Topics in the Value Distribution of Random Analytic Functions

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Abstract

This thesis deals with the behavior of random analytic functions. Our approach to this subject is the classical one, that is, we study random Taylor series with independent coefficients. In particular, we are interested in the value distribution of these functions, this basically means counting the number of certain values that these functions take, in some domains. Special attention is devoted to studying the relation between the number of zeros of these functions and the growth of their coefficients.

The contents of this thesis can be partitioned into two parts: value distribution of Rademacher Taylor series and asymptotics of the hole probability for Gaussian entire functions. A certain philosophy of this work, which is relevant to both parts, is to avoid making any assumptions on the magnitude (variance) of the Taylor coefficients, while studying a particular distribution of the phases. In the first part, we study Taylor series with random signs and in the second we study coefficients with a complex Gaussian distribution. As it turns out, in both cases it is possible to prove general results without making any assumptions on the ‘regularity’ of the growth of the coefficients.

We start this thesis with a study of the properties of Rademacher Fourier series, the main result states that the logarithm of such series is integrable (to any power). The methods that are used are related to those that were originally developed by F. Nazarov to tackle similar problems for lacunary Fourier series. This result then enables us to study in detail the behavior of Rademacher Taylor series. First we give an answer to an old question of J.-P. Kahane, concerning the range of Rademacher Taylor series in the unit disk. Then we prove a theorem that relates the number of zeros of Rademacher entire functions in large disks to the rate of growth of their coefficients. Finally, we prove that the zeros of these entire functions are equidistributed in sectors centered at the origin. This improves on some classical works of Littlewood and Offord.

The main motivation of the second part is to study ‘exceptional’ behavior of Gaussian entire functions (given by a Taylor series). It is well known that these functions are expected to have many zeros in large disks around the origin. We are interested in characterizing the logarithmic asymptotics of the probability of a ‘hole event’, that is, when the function has no zeros at all in such disks. We give a full solution for this problem, by introducing a new function which measures the relation between the rate of decay of the ‘hole probability’ and the magnitude of the coefficients of the Gaussian Taylor series.
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Part I. Introduction

This work is concerned with ‘typical’ and ‘exceptional’ behavior of analytic functions in general, and entire functions in particular. Here the words typical and exceptional are interpreted in the probabilistic sense. Our main objects of study are random analytic functions, more precisely, random Taylor series. This kind of functions have a long history, as they first appear at the end of the 19th century. It could be argued that this notion first began with the comments of Borel in 1896 on “General Taylor series”, and it concerned the natural boundary of Taylor series. The precise statement of this result is due to Steinhaus in 1929. Further work on random series has been taken in the 1930s, by Paley and Zygmund, Wiener, Lévy and others, see the excellent historical survey by [Ka1], and also the book [Ka2].

In additional to random Taylor series we are also interested in this work in random Fourier series, as the two are evidently closely related. There are several natural reasons for studying these two types of random series. First, one can use random Fourier series to give examples of functions with desired and undesired properties. In addition, random Fourier series have strong connections to Brownian motion. Furthermore, Fourier series are a useful tool for the study of entire functions, a fact which we use in this work. Second, random polynomials were studied for a large variety of reasons, including some motivation from mathematical physics, and random analytic functions are their natural generalization. A third reason is the relation between lacunary series and random series. It is well known since the works of Khintchin and Kolmogorov and of Paley and Zygmund that there are many similarities between them. Some of these similarities will be explored in this work. As it turns out, methods that were originally developed for lacunary series can be applied to random series as well (we will not present any new results relating to lacunary series).

One of the main interests of this work is in the behavior of the zero set of random analytic functions. The study of zeros of random polynomials and random analytic functions started with the pioneering works of Wiener, Littlewood and Offord, Kac, and Rice, and was later continued by Erdős, Ibragimov, Hammersley, Kahane, Maslova, Offord, and many others. During the last two decades, the subject was revived by several groups of researchers who came from very different areas and established new links to mathematical physics, probability theory, complex analysis, and complex geometry. Some of the recent works were surveyed in various places [NS3, Zel]; see also the introductory article [NS2] and the recent book [BKPV]. The field of random entire functions is still evolving, and contains many different and interesting open problems. In addition, the problems usually involve a unique combination of analytic and probabilistic techniques.
With only few exceptions, almost all of the interesting results proven in the last two decades deal with Gaussian analytic functions. The special properties of the Gaussian distribution usually play an absolutely indispensable role in these works. At the same time, there are several basic long-standing questions pertaining to zeros of random Taylor series and random polynomials with independent coefficients having non-Gaussian distribution, which remain open for about half a century, sometimes even more. For some of these questions ‘the Rademacher case’ (or Bernoulli case), when the random coefficients take the values ±1 with equal probability already contains all principal difficulties\(^1\). Moreover, in many instances, the Rademacher case yields similar results for more or less arbitrary random symmetric coefficients.

1 Contents of this work

In the first part of this work we discuss Rademacher Fourier series and prove that the logarithm of such a series is integrable (to any positive power). This will be in turn the main technical tool for the second part.

In the second part of the work we study the value distribution of random analytic functions. The first result is an answer to an old question from J.-P. Kahane book, about the range of Rademacher Taylor series in the unit disk. We then use the results of the first part to prove the almost sure convergence of the number of zeros for Rademacher entire functions. We also prove the equidistribution of the zeros of Rademacher entire functions, under a certain growth condition on the function. We then show that this condition is close to being necessary, by giving an example of Rademacher entire function with ‘too many’ real zeros. If one takes for granted the main results of the first part, then this part can be read independently from it.

The third part of this work is concerned with the exceptional behavior of the zero set of Gaussian entire functions. We find the logarithmic asymptotics for the probability of the event when such a function has no zeros in a large disk around the origin. We also show that this result depends in a crucial way on the nature of the random variables, by giving an example of a random entire function that always has zeros outside of some fixed disk. This part is independent from the first two.

In each part we give some historical background as well as a list of some open problems. We end this thesis with a list of references (which is joint for all the parts). We now give a formal description of the theorems which we are going to prove.

\(^1\) To quote M. Kac: “Upon closer examination it turns out that the proof I had in mind [in a previous paper]. . . . is inapplicable to the [case when the coefficients have a discrete distribution]. . . . This situation tends to emphasize the particular interest of the discrete case, which surprisingly enough turns out to be the most difficult.”
1.1 Random Taylor series

As was already mentioned, the main questions of interest in this work involve the statistical properties of the zero set of random analytic functions, given by a Taylor series of the form

\[ f(z) = \sum_{n \geq 0} \xi_n a_n z^n. \]  

(\*)

Here \( \{\xi_n\} \) is a sequence of independent and identically distributed random variables, and the coefficients \( a_n \) are deterministic (i.e. non-random). The classes of functions that are studied will be either entire functions or analytic functions in the unit disk. It is known that the domain of convergence for functions of the form (\*) is the disk where the (non-random) series \( \sum a_n z^n \) converges\(^2\) ([BKPV, Lemma 2.2.3]). Note that when dealing with random Taylor series of the form (\*), one usually arrives at the following ‘trade off’: either to impose regularity assumptions on \( a_k \)’s trying to achieve as much generality as possible with the \( \xi_k \)’s, or to consider general non-random coefficients \( a_k \)’s, imposing certain assumptions on the i.i.d. factors \( \xi_k \)’s. In general, in this work we chose the latter. For an example of papers that deal with the first alternative, see [IZ, KZ].

In order to count the number of zeros of a random analytic function \( f(z) \), we define the zero counting measure \( \nu_f \), which has an atom for each zero of \( f \) (weighted according to the multiplicity of the zero). By Green’s formula, we have the following expression for this measure in terms of \( f \):

\[ \nu_f (z) = \frac{1}{2\pi} \Delta \log |f(z)|, \]

where the Laplacian on the RHS is taken in the sense of distributions. In principle, it is difficult to analyze this measure directly, unless one makes some further assumptions (see [IZ, KZ]). We will now explain briefly the most well-studied case, where the random variables \( \xi_n \) are standard complex Gaussians. Such functions are called Gaussian analytic functions (GAFs) and a lot is known about their zero counting measure \( \nu_f \). Part of the motivation behind this work is to extend this knowledge to other types of random variables.

1.2 Gaussian analytic functions (GAFs)

A random variable \( \xi \) is a standard complex Gaussian, if \( \xi \) has density \( \frac{1}{\pi} e^{-|z|^2} \) with respect to Lebesgue measure \( m \) in the complex plane. Therefore \( \mathbb{E} \{ \xi \} = 0 \) and

\(^2\)Some minimal assumptions on the distribution of the random variables \( \xi_n \) are required. For example, \( \mathbb{E} \{ \log^+ |\xi_n| \} < \infty \) is sufficient.
\( \mathcal{E} \left\{ |\xi|^2 \right\} = 1 \), where here and in the rest of this work \( \mathcal{E} \) stands for the expected value.

As a Gaussian random process, the behavior of a GAF, and thus its zero set, is determined by its covariance kernel

\[
K_f(z, w) = \mathcal{E} \left\{ f(z) \overline{f(w)} \right\} = \sum_{n \geq 0} |a_n|^2 (zw)^n;
\]

for details, see for example [BKPV, p. 16]. A particularly nice example is the classical Edelman-Kostlan formula ([BKPV, p. 25]), which tells us that

\[
\mathcal{E} \{ \nu_f(z) \} = \frac{1}{2\pi} \Delta \log \sqrt{K_f(z, \overline{z})} dm(z).
\]

Now, if we write \( K_f(r) = K_f(r, r) \) and denote by \( n_f(r) \) the number of zeros of \( f \) inside the disk \( r \mathbb{D} = \{ |z| \leq r \} \), we conclude that

\[
\mathcal{E} \{ n_f(r) \} = \int_{r \mathbb{D}} \mathcal{E} \{ \nu_f(z) \} \ dm(z) = \frac{d \log \sqrt{K_f(r)}}{d \log r} = \frac{r}{2} \cdot \frac{K'(r)}{K(r)}.
\]

One of the goals of this work is to prove a similar result for other types of random variables.

### 1.3 Rademacher Fourier series - Logarithmic integrability

The extension of (1.2) for Bernoulli random variables is reduced, using Jensen’s formula, to the study of the behavior of Rademacher Fourier series. These are series of the form

\[
g(\theta) = \sum_{n \in \mathbb{Z}} b_n \xi_n e^{2\pi in\theta}, \quad \theta \in [0, 1],
\]

where the coefficients \( b_n \) satisfy \( \sum |b_n|^2 < \infty \), and the \( \xi_n \)s are i.i.d. random variables taking the values \( \pm 1 \) with probability \( \frac{1}{2} \) each. We denote by \( \mathbb{T} \) the circle (or \( \mathbb{R} \setminus \mathbb{Z} \)) with normalized Lebesgue measure \( m \). It is not difficult to check that

\[
\| g \|_2 \overset{\text{def}}{=} \mathcal{E} \left\{ \int_{\mathbb{T}} |g(\theta)|^2 \ d\theta \right\} = \sum_{n \in \mathbb{Z}} |b_n|^2 = \| \{ b_n \} \|_{\ell^2(\mathbb{Z})}^2.
\]

Intuitively one should expect that a random Fourier series will not have any ‘local’ structure, and, in particular, it should not be too small on relatively large sets. The main theorem of the first part of this work is the following quantitative version of this idea.
Theorem 1. [NNS1] Let $g$ be a Rademacher Fourier series with $\|g\|_2 = 1$. For every $q \geq 1$ we have

$$E\left\{ \int_{\pi} |\log |g(\theta)||^q d\theta \right\} \leq C^q q^{6q},$$

where $C > 0$ is some numerical constant.

Notice that this theorem is general, in the sense that the constant $C$ does not depend on the specific values of the coefficients $b_n$, a fact that will be important for our applications.

1.4 Solution to a question of Kahane

One of the consequences of Theorem 1 is the answer to a well-known old question from Kahane’s book [Ka2, p. xii]: Suppose that $F(z) = \sum_{n \geq 0} a_n \xi_n z^n$ is a Rademacher Taylor series with radius of convergence one and such that $\sum_{n \geq 0} |a_n|^2 = +\infty$. Is it true that, almost surely (a.s.), the range $F(\{|z| < 1\})$ is the whole complex plane?

In the second part of the work we will answer this question, and in fact prove even more. Given $b \in \mathbb{C}$, denote by $\Lambda_F(b)$ the collection of all solutions to the equation $F(z) = b$, repeated according to their multiplicity. Let $\{\zeta_n\}_{n \geq 0}$ be a sequence of independent complex-valued symmetric random variables, and write $F(z) = \sum_{n \geq 0} \zeta_n z^n$. We prove

Theorem 2. [NNS1] If the sequence $\{\zeta_n\}$ satisfies

$$\limsup_{n \to \infty} |\zeta_n|^{1/n} = 1 \quad \text{and} \quad \sum_{n \geq 0} |\zeta_n|^2 = +\infty, \quad \text{a.s.,}$$

then, almost surely,

$$\forall b \in \mathbb{C}, \quad \sum_{w \in \Lambda_F(b)} (1 - |w|) = +\infty.$$

The special case of Rademacher analytic functions is clearly implied by setting $\zeta_n = a_n \xi_n$. The case of Gaussian coefficients also follows (for another proof of this case see [Ka2, chap. 13]).

Remark. It should be mentioned that the proof of Theorem 2 requires a stronger version of Theorem 1. We will give the statement and the proof of this version in Part II Section 2.2.
1.5 Equidistribution of the zeros of random entire functions

Next, we apply the tools developed in the first part to show that the zeros of Rademacher entire functions are equidistributed in sectors (for GAFs this is suggested by the Edelman-Kostlan formula (1.1)).

Let
\[ F(z) = \sum_{n \geq 0} \xi_n a_n z^n \]
be a Rademacher Taylor series with an infinite radius of convergence (i.e., \( \sum a_n z^n \) is an entire function). We will assume that \(|F(0)| = 1\); the general case is reduced to this one by letting \( F(z) = a_\nu z^\nu F_1(z) \), where \( \nu \in \mathbb{N} \) is the multiplicity of the zero of \( F \) at the origin. Similarly to the case of GAFs, we use the notation
\[ \sigma_F^2(r) = \mathcal{E}\{|F(re^{i\theta}|^2\} = \sum_{n \geq 0} |a_n|^2 r^{2n}, \]
and also put
\[ s_F(r) = \frac{d \log \sigma_F(r)}{d \log r} = r \cdot \frac{d \log \sigma_F(r)}{dr}. \]

By \( n_F(r, \alpha, \beta) \) we denote the number of zeros (counted with multiplicities) of \( F \) in the sector \( \{z: \alpha \leq \arg z < \beta, |z| \leq r\} \). The integrated zero counting function is given by
\[ N_F(r, \alpha, \beta) = \int_0^r \frac{n_F(r, \alpha, \beta)}{t} \, dt. \]

A set \( E \subset [1, \infty) \) is a set of finite logarithmic measure if
\[ \int_E \frac{dt}{t} < \infty. \]

The main result of this section shows that under certain growth conditions on the function \( \sigma_F(r) \) the zeros of \( F \) are equidistributed in sectors.

**Theorem 3.** Let \( F \) be a Rademacher Taylor series with an infinite radius of convergence.

(i) Suppose that \( a > 2 \) and \( \gamma \in \left(\frac{1}{2} + \frac{1}{a}, 1\right) \). Then, a.s. and in mean,
\[ \sup_{0 \leq \alpha < \beta \leq 2\pi} \left| N_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r) \right| = O(\log^\gamma (\sigma_F(r))) \]
when \( r \to \infty \) and \( \log \sigma_F(r) > \log^a r \).
Suppose that $a > 4$ and $\gamma \in \left(\frac{3}{4} + \frac{1}{a}, 1\right)$. Then there exists a set $E \subset [1, \infty)$ of a finite logarithmic measure such that, a.s. and in mean,

$$\sup_{0 \leq \alpha < \beta \leq 2\pi} \left| n_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} s_F(r) \right| = O\left((s_F(r))^\gamma\right)$$

when $r \to \infty$, $r \notin E$, and $\log \sigma_F(r) > \log a r$.

In this theorem, some lower bound on the growth of $\sigma_F$ is necessary. We will show in Section 10 that if $\beta > \log 3$, then all the zeros of the Rademacher Taylor series $F(z) = \sum_{k \geq 0} \xi_k e^{-\beta k^2 z^2}$ are real (we have that $\log \sigma_F(r) \sim C_\beta \cdot \log^2 r$ as $r \to \infty$, for this function).

**Remark.** It should be mentioned that the exceptional set $E$ in the theorem is unavoidable, since we impose no regularity assumptions on the sequence $\{a_n\}$. On the other hand, if the sequence $\{a_n\}$ satisfies some very mild regularity condition, the exceptional set $E$ is not needed. This exceptional set will reappear, for similar reasons, in the third part of this work.

### 1.6 The hole probability for Gaussian entire functions

The third part of this work is devoted to the study of a special kind of ‘exceptional’ behavior of the zero set. The ‘hole’ event that we are interested in is when the function $f(z)$ has no zeros inside some domain $D$. We consider this problem for Gaussian entire functions with general coefficients $a_n$. We are interested in the logarithmic asymptotics of the probability of the hole event, in the case where the domain $D$ is the disk around zero with radius $r$, as $r$ tends to infinity. The following theorem gives a rather precise answer.

**Theorem 4.** [Ni3] Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and let

$$f(z) = \sum_{n \geq 0} a_n \xi_n z^n,$$

where $\xi_n$ are i.i.d. standard complex Gaussians and $a_0 \neq 0$. Assume that $\sum_{n \geq 0} a_n z^n$ is a non-constant entire function. We have that

$$\log - \mathbb{P}(f(z) \neq 0 \text{ in } |z| \leq r) = S(r) + O\left(S(r)^{\frac{1}{2} + \epsilon}\right), \quad r \notin E,$$

as $r \to \infty$. Here

$$S(r) = 2 \cdot \sum_{n \geq 0} \log^+(a_n r^n)$$

and $E \subset [1, \infty)$ is a non-random exceptional set of finite logarithmic measure.

**Remark.** The set $E$ depends only on the sequence $\{a_n\}$ and on $\epsilon$. 
1.7 Notation

Throughout this work the letters $C$ and $c$ denote various positive numerical constants; usually the first is reserved for ‘large’ ($\geq 1$) numbers, while the second is used for ‘small’ numbers. We use $C_\alpha$ for constants that depend on another quantity $\alpha$.

Several types of symbols are used to indicate the asymptotic relations between different quantities. For positive quantities $\Phi, \Psi$, the notation $\Phi \lesssim \Psi$ means that there exists a positive constant $C$ so that $\Phi \leq C \cdot \Psi$. Another notation for the same relation is $\Phi = O(\Psi)$ (the usage depends on the convenience at each point). The notation $\Phi = O_\alpha(\Psi)$ means that the implied constant depends only on $\alpha$. For positive sequences or functions the notation $\Phi = o(\Psi)$ indicates that the (appropriate) limit $\Phi/\Psi$ is zero.

We use $\mathcal{P}\{E\}$ or $\mathcal{P}(E)$ to denote the probability of an event $E$. The notation $\mathcal{E}\{X\}$ or $\mathcal{E}(X)$ is used for the expected value of the random variable $X$.

We use $r\mathbb{D}$ for the closed disk $\{|z| \leq r\}$, while $r\mathbb{T}$ is used for the circle $\{|z| = r\}$. Finally, the log-derivative $\frac{df(r)}{d \log r}$ is the same as $r \cdot \frac{df(r)}{dr}$. 
Part II. Logarithmic Integrability of Rademacher Fourier series

We start by explaining the central role played by the logarithmic integrability of Rademacher Fourier series. Consider a Rademacher Taylor series

\[ f(z) = \sum_{n \geq 0} a_n \xi_n z^n, \]

with radius of convergence \( R, 0 < R \leq \infty \), and for \( r = |z| \), put

\[ \sigma_f^2(r) = E \{ |f(z)|^2 \} = \sum_{n \geq 0} |a_n|^2 r^{2n}. \]

We will assume that \( \sigma_f(r) \to \infty \) as \( r \to R \). We are interested in the asymptotics of the random counting function \( n_f(r) \), which counts the number of zeros of \( f \) in the disk \( \mathbb{D} = \{ |z| \leq r \} \), as \( r \to R \). To simplify the notation, assume that \( a_0 = 1 \), and thus \( |f(0)| = 1 \). Denote by

\[ N_f(r) \overset{\text{def}}{=} \int_0^r \frac{n_f(t)}{t} \, dt \]

the integrated counting function. Then, by Jensen’s formula,

\[ N_f(r) = \int_{\mathbb{T}} \log |f(re^{2\pi i \theta})| \, d\theta - \log |f(0)| = \log \sigma_f(r) + \int_{\mathbb{T}} \log |\hat{f}(re^{2\pi i \theta})| \, d\theta, \]

where

\[ \hat{f}(re^{2\pi i \theta}) \overset{\text{def}}{=} \frac{f(re^{2\pi i \theta})}{\sigma_f(r)}. \]

Note that for a fixed \( r \) the function \( \hat{f}(re^{2\pi i \theta}) = \sum_{n \geq 0} \hat{a}_n(r) \xi_n e^{2\pi i n \theta} \) is a random Fourier series normalized by the condition \( \sum_{n \geq 0} |\hat{a}_n(r)|^2 = 1 \).

In order to explain the reduction to logarithmic integrability we start by considering a simpler case, where \( \xi_n \) are standard complex Gaussian random variables. Then, for every \( \theta \in \mathbb{T} \), the random variable \( \hat{f}(re^{2\pi i \theta}) \) is again a standard complex Gaussian. Thus \( \mathcal{E} \left\{ |\log |\hat{f}|| \right\} \) is a positive numerical constant and therefore, \( \sup_{r<R} \mathcal{E} \left\{ |N_f(r) - \log \sigma_f(r)| \right\} \leq C \). We can then ‘differentiate’ this asymptotic relation, and get

\[ n_f(r) = \frac{d \log \sigma_f(r)}{d \log r} + e(r), \text{ with } \mathcal{E} |e(r)| = o(1) \cdot \frac{d \log \sigma_f(r)}{d \log r}, \quad r \to R, \ r \notin E, \]
where $E$ is an exceptional set of $r$’s of finite logarithmic measure, which is unavoidable when the non-random coefficients $a_k$ have an irregular behaviour. Similarly, one gets $\sup_{r<R} \mathcal{E} \{ |N_f(r) - \log \sigma_f(r)|^p \} \leq (Cp)^{p}$, for $p \geq 1$. Combined with the classical Borel-Cantelli lemma, this leads to a good almost sure estimate for the error term $e(r)$.

The same approach works, albeit with more technical difficulties, for the Steinhaus coefficients $\xi_k = e^{2\pi i \gamma_k}$, where $\gamma_k$ are independent and uniformly distributed on $[0,1]$. In this case, one needs to estimate the $L^p$ norms of the logarithm of the absolute value of a normalized linear combination of independent Steinhaus variables. This was done by Offord in [Of2]; twenty years later, Favorov and Ullrich independently rediscovered this idea and gave new applications (see [Fa1, Fa2, Ul1, Ul2], and also the recent paper of Mahola and Filevich [MF]). However, these techniques fail for Rademacher random variables (in principle because linear combinations of Rademacher random variables can vanish with positive probability). Still, not everything is lost: note that in order to estimate the error term in Jensen’s formula we do not need a pointwise estimate for $\mathcal{E} \left\{ |\log \hat{f}(re^{2\pi i \theta})| \right\}$ that is uniform in $\theta \in \mathbb{T}$. For our purposes, the integral estimate $\mathcal{E} \left\{ \int_{\mathbb{T}} |\log \hat{f}(re^{2\pi i \theta})|^p \, d\theta \right\}$ is not worse than the uniform one.

In the first section we give some background and an outline of the proof. We also state the main tools used in the proof and some important notations. In the second and third sections we give the proof of the logarithmic integrability. In the last two section we give an example that illustrates the sharpness of the result and discuss some further related problems. In the next part we will apply this theorem to the study of the value distribution of random analytic functions.

## 2 Logarithmic Integrability - Background and the Main Result

In this section we describe the main result of this part of the work. It shows that an arbitrary Rademacher Fourier series cannot be too close to a random constant. The version we gave in the introduction corresponds to the case when the random constant is the zero function. The extension that we prove is needed for the proof of Theorem 2 on the range of random Taylor series in the unit disk.

### 2.1 Notation

We denote by $\mathbb{T}$ the interval $[0,1) \subset \mathbb{R}$, which we treat as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. $m$ will be either the Lebesgue measure on $\mathbb{T}$ normalized by $m(\mathbb{T}) = 1$, or the Lebesgue measure on $\mathbb{R}$, depending on the context. We also write $e(\theta) = e^{2\pi i \theta}$, $\theta \in \mathbb{T}$. 
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a standard probability space, and let \(\xi_k: \Omega \to \{\pm 1\}, k \in \mathbb{Z}\) be an independent Rademacher (or Bernoulli) sequence, that is, each \(\xi_k\) takes the values \(\pm 1\) with probability \(\frac{1}{2}\) each. Thus, we have a natural product measure space \(Q = \Omega \times \mathbb{T}\), with square integrable functions \(L^2(Q) = L^2(Q, \mu)\).

Our main interest here is in the (closed) subspace \(L^2_{RF} \subset L^2(Q)\) of Rademacher Fourier series
\[
    f = \sum_{k \in \mathbb{Z}} a_k \phi_k, \quad \sum_{k \in \mathbb{Z}} |a_k|^2 < \infty,
\]
where \(\phi_k(\omega, \theta) = \xi_k(\omega)e(k\theta), k \in \mathbb{Z}, (\omega, \theta) \in Q\). We note that the system \(\{\phi_k\}\) is an orthonormal basis for the space \(L^2_{RF}\), and for \(f \in L^2_{RF}\) one can easily check that
\[
    \|f\|_2^2 \overset{\text{def}}{=} \int_Q |f|^2 d\mu = \int_\Omega |f(\omega, \cdot)|^2 d\mathbb{P}(\omega) = \int_T |f(\cdot, \theta)|^2 d\mathbb{m}(\theta) = \sum_{k \in \mathbb{Z}} |a_k|^2 = \|\{a_k\}\|_{\ell^2(\mathbb{Z})}^2.
\]

A measurable function \(b\) on \(Q\) that does not depend on \(\theta\) will be called a random constant (note that \(b\) is not necessarily in \(L^2_{RF}\)).

### 2.2 The result

The goal of this part of the work is to prove that for any \(f \in L^2_{RF}\) with \(\|f\|_2 = 1\), any \(b \in L^\infty(\Omega)\) with \(\|b\|_\infty < \frac{1}{20}\), and every \(p \geq 1\), we have
\[
    \int_Q |\log |f - b||^p d\mu \leq (Cp)^6^p, \tag{2.1}
\]
where \(C\) is some positive numerical constant. We note that the condition on the function \(b\) is a technical one. Its purpose is to avoid some degenerate cases, for example, the case when the functions \(f\) and \(b\) are both equal to \(\xi_0\). The logarithmic integrability is a straightforward corollary of the following distribution inequality for \(L^2_{RF}\) functions.

**Theorem 5.** For any \(f \in L^2_{RF}\), any random constant \(b \in L^\infty(\Omega)\) with \(\|b\|_\infty < \frac{1}{20}\|f\|_2\), and any set \(E \subset Q\) of positive measure,
\[
    \|f\|_2^2 = \int_Q |f|^2 d\mu \leq \exp \left( C \log^6 \left( \frac{2}{\mu(E)} \right) \right) \int_E |f - b|^2 d\mu,
\]

\(^3\) It follows by applying Theorem 5 below for the sub-level sets \(E_\varepsilon = \{ (\omega, \theta) \in Q : |f - b| < \varepsilon \}\) and then integrating the LHS of (2.1) by parts.
where \( C \) is some positive numerical constant.

We note that the power 6 on the RHS is (probably) not the best possible, but in Section 5 we will give an example which shows that it cannot be replaced by any number less than 2. Since the proof of Theorem 5 is long and consists of several parts, we give a general description of some of the ideas behind it in this section. The next two sections contain the proof, and in the last section we discuss some further problems (as well as give the aforementioned example).

### 2.3 Background

The proof of Theorem 5 is based on ideas from harmonic analysis developed by Nazarov [Na1, Na2] to treat lacunary Fourier series. It uses a Turán-type lemma from [Na1, Chapter 1], and the technique of small shifts introduced in [Na1, Chapter 3]. It is worthwhile mentioning that the study of lacunary series has a long history. Already in 1872, Weierstrass gave a famous example of a continuous functions which is not differentiable at any point, in the form of a lacunary trigonometric series. Kolmogorov in the 1920s and Zygmund in the 1930s studied the convergence and integrability properties of lacunary series before they gave the counterpart (and more famous) results for random series.

Let \( \Lambda = \{m_k\}_{k \in \mathbb{Z}} \subset \mathbb{Z} \) be the ‘lacunary’ spectrum of some Fourier series, that is, we consider series of the form
\[
g(\theta) = \sum_{k \in \mathbb{Z}} a_{m_k} e(m_k \theta), \quad \theta \in \mathbb{T}, \quad \sum_{k \in \mathbb{Z}} |a_{m_k}|^2 < \infty, \tag{2.2}
\]
where the set \( \Lambda \) is ‘small’ in some sense. In 1948 Zygmund proved the following uniqueness result

**Theorem.** Suppose that the set \( \Lambda = \{m_k\}_{k \in \mathbb{Z}} \subset \mathbb{Z} \) satisfies
\[
R(\Lambda) \overset{\text{def}}{=} \sup_{r \neq 0} \# \{(k', k'') : m_{k'} - m_{k''} = r\} < \infty.
\]

For every measurable set \( E \subset \mathbb{T} \) of positive measure there exists a constant \( C(\Lambda, E) \) such that, for every \( g \in L^2(\mathbb{T}) \) of the form (2.2),
\[
\|g\|_{L^2(\mathbb{T})}^2 \leq C(\Lambda, E) \int_E |g|^2 \, dm.
\]
In the 1990s (see [Na1]), Nazarov significantly improved the previous theorem by giving an effective bound for the constant $C(\Lambda, E)$. More precisely, he proved that for any $\varepsilon > 0$, there exists a constant $D(\varepsilon, R(\Lambda))$ such that

$$C(\Lambda, E) \leq \exp\left(\frac{D(\varepsilon, R(\Lambda))}{m(E)^{2+\varepsilon}}\right);$$

in particular, this implies the logarithmic integrability of such lacunary series for every power less than 2.

**Remark.** Notice that in Nazarov’s result the constant $C(\Lambda, E)$ depends only on the measure of $E$.

In a later work ([Na2]), Nazarov considered the case of Hadamard lacunary series, that is, series whose spectrum $\Lambda$ satisfies

$$\lim\inf_{|k| \to \infty} \frac{m_{k+\text{sign}(k)}}{m_k} > 1.$$

Using an improved version of the results in [Na1], he proved that

$$\|g\|^2_{L^2(\mathbb{T})} \leq \exp\left(C\log^{10}\left(\frac{2}{m(E)}\right)\right) \int_E |g|^2 \, dm,$$

where $C$ is some positive numerical constant. In particular, this result implies the logarithmic integrability of such series for every positive power. Our proof is motivated by the ideas contained in both of these papers. It exemplifies — perhaps in a spirit similar to Kolmogorov and Zygmund’s results — the close connection between lacunary and random series. Just to give a particular example, the $\Lambda_p$ property of Hadamard lacunary series (which is central to the results of [Na2]) is replaced by the Khinchin and bilinear Khinchin inequalities.

### 2.4 The basic tools and some further notations

Here is the list of the tools we will be using in the proof of Theorem [5]

#### 2.4.1 Turán-type lemma

Let

$$p(z) = \sum_{k=0}^{n} a_k e^{i\lambda_k t}, \quad a_k \in \mathbb{C}, \quad \lambda_0 < \cdots < \lambda_n \in \mathbb{R},$$
be an exponential polynomial. Then for any interval $J \subset \mathbb{R}$ and any measurable subset $E \subset J$ of positive measure,

$$\sup_{J} |p| \leq \left( \frac{Cm(J)}{m(E)} \right)^{n} \sup_{E} |p|.$$ 

For the proof, see [Na1, Chapter I]. We will also use the $L^2$-bound that follows from this estimate (see [Na1, Chapter III, Lemma 3.3]). It states that, in the same setting,

$$\|p\|_{L^2(J)} \leq \left( \frac{Cm(J)}{m(E)} \right)^{n+\frac{1}{2}} \|p\|_{L^2(E)}.$$  \hspace{1cm}(2.3)

2.4.2 Khinchin’s inequality

Let $\{\xi_k\}$ be independent Rademacher random variables, and let $\{a_k\}$ be complex numbers. Then for each $p \geq 2$, we have

$$\left( \mathcal{E} \left| \sum_{k} a_k \xi_k \right|^p \right)^{1/p} \leq C \sqrt{p} \cdot \sqrt{\sum_{k} |a_k|^2}.$$ 

2.4.3 Bilinear Khinchin’s inequality

Let $\{\xi_k\}$ be independent Rademacher random variables, and let $\{a_{k,\ell}\}$ be complex numbers. Then for each $p \geq 2$, we have

$$\left( \mathcal{E} \left| \sum_{k \neq \ell} a_{k,\ell} \xi_k \xi_\ell \right|^p \right)^{1/p} \leq Cp \cdot \sqrt{\sum_{k \neq \ell} |a_{k,\ell}|^2}.$$ 

A simple and elegant proof of this inequality can be found in a recent preprint by L. Erdős, A. Knowles, H.-T. Yau, and J. Yin [EKYY, Appendix B].

2.4.4 Notations

For a set $E \subset Q$, we denote its sections by $E_\omega \stackrel{\text{def}}{=} \{\theta \in \mathbb{T} : (\omega, \theta) \in E\}$, $\omega \in \Omega$. The set $E \subset Q$ ‘shifted’ by $t \in \mathbb{T}$ is denoted by $E + t \stackrel{\text{def}}{=} \{(\omega, \theta) : (\omega, \theta - t) \in E\}$. Then

$$E_\omega + t = \{\theta : \theta - t \in E_\omega\} = (E + t)_\omega.$$ 

The function $g \in L^2(Q)$ shifted by $t$ is denoted by $g_t$: $g_t(\omega, \theta) = g(\omega, \theta + t)$. Note that for the indicator function of $E$, we have that $(\mathbb{1}_E)_t = \mathbb{1}_{E - t}$.

We write $[x]$ for the integral part of $x$. Also, $\Delta_t(E) \stackrel{\text{def}}{=} \mu((E + t) \setminus E)$, a notation that will appear many times during the proof.
2.5 An overview of the proof of Theorem 5

For the sake of simplicity we will assume here that \( b \equiv 0 \); the proof of the general case requires some technical modifications.

Let \( f \in L^2_{RF} \) be a Rademacher Fourier series, and let \( E \subset Q \) be a set of positive measure. We begin with the following observation, due to Zygmund: using the expansion of \(|f|^2\) as a diagonal and an off-diagonal sums, we can write

\[
\int_{E} |f|^2 \, d\mu = \mu(E) \|f\|^2_2 + \langle A_E f, f \rangle,
\]

where \( A_E \) is a certain (compact and self-adjoint) operator (on the subspace \( L^2_{RF} \)). Now if the operator norm of \( A_E \) is small, say \(|A_E|_{\text{op}} < \frac{\mu(E)}{2}\), then this immediately yields the conclusion of Theorem 5 (with a better dependence on \( \mu(E) \)). We will find upper bounds of this form in two special cases: the first and simpler case is when the measure of the set \( E \) is close to 1, the second case is when \( E \) is a set with many ‘long’ sections. We say that \( E \) has many ‘long’ sections, if there are sufficiently many \( \omega \)'s for which the \( m \)-measure of \( E_\omega \) is close to 1. It should be mentioned that these bounds use the random nature of the function \( f \), as non-random Fourier series can be norm-concentrated on an arbitrarily small set. Handling other types of sets \( E \) is much more difficult, and constitutes the main part of the proof; we will call such sets ‘spreadable’, for a reason that will soon become clear. The function \( \Delta_t(E) \) gives a quantitative measure for the spreadability of a set \( E \).

