Equivalent Codes, Optimality, and Performance Analysis of OSTBC: Textbook Study

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Abstract

An equivalent model for a multi-input, multi-output (MIMO) communication system with orthogonal space-time block codes (OSTBCs) is proposed based on a newly revealed connection between OSTBCs and Euclidean codes. Examples of distance spectra, signal constellations, and signal coordinate diagrams of Euclidean codes equivalent to simplest OSTBCs are given. A new asymptotic upper bound for the symbol error rate (SER) of OSTBCs, based on the distance spectra of the introduced equivalent Euclidean codes, is derived and new general design criteria for signal constellations of the optimal OSTBC are proposed. Some bounds relating distance properties, dimensionality, and cardinality of OSTBCs with constituent signals of equal energy are given, and new optimal signal constellations with cardinalities \( M = 8 \) and \( M = 16 \) for Alamouti’s code are designed. Using the new model for MIMO communication systems with OSTBCs, a general methodology for performance analysis of OSTBCs is developed. As an example of the application of this methodology, an exact evaluation of the SER of any OSTBC is given. Namely, a new expression for the SER of Alamouti’s OSTBC with binary phase shift keying (BPSK) signals is derived.

Index Terms

Euclidean codes, group codes, MIMO, optimal constellations, OSTBC, SER, signal coordinate diagrams, spherical codes.
I. Introduction

The simplicity of mathematical description, low complexity of maximum likelihood (ML) decoding, and unique properties allowing for noncoherent detection make orthogonal space-time block codes (OSTBCs) [1]–[4] the most attractive and well studied class of space-time codes. Any OSTBC can be described mathematically by its corresponding code matrix and a constituent signal constellation. Although the code matrices of OSTBCs have been tabulated for many important cases [1], [2], [5]–[11] using the theory of complex orthogonal design [2], results on the optimal constituent signal constellations of OSTBCs are extremely limited. In fact, almost all the investigations of OSTBCs are based on a restricted group of constituent signals which belong to a class of the constellations with independent signals. It is, however, unknown and questionable whether such signal constellations are actually optimal. Moreover, no general results for guaranteeing the optimality of OSTBCs are available. It has been stressed, for example, in [12] that general design criteria for optimal OSTBCs are unknown. Even the problem of finding constellations optimal in the sense of minimizing an average error probability of ML decoding on Rayleigh flat fading channels is an open problem of great interest for multi-input, multi-output (MIMO) communication systems. The latter problem has been investigated in [12] for the smallest possible constellations (up to \( M = 5 \), where \( M \) is the cardinality of an OSTBC signal constellation). Particularly, OSTBCs that are optimal in the sense of minimizing the symbol error rate (SER) of ML decoding have been designed only for constellations with \( M = 2 \sim 5 \). In these cases, the SER minimization is equivalent to the minimization of the average error probability of ML decoding. Despite the aforementioned limitation of the results in [12], this work is, to the best of the authors’ knowledge, the only source of information available on the design of optimal OSTBCs.

Similarly, although the distance properties of OSTBCs have been investigated in some previous research works [13]–[15], the distance properties of OSTBC signal constellations have not

\(^1\)Hereafter, the (OSTBC) signal constellation refers to the set of all realizable samples of the OSTBC matrix, each transmitted in a number of consecutive time slots, and the constituent signal constellation refers to the set of signals that constitutes the components of the OSTBC matrix. The cardinality of the former constellation is denoted \( M \), and of the latter, is denoted \( L \), as defined in Section II.
attracted any attention. Indeed, existing results on the distance properties of OSTBCs aim at verifying the resilience properties of OSTBCs, where a multidimensional constellation is said to be resilient in flat fading if it retains its shape when its points are subject to the multiplicative distortion associated with fading coefficients [16]. However, it is specifically the full understanding of the distance properties of OSTBC signal constellations that can enable formulating requirements or design criteria for OSTBC signal constellations.

In this paper, the aforementioned distance properties of OSTBCs with arbitrary signal constellations are analyzed. Based on the analysis, a new equivalent model for a communication system with orthogonal space-time block coding is proposed. The model is based on a connection found between the distance properties of OSTBCs and the distance properties of Euclidean codes, which allows viewing certain Euclidean codes as equivalent codes to OSTBCs. This connection brings important insights into OSTBCs since Euclidean codes fall under the classic theory of error correcting codes [13] and, thus, the OSTBCs can now be viewed as a part of the classic theory. Particularly, the connection between OSTBCs and Euclidean codes enables one to formulate a new general criterion for designing optimal OSTBCs with arbitrary constituent signals for the case of large signal-to-noise ratio (SNR). Indeed, OSTBCs can be viewed as a subclass of error correcting codes having a specific design criterion that enables searching for new existence conditions for optimal OSTBC signal constellations with constant envelope constituent signals. Such conditions are based on a connection between the optimal OSTBC signal constellations with equal energies and a class of spherical codes [18]. For example, we derive two new optimal biorthogonal signal constellations with cardinalities $M = 8$ and $M = 16$ for the Alamouti OSTBC with constant energy signals.

The model introduced for the OSTBC MIMO system can be used for performance analysis of OSTBCs and enables one to develop a new performance analysis methodology. Existing results on OSTBC performance analysis (see [19]–[26] and the references therein) aim at deriving exact solutions only for the SER of the constituent signals of the OSTBC, and there are no results on the exact solution for the true SER of any OSTBC. As an example of applying our methodology,

\footnote{Some initial results have been reported in [17].}
we derive a closed-form solution for the SER of the Alamouti OSTBC with constituent binary phase shift keying (BPSK) signals. To the best of the authors’ knowledge, this is a new expression for the SER of the Alamouti OSTBC with constituent BPSK signals. Moreover, this result is the only exact expression available for the SER of any OSTBC.

The remainder of this paper is organized as follows. In Section II, the distance properties of OSTBC signal constellations are analyzed and a new equivalent model for a MIMO communication system with orthogonal space-time block coding on a quasistatic fading channel is given. Some examples of signal coordinate diagrams and distance spectra of some simplest OSTBCs are also given. A new union bound on the SER of an OSTBC based on the distance properties of equivalent Euclidean codes is derived in Section III. Using this bound, a new general design criterion for optimal OSTBC constellations and a new design criterion for optimal constant envelope OSTBC signals as well as some existence conditions are formulated. As an example of applying this new design criterion, two new optimal biorthogonal constellations for the Alamouti OSTBC with \( M = 8 \) and \( M = 16 \) are designed. In Section IV, a new general OSTBC performance analysis methodology based on the equivalent model for MIMO systems introduced in Section II is described. A new closed-form solution for the SER of the Alamouti OSTBC with constituent BPSK signals is also derived. Section V presents some numerical examples and is followed by some conclusions in Section VI.

