The spectral norm of a Horadam circulant matrix

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Abstract

Let $a, b, p, q$ be integers and $(h_n)$ defined by $h_0 = a, h_1 = b, h_n = ph_{n-1} + qh_{n-2},$ $n = 2, 3, \ldots$. Complementing to certain previously known results, we study the spectral norm of the circulant matrix corresponding to $h_0, \ldots, h_{n-1}$.

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1. Introduction

Throughout this paper, let $a, b, p, q \in \mathbb{Z}$. We define the Horadam sequence $(h_n) = (h_n(a, b; p, q))$ via

$$h_0 = a, \quad h_1 = b, \quad h_n = ph_{n-1} + qh_{n-2}, \quad n = 2, 3, \ldots.$$

We also use the following abbreviations:

$(f_n) = (h_n(0, 1; 1, 1))$, the Fibonacci sequence;
$(\tilde{f}_n) = (h_n(0, 1; p, q))$, a generalization of the Fibonacci sequence;
$(l_n) = (h_n(2, 1; 1, 1))$, the Lucas sequence;

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\((\tilde{l}_n) = (h_n(2, p; p, q))\), a generalization of the Lucas sequence.

Some references call \((\tilde{l}_n)\) the Lucas sequence. In order to keep the language simple, we follow the custom in [7, p. 8] and call the sequence of Luca’s numbers briefly the Lucas sequence.

For \(n \geq 1\), we write

\[
\mathbf{f} = (f_0, \ldots, f_{n-1}), \quad \tilde{\mathbf{f}} = (\tilde{f}_0, \ldots, \tilde{f}_{n-1}), \\
\mathbf{l} = (l_0, \ldots, l_{n-1}), \quad \tilde{\mathbf{l}} = (\tilde{l}_0, \ldots, \tilde{l}_{n-1}), \\
\mathbf{h} = (h_0, \ldots, h_{n-1}).
\]

Let \(\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n\). The corresponding circulant matrix \(C(\mathbf{x})\) is defined as

\[
C(\mathbf{x}) = \begin{pmatrix}
x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\
x_{n-1} & x_0 & \cdots & x_{n-3} & x_{n-2} \\
x_{n-2} & x_{n-1} & \cdots & x_{n-4} & x_{n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_2 & x_3 & \cdots & x_0 & x_1 \\
x_1 & x_2 & \cdots & x_{n-1} & x_0
\end{pmatrix}.
\]

We let \(\| \cdot \|\) stand for the spectral norm. Our problem is to compute \(\|C(\mathbf{h})\|\) under suitable assumptions. Recently, Kocer et al. [6], ˙Ipek [5], Liu [8], and Bahşi [1] have already studied this question. We will survey their results in Section 2 and give further results in Sections 3 and 4. Finally, we will complete our paper with some remarks in Section 5.

2. Previous results

Let us first study the eigenvalues and singular values of \(C(\mathbf{x})\).

**Theorem 1.** The eigenvalues of \(C(\mathbf{x})\) are

\[
\lambda_i = \sum_{j=0}^{n-1} x_j \omega^{-ij}, \quad i = 1, \ldots, n,
\]

where \(\omega\) is the \(n\)'th primitive root of unity.
Proof. See [2, Theorem 3.2.2].

**Corollary 1.** The singular values of $C(x)$ are

\[ \sigma_i = \left| \sum_{j=0}^{n-1} x_j \omega^{-ij} \right|, \quad i = 1, \ldots, n. \]

Therefore

\[ \|C(x)\| = \max_{1 \leq i \leq n} \left| \sum_{j=0}^{n-1} x_j \omega^{-ij} \right|. \]

**Proof.** Since $C(x)$ is normal, its singular values are the absolute values of eigenvalues. \hfill \blacksquare

Applying this corollary, Kocer et al. [6, Theorem 2.2] proved that

\[ \|C(h)\| = \max_{0 \leq i \leq n-1} \left| \frac{h_n + (pa - b + qh_{n-1})\omega^{-i} - a}{q\omega^{-2i} + p\omega^{-i} - 1} \right|. \]

