Integration by Parts Formula and Smoothness of Densities of Solutions to SDE’s with Locally Lipschitz Coefficients

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Abstract

In this work we prove the existence of a smooth density for the solution to an SDE with locally Lipschitz and semi-monotone drift, and will derive an exponential decay for this density and all of its derivatives as well. Our main tool in this paper is an integration by parts formula for the solution of the mentioned SDE in the Wiener space. We construct an approximating sequence of SDE’s with globally Lipschitz drifts and obtain a uniform bound for the integral of their solutions from which we derive the exponential decay for the derivatives of the density of the original SDE.

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1. Introduction

Several studies have focused on the existence of smooth density and Integration by parts formula for the solutions of SDEs. Kusuoka and Stroock \cite{1} show that an SDE whose coefficients are $C^\infty$-globally Lipschitz and have polynomial growth, has a strong Malliavin differentiable solution of any order. Also, in \cite{2}, assuming some nondegeneracy condition they find an upper bound for all the derivatives of the density of solution, depending on the coefficients of SDE and their derivatives. This nondegeneracy condition could be also used to show the absolute continuity of the law of the solution of SDEs with respect to the Lebesgue measure and the smoothness of its density (see e.g. \cite{3, 4}). In recent years, there were attempts to generalize these results to SDEs with non-globally Lipschitz coefficients. For example, Yuki \cite{5} derive some local Hölder continuity behaviour of the densities of the solutions to SDEs with singular drifts and locally bounded coefficients. Using a Fourier transform argument and the Malliavin Calculus, Marco \cite{6} shows that the solution of an SDE with smooth coefficients for which the derivatives of coefficients are bounded on a domain $D$, has a strong solution with a smooth density on $D$. For other references on this subject, we refer the reader to \cite{7, 8, 9}.

Assuming the nondegeneracy condition one can derive some integration by parts formula on the Wiener space (see e.g. \cite{3}). This formula has many applications for example in financial mathematics. It is often of interest to investors to derive an option pricing formula and to know its sensitivity with respect to various parameters. The integration by parts formula obtained from Malliavin calculus can transform the derivative of the option price into weighted integral of random variables. This gives much more accurate and fast converging numerical solution estimates than obtained by the classical methods \cite{10, 11}. The

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interested reader could see [12, 13, 14, 6, 16, 17].

The SDE we consider has not global Lipschitz coefficients. Such equations mostly come from finance and biology and also dynamical systems and are more challenging when considered on infinite dimensional spaces. (see e.g. [18, 19, 20])

In this paper, we consider an SDE with locally Lipschitz coefficients and uniformly elliptic diffusion. In [15], we have proved the uniqueness and existence of a solution $X_t$ to this equation. Here we are going to prove the existence of a smooth density for $X_t$, and derive an exponential decay for the derivatives of this density, at infinity.

Since the drift of the SDE is not globally Lipschitz, we will construct a sequence of SDEs with globally Lipschitz drifts whose solutions have uniformly bounded Malliavin derivatives with respect to $n$, and converge to $X_t$ almost everywhere. In this way we can apply the classical Malliavin calculus to these solutions, and by the uniform boundedness of the moments of inverses of Malliavin covariance matrices and the convergence result we are able to prove an integration by parts formula in the Wiener space. Then we will prove that the densities of laws of the solutions to the constructed sequence of SDEs converge to the solution of the original SDE and derive the exponential decay of the density of the solution to it.

The organization of the paper is as follows. In section 2, we present some notions of Malliavin calculus, as stated in [3], and by use of them we formulate our main results. In section 3, we will prove the integration by parts formula in the Wiener space. In section 4, we show that the densities of the approximating processes converge almost surely to the density of the solution to the original SDE. Section 5 is devoted to finding a uniform bound for the integrals of the approximating processes and deriving the exponential decay of the density to the solution.

2. Formulation of main results

Let $\Omega$ denote the Wiener space $C_0([0,T];\mathbb{R}^d)$. We furnish $\Omega$ with the $\|\cdot\|_\infty$-norm making it a (separable) Banach space. Consider $(\Omega, \mathcal{F}, P)$ a complete probability space, in which $\mathcal{F}$ is generated by the open sets of the Banach space, $W_t$ is a $d$-dimensional Brownian motion, and $\mathcal{F}_t$ is the filtration generated by $W_t$. By $H := L^2([0,T];\mathbb{R}^d)$ we denote a Hilbert space. For $k, p \geq 1$, denote by $\mathbb{D}^{k,p}$ the domain of the $k$th order Malliavin derivative operator with respect to the norm

$$
\|F\|_{k,p} = \left[\mathbb{E}|F|^p + \|D^{i_1,\ldots,i_k}F\|_{L^p(\Omega; H^{\otimes k})}^p\right]^{\frac{1}{p}},
$$

and define $\mathbb{D}^\infty := \bigcap_{k,p} \mathbb{D}^{k,p}$.

