Synthesis of Maximally Entangled States by Entanglement-based Lyapunov Control

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Abstract—Quantum entanglement between particles is a prerequisite for quantum information transmission. The higher the entanglement intensity, the less the transmission of information will be disturbed by the environment. The quantum state with the greatest degree of entanglement thus has the highest value of quantum information and is to be created by means of control methods in this paper. Instead of using Lyapunov function based on state distances, this paper employs quantum entanglement measures to construct Lyapunov functions, from which Lyapunov entanglement control law is designed to synthesize the maximally entangled states (MES). Taking the scalar entanglement measure as the control objective will greatly simplify the synthesis of MES and provide a powerful tool for quantum entanglement control. For two-bit pure and mixed states, we propose a generalized entanglement measure, according to which a general Lyapunov control law is designed as the only control approach to synthesizing maximally entangled mixed states (MEMS) for mixed states at present. If entanglement measures for multi-bit systems are employed, the same entanglement control law can be applied to synthesize multi-bit MESs.

Index Terms—Quantum control, entanglement measure, Lyapunov function, maximally entangled state.

I. Introduction

In recent years, with the exponential increase in computing speed relative to traditional computers, fields related to quantum computers, such as quantum communications, quantum satellites, and quantum control, have become more popular. The quantum computer is composed of quantum logic gates and qubits [1], and quantum control deals with the evolution of the quantum state within the quantum computer. Quantum systems can be divided into closed systems and open systems. In a closed quantum system, the pure-state evolution is described by Schrödinger equation and the mixed-state evolution is described by Liouville equation. On the other hand, Lindblad equation is most commonly used to describe the evolution of open quantum systems.

The current methods of quantum control are π-pulse [2], Lie group decomposition [3], learning calculus [4], [5], optimal control [6-11], sliding control [12], adiabatic control [13-15], Lyapunov control [16-40] and optimal control based on Lyapunov [41]. Among them, optimal control and Lyapunov control are the most commonly used. The optimal control method has proposed Hamilton Jacobi Bellman (HJB) equations suitable for quantum systems [1], and proposed an improved method of nesting Schrodinger equations into nonlinear systems [7], which also has practical applications [8]. Although the optimal control method can obtain good control effects, it requires a lot of calculations and is difficult to apply in practice. For quantum control based on Lyapunov, it is relatively easy to design and can quickly calculate the control law. The main idea of Lyapunov control is to select the appropriate Lyapunov function to design the control law. Currently in quantum control, there are mainly three types of Lyapunov functions as the design of control laws. According to the state distance [18-20], [25]-[31], [38-40], based on the state error [21], [22], [32], [36], [37], the average value of an imaginary mechanical quantity [23], [24], [38], [40], the so-called imaginary mechanical quantity means that it is a linear Hermitian operator to be designed, which may not be an observable measure with physical meaning, such as coordinates and energy. Among the three Lyapunov functions, the distance and state error of the quantum control method based on the Lyapunov state only need to adjust the proportional factor control law. These two Lyapunov control methods are relatively simple and easy to master. The quantum-controlled mechanical quantity based on Lyapunov’s imaginary number average value contains more adjustable parameters. Thus, it is more flexible and more complex at the same time.

At present, the control method of Lyapunov theory has successfully simulated energy level state preparation [17]; it is more dynamic programming for drift-free systems [16]. However, although Lyapunov theory can guarantee the stability of the system, it cannot guarantee that the system converges to any desired goal. Probabilistic control in a quantum system requires us to design a control strategy that can make the system converge, because a stable quantum control method may cause the control system to fail to reach the desired target state. Therefore, another main focus of this control strategy is that the closed quantum system is a convergent system of control. So far, there have been the following research results on the convergence of closed quantum systems [18], [22-24], [26-28], [32], [35]. As long as the target state is the eigenstate of or the target density matrix is an arbitrary arrangement of the system energy on the diagonal, the system must be able to converge [24], the convergence problem is proved by LaSalle theorem [27], and the Lyapunov theoretical control method is converged Speed analysis [42]. In this article, we mainly consider using Lyapunov function to generate entanglement.

In quantum mechanics, entanglement is one of the most surprising characteristics, which have no counterpart in classical physics. With the development of quantum information theory in recent years, entanglement has become a valuable resource [43], which has been applied in many fields [44]-[47]. Therefore, how to generate the maximally entangled state becomes a key task. In the work of quantum information, local operations and classical communication (LOCC) are generally used as the main operation methods, because entangled quantum pairs are generally divided into two remote

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places, free operation is LOCC naturally. When constrained by LOCC, only quantum separable states can be prepared, so the entangled state becomes an important resource to overcome LOCC manipulation. Nielsen described the possible LOCC transitions between pure separable state in [48], which shows that the ordering of LOCC is reduced to majorization [49]. That is, there is a maximally entangled state that can be converted to any state of this dimension by LOCC, but any state in this dimension cannot be transformed into the maximally entangled state by LOCC. Therefore, the maximally entangled state is an important criterion for the degree of entanglement. Obviously, it is the most valuable state for two-way entanglement applications (for example, teleportation). Therefore, the preparation of the maximally entangled state as the control goal in this article.

Section II will introduce the system the control will follow and some useful mathematical techniques. In Section III, we will introduce a Lyapunov function-based entanglement measure, which we call Lyapunov entangled function. With Lyapunov entangled function, we will design control laws for pure bipartite in Section IV, a control law for arbitrary bipartite in Section V, and a control law for multipartite in Section VI. Lastly, in Section VII, we will use numerical simulation to verify the aforementioned control.

II. PRELIMINARIES

A. Pure states and Mixed states in a bipartite system

The pure state of a bipartite system is described by a complex vector \( |\psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \), whose time evolution satisfies the Schrödinger equation:

\[
i\hbar \dot{|\psi(t)\rangle} = H |\psi(t)\rangle, \quad H = H_0 + H_c,
\]

where \( H_0 \) is the internal Hamiltonian, and \( H_c = \sum_{k=1}^{r} H_k u_k(t) \) is the time-dependent control Hamiltonian that represents the interaction of the system with the external fields \( u_k(t) \). The Hamiltonians \( H_0 \) and \( H_c \) have to be Hermitian operators. Conveniently, we set \( \hbar \) to 1 by rewriting (1) as a dimensionless equation.

When considering quantum control problems, it is easier to examine the evolution of a closed quantum system under the interaction picture defined by \( |\psi(t)\rangle = e^{i\int_{0}^{t}H dt}|\psi(t)\rangle \), whose time evolution satisfies

\[
i\hbar \dot{|\psi(t)\rangle} = H_c \int_{\psi(t)} H \rho(t) \frac{|\psi(t)\rangle \langle \psi(t)|}{|\psi(t)\rangle \langle \psi(t)|}, \quad H_c = \sum_{k=1}^{r} A_k u_k(t)
\]

where \( A_k \) is related to \( H_k \) as \( A_k = e^{i\int_{0}^{t}H_k dt} \). Generally, a physical operator \( M \) in the interaction picture is defined as \( M = e^{i\int_{0}^{t}H_k dt} M e^{-i\int_{0}^{t}H_k dt} \) with \( M \) being the related operator in the Schrödinger picture. It can be shown that the expectation in the interaction picture \( \langle M(t) \rangle = \langle \psi(t) | M | \psi(t) \rangle \) is equal to the expectation (\( M \)) in the Schrödinger picture.

Because of the decoherence phenomenon, we inevitably have to deal with the mixed state, which is described by the density matrix

\[
\rho = \sum \rho_k |\psi_k\rangle \langle \psi_k| = \sum \rho_k \rho_k^*,
\]

where \( \rho_k \) represents the weight of the component state \( |\psi_k\rangle \) in the mixed state. The time evolution of the density matrix \( \rho \) satisfies the von Neumann equation,

\[
i\hbar \dot{\rho}(t) = H \rho(t) - \rho(t) H \Delta [H, \rho(t)],
\]

where \( H = H_0 + \sum_{k=1}^{r} H_k u_k(t) \). Under the interaction picture, (4) becomes

\[
i\hbar \dot{\rho}(t) = \sum_{k=1}^{r} A_k u_k(t) \rho_k \]

B. Spectral decomposition

Matrix \( A \) is said to be Hermitian, if \( A = (A^*)^T = A^T \). The eigenvalues of a Hermitian matrix are all real. Matrix \( A \) is said to be skew-Hermitian if it satisfies \( A^* = -A \). The eigenvalues of a skew-Hermitian matrix are all on the imaginary axis. The density matrix \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \) is Hermitian and positive semi-definite with unit trace \( \text{Tr}(\rho) = 1 \). If \( \rho \) contains only one state (pure state), we have \( \text{Tr}(\rho^2) = 1 \); otherwise, \( \text{Tr}(\rho^2) < 1 \). This property is useful to distinguish whether a density matrix \( \rho \) represents a pure state or a mixed state.

Matrix \( A \) is said to be normal, if \( A \) and \( A^T \) are commutative, i.e., \( [A, A^T] = AA^T - A^T A = 0 \). A normal matrix has a spectral decomposition \( A = VAV^T \), where \( A \) is a diagonal matrix composed by the eigenvalues \( \lambda_k \) of \( A \), and \( V = [v_1, v_2, \cdots, v_n] \) is a unitary matrix containing the eigenvectors \( v_k \) of \( A \). When expressed by \( v_k \), the spectral decomposition \( A = VAV^T \) becomes \( A = \sum_{k=1}^{n} \lambda_k v_k v_k^* \), which has an alternative expression in terms of the Dirac notation:

\[
A = \sum_{k=1}^{n} \lambda_k |v_k\rangle \langle v_k|,
\]

where \( |\lambda_k\rangle \) corresponds to the eigenvector \( v_k \) and \( \langle \lambda_k| \) to its conjugate transpose \( v_k^T \). Both Hermitian and skew-Hermitian matrices are normal and possess spectral decompositions, which provides a convenient way to find the function value of a normal matrix:

\[
f(A) = \sum_{k=1}^{n} f(\lambda_k) |\lambda_k\rangle \langle \lambda_k|,
\]

C. Schmidt decomposition

A bipartite pure state described by \( |\psi_{AB}\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \) is said to be separable, if and only if it can be expressed as a direct product of the states in the two subsystems:

\[
|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle,
\]

where \( |\psi_A\rangle \in \mathcal{H}_A \) and \( |\psi_B\rangle \in \mathcal{H}_B \). In terms of the orthogonal basis \( |e_k\rangle \in \mathcal{H}_A \) and \( |e_k^B\rangle \in \mathcal{H}_B \), any pure state \( |\psi_{AB}\rangle \) has a Schmidt decomposition [43] as

\[
|\psi_{AB}\rangle = \sum_{k=1}^{n} \alpha_k |e_k\rangle \otimes |e_k^B\rangle,
\]

where \( \alpha_k > 0 \) is the Schmidt coefficient, and \( r \) is the Schmidt rank of \( |\psi_{AB}\rangle \). A pure state \( |\psi_{AB}\rangle \) is separable, if and only if its Schmidt rank \( r \) is one, for which (9) reduces to (8).

D. LOCC operations

Local operations and classical communication (LOCC) refer to a local quantum operation performed on part of the system, and the result of that operation is communicated classically to another part of the system, where another local operation is performed on the information received. Since LOCC does not provide any quantum operation connecting simultaneously two subsystems in two different places, entanglement of the system does not increase by LOCC operation.

