Abstract

We consider the problem of optimal singular control of a stochastic partial differential equation (SPDE) with space-mean dependence. Such systems are proposed as models for population growth in a random environment. We obtain sufficient and necessary maximum principles for such control problems. The corresponding adjoint equation is a reflected backward stochastic partial differential equation (BSPDE) with space-mean dependence. We prove existence and uniqueness results for such equations. As an application we study optimal harvesting from a population modelled as an SPDE with space-mean dependence.

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1 Introduction

We start by a motivation for the problem that will be studied in this paper:
Consider a problem of optimal harvesting from a fish population in a lake $D$. We assume that the density $u(t,x)$ of the population at time $t \in [0,T]$ and at the point $x \in D$ is modelled by a stochastic reaction-diffusion equation with neighbouring interactions. By this we mean...
a stochastic partial differential equation of the form

\[
\begin{cases}
du(t, x) &= \left[\frac{1}{2} \Delta u(t, x) + \alpha \overline{u}(t, x)\right] dt + \beta \overline{u}(t, x) dB(t) - \lambda_0 \xi(dt, x); \\
&\quad (t, x) \in (0, T) \times D, \\
u(0, x) &= u_0(x) > 0; \quad x \in D, \\
u(t, x) &= u_1(t, x) \geq 0; \quad (t, x) \in (0, T) \times \partial D,
\end{cases}
\]

where \( D \) is a bounded Lipschitz domain in \( \mathbb{R}^n \) and \( \overline{u}(t, x) \) is the space-averaging operator

\[
\overline{u}(t, x) = \frac{1}{V(K_0)} \int_{K_0} u(x + y) dy.
\]  

(1.1)

Here \( V(\cdot) \) denotes Lebesgue volume and

\[
K_0 = \{y \in \mathbb{R}^n; |y| < \theta\}
\]

is the ball of radius \( r > 0 \) in \( \mathbb{R}^d \) centred at 0, and \( u_0(x) \), \( u_1(t, x) \) are given deterministic functions; \( (t, x) \in [0, T] \times D \).

In the above \( B(t) = B(t, \omega); (t, \omega) \in [0, \infty) \times \Omega, \) is an \( m \)-dimensional Brownian motion on a filtered probability space \( (\Omega, \mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, \infty)}, \mathbb{P}) \). Moreover, the coefficients \( \alpha, \beta \) and \( \lambda_0 > 0 \) are given constants and

\[
\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}
\]

is the Laplacian differential operator on \( \mathbb{R}^n \).

We may regard \( \xi(dt, x) \) as the harvesting effort rate, and \( \lambda_0 > 0 \) as the harvesting efficiency coefficient. The coefficients \( \alpha, \beta \) represent the relative growth rate and the relative noise rate with respect to \( \overline{u} \).

The performance functional is assumed to be of the form

\[
J(\xi) = \mathbb{E} \left[ \int_{D} \int_{0}^{T} (h_0(t, x)u(t, x) - c(t, x))\xi(dt, x)dx + \int_{D} h_0(T, x)u(T, x)dx \right],
\]  

(1.2)

\{eq248\}

where \( h_0(t, x) > 0 \) is the unit price of the fish and \( c(t, x) \) is the unit cost of energy used in the harvesting and \( T > 0 \) is a fixed terminal time. Thus \( J(\xi) \) represents the expected total net income from the harvesting. The problem is to maximise \( J(\xi) \) over all (admissible) harvesting strategies \( \xi(t, x) \).

Remark 1.1 This population growth model, which was first introduced in Agram et al [1], is a generalisation of the classical stochastic reaction-diffusion model, in that we have added the term \( \overline{u}(t, x) \) which represents an average of the neighbouring densities. Thus our model allows for the growth at a point to depend on interactions from the whole vicinity. This space-mean interaction is different from the pointwise interaction represented by the Laplacian.
The problem above turns out to be related to a problem of the following form:
Let \( \phi(x) = \phi(x, \omega) \) be an \( \mathcal{F}_T \)-measurable \( H = L^2(D) \)-valued random variable. Let 
\[
g : [0, T] \times D \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}
\]
be a given measurable mapping and \( L(t, x) : [0, T] \times D \rightarrow \mathbb{R} \) a given continuous function. Consider the problem to find an \( \mathbb{F} \)-adapted random fields \( Y(t, x) \in \mathbb{R}, Z(t, x) \in \mathbb{R}^m, \xi(t, x) \in \mathbb{R}^+ \) is left-continuous and increasing with respect to \( t \), such that
\[
\begin{aligned}
\frac{d}{dt}Y(t, x) &= -A Y(t, x) dt - F(t, x, Y(t, x), Y(t, \cdot), Z(t, x)) dt + Z(t, x) dB(t) \\
Y(t, x) &\geq L(t, x), \\
\int_0^T \int_D (Y(t, x) - L(t, x)) \xi(dt, x) dx = 0, \\
Y(T, x) &= \phi(x) \quad \text{a.s.},
\end{aligned} \tag{1.3} \{a\}
\]
where \( A \) is a second order linear partial differential operator. We call the equation (1.3) a reflected stochastic partial differential equation (SPDE) with space-mean dynamics. We will come back to this equation in the last section.

For a general introduction to stochastic partial differential equations (SPDEs) we refere to e.g. [5], [6], [7], [10], [16], [17], [18] and [19]. A maximum principle for the optimal control of SPDEs was introduced in [9]. See also [12], [14] and the presentation in [13], Chapter 13. Singular control of SPDEs has been studied in [15]. Our current paper differs from that paper and other papers on SPDEs in that we include a space-mean dependence in the dynamics. This is a new type of optimal stochastic control problem which, to the best of our knowledge, has not been studied before. As we will see in the next sections, it leads to the addition of a space-mean term in the Hamiltonian and in the corresponding new variational inequalities for the optimal singular control.

\section{The optimization problem}

We now give a general formulation of the problem discussed in the Introduction:

Let \( T > 0 \) and let \( D \subset \mathbb{R}^n \) be an open set with \( C^1 \) boundary \( \partial D \). Specifically, we assume that the state \( u(t, x) \) at time \( t \in [0, T] \) and at the point \( x \in \overline{D} := D \cup \partial D \) satisfies
\[
\begin{aligned}
\frac{du(t, x)}{dt} &= Au(t, x) dt + b(t, x, u(t, x), u(t, \cdot)) dt + \sigma(t, x, u(t, x), u(t, \cdot)) dB(t) \\
&\quad + f(t, x, u) \xi(dt, x); \quad (t, x) \in (0, T) \times D, \\
u(0^-, x) &= u_0(x); \quad x \in \overline{D}, \\
u(t, x) &= u_1(t, x); \quad (t, x) \in (0, T) \times \partial D.
\end{aligned} \tag{2.1} \{eq2.1\}
\]
Here \( B = \{B(t)\}_{t \in [0, T]} \) is a \( d \)-dimensional Brownian motion, defined in a complete filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \). The filtration \( \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0} \) is assumed to be the \( \mathbb{P} \)-augmented
filtration generated by $B$. We denote by $A$ the second order partial differential operator acting on $x$ given by

$$A\phi(x) = \sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \beta_i(x) \frac{\partial \phi}{\partial x_i}; \quad \phi \in C^2(\mathbb{R}^n),$$

where $(\alpha_{ij}(x))_{1 \leq i,j \leq n}$ is a given nonnegative definite $n \times n$ matrix with entries $\alpha_{ij}(x) \in C^2(D) \cap C(\overline{D})$ for all $i, j = 1, 2, \ldots, n$ and $\beta_i(x) \in C^2(D) \cap C(\overline{D})$ for all $i = 1, 2, \ldots, n$.

