RIGIDITY FOR GENERAL SEMICONVEX ENTIRE SOLUTIONS TO THE SIGMA-2 EQUATION

RAVI SHANKAR AND YU YUAN

Abstract. We show that every general semiconvex entire solution to the sigma-2 equation is a quadratic polynomial. A decade ago, this result was shown for almost convex solutions.

1. Introduction

In this paper, we show that every general semiconvex entire solution in $\mathbb{R}^n$ to the Hessian equation
\[
\sigma_k (D^2 u) = \sigma_k (\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} = 1
\]
with $k = 2$ must be quadratic. Here $\lambda_i$s are the eigenvalues of the Hessian $D^2 u$.

Theorem 1.1. Let $u$ be a smooth semiconvex solution to $\sigma_2 (D^2 u) = 1$ on $\mathbb{R}^n$ with $D^2 u \geq -KI$ for a large $K > 0$. Then $u$ is quadratic.

Recall the classical Liouville theorem for the Laplace equation $\sigma_1 (D^2 u) = \Delta u = 1$ or Jörgens-Calabi-Pogorelov theorem for the Monge-Ampère equation $\sigma_n (D^2 u) = \det D^2 u = 1$: all convex entire solutions to those equations must be quadratic. Theorem 1.1 has been settled under an almost convexity condition $D^2 u \geq \left( \delta - \sqrt{2/|n(n-1)|} \right) I$ for general dimension in the joint work with Chang [ChY]; and under the general semiconvexity condition $D^2 u \geq -KI$ in three dimensions by taking advantage of the special Lagrangian form of the equation in this case [Y]. Assuming a super quadratic growth condition, Bao-Chen-Ji-Guan [BCJG] demonstrated that all convex entire solutions to $\sigma_k (D^2 u) = 1$ with $k = 1, 2, \cdots, n$ are quadratic polynomials; and Chen-Xiang [CX] showed that all “super quadratic” entire solutions to $\sigma_2 (D^2 u) = 1$ with $\sigma_1 (D^2 u) > 0$ and $\sigma_3 (D^2 u) \geq -K$ are also quadratic polynomials. Warren’s rare saddle entire solutions for the $\sigma_2 (D^2 u) = 1$ case [W] confirm the necessity of the semiconvexity assumption. It was “guessed” in the 2009 paper [ChY] that Theorem 1.1 should hold true.

The equation $\sigma_2 (\kappa) = 1$ prescribes the intrinsic scalar curvature of a Euclidean hypersurface $(x, u(x))$ in $\mathbb{R}^n \times \mathbb{R}^1$ with extrinsic principal curvatures.
The function of the Schouten tensor arises in conformal geometry, and complex \( \sigma_2 \)-type equations arise from the Strominger system in string theory.

Our current work, as well the previous ones [ChY] [Y], has been inspired by Nitsche's classical paper [N], where the Legendre-Lewy transform was employed to produce an elementary proof of Jörgens' rigidity for the two dimensional Monge-Ampère equation, and in turn, Bernstein's rigidity for the two dimensional minimal surface equation.

The Legendre-Lewy transform of a general semiconvex solution satisfies a uniformly elliptic, saddle equation. In the almost convex case [ChY], the new equation becomes concave, thus Evans-Krylov-Safonov theory yields the constancy of the bounded new Hessian, and in turn, the old one. To beat the saddle case, one has to be “lucky”. Recall that, in general Evans-Krylov-Safonov fails as shown by the saddle counterexamples of Nadirashvili-Vlăduț [NV]. Our earlier trace Jacobi inequality, as an alternative log-convex vehicle, other than the maximum eigenvalue Jacobi inequality, in deriving the Hessian estimates for general semiconvex solutions in [SY], could rescue the saddleness. But the trace Jacobi only holds for large enough trace of the Hessian. It turns out that the trace added by a large enough constant satisfies the elusive Jacobi inequality (Proposition 2.1).

