Partial differential equations/Mathematical physics

On maximizing the fundamental frequency of the complement of an obstacle

Sur la maximisation de la fréquence fondamentale du complément d’un obstacle

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A B S T R A C T

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying a Hayman-type asymmetry condition, and let $D$ be an arbitrary bounded domain referred to as an “obstacle”. We are interested in the behavior of the first Dirichlet eigenvalue $\lambda_1(\Omega \setminus (x + D))$.
First, we prove an upper bound on $\lambda_1(\Omega \setminus (x + D))$ in terms of the distance of the set $x + D$ to the set of maximum points $x_0$ of the first Dirichlet ground state $\phi_{\lambda_1} > 0$ of $\Omega$. In short, a direct corollary is that if
\[
\mu_{\Omega} := \max_x \lambda_1(\Omega \setminus (x + D))
\]
is large enough in terms of $\lambda_1(\Omega)$, then all maximizer sets $x + D$ of $\mu_{\Omega}$ are close to each maximum point $x_0$ of $\phi_{\lambda_1}$.
Second, we discuss the distribution of $\phi_{\lambda_1}(\Omega)$ and the possibility to inscribe wavelength balls at a given point in $\Omega$.
Finally, we specify our observations to convex obstacles $D$ and show that if $\mu_{\Omega}$ is sufficiently large with respect to $\lambda_1(\Omega)$, then all maximizers $x + D$ of $\mu_{\Omega}$ contain all maximum points $x_0$ of $\phi_{\lambda_1}(\Omega)$.

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R É S U M É

Soit $\Omega \subset \mathbb{R}^n$ un domaine borné satisfaisant une condition de type Hayman asymétrique et soit $D$ un domaine borné arbitraire, dénommé « obstacle ». Nous nous intéressons au comportement de la première valeur propre de Dirichlet $\lambda_1(\Omega \setminus (x + D))$.
Nous établissons, dans un premier temps, une borne supérieure pour cette valeur propre en termes de la distance de l’ensemble $x + D$ à l’ensemble des points $x_0$ où la fonction propre du premier état de base de Dirichlet $\phi_{\lambda_1} > 0$ de $\Omega$ atteint son maximum. En bref, un corollaire immédiat est que, si
\[
\mu_{\Omega} := \max_x \lambda_1(\Omega \setminus (x + D))
\]
is large enough in terms of $\lambda_1(\Omega)$, then all maximizer sets $x + D$ of $\mu_{\Omega}$ are close to each maximum point $x_0$ of $\phi_{\lambda_1}$.
Second, we discuss the distribution of $\phi_{\lambda_1}(\Omega)$ and the possibility to inscribe wavelength balls at a given point in $\Omega$.
Finally, we specify our observations to convex obstacles $D$ and show that if $\mu_{\Omega}$ is sufficiently large with respect to $\lambda_1(\Omega)$, then all maximizers $x + D$ of $\mu_{\Omega}$ contain all maximum points $x_0$ of $\phi_{\lambda_1}(\Omega)$.

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1. Introduction and background

We consider the natural problem (seemingly first posed by Davies) of placing an obstacle in a domain so as to maximize the fundamental frequency of the complement of the obstacle. To be more precise, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $D$ be another bounded domain referred to as an “obstacle”. The problem is to determine the optimal translate $x + D$ so that the fundamental Dirichlet Laplacian eigenvalue $\lambda_1(\Omega \setminus (x + D))$ is maximized/minimized.

In case the obstacle $D$ is a ball, physical intuition suggests that for sufficiently regular domains and sufficiently small balls, $\Omega$, $\lambda_1(\Omega \setminus B_1(x))$ will be maximized when $x = x_0$, a point of maximum of the ground state Dirichlet eigenfunction $\phi_{1,1}$ of $\Omega$. Heuristically, such maximum points $x_0$ seem to be situated deeply in $\Omega$, hence removing a ball around $x_0$ should be an optimal way of truncating the lowest possible frequency. Our methods give equally good results for Schrödinger operators on a large class of bounded domains sitting inside Riemannian manifolds (see the remarks at the end of Section 2).

