The $\gamma$-Dimension of Images the Integral Staircase

S Wibowo$^1$, V Y Kurniawan$^2$, Siswanto$^3$

$^1,2,3$ Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Sebelas Maret Surakarta, Indonesia

e-mail: $^1$supriyadi_w@staff.uns.ac.id, $^2$vikayugi@staff.uns.ac.id, $^3$sis.mipa@staff.uns.ac.id

Abstract. In this paper, we have proved that the integral staircase function defined in the $\gamma$-dimensional compact set $F$ satisfies the bi-Lipschitz condition of order $\alpha \in (0,1)$ and as a consequence, the image of the integral staircase function does not preserve $\gamma$-dimension of its domain.

1. Introduction
Fractals are the geometrical shapes with the fractal dimension which is larger than their topological dimension. Calculus on fractals or $F^\alpha$-calculus is a calculus based on fractal set $F \subset R$. The integral staircase function plays a key role in fractal $F^\alpha$-calculus. The integral staircase function $S_F^\alpha(x)$ for any $x \in F$ of order (exponent) $\alpha \in (0,1)$ is a generalization of the Lebesgue-Cantor staircase function $S_C^\alpha(x)$ for any $x \in C$, where $C$ is the triadic Cantor set (Cantor ternary set) which is created by repeatedly deleting the open middle thirds of a set of line segments. Many results regarding the $F^\alpha$-calculus and its application on Cantor set or generalized Cantor set are available in the scientific literature (see, e.g., [1], [4], [5], [6] and [7]).

Parvate and Gangal (2009) showed if $C$ is the triadic Cantor set then it is known that $|S_C^\alpha(y) - S_C^\alpha(x)|$ is bounded by $|y - x|^{\alpha}$ from below and above such that

$$c_1|y - x|^{\alpha} \leq |S_C^\alpha(y) - S_C^\alpha(x)| \leq c_2|y - x|^{\alpha}, (x, y \in C)$$

where $0 < c_1 \leq c_2 < \infty$ and $\alpha = \ln 2/\ln 3$ is the $\gamma$-dimension of $C$. A fractal dimension is an index for characterizing fractal patterns or sets by quantifying their complexity as a ratio of the change in detail to the change in scale [8]. The functions that satisfies condition (1) is known as a bi-Lipschitz condition of order $\alpha$. It can also be showed that the image of the Lebesgue-Cantor staircase function does not preserve $\gamma$-dimensions of $C$, because the Lebesgue-Cantor staircase function maps the Cantor set $C$ onto $[0,1]$ so that the $\gamma$-dimensions of $[0,1]$ is 1 ([2],[3]).

In this paper, we have proved the bi-Lipschitz condition for the integral staircase function $S_F^\alpha$ is defined on the $\gamma$-dimensional compact set and $F$ -perfect set $F$ with a dimension $\alpha \in (0,1)$. Furthermore, it has been proved the image of the integral staircase function does not preserve $\gamma$-dimensions of $F$.

2. $F^\alpha$-Calculus
Following definition 2.1-2.5 about the coarse-grained mass, $\gamma$-dimension, the integral staircase function, $F$ –limit, and $F$-continuity [9] and definition 2.6 as regards the bi-Lipschitz (bi-Holder) condition of order $\alpha \in (0,1)$, we construct definition below.

Definition 2.1 Given $\delta > 0$ and $a \leq b$, the coarse-grained mass $\gamma^\alpha(F, a, b)$ of $F \cap [a, b]$ is given by
\[
\gamma^\alpha(F,a,b) = \lim_{\delta \to 0} \inf_{|P| = \delta} \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} \theta(F, [x_i, x_{i+1}])
\]
where \( |P| = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i) \) and \( \theta(F, [x_i, x_{i+1}]) = 1 \) if \( F \cap [x_i, x_{i+1}] \neq \emptyset \), \( \theta(F, [x_i, x_{i+1}]) = 0 \) otherwise.

Due to the similarity of the definitions of the mass function and the Hausdorff outer measure, the mass function can be used to define a fractal dimension. We call this number the \( \gamma \)-dimension of \( F \).

**Definition 2.2** The \( \gamma \)-dimension of \( F \cap [a,b] \), denoted by \( \text{dim}_\gamma(F \cap [a,b]) \), is
\[
\text{dim}_\gamma(F \cap [a,b]) = \inf \{ \beta; \gamma^\beta(F,a,b) = 0 \} = \sup \{ \beta; \gamma^\beta(F,a,b) = \infty \}.
\]

In definition 2.3 we introduce one of the central notions of this paper, the integral staircase function for a set \( F \) of the order \( \alpha \). This function, which is a generalization of functions like the Lebesgue-Cantor staircase function describes how the mass of \( F \cap [a,b] \).

