Convergence of Non-Perturbative Approximations to the Renormalization Group

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We provide analytical arguments showing that the non-perturbative approximation scheme to Wilson’s renormalisation group known as the derivative expansion has a finite radius of convergence. We also provide guidelines for choosing the regulator function at the heart of the procedure and propose empirical rules for selecting an optimal one, without prior knowledge of the problem at stake. Using the Ising model in three dimensions as a testing ground and the derivative expansion at order six, we find fast convergence of critical exponents to their exact values, irrespective of the well-behaved regulator used, in full agreement with our general arguments. We hope these findings will put an end to disputes regarding this type of non-perturbative methods.

Wilson’s renormalization group (RG) is an extraordinary means of understanding quantum and statistical field theories. Its perturbative implementation1–3, in particular under the form of the $\epsilon$-expansion, has been a very efficient toolbox. The last twenty-five years, however, have witnessed a revival of Wilson’s RG because an alternative formulation4–8 has allowed for new and nonperturbative approximation schemes (which are anyway needed to solve the exact RG equation). Using this nonperturbative approach to the RG (NPRG hereafter), remarkable results have been obtained on problems that were either very difficult or fully out of reach of the perturbative approach.

The method is versatile, allowing to treat equilibrium and non-equilibrium problems, disordered systems, with access to both universal and non-universal quantities. To list a few successes just within statistical physics, let us mention the random field Ising model (spontaneous supersymmetry breaking and the associated breaking of dimensionality reduction in a nontrivial dimension)3,10, the Kardar-Parisi-Zhang equation in dimensions larger than one (identification of the strong coupling fixed point)11–13, the glassy phase of crystalline membranes14, systems showing different critical exponents in their high and low temperature phases13, the phase diagram of reaction-diffusion systems16,17, etc.

Most of these results were obtained using an NPRG approximation scheme known as the derivative expansion (DE). In a nutshell, the underlying ideas are as follows: The exact NPRG equation governs the evolution of an effective action $\Gamma_k$ (in the field theory language the generating functional of one-particle irreducible correlation functions) with the RG momentum scale $k$. In the NPRG approach, a regulator function $R_k(q^2)$ ensures that the large wavenumber modes (with $q^2 > k^2$) are progressively integrated over while the others are frozen. When $k = 0$, all statistical fluctuations have been integrated and $\Gamma_{k=0} = \Gamma$, the Gibbs free energy of the model. The DE consists in approximating the functional $\Gamma_k[\phi]$, where $\phi$ represents all the fields of the problem, by its Taylor expansion in gradients of $\phi$ truncated at a finite order.

In spite of its undeniable successes, the DE —and the NPRG in general— has often been criticized. Two main points are usually raised, the (apparent) lack of a small parameter controlling its convergence and the arbitrariness induced by the choice of the regulator function $R_k$. Indeed, within any approximation scheme, the end results do carry a residual influence of $R_k$. This has been often invoked against the NPRG approach, even though the dependence on $R_k$ is similar to the renormalization scheme dependence in perturbation theory18.

In this Letter, we aim to put an end to this controversy. We use the Ising model as a testing ground both because its relative simplicity allows us to study the sixth order of the DE and because its critical exponents are accurately known19,20. We provide numerical evidence and analytical arguments showing that the DE not only converges, but does so rapidly. This conclusion holds beyond the Ising model and is most likely generic. Contrary to usual perturbative approaches, we find that the DE has (i) a finite radius of convergence and (ii) a fast convergence, even at low orders, when the anomalous dimension is small. We also discuss the respective quality of regulators $R_k(q^2)$ and propose empirical rules for selecting optimal ones, without prior knowledge about the problem at stake.

We start with a brief review of the NPRG, specialized here to the $\phi^4$ model for convenience. A one-parameter family of models indexed by a scale $k$ is defined such that only the short wavelength fluctuations, with wavenumbers $q = |q| > k$, are summed over in the partition function $Z_k$. The decoupling of the slow modes, $\varphi(|q| < k)$, in $Z_k$ is performed by adding to the original Hamiltonian $H$ a quadratic (mass-like) term which is nonvanishing only for these modes:

$$Z_k[J] = \int D\varphi \exp \left[-H[\varphi] - \Delta H_k[\varphi] + \int x J_{\varphi} \right]$$ (1)
where $\Delta H_k[\phi] = 1/2 \int_0^\infty R_k(q^2)\phi(q)\phi(-q)$. The form of the regulator function $R_k(q^2)$ is discussed in detail below (see Eqs. \textbf{[7]} for examples used here). The $k$-dependent Gibbs free energy $\Gamma_k[\phi]$ (with $\phi(x) = \langle \phi(x) \rangle$) is defined as the (slightly modified) Legendre transform of $\log Z_k[J]$:

$$\Gamma_k[\phi] + \log Z_k[J] = \int_x J\phi - \frac{1}{2} \int_q R_k(q^2)\phi(q)\phi(-q). \quad (2)$$