II. NEW SYSTEM MODEL AND EQUIVALENT CODES

In this section, a new model for a communication system with an OSTBC having an arbitrary signal constellation is introduced. Toward this end, a modified description of an OSTBC with an arbitrary signal constellation, its distance properties and its connections to the class of Euclidean codes are of interest. Based on the new model, examples of signal constellations and signal coordinate diagrams of Euclidean codes equivalent to the simplest OSTBC are developed.

A. OSTBCs With Arbitrary Constituent Signals

An OSTBC can be defined by a generalized complex orthogonal design \([2]\), i.e., by an \( N_T \times N_T \) code matrix \( G_u \) with orthogonal columns. The entries \( g_{i,j}^u \) \( (i, j = 1, \ldots, N_T) \) of the code
matrix $G_u$ are the elements $s_{t,u} (t = 1, \ldots, N_t; u = 0, \ldots, M - 1)$ of the codewords (signal constellations)

$$s_u = [s_{1,u}, s_{2,u}, \ldots, s_{NT,u}]^T, \quad u = 0, \ldots, M - 1$$

as well as the complex conjugates $s_{t,u}^* (t = 1, \ldots, N_T; u = 0, \ldots, M - 1)$, linear combinations of $s_{t,u} (t = 1, \ldots, N_T; u = 0, \ldots, M - 1)$ and $s_{t,u}^* (t = 1, \ldots, N_T; u = 0, \ldots, M - 1)$, or zeros. Here $[\cdot]^T$ is the matrix transpose and $M$ is the cardinality of the OSTBC signal constellation.

The codewords (1) belong to a block code with $J$ constituent 1- or 2-dimensional (1- or 2-D) signals $s_{t,u} (t = 1, \ldots, J; u = 0, \ldots, M - 1)$ and $N_T - J$ zero signals $s_{t,u} = 0 (t = J + 1, \ldots, N_T; u = 0, \ldots, M - 1)$ with $J \leq N_T$ denoting the number of information bearing constituent signals of the OSTBC. Since the multidimensional signal constellations (1) belong to a block code constructed of modulated symbols from its alphabet, such a code corresponds to the so-called Euclidean code known from the classic theory of error correcting coding. This connection helps to define a complex structure of $M$-ary constellations belonging to OSTBCs.

**Definition 1 [27]:** The Euclidean code is a finite set of $M$ points (codewords) in $n$-D Euclidean space $R^n$.

The constituent signals $s_{t,u} (t = 1, \ldots, N_T; u = 0, \ldots, M - 1)$ of a canonical OSTBC [1], [2] use the same, typically $L$-PSK or $L$-QAM, modulation with $L$ being the cardinality of the constellation. Note, however, that in the general case, the constituent symbols of the Euclidean code $s_{t,u} (t = 1, \ldots, N_T; u = 0, \ldots, M - 1)$ can have arbitrary signal constellations including correlated constellations.

Assuming a flat fading channel, the signal received by the $j$th receiving antenna ($j = 1, \ldots, N_R$) of the OSTBC MIMO system can be expressed as

$$r_j = G_u h_j + n_j, \quad j = 1, \ldots, N_R$$

where $h_j = [h_{1,j}, \ldots, h_{NT,j}]^T$ is the $N_T \times 1$ vector of fading channel coefficients, which are assumed to be independent, identically distributed zero-mean complex Gaussian variables with variance $\rho/2$ per dimension and are assumed constant over $N_T$ (or some multiple of $N_T$) time periods; $n_j = [n_{1,j}, \ldots, n_{NT,j}]^T$ is an $N_R \times 1$ vector of complex Gaussian additive noises.
consisting of independent samples of zero-mean complex Gaussian random variables each of variance $N_0/2$ per dimension; and $r_j = [r_{1,j}, \ldots, r_{N_T,j}]^T$ is the $N_T \times 1$ received signal vector at the $j$th receiving antenna. It can be observed from (2) that if the OSTBC codewords occur with equal probability, the average received SNR per antenna is given by

$$\text{SNR}_R \triangleq \frac{\rho}{M N_T N_0} \sum_{u=0}^{M-1} \| G_u \|^2_F$$

(3)

where $\| \cdot \|_F$ is the Frobenius norm of a matrix [28].

B. Distance Properties and Equivalent Model

According to the classic approach of analyzing any type of modulation or coding schemes, the distance properties (signal coordinate diagrams) of the signals under consideration should be first studied. To study the distance properties, we assume the noise-free case. Then the Euclidean distance between received noise-free codeword vectors $G_u h_j$ and $G_t h_j$ ($u \neq t$) of an OSTBC, denoted as $d_{u,t,\text{OSTBC}}^j$, can be expressed in terms of the distance for the equivalent Euclidean code $d_{u,t,\text{EC}} \triangleq \| s_u - s_t \|$ as

$$d_{u,t,\text{OSTBC}}^j = \| h_j \| d_{u,t,\text{EC}}, \quad j = 1, \ldots, N_R; \quad u, t = 0, \ldots, M - 1; \quad u \neq t$$

(4)

where $\| \cdot \|$ denotes the Euclidean norm of a vector.

Indeed, the distances $d_{u,t,\text{OSTBC}}^j$ are defined as

$$d_{u,t,\text{OSTBC}}^j \triangleq \| G_u h_j - G_t h_j \| = \| (G_u - G_t) h_j \|, \quad j = 1, \ldots, N_R; \quad u, t = 0, \ldots, M-1; \quad u \neq t.$$  

(5)

Using the OSTBC orthogonality property, and the property that [14, p. 120]

$$(G_u - G_t)^H (G_u - G_t) = \| s_u - s_t \|^2 I_{N_T}$$

(6)

where $(\cdot)^H$ denotes the Hermitian transpose, the distances (5) can be written as in (4) by also noting that $\| s_u - s_t \| \triangleq d_{u,t,\text{EC}}$. Here $I_{N_T}$ is the $N_T \times N_T$ identity matrix. Moreover, using the OSTBC orthogonality property, the norms of the received noise-free codeword vectors $G_u h_j$
\[(u = 0, \ldots, M - 1; \ j = 1, \ldots, N_R)\text{ can be computed as}

\[
\|G_u h_j\| = \sqrt{h_j^H G_u^H G_u h_j} = \|s_u\| \cdot \|h_j\|, \quad u = 0, \ldots, M - 1, \ j = 1, \ldots, N_R. \tag{7}
\]

On the other hand, let \(\Phi_j\) be an \(N_T \times N_T\) arbitrary unitary matrix, i.e. \(\Phi_j^H \Phi_j = \Phi_j \Phi_j^H = I_{N_T}\). Now consider the new constellation \(\|h_j\| \Phi_j s_u, u = 0, \ldots, M - 1\). It can be observed that this constellation has exactly the same Euclidean distance properties (4) and (7) of the fading-inflicted constellation \(G_u h_j, u = 0, \ldots, M - 1\).

The following theorem can now be formulated based on the distance properties cited above.

