Highest Trees of Random Mappings

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Abstract. We prove the exact asymptotic $1 - \left( \frac{2\pi}{3} - \frac{827}{288\pi} + o(1) \right) / \sqrt{n}$ for the probability that the underlying graph of a random mapping of $n$ elements possesses a unique highest tree. The property of having a unique highest tree plays a crucial role in the solution of the famous Road Colouring Problem [8] as well as its generalization in the proof of the author's result about the probability of being synchronizable for a random automaton [1].

1 Preliminaries

Given an integer $n > 0$, by the probability space $\Sigma_n$ of random mappings of $n$ elements we mean the set of all $n^n$ mappings from $Q = \{1, 2, \ldots, n\}$ to itself with the uniform probability distribution. Each mapping $f \in \Sigma_n$ can be represented via the directed graph $g(f)$ with constant outdegree 1. The graph $g(f)$ has the vertex set $Q$ and the edge set $E = \{(p, f(p)) \mid p \in Q\}$. Since this is a one-to-one correspondence, we identify $\Sigma_n$ with the probability space of random digraphs with $n$ vertices and constant outdegree 1.

Each digraph having common outdegree 1 consists of cycles and trees rooted at these cycles. Let $T$ be a highest tree of $g$ and $h$ be the height of a second by height tree. Let us call the crown of $g$ the (probably empty) forest consisting of all vertices of height at least $h + 1$ in $T$. A digraph $g$ on Figure 1 has the unique highest tree rooted at state 2. The height of a second by height tree is 2 whence the crown of $g$ is the forest consisting of the states 7, 8, 9, 14 of height at least 3.

Denote $\rho = \frac{1}{2\pi} \left( \frac{4\pi^2}{3} - \frac{827}{288\pi} \right) = \frac{2\pi}{3} - \frac{827}{288\pi}$. Our main results are as follows.

Theorem 1. Let $g \in \Sigma_n$ be a random digraph and $H$ be the crown of $g$ having $r$ roots. Then $|H| > 2r > 0$ with probability $1 - \Theta(1/\sqrt{n})$, in particular, a highest tree is unique and higher than all other trees of $g$ by 2 with probability $1 - O(1/\sqrt{n})$.

Theorem 2. The probability for random mapping on $n$ elements of having exactly two highest trees is $\frac{1}{\sqrt{n}}(1 + o(1))$.

Corollary 1. The probability that the underlying graph of a random mapping of $n$ states has a unique highest tree is $1 - \frac{1}{\sqrt{n}}(1 + o(1))$. 
In order to apply aforementioned results in [1], we need an easy generalization of Theorem 1. Given a digraph \( g \in \Sigma_n \) and an integer \( c > 0 \), let us call a \( c \)-branch of \( g \) any sub-tree of a tree of \( g \) with the root of height \( c \) in \( g \). For instance, the original trees are 0-branches. Let \( T \) be a highest \( c \)-branch of \( g \) and \( h \) be the height of the second by height \( c \)-branch. Let us call the \( c \)-crown of \( g \) the forest consisting of all the vertices of height at least \( h + 1 \) in \( T \). For example, the digraph \( g \) presented on Figure 1 has two highest 1-branches rooted in states 6,12. Without the state 14, the digraph \( g \) would have the unique highest 1-branch having the state 8 as its 1-crown.

**Theorem 3.** Let \( g \in \Sigma_n \) be a random digraph, \( c > 0 \), and \( H \) be the \( c \)-crown of \( g \) having \( r \) roots. Then \( |H| > 2r > 0 \) with probability \( 1 - \Theta(1/\sqrt{n}) \), in particular, a highest \( c \)-branch is unique and higher than all other \( c \)-branches of \( g \) by \( 2 \) with probability \( 1 - O(1/\sqrt{n}) \).

![Fig. 1. A digraph with a one cycle and a unique highest tree.](image)

We start from the proof of Theorem 2 and obtain the rest as consequences of the proof. To prove Theorem 2 we use the following scheme. The probability distributions (p.d.) on random digraphs from \( \Sigma_n \) having \( N \) cycle vertices can be considered as the p.d. on the set of random forests \( F_{n,N} \) with \( N \) roots and \( n \) vertices averaged by the p.d. of the number \( N \) of the cyclic states of random mappings. In its turn p.d. on random forests can be considered as p.d. on critical Galton-Watson branching processes under a condition on the total number of particles in the process. This idea of using Galton-Watson branching processes plays a crucial role in probability analysis of such combinatorial objects like random trees, forests and mappings. For an introduction to this theory, we refer the reader to [4, Section 2].

Unfortunately, in such a reduction there is an obstacle of using the theory of Galton-Watson branching processes out of the main range, namely, formulas become too rough when the number of cycle vertices is “big” and the height of random forests is “small” with respect to \( n \).

Our proof is organized as follows. First, in Section 2 we represent p.d. on mappings via p.d. on forests averaged by the number of cycle vertices \( N \), and then reduce the range for \( N \). Next, in Section 3 we use the standard representation of p.d. on random forests via branching processes. Using this representation and an independent approach, in Section 4 we reduce the range for the height of the
forests with respect to both \( n \) and \( N \). Then in Section \( 5 \) we present a theorem concerning branching processes for our case in the main range of parameters, and based on this theorem and results from the previous sections, we prove Theorem \( 2 \). Finally, in Section \( 6 \) we get remained results as consequences of the Theorem \( 2 \) proof.

2 Number of the Cyclic Vertices

Given a digraph \( g \in \Sigma_n \), denote by \( \lambda(g) \) the number of the cycle vertices and by \( ft(g) \in F_{n-\lambda(g),\lambda(g)} \) its forest, that is, the union of the trees of \( g \). Given \( d \geq 1 \), denote by \( B_n^d \) the set of digraphs \( g \in \Sigma_n \) having exactly \( d + 1 \) highest trees, by \( B_n \) the set of digraphs having at least two highest trees and by \( B_0^d \) the set of digraphs which do not satisfy Theorem \( 1 \). Given \( 0 < t < n, d \geq 0 \), denote by \( B_n^d, t \) the subset of mappings from \( B_n^d \) whose second by height tree has height \( t \). Given \( 0 < N \leq n \), denote by \( B_{n,N,t}^d \subseteq F_{n,N} \) the set of forests of digraphs from \( B_{n,t}^d \) with \( N \) cycle vertices. Due to the definitions, we have

\[
P_{g \in \Sigma_n} (B_n^d) = \sum_{t=0}^{n-1} P(B_n^d, t).
\]

Since the definition of \( B_n^d \) depends only on the forest of a digraph, we also have

\[
P_{g \in \Sigma_n} (B_n^d) = \sum_{N=0}^{n} P_{g \in \Sigma_n} (\lambda(g) = N) P(B_n^d, N, t). \tag{1}
\]

The following formula is well known (see e.g., \cite{3} Lemma 3, Section 3).

\[
P(\lambda(n) = N) = \frac{N(n-1)!}{n^N (n-N)!}. \tag{2}
\]

The following corollary along with \( \text{(1)} \) allows us to reduce the range we have to consider for the cyclic vertices number.

