EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FREE BOUNDARY PROBLEMS IN HETEROGENEOUS ENVIRONMENTS

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Abstract. In this short paper we study the existence and uniqueness of solutions of free boundary problems coming from ecology in heterogeneous environments.

1. Introduction and the main results. Recent years, free boundary problems coming from ecology in the spatial heterogeneous environments had been studied by many authors, see [1]-[3], [5]-[7], [9, 10, 12], [14]-[17], for example. In these references, proofs of the existence and uniqueness of local solutions were omitted and these authors claimed that the proofs are parallel to the cases of [4, 13] with constant coefficients. However, in the spatial heterogeneous environments, the uniqueness is different from the case with constant coefficients completely. Moreover, in the previous works, the application of the imbedding theorem

\[ \|v\|_{C^{1+\alpha}(\Delta_T)} \leq C\|v\|_{W^{1,2}_{p}(\Delta_T)} \]

is not appropriate in the proof of the existence and uniqueness of local solution because the imbedding constant \( C = C(T^{-1}) \) depends on \( T^{-1} \) and \( C(T^{-1}) \to \infty \) as \( T \to 0 \). In this short paper, for the convenience to readers, we shall give a general conclusion.

Consider the following free boundary problem

\[
\begin{cases}
  u_t - du_{xx} = f(t, x, u), & t > 0, \quad 0 < x < h(t), \\
  Bu(t, 0) = 0, \quad u(t, h(t)) = 0, & t > 0, \\
  h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
  h(0) = h_0, \quad u(0, x) = u_0(x), & 0 < x < h_0,
\end{cases}
\]

(1)

where, \( h_0 \) denotes the size of initial habitat, \( Bu = \beta u - (1 - \beta)u_x, 0 \leq \beta \leq 1 \) is a constant, the initial function \( u_0(x) \) satisfies

\[ u_0 \in W^2_p((0, h_0)) \text{ with } p > 3, \quad u_0 \geq 0 \text{ in } (0, h_0), \quad Bu_0(0) = u_0(h_0) = 0. \]

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The function $f$ satisfies

(F) $f(t, x, 0) \geq 0$. For any given $\tau, l, k > 0$, $f \in L^\infty((0, \tau) \times (0, l) \times (0, k))$ and there exists a constant $L_1(\tau, l, k)$ such that

$$|f(t, x, u) - f(t, x, v)| \leq L_1(\tau, l, k)|u - v|, \quad \forall \ t \in [0, \tau], \ x \in [0, l], \ u, v \in [0, k].$$

Denote $h^* = -\mu u_0(h_0)$ and

$$\Lambda = \{h_0, h^*, \|u_0\|_{W^2_0([0, h_0])}\}.$$

In order to save space, for the given interval $I \subset \mathbb{R}_+ = [0, \infty)$, in the case of not causing confusion we write simply

$$I \times [0, h(t)] = \bigcup_{t \in I} \{t\} \times [0, h(t)], \quad I \times (0, h(t)) = \bigcup_{t \in I} \{t\} \times (0, h(t)).$$

For the given $T > 0$, we set

$$D_T = (0, T) \times (0, h(t)).$$

**Theorem 1.1.** Assume that the condition (F) holds and fix $0 < \alpha < 1 - 3/p$.

(i) There is a constant $0 < T \ll 1$ for which (1) admits at least one solution $(u, h)$ and $(u, h)$ has the following properties:

(P$_T$) $u > 0$ in $D_T$, $h'(t) > 0$ in $(0, T)$, $u \in W^{1,2}_p(D_T) \cap C^{1+\frac{\alpha}{2}, 1+\alpha}(\overline{D_T})$, $h \in C^{1+\frac{\alpha}{2}, 1}([0, T])$,

where $T$ depends only on $\Lambda$.

