Complex Electronic Behavior in a 1D Correlated Quantum Liquid

J. M. P. Carmelo
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139-4307, USA
(Dated: 5 August 2005)

The relation of the rotated-electron site distribution configurations that describe the energy eigenstates of the one-dimensional Hubbard model to the momentum occupancy configurations of the same states associated with the Bethe-ansatz quantum numbers is clarified. This allows an explicit study of the complex behavior that occurs for the electronic degrees of freedom, following the electron - rotated-electron unitary transformation. Such a behavior plays a central role in the non-perturbative mechanisms associated with the organization of the electronic degrees of freedom in terms of exotic objects whose momentum occupancy configurations describe the energy eigenstates. Our study involves the introduction of the concepts of local pseudoparticle and effective pseudoparticle lattice, needed for the relation of both the local pseudoparticle internal structure and the pseudoparticle occupancy configurations to the rotated-electron site distribution configurations. Our results provide further useful information about the microscopic mechanisms behind the anomalous finite-energy spectral properties of low-dimensional complex materials.

PACS numbers: 03, 70

I. INTRODUCTION

The one-dimensional (1D) Hubbard model describes successfully the role of electronic correlations in the unusual electronic and optical properties observed in quasi-1D Mott-Hubbard insulators 1, doped Mott-Hubbard insulators 2,3, and metals 4. It is one of the few realistic models for which one can exactly calculate all the energy eigenstates and their energies 5,6.

Recently, the quantum numbers provided by the model Bethe-ansatz solution 5,6 and η-spin and spin SU(2) symmetries 7,8 were shown to correspond to occupancy configurations of exotic quantum objects 2. However, the relation of the local rotated-electron site distribution configurations that describe the energy eigenstates of the model to the momentum occupancy configurations of such objects was not fully clarified. The main goal of this paper is to present a detailed study of the relation between the rotated-electron site distribution configurations that emerge from the electron - rotated-electron unitary transformation 6 and the momentum occupancy configurations of such quantum objects. Clarification of this problem provides important information about the microscopic mechanisms that control the unusual finite-energy spectral properties of the model 10,11. The solution of the problem reported here involves the introduction of the concepts of a local pseudoparticle and an effective pseudoparticle lattice. It is found that the bare-momentum pseudoparticle description associated with the Bethe-ansatz Takahasi’s thermodynamic equations 5,6 is related by Fourier transforms to such a local-pseudoparticle representation. The latter representation refers to local-pseudoparticle occupancy configurations of spatial coordinates that correspond to an effective pseudoparticle lattice. Such configurations are expressed here in terms of the corresponding rotated-electron site distribution configurations. The research presented in this paper follows the introduction of the complete holon, spinon, and c pseudoparticle description of Ref. 6 and is complementary to the studies on the finite-energy spectral-weight distributions of Ref. 11, which are used in Refs. 12,13 to the description of the unusual properties observed in quasi-1D compounds. Indeed, the results obtained here provide important and useful information for the identification of the scatterer and scattering centers that control the spectral properties of the model 14. Furthermore, such results are also of interest for the understanding of the spectral properties of the new quantum systems described by cold fermionic atoms in optical lattices with on-site repulsion 14.

The results of this paper also provide further useful information about the complex behavior that occurs for the rotated electrons that emerge from the electron - rotated-electron unitary transformation. Such a behavior was identified in Ref. 6 and refers to an exotic spin-charge separation: For the whole Hilbert space and energies, the spin and charge degrees of freedom of the spin-projection σ-rotated electrons of the singly occupied sites separate, giving rise to chargeless spin-projection σ spinons and η-spinless and spinless c pseudoparticles of electronic charge −e, respectively. This is an example of the behavior observed in many complex physical and biological systems: their parts interact, leading to new properties, absent in the individual parts.

The paper is organized as follows: In Sec. II we introduce the 1D Hubbard model and summarize the concept of rotated electron as well as the description of the model in terms of holons, spinons, and pseudoparticles. In Sec. III we discuss the issue of the choice of the suitable complete set of energy eigenstates of the model in the limit U/t → ∞, where U is the on-site repulsion and t the first-neighbor transfer integral. This includes the introduction of a set of basic properties which are useful for finding the internal structure of the local composite αν pseudoparticle, which
is one of the subjects of Sec. IV. In that section we introduce a complete basis of local states which we express in terms of rotated-electron local charge, spin, and \( c \) pseudoparticle sequences. Moreover, we find the rotated-electron site distribution configurations which describe the internal structure of the local \( \alpha \nu \) pseudoparticle. The concepts of a local \( c \) pseudoparticle and a \( \alpha \nu \) effective pseudoparticle lattice are introduced in Sec. V, where we also express the energy eigenstates in terms of the Fourier-transform superpositions of local charge, spin, and \( c \) pseudoparticle sequences introduced in the previous section. Finally, in Sec. VI we present the discussion and concluding remarks.

II. THE 1D HUBBARD MODEL, ROTATED ELECTRONS, AND SUMMARY OF THE PSEUDOPARTICLE, HOLON, AND SPINON DESCRIPTION

In a chemical potential \( \mu \) and magnetic field \( H \) the 1D Hubbard Hamiltonian can be written as,

\[
\hat{H} = \hat{H}_{SO(4)} + \sum_{\alpha=c,s} \mu_\alpha \hat{S}_\alpha^z; \quad \hat{H}_{SO(4)} = \hat{H}_H - (U/2)(\hat{N} + N_a/2); \quad \hat{H}_H = \hat{T} + U \hat{D},
\]

where \( \hat{H}_{SO(4)} \) has \( SO(4) \) symmetry \( \mathbb{R}, \mathbb{R}, \mathbb{R} \) and \( \hat{H}_H \) is the simple Hubbard model. The expressions of the kinetic-energy operator \( \hat{T} \) and electron double-occupation operator \( \hat{D} \) are given in Eqs. (5) and (6), respectively, of Ref. \( \mathbb{R} \). Moreover, \( \hat{N} \) and \( \hat{N}_a \) are the electronic number operators and \( \alpha_j, \sigma = c_j, \sigma, c_j, \sigma \). The number of lattice sites \( N_a \) is even and large and we consider periodic boundary conditions. We denote by \( N_{\uparrow} \) and \( N_{\downarrow} \) the number of spin-up electrons and spin-down electrons, respectively, and by \( N = N_{\uparrow} + N_{\downarrow} \) the number of electrons. The lattice constant is denoted by \( a \) and thus the length of the system is \( L = N_a a \). We consider electronic densities and spin densities given by \( n = n_{\uparrow} + n_{\downarrow} \) and \( m = n_{\uparrow} - n_{\downarrow} \), respectively, where \( n_{\sigma} = N_a/L \) and \( n = N/L \). These densities belong to the domains defined by the following inequalities \( 0 \leq na \leq 1 ; 1 \leq na \leq 2 \) and \( -na \leq ma \leq na ; -(2-na) \leq ma \leq (2-na) \), respectively. The momentum operator reads,

\[
\hat{P} = \sum_{\sigma=\uparrow, \downarrow} \sum_{k} \hat{N}_\sigma(k) = \frac{L}{2\pi} \sum_{\sigma=\uparrow, \downarrow} \int_{-\pi/a}^{\pi/a} dk \hat{N}_\sigma(k),
\]

and commutes with the Hamiltonians introduced in Eq. \( \mathbb{1} \). The spin-projection \( \sigma \) momentum distribution operator in Eq. \( \mathbb{2} \) is given by \( \hat{N}_\sigma(k) = c_{k, \sigma}^\dagger c_{k, \sigma} \). The operators \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma} \) (and \( c_{k, \sigma}^\dagger \) and \( c_{k, \sigma} \)) which appear in the expressions of the above operators are the spin-projection \( \sigma \) electron creation and annihilation operators at site \( j \) (and carrying momentum \( k \)), respectively. Moreover, on the right-hand side of Eq. \( \mathbb{1} \), \( \mu_c = 2\mu, \mu_s = 2\mu_0 H, \mu_0 \) is the Bohr magneton, and \( \hat{S}_\sigma^z = -\frac{1}{2}[\hat{N}_a - \hat{N}] \) and \( \hat{S}_\sigma^z = -\frac{1}{2}[\hat{N}_a - \hat{N}_\sigma] \) are the diagonal generators of the \( \eta \)-spin and spin \( SU(2) \) algebras \( \mathbb{R}, \mathbb{R} \), respectively. We denote the corresponding \( \eta \)-spin and spin state eigenvalues by \( S_\sigma \) and \( S_a \), respectively. The Hamiltonian \( \hat{H}_{SO(4)} \) defined in Eq. \( \mathbb{1} \) commutes with the six generators of these algebras. The off-diagonal generators are given in Eqs. (7) and (8) of Ref. \( \mathbb{R} \). We note that the Bethe-ansatz solution of the model refers to the Hilbert subspace spanned by the lowest-weight states (LWSs) of the \( \eta \)-spin and spin algebras, i.e. such that \( S_\alpha = -S_a^\alpha \) where \( \alpha = c, s \).

The rotated-electron operator \( \tilde{c}_{j, \sigma}^\dagger \) is related to the corresponding electronic operator \( c_{j, \sigma}^\dagger \) in Eq. (19) of Ref. \( \mathbb{R} \). This relation involves the electron - rotated-electron unitary operator \( \hat{V}(U/t) \), which is uniquely defined by Eqs. (21)-(23) of the same paper. (Expressions for the unitary operator \( \hat{V}(U/t) \) in terms of electronic elementary operators order by order in \( t/U \) are provided in Ref. \( \mathbb{12} \).) The electron and rotated-electron operators \( c_{j, \sigma}^\dagger \) and \( \tilde{c}_{j, \sigma}^\dagger \), respectively, are only identical in the \( U/t \to \infty \) limit where electron double occupation becomes a good quantum number. The rotated-electron double occupation operator is given in Eq. (20) of Ref. \( \mathbb{R} \). Operators which commute with the electron - rotated-electron unitary operator \( \hat{V}(U/t) \) have the same expressions in terms of both elementary electronic operators \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma} \) and rotated-electron operators \( \tilde{c}_{j, \sigma}^\dagger \) and \( c_{j, \sigma} \). Examples of such operators are the six generators of the \( \eta \)-spin and spin \( SU(2) \) algebras and the momentum operator \( \mathbb{2} \).

The holons, spinons, and pseudoparticles studied in Ref. \( \mathbb{R} \) emerge from the electron - rotated-electron unitary transformation which is such that rotated-electron double occupation is a good quantum number. The holons have \( \eta \) spin \( 1/2 \) and spin zero, whereas the spinons have spin \( 1/2 \) and no \( \eta \)-spin degrees of freedom. Throughout this paper we denote the holons and spinons according to their \( \pm 1/2 \) \( \eta \)-spin and spin projections, respectively. For the description of the transport of charge in terms of electrons and electronic holes the -1/2 and +1/2 holons carry charge \(-2e\) and \(+2e\), respectively. Based on symmetry considerations \( \mathbb{2} \), we can classify the \( \pm 1/2 \) holons and \( \pm 1/2 \) spinons into two classes: those which remain invariant under the electron - rotated-electron unitary transformation, and those which...
The former are called ±1/2 Yang holons and ±1/2 HL spinons, with numbers reading $L_{c,±1/2} = [S_c + S_z^2]$ and $L_{s,±1/2} = [S_s + S_z^2]$, respectively, for all energy eigenstates. The latter are part of η-spin-zero 2ν-holon composite $c\nu$ pseudoparticles and spin-zero 2ν-spinon composite $s\nu$ pseudoparticles, respectively, where $\nu = 1, 2, \ldots$ is the number of $+1/2$ and $−1/2$ holon or $+1/2$ and $−1/2$ spinon pairs. Thus, the total number of $±1/2$ holons ($\alpha = c$) and $±1/2$ spinons ($\alpha = s$) reads $M_{\alpha,±1/2} = L_{\alpha,±1/2} + \sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}$, where $N_{\alpha\nu}$ denotes the number of composite $\alpha\nu$ pseudoparticles. The total number of holons ($\alpha = c$) and spinons ($\alpha = s$) is then given by $M_{\alpha} = L_{\alpha} + 2\sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}$ where $L_{\alpha} = 2S_{\alpha}$ denotes the total number of Yang holons ($\alpha = c$) and HL spinons ($\alpha = s$). These numbers are such that $M_c = [N_c − N_s]$ and $M_s = N_c$, where $N_c$ is the number of rotated-electron singly occupied sites and thus of $c$ pseudoparticles and $[N_c − N_s]$ the number of rotated-electron doubly occupied plus unoccupied sites and of $c$ pseudoparticle holes. (In the above designations HL spinon and Yang holon, HL stands for Heilmann and Lieb and Yang refers to C. N. Yang, respectively, who are the authors of Refs. [7, 8].) An useful concept introduced in Ref. [10] is that of a CPHS ensemble subspace where CPHS stands for $c$ pseudoparticle, holon, and spinon. This is a Hilbert subspace spanned by all energy eigenstates with fixed values for the $−1/2$ Yang holon number $L_{c,−1/2}$, $−1/2$ HL spinon number $L_{s,−1/2}$, $c$ pseudoparticle number $N_c$, and for the sets of $\alpha\nu$ pseudoparticle numbers $\{N_{\alpha\nu}\}$ and $\{N_{s\nu}\}$ corresponding to the $\nu = 1, 2, 3, \ldots$ branches.

II. ENERGY EIGENSTATES AND BASIC PROPERTIES ASSOCIATED WITH THE PSEUDOPARTICLE INTERNAL STRUCTURE AND UNOCCUPIED SITES

The studies of Ref. [6] reveal that the energy eigenstates of the model are described by the the same rotated-electron site distribution configurations for all values of $U/t$ but do not provide such configurations. Since the electron - rotated-electron unitary transformation becomes the unit transformation as $U/t → ∞$, in this section we study the $U/t$ independent rotated-electron site distribution configurations which describe these states by considering the corresponding electron site distribution configurations for the model in the limit $U/t → ∞$.

In such a limit there is a huge degeneracy of η-spin and spin occupancy configurations. Such a degeneracy results from the simple form that the kinetic energy $T$ and potential energy $V$ spectra associated with the operators (5) and (6) of Ref. [6], respectively, have in the limit $U/t → ∞$,

$$E_H = T + V ; \quad T = 2t \sum_{j=1}^{N_c} N_c(q_j)[−2t \cos q_j] ; \quad V/U = D .$$

(3)
Here \( E_H \) is the energy spectrum of the Hamiltonian \( \hat{H}_H \) of Eq. (11) and the electron double occupation \( D \) is a good quantum number which equals rotated-electron double occupation in that limit. The kinetic energy equals that of a system of free spin-less fermions \([10,18]\). The corresponding momentum spectrum is given in Eq. (36) of Ref. [9], with \( M_0, -1/2 \) replaced by electron double occupation \( D \). Since the energy spectrum \( \delta \) is independent of the \( \eta \)-spin and spin occupancy configurations, there are several choices for complete sets of energy eigenstates with the same energy and momentum spectra. However, only one of these choices is associated with the rotated-electron site distribution configurations which describe the energy eigenstates for all values of \( U/t \). A short discussion of the relation of our results to some of the well known concepts of the \( U/t \to \infty \) physics is presented in the Appendix.

The pseudoparticle bare momentum obeys well defined boundary conditions which are a necessary condition for the fulfillment of the periodic boundary conditions for the original electrons. However, such a pseudoparticle bare-momentum boundary conditions are not a sufficient condition to ensure the electronic periodic boundary conditions. A second condition imposes that the internal structure of the local composite \( \alpha \nu \) pseudoparticles introduced in later sections must be of a specific form. In this section we introduce a set of properties which are rather useful for the following two issues: First, for the construction of the specific rotated-electron site distribution configurations which describe the internal structure of such local composite \( \alpha \nu \) pseudoparticles; Second, for the definition of the corresponding pseudoparticle unoccupied sites in terms of occupancy configurations of rotated-electron sites.

Our suitable basis choice corresponds to the bare-momentum energy eigenstates associated with the thermodynamic Bethe-ansatz equations introduced by Takahashi \([6,9]\). For \( U/t \to \infty \) there are other choices for complete sets of energy eigenstates \([19]\). Both the bare-momentum energy eigenstates and the symmetrized energy eigenstates used in Ref. \([19]\) are superpositions of charge (and spin) sequences formed by local electron distribution configurations of doubly occupied and unoccupied sites (and spin-down and spin-up singly occupied sites). The expression of both these two sets of energy eigenstates ensures the periodic boundary conditions for the original electronic problem. However, only the former complete set of energy eigenstates is of interest for the study of the finite \( U/t \) problem studied here.

Throughout this paper we denote the rotated-electron doubly occupied and unoccupied sites by \( \bullet \) and \( \circ \) (and the spin-down and spin-up rotated-electron singly occupied sites by \( \downarrow \) and \( \uparrow \)), respectively. In the \( U/t \to \infty \) limit such a concept also refers to electrons. Often we add an index to these symbols which defines the position of the doubly occupied site or unoccupied site (and spin-down singly occupied site or spin-up singly occupied site). For the symmetrized energy eigenstates the charge and spin sequences are properly symmetrized owing to the periodic boundary conditions of the original problem. (This justifies the designation of symmetrized energy eigenstates.) The procedure used in such a symmetrization involves powers of suitable charge and spin operators \( \hat{T}_C \) and \( \hat{T}_S \), respectively, for cyclic permutation of the electron site distribution configurations of the local charge and spin sequences \([19]\). For instance, let

\[
(\bullet, \bullet, \circ, \bullet, \circ, ... \circ, \bullet, \bullet),
\]

and

\[
(\downarrow, \uparrow, \downarrow, \uparrow, \downarrow, ..., \downarrow, \uparrow, \uparrow),
\]

be a charge and a spin sequence, respectively. Then the operators \( \hat{T}_C \) and \( \hat{T}_S \) are such that,

\[
\hat{T}_C (\bullet, \bullet, \circ, \bullet, \circ, ... \circ, \bullet, \bullet) = (\circ, \bullet, \bullet, \circ, \bullet, \circ, ... \circ, \bullet),
\]

and

\[
\hat{T}_S (\downarrow, \uparrow, \downarrow, \uparrow, \downarrow, ..., \downarrow, \uparrow, \uparrow) = (\uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \uparrow, ..., \downarrow, \downarrow),
\]

respectively. The powers \( [\hat{T}_C]^{K_C} \) and \( [\hat{T}_S]^{K_S} \) were called in Ref. \([19]\) \( K_C \) and \( K_S \), respectively. This introduces the charge momentum \( k_C \) and spin momentum \( k_S \), respectively, such that,

\[
k_C = \frac{2\pi}{K_C a} m_C; \quad m_C = 0, 1, ..., K_C - 1; \quad k_S = \frac{2\pi}{K_S a} m_S; \quad m_S = 0, 1, ..., K_S - 1.
\]

The symmetrized energy eigenstates are classified according to their charge and spin sequence, their charge momentum \( k_C \) and spin momentum \( k_S \), and a number \( N_C \) of momenta which in the notation of Ref. \([19]\) equals the
number of charges. The latter discrete momenta are closely related to the discrete bare-momentum values occupied by \( c \) pseudoparticle holes in the pseudoparticle representation of the bare-momentum energy eigenstates considered in Ref. 9. Moreover, the number \( N_C \) of charges equals both the number \( N^h_c = [N_a - N_s] \) of \( c \) pseudoparticle holes and the number \( M_c \) of holons. Thus, this is a good quantum number for both the bare-momentum and symmetrized energy eigenstates. However, the charge momentum \( k_C \) and spin momentum \( k_S \) are eigenvalues of the charge momentum operator \( \hat{k}_C \) and spin momentum operator \( \hat{k}_S \), respectively, which in general do not commute with the set of operators \( \{ \bar{N}_{\alpha \nu}(q) \} \) of the bare-momentum basis considered in Ref. 9. Thus, in general the bare-momentum energy eigenstates are not eigenstates of the charge momentum operator \( \hat{k}_C \) and spin momentum operator \( \hat{k}_S \). Exceptions are the bare-momentum energy eigenstates with occupancy of a single \( c\nu \) pseudoparticle (and a single \( sv \) pseudoparticle) and with no finite occupancy of Yang holons (and HL spinons) and of \( c\nu' \) pseudoparticles (and \( sv' \) pseudoparticles) belonging to other branches such that \( \nu' \neq \nu \). We find below that in this case the corresponding bare-momentum energy eigenstate is also an eigenstate of the charge (and spin) momentum operator of eigenvalue \( k_C = \pi/a \) (and \( k_S = \pi/a \)).