2.5.1 The iterative procedure

The proof of Theorem 5 for spreadable sets is iterative in nature. Given such a set \( E \subset Q \), we wish to find a larger set \( E' \), which contains \( E \), and for which we still have a good estimate for \(|f|^2\). We call this procedure spreading.

We continue this procedure until the new set \( E' \) is either of measure close to 1, or contains many long sections. At this point, the procedure is stopped, by the methods that we have previously mentioned. In order to show that this can be done, it is essential to prove that we never ‘get stuck’ during this procedure. In mathematical terms, we show that there exist two positive functions \( \delta(s) \) and \( D(s) \), with \( \delta(s) \downarrow 0 \), \( D(s) \uparrow \infty \) as \( s \downarrow 0 \), such that for every measurable set \( E \) of measure at least \( s \), one can find a set \( E \supset E' \) of measure at least \( s + \delta(s) \) for which

\[
\int_{E} |f|^2 \, d\mu \leq D(s) \int_{E} |f|^2 \, d\mu.
\]
If we have good control on the decay of $\delta(s)$ and the growth of $D(s)$ near 0, then this procedure will give Theorem 5 as the conclusion. As it turns out, for spreadable sets we indeed have good control over these functions.

Remark. During the proof we will show that if a set is not spreadable, then it contains many long sections.

2.5.2 The Spreading Lemma

The heart of the iterative procedure is the Spreading Lemma, which guarantees the existence of the set $\tilde{E}$ above. The basic ‘mechanism’ is the Turán-type inequality (2.3) stated above. For exponential polynomials it gives an estimate of the type (2.5). It is reasonable to assume that if a function $f$ is ‘close’ (say in $L^2$ norm) to an exponential polynomial, then one can use this inequality to get such an estimate for $f$ as well. We will show that a function $f \in L^2_{RF}$ has a good local approximation by (random) exponential polynomials, which is sufficient for our purpose.

For an effective use of the Turán-type inequality we want that, first, the degree of the approximating polynomial (the number of frequencies) is sufficiently small, and second, that the ratio $m(J)/m(E)$ in (2.3) is not too large. The proof shows that $f$ can be approximated by (random) exponential polynomials of fixed degree, which depends only on the measure of the set $E$. We then make use of the fact that spreadable sets contain many ‘good’ sections. Good sections are those sections $E_\omega$ that contain small intervals $J$, where the ratio $m(J)/m(J \cap E_\omega)$ is not too large, while there is still ‘room’ for spreading (notice that the local approximation should be sufficiently accurate on the scale of the interval $J$). Thereby, the set $\tilde{E}$ is made up of a union of such small intervals. The existence of good sections follows from the fact that the quantity $\Delta_t(E)$ is not too small for spreadable sets, for some value of $t$.

2.5.3 Local approximation by exponential polynomials

Exponential polynomials of the form $\sum_{j=1}^n c_j e(\lambda_j x)$, $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$, are the (homogenous) solutions of the linear differential equation (of order $n$) $Dg = 0$, where $D = \prod_{j=1}^n e(\lambda_j x) \frac{d}{dx} e(-\lambda_j x)$.\footnote{This notation means: $\{e(\lambda x) \frac{d}{dx} e(-\lambda x) \} \{g(x)\} = e(\lambda x) \left[ \frac{d}{dx} (e(-\lambda x)g(x)) \right]$.} In order to obtain the approximation of $f$ by exponential polynomials we proceed as follows: We write $f$ as a sum $f = f_0 + f_1 + \cdots + f_m$, where $f_1, \ldots, f_m$ are certain trigonometric polynomials (with many frequencies), and $f_0$ is a (small) error term. Each of these trigonometric polynomials is a non-homogeneous solution to an equation $D_k f_k = g_k$, where the number of frequencies
in $D_k$ is relatively small, and the norm of $g_k$ is small. Since the norm of $g_k$ is small, the homogeneous solution $h_k$ of $D_k h_k = 0$ is a good approximation for $f_k$, and is of small degree.

Our method is similar to the description above, but since $f$ is random we approximate it by random exponential polynomials, with fixed (non-random) frequencies and random coefficients $c_j$. The key for the success of this method is to show the spectrum of $f$ is concentrated on some (finite and ‘small’) set of intervals, depending on the set $E$. This set of intervals will determine the partition of $f$ as mentioned above.

2.5.4 Spectral description of $L^2_{\mathbb{R}^n}$ functions - The $\text{Exp}_{1\text{oc}}$ property

An exponential polynomial has the important property that a finite linear combination of its translates (which we call ‘shifts’) vanishes. Actually, it is readily seen that we only need a linear combination of $m + 1$ different shifts, where $m$ is the degree of the polynomial.

A crucial ingredient of the proof is an analogous property for $L^2(Q)$ functions. To keep the formulation simple, we give here a non-random version of the

**Definition.** [Exp$_{1\text{oc}}$ property] Let $g \in L^2(\mathbb{T})$ and fix $n \in \mathbb{N}$, $\tau > 0$ and $\varkappa > 0$. We say that $g$ has the $\text{Exp}_{1\text{oc}}(n, \tau, \varkappa)$ property (or $g \in \text{Exp}_{1\text{oc}}(n, \tau, \varkappa)$) if for every $t \in (0, \tau)$ there exists complex numbers $a_k = a_k(t), k \in \{0, \ldots, n\}$, with $\sum_{k=0}^{n} |a_k|^2 = 1$, such that $\|\sum_{k=0}^{n} a_k g_{kt}\|_{L^2(\mathbb{T})} < \varkappa$.

Functions $f$ with the $\text{Exp}_{1\text{oc}}(n, \tau, \varkappa)$ property are in a sense a generalization of exponential polynomials. The above definition measures this in a quantitative way: $n$ is the degree of exponential polynomials that are ‘close’ to $f$, $\tau$ is the ‘resolution’ of the local approximation, and $\varkappa$ is the error in the approximation.

In order to explain why this definition implies some special spectral properties, we first note that if $h = \sum a_k g_{kt} \in L^2(\mathbb{T})$, then by Parseval’s theorem we have

$$\int_{\mathbb{T}} |h|^2 \, dx = \int_{\mathbb{T}} \left| \sum a_k g_{kt} \right|^2 \, dx = \sum_{m \in \mathbb{Z}} |\hat{g}(m)|^2 |q_t(e(tm))|^2,$$

where $\hat{g}$ are the Fourier coefficients of $g$, and $q_t$ is a certain polynomial that depends on the numbers $a_k(t)$. By the definition of property $\text{Exp}_{1\text{oc}}(n, \tau, \varkappa)$, the sum on the LHS is smaller than $\varkappa^2$ for $t \in (0, \tau)$. The proof uses averaging over $t$, to show that

---

5 This is somewhat counterintuitive, since the Fourier coefficients of $f$ are arbitrary. It means that for each set $E$ there are some ‘critical’ frequencies specific for this set.
$|\tilde{g}|$ is small outside a certain collection of intervals. The two main observations are that the polynomial $q_t$ is large outside some small exceptional set, and that there is some point where there is a good lower bound estimate for this polynomial inside this exceptional set.

The proofs of the spectral description, the local approximation, and the Spreading Lemma are stated in terms of functions with the $\text{Exp}_{\text{loc}}$ property. Thus, the main goal of the rest of the proof is to show that $L^2_{\text{RF}}$ functions have this property (the parameters will depend on the function and the set $E$).

### 2.5.5 Almost linear dependence for $L^2_{\text{RF}}$ functions

Let $\tilde{E} \subset Q$ be some set of positive measure, and recall the operator $A_{\tilde{E}}$ defined by the equation (2.4). Using the bilinear Khinchine inequality we can get an upper bound for the Hilbert-Schmidt norm of $A_{\tilde{E}}$ of the form

$$\|A_{\tilde{E}}\|_{\text{HS}} \leq B \left( \mu(\tilde{E}) \right),$$

where $B (t) \downarrow 0$ as $t \downarrow 0$ (actually we use an additional parameter $p$, which gives us some flexibility later in the proof). Now, let $V_{\tilde{E}} \subset L^2_{\text{RF}}$ be a subspace of finite dimension, which is the span of the eigenspaces corresponding to the largest eigenvalues of $A_{\tilde{E}}$. If the dimension of $V_{\tilde{E}}$ is large enough, so that

$$\|A_{\tilde{E}} |_{V_{\tilde{E}}^\perp} \|_{\text{HS}} \leq \frac{\mu(\tilde{E})}{2},$$

then for any $g \in L^2_{\text{RF}} \ominus V_{\tilde{E}}$

$$\int_{\tilde{E}} |g|^2 \, d\mu \geq \frac{\mu(\tilde{E})}{2} \|g\|_2^2.$$

Now if $f \in L^2_{\text{RF}}$, then we can find numbers $a_k = a_k(t)$ such that $g = \sum_{k=0}^n a_k f_{kt} \in L^2_{\text{RF}} \ominus V_{\tilde{E}}$. The problem in this case is how to relate the integrals $\int_E |g|^2 \, d\mu$ and $\int_E |f|^2 \, d\mu$. In order to circumvent this problem we introduce the set

$$E' = E'(t) = \bigcap_{k=0}^n (E - kt) \subset E.$$

The fact that $E$ is a spreadable set guarantees that there is some $\tau > 0$ such that $\mu(E') \geq \frac{1}{2} \mu(E)$ for $t \in (0, \tau)$. Here we use that $\Delta_t(E)$ is not too large for these
sets, for $t < n\tau$. Now, using the Cauchy-Schwarz inequality and the invariance of the integral under translations, we get that for $g \in L^2_{RF} \ominus V_{E'}$,

$$
\|g\|^2 \leq \frac{2}{\mu(E')} \int_{E'} \sum_{k=0}^{n} |f_{kt}|^2 \, d\mu \leq D(\mu(E)) \int_{E} |f|^2 \, d\mu,
$$

where $D(t) \uparrow \infty$ as $t \downarrow 0$. In particular, this implies that $f \in \text{Exp}_{\text{loc}}(n, \tau, \kappa)$ with a certain $\kappa$ depending on $\mu(E)$ and $\int_{E} |f|^2 \, d\mu$. For the method to succeed, it is important that we achieve good control over the size of the parameters $n$, $\kappa$, and $\Delta_{n\tau}(E)$.

### 2.6 The proof of Theorem 5

We start with a formal definition of the $\text{Exp}_{\text{loc}}$ property, in a general setting. In Section 3 we prove the spectral description, the local approximation and the Spreading Lemma for functions with the $\text{Exp}_{\text{loc}}$ property. In Section 4 we prove that Rademacher Fourier functions have this property (with effective control of the parameters), and finish the proof by analyzing the iterative procedure. In this section we also analyze the cases of set of large measure as well as sets with ‘long’ sections.

#### 2.6.1 The $\text{Exp}_{\text{loc}}$ property - functions with almost linearly dependent small shifts

Let $\mathcal{H}$ be a Hilbert space. By $L^2(\mathbb{T}, \mathcal{H})$ we denote the Hilbert space of square integrable $\mathcal{H}$-valued functions on $\mathbb{T}$ (in the sense of Bochner). Note that the space $L^2(\mathbb{T}, L^2(\Omega))$ can be identified with $L^2(Q)$. To define the property of functions in $L^2(\mathbb{T}, \mathcal{H})$ having almost linearly dependent small shifts, we introduce the following set of parameters:

- the order $n \in \mathbb{N}$ (a large parameter);
- the localization parameter $\tau > 0$ (a small parameter);
- the error $\kappa > 0$ (a small parameter).

**Definition 6.** [$\text{Exp}_{\text{loc}}$] We say that $g \in L^2(\mathbb{T}, \mathcal{H})$ has the $\text{Exp}_{\text{loc}}(n, \tau, \kappa, \mathcal{H})$ property if for each $t \in (0, \tau)$ there exist complex numbers $a_k = a_k(t)$, $k \in \{0, \ldots, n\}$, with $\sum_{k=0}^{n} |a_k|^2 = 1$, such that

$$
\left\| \sum_{k=0}^{n} a_k g_{kt} \right\|_{L^2(\mathbb{T}, \mathcal{H})} < \kappa.
$$
In the case $\mathcal{H} = \mathbb{C}$, this property was introduced in [Na1, Chapter III]. If $g$ has the $\text{Exp}_{1\text{oc}}(n, \tau, \kappa, \mathcal{H})$ property we will write $g \in \text{Exp}_{1\text{oc}}(n, \tau, \kappa, \mathcal{H})$. ‘In small’ (i.e., on intervals of length comparable with $\tau$), the functions with this property behave similarly to exponential sums with $n$ frequencies and with coefficients in $\mathcal{H}$. On the other hand, since the translations act continuously in $L^2(\mathbb{T}, \mathcal{H})$, for any given $g \in L^2(\mathbb{T}, \mathcal{H})$, $n \in \mathbb{N}$, $\kappa > 0$, one can choose the parameter $\tau > 0$ so small that $g \in \text{Exp}_{1\text{oc}}(n, \tau, \kappa, \mathcal{H})$.

3 Logarithmic Integrability - The Spreading Lemma

In this section, we extend the main results about functions with the $\text{Exp}_{1\text{oc}}$ property (the spectral description, the local approximability by exponential sums, and the spreading lemma) from the scalar case (as proved in [Na1, Chapter III]) to the case considered here.

3.1 Spectral description - The Approximate Spectrum Lemma

The first lemma shows that each function $g \in \text{Exp}_{1\text{oc}}(n, \tau, \kappa, \mathcal{H})$ has an ‘approximate spectrum’ $\Lambda_g$, which consists of $n$ frequencies, so that the Fourier transform of $g$ is small in the $\ell^2$-norm away from these frequencies.

For $m \in \mathbb{Z}$, $\Lambda \subset \mathbb{R}$, let

$$\theta_{\tau}(m) = \min(1, |\tau|m|), \quad \Theta_{\tau, \Lambda}(m) = \prod_{\lambda \in \Lambda} \theta_{\tau}(m - \lambda).$$

**Lemma 7.** Given $g \in \text{Exp}_{1\text{oc}}(n, \tau, \kappa, \mathcal{H})$, there exists a set $\Lambda = \Lambda_g \subset \mathbb{R}$ of $n$ distinct frequencies such that

$$\sum_{m \in \mathbb{Z}} \| \hat{g}(m) \|^2_{\mathcal{H}} \Theta^2_{\tau, \Lambda}(m) \leq (Cn)^4 n^{\kappa^2}.$$

The proof of Lemma 7 with small modifications, follows [Na1, Section 3.1]. We start with the following observation: if $g \in L^2(\mathbb{T}, \mathcal{H})$ and $a_0(t), \ldots, a_n(t)$ are complex numbers, then the $m$-th Fourier coefficient of the function

$$x \mapsto \sum_{k=0}^{n} a_k(t)g_{kt}(x) = \sum_{k=0}^{n} a_k(t)g(x + kt)$$

equals

$$\hat{g}(m) \cdot \sum_{k=0}^{n} a_k(t)e(km) = \hat{g}(m) \cdot q_t(e(tm)),$$
where \( q_t(z) = \sum_{k=0}^{n} a_k(t) z^k \). Slightly perturbing the coefficients \( a_k(t) \), we may assume, without loss of generality, that the coefficients \( a_0(t) \) and \( a_n(t) \) do not vanish for \( 0 < t < \tau \) (so that, for every \( t \) in this range, the polynomial \( q_t \) is exactly of degree \( n \) and does not vanish at the origin), and that the arguments of the roots of \( q_t \) are all distinct.

By Parseval’s theorem,
\[
\int_{\mathbb{T}} \left\| \sum_{k=0}^{n} a_k(t) g_{kt}(x) \right\|_{H}^2 \, dx = \sum_{m \in \mathbb{Z}} \left\| \hat{g}(m) \right\|_{H}^2 \left| q_t(e^{itm}) \right|^2. \tag{3.1}
\]

If \( g \in \text{Exp}_{\text{loc}}(n, \tau, \kappa, \mathcal{H}) \), then we can choose \( a_0, \ldots, a_k \) so that the LHS of (3.1) will be small for each \( t \in (0, \tau) \). On the other hand, whenever the norm of \( \hat{g}(m) \) is large, the RHS of (3.1) can be small only when \( q_t(e^{itm}) \) is small. The proof of Lemma 7 will be based on two facts. The first is that, on average, \( |q_t(e^{itm})| \) is relatively large outside some exceptional set, which can be covered by at most \( n \) intervals of length \( \frac{1}{4n(n+1)\tau} \). The second is that there exists a \( t_0 \) such that \( q_{t_0}(e^{itm}) \) can be effectively bounded from below on this exceptional set.

We start with a lemma on arithmetic progressions.

**Lemma 8.** Given a measurable set \( G \subset \mathbb{R}_+ \), put
\[
V_G = \left\{ t \in \left( \frac{1}{2} \tau, \tau \right) : \exists k \in \mathbb{N} \text{ s.t. } \frac{k}{t} \in G \right\}.
\]

Then \( m(V_G) < \tau^2 m(G) \).

This lemma shows that if \( m(G) < \frac{1}{2\tau} \), then there are significantly many points \( t \in (\frac{1}{2} \tau, \tau) \) such that no point \( k/t, k \in \mathbb{N} \), belongs to \( G \).

**Proof of Lemma 8.** We have
\[
\sum_{k \in \mathbb{N}} \mathbb{1}_G\left( \frac{k}{t} \right) \geq \mathbb{1}_{V_G}(t).
\]

Integrating over \( t \in \left( \frac{1}{2} \tau, \tau \right) \), we get
\[
m(V_G) \leq \int_{\tau/2}^{\tau} \sum_{k \in \mathbb{N}} \mathbb{1}_G\left( \frac{k}{t} \right) \, dt = \sum_{k \in \mathbb{N}} k \int_{k/\tau}^{2k/\tau} \mathbb{1}_G(s) \frac{ds}{s^2} \leq \int_{0}^{\infty} \mathbb{1}_G(s) \left( \sum_{s \tau/2 < k < s \tau} k \right) \frac{ds}{s^2} < \tau^2 \int_{0}^{\infty} \mathbb{1}_G(s) \, ds = \tau^2 m(G),
\]
because $\sum_{s\tau/2 < k < s\tau} k < \tau^2 s^2$. □

The following lemma shows that the Fourier coefficients $\hat{g}(m)$ are small outside $n$ intervals of controlled length. Put

$$
\delta = \frac{1}{8n(n+1)}.
$$

This choice of $\delta$ will stay fixed till the end of the proof of Lemma 7.

**Lemma 9.** There exist $n$ intervals $I_1, \ldots, I_n$ of length $\frac{2\delta}{\tau}$ such that

$$
\sum_{m \in \mathbb{Z} \setminus \bigcup I_j} \|\hat{g}(m)\|_\mathcal{H}^2 < \left(\frac{C}{\delta}\right)^2 n^2.
$$

**Proof of Lemma 9.** By the continuity of the shift in $L^2(T, \mathcal{H})$, we can assume that the coefficients $a_k(t)$ are piecewise constant functions of $t$, and hence measurable. Then we can integrate Parseval’s formula (3.1) over the interval $(0, \tau)$. Recalling that the LHS of (3.1) is less than $\kappa^2$, we get

$$
\sum_{m \in \mathbb{Z}} \|\hat{g}(m)\|_\mathcal{H}^2 \rho^2(m) < \kappa^2,
$$

where

$$
\rho^2(m) = \frac{1}{\tau} \int_0^\tau |q_t(e(tm))|^2 dt.
$$

Introduce the set

$$
S = \left\{ m \in \mathbb{Z} : \rho^2(m) < \frac{1}{4(n+1)} \left(\frac{\delta}{A}\right)^2 n \right\}.
$$

Here and elsewhere in this section, $A$ is the positive numerical constant from the RHS of the Turán-type Lemma 2.4.1. Then Lemma 9 will follow from the following claim:

$S$ cannot contain $n+1$ integers $m_1 < \cdots < m_{n+1}$ such that

$$
m_{j+1} - m_j > \frac{2\delta}{\tau}, \quad \forall j \in \{1, \ldots, n\}. \quad (3.2)
$$

Indeed, this condition ensures that the set $S$ can be covered by at most $n$ intervals $I_1, \ldots, I_n$ of length $2\delta/\tau$ and

$$
\rho^2(m) \geq \frac{1}{4(n+1)} \left(\frac{\delta}{A}\right)^2 n, \quad m \in \mathbb{Z} \setminus \bigcup_j I_j,
$$
whence
\[ \sum_{m \in \mathbb{Z} \setminus I_j} \| \hat{g}(m) \|_2^2 \leq 4(n + 1) \left( \frac{A}{\delta} \right)^{2n} \tau^2 < \left( \frac{C}{\delta} \right)^{2n} \tau^2 \]
with some numerical constant \( C \). Thus, we need to prove the claim (3.2).

Suppose that (3.2) does not hold, i.e., there are \( n + 1 \) integers \( m_1 < \cdots < m_{n+1} \) with \( m_{j+1} - m_j > 2\delta/\tau \) that belong to the set \( S \). Then
\[
\int_{\tau/2}^{\tau} \sum_{j=1}^{n+1} |q_t(e(tm_j))|^2 \, dt < \frac{\tau}{4} \left( \frac{\delta}{A} \right)^{2n}.
\] (3.3)
We call the value \( t \in \left( \frac{1}{2} \tau, \tau \right) \) bad if
\[
\sum_{j=1}^{n+1} |q_t(e(tm_j))|^2 < \left( \frac{\delta}{A} \right)^{2n}.
\]
Otherwise, the value \( t \) is called good. By (3.3), the measure of good \( t \)'s is less than \( \tau/4 \). In the rest of the proof we will show that the measure of bad \( t \)'s is also less than \( \tau/4 \), and this will lead us to a contradiction, which will prove Lemma 9.

We will use the following

Claim 10. Let \( q(z) = \sum_{k=0}^{n} a_k z^k \) with \( \sum_{k=0}^{n} |a_k|^2 = 1 \). Given \( \Delta \in (0, 1) \), let
\[
U = \left\{ s \in \mathbb{T} : |q(e(s))| < \left( \frac{\Delta}{A} \right)^n \right\}.
\]
Then the set \( U \) is a union of at most \( n \) intervals of length at most \( \Delta \) each.

Proof. \( U \) is an open subset of \( \mathbb{T} \) which consists of open intervals (since \( \Delta < 1 \) and \( A \geq 1 \), we have that \( U \neq \mathbb{T} \)). The boundary points of these intervals satisfy the equation \( |q(e(s))|^2 = \left( \frac{\Delta}{A} \right)^{2n} \), which can be rewritten as
\[
\left( \sum_{k=0}^{n} a_k z^k \right) \left( \sum_{k=0}^{n} \overline{a_k} z^{-k} \right) = \left( \frac{\Delta}{A} \right)^{2n}, \quad z = e(s).
\]
The LHS of this equation is a rational function of degree at most \( 2n \), and therefore the number of solutions is at most \( 2n \). Hence \( U \) consists of \( l \leq n \) intervals \( J_1, \ldots, J_l \).
Next, note that since the sum of squares of the absolute values of the coefficients of \( q \) equals 1, we have \( \max_{s \in T} |q(e(s))| \geq 1 \). Then, applying Lemma 2.4.1 to the exponential polynomial \( s \mapsto q(e(s)) \), we get

\[
1 \leq \sup_{s \in T} |q(e(s))| \leq \left( \frac{A}{m(J_i)} \right)^n \sup_{s \in J_i} |q(e(s))| \leq \left( \frac{\Delta}{m(J_i)} \right)^n.
\]

Hence, \( m(J_i) \leq \Delta \), proving the claim. \( \square \)

Note that in the proof of this claim we did not use the full strength of Turán’s lemma. For instance, we could have used the much simpler Remez’ inequality.

Now for \( t \in \left( \frac{1}{2} \tau, \tau \right) \) consider the set

\[
S_t = \left\{ m \in \mathbb{Z} : |q_t(e(tm))| < \left( \frac{\delta}{A} \right)^n \right\}.
\]

By the previous claim (applied with \( \Delta = \delta \)), there are points \( \xi_1, \ldots, \xi_n \in \mathbb{R} \) (centers of the intervals \( J_i \)) such that, for each \( m \in S_t \), there exist \( i \in \{1, \ldots, n\} \) and \( l \in \mathbb{Z} \) such that

\[
|tm - l - \xi_i| < \frac{1}{2} \delta. \tag{3.4}
\]

Suppose that the value \( t \) is bad. Then the \( n + 1 \) integers \( m_1, \ldots, m_{n+1} \) belong to the set \( S_t \), and by the Dirichlet box principle, there are two of these integers, say \( m_{j'} \) and \( m_{j''} \) with \( j' < j'' \), which satisfy (3.4) with the same value \( i \). Then for this pair \( |t(m_{j''} - m_{j'}) - k| < \delta \), with some non-negative integer \( k \). Thus,

\[
\left| \frac{k}{t} - (m_{j''} - m_{j'}) \right| < \frac{\delta}{t} < \frac{2\delta}{\tau}.
\]

Note that since \( m_{j''} - m_{j'} > \frac{2\delta}{\tau} \), the integer \( k \) must be positive. We conclude that the set of bad values \( t \) is contained in the set \( V_G \), where \( G \) is the union of \( \frac{1}{2}n(n+1) \) intervals of length \( \frac{4\delta}{\tau} \) centered at all possible differences \( m_{j''} - m_{j'} \) with \( j'' > j' \).

The measure of the set \( G \) is \( \frac{n(n+1)}{2} \cdot \frac{4\delta}{\tau} \), which, due to the choice of \( \delta \), equals \( \frac{1}{4} \tau \). By Lemma \( \square \) \( m(V_G) < \tau^2 m(G) \leq \frac{1}{4} \tau \). Thus, the measure of the set of bad \( t \)'s is also less than \( \frac{1}{4} \tau \), which completes the proof of Lemma 9. \( \square \)

**Proof of Lemma** We need to find a set \( \Lambda = \Lambda_g \subset \mathbb{R} \) of \( n \) frequencies such that

\[
\sum_{m \in \mathbb{Z}} \| \tilde{g}(m) \|_{H}^2 \Theta^2_{\tau, \Lambda}(m) \leq \left( Cn \right)^{4n} \delta^2.
\]
where
\[ \Theta_{\tau,\Lambda}(m) = \prod_{\lambda \in \Lambda} \theta_{\tau}(m - \lambda), \quad \theta_{\tau}(m) = \min(1, \tau|m|). \]

By Lemma 9, there exists a collection of \( n \) intervals \( \{I_j\} \), each of length \( \frac{2\delta}{\tau} \), such that
\[
\sum_{m \in \mathbb{Z} \setminus \bigcup_j I_j} \| \hat{g}(m) \|_H^2 \Theta_{\tau,\Lambda}^2(m) \leq \sum_{m \in \mathbb{Z} \setminus \bigcup_j I_j} \| \hat{g}(m) \|_H^2 \leq (Cn)^n \gamma^2.
\]

Therefore, it remains to estimate the sum
\[
\sum_{m \in \bigcup_j I_j} \| \hat{g}(m) \|_H^2 \Theta_{\tau,\Lambda}^2(m).
\]

By Parseval’s identity (3.1), for every \( t \in (0, \tau) \),
\[
\sum_{m \in \bigcup_j I_j} \| \hat{g}(m) \|_H^2 |q_t(e(tm))|^2 < \gamma^2.
\]

Hence, it suffices to show that there exist a value \( t_0 \in (0, \tau) \) and a set \( \Lambda \) of \( n \) real numbers such that \( |q_{t_0}(e(t_0m))| \geq \delta^n \Theta_{\tau,\Lambda}(m) \) for every \( m \in \bigcup_j I_j \).

First, we bound the absolute value of the polynomial \( q_t \) from below by the absolute value of another polynomial \( p \) whose zeros are obtained from the zeros of \( q_t \) by radial projection to the unit circle.

Claim 11. Let \( z_j \neq 0 \) for \( 1 \leq j \leq n \), and let \( g(z) = c \cdot \prod_{j=1}^n (z - z_j) \) be a polynomial of degree \( n \) such that \( \sup_{|z|=1} |g(z)| \geq 1 \). Let \( h(z) = \prod_{j=1}^n (z - \zeta_j) \), where \( \zeta_j = z_j/|z_j| \).

Then, for every \( z \in \mathbb{T} \),
\[ |h(z)| \leq 2^n |g(z)|. \]

Proof. The ratio \( \left| \frac{z - \zeta_j}{z - z_j} \right| \) attains its maximum on \( \{|z|=1\} \) at the point \( z = -\zeta_j \), where it is equal to \( \frac{1}{1+|z_j|} \). Therefore,
\[ \left| \frac{h(z)}{g(z)} \right| \leq \frac{1}{|c|} \prod_{j=1}^n \frac{2}{1+|z_j|}. \]

By our assumption, there is some \( z', |z'| = 1 \), such that \( |g(z')| \geq 1 \). Hence,
\[ 1 \leq |c| \prod_{j=1}^n |z' + z_j| \leq |c| \prod_{j=1}^n (1 + |z_j|). \]
Overall, we have
\[
|h(z)| \leq 2^n |g(z)| \cdot \frac{1}{|c|} \cdot \prod_{j=1}^{n} \frac{1}{1 + |z_j|} \leq 2^n |g(z)| ,
\]
proving the claim.

Recall that \( \sup_{|z|=1} |q_t(z)| \geq 1. \) Hence, applying Claim [11] we conclude that \( |q_t(z)| \geq 2^{-n} |p_t(z)| \) for \( |z| = 1 \), where \( p_t \) is a monic polynomial of degree \( n \) with all its zeros on the unit circle.

To choose \( t_0 \), we consider \( n \) intervals \( I_j \) of length \( 4\delta \tau^{-1} \) with the same centers as the intervals \( I_j \) of Lemma [9] and put \( \bar{S} = \bigcup_j I_j \). Let \( G = S - \bar{S} \) be the difference set, with \( m(G) \leq 8\delta \tau^{-1} \cdot n^2 \). We call the value \( t \in \left( \frac{1}{2} \tau, \tau \right) \) bad if there exists an integer \( k \neq 0 \) such that \( k/t \in \bar{G} \). Since the set \( \bar{G} \) is symmetric with respect to \( 0 \), we can estimate the measure of bad \( t \)'s by applying Lemma [8] to the set \( G \cap \mathbb{R}_+ \). Then the measure of bad values of \( t \) is less than \( \tau^2 \cdot \frac{1}{2} m(\bar{G}) \leq 4\delta \tau \cdot n^2 < \frac{1}{2} \tau \), since \( \delta \cdot 8n^2 < 1 \). Therefore, there exists at least one good value \( t_0 \in \left( \frac{1}{2} \tau, \tau \right) \) for which every arithmetic progression with difference \( t_0^{-1} \) has at most one point in \( \bar{S} \). We fix this value \( t_0 \) till the end of the proof.

To simplify notation, we put \( p = p_{t_0} \). The zero set of the function \( x \mapsto p(e(t_0 x)) \) consists of \( n \) arithmetic progressions with difference \( t_0^{-1} \). By the choice of \( t_0 \), at most \( n \) zeros of this function belong to the set \( \bar{S} \). We denote these zeros by \( \lambda_1, \ldots, \lambda_l, \) \( l \leq n \). If \( l < n \), we choose \( n - l \) zeros \( \lambda_{l+1}, \ldots, \lambda_n \) in \( \mathbb{R} \setminus \bar{S} \) so that \( \{e(t_0 \lambda_j)\}_{1 \leq j \leq n} \) is a complete set of zeros of the algebraic polynomial \( p \); we recall that these zeros are all distinct.

It remains to define a set \( \Lambda \) of \( n \) numbers, and to estimate from below \( |p(e(t_0 m))| \) when \( m \in \bigcup_j I_j \). Denote by \( d_j(m) \) the distance from the integer \( m \) to the nearest point in the arithmetic progression \( \{\lambda_j + kt_0^{-1}\}_{k \in \mathbb{Z}} \). We have
\[
|p(e(t_0 m))| = 2^n \prod_{j=1}^{n} |\sin(\pi t_0 (m - \lambda_j))| \geq 2^n \prod_{j=1}^{n} (2t_0 d_j(m)) \geq 2^n \tau^n \prod_{j=1}^{n} d_j(m) .
\]
We put \( \Lambda = \{\lambda_j\}_{1 \leq j \leq n} \). Recall that here \( m \in \bigcup_j I_j \), \( \bar{S} = \bigcup_j I_j \), and that the arithmetic progression \( \{\lambda_j + kt_0^{-1}\}_{k \in \mathbb{Z}} \) either misses the set \( \bar{S} \), or has at most one element in \( \bar{S} \). In the first case, we get \( d_j(m) \geq \delta \tau^{-1} \), while in the second case,
\[ d_j(m) \geq \min \left\{ \frac{\delta}{\tau}, |m - \lambda_j| \right\}. \] Therefore, in both cases,

\[ d_j (m) \geq \min \left\{ \frac{\delta}{\tau}, |m - \lambda_j| \right\} \geq \frac{\delta}{\tau} \min \left\{ 1, \tau |m - \lambda_j| \right\} = \frac{\delta}{\tau} \cdot \theta_{\tau, \Delta}(m - \lambda_j). \]

Tying the ends together, we get

\[ \left| q_{t_0}(e_{\tau_0}m) \right| \geq 2^{-n} \left| p(e_{\tau_0}m) \right| \geq 2^{-n} \cdot 2^n \tau^n \prod_{j=1}^{n} d_j(m) \]

\[ \geq \tau^n \cdot \left( \frac{\delta}{\tau} \right)^n \Theta_{\tau, \Delta}(m) = \delta^n \Theta_{\tau, \Delta}(m). \]

This completes the proof of Lemma 7. \(\square\)

### 3.2 Local approximation by exponential polynomials with \(n\) terms

Henceforth we will assume that \( \mathcal{H} = L^2(\Omega) \). Then \( \text{Exp}_{1\text{oc}}(n, \tau, \mathcal{X}, L^2(\Omega)) \subset L^2(Q) \).

For a finite set \( \Lambda \subset \mathbb{R} \), denote by \( \text{Exp}(\Lambda, \Omega) \) the linear space of exponential polynomials with frequencies in \( \Lambda \) and with coefficients depending on \( \omega \). The next lemma shows that, for a.e. \( \omega \in \Omega \), the function \( \theta \mapsto g(\omega, \theta) \), \( g \in \text{Exp}_{1\text{oc}}(n, \tau, \mathcal{X}, L^2(\Omega)) \), can be well approximated by exponential polynomials from \( \text{Exp}(\Lambda, \Omega) \), on intervals \( J \subset [0,1) \) of length comparable with \( \tau \).

Suppose that \( M > 1 \) satisfies

\[ \ell = \frac{1}{M \tau} \in \mathbb{N}, \]

and partition \( \mathbb{T} \) into \( l \) intervals of length \( M \tau \):

\[ \mathbb{T} = \bigcup_{k=0}^{\ell-1} \left[ \frac{k}{\ell}, \frac{k + 1}{\ell} \right]. \]

**Lemma 12.** Let \( M \) be as above and let \( g \in \text{Exp}_{1\text{oc}}(n, \tau, \mathcal{X}, L^2(\Omega)) \). There exists a non-negative function \( \Phi \in L^2(Q) \) with

\[ \| \Phi \|_2 \leq (Cn)^{2n} \mathcal{X}, \]

and with the following property: for every interval \( J \subset \mathbb{T} \) in the above partition there exists an exponential polynomial \( p^J \in \text{Exp}(\Lambda_g, \Omega) \) such that, for a.e. \( \omega \in \Omega \) and a.e. \( \theta \in J \),

\[ |g(\omega, \theta) - p^J(\omega, \theta)| \leq M^n \Phi(\omega, \theta). \]
The proof of Lemma 12 is very close to the proof of the corresponding result in [Na1, Section 3.2]. We start with a lemma on solutions of ordinary differential equations (cf. Lemma 3.2 in [Na1]).

