Privately detecting changes in unknown distributions

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Abstract

The change-point detection problem seeks to identify distributional changes in streams of data. Increasingly, tools for change-point detection are applied in settings where data may be highly sensitive and formal privacy guarantees are required, such as identifying disease outbreaks based on hospital records, or IoT devices detecting activity within a home. Differential privacy has emerged as a powerful technique for enabling data analysis while preventing information leakage about individuals. Much of the prior work on change-point detection—including the only private algorithms for this problem—requires complete knowledge of the pre-change and post-change distributions. However, this assumption is not realistic for many practical applications of interest. This work develops differentially private algorithms for solving the change-point problem when the data distributions are unknown. Additionally, the data may be sampled from distributions that change smoothly over time, rather than fixed pre-change and post-change distributions. We apply our algorithms to detect changes in the linear trends of such data streams.

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1 Introduction

The change-point detection problem seeks to identify distributional changes in streams of data. It models data points as initially being sampled from a pre-change distribution $P_0$, and then at an unknown change-point time $k^*$, data points switch to being sampled from a post-change distribution $P_1$. The task is to quickly and accurately identify the change-point time $k^*$. The change-point problem has been widely studied in theoretical statistics [She31, Pag54, Shi63, Pol87, Mei08] as well as practical applications including climatology [LR02], econometrics [BP03], and DNA analysis [ZS12].

Much of the previous work on change-point detection focused on the parametric setting, where the distributions $P_0$ and $P_1$ are perfectly known to the analyst. In this structured setting, the analyst could use algorithms tailored to details of these distributions, such as computing the maximum log-likelihood estimator (MLE) of the change-point time. In this work, we consider the nonparametric setting, where these distributions are unknown to the analyst. This setting is closer to practice, as it removes the unrealistic assumption of perfect distributional knowledge. In practice, an analyst may only have sample access to the current (pre-change) distribution, and may wish to detect a change to any distribution that is sufficiently far from the current distribution without making specific parametric assumptions on the future (post-change) distribution. The nonparametric setting requires different test statistics, as common approaches like computing the MLE do not work without full knowledge of $P_0$ and $P_1$.

In many applications, change-point detection algorithms are applied to sensitive data, and may require formal privacy guarantees. For example, the Center for Disease Control (CDC) may wish to analyze hospital records to detect disease outbreaks, or the Census Bureau may wish to analyze income records to detect changes in employment rates. We will use differential privacy [DMNS06] as our privacy notion, which has been well-established as the predominant privacy notion in theoretical computer science. Informally, differential privacy bounds the effect of any individual’s data in a computation, and ensures that very little can be inferred about an individual from seeing the output of a differentially private analysis. Differential privacy is achieved algorithmically by adding noise that scales with the sensitivity of a computation, which is the maximum change in the function’s value that can be caused by changing a single entry in the database. High sensitivity analyses require large amounts of noise, which imply poor accuracy guarantees (see Section 2.2 for more details).

Unfortunately, most nonparametric estimation procedures are not amenable to differential privacy. Indeed, all prior work on private change-point detection has been in the parametric setting, where $P_0$ and $P_1$ are known [CKM+18b, CKM+18a]. A standard approach in the nonparametric setting is to first estimate a parametric model, and then perform parametric change-point detection using the estimated model. Common nonparametric estimation techniques include kernel methods and spline methods [Par62, Ros56] or nonparametric regression [ABH89]. These methods are difficult to make private in part because of the complexity of finite sample error bounds combined with the effect of injecting additional noise for privacy. In contrast, simple rank-based statistics—which order samples by their value—have easy-to-analyze sensitivity.

In this work, we estimate nonparametric change-points using the Mann-Whitney test [Wil45, MW47], which is a rank-based test statistic, presented formally in Section 2.1. This test picks an index $k$ and measures the fraction of points before $k$ that are greater than points after $k$. For the change-point problem, this statistic should be largest around the true change-point $k^*$, and smaller elsewhere (under mild non-degeneracy conditions on the pre- and post-change distributions). Also note that this statistic simply computes pairwise comparisons of the observed data, and it does not require any additional knowledge of $P_0$ or $P_1$ beyond the assumption that a data point from $P_0$ is larger than a data point from $P_1$ with probability $> 1/2$. The test statistic has sensitivity $O(1/n)$ for a database of size $n$, and is known to have lower sensitivity than most other test statistics for the same task [MW47].

1.1 Our Results

In this paper, we provide differentially private algorithms for accurate nonparametric change-point detection in both the offline and online settings. We also show how our results can be applied to settings where data are not sampled i.i.d., but are instead sampled from distributions changing smoothly over time.
In the offline case, the entire database $X = \{x_1, \ldots, x_n\}$ is given up front, and the analyst seeks to estimate the change-point with small additive error. We use the Mann-Whitney rank-sum statistic and its extension to the change-point setting due to [Dar76]. At every possible change-point time $k$, the test measures the fraction of points before $k$ that are greater than points after $k$, using statistic $V(k) = \frac{\sum_{i=k+1}^{n} \sum_{j=1}^{k} I(x_i > x_j)}{k(n-k)}$. The test then outputs the index $\hat{k}$ that maximizes this statistic. Even before adding privacy, we improve the best previously-known finite sample accuracy guarantees of this estimation procedure. The previous non-private accuracy guarantee has $O(n^{2/3})$ additive error [Dar76], whereas our Theorem 6 in Section 3.1 achieves $O(1)$ additive error.

With these improved accuracy bounds, we give Algorithm 3 PNCPD in Section 3.2 to make this estimation procedure differentially private. Our algorithm uses the REPORTMAX framework of [DRL14]. The REPORTMAX algorithm takes in a collection of queries, computes a noisy answer to each query, and returns the index of the query with the largest noisy value. We instantiate this framework with our test statistics $V(k)$ as queries, to privately select the argmax of the statistics. One challenge is ensuring that the test statistics $V(k)$ have low enough sensitivity that the additional noise required for privacy does not harm the estimation error by too much. We show that our PNCPD algorithm is differentially private (Theorem 7) and has $O(\frac{1}{n\epsilon^2})$ additive accuracy (Theorem 8), meaning that adding privacy does not create any dependence on $n$ in the accuracy guarantee.

In the online case, the analyst starts with an initial database of size $n$, and receives a stream of additional data points, arriving online. The analyst’s goal here is to accurately estimate the change-point quickly after it occurs. This is a more challenging setting because the analyst will have very little post-change data if they want to detect changes quickly. In this setting, we give Algorithm 4 ONLINEPNCPD in Section 4. This algorithm uses the ABOVE_THRESHOLD framework of [DNR09, DNPR10]. The ABOVE_THRESHOLD algorithm takes in a potentially unbounded stream of queries, compares the answer of each query to a fixed noisy threshold, and halts when it finds a noisy answer that exceeds the noisy threshold. Our algorithm computes the test statistic $V(k)$ for the middle index $k$ of each sliding window of the last $n$ data points. Once the algorithm finds a window with a high enough test statistic, it waits for enough additional data points to meet the requirements of our offline algorithm PNCPD for accuracy, and then calls PNCPD on the $n$ most recent data points to estimate the change-point time. One technical challenge in the online setting is finding a threshold that is high enough to prevent false positives before a change occurs, and low enough that a true change will trigger a call to the offline algorithm. We show that our ONLINEPNCPD algorithm is differentially private (Theorem 9) and has $O(\log n)$ additive error (Theorem 10).

Finally, in Section 5 we apply our results to privately solve the problem of drift change detection, where points are not sampled i.i.d. pre- and post-change, but instead are sampled from smoothly changing distributions whose means are shifting linearly with respect to time, and the linear drift parameter changes at an unknown change-time $k^*$. We show how to reduce an instance of the drift change detection problem with non-i.i.d. samples to an instance of the change-point detection problem to which our algorithms can be applied. We show in Corollary 11 that our algorithms also provide differential privacy and accurate estimation for the drift change detection problem. We also suggest extensions of this reduction technique so that our algorithms may also be applied for non-linear drift change detection for other smoothly changing distributions that exhibit sufficient structure.