The exact RG flow equation of $\Gamma_k$ reads \textbf{[4]}:

$$\partial_t \Gamma_k = \frac{1}{2} \int_q \partial_t R_k(q^2) (\Gamma^{(2)}_k + R_k)^{-1}[q, -q; \phi] \quad (3)$$

where $t = \log(k/\Lambda)$ and $\Gamma^{(2)}_k[q, -q; \phi]$ is the Fourier transform of the second functional derivative of $\Gamma_k[\phi]$. The DE consists in solving Eq. \textbf{(3)} in a restricted functional space where $\Gamma_k[\phi]$ involves a limited number of gradients of $\phi$ multiplied by ordinary functions of $\phi$. Zeroth order is the commonly-used local potential approximation (LPA): only the momentum dependences present in $H$ are kept in the correlation functions. For the $\phi^4$ model, $\Gamma_k[\phi]$ is then approximated by: $\int_q \left( \bar{U}_k(\phi) + \frac{1}{2}(\nabla \phi)^2 \right)$, only a running priming order is retained. At order $s = 6$, the ansatz for $\Gamma_k$ involves thirteen functions (see section I of the Supplemental Material):

$$\Gamma_k[\phi] = \int d^d x \left[ U_6(\phi) + \frac{1}{2} Z_4(\phi)(\partial_\mu \phi)^2 + \frac{1}{2} W_8(\phi)(\partial_\mu \partial_\nu \phi)^2 + \cdots + \frac{1}{4!} X_6(\phi) (\langle \partial_\mu \phi \rangle^2)^3 \right]. \quad (4)$$

The flow of all functions is obtained by inserting the ansatz \textbf{(4)} in Eq. \textbf{(3)} and expanding and truncating the right hand side on the same functional subspace. In practice, this is implemented in Fourier space. For instance, we obtain from Eq. \textbf{(4)} that $Z_k(\phi) = \partial_\phi Z_k^{(2)}(\phi, \phi)|_{\phi=0}$ with $\phi$ a constant field. Thus, the flow of $Z_k(\phi)$ is given by the $p^2$ term of the flow of $\Gamma_k^{(2)}(\phi, \phi)$.

At criticality —the regime of interest here—, the system is self-similar: its RG flow reaches a fixed point. In practice, the fixed point is reachable when using dimensionless and renormalized functions denoted below by lowercase letters $(u_k, z_k, \ldots, x_k, k)$. We proceed as usual \textbf{[3]} by rescaling fields and coordinates. Here $\tilde{x} = k x$, $\phi(\tilde{x}) = \sqrt{Z_k^{(n)}(2-\delta)/2} \phi(x)$. Functions are then rescaled according to their canonical dimension and renormalized by $Z_k^{(n)}(n/2)$ where $n$ is the number of fields they multiply in the ansatz \textbf{(4)}. This leads to $Z_k(\phi) = Z_k^n z_k(\phi)$. The absolute normalization of both $Z_k^n$ and $z_k(\phi)$ is defined only once their value is fixed at a given point. We use the (re)normalization condition: $z_k(\phi_0) = 1$ for a fixed value of $\phi_0$. The running anomalous dimension $\eta_k$ is then defined by $\eta_k = -\partial_\phi \log Z_k^n$. It becomes the anomalous dimension $\eta$ at the fixed point \textbf{[5]}.

