Some examples of Picard groups of blocks *

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Abstract

We calculate the Picard groups for 2-blocks with abelian defect groups of 2-rank at most three with respect to a complete discrete valuation ring. In particular this shows directly that all such Picard groups are finite and the subgroup Piccent of Morita equivalences fixing the centre is trivial. These are amongst the first calculations of this kind. Further we prove some general results concerning Picard groups of blocks with normal defect groups as well as some other cases.

1 Introduction

Let $\mathcal{O}$ be a complete discrete valuation ring with $k := \mathcal{O}/J(\mathcal{O})$ algebraically closed of prime characteristic $p$. Let $K$ be the field of fractions of $\mathcal{O}$, of characteristic zero. Let $G$ be a finite group and $B$ be a block of $\mathcal{O}G$. The Picard group $\text{Pic}(B)$ of $B$ consists of isomorphism classes of $B$-$B$-bimodules which induce Morita self-equivalences of $B$. The group multiplication is given by $M \otimes_B N$. As yet relatively few examples have been calculated and there are many open questions regarding their structure, as raised in [1]. One is that, whilst the Picard group of a $k$-block is usually infinite, it is not clear whether the Picard group of an $\mathcal{O}$-block of a finite group must be finite. A related question is whether every element of $\text{Pic}(B)$ can be taken to be a bimodule with endopermutation source. There are also no known examples of blocks $B$ where the subgroup $\text{Piccent}(B)$ of $\text{Pic}(B)$ inducing the identity map on $Z(B)$ is nontrivial.

The purpose of this article is to find the Picard groups of some classes of examples, both to provide evidence for the main open questions, but also as tools for the classification of Morita equivalence classes. Previous examples where the Picard groups have been calculated are the blocks with cyclic or Klein four defect groups, and blocks of groups $P \rtimes E$ where $P$ is an abelian

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Let $P$ be a finite abelian 2-group.

(i) If $B = \mathcal{O}(P \times G_m)$, then $\text{Pic}(B) = \mathcal{L}(B) \cong (P \rtimes \text{Aut}(P)) \times S_3$.

(ii) If $B = B_0(\mathcal{O}(P \times A_5))$, then $\text{Pic}(B) = \mathcal{L}(B) \cong (P \rtimes \text{Aut}(P)) \times C_2$.

(iii) If $B = B_0(\mathcal{O}(G_m \times A_5))$, where $m \geq 1$, then $\text{Pic}(B) = T(B) \cong S_3 \times C_2$.

(iv) If $B = \mathcal{O}((C_2)^3 \rtimes (C_7 \times C_3))$, then $\text{Pic}(B) = T(B) \cong C_3$.

(v) If $B = B_0(\mathcal{O}SL_2(2^m))$, where $m \geq 2$, then $\text{Pic}(B) = T(B) \cong C_m$.

(vi) If $B = B_0(\mathcal{O}J_1)$, then $\text{Pic}(B) = 1$.

(vii) If $B = B_0(\mathcal{O}Aut(SL_2(8)))$, then $\text{Pic}(B) = T(B) \cong C_3$.

In each case $\text{Piccent}(B) = 1$.

Corollary 1.2. Let $G$ be a finite group and $B$ a 2-block of $\mathcal{O}G$ with abelian defect group $D$ of 2-rank at most three. Then the isomorphism type of $\text{Pic}(B)$ is determined and $|\text{Pic}(B)| \leq |\mathcal{L}(C)|$ for some block $C$ (of $\mathcal{O}H$ for some finite group $H$) Morita equivalent to $B$ with isomorphic defect group. In particular, $|\text{Pic}(B)|$ is bounded in terms of $D$. Further $\text{Piccent}(B) = 1$.

It is observed in [9] that derived equivalence preserves finiteness of Picard groups. In particular as by [10] Broué's conjecture holds for 2-blocks with abelian defect groups of 2-rank at most three, finiteness also holds by Proposition 4.5.

The structure of the paper is as follows. In Section 2 we define Picard groups and certain distinguished subgroups, and give some background results from [1]. The groups of perfect self-isometries of blocks play a major role in much of this paper, and Section 3 contains the relevant definitions and calculations. In Section 4 we apply the results of the previous section together with Weiss' criterion to prove most of Theorem 1.1. In Section 5 we calculate the Picard groups in the final three cases of Theorem 1.1.
2 Picard groups of blocks

The following is based on [1]. For further detail we also recommend [16, 17]. Let $G$ be a finite group and $B$ be a block of $OG$ with defect group $D$. Let $\mathcal{F}$ be the fusion system for $B$ on $D$, defined using a maximal $B$-subpair $(D, b_D)$. Write $E = N_G(D, b_D)/DC_G(D)$, the inertial quotient. $\text{Aut}(D, \mathcal{F})$ denotes the subgroup of $\text{Aut}(D)$ of automorphisms stabilizing $\mathcal{F}$. Write $\text{Out}(D, \mathcal{F}) = \text{Aut}_\mathcal{F}(D)$.

Now let $A$ be a source algebra for $B$, so $A$ is a $D$-algebra and we may consider the fixed points $A^D$ under the action of $D$. Write $\text{Aut}_D(A)$ for the group of algebra automorphisms of $A$ fixing each element of the image of $D$ in $A$, and $\text{Out}_D(A)$ for the quotient of $\text{Aut}_D(A)$ by the subgroup of automorphisms given by conjugation by elements of $(A^D)^\times$.

As noted in [1], by [20, 14.9] $\text{Out}_D(A)$ is isomorphic to a subgroup of $\text{Hom}(E, k^\times)$.

The Picard group $\text{Pic}(B)$ of $B$ consists of isomorphism classes of $B$-$B$-bimodules which induce Morita self-equivalences of $B$. For $B$-$B$-bimodules $M$ and $N$, the group multiplication is given by $M \otimes_B N$. Write $\mathcal{T}(B)$ for the subset of $\text{Pic}(B)$ consisting of bimodules with trivial source and $\mathcal{L}(B)$ for the subset consisting of linear source modules. It is shown in [1] that $\mathcal{T}(B)$ and $\mathcal{L}(B)$ form subgroups of $\text{Pic}(B)$ and are described by the exact sequences

$$1 \rightarrow \text{Out}_D(A) \rightarrow \mathcal{T}(B) \rightarrow \text{Out}(D, \mathcal{F}),$$

$$1 \rightarrow \text{Out}_D(A) \rightarrow \mathcal{L}(B) \rightarrow \text{Hom}(D/\text{foc}(D), O^\times_X) \times \text{Out}(D, \mathcal{F}),$$

where $\text{foc}(D)$ is the focal subgroup of $D$ with respect to $\mathcal{F}$, generated by the elements $\varphi(x)x^{-1}$ for $x \in D$ and $\varphi \in \text{Hom}_\mathcal{F}(x, D)$.

Automorphisms $\alpha$ of $B$ give rise to elements of the Picard group as follows. Define the $B$-$B$-bimodule $_\alpha B$ by taking $_\alpha B = B$ as sets and defining $a_1 \cdot m \cdot a_2 = \alpha(a_1)ma_2$ for $a_1, a_2, m \in B$. Inner automorphisms give isomorphic bimodules and $\alpha \mapsto _\alpha B$ gives rise to an injection $\text{Out}(B) \rightarrow \text{Pic}(B)$.