The maximization problem restricts the use of this formula. The same authors also proved [6, Corollary 2.3] that

\[ \|C(h)\| = \frac{h_n + qh_{n-1} + (p - 1)a - 1}{p + q - 1}, \quad (1) \]

assuming that $p, q \geq 1$ and $b = 1$. Doing so, they suppose nothing on $a$, but apparently $a \geq 0$ must hold. (To see this, take $n = 1$.)

Further, İpek [5, Theorem 1] proved (independently of (1)) that

\[ \|C(f)\| = f_{n+1} - 1 \]

and [5, Theorem 2]

\[ \|C(l)\| = f_{n+2} + f_n - 1. \]

Liu [8, Theorem 9] extended (1) to

\[ \|C(h)\| = \frac{h_n + qh_{n-1} + (p - 1)a - b}{p + q - 1}, \quad (2) \]

whenever $p + q \neq 1$, and to

\[ \|C(h)\| = \frac{qh_{n-1} + (n - 1)(qa + b) + a}{q + 1}. \]
as \( p + q = 1 \), but assumed nothing about \( a, b, p, q \).

Bašić [1, Theorem 2.1] proved (independently of (2)) that, if \( p, q \geq 1 \), then

\[
\|C(\tilde{f})\| = \frac{\tilde{f}_n + q\tilde{f}_{n-1} - 1}{p + q - 1}
\]

and [1, Theorem 2.2]

\[
\|C(\tilde{l})\| = \frac{\tilde{l}_n + q\tilde{l}_{n-1} + p - 2}{p + q - 1}.
\]

3. Computation of \( \|C(h)\|, h \geq 0 \)

We first take a more general viewpoint and verify a theorem that applies also to other matrices than circulant ones or those having elements from a recurrence sequence. If a matrix \( A \) and a vector \( x \) are entrywise nonnegative (respectively, positive), we denote \( A \geq 0 \) and \( x \geq 0 \) (respectively, \( A > 0 \) and \( x > 0 \)). We let \( \lambda(A) \) denote the Perron root of a square matrix \( A \geq 0 \).

**Theorem 2.** Assume that an \( n \times n \) matrix \( A \geq 0 \) has all row sums and column sums equal; let \( s \) be their common value. Then \( \lambda(A) = \|A\| = s \).

**Proof.** Denoting \( e = (1, \ldots, 1) \in \mathbb{R}^n \), we have \( Ae = A^T e = se \). So, \( s \) is an eigenvalue of \( A \) and \( A^T \), and \( e \) is a corresponding eigenvector. Since \( e > 0 \), actually \( s = \lambda(A) = \lambda(A^T) \), see [4, Theorem 8.3.4]. Because

\[
A^T A e = A^T s e = s A^T e = s^2 e,
\]

we similarly see that \( s^2 = \lambda(A^T A) = \|A\|^2 \).

**Corollary 2.** If \( x = (x_0, \ldots, x_{n-1}) \geq 0 \), then

\[
\|C(x)\| = x_0 + \cdots + x_{n-1}.
\]

In order to apply this corollary in the case \( x = h \), we must compute \( h_0 + \cdots + h_{n-1} \).
Lemma 1. If \( p + q \neq 1 \), then
\[
h_0 + \cdots + h_{n-1} = \frac{h_n + qh_{n-1} + (p-1)a - b}{p + q - 1}.
\] (3)

If \( p + q = 1 \) and \( p \neq 2 \), then
\[
h_0 + \cdots + h_{n-1} = \frac{qh_{n-1} + (n-1)(qa + b) + a}{q + 1}.
\] (4)

If \( p = 2 \) and \( q = -1 \), then
\[
h_0 + \cdots + h_{n-1} = n \frac{h_{n-1} + a}{2}.
\] (5)

Proof. Claim (3) is equivalent to [3, Equation (3.5)] and to [8, Lemma 5(1)]. Claim (4) is equivalent to [8, Lemma 5(2)]. Claim (5) is trivial, because the sequence \((h_n)\) is arithmetic.

