Pseudo-Character Expansions for $U(N)$-Invariant Spin Models on $CP^{N-1}$

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Abstract

We define a set of orthogonal functions on the complex projective space $CP^{N-1}$, and compute their Clebsch-Gordan coefficients as well as a large class of 6–$j$ symbols. We also provide all the needed formulae for the generation of high-temperature expansions for $U(N)$-invariant spin models defined on $CP^{N-1}$.

**KEY WORDS:** $\sigma$-model, $CP^{N-1}$ model, hyperspherical harmonics, spherical functions, Clebsch-Gordan coefficients, 6–$j$ symbols.
1 Introduction

Spin systems have been the subject of intense research for a long time. Indeed they describe the universal features of many different phenomena. For instance, the $N$-vector model — a generalization of the Ising model — describes the critical behavior of dilute polymers, ferromagnets, binary fluids, liquid helium, and of the Higgs sector of the Standard Model at finite temperature [1, 2, 3]. From the point of view of quantum field theory, the $N$-vector model provides the simplest example for the realization of a non-Abelian global symmetry. In particular, its two-dimensional version has been extensively studied because it shares with four-dimensional gauge theories the property of being asymptotically free in the weak-coupling perturbative expansion [4, 5, 6, 7].

A generalization of the $N$-vector model is provided by $CP^{N-1}$-models [8, 9, 10, 11, 12, 13] (for recent work see [14, 15, 16] and references therein). Their two-dimensional version, besides being asymptotically free and invariant under global $U(N)$-transformations, shows some additional features that are expected to hold (though very difficult to prove) in the QCD case. Indeed, $CP^{N-1}$-models have a local $U(1)$-gauge invariance, dynamical appearance of a linear confining potential between non-gauge-invariant states, and a non-trivial topological structure with instantons, anomalies and $\theta$-vacua. Besides being of interest in high-energy physics, the $CP^{N-1}$-models have applications in condensed-matter physics. Indeed, they describe models that have a complex vector order parameter (see e.g. [17]).

Spin systems can be studied in a variety of ways: predictions can be obtained using field-theory methods — based essentially on perturbation theory [3] — or using numerical techniques, e.g. Monte Carlo simulations or extrapolations of high-temperature series. The latter method (see for instance [18]) has proved very powerful in providing extremely accurate estimates of critical parameters. For spin systems, such as the $N$-vector model, precise results [19, 20] are obtained even in the case of asymptotically free models, for which $\beta_c = +\infty$.

Various methods have been used to generate high-temperature expansions for the $N$-vector model (for a review, see [21, 22]). The most straightforward method is the cumulant approach, or the closely related finite-cluster approach [23, 24, 25]. An improved method is the star-graph expansion, in which only star graphs need to be generated [26]. A further improvement was originally introduced by Joyce [27]. He proposes to expand the interaction term in hyperspherical harmonics. This allows a fast computation of the partition function for all star graphs. The main difficulty of the method is the computation of the Clebsch-Gordan coefficients, $6–j$ symbols and higher-order group invariants. Domb and coworkers [28, 29, 30] developed an algebraic method to compute the group factors associated to each graph. In Ref. [31] explicit expressions for the Clebsch-Gordan and the $6–j$ symbols were presented. Making use of simple algebraic rules [32] these general results allow the determination of all the group factors that are needed in the generation of high-temperature series of present-day length. These techniques have been recently used to generate long series in two and three dimensions [19, 32, 33]. We should mention that there exists another method that is extremely efficient in the generation of high-temperature series, the
linked-cluster technique [34, 35, 36, 37], which is an expansion in terms of free graphs. In this paper we will generalize the results of Ref. [31] to $CP^{N-1}$-models. We will use a representation of the hyperspherical harmonics\(^1\) in terms of completely symmetric and traceless tensors. The computation of the Clebsch-Gordan coefficients is then a straightforward combinatoric exercise. Moreover, using the same technique, we are able to compute a very large class of 6–j symbols, essentially all symbols that are needed in the generation of high-temperature series of present-day length.

The paper is organized as follows: in Sect. 2 we introduce the hyperspherical harmonics on $CP^{N-1}$ and discuss some general properties. General results and formulae for Clebsch-Gordan coefficients and 6–j symbols are reported in Sect. 3. In Sect. 4 we discuss the applications and report the high-temperature expansions of the Gibbs weight for the Hamiltonians that are used in studies of $CP^{N-1}$ models. The derivations of the results are reported in the Appendix.

2 $CP^{N-1}$-Hyperspherical Harmonics

In this section we introduce a class of orthogonal functions, which we call hyperspherical harmonics, defined on $CP^{N-1}$. These functions will provide the basis for expanding the Gibbs weight for $U(N)$-invariant spin models.

Let us first define $CP^{N-1}$. Its usual definition is the following: consider $w, z \in \mathbb{C}^N - \{0\}$ and introduce the following equivalence relation:

\[
\text{[Rel1 \ }] \ w \sim z \text{ if and only if there exists } a \in \mathbb{C} \text{ such that } w = az.
\]

Then

\[
CP^{N-1} = (\mathbb{C}^N - \{0\})/\sim. \tag{2.1}
\]

A second definition is the following: consider vectors $w$ belonging to the $(2N - 1)$-dimensional complex sphere, i.e. $w \in \mathbb{C}^N$ such that $\overline{w} \cdot w = 1$, and introduce the following equivalence relation:

\[
\text{[Rel2 \ }] \ w \sim z \text{ if and only if there exists } a \in \mathbb{C}, |a| = 1, \text{ such that } w = az.
\]

Then

\[
CP^{N-1} = \{z \in \mathbb{C}^N : \overline{z} \cdot z = 1\}/\sim. \tag{2.2}
\]

This second definition is obviously equivalent to the previous one. In our work we will use this representation.

We mention a third definition that immediately shows that $CP^{N-1}$ is a symmetric space of rank one (see e.g. Refs. [39, 42]). Given $\rho = (1,0,\ldots,0)$, notice that any $z \in \mathbb{C}^N$ such that $\overline{z} \cdot z = 1$ can be written as $z = U \rho$, with $U \in U(N)$. This representation is not unique: if $U, V \in U(N)$ are such that $z = U \rho = V \rho$, then we

\(^1\)These functions generalize the usual spherical harmonics for the three-dimensional rotation group [38]. Such a generalization is well-known for any symmetric space [39, 40, 41].
can write $\rho = U^+ V \rho$. It is trivial to see that the set of $R \in U(N)$ such that $\rho = R \rho$ is isomorphic to $U(N - 1)$. Therefore we obtain the result

$$\{z \in C^N : \bar{z} \cdot z = 1\} \simeq U(N)/U(N - 1).$$

(2.3)

It follows easily

$$CP^{N-1} = SU(N)/S(U(N - 1) \times U(1)).$$

(2.4)

It is interesting to observe that for $N = 2$, $CP^1 \simeq S^2$. The mapping is obtained as follows. Given $z \in C^2$ such that $\bar{z} \cdot z = 1$, consider

$$s^i \equiv \bar{z}_i (\sigma^i)^{\alpha}_\beta z^\beta,$$

(2.5)

where $\sigma^i$ are the Pauli matrices. Using the completeness relation

$$(\sigma^i)^{\alpha}_\beta (\sigma^i)^{\gamma}_\delta = 2 \delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma,$$

(2.6)

it is easy to verify that $s^i s^i = 1$, i.e. $s \in S^2$. Let us now show that the mapping is bijective. Given $s \in S^2$, if $t = (0,0,1)$, there exists $R \in SO(3)$, such that $R^{\alpha}_i t^\alpha = R^3 = s^i$. Now given $R \in SO(3)$, it is well known that there exists $U \in SU(2)$ such that $(U + \sigma^i U)^{\alpha}_\beta = R^\alpha (\sigma^i)^{\alpha}_\beta$. Let us consider $z^\alpha \equiv U^\alpha_\beta x^\beta$, with $x = (1,0)$. Substituting this expression for $z$ in Eq. (2.5), it is easy to see that one obtains $s^i = R^3$. Finally, suppose that $z_1$ and $z_2$ are unimodular and both satisfy Eq. (2.5) with the same $s^i$. Using Eq. (2.6) it is then easy to verify that

$$|z_1 \cdot z_2|^2 = \frac{1}{2} (1 + s^i s^i) = 1 = |z_1|^2 |z_2|^2.$$

(2.7)

This implies $z_1 = e^{i\alpha} z_2$, for some $\alpha \in \mathbb{R}$. Thus, $z_1$ and $z_2$ are equivalent in $CP^{N-1}$.

