Lower bound on the radii of black-hole photonspheres

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Abstract

The existence of closed null circular geodesics around black holes is one of the most intriguing predictions of general relativity. It has recently been conjectured that the radii of black-hole photonspheres are bounded from below by the simple relation $r_{ph} \geq \frac{3}{2} r_H$, where $r_H$ is the radius of the outer black-hole horizon. We here prove the validity of this conjecture for spherically symmetric hairy black-hole configurations whose radial pressure function $P \equiv |r^3 p|$ decreases monotonically.
I. INTRODUCTION

Null circular geodesics, closed orbits on which massless particles (photons, gravitons) can orbit a central black hole, provide valuable information about the physical characteristics of the corresponding curved spacetime \[1–4\]. Due to their importance in astrophysical \[5, 6\], cosmological \[7\], and theoretical \[8–16\] studies of black-hole spacetimes, light-like circular geodesics have attracted over the years a good deal of attention from physicists and mathematicians.

In an astrophysical context, the photonsphere (a compact null hypersurface around the central black hole) determines the optical appearance of a compact collapsing object as seen by external asymptotic observers \[5, 6\]. Likewise, the intriguing general relativistic phenomenon of strong gravitational lensing by black holes is closely related to the presence of null circular geodesics in these curved spacetimes \[7\].

In addition, the physical properties of unstable null circular geodesics in black-hole spacetimes are known to determine the complex resonant spectra (the quasinormal frequencies) that characterize the corresponding curved spacetimes in the eikonal (geometric-optics) regime (see \[8–11\] and references therein).

Interestingly, it has been proved in \[12\] that, in spherically symmetric hairy black-hole spacetimes, the radius \(r_\gamma\) of the innermost null circular geodesic sets a lower bound on the effective length of the external non-linear matter fields. In particular, it has been proved that in curved black-hole spacetimes that possess hair, the effective radius of the hair is bounded from below by the compact relation \(r_\text{hair} \geq r_\gamma\) \[12\].

In addition, it has been proved that, in spherically symmetric \[13\] as well as in axisymmetric \[14\] black-hole spacetimes, the innermost null circular orbit provides the fastest way to circle the central compact black hole as measured by asymptotic observers.

Using analytical techniques, it has been proved in \[15\] that, for spherically symmetric hairy black-hole spacetimes whose external matter fields are characterized by a non-positive energy-momentum trace \[17\], the radius \(r_\gamma\) of the innermost null circular geodesic of the curved black-hole spacetime is bounded from above by the remarkably compact relation \[18\]

\[
r_\gamma \leq 3M ,
\]

(1)

where \(M\) is the total ADM mass of the spacetime.
Intriguingly, based on the analysis of the physical properties of some non-trivial black-hole spacetimes, it has recently been conjectured \cite{16} that the radii of black-hole photonspheres are bounded from below by the simple relation

\begin{equation}
    r_\gamma \geq \frac{3}{2} r_H ,
\end{equation}

where \( r_H \) is the radius of the outer black-hole horizon.

The main goal of the present compact paper is to explore the regime of validity of this physically interesting conjecture. In particular, below we shall explicitly prove that the null circular geodesics of spherically symmetric hairy black-hole spacetimes whose radial pressure function \( P \equiv |r^3 p| \) decreases monotonically are characterized by the conjectured \cite{16} lower bound (2).

\section{Description of the System}

We shall analyze the null circular geodesics of spherically-symmetric non-vacuum black-hole spacetimes. The curved line element can be expressed in the form \cite{13, 19, 20}

\begin{equation}
    ds^2 = -e^{-2\delta} \mu dt^2 + \mu^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\end{equation}

where \( \delta = \delta(r) \) and \( \mu = \mu(r) \) are the radially-dependent metric functions.

In terms of the line element (3), the Einstein differential equations \( G^\mu_\nu = 8\pi T^\mu_\nu \) can be expressed in the form \cite{13, 19, 21}

\begin{equation}
    \mu' = -8\pi r \rho + (1 - \mu)/r
\end{equation}

and

\begin{equation}
    \delta' = -4\pi r (\rho + p)/\mu .
\end{equation}

Here \( \rho \equiv -T^t_t, \ p \equiv T^r_r, \) and \( p_T \equiv T^\theta_\theta = T^\phi_\phi \), where \( \rho, \ p, \) and \( p_T \) are respectively the energy density, radial pressure, and tangential pressure of the external matter fields \cite{17}.