Let $L(\mathbb{R}^n)$ denote the set of real measurable functions on $\mathbb{R}^n$. For each $t, x, u, \zeta$ the functions

$$\varphi \mapsto b(t, x, u, \varphi) : [0, T] \times D \times \mathbb{R} \times L(\mathbb{R}^n) \to \mathbb{R},$$

$$\varphi \mapsto \sigma(t, x, u, \varphi) : [0, T] \times D \times \mathbb{R} \times L(\mathbb{R}^n) \to \mathbb{R},$$

$$u \mapsto f(t, x, u) : [0, T] \times D \times \mathbb{R} \to \mathbb{R},$$

are $C^1$ functionals on $L^2(D) = L^2(D, m)$, where $dm(x) = dx$ is the Lebesgue measure on $\mathbb{R}^n$. Here $Au(t, x)$ is interpreted in the sense of distribution. Thus $u$ is understood as a weak (mild) solution to (2.1), in the sense that

$$u(0, x) = u_0(x) + \int_0^t P_s^A b(s, \cdot, u(s, x), u(s, \cdot))(x)ds + \int_0^t P_s^A \sigma(s, \cdot, u(s, x), u(s, \cdot))(x)dB(s)$$

$$+ \int_0^t P_s^A f(s, \cdot, u(s, x))(x)dx,$$

where $P_s^A$ is the semigroup associated to the operator $A$. Thus we see that we can in the usual way apply the Itô formula to such SPDEs.

Moreover, the adjoint operator $A^*$ of an operator $A$ on $C_0^2(\mathbb{R}^n)$ is defined by the identity

$$\langle A\phi, \psi \rangle = \langle \phi, A^* \psi \rangle, \quad \text{for all } \phi, \psi \in C_0^2(\mathbb{R}^n),$$

where $\langle \phi_1, \phi_2 \rangle_{L^2(\mathbb{R})} := \int_{\mathbb{R}} \phi_1(x)\phi_2(x)dx$ is the inner product in $L^2(\mathbb{R})$. In our case we have

$$A^*_x \phi(x) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} (\alpha_{ij}(x)\phi(x)) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\beta_i(x)\phi(x)); \quad \phi \in C^2(\mathbb{R}^n).$$

We interpret $u$ as a weak (variational) solution to (2.1), in the sense that for $\phi \in C_0^2(D)$,

$$\langle u(t), \phi \rangle_{L^2(D)} = \langle \eta(\cdot), \phi \rangle_{L^2(D)} + \int_0^t \langle u(s), A^*_s \phi \rangle ds + \int_0^t \langle b(s, u(s)), \phi \rangle_{L^2(D)} ds$$

$$+ \int_0^t \langle \sigma(s, u(s)), \phi \rangle_{L^2(D)} dB(s) + \int_0^t \langle f(s, u(s)), \phi \rangle_{L^2(D)} dx,$$

where $\langle \cdot, \cdot \rangle$ represents the duality product between $W^{1,2}(D)$ and $W^{1,2}(D)^*$, and $W^{1,2}(D)$ the Sobolev space of order 1. In the above equation, for simplicity we have not written all the
arguments of $b, \sigma, \gamma$.

We want to maximize the performance functional $J(\xi)$, given by

$$J(\xi) = \mathbb{E}\left[\int_0^T \int_D h_0(t, x, u(t, x), u(t, \cdot)) dx dt + \int_0^T \int_D h_1(t, x, u(t, x), u(t, \cdot)) \xi(dt, dx) + \int_D g(x, u(T, x), u(T, \cdot)) \xi(dx, \cdot)\right],$$  \hspace{1cm} (2.4) \{ju\}

over all $\xi \in \mathcal{A}$, where $\mathcal{A}$ is the set of all adapted processes $\xi(t, x)$ that are nondecreasing and left continuous with respect to $t$ for all $x$, with $\xi(0, x) = 0$, $\xi(T, x) < \infty$ and such that $J(\xi) < \infty$. We call $\mathcal{A}$ the set of admissible singular controls. Thus we want to find $\hat{\xi} \in \mathcal{A}$, such that

$$J(\hat{\xi}) = \sup_{\xi \in \mathcal{A}} J(\xi).$$ \hspace{1cm} (2.5) \{eq2.4\}

For each $t, x, u$ we assume that the functions $\varphi \mapsto h_0(t, x, u, \varphi) : [0, T] \times D \times \mathbb{R} \times L(\mathbb{R}^n) \to \mathbb{R}$, and $\varphi \mapsto g(x, u, \varphi) : D \times \mathbb{R} \times L(\mathbb{R}^n) \to \mathbb{R}$, are $C^1$ functionals on $L^2(D)$.

The Hamiltonian $H$ is defined by

$$H(t, x, u, \varphi, p, q)(dt, \xi(dt, x)) = H_0(t, x, u, \varphi, p, q)dt + H_1(t, x, u, \varphi, p)\xi(dt, x);$$ \hspace{1cm} (2.6) \{Ham\}

where

$$H_0(t, x, u, \varphi, p, q) = h_0(t, x, u, \varphi) + b(t, x, u, \varphi)p + \sigma(t, x, u, \varphi)q$$ \hspace{1cm} (2.7) \{H_0\}

and

$$H_1(t, x, u, \varphi, p) = f(t, x)p + h_1(t, x, u, \varphi).$$ \hspace{1cm} (2.8) \{H_1\}

We assume that $H, f, b, \sigma, \gamma$ and $g$ admit Fréchet derivatives with respect to $u$ and $\varphi$.

In general, if $h : L^2(D) \mapsto L^2(D)$ is Fréchet differentiable, we denote its Fréchet derivative (gradient) at $\varphi \in L^2(D)$ by $\nabla_\varphi h$, and we denote the action of $\nabla_\varphi h$ on a function $\psi \in L^2(D)$ by $\langle \nabla_\varphi h, \psi \rangle$.