Each element of $\text{Pic}(B)$ induces an automorphism of $Z(B)$. The subgroup consisting of those which induce the identity morphism is denoted $\text{Picc}(B)$. Note $\text{Picc}(B)$ is precisely the subgroup of bimodules in $\text{Pic}(B)$ that fix every irreducible character.

The following is clear from [1] but we state it here for convenience as it will be used frequently.

Lemma 2.1. Let $B$ be a block of $OG$ for a finite group $G$ and let $(D, b_D)$ be a maximal $B$-subpair. Suppose $N \triangleleft G$ with $G/N$ abelian, $C_G(D) \leq N$ and $G = N_G(D, b_D)N$. Let $A$ be a source algebra for $B$. Then $\text{Out}_D(A)$ has a subgroup isomorphic to $G/N$.

Proof. Let $\varphi$ be the inflation of an irreducible character of $G/N$. An element of $\text{Out}_D(A)$ is realised by taking the bimodule $_\alpha B$ inducing the Morita equivalence given by the automorphism $\alpha$ of $B$ given by $\alpha(x) = \varphi(x)x$. It is clear from the permutation of $\text{Irr}(G)$ given by $_\alpha B$ that distinct characters of $G/N$ give rise to distinct elements of $\text{Pic}(B)$.

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In order to find $\mathcal{T}(B)$ when $G$ is a direct product of groups we need to show that $\text{Out}_D(A)$ factorises according to the factorisation of $G$.

Write $i$ for the identity element of the source algebra $A$, so $A = iOG_i$. As described in [11, Remark 1.2] elements of $\text{Out}_D(A)$ correspond to direct summands of $OGi \otimes_{OD} iOG$ as $B-B$-bimodules inducing Morita equivalences. In the following we will have to pass temporarily to the source algebra defined with respect to $k$ in order to apply [3, Lemma 10.37]. We use the notation $kB$ (or $KB$) to denote $B \otimes_\mathcal{O} k$ (or $B \otimes_\mathcal{O} K$), and use similar notation for related objects.

**Lemma 2.2.** Let $G_1$ and $G_2$ be finite groups, $B_j$ a block of $OG_j$ and $B = B_1 \otimes \mathcal{O} B_2$ a block of $O(G_1 \times G_2)$. Let $D$ be a defect group of $B$ with $D = D_1 \times D_2$, where $D_j$ is a defect group of $B_j$. Let $i_j$ be a source idempotent of $B_j$. Then $i := i_1 \otimes i_2$ is a source idempotent of $B$ and $iBi \cong i_1B_1i_1 \otimes \mathcal{O} i_2B_2i_2$.

Further $\text{Out}_D(A) \cong \text{Out}_{D_1}(A_1) \times \text{Out}_{D_2}(A_2)$, where $A = iBi$ and $A_j = i_jB_ji_j$.

**Proof.** We have $B^D = B_1^{D_1} \otimes \mathcal{O} B_2^{D_2}$ and $C_G(D) = C_G(D_1) \times C_G(D_2)$. Now [3, Lemma 10.37] implies that $k(B_1^{D_1}i_1) \otimes_k k(B_2^{D_2}i_2)$ is an indecomposable $k(B_1^{D_1}) \otimes_k k(B_2^{D_2})$-module. So the image of $i$ in $k(B^D)$ is primitive hence so is $i$ itself. In addition

$$\text{Br}_{D_1 \times D_2}(i) = \text{Br}_{D_1}(i_1) \otimes_{\mathcal{O}} \text{Br}_{D_2}(i_2) \neq 0$$

and so we have shown that $i$ is a source idempotent of $B$. Now every indecomposable $B-B$-summand of

$$OGi \otimes_{OD} iOG \cong (OG_1i_1 \otimes_{OD_1} i_1OG_1) \otimes_{\mathcal{O}} (OG_2i_2 \otimes_{OD_2} i_2OG_2)$$

is isomorphic to $M_1 \otimes \mathcal{O} M_2$, for an indecomposable $B_1-B_1$-summand $M_1$ of $OG_1i_1 \otimes_{OD_1} i_1OG_1$ and an indecomposable $B_2-B_2$-summand $M_2$ of $OG_2i_2 \otimes_{OD_2} i_2OG_2$ (since another application of [3, Lemma 10.37] gives the analogous statement over $k$ and trivial source modules can be lifted uniquely to $\mathcal{O}$). Finally we note that $K(M_1 \otimes \mathcal{O} M_2)$ induces a bijection of simple $KB$-modules if and only if each $KM_j$ induces a bijection of simple $KB_j$-modules. The claim now follows from [11, Théorème 1.2].

## 3 Perfect isometries

Before we calculate some Picard groups of blocks it will be necessary to determine some perfect self-isometry groups. We first introduce some notation.

Let $G$ be a finite group and $B$ a block of $OG$. We write $\text{Irr}(B)$ for the set of irreducible characters in $B$ (with respect to $K$). Write $G'_p$ for the set of $p$-regular elements of $G$, $\text{IBr}(B)$ for the set of irreducible Brauer characters of $B$ and $\text{prj}(B)$ for the set characters of projective indecomposable $B$-modules. $B_0(OG)$ will denote the principal block of $OG$. 


Definition 3.1 ([5]). We denote by $\text{CF}(G, B, K)$ the $K$-subspace of class functions on $G$ spanned by $\text{Irr}(B)$, by $\text{CF}(G, B, \mathcal{O})$ the $\mathcal{O}$-submodule
\[ \{ \phi \in \text{CF}(G, B, K) : \phi(g) \in \mathcal{O} \text{ for all } g \in G \} \]
of $\text{CF}(G, B, K)$ and by $\text{CF}_p'(G, B, \mathcal{O})$ the $\mathcal{O}$-submodule
\[ \{ \phi \in \text{CF}(G, B, \mathcal{O}) : \phi(g) = 0 \text{ for all } g \in G \setminus G_p' \} \]
of $\text{CF}(G, B, \mathcal{O})$.

Now in addition let $H$ be a finite group and $C$ a block of $\mathcal{O}H$. A perfect isometry between $B$ and $C$ is an isometry $I : \mathbb{Z}\text{Irr}(B) \to \mathbb{Z}\text{Irr}(C)$, such that
\[ I_K := I \otimes_{\mathbb{Z}} K : K\text{Irr}(B) \to K\text{Irr}(C), \]
induces an $\mathcal{O}$-module isomorphism between $\text{CF}(G, B, \mathcal{O})$ and $\text{CF}(H, C, \mathcal{O})$ and also between $\text{CF}_p'(G, B, \mathcal{O})$ and $\text{CF}_p'(H, C, \mathcal{O})$. (Note that by an isometry we mean an isometry with respect to the usual inner products on $\mathbb{Z}\text{Irr}(B)$ and $\mathbb{Z}\text{Irr}(C)$, so for all $\chi \in \text{Irr}(B)$, $I(\chi) = \pm \psi$ for some $\psi \in \text{Irr}(C)$).

If $H = G$ and $C = B$ then we describe $I$ as a perfect self-isometry of $B$. We denote by $\text{Perf}(B)$ the group of perfect self-isometries of $B$.

Remark 3.2. An alternative way of phrasing the condition that $I_K$ induces an isomorphism between $\text{CF}_p'(G, B, \mathcal{O})$ and $\text{CF}_p'(H, C, \mathcal{O})$ is that $I$ induces an $\mathbb{Z}\text{prj}(B) \cong \mathbb{Z}\text{prj}(C)$.

The following two well-known lemmas are both proved in [5].