Equivalently, the reciprocal of the shifted trace Jacobi quantity is superharmonic, and it remains so in the new vertical coordinates under the Legendre-Lewy transformation by a transformation rule (Proposition 2.2). Then the iteration arguments developed in the joint work with Caffarelli [CaY] show the “vertical” solution is close to a “harmonic” quadratic at one small scale (Proposition 3.1, two steps in the execution: the superharmonic quantity concentrates to a constant in measure by applying Krylov-Safonov’s weak Harnack; a variant of the superharmonic quantity, as a quotient of symmetric Hessian functions of the new potential, is very pleasantly concave and uniformly elliptic, consequently, closeness to a “harmonic” quadratic is possible by Evans-Krylov-Safonov theory), and the closeness improves increasingly as we rescale (this is a self-improving feature of elliptic equations, no concavity/convexity needed). Thus a Hölder estimate for the bounded Hessian is realized, and consequently so is the constancy of the new and then the old Hessian. See Section 3.

In closing, we remark that, in three dimensions, our proof provides a “pure” PDE way to establish the rigidity, distinct from the geometric measure theory way used in the earlier work on the rigidity for special Lagrangian equations [Y, Theorem 1.3].
2. **Shifted trace Jacobi inequality and superharmonicity under Legendre-Lewy transform**

Taking the gradient of both sides of the quadratic Hessian equation

\[(2.1) \quad F (D^2 u) = \sigma_2 (\lambda) = \frac{1}{2} \left[ (\Delta u)^2 - |D^2 u|^2 \right] = 1,\]

we have

\[(2.2) \quad \Delta F Du = 0,\]

where the linearized operator is given by

\[(2.3) \quad \Delta F = \sum_{i,j=1}^n F_{ij} \partial_{ij} = \sum_{i,j=1}^n \partial_i (F_{ij} \partial_j),\]

with

\[(2.4) \quad (F_{ij}) = \begin{bmatrix} \sigma_1 + J \end{bmatrix} = \begin{bmatrix} \Delta u + J \end{bmatrix} = \begin{bmatrix} \sqrt{2 + |D^2 u|^2} \end{bmatrix} \begin{bmatrix} I \end{bmatrix} - D^2 u > 0.\]

Here without loss of generality, we assume \( \Delta u > 0 \) in the remaining. Otherwise the smooth Hessian \( D^2 u \) would be in the \( \Delta u < 0 \) branch of the equation \((2.1)\). Given the semiconvexity condition, the conclusion in Theorem 1.1 would be straightforward by Evans-Krylov-Safonov.

The gradient square \( |\nabla F v|^2 \) for any smooth function \( v \) with respect to the inverse “metric” \((F_{ij})\) is defined as

\[ |\nabla F v|^2 = \sum_{i,j=1}^n F_{ij} \partial_i v \partial_j v.\]

### 2.1. Shifted trace Jacobi inequality.

**Proposition 2.1.** Let \( u \) be a smooth solution to \( \sigma_2 (\lambda) = 1 \) with \( D^2 u \geq -KI \). Set \( \sqrt{b} = \ln ( \Delta u + J ) \). Then we have

\[(2.5) \quad \Delta_F b \geq \varepsilon |\nabla F b|^2\]

for \( J = 8nK/3 \) and \( \varepsilon = 1/3 \).

**Proof.** Step 1. Differentiation of the trace.

We derive the following formulas for function \( b = \ln ( \sigma_1 + J ) = \ln ( \Delta u + J ) \):

\[(2.6) \quad |\nabla F b|^2 = \sum_i f_i (\Delta u_i)^2 (\sigma_1 + J)^2,\]

and

\[(2.7) \quad \Delta_F b = \frac{1}{(\sigma_1 + J)} \left\{ 6 \sum_{i,j,k} w_{ijk}^2 + \sum_{j \neq i} w_{jii}^2 + \sum_i \left( 1 + \frac{f_i}{\sigma_1 + J} \right) (\Delta u_i)^2 \right\} \]

at \( x = p \), where, without loss of generality, \( D^2 u (p) \) is assumed to be diagonalized and \( f (\lambda) = \sigma_2 (\lambda) \).
Noticing (2.4), it is straightforward to have the identity (2.6) and at \( p \)

\[
\Delta F b = \sum_{i=1}^{n} f_i \left[ \frac{\partial_{ii} \Delta u}{(\sigma_1 + J)} - \frac{(\partial_i \Delta u)^2}{(\sigma_1 + J)^2} \right].
\]

