Role of particle-number statistics in interference of independent Bose fields

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We elucidate generally the interference of independent Bose fields in view of the conditional probability for the particle number measurements, and clarify its relation to the source number statistics. Despite lack of intrinsic phases, the interference phase can be inferred from the particle number registered at one detector by using the classical mean fields. If the conditional number distributions for the other detectors, given the outcome of the first detector, exhibit sufficiently narrow peaks around the values specified by the estimated phases, the mean field description is valid in a single run of interference. The widths in the conditional distribution are determined by the number statistics of the sources, among which notable scaling behavior is found depending on the detector configurations with the boundary at the Poissonian. The mean field description is found to be applicable to Poissonian and sub-Poissonian sources, whereas for super-Poissonian sources it is likely invalidated with the rather broad conditional distribution.

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Interference is often considered as a signature of superposition in quantum systems. In particular, interference in many-body systems as a macroscopic quantum effect has been attracting many interests. In usual experiments, two states originating from a common source are subject to interfere, namely, each particle interferes with itself. However, in many-boson systems including lasers [1, 2] and atomic Bose-Einstein condensates (BECs) [3], interference between independently prepared particles has also been observed. Such interference is often explained with the spontaneous symmetry breaking for the relative phase, which gives nonvanishing expectation values of the field operators or mean fields. In BECs, a U(1) symmetry is relevant for the global phase rotation of atomic wavefunctions, the breakdown of which relies on a nonphysical interaction [4, 5]. In optical systems, a U(1) symmetry also arises from lack of an absolute phase reference, which is ensured by the effective photon-number conservation in optical processes [6–8]. The U(1) symmetry breaking hence seems problematic in the absence of real mechanism.

The interference observed for independent sources under the U(1) symmetry has been attributed to the back-action of particle detection on the systems, which causes localization of the relative phase in a single run [6, 7, 9]. Another approach to the interference is to calculate the correlation functions of the particle numbers measured by the different detectors, which show the spatial modulation. By evaluating the statistical moments of the Fourier components of the spatial modulation up to the fourth order, the plane-wave interference of atomic BECs is predicted in a single run with a random phase [4, 10]. This analysis exploits the nature of the plane-wave mode functions. Generally, some common understanding will be presented for the interference appearing under various configurations, which is based on the probability theory on quantum measurement. Moreover, there will be some intimate relationship between the interference and the particle-number statistics of sources, by considering the fact that the interference is observed so far for lasers (Poissonian states) and BECs (sub-Poissonian states).

In this paper, we investigate the interference of independent Bose fields under general configurations for sources and detectors, and clarify its relation to the source statistics. We examine the joint probabilities of the particle numbers registered by the detectors directly, rather than the correlations, to see the interference in a single run. The outcome at one detector provides information about the relative phase, despite lack of intrinsic phases due to the U(1) symmetry. This information appears in the conditional distributions for the particle numbers at the other detectors, which are derived from the joint probabilities with the given outcome of the first detector. The relative phase is estimated by applying the mean field description to the measurement outcome. If the conditional distribution has sufficiently narrow peaks around the values predicted by the estimated phases, it is almost certain that the outcome at the second detector takes a value close to one of the mean field predictions. Hence, the conditional distribution provides a quantitative criterion for the validity of the mean field description. The mean field description is found to be applicable to Poissonian and sub-Poissonian sources, whereas for super-Poissonian sources it is likely invalidated with the rather broad conditional distribution.

We consider a system of noninteracting Bose particles, photons or cold atoms, where two independent sources are contained. The positive-frequency field operator \( \hat{\psi}(\mathbf{x}, t) \) is given generally in terms of the annihilation operators \( \hat{a}_l \) for a complete set of mode functions \( \{ \phi_l \} \):

\[
\hat{\psi}(\mathbf{x}, t) = \sum_l \hat{a}_l \phi_l(\mathbf{x}, t),
\]

