THE \(L\)-SERIES OF A CUBIC FOURFOLD

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Abstract. We study the \(L\)-series of cubic fourfolds. Our main result is that, if \(X/\mathbb{C}\) is a special cubic fourfold associated to some polarized \(K3\) surface \(S\), defined over a number field \(K\) and satisfying \(S^{[2]}(K) \neq \emptyset\), then \(X\) has a model over \(K\) such that the \(L\)-series of the primitive cohomology of \(X/K\) can be expressed in the \(L\)-series of \(S/K\). This allows us to compute the \(L\)-series for a discrete dense subset of cubic fourfolds in the moduli spaces of certain special cubic fourfolds. We also discuss a concrete example.

1. Introduction

After the proof of the Taniyama-Shimura-Weil conjecture by Wiles, Taylor and others, many efforts were made to prove similar results for other classes of varieties. The most natural class to investigate is that of Calabi-Yau varieties. In the case of \(K3\) surfaces Shioda and Inose [18] had already determined the zeta-function of singular \(K3\) surfaces defined over some number field \(K\). In the case that the singular \(K3\) surface \(S\) is defined over \(\mathbb{Q}\), Livné [13] has shown that the \(L\)-series associated to the 2-dimensional transcendental lattice of \(S\) comes from a weight 3 modular form with complex multiplication in some quadratic imaginary field \(L\). For rigid Calabi-Yau threefolds defined over \(\mathbb{Q}\), Dieulefait and Manoharmayum [9] have proved modularity under mild restrictions on the primes of bad reduction. Further examples and results are now also known for certain non-rigid Calabi-Yau threefolds and some higher dimensional examples. For a survey see [11].

The main purpose of this paper is to show that there are many cubic fourfolds whose \(L\)-series can be computed. A cubic fourfold \(X\) is called special if it contains a surface \(T\) which is not homologous to a multiple of the class \(h^2\) where \(h\) is the class of the hyperplane section. The discriminant of \(X\) is the discriminant of the saturated rank 2 sublattice of \(H^4(X, \mathbb{Z})\) spanned by \(h^2\) and \(T\). These cubic fourfolds were studied extensively by Hassett [10]. He proved the following result: if the discriminant \(d\) is greater than 6 and if \(d \equiv 0, 2 \mod 6\), then the special fourfolds of discriminant \(d\) are parameterized by a non-empty 19-dimensional quasi-projective variety \(\mathcal{C}_d\). These cubic fourfolds are closely related to \(K3\) surfaces, more precisely if \(d = 2(n^2 + n + 1)\) for some \(n \in \mathbb{Z}_{>1}\), then there exists a non-empty open subset of \(\mathcal{C}_d\) such that the Fano variety \(F(X)\) of a cubic fourfold belonging to this open

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set is isomorphic to the desingularized second symmetric product $S^{[2]}$ of some $K3$ surface $S$.

Fix such a $K3$-surface $S$. Consider the composition

$$\psi : S^{[2]} \hookrightarrow \text{Gr}(1, 5) \hookrightarrow \mathbb{P}^{14}.$$  

The first inclusion is given by $F(X) \cong S^{[2]}$, the second by the Plücker embedding. To $\psi$ we can associate a line bundle $L$.

It turns out that $L$ is isomorphic to $O(n\Delta + f)$, where $\Delta$ is the exceptional divisor on $S^{[2]}$ and $O(f)$ is a line bundle on $S$ pushed forward via the diagonal embedding. We call $O(f)$ the associated line bundle to the isomorphism $F(X) \cong S^{[2]}$.

Our main result (Theorem 4.11) now says the following:

**Theorem 1.1.** Let $K$ be a number field. Let $S/K$ be a $K3$ surface. Suppose $S^{[2]}(K) \neq \emptyset$. Assume there is a cubic fourfold $X/\mathbb{C}$ such that $F(X) \cong S^{[2]}_\mathbb{C}$ and that the associated line bundle on $S$ descends to $K$.

Then $X$ has a model over $K$, $F(X) \cong S^{[2]}$ and 

$$H^4_{\text{ét}}(X, \mathbb{Q}_\ell) = H^2_{\text{ét}}(S, \mathbb{Q}_\ell)(1) \oplus \mathbb{Q}_\ell[\Delta](1).$$

In the case where the $K3$ surface $S$ is singular (i.e., has Picard number 20), we can then determine the $L$-series of $X$ explicitly (see Corollary 4.14). Since the singular $K3$ surfaces are everywhere dense (in the $\mathbb{C}$-topology) in the 19-dimensional moduli space $\mathcal{C}_d$ of degree $d$ polarized $K3$ surfaces, we have in this way found a countable number of points in a non-empty Zariski open subset of $\mathbb{C}^{n^2 + n + 1}$ such that the corresponding fourfolds have a model over an explicitly known number field $K$ and where the $L$-function can be computed in terms of Hecke Grössencharakters.

The organization of this paper is as follows. In Section 2 we rephrase some results on the cohomology of cubic fourfolds, their Fano varieties, and symmetric products in terms of étale cohomology. In Section 3 we recall Hassett’s results from [10] on special cubic fourfolds. In Section 4 we prove that the isomorphism $F(X) \cong S^{[2]}$ descends to any field $K$ for which $S^{[2]}(K) \neq \emptyset$. The proof uses a classical description of morphisms onto a Grassmannian variety in terms of vector bundles and descent theory for vector bundles. In Section 5 we relate the $L$-function of the Fermat cubic fourfold with the $L$-function of a weight 3 Hecke eigenform.

1.1. **Convention:** Suppose $X/S$ is a scheme. Given a morphism $T \to S$, we denote by $X_T$ the base change of $X$ with respect to $T \to S$. In the case that $T = \text{Spec } R$ we write $X_R$ instead of $X_{\text{Spec } R}$.

2. **Geometry of cubic fourfolds**

In this section we recall basic facts about the geometry of cubic fourfolds. Fix a number field $K$ and a cubic fourfold $X \subset \mathbb{P}^5_K$.

The general cubic fourfold contains only surfaces whose cycle class is a multiple of $h^2$, where $h$ is the hyperplane section. Our main interest, however, lies in so-called special cubic fourfolds.