**Lemma 13.** Let

\[ D = \prod_{j=1}^{n} e(\lambda_j x) \frac{d}{dx} e(-\lambda_j x) \quad \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \quad \lambda_i \neq \lambda_j \text{ for } i \neq j, \]

be a differential operator of order \( n \geq 1 \), and let \( J \subset [0, 1] \) be an interval. Suppose that \( f \in L^2(\Omega \times J) \) and, for a.e. \( \omega \in \Omega \), \( x \mapsto f(\omega, x) \) is a \( C^n(J) \)-function satisfying the differential equation \( Df = h \) with \( h \in L^2(\Omega \times J) \). Then there exists an exponential polynomial \( p \) with spectrum \( \lambda_1, \ldots, \lambda_n \), such that, for a.e. \( \omega \in \Omega \),

\[
\sup_{x \in J} |f(\omega, x) - p(\omega, x)| \leq m(J)^n \frac{1}{m(J)} \int_J |h(\omega, x)| \, dx.
\]

**Proof.** Let \( \phi \) be a particular solution of the equation \( D\phi = h \) constructed by repeated integration:

\[
\phi = \left( \prod_{j=1}^{n} e(\lambda_j x) \mathcal{J} e(-\lambda_j x) \right) h
\]

where \( \mathcal{J} \) is the integral operator

\[
(\mathcal{J} \psi)(\omega, x) = \int_a^x \psi(\omega, t) \, dt
\]

and \( a \) is the left end-point of the interval \( J \). Then, for a.e. \( \omega \)

\[
|\phi(\omega, x)| \leq m(J)^n \frac{1}{m(J)} \int_J |h(\omega, x)| \, dx.
\]

The function \( f - \phi \) satisfies the homogeneous equation \( D(f - \phi) = 0 \). Hence, \( p = f - \phi \) is an exponential polynomial with coefficients depending on \( \omega \):

\[
p(\omega, x) = \sum_{j=1}^{n} c_j(\omega) e(\lambda_j x).
\]

\[ \square \]
Now we turn to the proof of Lemma 12. Fix \( g \in \text{Exp}_{\text{loc}}(n, \tau, \mathcal{K}, L^2(\Omega)) \). By Lemma 7, the function \( g \) has an ‘approximate spectrum’ \( \Lambda = \Lambda_g = \{\lambda_j\}_{1 \leq j \leq n} \) so that

\[
\sum_{m \in \mathbb{Z}} \| \hat{g}(m) \|_{L^2(\Omega)}^2 \Theta^2_{\tau, \Lambda}(m) \leq (Cn)^{4n} \tau^2 ,
\]

with

\[
\Theta_{\tau, \Lambda}(m) = \prod_{\lambda \in \Lambda} \theta(\tau - \lambda) , \quad \theta(\tau) = \min(1, \tau |m|) .
\]

We fix \( M > 1 \) so that \( 1/(M \tau) \) is a positive integer, and partition \( \mathbb{T} \) into intervals \( J \) of length \( M \tau \).

Put

\[
I_k = \left( \lambda_k - \frac{1}{\tau}, \lambda_k + \frac{1}{\tau} \right) , \quad \bar{I}_k = \left( \lambda_k - \frac{2}{\tau}, \lambda_k + \frac{2}{\tau} \right) , \quad E_0 = \mathbb{R} \setminus \bigcup_{k=1}^{n} I_k , \quad E_k = I_k \setminus \bigcup_{j=1}^{k-1} I_j .
\]

The sets \( E_k , 0 \leq k \leq n \), form a partition of the real line. Accordingly, we decompose \( g \) into the sum \( g = \sum_{k=0}^{n} g_k \), where \( g_k \) is the projection of \( g \) onto the closed subspace of \( L^2(Q) \) that consists of functions with spectrum contained in \( E_k \). For each \( k = 0, \ldots, n \), we have

\[
\sum_{m \in \mathbb{Z}} \| \hat{g}_k(m) \|_{L^2(\Omega)}^2 \Theta^2_{\tau, \Lambda}(m) < (Cn)^{4n} \tau^2 \overset{\text{def}}{=} \tilde{\tau}^2 .
\]

Since, \( \Theta^2_{\tau, \Lambda}(m) \equiv 1 \) for \( m \in E_0 \), we get \( \|g_0\|_{L^2(Q)} \leq \tilde{\tau} \).

Now let \( 1 \leq k \leq n \). Let \( n_k \) denote the number of points \( \lambda_j \) lying in \( \bar{I}_k \). We define a differential operator \( D_k \) of order \( n_k \) by

\[
D_k \overset{\text{def}}{=} \prod_{\lambda_j \in \bar{I}_k} e(\lambda_j x) \frac{d}{dx} e(-\lambda_j x) .
\]

The function \( g_k(x) \) is a trigonometric polynomial with coefficients depending on \( \omega \), hence, for a.e. \( \omega \), it is an infinitely differentiable function of \( x \). We set \( h_k \overset{\text{def}}{=} D_k g_k \). Note that this is a trigonometric polynomial with the same frequencies as \( g_k \):

\[
\hat{h}_k(\omega, m) = (2\pi i)^{n_k} \hat{g}_k(\omega, m) \prod_{\lambda_j \in \bar{I}_k} (m - \lambda_j) .
\]

Consequently,

\[
\| \hat{h}_k(\omega, m) \| = (2\pi)^{n_k} \| \hat{g}_k(\omega, m) \| \prod_{\lambda_j \in \bar{I}_k} |m - \lambda_j| .
\]
In the product on the RHS, \( m \in E_k \subset I_k \) and \( \lambda_j \in \tilde{I}_k \). Recalling the definition of the function \( \theta_\tau \), we see that
\[
|m - \lambda_j| \leq \frac{3}{\tau} \theta_\tau(m - \lambda_j) \quad \text{for } m \in I_k, \lambda_j \in \tilde{I}_k .
\]
Therefore,
\[
|\hat{h}_k(\omega, m)| \leq \left( \frac{6\pi}{\tau} \right)^n |\hat{g}_k(\omega, m)| \prod_{\lambda_j \in \tilde{I}_k} \theta_\tau(m - \lambda_j) .
\]
Note that for \( m \in E_k \) and for \( \lambda_j \in \mathbb{Z} \setminus \tilde{I}_k \), we have \( \theta_\tau(m - \lambda_j) = 1 \). Thus,
\[
|\hat{h}_k(\omega, m)| \leq \left( \frac{6\pi}{\tau} \right)^n |\hat{g}_k(\omega, m)| \Theta_{\tau, \Lambda}(m) , \quad \omega \in \Omega ,
\]
whence, recalling estimate \((3.5)\), we obtain
\[
\|h_k\|_{L^2(Q)} \leq \left( \frac{6\pi}{\tau} \right)^n \hat{h} .
\]

Applying Lemma 13 to an interval \( J \) of length \( M\tau \), we obtain an exponential polynomial \( p^J_k \) with spectrum consisting of frequencies \( \lambda_j \in \tilde{I}_k \) and with coefficients depending on \( \omega \), such that, for every \( x \in J \) and almost every \( \omega \in \Omega \),
\[
|g_k(\omega, x) - p^J_k(\omega, x)| \leq (M\tau)^n \cdot \frac{1}{M\tau} \int_J |h_k(\omega, t)| \, dt .
\]

We denote by
\[
\mathcal{M}f(\omega, x) = \sup_{L: x \in L} \frac{1}{m(L)} \int_L |f(\omega, t)| \, dt
\]
the Hardy-Littlewood maximal function. The supremum is taken over all intervals \( L \subset [0, 1] \) containing \( x \), but it is easy to see that it is enough to restrict ourselves to the intervals with rational endpoints, which allows us to rewrite \( \mathcal{M}f \) as \( \sup \{ F_{\alpha, \beta} : \alpha, \beta \in \mathbb{Q} \} \), where
\[
F_{\alpha, \beta}(\omega, x) = \mathbb{I}_{[\alpha, \beta]}(x)G_{\alpha, \beta}(\omega) \quad \text{and} \quad G_{\alpha, \beta}(\omega) = \frac{1}{\beta - \alpha} \int_\alpha^\beta |f(t, \omega)| \, dt .
\]

By the Fubini theorem, \( G_{\alpha, \beta} \) are measurable functions on \( \Omega \), so \( F_{\alpha, \beta} \) are measurable functions on \( Q \), and consequently, \( \mathcal{M} \) is measurable on \( Q \) as well.

Let \( \tilde{h}_k = \tau^{n_k} h_k \). Then
\[
|g_k(\omega, x) - p^J_k(\omega, x)| \leq M^{n_k} \cdot \mathcal{M}\tilde{h}_k(\omega, x) \leq M^{n_k} \cdot \mathcal{M}\tilde{h}_k(\omega, x) .
\]
Using the classical estimate for the $L^2$-norm of the maximal function, we get, for a.e. \( \omega \),
\[
\int_{\omega} \left[ \mathcal{M} \tilde{h}_k(\omega, x) \right]^2 \, dx \leq C \int_{\omega} \left| \tilde{h}_k(\omega, x) \right|^2 \, dx.
\]
Recalling that \( \| \tilde{h}_k \|_{L^2(\Omega)} < C^{m_k} \tilde{\kappa} \), we obtain
\[
\| \mathcal{M} \tilde{h}_k \|^2_{L^2(\Omega)} = \int_{\Omega \times T} \left[ \mathcal{M} \tilde{h}_k(\omega, x) \right]^2 \, dx \, d\mathcal{P}(\omega) \leq C \| \tilde{h}_k \|^2_{L^2(\Omega)} \leq C^{2m_k} \tilde{\kappa}^2.
\]
We now set \( p^J \defeq \sum_{k=1}^n p_k^J \). Notice that all the frequencies of the polynomial \( p^J \) belong to the set \( \Lambda_g \). Then, for every \( x \in J \),
\[
|g(\omega, x) - p^J(\omega, x)| \leq |g_0(\omega, x)| + \sum_{k=1}^n \left| g_k(\omega, x) - p_k^J(\omega, x) \right|
\leq |g_0(\omega, x)| + M^n \sum_{k=1}^n \mathcal{M} \tilde{h}_k(\omega, \theta)
\leq M^n \left( |g_0(\omega, x)| + \sum_{k=1}^n \mathcal{M} \tilde{h}_k(\omega, x) \right) \defeq M^n \Phi(\omega, x).
\]
It remains to bound the norm of the ‘error function’ \( \Phi \):
\[
\| \Phi \|^2_{L^2(\Omega)} \leq \| g_0 \|^2_{L^2(\Omega)} + \sum_{k=1}^n \| \mathcal{M} \tilde{h}_k \|^2_{L^2(\Omega)} \leq \tilde{\kappa} + \sum_{k=1}^n C^{m_k} \tilde{\kappa} \leq C^{m} \tilde{\kappa} \leq (Cn)^{2n} \tilde{\kappa}.
\]
This proves the desired result.

**3.3 The Spreading Lemma**

The next lemma is the crux of the proof of Theorem 3. Given a set \( E \subset Q \) of positive measure, we put \( \Delta_t(E) = \mu((E + t) \setminus E) \).

**Lemma 14.** Suppose \( g \in \text{Exp}_{loc}(n, \tau, \kappa, L^2(\Omega)) \) and \( E \subset Q \) is a set of positive measure. There exists a set \( \bar{E} \supset E \) of measure \( \mu(\bar{E}) \geq \mu(E) + \frac{1}{2} \Delta_{\tau r}(E) \) such that, for each \( b \in L^2(\Omega) \),
\[
\int_{\bar{E}} |g - b|^2 \, d\mu \leq \left( \frac{C n^3}{\Delta_{\tau r}(E)} \right)^{2n+1} \left( \int_{E} |g - b|^2 \, d\mu + \kappa^2 \right).
\]
This lemma follows from Lemma 12 combined with the Turán-type estimate (2.3). We now turn to the proofs of the lemmas. Till the end of this section, we fix the function \( g \in \text{Exp}_{\text{loc}}(n, \tau, \kappa, L^2(\Omega)) \), the set \( E \subset Q \) of positive measure, and the ‘random constant’ \( b \in L^2(\Omega) \).

We will use two parameters, \( M > 1 \), with \( \frac{1}{M \tau} \in \mathbb{N} \), and \( \gamma \in (0, 1) \); their specific values will be chosen later in the proof.

**Definition.** Let \( J \) be an interval of length \( M \tau \) in the partition of \( \mathbb{T} \). The interval \( J \) is called \( \omega \)-white if \( m(J \cap E_\omega) \geq \gamma m(J) \); otherwise, it is called \( \omega \)-black.

Given \( \omega \), the union of all \( \omega \)-white intervals will be denoted by \( W_\omega \). By \( W \subset Q \) we denote the union of all sets \( W_\omega \). Similarly, we denote by \( B_\omega \) the union of all \( \omega \)-black intervals and by \( B \subset Q \) the union of all sets \( B_\omega \). Since we can write the set \( W \) as

\[
\bigcup_J \left\{ \omega : m(J \cap E_\omega) \geq \gamma m(J) \right\} \times J
\]

and the function \( \omega \mapsto m(J \cap E_\omega) \) is measurable on \( \Omega \) for every interval \( J \) in the partition, we see that \( W \) and \( B = Q \setminus W \) are measurable subsets of \( Q \).

Let \( \Phi \) be the error function given by the Local Approximation Lemma. The next lemma enables us to extend our estimates for \( g - b \) from the set \( E \) to the set \( W \).

**Lemma 15.** We have

\[
\int_W |g - b|^2 \, d\mu \leq \left( \frac{C}{\gamma} \right)^{2n+1} \left[ \int_{W \cap E} |g - b|^2 \, d\mu + M^{2n+1} \int_W \Phi^2 \, d\mu \right].
\]

**Proof.** Let \( J \) be one of the \( \omega \)-white intervals of length \( M \tau \). By Lemma 12 for almost every \( \omega \in \Omega \) and every \( \theta \in J \), we have

\[
\left| (g(\omega, \theta) - b(\omega)) - (p^J(\omega, \theta) - b(\omega)) \right| = \left| g(\omega, \theta) - p^J(\omega, \theta) \right| \leq M^n \Phi(\omega, \theta),
\]

where \( p^J \) is an exponential polynomial with \( n \) frequencies and coefficients depending on \( \omega \). Therefore,

\[
\int_J |g - b|^2 \, d\theta \leq 2 \left( \int_J |p^J - b|^2 \, d\theta + M^{2n} \int_J \Phi^2 \, d\theta \right).
\]

(3.6)
Applying the $L^2$-version of the Turán-type lemma to the exponential polynomial $p^J - b$, which has at most $n+1$ frequencies, we get
\[
\int_J |p^J - b|^2 \, d\theta \leq \left( \frac{C}{\gamma} \right)^{2n+1} \int_{J \cap E_\omega} |p^J - b|^2 \, d\theta
\]
\[
\leq \left( \frac{C}{\gamma} \right)^{2n+1} \int_{J \cap E_\omega} |p^J - b|^2 \, d\theta.
\]
Plugging this into (3.6), we find that
\[
\int_J |g - b|^2 \, d\theta \leq \left( \frac{C}{\gamma} \right)^{2n+1} \int_{J \cap E_\omega} |p^J - b|^2 \, d\theta + 2M^2n \int_J \Phi^2 \, d\theta.
\]
Summing these estimates over all $\omega$-white intervals $J$, and using that
\[
|p^J - b| \leq |g - b| + |g - p_J| \leq |g - b| + M^n \Phi,
\]
we get
\[
\int_{W_\omega} |g - b|^2 \, d\theta \leq \left( \frac{C}{\gamma} \right)^{2n+1} \left[ \int_{W_\omega \cap E_\omega} |g - b|^2 \, d\theta + M^2n \int_{W_\omega} \Phi^2 \, d\theta \right]
\]
Integrating over $\omega$, we get the result.

The effectiveness of this lemma depends on the size of the set $W \cap E^c$. The following lemma is very similar to Lemma 3.4 from [Na1]. For the reader’s convenience, we reproduce its proof. Recall that $\Delta_{n\tau}(E) = \mu((E + n\tau) \setminus E)$.

**Lemma 16.** For $\gamma < \frac{1}{2}$,
\[
\mu(W \cap E^c) \geq \Delta_{n\tau}(E) - \left( \gamma + \frac{n}{M} \right).
\]

**Proof.** We have
\[
m\left((E_\omega + n\tau) \setminus E_\omega \right) = m\left((E_\omega + n\tau) \cap E^c_\omega \right)
\]
\[
= m\left((E_\omega + n\tau) \cap E^c_\omega \cap W_\omega \right) + m\left((E_\omega + n\tau) \cap E^c_\omega \cap B_\omega \right)
\]
\[
\leq m\left(W_\omega \cap E^c_\omega \right) + m\left((E_\omega + n\tau) \cap E^c_\omega \cap B_\omega \right).
\]
We need to estimate the second term on the RHS. If the interval $J$ is $\omega$-black, then
\[
m(J \cap E_\omega^c \cap (E_\omega + n\tau)) \leq m(J \cap (E_\omega + n\tau)) \leq m(J \setminus (J + n\tau)) + m((E_\omega + n\tau) \cap (J + n\tau)) \leq n\tau + m(E_\omega \cap J) < n\tau + \gamma m(J) = \left(\frac{n\tau}{m(J)} + \gamma\right)m(J).
\]

Summing this inequality over all $\omega$-black intervals $J$, and recalling that $m(J) = M\tau$, we obtain
\[
m((E_\omega + n\tau) \cap E_\omega^c \cap B_\omega) \leq \left(\frac{n\tau}{M\tau} + \gamma\right) \cdot m(B_\omega) \leq \frac{n}{M} + \gamma.
\]
Integrating over $\Omega$ we get the required result.

**Proof of Lemma 14**: We write $\Delta = \Delta_{n\tau}(E)$ and put
\[M_1 = \frac{8n}{\Delta}.
\]
We consider two cases, according to whether $M_1\tau \leq 1$ or not.

In the first case, we choose $M \in [M_1, 2M_1]$ so that $1/(M\tau)$ is an integer. Notice that $M > 1$. We set $\gamma = \frac{1}{8}\Delta < \frac{1}{2}$ and let $\tilde{E} = E \cup (W \cap E^c) = E \cup W$, where $W$ is the union of the corresponding white intervals. By Lemma 16,
\[
\mu(W \cap E^c) \geq \Delta - \left(\gamma + \frac{n}{M}\right) \geq \Delta - \left(\frac{\Delta}{8} + \frac{\Delta}{8}\right) > \frac{\Delta}{2}.
\]
Furthermore, using Lemma 15, we get
\[
\int_W |g - b|^2 \, d\mu \leq \left(\frac{C}{\gamma}\right)^{2n+1} \left[\int_{E \cap W} |g - b|^2 \, d\mu + M^{2n} \int_W \Phi^2 \, d\mu\right].
\]
Inserting here the values of the parameters $\gamma$ and $M$ and taking into account the bound on the norm of $\Phi$, we find that the RHS is
\[
\leq \left(\frac{C}{\Delta}\right)^{2n+1} \left[\int_{E \cap W} |g - b|^2 \, d\mu + \left(\frac{Cn}{\Delta}\right)^{2n} \int_W \Phi^2 \, d\mu\right]
\leq \left(\frac{C}{\Delta}\right)^{2n+1} \left[\int_{E} |g - b|^2 \, d\mu + \left(\frac{Cn^3}{\Delta^2}\right)^{2n} \Phi^2\right]
\leq \left(\frac{Cn^3}{\Delta^2}\right)^{2n+1} \left[\int_{E} |g - b|^2 \, d\mu + \Phi^2\right].
\]
Now we consider the second case, when $M_1\tau > 1$. We set $M = \frac{1}{\tau}$ (that is, there is only one interval in the ‘partition’) and note that

$$M = \frac{1}{\tau} < M_1 = \frac{8n}{\Delta}.$$  

We set $\gamma = \frac{A}{\Delta}$, and once again $\tilde{E} = E \cup (W \cap E^c) = E \cup W$. Similarly to the first case, Lemma 15 gives us

$$\int_W |g - b|^2 \, d\mu \leq \left( \frac{C n^3}{\Delta^2} \right)^{2n+1} \left[ \int_E |g - b|^2 \, d\mu + \kappa^2 \right].$$

We now show that there are sufficiently many $\omega$-white intervals that contain a noticeable portion of $E^c$. We define the function $\delta(\omega) = m((E_\omega + n\tau) \setminus E_\omega)$ and notice that

$$\int_\Omega \delta(\omega) \, d\mathcal{P}(\omega) = \Delta.$$  

Let $L = \{ \omega \in \Omega: \delta(\omega) > \frac{1}{2} \Delta \}$. It is clear that

$$\int_L \delta(\omega) \, d\mathcal{P}(\omega) \geq \frac{\Delta}{2}.$$  

For $\omega \in L$ we have that $m(E_\omega^c), m(E_\omega) \geq \delta(\omega) > \frac{1}{2} \Delta = \gamma$, and therefore $L \times \mathbb{T} \subset W$. Thus $(L \times \mathbb{T}) \cap E^c \subset W \cap E^c$ and

$$m(W \cap E^c) \geq m((L \times \mathbb{T}) \cap E^c) = \int_L m(E_\omega^c) \, d\mathcal{P}(\omega) \geq \int_L \delta(\omega) \, d\mathcal{P}(\omega) \geq \frac{\Delta}{2},$$

proving the lemma. \qed

4 Logarithmic Integrability - $\text{Exp}_{10c}$ property for Rademacher Fourier series

In this section we explain how to effectively show that functions from the subspace $L^2_{\text{RF}}$ have the $\text{Exp}_{10c}$ property. We finish the proof by analyzing the iterative process.

4.1 Zygmund’s premise and the operator $A_E$

Suppose that

$$f = \sum_{k \in \mathbb{Z}} a_k \phi_k, \quad \phi_k(\omega, \theta) = \xi_k(\omega) e(k\theta), \quad \{a_k\} \in \ell^2(\mathbb{Z}).$$
and that $b \in L^2(\Omega)$. Let $E \subset Q$ be a measurable set of positive measure. Then

$$\int_E |f - b|^2 \, d\mu = \int_E \left[ \sum_{k,\ell} a_k \bar{a}_\ell \phi_k \bar{\phi}_\ell - 2 \Re(f \bar{b}) + |b|^2 \right] \, d\mu$$

$$\geq \int_E \left[ \sum_k |a_k|^2 |\phi_k|^2 \right] \, d\mu + \int_E \left[ \sum_{k \neq \ell} a_k \bar{a}_\ell \phi_k \bar{\phi}_\ell \right] \, d\mu - 2 \Re(\langle f, 1E b \rangle)$$

$$= \mu(E) \| f \|^2 + \langle A_E f, f \rangle - 2 \Re(\langle f, 1E b \rangle),$$

where $A_E$ is a bounded self-adjoint operator on $L^2_{RF}$, whose matrix $(A_E(k, \ell))_{k,\ell \in \mathbb{Z}}$ in the orthonormal basis $\{\phi_k\}$ is given by

$$A_E(k, \ell) = \begin{cases} \langle 1E, \phi_k \bar{\phi}_\ell \rangle, & k \neq \ell; \\ 0, & k = \ell. \end{cases}$$

To estimate the Hilbert-Schmidt norm of $A_E$, we observe that the functions $\{\phi_k \bar{\phi}_\ell\}_{k \neq \ell}$ form an orthonormal system in $L^2(Q)$, and that each function from this system is orthogonal to the function $1E$. Then

$$\sum_{k \neq \ell} |A_E(k, \ell)|^2 + |\langle 1E, 1E \rangle|^2 \leq \| 1E \|^2 = \mu(E),$$

and therefore,

$$\| A_E \|_{\text{HS}} = \sqrt{\sum_{k \neq \ell} |A_E(k, \ell)|^2} \leq \sqrt{\mu(E) - \mu(E)^2}.$$  

This estimate is useful for sets $E$ of large measure.

### 4.2 The sets $E$ of large measure

For each $\mu \in (0, 1)$, let $D(\mu) \in (1, +\infty]$ be the smallest value such that the inequality

$$\int_Q |f|^2 \, d\mu \leq D(\mu) \int_E |f - b|^2 \, d\mu$$

holds for every $E \subset Q$ with $\mu(E) \geq \mu$, every $f \in L^2_{RF}$, and every random constant $b \in L^\infty(\Omega)$ with $\| b \|_{\infty} < \frac{1}{20} \| f \|_2$.

Using the estimates from the previous section, we get

$$\int_E |f - b|^2 \, d\mu \geq (\mu(E) - \| A \|) \| f \|^2 - 2 \| 1E b \|_2 \| f \|_2$$

$$\geq \left( \mu(E) - \sqrt{\mu(E) - \mu(E)^2} - \frac{1}{10} \right) \| f \|^2 \geq \frac{1}{2} \| f \|^2,$$
provided that $\mu(E) \geq \frac{9}{10}$. That is, $D(\mu) \leq 2$ for $\mu \geq \frac{9}{10}$.

In order to get an upper bound for $D(\mu)$ for smaller values of $\mu$, we first of all need to get a better bound for the Hilbert-Schmidt norm of the operator $A_E$.

### 4.3 A better bound for the Hilbert-Schmidt norm of $A_E$

Here, using the bilinear Khinchin inequalities \[2.4.3\] we show that for each $p \geq 1$,
\[
\|A_E\|_{HS} \leq C_p \cdot \mu(E)^{1 - \frac{1}{2p}}.
\]

For sets $E$ of small measure, this bound is better than the one we gave in \[4.1\].

**Proof.** First, using duality and then Hölder’s inequality, we get
\[
\sqrt{\sum_{k \neq \ell} |A_E(k, \ell)|^2} = \sup\left\{ \sum_{k \neq \ell} A_E(k, \ell) g_{k, \ell} : \sum_{k \neq \ell} |g_{k, \ell}|^2 \leq 1 \right\}
= \sup\left\{ \left| \int_{\Omega} \mathbb{1}_E \tilde{g} \, d\mu \right| : g \in \text{span} \left\{ \phi_k \tilde{\phi}_\ell \right\}_{k \neq \ell}, \|g\|_2 \leq 1 \right\}
\leq \mu(E)^{1 - \frac{1}{2p}} \cdot \sup\left\{ \|g\|_{2p} : g \in \text{span} \left\{ \phi_k \tilde{\phi}_\ell \right\}_{k \neq \ell}, \|g\|_2 \leq 1 \right\}.
\]

Now, using the bilinear Khinchin inequality, we will bound $\|g\|_{2p}$ by $C_p \|g\|_2$. Since $g \in \text{span} \left\{ \phi_k \tilde{\phi}_\ell \right\}_{k \neq \ell}$,
\[
g(\omega, \theta) = \sum_{k \neq \ell} g_{k, \ell} \xi_k(\omega) \xi_\ell(\omega) e((k - \ell)\theta),
\]
whence,
\[
\int_Q |g|^{2p} \, d\mu = \int_T d\mu(\theta) \int_\Omega d\mathcal{P}(\omega) \left| \sum_{k \neq \ell} g_{k, \ell} \xi_k(\omega) \xi_\ell(\omega) e((k - \ell)\theta) \right|^{2p}
\leq \int_T d\mu(\theta) (C_p)^{2p} \left( \sum_{k \neq \ell} |g_{k, \ell} e((k - \ell)\theta)|^2 \right)^p
= (C_p)^{2p} \left( \sum_{k \neq \ell} |g_{k, \ell}|^2 \right)^p = (C_p)^{2p} \|g\|_2^{2p},
\]
completing the proof. \qed
4.4 The subspace $V_{E,b}$

Let $p \geq 1$. We now show that there exists a positive numerical constant $C'$ with the following property. If $E \subset Q$ is a set of positive measure and $b \in L^2(Q)$, then there exists a subspace $V_{E,b} \subset L^2_{RF}$ of dimension at most

$$n = \left\lfloor \frac{C'p^2}{\mu(E)^{1/p}} \right\rfloor$$

such that for each function $g \in L^2_{RF} \ominus V_{E,b}$ and each $b_1 = c \cdot b$ with $c \in \mathbb{C}$, we have

$$\int_Q |g|^2 \, d\mu \leq \frac{2}{\mu(E)} \int_E |g - b_1|^2 \, d\mu.$$

**Proof.** This result is a rather straightforward consequence of the estimates from Subsections 4.1 and 4.3. We enumerate the eigenvalues of the operator $A_E$ so that their absolute values form a non-increasing sequence: $|\sigma_1| \geq |\sigma_2| \geq \cdots$. Let $h_1, h_2, \ldots$ be the corresponding eigenvectors. Let $m \in \mathbb{Z}$ and denote by $\tilde{V}_E$ the linear span of $h_1, h_2, \ldots, h_m$. Then the norm of the restriction $A_E$ to $L^2_{RF} \ominus \tilde{V}_E$ equals $|\sigma_{m+1}|$. Therefore, if the function $g \in L^2_{RF} \ominus \tilde{V}_E$, then $|\langle A_E g, g \rangle| \leq |\sigma_{m+1}| \cdot \|g\|_2^2$.

Next,

$$\sigma_{m+1}^2 \leq \frac{1}{m+1} \sum_{j=1}^{m+1} \sigma_j^2 \leq \frac{1}{m+1} \sum_{j=1}^{\infty} \sigma_j^2 = \frac{1}{m+1} \|A_E\|_{HS}^2 \leq \frac{Cp^2}{m+1} \cdot \mu(E)^{2-\frac{1}{p}} < \frac{1}{4} \mu(E)^2,$$

provided that

$$m \geq \left\lfloor \frac{C'p^2}{\mu(E)^{1/p}} \right\rfloor - 1$$

and $C'$ is chosen large enough.

Denote by $U_{E,b}$ the one-dimensional space spanned by the projection of the function $1 \cdot b$ to $L^2_{RF}$, and put $V_{E,b} = \tilde{V}_E + U_{E,b}$. Then, assuming that $g \in L^2_{RF} \ominus V_{E,b} \subset L^2_{RF} \ominus \tilde{V}_E$ and applying the estimate from 4.1, we get

$$\int_E |g - b_1|^2 \, d\mu \geq \mu(E)\|g\|_2^2 + \langle A_E g, g \rangle - 2 \Re \langle g, 1 \cdot b_1 \rangle \geq \mu(E)\|g\|_2^2 - \frac{1}{2} \mu(E)\|g\|_2^2 = \frac{\mu(E)}{2} \|g\|_2^2.$$

Since $\dim V_{E,b}$ is at most $\left\lfloor C'p^2 \mu(E)^{-1/p} \right\rfloor$, the proof is complete. \qed
Note that it suffices to take $C' = 4C^2 + 1$, where $C$ is the constant that appears in the bilinear Khinchin inequality 2.4.3 though this is not essential for our purposes.

4.5 Functions $f \in L^2_{RF}$ have the $\text{Exp}_{1oC}$ property. Condition $(C_\tau)$

Introduce the function

$$n(p, \mu) \overset{\text{def}}{=} \left[ C'' p^2 \cdot \mu - \frac{1}{\rho} \right]$$

where $C'' > C'$ is a sufficiently large numerical constant. Fix $p \geq 1$ and let $E \subset Q$ be a given set of positive measure. Put $n = n\left(p, \frac{1}{2}\mu(E)\right)$ and choose the small parameter $\tau$ so that, for every $t \in (0, \tau]$,

$$\mu\left(\bigcap_{k=0}^{n} (E - kt)\right) \geq \frac{1}{2}\mu(E). \quad (C_\tau)$$

This is possible since the function $t \mapsto \mu((E - t) \cap E)$ is continuous and equals $\mu(E)$ at 0.

Now we prove that given a set $E \subset Q$ of positive measure, $b \in L^2(Q)$, and $p \geq 1$, each function $f \in L^2_{RF}$ has the $\text{Exp}_{1oC}(n, \tau, \epsilon, L^2(\Omega))$ property with

$$n = n\left(p, \frac{1}{2}\mu(E)\right), \quad \epsilon^2 = \frac{4(n+1)}{\mu(E)} \int_E |f - b|^2 \, d\mu,$$

and arbitrary $\tau$ satisfying condition $(C_\tau)$.

**Proof.** To shorten the notation, we put

$$E' = E'_t = \bigcap_{k=0}^{n} (E - kt).$$

Then for every $k \in \{0, \ldots, n\}$,

$$\int_{E'} |f_{kt} - b|^2 \, d\mu \leq \int_{E - kt} |f_{kt} - b|^2 \, d\mu = \int_{E} |f - b|^2 \, d\mu,$$

since $b$ depends only on $\omega$, and so, $b_{kt} = b$.

Given $t \in (0, \tau]$, we choose $a_0, \ldots, a_n \in \mathbb{C}$ with $\sum_{k=0}^{n} |a_k|^2 = 1$ so that the function $g = \sum_{k=0}^{n} a_k f_{kt}$ belongs to the linear space $L^2_{RF} \ominus V_{E', b}$. This is possible since

$$\dim V_{E', b} \leq n(p, \mu(E')) \leq n\left(p, \frac{1}{2}\mu(E)\right) = n.$$
Since the function $g$ is orthogonal to the subspace $V_{E',b}$, we can control its norm applying the estimate from 4.4 with $b_1 = b \cdot \sum_k a_k$:

$$
\left\| g \right\|_{2}^{2} \leq 2 \frac{\mu(E')}{\mu(E)} \int_{E'} \left| g - b_1 \right|^{2} \, d\mu
\leq 4 \frac{\mu(E)}{\mu(E')} \int_{E'} \left| \sum_{k=0}^{n} a_k \left( f_{kt} - b \right) \right|^{2} \, d\mu
\leq 4 \frac{\mu(E)}{\mu(E')} \int_{E'} \sum_{k=0}^{n} f_{kt} - b \right|^{2} \, d\mu \leq \frac{4(n+1)}{\mu(E)} \int_{E} |f - b|^{2} \, d\mu.
$$

That is,

$$
\left\| \sum_{k=0}^{n} a_k f_{kt} \right\|_{2} \leq \varepsilon,
$$

and we are done. \qed

### 4.6 Spreading the $L^2$-bound. Condition $(C_E)$

We apply the spreading Lemma 14 to the function $f$ and the set $E$. It provides us with a set $\tilde{E} \supset E$, such that $\mu(\tilde{E}) \geq \mu(E) + \frac{1}{2} \Delta_{n\tau}(E)$ and

$$
\int_{E} \left| f - b \right|^{2} \, d\mu \leq \left( \frac{C n^3}{\Delta_{n\tau}(E)} \right)^{2n+1} \left( \int_{E} |f - b|^{2} \, d\mu + \varepsilon^{2} \right)
\leq \left( \frac{C n^3}{\Delta_{n\tau}(E)} \right)^{2n+1} \cdot \frac{C(n+1)}{\mu(E)} \int_{E} |f - b|^{2} \, d\mu,
$$

where $n = n \left( p, \frac{1}{2} \mu(E) \right) \leq 2C' n^{2} \cdot \mu(E)^{-\frac{1}{2}}$. There is not much value in this spreading until we learn how to control the parameter $\Delta_{n\tau}(E)$ in terms of our main parameters $\mu(E)$ and $p$. Clearly, the bigger $\Delta_{n\tau}(E)$, the better our spreading estimate is. Recall that till this moment, our only assumption on the value of $\tau$ has been condition $(C_\tau)$ at the beginning of Section 4.5.

