Doubly noncentral singular matrix variate beta distributions

José A. Díaz-García *
Department of Statistics and Computation
25350 Buenavista, Saltillo, Coahuila, Mexico
E-mail: jadiaz@uaaan.mx

Ramón Gutiérrez Jáimez
Department of Statistics and O.R.
University of Granada
Granada 18071, Spain
E-mail: rgjaimez@ugr.es

Abstract
In this paper, we determine the density functions of doubly noncentral singular matrix variate beta type I and II distributions.

1 Introduction
Matrix variate beta type I and II distributions have been studied by many authors using different definitions in the nonsingular case, see Kshirsagar (1961), Constantine (1963), Olkin and Rubin (1964), James (1964), Srinastava (1968), Khatri (1970), Srinastava and Khatri (1979), Cadell (1996) and Díaz-García and Gutiérrez-Jáimez (2007), among many others. Recently, these distributions have been studied in the singular cases by Uhlig (1994), Díaz-García and Gutiérrez (1997) and Díaz-García and Gutiérrez-Jáimez (2008a). Beta type I and II distributions play a very important role in several areas of multivariate statistics, such as canonical correlation analysis, the general linear hypothesis in MANOVA and shape theory, see Muirhead (1982), Srinastava (1968) and Goodall and Mardia (1993).

In the nonsingular case in particular, the doubly noncentral density functions of matrix variate beta type I and II distributions have been studied by diverse authors, but with special emphasis on symmetrised density functions and their application to the theory of matrix variate distribution, the multivariate Behrens-Fisher problem and the generalised regression coefficient, see Chikuse (1980, 1981), Davis (1980), Chikuse and Davis (1980) and Díaz-García and Gutiérrez-Jáimez (2008a).

Using Greenacre’s definition of the symmetrised density function (Greenacre, 1973), in an inverse way, we obtain the doubly noncentral nonsymmetrised density functions or simply, the doubly noncentral density functions of the singular matrix variate beta type I and II distributions, see Section 3. Moreover, as particular cases we find the noncentral

*Corresponding author

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Consider the following definition and notation.

2.1 Singular beta distributions

In this section we give some definitions and notations for the singular matrix variate beta type I and II distributions. We also include two results for the symmetrised function and invariant polynomials with matrix arguments.

2 Preliminary results

In this section we give some definitions and notations for the singular matrix variate beta type I and II distribution. We also include two results for the symmetrised function and invariant polynomials with matrix arguments.

Definition 2.1. Let \( A \) and \( B \) be independent, where \( A \sim \mathcal{PW}_m(r,\Sigma) \) and \( B \sim \mathcal{W}_m(s,\Sigma) \). We define \( U = (A + B)^{-1/2} \mathcal{A} ((A + B)^{-1/2}) \). Then its density function is given and denoted as (see Díaz-García and Gutiérrez [1997])

\[
BI_m(U; q, r/2, s/2) = cL^{(r-m-1)/2}I_m - U^{(s-m-1)/2}(dU), \quad 0 \leq U < I_m. \tag{1}
\]

\( U \) is said to have a singular matrix variate beta type I distribution, and this is denoted as \( U \sim BI_m(q, r/2, s/2) \). Re(s) > (m - 1); where \( U = H_LH_I \), with \( H_L \in \mathcal{V}_{q,m} \); \( V_{q,m} = \{ H_L \in \mathbb{R}^{m \times q} : H_LH_I = I_q \} \) denotes the Stiefel manifold; \( L = \text{diag}(l_1, \ldots, l_q) \), \( 1 > l_1 > \cdots > l_q > 0 \); \( q = m \) (nonsingular case) or \( q = r < m \) (singular case);

\[
c = \frac{\pi^{-(mr+rq)/2} \Gamma_m[(r+s)/2]}{\Gamma_q[r/2] \Gamma_m[s/2]}. \tag{2}
\]

\((dU)\) denotes the Hausdorff measure on \((mq-q(q-1)/2)\)-dimensional manifold of rank-q positive semidefinite \( m \times m \) matrices \( U \) with \( q \) distinct nonnull eigenvalues, given by (see Uhlig [1994] and Díaz-García and Gutiérrez [1997])

\[
(dU) = 2^{-q} \prod_{i=1}^{q} l_i^{m-q} \prod_{i<j} (l_i - l_j) \left( \prod_{i=1}^{q} dl_i \right) \wedge (H_I' dH_I), \tag{3}
\]

where \((H_I' dH_I)\) denotes the invariant measure on \( \mathcal{V}_{q,m} \) and where \( \Gamma_m[a] \) denotes the multivariate gamma function, this being defined as

\[
\Gamma_m[a] = \int_{\mathbb{R}^+} \text{etr}(-R) |R|^{a-(m+1)/2}(dR),
\]

