Dirac Wire

Farhad Khosravi\textsuperscript{1,2}, Todd Van Mechelen\textsuperscript{2}, and Zubin Jacob\textsuperscript{1,4}

\textsuperscript{1}Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta T6G 1H9, Canada and \textsuperscript{2}Birck Nanotechnology Center and Purdue Quantum Center, Purdue University, West Lafayette, IN 47906, USA

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The interplay of photon spin and orbital angular momentum (OAM) in the optical fiber (1D waveguide) has recently risen to the forefront of quantum nanophotonics. Here, we introduce the fermionic dual of the optical fiber, the Dirac wire, which exhibits unique electronic spin and OAM properties arising from confined solutions of the Dirac equation. The Dirac wires analyzed here represent cylindrical generalizations of the Jackiw-Rebbi domain wall and the minimal topological insulator, which are of significant interest in spintronics. We show the unique longitudinal spin arising from electrons confined to propagation in a wire, an effect which is fundamentally prohibited in planar geometries. Our work sheds light on the universal spatial dynamics of electron spin in confined geometries and the duality between electronic and photonic spin.

Introduction. — Confined solutions of Maxwell’s equations exhibit unique phenomena such as transverse photon spin and universal spin-momentum locking \[ \mathbb{1}, \mathbb{2} \]. These effects do not occur in conventional circularly polarized propagating plane waves where the spin is always directed longitudinally along the momentum vector \[ \mathbb{7} \]. One striking example is an optical fiber where Zeeman transitions in a cold atom shows spin-dependent directional photon transport \[ \mathbb{1}, \mathbb{2} \]. The goal of this paper is to introduce the concept of Dirac waveguides and understand the intriguing spin characteristics of confined electronic waves. Our work is motivated by the Dirac-Maxwell correspondence \[ \mathbb{8}, \mathbb{9} \] which studies the relativistic parallels between photons and electrons.

Here, we introduce the Dirac wire [see Fig. 1(a)], the fermionic dual of the optical fiber. This system is the cylindrical generalization of the \( m > 0, m < 0 \) domain wall introduced by Jackiw and Rebbi \[ \mathbb{10} \]; the canonical planar system which spurred the field of topological materials. Pioneering work has shown a null expectation value for the relativistic electron spin in the planar Jackiw-Rebbi problem \[ \mathbb{11} \]. In stark contrast, we find that the Dirac wire supports longitudinal fermionic spin along its axis. For completeness, we also mention that the two-dimensional (2D) photonic dual of the Jackiw-Rebbi domain wall was discovered only recently \[ \mathbb{12} \], and is described by the interface of positive/negative gyrotropic media. Comparing Maxwell’s equations to the 2D Dirac equation, the gyrotropic non-reciprocity coefficient was shown to play the role of photonic mass \[ \mathbb{13} \mathbb{14} \].

The radius of the proposed Dirac wire is on the order of the Compton wavelength of the electron \( \lambda_c \approx \hbar/(m_1 v_F) \), where \( \hbar \), \( m_1 \), and \( v_F \) are the Planck constant, electron mass, and Fermi velocity within the wire, respectively. We introduce three distinct classes of Jackiw-Rebbi (JR) domains labeled as JR\textsuperscript{+}, JR\textsuperscript{−}, and JR-D [Fig. 1(a)]. We also show important fundamental differences between cylindrical JR solutions (Dirac wires) and the conventional planar interface problem \[ \mathbb{10} \] widely studied in the field of topological insulators and majorana physics \[ \mathbb{23} \mathbb{25} \]. The main differences between the cylindrical and planar JR problems are the emergence of a longitudinal component of spin and the existence of confined solutions for all-positive electronic mass.