**Lemma 1** (see Sec. \( 7 \) for the proof).

\[
P(\sqrt{n} < \lambda(n) < 4\sqrt{n \ln n}) = 1 - o(1/\sqrt{n}).
\]

The following corollary easily follows from the proof of Lemma \( \text{(1)} \).

**Corollary 2.**

\[
P_{g \in \Sigma_n} (\lambda(n) = N) = \frac{ze^{-z^2/2}}{\sqrt{n}} (1 + o(1)),
\]

where \( z(N, n) = N/\sqrt{n} \) varies in the range \((n^{-0.25}, 4\sqrt{\ln n})\).
3 Random Forests via Branching Processes

We use the standard representation of random forests distribution via critical Galton-Watson branching processes following [3]. Denote by $G^N$ the probability space of all critical Galton-Watson branching processes with the offspring p.d. $(e^{-1}/r!) r \geq 0$ starting with $N$ founding ancestors (or, equivalently, consisting of $N$ independent branching processes $\mu^1, \mu^2, \ldots, \mu^N$ from $G^1$) where each particle independently has probability $e^{-1}/r!$ of producing $r$ offsprings in the next generation. The idea of such representation is that after a branching process with $N$ founding ancestors stabilizes, we obtain a forest having $N$ trees corresponding to the processes $\mu^1, \mu^2, \ldots, \mu^N$ and the induced p.d. on the set of random forests is uniform (see [3, Section 2] for details).

Following [3], denote by $F(z)$ the generating function for the offspring p.d.

$$F(z) = \sum_{r=0}^{+\infty} \frac{z^r}{er!} = e^{-1}. \quad (3)$$

Given $\mu \in G^N$ and $t \geq 0$, $\mu(t)$ (resp. $\nu(t)$) denote the number of particles in the process $\mu$ in generation $t$ (resp. less than $t$) and by $\nu$ denote the number of particles in $\mu$ after the extinction moment of $\mu$, that is, $\nu = \nu(\tau(\mu))$, where $\tau(\mu)$ is the index of the first empty generation of $\mu$. Notice that the height of the induced forest exceeds the extinction moment of $\mu$ by one.

For processes with multiple founding ancestors, the subscript is used to denote the number of founding ancestors (0-generation particles) and the superscript is used to denote the index of the ancestor in 0 generation. In particular, $\nu_N \in G^N$ denotes the total number of particles in Galton-Watson process starting from $N$ founding ancestors and $\nu^i$ denotes the number of particles of the $i$-th founding ancestor. By the definitions,

$$\nu_N = \sum_{i=1}^{N} \nu^i, \quad \nu_{N,t} = \nu_N(t) = \sum_{i=1}^{N} \nu^i(t)$$

where all $\nu^i$ are independent.

Now we determine the conditions on branching processes corresponding to the sets of digraphs $B_{n,N,t}^d$. To simplify notations, suppose that $\mu^1, \mu^2, \ldots$ are ordered by descending of extinction moments $\tau^1, \tau^2, \ldots, \tau$. Given $N > 0, t > 0$, denote by $A_{N,t}^0$ the set of processes from $G^N$ such that

$$\mu^1(t) > 0, \quad \nu^1 - \nu^1(t) \leq \mu^1(t), \quad \tau^2 = t + 1, \quad \mu^j(t + 1) = 0 \text{ for } j \geq 3 \quad (4)$$

and for $d > 0$ by $A_{N,t,r}^d$ the set of processes from $G^N$ such that

$$\mu^1(t) > 0, \quad \mu^1(t+r) = 0, \quad \tau^i = t+1, \quad \tau^j \leq t \text{ for } 2 \leq i \leq d+1, j > d+1. \quad (5)$$

Also denote $A_{N,t}^d = A_{N,t,1}^d$ and $A_{N,t} = \bigcup_{d=1}^{N-1} A_{N,t}^d$.

\footnote{For calculations, we cannot assume this and the order matters.}
For \(d \in \{0, 1, \ldots, N - 1\}\), due to the definitions of \(B_{\nu, \lambda}^d\) and \(A_{\lambda}^d\), the standard reduction to branching processes yields

\[
P(B_{\nu, \lambda}^d) = P(A_{\lambda}^d | \nu = \nu_N = n + N) = \frac{P(A_{\lambda}^d; \nu = n + N)}{P(\nu = n + N)},
\]

whenever \(P(\nu = n + N) > 0\). The following lemma is given in [3].

**Lemma 2 (Lemma 6, Section 3 [3]).** \(P(\nu = n + N) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}(1 + o(1))\), for \(0 < z_0 \leq \frac{N}{\sqrt{n}} \leq z_1\).

The proof is based on the following equation

\[
P(\nu = n + N) = \sqrt{2\pi B(n + N)} e^{-n^2/(2B(n + N))}(1 + o(1)),
\]

where \(B = F''(1) = 1\) in our case (see (3)). In its turn, this equation is based on [3, Theorem 2, Section 1] which holds for any \(n, N \to +\infty\). Hence from (7) for \(N = o(n)\), we have the following corollary.

**Corollary 3.** \(P(\nu = n + N) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}(1 + o(1))\), where \(z(n) = \frac{N}{\sqrt{n}}\) varies in the range \((n^{-0.25}, 4\sqrt{\ln n})\).

By Corollary 3, we have

\[
P_{\nu \in \Sigma_{n}^t}(B_{\nu, \lambda}^d) = \frac{P(\nu = n; A_{\lambda}^d)}{P(\nu = n)} = (1 + o(1))n \frac{e^{z^2/2}}{z} P(\nu = n; A_{\lambda}^d),
\]

for \(z = \frac{N}{\sqrt{n}} \in (n^{-0.25}, 4\sqrt{\ln n})\) and by Lemma 1 we have

\[
P_{\nu \in \Sigma_{n}^t}(B_{\nu, \lambda}^d) = \sum_{N = \frac{z}{\sqrt{n}}} \frac{4\sqrt{\ln n}}{\sqrt{n}} P(\lambda(n) = N)P(B_{\nu, \lambda}^d)\varepsilon_1(n),
\]

where \(\varepsilon_1(n) = (1 + o(1/\sqrt{n}))\).

Putting together (9), (8) and Corollary 2 we have

\[
P_{\nu \in \Sigma_{n}^t}(B_{\nu, \lambda}^d) = \sum_{N = \frac{z}{\sqrt{n}}} \frac{4\sqrt{\ln n}}{\sqrt{n}} \varepsilon_1(n) \frac{e^{-x^2/2}}{\sqrt{n}} P_{\nu \in \Sigma_{n}^t}(B_{\nu, \lambda}^d) = \varepsilon_1(n) \sqrt{n} \sum_{N = \frac{z}{\sqrt{n}}} P(\nu = n; A_{\lambda}^d).
\]

Thus in order to prove Theorem 2 it remains to show that

\[
\sum_{t=1}^{n} \sum_{N = \frac{z}{\sqrt{n}}} P(\nu = n; A_{\lambda}^d) = \frac{p}{n}(1 + o(1)).
\]
4 Height of Random Forests

Now we want to reduce the range we have to consider for the parameter $t$. For this purpose, we use the following basic equation (see e.g. [3]):

$$P(\mu_1(t) = 0) = 1 - \frac{2}{t}(1 + o(1)).$$

(11)

Lemma 3. If $t > 0, N > 2t \ln n$, then $P(\nu_n = n; A_{N,t}) = o(\frac{1}{n^3})$.

