A SIMPLE PROOF OF UNIQUE CONTINUATION FOR
J-HOLOMORPHIC CURVES

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Abstract. In this expository paper, we prove strong unique continuation for J-holomorphic curves by first giving a simple proof of Aronszajn’s theorem in the special case of the two-dimensional flat Laplacian.

1. Introduction

In the study of J-holomorphic curves and symplectic topology as presented by McDuff and Salamon [7], a basic fact is the strong unique continuation property for J-holomorphic curves. In their book, the strong unique continuation property is a first step in a chain of events leading to the proof that, for a generic almost complex structure J, the moduli space \( \mathcal{M}^*(A, \Sigma; J) \) of simple J-holomorphic A-curves is a smooth finite dimensional manifold, and from there to the construction of the Gromov-Witten invariants for a suitable class of symplectic manifolds (see pages 4 and 38 of [7] for the outline of this approach).

McDuff and Salamon give three proofs of the unique continuation property. The first proof is a few lines long but cites Aronszajn’s theorem as proven in [2]. The second and third proofs are given self-contained treatments, and, moreover, the methods find further application in their book. The second proof uses the Hartman–Wintner theorem [6] (proven in McDuff and Salamon’s Appendix E.4), which in fact implies the needed special case of Aronszajn’s theorem, and the third proof uses the Carleman similarity principle and the Riemann-Roch theorem (proven in their Appendix C).

Here we return to the first method of proof, but give a simplified argument. The method is well known in certain branches of partial differential equations; it is the method of weighted integral estimates depending on a parameter. This is also the approach of Aronszajn [2], but it goes back even further, to Carleman [3]. For a general treatment with some historical comments, one may consult Sections 17.1 and 17.2 of Hörmander’s book [9] or his corresponding paper [8]. However, all these references give much more than is needed for our application. Here we present only what is needed for J-holomorphic curves.

We give the full details for the case of \( C^\infty \) J-holomorphic curves; for J-holomorphic curves in Sobolev spaces with minimal assumptions, discussed in McDuff and Salamon’s book, one may find the appropriate modifications in Sections 17.1 and 17.2 of Hörmander’s book [9]. Here we focus on the \( C^\infty \) case, for ease of exposition and since the \( C^\infty \) case is sufficient for

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many purposes; after all, in Gromov’s original definition all $J$-holomorphic curves are $C^\infty$ [5].

The weighted integral estimates will depend on a parameter $0 < h \ll 1$ which may be interpreted as “Planck’s constant”, as appearing in the correspondence principle of the old quantum theory, or, more generally, as appearing in semiclassical analysis [4]. The general idea is that as $h$ tends to zero, asymptotic analysis reveals the classical mechanics of the operator’s symbol, interpreted as a Hamiltonian function. Hence symplectic geometry plays a role beneath the surface.

We begin by recalling the basic definitions, so that our presentation is self-contained. Let $(\Sigma, j)$ be a Riemann surface and $(M, J)$ an almost complex manifold. A smooth function $u : \Sigma \rightarrow M$ is called a $J$-holomorphic curve if its differential $du$ is a complex linear map with respect to $j$ and $J$; that is, if

\[ J \circ du = du \circ j, \]

or, equivalently,

\[ \bar{\partial}_J(u) := \frac{1}{2} (du + J \circ du \circ j) = 0. \]

Unique continuation is a local problem, so for our purposes we may take the domain of $u$ to be a connected neighborhood $X \subset \mathbb{C}$ of the origin, writing the elements of $X$ as $x = x_1 + ix_2$, and we may take $M$ to be $\mathbb{C}^n$. Hence we are interested in those $u \in C^\infty(X, \mathbb{C}^n)$ satisfying

\[ (1) \quad \partial_{x_1} u + J(u) \partial_{x_2} u = 0, \]

where $J : \mathbb{C}^n \rightarrow GL(2n, \mathbb{R})$ is, say, a $C^1$ function such that $J^2 = -I$.

