ADELIC COHOMOLOGY.

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Abstract. The characteristic feature of the adeles is that they involve localizations of products (or equivalently restricted products of localizations). The point of this paper is to introduce an adelic style cohomological invariant of a partially ordered set with auxiliary structure which covers several examples of established interest in commutative algebra and stable equivariant homotopy theory.

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1. Introduction

1.A. Motivation. The characteristic feature of the adeles is that they involve localizations of products (or equivalently restricted products of localizations). The point of this paper is to introduce an adelic style cohomological invariant of a partially ordered set with auxiliary structure. The construction covers several special cases of established interest, and gives a language and method of calculation in many more.

The first example is in commutative algebra. The Hasse square for \( \mathbb{Z} \) came from number theory, but there is a counterpart for any one dimensional Noetherian ring \( R \). This says that \( R \) is the pullback of a square formed using completions at primes and localizations at primes. Furthermore, this cube is also a pushout. Similarly, for a \( d \)-dimensional catenary Noetherian ring, the ring \( R \) is the pullback of a \((d+1)\)-cube, and in fact it is also a homotopy pullback. In our terms this states that the adelic cohomology is \( R \) in degree 0. It will be shown elsewhere \cite{BG19} that this is also the basis for understanding the category of \( R \)-modules. A variant of this construction gives the Beilinson-Parshin adeles \cite{Hub91}.

I am grateful Bhargav Bhatt for suggesting the connection with the Beilinson-Parshin complex.
The second established case comes up in equivariant homotopy. If $G$ is an $r$-dimensional torus a main result of [GS18] states that the rational equivariant sphere spectrum is the pullback of an $(r + 1)$-cube of commutative ring spectra. This gives an approach to the category of $S$-modules (i.e., to the category of rational $G$-spectra). Taking homotopy, we obtain a spectral sequence calculating the stable homotopy groups of the sphere, $\pi_*^G(S)$. In fact if we take the fixed points of the punctured cube, the ring spectra are all formal, and the sphere is determined by a diagram of graded rings. This diagram of rings gives a cochain complex for the adelic cohomology, so that the $E_2$-term of the spectral sequence for $\pi_*^G(S)$ is the adelic cohomology.

There are other closely related examples coming out of homotopy theory which are not covered by the construction here. The best known of these is the chromatic fracture square in stable homotopy theory, but there are many others of this type. Indeed, the work of [BG19] takes the present constructions at the level of a homotopy category and uses this as the basis of an adelic model of a symmetric monoidal model category. The main ingredient in this is to show that the unit is the homotopy pullback of a suitable cube of rings. In some cases the completions and localizations are functors of the homotopy of the unit, and hence the cube gives a spectral sequence starting with the adelic cohomology we describe here. Curiously, the formal framework for algebra is a little more elaborate than the homotopy theory because taking homotopy of different types of objects is described by different algebraic functors. Nonetheless, the constructions here are essentially abelian category level versions of the homotopical constructions of [BG19] and provide motivation as well as calculation for that case.

1.B. Context. The basic substrate is a partially ordered set (poset) $\mathcal{X}$, which will usually be infinite. The order relation will always be written $\leq$. This needs to be endowed with additional coefficient data to define cohomology. One of the main messages of this note is that we need one piece of data depending contravariantly on points $x \in \mathcal{X}$ (as completion does) and one piece of data depending covariantly on points $x \in \mathcal{X}$ (as localization does); this will be illustrated by a range of examples.

1.C. Spectral examples. We have in mind a number of examples arising from a tensor triangulated category $\mathcal{C}$. We begin by taking the Balmer spectrum $\text{Spcc}(\mathcal{C})$, consisting of the tensor ideals $\wp$ of the subcategory $\mathcal{C}^c$ of compact objects (i.e., $\wp$ is closed under completing triangles, and tensoring with an arbitrary object) which are prime (in the sense that if $x \otimes y \in \wp$ then either $x$ or $y$ is in $\wp$). To start with, $\text{Spcc}(\mathcal{C})$ is a partially ordered set under inclusion. The formalism we need for our cohomology is not restricted to this setting, but it will colour our choice of terminology.

1.C.1. Commutative rings. We start with a commutative Noetherian ring $R$ and we are interested in the category of $R$-modules. The category $\mathcal{C}$ is the derived category $D(R)$ of $R$-modules. There is a natural bijection

$$\text{Spec}(R) \xrightarrow{\cong} \text{Spcc}(D(R))$$

where the algebraic prime $\wp_a$ corresponds to the Balmer prime

$$\wp_b = \{ M \mid M_{\wp_a} \simeq 0 \}.$$  

This is an order reversing bijection, and we will always use the Balmer ordering.
We take $\mathfrak{X} = \text{Spec}(R)$ and note that the the minimal elements of $\mathfrak{X}$ (in the Balmer ordering) correspond to closed points.

Associated to a prime $\wp$ we have $\wp$-adic completion and localization at $\wp$.

1.C.2. Rational torus-equivariant spectra. If $G$ is an $r$-dimensional torus we may consider the category $\mathcal{C}$ of rational $G$-spectra. The category $\mathcal{C}^c$ is then the stable homotopy category of rational finite $G$-complexes. Equivalently, we can take $\mathcal{C}$ to be the derived category of objects from the category of differential graded objects of the algebraic category $\mathcal{A}(G)$ of $\text{Gre}08$.

In either case, $\mathfrak{X}_a = \text{Spcc}(\mathcal{C})$ is the set of closed subgroups of $G$, but the partial order is that of cotoral inclusion (i.e., $K \leq H$ when $K$ is a subgroup of $H$ with $H/K$ a torus) $\text{Gre}19$.

At the prime corresponding to the subgroup $K$, the relevant completion is given by a function spectrum (if $K = 1$ the completion of $X$ is the cofree spectrum $F(EG_+, X)$) and the localization corresponding to inverts the Euler classes of complex representations $V$ with $V^K = 0$.

1.C.3. Chromatic homotopy theory. It has been shown by Hopkins and Smith $\text{HS}98$ that the Balmer spectrum of finite spectra has primes corresponding to the Morava $K$-theories $K(n, p)$ for $0 \leq n \leq \infty$ for non-zero primes $p$, where $K(\infty, p) = H\mathbb{F}_p$ and $K(0, p) = H\mathbb{Q}$ independent of $p$. We write $\wp(p, n) := \{ X | K(n, p)_\ast X = 0 \}$, and then $\wp(0, p) > \wp(1, p) > \wp(2, p) > \cdots > \wp(\infty, p)$.

2. Terminology

We will be introducing some constructions that extend certain standard ones, so it will be helpful to explain our notation and terminology first in a familiar case.

2.A. Coefficient systems. If $\mathfrak{X}$ is partially ordered set, a coefficient system on $\mathfrak{X}$ with values in an abelian category $\mathcal{C}$ is a functor $M : \mathfrak{X}^{\text{op}} \to \mathcal{C}$. A dual coefficient system on $\mathfrak{X}$ is a functor $N : \mathfrak{X} \to \mathcal{C}$.

Remark 2.1. It is more usual to call these local coefficient systems, on the grounds that a coefficient system takes the same value on all simplices (but perhaps allows for monodromy) whereas a local system varies with the simplex. We have simplified this for brevity.

2.B. Simplicial complexes. A simplicial complex $K$ on a vertex set $V$ is a set of non-empty finite subsets of $V$ so that $\emptyset \neq \tau \subseteq \sigma \in K$ implies $\tau \in K$. An element $\sigma$ of $K$ with $n + 1$ vertices is said to be an $n$-simplex of $K$, and $\mathfrak{X}_n$ is the set of $n$-simplices of $K$.

Remark 2.2. Note that we have explicitly declared that the empty set is not a simplex, to fit with our applications below.

We may think of a simplicial complex $K$ as a partially ordered set, ordered by inclusion.
2.C. Coefficient systems on $\mathfrak{X}$ and its subdivision. Suppose now $\mathfrak{X}$ is a partially ordered set, and consider the poset $\mathfrak{X}'$ of non-empty flags $F = (\varphi_0 \supset \cdots \supset \varphi_s)$. Indeed, $\mathfrak{X}'$ is a simplicial complex whose vertex set consists of objects of $\mathfrak{X}$.

We note that a coefficient system $M : \mathfrak{X}' \to C$ induces a coefficient system $M^\ast$ on $\mathfrak{X}'$ by

$$M^\ast(\varphi_0 \supset \cdots \supset \varphi_s) = M(\varphi_0).$$

It also defines a dual coefficient system on $\mathfrak{X}'$ defined by

$$M^\ast(\varphi_0 \supset \cdots \supset \varphi_s) = M(\varphi_s).$$

Similarly a dual coefficient system $N : \mathfrak{X}' \to C$ induces a dual coefficient system $N^\ast$ by using the first term in the flag, and a coefficient system $N^\ast$ by using the last term. These are named so that lower star indicates the variance is the same and an upper star implies the variance is reversed.

2.D. Simplicial cohomology. If we have a simplicial complex $K$, the simplicial cochain complex with coefficients in $M$ is defined by

$$C^\ast\text{simp}(K; M) = \left( \prod_{\sigma_0 \in K_0} M \to \prod_{\sigma_1 \in K_1} M \to \prod_{\sigma_2 \in K_2} M \to \cdots \right),$$

non-zero in cohomological degrees $\geq 0$ only. The differential is defined by $\delta = \sum_i (-1)^i \delta_i$, where $\delta_i$ is obtained by omitting the $i$th vertex.