**Definition 2.1** We say that the Fréchet derivative $\nabla_\varphi h$ of a map $h : L^2(D) \mapsto L^2(D)$ has a dual function $\nabla_\varphi^* h \in L^2(D \times D)$ if

$$\langle \nabla_\varphi h, \psi \rangle(x) = \int_D \nabla_\varphi^* h(x, y)\psi(y)dy; \quad \text{for all } \psi \in L^2(D).$$ \hspace{1cm} (2.9) \{eq2.6a\}

**Remark 2.2** • Note in particular that if $h : L^2(D) \mapsto L^2(D)$ is linear, then $\nabla_\varphi h = h$ for all $\varphi \in L^2(D)$ and hence $\nabla_\varphi^* h(x, y) = h(x, y)$.
Also note that from (2.9) it follows by the Fubini theorem that
\[
\int_D \langle \nabla \phi h, \psi \rangle(x)dx = \int_D \int_D \nabla^* \phi h(x,y)\psi(y)dydx = \int_D \int_D \nabla^* \phi h(y,x)\psi(x)dydx
\]
where
\[
\nabla^* \phi h(x) := \int_D \nabla^* \phi h(y,x)dy.
\]

We associate to the Hamiltonian the following reflected BSPDE
\[
\begin{align*}
\{ & dp(t,x) = -A^*_x p(t,x)dt + \left\{ \frac{\partial H_0}{\partial u}(t,x) + \nabla^*_\phi H_0(t,x) \right\} \xi(dt,x) \\
& + q(t,x)dB(t); \quad (t,x) \in (0,T) \times D, \\
& p(T,x) = \frac{\partial g}{\partial u}(T,x) + \nabla^*_\phi g(T,x); \quad x \in D, \\
& p(t,x) = 0; \quad (t,x) \in (0,T) \times \partial D,
\end{align*}
\]
where we have used the simplified notation
\[
H_i(t,x) = H_i(t,x,\phi,\psi,p,q)|_{u=\hat{u}(t,x),\phi=\hat{u}(t,\cdot),p=p(t,x),q=q(t,x)}, \quad i = 0,1
\]
and similarly with
\[
g(T,x) = g(x,u(T,x),u(T,\cdot)).
\]

2.1 A sufficient maximum principle

We now formulate a sufficient version (a verification theorem) of the maximum principle for the optimal control of the problem (2.1)-(2.5).

**Theorem 2.3 (Sufficient Maximum Principle)** Suppose \( \hat{\xi} \in \mathcal{A} \), with corresponding solutions \( \hat{u}(t,x), \hat{p}(t,x), \hat{q}(t,x) \). Suppose the functions \( (u,\phi) \mapsto g(x,u,\phi) \) and \( (u,\phi,\xi) \mapsto H(t,x,u,\phi,\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,dx)) \) are concave for each \( (t,x) \in (0,T) \times D \). Moreover, suppose that
\[
\hat{\xi}(dt,x) \in \arg \max_{\xi \in \mathcal{A}} H(t,x,\hat{u}(t,x),\hat{u}(t,\cdot),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,dx));
\]
i.e.,
\[
\{ H(t,x,\hat{u}(t,x),\hat{u}(t,\cdot),\hat{p}(t,x),\hat{q}(t,x)) \} \xi(dt,dx) \leq \{ H(t,x,\hat{u}(t,x),\hat{u}(t,\cdot),\hat{p}(t,x),\hat{q}(t,x)) \} \hat{\xi}(dt,dx); \quad \text{for all } \xi \in \mathcal{A}.
\]

Then \( \hat{\xi} \) is an optimal singular control.
Proof. Consider

\[ J(\xi) - J(\tilde{\xi}) = I_1 + I_2 + I_3, \]

where

\[ I_1 = \mathbb{E} \left[ \int_0^T \int_D \left\{ h_0(t, x, u(t, x), u(t, \cdot)) - h_0(t, x, \tilde{u}(t, x), \tilde{u}(t, \cdot)) \right\} dxdt \right], \]

\[ I_2 = \mathbb{E} \left[ \int_0^T \int_D \left\{ h_1(t, x, u(t, x), u(t, \cdot))\xi(dt, x) - \int_0^T \int_D h_1(t, x, \tilde{u}(t, x), \tilde{u}(t, \cdot))\tilde{\xi}(dt, x) \right\} dxdt \right], \]

and

\[ I_3 = \int_D \mathbb{E}\left[ g(x, u(T, x), u(T, \cdot)) - g(x, \tilde{u}(T, x), \tilde{u}(T, \cdot)) \right] dx. \quad \{13\} \]

By concavity on \( g \) together with the identities (2.9)-(2.10), we get

\[ I_3 \leq \int_D \mathbb{E}\left[ \frac{\partial g}{\partial u}(T, x)(u(T, x) - \tilde{u}(T, x)) + \langle \nabla \varphi \hat{g}(T, x), u(T, \cdot) - \tilde{u}(T, \cdot) \rangle \right] dx \]

\[ = \int_D \mathbb{E}\left[ \frac{\partial g}{\partial u}(T, x)(u(T, x) - \tilde{u}(T, x)) + \nabla \varphi \hat{g}(T, x)(u(T, x) - \tilde{u}(T, x)) \right] dx \]

\[ = \int_D \mathbb{E}[\hat{p}(T, x)(u(T, x) - \tilde{u}(T, x))] dx \]

\[ = \int_D \mathbb{E}[\hat{p}(T, x)\tilde{u}(T, x)] dx, \]

where \( \tilde{u}(t, x) = u(t, x) - \hat{u}(t, x); t \in [0, T]. \)

Applying the Itô formula to \( \hat{p}(t, x)\tilde{u}(t, x) \), we have

\[ I_3 \leq \mathbb{E}\left[ \int_0^T \int_D \left\{ A_x \tilde{u}(t, x) + \tilde{b}(t, x) \right\} - \tilde{u}(t, x)A_x^*\hat{p}(t, x) \right. \]

\[ - \tilde{u}(t, x)\left\{ \frac{\partial \hat{H}_0}{\partial u}(t, x) + \nabla^* \hat{H}_0(t, x) \right\} + \tilde{q}(t, x)\tilde{\sigma}(t, x) \right)dxdt \]

\[ + \int_0^T \int_D \left( \hat{p}(t, x)f(t, x)\left( \xi(dt, x) - \tilde{\xi}(dt, x) \right) - \tilde{u}(t, x)\left\{ \frac{\partial \hat{H}_1}{\partial u}(t, x) + \nabla^* \hat{H}_1(t, x) \right\} \right)\xi(dt, x) \right]. \quad \{1\} \]

By the first Green formula (see e.g. Wloka [20] page 258), there exist first order boundary differential operators \( A_1, A_2 \), such that

\[ \int_D \{ \hat{p}(t, x)A_x \tilde{u}(t, x) - \tilde{u}(t, x)A_x^*\hat{p}(t, x) \} dx \]

\[ = \int_{\partial D} \{ \hat{p}(t, x)A_1 \tilde{u}(t, x) - \tilde{u}(t, x)A_2\hat{p}(t, x) \} dS, \]

where the last integral is the surface integral over \( \partial D \). We have that

\[ \tilde{u}(t, x) = \hat{p}(t, x) \equiv 0, \quad \{**\} \]
for all \((t, x) \in (0, T) \times \partial D\).

