ON THE GALOIS THEORY OF GROTHENDIECK

by Eduardo J. Dubuc and Constanza S. de la Vega

Spanish Abstract.

En este articulo tratamos la interpretacion hecha por Grothendieck de la teoria de Galois (y su relacion con el grupo fundamental y la teoria de cubrimientos) en Expose V section 4, ”Conditions axiomatiques d’une theorie de Galois” en el SGA1 1960/61.

Esta es una hermosa muestra de matematicas muy rica en conceptos categoricos, y su alcance es mucho mas vasto que el del trabajo original de Galois (asi como este llegaba mucho mas lejos que la simple no resubilidad de la quintica).

Aqui introducimos algunos axiomas y demostramos un teorema de caracterizacion de la categoria (topos) de acciones de un grupo discreto. Este teorema corresponde exactamente al resultado fundamental de Galois. El teorema de Grothendieck caracteriza la categoria (topos) de acciones continuas de un grupo topologico profinito. Desarrollamos una demostracion de este resultado como un ”paso al limite” (en un limite inverso de topos) de nuestro teorema de caracterizacion del topos de acciones de un grupo discreto. Tratamos el limite inverso de topos trabajando con un simple colimite filtrante (union) de las categorias que son sus (respectivos) sitios de definicion. No se necesitan conocimientos avanzados de teoria de categorias para leer este articulo.

Introduction.

In this paper we deal with Grothendieck’s interpretation of Galois’s Galois Theory (and its natural relation with the fundamental group and the theory of coverings) as he developed it in Exposé V, section 4, “Conditions axiomatiques d’une theorie de Galois” in the SGA1 1960/61, [6].

This is a beautiful piece of mathematics very rich in categorical concepts, and goes much beyond the original Galois’s scope (just as Galois went much further than the non resubility of the quintic equation). We show explicitly how Grothendieck’s abstraction corresponds to Galois work.

We introduce some axioms and prove a theorem of characterization of the category (topos) of actions of a discrete group. This theorem corresponds exactly to Galois fundamental result. The theorem of Grothendieck characterizes the category (topos) of continuous actions of a profinite topological group. We develop a proof of this result as a
"passage into the limit" (in an inverse limit of topoi) of our theorem of characterization of the topos of actions of a discrete group. We deal with the inverse limit of topoi just working with an ordinary filtered colimit (or union) of the small categories which are their (respective) sites of definition.

We do not consider generalizations of Grothendieck’s work in [6], except by commenting briefly in the last section how to deal with the prodiscrete (not profinite) case. We also mention the work of Joyal-Tierney, which falls naturally in our discussion.

There is no need of advanced knowledge of category theory to read this paper, except for the comments in the last section.

The paper has five sections, and a last one with comments and possible generalizations.

I. Examples.

II. Transitive actions of a discrete group.

III. Continuous transitive actions of a profinite group.

IV. All continuous actions of a profinite group.

V. All actions of a discrete group (and of a discrete monoid).

VI. Comments on this paper.

I - Examples.

A - Classical Galois Theory.

The ideas behind Galois Theory were developed through the work of Newton, Lagrange, Galois, Kronecker, Artin and Grothendieck.

Before Galois it was known the following, that we write here using modern notation:

- Given a field $k$ of numbers (of characteristic 0), a polynomial of degree $n$, $f(x) \in k[x]$ with all roots $\theta_1, \theta_2, \cdots, \theta_n$ different, and a polynomial $s \in k[x_1, x_2, \cdots, x_n]$, then:

$$s(\theta_1, \cdots, \theta_n) = s(\sigma \theta_1, \cdots, \sigma \theta_n) \quad \forall \sigma \in S_n \Rightarrow s(\theta_1, \cdots, \theta_n) \in k$$

where $S_n$ is the symmetric group in $n$ elements.

Galois in “Memoire sur les conditions de resolubilité des equations par radicaux” (see [4]), proved a sharper result, he showed that:

- There exists a group $G \subseteq S_n$ such that

$$s(\theta_1, \cdots, \theta_n) = s(\sigma \theta_1, \cdots, \sigma \theta_n) \quad \forall \sigma \in G \Rightarrow s(\theta_1, \cdots, \theta_n) \in k.$$

He also showed the reverse implication.

Today this group is known as the Galois group $G(L/k)$ of the splitting field $k \rightarrow L$ of $f$ over $k$, and the statement above, takes the form:

Given any $\alpha \in L$, then:
\[ \sigma \alpha = \alpha \quad \forall \sigma \in G(L/k) \Rightarrow \alpha \in k. \]

Notice that \( \alpha = s(\theta_1, \cdots, \theta_n) \) for some \( s \in k[x_1, x_2, \cdots, x_n] \).

See [4].

The whole statement known as the Fundamental Theorem of Galois Theory, after Artin, is the following:

Let \( k \to N \to L \) be an intermediate extension of \( L \) and \( H \subseteq G(L/k) \) any subgroup. Then the assignments:

\[
\begin{align*}
N &\to G(L/N) \subseteq G(L/k) \\
L^H &\leftarrow H \subseteq G(L/k)
\end{align*}
\]

where \( G(L/N) = \{\sigma/\sigma \alpha = \alpha \quad \forall \alpha \in N\} \) and \( L^H = \{\alpha \in L/\sigma \alpha = \alpha \quad \forall \sigma \in H\} \), establish a one-one (contravariant) correspondence between the lattices of subgroups of \( G(L/k) \) and the subextensions of \( L \).

In particular, the Galois group \( G(L/N) \) completely determines the extension \( N \).

**B - Classical Theory of Coverings and the Fundamental Group.**

A local homeomorphism \( X \xrightarrow{p} B \) (sheaf over \( B \)) is a covering space of \( B \) if it is locally constant. That is, there exist a set \( S \), an (open) covering \( U_\alpha \hookrightarrow B \) and homeomorphisms \( \varphi_\alpha \) such that:

\[
\varphi_\alpha : U_\alpha \times S \xrightarrow{\cong} X |_{U_\alpha} \xrightarrow{p} X
\]

The set \( S \) is the “fiber” of \( p \). Given \( b_0 \in U_{\alpha_0} \), the homeomorphism \( \varphi_{\alpha_0} \) establishes a bijection \( S \cong p^{-1}(b_0) \).

It is assumed that \( B \) is path connected, locally path connected and semi locally simply connected (see [11]) so that the existence of the universal covering \( \tilde{B} \to B \) is guaranteed.

The fundamental group \( \pi_1(B, b_0) \) acts on the fiber \( S \cong p^{-1}(b_0) \). We shall consider now only connected coverings which in this context is equivalent to the fact that this action is transitive.

The fundamental theorem of covering theory establishes:

1) The \( \pi_1(B, b_0) \)-set \( S \) completely determines the covering.
2) For any transitive \( \pi_1(B, b_0) \)-set \( E \), there is a covering with fiber \( E \).

The covering corresponding to \( E \) is constructed as follows:
Take an \( x \in E \) and let \( H \subseteq \pi_1(B,b_0) \) be the subgroup \( H = \{g/gx = x\} \). This subgroup \( H \) acts on the fibers of the universal covering and if we divide fiber by fiber by this action we obtain a set \( \tilde{B}/H \) that can be given a topology making it a quotient space \( \tilde{B} \to \tilde{B}/H \) which is a covering \( \tilde{B}/H \to B \).

Recall that given any group \( G \) there is a one-one correspondence between the subgroups of \( G \) (actually conjugacy classes of subgroups) and the transitive \( G \)-sets \( E \) (actually equivalent classes of transitive \( G \)-sets under isomorphism). Briefly, given \( E \) take any \( x \in E \) and set \( H = Fix(x) \), on the other hand, given \( H \) set \( E = G/H \).

Under this light, the fundamental theorem of coverings says the following: Let

\[
\begin{array}{ccc}
\tilde{B} & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & \end{array}
\]

be any covering of \( B \) and \( H \subseteq \pi_1(B,b_0) \) any subgroup.

Let \( Fix(x) \subseteq \pi_1(B,b_0) \), \( Fix(x) = \{g/gx = x \text{ for the action of } \pi_1(B,b_0) \text{ on } S\} \) and \( \tilde{B}/H \) be the quotient of \( \tilde{B} \) by the action of \( H \). Then the assignment:

\[
X \to B \quad \longrightarrow \quad Fix(x) \quad \text{where} \quad x \in S
\]

\[
\tilde{B}/H \quad \longleftarrow \quad H \subseteq \pi_1(B,b_0)
\]

establishes a one-one (covariant) correspondence between the lattices of subgroups of \( \pi_1(B,b_0) \) and the connected coverings over \( B \).

The above are two examples of a common theory that we call the representable case of the Grothendieck axiomatic approach to Galois Theory. This theory characterizes the category of all transitive actions of a discrete (non necessarily finite) group.

II - Transitive actions of a discrete group.

Let \( \mathcal{C} \) be any category.

Definition 2.1.

1) An arrow \( X \xrightarrow{f} Y \) in any category \( \mathcal{C} \) is an strict epimorphism if given any compatible arrow \( X \xrightarrow{g} Z \), there exists a unique \( Y \xrightarrow{h} Z \) such that \( g = h \circ f \). \( g \) is compatible if for all \( \mathcal{C} \xrightarrow{x} X \), \( \mathcal{C} \xrightarrow{y} X \) with \( g \circ x = g \circ y \), then also \( f \circ x = f \circ y \).

It immediately follows that: strict epi + mono = iso

Let now \( A \in \mathcal{C} \) be any object, and let \( G = (Aut(A))^\text{op} \) the opposite group of the group of automorphisms of \( A \).
Definition 2.2. Given any group $H$, a (left) action of $H$ on $A$ is a morphism of groups $H \rightarrow G = (\text{Aut}(A))^\text{op}$. The categorical quotient $A \rightarrow A/H$ is defined by the following universal property:

i) $\forall h \in H, q \circ h = q$ 
ii) Given an arrow $A \rightarrow X$ such that $\forall h \in H, xh = x$, there exists a unique $A/H \rightarrow X$ such that $\varphi \circ q = x$.

In a diagram:

We abuse the language and write ‘$h$’ for the automorphism of $A$ corresponding to the element $h \in H$. Notice that different elements may define the same automorphism.

Notice that it clearly follows that the arrow $A \rightarrow A/H$ is an strict epimorphism.

Proposition 2.3. Given any $H \subseteq G$, $H$ acts on the left on the sets $[A, X]$ of morphisms in $\mathcal{C}$ from $A$ to any other object $X \in \mathcal{C}$:

$$H \times [A, X] \rightarrow [A, X]$$

where “$\circ$” is composition of arrows in $\mathcal{C}$.

Assume for the moment that this action $G \times [A, X] \rightarrow [A, X]$ is transitive. We have in this way a functor:

$$\mathcal{C} \rightarrow t\text{Ens}^G$$

where by $[A, X]_G$ we indicate the set $[A, X]$ with this action of $G$ and $t\text{Ens}^G$ indicates the category of transitive $G$-sets.

Remark 2.4. 

