LIMITS OF TANGENTS OF SURFACES

JOÃO CABRAL AND ORLANDO NETO

ABSTRACT. We compute the limit of tangents of an arbitrary surface. We obtain as a byproduct an embedded version of Jung’s desingularization theorem for surface singularities with finite limits of tangents.

1. Introduction

Let $S$ be a singular surface of the germ of a complex analytical manifold $M$ at a point $o$. Theorem 2.3.7 of [5] states that the limit of tangents $\Sigma_o(S)$ is the union of the dual of the tangent cone $C_o(S)$ of $S$ with a finite set of projective lines of $\mathbb{P}^*M$.

Theorem 1.4.4.1 of [6] gives a criteria to decide if a certain projective line is contained or not in $\Sigma_o(S)$, provided a non degeneracy condition is verified. The proof relies on a commutative diagram

\[
\begin{array}{ccc}
\Gamma & \leftarrow & \tilde{\Gamma} \\
\downarrow & & \downarrow \\
S & \leftarrow & \tilde{S}
\end{array}
\]

where $\tilde{S}$ is the strict transform of $S$ by the blow up $\pi : \tilde{M} \to M$ of $M$ at $o$, and $\Gamma$ is the conormal of $S$. Hence $S \subset M$, $\tilde{S} \subset \tilde{M}$ and $\Gamma \subset \mathbb{P}^*M$; the surface $\tilde{\Gamma}$ has a more enigmatic status. If we had an immersion of $\tilde{\Gamma}$ into a manifold $\mathcal{X}$ endowed with a symplectic structure we could iterate the process, blowing up $\tilde{M}$ and eventually lifting the need for the non degeneracy condition. Unfortunately there can be no such symplectic structure on $\mathcal{X}$.

Theorem 8 presents a new proof of Theorem 2.3.7 of [5]. It sheds a new light on the problem. Set $E = \pi^{-1}(o)$. There is a vector bundle $T^*\langle \tilde{M}/E \rangle$ on $\tilde{M}$ with sheaf of sections the locally free $O_{\tilde{M}}$-module of logarithmic differential forms on $\tilde{M}$ with poles along $E$. Let $\pi_{\tilde{M}} : \mathbb{P}^*\langle \tilde{M}/E \rangle \to \tilde{M}$ be the associated projective bundle. There is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^*M & \leftarrow & \mathbb{P}^*\langle \tilde{M}/E \rangle \\
\pi_M \downarrow & & \downarrow \pi_{\tilde{M}} \\
M & \leftarrow & \tilde{M}
\end{array}
\]

where $\tilde{\pi}$ is blow up of $\mathbb{P}^*M$ along $\pi^{-1}_M(o)$. The projectivization $\mathbb{P}^*\langle \tilde{M}/E \rangle$ of the vector bundle $T^*\langle \tilde{M}/E \rangle$ provide an ambient space for $\tilde{\Gamma}$, which is

Date: November 26, 2015.
a Legendrian variety of $\mathbb{P}^*(\tilde{M}/E)$, the strict transform of $\Gamma$ by $\tilde{\pi}$ and the conormal of $\tilde{S}$.

There is a canonical embedding of the projective cotangent bundle $\mathbb{P}^*E$ into $\mathbb{P}^*(\tilde{M}/E)$. Moreover, $\tilde{\Gamma}' = \tilde{\Gamma} \cap \pi_{\tilde{M}}^{-1}(E)$ is contained in $\mathbb{P}^*E$ and $\tilde{\Gamma}'$ is a Legendrian variety of $\mathbb{P}^*E$. Since $\pi_{\tilde{M}}(\Gamma') = \tilde{S}\cap E$, there are $\sigma_1, ..., \sigma_n \in \tilde{S}\cap E$ such that

$$\tilde{\Gamma}' = \mathbb{P}^*_E(\tilde{S}\cap E) \cup \cup_i \mathbb{P}^*_E(\sigma_i).$$

Here $\mathbb{P}^*_E(\tilde{S}\cap E)$ denotes the conormal of the curve $\tilde{S}\cap E$, the smallest Legendrian curve of $\mathbb{P}^*E$ that projects on $\tilde{S}\cap E$, and $\mathbb{P}^*_E(\sigma_i)$ denotes the fiber at $\sigma_i$ of $\mathbb{P}^*E$. The restriction of $\tilde{\pi}$ to $\mathbb{P}^*E$ defines a map

$$\tilde{\pi} : \mathbb{P}^*E \to \mathbb{P}^*\tilde{M}.$$

Moreover,

$$\Sigma_o(S) = \tilde{\pi}(\mathbb{P}^*_E(\tilde{S}\cap E) \cup \cup_i \mathbb{P}^*_E(\sigma_i)).$$

More precisely, the dual curve of $\tilde{S}\cap E$ is the image of the Legendrian curve $\mathbb{P}^*_E(\tilde{S}\cap E)$ and the pencils of planes are the images of the projective lines $\mathbb{P}^*_E(\sigma_i)$.

There is a natural logarithmic generalization of the notion of limit of tangents. The surface $\tilde{\Gamma}$ is the conormal of the surface $\tilde{S}$, in a sense that will be precised in section 2. We call $\Sigma_o^E(\tilde{S}) = \tilde{\Gamma} \cap \pi_{\tilde{M}}^{-1}(\sigma)$ the logarithmic limit of tangents of $\tilde{S}$ at $\sigma$, with poles along $E$.

We reduce in this way the computation of the limit of tangents of a surface to the problem of deciding if, given $\sigma \in \tilde{S}\cap E$, $\Sigma_o^E(\tilde{S}) = \mathbb{P}^*_E(\sigma)$ or $\Sigma_o^E(\tilde{S})$ is finite. This problem is solved by Theorem 24.

The introduction of the notion of logarithmic limit of tangents allows us to iterate the construction that gives us a new proof of Theorem 2.3.7 of [5]. In order to compute the limit of tangents we need to introduce a canonical process of reduction of singularities for surfaces. In general this process will terminate before we desingularize the surface $S$, giving us enough information to compute the limit of tangents of $S$. If the limit of tangents of $S$ is finite, the process will terminate when all singular points of some strict transform of $S$ are quasi ordinary. See Theorem 25. We obtain in this way an embedded version of Jung’s desingularization algorithm.

Let $C$ be the singular locus of $S$. Roughly speaking, the algorithm of reduction of singularities proceeds in the following way:

(a) We blow up $M$ at $o$;

(b) Given a regular point $\sigma$ of the inverse image $N$ of $o$ by the sequence of blow ups, we blow up $\sigma$ if $\sigma$ is a singular point of the strict transform $\tilde{C}$ of $C$ or $\tilde{C}$ is not transversal to $N$ at $\sigma$.

(c) If a strict transform $\tilde{S}$ of $S$ by the sequence of blow ups contains a connected component $W$ of the singular locus of $N$, we blow up $W$. 
(d) If $\sigma$ is an isolated point of $\tilde{S} \cap W$, there is a local plane projection $\rho$ such that $\rho^{-1}(\rho(N)) = N$. If the germ at $\rho(\sigma)$ of the discriminant $\Delta_{\rho}S$ is not contained in $\rho(N)$, we blow up $W$.

In cases (c), (d) this algorithm works in a very similar way to Jung’s algorithm. In case (b) the singular locus of $S$ takes the place of the discriminant of $S$.

In section 2 we introduce some basic notions of Logarithmic Contact Geometry. In section 3 we introduce the notion of logarithmic limits of tangents and compute logarithmic limits of tangents of quasi-ordinary surfaces. The fact that a surface $S$ is quasi-ordinary relative to a given projection does not mean that it is quasi ordinary relatively to another projection. The situations changes if the limit of tangents of the surface is finite. This fact is a key argument in the proofs of Theorems 6 and 7. These Theorems relate the logarithmic limit of tangents of a surface with its discriminant, in the spirit of one of the statements of Theorem 1.4.4.1 of [6]. Section 5 studies the logarithmic limits of tangents $\Sigma^N_\sigma(S)$ when $\sigma$ is a singular normal crossings point of $N$. Here the main tool is the first sequence of blow ups. It reduces the computation of $\Sigma^N_\sigma(S)$ when $\sigma$ is a singular point of $N$ to the computation of $\Sigma^N_\sigma(S)$ when $\sigma$ is a regular point of $N$. In section 6 we study the behaviour of $\Sigma^N_\sigma(S)$ by blow up when the non degeneracy condition $C_\sigma(S)$ does not contain $C_\sigma(N)$ is verified.

In section 7 we study the behaviour of $\Sigma^N_\sigma(S)$ by blow up without assuming the non degeneracy condition. This is the longest and the more technical section of the paper. In Differential Geometry it is sometimes unavoidable the use of long computations involving systems of local coordinates. Here the main tool is the second sequence of blow ups, that will be the building block of the reduction of singularities procedure that decides if $\Sigma^N_\sigma(S) = \Sigma^N_\sigma$, when $\sigma$ is a regular point of $N$.

In section 8 we state the main results and use them to compute several limits of tangents.

The second named author would like to thank Bernard Teissier and Le Dung Trang, who showed him the beauty of Singularity Theory.

2. Logarithmic Contact Geometry

All manifolds considered in this paper are complex analytic manifolds. Let $N$ be a normal crossings divisor of a manifold $M$. We will denote by $\Omega^1_M(N)$ the sheaf of logarithmic differential forms on $M$ with poles along $N$. If $N = \emptyset$, $\Omega^1_M(N)$ equals the sheaf $\Omega^1_M$ of differential forms on $M$.

We will denote by $T^*\langle M/N \rangle$ the vector bundle with sheaf of sections $\Omega^1_M(N)$. We call $\pi_M : T^*\langle M/N \rangle \to M$ the logarithmic cotangent bundle of $M$ with poles along $N$. If $N = \emptyset$, $T^*\langle M/N \rangle$ equals the cotangent bundle $T^*M$ of $M$.

There is a canonical logarithmic 1-form $\theta_N$ on $T^*\langle M/N \rangle$ that coincides with the canonical 1-form $\theta$ of $T^*M$ outside of $\pi_M^{-1}(N)$.
We call the projectivization \( \pi_M : \mathbb{P}^*(M/N) \to M \) of \( T^*(M/N) \) the logarithmic projective cotangent bundle of \( M \) with poles along \( N \).

**Example 1.** Assume \((x_1, \ldots, x_n)\) is a system of coordinates on an open set \( U \) of \( M \) such that \( N \cap U = \{x_1 \cdots x_\nu = 0\} \). Given a differential form \( \omega \) on \( U \) there are holomorphic functions \( a_1, \ldots, a_n \in \mathcal{O}_M(U) \) such that
\[
\omega = \sum_{i=1}^\nu a_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^n a_i dx_i.
\]
There are \( \xi_1, \ldots, \xi_n \in \mathcal{O}_{T^*(M/N)}(\pi^{-1}_M(U)) \) such that
\[
\theta_N = \sum_{i=1}^\nu \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^n \xi_i dx_i.
\]

Assume \( \dim M = n \). Let \( \Gamma \) be an analytic subset of dimension \( n \) of \( T^*M \). We say that \( \Gamma \) is **conic** if \( \Gamma \) is invariant by the action of \( \mathbb{C}^* \) on the fibers of \( T^*M \). We say that \( \Gamma \) is a **Lagrangian variety** if the symplectic form \( d\theta \) of \( T^*M \) vanishes on the regular part of \( \Gamma \). Notice that \( \Gamma \) is a conic Lagrangian variety if and only if \( \theta \) vanishes on the regular part of \( \Gamma \).

