I. INTRODUCTION

In the last decades, we have assisted to an outstanding progress of statistical mechanics as a powerful mathematical tool, physically oriented, to analyze complex systems of the most diverse nature. A special impulse was later given to the interplay between concepts close to the realm of statistical physics, mostly within the spin glass theory [1], and the mathematical theory of complexity for optimization problems [2, 3]. On the other hand, more recently, the developments in the complex network theory and its numerous applications have been made even stronger the importance of statistical mechanics as a very interdisciplinary mathematical tool [4, 9]. In network theory, elements and interactions are simplified as node and links between them. Many models for growing or static networks have been proposed, and turned out to be able to reproduce the main universal features observed in real networks, first of all a scale-free character in the degree distribution, a feature which is responsible for several non classical phenomena (like a zero percolation threshold, i.e. the strongest resilience of a network to a random deletion of its links) not possible in the older models of random graphs. One of the most promising frameworks to analyze networks is provided by hidden variables models [10, 13]. The general idea behind hidden variable models relies on the possibility that each node is attributed a value that expresses its propensity to be connected to another node with hidden variable via a formula that relates these variables. It turns out that this scheme is both very general and very powerful in the description of many diverse networks, especially with respect to the possibility to have analytic results.

In this paper, we use statistical mechanics to face and solve two concatenate problems in hidden variable models. We first develop a general approach to extrapolate the typical configurations of a generic hidden-variable model, i.e., we use statistical mechanics to derive, in the thermodynamic limit, the equations that describe, in the space of the hidden variables, the node distribution corresponding to very rare graph realizations. The phase transition is triggered by the underlying geometry whose strength can be tuned by a parameter with $a = 1$ for rigid geometry (only close nodes are connected) and $a = 0$ for rigid anti-geometry (only distant nodes are connected). Consistently, when $a = 1/2$ there is no geometry and no phase transition. After discussing the numerical analysis, we provide a combinatorial argument to fully explain the mechanism inducing this phase transition and recognize it as an easy-hard-easy transition. Our result shows that, in general, ad hoc optimized networks can hardly be designed, unless to rely to specific heterogeneous constructions, not necessarily scale free.

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Statistical Mechanics of Random Geometric Graphs. Geometry-induced first order phase transition

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Random Geometric Graphs (RGG) can be cast within the formalism of hidden-variables models. For RGG, the hidden-variables coincide with the coordinates of the nodes. We develop a general approach to extrapolate the typical configurations of a generic hidden-variable model and apply it to RGG. For any RGG, defined through a rigid or a soft geometric rule, the method allows to face a non trivial optimization problem: Given $N$ nodes, a domain $D$, and a desired average connectivity ($k$), find - if any - the distribution of nodes having support in $D$ and average connectivity ($k$). However, we find out that, in the thermodynamic limit, nodes can be only either uniformly distributed, or highly condensed, the two regimes being separated by a first order phase transition characterized by a finite jump of $\langle k \rangle$. Other intermediate values of $\langle k \rangle$ correspond to very rare graph realizations. The phase transition is triggered by the underlying geometry whose strength can be tuned by a parameter $a \in [0, 1]$, with $a = 1$ for rigid geometry (only close nodes are connected) and $a = 0$ for rigid anti-geometry (only distant nodes are connected). Consistently, when $a = 1/2$ there is no geometry and no phase transition. After discussing the numerical analysis, we provide a combinatorial argument to fully explain the mechanism inducing this phase transition and recognize it as an easy-hard-easy transition. Our result shows that, in general, ad hoc optimized networks can hardly be designed, unless to rely to specific heterogeneous constructions, not necessarily scale free.
the percolation problem [20] which consists on mapping the counting of connected clusters toward a mean-field Potts model; an outstanding example of how statistical mechanics can be used to solve combinatorial problems (see also [2]). However, when we are dealing with RGG or small world models, the Kasteleyn and Fortuin approach would require to solve a general (i.e., non mean-field [29]) Potts model, which remains analytically unfeasible.

In this paper we change point of view. We do not aim at analyzing the percolation problem in RGG, nor to have a full descriptive solution of RGG (which would allows to know all the graph metrics). Rather, we look for reduced descriptive, yet exact, solutions. By using the hidden variable framework, and the general result obtained for the typical configurations, we face a practical optimization problem: Given \( N \) nodes, a domain \( D \), and a desired average connectivity \( \langle k \rangle \), find - if any - the distribution of nodes having support in \( D \) and average connectivity \( \langle k \rangle \).

Notice that this is a typical problem in ad hoc wireless optimized networks [21]: embedded in a space with \( d = 2 \) or \( d = 3 \) dimensions, one has \( N \) mobile devices that wants to be connected, in average, with a certain number of neighbors at a minimal cost. Furthermore, it might also be required that whole structure be a connected graph. However, we find out that, in the thermodynamic limit, nodes can be only either uniformly distributed, or highly condensed, the two regimes being separated by a first order phase transition characterized by a finite jump of \( \langle k \rangle \).

Other intermediate values of \( \langle k \rangle \) correspond only to very rare graph realizations. The phase transition is triggered by the underlying geometry whose strength can be tuned by a parameter \( a \in [0, 1] \), with \( a = 1 \) for rigid geometry (only close nodes are connected) and \( a = 0 \) for rigid anti-geometry (only distant nodes are connected). Consistently, when \( a = 1/2 \) there is no geometry and no phase transition. After discussing the numerical solution for RGG in \( d = 1 \) dimensions, we provide a combinatorial argument to fully explain the mechanism inducing this phase transition in general RGG. In turn, the combinatorial argument leads us to recognize the phase transition as an easy-hard-ease transition, the hard solutions of the problem being located in between the uniform and condensed regimes, i.e., at the critical point. Furthermore, we show that the hard solutions corresponding to connected graphs are networks quite heterogeneous. Our result shows that, in general, ad hoc optimized networks can hardly be designed, unless to rely to strongly heterogeneous constructions, not necessarily power law.

II. HIDDEN-VARIABLE MODELS

Given \( N \) nodes, hidden variable models are defined in the following way: i) to each node we associate a hidden variable \( h \) drawn from a given probability density function (PDF) \( \rho(h) \); ii) between any pair of nodes, we assign a link according to a given probability \( p(h, h') \), where \( h \) and \( h' \) are the hidden variables associated to the two nodes. The probability \( p(h, h') \) can be any function of the \( h \)'s, the only requirement being that \( 0 \leq p(h, h') \leq 1 \). The hidden variables can be real numbers or also vectors. Many networks can be embedded in a hidden-variable scheme. For example, in random geometric graphs the \( h \)'s are vectors representing the positions of the nodes, and the \( p(h, h') \)'s are defined in terms of geometric rules (non deterministic if \( 0 < p(h, h') < 1 \), or deterministic if \( p(h, h') \) takes only the values 0 or 1). Particular attention has been paid to the "configuration model". In this case \( p(h, h') \) has the following Fermi-Dirac form (or similar generalizations)

\[
P(h, h') = \left( 1 + \frac{k_s^2}{h h'} \right)^{-1}, \quad k_s = \sqrt{Nk},
\]  

FIG. 1: Typical node-distribution (associated to the typical graphs) for a RGG defined over the circle according to a simple geometric rule \([43]-[44]\) with a rigid constrain \((a = 1)\). The plot gives the distribution \( n_s \) as a function of the angle \( \theta \) for each value of the chemical potential \( \mu \). As the number of nodes \( N \) increases, there emerges a separation of two phases in correspondence of a critical value of the chemical potential \( \mu_c \approx 0.25 \). For \( \mu < \mu_c \) the nodes tend to be uniformly distributed and have a minimal connectivity, whereas, for \( \mu > \mu_c \), a condensation of nodes takes place and tend to form a fully connected graph. Top to bottom: \( N = 10, 20, 40 \).
where, for large $N$, $\bar{k}$ coincides with the average degree. In general, the actual degree $k$ of the nodes of the network realized with the above scheme are distributed according to $\rho$ and average degree equal to $\bar{k}$. In particular, if we choose the following PDF having support in $[h_{\min}, h_{\max}]$

$$\rho(h) = a \ h^{-\gamma}, \quad h_{\max} \geq h \geq h_{\min} > 0, \quad (2)$$

with $\gamma > 2$, the degree-distribution $p(k)$ of the resulting network will be a power law with exponent $\gamma$.

