Model Predictive Control for rotary inverted pendulum using LabVIEW

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Abstract. Inverted pendulum is a popular nonlinear system, extensively used in the study and analysis of various control techniques. Inverted pendulum has got variety of applications in robotics, aerospace, Segway etc. The paper deals with mathematical modelling, parameter estimation, disturbance rejection and tracking control of QNET 2.0 Rotary Inverted Pendulum Board for NI ELVIS in LabVIEW platform. A Model Predictive Controller is designed to maintain the pendulum in upright position and to handle disturbances. The tracking control problem for arm angle is also considered. A comparison between the disturbance rejections using MPC with standard LQR is carried out. Finally experimental results are described and compared based on the two control techniques designed for the stabilization of rotary inverted pendulum.

1. Introduction
A rotary inverted pendulum (IP) is like a man holding a stick in his palm and balancing it by moving his palm horizontally without falling. The study of rotary IP motivates the design of controllers for rockets to stay upright and automated landing of air crafts. A normal pendulum is stable in downward position but it is unstable at upright position. Hence it requires active balance control to maintain the pendulum in upright position and sweeping through a set trajectory. This is achieved by generating control actions to the servo motor to move the pivot horizontally to balance the pendulum in upright position. The Segway, a two wheeled platform, self-balancing vehicle uses the concept of rotary IP [1]. Missile launchers, pendubots, earthquake resistant buildings etc. are other applications of rotary IP [2].

Model Predictive Control (MPC) has been used as an overall process control in process industries, oil plants and real time applications [3], [4]. MPC uses the dynamic model of a system, whether it is linear or nonlinear, for control model subject to input and state constraints. Moving Horizon Control or Receding Horizon Control are some of the other known names of MPC. MPC uses the process model and prediction model to predict the future response of the plant. MPC relies on building the controllers that adjust the control action before the set point occurs. MPC uses finite horizon method for handling of process and the model constraints [5]. The first input is given to the plant and the entire calculation is iterated in subsequent control actions. At every iteration the prediction horizon moves forward to predict the future control action for the subsequent sequences.

A Strong emphasis is needed for a generalized approach to the rotary IP modelling [6], [7]. Vishwa Nath and R Mithra [8] describes the steps for designing the controllers for stabilization and swing up control of rotary IP. MPC mainly concentrates on constructing and optimizing feedback controllers at each discrete time instant [9]. An extensive use of model is the key element of MPC that may not represent the reality accurately. Therefore, predicted state may differ from future plant inputs. In this case robust control can be able to provide desired performance. It is able to provide a solution by considering the uncertainties [10-12]. Disturbance rejection is one of the main part in MPC which requires a special consideration [13]. MPC has an efficient disturbance rejection properties but less effective in unmeasured disturbances. A challenging real-time problem can be dealt with MPC [14].
This paper is an application of MPC applied to QNET 2.0 Rotary IP, which is an open loop unstable system. Section 2 deals with the mathematical modelling and parameter identification of the system. The design of MPC is discussed in Section 3. Section 4 outlines the simulation and experimental results of MPC and LQR applied to rotary IP system.

2. Mathematical modelling
The rotary IP consists of arm, mounted on a servo motor and an encoder housing, in which the pendulum is hanging. The rotary IP made by QUANSER [15] is shown in the figure 1. The device is add-on application board for the NI Educational Laboratory virtual instrumentation suit. It is developed for learning fundamentals of nonlinear control, parameter estimation, damping and stability analysis.

![Figure 1. QNET Rotary Inverted Pendulum board [15]](image1)

Figure 1 shows the free body diagram of rotary IP with mass $M_p$ and length $L_p$. The pendulum is connected to the DC motor through an arm of mass $M_{arm}$. $\theta$ and $\alpha$ are pendulum arm angle and pendulum angle respectively. Table 1 lists the terminologies used in this paper.

