Weak and Repulsive Casimir Force in Piston Geometries

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We study the Casimir force in piston-like geometries semiclassically. The force on the piston is finite and physical, but leading semiclassical approximation depends strongly on the shape of the surrounding cavity. Whereas this force is attractive for pistons in a parallelepiped with flat cylinder head, for which the semiclassical approximation by periodic orbits is exact, this approximation to the force on the piston vanishes for a semi-cylindrical head and becomes repulsive for a cylinder of circular cross section with a hemispherical head. In leading semiclassical approximation the sign of the force is related to the generalized Maslov index of short periodic orbits between the piston and its casing.

I. INTRODUCTION

Quite contrary to intuition derived from the Casimir force between two conducting plates\cite{1}, Boyer\cite{2} found that the zero-point contribution to the surface tension of a perfectly conducting spherical shell is negative. Until recently\cite{3}, there was no qualitative explanation of this result. However, the finite negative surface tension of a metallic spherical shell cannot by itself be measured. Any change in cavity radius necessarily involves material properties in a non-trivial way. The negative Casimir tension of a spherical shell in this sense is a mathematical result without physical consequences. It was later found that Casimir self-energies of many closed cavities are in fact plagued by divergences that cannot be removed without appealing to material properties of the cavity walls\cite{4}.

The Casimir force between distinct bodies on the other hand is in principle observable and must be finite. For some simple geometries this force between uncharged conductors has now been measured quite accurately\cite{5}. Experimentally as well as theoretically the force between conductors is attractive in all cases studied so far. A theorem by Kenneth and Klich\cite{6} and its recent generalization by Bachas\cite{7} states that the electromagnetic interaction between any mirror-pair of distinct (charge-conjugate) bodies is attractive. This theorem in particular implies that contrary to previous suggestions\cite{8,9}, the force between two half-spheres is attractive\cite{6}. The attractive Casimir force between polarizable atoms furthermore suggests that the force might be attractive for any geometry of conductors. Such considerations, as well as failed attempts\cite{8,9,10} to finding geometries with repulsion, could give the impression that repulsive Casimir forces between distinct bodies arise only for suitable (mixed) boundary conditions\cite{11,12,13}.

Although some of these arguments are very suggestive, neither the long list of examples nor the restrictive theorems by Kenneth, Klich and Bachas\cite{6,7} imply that the Casimir force is attractive between any two conductors. Intuition based on the Casimir-Polder\cite{14} force between atoms can be misleading\cite{15}. Polarizable atoms attract just as any distant conducting spheres would and as such do not give the Casimir force for other geometries. [If two-body forces were all one had to consider, a metallic spherical shell should, for instance, have positive surface tension.]

Although no general explanation for the sign of Casimir forces has been given so far, the semiclassical approximation tends to correctly predict the sign of Casimir energies and even provides reasonably accurate estimates of their magnitude whenever the leading contribution from periodic classical rays does not happen to vanish. Semiclassically, the Casimir force is related to optical properties of the cavity. From the semiclassical point of view, the Casimir energy of two parallel plates and of a spherical shell are as different as optical properties of flat and curved mirrors\cite{16}. The positive Casimir energy of the spherical shell in this approximation is due to the presence of caustic surfaces\cite{3}. These lead to a relative phase lag of the contributions from classical periodic rays that ultimately determines the sign of the Casimir energy.

However, the contribution of periodic rays to the Casimir force/energy does not always tell the whole and sometimes does not even tell the main story. The Casimir energy/force may vanish in this approximation. A cylindrical conducting shell in three spatial dimensions perhaps is the best known example\cite{3,17}, but geometries without any periodic orbits, such as the Casimir pendulum\cite{18}, are among these as well. The Casimir force/energy and in these cases depends entirely on contributions to the spectral density of higher semiclassical order. These include lower dimensional\cite{19} and diffractive\cite{13,20,21} contributions to the spectral density that are associated with the presence of a boundary. Much of the elegance and predictive power of the semiclassical approach is lost when periodic rays do not contribute and even the sign of the Casimir force/energy may be difficult to estimate in this case. The semiclassical approach is more predictive when the leading approximation due to classical periodic orbits does not vanish.
In integrable systems the latter provide a description of the spectral density that is dual to that of the cavity modes\cite{10}. The following investigation assumes that corrections of higher semiclassical order will not dominate the leading estimate to the Casimir force due to periodic orbits when the latter is appreciable. The corrections in particular should not change the sign of the leading order estimate. Our semiclassical analysis of three piston geometries singles out systems with interesting (because somewhat counterintuitive) semiclassical properties that in principle could be verified numerically\cite{22} – or perhaps even experimentally.