The following well-known result of Harrell–Kröger–Kurata treats the case when $\Omega$ satisfies convexity and symmetry conditions.

**Theorem 1.1** ([11]). Let $\Omega$ be a convex domain in $\mathbb{R}^n$ and $B$ a ball contained in $\Omega$. Assume that $\Omega$ is symmetric with respect to some hyperplane $H$. Then,

(a) at the maximizing position, $B$ is centered on $H$, and

(b) at the minimizing position, $B$ touches the boundary of $\Omega$.

The last result of Harrell–Kröger–Kurata seems to work under a rather strong symmetry assumption. We also recall that the proof of Harrell–Kröger–Kurata proceeds via a moving planes method, which essentially measures the derivative of $\lambda_1(\Omega \setminus B)$ when $B$ is shifted in a normal direction to the hyperplane (also see p. 58 of [13]). See also related work in [4], [14].

There does not seem to be any result in the literature treating domains without symmetry or convexity properties.

In our note, we consider bounded domains $\Omega \subset \mathbb{R}^n$ that satisfy an asymmetry assumption in the following sense.

**Definition 1.2.** A bounded domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the asymmetry assumption with coefficient $\alpha$ (or $\Omega$ is $\alpha$-asymmetric) if for all $x \in \partial \Omega$, and all $r_0 > 0$,

$$\frac{|B_{r_0}(x) \setminus \Omega|}{|B_{r_0}(x)|} \geq \alpha.$$

(2)

This condition seems to have been introduced in [12]. Further, the $\alpha$-asymmetry property was utilized by D. Mangoubi in order to obtain inradius bounds for Laplacian nodal domains (cf. [16]) as nodal domains are asymmetric with $\alpha = \frac{\pi}{\sqrt{\text{vol}(\Omega)}}$.

From our perspective, the notion of asymmetry is useful as it basically rules out narrow “spikes” (i.e. with relatively small volume) entering deeply into $\Omega$. For example, let us also observe that convex domains trivially satisfy our asymmetry assumption with coefficient $\alpha = \frac{1}{2}$.

2. The basic estimate for general obstacles

With the above in mind, we consider any bounded $\alpha$-asymmetric domain $\Omega \subset \mathbb{R}^n$ and a bounded obstacle domain $D$. We denote the first positive Dirichlet eigenvalue and eigenfunction of $\Omega$ by $\lambda_1$ and $\phi_{\lambda_1}(\Omega)$ respectively and let

$$M := \{x \in \Omega \mid \phi_{\lambda_1}(x) = \|\phi_{\lambda_1}(\Omega)\|_{L^\infty(\Omega)}\}$$

be the set of maximum points of $\phi_{\lambda_1}(\Omega)$.
Let us also put
\[
\mu_\Omega := \max_x \lambda_1(\Omega \setminus (x + D)).
\] (4)

Finally, for a given translate \(x + D\) of the obstacle, let us set
\[
\rho_x := \max_{y \in M} d(y, x + D),
\] (5)
measuring the maximum distance from a maximum point of \(\phi_{\lambda_1(\Omega)}\) to the translate \(x + D\).

Our main estimate is the following.

**Theorem 2.1.** Let us fix a translate \((x + D)\) and assume that \(\rho_x > 0\). Then
\[
\lambda_1(\Omega \setminus (x + D)) \leq \beta(\rho_x) \lambda_1(\Omega),
\] (6)
where \(\beta\) is a continuous decreasing function defined as
\[
\beta(\rho) = \begin{cases}
\beta_0 = \beta_0(n, \alpha), & \rho \sqrt{\lambda_1(\Omega)} > r_0 := r_0(n, \alpha), \\
\frac{c_0}{\rho^{\alpha_1(\Omega)}}, & \rho \sqrt{\lambda_1(\Omega)} \leq r_0, \quad c_0 = c_0(n),
\end{cases}
\] (7)
where \(\beta_0 r_0 = c_0\).

We remark that, in particular, if \(\rho_x\) is of sub-wavelength order (i.e. \(\lesssim \frac{1}{\sqrt{\lambda_1(\Omega)}}\)), then \(\lambda_1(\Omega \setminus (x + D)) \lesssim \frac{1}{\rho^2}\). If the obstacle \(D\) is convex, we can say more (see Theorem 4.1 below).