The group $G$ has a canonical action on itself (given by left multiplication), thus we can consider $G \in t\text{Ens}^G$. $G$ furnished with this action is the free $G$-set on one generator $e \in G$ ($e$ the neutral element). This means that given any $E \in t\text{Ens}^G$ there is a one-one correspondence:

$$x \in E$$

$$\varphi : G \rightarrow E, \quad \varphi(g) = gx$$

$$e \mapsto x$$
where the arrow $\varphi$ is a morphism of action (homogeneous map).

Given any $x \in E$, the corresponding map $G \xrightarrow{\varphi} E$ makes of $E$ a categorical quotient $E = G/H$ where $H = \text{Fix}(x) = \{ g \in G/\, gx = x \}$.

Consider any object $X \in \mathcal{C}$ and the $G$-set $[A, X]_G$. The bijection in the remark above takes the form:

\[
\begin{array}{c}
A \xrightarrow{x} X \\
\varphi : G \longrightarrow [A, X]_G
\end{array}
\]

This bijection means exactly that the object $A$ is the value of the left adjoint of the functor $[A, -]^G$ evaluated in $G$. This left adjoint is denoted:

\[
\mathcal{C} \xleftarrow{A \times_G (-)} \text{tEns}^G
\]

Thus $A \times_G G = A$.

Consider now any $E \in \text{tEns}^G$ and an element $x_0 \in E$. Let $H = \text{Fix}(x_0)$, and assume that the quotient $A/H$ exists in $\mathcal{C}$. Then, it follows immediately that in the correspondence (2.1) above, an arrow $A \longrightarrow X$ factors through $A/H$ if and only if the corresponding arrow $G \rightarrow [A, X]_G$ factors through $G \rightarrow G/H \cong E$. Thus, there is a one to one correspondence:

\[
\begin{array}{c}
A/H \longrightarrow X \\
E \longrightarrow [A, X]_G
\end{array}
\]

Therefore, the value of the left adjoint evaluated in the object $E$ is equal to $A/H$:

\[
A \times_G E = A/H
\]

Thus, if quotients of $A$ by subgroups of $\text{Aut}(A)$ exists in $\mathcal{C}$, the functor $[A, -]^G$ has a left adjoint given by the formula above.

We shall examine now conditions on $\mathcal{C}$ that will ensure that this left adjoint together with $[A, -]^G$ establishes an equivalence of categories.

### 2.5. Axioms for the representable connected case

Consider a category $\mathcal{C}$ and an object $A \in \mathcal{C}$:

**RC0)** For all $X$ in $\mathcal{C}$ there exist $A \rightarrow X$, and every arrow $A \rightarrow X$ is a strict epimorphism (see 2.1).

**RC1)** The quotient $A \xrightarrow{q} A/H$ of $A$ by any subgroup of $\text{Aut}(A)$, $H \subseteq \text{Aut}(A)$, exists in $\mathcal{C}$ and it is preserved by the functor $[A, -]$.

**RC2)** Every endomorphism of $A$ is an automorphism. That is $[A, A] = \text{Aut}(A)$. 


Remark that the axiom RC2) follows (from 2.6) if we assume that the functor $[A, -]$ preserves strict epimorphisms and that $[A, A]$ is a finite set (see 3.3 below).

Notice that it follows immediately from RC0 that every arrow $X \to Y$ in $C$ is a strict epimorphism. It also follows only from RC0 the following:

**Proposition 2.6.** Axiom RC0) implies that the functor $[A, -]$ is faithful, reflects monomorphisms and reflects isomorphisms.

**Proof.**

See 3.6 where the statement is proved for any functor $F$ in place of $[A, -]$.

**Remark 2.7.**

.1 On RC1)

The map $[A, A] \xrightarrow{q_*} [A, A/H]$ factors $q_* = \eta \circ \rho$

\[
\begin{array}{ccc}
[A, A] & \xrightarrow{\rho} & [A, A/H] \\
\downarrow q_* & & \downarrow \eta \\
\downarrow \rho & & \downarrow \eta \\
[A, A/H] & \xrightarrow{\eta} & [A, A/H]
\end{array}
\]

RC1) means that $\eta$ is a bijection. This is divided in two parts:

i) if $q \circ f = q \circ g$ then there exists $h \in H$ such that $f = h \circ g$

ii) For all $A \xrightarrow{x} A/H$, there exists $A \xrightarrow{f} A$ such that $q \circ f = x$.

.2 On RC2)

In the presence of axiom RC2 condition i) above becomes equivalent to:

iii) $q \circ f = q$ implies $f \in H$

Also, given any arrow $A \xrightarrow{x} X$, if we consider the surjective-injective factorization $x_* = \psi \circ \rho$:

\[
\begin{array}{ccc}
[A, A] & \xrightarrow{\rho} & I \\
\downarrow x_* & & \downarrow \psi \\
\downarrow \rho & & \downarrow \psi \\
[I, A] & \xrightarrow{\psi} & [A, X]
\end{array}
\]

axiom RC2 implies that $I = [A, A]/H$, where $H = Fix(x) \subset Aut(A)$

In the next three propositions of this section all the three axioms are needed.

**Proposition 2.8.** Any arrow $A \xrightarrow{x} X$ is a quotient of $A$ by the subgroup $H \subset Aut(A)$, $H = Fix(x)$

**Proof.**

Consider the factorization:

\[
\begin{array}{ccc}
A & \xrightarrow{q} & A/H \\
\downarrow x & & \downarrow \epsilon \\
A/H & \xrightarrow{\epsilon} & X
\end{array}
\]
If we apply the functor \([A, -]\) to this diagram, we have:

\[
\begin{array}{c}
\xymatrix{
[A, A] \ar[r]^{q_*} \ar[rd]_{\rho} & [A, A/H] \ar[d]^{\eta} \ar[r]^{\varepsilon_*} & [A, X] \\
& [A, A]/H & \\
}
\end{array}
\]

Where \(\rho\), \(\eta\) and \(\psi\) are as in 2.7.1 and 2.7.2.

Since \(\eta\) is a bijection and \(\psi\) is injective, it follows that \(\varepsilon_*\) is also injective. Thus by proposition 2.6, \(\varepsilon\) is a monomorphism. Since every arrow is an strict epimorphism, it follows that it is an isomorphism (see 5.7).

The statement in this proposition is actually equivalent to the fact that the functor \([A, -]\) reflects isomorphisms. In fact, given any arrow \(X \xrightarrow{f} Y\), take \(A \xrightarrow{x} X\). We have \(\text{Fix}(x) \subseteq \text{Fix}(f \circ x)\). If \(f_*\) is a bijection, given any \(A \xrightarrow{h} A\), the equation \(f \circ x \circ h = f \circ x\) implies \(x \circ h = x\). Thus \(\text{Fix}(x) = \text{Fix}(f \circ x)\). It follows from the statement of proposition 2.8 that \(f\) is an isomorphism.

**Proposition 2.9.** The functor \([A, -]\) preserves strict epimorphisms.

**Proof.**

By RC0) it clearly suffices to consider the case \(A \xrightarrow{x} X\). Then the statement follows immediately from proposition 2.8.

**Proposition 2.10.** The action of \(\text{Aut}(A)\) on \([A, X]\) is transitive for all \(X\).

**Proof.**

Since every arrow \(A \rightarrow X\) is an strict epimorphism (RC0), proposition 2.9 means exactly that the action of the monoid \([A, A]\) is transitive. Then RC2) finishes the proof.

**Theorem 2.11.** Let \(\mathcal{C}\) be any category and \(A \in \mathcal{C}\) as above. If the axioms RC0) to RC2) hold then the left adjoint \(A \times_G (-) \dashv [A, -]_G\) exists and the maps:

a) \(E \cong [A, A/H] \xrightarrow{\eta}[A, A/H]

b) \(A/H \xrightarrow{\varepsilon} X\)

are isomorphisms. Thus, they establish an equivalence of categories:

\[
\begin{array}{c}
\xymatrix{
\mathcal{C} \ar[r]^{[A, -]_G} & \text{Ens}^G \\
A \times_G (-) \ar[u]_{A \times_G (-)} &
}
\end{array}
\]

Notice that \(A/H = A \times_G [A, X]_G\) and \([A, A/H] = [A, A \times_G E]_G\), and that \(\eta\) and \(\varepsilon\) are the unit and counit of the adjunction.
Proof.

That $\eta$ and $\varepsilon$ are isomorphisms is proved in 2.7.1 and 2.8 respectively.

It is not difficult but not trivial to check that the classical case of Galois Theory (A in section I) and the covering example (B in section I) satisfy axioms RC0) to RC2).

In the first case $\mathcal{C}$ is the category dual to the category of subextension of the splitting field $L$ of $f$ and the object $A$ is the extension $\tilde{L}$, and in the second case $\mathcal{C}$ is the category of path connected coverings of the space $B$ and the object $A$ is the universal covering. The reader can also check that the recipes a),b) in theorem 2.11 are also exactly the ones given in the examples. Notice that in example A, the subextension $L^H = \{\alpha \in L/\sigma \alpha = \alpha \; \forall \alpha \in H\}$ is precisely the quotient of $L$ by $H$ in the dual category.

There are two conspicuous non-examples of the above theorem:

First take $k \to \tilde{k}$ be the algebraic closure of $k$ (we assume for simplicity that $k$ is of characteristic 0), and $\mathcal{C}$ the dual category of all intermediate extensions. In this case it is known that the theorem is false, thus some of the axioms must not hold.

The other non example is the case like above of all path connected coverings of $B$, a topological space path connected, locally path connected but not necessarily semilocally simply connected. In this case there is no universal covering and we don’t have the object $A \in \mathcal{C}$.

These two non-examples can be suitable modified so that a similar theorem will hold. In the first case there have to be taken only finite extension, and in the second case only coverings with finite fibers. So, the algebraic closure or the universal covering, even when the latter exists, are not any more objects in the category.

Both these cases share the fact that there is no universal object $A$ in the sense of axiom RC0 in the category $\mathcal{C}$. Grothendieck in [6] replaces the representable functor $[A,-]$ by a functor $F : \mathcal{C} \to \mathcal{E}ns$, which is a filtered colimit of representables and establishes a theorem which has as particular cases the two situations described above. In addition, he assumes that the functor $F$ takes its values on finite sets. Due to this assumption, his theorem is not a generalization of the theorem above, and in particular it does not has as a particular case the example B. Now, the group is not longer the usual fundamental group of loops, but it is its profinite closure, and the coverings included in the theorem are only those with finite fibers. Nevertheless, it can also be proved a theorem that includes the case of all covering spaces, even when there is no universal covering, as it is said (but not proved) in [[1], 2.7.4 and 2.7.5] (see 6.2).

III - Continuous transitive actions of a profinite group.

Consider a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{E}ns$.

Recall that the diagram of $F$, that we denote $\mathbf{G}_F$ is the category whose objects are pairs $(a, A)$ where $a \in F(A)$ and whose arrows $(a, A) \xrightarrow{f} (a', A')$ are maps $A \xrightarrow{f} A'$ such
that $F(f)(a) = a'$.