**Definition 2.1.** A cubic fourfold $X$ is called (geometrically) special if $X_K$ contains an algebraic surface $T/K$ which is not homologous to $h^2$, where $h$ is the class of the hyperplane section.

Let $X$ be a special cubic fourfold and $T \subset X$ a surface which is not homologous to a multiple of $h^2$. We denote by $L_T$ the saturated sublattice of $H^4(X, \mathbb{Z})$ spanned
by the classes of $T$ and $h^2$. The discriminant $d(X, T)$ of the pair $(X, T)$ is defined as the discriminant of the lattice $L_T$. We call a cubic fourfold $X$ a special cubic fourfold of discriminant $d$ if it contains a surface $T$ such that $d(X, T) = d$.

It was shown by Hassett [10] Theorem 1.0.1] that $d > 6$ and $d \equiv 0, 2 \mod 6$ is a necessary and sufficient condition for the existence of special cubic fourfolds. (See also Proposition 5.1.) Note that the discriminant of a special cubic fourfold is not uniquely determined by $X$ itself. In general, $X$ can (and will) often be special with respect to more than one discriminant.

It is a classical fact (cf. [1]) that $X$ is covered by lines. We denote by $F(X)$ the Fano variety of $X$, i.e.,

$$F(X) = \{ \ell \in \text{Gr}(1, 5); \ell \subset X \}.$$ 

Let $U \subset \text{Gr}(1, 5) \times \mathbf{P}^5$ be the universal line and let $p$ and $q$ denote the projection from $U$ onto the first and second factor respectively. Especially, since $X$ is covered by lines, the morphisms $p$ and $q$ give a correspondence between $X$ and $F(X)$.

By Lefschetz’ hyperplane theorem, we have that $H^i_{\text{et}}(X, \mathbb{Q}_\ell) \cong H^i_{\text{et}}(\mathbf{P}^5, \mathbb{Q}_\ell)$ for $i \neq 4$. From Bott’s formula and the comparison theorem between sheaf and étale cohomology we obtain that $H^4_{\text{et}}(X, \mathbb{Q}_\ell)$ is a 23-dimensional Galois module. We proceed by proving some results on $H^4_{\text{et}}(X, \mathbb{Q}_\ell)$.

For any Galois module $M$, we denote by $M(k)$ the $k$-th Tate twist of $M$.

**Proposition 2.2.** We have that $p_* q^*: H^4_{\text{et}}(X, \mathbb{Q}_\ell) \to H^2_{\text{et}}(F(X), \mathbb{Q}_\ell)(1)$ is an isomorphism of Galois modules.

**Proof.** It follows from [1] Proposition 6] that the map between the sheaf cohomology groups $p_* q^*: H^4(X, \mathbb{C}) \to H^2(F(X), \mathbb{C})$ is an isomorphism, hence has no kernel and cokernel. From the comparison theorem between sheaf and étale cohomology it follows that $p_* q^*: H^4_{\text{et}}(X, \mathbb{Q}_\ell) \to H^2_{\text{et}}(F(X), \mathbb{Q}_\ell)$ is an isomorphism of $\mathbb{Q}_\ell$-vector spaces.

Note that $p_*$ is given by taking the inverse of the Poincaré dual of the map $p^*: H^6_{\text{et}}(F(X), \mathbb{Q}_\ell) \to H^6_{\text{et}}(U, \mathbb{Q}_\ell)$. By Poincaré duality $H^6_{\text{et}}(U, \mathbb{Q}_\ell) \cong H^0_{\text{et}}(U, \mathbb{Q}_\ell)^{\vee}(5)$ and $H^6_{\text{et}}(F(X), \mathbb{Q}_\ell)^{\vee}(5) \cong H^2_{\text{et}}(F(X), \mathbb{Q}_\ell)(1)$ which shows that $p_* q^*: H^4_{\text{et}}(X, \mathbb{Q}_\ell) \to H^2_{\text{et}}(F(X), \mathbb{Q}_\ell)(1)$ is Galois-equivariant. Combining this with the fact that it is an isomorphism of vector spaces, we obtain that $p_* q^*$ is an isomorphism of Galois-modules. \qed

**Remark 2.3.** The map $p_* q^*$ is called the Abel-Jacobi map.

**Remark 2.4.** Beauville and Donagi [4] also proved that the Fano variety $F(X)_C$ is a symplectic 4-manifold. More precisely, they show that $F(X)_C$ is a deformation of a variety $S^{[2]}$ where $S$ is a complex $K3$ surface and $S^{[2]}$ is the Hilbert scheme of 0-cycles of length 2. It is easy to see that $S^{[2]}$ is the blow-up of the second symmetric product of the $K3$ surface $S$ along its (singular) diagonal. Alternatively, we can blow up the product $S \times S$ along the diagonal and then take the quotient with respect to the involution $\iota$ given by interchanging the two factors.

The above remark shows that there is a close relation between symmetric products of some $K3$ surfaces $S$ and cubic fourfolds $X$. 
Fix a $K3$-surface $S/K$ and consider the diagram

$$\begin{array}{ccc}
\widetilde{S} \times S & \xrightarrow{r} & S \times S \\
\downarrow g & & \downarrow f \\
S^{[2]} & \xrightarrow{g'} & \text{Sym}^2 S,
\end{array}$$

where $r$ is the blowup of $S \times S$ along $f(S)$ and $f$ is the diagonal embedding $S \to S \times S$. Let $S^{[2]}$ be the minimal resolution of $\text{Sym}^2 S$ given by resolving the transversal $A_1$-singularity along the diagonal and denote by $E$ the exceptional divisor of $r$. Let $\Delta$ denote $g_*(E)_{\text{red}}$. The analogue of the following proposition is well known for singular cohomology (see [4, Proposition 6]).