Substituting (2.16) in (2.15), yields

\[
I_3 \leq \mathbb{E} \left[ \int_0^T \int_D \left( \hat{p}(t, x) \tilde{b}(t, x) - \tilde{u}(t, x) \left\{ \frac{\partial \hat{H}_0}{\partial u}(t, x) + \nabla^* \phi \hat{H}_0(t, x) \right\} + \tilde{q}(t, x) \tilde{\sigma}(t, x) \right) dx \right] \\
+ \int_0^T \int_D \left( \hat{p}(t, x) f(t, x) \left( \xi(dt, x) - \hat{\xi}(dt, x) \right) - \tilde{u}(t, x) \left\{ \frac{\partial \hat{H}_1}{\partial u}(t, x) + \nabla^* \phi \hat{H}_1(t, x) \right\} \hat{\xi}(dt, x) \right) dx.
\]

(2.17) \{2\}

Using the definition of the Hamiltonian \(H\), we get

\[
I_1 = \mathbb{E} \left[ \int_0^T \int_D \left( H_0(t, x) - \hat{H}_0(t, x) \right) dx dt - \int_0^T \int_D \left\{ \hat{p}(t, x) \tilde{b}(t, x) + \tilde{q}(t, x) \tilde{\sigma}(t, x) \right\} dx dt \right].
\]

Summing the above we end up with

\[
J(\xi) - J(\hat{\xi}) = I_1 + I_2 + I_3 \\
\leq \mathbb{E} \left[ \int_0^T \int_D \left( H_0(t, x) - \hat{H}_0(t, x) - \tilde{u}(t, x) \left\{ \frac{\partial \hat{H}_0}{\partial u}(t, x) + \nabla^* \phi \hat{H}_0(t, x) \right\} \right) dx dt \\
+ \left( H_1(t, x) \xi(dt, x) - \hat{H}_1(t, x) \hat{\xi}(t, x) - \tilde{u}(t, x) \left\{ \frac{\partial \hat{H}_1}{\partial u}(t, x) + \nabla^* \phi \hat{H}_1(t, x) \right\} \hat{\xi}(dt, x) \right) dx \right] \\
\leq \mathbb{E} \left[ \int_0^T \int_D \left\langle \nabla \xi \hat{H}(t, x), \xi(dt, x) - \hat{\xi}(dt, x) \right\rangle dx \right].
\]

By the maximum condition of \(H\) (2.13), we have

\[
J(\xi) - J(\hat{\xi}) \leq 0.
\]

\[\Box\]

### 2.2 A necessary maximum principle

The concavity conditions in the sufficient maximum principle imposed on the involved coefficients are not always satisfied. Hence, we will derive now a necessary optimality conditions which do not require such an assumptions. We shall first need the following Lemmas:

For \(\xi \in \mathcal{A}\), we let \(\mathcal{V}(\xi)\) denote the set of adapted processes \(\zeta(dt, x)\) of finite variation with respect to \(t\) for each \(x\), such that there exists \(\delta = \delta(\xi) > 0\), such that \(\xi + y\zeta \in \mathcal{A}\) for all \(y \in [0, \delta]\).

**Lemma 2.4** Let \(\xi(dt, x) \in \mathcal{A}\) and choose \(\zeta(dt, x) \in \mathcal{V}(\xi)\). Define the derivative process

\[
\mathcal{Z}(t, x) := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( u^{\xi+\epsilon \zeta}(t, x) - u^\xi(t, x) \right).
\]

(2.18) \{z\}
Then $\mathcal{Z}$ satisfies the following singular linear SPDE

$$
\begin{aligned}
&d\mathcal{Z}(t,x) = A_{\mathcal{Z}}\mathcal{Z}(t,x)dt + \left( \frac{\partial b}{\partial u}(t,x)\mathcal{Z}(t,x) + \langle \nabla \varphi b(t,x), \mathcal{Z}(t,\cdot) \rangle \right) dt \\
&+ \left( \frac{\partial \sigma}{\partial u}(t,x)\mathcal{Z}(t,x) + \langle \nabla \varphi \sigma(t,x), \mathcal{Z}(t,\cdot) \rangle \right) dB(t) \\
&+ f(t,x)\zeta(dt,x) ; \quad (t,x) \in [0,T] \times D, \\
\mathcal{Z}(t,x) = 0; & \quad (t,x) \in (0,T) \times \partial D, \\
\mathcal{Z}(0,x) = 0 ; & \quad x \in D.
\end{aligned}
$$

(2.19) \hspace{0.5cm} \{\text{der-proc}\}

Lemma 2.5 \hspace{0.5cm} \{\text{der-j}\}

Let $\xi(dt,x) \in \mathcal{A}$ and $\zeta(dt,x) \in \mathcal{V}(\xi)$. Put $\eta = \xi + \epsilon \zeta; \epsilon \in [0, \delta(\xi)]$. Then

$$
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (J(\xi + \epsilon \zeta) - J(\xi))
= \mathbb{E} \left[ \int_0^T \int_D \left\{ f(t,x)p(t,x) + h_1(t,x,u(t,x),u(t,\cdot)) \right\} \zeta(dt,x) \right].
$$

Proof. By (2.4) together with (2.9),(2.10), we have

$$
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (J(\xi + \epsilon \zeta) - J(\xi))
= \mathbb{E} \left[ \int_0^T \int_D \left\{ \frac{\partial h_0}{\partial u}(t,x)\mathcal{Z}(t,x) + \langle \nabla \varphi h_0(t,x), \mathcal{Z}(t,\cdot) \rangle \right\} dxdt \\
+ \int_D \left\{ \frac{\partial g}{\partial u}(T,x)\mathcal{Z}(T,x) + \langle \nabla \varphi g(T,x), \mathcal{Z}(T,\cdot) \rangle \right\} dx \\
+ \int_0^T \int_D \left\{ \frac{\partial h_1}{\partial u}(t,x)\mathcal{Z}(t,x) + \langle \nabla \varphi h_1(t,x), \mathcal{Z}(t,\cdot) \rangle \right\} d\zeta(t,x) \\
+ \int_0^T \int_D h_1(t,x)d\zeta(t,x) \right]
= \mathbb{E} \left[ \int_0^T \int_D \left\{ \frac{\partial h_0}{\partial u}(t,x)\mathcal{Z}(t,x) + \nabla^* \varphi h_0(t,x)\mathcal{Z}(t,x) \right\} dxdt \\
+ \int_D \left\{ \frac{\partial g}{\partial u}(T,x)\mathcal{Z}(T,x) + \nabla^* \varphi g(T,x)\mathcal{Z}(T,x) \right\} dxdt \\
+ \int_0^T \int_D \left\{ \frac{\partial h_1}{\partial u}(t,x)\mathcal{Z}(t,x) + \nabla^* \varphi h_1(t,x)\mathcal{Z}(T,x) \right\} \xi(dt,x)dx \\
+ \int_0^T \int_D h_1(t,x)\zeta(dt,x)dx \right].
$$