**Proof of Theorem 2.1.** The proof essentially exploits the fact that there are “almost inscribed” wavelength balls centered at maximum points of \(\phi_{\lambda_1(\Omega)}\). To make this statement precise, we recall the following theorem from [6], which works for all domains in compact Riemannian manifolds of dimension \(n \geq 3\) (planar domains are known to have wavelength inradius from the work of Hayman [112]).

**Theorem 2.2.** Let \(\dim M \geq 3, \epsilon_0 > 0\) be fixed, \(\Omega\) a domain inside \(M\), and \(x_0 \in \Omega\) be such that \(|\phi_\lambda(x_0)| = \max_\Omega |\phi_\lambda|\), where \(\phi_\lambda\) is the ground-state Dirichlet eigenfunction of \(\Omega\). There exists \(r_0 = r_0(\epsilon_0)\), such that
\[
\frac{|B_{r_0} \cap \Omega|}{|B_{r_0}|} \geq 1 - \epsilon_0,
\] (8)
where \(B_{r_0}\) denotes \(B\left(x_0, \frac{r_0}{\sqrt{\lambda_1(\Omega)}}\right)\).

We note that the existence of such an “almost-inscribed” wavelength ball was first established by Lieb (see [15]), and followed by further contributions from Maz’ya–Shubin (see [18]). The latter brings to light the importance of small or “negligible capacities” in quantifying the “almost-inscribed”-ness (see in particular Theorem 1.1 and Subsection 5.1 of [18]). The main contribution of Theorem 2.2 is the specification of the location of the “almost-inscribed” wavelength ball. For completeness, recall that Theorem 2.2 relies on two main ingredients – namely, the Feynman–Kac formula and certain capacity estimates related to hitting probabilities of Brownian motion. We first establish that a Brownian particle starting at any max point of the ground-state eigenfunction has low probability of hitting the boundary of the domain; more precisely, such a probability is bounded above by \(1 - e^t\) at time scales \(\sim \frac{1}{\lambda_1(\Omega)}\). On the other hand, by reducing \(t\) and \(r\) and keeping \(\frac{t}{r^2}\) constant, we are able to show that the particle has comparatively high probability of escaping a ball of radius \(\sim \frac{1}{\sqrt{\lambda_1(\Omega)}}\) around the max point, which tells us that the “size” of the ball \(B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}})\) outside the domain \(\Omega\) is fairly small. This gives us a comparison of “sizes” of \(B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}})\) and \(B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}}) \setminus \Omega\) in terms of probability. Using the fact that the heat kernel is the transition density for Brownian motion, in [10] Grigor’yan and Saloff-Coste are able to estimate the hitting probabilities of relatively compact sets \(K \subset M\) by a Brownian particle, in terms of pointwise heat kernel bounds on \(M\) and capacity of \(K\). In our setting, we wish to use their results on the set \(K := B(x_0, \frac{r}{\sqrt{\lambda_1(\Omega)}}) \setminus \Omega\). Using in particular Remark 4.1 of [10], and isocapacitary inequalities due to Maz’ya (see [17], Section 2.2.3), we are able to translate a comparison of size in terms of probability into a comparison of size in terms of capacity (which fits nicely with the insights of [18]) and then in terms of volume, respectively. We refer to [6] for more details (see also [19] for an extension to Schrödinger operators along similar lines). We also note that it follows from the proof that in Theorem 2.2, \(r_0\) can be taken as \(r_0 = \epsilon_0^{\alpha_{Mn}}\), which is slightly better than the scaling in [15]. This has applications to the inner radius problem of nodal domains of Laplace eigenfunctions, see [5], [7] for more details.
Now, it is clear that under the $\alpha$-asymmetry assumption, there exists an $r_0 := r_0(\alpha, n)$, such that around each maximum point $x_0 \in \Omega$ of $\phi_{\lambda_1(\Omega)}$ one can find a fully inscribed ball $B_{r_0/\sqrt{\lambda_1(\Omega)}}(x_0) \subseteq \Omega$. By the definition of $\rho_0$, it follows that we can find a maximum point $x_0 \in (\Omega \setminus (x + D))$ and an inscribed ball $B_{\rho_0}(x_0)$ where
\[
\rho_0 := \min \left( \frac{r_0}{\sqrt{\lambda_1(\Omega)}}, \rho_{\lambda_1} \right).
\]

As the first eigenvalue is monotonic with respect to inclusion, we see that
\[
\lambda_1(\Omega \setminus (x + D)) \leq \lambda_1(B_{\rho_0}(x_0)) = C\frac{\rho_0}{\rho_0},
\]
where $C = C(n)$ is a universal constant.