1.2 Related work

Change-point detection is a canonical problem in statistics that has been studied for nearly a century; selected results include [She31, Pag54, Shi63, Rob66, Lor71, Pol85, Pol87, Mei06, Mei08, Cha17]. The problem originally arose from industrial quality control, and has since been applied in a wide variety of other contexts including climatology [LR02], econometrics [BP03], and DNA analysis [ZS12]. In the parametric setting where pre-change and post-change distributions $P_0$ and $P_1$ are perfectly known, the Cumulative Sum (CUSUM) procedure [Pag54] is among the most commonly used algorithms for solving the change-point detection problem. It follows the generalized log-likelihood ratio principle, calculating $\ell(k) = \sum_{i=k}^{n} \log \frac{P_1(x_i)}{P_0(x_i)}$ for each $k \in [n]$, and declaring that a change occurs if and only if $\ell(\hat{k}) \geq T$ for MLE
\( \hat{k} = \arg\max_k \ell(k) \) and appropriate threshold \( T > 0 \). Nonparametric change-point detection has also been well-studied in the statistics literature \cite{Dar76, Car88, B168}, and requires different test statistics that do not rely on exact knowledge of the distributions \( P_0 \) and \( P_1 \).

The only two prior works on differentially private change-point detection \cite{CKM18b, CKM18a} both considered the parametric setting and employed differentially private variants of the CUSUM procedure and the change-point MLE underlying it. \cite{CKM18b} directly privatized non-private procedures for the offline and online settings. \cite{CKM18a} gave private change-point detection as an instantiation of a solution to the more general problem of private hypothesis testing, partitioning time series data into batches of size equal to the sample complexity of the hypothesis testing problem, and then outputs the batch number most consistent with a change-point. Both works assumed that the pre- and post-distributions were fully known in advance.

In our nonparametric setting, we use the Mann-Whitney test \cite{Wil45, MW47} instead of the MLE that the CUSUM procedure is built on. The Mann-Whitney test was originally proposed as a rank-based nonparametric two-sample test, to test whether two samples were drawn from the same distribution using the null hypothesis that after randomly selecting one point from each sample, each point is equally likely to be the larger of the two. It was extended to the change-point setting by \cite{Dar76}, for testing whether samples from before and after the hypothesized change-point were drawn from the same distribution. Given a database \( X = \{x_1, \ldots, x_n\} \), for each possible change-point \( k \), the test statistic \( V(k) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{k} I(x_i > x_j)}{k(n-k)} \) counts the proportion of index pairs \( (i, j) \) with \( i \leq k < j \) for which \( x_i > x_j \). This is a nonparametric test because it does not require any additional knowledge of the distributions from which data are drawn. Additionally, the Mann-Whitney test is known to be more efficient \cite{GCT11} and have lower sensitivity \cite{MW47} than most other test statistics for the same task, including the Wald statistic \cite{WW40} and the Kolmogorov-Smirnov test \cite{Li67}. Differentially private versions of related test statistics have been used in recent unpublished work in the context of hypothesis testing, but they have not been applied to the change-point problem \cite{CKS18, CKS19}.

Although the current paper largely follows the same structure as \cite{CKM18b} for privatizing the change-point procedure, the analysis of the algorithm is vastly different, due to new challenges introduced by the nonparametric setting. Most test statistics for nonparametric estimation have high sensitivity, and therefore require large amounts of noise to be added to satisfy differential privacy. This means that off-the-shelf applications of nonparametric test statistics to the differentially private change-point framework of \cite{CKM18b} would result in high error. Indeed, even with our use of the Mann-Whitney test statistic which was chosen for its low sensitivity, an immediate application of the best known finite-sample accuracy bounds \cite{Dar76} yielded additive error \( O(n^{2/3}) \) in the offline setting for databases of size \( n \). To achieve our much tighter \( O(\epsilon^{-1.01}) \) error bounds required a new analysis.

## 2 Preliminaries

This section provides the necessary background for interpreting our results for the problem of private nonparametric change-point detection. Section 2.1 defines the nonparametric change-point detection problem, Section 2.2 describes the differentially private tools that will be brought to bear, and Section 2.3 gives the concentration inequality which will be used in our proofs.

### 2.1 Change-point background

Let \( X = \{x_1, \ldots, x_n\} \) be \( n \) real-valued data points. The change-point detection problem is parametrized by two distributions, \( P_0 \) and \( P_1 \). The data points in \( X \) are hypothesized to initially be sampled i.i.d. from \( P_0 \), but at some unknown change time \( k^* \in [n] \), an event may occur (e.g., epidemic disease outbreak) and change the underlying distribution to \( P_1 \). The goal of a data analyst is to announce that a change has occurred as quickly as possible after \( k^* \). Since the \( x_i \) may be sensitive information—such as individuals’ medical information or behaviors inside their home—the analyst will wish to announce the change-point time in a privacy-preserving manner.
In the standard non-private offline change-point literature, the analyst wants to test the null hypothesis $H_0 : k^* = n$, where $x_1, \ldots, x_n \sim_{\text{iid}} P_0$, against the composite alternate hypothesis $H_1 : k^* < n$, where $x_1, \ldots, x_{k^*} \sim_{\text{iid}} P_0$ and $x_{k^*+1}, \ldots, x_n \sim_{\text{iid}} P_1$. If $P_1$ and $P_0$ are known, the log-likelihood ratio of $k^* = \infty$ against $k^* = k$ will be given by

$$\ell(k, X) = \sum_{i=k+1}^{n} \log \frac{P_1(x_i)}{P_0(x_i)}.$$  

The maximum likelihood estimator (MLE) of the change time $k^*$ is given by $\text{argmax}_{k \in [n]} \ell(k, X)$. However, note that to perform this test, the analyst must have complete knowledge of distributions $P_0$ and $P_1$ to compute the log-likelihood ratio.

In this paper, we consider the situation that we do not know both the pre-change distribution and the post-change distribution. We require no knowledge of the pre- and post-change distributions, and assume only that the probability of an observation from $P_0$ exceeding an observation from $P_1$ is different than the probability of an observation from $P_1$ exceeding an observation from $P_0$, which is necessary for technical reasons. The Mann-Whitney test [Wil45] is a commonly used nonparametric test of the null hypothesis $H_0$, against the composite alternate hypothesis $H_1$. The Mann-Whitney test takes the form of a test statistic $V(k, X)$, which is calculated as follows:

$$V(k, X) = \frac{\sum_{j=k+1}^{n} \sum_{i=1}^{k} I(x_i > x_j)}{k(n-k)}$$  

(1)

For data $X$ drawn according to the change-point model with distributions $P_0, P_1$, this statistic is largest or smallest in expectation at the true change-point $k^*$ depending on the value $a = \Pr_{x_0 \sim P_0, x_1 \sim P_1}[x_0 > x_1]$. If $a > 1/2$, we estimate the change-point by taking the arg max of the Mann-Whitney statistics; otherwise we take the arg min. When $X$ is clear from context, we will simply write $V(k)$. The estimator $\hat{k}$ is understood to denote the argmax or argmin of $V(k)$ depending on whether $a > 1/2$.

We will measure the additive error of our estimations of the true change-point as follows.

**Definition 1** (($\alpha, \beta$)-accuracy). A change-point detection algorithm that produces a change-point estimator $\hat{k}$ is ($\alpha, \beta$)-accurate if $\Pr[|\hat{k} - k^*| > \alpha] \leq \beta$, where the probability is taken over randomness of the data $X$ sampled according to the change-point model with true change-point $k^*$ and (possibly) the randomness of the algorithm.

### 2.2 Differential privacy background

Differential privacy bounds the maximum amount that a single data entry can affect analysis performed on the database. Two databases $X, X'$ are neighboring if they differ in at most one entry.