Let us now give analytical arguments in favor of the convergence of the DE and about the rapidity of this convergence. We continue using the $\phi^4$ theory here, but our results are more general. The key remark is that the momentum expansion applied to the theory away from criticality, either in the symmetric or broken phase, is known to be convergent with a finite radius of convergence. For instance, calling $m$ the mass, that is, the inverse correlation length, the $c_n$ in

$$\frac{\Gamma^{(2)}(p, m)}{\Gamma^{(2)}(0, m)} = \frac{\Gamma^{(2)}(0, m)}{\Gamma^{(2)}(0, m)} = 1 + \frac{p^2}{m^2} + \sum_{n=2}^\infty c_n \left( \frac{p^2}{m^2} \right)^n \quad (5)$$

are universal close to criticality and behave at large $n$ as $c_{n+1}/c_n \sim -1/9$ and $-1/4$ in the symmetric and broken phases respectively (see, for example, \textbf{[3]}). These behaviors follow from the fact that the singularity nearest to the origin in the complex $p^2$ plane is $9m^2 (4m^2)$ because the Minkowskian version of the theory has a three- (two-) particle cut in the symmetric (broken) phase respectively \textbf{[22]}. Our argument relies on the fact that any regulator acts as a (momentum dependent) mass term. Thus, the critical theory regularized by $R_k(q^2)$ should be similar to the non-critical (massive) theory and should therefore also have a convergent expansion in $p^2/k^2$—which is nothing but the DE— with a finite radius of convergence that we call $R$ typically between 4 and 9 as we show below.

At criticality and for $s = 6$, the analogue of Eq. \textbf{(5)} is

$$\frac{\Gamma_k^{(2)}(p, \phi) + R_k(0)}{\Gamma_k^{(2)}(0, \phi) + R_k(0)} = 1 + \frac{Z_k p^2 + W_8^p 4 + X_6^p 6}{U_k^p + R_k(0)} \quad \rightarrow 1 + \frac{p^2}{m_{\text{eff}}^2} + \frac{w_8^p v_{8}^1}{z^2} \frac{p^4}{m_{\text{eff}}^4} + \frac{X_6^p v_{10}^3}{z^3} \frac{p^6}{m_{\text{eff}}^6} \quad (6)$$

where the field dependence in the r.h.s. has been omitted, primes denote derivation wrt $\phi$ or $\phi^\prime$, $u^+, z^+, w_8^p, s_6^p$ stand for the dimensionless functions of $\phi$ at the fixed point, and $m_{\text{eff}}^2 = k^2 v''/z^2$ with $v'' = u'' + R_k(0)/Z_k^2 k^2$.

If the two expansions \textbf{(5)} and \textbf{(6)} are indeed similar then $m_{\text{eff}}^2$ must be the mass generated by the regulator and the coefficients of $p^2/m_{\text{eff}}^2$, $p^4/m_{\text{eff}}^4$, and $p^6/m_{\text{eff}}^6$ must be analogous to $c_2$ and $c_3$ in Eq. \textbf{(5)}. As for $m_{\text{eff}}^2$, if it is indeed generated by the regulator, it must be of order $R_k(q^2 = 0) \simeq k^{2}$ (see discussion below and Eq. \textbf{(3)}). It is known that the $c_n$ in Eq. \textbf{(6)} form an alternating series and that they are very small \textbf{[3]}: In the symmetric phase, $c_2 = -4 \times 10^{-4}$ and $c_3 = 0.9 \times 10^{-5}$ and in the broken phase, $c_2 \simeq -10^{-2}$ and $c_3 \simeq 4 \times 10^{-3}$. Together with the fact that the series in Eq. \textbf{(6)} has a finite radius of convergence, this suggests that it not only converges but that it does so rapidly.

Let us now discuss the role and shape of the regulator function $R_k(q^2)$. If no approximation of the exact flow equation \textbf{(3)} were made, all physical quantities would be independent of the choice of regulator. However the DE —like any approximation scheme— introduces an influence of the choice of $R_k$ on the end results \textbf{[22]}. There exist some constraints and a priori guidelines to choose $R_k$ so that its influence stays minimal. First, $R_k$ must freeze
that the small momentum modes $\varphi(|q| < k)$ in $Z_k[j]$ (Eq. (1)), so that they decouple from the long distance physics. It must also leave unchanged the large momentum modes $\varphi(|q| > k)$. Second, the DE being a Taylor expansion of the $\Gamma_k^{(n)}(\{p_i\})$ in powers of $p_i \cdot p_j/k^2$ (around 0), it is valid provided $p_i \cdot p_j/k^2 < R$. This implies that whenever the $\Gamma_k^{(n)}$'s are replaced in a flow equation by their DE, the momentum region beyond $R$ must be efficiently cut off. This is the role of the $\partial_i R_k(q^2)$ term in Eq. (8). It suppresses this kinematic sector in the integral over the internal momentum $q$ if $R_k(q^2)$ almost vanishes for $|q| \gtrsim k$. On the other hand, modes $\varphi(|q| < k)$ are almost frozen if $R_k(q^2)$ is of order $k^2$ for $|q| < k$. These two characteristic features give the overall shape of $R_k(q^2)$. Note also that if a non-analytic regulator is chosen, one must make sure that the non-analyticities thus introduced in the complex plane of $q^2$ are further than $R$ from the origin. Finally, at order $s$ of the DE the flow equations of the functions involve $\partial_i R_k(q^2)$ and $\partial^a R_k(q^2)$ from order $n = 1$ to $s/2$. Since the DE is performed around $q = 0$, it is important that these derivatives decrease monotonically: if not, a “bump” at a finite value $q^2 = q_0^2 > 0$ would magnify a region around $q_0$ which is less accurately described by the DE (24).