The main point of this paper is to give a simple, elementary proof of the following strong unique continuation result:

**Theorem 1.** Let $X \subset \mathbb{C}$ be a connected neighborhood of 0, and suppose $u, v \in C^\infty(X, \mathbb{C}^n)$ satisfy (1) for some $C^1$ almost complex structure $J : \mathbb{C}^n \rightarrow GL(2n, \mathbb{R})$. If $u - v$ vanishes to infinite order at 0, then $u = v$ in $X$.

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2. **Proof of Unique Continuation**

Let $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ be the standard Laplacian. Since $(\partial_{x_1} J) J + J \partial_{x_1} J = 0$, we have that any solution $u$ of (1) is also a solution of

\[ \Delta u = (\partial_{x_2} J(u)) \partial_{x_1} u - (\partial_{x_1} J(u)) \partial_{x_2} u. \]
If \( v \) is another such function, then
\[
\Delta(u - v) = (\partial_{x_2}J(u))\partial_{x_1}(u - v) + [\partial_{x_2}(J(u) - J(v))]\partial_{x_1}v
- (\partial_{x_1}J(u))\partial_{x_2}(u - v) - [\partial_{x_1}(J(u) - J(v))]\partial_{x_2}v.
\]
Also, of course,
\[
J(u) - J(v) = \int_0^1 dJ(v + \tau(u - v))d\tau \cdot (u - v).
\]
So, if \( w := u - v \), then for some constant \( C > 0 \) we have
\[
|\Delta w| \leq C(|w| + |\partial_{x_1}w| + |\partial_{x_2}w|).
\]
Since we are considering fixed functions \( u \) and \( v \), the constant is allowed to depend on \( u, v \), and their derivatives.

Thus Theorem 1 is a consequence of the following unique continuation result, a special case of Aronszajn’s theorem [2]. (We follow the presentation of Theorem 17.2.6 in Hörmander’s book [9].)

**Theorem 2.** Let \( X \subset \mathbb{R}^2 \) be a connected neighborhood of 0, and let \( u \in C^\infty(X, \mathbb{C}^n) \) be such that
\[
|\Delta u| \leq C(|u| + |\partial_{x_1}u| + |\partial_{x_2}u|).
\]
If \( u \) vanishes to infinite order at 0, then \( u = 0 \) in \( X \).

**Proof.** For notational purposes, we assume \( n = 1 \). The proof works line-by-line for the general case.

We first introduce conformal polar coordinates in \( \mathbb{R}^2 \setminus \{0\} \),
\[
(x_1, x_2) = (e^t \cos \theta, e^t \sin \theta)
\]
with \( t \in \mathbb{R} \) and \( \theta \in S^1 \). Then, in these coordinates,
\[
\partial_{x_1} = e^{-t} \cos \theta \partial_t - e^{-t} \sin \theta \partial_\theta,
\]
\[
\partial_{x_2} = e^{-t} \sin \theta \partial_t + e^{-t} \cos \theta \partial_\theta,
\]
and
\[
\Delta = e^{-2t}(\partial_t^2 + \partial_\theta^2).
\]
Next, we “convexify” the coordinates. Let \( 0 < \epsilon < 1 \), and let \( T \) be such that
\[
t = T + e^{\epsilon T}.
\]
As noted by Hörmander [8], this change of coordinates comes from the work of Alinhac and Baouendi [1]. Then
\[
\frac{\partial t}{\partial T} = 1 + \epsilon e^{\epsilon T} > 0,
\]
and \( T < t < T + 1 < T/2 \) when \( T < -2 \). In these coordinates,
\[
\partial_t^2 + \partial_\theta^2 = (1 + \epsilon e^{\epsilon T})^{-2} \partial_T^2 - \epsilon^2(1 + \epsilon e^{\epsilon T})^{-3} e^{\epsilon T} \partial_T + \partial_\theta^2.
\]
Multiplying by \((1 + \epsilon e^{cT})^2\), we get the operator
\[
Q := \partial_T^2 + c(T)\partial_T + (1 + \epsilon e^{cT})^2\partial_\theta^2,
\]
with
\[
c(T) := -\epsilon^2(1 + \epsilon e^{cT})^{-1}e^{cT}.
\]