Nielsen [48] gave a convenient criterion to judge whether a given operation \( \Lambda \) is a LOCC operation. Let \( \Lambda |\psi_1\rangle = |\psi_2\rangle \) and express \( |\psi_1\rangle \) and \( |\psi_2\rangle \) by Schmidt decomposition:

\[
|\psi_1\rangle = \sum_k \sqrt{\alpha_k} |k\rangle \otimes |k\rangle, \quad |\psi_2\rangle = \sum_k \sqrt{\alpha_k'} |k\rangle \otimes |k\rangle,
\]

where the Schmidt coefficients \( \alpha_k \) and \( \alpha_k' \) are in descending order.
\[ \alpha_1 > \alpha_2 > \cdots > \alpha_n, \quad \alpha_1' > \alpha_2' > \cdots > \alpha_n'. \]

with normalization condition \[ \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \alpha_k' = 1. \]

Then \( \Lambda \) is a LOCC operation, if and only if \( \{ \alpha_k \} \) is majorized by \( \{ \alpha_k' \} \), i.e., for every \( l \) with \( 1 \leq l < n \), we have \[ \sum_{k=1}^{l} \alpha_k \leq \sum_{k=1}^{l} \alpha_k'. \]

According to the Nelson’s LOCC criterion, once we obtain a maximally entangled state (MES), we can transform it to any target state \( |\sigma\rangle \in \mathcal{H}_{AB} \) by a LOCC operation \( \Lambda \).

III. LYAPUNOV ENTANGLEMENT FUNCTION (LEF)

In this section, we would propose an class of entanglement measures suitable for Lyapunov entanglement control by axiomatic approach [51,52]. A bipartite entanglement measure \( E(\rho) \) is a mapping from density matrices into positive real numbers, i.e. \( E(\rho) \in \mathbb{R}^+ \), which represents the intensity of entanglement of \( \rho \). If the density matrix \( \rho \) is separable, then \( E(\rho) = 0 \). Entanglement measure \( E \) does not increase on average under LOCC, i.e., \( E(\rho) \geq E(\Lambda(\rho)) \). Vidal [52] characterized the entanglement measure \( E(\rho) \) in terms of entanglement monotone functions, which satisfy the following two properties.

1. \( h(\rho) \) is invariant under any unitary transformation \( U \), i.e.
   \[ h(U\rho U^\dagger) = h(\rho). \]
2. \( h(\rho) \) is concave, i.e. \( h(\rho) \geq \lambda h(\rho_1) + (1-\lambda) h(\rho_2) \), in which \( \lambda \in [0,1] \) and \( \rho_1, \rho_2 \) are matrices so that \( \rho = \lambda \rho_1 + (1-\lambda) \rho_2 \).

Given a qualified entanglement monotone function \( h(\rho) \), the entanglement measure for a pure state can be defined as \( E(\rho) = h(\rho_{\text{PTM}}) \), where \( \rho_{\text{PTM}} \) is the reduced density matrix of \( \rho \). A class of entanglement measure satisfying the property of entanglement monotone can be characterized explicitly as

\[ E(\rho) = G(\text{Tr}(f(\rho_{\text{PTM}}))), \]

where the trace operator \( \text{Tr}(\cdot) \) makes \( E(\rho) \) invariant under unitary transformation. The remaining properties of a qualified entanglement measure \( E_G(\rho) \) are satisfied by imposing proper conditions on the continuously differentiable functions \( G \) and \( f \), as will be derived in the following.

Firstly, consider a bipartite pure state described by \( \rho = |\psi\rangle\langle\psi| \) and express its reduced matrix by the spectral decomposition:

\[ \rho_{\text{PTM}} = \sum_{i} \lambda_i |\lambda_i\rangle\langle\lambda_i| = \lambda_1 |\lambda_1\rangle\langle\lambda_1| + \lambda_2 |\lambda_2\rangle\langle\lambda_2| \]

(11)

With the condition \( \lambda_1 + \lambda_2 = 1 \), it is convenient to denote \( \lambda_1 = \lambda \) and \( \lambda_2 = 1 - \lambda \) with \( 0 \leq \lambda \leq 1 \) so that the general entanglement measure \( E_G(\rho) \) in Eq. (14) becomes a function of the eigenvalue \( \lambda \):

\[ E_G(\rho) = G(X(\lambda)), \quad X(\lambda) = \text{Tr}(f(\rho_{\text{PTM}})). \]

Using (5b) and (12), the function \( X(\lambda) \) can be evaluated explicitly as

\[ X(\lambda) = \text{Tr}(f(\rho_{\text{PTM}})) = \text{Tr}(f(\lambda_1)|\lambda_1\rangle\langle\lambda_1| + f(\lambda_2)|\lambda_2\rangle\langle\lambda_2|) = f(\lambda) + f(1-\lambda). \]

(13)

where we note \( \text{Tr}(|\lambda_2\rangle\langle\lambda_2|) = |\lambda_2|^2 |\lambda_2|^2 = 1 \). As a result, we obtain a simple expression for the general entanglement measure \( E_G(\rho) \) as

\[ E_G(\rho) = G(X(\lambda)) = G(f(\lambda) + f(1-\lambda)). \]

(14)

Based on this concise expression, the required conditions on \( G \) and \( f \) to ensure \( E_G(\rho) \) as a qualified entanglement measure can be derived straightforwardly as follows.

1. \( E_G(\lambda) = 0 \) for separable state. When the quantum state is separable, the rank of \( \rho_{\text{PTM}} \) is 1, corresponding to \( \lambda = 0 \) or \( \lambda = 1 \). With (14), the requirement of \( E_G(0) = E_G(1) = 0 \) turns out to be

\[ G(f(0) + f(1)) = 0. \]

(15)

2. The positivity of \( E_G(\lambda) \). \( E_G(\lambda) \) must be positive for all entangled states, i.e. \( E_G(\lambda) > 0, \forall \lambda \in (0,1) \). This requirement is equivalent to

\[ G(X) > 0, \quad \forall X \in \mathbb{D}_Y - \{ f(0) + f(1) \}. \]

(16)

where \( \mathbb{D}_Y \) is the domain of \( X \).

3. \( E_G(\lambda) = 0 \) at the maximally entangled state (MES). When the quantum state \( \rho \) is the MES, its reduced density matrix becomes [53]

\[ \rho_{\text{PTM}} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \]

(17)

i.e., \( \lambda_1 = \lambda_2 = 1/2 \). Thus, \( E_G(\lambda) \) achieves its maximum at \( \lambda = 1/2 \), where the derivative of \( E_G(\lambda) \) must be zero. This condition is satisfied automatically by evaluating

\[ E_G'(\lambda) = \frac{dG(X)}{d\lambda} f(\lambda) - f'(1-\lambda), \]

(18)

at \( \lambda = 1/2 \) to give \( E_G'(1/2) = 0 \).

4. \( E_G(\lambda) \neq 0, \forall \lambda \neq 1/2 \). This property is to ensure that \( E_G(\lambda) \) has only one extreme point in the range \( 0 \leq \lambda \leq 1 \). From (20), this property requires \( G'(X) \neq 0, \forall X \in \mathbb{D}_Y \) and \( f''(\lambda) \neq f'(1-\lambda) \). Hence, \( G'(X) \) and \( f''(\lambda) \) have to be strictly increasing or decreasing in their definition domains.

5. \( E_G'(\lambda) < 0 \) at \( \lambda = 1/2 \). This property ensures that \( E_G(\lambda) \) is concave at the extreme point. From (18), the second derivative of \( E_G(\lambda) \) reads

\[ \frac{d^2 E_G(\lambda)}{d\lambda^2} = G''(X)(f''(\lambda) - f'(1-\lambda))^2 \]

(19)

The evaluation of \( d^2 E_G/\partial \lambda^2 \) at \( \lambda = 1/2 \) then gives \( E_G''(1/2) = G''(2f(1/2)) - f''(1/2) \). Therefore, the concavity of \( E_G(\lambda) \) at \( \lambda = 1/2 \) requires

\[ G''(2f(1/2)) - f''(1/2) < 0. \]

(20)

Together with condition 4, we then come to a conclusion that if \( f \) is a concave function, \( G \) must be strictly increasing; on the contrary, if \( f \) is a convex function, \( G \) must be strictly decreasing.

Condition 1 and condition 2 are the basic requirements for the entanglement measure \( E_G(\lambda) \) to ensure that except for the separable states, \( E_G(\lambda) \) must be a positive function. Condition 3 to condition 5 require that \( E_G(\lambda) \) must have a unique extreme point at \( \lambda = 1/2 \), which is the global maximum in the range \( 0 \leq \lambda \leq 1 \). When all the five conditions are satisfied, the general entanglement measure \( E_G(\rho) \) achieves its global maximum at \( \lambda = 1/2 \):

\[ \max_{\rho \text{ pure}} E_G(\rho) = \max_{\lambda \in [0,1]} G(X(\lambda)) = G(2f(1/2)). \]

(21)

The general entanglement measure \( E_G(\rho) \) characterizes a wide class of entanglement measures, including concurrence, Renyi entropy, and entropy of entanglement, etc.

**Example 1: Concurrence**

Concurrence [54] is a common entanglement measure defined by

\[ E_C(\rho) = \sqrt{2(1 - \text{Tr}(\rho_{\text{PTM}}^2))} \]

(22)

The corresponding \( G(X) \) and \( f(\lambda) \) functions for \( E_C(\rho) \) are

\[ G(X) = \sqrt{2(1 - X)}, \quad f(\lambda) = \lambda^2 \]
Expressed by $G(X)$ and $f(\lambda)$, $E_c(\rho)$ becomes a scalar function of $\lambda$:

$$E_c(\rho) = G(f(\lambda) + f(1 - \lambda)) = 2\sqrt{\lambda(1 - \lambda)}$$

It can be checked that $G(X)$ and $f(\lambda)$ satisfy the above five conditions. The maximum value of $E_c(\rho)$ is found from (21) as $E_c(\rho_M) = G(2f(1/2)) = G(1/2) = 1$, and the concavity of $E_c(\rho)$ at the extreme point is confirmed by $E_c''(1/2) = 2G'(2f(1/2)) \cdot f''(1/2) = -4 < 0$.

**Example 2: Renyi entropy**

Renyi entropy is defined by

$$E_a(\rho) = \frac{1}{1-a} \ln \Tr(\rho^a) , \quad a > 0$$

(23)

The related functions of $G(X)$ and $f(\lambda)$ are

$$G(X) = \frac{1}{1-a} \ln X , \quad f(\lambda) = \lambda^a ,$$

which satisfy the above five conditions and ensure $E_a(\rho)$ as a qualified entanglement measure. In terms of $\lambda$, $E_a(\rho)$ becomes

$$E_a(\lambda) = \frac{1}{1-a} \ln(\lambda^a + (1 - \lambda)^a)$$

The maximum of $E_a(\lambda)$ is $E_a(1/2) = G(2f(1/2)) = \ln 2$ and its concavity is confirmed by $E_a''(1/2) = -4\lambda < 0$. A special case of Renyi entropy is the entropy of entanglement $E_F(\rho)$, which is the limit value of $E_a(\rho)$ at $a = 1$ obtained by the L’Hospital’s rule,

$$\lim_{a \to 1} E_a(\rho) = -\frac{1}{\ln 2} \Tr(\rho_M \ln \rho_M) = E_F(\rho).$$

(24)

Based on the general entanglement measure $E_G(\rho)$, now we can construct a class of Lyapunov functions for entanglement control as

$$V_G(\rho) = N - E_G(\rho) = N - G(\Tr(f(\rho_M))).$$

(25)

where $N = G(2f(1/2))$ is the maximum of $E_G(\rho)$ to ensure $V_G(\rho) \geq 0$. The Lyapunov function $V_G(\rho)$ constructed from $E_G(\rho)$ is called Lyapunov entanglement function (LEF) to highlight its dual role. On the one hand, LEF plays the role of a Lyapunov function and determines the control strategy to make $V_G(\rho) < 0$. On the other hand, it plays the role of an entanglement measure, guiding the control process toward the direction of maximum entanglement. Combining the two roles together, the control strategy $V_G(\rho) < 0$ drives $\rho$ to the equilibrium state $\rho_{eq}$ with $V_G(\rho_{eq}) = 0$, which then gives $E_G(\rho_{eq}) = N$ from (25), indicating that the achieved equilibrium state $\rho_{eq}$ is the MES.