Now consider the following stochastic differential equation

$$
dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0, \quad (2.1)
$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to M_{d \times d}(\mathbb{R})$ are $C^\infty$ measurable functions. The function $\sigma$ is $C^\infty$ and all of its derivatives of order greater or equal 1 are bounded. The function $\sigma$ is globally Lipschitz and consider $k_1$ as its Lipschitz constant. $\sigma^*$ is uniformly continuous with modulus of continuity $\theta(\cdot)$ and there exist $\lambda, \tilde{\lambda} > 0$ such that for every $x, u \in \mathbb{R}^d$

$$
\lambda|u|^2 \leq \langle \sigma \sigma^*(x)u, u \rangle \leq \tilde{\lambda}|u|^2, \quad (2.2)
$$

where $*$ denotes transpose. Also for some positive constant $C_2$

$$
|\sigma^*(x)u|^2 \leq C_2|u|^2 \quad (2.3)
$$

Note that $\sigma$ is called uniformly elliptic if the first inequality in (2.2) holds. Notice that in this paper, the function $b$ is not considered globally Lipschitz.

In [3, Theorem 2.2.2 and Corollary 2.2.1] Nualart shows that SDEs which have globally Lipschitz coefficients with polynomial growth for all of their derivatives, have the strong unique solutions in $\mathbb{D}^\infty$. 

Also, he stated there the linear equation in which Malliavin derivative satisfy. In [15] we have shown that assuming monotonicity for the drifts existence of a unique strong solution to SDE (2.1). In this paper, we assume that the SDE (2.1) has a strong unique solution in $D^\infty$. We denote by $\mathcal{L}$ the second-order differential operator associated to SDE (2.1):

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma \sigma^*)_j^i(x) \partial_i \partial_j + \sum_{i=1}^{d} b_i(x) \partial_i$$

Consider the following stochastic differential equations

$$dX^n_t = b_n(X^n_t)dt + \sigma(X^n_t)dW_t, \quad X^n_0 = x_0. \tag{2.4}$$

where the functions $b_n$ are globally Lipschitz and all of their derivatives have polynomial growth; i.e., for each $x \in \mathbb{R}^d$ and each multiindex $\alpha$ with $|\alpha| = m$, there exist positive constants $q_m$ and $\Gamma_m$ which are independent of $n$ and

$$|\partial_\alpha b_n(x)|^2 \leq \Gamma_m(1 + |x|^{q_m}) \tag{2.5}$$

and $b_n(x) = b(x)$ for every $x \in \Omega_n$ where $\Omega = \Omega_n$.

As we pointed out, by [3] there exists unique strong solutions for SDEs (2.4) and $X^n \in D^\infty$. Also, for $r \leq t$,

$$D_r(X^n_t)^i = \sigma^i(X^n_t) + \int_r^t \nabla b^n_i(X^n_s).D_rX^n_s ds + \int_r^t \nabla \sigma^i(X^n_s).D_rX^n_s dW^i_s \tag{2.6}$$

and for $r > t$, $D_r(X^n_t)^i = 0$. Here $u.C$ denotes the product $C^*u$ for a vector $u$ and a matrix $C$, for example $\nabla f(x).D_rX^n = \sum_{i=1}^{d} \nabla f(x)_i D_r(X^n)^i$. We used the upper index to show a specific row, and the subindex to show a specific column of a matrix. As before denote the infinitesimal operator associated to these SDE’s by $\mathcal{L}_n$.

Throughout the paper we assume the following Hypothesis.

**Hypothesis 2.1.** For each $p \geq 1$, there exist some positive constants $c_p$, $\alpha_p$ and $\gamma_p$ such that

1. The sequence $\{X^n_t\}_{n \geq 1}$ converges to $X_t$ in $L^p(\Omega)$.

2. 

$$\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|DX^n_t|^p] \leq c_p. \tag{2.7}$$

3. 

$$\mathcal{L}_n|X^n_t - x_0|^p \leq \alpha_p|X^n_t - x_0|^p + \gamma_p \tag{2.8}$$

**Remark 1.** Applying Itô’s formula and Inequality (2.8) and then using Gronwall’s inequality, for maybe rewriting $c_p$, imply that

$$\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|X^n_t|^p] \leq c_p, \tag{2.9}$$

For every $0 \leq t \leq T$, denote the Malliavin covariance matrix of $X^n_t$ and $X_t$ by $Q_n(t)$ and $Q(t)$, respectively. We know that the diffusion coefficient $\sigma$ in (2.1) is uniformly elliptic. So that, the Hörmander condition holds for $\sigma$ and by Nualart [3, Theorem 2.3.3 and its proof] the solutions $X^n_t$ not only have a.s. invertible Malliavin covariance matrices and the nondegeneracy condition holds for them, but also they have infinitely differentiable densities. Concerning the moments of determinants of inverse Malliavin covariance matrices, assume that

**Hypothesis 2.2.** There exist some positive constants $c$ and $\lambda_0$ such that for each $p \geq 1$,

$$\sup_n \mathbb{E} \left[ \det(Q_n(t))^{-p} \right] < c(\lambda_0 t)^{-d(p - \frac{1}{2})} \tag{2.10}$$

In Appendix we will show that this assumption is satisfied by some weak assumptions on $\sigma$.