For what follows, is important to note that Eqs. (1) and (2) represent a very particular Hidden-Variable Model. In fact, for such a specific example we have full control of the average number of links via the parameter $k$ inside the function $p(h, h')$. Notice that, given any couple $p(h, h')$ and $\rho(h)$, the hidden variable scheme defined through the steps i)-ii) determines an ensemble of graphs whose properties, like average connectivity, degree distribution, density of triangles etc., are all mathematically encoded in $p(h, h')$ and $\rho(h)$. However, only in a few cases the relation between these ensemble properties and the parameters entering the functions $p(h, h')$ and $\rho(h)$ turn out to be relatively simple. In general, given $p(h, h')$ and $\rho(h)$, we are not able to control directly the ensemble properties via these parameters, not even the average number of links. In fact, this will be the case for RGG. Therefore, for the sake of generality, we will allow for the presence of a chemical potential $\mu$ to be adjusted toward a suitable value in order to have a desired average number of links.

### III. TYPICAL CONFIGURATIONS IN HIDDEN-VARIABLES MODELS

#### A. Motivation, formulation of the problem and synthesis of the method

Let be given a hidden-variable model through the functions $p(h, h')$ and $\rho(h)$. If we make a una tantum sampling of the $h$’s and these are kept fixed, we are dealing with a canonical ensemble of graphs where the $h$’s play the role of parameters. In such a case, by using saddle point techniques is possible to extrapolate the typical number of graphs of the ensemble as well as many characteristics which we can define “typical” [5, 7]. In fact, when the $h$’s are kept constant any motif has a well defined density in thermodynamic limit and fluctuations can be neglected. The ensemble where the $h$’s are random has instead a quite different nature. In this ensemble fluctuations in general cannot be neglected. In particular, it has been shown that, in the scale-free model [1, 2], the link-degree and the entropy are the only self-averaging observables [22, 24]. As a reflection of the lack of self-averaging, it turns out that in this enlarged ensemble the probability $P$ to have a particular graph with a particular hidden variable realization $(G; \{h_i\})$ (or equivalently $\{a_{h_i,h_{i'}}\}, \{a_{h_i,h_j}\}$ being the adjacency matrix for two nodes whose hidden variables are $h_i$ and $h_j$), cannot be expressed in the form $P \propto \exp[\mathcal{H}(G; \{h_i\})]$ (i.e., an exponential random graph [25]; see appendix for details). Nevertheless, in the limit of large $N$, we can find an effective expression for the probability to find some features of the graph. In fact, if we limit the description of a graph to its hidden-variables sequence, such a plan is mathematically possible (it has solutions). To this aim, we analyze the generating function for $P$, $Z(\mu)$ (see Eq. (9)), and perform the change of variables $\{h_i\} \to \{N_{h_i}\}$, where $N_{h_i} = \sum_j \delta_{h_i, h_j}$. With this change of variables, in the limit of large $N$, $Z(\mu)$ takes the form $Z(\mu) \propto \exp(-F(\{N_{h_i}\}))$, where $F(\{N_{h_i}\})$ plays the role of a free energy with respect to the variables $\{N_{h_i}\}$, and the $N_{h_i} = N n_{h_i}$ are solution of the saddle point Eqs. (24). The plan does not instead work when we want to keep a full description of the graph $(G; \{h_i\})$ (because the saddle-point Eqs. with respect to the $h$’s do not have solution, see appendix). Yet, the reduced graph description via the $\{N_{h_i}\}$ is very interesting and adds new crucial knowledge. As we will show, given $N$, $p = \{p_{h,h'}\}$, $\rho = \{\rho_h\}$, and a desired average connectivity $\langle k \rangle$, once we solve the saddle point Eqs. for $n_h$ and for $\mu$, the main result of this Section is a duality relation between the ensemble characterized by $p = \{p_{h,h'}\}$, $\rho = \{\rho_h\}$ and $\mu$, and an ensemble characterized by $\tilde{p}_\mu$, $n$, and $\mu = 0$

$$\text{Ensemble } \{p, \rho, \mu\} = \text{Ensemble } \{\tilde{p}_\mu, n, 0\}, \quad (3)$$

where $\tilde{p}_\mu$ is a proper modification of the probability $p$, Eq. (42). The duality relation (3) allows us to draw, in principle, networks with arbitrary $\langle k \rangle$ without chang-
ing $N$. The approach is particularly appealing for RGG where the $h$’s are vector-positions of the nodes.

### B. Analysis

Let be given a hidden-variable model through the functions $p(h, h')$ and $\rho(h)$. The conditional probability to realize a graph $G$ with frozen hidden variables $\{h_i\}$ is

$$P(G\{h_i\}) = \prod_{i<j} [p(h_i, h_j)]^{a_{i,j}} [1 - p(h_i, h_j)]^{1-a_{i,j}}$$

while the (unconditioned) probability to have $G$ is

$$P(G) = \int \prod_i dh_i \rho(h_i) \prod_{i<j} [p(h_i, h_j)]^{a_{i,j}} \times [1 - p(h_i, h_j)]^{1-a_{i,j}}.$$ \hspace{1cm} (5)

When we deal with the degree-degree uncorrelated configuration model we can choose the connection probabilities $p(h_i, h_j)$ already normalized in such a way that they provide the wanted average degree $\langle k \rangle$. In particular, in the case of the model given by Eqs. [1] and [2], for large $N$ the parameter $\tilde{k}$ coincides with the ensemble average $\langle k \rangle$. As we have already mentioned in Sec. II, in such a case we do not need to introduce a chemical potential. However, in general, we will not be able to control directly the average number of links. In fact, this will be the case for RGG. Therefore, for the sake of generality we now allow for the presence of a general chemical potential $\mu$ to control the average number of links. The probabilistic and operative meaning of $\mu$ will be made clearer later.

The generating function of the probabilities can be obtained from

$$Z(\mu) = \int \prod_i dh_i e^{\mu \sum_i a_{i,j}} P(G\{h_i\})$$ \hspace{1cm} (6)

and the average number of links $\langle L \rangle$ is given by

$$\langle L \rangle = \frac{\partial}{\partial \mu} \log Z(\mu).$$ \hspace{1cm} (7)

By performing the sum over the $\{a_{i,j}\}$ we obtain

$$Z(\mu) = \int \prod_i dh_i e^{-G(h_i)},$$ \hspace{1cm} (8)

where

$$-G(h_i) = \sum_i \log(p(h_i)) + \sum_{i<j} \log(p(h_i, h_j)) e^{\mu} + 1 - p(h_i, h_j).$$ \hspace{1cm} (9)

For the sake of clarity, we will suppose to deal with a discrete distribution for the $h$’s, on a finite set $\mathcal{D}$, and to stress this, we will write $\rho_h$ instead of $\rho(h)$:

$$\sum_{h \in \mathcal{D}} \rho_h = 1.$$ \hspace{1cm} (10)

Later on we will relax this hypothesis on a few easy cases. We notice now that $G\{h_i\}$ depends only on the multiplicities of the $h$’s:

$$-G(h_i) = -\tilde{G}(\{h_i\})$$

$$= \sum_h N_h \log(\rho_h) + \frac{1}{2} \sum_{h,h'} N_h N_{h'} \sigma_{h,h'},$$ \hspace{1cm} (11)

where $(\delta(x, y))$ stands for Kronecker delta function

$$N_h = \sum_i \delta(h_i, h),$$ \hspace{1cm} (12)

i.e., given any ensemble realization $\{h_i\}$, for any $h \in \mathcal{D}$, $N_h$ provides the number of nodes with hidden variable $h$, and

$$\sigma_{h,h'} = \log(p(h, h') e^{\mu} + 1 - p(h, h')).$$ \hspace{1cm} (13)