(2.20) \hspace{0.5cm} \{j\}
Using the definition (2.6) of the Hamiltonian, yields

\[
\mathbb{E} \left[ \int_0^T \int_D \left\{ \frac{\partial h_0}{\partial u}(t, x) \mathcal{Z}(t, x) + \nabla^*_\varphi h_0(t, x) \mathcal{Z}(t, x) \right\} \, dxdt \right]
\]

\[= \mathbb{E} \left[ \int_0^T \int_D \left\{ \frac{\partial H_0}{\partial u}(t, x) \mathcal{Z}(t, x) + \nabla^*_\varphi H_0(t, x) \mathcal{Z}(t, x) \right\} \, dxdt \right.
\]

\[- \int_0^T \int_D \left\{ p(t, x) \left( \frac{\partial b}{\partial u}(t, x) \mathcal{Z}(t, x) + \nabla^*_\varphi b(t, x) \mathcal{Z}(t, x) \right) \right.
\]

\[+ q(t, x) \left( \frac{\partial \sigma}{\partial u}(t, x) \mathcal{Z}(t, x) + \nabla^*_\varphi \sigma(t, x) \mathcal{Z}(t, x) \right) \left\} \, dxdt, \tag{2.21} \right. \}

where we have used the simplified notation

\[\frac{\partial H}{\partial u}(t, x) = \frac{\partial H}{\partial u}(t, x, u(t, x), u(t, \cdot), p(t, x), q(t, x))\]

etc.

Applying the Itô formula to \(p(T, x)\mathcal{Z}(T, x)\), we get

\[
\mathbb{E} \left[ \int_D \left\{ \frac{\partial g}{\partial u}(T, x) \mathcal{Z}(T, x) + \langle \nabla^*_\varphi g(T, x), \mathcal{Z}(T, \cdot) \rangle \right\} \, dx \right]
\]

\[= \mathbb{E} \left[ \int_D p(T, x) \mathcal{Z}(T, x) \, dx \right]
\]

\[= \mathbb{E} \left[ \int_0^T \int_D \left\{ p(t, x) \left( A_x \mathcal{Z}(t, x) + \frac{\partial b}{\partial u}(t, x) \mathcal{Z}(t, x) + \langle \nabla^*_\varphi b(t, x), \mathcal{Z}(t, \cdot) \rangle \right) \right.
\]

\[\left. - A^*_x p(t, x) \mathcal{Z}(t, x) + \left( \frac{\partial \sigma}{\partial u}(t, x) \mathcal{Z}(t, x) + \langle \nabla^*_\varphi \sigma(t, x), \mathcal{Z}(t, \cdot) \rangle \right) q(t, x) \right\} \, dxdt \right.
\]

\[+ \int_0^T \int_D f(t, x)p(t, x)\zeta(dt, x) \, dx
\]

\[- \int_D \int_0^T \left( \frac{\partial H_0}{\partial u}(t, x) + \nabla^*_\varphi H_0(t, x) \right) \mathcal{Z}(t, x) \, dxdt
\]

\[- \int_D \int_0^T \left( \frac{\partial H_1}{\partial u}(t, x) + \nabla^*_\varphi H_1(t, x) \right) \mathcal{Z}(t, x) \zeta(dt, x) \, dx \right]. \tag{2.22} \}

Since \(p(t, x) = \mathcal{Z}(t, x) = 0\) for \(x \in \partial D\), we deduce that

\[
\int_D p(t, x)A_x \mathcal{Z}(t, x) \, dx = \int_D A^*_x p(t, x) \mathcal{Z}(t, x) \, dx.
\]

Therefore, substituting (2.22) and (2.21) into (2.20), we get

\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( J(\xi + \epsilon \zeta) - J(\xi) \right)
\]

\[= \mathbb{E} \left[ \int_0^T \int_D \left\{ f(t, x)p(t, x) + h_1(t, x) \right\} \zeta(dt, x) \, dx \right]. \]
The desired result follows.

We can now state our necessary maximum principle:

\textbf{Theorem 2.6 (Necessary Maximum Principle)} (i) Suppose \( \xi^* \in A \) is optimal, i.e.

\[
\max_{\xi \in A} J(\xi) = J(\xi^*). \tag{2.23} \]

Let \( u^*, (p^*, q^*) \) be the corresponding solution of (2.1) and (2.11), respectively, and assume that (2.18) holds with \( \xi = \xi^* \). Then

\[
f(t,x)p^*(t,x) + h_1(t,x,u^*(t,x),u^*(t,\cdot)) \leq 0 \quad \text{for all } t,x \in [0,T] \times D, \ a.s. \tag{2.24} \]

and

\[
\left\{ f(t,x)p^*(t,x) + h_1(t,x,u^*(t,x),u^*(t,\cdot)) \right\} dt, dx = 0 \quad \text{for all } t,x \in [0,T] \times D, \ a.s. \tag{2.25} \]

(ii) Conversely, suppose that there exists \( \hat{\xi} \in A \), such that the corresponding solutions \( \hat{u}(t,x), (\hat{p}(t,x), \hat{q}(t,x)) \) of (2.1) and (2.11), respectively, satisfy

\[
f(t,x)\hat{p}(t,x) + h_1(t,x,\hat{u}(t,x),\hat{u}(t,\cdot)) \leq 0; \quad \text{for all } t,x \in [0,T] \times D, \ a.s. \tag{2.26} \]

and

\[
\left\{ f(t,x)\hat{p}(t,x) + h_1(t,x,\hat{u}(t,x),\hat{u}(t,\cdot)) \right\} \hat{\xi}(dt, dx) = 0; \quad \text{for all } t,x \in [0,T] \times D, \ a.s. \tag{2.27} \]

Then \( \hat{\xi} \) is a directional sub-stationary point for \( J(\cdot) \), in the sense that

\[
\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( J(\hat{\xi} + \epsilon \zeta) - J(\hat{\xi}) \right) \leq 0; \quad \text{for all } \zeta \in \mathcal{V}(\hat{\xi}). \tag{2.28} \]

\textbf{Proof.} The proof is just a consequence of Lemma 2.5 and Theorem 3 in Øksendal \textit{et al} [15]. ~ \Box

\section{3 Application to Optimal Harvesting}

We now return to the problem of optimal harvesting from a fish population in a lake \( D \) stated in the Introduction. Thus we suppose the density \( u(t,x) \) of the population at time \( t \in [0,T] \) and at the point \( x \in D \) is given by the stochastic reaction-diffusion equation

\[
\begin{cases}
du(t,x) = \left[ \frac{1}{2} \Delta u(t,x) + \alpha u(t,x) \right] dt + \beta u(t,x) dB(t) - \lambda u(t,x) \xi(dt, x); & (t,x) \in (0,T) \times D, \\
u(0,x) = u_0(x) > 0; & x \in D, \\
u(t,x) = u_1(t,x) \geq 0; & (t,x) \in (0,T) \times \partial D,
\end{cases} \tag{3.1} \]

\textbf{Proof.} The proof is just a consequence of Lemma 2.5 and Theorem 3 in Øksendal \textit{et al} [15]. ~ \Box
where $\lambda_0 > 0$ is representing the harvesting efficiency constant and, as in (1.1),

$$
\bar{u}(t, x) = \frac{1}{V(K_\theta)} \int_{K_\theta} u(x + y) dy.
$$

The performance criterion is assumed to be

$$
J(\xi) = \mathbb{E} \left[ \int_D \int_0^T h_{10}(t, x) u(t, x) \xi(dt, x) dx + \int_D g_0(T, x) u(T, x) dx \right],
$$

where $h_{10} > 0$ and $g_0 > 0$ are given deterministic functions. We can interpret $\xi(dt, x)$ as the harvesting effort at $x$.