(ii) Let $\tau > 0$ and $(u, h)$ be a solution of (1) defined in $\overline{D}_\tau$. If $u \in L^\infty(D_\tau)$ and $h \in C^1([0, \tau])$, then $(u, h)$ has the property (P$_\tau$). Moreover, there exists a constant $C(\Lambda) > 0$ such that

$$\|u\|_{W^{1,2}_p(D_\tau)} + \|u\|_{C^{1+\frac{\alpha}{2}, 1+\alpha}(\overline{D}_\tau)} + \|h\|_{C^{1+\frac{\alpha}{2}, 1}([0, \tau])} \leq C(\Lambda)$$

(2) provided $0 < \sigma < 1$.

(iii) In addition, if $f$ is also Lipschitz continuous in $x$, i.e., for any given $\tau, l, k > 0$, there exists a constant $L_2(\tau, l, k)$ such that

$$|f(t, x, u) - f(t, y, u)| \leq L_2(\tau, l, k)|x - y|, \quad \forall \ t \in [0, \tau], \ x, y \in [0, l], \ u \in [0, k],$$

then the solution $(u, h)$ of (1) is unique in the class $L^\infty(D_T) \times C^1([0, T])$ as long as $0 < T \leq T_\tau$, where $T_\tau$ depends only on $\Lambda$.

(iv) Under the above assumptions, if we further assume that, for any given $\tau > 0$, $f(\cdot, x, u) \in C^\frac{\alpha}{2}([0, \tau])$ uniformly in $(x, u)$ when $(x, u)$ varies in a bounded domain of $\mathbb{R}_+ \times \mathbb{R}_+$, i.e., for any given $\tau, l, k > 0$, there exists a constant $L_3(\tau, l, k)$ such that

$$\|f(\cdot, x, u)\|_{C^\frac{\alpha}{2}([0, \tau])} \leq L_3(\tau, l, k), \quad \forall \ x \in [0, l], \ u \in [0, k],$$

(4) then the unique solution obtained in the above satisfies $u \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T) \times [0, h(t)])$, and for any given $0 < \varepsilon < T$, there holds:

$$\|u\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([\varepsilon, T] \times [0, h(t)])} \leq C_3(\varepsilon, \Lambda).$$

Assume that the conditions (F), (3) and (4) hold. According to Theorem 1.1 (iv), we can extend the solution to the right. Define

$$T_0 = \sup\{T > 0 : (u, h) \in C(\overline{D_T}) \times C^1([0, T]) \text{ solves (1)}\}.$$
Theorem 1.2. Assume that conditions (F), (3) and (4) hold, and that $f$ is bounded in $\mathbb{R}_+ \times \mathbb{R}_+ \times (0, k)$ for any given $k > 0$. Let $T_0$ be given by (5). Then either $T_0 = \infty$, or $T_0 < \infty$ and

$$\limsup_{t \to T_0} \max_{0 \leq x \leq h(t)} u(t, \cdot) = \infty.$$ 

Such a $T_0$ is called the maximal existence time of the solution $(u, h)$ of (1).

Corollary 1. Under the conditions of Theorem 1.2, if we assume further that there exists a constant $M' > 0$ such that $f(t, x, u) \leq 0$ for all $t, x \in \mathbb{R}_+$ and $u > M'$. Then $T_0 = \infty$.

2. Proofs of the results. In this section we shall give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. (i) Let

$$y = x/h(t), \quad v(t, y) = u(t, h(t)y).$$

Then $\mathcal{B}v(0, 0) = v(0, 1) = 0$, and $(v, h)$ satisfies

$$\begin{cases}
\begin{aligned}
vt - \rho(t)v_{yy} - \xi(t)vy &= f(t, h(t)y, v), &0 < t \leq T, &0 < y < 1, \\
[\beta h(t)v - (1 - \beta)v](t, 0) &= v(t, 1) = 0, &0 < t \leq T,
\end{aligned}
\end{cases}$$

(6)

and

$$h'(t) = -\mu \frac{1}{h(t)} v_y(t, 1), \quad 0 < t \leq T; \quad h(0) = h_0, \quad (7)$$

where $\rho(t) = dh^{-2}(t)$, $\xi(t) = h'(t)/h(t)$. The problem (6) is an initial-boundary value problem with fixed boundary. We shall use the fixed point theorem to prove the existence of solution $(v, h)$ to (6) and (7).