**Definition 2.3** Let \( a_0 \) be an arbitrary but fixed real number. The integral staircase function \( S_F^\alpha(x) \) of order \( \alpha \in (0,1) \) for a set \( F \) is given by
\[
S_F^\alpha(x) = \begin{cases} 
\gamma^\alpha(F,a_0,x), & x \geq a_0 \\
-\gamma^\alpha(F,a_0,x), & x < a_0
\end{cases}
\]
The number \( a_0 \) can be chosen according to convenience.

**Definition 2.4** Let \( F \subset R^1 \) and \( x \in F \). A number \( l \) is said to be the limit of \( f \) through the points of \( F \), or simply \( F \)-limit, as \( y \to x \), if given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
y \in F \text{ and } |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon.
\]
If such a number exists, then it is denoted by
\[
l = F - \lim_{y \to x} f(y).
\]
This definition does not involve values of the function at \( y \) if \( y \notin F \). Also, \( F \)-limit is not defined at points \( x \notin F \).

All points of change of \( x \) is named these to \( f \) change of \( f(x) \) and is denoted by \( \text{Sch} (f) \). If \( \text{Sch} (S_F^\alpha) \) is a closed set and every point in it is a limit point, then \( \text{Sch} (S_F^\alpha) \) is called \( \alpha \)-perfect.

**Definition 2.5** A function \( f: R \to R \) is said to be \( F \)-continuous at \( x \in F \) if
\[
f(x) = F - \lim_{y \to x} f(y).
\]
We note that the notion of \( F \)-continuity is not defined at \( x \notin F \).

Based on the definition of the Holder condition with order \( \alpha \in (0,1) \) and the definition of bi-Lipschitz condition with \( \alpha = 1 \) [3] it can be defined as the bi-Lipschitz (bi-Holder) condition with the order \( \alpha \in (0,1) \) as follows.

**Definition 2.6** If \( F \subset R \) and \( f: F \to R \) satisfies a bi-Lipschitz condition, or is bi-Lipschitz continuous, then there exists real numbers \( c_1, c_2, 0 < c_1 \leq c_2 < \infty \) such that
\[
c_1 |y - x|^\alpha \leq |f(y) - f(x)| \leq c_2 |y - x|^\alpha
\]
for all \( x, y \in F \).

3. The Main Result
We will prove that \( |S_F^\alpha(y) - S_F^\alpha(x)| \) bounded by \( |y - x|^\alpha \) from below and above where \( F \) are the compact, \( \alpha \)-perfect sets, and for all \( x, y \in F \).

**Theorem 3.1** If \( F \) be a compact and \( \alpha \)-perfect sets and let \( S_F^\alpha: F \subset [a,b] \to R, \alpha \in (0,1) \) be an integral staircase function, then there exists real numbers \( c_1, c_2, 0 < c_1 \leq c_2 < \infty \) such that
\[
c_1 |y - x|^\alpha \leq |S_F^\alpha(y) - S_F^\alpha(x)| \leq c_2 |y - x|^\alpha
\]
for all \( x, y \in F \cap [a,b] \).

**Proof.** Given \( F \) is a compact set and \( [x,y] \) is closed interval for all \( x, y \in F \), then \( [x,y] \cap F \) is also a compact set in \( F \). We devide the proof into two cases.

i. Case 1, for \( c_1 |y - x|^\alpha \leq |S_F^\alpha(y) - S_F^\alpha(x)|, \alpha \in (0,1) \).
Given a subdivision \( P \) of \( [x,y] \) which is a finite set of points \( x = x_0, x_1, ..., x_n = y \), \( x_i < x_{i+1}, i = 0,1, ..., n - 1 \), we have
\[
\frac{1}{\Gamma(\alpha+1)}|y - x|^{\alpha} = \left| \sum_{i=0}^{n-1} \left( \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(\alpha+1)} \right) \right| \leq \sum_{i=0}^{n-1} \frac{|x_{i+1} - x_i|^{\alpha}}{\Gamma(\alpha+1)} \\
= \sum_{i=0}^{n-1} \left(\frac{\alpha}{\Gamma(\alpha+1)} \right) \theta([x,y] \cap F, [x_i, x_{i+1}]) \\
= \sigma^\alpha([x,y] \cap F, P)
\]

where \(|P| = \max\{x_{i+1} - x_i : i = 0, 1, 2, ..., n - 1\}\) for a subdivision \(P\) and the infimum is taken over all subdivisions \(P\) of \([x,y]\) satisfying \(|P| \leq \delta\).