By our assumptions, Hypotheses (2.1) and (2.2) we can derive an integration by parts formula in the Wiener space. By use of which and the uniform exponential boundedness of $X^n_t$ with respect to $n$, we will show the exponential decay of the density of the solution to SDE (2.1).
3. Integration by parts formula

In this section, we prove Theorem 3.1. By the integration by parts formula in [3, Proposition 2.1.4] and Hypothesis 2.2, there exists a family of random variables \( \{L_n\} \) depending on multiindices \( \beta \) of length strictly larger than 1 with coordinates \( \beta_j \in \{1, \ldots, d\} \), such that for every \( G \in \mathbb{D}^\infty \)

\[
E[\partial_\alpha g(X^n_t)G] = E[g(X^n_t)L^n_\alpha(X^n_t),G],
\]

and

\[
\| L^n_\alpha(X^n_t,G) \|_p \leq c_{p,q} \| \det((Q_n(t))^{-1}) \|^{m} \| DX^n_t \|_q \| G \|_q
\]

where

\[
L^n_\alpha(X^n_t,G) = \sum_{j=1}^m \delta \left( G(Q_n(t))^{-1} D(X^n_t)^j \right),
\]

and \( \delta \) denotes the adjoint of the Malliavin derivative operator \( D \).

**Theorem 3.1.** Let \( g \in C^{m+1} \), and all of its derivatives be bounded and have polynomial growth. If Hypotheses 2.1 and 2.2 hold, then for every \( G \in \mathbb{D}^\infty \) and every multiindex \( \alpha = (\alpha_1, \ldots, \alpha_d) \) with \( |\alpha| = m \), there exists a function \( H_\alpha(X_t,G) \) such that

\[
E(\partial_\alpha g(X_t)G) = E(g(X_t)H_\alpha(X_t,G)),
\]

where \( H_\alpha = H_{\alpha_k}(H_{(\alpha_1, \ldots, \alpha_{k-1})}) \).

**Proof.** Fix \( p \geq 1 \). Applying (2.7) and (2.2) to (3.2) we conclude that \( L^n_\alpha(X^n_t,G) \) is uniformly bounded and consequently weakly convergent in \( L^p(\Omega) \). Let us denote its limit by \( H_\alpha(X_t,G) \). We are going to show that

\[
E\left[ g(X^n_t)L^n_\alpha(X^n_t,G) \right] \to E\left[ g(X_t)H_\alpha(X_t,G) \right] \quad \text{as} \quad n \to \infty.
\]

By Hypothesis 2.1 part (1), there exists a uniformly integrable subsequence of \( X^n_t \) which converges to \( X_t \) a.s.. Since \( \partial_\alpha g(\cdot) \) has polynomial growth, \( \partial_\alpha g(X^n_t) \) are uniformly integrable and a.s. convergent to \( \partial_\alpha g(X_t) \). Choose \( p_1 \geq 1 \) such that \( \frac{1}{p} + \frac{1}{p_1} = 1 \). Thus

\[
\partial_\alpha g(X^n_t) \to \partial_\alpha g(X_t) \quad \text{in} \quad L^{p_1}(\Omega),
\]

Now by Cauchy-Schwarz inequality

\[
\left| E\left[ g(X^n_t)L^n_\alpha(X^n_t,G) \right] - E\left[ g(X_t)H_\alpha(X_t,G) \right] \right| \\
\leq \left| E\left[ g(X^n_t)L^n_\alpha(X^n_t,G) \right] - E\left[ g(X_t)L^n_\alpha(X^n_t,G) \right] \right| \\
+ \left| E\left[ g(X_t)L^n_\alpha(X^n_t,G) \right] - E\left[ g(X_t)H_\alpha(X_t,G) \right] \right| \\
\leq E\left[ |g(X^n_t) - g(X_t)|^{p_1} \right]^{\frac{1}{p_1}} E\left[ |L^n_\alpha(X^n_t,G)|^{p_1} \right]^{\frac{1}{p_1}} \\
+ \left| E\left[ g(X_t)L^n_\alpha(X^n_t,G) \right] - E\left[ g(X_t)H_\alpha(X_t,G) \right] \right|.
\]

By (3.5) and the uniform boundedness of \( \| L^n_\alpha(X^n_t,G) \|_p \), the first term in the right hand side of the above inequality tends to 0. Since \( g(X_t) \in L^{p_1}(\Omega) \) and \( L^n_\alpha(X^n_t,G) \to H_\alpha(X_t,G) \) weakly in \( L^p(\Omega) \), the second term also tends to 0 and (3.4) holds.