Let us now define functions on $CP^{N-1}$. It is obvious that a function $f$ on $CP^{N-1}$ (defined in Eq. (2.2)) extends naturally to a function $\hat{f}$ on the complex sphere. Indeed, it is enough to define $\hat{f}(z) = f(\{z\})$, where $\{z\}$ is the equivalence class (defined by [Rel2]) to which $z$ belongs. On the other hand, a given function $f$ on the complex sphere defines a function on $CP^{N-1}$ if and only if $f(z) = f(w)$ for $z \sim w$, or — in physicists’ language — if $f(z)$ is invariant under the $U(1)$-gauge transformations $z \rightarrow e^{i\alpha} z$, i.e.

$$f(e^{i\alpha} z, e^{-i\alpha} \bar{z}) = f(z, \bar{z}),$$

(2.8)

for any $\alpha$. In the following we will identify the space of functions on $CP^{N-1}$ with the space of $U(1)$-gauge-invariant functions defined on the complex $(2N-1)$-dimensional sphere. We also introduce a normalized $U(N)$-invariant measure

$$d\Omega(z, \bar{z}) = \frac{2^{-N}}{S_{2N}} d\bar{z} d\bar{z} \delta(\bar{z} \cdot z - 1),$$

(2.9)

where $S_{2N}$ is the surface area of the sphere $S^{2N-1}$

$$S_{2N} = \frac{2\pi^N}{(N-1)!}.$$

(2.10)

\footnote{In this paper we will use the usual summation convention over repeated indices.}
Finally, we consider the unitary representation $T(R)$ of $SU(N)$ on $L^2(CP^{N-1})$ defined by $(T(R)f)(z) = f(R^{-1}z)$. We want to find an orthogonal Hilbert-space decomposition of $L^2(CP^{N-1})$ into subspaces such that the representation $T(R)$ restricted to each subspace is irreducible. The needed decomposition turns out to be precisely the decomposition of $L^2(CP^{N-1})$ into eigenspaces of the Laplace-Beltrami operator $\mathcal{L} = \mathcal{L}_{CP^{N-1}}$. In fact, it can be proved that:

(a) The eigenvalues of $\mathcal{L}$ are

$$\lambda_{N,k} = k(N + k - 1) \geq 0,$$

where $k = 0, 1, 2, \ldots$. The corresponding eigenspace $E_{N,k}$ has dimension

$$\mathcal{N}_{N,k} \equiv \dim E_{N,k} = (N-1)(N-1+2k)\left[\frac{(N+k-2)!}{k!(N-1)!}\right]^2,$$

and can be given several equivalent descriptions:

(i) $E_{N,k}$ consists of the restrictions to the complex sphere $\overline{z} \cdot z = 1$ of the $U(1)$-gauge invariant polynomials $P_k(z, \overline{z})$ that are homogeneous of degree $k$ in $z, \overline{z} \in \mathbb{C}^N$ and that satisfy Laplace’s equation $\partial \overline{\partial} P_k = 0$ in $\mathbb{C}^N$.

(ii) Consider the tensors $Y_{N,k,\alpha_1,\ldots,\alpha_k}^{(N,k)}(z)$ of rank $k$ in $z$ and $\overline{z}$ that are completely symmetric under exchange of the $\alpha$ and $\beta$ indices and that are traceless, i.e. vanish under any contraction of an $\alpha$-index with a $\beta$-index: $Y_{N,k,\gamma\beta_1^{\gamma},\ldots,\beta_k^{\gamma}}^{(N,k)}(z) = 0$. We call these tensors spin-$k$ hyperspherical harmonics (they are described in more detail below). $E_{N,k}$ is then spanned by the tensors $Y_{N,k,\alpha_1,\ldots,\alpha_k}^{(N,k)}$ as the indices range over the $N^{2k}$ allowable values.

(b) Each eigenspace $E_{N,k}$ is left invariant by $T(R)$. Moreover, the representation $T(R)|_{\mathcal{E}_{N,k}}$ of $SU(N)$ is irreducible.

(c) $L^2(CP^{N-1}) = \bigoplus_{k=0}^{\infty} E_{N,k}$ (orthogonal Hilbert space decomposition).

To make all this concrete, we can write:

$$Y_{N,k,\alpha_1,\ldots,\alpha_k}^{(N,k)}(z) \equiv \mu_{N,k} \left[z^{\alpha_1} \ldots z^{\alpha_k} \overline{z}_{\beta_1} \ldots \overline{z}_{\beta_k} - \text{Traces}\right],$$

where $z \in \mathbb{C}^N$ with $\overline{z} \cdot z = 1$, “Traces” is such that $Y_{N,k}$ is completely symmetric and traceless and

$$\mu_{N,k} = \frac{1}{k! \left[\frac{(N-1+2k)!}{(N-1)!}\right]^{1/2}}.$$

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3See e.g. Refs. [39, 40, 41, 43].

4 The Laplace-Beltrami operator on $CP^{N-1}$ can be defined as the Casimir operator of the unitary representation of $SU(N)$ on $L^2(CP^{N-1})$. See e.g. Ref. [43].

5See [41], Theorem 13.1, p. 231 and Corollary 14.4, p. 243.

6Note that our $\mathcal{L}$ is the negative of the usual Laplacian, i.e. it is a positive-semidefinite operator.

7 For a proof see Appendix A.1 below. Note that $\mathcal{N}_{N,0} = 1$ and $\mathcal{N}_{N,1} = N^2 - 1$ for all $N$. 
Explicit examples are [the general formula is given in equation (A.12)] :

\[ Y_{N,0}(z) = 1; \]  
\[ Y_{N,1,\beta}(z) = \sqrt{N(N+1)} \left( z^\alpha z_\beta - \frac{1}{N} \delta^\alpha_\beta \right); \]  
\[ Y_{N,2,\gamma,\delta}(z) = \frac{1}{2} \sqrt{N(N+1)(N+2)(N+3)} \left[ z^\alpha z^\beta z_\gamma z_\delta - \frac{1}{N+2} \left( \delta^\alpha_\gamma z^\beta z_\delta + 3 \text{ permutations} \right) \right. \right. \left. \left. + \frac{1}{(N+1)(N+2)} \left( \delta^\alpha_\gamma \delta^\beta_\delta + \delta^\alpha_\delta \delta^\beta_\gamma \right) \right] . \]  

We note that for \( N = 2 \) we have \( \lambda_{2,k} = k(k+1) \) and \( \mathcal{N}_{2,k} = 2k+1 \), as expected since \( CP^1 \simeq S^2 \). Moreover, using the mapping (2.5), one can show that the \( Y \)'s are linear combinations of the usual spherical harmonics. They can also be related to the real-valued hyperspherical harmonics \( Y_{\text{sphere}} \) introduced in Ref. [31] by

\[ (Y_{\text{sphere}})^{i_1 \ldots i_k}_{3,k}(s) = 2^{-k/2} Y_{2,k+i_1 \ldots i_k}(z)(\sigma_i^{i_1})^\beta_\alpha \ldots (\sigma_i^{i_k})^\beta_\alpha , \]  

where \( s \) is related to \( z \) by Eq. (2.5) and we have used multiindex notation \( (\alpha)_k = \alpha_1 \ldots \alpha_k \). For \( N = 2 \) we can also prove the identity

\[ Y_{2,k+i_1 \ldots i_k}(z) = (-1)^k Y_{2,k+i_1 \ldots i_k}(w), \]  

where \( w \) is defined by: \( w^\alpha = \epsilon^\alpha_\beta \overline{z}_\beta, \overline{w}_\alpha = \epsilon_\alpha_\beta z^\beta \). Here \( \epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} \) is the completely antisymmetric tensor satisfying \( \epsilon^{12} = \epsilon_{12} = 1 \). In group-theoretical terms, Formula (2.19) is related to the fact that \( SU(2) \)-representations are self-conjugate.

The normalization \( \mu_{N,k} \) is chosen so that the following orthogonality relation holds (see Appendix A.1):

\[ \int d\Omega(z,\overline{z}) \ Y_{N,k+i_1 \ldots i_k}(z)Y_{N,k+i_1 \ldots i_k}(\overline{z}) = \delta_{kl} I_{N,k+i_1 \ldots i_k}, \]  

where \( I_{N,k} \) is the “identity” in the space of spin-\( k \) symmetric and traceless tensors, i.e. the unique orthogonal projector onto the space of completely symmetric and traceless tensors of rank \( k \). The trace of this operator is given by

\[ I_{N,k+i_1 \ldots i_k} = \mathcal{N}_{N,k} \equiv \dim E_{N,k} , \]  

as of course it must be. We remark that \( Y_{N,k}(z) \cdot Y_{N,k}(\overline{z}) \equiv Y_{N,k+i_1 \ldots i_k}(z)Y_{N,k+i_1 \ldots i_k}(\overline{z}) \) is independent of \( z \) [by \( U(N) \) invariance], and hence

\[ Y_{N,k}(z) \cdot Y_{N,k}(z) = \mathcal{N}_{N,k} \]  

by (2.20) and (2.21).
As stated in the theorem given at the beginning of this section, the hyperspherical harmonics are a complete set of functions on $L^2(CP^{N-1})$. Thus any $U(1)$-gauge-invariant function $f(z)$ can be expanded as

$$f(z) = \sum_{k=0}^{\infty} \tilde{f}_{k,(\beta)_k}^{(\alpha)_k} Y_{N,k,(\alpha)_k}(z),$$

where

$$\tilde{f}_{k,(\beta)_k}^{(\alpha)_k} = \int d\Omega(z, \bar{z}) f(z) Y_{N,k,(\beta)_k}^{(\alpha)_k}(z).$$

For smooth functions this expansion converges very fast. Indeed, if $f(z)$ is infinitely differentiable, then as $k \to \infty$ the coefficients of the expansion go to zero faster than any inverse power of $k$ (see Appendix A.2).