where the time evolution of the free field is represented in the mode functions \( \phi_l(\mathbf{x}, t) \), which is determined in practice by expanding \( \hat{\psi} \) alternatively in terms of the plane-wave modes. In order to describe an interference experiment, the mode functions are
chosen suitably to provide the two independent sources as $\hat{a}_1 \equiv \hat{a}$ and $\hat{a}_2 \equiv \hat{b}$. For example, in interference between two wavepackets of light the wavevector distributions are localized around the central wavevector of the respective sources. In the case of two atomic BECs [3], the initial mode functions $\phi_i^0(x,0)$ are divided into two groups consisting of the eigenstates of the respective one-particle Hamiltonians with harmonic traps. In the following we assume for simplicity that all the particles are populated in the two source modes ($l = 1, 2$), while the other modes ($l \geq 3$) are in the vacuum states. (This will be almost valid in typical interference experiments.) Then, the density matrix for the sources is given by $\hat{\rho} = \hat{\rho}_a \otimes \hat{\rho}_b$, where each source state, respecting the U(1) symmetry, is given with the particle-number statistics $p_s(N)$ [7] as

$$\hat{\rho}_s = \sum_{N=0}^{\infty} p_s(N)|N\rangle \langle N| \quad (s = a, b). \quad (1)$$

In the photon measurement for optical interference experiments, a commonly used photodetector records the number of photoelectrons emitted from the detector surface during a time interval $T$. The time and surface integrated photon-flux operator for the photoelectron emission at the detector $m$ is given [11] by

$$\hat{I}_m = n_m \int_0^T dt \int_{S_m} dxdy \hat{\psi}^\dagger(x,t) \hat{\psi}(x,t), \quad (2)$$

where $n_m$ is the quantum efficiency, and the $z$ axis is taken normal to the detector surface $S_m$. The bandwidth $\Delta \omega$ of the incident radiation is assumed to be small enough compared with the central frequency $\omega_0$. The photon-flux operators in Eq. (2) are specifically expressed as bilinear forms of the mode operators, $\hat{I}_m = \sum_{\nu \mu} R_{\nu \mu}^{(m)} \hat{a}^\dagger \hat{a}_\nu \hat{b}^{\mu} \hat{b}$, with the Hermitian matrices $R_{\nu \mu}^{(m)}$ obtained from Eq. (2) by substitution $\hat{\psi}^\dagger \hat{\psi} \rightarrow \phi_i^\dagger \phi_i$. For the detection of cold atoms, we may take a resonant interaction between the atomic internal levels and the probe light, which transfers the information of the atomic density to $\hat{\psi}^\dagger \hat{\psi}$ of the probe light [12]. Hence, the detection of cold atoms is treated in the same way as the photon number detection.

The joint probabilities of the photon counts $n_1, \ldots, n_M$ by the $M$ detectors ($1 \leq M \leq M_{\text{end}}$), which characterize the full statistics of interference, are given by

$$P(n_1, \ldots, n_M) = \left\langle \prod_{m=1}^{M} \frac{1}{n_m!} (\hat{I}_m)^{n_m} e^{-\hat{I}_m} \right\rangle, \quad (3)$$

where $::$ stands for normal ordering [11]. The flux operators are presented explicitly as

$$\hat{I}_m = R_{aa}^{(m)} \hat{a}^\dagger \hat{a} + R_{bb}^{(m)} \hat{b}^\dagger \hat{b} + R_{ab}^{(m)} \hat{a}^\dagger \hat{b} + R_{ba}^{(m)} \hat{b}^\dagger \hat{a}. \quad (4)$$

Here, it should be noted that the terms involving the vacuum modes ($l \geq 3$) are dropped in $\hat{I}_m$ since they provide null contributions to Eq. (3) as the normal-ordered expectation values. The mean particle number measured at each detector is given by

$$\langle n_m \rangle = \langle \hat{I}_m \rangle = R_{aa}^{(m)} \hat{N}_a + R_{bb}^{(m)} \hat{N}_b. \quad (5)$$

Here, $\hat{N}_s = \text{Tr}[\hat{\rho}_s \hat{a}^\dagger \hat{a}]$ are the mean particle numbers initially contained in the sources, which are assumed to be large enough to produce $\langle n_m \rangle \gg 1$ for high accuracy statistics. The coefficients $R_{aa}^{(m)}$ and $R_{bb}^{(m)}$ indicate the probabilities for each particle from the respective sources to fall into the detector $m$. They may represent the resolution of interference. Specifically, $R_{\nu \mu}^{(m)} \propto 1/M_{\text{end}} \rightarrow 0$, but keeping $R_{\nu \mu}^{(m)} \hat{N}_s \gg 1$ for $\langle n_m \rangle \gg 1$, when the particles are measured by almost continuously distributed many detectors, resulting in a fine interference pattern, e.g., spatial interference fringes [1, 3].