Now, since \( \partial_\alpha g(X^n_t) \) converges a.s. to \( \partial_\alpha g(X_t) \), and \( \partial_\alpha g \in C^1 \) is a bounded function, by Lebesgue’s dominated convergence theorem the following holds also true.

\[
E\left[ \partial_\alpha g(X^n_t)G \right] \to E\left[ \partial_\alpha g(X_t)G \right] \quad \text{as} \quad n \to \infty.
\]
Therefore, letting \( n \) tend to \( \infty \) in both sides of (3.1) and using (3.6) and (3.4) the integration by parts formula (3.1) results.

Notice that by (3.3) for every \( k \geq 2 \) and every multiindex \( \alpha := (\alpha_1, \ldots, \alpha_k) \), there exist some functions \( H_\alpha(X_t, G) \) and \( H_{(\alpha_1, \ldots, \alpha_{k-1})}(X_t, G) \) such that

\[
\mathbb{E}\left( \partial_{\alpha}g(X_t)G \right) = \mathbb{E}\left( g(X_t)H_\alpha(X_t, G) \right),
\]

and
\[
\mathbb{E}\left( \partial_{(\alpha_1, \ldots, \alpha_{k-1})}g(X_t)G \right) = \mathbb{E}\left( g(X_t)H_{(\alpha_1, \ldots, \alpha_{k-1})}(X_t, G) \right).
\]

Also there exist a function \( H_{\alpha_k} \) such that
\[
\mathbb{E}\left( \partial_{(\alpha_1, \ldots, \alpha_{k-1})}\partial_{\alpha_k}g(X_t)G \right) = \mathbb{E}\left( \partial_{\alpha_k}g(X_t)H_{(\alpha_1, \ldots, \alpha_{k-1})}(X_t, G) \right)
= \mathbb{E}\left( g(X_t)H_{\alpha_k}(X_t, H_{(\alpha_1, \ldots, \alpha_{k-1})})(X_t, G) \right)
= \mathbb{E}\left( g(X_t)H_{\alpha_k}H_{(\alpha_1, \ldots, \alpha_{k-1})}(X_t, G) \right)
\]

Since this equality holds for every \( G \in \mathcal{F}_t \), using (3.7) implies that \( H_\alpha = H_{\alpha_k}(H_{(\alpha_1, \ldots, \alpha_{k-1})}) \).

4. Pointwise Convergence of the sequence of densities

The existence of a density for \( X_t \), called \( \rho(t, x, \cdot) \), is a result of Theorem 2.1.1 in [3]. But in order to find a \( C^\infty \) density, note that there exists a sequence \( \{\rho^n(t, x, \cdot)\}_n \) of \( C^\infty \) densities associated to \( \{X^n_t\}_t \). Hereby we are going to prove the convergence of \( \rho^n(t, x, \cdot) \) to \( \rho(t, x, \cdot) \) in some sense, thereby finding some bounds for the derivatives of \( \rho(t, x, \cdot) \).

To this end, we will prove the following lemma by which and Lemma 11.4.1 in [21] the convergence of densities would results in \( L^1(\Omega) \).

**Lemma 4.1.** If Hypothesis 2.2 holds, then there exists a non-decreasing function \( \psi : (0, \infty) \rightarrow (0, \infty) \), depending only on \( d \), \( \lambda \) and \( \Lambda \) such that \( \lim_{\epsilon \rightarrow 0}\psi(\epsilon) = 0 \) and for every \( R > 0 \)

\[
\sup_{n \geq 1} \mathbb{E}\left( \int_{B(0, R)} |\rho^n(t, x, y + h) - \rho^n(t, x, y)|dy \right) \leq t^{-\nu}\psi(|h|),
\]

where \( B(0, R) \) is the open ball with radius \( R \) centered at origin.

**Proof.** This proof is motivated by the proof of Theorem 9.1.15 in [21]. We will use Girsanov theorem to omit the drift terms of SDEs associated to \( X^n_t \).

Let \( P^n(t, x_0, \cdot) \) be the transition probability associated to \( X^n_t \). By Girsanov Theorem [24], for every \( g \in C^\infty_0[0, T] \times \mathbb{R}^d \)

\[
\int_{B(0, R)} \left[ \int_0^T g_h(X^n_s) - g(X^n_s)ds \right] dP^n = \int_{B(0, R)} \left( \int_0^T g_h(Z_s) - g(Z_s)ds \right) S^n_T(1)dP
\]

where \( g_h(x) = g(x + h), \ x \in \mathbb{R}^d \) and

\[
S^n_T(c) = [S^n_T(c)]_{t=\tau} = \left[ \exp\left( -\frac{c^2}{2} \int_0^T |u_{t,n}(w)|^2dt + c \int_0^T \langle u_{t,n}(w), dW_t \rangle \right) \right]_{t=\tau},
\]

in which \( u_{t,n} \) and \( Z_t \) satisfy

\[
\sigma(Z_t)u_{t,n}(w) = b_n(Z_t), \quad \text{and} \quad dZ_t = \sigma(Z_t)dW_t.
\]