**Theorem 1:** A communication system with \(N_T\) transmitting and \(N_R\) receiving antennas, OSTBC, and maximum likelihood decoding of received signals is equivalent to the system given in Fig. 1 for Gaussian noise channels.

**Proof:** The statement of the theorem directly follows from the properties (4) and (7), and the fact that two codes (signals) with the same Euclidean distance properties provide the same performance with maximum likelihood decoding in the Gaussian noise channel \(\text{[29]}\).

Note that the matrix \(\Phi_j\) is, in fact, a rotation matrix in \(N_T\) dimensions. What can be seen from Theorem 1 and Fig. 1 is that the OSTBC effectively transforms the fading MIMO channel into an equivalent coded single-input, multi-output (SIMO) channel with the corresponding fading coefficients \(c_j \triangleq \|h_j\| (j = 1, \ldots, N_R)\) (and, therefore, the probability of deep fading in the channels of the equivalent SIMO system is lower than that in the actual channels of the MIMO system). This SIMO channel is invariant to phase rotation in the sense that different arbitrary rotation matrices \(\Phi_j (j = 1, \ldots, N_R)\) give rise to the same ML performance.

Note also that the resulting SNR in the equivalent model, Fig. 1, is not always equal to the original average SNR \(\text{[3]}\), if we exclude the zeros, i.e. non-information-bearing components, of \(s_u\) in the equivalent Euclidean code of the equivalent model (see Fig. 1). In fact in this case, recalling the definition of \(J\) in Section 11-A we can show by energy conservation that the average SNR in the equivalent model is \(N_T/J\) times the SNR \(\text{[3]}\).

It is also worth stressing that the system model in Fig. 1 is a special case of a receiver

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Note: The claim of the theorem may not hold in channels with other types of noise, such as Laplacian noise \(\text{[30]}\).
Data Encoder of Euclidean code \((s_u, u = 0, \ldots, M - 1)\)

\[
c_j = \sqrt{\sum_{i=1}^{N_R} |h_{i,j}|^2}
\]

\[
\Phi_j^H \Phi_j = \Phi_j \Phi_j^H = I_{N_T}
\]

Fig. 1. Equivalent model of a communication system with OSTBC and ML decoding.
However, it is also important to note that the proposed coded SIMO model is fundamentally different from the well-known single-input, single-output (SISO) model of [19] because (i) it represents the actual multidimensional structure of $M$-ary OSTBCs; (ii) it is applicable to arbitrary constituent signal constellations of an OSTBC; and (iii) it allows using any existing receiver diversity schemes.

It is worth highlighting as well that as follows from Fig. 1, Euclidean codes are equivalent to OSTBCs in the sense that the parameters of the Euclidean codes are the only available optimization parameters for optimizing the MIMO communication system with OSTBC (see Fig. 1). Therefore, the OSTBC design is equivalent to Euclidean code design from the system point of view.

Most of the known OSTBCs belong to a subclass of canonical OSTBCs, i.e., to the class of codes based on signal constellations with uncorrelated constituent signals. However, this condition is extremely restrictive for designing good signal constellations of OSTBCs, while it is clearly not necessary or particularly appealing from a practical (decoder complexity) viewpoint. As a result, OSTBCs with different or correlated constellations for the constituent signals and their properties are essentially overlooked and have not been studied. Therefore, we aim at correcting this deficiency in the existing literature by providing a detailed analysis and design criteria for such codes in Section III.

Toward this end, we first explicitly connect the terminology used to describe the signal constellations of OSTBC with the terminology commonly used for describing error correction codes. Particularly, we define the Euclidean code equivalent to a given OSTBC as follows.

**Definition 2:** The Euclidean code $\tilde{s}_u = [\tilde{s}_{1,u}, \ldots, \tilde{s}_{K,u}]$ is called equivalent to an OSTBC with “proxy” Euclidean code $s_u = [s_{1,u}, \ldots, s_{N_{T,u}}]$ if the Euclidean distance between two arbitrary codewords $s_u$ and $s_t$ of the OSTBC coincide with the distance between $\tilde{s}_u$ and $\tilde{s}_t$ for all $u, t \in \{0, \ldots, M - 1\}$. In other words, the distances between two codeword vectors $G_u h_j$ and $G_t h_j$ of the OSTBC coincide with the distances between $\tilde{s}_u$ and $\tilde{s}_t$ for all $u, t \in \{0, \ldots, M - 1\}$, that is, $d_{u,t,EC}$ and $d_{u,t,OSTBC}$ satisfy (4) for all $j = 1, \ldots, N_{R}$.

It follows from Definition 2 that if an OSTBC code matrix does not contain any zeros, then the
equivalent Euclidean code coincides with the “proxy” Euclidean code of the OSTBC. The salient example is Alamouti’s code. However, if the code matrix contains zeros, the dimensionality of the equivalent Euclidean code is smaller. For example, the equivalent Euclidean code to the rate $3/4$ OSTBC with the following code matrix

$$G_u = \begin{bmatrix} s_{1,u} & 0 & s_{2,u} & -s_{3,u} \\ 0 & s_{1,u} & s_{3,u}^* & s_{2,u}^* \\ -s_{2,u}^* & -s_{3,u} & s_{1,u}^* & 0 \\ s_{3,u}^* & -s_{2,u} & 0 & s_{1,u}^* \end{bmatrix}, \quad u = 0, \ldots, M - 1$$

(8)

and the Euclidean code $s_u = [s_{1,u}, s_{2,u}, s_{3,u}, 0]$ is $\tilde{s}_u = [\tilde{s}_{1,u}, \tilde{s}_{2,u}, \tilde{s}_{3,u}]$.

**C. Examples of Euclidean Codes Equivalent to Some Simplest OSTBCs**

Traditionally, only distance properties of OSTBC matrices have been investigated, and this was believed to be sufficient (see, for example, [14]). Geometrical properties of OSTBCs are discussed in several papers such as [13]–[15]. However, to the best of the authors’ knowledge, there is no research which reports the distance properties of signal coordinate diagrams even for the simplest OSTBCs. Thus, in this subsection, the distance properties of Euclidean codes equivalent to some simplest OSTBCs are studied.

The following definitions are needed for further discussion.

**Definition 3:** A distance profile $D_u$ of a codeword $s_u$ is a set of Euclidean distances $d_{u,j,EC}$ between the codeword $s_u$ and all other codewords $s_t$ ($t = 0, \ldots, M - 1; t \neq u$).

**Definition 4 [32]:** A code has a uniform constellation if all its codewords have the same distance profile. This means that all sets of distances between any codewords of the code are the same, and therefore, the corresponding average error probabilities under maximum likelihood decoding are the same for all codewords.