**IV. LYAPUNOV ENTANGLEMENT CONTROL BASED ON LEF**

In this section, we will derive the control law $u_k$ in (5) to make $V_G(\rho) < 0$. First we will discuss the entanglement control of pure states in this section, and then the control of mixed states in the following section.

The first step of control law design is to find the time derivative of $V_G(\rho)$ from Eq. (34):

$$\dot{V}_G(\rho) = -G(\Tr(f(\rho_M)))\Tr(f'(\rho_M)\dot{\rho}_M)$$

$$= iG'\Tr(f(\rho_M))\Tr(f'(\rho_M) \cdot (\dot{\rho}_M - \rho H_M))$$

(26)

where $\rho$ is given by Eq. (4a) and (4b) denotes reduced-matrix operation. Next, we apply the interaction picture introduced in Eq. (4) to express the Hamiltonian $H$ in Eq. (35) as:

$$\dot{V}_G(\rho) = iG'\Tr(f(\rho_M))\cdot \sum_k u_k \Tr(f'(\rho_M) \cdot (A_k \rho - \rho A_k)_M)$$

(27)

On designing the control law $u_k$ to render $V_G(\rho) < 0$, the following theorem is helpful.

**Theorem 4.1.** $\Tr(f'(\rho_M) \cdot (A_k \rho - \rho A_k)_M)$ is an imaginary number for pure states in a bipartite system.

**Proof.** Since $\rho_M$ is a Hermitian matrix for a pure bipartite system, $f'(\rho_M)$ is also a Hermitian matrix that can be expressed generally as

$$f'(\rho_M) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

(28)

where $H_{11}$ and $H_{22}$ are real numbers, and $H_{12}$ is a complex number. With the Hermitian property of $A_k$ and $\rho$, we have

$$(A_k \rho - \rho A_k) = -\rho A_k - A_k \rho = -(A_k \rho - \rho A_k)$$

(29)

In other words, $A_k \rho - \rho A_k$ is a skew-Hermitian matrix. In the next step of proof, we apply the rules of trace operation: $\Tr(A + B) = \Tr(A) + \Tr(B)$ and $\Tr(AB) = \Tr(BA)$ to obtain $\Tr(A_k \rho - \rho A_k) = 0$. With the skew-Hermitian and zero-trace properties of $A_k \rho - \rho A_k$, we can now express $(A_k \rho - \rho A_k)_M$ explicitly as

$$(A_k \rho - \rho A_k)_M = \begin{bmatrix} \eta_k & \zeta_k \\ -\zeta_k & -\eta_k \end{bmatrix}$$

(30)

where $\eta_k$ is a pure imaginary number, and $\zeta_k$ is a complex number. Therefore, the combination of Eq. (37) and Eq. (39) yields

$$\Tr(f'(\rho_M) \cdot (A_k \rho - \rho A_k)_M) = \Tr(\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \eta_k & \zeta_k \\ -\zeta_k & -\eta_k \end{bmatrix})$$

$$= (H_{11} - H_{22})\eta_k + (H_{12}^* \zeta_k - (H_{12}^2 \eta_k))$$

Noting that $H_{11} - H_{22}$ is real, and $\eta_k, \zeta_k, (H_{12}^* \zeta_k - (H_{12}^2 \eta_k))$ are pure imaginary, we then prove $\Tr(f'(\rho_M) \cdot (A_k \rho - \rho A_k)_M)$ to be a pure imaginary number.

The other factor affecting the sign of $V_G(\rho)$ in Eq. (35) is $G(\Tr(f(\rho_M)))$. We have shown in Section III that for a qualified entanglement measure $E_G(\rho) = G(\Tr(f(\rho_M)))$, the function $G(X)$ must be either strictly increasing, or strictly decreasing. In either case, it can be sure that $G'(X)$ will not change its sign in its domain of definition $D_G$.

With Theorem 4.1 and the monotonic property of $G'(X)$, we now can design the Lyapunov control law $u_k$ in terms of a new variable

$$x_k = i \cdot \Tr(f'(\rho_M) \cdot (A_k \rho - \rho A_k)_M)$$

(31)

According to Theorem 4.1, $x_k$ is a real variable and can be physically realized. Let $h_k(x_k)$ be a real function of $x_k$ satisfying the relation

$$h_k(x_k) \cdot x_k \geq 0, \quad h_k(x_k) = 0, \text{iff } x_k = 0$$

(32)

Obviously, the curve $y_k = h_k(x_k)$ passes through the origin of the $x_k - y_k$ plane and is located in the first or third quadrant. Then, the real function $h_k(x_k)$ serves as a feedback signal in the proposed control law

$$u_k = -\text{sgn}(G'(X))\rho_k h_k(x_k).$$

(33)

where $\rho_k$ is a positive gain to adjust the control amplitude. According to the control law (43) to Eq. (36), we achieve the goal of Lyapunov control

$$V_G(\rho) = -|G'(X)| \sum_k \rho_k h_k(x_k) \cdot x_k \leq 0$$

by noting $h_k(x_k) \cdot x_k \geq 0$ from Eq. (42). Out next task is to show that $V_G(\rho) = 0$ occurs only at the equilibrium state $\rho = \rho_{eq}$ and $V_G(\rho) > 0, \forall \rho \neq \rho_{eq}$. The dual role of the LEF $V_G(\rho)$ ensures that the minimum of $V_G$ and the maximum of $E_G$ are achieved simultaneously at $\rho = \rho_{eq}$.
Theorem 4.2 Under the Lyapunov control law (43), solutions of the von Neumann equation (4b) for a pure bipartite system asymptotically converge to the equilibrium state \( \rho_{eq} \) such that the Lyapunov function \( V_C(\rho) \) reaches its minimum \( V_C(\rho_{eq}) = 0 \) and meanwhile the general entanglement measure \( E_G(\rho) \) reaches its maximum \( E_G(\rho_{eq}) = 1 \).

Proof. According to the LaSalle’s invariance principle, the condition \( V_C(\rho) \leq 0 \) guarantees that the state trajectory \( \rho(t) \) converges to the invariant set

\[ \Omega_G = \{ \rho | V_C(\rho) = 0, \rho \in \mathcal{H}_{\rho} \}. \]

From the property of \( h_k(x_k) \cdot x_k \geq 0 \) in Eq. (42), the condition of \( V_C(\rho) = -|G'(X)| \sum_k \gamma_k h_k(x_k) \cdot x_k = 0 \) occurs only at \( x_k = 0, k = 1,2,\ldots,m \), where \( m \) is the number of control \( u_k \) used in (5). With \( x_k = 0 \), Eq. (43) gives \( u_k = -\text{sgn}(G'(X)) \gamma_k h_k(x_k) = 0 \) because of \( h_k(0) = 0 \). Applying \( u_k = 0 \) to Eq. (4b) then yields the equilibrium state \( \rho(t) = -i[ \sum_{k=1}^m A_k u_k(t), \rho(t) ] = 0 \). Therefore, the invariant set \( \Omega_G \) defined in Eq. (45) contains only the equilibrium state of the von Neumann equation (4b), to which the state trajectory \( \rho(t) \) converges asymptotically according to the LaSalle’s invariance principle. In other words, as \( \rho(t) \) approaches the equilibrium state \( \rho_{eq} \), \( V_C(\rho) \) approaches \( V_C(\rho_{eq}) = 0 \).

The proof of the other half of the theorem is about the properties of the equilibrium state \( \rho_{eq} \), which can be derived from the equilibrium condition \( x_k = 0 \). In terms of Eq. (40) in Theorem 4.1, the equilibrium condition \( x_k \) is \( \pm \text{Tr}(f'(\rho_{eq}) \cdot (A_k - \rho_{eq} A_k)) = 0 \) can be expressed by

\[ (H_{11} - H_{22}) \gamma_k + H_{12} \gamma_k - (H_{12} \gamma_k) = 0 \]  

Eq. (47) has to be satisfied for all Pauli matrices \( \sigma_x \) and \( \sigma_z \), \( k = 1,2,\ldots,m \), in order to achieve the condition \( V_C(\rho_{eq}) = 0 \), and the only possibility is \( H_{11} = H_{22} \) and \( H_{12} = H_{21} \). In turn, is substituted into Eq. (37) to yield

\[ f'(\rho_{eq}) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} H_{11} \gamma_k & 0 \\ 0 & H_{11} \gamma_k \end{bmatrix}. \]

Because \( f' \) is either strictly increasing or strictly decreasing as proved in Section 3, its inverse function \( f'^{-1} \) always exists and the equilibrium state can be solved as

\[ \rho_{eq} = \begin{bmatrix} f'^{-1}(H_{11}) & 0 \\ 0 & f'^{-1}(H_{11}) \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \]

where the identity \( \text{Tr}(\rho_{eq}) = 1 \) has been used to determine the value of \( f'^{-1}(H_{11}) \). Comparing Eq.(50) with Eq. (21), we obtain the main result of this theorem that the equilibrium state \( \rho_{eq} \) achieved by the Lyapunov control law (43) is identical to the maximum entanglement state \( \rho_{max} \). Because of \( E_G(\rho_{max}) = 1 \), the Lyapunov function evaluated at the equilibrium state becomes \( V_C(\rho_{eq}) = 1 - E_G(\rho_{eq}) = 1 - E_G(\rho_{max}) = 0 \).

This theorem shows that the proposed Lyapunov control law (43) can drive the quantum state to converge to the maximum entangled state \( \rho_{max} \), which maximizes a class of entanglement measure \( E_G(\rho) \). It is noted that the maximum entangled state achieved by the Lyapunov entanglement control is not unique and not necessary in the form of Bell states; nevertheless, they can all be made equivalent via LOCC operation.

V. NUMERICAL VERIFICATION OF MAXIMUM ENTANGLEMENT CONTROL

In this section, we will numerically verify the Lyapunov entanglement control method discussed in the previous section. We will consider a model representing two atoms or two quantum dots each located in a remote cavity connected by a closed-loop optical fiber. One of the two atoms is given a coherent input field of amplitude \( A_m \), and the output of each cavity enters the other. By eliminating the radiation field, the internal Hamiltonian is chosen as \( H_0 = 2|\sigma_x \otimes \sigma_z| \), where the spin-spin coupling constant \( J \) changes with the frequency of the applied radiation field and here \( J = 0.5 \) is used in the computation.

