Abelian functional equations, planar web geometry and polylogarithms

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Abstract: In this paper we study abelian functional equations (Afe), which are equations in the $F_i$’s of the type $F_1(U_1) + \cdots + F_N(U_N) = 0$. Here we restrict ourselves to the cases when the $U_i$’s are rational functions in two variables. First we prove that local measurable solutions actually are analytic and their components are characterized as solutions to linear differential equations constructed from the $U_i$’s. Then we propose two “methods” for solving (Afe). Next we apply these methods to the explicit resolution of generalized versions of classical (inhomogeneous) Afe satisfied by low order polylogarithms. Interpreted in the framework of web geometry, these results give us new non linearizable maximal rank planar webs (confirming some results announced by G. Robert about one year ago). Then we observe that there is a relation between these webs and certain configurations of points in $\mathbb{CP}^2$, which leads us to define the notion of “web associated to a configuration”: all these webs seems to be of maximal rank. Finally, we apply the preceding results to the problem of characterizing the dilogarithm and the trilogarithm by the classical functional equation they respectively satisfy. In particular, we show that, under weak regularity assumptions, the trilogarithm is the only function which verifies the Spence-Kummer equation.

1 Introduction and notations

1.1 Introduction

In this paper, we undertake a general study of the general solutions $(F_1, \ldots, F_N)$ of functional equations of the form

$$F_1(U_1(x, y)) + F_2(U_2(x, y)) + \cdots + F_N(U_N(x, y)) = 0 \quad (E)$$

where the $U_i$’s are real rational functions. We will call then “abelian functional equations” with real rational inner functions. Such equations have appeared in mathematics a long time ago: the equations

$$L(x + y) = L(x) + L(y) \quad x, y \in \mathbb{R} \quad (C)$$
$$L(xy) = L(x) + L(y) \quad x, y > 0 \quad (C')$$

are respectively satisfied by any linear function and by the logarithm. From a historical point of view, equation (C) is closely related to the definition of the logarithm itself and goes back to the 17th century.

From the early 19th century onwards, many mathematicians have gradually discovered a particular class of special functions, the polylogarithms, which verify some (inhomogeneous) functional equations of the type (E) (see [Lew]). Spence, Abel, Kummer (and others...) have established numerous versions of the following functional equation verified by the bilogarithm $L_{i2}$ for $0 < x < y < 1$:

$$L(x) - L(y) - L(\frac{x}{y}) - L(\frac{1-y}{1-x}) + L(\frac{x(1-y)}{y(1-x)}) = -\frac{\pi^2}{6} + \log(y) \log(\frac{1-y}{1-x}) \quad (L_2)$$

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*This is a preliminary version. Any comments or remarks will be welcome.
(this is Schaffer’s form, see [Scha]). Spence and (mostly) Kummer have discovered many functional equations satisfied by polylogarithms of order less than 5, such as $L_i$, which verifies the following “Spence-Kummer equation”, for $0 < x < y < 1$,

$$2L(x) + 2L(y) - L\left(\frac{x}{y}\right) + 2L\left(\frac{1-x}{1-y}\right) + 2L\left(\frac{x(1-y)}{y(1-x)}\right) - L(xy)$$

$$+ 2L\left(-\frac{x(1-y)}{(1-x)}\right) + 2L\left(-\frac{(1-y)}{y(1-x)}\right) - L\left(x(1-y)^2\right)$$

$$= 2L_i(1) - \log(y)^2 \log\left(\frac{1-x}{1-y}\right) + \frac{\pi^2}{3} \log(y) + \frac{1}{3} \log(y)^3$$

(we note $R_3(x/y)$ the right hand side of this equation).

The bilogarithm and some of its “cousins”, such as the Rogers dilogarithm or the single-valued Bloch-Wigner dilogarithm, are special functions which have appeared in various branches of mathematics from the 1830’s onwards. Here are a few examples: in 1836, a result by Lobachevsky expressed the volume of an ideal geodesic simplex in the 3-dimensional hyperbolic space $\mathbb{H}^3$ with vertices in $\partial \mathbb{H}^3$ through the dilogarithm. In 1935, G. Bol obtained the first example of a non-linearizable maximal rank planar 5-web by considering the web associated to a (homogeneous) version of the equation ($L_2$) of the bilogarithm (see part 4.1)

After a long period of neglect, for thirty years there has been an explosion of the occurrences of polylogarithmic functions in many areas of mathematics (see, for instance, [Gon1], [Ost], [Pol], [Za1], [Za2], .. ). Some mathematicians have generalized the construction of the Bloch-Wigner dilogarithm to polylogarithms of any order: they have constructed real univalued versions $\mathcal{L}_n$ of $\text{Li}_n$, defined and continuous on the whole $\mathbb{C}P^1$. A theorem due to Osterl´ e, Wojtkowiak and Zagier (Théorème 2 in [Ost]) says that the $\mathcal{L}_n$’s verify “clean” versions of the functional equations satisfied by the classical polylogarithms: if for $\text{Li}_n$ we have an equation of the form

$$\sum_{k=1}^N a_k \text{Li}_n(U_i) = \text{elem}_n$$

with $a_k \in \mathbb{C}, U_k \in \mathbb{R}(x,y)$, and $\text{elem}_n$ denoting a complex polynomial in some functions of the form $\text{Li}_j \circ g_k$ with $1 \leq j_k < n$ and $g_k \in \mathbb{R}(x,y)$, then $\sum_{k=1}^N a_k (\mathcal{L}_n \circ U_i)$ is constant. This shows that the theory of the functional equations of the polylogarithms can be considered a particular case of the general study undertaken here.

Determining continuous functions $L$ (resp. $\text{Li}$) satisfying (C) (or in an equivalent way (C)) was important for early 19th century mathematicians: it allowed them to justify in a rigorous manner the summation of “Newton’s binomial series”: $(1 + x)^n = 1 + \alpha x + \alpha(\alpha - 1)x^2/2 + \ldots$ (with $\alpha \in \mathbb{R}$ and $|x| < 1$). Cauchy was the first to rigorously determine continuous solutions for these equations (now known as “Cauchy equations”). It was an application of the formalism that he had introduced into analysis (see section 21.5 of [Acz-Dh]). The problem of characterizing the solutions of homogeneous versions of the equations satisfied by polylogarithms is an interesting one. Few results have been obtained in this direction (see part 4.2 and the conjecture (1.6) in [Gan]) although it could be an useful way to retrieve certain results: see, for instance, the remark 4.1.2 in [Ge-McPh] or the proof of Lobachevsky’s result stated above which is sketched in [Gon] (page 7).

In this paper we study local solutions ($F_1, \ldots F_N$) of the general equation ($E$) at $\omega \in \mathbb{R}^2$ and, in the spirit of the second part of Hilbert’s 5th problem (see [Acz]), we want to make minimal assumptions of regularity on the $F_i$’s so as to have “nice properties” for these solutions: measurability will appear natural (see 2.1). Under this assumption, we first prove (in proposition 1 of part 2.2.1) that any local solution of ($E$) is in fact analytic (modulo a condition of genericity on $\omega$): this allows us to complexify the problem and to restrict ourselves to the study of local holomorphic solutions of the complex version of ($E$). As already noticed by Abel, one functional equation in several variables can determine several unknown functions which must be very specific. In our case, this “philosophy” works very well and gives
us the

**Theorem A** Let be \( \mathbb{R} = (U_i) \in \mathbb{R}(x,y)^N \) such that \( W_{\mathbb{R}} \) is a web (i.e, the singular locus \( \Sigma_{\mathbb{R}} \subset \mathbb{C}P^2 \) of \( \mathbb{R} \) is proper, see 1.2 for definitions). Let be \( \omega \in \mathbb{R}^2 \setminus \Sigma_{\mathbb{R}} \) fixed. Then for each \( i \in \{1, ..., N\} \) there exists a linear differential equation \( (Lde_i) \), the coefficients of which are algebraic functions such that if \( F_1, ..., F_N \) are measurable germs satisfying the equation \( F_1(U_1) + .. + F_N(U_N) = 0 \) in a neighbourhood of \( \omega \), then every \( F_i \) is analytic and generically satisfies the equation \( (Lde_i) \). The germ \( F_i \) admits analytic continuation along any path in the Zariski open set \( X_i = U_i(\mathbb{C}P^2 \setminus \Sigma) \subset \mathbb{C}P^1 \).

Our result is explicit: for \( \Sigma_{\mathbb{R}} \), we have an explicit formula in terms of the functions \( U_i \)'s. And, given a \( N \)-uptlet \( \mathbb{R} \), we can explicitly construct the equation \( (Lde_i) \) for every \( i \) in terms of the \( U_i \)'s again.

We prove this theorem by using mostly elementary methods of complex analysis. The proof can be divided into 3 parts: from proposition 1, we know that the \( F_i \)'s are analytic germs. Then we complexify the setting. By successive differentiations along the level curves of the functions \( U_i \)'s, we construct for each \( i \) the linear differential equation \( (Lde_i) \) from the equation \( (E) \). This method is essentially an application to our case of Abel’s method for solving functional equations in several variables, described in [13]. Finally we prove the analytic continuation along any path in \( X_i \) by using a simple and general geometrical argument (see proposition 3).

From the proof of this theorem, we deduce two methods to solve equations of the form \( (E) \). The first, called “Abel’s method”, is explained in 2.3.1. It is effective and can be implemented on a computer: it consists in solving the equation \( Lde \), given by theorem A in order to reconstruct the solutions of \( (E) \). The second method, exposed in 2.3.2, is not so general. It is based on the fact that (modulo suitable condition on the \( U_i \)'s) solutions of \( (E) \) with logarithmic growth are characterized by their monogromy, which can be determined a priori. In the third part, we first explicitly solve equations associated to the classical equations of polylogarithms \( (L2) \) and \( (SK) \) stated above. Then in 3.5 we apply Abel’s method to an equation noted \( (E_c) \) associated to a degenerate configuration \( c \) of 5 points in \( \mathbb{C}P^2 \) (see figure 3). In part 4.1, we interpret the preceding results in the framework of planar web geometry: we obtain new “exceptional webs”. In particular we prove the

**Theorem B** The Spence-Kummer web \( W_{SK} \) associated to the equation \( (SK) \) is an exceptional 9-web.

The fact that we have found an explicit equivalent of the space of abelian relations for this web in 3.4, allows us to study its sub-webs. Thus we discover two non-equivalent exceptional 6-webs, and an exceptional 7-web. As in the case of Bol’s web, numerous abelian relations for these exceptional webs are constructed from polylogarithms. Then we observe that, modulo a suitable change of coordinates, all these exceptional webs are related to certain configurations of points in \( \mathbb{C}P^2 \).

This remark leads us to define (see definition 4) the notion of “web associated to a configuration of \( n \) points in the complex projective plane”. Next we consider the web \( W_c \) associated to the configuration \( c \). From the explicit basis of solutions of \( (E_c) \) obtained in 3.5, we can now construct a basis of the space of abelian relations of \( W_c \) showing that this web is exceptional. Then we state some general results about webs associated to configurations of \( n \) points, for \( n = 3,4,5 \):

**Theorem C** Let be \( n = 3,4 \) or 5. The web associated to any (degenerate if \( n = 5 \)) configuration of \( n \) points in \( \mathbb{C}P^2 \) is of maximal rank. Therefore it is exceptional if it contains a sub-configuration of 4 points in general position.

This allows us to formulate a conjecture which could give numerous exceptional webs and therefore numerous equations of the form \( (E) \). Since the equations in part 3 (which are related to webs associated to configurations) are mostly constructed by using iterated integrals, this conjecture could give functional equations for higher order polylogarithms.

In part 4.2 we apply the preceding results to the problem of characterizing measurable functions \( L \) satisfying equations \( (L2) \) or \( (SK) \). We prove, with weak regularity assumptions, that \( Li_2 \) and \( Li_3 \) are characterized by these equations. In the case of the trilogarithm, the result is new ( \( Li_3 \) is considered
here as an analytic function on $| - \infty, 1|$

**Theorem D** Let $F : [ - \infty, 1[ \to \mathbb{R}$ be a measurable function such that for $0 < x < y < 1$, we have
\[
2F(U_1(x,y)) + 2F(U_2(x,y)) - F(U_3(x,y)) + 2F(U_4(x,y)) + 2F(U_5(x,y)) - F(U_6(x,y)) + 2F(U_7(x,y)) + 2F(U_8(x,y)) - F(U_9(x,y)) = R_3(x,y)
\]

If $F$ is derivable at 0, then $F = L_{i_3}$.

This gives a proof of Goncharov’s “remark” about the problem of characterizing $L_{i_3}$ by the Spence-Kummer equation, stated in [Gon3] (page 209).

**remark : 1.** This paper is an extended version of the preprint [P].

2. While the author was working on the subject, he was told by G. Henkin that in a personal communication to him (nov. 2001), A. Hénaut announced that his colleague G. Robert had found that the Spence-Kummer’s web is of maximal rank by constructing an explicit basis of the space of abelian relations, which is equivalent to part 3.4 of this paper. G. Robert had interpreted this in the framework of web geometry and had obtained new exceptional $d$-webs for $d = 6, 7$ and 8. But no additional information about this has been given until now.

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1.2 Notations

We introduce here some notations which we will use in the paper.

Throughout this paper, $N$ will be a fixed integer bigger than 3.

If no precision is given, for every $i = 1, ..., N$, $U_i$ will denote a non-constant element of $\mathbb{R}(x, y)$ considered as a holomorphic map $\mathbb{C}P^2 \setminus S_i \to \mathbb{C}P^1$, where $S_i$ denotes the locus of indetermination of $U_i$; it is a finite set.

A functional equation of the form $F_1(U_1) + ... + F_N(U_N) = 0$ will be called “an abelian functional equation” (ab. Afe) with real rational inner functions. The name comes from the notion of abelian relation in web geometry, itself related to the notion of abelian sum in algebraic geometry (see part 4.1.1 or part 2.2 in the expository paper [He]).

In the whole text, $(\mathcal{E})$ will denote a general Afe $\sum_{i=1}^{N} F_i(U_i) = 0$.

The foliation $\mathcal{F}\{U_i\}$ (or more shortly $\mathcal{F}_i$) will be the global singular foliation of $\mathbb{C}P^2$, the leaves of which are the level curves of $U_i$. Let be $R = (U_1, ..., U_N)$ a $N$-uplet of real rational functions. To the unordered set of foliations $\mathcal{F}_R = \{F_i\mid i = 1, ..., N\}$, we associate the following algebraic subset of $\mathbb{C}P^2$: ( $S_i$ denotes the singular locus of the foliation $\mathcal{F}_i$)

$$
\Sigma_R := (\cup_{i=1}^{N} S_i) \cup \left( \cup_{i\neq j} \{ \eta \in \mathbb{C}P^2 \setminus (S_i \cup S_j) \mid (dU_i \wedge dU_j)(\eta) = 0 \} \right)
$$

By definition, $\mathcal{F}_R$ is a web if $\Sigma_R$ is proper in $\mathbb{C}P^2$. In this case we note $\mathcal{W}\{U_i\}$ or $W_R$ for $\mathcal{F}_R$, and $\Sigma_{W_R}$ for $\Sigma_R$ and the latter will be called the singular locus of the web. Because $\Sigma_{W_R}$ is the union of the singular locus of the foliations $\mathcal{F}_i$ with the locus in which the leaves of the foliations are not in general position, it depends only on the web and not on the functions $U_i$.
The web $\mathcal{W}\{\mathcal{E}\}$ associated to $\mathcal{E}$ will be the web $\mathcal{W}\{U_i\}$.