**Definition 5:** If a code has a uniform constellation, the corresponding distance profile is called a distance spectrum.
Let the normalized Euclidean distance be defined as

\[
\tilde{d}_{u,t,EC} \triangleq \frac{d_{u,t,EC}}{\bar{E}_{EC}}
\]  

(9)

where \(\bar{E}_{EC}\) is defined as the average energy of a codeword of the Euclidean code. The normalized distance spectra of two Euclidean codes equivalent to the simplest Alamouti OSTBC with constituent BPSK and quadrature PSK (QPSK) signals are given in Tables I and II respectively. These spectra are calculated according to (9). In Table III the normalized distance spectrum of the rate 3/4 OSTBC with the code matrix (8) and the equivalent Euclidean code with constituent BPSK signals is also given. Moreover, the average energies of the codes are \(E_{EC} = 2E\) for the Alamouti OSTBC with constituent BPSK and QPSK signals and \(E_{EC} = 3E\) for the rate 3/4 OSTBC (8) with constituent BPSK signals, where \(E\) is the energy of a constituent signal of the code. All these codes have uniform signal constellations in the equivalent Euclidean codes.

The corresponding signal constellations and signal coordinate diagrams (graphical representations) of the Euclidean codes equivalent to OSTBCs with the spectra given in Tables I–III are illustrated in Figs. 2–6 where Figs. 2 and 3 correspond to Table I; Figs. 4 and 5 correspond to Table II; and Fig. 6 correspond to Table III while the corresponding signal coordinate diagram is a simple cube with vertices corresponding to 8 codewords and edges of length \(2/\sqrt{3}\). Note that the codes are represented using the notation introduced in [33] for 4-D group codes. In these
TABLE III
DISTANCE SPECTRUM OF THE EUCLIDEAN CODE EQUIVALENT TO THE RATE 3/4 OSTBC \((8)\) WITH CONSTITUENT BPSK SIGNALS

| Normalized Euclidean distance | \(2\sqrt{\frac{1}{3}}\) | \(2\sqrt{\frac{2}{3}}\) | 2 |
|------------------------------|----------------|----------------|---|
| Number of codewords          | 3             | 3              | 1 |

![Signal constellation of the Euclidean code equivalent to Alamouti’s code with \(M = 4\) and constituent BPSK signals.](image)

Fig. 2. Signal constellation of the Euclidean code equivalent to Alamouti’s code with \(M = 4\) and constituent BPSK signals.

Figures, \(s_u\) \((u = 0, \ldots, M - 1)\) is the codeword transmitted for binary \(u\). In Figs. 2, 4, and 6, the constellation points transmitted for \(s_u\) are labeled by \(s_u\) itself, for simplicity. Note that in all figures, a Grey mapping scheme is followed, where the Euclidean distance between codewords \(s_u\) and \(s_t\) is nondecreasing as the Hamming distance between binary \(u\) and binary \(t\) increases.

For example for \(M = 8\), \(s_0\) and \(s_7\) or \(s_2\) and \(s_5\) have the largest distance. Fig. 5 and the cube in the 3-D space representing the Euclidean code with \(M = 8\) indicated in Fig. 6, equivalent to OSTBC \((8)\) with constituent BPSK signals use, in fact, the Schlegel diagram \[34\] to provide a geometrical representation of the signals and codes.

The tesseract depicted in Fig. 5 is an example of the 4-D Euclidean code (group code) well defined in 4-D geometry \[35\]. Note that the Euclidean codes given in Figs. 2, 4, and 6 belong to the class of spherical codes \[18\] and are also group codes \[35\].

**Remark 1:** Typically, the canonical OSTBCs are defined in the literature as OSTBCs with independent information-bearing PSK signals, as is the case with the examples given above.
Fig. 3. Normalized signal coordinate diagram of the Euclidean code equivalent to Alamouti’s code with \( M = 4 \) and constituent BPSK signals.

Fig. 4. Signal constellation of the Euclidean code equivalent to Alamouti’s code with \( M = 16 \) and constituent QPSK signals with Grey mapping.

However, it should be noted that independent information-bearing signals are just the \( L \)-ary auxiliary ‘components’ of a spatial modulator generating the actual \( M \)-ary multidimensional signal constellation of the OSTBC with correlated signals.

III. Optimality of OSTBCs

A cornerstone of designing ‘optimal’ codes is a proper definition of the design optimality criterion. Although there is a number of different definitions of OSTBC optimality (see, for example, [2], [12]), the most natural one is the following definition which is commonly used for modulated/coded signals in communication systems.
Definition 6: An OSTBC with given constituent signals and $M$ codewords is called optimal for a given type of channel if it provides the smallest SER under ML decoding among all OSTBCs with the same number of codewords, $M$, and arbitrary constituent signals.

Note that the design criteria for optimal codes are typically based on connecting the asymptotic SER behavior of a code for a given channel under a given decoding algorithm, with distance properties of this code. Thus, in this section, we aim at deriving a general design criterion for the optimal signal constellations of OSTBCs by connecting the distance properties of the Euclidean
codes equivalent to OSTBCs to the asymptotic SER performance of OSTBCs. Then the design
criterion for the particular case of large SNR and a large number of antennas is analyzed and
existence conditions for the optimal signal constellations of OSTBCs with constituent signals
having constant energy are provided, exploiting the connection of such OSTBCs with the optimal
spherical codes. A new biorthogonal constellation, which is an example of the optimal signal
constellation for the Alamouti OSTBC, is also given.

A. Union Bound on the SER of OSTBCs for Rayleigh Fading Channels

The interest in the union bound on the SER of OSTBCs in Rayleigh fading channels is
motivated by the need to connect the distance spectra of equivalent Euclidean codes with the
asymptotic properties of OSTBCs. As noted in the beginning of this section, this connection
will be used for formulating design criteria for equivalent Euclidean codes (i.e. constituent
multidimensional signals) of the optimal OSTBC for the Rayleigh fading channel.

Different upper bounds on the SER of OSTBCs have been previously derived in, for example,
[4], [12], [23]–[25]. However, one of the most often used upper bounds on the SER of OSTBCs
is a union bound, which can be written for codes with uniform constellations as

$$
\Pr_{s,\text{OSTBC}} \leq \sum_{t=0, t \neq u}^{M-1} \Pr(G_u \rightarrow G_t)
$$

(10)

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where \( \Pr(G_u \rightarrow G_t) \) is the pairwise error probability (PEP) of the OSTBC, i.e. the probability of detecting \( G_t \) when \( G_u \) is transmitted.

The closed-form solution for the PEP of ML decoding for OSTBCs with arbitrary constituent signals in the Rayleigh fading channel is well known (e.g., see [36] and the references therein). Indeed, the PEP is calculated as the expectation of \( \Pr(G_u \rightarrow G_t \mid H) \) over \( H \), where 
\[
H \triangleq [h_1, \ldots, h_{N_R}]
\]
is the matrix of channel coefficients. Using our notation for the equivalent Euclidean codes, the PEP of ML decoding for OSTBCs with arbitrary constituent signals can be obtained as
\[
\Pr(G_u \rightarrow G_t) = \frac{1}{2} - \frac{\mu_{u,t}}{2} \sum_{r=0}^{K-1} \binom{2r}{r} \left( \frac{1 - \mu_{u,t}^2}{4} \right)^r
\]
(11a)
where
\[
K \triangleq N_T N_R
\]
(11b)
and where
\[
\mu_{u,t} = \sqrt{\frac{d_{u,t,EC}^2 \tilde{\gamma}_c}{4 + d_{u,t,EC}^2 \tilde{\gamma}_c}}
\]
(11c)
where
\[
\tilde{\gamma}_c \triangleq \frac{\rho E_{EC}}{N_0}
\]
is the average received Euclidean code-to-noise ratio (cf. (3)).