The control Hamiltonian \( H_c = \sum_{k=1}^n H_k u_k(t) \) is synthesized by a local laser and the coupling Hamiltonian \( H_k \) is a combination of Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \). Here we choose \( H_1 = \sigma_x \otimes \sigma_y + \sigma_z \otimes \sigma_x \), \( H_2 = \sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_y \), and \( H_3 = \sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_y \). With the given \( H_k \), the time evolution of the density matrix equation is described by Eq. (4), and the control law \( u_k \) is given by Eq. (43), where the feedback signal is chosen to be the simplest form \( h_k(x_k) = x_k \) with gain \( \eta_k = 1 \). The density matrix \( \rho \) for pure states is described by \( \rho = |\psi(\psi)\rangle \) with the quantum state \( |\psi(\psi)\rangle \) expressed in terms of the basis as

\[ |\psi(t)\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \]  

The reduced maxit of the pure state \( \rho = |\psi(\psi)\rangle \) can be computed in terms of the coefficients of \( |\psi(t)\rangle \) as

\[ \rho_M = \begin{bmatrix} |a|^2 + |b|^2 & \beta \delta + a \gamma \\ \beta \delta + a \gamma & |c|^2 + |d|^2 \end{bmatrix} \]

which can be used in \( E_G(\rho) = G(\text{Tr}(f(\rho_{eq}))) \) to compute the entanglement measure. As mentioned in Section III, any qualified entanglement measure \( E_G(\rho) \) has a common property of \( E_G(\rho_0) = 0 \) for all separable states \( \rho_0 \). This property causes a numerical problem that if the entanglement control starts from a separable state \( \rho_0 \), the initial control signal \( u_0(0) \) will evitably become zero and nullify the control process. Therefore, a small perturbation is required to excite the control process, if a separable initial state is given. Once the control process is actuated, it will asymptotically converge to a maximum entanglement state regardless of the initial states. What we are interested in is, from what initial states, the maximum entangled state obtained just has the form of Bell states, i.e.,

\[ |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \]

\[ |\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \]

Firstly, we will consider the initial state \( |\psi_0\rangle = (1 + \epsilon)\rho_{00} + \rho_{01} \approx \rho_0 \), which has a small perturbation \( \epsilon \) from the separable state \( \rho_0 \). To other purpose of introducing \( \epsilon \) is to examine the influence of the perturbation of initial states on the convergence to Bell states.
In Fig. 1, we show the time response of the population in each basis state under the maximum entanglement control by using three entanglement measures: the concurrence $E_C(\rho)$, entropy of entanglement $E_E(\rho)$, and Renyi entropy $E_\alpha(\rho)$ with $\alpha = 1.5$, as introduced in the previous section. It can be seen that the quantum state $|\psi(t)\rangle$ starting from $|00\rangle$ converges asymptotically to the Bell state $|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The control results based on the three entanglement measures exhibit similar tendency of convergence, if we note that the entropy of entanglement $E_E(\rho)$ is a special case of Renyi entropy $E_\alpha(\rho)$ with $\alpha = 1$ and the concurrence $E_C(\rho)$ can be well approximated by Renyi entropy $E_\alpha(\rho)$ with $\alpha = 2$.

![Figure 1](image1.jpg)

**Figure 1:** Asymptotical convergence of the state population from the initial state $|00\rangle$ to the Bell state $|\beta_{00}\rangle$ based on three entanglement measures: the concurrence $E_C(\rho)$, entropy of entanglement $E_E(\rho)$, and Renyi entropy $E_\alpha(\rho)$ with $\alpha = 1.5$.

The different convergent speeds observed from Fig. 1 can be explained by the time responses of the related control signals shown in Fig. 2. The control signal generated by the entropy of entanglement $E_E(\rho)$ activates first and drives the quantum state to the Bell state faster than that by using the control signal generated by the concurrence $E_C(\rho)$, which is the latest of the three control signals to be activated. Although the three control signals are activated at different moments, their maximum control magnitudes are the same.

Next we consider a different perturbation applied to the initial state $|00\rangle$. The initial state $|\psi_0\rangle = (1 + \epsilon)|\beta_{00}\rangle + |\beta_{01}\rangle \approx |00\rangle$ considered previously has a slightly larger weight on $|\beta_{00}\rangle$ than $|\beta_{01}\rangle$ and causes the system eventually converges to $|\beta_{00}\rangle$. Now we add the perturbation $\epsilon$ to $|\beta_{01}\rangle$, instead of $|\beta_{00}\rangle$, to form a different initial state $|\psi_0\rangle = |\beta_{00}\rangle + (1 + \epsilon)|\beta_{01}\rangle \approx |00\rangle$. By applying the same Lyapunov entanglement control, the terminal state turns out to be $|\beta_{10}\rangle$. It appears that the terminal Bell state is very sensitive to the way how the quantum state departs from the initial state. Table 1 lists the initial states with different perturbation and their corresponding terminal states under Lyapunov entanglement control. The results of this table show that if the initial state is close to the four basis states, the achieved maximum entangled state appears to be one of the Bell states. Furthermore, if the initial state is perturbed towards a certain Bell state, it will cause the quantum state to evolve towards this Bell state.

![Figure 2](image2.jpg)

**Figure 2:** The time responses of the control magnitude used to derive the state from $|00\rangle$ to $|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ based on three entanglement measures.

In order to understand the global convergence range of Bell states, we select a large number of initial states at random and then determine their converging states after maximum entanglement control. For this purpose, a quantum state is represented as a linear combination of four Bell states

$$|\psi(t)\rangle = \beta_{a0}|\psi_{00}\rangle + \beta_{b0}|\psi_{01}\rangle + \beta_{b1}|\psi_{10}\rangle + \beta_{b2}|\psi_{11}\rangle.$$  \hspace{1cm} (55)

where the coefficients satisfy the normalization condition $|\beta_{a0}|^2 + |\beta_{b0}|^2 + |\beta_{b1}|^2 + |\beta_{b2}|^2 = 1$. Graphically, the four Bell states can be thought of as the four vertices of a regular tetrahedron, and a coefficient set $\{\beta_{a0}, \beta_{b0}, \beta_{b1}, \beta_{b2}\}$ determines the position of the corresponding quantum states in the tetrahedron, as shown in Fig. 3. We make use of different colors to distinguish regions converging to different Bell states in such a way that a blue dot represents an initial state, which converges to the Bell state $|\beta_{00}\rangle$ by the maximum entanglement control, and the cyan, yellow, and green dots represent, respectively, those initial states converging to $|\beta_{10}\rangle$, $|\beta_{01}\rangle$, and $|\beta_{11}\rangle$. The red dots, which covers most of the tetrahedron, correspond to the initial states converging to the maximum entangled states $\{MES\}$ in the form of Bell states, which are called Bell equivalent states.

![Figure 3](image3.jpg)

**Figure 3:** Graphically, the four Bell states can be thought of as the four vertices of a regular tetrahedron, and a coefficient set $\{\beta_{a0}, \beta_{b0}, \beta_{b1}, \beta_{b2}\}$ determines the position of the corresponding quantum states in the tetrahedron.

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where the coefficients satisfy the normalization condition $|\beta_{a0}|^2 + |\beta_{b0}|^2 + |\beta_{b1}|^2 + |\beta_{b2}|^2 = 1$. Graphically, the four Bell states can be thought of as the four vertices of a regular tetrahedron, and a coefficient set $\{\beta_{a0}, \beta_{b0}, \beta_{b1}, \beta_{b2}\}$ determines the position of the corresponding quantum states in the tetrahedron, as shown in Fig. 3. We make use of different colors to distinguish regions converging to different Bell states in such a way that a blue dot represents an initial state, which converges to the Bell state $|\beta_{00}\rangle$ by the maximum entanglement control, and the cyan, yellow, and green dots represent, respectively, those initial states converging to $|\beta_{10}\rangle$, $|\beta_{01}\rangle$, and $|\beta_{11}\rangle$. The red dots, which covers most of the tetrahedron, correspond to the initial states converging to the maximum entangled states $\{MES\}$ in the form of Bell states, which are called Bell equivalent states.

| Initial state | Initial deviation | final state |
|---------------|------------------|-------------|
| 1 | $|00\rangle$ | $|\beta_{00}\rangle$ |
| 2 | $|00\rangle$ | $|\beta_{01}\rangle$ |
| 3 | $|11\rangle$ | $|\beta_{10}\rangle$ |
| 4 | $|11\rangle$ | $|\beta_{11}\rangle$ |
| 5 | $|01\rangle$ | $|\beta_{10}\rangle$ |
| 6 | $|01\rangle$ | $|\beta_{11}\rangle$ |
| 7 | $|10\rangle$ | $|\beta_{00}\rangle$ |
| 8 | $|10\rangle$ | $|\beta_{01}\rangle$ |

**Table 1:** Initial states and the related final states under Lyapunov entanglement control.
When the Lyapunov entanglement control converges to an equivalent Bell state, all the populations of the four basis states are not zero according to Eq. (56). This is different from the case of a Bell state, which has only two basis states with non-zero populations.

Figure 4: Time evolution of the component populations from a random initial state to a terminal Bell equivalent state with components \(|\alpha|^2 = |\delta|^2\) and \(|\beta|^2 = |\gamma|^2\) for three entanglement measures: concurrence (solid line), entropy of entanglement (dashed line), and Renyi entropy with \(\alpha = 1.5\) (dotted line).

The convergence of Lyapunov entanglement control to a Bell equivalent state is shown in Fig. 4, where the initial state is randomly chosen from the red region in Fig. 3 so that the achieved terminal state is a Bell equivalent state. The obtained Bell equivalent state has four non-zero components with \(|\alpha|^2 = |\delta|^2\) and \(|\beta|^2 = |\gamma|^2\) are not zero according to Eq. (56). This is different from the terminal Bell state with \(|\alpha|^2 = |\delta|^2 = 1/2\) and \(|\beta|^2 = |\gamma|^2 = 0\) as shown in Fig. 1.

Figure 5: The convergence of three entanglement measures under the Lyapunov control to the Bell equivalent state considered in Fig. 4.

Bell equivalent states mean that they can be made identical to Bell states by LOCC operations and they share the same Schmidt decomposition with Bell states. Regardless of whether the terminal state is a Bell state or a Bell equivalent state, the Lyapunov entanglement control method proposed here ensures that the controlled quantum state can converge to the maximum...
entangled state. A numerical verification of Theorem 4.2 is shown in Fig. 5, where the degree of entanglement of the quantum state undergoing Lyapunov entanglement control is monitored by three entanglement measures. The initial state is chosen the same as that in Fig. 4 so that the terminal state is a Bell equivalent state instead of a Bell state. It can be seen that all the three entanglement measures converge to their maximum value, confirming the role of the Bell equivalent state as a maximum entangled state.

If the maximum entangled state is to be used as a medium for transmitting quantum information, then any terminal state obtained by the Lyapunov entanglement control can meet this requirement. If the maximum entangled state is to realize some special qubits or quantum logic gates, the maximum entangled state must have the form of a Bell state. For this case, the initial state has to be selected appropriately according to Table 1 and Fig. 3 to make the Lyapunov entanglement control converge to the specified Bell state. Alternatively, we can start from an arbitrary initial state to obtain a maximum entangled state by Lyapunov entanglement control and then transform it to the specified Bell state by LOCC operation.

VI. LYAPUNOV ENTANGLEMENT CONTROL FOR MIXED STATES

For pure-state quantum control of a bipartite system, although our method can reach the maximum entangled state, the Lyapunov control methods proposed in the literature can also obtain the same result. The difference is that our method can reach the maximum entangled state spontaneously without specifying its final state. Although the maximum entangled state obtained cannot be specified in advance, this is not a shortcoming of the proposed method, because all the maximum entangled states are equivalent under local operations. As to a mixed bipartite state, the current literature does not have a complete analytical description for the mixed state having maximum entanglement, but the method we propose here can be used to automatically search for the maximum entangled mixed state.

For a mixed bipartite described by $\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, its degree of entanglement can be measured by the concurrence as

$$E_C(\rho) = \min_{\{|\psi_k\rangle\}} \sum_k p_k E_C(|\psi_k\rangle),$$  \hspace{1cm} (58)

where the minimization is over all possible ways of decomposing the mixed state $\rho$ to its component pure states. In other words, the entanglement measure of a mixed state is defined to be smallest sum of the entanglement measure of its component pure states. This idea also applies to the entanglement of formation, which is defined as

$$E_F(\rho) = \min_{\{|\psi_k\rangle\}} \sum_k p_k E_F(|\psi_k\rangle),$$  \hspace{1cm} (59)

where $E_F(|\psi_k\rangle)$ is the entropy of entanglement of the component pure state $|\psi_k\rangle$. The same idea can be applied to other entanglement measures by extending their definitions from pure states to mixed states.