If $\mathcal{F}$ is a sheaf of function germs on $\mathbb{K}^d$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $d = 1, 2$, $F_\omega$ will denote the function germs of this sheaf at $\omega \in \mathbb{K}^d$ and we will note $\mathcal{F}_\omega(X)$ the space of determinations at $\omega$ of the elements of $\mathcal{F}(X)$. In this paper, we will consider mostly the sheaf $\mathcal{M}$ of measurable real valued function germs and the sheaf $\mathcal{O}_X$ (ab. $\mathcal{O}$) of holomorphic germs on a complex manifold generally noted $X$. Then $\tilde{X}$ will be the analytic universal covering of $X$, and $\mathcal{O}_X$ (ab. $\mathcal{O}$) will be the sheaf of multivalued holomorphic functions on $X$. In the paper, $X$ will be a Zariski open set in $\mathbb{C}^p$ with $k = 1, 2$. In part 2.2.2, we will use the sheaf of multivalued holomorphic functions on $X$, with logarithmic growth at infinity, noted $\mathcal{O}_{X,\log}$ (ab. $\tilde{\mathcal{O}}_{X,\log}$).

If $\gamma$ is a path linking $\omega$ to $\tilde{\omega}$ in a complex manifold $X$ and if $\mathcal{K} \in \mathcal{O}_\omega$ admits an analytic continuation along $\gamma$, then we note $\mathcal{K}^{(\gamma)}$ or $\mathcal{M}, \mathcal{K}$ the holomorphic germ at $\omega$ obtained by this analytic continuation.

If $\omega \in \mathbb{R}^2$, then in the whole paper, we set $\omega_i := U_i(\omega) \in \mathbb{R}^p$ when it is well defined. Then a “local solution of equation $\mathcal{E}$ at $\omega$ in the class $\mathcal{F}$” will denote an element of the space

$$\mathcal{S}_\mathcal{E}^p(\mathcal{E}) = \left\{ (F_1, .., F_N) \in \prod_{i=1}^N \mathcal{F}_\omega \mid \sum_{i=1}^N F_i(U_i) = 0 \text{ in } \mathcal{F}_\omega \right\}$$

We remark that if $\mathcal{F} = \mathcal{O}$, then $\mathcal{S}_\mathcal{E}^p(\mathcal{E})$ is the space of the local holomorphic solutions at $\omega$ of “the complex version” of $(\mathcal{E})$.

In the whole paper, to any $\mathcal{H} = (H_1, .., H_N) \in \prod_i \mathcal{F}_\omega$ such that the sum $\sum F_i(U_i)$ is constant and equal to $c$, we associate the element $(H_1 - c, .., H_N)$ of $\mathcal{S}_\mathcal{E}^p(\mathcal{E})$ again noted $\mathcal{H}$.

If $J$ is a subset of $\{1, .., N\}$, we note $(\mathcal{E}_J)$ the equation $\sum_{j \in J} F_j(U_j)$.

For $\omega \notin \Sigma_{\mathcal{E}}$ we have $\omega \notin \Sigma_{\mathcal{E}_J}$ and there is a linear embedding $\mathcal{S}_\mathcal{E}^p(\mathcal{E}_J) \hookrightarrow \mathcal{S}_\mathcal{E}^p(\mathcal{E})$. So we will consider the local solutions of $(\mathcal{E}_J)$ as particular local solutions of $(\mathcal{E})$. For $p \in \{3, .., N\}$ we note $\mathcal{F}_p^p(\mathcal{E})$ the sum $\sum_{J \in P} \mathcal{S}_\mathcal{E}^p(\mathcal{E}_J)$ where $P$ runs over all the subsets of $p$-elements in $\{1, .., N\}$. An element of $\mathcal{F}_p^p(\mathcal{E}) \setminus \mathcal{F}_p^p(\mathcal{E})$ (with $q = p - 1$) will be called a solution of order $p$ of the equation $(\mathcal{E})$. A solution of order $p < N$ will be called a “sub-solution”, when a solution of order $N$ will be a “genuine solution” of $(\mathcal{E})$.

By definition “a solution with logarithmic growth” of $(\mathcal{E})$ will be an element of $\mathcal{S}_\mathcal{E}^{\log}(\mathcal{E})$.

The components of most known solutions of $\mathcal{A}$ with rational inner functions are constructed from iterated integrals. This notion goes back to the work of K.T. Chen, in the 60’s. We state here the notations about iterated integrals used in the paper.

Let us note $X = \mathbb{C}^p \setminus \Sigma_W$ and $Z = \mathbb{C}^p \setminus U_i(\Sigma_W)$ where $i$ is a fixed element of $\{1, .., N\}$. There exists a finite number of distinct points $a_1, .., a_{M_i+1}$ in $\mathbb{C}^p$ such that we have $Z = \mathbb{C}^p \setminus \{a_1\}$. We can always assume that $a_{M_i+1} = \infty$ (we can substitute $g \circ U_i$ for $U_i$ with $g \in PGL_2(\mathbb{C})$ such that $g(a_{M_i+1}) = \infty$). This doesn’t change the nature of the problem.) We inductively define the iterated integrals which are functions noted $L_{x_1, .., x_m}$ with $i_k \in \{1, .., M_i\}$: if $z \in Z$ and $\gamma$ is a path in $Z$ from $\omega_i$ to $z$ defining a point over $z$ in $\tilde{Z}$, then we set

$$L_{x_0, x_1, .., x_m}(z, \gamma) := \int_{\omega_i, \gamma} \frac{L_{x_1, x_2, .., x_m}(\xi)}{a_{i_0} - \xi} d\xi, \quad i_0, .., i_m \in \{1, .., M_i\}$$

These functions are holomorphic functions on the analytic universal covering $\tilde{Z}$ of $Z$.

We note $\mathcal{I}(Z)$ (or $\mathcal{I}(a_1)$) the subspace of $\tilde{\mathcal{O}}(Z)$ spanned by the constants and the iterated integrals defined above. It is well defined: it doesn’t depend of the base point $\omega_i$.

In part 3, we will use special notations for some elements of $\mathcal{I}_{\{-1,0,1\}}$ that we describe now: let be $\Omega := \mathbb{C} \setminus (\Delta_0 \cup \Delta_1 \cup \Delta_{-1})$ where $\Delta_0$, $\Delta_1$ and $\Delta_{-1}$ are respectively the half-lines $iR^-$, $1 + iR^+$ and
First, let us consider the “generalized Cauchy equation” classical examples: A polylogarithmic function will be a function constructed from elements of the singular locus of $W$. This comes from the fact that $0$ belongs to the singular locus of $W$. The assumption, about $\omega$, is analytic indeed. But obtaining a precise local version of this statement needs to make another assumption, about $\omega$. If we take $\omega = 0 \in \mathbb{R}^2$, then $\dim_\mathbb{R} \mathcal{S}_\omega^M(C) = 2$ : actually $(\mathcal{C})$ doesn’t admit any non-constant analytic (and even continuous) solution at the origin. This comes from the fact that $0$ belongs to the singular locus of $W\{x, y, xy\}$. Therefore the point $\omega$ must not belong to this singular locus if we want it to have nice properties for the space $\mathcal{S}_\omega^F(C)$.

Another (more trivial) example of the pathologies which appear if we don’t make any assumption of genericity on $\omega$ is given by the functional equation $G_1(x) + G_2(y) + G_3(x) = 0$ noted $(\mathcal{T})$. Here, the singular locus is the whole $\mathbb{CP}^2$ and the local solutions of $(\mathcal{T})$ a priori don’t admit any analytic continuation and form an infinite dimensional linear space.

2 General properties of the solutions of $(\mathcal{E})$

2.1 preliminary remarks

Our object is to study the solutions of an abelian functional equation $(\mathcal{E})$ with real rational inner functions. Using the notations introduced in the preceding part, we want to study (and possibly determine) the space $\mathcal{S}_\omega^F(\mathcal{E})$ of local solutions of $(\mathcal{E})$ around $\omega$ in the class $\mathcal{F}$. What we want to prove is that, roughly speaking, the solutions of $(\mathcal{E})$ are analytic, admit analytic continuation on a Zariski open set of $\mathbb{CP}^1$ and form a finite dimensional linear space.

But we have to make some restrictions on $\mathcal{F}$ and $\omega$ to avoid pathological situations for the space $\mathcal{S}_\omega^F(\mathcal{E})$: we have to deal with at least measurable functions and we have to take $\omega$ outside of the singular locus $\Sigma_\mathcal{W}$ of the web $\mathcal{W}\{U_i\}$.

These two assumptions appear reasonable and quite natural if we consider the following simple and classical examples: First, let us consider the “generalized Cauchy equation”

$$(\mathcal{C}) \quad F_1(x) + F_2(y) + F_3 \left( \frac{x}{y} \right) = 0$$

It is well known that the space of multiplicative functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is infinite dimensional. To any such function corresponds a solution $(F, F, -F)$ of $(\mathcal{C})$. Such functions generally are not measurable: if the function actually is, then it is constructed from the logarithm. So, if no restriction on the regularity of the $F_i$’s is made, the space of solutions can be infinite dimensional, which contrasts with the measurable setting in which we have $\dim_\mathbb{R} \mathcal{S}_\omega^M(\mathcal{C}) = 3$. The assumption of measurability of the solutions of the general equation $(\mathcal{E})$ appears natural.

According to this assumption we can expect the solutions to have some good regularity properties such as analyticity: in our case, the “only” non constant measurable solution of $(\mathcal{C})$ is $(\log, \log, -\log)$, which is analytic indeed. But obtaining a precise local version of this statement needs to make another assumption, about $\omega$. If we take $\omega = 0 \in \mathbb{R}^2$, then $\dim_\mathbb{R} \mathcal{S}_\omega^M(\mathcal{C}) = 2$ : actually $(\mathcal{C})$ doesn’t admit any non-constant analytic (and even continuous) solution at the origin. This comes from the fact that $0$ belongs to the singular locus of $\mathcal{W}\{x, y, xy\}$. Therefore the point $\omega$ must not belong to this singular locus if we want it to have nice properties for the space $\mathcal{S}_\omega^F(C)$.

A polylogarithmic function will be a function constructed from elements of $\mathcal{I}_{(0,1)}$. 

-1 + i\mathbb{R}^-$ of $\mathbb{C}$. Now $\Omega$ is simply connected and does not contain $0$, $1$ and $-1$, so for any $z \in \Omega$ the value of any function defined by the expression below is well defined if we integrate along any path in $\Omega$:

\[
\begin{align*}
L_{x_0}(\bullet) &= \log(\bullet) \\
L_{x_1}(\bullet) &= -\log(1 - \bullet) \\
L_{x_{-1}}(\bullet) &= \log(1 + \bullet) \\
L_{x_{0,1}}(\bullet) &= \log_{i2}(\bullet) \\
L_{x_{x,1}}(\bullet) &= \int_1^0 \frac{L_{x_0}(\zeta)}{1 - \zeta} d\zeta \\
L_{x_{-1,x_0}}(\bullet) &= \int_0^1 \frac{L_{x_{-1}}(\zeta)}{1 - \zeta} d\zeta \\
L_{x_{x,x}}(\bullet) &= \int_0^1 \frac{L_{x_1}(\zeta)}{1 + \zeta} d\zeta \\
L_{x_{x,x}}(\bullet) &= \frac{1}{2}(L_{x_1}(\bullet))^2 \quad \text{for} \quad \epsilon = -1, 0, 1
\end{align*}
\]
These two elementary examples show that both the hypotheses of measurability for the $F_i$’s and of genericity for $\omega$ are quite natural and reasonable. From now on, we will always suppose that these hypotheses are satisfied.

### 2.2 General properties of the measurable solutions of $(\mathcal{E})$

#### 2.2.1 Analyticity of the measurable solutions

We prove now that any measurable local solution at a generic point $\omega$ of $(\mathcal{E})$ is in fact analytic.

**Proposition 1** Let $\omega \in \mathbb{R}^2 \setminus \Sigma$ and $F = (F_1, \ldots, F_N) \in \mathcal{S}_M^M(\mathcal{E})$. Then each $F_i$ is in fact an analytic germ at $\omega_i$. Its complexification gives a germ $F_i^c \in \mathcal{O}_{\omega_i}$ such that $F^c := (F_1^c, \ldots, F_N^c)$ is a holomorphic solution of $(\mathcal{E})$ at $\omega$.

*proof:* By hypothesis we have $\omega \notin \Sigma$, so it comes from Theorem 3.3. of [1a] that the $F_i$’s are continuous germs at $\omega_i$. By elementary tools of integration it comes next that they are $C^\infty$ smooth germs, so we have to prove that they are in fact analytic.

We obtain analyticity through the same method as J.L. Joly and J. Rauch in [Jo-Ra], by formulating the equation $(\mathcal{E})$ in the form of a linear elliptic $(N + 2) \times N$ differential system. Then the analyticity of the $F_i$’s follows from classical results on the regularity of solutions of elliptic systems (see [Pet]). Finally the unicity principle implies that $F^c \in \mathcal{S}_C^G(\mathcal{E})$. \(\blacksquare\)

**Remark:** Under the assumption that the $F_i$’s are smooth enough, we can show (see the next section) that each $F_i$ generically satisfies a linear differential equation with analytic coefficients and so is analytic by a classical result of ordinary differential equations. But this more elementary way to prove analyticity is not useful because thus it is not easy to deal with the genericity condition.

So we have two $\mathbb{R}$-linear morphisms:

the first is just the restriction of the real part of the holomorphic solutions of the complex version of $(\mathcal{E})$ to $\mathbb{R}^2$:

$$\rho : \mathcal{S}_C^G(\mathcal{E}) \rightarrow \mathcal{S}_C^G(\mathcal{E})$$

and the second is the complexification of the solutions given by proposition 1

$$\varphi : \mathcal{S}_M^M(\mathcal{E}) \rightarrow \mathcal{S}_C^G(\mathcal{E})$$

It is clear that $\rho \circ \rho = \text{Id}_{\mathcal{S}_M^M(\mathcal{E})}$, so the study of the measurable solutions of $(\mathcal{E})$ at $\omega$ amounts to the study of the holomorphic local solutions of $(\mathcal{E})$.

#### 2.2.2 characterization of the components of the holomorphic solutions of $(\mathcal{E})$

It is well known that, in the generic case, there is no non-constant holomorphic solution of a general abelian functional equation.

Let us consider now the very specific case when $(\mathcal{E})$ has a non trivial local holomorphic solution $F = (F_1, \ldots, F_N)$. Any non-constant component germ $F_i$ of $F$ must be a function of a very specific kind. The point is that any germ $F_i$ is just a local determination of a globally defined but ramified function which satisfies a linear differential equation with algebraic coefficients. We formulate this in the following

**Theorem 1** Let $N \geq 3$ be an integer and $R = (U_1, \ldots, U_N) \in \mathbb{R}(x, y)^N$ be such that $\Sigma_R$ is proper. Let $\omega \in \mathbb{R}^2 \setminus \Sigma_R$ be fixed. Then for every $i \in \{1, \ldots, N\}$ there exists a linear differential equation $(\text{Lde}_i)$, the coefficients of which are algebraic functions (meromorphic in a neighbourhood of $\omega_i$), such that for all $(F_1, \ldots, F_N) \in \mathcal{S}_C^G(\mathcal{E})$, the germ $F_i$ satisfies $(\text{Lde}_i)$ in a neighbourhood of $\omega_i$. The germ $F_i$ is a local determination at $\omega_i$ of a globally defined multivalued function on $\mathbb{C}P^1$, the ramification points of which belong to the finite set $U_i(\Sigma_R) \subset \mathbb{C}P^1$. 

7
**proof:** Without any loss of generality, we can assume that \( \omega = (0, 0) \notin \Sigma_R \) and \( U_i(\omega) = 0 \) for \( i = 1, \ldots, N \). Let be \( N \) germs \( F_i \in \mathcal{O}(\mathbb{C}, 0) \) such that \( \sum_1^N F_i(U_i) = 0 \) in a neighbourhood of \( \omega \). For \( \rho > 0 \) let's note \( \mathcal{D}_\rho = \{ z \in \mathbb{C} \mid |z| < \rho \} \).