According to Eqs. (B.1) and (B.2) of Ref. 9, the discrete values of the bare momentum \( q_j \) are such that \( q_{j+1} - q_j = 2\pi/L \) and \( q_j = [2\pi/L] \sum_{\nu=1}^{N_{\alpha \nu}} N_{\nu^*}^\alpha \) or \( q_j = [2\pi/L] \sum_{\nu=1}^{N_{\alpha \nu}} N_{\nu^*}^\alpha \) where the numbers \( I_j^\alpha \) and \( I_{\nu^*}^\alpha \) with \( j = 1, 2, ..., N_{\alpha}\) and \( j = 1, 2, ..., N_{\nu^*}^\alpha \), respectively, are integers or half-odd integers as a result of the following boundary conditions,

\[
e^{iq_j L} = (e^{i\pi})^{\sum_{\alpha=c,s} \sum_{\nu=1}^{N_{\alpha \nu}}} N_{\nu^*}^\alpha, \tag{9}\]

in the case of the \( c \) pseudoparticle branch and,

\[
e^{iq_j L} = (e^{i\pi})^{[1+N_{\nu^*}^\alpha]} = (e^{i\pi})^{[1+N_{\nu^*}^\alpha]} = (e^{i\pi})^{[1+N_{\nu^*}^\alpha]}; \quad \alpha = c, s, \quad \nu = 1, 2, ..., \tag{10}\]

for the \( \alpha\nu \) pseudoparticle branches. Here \( N_{\nu^*}^{\alpha} = N_{\nu^*}^{\alpha} + N_{\nu^*}^{\alpha} \) where the number \( N_{\nu^*}^{\alpha} \) is defined by Eqs. (B.7) and (B.11) of Ref. 9. According to Eq. (B.14) of the same paper, for the \( \alpha\nu \) pseudoparticle branches the \( j = 1 \) minimum and \( j = N_{\nu^*}^{\alpha} \) maximum index of the bare-momentum values \( q_j \) are such that,

\[
-q_1 = q_{N_{\nu^*}^{\alpha}} = q_{\nu^*} = \pi L \left[ N_{\nu^*}^{\alpha} - 1 \right]. \tag{11}\]

For the \( c \) pseudoparticles the limiting values are \( q_1 = q_c^- \) and \( q_{N_a} = q_c^+ \) where the bare momenta \( q_c^\pm \) are defined in Eqs. (B.15)-(B.17) of Ref. 9.

There is a holon, spinon, \( c \) pseudoparticle separation for the whole parameter space of the 1D Hubbard model. For the \( t/U \to 0 \) limit the description of the \( c \) pseudoparticle excitation sector is very similar for both the representations in terms of bare-momentum energy eigenstates and symmetrized energy eigenstates. The separation of the charge and spin excitation sectors occurs for these two alternative representations. The construction of the bare-momentum energy eigenstates also involves superpositions of charge (and spin) sequences associated with rotated-electron distribution configurations of doubly occupied and unoccupied sites (and spin-down and spin-up singly occupied sites). Such superpositions are also due to the periodic boundary conditions but are not in general generated by the above symmetrization procedure. In spite of these similarities, the complete sets of symmetrized and bare-momentum energy eigenstates of the 1D Hubbard model in the limit of \( t/U \to 0 \) correspond in general to different states. The difference between the symmetrized and bare-momentum energy eigenstates refers in general both to the form of the local electronic distribution configurations of doubly occupied and unoccupied sites (and spin-down and spin-up singly occupied sites) which describe the charge (and spin) sequences and to the form of the superposition of these local sequences which describes the energy eigenstates.

It is useful for the introduction of the concepts of a local pseudoparticle and an effective pseudoparticle lattice to consider the eight properties given below. However, the precise definition of these concepts involves clarification of several other issues beyond these properties and is only fulfilled in Sec. V. These properties are used in the ensuing section in finding the rotated-electron site distribution configurations of the local charge and spin sequences whose Fourier-transform superpositions describe the bare-momentum energy eigenstates of the model. They follow in part from symmetries and features of the pseudoparticle, holon, and spin description and related rotated-electron representation studied in Refs. 9, 11, 13 and from well known properties associated with the \( t/U \to 0 \) physics. Although some of the results used in the following properties are specific to the \( t/U \to 0 \) physics, we emphasize that some of the conclusions also apply to finite values of \( t/U \) provided that electrons are replaced by rotated electrons. These useful properties read:
1-III The numbers of electron doubly occupied sites, unoccupied sites, spin-down singly occupied sites, and spin-up singly occupied sites are good quantum numbers whose values are equal to the total numbers of $-1/2$ holons, $+1/2$ holons, $-1/2$ spinons, and $+1/2$ spinons, respectively, of the bare-momentum energy eigenstates.

2-III The kinetic energy $T$ given in Eq. 4 arises from the movements of the singly occupied sites relative to the doubly occupied and unoccupied sites which do not change double occupation. These movements are fully described by the $c$ pseudoparticles which are associated with the charge degrees of freedom of these sites. In the limit $t/U \to 0$ these quantum objects acquire a spin-less fermion spectrum. On the other hand, the electron distribution configurations of doubly occupied and unoccupied sites and of spin-down and spin-up singly occupied sites do not contribute to the kinetic energy $T$. Moreover, these electron site distribution configurations must remain unchanged in spite of the movements of the $c$ pseudoparticles. Alternatively, we can consider that the electron doubly occupied and unoccupied sites move relative to the singly occupied sites. In this case one describes the movements of the $c$ pseudoparticles in terms of the movements of the corresponding $c$ pseudoparticle holes. This is the choice of Ref. 19 for the case of the symmetrized energy eigenstates. We recall that the numbers $N_c$ of $c$ pseudoparticles, $N^{h}_c$ of $c$ pseudoparticle holes, $M_s$ of spinons, and $M_c$ of holons are such that $N_c + N^{h}_c = N_s$, $N_c = M_s$, and $N^{h}_c = M_c$ and thus these two alternative descriptions are fully equivalent.

3-III We call local charge sequences and local spin sequences the occupancy configurations of the $\pm 1/2$ holons (and corresponding electron distribution configurations of doubly occupied and unoccupied sites) and the occupancy configurations of the $\pm 1/2$ spinons (and corresponding electron distribution configurations of singly occupied sites of spin projection $\pm 1/2$), respectively. These local charge (and spin) sequences can also be expressed in terms of occupancy configurations of $\pm 1/2$ Yang holons and local $c\nu$ pseudoparticles (and $\pm 1/2$ HL spinons and local $s\nu$ pseudoparticles) belonging to $\nu = 1, 2, 3 ...$ branches. From the general properties introduced in Ref. 4, one finds that the electronic description of these quantum objects in terms of distribution configurations of electronic doubly occupied and unoccupied sites (and spin-down and spin-up singly occupied sites) is as follows: A $-1/2$ Yang holon (and $-1/2$ HL spinon) is described by a doubly occupied site (and a spin-down singly occupied site); A $+1/2$ Yang holon (and $+1/2$ HL spinon) is described by an unoccupied site (and a spin-up singly occupied site); A local $c\nu$ pseudoparticle (and a local $s\nu$ pseudoparticle) is described by a number $\nu$ of doubly occupied sites and a number $\nu$ of unoccupied sites (and a number $\nu$ of spin-down singly occupied sites and a number $\nu$ of spin-up singly occupied sites). The electron distribution configurations of doubly occupied and unoccupied sites (and of spin-down and spin-up singly occupied sites) of any local charge (and spin) sequence can be expressed in terms of a corresponding occupancy configuration of $\pm 1/2$ Yang holons and local $c\nu$ pseudoparticles (and $\pm 1/2$ HL spinons and local $s\nu$ pseudoparticles) belonging to the $\nu = 1, 2, 3 ...$ branches. The form of Eq. (36) of Ref. 4 reveals that a $-1/2$ holon carries momentum $\pi/a$ and that in the limit $t/U \to 0$ each doubly occupied site must also be associated with a momentum $\pi/a$, respectively. Therefore, the expressions of the local charge sequences must include a phase factor operator $\exp(i\pi \sum_j j \hat{D}_j)$ such that the $j$ summation runs over the sites doubly occupied and unoccupied and the local double occupation operator has eigenvalues 1 and 0 if the site $j$ is doubly occupied and unoccupied, respectively.

4-III The number of different possible spatial discrete positions of a local $c\nu$ pseudoparticle equals the number $N^{*}_{c\nu} = N_{c\nu} + N^{h}_{c\nu}$, with $N^{h}_{c\nu}$ given in Eqs. (B.7) and (B.11) of Ref. 4. The number $N^{*}_{c\nu}$ is directly provided by the Bethe-ansatz solution 4 since it also equals the number of different discrete bare momentum values $q_j$, where $j = 1, 2, ..., N^{*}_{c\nu}$, of the bare-momentum $c\nu$ pseudoparticle band. Such a local $c\nu$ pseudoparticle has an internal structure corresponding to the electron distribution configurations of $\nu$ doubly occupied sites and $\nu$ unoccupied sites ($\alpha = c$) or of $\nu$ spin-down and $\nu$ spin-up singly occupied sites ($\alpha = s$). Thus, each of the $N^{*}_{c\nu}$ occupied locations does not correspond a single lattice site but instead involves $2\nu$ lattice sites.

5-III For energy eigenstates with finite occupancy of $-1/2$ and $+1/2$ Yang holons (and $-1/2$ and $+1/2$ HL spinons) the electron distribution configurations of doubly occupied and unoccupied sites (and spin-down and spin-up singly occupied sites) describing the local $c\nu$ pseudoparticle (and local $s\nu$ pseudoparticle) must remain unchanged under the application of the off-diagonal generators of the $\eta$-spin SU(2) algebra given in Eqs. (7) of Ref. 4 (and off-diagonal generators of the spin SU(2) algebra given in Eq. (8) of the same reference). Moreover, for states with no $+1/2$ or $-1/2$ Yang holons (and no $+1/2$ or $-1/2$ HL spinons) application of the operators $\hat{S}_0 = \sum_j (-1)^j c_j,\downarrow c_j,\uparrow$ or $\hat{S}_b = \sum_j (-1)^j c_j,\uparrow c_j,\downarrow$ (and $\hat{S}_s = \sum_j c_j,\downarrow c_j,\uparrow$ or $\hat{S}_t = \sum_j c_j,\uparrow c_j,\downarrow$) onto these states must give zero. These requirements result from the $\eta$-spin singlet character of the $c\nu$ pseudoparticles (and spin singlet character of the $s\nu$ pseudoparticles). On the other hand, the transformations generated by application of these off-diagonal generators onto the electron distribution configurations of doubly occupied and unoccupied sites (and spin-down and spin-up
singly occupied sites) describing the $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons) must be the ones defined by the $\eta$-spin (and spin) algebra.

6-III Let us consider a local charge (and spin) sequence with no Yang holons (and no HL spinons) and consisting of $\nu$ electron doubly occupied sites and $\nu$ electron unoccupied sites (and $\nu$ electron spin-down singly occupied sites and $\nu$ electron spin-up singly occupied sites). If such a sequence describes a single local $cv$ pseudoparticle (and local $sv$ pseudoparticle) it is properly symmetrized in such way that the distribution configurations associated with the internal structure of that quantum object remain unchanged under cyclic permutations. Furthermore, property 5-III imposes that a bare-momentum energy eigenstate with the above Yang holon and $cv$ pseudoparticle (and HL spinon and $sv$ pseudoparticle) numbers is an eigenstate of the charge (and spin) momentum operator $\hat{k}_C$ (and $\hat{k}_S$) of eigenvalue $k_C = \pi/a$ (and $k_S = \pi/a$).

7-III The local $cv$ pseudoparticle (and a $sv$ pseudoparticle) is a quantum object whose internal structure involves $\nu$ doubly occupied sites and $\nu$ unoccupied sites (and $\nu$ spin-down singly occupied sites and $\nu$ spin-up singly occupied sites). The electron distribution configurations of the $2\nu$ sites which describe the internal structure a local $cv$ pseudoparticle (and $sv$ pseudoparticle) are the same for all local charge ($\alpha = c$) or spin ($\alpha = s$) sequences involved in the description of the $4^\nu$ bare-momentum energy eigenstates. Such a property follows from the indiscernible character of local $\alpha\nu$ pseudoparticles with the same spatial position but involved in occupancy configurations describing different local charge ($\alpha = c$) or spin ($\alpha = s$) sequences. This indiscernible character of the local $\alpha\nu$ pseudoparticles results from the corresponding indiscernible character of the $\alpha\nu$ pseudoparticles of bare-momentum $q_j$, which are indistinguishable quantum objects.

8-III Since in the limit $t/U \to 0$ there is nearest-neighbor hopping only and it does not change double occupation, the charge, spin, $c$ pseudoparticle separation studied in Ref. [9] implies that both the local charge and spin sequences of the energy eigenstates must be separately conserved. Moreover, the periodic boundary conditions of the original electronic problem are ensured if both the pseudoparticle bare-momentum discrete values obey Eqs. (9) and (10) and the energy eigenstates must be separately conserved. This implies a complex behavior for all electrons that singly occupy lattice sites, whose charge degrees of freedom correspond to the $c$ pseudoparticles and separate from the spin degrees of freedom, which refer to the spinons associated with the $sv$ pseudoparticles and HL spinons.

These eight basic properties play an important role in the mechanisms which in the $t/U \to 0$ limit determine the choice of the electron site distribution configurations of the local charge, spin, and $c$ pseudoparticle sequences whose Fourier-transform superpositions describe the bare-momentum energy eigenstates. Property 1-III follows from well known properties of the model in the limit $t/U \to 0$. Property 2-III is consistent with the finding of Ref. [10] that in the limit $t/U \to 0$ only the $c$ pseudoparticles move and carry kinetic energy, whereas the $\pm 1/2$ holons and $\pm 1/2$ spinons correspond to unchanged occupancy configurations in that limit. Such a property results from well established features of the Bethe-ansatz solution [10, 11, 12, 13]. Property 3-III is a consequence of the combination of property 1-III with the relation of $\pm 1/2$ Yang holons, $\pm 1/2$ HL spinons, and $\alpha\nu$ pseudoparticles to $\pm 1/2$ holons and $\pm 1/2$ spinons. In property 4-III the number $N^*_{\alpha\nu}$ plays a central role. The expression of that number is valid for all values of $U/t$ and is provided by the Bethe-ansatz solution. This is consistent with the electronic site distribution configurations which describe the local $\alpha\nu$ pseudoparticles in the limit of $t/U \to 0$ being the same as the rotated-electron site distribution configurations which describe these local $\alpha\nu$ pseudoparticles for finite values of $t/U$. Property 5-III follows from general symmetries of the model which are also valid for all values of $U/t$. This basic property is also consistent with the above equivalence of the electronic site distribution configurations in the limit of $t/U \to 0$ and of the corresponding rotated-electron configurations for finite values of $t/U$. Property 6-III is a consequence of the periodic boundary conditions in the particular case when a local charge or spin sequence corresponds to a single $cv$ or $sv$ pseudoparticle, respectively. The property 7-III results from the indiscernible character of the $\alpha\nu$ pseudoparticles. In the case of bare-momentum pseudoparticles such an indiscernible character is implicit in the description of the bare-momentum
energy eigenstates in terms of pseudoparticle occupancy configurations [9]. Finally, property 8-III is related to the periodic boundary conditions of the original electronic problem and to the existence in this limit of nearest-neighbor hopping only which does not change the value of double occupation. Such a property is also related to the charge, spin, pseudo-particle separation and associated complex behavior of the singly-occupied-sites rotated electrons for all energy scales and for the whole parameter space of the model. In addition, for the composite configurations whose total number is given in Eq. (12) by the coordinates in units of the lattice constant.

Below we often refer to the rotated-electron site distribution configurations only. However, whenever referring to rotated electrons we mean implicitly that in the limit $t/U \to 0$ the electronic site distribution configurations which describe the bare-momentum energy eigenstates are the same.

IV. THE PSEUDOPARTICLE INTERNAL STRUCTURE AND COMPLETE SET OF LOCAL STATES IN TERMS OF CHARGE, SPIN, AND $c$ PSEUDOPARTICLE SEQUENCES

In this section we find the rotated-electron site distribution configurations which describe the local charge, spin, and $c$ pseudoparticle sequences. This requires the study of the internal structure of the local $\alpha\nu$ pseudoparticles. In addition, such an analysis requires the introduction of the concept of an effective pseudoparticle lattice. The first step for the introduction of such a concept is the definition of the spatial positions of the rotated-electron site distribution configurations which describe both the local $c$ and $\alpha\nu$ pseudoparticles and their unoccupied sites. We find that each specific site distribution configurations of a local charge, spin, and $c$ pseudoparticle sequence defines a local state. The set of all different possible such states constitutes a complete set of local states. In the ensuing section we express the energy eigenstates of the model as Fourier-transform superpositions of these local states.

A. THE LOCAL $\alpha\nu$ PSEUDOPARTICLE INTERNAL STRUCTURE

We start by restricting our study to rotated-electron site distribution configurations which describe the lowest-weight states of both the $\eta$-spin and spin $SU(2)$ algebras. These states have no $-1/2$ Yang holons and no $-1/2$ HL spinons. Thus, in this case the number of Yang holons (and of HL spinons) is such that $L_c = L_{c,+1/2}$ (and $L_s = L_{s,+1/2}$).

In the limit $U/t \to \infty$ the $N_c c$ pseudoparticles behave as spin-less fermions in a lattice of $N_\alpha = N_c + N^h_c = M_s + M_c$ sites. Such a lattice is nothing but the effective $c$ pseudoparticle lattice mentioned in property 8-III. As for the bare-momentum $q_j$, such an effective $c$ pseudoparticle lattice and its site coordinates $x_j$ remain the same for the whole parameter space of the model. Also the $c$ pseudoparticle occupancy configurations of such a lattice describing a given energy eigenstate are the same for all values of $U/t$. At a fixed value $N_c = M_s$ of the numbers $N_c$ of $c$ pseudoparticles and $M_s$ of spinons the number of occupancy configurations of the local $c$ pseudoparticles in such an effective lattice is given by,

$$
\left(\frac{N_a}{N_c}\right) = \frac{N_a!}{N_c!N^h_c!} = \frac{N_a!}{M_s!M_c!}.
$$

Each of these $c$ pseudoparticle occupancy configurations of the effective $c$ pseudoparticle lattice defines a local $c$ pseudoparticle sequence.

Let $x_{j_l} = a j_l$ where $l = 1, 2, ..., N_c$ be the actual set of occupied coordinates of the effective $c$ pseudoparticle lattice out of the available $x_j = a j$ coordinate sites such that $j = 1, 2, ..., N_a$. We note that these $N_c$ sites have the same coordinates and lattice constant $a$ as the $N_c$ sites of the rotated-electron lattice. Moreover, the set of coordinates $x_{j_l} = a j_l$ where $l = 1, 2, ..., N_c$ corresponding to the occupied sites of the effective $c$ pseudoparticle lattice are precisely the same as the coordinates of the rotated-electron singly occupied sites. Thus, we can define each of the occupancy configurations whose total number is given in Eq. (12) by the coordinates in units of the lattice constant $a$ of the rotated-electron singly occupied sites,

$$
(j_1, j_2, ..., j_{N_c}).
$$

The locations occupied by the rotated-electron doubly occupied and unoccupied sites are then these left over by the rotated-electron singly occupied sites. Since the sequence (13) also gives the locations of the local $c$ pseudoparticles in their effective lattice, the sites left over by these pseudoparticles define the locations of the local $c$ pseudoparticle...
holes. Thus, the relation of the numbers $N_c$ and $N^h_c$ of local $c$ pseudoparticles and local $c$ pseudoparticle holes, respectively, to the rotated-electron site distribution configurations of a given energy eigenstate confirms that there are $N_c$ rotated-electron singly occupied sites and $N^h_c = [N_a - N_c]$ rotated-electron doubly-occupied and unoccupied sites, as stated in Ref. 9. From the use of the results of that reference we find that these numbers can be expressed as follows,

$$N^h_c = N_a - N_c = L_c + 2 \sum_{\nu=1}^{\infty} \nu N_{c\nu}; \quad N_c = L_s + 2 \sum_{\nu=1}^{\infty} \nu N_{s\nu}. \quad (14)$$

The local charge sequences (and spin sequences) introduced in property 3-III involve only the rotated-electron doubly-occupied and unoccupied site distribution configurations (and spin-down and spin-up rotated-electron singly occupied site distribution configurations) of $N^h_c = [N_a - N_c]$ sites (and $N_c$ sites) out of a total number $N_a$ of sites. As a result of the independent conservation of the charge and spin sequences, the rotated-electron site distribution configurations of a charge (and spin) sequence is for any energy eigenstate obtained simply by omitting the $N_c$ singly occupied sites (and $N^h_c = [N_a - N_c]$ doubly-occupied and unoccupied sites). Thus, a local charge (and spin) sequence corresponds to the rotated-electron occupancy configurations of a site domain involving $N^h_c = [N_a - N_c]$ sites (and $N_c$ sites). It follows that the number of sites of such a domain depends on the specific state under consideration. For the ground-state numbers provided in Appendix C of Ref. 9, the number of sites of the rotated-electron lattice which belong to the charge and spin sequences is $[N_a - N]$ and $N$, respectively.

Also the effective $\alpha \nu$ pseudoparticle lattices mentioned in property 8-III and introduced below, result from independent conservation laws associated with each $\alpha \nu$ pseudoparticle branch such that $\alpha = c, s$ and $\nu = 1, 2, ...$. These effective $\alpha \nu$ pseudoparticle (and $s\nu$ pseudoparticle) lattices are generated below by omission of a number $\sum_{\nu'=1}^{\infty} [\nu + \nu' - |\nu - \nu'|] N_{c\nu'} - N_{c\nu}$ (and $\sum_{\nu'=1}^{\infty} [\nu + \nu' - |\nu - \nu'|] N_{s\nu'} - N_{s\nu}$) of sites out of the total number $N^h_c = [N_a - N_c]$ (and $N_c$) of sites of the local charge sequence (and spin sequence). Again, the number of sites of the local charge (and spin) sequence which contribute to an effective $\alpha \nu$ pseudoparticle (and $s\nu$ pseudoparticle) lattice depends on the specific state under consideration. For instance, for a ground state corresponding to densities in the ranges $0 \leq na \leq 1$ and $0 \leq ma \leq na$ the number of sites of the local charge sequence (and spin sequence) which contribute to an effective $\alpha \nu$ pseudoparticle (and $s\nu$ pseudoparticle) lattice is $[N_a - N]$ (and $N_1$ for the $s1$ pseudoparticle branch and $[N_1 - N_s]$ for the $s\nu$ pseudoparticle branches such that $\nu > 1$) out of the $[N_a - N]$ sites (and $N = [N_1 + N_s]$ sites) of such a sequence. On the other hand, the number of sites of the effective $c$ pseudoparticle lattice equals the number $N_a$ of sites of the rotated-electron lattice and is the same for all energy eigenstates.