Lemma 3.3. Let $G$ and $G'$ be finite groups, $B$ and $B'$ blocks of $\mathcal{O}G$ and $\mathcal{O}G'$ respectively and $I : \mathbb{Z}\text{Irr}(B) \to \mathbb{Z}\text{Irr}(B')$ a perfect isometry. The $K$-algebra isomorphism between $Z(KB)$ and $Z(KB')$ given by the bijection of character idempotents induced by $I$ induces an $\mathcal{O}$-algebra isomorphism $\phi_I : Z(B) \to Z(B')$.

Lemma 3.4. Any Morita equivalence of blocks induces a perfect isometry.

Before proceeding with some specific examples we need a lemma about Picard groups and perfect self-isometry groups of group algebras of $p$-groups.

Lemma 3.5. Let $P$ be a finite $p$-group. Then we have the following isomorphisms of groups.

(a) $\text{Pic}(\mathcal{O}P) \cong \text{Out}(\mathcal{O}P) \cong \text{Hom}(P, \mathcal{O}^\times) \times \text{Out}(P)$

(b) If $P$ is abelian, then $\text{Perf}(\mathcal{O}P) \cong \text{Aut}(\mathcal{O}P) \times C_2$.

Proof.
(a) This is well-known.

(b) Since there is only one indecomposable projective module for $OP$, every perfect self-isometry of $OP$ must have all positive or all negative signs. Now by Lemma 3.3 the induced permutation of $\text{Irr}(OP)$ induces an automorphism of $\text{Aut}(Z(OP)) = \text{Aut}(OP)$. Hence the result.

We recall the character table of $A_4$, where we also set up some labelling of characters.

|   | $()$ | $(12)(34)$ | $(123)$ | $(132)$ |
|---|------|------------|--------|--------|
| $\chi_1$ | 1    | 1          | 1    | 1     |
| $\chi_2$ | 1    | 1          | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1    | 1          | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3    | $-1$       | 0    | 0     |

For the rest of this section we assume $p = 2$.

**Proposition 3.6.** [10, Proposition 2.8] The perfect self-isometries of $OA_4$ are precisely the isometries of the form:

$$I_{\sigma, \epsilon} : \mathbb{Z}\text{Irr}(A_4) \rightarrow \mathbb{Z}\text{Irr}(A_4)$$

$$\chi_j \mapsto \epsilon \delta_j \delta_{\sigma(j)} \chi_{\sigma(j)} ,$$

for $1 \leq j \leq 4$, where $\sigma \in S_4$, $\epsilon \in \{ \pm 1 \}$ and $\delta_1 = \delta_2 = \delta_3 = -\delta_4 = 1$. Hence $\text{Perf}(OA_4) \cong S_4 \times C_2$.

Let $P$ be a finite abelian 2-group.

**Theorem 3.7.** Every perfect self-isometry of $O(P \times A_4)$ is of the form $(J, I_{\sigma, \epsilon})$, where $J$ is a perfect isometry of $OP$ induced by an $O$-algebra automorphism, $\sigma \in S_4$ and $\epsilon \in \{ \pm 1 \}$.

**Proof.** We proceed as in the proof of [10, Theorem 2.11]. The projective indecomposable characters are

$$\text{prj}(O(C_2^n \times A_4)) = \{ \chi_{P_1}, \chi_{P_2}, \chi_{P_3} \}, \text{ where } \chi_{P_j} = \left( \sum_{\theta \in \text{Irr}(P)} \theta \right) \otimes (\chi_j + \chi_4).$$

Let $I$ be a perfect self-isometry of $O(P \times A_4)$. By counting constituents we see that

$$I(\chi_{P_1}) = \pm \chi_{P_1}, \pm \chi_{P_2}, \pm \chi_{P_3}, \pm (\chi_{P_1} - \chi_{P_2}), \pm (\chi_{P_1} - \chi_{P_3}), \pm (\chi_{P_2} - \chi_{P_3}),$$

(1)
for $1 \leq l \leq 3$. Consider the set

$$X_m := \left\{ j \left| j \in \chi_j, I \left( \sum_{\theta \in \text{Irr}(P)} \theta \otimes \chi_m \right) \right. \right\} \neq 0, \text{ for some } l \right\},$$

for $1 \leq m \leq 4$. By [1] we have shown that $|X_m| = 1$ or $2$ for every $1 \leq m \leq 4$. If $|X_1| = 2$, then by considering [1] for $l = 1$ we see that $X_4 = X_1$. Similarly by considering $I(\chi_{P_1})$, we get that $X_2 = X_4$. This is now a contradiction as then $I\left( \sum_{\theta \in \text{Irr}(P)} \theta \otimes (\chi_1 + \chi_2 + \chi_4) \right)$ has at most $2|P|$ constituents with non-zero multiplicity. Therefore $|X_1| = 1$ and so by considering $I(\chi_{P_1})$ we get that $|X_4| = 1$ and then by considering $I(\chi_{P_2})$ and $I(\chi_{P_3})$ we get that $|X_2| = |X_3| = 1$. Moreover, $X_1, X_2, X_3, X_4$ must all be disjoint. By composing $I$ with the perfect isometry $(\text{Id}, I_{\sigma,1})$, for some appropriately chosen $\sigma \in S_4$, we may assume $X_m = \{ m \}$ for all $1 \leq m \leq 4$. Therefore $I(\chi_{P_l}) = \pm \chi_{P_l}$ for $1 \leq l \leq 3$ and by considering

$$I \left( \sum_{\theta \in \text{Irr}(P)} \theta \otimes \chi_4 \right),$$

we see that in fact all these signs are the same and we may assume, possibly by composing $I$ with $(\text{Id}, I_{\text{id},-1})$, that

$$I \left( \sum_{\theta \in \text{Irr}(P)} \theta \otimes \chi_4 \right) = \left( \sum_{\theta \in \text{Irr}(P)} \theta \right) \otimes \chi_4,$$

for $1 \leq m \leq 4$. Next we note that

$$\frac{1}{12} \chi_1 + \chi_2 + \chi_3 + 3\chi_4) \in \text{CF}(P \times A_4, \mathcal{O}(P \times A_4), \mathcal{O})$$

for all $\theta \in \text{Irr}(P)$. As $3$ is invertible in $\mathcal{O}$, this implies

$$\theta \otimes \left( \sum_{m=1}^{4} \delta_m \chi_m \right) \in 4 \text{CF}(P \times A_4, \mathcal{O}(P \times A_4), \mathcal{O})$$

(see Proposition 3.6 for the definition of $\delta_m$), and so

$$I \left( \theta \otimes \left( \sum_{m=1}^{4} \delta_m \chi_m \right) \right) \in 4 \text{CF}(P \times A_4, \mathcal{O}(P \times A_4), \mathcal{O}) \quad (2)$$
for all $\theta \in \text{Irr}(P)$. Now set $\theta_m \otimes \chi_m := I(\theta \otimes \chi_m)$, for $1 \leq m \leq 4$. Evaluating \(2\) at $(x, 1)$, $(x, (123))$ and $(x, (132))$, for some $x \in P$, gives

\[
\begin{align*}
\theta_1(x) + \theta_2(x) + \theta_3(x) + \theta_4(x) & \in 4\mathcal{O}, \\
\theta_1(x) + \omega \theta_2(x) + \omega^2 \theta_3(x) & \in 4\mathcal{O}, \\
\theta_1(x) + \omega^2 \theta_2(x) + \omega \theta_3(x) & \in 4\mathcal{O}.
\end{align*}
\]

(3)\(\) (4)\(\) (5)

Proceeding as in the proof of \([10, \text{Theorem 2.11}]\) we have $\theta_1(x) = \theta_2(x) = \theta_3(x) = \theta_4(x)$ for all $x \in P$. In other words $\theta_1 = \theta_2 = \theta_3 = \theta_4$.