Next we substitute the fourth order derivative terms \( \partial_{ii} \Delta u = \sum_{k=1}^{n} \partial_{ii} u_{kk} \) in the above by lower order derivative terms. Differentiating equation (2.2)

\[
\sum_{i,j=1}^{n} F_{ij} \partial_{ij} u_k = 0
\]

and using (2.4), we obtain at \( p \)

\[
\sum_{i,j=1}^{n} f_i \partial_{ii} \Delta u = \Delta F u_{kk} = \sum_{i,j=1}^{n} -\partial_k F_{ij} \partial_{ij} u_k
\]

\[
= \sum_{i,j=1}^{n} - (\Delta u_k \delta_{ij} - u_{kij}) u_{kij} = \sum_{i,j=1}^{n} \left[ u_{ij}^2 - (\Delta u_k)^2 \right].
\]

Plugging the above identity in (2.8), we have at \( p \)

\[
\Delta F b = \frac{1}{(\sigma_1 + J)} \left[ \sum_{i,j,k=1}^{n} u_{ijk}^2 - \sum_{k=1}^{n} (\Delta u_k)^2 - \sum_{i=1}^{n} \frac{f_i}{\sigma_1 + J} (\Delta u_i)^2 \right]
\]

Regrouping those terms \( u_{\bigtriangleup \bigtriangleup}, u_{\bigtriangledown \bigtriangleup}, u_{\bigtriangledown \bigtriangleup \bigtriangledown}, \) and \( \Delta u_{\bigtriangledown} \) in the last two expressions, we obtain (2.7).

Subtracting (2.6) \( \varepsilon \) from (2.7), we have

\[
(\Delta F b - \varepsilon |\nabla F b|^2) (\sigma_1 + J) \geq 3 \sum_{i \neq j} u_{jii}^2 + \sum_{i} u_{iii}^2 - \sum_{i} \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) (\Delta u_i)^2
\]

with \( \delta = 1 + \varepsilon \).

Fix \( i \) and denote \( t = (u_{11i}, \ldots, u_{nmi}) \) and \( e_i \) the \( i \)’th basis vector in \( \mathbb{R}^n \), then the \( i \)’th term above can be written as

\[
Q = 3 |t|^2 - 2 \langle e_i, t \rangle^2 - \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \langle (1, \ldots, 1), t \rangle^2
\]

**Step 2. Tangential projection**

Equation (2.2) at \( p \) yields that \( t \) is tangential to the level set of the equation \( \sigma_2(\lambda) = 1 \), \( \langle Df, t \rangle = 0 \). Then by projecting \( e_i \) and \( (1, \ldots, 1) \) to the tangential space,

\[
E = (e_i)_T = e_i - \frac{f_i}{|Df|^2} Df \quad \text{and} \quad L = (1, \ldots, 1)_T = (1, \ldots, 1) - \frac{(n - 1) \sigma_1}{|Df|^2} Df.
\]

The coefficients of the two negative terms in the quadratic form (2.4)

\[
Q = 3 |t|^2 - 2 \langle E, t \rangle^2 - \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \langle L, t \rangle^2
\]
decrease, as simple symmetric computation shows

\[ |E|^2 = 1 - \frac{f_i^2}{|Df|^2} < 1, \quad |L|^2 = 1 - \frac{2(n-1)}{|Df|^2} < 1, \]

and \[ E \cdot L = 1 - \frac{(n-1)\sigma_1 f_i}{|Df|^2}. \]