**Proposition 2.5.** The diagonal embedding $f$ induces an isomorphism of Galois modules

$$H^2_{\text{ét}}(S^{[2]}, \mathbb{Q}_\ell) \cong H^2_{\text{ét}}(S, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell[\Delta].$$

**Proof.** Since $S$ is a $K3$ surface we have that $H^1_{\text{ét}}(S, \mathbb{Q}_\ell) = 0$. Hence the Künneth decomposition implies

$$H^2_{\text{ét}}(S \times S, \mathbb{Q}_\ell) \cong H^0_{\text{ét}}(S, \mathbb{Q}_\ell) \otimes H^2_{\text{ét}}(S, \mathbb{Q}_\ell) \oplus H^2_{\text{ét}}(S, \mathbb{Q}_\ell) \otimes H^0_{\text{ét}}(S, \mathbb{Q}_\ell).$$

Let $\iota$ be the involution of $S \times S$ sending $(x, y)$ to $(y, x)$. Clearly $\iota^*$ acts on $H^2(S \times S, \mathbb{Q}_\ell)$ as (with respect to the Künneth decomposition)

$$\left( \begin{array}{cc} 0 & I_{22} \\ I_{22} & 0 \end{array} \right).$$

Hence $\iota^*$ has 22 eigenvalues 1 and 22 eigenvalues $-1$. Since $\iota^*$ fixes the 22-dimensional module $H^2_{\text{ét}}(S, \mathbb{Q}_\ell)$ (embedded via $f$) we obtain that

$$H^2_{\text{ét}}(S \times S, \mathbb{Q}_\ell)^{\iota^*} \cong H^2_{\text{ét}}(S, \mathbb{Q}_\ell).$$

Let $\tau$ be the involution on $\widetilde{S} \times S$ associated to $g$. One easily sees that

$$H^2_{\text{ét}}(S^{[2]}, \mathbb{Q}_\ell) \cong H^2_{\text{ét}}(\widetilde{S} \times \widetilde{S}, \mathbb{Q}_\ell)^{\tau^*} \cong H^2_{\text{ét}}(S, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell[\Delta].$$

**Corollary 2.6.** Let $S/K$ be a $K3$ surface. Let $X/K$ be a cubic fourfold. Assume that $F(X) \cong S^{[2]}$. Then

$$H^4_{\text{ét}}(X, \mathbb{Q}_\ell) \cong H^2_{\text{ét}}(S, \mathbb{Q}_\ell)(1) \oplus \mathbb{Q}_\ell[\Delta](1)$$

as Galois-modules.

In Section 4 we prove that there exist $K3$ surfaces $S$ satisfying the assumptions of this corollary. Using this corollary and the Hodge conjecture (which is proven for cubic fourfolds), one obtains that if $F(X) \cong S^{[2]}$ then $X$ is a special cubic fourfold. Using lattice theory one can show that if $X$ is special of discriminant $d$ then $S$ admits a polarization of degree $d$. (See [10].)
3. Moduli of cubic fourfolds

The coarse moduli space $C$ of cubic fourfolds is the GIT-quotient $C = V//SL(6;\mathbb{C})$, where $V \subset P(H^0(P^5, O_{P^5}))$ is the Zariski-open subset of cubic equations whose zero-locus is smooth. Recall that all points in $V$ are properly stable in the sense of GIT. The variety $C$ is a quasi-projective variety of dimension 20. We define $C_d$ as the set of all special cubic fourfolds $X$ that contain a surface $T$ such that $d(X,T) = d$.

**Proposition 3.1.** Let $d > 6, d \equiv 0, 2 \mod 6$. Then $C_d$ is a non-empty irreducible divisor in the variety $C$.

**Proof.** This is [10, Theorem 1.0.1]. □

We say that a $K3$ surface $S$ is of degree $d$ if $S$ has a polarization of degree $d$ (here $d$ is necessarily even).

**Theorem 3.2.** Assume that $d$ equals $2(n^2 + n + 1)$ where $n$ is an integer at least 2. Then there exists an open set $U_d$ of $C_d$ such that for every $X \in U_d$ there exists a $K3$ surface $S$ of degree $d$ such that $F(X) \sim S^{[2]}$.

**Proof.** This is [10, Theorem 1.0.3]. □

4. Descent

Fix a number field $K$. There are many examples of fourfolds $X$ such that $F(X)_C \equiv S^{[2]}$ where $S/C$ is a $K3$ surface and $S^{[2]}$ is the minimal desingularization of $\text{Sym}^2 S$ (see Proposition 5.1). Suppose $S$ is defined over $K$. In this section we prove that then both $X$ and the isomorphism $S^{[2]} \rightarrow F(X)$ descend to $K$, provided that $S^{[2]}(K) \neq \emptyset$ and the associated line bundle to $S^{[2]} \rightarrow F(X)_C$ descends.

**Definition 4.1.** Let $T$ be a Noetherian scheme. A family of quotients of $O_T^r$ parameterized by $T$ is a coherent sheaf $F$ flat over $T$, together with a surjective $O_T$-linear homomorphism $q : O_T^r \rightarrow F$. Two pairs $(F, q)$ and $(F', q')$ are called isomorphic if $\text{ker}(q) = \text{ker}(q')$. Consider the functor from (Noetherian) schemes to sets

$$T \mapsto \left\{ \text{Isomorphism classes of locally free quotients of } O_T^{r+1} \xrightarrow{q} F, \right. \text{parameterized by } T, \text{ with } \text{rank } F = k + 1. \right\}.$$  

This functor is representable ([15]) by a scheme which we denote by $\text{Gr}(k, n)$. We call $\text{Gr}(k, n)$ the Grassmannian of $k$-dimensional subspaces in $P^n$.

A vector bundle on $X/T$ is a locally free sheaf on $X$, flat over $T$.

Suppose $L$ is a field. Then one easily shows that the variety $\text{Gr}(r, n)_L$ parameterizes dimension $r$ linear subspaces of $P^n_L$.

The following classical result describes all morphisms to Grassmann varieties:

**Proposition 4.2.** Let $R$ be a ring with unit. Let $Y$ be a projective scheme over $\text{Spec } R$. Giving a morphism $\varphi : Y \rightarrow \text{Gr}(k, n)_R$ is equivalent to giving a rank $k + 1$ vector bundle $V$ on $Y$ together with a surjective morphism of schemes $R^{n+1} \otimes O_Y \rightarrow V$ up to multiplication by an element in $R$. The composition of $\varphi$ with the Plücker embedding is given by the morphism associated to $\wedge^{k+1} R^{n+1} \otimes O_Y \rightarrow \wedge^{k+1} V$. 
Proof. See [2] Proposition 2.1] for the case that \( R \) is a field. The general case is a formal consequence of the fact that Quot-functors on Noetherian relative projective schemes are representable. For this formal deduction we refer to [19], Example 2.7], for a proof of the representability of the Quot-functor we refer to [15]. \( \square \)

**Lemma 4.3.** Let \( L \) be a field. Fix positive integers \( k, n \). Suppose \( 2k < n - 1 \). Let \( m = \binom{n+1}{k+1} - 1 \). Suppose \( P \) is a plane contained in the image of the Plücker embedding of \( \text{Gr}(k, n)_K \) in \( \mathbf{P}^m_K \). Then \( \dim P \leq n - k \).