The completeness of the hyperspherical harmonics can be expressed through the relation

$$\sum_{k=0}^{\infty} Y_{N,k,(\alpha)_k}(z) Y_{N,k,(\beta)_k}^{(\alpha)_k}(w) = \delta(z, w)$$

where the $\delta$-function is defined with respect to the measure $d\Omega(z, \bar{z})$.

Finally, let us consider a $U(N)$-invariant function $f$ defined on $CP^{N-1}$. Let $f$ be a function of two “spins” $z, w$, i.e. a function of $|z \cdot w|$. We want now to compute its expansion in terms of hyperspherical harmonics. Using Schur’s lemma we can write

$$f(|z \cdot w|) = \sum_{k=0}^{\infty} F_{N,k} Y_{N,k}(z) \cdot Y_{N,k}(w).$$

The “Traces” terms can be dropped from either one of the $Y$’s in the scalar product above, since the other $Y$ is traceless. Also, since the scalar product is $U(N)$-invariant, we can rotate $z$ to $\rho = (1, 0, \ldots, 0)$, and correspondingly rotate $w$ to some $v$ with $|z \cdot w| = |\rho \cdot v| = |v^1|$. In this way we obtain

$$Y_{N,k}(z) \cdot Y_{N,k}(w) = Y_{N,k}(\rho) \cdot Y_{N,k}(v) = \mu_{N,k} Y_{N,k,1,\ldots,1}(v).$$

Now $Y_{N,k,1,\ldots,1}(v)$ can be expressed in terms of Jacobi polynomials as (see Appendix A.1)

$$Y_{N,k,1,\ldots,1}(v) = \frac{N_{N,k}}{\mu_{N,k}} \frac{P_k^{(N-2,0)}(2|v^1|^2 - 1)}{P_k^{(N-2,0)}(1)},$$

and therefore

$$Y_{N,k}(z) \cdot Y_{N,k}(w) = \frac{N_{N,k}}{\mu_{N,k}} \frac{P_k^{(N-2,0)}(2|\bar{z} \cdot w|^2 - 1)}{P_k^{(N-2,0)}(1)}.$$

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8Note that the normalization here follows directly from the one defined for (2.20).

9This expression corresponds to the relation between $Y_{l0}$ and Legendre polynomials for the usual spherical harmonics. For definitions and properties for Jacobi polynomials see [44], pp. 1035–1037.
If we now invert Eq. (2.26) using the orthogonality relation (2.20) and Eq. (2.22) we get
\[ F_{N,k} = \int d\Omega(z, w) f(|z \cdot w|) \frac{Y_{N,k}(z) \cdot Y_{N,k}(w)}{N_{N,k}}. \] (2.30)

This can be rewritten, using the $U(N)$-invariance of the measure and Eq. (2.29), as
\[ F_{N,k} = \int d\Omega(z, w) f(|z|^1) \frac{P_k^{(N-2,0)}(2|z|^2 - 1)}{P_k^{(N-2,0)}(1)}. \] (2.31)

Now the integrand depends only on $z^1$ and we can integrate out the other coordinates. We finally obtain
\[ F_{N,k} = (N - 1) \int_0^1 dt \, (1 - t)^{N-2} f(\sqrt{t}) \frac{P_k^{(N-2,0)}(2t - 1)}{P_k^{(N-2,0)}(1)}. \] (2.32)

From the general properties of the hyperspherical harmonics we can derive the following properties of the coefficients $F_{N,k}$ (for the proofs of properties 1 and 2, see Appendix A.2):

1. If $f(t)$ is positive\(^{10}\) for $t \in [0, 1]$, then $|F_{N,k}| < F_{N,0}$ for all $k \neq 0$.

2. If $f(\sqrt{t})$ is smooth (i.e. $C^\infty$), then $\lim_{k \to \infty} k^n F_{N,k} = 0$ for every $n$.

3. If $f(t) = t^{2l}$, then the integral in (2.32) can be performed explicitly and the coefficients $F_{N,k}$ are given by\(^{11}\)
\[ F_{N,k}^{(l)} = \begin{cases} \frac{(N - 1)! \, (l)!^2}{(N - 1 + l + k)! \, (l - k)!} & \text{if } k \leq l \\ 0 & \text{otherwise} \end{cases} \] (2.33)

and are, in particular, always nonnegative. It immediately follows that for a generic function of the form
\[ f(t) = \sum_{l=0}^{\infty} f_l t^{2l}, \] (2.34)

the coefficients $F_{N,k}$ are given by
\[ F_{N,k} = \sum_{l=k}^{\infty} f_l F_{N,k}^{(l)}. \] (2.35)

Therefore, if all the coefficients $f_l$ are nonnegative, then so are the $F_{N,k}$.

\(^{10}\)More precisely, it suffices that $f$ be nonnegative and not almost-everywhere-vanishing.

\(^{11}\)See Eq. 11, p. 583, vol. 2 of Ref. [45]. This result can also be obtained using Formula 7.391.3 of [44], once one corrects the following misprint: $\Gamma(\alpha + 1)$ in the numerator should be replaced by $\Gamma(n + \alpha + 1)$. 

8
3 Clebsch-Gordan and $6-j$ Symbols

Let us now compute the Clebsch-Gordan coefficients for the decomposition into irreducible representations of the symmetric product of two generic representations. In general we can write

$$ Y_{N,k;\delta_4}^{(\alpha)_k}(z) Y_{N,l;\delta_4}^{(\gamma)_l}(z) = \sum_m C_{N;k,l,m;\delta_4}^{(\alpha)_k;(\gamma)_l;m} Y_{N,m;\delta_4}^{(\tau)_m}(z). \quad (3.1) $$

Using the orthogonality relations (2.20) we obtain

$$ C_{N;k,l,m;\delta_4}^{(\alpha)_k;(\gamma)_l;m} = \int d\Omega(z,\overline{z}) Y_{N,k;\delta_4}^{(\alpha)_k}(z) Y_{N,l;\delta_4}^{(\gamma)_l}(z) Y_{N,m;\delta_4}^{(\tau)_m}(z). \quad (3.2) $$

This integral can be computed explicitly. We get (see Appendix A.4)

$$ C_{N;k,l,m;\delta_4}^{(\alpha)_k;(\gamma)_l;m} = \frac{(N-1)! \mu_{N,k} \mu_{N,l} \mu_{N,m} k! l! m!}{(N-1+k+l+m)!} \times \sum_{i}^{i_{\max}} \binom{k}{i} \binom{l}{i} \binom{m}{j} \frac{I_{N,k;\delta_4}^{(\alpha)_k;(\nu)_k}(\nu)_{k-i} I_{N,l;\delta_4}^{(\gamma)_l;(\sigma)_l}(\sigma)_{k-h} I_{N,m;\delta_4}^{(\tau)_m;(\rho)_m}(\rho)_{m-i}}{I_{N,l;\delta_4}^{(\gamma)_l;(\eta)_l}(\eta)_{l-i}} \quad (3.3) $$

if $|l-k| \leq m \leq l+k$; the Clebsch-Gordan coefficient vanishes otherwise. In Eq. (3.3) $h = m - l + i$, $j = m - k + i$, $i_{\max} = \min(k,l,l+k-m)$ and $i_{\min} = \max(0,l-m,k-m)$.

In the following we will be interested in the scalar quantity

$$ C_{N;k,l,m}^{2} = C_{N;k,l,m} \cdot C_{N;k,l,m}. \quad (3.4) $$

The general formula is reported in Appendix A.4 [see (A.60)]. A particularly simple case is $m = l + k$:

$$ C_{N;k,l,l+k}^{2} = N_{N,l+k} \frac{\mu_{N,k}^{2} \mu_{N,l}^{2}}{\mu_{N,l+k}^{2}}, \quad (3.5) $$

which can be obtained directly from (3.3), using (2.21). If $k = 1$ this gives

$$ C_{N;1,l,l+1}^{2} = \frac{N(N^{2} - 1)}{(N + 2l)} \left( N + l - 1 \right)^{2}. \quad (3.6) $$

Notice that for $N = 2$, because of the symmetry (2.19), the Clebsch-Gordan coefficients $C_{N;k,l,m;\delta_4}^{(\alpha)_k;(\gamma)_l;m}$ — and therefore also the scalar invariants $C_{N;k,l,m}^{2}$ — vanish if $k + l + m$ is odd.