Our main tool is the following estimate:

**Proposition 3.** For some \(T_0 < 0\) and some \(h_0 > 0\) we have

\[
h \int \int (|U|^2 + |h\partial_T U|^2 + |h\partial_\theta U|^2 + |h^2 \partial_T^2 U|^2 + |h^2 \partial_T \partial_\theta U|^2 + |h^2 \partial_\theta^2 U|^2) e^{-2T/h + \epsilon T} d\theta dT 
\leq C \int \int |P^2 \partial_T U|^2 e^{-2T/h} d\theta dT
\]

for all \(U \in C_0^\infty((-\infty, T_0) \times S^1)\), and for all \(h \in (0, h_0)\). (The constant \(C > 0\) is independent of \(h\).)

**Proof.** (of the Proposition.) We set \(U := e^{T/h}V\) and let
\[
\tilde{Q} := h^2 e^{-T/h} \circ Q \circ e^{T/h}.
\]
That is,
\[
\tilde{Q} = (h\partial_T + 1)^2 + hc(T)(h\partial_T + 1) + (1 + \epsilon e^{cT})^2h^2\partial_\theta^2.
\]

Then the estimate (3) is equivalent to

\[
h \int \int (|V|^2 + |h\partial_T V|^2 + |h\partial_\theta V|^2 + |h^2 \partial_T^2 V|^2 + |h^2 \partial_T \partial_\theta V|^2 + |h^2 \partial_\theta^2 V|^2) e^{cT} d\theta dT 
\leq C \int \int |\tilde{Q}V|^2 d\theta dT
\]

for all \(V \in C_0^\infty((-\infty, T_0) \times S^1)\).

For bookkeeping purposes, we write \(\tilde{Q}\) as the sum of its symmetric and antisymmetric parts,
\[
\tilde{Q} = A + B,
\]
where
\[
A = h^2 \partial_T^2 + (1 + hc - \frac{1}{2}h^2 c') + (1 + \epsilon e^{cT})^2h^2\partial_\theta^2,
\]
and
\[
B = (2 + hc)h\partial_T + \frac{1}{2}h^2 c'.
\]
Hence, using the usual inner product notation on \(L^2\), and with \([A, B] = AB - BA\) denoting the commutator,
\[
\int \int |\tilde{Q}V|^2 d\theta dT = ||AV||^2 + ||BV||^2 + \langle [A, B]V, V \rangle.
\]
Repeated integration by parts gives
\[ ||AV||^2 = ||h^2 \partial^2_T V||^2 \]
(5)
\[ + ||(1 + hc - \frac{1}{2} h^2 c')V||^2 \]
(6)
\[ + ||(1 + \epsilon e^T)^2 h^2 \partial_\theta^2 V||^2 \]
\[ + h^3 \langle V, (c'' - \frac{1}{2} hc''' - V) \rangle \]
\[ - 2 \langle h \partial_T V, (1 + hc - \frac{1}{2} h^2 c') h \partial_T V \rangle \]
\[ - 2 \epsilon^3 h^2 \langle h \partial_\theta V, (1 + 2 \epsilon e^T) e^T h \partial_\theta V \rangle \]
\[ + 2 \langle h^2 \partial^2_\theta V, (1 + \epsilon e^T)^2 h^2 \partial^2_\theta V \rangle \]
(7)
\[ - 2 \langle h \partial_\theta V, (1 + hc - \frac{1}{2} h^2 c') (1 + \epsilon e^T)^2 h \partial_\theta V \rangle, \]
and
\[ ||BV||^2 = ||(2 + hc) h \partial_T V||^2 - \frac{1}{4} h^4 ||c' V||^2 - \frac{1}{2} h^4 \langle V, c'' V \rangle \]
\[ - h^3 \langle V, c''' V \rangle, \]
and
\[ \langle [A, B] V, V \rangle = -2 h^2 \langle c' h \partial_T V, h \partial_T V \rangle \]
\[ - 2 h^2 \langle c' V, V \rangle + h^3 \langle (c'' - cc') V, V \rangle + \frac{1}{2} h^4 \langle (cc'' + c''' V, V \rangle \]
(8)
\[ + 2 h \epsilon^2 \langle (2 + hc) (1 + \epsilon e^T) e^T h \partial_\theta V, h \partial_\theta V \rangle. \]