\( \text{Re}(a) > (m - 1)/2 \) and \( \text{etr}(\cdot) \equiv \exp(\text{tr}(\cdot)) \).
Similarly, we have:

**Definition 2.2.** Let \( A \sim \mathcal{PW}_m(r, I) \) and \( B \sim \mathcal{W}_m(s, I) \) be independent. Furthermore, let \( F = B^{-1/2}A(B^{-1/2})' \). \( F \) is then said to have a singular matrix variate beta type \( II \) distribution, denoted by \( F \sim BII_m(q, r/2, s/2) \) and if \( F = H_1G_1H_2, \) with \( H_1 \in \mathcal{Y}_{q,m} \) and \( G = \text{diag}(g_1, \ldots, g_q); g_1 > \cdots > g_q > 0 \), its density function is given and denoted as (see Díaz-García and Gutiérrez-Jáimez [1997])

\[
BII_m(F; q, r/2, s/2) = c|G|^{(r-m-1)/2}|I_m + F|^{-(r+s)/2}(dF), \quad F \geq 0.
\]

where \( c \) is given by \( 2 \), Re\( (s) > (m-1) \) and \( (dF) \) is given in an analogous form to \( 3 \).

Let us now extend these ideas to the doubly noncentral case, i.e. when \( A \sim \mathcal{PW}_m(r, I, \Omega_1) \) and \( B \sim \mathcal{W}_m(s, I, \Omega_2) \). In other words, \( A \) has a noncentral Pseudo-Wishart distribution with a matrix of noncentrality parameters \( \Omega_1 \) and \( B \) has a noncentral Wishart distribution with a matrix of noncentrality parameters \( \Omega_2 \), see Díaz-García et al. [1997]. Subsequently, Díaz-García and Gutiérrez-Jáimez [2008a] reported the following:

**Lemma 2.1.** Suppose that \( U \) has a doubly noncentral matrix singular variate beta type \( I \), which is denoted as \( U \sim BII_m(q, r/2, s/2, \Omega_1, \Omega_2) \). Then using the notation for the operator sum as in Davis [1980] its symmetrised density function is found to be

\[
dF_s(U) = BII_m(U; q, r/2, s/2) \text{etr} \left( -\frac{r}{2}(\Omega_1 + \Omega_2) \right) \times \sum_{\kappa, \lambda, \phi} \frac{\left( \frac{1}{2}(r+s) \right)_\phi}{\kappa! \lambda!} C^{\kappa,\lambda}_\phi(\frac{1}{2}r, \frac{1}{2}s) \frac{C^{\kappa,\lambda}_\phi(U, (I - U))}{C^{\kappa,\lambda}_\phi(I)} (dU),
\]

with \( 0 \leq U < I \), Re\( (s) > (m-1) \), \( (n) \) is the generalised hypergeometric coefficient or product of Pochhammer symbols and \( C^{\kappa,\lambda}_\phi(\cdot, \cdot) \) denotes the invariant polynomials with matrix arguments defined in Davis [1980], see also Chikuse [1998a] and Chikuse and Davis [1986].

Moreover:

**Lemma 2.2.** Suppose that \( F \geq 0 \) has a doubly noncentral singular matrix variate beta type \( II \), which is denoted as \( F \sim BII_m(q, r/2, s/2, \Omega_1, \Omega_2) \). Then its symmetrised density function is

\[
dG_s(F) = BII_m(F; q, r/2, s/2) \text{etr} \left( -\frac{r}{2}(\Omega_1 + \Omega_2) \right) \times \sum_{\kappa, \lambda, \phi} \frac{1}{2}(r+s)_\phi C^{\kappa,\lambda}_\phi(\frac{1}{2}r, \frac{1}{2}s) \frac{C^{\kappa,\lambda}_\phi((I + F)^{-1}F, (I + F)^{-1})}{C^{\kappa,\lambda}_\phi(I)} (dF),
\]

where \( F > 0 \) and Re\( (s) > (m-1) \).