**Definition 2** (Differential Privacy [DMNS06]). An algorithm $M : \mathbb{R}^n \rightarrow \mathcal{R}$ is $\epsilon$-differentially private if for every pair of neighboring databases $X, X' \in \mathbb{R}^n$, and for every subset of possible outputs $\mathcal{S} \subseteq \mathcal{R}$,

$$\Pr[M(X) \in \mathcal{S}] \leq \exp(\epsilon) \Pr[M(X') \in \mathcal{S}]$$

One common technique for achieving differential privacy is by adding Laplace noise. The Laplace distribution with scale $b$ is the distribution with probability density function: $\text{Lap}(x|b) = \frac{1}{b} \exp \left( -\frac{|x|}{b} \right)$. We will write $\text{Lap}(b)$ to denote the Laplace distribution with scale $b$, or (with a slight abuse of notation) to denote a random variable sampled from $\text{Lap}(b)$. The sensitivity of a function or query $f$ is defined as $\Delta(f) = \max_{\text{neighbors } X, X'} |f(X) - f(X')|$, and it determines the scale of noise that must be added to satisfy differential privacy. The Laplace Mechanism of [DMNS06] takes in a function $f$, database $X$, and privacy parameter $\epsilon$, and outputs $f(X) + \text{Lap}(\Delta(f)/\epsilon)$.

One helpful property of differential privacy is that it composes, meaning that the privacy parameter degrades gracefully as additional computations are performed on the same database.
Theorem 1 (Basic Composition [DMNS06]). Let $M_1$ be an algorithm that is $\epsilon_1$-differentially private, and let $M_2$ be an algorithm that is $\epsilon_2$-differentially private. Then their composition $(M_1, M_2)$ is $(\epsilon_1 + \epsilon_2)$-differentially private.

Our algorithms rely on the existing differentially private algorithms ReportMax [DR14]. The ReportMax algorithm takes in a collection of queries, computes a noisy answer to each query, and returns the index of the query with the largest noisy value. We use this as the framework for our offline private nonparametric change-point detector PNCPD in Section 3 to privately select the time $k$ with the highest Mann-Whitney statistics $V(k)$.

Algorithm 1 Report Noisy Max: ReportMax($X, \Delta, \{f_1, \ldots, f_m\}, \epsilon$)

**Input:** database $X$, set of queries $\{f_1, \ldots, f_m\}$ each with sensitivity $\Delta$, privacy parameter $\epsilon$

for $i = 1, \ldots, m$ do
  Compute $f_i(X)$
  Sample $Z_i \sim \text{Lap}(\frac{\Delta}{\epsilon})$
end for

Output $i^* = \arg\max_{i \in [m]} (f_i(X) + Z_i)$

Theorem 2 ([DR14]). ReportMax is $\epsilon$-differentially private.

The AboveThreshold algorithm of [DNR+09, DNPR10], refined to its current form by [DR14], takes in a potentially unbounded stream of queries, compares the answer of each query to a fixed noisy threshold, and halts when it finds a noisy answer that exceeds the noisy threshold. We use this algorithm as a framework for our online private nonparametric change-point detector ONLINEPNCPD in Section 4 when new data points arrive online in a streaming fashion.

Algorithm 2 Above Noisy Threshold: AboveThreshold($X, \Delta, \{f_1, f_2, \ldots\}, T, \epsilon$)

**Input:** database $X$, stream of queries $\{f_1, f_2, \ldots\}$ each with sensitivity $\Delta$, threshold $T$, privacy parameter $\epsilon$

Let $\hat{T} = T + \text{Lap}(\frac{2\Delta}{\epsilon})$

for each query $i$ do
  Let $Z_i \sim \text{Lap}(\frac{\Delta}{\epsilon})$
  if $f_i(X) + Z_i > \hat{T}$ then
    Output $a_i = \top$
    Halt
  else
    Output $a_i = \bot$
  end if
end for

Theorem 3 ([DNR+09]). AboveThreshold is $\epsilon$-differentially private.

Theorem 4 ([DNR+09]). For any sequence of $m$ queries $f_1, \ldots, f_m$ with sensitivity $\Delta$ such that $|\{i < m : f_i(X) \geq T - \alpha\}| = 0$,

AboveThreshold outputs with probability at least $1 - \beta$ a stream of $a_1, \ldots, a_m \in \{\top, \bot\}$ such that $a_i = \bot$ for every $i \in [m]$ with $f(i) < T - \alpha$ and $a_i = \top$ for every $i \in [m]$ with $f(i) > T + \alpha$ as long as

$$\alpha \geq \frac{8\Delta \log(2m/\beta)}{\epsilon}$$

2.3 Concentration inequalities

Our proofs will also use the following concentration inequality.
Theorem 5 (McDiarmid [McD89]). Define the discrete derivatives of the function \( f(X_1, \ldots, X_n) \) of independent random variables \( X_1, \ldots, X_n \) as
\[
D_i f(x) := \sup_z f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) - \inf_z f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).
\]
Then for \( X_1, \ldots, X_n \) independent, \( f(X_1, \ldots, X_n) \) is subgaussian with variance proxy \( \frac{1}{2} \sum_{i=1}^n \| D_i f \|_\infty^2 \). In particular,
\[
\Pr[f(X_1, \ldots, X_n) - \mathbb{E}[f(X_1, \ldots, X_n)] \geq t] \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^n \| D_i f \|_\infty^2}\right).
\]

3 Offline private nonparametric change-point detection

In this section, we give an offline private algorithm for change-point detection when the pre- and post-change distributions are unknown. In Section 3.1, we first offer the finite sample accuracy guarantee for the non-private nonparametric algorithm given by \( k = \arg\max V(k) \) for the test statistic \( V(k) \) given in Equation (1), which will serve as the baseline for evaluating the utility of our private algorithm. Then in Section 3.2, we present our private algorithm, and give privacy and accuracy guarantees.

3.1 Finite sample accuracy guarantee for the non-private nonparametric estimator

In this section, we provide error accuracy results for the offline non-private nonparametric change-point estimator when the data are drawn from two unknown distributions \( P_0, P_1 \) with true change-point \( k^* \in \{\lceil \gamma n \rceil, \ldots, \lfloor (1 - \gamma) n \rfloor \} \), for some known \( \gamma < 1/2 \). This \( \gamma \) bounds away from the change-point occurring too early or too late in the sample, and is necessary to ensure sufficient number of samples from both the pre-change and post-change distributions. Without loss of generality, we assume that \( a := \Pr_{x_0 \sim P_0, x_1 \sim P_1}[x_0 > x_1] > 1/2 \).

For the non-private task, we use the following estimation procedure of [Dar76], which calculates the estimated change-point \( \hat{k} \) as the argmax of \( V(k) \) over all \( k \) in the range permitted by \( \gamma \):
\[
\hat{k} = \arg\max_{k \in \{\lceil \gamma n \rceil, \ldots, \lfloor (1 - \gamma) n \rfloor \}} V(k),
\]
for test statistic \( V(k) \) defined in Equation (1). We show in Theorem 6 that the additive error of this procedure is constant with respect to the sample size \( n \).

Our result is much tighter than the previously known finite-sample accuracy result in [Dar76], which gave an estimation error bound of \( O(n^{2/3}) \). This sublinear result comes from a connection between the accuracy and the maximal deviation of \( V(k) \) from the expected value over \( [\gamma n, (1 - \gamma) n] \). To bound the maximal deviation, [Dar76] first analyzed the variance approximation of \( V(k) \) to bound the deviation for a single point \( k \). Then they used a Lipschitz property to partition \( [\gamma n, (1 - \gamma) n] \) to small intervals, and took a union bound over these intervals to yield a high probability guarantee. In contrast, we better leverage the connection between \( V(k) \) and \( V(k^*) \) for improved accuracy and a simplified proof. At a high level, we show that the expectation of \( V(k) \) is single-peaked around \( k^* \), and \( V(k) - V(k^*) \) is subgaussian. We carefully analyze the discrete derivative as a function of \( |k^* - k|, \gamma, \) and \( n \) to use a concentration bound yielding our constant error result.