Now we will need the following condition on our set $E$:

$$
\max_{t} \Delta_{t}(E) \geq \frac{1}{2n} \mu(E). \quad (C_E)
$$

If condition $(C_E)$ holds, then we can find $\tau > 0$ such that $\Delta_{n\tau}(E) = \frac{1}{2n} \mu(E)$, while for all $t \in (0, n\tau)$, $\Delta_{t}(E) < \frac{1}{2n} \mu(E)$.
Such $\tau$ will automatically satisfy condition $(C_{\tau})$ used in the derivation of the spreading estimate. Indeed, 

$$
\mu\left(\bigcap_{k=0}^{n} (E - kt) \right) = \mu\left( E \setminus \bigcup_{k=1}^{n} (E \setminus (E - kt)) \right)
\geq \mu(E) - \sum_{k=1}^{n} \mu\left( E \setminus (E - kt) \right)
= \mu(E) - \sum_{k=1}^{n} \mu\left( (E + kt) \setminus E \right)
= \mu(E) - \sum_{k=1}^{n} \Delta_{kt}(E) \geq \mu(E) - \sum_{k=1}^{n} \frac{\mu(E)}{2n} = \frac{1}{2} \mu(E).
$$

It is easy to see that there are sets $E \subset Q$ of arbitrary small positive measure that do not satisfy condition $(C_{E})$. We assume now that condition $(C_{E})$ is satisfied, putting aside the question “What to do with the sets $E$ for which $(C_{E})$ does not hold?” till the next section.

Substituting the value $\Delta_{n\tau} = \frac{1}{2n} \mu(E)$ into the spreading estimate and taking into account that $n \leq 2C^p p^2 \mu(E)^{-\frac{1}{p}}$, we finally get

$$
\int_{E} |f - b|^2 d\mu \leq \left( \frac{Cn^5}{\mu(E)^2} \right)^{2n+1} \cdot \frac{n + 1}{n} \int_{E} |f - b|^2 d\mu
\leq \left( \frac{Cp}{\mu(E)} \right)^{Cp^2 \mu(E)^{-1/p}} \int_{E} |f - b|^2 d\mu,
$$

while

$$
\mu(\tilde{E}) \geq \mu(E) + \frac{c}{p^2} \mu(E)^{1+\frac{1}{p}}.
$$

This is the spreading estimate that we will use for the sets $E$ satisfying condition $(C_{E})$.

### 4.7 The case of sets $E$ that do not satisfy condition $(C_{E})$

Now, let us assume that $E \subset Q$ is a set of positive measure that does not satisfy condition $(C_{E})$, that is, for each $t \in [0, 1]$, $\Delta_t(E) < \frac{1}{2n} \mu(E)$. The simplest example is any set of the form $E = \Omega_1 \times \mathbb{T}$, $\Omega_1 \subset \Omega$. For these sets, $\Delta_t(E) = 0$ for every $t$. We will show that this example is typical, i.e., the sets $E$ that do not satisfy condition $(C_{E})$ must have sufficiently many ‘long’ sections $E_\omega$. More precisely, let

$$
\Omega_1 = \{ \omega \in \Omega : m(E_\omega) > 1 - \frac{1}{n} \}. 
$$
We show that $\mathcal{P}\{\Omega_1\} > \frac{1}{2} \mu(E)$.

**Proof.** Let

$$\Omega_2 = \Omega \setminus \Omega_1 = \{\omega \in \Omega : m(E_\omega) \leq 1 - \frac{1}{n}\}.$$ 

Since condition $(C_E)$ is not satisfied, we have

$$\int_0^1 \Delta_t(E) \, dt < \frac{1}{2n} \mu(E).$$

A straightforward computation shows that

$$\int_0^1 m((E_\omega + t) \setminus E_\omega) \, dt = m(E_\omega) (1 - m(E_\omega)).$$

Since $m(E_\omega) \leq 1 - \frac{1}{n}$ implies that $m(E_\omega) \leq nm(E_\omega)(1 - m(E_\omega))$, we get

$$\int_{\Omega_2} m(E_\omega) \, d\mathcal{P}(\omega) \leq n \int_{\Omega_2} m(E_\omega) (1 - m(E_\omega)) \, d\mathcal{P}(\omega)$$

$$\leq n \int_{\Omega} m(E_\omega) (1 - m(E_\omega)) \, d\mathcal{P}(\omega)$$

$$= n \int_0^1 \Delta_t(E) \, dt < \frac{1}{2} \mu(E).$$

Therefore,

$$\mathcal{P}\{\Omega_1\} \geq \int_{\Omega_1} m(E_\omega) \, d\mathcal{P}(\omega) = \mu(E) - \int_{\Omega_2} m(E_\omega) \, d\mathcal{P}(\omega) > \frac{1}{2} \mu(E).$$

**Remark.** Since $n = n \left(p, \frac{1}{2} \mu(E)\right) \geq 2$ if $C'' \geq 2$, we trivially have

$$\mathcal{P}\{\Omega_1\} \leq \frac{n}{n - 1} \int_{\Omega_1} m(E_\omega) \, d\mathcal{P}(\omega) \leq \frac{n}{n - 1} \mu(E) \leq 2 \mu(E).$$
4.8 Many ‘long’ sections

Assume that the set $E$ does not satisfy condition $(C_E)$. We will show that

$$\int_Q |f|^2 \, d\mu \leq \frac{4}{\mu(E)} \int_E |f - b|^2 \, d\mu,$$

where, as above, $b = b(\omega)$ is a random constant, $\|b\|_\infty < \frac{1}{20} \|f\|_2$.

Let $\mu = \mu(E)$ and $\Omega_1$ be as above. We have

$$\int_E |f - b|^2 \, d\mu \geq \int_{\Omega_1} \left( \int_{E_\omega} |f - b|^2 \, dm \right) \, dP(\omega) = \int_{\Omega_1} \int_T |f - b|^2 \, dm \, dP(\omega) - \int_{\Omega_1} \int_{T \setminus E_\omega} |f - b|^2 \, dm \, dP(\omega) = (I) - (II).$$

Notice that by the result of Section 4.6, we have $2\mu \geq P(\Omega_1) \geq \frac{1}{2} \mu$.

Bounding integral $(I)$ from below is straightforward: we have

$$\int_T |f - b|^2 \, dm \geq (\|f\|_2 - \|b\|_\infty)^2 \geq \frac{9}{10} \|f\|_2^2,$$

whence,

$$(I) \geq \frac{9}{10} \|f\|_2^2 P(\Omega_1) \geq \frac{9}{20} \mu \|f\|_2^2.$$

Now let us estimate the integral $(II)$ from above. We have

$$(II) \leq 2 \int_{\Omega_1} \int_{T \setminus E_\omega} |f|^2 \, dm \, dP(\omega) + 2 \int_{\Omega_1} \int_{T \setminus E_\omega} |b|^2 \, dm \, dP(\omega) = (II_a) + (II_b).$$

Estimating the second integral is also straightforward:

$$(II_b) \leq \frac{4\mu}{n} \|b\|_\infty^2 < \frac{1}{10} \mu \|f\|_2^2$$

(recall that $n \geq 2$ and $\|b\|_\infty < \frac{1}{20} \|f\|_2$). Furthermore,

$$(II_a) = 2 \int_{\Omega_1} \int_T 1_{T \setminus E_\omega} |f|^2 \, dm \, dP(\omega) \leq 2 \left( \int_{\Omega_1} \int_T 1_{T \setminus E_\omega} \right)^{\frac{1}{r}} \left( \int_{\Omega_1} \int_T |f|^{2s} \right)^{\frac{1}{s}}$$

with $\frac{1}{r} + \frac{1}{s} = 1$. By Khinchin’s inequality,

$$\left( \int_{\Omega} \int_T |f|^{2s} \right)^{\frac{1}{s}} \leq C_s \|f\|_2^2.$$
Hence,

\[(\Pi_a) \leq \left( \frac{2\mu}{n} \right) ^{\frac{1}{2}} Cs \|f\|_2^2 .\]

Letting \( \frac{1}{p} = \frac{p}{p+1} \), \( \frac{1}{s} = \frac{1}{p+1} \) and recalling that \( n \geq \frac{1}{2} C'' p^2 \mu^{-1/p} \) and that \( p \geq 1 \), we continue the estimate as

\[(\Pi_a) \leq \left( \frac{4\mu^{1+\frac{1}{p}}}{C'' p^2} \right) ^{\frac{p}{p+1}} 2C p \|f\|_2^2 < \frac{8C}{C''} \mu p^{-\frac{p-1}{p+1}} \|f\|_2^2 < \frac{1}{10} \mu \|f\|_2^2 , \]

provided that the constant \( C'' \) in the definition of \( n \) was chosen sufficiently big. Finally,

\[
\int_E |f - b|^2 \geq (I) - (\Pi_a) - (\Pi_b) \geq \left( \frac{9}{20} - \frac{4}{20} \right) \mu \|f\|_2^2 = \frac{1}{4} \mu \|f\|_2^2 ,
\]

completing the argument. \( \square \)

### 4.9 End of the proof of Theorem 5: solving a difference inequality

Recall that by \( D(\mu) \) we denote the smallest value such that the inequality

\[
\int_Q |f|^2 \, d\mu \leq D(\mu) \int_E |f - b|^2 \, d\mu
\]

holds for every \( E \subset Q \) with \( \mu(E) \geq \mu \), every \( f \in L^2_{RF} \), and every random constant \( b \in L^\infty(\Omega) \) satisfying \( \|b\|_\infty < \frac{1}{20} \|f\|_2 \).

By 4.2 \( D(\mu) \leq 2 \) for \( \mu \geq \frac{9}{10} \), and by the estimates proven in 4.6 and 4.8 for \( 0 < \mu < \frac{9}{10} \) we have

\[
D(\mu) \leq \max \left\{ \left( \frac{Cp}{\mu} \right) ^{c p^2 \mu^{-\frac{1}{p}}} D \left( \mu + \frac{c}{p^2} \mu^{1+\frac{1}{p}} \right) , \frac{4}{\mu} \right\} .
\]

Increasing, if needed, the constant \( C \) in the exponent, and taking into account that \( \frac{p}{\mu} \geq \frac{1}{p/\mu} > 1 \) and \( D \geq 1 \), we simplify this to

\[
D(\mu) \leq \left( \frac{p}{\mu} \right) ^{c p^2 \mu^{-\frac{1}{p}}} D \left( \mu + \frac{c}{p^2} \mu^{1+\frac{1}{p}} \right) .
\]
Put
\[ \delta(\mu) = \frac{c}{p^2} \mu^{1+\frac{1}{p}}. \]

Making the constant \( c \) on the right-hand side small enough, we assume that \( \delta\left(\frac{9}{10}\right) < \frac{1}{10} \) (it suffices to take \( c < \frac{1}{10} \)). Then, for \( 0 < \mu < \frac{9}{10} \),
\[
\log D(\mu) - \log D(\mu + \delta(\mu)) < C p^2 \mu^{-\frac{1}{p}} \log \left( \frac{p}{\mu} \right) < C \delta(\mu) p^4 \mu^{-\frac{2}{p}} \log \left( \frac{p}{\mu} \right).
\]

To solve this difference inequality, we define the sequence
\[
\mu_0 = \mu, \quad \mu_{k+1} = \mu_k + \delta(\mu_k), \quad k \geq 0,
\]
and stop when \( \mu_{s-1} < \frac{9}{10} \leq \mu_s \). Since we assumed that \( \delta\left(\frac{9}{10}\right) < \frac{1}{10} \), the terminal value \( \mu_s \) will be strictly less than 1. We get
\[
\log D(\mu) = \log D(\mu_s) + \sum_{k=0}^{s-1} \left[ \log D(\mu_k) - \log D(\mu_{k+1}) \right]
< 1 + C p^4 \sum_{k=0}^{s-1} \delta(\mu_k) \mu_k^{-\frac{1}{p}} \log \left( \frac{p}{\mu_k} \right)
< 1 + C p^4 \log \left( \frac{p}{\mu} \right) \sum_{k=0}^{s-1} \delta(\mu_k) \mu_k^{-\frac{1}{p}}.
\]

Since \( \mu_{k+1} = \mu_k + c p^{-2} \mu_k^{1+\frac{1}{p}} < C \mu_k \), we have \( \mu_k^{-\frac{1}{p}} < C \mu_{k+1}^{-\frac{2}{p}} \). Therefore,
\[
\sum_{k=0}^{s-1} \delta(\mu_k) \mu_k^{-\frac{1}{p}} < C \sum_{k=0}^{s-1} \delta(\mu_k) \mu_{k+1}^{-\frac{2}{p}}
< C \sum_{k=0}^{s-1} \int_{\mu_k}^{\mu_{k+1}} \frac{dx}{x^{1+\frac{1}{p}}} < C \int_{\mu}^{1} \frac{dx}{x^{1+\frac{1}{p}}} < C p \mu^{-\frac{2}{p}},
\]
whence
\[
\log D(\mu) < 1 + C p^5 \mu^{-\frac{2}{p}} \log \left( \frac{p}{\mu} \right).
\]

This holds for any \( p \geq 1 \). Letting \( p = 2 \log \left( \frac{2}{\mu} \right) \), we finally get \( \log D(\mu) < C \log^6 \left( \frac{2}{\mu} \right) \).

This completes the proof of Theorem \([5]\).

**5 On the power 6 in the statement of Theorem [5]**

In this section we will present an example that shows that the constant 6 in the exponent on the RHS of the inequality proven in Theorem [5] cannot be replaced by any number smaller than 2.
Let \( g_N(\theta) = (\sin(2\pi\theta))^{2N} = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^{2N} = \sum_{|n| \leq N} a_n e^{(2n\theta)} \). The function \( g_N \) satisfies

\[
|g_N(\theta)| \leq e^{-cN^2} \quad \text{for } |\theta| \leq e^{-CN},
\]

provided that \( C \) is large enough.

Now consider the Rademacher trigonometric polynomial

\[
f_N(\theta) = \sum_{|n| \leq N} \xi_n a_n e^{(2n\theta)},
\]

denote by \( X_N \) the event that \( \xi_n = +1 \) for all \( n \in \{-N, \ldots, N\} \), and put \( E_N = X_N \times T_N \), where \( T_N = [-e^{CN}, e^{CN}] \subset \mathbb{T} \) is the set from (5.1). Then

\[
\mu(E_N) \geq 2^{-(2N+1)} \cdot e^{-CN} \geq e^{-CN},
\]

while

\[
\int_{E_N} |f_N|^2 \, d\mu \leq e^{-cN^2} \mu(E_N) \leq e^{-cN^2}
\]

and

\[
\int_{\Omega \times T} |f_N|^2 \, d\mu = \int_T |g_N|^2 \, dm.
\]

It is not difficult to see that the integral on the RHS is not less than \( \frac{c}{N} \), for some constant \( c > 0 \). Recalling that \( |\log \mu(E_N)| \leq CN \), we see that for every \( \varepsilon > 0 \), \( C > 0 \), the inequality

\[
\int_Q |f_N|^2 \, d\mu \leq e^{C|\log \mu(E_N)|} \int_{E_N} |f_N|^2 \, d\mu
\]

fails when \( N \geq N_0(\varepsilon, C) \). This shows that one cannot replace 6 by any number less than 2. \( \square \)

### 6 Some open problems

Following the last section, it is not clear what is the optimal constant in the statement of Theorem 5. Proving that the constant is strictly larger than 2 seems to be an interesting problem (in case it is true).
It should be mentioned that the result of the previous section (the optimal constant in the power of the logarithm) does not contradict the possibility that the distributional inequality can be improved if one is ready to discard an event of small probability. Here is a sample

**Problem 17.** Consider Rademacher trigonometric polynomials $f_n$ of large degree $n$. Given a small $\delta > 0$, does there exist an event $\Omega'(n, \delta) \subset \Omega$ with $\mathcal{P}(\Omega \setminus \Omega') \to 0$ as $n \to \infty$, such that

$$
\int_{\Omega' \times \mathbb{T}} |f_n|^{-\delta} \, d\mu < \infty?
$$

It would be interesting to find a simple probabilistic proof of an inequality of the form

$$
\|f\|^2_2 \leq C(\mu(E)) \int_E |f|^2 \, d\mu,
$$

even without a quantitative dependence on the measure of the set $E$.

A (related) very natural problem is to prove a generalization of Theorem 5 for other types of random variables. It is possible that one should modify the statement of theorem to consider the event where the sum $\sum_{n \in \mathbb{Z}} |\xi_n|^2 |a_n|^2$ is small.

A known open problem for non-random, ‘half-lacunary’ Fourier series is the following

**Problem 18.** Consider the random Fourier series of the form

$$
g(\theta) = \sum_{k>0} a_{nk} e^{2\pi i k \theta} + \sum_{n \geq 0} a_n e^{2\pi i n \theta},
$$

where $\{a_n\} \in \ell^2(\mathbb{Z})$, and $\{n_k\} \subset \mathbb{Z}^-$ is lacunary in the sense of Hadamard (i.e., $\liminf_{k \to \infty} \frac{n_{k+1}}{n_k} > 1$). Is it true that $g(\theta) = 0$ only on a set of measure 0? Is it true that $\log |g| \in L^p(\mathbb{T})$, for some $p > 0$?

The same question can be asked for ‘half-Rademacher’ Fourier series of the form

$$
g(\theta) = \sum_{n < 0} \xi_n a_n e^{2\pi i n \theta} + \sum_{n \geq 0} a_n e^{2\pi i n \theta},
$$

where, as usual, $\xi_n$ is a sequence of independent Bernoulli random variables, which take the values $\pm 1$, with probability $\frac{1}{2}$ each.

**Problem 19.** Let $g$ be a ‘half-Rademacher’ Fourier series. Is it true that $\log |g| \in L^p(\mathbb{Q})$ for some $p > 0$?
Part III. The value distribution of random analytic functions

The theory that is known today as the value distribution theory of holomorphic (or meromorphic) functions, was originally developed by Nevanlinna in the 1920s (see [GO]). Compared to the more classical and qualitative Picard Theorem, it tries to get a more quantitative measure of the number of times an analytic function $f(z)$ assumes some value $b$, as $z$ grows. This theory can also be generalized to several complex variables. One can think about the value distribution of random analytic functions as a representation of the behavior of ‘standard’ analytic functions. In particular, it is possible to give very precise versions of the general results of the theory.

One of the basic notions of the general theory is that of an exceptional value. Little Picard Theorem is a classical example, as it states that every entire function must take every complex value, except maybe one exceptional value. In general we expect that random functions will be “unable to distinguish” between different values, and thus that they will not have any exceptional values (one must define the correct model of course, as $e^{f(z)} \neq 0$ always). In the first section of this part we prove a result of this kind, which is an answer to an old question of J.-P. Kahane.

A central part of the theory of entire functions was developed for the goal of counting the number of zeros of entire functions, in large domains (motivated, in no small part, by Riemann’s Hypothesis). In the second section we give a short proof of a strong law of large numbers for the number of zeros of Rademacher entire functions in large disks around 0, as the radius of the disk tends to infinity. This is the analog of the simpler result for Gaussian analytic functions.

In the third section, we use more accurate tools than the ones used in the second section to show that random entire functions grow very regularly. In particular, their zeros are equidistributed in sectors centered at zero. Under certain restrictions functions with equidistributed zeros are a special case of a certain class of entire functions — those of completely regular growth in the sense of Levin and Pfluger (see [Le1]). We note that our result lies beyond the scope of the (classical) theory of functions of completely regular growth.

In the last two sections we give an example which demonstrates the tightness of the results of the third section, as well as some further work and related open problems.
We begin with a short description of some of the previous results. In 1972, Offord [O3] proved the analog of Theorem 2 in the case where $\zeta_k = u_k a_k$, and $u_k$ are uniformly distributed on the unit circle. The proof he gave also works for Gaussian Taylor series; see also Kahane [Ka2, Section 12.3] for a different proof. In the special case $\zeta_k = \xi_k a_k$, where $\xi_k$ are independent Rademacher random variables, the result was known under some additional restrictions on the growth of the deterministic coefficients $a_k$. In 1981, Murai [Mu2] proved it assuming that $\lim \inf |a_k| > 0$. Soon afterwards, Jacob and Offord [JO] weakened this assumption to

$$\lim \inf \frac{1}{\log N} \sum_{k=0}^{N} |a_k|^2 > 0.$$ 

To the best of our knowledge, since then there was no further improvement.

### 7.1 Solution to Kahane’s problem - Proof of Theorem 2

We start by proving the theorem in the special case when $\zeta_n = \xi_n a_n$, where $\xi_n$ are independent Rademacher random variables, and $\{a_n\}$ is a non-random sequence of complex numbers satisfying the conditions $\lim \sup_n |a_n|^{1/n} = 1$ and $\sum_n |a_n|^2 = \infty$. The proof is based on the logarithmic integrability of the Rademacher Fourier series (corollary (2.1) of Theorem 5) combined with Jensen’s formula. Then using ‘the principle of reduction’ as stated in Kahane’s book [Ka2, Section 1.7], we get the result in the general case.

Let us introduce some notation. For $b \in \mathbb{C}$, $0 < r < 1$, we denote by $n_F(r, b)$ the number of solutions to the equation $F(z) = b$ in the disk $r \mathbb{D}$, the solutions being counted according to their multiplicities. In this section it will be convenient to set

$$N_F(r, b) \overset{\text{def}}{=} \int_{1/2}^{r} \frac{n_F(t, b)}{t} \, dt.$$ 

By Jensen’s formula,

$$N_F(r, b) = \int_{0}^{1} \log |F(re(\theta)) - b| \, dm(\theta) - \int_{0}^{1} \log |F(\frac{1}{2} e(\theta)) - b| \, dm(\theta).$$

(7.1)

We will prove that a.s. we have

$$\lim_{r \to 1} N_F(r, b) = \infty, \forall b \in \mathbb{C},$$

which is equivalent to Theorem 2.
7.1.1 Proof of Theorem 2 in the Rademacher case

We define the functions \( \sigma_F^2 \) and \( \hat{F} \) by

\[
\sigma_F^2(r) \overset{\text{def}}{=} \sum_{n \geq 0} |a_n|^2 r^{2n}, \quad \hat{F}(z) \overset{\text{def}}{=} \frac{F(z)}{\sigma_F(|z|)},
\]

and note that \( \|\hat{F}(re(\theta))\|_{L^2(T)} = 1 \).

Let \( M \in \mathbb{N} \). For every \( r \in (\frac{1}{2}, 1) \), the function \((\omega, b) \mapsto N_F(r, b)\) on \( \Omega \times \mathbb{C} \) is measurable in \( \omega \) for fixed \( b \) and continuous in \( b \) for fixed \( \omega \). Therefore, we can find a measurable function \( b^* = b^*(\omega) \) such that \( |b^*| \leq M \) and

\[
\inf_{|b| \leq M} N_F(r, b) \geq N_F(r, b^*) - 1.
\]

Then

\[
\inf_{|b| \leq M} N_F(r, b) \geq \int_T \log |F(re(\theta)) - b^*| \, dm(\theta) - \int_T \log |F(\frac{1}{2}e(\theta)) - b^*| \, dm(\theta) - 1
\]

\[
= (I_1) - (I_2) - 1.
\]

Note that

\[
(I_2) \leq \frac{1}{2} \log \left( \int_T |F(\frac{1}{2}e(\theta)) - b^*|^2 \, dm(\theta) \right) \leq \frac{1}{2} \log \left( 2\sigma_F^2(\frac{1}{2}) + 2M^2 \right).
\]

For the integral \((I_1)\), we have the following lower bound:

\[
(I_1) = \log \sigma_F(r) + \int_T \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta)
\]

\[
\geq \log \sigma_F(r) - \int_T \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta).
\]

If we assume that \( r \) is so close to 1 that \( \sigma_F(r) \geq 20M \), then, using our result \((2.1)\) on the logarithmic integrability of the Rademacher Fourier series, we get

\[
P\left\{ \int_T \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta) > T \right\}
\]

\[
\leq \frac{1}{T} \mathcal{E}\left( \int_T \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta) \right) \leq \frac{C}{T},
\]

for all \( T > 0 \).
Taking $r = r_m$ so that $\log \sigma_F(r_m) = 2m^2$ and $T = m^2$, and applying the Borel-Cantelli lemma, we see that, for a.e. $\omega \in \Omega$, there exists $m_0 = m_0(\omega, M)$ such that, for each $m \geq m_0$,
\[
\int_{T} \left| \log \left| \hat{F}(r_me(\theta)) - \sigma_F^{-1}(r_m) \cdot b^* \right| \right| dm(\theta) < m^2,
\]
whence,
\[
\inf_{|b| \leq M} N_F(r_m, b) \geq m^2 - \frac{1}{2} \log \left( 2\sigma_F^2(\frac{1}{2}) + 2M^2 \right) - 1, \quad \forall m \geq m_0.
\]
Therefore, for every $M \in \mathbb{N}$, there is a set $A_M \subset \Omega$ with $\mathcal{P}(A_M) = 1$ such that, for every $\omega \in A_M$ and every $b \in \mathbb{C}$ with $|b| \leq M$, we have
\[
\lim_{r \to 1} N_F(r, b) = \infty. \tag{7.2}
\]
Let $A = \bigcap_{M} A_M$. Then $\mathcal{P}(A) = 1$, and for every $\omega \in A$, $b \in \mathbb{C}$, we have $\text{(7.2)}$. Thus, the theorem is proved in the Rademacher case.

### 7.1.2 Proof of Theorem 2 in the general case

For every $M \in \mathbb{N}$, consider the event
\[
B_M = \left\{ \omega : \lim_{r \to 1} \inf_{|b| \leq M} N_F(r, b) = +\infty \right\}.
\]
Given $r \in \left( \frac{1}{2}, 1 \right)$, the function $\inf_{|b| \leq M} N_F(r, b)$ is measurable in $\omega$ (note that here the infimum can be taken over any dense countable subset of the disk $\{|b| \leq M\}$). Thus, the set $B_M$ is measurable and so is the set $B = \bigcap_{M} B_M$, and for every $\omega \in B$, $b \in \mathbb{C}$, we have $\text{(7.2)}$. It remains to show that $B$ holds almost surely.

To that end, we extend the probability space to $\Omega \times \Omega'$ and introduce a sequence of independent Rademacher random variables $\{\xi_n(\omega')\}$, $\omega' \in \Omega'$, which are also independent from the random variables $\{\zeta_n(\omega)\}$, $\omega \in \Omega$, and consider the random analytic function
\[
G(z) = G(z; \omega, \omega') = \sum_{n \geq 0} \xi_n(\omega') \zeta_n(\omega) z^n, \quad (\omega, \omega') \in \Omega \times \Omega'.
\]
By the previous section, for fixed $\zeta_n$’s (outside a set of probability zero in $\Omega$), the event
\[
\left\{ \omega' \in \Omega' : \lim_{r \to 1} \inf_{|b| \leq M} N_G(r, b) = +\infty \right\}
\]
occurs with probability 1. Hence, by Fubini’s theorem, the event $B_M$ occurs a.s. and so does the event $B$. Note that due to the symmetry of the distribution of $\zeta_n$’s, the random variables $\{\xi_n(\omega')\zeta_n(\omega)\}$ are equidistributed with $\{\zeta_n(\omega)\}$. This yields the theorem in the general case of symmetric random variables.

\[ \square \]

7.2 Some further problems

Curiously enough, even in the case when $\zeta_k = \xi_k a_k$ with the standard complex Gaussian $\xi_k$’s, the question when $F(D) = C$ almost surely is not completely settled. In [Mu1] Murai proved Paley’s conjecture, which states that if $F$ is a (non-random) Taylor series with Hadamard gaps and with the radius of convergence 1, then $F$ assumes every complex value infinitely often, provided that $\sum_{k \geq 0} |a_k| = +\infty$. Therefore, the same holds for random Taylor series with Hadamard gaps. However, even the case of sequences $a_k$ with a regular behaviour remains open:

Problem 20. Suppose that $\xi_k$ are independent standard complex Gaussian random variables. In addition, assume that the non-random sequence $\{a_k\}$ is decaying regularly and satisfies

\[ \sum_{k \geq 0} |a_k|^2 < \infty, \quad \sum_{k \geq 0} \left| a_k \right| = \infty. \]  \tag{7.3} \]

Does the range of the random Taylor series $F(z) = \sum_{k \geq 0} \xi_k a_k z^k$ fill the whole complex plane $\mathbb{C}$ a.s.?

Note that convergence of the first series in (7.3) yields that, a.s., the function $F$ belongs to all Hardy spaces $H^p$ with $p < \infty$. Moreover, by the Paley-Zygmund theorem [Kn2, Chapter 5], a.s. we have $e^{\lambda|\hat{F}|^2} \in L^1(\mathbb{T})$ for every positive $\lambda$, where $\hat{F}$ denotes the non-tangential boundary values of $F$ on $\mathbb{T}$. On the other hand, by Fernique’s theorem [Kn2, Chapter 15], divergence of the second series in (7.3) yields that, a.s., $F$ is unbounded in $D$.

8 Almost sure convergence of the number of zeros

In this section, we explain how to use Theorem 1 to find almost sure asymptotics for the number of zeros of a Rademacher entire function $f$ in a disk $rD = \{|z| \leq r\}$, as $r$ tends to infinity (this can be viewed as a strong law of large numbers for the number of zeros). In the next section we use a more elaborate argument to prove the equidistribution of the zeros.
8 Almost sure convergence of the number of zeros

8.1 Short Background

There is a large body of work about the real zeros of random polynomials see the books [BS, Far]. However, there are not too many works that are concerned with complex zeros of random analytic functions. Hammersley [Ham] gave an expression for the joint distribution of the complex (and real) zeros of random polynomials with general Gaussian coefficients. However, it is not easy to deduce the asymptotic behavior of the zeros from his results. Shepp and Vanderbei [SV] studied the very special case of \( \sum_{k=0}^{n} \xi_k z^k \), with \( \xi_n \) i.i.d. standard real Gaussians. They extended the results of Kac and Rice about the distribution of the real zeros to the distribution of the complex ones. Using the special structure of these polynomials they found asymptotic expressions for the distribution of the zeros. Ibragimov and Zeitouni [IZ] then proved generalizations of these results for random variables \( \xi_k \) that belong to the domain of attraction of a stable law. Among several other results relating to the above polynomials, Edelman and Kostlan also proved in [EK, Theorem 8.2] the formula (1.2) for the expected number of zeros of a (Gaussian) entire function in the disk \( r \mathbb{D} \) that we have mentioned in the introduction.

8.2 Notations

Let

\[
f(z) = \sum_{n \geq 0} a_n \xi_n z^n
\]

be a Rademacher entire function. In order to simplify the argument, we will assume in this section that

\[
a_0 = 1.
\]

If \( n_f(r) \) is the number of zeros of \( f \) inside the disk \( r \mathbb{D} = \{ |z| \leq r \} \) (including multiplicities), then by Jensen’s formula

\[
\int_0^1 \log |f(re^{2\pi i \theta})| \, d\theta = \int_0^r \frac{n_f(t)}{t} \, dt \overset{def}{=} N_f(r),
\]

It should be mentioned, however, that we are not aware of any ‘asymptotic rate of growth’ result about the real zeros of polynomials with general (say) Gaussian coefficients.
We thus have \( n_f(r) = \frac{dN_f(r)}{d\log r} = r \cdot \frac{dN_f(r)}{dr} \). We also define the following functions:

\[
\sigma_f^2(r) = \sum_{k \geq 0} |a_n|^2 r^{2n},
\]

\[
s_f(r) = \frac{d\log \sigma_f(r)}{d\log r} = r \cdot \frac{\sigma_f'(r)}{\sigma_f(r)}.
\]

The first of these functions measures the ‘expected’ rate of growth of \( N_f(r) \), while the second one measures the rate of growth of \( n_f(r) \). A set \( E \subset [1, \infty) \) is called a set of finite logarithmic measure if

\[
\int_E \frac{dt}{t} < \infty.
\]

We write

\[
\hat{f}(re^{2\pi i \theta}) = \frac{f(re^{2\pi i \theta})}{\sigma_f(r)},
\]

and notice that

\[
\mathcal{E} \left\{ \int_T \left| \hat{f}(re^{2\pi i \theta}) \right|^2 d\theta \right\} = 1,
\]

Hence, by Theorem [1]

\[
\mathcal{E} \left\{ |N_f(r) - \log \sigma_f(r)|^p \right\} \leq \mathcal{E} \left\{ \int_T \log \left| \hat{f}(re^{2\pi i \theta}) \right|^p d\theta \right\} \leq (Cp)^6 p,
\]  

(8.1)

for every \( p \geq 1 \). Using this result we prove

**Theorem 21.** Almost surely, for every \( r \geq r_0 (\omega) \geq 1 \)

\[
|n_f(r) - s_f(r)| \leq C \sqrt{s_f(r) \log^4 s_f(r)}, \quad r \notin E,
\]

where \( E \) is a (non-random) set of finite logarithmic measure.

### 8.3 Proof of Theorem 21

After the change variables \( \rho = \log r \) the exceptional set that will appear has to be of finite (Lebesgue) measure in \( \rho \).
8.3.1 The sequences \( \{\rho_m\} \) and \( \{\rho_m^\pm\} \)

Since the function \( s_f(r) \) tends monotonically to infinity, we can choose a sequence \( \rho_m \uparrow \infty \) such \( s_f(e^{\rho_m}) = m^\alpha \log^\beta m \), where \( \alpha > 1, \beta > 0 \) are some constants that will be chosen later. We write \( \Delta_m = \rho_{m+1} - \rho_m \) and let \( \{\delta_m\} \) be a sequence such that \( 0 \leq \delta_m \leq \frac{1}{2}\Delta_m \), again to be chosen later.

Let \( \{\rho_m'\} \) be the sequence \( \rho_{m-1}, \rho_1, \rho_2, \rho_3, \rho_4, \ldots \), where \( \rho_m' = \rho_m - \delta_m, \rho_{m+1}' = \rho_m + \delta_m \).

The sequence \( \{\rho_m'\} \) is where we have ‘good sampling’ of the function \( N_f(e^t) \) (that is, it is close to \( \log \sigma_f(e^t) \)), then using the convexity of these functions we show that they cannot be far apart for any \( t \), outside an exceptional set of finite measure.