**Step 3. Two anisotropic and non-orthogonal directions**

We proceed to show that the quadratic form \( Q \) is positive definite. When \( t \) is perpendicular to both \( E \) and \( L \), \( Q = 3|t|^2 \geq 0 \). So we only need to deal with the anisotropic case, when \( t \) is along \( \{E, L\} \)-space. The corresponding matrix of the quadratic form \( Q \) is

\[ Q = 3I - 2E \otimes E - \eta L \otimes L \]

with \( \eta = 1 + \delta \frac{f_i}{\sigma_1 + J} = 1 + (1 + \varepsilon) \frac{f_i}{\sigma_1 + J} \). The real \( \xi \)-eigenvector equation for (symmetric) \( Q \) under non-orthogonal basis \( \{E, L\} \) is

\[ \begin{pmatrix} 3 - 2|E|^2 & -2E \cdot L \\ -\eta L \cdot E & 3 - \eta |L|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \xi \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \]

where corresponding real eigenvalues

\[ \xi = \frac{1}{2} \left( tr \pm \sqrt{tr^2 - 4 \det} \right) \quad \text{with} \]

\[ tr = 6 - 2|E|^2 - \eta |L|^2 \quad \text{and} \quad \det = 9 - 6|E|^2 - 3\eta |L|^2 + 2\eta \left[ |E|^2 |L|^2 - (E \cdot L)^2 \right]. \]

Now by (2.10)

\[ tr = 6 - 2 \left( 1 - \frac{f_i^2}{|Df|^2} \right) - \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \left( 1 - \frac{2(n-1)}{|Df|^2} \right) \]

\[ > 3 - \delta \frac{f_i}{\sigma_1 + J} = \frac{(3 - \delta) \sigma_1 + \delta \lambda_i + 3J}{\sigma_1 + J} > 0 \]

for any \( \delta \leq 1.5 \) and \( J \geq 0 \), given \( \sigma_1 = \sqrt{|\lambda|^2 + 2} > |\lambda| \) in the nontrivial remaining case.

Next again by (2.10)

\[ \det = 6 \frac{f_i^2}{|Df|^2} - 3\delta \frac{f_i}{\sigma_1 + J} + 3 \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \left( 1 - \frac{2(n-1)}{|Df|^2} \right) \]

\[ + 2 \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \left[ \frac{2(n-1)}{|Df|^2} - \frac{nf_i^2}{|Df|^2} - \frac{2(n-1)}{|Df|^2} \right] \]

\[ > -3\delta \frac{f_i}{\sigma_1 + J} + 4 \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \left( n-1 \right) \frac{f_i}{|Df|^2} \left( 1 + \frac{f_i}{\sigma_1 + J} \right) \left( 6 - 2n \left( 1 + \delta \frac{f_i}{\sigma_1 + J} \right) \right) \frac{f_i}{|Df|^2}. \]
Then for $\delta = 1 + \varepsilon = 4/3$, we have
\[
\det \cdot \frac{(\sigma_1 + J) |Df|^2}{f_i} \geq -3\delta \left[(n-1)\sigma_1^2 - 2\right] + \left\{ \left\{ \frac{4 (\sigma_1 + J + \delta f_i) (n-1) \sigma_1 + }{[6 - 2n] (\sigma_1 + J) - 2n\delta f_i} f_i \right\} \right\}
\]
\[
\begin{align*}
-3\delta \left[(n-1)\sigma_1^2 - 2\right] &+ \left\{ \left\{ \frac{4 (\sigma_1 + J + \delta f_i) (n-1) \sigma_1 + }{[6 - 2n] (\sigma_1 + J) - 2n\delta f_i} f_i \right\} \right\} \\
&= \left\{ \left\{ \frac{6\delta + 4(n-1)J\sigma_1 + 2(3-n)J\frac{\sigma_1 - \lambda_i}{f_i}}{\sigma_1 - \lambda_i} \right\} + (n-1)(4-3\delta)\sigma_1^2 + [2n(2\delta - 1) + 6 - 4\delta] \sigma_1 f_i - 2n\delta f_i \right\} f_i
\end{align*}
\]
(2.12)
\[
\begin{align*}
\delta = 4/3 &\quad 8 + 2(n+1)J\sigma_1 + 2(n-3)J\lambda_i + \frac{2(n+1)}{3} \sigma_1 f_i + \frac{8}{3}n\lambda_i f_i \\
(2.13) &\quad > 2(n+1)J\sigma_1 + 2(n-3)J\lambda_i + \frac{8}{3}n\lambda_i f_i.
\end{align*}
\]