Let $T_1 = \min \left\{1, \frac{h_0}{\sqrt[2]{2 + h^*}} \right\}$. For $0 < T \leq T_1$, we set

$$\Omega_T = \left\{h \in C^1([0, T]) : h(0) = h_0, \quad h'(0) = h^*, \quad \|h' - h^*\|_\infty \leq 1 \right\}.$$ 

Clearly, $\Omega_T$ is a bounded and closed convex set of $C^1([0, T])$. For the given $h \in \Omega_T$, we can extend $h$ to a new function, denoted by itself, such that $h \in \Omega_{T_1}$, where

$$\Omega_{T_1} = \left\{h \in C^1([0, T_1]) : h(0) = h_0, \quad h'(0) = h^*, \quad \|h' - h^*\|_\infty \leq 2 \right\}.$$ 

Therefore, when $h \in \Omega_T$ we have $h \in \Omega_{T_1}$ and

$$|h(t) - h_0| \leq T_1 \|h'\|_\infty \leq T_1(2 + h^*) \leq h_0/2, \quad \forall t \in [0, T_1]$$

which causes to

$$h_0/2 \leq h(t) \leq 3h_0/2, \quad \forall t \in [0, T_1].$$

Thus, for the given $h \in \Omega_T$, the functions $\rho(t)$ and $\xi(t)$ are well defined on $[0, T_1]$. By the standard method (the upper and lower solutions method, or the contraction mapping theory, for example) we can show that there exists $0 < T_* \leq T_1$, depending only on $\Lambda$ and the bound of $f$ on $[0, T_1] \times [0, \frac{3h_0}{2}] \times [0, \|u_0\|_\infty + 1]$, such that (6) has a unique solution $v(t, y) = v(t, y; h) \in W^{1, 2}(\Delta_{T_*}) \hookrightarrow C^{\frac{1 + \alpha}{2}, 1 + \alpha}(\Delta_{T_*})$ and

$$\|v\|_{W^{1, 2}_{p}(\Delta_{T_*})} + \|v\|_{C^{\frac{1 + \alpha}{2}, 1 + \alpha}(\Delta_{T_*})} \leq C_1(\Lambda, T_*, T_*^{-1}),$$

where $\Lambda = \max\{M, M'\}$.
Moreover, as \( v \) depends only on \( \Lambda \), we may think that \( T_\ast \) depends only on \( \Lambda \) and write \( C_1(\Lambda, T_\ast, T_\ast^{-1}) \) as \( C_1(\Lambda) \). That is,

\[
\|v\|_{W^{1,2}_p(\Delta_{T_\ast})} + \|v\|_{C^{\frac{1+\alpha}{\gamma}}(\Delta_{T_\ast})} \leq C_1(\Lambda),
\]

Therefore, when \( 0 < T \leq T_\ast \), the unique solution \( v(t, y) \) of (6) satisfies

\[
\|v\|_{W^{1,2}_p(\Delta_{T})} + \|v\|_{C^{\frac{1+\alpha}{\gamma}}(\Delta_{T})} \leq \|v\|_{W^{1,2}_p(\Delta_{T_\ast})} + \|v\|_{C^{\frac{1+\alpha}{\gamma}}(\Delta_{T_\ast})} \leq C_1(\Lambda). \tag{8}
\]

Notice that \( v(0, y) \geq 0 \), \( f(t, h(t), y, 0) \geq 0 \) and \( f(t, h(t), y, \cdot) \) is Lipschitz continuous in \( v \), we have \( v > 0 \) in \( (0, T) \times (0, 1) \) by the positivity lemma ([11, Lemma 1.26]).