There is a functor

$$
\Gamma_F \longrightarrow \mathcal{E}_{ns}^C \\
(a, A) \longrightarrow [A, -]
$$

with the obvious definition on arrows, and $F$ is the colimit of this diagram, that is:

$$
F = \lim_{(a, A) \in \Gamma_F} [A, -]
$$

Recall also that there is a one-one correspondence:

$$
a \in F(A) \quad \frac{[A, -] \longrightarrow aF}
$$

where the arrow $a$ is the natural transformation completely characterized by the equation $a(id_A) = a$ (notice the abuse of notation).

### 3.1 . Axioms for the connected case

Consider the following axioms:

C0) For all $A \in \mathcal{C}$, $F(A) \neq \emptyset$ and every arrow $A \xrightarrow{x} B$ is an strict epimorphism (in this case the category $\Gamma_F$ is actually a poset, see remark 3.2).

C1) Given any object $A$ and a finite group $H$ acting on $A$, $H \rightarrow (Aut(A))^\text{op}$, the quotient $A \xrightarrow{q} A/H$ exists and it is preserved by $F$ (see definition 2.2).

C2) $F(A)$ is finite $\forall A \in \mathcal{C}$ and $F$ preserves strict epimorphisms.

C3) The poset $\Gamma_F$ has all finite meets, in particular it is filtered.

**Remark 3.2** . Since all maps $A \xrightarrow{x} B$ in $\mathcal{C}$ are epimorphisms, for any object $X \in \mathcal{C}$ the transition morphisms corresponding to an arrow $(a, A) \xrightarrow{x} (b, B)$ in $\Gamma_F$, $x^* : [B, X] \longrightarrow [A, X]$ are all injective functions. By construction of filtered colimits in $\mathcal{E}_{ns}$ it follows that the canonical maps of the colimit $[A, -] \xrightarrow{a} F$ are injective natural transformations (thus monomorphisms in the category $\mathcal{E}_{ns}^C$). This implies that $\Gamma_F$ is a poset.

**Remark 3.3** . Axiom C2 implies that for all $A \in \mathcal{C}$ every endomorphism of $A$ is an automorphism. That is $[A, A] = Aut(A)$.

**Proof.**

Given $A \xrightarrow{h} A$, the function $F(A) \xrightarrow{h_*} F(A)$ is surjective, then if $F(A)$ is finite it is a bijection, and thus by 3.6 $h$ is an isomorphism.
Remark 3.4. Axiom C1) means that the factorization of the map $F(q)$ through $F(A)/H$ is a bijection. In a diagram:

\[
\begin{array}{c}
F(A) \xrightarrow{F(q)} F(A/H) \\
\downarrow \cong \\
F(A)/H
\end{array}
\]

Remark 3.5. Notice that the condition of having all finite meets on the poset $\Gamma_F$ clearly implies it is filtered. Filterness means for posets that given any pair $[A, -] \xrightarrow{a} F$, $[B, -] \xrightarrow{b} F$, there exist $[C, -] \xrightarrow{c} F$, $C \xrightarrow{x} A$ and $C \xrightarrow{y} B$ such that

\[
\begin{array}{c}
A \\
\downarrow x \\
C \\
\downarrow y \\
B
\end{array} \quad [A, -] \quad \begin{array}{c}
\downarrow x^* \\
\downarrow y^* \\
C \\
\downarrow b \\
[B, -]
\end{array} \quad \begin{array}{c}
\xrightarrow{a} \\
\xrightarrow{c} \\
F
\end{array}
\]

commutes.

Existence of meets means that there is a last such $C$. In this context, this is a strictly stronger condition, which is actually needed in proposition 3.13.

Remark 3.5 also says that $F$ is an strictly pro-representable functor in the sense of Grothendieck [1], and its formal dual is a pro-object that we call $P$. We have the following diagram:

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \leftarrow \\
\mathcal{P}(\mathcal{C})
\end{array} \quad \begin{array}{c}
\xrightarrow{\mathcal{E}_{ns}^c}_{op} \\
\downarrow \leftarrow \\
\mathcal{P}(\mathcal{C})
\end{array}
\]

where all the arrows are full and faithful functors, and to $F \in (\mathcal{E}_{ns}^c)$ it correspond $P \in \mathcal{P}(\mathcal{C})$. To the colimit diagram $[A, -] \xrightarrow{a} F$ (indexed by $\Gamma_F$) in $(\mathcal{E}_{ns}^c)$ it correspond a limit diagram (indexed by $\Gamma_F$) $P \xrightarrow{a} A$ in $\mathcal{P}(\mathcal{C})$. Here by abuse, given an object $A \in \mathcal{C}$ we also denote by $A$ the corresponding (representable) pro-object in $\mathcal{P}(\mathcal{C})$, $A \in \mathcal{P}(\mathcal{C})$. Thus, by this trick of Grothendieck, the functor $F$ becomes representable, but from the outside of $\mathcal{C}$, by a pro-object $P$. Notice that tautologically we have $F(X) = [P, X]$. 
Notice also that (tautologically) the diagram of $F$, $\Gamma_F$ becomes the category of all objects of $C$ below $P$: $P \xrightarrow{a} A$, with morphisms $A \xrightarrow{x} B$ in $C$ such that the triangle

\[
\begin{array}{ccc}
P & \xrightarrow{b} & B \\
\downarrow{a} & & \downarrow{f} \\
A & \xrightarrow{x} & B
\end{array}
\]

commutes. Furthermore given any object $A \in C$ all maps $P \xrightarrow{a} A$ are epimorphisms in the category $\mathcal{P}ro(C)$ (since they are monomorphisms in $\mathcal{E}ns^C$).

**Proposition 3.6.** From axiom C0 it follows that the functor $F$ is faithful, reflects monomorphisms and reflects isomorphisms.

**Proof.**

Let $A \xrightarrow{f} B, A \xrightarrow{g} B$ be such that $F(f) = F(g)$. Take $a \in F(A)$ and let $b = F(f)(a) = F(g)(a)$. We have a commutative diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{b} & B \\
\downarrow{a} & & \downarrow{f} \\
A & \xrightarrow{g} & B
\end{array}
\]

Since $a$ is an epimorphism, it follows $f = g$. This shows the first assertion.

Now, let $A \xrightarrow{f} B$ be such that $F(f)$ is an injective function, and let $C \xrightarrow{x} A, C \xrightarrow{y} A$ be any two morphisms such that $f \circ x = f \circ y$. Clearly $F(f) \circ F(x) = F(f) \circ F(y)$. Thus $F(x) = F(y)$, and by the first assertion $x = y$. This shows the second assertion. Clearly C0 implies that the third assertion follows from the second (see 5.7).

**Definition 3.7.** An object below $P \xrightarrow{a} A$ is Galois $\iff$ the arrow $\text{Aut}(A) \xrightarrow{a^*} [P, A]$ is a bijection. That is:

\[
\begin{array}{ccc}
P & \xrightarrow{\forall y} & A \\
\downarrow{a} & & \downarrow{\exists! h} \\
A
\end{array}
\]

From 3.3 this is equivalent to the fact that the arrow $[A, A] \xrightarrow{a^*} [P, A]$ is a bijection.

Notice that it follows immediately that $P \xrightarrow{a} A$ is Galois $\iff P \xrightarrow{b} A$ is Galois $\forall b$. We are justified in saying then that the object $A$ is Galois.

Referring to section I, Galois objects are exactly the Galois extensions in example A and the regular coverings in example B.
Definition 3.8. Given any $P \rightarrow A$ Galois, we define the full subcategory $C_A \hookrightarrow \mathcal{C}$ as follows:

$X \in C_A \iff [A, X] \overset{a^*}{\rightarrow} [P, X] = F(X)$ is a bijection, that is:

\[
\begin{array}{ccc}
P & \xrightarrow{a} & A \\
\downarrow \forall x & & \downarrow \exists!
\end{array}
\]

Observe that $A \in C_A$ since $A$ is Galois and that $C_A$ does not depend on $a$ but only on the object $A$.

**Proposition 3.9.** Given any transition morphism between Galois objects in $\Gamma_F$,

\[
\begin{array}{ccc}
P & \xrightarrow{a} & A \\
\downarrow \forall x & & \downarrow \exists!
\end{array}
\]

there is an inclusion of full subcategories $C_B \hookrightarrow C_A$.

**Remark 3.10.** The restriction of the functor $F$ on $C_A$ is the representable functor by $A$, that is

\[
\begin{array}{ccc}
C_A & \hookrightarrow & C \\
\downarrow \vdash A \dashv - & & \downarrow F \\
\mathcal{E}_{ns} & \xrightarrow{a^*} & \mathcal{E}_{ns}
\end{array}
\]

commutes.

**Proposition 3.11.** The category $C_A$ is closed by quotients of actions of finite groups.

**Proof.**

Suppose we have a quotient $A \rightarrow^q A/H$. Consider the commutative diagram:

\[
\begin{array}{ccc}
[A, A] & \xrightarrow{a^*} & [P, A] \\
\downarrow q_* & & \downarrow q_*
\end{array}
\]

\[
\begin{array}{ccc}
[A, A/H] & \xrightarrow{a^*} & [P, A/H]
\end{array}
\]

We have to show that the lower $a^*$ arrow is a bijection. This is immediate since the upper $a^*$ is bijective and the right $q_*$ is surjective since $[P, A/H] \cong [P, A]/H$ by axiom C1, see 3.4.
Theorem 3.12. The category $C_A$ with the object $A \in C_A$ satisfies the axioms $RC0)$ to $RC2)$. Thus by theorem 2.11 we have an equivalence of categories:

$$C_A \xrightarrow{[A,-]_G} t\mathcal{E}ns^G$$

where $G = Aut(A)^{op}$.

Proof.

Consider the remark 3.10, then: axiom C0) gives axiom RC0), axiom C1) together with proposition 3.11 give axiom RC1) and proposition 3.3 give axiom RC2).

Proposition 3.13. (Existence of Galois closure) Given any $X \in C$, there exist a Galois object $P \xrightarrow{a} A$ such that $\forall P \xrightarrow{x} X$, there exist a unique $\pi_x$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
P & \xrightarrow{a} & A \\
\forall x & \downarrow & \exists! \pi_x \\
X & \downarrow & \\
\end{array}
$$

In other words, the map $a^* : [A, X] \rightarrow [P, X]$ is bijective.

Proof.

Consider the finite set of all $P \xrightarrow{x} X$ and let

$$
\begin{array}{ccc}
P & \xrightarrow{a} & A \\
x & \downarrow & \pi_x \\
X & \downarrow & \\
\end{array}
$$

be the infima in $\Gamma_F$ which exists by C3). We shall see that $A$ is Galois.

Given any $P \xrightarrow{y} A$, consider the map $[P, X] \rightarrow [P, X]$, that sends $x \rightarrow \pi_x \circ y$. This map is an injective map ($\pi_x \circ y = \pi_{x'} \circ y \Rightarrow \pi_x = \pi_{x'} \Rightarrow x = x'$) between finite sets. Thus it is a bijection. Given any $x$, $x$ is then of the form $x = \pi_{x'} \circ y$ for some $x'$. Consider the diagram:
The unique existence indicated in the diagram follows by the universal property of the infima. This shows that $A$ is Galois.