By Cauchy-Schwarz inequality

\[
\mathbb{E}\left( \int_{B(0, R)} |\rho^n(t, x, y + h) - \rho^n(t, x, y)|dy \right) \leq t^{-\nu}\psi(|h|),
\]
\[
\int_{B(0,R)} \left( \int_0^T g_h(Z_s) - g(Z_s) ds \right) S^n_T(1) dP \leq \sqrt{T} \left[ \mathbb{E} \left( \int_0^T |g_h(Z_s) - g(Z_s)|^2 ds \right) \right]^{\frac{1}{2}} \left[ \int_{Z_t \in B(0,R)} [S^n_T(1)]^2 dP \right]^{\frac{1}{2}}.
\] (4.2)

Now we are going to find some appropriate bound for \( \int_{Z_t \in B(0,R)} [S^n_T(1)]^2 dP \) not dependent on \( n \). Set
\[
dK_i(w) := (u_{t,n}(w), dW_i(w))
\]
and use Cauchy-Schwarz inequality and the submartingale property of \( S^n_T(4) \), then
\[
\int_{Z_t \in B(0,R)} [S^n_T(1)]^2 dP \leq \int_{Z_t \in B(0,R)} \left[ \exp \left( -4 \int_0^T |u_{t,n}(w)|^2 dt + 2 \int_0^T dK_i \right) \right. \\
\times \left. \exp \left( 3 \int_0^T |u_{t,n}(w)|^2 dt \right) \right] dP \leq \mathbb{E} \left[ S^n_T(4) \right]^{\frac{1}{2}} \int_{Z_t \in B(0,R)} \left[ \exp \left( 6 \int_0^T |u_{t,n}(w)|^2 dt \right) \right]^{\frac{1}{2}} dP \leq \left[ \int_{Z_t \in B(0,R)} \exp \left( 6 \int_0^T |u_{t,n}(w)|^2 dt \right) dP \right]^{\frac{1}{2}}.
\]

Now, for \( n \) large enough and every \( x \in B(0,R) \) we know that \( b_n(x) = b(x) \). So, multiplying two sides of (4.4) by \( u_{t,n}(w) \) and using Hypothesis 2.4 for \( B_T := \sup_{x \in B(0,R)} b(x) \) we have
\[
\left[ \int_{Z_t \in B(0,R)} \exp \left( 6 \int_0^T |u_{t,n}(w)|^2 dt \right) dP \right]^{\frac{1}{2}} \leq e^{\frac{2}{3} T(B_T)^2}.
\]

Putting this bound into (4.2) and using Theorem 9.2.12 in [21], we will have
\[
\int_{B(0,R)} \left( \int_0^T g_h(X^n_t) - g(X^n_t) ds \right) dP^n \leq 4e^{\frac{2}{3} T(B_T)^2} \| g \| \| t^{-\nu} \psi(|h|).
\]

Since for every \( g \in C_b([0,T] \times \mathbb{R}^d) \) there exist an increasing sequence \( \{g_n\} \) in \( C^\infty_0([0,T] \times \mathbb{R}^d) \) which converges to \( g \), the monotone convergence theorem completes the proof.

Here, we are ready to show the pointwise convergence of densities \( \rho^n(t,x_0,.) \) to \( \rho(t,x_0,.) \).

**Lemma 4.2.** For every \( R > 0 \) if \( y \in B(0,R) \), then
\[
\rho^n(t,x_0,y) \to \rho(t,x_0,y).
\] (4.3)

**Proof.** By Lemma 4.1 there exists a non-decreasing function \( \phi : (0, \infty) \to (0, \infty) \) not dependent on \( n \) such that \( \lim_{t \to 0} \phi(t) = 0 \) and
\[
\sup_{n \geq 1} \int_{B(0,R)} |\rho^n(t,x_0,y + h) - \rho^n(t,x_0,y)| dy \leq t^{-\nu} \phi(|h|),
\] (4.4)
where \( \nu \) depends only on \( d \). On the other hand, we have already proved that \( X^n_t \to X_t \) in \( L^2(\Omega) \). So that for every \( \psi \in C_b(\mathbb{R}^d) \),
\[
\int \rho(t,x_0,y) \psi(y) dy = \lim_{n \to \infty} \int \rho^n(t,x_0,y) \psi(y) dy
\] (4.5)

By (4.4) and (4.5), the requirements of Lemma 11.4.1 in [21] hold true, hence \( \rho^n(t,x_0,.) \to \rho(t,x_0,.) \) in \( L^1(B(0,R)) \). This implies the existence of a subsequence of \( \rho^n(t,x_0,.) \) which converges a.s. to \( \rho(t,x_0,.) \). Finally, continuity of \( \rho(t,x_0,.) \) completes the proof.
5. Exponential decay at infinity

In this section we will find some bounds for the expectations of solutions. In the next section, we will use these bounds to prove Theorem 5.2.