We have now proved the following theorem.

Theorem 3. If \( h \geq 0 \), then
\[
\|C(h)\| = h_0 + \cdots + h_{n-1},
\]
where \( h_0 + \cdots + h_{n-1} \) is as in Lemma 1.

4. Generalization of Theorem 3

Can the assumption \( h \geq 0 \) be weakened? Again, we begin by taking a more general viewpoint. For \( m \in \mathbb{Z} \), we set
\[
m_n = m - \left\lfloor \frac{m}{n} \right\rfloor n.
\]

Theorem 4. Let \( x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \). If
\[
\sum_{i=0}^{n-1} x_i x_{(i+j-1)n} \geq 0
\]
for all \( j = 1, \ldots, n \), then
\[
\|C(x)\| = |x_0 + \cdots + x_{n-1}|.
\] (6)
Proof. Write $B = (b_{ij}) = C(x)^T C(x)$. Letting $c_1, \ldots, c_n$ to denote the column vectors of $C(x)$, we have

$$b_{1j} = c_1 \cdot c_j = \sum_{i=0}^{n-1} x_i x_{(i+j-1)_n}$$

for all $j = 1, \ldots, n$. So, the first row of $B$ is nonnegative. Summing its elements gives us

$$r_1 = \sum_{j=1}^{n} \sum_{i=0}^{n-1} x_i x_{(i+j-1)_n} = \sum_{i=0}^{n-1} x_i \sum_{j=1}^{n} x_{(i+j-1)_n} = \left( \sum_{i=0}^{n-1} x_i \right)^2.$$ 

The last equation follows from the fact that

$$\{i_n, \ldots, (i+n-1)_n\} = \{0, \ldots, n-1\}$$

for all $i = 0, \ldots, n-1$.

A simple modification of the above reasoning applies to all rows of $B$. Consequently, $B \geq O$ with row sums

$$r_1 = \cdots = r_n = \left( \sum_{i=0}^{n-1} x_i \right)^2.$$ 

Since $B$ is symmetric, every of its column sums has this value, too. Applying Theorem 2 to $B$, we therefore obtain

$$\|C(x)\|^2 = \lambda(B) = \left( \sum_{i=0}^{n-1} x_i \right)^2,$$

and (6) follows.

Corollary 3. If

$$\sum_{i=0}^{n-1} h_i h_{(i+j-1)_n} \geq 0$$

for all $j = 1, \ldots, n$, then

$$\|C(h)\| = |h_0 + \cdots + h_{n-1}|,$$

where $h_0 + \cdots + h_{n-1}$ is as in Lemma 1.
5. Concluding remarks

In Section 2, we saw that, in the previous literature, \( \| C(h) \| \) is computed under various assumptions on \( h \). For example, in [6], the Horadam numbers were involved requiring that \( a \geq 0, b = 1 \) and \( p, q \geq 1 \). We assumed first only that \( h \geq 0 \), and then, even more generally, that \( (7) \) holds. As byproducts, Corollary 2 and Theorem 4 provided us with the corresponding results on \( \| C(x) \| \), too.

We also mention that Yazlik and Taskara [9] defined the notion of a generalized \( k \)-Horadam sequence \( (H_{k,n})_{n \in \mathbb{N}} \). In fact, Liu [8] ended up with (2) by studying a circulant matrix corresponding to such a sequence. However, since \( k \) is fixed in [9, Definition 1], this sequence is nothing but an ordinary Horadam sequence \( (h_n) = (h_n(a, b; p, q)) \) with \( p = f(k) \) and \( q = g(k) \).

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