We identify \( M \) with the graph of the zero section of \( T^*M \). There is a canonical map \( \gamma : T^*M \setminus M \to \mathbb{P}^*M \). We say that an analytic subset \( \Gamma \) of \( \mathbb{P}^*M \) is a **Legendrian variety** of \( \mathbb{P}^*M \) if \( \gamma^{-1}(\Gamma) \) is a (conic) Lagrangian variety of \( T^*M \).

Let \( S \) be a closed irreducible analytic subset of \( M \). We call **conormal** of \( S \) to the smallest Legendrian variety \( \Gamma \) of \( \mathbb{P}^*M \) such that \( \pi_M(\Gamma) = S \). We will denote it by \( \mathbb{P}^*_S M \). If \( S \) has irreducible components \( S_i, i \in I \), we set \( \mathbb{P}^*_S M = \bigcup_i \mathbb{P}^*_{S_i} M \).

Let \( \Gamma \) be a closed analytic subset of \( \mathbb{P}^*(M/N) \). Set \( \Gamma' = \Gamma \cap \mathbb{P}^*(M \setminus N) \). We say that \( \Gamma \) is a **Legendrian variety** of \( \mathbb{P}^*(M/N) \) if \( \Gamma' \) is a Legendrian variety of \( \mathbb{P}^*(M \setminus N) \) and \( \Gamma \) is the closure of \( \Gamma' \).

Let \( S \) be an analytic subset of \( M \) such that \( S \) equals the closure of \( S \setminus N \). We call **conormal** of \( S \) to the closure \( \mathbb{P}^*_S(M/N) \) of \( \mathbb{P}^*_{S\setminus N} M \) of \( \mathbb{P}^*_S(M/N) \).

Given an irreducible Legendrian variety \( \Gamma \) of \( \mathbb{P}^*(M/N) \), \( \Gamma = \mathbb{P}^*_S(M/N) \), where \( S = \pi_M(\Gamma) \). The proof follows the arguments of the equivalent proof in the classical case.

**Lemma 2.** (see [8]) Assume \( N \) smooth. Then there is a canonical immersion of \( \mathbb{P}^*N \) into \( \mathbb{P}^*(M/N) \). If \( \Gamma \) is a Legendrian variety of \( \mathbb{P}^*(M/N) \), then
\[
\Gamma_0 = \Gamma \cap \pi^{-1}_M(N) \subset \mathbb{P}^*N.
\]
Moreover, \( \Gamma_0 \) is a Legendrian variety of \( \mathbb{P}^*N \).

Section [5] studies the intersection of \( \Gamma \) with \( \pi^{-1}_M(N) \) when \( N \) is a singular normal crossings divisor.
3. Logarithmic Limits of Tangents

Let $M$ be a germ of a complex manifold of dimension 3 at a point $o$. Let
$N$ be a normal crossings divisor of $M$. Let $S$ be a surface of $M$. Let
$$
\Sigma^N_o(S) = \mathbb{P}_S^*(M/N) \cap \mathbb{P}_o^*(M/N)
$$
be the logarithmic limit of tangents of $S$ along $N$ at the point $o$.

If $N$ is empty we get the usual definition of limit of tangents of $S$ at $o$. If
$N$ is smooth, set
$$
\Sigma^N_o = \mathbb{P}_o^* N \subset \mathbb{P}_o^*(M/N).
$$

**Lemma 3.** If $N$ is smooth, $\Sigma^N_o(S) \subset \Sigma^N_o$.

**Proof.** It follows from Lemma 2. \hfill $\Box$

Let $\rho$ be a submersion of $M$ into a smooth surface $X$. We say that $\rho$ is
compatible with $N$ if $\rho^{-1}(\rho(N)) = N$.

Let $\Xi_\rho(S)$ be the apparent contour of $S$ relatively to the projection $\rho$. Let
$\Delta_\rho(S) = \rho(\Xi_\rho(S))$ be the discriminant of $S$ relatively to $\rho$.

Assume $\rho$ is compatible with $N$. We call the closure $\Xi^N_\rho(S)$ of $\Xi_\rho(S) \setminus N$
the logarithmic apparent contour along $N$ of $S$ relatively to $\rho$.

We call logarithmic singular set of $S$ with respect to $N$ to the closure
$\text{Sing}^N(S)$ of $\text{Sing}(S) \setminus N$, where $\text{Sing}(S)$ is the singular locus of $S$.

We will fix systems of local coordinates $(x, y, z)$, $[(x, y)]$ on $M$ $[X]$ such
that $o = (0, 0, 0)$ and $\rho(x, y, z) = (x, y)$. We set $\Delta_\rho(S) = \Delta_\rho(S)$.

**Lemma 4.** If $\rho$ is compatible with $N$ and $\Delta_\rho(S) \subset \rho(N)$,
$$
\Sigma^N_o(S) = \{(0 : 0 : 1)\}.
$$

**Proof.** We can assume $N = \{x = 0\}$ or $N = \{xy = 0\}$.

Consider the first case. The surface $S$ admits a parametrization
(2)
$$
x = t^k, \quad z = t^n \varphi(t, y),
$$
where $\varphi \in \mathbb{C}\{t, y\}$ and $\varphi(0, 0) \neq 0$. Replacing (2) in
(3)
$$
\xi \frac{dx}{x} + \eta dy + \zeta dz
$$
we conclude that $\mathbb{P}_S^*(M/N)$ is contained in the image, by $(t, y; \xi : \eta : \zeta) \mapsto
(t^k, y, t^n \varphi(t, y); \xi : \eta : \zeta)$, of the set defined by the equations
$$
k \xi + t^n (n \varphi + t \partial_t \varphi) \zeta = 0, \quad \eta + t^n \partial_y \varphi \zeta = 0.
$$
The proof in the second case is similar. \hfill $\Box$

Given $\sigma \in S$, let $m_\sigma(S)$ denote the multiplicity of $S$ at $\sigma$.

**Lemma 5.** Assume $N = \{x = 0\}$, $S$ admits a fractional power expansion
$$
z = x^\lambda y^\mu \varphi(x^{1/d}, y^{1/d}), \text{ with } \varphi(0, 0) \neq 0, (\lambda, \mu) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2, \mu \neq 0 \text{ and } \text{Sing}^N(S) = \{y = z = 0\}$. The following statements are equivalent:
(a) $\Sigma^N_o(S) = \{(0 : 0 : 1)\}$,
(b) $\Sigma^N_o(S) \neq \Sigma_o^N$, 
Proof. The equivalence between (c) and (d) follows from well known facts on quasi-ordinary surfaces. The prove that (a) follows from (c) is similar to the proof of Lemma 4. Moreover, (a) implies (b).

Assume $\mu < 1$. There are positive integers $m, n, k$ such that $n < k$ and
\begin{equation}
    x = t^k, \quad y = s^k, \quad z = t^m s^n \varphi(t, s)
\end{equation}
is a parametrization of $S$. Replacing (4) in (3) we conclude that
\begin{equation}
    k\eta + t^m s^n (m\varphi + s\partial_s \varphi) = 0.
\end{equation}
Setting $s = C t^{m/(k-n)}$ in (5) and taking limits we conclude that
\begin{equation}
    k\eta + C^{n-k} m\varphi(0,0) = 0.
\end{equation}
Hence the limits of tangents along the curves
\begin{equation}
    x = t^k, \quad y = C t^{mk/(k-n)}, \quad z = t^{mk/(k-n)} \varphi(t, C t^{m/(k-n)})
\end{equation}
of $S$ are dense in $\Sigma^N$. Therefore (b) is false. $\square$

**Theorem 6.** Assume $o \in S \cap N$ is a smooth point of $N$ and the singular locus of $S$ is contained in $N$ in a neighbourhood of $o$. Let $\rho$ be a projection compatible with $N$. Then $\Sigma^N(o) = \Sigma^N$ if and only if $o \in \Xi^N(o)$.

Proof. We can assume $S$ irreducible. There is a system of local coordinates $(x, y, z)$ centered at $o$ such that $N = \{x = 0\}$ and $S = \{F = 0\}$, for some $F \in \mathbb{C}\{x, y, z\}$.

Assume $o \notin \Xi^N(o)$. Then $S$ admits a fractional power series expansion
\begin{equation}
    z = \varphi(x^{1/d}, y), \quad \text{where } \varphi \in \mathbb{C}\{x^{1/d}, y\},
\end{equation}
for some positive integer $d$, and there are $a_\ell \in \mathbb{C}\{y\}$ such that $x = t^k$,
\begin{equation}
    \varphi(t, y) = \sum_{\ell} a_\ell(y) t^\ell \quad \text{and} \quad a_\ell = 0 \text{ or } a_\ell(0) \neq 0,
\end{equation}
for each $\ell$. By Lemma 4, $\Sigma^N(o) = \langle dz \rangle$.

Set $\rho_\lambda(x, y, z) = (x, y + \lambda z)$. In order to prove the other implication it is enough to show that, for each $\lambda \in \mathbb{C}$,
\begin{equation}
    o \notin \Xi^N(o) \text{ if and only if } o \notin \Xi^N_{\rho_\lambda}(o).
\end{equation}
Notice that if $o \in \Xi^N_{\rho_\lambda}(o)$ for each $\lambda \in \mathbb{C}$,
\begin{equation}
    \langle dy - \lambda dz \rangle \in \Sigma_o(S), \quad \text{for each } \lambda \in \mathbb{C}.
\end{equation}
Let us prove (7). Assume $o \notin \Xi^N_{\rho}(o)$. Set $\Phi(x, y, z) = (x, y - \lambda z, z)$, $G = F \circ \Phi$. Set $h(t, y, z) = z - \varphi(t, y + \lambda z)$. There is $\psi \in \mathbb{C}\{t, y\}$ such that
\begin{equation}
    h(t, y + \lambda \psi(t, y), z) = 0.
\end{equation}
Hence
\begin{equation}
    \psi(t, y) = \varphi(t, y + \lambda \psi(t, y))
\end{equation}
and $G$ admits the fractional power series expansion $z = \psi(x^{1/d}, y)$. Moreover, there are $b_k \in \mathbb{C}\{y\}$ such that

$$\psi(t, y) = \sum b_k(y)t^k \quad \text{and} \quad b_k = 0 \text{ or } b_k(0) \neq 0.$$ 

Therefore $\Delta_s G \subset \{x = 0\}$. 

**Theorem 7.** Assume $\text{Sing}^N(S)$ is transversal to $N$ at $o$. Let $\rho$ be a projection compatible with $N$. Then $\Sigma_o^N(S) \neq \Sigma_o^N$ if and only if $o \notin \Xi_{\rho}^N(S) \setminus \text{Sing}^N(S)$ and there is an open neighbourhood $U$ of $o$ such that $m_\sigma(S) = m_o(S)$, for each $\sigma \in \text{Sing}^N(S) \cap U$.

**Proof.** We can assume that $S$ irreducible, $N = \{x = 0\}$ and $\text{Sing}^N(S) = \{y = z = 0\}$. Therefore $S$ admits a fractional power expansion

$$z = x^\mu y^\nu \varphi(x^{1/d}, y^{1/d}),$$

where $\varphi \in \mathbb{C}\{x, y\}$, $\varphi(0, 0) \neq 0$ and $\mu, \nu \in \mathbb{Q}$, for some positive integer $d$. After an eventual a change of coordinates, we can assume $(\mu, \nu) \notin \mathbb{Z}^2$.

Let us show that the condition is sufficient. Since $\text{Sing}^N(S)$ is transversal to $N$ at $o$, $\nu \neq 0$. Since $\sigma \in \text{Sing}^N(S) \cap U$ implies $m_\sigma(S) = m_o(S), \nu \geq 1$. Hence, by Lemma 5, $\Sigma_o^N(S) = (dz)$.