On the other hand, by the continuous dependence on the given data, in the space \( C^{\frac{1+\alpha}{\gamma}}(\Delta_T), \) \( v \) depends continuously on \( h \in \Omega_T \). For such a definite function \( v \), the initial value problem (7) has a unique solution, denoted by \( \tilde{h}(t) = \tilde{h}(t; h) \). Then \( \tilde{h}(0) = h_0, \tilde{h}'(0) = h^* \) and

\[
\tilde{h}'(t) > 0, \quad \tilde{h}' \in C^2([0, T]), \quad \|\tilde{h}'\|_{C^2([0, T])} \leq C_2(\Lambda), \quad \forall h \in \Omega_T. \tag{9}
\]

Clearly, in the space \( C^1([0, T]), \) \( \tilde{h} \) continuously depends on \( v \in C^{\frac{1+\alpha}{\gamma}}(\Delta_T) \), so does on \( h \in \Omega_T \). Now we define a mapping \( \mathcal{F} : \Omega_T \to C^1([0, T]) \) by

\[
\mathcal{F}(h) = \tilde{h}.
\]

Obviously, \( \mathcal{F} \) is continuous in \( \Omega_T \), and \( h \in \Omega_T \) is a fixed point of \( \mathcal{F} \) if and only if \( (v, h) \) solves (6) and (7).

According to (9), we know that \( \mathcal{F} \) is compact and

\[
\|\tilde{h}' - h^*\|_{C([0, T])} \leq \|\tilde{h}'\|_{C^2([0, T])} T^\frac{2}{\gamma} \leq C_2(\Lambda)T^\frac{2}{\gamma}.
\]

Hence \( \mathcal{F} \) maps \( \Omega_T \) into itself if

\[
T \leq \min \left\{ 1, \frac{h_0}{2(2 + h^*)}, C_2^{-2/\alpha}(\Lambda) \right\}.
\]

Consequently, \( \mathcal{F} \) have at least one fixed point \( h \in \Omega_T \) by the Schauder fixed point theorem, and then (6) and (7) have at least one solution \( (v, h) \) defined in \( [0, T] \). Moreover, as \( v(t, y) > 0 \) and \( v(t, 1) = 0 \), it deduces by the Hopf boundary lemma that \( v_y(t, 1) < 0 \), which implies \( h'(t) > 0 \) for \( t > 0 \).

Obviously, the function \( u(t, x) = v(t, h^{-1}(t)x) \) satisfies

\[
u \in W^{1,2}_p(D_T) \cap C^{\frac{1+\alpha}{\gamma}}(\Delta_T), \quad u > 0 \quad \text{in} \quad D_T,
\]

and \( (u, h) \) solves (1) and

\[
h \in C^{1+\frac{2}{\gamma}}([0, T]).
\]

The conclusion (ii) can be proved by the \( L^p \) theory and embedding theorem. We omit the details and left them to the reader.

(iii) Let \( (u_i, h_i) \in L^\infty(D_T) \times C^1([0, T]), \ i = 1, 2, \) be two solutions of (1) and \( 0 < T \ll 1 \). Then \( (u_i, h_i) \) has the property \( \mathcal{P}_T \) and satisfies (2). Notice \( h_i'(t) > 0 \). We may assume that