Theorem 6. For data \( X = \{x_1, \ldots, x_n\} \) drawn according to the change-point model with any distributions \( P_0, P_1 \) with \( a = \Pr_{x \sim P_0, y \sim P_1}[x > y] > 1/2 \), constraint \( \gamma \in (0, 1/2) \), and change-point \( k^* \in \{\lceil \gamma n \rceil, \ldots, \lfloor (1 - \gamma) n \rfloor \} \), we have that the estimator
\[
\hat{k} = \arg\max_{k \in \{\lceil \gamma n \rceil, \ldots, \lfloor (1 - \gamma) n \rfloor \}} \frac{\sum_{j=k+1}^n \sum_{i=1}^k f(x_i > x_j)}{k(n-k)}
\]
is \((\alpha, \beta)\)-accurate for any \(\beta > 0\) and

\[
\alpha = C \cdot \left( \frac{1}{\gamma^4 (a - 1/2)^2} \right)^c \cdot \log \frac{1}{\beta}
\]

for any constant \(c > 1\) and some constant \(C > 0\) depending on \(c\).

If \(a < 1/2\) we achieve the same error bound using \(\hat{k} = \text{argmin} \sum_{i=k+1}^n \frac{k(n-k)}{k(n-k)} I(x_i > x_j)\).

**Proof.** We will show that for \(\hat{k} = \text{argmax} V(k)\) and \(\alpha\) as in the theorem statement,

\[
\Pr[|\hat{k} - k^*| > \alpha] \leq \sum_{k:|k-k^*| > \alpha} \Pr[V(k) > V(k^*)] \leq \beta.
\]

To do this, we fix any \(\pm \alpha\) by application of Theorem 5. First we give a lower bound the difference in expectation of \(V(k^*)\) and \(V(k)\). Observe that

\[
\mathbb{E}[V(k)] = \sum_{i<k, j>k} \Pr[x_i > x_j] \frac{k(n-k)}{k(n-k)} = \begin{cases} \frac{1}{k^*(n-k)} & k \leq k^* \\ \frac{ak^*+\frac{k^*}{k}}{k^*} & k > k^* \end{cases},
\]

achieving its maximum at \(\mathbb{E}[V(k^*)] = a\). Therefore, we can bound

\[
\mathbb{E}[V(k^*) - V(k)] = \begin{cases} (a - \frac{1}{2}) \frac{k^*-k}{n-k} & k \leq k^* \\ (a - \frac{1}{2}) \frac{k^*-k}{n-k} & k > k^* \end{cases} \geq (a - \frac{1}{2}) \frac{|k^*-k|}{n}.
\]

In the following bounds on the discrete derivative of \(f(X) = V(k) - V(k^*)\), we will make use of the fact that \(f\) can be written as:

\[
f(X) = \frac{\sum_{j=k+1}^n \sum_{i=1}^k I(x_i > x_j)}{k(n-k)} - \frac{\sum_{j=k^*+1}^n \sum_{i=1}^{k^*} I(x_i > x_j)}{k^*(n-k^*)} = \left( \frac{1}{k(n-k)} - \frac{1}{k^*(n-k^*)} \right) \left( \sum_{j=1}^{k^*} I(x_i > x_j) \right) + \frac{1}{k^*(n-k^*)} \left( \sum_{j=k+1}^n \sum_{i=1}^{k^*} I(x_i > x_j) - \sum_{j=k^*+1}^n I(x_i > x_j) \right)
\]

We bound the discrete derivative \(D_i f\) separately for \(i \leq \min \{k, k^*\}\), \(i \in (\min \{k, k^*\}, \max \{k, k^*\}]\), and \(i > \max \{k, k^*\}\). When \(x_i\) changes arbitrarily for \(i \leq \min \{k, k^*\}\), we note that \(\sum_{j=k+1}^n I(x_i > x_j)\) can change by at most \(\pm(n - k)\) and \(\sum_{j=k+1}^n I(x_i > x_j)\) can change by at most \(\pm(k^* - k)\). These counts are normalized in \(f\), and the normalization ensures this former count contributes at most \(\frac{|k^* - k|}{k^*(n-k^*)} + \frac{k^* - k}{kk^*}\) to
the discrete derivative. We bound the discrete derivative for \( i \leq \max \{k, k^*\} \) as follows:

\[
D_i f \leq \left| \frac{1}{k(n-k)} - \frac{1}{k^*(n-k^*)} \right| \cdot (n-k) + \frac{|k^* - k|}{k^*(n-k^*)}
\]

\[
= \frac{1}{k} \cdot \frac{n-k}{k^*} \cdot \frac{|k^* - k|}{k^*(n-k^*)}
\]

\[
= \frac{|k - k^*|}{k^*} + \frac{|k - k^*|}{k^*(n-k^*)} + \frac{|k - k^*|}{k^*(n-k^*)}
\]

\[
\leq \frac{|k - k^*|}{\gamma^2 n^2} + \frac{2 |k - k^*|}{\gamma(1 - \gamma)n^2}
\]

\[
\leq \frac{3 |k - k^*|}{\gamma^2 n^2}
\]

Finally we bound the discrete derivative for \( \min \{k, k^*\} < i \leq \max \{k, k^*\} \). To do this, we note that the first summation in \( f \) changes by \( k \) if \( k < k^* \) or \( n-k \) if \( k > k^* \), and the difference of summations in the second term changes by at most \( n-(k+k^*) \) in either case. Then we achieve our bound as follows:

\[
D_i f \leq \left| \frac{1}{k(n-k)} - \frac{1}{k^*(n-k^*)} \right| \cdot \max \{k, n-k\} + \frac{n-(k+k^*)}{k^*(n-k^*)}
\]

\[
\leq \frac{|k - k^*|}{\gamma^2 n^2} + \frac{n}{\gamma(1 - \gamma)n^2}
\]

\[
\leq \frac{2}{\gamma^2 n}
\]

Then since \( D_i f \) is finite for each \( i \), we have that \( f \) is subgaussian with variance proxy as follows:

\[
\frac{1}{4} \sum_{i=1}^{n} (D_i f)^2 \leq \frac{n}{4} \cdot \frac{9 |k - k^*|^2}{\gamma^4 n^4} + \frac{|k^* - k|}{\gamma^4 n^2} \left( \frac{|k - k^*|}{\gamma^2 n^2} + \frac{1}{\gamma(1 - \gamma)n} \right)^2
\]

\[
\leq \frac{9 |k - k^*|^2}{4\gamma^4 n^3} + \frac{|k^* - k|}{\gamma^4 n^2}
\]

\[
\leq \frac{13 |k^* - k|}{4\gamma^4 n^2}
\]

We can now bound the probability of outputting any particular \( k = \lfloor \gamma n \rfloor, \ldots, \lfloor (1 - \gamma)n \rfloor \) as a function
of $|k - k^*|$ by applying Theorem 5, recalling our bound on $\mathbb{E}[V(k^*) - V(k)]$ from Equation 3.

$$Pr[V(k) > V(k^*)] = Pr[V(k) - V(k^*) - \mathbb{E}[V(k) - V(k^*)] > \mathbb{E}[V(k^*) - V(k)]]$$

$$\leq Pr \left[ V(k) - V(k^*) - \mathbb{E}[V(k) - V(k^*)] > (a - \frac{1}{2}) \frac{|k - k^*|}{n} \right]$$

$$\leq \exp(-\frac{2\gamma^4}{13}(a - \frac{1}{2})^2|k - k^*|).$$

We complete the proof by bounding the probability of any incorrect $\hat{k}$ such that $|\hat{k} - k^*| > a$ by $\beta$.

$$Pr[|\hat{k} - k^*| > a] \leq 2 \sum_{|k-k^*|=\alpha}^{n} \exp(-\frac{2\gamma^4}{13}(a - \frac{1}{2})^2|k - k^*|)$$

$$\leq \frac{2\exp(-\frac{2\gamma^4}{13}(a - \frac{1}{2})^2a)}{1 - \exp(-\frac{2\gamma^4}{13}(a - \frac{1}{2})^2)}$$

$$\leq \beta$$

Rearranging shows that our accuracy result will hold for

$$\alpha \geq \frac{13}{2\gamma^4(a - 1/2)^2} \left( \log \frac{2}{\beta} + \log \frac{1}{1 - \exp(-\frac{2\gamma^4}{13}(a - \frac{1}{2})^2)} \right)$$