If \( i \neq j \), since \( \omega \notin \Sigma_R \), \((U_i, U_j)\) defines a system of holomorphic coordinates on a neighbourhood \( \Omega_{ij} \) of \( \omega \). It is clear that we can find \( \epsilon > 0 \) such that each \( F_i \) is holomorphic on the whole \( \mathcal{D}_\epsilon \), and such that \( \Omega := \bigcap_k U_k^{-1}(\mathcal{D}_\epsilon) \subset \Omega_{ij} \) for all \( i \neq j \).

We now want to deduce from the functional equation \((\mathcal{E})\) a linear differential equation \((\mathcal{L}_{de})\) satisfied by \( F_N \) (or by any other \( F_i \), the process remaining the same). To do this, we will find it useful to introduce a more general class of equations than \( \text{Adfe} \):

**Definition 1** Let be \((N, M_1, \ldots, M_N) \in \mathbb{N}^* \times \mathbb{N}^N\), and let \( V_i, A_{ij} (1 \leq i \leq N, 0 \leq j \leq M_i) \) be holomorphic functions on an open set \( \Theta \subset \mathbb{C}^2 \). An “Abelian Differential Functional Equation” (ab. \( \text{Adfe} \)) is an equation of the type

\[
\sum_{i=1}^{M_i} \sum_{j=0}^{M_i-1} A_{ij} G_i^{(j)}(V_i) = 0 \quad (\text{Adfe})
\]

where the unknowns are the function germs \( G_1, \ldots, G_N \) which are supposed smooth enough (\( G_i^{(k)} \) denoting the \( k \)-th derivative of \( G_i \) for \( k \in \mathbb{N} \)).

If \( A_{iM_i} \neq 0 \) for all \( i \)'s, then the \( N \)-plet \((M_1, \ldots, M_N)\) is called “the true type” of the equation \((\text{Adfe})\), and its “type” if not.

The notion of \( \text{Adfe} \) generalizes \( \text{Afe} \) and \( \text{Lde} \): \( \text{Afe} \) are \( \text{Adfe} \) of the type \((0, \ldots, 0)\) and \( \text{Lde} \) are \( \text{Adfe} \) of the type \((M_1)\) with \( M_1 > 0 \).

Let us assume that for all \( i \neq j \), the couple \((V_i, V_j)\) (noted after definition 1) defines holomorphic coordinates on \( \Theta \). We now describe a process to obtain an \( \text{Adfe} \) of the type \((M_1 - 1, M_2 + 1, \ldots, M_N + 1)\), or \((M_2 + 1, \ldots, M_N + 1)\) if \( M_1 = 0 \), from an \( \text{Adfe} \) of the true type \((M_1, \ldots, M_N)\).

To begin with, let’s study the case when \( M_1 > 0 \).

By definition we have \( A_{1M_1} \neq 0 \) on \( \Theta \), therefore the equation \((\text{Adfe})\) implies that on \( \Theta' = \Theta \setminus \{ A_{1M_1} = 0 \} \), so we have

\[
G_1^{(M_1)}(V_1) + \sum_{j=0}^{M_1-1} \frac{A_{1j}}{A_{1M_1}} G_1^{(j)}(V_1) + \sum_{i=2}^{N} \sum_{j=0}^{M_i} \frac{A_{ij}}{A_{1M_1}} G_i^{(j)}(V_i) = 0
\]

Let \( \partial \) be the vector field on \( \Theta \) which corresponds to the differentiation with respect to \( V_2 \) in the coordinate system \((V_1, V_2)\).

By application of this derivation to this last form of \((\text{Adfe})\) we get a new \( \text{Adfe} \) on \( \Theta' \):

\[
\sum_{j=0}^{M_1-1} \partial \left( \frac{A_{1j}}{A_{1M_1}} G_1^{(j)}(V_1) \right) + \sum_{i=2}^{N} \sum_{j=0}^{M_i} \left( \partial \left( \frac{A_{ij}}{A_{1M_1}} G_i^{(j)}(V_i) \right) + \frac{A_{ij}}{A_{1M_1}} \partial (V_i) G_i^{(j+1)}(V_i) \right) = 0
\]

which can be written

\[
\sum_{i=1}^{N} \sum_{j=0}^{\tilde{M}_i} \tilde{A}_{ij} G_i^{(j)}(V_i) = 0 \quad (\text{Adfe}^2)
\]

where \( \tilde{M}_i = M_i - 1 \) (resp. \( M_i + 1 \)) if \( i = 1 \) (resp. \( i > 1 \)) and

\[
(\ast) \quad \tilde{A}_{ij} = \begin{cases} 
\partial \left( \frac{A_{1j}}{A_{1M_1}} \right) & \text{if } i = 1 \\
\partial \left( \frac{A_{ij}}{A_{1M_1}} \right) & \text{if } i > 1 \text{ and } j = 0 \\
\partial \left( \frac{A_{ij}}{A_{1M_1}} \right) + \frac{A_{ij}}{A_{1M_1}} \partial (V_i) & \text{if } 1 < i \text{ and } 0 < j \leq M_i \\
\frac{A_{ij}}{A_{1M_1}} \partial (V_i) & \text{if } 1 < i \text{ and } j = M_i + 1 
\end{cases}
\]

We remark that, because \( \partial (V_i) \neq 0 \) for \( i > 1 \), no \( \tilde{A}_{iM_i} \) is a null function, so the equation that we obtain is of the true type \((K, M_2 + 1, \ldots, M_N + 1)\), \( K \) being an integer smaller than \( M_1 - 1 \).
If \( M_1 = 0 \), then we similarly get an equation of the form
\[
\sum_{i=2}^{N} \sum_{j=0}^{\tilde{M}_i} \tilde{A}_{ij} G_i^{(j)}(V_i) = 0
\]
(A_{dfe}^2)
where \( \tilde{M}_i = M_i + 1 \) for \( 2 \leq j \leq N \) and
\[
(\star\star) \quad \tilde{A}_{ij} = \begin{cases} 
\frac{\partial (A_{ij}^2)}{A_{i+1,j}} & \text{if } i \geq 2 \text{ and } j = 0 \\
\frac{\partial (A_{ij}^2)}{A_{i+1,j}} + \frac{A_{i-1,j}}{A_{i+1,j}} \partial (V_i) & \text{if } 2 \leq i \text{ and } 0 < j \leq M_i \\
\frac{A_{i+1,j}}{A_{i,M_i}} \partial (V_i) & \text{if } 2 \leq i \text{ and } j = M_i + 1
\end{cases}
\]
As in the preceding case, it’s quite obvious that we obtain an Adfe of the true type \( (M_2 + 1, \ldots, M_N + 1) \).

In both cases \( (M_1 = 0) \) or \( (M_1 > 0) \), we can apply these operations again to \( (A_{dfe}^2) \).

After several applications of this process on \( \Omega \) to the Adfe \( (E) \) we obtain an Adfe of the type \( (K) \) (with \( K \in \mathbb{N}^* \)) on \( \Omega' = \Omega \setminus \Lambda \), where \( \Lambda \) is an analytic subset of \( \Omega \). This equation can be written in the coordinate system \( (U, V) = (U_{N-1}, U_N) \) in the following form:
\[
A_1(U, V)F_{N}^{(1)}(V) + A_2(U, V)F_{N}^{(2)}(V) + \ldots + A_K(U, V)F_{N}^{(K)}(V) = 0
\]
(Lde_N)
Let us take now \( (U_0, V_0) \in \Omega' \). By fixing \( U = U_0 \) in the preceding equation, we get, in a neighbourhood of \( V_0 \), a linear differential equation of order \( K \) in the variable \( V \), the solutions of which contain \( F_N \):
\[
A_1(V)F_{N}^{(1)}(V) + A_2(V)F_{N}^{(2)}(V) + \ldots + A_K(V)F_{N}^{(K)}(V) = 0
\]
It is clear that this equation \( (Lde_N) \) doesn’t depend on the solution \( (F_1, \ldots, F_N) \) but only on the \( U_i \)'s. Then the \( N \)th component of every solution \( F \in \mathbb{S}^N(E) \) will verify this equation, at least generically in a neighbourhood of \( \omega_i \).

From now on, we assume that we can take \( (U_0, V_0) = (0, 0) \). We now prove by “induction on the type” that the coefficients of the preceding equation are algebraic functions of \( V \). Let be \( \mathbb{C}\{U, V\}^{alg} = \{ h \in \mathbb{C}\{U, V\} \mid \exists Q \in \mathbb{C}\{U, V, W\} \forall (U, V, h(U, V)) = 0 \} \).
It is well known that this space has some strong properties of closure:

**Proposition 2** Let be \( F, G \in \mathbb{C}\{U, V\}^{alg} \). Then \( F + G, \ F \times G, \ \partial_U F, \ \partial_V F \) and \( 1/F \) (if \( F(0, 0) \neq 0 \)) are still elements of \( \mathbb{C}\{U, V\}^{alg} \). If \( \Phi = (F, G) \) defines a germ of diffeomorphism of \( \mathbb{C}^2 \) at the origin, then the components of the local inverse \( \Phi^{-1} \) are algebraic functions too.

Let us note \( \partial_k^{kl} \) the derivation on \( U_k \) with respect to \( U_k \) in the coordinate system \( (U_k, U_l) \). One can easily prove that we have \( \partial_k^{kl} = U_k^{kl} \partial_U + V_k^{kl} \partial_V \) with \( U_k^{kl}, V_k^{kl} \in \mathbb{C}\{U, V\}^{alg} \). Then by proposition 5, the latter is closed under the action of the \( \partial_k^{kl} \)’s. By proposition 5 again and from the above relations (\( \star \)) and (\( \star\star \)), if the \( A_{ij} \)'s of \( Adfe \) are algebraic functions, then the \( \tilde{A}_{ij} \)'s ( or the \( \tilde{A}_{ij} \)'s) of \( (A_{dfe}^N) \) are still algebraic. Because all the coefficients of \( Adfe \) are equal to 1, we get, by induction, that the \( A_i \)'s of \( (A_{dfe}^N) \) are elements of \( \mathbb{C}\{U, V\}^{alg} \). Therefore the \( A_i \)'s of \( (Lde_N) \) are algebraic functions of \( V \).

Because the \( A_i \)'s are algebraic, they are globally defined but ramified. A classical result of the theory of linear differential equations of a complex variable implies that the germ \( F_N \) can be analytically extended along any curve in \( \mathbb{C}P^1 \setminus R \), where \( R \) is the union of the poles with the ramification points of the \( A_i \)'s.

But this argument didn’t allow us to prove that \( F_N \) admits analytic continuation along any path in the whole \( U_N(\mathbb{C}P^2 \setminus \Sigma_R) \) because, if it’s not hard to see that the ramification points of the \( A_i \)'s are in \( U_N(\Sigma_R) \), it is not the same for their possible poles, which can generate some ramification for any solution of \( (Lde_N) \).
The last part of the theorem comes from the following proposition 3.
Proposition 3 Let $X$ be a connected paracompact complex manifold of dimension 2 and let $U_i : X \to \mathbb{C}$, $(i = 1, \ldots, N)$ be holomorphic functions such that, if $i \neq j$, we have $dU_i \wedge dU_j \neq 0$ on $X$. If for $\omega \in X$ we have $N$ holomorphic germs $F_i$ such that $\sum_1^N F_i \circ U_i$ is a holomorphic germ at $\omega$ which can be analytically continued along any path in the whole $X$, then every $F_i$ can be analytically continued along any path in $U_i(X)$.

Proof: we will prove this proposition under the assumption that $\sum_1^N F_i \circ U_i = 0$. The proof in the general case is similar. For $i = 1, \ldots, N$, let’s note $\Psi_i := F_i \circ U_i \in \mathcal{O}_\omega$. Because $X$ is supposed paracompact, it is metrisable as a topological space. We fix a metric on $X$, compatible with its topology. Then there exists $\epsilon > 0$ such that each $\Psi_i$ is defined on $B(\omega, \epsilon) \subset X$. First we prove the following.

Lemma 1 Let us assume that $X$ is an open ball in $\mathbb{C}^2$ centered in $\omega$, of radius $\rho \geq \epsilon$. Then each $\Psi_i$ can be analytically extended to $X = B_\rho := B(\omega, \rho)$.

Proof of the lemma: Let be $\tau := \sup \{ \delta \in [\epsilon, \rho] \mid \text{each } \Psi_i \text{ extends to } B_\delta \}$. We want to prove that $\tau = \rho$. Let us suppose that $\tau < \rho$: by definition each $\Psi_i$ extends analytically to $B_\tau$. We note again $\Psi_i$ this extension.

Let us choose arbitrarily $\eta \in \partial B_\tau$. We are going to prove that all the $\Psi_i$’s have a holomorphic extension in a neighbourhood of $\eta$. By compactness, it will imply that each $\Psi_i$ extends to a neighbourhood of the closure $\overline{B_\tau}$, which will contradict the definition of $\tau$.

Let $(x, y)$ denote the standard complex coordinates on $\mathbb{C}^2$. We introduce the holomorphic vector fields of differentiation along the level curves of the $U_i$’s: $X_i := (\frac{dU_i}{\partial \overline{y}}) \partial_x - (\frac{dU_i}{\partial y}) \partial_y$.

According to the definition of $\Psi_i$, we have $X_i \Psi_i = 0$ on $B_\tau$ and therefore on $B_\tau$ by unicity theorem: $\Psi_i$ is constant along the level curves of $U_i$ in $B_\tau$. But these level curves are globally defined on $X$ and in particular in a neighbourhood of $\eta$. This fact combined with the general position assumption on these level curves at $\eta$ (formulated by $dU_i \wedge dU_j(\eta) \neq 0$ according to the hypothesis of the theorem) will allow us to extend each $\Psi_i$ near $\eta$.

But we have to make it more precise:

We note $T_\eta \partial B_\tau$ the real tangent space of $\partial B_\tau$ in $\eta$. It is a real subspace of real dimension 3 of the complex tangent space to $\mathbb{C}^2$ at $\eta$, noted $T_\eta \mathbb{C}^2$. It contains an unique complex line noted $T_\eta^C \partial B_\tau$.

Let us extend $\Psi_i$ in a neighbourhood of $\eta$.

Let $C_\eta^1$ be the level curve of $U_j$ through $\eta$. Since $dU_1(\eta) \neq 0$, we know that there exists a neighbourhood $\mathcal{V}$ of $\eta$ such that $C_\eta^1 \cap \mathcal{V}$ is a complex 1-dimensional manifold. Let $T_\eta^C \mathbb{C}^1$ be its holomorphic tangent space in $\eta$.

Let us assume that $T_\eta^C \partial B_\tau$ and $T_\eta^C \mathbb{C}^1$ are transverse (i.e. their intersection in $T_\eta \mathbb{C}^2$ is null). Because all geometrical objects considered here are analytic, therefore smooth, this condition of transversality, called “condition (T)”, is open: there exists an open connected neighbourhood $V_\eta \subset X$ of $\eta$ such that for all $\zeta \in V_\eta \cap \partial B_\tau$, the transversality condition between $C_\zeta^1$ and $\partial B_\tau$ remains satisfied.

Let us note $W_\eta := V_\eta \cap U_1^{-1}(U_1(V_\eta \cap \partial B_\tau))$. It is an open neighbourhood of $\eta$.

For $\zeta \in W_\eta$, then $C_\zeta^1 \cap \partial B_\tau \neq \emptyset$. On the other hand, $C_\zeta^1$ verifies the transversality condition (T). The fact that $T_\zeta \partial B_\tau$ contains an unique complex line implies, for dimensional reasons, that $T_\zeta \partial B_\tau \cap T_\zeta^C \mathbb{C}^1 \neq (0)$.