Lemma 5.1. Assuming the first and the second part of Hypothesis 2.1 for every \( \zeta > 0 \) and \( q > 1 \)

\[
E\left[\exp\left(\sup_{0 \leq t \leq T} |\zeta e^{-\eta t}|X^n_t - x_0|^2\right)\right] \leq (8C_2\zeta^2 + 2)e^{\frac{\zeta^2}{\alpha_2}},
\]

(5.1)

where \( \eta = \alpha_2 + 2C_2\zeta + 1/\zeta \).

Proof. Set

\[
\Gamma(x) := |x - x_0|^2 + \frac{\zeta^2}{\alpha_2},
\]

and

\[
Z^n_t := e^{-\eta t}|X^n_t - x_0|^2 + e^{-\eta t}\frac{\zeta^2}{\alpha_2}.
\]

By Itô’s formula,

\[
Z^n_t = \Gamma(x_0) + \int_0^t e^{-\eta s}[\mathcal{L}_n \Gamma - \eta \Gamma](X^n_s)ds + \int_0^t e^{-\eta s}\langle \nabla \Gamma(X^n_s), \sigma(X^n_s)dW_s \rangle.
\]

Now consider the function

\[
g_\zeta(r) := \exp[r], \quad r > 0
\]

again by Itô’s formula, (2.3) and (2.3),

\[
g_\zeta(Z^n_t) = g_\zeta(Z_0) + \zeta \int_0^t g_\zeta(Z^n_s)e^{-\eta s}[\mathcal{L}_n \Gamma - \eta \Gamma](X^n_s)ds
\]

\[
+ 2\zeta^2 \int_0^t g_\zeta(Z^n_s)e^{-2\eta s}[\sigma^*(X^n_s)(X^n_s - x_0)]^2 + 2\zeta \int_0^t g_\zeta(Z^n_s)e^{-\eta s}(X^n_s - x_0, \sigma(X^n_s)dW_s)
\]

\[
\leq g_\zeta(Z_0) + \zeta \int_0^t g_\zeta(Z^n_s)e^{-\eta s}[\alpha_2 - \eta + 2\zeta C_2]\Gamma(X^n_s)ds
\]

\[
+ 2\zeta \int_0^t g_\zeta(Z^n_s)e^{-\eta s}(X^n_s - x_0, \sigma(X^n_s)dW_s).
\]

setting \( \eta = \alpha_2 + 2C_2\zeta + 1/\zeta \), one has

\[
g_\zeta(Z^n_t) \leq g_\zeta(Z_0) - \int_0^t g_\zeta(Z^n_s)Z^n_s ds + 2\zeta \int_0^t g_\zeta(Z^n_s)e^{-\eta s}(X^n_s - x_0, \sigma(X^n_s)dW_s).
\]

Taking expectations, for every \( \zeta, t > 0 \), we have

\[
E\left[\int_0^t g_\zeta(Z^n_s)Z^n_s ds\right] \leq g_\zeta(Z_0).
\]

(5.2)

On the other hand, by Doob’s maximal inequality,

\[
E\left[\sup_{t \in [0, T]} |g_\zeta(Z^n_t)|^2\right] \leq 2|g_\zeta(Z_0)|^2 + 8.4\zeta^2E \left[ \int_0^T e^{-2\eta s}|g_\zeta(Z^n_s)|^2|\sigma^*(X^n_s)(X^n_s - x_0)|^2ds \right]
\]

\[
\leq 2|g_\zeta(Z_0)|^2 + 8.4\zeta^2C_2E \left[ \int_0^T g_\zeta(Z^n_s)Z^n_s ds \right],
\]

where for the last inequality we make use of (2.3). Using (5.2), we obtain

\[
E\left[\sup_{t \in [0, T]} |g_\zeta(Z^n_t)|^2\right] \leq (32C_2\zeta^2 + 2)g_\zeta(Z_0).
\]

By replacing \( \zeta \) by \( \zeta/2 \) the proof is complete. \( \square \)
Now, we are going to derive the exponential decay for the density.

**Theorem 5.2.** By Hypotheses 2.1 and Hypothesis 2.2, the density of $X_t \mid \rho_t(x_0,.), \text{ is infinitely differentiable and there exist constants } \eta, q, m \text{ and } c \text{ such that for every } y \in \mathbb{R}^d$

\[
\max_{|\alpha| \leq n} \left| \frac{\partial^\alpha \rho^n (x_0,y)}{\partial y} \right| \leq \left( \frac{32C_2 + 2}{\lambda_0 t} \right)^q \exp \left\{ -\frac{2\gamma_2}{\alpha_2} - 2e^{-\eta t}|y - x_0|^2 \right\} \tag{5.3}
\]