For so many different definitions of mixed-state entanglement measures, which one is the most representative and can be used in the maximum entanglement control. In fact, any qualified mixed-state entanglement measure can be used in the maximum entanglement control, and the maximum entangled state obtained by them is the same. The reason is that there exists a monotonic mapping between any two qualified entanglement measures. For example, the entanglement of formation $E_F(\rho)$ can be expressed as a function of the concurrence $E_C(\rho)$:

$$E_F(\rho) = E(C(\rho)), \hspace{1cm} (60)$$

where the monotonic function $E$ is defined as

$$E(C) = -\frac{1 + (1-\varepsilon^2)}{2} \log_2 \frac{1 + (1-\varepsilon^2)}{2}$$

$$-\frac{1 - (1-\varepsilon^2)}{2} \log_2 \frac{1 - (1-\varepsilon^2)}{2}$$  \hspace{1cm} (61)

Due to the monotonic relation established by Eq. (61), the two entanglement measures $E_C(\rho)$ and $E_F(\rho)$ have the same minimal decomposition of the density matrix $\rho$ in Eq. (58) and Eq. (59), and if they are used as the Lyapunov entanglement functions, the Lyapunov entanglement control will converge to the same maximum entanglement state.

Similar to Eq. (58) and Eq. (59), the entanglement measure for mixed state can also be defined in terms of the general entanglement measure $E_C(\rho)$ as

$$E_G(\rho) = \min_{\{|\psi_k\rangle\}} \sum k p_k E_C(|\psi_k\rangle).$$  \hspace{1cm} (62)

It is not surprising that the minimal decomposition involved in Eq. (62) is the same as Eq. (58), because monotonic mapping also exists between the concurrence $E_C$ and the general entanglement measure $E_G$, which can be derived from Eq. (30) as

$$E_G = G(f(X) + f(1-X)) = (1 + \sqrt{1-\varepsilon^2})/2,$$  \hspace{1cm} (63)

where $G$ is a monotonic function as verified in Section III. Therefore, the result of maximum entanglement control based on the measure $E_G$ will be identical to that based on the measure $E_C$. Since the acquisition of the maximally entangled state is independent of the entanglement measure used, we will employ the concurrence $E_C$ as a demonstration of applying Lyapunov entanglement control to two-qubit mixed states.

For mixed states, the maximally entangled states over the entire $\mathcal{H}_{AB}$ space is still unknown in the literature. Ishizaka [58] proposed a class of maximally entangled mixed states for two-qubit bipartite systems, whose concurrence $E_C$ is maximized over all mixed states with given spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. This class of maximally entangled mixed states is generated by applying any local unitary transformation to the kernel mixed state

$$\rho_{\text{max}} = \lambda_1 |\beta_{11}\rangle \langle \beta_{11}| + \lambda_2 |00\rangle \langle 00| + \lambda_3 |\beta_{10}\rangle \langle \beta_{10}| + \lambda_4 |11\rangle \langle 11|$$  \hspace{1cm} (64)

where $\lambda_i$'s are the eigenvalues of $\rho_{\text{max}}$ in decreasing order with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. All the states in this class have the same concurrence given by

$$E_C(\rho_{\text{max}}) = E_C^* = \max\{0, \lambda_1 - \lambda_3 - 2\sqrt{\lambda_3 \lambda_4}\}$$  \hspace{1cm} (65)

which is proved to be the maximum concurrence that can be achieved for all mixed states with given spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. The role of the maximally entangled state $\rho_{\text{max}}$ in the mixed state is similar to that of the Bell state in the pure state; however, no quantum control has been proposed to realize this Bell mixed state till now. The Lyapunov entanglement control developed in Section IV is particularly suitable for this task, because the operation involved in von Neumann equation (4) is just a unitary transform for density matrix so that the spectrum of density matrix remains unchanged during the control process. In the following, we will apply the Lyapunov entanglement control to drive an initial state $\rho_0$ with specified
spectrum \{\lambda_j\} to the maximally entangled state \(\rho_{\text{max}}\), which maximize the entanglement measure \(E_c(\rho)\).

The Lyapunov entanglement function is chosen as

\[ V_c(\rho) = \mathcal{N} - E_c(\rho) = \mathcal{N} - \min_{(\rho_0, \rho_1)} \sum_k p_k E_c(\ket{\psi_k}) \]  \quad (66)

where \(\mathcal{N}\) is a constant, which can be set to 1, if the concurrence \(E_c(\rho)\) is normalized by its maximum value \(E_c^* = \lambda_1 - \lambda_3 - 2\sqrt{\lambda_2\lambda_4}\). However, normalization of \(E_c(\rho)\) with respect to \(E_c^*\) is not a necessary step, because the maximally entangled state is determined by the condition \(V_c(\rho) = 0\) and is independent of the absolute magnitude of \(\mathcal{N}\) and \(E_c(\rho)\). In other words, the maximum entanglement measure \(E_c^*\) need not be specified in advance in the design process of Lyapunov entanglement control. Once the maximally entangled state \(\rho_{\text{max}}\) is obtained by the entanglement control, \(E_c(\rho_{\text{max}})\) automatically gives the value of \(E_c^*\). The analytical expression of \(E_c^*\) introduced in (65) is only for the purpose of comparing with the \(E_c^*\) obtained by the proposed entanglement control.

For a given density matrix \(\rho\), the evaluation of the concurrence \(E_c(\rho)\) involves a minimum decomposition process (66), which causes a difficulty in expressing \(V_c(\rho)\) as an explicit function of \(\rho\). Fortunately, this difficulty can be overcome by the method of tilded decomposition introduced by Wootters [45]. In terms of the tilded orthogonal basis \(|y_k\rangle\), the minimum decomposition of \(\rho\) can be expressed by

\[ \rho = p_k|y_k\rangle\langle y_k| + \sum_{k=2}^n p_k|y_k\rangle\langle y_k|, \]  \quad (67)

where \(p_k\) is the weight corresponding to the states \(|y_k\rangle\). The concurrence of \(\rho\) turns out to be the summation of the concurrence of the component pure state \(Y_k = |y_k\rangle\langle y_k|\) as

\[ E_c(\rho) = \rho_k E_c(Y_k) - \sum_{k=2}^n p_k E_c(Y_k). \]  \quad (68)

For the convenience of expression, we define the new states \(|z_k\rangle = \sqrt{p}|y_k\rangle\) and \(|z_k\rangle = \sqrt{p_k}|y_k\rangle\), \(k = 2, 3, 4\), to rewrite (67) as

\[ \rho = \sum_k |z_k\rangle\langle z_k|, \]  \quad (69)

Substituting (69) into (66) and using the definition of concurrence of pure states given by (29), we obtain

\[ V_c(\rho) = \mathcal{N} - \sum_k E_c(Z_k) = \mathcal{N} - \sum_k \sqrt{2(1 - \text{Tr}(Z_k Z_k))} \]  \quad (70)

In the following, the Lyapunov control law \(u_k(\rho)\) will be derived from (68) in terms of \(\rho\)’s component pure state \(Y_k\) to achieve the control goal \(V_c(\rho) < 0\), \(\forall \rho \neq \rho_{\text{eq}}\) and \(V_c(\rho_{\text{eq}}) = 0\).

### A. Control Law Design

According to (70), the first-order time derivative of \(V_c(\rho)\) is

\[ \dot{V}_c(\rho) = -2i \sum_j [E_c(Z_j)]^{-1} \text{Tr}(Z_j M) \cdot (HZ_j - Z_j H) M. \]  \quad (71)

With the Hamiltonian \(H\) expressed by the interaction picture (4b), \(\dot{V}_c(\rho)\) can be further simplified to

\[ \dot{V}_c(\rho) = -2i \sum_k u_k \sum_j [E_c(Z_j)]^{-1} \text{Tr}(Z_j M) \cdot (A_k Z_j - Z_j A_k) M. \]  \quad (72)

By a similar way taken by Theorem 4.2, we can show that \(\text{Tr}((Z_j M) \cdot (A_k Z_j - Z_j A_k) M)\) is a pure imaginary number and that \(E_c(Z_j)\) is a real number. Thus the following quantity appears to be real-valued:

\[ x_k = i \sum_j [E_c(Z_j)]^{-1} \text{Tr}((Z_j M) \cdot (A_k Z_j - Z_j A_k) M). \]  \quad (73)

Like the case of pure-state control, a real function \(h_k(x_k)\) of \(x_k\) can be introduced to satisfy the conditions \(h_k(x_k) = 0\) if \(x_k = 0\), which can provide the feedback signal for the mixed-state Lyapunov control law \(u_k\) as

\[ u_k = v_x h_k(x_k), \quad v_x > 0 \]  \quad (74)

The substitution of (72) into (70) yields the desired goal of Lyapunov control

\[ \dot{V}_c(\rho) = -2\sum_k v_x h_k(x_k) \cdot x_k \leq 0 \]  \quad (75)

Therefore, the control law (74) ensures that the LEF \(V_c(\rho)\) is decreasing and meanwhile the entanglement measure \(E_c(\rho)\) is increasing due to the relation \(\dot{V}_c(\rho) = \mathcal{N} - E_c(\rho)\).

### B. Asymptotical Stability

The density matrix \(\rho\) is the control law characterized by \(\dot{V}_c(\rho) = 0\), to which the only solution is \(x_k = 0\) because of \(h_k(x_k) = 0\). For arbitrary \(x_k \neq 0\), we have \(\dot{V}_c(\rho) < 0\) from (73). With \(x_k = 0\), the control law (72) then gives \(u_k = 0\), which in turn yields \(\dot{\rho} = 0\) from (4b). Hence, the invariant set contains only the equilibrium states \(\rho_{\text{eq}}\) of the von Neumann equation (4), which implies that the density matrix \(\rho\) controlled by (72) converges asymptotically to the equilibrium state \(\rho_{\text{eq}}\) with \(\dot{V}_c(\rho_{\text{eq}}) = 0\). According to the properties \(\dot{V}_c(\rho) < 0\) and \(E_c(\rho) > 0\), \(\forall \rho \neq \rho_{\text{eq}}\) and \(\dot{V}_c(\rho_{\text{eq}}) = 0\), which implies that the equilibrium state \(\rho_{\text{eq}}\) is the extremal state with respect to time, which minimizes the LEF \(V_c(\rho)\) and meanwhile maximizes the entanglement measure \(E_c(\rho)\). Of significance is that the value of \(E_c(\rho_{\text{max}})\) automatically gives the maximum entanglement measure \(E_c^*\), and we do not need to specify it in advance in the design of Lyapunov entanglement control. In the following numerical verification, the maximally entangled state \(\rho_{\text{max}}\) will be realized by the Lyapunov entanglement control and the achieved maximum entanglement measure \(E_c = E_c(\rho_{\text{max}})\) will be compared with the analytical solution (65).