### Table 1. Rotary Inverted Pendulum parameters

| Symbol | Description                              | Value     | Unit  |
|--------|------------------------------------------|-----------|-------|
| $M_p$  | Mass of pendulum                         | 0.024     | Kg    |
| $l_p$  | Length of pendulum from centre of mass to pivot | 0.129     | m     |
| $r$    | Length of arm pivot to pendulum pivot    | 0.085     | m     |
| $J_p$  | Pendulum MI about pivot axis             | 0.0001    | Kg.m$^2$ |
| $J_{eq}$ | Equivalent MI about motor shaft axis    | 5.7x10$^4$ | Kg.m$^2$ |
| $K_t$  | Motor torque constant                    | 0.042     | N.m   |
| $K_m$  | Motor back EMF constant                  | 0.042     | V/(rad/s) |
| $R_m$  | Motor armature resistance                | 8.4       | Ω     |
| $M_{arm}$ | Mass of arm                                 | 0.095     | Kg    |
| $g$    | Gravitational constant                   | 9.81      | m/s$^2$ |

From figure 2, the set of nonlinear equations of rotary IP can be derived [6] as,
\[
\frac{d^2 \theta}{dt^2} = \frac{M_p^2 g l_p^2 r \cos(\theta(t)) \alpha(t)}{J_p r (\theta(t))^2 - J_{eq} - M_p r^2} - \frac{J_p M_p r^2 \cos(\theta(t)) \sin(\theta(t)) \dot{\theta}}{M_p r (\theta(t))^2 - J_{eq} - M_p r^2} \bigg( \frac{1}{J_p} - \frac{1}{l_p J_{eq}} \bigg) + \frac{J_p r \tau_{output} + M_p l_p^2 r_{output}}{M_p r (\theta(t))^2 - J_{eq} - M_p r^2} \bigg( \frac{1}{J_p} - \frac{1}{l_p J_{eq}} \bigg) \]

(1)

\[
\frac{d^2 \alpha}{dt^2} = \frac{l_p M_p \left( -J_{eq} g + M_p \sin(\theta(t)) r^2 g - M_p r^2 g \right) \alpha(t)}{M_p r (\theta(t))^2 - J_{eq} - M_p r^2} + \frac{l_p M_p r \sin(\theta(t)) J_p \dot{\theta}}{M_p r (\theta(t))^2 - J_{eq} - M_p r^2} \bigg( \frac{1}{J_p} - \frac{1}{l_p J_{eq}} \bigg) + \frac{l_p M_p r \tau_{output} \cos(\theta(t))}{M_p r (\theta(t))^2 - J_{eq} - M_p r^2} \bigg( \frac{1}{J_p} - \frac{1}{l_p J_{eq}} \bigg) \]

(2)

where \( \tau_{output} = \frac{K_v \left( V_m - K_u \left( \frac{d}{dt} (\theta(t)) \right) \right)}{R_m} \)  

(3)

Linearizing (1) and (2) by approximating \( \cos(\theta(t)) = 1 \) and \( \sin(\theta(t)) = 0 \),

\[
\frac{d^2 \theta}{dt^2} = \frac{M_p^2 g l_p^2 r}{J_p r + J_p M_p r^2 + M_p l_p^2 J_{eq}} \alpha(t) - \frac{J_p K_v K_m - M_p l_p K_v K_m}{R_m \left( J_{eq} J_p + J_p M_p r^2 + M_p l_p^2 J_{eq} \right)} \theta(t) + \frac{J_p K_v + M_p l_p^2 K_v}{R_m \left( J_{eq} J_p + J_p M_p r^2 + M_p l_p^2 J_{eq} \right)} V_m \]

(4)

\[
\frac{d^2 \alpha}{dt^2} = \frac{M_p J_p \left( -J_{eq} g + M_p r^2 g \right)}{J_{eq} J_p + J_p M_p r^2 + M_p l_p^2 J_{eq}} \alpha(t) - \frac{M_p J_p K_v K_m}{R_m \left( J_{eq} J_p + J_p M_p r^2 + M_p l_p^2 J_{eq} \right)} \dot{\theta}(t) - \frac{M_p J_p r K_v}{R_m \left( J_{eq} J_p + J_p M_p r^2 + M_p l_p^2 J_{eq} \right)} V_m \]

(5)