Proof. Due to the definition of $A_{N,t}$, we have

$$P(\nu_n = n; A_{N,t}) \leq P(A_{N,t}) \leq P(\mu_1(t) = 0)^{N-2}.$$ 

It follows from (11) that there exists $t_0 > 0$ such that $P(\mu_1(t) = 0) \leq 1 - \frac{1.99}{t}$ for $t > t_0$. Denote $p_0 = \min_{j \leq t_0} P(\mu_1(j) = 0)$. Clearly $p_0 < 1$. Hence for $t \leq t_0$, the lemma trivially holds. For $t > t_0$, we have

$$P(\nu_n = n; A_{N,t}) \leq O(1)(1 - \frac{1.99}{t})^{2t \ln n} \leq O(1)e^{-3.8 \ln n} \leq o(\frac{1}{n^3}).$$

Following [7] denote by $T_{n,h}$ the number of mappings of $n$ element set of height at most $h$, by $L_j(x)$ the $j$-th iteration of the function $xe^x$ and by $\rho_j$ the unique real positive solution of the equation $L_j(x) = 1$.

Corollary 4 ([7], [3]). $T_{n,h} = n!\rho_h^{-n}(1 + \rho_1 + \rho_1\rho_2 + \cdots + \rho_1\rho_2 \cdots \rho_h)^{-1}$.

In another work, Grusho [2, Lemma 4] found the following asymptotic for $\rho_m$ where $m/n^3 \to C > 0$ for $0 < \beta < \frac{1}{2}$:

$$\rho_m = \frac{1}{e} \left(1 + \frac{2}{m^2} \left(\frac{\pi}{2}\right)^2 + o\left(\frac{1}{m^2}\right)\right).$$

(12)

Lemma 4. For each $\epsilon > 0$, $\sum_{t=1}^{n^{0.5-\epsilon}} \sum_{N=1}^{n} P_{g \in \Sigma_n}(B_{n,t}) = O(\frac{1}{\sqrt{n}})$.

Proof. It follows from Lemma [11] and Lemma [3] that

$$\sum_{t=1}^{\sqrt{n}(2\ln n)} \sum_{N=1}^{n} P(B_{n,N,t}) = O\left(\frac{1}{\sqrt{n}}\right).$$

Using that $\rho_m$ is monotonic descending, for $n^{0.24} < t < n^{0.5-\epsilon}$, due to (12) and Corollary [3] for $m = [n^{(1-\epsilon)/2}]$ and $n$ big enough (such that $t + r \leq m$), we have

$$P(B_{n,t}) \leq \frac{T_{n,t+r}}{n^r} \leq \frac{n!(\frac{1}{e} (1 + \frac{2}{m^2} (\frac{\pi}{2})^2 + o(\frac{1}{m^2}))))^{-n}}{n^r} \leq \frac{O(1)\sqrt{2\pi n}\left(\frac{1}{e} (1 + \frac{2}{m^2} (\frac{\pi}{2})^2 + o(\frac{1}{m^2}))\right)^{-n}}{e^n} \leq O(1)\sqrt{n} e^{-\frac{m}{e^2}(1+o(1))} \leq O(1)\sqrt{n} e^{-n(1+o(1))} = o\left(\frac{1}{n^3}\right).$$

2 The Corollary [4] is given for constant $h$ but it is not used anyhow in the proof.
5 The Main Range

We write \( \phi_1(n) \sim \phi_2(n) \) if \( \phi_1(n) = (1 + o(1)) \phi_2(n) \). The following theorem is based on an analysis of [5] presented in Section 7.

**Theorem 4.** Denote \( \beta = \beta(\theta, n) = \sqrt{-2 i \theta / n} \); Then for \( N \in (\sqrt{n}, 4 \sqrt{n \ln n}) \), \( t \in (n^{0.49}, n) \), \( N/t \leq 2 \ln n \), we have

\[
P(\nu_N = n; A_{N,t}) \sim \frac{N^2}{4 \pi n} \int_{-\frac{n \ln 4}{t}}^{\frac{n \ln 4}{t}} e^{-i \theta} \beta^4 e^{-2i \beta} \frac{e^{-N \beta^4 + i t \beta}}{1 - e^{-i t \beta}} d\theta.
\]

where \( |\epsilon_3(t, u)| \leq O(1) \left( \ln n \left( \frac{\sqrt{u} + \frac{1}{\sqrt{t}}}{} \right)^2 \right) \); and for any constant \( r > 0 \)

\[
P(\nu_N = n; A_{N,t,r}) \leq O(1) P(\nu_N = n; A_{N,t}).
\]

**Proof.** Let \( f(z), f_t(z) \) be the probability generating functions (pgf) of \( \nu \) and \( \nu; \mu(t) = 0 \) resp., that is,

\[
f_t(z) = \sum_{n=0}^{+\infty} P(\nu(t) = n, \mu(t) = 0) z^n, \quad f(z) = \sum_{n=0}^{+\infty} P(\nu = n) z^n.
\]

First let us make the following remarks.

**Remark 1.** For given original indices of \( \mu^1, \mu^2 \), the pgf of \( (\nu_N; A_{N,t,r}) \) (and \( P(\nu_N = n; A_{N,t}) \)) is equal to \( f_t^{N-2}(z) \phi_1(z) \phi_2(z) \), where \( \phi_1, \phi_2 \) are the pgfs of \( (\nu^1; A_{N,t,r}) \), \( (\nu^2; A_{N,t,r}) \) (for \( r = 1 \)) resp.

**Remark 2.** For any \( r \geq 1 \) there are at most \( N(N-1) \) choices for the indices of \( \nu^1 \) and \( \nu^2 \) for \( \mu \in A_{N,t,r} \) and there are exactly \( \frac{N(N-1)}{2} \) choices for \( \nu^1 \) and \( \nu^2 \) for \( \mu \in A_{N,t}^1 \).

Notice that if \( g(z) \) is the generating function of a random variable \( X \) taking positive integer values, then \( \chi(\theta) = g(e^{i \theta}) \) is the characteristic function of \( X \).