Let us now derive two important properties of the Clebsch-Gordan coefficients. Using their definition in terms of hyperspherical harmonics and the completeness relation (2.25) we can easily prove the crossing relation

$$ \sum_{p=0}^{\infty} C_{N;p,k,l;\delta_4}^{(\alpha)_p;(\beta)_k;\gamma_l;m} C_{N;p,m,n;\alpha_\mu;\sigma_m;\tau_n}^{(\delta)_p;(\mu)_m;\nu_n} = \sum_{p=0}^{\infty} C_{N;p,k,m;\delta_4}^{(\alpha)_p;(\beta)_k;\gamma_l;m} C_{N;p,l,n;\alpha_\mu;\sigma_m;\tau_n}^{(\delta)_p;(\mu)_m;\nu_n} \quad (3.7) $$
A second relation, which follows immediately from Schur’s lemma, is
\[
\mathcal{C}_{N; k, l, m, (\alpha)k; (\beta)l; (\gamma)m} \mathcal{C}_{N; k, l, n, (\alpha)k; (\beta)l; (\nu)n} = \delta_{nm} \frac{1}{N_{N,m}} \mathcal{I}_{N_{N,m}, (\zeta)m; (\mu)m} \mathcal{C}_{N; k, l, m}^2 .
\] (3.8)

Finally, using the completeness relation (2.25) and Eqs. (3.2), (2.22), it is easy to verify the identity
\[
\sum_{k=0}^{\infty} \mathcal{C}_{N; k, l, m}^2 = N_{N,l} N_{N,m} .
\] (3.9)

Let us now introduce the 6–j symbols (also called Racah symbols). They are \(U(N)\)-scalars defined by
\[
\mathcal{R}_N(a, b, c; d, e, f) = \mathcal{C}_{N; a, d, c; (\alpha)a; (\beta)d; (\gamma)c} \mathcal{C}_{N; b, c, e; (\eta)b; (\lambda)d; (\rho)e}
\times \mathcal{C}_{N; c, f, e; (\zeta)c; (\sigma)f} \mathcal{C}_{N; b, c, f; (\sigma)f; (\mu)b; (\nu)c} .
\] (3.10)

The tetrahedral symmetry, which is enjoyed by the standard 6–j symbols [38], is trivially true also for our definition.

We have not been able to compute a general formula for the 6–j symbols, but we have computed a very large class of special cases. This class is sufficient for computing the high-temperature expansion of the \(U(N)\)-invariant \(\sigma\)-models, in general dimension \(d\), up to rather high order.

We begin by deriving a completeness relation for the 6–j symbols. Using Eqs. (3.7) and (3.8), and the definition (3.10) we get
\[
\sum_a \mathcal{R}_N(a, b, c; d, e, f) = \frac{1}{N_{N,f}} \mathcal{C}_{N; d, e, f}^2 \mathcal{C}_{N; b, c, f}^2 .
\] (3.11)

A general formula for \(\mathcal{R}_N(l+p, l, p; r, m, k)\) for arbitrary \(l, p, r, m, k\) is reported in App. A.4 [see Eq. (A.70)]. It is particularly simple to derive all 6–j symbols in which one of the spins is 1. In this case we need to compute the coefficients \(\mathcal{R}_N(1, l+a, k+b; k, l, m)\) with \(a, b = \pm 1, 0\). Using the tetrahedral symmetry one can immediately recognize that all cases can be computed using Eq. (A.70) of App. A.4, except when \(a = b = 0\). In this case, however, we can use the completeness relation (3.11) to write
\[
\mathcal{R}_N(1, l, k; k, l, m) = \frac{1}{N_{N,k}} \mathcal{C}_{N; k, l, m}^2 \mathcal{C}_{N; 1, k, k}^2
\times \mathcal{R}_N(1, l, k; k, l, m) - \mathcal{R}_N(1, l + 1, k; k, l, m) .
\] (3.12)

4 Applications

The formalism we have presented in the previous sections can be applied to the generation of high-temperature series for \(U(N)\)-invariant spin models defined on \(CP^{N-1}\).

\(^{12}\)In the terminology used in works dealing with high-temperature expansions [26], the 6–j symbols are the group factors associated to the so-called \(\alpha\)-topology.
The idea is to expand the Gibbs weight $e^{-\beta H}$ for the given spin model in terms of hyperspherical harmonics, and then use their orthogonality to reduce the number of non-vanishing terms. We will consider here the Hamiltonians that are commonly used in the study of these models.

The “quartic” Hamiltonian is given by

$$H = -\sum_{\langle xy \rangle} |\mathbf{z}_x \cdot \mathbf{z}_y|^2,$$  \hspace{1cm} (4.1)

where $\mathbf{z}_x \cdot \mathbf{z}_x = 1$ and the sum is over all lattice links $\langle xy \rangle$. The Hamiltonian (4.1) is invariant under global $U(N)$ transformations and under the local $U(1)$-gauge transformations defined by

$$\mathbf{z}_x \rightarrow e^{i\alpha_x} \mathbf{z}_x,$$  \hspace{1cm} (4.2)

and therefore defines a theory on $CP^{N-1}$. Notice that for $N = 2$, using Eq. (2.5), we have $|\mathbf{z}_x \cdot \mathbf{z}_y|^2 = (1 + s_x \cdot s_y)/2$. Thus the model with Hamiltonian (4.1) and $N = 2$ is equivalent to the $N$-vector model with $N = 3$.

A second possibility is the so-called “gauge” Hamiltonian. In this case one associates to each lattice link a real gauge field $\theta_{xy} \in [0, 2\pi]$ and considers

$$H = -\text{Re} \sum_{\langle xy \rangle} (\mathbf{z}_x \cdot \mathbf{z}_y) e^{i\theta_{xy}},$$  \hspace{1cm} (4.3)

where $\mathbf{z}_x \cdot \mathbf{z}_x = 1$ and the sum is over all lattice links $\langle xy \rangle$. The Hamiltonian (4.3) is invariant under the local transformations

$$\mathbf{z}_x \rightarrow e^{i\alpha_x} \mathbf{z}_x, \quad \theta_{xy} \rightarrow \theta_{xy} + \alpha_x - \alpha_y,$$  \hspace{1cm} (4.4)

and therefore defines a theory on $CP^{N-1}$. If one considers only correlations of the $\mathbf{z}$-field, one can integrate out the gauge field $\theta_{xy}$ obtaining an effective Hamiltonian

$$H = -\frac{1}{\beta} \sum_{\langle xy \rangle} \log [I_0 (\beta |\mathbf{z}_x \cdot \mathbf{z}_y|)].$$  \hspace{1cm} (4.5)

For the Hamiltonians (4.1) and (4.3), using (2.32)–(2.35), it is possible to compute the expansion coefficients $F_{N,k}$. We need to compute $F_{N,k}$ for the functions $\exp \left[\beta |\mathbf{z} \cdot \mathbf{w}|^2 \right]$ and $I_0 (\beta |\mathbf{z} \cdot \mathbf{w}|)$. In the first case we obtain

$$F_{N,k} = \frac{(N - 1)! k!}{(N - 1 + 2k)!} \beta^k \ _1F_1(k + 1; N + 2k; \beta),$$  \hspace{1cm} (4.6)

where $_1F_1$ is the confluent (degenerate) hypergeometric function. In the second case the integration gives

$$F_{N,k} = (N - 1)! \left(\frac{\beta}{2}\right)^{1-N} I_{N+2k-1}(\beta),$$  \hspace{1cm} (4.7)

where $I_r$ is the modified Bessel function.

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13See [44], pp. 1058–1059.
A Properties of Hyperspherical Harmonics

A.1 Some basic formulae

Let us begin by computing the dimension of the linear space $E_{N,k}$. This can be done by computing the dimension of the space of all completely symmetric tensors of rank $k$ in two sets of indices, and then subtracting from it the number of independent trace conditions that have to be imposed in order to ensure the tracelessness of these tensors. The number of linearly independent symmetric tensors is given by $\binom{N+k-1}{k}$ and the number of traces is given by $\binom{N+k-2}{k-1}$. Therefore we obtain

$$N_{N,k} \equiv \dim E_{N,k} = \binom{N+k-1}{k} - \binom{N+k-2}{k-1}$$

This proves Formula (2.12). For $N = 2$ we have $N_{2,k} = 2^k + 1$, as expected on the basis of the isomorphism $S^2 \simeq \mathbb{C}P^1$.

