Kan extensions and cartesian monoidal categories

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Abstract
The existence of adjoints to algebraic functors between categories of models of Lawvere theories follows from finite-product-preservingness surviving left Kan extension. A result along these lines was proved in Appendix 2 of Brian Day’s PhD thesis [1]. His context was categories enriched in a cartesian closed base. A generalization is described here with essentially the same proof. We introduce the notion of cartesian monoidal category in the enriched context. With an advanced viewpoint, we give a result about left extension along a promonoidal module and further related results.

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An advanced viewpoint

1 Introduction

The pointwise left Kan extension, along any functor between categories with finite products, of a finite-product-preserving functor into a cartesian closed category is finite-product-preserving. This kind of result goes back at least to Bill Lawvere’s thesis \[8\] and some 1966 ETH notes of Fritz Ulmer. Eduardo Dubuc and the author independently provided Saunders Mac Lane with a proof along the lines of the present note at Bowdoin College in the Northern Hemisphere Summer of 1969. Brian Day’s thesis \[1\] gave a generalization to categories enriched in a cartesian closed base. Also see Kelly-Lack \[7\] and Day-Street \[3\]. Our purpose here is to remove the restriction on the base and, to some extent, the finite products.

2 Weighted colimits

We work with a monoidal category \(\mathcal{V}\) as used in Max Kelly’s book \[9\] as a base for enriched category theory.

Recall that the colimit of a \(\mathcal{V}\)-functor \(F : \mathcal{A} \rightarrow \mathcal{X}\) weighted by a \(\mathcal{V}\)-functor \(W : \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}\) is an object

\[\text{colim}(W, F) = \text{colim}_{\mathcal{A}}(WA, FA)\]

of \(\mathcal{X}\) equipped with an isomorphism

\[\mathcal{X}(\text{colim}(W, F), X) \cong [\mathcal{A}^{\text{op}}, \mathcal{V}](W, \mathcal{X}(F, X))\]

\(\mathcal{V}\)-natural in \(X\).

Independence of naturality in the two variables of two variable naturality, or Fubini’s theorem \[9\], has the following expression in terms of weighted colimits.

Nugget 1. For \(\mathcal{V}\)-functors

\[W_1 : \mathcal{A}_1^{\text{op}} \rightarrow \mathcal{V}, \quad W_2 : \mathcal{A}_2^{\text{op}} \rightarrow \mathcal{V}, \quad F : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow \mathcal{X},\]

if \(\text{colim}(W_2, F(A, -))\) exists for each \(A \in \mathcal{A}\) then

\[\text{colim}(W_1, \text{colim}(W_2, F)) \cong \text{colim}(W_1 \otimes W_2, F)\]

Here the isomorphism is intended to include the fact that one side exists if and only if the other does. Also \((W_1 \otimes W_2)(A, B) = W_1 A \otimes W_2 B\).
Proof. Here is the calculation:

\[
\mathcal{L}(\text{colim}(W_1 \otimes W_2, F), X) \cong [(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\text{op}}, \mathcal{V}](W_1 \otimes W_2, \mathcal{L}(F, X)) \\
\cong [\mathcal{A}_1^{\text{op}}, \mathcal{V}](W_1, [\mathcal{A}_2^{\text{op}}, \mathcal{V}](W_2, \mathcal{L}(F, X))) \\
\cong [\mathcal{A}_1^{\text{op}}, \mathcal{V}](W_1, \mathcal{L}(\text{colim}(W_2, F), X)) \\
\cong \mathcal{L}(\text{colim}(W_1, \text{colim}(W_2, F)), X).
\]

Here is an aspect of the calculus of mates expressed in terms of weighted colimits. Note that \( S \dashv T : A \to C \) means \( T^{\text{op}} \dashv S^{\text{op}} : A^{\text{op}} \to C^{\text{op}} \).