The expressions of Eq. (14) show that the value of the number $L_c$ of Yang holons (and $L_s$ of HL spinons) of the local charge (and spin) sequence is uniquely determined by the values of the numbers of $c$ pseudoparticles and $\alpha \nu$ pseudoparticles (c pseudoparticles and $s\nu$ pseudoparticles) of the same sequence. Thus, for given number $N^h_c = [N_a + N_c]$ of rotated-electron doubly-occupied and unoccupied sites (and $N_c$ of rotated-electron singly occupied sites) we can uniquely define the rotated-electron site distribution configurations of the local charge (and spin) sequence by providing the sites occupied by local $c\nu$ pseudoparticles (and $s\nu$ pseudoparticles) belonging to the branches $\nu = 1, 2, ...$ with finite occupancy. The charge-sequence (and spin-sequence) sites of the rotated-electron lattice left over by these local pseudoparticles define the positions of the Yang holons (and HL spinons). It is useful to introduce the charge-sequence site index $h$ and the spin-sequence site index $l$ such that,

$$h = 1, 2, ..., [N_a - N_c]; \quad l = 1, 2, ..., N_c. \quad (15)$$

The ordering of the charge-sequence (and spin-sequence) index corresponds to the spatial-position order from the left to the right-hand side of the corresponding site in the rotated-electron lattice. The position in such a lattice of a spin-sequence site of index $l$ and of a charge-sequence site of index $h$ are given by,

$$x_{jl} = j_l a, \quad l = 1, 2, ..., N_c; \quad x_{jh} = j_h a, \quad h = 1, 2, ..., [N_a - N_c], \quad (16)$$

where $j_l$ are the indices of Eq. (13) which define the position of the rotated-electron singly occupied sites and $j_h$ are the indices which define the position of the rotated-electron doubly occupied and unoccupied sites. The spatial position of the latter sites corresponds to the sites left over by the rotated-electron singly occupied sites.

From both the above analysis and property 3-III we conclude that one can uniquely specify a given rotated-electron site distribution configuration by providing the position of the $N_c$ sites occupied by local $c$ pseudoparticles, $2 \sum_{\nu=1}^{\infty} \nu N_{c\nu}$ sites occupied by local $c\nu$ pseudoparticles, and $2 \sum_{\nu=1}^{\infty} \nu N_{s\nu}$ sites occupied by local $s\nu$ pseudoparticles.
The $N_c$ sites occupied by $c$ pseudoparticles define the position of the rotated-electron singly occupied sites. The $[N_h - N_c]$ rotated-electron doubly-occupied and unoccupied sites are the sites left over by the rotated-electron singly occupied sites. The $L_c = 2S_c$ sites occupied by Yang holons and the $L_s = 2S_s$ sites occupied by HL spinons are the sites left over in the charge and spin sequences, respectively, by the sites occupied by local $cv$ pseudoparticles and local $s\nu$ pseudoparticles, respectively. As mentioned above, to start with we consider that all Yang holons (and HL spinons) have $\eta$-spin projection (and spin projection) $+1/2$ and thus correspond to rotated-electron unoccupied sites (and rotated-electron spin-up singly occupied sites). The generalization to rotated-electron site distribution occupancies associated with states containing both $\pm 1/2$ Yang holons (and $\pm 1/2$ HL spinons) is straightforward and is introduced later in this section.

The Yang holons (and HL spinons) have no internal structure and are the simplest of the quantum objects which occupy the charge (and spin) sequence of the rotated-electron lattice. According to property 1-III, the $+1/2$ Yang holons (and $+1/2$ HL spinons) correspond to rotated-electron unoccupied sites (and spin-up rotated-electron singly occupied sites) of these sequences. On the other hand, a local $\alpha\nu$ pseudoparticle has internal structure and thus is a more involved quantum object than a Yang holon or a HL spinon. This internal structure corresponds to the rotated-electron distribution configurations of the $\nu$ doubly occupied sites and $\nu$ unoccupied sites ($\alpha = c$) or $\nu$ spin-down singly occupied sites and $\nu$ spin-up singly occupied sites ($\alpha = s$) which according to property 3-III describe such a local $\alpha\nu$ pseudoparticle. If in the rotated-electron site distribution configurations which describe the local $cv$ pseudoparticles we replace doubly occupied sites $\bullet$ and unoccupied sites $\circ$ by spin-down singly occupied sites $\uparrow$ and spin-up singly occupied sites $\uparrow^\uparrow$, respectively, and omit the phase factors generated by the operator $\exp(i\pi\sum_j D_j)$ mentioned in property 3-III, we obtain the corresponding distribution configurations of the $s\nu$ pseudoparticles. Therefore, often we consider the rotated-electron site distribution configurations which describe the local $cv$ pseudoparticles only.

It is useful to classify the $2\nu$ lattice sites of the local charge sequence involved in the description of a local $cv$ pseudoparticle into two sets of $\nu$ sites each. Let us denote the index of these two sets of $\nu$ sites by $h_j, g$ and $h_j, v+g$, respectively. Here $g = 1, 2, \ldots, \nu$ refers to the internal site structure of the local $cv$ pseudoparticle and the index $j = 1, 2, \ldots, N_{cv}$ corresponds to its position $[h_j a]$ in the rotated-electron lattice where,

$$h_j = \frac{j h_{j, \nu} + j h_{j, \nu+1}}{2}, \quad (17)$$

and the numbers $j_{h, \nu}$ are the indices $j_h$ of Eq. \([13]\) which define the position of the rotated-electron doubly occupied/unoccupied sites in such a lattice. Combination of the values of the indices $j = 1, 2, \ldots, N_{cv}$ and $g = 1, 2, \ldots, \nu$ fully defines the position of the above $2\nu$ sites. The internal-structure index $g$ is such that $h_{j, 1} < h_{j, 2} < \ldots < h_{j, \nu}$ and $h_{j, \nu+1} < h_{j, \nu+2} < \ldots < h_{j, 2\nu}$, respectively, where $h_{j, \nu} < h_{j, \nu+1}$. Equivalently, often we denote these $2\nu$ internal-structure indices simply by $h_{j, x}$, where $x = 1, 2, \ldots, 2\nu$ and $h_{j, 1} < h_{j, 2} < \ldots < h_{j, 2\nu}$. The set of $2\nu$ internal-structure indices $\{h_{j, 1}, h_{j, 2}, \ldots, h_{j, 2\nu}\}$ is in general a sub-set of the $[N_h - N_c]$ charge-sequence indices $h$ given in Eq. \([13]\). The latter indices define the position of the charge-sequence rotated-electron doubly occupied and unoccupied sites. We can also define the charge-sequence position of the local $cv$ pseudoparticle which is defined as,

$$\tilde{h}_j = \frac{h_{j, \nu} + h_{j, \nu+1} - 1}{2}. \quad (18)$$

The same definitions hold for the local $s\nu$ pseudoparticles with the indices $h_{j, g}$ and $h_{j, \nu+g}$ replaced by the indices $l_{j, g}$ and $l_{j, \nu+g}$, respectively, and thus the equivalent indices $h_{j, x}$ replaced by $l_{j, x}$. The position of the local $s\nu$ pseudoparticle in the rotated-electron lattice is defined as $[l_j a]$ where,

$$l_j = \frac{j l_{j, \nu} + j l_{j, \nu+1}}{2}, \quad (19)$$

$j = 1, 2, \ldots, N_{s\nu}$, and $j_l$ are the indices of Eq. \([13]\) which define the position of the rotated-electron singly occupied sites. On the other hand, the spin-sequence position of the local $s\nu$ pseudoparticle is defined as,

$$\tilde{l}_j = \frac{l_{j, \nu} + l_{j, \nu+1} - 1}{2}, \quad (20)$$

where again $j = 1, 2, \ldots, N_{s\nu}$. We note that the indices $\tilde{h}_j$ (and $\tilde{l}_j$) given in Eq. \([18]\) (and Eq. \([20]\)) which define the charge-sequence (and spin-sequence) position of the local $cv$ pseudoparticle (and local $s\nu$ pseudoparticle) are always positive integer numbers $1, 2, 3, \ldots$. According to Eq. \([17]\) (and Eq. \([19]\)) the position $[h_j a]$ (and $[l_j a]$) of the local
a \nu \nu \nu \nu pseudoparticle (and local s \nu pseudoparticle) in the N_a-site rotated-electron lattice refers to a single point inside the 2\nu-site domain associated with such a quantum object. On the other hand, the charge-sequence position (and spin-sequence position) of the local c\nu pseudoparticles (and local s\nu pseudoparticles) defined by the index of Eq. (18) (and Eq. (20)) refers to the position of that quantum object relative to the \([N_a - N_c]\) sites (and N_c sites) of the charge (and spin) sequence only.

Below we clarify the following two issues: First, we find the rotated-electron distribution configurations of the 2\nu sites which describe the internal structure of a local c\nu pseudoparticle; Second, we find the rotated-electron site distribution configurations which define the \(N_a^c\) charge-sequence (\(\alpha = c\)) or spin-sequence (\(\alpha = s\)) unoccupied sites corresponding to the local c\nu pseudoparticle branch. This study reveals that the position in the rotated-electron lattice of the sites associated with the \(N_a^c\) charge-sequence or spin-sequence unoccupied sites of each local c\nu pseudoparticle branch is uniquely determined by the position of the local c pseudoparticles, local c\nu pseudoparticles, and local s\nu pseudoparticles. The same holds for the positions of the Yang holons and HL spinons, as discussed above. Below we also confirm that for fixed values of the numbers \(N_a^c\) and \(N_a^s\), the number of occupancy configurations of the local c\nu pseudoparticles is given by,

\[
\frac{N_a^c}{N_a^\nu} = \frac{N_a^c}{N_a^\nu} \prod_{\alpha = c, a}^\infty \left( \frac{N_a^\nu}{N_a^\alpha} \right).
\]

We classify each of the local c\nu pseudoparticle occupancy configurations by providing the indices \(h_j\) or \(l_j\) given in Eqs. (17) or (19), respectively, corresponding to the \(N_a^c\) and \(N_a^s\) pseudoparticle positions in the rotated-electron lattice,

\[
(h_1, h_2, ..., h_{N_a^c}),
\]

and

\[
(l_1, l_2, ..., l_{N_a^s}),
\]

respectively. From the combination of Eqs. (12) and (21), it follows that the number of occupancy configurations of the local c and c\nu pseudoparticles of a CPHS ensemble subspace is given by,

\[
N_{\text{CPHS-ES}} = \left( \frac{N_a}{N_c} \right) \prod_{\alpha = c, a} \prod_{i = 1}^\infty \left( \frac{N_a^\nu}{N_a^\alpha} \right).
\]

Let us denote each local state representing a specific local charge, spin, and c pseudoparticle sequence whose number for a given CPHS ensemble subspace is given in Eq. (24) by,

\[
| (j_1, j_2, ..., j_{N_c}); (h_1, h_2, ..., h_{N_a^c}); (l_1, l_2, ..., l_{N_a^s}) \rangle.
\]

Here

\[
| (h_1, h_2, ..., h_{N_a^c}) \rangle = (h_1, h_2, ..., h_{N_a^c}); (h_1, h_2, ..., h_{N_a^c}); (h_1, h_2, ..., h_{N_a^c}); ...,
\]

gives the positions in the rotated-electron lattice of the local c\nu pseudoparticles belonging to branches with finite occupancy in the state under consideration and,

\[
| (l_1, l_2, ..., l_{N_a^s}) \rangle = (l_1, l_2, ..., l_{N_a^s}); (l_1, l_2, ..., l_{N_a^s}); (l_1, l_2, ..., l_{N_a^s}); ...,
\]

gives the positions in the same lattice of the local s\nu pseudoparticles belonging to branches with finite occupancy in the same state.

We emphasize that the number of such local states which equals the number of different local pseudoparticle occupancy configurations of a CPHS ensemble subspace indeed equals the dimension of such a subspace. It is straightforward to confirm from the results of Ref. [4] that the number of bare-momentum energy eigenstates that span such a CPHS ensemble subspace also has the same value. This suggests that the set of local states of form (25) is complete.
in that subspace, as is confirmed below. We find later on in this section that the generalization of the present analysis to rotated-electron site distribution configurations describing energy eigenstates with finite occupancies of ±1/2 Yang holons and ±1/2 HL spinons requires the introduction of the numbers $L_c, -1/2$ of $-1/2$ Yang holons and $L_n, -1/2$ of $-1/2$ HL spinons in the labeling of the local states \( \Omega \).

In order to find the rotated-electron site distribution configurations which describe the internal structure of a local $c\nu$ pseudoparticle, we use mainly the basic properties 3-III, 5-III, 6-III, and 7-III. Often we consider the specific case $\alpha$ and $\pm$ sequence constituted by $\nu$ rotated-electron doubly occupied sites and $\nu$ rotated-electron unoccupied sites. The use of properties 5-III and 6-III leads to the finding of the specific rotated-electron site distribution configurations which describe the local $c\nu$ pseudoparticle. Property 7-III then states that the obtained rotated-electron site distribution configurations describe a local $c\nu$ pseudoparticle in any charge sequence.

According to property 3-III, the internal structure of a local $c\nu$ pseudoparticle involves a number $\nu$ of rotated-electron doubly occupied sites and an equal number of rotated-electron unoccupied sites. There is a number $\binom{2\nu}{\nu}$ of different distribution configurations of these $\nu$ rotated-electron doubly occupied sites and $\nu$ rotated-electron unoccupied sites. For simplicity, let us restrict the present preliminary analysis to charge sequences with $\nu$ rotated-electron doubly occupied sites, an equal number of rotated-electron unoccupied sites, and no Yang holons. Below we find that a local $c\nu$ pseudoparticle is a superposition of a number $2\nu$ of the above rotated-electron site distribution configurations. For $\nu = 1$ the local $c1$ pseudoparticle is a properly symmetrized superposition of the two available rotated-electron site distribution configurations. For $\nu = 2$ the local $c2$ pseudoparticle is a properly symmetrized superposition of four out of the six available rotated-electron site distribution configurations. Thus, states including all possible six different distribution configurations are a superposition of two states including two local $c1$ pseudoparticles and a local $c2$ pseudoparticle, respectively. For $\nu = 3$ the local $c3$ pseudoparticle is a properly symmetrized superposition of eight out of the twenty available rotated-electron site distribution configurations. States including all possible twenty different distribution configurations are a superposition of three states including (i) three local $c1$ pseudoparticles, (ii) a local $c1$ pseudoparticle and a local $c2$ pseudoparticle, and (iii) a local $c3$ pseudoparticle. In the general case one has that the local $c\nu$ pseudoparticle is a properly symmetrized superposition of $2\nu$ out of the $\binom{2\nu}{\nu}$ available rotated-electron site distribution configurations. States including all possible $\binom{2\nu}{\nu}$ different distribution configurations are a superposition of several states including different numbers $N_{c\nu'}$ of $c\nu'$ pseudoparticles such that $\nu' = 1, 2, \ldots, \nu$ and $\sum_{\nu'=1}^{\nu} \nu' \cdot N_{c\nu'} = \nu$.

Among the $2\nu$ rotated-electron site distribution configurations of a $c\nu$ pseudoparticle there is always a distribution configuration where the first $\nu$ sites of index $h_{j,g}$ and $g = 1, 2, \ldots, \nu$ are doubly occupied by rotated electrons and the last $\nu$ sites of index $h_{j,\nu+g}$ and $g = 1, 2, \ldots, \nu$ are free of rotated electrons. We represent such a local $c\nu$ pseudoparticle rotated-electron site distribution configuration of a charge sequence by,

$$\left(\bullet_{h_{j,1}}, \ldots, \bullet_{h_{j,\nu}}, \circ_{h_{j,1+\nu}}, \ldots, \circ_{h_{j,2\nu}}\right).$$ \(28\)

The same applies to a local $s\nu$ pseudoparticle, the rotated-electron site distribution configuration of a spin sequence corresponding to the one given in Eq. \(28\) being,

$$(\downarrow_{l_{j,1}}, \ldots, \downarrow_{l_{j,\nu}}, \uparrow_{l_{j,1+\nu}}, \ldots, \uparrow_{l_{j,2\nu}}).$$ \(29\)

We confirm below that the $c\nu$ pseudoparticle located in the rotated-electron lattice at position $[h_{j,a}]$ and the $s\nu$ pseudoparticle located in the same lattice at position $[l_{j,a}]$ are described by the following properly symmetrized superposition of $2\nu$ rotated-electron site distribution configurations,

$$\left[\prod_{x=1}^{2\nu} e^{i\pi h_{j,x} \hat{D}_{j,x}}\right] \left[\prod_{g=1}^\nu (1 - \hat{T}_{c\nu, j,g})\right] \left(\bullet_{h_{j,1}}, \ldots, \bullet_{h_{j,\nu}}, \circ_{h_{j,1+\nu}}, \ldots, \circ_{h_{j,2\nu}}\right); \quad j = 1, 2, \ldots, N_{c\nu},$$ \(30\)

and

$$\left[\prod_{g=1}^\nu (1 - \hat{T}_{s\nu, j,g})\right] \left(\downarrow_{l_{j,1}}, \ldots, \downarrow_{l_{j,\nu}}, \uparrow_{l_{j,1+\nu}}, \ldots, \uparrow_{l_{j,2\nu}}\right); \quad j = 1, 2, \ldots, N_{s\nu},$$ \(31\)

respectively. According to property 3-III, the site-$h_{j,x}$ rotated-electron double occupation operator $\hat{D}_{j,x}$ appearing here has eigenvalue 1 and 0 when such a site is doubly occupied by rotated electrons and free of rotated electrons,
respectively, and the operator $\hat{T}_{c\nu, j, g}$ (and $\hat{T}_{s\nu, j, g}$) acts onto the pair of sites of indices $h_{j, g}$ and $h_{j, g+\nu}$ (and $l_{j, g}$ and $l_{j, g+\nu}$) only. This operator always acts onto rotated-electron site distribution configurations of the particular form illustrated in Eq. (28) (and in Eq. (29)). From the application of this operator onto such a rotated-electron site distribution configuration, a new distribution configuration is generated where the site of index $h_{j, g}$ is free of rotated electrons, the site of index $h_{j, g+\nu}$ is doubly occupied by rotated electrons, and the occupancy of the other $2(\nu - 1)$ sites remains unchanged, i.e.

$$\hat{T}_{c\nu, j, g}(..., \circ_{h_{j, g}}, ..., \circ_{h_{j, g+\nu}}, ...) = (..., \circ_{h_{j, g}}, ..., \bullet_{h_{j, g+\nu}}, ...).$$

The same transformation law,

$$\hat{T}_{s\nu, j, g}(..., \uparrow_{h_{j, g}}, ..., \uparrow_{h_{j, g+\nu}}, ...) = (..., \uparrow_{h_{j, g}}, ..., \downarrow_{h_{j, g+\nu}}, ...),$$

is associated with the application of the operator $\hat{T}_{s\nu, j, g}$ onto a rotated-electron site distribution configuration of the type illustrated in Eq. (29).

The $2\nu$ internal rotated-electron site distribution configurations on the right-hand side of Eqs. (30) (and Eq. (31)) are generated by considering that in each of the $g = 1, 2, ..., \nu$ pairs of sites of indices $h_{j, g}$ and $h_{j, g+\nu}$, the site of index $h_{j, g}$ is doubly occupied by rotated electrons and the site of index $h_{j, g+\nu}$ is free of rotated electrons and vice versa. In general, we call $h_{j, x} \leftrightarrow h_{j, x'}$ site pair a superposition of two rotated-electron distribution configurations of a rotated-electron doubly occupied site and a rotated-electron unoccupied site where $h_{j, x}$ and $h_{j, x'}$, such that $x, x' = 1, ..., 2\nu$ and $x' > x$, are the indices of the two lattice sites involved. In the first rotated-electron site distribution configuration, the sites of indices $h_{j, x}$ and $h_{j, x'}$ are doubly occupied by rotated electrons and free of rotated electrons, respectively. In the second rotated-electron site distribution configuration, the sites of indices $h_{j, x}$ and $h_{j, x'}$ are free of rotated electrons and doubly occupied by rotated electrons, respectively. The rotated-electron distribution configurations of all the remaining sites of the charge sequence except these two are identical. The

$$h_{j, x} \leftrightarrow h_{j, x'}; \quad x, x' = 1, ..., 2\nu; \quad x' > x,$$

site pair is defined as a superposition of these two rotated-electron site distributions configurations where the first and the second distributions are multiplied by a phase factor of 1 and $-1$, respectively.

In figure 1 we represent a $h_{j, x} \leftrightarrow h_{j, x'}$ site pair by two vertical lines connected by a horizontal line. The two vertical lines have the same height and connect the corresponding site of the charge sequence to the horizontal line. Thus, the two sites of a charge sequence connected by such lines are assumed to be involved in a superposition of two rotated-electron site distribution configurations multiplied by the phase factors of 1 and $-1$, respectively, where according to the above recipe one of these sites is doubly occupied by rotated electrons and the other one is free of rotated electrons and vice versa. Similarly, we call

$$l_{j, x} \leftrightarrow l_{j, x'}; \quad x, x' = 1, ..., 2\nu; \quad x' > x,$$

site pair an equivalent superposition of two rotated-electron site distribution configurations of a spin sequence. The definitions are the same provided that we replace rotated-electron doubly occupied sites and rotated-electron unoccupied sites by spin-down rotated-electron singly occupied sites and spin-up rotated-electron singly occupied sites, respectively. Such a rotated-electron singly occupied site pair is also graphically represented by the two vertical lines connected by a horizontal line plotted in Fig. 1. The concept of a site pair plays an important role in the description of the rotated-electron site distribution configurations of a local $\alpha\nu$ pseudoparticle.