Due to the inversion formula, we have

\[
P(\nu_N = n; A_{N,t}) \sim \frac{N(N-1)}{4\pi} \int_{-\pi}^{\pi} e^{-i \theta n} f_t^{N-2}(e^{i \theta})(f_{t+1}(e^{i \theta}) - f_t(e^{i \theta}))^2 d\theta \sim \frac{N^2}{4 \pi n} \int_{-\pi}^{\pi} e^{-i \theta n} f_t^N(e^{i \theta})(f_{t+1}(e^{i \theta}) - f_t(e^{i \theta}))^2 d\theta.
\]

Notice that

\[
|f_{t+r}(e^{i \theta}) - f_t(e^{i \theta})| \leq P(t \leq \tau(\mu) \leq t + r) \leq \frac{O(1)}{t^2}.
\]

(16)
We consider three ranges for \( \theta \). Due to Lemma 9, there are positive constants \( \varepsilon, \delta, c_1, c_2 \) such that

\[
|f_t(e^{i\frac{\theta}{n}})| \leq 1 - \delta_2 \sqrt{\frac{|\theta|}{n}} + c_1 t^{-1} e^{-c_2 t \sqrt{n}},
\]

for \( |\theta| \leq \varepsilon \). Denote \( A = \frac{n \ln^4 n}{\varepsilon^2} \). Then \( \delta_2 \sqrt{\frac{|\theta|}{n}} \geq 2c_1 t^{-1} e^{-c_2 t \sqrt{n}} \) for \( |\theta| \geq A \) and \( n \) big enough. Hence using (17) and (16), we have that

\[
\left| \int_A^{\pi A} e^{-i\theta} f_t^N (e^{i \frac{\theta}{n}}) (f_{t+\varepsilon} (e^{i \frac{\theta}{n}})) - f_t (e^{i \frac{\theta}{n}}) \right| \leq 1 \sqrt{\frac{\varepsilon}{A^2}} \left[ 1 - 0.5 \delta_2 \sqrt{\frac{|\theta|}{n}} \right] e^{-c_3 \sqrt{n} \sqrt{\theta}} \left( 1 + c_3 \sqrt{\frac{\theta}{n}} \right).
\]  

Due to Theorem 4, it remains to compute the sum for all integers \( t, N \) such that \( t \in (n^{0.49}, n), N \in (\sqrt{n}, min(4 \sqrt{n} \ln n, 2 \ln n)) \) of

\[
\phi(N) = \int_{-\frac{n \ln^4 n}{\varepsilon^2 n}}^{\frac{n \ln^4 n}{\varepsilon^2 n}} e^{-i\theta} N^2 \beta^4 e^{-2t\beta} e^{-N^{\frac{1}{2} + \varepsilon} \frac{1}{4\pi(1 - e^{-t\beta})^4} (1 + \varepsilon_3(t, \theta, n))} d\theta.
\]
First let us sum it up by \(N\). Since \(\phi(N)\) is smooth and positive for \(N > 0\), we have that
\[
\sum_{N = \sqrt{\pi}}^{2t\ln n} \phi(N) = \int_{\sqrt{\pi}}^{2t\ln n} \phi(z) \, dz + \sum_{N = \sqrt{\pi}}^{2t\ln n} \left( \phi(N) - \int_{N}^{N+1} \phi(z) \, dz \right) =
\]
\[
= \int_{\sqrt{\pi}}^{2t\ln n} \phi(z) \, dz + \int_{\sqrt{\pi}}^{2t\ln n} \phi'(N + \delta'_N) =
\]
\[
= \int_{\sqrt{\pi}}^{2t\ln n} \phi(z) \, dz + \int_{\sqrt{\pi}}^{2t\ln n} \phi'(z) \, dz + \sum_{N = \sqrt{\pi}}^{2t\ln n} \phi''(N) =
\]
\[
= \left( \int_{\sqrt{\pi}}^{2t\ln n} \phi(z) \, dz + \phi(2t\ln n) - \phi(\sqrt{\pi n}) \right) + \sum_{N = \sqrt{\pi}}^{2t\ln n} \phi''(N) \sim
\]
\[
\sim \int_{\sqrt{\pi}}^{2t\ln n} \phi(z) \, dz + \sum_{N = \sqrt{\pi}}^{2t\ln n} \phi''(N),
\]
(21)
where \(\delta'_N \in (0, 1), N \leq N_\delta \leq N + 1\).

Denote \(u(\theta) = t\sqrt{\frac{\ln n}{n}}\), \(\alpha(\theta) = \frac{\beta t}{1 - e^{-\theta}}\). The following lemma follows from [3][Lemma 5, Section 3] but we reproduce it here for completeness.

**Lemma 5.** For some positive constant \(c\), \(\text{Re}(\alpha(\theta)) \geq c \frac{u(\theta)}{t}\).

**Proof.** One can check that
\[
t \frac{\text{Re}(\alpha)}{u} = \frac{1 - e^{-2u} + 2e^{-u}}{1 - 2e^{-u}\cos(u) + e^{-2u}}.
\]
It follows from here that \(\phi(u) = \frac{\text{Re}(\alpha)}{u}\) as the function of \(u\) is positive, continuous and \(\lim_{u \to 0} \phi(u) = +\infty\) and \(\lim_{u \to 0} \phi(u) = 1\). The lemma follows.

From the other hand, we have that
\[
|\alpha(\theta)| \leq \frac{u(1 + e^{-u})}{t(1 - e^{-u})} \leq \frac{3(u + 1)}{t}.
\]
(22)
The latter inequality follows from
\[
1 \leq \frac{x}{1 - e^{-x}} \leq 3(x + 1)
\]
(23) for each \(x \geq 0\). Since for \(|\theta| \leq \frac{n \ln n}{t^2}\),
\[
|e_3(t, \frac{\theta}{n})| \leq O(1) \left( \ln n \left( \sqrt{\frac{\theta}{n}} t + \frac{1}{\sqrt{t}} \right) \right)^2 \leq O(1),
\]

for the second term of (21), we have

\[
\left| \sum_{N = \sqrt{n}}^{2t \ln n} \phi''(N_\delta) \right| \leq \leq O(1) \sum_{N = \sqrt{n}}^{2t \ln n} \int \frac{\ln^4 n}{t^4} \left| (-2N_\delta - N_\delta^2 \alpha) + 2 - 2N_\delta \alpha \right| \frac{\beta^4 e^{-2t \beta}}{(1 - e^{-t \beta})^4} e^{-N_\delta \alpha} \, d\theta \leq \leq O(1) \sum_{N = \sqrt{n}}^{2t \ln n} \int \frac{\ln^4 n}{t^4} \left| (N_\delta \alpha - 2)^2 \frac{\beta^4 e^{-2t \beta}}{(1 - e^{-t \beta})^4} e^{-N_\delta \alpha} \right| \, d\theta \leq \leq O(1) \sum_{N = \sqrt{n}}^{2t \ln n} \int \frac{\ln^4 n}{t^4} \left| (\ln n(u + 1) + 2)^2 \frac{\beta^4 e^{-2t \beta}}{(1 - e^{-t \beta})^4} e^{-N_\delta \alpha} \right| \, d\theta \leq [\text{Lemma 5}]
\]

\[
\leq O(1) \ln^2 n \sum_{N = \sqrt{n}}^{2t \ln n} \int \frac{\ln^4 n}{t^4} \left| (u + 1)^2 u^4 e^{-2u} \frac{\beta^4 e^{-2t \beta}}{(1 - e^{-t \beta})^4} e^{-N_\delta \alpha} \right| \, d\theta \leq \left[ \frac{u}{1 - e^{-u}} \right] \leq u + 1
\]

\[
\leq O(1) \ln^2 n \sum_{N = \sqrt{n}}^{2t \ln n} \int \frac{\ln^4 n}{t^4} \left| (u + 1)^6 \frac{\beta^4 e^{-2t \beta}}{(1 - e^{-t \beta})^4} e^{-N_\delta \alpha} \right| \, d\theta \leq [e^{-N_\delta \alpha} \leq 1]
\]

\[
\leq O(1) \ln^3 n \int \frac{\ln^4 n}{t^4} \left( u + 1 \right)^6 \frac{\beta^4 e^{-2t \beta}}{(1 - e^{-t \beta})^4} e^{-N_\delta \alpha} \, d\theta = [d\theta = \frac{2nu}{t^2} du] = \leq O(1) \ln^3 n \int \frac{\ln^2 n}{t^5} n(u + 1)^6 e^{-2u} du \leq O(1) n \ln^3 n \frac{\ln n}{t^5}.
\]

(24)

Summing up \(O(1) n \ln^3 n \frac{\ln n}{t^5}\) by \(t \in (n^{0.49}, n)\), we get that it is bounded by \(O(\frac{1}{n^{0.50}})\).