Of greater interest to our investigation is that the force on the piston may be \textit{exactly} computed semiclassically\cite{19} for any position of the wall and any dimension of the parallelepiped. The force gives the dependence of a suitably subtracted zero-point energy on the height \(d\) of the piston,

\[
F_p(d, H, l_2, l_3) = \frac{\partial}{\partial d} \tilde{E}_p(d, H, l_2, l_3)
\]

\[
\tilde{E}_p(d, H, l_2, l_3) = \left[ E_p(d, l_2, l_3) + E_p\left(H - d, l_2, l_3\right) - 2E_p\left(H/2, l_2, l_3\right) \right].
\]

Here \(E_p(l_1, l_2, l_3)\) is the (formal) zero-point energy of a parallelepiped with dimensions \(l_1 \times l_2 \times l_3\). Note that the (infinite, but \(d\)-independent) vacuum-energy of the Casimir piston at \(d = H/2\) has been subtracted. Svaiter emphasizes that the subtracted zero point energy in Eq.\(2\) is finite and does not depend on the cutoff procedure. These general considerations are quite independent of the nature of the walls and of the exterior to the system and apply equally well to pistons and walls of finite thickness and finite conductivity.

The force on the piston can be computed semiclassically for idealized boundary conditions, essentially because the duality transformation from cavity modes to classical periodic paths (using Poisson’s formula) can be performed \textit{exactly} \cite{19}. Contributions to the spectral density proportional to the total volume of the parallelepiped, its total surface area and (total) edge length as well as contributions that reflect topological features of this geometry, such as the number and type of corners\cite{26}, do not depend on \(d\) and do not contribute to the subtracted Casimir energy in Eq.\(2\). The force on the piston is entirely due to families of periodic classical rays that touch the piston as well as the enclosing parallelepiped. Their contribution to the Casimir energy \textit{exactly} matches the field theoretic result\cite{27, 28} for this system. Some representative periodic rays in the bulk of the cavity are shown in Fig.\(1a\). All rays are of finite length and the semiclassical expression for the (subtracted) Casimir energy is inherently finite. For \(l_2 = L \gg l_3 = 2R\) it is readily seen that the plate is attracted\cite{29} toward the nearer end of the parallelepiped in the electromagnetic case\cite{27, 28}. Boundary corrections\cite{19} due to periodic rays that lie within the surfaces of the parallelepiped cancel for the electromagnetic case\cite{27, 28, 30} and the contribution to the \(d\)-dependent part of the Casimir energy from edges may be ignored\cite{31} for \(l_2 \gg l_3\). Any family of periodic rays in the bulk of the cavity contributes negatively to the Casimir energy because its Maslov index vanishes\cite{30} – there are no caustics or focal points and the number of reflections is even. Because the (negative) contribution to the Casimir energy of longer rays decreases in magnitude, the piston is attracted toward the closer end of the cavity in this case.

This example of a parallelepiped-piston already hints at the possibility that the Casimir force could be repulsive if the Maslov index of dominant (preferably all)

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**II. CASIMIR PISTONS**

**A. Parallelepiped with Flat Head**

The geometry of the Casimir piston shown in Fig.\(1a\) apparently was first used by Power\cite{23} and clearly demonstrates that the Casimir force between two flat and perfectly conducting mirrors is physical and does not depend on calculational details such as cutoffs\cite{24}. We here refer to any cavity with an internal dividing surface that is freely movable by Casimir forces as a "Casimir piston", a terminology coined in\cite{22}. The force \(F_p(d < H, l_2, l_3)\) on the dividing wall at a distance \(d\) from one end of the parallelepiped in general depends non-trivially on the dimensions of the parallelepiped and the position of the piston. For \(d \ll H, l_2, l_3\) the force approaches the one obtained by Casimir for two parallel conducting plates\cite{1},