Case $\lambda_i \geq 0$: (2.13) is positive by the ellipticity $f_i > 0$ from (2.4).
Case $0 > \lambda_i \geq -K$:
\[
(2.13) = 2nJ \left( \frac{(\sigma_1 + \lambda_i)}{\sqrt{2 + |\lambda_i|^2 + \lambda_i > 0}} \right) > 6J\lambda_i + 2J\sigma_1 + \frac{8}{3}n\lambda_i \frac{f_i}{\sigma_1 - \lambda_i < 2\sigma_1}
\]
if $J = 8nK/3$.

Therefore, the quadratic form $Q$ is positive definite, and we have derived the shifted Jacobi inequality (2.5) in the semiconvex case. \qed

**Remark.** In three dimensions, the Jacobi inequality (2.5) still holds for any $J \geq 0$ and $\varepsilon = 1/3$ without the semiconvexity assumption $D^2 u \geq -K1$.
Actually, we only need to show that, in Step 3, (2.12) with $\delta = 1 + \varepsilon = 4/3 < 1.5$ is also positive for negative $\lambda_i$. We would have the desired lower bound for (2.12)
\[
\det \cdot \frac{(\sigma_1 + J) |Df|^2}{f_i} > 8f_i \left( \frac{\sigma_1}{3} + \lambda_i \right) > 0,
\]
if we know $\lambda_i > -\sigma_1/3$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Because $\lambda_2 + \lambda_3 = f_1 > 0$, only the smallest eigenvalue $\lambda_3$ could be negative. In such a negative case $\lambda_3 = \frac{1 - \lambda_1 \lambda_3}{\lambda_1 + \lambda_2}$ with $\lambda_1 \lambda_2 > 1$, we do have $\lambda_i > -\sigma_1/3$ or $\frac{\sigma_1}{\lambda_3} > 3$. Because
\[
\frac{\sigma_1}{\lambda_3} = -1 + \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2 - 1} \geq -1 + \frac{4\lambda_1 \lambda_2}{\lambda_1 \lambda_2 - 1} > 3.
\]

Note that, in three dimensions, the Jacobi inequality for the log-convex $b = \Delta u = \ln \sqrt{2 + |\lambda|^2} \ (with \ \varepsilon = 1/100)$ was derived by Qiu \cite[Lemma}{Q, Lemma}
3] for solutions to (2.1) along with variable right hand side; and the Jacobi inequality with \( \varepsilon = 1/3 \) for the log-max \( b = \ln \lambda_{\text{max}} \) (with \( \varepsilon = 1/3 \)) was derived for solutions to (2.1) in [WY, Lemma 2.2].

In general dimensions, a Jacobi inequality for sufficiently large \( b = \ln u_{11} \), at points where \( u_{11} = \lambda_{\text{max}} \), was obtained for solutions having \( \sigma_3 \left(D^2u\right) \) lower bound to (2.1) along with variable right hand side by Guan-Qiu [GQ, p.1650]; and another Jacobi inequality for sufficiently large \( b = \ln \lambda_{\text{max}} \) was derived for semiconvex solutions to (2.1) in [SY, Proposition 2.1], as mentioned in the introduction.