We have shown that we may assume $I$ is of the form

$$I(\theta \otimes \chi_m) \mapsto \sigma(\theta) \otimes \chi_m,$$

for all $\theta \in \text{Irr}(P)$, where $\sigma$ is a permutation of $\text{Irr}(P)$. In particular the $\mathcal{O}$-algebra automorphism of $Z(\mathcal{O}(P \times A_4))$ induced by $I$ leaves $\mathcal{O}P$ invariant. Therefore the permutation $\sigma$ of $\text{Irr}(P)$ must induce an automorphism of $\mathcal{O}P$ and the theorem is proved.

For $n \in \mathbb{N}$, set $H_n$ to be $(C_{2^n} \times C_{2^n}) \times S_3$, where the action of $S_3$ is given by permuting $a, b$ and $ab$, where $a$ and $b$ are generators for the two cyclic factors. In addition set $G_n \leq H_n$ to be $(C_{2^n} \times C_{2^n}) \times C_3$, where the action of $C_3$ is given by cyclically permuting $a, b$ and $ab$. For $1 \leq i \leq 4$ set $\chi_i \in \text{Irr}(G_n)$ to be the character of $A_4$ with the same label inflated to $G_n$ and $\text{IBr}(G_n) = \{\phi_1, \phi_2, \phi_3\}$ such that $\chi_i$ reduces to $\phi_i$ for $1 \leq i \leq 3$.

**Proposition 3.8.** Let $P$ be a finite abelian 2-group and $n \in \mathbb{N}$. Suppose $I$ is a permutation of $\text{Irr}(P \times G_n)$ induced by a Morita self-equivalence of $\mathcal{O}(P \times G_n)$. Then there exists $\sigma \in S_3$ and $J \in \text{Perf}(\mathcal{O}P)$, with all signs positive, such that $I$ satisfies $I(\theta \otimes \chi_i) = J(\theta \otimes \chi_{\sigma(i)})$ for all $\theta \in \text{Irr}(P)$ and $1 \leq i \leq 3$.

**Proof.** For $1 \leq i \leq 3$ set $X_i := \{\theta \otimes \chi_i | \theta \in \text{Irr}(P)\}$. $I$ must permute the $X_i$’s as each $X_i$ is exactly the subset of $\text{Irr}(P \times G_n)$ of characters that reduce to $\phi_i$. By composing with a Morita equivalence induced by tensoring with a linear character of $G_n$ and/or conjugation by some element of $H_n$, we may assume that $I$ leaves each $X_i$ invariant. Now for $1 \leq i \leq 3$, we define $J_i : \text{Irr}(P) \to \text{Irr}(P)$ by

$$I(\theta \otimes \chi_i) = J_i(\theta) \otimes \chi_i.$$  

By Lemma 3.4 $I$ must be a perfect isometry. As

$$\sum_{\theta \in \text{Irr}(P)} \alpha_\theta(\theta \otimes \chi_i) \in \text{CF}(P \times A_4, \mathcal{O}(P \times A_4), \mathcal{O})$$

if and only if

$$\sum_{\theta \in \text{Irr}(P)} \alpha_\theta \in \text{CF}(P, \mathcal{O}P, \mathcal{O}),$$

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each $J_i$ must be a perfect isometry with all signs positive. Now for each $\theta \in \text{Irr}(P)$

$$\frac{1}{12} \theta \otimes (\chi_1 + \chi_2 + \chi_3 + 3\chi_4) \in \text{CF}(P \times A_4, \mathcal{O}(P \times A_4), \mathcal{O}).$$

Proceeding exactly as in the proof of Theorem 3.7 proves that $J_1 = J_2 = J_3$ and hence the result is proved.

We recall the decomposition matrix of $B_0(\mathcal{O}A_5)$, where we also set up some labelling of characters and Brauer characters.

|   | $\mu_1$ | $\mu_2$ | $\mu_3$ |
|---|---------|---------|---------|
| $\psi_1$ | 1       | 0       | 0       |
| $\psi_2$ | 1       | 1       | 0       |
| $\psi_3$ | 1       | 0       | 1       |
| $\psi_4$ | 1       | 1       | 1       |

**Proposition 3.9.** Let $n \in \mathbb{N}$. Suppose $I$ is a permutation of $\text{Irr}(B_0(\mathcal{O}(G_n \times A_5)))$ induced by a Morita self-equivalence of $B_0(\mathcal{O}(G_n \times A_5))$. Then the set $\{\chi_i \otimes \psi_j | 1 \leq i \leq 3, 1 \leq j \leq 4\}$ is left invariant by $I$.

**Proof.** We first note that $I$ leaves $\{\chi_i \otimes \psi_j | 1 \leq i \leq 3, 1 \leq j \leq 3\}$ invariant as this is exactly the subset of $\text{Irr}(B_0(\mathcal{O}(G_n \times A_5)))$ consisting of characters that reduce to 1 or 2 Brauer characters. Similarly

$$\{\chi_i \otimes \psi_4 | 1 \leq i \leq 3\} \cup \{\alpha \otimes \psi_1 | \alpha \in \text{Irr}(G_n) \backslash \{\chi_1, \chi_2, \chi_3\}\}$$

is exactly the subset of $\text{Irr}(B_0(\mathcal{O}(G_n \times A_5)))$ consisting of characters that reduce to a sum of 3 distinct irreducible Brauer characters and so is also left invariant by $I$. Now for $1 \leq i \leq 3$

$$\sum_{\phi \in \text{IBr}(B_0(\mathcal{O}(G_n \times A_5)))} d_{\chi_i \otimes \psi_4, \phi} d_{\chi_i, \phi} = \sum_{j=1}^{3} d_{\chi_i \otimes \psi_i, \mu_j} = \frac{2^{2n} + 2}{3} \frac{8}{3}$$

and for $\alpha \in \text{Irr}(G_n) \backslash \{\chi_1, \chi_2, \chi_3\}$

$$\sum_{\phi \in \text{IBr}(B_0(\mathcal{O}(G_n \times A_5)))} d_{\alpha \otimes \psi_1, \phi} d_{\chi_i, \phi} = \sum_{j=1}^{3} d_{\chi_i \otimes \psi_i, \mu_j} = \frac{3^{2n} + 2}{3} \frac{4}{3},$$

where $(d_{\chi, \phi})$ is the decomposition matrix of $B_0(\mathcal{O}(G_n \times A_5))$. Therefore $\{\chi_i \otimes \psi_4 | 1 \leq i \leq 3\}$ is left invariant by $I$ and the claim is proved. \[\square\]
4 Picard groups of blocks with normal defect group and related blocks

An important tool in determining many of the Picard groups in this section will be Weiss’ criterion, originally stated over the ring of \( p \)-adic integers. See [1, Remark 1.8] for a discussion of the generalisation of the ground ring. For all of this section up until Theorem 4.6 the prime \( p \) may be taken to be arbitrary.