For \(k > 0\) and \(\alpha\) having the positive real part, we have

\[
\int_0^{+\infty} x^k e^{-\alpha x} \, dx = \frac{1}{\alpha^{k+1}} \int_0^{+\infty} x^k e^{-\alpha x} \, dx = \frac{e_k}{\alpha^{k+1}}.
\]

(25)

where \(e_k\) is a constant which depends on \(k\) and for \(k = 2\)

\[
\int_0^{+\infty} x^2 e^{-\alpha x} \, dx = e^{-x^2/\alpha^{k+1}} \frac{2 x + 2}{\alpha^{k+1}} \bigg|_0^{+\infty} = \frac{2}{\alpha^{k+1}}.
\]

(26)
For the first term of (21), by (25), we have

\[
\int_{\sqrt{n}}^{2t \ln n} \phi(z) dz \sim \int_0^{+\infty} \phi(z) dz \sim \\
\sim \int_{\frac{\ln n}{t^{1/2}}}^{\frac{\ln n}{t^{1/2}}} e^{-i\theta} \frac{\beta^4 e^{-2i\beta}}{2\pi(1-e^{-i\beta})^4} \left( 1 + \epsilon_3(t, \frac{\theta}{n}) \right) d\theta = \\
= \int_{-\frac{\ln n}{t^{1/2}}}^{\frac{\ln n}{t^{1/2}}} e^{-i\theta} \frac{\beta e^{-2i\beta}(1+e^{-i\beta})^3}{2\pi(1-e^{-i\beta})} \left( 1 + \epsilon_3(t, \frac{\theta}{n}) \right) d\theta \sim \\
\sim \int_{-\frac{\ln n}{t^{1/2}}}^{\frac{\ln n}{t^{1/2}}} e^{-i\theta} \frac{\beta e^{-2i\beta}(1+e^{-i\beta})^3}{2\pi(1-e^{-i\beta})} d\theta + \\
+ \int_{-\frac{\ln n}{t^{1/2}}}^{\frac{\ln n}{t^{1/2}}} e^{-i\theta} \frac{\beta e^{-2i\beta}(1+e^{-i\beta})^3}{2\pi(1-e^{-i\beta})} \epsilon_3(t, \frac{\theta}{n}) d\theta. \quad (27)
\]

Recall that \( u(\theta) = t \sqrt{\frac{\theta}{n}} \). We have

\[
|\epsilon_3(t, \frac{\theta}{n})| \leq O(1) \left( \ln n \left( \sqrt{\frac{\theta}{n}} |t + \frac{1}{\sqrt{t}}| \right) \right)^2 \leq O(1) \frac{\ln^2 n}{t} (|u| + 1)^2.
\]

Using that \( \text{Re}(t\beta) = u \), we can upper bound the second term of (27) as follows.

\[
O(1) \int_{-\frac{\ln n}{t^{1/2}}}^{\frac{\ln n}{t^{1/2}}} \left| \frac{\beta e^{-2i\beta}(1+e^{-i\beta})^3}{(1-e^{-i\beta})} \epsilon_3(t, \frac{\theta}{n}) \right| d\theta \leq \\
\leq O(1) \int_{0}^{\ln^2 n} \left| \frac{ue^{-2u}(1+e^{-u})^3 \ln^2 n}{t(1-e^{-u})} (u+1)^2 \right| d\theta \leq [\theta = \frac{2nu}{t^2} du] \\
\leq O(n \ln^2 n) \int_{0}^{\ln^2 n} \left| \frac{u^2 e^{-2u}(1+e^{-u})^3}{t^4(1-e^{-u})} (u+1)^2 \right| du \leq \left[ \frac{u}{(1-e^{-u})} \leq u + 1 \right] \leq \\
\leq O\left( \frac{n \ln^2 n}{t^4} \right) \int_{0}^{\ln^2 n} (u+1)^4 e^{-2u}(1+e^{-u})^3 du \leq O\left( \frac{n \ln^2 n}{t^4} \right), \quad (28)
\]

The sum of \( \frac{n \ln^2 n}{t^4} \) by \( t \in (n^{0.49}, n) \) is upper bounded by

\[
n \ln^2 n \int_{n^{0.49}}^{n} \frac{1}{t^4} dt \leq O\left( \frac{\ln^2 n}{n^{0.49}} \right).
\]
Let us now consider the first term of (27) for positive $\theta$.

\[
\int_0^{\pi} e^{-i\theta} \frac{e^{-2i\beta(1 + e^{-t\beta})^3}}{2\pi(1 - e^{-t\beta})} d\theta = \\
= \int_0^{\pi} e^{-i\theta} \frac{(1 - i)u^2 e^{-2u(1 - i)(1 + e^{-u(1 - i)})^3}}{2\pi t(1 - e^{-u(1 - i)})} d\theta = [d\theta = \frac{2nu}{t^2} du] \\
= \frac{\pi}{t^3} \int_0^{\pi} e^{-i\theta} (1 - i)u^2 e^{-2u(1 - i)(1 + e^{-u(1 - i)})^3} (1 - e^{-u(1 - i)}) \frac{du}{1 - e^{-u(1 - i)}} = \psi(t). \quad (29)
\]

Using the same trick as in (21) for $\psi(t)$, we have

\[
\sum_{t=n^{0.49}}^{n} \psi(t) \sim \int_{n^{0.49}}^{n} \psi(t) dt + \sum_{t=n^{0.49}}^{n} \psi''(t_\delta), 
\quad (30)
\]

where $t_\delta \in (t, t + 1)$.