Moreover if \( \dim P = n - k \) then \( P \) corresponds to the family of \( k \)-dimensional planes containing a fixed \( k - 1 \)-dimensional plane. Conversely, every \( k - 1 \)-dimensional plane gives such a family.

Proof. The first claim is an easy exercise in Plücker relations.

The second claim is an exercise in decomposable tensors: we illustrate this for the case \( k = 1 \). In this case we need to show that a maximal linear subspace has dimension \( n - 1 \) in \( \text{Gr}(1, n) \), and that this subspace correspond to lines throug a fixed point.

First let \( v_1 \) and \( v_2 \) be two decomposable \( 2 \)-tensors. If the corresponding lines \( l_1 \) and \( l_2 \) in \( \mathbf{P}^2_L \) are skew, then we can assume \( v_1 = e_0 \wedge e_1 \) and \( v_2 = e_2 \wedge e_3 \). But then \( v_1 \wedge v_2 \neq 0 \) and the Plücker relations are, therefore, not satisfied, i.e., the line spanned by \( v_1 \) and \( v_2 \) is not in the Grassmannian. For this to happen, we must have \( v_1 = e_0 \wedge e_1 \) and \( v_2 = e_0 \wedge e_2 \) (after choosing a suitable basis). Assume we have a third decomposable \( 2 \)-tensor \( v_3 \) such that every tensor in the plane spanned by \( v_1, v_2 \) and \( v_3 \) is decomposable. Then the line \( l_3 \) corresponding to \( v_3 \) must meet \( l_1 \) and \( l_2 \). This can happen in two ways. If \( l_3 \) is in the plane spanned by \( l_1 \) and \( l_2 \), then we can assume \( v_3 = e_1 \wedge e_2 \). This gives us a plane in the Grassmannian, namely the plane parametrizing all lines in a given \( \mathbf{P}^2_L \). If \( n = 3 \) this is indeed a maximal dimensional subspace of \( \text{Gr}(1, 3)_L \), but then \( 2k = n - 1 \). Again using the Plücker relations one sees that the plane spanned by \( v_1 = e_0 \wedge e_1 \), \( v_2 = e_0 \wedge e_2 \) and \( v_3 = e_1 \wedge e_2 \) is not contained in any higher dimensional linear subspace contained in the Grassmannian. The second possibility for \( l_3 \) is that it meets the point of intersection of \( l_1 \) and \( l_2 \), in which case \( v_3 = e_0 \wedge e_3 \), again in a suitable basis. We can now complete this argument to conclude that any \( (n - 1) \)-dimensional linear subspace of the Grassmannian is spanned by tensors of the form \( e_0 \wedge e_1, \ldots, e_0 \wedge e_n \), i.e., all lines containing the point given by \( e_0 \). \( \square \)

**Lemma 4.4.** Let \( L \) be a field. Fix non-negative integers \( k, n \). Suppose that \( 2k < n - 1 \). Then \( \text{Aut}(\text{Gr}(k, n)_{L}) = \text{Aut}(\mathbf{P}^2_L) \).

Proof. Let \( m = \binom{n+1}{k+1} - 1 \) and let \( \varphi : \text{Gr}(k, n)_K \to \mathbf{P}^m_K \) be the Plücker embedding.

We shall prove the lemma by induction on \( k \). The statement is clear for \( k = 0 \). From Lemma [4.3] it follows that an automorphism of \( \text{Gr}(k, n)_L \) induces an automorphism of \( \text{Gr}(k - 1, n)_L \) by considering its action on maximal dimensional subspaces. By induction, this automorphism is induced from an automorphism of \( \mathbf{P}^2_L \). Now suppose that we have an automorphism \( \sigma \) of \( \text{Gr}(k, n)_L \) inducing the identity on \( \text{Gr}(k - 1, n)_L \). We claim that \( \sigma \) is the identity on \( \text{Gr}(k, n)_L \). Indeed, consider a subspace \( \mathbf{P}^k_L \). For any subspace \( \mathbf{P}^{k-1}_L \) we know that \( \sigma(\mathbf{P}^{k-1}_L) \) also contains \( \mathbf{P}^{k-1}_L \). Hence \( \sigma(\mathbf{P}^k_L) = \mathbf{P}^k_L \). \( \square \)
Remark 4.5. The Grassmannian $\text{Gr}(1, 3)_K$ contains two families of planes (they are usually called $\alpha$-planes and $\beta$-planes). There is an automorphism of the Grassmannian $\text{Gr}(1, 3)_K$ (which is a quadric in $\mathbb{P}^5_K$), which interchanges the two types of planes and is, therefore, not induced by a linear map of $\mathbb{P}^1_K$.

Fix a $K3$ surface $S/K$. Assume that there exists a number field $L \supset K$ and a cubic fourfold $X/L$ such that $F(X) \cong S^{[2]}_L$, where $F(X) \subset \text{Gr}(1, 5)$ denotes the Fano variety of lines of $X$.

For the rest of the section fix a rank 2 vector bundle $V$ on $S^{[2]}_L$ yielding an isomorphism $S^{[2]}_L \cong F(X) \subset \text{Gr}(1, 5)$. We proceed now by proving that $V$ descends to $S^{[2]}$ under the assumption that $\det(V)$ descends to $S^{[2]}$.