For a cylindrical Dirac waveguide, the difference in electronic mass inside and outside the wire gives rise to bound fermionic waves. These solutions can be derived from the time-independent Dirac equation,

\[ H \psi_\mu = (v_F \alpha \cdot p + m v_F^2 \beta) \psi_\mu = E \psi_\mu. \]  \hspace{1cm} (1)

Eigenstates of the Dirac equation can be identified by five good quantum numbers which correspond to five commuting operators. In cylindrical coordinates, these op-
where $k_\Sigma = \text{fined as an operator, outside the fiber}$ [28]. For the JR optical fiber, which necessitates a lower refractive index, $s = \pm 1$, where $\mu \in \mathbb{Z} + \frac{1}{2}$ is half-integer due to the fermionic nature of electrons. The two solutions corresponding to the two eigenvalues of transverse helicity $s = \pm 1$ are (see supp. info. [27]),

$$\begin{align*}
\mathbf{u}^{(\pm)}_{\mu, M}(k) &= \frac{C_\mu e^{ik_\Sigma z} e^{i\mu \phi}}{\sqrt{2}} \begin{pmatrix} Z_{n_+}(k_\perp \rho)e^{-i\phi/2} \\
\pm Z_{n_-}(k_\perp \rho)e^{i\phi/2} \\
\mp ik_\perp \mu e^{ik_\Sigma z} Z_{n_+}(k_\perp \rho)e^{-i\phi/2} \\
+ ik_\perp \mu e^{ik_\Sigma z} Z_{n_-}(k_\perp \rho)e^{i\phi/2} \end{pmatrix}
\end{align*}$$

(2)

where $C_\mu$ is the normalization factor, $M = E + m_1 v_0^2$, $k_\Sigma = \sqrt{k_1^2 - k_\perp^2}$, and $n_+ - \frac{1}{2} = n_- + \frac{1}{2} = \mu$. Here, $h^2 k_\perp^2$ are the eigenvalues of total momentum operator $\mathbf{p}^2$, and $n_{\pm} \in \mathbb{Z}$ are integers. The $s = \pm 1$ signs appearing in Eq. (2) refer to the eigenvalues of the transverse helicity operator, $h_\perp$. $Z_{n}(k_\perp \rho)$ is a Bessel function of order $n$ and argument $k_\perp \rho$, where $\rho$ is the radial coordinate.

The vector spin operator of the Dirac equation is defined as $\mathbf{S} = \frac{\hbar}{2} \sigma$, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices expressed in vector operator form. The longitudinal component of the orbital angular momentum (OAM) operator is $L_z = -i \hbar \frac{\partial}{\partial \phi}$. Together with the spin operator, we obtain the longitudinal total angular momentum $J_z = \Sigma_z + L_z$. In the subsequent sections we will use these operators to find the expectation values of the spin and orbital angular momentum of the modes.

**Cylindrical Jackiw-Rebbi domain wall.** We now solve the cylindrical wire geometry with an effective electronic mass $m_1$ surrounded by a medium with an effective electronic mass $m_2$. This is the cylindrical analogue of the 1D Jackiw-Rebbi (JR) domain wall [10, 21, 22]. Unlike the 1D problem, however, solutions of the cylindrical geometry are not limited to the condition $m_1 m_2 < 0$. Therefore, we analyze two separate cases; the case when $m_1 m_2 > 0$ and label it as JR$^+$, and the case when $m_1 < 0$, $m_2 > 0$ and label it as JR$^-$. For the case of $m_1, m_2 > 0$ (JR$^+$), solutions of Eq. (11) only exist when $m_2 > m_1$ which requires a larger mass (bandgap) outside the wire to confine the waves. This condition is analogous to total internal reflection in an optical fiber, which necessitates a lower refractive index outside the fiber [28]. For the JR$^+$ problem, the solutions are characterized by $k_\perp$ real and $k_\Sigma$ imaginary where $k_\perp = \sqrt{k_1^2 - k^2}$ are the transverse (to the $z$-axis) propagation constants. $k_1$ and $k_2$ being the characteristic wavelengths inside and outside of the wire, respectively. Being comprised of evanescent waves outside the wire and standing waves inside, we denote these solutions as hybrid modes $H_{\mu, \nu}$. The subscripts $\mu$ and $\nu$ correspond to the total angular momentum eigenvalue and the order of the radial zero of the Bessel function.
The spin and OAM are not individually conserved but their integrals over the entire cross-section of the problem. Note that the wavefunction on the surface of the wire, irrespective of the mass outside. This is confirmed by the plot of the probability density in the right panel in Fig. 1(b), where the wavefunction is identically zero for $\rho > a$. The dispersive mass considered here is the simplest model that produces the gapless edge states on the surface of a topological insulator [21, 31]. It can be shown that the bulk $Z_2$ invariant is nontrivial ($-1) = \text{sgn}(-m_0B)$ whenever $m_0B > 0$. Hence, the medium $\rho < a$ is a topological insulator $\zeta = 1$. Note, we do not consider the inverse problem in this paper, where the medium $\rho > a$ is topological and the wire is treated as a cylindrical defect.