More generally, if we have a dual coefficient system $M' : K' \to C$ on $K'$, we may use the same method.

$$H^\ast\text{simp}(K; M') = H^\ast\left( \prod_{\sigma_0 \in K_0} M'(\sigma_0) \to \prod_{\sigma_1 \in K_1} M'(\sigma_1) \to \prod_{\sigma_2 \in K_2} M'(\sigma_2) \to \cdots \right)$$

In any case, if $M$ is a coefficient system on $K$ and $N$ is a dual coefficient system on $K$ this gives us dual coefficient systems $M^\ast$ and $N^\ast$ on $K'$ and we may define

$$H^\ast\text{simp}(K'; M^\ast) \text{ and } H^\ast\text{simp}(K'; N^\ast).$$

Remark 2.3. We will generally omit the subscript $\text{simp}$ unless required for emphasis.

3. Towards adelic cohomology

We now start with a poset $\mathfrak{X}$, and use the above ideas to define cohomology at various levels of generality. The constructions are all familiar, but running through them is a good way to introduce the relevant notation and structure, and to emphasize variance of constructions.

3.A. Constant coefficients. To start with we may form the poset $\mathfrak{X}'$ of flags. This is a simplicial complex, so that given an object $M$ in an abelian category with products we may consider the simplicial cochain complex of Subsection 2.D given by

$$C^s(\mathfrak{X}'; M) = \prod_{\varphi_0 \supset \cdots \supset \varphi_s} M.$$
is then defined as an alternating sum

\[ \delta = \sum_i (-1)^i \delta_i \]

where \( \delta_i \) is induced by deleting the \( i \)th term in a flag. Taking cohomology, we have

\[ H^*(\mathcal{X}'; M) = H^*(C^*(\mathcal{X}'; M)). \]

**Remark 3.1.** Note that we have displayed the simplicial complex of flags \( \mathcal{X}' \) (rather than the poset \( \mathcal{Y} \) itself) in the notation for consistency with ordinary usage. This avoids ambiguity when \( \mathcal{X} \) itself is already a simplicial complex.

3.B. **Coefficient systems.** Next, we suppose that rather than a single object \( M \), we have a coefficient system \( M : \mathcal{X}^{op} \to \mathbb{C} \).

Note that \( M \) induces a dual coefficient system \( M^* \) on \( \mathcal{X}' \). Accordingly, we may then define a cochain complex on objects by

\[ C^s_{fl}(\mathcal{X}; M) = C^s(\mathcal{X}'; M^*) = \prod_{\varphi_0 > \cdots > \varphi_s} M(\varphi_s), \]

with the subscript \( fl \) indicating that the complex formed from flags in \( \mathcal{X} \). The coboundary

\[ \delta : C^s_{fl}(\mathcal{X}; M) \to C^{s+1}_{fl}(\mathcal{X}; M) \]

is again defined as an alternating sum

\[ \delta = \sum_i (-1)^i \delta_i. \]

Now, if \( i < s + 1 \) the map \( \delta_i \) is still induced by deleting the \( i \)th term in a flag. However \( \delta_{s+1} \)

is defined to be the product over \( s \)-simplices \( (\varphi_0 > \cdots > \varphi_s) \) with final vertex \( \varphi_s \) of the maps

\[ M(\varphi_s) \to \prod_{\varphi_{s+1} < \varphi_s} M(\varphi_{s+1}) \]

whose components are given by the functor \( M \). Once again, one finds that the composites \( \delta_i \delta_j \) only depend on the vertices \( i \) and \( j \) omitted, and hence \( \delta^2 = 0 \). We may then define the cohomology with coefficient system \( M \) by the formula

\[ H^s_{fl}(\mathcal{X}; M) = H^s(C^s_{fl}(\mathcal{X}; M)). \]

**Remark 3.2.** (i) If \( \mathcal{X} \) is itself a simplicial complex, the notation \( C^*(\mathcal{X}; N) \) is only defined when \( N \) is a dual coefficient system. However, if \( N \) is constant for example, we may treat it as a coefficient system and so define both \( C^*(\mathcal{X}; N) \) and \( C^s_{fl}(\mathcal{X}; N) = C^s(\mathcal{X}'; N^*) \). We note that the subdivision map

\[ Sbd : C^*(\mathcal{X}; N) \to C^*(\mathcal{X}'; N^*) = C^s_{fl}(\mathcal{X}; N^*) \]

is then a chain homotopy equivalence, so the similarity in notation should cause no serious confusion.

(ii) When \( \mathbb{C} \) is symmetric monoidal, and \( M \) is a diagram of commutative ring objects in \( \mathbb{C} \), the cohomology will be ring valued if the images of \( \delta \) are ideals, as happens if the maps \( \delta_i \) are ring maps.
Example 3.3. (i) Taking $\mathfrak{X} = \text{Spec}(R)$ with the Bahner ordering, completion defines a coefficient system, by

$$M(\emptyset) = M^\wedge.$$

(ii) We might take $\mathfrak{X}_a$ to be the poset of all closed subgroups of a torus $G$. We then have the inflation coefficient system $R$, whose value at $K$ is $H^*(BG/K)$. If $L \subseteq K$ we have an inflation map $R(K) = H^*(BG/K) \to H^*(BG/L) = R(L)$. In particular this applies to cotoral inclusions $L \leq K$ in the sense of Subsubsection 1.C.2.

(iii) On the other hand we may take $\mathfrak{X}_c$ to be the poset of connected subgroups of a torus $G$. This gives a somewhat a more complicated example.

Then we define a coefficient system $\tilde{R}$ as follows. At a connected subgroup $K$ it has value

$$\tilde{R}(K) = \prod_{\tilde{K} \in \mathcal{F}/K} H^*(BG/\tilde{K}),$$

where $\mathcal{F}/K$ is the set of subgroups $\tilde{K}$ with identity component $K$. To see this is a coefficient system, suppose $L \subseteq K$. We note that if $\tilde{L}$ has identity component $L$ then $\tilde{K} = \tilde{L} \cdot K$ is the unique subgroup of $G$ so that (a) $\tilde{K}$ has identity component $K$ and (b) $\tilde{K}/\tilde{L}$ is a torus. This means that if we take the product over $\tilde{K}$ with identity component $K$ of the maps

$$H^*(BG/\tilde{K}) \to \prod_{L < \tilde{K}} H^*(BG/\tilde{L})$$

then we get precisely a map

$$\tilde{R}(K) \to \tilde{R}(L)$$

as required.

We note that the map $q : \mathfrak{X}_a \to \mathfrak{X}_c$ taking a subgroup to its identity component has the requisite properties that the coefficient system $R$ on $\mathfrak{X}_a$ gives the coefficient system $\mathcal{O}_\mathcal{F} = q_* R$ in the notation of [Gre16].

(iv) If we start with a spectrum $M$, then for a prime $\wp(p, n)$ we obtain a coefficient system by taking the homotopy of the Bousfield localization at the corresponding Morava $K$-theory:

$$M(\wp(p, n)) = \pi_*(L_{K(p, n)} M).$$

3.C. $\mathfrak{X}$-collections of localizations. Rather than just consider the individual localizations or completions, we will consider collections indexed by the poset $\mathfrak{X}$.

Definition 3.4. (i) An idempotent localization $L : \mathbb{C} \to \mathbb{C}$ is an idempotent monad. It consists of the functor $L$, together with a natural transformation $\eta : 1 \to L$ giving a natural equivalence

$$L\eta = \eta_L : L = L \circ 1 \to L \circ L$$

(ii) An $\mathfrak{X}$-collection of localizations is a collection $\{L_\wp\}$ of localizations for $\wp \in \mathfrak{X}$. An $\mathfrak{X}$-collection is said to be left absorbative if, whenever $\wp_1 \geq \wp_2$, the natural map

$$A_{\wp_1}(\eta_{\wp_2}) : A_{\wp_1} = A_{\wp_1} \circ 1 \to A_{\wp_1} \circ A_{\wp_2}$$

is an isomorphism. An $\mathfrak{X}$-collection is right absorbative if, whenever $\wp_1 \geq \wp_2$, the natural map

$$\eta_{\wp_1} : A_{\wp_2} \to A_{\wp_1} \circ A_{\wp_2}$$

is an isomorphism.
Lemma 3.5. (i) A left absorbative $\mathcal{X}$-collection $L_\bullet$ gives idempotent localizations $L_\wp$ which fit together to give a functor

$$\mathcal{X} \to \mathcal{C}, \mathcal{C}$$

(i.e., an inequality $\wp_2 \leq \wp_1$ gives rise to a natural transformations $L_{\wp_2} \to L_{\wp_1}$, and these are closed under composition).

(ii) A right absorbative $\mathcal{X}$-collection $\Lambda_\bullet$ gives idempotent localizations $\Lambda_\wp$ which fit together to give a functor

$$\mathcal{X}^{\text{op}} \to \mathcal{C}, \mathcal{C}$$

(i.e., an inequality $\wp_2 \leq \wp_1$ gives rise to a natural transformations $\Lambda_{\wp_1} \to \Lambda_{\wp_2}$, and these are closed under composition).

Remark 3.6. (i) Following Part (i) we will usually refer to a left absorbative system as a dual system of localizations.

The motivating example is the collection of localizations at a prime in commutative algebra. Writing $\wp_a$ for prime ideal in the algebraic sense, and $\wp_b$ for the associated Balmer prime, we have $L_{\wp_b}M = M_{\wp_a}$. The variance is covariant in $\wp_b$.

We will usually use the letter $L$ for a dual system of localizations.

(ii) Following Part (ii) we refer to a right absorbative system as a (direct) system of localizations.

The motivating example is the collection of completions at a prime in commutative algebra. Writing $\wp_a$ for prime ideal in the algebraic sense, and $\wp_b$ for the associated Balmer prime, we have $\Lambda_{\wp_b}M = M_{\wp_a}^{\Lambda}$. The variance is contravariant in $\wp_b$.

We will usually use the letter $\Lambda$ for a direct system of localizations.

(iii) The composite of idempotent localizations (such as $A_\wp = L_\wp \Lambda_\wp$ or $\Lambda_\wp L_\wp$) need not be an idempotent localization, but it does come equipped with a natural transformation $1 \to A_\wp$, which will be the essential input into constructing the adelic cochain complex below.