From here on we will assume that $\det(V)$ on $S^{[2]}_L$ is isomorphic to the pullback of a line bundle on $S^{[2]}$. We express this by saying that $\det(V)$ is defined over $K$. This is equivalent to stating that the associated line bundle (cf. Section 1) on $S$ descends to $K$. One can show that $\det(V)$ is a combination of the associated line bundle and $\delta$, the square root of the exceptional divisor $\Delta$. So we need to prove that $\delta$ is defined over $K$. For any element $\sigma \in \text{Gal}(K/K)$ we have $\delta \otimes \delta = \Delta = \delta^\sigma \otimes \delta^\sigma$. Since $\text{Pic}(S^{[2]})_{\text{tor}}$ is trivial this implies that $\delta^\sigma \cong \delta$. Since $S^{[2]}(K)$ is non-trivial we can ‘rigidify’ $\delta$. Using descent theory as in [12, Theorem 2.5] one can show that $\delta$ is defined over $K$.

To assume that $\det(V)$ is defined over $K$ is not a severe restriction: one can show that for all examples produced by Hassett [10] $\det(V)$ is a linear combination of the polarizing class of $S$ and the square root $\delta$. The exceptional divisor is defined over $K$. In general it is not hard to check whether the polarizing class is defined over $K$.

Diagram 4.6. Let $R := L \otimes_K L$ and $R' := L \otimes_K L \otimes_K L$. Denote by $h_i : \text{Spec } L \to \text{Spec } R$ and by $h_{j,k} : \text{Spec } R' \to \text{Spec } R$ the morphism induced by the inclusion on the $i$-th and on the $j$-th and $k$-th factor, respectively. Similarly, denote by $s_n : \text{Spec } R' \to \text{Spec } R$ the morphism induced by the inclusion of the $n$-th factor. The morphism $p_i$ is the morphism $S^{[2]}_R \to S^{[2]}_L$ induced by the base change $h_i$, the morphism $p_{j,k}$ is defined analogously. Set $n = j$ if $i = 1$, or $n = k$ if $i = 2$. Then we obtain the following commutative diagram:

Remark 4.7. Suppose for the moment that $L/K$ is Galois. Write $L := K[t]/(f)$, for some polynomial $f \in K[t]$ of degree $d$. Then

$$R = L \otimes_K L = (K[t]/(f)) \otimes_K L = L[t]/(f) \cong L^d,$$

where the isomorphism follows from the Chinese remainder theorem. In particular, $\text{Spec } R$ consists of $d$ closed points.

One can show that we can choose the isomorphism $L \otimes_K L \to L^d$ such that

$$w \otimes 1 \mapsto (w, w, \ldots, w) \text{ and } 1 \otimes w \mapsto (w, g_1(w), g_2(w), \ldots, g_{d-1}(w)),$$
where $\text{Gal}(L/K) = \{1, g_1, \ldots, g_{d-1}\}$.

Similarly, one can show that $S^{[2]}_R$ is the disjoint union of $d$ copies of $S^{[2]}_L$.

**Remark 4.8.** Suppose we sheafify the functor $B$ from $K$-schemes to sets given by

$$T \mapsto \{\text{Isomorphism classes of locally free sheaves of rank } r \text{ on } X_T\}/\text{Pic}(T)$$

in the étale, fpqc or fppf topology. Denote the sheafified functor by $B'$. The following lemma shows that $V \in B'(\text{Spec } K)$. (In this context, our aim is to prove that $V \in B(\text{Spec } K)$.)

**Remark 4.9.** Let $f : S^{[2]} \to \text{Spec } K$ be the structure map. We say that $f_*\mathcal{O}_{S^{[2]}} \cong \mathcal{O}_{\text{Spec } K}$ holds universally, if there is an isomorphism $\psi : f_*\mathcal{O}_{S^{[2]}} \to \mathcal{O}_{\text{Spec } K}$ such that for any morphism of schemes $h : T \to \text{Spec } K$ we have that the base-changed morphism of $\mathcal{O}_T$-modules $\psi_T : f_{T*}\mathcal{O}_{S^{[2]}} \to \mathcal{O}_T$ is an isomorphism.

In [12] Answer to exercise 3.11 it is shown that if $X/B$ is a proper and flat family, with connected and reduced fibers, then $f_*\mathcal{O}_X \cong \mathcal{O}_B$ holds universally. Since we are working with a projective variety over a field, we are in this situation.

**Proposition 4.10.** Suppose $S^{[2]}(K) \neq \emptyset$ and $\det V$ descend to $S^{[2]}$. Then the vector bundle $V$ on $S^{[2]}_L$ descends to $S^{[2]}_R$.

**Proof.** Assume the notation from Diagram [4.6]. Without loss of generality we may assume that $L/K$ is Galois. This implies that $\text{Spec } R \to \text{Spec } L$ is a finite union of points.

We start by constructing an isomorphism $w : p_1^*V \to p_2^*V$. Consider the embedding $\varphi_1$ associated with $\wedge^2 p_1^*V$. Since $\wedge^2 p_1^*V \cong \wedge^2 p_2^*V$ we obtain that the image of $\varphi_1$ and $\varphi_2$ differ by an automorphism of $\mathbf{P}^{14}_R/\text{Spec } (R)$ fixing the Plücker embedding of $\text{Gr}(1,5)_R$. Since $\text{Gr}(1,5)_R = \sqcup \text{Gr}(1,5)_L$, we obtain from Lemma [4.34] that $\text{Aut}(\text{Gr}(1,5)_R/\text{Spec } (R)) \cong \text{Aut}(\mathbf{P}^3_R/\text{Spec } R) = \text{PGL}_6(R)$. In particular there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}^6_{S^{[2]}_R} & \longrightarrow & p_1^*V \\
\downarrow w_1 & & \downarrow w \\
\mathcal{O}^6_{S^{[2]}_R} & \longrightarrow & p_2^*V \\
\end{array}
$$

where the vertical arrows are isomorphisms.

By descent theory (e.g., [13] Theorem I.2.23) it suffices to show that the automorphism

$$\tau := (p_{13}^*w)^{-1} \circ p_{23}^*w \circ p_{12}^*w$$

of $q_1^*V$ is trivial.

Consider the quotient

$$s : \mathcal{O}^6_{S^{[2]}_R} \longrightarrow V_R$$

induced by the embedding $S^{[2]}_R \to \text{Gr}(1,5)_R$. Then $\tau \circ s$ gives another embedding $S^{[2]}_R \to \text{Gr}(1,5)_R$. As above we can compose these two embedding with the Plücker embedding and we get two embeddings into $\mathbf{P}^{14}_R$ that differ by an automorphism of $\text{Gr}(1,5)_R/\text{Spec } R$. Each such automorphism is coming from an automorphism
Proof. By [10, Theorem 1.0.3] there is a set $S/K$ such that for every $K3$ surface $S$ over $K$ there exists a cubic fourfold $X$ such that $S/C$ corresponds to a point in $V_d$ there exists a cubic fourfold $X/C$ such that $S/C \cong F(X)$.