**Proof.** By Hypothesis 2.2 we know that the nondegeneracy condition holds for $X_t^n$, so that from Proposition 2.1.5 in [3] we have

\[
\partial_\alpha \rho^n (t,x_0,y) = (-1)^{|\alpha|} E \left[ 1_{X^*_t > y} L_\alpha L^n_{1,2,\ldots,d} (X_t^n,1) \right]
\]

for every multiindex $\alpha \in \{1, \ldots, d\}^k$ and $y \in \mathbb{R}^d$. Consider the Cauchy-Schwarz inequality

\[
|\partial_\alpha \rho^n (t,x_0,y)| \leq E \left[ 1_{X^*_t > y} \right]^{\frac{q}{2}} \left\| L^n_{1,2,\ldots,d} (X_t^n,1) \right\|_P
\]

where $\frac{1}{p} + \frac{1}{q} = 1$. Applying (2.7) and (2.2) to (3.2) implies the existence of constants $c_\alpha > 0$, $\beta_0 > 1$, and $m > 1$ not dependent on $n$ such that

\[
|\partial_\alpha \rho^n (t,x_0,y)| \leq E \left[ 1_{X^*_t > y} \right]^{\frac{q}{2}} \lambda_0 t^{-dm(\beta_0 - \frac{1}{2})} \tag{5.4}
\]

Set $\eta := \alpha_2 + 4C_2 + 1/2$, then the Markov’s inequality and inequality (5.4) imply that

\[
P \left( 1_{X^*_t > y} \right) \leq P \left( \exp \{ e^{-\eta t}|X_t^n - x_0|^2 \} > \exp \{ e^{-\eta t}|y - x_0|^2 \} \right)
\]

\[
\leq \frac{\exp \left\{ 2 \sup_{0 \leq t \leq T} [e^{-\eta t}|X_t^n - x_0|^2] \right\}}{\exp \{ 2e^{-\eta t}|y - x_0|^2 \}}
\]

\[
\leq (32C_2 + 2) \exp \left\{ -\frac{2\gamma_2}{\alpha_2} - 2e^{-\eta t}|y - x_0|^2 \right\} \tag{5.5}
\]

Now, substituting (5.5) in (5.4) we have

\[
|\partial_\alpha \rho^n (t,x_0,y)| \leq c_\alpha (32C_2 + 2)^q \lambda_0 t^{-dm(\beta_0 - \frac{1}{2})} \exp \left\{ -\frac{2\gamma_2}{\alpha_2} - 2e^{-\eta t}|y - x_0|^2 \right\} 
\]

(5.6)

The latter inequality implies that the functions $\partial_\alpha \rho^n (t,x_0,.)$ are equicontinuous on $\mathbb{R}^d$. Thus on every compact subset $V$ in $\mathbb{R}^d$, $\partial_\alpha \rho^n (t,x_0,.)$ contains a uniformly convergent subsequence, which we denote by $\partial_\alpha \rho^n (t,x_0,.)$, too. This is true especially for $|\alpha| = 2$. Hence, $(\rho^n (t,x_0,))$ converges uniformly on $V$ and $(\rho^n (t,x_0,))' \rightarrow (\rho(t,x_0,.))'$. Using (5.6) for every multiindex $\alpha$,

\[
\partial_\alpha \rho^n (t,x_0,.) \rightarrow \partial_\alpha \rho(t,x_0,.) \quad \text{uniformly on } V.
\]

Therefore, $\rho(t,x_0,.)$ is a Schwartz distribution on $\mathbb{R}^d$ and by (5.6), inequality (5.3) is satisfied.

**Appendix A. The nondegeneracy condition for $X_t^n$**

In this section, we will show that the nondegeneracy condition (Hypothesis 2.2) holds if the function $\sigma$ depend only on $t$ instead of $x$ and for $0 \leq t \leq T$ the exists some function $\psi(.)$ such that $\int_0^t |\psi(s)|^2 \exp \{ 2s \} ds < \infty$ and for each $x \in \mathbb{R}^d$,

\[
\langle \sigma(t)\sigma^*(t)x,x \rangle \leq \psi(t)|x|^2. \tag{A.1}
\]
By Theorem 1.9. in [20], (2.1) and Itô’s formula, for every 0 ≤ i, j ≤ d we have

\[ d⟨D_ξ X_i^t, D_ξ X_j^t⟩ = ⟨D_ξ X_i^t, ∇b^i(X_t).D_ξ X_i⟩ + ⟨∇b^i(X_t).D_ξ X_i, D_ξ X_j⟩ dt := N_{ij}^t(r) dt. \]

By Fubini’s theorem, the components of \( Q_{ij}(t) \) (ij-component of the matrix \( Q \)), satisfies

\[ Q_{ij}^t(t) = \int_0^t ⟨\sigma^i(r), \sigma^j(r)⟩ dr + \int_0^t \int_r^s N_s(r) ds dr \]

**Lemma A.1.** Assuming Hypothesis (2.1) and (A.1) for a bounded diffusion \( \sigma \), there exists a constant \( C_Q \) not dependent on \( n \) such that

\[ \sup_n \sup_{0 ≤ t ≤ T} E[|Q_n(t)|^2] < C_Q. \]