By using Eqs. [9][13], $Z(\mu)$ becomes

$$Z(\mu) = \sum_{\{N_h\}} e^{-F(\{N_h\})} \prod_{h,h'} \sum_{N_{h'}} \frac{1}{N_h N_{h'}}!$$ \hspace{1cm} (14)

where $\mathcal{N}(\{N_h\})$ is the number of ways in which we can arrange $N$ nodes such that the multiplicities of the $h$’s are $\{N_h\}$, and $\sum(\{N_h\})$ stands for sum over all the normalized multiplicities $\sum_h N_h = 1$. The number $\mathcal{N}(\{N_h\})$ is given by

$$\mathcal{N}(\{N_h\}) = \frac{N!}{\prod_h N_h!},$$ \hspace{1cm} (15)

therefore, by using the Stirling approximation, Eq. (14) becomes

$$Z(\mu) = \sum_{\{N_h\}} e^{-F(\{N_h\})},$$ \hspace{1cm} (16)

where now

$$-F(\{N_h\}) = N \log(N) + \sum_h N_h (\log(\rho_h) - \log(N_h))$$

$$+ \frac{1}{2} \sum_{h,h'} N_h N_{h'} \sigma_{h,h'}.$$ \hspace{1cm} (17)

Notice that, since we have put the “original” probabilistic terms $\{\rho_h^{N_h}\}$ in the weight $\exp(-F(\{N_h\}))$, in the rhs of Eq. (16) the $\{N_h\}$’s in the sum are to be thought as independent random variables uniformly distributed, i.e., formally, the probability to take at random from the ensemble any realization $\{N_h\}$, is a constant. We will evaluate the sum in Eq. (16) by making use of

$$\sum_{\{N_h\}} \sim \int \prod_h dN_h = \int \prod_h dN_h \delta \left(N - \sum_h N_h\right)$$

$$= \frac{1}{2\pi} \int \prod_h dN_h \int dx e^{ix(N-\sum_h N_h)}. \hspace{1cm} (18)$$
The \( N_h \)'s are extensive. Let us rewrite them as
\[
N_h = N n_h.
\]

With these notations we have
\[
Z(\mu) = \sum_{\{ N_h \}} e^{-F(\{ N_h \})}
\]
\[
\sim \frac{N^{\left| D \right|}}{2\pi} \int \prod_h d n_h \int dx e^{-N f(\{ n_h \}; x)},
\]
where \( |D| \) is the cardinality of the set of \( h \)'s, \( D \), and
\[
-f(\{ n_h \}; x) = \log(N) \left( 1 - \sum_n n_h \right)
\]
\[
+ i x \left( 1 - \sum_n n_h \right) + \sum_n n_h (\log(\rho_h) - \log(n_h))
\]
\[
+ \frac{1}{2} N \sum_{h,h'} n_h n_{h'} \sigma_{h,h'}.
\]

(21)

Of course, in the continuum limit \( |D| \to \infty \), so that \( Z(\mu) \)
is ill defined, but the averages are all well defined. We have
\[
- \frac{\partial f}{\partial n_h} = - \log(N) - i x + \log(\rho_h) - \log(n_h) - 1
\]
\[
+ N \sum_{h'} n_{h'} \sigma_{h,h'}
\]

(22)

and
\[
- \frac{\partial f}{\partial x} = i \left( 1 - \sum_n n_h \right).
\]

(23)

The saddle point Eqs. \( \nabla f = 0 \) imply that Eq. (23) fixes the normalization of the \( n_h \)'s: \( \sum_h n_h = 1 \), so that
\[
n_h = \frac{\rho_h e^{N \sum_{h'} n_{h'} \sigma_{h,h'}}}{\sum_{h'} \rho_{h'} e^{N \sum_{h''} n_{h''} \sigma_{h'',h'}}}.
\]

(24)

These Eqs. remain well defined also in the continuum limit \( |D| \to \infty \), being
\[
n(h) = \frac{\rho(h) e^{N \int dh'' n(h') \sigma(h,h')}}{\int dh'' \rho(h'') e^{N \int dh'' n(h') \sigma(h,h')}}.
\]

(25)

Once solved, Eqs. (24) or Eqs. (25), give the solution \( n_h \) to be plugged into the expression for \( Z(\mu) \), and provide the most likelihood sequences \( \{ n_h \} \) that maximizes \( \bar{P}_\mu(\{ n_h \}) \), defined as the probability to pick up at random any graph with given sequence \( \{ n_h \} \) in the presence of a chemical potential \( \mu \):
\[
Z(\mu) \propto e^{-N f(\{ n_h \})}
\]
\[
\sim \max_{\{ n_h \}} \left\{ N(\{ n_h \}) \prod_h \rho_h^{N_h} P_\mu(\{ N_h \}) \right\}
\]
\[
\propto \max_{\{ n_h \}} \left\{ \bar{P}_\mu(\{ N_h \}) \right\},
\]

(26)

where (with an abuse of notation, on noticing that \( f(\{ n_h^* \}; x) \) does not depend on \( x \), we write \( f(\{ n_h^* \}) \) instead of \( f(\{ n_h^* \}; x) \))
\[
-f(\{ n_h^* \}) = \sum_h n_h^* \left( \log(\rho_h) - \log(n_h^*) \right)
\]
\[
+ \frac{1}{2} N \sum_{h,h'} n_h^* n_{h'}^* \sigma_{h,h'}.
\]

(27)

Numerically, we can solve Eqs. (24) only for \( h \)'s discrete. In some cases we can manage the continuum limit analytically. The first example is the classical random graph. In this case \( \sigma(h,h') = \sigma \) is a constant, so that Eqs. (25) immediately gives \( n(h) = \rho(h) \).

C. The role of the chemical potential \( \mu \); complete system of Eqs.

From Eq. (13) we see that
\[
\text{sgn}(\sigma_{h,h'}) = \text{sgn}(\mu).
\]

(28)

Now we see that, when \( \mu \to 0 \), we have \( \sigma_{h,h'} \to 0 \) and, as a consequence, Eqs. (24) give \( \rho_n^\# \to \rho_n \). As we mentioned in Sec. II, in the case of the scale-free model (1)-(2), we are able to control the average connectivity \( \langle k \rangle = 2L/N \) via the parameter \( k \), and we do not need to operate with a \( \mu \neq 0 \), so that \( n_h^* = \rho_h \), consistently with the fact that the distribution of the expected degree \( h \)'s, is equal to \( \rho_h \).

However, if we consider more general cases as, like e.g., RGG, \( p_{i,j} \) is a “rigid” or “soft” function of the relative distances between nodes \( i \) and \( j \) and we are not able to directly control \( \langle L \rangle \) from the parameters of \( p_{i,j} \). We are then forced to use a finite chemical potential \( \mu \). If \( \mu \neq 0 \), we have \( \sigma_{h,h'} \neq 0 \) and then \( n_h \neq \rho_h \); i.e., the system is not in equilibrium with respect to \( \rho_h \). To evaluate \( \mu \), from Eqs. (7), (13) and (24), we have
\[
\langle L \rangle = -N \frac{\partial f(\{ n_h^* \})}{\partial \mu}, \quad \text{where}
\]
\[
\frac{\partial f(\{ n_h^* \})}{\partial \mu} = \sum_h \frac{\partial f}{\partial n_h} \frac{\partial n_h}{\partial \mu} + \frac{1}{2} N \sum_{h,h'} n_h n_{h'} \frac{\partial \sigma_{h,h'}}{\partial \mu},
\]

which, when calculated at the saddle point \( n_h = n_h^* \), simplifies in
\[
\frac{\partial f(\{ n_h^* \})}{\partial \mu} = \frac{1}{2} N \sum_{h,h'} n_h^* n_{h'}^* \frac{\partial \sigma_{h,h'}}{\partial \mu}, \quad \text{with}
\]
\[
\frac{\partial \sigma_{h,h'}}{\partial \mu} = \rho_{h,h'} e^\mu - \sigma_{h,h'}.
\]

