\Psi \rangle.
\]

Since \( iG - iL^*L = (iG)^* \), we obtain the following identity, which means that the operator \( \hat{H} \) is symmetric in \( D_{W,L} \):
\[
\langle \Phi, \hat{H}\Psi \rangle = ((I \otimes \hat{E}\Phi, \Psi) + \langle \Phi, iG \otimes I + iL^*W \otimes A(\delta_-)\Psi \rangle
\]
\[- i\langle \Phi, I \otimes (\Lambda(\delta_+) - \Lambda(\delta_-))\Psi \rangle
\]
\[= ((I \otimes \hat{E}\Phi, \Psi) + \langle \Phi, (iG)^* \otimes I \Psi \rangle + i\langle L^*W \otimes A(\delta_-)\Psi, \Psi \rangle = \langle \hat{H}\Phi, \Psi \rangle.
\]

Let us find how the generator of the group \( U_t \) acts on the range of the resolvent. Set \( \Psi \in \mathfrak{h} \) and \( \Phi = R_{\theta}\psi \otimes \psi(v) \). By the definition of the generator, we have
\[
\langle \Psi, \hat{H}\Phi \rangle = \lim_{s \to +0} \frac{1}{i} \frac{d}{ds} \langle \Psi, U_s\Phi \rangle = \frac{1}{i} \int_{0}^{\infty} dt \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}\setminus\{0\})^n} d\tau
\]
\[\times \left( \tilde{\varphi}_{n,t}(\tau), \frac{d}{ds} e^{-(G+\psi)t-Gs+it\psi} \sum_{m=1}^{n} \frac{\partial}{\partial \tau_k} \tilde{\varphi}_{n,t}(\tau) \right) \bigg|_{s=0}.
\]
(4.9)

Note that the functions \( \tilde{\varphi}_{n,t}(\tau) \) depend on the differences \( \tau_k - t \). Therefore,
\[
\frac{d}{dt} \tilde{\varphi}_{n,t}(\tau) = - \sum_{k=1}^{n} \frac{\partial}{\partial \tau_k} \tilde{\varphi}_{n,t}(\tau) = i\hat{E}\tilde{\varphi}_{n,t}(\tau).
\]
(4.10)

On the other hand,
\[
A(\delta_-)\varphi_{n,t}(\tau) = n\tilde{\psi}(-t)\varphi_{n-1,t}(\tau)
\]
(4.11)

by the definition of \( A(\delta_-) \). Now, taking into account definition (4.9) and identities (4.10) and (4.11), we obtain
\[
\langle \Psi, \hat{H}\Phi \rangle = \int_{0}^{\infty} dt \left( \psi_0, iGe^{-(G+\psi)t-iL^*Wf_0\tilde{\psi}(-t)d\tau}\mathfrak{h} \right) + \int_{0}^{\infty} dt \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}\setminus\{0\})^n} d\tau
\]
\[\times \left( \tilde{\varphi}_n(\tau), e^{-(G+\psi)t-iL^*Wf_0\tilde{\psi}(-t)d\tau} \left( iG + iL^*W\tilde{\psi}(-t) + i\sum_{k=1}^{n} \frac{\partial}{\partial \tau_k} \right) \tilde{\varphi}_{n,t}(\tau) \right) \mathfrak{h}
\]
\[= \langle \Psi, (iG + iL^*W \otimes A(\delta_-) + I \otimes \hat{E})\Phi \rangle \mathfrak{h}.
\]

that is, the generator \( \hat{H} \) of the group \( U_t \) has the form (4.6). Thus, we have proved the following theorem.
Theorem 4.1. The symmetric operator

$$\hat{H} = iG \otimes I + I \otimes \hat{E} + iL^*W \otimes A(\delta_-)$$

in $D_{W,L}$, where $G = iH + \frac{1}{4}L^*L$, $H = \frac{1}{4}L^*KL - H_0$, is the generator of the one-parameter unitary group $U_t$.