Let us now compute the integral of monomials in $z$ and $\bar{z}$, i.e. of

$$\int d\Omega(z,\bar{z}) z^{\alpha_1} \ldots z^{\alpha_k} \bar{z}_{\beta_1} \ldots \bar{z}_{\beta_l}.$$  \hspace{1cm} (A.3)

It is trivial to see that the integral is zero if $k \neq l$. To compute its value for $k = l$, we introduce an arbitrary complex vector $A^\alpha$, and define

$$I_k(A) = \int d\Omega(z,\bar{z}) (\bar{A} \cdot z)^k (\bar{z} \cdot A)^k.$$  \hspace{1cm} (A.4)

Since the measure $d\Omega(z,\bar{z})$ is $U(N)$-invariant, we have $I_k(A) = I_k(UA)$ for any $U \in U(N)$, so that $I_k(A)$ depends only on $\bar{A} \cdot A$. Moreover, $I_k(A)$ is manifestly a homogeneous function of degree $k$ in $A$ and $\bar{A}$. Hence we must have $I_k(A) = J_k(\bar{A} \cdot A)^k$ for some constant $J_k$. Now, since $\bar{z} \cdot z = 1$, we have

$$\frac{\partial}{\partial A_\alpha} \frac{\partial}{\partial A^\alpha} I_k(A) = k^2 I_{k-1}(A).$$  \hspace{1cm} (A.5)

A recursion relation for $J_k$ immediately follows:

$$J_k = \frac{k}{N+k-1} J_{k-1}.$$  \hspace{1cm} (A.6)

Using $J_0 = 1$ we obtain the general solution

$$J_k = \frac{k!(N-1)!}{(N-1+k)!}.$$  \hspace{1cm} (A.7)
Taking then $k$ derivatives with respect to $A$ and with respect to $\bar{A}$ in Eq. (A.4) we obtain the well-known result

$$
\int d\Omega(z, \bar{z}) z^{\alpha_1} \ldots z^{\alpha_k} \bar{z}_{\beta_1} \ldots \bar{z}_{\beta_k} = \frac{(N-1)!}{(N-1+k)!} \left( \delta_{\beta_1}^{\alpha_1} \ldots \delta_{\beta_k}^{\alpha_k} + \ldots \right)
$$

(A.8)

where the terms in parentheses correspond to the $k!$ different pairings of the indices.

Let us now prove the orthogonality relation (2.20). This is completely equivalent to proving that for arbitrary completely symmetric (in two sets of indices) and traceless tensors $T_{N,k}$ and $U_{N,l}$ we have

$$
\int d\Omega(z, \bar{z}) \ (Y_{N,k} \cdot T_{N,k})(Y_{N,l} \cdot U_{N,l}) = \delta_{kl} T_{N,k} \cdot U_{N,k} .
$$

(A.9)

To see this, let us first use the definition (2.13) and let us notice that the “Traces” terms do not give any contribution due to the tracelessness of $T_{N,k}$ and $U_{N,l}$. Thus the l.h.s. in Eq. (A.9) becomes simply

$$
\mu_{N,k} \mu_{N,l} \int d\Omega(z, \bar{z}) \ z^{\alpha_1} \ldots z^{\alpha_k} \bar{z}_{\gamma_1} \ldots \bar{z}_{\gamma_l} \ T^{(\gamma)}_{N,k,\alpha} T^{(\delta)}_{N,k,\beta} U_{N,l,\beta} U_{N,l,\delta} .
$$

(A.10)

We can then use Eq. (A.8). The only non-vanishing contributions come from those terms which do not contain $\delta_{\gamma_j}^{\alpha_i}$ or $\delta_{\beta_j}^{\beta_i}$; such terms exist only if $l = k$. In this last case there are $(k!)^2$ equivalent contractions and we end up with

$$
\delta_{kl} \mu_{N,k}^2 \frac{(N-1)!(k!)^2}{(N-1+2k)!} T_{N,k} \cdot U_{N,k} = \delta_{kl} T_{N,k} \cdot U_{N,k} .
$$

(A.11)

Using the expression (2.14) for the normalization factor $\mu_{N,k}$, we obtain the orthogonality relation (2.20) for the $Y$'s.

The general formula for the hyperspherical harmonics [13] can be obtained by using the fact that they are completely symmetric and traceless. The complete symmetry, together with the needed transformation properties under $U(N)$, implies an expansion of the form

$$
Y_{N,k,(\alpha)}(z) = \mu_{N,k} \sum_{s=0}^{k} A_{N,k:s} P_{(k:s)(\beta)}^{(\alpha)}(z) ,
$$

(A.12)

where

$$
P_{(k:s)(\beta)}^{(\alpha)}(z) \equiv \delta_{\beta_1}^{\alpha_1} \ldots \delta_{\beta_s}^{\alpha_s} z^{\alpha_{s+1}} \ldots z^{\alpha_k} \bar{z}_{\beta_{s+1}} \ldots \bar{z}_{\beta_k} + \text{permutations}
$$

(A.13)

and the number of permutations necessary to make $P_{(k:s)}$ completely symmetric is $s! \left(\begin{array}{c} k \\ s \end{array}\right)^2$. Now we impose the tracelessness. We first note that

$$
P_{(k:s)(\alpha)(\beta)}^{(\gamma)}(z) = P_{(k-1:s)(\gamma)(\beta)}^{(\alpha)}(z) + (N+2k-s-1)P_{(k-1:s-1)(\beta)}^{(\gamma)(\alpha)}(z) ,
$$

(A.14)

13
with the understanding that $P_{(k;s)} = 0$ for $s > k$ and $s < 0$. From the tracelessness of the hyperspherical harmonics we obtain the recursion relation

$$A_{N,k;s} + (N + 2k - s - 2) A_{N,k;s+1} = 0 .$$

(A.15)

Imposing the normalization $A_{N,k;0} = 1$, we find

$$A_{N,k;s} = (-1)^s \frac{(N + 2k - s - 2)!}{(N + 2k - 2)!} .$$

(A.16)

Let us now discuss the relation between the hyperspherical harmonics and the Jacobi polynomials. From Section 2 we know that $Y_{N,k,1\ldots,1}(z)$ is the restriction to the complex unit sphere of a harmonic polynomial of degree $k$ in $z$ and $\overline{z}$. Moreover, it depends only on $|z^1|$ because of the gauge invariance $z \to e^{i\alpha}z$. Therefore it can be written as $r^{2k}P_k(|z^1|^2/r^2)$ where $r = |z|$. Requiring the polynomial to satisfy Laplace’s equation we get for $P_k(x)$ the equation

$$x(1-x) \frac{d^2P_k}{dx^2} + (1-Nx) \frac{dP_k}{dx} + k(N + k - 1) P_k = 0 .$$

(A.17)

The regular solution of this equation is the Jacobi polynomial $P_k^{(N-2,0)}(2x - 1)$ (see [44, p. 1036]). The normalization is fixed by the requirement that

$$Y_{N,k,1\ldots,1}(z) = \mu_{N,k} |z^1|^{2k} + \text{lower-order terms} .$$

(A.18)

We thus get (see Formula 8.962.1b in [44, p. 1036])

$$Y_{N,k,1\ldots,1}(z) = \mu_{N,k} \frac{k!(N - 2 + k)!}{(N - 2 + 2k)!} P_k^{(N-2,0)}(2|z^1|^2 - 1)$$

(A.19)

which, using the fact that

$$P_k^{(N-2,0)}(1) = \binom{N - 2 + k}{k}$$

(A.20)

gives (2.28).

Note that we could have derived (A.12) by using (2.28) and the expansion of the Jacobi polynomials.

### A.2 Expansions in Terms of Hyperspherical Harmonics

We want now to discuss the convergence of the expansion (2.23). We will begin by noticing that, given a generic complex tensor $T_{N,k,(\alpha)k}$, the hyperspherical harmonics satisfy the inequality

$$|T_{N,k} \cdot Y_{N,k}(z)|^2 \leq (T_{N,k} \cdot T_{N,k}^*) N_{N,k} .$$

(A.21)
where the adjoint tensor $T^*_{N,k}$ is given by
\[
\left(T^*_{N,k}\right)_{(\alpha)k}^{(\beta)k} = \overline{(T_{N,k})_{(\beta)k}^{(\alpha)k}}, \tag{A.22}
\]
and the overline indicates complex conjugation. Eq. (A.21) follows immediately from Schwarz’s inequality and (2.22). Moreover, equality in (A.21) is possible only for those $z$’s for which
\[
T_{N,k} = \gamma Y_{N,k}(z) \tag{A.23}
\]
for some constant $\gamma$. This requires first of all that $T_{N,k}$ be completely symmetric in both sets of indices and traceless. The constant $\gamma$ is easily obtained squaring the previous relation:
\[
|\gamma|^2 = \frac{(T^*_{N,k} \cdot T_{N,k})}{N_{N,k}}. \tag{A.24}
\]

Now let us consider the special case $T_{N,k} = Y_{N,k}(x) \Rightarrow x = (1,0,\ldots,0)$ and $k \geq 1$. In this case we can rephrase Eq. (A.21) in terms of Jacobi polynomials [using Eqs. (2.22), (2.28) and (2.27)] as
\[
|P_{k}^{(N-2,0)}(t)|^2 \leq |P_{k}^{(N-2,0)}(1)|^2. \tag{A.25}
\]
Equality in this case is possible only if
\[
Y_{N,k}(x) = \gamma Y_{N,k}(z), \tag{A.26}
\]
with $|\gamma| = 1$. We will now prove that, for $N > 2$ this equation is satisfied only if $\gamma = 1$ and $z = e^{i\theta} x$. For $N = 2$ there is a second solution: $\gamma = (-1)^k$ and $z = e^{i\theta} \tilde{x}$, with $\tilde{x} = (0,1)$.