\[
h_0 \leq h_i(t) \leq h_0 + 1 \quad \text{in} \quad [0, T], \quad u_i \leq \|u_0\|_{\infty} + 1 \quad \text{in} \quad [0, T] \times [0, h_i(t)], \quad i = 1, 2.
\]
Define
\[ v_i(t, y) = u_i(t, h_i(t)y), \quad 0 \leq t \leq T, \quad 0 \leq y \leq 1 \]
for \( i = 1, 2 \), and set \( v = v_1 - v_2, \ h = h_1 - h_2 \). Then we have
\[
\begin{aligned}
&v_i - \rho_i(t)v_{yy} - \xi_i(t) y v_y - a(t, y)v = [\rho_1(t) - \rho_2(t)] v_{2yy} \\
&\qquad + [\xi_1(t) - \xi_2(t)] y v_{2y} + b(t, y)y\ h, \quad 0 \leq t \leq T, \quad 0 < y < 1, \\
&\mathcal{B}_i v(t, 0) = -\beta h v_2(t, 0), \ v(t, 1) = 0, \quad 0 < t \leq T, \\
v(0, y) = 0, \quad 0 < y < 1,
\end{aligned}
\]
and
\[
h'(t) = \mu \left( \frac{1}{h_2} v_2(t, 1) - \frac{1}{h_1} v_1(t, 1) \right), \quad 0 < t \leq T; \quad h(0) = 0,
\]
where \( \mathcal{B}_i v = \beta h_1(t)v - (1 - \beta)v_y \) and
\[
a(t, y) = \int_0^1 f_u(t, h_1y, v, s(v_1 - v_2))ds, \\
b(t, y) = \int_0^1 f_x(t, h_2 + s(h_1 - h_2))y, v)ds.
\]
The assumptions on \( f \) implies \( a, b \in L^\infty(\Delta_T) \), and the \( L^\infty(\Delta_T) \) norms of \( a, b \) depend only on \( h_0 \) and \( ||u_0||_\infty \). Remember \( ||v_2||_{W^1, 2(\Delta_T)} \leq C_1(\Lambda) \) and \( 0 < h_1(t) \leq C_2(\Lambda), \ h_0 \leq h_i(t) \leq h_0 + 1 \). Applying the \( L^p \) theory to (10) we achieve
\[
||v||_{W^1, 2(\Delta_T)} \leq C_3(\Lambda)(h_2^{-2} - h_2^{-2})v_{2yy}||L^p(\Delta_T) + ||(h_1'/h_1 - h_2'/h_2)yv_{2y}||L^p(\Delta_T)
\]
\[
+ ||bh||L^p(\Delta_T) + ||\beta h v_2||_{W^1, 2(\Delta_T)}
\]
\[
\leq C_4(\Lambda)||h||_{C^1([0, T])}.
\]
Noting the dimension of space is 1, follow the proofs of [8, (5.4.3) and Theorem 5.5.4] for \( v_y(t, y) \) without the extension of \( v \) to a large domain, we can show
\[
[v]_{C^{1/2, \alpha}(\Delta_T)}, \ [v_y]_{C^{1/2, \alpha}(\Delta_T)} \leq C||v||_{W^1, 2(\Delta_T)}
\]
for some positive constant \( C \) independent of \( T^{-1} \), here \([\cdot]_{C^{1/2, \alpha}} \) is the H"older seminorm. Thus
\[
[v_y]_{C^{1/2, \alpha}(\Delta_T)} \leq CC_4(\Lambda)||h||_{C^1([0, T])}.
\]
This combines with (11) yields
\[
[h']_{C^{1/2}(\Delta_T)} \leq \mu[h_1^{-1}v_y(t, 1)]_{C^{1/2}(\Delta_T)} + \mu\left( [h_1^{-1} - h_2^{-1}]v_{2y}(t, 1) \right)_{C^{1/2}(\Delta_T)}
\]
\[
\leq C_5(\Lambda)||h||_{C^1([0, T])}.
\]
Because of \( h(0) = h'(0) = 0 \), it is deduced that
\[
||h||_{C^1([0, T])} \leq 2T^{1/2}||h'||_{C^{1/2}(\Delta_T)} \leq 2C_5(\Lambda)T^{1/2}||h||_{C^1([0, T])}.
\]
We can choose \( 0 < \hat{T}(\Lambda) \ll 1 \) such that when \( 0 < T \leq \hat{T}(\Lambda) \), \( h = 0 \) and hence \( v = 0 \).

(iv) Since \( f \) satisfies conditions (F), (3) and (4), using (8) and (9) we see that the function \( \hat{F}(t, y) := f(t, h(t)y, v(t, y)) \in C^{3/2, \alpha}(\Delta_T) \). Apply the Schauder theory
to (6) to deduce that \( v \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T] \times [0, 1]) \), and for any given \( 0 < \varepsilon < T \), there holds
\[
\|v\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([\varepsilon, T] \times [0, 1])} \leq C_3(\varepsilon, \Lambda).
\]
Consequently, \( u \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T] \times [0, h(t)]) \) and
\[
\|u\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([\varepsilon, T] \times [0, h(t)])} \leq C_3(\varepsilon, \Lambda).
\]
The proof is complete. \( \square \)

In order to study the property of \( T_0 \) determined by (5), we shall give the following estimate.