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of $P^5_{R'}$. Equivalently, we have a vector bundle $V$ on $S[K]$ such that for every $K3$ surface $S$ over $K$, $S/K \cong H^0(Gl_6(O_S^{[2]}))$. Fix a point in $S^{[2]}(K)$ and let $g : Spec K \hookrightarrow S^{[2]}$ be the corresponding morphism. Then by the definition of fibre products and the fact that $g_{R'}$ corresponds to a $K$-rational point we obtain that $g_{R'}^* \tau_1$ is trivial. Now, $\tau_1 \in H^0(Gl_6(O_S^{[2]})) = H^0(Gl_6(f_*O_{S^{[2]}_{R'}})) \psi^* H^0(Gl_6(O_{Spec R'}))$ (as groups). Since $\psi_{R'} : f_*O_{S^{[2]}_{R'}} \cong O_{Spec R'}$, we have that the induced map $\psi^*_{R'}$ is an isomorphism of groups. Since $g_{R'}^* \tau_1$ is trivial we obtain that $\psi^*_{R'}(\tau_1)$ is also trivial. This implies that $\tau$ is trivial, hence $V$ descends to $S^{[2]}$. □

Theorem 4.11. Let $S/K$ be a $K3$ surface such that $S^{[2]}(K)$ is non-empty and such that over some finite extension $L/K$ we have $S^{[2]}_L \cong F(X)$ with $X/L$ a cubic fourfold. Suppose the associated line bundle (cf. Section 1) descends to $X/K$. Then the cubic fourfold $X$ can be defined over $K$ in such a way that $S^{[2]} \cong F(X)$.

Proof. It follows from Proposition 4.10 that $V$ is vector bundle on $S^{[2]}$. Consider the evaluation map $\omega : H^0(V) \otimes O_{S^{[2]}} \to V$. Since forming cohomology commutes with flat base change, and $\omega_L$ is surjective, it follows that $\omega$ is surjective. This implies that the vector bundle $V$ is globally generated.

By Proposition 4.22 we obtain that the associated map $\psi : S^{[2]} \to Gr(1,5)$ is defined over $K$. Let $U \subset Gr(1,5) \times P^5$ be the universal line and $p$ and $q$ the projections. If $X'/K$ is the image of $p^{-1}(\psi(S^{[2]}))$ under $q$, then, by construction, we have $X' \cong X$. Hence $X'$ is a model of $X$, so $X$ has a model over $K$ and $\psi : S^2 \to F(X')$. □

Corollary 4.12. Fix $n \geq 2$ and let $d = 2(n^2 + n + 1)$. There is a non-empty Zariski open set $V_d$ of the moduli space $F_d$ of polarized $K3$ surfaces of degree $2d$ such that if

(1) $K$ is a subfield of $C$;
(2) $S/K$ a $K3$ surface such that $S_C$ corresponds to a point in $V_d$;
(3) the set $S^{[2]}(K)$ is non-empty;
(4) all ample line bundles of degree $d$ on $S$ descend to $K$;

then there exists a cubic fourfold $X/K$ such that $F(X) \cong S^{[2]}$.

Proof. By [10, Theorem 1.0.3] there is a set $V_d$ such that for every $K3$ surface $S/C$ corresponding to a point in $V_d$ there exists a cubic fourfold $X/C$ such that $S/C \cong F(X)$. 


Assume $S$ can be defined over $K$. Consider the subscheme $T$ of the Hilbert scheme of $\text{Gr}(1, 5)$ corresponding to subvarieties of $\text{Gr}(1, 5)$ that are geometrically isomorphic to $S^{[2]}$, and the isomorphism can be given by an element of $\text{Aut}(\mathbb{P}^5_\mathbb{C}) \subset \text{Aut}(\text{Gr}(1, 5)_\mathbb{C})$. It easy to see that $T$ is defined over $K$. By assumption $T$ is not empty, hence $T(\overline{K}) \neq \emptyset$. In particular, there exists a finite extension $L/K$ for which $T(L) \neq \emptyset$. Hence $S_L^{[2]}$ can be embedded in $\text{Gr}(1, 5)_L$. Since $q^p - (S_L^{[2]})$ is a cubic fourfold, we obtain that $S_L^{[2]} \cong F(X)$ for some cubic fourfold $X/L$. From [10] Theorem 1.0.3 it follows that the associated line bundle is an ample line bundle of degree $d$. Since all ample line bundles of degree $d$ are defined over $K$, we have that the associated line bundle to $F(X) \cong S^{[2]}$ is defined over $K$. Now apply Theorem 4.11.

We can now relate the zeta function of the cubic fourfold $X$ to the zeta function of the $K3$ surface $S$.

**Corollary 4.13.** Let $S, X, K$ be as before. Then we have the following equality of zeta functions

$$Z_K(X, s) \cong Z_K(S, s - 1)\zeta_K(s) \zeta_K(s - 2) \zeta_K(s - 4).$$

Here $\cong$ means equality up to, possibly, finitely many local Euler factors.