\[
F_p(d \ll l_2, l_3 \ll H) \sim \frac{\hbar c n^2 l_2 l_3}{240 d^3}.
\]
classes of periodic rays would not vanish. The merit of this conjecture is more apparent when the perfectly conducting piston is replaced by a perfectly permeable one. Huslwater\textsuperscript{11} showed that the Casimir force between a perfectly conducting and a perfectly permeable plate is repulsive and Fuller\textsuperscript{13} recently extended these considerations to the piston geometry. The change in sign of the Casimir force is readily explained semiclassically\textsuperscript{31} by a change in the Maslov index of the relevant periodic orbits. Since the reflection coefficients of a classical ray are of opposite sign on the permeable piston and the conducting parallelepiped, the Maslov index of a class of periodic rays that reflect \( n \) times off the piston is \( 2n \) modulo 4. It corresponds to an overall phase lag of \( n\pi \). Periodic rays that reflect an odd number of times off the piston thus contribute positively to the Casimir energy. Since rays with \( n = 1 \) reflections off the piston are by far the shortest relevant periodic orbits, their contribution dominates the Casimir energy and the force on a perfectly permeable piston surrounded by a perfectly conducting parallelepiped is repulsive\textsuperscript{31}.

A highly permeable plate may be difficult to realize but this simple example links the sign of the semiclassical estimate of the Casimir force to the Maslov index of the dominant (usually the shortest) classical periodic rays\textsuperscript{31}. We now consider Casimir pistons for which the Maslov index of \textit{all} classes of periodic rays differs from that in the parallelepiped geometry. The change in Maslov index in this case is of geometrical (optical) origin rather than due to a change in boundary conditions.

We will only consider Neumann, Dirichlet and perfectly conducting surfaces.

### B. Parallelepiped with Semi-Cylindrical Head

We first replace the flat head of the parallelepiped by a half-cylinder of radius \( R \) as shown in Fig. 1b). The rectangular cross section of this Casimir piston remains \( 2R \times L \). To simplify the analysis, we again study the limit of large \( L \gg R \).

In this limit we have a translational symmetry (chosen along the \( y \)-axis). Volume, surface and topological parts of the spectral density cancel as before and we are again led to only consider contributions from classical periodic rays of finite length that \textit{depend} on the height \( d \geq R \) of the piston above the half-pipe. As for a rectangular cavity, (classes of) periodic orbits that do not reflect off the piston contribute to the spectral density and Casimir energy, but not to the Casimir force on the piston.

The translational symmetry in \( y \)-direction essentially reduces our problem to the 2-dimensional one of Fig. 2a). The spectral density of this 2-dimensional cavity decomposes into that of half a stadium and that of a rectangle. The stadium billiard is a classically chaotic system that has been studied intensely and for general \( d > R \) the spectral density fluctuations appear to be best described by random matrix theory\textsuperscript{32}. At \( d = R \) the stadium is a half-disc and most periodic orbits are degenerate. As shown in Figs. 2b) and c), the degenerate periodic orbits of the half-cylinder (half-disc) are in direct correspondence with those of the cylinder (disc). The latter may be identified by a pair of integers \( (n, m) \), where \( n \) is the number of reflections off the cylinder and \( m \) denotes the number of turns of the periodic orbit around the cylinder axis. Evidently \( 2 \leq 2m \leq n \) and only classes of periodic orbits with \( n = 2m \) pass through the axis of the cylinder. A half-cylinder can be viewed as a cylinder where points reflected about a plane through the cylinder axis have been identified. All periodic orbits of the cylinder thus correspond to periodic orbits of the half-cylinder, but the half-cylinder has some additional ones. Just as for the full cylinder, a class of periodic orbits of the half-cylinder is identified by just two numbers: the integer number \( n \) of reflections off the half-pipe and the number of times \( (m) \) the periodic orbit cycles between the two quadrants of the half-cylinder. This classification fails only for two types of periodic orbits – the up-down orbits on the radian dividing the two quadrants of the half-cylinder and periodic orbits that touch an edge of the half-cylinder. The latter are limiting cases within a family of periodic orbits that generally do not touch either corner – they therefore belong to these classes. However, the up-down periodic orbits of the half-cylinder with an \textit{odd} number of reflections off the half-pipe have no analog among those of the full cylinder. We will associate these, in the half-disc, isolated periodic orbits with a pair \( (n, n/2) \) where \( n \) is an odd integer. This assignment is consistent with the previous one insofar as any \textit{even} number of repetitions of the primitive \((1, 1/2)\) orbit belongs to the degenerate class \((2m, m)\) with integer \( m \), whereas an \textit{odd} number of repetitions of this orbit results in an up-down orbit that has no analog in the full cylinder.