Nugget 2. For \( \mathcal{V} \)-functors \( W : \mathcal{A}^{\text{op}} \to \mathcal{V}, G : \mathcal{C} \to \mathcal{L} \), and a \( \mathcal{V} \)-adjunction \( S \dashv T : \mathcal{A} \to \mathcal{C} \), there is an isomorphism

\[
\text{colim}(WS^{\text{op}}, G) \cong \text{colim}(W, GT).
\]

Proof. Here is the calculation:

\[
\mathcal{L}(\text{colim}(W, GT), X) \cong [\mathcal{A}^{\text{op}}, \mathcal{V}](W, \mathcal{L}(GT, X)) \\
\cong [\mathcal{A}^{\text{op}}, \mathcal{V}](W, \mathcal{L}(G, X)^{\text{op}}) \\
\cong [\mathcal{G}^{\text{op}}, \mathcal{V}](WS^{\text{op}}, \mathcal{L}(G, X)) \\
\cong \mathcal{L}(\text{colim}(WS^{\text{op}}, G), X).
\]

3 Cartesian monoidal enriched categories

A monoidal \( \mathcal{V} \)-category \( \mathcal{A} \) will be called cartesian when the tensor product and unit object have left adjoints. That is, \( \mathcal{A} \) is a map pseudomonoid in the monoidal 2-category \( \mathcal{V} \)-Cat\(^{\text{co}} \) in the sense of [5].

Let us denote the tensor product of \( \mathcal{A} \) by \( \otimes : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) with left adjoint \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) and the unit by \( N : I \to \mathcal{A} \) with left adjoint \( E : \mathcal{A} \to I \). (Here \( \mathcal{I} \) is the unit \( \mathcal{V} \)-category: it has one object 0 and \( \mathcal{I}(0, 0) = I \).) It is clear that these right adjoints make \( \mathcal{A} \) a comonoidal \( \mathcal{V} \)-category; that is, a pseudomonoid
in \( \mathcal{V}\text{-Cat}^{\text{op}} \). Since \( \text{ob} : \mathcal{V}\text{-Cat} \to \text{Set} \) is monoidal, we see that \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) is given by the diagonal on objects. We have
\[
\mathcal{A}(A, A_1 \star A_2) \cong \mathcal{A}(A, A_1) \otimes \mathcal{A}(A, A_2),
\]
where \( \mathcal{V}\text{-functoriality in } A \) on the right-hand side uses \( \Delta \).

If \( \mathcal{A} \) is cartesian, the \( \mathcal{V}\text{-functor category} \ [\mathcal{A}, \mathcal{V}] \) becomes monoidal under convolution using the comonoidal structure on \( \mathcal{A} \). This is a pointwise tensor product in the sense that, on objects, it is defined by:
\[
(M \star N)A = MA \otimes NA.
\]
On morphisms it requires the use of \( \Delta \). Indeed, the Yoneda embedding
\[
Y : \mathcal{A}^{\text{op}} \to \mathcal{A}, \mathcal{V}
\]
is strong monoidal.

4 Main result

**Theorem 3.** Suppose \( J : \mathcal{A} \to \mathcal{B} \) is a \( \mathcal{V}\text{-functor between cartesian monoidal } \mathcal{V}\text{-categories} \). Assume also that \( J \) is strong comonoidal. Suppose \( \mathcal{X} \) is a monoidal \( \mathcal{V}\text{-category such that each of the } \mathcal{V}\text{-functors } - \otimes X \) and \( X \otimes - \) preserves colimits. Assume the \( \mathcal{V}\text{-functor } F : \mathcal{A} \to \mathcal{X} \) is strong monoidal. If the pointwise left Kan extension \( K : \mathcal{B} \to \mathcal{X} \) of \( F \) along \( J \) exists then \( K \) too is strong monoidal.

**Proof.** Using that tensor in \( \mathcal{X} \) preserves colimits in each variable, the Fubini Theorem 1, that \( F \) is strong monoidal, Theorem 2 with the cartesian property of \( \mathcal{A} \), and the cartesian property of \( \mathcal{B} \), we have the calculation:
\[
K(B_1 \otimes B_2) \cong \text{colim}_{A_1}(\mathcal{B}(JA_1, B_1), FA_1) \otimes \text{colim}_{A_2}(\mathcal{B}(JA_2, B_2), FA_2)
\]
\[
\cong \text{colim}_{A_1,A_2}(\mathcal{B}(JA_1, B_1) \otimes \mathcal{B}(JA_2, B_2), FA_1 \otimes FA_2)
\]
\[
\cong \text{colim}_{A_1,A_2}(\mathcal{B}(JA_1, B_1) \otimes \mathcal{B}(JA_2, B_2), F(A_1 \star A_2))
\]
\[
\cong \text{colim}_{A}(\mathcal{B}(JA, B_1) \otimes \mathcal{B}(JA, B_2), FA)
\]
\[
\cong \text{colim}_{A}(\mathcal{B}(JA, B_1 \star B_2), FA)
\]
\[
\cong K(B_1 \star B_2).
\]
For the unit part, for similar reasons, we have:
\[
N \cong FN0
\]
\[
\cong \text{colim}_{0}(\mathcal{F}(0,0), FN0)
\]
\[
\cong \text{colim}_{A}(\mathcal{F}(EA,0), FA)
\]
\[
\cong \text{colim}_{A}(\mathcal{F}(EJA,0), FA)
\]
\[
\cong KN.
\]
\(\square\)
5 An advanced viewpoint