**Theorem 3.14.** (Corollary of 3.13) Galois objects are cofinal in the diagram of $F$. Thus, $F$ is a filtered colimit of representables $[A, -]$ with $A$ Galois. Let $\Lambda_F$ be the cofinal subdiagram of $\Gamma_F$ whose objects are Galois. We have $P \xrightarrow{a} A$, $(a, A) \in \Lambda_F$ is a filtered inverse limit diagram in such a way that $C$ becomes a filtered colimit of the full subcategories $C_A \hookrightarrow C$.

**Proof.**

It follows straightforward from proposition 3.13.

Consider $(a, A) \xrightarrow{x} (b, B)$ in $\Lambda_F$. We define $\rho_x : [A, A] \to [B, B]$ by means of the following commutative diagram:

$$
\begin{array}{c}
[P, A] \xrightarrow{x_*} [P, B] \\
\cong \downarrow_{a^*} \cong \downarrow_{b^*} \\
[A, A] \xrightarrow{\rho_x} [B, B]
\end{array}
$$

We define also $[P, P] \xrightarrow{\pi_a} [A, A]$ by the following commutative diagram:

$$
\begin{array}{c}
[P, P] \xrightarrow{\pi_a} [A, A] \\
\downarrow_{a_*} \downarrow_{a^*} \\
[P, A]
\end{array}
$$

It follows that we have a cone over $\Lambda_F$: 

![Diagram](image-url)
and the diagram \([P, P] \xrightarrow{\pi_a} [A, A]\) is an inverse limit diagram of finite groups with the transition morphisms \([A, A] \xrightarrow{\rho_x} [B, B]\) surjective. It follows immediately that \(\text{Aut}(P) = [P, P]\). The fact that \([P, P]\) is an inverse limit of finite groups means by definition that it is a profinite group. Moreover, the projections in the limit diagram \([P, P] \xrightarrow{\pi_a} [A, A]\) are also surjective (see 1) in proposition 3.17).

**Definition 3.15.** Let \(\Gamma \to Gr\) be an inverse filtered limit of finite groups. The inverse limit group \(\pi \xrightarrow{\rho_\lambda} G_\lambda\) is called a profinite group when considered as a topological group with the product topology of the discrete finite groups \(G_\lambda\).

If \(\pi\) acts continuously and transitively on a set (discrete space) \(E\), given any \(x \in E\) there is a continuous surjective function \(\pi \xrightarrow{(-) \cdot x} E\). Thus, since \(\pi\) is compact, \(E\) must be finite. But more than that is true:

**Proposition 3.16.** A transitive action \(\pi \times E \to E\) on a set (discrete space) \(E\) is continuous if and only if it factors:

\[
\begin{array}{ccc}
\pi \times E & \xrightarrow{\rho_\lambda \times \text{id}} & E \\
\downarrow & & \downarrow \\
G_\lambda \times E & & \end{array}
\]

for some \(\lambda\). In particular \(E\) must be a finite set.

**Proof.**

In fact, consider any element \(x \in E\), and let \(H \subset \pi\) be the subgroup \(H = \text{Fix}(x)\). \(H\) is open because \(E\) is discrete. Since the subgroups \(K_\lambda = \text{Ker}(\rho_\lambda)\) are a neighborhood base of the unit \(e\), there is a \(K_\lambda \subset H\). Since \(K_\lambda\) is normal, it follows that \(K_\lambda \subset \text{Fix}(y)\) for all other elements \(y \in E\). It follows that the given action factors through \(G_\lambda\).

**Proposition 3.17.** Let \(\pi \xrightarrow{\rho_\lambda} G_\lambda\) be a profinite group. Recall that the transition morphisms \(G_\lambda \xrightarrow{\rho_\lambda \rho_\mu} G_\mu\) (for any \(\lambda \to \mu\) in \(\Gamma\)) are surjective. We have:

1) The projections \(\pi \xrightarrow{\rho_\lambda} G_\lambda\) are surjective.
2) Given any \(\lambda \in \Gamma\), there is a full and faithful inclusion of categories:

\[
\begin{array}{ccc}
t\mathcal{E}n_s^{G_\lambda} & \xrightarrow{ct\mathcal{E}n_s} & \mathcal{E}n_s^\pi \\
\end{array}
\]

where the “t” stands for transitive and the “c” stands for continuous.
3) Given any arrow \( \lambda \to \mu \) in \( \Gamma \), there is a full and faithful inclusion of categories:

\[
\begin{array}{ccc}
\mathcal{E}_{ns}^{G_{\mu}} & \longrightarrow & \mathcal{E}_{ns}^{G_{\lambda}} \\
\end{array}
\]

4) The category \( \mathcal{E}ns^{\pi} \) is a filtered colimit (indexed by \( \Gamma \)) of the subcategories \( \mathcal{E}_{ns}^{G_{\lambda}} \).

**Proof.**

The statement 1) is by no means easy to proof, but it is systematically assumed to be true in the literature. We have not found yet a proof in print to give as a reference. The rest of the statements 2), 3) and 4) are straightforward.

We establish now the compatibilities arising from a transition morphism in \( \mathbf{\Pi}_{F} \):

\[
\begin{array}{ccc}
P & \xrightarrow{a} & B \\
\downarrow{a} & & \downarrow{b} \\
A & \xrightarrow{x} & B
\end{array}
\]

**Proposition 3.18.** The following diagram commutes (up to a natural isomorphism):

\[
\begin{array}{ccc}
\mathcal{C}_{A} & \xrightarrow{[A,-]_{G}} & \mathcal{E}_{ns}^{G} \\
\uparrow & & \uparrow \\
\mathcal{C}_{B} & \xrightarrow{[B,-]_{L}} & \mathcal{E}_{ns}^{L}
\end{array}
\]

where \( G = Aut(A)^{op} \) and \( L = Aut(B)^{op} \).

**Proof.**

Let \( X \in \mathcal{C}_{B} \) and consider the morphism in \( \mathcal{E}ns^{G} \), \( [B,X] \xrightarrow{x^*} [A,X] \) induced by \( x \). Is straightforward to check that it is actually a morphism of \( G \)-actions (where \( [B,X] \) is considered with the action induced by \( G \xrightarrow{p_{x}} L \)). On the other hand, the diagram:

\[
\begin{array}{ccc}
[A,X] & \xrightarrow{a^*} & [P,X] \\
\uparrow{x^*} & & \uparrow{b^*} \\
[B,X]
\end{array}
\]

clearly commutes.

Since \( a^* \) and \( b^* \) are bijections, \( x^* \) is a bijection.
Proposition 3.19. The following diagram commutes (up to natural isomorphism):

\[
\begin{array}{ccc}
C_A & \xleftarrow{A \times_G (-)} & t\mathcal{E}ns^G \\
\uparrow & & \uparrow \\
C_B & \xleftarrow{B \times_L (-)} & t\mathcal{E}ns^L
\end{array}
\]

Proof.

Since every object in \( t\mathcal{E}ns^L \) is a quotient of \( L \) (2.4), and the functors in the diagram preserves quotients (the horizontal ones are left adjoints, for the vertical inclusions use 3.11) it will be enough to show the proposition for the object \( L \). Recall that given \( A \xrightarrow{x} B \), then \( B \cong A/H \) where \( H \hookrightarrow G, H = Fix(x) \). It is immediate to check that \( G \to L \) is the quotient \( L \cong G/H \). With this in hand, it is clear that the object \( L \) in \( t\mathcal{E}ns^L \) goes into \( B \) in \( C_A \) by any of the two paths in the diagram.

Proposition 3.20. Given any \( X \in C \), there is a continuous transitive left action of the group \( \pi = [P, P]^{op} \) on the set \([P, X] \):

\[
\pi \times [P, X] \rightarrow [P, X] \\
h \times x \rightarrow x \circ h
\]

given by composition in \( \mathcal{P}ro(C) \).

Proof.

Let \( P \xrightarrow{a} A \) be such that \( a^{\ast} : [A, X] \rightarrow [P, X] \) is a bijection (given by proposition 3.13). Consider the following commutative diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{h} & P \\
\downarrow{a} & & \downarrow{a} \\
A & \xrightarrow{\tilde{h}} & A
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{\tilde{x} = (a^{\ast})^{-1}(x)} \\
\xrightarrow{\pi_a(h) \text{ (see definition of } \pi_a)}
\end{array}
\]

where \( \tilde{x} = (a^{\ast})^{-1}(x) \) and \( \tilde{h} = \pi_a(h) \) (see definition of \( \pi_a \)). Thus, we have an action \( x \circ h = \tilde{x} \circ \tilde{h} \circ a \).

Let \( G = [A, A]^{op} \) and consider the following diagram:
It only remains to see that this action is transitive. That the action $G \times [A, X] \to [A, X]$ is transitive has been proved in proposition 2.10. Now the statement follows immediate since the morphism $\pi_a : \pi \to G$ is surjective.

For any $X \in \mathcal{C}$, let $[P, X]_\pi$ be the set $[P, X]$ enriched with this action. In this way we have a functor:

$$
\mathcal{C} \xrightarrow{[P, -]_\pi} \text{ctEns}_\pi
$$

into the category of continuous transitive $\pi$-actions.

We shall show now how from our theorem for the representable case and the structure results in proposition 3.9 and proposition 3.17, it follows that the functor $[P, -]_\pi$ has a left adjoint and establishes an equivalence of categories.

**Theorem 3.21.** The functor

$$
\mathcal{C} \xrightarrow{[P, -]_\pi} \text{ctEns}_\pi
$$

has a left adjoint $P \times_\pi (-)$ which establishes an equivalence of categories.

**Proof.**

We have filtered colimits of full subcategories:

$$
\mathcal{C} \xrightarrow{[P, -]_\pi} \text{ctEns}_\pi
$$

$$
\mathcal{C}_A \xrightarrow{[A, -]_G} \text{tEns}_G
$$

$$
\xymatrix{ \mathcal{C} \ar[r]^{[P, -]_\pi} \ar[dr]_{[A, -]_G} & \text{ctEns}_\pi \ar[d] \ar[dl] \ar[r]_{A \times G (-)} & \text{tEns}_G \\ \mathcal{C}_A }$$
which are compatible in the sense of propositions 3.18 and 3.19. Given \( E \in \text{ctEns}^\pi \),
to define \( P \times_\pi E \), observe that \( E \in \text{tEns}^G \) for some \( G = \text{Aut}(A)^{\text{op}} \) and define \( P \times_\pi E = A \times_G E \). That this is well defined and actually determines a left adjoint of \( [P, -]_\pi \) follows immediately from the fact that the subcategories are full and the quoted compatibilities.

Finally, since in all \( A \)-levels we have an equivalence, it follows that we also have an equivalence in the limit.

**IV - All continuous actions of a profinite group.**

We consider now Grothendieck axioms as he wrote them in [6] and prove its fundamental theorem. He considered a category with, in particular, finite sums, and proved that it is equivalent to the category of finite continuous actions on a profinite group. However, in our proof of this result it can be seen clearly that the argument goes through if one assume arbitrary sums, and the conclusion is now that the category is equivalent to the category (in this case a topos) of all continuous actions of the same profinite group. We shall write this two results in parallel:

**Convention.** The word [finite] between brackets will mean that the statement stands in fact for two statements: one assuming finite and the other not assuming finite. When the word finite appears not in between brackets it has its usual meaning, and there is only one statement (which assumes finite).