(iv) Given an object $M$ of $\mathcal{C}$, any coefficient system $\Lambda_\bullet$ of localizations defines a coefficient system

$$\Lambda_\bullet M : \mathcal{X}^{\text{op}} \to \mathcal{C}.$$ 

Similarly a dual system $L_\bullet$ of localizations, defines a dual coefficient system

$$L_\bullet M : \mathcal{X} \to \mathcal{C}.$$ 

3.D. The dual system from commutative algebra. The simplest type of example might come from taking $\mathcal{X} = \text{Spec}(R)$ with the Balmer partial order (i.e., the reverse of the classical order), and defining the system of localizations

$$L_\wp M := M_{\wp}$$

to be classical localization. Localization $M_\wp$ simply inverts the elements of the complement $\wp^c$ so that if $\wp_1 \geq \wp_2$ in the Balmer order, we have $(M_{\wp_2})_{\wp_1} = M_{\wp_1}$.

3.E. Adelic cochains and adelic cohomology. To define adelic cohomology we need a coefficient system $M$ and an $\mathcal{X}$-collection $A$. In this subsection we define the adelic cochain complex $C^*_\text{ad}(\mathcal{X}; A, M)$ and the adelic cohomology

$$H^*_\text{ad}(\mathcal{X}; A, M) := H^*(C^*_\text{ad}(\mathcal{X}; A, M)).$$
**Definition 3.7.** For \( s \geq 0 \), the *adelic s-cochains* may be thought of as functions on \( s \)-simplices of the poset \( \mathfrak{X} \):

\[
C^s(\mathfrak{X}; A, M) = \prod_{\varphi_0} A_{\varphi_0} \prod_{\varphi_1 < \varphi_0} A_{\varphi_1} \prod_{\varphi_2 < \varphi_1} \cdots \prod_{\varphi_{s-1} < \varphi_{s-2}} A_{\varphi_{s-1}} \prod_{\varphi_s < \varphi_{s-1}} A_{\varphi_s} M(\varphi_s).
\]

The differential

\[
\delta: C^s(\mathfrak{X}; A, M) \to C^{s+1}(\mathfrak{X}; A, M).
\]

is given as a sum \( \delta = \sum_i (-1)^i \delta_i \), where \( \delta_i \) is based on omitting the \( i \)th term in an \((s+1)\)-flag. In more detail, if \( i < s + 1 \), we define

\[
M_{i+1}(\varphi_i) = \prod_{\varphi_{i+1} < \varphi_i} A_{\varphi_{i+1}} \prod_{\varphi_{i+2} < \varphi_{i+1}} \cdots \prod_{\varphi_{s-1} < \varphi_{s-2}} A_{\varphi_{s-1}} \prod_{\varphi_s < \varphi_{s-1}} A_{\varphi_s} M(\varphi_{s+1}).
\]

(this is just \( C^{s-i}(\Lambda(\varphi_i)^*; A, M) \) in the previous notation, where \( \Lambda(\varphi_i) \) consists of specializations of \( \varphi_i \) and the star indicates that \( \varphi_i \) itself is omitted). Then the map \( \delta_i \) is simply given by taking

\[
M_{i+1}(\varphi_i) \to A_{\varphi_i} M_{i+1}(\varphi_i)
\]

at the \( i \)th spot, and then applying the same sequence of products and localizations to both domain and codomain.

If \( i = s + 1 \) we take the map

\[
M(\varphi_s) \to \prod_{\varphi_{s+1} < \varphi_s} M(\varphi_{s+1}) \to \prod_{\varphi_{s+1} < \varphi_s} A_{\varphi_{s+1}} M(\varphi_{s+1})
\]

with components \( M(\varphi_s) \to M(\varphi_{s+1}) \to A_{\varphi_{s+1}} M(\varphi_{s+1}) \) given by the coefficient system, and then apply \( A_{\varphi_s} \) and the same sequence of products and localizations to both domain and codomain.

To see we get a cochain complex we need only observe that the composite of two \( \delta_i \)'s depends only on the dimensions omitted. More precisely, for

\[
C^{s-2} \xrightarrow{\delta} C^{s-1} \xrightarrow{\delta} C^s
\]

if the numbers omitted are \( 0 \leq a < b < s \), then we may omit \( a \) and \( b \) in either order and we need to know that \( \delta_a \delta_b = \delta_{b-1} \delta_a \).

If \( a < b < s \) then the verification is immediate from the naturality of the transformations \( \eta_\varphi \), together with the categorical properties of the product.

If \( b = s \) there are two cases. The simplest is when \( a < s - 2 \). Then the diagram

\[
\begin{array}{ccc}
A_{\varphi_{a+1}} \cdots A_{\varphi_{s-1}} M(\varphi_{s-1}) & \xrightarrow{A_{\varphi_{a+1}} \cdots A_{\varphi_{s-1}} A_{\varphi_{s}} M(\varphi_{s})} & A_{\varphi_{a+1}} A_{\varphi_{s+1}} \cdots A_{\varphi_{s-1}} A_{\varphi_{s}} M(\varphi_{s}) \\
\downarrow & & \downarrow \\
A_{\varphi_{a}} A_{\varphi_{a+1}} \cdots A_{\varphi_{s-1}} M(\varphi_{s-1}) & \xrightarrow{A_{\varphi_{a}} A_{\varphi_{a+1}} \cdots A_{\varphi_{s-1}} A_{\varphi_{s}} M(\varphi_{s})} & A_{\varphi_{a}} A_{\varphi_{a+1}} \cdots A_{\varphi_{s-1}} A_{\varphi_{s}} M(\varphi_{s})
\end{array}
\]

commutes since \( \eta : 1 \to A_{\varphi_{a}} \) is a natural transformation. The required commutation then follows from the categorical properties of the product.
The case \( b = s - 1, a = s \) is the most complicated. We will abbreviate \( M(\varphi_s) = M(s) \) and \( A_{\varphi_s} = A_s \) for readability. The following diagram has \( A_0 A_1 \ldots A_{n-2} \) applied to it.

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The left and right faces commute since the unit for \( A_{n-1} \) is a natural transformation. The front and back faces commute since the unit for \( A_n \) is a natural transformation. The top and bottom faces commute because the unit for \( A_{n-1} \) is a natural transformation. The relevant square involves \( M(n-2), A_{n-1} A_n M(n-1), A_n M(n) \) and \( A_{n-1} A_n M(n) \). The required commutation then follows from the categorical properties of the product.

3.F. **Local meromorphic coefficients.** In this subsection we focus on the important example where the \( \mathcal{X} \)-collection \( L \) is left absorbative (i.e., it is a dual system of localizations).

If the poset is finite, the products in the definition of the cochain complex are finite. This means that all the localizations can be collected on the value \( M(\varphi_s) \) of the coefficient system:

\[
L_{\varphi_0} \cdots L_{\varphi_s} M(\varphi_s) = L_{\varphi_0} M(\varphi_s).
\]

Actually, we need to deal with infinite posets, so the localizations will usually not commute with the products. Nonetheless, we think of \( L_{\varphi_0} \) as specifying ‘permitted denominators in the completed stalk \( M(\varphi_s) \)’, and arrange terminology accordingly.

**Definition 3.8.** An adelic coefficient system is a local coefficient system \( M \), together with a dual system of localizations \( L \).
There are two degenerate cases. If $M$ is constant, we just have a dual coefficient system $L_\wp M$. If the dual system of localizations all consist of the identity functor, we just have a coefficient system $M(\wp)$.

**Example 3.9.** The motivating example of an adelic coefficient system is $\mathfrak{X} = \text{Spec}(R)$ with the Balmer partial order (i.e., the reverse of the classical order). We then take the completion coefficient system $M(\wp) = M^\wedge_\wp$, and the dual system of localizations

$$L_\wp N := N_\wp$$

defined to be classical localization.

### 4. Catenary posets $\mathfrak{X}$

If the poset $\mathfrak{X}$ is well behaved we can organize the adelic complex into a cube. This can be helpful in examples, but it is an unnecessary detour in the general development.

#### 4.A. Dimensions.

We want to suppose that the poset $\mathfrak{X}$ is catenary in the sense that for any prime $\wp$ there is a bound on the length $s$ of flags

$$F = (\wp = \wp_0 \supset \wp_1 \supset \ldots \supset \wp_s)$$

of primes, and all maximal chains of this form starting at $\wp$ have the same length. If the displayed chain is a maximal chain starting at $\wp = \wp_0$ we will say that $\wp$ is of dimension $s$. Thus closed points (Balmer-minimal primes) are of dimension 0.

The dimension of $\mathfrak{X}$ is defined by

$$r := \text{dim}(\mathfrak{X}) := \max\{\text{dim}(\wp) \mid \wp \in \mathfrak{X}\},$$

which may be finite or $\infty$.

We write $\Delta^r := \{0, 1, \ldots, r\}$ if $r$ is finite and $\Delta^\infty = \mathbb{N}$. The dimension function $\text{dim} : \mathfrak{X} \to \Delta^r$ is a function of posets, and therefore induces a function on flags

$$\text{dim} : \mathfrak{X}' \to (\Delta^r)'$$

When $r$ is finite, we will usually identify $(\Delta^r)'$ (the set of non-empty subsets of $\{0, 1, \ldots, r\}$) with the punctured $r$-cube by identifying a subset with its characteristic function. More concretely any chain of the form displayed can be viewed as an $s$-simplex of $\mathfrak{X}$, and we write

$$\text{dim}(F) = (d_0 > d_1 > \ldots > d_s);$$

the flag is maximal if $d_i = s - i$ where $\text{dim}(\wp) = s$.