In terminology of [4], suppose $H : \mathcal{M} \to \mathcal{N}$ is a monoidal pseudofunctor between monoidal bicategories. The main point to stress here is that the constraints

$$\Phi_{A,B} : HA \otimes HB \to H(A \otimes B)$$

are pseudonatural in $A$ and $B$. Then we see that $H$ takes pseudomonoids (= monoidales) to pseudomonoids, lax morphisms of pseudomonoids to lax morphisms, oplax morphisms of pseudomonoids to oplax morphisms, and strong morphisms of pseudomonoids to strong morphisms.

In particular, this applies to the monoidal pseudofunctor

$$\mathcal{V} \text{-Mod}(-, \mathcal{X}) : \mathcal{V} \text{-Mod}^{\text{op}} \to \mathcal{V} \text{-CAT}$$

which takes the $\mathcal{V}$-category $\mathcal{A}$ to the $\mathcal{V}$-functor $\mathcal{V}$-category $[\mathcal{A}, \mathcal{V}]$. Now pseudomonoids in $\mathcal{V}$-Mod$^{\text{op}}$ are precisely promonoidal (= premonoidal) $\mathcal{V}$-categories in the sense of Day [1, 2]. Therefore, for each promonoidal $\mathcal{V}$-category $\mathcal{A}$, we obtain a monoidal $\mathcal{V}$-category

$$\mathcal{V} \text{-Mod}(\mathcal{A}, \mathcal{X}) = [\mathcal{A}, \mathcal{V}]$$

which is none other than what is now called Day convolution since it is defined and analysed in [1, 2].

A lax morphism of pseudomonoids in $\mathcal{V}$-Mod$^{\text{op}}$, as written in $\mathcal{V}$-Mod, is a module $K : \mathcal{B} \to \mathcal{A}$ equipped with module morphisms

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{P} & \mathcal{B} \otimes \mathcal{B} \\
K & \downarrow & \downarrow K \otimes K \\
\mathcal{A} & \xrightarrow{P} & \mathcal{A} \otimes \mathcal{A}
\end{array}
$$

satisfying appropriate conditions. In other words, we have

$$
\phi_{A_1,A_2,B} : \text{colim}_{B_1,B_2}(K(A_1, B_1) \otimes K(A_2, B_2), P(B_1, B_2, B)) \to \text{colim}_{A}(K(A, B), P(A_1, A_2, A))
$$

and

$$
\phi_0 : JB \to \text{colim}_{A}(K(A, B), JA)
$$

We call such a $K$ a promonoidal module. It is strong when $\phi$ and $\phi_0$ are invertible.

We also have the $\mathcal{V}$-functor

$$\exists K : [\mathcal{A}, \mathcal{X}] \to [\mathcal{B}, \mathcal{X}]$$

defined by

$$(\exists K)B = \text{colim}_{A}(K(A, B), FA)$$.
By the general considerations on monoidal pseudofunctors, \( \exists_K \) is a monoidal \( V \)-functor when \( \mathcal{X} = \mathcal{Y} \). However, the same calculations needed to show this explicitly show that it works for any monoidal \( V \)-category \( \mathcal{X} \) for which each of the tensors \( X \otimes - \) and \( - \otimes X \) preserves colimits.

**Theorem 4.** If \( K : \mathcal{B} \to \mathcal{A} \) is a promonoidal \( V \)-module then \( \exists_K : [\mathcal{A}, \mathcal{X}] \to [\mathcal{B}, \mathcal{X}] \) is a monoidal \( V \)-functor. If \( K \) is strong promonoidal then \( \exists_K \) is strong monoidal.