For \( G \) a finite group, \( H \) a subgroup and \( M \) an \( O_G \)-bimodule, we define

\[
H^M = \{ m \in M | g.m = m \text{ for all } g \in H \},
\]

\[
M^H = \{ m \in M | m.g = m \text{ for all } g \in H \}.
\]

In particular, if \( M \) is a left \( O_G \)-module we adopt the above notation by viewing \( M \) as an \( O_G \)-\( O \{1\} \)-bimodule.

**Proposition 4.1** (Weiss [22]). Let \( P \) be a finite \( p \)-group, \( M \) a finitely generated \( OP \)-module and \( Q < P \) such that Res\(_P^Q\)(\( M \)) is free and \( Q^M \) is a permutation \( O(P/Q)\)-module. Then \( M \) is a permutation \( OP \)-module.

Before we apply Weiss’ criterion we need a proposition. Let \( G \) be a finite group, \( P \) a normal \( p \)-subgroup and \( B \) a block of \( O_G \). We denote by \( B_P \) the sum of blocks of \( O(G/P) \) dominated by \( B \), that is those blocks not annihilated by the image of \( e_B \) under the natural \( O\)-algebra homomorphism \( p_P : OG \to O(G/P) \), where \( e_B \in OG \) is the block idempotent corresponding to \( B \). With this notation we have

**Proposition 4.2.**

(a) \( \text{Irr}(B^P) \) is precisely the set of irreducible characters in \( \text{Irr}(G/P) \) that inflate to characters in \( B \).

(b) Suppose \( M \) is a \( B-B \)-bimodule inducing a Morita self-equivalence of \( B \) that permutes the elements of

\[
\text{Irr}(B)^P := \{ \chi \in \text{Irr}(B) | P \text{ is contained in the kernel of } \chi \}.
\]

Then \( P^M = M^P \) induces a Morita self-equivalence of \( B^P \).

**Proof.**

(a) First note that for each \( \chi \in \text{Irr}(G) \), \( \chi \downarrow_P \) is a sum of trivial or a sum of non-trivial irreducible characters of \( P \). Now we can identify \( O(G/P) \) with \( O(G(\sum_{g \in P} g)) \) and view \( p_P \) as multiplication by \( \sum_{g \in P} g \). The claim now follows by applying \( p_P \otimes K \) to \( KGe_{\chi} \) for each \( \chi \in \text{Irr}(B) \), where \( e_{\chi} \in KB \) is the character idempotent corresponding to \( \chi \).

(b) \( M \) is projective as a left \( OG \)-module, so \( M \downarrow_{S \times \{1\}} \) is a free \( OS \)-module, where \( S \in \text{Syl}_p(G) \). So \( P^M \) is an \( O \)-summand of \( M \) and \( P^M \) is a free left
$\mathcal{O}(S/P)$-module. Hence $^P M$ is projective as a left $\mathcal{O}(G/P)$-module. We have analogous statements for $M^P$ as a right $\mathcal{O}(G/P)$-module. Now

$$K M \cong \bigoplus_{\chi \in \text{Irr}(B)} V_{\chi} \otimes_K V_{f(\chi)}^*,$$

where $V_{\chi}$ is the simple $KB$-module corresponding to $\chi$ and $f$ is the permutation of $\text{Irr}(B)$ induced by $M$. So

$$K(^P M) = K(M^P) = ^P (K M) \cong \bigoplus_{\chi \in \text{Irr}(B)^P} V_{\chi} \otimes_K V_{f(\chi)}^*,$$

where the first equality follows from the fact that $f$ permutes $\text{Irr}(B)^P$ and the second equality from that $^P M$ is an $\mathcal{O}$-summand of $M$. Again, since $^P M$ and $M^P$ are $\mathcal{O}$-summands of $M$, we have that $^P M = M^P$. So $M^P$ is an $\mathcal{O}(G/P)\otimes \mathcal{O}(G/P)$-bimodule, projective on the left and right that induces a permutation of characters of $\text{Irr}(B)^P$ which we can identify with $\text{Irr}(B^P)$ by part (a). The claim now follows from \cite[Théorème 1.2]{1}.

\[\square\]

The following corollary is a consequence of Weiss’ criterion and Proposition 4.2 and will be the main tool used in proving Theorem 4.6.

**Corollary 4.3.** Suppose further to the hypotheses of Proposition 4.2 that $^P M$ has trivial source when considered as an $\mathcal{O}(G/P)\otimes \mathcal{O}(G/P)$-bimodule. Then $M$ has trivial source.

**Proof.** Due to Proposition 4.3 it is enough to show that $^P M$ has trivial source when considered as an $\mathcal{O}((G/P)\times G^o)$-module, where $G^o$ is the opposite group. Equivalently we need only show that $^P M$ is an $\mathcal{O}(S/P\times S^o)$-permutation module for some $S \in \text{Syl}_p(G)$. However, this follows immediately from the fact that we already know $^P M$ is an $\mathcal{O}(S/P \times (S/P)^o)$-permutation module by Proposition 4.2. \[\square\]

This allows us to generalize part of \cite[Proposition 4.3]{1}.

**Corollary 4.4.** Let $G$ be a finite group and $B$ a block of $\mathcal{O}G$ with normal defect group $D \trianglelefteq G$. If

\[\{\chi \in \text{Irr}(B)|D \leq \ker(\chi)\} = \{\chi \in \text{Irr}(B)|\chi \text{ is a lift of some } \phi \in \text{IBr}(B)\},\]

then $\text{Pic}(B) = T(B)$. In particular, if $G/D$ is abelian and $Z(G)\cap D = \{1\}$, then $\text{Pic}(B) = T(B)$.

**Proof.** We first note that $|\text{Irr}(B^D)| = |\text{IBr}(B^D)|$ and so $B^D$ is a sum of defect zero blocks and so we trivially have $\text{Pic}(B^D) = T(B^D)$. Now since

\[\{\chi \in \text{Irr}(B)|\chi \text{ is a lift of some } \phi \in \text{IBr}(B)\},\]
is left invariant by any Morita self-equivalence of $B$, we can apply Corollary 4.3 and the first statement follows. The second statement follows from the fact that when $G/D$ is abelian and $Z(G) \cap D = \{1\}$,

$$\{\chi \in \text{Irr}(B) | D \leq \ker(\chi)\} = \{\chi \in \text{Irr}(B) | \chi(1) = 1\} = \{\chi \in \text{Irr}(B) | \chi \text{ is a lift of some } \phi \in \text{IBr}(B)\}.$$  

Our first consequence is the boundedness of Picard groups for general blocks with normal defect groups.

**Proposition 4.5.** Let $G$ be a finite group and $B$ a block of $\mathcal{O}G$ with normal defect group $D$, then $\text{Piccent}(B) \leq \mathcal{T}(B)$. In particular $|\text{Pic}(B)| \leq |D|^2|\mathcal{T}(B)|$.

**Proof.** As in Corollary 4.4, we have $|\text{Irr}(BD)| = |\text{IBr}(BD)|$ and so $\text{Pic}(BD) = \mathcal{T}(BD)$. Let $M \in \text{Piccent}(B)$, then certainly $M$ satisfies the conditions of Proposition 4.2 and so the first part follows from Corollary 4.3. Now recall that $\text{Piccent}(B)$ is precisely the subgroup of bimodules in $\text{Pic}(B)$ that fix every irreducible character. So by the Brauer-Feit theorem [3], $\text{Piccent}(B)$ is a subgroup of $\text{Pic}(B)$ of index at most $|D|^2!$ and the second part follows.