Also, we recall that
\[ c(T) = -\epsilon^2 e^T (1 + \epsilon e^T)^{-1} \]
so that
\[ c'(T) = -\epsilon^3 e^T (1 + \epsilon e^T)^{-2} \]
is also a negative quantity.

Most of the terms in the above expansions may be absorbed into other terms when we take \( 0 < h \) to be sufficiently small. It is only the term (7) that gives some difficulty. We write (7) as
\[ (7') - 2 \langle h \partial_\theta V, (1 + \lambda hc) (1 + \epsilon e^T)^2 h \partial_\theta V \rangle - 2 \langle h \partial_\theta V, ((1 - \lambda) hc - \frac{1}{2} h^2 c') (1 + \epsilon e^T)^2 h \partial_\theta V \rangle. \]
Here \( \lambda \in \mathbb{R} \) is to be determined; as we will see, any \( 2 < \lambda < 3 \) will suffice.
For the first term of (7), we use the elementary inequality
\[ 2 \langle (1 + \lambda hc)^{1/2} V, (1 + \lambda hc)^{1/2} (1 + \epsilon e^T)^2 h^2 \partial_\theta^2 V \rangle \geq -\langle (1 + \lambda hc) V, V \rangle \]
(9)
\[ - \langle (1 + \lambda hc) (1 + \epsilon e^T)^2 h^2 \partial_\theta^2 V, (1 + \epsilon e^T)^2 h^2 \partial_\theta^2 V \rangle. \]
(10)
Now (9) is absorbed into (5) when \( \lambda > 2 \), and (10) may be absorbed into (6) when \( \lambda > 0 \) (in both cases we are left with an order \( h \) term).

As for the second term in (7), it may be absorbed into (8) as long as \( \lambda < 3 \). All the terms are thus accounted for, completing the proof of (4) and of the proposition. \( \square \)

End of proof of Theorem 2. We write \( U(T, \theta) := u(x_1, x_2) \). Since we are only considering \( T < T_0 (\ll 0) \), our hypothesized upper bound (2) gives
\[ |QU| \leq C e^T (|U| + |\partial_T U| + |\partial_\theta U|). \]
Now we let \( \psi \in C^\infty(\mathbb{R}) \) be such that
\[ \begin{align*}
\psi &= 1 \quad \text{in } (-\infty, T_0 - 1) \\
\psi &= 0 \quad \text{in } (T_0, \infty),
\end{align*} \]
and we set
\[ U^\psi(T, \theta) = \psi(T) U(T, \theta). \]