### C. Numerical Verification

The Lyapunov entanglement control (74) with \(h_k(x_k) = 0\) will be employed to obtain the maximally entangled mixed state (MEMS). The feedback signal \(x_k\) defined by (73) is generated by the von Neumann equation (4), where the internal Hamiltonian is chosen as \(H_0 = \sigma_x \otimes \sigma_x\) and the control Hamiltonian \(H_k\) is constructed in the form of

\[ H_k = i(m)n - |n\rangle\langle m|, \quad k = 1, 2, \ldots, 6, \]  \quad (76)

where \(\{|m\rangle, |n\rangle\} \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}\). The number of the available control Hamiltonian \(H_k\) is related to the degree of freedom the controller can manipulate. Compared with the control of the pure state, the control of the mixed state requires more controller degree of freedom. It can be seen that the control Hamiltonian \(H_k\) (76) deals with the intertransfer between the two bases \(|m\rangle\) and \(|n\rangle\). A general state transfer may comprise all possible intertransfers between any two bases in the set \(|00\rangle, |01\rangle, |10\rangle, |11\rangle\}. Since there are six different ways of intertransfer between the two bases, the number of \(H_k\) must be at least six to cover the entire range of state transfer. An equivalent matrix expression of the control Hamiltonian \(H_k\) (76) is given by the tensor products of Pauli matrices

\[ H_1 = \sigma_x \otimes \sigma_x, \quad H_2 = \sigma_x \otimes \sigma_y, \quad H_3 = \sigma_y \otimes \sigma_x, \]  \quad (77)

\[ H_4 = \sigma_x \otimes \sigma_y, \quad H_5 = \sigma_y \otimes \sigma_y, \quad H_6 = \sigma_y \otimes \sigma_z, \]  \quad (77)

which are the linear combinations of the \(H_k\) in (76). With the specified \(H_k\), the time evolution of the density matrix under the
interaction picture is described by (4) as 
\[ i\dot{\rho}(t) = [\Sigma_{\kappa=1}^{6} u_{k}\Lambda_{k}, \rho(t)] \]
where the control signal \( u_{k} \) is determined by (74) with gain \( r_{k} = 5 \). Figure 6 shows the time responses of the concurrence by the proposed Lyapunov entanglement control for three initial density matrices \( \rho_{0} \) with the same spectrum \( \{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\} = \{0.4932, 0.3485, 0.1301, 0.0282\} \).

\[ E_{c}(Y_{1}) = E_{c}(Y_{2}) = 1, \]
\[ E_{c}(Y_{3}) = E_{c}(Y_{4}) = 2\sqrt{\lambda_{2}\lambda_{4}/(\lambda_{2} + \lambda_{4})}. \]

Meanwhile, the steady-state weight \( p_{k} \) of \( E_{c}(Y_{k}) \) shows the following regularity

\[ p_{1} = \lambda_{1}, p_{2} = \lambda_{3}, p_{3} = p_{4} = (\lambda_{2} + \lambda_{4})/2 \]

The combination of (78) and (79) gives an error-free preiction of \( E_{c}(\rho_{ss}) \) as

\[ E_{c}(\rho_{ss}) = p_{1}E_{c}(Y_{1}) - p_{2}E_{c}(Y_{2}) - p_{3}E_{c}(Y_{3}) - p_{4}E_{c}(Y_{4}) = \lambda_{1} - \lambda_{3} - 2\sqrt{\lambda_{2}\lambda_{4}}, \]

which recovers the theoretical result (65).

Table 2: The comparison of the steady-state concurrence \( E_{c}(\rho_{ss}) \) with the theoretical value \( E_{c}^{*} \) for ten sets of spectrum.

| No | \( \lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\) | \( p_{1} \) | \( p_{2} \) | \( p_{3} \) | \( p_{4} \) | \( E_{c}^{*} \) |
|----|----------------|--------|--------|--------|--------|--------|
| 1  | 0.4497, 0.2978, 0.2498, 0.0026 | 0.4497 | 0.2502 | 0.1548 | 0.1544 | 0.1442 |
| 2  | 0.5326, 0.2953, 0.1624, 0.0096 | 0.5323 | 0.1516 | 0.1536 | 0.2637 |        |
| 3  | 0.5939, 0.2516, 0.1266, 0.0278 | 0.5935 | 0.1431 | 0.1344 | 0.3000 |        |
| 4  | 0.5467, 0.3363, 0.1099, 0.0070 | 0.5466 | 0.1743 | 0.1679 | 0.3398 |        |
| 5  | 0.5155, 0.3716, 0.1118, 0.0010 | 0.5155 | 0.1907 | 0.1815 | 0.3651 |        |
| 6  | 0.6607, 0.1901, 0.1083, 0.0409 | 0.6605 | 0.1177 | 0.1168 | 0.3761 |        |
| 7  | 0.5884, 0.2693, 0.1398, 0.0024 | 0.5884 | 0.1419 | 0.1409 | 0.3978 |        |
| 8  | 0.6465, 0.2604, 0.0659, 0.0271 | 0.6465 | 0.1397 | 0.1476 | 0.4126 |        |
| 9  | 0.6122, 0.3039, 0.0714, 0.0125 | 0.6120 | 0.1566 | 0.1596 | 0.4175 |        |
| 10 | 0.7760, 0.1800, 0.0294, 0.0146 | 0.7756 | 0.0870 | 0.1077 | 0.6441 |        |

By comparing with the analytical solution, the above numerical results confirm that the proposed Lyapunov control method can automatically converge to the MEMS. More importantly, we do not need to specify in advance the MEMS to be reached during the control process. It is precisely because of this property of the proposed control method that we have discovered more different forms of MEMS not belonging to the class generated by the kernal mixed state (64).

The tilde decomposition of the MEMS in the class generated by (64) possesses the properties expressed by (78) and (79), as shown in Tab. 2. However, there are many MEMS outside this class. For example, considering the following MEMS

\[ \rho = \lambda_{1}\vert\beta_{0}\rangle\langle\beta_{0} \vert + \sqrt{\lambda_{2}\lambda_{4}}\vert\beta_{10}\rangle\langle\beta_{10} \vert + \lambda_{3}\vert\beta_{01}\rangle\langle\beta_{01} \vert + \sqrt{\lambda_{2}\lambda_{4}}\vert\beta_{11}\rangle\langle\beta_{11} \vert, \]  

we find that its tilde decomposition has the property \( E_{c}(Y_{1}) = E_{c}(Y_{2}) = E_{c}(Y_{3}) = E_{c}(Y_{4}) = 1 \), which is different from the pattern specified by (78). Obviously, the MEMS given by (81) does not belong to the class generated by (64); however, it still possesses the maximum concurrence \( E_{c}^{*} \) given by (65).

Table 3 lists the MEMS achieved by the Lyapunov entanglement control law (74), which otherwise can not be obtained by applying any local unitary transformation to (64).
It can be checked that the steady-state values of $p_1$ and $E_c(Y_1)$ listed in Tab. 3 do not have the regularities expressed by (78) and (79), indicating that the class of MEMS in Tab. 3 is different from the class covered by Tab. 2. Nevertheless, we note that although the MENS in Tab.2 and Tab. 3 belong to different classes, they all attain the maximum concurrence $E_c^* = \lambda_1 - \lambda_3 - 2\sqrt{\lambda_2} \lambda_3$ within the numerical accuracy. The last column in Tab. 3 compares the computed $E_c(p_{ss}) = p_1 E_c(Y_1) - \sum_{k=2}^{r} p_k E_c(Y_k)$ with the theoretical value $E_c^*$.

Table 3: Several MES’s obtained by the Lyapunov entanglement control law (74) but not belonging to the class covered by Tab. 2.

| No | $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ | $E_c(Y_1)$ | $E_c(Y_2)$ | $E_c(Y_3)$ | $E_c(Y_4)$ | $E_c(p_{ss})$ |
|----|----------------------------------|------------|------------|------------|------------|--------------|
| 1  | 0.6523, 0.2515, 0.0768, 0.0194    | 0.6516     | 0.1108     | 0.1223     | 0.1153     | 0.4358       |
| 2  | 0.6099, 0.3298, 0.0522, 0.0082    | 0.6097     | 0.1385     | 0.1402     | 0.1116     | 0.4537       |
| 3  | 0.6385, 0.3130, 0.0433, 0.0052    | 0.6378     | 0.1089     | 0.1424     | 0.1109     | 0.5145       |
| 4  | 0.7336, 0.2303, 0.0321, 0.0040    | 0.7330     | 0.0867     | 0.1138     | 0.0665     | 0.6412       |
| 5  | 0.8069, 0.1686, 0.0229, 0.0016    | 0.8064     | 0.0478     | 0.0678     | 0.0780     | 0.7516       |
| 6  | 0.8428, 0.1348, 0.0210, 0.0011    | 0.8422     | 0.0391     | 0.0463     | 0.0724     | 0.7974       |
| 7  | 0.8437, 0.1507, 0.0050, 0.0006    | 0.8433     | 0.0752     | 0.0764     | 0.0502     | 0.8197       |

Figure 7 shows the convergence of the concurrence $E_c(p(t))$ to $E_c(p_{max})$ with the MEMS $p_{max}$ not belonging to class generated by (64) for three sets of spectrum listed in Tab. 3. It can be seen that the steady-state concurrence $E_c(p_{ss})$ approaches the theoretical prediction $E_c^*$ labelled by the dashed line within the numerical accuracy.

![Figure 7: The convergence of the concurrence $E_c(p(t))$ to the MES $p_{max}$ not belonging to the class generated by (64) for the three sets of spectrum listed in Tab. 3.](image)

Regarding the entanglement control of mixed states, Table 2 and Table 3 present two of the major results of this paper. The former shows that the MEMS obtained by the proposed method is completely consistent with the analytical solution mentioned in the literature, and the latter shows that our control method can also be used to generate new forms of MEMS not known in the literature.

VII. LYAPUNOV ENTANGLEMENT CONTROL FOR MULTIPARTITE SYSTEMS

The same control strategy that has been used in the maximum entanglement control for the pure state and the mixed state of bipartite systems can be equally applied to multipartite systems. For multipartite systems, the biggest difficulty lies not in the formulation of the Lyapunov entanglement control strategy, but in that the currently available entanglement measures can only determine the lower bound of the quantum entanglement, not the true entanglement of a multipartite state. Furthermore, the maximally entangled state of multipartite systems is not unique, because using different entanglement measures may result in different maximum entangled states, such as W state or GHZ state, between which there is no local unitary transformation.

Since Lyapunov maximum entanglement control is only responsible for the adopted entanglement measure, the MES of multipartite systems obtained by the Lyapunov entanglement control can only maximize the entanglement measure adopted in the Lyapunov function, but it may not be the MES, when evaluated by other entanglement measures; meanwhile, the entanglement measure that has been maximized by the Lyapunov control may merely represents a lower bound of the real quantum entanglement. In this section, two entanglement measures for multipartite systems, i.e., generalized concurrence and genuine multipartite entanglement, will be employed in the Lyapunov entanglement control. The Lyapunov control law will be designed to drive the multi-bit state to the MES maximizing the two entanglement measures, which is achieved theoretically by the GHZ state.

A. Two Entanglement measures for multipartite states

Generalized concurrence [55] provides a lower bound of the degree of multipartite entanglement. Let $H$ denote a Hilbert space with dimension $d$, whose base is given by $|i\rangle$, $i = 1, 2, ..., d$. A pure multipartite state in the space of $H \otimes \cdots \otimes H$ is represented by

$$|\psi\rangle = \sum_{i_1,i_2,...,i_N=1}^N a_{i_1,i_2,...,i_N}|i_1,i_2,...,i_N\rangle,$$

(82)

where $N$ is the number of particles involved in multipartite system. Let $\alpha$ and $\alpha'$ (resp. $\beta$ and $\beta'$) be the subsets of the sub-indices of $a$, which are associated with the same sub-Hilbert spaces but with different summing indices, so that $\alpha$ (or $\alpha'$) and $\beta$ (or $\beta'$) span the whole space of the given sub-index of $a$.