In conclusion, we have to solve the systems of Eqs.
\[
n_h = \frac{\rho_h e^{N \sum_{h'} n_{h'} \sigma_{h,h'}}}{N \sum_{h'} \rho_{h'} e^{N \sum_{h''} n_{h''} \sigma_{h'',h'}}},
\]

(29a)

\[
\langle L \rangle = \frac{1}{2} N^2 \sum_{h,h'} n_h n_{h'} \rho_{h,h'} e^{\mu - \sigma_{h,h'}}.
\]

(29b)
or, when $\mathcal{D}$ is continuum

$$n(h) = \frac{\rho(h)e^N \int dh'n(h')\sigma(h,h')}{\int dh''\rho(h'')e^N \int dh'n(h')\sigma(h,h')} = \langle n(h) \rangle.$$  

(30a)

$$\langle L \rangle = \frac{1}{2}N^2 \int dhdh'n(h)n(h')p(h,h')e^{\mu - \sigma(h,h')}.$$  

(30b)

The sums in (29) and the integrals in (30) are understood to run over the set $\mathcal{D}$. In the continuum, the system (30) is a functional system. In such a case the discrete system (29) provides a numerical approximation to (30) whose accuracy grows with the number of points $N_s$ used to discretize $\mathcal{D}$. The system (29) consists of $|\mathcal{D}| + 1$ Eqs. in the $|\mathcal{D}| + 1$ unknowns $\{(n_h); \mu\}$, with the $|\mathcal{D}| + 1$ inputs $\{(\rho_h); \langle L \rangle\}$. In the following, with some abuse of notation, sometimes we will write $\{(\rho_h); \langle L \rangle\}$ to favor $(\rho, L)$ or unfavor $(\mu, L)$.

It is useful to observe that Eq. (30b) (and similarly Eq. (29b)) can also be written as

$$\langle L \rangle = \frac{1}{2}N^2 \int dhdh'n(h)n(h')\tilde{p}_\mu(h,h'),$$  

(31)

where we have introduced the normalized probability

$$\tilde{p}_\mu(h,h') = \frac{p(h,h')e^{\mu - \sigma(h,h')}}{p(h,h)e^{\mu} + 1 - p(h,h')}.$$  

(32)

Eqs. (29b, 32) make clear that we are dealing with an ensemble duality:

$$\text{Ensemble } \{p, \rho, \mu\} = \text{Ensemble } \{\tilde{p}_\mu, n, 0\}.$$  

(33)

The duality (33) says that, given $N$ and $\mathcal{D}$, in order to draw a network with a desired average connectivity $\langle k \rangle = 2\langle L \rangle/N$, we need to look for a proper distribution of nodes $n(h)$ and for a proper distortion of $p(h,h')$ via $\tilde{p}_\mu(h,h')$ to favor $(\mu > 0)$ or unfavor $(\mu < 0)$ the probability to have a link between $h$ and $h'$. Since (33) has been obtained by using a saddle point technique, it is exact only in the thermodynamic limit $N \to \infty$ and, the Ensemble $\{\tilde{p}_\mu, n, 0\}$ contains only the typical graphs of the Ensemble $\{p, \rho, \mu\}$.

IV. APPLICATION TO RANDOM GEOMETRIC GRAPHS

A. General definition of RGG

In RGG the $h$’s represent the positions of the nodes and two nodes are connected or not according to

$$p_{h,h'} = \begin{cases} 1, & \text{if } h \text{ and } h' \text{ satisfy a geometric rule}, \\ 0, & \text{otherwise}. \end{cases}$$  

(34)

Eqs. (34) plugged into Eqs. (13) give

$$\sigma_{h,h'} = \begin{cases} \mu, & \text{if } h \text{ and } h' \text{ satisfy a geometric rule}, \\ 0, & \text{otherwise}. \end{cases}$$  

(35)

Eq. (34) represents rigid RGG. More in general, we might consider soft RGG where Eq. (34) is replaced by

$$p_{h,h'} = \begin{cases} a, & \text{if } h \text{ and } h' \text{ satisfy a geometric rule}, \\ 1-a, & \text{otherwise}, \end{cases}$$  

(36)

where $0 < a < 1$ can be a constant or any function of the node positions $h$ and $h'$.

Particularly important are the RGG defined over a continuous domain $\mathcal{D}$ equipped with a distance. In the deterministic case ($a = 1$ in Eq. (36), for any sprinkle of nodes, there is one single graph obtained by connecting all pairs of nodes whose distance is at most $r$, $r > 0$ being a fixed parameter. Within our approach, the nodes are sprinkled according to the given PDF $\rho(h)$, and the support of $\rho(h)$ defines the domain $\mathcal{D}$. A natural choice for $\rho(h)$ is the uniform PDF $\rho(h)\equiv\text{constant}$, and this will be the case in the following numerical examples. However, for practical and theoretical reasons it is convenient to think at $\rho(h)$ as an arbitrary PDF.

B. Percolation in RGG

Percolation in RGG has been studied for Euclidean RGG’s and also rigorously in $d = 1$ and $d = 2$ dimensions [14, 15]. It is not the aim of this paper to analyze in detail the percolation in RGG, however, it should be clear that the solution $n(h)$ of the saddle point Eqs. (29a) contains crucial information about the percolation problem. In particular, we can analyze the percolation as follows. Let us consider a $d$-dimensional rigid ($a = 1$) Euclidean RGG with domain $\mathcal{D} = [0,1]^d$. Here $h$ represents a position vector in $[0,1]^d$. Given $r > 0$, two nodes are connected if and only if their euclidian distance is at most $r$. Given the number of nodes $N$, the initial density $\rho(h)$ (which in turn defines $\mathcal{D}$), and the solution $n(h)$ of Eqs. (29a), let us define the following set in the continuum

$$\mathcal{D}_1 = \{ h \in \mathcal{D} : N \int_{B(h,r)} n(h')dh' \geq 1 \}.$$  

(37)

where $B(h,r)$ stands for the $d$-ball of radius $r$ centered at $h$. The set $\mathcal{D}_1$ provides the positions $h$’s where, in average, there is at least one node within the balls $B(h',r)'s$. It is then clear that, in the limit in which we can neglect fluctuations of the node positions, the RGG will be percolating if $\mathcal{D}_1$ is a connected set in $\mathcal{D} = [0,1]^d$ (in the topological sense). Let us consider the standard case where nodes are uniformly sprinkled over $[0,1]^d$, i.e., $\rho(h) = 1$. In this case we have also $n(h) = 1$, from which we get that, depending on the value of $r$, we have either $\mathcal{D}_1 = \emptyset$, or $\mathcal{D}_1 = [0,1]^d$. Therefore, since $[0,1]^d$ is connected,
the RGG is percolating if \( r > r_p \), and not percolating if \( r < r_p \), where

\[
 r_p = \frac{1}{(\Omega_d N)^{1/d}},
\]

(38)

where \( \Omega_d \) is the solid angle in \( d \) dimension. Despite our crude approximation in neglecting the node fluctuations, the dependence of \( r_p \) on \( N \) turns out to be correct. In fact, in \( d = 2 \) dimension the percolation threshold \( r_1 \), i.e., the minimal value of \( r \) where a giant connected component appears, is rigorously known to be \( r_1 \propto (1/N)^{1/2} \). However, the critical value of \( r \) above which the RGG is also connected, is rigorously known to be \( r_c \propto (\log(N)/N)^{1/2} > r_1 \), therefore the argument of the above approximation is not fully consistent. However, it has the advantage that it is general and can be applied to any sprinkle \( \rho(h) \), not necessarily uniform or nearly uniform. Clearly, the smaller is the variance of \( \rho(x) \), the better is the approximation. We stress however that the following core paragraphs do not have any direct relation with the percolation problem.