**Example 4.1.** If $R$ is a catenary commutative Noetherian ring and $\wp_a$ is a prime in the algebraic sense, with associated Balmer prime $\wp_b$ then $\text{dim}(\wp_b) = \text{dim}(A/\wp_a)$. The dimension of $\mathfrak{X} = \text{Spec}(R)$ is the usual Krull dimension of $R$.

**Example 4.2.** If $G$ is a torus then $\text{dim}(\wp_K) = \text{dim}(K)$. In this case there is a unique prime $\wp_G$ which is Balmer-maximal (i.e., corresponding to an irreducible component), and infinitely many closed points $\wp_F$ where $F$ is finite. The dimension of the poset of subgroups is the dimension of $G$. 

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4.B. **Collecting cochains.** If $\mathfrak{X}$ is catenary, we may then divide $s$-simplices into those of different dimensions $d_0 > d_1 > \cdots > d_s$ of the vertices of the $s$-simplex. Provided $\mathfrak{X}$ is finite dimensional, there are only finitely many possible dimension vectors of $s$-simplices. Since $A_\wp$ is additive and the product is a categorical sum

$$A_\wp(M \times N) = A_\wp M \times A_\wp N,$$

we may then break cochains up by dimension.

Thus

$$C^s(\mathfrak{X}; A, M) = \prod_{d_0 > d_1 > \cdots > d_s} C^{d_0 > d_1 > \cdots > d_s}(\mathfrak{X}; A, M)$$

and

$$C^{d_0 > d_1 > \cdots > d_s}(\mathfrak{X}; A, M) = \prod_{\dim \wp_0 = d_0} A_{\wp_0} \prod_{\wp_1 < \wp_0, \dim \wp_1 = d_1} A_{\wp_1} \cdots \prod_{\wp_{s-1} < \wp_{s-2}, \dim \wp_{s-1} = d_{s-1}} A_{\wp_{s-1}} \prod_{\wp_s < \wp_{s-1}, \dim \wp_s = d_s} A_{\wp_s} M(\wp_s).$$

Since the dimension of a face of a flag is the corresponding face of its dimension, the functions $\delta_i$ are compatible with this decomposition.

Writing $d = (d_0 > \cdots > d_s)$, thought of as a face of $\Delta^r$, we may think of

$$d \mapsto C^d(\mathfrak{X}; A, M)$$

as a dual coefficient system on the subdivision of the simplex $\Delta^r$. We will often display it as a diagram on the punctured $(r + 1)$-cube. The cochain complex $C^*(\mathfrak{X}; A, M)$ is obtained from the diagram by totalizing.

5. **The classical Hasse square**

We suppose given a catenary Noetherian commutative ring $R$. There are two interesting points here. The first is the straightforward utility: we will show (Theorem 5.1) that the adelic cohomology is entirely in degree 0 where it is equal to the ring (this corresponds to the fact that the classical Hasse square is both a pullback and a pushout).

Deferring discussion of coefficients, we have

$$H^*_\text{ad}(\text{Spec}(R); L, M) = H^0_\text{ad}(\text{Spec}(R); L, M) = R.$$

The second striking thing is that there are numerous different choices of coefficients $L, M$ for which this is true. We could crudely say that they are all based on localization and completion at primes $\wp$. Our preferred variation is that the input is the rings $L_\wp \Lambda_\wp, R = (R_\wp^\wedge)_\wp$ and the Beilinson-Parshin approach [Hub91] is based on the rings $\Lambda_\wp L_\wp, R = (R_\wp^\wedge)$. Even within these two categories there is some variation in the chain complex.

5.A. **Completion.** The dual system of localizations is $L_\wp M = M_\wp$ but there are several variations on what is done with it. In this subsection we consider the choice of whether $\Lambda_\wp R = R_\wp^\wedge$ (which we view as the true adelic approach) or $\Lambda_\wp = id$ for non-closed points.

This binary choice can each be further multiplied. We may divide the poset $(\Delta^r)'$ into an initial part $I$ and the complementary terminal part $J$, where there are no maps from a point of $J$ to a point of $I$. Then $\Lambda'_\wp = id$ at points of $J$ and $\Lambda'_\wp M = M_\wp^\wedge$ for points of $I$. The theory that has been called ‘isotropic’ in [GS18] is the one in which $I$ consists of the finite subgroups, and we call it ‘local’ in the present context. We restrict the use of $\Lambda'$ to this case.
5.B. **Products.** The second choice is about the size of the products. At dimension \( d_0 > \cdots > d_s \), if \( d_s = 0 \) and for a fixed prime \( \wp \) of dimension \( d_s - 1 \) the choice is whether we use

\[
\prod_{m \leq \wp - 1} R_m^\wedge \text{ or } \prod_{m} R_m^\wedge.
\]

The first follows Definition 3.7 which is formally most natural and most geometric. The second is what is used in equivariant topology [GS18], where it was forced by the need to have a commutative ring spectrum at that point.

Again this binary choice may be further multiplied. We could replace the product over primes contained in \( \wp - 1 \) by a larger product even when \( d_s \neq 0 \), provided this is compatible with the maps. We will not attempt to axiomatize this in the absence of applications.

5.C. **Beilinson-Parshin adeles.** The description in [Hub91, Proposition 2.1.1] is inductive, but it amounts to taking the coefficient system \( M(\wp) = R \) and the system of localizations \( \Lambda_\wp L_\wp \):

\[
H^*_\text{BP}(R) = H^*_\text{ad}(\text{Spec}(R); \Lambda L, R).
\]

5.D. **The statement.** In all of the situations identified above, the adelic cohomology recovers the ring in degree 0.

**Theorem 5.1.** If \( R \) is a commutative Noetherian catenary ring and we take one the three coefficient systems \((L, \Lambda R)\), \((L, \Lambda' R)\), \((\Lambda L, R)\) then there is no higher cohomology and \( H^0 \) is the original ring:

\[
H^*_\text{ad}(\text{Spec}(R)) = R.
\]

This also holds when all products involving maximal ideals involve all maximal ideals.

**Proof:** For Beilinson-Parshin adeles, this is [Hub91, Theorem 4.1.1], but our proof in the other cases gives an alternative proof.

Our method is to establish that in the derived category, the ring \( R \) is the homotopy pull-back of a suitable cube of rings following the method of [GS18] (see Section 11): this is done in Section 11. We then apply the results of [GM92] to see that the 0th left derived functor of completion for the Noetherian ring \( R \) is the ordinary completion and the higher derived functors vanish; it then follows that the derived completion coincides with the ordinary completion.

6. **Subdivision and constant coefficients**

We observe here that the adelic cohomology of a constant coefficient system \( M \), can sometimes be calculated from a much smaller cochain complex. Indeed, the case that we will use is when \( \mathfrak{X} \) is itself a simplicial complex, and the result is simply that the cohomology of a complex and its subdivision agree.
6.A. Simplicial cohomology. By the time we have explained the statement, the reader will agree with it, but there are several notational delicacies worth clarifying on the way.

We are given an object $M$ in $C$ (defining a constant coefficient system) and a dual system $L$ of localizations. This gives a dual coefficient system $LM : \mathfrak{X} \to C$, and a corresponding dual coefficient system $LM^* : \mathfrak{X}' \to C$ on the flag complex. If $\mathfrak{X}$ is itself a simplicial complex, the simplicial cohomology of $\mathfrak{X}$ is defined by

$$H^*_{\text{simp}}(\mathfrak{X}; LM) = H^* \left( \prod_{\sigma_0 \in \mathfrak{X}_0} L_{\sigma_0} M \to \prod_{\sigma_1 \in \mathfrak{X}_1} L_{\sigma_1} M \to \prod_{\sigma_2 \in \mathfrak{X}_2} L_{\sigma_2} M \to \cdots \right)$$

**Lemma 6.1.** If the partially ordered set $\mathfrak{X}$ is a finite simplicial complex then

$$H^*_{\text{ad}}(\mathfrak{X}; L, M) \cong H^*_{\text{simp}}(\mathfrak{X}; LM)$$

**Proof:** Using the fact that localizations commute with finite products, the adelic $s$-cochains are

$$\prod_{\sigma_0} L_{\sigma_0} \prod_{\sigma_1 \subseteq \sigma_0} L_{\sigma_1} \prod_{\sigma_2 \subseteq \sigma_1} L_{\sigma_2} \cdots \prod_{\sigma_s \subseteq \sigma_{s-1}} L_{\sigma_s} M = \prod_{\sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_s} L_{\sigma_0} M.$$

The cohomology of this is the cohomology of the subdivided complex $\mathfrak{X'}$:

$$H^*_{\text{ad}}(\mathfrak{X}; L, M) = H^*_{\text{simp}}(\mathfrak{X}'; LM_s)$$

Finally, simplicial homology is invariant under passage to subdivisions.

$$H^*_{\text{simp}}(\mathfrak{X}; LM) \cong H^*_{\text{simp}}(\mathfrak{X}'; LM_s).$$

\[\square\]

6.B. The Cech complex. One instance of Lemma 6.1 is very important to us.

We suppose given a ring $R$ consider some elements $x_a$ as the subscript $a$ runs through a set $A$. Now we take $\mathfrak{X} = \Delta(A)$ to be the partially ordered set of non-empty finite subsets of $A$, and write $\mathfrak{X}^+$ for the poset of all finite subsets. If $A$ is a finite set with $r$ elements $\mathfrak{X}^+$ is the $r$-cube and $\mathfrak{X} = \Delta(A)$ is the punctured $r$-cube.

An $R$-module $M$ defines a constant coefficient system on $\mathfrak{X}$. It is then natural to consider the dual coefficient system $M_A$ on $\mathfrak{X}^+$, whose value on a finite subset $\sigma$ of $A$ is the localization $M[1/x_\sigma]$, where $x_\sigma = \prod_{a \in \sigma} x_a$. If $A$ is finite with $r$ elements, $\mathfrak{X}^+$ is an $r$-cube, and the diagram $M$ is commutative so we may totalize this diagram to get the usual stable Koszul complex.