We deduce that $C_\zeta^1 \cap B_\tau \neq \emptyset$. Then let us consider $\zeta' \in C_\zeta^1 \cap B_\tau$: we define the value of $\Psi_1$ in $\zeta$ by setting $\Psi_1(\zeta) := \Psi_1(\zeta')$. Because $\Psi_1$ is constant along the level curves of $U_1$ in $\mathbb{W}_\eta \cap B_\tau$, it comes that $\Psi_1(\zeta)$ is well defined.

We remark that we have $X_1 \Psi_1 = 0$ near $\eta$ again for this extension, so we have holomorphically extended $\Psi_1$ to $B_\tau \cup W_\eta$.  

10
Let us suppose now that the condition (T) is not satisfied by $C^1_\eta$.
It means that $T^\eta_d B_r = T^\eta_d C^1$. But the hypothesis $dU_1 \wedge dU_2(\eta) \neq 0$ for $j \geq 2$ has the geometrical interpretation that the curves $C^1_\eta$ and $C^1_\eta$ are transverse in $\eta$ (for $j \geq 2$). Therefore all the level curves $C^1_\eta$ (for $j \geq 2$) satisfy the transversality condition (T) at $\eta$. By the same argument than above, we can extend analytically each $\Psi_j$ ($j \geq 2$) to a neighborhood $W$ of $\eta$. To extend $\Psi_1$ close to $\eta$ we will set $\Psi_1 := - \sum_{j=2}^N \Psi_j$ on $W$, which will do.

**end of the lemma’s proof**

Now let’s prove proposition 3.

Let $\gamma : [0,1] \to X_1 := U_1(X)$ be a path with $\omega_1$ as its origin. We want to extend $\Psi_1$ along $\gamma$. Because $dU_1 \wedge dU_2 \neq 0$ on $X$, $\Gamma = (U_1,U_2)$ defines holomorphic coordinates in a neighborhood of any point of $X$. This implies first that $dU_1 \neq 0$ on $X$, so we can find a lift $\tilde{\gamma}$ of $\gamma$ to $U_1$ with $\omega$ as its origin.

Since the support $|\gamma|$ of $\tilde{\gamma}$ is compact, it comes too that we can find a subdivision $\alpha_{-1} < 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_M = 1$ of $[0,1]$ such that, for every $j \in \{0,\ldots, M-1\}$, there exists a holomorphic chart $(\Theta_j,\Gamma_{\Theta_j})$ centered at $\tilde{\gamma}(\alpha_j)$, such that $\tilde{\gamma}(\alpha_j,\alpha_{j+1}) \subset \Theta_j$, and $\Gamma(\Theta_j)$ is a ball in $C^2$ with $\Gamma(\alpha_j)$ as its center. We can apply the preceding lemma when taking $X = \phi_0(\Theta_0)$ and considering the functions $U_1 \circ \phi_0^{-1}$ instead of the functions $U_i$. So it comes that the $\Psi_i$ can be extended along $\tilde{\gamma}|_{\{\alpha_0,\alpha_1\}}$.

By iterating this process $M-1$ times, we finally get an extension of each $\Psi_i$ along $\tilde{\gamma}$, again noted $\Psi_i$.

This gives us the analytic extension of $\Psi_1$ along $\gamma$: for each chart $(\Theta_j,\Gamma_{\Theta_j})$ we have on $\Theta_j$ some holomorphic vector fields $X^j_\omega$ of differentiation along the level curves of $U_i$. There is $g^j_\omega \in \mathcal{O}(\Theta_j \cap \Theta_{j+1})$ such that $X^j_\omega = g^j_\omega X^j_{\omega+1}$ on $\Theta_j \cap \Theta_{j+1}$ for all $i$ and $j < M$.

By the construction of the lemma, we have $X^i_\omega \Psi_i = 0$ on each $\Theta_j$. The holomorphic inverse function theorem implies that we can write $\Psi_1 = F^j_1(U_1)$ on each $\Theta_j$, where $F^j_1$ is holomorphic in a neighborhood of $(U_1 \circ \tilde{\gamma})(\alpha_{j-1},\alpha_{j+1}) = \gamma(\alpha_{j-1},\alpha_{j+1})$. It is not difficult now to see that on $\gamma(\alpha_{j},\alpha_{j+1})$ we have $F^j_1 = F^j_{\omega+1}$ (for $0 < j < M-2$).

Now $F^0_1$ is the extension to $U_1(\Theta_0)$ of the original germ $F_1$. By setting $F_1 := F^0_1$ on $U_1(\Theta_j)$ for $j = 1,\ldots, M-1$, we get an analytic extension of $F_1$ along $\gamma$.

By the same way we can construct such an analytic continuation for every $F_j$. ■

**end of the proof of proposition 3**

**remarks:**

1. From the preceding proof, we get, using the same notations:

**corollary 1** In the generic case, there are no non-constant local holomorphic solutions of $(E)$ at any $\omega \notin \Sigma_R$.

**proof:** Because equation $(A^N_{\text{def}})$ is of the true type $(K)$ with $K > 1$, we can assume that $A_K \equiv 1$. Let us assume that one of the $A_i$’s ($i = 1,\ldots, K-1$) depends on the variable $U$. Then by differentiating with respect to $U$, we reduce $(A^N_{\text{def}})$ to an equation $(A^N_{\text{def}}')$ of the true type $(K')$ with $0 \leq K' < K$. The obstacle to this process of reduction is when $K' = 0$: the differentiation with respect to $U$ gives a trivial equation. It corresponds to the cancellation of all the $\partial_UA_i$. It corresponds to $K-1$ (non-linear) differential conditions on the $U_i$’s. Then the possibility to reduce $(A^N_{\text{def}})$ to an Adfe of the form $\widehat{A}_N(U,V)F^N_N(V) = 0$, with $\widehat{A}_N \neq 0$, corresponds to the non-vanishing of a finite number of differential expressions in the $U_i$’s. An element $(U_i) \in \mathbb{R}(x,y)^N$ satisfying these conditions will be said $N$-generic, and generic if $(U_{\sigma(i)})$ is $N$-generic for every $\sigma \in S_N$. It is clear that the genericity condition described here implies that any holomorphic solution of $(E)$ is constant. ■

2. Theorem 1 implies that $S^G(E)$ is finite dimensional. The following proposition (due to G. Bol, see [Bol1]) gives an effective bound to its dimension and will be important in part 4.

**Proposition 4** If $\omega \notin \Sigma_R$, then $\dim_C S^G(E) \leq N(N-1)/2$ and this majoration is optimal.

The majoration is just a particular case (for rational inner functions) of one of the first basic results of web geometry (see [Ha-Bol]). If we consider the case when $U_i(x,y) = x - a_iy$ for $i = 1,\ldots, N$, with
0 = a_1 < a_2 < ... < a_N = 1, we easily see that the bound \( N(N-1)/2 \) is reached in this case.

3. Some of the preceding results remain valid in a more general situation. For instance, if instead of taking the \( U_i \)'s rational, we consider some analytic germs, then the \( H_i \)'s solutions of \( \sum H_i(U_i) = 0 \) generically verify a linear differential equation which can be constructed from the \( U_i \)'s, and we find again that, in the generic case, there is no non-constant solution.

4. The method used here to obtain a linear differential equation from a functional equation is the one described by Abel in his first publication \( [Ab] \). We will call it “Abel’s Method”.

5. The point is that this method is effective: for a given \( \omega \) we can choose \( \sum H_i(U_i) = 0 \) generically verify a linear differential equation which can be constructed from the \( U_i \)'s, and we find again that, in the generic case, there is no non-constant solution.

6. Through the process used to extend the \( F \)'s in proposition 3, it could be possible to obtain some properties of (moderate) growth on the \( F \)'s.

7. We can assume that equation \( Lde \) is “totally reduced” i.e. that another application of one step of Abel’s method gives a null equation. In this case, it is interesting to study the quotient \( S^C_{\omega}(Lde_i)/[S^C] \).

We conjecture that it is trivial. Combined with remark 6, this could give supplementary informations about the nature of equation \( (Lde_i) \).

2.3 Two Methods to solve \( \text{Afe} \) with real rational inner functions

The proof of the preceding theorem contains some useful tools to construct two “methods” of solving \( \text{Afe} \) of the type \( (\mathcal{E}) \).

The first tool is essentially based on Abel’s method. It is well formalized and appears very general: its only defect is to be computational.

The second only consists in a remark and is not well established as a general method. Meanwhile, this remark allows us in part 3 to solve the two \( \text{Afe} \) (\( \mathcal{R} \)) and (\( \mathcal{SK} \)) associated respectively to equation \( (L_2) \) and Spence-Kummer equation \( (SK) \).

It is based on the idea that certain solutions of \( (\mathcal{E}) \) are determined by their monodromy.

2.3.1 Abel’s method of resolution of \( \text{AFE} \) with rational inner functions

Let us assume that \( (U_1, ..., U_N) \in \mathbb{R}(x, y)^N \) is such that there exists a non-constant holomorphic genuine solution \( F = (F_1, ..., F_N) \in S^C_{\omega}(\mathcal{E}) \). Then let \( [S^C]_{\omega,i} \) be the subspace of holomorphic germs at \( \omega_i \) spanned by the \( i \)-th components of solutions \( F \in S^C_{\omega}(\mathcal{E}) \).

We can choose \( \omega \notin \Sigma_R \) such that for each \( i \in \{1, ..., N\} \), there is a non-trivial linear differential equation \( (Lde_i) \) having algebraic coefficients which are well defined at \( \omega \). This equation is such that every component \( F_i \) of \( F \in S^C_{\omega}(\mathcal{E}) \) satisfies \( (Lde_i) \). We note \( S^C_{\omega}(Lde_i) \supset [S^C]_{\omega,i} \) the linear space of the holomorphic germs at \( \omega_i \) which are solutions of this equation. Let \( \{G^\nu_i \mid \nu = 1, ..., \nu_i \} \) be a basis of this space.

Then we have

\[
S^C_{\omega}(\mathcal{E}) = \left\{ \sum_{\nu=1}^{\nu_1} a_1^\nu G_1^\nu, ..., \sum_{\nu=1}^{\nu_N} a_N^\nu G_N^\nu \in \bigoplus_{i=1}^{N} S^C_{\omega,i}(Lde_i) \mid \sum_{i=1}^{N} \sum_{j=1}^{\nu_i} a_i^\nu G_i^\nu(U_i) = 0 \right\}
\]

so, in a certain way, the explicit resolution of \( (\mathcal{E}) \) at \( \omega \) amounts to some linear algebra in a finite dimensional space.

It is easy to prove that, in the standard coordinates system \( (x, y) \) on \( \mathbb{C}^2 \), the derivations \( \partial^k_p \) (where \( p = l, k \)) are elements of \( \mathbb{C}(x, y)\partial_x + \mathbb{C}(x, y)\partial_y \).

Then the coefficients \( A_{ij} \) of any \( \text{Adfe} \) obtained through the application of several steps of Abel’s method to \( (\mathcal{E}) \) belong to \( \mathbb{C}(x, y) \), therefore the process to obtain \( (Lde_i) \) from the \( \text{Afe} \) \( (\mathcal{E}) \) can be performed within \( \mathbb{C}(x, y) \). This fact allows us to easily implement an algorithm on a computer algebra system which constructs \( Lde_1 \) from \((U_1, U_2, ..., U_N)\).

The author has used this method to solve the equation \((\mathcal{E}_c)\) of part 3.5, and it seems possible to apply it to all the equations of part 3.
2.3.2 Method of monodromy “a priori”

Contrarily to the preceding method, the one described here doesn’t seem to be valid in the general case, but its interest lies in the fact that it works for at least three \( A \) \( \pi \) associated to classical functional equations of polylogarithms \( L_k \) with \( k \leq 3 \) (see part 3).

It is a “method” to find solutions with logarithmic growth of an \( A \) \( \pi \) when the \( U_i \)'s verify a certain condition called “condition (C)”, which is defined below. Roughly speaking, it is based on the fact that solutions with logarithmic growth are determined by their monodromy, which can be determined “a priori” when the solutions of some sub-equations of \( (E) \) are known.

We now define “condition (C)”:

**Definition 2** The set of rational functions \( \{ U_i \} \) verifies “condition (C)” if for all \( i \in \{1, \ldots, N\} \) there exists \( l(i) \neq i \) such that \((U_i, U_{l(i)})\) is a global system of coordinates on \( X := \mathbb{C}P^2 \setminus \Sigma \).

In the following pages, we will assume this strong condition verified.

Let \( F = (F_1, \ldots, F_N) \) be a genuine solution of \( (E) \) at a generic point \( \omega \in \mathbb{R}^2 \setminus \Sigma \).

Let be \( i \in \{1, \ldots, N\} \). There exists an integer \( m_i \) and a finite number of distinct points \( a_k^i \in \mathbb{C}P^1 \) (\( 1 \leq k \leq m_i \)) such that \( X_i = U_i(X) = \mathbb{C}P^1 \setminus \{a_k^i\} \). Theorem 1 implies that every germ \( F_i \) at \( \omega_i \) can be analytically extended along any path in \( X_i \).

Let be \( A_i = \{ \gamma_i^\lambda \}_{\lambda \leq m_i} \), a minimal family of loops of basepoint \( \omega_i \) in \( X_i \), such that their homotopy classes and their inverse span \( \Pi_1(X_i, \omega_i) \) (a suitable choice is to take for \( \gamma_i^\lambda \) a loop in \( X_i \), of index 1 with respect to \( a_j^i \) if \( j = \lambda \), and of index 0 otherwise).

Now we fix \( i \) and we note \( l \) for \( l(i) \). Condition (C) implies that we can find a loop \( \gamma_i^\lambda \) of basepoint \( \omega \) in \( X \) such that \([U_i \circ \gamma_i^\lambda] = [1] \) in \( \Pi_1(X_i, \omega_i) \) and \([U_i \circ \gamma_i^\lambda] = [\gamma_i^\lambda] \) in \( \Pi_1(X_i, \omega_i) \).

Because we have \( F_1(U_1) + F_2(U_2) + \ldots + F_N(U_N) = 0 \) in a neighbourhood of \( \omega \), then by analytic continuation along \( \gamma_i^\lambda \) we get a new functional relation in \( \mathcal{O}_{\omega} \):

\[
F_1^{[U_i \circ \gamma_i^\lambda]} + \ldots + F_1^{[\gamma_i^\lambda]}(U_1) + \ldots + F_1^{[U_i \circ \gamma_i^\lambda]}(U_N) = 0
\]

which can be summarized by \( F[\gamma_i^\lambda] \in \mathcal{S}_{\omega}(E) \) where \( F[\gamma_i^\lambda] := (F_k^{[U_i \circ \gamma_i^\lambda]})_{k=1..N} \).

By taking the difference between the above equations, we get a new one

\[
(F_1^{[U_i \circ \gamma_i^\lambda]}(U_1) - F_1(U_1)) + \ldots + (F_1^{[\gamma_i^\lambda]}(U_i) - F_1(U_i)) + \ldots + (F_1^{[U_i \circ \gamma_i^\lambda]}(U_N) - F_1(U_N)) = 0
\]

Now the germ \( F_1(U_1) - F_1 \) is null in \( \mathcal{O}_{\omega} \), therefore \( F - F[\gamma_i^\lambda] \) is not a genuine solution of \( (E) \) any more.

Let be \( K_i^\lambda = \{ k \mid [U_k \circ \gamma_i^\lambda] \neq [1] \} \) in \( \Pi_1(X_k, \omega_k) \). We have \( K_i^\lambda \subset \{1, \ldots, N\} \).