8.3.2 How to ‘differentiate’ \( N_f(e^t) \)?

By \([8.1]\) and Chebyshev’s inequality,

\[
P\left( \left| N_f(e^{\rho_m'}) - \log \sigma_f(e^{\rho_m'}) \right| > t \right) = P\left( \left| N_f(e^{\rho_m'}) - \log \sigma_f(e^{\rho_m'}) \right|^p > t^p \right) \leq \left( \frac{Cp^6}{t^p} \right),
\]

and choosing \( t = Cpe^6, p = 2 \log m \) we get

\[
P\left( \left| N_f(e^{\rho_m'}) - \log \sigma_f(e^{\rho_m'}) \right| > C \log^6 m \right) \leq \frac{1}{m^2}.
\]

Therefore, by the Borel-Cantelli lemma we have that a.s. for every \( m \geq m_0(\omega) \),

\[
\left| N_f(e^{\rho_m'}) - \log \sigma_f(e^{\rho_m'}) \right| \leq C \log^6 m. \tag{8.2}
\]

Now, since \( t \mapsto N_f(e^t) \) and \( t \mapsto \log \sigma_f(e^t) \) are convex functions, we get for \( \rho \in (\rho_m', \rho_{m+1}') \),

\[
n_f(e^\rho) \leq n_f(e^{\rho_{m+1}'}) \leq \frac{N_f(e^{\rho_{m+1}}) - N_f(e^{\rho_{m+1}'})}{\rho_{m+1} - \rho_{m+1}'} \leq \frac{\log \sigma_f(e^{\rho_{m+1}}) - \log \sigma_f(e^{\rho_{m+1}'})}{\rho_{m+1} - \rho_{m+1}'} + \frac{C \log^6 m}{\delta_m} \leq s_f(e^{\rho_{m+1}}) + \frac{C \log^6 m}{\delta_m}.
\]
In the other direction, using a similar argument, we get

\[ n_f(e^\rho) \geq s_f(e^\rho) \geq \frac{C \log^b m}{\delta_m}. \]

In addition, by the choice of the sequence \( \rho_m \),

\[ s_f(e^{\rho_{m+1}}) \leq s_f(e^{\rho_m}) \left(1 + \frac{C}{m}\right) \leq s_f(e^\rho) \left(1 + \frac{C}{m}\right), \]

and

\[ s_f(e^{\rho_m}) \geq s_f(e^{\rho_{m+1}}) \left(1 - \frac{C}{m}\right) \geq s_f(e^\rho) \left(1 - \frac{C}{m}\right). \]

Therefore, for \( \rho \in (\rho_m^+, \rho_{m+1}^-) \) we have

\[ |n_f(e^\rho) - s_f(e^\rho)| \leq \frac{C \cdot s_f(e^\rho)}{m} + \frac{C \log^b m}{\delta_m} m^{a-1} + C m^{a-1} \log^b m + \frac{C \log^b m}{\delta_m}. \]

The exceptional set is contained in \( \bigcup_m (\rho_m, \rho_m^+) \), and thus we have to require \( \sum \delta_m < \infty \). Hence, we have the conditions:

\[ C m^{a-1} \log^b m + \frac{C \log^b m}{\delta_m} = o \left(m^a \log^b m\right), \quad (8.3) \]

\[ \sum \delta_m < \infty, \quad (8.4) \]

and

\[ 0 \leq \delta_m \leq \frac{1}{2} \Delta_m. \]

### 8.3.3 Choice of the parameters

Recall that \( \sum \Delta_m = \infty \). We balance the requirement \( (8.3) \) by choosing

\[ \delta_m = \frac{C \log^{6-\beta} m}{m^{a-1}}. \]

Consequently, if we want to satisfy the requirement \( (8.4) \), we can select for example \( \alpha = 2 \) and \( \beta = 8 \). Now we must have

\[ \delta_m = \frac{C}{m \log^2 m} \leq \frac{1}{2} \Delta_m. \quad (8.5) \]
Almost sure convergence of the number of zeros

Notice that in principle $\Delta_m$ might be very small for a lot of values of $m$, but then we can discard the corresponding intervals! (that is, add them to the exceptional set). We denote by $E = \{m_k\}$ the sequence of indices such that following holds

$$\Delta_{m_k} < \frac{2C}{m_k \log^2 m_k},$$

and we add the sequence of intervals $[\rho_m, \rho_{m+1}]$, $m \in E$, to the exceptional set (the sum of their lengths is finite). For the rest we can choose $\delta_m$ so that $\delta_m$ holds (it is clear that there are infinitely many indices that satisfy (8.5), since $\sum \Delta_m = \infty$).

We conclude that for $\rho \in (\rho_m, \rho_{m+1})$, $m \notin E$, and $m \geq m_0(\omega)$, we have, by the choice of the sequence $\{\rho_m\}$,

$$|n_f (e^\rho) - s_f (e^\rho)| \leq \frac{C \cdot s_f (e^{\rho_m})}{m} \leq C \sqrt{s_f (e^{\rho_m}) \log^4 s_f (e^{\rho_m})}.$$

Finally, for $r \geq r_0 (\omega)$ and outside a set of finite logarithmic measure

$$|n_f (r) - s_f (r)| \leq C \sqrt{s_f (r) \log^4 s_f (r)}.$$

Remark. By a similar argument to the above, and using (8.2) (with a different choice of the sequence $\rho'_k$), one can prove that a.s., for $r \geq r_0 (\omega)$ we have

$$N_f (r) = \log \sigma_f (r) + O \left(\log^6 (\log \sigma_f (r))\right).$$

No exceptional set is required in this case.

8.4 On The Exceptional Set in Theorem [21]

In this section we construct an example which shows that the exceptional set in Theorem [21] cannot always be avoided. The idea is to construct a lacunary entire function $f$, so that the number of zeros ‘jumps’ at certain radii, while the function $s_f (r)$ increases more smoothly. This is an example which is essentially non-random.

Let $f$ be the following lacunary series

$$f (z) = \sum_{n \geq 0} \xi_n \frac{z^{2^n}}{\exp (2^n n)}, \quad (8.7)$$

with $\xi_n \in \{\pm 1\}$ (they need not be random). We thus have

$$\sigma_f^2 (r) = \sum_{n \geq 0} \frac{r^{2^{n+1}}}{\exp (2^{n+1} n)}.$$
and

\[ s_f (r) = \frac{d \log \sigma_f}{d \log r} = \frac{r}{2 \sigma_f^2 (r)} \cdot \sum_{n \geq 0} \frac{(2^{n+1}) r^{2n+1}}{\exp (2^{n+1} n)} \]

\[ = \frac{1}{\sigma_f^2 (r)} \cdot \sum_{n \geq 0} \frac{(2^n) r^{2n+1}}{\exp (2^{n+1} n)}. \]

We show that there exists a set \( E \subset [1, \infty) \) of infinite Lebesgue measure, such that for any choice of signs \( \xi_n \in \{ \pm 1 \} \), we have

\[ |n_f (r) - s_f (r)| \geq c s_f (r), \quad r \in E, \]

with some numerical constant \( c > 0 \).

To simplify notation let us write \( s = \log r \), and for \( |z| = r = e^s \)

\[ B_n (s) = \frac{r^{2^{n+1}}}{\exp (2^n n)} = \exp (2^n (s - n)), \]

\[ b_n (s) = \log B_n (s) = 2^n (s - n), \]

so now we have

\[ s_f (r) = \sum_{n \geq 0} \left[ 2^n B_n^2 (s) \right] / \sum_{n \geq 0} B_n^2 (s). \]

We note that for \( s \) which is near an integer \( k \in \mathbb{N} \) the largest terms in \([8,7]\) are those with indices \( k - 2 \) and \( k - 1 \). The rest we can put to the error term. We have

\[ b_{k-1} (s) = 2^{k-1} (s - k + 1), \]

\[ b_{k-2} (s) = 2^{k-2} (s - k + 2) = 2^{k-1} \cdot \frac{s - k + 2}{2}, \]

we also use the following notation for the sum of the remaining terms

\[ E = E (z) = \sum_{n \notin \{k-2,k-1\}} \xi_n \frac{z^{2n}}{\exp (2^n n)}, \]

\[ E' = E' (s) = \sum_{n \notin \{k-2,k-1\}} B_n^2 (s), \]

\[ E'' = E'' (s) = \sum_{n \notin \{k-2,k-1\}} 2^n B_n^2 (s). \]

We are going to find \( \delta = \delta (k) \) such that \( f (z) \) has at most \( 2^{k-2} \) zeros in \((k - \delta (k)) \mathbb{D} \) and \( f (z) \) has at least \( 2^{k-1} \) zeros in \((k + \delta (k)) \mathbb{D} \). The exceptional set will be around the points \( k - \delta (k) \) (too few zeros) and \( k + \delta (k) \) (too many zeros).

We start with a bound for the error terms, first for \( E \) and \( E' \).
Claim 22. Let $k \in \mathbb{N}$ be sufficiently large, and let $|z| = e^s$. If $|s - k| < \frac{1}{4}$, then

$$|E| \leq k \exp \left( \frac{s - k + 3}{4} \cdot 2^{k-1} \right),$$

$$E' \leq k \exp \left( \frac{s - k + 3}{4} \cdot 2^k \right).$$

**Proof.** The modulus of $E$ is bounded as follows

$$|E| \leq \sum_{0 \leq n \leq k-3} \exp (b_n (s)) + \sum_{n \geq k} \exp (b_n (s)) = S_1 + S_2.$$ 

We write $m = k - n$. For the first sum we have

$$S_1 = \sum_{3 \leq m \leq k} \exp \left( 2^k \cdot 2^{-m} (s - k + m) \right) \leq (k - 2) \exp \left( \frac{s - k + 3}{4} \cdot 2^{k-1} \right)$$

and for the second one

$$S_2 = \sum_{m \leq 0} \exp \left( 2^k \cdot 2^{-m} (s - k + m) \right) = \sum_{l \geq 0} \exp \left( 2^k \cdot 2^l (s - k - l) \right) = (\star).$$

Now, if $|s - k| < 1$ we have

$$\star \leq \exp \left( 2^k (s - k) \right) + \sum_{l \geq 1} \exp \left( 2^k \cdot 2 (s - k - l) \right) \leq \exp \left( 2^k (s - k) \right) + 2$$

$$\leq 2 \exp \left( 2 (s - k) \cdot 2^{k-1} \right).$$

Since for $|s - k| < \frac{1}{4}$ we have $\frac{s - k + 3}{4} > 2 (s - k)$ we get the required result for $E$. It is clear that exactly the same method leads to the estimate for $E'$. \hfill \square

We now need similar estimates for the error term in the numerator of $s_f (r)$.

Claim 23. Let $k \in \mathbb{N}$ be sufficiently large. If $|s - k| < \frac{1}{4}$, then

$$E'' \leq 4^{k+1} \exp \left( \frac{s - k + 3}{4} \cdot 2^k \right).$$

**Proof.** Let us write

$$E'' = \sum_{0 \leq n \leq k-3} 2^n B_n^2 (s) + \sum_{n \geq k} 2^n B_n^2 (s) = S_1 + S_2,$$
and again write \( m = k - n \). Thus we have the following upper bounds

\[
S_1 = \sum_{3 \leq m \leq k} 2^{k-m} \exp\left(2^{k+1} \cdot 2^{-m} (s - k + m)\right) \leq 4^k \exp\left(\frac{s - k + 3}{4} \cdot 2^k\right),
\]

and similarly

\[
\begin{align*}
S_2 &= \sum_{m \leq 0} 2^{k-m} \exp\left(2^{k+1} \cdot 2^{-m} (s - k + m)\right) \\
&= \sum_{l \geq 0} 2^{k+l} \exp\left(2^{k+1} \cdot 2^l (s - k - l)\right) \\
&\leq 2^k \exp\left(2^{k+1} (s - k)\right) + \sum_{l \geq 1} 2^{k+l} \exp\left(2^{k+1} \cdot 2 \left(\frac{1}{4} - l\right)\right) \\
&\leq 2^{k+1} \exp\left(2 (s - k) \cdot 2^k\right) + 2^{2+k} \exp\left(-3 \cdot 2^{k+2}\right) \\
&\leq 2^{k+2} \exp\left(2 (s - k) \cdot 2^k\right).
\end{align*}
\]

Again using the fact that for \(|s - k| < \frac{1}{4}\) we have that \(\frac{s-k+3}{4} > 2 (s - k)\) we get the required estimate for \(E''\).

The conclusion of these claims is that if \(|z| = e^s, |s - k| < \frac{1}{4}\), then

\[
f(z) = \xi_{k-2} \frac{z^{2k-2}}{\exp(2^{k-2} (k - 2))} + \xi_{k-1} \frac{z^{2k-1}}{\exp(2^{k-1} (k - 1))} + O \left( k \cdot \exp\left(\frac{s - k + 3}{4} \cdot 2^{k-1}\right)\right),
\]

and

\[
s_f(e^s) = \frac{2^{k-2}B_{k-2}^2(s) + 2^{k-1}B_{k-1}^2(s) + O\left(4^{k+1} \exp\left(\frac{s-k+3}{4} \cdot 2^k\right)\right)}{B_{k-2}^2(s) + B_{k-1}^2(s) + O \left( k \cdot \exp\left(\frac{s-k+3}{4} \cdot 2^k\right)\right)}.
\]

Now let us set \(\delta_k = 2^{2-k} \log 2\). The following claims show that \(n_f(r)\) has a ‘jump’ somewhere between the points \(s = k - \delta_k\) and \(s = k + \delta_k\), while the function \(s_f(r)\) increases more smoothly. Therefore, there is some discrepancy between these functions near those points.

**Claim 24.** Let \(k \in \mathbb{N}\) be sufficiently large. We have

\[
n_f(e^s) = \begin{cases}
2^{k-2}, & s \in (k - 2\delta_k, k - \delta_k), \\
2^{k-1}, & s \in (k + \delta_k, k + 2\delta_k).
\end{cases}
\]
Proof. We will consider only the case \( s \in (k - 2\delta_k, k - \delta_k) \), the second case being similar. By Rouché’s Theorem it is sufficient to show that

\[
\left| \xi_{k-2} \frac{z^{2k-2}}{\exp(2^{k-2}(k-2))} \right| > \left| \xi_{k-1} \frac{z^{2k-1}}{\exp(2^{k-1}(k-1))} \right| + |E|,
\]

for \( |z| = e^s \), which in our notation translates to the inequality

\[
B_{k-2}(s) > B_{k-1}(s) + |E|,
\]

which is equivalent to

\[
\exp(b_{k-2}(s)) > \exp(b_{k-1}(s)) + |E|.
\]

Let us write \( s = k - \delta \), for some \( \delta \in (\delta_k, 2\delta_k) \). Thus,

\[
b_n(s) = 2^n (s - n) = 2^n (k - n - \delta),
\]

and so

\[
b_{k-2}(s) - b_{k-1}(s) = 2^{k-2} (2 - \delta) - 2^{k-1} (1 - \delta),
\]

\[
= 2^{k-2} \delta.
\]

By our choice of \( \delta_k \) it now follows that

\[
B_{k-2}(s) \geq 2B_{k-1}(s), \quad s \in (k - 2\delta_k, k - \delta_k),
\]

we also note that for all such \( s \) we have

\[
B_{k-2}(s) \geq \exp \left( 2^{k-2}(2 - 2\delta_k) \right) = \frac{1}{4} \exp \left( 2^{k-1} \right).
\]

Now, by Claim [22] for any \( s \in (k - 2\delta_k, k - \delta_k) \) we have for \( k \) sufficiently large

\[
|E| \leq k \cdot \exp \left( \frac{s - k + 3}{4} \cdot 2^{k-1} \right) \leq k \cdot \exp \left( \frac{3}{4} \cdot 2^{k-1} \right) < \frac{1}{2} B_{k-2}(s).
\]

This concludes the proof of the claim. \( \square \)

On the other hand
Claim 25. If \( k \in \mathbb{N} \) is sufficiently large, then
\[
s_f(e^s) \geq \frac{18}{17} \cdot 2^{k-2} (1 + o(1)), \quad s \in (k - 2\delta_k, k - \delta_k)
\]
and
\[
s_f(e^s) \leq \frac{33}{34} \cdot 2^{k-1} (1 + o(1)), \quad s \in (k + \delta_k, k + 2\delta_k).
\]

Proof. We again consider only the case \( s \in (k - 2\delta_k, k - \delta_k) \). Note that
\[
        b_{k-2} (k - 2\delta_k) - b_{k-1} (k - 2\delta_k) = 2^{k-2} (2\delta_k) = 2 \log 2,
\]
and thus
\[
4B_{k-1} (k - 2\delta_k) = B_{k-2} (k - 2\delta_k),
\]
in addition,
\[
B_{k-2} (k - 2\delta_k) = \exp \left( 2^{k-2} (2 - 2\delta_k) \right) = \frac{1}{4} \exp \left( 2^{k-1} \right).
\]
By combining the estimates for the error terms from Claim 22 and Claim 23 we get
\[
s_f(e^{k-2\delta_k}) = \frac{2^{k-2}B_{k-2}^2 (k - 2\delta_k) + 2^{k-1}B_{k-1}^2 (k - 2\delta_k) + O \left( k \exp \left( \frac{3}{4} \cdot 2^k \right) \right)}{B_{k-2}^2 (k - 2\delta_k) + B_{k-1}^2 (k - 2\delta_k) + O \left( 4^{k+1} \exp \left( \frac{3}{4} \cdot 2^k \right) \right)}
\]
\[
= \frac{2^{k-2} \cdot 16 + 2^{k-1} + o(1)}{16 + 1 + o(1)} = \frac{18}{17} \cdot 2^{k-2} (1 + o(1)).
\]
Since the function \( s_f(r) \) is increasing we finished the proof of this claim. \( \Box \)

This concludes the example for the necessity of an exceptional set.

Remark. It should be mentioned that the Lebesgue measure of the exceptional set is infinite. This follows from the fact that the series
\[
\sum_{k \geq 1} [\exp (k + 2\delta_k) - \exp (k + \delta_k)], \quad \sum_{k \geq 1} [\exp (k - \delta_k) - \exp (k - 2\delta_k)],
\]
are both divergent (in fact their terms tend to infinity).
9 Equidistribution of the zeros - Proof of Theorem \textbf{1.2}

In this part we are interested in employing the tools developed in the previous part of the work to study in more detail the distribution of the zeros of Rademacher entire functions

\[ F(z) = \sum_{n \geq 0} \xi_n a_n z^n. \]

By our knowledge on the behavior of GAFs, and specifically by the Edelman-Kostlan formula (1.1), we expect that the function

\[ \sigma_F^2(r) = \mathcal{E}\{ |F(re^{i\theta})|^2 \} = \sum_{n \geq 0} |a_n|^2 r^{2n}, \]

and its logarithmic derivative

\[ s_F(r) = \frac{d \log \sigma_F(r)}{d \log r}, \]

will determine the rate of growth of the number of zeros of \( F \), and this is confirmed by the results of the previous section. In the same way as we did in the Introduction, let us denote by \( n_F(r, \alpha, \beta) \) the number of zeros of \( F \) in the sector \( \{ z : \alpha \leq \arg z < \beta, |z| \leq r \} \), counted with multiplicities. Then the integrated counting function is given by

\[ N_F(r, \alpha, \beta) = \int_0^r \frac{n_F(r, \alpha, \beta)}{t} \, dt. \]

In the pioneering paper \cite{LO2}, Littlewood and Offord considered Rademacher Taylor series that represent entire functions of a finite and positive order of growth. That is, they assumed that

\[ 0 < \limsup_{n \to \infty} \frac{n \log n}{\log(1/|a_n|)} < \infty, \]

which is equivalent to

\[ 0 < \limsup_{r \to \infty} \frac{\log \log \sigma_F(r)}{\log r} < \infty. \]

Under this assumption, they discovered that, for every \( \varepsilon > 0 \), a.s.,

\[ \log |F(re^{i\theta})| \geq \log \max_{n \geq 0} \left( |a_n|r^n \right) - O(r^\varepsilon) \quad (9.1) \]

everywhere in the plane, except for the union of simply connected domains of small diameter, in the interior of which the function \( F \) has one or more zeros, and on
the boundary of which \( |F| \) is a constant. They called these domains ‘pits’. Later, Offord \([Of1, Of4]\) extended this result to random Taylor series with rather general independent coefficients that represent entire functions of positive or infinite order of growth. Other results obtained by Littlewood and Offord provided additional information about the size of the pits and their location (see \([LO2, \text{Theorem 4}]\)). However, it appears that they could not deduce from (9.1) the asymptotics for the counting function 

\[
N_F(r, \alpha, \beta)
\]

similar to the one we obtain in this part of the work.

In the next section we will use a more general form of Jensen’s formula to show that, asymptotically, the size of \( N_F(r, \alpha, \beta) \) is proportional to \( \frac{\beta - \alpha}{2\pi} \pi \), and that, actually, the error can be bounded uniformly. This is the conclusion of the first result:

\[
\text{Suppose that } a > 2 \text{ and } \gamma \in \left( \frac{1}{2} + \frac{1}{a}, 1 \right). \text{ Then, a.s. and in mean, when } r \to \infty \text{ and } \log \sigma_F(r) > \log^a r, \]

\[
\sup_{0 \leq \alpha < \beta \leq 2\pi} \left| N_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r) \right| = O \left( \left( \log \sigma_F(r) \right)^\gamma \right).
\]

We then ‘differentiate’ the above estimate (in a similar spirit to the proof of Theorem \([21]\)) to get the following asymptotic estimate for \( n_F(r, \alpha, \beta) \)

\[
\text{Suppose that } a > 4 \text{ and } \gamma \in \left( \frac{3}{4} + \frac{1}{a}, 1 \right). \text{ Then there exists a set } E \subset [1, \infty) \text{ of finite logarithmic measure such that, a.s. and in mean,}
\]

\[
\sup_{0 \leq \alpha < \beta \leq 2\pi} \left| n_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} s_F(r) \right| = O \left( \left( s_F(r) \right)^\gamma \right)
\]

when \( r \to \infty \), \( r \notin E \) and \( \log \sigma_F(r) > \log^a r \).

We mention here again that some lower bound on the growth of \( \sigma_F \) is necessary, as we will show in Section \([10]\) that certain functions have only real zeros (deterministically).

**Regular coefficients** \( a_n \)

It is also worth mentioning that if the coefficients \( a_n \) have a very regular behaviour, namely, for some \( \rho > 0 \) and \( \Delta > 0 \),

\[
a_n = (\Delta + o(1))^n e^{-\frac{\rho}{\Delta} \log n} \quad \text{as } n \to \infty,
\]

then the results of Theorem \([3]\) follow from the lower bound (9.1) of Littlewood and Offord combined with some classical results on entire functions of completely regular growth (see \([Le1, \text{Chapter III}]\)). In this case, the exceptional set \( E \subset [1, \infty) \) in the statement of the theorem is not needed. In a recent work \([KZ]\), Kabluchko and...
Zaporozhets gave an elegant proof of similar results for the same class of very regular non-random sequences \( a_n \) and for arbitrary i.i.d. random variables \( \xi_k \) satisfying \( \mathbb{E} \{ \log^+ |\xi| \} < \infty \). Their proof relies upon estimates of a concentration function combined with some potential theory. Apparently, it would not work for sequences \( a_n \), that do not behave very regular.

**Equidistribution for zeros of polynomials**

Let us denote (only for the purpose of this explanation) the number of zeros of the polynomial \( g_m(z) = \sum_{k=0}^{m} c_k z^k \) inside the sector \( 0 \leq \alpha < \arg z < \beta \leq 2\pi \) by \( n(\alpha, \beta) \). Erdős and Turán considered the uniform distribution of the zeros of \( g_m \). They showed in [ET] that

\[
\left| n(\alpha, \beta) - \frac{(\beta - \alpha) m}{2\pi} \right| < 16 \sqrt{m \cdot \log \left( \frac{\sum_{k=0}^{m} |c_k|}{\sum |c_0 c_m|} \right)}, \quad \forall 0 \leq \alpha < \beta \leq 2\pi,
\]

which, for example, implies the equidistribution of the zeros in the case where all the coefficients are bounded: \( c < |c_k| < C \). Unfortunately, it seems that this result cannot be used to prove the equidistribution of the zeros of entire functions. The key difference is that in [ET], the limiting measure of the zero counting measure is \( m_T \), the Lebesgue measure on the circle. In that case equidistribution can be phrased in terms of the absolute values \( |c_k| \). In other cases (such as \( m_D \)), this is not the case. In general, for entire functions the arguments of the coefficients are important for determining the equidistribution of the zeros.

### 9.1 Proof of Theorem 3

In the rest of this section we will prove Theorem 3 by estimating the ‘error functions’

\[
E_F(r) = \sup_{0 \leq \alpha < \beta \leq 2\pi} \left| N_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r) \right|
\]

\[
e_F(r) = \sup_{0 \leq \alpha < \beta \leq 2\pi} \left| n_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} s_F(r) \right|
\]

The proof starts with a Jensen-type estimate for \(|E_F(r)|\). This part uses some arguments that are customary in the theory of entire functions, and is non-probabilistic. Then, combining the Jensen-type estimate with the log-integrability of Rademacher Fourier series, we estimate the error function \( E_F \), first in mean, and then almost surely. We proceed with a growth lemma about functions of a real variable. This
lemma is a version of the classical Borel-Nevanlinna lemma, which is widely used in the theory of entire and meromorphic functions. This part is also non-probabilistic. At last, applying the growth lemma, we ‘differentiate’ estimates for $E_F(r)$ and obtain their counterparts for $e_F(r)$.

**9.2 Jensen-type estimate**

In this section

$$G(z) = \sum_{n \geq 0} g_n z^n$$

is a non-random analytic function in the closed disk $\{ |z| \leq r \}$ with $r \geq 2$. We assume that $|G(0)| = 1$ and fix the value of $\arg G(0)$. If $G$ does not vanish on the segment $\{ z = te^{i\theta} : 0 \leq t \leq r \}$, then we take a continuous branch of $\arg G(te^{i\theta})$ and let

$$v(t, \theta) = \arg G(te^{i\theta}) - \arg G(0).$$

By $A_G$ we denote various positive constants, which depend only on the sequence of absolute values $|g_n|$ of the Taylor coefficients of $G$.

Put

$$\sigma^2_G(t) = \sum_{n \geq 0} |g_n|^2 t^{2n}, \quad L_G(t, \theta) = \log |G(te^{i\theta})| = \hat{L}_G(t, \theta) + \log \sigma_G(t),$$

and let

$$L^*_G(t, \delta) = \sup_{m(t)=\delta} \int_I |\hat{L}_G(t, \theta)| \, dm(\theta),$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ of length $\delta$.

**Lemma 26.** [Jensen-type estimate] For every $r \geq 2$ and every $\delta \in (0, 2\pi]$, we have

$$\sup_{0 \leq \alpha < \beta \leq 2\pi} \left| N_G(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_G(r) \right| \lesssim \delta \log \sigma_G(r) + \frac{1}{\delta^2} \int_1^r L^*_G(t, \delta) \frac{\log(r)}{t} \, dt + \int_{\mathbb{T}} |\hat{L}_G(t, \theta)| \, dm(\theta) + A_G \log r.$$

Our starting point is a Jensen-type integral formula [Le1 Chapter III, Eq. (3.04)], which is a straightforward consequence of the argument principle and the Cauchy-Riemann equations.
Lemma 27. [Jensen-type integral formula] Suppose the function $G$ does not vanish on the segments $\{z = te^{i\alpha} : 0 \leq t \leq r\}$ and $\{z = te^{i\beta} : 0 \leq t \leq r\}$. Then

$$N_G(r, \alpha, \beta) = \frac{1}{2\pi} \int_{\alpha}^{\beta} L_G(r, \theta) \, d\theta + \frac{1}{2\pi} \int_{0}^{r} \frac{v(t, \alpha) - v(t, \beta)}{t} \, dt.$$

Proof of Lemma 26: First, we observe that in order to prove Lemma 26, it suffices to prove the upper bound

$$N_G(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_G(r) \lesssim \delta \log \sigma_G(r) + \frac{1}{\delta} \int_{1}^{r} L_G^*(t, \delta) \log \left( \frac{r}{t} \right) \frac{dt}{t} + \int_{1}^{\hat{L}_G(t, \theta)} \, dm(\theta) + A_G \log r,$$

uniformly with respect to $\alpha$ and $\beta$. Indeed, having (9.2), observing that

$$N_G(r, 0, 2\pi) = \int_{1}^{\hat{L}_G(t, \theta)} \log |G(re^{i\theta})| \, dm(\theta),$$

we get the lower bound for $N(r, \alpha, \beta)$, which matches the upper one.

By Lemma 27

$$N_G(r, \alpha, \beta) - N_G(1, \alpha, \beta) = \frac{1}{2\pi} \int_{\alpha}^{\beta} L_G(r, \theta) \, d\theta - \frac{1}{2\pi} \int_{\alpha}^{\beta} L_G(1, \theta) \, d\theta + \frac{1}{2\pi} \int_{1}^{r} \frac{v(t, \alpha) - v(t, \beta)}{t} \, dt.$$

Once again, using the classical Jensen formula (and recalling that $|G(0)| = 1$), we see that the terms $N_G(1, \alpha, \beta)$ and $\int_{1}^{\hat{L}_G(1, \theta)} L_G(1, \theta) \, dm(\theta)$ are bounded by $C \cdot \log \sigma_G(2)$, whence

$$N_G(r, \alpha, \beta) \leq \frac{1}{2\pi} \int_{\alpha}^{\beta} L_G(r, \theta) \, d\theta + \frac{1}{2\pi} \int_{1}^{r} \frac{v(t, \alpha) - v(t, \beta)}{t} \, dt + A_G.$$

By the Cauchy-Riemann equations, for $t \geq 1$ we have

$$v(t, \theta) = v(1, \theta) - \frac{d}{d\theta} \int_{1}^{t} L_G(s, \theta) \, ds,$$
whence
\[
\frac{1}{2\pi} \int_1^r \frac{v(t, \alpha) - v(t, \beta)}{t} \, dt = \left[ v(1, \alpha) - v(1, \beta) \right] \log r + \frac{1}{2\pi} \left[ \frac{d}{d\theta} \int_1^r L_G(t, \theta) \log \left( \frac{r}{t} \right) \frac{dt}{t} \right]_\alpha^\beta.
\]

By a standard complex analysis estimate (see, for instance, [Le1, Lemma 6, Chapter VI]), for every \( \theta \) such that \( G \) does not vanish on the segment \( \{te^{i\theta} : 0 \leq t \leq 1\} \), we have \( |v(1, \theta)| \leq A_G \). Therefore,

\[
N_G(r, \alpha, \beta) \leq \frac{1}{2\pi} \int_\alpha^\beta L_G(r, \theta) \, d\theta + \frac{1}{2\pi} \left[ \frac{d}{d\theta} \int_1^r L_G(t, \theta) \log \left( \frac{r}{t} \right) \frac{dt}{t} \right]_\alpha^\beta + A_G \log r. \quad (9.3)
\]

Given \( \delta \in (0, 2\pi] \), we have

\[
N_G(r, \alpha, \beta) \leq \frac{1}{\delta} \int_0^\delta N_G(r, \alpha - \phi, \beta + \phi) \, d\phi.
\]

Without loss of generality we assume that the function \( G \) does not vanish on the unit circle \( \mathbb{T} \). Then the derivative

\[
\frac{d}{d\theta} \int_1^r L_G(t, \theta) \log \left( \frac{r}{t} \right) \frac{dt}{t} = \Re \left\{ ie^{i\theta} \int_1^r \frac{G'(te^{i\theta})}{G(te^{i\theta})} \log \left( \frac{r}{t} \right) \frac{dt}{t} \right\}
\]

is a bounded function of \( \theta \), which may have at most finitely many points of discontinuity\(^7\). Therefore, we can apply the Newton-Leibnitz formula when averaging the RHS of inequality (9.3). We get

\[
N_G(r, \alpha, \beta) \leq \frac{1}{\delta} \int_0^\delta N_G(r, \alpha - \phi, \beta + \phi) \, d\phi.
\]

\(^7\) This follows from the fact that, for \( r > 1 \) and \( 1 < \rho \leq r \),

\[
\phi \mapsto \int_1^r \frac{1}{t - \rho e^{i\phi}} \log \left( \frac{r}{t} \right) \frac{dt}{t}
\]

is a bounded function continuous everywhere except at the point \( \phi = 0 \).
Applying the same average one more time, we get

\[
N_G(r, \alpha, \beta) \leq \frac{1}{\delta^2} \int_0^\delta d\phi_1 \int_0^\delta d\phi \frac{1}{2\pi} \int_{\alpha-\phi_1-\phi}^{\beta+\phi_1+\phi} L_G(r, \theta) d\theta + A_G \log r
+ \frac{1}{\delta^2} \frac{1}{2\pi} \int_0^r \log\left(\frac{r}{t}\right) \frac{dt}{t} \left[\int_{\beta+\delta}^{\beta+2\delta} - \int_{\beta}^{\beta+\delta} + \int_{\alpha-2\delta}^{\alpha-\delta} - \int_{\alpha-\delta}\right] L_G(t, \theta) d\theta
\leq \frac{1}{\delta^2} \int_0^\delta d\phi_1 \int_0^\delta d\phi \frac{\beta - \alpha + 2\phi_1 + 2\phi}{2\pi} \log \sigma_G(r) + A_G \log r
+ \frac{1}{\delta^2} \frac{1}{2\pi} \int_0^\delta d\phi_1 \int_0^\delta d\phi \frac{1}{2\pi} \int_{\alpha-\phi_1-\phi}^{\beta+\phi_1+\phi} \hat{L}_G(r, \theta) d\theta
+ \frac{1}{\delta^2} \frac{1}{2\pi} \int_0^r 4L_G^*(t, \delta) \log\left(\frac{r}{t}\right) \frac{dt}{t}
\leq \frac{\beta - \alpha}{2\pi} \log \sigma_G(r) + \text{RHS of (9.2)},
\]

completing the proof of Lemma 26. \(\square\)

### 9.3 Proof of the first part of Theorem 3

In what follows, we denote by \(A\) various positive constants depending on the absolute values of the non-random coefficients \(|a_n|\) in the Taylor expansion of \(F(z)\).

#### 9.3.1 Mean estimate

We start with an estimate for the random maximal function

\[
L_F^*(t, \delta) = \sup \left\{ \int_I |\hat{L}_G(t, \theta)| dm(\theta) : I \subset \mathbb{T} \text{ is an arc, } m(I) = \delta \right\}.
\]

**Claim 28.** (i) For every \(t \geq 1\) and every \(0 < \delta \leq 2\pi\), \(E\{L_F^*(t, \delta)\} \lesssim \delta \log^6(C\delta^{-1})\).

(ii) For every \(p \geq 1\), every \(0 < \delta \leq 2\pi\), and every \(r \geq 2\),

\[
\left\| \int_1^r L_F^*(t, \delta) \log\left(\frac{r}{t}\right) \frac{dt}{t} \right\|_{L^p(\Omega)} \leq C p^6 \cdot \delta^{1-\frac{1}{p}} \cdot \log^2 r.
\]

**Proof.** Fix arbitrary \(p \geq 1\), \(\delta \leq 1\), and \(t \geq 1\). Then, for every arc \(I \subset \mathbb{T}\) of measure \(\delta\), we have

\[
E\left( \int_I |\hat{L}_F(t, \theta)| dm(\theta) \right)^p \leq E\left( \int_{\mathbb{T}} |\hat{L}_F(t, \theta)|^p dm(\theta) \cdot |I|^{p-1} \right) \leq (Cp)^6 p \delta^{p-1}.
\]
This follows from Hölder’s inequality combined with the log-integrability of Rademacher Fourier series (Theorem 1). Therefore,

$$\|L_F^*(t, \delta)\|_{L^p(\Omega)} \leq C p^6 \delta^{1-\frac{1}{p}}.$$ 

Letting $p = \log(C/\delta)$, we get estimate (i).

Estimate (ii) follows by application of the integral Minkowski inequality:

$$\left\| \int_1^r L_F^*(t, \delta) \log \left( \frac{r}{t} \right) \frac{dt}{t} \right\|_{L^p(\Omega)} \leq \int_1^r \left\| L_F^*(t, \delta) \right\|_{L^p(\Omega)} \log \left( \frac{r}{t} \right) \frac{dt}{t} \leq (i) \leq C p^6 \delta^{1-\frac{1}{p}} \cdot \log^2 r,$$

completing the proof.