Assume $\Sigma_o^N(S) \neq \Sigma_o^N$. By Lemma 5, $\nu \geq 1$. Hence the condition on $m_\sigma(S)$ is verified. Moreover, there is $\lambda \in \mathbb{C}$ such that

$$o \notin \Xi_{\rho \lambda}^N(S) \setminus \text{Sing}^N(S).$$

Therefore it is enough to show that, for each $\lambda \in \mathbb{C}$,

$$o \notin \Xi_{\rho}^N(S) \setminus \text{Sing}^N(S) \quad \text{if and only if} \quad o \notin \Xi_{\rho \lambda}^N(S) \setminus \text{Sing}^N(S).$$

Set $x_s = x$, $y_s = y - \lambda z$, $z_s = z$, $x = t^d$, $y = s^d$, $y_s = s_s^d$,

$$f(t, s) = t^{\mu d}s^{(\nu - 1)d}\varphi(t, s).$$

Assume that there is $\psi \in \mathbb{C}\{t, s_s\}$ such that

$$s = s_s(1 + t^{\mu d}s_s^{(\nu - 1)d}\varphi(t, s_s)\psi(t, s_s)).$$

There are $a_k \in \mathbb{C}^*$, $b_{\alpha, \beta} \in \mathbb{C}$, with $k \geq 1$, $\alpha, \beta \geq 0$ such that $b_{0, 0} = 0$,

$$(1 - u)^{1/d} = \sum a_k u^k \quad \text{and} \quad f(t, s) = \sum b_{\alpha, \beta} t^\alpha s^\beta.$$ 

Replacing (9) in $y_s = y - \lambda z$, we conclude that

$$(12) \quad s_s = s(1 + \sum a_k \lambda^k f^k(t, s)).$$

Replacing (11) in (12), we show that

$$(13) \quad \psi f + (1 + \psi f)(1 + \sum_{k \geq 1} a_k \lambda^k f^k(t, s_s(1 + \psi f))) = 0.$$
There are $c_{\alpha,\beta} \in \mathbb{C}$, depending only on the $b_{\alpha,\beta}$'s, such that

$$f(t, s_0(1 + \psi f(t, s_0))) = f(t, s_0)[1 + \sum_{\alpha, \beta \geq 0} \sum_{j=1}^{\beta} c_{\alpha,\beta} t^{\alpha} s_0^{\beta} \psi^j f^{j-1}(t, s_0)].$$

Setting

$$\epsilon_\psi(t, s_0) = 1 + \sum_{\alpha, \beta \geq 0} \sum_{j=1}^{\beta} c_{\alpha,\beta} t^{\alpha} s_0^{\beta} \psi^j f^{j-1}(t, s_0).$$

we can rewrite equality (13) as

$$\psi + (1 + \psi f) \sum_{k \geq 1} a_k \lambda k f^{k-1} \epsilon_{\psi} = 0,$$

Hence $\psi$ is well defined. Furthermore, $\psi(0,0) = \lambda/d$. Hence, for each $\lambda \in \mathbb{C}^*$, $S$ admits a fractional power series expansion $z_0 = x_0^{1/d} y_0^{1/d} \phi(x_0, y_0, z_0)$ with $\phi(0,0) \neq 0$. Therefore $o \notin \Xi_N^*(S) \setminus \text{Sing}^N(S)$.

4. A LOGARITHMIC VERSION OF A CLASSICAL RESULT

Let $M$ be a germ of a complex manifold of dimension 3 at a point $o$. Let $\pi : \tilde{M} \to M$ be the blow up of $M$ at $o$. Set $E = \pi^{-1}(o)$. Let $\tilde{\pi} : \mathbb{P}^*(\tilde{M}/E) \to \mathbb{P}^*M$ be the blow up of $\mathbb{P}^*M$ along the Legendrian variety $\pi_\alpha^1(o)$. It follows from Proposition 9.4 of [S] that diagram (11) commutes. We will also denote by $\tilde{\pi}$ the bimeromorphic map from a dense open set of $T^*(\tilde{M}/E)$ into $T^*M$ that induces $\tilde{\pi}$.

Let $\ell$ be a line of $T_oM$ that contains the origin. Let $\Sigma_\ell$ be the set of planes of $T_oM$ that contain $\ell$. Let $\gamma$ be the germ at $o$ of a smooth curve of $M$ with tangent space $\ell$. The point $o_\ell$ where the strict transform of $\gamma$ intersects $E$ does not depend on $\gamma$.

**Theorem 8.** Let $S$ be a surface of $M$. Then $\Sigma_o(S)$ is the union of the dual of the projectivization of $C_o(S)$ and a finite set of projective lines of $\mathbb{P}_o^*M$.

Moreover, $\Sigma_\ell \subset \Sigma_o(S)$ if and only if $\Sigma_{o_\ell}^E(\tilde{S}) = \Sigma_{o_\ell}^E$.

**Proof.** Let $\ell \in \mathbb{P}(T_oM)$. Choose local coordinates $(x, y, z)$ of $M$ such that $\ell = \{y = z = 0\}$. Setting $x_0 = x, y_0 = y/x, z_0 = z/x, (x_0, y_0, z_0)$ is a system of local coordinates on an affine set $U$ of $\tilde{M}$, centered at $o_\ell$ such that $E \cap U = \{x_0 = 0\}$. Let

$$\xi_0 \frac{dx_0}{x_0} + \eta_0 dy_0 + \zeta_0 dz_0$$

be the canonical 1-form of $T^*(\tilde{M}/E)$ in a neighbourhood of $o_\ell$. Since

$$\tilde{\pi}^*(\xi dx + \eta dy + \zeta dz) = (x\xi + y\eta + z\zeta) \frac{dx_0}{x_0} + x\eta dy_0 + x\zeta dz_0,$$

in a neighbourhood of $o_\ell$, $\tilde{\pi}$ induces a map $\tilde{\pi}_\ell : \mathbb{P}^*_{o_\ell}(\tilde{M}/E) \to \mathbb{P}^*_{o_\ell}M$ given by

$$\tilde{\pi}_\ell(\xi_0 : \eta_0 : \zeta_0) = (\xi_0 : \eta_0 : \zeta_0).$$
Therefore
\[ \hat{\pi}(\mathbb{P}_o^* E) = \Sigma_{\ell}. \]

Since \( \hat{\pi}(\mathbb{P}_S^* (\widetilde{M}/E)) = \mathbb{P}_S^* M, \)
\[ \hat{\pi}(\mathbb{P}_S^* (\widetilde{M}/E) \cap \mathbb{P}^* E) = \Sigma_o(S). \]

Since \( \mathbb{P}_S^* (\widetilde{M}/E) \cap \pi_{-1}^* (E) = \mathbb{P}_S^* (\widetilde{M}/E) \cap \mathbb{P}^* E \) is a Legendrian variety of \( \mathbb{P}^* E \) and
\[ \pi_{\widetilde{M}}(\mathbb{P}_S^* (\widetilde{M}/E) \cap \mathbb{P}^* E) = \widetilde{S} \cap E, \]
there are \( \ell_1, \ldots, \ell_k \in \mathbb{P}(T_o M) \) such that
\[ \mathbb{P}_S^* (\widetilde{M}/E) \cap \mathbb{P}^* E = \mathbb{P}_S^* E \cup \cup_{i=1}^k \mathbb{P}_o^* E. \]

Since \( \widetilde{S} \cap E \cong \text{Proj}(C_o(S)), \) \( \hat{\pi}(\mathbb{P}^*(\widetilde{S} \cap E)) \) equals the dual of \( \text{Proj}(C_o(S)). \)
The Theorem follows from (16).

5. The First Sequence of Blow-Ups

Let \( M \) be a germ of a complex manifold of dimension 3 at a point \( o. \) Let \( \rho \) be a submersion of \( M \) into a smooth surface \( X. \) We will fix systems of local coordinates \((x, y, z), [(x, y)]\) on \( M [X]\) such that \( o = (0, 0, 0) \) and \( \rho(x, y, z) = (x, y, z). \) Let
\[ \xi dx + \eta dy + \zeta dz \]
be the canonical 1-form \( \theta_N \) of \( T^* (M/N). \) Given \((a : b) \in \mathbb{P}^1, \) set
\[ \Sigma_o^n(a : b) = \{(\xi : \eta : \zeta) \in \mathbb{P}_o^* (M/N) : b\xi + a\eta = 0\}. \]

Let \( \pi : \widetilde{M} \to M \) be the blow up of \( M \) along the singular locus \( N^\sigma \) of \( N. \) Set \( \widetilde{N} = \pi^{-1}(N). \) Let \( \tilde{\pi} : T^* (\widetilde{M}/\widetilde{N}) \to T^* (M/N) \) be the blow up of \( T^* (M/N) \) along \( \pi_{\widetilde{M}}^{-1}(N^\sigma). \) There is a commutative diagram
\[ \begin{array}{ccc}
T^* (M/N) & \leftarrow & T^* (\widetilde{M}/\widetilde{N}) \\
\downarrow & & \downarrow \\
M & \leftarrow & \widetilde{M}
\end{array} \]

Set \( x_0 = x, y_0 = y/x \) and \( z_0 = z. \)

Lemma 9. Let \( o_1 \in \pi^{-1}(o). \) The following statements hold:

(a) The map \( \hat{\pi} \) induces a linear isomorphism \( \tilde{\pi} : T_{o_1}^*(\widetilde{M}/\widetilde{N}) \to T_{o_1}^*(M/N) \) given by
\[ (\xi_0, \eta_0, \zeta_0) \mapsto (\xi_0 - \eta_0, \eta_0, \zeta_0). \]
Proof. Statement (a) follows from the fact that
\[\hat{\pi}^* \theta_N = (\xi + \eta) \frac{dx_0}{x_0} + \eta \frac{dy_0}{y_0} + \zeta dz_0.\]
in a neighbourhood of \(o_1\). Statements (b) and (c) follow from statement (a).
Since \(\mathbb{P}^*_S(M/N) \subset \hat{\pi} \left( \mathbb{P}^*_S(\tilde{M}/\tilde{N}) \right)\),
\[\mathbb{P}^*_S(M/N) \cap \pi_1^{-1}(o) \subset \hat{\pi} \left( \mathbb{P}^*_S(\tilde{M}/\tilde{N}) \cap \pi_1^{-1}(\pi^{-1}(o)) \right).\]

Therefore
\[\Sigma^N_o(S) \subset \bigcup_{s \in \tilde{S} \cap \pi^{-1}(o)} \hat{\pi} \left( \Sigma^N_s(\tilde{S}) \right).\]
Statement (d) follows from (22) and statements (b) and (c).

10. Set \(M_0 = M, N_0 = N, X_0 = X, S_0 = S, \rho_0 = \rho\).
Let \(\rho_\ell : M_\ell \to X_\ell\) be a holomorphic submersion, Let \(N_\ell\) be a normal crossings divisor of \(M_\ell\). Let \(S_\ell\) be a surface of \(M_\ell\).
Let \(D_\ell\) be the intersection of the closure of \(\Delta_{\rho_\ell}(S_\ell) \setminus (\rho_\ell(N_\ell))\) and \(\rho_\ell(N^\sigma_\ell)\).
Let \(\tau_{\ell+1} : X_{\ell+1} \to X_\ell\) be the blow up of \(X_\ell\) along \(D_\ell\). Let \(\rho_{\ell+1} : M_{\ell+1} \to M_\ell\) be the blow up of \(M_\ell\) along \(\rho_\ell^{-1}(D_\ell)\). Let \(S_{\ell+1}\) be the strict transform of \(S_\ell\) by \(\tau_{\ell+1}\).
By the universal property of the blow-up there is a map \(\rho_{\ell+1} : M_{\ell+1} \to X_{\ell+1}\) such that \(\rho_\ell \tau_{\ell+1} = \tau_{\ell+1} \rho_{\ell+1}\). Moreover, \(\rho_{\ell+1}\) is a submersion. Hence we can iterate the process.