Taking into account all the prerequisites above, we have considered either regulators that are $C^\infty$ in the complex plane of $q^2$, decay rapidly but do not vanish for $q > k$, or functions that vanish identically for $q > k$, are not $C^\infty$ but are sufficiently differentiable for regularizing the DE at the order $s$ studied, and have their derivatives as small as possible for $q \approx k$. Specifically, we used:

\[ W_k(q^2) = a Z_k^n k^2 y/(\exp(y) - 1) \]  
\[ \Theta_k(q^2) = a Z_k^n k^2 (1 - y)^n \theta(1 - y) \quad n \in \mathbb{N} \]  
\[ E_k(q^2) = a Z_k^n k^2 \exp(-y) \]  

where $y = q^2/k^2$. We show in the Supplemental Material that $\Theta_k(q^2)$ is equivalent to $E_k(q^2)$.

We now present our results (27). We focus here on the three-dimensional (3D) case, for which near-exact results are provided by conformal bootstrap (13 21), but we have obtained similar in two dimensions (28). For each regulator function $R_k$, we have calculated the critical exponents $\nu$ (associated with the divergence of the correlation length) and $\eta$, as well as the different ratios appearing in Eq. (9), at orders $s = 0$ (LPA), 2, 4, and 6. (Numerical details can be found in Section D of the Supplemental Material.) These quantities depend on the parameters of $R_k$, that is, for the regulators (7), on $\alpha$ that we typically vary in the range $[0.1, 10]$.

Each regulator function we studied yielded very similar results. We first discuss those obtained with (7). In Fig. 1 we show the curves $\nu(\alpha)$ and $\eta(\alpha)$ for orders $s = 2$ to 6. At each order, exponent values exhibit a maximum or a minimum at some value $\alpha_{\text{opt}}$ as $\alpha$ varies. Following a “principle of minimal sensitivity” (29 31), we select the values $\nu(\alpha_{\text{opt}})$ and $\eta(\alpha_{\text{opt}})$ taken at the extrema as our best estimates. Note that this is the situation closest to the exact theory, for which there is no dependence on the regulator. At a given order $s$, $\alpha_{\text{opt}}^{(\nu)}$ and $\alpha_{\text{opt}}^{(\eta)}$ are close but different, and their difference decreases fast with increasing $s$, see Fig. 1.

Important remarks are in order. For each exponent, increasing the order $s$: (i) extrema alternate between being given by a maximum and a minimum; (ii) the local curvature at $\alpha_{\text{opt}}$ increases; (iii) strikingly, the exponent values at $\alpha_{\text{opt}}$ essentially alternate around and converge very fast to values very close to the conformal bootstrap “exact” ones. (At order 6, the optimal value of $\nu$ ‘crosses’ the exact value, but these 2 numbers coincide up to 3 or 4 significant digits, see Fig. 1 and Table I.) The increase of curvature at $\alpha_{\text{opt}}$ and the accompanying faster variations of exponent values with $\alpha$ as $s$ is increased imply that it is crucial to work with the optimal values given by the extrema, that is $\nu(\alpha_{\text{opt}}^{(\nu)})$ and $\eta(\alpha_{\text{opt}}^{(\eta)})$. This fast, alternating convergence is due to the alternating nature of curvature at $\alpha_{\text{opt}}$. (At order 6, the optimal value of $\nu$ ‘crosses’ the exact value, but these 2 numbers coincide up to 3 or 4 significant digits, see Fig. 1 and Table I.) The increase of curvature at $\alpha_{\text{opt}}$ and the accompanying faster variations of exponent values with $\alpha$ as $s$ is increased imply that it is crucial to work with the optimal values given by the extrema, that is $\nu(\alpha_{\text{opt}}^{(\nu)})$ and $\eta(\alpha_{\text{opt}}^{(\eta)})$. This fast, alternating convergence is due to the alternating nature