Now, combining Lemma 29 with Theorem 1 and using the previous claim, we get

$$\mathcal{E}\{E_F(r)\} \lesssim \delta \log \sigma_F(r) + \frac{1}{\delta^2} \int_1^r \mathcal{E}\{L_F^*(t, \delta)\} \log \left( \frac{r}{t} \right) \frac{dt}{t} + A \log r \lesssim \delta \log \sigma_F(r) + \frac{1}{\delta} \log^6 \left( \frac{C}{\delta} \right) \cdot \log^2 r + A \log r.$$

Choosing $\delta$ that balances the RHS and recalling that $\log \sigma_F(r) \geq \log^a r$ with some $a > 2$, we obtain

$$\mathcal{E}\{E_F(r)\} \leq A \left( \log \sigma_F(r) \right)^{\frac{1}{2}} \cdot \left( \log \log \sigma_F(r) \right)^{3} \cdot \log r,$$

which does not exceed $A \log^\gamma \sigma_F(r)$ with $\frac{1}{2} + \frac{1}{a} < \gamma < 1$ and $r$ sufficiently large. □

### 9.3.2 Almost sure estimate

The proof of the a.s. estimate uses an $L^p(\Omega)$-estimate of $E_F(r)$ with large values of $p$. Put

$$I_p(r) = \left( \log \sigma_F(r) \right)^{\frac{p+1}{2p+1}} \cdot \left( \log^2 r \right)^{\frac{p}{2p+1}}.$$

**Lemma 29.** For every $p \geq 1$ and every $r \geq 2$ with $\log \sigma_F(r) \geq \log^2 r$, we have

$$\|E_F(r)\|_{L^p(\Omega)} \leq A p^6 \cdot I_p(r).$$

(9.5)
Proof. Combining Lemma 26, Theorem 1 and Claim 28, we get

$$\| E_F(r) \|_{L^p(\Omega)} \lesssim \delta \log \sigma_F(r) + \frac{1}{\delta^2} \left\| \int_1^r L_F^*(t, \delta) \log \left( \frac{r}{t} \right) \frac{dt}{t} \right\|_{L^p(\Omega)} + p^6 + A \log r \tag{9.5}\]$$

To balance the RHS, we let

$$\delta = \left( \frac{p^6 \log^2 r}{\log \sigma_F(r)} \right)^{\frac{1}{2p+1}};$$

this choice is possible provided that $\log \sigma_F(r) \geq p^6 \cdot \log^2 r$. This gives us

$$\| E_F(r) \|_{L^p(\Omega)} \leq A p^6 \cdot I_p(r).$$

If $\log \sigma_F(r) \leq p^6 \log^2 r$, we let $\delta = 1$. In this case,

$$\| E_F(r) \|_{L^p(\Omega)} \leq A p^6 \log^2 r,$$

which is again consistent with the RHS of (9.5). This proves Lemma 29. \(\square\)

Now we are ready to prove the a.s. estimate for $E_F(r)$. Using Chebyshev’s inequality and Lemma 29, for every $T > 0$, we get

$$\mathcal{P}\{ E_F(r) > T \} \leq T^{-p} \| E_F(r) \|_{L^p(\Omega)}^p \leq T^{-p} \left( A p^6 \cdot I_p(r) \right)^p,$$

whence, for every $p \geq 1$ and every $r \geq 2$,

$$\mathcal{P}\{ E_F(r) > e \cdot A p^6 \cdot I_p(r) \} \leq e^{-p}.$$

For sufficiently large $m \in \mathbb{N}$, we let $p_m = 2 \log m$, and choose $r_m$ so that $\log \sigma_F(r_m) = m$. Then

$$\mathcal{P}\{ E_F(r_m) \geq A \log^6 m \cdot I_{p_m}(r_m) \} \leq \frac{1}{m^2}.$$

Therefore, by the Borel-Cantelli lemma, there exists an event $\Omega_0 \subset \Omega$ of probability zero such that, for each $\omega \in \Omega \setminus \Omega_0$, there exists $m_0(\omega) \in \mathbb{N}$ with the following property:

$$E_F(r_m) \leq A I_{p_m}(r_m) \cdot \log^6 m, \quad m \geq m_0(\omega). \tag{9.6}$$
Fix \( \omega \in \Omega \setminus \Omega_0 \), and assume that \( r_m \leq r \leq r_{m+1} \) and \( m \geq m_0(\omega) \). For each \( 0 \leq \alpha < \beta \leq 2\pi \), we have

\[
N_F(r_m, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r_m) - 1 \leq N_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r) \leq N_F(r_{m+1}, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r_{m+1}) + 1.
\]

Therefore,

\[
E_F(r) \leq \max\{E_F(r_m), E_F(r_{m+1})\} + 1 \leq A \max\{I_{p_m}(r_m), I_{p_{m+1}}(r_{m+1})\} \cdot (\log \log \sigma_F(r_m))^6.
\]

Note that

\[
I_{p_m}(r_m) = \left(\log \sigma_F(r_m)\right)^{2 \log \log r_m + 1} \cdot \left(\log^2 r_m\right)^{2 \log \log r_m + 1} = m \frac{1}{\log \log r_m + 1} \cdot \left(\log \sigma_F(r_m)\right)^{\frac{1}{2}} \cdot \left(\log^2 r_m\right)^{1 - \frac{1}{\log \log r_m + 1}} \lesssim \left(\log \sigma_F(r_m)\right)^{\frac{1}{2}} \log r_m.
\]

Due to the convexity of the function \( t \mapsto \log \sigma_F(e^t) \), we have

\[
\frac{\log \sigma_F(r_m) - \log \sigma_F(1)}{\log r_m} \leq \frac{\log \sigma(r_{m+1}) - \log \sigma(r_m)}{\log r_{m+1} - \log r_m},
\]

that is, \((m - c)(\log r_{m+1} - \log r_m) \leq \log r_m\), whence \( \log r_{m+1} \leq 2 \log r_m \), at least when \( m \) is large enough. Hence, \( I_{p_{m+1}}(r_{m+1}) \lesssim \left(\log \sigma_F(r_m)\right)^{\frac{1}{2}} \log r_m \). Recalling that \( \log \sigma_F(r) > \log^a r \), we obtain

\[
E_F(r) \leq A \left(\log \sigma_F(r)\right)^{\frac{1}{2}} \cdot (\log \log \sigma_F(r))^6 \cdot \log r \leq A \log^\gamma \sigma_F(r), \quad (9.7)
\]

as above, with \( \frac{1}{2} + \frac{1}{a} < \gamma < 1 \). This concludes the proof of the first part of Theorem 3.

\[\square\]

### 9.4 A growth lemma

Given \( b > 1 \), a continuous non-decreasing function \( \Phi : [1, \infty) \to (1, \infty) \), \( \lim_{r \to \infty} \Phi(r) = \infty \), and a continuous non-increasing function \( \varepsilon : [1, \infty) \to (0, 1] \), we define \( X \subset [1, \infty) \) as the set of \( r \)'s for which

\[
(1 - \varepsilon(r)) \Phi(r) < \Phi\left((1 + \varepsilon(r) \log^{-b} \Phi(r))^{-1} r\right) \leq \Phi\left((1 + \varepsilon(r) \log^{-b} \Phi(r))r\right) < (1 + \varepsilon(r)) \Phi(r).
\]
Lemma 30. The complement $[1, \infty) \setminus X$ has finite logarithmic measure.

Proof. Letting

$$E = \left\{ r \in [1, \infty) : \Phi \left( r + r \varepsilon(r) \log^{-b} \Phi(r) \right) \geq (1 + \varepsilon(r)) \Phi(r) \right\},$$

we show that the set $E$ has finite logarithmic measure. This gives half of the statement. The proof of the other half is very similar and we skip it. We suppose that the set $E$ is non-empty and unbounded, otherwise, there is nothing to prove. We construct inductively sequences $r_n < r'_n < r_{n+1}$ such that $E$ is contained in the union of the intervals $[r_n, r'_n]$, and then show that the series $\sum_n (r'_n - r_n)/r_n$ converges.

Let $r_1 = \inf \{ r : r \in E \}$. Since $E$ is closed, $r_1 \in E$. Suppose that the values $r_1, \ldots, r_n$ have already been defined. Put

$$r'_n = \left(1 + \varepsilon(r_n) \log^{-b} \Phi(r_n) \right) \cdot r_n, \quad r_{n+1} = \inf \left\{ r \geq r'_n : r \in E \right\},$$

and let

$$P_{n+1} = \sum_{1 \leq j \leq n} \log \left(1 + \varepsilon(r_j)\right) \leq \sum_{1 \leq j \leq n} \varepsilon(r_j).$$

Note that $r_n \uparrow \infty$. Otherwise, $r_n \uparrow r^*$, and then also $r'_n \uparrow r^*$. Hence,

$$\varepsilon(r_n) \log^{-b} \Phi(r_n)r_n = r'_n - r_n \to 0.$$

On the other hand, by the continuity of $\Phi$ and $\varepsilon$, the LHS converges to $\varepsilon(r^*) \log^{-b} \Phi(r^*)r^*$, which cannot be zero. By construction, $E \subset \bigcup_{n \geq 1} [r_n, r'_n]$.

Next,

$$\Phi(r_{n+1}) \geq \Phi(r'_n) = \Phi \left( r_n + \varepsilon(r_n) \log^{-b} \Phi(r_n)r_n \right)_{r_n \in E} \geq \left(1 + \varepsilon(r_n)\right) \Phi(r_n) \geq \cdots \geq \prod_{1 \leq j \leq n} \left(1 + \varepsilon(r_j)\right) \cdot \Phi(r_1) \geq e^{P_{n+1}}.$$

Using this estimate, we get

$$\int_E \frac{dr}{r} \leq \sum_{n \geq 1} \frac{r'_n - r_n}{r_n} = \sum_{n \geq 1} \varepsilon(r_n) \log^{-b} \Phi(r_n) \leq \sum_{n \geq 1} \varepsilon(r_n) P^{-b}. $$

Now we consider two cases. If the $P_n$’s converge to a finite limit, then, by the definition of that sequence, the series $\sum_n \varepsilon(r_n)$ also converges, and therefore, the set
9 Equidistribution of the zeros - Proof of Theorem 1.2

$E$ has finite logarithmic measure. If the $P_n$'s increase to $\infty$, then we note that, for $n$ large enough,

$$P_n^b = \left( \sum_{1 \leq j \leq n-1} \log \left( 1 + \varepsilon(r_j) \right) \right)^b \geq c_b \left( \sum_{1 \leq j \leq n} \varepsilon(r_j) \right)^b.$$  

Since $b > 1$, the series

$$\sum_{n \geq 1} \varepsilon(r_n) \left( \sum_{1 \leq j \leq n} \varepsilon(r_j) \right)^{-b}$$

always converges. Once again, we conclude that the set $E$ also has finite logarithmic measure.

\[ \square \]

9.5 Proof of the second part of Theorem 3

Fix $\alpha$ and $\beta$ such that $0 \leq \alpha < \beta \leq 2\pi$, and fix $b > 1$. Let $r_1 = r \left( 1 - \varepsilon(r) \log^{-b} s_F(r) \right) \geq 2$, and $r_2 = r \left( 1 + \varepsilon(r) \log^{-b} s_F(r) \right)$. The continuous decreasing function $\varepsilon(r)$ will be specified later.

Since the functions $t \mapsto N_F(e^t)$ and $t \mapsto \log \sigma_F(e^t)$ are convex, we have

$$n_F(r_1, \alpha, \beta) \leq \frac{N_F(r, \alpha, \beta) - N_F(r_1, \alpha, \beta)}{\log r - \log r_1} \leq n_F(r, \alpha, \beta) \leq \frac{N_F(r_2, \alpha, \beta) - N_F(r, \alpha, \beta)}{\log r_2 - \log r} \leq n_F(r_2, \alpha, \beta),$$

and

$$s_F(r_1) \leq \frac{\log \sigma_F(r) - \log \sigma_F(r_1)}{\log r - \log r_1} \leq s_F(r) \leq \frac{\log \sigma_F(r_2) - \log \sigma_F(r)}{\log r_2 - \log r} \leq s_F(r_2).$$

Therefore,

$$n_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} s_F(r)$$

$$= n_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} s_F(r_2) + \frac{\beta - \alpha}{2\pi} s_F(r_2) - \frac{\beta - \alpha}{2\pi} s_F(r)$$

$$\leq \left[ N_F(r_2, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r_2) \right] - \left[ N_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} \log \sigma_F(r) \right]$$

$$+ \left[ s_F(r_2) - s_F(r) \right]$$

$$\leq \frac{E_F(r_2) + E_F(r)}{\log r_2 - \log r} + \left[ s_F(r_2) - s_F(r) \right].$$
Applying the growth Lemma 30 with the function $\Phi(r) = s_F(r)$, we conclude that

$$n_F(r, \alpha, \beta) - \frac{\beta - \alpha}{2\pi} s_F(r) \leq C \frac{E_F(r) + E_F(r_2)}{\varepsilon(r)} \left( \log s_F(r) \right)^b + \varepsilon(r) s_F(r),$$

provided that $r \in [2, \infty) \setminus E$, where $E$ is a set of finite logarithmic measure. Similarly,

$$n(r) - \frac{\beta - \alpha}{2\pi} s_F(r) \geq -\frac{S(r) + S(r_1)}{\log r - \log r_1} - \left[ s_F(r) - s_F(r_1) \right]$$

$$\geq -C \frac{E_F(r) + E_F(r_1)}{\varepsilon(r)} \left( \log s_F(r) \right)^b - \varepsilon(r) s_F(r),$$

provided that $r \in [2, \infty) \setminus E$. Therefore,

$$e_F(r) \lesssim \frac{E_F(r) + E_F(r_1) + E_F(r_2)}{\varepsilon(r)} \left( \log s_F(r) \right)^b + \varepsilon(r) s_F(r) \quad (9.8)$$

provided that $r \in [2, \infty) \setminus E$, where $E$ is a set of finite logarithmic measure. The choice of the decreasing function $\varepsilon(r)$ is at our disposal. Now we can readily deduce estimates for $e_F(r)$ in mean and a.s. from the corresponding estimates (9.4) and (9.7)

for $E_F(r)$.

By estimates (9.4) and (9.7),

$$E_F(r) \leq A \left( \log \sigma_F(r) \right)^{\frac{1}{2}} \cdot \left( \log \log \sigma_F(r) \right)^{\frac{C}{\gamma_1}} \cdot \log r \quad \text{a.s.} \quad (9.9)$$

and also in mean, provided that $r$ is sufficiently big. Since $\log \sigma_F(r) \leq s_F(r) \cdot \log r + A$, and since we assume that $\log \sigma_F(r) \geq \log^a r$ with some $a > 4$, by (9.9) we have that a.s., $E_F(r) \leq A s_F^{\gamma_1}(r)$ with $\frac{1}{2} + \frac{2}{a} < \gamma_1 < 1$. The same bound holds for $E_F(r_1)$. Since $r \notin E$, this also holds for $E_F(r_2)$. Then, making use of (9.8), we see that $e_F(r)$ does not exceed $A \varepsilon(r)^{-1} \cdot s_F^2(r) + \varepsilon(r) s_F(r)$, both in mean and a.s., with the exponent $\gamma_2$ lying in the same range as $\gamma_1$. Letting $\varepsilon(r) = (s_F(r))^{-(1-\gamma_2)/2}$, we get part (ii) of Theorem 3 with $\gamma = \frac{1}{2} (1 + \gamma_2)$. 

\[ \square \]

10 Random entire functions with a fixed portion of real zeros

In this section we construct an example of a class of entire functions with only real zeros. In particular this implies that the zeros are not equidistributed in sectors.

The next theorem is based on an idea that goes back to Hardy [Har]. It is worth mentioning that Littlewood and Offord used a similar construction in [LO1, Theorem 3 (iii)].
Theorem 31. Let
\[ f(z) = \sum_{k \geq 0} \xi_k e^{-\alpha k^2} z^k, \]
be a Rademacher entire function. For \( \alpha \geq \log 3 \) the function \( f \) has only real zeros.

Remark. This result is not probabilistic, as it does not depend on the values of the sequence \( \{\xi_n\} \). It is possible to show that for every \( \alpha > 0 \) there is, almost surely, a positive portion (depending on \( \alpha \)) of the zeros in \( r\mathbb{D} \) that are real.

Let \( \alpha = \beta^2/2 \). We first show that \( \log \sigma_f(r) \) is of order \( \log^2 r \). Note that
\[ \sigma_f^2(r) = \sum_{k \geq 0} e^{-2\alpha k^2} r^{2k} = \sum_{k \geq 0} e^{-2\alpha k^2 + 2k \log r}, \]
and thus
\[ \sigma_f^2(r) = \exp \left( \frac{\log^2 r}{2\alpha} \right) \sum_{k \geq 0} \exp \left( -2\alpha \left( k - \frac{\log r}{2\alpha} \right)^2 \right). \]
Hence for \( r \geq e \), and using the fact that
\[ \sum_{k \geq 1} e^{-2\alpha k^2} \leq \sum_{k \geq 0} \exp \left( -2\alpha \left( k - \frac{\log r}{2\alpha} \right)^2 \right) \leq 2 \cdot \sum_{k \geq 0} e^{-2\alpha k^2}, \]
we get that
\[ C_1,\alpha \cdot \exp \left( \frac{\log^2 r}{2\alpha} \right) \leq \sigma_f^2(r) \leq C_2,\alpha \cdot \exp \left( \frac{\log^2 r}{2\alpha} \right), \]
where \( C_1,\alpha, C_2,\alpha \) are certain positive constants that depend only on \( \alpha \). Therefore,
\[ \log \sigma_f(r) = \frac{\log^2 r}{4\alpha} + O_\alpha(1), \]
and indeed the function \( f \) lies outside of the scope of Theorem 3.

Before we prove Theorem we first need a lemma that locates the zeros of \( f \). Without changing notation, we now assume that \( \{\xi_n\} \) is an arbitrary sequence, which consists of the numbers \( \pm 1 \) (there is no probability involved).

Lemma 32. Suppose that \( \alpha \geq \log 3 \). Then given any \( m \in \mathbb{N} \), the function \( f \) has exactly \( m \) zeros inside the disk \( \{|z| \leq e^{2\alpha m}\} \).
Proof. Let \( m \in \mathbb{N} \) and \( r_m = e^{2am} \). Then for any \( z \) on the circle \( \{|z| = r_m\} \) we have
\[
|f(z) - \xi_m e^{-\alpha m^2} z^m| < \sum_{k \neq m} e^{-ak^2} |z|^k = \exp(\alpha m^2) \sum_{k \neq m} e^{-\alpha(k-m)^2} \leq \exp(\alpha m^2) \cdot 2 \sum_{k \geq 1} e^{-ak^2}.
\]
We now observe that for \( \alpha \geq \log 3 \)\(^8\)
\[
\sum_{k \geq 1} e^{-ak^2} \leq \sum_{k \geq 1} 3^{-k^2} < \sum_{k \geq 1} 3^{-k} < \frac{1}{2}.
\]
Hence, we conclude that
\[
|f(z) - \xi_m e^{-\alpha m^2} z^m| < |\xi_m e^{-\alpha m^2} z^m| = \exp(\alpha m^2).
\]
Now Rouché’s Theorem yields the required result. \( \square \)

Proof of Theorem 31: By the lemma, each annulus \( \{e^{2am} < |z| \leq e^{2\alpha(m+1)}\} \), \( m \in \mathbb{N} \), contains exactly one zero of the function \( f \). Since the Taylor coefficients of the function \( f \) are real, this zero must be real as well. The same holds for the disk \( \{|z| \leq e^{2\alpha}\} \). Hence, all the zeros of \( f \) are real. \( \square \)

11 Further Work and Open Problems

In a new work \[NNS2\] we prove stronger versions of Theorem 3. Let \( F(z) = \sum_{n \geq 0} \xi_n a_n z^n \) be a Rademacher entire function, which is not a polynomial and such that \( a_0 \neq 0 \). Let \( \varphi \in C^2(\mathbb{R}) \), \( \varphi(\theta) \in [0, 1] \) be a \( 2\pi \)-periodic test function. Our zero counting function is
\[
n_F(r; \varphi) = \sum_{\zeta \in Z_F \cap r \mathbb{D}} \varphi(\arg \zeta), \quad Z_F = F^{-1}\{0\},
\]
where the zero set \( Z_F \) includes multiplicities. We mention that \( n_F(r) = n_F(r; 1) \).

We will use the following notation:
\[
F_r(\theta) = F(re^{i\theta}), \quad \hat{F}_r = \frac{F_r}{\sigma_F(r)},
\]
\(^8\)With a more precise calculation this can be improved to \( \alpha \geq 0.7855 \).
and for any integrable $g: [0, 2\pi] \to \mathbb{R}$
\[ \langle g \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(t) \, dt. \]

Fix some parameters $\gamma_0 \in \left( \frac{1}{2}, 1 \right)$, $q_0 > 1$ and put $p_0 = \frac{q_0}{q_0 - 1}$. Here $c$ and $C$ denote various positive constants that may depend (only) on these parameters. We denote by $A_F$ various positive constants that depend only on the sequence \{\left| a_n \right| \} of absolute values of the Taylor coefficients of the Rademacher entire function $F$, and on the parameters $\gamma_0, q_0$.

The first theorem is a more accurate version of Theorem 3, where we study the concentration of $n_F(r; \varphi)$ around its mean, for a certain test function $\varphi$.

**Theorem.** [Almost sure concentration for $n_F(r; \varphi)$] There exists a set $E = E(\varphi) \subset [1, \infty)$ of finite logarithmic measure, such that almost surely we have
\[ |n_F(r; \varphi) - E \{ n_F(r; \varphi) \}| \leq C (1 + \| \varphi'' \|_1) \left( E \{ n_F(r; \varphi) \} \right)^{\gamma_0} + A_F \| \varphi'' \|_{q_0} \log^{\gamma_0} r, \]
for $r \geq r_0$, and $r \notin E$.

**Remark.** Here $r_0$ is random. It is worth to mention that
\[ E \{ n_F(r; \varphi) \} = \langle \varphi \rangle s_F(r) + \int_0^r \left< \varphi \cdot E \log \left| \tilde{F}_t \right| \right> \frac{dt}{t} + \frac{d \left< \varphi \cdot E \log \left| \tilde{F}_t \right| \right>}{d \log r}. \]

Put
\[ \varepsilon_F(r) = \sup_{t \geq r} \left( \frac{\log t}{s_F(t)} \right). \]

Note that $\varepsilon_F(r)$ does not increase when $F$ does not grow too slow; that is, when log $r = O(s_F(r))$ (which is equivalent to log$^2 r = O(\log \sigma_F(r))$). Otherwise $\varepsilon_F(r) \equiv +\infty$.

The second theorem is an uniform equidistribution result.

**Theorem.** [Uniform equidistribution for $n_F(r; \varphi)$] There exists a set $E \subset [1, \infty)$ of finite logarithmic measure, such that almost surely we have
\[ |n_F(r; \varphi) - \langle \varphi \rangle s_F(r)| \leq A_F \left( 1 + \| \varphi'' \|_{q_0} \right) s_F^{2\gamma_0}(r) + C \| \varphi'' \|_{q_0} s_F(r) \varepsilon_F(r), \quad (*) \]
for any test function $\varphi$, and any $r \geq r_0$, such that $r \notin E$. Furthermore, for any $r \notin E$
\[ |E n_F(r; \varphi) - \langle \varphi \rangle s_F(r)| \leq \text{the RHS of } (*). \]
Remark. Here \( r_0 \) is random, but it does not depend on \( \varphi \). In particular, the last theorem implies the equidistribution of the zeros of \( F \), as long as \( \log r = o(s_F(r)) \).

**Functions with ‘many’ real zeros**

Let \( f \) be a Rademacher entire function of the form
\[
f(z) = \sum_{k\geq 0} \xi_k e^{-\alpha k^2} z^k,
\]
and let us denote by \( n_{\text{Re}}(r) \), \( n_{\text{R}}(r) \) the number of (complex non-real \( \setminus \) real) roots of \( f \) inside \( r \mathbb{D} \). Recall that for \( \alpha \geq \log 3 \approx 1.099 \) we proved that \( n_{\text{Re}}(r) = 0 \) always, for any \( r \geq 0 \). Some numerical experiments suggest that already for \( \alpha = 0.7853 \) we have that \( n_{\text{Re}}(r) > 0 \) with a positive probability. It seems that there is a critical ‘threshold’ value, the root of the equation
\[
\sum_{k\geq 1} e^{-\alpha k^2} = \frac{1}{2},
\]
which is approximately 0.785409. It is not clear if there are some values of \( \alpha \) such that \( n_{\text{Re}}(r) = 0 \) almost surely, but some instances (of measure 0) for which \( n_{\text{Re}}(r) > 0 \). Another possible situation is that there are values of \( \alpha \) for which the ratio \( n_{\text{Re}}(r)/n_{\text{R}}(r) \) almost surely tends to 0 as \( r \) tends to infinity (that is, there are ‘few’ non-real zeros).

**Distribution of zeros of random entire functions**

There are many possible generalization of Theorem \( \text{3} \) and related questions. A natural problem is to generalize the results to more general i.i.d. random variables, while trying to keep the assumptions on the sequence of coefficients \( \{a_k\} \) minimal. A result for not necessarily independent coefficients also seems very interesting.

Other related problems are to find bounds for the variance of the number of zeros (at least for regular sequences \( \{a_k\} \), or upper bounds for general coefficients), or to find accurate bounds for the number of real zeros (in terms of \( s_F(r) \)). Both problems are interesting even for (real) Gaussian coefficients.
Part IV. The hole probability for Gaussian entire functions

We begin this part with the background of the problem and a (short) history of previous results. We mention relations to questions in probability theory and in mathematical physics. We then give a proof of the main result and discuss, briefly, its sharpness. We then give an example that illustrates the significance of Gaussianity for the result. We finish with a list of related open problems.

12 Background and History

Let us consider first the following special case of a Gaussian entire function

$$F(z) = \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{n!}}, \quad (12.1)$$

which is sometimes referred to as the Gaussian Entire Function (or GEF), for reasons that will soon become clear. The random zero set of this function is known to be distribution invariant with respect to isometries of the plane. Furthermore, this is the only Gaussian entire function with distribution invariant zeros (see the book [BKPV, Section 2.5] for details). It is also known that the GEF and GAFs in general have simple zeros almost surely (unless the zero is non-random). Thus one can consider the zero set of a GAF as a simple point process in the plane, and compare it to other point processes.

We recall that if $\nu_F$ is the random zero counting measure of $F$, then by the Edelman-Kostlan formula (1.1) we have

$$\mathcal{E} \{ \nu_F(z) \} = \frac{1}{2\pi} \Delta \log \sqrt{K_F(z, \bar{z})} = \frac{1}{4\pi} \Delta \log \mathcal{E} \left\{ F(z) \overline{F(\overline{z})} \right\} = \frac{d m(z)}{\pi}$$

in the sense of distributions, where $m$ is the Lebesgue measure in the complex plane.

We conclude that the GEF is similar to the Poisson point process in the plane, in the sense that the expected number of points in a region is proportional to the area of the region. We mention that for the GEF there are many results concerning the linear statistics of the zeros, such as the variance, asymptotic normality and deviations from normality (see [NS1, NSV1, ST1] for explanations and proofs). There are also various geometric properties that are known about the zero set of the GEF (see for example [NSV2]).
12.1 Hole Probability

We now show a particular way in which the Poisson point process and the GEF zero set differ in a rather dramatic way. One of the interesting characteristics of a random point process is the probability that there are no points in some (say, simply connected) domain $D \subset \mathbb{C}$. In the case of the Poisson point process, we know exactly what is the distribution of the number of points, that is, the Poisson distribution with parameter $m(D)$. We immediately conclude that the probability that there are no points in $D$ is $\exp(-m(D))$. To simplify matters, let us now consider the probability of the event where there are no points of the process inside the disk $rD = \{|z| \leq r\}$, as $r$ tends to infinity. We will call this event the ‘hole’ event. Before returning to the GEF, let us give some other examples.

12.1.1 The Poisson point process

As we have already mentioned, this is the simplest process to analyze. The probability of the hole event is $\exp(-cr^2)$, where the constant $c$ depends on the normalization of the processes (i.e., the number of expected points of the process per unit area). We do not expect that this process will be a good model for the zeros of the GEF, as the points of this process tend to ‘clump’ together, while the zeros of the GEF have a more rigid (lattice like) structure.

12.1.2 A (Gaussian) perturbed lattice

Here we consider the perturbed lattice:

$$\left\{\frac{(m,n)}{\sqrt{\pi}} + \xi_{m,n} \mid n, m \in \mathbb{Z}\right\},$$

where $\xi_{m,n}$ are i.i.d. standard complex Gaussians. The hole probability is expected to be of order $\exp(-cr^4)$. Heuristically, the reason is that in order to have a hole around 0, we have to “push” the points which are at distance $s$ from 0, a distance of at least $r-s$. The probability of the last event is of order $\exp\left(- (r-s)^2 \right)$. Since there are approximately $2s$ points at distance $r-s$ from $0$, the hole probability is approximately of order

$$\exp\left(- \int_0^r (r-s)^2 \cdot 2s \, ds \right) = \exp\left(- \frac{r^4}{6}\right).$$

We will see that this probability is much closer to the hole probability of the GEF.
12.1.3 Infinite Ginibre ensemble
Let $M$ be an $N \times N$ matrix with entries which are i.i.d standard complex Gaussians, and let $\lambda_n$ be its eigenvalues. Kostlan showed that
\[
\{|\lambda_1|, \ldots, |\lambda_n|\} \overset{d}{=} \{Z_1, \ldots, Z_n\},
\]
where the random variables $Z_k$ are independent and $Z_k^2$ has the $\Gamma(k,1)$ distribution. The infinite Ginibre ensemble is the ‘limit’ of this process (see [BKPV, p. 69]). In this book there is a proof that the hole probability is of order $\exp\left(-\frac{r^4}{4}\right)$. This model is a special case of the one-component plasma model.

12.1.4 One-component plasma
This is a simple physical model, where particles with positive charge are embedded in a background of negative charge. The simplest case is the two-dimensional one, studied non-rigorously in [JLM]. They found the hole probability to be of order $\exp(-cr^4)$. This model has natural generalizations to higher dimensions, which are interesting to study.

12.1.5 The GEF - The plane invariant GAF
We now consider the probability of the event where $F(z) \neq 0$ in the disk $r\mathbb{D} = \{|z| \leq r\}$, as $r$ tends to infinity, and denote it by $P_H(F; r)$. Since the decay rate of this probability is known to be exponential in $r$, we will use the notation
\[
 p_H(F; r) = \log^- P_H(F; r) = \log^- \mathcal{P}(F(z) \neq 0 \text{ for } |z| \leq r).
\]
In the paper [ST3], Sodin and Tsirelson showed that for $r \geq 1$,
\[
c_1 r^4 \leq p_H(F; r) \leq c_2 r^4
\]
with some (unspecified) positive numerical constants $c_1$ and $c_2$. This result was extended in different directions by Ben Hough [BH1], Krishnapur [Kr], Zrebiec [Zt1, Zt2] and Shiffman, Zelditch and Zrebiec [SZZ].

In [ST3], Sodin and Tsirelson asked whether the limit
\[
\lim_{r \to \infty} \frac{p_H(F; r)}{r^4}
\]
exists and what is its value? We found an answer to this question in the paper [Ni1].
Theorem 33. For \( r \) large enough

\[
p_H(F; r) = \frac{e^2}{4} \cdot r^4 + O \left( r^{18/5} \right).
\] (12.3)

We mention that before this result the explicit constant for a GAF was known only for the very special case \( \sum_{n=0}^{\infty} \xi_n z^n \) (see [PV]), using completely different methods.

12.1.6 Gaussian entire functions

We extended Theorem 33 to a large class of GAFs, whose sequence of coefficients \( \{a_n\} \) satisfy some regularity condition (see [Ni2]). In this part of the work we prove Theorem 4 which generalizes the latter result to arbitrary GAFs given in a Taylor series form.

To recall the statement of the theorem, let

\[ f(z) = \sum_{n \geq 0} \xi_n a_n z^n \]

be a GAF such that \( \sum_{n \geq 0} a_n z^n f \) is a non-constant entire functions and \( a_0 \neq 0 \) (in order to avoid trivial situations). We write

\[ S(r) = 2 \cdot \sum_{n \geq 0} \log^+ (a_n r^n). \]

A set \( E \subset [1, \infty) \) is of finite logarithmic measure if \( \int_E \frac{dt}{t} < \infty \). Theorem 4 states that for \( \varepsilon \in \left( 0, \frac{1}{2} \right) \) there exists an exceptional set \( E \subset [1, \infty) \), of finite logarithmic measure, which depends only on \( \varepsilon \) and on the sequence \( \{a_n\} \) such that as \( r \to \infty \) outside of the set \( E \),

\[ p_H(f; r) = S(r) + O \left( S(r)^{1/2+\varepsilon} \right). \]

We remark that the exceptional set \( E \) does not appear if the coefficients of \( f \) satisfy some regularity conditions, but in general it is unavoidable (see Section 17). It turns out that Gaussianity is very important for Theorem 4 to hold. For the reader’s convenience we reproduce in Section 18 the proof of the next theorem (from [Ni1]).

Theorem 34. Let \( K \subset \mathbb{C} \) be a compact set and \( 0 \notin K \). Let \( \{\phi_n\} \) be any sequence, such that \( \phi_n \in K \) for each \( n \), and put

\[ g(z) = \sum_{n \geq 0} \phi_n \frac{z^n}{\sqrt{n!}}. \] (12.4)

Then there exists \( r_0 = r_0(K) < \infty \) so that \( g(z) \) must vanish somewhere in the disk \( \{|z| \leq r_0\} \).
13 Proof of Theorem 4 - Preliminaries

In this section we introduce some special notation for the proof. We also prove some estimates which we use later to control some error terms. In addition, we give a short description of the strategy of the proof.

13.1 Notation

For \( r \geq 1 \), we denote by \( r \mathbb{D} \) the disk \( \{ z \mid |z| \leq r \} \) and by \( r \mathbb{T} \) its boundary \( \{ z \mid |z| = r \} \). The letters \( c \) and \( C \) denote positive absolute constants (which can change from line to line). We also use the standard notation

\[
M(r) = \max_{z \in r \mathbb{D}} |f(z)|.
\]

13.2 A growth lemma by Hayman

We recall that a set \( E \subset [1, \infty) \) is of finite logarithmic measure if

\[
\int_E \frac{dt}{t} < \infty.
\]

The following lemma (taken from [Ha2]) is a general result about the growth of functions.

**Lemma 35.** Let \( \eta > 0 \) and suppose that \( D(r) \) is a positive increasing function of \( r \) for \( r \geq r_0 \). For all \( r \) outside a set of finite logarithmic measure and for any \( \delta \) such that \( |\delta| < D(r)^{-\eta} \), we have

\[
\left| D(re^{\delta}) - D(r) \right| < \eta D(r).
\]

13.3 Further notation, definition of the exceptional set

We use the following notations:

\[
b_n(r) = \begin{cases} \frac{1}{n} \log a_n + \log r, & \text{if } a_n > 0, \\ -\infty, & \text{if } a_n = 0, \end{cases}
\]

and

\[
N(r) = \{ n \mid b_n(r) \geq 0 \},
\]

\[
S(r) = 2 \cdot \sum_{n \in N(r)} \log (a_n r^n) = 2 \cdot \sum_{n \in N(r)} nb_n(r).
\]
Since the coefficients $a_n$ satisfy $\frac{\log a_n}{n} \to -\infty$, we have that $b_n \to -\infty$ as $n \to \infty$ and the set $N(r)$ is finite for every $r \geq 1$.

We now write

\[ n (r) = \#N(r), \]
\[ m (r) = 4 \cdot \sum_{n \in N(r)} n, \]
\[ N_{\delta} (r) = \{ n \mid b_n (r) \geq -\delta\}, \]
\[ n_{\delta} (r) = \#N_{\delta} (r). \]

Note that $b_n (r)$ is increasing with $r$ and therefore $n (r)$ and $m (r)$ are increasing functions of $r$. Moreover,

\[ N_{-\delta} (r) \subset N_0 (r) = N (r), \]

\[ N_{-\delta} (r) = N (re^{-\delta}). \]

Let $\eta \in \left(0, \frac{1}{4}\right]$. We apply Lemma 35 to the function $m (r)$, taking

\[ \delta = \delta (r) = m^{-\eta} (r). \]

We conclude that outside an exceptional set $E$ of finite logarithmic measure, we have

\[ m \left(re^{-\delta}\right) > (1 - \eta) m (r), \]
\[ m \left(re^\delta\right) < (1 + \eta) m (r). \]

From now on we will fix $r \notin E$. We also assume that $r$ is large enough, depending on the coefficients $a_n$ and on the value of $\eta$. In particular, we assume that $n (r) \geq 2$.