Then the generalized concurrence of $|\psi\rangle$ is given by

$$E_{GC} = \frac{1}{\sqrt{2^d(2d-1)}} \left| \sum_{a,a',\beta,\beta'} \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \left| \alpha_{ab}^\dagger \alpha_{a'b'}^\dagger - \alpha_{a'b} \alpha_{ab'}^\dagger \right| \right|^2$$

(83)

where $m = 2^{d-1} - 1$ and the summation is over all possible combinations of the indices of $\alpha$ and $\beta$. For a N-bit state $|\psi\rangle$, we have $d = 2$, for which (83) can be simplified to

$$E_{GC} = \frac{1}{\sqrt{2^{N-1}}} \left( 1 - \sum_{\alpha=1}^{N} \text{Tr}(\rho_{\alpha}^\dagger) \right)$$

(84)

where $\rho_{j} = \text{Tr}_{M_j}(\rho)$ is the reduced matrix of $\rho = |\psi\rangle\langle\psi|$ obtained by taking the partial traces for all the particles except the $j$ th particle. When $N = 2$, (84) reduces to (29) for bipartite
pure states. The Lyapunov entanglement function for \( E_{GC}(\rho) \) can be chosen as
\[
V_{GC}(\rho) = \mathcal{N} - E_{GC}(\rho), \tag{85}
\]
where \( \mathcal{N} \) is a trivial constant. The Lyapunov control law will be derived from the condition \( V_{GC} = -E_{GC} \leq 0 \) to drive \( \rho = |\psi\rangle \langle \psi| \) from an arbitrary state \( \rho_0 \) to the steady state \( \rho_{eq} \), at which \( E_{GC} \) achieves its maximum. The computed maximum then can be compared with the theoretical maximum achieved by the [GHZ] state to verify the correctness of the proposed Lyapunov control law.

The same control strategy can be applied to other qualified entanglement measure for multipartite systems, such as genuine-multipartite-entanglement (GME) concurrence [56], [57]. A pure multipartite state \( |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \) is said to be biseparable, if it can be written as \( |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \), where \( |\psi_1\rangle \in \mathcal{H}_1 = \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_k} \) and \( |\psi_2\rangle \in \mathcal{H}_2 = \mathcal{H}_{j_{k+1}} \otimes \cdots \otimes \mathcal{H}_n \); otherwise, it is said to be genuinely \( N \)-partite entangled. Supposing \( \{y_k|y_k\} = \{j_1, j_2, \ldots, j_k, j_{k+1}, \ldots, j_n\} \) is a bipartition of \( \{1, 2, \ldots, n\} \), GME concurrence searches for the particular bipartition of the system such that the resulting subsystem minimizes the concurrence:
\[
E_{GME}(\rho) = \min_{y_k} \sqrt{2(1 - \text{Tr}(\rho_{Y_k}^2))}, \tag{86}
\]
where \( \rho_{Y_k} \) is the reduced density matrix of the subsystem indexed by \( y_k \) and the minimization is over all possible subsystems. The Lyapunov entanglement function for \( E_{GC}(\rho) \) can be chosen as
\[
V_{GME}(\rho) = \mathcal{N} - E_{GME}(\rho). \tag{87}
\]
The aim of Lyapunov control is to drive \( E_{GME}(\rho) \) to its maximum and to compare with theoretical maximum achieved by the [GHZ] state. However, [GHZ] is not the unique MES of \( E_{GME}(\rho) \), and there are MES’s other than [GHZ], which still attain \( E_{GME}(\rho) = 1 \) and can be achieved by the Lyapunov control with \( V_{GME} < 0 \).

### B. Designing Lyapunov entanglement control laws

Once a qualified Lyapunov entanglement function is chosen for multipartite states, the design of Lyapunov entanglement control law is the same as that of bipartite states without additional difficulty. We start with the Lyapunov function of the general concurrence \( V_{GC}(\rho) \), whose first-order time derivative can be expressed as
\[
V_{GC} = \frac{E_{GC}}{2N-1} \sum_{j=1}^{N} \text{Tr}(\rho_j \Delta_j), \tag{88}
\]
The system Hamiltonian \( H \) under the interaction picture is given by (4b) with which the relation of \( V_{GC} \) to the control signal \( u_k \) can be derived as
\[
V_{GC} = -i \frac{E_{GC}}{2N-1} \sum_{k=1}^{N} u_k \sum_{j=1}^{N} \text{Tr}(\rho_j (A_k \rho - \rho A_k)). \tag{89}
\]
No matter how many qubits the system has, the Hermitian property of the density matrix \( \rho \) and its reduced form \( \rho_j \) does not change. As a result, we can show by the same way used in Theorem 4.1 that \( \sum_{j=1}^{N} \text{Tr}(\rho_j (A_k \rho - \rho A_k)) \) is an imaginary number. In terms of the real-valued variable,
\[
x_k = i \cdot \sum_{j=1}^{N} \text{Tr}(\rho_j (A_k \rho - \rho A_k)), \tag{90}
\]
the desired Lyapunov control law now can be constructed as \( u_k = \tau_k h_k(x_k) \), where \( h_k(x_k) \) satisfies the relation (42) and \( \tau_k \) is a positive control gain. With \( u_k = \tau_k h_k(x_k) \), the time derivative of \( V_{GC} \) becomes
\[
V_{GC} = -\frac{E_{GC}}{2N-1} \sum_{k=1}^{N} \tau_k x_k h_k(x_k) \leq 0, \forall t \geq 0. \tag{91}
\]
It can be shown from the property of \( h_k(x_k) \) that \( V_{GC} = E_{GC} = 0 \) occurs only at the equilibrium state \( \rho_{eq} \), and \( V_{GC} = -E_{GC} < 0, \forall \rho \neq \rho_{eq} \). Therefore, the proposed control law drives \( \rho \) to the equilibrium state \( \rho_{eq} \), where \( V_{GC} \) reaches its minimum, i.e., \( \rho_{eq} \) is the MES of \( E_{GC} \).

Next we consider the control law design of GME-concurrence defined by (86). Let \( \rho_{min} \) be the \( \rho \) that attains the minimum in Eq. (86) at time \( t \), then the first-order time derivative of \( V_{GME} \) at time \( t \) can be expressed as
\[
V_{GME} = -2iE_{GME} \sum_{k=1}^{N} u_k \text{Tr}(\rho_{min} \cdot (A_k \rho - \rho A_k)). \tag{92}
\]
To ensure \( V_{GME} \leq 0 \), the control law \( u_k = \tau_k h_k(x_k) \) is applied again with the real-valued feedback signal \( x_k \) defined by
\[
x_k = i \cdot \text{Tr}(\rho_{min} \cdot (A_k \rho - \rho A_k)). \tag{93}
\]
This control law yields
\[
V_{GME} = -2 \sum_{k=1}^{N} \tau_k x_k h_k(x_k) \leq 0, \forall t \geq 0 \tag{94}
\]
and drive \( E_{GME} \) to its maximum at the equilibrium state \( \rho_{eq} \).

### C. Numerical verification

In this section the convergence of the proposed Lyapunov entanglement control to the MES will be demonstrated through tripartite quantum states. Firstly, the internal Hamiltonian \( \hat{H} \) and the control Hamiltonian \( \hat{H}_g \) adopt the same setting values as in Section V. The Lyapunov functions for the two entanglement measures \( E_{GC} \) and \( E_{GME} \) with \( N = 3 \) are given, respectively by (85) and (87) as
\[
V_{GC} = 1 - \sqrt{\frac{3}{2}(1 - \text{Tr}(\rho_{
\rho_{min}}^2)))^2}, \tag{95a}
V_{GME} = 1 - \min_{k=1,2,3} \sqrt{\frac{2}{2}(1 - \text{Tr}(\rho_{
\rho_{min}}^2)))^2}. \tag{95b}
\]
The same Lyapunov control law \( u_k = \tau_k h_k(x_k) = 5x_k \) is applied to the two entanglement measures, where the feedback signal \( x_k \) for \( V_{GC} \) and \( V_{GME} \) is calculated from (90) and (93), respectively. Two initial states are tested in the numerical demonstration: one is the separable \( \rho_0 = |000\rangle \), and the other is a randomly selected inseparable \( \rho_0 \). The resulting time responses of \( E_{GC}(t) \) and \( E_{GME}(t) \) are shown in Fig. 8a. As expected, the proposed Lyapunov control law drives both \( E_{GC}(t) \) and \( E_{GME}(t) \) to their maximum, which is equal to one achieved theoretically by the [GHZ] state. \( E_{GC}(t) \) converges faster than \( E_{GME}(t) \), but consumes much more control energy, as shown in Fig. 8b.

The other noticeable observation from Fig. 8 is that the Lyapunov entanglement control starting from different initial state \( \rho_0 \) may converge to different MES with different convergence speed. Obviously, the convergence to the MES from the randomly selected \( \rho_0 \) is slower than that of \( \rho_0 = |000\rangle \). Collecting all the MES’s achieved by the Lyapunov entanglement control from different initial states forms a class of MES whose entanglement measure is equal to that of [GHZ]. Although the degree of entanglement of the obtained MES’s is the same as that of [GHZ], it does not mean that we can apply the LOCC operation to convert these MES’s into the [GHZ] state. Because the equivalent class is based on the equivalence of Schmidt decomposition, which is no longer applicable to 3-qubit states, we cannot classify all the MES’s obtained by the Lyapunov control into the equivalent class of the [GHZ] state.
This is the main difference from the 2-qubit entanglement control, for which all the MES’s can be made equivalent to the Bell state by LOCC operation.

Another advantage of this article is that our control is independent of the final state, so we can control state to some unknown maximally entangled mixed state.

Finally, we indicate that the Lyapunov entangled function can also be easily extended to multipartite, which refers that the applicability of the Lyapunov entangled function is comprehensive.

REFERENCES

[1] P. I. Hagouel and I. G. Karafyllidis, “Quantum computers: Registers, gates and algorithms,” 2012 28th International Conference on Microelectronics Proceedings, May 2012.

[2] V. S. Malinovsky and I. R. Sola, “Quantum control of entanglement by phase manipulation of time-delayed pulse sequences. II,” Phys. Rev. A, vol. 70, Oct. 2004.

[3] A. Islers, H. Z. Munthe-Kaas, S. P. Norsset, and A. Zanna, “Lie-group methods,” Acta Numerica, vol. 9, pp. 215–365, Jan. 2000.

[4] Botsinis, Panagiotsis, S. X. Ng, and L. Hanzo, “Quantum search algorithms, quantum wireless, and a low-complexity maximum likelihood iterative quantum multi-user detector design,” IEEE Access, vol. 1, pp. 94–122, 2013.

[5] R. Nevols, J. Jeong, and P. Hemmer, “Microwave simulation of Grover’s quantum search algorithm,” IEEE Antennas and Propagation Magazine, vol. 48, pp. 38–47, Oct. 2006.

[6] J. Gough, V. A. Belavkin and O. G. Smolyanov, “Hamilton-Jacobi-Bellman equations for quantum optimal control,” DAYS on DIFFRACTION 2006, 2006.

[7] Q.-F. Wang, “Quantum optimal control of nonlinear dynamics systems described by Klein-Gordon-Schrödinger equations,” 2006 American Control Conference, 2006.

[8] Q.-F. Wang, “Quantum optimal control of nuclei in the presence of perturbation in electric field,” IET Control Theory Applications, vol. 3, pp. 1175–1182, Sep. 2009.

[9] R. Schmidt, A. Negretti, J. Ankerhold, T. Calarco, and J. T. Stockburger, “Optimal control of open quantum systems: Cooperative effects of driving and dissipation,” Phys. Rev. Lett., vol. 107, Sep. 2011.

[10] M. Hintermüller, D. Marahrens, P. A. Markowich, and C. Sparber, “Optimal bilinear control of gross–pitaevskii equations,” SIAM Journal on Control and Optimization, vol. 51, pp. 2509–2543, Jan. 2013.

[11] S. Li, J. Ruths, T.-Y. Yu, H. Arthanari, and G. Wagner, “Optimal pulse design in quantum control: A unified computational method,” Proceedings of the National Academy of Sciences, vol. 108, pp. 1879–1884, Jan. 2011.