Regularity of the spacetime at the black-hole outer horizon, \( r = r_H \), enforces the boundary conditions \cite{22}

\begin{equation}
    \mu(r_H) = 0 \quad \text{with} \quad \mu'(r_H) \geq 0
\end{equation}

and

\begin{equation}
    \delta(r_H) < \infty \quad ; \quad \delta'(r_H) < \infty .
\end{equation}
From Eqs. (5), (6), and (7) one finds the simple boundary condition

\[ p_H = -\rho_H \tag{8} \]

at the black-hole horizon, where \( p_H \equiv p(r = r_H) \) and \( \rho_H \equiv \rho(r = r_H) \). In addition, from Eqs. (4), (6), and (8) one obtains the boundary condition

\[ -8\pi r_H^2 p_H \leq 1 \tag{9} \]

for the radial pressure function.

We shall assume that the components of the energy-momentum tensor satisfy the well-known dominant energy condition, according to which the energy density of the matter fields is positive semidefinite and it bounds the pressure components:

\[ \rho \geq 0 ; \quad \rho \geq |p|, |p_T| \tag{10} \]

In addition, following [17] we shall assume that the external matter fields of the black-hole spacetime are characterized by a non-positive trace of the energy-momentum tensor:

\[ T \leq 0 \tag{11} \]

where \( T = -\rho + p + 2p_T \).

For later purposes we note that the gravitational mass \( m(r) \) contained within a sphere of radius \( r \) is given by the integral relation

\[ m(r) = \frac{1}{2} r_H^2 + \int_{r_H}^{r} 4\pi r'^2 \rho(r')dr' \tag{12} \]

Here \( m(r_H) = r_H/2 \) is the mass contained within the black-hole horizon. Taking cognizance of Eqs. (11) and (12), one can express the radially-dependent metric function \( \mu(r) \) in terms of the mass function \( m(r) \):

\[ \mu(r) = 1 - \frac{2m(r)}{r} \tag{13} \]

III. THE LOWER BOUND ON THE RADII OF BLACK-HOLE PHOTON-SPHERES

In the present section we shall consider the following physically interesting question: How close can the innermost null circular geodesic of a black-hole spacetime be to its outer
horizon? Below we shall explicitly prove that in hairy black-hole spacetimes whose radial pressure function

\[ P \equiv |r^3 p(r)| \]  

(14)
decreases monotonically \[25, 26\], the null circular geodesics cannot lie arbitrarily close to the outer black-hole horizon. In particular, we shall show that the null circular geodesics of these black-hole spacetimes are characterized by the lower bound \[2\].

We shall first derive a lower bound on the mass \(m_{\text{hair}}\) of the matter fields (hair) outside the black-hole horizon. Taking cognizance of Eqs. (10) and (12), one finds the inequality

\[ m_{\text{hair}} = \int_{r_{\text{H}}}^{\infty} 4\pi r^2 \rho(r) dr \geq - \int_{r_{\text{H}}}^{\infty} 4\pi r^2 p(r) dr . \]  

(15)

Interestingly, and most importantly for our analysis, it has been proved in \[12\] that the pressure function \(r^4 p\) of hairy matter fields which satisfy the energy conditions (10) and (11) is non-positive and monotonically decreasing in the interval \(r \in [r_{\text{H}}, r_{\gamma}]\):

\[ \{p(r) \leq 0 \text{ and } (r^4 p)' \leq 0\} \text{ for } r_{\text{H}} \leq r \leq r_{\gamma} . \]  

(16)

In particular, from (16) one deduces the simple relation

\[ 0 \leq -r_{\text{H}}^4 p_{\text{H}} \leq -r^4 p(r) \text{ for } r_{\text{H}} \leq r \leq r_{\gamma} . \]  

(17)

Using the relation (17), one obtains from (15) the series of inequalities

\[ m_{\text{hair}} \geq - \int_{r_{\text{H}}}^{r_{\gamma}} 4\pi r^2 p(r) dr \geq - \int_{r_{\text{H}}}^{r_{\gamma}} 4\pi r_{\text{H}}^4 p_{\text{H}} \left( \frac{1}{r_{\text{H}}} - \frac{1}{r_{\gamma}} \right) \]  

for the mass \(m_{\text{hair}}\) of the external matter fields (hair).