There is an integer \(L\) such that \(D_L = \emptyset\). Hence the procedure described in paragraph 10 will terminate. Set \(\pi = \pi_1 \circ \cdots \circ \pi_L\). We call the map \(\pi : M_L \to M\) the first sequence of blow-ups.

Theorem 11. Assume \(N\) has two irreducible components and \(S \cap N^\sigma = \{o\}\). There are positive integers \(a_1, \ldots, a_n, b_1, \ldots, b_n\) such that
\[\Sigma^N_o(S) \subset \bigcup_{i=1}^{n} \Sigma^N_{a_i}(a_i : b_i)\]
Moreover, \(\Sigma^N_o(S)\) contains a projective line if and only if there is a regular point \(o_1\) of \(N_L\) such that \(\Sigma^{N_L}_{o_1}(S_L) = \Sigma^{N_L}_{o_1}\).
Proof. Let $\sigma \in S_L$. If $\sigma$ is a singular point of $N_L$,
\[
\Delta_{\rho_L}(S_L) = \rho_L(N_L).
\]
Assuming $N_L = \{xy = 0\}$ in a neighbourhood of $\sigma$, it follows from Lemma 4 that $\Sigma^N_\alpha = \{(0 : 0 : 1)\}$. Assume $L = 1$. It follows from statements (a) and (c) of Lemma 9 that $\Sigma^N_0(S) \subset \Sigma^N_0(1 : 1)$.
The induction step follows from statement (d) of Lemma 9. \qed

6. The Non Degenerated Case

Let $N$ be a smooth divisor of a manifold $M$ of dimension 3. Let $S$ be a surface of $M$ that does not contain $N$. Let $o \in S \cap N$. Let $\pi^0 : M_0 \to M$ be the blow up of $M$ with center $o$. Let $\tilde{N}$ be the strict transform of $N$. Let $S_0$ denote the strict transform of $S$. Set $N_0 = (\pi^0)^{-1}(N)$, $E = (\pi^0)^{-1}(o)$.

We say that $S$ is non degenerated at $o$ if $C_o(S)$ does not contain $C_o(N)$.

Lemma 12. There is an open set $\mathbb{P}^\circ(M_0/N_0)$ of $\mathbb{P}^*(M_0/N_0)$ and an holomorphic map $\pi_0 : \mathbb{P}^\circ(M_0/N_0) \to \mathbb{P}^*(M/N)$ such that the diagram
\[
\begin{array}{ccc}
\mathbb{P}^*(M/N) & \xleftarrow{\pi_0} & \mathbb{P}^\circ(M_0/N_0) \\
\pi M & \downarrow & \pi M_0 \\
M & \xrightarrow{\pi_0} & M_0
\end{array}
\]
commutes and
(a) $\mathbb{P}^\circ(M \setminus N) \hookrightarrow \mathbb{P}^\circ(M_0/N_0)$,
(b) $\pi_0|_{\mathbb{P}^*(M_0 \setminus E)} : \mathbb{P}^*(M_0 \setminus E) \to \mathbb{P}^*(M \setminus \{o\})$ is a contact transformation,
(c) for each Legendrian surface $\Gamma$ of $\mathbb{P}^*(M/N)$, $\Gamma \subset \pi_0(\mathbb{P}^\circ(M_0/N_0))$,
(d) for each Legendrian surface $\Gamma_0$ of $\mathbb{P}^\circ(M_0/N_0)$, $\Gamma_0 \subset \mathbb{P}^\circ(M_0/N_0)$.

Proof. Assume $M$ is an affine set with coordinates $(x, y, z)$ and $N = \{x = 0\}$. The manifold $M_0$ is the gluing of the open affine sets $V_i$, $i = 1, 2, 3$, with coordinates $(x_i, y_i, z_i)$ such that

(1) $x_1 = x, y_1 = y/x, z_1 = z/x$;
(2) $x_2 = x/y, y_2 = y, z_2 = z/y$;
(3) $x_3 = x/z, y_3 = z, z_3 = y/z$.

Let $\pi_0 : \mathbb{P}^*(M_0 \setminus E) \to \mathbb{P}^*(M \setminus \{o\})$ be the contact transformation such that $\pi_M \circ \pi'_0 = \pi^0 \circ \pi_{M_0}$.

Let $\pi_{0,i}$ be the restriction of $\pi_0$ to $\pi^{-1}_M(V_i \setminus N_0)$, $i = 1, 2, 3$. Since
\[
\pi_{0,1}^*(\xi \frac{dx}{x} + \eta dy + \zeta dz) = (\xi + y\eta + z\zeta) \frac{dx_1}{x_1} + x\eta dy_1 + x\zeta dz_1,
\]
\[
\xi_1 = \xi + y\eta + z\zeta, \eta_1 = x\eta \text{ and } \zeta_1 = x\zeta.
\]
Hence
\[
\pi_{0,1}(x_1, y_1, z_1; \xi_1 : \eta_1 : \zeta_1) = (x_1, x_1 y_1, x_1 z_1; x_1 (\xi_1 - y_1 \eta_1 - z_1 \zeta_1) : \eta_1 : \zeta_1).
\]
Therefore $\pi_{0,1}$ is defined outside of $B_1 = \{x_1 = \eta_1 = \zeta_1 = 0\}$.
The canonical 1-form of $T^* (V_2/N_0 \cap V_2)$ is
\[ \xi_2 dx_2 + \eta_2 dy_2 + \zeta_2 dz_2 \quad \left[ \xi_2' dx_2 + \eta_2' dy_2 + \zeta_2 dz_2, \right] \]
where $\xi_2 = x_2 \xi_2'$ if $x_2 \neq 0$. Since
\[ \pi_{0,2}(\frac{dx}{x} + \eta dy + \zeta dz) = \xi dx_2 + (\xi + \eta z) dy_2 + \eta_2 dz_2, \]
$\pi_{0,2}$ is given by $x = x_2y_2$, $y = y_2$, $z = y_2z_2$.

Therefore $\pi_{0,2}$ is defined outside of $\{ y_2 = \eta_2 - \xi_2 = \zeta_2 = 0 \}$. Since $\xi_2 = x_2 \xi_2'$, when $x_2 \neq 0$, $\pi_{0,2}$ is defined outside of the union of the sets
\[ B_2 = \{ x_2 = y_2 = \eta_2 - \xi_2 = \zeta_2 = 0 \}, \]
\[ B_2' = \{ y_2 = \eta_2 - x_2 \xi_2' = \zeta_2 = 0, x_2 \neq 0 \}. \]

Let $\Gamma_0$ be a Legendrian variety of $\mathbb{P}^*(M_0/N_0)$. Since $B_1 \cap \{ \xi_1 = 0 \} = \emptyset$, $\Gamma_0 \cap B_1 = \emptyset$ by Lemma 2. By a similar argument $\Gamma_0 \cap B_2 = \emptyset$. By Theorem 11, $\Gamma_0 \cap B_2 = \emptyset$. We apply the same reasoning to $V_3$.

**Lemma 13.** If $\Sigma_o^N(S)$ is finite, $C_o(S)$ is a union of planes.

**Proof.** Let $L$ be an irreducible component of $S_0 \cap E$. Notice that if $L = \tilde{N} \cap E$, $L$ is the projectivization of $C_o(N)$. Assume $L \neq \tilde{N} \cap E$. By Bezout’s Theorem, there is $\sigma \in L \cap \tilde{N}$. We can assume that $\sigma$ is the origin of $V_2$. Let $\gamma$ be an irreducible component of the germ of $L$ at $\sigma$. There is a local parametrization of $\gamma$ of the type
\[ x_2 = \varepsilon_1(t)t^{k_1}, \quad y_2 = 0, \quad z_2 = \varepsilon_2(t)t^{k_2}, \]
such that $k_1, k_2$ are positive integers, $(k_1, k_2) = 1$, $\varepsilon_1, \varepsilon_2 \in \mathbb{C}\{t\}$ and $\varepsilon_1 \neq 0$. Furthermore, we can assume
\begin{enumerate} 
  \item \text{(a)} if $\varepsilon_2 \equiv 0, \varepsilon_1 \equiv 1$ and $k_1 = 1$,
  \item \text{(b)} if $k_j \geq k_n, \varepsilon_n \equiv 1$ and $\varepsilon_j(0) \neq 0$,
\end{enumerate}
where $j, n \in \{1, 2\}$ and $j \neq n$. Therefore $\mathbb{P}^*_\gamma(N_0 \setminus \tilde{N})$ admits a parametrization
\[ x_2 = \varepsilon_1(t)t^{k_1}, \quad y_2 = 0, \quad z_2 = \varepsilon_2(t)t^{k_2}, \quad \zeta_2 = -\delta(t)t^{k_2}\zeta_2, \quad \eta_2 = 0, \]
where $\delta(t) = \varepsilon_1(t)(k_2\varepsilon_2(t) + \varepsilon'_2(t))t/(k_1\varepsilon_1(t) + \varepsilon'_1(t))t$.

Since $\Sigma_o^N(S) \neq \Sigma_o^N$, $\pi_0(\mathbb{P}^*_\gamma(N_0 \setminus \tilde{N}))$ is a point. By (24), $\delta = \varepsilon_2$. Therefore
\[ (k_2 - k_1)\varepsilon_1 \varepsilon_2 + t(\varepsilon_1\varepsilon'_2 - \varepsilon'_1 \varepsilon_2) = 0. \]
If $\varepsilon_2 = 0$, $\gamma$ is contained in a projective line. Hence $L$ is a projective line. Assume $\varepsilon_2(0) \neq 0$. Then $k_2 - k_1 = 0$. Hence we can assume $k_2 = 1, \varepsilon_1 = 1$. Replacing $k_1, k_2$ and $\varepsilon_1$ in (25), we conclude that $\varepsilon'_2 = 0$. Therefore $\varepsilon_2 \in \mathbb{C}^*$. Hence $L$ is a projective line. □
Theorem 14. Assume $S$ is non degenerated at $o$. Then $\Sigma_α^N(S)$ is finite if and only if $C_0(S)$ is a union of planes and for each $σ \in S_0 \cap E$, $\Sigma_σ^N(S_0)$ is finite.

Proof. By Lemma 13, $S_0 \cap E$ is a union of projective lines $L_i$, $i = 1, ..., n$. Since $S$ is non degenerated, there are $σ_1, ..., σ_n$ such that

$$S_0 \cap E \cap \tilde{N} = \{σ_1, ..., σ_n\}.$$  

Moreover, there are points $o_1, ..., o_m$ of $(S_0 \cap E) \setminus \tilde{N}$ such that

$$\mathbb{P}^*_S(M_o/N_0) \cap π^{-1}_{M_0}(E) = \cup_i \mathbb{P}^*_L E \cup \cup_i Σ_σ^N(S_0) \cup \cup_i Σ_α^N(S).$$

By Lemma 12, $Σ_α^N(S) \subset π_0(\mathbb{P}^*_L E \cup \cup_i Σ_σ^N(S_0) \cup \cup_i Σ_α^N(S_0)).$

By the arguments of Lemma 13, $π_0(\mathbb{P}^*_L E)$ is a point, for $i = 1, ..., n$. The type of arguments used in Theorem 8 show that $π_0(Σ_σ^N(S_0))$ is finite if and only if $Σ_σ^N(S_0)$ is finite, $i = 1, ..., n$ and $π_0(Σ_α^N(S_0))$ is infinite for $i = 1, ..., m$.