### 2.2 Symmetrised function and invariant polynomials with matrix arguments

Consider the follow extension of the definition given by Greenacre [1973], see also Roux [1973]:

**Definition 2.3.** The symmetrised density function of the non-negative definite matrix \( X : m \times m \), which has a density function \( f_X(X) \), is defined as

\[
f_s(X) = \int_{\mathcal{O}(m)} f_X(HXH')(dH), \quad H \in \mathcal{O}(m)
\]
where $O(m) = \{H \in \mathbb{R}^{m \times m} | (HH') = (H'H) = I_m\}$ and $(dH)$ denotes the normalised invariant measure on $O(m)$, then
\[
\int_{O(m)} (dH) = 1
\]
(Muirhead, 1982, p. 72).

Now consider the following theorem, which generalises eq. (5.4) of Davis (1980), and proof of which is given by Díaz-García (2006).

**Lemma 2.3.** Let $A, B, X$ and $Y$ be $m \times m$ symmetric matrices, then we have
\[
\int_{O(m)} C_{\phi, \lambda}^{*, \lambda}(AH'XH, BH'YH) (dH) = \frac{C_{\phi, \lambda}^{*, \lambda}(A, B)C_{\phi, \lambda}^{*, \lambda}(X, Y)}{\theta_{\phi, \lambda}^{*, \lambda}C_{\phi}(I)},
\]
with
\[
\theta_{\phi, \lambda}^{*, \lambda} = \frac{C_{\phi, \lambda}^{*, \lambda}(I, I)}{C_{\phi}(I)}.
\]

### 3 Doubly noncentral singular matrix variate beta distributions

Taking into account equation (6) it is now is possible to propose an expression for the (nonsymmetrised) density functions of doubly noncentral matrix variate beta type I and II distributions, applying the idea of Greenacre (1973) (see also Roux (1975)), but in an inverse way.

**Theorem 3.1.** Assume that $U \sim BI_m(q, r/2, s/2, \Omega_1, \Omega_2)$. Then its density function is
\[
dF_U(U) = BI_m(U; q, r/2, s/2) \text{etr}\left(-\frac{1}{2}(\Omega_1 + \Omega_2)\right)
\times \sum_{\kappa, \lambda} \frac{1}{k!} \phi_{\kappa, \lambda}(u) \frac{1}{\phi_{\kappa, \lambda}(1/2, \kappa, \lambda)} (U)(I-U) (dU)
\]
with $0 < U < I$, $\text{Re}(s) > (m-1)$.

**Proof.** First observe that $BI_m(U; q, r/2, s/2)$ is a symmetric function, then $BI_m(UH'U; q, r/2, s/2) = BI_m(U; q, r/2, s/2)$. Thus
\[
dF_U(U) = BI_m(U; q, r/2, s/2) \text{etr}\left(-\frac{1}{2}(\Omega_1 + \Omega_2)\right)
\times \sum_{\kappa, \lambda} \frac{1}{k!} \phi_{\kappa, \lambda}(u) \frac{1}{\phi_{\kappa, \lambda}(1/2, \kappa, \lambda)} (U)(I-U) (dU),
\]
for a function $h$. By (6) observe that
\[
\int_{O(m)} C_{\phi, \lambda}^{*, \lambda}(1/2, \Omega_1, \Omega_2, UH'U, I-UH') (dH) = \frac{C_{\phi, \lambda}^{*, \lambda}(1/2, \Omega_1, \Omega_2, U, I-U)}{\theta_{\phi, \lambda}^{*, \lambda}C_{\phi}(I)}.
\]
Then, by applying (5) in an inverse way, in (7) we have
\[
h(U) = \theta_{\phi, \lambda}^{*, \lambda} C_{\phi, \lambda}^{*, \lambda}(1/2, \Omega_1, \Omega_2, U, I-U),
\]
from where the desired result is obtained.
Theorem 3.2. Assume that 
\[ F \sim \mathcal{B}I_{m}(q, r/2, s/2, \Omega_1, \Omega_2). \]  
Then we find that its density function is 
\[ dG_{F}(F) = BII_{m}(F; q, r/2, s/2) \text{etr} \left( \frac{1}{2}(\Omega_1 + \Omega_2) \right) \times \sum_{\kappa, \lambda, \phi} \frac{1}{(\frac{1}{2}r)_{\kappa} (\frac{1}{2}s)_{\lambda}} \phi_{\kappa, \lambda}^{\phi} C_{\phi}^{\kappa, \lambda} \left( \frac{1}{2}\Omega_1 (I + F)^{-1} F, \frac{1}{2} \Omega_2 (I + F)^{-1} \right) (dF), \]
where \( F > 0 \) and \( \text{Re}(s) > (m - 1) \).