Let us assume that we know a basis \( \{ B_{i,k}^{\lambda} \mid k \in K_i^\lambda \} \) of \( \sum_{K_i^\lambda} \), with \( B_{i,k}^{\lambda} = (b_{i,k}^{\lambda,1}, \ldots, b_{i,k}^{\lambda,N}) \).

Then we get a relation

\[
F - F[\gamma_i^\lambda] = \sum_{\sigma \in \Delta_i^\lambda} \beta_{i,\sigma}^\lambda B_{i,\sigma}^{\lambda} \quad \text{with} \quad \beta_{i,\sigma}^\lambda \in \mathbb{C}
\]

from which we get the following relations for all \( \lambda \in A_i \):

\[
\mathcal{M}[\gamma_i^\lambda] F_i = F_i + \sum_{\sigma \in \Delta_i^\lambda} \beta_{i,\sigma}^\lambda b_{i,\sigma}^{\lambda,i}
\]

\[
\mathcal{M}[U_i \circ \gamma_i^\lambda] F_s = F_s + \sum_{\sigma \in \Delta_i^\lambda} \beta_{i,\sigma}^\lambda b_{i,\sigma}^{\lambda,s} \quad \text{for} \quad s \in K_i^\lambda \quad \text{and} \quad s \neq i
\]
If \( Y \) is a complex manifold, knowing the monodromy of \( G \in \tilde{O}(Y) \) means knowing a representation

\[
\Pi_{1}(Y, y) \rightarrow \text{End}_{\mathbb{C}}(Dy)
\]

\[\gamma \rightarrow T_{[\gamma]} : g \rightarrow g^{[\gamma]}\]

for at least one \( y \in Y \), where \( Dy \) denotes the linear space of the determinations of \( G \) at \( y \).

Because we have chosen the family \( \{ [\gamma^{i}] \} \) such that it spans \( \prod_{1}(X_{i}, \omega_{i}) \), the relations \((*)_{i}\) give us “a priori” the monodromy of each of the components \( F_{i} \) in function of the components \( b_{\lambda, \sigma}^{i} \) of the subsolutions \( B_{\lambda, \sigma}^{i} \) of \( (E) \) (for \( i = 1, \ldots, N \) and \( \lambda \leq m_{i} \)).

**Proposition 5** Under condition (C), the monodromy of each of the components \( F_{i} \) of a genuine solution of \( (E) \) can be expressed in terms of the components of some subsolutions of \( (E) \).

This transforms our point of view on equation \((E)\) : although considering it in a functional form we will now see relations \( (*) \) as “monodromy equations” for the components of the solutions, and relations \( (**) \) as “compatibility relations” between those equations of monodromy.

We now want to find some genuine solution of \( (E) \) by “solving” the monodromy equations \( (*) \).

Let us assume that there exists a genuine solution \( F = (F_{1}, \ldots, F_{N}) \) of \( (E) \) at \( \omega \). From the preceding lines, it comes that there exist complex constants \( \beta_{\lambda, \sigma}^{i}(F) \) satisfying both relations \( (*) \) and \( (**) \).

Let be \( \widetilde{H} = (\widetilde{H}_{1}, \ldots, \widetilde{H}_{N}) \in \prod_{i} \tilde{O}(X) \) such that each \( \widetilde{H}_{i} \) has a determination \( H_{i} \) at \( \omega_{i} \) satisfying the equations \( (*)_{i} \). Then the germ \( H_{i} - F_{i} \) can be extended analytically to \( X_{i} \) without ramifications. This implies that the germ \( \mathcal{H} = \sum(H_{i} - F_{i})U_{i} \) at \( \omega \) develops into a global holomorphic function : \( \mathcal{H} \in \mathcal{O}(\mathbb{C}P^{2} \setminus \Sigma) \).

Now let us suppose that we can choose \( \widetilde{H}_{i} \) with logarithmic growth. Then \( \mathcal{H} \) is a global holomorphic function on \( \mathbb{C}P^{2} \setminus \Sigma \) with logarithmic growth at infinity. By a Liouville type theorem, this implies that \( \mathcal{H} \) is constant. Then \( \mathcal{H} = (H_{1}, \ldots, H_{N}) \in \mathcal{S}_{\mathcal{O}}(\mathcal{E}) \).

We note \( \mathcal{S}_{\mathcal{O}}(\mathcal{E})^{\log} \) the subspace of the solutions of \( (E) \) with logarithmic growth. Then the problem of finding genuine solutions in \( \mathcal{S}_{\mathcal{O}}(\mathcal{E})^{\log} \) amounts to solving the equations \( (*) \) in the space \( \prod_{i} \mathcal{S}_{\mathcal{O}}^{\log}(X_{i}) \). One of the conceptual interests of this is that the problem is now reduced into a linear form.

Let be \( F \in \mathcal{S}_{\mathcal{O}}(\mathcal{E})^{\log} \). Then the subsolutions of the form \( F - F^{[\gamma]} \) which appear in the preceding discussion are now elements of \( \mathcal{S}_{\mathcal{O}}(\mathcal{E}_{K}^{\log}) \). Under suitable conditions on the \( U_{i} \)'s, the \( \{U_{j}\}_{j \in K} \) verify condition (C) again. In this case it could be possible to inductively determine the solutions with logarithmic growth of equation \( (E) \).

**Remarks 1.** Most components of most of the solutions of known \( (E) \)-form equations are constructed from iterated integrals (see part 3). Then it will appear interesting and useful to work in the subspace \( \prod_{i} \mathcal{I}_{\omega_{i}} \subset \prod_{i} \mathcal{S}_{\mathcal{O}}^{\log}(X_{i}) \) where \( \mathcal{I}_{\omega_{i}} \) denotes the space of the determinations of the elements of \( \mathcal{L}_{(X_{i})} \) at \( \omega_{i} \).

2. But not all the components of the solutions with logarithmic growth are constructed from iterated integrals: for instance the function Arctan(\( \sqrt{\omega} \)) is a component of a solution of the Abel with real rational inner functions (SK) considered in 3.4. This function cannot be expressed from iterated integrals although it is ramified with logarithmic growth on \( \mathbb{C}P^{1} \setminus \{0, 1, \infty\} \).

### 3 Examples of explicit resolution of abelian functional equations with real rational inner functions

In this part we apply the method sketched above to the resolution of some functional equations: to begin with, we solve some very classical equations which have already been treated by Abel using his own
method in [Ab]. Here we use some monodromy arguments to solve them. We finish with the “generalized Spence-Kummer equation of the trilogarithm” and with another one which will be interpreted in the framework of web geometry in part 4.1.

### 3.1 Cauchy equation revisited

Here we want to solve again the “generalized Cauchy equation” (C) in 3 unknowns

\[ F_1(x) + F_2(x) + F_3(x) = 0 \]  

by using monodromy arguments: we are interested in solutions the

We note \( U_1(x, y) = x, \ U_2(x, y) = y, \ U_3(x, y) := \frac{x}{y}, \) and \( W_C \) the web given by the three foliations, the leaves of which are respectively the level curves of \( U_1, U_2, \) and \( U_3. \) Its singular locus is \( \Sigma_C := \{(z, \zeta) \in \mathbb{C}^2 \mid z \zeta = 0 \}. \)

An easy computation gives us that \( U_i(\mathbb{C}^2 \setminus \Sigma_C) = \mathbb{C}^* \) for \( i = 1, 2, 3. \) We deduce that if \( (F_1, F_2, F_3) \in \mathcal{S}_C^3(C) \) where \( \omega = (1, 1) \notin \Sigma_C, \) then \( F_i \in \mathcal{O}(\mathbb{C}^*) \) for \( i = 1, 2, \) and \( 3. \)

In this case, condition (C) is verified. We are looking for the solutions of (C) the components of which are elements of the space \( \mathcal{I}_{(0)} \) of the iterated integrals on \( \mathbb{C}P^1 \setminus \{0, \infty\} \) relative to the rational 1-form \( \omega_0 := dz/z : \) we have \( \mathcal{I}_{(0)} := \text{Vect}_\mathbb{C} \{ \log^k(\zeta) \}_{k \in \mathbb{N}} \).

Let \( \gamma_1 \) be a loop with 1 as its base point turning around 0 in the direct sense: its homotopy class \( [\gamma] \) is a generator of \( \Pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}. \)

There are two loops in \( \mathbb{C}P^2 \setminus \Sigma_C, \) noted \( \gamma_1 \) and \( \gamma_2, \) such that we have \( U_i \circ \gamma_j = \gamma_1 \) if \( i = j, \) and \( U_i \circ \gamma_j = \gamma_2 \) is the constant path otherwise. By analytic continuation of (C) along \( \gamma_1, \) we get a new functional equation. By taking the difference between these two equations, it comes that in a neighbourhood of \( \omega, \) we have

\[ (F_1^{[\gamma]}(x) - F_1(x)) + (F_3^{[\gamma]}(x) - F_3(x)) = 0 \]

Both this equation and the one given by analytic continuation along \( \gamma_2 \) imply that there exists a constant \( a \in \mathbb{C} \) such that

\[ M_0 F_1 = F_1 + a, \quad M_0 F_2 = F_2 + a, \quad M_0 F_3 = F_3 - a \]

Considering these relations as equations of monodromy in the algebra \( \mathcal{I}_{(0)}, \) we get only one possible solution (modulo the constants)

\[ L := a \left( \int \omega_0, \int \omega_0, - \int \omega_0 \right) = a \left( \log(\cdot), \log(\cdot), - \log(\cdot) \right) \]

Using Bol’s bound of proposition 4, we obtain that, for all \( \omega \not\in \Sigma_C, \) modulo the constant solutions, \( \mathcal{S}_C^3(C) \) is spanned by any analytic continuation of \( L \) from \( \omega \) until \( \omega \) in \( \mathbb{C}P^1 \setminus \Sigma_C. \)

### 3.2 Arctangent equation revisited (see [Ab])

It is well known that the arctangent function \( \text{Arc}(\cdot) := \int_0^\cdot \frac{d\theta}{1+\theta^2} \) satisfies the functional equation

\[ \text{Arc}(x) + \text{Arc}(y) = \text{Arc} \left( \frac{x+y}{1-xy} \right) \]  

on the two real sets, \( \{xy < 1\} \) and \( \{xy > 1\}. \)

We consider a generalized version of (Arc) (with \( V_1(x, y) = \frac{x+y}{1-xy} \))

\[ G_1(U_1) + G_2(U_2) + G_3(V_1) = 0 \]  

(Arc)
Then the singular locus of the web associated to \((Arc)\) is

\[ \Sigma_{Arc} := \{(z, \zeta) \in \mathbb{C}^2 \mid (1-z \zeta)(1+z^2)(1+\zeta^2) = 0 \} \]

Let be \(\omega := (0,0) \notin \Sigma_{Arc}\). We want to determine \(S_0^\omega(Arc)\).

If \(A := (A_1, A_2, A_3) \in S_0^\omega(Arc)\), then, in the same way than in 3.1, we get that the \(A_i\)'s are global analytic functions ramified in \(+i\) and \(-i\) : \(A_j \in O_\infty(C \setminus \{\pm i\})\).

We are looking for solutions, the components of which are elements of the algebra \(\mathcal{I}_{(\pm i)}\).

By using the method of monodromy “a priori” we obtain the following relations of monodromy for the \(A_i\)’s:

\[
\begin{align*}
\mathcal{M}_i A_1 &= A_1 + a \\
\mathcal{M}_i A_2 &= A_2 + a \\
\mathcal{M}_i A_3 &= A_3 + a
\end{align*}
\]

where \(a \in \mathbb{C}\) is a constant.

The relations \((Mo)\) considered as equations in \(\mathcal{I}_{(\pm i)}^3\) admit a single possible solution (modulo the constants):

\[
A_1(\bullet) = A_2(\bullet) = -A_3(\bullet) = a \int_0^1 \omega_1 - a \int_0^1 \omega_2 = 2ia \int_0^1 \frac{dz}{1+z^2}
\]

For dimensional reasons, we obtain that, for all \(\omega \notin \Sigma_C\), modulo the constant solutions, \(\omega_0^\omega(Arc)\) is spanned by any analytic continuation in \(\mathbb{C}P^1 \setminus \Sigma_C\) of \((Arc, Arc, Arc)\) from \(\omega\) until \(\omega\).

### 3.3 Roger’s dilogarithm equation revisited (see [Bla-Bo], [Ro])

In [Ro], L. Rogers established a “clean version” of the equation \((L_2)\) verified by the Rogers dilogarithm \(d\), for \(0 < x < y < 1\):

\[
d(x) - d(y) - d\left(\frac{x}{y}\right) - d\left(\frac{1-y}{1-x}\right) + d\left(\frac{y(1-x)}{x(1-y)}\right) = 0 \quad (R)
\]

(here we have taken \(d(\bullet) := \log_2(\bullet) + \frac{1}{2}\log_2(1-\bullet)\log(1-\bullet) - \frac{1}{2}\pi^2\) : it is a normalized version of the original Rogers dilogarithm (by addition of \(-\pi^2/6\) in order to have 0 for the rhs of \((R)\)).

We consider the more general equation in 5 unknowns:

\[
D_1(x) + D_2(y) + D_3\left(\frac{x}{y}\right) + D_4\left(\frac{1-y}{1-x}\right) + D_5\left(\frac{y(1-x)}{x(1-y)}\right) = 0 \quad (R)
\]

We note \(W_R\) the singular web associated to the inner functions \(U_1, U_2, \ldots, U_4, U_5\) of \((R)\), where \(U_4(x,y) := \frac{1-x}{y} \) and \(U_5(x,y) := \frac{y(1-x)}{x(1-y)}\).

After computation we get that its singular locus is

\[ \Sigma_R := \{(z, \zeta) \in \mathbb{C}^2 \mid z \zeta (1-z)(1-\zeta)(z-\zeta) = 0 \} \]

We choose \(\omega := (\frac{i}{4}, \frac{i}{2}) \in \mathbb{R}^2 \setminus \Sigma_R\).

In [Bol], G. Bol found an equivalent of a basis of this space: in the framework of web geometry (see part 4.1. below), he determines a basis of the space of abelian relations of \(W_R\). We want to rediscover Bol’s results by application of our two “methods” described in part 2.5.

#### A) Resolution of \((R)\) by the method of “monodromy a priori”

By an easy computation we find that \(U_i(\mathbb{C}^2 \setminus \Sigma_R) = \mathbb{C} \setminus \{0, 1\}\). So, if \(D = (D_1, \ldots, D_5) \in S_0^\omega(\mathcal{R})\), then \(D_i \in O_{\infty}(\mathbb{C} \setminus \{0, 1\})\) for \(i = 1, \ldots, 5\).
In this case, equation \((\mathcal{R})\) can be solved by the method of monodromy a priori. We want to determine the solutions of \((\mathcal{R})\) the components of which are iterated integrals elements of \(I_{(0,1)}\).

We begin to search the 3-solutions of this type: we want to determine \(F^3\Sigma^3(\mathcal{R})\).

Our method of “monodromy a priori” works very well without difficulties and too many computations. It gives us the following 5 non-constant independent elements of \(F^3\Sigma^3(\mathcal{R})\):

\[
\begin{align*}
\Delta_1 & := \left( L_{x_0}, -L_{x_0}, -L_{x_0}, 0, 0 \right) & \Delta_2 & := \left( 0, 0, L_{x_0}, L_{x_0}, -L_{x_0} \right) \\
\Delta_3 & := \left( L_{x_1}, -L_{x_1}, 0, -L_{x_0}, 0 \right) & \Delta_4 & := \left( L_{x_1}, 0, -L_{x_1}, 0, L_{x_1} \right) \\
\Delta_5 & := \left( L_{x_1+x_0}, 0, -L_{x_1+x_0}, L_{x_1}, 0 \right)
\end{align*}
\]

Now we have to try to determine the last non-constant solution of \((\mathcal{R})\) if there is one. We will use our method again: we want to detail the computation to be well understood.