For $b \neq 1$ we have from Eq. (A.26)
\[
0 = Y_{N,k,b,\ldots,b}(z) = \mu_{N,k} (z^{1})^{k} (\overline{z_{b}})^{k}, \tag{A.27}
\]
that implies either $z^{1} = 0$ or $z^{b} = 0$. Now, if $z^{1} = 0$, from Eqs. (A.26) and (2.28), we obtain
\[
P_{k}^{(N-2,0)}(1) = \gamma P_{k}^{(N-2,0)}(-1). \tag{A.28}
\]
Now $P_{k}^{(N-2,0)}(-1) = (-1)^k$, while $P_{k}^{(N-2,0)}(1)$ is given in Eq. (A.20). For $N > 2$ and $|\gamma| = 1$, one immediately verifies that Eq. (A.28) is never satisfied. Therefore $z^{1} \neq 0$, so that $z^{b} = 0$ for all $b \neq 1$. The result immediately follows. For $N = 2$, using Eq. (2.19), it is easy to verify that both $e^{i\theta} x$ and $e^{i\theta} \tilde{x}$ satisfy Eq. (A.26).

Thus, we have shown that equality in Eq. (A.25) holds only for the case $t = 1$ (respectively $t = \pm 1$) for $N > 2$ (respectively $N = 2$). We therefore have [since $P_{k}^{(N-2,0)}(1) > 0$]
\[
|P_{k}^{(N-2,0)}(t)| < P_{k}^{(N-2,0)}(1) \tag{A.29}
\]
for $-1 < t < 1$ and $k \geq 1$. 

15
To discuss the convergence of the series (2.23) let us first notice that using the completeness relation (2.25) and Eq. (2.24) we get

$$\sum_{k=0}^{\infty} \tilde{f}_{k,(\beta)k} \tilde{f}_{k,(\alpha)k} = \int d\Omega(z,\bar{z}) \left|f(z)\right|^2,$$

(A.30)

which is the Plancherel identity for harmonic analysis in $\mathbb{C}P^{N-1}$. Now let us consider, instead of $f$, the function $\mathcal{L}^n f$ where $\mathcal{L}$ is the Laplace-Beltrami operator. In this case $\tilde{f}_{k,(\beta)k}^{(\alpha)k}$ is replaced by $(\lambda_{N,k})^n \tilde{f}_{k,(\beta)k}^{(\alpha)k}$, and thus we obtain

$$\sum_{k=0}^{\infty} (\lambda_{N,k})^{2n} \tilde{f}_{k,(\beta)k}^{(\alpha)k} \tilde{f}_{k,(\alpha)k}^{(\beta)k} = \int d\Omega(z,\bar{z}) |\mathcal{L}^n f(z)|^2.$$

(A.31)

If now $f$ is a $C^\infty$ function, the integral is finite for all $n$. Thus the sum on the l.h.s. is converging for all $n$. Since $\lambda_{N,k} \sim k^2$ for $k \to \infty$, we get that $k^{2n} \tilde{f}_k \cdot \tilde{f}_k \to 0$ in the same limit, for every $n$. This implies that all coefficients $\tilde{f}_k$ decrease faster than any inverse power of $k$. To prove the convergence of the series (2.23) it is then enough to notice that $|Y_{N,k}(z)| \leq (\mathcal{N}_{N,k})^{1/2}$ and that $\mathcal{N}_{N,k}$ behaves as $k^{2N-3}$ for large $k$.

Let us now discuss the properties of the coefficients $F_{N,k}$ in (2.32). The second property follows immediately from the previous discussion. We want now to prove that, if $f(t)$ is positive for $t \in [0, 1]$, then $|F_{N,k}| < F_{N,0}$ for $k \geq 1$. Indeed, from Eqs. (2.32) and (2.29) we get

$$|F_{N,k}| \leq (N-1) \int_0^1 dt (1-t)^{N-2} f(\sqrt{t}) \frac{P_k^{(N-2,0)}(t)}{P_k^{(N-2,0)}(1)} < (N-1) \int_0^1 dt (1-t)^{N-2} f(\sqrt{t}) = F_{N,0}.$$

(A.32)

### A.3 Some Useful Contractions

We are now going to report some useful formulae for the contractions of various products of hyperspherical harmonics. To simplify the results we will introduce some additional notation.

Given three irreducible (completely symmetric and traceless) tensors $T_{N,k}$, $U_{N,l}$, and $V_{N,m}$ — respectively of rank $k$, $l$, and $m$ — we want to construct a scalar, i.e. a quantity that is invariant under $U(N)$ transformations. It is easy to see that there are many ways of doing this, each one of them characterized by an integer $i$. We define

$$[T_{N,k}, U_{N,l}, V_{N,m}]_i \equiv$$

$$T_{N,k,(\alpha)k-i}^{(\alpha)\gamma (\beta)k} U_{N,l,(\gamma)k-i+l}^{(\gamma)\epsilon (\delta)l} V_{N,m,(\delta)m-i}^{(\epsilon)\iota (\zeta)m-i}.$$

(A.33)

This contraction is defined for $0 \leq i \leq (k,l) \leq m + i \leq k + l$. In the previous expression, $i$ counts the number of upper indices of $T_{N,k}$ that are contracted with the lower indices of $U_{N,l}$. Once $i$ is given, all other contractions are completely fixed.
The definition (A.33) can be generalized by writing

\[
[T_{N,l+1}^{(\mu)_{p}} : U_{N,l} : V_{N,m}^{(\nu)}]_{i} \equiv T_{N,k+1}^{(\mu)_{p}^{(\nu)_{p}}(\mu)_{p}} U_{N,l}^{(\gamma)_{k+i+l-m}(\delta)_{m+i-k}} V_{N,m}^{(\epsilon)_{l+i}(\zeta)_{l+i+k}}
\]  

(A.34)

We also introduce a second useful notation:

\[
Z_{N,k,i\beta}^{(\alpha)}(z) \equiv z^{\alpha_{1}} \cdots z^{\alpha_{k}} \beta_{1} \cdots \beta_{k}.
\]  

(A.35)

Let us now compute various contractions that will be used in the determination of the Clebsch-Gordan and 6–j coefficients.

(i) \(B_{1}(l, m; h, k)\).

Let us define

\[
B_{1}(l, m; h, k) = [Y_{N,l}(z) \cdot Y_{N,m}(z) \cdot Z_{N,l+m-k-h}(z)]_{h},
\]  

(A.36)

for \((l, m) \geq (h, k)\). By \(U(N)\)-invariance it is immediate to see that \(B_{1}\) does not depend on \(z\). Using the property\(^{14}\)

\[
z^{\alpha_{1}} \beta_{1} \cdots Y_{N,m,\alpha(\delta)_{m-1}}(z) = \frac{N_{N,m}}{N_{N,m-1}} \frac{\mu_{N,m-1}}{\mu_{N,m}} Y_{N,m,\alpha(\delta)_{m-1}}(z),
\]  

(A.39)

we obtain for \(h > k\)

\[
B_{1}(l, m; h, k) = \frac{N_{N,m}N_{N,l}^{2}N_{h}^{2}}{\mu_{N,m}^{2}N_{l}^{2}} [Y_{N,h}(z) \cdot Y_{N,h}(z) \cdot Z_{N,h-k}(z)]_{h}.
\]  

(A.40)

Now we expand the first \(Y_{N,h}(z)\) using Eq. (A.12), and then use repeatedly Eq. (A.39). We obtain finally \((h \geq k)\)

\[
B_{1}(l, m; h, k) = \frac{N_{N,m}N_{N,l}N_{h}^{2}}{\mu_{N,m}^{2}N_{l}^{2}} \sum_{s=0}^{h-k} \binom{h}{s} \binom{h-k}{s} s! A_{N,h; s}.
\]  

(A.41)

For \(h < k\) we can use the symmetry

\[
B_{1}(l, m; h, k) = B_{1}(m, l; k, h).
\]  

(A.42)

Since the final expression, Eq. (A.41), is symmetric in \(m\) and \(l\) and in \(h\) and \(k\), it is valid also for \(h < k\).