| D.E. | $\nu$ | $|\delta\nu|$ | $\eta$ | $|\delta\eta|$ |
|------|-------|--------------|-------|--------------|
| s = 0 | 0.65103 | 0.02106 | 0 | 0.03630 |
| s = 2 | 0.62752 | 0.00245 | 0.04551 | 0.00921 |
| s = 4 | 0.63057 | 0.00060 | 0.03357 | 0.00273 |
| s = 6 | 0.63007 | 0.00010 | 0.03648 | 0.00018 |

| conf. boot. | 0.629971(4) | 0.0362978(20) |
| 6-loop | 0.6304(13) | 0.0335(25) |
| High-T. | 0.63012(16) | 0.03639(15) |
| M.-C. | 0.63002(10) | 0.03627(10) |

**TABLE I.** 3D Ising critical exponents obtained with regulator (26) at orders $s = 0$ (LPA) to 6. Absolute distances between these values and the near-exact conformal bootstrap (21) ones are given by $|\delta\nu|$ and $|\delta\eta|$. Monte-Carlo (22). High-temperature expansion (26), and 6-loop perturbative RG values (3) are also given for comparison.
of the series of coefficients $c_n$. The speed of convergence is also in agreement with our considerations above about the radius of convergence $R$ of the DE at criticality: the amplitude of the oscillations of the optimal values considered as functions of $s$ decreases typically by a factor between 4 to 10 at each order $s$ (Table I).

As mentioned above, all regulators we studied yield very similar results. For each exponent, the dispersion of values (over all regulators studied) typically also decreases by a factor 4 to 8 when going from one order $s$ to the next, something we interpret as another manifestation of the radius of convergence of the DE, see Table I. We also noticed that regulators not satisfying our prerequisites very well typically yield “worse” results, somewhat away from those given by the set of good regulators $\chi$ [28]. Our extensive exploration of regulators, including some multi-parameter ones not described here, thus leads us to conjecture the existence of a “ceiling” —or a “floor”, depending on the exponent and the order considered—that cannot be passed by any regulator (taken at its optimal parameter value $\alpha_{\text{opt}}$), see Section E of the Supplemental Material.

We have also checked for all regulators and all $\phi$ that $u_s^* \sim s^2 < 0$ and $x_s^* \sim s^3 > 0$ in agreement with the signs of $c_2$ and $c_3$. The ratio $r = x_s^*/u_s^*$, which plays a role analogous to $c_3/c_2$, varies between $\frac{1}{36}$ for $\phi$ around $\tilde{\phi}_{\text{min}}$, the minimum of the potential, and $\frac{1}{6}$ at large $\phi$ and is largely regulator independent (see Section F of the Supplemental Material). These values correspond typically to what is found in the symmetric and broken phases respectively, which is expected for a regularized theory at criticality.

We now go a step further and explain the behavior of the coefficients of $p^4$ and $p^6$ in Eq. (6). We know that at criticality, when $p \gg k$, $\Gamma_k^{(2)}(p, \phi) \simeq \Gamma_k^{(2)}(p, 0) \propto p^{2-\eta}$. On the other hand, when $p \ll k$, $\Gamma_k^{(2)}(\phi)$ is given by Eq. (6) at $\phi = 0$. Matching these two expressions for $p \sim k$, we find a simple analytic representation of the form $\Gamma_k^{(2)}(p) \approx A p^2 (p^2 + b k^2)^{-\eta/2} + m_k^2$ where $A$ and $b$ are constants and $m_k = 0$. Expanded in powers of $p^2/k^2$, this expression yields an alternating series with a negative coefficient starting from $p^4$ and a positive one for $p^6$ as in Eq. (6). Moreover, all coefficients of the series from $p^4$ are proportional to $\eta$, which makes all of them naturally small, again as in Eq. (6). We therefore conclude that the DE is a convergent expansion with (i) a finite radius of convergence typically between 4 and 9 and, (ii) a rapid convergence because all the coefficients of the $(p^2/k^2)^n$ terms with $n \geq 2$ are proportional to $\eta$, which is small for the 3D Ising model [34].

In summary, we have shown that the derivative expansion often used in NPRG studies has a finite radius of convergence and we provided guidelines for choosing the regulator function at the heart of the procedure. Using the Ising model in three dimensions as a testing ground, we find fast convergence of critical exponents to their exact values, irrespective of the well-behaved regulator used, in full agreement with our general arguments. Our findings naturally extend to many other models –
those having a unitary Minkowskian extension—and to other NPRG approximations such as the Blaizot-Mendéz-Wschebor scheme [35–37]. This establishes firmly that the NPRG approach is not only versatile, being able to deal with any equilibrium or non-equilibrium model, but also quantitative, providing accurate results even at low orders.