![Fig. 2](image-url)  
\textit{Fig. 2:} a) Two-dimensional cavity with semi-circular head and a piston at \( d > R \). b) Representative primitive \((3, 1)\) and \((2, 1)\) orbits when \( d = R \). The corresponding reflected orbits in the \((3, 1)\) and \((2, 1)\) classes of the full cylinder are shown as dashed extensions. Note that two representatives of a class of periodic orbits of the full disc correspond to the same representative in the half-disc. c) The primitive \((1, 1/2)\) orbit of the half-disc.
thus may be written,

$$
\hat{\rho}_{\text{cyl},/2}(E) = \frac{1}{2} \hat{\rho}_{\text{cyl}}(E) + \sum_{k=1}^{\infty} \hat{\rho}^{\text{halfcyl.}}_{(2k-1,k-\frac{1}{2})}(E).
$$

(3)

The first term in Eq. (3) is just half the corresponding spectral density of a cylinder due to classes of periodic orbits \((n,m)\) to integer \(n\). The correction is from up-down orbits \(n = 2m\) to odd \(n\). The contribution to the zero-point energy of a half-cylinder from periodic orbits may be similarly decomposed,

$$
\mathcal{E}_{\text{cyl},/2} = \frac{1}{2} \int_{0}^{\infty} E \hat{\rho}_{\text{cyl},/2}(E) dE = \frac{1}{2} \mathcal{E}_{\text{cyl}} + \mathcal{E}_{\text{ud-cyl}},
$$

(4)

with

$$
\mathcal{E}_{\text{ud-cyl}} = \frac{1}{2} \sum_{k=1}^{\infty} \int_{0}^{\infty} E \hat{\rho}^{\text{halfcyl.}}_{(2k-1,k-\frac{1}{2})} dE
$$

(5)

The semiclassical contribution from periodic rays to the Casimir energy of a perfectly conducting cylinder vanishes \([33, 34]\). We here show that the semi-classical Casimir energy of a half-cylinder vanishes as well. We use the extension of Gutzwiller’s trace formula \([33]\) developed by Creagh and Littlejohn \([31, 48, 52]\) to include continuous symmetries. The translational symmetry is a particularly simple Abelian symmetry generated by the momentum \(p_y\) along the \(y\)-axis. For a periodic orbit \(\Gamma\) of length \(l_{\Gamma}\) in the symmetry-reduced space, the Jacobian is

$$
J_{\Gamma} = \det \frac{\partial y}{\partial p_y} = \frac{l_{\Gamma}}{\pi} = \frac{l_{\Gamma} c}{E}
$$

(6)

The volume of the translation group is just the total length \(V_{\Gamma} = L \gg R\) of the half-cylinder. The primitive period of an up-down ray is \(T_{\Gamma} = 2R/c\) and its classical action is simply \(S_{\Gamma} = \int p \cdot dx = El_{\Gamma}/c\). The semiclassical expression for the spectral density derived in \([34]\) (with \(f = 1\) constants of motion) for our special case is,

$$
\hat{\rho}_{\Gamma}(E) = \frac{T_{\Gamma} V_{\Gamma}}{\pi \hbar} \cos \left( \frac{S_{\Gamma}}{\hbar} - \sigma_{\Gamma} \frac{\pi}{2} - \frac{\pi}{4} \right) \cos \left( \frac{El_{\Gamma}}{\hbar c} - \sigma_{\Gamma} \frac{\pi}{2} - \frac{\pi}{4} \right),
$$

(7)

where we have used that the reduced stability matrix \(M_{\Gamma}\) of an up-down ray with an odd number of reflections off the half-pipe has two eigenvalues equal to \(-1\) (the orbit is inverse parabolic) and thus \(\sqrt{\det(M_{\Gamma} - 1)} = 2\). [The stability matrix of an up-down ray with an even number of reflections off the half-pipe has eigenvalues \(\lambda = 1\) and, as we have seen, represents a class of degenerate orbits of the cylinder.] The dependence of the spectral density of up-down orbits on \(E\) is explicit in Eq. (7) and one can perform the integration in Eq. (5) to obtain,

$$
\mathcal{E}_{\text{ud-cyl}} = \frac{3LRhc}{8\pi\sqrt{2}} \sum_{\Gamma \in \{2k-1,k-1/2\}} -\cos \left( \sigma_{\Gamma} \frac{\pi}{2} \right)
$$

(8)