It is confirmed below that the rotated-electron site distribution configurations which describe the internal structure of a local $\alpha\nu$ pseudoparticle always involve a number $\nu$ of site pairs such that $x = g$ and $x' = g + \nu$ where $g = 1, ..., \nu$. Since there are two possible rotated-electron site distribution configurations for each pair and the number of pairs of each local $\alpha\nu$ pseudoparticle is $\nu$, the total number of different internal rotated-electron site distribution configurations is indeed $2^\nu$. For example, for a $\alpha\nu$ pseudoparticle located in the rotated-electron lattice at the position $[h_{j, a}]$, the $2^\nu$ rotated-electron site distribution configurations superposed in expression (30) are the following,

$$\prod_{x=1}^{2} e^{i \pi h_{j, x} \partial_{j, x}} (1 - \hat{T}_{c\chi, j, g}) (\bullet_{h_{j, 1}}, \circ_{h_{j, 2}}) = -e^{i \pi h_{j, 1}} (\bullet_{h_{j, 1}}, \circ_{h_{j, 2}}) + e^{i \pi h_{j, 2}} (\circ_{h_{j, 1}}, \bullet_{h_{j, 2}}).$$


for a rotated-electron lattice at position \(l\) again a local charge sequence with no Yang holons and constituted by a number of the 2 double occupation operator phase factors by one.

Before using properties 3-III, 6-III, and 7-III to confirm the validity of the local pseudoparticle expressions (30)-(38), let us follow property 5-III and consider in the particular case of \(x = 1\) and \(x' = 2\) the figure represents the \(h_{j,1} \leftrightarrow h_{j,2}\) site pair of a c1 pseudoparticle or of the \(l_{j,1} \leftrightarrow l_{j,2}\) site pair of a s1 pseudoparticle.

for the c1 pseudoparticle,

\[
\prod_{x=1}^{4} e^{i\pi h_{j,x} \hat{D}_{j,x}} \prod_{g=1}^{2} (1 - \hat{T}_{c2,j,g}) (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \circ_{h_{j,3}}, \circ_{h_{j,4}}) = \\
+ e^{i\pi [h_{j,1} + h_{j,2}]} (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \circ_{h_{j,3}}, \circ_{h_{j,4}}) - e^{i\pi [h_{j,2} + h_{j,3}]} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}) \\
+ e^{i\pi [h_{j,3} + h_{j,4}]} (\circ_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}) - e^{i\pi [h_{j,1} + h_{j,4}]} (\bullet_{h_{j,1}}, \circ_{h_{j,2}}, \circ_{h_{j,3}}, \bullet_{h_{j,4}}),
\]

(37)

in the case of the c2 pseudoparticle, and

\[
\prod_{x=1}^{6} e^{i\pi h_{j,x} \hat{D}_{j,x}} \prod_{g=1}^{3} (1 - \hat{T}_{c3,j,g}) (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) = \\
- e^{i\pi [h_{j,1} + h_{j,2} + h_{j,3}]} (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) \\
+ e^{i\pi [h_{j,2} + h_{j,3} + h_{j,4}]} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) \\
- e^{i\pi [h_{j,3} + h_{j,4} + h_{j,5}]} (\circ_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \bullet_{h_{j,5}}, \circ_{h_{j,6}}) \\
+ e^{i\pi [h_{j,1} + h_{j,3} + h_{j,5}]} (\bullet_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \circ_{h_{j,6}}) \\
- e^{i\pi [h_{j,1} + h_{j,5} + h_{j,6}]} (\bullet_{h_{j,1}}, \circ_{h_{j,2}}, \circ_{h_{j,3}}, \bullet_{h_{j,4}}, \bullet_{h_{j,5}}, \circ_{h_{j,6}}) \\
+ e^{i\pi [h_{j,1} + h_{j,2} + h_{j,6}]} (\bullet_{h_{j,1}}, \bullet_{h_{j,2}}, \circ_{h_{j,3}}, \bullet_{h_{j,4}}, \circ_{h_{j,5}}, \bullet_{h_{j,6}}) \\
- e^{i\pi [h_{j,2} + h_{j,4} + h_{j,6}]} (\circ_{h_{j,1}}, \bullet_{h_{j,2}}, \bullet_{h_{j,3}}, \circ_{h_{j,4}}, \circ_{h_{j,5}}, \bullet_{h_{j,6}}) \\
+ e^{i\pi [h_{j,4} + h_{j,5} + h_{j,6}]} (\circ_{h_{j,1}}, \circ_{h_{j,2}}, \bullet_{h_{j,3}}, \bullet_{h_{j,4}}, \bullet_{h_{j,5}}, \bullet_{h_{j,6}}),
\]

(38)

for a c3 pseudoparticle. Similar expressions for local s1, s2, or s3 pseudoparticles are obtained by replacing in Eqs. (36), (37) the \(\bullet\) and \(\circ\) sites by \(\downarrow\) and \(\uparrow\) sites, respectively, the \(h_{j,g}\) indices by \(l_{j,g}\) indices, and the local rotated-electron double occupation operator phase factors by one.

Before using properties 3-III, 6-III, and 7-III to confirm the validity of the local pseudoparticle expressions (36), (37), let us check whether these expressions conform to the requirement of property 5-III. According to such a requirement, application of the off-diagonal generators of the 7-spin SU(2) algebra given in Eq. (7) of Ref. 4 onto the superposition of the 2\(n\) internal rotated-electron site distribution configurations of a \(c\) pseudoparticle located in the rotated-electron lattice at position \([h_{j,a}]\) must give zero. It is straightforward to confirm from the analysis of the changes in the rotated-electron site distribution configurations generated by these operators that this requirement is fulfilled. The same is valid for application of the off-diagonal generators of the spin SU(2) algebra given in Eq. (8) of Ref. 4 onto the superposition of the 2\(n\) internal rotated-electron site distribution configurations of a \(s\) pseudoparticle located in the rotated-electron lattice at position \([l_{j,a}]\).

In order to confirm the validity of the pseudoparticle expressions (36), (37), let us follow property 5-III and consider again a local charge sequence with no Yang holons and constituted by a number \(\nu\) of rotated-electron doubly occupied
sites and a number $\nu$ of rotated-electron unoccupied sites. If we request that the rotated-electron distribution configurations of these $2\nu$ sites describe the internal structure of a single $cv$ pseudoparticle, according to the restrictions imposed by the Bethe-ansatz solution, the associated local charge sequence must be characterized by the following numbers,

$$M_c = 2\nu; \quad N_{cv} = 1; \quad N_{cv}^0 = 0; \quad N_{cv'} = 0, \quad \nu' \neq \nu.$$  \hspace{1cm} (39)

Such a local charge sequence has a single $cv$ pseudoparticle and no $cv$ pseudoparticle unoccupied sites. This corresponds to an effective $cv$ pseudoparticle lattice constituted by a single site located in the rotated-electron lattice at $[h_1 a]$. That site is occupied by a $cv$ pseudoparticle. In this limiting case, in order to ensure periodic boundary conditions for the original electronic problem it is required that the charge sequence must be properly symmetrized by the method used in Ref. 9. Such a symmetrization involves the operator $\hat{T}_C$ of Eq. 6.

The fulfillment of the requirement of property 5-III imposes that the superposition of rotated-electron distribution configurations of the $\nu$ doubly occupied sites and $\nu$ unoccupied sites (and $\nu$ spin-down singly occupied sites and $\nu$ spin-up singly occupied sites) of such a $cv$ pseudoparticle (and $sv$ pseudoparticle) must be expressed in the form of a number $\nu$ of $h_{1,x} \leftrightarrow h_{1,x'}$ site pairs (and $l_{1,x} \leftrightarrow l_{1,x'}$ site pairs) where $x$ and $x'$ are such that $x' > x$ and have the values $x, x' = 1, 2, ..., 2\nu$. There is a number $(2\nu - 1)!$ of possible different choices for these $\nu h_{1,x} \leftrightarrow h_{1,x'}$ and $l_{1,x} \leftrightarrow l_{1,x'}$ site pairs. The use of requirement 6-III reveals that only one of these $(2\nu - 1)!$ choices corresponds to the $cv$ pseudoparticle (and $sv$ pseudoparticle). The remaining $(2\nu - 1)! - 1$ choices can be expressed as a superposition of several states described by different numbers $N_{cv'}$ of $cv'$ pseudoparticles such that $\sum_{\nu'} N_{cv'} = \nu$ (and $N_{cv'}$ of $sv'$ pseudoparticles such that $\sum_{\nu'} N_{sv'} = \nu$). Moreover, the fact that the symmetrized rotated-electron site distribution configuration which describes the internal structure of the $cv$ pseudoparticle (and $sv$ pseudoparticle) involves a number $\nu$ of site pairs implies that it corresponds to a state with charge (and spin) momentum $k_C = \pi/a$ (and $k_S = \pi/a$). The possible values of such a charge momentum are given in Eq. 5. For other values of charge momentum, such a state cannot be described in terms of a number $\nu$ of site pairs and thus the requirement introduced in property 5-III is not fulfilled.

For instance, when $\nu = 1$ application of the basic properties 3-III, 5-III, and 6-III leads uniquely to the following description of the $c1$ pseudoparticle,

$$\prod_{x=1}^{2} e^{i\pi h_{1,x} D_{1,x}} \sum_{\nu C = 0} \sum_{\nu C} e^{i\pi\nu C} \left( T_C \right)^{\nu C} (h_{1,1} \otimes h_{1,2}).$$  \hspace{1cm} (40)

This is the product of a rotated-electron double-occupation operator phase factor with a symmetrized charge sequence. The charge sequence defined by Eq. 10 can also be written as,

$$\prod_{x=1}^{2} e^{i\pi h_{1,x} D_{1,x}} (1 - T_{c1,1,1})(h_{1,1} \otimes h_{1,2}),$$  \hspace{1cm} (41)

in agreement with the general expression 10 and the local $c1$ pseudoparticle expression 36. For $j = 1, x = 1$, and $x' = 2$ Fig. 1 describes the $h_{1,1} \leftrightarrow h_{1,2}$ site pair associated with this $c1$ pseudoparticle.

Next, we consider the case $\nu = 2$. The fulfillment of the requirement introduced in property 5-III alone imposes in this case that the superposition of rotated-electron distribution configurations of the two doubly occupied sites and two unoccupied sites representative of a $c2$ pseudoparticle must be expressed in the form of two $h_{1,x} \leftrightarrow h_{1,x'}$ site pairs where $x$ and $x'$ are such that $x' > x$ and have the values $x, x' = 1, 2, 3, 4$. There are three possible different choices for the superposition of rotated-electron distribution configurations formed by two site pairs. These choices correspond to the following sets: (a) $\{h_{1,1} \leftrightarrow h_{1,1}; h_{1,2} \leftrightarrow h_{1,3}\}$, (b) $\{h_{1,1} \leftrightarrow h_{1,3}; h_{1,2} \leftrightarrow h_{1,4}\}$, and (c) $\{h_{1,1} \leftrightarrow h_{1,2}; h_{1,3} \leftrightarrow h_{1,4}\}$. According to property 7-III, the charge sequence (c) is excluded, because it describes two $c1$ pseudoparticles and thus there remain two possible choices for the $c2$ pseudoparticle. Figure 2 represents the charge sequences corresponding to those (a) and (b). Each of the $h_{1,1} \leftrightarrow h_{1,1'}$ site pairs are represented as in Fig. 1 with $j = 1$. The figure shows how the rotated-electron site distribution configurations of these two charge sequences change as a result of a cyclic permutation which transforms the first site of the charge sequence to the last site and multiplies the final state by a phase factor of $-1$. This phase factor arises from the required $k_C = \pi/a$ charge momentum. We note that the single $h_{1,1} \leftrightarrow h_{1,2}$ site pair descriptive of the $c1$ pseudoparticle given in expressions 40 and 41 transforms into itself under such a transformation. On the other hand, in the present case the rotated-electron site distribution configuration (a) transforms onto two $c1$ pseudoparticles whereas...
the rotated-electron site distribution configuration (b) transforms onto itself. Thus, according to property 6-III only the rotated-electron site distribution configuration (b) is properly symmetrized and describes the c2 pseudoparticle. Moreover, only this sequence can be written as a symmetrized charge sequence of charge momentum \( k_C = \pi/a \). Again, such a symmetrized charge sequence can also be written in the form given in the general Eq. \((30)\) and in the local c2 pseudoparticle expression \((37)\). The two equivalent expressions read,

\[
\prod_{x=1}^{4} e^{i \pi h_{1,x} D_{j,x}} \sum_{\nu C = 0}^{3} e^{i \pi \nu C} \left( \mathcal{T}_C \right)^{\nu C} \left( \circ h_{1,1}, \circ h_{1,2}, \circ h_{1,3}, \circ h_{1,4} \right) = \prod_{x=1}^{4} e^{i \pi h_{1,x} D_{j,x}} \left( \prod_{g=1}^{2} \left( 1 - \mathcal{T}_{C_{2,1,g}} \right) \right) \left( \circ h_{1,1}, \circ h_{1,2}, \circ h_{1,3}, \circ h_{1,4} \right).
\]

(42)

Let us now consider the case \( \nu = 3 \). Again, the fulfillment of the requirement introduced in property 5-III imposes that the superposition of rotated-electron distribution configurations of the three doubly occupied sites and three unoccupied sites representative of a c3 pseudoparticle should be expressed in the form of three \( h_{1,x} \leftrightarrow h_{1,x'} \) site pairs where \( x \) and \( x' \) are such that \( x' > x \) and have the values \( x, x' = 1, 2, 3, 4, 5, 6 \). There are fifteen different possible choices for superposition of rotated-electron site distribution configurations formed by three site pairs. However, only one of these charge sequences can be written in the form of symmetrized charge sequence of charge momentum \( k_C = \pi/a \). For example, in Fig. 3 we represent the charge sequences (a) \{ \( h_{1,1} \leftrightarrow h_{1,4} ; h_{1,2} \leftrightarrow h_{1,5} ; h_{1,3} \leftrightarrow h_{1,6} \) \} and (b) \{ \( h_{1,1} \leftrightarrow h_{1,3} ; h_{1,2} \leftrightarrow h_{1,5} ; h_{1,4} \leftrightarrow h_{1,6} \) \}. The figure shows how the rotated-electron site distribution configurations of these two charge sequences change as a result of a cyclic permutation which transforms the first site of the charge sequence to the last site and multiplies the final state by a phase factor of \(-1\). Both the rotated-electron site distribution configuration (b) and all remaining possible rotated-electron site distribution configurations constituted by three \( h_{1,x} \leftrightarrow h_{1,x'} \) site pairs except the rotated-electron site distribution configuration (a) do not transform onto themselves. The latter rotated-electron site distribution configuration transforms onto itself, as confirmed by analysis of the figure. Thus, according to property 6-III only the rotated-electron site distribution configuration (a) is properly symmetrized and describes the local c3 pseudoparticle. Only this sequence can be written in terms of a symmetrized charge sequence of charge momentum \( k_C = \pi/a \). Again, such symmetrized charge sequence can also be written in the form given in the general Eq. \((30)\) and in the local c3 expression \((38)\). The two equivalent expressions are the following.

![Graphical representation of charge sequences of a zero \( \eta \)-spin energy eigenstate with no Yang holons and two rotated-electron doubly occupied sites and two rotated-electron unoccupied sites only. The \( h_{1,g} \leftrightarrow h_{1,g+\nu} \) site pairs are represented as in Fig. 1. We note that the relative height of different site pairs has no physical meaning. The goal of using site pairs of different height is just to clearly define the two sites of each pair. In figures (a) and (b) two possible rotated-electron doubly occupied site/unoccupied site pair distribution configurations are represented. The figures show how these distribution configurations change as a result of a cyclic permutation which transforms the first site of the charge sequence onto its last site. The distribution configuration (b) transforms onto itself whereas the distribution configuration (a) transforms onto two c1 pseudoparticles. Thus, according to property 6-III the distribution configuration (b) describes a c2 pseudoparticle. Note that the distribution configuration (a) describes two c1 pseudoparticles. The distribution occupancies of the figure describe alternatively s1 and s2 pseudoparticles. In this case the spin sequences correspond to a zero spin energy eigenstate with no HL spinons and with two spin-down singly occupied sites and two spin-up singly occupied sites only.](image-url)
Note that in this case one needs two symmetrized charge sequences to reach the state obtained from the superposition of these two charge sequences is an eigenstate of the charge momentum operator with eigenvalue $\nu > 3$. According to property 6-III, this distribution configuration (a) describes a c3 pseudoparticle. The figures represent alternatively spin sequences of a zero spin energy eigenstate with no HL spinons and with three spin-down singly occupied sites and three spin-up singly occupied sites only.

\[
\begin{align*}
\prod_{x=1}^{6} e^{i\pi h_{1,x} D_{j,x}} \left\{ \sum_{\nu C=0}^{5} e^{i\pi\nu C} \left( \hat{T}_{C} \right)^{\nu C} \left( \bullet h_{1,1}, \bullet h_{1,2}, \bullet h_{1,3}, \circ h_{1,4}, \circ h_{1,5}, \circ h_{1,6} \right) \right\} + \\
1 \prod_{\nu C=0}^{5} e^{i\pi\nu C} \left( \hat{T}_{C} \right)^{\nu C} \left( \circ h_{1,1}, \bullet h_{1,2}, \circ h_{1,3}, \bullet h_{1,4}, \circ h_{1,5}, \bullet h_{1,6} \right) \right\} \\
= \prod_{x=1}^{6} e^{i\pi h_{1,x} D_{j,x}} \left\{ \prod_{g=1}^{3} \left( 1 - \hat{T}_{C3,1,g} \right) \left( \bullet h_{1,1}, \bullet h_{1,2}, \bullet h_{1,3}, \circ h_{1,4}, \circ h_{1,5}, \circ h_{1,6} \right) \right\}. 
\end{align*}
\]

Note that in this case one needs two symmetrized charge sequences to reach the c3 expression upon application of the operator $\hat{T}_{C}$ onto these charge sequences two and six times, respectively. However, in agreement with properties 5-III and 6-III, the state obtained from the superposition of these two charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_C = \pi/a$. Also the cν pseudoparticles belonging to branches such that $\nu > 3$ are a superposition of several such symmetrized charge sequences. However, always the state obtained from the superposition of several charge sequences is an eigenstate of the charge momentum operator with eigenvalue $k_C = \pi/a$. Such a state can also be expressed in the general form given in Eq. 10).

For the case of charge (and spin) sequences constituted by several local cν pseudoparticles (and local sν pseudoparticles) belonging to one or several $\nu = 1, 2, ...$ branches and by Yang holons and HL spinons the internal structure of the local cν pseudoparticles (and local sν pseudoparticles) is given by Eq. 10 (and Eq. 11). This result follows from the basic property 7-III. We will in the ensuing section confirm that, in agreement with property 8-III, this ensures the periodic boundary conditions of the original electronic problem provided that the bare-momentum energy eigenstates are suitable Fourier-transform superpositions of the local charge sequences with bare-momentum values obeying Eqs. 10 and 11.

However, whether inside the domain of $2\nu$ sites of a local cν pseudoparticle there can be sites with index $j_h$ such that $j_{h,1} < j_h < j_{h,2\nu}$ and which sites are occupied by Yang holons or other local cν pseudoparticles remains an open question. This problem is absent in the limiting case we considered here where the charge sequence is constituted by a number $2\nu$ of sites of the rotated-electron lattice. This issue is closely related to the definition of the local cν pseudoparticle charge-sequence unoccupied sites, which is discussed in the following. A similar problem occurs for the spin sequences.
B. THE LOCAL \( \nu \)-Pseudoparticle Unoccupied Sites and the Moving Pseudoparticle Elementary Steps

In the definition of the \( N^h_{\nu} \) charge-sequence unoccupied sites of a local \( \nu \)-pseudoparticle, we start by considering charge sequences where \( N_{\nu} = 1 \) and all remaining local \( \nu' \)-pseudoparticles except the specific local \( \nu \)-pseudoparticle keep their charge-sequence positions \( h_j \) defined in Eq. (13). Below we call steady local \( \nu' \)-pseudoparticles the former local pseudoparticles whose charge-sequence positions \( h_j \) remain unchanged. This concept does not apply for charge sequences where \( N_{\nu} > 1 \), when the specific local \( \nu \)-pseudoparticle passes a pseudoparticle belonging to the same branch. Indeed, in this case the two objects are indiscernible. However, also for such charge sequences the number of unoccupied sites remains well defined, as discussed below. Obviously, any local pseudoparticle can be chosen to be the moving pseudoparticle. This just refers to a set of different charge sequences where the charge-sequence position of that quantum object changes whereas the charge-sequence positions of the remaining local pseudoparticles remain unchanged. We note that according to Eq. (13), the charge-sequence position of a local \( \nu \)-pseudoparticle can remain unchanged in spite of the movements of some of its \( 2\nu \) sites, as we confirm below. Concerning the definition of the \( N^h_{\nu} \) charge-sequence unoccupied sites of a local \( \nu \)-pseudoparticle, from the use of properties 1-III to 8-III and of other properties of the model we find the following distribution configuration properties:

1-IV Let us consider a set of rotated-electron doubly occupied and/or rotated-electron unoccupied sites located at positions \([j_h a]\) according to Eqs. (13) and (14) the index \( h \) has the values \( h = 1, 2, \ldots, [N_a - N_e] \). Moreover, we consider that all of these sites are located inside the domain of sites of a local \( \nu \)-pseudoparticle and thus their indices \( j_h \) are such that \( j_{h,1} < j_h < j_{h,2\nu} \) and \( j_h \not= j_{h',x} \) where \( x = 1, 2, \ldots, 2\nu \). This property states that the set of rotated-electron doubly occupied/unoccupied sites of index \( j_h \) obeying these inequalities cannot correspond to sites occupied by Yang holons or to any of the \( 2\nu \) sites of the charge sequence associated with local \( \nu' \)-pseudoparticles belonging to branches such that \( \nu' > \nu \). Thus, such a set of rotated-electron doubly occupied or unoccupied sites can only correspond to sub-sets of \( 2\nu' \) sites of the charge sequence associated with local \( \nu' \)-pseudoparticles belonging to branches such that \( \nu' < \nu \).