As \(\delta \to 0\), we get
\[
\frac{1}{\Gamma(\alpha+1)}|y - x|^{\alpha} \leq \gamma^\alpha([x,y] \cap F, x, y) = \inf_{|P| \leq \delta} \sigma^\alpha([x,y] \cap F, P)
\]
\[
\frac{1}{\Gamma(\alpha+1)}|y - x|^{\alpha} \leq |S^\alpha_F(y) - S^\alpha_F(x)|.
\]

Taking \(c_1 = \frac{1}{\Gamma(\alpha+1)}\), we get
\[
\gamma^\alpha([x,y] \cap F, x, y) \leq c_2|y - x|^{\alpha}, \alpha \in (0,1).
\]

ii. Case 2, for \(|S^\alpha_F(y) - S^\alpha_F(x)| \leq c_2|y - x|^{\alpha}, \alpha \in (0,1)\).

As \(F\) is compact set, so \(F \cap [x,y]\) have finite sub open cover \(\{D_i\}\) such that \([x,y] \cap F \subseteq \bigcup_{i=1}^{n} D_i\). Suppose \(D_i = (x_i, y_i), i = 1, 2, ..., n,\) without loss of generality we can choose this finite subcover \(\{D_i\}\) such that \(D_i \supset D_j \) whenever \(i \neq j, i, j = 1, 2, ..., n\).

Furthermore, the sets are labeled such that \(x_1 \leq y_i \leq y_{i+1}\). But as \(D_i \supset D_{i+1}\) and \(D_{i+1} \supset D_i\), it implies that \(x_i < x_{i+1}\) and \(y_i \leq y_{i+1}\). Now we consider the closures \(\overline{D_i}\) of \(D_i\) for \(i = 1, 2, ..., n\). Let \(I_1 = \overline{D_1}\), \(I_i = \overline{D_i}/D_{i-1}\) for \(2 \leq i \leq n\). The collection \(\{I_i\}\) forms a finite cover of \([x,y] \cap F\) by closed intervals \(I_i\) share at the most endpoints. The set of all the end points of \(I_i, i = 1, 2, ..., n\) forms a subdivision \(P\) of \([x,y]\), thus we obtained
\[
\sigma^\alpha([x,y] \cap F, P) = \sum_{i=1}^{n} \frac{\text{diam}(I_i)}{\Gamma(\alpha+1)} \theta([x,y] \cap F, I_i)
\]
with \(\text{diam}(I_i) = \sup|s_i - r_i| : r_i, s_i \in I_i\).

For a positive integer \(n\), it can be chosen a real number \(c\) with \(c \geq c_2 = \frac{1}{\Gamma(\alpha+1)}\) such that
\[
\max\left\{\left(\text{diam}(I_i)\right)^{\alpha} : i = 1, 2, ..., n\right\} \leq c\frac{\gamma(y-x)^{\alpha}}{n}.
\]

From (3) and (4), we obtained
\[
\sigma^\alpha([x,y] \cap F, P) \leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \frac{c}{n} \frac{\gamma(y-x)^{\alpha}}{\theta([x,y] \cap F, I_i})}
\leq \frac{c}{\Gamma(\alpha+1)}|y - x|^{\alpha}.
\]

A subdivision \(P\) of \([x,y]\) which can be refined to a subdivision \(Q\) such that \(|Q| \leq \delta\) such that
\[
\sigma^\alpha([x,y] \cap F, Q) \leq \frac{c}{\Gamma(\alpha+1)}|y - x|^{\alpha}.
\]

Furthermore, we also obtained
\[
\gamma^\alpha([x,y] \cap F, x, y) = \inf_{|Q| \leq \delta} \sigma^\alpha([x,y] \cap F, Q) \leq \frac{c}{\Gamma(\alpha+1)}|y - x|^{\alpha}.
\]

As \(\delta \to 0\), we obtained
\[
\gamma^\alpha([x,y] \cap F, x, y) = F - \lim_{\delta \to 0} \gamma^\alpha([x,y] \cap F, x, y) \leq \frac{c}{\Gamma(\alpha+1)}|y - x|^{\alpha}.
\]
\[
|S^\alpha_F(y) - S^\alpha_F(x)| \leq \frac{c}{\Gamma(\alpha+1)}|y - x|^{\alpha}.
\]

Taking \(c_2 \geq \frac{c}{\Gamma(\alpha+1)}\), we have
\[
|S^\alpha_F(y) - S^\alpha_F(x)| \leq c_2|y - x|^{\alpha}.
\]

By using (2) and (5), we get
\[
c_1|y - x|^{\alpha} \leq |S^\alpha_F(y) - S^\alpha_F(x)| \leq c_2|y - x|^{\alpha}
\]
and the proof is complete.
It can be seen in Theorem 3.1 that the integral staircase function satisfies the bi-Lipschitz condition with the order $\alpha \in (0,1)$.