We also note that the choice of $\delta$ remains the same throughout the paper. The limit in Theorem 4 is taken over $E^c$. If, for some set of coefficients $a_n$, the inequalities (13.2) hold for all large values of $r$, then there is no exceptional set in Theorem 4.

### 13.4 Estimates for $S (r)$

Here we find relations between $S (r)$ and the functions $m (r), n (r)$, that will be used later in the proof.

**Lemma 36.** We have

\[ S(r) \geq \frac{1}{8} \cdot (m (r))^{1-\eta} \geq \frac{1}{8} \cdot n (r)^{2-2\eta}. \]
Proof. Recall that it is assumed that $n(r) \geq 2$. Notice that
\[ m(r) = 4 \cdot \sum_{n \in N(r)} n \geq n^2(r), \tag{13.3} \]
since $m(r)$ is minimal when $N(r) = \{0, 1, \ldots, n(r) - 1\}$. Now, using the inclusion $N_{-\delta}(r) \subset N(r)$, we get:
\[
\frac{S(r)}{2} = \sum_{n \in N(r)} nb_n(r) \geq \sum_{n \in N_{-\delta}(r)} nb_n(r) \\
\geq \sum_{n \in N(re^{-\delta})} n\delta \geq \frac{\delta}{4} \cdot m(re^{-\delta}) \\
\geq (1 - \eta) \cdot \frac{m(r)}{(m(r))^\eta} \geq \frac{1}{8} \cdot n(r)^{2-2\eta}.
\]

We now estimate the rate of growth of the function $S(r)$.

Lemma 37. For $\gamma \in (0, \frac{1}{2})$ we have,
\[ S((1 - \gamma) r) \geq S(r) - \gamma \cdot m(r). \]

Proof. Write $r' = (1 - \gamma) r$ and notice that for $\gamma < \frac{1}{2}$ we have
\[ \log (1 - \gamma) \geq -\gamma - \frac{\gamma^2}{2} \geq -2\gamma. \]

We will also use the inequality
\[ \log^+ a - \log^+ b \leq \log \frac{a}{b}, \quad a > b > 0. \]

It follows that (since $N(r') \subset N(r)$)
\[
S(r) - S(r') = 2 \cdot \sum_{n \in N(r)} \left\{ \log^+ (a_n r^n) - \log^+ [a_n (r')^n] \right\} \\
\leq 2 \cdot \sum_{n \in N(r)} n \cdot \log \frac{r}{r'} = \log \left( \frac{1}{1 - \gamma} \right) \cdot \frac{m(r)}{2} \\
\leq \gamma \cdot m(r).
\]
In the next lemma we show that the value of \( S(r) \) does not depend on the scaling of the coefficients (up to some error term).

**Lemma 38.** Let \( d > 0 \) and set \( \tilde{S}(r) = 2 \cdot \sum_{n \geq 0} \log^+(d \cdot a_n r^n) \). We have

\[
\left| S(r) - \tilde{S}(r) \right| \leq C \sqrt{m(r)},
\]

where \( C \) is a positive constant that might depend on \( d \).

**Proof.** If \( d < 1 \), then \( \tilde{S}(r) \) is at least \( S(r) - \log^+ \frac{1}{d} \cdot n(r) \geq S(r) - \log^+ \sqrt{m(r)} \). Now assume that \( d > 1 \). Denote by \( \tilde{b}_n(r), \tilde{N}(r), \tilde{n}(r) \) the corresponding functions for the set of coefficients \( \{d \cdot a_n\}_{n \geq 0} \). By Lemma 36, we have \( n(re^\delta) \leq C \sqrt{m(r)} \), so the contribution from terms in \( N(re^\delta) \) to \( \tilde{S}(r) \) is at most

\[
C \log d \cdot \sqrt{m(r)}.
\]

If \( n \notin N(re^\delta) \) and \( n \in \tilde{N}(r) \), then

\[
-\delta + \frac{\log d}{n} \geq \tilde{b}_n(r) + \frac{\log d}{n} = \frac{\log (d a_n)}{n} + \log r = \tilde{b}_n(r) \geq 0,
\]

so we have \( n \leq \frac{\log d}{\delta} \leq \log d \cdot m^n(r) \). So the contribution from these terms is at most \( \log^2 d \cdot m^n(r) \leq C \sqrt{m(r)} \).

**13.5 Gaussian distributions**

Many times we use the fact that if the random variable \( a \) has standard complex Gaussian distribution, then

\[
\mathcal{P}(|a| \geq \lambda) = \exp(-\lambda^2),
\]

and for \( \lambda \leq 1 \),

\[
\mathcal{P}(|a| \leq \lambda) \in \left[ \frac{\lambda^2}{2}, \lambda^2 \right].
\]
13.6 Strategy of the proof

The proof consists of two parts. We first show that

$$p_H (r) \leq S (r) + C \sqrt{m (r)} \log m (r)$$

(Proposition 39), and then we prove that

$$p_H (r) \geq S (r) - C n (r) \log S (r)$$

(Proposition 46). Combining these bounds with Lemma 36, we get Theorem 4. If the coefficients $a_n$ are such that for $r \geq 1$ there is no exceptional set $E$, then Theorem 4 holds for every $r$ large enough. In some cases, where the coefficients $a_n$ have regular asymptotic behavior, it is possible to prove that $m (r) \leq CS (r)$ and obtain

$$p_H (r) = S (r) + O \left( \sqrt{S (r)} \log S (r) \right), \quad r \to \infty.$$  

For instance, such is the case for the coefficients of the GEF, where $a_n = \frac{1}{\sqrt{n!}}$, or more generally where $a_n \sim \frac{1}{\Gamma (\alpha n + 1)}$, with some $\alpha > 0$.

**Remark.** By Lemma 38, we can scale $f (z)$ by a constant factor that will add a term of order $\sqrt{m (r)}$ to $S (r)$. By Lemma 36, this term is of order at most $S (r)^{1/2 + \varepsilon}$ and can be ignored. From now on, in order to simplify some of the expressions in the paper, we will assume that

$$a_0 = 1.$$  

14 Proof of Theorem 4 - Upper Bound for $p_H (f; r)$

In this section we will prove the following

**Proposition 39.** We have

$$p_H (f; r) \leq S (r) + C \cdot \sqrt{m (r)} \log m (r),$$  

with some positive absolute constant $C$.

**Remark.** Recall that $r$ is assumed to be large.
The simplest case where \( f(z) \) has no zeros inside \( r\mathbb{D} \) is when the constant term dominates all the others. Consider the event \( \Omega_r \), that is the intersection of the events (i),(ii), and (iii) (\( C \) will be selected in an appropriate way)

(i) : \( |\xi_0| \geq C (m(r))^{1/4} \),
(ii) : \( \bigcap_{n \in N(r) \setminus \{0\}} (ii)_n \),
(iii) : \( \bigcap_{n \in \tilde{N}_\delta(r) \setminus N(r)} (iii)_n \),
(iv) : \( \bigcap_{n \in (\tilde{N}_\delta(r))^c} (iv)_n \),

where \( \tilde{N}_\delta(r) = N_\delta(r) \cup \{ n \mid n < \sqrt{m(r)} \} \) and

(ii)_n : \( |\xi_n| \leq \frac{(a_n r^n)^{-1}}{\sqrt{m(r)}} \),
(iii)_n : \( |\xi_n| \leq \frac{1}{\sqrt{m(r)}} \),
(iv)_n : \( |\xi_n| \leq \exp \left( \frac{dn}{2} \right) \).

We notice that by (13.2) and (13.3) we have that \( \# \tilde{N}_\delta(r) \leq 2 \sqrt{m(r)} \).

**Lemma 40.** If \( \Omega_r \) holds, then \( f \) has no zeros inside \( r\mathbb{D} \).

**Proof.** Recall that \( \delta = m^{-\eta}(r) \). To see that \( f(z) \) has no zeros inside \( r\mathbb{D} \) we note that

\[
|f(z)| \geq |\xi_0| - \sum_{n=1}^{\infty} |\xi_n|a_n r^n. \tag{14.1}
\]

First, we estimate the sum over the indices in \( N(r) \setminus \{0\} \):

\[
\sum_{n \in N(r) \setminus \{0\}} |\xi_n|a_n r^n \leq \sum_{n \in N(r)} \frac{1}{\sqrt{m(r)}} \leq C_1,
\]

by (13.3). Second, we estimate the sum over the indices in \( \tilde{N}_\delta(r) \setminus N(r) \). Notice that here \( b_n(r) \leq 0 \) and \( a_n r^n = e^{\log(a_n r^n)} = e^{nb_n} \), so

\[
\sum_{n \in \tilde{N}_\delta(r) \setminus N(r)} |\xi_n|a_n r^n = \sum_{n \in \tilde{N}_\delta(r) \setminus N(r)} |\xi_n|e^{nb_n} \leq \sum_{n \in \tilde{N}_\delta(r) \setminus N(r)} \frac{1}{\sqrt{m(r)}} \leq C_2.
\]
Now the rest of the tail is bounded by
\[ \sum_{n \in \left( N \delta(r) \right)^c} |\xi_n| a_n r^n = \sum_{n \in \left( N \delta(r) \right)^c} |\xi_n| e^{nb_n} \leq \sum_{n \in \left( N \delta(r) \right)^c} e^{\frac{\delta n}{2}} e^{-\delta n} \]
\[ \leq \sum_{n \geq 0} \exp \left(-\frac{\delta n}{2}\right) \leq \frac{3}{\delta} \leq 3 \cdot (m(r))^\eta. \]

Here we used the fact that \( b_n(r) \leq -\delta \). Hence, (14.1) yields
\[ |f(z)| > C (m(r))^{1/4} - C_1 - C_2 - 3 (m(r))^\eta > 0, \]
provided that \( C > C_1 + C_2 + 3 \).

**Lemma 41.** The probability of the event \( \Omega_r \) is bounded from below as follows:
\[ \log P (\Omega_r) \geq -S(r) - C \cdot \sqrt{m(r) \log m(r)}. \]

**Proof.** By the properties of \( \xi_n \) (see Section 13.5),
\[ P ((i)) = \exp \left(-C^2 \cdot \sqrt{m(r)}\right) \]
and
\[ P ((ii)) \geq \frac{(a_n r^n)^{-2}}{2m(r)}. \]

Therefore,
\[ P ((iii)) \geq \prod_{n \in N(r)} \left( \frac{(a_n r^n)^{-2}}{2m(r)} \right) = \left( \prod_{n \in N(r)} e^{-2nb_n} \right) \cdot \exp (-n (r) \log (2m(r))) \]
\[ \geq \exp \left(-2 \cdot \sum_{n \in N(r)} nb_n \right) \cdot \exp (-Cn (r) \log m (r)) \]
\[ \geq \exp \left(-S(r) - C \sqrt{m(r)} \log m(r) \right). \]

Similarly, we have
\[ P ((iii)) \geq \frac{1}{2m(r)} \]
and so (by (13.2) and (13.3))
\[ P ((iii)) \geq \exp \left(-C \sqrt{m(r)} \log m(r) \right). \]
Finally,

\[ \mathcal{P}((iv)^c_n) = \exp\left(-e^{\delta n}\right). \]

Let \(\{A_n\}\) be some positive sequence. Using the inequality

\[ \mathcal{P}(\forall n : |\xi_n| \leq A_n) = 1 - \mathcal{P}(\exists n : |\xi_n| > A_n) \geq 1 - \sum \mathcal{P}(|\xi_n| > A_n), \]

we obtain

\[ 1 - \mathcal{P}((iv)) \leq \sum_{n \in (N \alpha(r))'} \exp\left(-e^{\delta n}\right) \]
\[ \leq \sum_{n \geq \sqrt{m(r)}} \exp\left(-e^{\delta n}\right) = \sum_{n \geq \sqrt{m(r)}} \exp(-\delta n) = \frac{\exp\left(-\delta \sqrt{m(r)}\right)}{1 - e^{-\delta}}. \]

Since \(\sqrt{m(r)} \delta \geq \frac{1}{\delta}\), we get

\[ \frac{\exp\left(-\delta \sqrt{m(r)}\right)}{1 - e^{-\delta}} \leq \frac{\exp\left(-\frac{1}{\delta}\right)}{1 - e^{-\delta}} \leq \frac{3}{4}, \]

and so

\[ \mathcal{P}((iv)) \geq \frac{1}{4}. \]

Since the events (i) – (iv) are independent, we have

\[ \mathcal{P}(\Omega_r) = \mathcal{P}((i)) \cdot \mathcal{P}((ii)) \cdot \mathcal{P}((iii)) \cdot \mathcal{P}((iv)) \]
\[ \geq \exp\left(-S(r) - C \sqrt{m(r) \log m(r)}\right). \]

Proposition [39] now follows from the previous lemmas.

### 15 Proof of Theorem 4 - Bounds for Gaussian Entire Functions

In this section we get some bounds for the moduli and the logarithmic derivatives of Gaussian entire functions, which hold with high probability (We use the term ‘high probability’ for events which occur with probability greater than \(1 - \exp(-2 \cdot S(r))\)). These results will be used in the next section in the proof of the lower bound.
15.1 Bounds on the modulus of Gaussian entire functions

We first bound the probability of the events where \( M(r) \) is relatively large or small.

**Lemma 42.** We have

\[
\mathcal{P} \left( M(r) \geq e^{3S(r)} \right) \leq C \cdot \exp \left( - \exp \left( S(r) \right) \right) \leq e^{-S^2(r)}.
\]

**Proof.** We set \( \widetilde{N}(r) = N_\delta(r) \cup \{ n < S^2(r) \} \). Notice that, by Lemma 36, we can assume that \( \#\widetilde{N}(r) \leq 2S^2(r) \) if \( r \) is large enough. The proof is similar to that of Proposition 39. We define the event \( \Omega_r \) as the intersection of the events (i) and (ii), where

\[
(i) : \bigcap_{n \in \widetilde{N}(r)} (i)_n, \\
(ii) : \bigcap_{n \in (\widetilde{N}(r))^c} (ii)_n,
\]

and

\[
(i)_n : \ |\xi_n| \leq (a_nr^n)^{-1} e^{2S(r)}, \\
(ii)_n : \ |\xi_n| \leq \exp \left( \frac{1}{2} \delta n \right).
\]

We have the following estimate for \( M(r) \):

\[
|f(z)| \leq \sum_{n \in \widetilde{N}(r)} |\xi_n|a_nr^n + \sum_{n \in (\widetilde{N}(r))^c} |\xi_n|a_nr^n \\
\leq \#\widetilde{N}(r) \cdot e^{2S(r)} + \sum_{n \geq S^2(r)} e^{\frac{\delta n}{2}} \cdot e^{-\delta n} \\
\leq 2S^2(r) \cdot e^{2S(r)} + \frac{C}{\delta} \cdot e^{-\frac{\delta S^2(r)}{2}} \\
\leq e^{3S(r)},
\]

provided that \( r \) is sufficiently large. Here we used \( \delta S^2(r) \geq c\delta (\delta m(r))^2 \geq \frac{\xi}{\delta} \).

Now we estimate the probability of the complement of \( \Omega_r \). We have:

\[
\mathcal{P} \left( |\xi_n| \geq \frac{e^{2S(r)}}{a_nr^n} \right) = \exp \left( -\frac{e^{4S(r)}}{(a_nr^n)^2} \right) \leq \exp \left( -e^{2S(r)} \right), \\
\mathcal{P} \left( |\xi_n| \geq e^{\frac{\delta n}{2}} \right) = \exp \left( -\exp (\delta n) \right).
\]

By the union bound,

\[
\mathcal{P} ((i)^c) \leq C S^2(r) \cdot \exp \left( -\exp (2S(r)) \right),
\]
and
\[ P \left( (\text{ii})^c \right) \leq \sum_{n \geq S^2(r)} \exp(-\exp(\delta n)) \leq C \cdot \exp(-\exp(S(r))) , \]
since the first term in the sum above dominates the others. Here we used
\[ \delta S^2(r) = S^2(r) / m^q(r) \geq S(r) , \]
for \( r \) large enough. So overall we have
\[ P \left( M(r) \geq e^{3S(r)} \right) \leq C \cdot \exp(-\exp(S(r))) \leq \exp(-S^2(r)) . \]

In the other direction we have the following

**Lemma 43.** Let \( \rho > 0 \) be sufficiently large. Then
\[ P \left( M(\rho) \leq \exp(-S(\rho)) \right) \leq \exp(-S(\rho) \cdot n(\rho)) . \]

**Proof.** By Cauchy’s estimate,
\[ |\xi_n|a_n\rho^n \leq M(\rho) \leq e^{-S(\rho)} . \]
For \( n \in N(\rho) \) we have
\[ P \left( |\xi_n| \leq (a_n\rho^n)^{-1} e^{-S(\rho)} \right) \leq e^{-2S(\rho)} , \]
and so we get
\[ P \left( M(\rho) \leq \exp(-S(\rho)) \right) \leq \prod_{n \in N(\rho)} e^{-2S(\rho)} = \exp(-2 \cdot S(\rho) \cdot n(\rho)) . \]

Notice that we do not assume that \( \rho \notin E \).

### 15.2 Bounds for the logarithmic derivative

We denote by \( m \) the normalized angular measure on \( r \mathbb{T} \). In this section we assume that \( f(z) \neq 0 \) inside \( r \mathbb{D} \), and therefore \( \log |f| \) is harmonic there. Under this condition we have the following bound for the average value of \( |\log |f|| \):
Lemma 44. Let $0 < \rho < r$. Outside a set of probability at most

$$2 \cdot \exp (-S(\rho) \cdot n(\rho)),$$

we have

$$\int_{rT} |\log |f|| \, dm \leq C \left(1 - \frac{\rho}{r}\right)^{-2} \cdot S(r).$$

Proof. Denote by $P(z, a)$ the Poisson kernel for the disk $r\mathbb{D}$, $|z| = r$, $|a| = \rho$. By Lemma 43, we can assume that there is a point $a \in \rho \mathbb{T}$ such that $\log |f(a)| \geq -S(\rho)$ (discarding an event of probability $\leq \exp (-S(\rho) \cdot n(\rho)))$. Then we have

$$-S(\rho) \leq \int_{rT} P(z, a) \log |f(z)| \, dm(z),$$

and hence

$$\int_{rT} P(z, a) \log^- |f(z)| \, dm(z) \leq \int_{rT} P(z, a) \log^+ |f(z)| \, dm(z) + S(\rho).$$

For $|z| = r$ and $|a| = \rho$ we have,

$$\frac{r - \rho}{2r} \leq \frac{r - \rho}{r + \rho} \leq P(z, a) \leq \frac{r + \rho}{r - \rho} \leq \frac{2r}{r - \rho}.$$

By Lemma 42, outside a set of very small probability (of the order $e^{-S^2(r)}$), we have $\log M(r) \leq 3 \cdot S(r)$. Therefore

$$\int_{rT} \log^+ |f| \, d\mu \leq 3 \cdot S(r).$$

Further,

$$\int_{rT} \log^- |f| \, d\mu \leq \frac{2r}{r - \rho} \cdot S(\rho) + \frac{12r^2}{(r - \rho)^2} \cdot S(r).$$

Finally, we get

$$\int_{rT} |\log |f|| \, d\mu \leq \frac{Cr^2}{(r - \rho)^2} \cdot S(r) = C \left(1 - \frac{\rho}{r}\right)^{-2} \cdot S(r) \quad (15.1)$$

Next we provide an upper bound for the (angular) logarithmic derivative of $\log |f|$ inside $r\mathbb{D}$.
Lemma 45. Let $0 < \rho < r$. Then

$$\left| \frac{d \log |f(\rho e^{i\phi})|}{d\phi} \right| \leq C \left(1 - \frac{\rho}{r}\right)^5 \cdot S(r)$$

outside a set of probability at most

$$2 \cdot \exp(-S(\rho) \cdot n(\rho)) .$$

Proof. We start with Poisson’s formula

$$\log |f(\rho e^{i\phi})| = \int_0^{2\pi} \frac{r^2 - \rho^2}{|re^{i\theta} - \rho e^{i\phi}|^2} \cdot \log |f(re^{i\theta})| \frac{d\theta}{2\pi} .$$

Differentiating under the integral we get

$$\frac{d \log |f(\rho e^{i\phi})|}{d\phi} = \int_0^{2\pi} \frac{pr^2 (r^2 - \rho^2) \sin(\theta - \phi)}{|re^{i\theta} - \rho e^{i\phi}|^4} \cdot \log |f(re^{i\theta})| \frac{d\theta}{2\pi} .$$

Taking the absolute value, we obtain

$$\left| \frac{d \log |f(\rho e^{i\phi})|}{d\phi} \right| \leq C \frac{(r + \rho)}{(r - \rho)^3} \int_0^{2\pi} |\log |f(re^{i\theta})|| \frac{d\theta}{2\pi} .$$

Using the previous lemma we get the required result. \hfill \Box

16 Proof of Theorem 4 - Lower Bound for $p_H(f; r)$

The goal of this section is to prove

Proposition 46. We have

$$p_H(f; r) \geq S(r) - Cn(r) \log S(r) ,$$

with some positive numerical constant $C$.

In order to find the lower bound for $p_H(f; r)$ we now assume that $f(z) \neq 0$ inside $r\mathbb{D}$. We take a small $\gamma > 0$ (that will depend on $r$), and write $\rho = r(1 - \gamma)$.

The function $\log |f(z)|$ is harmonic in $r\mathbb{D}$, therefore

$$\log |f(0)| = \int_0^{2\pi} \log |f(\rho e^{i\alpha})| \frac{d\alpha}{2\pi} .$$
Now if we select $n$ points $z_j = \rho e^{i\theta_j}$, we have
\[
\int_0^{2\pi} \frac{1}{n} \sum_{j=1}^{n} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha})| \frac{d\alpha}{2\pi} = \frac{1}{n} \sum_{j=1}^{n} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha})|
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} \log |f(0)|
\]
\[
= \log |f(0)|. 
\]
Since $\log |f(z)|$ is continuous inside $rD$, we conclude that there exists some $\alpha^*$ such that
\[
\frac{1}{n} \sum_{j=1}^{n} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha^*})| = \log |f(0)|. 
\]
Let $\Delta\alpha = \frac{c_4^5}{s(r)}$. By Lemma \ref{lemma}, if $\alpha$ satisfies
\[
|\alpha - \alpha^*| \leq \Delta\alpha, 
\]
then the logarithmic derivative of $f$ is not too large with high probability. Discarding this event, we get
\[
\frac{1}{n} \sum_{j=1}^{n} \log |f(\rho e^{i\theta_j} \cdot e^{i\alpha})| \leq \log |f(0)| + 1. 
\]
(16.1)
In this section we will show that, if we select the points $\{z_j\}$ is an appropriate way, then the probability of (16.1) is of order $\exp(-S(r))$. This will allow us to prove Proposition \ref{proposition}.

### 16.1 Reduction to an estimate of a multivariate Gaussian event

In this section, we reduce the problem to an estimate of the probability of an event in some finite-dimensional complex Gaussian space. We first note that we work in the product space $\{(\alpha, \omega) \in [0,2\pi] \times \Omega\}$, where $\alpha$ is chosen uniformly in $[0,2\pi]$ and $\Omega$ is the probability space for our Gaussian entire function $f$ (we denote the probability measures by $m$ and $\mu$, respectively). We define the following events (all depend on $r$):

- $H = \{(\alpha, \omega) \mid f(z) \neq 0 \text{ in } rD\}$ – the hole event.


• $L = \left\{ (\alpha, \omega) \mid \left| \frac{d}{d\phi} \log (p(\rho e^{i\phi})) \right| \leq C \cdot \left( 1 - \frac{\rho}{r} \right)^{-5} \cdot S(r), \forall \phi \in [0, 2\pi] \right\}$

  - The non-exceptional event of Lemma 45, w.r.t $r$ and $\rho$.

• $C = \left\{ (\alpha, \omega) \mid \frac{1}{n} \sum \log |f(z_j e^{i\alpha})| \leq \log |f(0)| + 1 \right\}$.

• $D = \left\{ (\alpha, \omega) \mid |\alpha - \alpha^*(\omega)| < \Delta \alpha \right\}$.

**Remark.** We defined $\alpha^*$ only when $f(z) \neq 0$ in $r\mathbb{D}$, in other cases we can arbitrarily select $\alpha^* = 0$.

We note that $\alpha^*$ can be chosen to be measurable with respect to $\Omega$ and that the events $H$ and $L$ do not depend on the choice of $\alpha$.

Notice that the event $D$ is independent from the event $H \cap L$. Indeed, since

$$1_{H \cap L \cap D}(\alpha, \omega) = 1_{H \cap L}(\alpha, \omega) \cdot 1_D(\alpha, \omega) = 1_{H \cap L}(0, \omega) \cdot 1_D(\alpha, \omega)$$

we get that

$$\mathcal{P}(H \cap L \cap D) = \int \left[ \left( \int_{|\alpha - \alpha^*(\omega)| < \Delta \alpha} dm(\alpha) \right) \cdot 1_{H \cap L}(0, \omega) \right] d\mu(\omega)$$

$$= 2 \Delta \alpha \cdot \int 1_{H \cap L}(0, \omega) d\mu(\omega) = \mathcal{P}(D) \cdot \mathcal{P}(H \cap L).$$

Notice also that by the discussion in the previous section we have $H \cap L \cap D \subset C$. Therefore,

$$\mathcal{P}(C) \geq \mathcal{P}(H \cap L \cap D) = \mathcal{P}(D) \cdot \mathcal{P}(H \cap L) \geq$$

$$\geq \Delta \alpha \cdot \left( \mathcal{P}(H) - \mathcal{P}(L^c) \right),$$

and so

$$\mathcal{P}(H) \leq \frac{1}{\Delta \alpha} \cdot \mathcal{P}(C) + \mathcal{P}(L^c).$$

We now introduce the following events:

• $A = \left\{ (\alpha, \omega) \mid \log |f(0)| \leq \log S(r) \right\}$.

• $B = \left\{ (\alpha, \omega) \mid M(r) = \max_{z \in r\mathbb{D}} |f(z)| \leq e^{3S(r)} \right\}$. 
To estimate $\mathcal{P}(C)$ from above we use the inequality

$$\mathcal{P}(C) \leq \mathcal{P}(A \cap B \cap C) + \mathcal{P}(A^c) + \mathcal{P}(B^c).$$

By (13.4) from Subsection 13.5 and Lemma 42,

$$\mathcal{P}(A^c) \leq e^{-S^2(r)},$$
$$\mathcal{P}(B^c) \leq e^{-cS^2(r)}.$$

Note that these probabilities are very small compared to $\exp(-2S(r))$.

In the next section, we will show that for a good selection of the set of points $\{z_j\}$ we have

$$\mathcal{P}(A \cap B \cap C) \leq \exp(-S(\rho) + Cn(r) \log S(r)).$$

(16.2)

Therefore, we get (using Lemma 45),

$$\mathcal{P}(H) \leq \mathcal{P}(L^c) + \frac{1}{\Delta \alpha} \cdot (\mathcal{P}(A \cap B \cap C) + \mathcal{P}(A^c) + \mathcal{P}(B^c))$$
$$\leq 2e^{-S(\rho)n(\rho)} + \exp\left(-S(\rho) + \log \frac{1}{\Delta \alpha} + Cn(r) \log S(r) + O(1)\right)$$
$$\leq \exp\left(-S(\rho) + c \log \frac{1}{\gamma} + Cn(r) \log S(r) + O(1)\right).$$

Finally, by Lemma 37, if we select $\gamma = \frac{1}{m(r)}$, we have

$$\mathcal{P}(H) \leq \exp(-S(r) + Cn(r) \log S(r)),$$

thus proving Proposition 46.

### 16.2 Estimates for the Probabilities

We now turn to finding an upper bound for the probability of the event $A \cap B \cap C$, that is the event when simultaneously

$$\log |f(0)| \leq \log S(r),$$
$$M(r) = \max_{z \in \rho T} |f(z)| \leq e^{3S(r)},$$
$$\frac{1}{n} \sum \log |f(z_j e^{i\alpha})| \leq \log |f(0)| + 1 \leq \log S(r) + 1.$$

Recall that the points $\{z_j\}$ were until now some $n$ arbitrary points on $\rho T$. 
For every $\alpha \in \mathbb{R}$, the vector $(f(e^{i\alpha}z_1), \ldots, f(e^{i\alpha}z_n))$ has the multivariate complex Gaussian distribution with the covariance matrix
\[ \Sigma_{ij} = \text{Cov}(f(e^{i\alpha}z_i), f(e^{i\alpha}z_j)) = \mathcal{E}(f(e^{i\alpha}z_i)f(e^{i\alpha}z_j)) = \sum_k a_k^2 (z_i \bar{z}_j)^k. \]

We see that the covariance matrix does not depend on rotation by $\alpha$. By Fubini,
\[
\mathcal{P}(A \cap B \cap C) = \int \int 1_{A \cap B \cap C} (\alpha, \omega) \, dm(\alpha) \, d\mu(\omega) = \int \left[ \int 1_{A \cap B \cap C} (\alpha, \omega) \, d\mu(\omega) \right] \, dm(\alpha) \\
\leq \int_{\Omega} \frac{1}{\pi^n \det \Sigma} \exp(-\zeta^* \Sigma^{-1} \zeta) \, d\zeta,
\]
where $\Omega = \Omega_r$ is the following set \{event\}:
\[
\Omega = \left\{ (\zeta_1, \ldots, \zeta_n) \mid \frac{1}{n} \sum_{j=1}^n \log |\zeta_j| \leq \log S(r) + 1, |\zeta_j| \leq e^{3S(r)}, 1 \leq j \leq n \right\}.
\]

We have the following upper bound for the probability of the event $\Omega$:
\[
\mathcal{P}(\Omega) = \int_{\Omega} \frac{1}{\pi^n \det \Sigma} \exp(-\zeta^* \Sigma^{-1} \zeta) \, d\zeta \leq \int_{\Omega} \frac{1}{\pi^n \det \Sigma} \, d\zeta = \frac{\text{vol}_{C^n}(\Omega)}{\pi^n \det \Sigma}.
\]

We start by finding a good lower bound for $\det \Sigma$. This depends on a special selection of the points $\{z_j\}$.

**Lemma 47.** Let $n \in \mathbb{N}$ and $j_1, \ldots, j_{n-1} \in \mathbb{N}$. There exist $n$ points $\{z_j\}$ on $\rho \mathbb{T}$ such that the determinant of the generalized Vandermonde matrix
\[
A = \begin{pmatrix}
1 & z_1^{j_1} & \ldots & z_1^{j_{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
1 & z_n^{j_1} & \ldots & z_n^{j_{n-1}}
\end{pmatrix}
\]
satisfies $|\det A| \geq \rho \sum_{k=1}^{n-1} j_k$.

**Proof.** Start with the formal expression for the determinant:
\[
\det A = \sum_{\sigma} \text{sgn}(\sigma) \prod_{m=1}^n z_m^{j_{\sigma(m)} - 1},
\]
where the sum is over all the permutations of the set \( \{1, \ldots, n\} \) and we set \( j_0 = 0 \).

If we write \( z_j = \rho e^{i\theta_j} \) then we have

\[
\int_{\mathbb{T}^n} |\det A|^2 \, d\theta_1 \ldots d\theta_n = \int_{\mathbb{T}^n} \left( \sum_{\sigma} \text{sgn} (\sigma) \prod_{m=1}^{n} z_{\sigma(m)-1}^{j_{\sigma(m)-1}} \right) \cdot \left( \sum_{\tau} \text{sgn} (\tau) \prod_{m=1}^{n} z_{\tau(m)-1}^{j_{\tau(m)-1}} \right) \, d\theta_1 \ldots d\theta_n
\]

\[
= \rho^2 \sum_{m=1}^{n-1} j_k (\Sigma_1 + \Sigma_2),
\]

where

\[
\Sigma_1 = \sum_{\sigma} \int_{\mathbb{T}^n} 1 \, d\theta_1 \ldots d\theta_n = (2\pi)^n \cdot n!,
\]

\[
\Sigma_2 = \sum_{\sigma \neq \tau} \text{sgn} (\sigma) \cdot \text{sgn} (\tau) \left[ \int_{\mathbb{T}^n} \prod_{m=1}^{n} \exp \left( i\theta_m \left( j_{\sigma(m)-1} - j_{\tau(m)-1} \right) \right) \, d\theta_1 \ldots d\theta_n \right].
\]

Notice that \( j_k \neq j_l \) if \( k \neq l \). Since \( \sigma \neq \tau \) in the sum \( \Sigma_2 \), for at least one \( m \in \{1, \ldots, n\} \) we have that \( j_{\sigma(m)-1} \neq j_{\tau(m)-1} \). The numbers \( j_k \) are all integers and therefore we get that \( \Sigma_2 = 0 \). Thus we conclude that there exist some \( n \) points \( \{z_j\} \) on \( \rho \mathbb{T} \) such that

\[
|\det A| \geq \rho \sum_{m=1}^{n-1} j_k.
\]

\[
\square
\]

We now have the following

**Corollary 48.** Using the configuration of \( n = n(r) \) points \( \{z_j\} \) given by the previous lemma, we have

\[
\log (\det \Sigma) \geq S(r).
\]

**Proof.** Notice that we can represent \( \Sigma \) in the form

\[
\Sigma = V \cdot V^*;
\]

where

\[
V = \begin{pmatrix}
a_0 & a_1 \cdot z_1 & \ldots & a_j \cdot z_{j_1}^i & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\
0 & 0 & \ldots & 0 & \ldots \\

a_0 & a_1 \cdot z_n & \ldots & a_j \cdot z_{j_n}^i & \ldots
\end{pmatrix}
\]
We can estimate \( \det \Sigma \) by projecting \( V \) on \( N(r) = \{a_0, a_{j_1}, \ldots, a_{j_{n-1}}\} \) coordinates (let’s denote this projection by \( P \)). Since \( \det \Sigma \) is the square of the product of the singular values of \( V \), and these values are only reduced by the projection, we have

\[
\det \Sigma \geq (\det PV)^2 = \frac{1}{\prod_{j \in N(r)} a_j^2} \cdot \begin{vmatrix} 1 & z_1^j & \cdots & z_{n-1}^j \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n^j & \cdots & z_{n-1}^j \end{vmatrix}^2,
\]

and so, by the previous lemma,

\[
\det \Sigma \geq \prod_{j \in N(r)} a_j^2 \cdot \begin{vmatrix} 1 & z_1^j & \cdots & z_{n-1}^j \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_n^j & \cdots & z_{n-1}^j \end{vmatrix}^2 \geq \prod_{j \in N(r)} a_j^2 \cdot r^2 \sum_{j \in N(r)} \frac{1}{j} \geq \prod_{j \in N(r)} a_j^2 r^{2j} = \exp(S(r))
\]

We now want to estimate

\[
I = \frac{1}{\pi^n} \cdot \text{vol}_{C^n}(\Omega),
\]

with respect to the Lebesgue measure on \( \mathbb{C}^n \), where

\[
\Omega = \left\{ \zeta \in \mathbb{C}^n \mid \frac{1}{n} \sum_{j=1}^{n} \log |\zeta_j| \leq \log S(r) + 1 \quad \text{and} \quad |\zeta_j| \leq e^{3S(r)}, \quad 1 \leq j \leq n \right\}.
\]

**Lemma 49.** We have

\[
I \leq \exp (Cn(r) \log S(r)).
\]

**Proof.** We use the following abbreviated notation: \( n = n(r), S = S(r) \). Let \( \zeta = (\zeta_1, \ldots, \zeta_n) \in \Omega \). For every \( \zeta_j \) we find the minimal integer \( k_j \) such that \( |\zeta_j| \leq e^{k_j} \). Note that \( k_j \leq 4S \). We also have

\[
\sum_{j=1}^{n} (k_j - 1) \leq \sum_{j=1}^{n} \log |\zeta_j| \leq n (\log S + 1),
\]
and so
\[ \sum_{j=1}^{n} k_j \leq n \log S + 2n. \]
If it happens that \( \sum_j k_j < 0 \), then we just increase some of the negative \( k_j \)’s until the sum is 0. We now observe that each \( \zeta \in \Omega \) is contained in some polydisk of the form \( \{ \xi \mid |\xi_j| \leq e^{k_j}, 1 \leq j \leq n \} \). What is left is to estimate the total volume of all polydisks for which
\[ 0 \leq \sum_{j=1}^{n} k_j \leq n \log S + 2n \]
and
\[ k_j \leq 4S. \]
It is clear that the volume of each polydisk is bounded by
\[ \pi^n \exp \left( \sum_j k_j \right) \leq \pi^n \exp (n \log S + 2n). \]
Also, the value of each \( k_j \) is between \(-4S(n-1)\) and \(4S\), so the total number of polydisks is at most \((4Sn)^n\). Overall, the total volume of all the polydisks is at most
\[ \pi^n \exp (n \log S + 2n) \cdot (4Sn)^n \leq \pi^n \exp (2n \log S + Cn \log n) \tag{16.3} \]
\[ \leq \pi^n \exp (Cn \log S). \]
Therefore we get the required bound for \( I \).