**Problem 3.1** We want to find $\hat{\xi} \in A$ such that $\sup_{\xi \in A} J(\xi) = J(\hat{\xi})$. \{problem\}

In this case the Hamiltonian takes the form

$$
H(t, x, u, \bar{u}, p, q)(dt, \xi(dt, x)) = (\alpha \bar{u} p + \beta u q) dt + [-\lambda_0 p + h_{10}(t, x)] u \xi(dt, x).
$$

Recall that for the map $L : L^2(D) \mapsto L^2(D)$ given by $L(u) = \bar{u}$, we know that

$$
\nabla_x^* L = \frac{V((x + K_\theta) \cap D)}{V(K_\theta)}.
$$

See Example 3.1 in Agram et al [1]. Therefore the adjoint equation is of the form

$$
\begin{cases}
   dp(t, x) = - \left[ \frac{1}{2} \Delta p(t, x) + \alpha p(t, x) \frac{V((x + K_\theta) \cap D)}{V(K_\theta)} + \beta q(t, x) \right] dt \\
   + [\lambda_0 - h_{10}(t, x)] \xi(dt, x) + q(t, x) dB(t, x); & (t, x) \in (0, T) \times D, \\
   p(T, x) = g_0(T, x); & x \in D, \\
   p(t, x) = 0; & (t, x) \in (0, T) \times \partial D.
\end{cases}
$$

(3.2) \{bspde\}

The variational inequality for an optimal control $\hat{\xi}(dt, x)$ and the associated $\hat{p}$ are:

$$
[-\lambda_0 \hat{p}(t, x) + h_{10}(t, x)] \hat{u}(t, x) \xi(dt, x)
\leq [-\lambda_0 \hat{p}(t, x) + h_{10}(t, x)] \hat{u}(t, x) \hat{\xi}(dt, x); & (t, x) \in [0, T] \times D, \text{ for all } \xi.
$$

We claim that

$$
u(t, x) > 0 \text{ for all } (t, x) \in [0, T] \times D.
$$

(3.3) \{eq3.5\}

Suppose this claim is proved. Then, choosing first $\xi = 2\hat{\xi}$ and then $\xi = \frac{1}{2} \hat{\xi}$ in the above, we obtain that

$$
\left[ \frac{1}{\lambda_0} h_{10}(t, x) \right] \hat{\xi}(dt, x) = 0; & (t, x) \in [0, T] \times D.
$$
In addition we get that
\[
\hat{p}(t, x) - \frac{1}{\lambda_0} h_{10}(t, x) \hat{\xi}(dt, x) \leq 0;
\]
which implies that \( \hat{p}(t, x) - \frac{1}{\lambda_0} h_{10}(t, x) \leq 0 \) always.

Summarising, we have proved the following:

**Theorem 3.2** Suppose that \( \hat{u} > 0 \) and \( (\hat{p}, \hat{\xi}) \) satisfies the following variational inequality
\[
\max \left\{ \hat{p}(t, x) - \frac{1}{\lambda_0} h_{10}(t, x), -\hat{\xi}(dt, x) \right\} = 0; \quad (t, x) \in [0, T] \times D. \tag{3.4} \text{ (eq3.8)}
\]

Then \( \hat{\xi} \) is an optimal singular control for the space-mean SPDE singular control problem (3.1)

We see that this, together with (3.2) constitute a reflected BSPDE, albeit of a slightly different type than the one that will be discussed in the next section.

We summerize the above in the following:

**Theorem 3.3 (a)** Suppose \( \xi(dt, x) \in \mathcal{A} \) is an optimal singular control for the harvesting problem
\[
\sup_{\xi \in \mathcal{A}} \mathbb{E} \left[ \int_D \int_0^T h_1(t, x) \xi(dt, x) dx + \int_D g_0(T, x) u(T, x) dx \right],
\]
where \( u(t, x) \) is given by the SPDE (3.1). Then \( \xi(dt, x) \) solves the reflected BSPDE (3.2), (3.4).

(b) Conversely, suppose \((p, q, \xi)\) is a solution of the reflected BSPDE (3.2), (3.4). Then \( \xi(dt, x) \) is an optimal control for the problem to maximize the performance (1.2).

Heuristically we can interpret the optimal harvesting strategy as follows:

- As long as \( p(t, x) < \frac{1}{\lambda_0} h_1(t, x) \), we do nothing.
- If \( p(t, x) = \frac{1}{\lambda_0} h_1(t, x) \), we harvest immediately from \( u(t, x) \) at a rate \( \xi(dt, x) \) which is exactly enough to prevent \( p(t, x) \) from dropping below \( \frac{1}{\lambda_0} h_1(t, x) \) in the next moment.
- If \( p(t, x) > \frac{1}{\lambda_0} h_1(t, x) \), we harvest immediately what is necessary to bring \( p(t, x) \) up to the level of \( \frac{1}{\lambda_0} h_1(t, x) \).

**Remark 3.4** Note that if \( p(t, x) = \frac{1}{\lambda_0} h_{10}(t, x) \) and \( \lambda_0 > h_{10}(t, x) \), then an immediate harvesting of an amount \( \Delta \xi > 0 \) from \( u(t, x) \) produces an immediate decrease in the process \( p(t, x) \) and hence pushes \( p(t, x) \) below \( \frac{1}{\lambda_0} h_{10}(t, x) \). This follows from the comparison theorem for reflected BSPDEs of the type (3.2). We do not study such a comparison theorem in this paper. It will be the subject of future research.
4 Existence and uniqueness of solutions of space-mean
reflected backward SPDEs

Let \( W, H \) be two separable Hilbert spaces such that \( W \) is continuously, densely imbedded in \( H \). Identifying \( H \) with its dual we have
\[
W \subset H \cong H^* \subset W^*,
\]
where we have denoted by \( W^* \) the topological dual of \( V \). Let \( A \) be a bounded linear operator from \( W \) to \( W^* \) satisfying the following Gårding inequality (coercivity hypothesis): There exist constants \( \alpha > 0 \) and \( \lambda \geq 0 \) so that
\[
2\langle Au, u \rangle + \lambda ||u||_H^2 \geq \alpha ||u||_W^2; \quad \text{for all } u \in W,
\]
where \( \langle Au, u \rangle = Au(u) \) denotes the action of \( Au \in W^* \) on \( u \in W \) and \( || \cdot ||_H \) (respectively \( || \cdot ||_W \)) the norm associated to the Hilbert space \( H \) (respectively \( W \)). We will also use the following spaces:

- \( L^2(D) \) is the set of all Lebesgue measurable \( Y : D \to \mathbb{R} \), such that
  \[
  ||Y||_{L^2(D)} := \left( \int_D |Y(x)|^2 dx \right)^{\frac{1}{2}} < \infty.
  \]

- \( L^2(H) \) is the set of \( \mathcal{F}_T \)-measurable \( H \)-valued random variables \( \varsigma \) such that \( \mathbb{E}[||\varsigma||_H^2] < \infty \).