This is a particular example of the sign of the contribution of an isolated classical period orbit is determined by its generalized Maslov index \([31]\). The Maslov index of an isolated orbit is the sum of two integers \([32]\), \(\sigma_{\Gamma} = \mu_{\Gamma} + \nu_{\Gamma}\). \(\nu_{\Gamma}\) gives the number of independent transverse directions in which a movement of the starting/endpoint reduces the classical action of the closed orbit and is the number of negative eigenvalues of Gutzwiller’s stability matrix \([19, 33]\). The following mechanical analogy often makes \(\nu_{\Gamma}\) “obvious” for billiard systems without much calculation. Since the classical action of any billiard is proportional to the length of the periodic orbit, one simply thinks of the trajectory as an elastic band. The index \(\nu_{\Gamma}\) at any point on the orbit then is the number of independent transverse directions in which a (slight) pull would cause the orbit to slip and not return to its original position upon release. From this mechanical analogy it is quite clear that \((2k-1,k-1/2)\)-rays have a stability index \(\nu_{(2k-1,k-1/2)} = 1\) that does not depend on the choice of starting point on the orbit. The integer \(\mu_{\Gamma}\) is determined from the number and order of conjugate points and the boundary conditions. The number of reflections on an up-down orbit is always even. For Dirichlet or Neumann conditions on both, the piston and the half-pipe, reflections thus contribute an irrelevant multiple of \(2\pi\) to the overall phase. Only the number of conjugate points on the orbit matters. \(\sigma_{\Gamma} = \mu_{\Gamma} + \nu_{\Gamma}\) is a topological characteristic of the periodic orbit \([36]\) that does not depend on the starting point and we have seen that \(\nu = 1\) for up-down trajectories also does not depend on this choice. One therefore can obtain the number of conjugate points using any starting point on the orbit. Since approximately paraxial rays remain approximately paraxial for many cycles and cross the \(y\)-axis near the focal point at \(z = R/2\) (they cross in the interval \([3-\sqrt{5}]R/2, R/2]\), there are \((2k-2)\) conjugate points on an \((2k-1,k-1/2)\)-ray. All conjugate points are of first order and the generalized Maslov index of up-down rays (with Dirichlet or Neumann boundary conditions) therefore is odd,

$$
\sigma_{(2k-1,k-1/2)} = \mu_{(2k-1,k-1/2)} + \nu_{(2k-1,k-1/2)} = 2k - 1 \mod 4.
$$

(9)