2-IV Let us consider a local \( \nu \)-pseudoparticle and a local \( \nu' \)-pseudoparticle belonging to the same charge sequence and to branches such that \( \nu < \nu' \). Moreover, we consider that there are no charge-sequence local pseudoparticles inside the \( 2\nu' \)-site domain of the \( \nu' \)-pseudoparticle other than the \( \nu \)-pseudoparticle. (The description of cases involving more than two local pseudoparticles can be easily reached by generalization of the two-pseudoparticle situation considered here.) Then the \( 2\nu \) sites of the charge sequence associated with the local \( \nu \)-pseudoparticle are located either outside the domain of \( 2\nu' \) sites of the local \( \nu' \)-pseudoparticle limited in the left and right ends by the sites of index \( h_{j,1} \) and \( h_{j,2\nu} \), respectively, or are all located inside such a domain. If the \( 2\nu \) sites are located inside the domain of these \( 2\nu \) sites there is a number \( 2\nu' - 1 \) of permitted positions for the \( \nu \)-pseudoparticle. In each of these permitted positions all of the \( 2\nu \) sites of the local \( \nu \)-pseudoparticle are located between two first neighboring sites of indices \( h_{j,x} \) and \( h_{j',x+1} \) of the local \( \nu' \)-pseudoparticle. These permitted positions correspond to the neighboring sets of indices \( h_{j,x} \) and \( h_{j',x+1} \) such that \( x = 1, 2, \ldots, 2\nu' - 1 \).

3-IV Let us consider again the local \( \nu \)-pseudoparticle and the local \( \nu' \)-pseudoparticle referred to in property 2-IV. The charge-sequence position \( h_j \) given in Eq. (13) of such a steady local \( \nu' \)-pseudoparticle remains unchanged in all possible \( 2\nu' - 1 \) positions of the \( \nu' \)-pseudoparticle while passing its \( 2\nu' \)-site domain. The inverse statement is even stronger. It is required that when a local \( \nu' \)-pseudoparticle belonging to a branch such that \( \nu' > \nu \) passes a steady local \( \nu \)-pseudoparticle, the position in the rotated-electron lattice of all the \( 2\nu \) sites of the latter pseudoparticle must remain unchanged. Moreover, the \( 2\nu' - 1 \) possible different positions of the \( 2\nu \) sites of the local \( \nu \)-pseudoparticle inside the site domain of the local \( \nu' \)-pseudoparticle must be the same when the local \( \nu \)-pseudoparticle passes the steady local \( \nu' \)-pseudoparticle from its left to its right-hand side and vice versa. Also the relative positions of the two quantum objects when the local \( \nu \)-pseudoparticle passes the steady local \( \nu' \)-pseudoparticle and when the local \( \nu' \)-pseudoparticle passes the steady local \( \nu \)-pseudoparticle must be the same. However, note that although these relative positions are the same, when a local \( \nu' \)-pseudoparticle belonging to a branch such that \( \nu' > \nu \) passes a steady local \( \nu \)-pseudoparticle it does not occupy any of the \( 2\nu \) sites of the rotated-electron lattice occupied by the latter local pseudoparticle. Thus, the positions in the lattice of these \( 2\nu \) sites remain indeed unchanged. A last requirement is that the positions of all \( 2\nu' \) sites of a steady local \( \nu' \)-pseudoparticle belonging to a branch such that \( \nu' > \nu \) must be the same in the rotated-electron site distribution configurations before and after it is passed by the moving local \( \nu \)-pseudoparticle. If the local \( \nu \)-pseudoparticle is moving from the left to the right-hand side of the steady local \( \nu' \)-pseudoparticle, before and after is meant here as the rotated-electron site distribution configurations where the \( 2\nu \) sites of the former quantum object are located outside the domain of \( 2\nu' \) sites of the steady local pseudoparticle and
FIG. 4: Graphical representation of a domain of a charge sequence of the rotated-electron lattice including thirteen sites. The $h_{j,g} \leftrightarrow h_{j,g+\nu}$ site pairs are represented as in Fig. 1. The six sites with no vertical lines correspond to Yang holons. There are two local c1 pseudoparticles described by $h_{j,1} \leftrightarrow h_{j,2}$ site pairs and a local c2 pseudoparticle described by a $h_{j,1} \leftrightarrow h_{j,3}$ and a $h_{j,2} \leftrightarrow h_{j,4}$ site pairs. Alternatively, if we replace the Yang holons by HL spinons and the cv pseudoparticles by sv pseudoparticles the figure represents the domain of a spin sequence with six HL spinons, two s1 pseudoparticles, and a s2 pseudoparticle.

on its right and left hand sides, respectively.

4-IV Corresponding rules are also valid for the case of the spin sequences and local sv pseudoparticles.

A simple example of a domain constituted by thirteen sites of a permitted charge sequence is represented in Fig. 4. In such a figure the $h_{j,g} \leftrightarrow h_{j,g+\nu}$ site pairs of the local cv pseudoparticles are represented as in Figs. 1 to 3. The six sites with no vertical lines correspond to unoccupied sites associated with Yang holons. There are two local c1 pseudoparticles and a local c2 pseudoparticle in the charge-sequence domain represented in the figure.

In order to confirm the validity of the counting of rotated-electron site distribution configurations leading to expression (21), let us use the properties 1-IV to 4-IV in the classification of the different possible positions of a local cv pseudoparticle in the rotated-electron lattice. Our analysis can be easily generalized to the case of a sv pseudoparticle. Our study includes the introduction of the concept of 2ν-leg caterpillar step, where the 2ν-leg caterpillar is nothing but the 2ν-site pseudoparticle block associated with the moving cv pseudoparticle. For simplicity, we consider a charge sequence such that $N_{cv} = 1$ for the ν branch. The case of sequences with $N_{cv} > 1$ is considered below. As mentioned above, the possible positions of such a pseudoparticle correspond to a number $N^*_{cv}$ of different charge sequences where the charge-sequence positions of all remaining quantum objects remain unchanged. Thus, these $N^*_{cv}$ charge sequences differ in the position of the cv pseudoparticle relative to the other quantum objects. The above property 3-IV states that the charge-sequence position $h_j$ given in Eq. (18) of all quantum objects except the position of the moving cv pseudoparticle must remain unchanged in all these $N^*_{cv}$ charge sequences. That is obviously true for steady local cv′ pseudoparticles belonging to branches such that $\nu' < \nu$. Let us confirm that the charge-sequence position of a steady local cv′ pseudoparticle belonging to a branch such that $\nu' > \nu$ indeed remains unchanged when it is passed by a local cv pseudoparticle, in spite of changes of the positions in the rotated-electron lattice of some of its 2ν′ sites.

We start by considering the case of a local cv′ pseudoparticle where $\nu = \bar{\nu}$ denotes the largest ν value of the cv pseudoparticle branches with finite occupancy in the charge sequence. According to Eq. (B.7) of Ref. [4], in this case the number of local cv′ pseudoparticle unoccupied sites is given by $N^h_{cv} = L_c$ and equals the number of Yang holons. The different possible positions of a cv′ pseudoparticle can be achieved by elementary steps in the charge sequence where each of its 2ν′ sites moves forward by a single site of the charge sequence. If the movement is from the left to the right-hand side, the 2ν′th pseudoparticle site of index $h_{j,2\nu}$ moves into the site previously occupied by a Yang holon. The remaining $2\bar{\nu} - 1$ pseudoparticle sites of index $h_{j,x}$ move into the site previously associated with the pseudoparticle site of index $h_{j,x+1}$ where $x = 1, \ldots, 2\bar{\nu} - 1$. Finally, the Yang holon removed from the site newly associated with the pseudoparticle site of index $h_{j,2\nu}$ moves into the site previously associated with the first pseudoparticle site of index $h_{j,1}$. We call such an elementary collective step of the 2ν′ pseudoparticle sites $2\nu$-leg caterpillar step. According to such an analogy, the 2ν′ legs refer to these sites and the caterpillar refers to the local pseudoparticle. While each of these legs moves forward by a charge-sequence site, the whole caterpillar itself also moves forward by a single charge-sequence site. The net result of such a 2ν′-leg caterpillar step is that the compact domain of 2ν′ pseudoparticle sites interchanges position with a Yang holon. When such a moving cv′ pseudoparticle passes a steady local cv pseudoparticle, each of its 2ν′ sites jump the 2ν sites of the latter local pseudoparticle. Thus,
the 2ν-leg caterpillar moves forward like the 2ν sites of such a steady local cv pseudoparticle do not exist in the charge sequence. As we discuss below, the construction of the corresponding effective cv′ pseudoparticle lattice mentioned in property 8-III involves omission of the sites of the rotated-electron lattice belonging to the 2ν-site domain of such a local cv pseudoparticles. Moreover, for such an effective cv′ pseudoparticle lattice the 2ν-site domain of the cv pseudoparticles is seen as a point-like occupied site. Thus, in the particular case of the ν branch such an effective lattice has \(N^*_v = L_v + N_{cv}\) sites and a number \(N^*_{cv} = L_v\) free of cv pseudoparticles, as we confirm below.

Let us consider now the case of a general local cv pseudoparticle belonging to a branch such that \(1 \leq \nu < \nu′\). For simplicity, we keep the assumption that \(N_{cv} = 1\) for the ν branch. We consider a given charge sequence where the cv pseudoparticle is located in the rotated-electron lattice at position \([h_j,a]\). It follows from property 4-III that the \(N^*_{cv}\) remaining possible positions of the local cv pseudoparticle in the charge sequence define the positions of the local cv pseudoparticle charge-sequence unoccupied sites. Out of these \(N^*_{cv}\) possible positions, \(L_v\) is the number of positions associated with Yang holons. Again, the different positions of the moving local cv pseudoparticle can be achieved by 2ν-leg caterpillar steps. The net result of such a 2ν-leg caterpillar step is that the compact domain constituted by the 2ν cv pseudoparticle sites interchanges position with a Yang holon or with a site of a \(h_j′, g′ h_{j′, g′ + ν′}\) pair of a steady local cv′ pseudoparticle belonging to a branch such that \(ν′ > ν\). A 2ν-leg caterpillar step is defined here as for the above moving cv pseudoparticle. If the movement is from the right to the left-hand side, the local cv pseudoparticle site of index \(h_{j,2\nu}\) moves into the site previously occupied by a Yang holon or into one of the two sites previously associated with a local cv′ pseudoparticle \(h_{j′, g′ h_{j′, g′ + ν′}}\) site pair. The remaining \(2ν−1\) sites of the local cv pseudoparticle of index \(h_{j, ν−1}\) such that \(x = 1, ..., 2ν−1\), move into the site previously associated with the local cv pseudoparticle site of index \(h_{j, x+1}\) where again \(x = 1, ..., 2ν−1\). Finally, the Yang holon or cv′ pseudoparticle site associated with the local cv pseudoparticle of index \(h_{j, x+1}\) moves into the site previously associated with the local cv pseudoparticle site of index \(h_{j, 2ν}\). Again, the net result of such a 2ν-leg caterpillar step is that the local cv pseudoparticle moves forward by a single charge-sequence site.

Let us consider that in the initial rotated-electron site distribution configuration the moving cv pseudoparticle is located on the left-hand side of the steady local cv′ pseudoparticle. Thus, in the final state just after passing the latter pseudoparticle, the cv pseudoparticle will occupy in the charge sequence a domain of 2ν sites located on the right-hand side of the cv′ pseudoparticle site domain. According to property 3-IV, in both the initial and final rotated-electron site distribution configurations all 2ν′ sites of the latter quantum object remain the same. For simplicity, we consider that in the initial rotated-electron site distribution configuration, the 2ν charge-sequence sites located just on the right-hand side of the steady local cv′ pseudoparticle are occupied by Yang holons. However, our results can be generalized to rotated-electron site distribution configurations where these sites are occupied by a third pseudoparticle. Another case which we could consider is when the moving local pseudoparticle passes two steady local cv′ and cv″ pseudoparticles such that one of them is located within the site domain of the other one. From the systematic use of the properties 1-IV to 4-IV one can describe the movements of a local cv pseudoparticle when it passes any rotated-electron site distribution configuration involving several steady local pseudoparticles.

According to property 2-IV, the 2ν sites of a local cv pseudoparticle can have 2ν′ − 1 different positions inside the 2ν′-site domain of a local cv′ pseudoparticle belonging to a branch such that \(ν′ > ν\). We consider that in the initial rotated-electron site distribution configuration, the local cv and cv′ pseudoparticles are located side by side in the charge sequence. Thus, these two quantum objects occupy together a compact \(2(ν + ν′)\)-site domain. By simple counting arguments it is straightforward to realize that in order to reach each of these permitted positions, each of the 2ν sites of the local cv pseudoparticle must jump 2ν sites of the charge sequence, somewhere inside the 2ν′-site domain of the steady local cv′ pseudoparticle. This result is consistent with the expression of the number \(2 \sum_{ν′+1}^{∞}(ν′ − ν)N_{cv′}\) of the available positions for the moving local cv pseudoparticle involving sites of steady local cv′ pseudoparticles belonging to branches such that \(ν′ > ν\). This expression reveals that when located inside the site domain of such a pseudoparticle, each of the 2ν sites of the local cv pseudoparticle can move into \(2(ν′ − ν)\) sites out of the 2ν′ total number of sites of the steady local cv′ pseudoparticle. This agrees with the above result obtained from counting arguments alone. We note that the number of charge-sequence sites jumped by each local cv caterpillar leg equals precisely 2ν, the number of sites of such a cv pseudoparticle. Thus, these jumps ensure that the position in the rotated-electron lattice of all the 2ν′ sites of the steady local cv′ pseudoparticle remains unchanged after the cv passes it, as requested by the property 3-IV. This result also applies to the general situation when a moving local cv pseudoparticle passes a domain containing several local steady cv′ pseudoparticles belonging to branches such that \(ν′ > ν\). Also in the general situation that each of the legs of the moving caterpillar uses \(2(ν′ − ν)\) possible sites of each steady local cv′ pseudoparticle, out of a total of \(2ν′\) sites.

The question which arises from the occurrence of the above jump mechanism is which are the 2ν sites of the steady local cv′ pseudoparticle that are jumped by each site of the moving local cv pseudoparticle. In the following analysis of the site distribution configurations of the cv pseudoparticle when it passes the steady local cv′ pseudoparticle we provide explicit information about the location of its 2ν sites. The corresponding location of the 2ν′ sites of the
steady local $c\nu'$ pseudoparticle is implicitly given if one recalls that in all occupancy configurations reported below both pseudoparticles occupy a compact charge-sequence domain of $2(\nu + \nu')$ sites. While these occupancy configurations involve movements of the $2\nu'$ sites of the $c\nu'$ pseudoparticle, according to the property 3-IV the steady character of that object follows from the invariance of its charge-sequence position $\bar{h}_{ij}$, as given by Eq. (13), which remains the same in all such occupancy configurations. The requirements mentioned in that property provide the needed information for the definition of the $2\nu' - 1$ permitted positions of the local $c\nu$ pseudoparticle inside the steady local $c\nu'$ pseudoparticle: To reach these permitted positions, each site of the local $c\nu$ pseudoparticle must perform two independent jumps of $\nu$ sites in the charge sequence, as described below. The possible positions of a local $c\nu$ pseudoparticle when it passes from the left to the right-hand side of a steady local $c\nu'$ pseudoparticle are the following: First the local $c\nu'$ pseudoparticle performs a $2\nu$-leg caterpillar step. The net result of such an elementary collective step is that the compact domain of $2\nu$ sites of the local $c\nu$ pseudoparticle interchanges position with the site of index $h_{j',1}$ of the local $c\nu'$ pseudoparticle. In the reached rotated-electron site distribution configuration all the $2\nu$ sites of the local $c\nu$ pseudoparticle are located between the sites of index $h_{j',1}$ and $h_{j',2}$ of the steady local $c\nu'$ pseudoparticle. This procedure is repeated $\nu' - 1$ times until the $2\nu$ sites of the $c\nu'$ pseudoparticle are located between the sites of index $h_{j',\nu'-1}$ and $h_{j',\nu'}$ of the steady local $c\nu'$ pseudoparticle. In each of the corresponding $2\nu$-leg caterpillar steps the local $c\nu$ pseudoparticle moves forward by a single lattice site of the charge sequence. However, in the $\nu'$th step each site of the $c\nu'$ pseudoparticle jumps $\nu$ sites of the charge sequence. This jump brings it to a rotated-electron site distribution configuration where the charge-sequence position $\bar{h}_1$ of the local $c\nu$ pseudoparticle given in Eq. (13) coincides with that of the steady local $c\nu'$ pseudoparticle. In order to reach the next permitted position each of the $2\nu$ local $c\nu$ pseudoparticle sites jump again a number $\nu$ of lattice sites of the charge sequence. This brings the local $c\nu$ pseudoparticle to a distribution configuration where its $2\nu$ sites are located between the sites of index $h_{j',1+\nu'}$ and $h_{j',2+\nu'}$ of the steady local $c\nu'$ pseudoparticle. These two jumps are followed by $\nu' - 1$ $2\nu$-leg caterpillar steps where the $c\nu'$ pseudoparticle moves again by a single lattice site of the charge sequence. The net result of each of these elementary collective steps is that the compact domain of $2\nu$ $c\nu$ pseudoparticle sites interchanges position with one of the sites of the $c\nu'$ pseudoparticle. Finally, the last step brings the local $c\nu$ pseudoparticle to a distribution configuration where it is located outside and on the right-hand side of the domain of $2\nu'$ sites of the steady local $c\nu'$ pseudoparticle. The same intermediate positions are reached when the $c\nu$ pseudoparticle passes the domain of $2\nu'$ sites of the $c\nu'$ pseudoparticle from the right to the left-hand side. Such events are illustrated in Fig. 5 for the case when a local $c1$ pseudoparticle passes a steady local $c2$ pseudoparticle from the left to the right-hand side. If instead we consider that the local $c1$ pseudoparticle is steady, Fig. 5 illustrates the possible positions of a local $c2$ pseudoparticle when it passes from the right to the left-hand side a steady local $c1$ pseudoparticle. In this case the position in the effective electronic lattice of the two sites of the $c1$ pseudoparticle remains unchanged and the different relative positions of the two quantum objects are reached by movements of the $c2$ pseudoparticle. Importantly, we emphasize that in this case the $c2$ pseudoparticle does not occupy the sites occupied by the $c1$ pseudoparticle. Again this is consistent with property 3-IV.

We have just studied a situation where a local $c\nu$ pseudoparticle passes a steady local $c\nu'$ pseudoparticle belonging to a branch such that $\nu' > \nu$. For simplicity, we assumed that in the initial rotated-electron site distribution configuration the domain of $2\nu$ charge-sequence sites located just on the right-hand side of the steady local $c\nu'$ pseudoparticle is occupied by Yang holons. However, our results can be generalized to more complicated situations. An example is a situation where a local $c\nu'$ pseudoparticle passes a steady local $c\nu'$ pseudoparticle which has as first neighbor in the charge sequence a second steady local $c\nu''$ pseudoparticle. Another example is when a local $c\nu$ pseudoparticle passes a steady local $c\nu'$ pseudoparticle containing inside its $2\nu'$-site domain a third steady local $c\nu''$ pseudoparticle belonging to a branch such that $\nu'' < \nu'$. These more general situations can be described by the use of the properties 1-IV to 4-IV. One always finds that in order to move forward, the $2\nu$ sites of the local $c\nu$ pseudoparticle use $2(\nu' - \nu)$ sites out of the total number of $2\nu$ sites of each steady local $c\nu'$ pseudoparticle belonging to branches such that $\nu' > \nu$. Moreover, when passing the central region of the $2\nu'$-site domain of such a local $c\nu'$ pseudoparticle, the $2\nu$ legs of the $c\nu$ caterpillar always perform the two $\nu$-site jumps described above.

Finally, we consider the general case of a charge sequence such that $\sum_{\nu' = \nu + 1}^{\nu''} (\nu' - \nu) > 1$ for the $c\nu$ branch of the moving local pseudoparticle. In this case such an object passes local pseudoparticles belonging to the same branch. We consider that in the initial rotated-electron site distribution configuration, the moving $c\nu$ pseudoparticle and a second $c\nu'$ pseudoparticle are located side by side in the charge sequence. Thus, these two quantum objects occupy together a compact $4\nu$-site domain. Let us suppose that the local $c\nu$ pseudoparticle is moving from the left to the right-hand side. It follows from the indiscernible character of the two objects that in this case they move side by side, both performing $2\nu$ $2\nu$-leg caterpillar steps. This brings such an objects to a distribution configuration where the $c\nu$ pseudoparticle on the left-hand side has the same position than that on the right-hand side had in the initial distribution configuration. Since the two quantum objects are indiscernible, the moving $c\nu'$ pseudoparticle is now that located on the right-hand side. By counting the involved site distribution configurations one realizes that they are consistent with the number of unoccupied sites being $N^h_{c\nu} = L_c + 2 \sum_{\nu' = \nu + 1}^{\nu''} (\nu' - \nu) N_{c\nu'}$. 
Moreover, for such an effective pseudoparticle, the charge and spin sequence. We note that each of the (rotated-electron doubly occupied/unoccupied sites). Such a procedure follows from the independent conservation rules. Alternatively, the figure represents the corresponding electron site distribution configurations of a steady local c1 pseudoparticle when it passes from the right to the left-hand side of a steady local c1 pseudoparticle. However, in this second case the charge-sequence position of the local c1 pseudoparticle should remain unchanged and the X point of figures (a)-(e) should be moved accordingly. In both cases these five distribution configurations correspond to five different charge sequences. Alternatively, the figure represents the corresponding electron site distribution configurations of a s1 pseudoparticle and a s2 pseudoparticle.