We let \( W := W^{1,2}(D) \) and \( H = L^2(D) \).

Denote by \( L(t, x) \) the barrier, which is assumed to be a measurable function that is differentiable in time \( t \) and twice differentiable in space \( x \), such that
\[
\int_0^T \int_D \left( \frac{\partial L}{\partial t}(t,x) \right)^2 dx dt < \infty, \quad \int_0^T \int_D |\Delta L(t,x)|^2 dx dt < \infty.
\]

\( \eta \) is a \( H \)-valued continuous process, nonnegative, nondecreasing in \( t \) and \( \eta(0,x) = 0 \).

We now consider the adjoint equation (2.11) as a reflected backward stochastic evolution equation
\[
\begin{aligned}
&dY(t, x) = -AY(t, x)dt - F(t, Y(t, x), Y(t, x), Z(t, x), Z(t, x))dt \\
&
+ Z(t, x)dB(t) - \eta(dt, x), t \in (0, T),

&Y(t, x) \geq L(t, x),

&\int_0^T \int_D (Y(t, x) - L(t, x))\eta(dt, x)dx = 0,

&Y(T, x) = \phi(x); \quad \text{a.s.},
\end{aligned}
\]
(4.2) \{r-BSPDE\}

where \( Y(t, x) \) stands for the \( W \)-valued continuous process \( Y(t, x) \) and the solution of equation (4.2) is understood as an equation in the dual space \( W^* \) of \( W \).
We mean by \(dY(t,x)\) the differential operator with respect to \(t\), while \(A_x\) is the partial differential operator with respect to \(x\), and in this section only we use the notation

\[
\nabla(t,x) = G(x,Y) = \frac{1}{V} \int_{K_0} Y(x+\rho)d\rho,
\]

\[
Z(t,x) = G(x,Z) = \frac{1}{V} \int_{K_0} Z(x+\rho)d\rho.
\]

The following result is essential due to Agram et al [1]:

\[\text{Lemma 4.1} \quad \text{For all } \varphi \in H \text{ we have}
\]

\[
\|G(\cdot, \varphi)\|_H \leq \|\varphi\|_H. \quad (4.3)
\]

We shall now state and prove our main result of existence and uniqueness of solutions to reflected BSPDE.

\[\text{Theorem 4.2 (Existence and uniqueness of solutions)} \quad \text{The space-mean reflected BSPDE (4.2) has a unique solution } (Y(t,x),Z(t,x),\eta(t,x)) \in W \times L^2(D,\mathbb{R}^m) \times H\text{-valued progressively measurable process, provided that the following assumptions hold:}
\]

(i) The terminal condition \(\phi\) is \(\mathcal{F}_T\)-measurable random variable and satisfies

\[
\mathbb{E}[\|\phi\|^2_H] < \infty.
\]

(ii) There exists a constant \(C > 0\) such that

\[
\|F(t,y_1,\overline{y}_1,z_1,\overline{z}_1) - F(t,y_2,\overline{y}_2,z_2,\overline{z}_2)\|_H \\
\leq C (\|y_1 - y_2\|_H + \|\overline{y}_1 - \overline{y}_2\|_H + \|z_1 - z_2\|_H + \|\overline{z}_1 - \overline{z}_2\|_H),
\]

for all \(t,y_i,\overline{y}_i,z_i,\overline{z}_i; i = 1,2\).

Proof. For the proof of the theorem, we introduce the penalized backward SPDEs:

\[
\begin{aligned}
    dY^n(t,x) &= -AY^n(t)dt - F(t,Y^n(t,x),Z^n(t,x),Z^n(t,x))dt \\
    &+ Z^n(t,x)dB(t) - n(Y^n(t,x) - L(t,x))^- dt; \quad t \in (0,T), \\
    Y^n(T,x) &= \phi(x) \quad \text{a.s.}
\end{aligned}
\]

(4.4) {2.2}

According to Agram et al [1], the solution \((Y^n,Z^n)\) of the above equation (4.4) exists and is unique. We are going to show that \((Y^n,Z^n)_{n \geq 1}\) forms a Cauchy sequence, i.e.,

\[
\lim_{n,m \to \infty} \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y^n(t) - Y^m(t)|^2_H\right] = 0,
\]

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\[
\lim_{n,m \to \infty} \mathbb{E} \left[ \int_0^T ||Y^n(t) - Y^m(t)||_W^2 dt \right] = 0,
\]
\[
\lim_{n,m \to \infty} \mathbb{E} \left[ \int_0^T |Z^n(t) - Z^m(t)|_{L^2(D,\mathbb{R}^m)}^2 dt \right] = 0.
\]

Applying the Itô formula we get

\[
|Y^n(t) - Y^m(t)|_H^2
= 2 \int_t^T \langle Y^n(s) - Y^m(s), A(Y^n(s) - Y^m(s)) \rangle ds
+ 2 \int_t^T \langle Y^n(s) - Y^m(s), F(s, Y^n(s), Y^n(s), Z^n(s), \overline{Z}^n(s)) - F(s, Y^m(s), Y^m(s), Z^m(s), \overline{Z}^m(s)) \rangle ds
- 2 \int_t^T \langle Y^n(s) - Y^m(s), Z^n(s) - Z^m(s) \rangle dB(s)
+ 2 \int_t^T \langle Y^n(s) - Y^m(s), n(u^n(s) - L(s))^+ - m(Y^m(s) - L(s))^+ \rangle ds
- \int_t^T |Z^n(s) - Z^m(s)|_{L^2(D,\mathbb{R}^m)}^2 ds.
\]

Now we estimate each of the terms on the right hand side of the above equality:

\[
2 \int_t^T \langle Y^n(s) - Y^m(s), A(Y^n(s) - Y^m(s)) \rangle ds
\leq \lambda \int_t^T ||Y^n(s) - Y^m(s)||_H^2 ds - \alpha \int_t^T ||Y^n(s) - Y^m(s)||_V^2 ds. \tag{4.5} \{4.5\}
\]

By the Lipschitz continuity (ii) of \(b\) and the inequality \(ab \leq \varepsilon a^2 + C_\varepsilon b^2\), together with inequality (4.3), we get

\[
2 \int_t^T \langle Y^n(s) - Y^m(s), \rangle
F(s, Y^n(s), Y^n(s), Z^n(s), \overline{Z}^n(s)) - F(s, Y^m(s), Y^m(s), Z^m(s), \overline{Z}^m(s)) \rangle ds
\leq C \int_t^T |Y^n(s) - Y^m(s)|_H^2 ds + \frac{1}{2} \int_t^T |Z^n(s) - Z^m(s)|_{L^2(D,\mathbb{R}^m)}^2 ds. \tag{4.6} \{4.6\}
\]
It follows from (4.5) and (4.6) that
\[
\mathbb{E}[|Y^n(t) - Y^m(t)|^2_W] + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z^n(s) - Z^m(s)|_{L^2(D,\mathbb{R}^m)}^2 ds \right]
+ \mathbb{E} \left[ \int_t^T ||Y^n(s) - Y^m(s)||^2_W ds \right]
\leq C \int_t^T \mathbb{E}[|Y^n(s) - Y^m(s)|^2_W] ds + C' \left( \frac{1}{n} + \frac{1}{m} \right).
\]