To analyze the asymptotic behavior of the PEP (11), the following new lemma is useful.

**Lemma 1:** The PEP of the OSTBC (11) satisfies the following identity
\[
\frac{1}{2} - \frac{\mu_{u,t}}{2} \sum_{r=0}^{K-1} \binom{2r}{r} \left( \frac{1 - \mu_{u,t}^2}{4} \right)^r = \left( \frac{1 - \mu_{u,t}^2}{2} \right) \sum_{r=0}^{K-1} \binom{K-1+r}{r} \left( \frac{1 + \mu_{u,t}}{2} \right)^r.
\]
(12)

**Proof:** Substituting \( z = (1 + \mu_{u,t})/2 \) into the combinatorial identity [37] eq. (5.138)]
\[
1 + \frac{1 - 2z}{2} \sum_{r=1}^{n} \binom{2r}{r} (z(1-z))^r = (1-z)^n \sum_{r=0}^{n} \binom{n+r}{r} z^r
\]
(13)
we immediately obtain (12).

Substituting (12) in (10) yields the union bound for the SER of OSTBCs with uniform signal
constellations in the Rayleigh fading channel in the form
\[
\Pr_{s,\text{OSTBC}} \leq \sum_{t=0,t \neq u}^{M-1} \left( \frac{1 - \mu_{u,t}}{2} \right)^K \sum_{r=0}^{K-1} \binom{K - 1 + r}{r} \left( \frac{1 + \mu_{u,t}}{2} \right)^r \tag{14}
\]
where \( K \) is given by (11b).

\section*{B. Optimality of the OSTBC Signal Constellation: Large SNR}

In the case of large SNR, i.e. \( \bar{\gamma}_c \gg 1 \), approximating the terms \((1 - \mu_{u,t})/2\) and \((1 + \mu_{u,t})/2\) using the first component of their Taylor series yields \((1 - \mu_{u,t})/2 \approx 1/\left(\tilde{d}_{u,t,EC}^2 \bar{\gamma}_c\right)\) and \((1 + \mu_{u,t})/2 \approx 1\). Furthermore, using the following combinatorial expression \cite[eq. (14.4-17)]
\[
\sum_{r=0}^{K-1} \binom{K - 1 + r}{r} = \binom{2K - 1}{K} \tag{15}
\]
we can approximate (14) for the case of large SNR as
\[
\Pr_{s,\text{OSTBC}} \leq C_{EC}(K) \left( \frac{2K - 1}{K} \right) \bar{\gamma}_c^{-K} \tag{16a}
\]
where \( K \) is defined by (11b) and where
\[
C_{EC}(K) \triangleq \sum_{t=0,t \neq u}^{M-1} \tilde{d}_{u,t,EC}^{-2K} \tag{16b}
\]
which is called here the normalized distance spectrum constant (NDSC) of the OSTBC. It is interesting to note that the NDSC is a fixed parameter of an OSTBC for a given \( K \). The NDSC is defined only by the distance properties of the Euclidean code equivalent to the OSTBC and it does not depend on SNR. Therefore, we can say that a Euclidean code with minimal \( C_{EC}(K) \) among all Euclidean codes with the same \( M, K \), and dimensionality \( n \) is optimal in the sense that it provides the smallest SER at large SNR. Here, identical dimensionality ensures the same requirement for time/frequency resources and the same required number of transmitted bits per dimension, needed for fair comparison of the SER. The following theorem gives a more precise statement of the optimality.

\textbf{Theorem 2: For a quasistatic fading channel, large SNR \( \bar{\gamma}_c \gg 1 \), and a given \( K \), an OSTBC with cardinality \( M \) is optimal if and only if the Euclidean code equivalent to this OSTBC has}
the minimal NDSC \(16b\) among all Euclidean codes with the same \(M\) and dimensionality.

**Proof:** Both necessity and sufficiency follow directly from (16). If an OSTBC is optimal, it has the minimal NDSC. Otherwise, a code with a smaller NDSC achieves smaller SER according to (16). Conversely, if an OSTBC has the minimal NDSC, it is not outperformed by any other OSTBC, as the latter has an equal or larger NDSC, and thus SER, based on (16). \(\square\)

The following, perhaps obvious, but important corollary follows from Theorem 2.

**Corollary 1:** For a quasistatic fading channel, large SNR, and given \(K\), an OSTBC signal constellation is optimal if and only if the Euclidean code equivalent to this OSTBC has the minimal NDSC \(16b\) among all Euclidean codes with the same \(M\) and dimensionality.

This corollary formulates the general criterion for designing optimal OSTBC signal constellations on quasistatic fading channels. To the best of the authors’ knowledge, this is a new general design criterion for optimal OSTBC signal constellations. Moreover, as also follows from (16b) and Theorem 2, the optimality of the Euclidean code equivalent to an OSTBC for a given number of receiving antennas, \(N_R\), is not a sufficient condition for the optimality of the same code for a different number of receiving antennas. This is due to the nonlinear behavior of the NDSC (16b) with respect to \(N_R\). The following remark formalizes the novelty of the results given above.

**Remark 2:** Methods of design for Euclidean codes with minimal NDSC are not known. Also, the results embodied in (16) have not appeared before in the literature in the context of OSTBCs. As a result, Euclidean codes satisfying the conditions of Theorem 2 have not yet been investigated for any OSTBC. Moreover, there is no regular method of design for any class of Euclidean codes which are optimal according to any design criterion.

**C. Optimality of the OSTBC Signal Constellation: Large SNR \(\bar{\gamma}_c \gg 1\) and for a Large Number of Antennas \(N_T, N_R \gg 1\)**

In this case, the NDSC (16b) can be approximated as

\[
C_{EC}(K) \approx N_{d_{\min,EC}} d_{\min,EC}^{-2K}
\]

(17)

where \(d_{\min,EC}\) is the minimal normalized Euclidean distance of the Euclidean code equivalent to the OSTBC and where \(N_{d_{\min,EC}}\) is the number of codewords with the minimal distance \(d_{\min,EC}\).
Approximation (17) simplifies bound (16a) to

\[ P_{s,\text{OSTBC}} \leq N_{d_{\text{min,EC}}} \tilde{d}_{\text{min,EC}}^{-2K} \left( \frac{2K - 1}{K} \right) \bar{\gamma}^{-K}. \]  

(18)

The bound (18) is well known (e.g., see [26]), and thus, can serve as a check on our previous derivations. However, we use this bound here to derive existence conditions for OSTBCs based on their connection with the equivalent Euclidean codes. To the best of the authors’ knowledge, such discussion has not appeared in the literature before. Note from (18) that the dominant parameter for OSTBC optimality in the case of large SNR and for a large number of antennas is the minimal distance of the equivalent Euclidean code.