A simple example will make the result clear, and explain our care with the empty set.

**Example 6.2.** If $A$ has just 2 elements, $\mathfrak{X} = \{ \{x\}, \{y\}, \{x, y\} \}$ has two vertices and one edge, and the dual coefficient system $M^+_A$ on $\mathfrak{X}^+$ is

$$M \quad \xrightarrow{1} \quad M[\frac{1}{x}] \quad \xrightarrow{1} \quad M[\frac{1}{y}]$$

$$M[\frac{1}{x}] \quad \xrightarrow{1} \quad M[\frac{1}{xy}]$$
which is totalized to
\[ M \to M[\frac{1}{x}] \oplus M[\frac{1}{y}] \to M[\frac{1}{xy}], \]
whose simplicial cohomology is (by definition) the local cohomology
\[ H^\ast_{\text{simp}}(\Delta(A)^+; M_A^+) = H^\ast_{(x,y)}(R; M). \]
If we restrict attention to \( \mathcal{X} \), then we omit the top left entry and regrade so that we obtain the Čech cohomology
\[ H^\ast_{\text{simp}}(\Delta(A); M_A) = \check{C}^\ast_{(x,y)}(R; M). \]

For the adelic cochain complex, we note \( \mathcal{X}' \) has three vertices (the two \( \{x\}, \{y\} \) of dimension 0 correspond to the vertices of \( \mathcal{X} \), and the one \( \{x, y\} \) of dimension 1 corresponds to the whole of \( \mathcal{X} \)), and two edges (\( \{x, y\} \supset \{x\} \) and \( \{x, y\} \supset \{y\} \)). The dual coefficient system \( M_A^+ \) on \( \mathcal{X}^+ \) induces \( (M_A^+)_\ast \) on \( (\mathcal{X}')^+ \). Using the decomposition by dimension vectors, we obtain the augmented complex
\[ M \to M[\frac{1}{xy}] \]
\[ M[\frac{1}{x}] \oplus M[\frac{1}{y}] \to M[\frac{1}{xy}] \oplus M[\frac{1}{xy}] \]

Lemma 6.1 shows the cohomology of the complexes obtained from the two displayed squares are the same, and in fact it is easy to construct a homotopy equivalence.

For a general set \( A \) of variables in a Noetherian ring \( R \), Lemma 6.1 gives the following calculation.

**Corollary 6.3.** With \( M \) an \( R \)-module and \( L_A \) being the localization away from a set \( A \) of elements of \( R \), we have
\[ H^\ast_{\text{ad}}(\Delta(A)^+; L_A, M) \cong H^\ast_{(x_a \mid a \in A)}(R; M) \]
\[ H^\ast_{\text{ad}}(\Delta(A); L_A, M) \cong \check{C}^\ast_{(x_a \mid a \in A)}(R; M) \]

Note that this also shows the adelic cohomology only depends on the radical of the ideal \( I(A) = (x_a \mid a \in A) \).

**Proof:** We will discuss the local cohomology case for definiteness, but a precisely similar argument applies to Čech cohomology.

The case when \( A \) is finite is given by Lemma 6.1 together applied to the adelic coefficient system \( (L_A, M) \).

If we add one element to \( A \) to form \( B = A \cup \{b\} \), we obtain a commutative square
\[ H^\ast_{\text{ad}}(\Delta(A)^+; L_A, M) \xrightarrow{\cong} H^\ast_{(x_a \mid a \in A)}(R; M) \]
\[ H^\ast_{\text{ad}}(\Delta(A)^+; L_A, M) \xrightarrow{\cong} H^\ast_{(x_a \mid a \in A)}(R; M) \]

Because the ring is Noetherian, any such chain of adding elements eventually gives vertical isomorphisms. The choice of chain is not important since any two particular collections of variables \( A_1, A_2 \) may be compared to \( A_1 \coprod A_2 \). □
7. Varying the ambient category

The definition of adelic cohomology above fails to cover some important examples, so we introduce a more flexible context.

7.A. Arrivals from homotopy theory. One motivation for this work is that homotopy theory provides a rich source of examples. As described in [BG19] we may obtain a coefficient system $M(\wp)$ and a dual system of localizations $A_\wp$ in a category $C$ of homotopical origin. This means that we can apply homotopy (or some other homology theory) to obtain a coefficient system $\pi_* (M(\wp))$ in some abelian category $A$. However, there is no reason to expect $\pi_* (L_q M(\wp))$ to be a functor of $\pi_* (M(\wp))$, so we do not automatically get a dual system of localizations.

Nonetheless, under catenary and finite dimensionality assumptions we obtain an $(r + 1)$-cube in $C$, and applying homotopy gives a cube in $A$, from which we may obtain a chain complex $C^d_{ad}(\mathfrak{X}; \pi_*, L, M)$ with

$$C^d_{ad}(\mathfrak{X}; \pi_*, L, M) = \pi_* (C^d_{ad}(\mathfrak{X}; L, M)) = \pi_* (\prod_{\wp_0} L_{\wp_0} \prod_{\wp_1} L_{\wp_1} \cdots \prod_{\wp_s} L_{\wp_s} M(\wp_s)).$$

We would like to consider cases in which this chain complex is an example of the adelic complex of Definition 3.7. Of course one example is that from commutative algebra, but many examples do not fit this pattern. On the other hand, a small variation will cover some more examples, and it is the purpose of this section to introduce the variation.

7.B. Relative localization. It may happen that $\pi_* (M(\wp))$ takes values in an abelian category $\mathcal{A}(\wp)$ depending on $\wp$, and that in that context there is an algebraic localization reflecting the homotopical one.

This applies to the examples from equivariant topology. In fact, there is a homotopy category level version which applies for $G$-spectra, but when one takes homotopy groups to move into algebra one needs to take account of the fact that at a subgroup $K$ we get a module over $H^*(BG/K)$ (i.e., the ambient category varies with the prime). In line with the algebraic focus of this paper we will restrict to variation controlled by a coefficient system $R$ of rings.

Thus we assume our dual system of localizations is monoidal so that if $R$ is a ring then the dual coefficient system $R(\wp)$ gives a diagram of rings. This means that each localization $L_\wp$ needs to have a version for $R(\wp)$-modules for all $\wp \leq \wp$.

**Definition 7.1.** A dual system of relative localizations is a left absorbative system of functors $L_{\wp_1/\wp_2} : R(\wp_2)-\text{modules} \to R(\wp_2)-\text{modules}$ (where $\wp_1 \geq \wp_2$), which are transitive in the sense that when $\wp_1 \geq \wp_2 \geq \wp_3$, the diagram

\[
\begin{array}{ccc}
R(\wp_2)-\text{modules} & \xrightarrow{L_{\wp_1/\wp_2}} & R(\wp_2)-\text{modules} \\
\downarrow{R_*} & & \downarrow{R_*} \\
R(\wp_3)-\text{modules} & \xrightarrow{L_{\wp_2/\wp_3}} & R(\wp_3)-\text{modules}
\end{array}
\]
commutes.

We will see that the equivariant examples satisfy this transitivity condition. Unfortunately this condition by itself does not seem to be enough to complete the algebraic construction, so we will restrict further.

7.C. **Multiplicative systems.** Once again, we return to the equivariant setting for motivation.

For a torus $G$ the complex representations of $G/K$ are the representations $V$ of $G$ which are $K$-fixed (i.e., $V^K = V$). Consider the $H$-essential representations of $G/K$:

$$\text{Rep}(G/K)_{H/K} := \{V \in \text{Rep}(G/K) \mid V^H = 0\}.$$ 

Now suppose we have a dual coefficient system $R(K)$ of rings and Euler class functions

$$e : \text{Rep}(G/K) \longrightarrow R(K)$$

which are multiplicative in the sense that $e(V \oplus W) = e(V)e(W)$, and compatible with the dual coefficient system in the sense that

$$e(\inf_{G/L}^{G/K} V) = R_* e(V).$$

Now we write

$$\mathcal{E}_{H/K} := \{e(V) \in R(K) \mid V^H = 0\}.$$ 

**Remark 7.2.** (i) If $L < K$ then $V^L = 0$ implies $V^K = 0$ so that $\mathcal{E}_L \subseteq \mathcal{E}_K$, and localization is transitive.

(ii) If $V$ is an arbitrary representation of $G$ then $V = V^K \oplus V'$ with $(V')^K = 0$ so that the multiplicative set of Euler classes of $H$-essential representations of various subgroups is partially generated by inflations:

$$\mathcal{E}_{H/1} = \langle R_* \mathcal{E}_{H/K}, \mathcal{E}_{K/1} \rangle.$$ 

Abstracting this example slightly we reach the definition.

**Definition 7.3.** A **relative system of Euler classes** for a coefficient system of rings is specified by one multiplicative set $\mathcal{E}_{\varphi_1/\varphi_2}$ in $R(\varphi_2)$ whenever $\varphi_1 \geq \varphi_2$ so that

$$\mathcal{E}_{\varphi_0/\varphi_2} = \langle R_*, \mathcal{E}_{\varphi_0/\varphi_1}, \mathcal{E}_{\varphi_1/\varphi_2} \rangle.$$ 

An any relative system of Euler classes gives a dual relative system of localizations by taking

$$L_{\varphi_1/\varphi_2} = \mathcal{E}^{-1}_{\varphi_1/\varphi_2}.$$ 

This gives a sufficiently general framework that we can cover the equivariant cases by our machinery.