Applying these methods, we have for example:

**Theorem 4.6.** Let $p = 2$, $P$ be a finite abelian 2-group and $n, n_1, n_2 \in \mathbb{N}$.

(a) $\text{Pic}(\mathcal{O}(P \times G_n)) = \mathcal{L}(\mathcal{O}(P \times G_n)) \cong (P \rtimes \text{Aut}(P)) \times S_3$.

(b) $\text{Pic}(B_0(\mathcal{O}(P \times A_5))) = \mathcal{L}(B_0(\mathcal{O}(P \times A_5))) \cong (P \rtimes \text{Aut}(P)) \times C_2$.

(c) $\text{Pic}(\mathcal{O}(G_{n_1} \times G_{n_2})) = \mathcal{T}(\mathcal{O}(G_{n_1} \times G_{n_2}))$

(i) $\cong \mathcal{T}(\mathcal{O}G_{n_1}) \rtimes C_2 \cong S_3 \rtimes C_2$ if $n_1 = n_2$.

(ii) $\cong \mathcal{T}(\mathcal{O}G_{n_1}) \rtimes \mathcal{T}(\mathcal{O}G_{n_2}) \cong S_3 \times S_3$, if $n_1 \neq n_2$.

(d) $\text{Pic}(B_0(\mathcal{O}(G_n \times A_5))) = \mathcal{T}(B_0(\mathcal{O}(G_n \times A_5))) \cong \mathcal{T}(\mathcal{O}G_n) \times \mathcal{T}(B_0(\mathcal{O}A_5)) \cong S_3 \times C_2$.

(e) $\text{Pic}(\mathcal{O}(C_2^3 \times (C_7 \rtimes C_3))) = \mathcal{T}(\mathcal{O}(C_2^3 \times (C_7 \rtimes C_3))) \cong C_3$.

For each block $B$ above Piccent$(B) = 1$.

**Proof.** Write $D$ for a defect group of the block $B$ of the group $G$ under consideration and $A$ for a source algebra. Let $E$ be the inertial quotient of the block and $\mathcal{F}$ for the fusion system associated to $B$. In the arguments that follow we will make free use of the results of [11] as presented in Section 2. We also make repeated use of Lemma 3.4 without further reference to it.
(a) Let $M \in \text{Pic}(O(P \times G_n))$. Recall that we are writing $\chi_1, \chi_2, \chi_3$ for the linear characters of $G_n$. By Proposition 3.8 and Lemma 3.6 part (2) we may compose $M$ with some element of $\text{Aut}_O(OP)$ such that the induced permutation $I$ of $\text{Irr}(P \times G_n)$ satisfies $I(\theta \otimes \chi_i) = \theta \otimes \chi_{\sigma(i)}$, where $\sigma \in S_3$, for all $\theta \in \text{Irr}(P)$ and $1 \leq i \leq 3$. Now assume, as we may, that $M$ satisfies the conditions of Proposition 4.2 with respect to $S \in \text{Syl}_p(G_n)$. Then $^SM$ induces a Morita self-equivalence of $O(P \times C_3)$, the sum of three blocks each isomorphic to $OP$. Moreover, by construction, this Morita equivalence is given by identifying each of these three blocks with $OP$ and permuting them. Any such Morita equivalence certainly has trivial source and so by Corollary 4.3 so does $M$. Noting that $\mathcal{T}(B) \leq \mathcal{L}(B)$ and that $\text{Aut}_O(OP)$ injects into $\text{Pic}(O(P \times G_n))$ we have shown that $\text{Pic}(O(P \times G_n)) = \mathcal{L}(B) \cong \text{Aut}_O(OP) \mathcal{T}(B)$. Now $\text{Out}_D(A) \cong C_3$ and $\text{Out}(D, F) \cong \text{Aut}(P) \times C_2$. By considering $G \leq \text{Aut}(G)$ the group $\text{Out}(D, F)$ is realised as a quotient of $\mathcal{T}(B)$ the result follows.

(b) Let $M \in \text{Pic}(B_0(O(P \times A_5)))$. We first note that by [5, A1.3], $O_{A_2}$ and $B_0(OA_5)$ are perfectly isomorphic. We can then apply Theorem 6.7 and compose $M$ with an appropriate element of $\text{Aut}_O(OP)$ to produce an element of $\text{Pic}(B)$ permuting the irreducible characters of $B$ with $P$ in their kernel. Suppose now that $M$ is of this type. By Proposition 4.2 $^SM$ induces a Morita self-equivalence of $B^P = B_0(OA_5)$. By ([11, Theorem 1.5] $\text{Pic}(B_0(OA_5)) = \mathcal{T}(B_0(OA_5)) \cong C_2$, where the non-trivial element interchanges the two irreducible characters of $A_5$ of degree three. By Corollary 4.3 $M$ must have trivial source. By Lemma 2.2 $\text{Out}_D(A) = 1$. It follows that $\mathcal{T}(B) \leq \text{Out}(D, F)$, and hence $\mathcal{T}(B) \cong \text{Aut}(P) \times C_2$. The result follows.

(c) That $\text{Pic}(O(G_{n_1} \times G_{n_2})) = \mathcal{T}(O(G_{n_1} \times G_{n_2}))$ follows immediately from the second part of Corollary 4.4. Now $\text{Out}_D(A)$ is isomorphic to a subgroup of $\text{Hom}(E, k^n) \cong C_3 \times C_3$. By Lemma 2.1 we do indeed have $C_3 \times C_3$. The maximal subgroup of $\text{Aut}(D)$ containing $C_3 \times C_3$ is $S_3 \times C_2$ if $n_1 = n_2$ and $S_3 \times S_3$ otherwise. Hence $\text{Out}(D, F) \cong D_8$ or $C_2 \times C_2$ according to whether $n_1 = n_2$ or $n_1 \neq n_2$. By considering $G \leq \text{Aut}(G)$ the result is then clear.

(d) Let $M \in \text{Pic}(B)$ and let $S \in \text{Syl}_p(G_n)$. By Proposition 4.2 and Proposition 4.3 $^SM$ induces a Morita self-equivalence of $B^S \cong OC_3 \otimes B_0(OA_5)$. Similarly to part (a), $B^S$ is the sum of three blocks each isomorphic to $B_0(OA_5)$ and $^SM$ permutes these three blocks, so that $^SM \in \text{Pic}(B_0(OA_5)) \cdot S_3$. By [11, Theorem 1.5] $\text{Pic}(B_0(OA_5)) = \mathcal{T}(B_0(OA_5)) \cong C_2$, so it follows that $^SM$ has trivial source. Hence by Corollary 4.3 $M \in \mathcal{T}(B)$. Now by Lemma 2.2 $\text{Out}_D(A) \cong \text{Out}_D(A_1) \times \text{Out}_D(A_2) \cong C_3$, where $A_1, A_2$ is a source algebra of $kG_n$, $B_0(kA_5)$ respectively, $D_1 = O_{A_2}(G_n)$ and $D_2$ is a Sylow 2-subgroup of $A_5$ so that $D = D_1 \times D_2$. It is obtained by tensoring with either of the non-trivial modules of dimension one. We have $\text{Out}(D, F) \cong C_2 \times C_2$. By considering $G_n \times A_5 < H_n \times S_5$ we have $S_3 \times C_2 \leq \mathcal{T}(B)$. Note that
Proposition 5.1. Let $G$ be a finite group and $B$ a block of $OG$ with defect group $D$ of 2-rank at most 3. Then $B$ is Morita equivalent to the principal block of one of: (i) $OD$; (ii) $O(D \rtimes C_3)$; (iii) $O(C_{2^n} \rtimes A_5)$ for $n \geq 0$; (iv) $O(D \rtimes C_7)$; (v) $OSL_2(8)$; (vi) $O(D \rtimes (C_7 \rtimes C_3))$; (vii) $OJ_1$; (viii) $O\text{Aut}(SL_2(8))$.