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SUPPLEMENTAL MATERIAL

A. Ansatz for $\Gamma_k$ at order 6

At order 6 of the derivative expansion, the ansatz for $\Gamma_k(\phi)$ reads:

\[
\begin{align*}
\Gamma_k(\phi) &= \int d^2x \left[ U_k(\phi) + \frac{1}{2} Z_k(\phi)(\partial \phi)^2 \
+ \frac{1}{2} W_k^\alpha(\phi)(\partial_\mu \partial_\nu \phi)^2 + \frac{1}{2} \phi W_k^\alpha(\phi)(\partial^2 \phi)(\partial \phi)^2 \
+ \frac{1}{2} W_k(\phi) \left( (\partial \phi)^2 \right)^2 + \frac{1}{2} \bar{X}_k^\alpha(\phi)(\partial_\mu \partial_\nu \phi)(\partial_\mu \partial_\nu \phi) \
+ \frac{1}{2} \phi \bar{X}_k^\alpha(\phi)(\partial \phi)^2 \left( (\partial^2 \phi)^2 + \frac{1}{2} \bar{X}_k^\alpha(\phi) \left( (\partial \phi)^2 \right)^2 + \frac{1}{2} \bar{X}_k^\alpha(\phi) \left( (\partial \phi)^2 \right)^3 \right). \end{align*}
\]

The vertex functions $\Gamma_k^{(n)}(\{p_i\}, \phi)$ where $\phi$ is a constant field can be computed in Fourier space from $\Theta_k$ in terms of $U_k(\phi), \ldots, \bar{X}_k^\alpha(\phi)$. The definition of these functions follows from this calculation and yields for instance:

\[
\begin{align*}
X_a &= X_a, \quad \bar{X}_b = X_b - 2X_c, \quad \bar{X}_c = X_c \\
\bar{X}_d &= \frac{1}{2}X_d - \frac{1}{2}(X_c + \phi X'_c) \\
\bar{X}_e &= \frac{1}{2}(X_d + X_e + X_f) - (X_e + \phi X'_e) \\
\bar{X}_f &= \frac{1}{2}X_d + \frac{1}{2}(X_e - X_f) - \frac{1}{2}(X_c + \phi X'_c) \\
\bar{X}_g &= 6X_g + 2(2X'_d + \phi X''_d) - X'_d - 9X_c - 9X_f \\
\bar{X}_h &= X_h + 20(3X''_d + \phi X'''_d) - 5X''_d - 7X_e - 9X'_f \quad (9)
\end{align*}
\]

The vertex functions $\Gamma_k^{(n)}(\{p_i\}, \phi)$ where $\phi$ is a constant field can be computed in Fourier space from $\Theta_k$ in terms of $U_k(\phi), \ldots, \bar{X}_k^\alpha(\phi)$. The definition of these functions follows from this calculation and yields for instance:

\[
\begin{align*}
X_a(\phi) &= \partial_{p^2} \Gamma_k^{(2)}(p; \phi)|_{p=0} \\
X_b(\phi) &= \partial_{(p_1, p_2)} \Gamma_k^{(6)}(p_1, p_2, \phi)|_{p_1 = p_2 = 0} \\
X_c(\phi) &= \partial_{(p_1, p_2, p_3, p_4, p_5)} \Gamma_k^{(10)}(p_1, \ldots, p_5; \phi)|_{p_1 = \ldots = p_5 = 0} \quad (10)
\end{align*}
\]

B. Flow equations, truncation issues

Let us consider the function $Z_k(\phi)$ defined by

\[
Z_k(\phi) = \partial_{p^2} \Gamma_k^{(2)}(p; \phi)|_{p=0} \quad (11)
\]

Its flow is given by the $p^2$ term of the flow of $\Gamma_k^{(2)}(p; \phi)$:

\[
\partial_t \Gamma_k^{(2)}(p; \phi) = \int_q \hat{R}_k(q^2) G_k^2(q) \left[ - \frac{1}{2} \Gamma_k^{(4)}(p - p, q, -q) + \Gamma_k^{(3)}(p, q, -p - q) G_k(p + q) \Gamma_k^{(3)}(-p, -q, p + q) \right]. \quad (12)
\]

Let us now consider the flow of $Z_k(\phi)$ at order $s = 2$ of the DE where:

\[
\Gamma_k(\phi) = \int_x \left[ U_k(\phi) + \frac{1}{2} Z_k(\phi)(\nabla \phi)^2 + O(\nabla^4) \right]. \quad (13)
\]

The value of $\Gamma_k^{(3)}$ computed from $\Theta_k$ is inserted in the term $\Gamma_k^{(3)}(p, q, -p - q) G_k(p + q) \Gamma_k^{(3)}(-p, -q, p + q)$ of Eq. (12).