We achieve our final bound by simplifying the above expression as follows. We observe that $\gamma < 1/2$, $a < 1$ implies $x = 2\gamma^4(a - 1/2)^2/13 \leq 1/416$, and for small $x$ we have $\log(1/(1 - \exp(-x))) \leq 2 \log(1/x)$. For any $c > 0$, we have $\log(1/x) \leq C(1/x)^c$ for any $1/x \geq 416$ and $C \geq (\log 416)/(416^c)$, which can be applied to get our final bound.

\[ \square \]

### 3.2 Private offline algorithm

We now give a differentially private version of the nonparametric estimation procedure of \cite{Dar76}, in Algorithm 3. Our algorithm uses REPORTMAX as a private subroutine, instantiated with queries $V(k)$ to privately compute argmax $V(k)$. We show that our algorithm is differentially private (Theorem 5) and produces an estimator with additive accuracy that is constant with respect to the sample size $n$ (Theorem 4).

The crux of the privacy proof involves analyzing the sensitivity of the Mann-Whitney statistic to ensure that sufficient noise is added for the REPORTMAX algorithm to maintain its privacy guarantees. The low sensitivity of this test statistic plays a critical role in requiring only small amounts of noise to preserve privacy. The accuracy proof extends Theorem 5 for the non-private estimator to incorporate the additional error due to the Laplace noise added for privacy. Since the event $V(k) > V(k^*)$ is less probable for $k$ that are further away from $k^*$, our analysis permits larger values of Laplace noise $Z_k$ for $k$ far from $k^*$, allowing privacy "for free" with respect to accuracy, for small constants $\epsilon$.

**Algorithm 3** Private Nonparametric Change-Point Detector: PNCPD($X, \epsilon, \gamma$)

**Input:** Database $X = \{x_1, \ldots, x_n\}$, privacy parameter $\epsilon$, constraint parameter $\gamma$.

**for** $k = \{\lceil \gamma n \rceil, \ldots, \lfloor (1 - \gamma) n \rfloor \}$ **do**

Compute $V(k) = \frac{1}{k(n-k)} \sum_{j=k+1}^{n} \sum_{i=1}^{k} I(x_i > x_j)$

Sample $Z_k \sim \text{Lap}(\frac{\epsilon}{\gamma n})$

**end for**

Output $\hat{k} = \arg\max_{k \in \{\lceil \gamma n \rceil, \ldots, \lfloor (1 - \gamma) n \rfloor \}} \{V(k) + Z_k\}$

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Theorem 7. For arbitrary data \( X = \{ x_1, \ldots, x_n \} \), privacy parameter \( \epsilon > 0 \), and constraint \( \gamma \in (0, 1/2) \), PNCPD(\( X, \epsilon, \gamma \)) is \( \epsilon \)-differentially private.

Proof. Privacy follows by instantiation of REPORTMAX with queries \( V(k) \) for \( k \in \{ [\gamma n], \ldots, [(1-\gamma)n] \} \), which have sensitivity \( \Delta(V) = 1/(\gamma n) \), with the observation that noise parameter \( 2\Delta(V)/\epsilon \) suffices for non-monotonic statistics. We include a proof for completeness.

Fix any two neighboring databases \( X, X' \) that differ on index \( t \). For any \( k \in \{ [\gamma n], \ldots, [(1-\gamma)n] \} \), denote the respective rank statistics as \( V(k) \) and \( V'(k) \). By the definition of \( V(k) \), we have

\[
|V(k) - V'(k)| = \begin{cases} \frac{1}{\log(1-\epsilon)} \sum_{j=k+1}^{n} (I(x_{j} > x_{j'}) - I(x_{j'} > x_{j})) & \text{if } t \leq k \\ \frac{1}{\log(1-\epsilon)} \sum_{t=1}^{k} (I(x_{i} > x_{i'}) - I(x_{i'} > x_{i})) & \text{if } t > k, \end{cases}
\]

and it follows that \( \Delta(V) = 1/(\gamma n) \).

Next, for a given \( 1 \leq t \leq n \), fix \( Z_{-t} \), a draw from \( \text{Lap}(2/(\gamma \epsilon n)) \) used for all the noisy rank statistics values except the \( t \)th one. We will bound from above and below the ratio of the probabilities that the algorithm outputs \( k = t \) on inputs \( X \) and \( X' \). Define the minimum noisy value in order for \( t \) to be selected with \( X \):

\[
Z_{t}^{*} = \min\{Z_{t} : V(t) + Z_{t} > V(k) + Z_{k}, \forall k \neq t \}
\]

For all \( k \neq t \) we have

\[
V'(t) + \Delta(V) + Z_{t}^{*} \geq V(t) + Z_{t}^{*} > V(k) + Z_{k} \geq V'(k) - \Delta(V) + Z_{k}.
\]

Hence, \( Z_{t}^{*} \geq Z_{t}^{*} + 2\Delta(V) \) ensures that the algorithm outputs \( t \) on input \( X' \), and the theorem follows from the following inequalities for any fixed \( Z_{-t} \), with probabilities over the choice of \( Z_{t} \sim \text{Lap}(2/(\gamma \epsilon n)) \).

\[
\Pr[k = t] = \Pr[Z_{t}^{*} \geq Z_{t}^{*} + 2\Delta(V) | Z_{-t}] \geq \epsilon^{-\epsilon} \Pr[Z_{t} \geq Z_{t}^{*} | Z_{-t}] = \epsilon^{-\epsilon} \Pr[k = t | X, Z_{-t}]
\]

Next we provide an accuracy guarantee for our private algorithm PNCPD when the data are drawn according to the change-point model. The first term in the error bound of Theorem comes from the randomness of the \( n \) data points, and the second term is the additional cost that comes from the randomness of the sampled Laplace noises, which quantifies the cost of privacy. To ensure that the cost of privacy is as small as possible, we use \( k \)-specific thresholds \( t_k \) in the proof to optimize the trade-off between how much to tolerate the Laplace noise added for privacy versus the randomness of the data. As \( |k - k^{*}| \) increases, \( V(k) \) is less likely to be close to \( V(k^{*}) \), so we can allow more Laplace noise rather than set a universal tolerance for all \( k \).

Theorem 8. For data \( X = \{ x_1, \ldots, x_n \} \) drawn according to the change-point model with any distributions \( P_0, P_1 \) with \( a = \Pr_{x \sim P_0, y \sim P_1} [x > y] > 1/2 \), constraint \( \gamma \in (0, 1/2) \), change-point \( k^{*} \in \{ [\gamma n], \ldots, [(1-\gamma)n] \} \), and privacy parameter \( \epsilon > 0 \), we have that PNCPD(\( X, \epsilon, \gamma \)) is \( (\alpha, \beta) \)-accurate for any \( \beta > 0 \) and

\[
\alpha = \max\{C_1 \cdot \frac{1}{\gamma^4(a-1/2)^2} \cdot \log \frac{1}{\beta}, C_2 \cdot \frac{1}{(\gamma^2(a-1/2)^2)} \cdot \log \frac{1}{\beta} \},
\]

for any constant \( c > 1 \) and some constants \( C_1, C_2 > 0 \) depending on \( c \).

As with our analysis of the non-private estimator, we can take the argmin and get the same error bounds (with \( a - 1/2 \) replaced by \( |a - 1/2| \) if \( \Pr_{x \sim P_0, y \sim P_1} [x > y] < 1/2 \).  