7.C.1. **The dual system for all closed subgroups of a torus.** In the toral example $\mathfrak{X}_a$ with all subgroups, we have $R(K) = H^*(BG/K)$ and for $L \leq K$ we define

$$L_{K/L}M = \mathcal{E}^{-1}_{K/L}M.$$ 

7.C.2. **The dual system for connected subgroups of a torus.** In the toral example $\mathfrak{X}_c$ with connected subgroups, we have $R(K) = \mathcal{O}_{\mathcal{F}/K}$ and for $L \leq K$ we define

$$L_{K/L}M = \mathcal{E}^{-1}_{K/L}M.$$
7.D. **Localizations of products.** There is a second problem with permitting \( M(\wp) \) to lie in an abelian category \( \mathcal{A}(\wp) \) depending on \( \wp \), because we need to take products of objects from different categories. We therefore assume that the categories are all enriched in an abelian category \( \mathcal{A} \), and that products in \( \mathcal{A}(\wp) \) are created in \( \mathcal{A} \). We may then hope that the adelic cohomology takes values in \( \mathcal{A} \).

Given this, we then need to take localizations of products in the form

\[
L_{\wp_1} \prod_{\wp_2 \leq \wp_1} M(\wp_2).
\]

Here we assume our relative system of localizations comes from a relative system of Euler classes.

\[
L_{\wp_1/\wp_2} M(\wp_1) = \mathcal{E}^{-1}_{\wp_1/\wp_2} M(\wp_1) = \operatorname{lim} \left( M(\wp_1) \xrightarrow{e_1} M(\wp_1) \xrightarrow{e_2} M(\wp_1) \rightarrow \cdots \right)
\]

We then define the localization of the product by a precisely similar colimit

\[
L_{\wp_1} \prod_{\wp_2 \leq \wp_1} M(\wp_2) = \mathcal{E}^{-1}_{\wp_1} \prod_{\wp_2 \leq \wp_1} M(\wp_2) = \operatorname{lim} \left( \prod_{\wp_2 \leq \wp_1} M(\wp_2) \xrightarrow{e_1} \prod_{\wp_2 \leq \wp_1} M(\wp_2) \xrightarrow{e_2} \prod_{\wp_2 \leq \wp_1} M(\wp_2) \rightarrow \cdots \right).
\]

This time there is some interpretation since if \( e \in \mathcal{E}_{\wp_1} \), for each \( \wp \leq \wp_1 \) we may write \( e = e'_q e''_q \), and in the \( \wp \) factor we interpret multiplication by \( e''_q \) as an isomorphism, so that in effect \( e \) is multiplication by \( e'_q \).

7.E. **Generalized adelic cohomology.** In the context that we have

- a coefficient system \( R \) of commutative rings
- a coefficient system \( M \) of \( R \)-modules
- a dual system of Euler classes

then we can define the adelic chain complex and adelic cohomology \( H^*_a(X; \mathcal{E}^{-1}, M) \) by the same formula as before.

8. **Examples of adelic cohomology**

None of following three examples are covered by the original definition of adelic cohomology, but the second is covered by the varying-category version of Section 7.

8.A. **Projective curves.** If \( C \) is a smooth irreducible projective curve we may take \( X \) to consist of the irreducible subvarieties (i.e., the closed points of \( C \) are minimal and the generic point is maximal; by a theorem of Thomason this is the Balmer spectrum of perfect complexes of quasi-coherent sheaves).

We take the coefficient system to be given by the completed stalks \( (\mathcal{O}_C)^\wedge \) of the structure sheaf. The adelic cochain complex is

\[
C^*_a(Spc(C)) = \left( \mathcal{K}_C \oplus \prod_x (\mathcal{O}_C)^\wedge_x \rightarrow \mathcal{K}_C \otimes \mathcal{O}_C \prod_x (\mathcal{O}_C)^\wedge_x \right)
\]

where \( \mathcal{K}_C \) is the ring of meromorphic functions, \( x \) runs through the closed points and \( (\mathcal{O}_C)^\wedge_x \) is the completed stalk at \( x \).

**Lemma 8.1.** For any locally free \( \mathcal{O}_C \)-module \( \mathcal{F} \) of finite rank, the adelic cohomology is the sheaf cohomology.

\[
H^*_a(Spc(C); \mathcal{F}) = H^*(C; \mathcal{F}).
\]
Remark 8.2. In effect the adelic complex is the embodiment of the residue approach to cohomology.

Proof: We may work in the category of sheaves and see that the square
\[ \begin{array}{ccc}
\mathcal{O}_C & \rightarrow & \mathcal{K}_C \\
\downarrow & & \downarrow \\
\prod_x (\mathcal{O}_C)_x & \rightarrow & \mathcal{K}_C \otimes \mathcal{O}_C \prod_x (\mathcal{O}_C)_x
\end{array} \]
is a homotopy pullback (where now \( \mathcal{K}_C \) is the constant sheaf of meromorphic functions and \( (\mathcal{O}_C)_x \) is a skyscraper sheaf concentrated at \( x \)). Indeed, the fibres of both horizontals are isomorphic because the local cohomology of a ring and its completion are isomorphic. Tensoring with a locally free sheaf \( \mathcal{F} \) preserves this property. The homotopy pullback square gives a cofibre sequence of sheaves. Taking cohomology gives the adelic complex: indeed, since \( \mathcal{K}_C \) is the constant sheaf \( H^*(\mathcal{C}; \mathcal{K}_C) = H^0(\mathcal{C}; \mathcal{K}_C) = \mathcal{K}_C \), and since \( (\mathcal{O}_C)_x \) is an injective skyscraper sheaf. \( H^*((\mathcal{O}_C)_x) = H^0((\mathcal{O}_C)_x) = (\mathcal{O}_C)_x \). □

8.B. Rational \( SO(2) \)-spectra. As mentioned above, \( \text{Spcc}(SO(2)\text{-spectra}/\mathbb{Q}) \) is a partially ordered set with one maximal (generic) prime \( \wp_{SO(2)} \) and closed points \( \wp_C \) corresponding to the finite cyclic subgroups \( C \). Each \( \wp_C \leq \wp_{SO(2)} \) and there are no other containments.

The structure sheaf has value \( \mathbb{Q} = H^*(BT/T) \) on \( \wp_{SO(2)} \) and \( \mathbb{Q}[c] = H^*(BT/C) \) at \( \wp_C \). The adelic complex is
\[ C^*_{ad}(SO(2); \mathcal{E}^{-1}, H^*(BG/\bullet)) = \left( \mathbb{Q} \oplus \prod_C \mathbb{Q}[c] \rightarrow \mathcal{E}^{-1} \prod_C \mathbb{Q}[c] \right), \]
and
\[ H^i_{ad}(SO(2); \mathcal{E}^{-1}, H^*(BG/\bullet)) = \left\{ \begin{array}{ll} \mathbb{Q} & \text{if } i = 0 \\
\bigoplus_C H^*(BT/C) & \text{if } i = 1 \end{array} \right. \]

8.C. Rational \( O(2) \)-spectra. We describe an example where it is clear how to define an appropriate cohomology but which is not covered by the version of adelic cohomology described here.

The point is that the Balmer spectrum of rational \( O(2) \)-equivariant cohomology theories is not topologically discrete. In fact there is a homeomorphism
\[ \text{Spcc}(O(2)\text{-spectra}/\mathbb{Q}) = \mathcal{C} \coprod \mathcal{D} \]
where
\[ \mathcal{C} = \text{Spcc}(SO(2)\text{-spectra}/\mathbb{Q}) \text{ and } \mathcal{D} = \{ (D_2), (D_4), (D_6), \ldots, O(2) \} \]
where \( \mathcal{D} \) is topologized as the subset \( \{ 1/n \mid n \geq 1 \} \cup \{ 0 \} \) of \( \mathbb{R} \).

We have already defined appropriate coefficients for \( \text{Spcc}(SO(2)\text{-spectra}/\mathbb{Q}) \), but the difference is that the structure sheaf has stalks \( H^*(BO(2)/C) = H^*(BSO(2)/C)^{C_2} = \mathbb{Q}[d] \) with \( d = c^2 \) of codegree 4. Hence
\[ H^i_{ad}(\mathcal{C}; \mathcal{E}^{-1}, H^*(BO(2)/\bullet)) = \left\{ \begin{array}{ll} \mathbb{Q} & \text{if } i = 0 \\
\bigoplus_C H^*(BO(2)/C) & \text{if } i = 1 \end{array} \right. \]
Over $\mathcal{D}$ it makes sense to treat the coefficients as a sheaf over $\mathcal{D}$ and to take sheaf cohomology rather than simply taking the product of stalks. The relevant sheaf for equivariant homotopy theory is the constant sheaf $\mathbb{Q}$, so that

$$H^0(\mathcal{D}; \mathbb{Q}) = C(\mathcal{D}, \mathbb{Q}).$$

The first sheaf cohomology is an uncountable vector space, which does not appear relevant to $\pi_*^{O(2)}(S^0) \otimes \mathbb{Q}$. On the other hand, the constant sheaf is injective in the category of realizible sheaves, reflecting the fact that understanding realizable objects is an important ingredient in constructing a model.

9. ADELIC COHOMOLOGY AND THE HOMOTOPY OF THE SPHERE

The author’s original motivation for the definition of adelic cohomology came from the occurrence of adelic cochains in the study of rational torus-equivariant cohomology theories. By tom Dieck splitting the rational stable homotopy groups of the sphere are well known additively. If $G$ is a torus we have

$$\pi_*^G(S) \cong \bigoplus_K \Sigma^{\text{codim}(K)} H_*(BG/K).$$

In some sense this is an input to the algebraic model of [GS18], so we are certainly not expecting an independent calculation of $\pi_*^G(S)$. On the other hand, the expression of $S^0$ as a homotopy pullback of a diagram of ring spectra does show that the adelic cohomology gives information about the ring structure. It is also interesting to see how information about completed objects (in particular uncountable vector spaces) feeds into the final answer (which is torsion and countable).