\(\mathcal{E}_{\text{ud}}\) defined in Eq. (8) thus vanishes for Dirichlet (D) as well as Neumann (N) boundary conditions. The results
for a cylinder then give,
\[
\mathcal{E}_{cyl./2}^{(N)} = \frac{1}{2} \mathcal{E}_{cyl.}^{\text{EM}}
\]
\[
= \left( \pm \right) \frac{Lhc}{32\pi R^2} \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} (-1)^k 2(k+1)^4 \sin^2 \left( \frac{m\pi}{2R} \right)
\]
\[
= \left( \mp \right) 0.0001209 \ldots \frac{Lhc}{R^2}
\]
where the upper sign corresponds to Dirichlet (D) and the lower to Neumann (N) boundary conditions. The electromagnetic Casimir energy of a cylinder (or any cavity with a one-dimensional translational symmetry) is just the sum of the contributions from two decoupled scalar fields satisfying Dirichlet and Neumann boundary conditions. From Eq.\((10)\) it is apparent that the contribution due to periodic rays to the electromagnetic Casimir (self-)energies of a cylinder and a half-cylinder both vanish,
\[
\mathcal{E}_{cyl./2}^{\text{EM}} = \frac{1}{2} \mathcal{E}_{cyl.}^{\text{EM}} = 0.
\]
By contrast, the field theoretic result for the Casimir energy of an infinitesimally thin but perfectly metallic half-cylinder diverges due to arbitrary short closed paths near the sharp edges. The field theoretic electromagnetic Casimir self-energy of a cylinder with metallic boundaries also does not vanish and has the finite negative value \(\mathcal{E}_{cyl.}^{\text{EM (fieldtheory)}} = -0.1356 \ldots Lhc/R^2\). Before dismissing the semi-classical result of Eq.\((11)\), note that the field theoretic calculation of the Casimir self-energy of a cylinder and half-cylinder includes contributions that are irrelevant to the force on the piston of Fig. 2b. Among these are contributions from exterior modes as well as ultraviolet divergent contributions to the self-energy of a half-cylinder due to its sharp edges. The latter do not change when the piston is moved and do not contribute to the force on it. The semiclassical contribution to the Casimir energy due to classical periodic orbits that we have calculated on the other hand certainly depends on the position of the piston. As for the rectangular piston of Fig. 1a, no net force results from periodic orbits that do not touch the piston and the surrounding cavity. The contribution to the Casimir energy from the corresponding periodic orbits of the \(2R \times H \times L\) dimensional parallelepiped on the other side of the piston is finite and negative, but readily seen to decrease with increasing height \(H\). For dimensional reasons this contribution to the vacuum energy of the cylinder vanishes as \(hLcR/H^3\). For \(d \gg 2R\), periodic orbits that touch the piston and the half-pipe at least have length \(2d\) and their contribution to the Casimir energy is proportional to \(hLcR/d^3\) on dimensional grounds. For \(H \sim \infty\) we conclude that the electromagnetic Casimir energy of the cavity in Fig. 1b) due to periodic orbits vanishes at two positions of the piston,
\[
\mathcal{E}_{1b}^{\text{EM}} (d = R) = \mathcal{E}_{1b}^{\text{EM}} (d = H/2 \sim \infty) = 0.
\]
Note that this equality holds only for the (semiclassical) contribution from periodic orbits. If it were the only \(d\)-dependent contribution, Eq.\((12)\) would imply that the Casimir force on a rectangular piston within a parallelepiped of dimensions \(2R \ll L, H\) capped by a half-cylindrical head either vanishes for any \(d > R\), or is repulsive for some \(d > R\). Diffractive corrections we have not computed may alter this conclusion, but it nevertheless is quite striking that the semiclassical estimate of the Casimir energy due to periodic orbits differs drastically when the half-cylinder of Fig. 1b) replaces the flat piston head of Fig. 1a). The situation is slightly more dramatic for a massless scalar field satisfying Neumann boundary conditions on all surfaces. The previous considerations for \(L, H \gg 2R\) together with Eq.\((10)\) in this case lead to the conclusion that,
\[
\mathcal{E}_{1b}^{N} (d = R) \sim 0.0001209 \frac{Lhc}{R^2} > \mathcal{E}_{1b}^{N} (d = H/2 \sim \infty),
\]
and imply that the leading semiclassical contribution to the Casimir force on the piston is repulsive for some \(d > R\). Diffractive corrections in this case would have to overwhelm the (numerically small) leading semiclassical contribution from periodic orbits for the Casimir force on the piston to be attractive at all \(d > R\).

### C. Cylinder with Hemispherical Head

Massless scalars satisfying Neumann boundary conditions are not readily available and our intuition is mainly based on the attractive nature of the Casimir force between polarizable atoms. The previous example of a semiclassical Casimir force that is repulsive for geometrical reasons thus is only of academic interest. However, it illustrates that the leading semiclassical contribution to the Casimir force may change sign for cavities with the appropriate optical properties. The metallic cylindrical cavity with a piston and a hemispherical head of Fig. 1c) still may be difficult to realize, but has the distinct advantage that in leading semiclassical approximation the force on the piston is repulsive also in the electromagnetic case.

The argument closely follows that for the half-cylindrical head, with just two (crucial) modifications. We again consider only the contribution of periodic orbits that reflect off the piston and the enclosing cylinder. For \(d = R\) their contribution to the semiclassical Casimir energy of the cavity is that for a half-sphere. The remaining contribution to the Casimir force due to periodic orbits in the upper cylinder becomes vanishingly small with increasing length \(\infty \sim H \gg 2R\) of the cylinder. As before, the force on the piston does not arise due to periodic orbits in planes perpendicular to the cylinder axis. When the piston is repositioned, the phase space of transverse orbits lost on one side is regained on the other side of the piston. Although these orbits give the dominant contribution to the oscillatory spectral density
of a long cylinder, they do not contribute to the net force on the piston.] For \( d = R \) we again may decompose the Casimir energy of the half-sphere into half that of the sphere and that due to (in this case) isolated up-down \( (2k - 1, k - 1/2), k = 1, 2, \ldots \)-orbits along the z-axis with an odd number of reflections off the hemisphere,

\[
E_{\text{sp./2}} = \frac{1}{2} E_{\text{sp.}} + E_{\text{ud-sp.}}, \tag{14}
\]

with

\[
E_{\text{ud-sp.}} = \frac{1}{2} \sum_{k=1}^{\infty} \int E_{\text{sp./2}} \left( \frac{\pi}{2} \right) dE. \tag{15}
\]

Contrary to the cylindrical case, periodic classical orbits contribute positively and give about 99% of the field theoretic Casimir self-energy of an infinitesimally thin but perfectly conducting spherical shell \[2, 3\]. The rather small difference to the field theoretic self-energy perhaps is due to the fact that the Casimir energy of the boundary of a 3-dimensional ball, that of a two-sphere, vanishes.