Hence $Σ_α^N(S)$ is finite if and only if $Σ_σ^N(S_0)$ is finite, $i = 1, ..., n$ and $m = 0$. □

7. The Second sequence of Blow-ups

We will introduce a generalization for surfaces of a construction for curves that was introduced in [3]. Given non negative integers $a_0, a_1, ..., a_g$ assume that $a_1, ..., a_g \geq 1$ or $g = 1, a_1 = 0$. Set $[a_0, 0] = ∞, [a_0] = a_0$. If $g, a_g \geq 1$, set

$$[a_0, ..., a_g] = a_0 + [a_1, ..., a_g]^{-1}.$$  

Assuming $a_g \geq 2$, $[a_0, ..., a_g] = [a_0, ..., a_g - 1, 1]$. It is well known that each positive rational number is described by exactly two continuous fractions.

If $α = [a_0, ..., a_g]$ we call length of $α$ to $|α| = a_0 + ... + a_g$. Set $n_∞ = 1, d_∞ = 0$. If $α = a/b$ where $a, b$ are positive integers such that $(a, b) = 1$, set $n_α = a, d_α = b, e_α = a + b$.

Let $α$ be a positive rational number, $α = [a_0, ..., a_g]$. If $α$ is an integer, set $α_ω = α - 1, α_π = ∞$. Otherwise, $g \geq 1$. Moreover, we can assume $a_g \geq 2$. Set $α_ω = [a_0, ..., a_g - 1]$ and $α_π = [a_0, ..., a_g - 1]$ if $g$ even, otherwise set $α_ω = [a_0, ..., a_g - 1]$ and $α_π = [a_0, ..., a_g - 1]$.

Assume $α = 1$ or $a_g \geq 2$. Set $α_s = [a_0, ..., a_g - 1, a_g - 1, 2]$ and $α_b = [a_0, ..., a_g - 1, a_g + 1]$ if $g$ even, otherwise set $α_s = [a_0, ..., a_g - 1, a_g + 1]$ and $α_b = [a_0, ..., a_g - 1, a_g - 1, 2]$. Notice that $α_s$ and $α_b$ are the only rationals such that $α_sπ = α_bω = α$. Moreover, $α_sω = α_ω, α_bπ = α_π$,

$$α_s = \frac{n_α + n_α_ω}{d_α + d_α_ω}, \quad α_b = \frac{n_α + n_α_π}{d_α + d_α_π}.$$  

$e_α + e_α_ω = e_α_b$ and $e_α + e_α_π = e_α_s$.  

Let $M$ be the germ of a complex analytic manifold of dimension 3 at a point $o$. Let $N$ be a smooth surface of $M$. Let $S$ be a singular surface of $M$. Assume that $N$ is not an irreducible component of $S$ and $C_o(S) \supset C_o(N)$.

Let $\pi^0 : M^0 \to M$ be the blow up of $M$ at $o$. Set $N^0 = (\pi^0)^{-1}(N)$ and $E^0 = (\pi^0)^{-1}(o)$. Let $S^0$ be the strict transform of $S$ by $\pi^0$. Notice that $S^0$ contains the singular locus $Z^0$ of $N^0$.

Let $N^k$ be a normal crossings divisor of a manifold $M^k$ of dimension 3. Let $S^k$ be a singular surface of $M^k$. Let $Z^k$ be the union of the connected components of the singular locus of $N^k$ that are contained in $S^k$.

We iterate the process defining $\pi^{k+1} : M^{k+1} \to M^k$ as the blow up of $M^k$ along $Z^k$, defining $S^{k+1}$ as the strict transform of $S^k$ by $\pi^{k+1}$ and setting $N^{k+1} = (\pi^{k+1})^{-1}(N^k)$, $E^{k+1} = (\pi^{k+1})^{-1}(Z^k)$.

This process will terminate after a finite number $k_0$ of steps. The intersection of $S^{k_0}$ with the singular locus of $N^{k_0}$ is a finite set. We will now perform the first sequence of blow-ups at each point of this intersection.

We obtain in this way a map $\pi : \TM \to M$, a normal crossings divisor $\TN$ of $M$ and a singular surface $\TS$ of $M$. We call $\pi : \TM \to M$ the second sequence of blow-ups.

Let $M^{(a)}$ be the gluing of the affine sets $U_{a,i}$, with coordinates $(u_{a,i}, v_{a,i}, w_{a,i})$, $i = 1, 2, 3, 4$, by the transformations

\begin{align*}
v_{a,3} &= v_{a,1} w_{a,1}^{e_{a}}; \\
v_{a,4} &= v_{a,2} w_{a,2}^{e_{a}}; \\
v_{a,2} &= v_{a,1}, \\
v_{a,1} &= v_{a,1} w_{a,1}^{e_{a}}; \\
v_{a,3} &= v_{a,1} w_{a,1}^{e_{a}}; \\
v_{a,4} &= v_{a,2} w_{a,2}^{e_{a}}; \\
v_{a,2} &= v_{a,1} w_{a,1}^{e_{a}}; \\
v_{a,1} &= v_{a,1} w_{a,1}^{e_{a}};
\end{align*}

Let $E^{(a)}$, $N^{(a)}_b$, $N^{(a)}_s$ be defined by

\begin{align*}
E^{(a)} \cap U_{a,i} &= \{u_{a,i} = 0\}, \\
N^{(a)}_b \cap U_{a,i} &= \{v_{a,i} = 0\}, \\
N^{(a)}_s \cap U_{a,i} &= \{v_{a,i} = 0\},
\end{align*}

for $a = b$, $i = 1, 2, 3, 4$; $i = 1, 3$; $i = 2, 4$.

Set

\begin{align*}
Z^{(a)}_b &= E^{(a)} \cap N^{(a)}_b, \\
Z^{(a)}_s &= E^{(a)} \cap N^{(a)}_s, \\
M^{(a)}_b &= M^{(a)} \setminus N^{(a)}_s, \\
M^{(a)}_s &= M^{(a)} \setminus N^{(a)}_b.
\end{align*}

We will denote by

\[N^{(a)}_b \subset [N^{(a)}_b, N^{(a)}_s, N^{(a)}_s]\]

the strict transform of $E^{(a)} \cap [N^{(a)}_b, N^{(a)}_s, E^{(a)}]$ by the blow up of $M^{(a)}_b \cap [M^{(a)}_s, M^{(a)}_b, M^{(a)}_s]$ along $Z^{(a)}_b \subset [Z^{(a)}_s, Z^{(a)}_s, Z^{(a)}_s]$.

\textbf{Lemma 16.} Assume $M$ is an affine set with coordinates $(x, y, z)$, $N = \{x = 0\}$, $o$ is the origin and $k \geq 1$. Then the following statements hold:

There are finite sets $I^k_b \subset \mathbb{Q}$ such that $M^k$ is a gluing of the manifolds

\[M^{(a)}_b, \alpha \in I^k_b, \quad M^{(a)}_s, \alpha \in I^k_s;\]

For each $\alpha \in I^k_b$, $E^k \cap M^{(a)}_b \subset E^{(a)}$, $N^k \cap M^{(a)}_b \subset E^{(a)} \cup N^k_b$.

For each $\alpha \in I^k_s$, $E^k \cap M^{(a)}_s \subset E^{(a)}$, $N^k \cap M^{(a)}_s \subset E^{(a)} \cup N^k_s$. 
For each $\alpha$, $Z^\alpha$ is a projective line. The restriction $\pi^\alpha : M^{(\alpha)} \to M^{(\alpha)} \setminus N^{\alpha}$ of $\pi^{k+1} : M^{k+1} \to M^k$ is given by

$$u_{\alpha,i} = u_{\alpha,i}, \quad v_{\alpha,i} = u_{\alpha,i}w_{\alpha,i}, \quad w_{\alpha,i} = w_{\alpha,i}, \quad i = 1, 3,$$

$$u_{\alpha,i-1} = u_{\alpha,i}v_{\alpha,i}, \quad v_{\alpha,i-1} = u_{\alpha,i} \quad w_{\alpha,i-1} = w_{\alpha,i}, \quad i = 2, 4,$$

The restriction $\pi^\alpha : M^{(\alpha)} \to M^{(\alpha)} \setminus N^{\alpha}$ of $\pi^{k+1} : M^{k+1} \to M^k$ is given by

$$u_{\alpha,i+1} = u_{\alpha,i}v_{\alpha,i}, \quad v_{\alpha,i+1} = u_{\alpha,i} \quad w_{\alpha,i+1} = w_{\alpha,i}, \quad i = 1, 3,$$

$$u_{\alpha,i} = u_{\alpha,i}, \quad v_{\alpha,i} = u_{\alpha,i}v_{\alpha,i}, \quad w_{\alpha,i} = w_{\alpha,i}, \quad i = 2, 4.$$

Proof. The manifold $M^0$ is the gluing of the affine sets $V_1$ introduced at the proof of Lemma [12]. Remark that $Z^1$ is a projective line contained in $V_2 \cup V_3$. Moreover, $I_b = I_s^2 = \{1\}$. Setting

$$u_{1,1} = x_2, \quad v_{1,1} = y_2/x_2, \quad w_{1,1} = z_2;$$

$$u_{1,2} = y_2, \quad v_{1,2} = x_2/y_2, \quad w_{1,2} = z_2;$$

$$u_{1,3} = x_3, \quad v_{1,3} = y_3/x_3, \quad w_{1,3} = z_3;$$

$$u_{1,4} = y_3, \quad v_{1,4} = x_3/y_3, \quad w_{1,4} = z_3,$$

we conclude that the lemma holds for $k = 1$.

Assume $|\alpha| = k$. If $Z^\alpha \supset S$, $[Z^\alpha \supset S]$ we withdraw $\alpha$ from $I^k_b$ and include $\alpha$ into $I^k_b$ and $I_s^{k+1}$. Defining $u_{\alpha,1}, v_{\alpha,1}, w_{\alpha,1}$ in such a way that $\pi^\alpha$ is as proposed in this Lemma, we conclude that

$$u_{\alpha,3} = u_{\alpha,1}w_{\alpha,1}^{-e_\alpha} = u_{\alpha,1}w_{\alpha,1}^{-e_\alpha};$$

$$v_{\alpha,3} = v_{\alpha,1}u_{\alpha,1}^{-1} = v_{\alpha,1}u_{\alpha,1}^{-1}w_{\alpha,1}^{-e_\alpha} = v_{\alpha,1}u_{\alpha,1}^{-1}w_{\alpha,1}^{-e_\alpha} = v_{\alpha,1}w_{\alpha,1}^{-e_\alpha}$$

and $w_{\alpha,3} = w_{\alpha,1} = w_{\alpha,1}^{-1}$. \qed

Let $\pi^{(\alpha)} : E^{(\alpha)} \to Z^\alpha$ be the restriction of $\pi^\alpha$. Let $C$ be an irreducible curve of $E^{(\alpha)}$. We say that $C$ is well behaved if $C$ is a fiber of $\pi^{(\alpha)}$ or $C$ is the graph of a section of $\pi^{(\alpha)}$ such that $C \cap Z_b^\alpha = \emptyset$ and $C$ intersects $Z^\alpha_s$ at exactly one point with multiplicity $e_\alpha$.

We are now able to state the main theorem of this section. We will prove it at the end of the section.

Theorem 17. Let $M$ be a germ of a complex analytic manifold at a point $o$. Let $N$ be a smooth surface of $M$. Let $S$ be a singular surface of $M$. Let $\pi : \tilde{M} \to M$ be the second sequence of blow-ups. Then $\Sigma^{\alpha}_0(S) = \Sigma^{\alpha}_0$ if and only if one of the following conditions is verified:

1. The tangent cone of $S$ is not a union of planes
2. There is an integer $k$ and an irreducible component of $S^k \cap E^k$ that is not well behaved.
3. There is a regular point $o_1$ of $\tilde{N}$ such that $\Sigma^{\tilde{N}}_0(S) \supset \Sigma^{\tilde{N}}_0$.