\(^{14}\)This property is easily obtained recursively, by writing

\[
z^{\alpha_{1}} \beta_{1} \cdots Y_{N,m,\alpha(\delta)_{m-1}}(z) = x_{m} Y_{N,m-1,\alpha(\delta)_{m-1}}(z),
\]  

(A.37)

and noticing that

\[
Z_{N,m}(z) \cdot Y_{N,m}(z) = \frac{N_{N,m}}{\mu_{N,m}} = \prod_{k=1}^{m} x_{k}.
\]  

(A.38)
Let us define

\[ B_2(s; i, l, m, n) = \left[ P_{(l:i)}(z) \cdot Y_{N,m}(z) \cdot Y_{N,n}(z) \right]_i, \quad (A.43) \]

for \( 0 \leq i \leq (m, l) \leq n + i \leq m + l \) and \( 0 \leq s \leq l \).

The computation is simple. Indeed, using the definition (A.13) of \( P_{(l:i)}(z) \) we can easily express \( B_2 \) in terms of \( B_1 \). If \( s \leq l - |m - n| \) we obtain

\[ B_2(s; i, l, m, n) = \sum_{r=r_{\text{min}}}^{r_{\text{max}}} \binom{i}{r} \binom{n-m+i}{r} \binom{l-i}{s-r} \binom{m-n+l-i}{s-r} \times r!(s-r)! B_1(m, n; n-l+i+s-r, r+m-i), \quad (A.44) \]

where \( r_{\text{min}} = \max(0, i+s-l, i+s-l+n-m) \), \( r_{\text{max}} = \min(s, i, i+n-m) \). Otherwise (if \( s > l - |m - n| \)) we get

\[ B_2(s; i, l, m, n) = 0. \quad (A.45) \]

(iii) \( B_3(i; l, m, n) \).

Let us define

\[ B_3(i; l, m, n) = \left[ Y_{N,l}(z) \cdot Y_{N,m}(z) \cdot Y_{N,n}(z) \right]_i, \quad (A.46) \]

for \( 0 \leq i \leq (m, l) \leq n + i \leq m + l \).

Using Eq. (A.12) we rewrite \( B_3 \) in terms of \( B_2 \). It is easy to see that

\[ B_3(i; l, m, n) = 0 \quad (A.47) \]

if \( l < |m - n| \) or \( l > m + n \); otherwise we obtain

\[ B_3(i; l, m, n) = \mu_{N,l} \sum_{s=0}^{l-|m-n|} A_{N,l,s} B_2(s; i, l, m, n). \quad (A.48) \]

Notice the following symmetries of \( B_3 \):

\[ B_3(i; l, m, n) = B_3(l-i; l, n, m) = B_3(n+i-l; m, n, l) = B_3(m+l-n-i; m, l, n). \quad (A.49) \]

These symmetries are very useful to check the correctness of our result (A.48) since our method of computation gives an expression that does not have any of these symmetries in an obvious way.

(iv) \( B_4(i, j; p, r, m) \).

Let us define

\[ B_4(i, j; p, r, m) = \int d\Omega(z, \bar{z}) d\Omega(w, \bar{w}) \left[ Y_{N,p}(w) \cdot Z_{N,r-p+i+j}(z) \cdot Y_{N,r}(w) \right]_i \left( Y_{N,m}(z) \cdot Y_{N,m}(w) \right), \quad (A.50) \]
for $0 \leq (i,j) \leq p \leq [(r+i),(r+j)]$.

Using the tracelessness of $Y_{N,m}(w)$ and Eq. (2.13) we can write

$$B_4(i,j;p,r,m) = \mu_{N,m} \int d\Omega(z) d\Omega(w) \left[ Y_{N,p}(w) \cdot Z_{N,r-p+j}(z) \cdot Y_{N,m}(w) \right]_i (Z_{N,m}(z) \cdot Y_{N,m}(w)).$$

(A.51)

We can now perform the integral over $z$ using Eq. (A.8). It is easy to see that $B_4(i,j;p,r,m)$ is non-vanishing only if $|r-p| \leq m \leq r-p+i+j$. If this condition is satisfied we obtain (the integral over $w$ can be dropped)

$$B_4(i,j;p,r,m) = \mu_{N,m} \frac{(N-1)! j! (m)!^2 (r-p+j)! (r-p+i)!}{(N-1+r-p+i+j+m)!} \sum_{a=a_{\min}}^{a_{\max}} \frac{1}{a! b! c! d! e! f!} \left[ Y_{N,p}(w) \cdot Y_{N,r}(w) \cdot Y_{N,m}(w) \right]_{r+j-m-a}.$$  

(A.52)

where $b = j - a$, $c = r - p - m - a + i + j$, $d = a + m - j$, $e = p + m + a - r - j$, $f = r + j - p - a$, $a_{\min} = \max(0, j-m, r-p+j-m)$, $a_{\max} = \min(j, r-p+j, r-p+i+j-m)$.

We can now obtain the final result by using Eqs. (A.46) and (A.48).

We note that, starting from the definition (A.50) and using the completeness relation (2.25) and Eq. (A.41), we obtain the sum rule

$$\sum_{m=|r-p|}^{r-p+i+j} B_4(i,j;p,r,m) = B_1(p,r;p-i,p-j)$$

$$= \frac{N_{r,N} N_{r,p} (N-2)! (N-2+2p-i-j)!}{\mu_{N,r} \mu_{N,p} (N-2+p-i)! (N-2+p-j)!}.$$  

(A.53)

(v) $B_5(i,l;p,k,r,m)$.

Let us define

$$B_5(i,l;p,k,r,m) = \int d\Omega(z) d\Omega(w) Y_{N,k}(z) Y_{N,l}(\alpha_i) (w)$$

$$\times \left[ Y_{N,l+p+i}(w) \cdot Y_{N,k}(z) \cdot Y_{N,r}(w) \right]_i (Y_{N,m}(z) \cdot Y_{N,m}(w)).$$

(A.54)

for $0 \leq i \leq (p,k) \leq r+i \leq k+p$.

We expand $Y_{N,k}(z)$ using Eq. (A.12) and notice that $Y_{N,l}(z)$ can be replaced by $\mu_{N,l} Z_{N,l}(z)$. In this way $B_5$ is expressed in terms of $B_4$. We obtain finally

$$B_5(i,l;p,k,r,m) = \mu_{N,k} \mu_{N,l} \sum_{s=0}^{k-[r-p]} \sum_{a=a_{\min}}^{a_{\max}} \binom{r-p+i}{s-a} \binom{k-r+p-i}{s-a} \binom{k-i}{s-a}$$

$$\times (s-a)! a! A_{N,k,s} B_4(i-a+l,k-r-p-i-s+a+l;p+l,r,m).$$

(A.55)

where $a_{\min} = \max(0, s-k+i, s-k+i+r-p)$ and $a_{\max} = \min(i, s, r-p+i)$.

Using the fact that $B_4(i,j;p,r,m)$ vanishes unless $|r-p| \leq m \leq r-p+i+j$, we see that $B_5(i,l;p,k,r,m) = 0$ if $m > k+l-s$ or $m < |r-p-l|$.
A.4 Clebsch-Gordan Coefficients and 6–j Symbols

Let us now discuss the computation of the Clebsch-Gordan coefficients (3.3). For arbitrary completely symmetric and traceless tensors $T_{N,k}$, $U_{N,l}$ and $V_{N,m}$, we want to compute

$$
\int d\Omega(z,\zeta) \ (T_{N,k} \cdot Y_{N,k}(z)) \ (U_{N,l} \cdot Y_{N,l}(z)) \ (V_{N,m} \cdot Y_{N,m}(z)) \ .
$$ (A.56)

Using the definition (2.13) of the hyperspherical harmonics this reduces to

$$
\mu_{N,k} \mu_{N,l} \mu_{N,m} \int d\Omega(z,\zeta) \ (T_{N,k} \cdot Z_{N,k}(z)) \ (U_{N,l} \cdot Z_{N,l}(z)) \ (V_{N,m} \cdot Z_{N,m}(z)) \ .
$$ (A.57)

We can now use Eq. (A.8). For $|l - m| \leq k \leq l + m$ the integral becomes

$$
\frac{(N - 1)! \mu_{N,k} \mu_{N,l} \mu_{N,m}}{(N - 1 + k + l + m)!} \sum_{i = i_{\min}}^{i_{\max}} \binom{k}{h} \binom{l}{i} \binom{m}{j} k! l! m! \ [T_{N,k} \cdot U_{N,l} \cdot V_{N,m}]_{i} \ ,
$$ (A.58)

where $h = m - l + i$, $j = m - k + i$, $i_{\max} = \min(k, l, l + k - m)$, $i_{\min} = \max(0, l - m, k - m)$ and we use the notation introduced in Eq. (A.33). If $k < |l - m|$ or $k > l + m$ the integral vanishes. Formula (3.3) immediately follows.