Let us consider the loop \(\gamma : [0,1] \to (\exp(2i\pi\sigma)/3,1/2) \in \mathbb{C}^2 \setminus \Sigma_\mathcal{R}\). The computations give

\[
\begin{align*}
[U_1 \circ \gamma] & = [c_0^1] & [U_2 \circ \gamma] & = [1] & [U_3 \circ \gamma] & = [c_0^2] \\
[U_4 \circ \gamma] & = [1] & [U_5 \circ \gamma] & = [c_0^5]
\end{align*}
\]

where these equalities are (respectively) in \(\Pi_1(\mathbb{C} \setminus \{0,1,\omega_i\}, \omega)\), for \(i = 1, \ldots, 5\).

So we have a new functional equation

\[
(D_1^{[c_0^1]}(U_1) - D_1(U_1)) + (D_3^{[c_0^2]}(U_3) - D_3(U_3)) + (D_5^{[c_0^5]}(U_5) - D_5(U_5)) = 0
\]

which corresponds to an element of \(F^3\Sigma^3(\mathcal{R})\). But we explicitly know this space, and in this case we obtain the following relations of monodromy for the components of any solution \(D\):

\[
\begin{align*}
\mathcal{M}_0 \, D_1 & = D_1 + a L_{x_1} + a_1 \\
\mathcal{M}_0 \, D_3 & = D_3 - a L_{x_1} + a_2 \\
\mathcal{M}_0 \, D_5 & = D_5 + a L_{x_1} - (a_1 + a_2)
\end{align*}
\]

where \(a, a_1, a_2, a_3\) are complex constants.

Now considering the path \(\sigma \in [0,1] \to \left(\frac{1}{3}, 1 - \frac{1}{2} \exp(2i\pi\sigma)\right)\) in \(\mathbb{C}^2 \setminus \Sigma_\mathcal{R}\) we get by the same way

\[
\begin{align*}
\mathcal{M}_1 \, D_2 & = D_2 + a' L_{x_0} + a'_1 \\
\mathcal{M}_0 \, D_4 & = D_4 + a' L_{x_1} + a'_2 \\
\mathcal{M}_0 \, D_5 & = D_5 - a' L_{x_1} - (a'_1 + a'_2)
\end{align*}
\]

From these relations it comes that \(a = -a'\) and \(a_1 + a_2 = a'_1 + a'_2\).

We can continue this type of computation and finally we get that the monodromy “a priori” of the components of holomorphic solutions of \((\mathcal{R})\) are \((a_i, b_j)\) being complex constants satisfying certain linear relations

\[
\begin{align*}
\mathcal{M}_0 D_j & = D_j - \epsilon_j a L_{x_1} + a_j \\
\mathcal{M}_1 D_j & = D_j + \epsilon_j a L_{x_0} + b_j
\end{align*}
\]

with \(\epsilon_j = 1\) for \(j = 1, 5\) and \(-1\) otherwise. It can be proved that the \(a_i\)’s and \(b_j\)’s are such that there exists a linear combination \(H = \sum \alpha_i \Delta_i\) such that the monodromy of the components of \(D' = (D'_j) := D + H\) verifies

\[
\begin{align*}
\mathcal{M}_0 D'_j & = D'_j - \epsilon_j a L_{x_1} \\
\mathcal{M}_1 D'_j & = D'_j + \epsilon_j a L_{x_0}
\end{align*}
\]
Now it is not difficult to prove that the function \( f = a \mathbf{d} \in \mathcal{I}_{\{0,1\}} \) satisfies the following monodromy equations

\[
\mathcal{M}_0 f = f - a \mathbf{L}_{x_1}, \quad \mathcal{M}_1 f = f + a \mathbf{L}_{x_0}
\]

We deduce that \( \Delta_k := (\mathbf{d} + c, -\mathbf{d}, -\mathbf{d}, -\mathbf{d}, \mathbf{d}) \in \mathcal{S}_0^C(\mathcal{R}) \), where \( c \) is a constant (in fact \( c = 0 \)).

We can easily construct a basis \( \{ \Delta_i \mid i = -3, -2, 0 \} \) of the constant solutions of \( (\mathcal{R}) \). It is not difficult to prove that the 10 elements \( \Delta_j \) described above are linearly independent.

On the other hand we have

\[
10 = \dim_C \left( \{ \Delta_j \} \right) \leq \dim_C \mathcal{S}_0^C(\mathcal{R}) \leq 5(5 - 1)/2 = 10
\]

where the last inequality is given by proposition 4. Then we deduce that

\[
\mathcal{S}_0^C(\mathcal{R}) = \left\{ \{ \Delta_j \mid j = -3, -2, 0 \} \right\}
\]

This solves \( (\mathcal{R}) \) at \( \omega \) in the holomorphic class. We get the local holomorphic solutions around \( \omega' \not\in \Sigma_\mathcal{R} \).

By analytic continuation of the \( \Delta_j \)'s along any path joining \( \omega \) to \( \omega' \) in \( X \).

**B) Resolution of \( (\mathcal{R}) \) by Abel's method**

A simple application of Abel's method implies that on the whole \( \Omega \), the first component of every solution of \( \mathcal{R} \) must verify the following linear differential equation

\[
\frac{d^4 g}{dv^4} + \frac{4(2v^3 - 3v^2 + v)}{v^2(1 - v)^2} \frac{d^3 g}{dv^3} + \frac{2(1 - 7v + 7v^2)}{v^2(1 - v)^2} \frac{d^2 g}{dv^2} + \frac{2(2v - 1)}{v^2(1 - v)^2} \frac{dg}{dv} = 0
\]

By integrating this equation, which can be done without great difficulty with a computer system, we find that it admits as general solutions the functions of the form \( c_1 \mathbf{d} + c_2 \mathbf{L}_{x_0} + c_3 \mathbf{L}_{x_1} + c_4 \), which is the form that any first component of any solution of \( (\mathcal{R}) \) can have.

**remark:** But even without integrating, the last equation gives us some informations: it admits three singular points, 0, 1 and \( \infty \). One can easily prove that they are regular points. By a classical theorem of the theory of linear differential equations with meromorphic coefficients, it comes that any solution of \( (\mathcal{R})_{\mathbb{C}^\mathcal{R}} \) a priori has moderate growth near 0, 1 and \( \infty \). Another remark is that the differential operator associated to this equation can be factorized into a product of differential operators of first order.

### 3.4 Spence-Kummer equation of the trilogarithm visited (see [Lew](#))

To the Spence-Kummer equation \( (SK) \) satisfied by \( \text{Li}_3 \) (with \( 0 < x < y < 1 \) ) we can associate the following abelian functional equation

\[
F_1(U_1) + F_2(U_2) + F_3(U_3) + \ldots + F_9(U_9) = 0 \quad (SK)
\]

where the \( U_i \)'s are the rational inner functions which appear in \( (SK) \): \( U_1, U_2, \ldots, U_5 \) have been defined above and we note

\[
U_6(x, y) = xy \quad U_7(x, y) = \frac{x(1 - y)}{x - 1}
\]

\[
U_8(x, y) = \frac{1 - y}{y(x - 1)} \quad U_9(x, y) = \frac{x(1 - y)^2}{y(1 - x)^2}
\]

We note \( \mathcal{W}_{SK} \) the planar web associated to \( U_1, \ldots, U_9 \). Its singular locus is

\[
\Sigma_{SK} = \{ (z, \zeta) \in \mathbb{C}^2 \mid z \zeta (1 - z)(1 - \zeta)(z - \zeta)(1 + \zeta)(1 + z) \times (1 - z\zeta - 2 - z - \zeta)(z\zeta - 2\zeta + 1)(z\zeta - 2\zeta - z) = 0 \}
\]

We choose again \( \omega = (\frac{1}{3}, \frac{1}{2}) \in \mathbb{R}^2 \setminus \Sigma_{SK} \). We want to find the local holomorphic solutions of \( (SK) \) at \( \omega \).

As in the case of the dilogarithm, we get that \( U_i(\mathbb{C}^2 \setminus \Sigma_{SK}) = \mathbb{C} \setminus \{0, 1\} \) so, if \( (F_1, \ldots, F_9) \in \mathcal{S}_{C}^C(\mathcal{SK}) \), then \( F_j \in \mathcal{O}_\omega(\mathbb{C} \setminus \{0, 1\}) \) for \( j = 1, \ldots, 9 \).
In this case, the method of monodromy a priori can be applied to find all the elements of $S_n^F(SK)$. Then by applying Abel’s method, we get next the missing solutions (noted $F_8, F_{10}, F_{15}, F_{16}$ and $F_{17}$ below).

One can verify that the 28 following 9-uplets of holomorphic germs are elements of $S_n^F(SK)$:

$$F_1 = \left( L_{x_0}, -L_{x_0}, -L_{x_0}, 0, 0, 0, 0, 0 \right)$$

$$F_2 = \left( L_{x_0 + x_1}, 0, -L_{x_0 + x_1}, L_{x_1}, 0, 0, 0, 0 \right)$$

$$F_3 = \left( L_{x_1}, L_{x_1}, 0, -L_{x_0}, 0, 0, 0, 0 \right)$$

$$F_4 = \left( 0, 0, L_{x_0}, L_{x_0}, -L_{x_0}, 0, 0, 0 \right)$$

$$F_5 = \left( L_{x_1}, 0, -L_{x_1}, 0, L_{x_1}, 0, 0, 0 \right)$$

$$F_6 = \left( L_{x_0}, L_{x_0}, 0, 0, 0, -L_{x_0}, 0, 0, 0 \right)$$

$$F_7 = \left( L_{x_0}, 0, 0, L_{x_0}, 0, 0, -L_{x_0} + i\pi, 0, 0 \right)$$

$$F_8 = \left( L_v, 0, 0, 0, L_v, 0, L_v - 1, 0, 0 \right)$$

$$F_9 = \left( L_{x_1}, 0, 0, 0, 0, -L_{x_1}, L_{x_1}, 0, 0 \right)$$

$$F_{10} = \left( 0, I_d, 0, I_d, 0, 0, I_d - 1, 0, 0 \right)$$

$$F_{11} = \left( 0, 0, 0, 0, 0, L_{x_0}, -L_{x_0}, L_{x_0}, 0 \right)$$

$$F_{12} = \left( 0, 0, L_{x_0}, 0, 0, 0, 0, L_{x_1}, -L_{x_1} \right)$$

$$F_{13} = \left( 0, 0, 0, 0, 0, L_{x_0}, L_{x_0}, -L_{x_0} - 2i\pi \right)$$

$$F_{14} = \left( 0, 0, 0, 0, L_{x_1}, 0, L_{x_1}, 0, -L_{x_1} \right)$$

$$F_{15} = \left( 0, I_v, 0, 0, I_d, 0, 0, I_d - 1, 0 \right)$$

$$F_{16} = \left( I_d, 0, 0, I_v, 0, 0, 0, L_v - 1, 0 \right)$$

$$F_{17} = \left( 0, 0, a, 0, 0, -a, 0, 0, -a \right)$$

$$F_{18} = \left( 2L_{x_0 x_0}, 2L_{x_0 x_0}, -L_{x_0 x_0}, 0, 0, -L_{x_0 x_0}, 0, 0, 0 \right)$$

$$F_{19} = \left( 0, 0, 0, 0, L_{x_0 x_0}, -2L_{x_0 x_0}, -2L_{x_0 x_0}, L_{x_0 x_0} + 4i\pi L_{x_0} - 4\pi^2 \right)$$

$$F_{20} = \left( 0, 0, L_{x_0 x_0}, -2L_{x_0 x_0}, -2L_{x_0 x_0}, 0, 0, 0, L_{x_0 x_0} \right)$$

$$F_{21} = \left( d, -d, -d, -d, d, 0, 0, 0, 0 \right)$$

19
\[
F_{22} = \left( d, d - \frac{i\pi}{2} L_{x_0}, 0, 0, 0, -d, d, -d, 0 \right)
\]
\[
F_{23} = \left( \pi^2, 0, 0, d - \frac{i\pi}{2} L_{x_0}, d, 0, d, d + \frac{i\pi}{2} L_{x_0} + i\pi L_{x_1}, -d \right)
\]
\[
F_{24} = \left( L_{x_0 x_1}, L_{x_0 x_1}, 0, L_{x_0 x_0}, 0, -L_{x_0 x_1}, L_{x_0 x_1}, -L_{x_0 x_1} - L_{x_0 x_0} + i\pi L_{x_0}, \frac{\pi^2}{3} \right)
\]
\[
F_{25} = \left( 0, L_{x_0 x_0}, 0, L_{x_0 x_1}, L_{x_0 x_1}, 0, L_{x_0 x_1}, L_{x_0 x_1}, -L_{x_0 x_1} \right)
\]
\[
F_{26} = \left( 2 L_{x_0 x_1}, 0, -L_{x_0 x_1}, 0, 2 L_{x_0 x_1}, -L_{x_0 x_1}, 2 L_{x_0 x_1}, 0, -L_{x_0 x_1} \right)
\]
\[
F_{27} = \left( 2 g_2, 2 g_2, -g_2, 2 g_2, 2 g_2, -g_2, 2 \hat{g}_2, 2 \hat{g}_2, -g \right)
\]
\[
F_{28} = \left( 2h(\bullet), 2h(\bullet) - \frac{2\pi^2}{3} L_{x_0}, -h(\bullet), 2h(\bullet), 2h(\bullet), -h(\bullet), 2h(\bullet), 2h(\bullet), -h(\bullet) \right)
\]

with
\[
I_d := \text{Id}_\mathbb{C}
\]
\[
I_c := 1/\text{Id}_\mathbb{C}
\]
\[
a : \bullet \to \text{arcth} (\sqrt{\bullet})
\]
\[
d := L_{x_0 x_1} - L_{x_1 x_0} - \frac{\pi^2}{6}
\]
\[
g := 2 L_{x_0 x_1} - L_{x_0 x_1} - L_{x_1 x_0} - \frac{2}{3} L_{x_1}(1)
\]
\[
\hat{g} := g + i\pi L_{x_0 x_1} - 4i\pi L_{x_1 x_0} - \pi^2 L_{x_1} + 2i\pi^3
\]
\[
h := L_{x_0 x_1} - L_{x_1 x_0}
\]
\[
\hat{h} := h - i\pi L_{x_0 x_1} + 2i\pi L_{x_1 x_0} + \frac{\pi^2}{2} L_{x_1} - \frac{2\pi^3}{6}
\]

Let \( \{ F_i \mid l = -7, ..., 0 \} \) be a basis of the space of the constant solutions of \( SK \). Then it’s just a tedious exercise of linear algebra to verify that the \( F_i \)'s (for \(-7 \leq i \leq 28\)) are 36 linearly independent elements of \( S^C_0(SK) \). Then it comes that
\[
36 = \dim_\mathbb{C} \langle \{ F_j \} \rangle \leq \dim_\mathbb{C} S^C_0(SK) \leq 9(9 - 1)/2 = 36
\]

(the last inequality comes from proposition 4).

So we have
\[
S^C_0(SK) = \langle \{ F_j \mid -7 \leq j \leq 28 \} \rangle
\]

This solves \( SK \) at \( \omega \) in the holomorphic class. We get the the local holomorphic solutions around \( \omega' \not\in \Sigma_{SK} \) by analytic continuation of the \( F_j \)'s.