Although the aforementioned design criterion based on (18) is known, what has not been exploited before is that such a criterion coincides with the standard one for the error correcting codes optimal for the Gaussian channel. Thus, results for the optimal Euclidean codes known from the classic theory of error correcting coding can be used to define the existence conditions of the optimal OSTBC for large SNR and for a large number of antennas.

An interesting special case of Euclidean codes is a spherical code, for which every symbol of the code has the same norm [18]. Since we are interested in designing optimal OSTBCs, the notion of optimality for spherical codes is of importance. The optimal spherical code [18] is the code with the maximal minimum normalized Euclidean distance \( \tilde{d}_{\text{min,EC}} \) among all spherical codes with the same cardinality \( M \) and dimensionality \( n \). Note that the Euclidean codes equivalent to OSTBCs with constant energy constituent signals belong to the class of spherical codes. This leads to interesting connections between OSTBCs and error correcting codes. Particularly, the bounds obtained for the spherical codes can be used to define parameters of the asymptotically optimal OSTBCs. Some of the strongest and deepest results on the existence conditions of spherical codes with small dimensionality and squared minimal Euclidean distance \( 0 < \tilde{d}_{\text{min}}^2 \leq 4 \) were obtained by Rankin [38] (see also [18, Ch. 1.4]) and Coxeter-Böröczky [39, p. 28]. Based on the results of Rankin and Coxeter-Böröczky for spherical codes, the following bounds for asymptotically optimal signal constellations of OSTBCs can be formulated for the case of large SNR and for a large number of antennas.
**Theorem 3** *(Similar to the Coxeter-Böröczky bound)*: For quasistatic fading channels at large SNR and for a large number of antennas, any asymptotically optimal OSTBC that uses constituent signals with equal energies in the $n \geq 2$ dimensional Euclidean space, satisfies the conditions

$$d_{\text{min}} = 2 \sin \frac{\alpha}{2}$$  \hspace{1cm} (19)

and

$$M \leq \frac{2F_{n-1}(\alpha)}{F_n(\alpha)}$$  \hspace{1cm} (20)

where $F_n(\alpha)$ is the Schläfli’s function defined as

$$F_n(\alpha) = \frac{2}{\pi} \int_{\frac{\alpha}{4\text{arcsec}(n-1)}}^{\alpha} F_{n-2}(\beta) d\alpha$$  \hspace{1cm} (21)

where $\sec(2\beta) = \sec(2\alpha) - 2$, $F_0(\alpha) = F_1(\alpha) = 1$, $0 < \alpha \leq \pi$.

**Proof:** An OSTBC with equal-energy constituent signals corresponds to a Euclidean spherical code. Under the asymptotic hypotheses of the theorem, the optimality of the OSTBC corresponds to the maximality of the minimum Euclidean distance of the equivalent spherical code. This maximality condition is satisfied under the claims of the theorem, based on [39, p. 28].

**Theorem 4** *(Similar to Rankin’s first bound)*: For quasistatic fading channels at large SNR and for a large number of antennas, any asymptotically optimal OSTBC that uses constituent signals with equal energies satisfies the inequality

$$d_{\text{min}}^2 \leq \frac{2M}{M - 1}.$$  \hspace{1cm} (22)

**Proof:** See the proof of Theorem 3, and we also refer to [38] and [18, Ch. 1.4].

**Remark 3:** An interesting fact about the bound (22) is that it does not depend on the dimensionality of the code.

**Theorem 5** *(Similar to Rankin’s second bound)*: For quasistatic fading channels at large SNR and for a large number of antennas, the largest $M$ of an OSTBC that uses constituent signals with equal energies satisfies the inequality

$$M \leq n + 1$$  \hspace{1cm} (23)
for \(2 < \bar{d}_{\text{min}}^2 \leq 4\).

**Proof:** See the proof of Theorem 3, and we also refer to [38] and [18, Ch. 1.4].

**Theorem 6 (Similar to Rankin’s third bound):** For quasistatic fading channels at large SNR and for a large number of antennas, the largest \(M\) of an OSTBC that uses constituent signals with equal energies satisfies the inequality

\[
M \leq 2n
\]  

for \(\bar{d}_{\text{min}}^2 = 2\).

**Proof:** See the proof of Theorem 3, and we also refer to [38] and [18, Ch. 1.4].

The importance of Theorems 3–6 is especially stressed by the fact that these theorems provide the only known general bounds on \(M\) as existence conditions for asymptotically optimal OSTBCs using constituent signals with equal energies, assuming coherent receivers, quasistatic fading channels, large SNR, and a large number of antennas.

**Remark 4:** Although OSTBCs are connected now to spherical codes, it is still worth noting that regular methods for designing spherical codes with constituent modulated signals are not known. Thus, the code design problem is still not a simple problem, but such connections allow us to exploit some results on the design of spherical codes, such as a number of results summarized in [18]. Moreover, an approach based on the theory of group codes [31], [33] can also be useful, although methods for regular design of group codes with optimal distance properties are not known either. A possible undesirable consequence of considering group codes is that the constituent signals of these codes have symmetric properties; this is a severe restriction for code design and can result in nonoptimal codes. Finally, it is noteworthy that some useful properties of group codes suitable for the signal constellations of OSTBCs have been exploited in the OSTBC literature (e.g., see the research works on unitary code design [40], [41]).

**D. New Asymptotically Optimal** \(M = 8\) and \(M = 16\)** Biorthogonal Signal Constellations for the Alamouti OSTBC**

As an example of code design based on our studies in this section, we consider biorthogonal spherical codes. Indeed, biorthogonal spherical codes can be constructed for almost any
TABLE IV
NORMALIZED DISTANCE SPECTRUM OF THE BIOORTHOGONAL SPHERICAL CODE EQUIVALENT TO ALAMOUTI’ S CODE WITH BIOORTHOGONAL 4-D CONSTITUENT SIGNAL CONSTELLATION AND $M = 8$

| Normalized Euclidean distance | 2 | \( \sqrt{2} \) |
|------------------------------|---|----------------|
| Number of codewords          | 6 | 1              |

Fig. 7. Optimal signal constellation of a 4-D biorthogonal code with $M = 8$ for Alamouti’s code.

multidimensional space [18].