For the second term of (30), we have

\[
\left| \sum_{t=n^{0.49}}^{n} \psi''(t_\delta) \right| \leq O(1)n \sum_{t=n^{0.49}}^{n} \int_{0}^{\ln^2 n} \left( \frac{\frac{1}{t^4} + \frac{nu}{t^6}}{t^3} \right) \frac{nu}{t^5} \left| \frac{(1 - i)u^2 e^{-2u(1 - i)(1 + e^{-u(1 - i)})^3}}{1 - e^{-u(1 - i)}} \right| du \leq \\
\leq O(1) \frac{n \ln^4 n}{t^5} \sum_{t=n^{0.49}}^{n} \int_{0}^{\ln^2 n} \left( \frac{n}{t^2} + \frac{n^2}{t^4} + 1 \right) u(u + 1)e^{-2u} du \leq \\
\leq O(1) \int_{n^{0.49}}^{n} \frac{n \ln^4 n}{t^5} \left( \frac{n}{t^2} + \frac{n^2}{t^4} + 1 \right) dt \leq \\
\leq O(1) \frac{n \ln^4 n}{t^4} \left( \frac{n}{t^2} + \frac{n^2}{t^4} + 1 \right) \left| n^{0.49} \right| = o(n^{0.91}). \quad (31)
\]
Finally, for the first term of (30), we have

\[
\int_{0.49}^{n} \psi(t) dt = [v = -\frac{1}{t^2}, dv = \frac{2dt}{t^3}] = \\
= n \int_{0}^{\ln n} e^{iu(v)} (1 - i)u^2 e^{-u(1-i)} (1 + e^{-u(1-i)})^3 du = \\
= \int_{0}^{\ln n} \frac{(1-i)u^2 e^{-u(1-i)} (1 + e^{-u(1-i)})^3}{2\pi(1 - e^{-u(1-i)})} du = [g = (1-i)u] \\
= \int_{0}^{\ln n} \frac{(1-i)u^2 e^{-u(1-i)} (1 + e^{-u(1-i)})^3}{2\pi(1 - e^{-u(1-i)})} du = A \\
= \frac{1 - i}{4\pi} \int_{0}^{\ln n} (1 + O(\frac{\ln n}{n}) - e^{(1-i)g/2}) ge^{-g(1+e^{-g})^3} \sum_{k=0}^{+\infty} e^{-kg} dg = B \\
= \frac{1 - i}{4\pi} \sum_{k=0}^{+\infty} \left( \frac{1}{(k+2)^2} + \frac{3}{(k+3)^2} + \frac{3}{(k+4)^2} + \frac{1}{(k+5)^2} \right) (1 + o(1)) \sim \\
\sim \frac{1 - i}{4\pi} \left( \frac{8\pi^2}{6} - \frac{7}{4} - \frac{9}{4} - \frac{1}{16} \right) = \frac{1 - i}{4\pi} \left( \frac{4\pi^2}{3} - \frac{827}{144} \right), \quad (32)
\]

where (A) follows from \( \frac{1}{1-g} = \sum_{k=0}^{+\infty} g^k \) and \( e^{(1-i)g/2} = 1 + O(\frac{\ln n}{n}) \) for \( |g| \leq 2 \ln^2 n \) and (B) follows from the equality \( \int_{0}^{\ln n} g e^{-\lambda g} = \frac{(g+1)}{\lambda^2} (1 - e^{-\lambda \ln^2 n}) \).

One can easily obtain for negative \( \theta \) the same expression with the factor \( 1+i \) instead of \( 1-i \). Thus we get that

\[
\sum_{i=1}^{n} \sum_{N=1}^{n} P(\nu_N = n; A_{N,t}^1) = \frac{\rho}{n} (1 + o(1)). \quad (33)
\]

Theorem 2 now follows from (33).

6 Conclusions

Notice that Theorem 1 corresponds to digraphs \( B_n \) which correspond to branching processes \( A_{N,t} \). Hence, the lower bound of Theorem 1 follows from the fact that \( A_{1,t}^1 \) implies \( A_{N,t} \). To prove the upper bound, we use inequality (14) of Theorem 4 and the following lemma.

**Lemma 6** (see Sec. 7 for the proof). There exists some constant \( r > 0 \), such that for each \( t > n^{0.49}, m > n^{0.9} \) the following inequality holds for \( n \) big enough.

\[
P(\nu = m; \nu - \nu(t) \leq \mu(t)) \leq \\
\leq O(1/n^{0.2} \ln n) + c_2 P(\nu = m(1 + o(1)); \mu(t) > 0; \mu(t+r) = 0).
\]
The above lemma allows to neglect the difference in the definition of $A_{N,t,r}$ and $A_{N,l,r}$ for proving the upper bound. The only thing we have to notice is that $P(\nu_N = n; A_{N,l,r})$ is continuous by $n$ in the main range of parameters, i.e. $P(\nu_N = n(1 + o(1)); A_{N,l,r}) \sim P(\nu_N = n; A_{N,l,r})$ (this easily follows from the proof). The precision of Lemma 6 is enough due to the reduction \(^4\) and Corollary 2.

In order to prove Corollary 1 consider the set of processes in $G^N$ having at least 3 highest trees. Then, we would get a factor \(\binom{N}{3} \left( \frac{\beta^2 e^{-t\beta}}{e^{-t\beta}} \right)^3\) instead of \(\binom{N}{3} \left( \frac{\beta^2 e^{-t\beta}}{e^{-t\beta}} \right)^2\). Since \(|\left( \frac{\beta^2 e^{-t\beta}}{e^{-t\beta}} \right)| \leq O(1/t^2)\), we would have the additional factor of order $\frac{N}{N^t}$. Due to Lemma 3 we consider $N \leq 2t \ln n$ whence $\frac{N}{N^t} = o(1)$. Corollary 1 now follows from Theorem 2.

Since the generating function of $\mu_N(1)$ is given by

$$\left( \sum_{j=0}^{+\infty} \frac{z^j}{e^j!} \right)^N = e^{N(z-1)} = \sum_{j=0}^{+\infty} \frac{z^j N^j}{j! e^N},$$

we get that $P(\mid \mu_N(1)/N - 1 \mid > 1/2) = o(1/N^5)$. Due to Lemma 1 this means that whp $\mu_N(1)$ is equal to $N$ up to the constant factor, and the Theorem 1 holds for 1-branches instead of the trees. By induction, one can easily generalize these arguments for any constant $c > 0$ and Theorem 3 follows.

7 Technical lemmas

**Lemma 1**. $P(\sqrt{n} < \lambda(n) < 4\sqrt{n \ln n}) = 1 - o(1/\sqrt{n})$.

**Proof.** For $N = n$, $P(\lambda(n) = N) = \frac{n!}{n^N} = O(\frac{1}{n^N})$. For $n \to +\infty$, $N < n$, by Stirling’s formula $k! = (\frac{k}{e})^k \sqrt{2\pi k}(1 + o(1))$, we have

$$P(\lambda(n) = N) = \frac{N(n-1)!}{n^N(n-N)!} = \frac{(1 + o(1))N((n-1)/e)^{n-1}\sqrt{2\pi n}}{n^N((n-N)/e)^{n-N}\sqrt{2\pi (n-N)}} =$$

$$= (1 + o(1))\frac{N\pi^{n-1}(1 - \frac{1}{n})^{n-1}}{\pi^{n-N}\pi^{n-N}(1 - \frac{1}{n})^{n-N}} \sqrt{\frac{n}{n-N}} =$$

$$= (1 + o(1))\frac{N\pi^{n-1}(1 - \frac{1}{n})^{n-1}}{\pi^{n-N}\pi^{n-N}(1 - \frac{1}{n})^{n-N}} \sqrt{\frac{n}{n-N}} = \frac{(1 + o(1))x}{e^{n(x+(1-x)\ln(1-x))\sqrt{1-x}}}, \quad (34)$$

where $x = \frac{N}{n}$. Denote $\phi(x) = x + (1-x)\ln(1-x)$ for $0 < x < 1$. Using Taylor’s expansion $\ln(1-x) = -x - \frac{x^2}{2} - O(x^3)$, we get that

$$\phi(x) = x + (x-1)(x + \frac{x^2}{2}) + O(x^3) = \frac{x^2}{2} + O(x^3). \quad (35)$$