The charge sequence (and the spin sequence) is obtained by omission of the rotated-electron singly occupied sites (and rotated-electron doubly occupied/unoccupied sites). Such a procedure follows from the independent conservation of the charge and spin sequence. We note that each of the cv pseudoparticle (and sv pseudoparticle) branch occupancy configurations are also independently conserved. Thus, it follows from our above analysis that the construction of the effective cv pseudoparticle lattice mentioned in property 8-I1 also involves omission of the sites of the rotated-electron lattice which are jumped by the movements of a cv pseudoparticle, since these sites are not seen by such a moving quantum object. The omitted sites correspond to the 2ν-site domains of local cv pseudoparticles of branches such that ν′ < ν and to 2ν-site domains belonging to the 2ν sites of each cv pseudoparticle such that ν′ > ν. Moreover, for such an effective cv pseudoparticle lattice the 2ν-site domain of the cv pseudoparticles is seen as a point-like occupied site. Thus, such an effective lattice has $N_{cv} = L_c + \sum_{\nu' > \nu} 2(\nu' - \nu) N_{cv} + N_{cv}$ sites, a number $N_{cv} = L_c + 2 \sum_{\nu' > \nu}(\nu' - \nu) N_{cv}$ of which are unoccupied of cv pseudoparticles, as we confirm below. A similar analysis holds for the sv pseudoparticles.

C. COMPLETE SET OF CHARGE, SPIN, AND c PSEUDOPARTICLE SEQUENCE LOCAL STATES

We have just found the rotated-electron distribution configurations of the 2ν sites which describe a local αν pseudoparticle. Moreover, we considered the $N_{cv}$ charge-sequence unoccupied sites of such a local αν pseudoparticle. Let us consider again the case of the local charge sequences and associated local cv pseudoparticles. A result which follows from our findings is that indeed the positions of the $N_{cv}$ charge-sequence unoccupied sites of each local cv pseudoparticle branch are fully determined by the position of the c pseudoparticles and cv pseudoparticles alone. Our results thus confirm that for fixed values of the numbers $N_{cv}$ and $N_{cv}$ the number of occupancy configurations of the local cv pseudoparticles is given by Eq. (21) for α = c. It follows that we can classify the local cv pseudoparticle occupancy configurations for ν = 1, 2, ... branches with finite occupancy in a given state by providing the indices $h_j$ given in Eqs. (17). These indices correspond to the $N_{cv}$ pseudoparticle positions provided in Eq. (22). Our results also confirm that the number of local pseudoparticle occupancy configurations of a CPHS ensemble subspace is indeed given by Eq. (21). Finally, since our results can be generalized to spin sequences as well, they also validate the use of the representation associated with the local state defined in Eqs. (25)-(27). Such a local state represents each of the different local pseudoparticle occupancy configurations whose number for a specific CPHS ensemble subspace is given in Eq. (21). Our results also confirm that the set of these local states whose number is given in Eq. (21) constitutes a complete set of states in the corresponding CPHS ensemble subspace.

Thus, for CPHS ensemble subspaces with no −1/2 Yang holons and no −1/2 HL spinons the local states defined in Eqs. (25)-(27) constitute a complete set of states. Our main goal here is the generalization of these local states to CPHS ensemble subspaces with finite occupancies of both ±1/2 Yang holons and ±1/2 HL spinons. We start by extending the analysis to these cases.
expressing the complete set of local states defined in Eqs. (45) in terms of the following more basic states,

\[ |(j_1, j_2, ..., j_N, \eta); \{(h_{j_1,1}, ..., h_{j_1,v_1}, \circ h_{j_1,j_1+1}, ..., \circ h_{j_1,2v_1})\}; \{(\psi_{j_2,1}, ..., \psi_{j_2,v_2}, \uparrow l_{j_2,1}, ..., \uparrow l_{j_2,2v_2})\} \],

where the charge sequence refers to the following sets of rotated-electron distribution configurations of doubly-occupied and unoccupied sites,

\[ \{(h_{j_1,1}, ..., h_{j_1,v_1}, \circ h_{j_1,j_1+1}, ..., \circ h_{j_1,2v_1})\} = (h_{1,1}, \circ h_{1,2}), (h_{2,1}, \circ h_{2,2}), ..., (h_{N_{c},1}, \circ h_{N_{c},2}); \\
(\psi_{j_2,1}, ..., \psi_{j_2,v_2}, \uparrow l_{j_2,1}, ..., \uparrow l_{j_2,2v_2}) \]

and the same follows for the spin sequence with the index \( h \) replaced by the index \( l \), rotated-electron doubly occupied sites replaced by spin-down rotated-electron singly occupied sites \( \downarrow \), and sites free of rotated electrons \( \circ \) replaced by spin-up rotated-electron singly occupied sites \( \uparrow \). The states defined in Eqs. (45) provide the position of the \( 2\nu \) sites associated with each of the \( \alpha\nu \) pseudoparticles whose branches have finite occupancy in the corresponding charge and spin sequence. However, we note that these states include only rotated-electron site distribution configurations of the simple type given in Eqs. (28) and (29). The general charge and spin sequences can be obtained from these states by application of suitable operators, as we discussed above. These are thus the most basic and simple states. Our first task is expressing the states defined in Eqs. (45) in terms of these simple states.

Each local \( \alpha\nu \) and \( \nu\nu \) pseudoparticle is described by the superposition of \( 2^\nu \) rotated-electron singly occupied sites \( \bullet \) replaced by spin-down rotated-electron singly occupied sites \( \downarrow \), and sites free of rotated electrons \( \circ \) replaced by spin-up rotated-electron singly occupied sites \( \uparrow \). The states defined in Eqs. (45) and (46) respectively. The type of configuration superposition given in these equations is common to the description of all local \( \alpha\nu \) pseudoparticles of a charge or spin sequence. Thus, the proper description of the charge and spin sequence of each of the states defined in Eqs. (45) involves the superposition of 

\[ 2^\sum_{\alpha=\nu}^{c} \sum_{\nu=1}^{\infty} \nu N_{\alpha\nu} \]

such a rotated-electron site distribution configurations. It follows that these normalized local states can be expressed as follows,

\[
|\{(j_1, j_2, ..., j_N, \eta); \{(h_{j_1,1}, ..., h_{j_1,v_1}, \circ h_{j_1,j_1+1}, ..., \circ h_{j_1,2v_1})\}; \{(l_1, l_2, ..., l_{N_{\nu}})\}\} = 2^{-\sum_{\alpha=\nu}^{c}} 2^{\sum_{\nu=1}^{\infty} \nu N_{\alpha\nu}} / 2 \cdot \prod_{\nu^\prime=1}^{\infty} \prod_{1 \neq 1 \leq x^\prime \leq 1} e^{i \pi h_{j_1} \cdot x^\prime} \cdot D_{j_1} \cdot x^\prime \cdot x^\prime^{\nu^\prime} \nonumber \times |\{(j_1, j_2, ..., j_N, \eta); \{(h_{j_1,1}, ..., h_{j_1,v_1}, \circ h_{j_1,j_1+1}, ..., \circ h_{j_1,2v_1})\}; \{(l_1, l_2, ..., l_{N_{\nu}})\}\} ,
\]

where the operator \( D_{j_1} \cdot x^\prime \cdot x^\prime^{\nu^\prime} \) measures the number of rotated-electron doubly occupied sites in the charge-sequence site index \( h_{j_1} \cdot x^\prime \) (whose value is given by 1 or 0) and the transformation laws for application of the operator \( T_{\alpha\nu^\prime} \cdot j^\prime \cdot g^{\nu^\prime} \) were given above and are illustrated in Eqs. (25) and (26).

The local states given in both Eqs. (45) and (46) provide a complete basis of states for CPHS ensemble subspace spanned by lowest-weight states of both the \( \eta \)-spin and spin \( SU(2) \) algebras. Let us now consider the general case of states associated with CPHS ensemble subspaces spanned by states with both finite occupancies of \( \pm 1/2 \) Yang holons and \( \pm 1/2 \) HL spinons. The description of the local charge and spin sequences with finite occupancies of \( \pm 1/2 \) Yang holons and \( \pm 1/2 \) HL spinons, respectively, is quite similar to the ones we studied above. For simplicity, let us consider first the case of the charge sequences. According to property 1-III the \( -1/2 \) Yang holons and \( +1/2 \) Yang holons correspond to single rotated-electron doubly occupied lattice sites and single rotated-electron unoccupied lattice sites of such a charge sequence, respectively. For a lowest weight state (and highest weight state) of the \( \eta \)-spin algebra all Yang holons have the same \( \eta \)-spin projection \( +1/2 \) (and \(-1/2 \)) and correspond to rotated-electron unoccupied sites (and rotated-electron doubly occupied sites). The application of the off-diagonal generators of the \( \eta \)-spin \( SU(2) \) algebra given in Eq. (7) of Ref. [29] leads to flips of the Yang holons \( \eta \)-spins. Following property 5-III, in order to achieve the Yang holon occupancy configurations required by the \( \eta \)-spin \( SU(2) \) algebra, we find that each local charge sequence is described by a superposition of \( C_\alpha \) Yang holon occupancy configurations where,

\[
C_\alpha = \left( \frac{L_\alpha}{L_{\alpha,-1/2}} \right) = \frac{L_\alpha!}{L_{\alpha,-1/2}! L_{\alpha,+1/2}!}; \quad \alpha = c, s.
\]

Here \( L_c = 2S_c \) (and \( L_s = 2S_s \)) stands for the total number of Yang holons (and HL spinons) in the charge (and spin) sequence. All the \( \left( \frac{L_c}{L_{c,-1/2}} \right) \) states associated with these configurations have the same set of \( L_c \) lattice sites occupied
by Yang holons and the same numbers of $-1/2$ Yang holons and $+1/2$ Yang holons. However, these states differ in the distributions of the $-1/2$ Yang holons and $+1/2$ Yang holons over the $L_c$ lattice sites. The same analysis holds for spin sequences with finite occupancies of $\pm1/2$ HL spinons. Thus, there are $\prod_{\alpha=c,s}(L_\alpha + 1)$ states with precisely the same occupancy configurations of local $c$ pseudoparticles and local $\alpha\nu$ pseudoparticles. Each of these occupancy configurations is described by a state of form given in Eq. (46). On the other hand, we denote each of the $\prod_{\alpha=c,s}(L_\alpha + 1)$ general states by,

$$|(L_{c,-1/2}, L_{s,-1/2}); (j_1, j_2, ..., j_{N_c}); \{(h_1, h_2, ..., h_{N_{dc}}); \{(l_1, l_2, ..., l_{N_{ds}})}\rangle,$$

where $L_{c,-1/2}$ and $L_{s,-1/2}$ are the corresponding numbers of $-1/2$ Yang holons and $-1/2$ HL spinons, respectively. We note that Eq. (47) remains valid for the general local states (48). Thus, the values of the numbers $L_c$ and $L_s$ are determined by the values of the numbers of $c$ pseudoparticles and of $\alpha\nu$ pseudoparticles. The values of the numbers $L_{c,-1/2}$ and $L_{s,-1/2}$ were added to the local states (48) because they are needed for the specification of the corresponding CPHS ensemble subspace. In contrast, the values of the numbers $L_{c,+1/2}$ and $L_{s,+1/2}$ are dependent and given by $L_{c,+1/2} = L_c - L_{c,-1/2}$ and $L_{s,+1/2} = L_s - L_{s,-1/2}$ and are not explicitly provided in the expression of the local states (48). Each of these normalized states is given by,

$$\langle \prod_{\beta_{c,s}} (\alpha^g_{\beta_{c,s}}) \rangle = 2^{N_{\alpha\nu}} \prod_{\alpha'=c,s} \prod_{\nu'=1}^{2 \nu'} e^{i \pi h_{j',s} \beta_{j',s}} \prod_{\alpha''=c,s} \prod_{\nu''=1}^{2 \nu''} (1 - \hat{T}_{\alpha'' \nu''}^{\nu''} . j'' , g'') \times |(j_1, j_2, ..., j_{N_c}); \{(h_{1,1}, ..., \alpha_{h_{j_2+1}, ... , \alpha_{h_{j,2}}})}; \{(l_{j_1,1}, ..., \alpha_{l_{j_2+1}, ... , \alpha_{l_{j,2}}})\rangle ,$$

where the operators $\hat{S}_i^\alpha$ with $\alpha = c, s$ are the off-diagonal generators defined in Eqs. (7) and (8) of Ref. 9 and the corresponding normalization constants $C_\alpha$ are given in Eq. (47). Application of these off-diagonal generators onto the local states given in Eq. (48) leaves the rotated-electron site distribution configurations of the $\alpha\nu$ pseudoparticles invariant and generates $\eta$-spin and spin flips in the $+1/2$ Yang holons and $+1/2$ HL spinons, respectively, as requested by property 5-III.

The set of all local states of general form given in Eq. (49) provides a complete basis of states in any CPHS ensemble subspace of the Hilbert space. However, these states do not ensure the periodic boundary conditions of the original electronic problem. Such conditions are ensured by the construction of the energy eigenstates as Fourier-transform superpositions of local states of general form given in Eq. (49), as discussed in the ensuing section.

V. SEPARATION OF THE $\alpha\nu$ PSEUDOPARTICLE TRANSLATIONAL - INTERNAL DEGREES OF FREEDOM: EFFECTIVE $\alpha\nu$ PSEUDOPARTICLE LATTICES AND THE ENERGY EIGENSTATES

In this section we finally reach a precise definition for the concept an effective $\alpha\nu$ pseudoparticle lattice. The precise definition of such a concept is a necessary condition for the construction of the energy eigenstates as Fourier-transform superpositions of the local states introduced in the previous section. The concept of an effective $\alpha\nu$ pseudoparticle lattice is related to the separation of the translational and internal degrees of freedom of these quantum objects, as we explain below. As mentioned in property 8-III, in addition to the effective $c$ pseudoparticle lattice, there is an effective $\alpha\nu$ pseudoparticle lattice for each $\alpha\nu$ pseudoparticle branch. The spatial coordinate associated with these lattices is the conjugate of the pseudoparticle bare-momentum $q_j$, which is a good quantum number of the many-electron problem. As discussed in Appendix B of Ref. 9, this quantum number is associated with the Bethe-ansatz solution. The Fourier transforms which relate the local pseudoparticles introduced in this paper to the bare-momentum pseudoparticles obtained from analysis of the Bethe-ansatz solution in Ref. 9 involve the spatial coordinate of the effective pseudoparticle lattices.

Besides the introduction of the effective pseudoparticle lattices, in this section we also provide explicit expressions for the energy eigenstates in terms of Slater determinants which involve Fourier-transform superpositions of the local states introduced in the previous section. This clarifies the relation of the energy eigenstates obtained from the use of the Bethe-ansatz solution 9 and $\eta$-spin and spin $SU(2)$ symmetries 2, 8 to the rotated-electron site distribution configurations that emerge from the electron - rotated-electron unitary transformation 8.
A. TRANSLATIONAL - INTERNAL DEGREES OF FREEDOM SEPARATION AND THE EFFECTIVE $c\nu$ PSEUDOPARTICLE LATTICES

For simplicity, let us again concentrate mainly on the case of the $c\nu$ pseudoparticles. The separation of the $c\nu$ pseudoparticle translational and internal degrees of freedom can be defined in terms of the following general properties which are obtained from the above properties 1-III to 8-III and 1-IV to 3-IV, and other features of the model:

1-V One can separate the $2\nu$-site internal structure of a local $c\nu$ pseudoparticle from the problem of the description of the movements of these quantum objects in the rotated-electron lattice. Such separation leads to the concept of effective pseudoparticle lattice, whose spatial coordinates are associated with the pseudoparticle translational degrees of freedom only. The motion of a $c\nu$ pseudoparticle involves $2\nu$-leg caterpillar steps, whereas the $2\nu$-site domains of these objects play the role of the occupied sites of an effective $c\nu$ pseudoparticle lattice. From the point of view of the motion of these objects in such an effective $c\nu$ pseudoparticle lattice, these $2\nu$-site domains are point-like sites without internal structure. Roughly speaking, these point-like pseudoparticle occupied sites correspond to the center of mass of the $2\nu$-site block associated with each local $c\nu$ pseudoparticle. This description corresponds to a separation of the pseudoparticle translational and internal degrees of freedom. The latter degrees of freedom are described by the $2\nu$-site rotated-electron distribution configurations studied in the previous section. That internal structure is such that the $c\nu$ pseudoparticle transforms as a $\eta$-spin zero quantum object under application of the off-diagonal generators of the $\eta$-spin $SU(2)$ algebra. In addition, such an internal structure is a necessary condition for the fulfillment of the periodic boundary conditions of the original electronic problem. On the other hand, the $c\nu$ pseudoparticle translational degrees of freedom are closely related to the spatial coordinates of the effective pseudoparticle lattice. These spatial coordinates play the role of conjugate variable relative to the bare momentum $q_j$. Such a bare momentum obeys Eq. (11) to ensure periodic boundary conditions for the original electronic problem. An important point is that the pseudoparticle spatial coordinate occupancy configuration of each $c\nu$ pseudoparticle branch with finite occupancy in a given energy eigenstate is independently conserved. As the charge (and spin) sequence corresponds to the rotated-electron doubly occupied and unoccupied sites (and rotated-electron singly occupied sites) only, one can also introduce an independent effective $c\nu$ pseudoparticle lattice for each of these occupied pseudoparticle branches, whose coordinates correspond to some of the sites of the local charge sequence only.

2-V Within the above separation of the pseudoparticle translational and internal degrees of freedom, the moving $c\nu$ pseudoparticle sees the $2\nu$-site domains of each of the local pseudoparticles belonging to the same branch as point-like occupied pseudoparticle sites. On the other hand, according to the property 3-IV a moving $c\nu$ pseudoparticle jumps the $2\nu'$-site domains representative of each local $c\nu'$ pseudoparticles belonging to branches such that $\nu' < \nu$. This property can be understood as follows: Since in its movements the $c\nu$ pseudoparticle sees the $2\nu$-site domains of the remaining $c\nu$ pseudoparticles of the same branch as point-like unreachable occupied pseudoparticle sites, such a $2\nu$-site domain width is the smallest width seen by the $c\nu$ pseudoparticle. Thus, such a width plays the role of a site domain width uncertainty. As a result, pseudoparticle site domains of width smaller than the $2\nu$-site domain are not seen by the $c\nu$ pseudoparticle. This property is consistent with the exact property that the moving $c\nu$ pseudoparticle does not see (i.e. jumps) smaller $2\nu'$-site $c\nu'$ pseudoparticle domains corresponding to branches such that $\nu' < \nu$. On the other hand, again according to property 3-IV, the $2\nu$-leg caterpillar step movements of the $c\nu$ pseudoparticle uses $2(\nu' - \nu)$ sites only, out of the $2\nu'$-site domain of each local $c\nu'$ pseudoparticle belonging to branches such that $\nu' > \nu$. Thus, again consistently with the above site domain width uncertainty, in their motion throughout the rotated-electron lattice, a local $c\nu$ pseudoparticle does not see a number $2\nu'$ of sites out of the $2\nu'$ sites of such a $\nu' > \nu$ local $c\nu'$ pseudoparticle.

3-V The concepts of an unoccupied site and a lattice constant of the effective $c\nu$ pseudoparticle lattice follow from the above analysis. For the moving $c\nu$ pseudoparticle all the $N^h_{c\nu'}$ charge-sequence unoccupied sites corresponding to a number $L_c$ of $\pm 1/2$ Yang holons and a number $2\sum_{\nu' = \nu+1}^{\infty} N^*_{c\nu'}$ of sites belonging to the $2\nu'$-site domains of local $c\nu'$ pseudoparticles of the charge sequence such that $\nu' > \nu$ are seen as equivalent and indiscernible point-like unoccupied sites of a effective $c\nu$ pseudoparticle lattice. Moreover, according to the property 1-V such a quantum object also sees the $N_{c\nu}$ $2\nu$-site domains of pseudoparticles belonging to the $c\nu$ pseudoparticle branch as equivalent and indiscernible point-like occupied sites. Thus, while running through all its possible positions in the rotated-electron lattice, the moving $c\nu$ pseudoparticle sees all the $N^h_{c\nu} = N^c_{c\nu} + N^h_{c\nu'}$ sites of the effective $c\nu$ pseudoparticle lattice as equivalent and indiscernible point-like sites. Therefore, these sites are equally spaced for the effective $c\nu$ pseudoparticle lattice. The value of the corresponding lattice constant is uniquely defined as follows: In its movements a $c\nu$ pseudoparticle does not see (i) the $2\sum_{\nu' = \nu+1}^{\infty} \nu' N^c_{c\nu'}$ sites occupied by $c\nu'$ pseudoparticles belonging to branches such that $\nu' \leq \nu$; (ii) a number $2\nu \sum_{\nu' = \nu+1}^{\infty} N^c_{c\nu'}$ of sites belonging to the $2\nu'$-site domains of $c\nu'$ pseudoparticles belonging to branches such that $\nu' > \nu$; and (iii) the $N_c$ sites singly occupied by rotated electrons. Thus, the $c\nu$
pseudoparticle jumps all the above sites of the rotated-electron lattice and sees the 2ν-site pseudoparticle domains of the $c\nu$ pseudoparticle branch as point-like occupied sites. On the other hand, in order to pass all the $N_{c\nu}^s = N_{c\nu} + N_{c\nu}^h$ sites of the effective $c\nu$ pseudoparticle lattice and return to its original position, the $c\nu$ pseudoparticle should run through a distance which equals the length $L$ of the rotated-electron lattice. Therefore, a necessary condition to ensure the periodic boundary conditions of the original electronic problem is that the length of the effective $c\nu$ pseudoparticle lattice must equal the length $L$ of the rotated-electron lattice. This determines uniquely the value of the lattice constant of the effective $c\nu$ pseudoparticle lattice which reads $a_{c\nu} = L/N_{c\nu}^s$. That the length of the effective $c\nu$ pseudoparticle lattice is $L$ is consistent with the spacing $q_{j+1} - q_{j} = 2\pi/L$ given in Eq. (B.2) of Ref. 9 for the corresponding pseudoparticle discrete bare-momentum values provided by the exact Bethe-ansatz solution. Such a discrete bare momentum $q_j$ is the conjugate of the coordinate $x_j = a_{c\nu} j$ of the effective $c\nu$ pseudoparticle lattice where $j = 1, 2, ..., N_{c\nu}^*$.