Only terms at order 2 in the internal $(q)$ and external $(p)$ momenta are kept. Similarly, at order $s$, only terms at order $s$ are kept on the rhs of flow equations.

Note that this is not what most previous studies did, since usually all terms on the rhs are kept, in contradiction with the spirit of a Taylor expansion. At order 6, cutting off higher-order terms on the rhs drastically simplifies expressions that would be otherwise very hard to handle numerically. The differences in flow equations result in our exponent values being slightly different from those given previous works (references [7-8, 30-31, 37-42] of the main text). The differences may become more significant when, in addition, the functions are Taylor expanded in powers of the field.

C. Relation between the $\Theta_k^n$ and exponential regulators

We show here that the $E_k(q^2)$ regulator is the $n \to \infty$ limit of the $\Theta_k^n(q^2)$ function. Recall first that two regulators that only differ by the definition of their scale $k$ lead to identical physical results when $k \to 0$. The regulator

\[
\Theta_k^n(\alpha, q^2) = \alpha Z_k^n k^2 \left( 1 - \frac{q^2}{k^2} \right)^n \theta \left( 1 - \frac{q^2}{k^2} \right), \quad (14)
\]

when we change its scale $k$ into $k'$ given by $k = \sqrt{n} k'$, becomes

\[
\Theta_k^n(\alpha, q^2) = \alpha Z_k^n n k'^2 \left( 1 - \frac{q^2}{nk'^2} \right)^n \theta \left( 1 - \frac{q^2}{nk'^2} \right) \quad (15)
\]

that we now consider as a new, although equivalent, regulator depending on the scale $k'$. When $n \to \infty$ we have

\[
\left( 1 - \frac{q^2}{nk'^2} \right)^n \to \exp \left( \frac{q^2}{k'^2} \right) \quad \theta \left( 1 - \frac{q^2}{nk'^2} \right) \to 1. \quad (16)
\]

Thus, at very large values of $n$, $\Theta_k^n(q^2)$ becomes equivalent to the regulator $\alpha n Z_{0,k'} k'^2 \exp(-q^2/k'^2)$. This imposes that when $n$ and $n'$ are large, the two regulators $\Theta_k^n(\alpha, q^2)$ and $\Theta_k^{n'}(\alpha', q^2)$ are (almost) equivalent if
FIG. 2. Exponent curves $\nu(\alpha)$ and $\eta(\alpha)$ at order $s = 6$ for different regulators. Conformal bootstrap values are indicated by the dashed horizontal lines.

$a n = a' n'$ because they are equivalent to the exponential regulator $E_k(q^2)$. We have numerically checked that, for $n, n' > 10$, this relation is satisfied to a very good accuracy when using the optimal values $\alpha_{\text{opt}}(n)$ and $\alpha_{\text{opt}}(n')$.

D. Numerical implementation

We directly solved for the fixed point solutions of the coupled integro-differential flow equations using a Newton-Raphson method. To obtain good enough initial guesses, we first simulated directly the equations (using explicit Euler time-stepping) from some bare initial condition, proceeding by “dichotomy” to get near the fixed point at large RG times. All our results have been checked against changing resolution, the extent of the field domain considered, the accuracy with which integrals are calculated, and the order at which derivatives are estimated. Typical choices are: discretization of the field $\rho = \phi^2$ over a regular grid, field domain extending from 0 to 3 times the minimum of the potential with free boundary conditions, derivatives calculated over 7 grid points, integrals computed with the Gauss-Kronod method and a relative accuracy of order $10^{-12}$.

E. Exponent values

At each order $s$, all regulators we used provide similar-looking $\nu(\alpha)$ and $\eta(\alpha)$ curves. Figure 2 shows them, centered around the $\alpha_{\text{opt}}$ position, at order 6. Note that at this order the dispersion of exponent values at $\alpha_{\text{opt}}$ is of the order of the distance to the exact values. The range of exponent values found at $\alpha_{\text{opt}}$ was used to define our ‘final’ estimates (taken to be the value at the middle of the range), and their error bars (given by the half-range) in Table II of the main text.