Proof. We will show that \( \hat{k} = \arg \max \{ V(k) + Z_k \} \) and \( \alpha \) as in the theorem statement,

\[
\Pr[|\hat{k} - k^{*}| > \alpha] \leq \sum_{k:|k-k^{*}| > \alpha} \Pr[V(k) + Z_k > V(k^{*}) + Z_{k^{*}}] \leq \beta
\]
by showing that $V(k) - V(k^*)$ is subgaussian as in Theorem 6 and we will additionally show that the Laplace noise does not introduce too much additional error. For the algorithm to output an incorrect $\hat{k}$, it must either be the case that the statistic $V(k)$ is nearly as large as $V(k^*)$ because of the randomness of the data points, or that $Z_k$ is much larger than $Z_{k^*}$. For each value of $k$, we choose a threshold $t_k$ increasing in $|k - k^*|$ specifying how much to tolerate bad Laplace noise versus bad data, and we bound the probability that the algorithm outputs $k$ as follows:

$$\Pr[V(k) + Z_k > V(k^*) + Z_{k^*}] \leq \Pr[V(k^*) - V(k) < t_k] + \Pr[Z_k - Z_{k^*} > t_k] \quad (4)$$

Setting $t_k = (a - 1/2)|k - k^*|/(2n)$, we can bound the first term as in Theorem 4 using Theorem 5 as follows:

$$\Pr[V(k) - V(k^*) > -t_k] = \Pr \left[ V(k) - V(k^*) - \mathbb{E}[V(k) - V(k^*)] > \left( a - \frac{1}{2} \right) \frac{|k - k^*|}{2n} \right]$$

$$\leq \exp \left( -\frac{\gamma^4 (a - \frac{1}{2})^2 |k - k^*|}{26} \right).$$

We bound the second term of (4) by analyzing the Laplace noise directly.

$$\Pr[Z_k - Z_{k^*} > t_k] \leq \Pr \left[ 2 |\text{Lap}(2/(c\gamma n))| > \left( a - \frac{1}{2} \right) \frac{|k - k^*|}{2n} \right]$$

$$\leq \exp \left( -\frac{(a - \frac{1}{2}) c \gamma |k - k^*|}{8} \right).$$

We complete the proof by bounding the probability of any incorrect $\hat{k}$ such that $|\hat{k} - k^*| > \alpha$ by $\beta$.

$$\Pr \left[ |\hat{k} - k^*| > \alpha \right] \leq 2 \sum_{k:|k - k^*| = \alpha} \exp \left( -\frac{\gamma^4 (a - \frac{1}{2})^2 |k - k^*|}{26} \right) + \exp \left( -\frac{(a - \frac{1}{2}) c \gamma |k - k^*|}{8} \right)$$

$$\leq \frac{2 \exp \left( -\frac{\gamma^4}{26} (a - \frac{1}{2})^2 \alpha \right)}{1 - \exp \left( -\frac{\gamma^4}{26} (a - \frac{1}{2})^2 \right)} + \frac{2 \exp \left( -\frac{c \gamma}{8} (a - \frac{1}{2}) \alpha \right)}{1 - \exp \left( -\frac{c \gamma}{8} (a - \frac{1}{2}) \right)}$$

$$\leq \beta.$$

We bound each term above by $\beta/2$. Rearranging shows that our accuracy result will hold for

$$\alpha \geq \max \left\{ \frac{26}{\gamma^4 (a - 1/2)^2} \left( \log \frac{4}{\beta} + \log \frac{1}{1 - \exp \left( -\frac{\gamma^4}{26} (a - \frac{1}{2})^2 \right)} \right), \right.$$

$$\frac{8}{c \gamma (a - 1/2)} \left( \log \frac{4}{\beta} + \log \frac{1}{1 - \exp \left( -\frac{c \gamma}{8} (a - \frac{1}{2}) \right)} \right) \right\}. $$

We achieve our final bound by simplifying the above expression as follows. For the first term, we observe that $\gamma < 1/2, a < 1$ implies $x = \gamma^4 (a - 1/2)^2/26 \leq 1/1664$, and for small $x$ we have $\log(1/(1 - \exp(-x))) \leq 2 \log(1/x)$. For any $c > 0$, we have $\log(1/x) \leq C(1/x)^c$ for any $1/x \geq 1664$ and $C \geq (log 1664)/(1664^c)$, which can be applied to get our final bound. For the second term, we observe that $x = c \gamma (a - 1/2)/8 \leq \epsilon/32$. When $\epsilon$ is small and the corresponding $x \leq 4/5$, we have $\log(1/(1 - \exp(-x))) \leq 2 \log(1/x)$, and for any $c > 0$, we have $\log(1/x) \leq C(1/x)^c$ for any $1/x \geq 5/4$ and $C \geq (log 4/5)/(4/5)^c$. When $\epsilon$ is large and the corresponding $x > 4/5$, we have $\log(1/(1 - \exp(-x))) \leq \log 2$, which can be incorporated into the constant in our final bound.

$\square$
4 Online change point detection

In this section, we show how to extend our results for change-point detection with unknown distributions to the online setting, where the database $X$ is not given in advance, but instead data points arrive one-by-one. We assume the analyst starts with a database of size $n$, and receives one new data point per unit time.

Our algorithm uses the Above Noisy Threshold algorithm of [DNR+09, DNPR10] (AboveThreshold, Algorithm 2) instantiated with queries of the Mann-Whitney test statistic $V(k)$ in the center of a sliding window of the most recent $n$ points. With each new data point $k > n$, the algorithm computes $V(k)$ for database $X = \{x_{k-n/2+1}, \ldots, x_{k+n/2}\}$, and compares this statistic against a noisy threshold for significance. When this statistic is sufficiently high, the online algorithm calls the offline algorithm PNCPD on this window to estimate $k^*$. For simplicity in indexing and to avoid confusion with the notation of the previous section, we define $U(k) = V(k)$ when $V(k)$ is taken over database $X$ for each $k > n/2$. Since the algorithm only checks for a change-point in the middle of the window, we assume that $k^* \geq n/2$ to ensure that the change-point does not occur too early to be detected.

We note that the offline subroutine PNCPD assumes that a change point occurs sometime after the first $\gamma n$ and before the last $\gamma n$ of the $n$ data points on which it is called. We will show that for an appropriate choice of $T$, OnlinePNCPD exceeds $T$ for some $k$ such that $k^* \in [k, k+n/2]$. Therefore, by waiting for an additional $\gamma n$ data points, we ensure that the assumptions of PNCPD are met as long as $\gamma < 1/4$.

\begin{algorithm}
\begin{algorithmic}
\State \textbf{Input:} Data stream $X$, starting size $n$, privacy parameter $\epsilon$, constraint parameter $\gamma$, threshold $T$.
\State Let $\hat{T} = T + \text{Lap}\left(\frac{8}{\epsilon n}\right)$.
\For {each new data point $x_{k+n/2}, k > n/2$} 
\State Compute $U(k) = \frac{1}{n^2} \sum_{j=k+1}^{k+n/2} \sum_{i=k-n/2+1}^{k} I(x_i > x_j)$
\State Sample $Z_k \sim \text{Lap}\left(\frac{16}{\epsilon n}\right)$
\If {$U(k) + Z_k > \hat{T}$} 
\State Wait for $\gamma n$ new data points to arrive
\State Output PNCPD ($\{x_{k+n/2+1+\gamma n}, \ldots, x_{k+n/2+2+\gamma n}\}, \epsilon/2, \gamma$)
\State Halt
\EndIf
\EndFor
\end{algorithmic}
\caption{Online Private Nonparametric Change-Point Detector: OnlinePNCPD($X, n, \epsilon, \gamma, T$)}
\end{algorithm}

Privacy follows immediately from the privacy guarantees of AboveThreshold and PNCPD.

**Theorem 9.** For arbitrary data stream $X$ with starting size $n$, privacy parameter $\epsilon > 0$, and constraint $\gamma \in (0, 1/2)$, OnlinePNCPD($X, n, \epsilon, \gamma$) is $\epsilon$-differentially private.