**Proposition 9.1.** For any torus $G$, using the generalized adelic coefficients of Example 7.C.1 there is a spectral sequence

$$H^*(X_c; \mathcal{E}^{-1}, O_{\mathcal{F}/*}) \Rightarrow \pi_*^G(S^0)$$

**Proof:** One of the main results of [GS18] is the fact that the equivariant sphere $S^0$ is the homotopy pullback of a cubical diagram of ring $G$-spectra. Filtering the cube by distance from the empty face gives a spectral sequence

$$H^*(\pi_*^G(R(\sigma))) \Rightarrow \pi_*^G(S^0),$$

where $\sigma$ runs through dimension vectors $d_0 > d_1 > \cdots > d_s$ (i.e., it runs through non-empty subsets of the dimension poset $\{0 < 1 < \ldots < r\}$). The definition of the adelic cochains was motivated by the isomorphisms

$$C^\sigma(X_c; \mathcal{E}^{-1}, O_{\mathcal{F}/*}) \cong \pi_*^G(R(\sigma)),$$

and by construction the maps in the cube correspond to the differentials. Accordingly the spectral sequence takes the form in the statement. $\square$

This raises the question of the significance of the individual cohomology groups, and the behaviour of the spectral sequence.

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Proposition 9.2. The adelic cohomology corresponds to the tom Dieck splitting

\[
H^s(\mathcal{X}_c; \mathcal{E}, \mathcal{O}_F) \cong \bigoplus_{\text{codim}(K) = s} H_s(BG/K),
\]

and the spectral sequence collapses to show

\[
\pi^G_s(S) = \bigoplus_s H^s(\mathcal{X}_c; \mathcal{E}, \mathcal{O}_F) = \bigoplus_{K} \Sigma^\text{codim}K H_s(BG/K).
\]

Remark 9.3. The present proof of the collapse of this spectral sequence depends on tom Dieck splitting.

Proof: We describe a filtration

\[
0 = F^{r+1} \subset F^r \subset \cdots \subset F^0 = \text{Whole-}(r+1)-\text{Cube}
\]
of the cube so that the subquotients \( \overline{F}^n = F^n/F^{n+1} \) have only one cohomology group

\[
H^n(\overline{F}^m) = \bigoplus_{\text{codim}(K) = n} H_s(BG/K).
\]

Indeed, we take

\[
F^n = \bigoplus_{\text{codim}(d_0 \geq d_s)} C^{d_0 \geq \cdots \geq d_s},
\]

noting that this is a subcomplex since differentials either retain the last term subgroup or replace it by a proper subgroup.

We note that the quotient

\[
\overline{F}^n = F^n/F^{n+1}
\]
is an \( n \)-cube, namely the one in which every term ends with a codimension \( n \) subgroup. Thus

\[
\overline{F}^0 = (\mathcal{O}_F/G), \overline{F}^1 = \left( \prod_{\text{codim}(H) = 1} \mathcal{O}_{F/H} \rightarrow \mathcal{E}_G^{-1} \prod_{\text{codim}(H) = 1} \mathcal{O}_{F/H} \right)
\]

and

\[
\overline{F}^2 = \left( \begin{array}{c}
\prod_{\text{codim}(K) = 2} \mathcal{O}_{F/K} \\
\prod_{\text{codim}(H) = 1} \mathcal{E}_H^{-1} \prod_{K,H} \mathcal{O}_{F/K}
\end{array} \right) \rightarrow \left( \begin{array}{c}
\mathcal{E}_G^{-1} \prod_{\text{codim}(K) = 2} \mathcal{O}_{F/K} \\
\mathcal{E}_H^{-1} \prod_{K,H} \mathcal{O}_{F/K}
\end{array} \right)
\]

It remains to show that \( \overline{F}^n \) has a single cohomology group in codegree \( s \), and to identify it. The collapse of the spectral sequence uses the tom Dieck splitting, and a verification that the factors in that decomposition correspond to those at \( E_2 \). \[ \square \]

The formal ingredient is as follows.
Lemma 9.4. Suppose $M_i$ is an $R_i$-module for each $i$, and that $\mathcal{E}$ is a multiplicative sequence consisting of elements $(r_i) \in \prod_i R_i$ almost all equal to 1. If each $M_i$ is $\mathcal{E}$-torsion free then the vertical map

$$
\bigoplus_i M_i \longrightarrow \mathcal{E}^{-1} \bigoplus_i M_i
$$

is a homology isomorphism. Both horizontals are injective, and the common cokernel is isomorphic to

$$
\bigoplus_i (\mathcal{E}^{-1} M_i)/M_i.
$$

Proof: Considering the Snake Lemma, it suffices to prove that the map

$$
(\mathcal{E}^{-1} \bigoplus_i M_i)/(\bigoplus_i M_i) \longrightarrow (\mathcal{E}^{-1} \prod_i M_i)/\prod_i M_i
$$

is an isomorphism. This follows since elements of $\mathcal{E}$ are almost all 1, and the $M_i$ are $\mathcal{E}$-torsion free.

We now apply the formal ingredient in two stages.

Corollary 9.5. If $M_L$ is a $\mathcal{E}_{G/L}$-torsion free $\mathcal{O}_{F/L}$-module for all $L$, there is a short exact sequence

$$
0 \longrightarrow \prod_{L<K} M_L \rightarrow \mathcal{E}_K^{-1} \prod_{L<K} M_L \rightarrow \bigoplus_{L<K} (\mathcal{E}_K^{-1} M_L)/M_L \longrightarrow 0.
$$

Proof: We need only observe that $\mathcal{E}_K$ satisfies the hypotheses for $\mathcal{E}$ in the lemma. Then we know that the cokernel of

$$
\prod_{L<K} M_L \rightarrow \mathcal{E}_K^{-1} \prod_{L<K} M_L
$$

is the same as that of

$$
\bigoplus_{L<K} M_L \rightarrow \mathcal{E}_K^{-1} \bigoplus_{L<K} M_L = \bigoplus_{L<K} \mathcal{E}_K^{-1} M_L
$$

□

Corollary 9.6. If $M_{K_s}$ is a $\mathcal{E}_{G/K_s}$-torsion free $\mathcal{O}_{F/K_s}$-module, there is a short exact sequence

$$
0 \longrightarrow \prod_{K_0<K} \mathcal{E}_{K_0}^{-1} \cdots \mathcal{E}_{K_{s-1}}^{-1} M_{K_s} \longrightarrow \mathcal{E}_K^{-1} \prod_{K_0<K} \mathcal{E}_{K_0}^{-1} \cdots \mathcal{E}_{K_{s-1}}^{-1} \prod_{K_s<K_{s-1}} M_{K_s} \longrightarrow \bigoplus_{K>K_0>\cdots>K_s} (\mathcal{E}_{K/K_s}^{-1} M_{K_s})/M_{K_s} \longrightarrow 0.
$$

Proof: We apply the previous corollary $s$ times.

□
10. Derived commutative algebra

Our calculation of the adelic cohomology in commutative algebra relies on an argument in the derived category $\mathcal{D}(R)$. In effect, it is the argument of [GS18] transposed into algebra. However there are some differences beyond the change of context. Firstly, the diagram of rings involved is slightly different (as detailed below), and secondly the proof has been packaged much better.

10.A. Height, dimension and partial order. We assume that our commutative ring $R$ is catenary and Noetherian and of finite dimension $r$. We use the Balmer ordering, so that $q \leq p$ if and only if $q \supseteq p$. Accordingly, $\dim(p)$ is the Krull dimension of $A/p$. Balmer-minimal primes are the closed points (which correspond to the ideals which are maximal under containment), and Balmer-maximal primes correspond to irreducible components.

10.B. Localization, completion and local cohomology. We write $L_p M = M_p$ for the usual localization from commutative algebra. If $p$ means the algebraic prime (a subset of the ring $R$), then the localization inverts all elements outside $p$. We write $p_b = \{ M \mid M_p \simeq 0 \}$ for the associated Balmer prime and the localization is nullification of all elements of $p_b$.

We use the traditional notation $\Gamma_p M$ for the derived $p$-power torsion functor. Thus if $p = (x_1, \ldots, x_n)$ we may use the model

$$\Gamma_p M = (R \to R[1/x_1]) \otimes_R \cdots \otimes_R (R \to R[1/x_n]) \otimes_R M.$$ 

The functor $\Gamma_p$ is the $R/p$-cellularization. (We note that [BIK08] uses this notation for $L_p \Gamma_p M = \Gamma_p L_p M$, which we will never do.) We note that the composite is smashing in the sense that $\Gamma_p L_p M \simeq (\Gamma_p L_p R) \otimes_R M$.

We use the traditional notation $\Lambda_p M$ for the derived $p$-completion functor, so that if $p = (x_1, \ldots, x_n)$ we may use the model

$$\Lambda_p M = \text{Hom}_R(\Gamma_p R, M).$$

We also write

$$V_p M = \text{Hom}_R(L_p R, M).$$

(We note that [BIK08] uses $\Lambda_p$ for the composite $V_p \Lambda_p M = \Lambda_p V_p M$, which we will never do.) We note that $\Lambda_p V_p M \simeq \text{Hom}_R(\Gamma_p L_p R, M)$.

Definition 10.1. The support and cosupport of an $R$-module $M$ are defined by

$$\text{supp}(M) = \{ p \mid \Gamma_p L_p R \otimes_R M \nrightarrow 0 \}.$$ 

$$\text{cosupp}(M) := \{ p \mid V_p \Lambda_p M \nrightarrow 0 \} = \{ p \mid \text{Hom}_R(L_p \Gamma_p R_p, M) \nrightarrow 0 \}.$$ 

Remark 10.2. (i) When $M$ is compact the support is

$$\{ p \mid M_p \nrightarrow 0 \} = \{ p \mid M \nsubseteq p_b \},$$ 

but in general the support is a proper subset of $\{ p \mid M_p \nrightarrow 0 \}$.