C. Solving the Saddle Point Eqs; Dynamics of RGG

Suppose for simplicity that \( \mu \) is known. Given any \( \rho_h \), which plays the role of an initial density \( n_h(t = 0) = \rho_h \), Eqs. (29a) leads to a natural dynamics over the \( n_h \)'s. For example, we can define a discrete-time dynamics by simple iteration of the \( n_h \)'s:

\[
 n_h(t + 1) = \frac{\rho_h e^{N \sum_{h', h''} \sigma(h', h'') \chi_D(h)}}{\sum_{h'} \rho_{h'} e^{N \sum_{h''} \sigma(h'', h')}},
\]

\[
 n_h(t = 0) = \rho_h.
\]

(39)

Under the hypothesis that there exists a stable fixed point, the above iteration offers a way to solve Eqs. (29a) iteratively. There are many ways to define the dynamics, but all make use of Eqs. (29a) and therefore have the same stationary solutions. In general, if \( \mu \neq 0 \), Eqs. (29a) can generate a “diffusion” with asymptotic solution toward \( n_h(t = \infty) = n_h \). This picture is very appealing for RGG. However, we have to keep in mind that we need to work with a large value of \( |D| = N_s \) in order to recover a diffusion-like trend. In fact, if we choose for \( \rho_h \) a point like distribution:

\[
 \rho_h = \delta_{h, h_0},
\]

(40)

it is immediate to check that Eqs. (29a) are solved with

\[
 n_h = \delta_{h, h_0}.
\]

(41)

More in general, \( n_h \) is always of the form

\[
 n_h = \chi_D(h)f_h,
\]

(42)

where \( \chi_D(h) \) is the characteristic function of the set \( D \) and \( f_h \) a suitable vector. It is then clear that for small values of \( |D| \), we cannot have a diffusion like trend. We expect to see a diffusion when \( |D| \sim N^\alpha \), with \( \alpha > 0 \). RGG however, are usually defined over a continuous domain \( D \) and for an exact solution we should solve the functional system (50). As anticipated, a numerical solution relies on a discretization of \( D \) by using a large enough number of points \( N_s \). Given \( N_s \), the numerical solution will approach the exact solution in the limit \( N_s/N \to \infty \). However, for all practical aims a finite ratio provides excellent approximations. Note that, solving (29) by using the iteration (39), produces only the solutions that are minima (local or global) of the free energy density (28).

Finally, we point out that, although there might exit smarter methods to solve (29), traditional population dynamics here cannot be used. In fact, population dynamics essentially differs from (39) for the fact that the iteration, instead being exact, is sampled a number of times to randomly selected sites and their respective neighbors (here two sites are neighbors if \( \sigma_{h, h'} \neq 0 \)). Population dynamics returns the exact iteration when \( N_s \to \infty \). However, it must be noted that such a sampling relies on the statistical knowledge of the first neighbors of each site, but in our case we do not have access to this knowledge when \( N_s \to \infty \). In fact, if there is a geometry, given \( h \), the number site neighbors of \( h \) diverges for \( N_s \to \infty \). Therefore, we are forced to rely on the full exact iteration (39) with finite \( N_s \).

D. RGG on the circle

We can deepen our understanding with the simplest RGG: the circle. In this case \( D = [0, 2\pi] \). Given the number of nodes \( N \), we choose a discretization of \( D \) with \( N_c \) equidistant sites; in other words the \( N \) nodes can occupy any of the \( N_c \) sites whose positions are given by their angles \( \theta_h \), \( h = 1, \ldots, N_c \). The \( N \) nodes are randomly sprinkled through a given distribution \( \rho_h \). Then we connect or not two points according to the following probability

\[
 p(\theta_h, \theta_{h'}) = \begin{cases} a, & ||\theta_h - \theta_{h'}|| < r, \\ 1 - a, & ||\theta_h - \theta_{h'}|| \geq r, \end{cases}
\]

(43)

where \( a \) and \( r \) are constant, and \( ||\theta - \theta'|| = \pi - ||\theta - \theta'|| \) is the distance on the circle. We set

\[
 r = \frac{4\pi}{N_c}.
\]

(44)

Eqs. (43) plugged into Eqs. (13) give

\[
 \sigma(\theta_h, \theta_{h'}) = \begin{cases} \mu, & ||\theta_h - \theta_{h'}|| < r, \\ 0, & ||\theta_h - \theta_{h'}|| \geq r. \end{cases}
\]

(45)
If for $\rho_h$ we choose the uniform distribution

$$\rho_h = \rho(\theta_h) = \frac{1}{N_s},$$

$$\theta_h \in \left\{ 0, \frac{2\pi}{N_s}, \frac{4\pi}{N_s}, \ldots, \frac{(N_s-1)2\pi}{N_s} \right\}, \quad (46)$$

it is immediate to check that $n_h = \rho(\theta_h)$, independently from $\mu$ (this is true for any RGG: if $\rho_h$ is the uniform distribution, $n_h = \rho_h$). However, for $\rho(\theta_h)$ we can choose a distribution which differs slightly from the uniform one:

$$\rho_h = A(1 + \epsilon_h) \quad (47)$$

where $\epsilon_h$ are uniformly distributed random variables with mean 0 and finite variance, and $A$ is a normalization constant. In general, if we use Eq. (39) with the initial distribution (47), the system may evolve toward a distribution which is different from the uniform one. In principle, any initial distribution (provided not equal to the uniform one) can be used to solve the system (29). However, for $\rho(\theta_h)$ we can choose a distribution which differs slightly from the uniform one:

$$\rho_h = A(1 + \epsilon_h) \quad (47)$$

where $\epsilon_h$ are uniformly distributed random variables with mean 0 and finite variance, and $A$ is a normalization constant. In general, if we use Eq. (39) with the initial distribution (47), the system may evolve toward a distribution which is different from the uniform one. In principle, any initial distribution (provided not equal to the uniform one) can be used to solve the system (29). In fact, we find that via (39) they all tend to the same solution for (29), provided they have the same support in $D$.

We expect that, the larger is $\langle k \rangle = \langle 2L \rangle / N$, the larger is $\mu$. However, it is a priori difficult to know whether, for any $\langle k \rangle$, there corresponds a solution for (29). In fact, in general this is not the case, furthermore, we find out that $\langle k \rangle$, as a function of $\mu$, undergoes a finite jump in correspondence of a critical $\mu_c$, as depicted in Figs. 2, 4.

**FIG. 2:** Distribution $n_\theta$ for the RGG (43)-(44) with $\alpha = 1$, as a function of the angle $\theta$ for each value of the chemical potential $\mu$. The plot is obtained by solving the discrete system (29) via the iteration (39) with a number of iterations $t = 200$. Here the number of nodes is $N = 100$, whereas the discretization parameter is set to $N_s = 1000$. The RGG is exactly reproduced only in the limit $N_s \to \infty$, however, for our numerical aims $N_s = 1000$ is large enough and, on this scale, larger values of $N_s$ do not produce serious differences. Note that, in the localized regime, there is an high degeneracy of solutions, since the $N$ points can get localized in any small region of the available interval in $[0, 2\pi)$. Finite size effects can be seen in Figs. 1.

**FIG. 3:** Analysis of the RGG (43)-(44) for several values of $\alpha$ (here $h$ stands for $\theta$). Average connectivity $\langle k \rangle$ vs the chemical potential $\mu$. Each plot is obtained with the same numerical parameters described in Fig. 2. $\text{Err} = \sum_h |n_h(t+1) - n_h(t)|$ provides an evaluation of the existence of solutions: for $\text{Err} > 0$ there are no solutions; $\text{Var} n_h$ is the variance of $n_h(t)$.
E. The general scenario: a geometry-induced condensation of nodes and links

The phase transition depicted in Figs. 2 and 3 is perfectly compatible with the geometric interpretation and we can summarize the general scenario as follows. Given $N$ nodes and a rigid ($a = 1$) RGG defined over a continuous domain $D$, we are free to arrange the $N$ nodes in any way, with two opposite limit regimes: if nodes are uniformly distributed over $D$, $(k) = O(1)$, while if nodes are localized in a small subset of $D$, $(k) = O(N)$. However, our analysis shows that, when $N \to \infty$, the probability to find a configuration out of these two regimes tends to 0, and the two regimes become two phases separated by a first order phase transition: a uniformly diluted/regular phase, and a condensate of nodes and links in the other phase. Such a transition is triggered by the geometry whose strength can be tuned by the value of the parameter $0 < a \leq 1$. For $a > 1/2$ the two regimes are separated by the geometry: localized nodes have a high number of links. For $a < 1/2$ the two regimes are separated by an anti-geometry: localized nodes have a low number of links. Finally, for $a = 1/2$ there is no geometry and no phase transition. See Fig. 5 for a qualitative phase diagram description. We stress that this phase transition scenario has nothing to share with the percolation phenomena. In fact, as we shall prove later by a combinatorial argument, the scenario does not depend on the details of the model, nor on the dimension $d$, or on the particular value $r$ chosen, which can be changed to scale in the very sparse or dense regimes (as we have numerically checked by replacing $r$ in Eq. (44) with $r \propto 1/N^2$ and $r \propto \log(N)/N$, respectively) without affecting the qualitative features of the phase transition.