### 3.5 An Afe associated to a degenerate configuration of 5 points

Here we are considering the following Afe
\[
G_1(x) + G_2(y) + G_3(y) + G_4(\frac{1-y}{1-x}) + G_5(\frac{x(1-y)}{y(1-x)}) \quad (\mathcal{E}_c)
\]
\[
+ G_6(\frac{1+x}{1+y}) + G_7(\frac{x(1+y)}{y(1+x)}) + G_8(\frac{(1-y)(1+x)}{(1-x)(1+y)}) = 0
\]

We set \( V_6(x, y) = \frac{1+x}{1+y} \), \( V_7(x, y) = \frac{x(1+y)}{y(1+x)} \), \( V_8(x, y) = \frac{(1-y)(1+x)}{(1-x)(1+y)} \) and \( V_i = U_i \) for \( i = 1, ..., 5 \).

We note \( \mathcal{W}_c \) the web associated to the \( V_i \)'s; we will see in the next part that it is associated to a configuration of 5 points. A simple computation gives us its singular locus \( \Sigma_c \).

We take \( \omega = (1/3, 1/2) \in \mathbb{R} \setminus \Sigma_c \). We want to determine the space \( S^C_0(\mathcal{E}_c) \).

By applying Abel’s method, the author has constructed the following 21 elements of \( S^C_0(\mathcal{E}_c) \):

```markdown
20
```
\begin{align*}
G_1 &= \left( 0, 0, 2j, j, 0, -j, 0, -1 \right) & G_2 &= \left( 0, 0, 2j, 0, -j, 0, -j, 0 \right) \\
G_3 &= \left( 1, 0, 0, 0, 0, j, -j, -2j \right) & G_4 &= \left( L_{x_0}, -L_{x_0}, -L_{x_0}, 0, 0, 0, 0, 0 \right) \\
G_5 &= \left( L_{x_0+x_1}, 0, -L_{x_0+x_1}, L_{x_1}, 0, 0, 0, 0 \right) & G_6 &= \left( -L_{x_1}, L_{x_1}, 0, L_{x_0}, 0, 0, 0, 0 \right) \\
G_7 &= \left( 0, 0, L_{x_0}, -L_{x_0}, 0, 0, 0 \right) & G_8 &= \left( L_{x_1}, 0, -L_{x_1}, 0, L_{x_1}, 0, 0, 0 \right) \\
G_9 &= \left( L_{x_0-x_1}, 0, -L_{x_0+x_1}, 0, 0, L_{x_0+x_1}, 0, 0 \right) & G_{10} &= \left( L_{x-1}, -L_{x-1}, 0, 0, -L_{x_0}, 0, 0 \right) \\
G_{11} &= \left( 0, 0, L_{x_0}, 0, -L_{x_0}, -L_{x_0}, 0 \right) & G_{12} &= \left( L_{x-1}, 0, L_{x_1}, 0, 0, 0, -L_{x_1}, 0 \right) \\
G_{13} &= \left( 0, 0, 0, L_{x_0}, 0, L_{x_0}, 0, -L_{x_0} \right) & G_{14} &= \left( -L_2, L_{x-1}, 0, L_{x_1}, 0, 0, 0, -L_{x_1} \right) \\
G_{15} &= \left( d, -d, -d, -d, d, 0, 0, 0, 0 \right) & G_{16} &= \left( L_{x-1-x_0-x_1}, -L_{x-1-x_0-x_1}, -2d, 0, 0, 2d, 2d, 0 \right) \\
G_{17} &= \left( 0, 0, 8c - 10j, -c - j, -c, -c - j, -c + 4j, 8c \right) & G_{18} &= \left( L_{x-1-x_1-x_1}, + L_2 L_{x_1+x_1}, -L_{x-1-x_1-x_1}, -L_{x_1+x_1}, 0, -2d, 0, -2d, 0, 2d \right) \\
G_{19} &= \left( L_{x_0 x_1+x_0 x_1}, L_{x_0 x_1+x_0 x_1}, 0, -L_{x_0 x_1+x_0 x_1}, L_{x_0 x_1+x_0 x_1}, L_{x_0 x_1+x_0 x_1}, -L_{x_0 x_1+x_0 x_1}, -\frac{\pi^2}{12} \right) \\
G_{20} &= \left( L_{x-1}, -L_{x-1}, -2b, b, b, b, b, -2b \right) & G_{21} &= \left( L_{x-1-x_0-x_1}, L_{x_1-x_1-x_1}, L_{x_1-x_1-x_1}, -L_2 L_{x_0+x_1}, c_{21}, -L_{x_1+x_0 x_1}, + L_2 L_{x_0+x_1}, -L_{x_1+x_0 x_1}, + L_2 L_{x_0+x_1}, 0, L_{x_1+x_0 x_1} \right) \\
\end{align*}

with

\[ j = \frac{1}{1 - \bullet} \quad b = \frac{L_{x_0}}{1 - \bullet} \quad c = \frac{1}{(1 - \bullet)^2} \]

\[ L_2 = \log(2) \quad c_{21} = \frac{\pi^2}{6} - \frac{1}{2} \log^2(2) \]

(\text{the constant } c_{21} \text{ had been determined numerically with the precision of } 10^3 \text{ digits}).

Let \( \{ G_i \mid i = -6, \ldots, 0 \} \) be a basis of the space of the constant solutions of \((E_c)\). Then it’s just a tedious exercise of linear algebra to verify that the \( G_i \)’s (for \(-6 \leq i \leq 21\)) are 28 linearly independent elements of \( S^C(E_c) \). Then it comes that

\[ 28 = \dim_C \langle \{ G_j \} \rangle \leq \dim_C S^C(E_c) \leq 8(8 - 1)/2 = 28 \]

So we have

\[ S^C(E_c) = \langle \{ G_j \mid -6 \leq j \leq 21 \} \rangle \]
4 Application to web theory and to the characterization of polylogarithmic functions of order $\leq 3$ by their functional equation

In the introduction we noticed that abelian functional equations arise in many areas of mathematics. We now give some applications of the material exposed in the preceding part to two of these areas: planar web geometry and theory of polylogarithmic functional equations.

4.1 Application to web theory

4.1.1 A brief introduction to planar web geometry

We now briefly recall, in the analytic setting, the basic notions of planar web geometry (the standard reference is [Bla-Bd]. See [Ak-Gc], [Chd], [Ch-Gr] or [Web] for more modern points of view).

A planar N-web $W$ on a domain $\Omega$ in a 2-dimensional complex manifold $X$, is the data of an unordered set $\{F_i\}$ of $N$ foliations of $\Omega$ such that their leaves are in general position. We are interested in the geometric local study of these webs to which we want to attach some invariants. A classical example of web is the "algebraic N-web $W_C$" associated to an algebraic reduced curve $C \subset \mathbb{CP}^2$ of degree $N$: let $L_0$ be a generic line in $\mathbb{CP}^2$ which transversally intersects the regular part of $C$ in $N$ points: $C.L_0 = P_1(L_0) + \cdots + P_N(L_0)$.

There exists an open neighbourhood $\Omega_0$ in the dual projective space $\mathbb{CP}^{2*}$ and there are $N$ holomorphic maps $P_i : \Omega_0 \to C$ such that all $L \in \Omega_0$ transversally intersect $C$ and $C.L = P_1(L) + \cdots + P_N(L)$. Let $F_i$ be the foliation of $\Omega_0$, the leaf of which at $L$ is the segment line $\{P_i = P_i(L)\}$. Then $W_C = \{F_i\}_{i=1..N}$ is a web on $\Omega_0$. Because the leaves of the foliations are segments of straight lines, $W_C$ is a "linear web".

The problem of the linearization of (germs of) planar webs was a central one. It has recently been solved in all its generality by M. Akivis, V. Goldberg and V. Lychagin: for a N-web, they give $N-2$ differential invariants which vanish if and only if the web is linearizable (see [Ak-Go-Ly]).

Let us consider our algebraic web $W_C$ again. Assume that $C$ is smooth (to simplify) and let $\omega$ be a differential of the first kind on $C$. Abel's theorem implies that $\sum P_i^* \omega = 0$ on $\Omega_0$: the abelian sums vanish.

Let $W = \{F_i\}_{i \leq N}$ be a (germ of) web at the origin in $\mathbb{C}^2$. Then there exists $N$ germs of holomorphic maps $U_i$ such that the leaves of $F_i$ are the level curves of $U_i$. We copy the notion of abelian sum for this general web $W$: an abelian relation for $W$ (relatively to the $U_i$'s) will be an equation of the form $\sum G_i(U_i) dU_i = 0$ in the space $\Omega_1^k$ of holomorphic germs of 1-form at the origin in $\mathbb{C}^2$. The space of abelian relations of $W$ relatively to the $U_i$'s will be noted $\mathcal{A}(W)$. It has a natural structure of linear space. By definition, the rank of $W$ will be

$$r_k(W) := \dim_{\mathbb{C}} \mathcal{A}(W)$$

The general version of proposition 4 is that the rank is always finite and that $r_k(W) \leq (N-1)(N-2)/2$.

The rank is a well defined invariant for webs (up to local diffeomorphisms). Using the notations of this paper, we see that, modulo the constant solutions, we have an isomorphism between $\sum E_i(U_i)$ and $\mathcal{A}(W)$ where $E_i(U_i)$ is the Abel $F_i(U_1) + \cdots + F_i(U_N) = 0$. Then the results of part 2 may be seen, in the framework of web geometry, as tools to study the abelian relations of webs the foliations of which are the level curves of real rational functions. We will see that such webs are of particular interest.

For our algebraic web $W_C$, because

$$\dim_{\mathbb{C}} H^0(C, \Omega_C^1) = \frac{(N-1)(N-2)}{2} \geq r_k(W_C)$$

we obtain that the following map is an isomorphism:

$$H^0(C, \Omega_C^1) \to \mathcal{A}(W_C)$$

$$\omega^\lambda \mapsto \sum P_i^* \omega^\lambda$$

Therefore an algebraic web is a linear web of maximal rank.

The Abel inverse theorem (due to Lie for $N = 4$, and generalized by Poincaré, Darboux, Griffiths [Gr] and Henkin [Hen]) tells us that a web of maximal rank is algebraic if and only if it is linearizable.
It is well known that N-webs of maximal rank are linearizable (therefore algebraic) when \( N = 3, 4 \) (the case \( N = 3 \) is easy, and \( N = 4 \) is due to some work by Lie on translation surfaces. A naive idea would be that all maximal rank web are linearizable and therefore algebraic. But this is no longer true for \( N \)-webs with \( N \geq 5 \); in \([\text{Bol2}]\), G. Bol gave an example of a non-linearizable 5-web of maximal rank which cannot be algebraic: this web (now known as “Bol’s web” and noted \( \mathcal{B} \)) is the web the foliations of which are the level curves of the \( U_i \)’s of equation \( (\mathcal{R}) \) in part 3.3. From the elements \( \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5 \) and \( \Delta_6 \) of \( \mathcal{S}_0^1(\mathcal{R}) \) we can construct a basis of the space \( \mathcal{A}(\mathcal{B}) \) of the abelian relations of \( \mathcal{B} \) at any generic point \( \omega_0 \).

So we have \( \dim \mathcal{A}(\mathcal{B}) = 6 \) and the web \( \mathcal{B} \) is of maximal rank 6, although it is not linearizable. From its discovery by Bol in the 30’s onwards, this has been the single known counterexample to the problem of linearization of planar webs of maximal rank.

Such webs, which looked very special, are called “exceptional webs” (see section 6 in \([\text{Ak-Ge}]\) and part 3.2 and 3.3 of \([\text{He}]\) for the problem of linearization of webs of maximal rank).

4.1.2 exceptional planar webs and configuration of points

According to Chern and Griffiths (see \([\text{Ch-Gr2}]\) page 83), classifying the non linearizable maximal rank webs is the fundamental problem in web geometry. Since Bol’s web is related to the functional equation of Rogers dilogarithm, the “Spence-Kummer web” \( \mathcal{W}_{SK} \) associated to the Spence-Kummer equation of the trilogarithm “seems to be a good candidate as an exceptional 9-web”, as noticed by A. Hénaut in part 3.3 of \([\text{He}]\). We now prove that this web actually is exceptional. The explicit resolution of the equations \( (\mathcal{S}_0^1) \) and \( (\mathcal{E}_2) \) done respectively in parts 3.4 and 3.5 allows us to find other new examples of such “exceptional webs”.

We first study \( \mathcal{W}_{SK} \) and its subwebs. For any subset \( J \subset \{1, .., 9\} \) we note \( \mathcal{W}_J \), the \( |J| \)-subweb of \( \mathcal{W}_{SK} \) given by the level curves of the function \( U_j \), with \( j \in J \). If \( j_1, ..., j_p \) are p distinct integers in \( \{1, .., 9\} \), then we note \( j_1, ..., j_p := \{1, .., 9\} \setminus \{j_1, ..., j_p\} \).

**Theorem 2:**

- \( \mathcal{W}_{69}^{238} \) is an exceptional 9-web
- \( \mathcal{W}_{39}^{23} \) is an exceptional 6-web

Those two exceptional 6-webs are not equivalent.

- \( \mathcal{W}_{369}^{679} \) is an hexagonal 6-web.

**proof:** For each of these webs, we have to prove two distinct things: the first is that the rank is maximal, the second is that the web is non-linearizable.

From the basis \( \{\mathcal{F}_j\} \) of \( \mathcal{S}_0^0(\mathcal{W}_{SK}) \) described in 3.4, we can easily construct 28 linearly independent abelian equations for \( \mathcal{W}_{SK} \), which is of maximal rank. For \( \omega \) generic, we have natural linear inclusions \( \mathcal{S}_0^0(\mathcal{W}_J) \rightarrow \mathcal{S}_0^0(\mathcal{W}_{SK}) \). From this we can easily deduce an explicit basis of the spaces \( \mathcal{S}_0^0(\mathcal{W}_J) \), and so of the space \( \mathcal{A}(\mathcal{W}_J) \) for any subset \( J \subset \{1, .., N\} \). So we can calculate the rank of any sub-web of \( \mathcal{W}_{SK} \): all the webs in proposition 4 have maximal rank.

Let us note \( \mathcal{W} \) an exceptional web of proposition 6 distinct from \( \mathcal{W}_{238}^{238} \). We remark that \( \mathcal{W} \) contains \( \mathcal{B} \) as a 5-subweb. Because \( \mathcal{B} \) cannot be linearized, the same is true for \( \mathcal{W} \) which thus is exceptional. We can’t use this argument to prove that \( \mathcal{W}_{238}^{238} \) is not linearizable: it’s easy (but tedious) to see that all the 5-subwebs of \( \mathcal{W}_{238}^{238} \) have rank 5 and so are not equivalent to Bol’s web (this already shows that \( \mathcal{W}_{238}^{238} \) and \( \mathcal{W}_{679}^{679} \) are not equivalent).

One can verify that its associated polynomial in \( \mathcal{P}_{\mathcal{W}_{238}^{238}} \) (see \([\text{He2}]\) for a definition) is of degree \( >3 \).

**Theorem 2** in \([\text{He2}]\) says that if \( \mathcal{W} \) is a web of maximal rank, then it is linearizable if and only if \( \mathcal{P}_{\mathcal{W}} \) is of degree smaller than 3. Because \( \mathcal{W}_{238}^{238} \) is of maximal rank, it implies that it is exceptional.

**remarques:**

1. The sub-webs \( \mathcal{W}_{369}^{369} \) and \( \mathcal{W}_{39}^{39} \) are exceptional too but equivalent to \( \mathcal{W}_{69}^{69} \).
2. The sub-webs \( \mathcal{W}_{689}^{689} \), \( \mathcal{W}_{429}^{429} \), \( \mathcal{W}_{236}^{236} \), \( \mathcal{W}_{136}^{136} \), and \( \mathcal{W}_{136}^{136} \) are exceptional too but equivalent to \( \mathcal{W}_{679}^{679} \).
3. The sub-webs \( \mathcal{W}_{147}^{147} \), \( \mathcal{W}_{257}^{257} \), and \( \mathcal{W}_{358}^{358} \) are exceptional too but equivalent to \( \mathcal{W}_{238}^{238} \).
4. The exceptional d-subwebs of \( \mathcal{W}_{SK} \) (with \( d \geq 6 \)) are those which are described in proposition 9 and
in the above remarks 1,2 and 3.