We must point out that the verification of the symmetry property did not rely on the assumption that $L$, $G$, and $W$ commute and can readily be extended to operators of the form

$$\hat{H} = iG + I \otimes \hat{E} + i \sum_{l,m} L_l^* W_{l,m} \otimes A_m(\delta_-) \quad (4.12)$$

with the boundary condition

$$(\hat{N} + 1)^{-1} \left( I \otimes A_l(\delta_+) - \sum_m W_{l,m} \otimes A_m(\delta_-) \right) \Psi = L_l \otimes I \Psi, \quad (4.13)$$

where $W = \{W_{l,m}\}$ is an $M \times M$-matrix with entries in $B(H)$ such that $W^*W = I$, and $\{A_m(g) : g \in L_2(\mathbb{R}), 1 \leq m \leq M\}$ are the annihilation operators in $\Gamma^S(L_2(\mathbb{R}^M))$, which commute for different $l$.

§5. The Markov evolution equation

In conclusion, we describe a derivation of the Markov evolution equation from the Cauchy problem for the Schrödinger equation

$$\frac{d}{dt} \Psi(t) = \left( -G + iI \otimes \hat{E} - \sum_{l,m} L_l^* W_{l,m} \otimes A_m(\delta_-) \right) \Psi(t)$$

with the boundary conditions (4.13). Let $B \in B(H)$ be a Hermitian operator, and let $h, g \in D$. Consider the equation

$$(G, P_t(B)h)_{\mathcal{H}} = (U_t g \otimes \Psi(0)|B \otimes I|U_t h \otimes \Psi(0))_h$$

for the mean value. From (4.12) we have

$$\frac{d}{dt}(g, P_t(B)h)_{\mathcal{H}} = -\left( (G + \sum_{l,m} L_l^* W_{l,m} A_m(\delta_-)) U_t g \otimes \Psi(0)|B \otimes I|U_t h \otimes \Psi(0) \right)$$

$$- (U_t g \otimes \Psi(0)|B \otimes I| \left( G + \sum_{l,m} L_l^* W_{l,m} A_m(\delta_-) \right) U_t h \otimes \Psi(0))$$

$$+ \left( U_t g \otimes \Psi(0)|B \otimes (\Lambda(\delta_+) - \Lambda(\delta_-))|U_t h \otimes \Psi(0) \right). \quad (5.1)$$

Now we can use identity (4.8), which in this case can be rewritten as follows:

$$\left( \Phi, B \otimes (\Lambda(\delta_+) - \Lambda(\delta_-)) \Psi \right) = \sum_{l,m} \left( \Phi, (W_{l,m}^* BW_{l,m} - B) A_m(\delta_-) \Psi \right)$$

$$+ \sum_l (L_l \Phi, BL_l \Psi) + \sum_{l,m} \left( (W_{l,m} A_m(\delta_-) \Phi, BL_l \Psi) + (L_l \Phi, BW_{l,m} A_m(\delta_-) \Psi) \right). \quad (5.2)$$
Taking into account the fact that \( A(\delta_-)\Psi(0) = 0 \) and \( \Lambda(\delta_-)\Psi(0) = 0 \), from Eqs. (5.1) and (5.2) we obtain
\[
\frac{d}{dt}(g, P_t(B)h)|_{t=0} = (g, \mathcal{L}(B)h) = -(Gg, Bh) - (g, B\mathcal{L}g) + \sum_l (L_lg, B\mathcal{L}l)g.
\]

Thus, we have obtained the generator \( \mathcal{L}(\cdot) \) of the Markov evolution equation in Lindblad form:
\[
\frac{d}{dt}P_t(B) = \mathcal{L}(P_t(B)), \quad \mathcal{L}(B) = -G^*B - BG + \frac{1}{2} \sum_l L_l^*B\mathcal{L}l, \quad G = iH + \frac{1}{2} \sum_l L_l^*L_l.
\]

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