Since $\phi'(x) = 1 - 1 - \ln 1 - x > 0$, $\phi(x)$ is growing. Hence by (35) for $x \geq 4\sqrt{\ln n}/n$,

$$n\phi(x) \leq n\phi(4\sqrt{\ln n}/n) = 8 \ln n(1 + o(1)). \quad (36)$$
Since also $\sqrt{1-x} \leq \sqrt{n}$ for $1 \leq N < n$, by (34) and (36), we get that

$$P(\lambda(n) > 4\sqrt{n \ln n}) \leq n\sqrt{ne^{-8\ln n}} = o(\frac{1}{n}).$$

For $N < 4\sqrt{n \ln n}$, $x = o(1)$ whence by (34) and (35), we have

$$P(\lambda(n) = N) = (1 + o(1))\frac{(1 + o(1))N}{n}e^{-\frac{N^2}{2n}}. \tag{37}$$

From here, we have also that

$$P(\lambda(n) < \sqrt{n}) \leq \sqrt{n}\frac{(1 + o(1))\sqrt{n}}{n} = o(\frac{1}{\sqrt{n}}).$$

The lemma follows.

**Lemma 6.** There exists some constant $r > 0$, such that for each $t > n^{0.49}, m > n^{0.9}$ the following inequality holds for $n$ big enough.

$$P(\nu = m; \nu - \nu(t) \leq \mu(t)) \leq O(1/n^{0.2\ln n}) + c_2P(\nu = m(1 + o(1)); \mu(t) > 0; \mu(t + r) = 0).$$

**Proof.** Since the distributions of $\nu(t)$ and $\mu(t + r)$ are determined by $\mu(t)$, for the left hand side, we have

$$P(\nu(t) = m; \nu - \nu(t) \leq \mu(t)) =$$

$$= \sum_{k=1}^{+\infty} P(\mu(t) = k)P(\nu(t) = m | \mu(t) = k)P(\nu - \nu(t) \leq \mu(t) | \mu(t) = k) =$$

$$= \sum_{k=1}^{+\infty} P(\mu(t) = k)P(\psi_{t,k} = m)P(\nu_k \leq 2k), \tag{38}$$

where $\psi_{t,k} = \nu(t) | \mu(t) = k$. For the right hand side, we have

$$P(\nu = m'; \mu(t) > 0; \mu(t + r) = 0) =$$

$$= \sum_{k=1}^{+\infty} P(\mu(t) = k)P(\nu = m' | \mu(t) = k; \mu(t+r) = 0)P(\mu(t+r) = 0 | \mu(t) = k) =$$

$$= \sum_{k=1}^{+\infty} P(\mu(t) = k)P(\psi_{t,k} + \nu_{k,r} = m')P(\mu(r) = 0)^k. \tag{39}$$

Due to Corollary 3, $P(\nu_k \leq 2k) \leq c_2e^{-k/5}$ and due to (11), $P(\mu(r) = 0) \geq e^{1/5}$ for some constant $r > 0$. Thus it remains to show that $\nu_{k,r} = o(m)$ with high probability. Suppose that $\nu_{k,r} \geq kD^r$ for some $D > 0$. Then at some generation
Lemma 7 (Lemma 1 [5]).

and denote \( \Delta \) For

Hence

\[ P(\nu_k \geq kD^r) \leq \frac{kD^r}{eD!}. \]

For \( k > \ln^2 n \), we have that

\[ P(\nu_k \leq 2k) \leq c_2 e^{-\ln^2 n/5} = O(1/n^{0.2\ln n}). \]

For \( k \leq \ln^2 n \), we have that

\[
P(\psi_{t,k} + \nu_{k,r} = m') = P(\psi_{t,k} + \nu_{k,r} = m' \mid \nu_{k,r} \leq kn^{0.8})P(\nu_{k,r} \leq kn^{0.8}) + \]
\[
P(\psi_{t,k} + \nu_{k,r} = m' \mid \nu_{k,r} > kn^{0.8})P(\nu_{k,r} > kn^{0.8}) \leq \]
\[
\leq P(\psi_{t,k} = m(1 + o(1))) + \frac{n^{0.8n} \ln^2 n}{e(n^{0.8})^1} = \]
\[
= P(\psi_{t,k} = m(1 + o(1))) + o(n^{-0.7n}). \]

So, we have

\[
P(\nu(t) = m; \nu - \nu(t) \leq \mu(t)) \leq O(1/n^{0.2\ln n}) + \]
\[
+ \sum_{k=1}^{\ln^2 n} P(\mu(t) = k)P(\psi_{t,k} = m)P(\nu_k \leq 2k) \leq \]
\[
\leq O(1/n^{0.2\ln n}) + o(n^{-0.7n}) + c_2 P(\nu = m(1 + o(1)); \mu(t) > 0; \mu(t + r) = 0). \]

The lemma follows.

Recall that \( f(z), f_t(z) \) are the probability generating functions (pgf) of \( \nu \) and 
\( \nu; \mu(t) = 0 \) resp., that is,

\[
f_t(z) = \sum_{n=0}^{+\infty} P(\nu(t) = n, \mu(t) = 0)z^n, \quad f(z) = \sum_{n=0}^{+\infty} P(\nu = n)z^n \]

and denote \( \Delta_t(z) = f(z) - f_t(z) \). The following lemma is proved in [5].

Lemma 7 (Lemma 1 [5]). If \( t \to +\infty, u \to 0 \) such that \( ut \to 0 \), then

\[
\Delta_t(e^{iu}) = \frac{2\alpha(u)e^{-i\alpha(u)}}{1 - e^{-i\alpha(u)}}(1 + \epsilon(t, u)), \quad (40) \]

where \( \alpha(u) = \sqrt{2iu} \) (the branch with a positive real part is always chosen) and 
\( |\epsilon(t, u)| \leq c \ln t(ut + 1/t) \).

Corollary 5 (Corollary 1, Section 2.3, Page 127 [3]). For each \( \epsilon > 0 \) there
exists constant \( q_2 < 1 \) such that \( |\Delta_t(e^{iu})| \leq q_2^2 \) for \( \epsilon \leq |u| \leq \pi \).