Now, the estimate \((16.2)\) follows, since the original points \( \{z_j\} \) satisfy \(|z_j| = \rho\), and by Corollary 48 and Lemma 49
\[ \mathcal{P}(\Omega) \leq \exp (-S(\rho) + Cn(r) \log S(r)). \]
This completes the proof of Proposition 46.

17 On The Exceptional Set in Theorem 4

We construct a GAF \( f(z) = \sum_{n \geq 0} \xi_n a_n z^n \) and a set \( E \) of infinite Lebesgue measure such that
\[ p_H(f; r) \geq 2 \cdot S(r) - C \sqrt{S(r)}, \quad r \in E. \]
This shows that in general, an exceptional set in the statement of Theorem 4 is necessary. The idea of the proof is that indices \( n \) such that \( a_n r^n = 1 \) do not increase
the value of $S(r)$, while they reduce the probability of a hole event (since they reduce the probability that the free term will dominate the others). Therefore, if there are sufficiently many indices with this property, then the hole probability can be reduced. It is intuitively clear that this phenomenon cannot occur for too many values of $r$. In fact, by Theorem 4 (and specifically Lemma 54), the set of such ‘bad’ $r$’s is of a finite logarithmic measure.

17.1 Parameters of the construction

The construction is relatively simple, we start by fixing some increasing sequence of radii $\{r_m\}$. At each radius $r_m$ we set a ‘block’ of terms $a_j r^j$ to be equal to 1, the length of the $m$th block is equal to $l_m$ (which is an increasing sequence of integers). All the blocks are consecutive, so that this construction determines all the coefficients of the function $f$. We note, that at the radius $r_m$ only the terms in the blocks 1 to $m - 1$ contribute to $S(r_m)$ (since all the other terms will be of modulus equal or smaller than 1).

We then choose a sequence $\delta_m \in (0, 1)$, such that the terms in the $m$th block are still sufficiently large at the radius $r_m e^{-\delta_m}$. Now we repeat the proof of the lower bound for the hole probability, with some changes. The most important of these changes, is that instead of using Corollary 48 to get a lower bound for the determinant of the covariance matrix, we use a more precise estimate. We note that the corollary has advantage over the lemma in the case where the coefficients $a_n$ are not regular (for example when the function $f$ is lacunary).

We will fix the following values of the parameters, for $m \geq 1$:

\[
\begin{align*}
r_0 &= 1, & r_m &= \exp(a^m), \\
k_0 &= 0, & k_m &= \lfloor \exp(b^m) \rfloor, \\
l_0 &= 1, & l_m &= k_m - k_{m-1} \\
\delta_0 &= \frac{1}{2}, & \delta_m &= \frac{1}{2k_m},
\end{align*}
\]

where $[x]$ denotes the integral part of $x$. We will choose the values of $a, b \in (1, \infty)$ later in the proof.

For every $m \in \mathbb{N}$ we choose $a_j$ for all the indices $j \in \{k_{m-1} + 1, \ldots, k_m\}$ such that they will satisfy $a_j r^j_m = 1$. In addition, we set $a_0 = 1$.

We now turn to look at the properties of this function. First it is worth to mention that this choice of coefficients guarantees that $f$ is an entire function, since $a_j^{1/j} = \frac{1}{r_m}$ for $j$’s in the $m$th block and $r_m \to \infty$ as $m \to \infty$. 
17.2 Some properties of the function $f$

We start by analyzing the size of $S(r)$ at and near the radius $r_m$.

**Claim 50.** Let $\eta \in (0,1)$ and $\mu \in \mathbb{N}$ sufficiently large. We have the following bounds

$$
S(r^{\mu}e^{-\eta}) \geq S(r^{\mu}) - 2\eta \exp(2b^{\mu-1}) ,
$$

$$
S(r^{\mu}e^{\eta}) \leq S(r^{\mu}) + 2\eta \exp(2b^{\mu})
$$

and

$$
S(r^{\mu}e^{-\eta}) \geq \frac{(a-1)}{2} \cdot a^{\mu-1} \exp(2b^{\mu-1}) - 2\eta \exp(2b^{\mu-1}) ,
$$

$$
S(r^{\mu}e^{\eta}) \leq 2(a-1)a^{\mu-1} \exp(2b^{\mu-1}) + 2\eta \exp(2b^{\mu}) .
$$

**Proof.** We recall that

$$
S(r) = 2 \cdot \sum_{n \geq 0} \log^+(a_n r^n) ,
$$

which, using the special properties of our construction, we can rewrite in the following way

$$
S(r) = 2 \cdot \sum_{m=1}^{\infty} \sum_{j=1}^{k_m} \log^+ \left( a_j r_m^j \left( \frac{r}{r_m} \right)^j \right) = 2 \cdot \sum_{m=1}^{\infty} \left[ \log^+ \left( \frac{r}{r_m} \right) \sum_{j=1}^{k_m - 1} j \right] + \sum_{m=1}^{\infty} \left[ \log^+ \left( \frac{r}{r_m} \right) \sum_{j=1}^{k_m} j \right].
$$

By our choice of the sequence $r_m$ we note that $r_{\mu-1} < r^{\mu}e^{-\eta}$, for sufficiently large $m$. We conclude that if $r = r^{\mu}e^{-\eta}$ only the first $\mu - 1$ terms in (17.1) do not vanish, while if $r = r^{\mu}e^\eta$ the first $\mu$ contribute to $S(r)$. This leads to the following estimates

$$
S(r^{\mu}e^{-\eta}) = \sum_{m=1}^{\mu-1} \left[ (\log r^{\mu} - \log r_m - \eta) \left( k_m^2 + k_m - k_{m-1} - k_{m-1} \right) \right]
$$

$$
= S(r^{\mu}) - \eta \sum_{m=1}^{\mu-1} \left( k_m^2 + k_m - k_{m-1} - k_{m-1} \right)
$$

$$
\geq S(r^{\mu}) - 2\eta k^{2}_{\mu-1} \geq S(r^{\mu}) - 2\eta \exp(2b^{\mu-1}) .
$$

and

$$
S(r^{\mu}e^{\eta}) = \sum_{m=1}^{\mu} \left[ (\log r^{\mu} - \log r_m + \eta) \left( k_m^2 + k_m - k_{m-1} - k_{m-1} \right) \right]
$$

$$
\leq S(r^{\mu}) + \eta \sum_{m=1}^{\mu} \left( k_m^2 + k_m - k_{m-1} - k_{m-1} \right)
$$

$$
\leq S(r^{\mu}) + 2\eta k^{2}_{\mu} \leq S(r^{\mu}) + 2\eta \exp(2b^{\mu}) .
$$
The other estimates follow
\[
S \left( r_\mu e^{-\eta} \right) \geq (a^\mu - a^{\mu-1}) \left( \exp \left( 2b^{\mu-1} \right) - \exp \left( 2b^{\mu-2} \right) \right) - 2\eta \exp \left( 2b^{\mu-1} \right) \\
\geq \frac{(a - 1)}{2} \cdot a^{\mu-1} \exp \left( 2b^{\mu-1} \right) - 2\eta \exp \left( 2b^{\mu-1} \right),
\]
and
\[
S \left( r_\mu e^{\eta} \right) \leq 2 \left( a^\mu - a^{\mu-1} \right) \exp \left( 2b^{\mu-1} \right) + 2\eta \exp \left( 2b^{\mu} \right) \\
= 2 (a - 1) a^{\mu-1} \exp \left( 2b^{\mu-1} \right) + 2\eta \exp \left( 2b^{\mu} \right).
\]

We recall the following definitions
\[
N_\delta (r) = \{ n : a_n r^n \geq \exp (-\delta n) \}, \\
N (r) = N_0 (r), \\
n (r) = \# N (r).
\]
It is clear from the construction that \( n (r_m) = k_m \). Moreover, for any \( \delta \in [0, 1] \) we have that \( \# N_\delta (r) = n (r) \) for \( r \) sufficiently large. Indeed if \( r \in [r_{m-1} , r_m] \), then it is enough to consider indices \( j \in \{ k_m + 1, k_{m+1} \} \), and for them
\[
a_j r^j = a_j r_{m+1}^j \left( \frac{r}{r_{m+1}} \right)^j \leq \left( \frac{r_m}{r_{m+1}} \right)^j < e^{-\delta j},
\]
where the last inequality holds (for \( m \) sufficiently large) by our choice of the sequence \( r_m \). A similar argument shows that \( \# N_\delta (re^{\eta}) = n (r) \) for any \( \eta \in (0, 1) \) and any \( \delta \in [0, 1] \).

In order to improve the lower bound estimate we need a ‘corrected’ version of Lemma 42, which will hold for every sufficiently large \( r \in [r_m , r_m e^{\eta}] \) (for our specific function \( f \)). For the convenience of the reader we reprove the lemma with the necessary changes. We recall our notation
\[
M (r) = \max_{z \in rD} |f (z)|.
\]

**Lemma 51.** Let \( \eta \in (0, 1) \) and let \( m \in \mathbb{N} \) be sufficiently large. For any \( r \in [r_m , r_m e^{\eta}] \)
\[
\mathcal{P} (M (r) \geq e^{3S(r)}) \leq C \cdot \exp (-\exp (S (r))) \leq e^{-S^2(r)}.
\]
Proof. We set \( \widetilde{N} (r) = \{ n < S^{b/2} (r) \} \). Notice that, by Claim 50 and by the previous discussion we have that \( S^{b/2} (r) \geq S^{b/2} (r_m) \geq N_1 (r_m e^\eta) = n (r_m) = k_m \) for every \( \eta \in [0, 1) \). The rest of the proof is very similar to that of Lemma 42. We define the event \( \Omega_r \) as the intersection of the events (i) and (ii), where

\[
(i) : \bigcap_{n \in N(r)} (i)_n,
(ii) : \bigcap_{n \in (\widetilde{N}(r))^c} (ii)_n,
\]

and

\[
(i)_n : |\xi_n| \leq (a_n r^n)^{-1} e^{2S(r)};
(ii)_n : |\xi_n| \leq \exp \left( \frac{n}{2} \right).
\]

We have the following estimate for \( M (r) \):

\[
|f(z)| \leq \sum_{n \in \widetilde{N}(r)} |\xi_n| a_n r^n + \sum_{n \in (\widetilde{N}(r))^c} |\xi_n| a_n r^n
\]

\[
\leq S^{b/2} (r) \cdot e^{2S(r)} + \sum_{n \geq S^{b/2} (r)} e^{\frac{n}{2}} \cdot e^{-n}
\]

\[
\leq S^{b/2} (r) \cdot e^{2S(r)} + O(1)
\]

\[
\leq e^{3S(r)};
\]

provided that \( r \) is sufficiently large.

Now we estimate the probability of the complement of \( \Omega_r \). We have:

\[
P \left( |\xi_n| \geq \frac{e^{2S(r)}}{a_n r^n} \right) = \exp \left( - \frac{e^{4S(r)}}{(a_n r^n)^2} \right) \leq \exp \left( - e^{2S(r)} \right),
\]

\[
P \left( |\xi_n| \geq e^{\frac{n}{2}} \right) = \exp \left( - \exp \left( n \right) \right).
\]

By the union bound,

\[
P ((i)^c) \leq S^{b/2} (r) \cdot \exp \left( - \exp \left( 2S (r) \right) \right),
\]

and

\[
P ((ii)^c) \leq \sum_{n \geq S^{b/2} (r)} \exp \left( - \exp \left( n \right) \right) \leq C \cdot \exp \left( - \exp \left( S^{b/2} (r) \right) \right),
\]

since the first term in the sum above dominates the others. So overall we have

\[
P \left( M (r) \geq e^{3S(r)} \right) \leq C \cdot \exp \left( - \exp \left( S^{b/2} (r) \right) \right) \leq \exp \left( - S^2 (r) \right),
\]

again for \( r \) large enough. \( \square \)
To make the reading of this proof easier, we quote here the statements of some of the lemmas that are used in the proof of the lower bound. The proofs can be found in Section 15.

**Lemma 52.** Let $\rho > 0$ be sufficiently large. Then

$$\mathcal{P} (M(\rho) \leq \exp (-S(\rho))) \leq \exp (-S(\rho) \cdot n(\rho)).$$

Note the proof of this lemma do not require any modification. The next two lemmas requires the use of Lemma 51 and Lemma 52 instead of the original lemmas used. The rest of the proofs is the same.

**Lemma 53.** Let $r \in [r_m, r_m e^\eta]$ and $0 < \rho < r$. Outside a set of probability at most

$$2 \cdot \exp (-S(\rho) \cdot n(\rho)),$$

we have

$$\int_{r_T} |\log |f|| \, dm \leq C \left(1 - \frac{\rho}{r}\right)^{-2} \cdot S(r).$$

**Lemma 54.** Let $r \in [r_m, r_m e^\eta]$ and $0 < \rho < r$. Then

$$\left|\frac{d}{d\phi} \log |f(\rho e^{i\phi})|\right| \leq C \left(1 - \frac{\rho}{r}\right)^{-5} \cdot S(r)$$

outside a set of probability at most

$$2 \cdot \exp (-S(\rho) \cdot n(\rho)).$$

### 17.3 Proof of the Lower Bound for $f$

We repeat step by step the general proof of the lower bound. The main difference being the better lower bound estimate for the determinant of the covariance matrix. Let $r \in [r_m, r_m e^\eta]$, where $\eta = \eta_m$. In the original proof, we got the following upper bound for the probability of the hole event

$$\mathcal{P} (H) \leq \mathcal{P} (L^c) + \frac{CS(r)}{\gamma^5} \cdot (\mathcal{P} (A \cap B \cap C) + \mathcal{P} (A^c) + \mathcal{P} (B^c)).$$

Here $\rho = r (1 - \gamma) = re^{-\delta}$, and the following events are used
\begin{itemize}
\item $A - \log |f(0)| \leq \log S(r)$,
\item $B - M(r) = \max_{z \in rD} |f(z)| \leq e^{3S(r)}$,
\item $C - \frac{1}{n} \sum \log |f(z_j e^{i\alpha})| \leq \log |f(0)| + 1$,
\item $L - \left| \frac{d \log|f(\rho e^{i\phi})|}{d\phi} \right| \leq C \cdot \left(1 - \frac{\rho}{r}\right)^{-5} \cdot S(r)$.
\end{itemize}

By (13.4), Lemma 51 and Lemma 54 we have the following bounds
\begin{align*}
P(A^c) &\leq e^{-S^2(r)}, \\
P(B^c) &\leq e^{-eS^2(r)}, \\
P(L^c) &\leq 2e^{-S(\rho)\cdot n(\rho)}.
\end{align*}

In addition, like in the original proof we have
\begin{equation*}
P(A \cap B \cap C) \leq \frac{\text{vol}_{\mathbb{C}^n}(\Omega)}{\pi^n \det \Sigma}.
\end{equation*}

Here $\Omega = \Omega_\rho$ is the following set (or event)
\begin{equation*}
\Omega = \left\{ (\zeta_1, \ldots, \zeta_n) \mid \frac{1}{n} \sum_{j=1}^{n} \log |\zeta_j| \leq \log S(r) + 1, |\zeta_j| \leq e^{3S(r)}, 1 \leq j \leq n \right\},
\end{equation*}

and $\Sigma$ is the covariance matrix of the Gaussian vector $(f(e^{i\alpha z_1}), \ldots, f(e^{i\alpha z_n}))$. Unlike the original proof, we choose the points $z_j$ to be equally distributed on $\rho \mathbb{T}$, we also choose $n$ to be smaller than $n(r) = k_m$. We set $n = \alpha k_m$ with some $\alpha = \alpha_m \in (0, 1)$ to be chosen later.

The upper bound for the volume of $\Omega$ is unchanged, but we need to use the precise estimate (16.3) that was proved in Lemma 16.3 that is
\begin{equation*}
\frac{\text{vol}_{\mathbb{C}^n}(\Omega)}{\pi^n} \leq \exp \left(2n \log S(r) + n \log n + Cn\right).
\end{equation*}

It remains to show that the lower bound for the determinant can be significantly improved beyond the original estimate $\log (\det \Sigma) \geq S(\rho)$. 

17.3.1 Some facts about circulant matrices

A matrix $C$ of the following form
\[
C = \begin{pmatrix}
  c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\
  c_{n-1} & c_0 & & c_{n-3} & c_{n-2} \\
  & & \ddots & \vdots & \vdots \\
  c_2 & c_3 & & c_0 & c_1 \\
  c_1 & c_2 & \cdots & c_{n-1} & c_0
\end{pmatrix}
\]
is called *circulant* (it is a special kind of a Toeplitz matrix). It is known that the eigenvalues of $C$ are given by
\[
\lambda_j(C) = c_0 + c_1 \omega_j + \ldots + c_{n-1} \omega_j^{n-1},
\]
where $\omega_j = \exp(2\pi ij/n)$, $j \in \{0, \ldots, n-1\}$ are the $n$th roots of unity.

Let $f(z) = \sum_{n \geq 0} \xi_n a_n z^n$ be a GAF, with $\{a_n\} \in \mathbb{R}^+$ and let $\{z_j\}$ be $N$ equidistributed points on the circle $rT$, say $z_j = re^{2\pi i j/N}$, $j \in \{0, \ldots, N-1\}$. We denote by $\Sigma$ the covariance matrix of the complex Gaussian vector $(f(z_0), \ldots, f(z_{N-1}))$, thus
\[
\Sigma_{jk} = \text{Cov}(f(z_j), f(z_k)) = E \left\{ f(z_j) f(z_k) \right\} = \sum_{n \geq 0} a_n^2 r^{2n} e^{2\pi i (j-k)n/N}.
\]

Then we have

**Lemma 55.** The eigenvalues of the covariance matrix $\Sigma$ are
\[
\lambda_k(\Sigma) = N \cdot \sum_{l \geq 0} a_{k+lN}^2 r^{2(k+lN)}, \quad k \in \{0, \ldots, N-1\}.
\]

**Proof.** We note that since the $z_j$s are equidistributed on $rT$ it follows that the matrix $\Sigma$ is circulant. Indeed $\Sigma_{jk} = C'_{k-j \mod N}$, where
\[
C_m = \sum_{n \geq 0} a_n^2 r^{2n} e^{-2\pi i mn/N}, \quad m \in \{0, \ldots, N-1\}.
\]

Therefore for $l \in \{0, \ldots, N-1\}$
\[
\lambda_l(\Sigma) = \sum_{m=0}^{N-1} C_m \omega_l^m = \sum_{m=0}^{N-1} \left[ \sum_{n \geq 0} a_n^2 r^{2n} e^{-2\pi i mn/N} \right] e^{2\pi i l m/N} = \sum_{n \geq 0} a_n^2 r^{2n} \sum_{m=0}^{N-1} \exp \left( \frac{2\pi i (l-n) m}{N} \right).
\]
Since
\[ \sum_{m=0}^{N-1} \exp \left( \frac{2\pi i (l-n)}{N} \cdot m \right) = \begin{cases} N, & \text{if } l = n \mod N \\ 0, & \text{otherwise,} \end{cases} \]
we get the result. \hfill \Box

### 17.3.2 Lower bound for the determinant of \( \Sigma \) and conditions on \( n \)

We set \( \rho = re^{-\delta_m} \). Now we can use Lemma 55 to improve our estimate for the determinant of \( \Sigma \).

**Claim 56.** We have
\[
\det \Sigma \geq \exp \left( S(\rho) \right) \exp \left( \alpha k_m (\log k_m - 2) \right).
\]

**Proof.** By Lemma 55 the eigenvalues of \( \Sigma \) are given by
\[
\lambda_k(\Sigma) = n \cdot \sum_{l=0}^{\infty} a_{k+ln}^2 \rho^{2(k+ln)}, \quad k \in \{0, \ldots, n-1\}.
\]
We trivially have for \( k \in \{0, \ldots, k_m-1\} \)
\[
\lambda_k(\Sigma) \geq n \cdot a_k^2 \rho^{2k}.
\]
For \( j \in \{k_m-1 + 1, \ldots, k_m\} \) we have
\[
a_j^2 \rho^{2j} = a_j^2 \left(re^{-\delta_m}\right)^{2j} \geq a_j^2 r_m^{2j} e^{-2\delta_m j} = e^{-2\delta_m j},
\]
thus by our choice \( \delta_m = \frac{1}{2k_m} \), we have \( a_j^2 \rho^{2j} \geq \frac{1}{e} \). Thus, for \( k \in \{k_m-1 + 1, \ldots, n\} \) we have
\[
\lambda_k(\Sigma) \geq \frac{n}{e} \sum_{l=0}^{\infty} \mathbb{I}_{\{k+ln \leq k_m\}} \geq \frac{n}{e} \cdot \left( \frac{k_m}{n} - 1 \right) \geq \frac{k_m}{e^2},
\]
therefore,
\[
\det \Sigma \geq \prod_{k \in \{0, \ldots, k_m-1\}} \left( n \cdot a_k^2 \rho^{2k} \right) \cdot \prod_{k=k_m-1+1}^{n} \frac{k_m}{e^2} \geq \exp \left( S(\rho) \right) \exp \left( k_m \log n + (\log k_m - 2) (n - k_m-1) \right).
\]
After setting \( n = \alpha k_m \) we get
\[
\det \Sigma \geq \exp \left( S(\rho) \right) \exp \left( k_m \log (\alpha k_m) + (\log k_m - 2) (\alpha k_m - k_m-1) \right) \geq \exp \left( S(\rho) \right) \exp \left( \alpha k_m (\log k_m - 2) \right).
\]
\hfill \Box
In order to improve on the original estimate we want to choose $k_m$ such that

$$\alpha k_m \log k_m \geq \varepsilon_1 S(r) \geq \varepsilon_1 S(\rho),$$

for some numerical constant $\varepsilon_1 > 0$. This gives the condition

$$\alpha \geq \frac{\varepsilon_1 S(r)}{k_m \log k_m}.$$

A non-trivial bound for the hole probability, we imply an upper bound for $\alpha$. We have

$$\frac{\text{vol}_C(\Omega)}{\pi^n \det \Sigma} \leq \exp \left( 2n \log S(r) + n \log n + C n \right) \frac{\exp \left( S(\rho) + \alpha k_m (\log k_m - 2) \right)}{\exp \left( \alpha k_m (2 \log S(r) + \log (\alpha) + C) - S(\rho) \right)}.$$

The natural condition on $\alpha$ is therefore

$$2 \log S(r) + \log (\alpha) + C \leq -\varepsilon_2 \log k_m,$$

with some numerical constant $\varepsilon_2 > 0$. Decreasing $\varepsilon_2$ by a little, we require

$$\log \left( \frac{1}{\alpha} \right) \geq 2 \log S(r) + \varepsilon_2 \log k_m,$$

which we can write as

$$\alpha \leq \frac{1}{S^2(r) k_m^{\varepsilon_2}} \leq 1.$$

Combining the two conditions on $\alpha$, we get

$$\frac{\varepsilon_1 S(r)}{k_m \log k_m} \leq \frac{1}{S^2(r) k_m^{\varepsilon_2}} \Rightarrow \varepsilon_1 S^3(r) \leq k_m^{1-\varepsilon_2} \log k_m.$$

### 17.3.3 Choosing the values of the parameters and finishing the proof

We now choose the parameters $a, b, \varepsilon_1, \varepsilon_2, \eta$ in an appropriate way. We have

$$k_m^{1-\varepsilon_2} \log k_m \geq \frac{1}{2} \exp \left( (1 - \varepsilon_2) b^m \right) b^m,$$

and

$$S(r) \leq 2 \left( a - 1 \right) a^{m-1} \exp \left( 2b^{m-1} \right) + 2 \eta \exp \left( 2b^m \right).$$
We first choose $\eta = \exp\left(-\left(2-\frac{2}{\varepsilon}\right)b^m\right)$ by balancing, which leads to the bound

$$S(r) \leq 2a^m \exp\left(2b^{m-1}\right), \quad r \in [r_m, r_m e^n].$$

Choosing $\varepsilon_1 = 2$ and $\varepsilon_2 = \frac{1}{2}$ we are left to select $a, b$ such that

$$16a^3m \exp\left(\frac{6}{b} \cdot b^m\right) \leq \frac{b^m}{2} \exp\left(\frac{1}{2} b^m\right).$$

It is clear that any $b > 12$ will be sufficient (for $m$ sufficiently large), so let us choose $b = 20$. We are left with some freedom to chose the value of $a$, in order to get an exceptional set of infinite Lebesgue measure we will require

$$\eta = \exp\left(-\frac{10}{10} \cdot 20^m\right) \geq \frac{1}{r_m} = \exp\left(-a^m\right),$$

so $a = 21$ will do. Notice that by this choice $\eta = o(\delta_m)$ and thus $\rho < r_m$.

Remark. We can check the validity of these choices, by noticing that

$$n \geq \alpha k_m \geq \frac{\varepsilon_1 S(r)}{\log k_m} \geq 20^{-m} \cdot \left(20^m \cdot \exp\left(2 \cdot 20^{m-1}\right)\right)$$

$$= \exp\left(2 \cdot 20^{m-1}\right) \geq \exp(20^{m-1}) \geq k_{m-1},$$

(here we used Claim 50).

Now, let us write $r = r_m e^\rho$, with $\rho \in [0, \eta]$. By the first two bounds in Claim 50 we have

$$S(\rho) = S\left(r e^{-\delta_m}\right) = S\left(r_m e^{\rho - \delta_m}\right) \geq S(r_m) - 2 (\rho - \delta_m) \exp\left(2 \cdot 20^{m-1}\right)$$

$$\geq S(r) - 2 (\rho - \delta_m) \exp\left(2 \cdot 20^{m-1}\right) - 2\eta \exp\left(2 \cdot 20^m\right)$$

$$\geq S(r) - 4\eta \exp\left(2 \cdot 20^m\right) \geq S(r) - 4 \exp\left(20^{m-1}\right)$$

$$\geq S(r) - \sqrt{S(r)}.$$

Wrapping things up

$$\frac{\text{vol}_{C_n}(\Omega)}{\pi^n \det \Sigma} \leq \exp(-\varepsilon_2 \alpha k_m \log k_m - S(\rho)) \leq \exp(-\varepsilon_1 \varepsilon_2 S(r) - S(\rho))$$

$$\leq \exp\left(-2S(r) + \sqrt{S(r)}\right).$$
Now, we use the fact that $\gamma \geq 2\delta_m = \frac{1}{k_m} \geq \exp (-20^m) \geq \frac{1}{S(r)^{10}}$, to conclude that
\[
\frac{CS(r)}{\gamma^5} \leq S(r)^{100},
\]
and so finally
\[
P(H) \leq \exp (-S(\rho) \cdot n(\rho)) + S(r)^{100} \cdot \exp \left(-2S(r) + C\sqrt{S(r)}\right)
\]
\[
\leq \exp \left(-2S(r) + C\sqrt{S(r)}\right).
\]
We remind that this upper bound holds for any $r \in [r_m, r_m e^\eta]$.

Remark. The exceptional set is of infinite Lebesgue measure. This follows from the fact that the following series
\[
\sum_{m=1}^{\infty} r_m (e^\eta - 1) \geq \sum_{m=1}^{\infty} \exp \left(21^m - \frac{19}{10} \cdot 20^m\right)
\]
are divergent.

18 (Counter-) Example for non-Gaussian random variables - Proof of Theorem 34

Let $K \subset \mathbb{C}$ be a compact set such that $0 \notin K$. Let $g(z) = \sum_{n \geq 0} \phi_n \frac{z^n}{\sqrt{n!}}$ be an entire function, with a sequence of coefficients $\phi_n \in K$. We want to prove that there exists an $r_0 = r_0(K) < \infty$ so that $g(z)$ must vanish somewhere in the disk $\{ |z| \leq r_0 \}$.

Suppose that the theorem is false, that is, there exists a sequence of entire functions
\[
g_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} \frac{z^n}{\sqrt{n!}}, \quad \phi_{n,k} \in K,
\]
and a sequence $r_k \to \infty$, such that $g_k$ does not vanish in $r_k \mathbb{D}$. Since $K$ is a compact set, we can find a subsequence, also denoted by $\{g_k\}$, such that $\phi_{n,k} \to \phi_n$ for each $n \in \mathbb{N}$. It is easy to see that the sequence $\{g_k\}$ converges locally uniformly to a limiting function $f$. Since $0 \notin K$, the limiting function $f$ is not identically zero. Now, using Hurwitz’s theorem (see [Ah, pg. 178]), $f$ does not vanish in any disk $r_k \mathbb{D}$; i.e. is does not vanish in the whole complex plane.

By known formulas expressing the order and type of entire functions in terms of its Taylor coefficients (see, for instance [Le2, pg. 6]), $f$ has order 2 and type $\frac{1}{2}$. Since
it does not vanish on $\mathbb{C}$, by Hadamard’s theorem, $f(z) = \exp(\alpha z^2 + \beta z + \gamma)$, with complex constants $\alpha, \beta, \gamma$, $|\alpha| = \frac{1}{2}$.

We want to prove that we cannot get a function $f$ of this form, using coefficients from the set $K$. We will use the asymptotics of the coefficients of $f$ to prove this. Denoting the Taylor coefficients of $f(z)$ by $b_n$, it is sufficient to show that the product $|b_n| \cdot \sqrt{n!}$ is not bounded between any two positive constants.

We first study the asymptotics of function of the form (12.4). Using Stirling’s approximation we get

$$\sqrt{n!} = \left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)\right)^{1/2}$$

$$= (2\pi n)^{1/4} \left(\frac{n}{e}\right)^{n/2} \left(1 + O\left(\frac{1}{n}\right)\right). \quad (18.1)$$

The asymptotics of $f$ are not as simple. Using rotation and scaling, we can assume $\alpha = \frac{1}{2}$ and $\gamma = 1$; moreover, it is easy to see that $\beta$ should not be zero. Therefore the problem is reduced to the study of the asymptotics of

$$f(z) = \exp \left(\frac{1}{2} \cdot z^2 + \beta \cdot z\right) = \sum_{n=0}^{\infty} b_n(\beta) \cdot z^n, \quad (18.2)$$

with $\beta \neq 0$, possibly a complex number. A standard application of the saddle point method\footnote{For the details see the version of the paper [Ni1] found in the arXiv.} shows that

$$b_{n-1}(\beta) = C_\beta \cdot \left(\frac{e}{n}\right)^{\frac{n}{2}} \cdot \left(\frac{e^{\beta \sqrt{n}}}{n} \cdot (-1)^n \cdot e^{-\beta \sqrt{n}}\right) \cdot \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \quad (18.3)$$

where $C_\beta$ is some constant. We can see that this is not the same rate of decay as in (18.1) for $n \to \infty$. We arrive at a contradiction, which completes the proof of Theorem 34.

19 Some open problems

One can ask if the error term in Theorem 4 is optimal for a regular sequence of coefficients $\{a_n\}$. This could already be interesting for the GEF. It is not clear if the
lower bound for \( p_{H,K} (f; r) \) requires an exceptional set. Another possibility is to try to find a ‘smoothed’ version of the function \( S (r) \), that will give rise to a result with no exceptional set \( E \).

A natural question in the case of the GEF is to study the hole probability in more general domains. Let \( K \subset \mathbb{C} \) be some connected compact set, with non-empty interior, and denote by \( rK \) the homothety by \( r \) of \( K \). We propose the following conjecture\(^\text{[10]}\)

**Conjecture 57.** For large values of \( r \),

\[
p_{H,K} (f; r) = \log^{-} \mathcal{P} (f (z) \neq 0 \text{ in } rK) = S (c (K) \cdot r) (1 + o (1)),
\]

where \( c (K) \) is the capacity (transfinite diameter) of \( K \).

The reason for proposing this conjecture is that in the case of the plane invariant model, the evaluation of the determinant in Corollary 48 leads to the Vandermonde determinant

\[
\begin{vmatrix}
1 & z_1 & \ldots & z_1^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & z_n & \ldots & z_n^{n-1}
\end{vmatrix}^2 = \prod_{1 \leq i \neq j \leq n} |z_i - z_j|^2.
\]

Taking the maximum of this expression over the possible positions of the \( z_j \)'s we can approximate the (appropriate power of) the capacity \( c (K) \). However, as the current method heavily uses the Taylor expansion, which works primarily in the case of the disk, some new ideas will be required.

Another very natural question is the asymptotics of the hole probability for disconnected domains. The most basic case is when our domain is the union of two disjoint disks. Notice that there are three parameters for this problem, the scale, the radius of one of the disk, and the distance between the center of the disks. In particular, it is not clear if the event that \( f (z) \neq 0 \) in one disk is ‘almost independent’ from the event that \( f (z) \neq 0 \) in the other disk, if they are sufficiently far from each other.

The third type of questions for the GEF is the question of hole probability for ‘thin domains’, that is, domains which are scaled only in one direction. For example, it is not clear what are the asymptotics of the hole probability in a rectangular domain with sides \( r \) and 1, as \( r \) tends to infinity. For an upper bound estimate that might be sharp see [GKP, Theorem 4.1]. Another interesting question is the asymptotics of the conditional hole probability, that is

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\(^{10}\) This problem is well-defined since the distribution of the zero set of the GEF is invariant with respect to the plane isometries.
Problem 58. What is the probability that there are no zeros in the disk $R \mathbb{D}$, conditioned on the event that there are no zeros in $r \mathbb{D}$, for $R > r$?

Still another question is to consider non-Gaussian random variables. We are interested in characterizing the asymptotics of the hole probability in terms of properties of the distribution function of the random variables. This can be viewed as a more accurate version of some results of Offord [OF2] (see also [BKPV, sect. 7.1]).

It should be mentioned that there are some partial results related to the asymptotics of the hole probability for GAFs in the unit disk ([BNPS]). However, it is still an open problem to find the analog of Theorem 4 for general GAFs in the unit disk.

It would be interesting to generalize Theorem 4 to large deviations of the number of zeros (for the GEF, see the papers [NSV1, ST3]).
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