The equations (4-5) can be represented as a single state space model as,

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad \text{(6)}
\]

\[
y(t) = Cx(t) \quad \text{(7)}
\]

where the state variables \([\theta \quad \alpha \quad \dot{\theta} \quad \ddot{\theta}]\), control action \( u \) is motor control voltage and output \( y \) is pendulum angle. The state space system matrices are derived as follows:
Moment of inertia, $J_p$ is unknown in (3) and (4). It can be found out by Lagrangian of pendulum using the potential energy and kinetic energy of the pendulum. The gravitational potential energy $U(t)$ is

$$U(t) = M_p g l_p (1 - \cos \alpha(t))$$

The gravitational potential energy in (9) depends on vertical position of the pendulum. The potential energy is zero when it is in rest position and equals to $2M_p g l_p$ when it is in vertical position.

The total kinetic energy, $T_t$ of pendulum is,

$$T_t = \frac{1}{2} J_p \left( \frac{d}{dt} \alpha(t) \right)^2$$

The Langrange of the pendulum for position and velocity is,

$$L(\alpha, \frac{d}{dt} \alpha(t)) = T_t - U$$

The nonlinear equation of pendulum using Euler-Lagrange is,

$$J_p \left( \frac{d^2}{dt^2} \alpha(t) \right) + M_p g l_p \sin \alpha(t) = \tau_{pend}$$

where $\tau_{pend}$ is the torque applied to the pendulum pivot.

The linearized form of (12) by approximating $\sin \alpha = \alpha$ is given by,

$$J_p \left( \frac{d^2}{dt^2} \alpha(t) \right) + M_p g l_p \alpha(t) = 0$$

solving the linear differential equation in (13) with initial conditions $\alpha(0) = 0$ and $\frac{d}{dt} \alpha(t) = 0$, the solution of (13) should be of the form,

$$\alpha(t) = \alpha_0 \cos(2\pi f t)$$

where $f$ in (15) is frequency of pendulum. The frequency, $f$ is calculated by,

$$f = \frac{n_{cyc}}{\Delta t}$$

$n_{cyc}$ is the number of cycles when a manual excitation is given to the pendulum and $\Delta t$ is the duration of these cycles.
The moment of inertia, $J_p$, from (14) by substituting (15) with initial conditions is given by,

$$J_p = \frac{M_p gl_p}{4 f^2 \pi^2}$$  \hspace{1cm} (17)

from figure 3 and by using (17) the value of $J_p$ is calculated as $1.69 \times 10^{-9} Kg.m^2$.

by substituting this value of $Jp$ in the continuous time state space model (8), the desired system matrices are obtained as,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 174.493 & -2.28951 & 0 \\ 0 & 305.542 & -2.413 & 0 \end{bmatrix} \hspace{1cm} B = \begin{bmatrix} 0 \\ 0 \\ 54.5121 \\ 57.4524 \end{bmatrix}$$  \hspace{1cm} (18)

3. Model predictive control
MPC is an online/offline optimized control algorithm that minimizes a constraint/unconstraint cost function over an infinite/finite horizon. With its predictive ability enables the actuator to make smooth adjustments closer to optimal control values when combined with traditional feedback operation. The basic structure of MPC is shown in figure 4. At each time instants a new control input vector $u_k$ is computed and fed into the system by considering the input and state constraints. MPC control algorithm consists of

I. A prediction model (19) which predicts the future outputs from present and past inputs and state variables.

$$y = GU + Fx(k)$$  \hspace{1cm} (19)

where $y$ is a column matrix of predicted outputs during the horizon $k+1$ to $k+N$ and $U$ is the optimized control actions in the horizon. The matrices $G$ and $U$ can be derived iteratively from [16].

II. A cost function, which is to be minimized by a sequence of future control actions. The cost function $J$ is given by equation :

$$J = \sum_{k=0}^{h_p} (\hat{y} - r)^T Q (\hat{y} - r) + \sum_{k=0}^{k_e} \Delta u^T R \Delta u$$  \hspace{1cm} (20)

where,

$h_p$ is prediction horizon
$h_u$ is control horizon

$\hat{y}$ is predicted process output

$r$ is set point

$\Delta u$ is predicted change in control input

$Q$ is output error penalization

$R$ is control rate penalization

The cost function (20) is in the form of standard quadratic programing problem and can be solved online/offline.