The Gronwall inequality yields
\[
\lim_{n,m \to \infty} \left\{ \mathbb{E}[|Y^n(t) - Y^m(t)|^2_H] + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z^n(s) - Z^m(s)|_{L^2(D,\mathbb{R}^m)}^2 ds \right] \right\} = 0, \quad (4.7) \{4.7\}
\]
and
\[
\lim_{n,m \to \infty} \mathbb{E} \left[ \int_t^T ||Y^n(s) - Y^m(s)||^2_H ds \right] = 0.
\]

By inequality (4.7) and the Burkholder inequality, we get
\[
\lim_{n,m \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n(t) - Y^m(t)|^2_H \right] = 0.
\]

Under the conditions of Theorem 4.2 and by Lemma 5 in Øksendal et al [15], there exists a constant \(C\), such that
\[
\mathbb{E} \left[ \int_0^T \int_D (|Y^n(t,x) - L(t,x)|^2)^2 dxdt \right] \leq \frac{C}{n^2}. \quad (4.8) \{dp\}
\]

Denote by \(Y(t,x), Z(t,x)\) the limit of \(Y^n\) and \(Z^n\), respectively. Put
\[
\overline{\eta}^n(t,x) = n(Y^n(t,x) - L(t,x)).
\]

Inequality (4.8) implies that \(\overline{\eta}^n(t,x)\) admits a non-negative weak limit, denoted by \(\overline{\eta}(t,x)\), in the following Hilbert space:
\[
\overline{H} = \left\{ h; h \text{ is a } H\text{-valued adapted process, such that } \mathbb{E} \left[ \int_0^T |h(s)|^2_H ds \right] < \infty \right\},
\]
with inner product
\[
\langle h_1, h_2 \rangle_{\overline{H}} = \mathbb{E} \left[ \int_0^T \int_D h_1(t,x)h_2(t,x)dxdt \right].
\]

Set \(\eta(t,x) = \int_0^t \overline{\eta}(s,x) ds\). Then \(\eta\) is a continuous \(H\)-valued process which is increasing in \(t\).

Letting \(n \to \infty\) in (4.4) we obtain
\[
Y(t,x)
= \phi(x) + \int_t^T AY(s,x)ds + \int_t^T F(s, Y(s,x), Y(s,x), Z(s,x), Z(s,x))ds
- \int_t^T Z(s,x)dB(s) + \eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T. \quad (4.9) \{2.48\}
\]
Inequality (4.8) and the Fatou Lemma imply that 
\[ \mathbb{E} \left[ \int_t^T \int_D ((Y(s, x) - L(s, x))^2) dx ds \right] = 0. \]
In view of the continuity of \( Y \) in \( t \), we conclude that \( Y(t, x) \geq L(t, x) \) a.e. in \( x \), for every \( t \geq 0 \). Combining the strong convergence of \( Y^n \) and the weak convergence of \( \bar{\eta}^n \), we also have
\[
\mathbb{E} \left[ \int_0^T \int_D (Y(s, x) - L(s, x)) \eta(t, x) dtdx \right]
\leq \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \int_D (Y^n(s, x) - L(s, x)) \bar{\eta}^n(t, x) dtdx \right] \leq 0. \tag{4.10} \quad \{2.49\}
\]
Hence,
\[
\int_0^T \int_D (Y(s, x) - L(s, x)) \eta(dt, x) dx = 0, \quad \text{a.s.}
\]
We have shown that \((Y, Z, \eta)\) is a solution to the reflected backward SPDE (4.2).

**Uniqueness.** Let \((Y_1, Z_1, \eta_1)\), \((Y_2, Z_2, \eta_2)\) be two such solutions to equation (4.2). By the Itô formula, we have
\[
\begin{align*}
|Y_1(t) - Y_2(t)|^2_H \\
= & \ 2 \int_t^T \langle Y_1(s) - Y_2(s), \Delta(Y_1(s) - Y_2(s)) \rangle ds \\
+ & \ 2 \int_t^T \langle Y_1(s) - Y_2(s), F(s, Y_1(s), Z_1(s)) - F(s, Y_2(s), Z_2(s)) \rangle ds \\
- & \ 2 \int_t^T \langle Y_1(s) - Y_2(s), Z_1(s) - Z_2(s) \rangle dB(s) \\
+ & \ 2 \int_t^T \langle Y_1(s) - Y_2(s), \eta_1(ds) \rangle ds \\
- & \ \int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D, \mathbb{R}^m)} ds.
\end{align*}
\tag{4.11} \quad \{2.50\}
\]
Similar to the proof of existence, we have
\[
2 \int_t^T \langle Y_1(s) - Y_2(s), A(Y_1(s) - Y_2(s)) \rangle ds \leq 0, \tag{4.12} \quad \{2.51\}
\]
and
\[
2 \int_t^T \langle Y_1(s) - Y_2(s), F(s, Y_1(s), \overline{Y}_1(s), Z_1(s), \overline{Z}_1(s)) - F(s, Y_2(s), \overline{Y}_2(s), Z_2(s), \overline{Z}_2(s)) \rangle \, ds \\
\leq C \int_t^T |Y_1(s) - Y_2(s)|^2_H ds + \frac{1}{2} \int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D, R^m)} ds
\]  \quad (4.13) \quad \{2.52\}

On the other hand,
\[
2 \mathbb{E} \left[ \int_t^T \langle Y_1(s) - Y_2(s), \eta_1(ds) - \eta_2(ds) \rangle \right] \\
= 2 \mathbb{E} \left[ \int_t^T \int_D (Y_1(s,x) - L(s,x)) \eta_1(ds,x) dx \right] \\
- 2 \mathbb{E} \left[ \int_t^T \int_D (Y_1(s,x) - L(s,x)) \eta_2(ds,x) dx \right] \\
+ 2 \mathbb{E} \left[ \int_t^T \int_D (Y_2(s,x) - L(s,x)) \eta_2(ds,x) dx \right] \\
- 2 \mathbb{E} \left[ \int_t^T \int_D (Y_2(s,x) - L(s,x)) \eta_1(ds,x) dx \right] \\
\leq 0. \tag{4.14} \quad \{2.53\}
\]

Combining (4.11)-(4.14) we arrive at
\[
\mathbb{E}[|Y_1(t) - Y_2(t)|^2_H] + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D, R^m)} ds \right] \\
\leq C \int_t^T \mathbb{E}[|Y_1(s) - Y_2(s)|^2_H] ds.
\]

Appealing to the Gronwall inequality, this implies
\[
Y_1 = Y_2, \quad Z_1 = Z_2
\]
which further gives \( \eta_1 = \eta_2 \) from the equation they satisfy. \qed

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