We have contact transformations

$$\pi_{k+1} : \mathbb{P}^s(M^{k+1}/N^{k+1}) \to \mathbb{P}^s(M^k/N^k),$$

$k \geq 0$, such that the diagrams
etc. Set $\tau_k = \pi_2 \circ \cdots \circ \pi_k : M^k \to M^1$.

Let $\alpha \in I^k_{\mathcal{B}} \cup I^k_{\mathcal{S}}$. Let $i \in \{1, 2, 3, 4\}$. There is a system of coordinates $(u_{\alpha,i}, v_{\alpha,i}, w_{\alpha,i}; \xi_{\alpha,i} : \eta_{\alpha,i} : \zeta_{\alpha,i})$ on $W_{\alpha,i} = \pi^{-1}_{M_k}(U_{\alpha,i})$ such that

\begin{equation}
\xi_{\alpha,i} \frac{du_{\alpha,i}}{u_{\alpha,i}} + \eta_{\alpha,i} \frac{dv_{\alpha,i}}{v_{\alpha,i}} + \zeta_{\alpha,i} dw_{\alpha,i}
\end{equation}

is the restriction to $W_{\alpha,i}$ of the canonical 1-form of $T^* \langle M^k/N^k \rangle$.

**Lemma 18.** If $|\alpha| = k$ and $\alpha > 1$, $\tau_k(W_{\alpha,i} \cap \pi^{-1}_M(E^k)) \subset W_{1,1} \cap \pi^{-1}_M(Z^1)$, $i = 1, 2$ and $\tau_k(W_{\alpha,i} \cap \pi^{-1}_M(E^k)) \subset W_{1,3} \cap \pi^{-1}_M(Z^1)$, $i = 3, 4$. Moreover, the restriction of $\tau_k$ to $W_{\alpha,i} \cap \pi^{-1}_M(E^k)$ is given by

\begin{align*}
w_{1,1} &= w_{\alpha,1} = w_{\alpha,2}; w_{1,3} = w_{\alpha,3} = w_{\alpha,4}; \xi_{1,1} = \eta_{\alpha,1} = \xi_{\alpha,1} = \eta_{\alpha,2}, \xi_{1,3} = \eta_{\alpha,3} = \xi_{\alpha,3}; \\
\eta_{1,i} &= \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}} - \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}} = \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}} - \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}}, \quad \text{if } i = 1, 3; \\
\xi_{1,i} &= \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}} - \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}} = \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}} - \frac{d\alpha_{\alpha,i}}{\alpha_{\alpha,i}}, \quad \text{if } i = 2, 4.
\end{align*}

**Proof.** If $i = 1, 3$, the pull-back by $\pi_k+1 |_{W_{\alpha,i}}$ of (26) equals

\begin{equation}
(\xi_{\alpha,i} + \eta_{\alpha,i}) \frac{du_{\alpha,i}}{u_{\alpha,i}} + \frac{dv_{\alpha,i}}{v_{\alpha,i}} + \zeta_{\alpha,i} dw_{\alpha,i}.
\end{equation}

Hence $\pi_{k+1} |_{W_{\alpha,i} \cap \pi^{-1}_M(E^k)}$ is given by the relations

\begin{align*}
w_{\alpha,i} &= w_{\alpha,i}; \xi_{\alpha,i} = \xi_{\alpha,i} - \eta_{\alpha,i}; \eta_{\alpha,i} = \eta_{\alpha,i}; \quad \zeta_{\alpha,i} = \zeta_{\alpha,i}.
\end{align*}

Therefore the restriction of $\tau_{k+1}$ to $W_{\alpha,i} \cap \pi^{-1}_M(E^k+1)$ is given by

\begin{align*}
\xi_{1,i} &= n_{\alpha_{\alpha,i}} \xi_{\alpha,i} - n_{\alpha_{\alpha,i}} \eta_{\alpha,i} = n_{\alpha_{\alpha,i}} \xi_{\alpha,i} - n_{\alpha_{\alpha,i}} \eta_{\alpha,i}; \\
\eta_{1,i} &= d\alpha_{\alpha,i} - d\alpha_{\alpha,i} = d\alpha_{\alpha,i} - d\alpha_{\alpha,i}; \\
\zeta_{1,i} &= w_{\alpha,i}; \xi_{1,i} = \zeta_{\alpha,i}.
\end{align*}

There is a canonical embedding of $\mathbb{P}^*(E^k/Z^k)$ into $\mathbb{P}^*(M^k/N^k)$. Moreover,

\begin{equation}
\pi^k(E^k) \subset Z^{k-1}, \quad \pi_k(\mathbb{P}^*(E^k/Z^k)) \subset \mathbb{P}^*(M^{k-1}/N^{k-1}) \cap \pi^{-1}_{M^{k-1}}(Z^{k-1}).
\end{equation}

Hence $\tau_k(\mathbb{P}^*(E^k/Z^k)) \subset \mathbb{P}^*(M^1/N^1) \cap \pi^{-1}_M(Z^1)$. Therefore $\pi_0 \circ \pi_1 \circ \tau_k$ defines a map $u_k : \mathbb{P}^*(E^k/Z^k) \to \Sigma^N$. Set $U_{\alpha,i} = E^k \cap U_{\alpha,i}$, $W_{\alpha,i} = \mathbb{P}^*(E^k/Z^k) \cap W_{\alpha,i}$. Notice that

\begin{equation}
W_{\alpha,i} = \{(u_{\alpha,i}, v_{\alpha,i}, w_{\alpha,i}; \xi_{\alpha,i} : \eta_{\alpha,i} : \zeta_{\alpha,i}) : u_{\alpha,i} = \xi_{\alpha,i} = 0\}
\end{equation}
and $U'_{\alpha,i} = \{(u_{\alpha,i}, v_{\alpha,i}, w_{\alpha,i}) : u_{\alpha,i} = 0\}$. Moreover,

$$\eta_{\alpha,i} \frac{dv_{\alpha,i}}{v_{\alpha,i}} + \zeta_{\alpha,i} dw_{\alpha,i}$$

is the restriction to $W'_{\alpha,i}$ of the canonical 1-form of $T^*\langle E^k/Z^k \rangle$.

**Lemma 19.** If $|\alpha| = k$, $v_k$ is given by

$$\eta = (-1)^{i+1}e_\alpha \eta_{\alpha,i} - w_{\alpha,i} \zeta_{\alpha,i}, \quad \zeta = \zeta_{\alpha,i}, \quad \text{if } i = 1, 2,$$

$$\zeta = (-1)^{i+1}e_\alpha \eta_{\alpha,i} - w_{\alpha,i} \zeta_{\alpha,i}, \quad \eta = \zeta_{\alpha,i}, \quad \text{if } i = 3, 4.$$ 

**Proof.** Assume $\alpha \geq 1$. Let

$$\frac{dx}{x} + \eta dy + \zeta dz, \quad \frac{dx_2}{x_2} + \eta_2 dy_2 + \zeta_2 dz_2, \quad \frac{dx_3}{x_3} + \eta_3 dy_3 + \zeta_3 dz_3$$

be the canonical 1-form of $T^*\langle M/N \rangle$, the restriction to $\pi_{M_0}^{-1}(V_i)$ of the canonical 1-form of $T^*\langle M^0/N^0 \rangle$, $i = 2, 3$. By Lemma 19, the restriction of $\pi_k$ to $W'_{\alpha,i}$ is given by

$$w_{1,i} = w_{\alpha,i}, \quad \xi_{1,i} = (-1)^{i+1}e_\alpha \eta_{\alpha,i}, \quad \eta_{1,i} = (-1)^{i+1}d_\alpha \eta_{\alpha,i}, \quad \zeta_{1,i} = \zeta_{\alpha,i},$$

for $i = 1, 2, 3, 4$. The result follows from the fact that the restriction of $\pi_{0}$ to $\mathbb{P}^*\langle M^0/N^0 \rangle \cap \pi_{M_0}^{-1}(E^0)$ is given by

$$\eta = \eta_2 - \xi_2 - z_2 \zeta_2, \quad \zeta = \zeta_2; \quad \eta = \zeta_3, \quad \zeta = \eta_3 - \xi_3 - z_3 \zeta_3,$$

and $\pi_1$ is given by

$$z_2 = w_{1,1}, \xi_2 = \xi_{1,1} - \eta_{1,1}, \eta_2 = \eta_{1,1}, \zeta_2 = \zeta_{1,1};$$

$$z_3 = w_{1,3}, \xi_3 = \xi_{1,3} - \eta_{1,3}, \eta_3 = \eta_{1,3}, \zeta_3 = \zeta_{1,3};$$

$$z_4 = w_{1,4}, \xi_3 = \xi_{1,4}, \eta_3 = \eta_{1,4}, \zeta_3 = \zeta_{1,4}.$$

The proof in the case $\alpha > 1$ is similar. Remark that $e_{\alpha-1} = e_\alpha$. \qed

**Lemma 20.** For each $\alpha$ and each curve $C$ of $E^{(\alpha)}$, $C$ intersects $Z^{\alpha}$.

**Proof.** Assume that $C$ does not intersect $Z^{\alpha}$. The intersection of $C$ with $U'_{\alpha,3}$ is defined by a polynomial $\sum_{i=0}^{\ell} a_i(w_{\alpha,3}) v_{\alpha,3,1}$. Hence $a_0 \in \mathbb{C}^*$. There is an integer $\mu \geq 0$ such that $C \cap U'_{\alpha,1}$ is given by the polynomial

$$w_{\alpha,1}^{\mu} \left( \sum_{i=0}^{\ell} a_i(w_{\alpha,1}) u_{\alpha,1}^{e_{\alpha,i} i} v_{\alpha,1, i} \right).$$

Since $C$ does not intersect $Z^{\alpha}$, $\mu = 0$. The intersection of $C$ with $U'_{\alpha,2}$ is defined by

$$\sum_{i=0}^{\ell} a_i(w_{\alpha,1}) u_{\alpha,2}^{e_{\alpha,i} i} v_{\alpha,1}^{\ell - i}.$$ 

Hence there is $\lambda \in \mathbb{C}^*$ such that $a_\ell(t) = \lambda t^{e_{\alpha,\ell}}$. Finally, $C \cap U'_{\alpha,4}$ is given by $\sum_{i=0}^{\ell} a_i(w_{\alpha,4}) v_{\alpha,4, i}^{\ell - i}$. Therefore $a_\ell \in \mathbb{C}^*$, which leads to a contradiction. \qed
Lemma 21. Let \( C \) be an irreducible curve of \( E^{(a)} \). The image by \( v_k \) of \( \Gamma = \mathbb{P}_C^s(E^k/N^k) \) is different from \( \Sigma^N_0 \) if and only if \( C \) is well behaved.

Proof. Assume \( v_k(\Gamma) \) is different from \( \Sigma^N_0 \). Set
\[
p = -\eta \zeta^{-1}, \quad p_i = -\eta_{a,i} \zeta_{a,i}^{-1}, \text{ if } i = 1, 2;
p = -\zeta \eta^{-1}, \quad p_i = -\eta_{a,i} \zeta_{a,i}^{-1}, \text{ if } i = 3, 4.
\]
The restriction of \( C \) is the germ of \( C \) at \( o_1 \). Then \( C_1 = \{ w_{\alpha,i} = c \} \) or \( C_1 \) admits one of the following parametrizations
\[
(27) \quad v_{\alpha,i} = t^a, \quad w_{\alpha,i} = c + t^b \varepsilon; \quad v_{\alpha,i} = t^b \varepsilon, \quad w_{\alpha,i} = c + t^a;
\]
where \( b \geq a \) and \( \varepsilon \) is a unit of \( \mathbb{C}\{t\} \). In the first case \( C_1 \) is well behaved.