**Lemma 2.1.** Assume that conditions (F), (3) and (4) hold. Let \( T > 0 \), \((u, h) \in C(D_T) \times C^1([0, T])\) be a solution of (1) and \( u \leq M \) for some \( M > 0 \). If \( f \) is bounded in \( \mathbb{R}_+ \times \mathbb{R}_+ \times (0, M) \), then there exists a constant \( K > 0 \), which depends on \( M \) but not on \( T \), such that
\[
h'(t) \leq 2\mu MK := C(M) \quad \text{in} \quad [0, T).
\]

**Proof.** Set \( A = \|f\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times (0, M))} \) and take
\[
K = \max \left\{ \sqrt{\frac{A}{2dM}}, \frac{1}{h_0}, -\frac{1}{M} \min_{[0, h_0]} u'(x) \right\}.
\]
The proof is the same as that of [4, Lemma 2.2] and we shall omit the details. \( \square \)

**Proof of Theorem 1.2.** Assume on the contrary that \( T_0 < \infty \) and there exists \( M > 0 \) such that
\[
u \leq M \quad \text{in} \quad [0, T_0) \times [0, h(t)].
\]
Due to (12), we have \( h'(t) \leq C(M) \) in \((0, T_0)\) and so \( h_0 \leq h(t) \leq h_0 + C(M)T_0 \). Let \( v(t, y) = u(t, h(t)y) \). For any given \( T < T_0 \), applying the \( L^p \) theory to (6) we get
\[
\|v\|_{W^{1,2}_{p}(\Delta_T)} \leq C_1(\Lambda, M, T_0) \quad \text{which is independent of} \ T, \quad \text{thus} \ v \in W^{1,2}_{p}(\Delta_{T_0})
\]
and
\[
\|v\|_{W^{1,2}_{p}(\Delta_{T_0})} + \|v\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}(\Delta_{T_0})} \leq C_1(\Lambda, M, T_0).
\]
It follows from (7) that \( h \in C^{1+\frac{\alpha}{2}}([0, T_0]) \) and
\[
\|h\|_{C^{1+\frac{\alpha}{2}}([0, T_0])} \leq C_2(\Lambda, M, T_0).
\]
(13)

Similar to the proof of Theorem 1.1 (iv), we can apply the Schauder theory to (6) to get that \( v \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T_0] \times [0, 1]) \), and for any given \( 0 < \varepsilon < T_0 \), there holds
\[
\|v\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([\varepsilon, T_0] \times [0, 1])} \leq C_3(\varepsilon, \Lambda, M, T_0).
\]
Therefore, \( u \in C^{1+\frac{\alpha}{2}, 2+\alpha}((0, T_0] \times [0, h(t)]) \) and
\[
\|u\|_{C^{1+\frac{\alpha}{2}, 2+\alpha}([\varepsilon, T_0] \times [0, h(t)])} \leq C_3(\varepsilon, \Lambda, M, T_0).
\]
(14)
This shows that the solution \((u, h)\) exists on \([0, T_0]\). Choosing \( t_n \in (0, T_0) \) with \( t_n \searrow T_0 \), and regarding \( t_n \) and \((u(t_n, \cdot), h(t_n))\) as the initial time and initial datum. Similar to the proof of Theorem 1.1 we can find a constant \( 0 < T \ll 1 \), which depends only on \( \Lambda, M, T_0 \) and \( h(T_0) = h_0 \), such that (1) has a unique solution \((u_n, h_n)\) in \([t_n, t_n + T]\). By the uniqueness, \((u, h) = (u_n, h_n)\) for \( t_n \leq T \). This shows that the solution \((u, h)\) of (1) can be extended uniquely to \([0, t_n + T]\). According to (13) and (14), we can choose \( T \) independent of
n. Hence, $t_n + T > T_0$ when $n$ is large because of $t_n \nearrow T_0$. This is a contradiction with the definition of $T_0$ and the proof is complete.

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