F. Combinatorial argument

From a graph construction viewpoint, one might wonder why there is this finite jump in $(k)$. In other words, if we are free to lay down nodes as we like, why only certain values of $(k)$ are visible when $N \to \infty$? Given a RGG with domain $D$, let us consider for simplicity the rigid case $a = 1$. In this case, given the connectivity $(k)$ (or, equivalently, given $\mu$), the configurations are all equiprobable. It is also convenient to proceed with a discretized version of $D$, with $|D| = N_s = \alpha N$, where $\alpha \gg 1$ is an arbitrarily large constant. When we look at all the possible realizations of the graphs, we start with the graph with minimal connectivity $(k) = O(1)$, where the nodes are uniformly distributed and this configuration is unique (0 degeneracy). Next, we consider the opposite case where the graph has maximal connectivity $(k) = N$, where the nodes are densely localized over a small region (average distance smaller than $r$), and this configuration has degeneracy which goes like $N$. Let us now consider the configurations in which there are two separated highly dense regions having $N/2$ nodes each.

FIG. 4: Analysis of the RGG [43]–[44] for several values of $a$ (here $h$ stands for $\theta$). Enlargements of the plots of Figs. 3
In this latter case we have \( \langle k \rangle = N/2 \), and the degeneracy of such possible configurations goes like \( N^2 \). According to this counting, we should then expect to see this latter case, where \( \langle k \rangle = N/2 \), as highly more probable than the former case, where \( \langle k \rangle = N \), which is the opposite of what we have seen in the previous paragraph (see upper panel of fig. 3). The apparent paradox is due to the fact that such a naive counting does not take into account the perturbations from these ideal cases. Suppose to perturb slightly the case with \( \langle k \rangle = N \) in the following way: we lay down all the nodes again in a single small region, yet this region has a slightly larger extension such that the total number of links is \( L_1 = x \cdot N_1 \), where \( N_1 = \binom{N}{2} \), and where \( 0 < x < 1 \), with \( x \approx 1 \). Due to the fact that \( N_2 \) can be set as large as we like (\( D \) is continuous), this is always possible, for any \( x \). When \( x \to 1 \) we recover the ideal case having only one possible type of realization: the fully connected graph. However, when \( x < 1 \), the number of possible graphs \( N_1 \) that we can construct in this way goes like

\[
N_1(x) = \alpha N \left( \frac{N_1}{L_1} \right) \simeq \alpha N e^{N_1 s(x)},
\]

with \( \langle k \rangle = \frac{2L_1}{N} = Nx \) \([48]\)

where the approximation holds for \( N \) large, and \( s(x) \geq 0 \) is a “1-particle entropic factor”

\[
s(x) = -\log(x)x - (1-x) \log(1-x).
\]

Similarly, if we perturb slightly the ideal case with \( \langle k \rangle = N/2 \), in at least one of the two highly dense and separated regions, by introducing \( N_2 = \binom{N/2}{2} \), and \( L_2 = x \cdot N_2 \), we see that the total number of possible graphs \( N_2 \) that we can construct in this way goes like

\[
N_2(x) = \frac{\alpha N}{2} \left( \frac{N_2}{L_2} \right)^2 \simeq \left( \frac{\alpha N}{2} \right) e^{2N_2 s(x)},
\]

with \( \langle k \rangle = \frac{2L_2}{N} = \frac{N}{2} \cdot x \), \([50]\)

where \( s(x) \) is again given by Eq. \([49]\). If we put at ratio Eqs. \([48]\) and \([50]\) we get

\[
\frac{N_2(x)}{N_1(x)} \simeq \frac{\alpha N}{2} e^{(2N_2-N_1) s(x)} = \frac{\alpha N}{2} e^{-\frac{N^2}{2} s(x)}.
\]

We can iterate the above argument for other values of \( \langle k \rangle \). In particular, we can consider the configurations having \( 2^q \) small equally populated regions, each having an average connectivity \( \langle k \rangle = N/2q \), where \( q \) is an integer. In this case the leading term of the ratio between \( N_1(x) \) and \( N_2(x) \) goes like

\[
\frac{N_2(x)}{N_1(x)} \approx \left( \frac{\alpha N}{2q-1} \right) e^{-\frac{N^2}{2q} s(x)},
\]

It is then clear that, as far as \( q \) is finite, the configurations with \( \langle k \rangle \approx N \) highly dominate the others with \( \langle k \rangle \approx N/2q \). We might have a different situation only when \( q = O(\log(N)) \), where \( \langle k \rangle = O(1) \), consistently with what we have shown numerically in the previous paragraphs. Similar arguments hold for other intermediate values of \( \langle k \rangle \).

Remarkable, this combinatorial argument does not depend on the details of the RGG, nor on the dimension \( d \), and neither on the nature of the underlying geometry, which, in principle, can also be non-Euclidean. As soon as we are dealing with a continuous set \( D \) equipped with some distance, our combinatorial argument can be equally applied.

**G. An easy-hard-easy transition**

We can better formalize the findings of the previous paragraphs as follows. Let us consider a rigid \( \langle a = 1 \rangle \)
RGG. For finite $N$ and $N_s$, let $\mathcal{N}(\langle k \rangle)$ be the number of graphs having average connectivity $\langle k \rangle$ each, and let $\mathcal{N} = \sum (\langle k \rangle) \mathcal{N}(\langle k \rangle)$ be the total number of graphs (for $N_s$ finite the sum runs over a finite set of possible values of $\langle k \rangle$). If we introduce the PDF

$$f(\langle k \rangle) = \lim_{N_s \to \infty} \frac{\mathcal{N}(\langle k \rangle)}{\mathcal{N}}, \quad (53)$$

our approach shows that, for $N$ large

$$f(\langle k \rangle) \to b\delta(\langle k \rangle - \Omega_d \rho^d(N - 1)) + (1 - b)\delta(\langle k \rangle - (N - 1)), \quad (54)$$