In the above sense, as seen in Eq. (5), a change of $R_{\nu \mu}^{(m)}$ (or resolution) for the detectors may be viewed alternatively as an modification of the source statistics. Here, consider scaling of the detector matrices (by removing several detectors and changing the quantum efficiencies),

$$\bar{R}_{\nu \mu}^{(m)}(q; M) = R_{\nu \mu}^{(m)} / q \quad (m = 1, \ldots, M) \quad (6)$$

with $\bar{R}_{\nu \mu}^{(m)}(q; M) = 0$ ($M > M_{\text{end}}$), and define the binomial distribution

$$B_N^N(q) = \binom{N}{N'} q^{N'} (1 - q)^{N - N'} \quad (0 \leq N' \leq N). \quad (7)$$

In evaluating the joint probabilities, $\langle \langle \hat{I}_m^{k_1} \cdots \hat{I}_M^{k_M} \rangle \rangle$ contained in Eq. (3) are calculated for a Fock state $|N_a, N_b\rangle$ with the normal-ordered expectation values $\langle \langle \hat{a}^{k_1} \hat{b}^{k_2} \rangle \rangle = |N_a!/(N_a - k_a)!| |N_b!/(N_b - k_b)!| (k_a + k_b = k_1 + \cdots + k_M)$, which are multiplied by $q^{k_a} q^{k_b}$ under the scaling. Then, by considering the relation $q^{N!/(N - k)!} = \sum_{N'=0}^{N} B_N^N(q) |N'!/(N' - k)!|$, the effects of this scaling can be renormalized to the source statistics without changing the calculations in Eq. (3) as

$$\tilde{p}_s(N; q) = \sum_{N'=N}^{\infty} p_s(N') B_N^N(q), \quad (8)$$

which is also normalized as the original $p_s(N)$. Hence, the number statistics of the sources may be replaced with the effective ones in Eq. (8) for any scaling of $q$, reproducing the same joint probability for the measurement by the $M$ detectors (namely the $M$-detector model):

$$\{ \tilde{R}_{\nu \mu}^{(m)}(q; M), \tilde{p}_s(N; q) \} \rightarrow P(n_1, \ldots, n_M). \quad (9)$$

This may be viewed as a renormalization transformation among the number statistics. It indicates universal relation for various interference phenomena, ranging from two-mode homodyne detection ($M = 2$) to measurement of spacial fringes ($M = M_{\text{end}} \gg 1$). According to Eq. (8), the mean $\bar{N}_s$ and variance $\bar{V}_s$ for the effective statistics
are given in terms of the original ones as \( \tilde{N}_s = q\tilde{N}_s \) and \( \tilde{V}_s = q^2V_s + (1-q)q\tilde{N}_s \). Then, for a sub-Poissonian distribution \( (\tilde{V}_s < \tilde{N}_s) \), the effective one is still sub-Poissonian \( (\tilde{V}_s < \tilde{N}_s) \) as

\[
\tilde{V}_s/\tilde{N}_s = q(V_s/\tilde{N}_s) + 1 - q. \tag{10}
\]

The Poissonian form is preserved under the renormalization up to the scaling of mean as \( \tilde{N}_s = q \tilde{N}_s \). On the other hand, for a super-Poissonian distribution the effective one is still super-Poissonian.

We now examine the validity of the mean field description for interference phenomena, where the field operators are replaced with c-numbers as \( \hat{a} \to \alpha \) and \( \hat{b} \to \beta \) (expectation values for coherent states \( |\alpha, \beta\rangle \)). Specifically, we have

\[
\tilde{n}_m = \langle \alpha, \beta | \hat{f}_m | \alpha, \beta \rangle = \langle n_m \rangle + 2|R_{ab}^{(m)}|^2 \tilde{N}^{1/2}_a \tilde{N}^{1/2}_b \cos(\delta_{ab} + \theta_m), \tag{11}
\]

where \( \tilde{N}_a = |\alpha|^2 \), \( \tilde{N}_b = |\beta|^2 \), \( \delta_{ab} = \arg \alpha - \arg \beta \), \( \theta_m = \arg R_{ab}^{(m)} \), and \( \langle n_m \rangle \) is the same as Eq. (5) for the U(1)-invariant sources. The set \( \{ \tilde{n}_m \} \) exhibits the interference pattern with the cosine term in Eq. (11), which oscillates with \( \theta_m \) depending on the detector location. The mean field description is, however, not directly applicable to the U(1)-invariant sources in Eq. (1) with \( \langle \hat{a} \hat{b} \rangle = 0 \), eliminating the cosine term in Eq. (11). Nevertheless, by experiments and theoretical calculations the interference fringes are observed in a single run with a random relative phase for Poissonian sources (laser fields [1]) and sub-Poissonian sources (optical number states [6, 7] and BECs [3, 4, 10]).