The up-down orbits in this case are isolated and the corresponding semiclassical contribution to the spectral density of a half-sphere is given by Gutzwiller’s trace formula \[32\],

\[
\tilde{\rho}^r = \frac{1}{\hbar \pi} \frac{T_r \cos \left( S_r / \hbar - \sigma_r \pi \frac{1}{2} \right)}{|\det(M_r - 1)|^{1/2}}. \tag{16}
\]

As before, the primitive period of any up-down ray is \( T_{\text{ud}} = 2R/c \), but the stability matrix \( M_r \) here is 4-dimensional. For \((2k - 1, k - 1/2)\)-rays all four eigenvalues equal \(-1\) and \(|\det(M_{(2k - 1, k - 1/2) - 1})|^{1/2} = 4\). The integral over the energy \( E \) of Eq.\((15)\) can again be performed with the result,

\[
E_{\text{ud-sp.}}^{(\nu)} = -\frac{\hbar c R}{4\pi} \sum_{G \in \{2k - 1, k - 1/2\}} \frac{\cos(\sigma_G \pi \frac{1}{2})}{l_G^2}, \tag{17}
\]

irrespective of whether Neumann or Dirichlet boundary conditions hold. The other difference to the cylindrical case is in the generalized Maslov index \( \sigma_r \) of the up-down orbits. It is computed as before, but there now are two independent unstable directions and \( \nu = 2 \) for all up-down orbits. In addition, the conjugate points of an \((2k - 1, k - 1/2)\)-orbit are of second order for the hemisphere and each of the \( 2k - 2 \) conjugate points thus increase \( \mu \) by 2. The overall generalized Maslov index of an \((2k - 1, k - 1/2)\)-orbit thus is,

\[
\sigma_{(2k - 1, k - 1/2)} = \mu(2k - 1, k - 1/2) + \nu(2k - 1, k - 1/2) = 2 \mod 4. \tag{18}
\]

Up-down orbits therefore contribute positively to the Casimir energy of a half-sphere,

\[
E_{\text{ud-sp.}}^{(\nu)} = \frac{\hbar c}{16\pi R} \sum_{k=1}^{\infty} (2k - 1)^{-2} \approx \frac{\hbar c}{128 R} \sim 0.02454 \ldots. \tag{19}
\]

Since the semiclassical Casimir energy due to periodic rays of a spherical cavity is \[3\],

\[
E_{\text{sp.}}^{(\nu)} = \frac{\hbar c}{32\pi R} \left[ \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=2}^{\infty} \frac{15\sqrt{2}}{16k^2} \sum_{m=1}^{k-1} \frac{\cos\left( \pi \frac{m}{2k} \right)}{\sin^2\left( \pi \frac{m}{2k} \right)} \right], \tag{20}
\]

that of the half-sphere becomes,

\[
E_{\text{sp./2}}^{(\nu)} = \frac{\hbar c\pi}{128 R} \left[ 1 + \frac{\pi^2}{45} + \sum_{k=2}^{\infty} \frac{15\sqrt{2}}{8\pi k^2} \sum_{m=1}^{k-1} \frac{\cos\left( \pi \frac{m}{2k} \right)}{\sin^2\left( \pi \frac{m}{2k} \right)} \right],
\]

\[
\approx \frac{1}{2} E_{\text{EM}}^{\text{sp./2}} \sim 0.03621 \ldots \frac{\hbar c}{R}. \tag{21}
\]

The contribution of periodic orbits to the Casimir energy of a half-sphere does not depend on whether Dirichlet or Neumann conditions are imposed. To leading semiclassical order, the electromagnetic spectral density due to periodic rays for a perfectly conducting cavity again is just that of two massless scalars satisfying Neumann and Dirichlet boundary conditions respectively. The Casimir energy from periodic rays entirely within the circular edge where the piston joins the cavity does not depend on \( d \) and therefore does not contribute to the force. Note that up-down orbits give about \( 2/3 \) of the total. This is consistent with the fact that the primitive up-down ray is the shortest and all others have at least twice its length. Assuming that the energy scales inversely with the length of the shortest primitive orbit when the piston is (slightly) moved, gives an order-of-magnitude estimate of the repulsive force on the piston,

\[
F_{1c}^{\text{EM}}(d = R) \sim 0.07 \frac{\hbar c}{R^2} \sim 2 \times 10^3 \frac{pN}{(R \text{ in nm})^2}. \tag{22}
\]

The force is extremely weak and can barely support the weight(!) of a 75nm diameter and a few nanometer thick graphite "piston".