Consider a 4-D biorthogonal code with $M = 8$ and an 8-D biorthogonal code with $M = 16$. Such codes satisfy the upper bound (24) with equality. Therefore, the signal constellations of the new biorthogonal spherical codes depicted in Figs. 7 and 8 based on QPSK signaling can serve as examples of new asymptotically optimal signal constellations for Alamouti’s codes with $M = 8$ and $M = 16$. The spectrum and graphical representation of the code with $M = 8$ are given in Table IV and Fig. 9 respectively. The spectrum of the code with $M = 16$ is similar to that of the code with $M = 8$; only the number 6 in Table IV changes to 14. The signal coordinate diagram of the code with $M = 16$ (in 8-D space) has not been depicted as it is cumbersome. Note from Figs. 7 and 8 that codewords $s_0, \ldots, s_{M/2-1}$ are orthogonal, and are respectively the complements of $s_{M-1}, \ldots, s_{M/2}$ to ensure Grey mapping. Performance simulations for these two codes are presented in Section V.
IV. OSTBC PERFORMANCE ANALYSIS

The existing results on OSTBC performance analysis (see [19]–[26] and the references therein) aim at deriving exact expressions only for the SER of the constituent signals of the OSTBCs, while there are no results on the SER of an OSTBC in the sense of the probability that a codeword (code matrix) is transmitted but another codeword is detected. However, it is the latter SER for all types of modulation and coding, including orthogonal space-time coding, that is a common and important performance evaluation measure in communication systems.

Performance analysis of orthogonal space-time coding MIMO communication systems in a
Fig. 9. The Schlegel diagram of the hexadecachoron (16-cell) in the 4-D space [after http://en.wikipedia.org/wiki/16-cell] showing a geometrical representation of the optimal signal constellation with $M = 8$ (the biorthogonal spherical code given in Fig. 7) for Alamouti’s code.

Fading channel can be performed using the proposed equivalent model for MIMO systems given in Fig. 1. This model is connected to the classic receiver diversity system (e.g., see [31] and [42, Fig. 1]), which has long been of interest. However, the significant difference between our model in Fig. 1 and the classic receiver diversity system is that our model is, in fact, a form of receiver diversity of the block coded signals. This difference is especially useful from the performance analysis point of view.
A. Methodology for the General Case of Arbitrary Constituent Signals

Using the model in Fig. 1, the bit error rate (BER), i.e. \( \Pr_b \), and SER, i.e. \( \Pr_s \), of an OSTBC with arbitrary constituent signals over the Rayleigh fading channel can be evaluated based on the classic approach of estimating the error performance of a digital communication system over fading channels. That is, the BER and SER of the equivalent Euclidean code are evaluated by statistically averaging the conditional BER \( \Pr_{b,EC}^{EC}(\gamma_b) \) and SER \( \Pr_{s,EC}^{EC}(\gamma_b) \) at the output of a coherent receiver of this code for the Gaussian channel over the joint probability density function (PDF) of the fading amplitudes \( f_{\gamma_b}(\gamma_b) \) as

\[
\Pr_{b,OSTBC} = \int_0^\infty \Pr_{b,EC}^{EC}(\gamma_b) f_{\gamma_b}(\gamma_b) d\gamma_b \tag{25}
\]

\[
\Pr_{s,OSTBC} = \int_0^\infty \Pr_{s,EC}^{EC}(\gamma_b) f_{\gamma_b}(\gamma_b) d\gamma_b \tag{26}
\]

where \( \gamma_b \) is the total instantaneous SNR per bit at the output of the ML receiver given by

\[
\gamma_b \triangleq \sum_{j=1}^{N_t} \gamma_j \tag{27}
\]

where \( \gamma_j \) is the instantaneous SNR per bit in \( j \)th channel.

Towards evaluating (25) and (26), it is first required to obtain the PDF of the combined fading coefficient \( f_{\gamma_b}(\gamma_b) \), conditional BER \( \Pr_{b,EC}^{EC}(\gamma_b) \), and SER \( \Pr_{s,EC}^{EC}(\gamma_b) \) of the equivalent Euclidean code on the Gaussian channel. As also follows from (25) and (26), the main problem of performance analysis of OSTBCs based on the proposed model in Fig. 1 can be reduced to the evaluation of BER/SER of the corresponding Euclidean code over the channel with Gaussian noise. For example, the problem is reduced to the BER/SER evaluation of a 4-D Euclidean code in the case of the canonical Alamouti code with 2-D constituent signals and to the evaluation of a 6-D Euclidean code in the case of the rate 3/4 OSTBC with 2-D constituent signals. Although this methodology for exact BER/SER evaluation for the multidimensional signal constellations of interest is straightforward after the model in Fig. 1 is introduced, it cannot be found in the available literature and appears for the first time here. Another general methodology for performance analysis of OSTBCs with arbitrary constituent signals has been formulated in [22].
However, our methodology based on the classic performance analysis approach and the equivalent model in Fig. 1 is more straightforward and appears to be significantly simpler than the approach of [22].

B. BER and SER of Alamouti’s Code With Constituent BPSK Signals Over the Rayleigh Fading Channel

As an example of applying our performance analysis methodology, we derive a closed-form solution for the BER and SER of the Alamouti OSTBC with constituent BPSK signals. The signal coordinate diagram of the equivalent Euclidean code for Alamouti’s code with constituent BPSK signals is given in Fig. 3. Note that this diagram coincides with the signal coordinate diagram of QPSK with Grey mapping. Therefore, the Alamouti scheme with constituent BPSK signals corresponds to the receiver diversity scheme of Fig. 1 with QPSK signaling and with the ML receiver simplifying to the maximal-ratio combining receiver.

The expressions for $P_{r_{b}}^{EC}(\gamma_{b})$ and $P_{r_{s}}^{EC}(\gamma_{b})$ of a coherent receiver for QPSK with Grey mapping in the Gaussian channel are well known and can be found, for example, in [36] as

\begin{align}
P_{r_{b}}^{EC}(\gamma_{b}) &= Q\left(\sqrt{2\gamma_{b}}\right) \\
P_{r_{s}}^{EC}(\gamma_{b}) &= 2Q\left(\sqrt{2\gamma_{b}}\right) - Q^{2}\left(\sqrt{2\gamma_{b}}\right)
\end{align}

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^{2}/2} dt$ is the Gaussian $Q$-function. Also, the average SNR per bit is

$$
\gamma_{b} = \frac{\bar{E} \eta}{N_{T}N_{0} \log_{2} M}
$$

where, as follows from the model in Fig. 1

\begin{align*}
\eta &\triangleq \sum_{j=1}^{N_{R}} \|c_{j}\|^{2} = \sum_{i=1}^{N_{T}} \sum_{j=1}^{N_{R}} \|h_{i,j}\|^{2},
\bar{E} &\triangleq 2E_{b},
\end{align*}

and $M = 4$ since the signaling is quaternary. Then, the average SNR per bit can be expressed as

$$
\gamma_{b} = \frac{E_{b}}{N_{T}N_{0} \sum_{i=1}^{N_{T}} \sum_{j=1}^{N_{R}} \|h_{i,j}\|^{2}}.
$$