We now discuss the computation of $C_{N; k,l,m}^{2}$ for $|l - m| \leq k \leq l + m$. Using (3.2) we rewrite

$$
C_{N; k,l,m}^{2} = \int d\Omega(z,\zeta) \ C_{N; k,l,m,\alpha; \beta; \gamma; \delta; \epsilon; \zeta}^{(\alpha)(\beta)(\gamma)(\epsilon)(\zeta)} \ Y_{N,k}^{(\alpha)}(z) Y_{N,l}^{(\beta)}(z) Y_{N,m}^{(\epsilon)}(z) \ .
$$ (A.59)

Using (A.58) we obtain

$$
C_{N; k,l,m}^{2} = \frac{(N - 1)! \mu_{N,k} \mu_{N,l} \mu_{N,m}}{(N - 1 + k + l + m)!} \sum_{i = i_{\min}}^{i_{\max}} \binom{k}{h} \binom{l}{i} \binom{m}{j} k! l! m! \ [Y_{N,k}(z) \cdot Y_{N,l}(z) \cdot Y_{N,m}(z)]_{i} \ ,
$$ (A.60)

where $h$, $j$, $i_{\min}$, $i_{\max}$ are defined as in Eq. (A.58). The final result can be expressed in terms of $B_{3}(i; k, l, m)$ computed in Eq. (A.48).

We now describe another way of computing $C_{N; k,l,m}^{2}$. Using (3.2) we can write

$$
C_{N; k,l,m}^{2} = \int d\Omega(z,\zeta) d\Omega(w,\bar{w}) \ (Y_{N,k}(z) \cdot Y_{N,k}(w)) (Y_{N,l}(z) \cdot Y_{N,l}(w)) (Y_{N,m}(z) \cdot Y_{N,m}(w)) \ .
$$ (A.61)

The integrand is only a function of $|\bar{w} \cdot z|$. Thus, using the $U(N)$- invariance of the measure, we can fix one of the two spins to an arbitrary value. Let us set $w = x \equiv (1, 0, \ldots, 0)$. We obtain, after integrating in $d\Omega(w)$,

$$
C_{N; k,l,m}^{2} = \int d\Omega(z,\zeta) \ (Y_{N,k}(z) \cdot Y_{N,k}(x)) (Y_{N,l}(z) \cdot Y_{N,l}(x)) (Y_{N,m}(z) \cdot Y_{N,m}(x)) \ .
$$ (A.62)
and by using (2.29) we end up with

\[ C_{N; k, l, m}^2 = (N - 1) \mathcal{N}_{N, k} \mathcal{N}_{N, l} \mathcal{N}_{N, m} \times \]
\[ \times \int_0^1 dt (1 - t)^{N - 2} \frac{P_k^{(N-2,0)}(2t - 1)}{P_k^{(N-2,0)}(1)} \frac{P_l^{(N-2,0)}(2t - 1)}{P_l^{(N-2,0)}(1)} \frac{P_m^{(N-2,0)}(2t - 1)}{P_m^{(N-2,0)}(1)}. \]  \quad (A.63)

Using the expansion of \( P_k^{(N-2,0)}(x) \) in powers of \((1 - x)\) (cf. [44], p. 1036) we can write for \(|l - m| \leq k \leq l + m\)

\[ C_{N; k, l, m}^2 = (N - 1) \mathcal{N}_{N, k} \mathcal{N}_{N, l} \mathcal{N}_{N, m} \]
\[ \times \sum_{s=0}^{k} (-1)^s \binom{k}{s} \frac{(N - 2)! (N - 2 + k + s)!}{(N - 2 + k)!(N - 2 + s)!} I(s; l, m), \]  \quad (A.64)

where

\[ I(s; l, m) = 2^{1-N-s} \int_{-1}^{1} dx (1 - x)^{N-2+s} \frac{P_l^{(N-2,0)}(x)}{P_l^{(N-2,0)}(1)} \frac{P_m^{(N-2,0)}(x)}{P_m^{(N-2,0)}(1)}. \]  \quad (A.65)

This integral can be evaluated explicitly (cf. Formula 2.22.18.2 of Ref. [45], vol. 2). For \( s < |l - m| \) we have \( I(s; l, m) = 0 \), while for \( s \geq |l - m| \) we obtain

\[ I(s; l, m) = \frac{2^{1-N-s}}{(N - 2 + l)! (N - 2 + m)!} \times \sum_{t=t_{\text{min}}}^{m} (-1)^{l+t} \frac{(s + t)!}{t!(m - t)!(s - l + t)!} \frac{(N - 2 + t + m)! (N - 2 + t + s)!}{(N - 2 + t)! (N - 1 + t + s + l)!}. \]  \quad (A.66)

where \( t_{\text{min}} = \max(0, l - s) \).

Finally, we report the computation of \( \mathcal{R}_N(l + p, p, r; m, l, k) \), defined in Eq. (3.10), for \(|l + p - r| \leq m \leq l + p + r, \ |l - k| \leq m \leq l + k \) and \(|p - k| \leq r \leq p + k\). If one of these inequalities is not satisfied \( \mathcal{R}_N(l + p, p, r; m, l, k) = 0 \), since \( C_{N; s, t, u} = 0 \) if \( s > t + u \) or if \( s < |t - u| \).

Using Eq. (3.2), we rewrite \( \mathcal{R}_N(l + p, p, r; m, l, k) \) as

\[ \int \Omega(z; \overline{z}) \Omega(w; \overline{w}) \Omega(x; \overline{x}) Y_{N, l+p, (\alpha)}(w) Y_{N, l, (\gamma)}(z) Y_{N, p, (\beta)}(x) C_{N, l+p, l, p, (\alpha)_{(\beta)}} \]
\[ \times (Y_{N, m}(w) \cdot Y_{N, m}(z)) (Y_{N, k}(x) \cdot Y_{N, k}(z)) (Y_{N, r}(x) \cdot Y_{N, r}(w)). \]  \quad (A.67)

We can now use Eq. (3.3) to rewrite the above equation as

\[ \frac{\mu_{N, p} \mu_{N, l}}{\mu_{N, p+l}} \int \Omega(z; \overline{z}) \Omega(w; \overline{w}) \Omega(x; \overline{x}) \left[ Y_{N, l+p}(w) \cdot Y_{N, l}(z) \cdot Y_{N, p}(x) \right] \]
\[ \times (Y_{N, m}(w) \cdot Y_{N, m}(z)) (Y_{N, k}(x) \cdot Y_{N, k}(z)) (Y_{N, r}(x) \cdot Y_{N, r}(w)). \]  \quad (A.68)
Now we integrate over $\mathbf{x}$ using Eqs. (3.2) and (3.3). We obtain

$$
\frac{(N-1)! \mu_{N,p}^2 \mu_{N,l} \mu_{N,k} \mu_{N,r}}{(N-1+p+k+r)! \mu_{N,p+l}} \int d\Omega(z, \mathbf{z}) \, d\Omega(w, \mathbf{w}) \sum_{i=i_{\text{min}}}^{i_{\text{max}}} \left( \begin{array}{c} k \\ h \end{array} \right) \left( \begin{array}{c} r \\ i \end{array} \right) \left( \begin{array}{c} p \\ j \end{array} \right) k! \, r! \, p! \, Y_{N,l,(\alpha)}(z) \times \left[ Y_{N,l+p,(\alpha)}(w) \cdot Y_{N,k}(z) \cdot Y_{N,r}(w) \right]_{p-r+i} \right)
$$

where $h = p - r + i$, $j = p - k + i$, $i_{\text{max}} = \min(k, r, k + r - p)$ and $i_{\text{min}} = \max(0, r - p, k - p)$.

At this point we notice that the remaining term corresponds to $B_5$, cf. Eq. (A.54). We have therefore

$$
\mathcal{R}_N(l+p,p,r;m,l,k) = \frac{(N-1)! \mu_{N,p}^2 \mu_{N,l} \mu_{N,k} \mu_{N,r}}{(N-1+p+k+r)! \mu_{N,p+l}} \sum_{i=i_{\text{min}}}^{i_{\text{max}}} \left( \begin{array}{c} k \\ h \end{array} \right) \left( \begin{array}{c} r \\ i \end{array} \right) \left( \begin{array}{c} p \\ j \end{array} \right) k! \, r! \, p! \, B_5(p - r + i; l; p, k, r, m). \quad (A.69)
$$

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