**Proof.** Although the result should be expected from our earlier remarks, here is a direct calculation.

\[
(\exists_K F_1 \ast \exists_K F_2)B \cong \text{colim}_{B_1, B_2} (P(B_1, B_2, B), (\exists_K F_1)B_1 \otimes (\exists_K F_2)B_2)
\cong \text{colim}_{B_1, B_2} (P(B_1, B_2, B), \text{colim}_{A_1} (K(A_1, B_1), F_1A_1) \otimes \\
\text{colim}_{A_2} (K(A_2, B_2), F_2A_2))
\cong \text{colim}_{B_1, B_2, A_1, A_2} (K(A_1, B_1) \otimes K(A_2, B_2) \otimes P(B_1, B_2, B), \\
F_1A_1 \otimes F_2A_2)
\implies \text{colim}_{A_1, A_2} (K(A, B) \otimes P(A_1, A_2, A), F_1A_1 \otimes F_2A_2)
\cong \text{colim}_{A} (K(A, B), \text{colim}_{A_1, A_2} (P(A_1, A_2, A), F_1A_1 \otimes F_2A_2))
\cong \text{colim}_{A} (K(A, B), (F_1 \ast F_2)A))
\cong \exists_K F_1 \ast \exists_K F_2 B.
\]

The morphism on the fourth line of the calculation is induced by \( \phi_{A_1, A_2, B} \) and so is invertible if \( K \) is strong promonoidal. We also have \( \phi_{0B} : JB \implies (\exists_K J)B. \]

For the corollaries now coming, assume as above that \( \mathcal{X} \) is a monoidal \( V \)-category such that \( X \otimes - \) and \( - \otimes X \) preserve existing colimits. Also \( \mathcal{A} \) and \( \mathcal{B} \) are monoidal \( V \)-categories. The monoidal structure on \( [\mathcal{A}^{\text{op}}, \mathcal{X}] \) is convolution with respect to the promonoidal structure \( \mathcal{A}(A_1 \ast A_2) \) on \( \mathcal{A}^{\text{op}} \); similarly for \( [\mathcal{B}^{\text{op}}, \mathcal{X}] \).

**Corollary 5.** If \( J : \mathcal{A} \to \mathcal{B} \) is strong monoidal then so is

\[
\text{Lan}_{J^{\text{op}}} : [\mathcal{A}^{\text{op}}, \mathcal{X}] \to [\mathcal{B}^{\text{op}}, \mathcal{X}] .
\]

**Proof.** Apply Theorem 4 to the module \( K : \mathcal{B}^{\text{op}} \to \mathcal{A}^{\text{op}} \) defined by \( K(A, B) = \mathcal{B}(B, JA) \). We see that \( K \) is strong promonoidal using Yoneda twice and strong monoidalness of \( J. \)

**Corollary 6.** If \( W : \mathcal{A} \to \mathcal{V} \) is strong monoidal then so is

\[
\text{colim}(W, -) : [\mathcal{A}^{\text{op}}, \mathcal{X}] \to \mathcal{X} .
\]

**Proof.** Take \( \mathcal{B} = \mathcal{A} \) in Theorem 4.
Corollary 7. Suppose \( \mathcal{A} \) is cartesian monoidal. If \( F : \mathcal{A} \rightarrow \mathcal{X} \) is strong monoidal then so is
\[
\colim(-, F) : [\mathcal{A}^{\text{op}}, \mathcal{Y}] \rightarrow \mathcal{X}.
\]

Proof. Here is the calculation for binary tensoring:
\[
\begin{align*}
\colim(W_1 \otimes W_2, F) &\cong \colim_A((W_1 \otimes W_2) \Delta A, FA) \\
 &\cong \colim_{A_1,A_2}(W_1 A_1 \otimes W_2 A_2, F(A_1 \ast A_2)) \\
 &\cong \colim_{A_1,A_2}(W_1 A_1 \otimes W_2 A_2, FA_1 \otimes FA_2) \\
 &\cong \colim_A(W_1 A_1, FA) \otimes \colim_A(W_2 A_2, FA_2) \\
 &\cong \colim(W_1, F) \otimes \colim(W_2, F).
\end{align*}
\]
The unit preservation is easier. \( \Box \)

Corollary 8. Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are cartesian monoidal and \( J : \mathcal{A} \rightarrow \mathcal{B} \) is strong comonoidal. If \( F : \mathcal{A} \rightarrow \mathcal{X} \) is strong monoidal then so is \( \mathrm{Lan}_J F : \mathcal{B} \rightarrow \mathcal{X} \).

Proof. Notice that \( \mathrm{Lan}_J F \) is the composite of \( \mathcal{B}(J, 1) : \mathcal{B} \rightarrow [\mathcal{A}^{\text{op}}, \mathcal{Y}] \) and \( \colim(-, F) : [\mathcal{A}^{\text{op}}, \mathcal{Y}] \rightarrow \mathcal{X} \). The first is strong monoidal by hypothesis on \( J \). The second is strong monoidal by Corollary 7. \( \Box \)

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