We hence consider the relationship between the interference phenomena and the source number statistics. Specifically, we examine the validity of the mean field description by inspecting the joint probability \( P(n_1, n_2) \) for any pair of detectors, say 1 and 2, depending on the source statistics. Given the outcome \( n_1 \) at detector 1, the mean field description in Eq. (11) provides an estimate for the relative phase, generally with two possibilities \( \delta_{ab}^\pm \) due to the cosine. Then, the outcome \( n_2 \) at detector 2 is inferred with the estimated phases:

\[
\tilde{n}_1 = n_1 \to \delta_{ab}^+(n_1) \to \tilde{n}_2[\delta_{ab}^+(n_1)]. \tag{12}
\]

If the actual count \( n_2 \) is close to one of \( \tilde{n}_2[\delta_{ab}^+(n_1)] \), fixing the estimation of \( \delta_{ab} \), we find that the interference occurs as described by the mean (classical) fields. This criterion for the interference can be checked readily by calculating the conditional distribution \( P_c(n_2 | n_1) \) from \( P(n_1, n_2) \) with given \( n_1 \). If \( P_c(n_2 | n_1) \) has sufficiently narrow peaks at \( \tilde{n}_2[\delta_{ab}^+(n_1)] \), the second outcome \( n_2 \) should be close to either of the peaks with high probability. Specifically, the width of the peak should be no greater than that of the Poisson distribution \( e^{-\tilde{n}_2} \tilde{n}_2^{n_2} / n_2! \), which is the shot noise level for the coherent states \( |\alpha, \beta\rangle \). Here, we conjecture that sub-Poissonian sources lead to the narrow peaks, showing the interference pattern. It is pointed out [13] that wavepackets emitted from a cavity maintain a pronounced relative phase coherence when the intracavity field has a narrow number distribution. Light beams from such sub-Poissonian cavities will exhibit the interference. This phase coherence of each source is essential to fix the interference phase in the number measurements.

Consider first the case of fine detector resolution with \( |R^{(m)}| \ll 1 \) in the usual measurement of spatial interference fringes. This case can be treated by scaling as the two-detector model with \( \bar{R}^{(1,2)} = R^{(1,2)} / q \sim 1 \) and \( q \to 0 \), which provides the same \( P(n_1, n_2) \) with the effective statistics in Eq. (9). Then, as seen in Eq. (10), the effective statistics of sub-Poissonian sources approach the Poissonian for \( q \to 0 \). Hence, by using any sub-Poissonian sources, essentially the same result is obtained for the interference fringes as the Poissonian case, where the mean field description is valid as numerically confirmed in the following. This is not the case for super-Poissonian sources. For \( R^{(m)} \sim qR^{(m)} \to 0 \), with \( \bar{R}^{(m)} \sim 1 \) fixed, the large \( \tilde{N}_s = \tilde{N}_s/q \propto 1 / |R^{(m)}| \), which is required to produce \( \langle n_m \rangle \gg 1 \), may derive even the larger \( V_s \), e.g., \( V_s \propto \tilde{N}_s^2 \), for a super-Poissonian source, giving a nonzero \( q(V_s/\tilde{N}_s) \) for \( q \to 0 \) in Eq. (10).