**Proof.** First recall that $H^1(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = 0$ and that $H^2(X, \mathcal{O}_X) = \mathbb{Q}h$ where $h$ is the class of the hyperplane section. Moreover $H^1(S, \mathcal{O}_X) = 0$. It follows from Corollary 2.6 that

$$H^3_{et}(X, \mathbb{Q}_l) \cong H^3_{et}(S, \mathbb{Q}_l)(1) \oplus \mathbb{Q}_l[\Delta](1).$$

Hence we have that

$$\frac{Z_K(X, s)}{\zeta_K(s) \zeta_K(s - 1) \zeta_K(s - 3) \zeta_K(s - 4)} \cong \frac{Z_K(S, s - 1) \zeta_K(s - 2)}{\zeta_K(s - 1) \zeta_K(s - 3)}.$$

which gives the claim. \qed

If the surface $S$ is singular, we can determine the zeta function explicitly. Recall that a $K3$ surface $S/K$ is called singular if its geometric Picard number $\rho(S_K)$ equals 20. Singular $K3$ surfaces were classified by Shioda and Inose [13]. The isomorphism classes of these surfaces are in 1-to-1 correspondence with $\text{SL}(2, \mathbb{Z})$-isomorphism classes of even binary quadratic forms. This correspondence is given by associating to a singular $K3$ surface $S$ its transcendental lattice $T(S)$. The discriminant of the surface $S$ is defined as the discriminant of the lattice $T(S)$. Shioda and Inose showed that every singular $K3$ surface $S$ is either the Kummer surface of a product $E \times F$, where $E$ and $F$ are isogenous elliptic curves with complex multiplication (CM), or $S$ has a (rational) double cover by such a surface. The elliptic curve $E$ has CM in the field $\mathbb{Q}(\sqrt{-d_0})$ where $d_0$ is the discriminant of the surface $S$. The construction of Shioda and Inose allows one to have control over the field of definition $K$.

Using this classification, Shioda and Inose have computed the zeta function of singular $K3$ surfaces ([13] Theorem 6). If the field $K$ is chosen big enough, namely such that the Néron-Severi group can be generated by elements defined over $K$ (this means essentially that the points of order 2 of the curves $E$ and $F$ are defined over $K$), then the zeta function of $S$ is a product whose factors are either (twisted) Dedekind zeta functions or the $L$-function of a suitable Hecke Grössencharakter. For
a recent account of Hecke Grössencharakters and their associated Hecke eigenforms, in particular in connection with $K3$ surfaces, we refer the reader to [10]. It is a consequence of our descent Theorem 4.14 that there is a cubic fourfold $X$ defined over $K$ with $S^{[2]} \cong F(X)$. As a consequence we can compute the zeta function of the variety $X$.

We recall that the singular $K3$ surfaces form a countable set in the moduli space $\mathcal{F}_d$ which is dense in the $\mathbb{C}$-topology. This can be proved in the same way as the density in the period domain of all $K3$ surfaces (see the proof of [3 Corollary VIII.8.5]). This shows, using Theorem 3.2, that for every value of $\varepsilon_3$.

**Corollary 4.14.** Assume $S$ satisfies the assumptions of Corollary [4.12]. In particular, there exists a cubic fourfold $X/K$ with $S^{[2]} \cong F(X)$. Moreover assume that $\rho(S) = \rho(SR) = 20$, that the square root of the negative of the discriminant of the transcendental lattice of $S$ is contained in $K$ and that the Néron-Severi group of $S$ can be generated by divisors defined over $K$. Then there exists a Grössencharakter $\psi$ such that

$$Z_K(X, s) = \zeta_K(s) \zeta_K(s - 1) \zeta_K(s - 2)^{21} \zeta_K(s - 3) \zeta_K(s - 4) L(\psi^2, s - 1) L(\overline{\psi}, s - 1).$$

**Proof.** From [18, Theorem 6] it follows that there exists a Grössencharakter $\psi$ such that

$$L(S, s) = \zeta_K(s) \zeta_K(s - 1)^{20} \zeta_K(s - 2) L(\psi^2, s) L(\overline{\psi}, s).$$

Combine Corollaries [4.12] and [4.13] to conclude the proof. □

If a singular $K3$ surface $S$ can be defined over $\mathbb{Q}$, then the transcendental lattice $T(S)$ is a 2-dimensional Galois module. Livné [13, Example 1.6] has shown that in this case the $L$-function defined by $T(S)$ is that of a Hecke eigenform of weight 3. Assume that $S^{[2]} \cong F(X)$ for some cubic fourfold. Then we can, by our descent theorem, assume that $X$ is defined over $\mathbb{Q}$. The sublattice of $H^4(X, \mathbb{Z})$ generated by algebraic cycles has rank 21. We denote its orthogonal complement by $T(X)$ and refer to this as the transcendental lattice of $X$.

**Corollary 4.15.** Assume that $S$ is a singular $K3$ surface defined over $\mathbb{Q}$, and that it satisfies the assumptions of Corollary [4.12] in particular $S^{[2]} \cong F(X)$ for some cubic fourfold $X$ defined over $\mathbb{Q}$. Then there exists a weight 3 Hecke eigenform $f$ such that

$$L(T(X), s) = L(f, s - 1).$$

**Proof.** Proposition [22] implies that $T(X)$ and $T(S)(1)$ are isomorphic. □

5. The Fermat cubic

We shall now present an explicit example, showing that the weight 3 form associated to the Fermat cubic is the level 27 newform.

Our example will be a special case of the Pfaffian construction due to Beauville and Donagi [5]. For this we shall consider a cubic fourfold $X$ containing two planes $P$ and $Q$, which we can assume to be $P = \{u = v = w = 0\}$ and $Q = \{x = y = z = 0\}$.
The defining polynomial for $X$ can be written in the form $F - G$ where $F, G$ are bihomogeneous in $\mathbb{C}[u, v, w; x, y, z]$ of bidegrees $(2, 1)$ and $(1, 2)$ respectively. Such a fourfold is always rational. Indeed, a general line joining the two planes $P$ and $Q$ will meet $X$ in a third point. In this way we obtain a birational map $\pi_1 \times \pi_2 : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$. The lines contained in $X$ correspond to the complete intersection $S$ given by $F = G = 0$. The surface $S$ is a complete intersection of bidegree $(2, 1), (1, 2)$ and hence a $K3$-surface. Via the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^8$, where it is of degree $14$. This corresponds to the case $n = 2$ in Theorem 3.2.

The inverse birational map is given by the linear system $|L|$ of bidegree $(2, 2)$ forms on $\mathbb{P}^2 \times \mathbb{P}^2$ vanishing along $S$. In this case, as was shown by Hassett (see [2] page 41), the map $S^{[2]} \dashrightarrow F(X)$ which was described in [4] Proposition 5 is no longer an isomorphism. However, there is a birational map $S^{[2]} \dashrightarrow F(X)$, which will be good enough for our purposes: Take a point $s + t \in S^{[2]}$ and consider the lines $l_i$ spanned by $\pi_i(s)$ and $\pi_i(t)$ for $i = 1, 2$. If $s$ and $t$ are sufficiently general, then $l_1 \times l_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ intersects $S$ in precisely five points. Two of these points are $s$ and $t$, call the other three points $u, v, w$. Let $C$ be the $(1, 1)$ curve on $l_1 \times l_2$ passing through $u, v, w$. Since the three points $u, v, w$ lie on $S$, the linear series $|L|$ restricted to $C$ has degree $1$, and hence $C$ is mapped to a line in $\mathbb{P}^5$. One can show that the general line on $X$ is obtained in this way, giving the desired birational map.