2.2. Superharmonicity under Legendre-Lewy transform. Set \( \tilde{u} \left(x\right) = u \left(x\right) + K |x|^2 / 2 \) for our \( K \)-semiconvex entire solution \( u \) and say, \( \bar{K} = J/n > K + 1 \), where \( J = 8nK/3 \) is from Proposition 2.1. The \( \bar{K} \)-convexity of \( \tilde{u} \) ensures that the smallest canonical angle of the “Lewy-sheared” “gradient” graph is larger than \( \pi/4 \). This means we can make a well defined Legendre reflection about the origin,

\[
(x, D\tilde{u}(x)) = (Dw(y), y) \in \mathbb{R}^n \times \mathbb{R}^n
\]

where \( w(y) \) is the Legendre transform of \( u + \bar{K} |x|^2 \); see [L]. Note that \( y(x) = Du(x) + Kx \) is a diffeomorphism from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) and

\[
0 < D^2w = (D^2u + \bar{K}I)^{-1} < I.
\]

More precisely, by [ChY, p.663] or [SY, (2.11)], the eigenvalues \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \) of \( D^2w \) satisfy

\[
0 < \mu_1 \leq c(n) < 1 \quad \text{and} \quad 0 < c(n, K) \leq \mu_i < 1 \quad \text{for} \quad i \geq 2.
\]

As shown in [ChY, p.663] or [SY, proof of Proposition 2.4], the equation solved by the vertical coordinate Lagrangian potential \( w(y) \),

\[
G(D^2w) = -F(D^2u) = -\sigma_2(-\bar{K}I + (D^2w)^{-1}) = -1,
\]

is conformally, uniformly elliptic for \( K \)-convex solutions \( u \), in the sense that for \( H_{ij} := \sigma_n(\mu(D^2w))G_{ij} \), the linearized operator \( H_{ij}\partial_{ij} \) of equation

\[
0 = H \left(D^2w\right) = \sigma_n \left(D^2w\right) \left[G \left(D^2w\right) + 1\right]
\]

(2.16)

\[
= -\sigma_{n-2}(\mu) + \underbrace{\left(n-1\right)\bar{K}}_{A_1} \sigma_{n-1}(\mu) - \underbrace{n\left(n-1\right)\bar{K}^2 - 1}_{A_2} \sigma_n(\mu)
\]

is uniformly elliptic:

\[
c(n, K)I \leq (H_{ij}) = \sigma_n(\mu)(G_{ij}) \leq C(n, K)I.
\]

**Proposition 2.2.** Let \( u(x) \) be a smooth solution to \( \sigma_2(\lambda) = 1 \) with \( D^2u \geq -K_I \). Set

\[
a(y) = \left(\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}\right)^{-1/3} = \left[\frac{\sigma_n}{\sigma_{n-1}(\mu)}\right]^{1/3}
\]
with \( \mu_is \) being the eigenvalues of the Hessian \( D^2 w(y) \) of the Legendre-Lewy transform of \( u(x) + K |x|^2 / 2 \). Then we have

\[ \triangle H a \leq 0. \]

**Proof.** The trace Jacobi inequality (2.5) in Proposition 2.1 with \( J = nK \) is equivalent to

\[ \triangle F (\triangle u + J)^{-1/3} = \triangle F e^{-b/3} \leq 0. \]

Noticing that \( \triangle u + nK = \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n} \), and applying the transformation rule [SY, Proposition 2.3], we immediately obtain the desired superharmonicity

\[ \triangle H a = \sigma_n(\mu) \triangle G a \leq 0. \]

\( \square \)

3. H"older Hessian estimate for saddle equation and rigidity

The Hessian bound \( 0 < D^2 w(y) \leq I \) ensures that establishing a local \( C^{2,\alpha} \) estimate for such solutions to (2.16) will prove, by scaling, that \( w(y) \) is a quadratic polynomial. By the iteration arguments developed in [CaY] for such smooth PDEs \( H(D^2 w) = 0 \) with solutions satisfying Hessian bounds, proving \( C^{2,\alpha} \) regularity at a point, say the origin, reduces to showing that \( w(y) \) is close to a uniform quadratic polynomial, namely:

**Proposition 3.1.** Let \( u(x) \) be a smooth solution to \( \sigma_2(\lambda) = 1 \) with \( D^2 u \geq -KI \) in \( \mathbb{R}^n \). Let \( w(y) \) be its Legendre-Lewy transform defined in (2.14) solving (2.16) in \( \mathbb{R}^n \) with \( 0 < D^2 w \leq I \). Given any \( \theta > 0 \), there exists small \( \eta = \eta(n,K,\theta) > 0 \) and a quadratic polynomial \( P(y) \) whose coefficients only depend on \( n, K, \theta \) such that

\[ \left| \frac{1}{\eta^2} w(\eta z) - P(z) \right| \leq \theta \]

is valid for \( |z| \leq 1 \).