### 4.1 . Grothendieck’s axioms

Consider a category \( \mathcal{C} \) and a functor \( F : \mathcal{C} \to \text{Ens} \) and assume the following axioms:

First we introduce the axiom G0) necessary to deal with the not finite case:

An object \( X \in \mathcal{C} \) is called finite if \( F(X) \) is a finite set. Then:

G0) The subdiagram of the diagram of \( F \) consisting of those pairs \( (x, X), x \in F(X) \) with \( X \) finite is cofinal.

**Axioms on \( \mathcal{C} \):**

G1) \( \mathcal{C} \) has final object 1 and fiber products (notice that this implies the existence of all finite limits).

G2) \( \mathcal{C} \) has initial object 0, [finite] coproducts and quotient of objects by a finite group.

G3) \( \mathcal{C} \) has strict epi-mono factorizations. That is, given any \( f : X \to Y \), there is a factorization \( X \xrightarrow{e} I \xrightarrow{i} Y \) where \( e \) is a strict epimorphism and \( i \) is a monomorphism. Furthermore it is assumed that \( I \) is isomorphic to a direct summand of \( Y \). That is, there exist a subobject \( J \hookrightarrow Y \) and an isomorphism \( I \amalg J \cong Y \).

**Axioms on \( F \):**

G4) \( F \) is left exact, that is, it preserves finite limits.

G5) \( F \) preserves initial object, [finite] sums, quotients by actions on finite groups and sends strict epimorphisms to surjections.

G6) \( F \) reflects isomorphisms.
We shall see that every object of $\mathcal{C}$ is a finite direct sum of connected objects (see definition 5.12) and that the full subcategory of non empty connected objects satisfy axioms C0) to C3) of section III.

We start with an observation:

**Proposition 4.2.**

1) The initial object $0$ is empty. That is, if we have an arrow $X \to 0$, then $X = 0$. We write $0 = \emptyset$.

Coproducts are disjoint and stable by pulling back.

2) Given any object $X \in \mathcal{C}$, $F(X) \cong \emptyset$ if and only if $X \cong \emptyset$.

3) Given any finite objects $A, A_1, A_2$ and an isomorphism $A \cong A_1 \amalg A_2$, if $A_1 \cong A$ then $A_2 \cong \emptyset$.

4) The functor $F$ preserves and reflects monomorphisms.

**Proof.**

1) and 2) follows immediately by G5) and G6).

To see 3) we apply the functor $F$ and by G5) we get a bijection of finite sets, $F(A) \cong F(A_1) \amalg F(A_2)$. It follows that $F(A_2) \cong \emptyset$. Thus by 2) $A_2 \cong \emptyset$.

Finally, 4) follows from G4) and G6) since in any category an arrow $u : X \to Y$ is a monomorphism if and only if the square

\[
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{id} & & \downarrow{u} \\
X & \xrightarrow{u} & Y
\end{array}
\]

is a pullback.

**Remark 4.3.** Since pulling back along any arrow preserves coproducts, it follows immediately that if $A$ is connected, the representable functor $[A, -]$ preserves coproducts.

We consider now the diagram of the functor $F$. By G0) $F$ will also be the colimit of the subdiagram $\Theta_F$ consisting of the finite objects. More precisely, the objects of $\Theta_F$ are pairs $(x, X)$ where $x \in F(X)$ and $X$ is finite, and the arrows $(x, X) \xrightarrow{f} (y, Y)$ are maps $X \xrightarrow{f} Y$ such that $F(f)(x) = y$. Recall also that since $\mathcal{C}$ has finite limits and $F$ preserves them, $\Theta_F$ is a filtered category. Thus $F$ is a pro-representable functor and we denote $P$ the pro-object associated to $F$. Therefore we have (by Yoneda) a correspondence

\[
\begin{align*}
\begin{array}{ccc}
P & \xrightarrow{x} & X \\
\xrightarrow{[X, -]} & & \xrightarrow{x} \ F \\
& x \in F(X)
\end{array}
\end{align*}
\]
and the functor $F$ is “representable” by $P$. We write $F(X) \cong [P, X]$.

**Remark 4.4.** The fact that $F$ preserves [finite] coproducts (G5) means in a sense that $P$ is connected.

**Proposition 4.5.** Every arrow $A \to B$ between connected objects, with $A \not\cong \emptyset$ is an strict epimorphism.

**Proof.**

Consider the factorization of $f$ given by G3):

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
I & \xrightarrow{g} & I
\end{array}
\]

Since $I$ is complemented and non empty, it follows that it’s complement is empty. Thus $I \cong B$.

**Proposition 4.6.** Given any strict epimorphism $A \xrightarrow{f} B$, if $A$ is connected so is $B$.

**Proof.**

Let $B \cong B_1 \amalg B_2$. Then by 4.2 1) $A \cong f^*B_1 \amalg f^*B_2$, where by $f^*$ we indicate pulling back along $f$. It follows that $f^*B_1 \cong \emptyset$ or $f^*B_2 \cong \emptyset$. Let $f^*B_1 \cong \emptyset$. We have then an strict epimorphism $\emptyset \to B_1$. Since every arrow that starts in $\emptyset$ is a monomorphism (because $\emptyset$ is empty), it follows that $B_1 \cong \emptyset$.

**Definition 4.7.** An object of $\Theta F$, $P \xrightarrow{a} A$ is minimal if and only if it does not admit proper subobjects in $\Theta F$. That is, each time we have

\[
\begin{array}{ccc}
P & \xrightarrow{a} & A \\
\downarrow & & \downarrow \\
x & \xleftarrow{u} & u
\end{array}
\]

with $u$ monomorphism, it follows that $u$ is an isomorphism, thus $X \cong A$.

**Proposition 4.8.** The following statements are equivalent:

i) $A$ is a finite non-empty connected object in the category $C$.
ii) Every $P \xrightarrow{a} A$ is minimal and $[P, A] \not\cong \emptyset$.
iii) There exist $P \xrightarrow{a} A$ minimal.
Proof.

i) ⇒ ii)
Consider the following diagram

\[
P \xrightarrow{a} A \quad \xleftarrow{x} X \quad \xrightarrow{u} X
\]

with \(u\) monomorphism.
Since by G3) every subobject is complemented and \(A\) is a non empty connected object, it follows that \(X \cong A\).

ii) ⇒ iii) is clear.

iii) ⇒ i)
Let \(A \cong A_1 \amalg A_2\) and take \(P \xrightarrow{a} A\) minimal. Since \(F\) preserves coproducts, the arrow \(a\) factors through \(A_1\) or \(A_2\). Assume that it factors through \(A_1\), thus \(A \cong A_1\). Then by proposition 4.2 item 3) it follows that \(A_2 \cong \emptyset\).

We remark that to show i) ⇒ ii) in this equivalence it is essential the whole strength of axiom G3), that is we need in the factorization that the subobject be complemented.

Proposition 4.9. The minimal objects are cofinal in the diagram \(\Theta_F\).

Proof.

Let \(P \xrightarrow{x} X\) be an object of \(\Theta_F\). If it is minimal we are done. If not, we have a subobject \(P \xrightarrow{x_1} X_1\) such that

\[
P \xrightarrow{x} X \quad \xleftarrow{x_1} X_1
\]

If \(P \xrightarrow{x_1} X_1\) if minimal we are done. If not, we have an \(X_2\), etc... It follows by proposition 4.2 item 4) and axiom G6) that a chain like this stabilizes in a finite number of steps. We reach thus an minimal object in \(\Theta_F\) above \(P \xrightarrow{x} X\).

Corollary 4.10. The subdiagram \(\Gamma_F\) of \(\Theta_F\) consisting of those objects \(P \xrightarrow{a} A\) with \(A\) finite connected is cofinal in \(\Theta_F\) and thus by G0) it is also cofinal in the whole diagram of \(F\). Notice that this subdiagram is a poset by proposition 4.5, see remark 3.2.

Corollary 4.11. Every connected object is finite.
Proof.

Given $P \to X$, take

\[ P \to A \to X \]

with $A$ finite connected. If $X$ is connected, then $A \to X$ is an strict epimorphism, thus $F(A) \to F(X)$ is surjective. It follows that $F(X)$ is finite because $F(A)$ is.

**Proposition 4.12.** The subdiagram $\Gamma_F$ has meets (finite infima).

**Proof.**

Let $P \overset{a}{\to} A, P \overset{b}{\to} B$ and take $C$ connected such that

\[ P \to C \to I \]

\[ A \quad C \quad B \]

\[ a \quad a \quad b \]

By abuse of notation we indicate by the same label the arrows from $C$. Take then the strict epi-mono factorization

\[ C \overset{a,b}{\to} A \times B \]

\[ I \]

Since $C$ is connected, by proposition 4.6, so is $I$. We shall see that $P \to C \to I$ is the infima in $\Theta_F$ of $P \overset{a}{\to} A$ and $P \overset{b}{\to} B$. In fact, given

\[ P \to Z \to B \]

\[ A \]

\[ a \quad b \]

\[ x \quad y \]

with $Z$ connected, take the pullback $H$: 


Since $P \to Z$ is minimal it follows that $H \cong Z$. This finishes the proof.

Theorem 4.13. The full subcategory of non empty connected objects $\text{Con}(\mathcal{C}) \hookrightarrow \mathcal{C}$ together with the functor $F$ satisfies the axioms C0) to C3).

Proof.

C0) By proposition 4.5.
C1) By axioms G2) and G5).
C2) By corollary 4.11.
C3) By proposition 4.12.

We pass now to prove that every object in $\mathcal{C}$ is a [finite] coproduct of connected objects.

Proposition 4.14. Given any two disjoint subobjects of $X$, $A \to X$ and $B \to X$, the map $A \amalg B \to X$ is a monomorphism.

Proof.

Apply the functor $F$ and use the corresponding result which holds in $\mathcal{E}ns$. Use then the fact that $F$ reflects monomorphisms (proposition 4.2 item 4).

Proposition 4.15. Given any two connected subobjects $A \hookrightarrow X$ and $B \hookrightarrow X$, then $A \cap B \cong \emptyset$ or there exist an isomorphism $A \cong B$ such that

\[
\begin{array}{ccc}
A & \cong & B \\
\downarrow & & \downarrow \\
X & & \\
\end{array}
\]

Proof.

Consider the pullback diagram and the strict epi-mono factorization given by G3) as indicated in the following diagram:
Since $A$ is connected, $I \cong \emptyset$ or $I \cong A$. In the first case, $A \cap B \cong \emptyset$ and in the second the map $A \cap B \rightarrow A$ is a monomorphism and a strict epimorphism, thus is an isomorphism.

**Theorem 4.16.** Every [finite] object $X \in C$ is a [finite] coproduct of connected objects of $C$.

**Proof.**

Assume $X \not\cong \emptyset$. Given $P \xrightarrow{x} X$, let $P \xrightarrow{\alpha_x} A_x$ be a factorization:

$$
\begin{array}{ccc}
P & \xrightarrow{\alpha_x} & A_x \\
x \downarrow & \downarrow & \downarrow \theta_x \\
X & & X
\end{array}
$$

with $A_x$ connected. This factorization always exists by corollary 4.10.