Expressing the right-hand side of the last inequality in terms of $\lambda_1(\Omega)$, we define the function $\beta(\rho)$ as above. This concludes the proof. $\square$

Here, we have considered the obstacle problem in the case of Euclidean spaces, on reasonably well-behaved domains, and for the operator $-\Delta + \lambda_1(\Omega)$, as that seems to be the primary case of interest. However, we also include some remarks outlining some straightforward generalizations.

**Remark 2.3.** It is clear that removing capacity zero sets from $\alpha$-asymmetric domains considered in Definition 1.2 will lead to the same conclusions. Indeed, in this situation, we will not be dealing with fully inscribed balls as above; instead, we will have balls whose first eigenvalue is comparable to the one of an inscribed one.

**Remark 2.4.** Also, in the setting of curved spaces, one has absolutely similar results for $\Omega \subseteq M$, where $(M, g)$ is a smooth compact Riemannian manifold, if we allow the constants to depend on the dimension, asymmetry, and the metric $g$.

**Remark 2.5.** Lastly, it is clear that the results of [19] allow us to extend our discussion here from operators of the form $-\Delta + \lambda_1(\Omega)$ to Schrödinger operators of the form $-\Delta + V$, where $V$ is bounded above. The conclusions are analogous with $\lambda_1(\Omega)$ replaced by $\|V\|_{\infty}$ and the proofs are identical.

Now, as an immediate implication of Theorem 2.1, we have the following corollary.

**Corollary 2.6.** Suppose that $\mu_\Omega = C_0 \lambda_1(\Omega)$, where $C_0 > \frac{\mu_\Omega}{r_0} \Omega$ is a given fixed constant and $c_0, r_0$ are the constants in Theorem 2.1. Then, for a maximizer $\tilde{x} + D$ of $\mu_\Omega$, we have
\[
\rho_{\tilde{x}} \leq \beta^{-1}(C_0).
\]
In particular, if $C_0$ is large,
\[
\rho_{\tilde{x}} \lesssim \frac{1}{\sqrt{C_0 \lambda_1(\Omega)}}.
\]

In other words, the above corollary can be interpreted as follows: either $\mu_\Omega$ is comparable to $\lambda_1(\Omega)$, or the maximum points of $\phi_{\lambda_1(\Omega)}$ are near the maximizer sets $\tilde{x} + D$ of $\mu_\Omega$.

We note that the localization in the Corollary above gets better when $C_0$ is large. By Faber–Krahn’s inequality, straightforward examples with large $C_0$ are domains $\Omega$ for which $\Omega \setminus (x + D)$ is sufficiently small for some $x$.

Particularly, for bounded convex domains in $\mathbb{R}^n$, by a theorem of Brascamp–Lieb (see Section 6 of [1] in particular), the level sets of $\phi_{\lambda_1(\Omega)}$ are convex. Since $\phi_{\lambda_1(\Omega)}$ is real analytic and it can be assumed positive on $\Omega \setminus \partial \Omega$ without loss of generality, this means that it has a unique point of maximum. So, in this setting, our result heuristically says that if the removal of a ball $B_r$ has a “significant effect” on the vibration of $\Omega \setminus B_r$, then $B_r$ must be centered quite close to the max point of the ground-state Dirichlet eigenfunction $\phi_{\lambda_1}$ of the domain $\Omega$, where the bound on $\rho_0$ gives the quantitative relation between the “effect” and the order of “closeness”. In a sense, this can be seen to be complementary to Corollary II.3 of [11].