In order to prove Corollary 1.2, following Lemma 3.5 and Theorem 4.6 it remains to calculate the Picard groups for $SL_2(8)$, $\text{Aut}(SL_2(8))$ and $J_1$.

5 Blocks with abelian defect groups of 2-rank at most three

In [10], which uses also results of [23], the Morita equivalence classes of 2-blocks with abelian defect groups of 2-rank at most three were classified. In this section we complete the computation of the Picard groups of a representative from each Morita equivalence class, so that the isomorphism type of the Picard group is known for every such block.

Proposition 5.1 ([10, 23]). Let $G$ be a finite group and $B$ a block of $OG$ with defect group $D$ of 2-rank at most 3. Then $B$ is Morita equivalent to the principal block of one of: (i) $OD$; (ii) $O(D \rtimes C_3)$; (iii) $O(C_{2^n} \rtimes A_5)$ for $n \geq 0$; (iv) $O(D \rtimes C_7)$; (v) $OSL_2(8)$; (vi) $O(D \rtimes (C_7 \rtimes C_3))$; (vii) $OJ_1$; (viii) $O\text{Aut}(SL_2(8))$.

Proof. This follows from [11 4.3]. That Piccent$(OG) = 1$ follows since element of Pic$(OG)$ induces a nontrivial permutation of Irr$(G)$. □

The calculation of the Picard group of the principal 2-block of $SL_2(2^n)$ is a generalisation of the arguments in [11] for $A_5 \cong SL_2(4)$.

Proposition 5.3. Let $B = B_0(OSL_2(2^n))$, where $n > 1$. Then Pic$(B) = T(B) \cong C_n$ and Piccent$(B) = 1$.

Proof. Let $G = SL_2(2^n)$, where $n > 1$, and let $B$ be the principal block of $OG$. Write $q = 2^n$. Let $D \in \text{Syl}_2(G)$, so $D$ is elementary abelian of order $q$ and $H := NG(D) = D \rtimes E$ where $E \cong C_{q-1}$. Let $b = OH$. By Lemma 5.2 Pic$(b) = T(b) = C_{q-1} \rtimes C_n$, where the elements of $C_{q-1}$ permute the linear characters of $H$ and $C_n$ acts as group automorphisms obtained from $\text{Aut}(G)$ (note that $N_{\text{Aut}(G)}(D) \cong H \rtimes C_n$). Now $D$ is a trivial intersection subgroup of
$G$, so induction and restriction gives a splendid stable equivalence of Morita type $F: \text{mod } (b) \to \text{mod } (B)$. Let $M \in \text{Pic}(B)$. By [14] the trivial module for $B \otimes O k$ is distinguished as the unique simple module $V$ with $\text{Ext}^1_{kG}(V,W) \neq 0$ for three distinct simple $kG$-modules $W$. It follows that $M_H := F^* \circ M \circ F: \text{mod } (b) \to \text{mod } (B)$ gives a stable self-equivalence of Morita type preserving the trivial $b$-module. By [6, Corollary 3.3] $M_H$ must then preserve every simple $b$-module, and so it is a Morita equivalence. The subgroup of $\text{Pic}(b)$ consisting of bimodules fixing the trivial module is the subgroup of bimodules induced by group automorphisms of $G$ as described above, and is isomorphic to $C_n$. Since $M_H \in T(b)$ and we have a splendid stable equivalence between $b$ and $B$, it follows that $M \in T(B)$. By the description of $T(B)$, it follows that the Morita equivalence given by $M$ is also induced by an automorphism of $G$. Since $\text{Out}(G) \cong C_n$, the result follows, noting that every such automorphism acts nontrivially on $\text{Irr}(B)$ so $\text{Piccent}(B) = 1$. 

\section*{5.2 The principal 2-block of $\text{Aut}(SL_2(8))$}

The calculation of $\text{Pic}(B_0(O \text{Aut}(SL_2(8))))$ is complicated by the fact that we may no longer use [6, Corollary 3.3], which requires the inertial quotient to be cyclic. Instead we must show directly that any Morita equivalence of $B_0(O \text{Aut}(SL_2(8)))$ restricted to the principal block $b$ of the normalizer of a Sylow 2-subgroup gives rise to a self-equivalence of $b$ permuting the simple modules.

**Proposition 5.4.** Let $B = B_0(O \text{Aut}(SL_2(8)))$ or $B_0(O^2G_2(3^{2m+1}))$ for $m \geq 1$. Then $\text{Pic}(B) = T(B) \cong C_3$ and $\text{Piccent}(B) = 1$.

**Proof.** Let $G = \text{Aut}(SL_2(8))$ and $N \leq G$ with $N \cong SL_2(8)$. Let $D \in \text{Syl}_2(G)$ and write $H = N_G(D)$. We have $H \cong D \ltimes (C_7 \ltimes C_3) = (D \ltimes C_7) \ltimes C_3$. Write $B$ for the principal block of $OG$ and $b$ for the principal block of $OH$.

Let $M \in \text{Pic}(B)$. Since $D$ is a trivial intersection subgroup of $G$, induction and restriction gives a splendid stable equivalence of Morita type $F: \text{mod } (b) \to \text{mod } (B)$. Hence $M_H := F^* \circ M \circ F: \text{mod } (b) \to \text{mod } (b)$ induces a stable self-equivalence of Morita type of $b$. We show that this stable equivalence sends simple modules to simple modules.

Using GAP, with output given in [11], $B$ has irreducible characters of degrees $1, 1, 1, 1, 7, 7, 7, 21, 27$, irreducible Brauer characters of degrees $1, 1, 1, 6, 12$ (which we write $I, 1, 1^*, 6, 12$) and decomposition matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1
\end{pmatrix}
$$

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From examination of the decomposition matrix, $M \otimes k$ permutes the modules of dimension one, fixing the other simple modules.

$b$ has simple modules of dimensions 1, 1, 1, 3, 3, labelled 1, 1, 2, 3, 3. We may choose our labelling so that $I$, 1, 1* has Green correspondent 1, 1, 2, 3, 3 respectively. By [15] $\text{Res}_H^G(6) = \begin{pmatrix} 3_1 \\ 3_2 \end{pmatrix}$ and

$$\text{Res}_H^G(12) = \begin{pmatrix} 3_2 \\ 3_1 \\ 3_1 \end{pmatrix}.$$  

Writing $PS$ for the projective cover of the simple module $S$, examination of the structure of the projective indecomposable modules given in [15] yields that $\text{Ind}_G^H(3_1) \cong P_{12}/\text{rad}^4(P_{12}) = \begin{pmatrix} 12 \\ 6 \\ 6 \\ 6 \\ 6 \\ 6 \\ 12 \end{pmatrix}$ and $\text{Ind}_G^H(3_2) \cong \text{rad}^3(P_{12}) = \begin{pmatrix} 1 \\ 1 \\ 1* \\ 1 \\ 1* \end{pmatrix}$.