In the case that the level set \( \{ H(D^2 w) = 0 \} \) were convex (in fact saddle from [ChY, p.661]), the alternative way in [CaY] other than Evans-Krylov-Safonov is the following. The Laplacian \( \Delta w(y) \) is a sub or supersolution of the linearized operator \( \Delta_H = H_{ij} \partial^2 / \partial y_i \partial y_j \) of \( H(D^2 w) \). The weak Harnack inequality shows that \( \Delta w(y) \) concentrates in measure at a level \( c \) on a small ball \( B = B_r(0) \). Solving the equation \( \Delta v = c \) on \( B \) with \( v = w \) on \( \partial B \) furnishes the desired smooth approximation, which is uniform by the ABP estimate. The Laplacian can be replaced with any elliptic slice of the Hessian, such that the elliptic slice is a supersolution of \( \Delta_H \), and the corresponding elliptic equation both has \( C^{2,\alpha} \) interior regular solutions and allows for the ABP estimate.

However, it is not clear if the saddle level set \( \{ H(D^2 w) = 0 \} \) of (2.16) is any of trace-convex [CaY], max-min [CC], or twisted [CS] [C], so it is not clear if there are good PDEs which super-solve \( \Delta_H \). Now that the remarkable
superharmonic quantity \( \sigma_n(\mu)/\sigma_{n-1}(\mu) \) in Proposition 2.2 is available, the core method in [CaY, pg 687-690] becomes more realistic.

There is still one more hurdle to overcome. The superharmonic, “one-step” Hessian quotient \( a^3 = \sigma_n(\mu)/\sigma_{n-1}(\mu) \) is well known to be concave, but not uniformly elliptic, because \( \sigma_n(\mu) \) could be arbitrarily close to zero. This prevents applying Evans-Krylov-Safonov theory. We resolve this by substituting the concentration of \( a \) into the “conformal” equation (2.16). This implies concentration of a better quantity. Observe that equation (2.16) can be written as

\[
q(\mu) := \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}(\mu)} = \left( A_1 - A_2 \frac{\sigma_n(\mu)}{\sigma_{n-1}(\mu)} \right)^{-1}.
\]

Thus, the concentration of the higher quotient \( a^3 = \sigma_n/\sigma_{n-1} \) implies concentration of the lower quotient \( \sigma_{n-1}/\sigma_{n-2} \), which is also a concave operator [L, Theorem 15.18]. The almost-convex case, \( D^2u \geq (-K + \delta)I \) for \( K^{-2} = n(n-1)/2 \) and any \( \delta > 0 \) considered in [ChY], corresponds to \( A_2 = 0 \). There, it was shown that (2.16) is uniformly elliptic for arbitrarily large \( K \), in particular, the lower quotient \( \sigma_{n-1}/\sigma_{n-2} = A_1^{-1} \) for \( K^{-2} = n(n-1)/2 \). For arbitrary \( K \), using the bound for \( \mu \) in (2.15) and the result in [L, Theorem 15.18], we deduce the uniform ellipticity of \( q(\mu) \)

\[
\partial_\mu q \in \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}^2(\mu)} \sigma_{n-2, i}(\mu) [c(n), 1] \subset [c(n, K), C(n, K)].
\]