The vanishing hypothesis on \( u \) says that for every \( N \) there exists a constant \( C_N \) such that
\[ |u(x)| \leq C_N |x|^N \]
in a neighborhood of the origin, so that, in the new coordinates, for any \( N \) we have
\[ |U(T, \theta)| \leq C_N e^{NT} \]
for \( T \) in a neighborhood of \( -\infty \). Therefore
\[ \int \int |U^\psi|^2 e^{-NT} d\theta dT < \infty \]
for any \( N \). The same argument holds for all derivatives of \( U^\psi \). We then let \( \chi \in C^\infty(\mathbb{R}) \) be such that
\[ \begin{align*}
\chi &= 0 \quad \text{in } (-\infty, -2) \\
\chi &= 1 \quad \text{in } (-1, \infty),
\end{align*} \]
and for \( R > 0 \) we let \( \chi_R(T) = \chi(T/R) \). We may apply Proposition 3 to \( \chi_R(T) U^\psi(T, \theta) \) and take the limit as \( R \to \infty \); by the Dominated Convergence Theorem, Proposition 3 thus holds for \( U^\psi \).
The righthand side of (3) is then
\[ \int \int \left| h^2 QU^{\psi} \right|^2 e^{-2T/h} d\theta dT = h^4 \int \int \left| \psi QU + \psi'' U + 2 \psi' \partial_T U + c \psi' U \right|^2 e^{-2T/h} d\theta dT \]
(11)
\[ \leq Ch^4 \int \int e^{2T} (\left| U^{\psi} \right|^2 + \left| \partial_T U^{\psi} \right|^2 + \left| \partial_\theta U^{\psi} \right|^2) e^{-2T/h} d\theta dT \]
(12)
\[ + Ch^4 \int \int_{T_0-1} T_0 \left( \left| U \right|^2 + \left| \partial_T U \right|^2 \right) e^{-2T/h} d\theta dT. \]

Since \( 2T < \epsilon T \), the term (11) is bounded by
\[ Ch^2 \int \int (\left| U^{\psi} \right|^2 + \left| h \partial_T U^{\psi} \right|^2 + \left| h \partial_\theta U^{\psi} \right|^2) e^{-2T/h + \epsilon T} d\theta dT, \]
and hence can be absorbed into the lefthand side of (3) when \( h > 0 \) is sufficiently small.

Since \( U \) and \( \partial_T U \) are bounded, the term (12) is bounded by
\[ Ch^5 e^{-2(T_0-1)/h}. \]

Hence we have
\[ h \int \int \left( \left| U^{\psi} \right|^2 + \left| h \partial_T U^{\psi} \right|^2 + \left| h \partial_\theta U^{\psi} \right|^2 + \left| h^2 \partial_T^2 U^{\psi} \right|^2 + \left| h^2 \partial_\theta^2 U^{\psi} \right|^2 + \left| h^2 \partial_T \partial_\theta U^{\psi} \right|^2 \right) e^{-2T/h + \epsilon T} d\theta dT \]
\[ \leq Ch^5 e^{-2(T_0-1)/h}. \]
Letting \( h \to 0 \), we see that \( U = 0 \) when \( T < T_0 - 1 \), as otherwise the left side grows faster than the right side. Hence the original function \( u \) vanishes in a neighborhood of the origin.

We have thus shown that the set of points where \( u \) vanishes to infinite order is an open set. The complement is obviously also an open set, so by the connectedness of \( X \) we have that \( u = 0 \) in \( X \). This concludes the proof of the theorem. \( \square \)

References
[1] S. Alinhac and M. S. Baouendi. Uniqueness for the characteristic Cauchy problem and strong unique continuation for higher order partial differential inequalities. Amer. J. Math. 102 (1980), no. 1, 179–217.
[2] N. Aronszajn. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pures Appl. (9) 36 (1957), 235–249.
[3] T. Carleman. Sur un problème d’unicité pour les systèmes d’équations aux dérivées partielles à deux variables indépendantes. (French) Ark. Mat., Astr. Fys. 26, (1939), no. 17, 9 pp.
[4] L. C. Evans and M. Zworski. Semi-classical analysis, Edition 0.3. www.math.berkeley.edu/~zworski/semiclassical.pdf, 2007.
[5] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. Invent. Math. 82 (1985), no. 2, 307–347.
[6] P. Hartman and A. Wintner. On the local behavior of solutions of non-parabolic partial differential equations. Amer. J. Math. 75 (1953), 449–476.
[7] D. McDuff and D. Salamon. $J$-holomorphic curves and symplectic topology. American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.

[8] L. Hörmander. Uniqueness theorems for second order elliptic differential equations. Comm. Partial Differential Equations 8 (1983), no. 1, 21–64.

[9] L. Hörmander. The analysis of linear partial differential operators. III. Pseudodifferential operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 274. Springer-Verlag, Berlin, 1985.

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