Take the strict epi-mono factorization of each $\theta_x$ given by G3):

$$
\begin{array}{ccc}
A_x & \xrightarrow{\theta_x} & X \\
& \searrow & \nearrow \theta_x \\
& I_x & X
\end{array}
$$

By proposition 4.6, the objects $I_x$ are connected and by proposition 4.15 we can take a subfamily $I_l$ with $l \in J \subseteq [P, X]$ such that:

i) If $l \neq s$ then $I_l \cap I_s \cong \emptyset$

ii) $\forall x \in [P, X]$, there exists $l \in J$ and an isomorphism such that

$$
\begin{array}{ccc}
I_x & \cong & I_l \\
& \searrow & \nearrow \theta_x \\
& X & X
\end{array}
$$

From i) and proposition (4.14) it follows that the map $\coprod_{l \in J} I_l \xrightarrow{\lambda} X$ is a monomorphism, thus $F(\lambda)$ is injective. We shall see now that $F(\lambda)$ is surjective. Then, by G6) it will follow that $\lambda$ is an isomorphism.
In fact, let $x \in F(X)$, that is $P \xrightarrow{x} X$. We have $P \rightarrow I_x \rightarrow X$. It follows immediately from ii) above that $x$ comes from some $l \in F(I_x)$,

$$
\begin{array}{c}
P \\
\downarrow \quad \downarrow\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
I_l \quad X
\end{array}
$$

This finishes the proof. Clearly, if $X$ is finite, this coproduct is finite.

**Proposition 4.17.** Given any $X \in \mathcal{C}$, the set $[P, X] = F(X)$ has a continuous left action by the group $\pi = \text{Aut}(P)_{\text{op}}$ (which now is not transitive when $X$ is not connected).

**Proof.**

Write $X \cong \bigsqcup_{i \in I} X_i$ with $X_i$ connected. It follows from theorem 4.13 (and proposition 2.3) that each $[P, X_i]$ has a continuous left action by the group $\pi$. The proof follows immediately since the functor $[P, -]$ preserves coproducts. In fact, consider the following diagram:

$$
\begin{array}{c}
\pi \times [P, X] \\
\downarrow \quad \downarrow \\
\pi \times [P, X_i] \quad \longrightarrow \quad [P, X_i]
\end{array}
$$

where the vertical arrows are coproduct diagrams in $\mathcal{E}_\text{ns}$. The horizontal dotted arrow exist by the universal property, and it is continuous since the left vertical arrows are also a coproduct diagram in the category of topological spaces because $\pi$ is compact Hausdorff.

We are now in conditions to establish Grothendieck’s Theorem.

Let $[P, X]_{\pi}$ be the $\pi$-set defined in proposition 4.17. Then:

**Theorem 4.18.** Let $\mathcal{C}$ be a category such that axioms 4.1 hold. Consider the functor:

$$
\mathcal{C} \xrightarrow{[P, -]_{\pi}} \mathcal{E}_\text{ns}^\pi
$$

defined in proposition 4.17. Then this functor is an equivalence of categories.

Let $f\mathcal{C}$ be the full subcategory of finite objects. Then the functor $[P, -]_{\pi}$ restricts into the functor:

$$
f\mathcal{C} \xrightarrow{[P, -]_{\pi}} f\mathcal{E}_\text{ns}^\pi
$$

which also establishes an equivalence of categories.
Proof.

We have commutative diagrams of categories and functors:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{[P,-]_\pi} & c\mathcal{E}ns^\pi \\
\uparrow & & \uparrow \\
\text{Con}(\mathcal{C}) & \xrightarrow{[P,-]_\pi} & ct\mathcal{E}ns^\pi \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{[P,-]_\pi} & f\mathcal{E}ns^\pi \\
\uparrow & & \uparrow \\
\text{Con}(\mathcal{C}) & \xrightarrow{[P,-]_\pi} & ct\mathcal{E}ns^\pi
\end{array}
\]

(4.1)

where by $\text{Con}(\mathcal{C})$ we indicate the full subcategory of non empty connected objects.

It is immediate to see that every object in $c\mathcal{E}ns^\pi$ (resp. $f\mathcal{E}ns^\pi$) is a coproduct (respectively finite coproducts) of transitive actions. Define the functors:

\[
\begin{array}{ccc}
\mathcal{C} & \leftarrow \mathcal{C} \times_\pi (\mathcal{E}ns^\pi) \\
\uparrow & & \uparrow \\
\text{Con}(\mathcal{C}) & \leftarrow \text{Con}(\mathcal{C}) \times_\pi (\mathcal{E}ns^\pi)
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C} & \leftarrow \mathcal{C} \times_\pi (\mathcal{E}ns^{<\infty}) \\
\uparrow & & \uparrow \\
\text{Con}(\mathcal{C}) & \leftarrow \text{Con}(\mathcal{C}) \times_\pi (\mathcal{E}ns^{<\infty})
\end{array}
\]

using coproducts in $\mathcal{C}$ (resp. finite coproducts in $f\mathcal{C}$) and the fact that it is already defined in $ct\mathcal{E}ns^\pi$. Since the functors:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{[P,-]_\pi} & c\mathcal{E}ns^\pi \\
\uparrow & & \uparrow \\
\mathcal{C} & \xrightarrow{[P,-]_\pi} & c\mathcal{E}ns^{<\infty}
\end{array}
\]

preserve coproducts (resp. finite coproducts), the proof follows from theorem 4.16 and the fact that the arrows below in the diagrams (4.1) are an equivalence of categories by theorem 4.13 and theorem 3.21 in section III.

V - All actions of a discrete group.

In section II we developed the representable and connected case, which characterize the category of transitive actions of a discrete group. We restrict to the connected case for two reasons. First, it develops in an abstract setting exactly the same techniques that are utilized by topologists in the theory of covering spaces and its relation with the fundamental group. This clearly shows where the ideas come from. Second, it is the only case needed for Grothendieck’s development of the general case (non representable non connected), as we show in sections III and IV. Here quotients by group actions on connected objects play a central role, and coproducts are only used to extend the constructions to the non connected objects.

However, it seems to us that in the representable case different phenomena are also behind the result. Concretely, a direct characterization of the category (now a topos) of all actions of a discrete group can be obtained, rather than derive it from the connected case (as we do for the non representable case in section IV). The proof uses some general categorical techniques that have its own interest and apply to many other situations (like Beck’s tripability theorem, Giraud theorem on characterization of topoi,
and in the additive case, representation theorems for abelian categories and the Morita
 equivalences).

Let $C$ be any category and $A \in C$ an object. Let $G = [A, A]^{op}$ the opposite monoid
of the monoid of endomorphisms of $A$.

**Proposition 5.1.** The monoid $G$ acts on the left on the sets $[A, X]$ of morphisms in
$C$ from $A$ to any other object $X \in C$:

\[
G \times [A, X] \rightarrow [A, X]
\]

\[
g \times x \rightarrow gx = x \circ g
\]

where “$\circ$” is composition of arrows in $C$.

We have in this way a functor:

\[
C \xrightarrow{[A, -]} E_{ns}^G
\]

where by $[A, X]_G$ we indicate the set $[A, X]$ with this action of $G$ and $E_{ns}^G$ indicates
the category of $G$-sets.

We assume now that the category $C$ has coproducts and coequalizers. It is well known
and easy to see that the representable functor $[A, -]$ has a left adjoint (the “tensor” in
closed category terminology) that we denote here $A \bullet (-)$, and that it is given by the
formula

\[
A \bullet S = \coprod_S A
\]

Clearly, by definition of coproducts, there is a bijection:

\[
\frac{A \bullet S \rightarrow X}{S \rightarrow [A, X]} \text{ natural in } X \tag{5.1}
\]

In the case that $C = E_{ns}^G$, for any monoid $G$, and $A = G$ with its canonical action
on itself, the representable functor $[G, -]$ is just the underline set functor ($G$ is the free
action on one generator), $[G, E] = |G|$, and the functor $G \bullet (-)$ is conveniently given by
the formula:

\[
G \bullet S = G \times S \quad \text{with the action} \quad g(f, s) = (gf, s)
\]

We have then the following diagram of categories and functors:
where the triangle formed by the right adjoints commutes.

In this situation, if the category $\mathcal{C}$ has coequalizers, there is a left adjoint $A \times_G (-)$ for the functor $[A, -]_G$ that it is given by an specific construction in terms of the other two adjointness in the triangle (see [3]). In the particular situation considered here we have:

**Proposition 5.2.** Let $\mathcal{C}$ be a category with coproducts and coequalizers (thus, all colimits), and let $A$ in $\mathcal{C}$. Then, the functor $[A, -]_G$ has a left adjoint $A \times_G (-)$ defined, given $E$ in $\text{Ens}^G$, as the following collective coequalizer:

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & A \times_G E \\
\downarrow{g} & & \downarrow{q} \\
A & \xrightarrow{\lambda_g} & A \times_G E
\end{array}
\]

That is, the arrow $A \bullet [E] \xrightarrow{q} A \times_G E$ is universal with respect to the equations $q \circ \lambda_x \circ g = q \circ \lambda_{gx}$, one for each $x \in E$ and $g \in [A, A]$.

**Proof.**

The proof is immediate, just check that in the bijection (5.1), an arrow $A \bullet [E] \xrightarrow{f} X$ factors through $A \times_G E$ (that is, it satisfies all equations $f \circ \lambda_x \circ g = f \circ \lambda_{gx}$) if and only if the corresponding arrow $E \rightarrow [A, X]_G$ is a morphism of actions.

We set now the following definition:

**Definition 5.3.** An object $A$ in a category with coproducts $\mathcal{C}$ is called a generator if for every $X \in \mathcal{C}$ the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & A \bullet [A, X] \\
\downarrow{g} & & \downarrow{\varepsilon} \\
A & \xrightarrow{\lambda_g} & X
\end{array}
\]

where $\varepsilon$ is the unique map such that $\varepsilon \circ \lambda_x = x$, is a collective coequalizer. In particular, every object is a quotient of a sum of copies of $A$. 

Notice that the map $A \cdot [A, X] \to X$ induces precisely the counit of the adjunction $A \times_G [A, X] \to X$. Thus, $A$ is a generator if and only if this counit is an isomorphism. This notion is sometimes called “dense generator”.

**Proposition 5.4**. Let $C$ be a category with coproducts and coequalizers and $A \in C$. Then:

i) If $A$ is a generator then the representable functor $[A, -]$ reflects isomorphisms.

ii) Assume the representable functor $[A, -]$ preserves coproducts and coequalizers. Then, if it reflects isomorphisms, $A$ is a generator.