**Proof.** By Theorem 8, AboveThreshold is $\epsilon$-differentially private, and by Theorem 7, the statistics $V(k)$ and $U(k)$ have sensitivity $2/n$. Also by Theorem 4 PNCPD is $\epsilon$-differentially private. Thus the algorithm OnlinePNCPD is simply AboveThreshold instantiated with privacy parameter $\epsilon/2$, composed with PNCPD also instantiated with privacy parameter $\epsilon/2$. By Basic Composition (Theorem 4), OnlinePNCPD($X, n, \epsilon, \gamma$) is $\epsilon$-differentially private.

To give accuracy bounds on the performance of OnlinePNCPD, we need to bound several sources of error. First we need to set the threshold $T$ such that the algorithm will not raise a false alarm before the change-point occurs (i.e., control the false positive rate) and that the algorithm will not fail to raise an alarm on a window containing the true change-point (i.e., control the false negative rate). This must be done taking into account the additional error from the private AboveThreshold subroutine. Finally, we can use the accuracy guarantees of PNCPD to show that conditioned on calling a window that contains the true change-point, we are likely to output an estimator $\hat{k}$ that is close to the true change-point $k^*$. 

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Theorem 10. For data stream $X$ with starting size $n$ drawn according to the change-point model with any distributions $P_0, P_1$ with $\alpha = \Pr_{x \sim P_0, y \sim P_1}[x > y] > 1/2$, constraint $\gamma \in (0, 1/4)$, change-point $k^* \geq n/2$, privacy parameter $\epsilon > 0$, and threshold $T \in [T_L, T_U]$ such that

$$T_L = \frac{1}{2} + \sqrt{\frac{2}{n} \log \left( \frac{8(k^* - n/2)}{\beta} \right)} + \frac{32 \log((k^* - n/2)/\beta)}{n \epsilon}$$

$$T_U = a - \sqrt{\frac{2}{n} \log \left( \frac{8}{\beta} \right)} - \frac{32 \log(8(k^* - n/2)/\beta)}{n \epsilon},$$

we have that ONLINEPNCPD$(X, n, \epsilon, \gamma, T)$ is $(\alpha, \beta)$-accurate for any $\beta > 0$ and

$$\alpha = \max \left\{ C_1 \cdot \left( \frac{1}{\gamma^4(a-1/2)^2} \right) \cdot \log \frac{n}{\beta}, C_2 \cdot \left( \frac{1}{c \gamma(a-1/2)} \right) \cdot \log \frac{n}{\beta} \right\},$$

for any constant $c > 1$ and some constants $C_1, C_2 > 0$ which depend only on $c$.

Proof. First, we find an interval $[T_L, T_U]$ for the threshold $T$ that ensures that the algorithm neither calls PNCPD before the true change-point has occurred nor fails to call PNCPD on the window containing $k^*$ somewhere in the middle $(1 - 2\gamma)n$ data points. For now we will ignore the error from ABOVETHRESHOLD, and use $T_L, T_U$ to denote the desired thresholds ignoring this additional source of noise. For ease of notation and reindexing, we define $U(k) = V(k)$ when $V(k)$ is computed over database $X = \{ x_{k-n/2+1}, \ldots x_{k+n/2} \}$ for the Mann-Whitney test statistic $V(\cdot)$ as defined in Equation (1).

Thus we aim to find a range $[T_L', T_U']$ such that

$$\Pr[U(k) > T_L'|X_{k-n/2+1}, \ldots x_{k+n/2} \sim P_0] \leq \frac{\beta}{8(k^* - n/2)},$$

$$\Pr[U(k) < T_U'|X_{k-n/2+1}, \ldots x_{k+n/2} \sim P_0, x_{k+1}, \ldots x_{k+n/2} \sim P_1] \leq \frac{\beta}{8}. \quad \text{(5)}$$

Condition (5) means that after taking a union bound over all the windows that do not contain $k^*$, the probability that ABOVETHRESHOLD raises the alarm on the window that does not contain the true change point $k^*$ does not exceed $\beta/8$. Condition (6) means that on the window containing the true change-point $k^*$ in the center of the window, ABOVETHRESHOLD will fail to raise the alarm with probability at most $\beta/8$.

It will be helpful to have high probability bounds that the test statistics $U(k)$ are close to their means. Using McDiarmid’s Inequality (Theorem 5) we can obtain that for any $k > n$

$$\Pr[U(k) - \mathbb{E}[U(k)] > t] \leq \exp(-t^2 n/2), \quad \text{(7)}$$

$$\Pr[U(k) - \mathbb{E}[U(k)] < -t] \leq \exp(-t^2 n/2). \quad \text{(8)}$$

Using these bounds, we will first find $T_L'$. Note that Condition (5) on $T_L'$ considers the setting where all points in the current window are drawn from $P_0$. Under this condition, $\mathbb{E}[U(k)] = 1/2$. Then by plugging in $t = T_L' - 1/2$ into Inequality (7), we get the following expression:

$$\Pr \left[ U(k) \geq T_L'|X_{k-n/2+1}, \ldots x_{k+n/2} \sim P_0 \right] \leq \exp \left( -\frac{n}{2} \left( T_L' - \frac{1}{2} \right)^2 \right)$$

Setting the right hand side of this to less than or equal to $\frac{\beta}{8(k^* - n/2)}$ and solving for $T_L'$ gives the following lower bound, which satisfies Condition (5):

$$T_L' = \frac{1}{2} + \sqrt{\frac{2}{n} \log \left( \frac{8(k^* - n/2)}{\beta} \right)}.$$
Next we find the upper bound \( T_U \). Note that Condition (6) on \( T_U' \) considers the setting where the first \( n/2 \) points in the window are drawn from \( P_0 \) and the remaining \( n/2 \) points are drawn from \( P_1 \). Under this condition, \( \mathbb{E}[U(k)] = a \). Then plugging \( t = a - T_U' \) in Inequality (8) and using Condition (6), we get the following bound:

\[
\Pr[U(k) \leq T_U'|X, \ldots X_{n/2} \sim P_0, X_{n/2+1}, \ldots X_n \sim P_1] \leq \exp\left(-\frac{(a - T_U')^2 n/2}{2}\right) \leq \frac{\beta}{8}.
\]

Solving this for \( T_U' \) gives the following Inequality which satisfies Condition (6):

\[
T_U' \leq a - \sqrt{\frac{2}{n} \log \left(\frac{8}{\beta}\right)}.
\]

We now return to account for the error from ABOVE_THRESHOLD. To ensure that this error does not cause a window to be called before the true change-point and also does not skip the window with the true change-point, we require the following conditions to both hold with probability \( \frac{1}{4} \):

- For \( T \geq T_L \), \( U_k < T - \alpha' \) when \( k < k^* \)
- For \( T \leq T_U \), \( U_k > T + \alpha' \)

Thus we obtain that the new interval for \( T \) is \([T_L, T_U] \), where \( T_L = T_L' + \alpha' \), and \( T_U = T_U' - \alpha' \). If both those conditions hold then for \( \alpha' = \frac{32 \log(8(k^* - n/2)/\beta)}{n \epsilon \gamma} \), ABOVE_THRESHOLD will identify the window which contains the true change point with probability \( 1 - \beta/4 \) by Theorem 4. Taking a union bound over the failure probabilities of Conditions (5) and (6), and the statement above, we can see that ONLINE_PNCPD will call PNCPD on the right window except with small probability \( \beta/2 \).