(ii) Since the ring is Noetherian, for any prime $p$ we may choose a compact object $K_p$ so that $\text{supp}(K_p) = \{ p \}$. For example if $p = (x_1, \ldots, x_n)$ we may take $K_p = (R \xrightarrow{x_1} R) \otimes_R \cdots \otimes_R (R \xrightarrow{x_n} R)$. 

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The main fact we use is that an object is trivial if it has empty support or if it has empty cosupport. It is helpful to bear in mind that the support and the cosupport have the same Balmer-minimal elements [BIK08, Theorem 4.13].

10.C. Semiorthogonal decompositions by support. We will want to consider collections \( F \) of primes closed under specialization (‘families’) and collections \( G \) of primes closed under generalization (‘cofamilies’). If \( F \) is a family, we write \( \bar{F} \) for the complementary cofamily.

In particular we consider the subgroups above and below a fixed prime \( q \):
\[
\Lambda(q) = \{ p \mid p \leq q \} \quad \text{and} \quad V(q) = \{ p \mid q \leq p \}.
\]
The first is a family (the closure of \( \{ q \} \)) and the second is a cofamily.

Given a family \( F \), we may consider the set of Koszul objects for primes in \( F \). Taking the cellularization with respect to these small objects gives \( \Gamma_{F}X \) (so that \( \Gamma_{p} = \Gamma_{A(p)} \)) and the nullification gives \( L_{\bar{F}}X \) (so that \( L_{p} = L_{V(p)} \)). We then have a natural cofibre sequence
\[
\Gamma_{F}X \to X \to L_{\bar{F}}X
\]
with
\[
\text{supp}(\Gamma_{F}X) = \text{supp}(X) \cap F \quad \text{and} \quad \text{supp}(L_{\bar{F}}X) = \text{supp}(X) \setminus F.
\]

If \( F \) is the family of primes of dimension \( \leq i \) and \( \bar{F} \) is the complementary cofamily of primes of dimension \( \geq i + 1 \) we write
\[
M_{\leq i} \to M \to M_{\geq i+1}
\]
for the cellularization and nullification. We say that a map \( X \to Y \) is a \((\leq i)\)-equivalence if it induces an equivalence \( X_{\leq i} \to Y_{\leq i} \), or equivalently if it is an equivalence when tensored with any compact object \( K \) with \( \text{supp}(K) \) consisting of primes of dimension \( \leq i \).

11. The adelic homotopy pullback cube

Our analysis is based on expressing an \( r \)-dimensional ring as the homotopy pullback of an \((r + 1)\)-cube \( C \) of simpler rings. For the integers \( R = \mathbb{Z} \) we have the classical Hasse square

\[
\begin{array}{ccc}
\mathbb{Z} & \to & \mathbb{Q} \\
\downarrow & & \downarrow \\
\prod_{p} \mathbb{Z}_{p}^\wedge & \to & \mathbb{Q} \otimes \prod_{p} \mathbb{Z}_{p}^\wedge,
\end{array}
\]
and in general it is a suitably arranged version of the adelic chain complex.

11.A. The local and adelic cubes. We are going to describe the diagram \( R_{ad} \) of rings which is cubical in the sense that it is a functor from an \((r + 1)\)-cube \( C \) to commutative rings. There are two natural views of the \((r + 1)\)-cube: as a product of copies of \( I = (0 < 1) \) and as the set of subsets of \( \{0, 1, \ldots, r \} \). The latter point of view focuses on the important features, whilst the former helps us draw pictures.

We consider subsets \( d = (d_0 > d_1 > \cdots > d_s) \) of \( \{0, 1, \ldots, r\} \) and view this as a flag of dimensions. We will display this subset at a point of the cube with coordinates \((a_0, a_1, \ldots, a_r)\) where each coordinate \( a_i \) takes the value 1 if one of the dimensions is \( i \), and otherwise takes the value 0.
Definition 11.1. The *adelic* diagram is defined by

\[ R_{ad}(d) := C^d(\text{Spec}(R); L, \Lambda R). \]

If \( R \) is 1-dimensional \( R_{ad} \) is a minor variation on the Hasse square. It is worth writing the diagram completely in the 2-dimensional irreducible case. The layout is

\[
\begin{array}{c}
(010) \quad \rightarrow \quad (110) \\
(000) \quad \rightarrow \quad (100) \\
(011) \quad \rightarrow \quad (111) \\
(001) \quad \rightarrow \quad (101)
\end{array}
\]

and the diagram \( R_{ad} \) of rings is

\[
\begin{array}{c}
\prod_p (R_{ad})_p \\
R \quad \rightarrow \quad R(0) \\
\prod_p R_p \otimes \prod_{m \leq p} R_m \\
\prod_m R_m
\end{array}
\]

\[
\begin{array}{c}
R(0) \otimes \prod_p (R_{ad})_p \\
R(0) \otimes \prod_p R_p \otimes \prod_{m \leq p} R_m \\
R(0) \otimes \prod_m R_m
\end{array}
\]

11.B. **Strategy.** First we recall the Cubical Reduction Principle for homotopy pullbacks. A cubical diagram \( X : C \longrightarrow \mathbb{D} \) is a homotopy pullback if the initial point \( X(\emptyset) \) is the homotopy inverse limit over the punctured cube \( PC \). It is thus clear that a 0-cube is a homotopy pullback if \( X(\emptyset) \simeq \ast \). For a 1-cube \( X : I \longrightarrow \mathbb{D} \) write \( X_f = \text{fibre}(X(0) \longrightarrow X(1)) \) for the homotopy fibre. This diagram is a homotopy pullback if and only if the map \( X(0) \longrightarrow X(1) \) is an equivalence which happens if and only if \( X_f \simeq \ast \).

Now suppose \( C = I \times C' \), and note that \( X : C \longrightarrow \mathbb{D} \) induces a cube \( X_f^1 : C' \longrightarrow \mathbb{D} \) of homotopy fibres, where the 1 refers to the fact that the fibre has been taken with respect to the first coordinate. The Cubical Reduction Principle states that the diagram \( X \) is a homotopy pullback if and only if \( X_f^1 \) is a homotopy pullback.

**Theorem 11.2.** The diagram \( R_{ad} \) is a homotopy pullback.

**Proof:** For each \( n \)-dimensional prime we may consider the set \( \Lambda(p) \) of primes in \( p \), and form the \((n+1)\)-cube indexed by subsets of \( \{0, 1, \ldots, n\} \). We consider the \((n+1)\)-cube \( R_{ad}(p) \), with the same definition as \( R_{ad} \), but the primes are restricted to \( \Lambda(p) \) and hence the dimensions are restricted to \( \{0, 1, \ldots, n\} \). Evidently if \( q \leq p \) we have maps of diagrams

\[
R_{ad}(q) \longrightarrow R_{ad}(p) \longrightarrow R_{ad}.
\]
Note that $R_{ad}$ is a homotopy pullback if and only if $R_{ad} \otimes K_p$ is a pullback for all $p$. Since $K_p \otimes R_q \simeq 0$ unless $q \leq p$ we see

$$K_p \otimes_R R_{ad} \simeq K_p \otimes_R R_{ad}(p),$$

so that it suffices to show $K_p \otimes_R R_{ad}(p)$ is a pullback for all primes $p$.

We will prove by induction that $\dim(p) = n$ then $R_{ad}(p)$ is a homotopy pullback in dimension $\leq n$. The base of the induction is the trivial case $n = -1$.

For the inductive step we suppose that $\dim(p) = n$ and if $q \leq p$ with $\dim(q) = i \leq n - 1$ then $R_{ad}(q)$ is a homotopy pullback in dimension $\leq i$. By the Cubical Reduction Principle, $R_{ad}(p)$ is a homotopy pullback if and only if $(R_{ad}(p))^n$ is a homotopy pullback.

Since $p$ is the only $n$-dimensional prime in $R_{ad}(p)$, the cubical reduction takes the fibre of localization at $p$, and in view of the fibre sequence $\Gamma_{V(p)^c} R \to R \to L_{V(p)} R$ we have

$$R_{ad}(p)(d_0 \to \cdots \to d_s) = (\Gamma_{V(p)^c} R) \otimes_R (R_{ad}(p)(d_0 \to \cdots \to d_s)).$$

Any prime $q \leq p$ of dimension $\leq n$ in $V(p)^c$ is actually of dimension $\leq n - 1$. Next note that

$$K_q \otimes_R R_{ad}(p)(q_0 \to \cdots \to q_s) \simeq 0$$

unless $q \succeq q_0$: this uses the fact that $K_q \otimes_R R_{q_0} \simeq 0$ unless $q_0 \leq q$, and the fact that $K_q$ is compact so that it passes inside the products. Accordingly,

$$K_q \otimes_R R_{ad}(p)(q_0 \to \cdots \to q_s) \simeq K_q \otimes R_{ad}(q),$$

which is a pullback cube by the induction hypothesis, completing the inductive step.

By induction we see that $K_p \otimes_R R_{ad}(p)$ is a homotopy pullback for all primes of dimension $r$, and hence $R_{ad}$ is a homotopy pullback as required. 

\[\square\]

Remark 11.3. This inductive scheme would apply equally well to other localization systems provided $K_p \otimes A_q \simeq 0$ unless $q \leq p$, and provided the support of the fibre of $1 \to A_p$ does not contain $p$.

If $A_p = \Lambda_p L_p$ as for the Beilinson-Parshin case the first condition is clear since $K_p$ is small and $K_p \otimes_R L_q R \simeq 0$ unless $q \leq p$. For the second condition we factor it as $1 \to L_p \to \Lambda_p L_p$, and it suffices to show that the fibres of both factors are supported in dimension $\leq n - 1$. This is true as before for the first map. For the second the fibre is of the form $\text{Hom}(L_{\Lambda(p)^c} R, L_p M)$, and since $K_p$ is small and $p \not\in \Lambda(p)^c \cap V(p)$.

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