3. **Inscribed balls and distribution of $\phi_{\lambda_1(\Omega)}$**

Further, we specify our results to the obstacle being a ball $D$. We point out a few statements related to the connection between the distribution of $\phi_{\lambda_1(\Omega)}$ and the possibility to inscribe a large ball at a given point $x$ in $\Omega$.

First, by Theorem 2.2 above, we immediately have the following observation.
Proposition 3.1. Let $\Omega$ be $\alpha$-asymmetric and let $\text{inrad}(\Omega)$ denote the inner radius of $\Omega$. If $x_0$ is a point of maximum of $\phi_{\lambda_1}(\Omega)$, then there exists an inscribed ball $B_C \text{inrad}(\Omega) \subseteq \Omega$, where $C = C(n, \alpha)$.

Proof of Proposition 3.1. We observe that by the results of [16], $\alpha$-asymmetric domains $\Omega$ satisfy

$$\frac{C_1(\alpha, n)}{\sqrt{\lambda_1(\Omega)}} \leq \text{inrad}(\Omega) \leq \frac{C_2(n)}{\sqrt{\lambda_1(\Omega)}}. \quad (13)$$

Now, it follows from our Theorem 2.2 (see [6]) that there exists an inscribed wavelength ball at the max point $x_0$, which concludes the proof. □

In particular, the last proposition applies for convex domains. We mention in this connection that localization results for maximum points of $\phi_{\lambda_1}(\Omega)$ in case $\Omega$ in plane convex domains can be found in the work of Grieser–Jerison (see [9]).

On the other hand, it is natural to ask how large is $\phi_{\lambda_1}(\Omega)$ at points admitting a large inscribed ball. For reasonably nicely behaved domains, we have the following:

Corollary 3.2. Let $\Omega$ be a $C^{2, \beta}$-regular $\alpha$-asymmetric domain and let $\phi_{\lambda_1}(\Omega)$ be normalized so that $\|\phi_{\lambda_1}(\Omega)\|_L^\infty(\Omega) = 1$. Suppose that for $x \in \Omega$ there exists a maximal inscribed ball $B_r(x) \subseteq \Omega$ where $r := c \text{inrad}(\Omega)$ for some $0 < c \leq 1$, such that $\frac{|\Omega \setminus B_r(x)|}{|\Omega|}$ is sufficiently small. Then

$$\phi_{\lambda_1}(x) > C,$$

where $C = C(\Omega, \partial\Omega, c, n)$. \hspace{1cm} (14)

Analogously, one can show a similar statement by demanding that $|B_r(x) \cap \Omega|$ is sufficiently large in comparison to $|\Omega|$.

Proof of Corollary 3.2. Let us first suppose that

$$|\Omega| = \kappa r^n, \quad \kappa > \omega_n,$$

where $\omega_n$ is the volume of a ball of radius 1. We use the Faber–Krahn inequality to obtain

$$\lambda_1(\Omega \setminus B_r(x)) \geq \frac{C}{|\Omega \setminus B_r(x)|^{2/n}} = \frac{C}{(|\Omega| - \omega_n r^n)^{2/n}} = \frac{C}{(\kappa - \omega_n)^{2/n}} \geq \frac{C \sqrt{C_\gamma(n)}}{\sqrt{C_\gamma(n)^2} \lambda_1(\Omega)} = \tilde{C_\gamma} \lambda_1(\Omega). \quad (16)$$

By assumption, $\tilde{C_\gamma}$ is sufficiently large, i.e., in particular, $\tilde{C_\gamma} > \frac{C_\gamma}{\tilde{C_\gamma}}$, so we may apply Corollary 2.6 to obtain that

$$\rho_k \leq \beta^{-1}(\tilde{C_\gamma}) = \sqrt{\frac{C_0}{\tilde{C_\gamma} \lambda_1(\Omega)}}. \quad (17)$$

On the other hand, the Schauder a priori estimates up to the boundary for $\phi_{\lambda_1}(\Omega)$ (see [8], Theorem 6.6) yield the existence of $\gamma = \gamma(\Omega, n)$, such that

$$\|\nabla \phi_{\lambda_1}(\Omega)\|_{L^\infty(\Omega)} \leq \gamma(\Omega, n) \sqrt{\lambda_1(\Omega)}. \quad (18)$$

