An Application of a Poisson Distribution Series on Certain Analytic Functions

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Received 25 November 2013; Accepted 7 January 2014; Published 18 February 2014

Academic Editor: Janne Heittokangas

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The purpose of the present paper is to introduce a Poisson distribution series and obtain necessary and sufficient conditions for this series belonging to the classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$. We also consider an integral operator related to this series.

1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by $S$ the subclass of $A$ consisting of functions of the form (1) which are also univalent in $U$ and let $T$ be the subclass of $S$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n.$$

Let $T(\lambda, \alpha)$ be the subclass of $T$ consisting of functions which satisfy the condition

$$\text{Re} \left\{ \frac{zf'(z)}{\lambda zf'(z) + (1 - \lambda) f(z)} \right\} > \alpha,$$

for some $\alpha (0 \leq \alpha < 1), \lambda (0 \leq \lambda < 1)$ and for all $z \in U$.

Also, we let $C(\lambda, \alpha)$ denote the subclass of $T$ consisting of functions which satisfy the condition

$$\text{Re} \left\{ \frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} \right\} > \alpha,$$

for some $\alpha (0 \leq \alpha < 1), \lambda (0 \leq \lambda < 1)$ and for all $z \in U$.

From (3) and (4) it is easy to verify that

$$f(z) \in C(\lambda, \alpha) \iff zf'(z) \in T(\lambda, \alpha).$$

The classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$ were extensively studied by Altintas and Owa [1] and certain conditions for hypergeometric functions and generalized Bessel functions for these classes were studied by Mostafa [2] and Porwal and Dixit [3].

It is worthy to note that $T(0, \alpha) \equiv T^*(\alpha)$, the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ and $C(0, \alpha) \equiv C(\alpha)$, the class of convex functions of order $\alpha (0 \leq \alpha < 1)$ (see [4]).

A variable $x$ is said to have Poisson distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $e^{-m}, m e^{-m}/1!, m^2 e^{-m}/2!, m^3 e^{-m}/3!, \ldots$, respectively, where $m$ is called the parameter.

Thus

$$P(x = k) = \frac{m^k e^{-m}}{k!}, \quad k = 0, 1, 2, \ldots.$$  

Now, we introduce a power series whose coefficients are probabilities of the Poisson distribution:

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$  

We note that, by ratio test, the radius of convergence of the above series is infinity.

Now, we introduce the series

$$F(m, z) = 2z - K(m, z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$
Motivated by results on connections between various sub-classes of analytic univalent functions by using hypergeometric functions (see [5–10]) and generalized Bessel functions (see [3, 11–13]), we obtain necessary and sufficient conditions for function \( F(m, z) \) belonging to the classes \( T(\lambda, \alpha) \) and \( C(\lambda, \alpha) \). Finally, we give conditions for an integral operator \( G(m, z) \) belonging to the classes \( T(\lambda, \alpha) \) and \( C(\lambda, \alpha) \).

2. Main Results

To establish our main results, we will require the following Lemmas according to Altintas and Owa [1].

**Lemma 1** (see [1]). A function \( f(z) \) defined by (2) is in the class \( T(\lambda, \alpha) \) if and only if
\[
\sum_{n=2}^{\infty} [n - \lambda \alpha n - \alpha + \lambda \alpha] |a_n| \leq 1 - \alpha. \tag{9}
\]

**Lemma 2** (see [1]). A function \( f(z) \) defined by (2) is in the class \( C(\lambda, \alpha) \) if and only if
\[
\sum_{n=2}^{\infty} [n - \lambda \alpha n - \alpha + \lambda \alpha] |a_n| \leq 1 - \alpha. \tag{10}
\]

**Theorem 3.** If \( m > 0 \), then \( F(m, z) \) is in \( T(\lambda, \alpha) \), if and only if
\[
me^m (1 - \alpha \lambda) \leq 1 - \alpha. \tag{11}
\]

**Proof.** Since
\[
F(m, z) = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n, \tag{12}
\]
according to Lemma 1, we must show that
\[
\sum_{n=2}^{\infty} [n - \lambda \alpha n - \alpha (1 - \lambda)] \frac{e^{-m} m^{n-1}}{(n-1)!} \leq 1 - \alpha. \tag{13}
\]

Now
\[
\sum_{n=2}^{\infty} [n - \lambda \alpha n - \alpha (1 - \lambda)] \frac{e^{-m} m^{n-1}}{(n-1)!} = e^{-m} \left[ \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} + (1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] = e^{-m} \left[ (1 - \alpha) \sum_{n=3}^{\infty} \frac{m^{n-1}}{(n-3)!} + (1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] = e^{-m} \left[ (1 - \alpha) m + (1 - \alpha) \frac{m^{n-1}}{(n-1)!} \right] + (1 - \alpha) \left( e^{-m} - 1 \right). \tag{14}
\]

But this last expression is bounded above by \( 1 - \alpha \) if and only if (11) holds. This completes the proof of Theorem 3.

3. An Integral Operator

In the following theorem, we obtain similar results in connection with a particular integral operator \( G(m, z) \) as follows:
\[
G(m, z) = \int_{0}^{z} \frac{F(m, t)}{t} \, dt. \tag{19}
\]

**Theorem 5.** If \( m > 0 \), then \( G(m, z) \) defined by (19) is in \( C(\lambda, \alpha) \) if and only if
\[
me^m (1 - \alpha \lambda) \leq 1 - \alpha. \tag{20}
\]

**Proof.** Since
\[
G(m, z) = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{(n-1)!} z^n \tag{21}
\]

Thus the proof of Theorem 3 is established.
by Lemma 2, we need only to show that
\[
\sum_{n=2}^{\infty} n \left[ (1 - \lambda \alpha) - \alpha (1 - \lambda) \right] e^{-m} m^{n-1} n! \leq 1 - \alpha. \quad (22)
\]

Now
\[
\sum_{n=2}^{\infty} n \left[ (1 - \lambda \alpha) - \alpha (1 - \lambda) \right] e^{-m} m^{n-1} n! = \sum_{n=2}^{\infty} \left[ (n-1) (1 - \lambda \alpha) + (1 - \alpha) \right] m^{n-1} (n-1)!
\]
\[
= e^{-m} \left[ (1 - \alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} + (1 - \alpha) \sum_{n=2}^{\infty} \frac{n^{n-1}}{(n-1)!} \right]
\]
\[
= e^{-m} \left[ (1 - \alpha) m e^{m} + (1 - \alpha) (e^{-m} - 1) \right]
\]
\[
= (1 - \alpha) m + (1 - \alpha) \left( 1 - e^{-m} \right),
\]
(23)

which is bounded above by $1 - \alpha$, if and only if (20) holds. \(\square\)

**Theorem 6.** If $m > 0$, then $G(m, z)$ defined by (19) is in $T(\lambda, \alpha)$ if and only if
\[
\left( 1 - \lambda \alpha \right) - \frac{\alpha (1 - \lambda)}{m} (1 - e^{-m}) + \alpha (1 - \lambda) e^{-m} \leq 1 - \alpha.
\]
(24)

**Proof.** The proof of this theorem is similar to that of Theorem 5. Therefore we omit the details involved. \(\square\)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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