Assume \( C_1 \) admits the first parametrization. Setting \( \Gamma_1 = \mathbb{P}^s_{C_1}(E^k/Z^k) \), \( \Gamma_1 \) admits a local parametrization given by \( (27) \) and \( p_i = (t^b/a)(bc + t\varepsilon') \).

Therefore \( v_k(\Gamma_1) \) contains the set of points \( p \) such that
\[
p = c + t^b[(-1)^{i+1}e_{\alpha}(b/a) + 1] \varepsilon + (-1)^{i+1}e_{\alpha}(1/a)t\varepsilon'
\]
and \( |t| << 1 \).

This set is finite if and only if \( \varepsilon \) is the solution of an ODE \( t\varepsilon' + \lambda \varepsilon = 0 \). Since \( \varepsilon \) is a unit, \( \lambda = 0 \). Hence \( a = (-1)^i be_{\alpha} \). Since \( a, b, e_{\alpha} \) are positive, \( i \) is even. Hence \( C \) cannot intersect \( Z^a_\varepsilon \). Moreover, \( C \cap U'_{\alpha,i} \) is described by an equation of the type
\[
(28) \quad v_{\alpha,i} = (\mu w_{\alpha,i} + \nu)^\varepsilon_{\alpha},
\]
Hence \( C_1 \) is the graph of a section of \( \pi^{(a)} \). The remaining case can be treated in a similar way. Remark that in each case \( C = C_1 \).

Let \( C \) be a section of \( \pi^{(a)} \) verifying the statements of the lemma. Then \( C \) is a section of the restriction \( \pi^{[a]} \) of \( \pi^{(a)} \) to \( E^{(a)} \setminus Z^a_\varepsilon \). Since \( \pi^{[a]} \) is a line bundle of degree \( e_{\alpha} \) and \( C \) has a zero of order \( e_{\alpha} \), \( C \) is of the type \( (28) \).

Proof of Theorem 21. If \( (a) [(b), (c)] \) holds it follows from Lemma 20. If \( (b) \) holds from Lemma 13. If \( (c) \) holds from Lemma 12. If \( (d) \) holds from Lemma 11. If \( (e) \) holds from Lemma 14. If \( (f) \) holds from Lemma 15.

Let \( \pi_k : M_k \to M \) be the second sequence of blow ups. Let \( \ell \leq k \) and let \( F_\ell \) be an irreducible component of \( N_\ell \). Let \( F_\ell' \) be the intersection of \( F_\ell \) with the regular part of \( N_\ell \).

Assume \( (a), (b), (c) \) do not hold. Since \( (c) \) does not hold, the closure of
\[
\mathbb{P}^s_{S_k}(M_k/N_k) \cap \pi_{M_k}^{-1}(F_k')
\]
is the closure of \( \mathbb{P}^s_{S_k \cap F_k'} F_k' \). We show by induction in \( \ell \), using theorems 11 and 14 that
\[
\mathbb{P}^s_{S_{k-\ell}}(M_{k-\ell}/N_{k-\ell}) \cap \pi_{M_{k-\ell}}^{-1}(F_{k-\ell})
\]
is the closure of $\mathbb{P}_{S_k-\ell}\cap F'_{k-\ell}$ for each $\ell \leq k$.

Since (a) does not hold, it follows from Theorem 13 that $\pi_1(\mathbb{P}_1\cap F_1)\cap F'_1$ is finite. Since (b) does not hold, it follows from Lemma 21 that $\pi_1(\mathbb{P}_{S_k}\cap F_1)\cap F'_1$ is finite, for $2 \leq \ell \leq k$.

8. Main Results

22. Let $N_0$ be a smooth surface of a germ of a manifold $M_0$ of dimension 3 at a point $o$. Let $S_0$ be a surface of $M_0$ that does not contain $N_0$.

Let $S_k$ be a normal crossings divisor of a manifold $M_k$ of dimension 3. Let $S_k$ be a singular surface of $M_k$ that does not contain any irreducible component of $N_k$. Assume $\text{Sing}^N(S_k)$ does not intersect the singular locus of $N_k$.

Let $\sigma \in \text{Sing}^N(S_k) \cap N_k$. Assume the germ of $\text{Sing}^N(S_k) \cap N_k$ at $\sigma$ is not smooth or is not transversal to $N_k$. If $S_k$ is non degenerated at $\sigma$, we blow up $M_k$ at $\sigma$ followed by the first sequence of blow ups. Otherwise we perform the second sequence of blow ups at $\sigma$. After modifying $M_k$ at each point of the finite set $\text{Sing}^N(S_k) \cap N_k$, we obtain a map $\pi_{k+1} : M_{k+1} \rightarrow M_k$. Set $N_{k+1} = \pi_{k+1}^{-1}(N_k)$. Let $S_{k+1}$ be the strict transform of $S_k$ by $\pi_{k+1}$. Applying, accordingly, the first sequence or the second sequence of blow ups, we guarantee that $\text{Sing}^N(S_{k+1})$ does not intersect the singular locus of $N_{k+1}$.

Lemma 23. There is an integer $k$ such that each connected component of $\text{Sing}^N(S_k)$ is a smooth curve transversal to $N_k$. Hence the procedure described in paragraph 22 will terminate after a finite number of steps.

Proof. Notice that, for each $\ell$, $\text{Sing}^N(S_{\ell+1})$ is the strict transform by $\pi_{\ell+1}$ of $\text{Sing}^N(S_\ell)$.

Let $C$ be an irreducible singular curve of a germ of manifold $M$ of dimension 3 at a point $o$. Let $\gamma : (\mathbb{C},0) \rightarrow C$ be the normalization of $C$. Let $\Gamma$ be the semi group of the orders of the functions $\gamma^*f$, $f \in \mathcal{O}_{M,o}$. Let $m_C$ be the smallest positive integer that belongs to $\Gamma$. The integer $m_C$ equals the multiplicity of $C$. Let $n_C$ be the infimum of $\Gamma \setminus \{m_C\}$.

Let $\tilde{C}$ be the strict transform of $C$ by the blow up of $M$ along a smooth line that contains $o$. Then

$$m_{\tilde{C}} < m_C \quad \text{or} \quad m_{\tilde{C}} = m_C \quad \text{and} \quad n_{\tilde{C}} \leq n_C.$$ 

Hence the invariant does not get worse. Let $\tilde{C}$ be the strict transform of $C$ by the blow up of $M$ along $o$. Then

$$m_{\tilde{C}} < m_C \quad \text{or} \quad m_{\tilde{C}} = m_C \quad \text{and} \quad n_{\tilde{C}} < n_C.$$ 

Hence the invariant improves. The facts above show that there is an integer $k$ such that $\text{Sing}^N(S_k)$ is a union of smooth curves. Hence there is an integer $\ell$ such that $\text{Sing}^N(S_\ell)$ is a union of smooth curves transversal to $N_\ell$. 

Let \( C, C' \) be two curves of \( M \). Let \( m(C, C') \) be the number of blow ups necessary to separate \( C \) and \( C' \). Let \( \tilde{C} [\tilde{C}'] \) be the strict transform of \( C[C'] \) by the blow up of \( M \) along a smooth line that contains \( o \). Then
\[
m(\tilde{C}, \tilde{C}') \leq m(C, C').
\]

Let \( \tilde{C} [\tilde{C}'] \) be the strict transform of \( C[C'] \) by the blow up of \( M \) along \( o \). Then
\[
m(\tilde{C}, \tilde{C}') < m(C, C').
\]

Hence there is an integer \( m \) such that each connected component of \( \text{Sing}^{N_m}(S_m) \) is a smooth curve transversal to \( N_m \).

**Theorem 24.** Let \( M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_k \) be the sequence of morphisms described in paragraph \( \ref{para22} \). Then \( \Sigma^N_\sigma(S) = \Sigma^N_\sigma \) if and only if one of the following statements holds.

(a) somewhere along the process a curve that is not well behaved is produced,
(b) there is \( \sigma \in (S_k \cap N_k) \setminus \text{Sing}^{N_k}(S_k) \) such that \( \sigma \in \Xi^N_\rho(S) \) for some projection \( \rho \) compatible with \( N \),
(c) there is \( \sigma \in S_k \cap N_k \cap \text{Sing}^{N_k}(S_k) \) such that \( \sigma \in \Xi^N_\rho(S) \setminus \text{Sing}^N(S) \) for some projection \( \rho \) compatible with \( N \).

**Proof.** Assume (a) holds. Then there is an integer \( \ell \) such that a non well behaved curve is produced along the second sequence of blow ups \( M_\ell \leftarrow M_{\ell+1} \). By Theorem \( \ref{thm17} \) there is \( \sigma_\ell \in S_\ell \cap N_\ell \) such that \( \Sigma^N_{\sigma_\ell}(S_\ell) = \Sigma^N_{\sigma_\ell} \). We prove by induction in \( n \), using Theorem \( \ref{thm17} \) that for each \( n \leq \ell \) there is \( \sigma_{\ell-n} \in S_{\ell-n} \cap N_{\ell-n} \) such that \( \Sigma^N_{\sigma_{\ell-n}}(S_{\ell-n}) = \Sigma^N_{\sigma_{\ell-n}} \).

Assume (b) [(c)] holds. By Theorem \( \ref{thm6} \) [Theorem \( \ref{thm7} \)] there is \( \sigma_k \in S_k \cap N_k \) such that \( \Sigma^N_{\sigma_k}(S_k) = \Sigma^N_{\sigma_k} \). We repeat the argument of the previous paragraph.

Assume (a), (b), (c) do not hold. Since (b), (c) do not hold, \( \Sigma^N_{\sigma_k}(S_k) \) is finite for each \( \sigma \in S_k \cap N_k \). We can now show by induction in \( \ell \), using Theorem \( \ref{thm17} \) and the fact that (a) does not hold, that \( \Sigma^N_{\sigma_k-\ell}(S_{k-\ell}) \) is finite for each \( \sigma \in S_{k-\ell} \cap N_{k-\ell} \) and each \( \ell \leq k \).

**Theorem 25.** Let \( S \) be a surface of the germ of a complex manifold \( M \) at a point \( o \). Assume \( \Sigma_\sigma(S) \) is finite. Let us blow up \( M \) at \( o \). Let us apply the procedure described in paragraph \( \ref{para22} \) at each singular point \( \sigma \) of the strict transform \( S_0 \) of \( S \) that belongs to the exceptional divisor of the blow up. We obtain is this way a manifold \( M_n \), a normal crossings divisor \( N_n \) and a surface \( S_n \) such that at each point \( \sigma \) of \( S_n \cap N_n \), the germ of \( S_n \) at \( \sigma \) is a quasi ordinary singularity relative to a projection \( \rho \) compatible with \( N_n \). Moreover, \( \Delta_\rho S_n \cup \rho(N_n) \) is a normal crossings divisor at \( \rho(o) \).

**Proof.** By Lemma \( \ref{lem23} \), \( \text{Sing}^{N_k}(S_k) \) is smooth and transversal to \( N_k \) at smooth points of \( N_k \).

Let \( \sigma \in S_k \cap N_k^\prime \). The procedure of paragraph \( \ref{para22} \) relies on the procedure of paragraph \( \ref{para10} \). Therefore \( \Delta_{\rho_k} S_k \subset \rho_k(N_k) \) for each projection \( \rho_k \).
compatible with the germ of $N_k$ at $\sigma$. Therefore $S_k$ is quasi ordinary at the singular points of $N_k$.

By Theorem 24, $\Sigma^{N_k}_\sigma(S_k)$ is finite for each $\sigma \in S_k \cap N_k$. By Theorem 0, $S_k$ is quasi ordinary at the regular points of $N_k$ that do not belong to $\text{Sing}^{N_k}(S_k)$. By Theorem 7, $S_k$ is quasi ordinary at the points of $\text{Sing}^{N_k}(S_k) \cap N_k$.