Finally, we can use the accuracy guarantees of PNCPD to show that conditioned on raising an alarm in the correct window, we are likely to output an estimate \( \hat{k} \) that is close to the true change-point \( k^* \). Slightly more careful accounting is needed here, because conditioning on raising an alarm and calling PNCPD, the data points in the chosen window are no longer distributed according to the change-point model. Let \( W(k) \) denote the event that ONLINE_PNCPD calls PNCPD(\([x_{k-n/2+1+\gamma n}, \ldots x_{k+n/2+\gamma n}] \), \( \epsilon/2, \gamma \)) on the window centered at \( k \). Then

\[
\Pr[|\hat{k} - k^*| > \alpha] = \sum_{k > n/2} \Pr[W(k) \cap \{|\hat{k} - k^*| > \alpha\}] \\
\leq \sum_{k \notin (k^* - n/2, k^*)} \Pr[W(k)] + \sum_{k \in (k^* - n/2, k^*)} \Pr[W(k) \cap \{|\hat{k} - k^*| > \alpha\}] \\
\leq \frac{\beta}{2} + \frac{n}{2} \Pr[\text{PNCPD fails}] < \beta
\]

To achieve the inequality above, the probability of PNCPD fails to report the change point within the \( \alpha \)-window around \( k^* \) has to be bounded by \( \beta/n \). Thus by Theorem 8 we set the error to be,

\[
\alpha = \max \left\{ C_1 \cdot \left(\frac{1}{\gamma^4 (a - 1/2)^2}\right)^c \cdot \log \frac{n}{\beta}, C_2 \cdot \left(\frac{1}{\epsilon \gamma (a - 1/2)}\right)^c \cdot \log \frac{n}{\beta} \right\},
\]

for any constant \( c > 1 \) and some constant \( C_1, C_2 > 0 \) depending on \( c \). \( \Box \)

We have proved the theorem, but we should also show that the window \([T_L, T_U] \) is non-empty, and there exists a good range in which to choose the threshold \( T \). The condition that \( T_L < T_U \) is equivalent to,

\[
a - \frac{1}{2} > \sqrt{\frac{2}{n} \log \left(\frac{8(k^* - n/2)}{\beta}\right)} + \sqrt{\frac{2}{n} \log \left(\frac{8}{\beta}\right)} + \frac{64 \log(8(k^* - n/2)/\beta)}{n \epsilon}.
\]
We can simplify Inequality (9) as,
\[
\sqrt{\frac{2}{n} \log \left( \frac{8(k^* - n/2)}{\beta} \right)} + \sqrt{\frac{2}{n} \log \left( \frac{8}{\beta} \right)} + \frac{64 \log(8(k^* - n/2)/\beta)}{ne} < a - \frac{1}{2}.
\]

Finally, solving the right hand side for \( n \), we find the following bound on \( n \) that satisfies Inequality (9).

\[
n > \frac{1}{(a - 1/2)^2} \left( \sqrt{2 \log \left( \frac{8k^*}{\beta} \right)} + \sqrt{2 \log \left( \frac{8}{\beta} \right)} + \frac{64}{e} \log \left( \frac{8k^*}{\beta} \right) \right)^2.
\]

For any starting database size that is at least this large (only \( n = \Omega((\log(k^*/\beta))^2) \)), the acceptable region \([T_L, T_U]\) for a threshold \( T \) will be non-empty. Moreover, the \( \log k^* \) dependence of \( T_L \) and \( T_U \) means that only a rough estimate of the true change-point is necessary in practice to choose an acceptable threshold \( T \).

## 5 Application: Drift Change Detection

In this section, we extend our consideration of the change-point problem to the setting where data are not sampled i.i.d. from fixed pre- and post-change distributions, but instead are sampled from distributions that are changing smoothly over time. In particular, we consider distributions with drift, where the parameter of the distribution changes linearly with time, and the rate of linear drift changes at the change-point. Since the samples are not i.i.d., we consider differences between successive pairs of samples in order to apply the algorithms from the previous sections.

The drift change detection problem is parametrized by error terms \( e_t \) independently sampled from a mean-zero distribution \( S \), two drift terms \( \xi_0 \) and \( \xi_1 \), a drift change-point \( t^* \in [n] \), and a mean \( \eta \) associated with \( t^* \). Independent random variables \( X = \{x_1, \ldots, x_n\} \) are said to be drawn from the drift change detection model if we can write

\[
x_t = \mu_t + e_t,
\]

for \( \mu_t \) piecewise linear as follows:

\[
\mu_t = \begin{cases} 
\eta - (t^* - t)\xi_0 & t \leq t^* \\
\eta + (t - t^*)\xi_1 & t > t^* 
\end{cases}
\]

Our goal is to detect the drift change-point \( t^* \) with the smallest possible error.

In order to apply our algorithms which require i.i.d. samples, we will transform the sample \( X \) by considering differences of consecutive pairs of \( x_i \). These differences are i.i.d. with mean \( \xi_0 \) before \( t^* \), and i.i.d. with mean \( \xi_1 \) after \( t^* \), and we can now apply PNCPD to this instance of change-point detection. For ease of presentation, we will assume \( n \) is even and \( t^* \) is odd.

Formally, define a new sample \( Y = \{y_1, \ldots, y_{n/2}\} \) with sample points \( y_t = x_{2t} - x_{2t-1} \), for \( t = 1, \ldots, n/2 \). Then we have

\[
y_t = \begin{cases} 
\xi_0 + e_{2t} - e_{2t-1}, & \text{for } t = 1, \ldots, \frac{t^* - 1}{2}, \\
\xi_1 + e_{2t} - e_{2t-1}, & \text{for } t = \frac{t^* + 1}{2}, \ldots, \frac{n}{2}.
\end{cases}
\]

Note that random variables \( (e_{2t} - e_{2t-1}) \) are independent and identically distributed. Thus the \( y_t \) are independent, and they are sampled from a fixed distribution before the change point, and from another distribution after the change-point. We can then apply the PNCPD algorithm and privately estimate the
drift change-point $\hat{t}$ as twice the output of PNCPD($\{y_1, \ldots, y_{n/2}\}, \epsilon, \gamma$). This estimation procedure will inherit the privacy and accuracy results of Theorems 7 and 8.

As a concrete example, consider points sampled from a Gaussian distribution with mean $\mu_t = \xi_0 t + \eta_0$ and standard deviation $\sigma$ for $t \leq t^*$, and from a Gaussian distribution with mean $\mu_t = \xi_1 t + \eta_1$ and standard deviation $\sigma$ for $t > t^*$. Then $y_t = x_{2t} - x_{2t-1}$ will be Gaussian with variance $2\sigma^2$ and mean $\xi_0$ before the change-point and $\xi_1$ after it. If any of the parameters $\xi_0, \xi_1$, or $\sigma$ are unknown, this would require nonparametric change-point estimation.

**Corollary 11.** For data $X = \{x_1, \ldots, x_n\}$ drawn according to the drift change model with drift terms $\xi_0 > \xi_1$, constraint $\gamma \in (0,1/2)$, drift change time $t^* \in \left(\frac{n}{2}, \ldots, \frac{(1 - \frac{1}{2})n}{2}\right)$, and privacy parameter $\epsilon > 0$, there exists an $\epsilon$-differentially private nonparametric change point estimator that is $(\alpha, \beta)$-accurate for any $\beta > 0$ and

\[
\alpha = \max \left\{ C_1 \cdot \left( \frac{1}{\gamma^4(a - 1/2)^2} \right)^c \cdot \log \frac{1}{\beta}, C_2 \cdot \left( \frac{1}{\epsilon \gamma(a - 1/2)} \right)^c \cdot \log \frac{1}{\beta} \right\},
\]

for any constant $c > 1$ and some constant $C_1, C_2 > 0$ depending on $c$.

We note that this approach is not restricted solely to offline linear drift detection. The same reduction in the online setting would allow us to use ONLINE-PNCPD to detect drift changes online. Additionally, a similar approach could be used to find other types of smoothly changing data, as long as the smooth changes exhibited enough structure to allow for reduction to the i.i.d. setting. For example, if data were sampled of the form $x_t = f(\mu_t + \epsilon_t)$ for any one-to-one function $f : \mathbb{R} \rightarrow \mathbb{R}$, we could define $y_t = f^{-1}(x_{2t}) - f^{-1}(x_{2t-1})$, and these $y_t$s would again be i.i.d.. This includes random variables of the form $\exp(\mu_t + \epsilon_t)$, $\log(\mu_t + \epsilon_t)$, and arbitrary polynomials $(\mu_t + \epsilon_t)^k$ (where even-degree polynomials must be restricted to, e.g., only have positive range).

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