In order to present an explicit example, consider the surface $S$, given by the complete intersection of

$$F = xu^2 + yv^2 + zw^2 = 0 \text{ and } G = x^2u + y^2v + z^2w = 0.$$ 

This surface was considered in [2] as an example for a $K3$-surface in characteristic $2$ such that the automorphism group has a quotient that is not isomorphic to a subgroup of the Mathieu group $M_{23}$, proving that Mukai’s classification of finite groups acting symplectically on complex $K3$ surfaces does not extend to characteristic $2$.

The surface $S$ contains many lines. An easy calculation, using the intersection pairing, shows that $\rho(S) = 20$. Moreover, the discriminant of the transcendental lattice $T(S)$ equals $3$ up to a square. One can also show that $3$ is the only prime of bad reduction. In particular, $L(S, s) \cong L(f, s - 1)$, with $f$ a weight $3$ Hecke newform of level $3^b$, for some $b$. There are precisely three such forms, namely the newform of level $27$ and two twists of this newform, by the standard cubic character and the square of the standard cubic character. The Fourier coefficients at $7$ of these three forms are pairwise distinct.

We will determine $b$ and $f$: Let $p$ be a prime of good reduction and let $\mathcal{S}$ be the reduction of $S$ modulo $p$. Consider the action of Frobenius $\text{Fr}$ on the cohomology of $\mathcal{S}$. On $H^2_f(S, \mathbb{Q}_p)$ Frobenius acts as multiplication by $p^2$, on $H^1_f(S, \mathbb{Q}_p)$ as multiplication by $1$. It is well-known to the experts that the trace of Frobenius on $NS(\mathcal{S}) \otimes \mathbb{Q}_p \subset H^2_{et}(\mathcal{S}, \mathbb{Q}_p)$ is $0$ modulo $p$. From the Lefschetz trace formula it follows that

$$\text{Tr} \text{Fr}^* | T(\mathcal{S}) \equiv \#(\mathbb{F}_p) - 1 \mod p.$$ 

For primes $p \equiv 2 \mod 3$, one easily shows, using the form of the equations, that this trace is $0 \mod p$. For primes $p \equiv 1 \mod 3$ a straightforward calculation by
Since the form $f$ corresponds to the Galois representation of the transcendental lattice, the Fourier coefficients of $f$ coincide with $\text{Tr} \text{Fr}^* \mid T(S)$. Using the tables from [16] one concludes that $f$ is the unique newform of level 27 up to a twist by a cubic character, unramified outside 3. By class field theory there are three such characters, say $\chi, \chi^2$ and the trivial character. One easily sees that the Fourier coefficients of $f$, $f \otimes \chi$ and $f \otimes \chi^2$ at 7 are different. From this we conclude that $f$ is the correct form.

The corresponding fourfold $X$ can easily be determined explicitly. The linear system of forms of bidegree $(2, 2)$ which vanish on $S$ defines the map

$$(xF : yF : zF : uG : vG : wG) : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$$

and an easy calculation shows that the image satisfies the equation

$$F - G = xu^2 - ux^2 + vy^2 - vy^2 + zw^2 - z^2 w$$

which is the equation of the associated fourfold $X$. The birational map $\psi : S^{[2]} \to F(X)$ induces an isomorphism $T(S) \cong T(F(X))$ of Galois modules. (This follows from the fact that $\psi$ induces an isomorphism $\psi^* : H^{2,0}(S) \oplus H^{0,2}(S) \to H^{2,0}(F(X)) \oplus H^{0,2}(F(X))$.) As in Corollary 4.14 it now follows that

$$L(T(X), s) \cong L(f, s - 1)$$

where $T(X)$ is the transcendental lattice of $X$ and $f$ is the weight 3 Hecke eigenform of level 27 from above.

Since the equation of $X$ is the sum of three terms which only depend on two variables each, it follows that $X_{\mathbb{Q}(\sqrt{-3})}$ is isomorphic to the Fermat cubic $X'$

$$u^3 + v^3 + w^3 + x^3 + y^3 + z^3.$$ 

This isomorphism does not descend to $\mathbb{Q}$. This can be shown as follows. The automorphism group of $X'$ is generated by permutations of the coordinates and by multiplying the coordinates by a third root of unity (see [17]). This implies that $\# \text{Aut}(X')(\mathbb{Q}) = \#S_6 = 720 = 6^2 \cdot 20$. One can easily find many $\mathbb{Q}$-rational automorphisms of $X$. For example the automorphisms $[x, u, y, v, z, w] \mapsto [-u, -x, y, v, z, w]$ and $[x, u, y, v, z, w] \mapsto [u - x, -x, y, v, z, w]$ generate a subgroup of order 6. Considering the same automorphisms for the pairs $(y, v)$ and $(z, w)$ gives a subgroup of order $6^3$ in $\text{Aut}(X)(\mathbb{Q})$. In particular, $\# \text{Aut}(X)(\mathbb{Q}) \neq \text{Aut}(X')(\mathbb{Q})$.

This implies that $X$ and $X'$ are not isomorphic and that the K3 surface associated to $X'$ is a twist of our surface $S$. However, we shall show that the associated newform to $X'$ is the same form $f$.

The Galois representation on $T(X')$ is a twist of the Galois representation on $T(X)$ by the quadratic character $\chi_{-3}$. In particular, the Hecke newform $f'$ associated to $X'$ has level $3^2$, so it is one of the three above mentioned forms. From $\#X'(\mathbb{F}_7) = 3690$ one deduces, as above, that $f' = f$.

The zeta function of the Fermat cubic was also considered by Goto in [3]. His approach, however, is entirely different, since he considered this over the field $\mathbb{Q}(\sqrt{-3})$ and he used Weil’s classical approach using Jacobi sums.
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