Exponent values for the $\Theta^n$ regulators were found to vary monotonously with $n$, from the minimal value $n = 1 + s/2$ possible at order $s$ to infinity, represented by the exponential regulator $E$. In addition to these, we also used the $W$ regulator. Table III gives typical values found, together with the quasi-exact ones provided by conformal bootstrap. For simplicity, we only show, at each order, the digits that vary when changing the regulator.

These results show the reduction in the regulator dependent scatter, as the order of the derivative expansion is increased, despite the fact that the optimization curves become steeper with increasing orders, as seen in Fig. 1 of the main text.

F. Numerical results about the momentum expansion of $\Gamma_k^{(2)}(p, \phi) + R_k(0)$

The dependence of the dimensionless squared mass $m_{\text{eff}}^2(\phi)$ on the regulator parameter $\alpha$ is predominantly linear, for all regulators considered (Fig. 3). For every regulator, varying $\alpha$, we find a small deviation from the

| regulator | $\nu$ | $\eta$ |
|-----------|-------|-------|
| LPA       | 0.65059(2062) | 0     |
| $\Theta^1$ | 0.64956(1959) | 0     |
| $\Theta^2$ | 0.65003(2006) | 0     |
| $\Theta^3$ | 0.65020(2023) | 0     |
| $\Theta^4$ | 0.65056(2059) | 0     |
| $\Theta^8$ | 0.65103(2106) | 0     |
| $O(\partial^2)$ | 0.62779(218) | 0.04500(870) |
| $\Theta^1$ | 0.62814(183) | 0.04428(798) |
| $\Theta^3$ | 0.62802(195) | 0.04454(824) |
| $\Theta^4$ | 0.62793(204) | 0.04474(844) |
| $\Theta^8$ | 0.62775(222) | 0.04509(879) |
| $E$       | 0.62752(245) | 0.04551(921) |
| $O(\partial^4)$ | 0.63027(30) | 0.03581(57) |
| $\Theta^2$ | 0.63014(17) | 0.03507(123) |
| $\Theta^3$ | 0.63021(24) | 0.03480(150) |
| $\Theta^4$ | 0.63036(39) | 0.03426(204) |
| $\Theta^8$ | 0.63057(60) | 0.03357(272) |
| $E$       | 0.63007(10) | 0.03648(18) |

| conf. boot. | 0.629971(4) | 0.0362978(20) |
FIG. 3. Squared dimensionless mass generated by the regulator \( \tilde{m}_2^{\text{eff}}(\phi_{\min}) = v^{\ast''}/z^{\ast}|_{\phi_{\min}} \) as a function of the regulator parameter \( \alpha \) for several different regulators. Diamonds show the optimal values \( \alpha_{\text{opt}} \) for \( \eta \) (for a smaller value of \( \alpha \)) and \( \nu \) (for a larger value of \( \alpha \)). The dashed line has slope 1.65. These data have been obtained at \( \phi_{\min} \) the minimum of the dimensionless fixed point potential, but choosing another point yields similar results.

Strikingly, at the optimal values \( \alpha_{\text{opt}} \), \( \tilde{m}_2^{\text{eff}}(\phi) \) is exactly on a straight line passing through the origin.

In Figure 4 we show how the ratio \( r = x_3^u u^{\ast''}/(w_3^u z^{\ast}) \), which plays a role analogous to \( c_3/c_2 \), varies with \( \phi \). The ratio \( r \) is always negative as expected. For values of \( \tilde{\rho} \) close to the minimum of the potential \( \rho_{\min} \), the ratio is order \(-1/20\). On the other hand, when \( \tilde{\rho} \) goes to infinity \( r \) approaches \(-1/4\). This shows that \( r \) varies between the typical values of \( c_3/c_2 \) corresponding to the symmetric phase for \( \rho \sim \rho_{\min} \) and the value \(-1/4\) of the broken phase. This is consistent with the fact that for large values of the external field the symmetry is broken and the theory has a gap with a threshold for multiparticle states at \( p^2 = -4m_{\text{eff}}^2 \).

FIG. 4. The ratio \( r = x_3^u u^{\ast''}/(w_3^u z^{\ast}) \) as a function of \( \tilde{\rho} = \tilde{\phi}^2/2 \). The line \( r = 0.25 \) is a guide for the eyes.