III. Constraints, which are physical limits to the plant, e.g. voltage limits, actuator limits etc.

Constraints in output, input and control rate are

$$y_{\min} \leq y \leq y_{\max} \quad (21)$$

$$u_{\min} \leq u \leq u_{\max} \quad (22)$$

$$\Delta u_{\min} \leq \Delta u \leq \Delta u_{\max} \quad (23)$$

4. Results and discussion

The continuous time state space model (18) is discretized with a sampling time of $T_s=10ms$. The discretized model is MIMO system with outputs as pendulum angle and arm angle, states as $\theta$, $\alpha$, $\dot{\theta}$, $\dot{\alpha}$ and input as motor control voltage ($u$). The control objectives were i) to keep the pendulum angle in upright position ($\alpha=180^\circ$) and ii) to track the arm angle ($\theta$) with respected desired set-points ($\theta_r$). The MPC algorithm described above is implemented in LabVIEW with prediction horizon $h_p=120$ and control horizon $h_u=115$. Constraints were considered for control input $|u(t)| \leq 12V$ and output, $\theta(t) \leq 120^\circ$. A cost function in the form of (20) was considered which penalizes control error and control rate. The simulation results and experimental results of MPC applied to arm angle control were compared in the figure 5. A series of step changes (centre 0, left 20 and right -20) of arm angle were considered as set-points. The motor voltages, pendulum angle and angular velocities are shown in figure 6. The pendulum angle, $\alpha$ switches from $175^\circ$ to $185^\circ$ when the set point change occurs and the corresponding control inputs were within the desired limits.

![Figure 5. Set point tracking by MPC](image1.png)

![Figure 6. Control input and States](image2.png)
Table 2. Performance comparison

| Controller | Tuning parameters | ISE   | IAE   | ITAE  |
|------------|-------------------|-------|-------|-------|
| LQR        | \(Q=\text{diag}[200,55,0.1,0.1]\), \(R=3\) | 100.93 | 194.89 | 839.54 |
| MPC        | \(Q=\text{diag}[150,55,0.1,0.1]\), \(R=1.5\) | 6.17   | 53.27  | 273.00 |
| MPC        | \(Q=\text{diag}[200,55,0.1,0.1]\), \(R=3\) | 10.26  | 66.49  | 304.63 |

An infinite horizon LQR was implemented for disturbance rejection and set point tracking for comparison purpose. Figure 7 shows the responses of arm angle set-point tracking with MPC (\(\theta_{\text{MPC}}\)) and LQR (\(\theta_{\text{LQR}}\)) for \(\theta_r=\pm20^\circ\). Table 2 lists the performance matrices (ISE, IAE, ITAE) calculated for arm angle tracking experiments. It can be noted that, the performances of MPC were significantly better than LQR as expected, mainly because of prediction capability of MPC. Figure 8 shows the response of disturbance rejection control using LQR and MPC. A large manual disturbance to pendulum rod (\(\theta\)) and pendulum arm (\(\alpha\)) were applied. Both LQR and MPC were able to reject those disturbances and bring back to the steady state, although MPC was slightly better.

![Set point tracking MPC vs LQR](image)

![Disturbance rejection](image)

5. Conclusion
A MPC controller on LabVIEW platform was designed to provide real time control for rotary IP. The mathematical model for the system was derived into state space representation using Lagrange method. Unknown parameters of the state space model were identified experimentally. A Model Predictive Controller imparting constraints on inputs and outputs was designed by penalising the control error and control rate. Simulation and experimental validation were carried out for pendulum angle disturbance rejection and arm angle trajectory tracking. The results were analysed and compared with a standard LQR. The results showed the dominance of MPC over LQR for the chosen tuning parameters.

The application of MPC in faster systems is not preferred owing to its requirement for larger computational time. However, with the use of the MPC toolbox in LabVIEW, even faster systems can be controlled by MPC – 10ms sampling time in this case. As a future work, swing up control with adaptive MPC can be considered.
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