**Proof.**

i) Let $X \to Y$ and consider the commutative square:

$$
\begin{array}{ccc}
A \cdot [A, X] & \to & X \\
A \cdot f_* & \downarrow & f \\
A \cdot [A, Y] & \to & Y
\end{array}
$$

where the horizontal arrows are collective coequalizers with respect to the equations in definition 5.3. If $f_* = [A, f]$ is a bijection, $A \cdot f_*$ is an isomorphism, and one can readily check that the bijection $f_*$ establishes also a correspondence between the respective equations.

ii) Let $A \cdot [A, X] \to Q$ be the collective coequalizer, and consider:

$$
\begin{array}{ccc}
A \cdot [A, X] & \to & Q \\
& \nearrow & \\
& \eta & \searrow \\
& X
\end{array}
$$

We shall see that $\eta$ is an isomorphism. Applying the functor $[A, -]$ we have:

$$
\begin{array}{ccc}
[A, A \cdot [A, X]] & \to & [A, Q] \\
& \nearrow & \eta_* \\
& [A, X]
\end{array}
$$

Where $q_*$ is the respective coequalizer in the category of sets. Since for every $x \in [A, X]$ we have $\varepsilon_*(\lambda_x) = x$, we have that $\eta_*$ is surjective. To see that it is also injective we have to see that $\varepsilon_*$ does not identify more than $q_*$. Let $u, v \in [A, A \cdot [A, X]]$ be such that $\varepsilon \circ u = \varepsilon \circ v$. Since $[A, -]$ preserves coproducts, $u = \lambda_x \circ h, v = \lambda_y \circ g$, for some $x, y \in [A, X]$ and $h, g \in G$.

In the next chain of equations, each one follows immediately from the preceding one:

$$
\begin{align*}
\varepsilon \circ u = \varepsilon \circ v & \implies \varepsilon \circ \lambda_x \circ h = \varepsilon \circ \lambda_y \circ g \\
& \implies \varepsilon \circ \lambda_{hx} = \varepsilon \circ \lambda_{gy} \\
& \implies hx = gy \\
& \implies \lambda_{hx} = \lambda_{gy} \\
& \implies q \circ \lambda_{hx} = q \circ \lambda_{gy} \\
& \implies q \circ \lambda_x \circ h = q \circ \lambda_y \circ g \\
& \implies q \circ u = q \circ v.
\end{align*}
$$
5.5. Axioms for the representable case

Consider a category \( C \) and an object \( A \in C \).

Axioms on \( C \): 
R1) \( C \) has a terminal object and pullbacks (thus all finite limits).
R2) \( C \) has coequalizers.
R3) \( C \) has coproducts.

Axioms on \( A \) (in terms of the representable functor \( [A, -] \)): 
R4) \( [A, -] \) preserves coequalizers.
R5) \( [A, -] \) preserves coproducts.
R6) \( [A, -] \) reflects isomorphisms.

Theorem 5.6. Let \( C \) be any category and \( A \in C \) as in 5.5. Then the left adjoint \( A \times_G (-) \dashv [A, -]_G \) exists and establishes an equivalence of categories.

Proof.

By R6 and proposition 5.4 the counit of the adjunction in \( C \) is an isomorphisms. Since \( G \) is itself a generator (in the sense of definition 5.3) for the category \( \mathcal{E}ns^G \), and by R4 and R5 the monad (composite of the two functors) in \( \mathcal{E}ns^G \) preserves coproducts and coequalizers, it is enough to check that the unit is an isomorphisms in the object \( G \). But this is clear. In fact, using again that \( A \) is a generator, we have \( A \times_G G \) is isomorphic to \( A \), thus \( [A, A \times_G G] \) is isomorphic to \( [A, A] = G \).

It is interesting to observe that it is not necessary to assume any exactness properties in the category \( C \). However, if we want to write the axioms R4 and R5 as properties of the object \( A \) (projective and connected), then we need some of the exactness properties characteristic of a topos (or pretopos).

We recall (in 5.7, 5.8, 5.9, and 5.10) from [[1], I, 10] the definition of strict epimorphism and some of its properties. The interested reader can also consult [2].

Definition-Proposition 5.7.

1) An arrow \( X \xrightarrow{f} Y \) in any category \( C \) is an strict epimorphism if given any compatible arrow \( X \xrightarrow{g} Z \), there exists a unique \( Y \xrightarrow{h} Z \) such that \( g = h \circ f \). \( g \) is compatible if for all \( C \xrightarrow{x} X \), \( C \xrightarrow{y} X \) with \( g \circ x = g \circ y \), then also \( f \circ x = f \circ y \).

2) It immediately follows that:

\[ \text{strict epimorphism} + \text{monomorphism} = \text{isomorphism} \]

3) The kernel pair of an arrow \( X \xrightarrow{f} Y \) is the pull-back of \( f \) with itself, and it will be denoted \( R_f \). The subobject \( R_f \subset X \times X \) is always an equivalence relation.

4) An equivalence relation is called effective if the quotient exists and it becomes the kernel pair of this quotient.

5) An strict epimorphism with a kernel pair is called effective.
6) Given any arrow $X \xrightarrow{f} Y$, the diagram

$$
\begin{array}{ccc}
R_f & \xrightarrow{f} & X \\
\downarrow & & \downarrow \xrightarrow{f} \\
\end{array}
$$

is a coequalizer if and only if $f$ is an strict epimorphism.

7) A diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow & & \downarrow \xrightarrow{f} \\
\end{array}
$$

is a coequalizer if and only if $f$ is an strict epimorphism, and $R_f$ is the smallest effective equivalence relation that factories the arrow $(g,h) : Z \rightarrow X \times X$.

**Proposition 5.8.** (strict epi - mono factorizations, Tierny-Kelly). Let $C$ be any category with pull-backs and quotients of equivalence relations which are stable under pulling-back (equivalently, strict epimorphisms are stable under pulling back). Then $C$ has strict epi-mono factorizations. That is, given any $f : X \rightarrow Y$, there is a factorization $X \xrightarrow{e} I \xrightarrow{i} Y$ where $e$ is a strict epimorphism and $i$ is a monomorphism.

**Proof.**

Let $X \rightarrow Z$ be the quotient of $X$ by the kernel pair of $f$. Clearly there is a factorization $X \rightarrow Z \rightarrow Y$. From the stability of strict epimorphisms it readily follows that the arrow $Z \rightarrow Y$ is a monomorphism.

**Definition-Proposition 5.9.**

1) A category $C$ is Regular if it has finite limits and strict epi - mono factorizations (thus, direct images) stable under pull-backs.

2) A relation in $X$ is a subobject of $X \times X$. Using pull-backs and images the composition of relations is defined in the usual way as in the category of sets.

3) If $C$ has unions of denumerable chains of subobjects which are stable under pulling-back, the transitive closure of any relation is constructed as the union of iterated compositions. Thus, if in addition there are finite unions of subobjects (to make a relation reflexive and symmetric), any relation is contained in a smallest equivalence relation, which is generated as described above. Notice that denumerable stable coproducts are sufficient, since unions are the direct image of the induced map from the coproduct.

**Proposition 5.10.** Let $C$ be a Regular Category such that every equivalence relation has a quotient and every denumerable chain of subobjects has an stable union (or denumerable stable coproducts exists in $C$). Then, $C$ has coequalizers. If in addition, equivalence relations are effective, then any finite limit preserving functor $F : C \rightarrow E$ into any other such category will preserve coequalizers if it preserves strict epimorphisms and stable denumerable unions (or preserves stable denumerable coproducts).
Proof.

Given any pair of arrows \( Z \xrightarrow{g} X, Z \xrightarrow{h} X \), let \( R(g, h) \) be the equivalence relation generated (as in 5.9.3) by the image of \( X \xrightarrow{(g, h)} X \times X \), and \( X \xrightarrow{f} Y \) its quotient. If an arrow \( X \xrightarrow{s} H \) coequalizes \( g \) and \( h \), \((g, h)\) factors through its kernel pair \( R_s \subset X \times X \). Thus \( R(g, h) \subset R_s \), which implies there is a map \( Y \to H \) showing that \( X \xrightarrow{f} Y \) is the coequalizer of \( g \) and \( h \). For the second assumption, notice that if equivalence relations are effective, 7) in 5.7 says that \( R_f \) is just the smallest equivalent relation that factories \((g, h)\). Then, the proof is clear.

5.11. Axioms for the representable case (second version)

Consider a category \( C \) and an object \( A \in C \).

Axioms on \( C \):

E1) \( C \) has a terminal object and pullbacks.

E2) \( C \) has quotients of equivalent relations which are then the kernel pair of this quotient, and this quotient is stable by pulling back (i.e. equivalence relations are effective and universal).

E3) \( C \) has an initial object and coproducts stable by pulling back.

Notice that we do not require that coproducts be disjoint.

Axioms on \( A \):

E4) \( A \) is projective (see 5.14).

E5) \( A \) is connected (see 5.12).

E6) \( A \) is a generator (see 5.3).

Notice that E6) implies that \( A \) is non empty.

Definition-Proposition 5.12. An object \( A \) in a category \( C \) is connected if and only if it satisfies any of the two following equivalent statements:

i) If \( A \cong A_1 \amalg A_2 \), then \( A_1 \cong \emptyset \) or \( A_2 \cong \emptyset \).

ii) If \( A \cong \coprod_{i \in I} A_i \), with \( I \) any set, then there exists an \( i \in I \) such that \( A \cong A_i \) and \( A_j \cong \emptyset \) for every \( j \neq i \).

Proof.

We can assume \( A \not\cong \emptyset \). For each \( i \) we have \( A \cong A_i \amalg \coprod_{j \neq i} A_j \). Clearly, there exists \( i \) such that \( A_i \not\cong \emptyset \). Then \( \coprod_{j \neq i} A_j \cong \emptyset \). Thus, \( A_j \cong \emptyset \) for all \( j \) (consider the arrow \( A_j \xrightarrow{\lambda_j} \coprod_j A_j \)).

Remark 5.13. By axiom E3) pulling back along any arrow preserves coproducts, then it follows immediately that if \( A \) is connected, the representable functor \( [A, -] \) preserves coproducts. If the initial object is empty and coproducts are disjoint, then the other implication also holds.
Definition 5.14. By projective we mean the assumption that the functor $[A,-]$ preserves strict epimorphisms (this is weaker than the usual meaning of projective, which is defined with respect to all epimorphisms).

Theorem 5.15. Let $C$ be any category and $A \in C$ as in 5.11. Then the left adjoint $A \times_G (-) \dashv [A,-]_G$ exists and establishes an equivalence of categories.

Proof.

We shall see that the axioms in theorem 5.6 hold. R1) = E1), and from propositions 5.8 and 5.10 we have R2). E4) and E5) imply R4) by remark 5.13 and proposition 5.10, and E5) gives R5) also by remark 5.13. Finally, E6) implies R6 by proposition 5.4.

Group actions

Group actions come into the scenery when the monoid $[A,A]$ is actually a group. That is, every endomorphism of $A$ is an isomorphism, $\text{Aut}(A) = [A,A]$. In this case, the theorems above are much simpler. Notably, there is no need to develop all the calculus of relations in regular categories, neither to consider equivalence relations, transitive closures and coequalizers in general. In place of all this, quotients by group actions do all the work.