### III. CONCLUSIONS

The Casimir force on the pistons of the three geometries shown in Fig. 1 is finite for any (non-singular) boundary condition. In particular does not diverge for material surfaces of finite thickness and arbitrary reflection coefficients. Contrary to the Casimir self-energy of the cavity itself, the force on the piston in particular is finite for idealized Dirichlet, Neumann and perfectly conducting boundary conditions in all cases. Movement of the piston does not change the ultra-violet divergence of the zero-point energy of the cavity and such contributions to the self-energy are subtracted by referring to a standard configuration of the Casimir piston. One thus can study physical consequences of adiabatic changes in geometry in these simple geometries and avoid conclusions that depend on mathematical properties of idealized boundaries and/or regularization and subtraction schemes.
The rectangular piston of Fig. 1a) was discussed by Powel[23] in connection with the original Casimir force between conducting plates. The zero-point energy due to interior modes of a parallelepiped can be computed exactly[27, 28] and is reproduced by the semiclassical contribution due to periodic rays. Using Poisson’s formula, the duality transformation can be explicitly performed in this case[19]. However, only classical periodic trajectories in either parallelepiped that depend on the position of the piston contribute to the force on it. These periodic rays have finite length and give a finite contribution to the spectral density. For Neumann, Dirichlet and metallic boundary conditions, the internal piston is attracted[29] to the nearer end of the cavity when $H \gg L, 2R$. The force in the parallelepiped-system is entirely given semiclassically by classes of periodic rays that reflect off the piston.

For the Casimir pistons shown in Figs. 1b) and 1c) the duality transformation cannot be performed in closed form and the semiclassical approximation no longer is exact. However, this approximation presumably does not suddenly fail qualitatively when the flat cylinder head of Fig. 1a) is replaced by the half-cylindrical one of Fig. 1b) or the piston geometry is that of Fig. 1c). The length and number of classical periodic rays is similar for all three cases, but the character of the orbits changes. Although the integrable parallelepiped is compared to (for general $d > R$) chaotic systems, semiclassical periodic orbit theory should give reasonable estimates of the spectral density, and in particular the dependence on cavity dimensions of its lowest moments.

Assuming that the semiclassical approximation by periodic orbits remains qualitatively correct, we have seen how the Casimir force on a piston may be diminished and may even change sign for cavities with the appropriate optical properties. For a repulsive Casimir force, the dominant (short and relevant) periodic orbits must have a Maslov index that is an odd multiple of two. This can be achieved by either changing reflection properties of the boundary or by changes in geometry that introduce focal points or caustics. Since highly permeable surfaces are difficult to diffuse, the latter option could be more practical. However, the force on nano-scale Casimir pistons is extremely small, and it could prove difficult to measure it experimentally. Recent advances in the world-line approach[22] to Casimir effects potentially allow numerical studies of the Casimir pistons we have here investigated semiclassically. The piston geometry of Fig. 1c) is repulsive even for Dirichlet boundary conditions. The main numerical challenge appears to be the (accurate) subtraction of divergent contribution to the Casimir energy that arise from arbitrarily short closed world lines that span the edge between the piston and the cylinder, but it may be possible to directly compute the finite force. It may also be possible to study Casimir pistons by field theoretic methods, perhaps by perturbing about the degenerate $d = R$-limit in which these cavities reduce to composite systems with a high degree of symmetry. The field theoretic Casimir self-energy of a half-cylinder diverges[37], but this divergence evidently is due to the sharp edges and should not carry over to the force on the piston. Generalized Casimir pistons can help determine parts of Casimir self-energies that are relevant for adiabatic changes of the geometry of a cavity. Such physical Casimir energies do not depend on the contribution to the zero-point energy of exterior modes and are free of ultraviolet divergences. They furthermore ought not to drastically depend on the particular type of idealized boundary one might consider. Since the semiclassical contribution from periodic orbits incorporates all these characteristics, this approximation might provide a useful guide and perhaps even a reasonable estimate to physical Casimir forces.

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