where $\langle k \rangle = \Omega_d \rho^d(N - 1)$ coincides with the connectivity of a regular $d$-dimensional lattice, $\langle k \rangle = N - 1$ with the connectivity of a fully connected graph, $\delta(x)$ stands for Dirac delta function, and $0 < b < 1$ is a constant. Eq. (54) is consequence of the first order phase transition in $\langle k \rangle$ piloted by $\mu$. On the other hand, the combinatorial argument of the previous paragraph leads us to interpret the phase transition as an easy-hard-easy transition. In fact, when we want to draw graphs with a very low connectivity, $\langle k \rangle = O(1)$, or a very high connectivity, $\langle k \rangle = O(N)$, we are able to figure out how to locate the $N$ nodes in order to have such connectivities. These two opposite cases are indeed relatively easy to build, and their extreme ideal limits corresponding to $\langle k \rangle = \Omega_d \rho^d(N - 1)$, and to $\langle k \rangle = N - 1$, are even trivial. Note that, as the combinatorial argument makes clear, we are still able to build graphs with any desired connectivity $\langle k \rangle$. In order to do so, it is in fact enough to split the $N$ nodes into $m = N/(\langle k \rangle + 1)$ groups each containing $\langle k \rangle + 1$ nodes which, inside the group, are sufficiently close to each other so that each forms a fully connected graph of $\langle k \rangle + 1$ nodes. If the ratio $m = N/(\langle k \rangle + 1)$ is not an integer, it is still possible to consider slight variations of this construction in order to reach the desired average connectivity (for example by introducing some asymmetry among the groups). However, such solutions are very specific. An algorithm that aims at finding the distribution of nodes producing the target $\langle k \rangle$, and that is based on a uniform sampling over the space of graphs, would hardly converge (in a time that scales polynomially with $N$) if $\langle k \rangle \neq \Omega_d \rho^d(N - 1)$ or $\langle k \rangle \neq N - 1$. In fact, our approach shows that we are in the presence of an easy-hard-easy transition: the first order phase transition marks the boundary between two easy phases between which, in correspondence of the critical point $\mu_c$, there are the hard solutions. Furthermore, the above specific solution with $m$ groups has the bad aspect to be in general a disconnected graph (notice that $m < N$). In fact, in most of practical problems, like in ad hoc mobile-networks, one is interested to find solutions which are also connected graphs. Our guess is that such hard connected solutions correspond to specific heterogeneous graphs. Chosen any point of $D$ as a reference center, in these solutions, the highest connected nodes are located near the center of the domain $D$, whereas the lowest connected nodes are located near the boundaries of $D$. The idea is that, radially, nodes are distributed according to a density that decays exponentially with the radial distance with some exponent $\alpha$. The larger is $\alpha$, the smaller is the region occupied by the nodes and - as a consequence - the lager is $\langle k \rangle$. This construction is similar to the construction of hyperbolic RGG where (embedded in a hyperbolic space) nodes are instead uniformly distributed (with respect to the hyperbolic metric) [27]. The network so constructed is connected and can give rise to a wide spectra of cases according to $\gamma$ - the exponent characterizing the degree distribution $p(k)$ - which in turn depends on $\alpha$, but the network could also be non power law. Let us consider a $d = 1$ dimensional RGG with $D = [0, 1]$ and threshold $r$. For a normalized exponential PDF we have

$$n(x) = \frac{x^\alpha}{1 - e^{-\alpha}}e^{-\alpha x}. \quad (55)$$

It is easy to see that, given $N$ nodes, the expected degree $k(x)$ of a node located at $x$ is

$$k(x) = N\frac{e^{-\alpha x}}{1 - e^{-\alpha}}2\sinh(\alpha r), \quad (56)$$

therefore, for $\langle k \rangle$ we have

$$\langle k \rangle = \frac{1 - e^{-2\alpha}}{(1 - e^{-\alpha})^2}N \sinh(\alpha r). \quad (57)$$

Similar expressions hold in any dimension $d$ for a PDF $n(x)$ that decays exponentially with the radial distance: $n(x) = b\exp(-\alpha |x|)/\Omega_d$, with $b$ normalization constant. In particular, it is possible to show that $C_1 \exp(-\alpha |x|) \leq k(x) \leq C_2 \exp(-\alpha |x|)$, where $C_1$ and $C_2$ are two constants. Eq. (57) shows that, whatever is $r$, by properly choosing $\alpha$ we can achieve any desired $\langle k \rangle$. In particular, if we consider the standard choice $r \propto 1/N$, for $\alpha = O(1)$ we have $\langle k \rangle = O(1)$, while for $\alpha = O(N)$ we have $\langle k \rangle = O(N)$, i.e., the regimes corresponding to the two easy phases. Yet, the number of possible graphs that one can actually build by using this construction strongly depends on $\alpha$, or on $\gamma$, if the resulting degree distribution is scale free. In fact, in [25], Del Genio et al. have proved that scale free graphs with $0 < \gamma < 2$ are either very rare or do not exist, while they exist for $0 < \gamma < 2$. Graphs with $\gamma > 2$ are networks with $\langle k \rangle = O(1)$, and are in correspondence to one of the two easy phases of our optimization problem, whereas graphs with $\gamma < 0$ correspond to graph realizations nearly fully connected, and are in correspondence with the other easy phase where $\langle k \rangle = O(N - 1)$. Graphs with $0 < \gamma < 2$, if any, are instead networks that would be able to give rise to $\langle k \rangle = O(N^\beta)$ with $0 < \beta < 1$, i.e., graphs in the set of the hard solutions of our optimization problem (if not empty), but the result of [25] forbids their existence. On the other hand, Eqs. (55)-(57) are exact and define a way to build graphs for any $\alpha$ and therefore for any desired $\langle k \rangle$: even if the number of such graphs strongly depends on $\alpha$, it is never
zero. It is possible to show that this conclusion does not contradict the finding of [23]. In fact, if we indicate with \( p(k|x) \) the conditional probability that a node located at \( x \) has degree \( k \), from \( p(k) = \int dx n(x)p(k|x) \) and \( p(k|x) = (\bar{k}(x))^k/k! \exp(-\bar{k}(x)) \), by using the asymptotic behavior of the incomplete gamma function, for \( d = 1, r \propto 1/N \) and \( N \) large, one has

\[
p(k) \sim \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}, \quad \text{for} \quad \alpha = O(1), \quad (58)
\]

\[
p(k) \sim \text{Constant}, \quad \text{for} \quad \alpha = O(N^\beta), \quad 1 \geq \beta > 0. \quad (59)
\]

Similar expressions can be found as upper bounds for \( d > 1 \). Eqs. (58) and (59) can be used to tune the average number of triangles, or \( \langle k \rangle = O(1) \), the hard solutions correspond approximately to a Poissonian \( p(k) \), while, when we are in a dense regime, with \( \langle k \rangle = O(N^\beta) \), the hard solutions correspond approximately to a uniform \( p(k) \), where all nodes tend to form nearly fully connected structures. The two cases correspond, formally, to \( \gamma \to +\infty \) and to \( \gamma \to 0 \), respectively, compatibly with [28].

V. CONCLUSIONS

By making use of generating function and saddle point techniques, we have derived the equations for the typical node distributions \( n(h) \) of a generic hidden-variable model. We have then applied these equations to RGG to facing a non trivial optimization problem: Given \( N \) nodes, a domain \( D \), and a desired average connectivity \( \langle k \rangle \), find - if any - the distribution of nodes having support in \( D \) and average connectivity \( \langle k \rangle \). However, the numerical solutions of these equations for \( d = 1 \) Euclidean RGG shows that the typical \( n(h) \), in the thermodynamic limit, can be only either uniformly distributed, or highly condensed, the two regimes being separated by a first order phase transition characterized by a finite jump of \( \langle k \rangle \). Other intermediate values of \( \langle k \rangle \) correspond in fact to very rare graph realizations. We have then provided a combinatorial argument to fully explain the mechanism inducing this phase transition in general RGG and recognize it as an easy-hard-easy transition triggered by the geometry, and that the hard and connected solutions correspond to strongly heterogeneous constructions embedded in the geometrical space, but these are not necessarily scale free. Our result concludes that, in general, ad hoc optimized networks embedded in geometrical spaces can hardly be designed, unless to rely to very specific constructions.

In our approach, a crucial mathematical tool has been the use of a chemical potential \( \mu \) in order to tune the desired average connectivity \( \langle k \rangle \). Similarly, one could include in the approach other free parameters in order to control, for example, the average number of triangles, or other interesting graph metrics. Notice that, once the solution \( n(h) \) has been found, one has access not only to the averages of a graph metric, but also to its higher moments. It would be interesting, for certain practical problems described via hidden variable models, to investigate how to better exploit this approach, for example, by requiring that some metrics have also minimal fluctuations, or by requiring that certain correlations are reproduced, similarly to the analysis performed in [29], where clustered scale free models are tuned in such a way to reproduce metrics observed in real world networks.

A different urgent question concerns what this phase transition scenario implies for non Euclidean RGG. We have already mentioned the parallelism with the hyperbolic case [27]. Although our combinatorial argument suggests that this phase transition is expected to be present in any RGG defined over a continuous domain, the issue requires further thoughtful studies.