Using the Einstein field equations (4) and (5), it has been explicitly proved \[12\] that the black-hole null circular geodesics are characterized by the relation \[27\]

\[ N(r) \equiv 3\mu(r) - 1 - 8\pi r^2 p(r) = 0 \text{ for } r = r_{\gamma} . \]  

(19)

Substituting the lower bound (18) into Eq. (19) and using the relation (13), one obtains the inequality

\[ r_{\gamma} - \frac{3}{2} r_{\text{H}} + 12\pi r_{\text{H}}^4 p_{\text{H}} \left( \frac{1}{r_{\text{H}}} - \frac{1}{r_{\gamma}} \right) - 4\pi r_{\gamma}^3 p_{\gamma} \geq 0 , \]  

(20)

which characterizes the null circular geodesics of the spherically-symmetric hairy black-hole spacetime. In addition, using the inequality \(r_{\text{H}}^3 p_{\text{H}} \leq r_{\gamma}^3 p_{\gamma}\), which follows from the assumed
monotonic behavior of the radial pressure function \( p(r) \) and the fact that the radial pressure is non-positive between the black-hole horizon and the innermost null circular geodesic [see Eq. (16)], one deduces from (20) the inequality
\[
\frac{r_\gamma}{2} - 3r_H + 12\pi r_H^2 p_\gamma \left( \frac{1}{r_H} - \frac{1}{r_\gamma} \right) - 4\pi r_\gamma^3 p_\gamma \geq 0,
\]
which can be expressed in the remarkably compact form
\[
\left( \frac{r_\gamma}{2} - r_H \right) \cdot (1 + 8\pi r_\gamma^2 p_\gamma) \geq 0.
\]

Taking cognizance of the characteristic relation (19) for the null circular geodesics of the black-hole spacetime (3), one can write (22) in the form
\[
\left( \frac{r_\gamma}{2} - r_H \right) \cdot \mu(r_\gamma) \geq 0,
\]
which yields the lower bound
\[
r_\gamma \geq \frac{3}{2} r_H
\]
on the radii of the black-hole null circular geodesics.

IV. SUMMARY

Black-hole spacetimes are characterized by the presence of null circular geodesics on which massless particles (photons, gravitons) can orbit the central black hole. These closed light-like orbits play important physical roles in astrophysical [5, 6], cosmological [7], and theoretical [8–16] studies of curved black-hole spacetimes.

In the present compact paper we have addressed the following physically interesting question: In a spherically symmetric black-hole spacetime, how close can the innermost null circular geodesic be to the black-hole horizon? Interestingly, it has recently been conjectured [16] that, for a black hole of horizon radius \( r_H \), massless particles can orbit the central black hole on circular trajectories whose radii are bounded from below by the simple relation \( r_\gamma \geq \frac{3}{2} r_H \) [see Eq. (2)].

Using analytical techniques, we have presented a remarkably compact theorem that proves the validity of this intriguing conjecture for spherically symmetric hairy black-hole spacetimes whose radial pressure function \( |r^3 p(r)| \), which characterizes the spatial behavior of the external matter fields (hair), decreases monotonically.
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Here \(\{t, r, \theta, \phi\}\) are the Schwarzschild spacetime coordinates.

Here the prime denotes a spatial derivative with respect to the radial coordinate \(r\).

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The equality sign in (9) corresponds to the case of extremal black holes.

It is worth emphasizing that the radial pressure \(p(r)\) is non-positive in the interval \(r_H \leq r \leq r_\gamma\) between the outer black-hole horizon and the innermost null circular geodesic [see [12] and Eq. (16) below].

It is worth noting that the physical motivation for the assumption that the radial pressure function (14) is monotonically decreasing stems from the fact that, for asymptotically flat hairy black-hole configurations, this pressure function is known to be asymptotically decreasing [12]. In the present paper we shall prove that if this functional behavior is assumed to hold true in the interval \([r_H, r_\gamma]\), then the null circular geodesics of the hairy black-hole spacetimes are characterized by the physically interesting lower bound (2).

It is worth noting that our theorem would be valid for any matter theory in which the radial pressure function \(|r^3p(r)|\) decreases monotonically between the black-hole outer horizon and the innermost null circular geodesic. In particular, the behavior of the pressure function in the spacetime region \(r > r_\gamma\) outside the null circular geodesic is not important for our analysis.

We consider non-extremal black holes for which \(\mathcal{N}(r_H) < 0\) [12].