As by assumption $\phi_{\lambda_1}(\Omega)(x_0) = 1$ and $\tilde{C_\gamma}$ is sufficiently large, then

$$\phi_{\lambda_1}(\Omega)(x) \geq C = C_0(\tilde{C_\gamma}, \gamma), \quad (19)$$

which concludes the claim. □

4. Relation between maximum points and convex obstacles

Note that Theorem 2.1 holds for arbitrary obstacles and gives a bound on the distance $\rho_k$ to maximum points of $\phi_{\lambda_1}(\Omega)$. However, it is desirable to deduce that $\rho_k = 0$, i.e. maximizers actually contain the maximum points of $\phi_{\lambda_1}(\Omega)$.

From Proposition 3.1 and Theorem 2.1, we deduce the following:

Theorem 4.1. Let $D$ be a convex obstacle, and $\tilde{x} + D$ maximize $\lambda_1(\Omega \setminus (\tilde{x} + D))$. Then there exists a constant $C_0 = C_0(\alpha, n)$ such that if $\lambda_1(\Omega \setminus (\tilde{x} + D)) \geq C \lambda_1(\Omega)$ for some $C \geq C_0$, then $\rho_k = 0$.

In other words, either $\mu_\Omega \sim \lambda_1(\Omega)$ or $\rho_k = 0$. 
Proof. To the contrary let us suppose that $\rho_x = d(\bar{x} + D, x_0) > 0$ where $x_0$ is a maximum point of $\phi_{\lambda_1}(\Omega)$ and $\lambda_1(\Omega \setminus (\bar{x} + D)) \geq C_1 \lambda_1(\Omega)$ for an arbitrary large $C > 0$.

We apply the statement of Proposition 3.1 and deduce that there is a wavelength inscribed ball $B$ at $x_0$. As $D$ is a convex domain, we can find a wavelength half-ball $B^{1/2} \subset \Omega \setminus (\bar{x} + D)$ containing $x_0$. By the assumption and eigenvalue monotonicity with respect to inclusion:

$$C_1 \lambda_1(\Omega) \leq \lambda_1(\Omega \setminus (\bar{x} + D)) \leq \lambda_1(B^{1/2}) \leq \frac{C_1}{(\text{inrad}(\Omega))^2} = C_2 \lambda_1(\Omega),$$

where $C_2 = C_2(n, \alpha)$. Taking $C$ sufficiently large, we get a contradiction. \hfill \Box

It is clear that for explicit applications, particularly in the case of convex domains, Theorem 4.1 is dependent on a precise knowledge of the location of the maximum point of $\phi_{\lambda_1}(\Omega)$. Localization of the maximum point of $\phi_{\lambda_1}(\Omega)$ (or more generally, the “hot spot”) is a problem that is far from being settled. Here we take the space to augment Theorem 4.1 with the recent results of [2].

First we recall the definition of the “heart” of a convex body $\Omega$. The following intuitive definition appears in [3], and it is equivalent to the (more technical) definition presented in [2].

Definition 4.2. Let $P$ be a hyperplane in $\mathbb{R}^n$ that intersects $\Omega$ so that $\Omega \setminus P$ is the union of two components located on either side of $P$. The domain $\Omega$ is said to have the interior reflection property with respect to $P$ if the reflection through $P$ of one of these subsets, denoted $\Omega_s$, is contained in $\Omega$, and in that case $P$ is called a hyperplane of interior reflection for $\Omega$. When $\Omega$ is convex, the heart of $\Omega$, denoted by $\mathcal{H}(\Omega)$, is defined as the intersection of all such $\Omega \setminus \Omega_s$ with respect to the hyperplanes of interior reflection of $\Omega$.

The following result is contained in Proposition 4.1 of [2].

Proposition 4.3 ([2]). The unique maximum point $x_0$ of $\phi_{\lambda_1}(\Omega)$ is contained in $\mathcal{H}(\Omega)$. Furthermore, $x_0$ is contained in the interior of $\mathcal{H}(\Omega)$, if the latter is non-empty.

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