**Proof.** By Fubini’s theorem, the components of \( Q_n(t) \) denoted by \( Q_{ij}^n(t) \) satisfy

\[ Q_{ij}^n(t) = \int_0^t ⟨\sigma^i(r), \sigma^j(r)⟩ dr + \int_0^t \int_r^s N_{ij}^n(r) ds dr, \]

where

\[ N_{ij}^n(r) := ⟨D_ξ (X^n_i)^r, ∇b^i_n(X^n).D_ξ X^n_i⟩ + ⟨∇b^i_n(X^n_i).D_ξ X^n_i, D_ξ (X^n_i)^r⟩ \]

Hence, by Itô’s formula

\[ \frac{d}{dt} \mathbb{E}[|Q_{ij}^n(t)|^2] = 2 \mathbb{E}[Q_{ij}^n(t)(⟨\sigma^i(t), \sigma^j(t)⟩) + 2 \mathbb{E}[Q_{ij}^n(t) \int_0^t N_{ij}^n(r) dr] \]

\[ \leq 2 \mathbb{E}[|Q_{ij}^n(t)|^2] + |⟨σ^i(t), σ^j(t)⟩|^2 + \mathbb{E}[\int_0^t N_{ij}^n(r) dr]^2 \]

Thus, by (A.1)

\[ \frac{d}{dt} \sum_{i,j=1}^d \mathbb{E}[|Q_{ij}^n(t)|^2] \leq 2 \sum_{i,j=1}^d \mathbb{E}[|Q_{ij}^n(t)|^2] + |ψ(t)|^2 + \sum_{i,j=1}^d \mathbb{E}[\int_0^t N_{ij}^n(r) dr]^2. \]

Using Young inequality and (2.4), (2.5) and (2.9) we obtain some bounds for the last two terms in the right hand side of inequality (A.3), namely some constant \( c_1 \) such that

\[ \sup_{n ≥ 1} \mathbb{E}[\int_0^t N_{ij}^n(r) dr]^2 < c_1. \]

Thus by Lemma 1.1 in [22] we obtain the inequality (A.2).

Here, we are going to show that the nondegeneracy condition holds for the sequence \{X^n_t\}. To this end,

**Lemma A.2.** Assume that Hypothesis (2.1) holds and for vert 0 ≤ t ≤ T, \( ψ(t) > d + c_1 d^2 \), then there exist some positive constant \( \lambda \) such that for every \( p ≥ 1 \), and 0 ≤ t ≤ T

\[ \sup_n \mathbb{E}[\det(Q_n(t))^{-p}] < c_1 2d^{-\frac{p}{2}} \frac{\Gamma(2p + \frac{3}{2})}{\Gamma(p)} (λt)^{-d(p - \frac{1}{2}) - \frac{3}{2}} \]

**Proof.** Let \( x_i ≥ 0 \) for 1 ≤ i ≤ d and define \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) by

\[ f(y_{11}, \ldots, y_{dd}) = \prod_{1 ≤ i,j ≤ d} \exp{-x_i y_{ij} x_j} \]
Then \( f \) is in \( C^2_0(\mathbb{R}^d) \) and we can use Dynkin’s formula (see e.g. [23, p. 120]) and inequalities \( (2.7) \) and \( (A.2) \) to derive a suitable bound for the expectation of \( Y^n_t := \exp\{ -x^T Q_n(t)x \} \):

\[
\frac{d}{dt} E[Y^n_t] = E\left[ \sum_{i,j} -x_i x_j Y^n_t \langle \sigma^i(t), \sigma^j(t) \rangle \right] + E\left[ \sum_{i,j} -x_i x_j Y^n_t \int_0^t N^{n,ij}_n(r) dr \right]
\]

\[
\leq (-\psi(t) + c_1 d^2)|x|^2 E[Y^n_t]
\]

where we have used \( (A.1), (A.4) \) and the non-negative semidefiniteness of the covariance matrix. Therefore, from Lemma 1.1 in [22] we have:

\[
E[Y^n_t] \leq \exp\{ (-\int_0^t \psi(s) ds + c_1 d^2 t)|x|^2 \} \leq \exp\{ -td|x|^2 \}
\]

Finally, by use of Lemma 7-29 in [4, p. 92] and the symmetry of the integral of even functions:

\[
E[\det(Q_n(t))^{-p}] \leq c_1 2^d \Gamma(p) \int_{(\mathbb{R}^d)^{+}} |x|^{d(2p-1)} e^{x t - d|x|^2} dx
\]

\[
= c_1 2^d \Gamma(2p) \Gamma(p) \left( td \right)^{-d(p-\frac{1}{2})-2}
\]

Remark 2. If the Malliavin covariance matrix \( Q(t) \) is a.s. invertible, then we can derive Lemma \( A.2 \) and the nondegeneracy condition for \( X_t \).

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