4-V A similar analysis is valid for the local $s\nu$ pseudoparticles, provided that the above mentioned $N_c$ sites singly occupied by rotated electrons are replaced by the $[N_a - N_c]$ sites doubly occupied by rotated electrons and free of rotated electrons. The lattice constant of the effective $s\nu$ pseudoparticle lattice is given by $a_{s\nu} = L/N_{s\nu}^a$. On the other hand, the effective $c$ pseudoparticle lattice has $N_s$ lattice sites and its lattice constant $a$ is the same as for the rotated-electron lattice. The positions of the $N_c$ $c$ pseudoparticles and $N_c^h = [N_a - N_c]$ $c$ pseudoparticle holes in this lattice equal the corresponding positions of the rotated-electron singly occupied sites and rotated-electron doubly occupied/unoccupied sites, as already discussed in previous sections.

5-V The set of pseudoparticle occupancy configurations of the effective $c$ pseudoparticle, $c\nu$ pseudoparticle, and $s\nu$ pseudoparticle lattices belonging to branches $\nu = 1, 2, ...$ with finite occupancy in a given local state of form (14) together with the numbers $L_{c,-1/2}$ and $L_{c,1/2}$ of such a state contain the same information as the corresponding description of the same local state in terms of the rotated-electron site distribution configurations.

Since the value of the number $N_{a\nu}^s$, defined by Eqs. (B.6), (B.7), and (B.11) of Ref. 9 is distinct for different CPHS ensemble subspaces, it follows from the expression of the $a\nu$ pseudoparticle lattice constants,

$$a_{a\nu} = a \frac{N_a}{N_{a\nu}^s} = \frac{L}{N_{a\nu}^s},$$

that the value of such constants changes accordingly.

Let us introduce the following notation for the spatial coordinates of the effective $c$ pseudoparticle lattice,

$$x_j = a_{c\nu} j, \quad j = 1, 2, 3, ..., N_a,$$

where the index $j$ was called $j_l$ and $j_h$ in Eq. (14) for the $c$ pseudoparticle occupied sites ($l = 1, 2, 3, ..., N_c$) and unoccupied sites ($h = 1, 2, 3, ..., [N_a - N_c]$), respectively. Moreover, let us introduce the following notation for the spatial coordinates of the effective $a\nu$ pseudoparticle lattices,

$$x_j = a_{a\nu} j, \quad j = 1, 2, 3, ..., N_{a\nu}^s.$$  

The bare-momentum limiting values given in Eq. (B.14) of Ref. 9 for the $a\nu$ pseudoparticle bands can be expressed in terms of the corresponding lattice constants $a_{a\nu}$ as follows,

$$q_{a\nu} = \frac{\pi}{a_{a\nu}} [1 - 1/N_a] \approx \frac{\pi}{a_{a\nu}}.$$  

Also the $c$ pseudoparticle limiting bare-momentum values given in Eqs. (B.16) and (B.17) of the same paper can be written as,

$$q_c^\pm \approx \pm q_c; \quad q_c = \frac{\pi}{a}.$$  

Thus, the domain of available pseudoparticle bare-momentum values corresponds to an effective first-Brillouin zone associated with an underlying effective pseudoparticle lattice.
In reference [2] it is found that both the electronic lattice and the momentum operator [2], which is the generator of the spatial translations in such a lattice, remain invariant under the electron - rotated-electron unitary transformation. This justifies why the electronic and rotated-electron lattices are identical. Therefore, the lattice spatial coordinates occupied by the rotated electrons correspond to real-space coordinates. Since we have just shown that the effective pseudoparticle lattices refer directly to the rotated-electron lattice, also the space coordinates of the former lattices refer to real-space coordinates. The pseudoparticle bare momentum is the conjugate of the spatial coordinate of the corresponding effective pseudoparticle lattice. Thus, it follows from the above invariance and relations that the pseudoparticle bare-momentum refers to real momentum. This is confirmed by the form of Eq. (36) of Ref. [9] for the momentum in terms of the pseudoparticle bare-momentum occupancy configurations. Indeed, the expression of such an equation is additive in the bare momentum for all pseudoparticle branches.

The ground state plays an important role in the study of one- and two-electron spectral functions [11]. Thus, let us use the ground-state Eqs. (B.11), (C.12)-(C.14), (C.24), and (C.25) of Ref. [9] in order to find the corresponding values for the effective \( \alpha \nu \) pseudoparticle lattice constants in the case of electronic densities \( n \) and spin densities \( m \) such that \( 0 < \nu a < 1 \) and \( 0 < m a < n a \), respectively. We find that in the case of such a state these constants read,

\[
a^0_{\alpha \nu} = \frac{1}{\delta}; \quad a^0_{s\uparrow} = \frac{1}{n\uparrow}; \quad a^0_{s\downarrow} = \frac{1}{m},
\]

where \( \delta = [1 - \nu a]/a \) is the doping concentration away from half filling. Moreover, the ground-state number \( N^*_{\alpha \nu} \) given in Eqs. (B.6), (B.7), and (B.11) of Ref. [9] and the ground-state limiting bare-momentum values given in Eq. (C.12)-(C.14) of the same paper can be written in terms of the effective pseudoparticle lattice constants as follows,

\[
N^*_{\alpha \nu} = \frac{L}{a^0_{\alpha \nu}}; \quad q^0_{\alpha \nu} = \frac{\pi}{a^0_{\alpha \nu}}.
\]

We note that when the effective pseudoparticle lattice constants given in Eq. (55) diverge, as for \( a^0_{\alpha \nu} \) as \( \delta a = [1 - \nu a] \rightarrow 0 \) and \( a^0_{s\nu} \) as \( m a = [n\uparrow - n\downarrow]/a \rightarrow 0 \) when \( \nu > 1 \), the value of the corresponding number \( N^*_{\alpha \nu} \) given in Eq. (56) is zero. This just means that in these limits the corresponding effective pseudoparticle lattices do not exist for states belonging to the ground-state CPHS ensemble subspace.

The relation of the effective pseudoparticle lattices introduced here to previous results on the model in the limit \( U/t \rightarrow \infty \) is briefly discussed in the Appendix.

### B. CONSTRUCTION OF THE ENERGY EIGENSTATES

In order to relate the description of the energy eigenstates in terms of pseudoparticle occupancy configurations obtained from the use of the Bethe-ansatz solution and \( \eta \)-spin and spin symmetries to the representation of the same states in terms of rotated-electron site distribution configurations, there is a last goal to be reached. That is the construction of the energy eigenstates in terms of the local charge, spin, and \( c \) pseudoparticle sequences studied in previous sections. In section IV we expressed the local states given in both Eqs. (25)-(27) and Eq. (46) in terms of rotated-electron site distribution configurations and showed that they provide a complete basis of states. In order to relate these local states to the energy eigenstates we should first express the former states in terms of local \( c \) and \( \alpha \nu \) pseudoparticle occupancy configurations in the corresponding effective pseudoparticle lattices. We then Fourier transform the obtained states into bare-momentum space \( q_j \) with respect to the spatial coordinates of the effective \( c \) and \( \alpha \nu \) pseudoparticle lattices.

As in the case of Eq. (16) for the effective \( c \) pseudoparticle lattice, we denote by \( j_l \) and \( j_h \) where \( l = 1, 2, ..., N_{\alpha \nu} \) and \( h = 1, 2, ..., N_{\alpha \nu} \), respectively, the occupied and unoccupied sites, respectively, of the effective \( \alpha \nu \) pseudoparticle lattices. The spatial coordinate of these occupied and unoccupied pseudoparticle sites is given by,

\[
x_{j_l} = a_{\alpha \nu} j_l, \quad l = 1, 2, ..., N_{\alpha \nu}; \quad x_{j_h} = a_{\alpha \nu} j_h, \quad h = 1, 2, ..., N_{\alpha \nu}^{\alpha \nu},
\]

where \( j_l \) is the index which defines the position of the occupied pseudoparticle sites and \( j_h \) the position of the unoccupied pseudoparticle sites. We note that the unoccupied sites of the effective pseudoparticle lattices correspond to the sites left over by the occupied sites and vice versa. Thus, we can uniquely specify a given effective pseudoparticle lattice site distribution configuration by providing the position of the occupied sites (or unoccupied sites) only. Here...
we choose the representation in terms of the positions in the effective $c$ and $\alpha\nu$ pseudoparticle lattices of the occupied sites $x_{j_1}$ of Eqs. (16) and (17). In such a representation the states given in Eq. (19) are denoted as,

$$
|\{(L_{c,-1/2}, L_{s,-1/2}); (x_{j_1}, x_{j_2}, ..., x_{jN_c}); \{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}); \{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}) \} \} \rangle.
$$

(58)

Here $\{x_{j_1}, x_{j_2}, ..., x_{jN_c}\}$ is the set of spatial coordinates associated with the set of indices $(j_1, j_2, ..., j_{N_c})$ given in Eq. (13). These spatial coordinates correspond to the set of sites occupied by $c$ pseudoparticles in the corresponding effective lattice. Moreover,

$$
\{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}); \{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}); \{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}) \} \} \rangle.
$$

(59)

gives the spatial coordinates of each $\alpha\nu$ pseudoparticle effective lattice with finite occupancy.

We emphasize that from analysis of the specific rotated-electron site distribution configurations relative to a given state of form (16) one can construct by use of the above properties 1-V to 3-V the $c$ pseudoparticle, $c\nu$ pseudoparticle, and $\alpha\nu$ pseudoparticle occupancy configurations in the effective $c$, $c\nu$, and $\alpha\nu$ pseudoparticle lattices, respectively, of the corresponding state of form (16). This relation was already mentioned in the above property 5-V and the inverse statement is obviously also true. Thus, Eqs. (16) and (17) refer to two different representations of the same local states. These local states constitute a complete set of states. However, the form of these local states does not ensure the periodic boundary conditions of the original electronic problem.

The representation of the local states in terms of the effective pseudoparticle lattice occupancy configurations given in Eq. (16) is the most suitable starting point for construction of the energy and momentum eigenstates. Such a construction is fulfilled by Fourier transforming the local states given in Eq. (58). The obtained states ensures the periodic boundary conditions of the original electronic problem. Such procedures lead to the following expression for the energy eigenstates in terms of the local states of Eqs. (16) and (17),

$$
\begin{align*}
&\{|(L_{c,-1/2}, L_{s,-1/2}); (q_{j_1}, q_{j_2}, ..., q_{jN_c}); \{ (q_{j_1}, q_{j_2}, ..., q_{jN_{\alpha\nu}}); \{ (q_{j_1}, q_{j_2}, ..., q_{jN_{\alpha\nu}}) \} \rangle \\
&= N_a^{-N_c/2} \left[ \prod_{\alpha' = e, s} \prod_{\nu' = 1}^{\infty} [N_{\alpha'\nu'}^*]^{-N_{\alpha'\nu'}/2} \right] \sum_{j_1 < j_2 < ... < j_{N_c}} \sum_P (-1)^P e^{i q_a \sum_{l=1}^{N_c} jP(l) q_{j_1}} \\
&\times \left[ \prod_{\nu' = 1}^{\infty} \left( \sum_{j_1 < j_2 < ... < j_{N_{\alpha\nu}}} (-1)^P e^{i q_{a\nu} \sum_{l=1}^{N_{\alpha\nu}'} jP(l) (\frac{\pi}{a} - q_{j_1})} \right) \right] \\
&\times \left[ \prod_{\nu' = 1}^{\infty} \left( \sum_{j_1 < j_2 < ... < j_{N_{\alpha\nu}''}} (-1)^P e^{i q_{a\nu} \sum_{l=1}^{N_{\alpha\nu}''} jP(l) q_{j_1}} \right) \right] \\
&\times \{|(L_{c,-1/2}, L_{s,-1/2}); (x_{j_1}, x_{j_2}, ..., x_{jN_c}); \{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}); \{ (x_{j_1}, x_{j_2}, ..., x_{jN_{\alpha\nu}}) \} \} \rangle.
\end{align*}
$$

(60)

The permutations $P$ on the right-hand side of Eq. (60) generate a Slater determinant of the bare momenta and the spatial coordinates of the $N_c$ $c$ pseudoparticles and $N_{\alpha\nu}$ $\alpha\nu$ pseudoparticles belonging to $\alpha\nu$ branches with finite occupancy in the corresponding local charge and spin sequences. On the left-hand side of Eq. (60) the set of pseudoparticle occupied bare-momentum values specify the energy eigenstate. There, $(q_{j_1}, q_{j_2}, ..., q_{jN_c})$ are the set of $N_c$ occupied bare momentum values out of the $N_a$ available discrete $q_j$ values of the $c$ pseudoparticle band such that $j = 1, ..., N_a$. Moreover,

$$
\{ (q_{j_1}, q_{j_2}, ..., q_{jN_{\alpha\nu}}) \} = (q_{j_1}, q_{j_2}, ..., q_{jN_{\alpha\nu}}); (q_{j_1}, q_{j_2}, ..., q_{jN_{\alpha\nu}}); (q_{j_1}, q_{j_2}, ..., q_{jN_{\alpha\nu}}); \ldots
$$

(61)

are the set of $N_{\alpha\nu}$ occupied bare momentum values out of the $N_{\alpha\nu}^*$ available discrete $q_j$ values of the $\alpha\nu$ pseudoparticle bands of $\alpha\nu$ branches with finite occupancy. Such bare-momentum pseudoparticle occupancy configurations are discussed in Ref. [3].

Since equations (16) and (17) refer to different representations of the same states, the energy eigenstates (31) are Fourier-transform superpositions involving permutations of the local charge-sequence and spin-sequence states (58). The latter local states correspond to rotated-electron site distribution configurations. Thus, combination of the general expressions (19) and (60) provides important information about the relationship between the bare-momentum
pseudoparticles and the rotated-electron occupancy configurations. We emphasize that the permutations on the right-hand side of Eq. (60) are associated with different positions for the local $\alpha \nu$ pseudoparticles but that the internal structure of these quantum objects remains invariant under these permutations. Both the Fourier transforms and permutations of the general expression (60) do not touch the internal structure of the local $\alpha \nu$ pseudoparticles and refer only to their translational degrees of freedom. This results from the point-like character of the occupied sites of the effective pseudoparticle lattices. Thus, the local and bare-momentum $\alpha \nu$ pseudoparticles have the same internal structure.

Also the rotated electrons have alternative local and momentum representations. The creation operator for a rotated electron of momentum $k$ and spin $\sigma$ is such that $\hat{c}_{k, \sigma}^\dagger = \hat{V}^\dagger(U/t)\hat{c}_{k, \sigma} \hat{V}(U/t)$. Here $\hat{c}_{k, \sigma}^\dagger$ refers to the corresponding electronic operator and $\hat{V}(U/t)$ is the electron - rotated-electron unitary operator $\hat{V}(U/t)$ uniquely defined by Eqs. (21)-(23) of Ref. [3]. The rotated-electron site distribution configurations studied in previous sections of this paper refer to local rotated electrons. As the local rotated electrons and local pseudoparticles are directly related, one could also express the bare-momentum pseudoparticles in terms of momentum rotated electrons. However, for the energy eigenstates given in Eq. (60) the simplest way to relate the pseudoparticles to the rotated electrons is through the local representations of these quantum objects. In this paper we clarified how the local pseudoparticles are related to local rotated electrons. Since the bare-momentum pseudoparticle description is related to the corresponding local pseudoparticle representation by a simple Fourier transform, such a transform also relates the bare-momentum pseudoparticles to the local rotated electrons.

For the energy eigenstates given in Eq. (60), a first condition for the fulfillment of the periodic boundary conditions of the original electronic problem is that in all the local charge and spin sequences described by the states (49) and (51) for $\alpha = c$ and $\alpha = s$, respectively. These are obtained from the proper symmetrization of the corresponding single pseudoparticle 2ν-site sequences. A second condition to ensure these periodic boundary conditions is that the energy eigenstates given in Eq. (60) are Fourier transform superpositions of these charge and spin sequences of the form given on the right-hand side of such an equation. Provided that the discrete values of the bare momentum $q_j$ of these Slater determinant superpositions obey Eqs. (9) and (10), the periodic boundary conditions of the original electronic problem are ensured. The concept of an effective pseudoparticle lattice introduced above allows the following physical interpretation of the pseudoparticle bare-momentum boundary conditions (9) and (10) needed to ensure the periodic boundary conditions of the original electronic problem. When the number $\sum_{\alpha=c,s} \sum_{\nu=1}^{\infty} N_{\alpha \nu}$ is odd (and the number of sites $N_{s \nu}$ is even), Eq. (9) (and (10)) just imposes a twisted boundary condition such that each $c$ pseudoparticle (and $\nu$ pseudoparticle) hopping from site $N_\alpha$ to site 0 (and site $N_{s \alpha}$ to site 0) of the corresponding effective lattice will acquire a phase factor given by $e^{i\pi} = -1$ or $e^{i\pi} = 1$. Such a phase factor corresponds to the scattering-less part of the $c$ pseudofermion (and $\nu$ pseudofermion) $S$ matrix introduced in Ref. [13], which controls the unusual finite-energy spectral properties of the model (11).

Note that the general expression provided in Eq. (60) includes suitable permutations of the local sequences. In the same equation the rotated-electron site distribution configurations of the local charge and spin sequences associated with the states (49) or (51) are reexpressed in terms of pseudoparticle occupancy configurations of the corresponding effective pseudoparticle lattices. Finally, let us consider the specific case of a ground state with electronic densities and spin densities in the ranges $0 \leq na \leq 1$ and $0 \leq ma \leq na$, respectively. Such a state is a superposition of local normalized states (30) which have no $-1/2$ Yang holons, $-1/2$ HL spinons, $\nu$ pseudoparticles, and $s \nu$ pseudoparticles belonging to branches such that $\nu > 1/2$. Thus, for such a ground state these local states are of the following simplified form,

$$2^{N_{s \nu}} \left[ |(0, 0); (x_{j_1}, x_{j_2}, ..., x_{j_{N_\nu}}); (x_{j_1}, x_{j_2}, ..., x_{j_{N_{s \nu}}})\rangle \right]$$

$$= \prod_{j'=1}^{N_\nu} (1 - \hat{T}_{s \nu, j'; 1}) |(0, 0); (j_1, j_2, ..., j_{N_\nu}); \{(|\psi_{j'_1, 1}; \uparrow l_{j'_1, 1})\},$$

where

$$\{(|\psi_{j_1, 1}; \uparrow l_{j_1, 1})\} = (|l_{i_1, 1}; \uparrow l_{i_1, 2}), (|l_{i_2, 1}; \uparrow l_{i_2, 2}), ..., (|l_{i_{N_{s \nu}}, 1}; \uparrow l_{i_{N_{s \nu}}, 2})$$

and the operators $\hat{T}_{s \nu, j'; 1}$ are the same as in the general Eq. (49). The states defined in Eqs. (52) and (53) provide the position of the two rotated-electron singly occupied sites associated with the internal structure of each local $s \nu$ pseudoparticle. These local states have an alternative representation in terms of the $c$ and $s \nu$ pseudoparticle occupancy configurations of the effective $c$ and $s \nu$ pseudoparticle lattices, respectively. Within such an effective pseudoparticle lattice representation the local states (62) are denoted by $|(0, 0); (x_{j_1}, x_{j_2}, ..., x_{j_{N_\nu}}); (x_{j_1}, x_{j_2}, ..., x_{j_{N_{s \nu}}})\rangle$. Here
(x_{j1}, x_{j2}, ..., x_{jN_s1}) are the spatial coordinates which define the positions of the local s\textsubscript{1} pseudoparticles in the effective s\textsubscript{1} pseudoparticle lattice. These spatial coordinates correspond to the indices (l\textsubscript{1}, l\textsubscript{2}, ..., l\textsubscript{N_s1}) on the left-hand side of Eq. (62). The sites of such an effective s\textsubscript{1} pseudoparticle lattice are equally spaced, the corresponding ground-state lattice constant \( a_s^0 \) being given in Eq. (63). The ground state expression is a particular case of the Fourier-transform superpositions of local states given in Eq. (60). For the ground state such an expression simplifies to,

\begin{equation}
|GS\rangle = N_a^{N_s/2} (N_{\uparrow}^0)^{N_s/2} |(0, 0); (q_{j1}, q_{j2}, ..., q_{jN_s}); (q_{j1}, q_{j2}, ..., q_{jN_s})\rangle
\end{equation}

\begin{equation}
= \sum_{c_1 < c_2 < ... < N_s} \sum_P (-1)^P e^{i\sum_{i=1}^{N_s} c_P(q_{j1})} \sum_{g_1 < g_2 < ... < g_N} \sum_P (-1)^P e^{i\sum_{j'=1}^{N_s} \sum_P g_P(q_{j1})} \times |(0, 0); (x_{j1}, x_{j2}, ..., x_{jN_s}); (x_{j1}, x_{j2}, ..., x_{jN_s})\rangle
\end{equation}

where \( N_c = N, N_{s1} = N_\uparrow, \) and \( N_s^* = N_{\uparrow}. \)

VI. DISCUSSION AND CONCLUDING REMARKS

The electronic site distribution configurations which describe the energy eigenstates of the 1D Hubbard model are very complex and dependent on the value of \( U/t \). This is confirmed by the \( U/t \) dependence of the double-occupation quantities studied in Ref. [16]. However, the electron - rotated-electron unitary transformation is such that it maps these complex and \( U/t \) dependent electronic site distribution configurations onto the corresponding \( U/t \) independent rotated-electron site distribution configurations studied in this paper. We found here that the local pseudoparticle site distribution configurations of the effective pseudoparticle lattices which describe the same states are also independent of the value of \( U/t \). Moreover, since the spatial coordinate of these effective lattices is the conjugate of the pseudoparticle bare-momentum, the pseudoparticle bare-momentum occupancies which describe the energy eigenstates are also independent of the values of \( U/t \). As discussed in the previous section, this independence on the value of \( U/t \) is related to the invariance of the electronic lattice and the momentum operator of the system, which is the generator of the spatial translations in such a lattice, under the electron - rotated-electron unitary transformation.