5. We have a beautiful functional equation associated to \( W_{369} \) for \( L_1 \). It is given by the element \( F_{26} \) of part 3.4:

\[
2L_1(x) - L_1(\frac{x}{y}) + 2L_1(\frac{x(1-y)}{y(1-x)}) - L_1(xy) + 2L_1(-\frac{x(1-y)}{1-x}) - L_1(\frac{x(1-y)^2}{y(1-x)}) = 0
\]

It is equivalent to Newman’s functional equation of the bilogarithm (see formula (1.43) in [Lew], page 13).

By Bol’s theorem (see [Bla-Bo] page 108), the fact that \( W_{369} \) is hexagonal implies that it is linearizable into a web formed by 6 pencils of lines, therefore, by duality, it is associated to a configuration of 6 points on \( \mathbb{CP}^2 \). A linearization for this web is given by the quadratic Cremona transform \( C : (x, y) \to (1/(x-1), 1/(y-1)) \). It is natural to ask what is the action of \( C \) on the whole web \( W_{SK} \). We introduce some definitions. For \( d > 0 \), let \( \delta_d \) be the dimension of the space of algebraic curves of degree \( d \) in \( \mathbb{CP}^2 \).

**Definition 3** Let be \( d \geq 1 \). If \( K \) is a set of \( \delta_d - 1 \) points in general position in the complex projective plane, then the family of the curves of degree \( d \) through these \( \delta_d - 1 \) points is noted \( F_K \). It is a singular foliation of \( \mathbb{CP}^2 \).

For \( n \geq 3 \), we note \( \Delta_n = \bigcup_{i<j} \{(p_1, ..., p_n)\} \in (\mathbb{CP}^2)^n \mid p_i = p_j \}. We define the space of configuration of \( n \) points in \( \mathbb{CP}^2 \) as the set \( \mathbb{C}_n = (\mathbb{CP}^2)^n - \Delta_n \). If three distinct points \( p_i, p_j, p_k \) of a configuration \( (p_1, ..., p_n) \) lie on a same line, the configuration is said “degenerate”.

**Definition 4** Let be \( N \geq 3 \). The web \( W_p \) associated to a configuration \( p = (p_1, ..., p_N) \) of \( N \) points in \( \mathbb{CP}^2 \) is the singular web defined on the whole plane, the foliations of which are the \( F_J \)’s where \( J \) runs on the set of subsets of \( \delta_d - 1 \) points in \( \{p_1, ..., p_N\} \), in general position, with \( 1 \leq j \leq N \).

It is well known that Bol’s web \( B \) is associated to a configuration of 4 points in generic position in the projective plane. More precisely, the web given by the level curves of the functions \( U_1, ..., U_5 \) is the web associated (in the sense of definition 4) to the configuration \( b \) described by figure 1 below.

For the Spence-Kummer web \( W_{SK} \), we have the following

**Proposition 6** The web \( W_q \) associated to the degenerate configuration \( q \) of 6 points in \( \mathbb{CP}^2 \) given by figure 2 above is the image of \( W_{SK} \) by \( C \).

The web \( W_c \) in 3.5, which is of maximal rank 21, is also associated to a configuration noted \( c \) and defined by figure 3 below.

Because configuration \( b \) is a subconfiguration of configuration \( c \), Bol’s web is a sub-web of \( W_c \). Then this web is non-linearizable and since it is of maximal rank (see part 3.5), it comes the

**Proposition 7** The web \( W_c \) associated to \( c \) is an exceptional planar 8-web.
The exceptional 6-subweb $W_{679}$ of $W_{SK}$ is associated too with a sub-configuration of $q$.

**Proposition 8** The image of the exceptional web $W_{679}$ by $C$ is the web associated to the subconfiguration $(q_1, q_2, q_3, q_4, q_5)$ of $q$.

The other exceptional subwebs of $W_{SK}$ must also be associated to configurations of points but in a more complicated way than in definition 4.

The fact that the only known exceptional planar webs described above are related to configurations of points may be an important fact which should be studied.

In [Dam], D. Damiano considers some webs of curves in $\mathbb{R}^N$, $(N \geq 2)$ similarly associated to configurations of points. He shows that those webs are exceptional curvilinear webs.

All this results allow to think that it could exist a real link between configurations of points and exceptional webs. In this spirit we have the following general results:

**Proposition 9** The web associated to any configuration of 4 points in $\mathbb{CP}^2$ is of maximal rank. It is non linearizable only if the configuration is generic: then it is (projectively) equivalent to Bol’s web $B$.

and for configurations of 5 points in the plane:

**Proposition 10** The web associated to any degenerate configuration of 5 points in $\mathbb{CP}^2$ is of maximal rank.

**sketch of the proof:** We consider the stratification of $\mathbb{CP}^2$ described by figure 4:

- $S_0$ is the open subset of generic configurations.
- $S_1$ is the analytic strata of degenerate configurations such that three and only three points are lying on a same line.
- $S_2$ is the analytic strata of degenerate configurations such that exactly four points lie on a same line.
- $S_3$ is the analytic strata of degenerate configurations $(p_1, .., p_5)$ outside $S_2$ such that there exists a unique $p_j$ such that for all $i \neq j$ there exists $k$ distinct from $i$ and $j$ such that the three points $p_i, p_j$ and $p_k$ are aligned.
- $S_4$ is the analytic strata of degenerate configurations such that the five points are aligned.

Each strata $S_i$ is a smooth connected analytic subvariety of $\mathbb{CP}^2$.

We note $N_0 = 10$, $N_1 = 8$, $N_2 = 5$, $N_3 = 6$, and $N_4 = 5$. For each $i \in \{0, .., 4\}$, the web $W_p$ associated to a configuration $p \in S_i$ is a $N_i$-web.
Figure 4:
stratification of $C^2_5$ by degenerate configurations

An arrow $A \rightarrow B$ between two stratas means that $B \subset \partial A$ in $C^2_5$.

The natural action of $PGL_3(\mathbb{C})$ on $\mathbb{CP}^2$ induces a group action $q = (q_1, \ldots, q_5) \rightarrow q^g = (g(q_1), \ldots, g(q_5))$ on $C^2_5$. Two webs $W_q$ and $W_p$ are projectively equivalent if and only if $q$ and $p$ belong to the same orbit. Then for any orbit $O \subset C^2_5$, we consider a particular configuration $p_0 \in O$. We prove that the rank of $W_{p_0}$ is maximal by constructing a basis of the space $S^C_0(W_{p_0})$ at a generic $\omega$. Moreover the rank is a local invariant of the webs. All this implies proposition 10. We skip here the explicit determinations of the spaces of abelian relations of the webs $W_{p_0}$.

remarks:
1. For $a \in \mathbb{C} \setminus \{0, 1\}$, we note $c_5^a = [a : a : -1]$ and $c_a = (c_1, \ldots, c_4, c_5^a) \in S_1$.
   The web $W_{c_a}$ is exceptional: it is non-linearizable and, as in part 3.5, we can construct a basis of dimension 21 of its space of abelian relations. This gives us a family of exceptional webs non-projectively equivalent. It would be interesting to know if they are locally equivalent or not.

2. For a generic configuration of 5 points, this proposition is not proved for the moment.

All the results above allow us to state the

Conjecture 1. The web $W_q$ associated to any configuration $q$ of points in the projective plane is of maximal rank.

By an argument used above, we see that $W_q$ is non-linearizable as soon as $q$ contains a sub-configuration of 4 points in general position, so conjecture 1 may give us a list of exceptional webs.

It is under the inspiration of this conjecture than the author has studied the $A\text{fe}$ ($E_x$).

The results of part 3 show that most abelian functional equations of webs associated to configurations of points studied in part 4.1 are constructed from iterated integral functions. If conjecture 1 is true, there could exist numerous $A\text{fe}$ linked to the exceptional webs associated to configurations. We can expect that some of those $A\text{fe}$ may be constructed from iterated integrals too. This could be a way to find new functional equations for higher order polylogarithms, which would be useful for the K-theoretical study of algebraic number fields (see for instance [Za2] or [Gan]).
4.2 application to the problem of characterizing polylogarithmic functions by their functional equation

Our objective here is to study the function which satisfies the equation $(L_2)$ or $(SK)$. This kind of problem has been studied for a long time for the Cauchy equation $(C)$: we know that any non-constant measurable local solution of $(C)$ is constructed from the logarithm. The explicit resolution of equations $(R)$ and $(SK)$ done in part 3 allows us to get the same kind of results for the dilogarithm and the trilogarithm: these functions are respectively “characterized” (in the measurable class) by their functional equation $(L_2)$ and $(SK)$.

4.2.1 Characterization of the dilogarithm by Rogers equation $(R)$

We first have this result which comes easily from a result established by Rogers in the early 20th century (see [Ro] section 4)

**Proposition 11** If $F$ is a function of class $C^3$ on $]0,1[$ such that

\[(*) \quad F(x) - F(y) - F(\frac{x}{y}) - F(\frac{1-y}{1-x}) + F(\frac{x(1-y)}{y(1-x)}) = 0\]

for $0 < x < y < 1$, then we have $f = \alpha d$ where $\alpha \in \mathbb{R}$.

The proof is essentially an application of Abel’s method to this case. Proposition 1 allows us to see that proposition 11 is still valid under the weaker assumption of measurability on $F$.

The dilogarithm has a single valued version: the Bloch-Wigner function $L_2(z) = \Im \log(1-z) \log |z|$ which is real analytic on $CP^1 \setminus \{0,1,\infty\}$ and extends to the whole projective line by continuity. For this function, the functional equation $(L_2)$ becomes

\[\bigcirc \quad \sum_{i=1}^{4} (-1)^i L_2(c_r(z_0, \ldots, z_i, \ldots, z_4)) = 0, \quad z_i \in \mathbb{C}P^1\]

where $c_r(z_1, \ldots, z_4)$ denotes the cross ratio of 4 points. The equation $(\bigcirc)$ takes the form $(*)$ when we take $(z_1, \ldots, z_4) = (\infty, 0, 1, y, x)$.

In [Blo], Bloch proves the following characterization of $L_2$ in the measurable class by the equation $(\bigcirc)$.

**Proposition 12** Let $f : \mathbb{C}P^1 \to \mathbb{R}$ be measurable and satisfying $(\bigcirc)$. Then $f$ is proportional to $L_2$.

Using proposition 1 and Bol’s discovery of the space $A(R)$ (see 3.3), we can state

**Proposition 13** If $F, G$ are measurable functions on $]0,1[$ satisfying

\[F(x) - F(y) - F(\frac{x}{y}) - F(\frac{1-y}{1-x}) + G(\frac{x(1-y)}{y(1-x)}) = 0\]

for $0 < x < y < 1$, then we have $F = G = \alpha d$ with $\alpha \in \mathbb{R}$.

(We can prove this result by a direct application of Abel’s method, because by proposition 1, the functions $F$ and $G$ of proposition 14 are analytic). In the class of measurable functions, this result gives us a semi-local characterization of Roger’s dilogarithm by its functional equation $(R)$, for two unknown functions. In a certain sense, it’s stronger than the results by Rogers and Bloch. The explicit knowledge of $SO(R)$ allows us to state numerous variants of proposition 13.

Those results can be formulated in an inhomogeneous form to obtain some characterization of $Li_2$ by functional equations inspired from $(L_2)$. 27
4.2.2 characterization of the trilogarithm by Spence-Kummer equation (SK).

The fact that the logarithm and dilogarithm are characterized by the Afe with rational inner functions which they verify naturally leads us to ask if the same is true for any trilogarithmic function.

In his paper [Gon3], A. Goncharov obtains some results of this kind: he considers the real single-valued cousin of \(L_3\) introduced by Ramakrishnan and Zagier:

\[ L_3(z) := \Re \left( \text{Li}_3(z) - \text{Li}_2(z) \log |z| + \frac{1}{3} \text{Li}_1(z) \log |z|^2 \right) \]

defined on the whole \(\mathbb{CP}^1\) and extended to \(\mathbb{R}[\mathbb{CP}^1]\) by linearity.

When it is well defined, he considers the following element of \(\mathbb{Q}[\mathbb{CP}^1]\):

\[
R_3(\alpha_1, \alpha_2, \alpha_3) := \sum_{i=1}^{3} \left( \{ \alpha_{i+2} \alpha_i - \alpha_i + 1 \} + \{ \frac{\alpha_{i+2} \alpha_i - \alpha_i + 1}{\alpha_{i+2} \alpha_i} \} \right) + \{ \frac{\alpha_{i+2} \alpha_i - \alpha_i + 1}{\alpha_{i+2} \alpha_i} \} - \{ \frac{\alpha_{i+2} \alpha_i - \alpha_i + 1}{\alpha_{i+2} \alpha_i} \} - \{ 1 \}
\]

for \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{CP}^1\). (The indices \(i\) are taken modulo 3).

Next he proves that we have the functional equation in 22 terms

\[(** \quad L_3(R_3(a, b, c)) = 0 \quad a, b, c \in \mathbb{C})\]

Then he shows (part (a) of Theorem 1.10 in [Gon3]) that

"the space of real continuous functions on \(\mathbb{CP}^1 \setminus \{0, 1, \infty\}\) that satisfy the functional equation \((**)\) is generated by the functions \(L_3(z)\) and \(L_2(z) \log |z|\)."

(In fact, what he proves implies that this theorem is valid for measurable functions).

He had remarked before that, if we specialize this equation by setting \(a = 1, b = x, c = \frac{1-y}{x}\), the equation \((**\) simplifies and by using the inversion relation \(L_3(x^{-1}) = L_3(x), x \in \mathbb{CP}^1\), we obtain a homogeneous version (i.e. without the right hand side \(R_3(x, y)\)) of equation \((SK)\).

This leads him to ask if this specialization characterizes the solutions of \((**)\).

The explicit determination of a basis of \(S_0^0(SK)\) done in part 3.4. allows us to give a positive answer to this question: we have this real semi-local characterization of \(L_3\):

**Theorem 3** Let \(G : -\infty, 1[ \to \mathbb{R}\) be a measurable function such that for \(0 < x < y < 1\) we have

\[
2 G(x) + 2 G(y) - G \left( \frac{x}{y} \right) + 2 G \left( \frac{1-y}{1-x} \right) + 2 G \left( \frac{x(1-y)}{y(1-x)} \right) - G(xy) = 2 \text{Li}_3(1)
\]

Then if we suppose \(G\) continuous at 0, then there exists \(\alpha \in \mathbb{R}\) such that

\[
G = \alpha L_3 + \frac{2}{3} (1-\alpha) \text{Li}_3(1)
\]

With our results of part 2 and 3.4, the proof is just a tedious exercise of linear algebra. (It can be proved again by a suitable application of Abel’s method). It implies this result for \(\text{Li}_3\):

\[ \]
corollary 2 Let \( F : [\xi, \eta] \to \mathbb{R} \) be a measurable function such that for \( 0 < x < y < 1 \), we have

\[
2 F(U_1(x,y)) + 2 F(U_2(x,y)) - F(U_3(x,y)) + 2 F(U_4(x,y)) + 2 F(U_5(x,y)) - F(U_6(x,y)) + 2 F(U_7(x,y)) + 2 F(U_8(x,y)) - F(U_9(x,y)) = R_3(x,y)
\]

- If \( F \) is continuous at 0 then there exists \( \alpha \in \mathbb{R} \) such that

\[
F = L_{i3} + \alpha (L_3 - \frac{2}{9} L_{i5}(1))
\]

- If \( F \) is derivable at 0 then \( F = L_{i3} \).

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30