Corollary 6 (Corollary 2, Section 2.3, Page 128 [3]). There exists \( \epsilon > 0, c_1 > 0, c_2 > 0 \) such that \( |\Delta_t(e^{iu})| \leq c_1 t^{-1} e^{-c_2 \sqrt{\pi}} \) for \( |u| \leq \epsilon \).
It is noticed in [3, Lemma 2, Section 3, Page 182] that there exists a

\begin{equation}
    f(e^{iu}) = 1 - \alpha(u) + c(u)\alpha^2(u),
\end{equation}

Proof. \[ \text{Lemma 9.} \]

For each \[ \text{Lemma 10.} \]

There exists positive constants \[ \text{Lemma 11.} \]

Due to Lemma 7, we have that \[ \text{Lemma 8.} \]

\begin{equation}
    f_i^N(e^{iu}) = \left( 1 - \alpha(u) + c(u)\alpha^2(u) - \frac{2\alpha(u)e^{-\alpha(u)}}{1 - e^{-\alpha(u)}}(1 + \epsilon(t, u)) \right)^N = \left(1 + Nc_3(t, u)e^{\exp\left(-N\alpha(u)\frac{1 + e^{-\alpha(u)}}{1 - e^{-\alpha(u)}}\right)}\right).
\end{equation}

where \( |c_3(t, u)| \leq O(1)\left(|u| + |\Delta(t, u)|\epsilon(t, u)\right) \leq O(1)\ln t \left(|u| + \frac{1}{t} \right).\)

\textbf{Lemma 9.} For each \( \varepsilon > 0 \) there exists constants \( q_1 < 1, q_2 < 1 \) such that \( |f_i(e^{iu})| < q_1 + q_2^\delta \) for \( \varepsilon \leq |u| \leq \pi. \)

\textit{Proof.} It is noticed in [3] Lemma 2, Section 3, Page 182] that there exists a constant \( q_1 < 1 \) such that \( |f_i(e^{iu})| \leq q_1 \) for \( \varepsilon \leq |u| \leq \pi. \) The lemma now follows from Corollary 6 and the definition of \( \Delta \) (see (11)).

Due to (11) and Corollary 6 we have the following lemma.

\textbf{Lemma 10.} There exists positive constants \( \varepsilon, \delta_2, c_1, c_2 \) such that for \( |u| \leq \varepsilon, \)

\[ |f_i(e^{iu})| \leq 1 - \delta_2 \sqrt{|u| + c_1 t^{-1} e^{-c_2 t \sqrt{|u|}}}. \]

\textbf{Lemma 11.} For any \( r > 0, t \to +\infty, u \to 0 \) such that \( ut \to 0, \)

\[ \Delta_{t+r}(e^{iu}) - \Delta_t(e^{iu}) = f_i(e^{iu}) - f_{t+r}(e^{iu}) = \frac{2r\alpha^2(u)e^{-\alpha(u)}}{1 - e^{-\alpha(u)}} \left(1 + \epsilon_1(t, u)\right), \]

where \( |\epsilon_1(t, u)| \leq O(1)\left(|u| + \frac{\ln^2 t}{t} + \sqrt{\ln t}\right). \)

\textit{Proof.} Due to Lemma 7 we have that \( f_i(z) - f_{t+r}(z) = \Delta_t(z) - \Delta_{t+r}(z). \) In order to prove the lemma, we use the following ingredients from the proof of Lemma 7. From equations (8),(9),(10),(12) of Lemma 7 we have \( \Delta_t(z) = a^t(z)/\psi(t, z) + \varepsilon_5(t, z), \) where

\[ \psi(t, z) = \frac{1 - e^{-\alpha(u)}\left(1 + O(1)ut\right)}{2\alpha(u)}(1 + O(1)\sqrt{|u|}). \]

\[ \varepsilon_5(t, z) = 1/f(z) - \sum_{k=0}^{t-1} a^k(z)\Delta_k(z)O_k(1)(z). \]

\[ a^k(e^{iu}) = e^{-k\alpha(u)}(1 + O(1)ut). \]
From here we have that

\[ |\varepsilon_5(t + r, z) - \varepsilon_5(t, z)| \leq \sum_{k=t}^{t+r-1} |a^k(z)\Delta_k(z)O_k(1)(z)| \leq O(1) \frac{e^{-t\sqrt{|u|}}}{t}. \]

Since also \(|\psi(t, z)| \geq 0.5t(1 - O(t\sqrt{|u|})) > 2|\varepsilon_5(t, z)|\), we have that

\[
\left| \frac{a^{1+r}(z)}{\psi(t + r, z) + \varepsilon_5(t + r, z)} - \frac{a^{1+r}(z)}{\psi(t + r, z) + \varepsilon_5(t, z)} \right| \leq \frac{e^{-(t+r)\alpha(u)}(1 + O(1)ut)|\varepsilon_5(t + r, z) - \varepsilon_5(t, z)|}{(|\psi(t + r, z)| - |\varepsilon_5(t + r, z)|(|\psi(t + r, z)| - |\varepsilon_5(t, z)|)} \leq O(1) \frac{\alpha^2(u)e^{-t\sqrt{|u|}}}{t|1 - e^{-\alpha(u)(1 + O(1)ut)}|^2}. \quad (42)
\]

Now let us consider the function

\[ \gamma(x) = \frac{a^2(z)}{\psi(x, z) + \varepsilon_5(x, z)} = \frac{2\alpha(u)e^{-x\alpha(u)}(1 + O(1)ux)}{1 - e^{-x\alpha(u)}(1 + O(1)ux) + O(1)\alpha(u)\ln t}. \]

One can see that \(x\) appears equally as a factor before real and imaginary part of \(\gamma(x)\). Since \(\gamma(x)\) is also smooth for \(x > 0\), for some \(x \in (t, t + r)\) we get that

\[
\gamma(t + r) - \gamma(t) = r\gamma'(x) = 2\alpha(u)e^{-x\alpha(u)}(O(1)u - \alpha(u))(1 + O(1)ux)\cdot
\]

\[ + \frac{1}{1 - e^{-x\alpha(u)}(1 + O(1)ut) + O(1)\alpha(u)\ln t} + \frac{e^{-x\alpha(u)}}{(1 - e^{-x\alpha(u)})(1 + O(1)ut) + O(1)\alpha(u)\ln t} \]
\[= \frac{2\alpha^2(u)e^{-x\alpha(u)}}{(1 - e^{-x\alpha(u)})^2(1 + \varepsilon_1(x, u))}, \quad (43)\]

where \(|\varepsilon_1(x, u)| \leq O(1)(ux + \frac{\ln x}{x^2} + \sqrt{|u|}\ln x)\). The lemma now follows from \(11\) and \(42\) and the condition that \(t \to +\infty\).

We can get the following corollary from Lemma \(11\) and Lemma \(8\).

**Corollary 7.** For any \(r > 0\), \(t, N \to +\infty\), \(u \to 0\) such that \(ut \to 0\) and \(N \leq 2t\ln n\), we have

\[ f_i(t)(e^{iu})(f_{i+r}(e^{iu}) - f_i(e^{iu}))^2 = \frac{4\alpha^2(u)e^{-2t\alpha(u)}}{(1 - e^{-t\alpha(u)})^4} \exp \left(-N\alpha(u)\frac{1 + e^{-t\alpha(u)}}{1 - e^{-t\alpha(u)}}\right) \]

where \(|\varepsilon_3(t, u)| \leq O(1) \left(\ln n \left(\sqrt{|u|} + \frac{1}{\sqrt{t}}\right)^2 \right). \]
Proof. Due to Lemma 11 and Lemma 8, it remains to bound the residual term as follows.

\[
\left( ut + \frac{\ln^2 t}{t^2} + \sqrt{u} \ln t \right) + N \ln t \left( |u| + \frac{1}{t^2} \right) \leq O(1) \ln^2 t \left( |u| t + \frac{1}{t} + \sqrt{u} + 2 |u| t^2 + \frac{2}{t} \right) \leq O(1) \ln^2 t \left( |u| t + \sqrt{u} \right) \leq O(1) \ln n \left( \sqrt{|u| t} + \frac{1}{\sqrt{t}} \right)^2 .
\]

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