To guide the reasoning it is useful to consider now the elementwise description of coproducts which consist on writing informally:

$$A \cdot S = \{(a,x)/x \in S \text{ and } a \in \text{the x-copy of } A\}$$

Then the object $A \times_G E, G = \text{Aut}(A)^{\text{op}}$, can be described informally as the object $A \cdot |E|$ divided by the equivalence relation (clearly reminiscent of the tensor product construction) defined by pairs $(ag,x) = (a,gx)$ ($G$ acts on the right on "a" because it is the opposite group $\text{Aut}(A)^{\text{op}}$). This equivalence relation is thus $(b,x) = (a,y)$ if and only if there exists $g \in G$ such that $b = ag$ and $y = gx$. There is a left action of $G$ in $A \cdot |E|$ given by $g(a,x) = (g^{-1}a,gx)$, and the orbits of this action define precisely the equivalence classes of the relation above.

It is instructive to see directly the consistency of this construction with the one given for transitive actions in section II (both constructions must coincide by the universal property of the adjoint). Given an $x_0 \in E$, we have the equations $q \circ \lambda_{x_0} \circ h = q \circ \lambda_{x_0}$ for all $h \in H = \text{Fix}(x_0)$. Thus the map $\lambda_{x_0}$ induces an injective map $A/H \rightarrow A \times_G E$.

When the action is transitive, it is immediate to see using the description above that it is also surjective.

Formally:

Proposition 5.16. There is a left action of $G$ on $A \cdot |E|$ defined by the following diagrams (one for each $g \in G$):
and the quotient by this action (definition 2.2) is precisely the collective coequalizer that defines $A \times_G (-)$.

**Theorem 5.17.** Let $\mathcal{C}$ be any category and $A \in \mathcal{C}$ as in 5.5 but with axioms R2) and R4) respectively replaced by:

- R'2) $\mathcal{C}$ has quotients of objects by groups of automorphisms.
- R'4) $[A, -]$ preserves quotients of objects by groups of automorphisms.

Assume that $[A, A] = \text{Aut}(A)$, then the left adjoint $A \times_G (-) \dashv [A, -]_G$ exists and establishes an equivalence of categories.

**Proof.**

It follows the same lines that the proof of theorem 5.6. By the proposition above and R’2 we can construct the functor $A \times_G (-)$. From definition 5.3 it is clear that when $G$ is a group R’4 and R5 imply that it is enough to check that the unit is an isomorphisms in the object $G$, which we do as in theorem 5.6.

**Definition-Proposition 5.18.** Let $\mathcal{C}$ be any category with finite limits and coproducts and let $H$ be a group that acts by automorphisms in an object $A \in \mathcal{C}$. Then the quotient $A \overset{q}{\longrightarrow} A/H$ (see definition 2.2) fits into a coequalizer:

$$
\begin{array}{ccc}
A \cdot H & \overset{\nabla}{\longrightarrow} & A \\
\nabla_H & \searrow & q \\
A & \nearrow & A/H
\end{array}
$$

where $\nabla$ is the codiagonal and $\nabla_H$ is defined by:

$$
\begin{array}{ccc}
A \cdot H & \overset{\nabla_H}{\longrightarrow} & A \\
\lambda_h & \searrow & h \\
A & \nearrow & A
\end{array}
$$

We say that the action of $H$ on $A$ is effective if the quotient exists and the map

$$
\begin{array}{ccc}
A \cdot H & \overset{(\nabla, \nabla_H)}{\longrightarrow} & R_q \subseteq A \times A
\end{array}
$$

is an strict epimorphism.
Proposition 5.19. Let $\mathcal{C}$ be any category with finite limits, coproducts and where actions of groups by automorphisms are effective. Then any finite limit preserving functor $F : \mathcal{C} \to \mathcal{E}$ into any other such category will preserve quotients by actions of groups if it preserves strict epimorphisms and coproducts.

Proof.

Consider the diagram

\[
\begin{array}{ccc}
A \bullet H & \rightarrow & A \\
\downarrow \nabla H & & \downarrow q \\
A & \rightarrow & A/H \\
\end{array}
\]

By the assumptions made this diagram goes by $F$ into a diagram:

\[
\begin{array}{ccc}
FA \bullet H & \rightarrow & FA \\
\downarrow \nabla H & & \downarrow Fq \\
FA & \rightarrow & F(A/H) \\
\end{array}
\]

That $F(q)$ is the coequalizer of $\nabla$ and $\nabla H$ follows immediately from the fact that $F(e)$ and $F(q)$ are both strict epimorphisms.

Theorem 5.20. Let $\mathcal{C}$ be any category and $A \in \mathcal{C}$ as in 5.11 but with $E2$) replaced by:

$E’2)$ In $\mathcal{C}$ actions of groups by automorphisms are effective.

Assume that $[A, A] = \text{Aut}(A)$, then the left adjoint $A \times_G (-) \dashv [A, -]_G$ exists and establishes an equivalence of categories.

Proof.

Construct the functor $A \times_G (-)$ as in theorem 5.17. To finish the proof it remains to see that $[A, -]$ preserves quotients by group actions. But this is done in the proposition above.

VI - Final Comments.

We have seen how Grothendieck’s interpretation of the Galois’s Galois Theory (and its natural relation with the fundamental group and the theory of coverings) allows the development of a categorical theory much beyond the original Galois’s scope. Here are some further results essentially proved in the previous sections and some comments about them.
6.1. On sections II and V

We can easily generalize the results in section V by replacing the single object $A$ by a (small) set of objects and its associated full subcategory $A \subset \mathcal{C}$. In this case the category $\mathcal{E}ns^G$ becomes the presheaf category $\mathcal{E}ns^{A^{op}}$ and the restriction of the Yoneda embedding $h : \mathcal{C} \to \mathcal{E}ns^{A^{op}}$ corresponds to the functor $[A, -]_G$. This generalizes proposition 5.1. The left adjoint to $h$ is given by a Kan extension that we denote $k : \mathcal{E}ns^{A^{op}} \to \mathcal{C}$. Given $E \in \mathcal{E}ns^{A^{op}}$, consider its canonical diagram $\Gamma_E$ so that

$$E = \operatorname{colim}_{(a, A) \in \Gamma_E} [-, A]$$

Then $k(E)$ is the corresponding colimit taken in $\mathcal{C}$

$$k(E) = \operatorname{colim}_{(a, A) \in \Gamma_E} A$$

The adjointness is easily seen from this definition. This generalizes Proposition 5.2.

We pass now to the axiom set 5.11. Axioms E1), E2) and E3) are respectively conditions a), c) and b) in Giraud's Theorem [[1], IV, 1.2]. Axiom E6) says that the category $\mathcal{C}$ has an (small) generating set, which is condition d) in that theorem. We have in addition axioms E4) and E5), which say that the objects in the generating set are projective and (non-empty) connected. Thus, theorem 5.15 is Giraud’s Theorem in the particular case in which E4) and E5) hold. It gives a characterization of categories of presheaves. Compare also with [[1], IV, 7.6], where condition [ii): the full subcategory of (non empty) connected projective objects is generating] in exercise 7.6 d) consists precisely on the axioms E4), E5) and E6) together. This characterization is due to J. E. Ross.

Giraud’s Theorem can be analyzed as follows: Assume axioms E1), E2), E3) and E6) (not E4) and E5)), that is, Giraud’s assumptions. Then we have an adjunction as indicated above (proposition 5.2. Axiom E6) says (tautologically) that the counit of this adjunction is an isomorphism, making $\mathcal{C}$ a full subcategory of $\mathcal{E}ns^{A^{op}}$. How can we make the unit of the adjunction in $\mathcal{E}ns^{A^{op}}$ be an isomorphism without E4) and E5) ?, (as we do in theorem 5.11). Giraud’s answer is to force this unit to be an isomorphism by means of the appropriate Grothendieck topology (here it is probably needed that coproducts be disjoint in addition to stable).

All this relativices to the additive case, where, in the place of sets, the category of abelian groups is the base category. In fact, it was this case that was developed first. The statements and their proofs are essentially the same. A conspicuous difference is that now preservation of coproducts by the representable functor $[A, -]$ means that the object $A$ is “small”, instead of “connected”. A coproduct of two small objects is still small, and this implies that the category of modules over a ring does not characterize the ring (while the category of actions of a monoid does characterize the monoid). An earlier instance of the additive case of theorem 5.15 are the Morita equivalences. The
abstract additive theorem 5.15 was known I imagine by the end of the fifties. It is explicitly stated for example in \([5], \text{Ch 4, F}\). The generalization to several objects discussed above in \([5], \text{Ch 5, H}\).

6.2. On sections III and IV

In section III, theorem 3.21, we proved a characterization of the category of transitive actions of a profinite group by “passing into the limit” (theorem 2.11 in section II on transitive actions of a discrete group) in a filtered colimit of categories. Using this, we show in section IV Grothendieck’s theorem in \([6]\). This theorem (4.18 in section IV) implies that a connected locally connected \([1], \text{IV, 8.7, 1}\) boolean topos with a “profinite point” (that is, its inverse image functor satisfies axiom G0 in 4.1) is the topos of continuous \(\pi\)-sets for a profinite group \(\pi\) (locally connected + boolean = atomic, see \([8], \text{Ch. 8}\)). A careful analysis shows, in the light of axioms 3.1 and proposition 3.6, that axiom G6) can be omitted in 4.1.

Our proof shows that Grothendieck’ theorem is the result of “passing into the limit” (theorem 5.17 in section V on all actions of a discrete group) in a filtered inverse limit of topoi. Warning: such a limit is not a colimit of categories and inverse image functors, but, nevertheless, we have such a colimit on the sites, (see \([1], \text{VI, 8.2.3, 8.2.9}\), also \([10]\)).

This theorem can be generalized to the case in which the point is not any more “profinite”. Here it is necessary to consider the diagram of discrete groups without passing to its inverse limit group, since the projections from this limit can not any more be guaranteed to be surjective \([1], \text{IV, 2.7}\). Thus, the topological group is replaced by a whole system of discrete groups, called a “progroup” by Grothendieck. The corresponding theorem is a theorem of characterization of the category of all actions (as defined by Grothendieck) of a progroup. In this more general context the existence of the Galois Closure can not be proved as in proposition 3.13. It is necessary to introduce an stronger (than just finite limit) preservation property in the functor F. This condition (preservation of cotensors = arbitrary products of a same object) replace the finiteness condition G0. We will show how to do all this elsewhere.

In \([9], \text{VIII, 3, theorem 1}\] the finiteness condition in theorem 4.18 it is also removed. This is done following a different line that the one sketched above. As it was stressed by Joyal since the early seventies, topological spaces are replaced by “spaces”, which are the dual objects of locales (sup-complete distributive lattices). The profinite (topological) group becomes an spatial group, which in general will not have sufficiently many points, and thus it will not be a topological group in the classical sense. An spatial group is a more general concept that a progroup. The topos of continuous \(\pi\)-sets of an spatial group satisfy the condition on preservation of cotensors presicely when the spatial group corresponds to a progroup.

Finally, let us point out that all this should be generalized taking a category (grupoid) of appropriate points, instead of a single point, as it is already developed in \([6]\), and
continued in [7]. It is not clear to us the relation of this with the principal theorem in [9] concerning continuous actions of an spatial grupoid.

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