In order to examine the validity of the mean field description for general \( R^{(m)} \), we have calculated numerically \( P_c(n_2 | n_1) \) by using Eq. (3) for some typical sources. The detector matrices are chosen for instance as \( R_{aa}^{(1)} = R_{bb}^{(2)} = 0.6R, R_{bb}^{(1)} = R_{aa}^{(2)} = 0.4R, |R_{ab}^{(1,2)}|^2 = R_{aa}^{(1,2)} R_{bb}^{(1,2)} \), giving the maximum interference term in Eq. (11), with the relative phase \( \theta_2 - \theta_1 = 0.9\pi \). The first outcome is set as \( n_1 = 118 \), which corresponds to \( \delta_{ab}^+(n_1) + \theta_1 = \mp 1.39 \) and \( \tilde{n}_2[\delta_{ab}^+(n_1)] \approx 53, 113 \). Due to limitation on the numerical calculation, \( R\tilde{N}_a = R\tilde{N}_b = 100 \) are taken, giving \( \langle n_1 \rangle = \langle n_2 \rangle = 100 \) with \( R_{aa}^{(1,2)} = R_{bb}^{(1,2)} = R \), and consistently \( \langle n_1 \rangle + \langle n_2 \rangle = 200 \approx 118 + (53 + 113)/2 \). The scaling for the effective statistics is also used by taking \( R/q = \bar{R} = 0.867 \) to calculate \( P_c(n_2 | n_1) \) for the increasing \( \tilde{N}_{a,b} = 100/R \) with the smaller \( R \), after it is checked numerically for \( q \sim 0.5 \) with \( \tilde{n}_a,b = 25/R \). A bound on \( \theta_2 - \theta_1 \) may appear for the increasing \( R \) from the condition \( \tilde{n}_1 + \tilde{n}_2 \leq \tilde{N}_a + \tilde{N}_b \) (= 200/R) due to the unitarity or the total number conservation, e.g., \( 0.9\pi \leq \theta_2 - \theta_1 \leq \pi \) for \( R = 0.867 \). This is clearly seen in the familiar two-mode homodyne detection, where \( \hat{I}_{i,2} = (\hat{a} \pm \hat{b})(\hat{a} \pm \hat{b})^\dagger /2 \) with \( e^{i(\theta_2 - \theta_1)} = -1 \).

The results for number states \( |N/R, N/R \rangle \) with \( N = 100 \) and some values of \( R \) are shown in Fig. 1. The case of Poissonian source is also plotted for comparison, corresponding to \( R \to 0 \), where the Poisson distribution \( \propto (\tilde{n}_2)^{n_2} / n_2! \) for \( n_2 \) is confirmed around the peaks (though rather broad due to not so large \( \langle n_1, 2 \rangle = 100 \)).
The peaks agree with \( \bar{n}_2[\delta_{ab}^+(n_1)] \approx 53,113 \) (vertical dotted lines), and exhibit the narrower widths than the Poissonian case. Therefore, the mean field description is valid for these sub-Poissonian number states and also their effective statistics, i.e., the binomial distributions in Eq. (7). Here, the limit \( R \to 1 \) becomes unphysical with the dominating \( \bar{n}_2[\delta_{ab}^+(n_1)] \) to give \( n_1 + n_2 \approx 118 + 113 > 200(R = 1) \), violating the unitarity.

We have also considered a super-Poissonian source with a U(1)-invariant \( \mathcal{P} \)-representation as

\[
\mathcal{P}(\alpha) \propto (-|\alpha|^2/Q \bar{N})^{1/Q-1} \exp(-|\alpha|^2/Q \bar{N}), \tag{13}
\]

where \( Q = (V - \bar{N})/\bar{N}^2 \) with \( Q > 0 \). The limit \( Q \to 0 \) corresponds to the Poissonian, whereas \( Q = 1 \) to the thermal state. The conditional distribution is shown in Fig. 2, which does not depend on \( R \) in this case with \( R \bar{N}_{a,b} (=100) \) fixed. The increasing \( Q \) broadens the distribution, eventually washing out the peaks. We have further examined the single-photon-added thermal state [14]. This nonclassical super-Poissonian state has the variance smaller than the thermal case. Despite this fact, for the small \( R \) \( (R \bar{N}_{a,b} \text{ fixed}) \), the conditional distribution becomes flatter than that for the thermal sources. These results indicate that the behavior of interference is rather complicated for super-Poissonian sources, likely invalidating the mean field description.

To conclude, in view of the conditional probability for the number measurements, we have elucidated the common mechanism for the interference of independent Bose fields under various situations, ranging from two-mode homodyne interference to spatial fringes. The interference is determined by the source number statistics, among which the scaling behavior is present depending on the detector characteristics with the boundary at the Poissonian. For sub-Poissonian and Poissonian sources the interference pattern appears in a single run, consistently with the mean field description, whereas this is not the case for super-Poissonian sources. It will be a challenge for future experiments to confirm the role of source statistics with the scaling behavior, by preparing various source states and detector configurations.

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