In the JR-D case, the eigenfunctions are of similar form as Eq. (2) with the difference that instead of two, there are four eigenfunctions:

$$
\begin{align*}
\mathbf{u}_{\mu,M}^{(+)}(k^{(1)}), \quad & \mathbf{u}_{\mu,M}^{(-)}(k^{(1)}), \\
\mathbf{u}_{\mu,M}^{(+)}(k^{(2)}), \quad & \mathbf{u}_{\mu,M}^{(-)}(k^{(2)}),
\end{align*}
$$

where $M^{(i)} = E + m_0v_F^2 - B\hbar^2k^{(i)}2$ and $u_{\mu,M}^{(+)}(k^{(i)})$'s are given by Eq. (3). Here $k^{(i)} = \sqrt{(k^{(i)})^2 - \frac{4B^2E^2}{v_F^2}}$, where $m_0 = m_0 - B\hbar^2k^2$ are two possible propagation constants within the wire, resulting from the dispersive mass [27].

$$
k^{(1,2)} = \frac{v_F}{\sqrt{2B\hbar}} \left(2m_0B - 1\right) \pm \sqrt{\left(1 - 4m_0B\right) + \frac{4B^2E^2}{v_F^2}}.
$$

Unlike 1D solutions of the topological insulator [21, 22], solutions of the cylindrical JR-D problem exist irrespective of the sign of $m_0B$. In this paper, however, we only consider the scenario when $m_0B > 0$ since the solutions of the trivial case $\zeta = 0$ are similar to the JR$^+$ domain and are not particularly interesting.

Like the JR$^\pm$ states, we can label the modes depending on whether the two transverse propagation constants, $k^{(1)}_\perp$ and $k^{(2)}_\perp$, are real or imaginary. Note that $k^{(1)}_\perp$ and $k^{(2)}_\perp$ are real or imaginary. Note that $k^{(1)}_\perp$ and $k^{(2)}_\perp$ are real or imaginary.
FIG. 3. (a) Dispersion and group velocities (inset) for the dominant modes of JR\(^+\) (dashed blue), JR\(^-\) (dotted red), and JR-D (solid black). Group velocities are normalized to the Fermi velocity \(v_F \simeq 1.52 \times 10^7\) m/s. Wire radius dependence of (b) bandgaps, (c) spin, and (d) OAM for the three problems at \(k_z = 0\). Here \(\mu = 2/3\), \(k_x = 0\), \(m_1|\psi_1^N| = 1\) eV, \(m_2|\psi_2^N| = 2\) eV, and \(\lambda_c \simeq 8\) Å. For the topological insulator (JR-D), \(m_0|\psi_0^N| = 1\) eV and \(B\lambda^2 = 50\) eVÅ\(^2\). Due to confinement in the cylindrical geometry, the bandgap is opened for all three problems. For JR-D, however, the bandgap closes and reopens for certain values of \(a\) as seen in panel (b). Note that the summation \(J_z = S_z + L_z\) produces the conserved value of 1/2 in all three cases. In the limit \(a \to \infty\), OAM vanishes \(L_z \to 0\) for JR\(^+\), while spin vanishes \(S_z \to 0\) for JR\(^-\) and JR-D.