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Appendix A: Absence of typical detailed configurations in hidden-variables models

In this appendix we show the lack of typical detailed configurations in the ensemble in which the hidden-variables are not fixed. Let be given a hidden variable model via the PDF \( \rho(h) \) and the link probability \( p(h_i, h_j) \). The conditional probability to realize a graph \( G \) with frozen hidden variables \( \{h_i\} \) is

\[
P(G|\{h_i\}) = \prod_{i<j} [p(h_i, h_j)]^{a_{i,j}} [1 - p(h_i, h_j)]^{1-a_{i,j}},
\]

so that the PDF to have \( G \) together with the hidden-variable values \( \{h_i\} \) is

\[
P(G \cap \{h_i\}) = \prod_i \rho(h_i) \prod_{i<j} [p(h_i, h_j)]^{a_{i,j}} \times [1 - p(h_i, h_j)]^{1-a_{i,j}},
\]

where the (unconditioned) probability to have \( G \) is

\[
P(G) = \int \prod_i \rho(h_i) \prod_{i<j} [p(h_i, h_j)]^{a_{i,j}} \times [1 - p(h_i, h_j)]^{1-a_{i,j}}.
\]

The generating function of these probabilities can be obtained from

\[
Z(\{x_{i,j}\}; \{y_{i,j}\}) = \prod_i dh_i \rho(h_i) e^{h_i y_{i,j}} \sum_{\{a_{i,j}\}} P(G|\{h_i\}) e^{x_{i,j} a_{i,j}} \quad (A1)
\]
where \( \{x_{i,j}\} \) and \( \{y_i\} \) are link- and node-auxiliary fields, respectively. Actually, we do not need to use the \( y_i \)'s, since they do not have (at least here) an interesting graph meaning. However, just for completeness, for the moment we keep the \( y_i \)'s general, while we will set them to 0 later on. Let us rewrite \( Z \) as (compact notation: \( p_{i,j} = p(h_i, h_j) \))

\[
Z(\{x_{i,j}\}; \{y_i\}) = \int \prod_i dh_i \prod_i \rho(h_i) e^{y_i h_i} \prod_{i < j} \sum_{a_{i,j}=0,1} a_{i,j} \log(\rho(h_i)) + (1-a_{i,j}) \log(1-\rho(h_j))
\]

By summing over the \( a_{i,j} \)'s, we get

\[
Z(\{x_{i,j}\}; \{y_i\}) = \int \prod_i dh_i e^{Nf(\{x_{i,j}\};\{y_i\};\{h_i\})} (A3)
\]

where

\[
Nf(\{x_{i,j}\}; \{y_i\}; \{h_i\}) = \sum_i (y_i h_i + \log(\rho(h_i))) + \sum_{i<j} \log(1 - p_{i,j} + p_{i,j} e^{x_{i,j}}).
\] (A4)

For \( N \) large we can try to use a “saddle-point” technique by solving for the \( h_i \)'s the system of the \( N \) coupled Eqs.

\[
\frac{\partial f}{\partial h_i} = 0, \quad i=1,\ldots,N.
\] (A5)

From Eq. (A4) we have

\[
\frac{N \partial f}{\partial h_i} = y_i + \frac{1}{\rho(h_i)} \frac{\partial \rho(h_i)}{\partial h_i},
\]

\[
+ \sum_{j: j \neq i} \frac{p_{i,j}}{p_{i,j} (e^{x_{i,j}} - 1) + 1} \frac{\partial p_{i,j}}{\partial h_i}.
\] (A6)

Let us specialize now for the following family of hidden-variable models:

\[
\rho(h) = ah^{-\gamma}, \quad \gamma > 0 (A7)
\]

\[
p(h_i, h_j) = h_i h_j w(h_i h_j), \quad \gamma > 0 (A8)
\]

where \( a \) is a normalization constant, and \( w(x) \) is a positive damping factor such that \( x w(x) \leq 1 \) for any \( x \geq 0 \), which in particular implies \( w(x) \to 0, w'(x) \to 0 \), and \( w(x)/w'(x) \to 0 \) for \( x \to 0 \). For the model (A7 A8), the derivatives take simple forms:

\[
\frac{\partial p_{i,j}}{\partial h_i} = \frac{p_{i,j}}{h_i} + \frac{p_{i,j}^2}{h_i} \frac{w'(x_{i,j})}{w_{i,j}}.
\] (A9)

In particular, for the standard case \( w(x) = 1/(1 + x) \), we have

\[
\frac{\partial p_{i,j}}{\partial h_i} = \frac{p_{i,j}}{h_i} - \frac{p_{i,j}^2}{h_i}.
\] (A10)

For \( N \) large and \( \gamma > 2 \), we can neglect terms in \( p_{i,j}^2 \) w.r.t. to those in \( p_{i,j} \) (this approximation is however not essential here). In conclusion, for the model (A7 A8), Eqs. (A6) become

\[
\frac{N \partial f}{\partial h_i} = y_i - \gamma + \frac{1}{h_i} \sum_{j: j \neq i} \frac{p_{i,j} (e^{x_{i,j}} - 1)}{p_{i,j} (e^{x_{i,j}} - 1) + 1},
\] (A11)

which leads to the following system of saddle-point Eqs. for the \( h_i \)'s

\[
y_i h_i = \gamma - \sum_{j: j \neq i} \frac{p_{i,j} (e^{x_{i,j}} - 1)}{p_{i,j} (e^{x_{i,j}} - 1) + 1}, \quad i=1,\ldots,N.\] (A12)

For a given set of values of the auxiliary fields \( \{x_{i,j}\}; \{y_i\} \), Eqs. (A13) can have one or more solutions. We will indicate a solution of the saddle-point Eqs. (A13) with the superscript \( * \): \( \{h_i^*\} \). Note that \( h_i^* = h_i^* (\{x_{i,j}\}; \{y_i\}) \).

If we set \( y_i = 0 \), Eq. (A13) reduce to

\[
\gamma = \sum_{j: j \neq i} \frac{p_{i,j} (e^{x_{i,j}} - 1)}{p_{i,j} (e^{x_{i,j}} - 1) + 1}, \quad i=1,\ldots,N. \] (A13)

But we immediately see that Eqs. (A13) have no solution for \( x_{i,j} = 0 \), which is the value of the auxiliary fields where we have to set up our calculations at the end to get \( Z \), and its derivatives (in order to get also the averages). In other words, in the ensemble where the \( h_i \)'s are random variables there are no typical graphs.

We can alternatively try to make first the integral over the \( h_i \)'s and only later to sum over the \( \{a_{i,j}\} \).

\[
Z(\{x_{i,j}\}) = \sum_{\{a_{i,j}\}} \int \prod_i dh_i e^{N g(\{x_{i,j}\}; \{a_{i,j}\}; \{h_i\})}, \quad \gamma > 2
\]

where

\[
N g(\{x_{i,j}\}; \{a_{i,j}\}; \{h_i\}) = \sum_i \log(\rho(h_i)) + \sum_{i<j} [a_{i,j} \log(p_{i,j} + x_{i,j}) + (1 - a_{i,j}) \log(1 - p_{i,j})].\]

We have

\[
\frac{N \partial g}{\partial h_i} = -\frac{\gamma}{h_i} + \frac{1}{h_i} \sum_{j: j \neq i} \frac{p_{i,j} (e^{x_{i,j}} - 1)}{p_{i,j} (e^{x_{i,j}} - 1) + 1},
\] (A16)

which leads to the following system of saddle-point Eqs. for the \( h_i \)'s

\[
\gamma = \sum_{j: j \neq i} \frac{a_{i,j} - p_{i,j}}{1 - p_{i,j}}, \quad i=1,\ldots,N. \] (A17)

It is not possible to satisfy such saddle-point Eqs..
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[29] More precisely, one should solve a Potts model on graphs with an underlying geometry. Notice that the critical behavior on small world models is mean-field like, but this does not imply the possibility to solve exactly the model. [ES].