**Proof of Proposition 3.1.** Given any small \( \xi, \delta > 0 \), we denote \( a_k = \min_{B_{1/2k}} a \) and define a “bad set” \( E_k = \{ y \in B_{1/2k} : a > a_k + \xi \} \). Using Krylov-Safonov’s weak Harnack inequality [GT, Theorem 9.22] for supersolution \( a(y) \) from Proposition 2.2, the iteration argument in [CaY, pg 687-690] shows that there exists \( k_0 \) large enough such that \( |E_\ell| < \delta |B_{1/2\ell}| \) for some \( \ell \in [1, k_0] \). By applying the quadratic scaling \( 2^\ell w(2^{-\ell} y) \) and modifying the small constant \( \eta \) in Proposition 3.1, we may assume \( \ell = 0 \). This is without loss of generality and is possible, if we assume \( w(0) = 0 \) and \( Dw(0) = 0 \) in the beginning.

Using uniform ellipticity (3.2), let us extend \( \sigma_{n-1}/\sigma_{n-2} \) to a uniformly elliptic concave operator \( q(\mu) \) outside the eigenvalue rectangle (2.15), \( \mu \in [0, c(n)] \times [c(n, K), C(n, K)] \). The notation \( q \) in (3.1) was abused for the sake of notation simplicity. Let \( v(y) \in C^\infty(B_1) \) solve the concave equation \( q[v] = q(\mu(D^2v)) = (A_1 - A_2 a_\ell)^{-1} \) in \( B_1 \) with \( v = w \) on \( \partial B_1 \). Then from the quotient representation (3.1) for the equation that \( w \) solves, the ABP estimate [GT, Theorem 9.1] yields on \( B_1 \),

\[
|w - v| \leq C(n, K)||q[w] - q[v]||_{L^\infty(B_1)} \leq C(n, K)\delta + C(n, K) \left\| \frac{a^3 - a_\ell^3}{(A_1 - A_2 a^3)(A_1 - A_2 a_\ell^3)} \right\|_{L^\infty(E_\ell)} \leq C(n, K)(\delta + \xi),
\]

for arbitrary \( \lambda \), where \( k_0 = \mu(\partial B_1) \).
where, in the last inequality, we used the boundedness of $(A_1 - A_2 a)^{-1}$ via (3.1) and (2.15). By Evans-Krylov-Safonov theory applied to the smooth equation $q [v] = a_\ell$, $v(y)$ has uniform interior estimates, so $v$ can be replaced by its quadratic part at the origin up to a uniform $O(|x|^3)$ term, which is $O(r^3)$ on $B_r(0)$. The conclusion of this proposition follows for $\eta = r$ in a standard way by successively choosing the small constants $\xi, \delta, r$ depending on the small parameter $\theta$ and $n, K$. □

Proof of Theorem 1.1. As indicated in the beginning of Section 2, we only need to handle the positive branch $\triangle u > 0$ of the quadratic equation $\sigma_2 (D^2 u) = 1$. This is because the only other possibility is that $D^2 u$ is on the negative branch $\triangle u < 0$ of the still elliptic and concave equation $\sigma_2 (D^2 u) = 1$. Then the semiconvex solutions must have bounded Hessian, and consequently, the conclusion in Theorem 1.1 is straightforward by Evans-Krylov-Safonov.

Now armed with Proposition 3.1, the initial closeness of $w$ to a “harmonic” quadratic on the unit ball, and repeating the proof of Proposition 2 in [CaY] with the equation there replaced by our smooth uniformly elliptic equation (3.1), we see that the closeness to “harmonic” quadratics accelerates. As in [CaY, p.692], we obtain that $D^2 w$ is Hölder at the origin. Similarly, one proves that $D^2 w$ is Hölder in the half ball

$$[D^2 w]_{C^\alpha (B_{1/2})} \leq C(n, K),$$

where $\alpha = \alpha(n, K) > 0$.

By quadratic scaling $R^2 w(y/R)$, we get

$$[D^2 w]_{C^\alpha (B_R)} \leq \frac{C(n, K)}{R^{3\alpha}} \rightarrow 0, \quad \text{as } R \to \infty.$$

We conclude that $D^2 w$ is a constant matrix, and in turn, so is $D^2 u$. □

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University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195

Email address: shankarr@uw.edu, yuan@math.washington.edu