In this paper we profited from such a \( U/t \) independence of the pseudoparticle and rotated-electron occupancy configurations and extracted some of our results from analysis of the corresponding \( U/t \to \infty \) electronic problem. In such a limit the rotated electron and electron are the same object and the description of the quantum problem simplifies because both the \( \eta \)-spin and spin occupancy configurations are highly degenerated and electron double occupation is a good quantum number, as discussed in the Appendix. We presented a detailed study of the relation of the description of the energy eigenstates of the model in terms of rotated-electron site distribution configurations to the representation of the same states in terms of Yang holon, HL spinon, and pseudoparticle bare-momentum occupancy configurations. The latter representation of the energy eigenstates is obtained from the Bethe-ansatz relation and \( \eta \)-spin and spin algebras. The connection between these two descriptions follows from the relation of rotated electrons to the quantum numbers of the above solution and symmetries established in Ref. [20]. That connection involves complex rotated-electron behavior such that, for all energy eigenstates, the \( N_c \) rotated electrons of the singly occupied sites separate into \( M_e = N_c \) charge-less spin 1/2 spinons and \( N_c \) spin-less and \( \eta \)-spin-less \( c \) pseudoparticles of charge \(-e\). (The \(-1/2\) and \(+1/2\) holons are less exotic and correspond to the rotated-electron doubly-occupied and unoccupied sites, respectively.) To further study such a connection, in this paper we introduced the concepts of a local pseudoparticle and an effective pseudoparticle lattice. For an effective \( \alpha \nu \) pseudoparticle lattice the values of its length and lattice constant have an exotic dependence on the values of the numbers of quantum objects in a given state but are independent of the value of \( U/t \). The corresponding pseudoparticle spatial coordinate \( x_j \) is given in Eqs. (31) and (32) and is the conjugate of the pseudoparticle bare-momentum \( q_j \). The pseudoparticle bare-momentum is a quantum number which emerges from the Bethe-ansatz Takahashi’s thermodynamic equations and the introduction of the concept of an effective pseudoparticle lattice involved the study of the rotated-electron distribution configurations of doubly occupied and unoccupied sites (and spin-down and spin-up singly occupied sites) which describe the internal structure of a local \( \nu \nu \) pseudoparticle (and \( \alpha \nu \) pseudoparticle).

In what the translational degrees of freedom are concerned, the local \( \alpha \nu \) pseudoparticle is a point-like quantum object, its internal structure being the same as that of the corresponding bare-momentum pseudoparticle. We also found that the spatial coordinates of the occupied and unoccupied sites of the effective \( c \) pseudoparticle lattice are the same as the corresponding coordinates of the sites singly occupied by rotated electrons and doubly occupied by or free of rotated electrons, respectively. The energy eigenstates can be described in terms of local pseudoparticle site distribution configurations in the corresponding effective pseudoparticle lattice. Our results reveal that there is
an one-to-one correspondence between the local pseudoparticle site distribution configurations in the effective pseudoparticle lattices which describe a given energy eigenstate and the rotated-electron site distribution configurations which describe the same state.

The results of this paper provide useful information for the correct identification of the scatterers and scattering centers \( \mathbb{R} \) that control the model unusual spectral properties \( \mathbb{I} \). The use of the pseudoparticle description studied in this paper for the evaluation of the finite-energy spectral-weight distributions \( \mathbb{I} \) involves a second unitary transformation, which maps the \( c \) pseudoparticles (and composite \( cv \) or \( su \) pseudoparticles) onto \( c \) pseudofermions (and composite \( cv \) or \( su \) pseudofermions) and is defined in the Hilbert subspace where the one- and two-electron excitations are contained \( \mathbb{I} \). That transformation introduces shifts of order \( 1/L \) in the pseudoparticle discrete momentum values and leaves all other pseudoparticle properties invariant. (The Yang holons and HL spinons remain invariant under such a transformation.) Since the effective pseudoparticle lattices and the corresponding site distribution configurations that describe the energy eigenstates belonging to that subspace remain invariant under the pseudoparticle - pseudofermion unitary transformation, our results concerning the local site distribution configurations apply both to pseudoparticles and pseudofermions.

As for the pseudoparticles, the unoccupied sites of the effective pseudofermion lattices correspond to the sites left over by the occupied sites and vice versa. Thus, we can uniquely specify a given effective pseudoparticle lattice site distribution configuration by providing the position of the occupied sites (or unoccupied sites) only. Moreover, it follows from our detailed studies that although the number of occupied (and unoccupied) sites of the \( c \) pseudofermions equals the number of spinons, \( N_c = M_c \) (and the number of holons, \( N_{ch} = M_{ch} \)) a \( c \) pseudofermion (and \( c \) pseudofermion hole) is not a spinon (and a holon). Our results also reveal that the fact that the number of \( su \) pseudofermion holes (and \( cv \) pseudofermion holes) is given by \( N_{sh}^h = L_s + 2 \sum_{\nu' = \nu + 1}^{\infty} (\nu' - \nu) N_{s\nu'} \) (and \( N_{ch}^h = L_c + 2 \sum_{\nu' = \nu + 1}^{\infty} (\nu' - \nu) N_{c\nu'} \)) does not imply that the \( N_{sh}^h, s\nu \) pseudofermion holes (and \( N_{ch}^h, cv \) pseudofermion holes) of a given energy eigenstate are the same objects as the \( L_s \) HL spinons and \((\nu' - \nu) -1/2 \) spinons and \((\nu' - \nu) +1/2 \) spinons of the set of composite \( su' \) pseudofermions of the same state such that \( \nu' > \nu \) (and the \( L_c \) Yang holons and \((\nu' - \nu) -1/2 \) holons and \((\nu' - \nu) +1/2 \) holons of the set of composite \( cv' \) pseudofermions of the same state such that \( \nu' > \nu \)). Indeed, both the HL spinon and Yang holon occupancies and the \( \alpha \nu \) pseudofermion spatial coordinate occupancy configuration of each \( \alpha \nu \) pseudofermion branch with finite occupancy in a given energy eigenstate are independently conserved.

Since the Yang holons and HL spinons remain invariant under the pseudoparticle - pseudofermion unitary transformation they are neither scatterers nor scattering centers, whereas both the pseudofermion and pseudofermion holes are scatterers, and the pseudofermions and pseudofermion holes created in a ground-state - excited-state transition are the scattering centers \( \mathbb{R} \). The results of this paper confirm that the \( c \) pseudofermion holes are spin-less and \( \eta \)-spin-less objects, as the corresponding \( c \) pseudofermions, in spite of their number equaling the number of \( \eta \)-spin 1/2 holons. This is consistent with the complementary studies of Ref. \( \mathbb{R} \), which reveal that the \( c \) pseudofermion and hole occupancy configurations are not related to the \( \eta \)-spin irreducible representations. Our results also confirm that the \( s1 \) pseudofermion holes have zero spin, as the corresponding \( s1 \) pseudofermions, in spite of their number reading \( N_{s1}^h = L_s + 2 \sum_{\nu' = 2}^{\infty} (\nu' - 1) N_{s\nu'} \). This is again consistent with the complementary studies of Ref. \( \mathbb{R} \), which show that the \( s1 \) pseudofermion and hole occupancy configurations only contribute to the spin singlet occupancy configurations. That the \( c \) pseudofermions and \( c \) pseudofermion holes are \( \eta \)-spin-less and spin-less objects and the \( s1 \) pseudofermions and \( s1 \) pseudofermion holes carry zero spin justify the simple form of the corresponding \( S \) matrices, which have dimension one \( \mathbb{R} \). Such \( S \) matrices fully control the unusual spectral properties of the model through the pseudofermion anticommutators \( \mathbb{I} \). If they had dimension larger than one, the pseudofermion anticommutators would also be matrices of dimension larger than one and the study of the spectral properties would be much more involved. Furthermore, the concepts of a local pseudoparticle (and pseudofermion) and an effective pseudoparticle (and pseudofermion) lattice introduced here are applied in the studies of Ref. \( \mathbb{I} \) to the evaluation of finite-energy spectral function expressions. Therefore, the studies of this paper provide new insights and contribute to the further understanding of the unusual finite-energy properties observed in low-dimensional materials \( \mathbb{R} \).
APPENDIX A: PROPERTIES OF THE LARGE $U/t$ PHYSICS USEFUL FOR OUR STUDIES

In this Appendix we review some aspects of the large $U/t$ physics that are useful for the problems studied in this paper. By use of Eqs. (12-16) of Ref. [9], we find that in the limit $U/t \to \infty$ the momentum (2) and the energy $E_H$ associated with the simple 1D Hubbard Hamiltonian $H_H$ given in Eq. (1) simplify to Eq. (36) of Ref. [9] with $M_{c,-1/2} = D$ and expression (4), respectively. (Here the electronic double occupation $D$ is a good quantum number such that $D = M_{c,-1/2}$ and for finite values of $U/t$ the number $M_{c,-1/2}$ refers to the rotated-electron double occupation.) For zero spin density, electronic densities in the range $0 \leq na \leq 1$, and on-site repulsion $U \gg t$ the ground state energy $E_0$ of the 1D Hubbard model is given by (20),

$$E_0 = \frac{2Nt}{\pi} \sin(\pi na) - U D_0; \quad D_0 = \frac{\partial E_0}{\partial U} = \left(\frac{t}{U}\right)^2 4N n \ln 2 \left(1 - \frac{\sin(2\pi na)}{2\pi na}\right), \quad (A1)$$

where $D_0$ is the ground-state expectation value of the electronic double occupation (16). Here the term $-\frac{2Nt}{\pi} \sin(\pi na)$ is the kinetic energy associated with the hopping processes which do not change electronic double occupation. On the other hand, the energy term $-UD_0$ includes both kinetic and potential energy contributions, $-U/2D_0$ and $UD_0$, respectively. It arises from hopping processes which change electronic double occupation and lead to the ground-state expectation value $D_0$ given in Eq. (A1). Thus, the physics associated with this second energy term corresponds to excitation processes of higher order in $t/U$ relative to the $t/U \to 0$ limit where electronic double occupation is a good quantum number and the ground-state electronic double occupation of the model is exactly zero. If we include energy contributions of the order $t(t/U)^1$, the ground state contains a small but finite electronic double occupation expectation value given in Eq. (A1) and the corresponding quantum problem is not equivalent to the physical situation of interest for the rotated-electron studies of this paper. Indeed, electrons equal rotated electrons when the limit $t/U \to 0$ is reached and double occupation is a good quantum number. Only in such a limit do the electronic occupancy configurations which describe the bare-momentum energy eigenstates equal the corresponding rotated-electron configurations that are valid for all values of $U/t$.

Note that at half-filling the electronic density reads $na = 1$, the energy term of order $t(t/U)^0$ on the right-hand side of Eq. (A1) vanishes, and the ground-state energy and electronic double occupation expressions given in Eq. (A1) simplify to $E_0 = -[t^2/U] 4N n \ln 2$ and $D_0 = (t/U)^2 4N n \ln 2$, respectively. It is well known that this energy can be associated with an isotropic Heisenberg model (21). The corresponding ground state leads to energy contributions of the order $t(t/U)^1$ and thus contains a small but finite electronic double occupation expectation value, $D_0 = (t/U)^2 4N n \ln 2$. It follows that the usual description of the large-$U/t$ half-filling Hubbard model in terms of an isotropic Heisenberg model is not equivalent to our limit. In such a limit only hopping processes which do not change the value of electronic double occupation must be considered.

To leading order in the parameter $t/U$, the energy spectrum of the 1D Hubbard model in the limit $t/U \to 0$ is of the form given in Eq. (3). The permitted hopping processes lead to contributions in the eigenstates energies of order $t(t/U)^{-1}$ and $t(t/U)^0$. These contributions are associated with the energy terms $[N\alpha/2\pi] \int_{\alpha\nu} dq N_c(q)[-2\nu \cos q]$ and $UD$, respectively, on the right-hand side of Eq. (3). In the particular case of the ground state there are no contributions of order $t(t/U)^{-1}$ because the electronic double occupation eigenvalue $D_0$ is zero. In the limit $t/U \to 0$ only hopping processes such that the electronic singly occupied sites can move relatively to the electronic doubly-occupied and unoccupied site distribution configurations without changing these configurations are permitted. In such a limit there is a huge degeneracy of $\eta$-spin and spin occupancy configurations. Thus, there are several choices for complete sets of energy eigenstates with the same energy and momentum spectra given in Eq. (3) and Eq. (36) of Ref. [9] with $M_{c,-1/2}$ replaced by electronic double occupation, respectively. This justifies why in this limit there are many choices for complete sets of compatible observables. The 1D Hubbard model in the limit of $U/t \to \infty$ has been studied in the literature by many authors [10, 13, 18, 19, 20, 22, 24]. For the evaluation of quantities describing the physics of the model in the limit of $U/t \to \infty$, the alternative use of different complete sets of states leads to the same final expressions for correlation functions and other quantities of physical interest.

In this paper we are interested in one of these choices of energy eigenstates only. It corresponds to the complete set of $4^{Na}$ energy eigenstates generated from the corresponding energy eigenstates of the finite-$U/t$ 1D Hubbard model by turning off adiabatically the parameter $t/U$. Only such a set of states corresponds to the states obtained from the finite $t/U$ states by the electron - rotated-electron unitary transformation. These states are common eigenstates of both the 1D Hubbard model as $U/t \to \infty$ and of the set of number operators $\{\hat{L}_{\alpha,-1/2}\}$, $\{\hat{N}_c(q)\}$, and $\{\hat{N}_{\alpha\nu}(q)\}$ with $\alpha = c, s$ and $\nu = 1, 2, 3, ...$ considered in Ref. [9] in the same limit. Together with the Hamiltonian, these operators constitute a complete set of compatible and commuting hermitian operators. These operators also commute with the momentum operator $\hat{P}$. For these energy eigenstates the bare momentum $q_j$ of the $c$ pseudoparticles and $cv$ and $sv$ pseudoparticles such that $\nu = 1, 2, ...$ is a good quantum number. This justifies the designation of bare-momentum...
energy eigenstates. However, in the limit $t/U \rightarrow 0$ there are other choices for complete sets of energy and momentum eigenstates which are due to the periodic boundary conditions of the original electronic problem. In section III we discuss an alternative set of energy eigenstates used in the studies of Ref. [19].

Finally, we briefly discuss the relation of the effective pseudoparticle lattices introduced in Sec. V to previous results on the model in the limit $U/t \rightarrow \infty$. It is well known that in such a limit and for the subspace with no electronic double occupancy and thus such that $N_c = N$ and at zero spin density the charge and spin degrees of freedom of the model can be described by two independent systems of $N$ spin-less fermions and $N/2$ spin-down spins $\Psi_{\nu \sigma}$. The spin-less fermions can be associated with an effective lattice with $j = 1, 2, 3, \ldots, N_{\text{a}}$ sites, whereas the $N/2$ spin-down spins correspond to a squeezed effective lattice with $j = 1, 2, 3, \ldots, N/2$ sites. The spin-less fermion and spin occupancy configurations of these effective lattices describe electronic site distribution configurations. In the limit $U/t \rightarrow \infty$ such spin-less fermion and spin effective lattices are directly related to the effective $c$ pseudoparticle and $s$ pseudoparticle lattices, respectively, introduced in Sec. V. In the limit $U/t \rightarrow \infty$ the energy eigenstates with finite occupancies in the effective $c_\nu$ pseudoparticle lattices have an infinite energy relative to the ground state and do not contribute to the finite-energy physics. This explains why these states are unimportant for the $U/t \rightarrow \infty$ physics and are in general not considered [10, 18]. In the limit $U/t \rightarrow \infty$ the spin excitations involving both occupancy configurations of $s_1$ pseudoparticles and $s_{\nu}$ pseudoparticles belonging to branches such that $\nu > 1$ are often described by the isotropic Heisenberg chain which describes these excitations [10, 18].

[1] Y. Mizuno, K. Tsutsui, T. Tohyama, S. Maekawa, Phys. Rev. B 62 (2000) R4769; H. Kishida, M. Ono, K. Miura, H. Okamoto, M. Izumi, T. Manako, M. Kawasaki, Y. Taguchi, Y. Tokura, T. Tohyama, K. Tsutsui, S. Maekawa, Phys. Rev. Lett. 87 (2001) 177401.
[2] V. Vescoli, L. Degiorgi, W. Henderson, G. Gruner, K. P. Starkey, L. K. Montgomery, Science 281 (1998) 1181.
[3] J. M. P. Carmelo, F. Guinea, K. Penc, P. D. Sacramento, Europhys. Lett. 68 (2004) 839.
[4] M. Sing, U. Schwingschogl, R. Claessen, P. Blaha, J. M. P. Carmelo, L. M. Martelo, P. D. Sacramento, M. Dressel, C. S. Jacobsen, Phys. Rev. B 68 125111, (2003).
[5] Elliott H. Lieb, F. Y. Wu, Phys. Rev. Lett. 20 (1968) 1445; M. J. Martins, P.B. Ramos, Nucl. Phys. B 522 (1998) 413.
[6] M. Takahashi, Prog. Theor. Phys. 47 (1972) 69 (1972).
[7] O. J. Heilmann, E. H. Lieb, Ann. N. Y. Acad. Sci. 172 (1971) 583.
[8] C. N. Yang, Phys. Rev. Lett. 63 (1989) 2144.
[9] J. M. P. Carmelo, J. M. Roman, K. Penc, Nucl. Phys. B 683 (2004) 387.
[10] Karlo Penc, Karen Hallberg, Frederic Mila, Hiroyuki Shiba, Phys. Rev. Lett. 77 (1996) 1390; ibid. Phys. Rev. B 55 (1997) 15475.
[11] J. M. P. Carmelo, K. Penc, D. Bozi, at press in Nucl. Phys. B [cond-mat/0505601] and references therein.
[12] J. M. P. Carmelo, K. Penc, L. M. Martelo, P. D. Sacramento, J. M. B. Lopes dos Santos, R. Claessen, M. Sing, U. Schwingschogl, Europhys. Lett. 67 (2004) 233; J. M. P. Carmelo, D. Bozi, P. D. Sacramento, K. Penc, submitted for publication (2005).
[13] J. M. P. Carmelo, D. Bozi, P. D. Sacramento, preprint; J. M. P. Carmelo, cond-mat/0504271.
[14] D. Jaksch, P. Zoller, Ann. of Phys. 315 (2005) 52; J. I. Cirac, P. Zoller, Science 301 (2003) 176.
[15] A. Brooks Harris, Robert V. Lange, Phys. Rev. 157 (1967) 295; A. H. MacDonald, S. M. Girvin, D. Yoshioka, Phys. Rev. B 37 (1988) 9753.
[16] J. M. P. Carmelo, P. D. Sacramento, Phys. Rev. B 68 (2003) 085104.
[17] J. M. P. Carmelo, cond-mat/0405411.
[18] Masao Ogata, Hiroyuki Shiba, Phys. Rev. B 41 (1990) 2326.
[19] P. Gebhard, K. Bott, M. Scheidler, P. Thomas, S. W. Koch, Phylo. Magaz. B 75 (1997) 13.
[20] Jos Carneiro, Dionys Baeriswyl, Phylo. Rev. B 37 (1988) 7541.
[21] V. Emery, Phys. Rev. B 14 (1975) 2989.
[22] G. Beni, T. Holstein, P. Pincus, Phys. Rev. B 8 (1973) 312; Douglas J. Klein, Phys. Rev. B 8 (1973) 3452.
[23] R. G. Dias, J. M. Lopes dos Santos, J. de Physique 12 (1992) 1889.
[24] Henk Eskes, Andrzej M. Oleś, Phys. Rev. Lett. 73 (1994) 1279; Henk Eskes, Andrzej M. Oleś, Marcel B. J. Meinders, Walter Stephan, Phys. Rev. B 50 (1994) 17980.