**Proof.** The proof is parallel to that given for Theorem 3.1.

In the following two corollaries we shall obtain, as particular cases, the noncentral density functions of singular matrix variate beta type I(A), I(B), II(A) and II(B) distributions.

**Corollary 3.1.** Under the hypothesis of Theorem 3.1:

i) If \( \Omega_1 = 0 \), i.e. \( A \sim \mathcal{P}W_{m}(r, I) \), then we obtain the noncentral singular matrix variate beta type I(A) distribution denoted as 
\[ U \sim B(A)_{m}(q, r/2, s/2, \Omega_2) \]
and its density function is then given by 
\[ dF_{U}(U) = B_{m}(U; q, r/2, s/2) \text{etr} \left( -\frac{1}{2} \Omega_2 \right) \times {}_{1}F_{1} \left( \frac{1}{2}(r + s); \frac{1}{2} s; \frac{1}{2} \Omega_2 (I - U) \right) (dU) \]

ii) Alternatively, if \( \Omega_2 = 0 \), i.e. \( B \sim W_{m}(s, I) \), then we obtain the noncentral singular matrix variate beta type I(B) distribution denoted as \( U \sim B(B)_{m}(q, r/2, s/2, \Omega_1) \), for which its density function is 
\[ dF_{U}(U) = B_{m}(U; q, r/2, s/2) \text{etr} \left( -\frac{1}{2} \Omega_1 \right) {}_{1}F_{1} \left( \frac{1}{2}(r + s); \frac{1}{2} s; \frac{1}{2} \Omega_1 U \right) (dU) \]
with \( 0 \leq U < I, \text{Re}(s) > (m - 1) \) and where \( {}_{1}F_{1}(\cdot) \) is the hypergeometric function with matrix arguments, see [Muirhead, 1982, definitions 7.3.1, p. 258].

**Proof.** The density functions in two items are a consequence of the basic properties of invariant polynomials, see [Davis, 1979, equations (2.1) and (2.3)], see also [Chikuse, 1980, equations (3.3) and (3.6)].

**Corollary 3.2.** Under the conditions of Theorem 3.2:

i) if \( \Omega_1 = 0 \), i.e. \( A \sim \mathcal{P}W_{m}(r, I) \), then we obtain the noncentral singular matrix variate beta type II(A) distribution denoted as 
\[ F \sim BII(A)_{m}(q, r/2, s/2, \Omega_2), \]
and its density function is 
\[ dG_{F}(F) = BII_{m}(F; q, r/2, s/2) \text{etr} \left( -\frac{1}{2} \Omega_2 \right) \times {}_{1}F_{1} \left( \frac{1}{2}(r + s); \frac{1}{2} s; \frac{1}{2} \Omega_2 (I + F)^{-1} \right) (dF) \]


ii) if $\Omega_2 = 0$, i.e. $B \sim \mathcal{W}_m(s, I)$, then we obtain the noncentral singular matrix variate beta type II(B) distribution denoted as

$$F \sim BII_1(B)_m(q, r/2, s/2, \Omega_1),$$

for which its density function is

$$dG_p(F) = BII_m(F; q, r/2, s/2) \text{etr} \left(-\frac{1}{2} \Omega_1 \right) \times _1 F_1 \left(\frac{1}{2}(r + s); \frac{1}{2}s; \frac{1}{2} \Omega_1(I + F)^{-1}F\right) (dF)$$

with $0 \leq F$ and $\text{Re}(s > (m - 1))$.

Proof. The proof is analogous to that given for Corollary 3.1.

Conclusions

Chikuse (1980), Chikuse (1981) and Davis (1979) have found the symmetrised doubly noncentral density functions of the nonsingular matrix variate beta type I and II distributions. However, the question of nonsymmetrised density functions (or simply density functions) remained to be resolved. In this paper, by applying Greenacre’s definition of symmetrised function of Greenacre (1973) in an inverse way, we respond to these two open problems with respect to singular and nonsingular cases. Furthermore, in another way to the method given in Díaz-García and Gutiérrez-Jáimez (2007) and in Díaz-García and Gutiérrez-Jáimez (2008a), we obtain the noncentral density functions of singular matrix variate beta type I(A), I(B), II(A) and II(B) distributions, from where, implicitly, we resolve the integral proposed by Constantine (1963), Khatri (1970) and reconsidered in Farrell (1985, p. 191), see also Díaz-García and Gutiérrez-Jáimez (2007), in the singular case, and in the nonsingular one, of course, by simply taking $q = m$.

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