**Theorem 3.2** If $F$ is a compact set and $\alpha$–perfect set and let $S_F^\alpha : F \subset [a,b] \rightarrow R, \alpha \in (0,1)$ be an integral staircase function, then there exists real numbers $c_1, c_2, 0 < c_1 \leq c_2 < \infty$ and any real number $s$ such that

$$(c_1)^{s/\alpha} \gamma^s(F,x,y) \leq \gamma^{s/\alpha}(S_F^\alpha(F), S_F^\alpha(x), S_F^\alpha(y)) \leq (c_2)^{s/\alpha} \gamma^s(F,x,y)$$

**Proof.** Let $P_{[a,b]} = \{a = x_0, x_1, ..., b = x_n\}$, $a < b, x_i < x_{i+1}, i = 0,1, ..., n-1$ be any subdivision of $[a,b]$. Therefore

$$S_F^\alpha(F \cap [x_i, x_{i+1}]) \subset [S_F^\alpha(x_i), S_F^\alpha(x_{i+1})], i = 0,1, ..., n-1$$

and with Theorem 3.1, we obtained

$$c_1|x_{i+1} - x_i|^{\alpha} \leq |S_F^\alpha(x_{i+1}) - S_F^\alpha(x_{i})| \leq c_2|x_{i+1} - x_i|^{\alpha}, i = 0,1, ..., n-1$$

for some real numbers $c_1, c_2, 0 < c_1 \leq c_2 < \infty$.

Let $P'_{[S_F^\alpha(a), S_F^\alpha(b)]}$ be any subdivision of $[S_F^\alpha(a), S_F^\alpha(b)]$, i.e.,

$$P'_{[S_F^\alpha(a), S_F^\alpha(b)]} = \{S_F^\alpha(a), S_F^\alpha(x_0), ..., S_F^\alpha(b) = S_F^\alpha(x_n)\},$$

for $i = 0,1, ..., n-1$.

We get

$$(c_1)^{s/\alpha}\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{n-1} |x_{i+1} - x_i|^{\alpha} \theta(F, [x_i, x_{i+1}]) \leq \sum_{i=0}^{n-1} |S_F^\alpha(x_{i+1}) - S_F^\alpha(x_{i})|^{s/\alpha} \theta(S_F^\alpha(F), S_F^\alpha(F \cap [x_i, x_{i+1}])$$

$$\leq (c_2)^{s/\alpha}\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{n-1} |x_{i+1} - x_i|^{\alpha} \theta(F, [x_i, x_{i+1}]).$$

By using Definition 2.1, we obtained

$$(c_1)^{s/\alpha} \inf_{P_{[a,b]}} \sigma^s[F,P] \leq \inf_{P'_{[S_F^\alpha(a), S_F^\alpha(b)]}} \sigma^{s/\alpha}[S_F^\alpha(F),P']$$

$$\leq (c_2)^{s/\alpha} \inf_{P'_{[a,b]}} \sigma^{s/\alpha}[F,P]$$

where $\delta' = c_2 \delta^\alpha$.

If we take infimum over all subdivisions $P$ and $P'$ respectively such that $|P| \leq \delta$ and $|P'| \leq \delta' = c_2 \delta^\alpha$, we get

$$(c_1)^{s/\alpha} \gamma^s(F,a,b) \leq \gamma^{s/\alpha}_\delta(S_F^\alpha(F), S_F^\alpha(a), S_F^\alpha(b)) \leq (c_2)^{s/\alpha} \gamma^s(F,a,b).$$

Taking the limit respectively as $\delta \rightarrow 0$ and $\delta' \rightarrow 0$, we have

$$(c_1)^{s/\alpha} \gamma^s(F,a,b) \leq \lim_{\delta \rightarrow 0} \gamma^{s/\alpha}_\delta(F,a,b) \leq \gamma^{s/\alpha}(S_F^\alpha(F), S_F^\alpha(a), S_F^\alpha(b)) \leq (c_2)^{s/\alpha} \gamma^s(F,a,b)$$

and we complete the proof. $

The following is given the theorem of the relationship between the $\gamma$–dimension of the fractal set $F$ with the $\alpha$–dimension of the image of the integral staircase function $S_F^\alpha(F) = \{S_F^\alpha(x) : x \in F\}$.