Let $M$ be an affine chart with coordinates $(x, y, z)$. Let $\rho : M \rightarrow X$ be the linear projection $(x, y, z) \mapsto (x, y)$. Set $o = (x, y, z)$, $\sigma = (0, 0)$, $\Delta = o$. By Theorem 1.4.4.1 of [6], $(0 : 0 : 1) \in C$.

Example 26. Let $S$ be the surface defined by the polynomial

$$z^2 - x^2(x + y^2).$$

Notice that $C_o(S) = \{z = 0\}$ and $C_o(\Delta_z(S)) = C_o(\rho(\text{Sing}(S))) = \{x = 0\}$. By Theorem 1.4.4.1 of [6], $(0 : 0 : 1) \in \Sigma_o(S) \subset \Sigma_\ell$, where

$$\ell = \{x = z = 0\} \quad \text{and} \quad \Sigma_\ell = \{\eta = 0\}.$$

Set $x = x_1y_1$, $y = y_1$, $z = y_1z_1$. Then $o_\ell$ is the origin of the chart $W_1$ with coordinates $(x_1, y_1, z_1)$. Moreover, $N_1 \cap W_1 = \{y_1 = 0\}$ and $S_1 \cap W_1$ is defined by the polynomial

$$z_1^2 - x_1^2y_1(x_1 + y_1).$$

Since $C_{o_\ell}(S_1) = \{z_1 = 0\}$, $S_1$ is non degenerated at $o_\ell$.

Let $\pi_2 : M_2 \rightarrow M_1$ be the blow up of $M_1$ at $o_\ell$.

Set $x_1 = x_2y_2$, $y_1 = y_2$, $z_1 = y_2z_2$. If $W_2$ is the affine chart with coordinates $(x_2, y_2, z_2)$, $N_2 \cap W_2 = \{y_2 = 0\}$ and $S_2 \cap W_2$ is defined by the polynomial

$$z_2^2 - x_2^2y_2(x_2 + 1).$$

Since $\Delta_{x_2}(S_2) = \{x_2y_2(x_2 + 1)\}$, $\Sigma^{N_2}_{(1, 0, 0)}(S_2) = \{\eta = 0\}$. Hence $C_o(S) = \Sigma_\ell$.

Example 27. Let $S$ be the swallowtail surface, defined by the polynomial

$$256z^3 - 27y^4 - 128x^2z^2 + 144xy^2z + 16x^4x - 4x^3y^2.$$

Notice that $C_o(S) = \{z = 0\}$ and $C_o(\Delta_z(S)) = C_o(\rho(\text{Sing}(S))) = \{y = 0\}$. By Theorem 1.4.4.1 of [6], $(0 : 0 : 1) \in \Sigma_o(S) \subset \Sigma_\ell$, where

$$\Sigma_\ell = \{\xi = 0\}.$$

Set $x = x_1$, $y = x_1y_1$, $z = x_1z_1$. Then $o_\ell$ is the origin of the chart $W_1$ with coordinates $(x_1, y_1, z_1)$. Moreover, $N_1 \cap W_1 = \{x_1 = 0\}$ and $S_1 \cap W_1$ is defined by the polynomial

$$256z_1^3 - 27x_1y_1^4 - 128x_1z_1^2 + 144x_1y_1^2z_1 + 16x_1^2z_1 - 4x_1^2y_1^2.$$
Since \( C_{o_1}(S_1) = \{ z = 0 \} \), \( S_1 \) is non degenerated at \( o_1 \).

Let \( \pi_2 : M_2 \rightarrow M_1 \) be the blow up of \( M_1 \) at \( o_1 \).

Set \( x_1 = x_2, y_1 = x_2y_2, z_1 = x_2z_2 \). If \( W_2 \) is the chart with coordinates \((x_2, y_2, z_2)\), \( N_2 \cap W_2 = \{ x_2 = 0 \} \) and \( S_2 \cap W_2 \) is defined by the polynomial

\[
256z_2^3 - 27x_2^2y_2^4 - 128z_2^2 + 144x_2y_2^2z_2 + 16z_2 - 4x_2y_2^2.
\]

Since \( S_2 \cap N_2 \cap W_2 = \{ x_2 = z_2(4z_2 - 1) = 0 \} \) and \( \Delta_{x_2}(S_2) = \{ x_2y_2(27x_2y_2^2 + 8) = 0 \} \), we need to analyse the points \( o_1 = (0, 0, 0) \) and \( o_2 = (0, 0, 1/4) \). Since \( \text{Sing}_{N_2}(S_2) \) equals \( \{ y_2 = 4z_2 - 1 = 0 \} \) in a neighbourhood of \( N_2 \), it follows from Theorem 6 that \( \Sigma_{o_2}^{N_2}(S_2) \) is finite. Since \( m_{(a, 0, 1/4)}(S_2) \) does not depend on \( a \), for \( |a| << 1 \), it follows from Theorem 7 that \( \Sigma_{o_2}^{N_2}(S_2) \) is finite.

Set \( x_1 = x_3y_3, y_1 = y_3, z_1 = x_3z_3 \). If \( W_3 \) is the chart with coordinates \((x_3, y_3, z_3)\), \( N_2 \cap W_3 = \{ x_3y_3 = 0 \} \) and \( S_2 \cap W_3 \) is defined by the polynomial

\[
256z_3^3 - 27x_3y_3^4 - 128x_3z_3^2 + 144x_3y_3z_3 + 16x_3^2z_3 - 4x_3^2y_3.
\]

Set \( o_3 = (0, 0, 0) \). Since the intersection of \( S_2 \) with the singular locus \( N_2^\sigma \) of \( N_2 \) equals \( o_3 \), we need to compute \( \Sigma_{o_3}^{N_2}(S_2) \).

Let \( \pi_3 : M_3 \rightarrow M_2 \) be the blow up of \( M_2 \) along \( N_2^\sigma \).

Set \( x_3 = x_4y_4, y_3 = y_4, z_3 = z_4 \). If \( W_4 \) is the chart with coordinates \((x_4, y_4, z_4)\), \( E_3 \cap W_4 = \{ y_4 = 0 \} \), \( N_3 \cap W_4 = \{ x_4y_4 = 0 \} \) and \( S_3 \cap W_4 \) is defined by the polynomial

\[
256z_4^3 - 27x_4y_4^3 - 128x_4y_4z_4^2 + 144x_4y_4^2z_4 + 16x_4^2y_4z_4 - 4x_4^2y_4.
\]

Since \( S_3 \cap E_3 \cap W_4 = \{ z_4 = 0 \} \) and \( \Delta_{z_4}(S_3) = \{ x_4y_4(8x_4 - 27) = 0 \} \), we need to analyse the points \( o_4 = (0, 0, 0) \) and \( o_5 = (-27/8, 0, 0) \). By Lemma 4, \( \Sigma_{o_3}^{N_3}(S_3) \) is finite.

Set \( x_5 = x_4 - 27/8, y_5 = y_4, z_5 = z_4 + 9y_4/32 + x_4y_4/36 \). There are integers \( a_1, ..., a_5 \) such that \( S_3 \) is defined near \( o_5 \) by the polynomial

\[
a_1z_5^3 + a_2y_5z_5^2 + a_3x_5^2y_5z_5 + a_4x_5^2y_5^2z_5 + a_5x_5y_5z_5^2.
\]

Now \( \text{Sing}_{N_3}(S_3) = \{ y_5 = z_5 = 0 \} \) and it is easy to check that \( m_\sigma(S_3) \) does not depend on \( \sigma \in \text{Sing}_{N_3}(S_3) \). By Theorem 7, \( \Sigma_{o_5}^{N_3}(S_3) \) is finite. By Theorem 24, \( \Sigma_{o}^{N}(S) = \{(0 : 0 : 1)\} \).

**Example 28.** Let \( N = \{ x = 0 \} \), \( o = (0, 0, 0) \) and \( \sigma = (0, 0) \). Let \( S \) be the surface defined by the polynomial

\[
z^5 - x^2y.
\]

The surface \( S \) is degenerated at \( o \). We will perform the second sequence of blow ups at \( o \). Let \( \pi_1 : M_1 \rightarrow M \) be the blow up of \( M \) at \( o \).

Set \( x = x_1y_1, y = y_1, z = y_1z_1 \). If \( W_1 \) is the affine open set of \( M_1 \) with coordinates \((x_1, y_1, z_1)\), \( S_1 \cap W_1 \) is defined by the polynomial

\[
x_1^2 - y_1^2z_1^5
\]

and \( S_1 \cap E_1 \cap W_1 = N_1^\sigma \cap W_1 = \{ x_1 = y_1 = 0 \} \).
Set $x = x_2 z_2$, $y = y_2 z_2$, $z = z_2$. If $W_2$ is the affine open set of $M_1$ with coordinates $(x_2, y_2, z_2)$, $S_1 \cap W_2$ is defined by the polynomial $z_2^2 - x_2^2 y_2$.

$S_1 \cap E_1 \cap W_2 = \{z_2 = x_2 y_2 = 0\}$ and $N_1^\sigma \cap W_2 = \{x_2 = z_2 = 0\}$.

The surface $S_1$ is smooth in the other chart.

Let $\pi_2 : M_2 \to M_1$ be the blow up of $M_1$ along $N_1^\sigma$. Set $E_2 = \pi_2^{-1}(N_1^\sigma)$. Notice that $\pi = \pi_1 \circ \pi_2 : M_2 \to M_1$ is the second sequence of blow ups of $S$. Since $S_2$ is quasi ordinary at each point it is quite easy to verify that $\Sigma_{\sigma}^{N_2}(S_2)$ is finite for each $\sigma \in S_2$.

It follows from Theorem [24] that $\Sigma_\sigma(S) = \Sigma_\sigma^N$ because the curve $S_2 \cap E_2$ has a singularity, hence one of its irreducible components is not well behaved.

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the projects UID/MAT/04561/2013 (CMAF-CIU) and UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

References

[1] A. Araujo and O. Neto - Limits of Tangents of a Quasi-Ordinary Hypersurface, Proc. Amer. Math. Soc., Vol. 143 (2013), pp 1-11.
[2] A. Araujo, J. Cabral and O. Neto - Desingularization of Legendrian Surfaces, Submitted, [arXiv:1510.09126] [math.AG]
[3] J. Cabral, O. Neto and P.C.Silva - On the Resolution Graph of a Plane Curve, Submitted, [arXiv:1409.3048] [math.AG]
[4] J.-P. Henry, Lê Dung Trang, Limites d’espaces Tangents, Seminaire Norguet, Springer Lecture Notes 482.
[5] Lê D. T., Limites d’espaces Tangents sur les surfaces, Proc. Symp. on Algebraic singularities, Rheinhardtsbrunn 1978, Academia Leopoldina.
[6] Lê D. T., B. Teissier, Sur la Géometrie des Surfaces Complexes I, Tangentes Exceptionnelles, Am. Journal of Math., Vol. 101, pp 420-452.
[7] J. Lipman - Quasi-ordinary singularities of embedded surfaces, Ph.D. thesis, Harvard University, 1965.
[8] O. Neto - Blow up for a Holonomic System, Publ. Res. Inst. Math. Sci. 29 (1993), no. 2, 167-233.
[9] O. Neto - Equisingularity and Legendrian Curves, Bull. London Math. Soc. 33 (2001), 527-534.

Centro de Matemática e Aplicações (CMA), FCT, UNL and Departamento de Matemática, FCT, UNL, 2829-516 Caparica, Portugal

E-mail address: jpb@fct.unl.pt

CMAF, Faculdade de Ciências, Universidade de Lisboa, Campo Grande 1749-016, Lisboa, Portugal

E-mail address: orlando60@gmail.com