We deduce that $M$ fixes the Green correspondents of 3_1 and 3_2, and permutes those of the linear modules, so $M_H$ permutes the simple modules. It follows by [16, Theorem 4.14.10] that $M_H$ induces a Morita self-equivalence of $b$. By Theorem 4.6 $\text{Pic}(b) = T(b)$, so $M_H$ has trivial source as a bimodule. Since $F$ is induced by a trivial source bimodule it follows that $M$ also has trivial source, i.e., $M \in T(B)$.

We now calculate $T(B)$. Let $A$ be a source algebra for $B$. Now $\text{Out}_D(A)$ is isomorphic to a subgroup of $\text{Hom}_k(C_7 \rtimes C_3, k^*) \cong C_3$. It follows from Lemma 2.1 that $\text{Out}_D(A) \cong C_3$. Since $C_7 \rtimes C_3$ is a maximal subgroup of $GL_2(8)$ it follows that $\text{Out}(D, F) = 1$. Hence $T(B) \cong C_3$ as required. Note that each element of $\text{Pic}(B)$ induces a nontrivial permutation of $\text{Irr}(B)$, so Piccent$(B) = 1$.

Now suppose that $B = B_0(O^G2_2(3^{2m+1}))$ for some $m \geq 1$. By [19] Example 3.3 there is a splendid Morita equivalence between $B_0(O^G2_2(3^{2m+1}))$ and $B_0(\text{Aut}(SL_2(8)))$, i.e., one induced by a trivial source bimodule, so that $T(B_0(O^G2_2(3^{2m+1}))) \cong T(B_0(\text{Aut}(SL_2(8))))$ and Pic$(B_0(O^G2_2(3^{2m+1}))) \cong \text{Pic}(B_0(\text{Aut}(SL_2(8))))$.

5.3 Pic$(B_0(OJ_1))$

In [12] derived equivalences, and so stable equivalences, are constructed between $B_0(kJ_1)$, $B_0(k^2G_2(3))$ and $B_0(k(C_2 \times C_2 \times C_2) \times (C_7 \rtimes C_3))$. In [21, §6.4] it is shown that there is a splendid stable equivalence of Morita type between any block and the Brauer correspondent block of the normalizer of a defect.
group when the defect group is $(G_2)^3$. In [7] it is shown that the stable equivalences constructed in [12] are of the type constructed in [21], and so they lift to stable equivalences with respect to $O$. The images of the simple modules under the stable equivalence between $B_0(kJ_1)$ and $B_0(k^2G_2(3))$ are given in [12], which we make use of in our calculation of $\text{Pic}(B_0(OJ_1))$. Note that $2^2G_2(3) \cong \text{Aut}(SL_2(8))$.

Florian Eisele has already shown that $\text{Pic}(B_0(OJ_1)) = 1$ using different methods in (at present) unpublished notes.

**Proposition 5.5.** $\text{Pic}(B_0(OJ_1)) = 1$.

**Proof.** Let $G = J_1$ and $H = 2^2G_2(3)$. Recall that the simple $B_0(kH)$-modules are labelled by $I, 1, 1^*, 6, 12$. The simple $B_0(kG)$-modules are labelled $I, 20, 56_1, 56_2$ and 76. Write $P_S$ for the projective cover of a simple $kG$-module $S$.

Write $F$ for the stable equivalence from between $B_0(kH)$ and $B_0(kG)$, called $gh$ in [12] where it is shown that images of the simple modules under $F$ have the following properties:

$$
F(I) \cong I, \quad F(20) \cong 12,
F(56_1)/\text{rad}(F(56_1)) \cong 1^*, \quad \text{soc}(F(56_1)) \cong 1 \oplus 1^*,
F(56_2)/\text{rad}(F(56_2)) \cong 1, \quad \text{soc}(F(56_2)) \cong 1^* \oplus 1,
F(76)/\text{rad}(F(76)) \cong \text{soc}(J(76)) \cong 6.
$$

By adjointness we get that $F^*(I) = I, F^*(12) = 20, F^*(1)/\text{rad}(F^*(1)) \cong 56_1 \oplus 56_2, \text{soc}(F^*(1)) \cong 56_1, F^*(1^*)/\text{rad}(F^*(1^*)) \cong 56_2 \oplus 56_1, \text{soc}(F^*(1^*)) \cong 56_1$ and $F^*(6)/\text{rad}(F^*(6)) \cong \text{soc}(F^*(6)) \cong 76$.

We use the computer algebra package MAGMA [2] to find $B_0(kG)$-modules which could occur as images of simple modules under $F^*$. We do so by calculating the submodules of projective indecomposable modules with the given head and socle.

Suppose that $U$ is a $kG$-module satisfying $U/\text{rad}(U) \cong 56_1 \oplus 56_2$ and $\text{soc}(U) \cong 56_1$. Then $U$ is isomorphic to the unique submodule of $\text{rad}^2(P_{56_1})$ or $\text{rad}^4(P_{56_1})$ whose quotient is the simple module 76.

Suppose that $U$ is a $kG$-module satisfying $U/\text{rad}(U) \cong 56_1 \oplus 56_2$ and $\text{soc}(U) \cong 56_2$. Then $U$ is isomorphic to the unique submodule of $\text{rad}^2(P_{56_2})$ or $\text{rad}^4(P_{56_2})$ whose quotient is the simple module 76.

Suppose that $U$ is a $kG$-module satisfying $U/\text{rad}(U) \cong \text{soc}(U) \cong 76$. Then $U$ is isomorphic to either 76, $P_{76}$ or the unique submodule of $P_{76}$ with structure

$$
\begin{array}{c}
76 \\
56_1 \\
I \\
76
\end{array}
$$

Let $M \in \text{Pic}(B_0(OG))$. It is clear from the structure of the projective indecomposable modules of $B_0(kJ_1)$ (or the decomposition matrix) that $M \otimes_O k$
must fix the simple modules \( I, 20 \) and 76, and so must permute \( F^*(I), F^*(1), F^*(1^*), F^*(6) \) and \( F^*(12) \).

By the discussion above \( F \) is a splendid stable equivalence of Morita type, hence \( F \) lifts to a stable equivalence with respect to \( \mathcal{O} \). Let \( M_H := F^* \circ M \circ F : \mod (B_0(\mathcal{O}H)) \to \mod (B_0(\mathcal{O}H)) \), inducing a stable self-equivalence of Morita type of \( B_0(\mathcal{O}H) \). We show that this stable equivalence sends simple modules to simple modules.

Since \( M \otimes \mathcal{O} k \) permutes the images of the simple modules under \( F^* \), it follows that \( M_H \otimes \mathcal{O} k \) must permute the simple \( B_0(kH) \)-modules. Hence \( M_H \) permutes the simple \( B_0(\mathcal{O}H) \)-modules and by [16, Theorem 4.14.10] \( M_H \) induces a Morita equivalence.

We have by Proposition 5.4 that \( \text{Pic}(B_0(\mathcal{O}H)) = \mathcal{T}(B_0(\mathcal{O}H)) \cong C_3 \). Note that \( M_H \) either induces the trivial equivalence or permutes cyclically the three simple modules of dimension one (fixing the other two). It is clear that \( M \otimes \mathcal{O} k \) cannot permute \( F^*(I), F^*(1) \) and \( F^*(1^*) \) transitively, so \( M_H \) and hence \( M \) must induce the trivial equivalence. We have shown that \( \text{Pic}(B_0(\mathcal{O}G)) = 1 \).

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