**Theorem 3.3** If $F$ is a compact set, $\alpha$–perfect set with $\dim_{\gamma}(F) = \alpha$ and let $S_F^\alpha : F \subset [a,b] \rightarrow R, \alpha \in (0,1)$ be an integral staircase function, then

$$\dim_{\gamma}(S_F^\alpha(F)) = 1.$$  

**Proof.** For case $s > \dim_{\gamma}(F)$ by using Theorem 3.2 and Definition 2.2, we get

$$\gamma^{s/\alpha}(S_F^\alpha(F), S_F^\alpha(x), S_F^\alpha(y)) \leq (c_2)^{s/\alpha} \gamma^s(F,x,y) = 0$$

so that it results

$$\dim_{\gamma}(S_F^\alpha(F)) \leq \frac{s}{\alpha} \text{ for } s > \dim_{\gamma}(F).$$
Thus
\[ \dim_{\gamma}(S_F^\alpha(F)) \leq \frac{1}{\alpha} \dim_{\gamma}(F). \]  
Conversely, for case \( s < \dim_{\gamma}(F) \) also by using Theorem 3.2 and Definition 2.2, we get
\[ \infty = \left(c_1\right)^{s/\alpha} \gamma^{s}(F,x,y) \leq \gamma^{s/\alpha}(S_F^\alpha(F),S_F^\alpha(x),S_F^\alpha(y)) \]
we have
\[ \frac{s}{\alpha} \leq \dim_{\gamma}(S_F^\alpha(F)) \text{ untuk } s < \dim_{\gamma}(F). \]
Hence
\[ \frac{1}{\alpha} \dim_{\gamma}(F) \leq \dim_{\gamma}(S_F^\alpha(F)). \]  
From (6) and (7), we get
\[ \dim_{\gamma}(S_F^\alpha(F)) = \frac{1}{\alpha} \dim_{\gamma}(F) = 1 \]
and the proof is complete. ■

Based on Theorem 3.3 we have obtained \( \dim_{\gamma}(S_F^\alpha(F)) = 1 \) for any \( \alpha \in (0,1) \). This value does not depend on \( \gamma \) – dimension of \( F \). This means the image of the integral staircase function does not preserve \( \gamma \) – dimensions of \( F \).

4. Conclusions

In this paper, we based on the discussion the integral staircase function that is defined on the \( \gamma \) – dimensional compact and \( F \) – perfect sets \( F \) satisfying the bi-Lipschitz condition. Then it is showed using \( F^\alpha \) – calculus that the image of the integral staircase function does not preserve \( \gamma \) – dimensions of \( F \).

Acknowledgment

The authors would like to thank the Institute for Research and Community Services of Universitas Sebelas Maret for funding to this research in the academic year of 2018.

References

[1] Balankin  A S, Golmankhanem A K, Patiño-Ortiz J, and Patiño-Ortiz M 2018 Noteworthy fractal features and transport properties of Cantor tartans. Phys. Lett. A, 382, 1534–1539
[2] Dovgoshey O, Martio O, Ryazanov V, and Vuorinen M 2006 Cantor Function. Expositiones Mathematicae Volume 24, Issue 1, Pages 1-37
[3] Falconer K 2003 Fractal geometry: Mathematical foundations and applications. second edition (Chichester, West Sussex,UK: John Wiley and Sons, Ltd)
[4] Golmankhanem A K and Balankin A S 2018 Sub-and super-diffusion on Cantor sets: Beyond the paradox. Phys. Lett. A, 382, 960–967
[5] Golmankhanem A K and Baleanu D 2016 Diffraction from fractal grating Cantor sets. J. Mod. Opt. 63, 1364–1369
[6] Golmankhanem A K and Baleanu D 2017 New heat and Maxwell’s equations on Cantor cubes Rom. Rep. Phys. 2017, 69, 109
[7] Golmankhanem A K, Fernandez A, Golmankhanem A K, and Baleanu D 2018 Diffusion on Middle-\( \xi \) Cantor Sets, Entropy, 20(7), 504
[8] Mandelbrot B B 1983 The fractal geometry of nature (New York: W. H. Freeman)
[9] Parvate A and Gangal A D 2009 Calculus on fractal subsets of real-line I: Formulation. Fractals, 17, 53–148