[12] D. Dong and I. R. Petersen, “Sampled-data control of two-level quantum systems based on sliding mode design,” 50th IEEE Conference on Decision and Control and European Control Conference, pp. 6236–6241, Dec. 2011.

[13] K. Mishima and K. Yamashita, “Free-time and fixed end-point optimal control theory in quantum mechanics: Application to entanglement generation,” The Journal of Chemical Physics, vol. 130, p. 034108, Jan. 2009.

[14] U. V. Boscaín, F. Chittaro, P. Mason, and M. Sigalotti, “Adiabatic control of the Schrödinger equation via conical intersections of the eigenvalues,” IEEE Transactions on Automatic Control, vol. 57, pp. 1970–1983, Aug. 2012.

[15] N. Smaoui, A. El-Kadi, and M. Zribi, “Adaptive boundary control of the unforced generalized korteweg–de vries–burgers equation,” Nonlinear Dynamics, vol. 69, pp. 1237–1253, Feb. 2012.

[16] H. B. Silveira, P. S. Pereira da Silva and P. Rouchon, “A time-periodic Lyapunov approach for motion planning of controllable driftless systems on SU(n),” Proceedings of the 48th IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference, Dec. 2009.

[17] Q. Zhang, W. Wang, and L. Wang, “Quantum system control based on lyapunov technology,” 2008 Fourth International Conference on Natural Computation, 2008.

[18] U. Boscaín, F. C. Chittaro, P. Mason, R. Pacqueau, and M. Sigalotti, “Motion planning in quantum control via intersection of eigenvalues,” 49th IEEE Conference on Decision and Control (CDC), Dec. 2010.

[19] S. Kuang and S. Cong, “Lyapunov control methods of closed quantum systems,” Automatica, vol. 44, pp. 98–108, Jan. 2008.

Figure 8: The time responses of $E_{GC}(t)$ and $E_{GME}(t)$ and their control magnitudes $u_{GC}(t)$ and $u_{GME}(t)$ starting from two initial states. Both $E_{GC}(t)$ and $E_{GME}(t)$ converge to the theoretical maximum equal to 1, with the former converging faster than the latter.

VIII. CONCLUSION

This article provides a series of new Lyapunov functions to search for the maximally entangled state. Instead of searching for the distance between states and final state, we use entanglement measure to control the quantum state. The concept of measurement is used to control the final stable state of the quantum state as the maximally entangled state. The maximally entangled state is the most valuable quantum state in quantum information. Since any state can be obtained by performing the LOCC operation on the maximally entangled state. As long as there is a way to prepare the maximally entangled state, which means that any state can be easily prepared.

Most importantly, the decisive value of this article is the introduction of the concept of entanglement measure into Lyapunov control. As long as there is an entanglement measure exactly describing multipartite, the maximally entanglement control can be carried out in multipartite, and Lyapunov control can be voluminously constructed.
[20] D. Dong and I. R. Petersen, “Sliding mode control of two-level quantum systems,” *Automatica*, vol. 48, pp. 725–735, 05 2012.

[21] X. X. Yi, S. L. Wu, C. Wu, X. L. Feng, and C. H. Oh, “Time delay effects and simplified control fields in quantum Lyapunov control,” *J. Phys. B: At. Mol. Opt. Phys.*, vol. 44, p. 195503, Sep. 2011.

[22] J. WEN and S. CONG, “Transfer from arbitrary pure state to target mixed state for quantum systems,” IFAC Proceedings Volumes, vol. 44, pp. 4638–4643, Jan. 2011.

[23] M. Mirrahimi, P. Rouchon, and G. Turinici, “Lyapunov control of bilinear Schrödinger equations,” *Automatica*, vol. 41, pp. 1987–1994, Nov. 2005.

[24] S. Kuang, S. Cong, and Y. Lou, “Population control of quantum states based on invariant subsets under a diagonal Lyapunov function,” *IEEE Xplore*, pp. 2486–2491, Dec. 2009.

[25] S. Grivopoulos and B. Bamieh, “Lyapunov-based control of quantum systems,” 42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475), vol.1, pp. 434–438, Dec. 2003.

[26] M. A. Nielsen, “Preparation of entanglement states in a two-spin system by Lyapunov-based method,” *Journal of Systems Science and Complexity*, vol. 25, pp. 451–462, Jun. 2012.

[27] X. Wang and S. Schirmer, “Analysis of Lyapunov control for Hamiltonian quantum systems,” *Proceedings of the ENOC*, Saint Petersburg, Russia, May 2008.

[28] X. Wang and S. G. Schirmer, “Analysis of Lyapunov method for control of quantum states,” *IEEE Transactions on Automatic Control*, vol. 55, pp. 2259–2270, Oct. 2010.

[29] X. Wang and S. Schirmer, “Analysis of effectiveness of Lyapunov control for non-generic quantum states,” *IEEE Transactions on Automatic Control*, vol. 55, pp. 1406–1411, Jun. 2010.

[30] Q. Fan, “Generation of bell states via Lyapunov control on a two-qubit system with an anisotropic XY Heisenberg interaction,” *Science China Physics, Mechanics and Astronomy*, vol. 54, pp. 474–478, Feb. 2011.

[31] P. Yang and S. Cong, “Purification of mixed state for two-dimensional systems via interaction control,” *IEEE Xplore*, vol. 2, p. 91–94, Oct. 2010.

[32] J. Liu and S. Cong, “Trajectory tracking of quantum states based on lyapunov method,” 2011 9th IEEE International Conference on Control and Automation (ICCA), Dec. 2011.

[33] W. Yang and J. Sun, “One Lyapunov control for quantum systems and its application to entanglement generation,” *Physics Letters A*, vol. 377, pp. 151–155, May 2013.

[34] S. Cong and F. Yang, “Control of quantum states in decoherence-free subspaces,” *Journal of Physics A: Mathematical and Theoretical*, vol. 46, p. 075305, Feb. 2013.

[35] Y. Lou, J. Yang, S. Kuang, and S. Cong, “Path programming control strategy of quantum state transfer,” *IET Control Theory Applications*, vol. 5, pp. 291–298, Jan. 2011.

[36] S. KUANG and S. CONG, “Population control of equilibrium states of quantum systems via Lyapunov method,” *Acta Automatica Sinica*, vol. 36, pp. 1257–1263, 12 2010.

[37] K. Beauchard, J. M. Coron, M. Mirrahimi, and P. Rouchon, “Implicit lyapunov control of finite dimensional Schrodinger equations,” *Systems Control Letters*, vol. 56, pp. 388–395, May 2007.

[38] A. Grigor’Iu, “Implicit Lyapunov control for Schrodinger equations with dipole and polarizability term,” *IEEE Conference on Decision and Control and European Control Conference*, Dec. 2011.

[39] S. Zhao, H. Lin, J. Sun, and Z. Xue, “An implicit Lyapunov control for finite-dimensional closed quantum systems,” *International Journal of Robust and Nonlinear Control*, vol. 22, pp. 1212–1228, May 2011.

[40] F. Meng, S. Cong, and S. Kuang, “Implicit Lyapunov control of multi-control Hamiltonian systems based on state distance,” *Proceedings of the 10th World Congress on Intelligent Control and Automation*, Jul. 2012.

[41] J. Sharifi and H. Memeni, “Lyapunov control of squeezed noise quantum trajectory,” *Physics Letters A*, vol. 375, pp. 522–528, Jan. 2011.

[42] F. Ticozzi, R. Lucchese, P. Cappellaro, and L. Viola, “Hamiltonian control of quantum dynamical semigroups: Stabilization and convergence speed,” *IEEE Transactions on Automatic Control*, vol. 57, pp. 1931–1944, Aug. 2012.

[43] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[44] N. J. Cerf and C. Adami, “Information theory of quantum entanglement and measurement,” *Physica D: Nonlinear Phenomena*, vol. 120, pp. 62–81, Sep. 1998.

[45] W. K. Wootters, “Entanglement of formation of an arbitrary state of two qubits,” *Phys. Rev. Lett.*, vol. 80, pp. 2245–2248, Mar. 1998.

[46] C. Xu, “Completely positive matrices,” *Linear Algebra and its Applications*, vol. 379, pp. 319–327, Mar. 2004.

[47] A. Younes, J. Rowe, and J. Miller, “Enhanced quantum searching via entanglement and partial diffusion,” *Physica D: Nonlinear Phenomena*, vol. 237, pp. 1074–1078, Jun. 2008.

[48] M. Nielsen, “Conditional entanglement and partial entanglement transformations,” *Phys. Rev. Lett.*, vol. 83, pp. 436–439, Jul. 1999.

[49] B. C. Arnold, A. W. Marshall, and I. Olkin, “Inequalities: Theory of majorization and its applications.” *Journal of the American Statistical Association*, vol. 76, p. 492, Jun. 1981.

[50] R. F. Werner, “Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model,” *Phys. Rev. A*, vol. 40, pp. 4277–4281, May 1989.

[51] M. Plenio and S. Virmani, “An introduction to entanglement measures,” *Quantum Information and Computation*, vol. 7, pp. 1–51, 01 2007.

[52] G. Vidal, “Entanglement monotones,” *Journal of Modern Optics*, vol. 47, pp. 355–376, 02 2000.

[53] J. Preskill, “Lecture notes for physics 219: Quantum computation,” Jan. 1999.

[54] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” *Rev. Mod. Phys.*, vol. 81, pp. 865–942, Jun. 2009.

[55] M. Li, S. M. Fei, and Z. X. Wang, “A lower bound of concurrence for multipartite quantum states,” *Journal of Physics A: Mathematical and Theoretical*, vol. 42, p. 145303, 03 2009.

[56] Z.-H. Ma, Z.-H. Chen, J.-L. Chen, C. Spengler, A. Gabriel, and M. Huber, “Measure of genuine multipartite entanglement with computable lower bounds,” *Phys. Rev. A*, vol. 83, Jun. 2011.

[57] Z.-H. Chen, Z.-H. Ma, J.-L. Chen, and S. Severini, “Improved lower bounds on genuine-multipartite-entanglement concurrence,” *Phys. Rev. A*, vol. 85, Jun. 2012.

[58] S. Ishizaka and T. Hiroshima, “Maximally entangled mixed states under nonlocal unitary operations in two qubits,” *Phys. Rev. A*, vol. 62, Jul. 2000.

[59] M. Christandl and A. Winter, “‘squashed entanglement’: An additive entanglement measure,” *Journal of Mathematical Physics*, vol. 45, pp. 829–840, Mar. 2004.

[60] K. Chen, S. Albeverio, and S.-M. Fei, “Entanglement of formation of bipartite quantum states,” *Phys. Rev. Lett.*, vol. 95, Nov. 2005.

[61] J. Osborne, “Entanglement measure for rank-2 mixed states,” *Phys. Rev. A*, vol. 72, Aug. 2005.

[62] R. Demkowicz-Dobrzański, A. Buchleitner, M. Ku’ s, and F. Mintert, “Evaluative multipartite entanglement measures: Multipartite concurrences as entanglement monotones,” *Phys. Rev. A*, vol. 74, Nov. 2006.

[63] C. Eltschka, T. Bastin, A. Osterloh, and J. Siewert, “Multipartite entanglement monotones and polynomial invariants,” *Phys. Rev. A*, vol. 85, Feb. 2012.

[64] Y. Most, Y. Shimoni, and O. Biham, “Formation of multipartite entanglement using random quantum gates,” *Phys. Rev. A*, vol. 76, 08 2007.

[65] A. Osterloh and J. Siewert, “Constructing N-qubit entanglement monotones from antilinear operators,” *Phys. Rev. A*, vol. 72, Jul. 2005.