\(k_\perp^{(2)}\) both belong to the interior of the wire \(\rho < a\) as there are now two characteristic wavelengths \[Eq. (3)\]. In addition to \(H_{\mu,\nu}\) and \(D_{\mu}\), two other types of modes labeled as \(R_{\mu,\nu}\) and \(C_{\mu,\nu}\) exist in the JR-D problem. These modes refer to real \((R_{\mu,\nu})\) and complex \((C_{\mu,\nu})\) solutions for \(k_\perp^{(1,2)}\), respectively. The third row of Fig. 2 shows the spin and orbital angular momenta densities for the dominant mode, \(R_{\mu,1}\), of the JR-D problem. Here also, the azimuthal and radial components of the spin and OAM are identically zero – only the longitudinal part is non-vanishing. Due to spin-orbit coupling, the spin and orbital angular momentum are not individually conserved. This means it is difficult to distinguish between the separate contributions of the total angular momentum in an experiment. The spatial distributions of spin and OAM in Fig. 2 however, suggest a way to observe the spin or orbital parts locally. Analyzing the spin and orbital parts of \(R_{\mu,1}\) for the JR-D problem, for instance, we observe that while the spin is dominantly at the center of the wire, the orbital angular momentum is zero here and is distributed closer to the perimeter. This shows that using a point contact at the center of the wire, one can exclusively couple to the local spin of the \(R_{\mu,1}\) mode where the orbital angular momentum vanishes.

The dispersion relation \(E = E(k_z)\) of the dominant modes is presented in Fig. 3(a) and shows significantly larger group velocities for JR and JR-D compared to JR\(^+\), which implies higher conductivity. Anomalous dispersion for JR-D can be explained by the fact that, due to the dispersive electronic mass, charge transport is dominated by holes rather than electrons. This means that, in the regions where the group velocity becomes negative [inset of Fig. 3(a)], charge currents propagate along the negative \(z\) direction for \(k_z > 0\). In the JR\(^+\) problem, the bandgap plateaus to \(m_1\) (1 eV) for large radii [Fig. 3(b)]. Since the \(H_{\mu,\nu}\) modes of the JR\(^+\) domain wall are mostly distributed within the bulk of the wire [Fig. 1(b)], these modes transform into bulk states of the Dirac equation when \(a \to \infty\). For JR\(^-\), on the other hand, the bandgap closes when \(a \to \infty\). This can be explained by the fact that the mode is predominantly distributed around the perimeter of the wire [middle panel of Fig. 3(b)]. Therefore, the \(D_{\mu}\) mode transforms into the edge states of the conventional 1D Jackiw-Rebbi problem \[10\] when \(a \to \infty\). The opening of the band-gap in the JR\(^-\) problem, for small wire radius, can be explained by the hybridization of the edge state modes \[32\].

More interesting is the bandgap of the topological insulator (JR-D) where for some finite values of radii, the bandgap closes and re-opens in an oscillatory fashion with \(a\). For JR-D, spin also exhibits oscillatory behavior and passes through regions of positive and negative \(S_z\) upon increasing the wire radius [Fig. 3(c)]. However, as \(a \to \infty\), angular momentum is dominated by spin for the JR\(^+\) problem and conversely dominated by OAM for JR\(^-\) and JR-D. This means the dominant JR\(^-\) and JR-D modes behave like edge states in the limit \(a \to \infty\) and circulate around the perimeter of the material. Another important observation in Fig. 3(c) is that, although the spin is not conserved in any problem, its absolute value never exceeds \(1/2\). This holds for all higher orders of \(\mu\) and \(\nu\) as well \[27\]. Note also, for all three cases, the total angular momentum is still conserved irrespective of the value of the wire radius.

**Conclusion.**— Our results show important differences between the 1D JR \[11, 21\] and the cylindrical JR domain walls. In contrast to the 1D problem, the confined geometry of JR\(^\pm\) and JR-D display non-zero longitudinal spin and orbital angular momentum. Moreover, we have shown that a sign change in mass is not necessary for the existence of confined cylindrical solutions of the Dirac equation. Labeled by JR\(^\pm\), these Dirac waveguide solutions are the electronic analogue of the guided
The Dirac wire is realizable using available materials [32]. Although the JR-D problem has the simplest model of radius $a = 20$ nm have been reported in the literature [28]. For $\text{Bi}_2\text{Se}_3$, for instance, a Fermi velocity of $v_F \approx 5.0 \times 10^5 m/s$, a band-gap of about 0.28 eV, and also dispersion factor of $56.6 \text{ eV}^2$ has been reported [29]. The parameters for other topological insulators such as $\text{Bi}_2\text{Te}_3$ and $\text{Sb}_2\text{Te}_3$ show that the Dirac wave is realizable using available materials [32].

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