It has been shown in [31] (see also [19]) that the PDF of the average per bit SNR (31) is
given as

\[ f_{\gamma_b}(\gamma) = \frac{1}{(K-1)!\bar{\gamma}_b^K} \gamma^{K-1} e^{-\gamma/\bar{\gamma}_b} \]  \hspace{1cm} (32)

where \( K \) is defined by (11b) and where \( \bar{\gamma}_b \triangleq E_b/N_T N_0 \). Substituting (28) and (32) into (25), the average BER of the Alamouti code with constituent BPSK signals can be expressed as

\[ \Pr_b = \int_0^\infty \frac{1}{(K-1)!\bar{\gamma}_b^K} \gamma^{K-1} e^{-\gamma/\bar{\gamma}_b} Q\left(\sqrt{2\gamma}\right) d\gamma. \]  \hspace{1cm} (33)

Moreover, after some computations, it can be derived that

\[ \Pr_b = \frac{1}{2} - \frac{\mu_b}{2} \sum_{r=0}^{K-1} \left(\frac{2r}{r}\right) \left(\frac{1 - \mu_b^2}{4}\right)^r \]  \hspace{1cm} (34)

where \( \mu_b \triangleq \sqrt{\bar{\gamma}_b/(1 + \bar{\gamma}_b)} \). Note that the solution (34) is not new and has been derived by Bauch et al. in [19] based on the SISO model and later also verified in [21]–[23].

The average SER of Alamouti’s code with BPSK constituent signals has not been obtained previously. Substituting (29) and (32) into (26), the average SER can be expressed as

\[ \Pr_s = \int_0^\infty \frac{1}{(K-1)!\bar{\gamma}_b^K} \gamma^{K-1} e^{-\gamma/\bar{\gamma}_b} \left(2Q\left(\sqrt{2\gamma}\right) - Q^2\left(\sqrt{2\gamma}\right)\right) d\gamma. \]  \hspace{1cm} (35)

Moreover, using [42, eqs. (2) and (6)] and performing some computations, the expression (35) can be rewritten as

\[ \Pr_s = \frac{2}{\pi} \int_0^{\pi/4} \left(\frac{\cos^2 \theta}{\cos^2 \theta + \bar{\gamma}_b}\right)^K d\theta + \frac{1}{\pi} \int_0^{\pi/4} \left(\frac{\sin^2 \theta}{\sin^2 \theta + \bar{\gamma}_b}\right)^K d\theta. \]  \hspace{1cm} (36)

To the best of the authors’ knowledge, (36) is a new expression for the SER of Alamouti’s code with the constituent BPSK signals. Moreover, this is the only available exact expression for the SER of any OSTBC. All other known results are for the SER of the constituent signals of the OSTBC, that is, obviously, not the same and less descriptive of system performance.

V. NUMERICAL EXAMPLES

New asymptotically optimal signal constellations for the Alamouti OSTBC were found in Section III-D. The performances of these codes represent best cases of interest and are determined
Fig. 10. The BERs of three codes, Alamouti’s code with constituent QPSK signals, the rate $3/4$ OSTBC with constituent QPSK signals, and Alamouti’s code with constituent biorthogonal 4-D signals, for the Rayleigh fading channel.

by simulation in this section.

Figs. [10] and [11] show the simulated BER performances of the OSTBC designs based on spherical codes presented in Section [III-D]. The receiver structure is ML decoding based on the equivalent model shown in Fig. [1]. In fact, the signaling for the equivalent model is $M$-ary biorthogonal [31]. The simulations have been done for Rayleigh fading with $10^7$ trials.

Fig. [10] also shows the BERs of two other conventional schemes for comparison, Alamouti’s code, and the rate $3/4$ OSTBC [8], both with constituent QPSK signals. Note that in these codes, the constituent signals are independent (in contrast to the new design) so that the Alamouti code has $M = 16$, and the rate $3/4$ code has $M = 64$. The BERs of these codes are known to be equivalent to the BER of QPSK signaling in Nakagami fading channels. The latter has been analytically obtained in [43, Section 5.1]. The choice of these two codes for comparison is justified as follows. The new code based on 4-D spherical codes uses two transmitting antennas and its rate is $3/2$ bits per 2-D degree of freedom (DoF). However, there is no conventional space-
time code with the same rate that uses two transmitting antennas. Nonetheless, the Alamouti code with constituent QPSK signals uses two transmitting antennas, but its rate is 2 bits per 2-D DoF. The rate 3/4 code uses four transmitting antennas, but has the same rate as the new code (note that the label “rate 3/4” only refers to the fact that the system transmits three symbols in four time slots).

It can be seen from Fig. 10 that the new design is superior to the conventional Alamouti scheme. However, the new design is outperformed by the OSTBC (8), especially for a smaller number of antennas. Note that the new code uses half as many transmitting antennas as the OSTBC (8), which translates into less complexity and smaller size.

Fig. 11 in a manner similar to Fig. 10 exhibits and compares the BER performances of three codes, including the new OSTBC design based on 8-D spherical codes, and two conventional schemes, Alamouti’s code with constituent BPSK signals and Alamouti’s code with constituent QPSK signals. The QPSK Alamouti code here is the same as the one used for comparison in
TABLE V

PERFORMANCE TRADEOFFS IN TRANSITION FROM ALAMOUTI’S CODE WITH CONSTITUENT BIORTHOGONAL 4-D SIGNALS TO THE CODE WITH BIORTHOGONAL 8-D SIGNALS

| Time DoFs | Frequency DoFs | Power | SNR | Bit Rate | BER          |
|-----------|---------------|-------|-----|----------|--------------|
| 1         | 2             | 1     | 0.5 | 4/3      | $\approx 1.6N_R^{0.33}$ |
| 1         | 2             | 2     | 1   | 4/3      | $\approx 0.50N_R^{-1.7}$ |
| 2         | 1             | 1     | 1   | 2/3      | $\approx 0.50N_R^{-1.7}$ |
| 2         | 1             | 2     | 2   | 2/3      | $\approx 0.11N_R^{-3.3}$ |

Fig. 10 The BER of the BPSK Alamouti scheme has been obtained from (34). Note that all the codes use two transmitting antennas, and their rates are respectively 1 bit, 1 bit, and 2 bits per 2-D DoF. Fig. 11 demonstrates the BER superiority of the new code over the two conventional schemes, i.e., even over the BPSK Alamouti scheme which has the same rate. The superiority is augmented as the number of receiving antennas increases.

It is noteworthy that the performances of the new designs based on 4-D and 8-D spherical codes in Figs. 10 and 11 are not directly comparable. In fact, in transition from the code with $n = 4$ and $M = 8$ (Fig. 10) to the code with $n = 8$ and $M = 16$ (Fig. 11), the number of dimensions or DoFs is doubled, which means that twice as many time and/or frequency resources are expended. The impact of this transition is shown in Table V in terms of tradeoffs between the utilized time DoFs, utilized frequency DoFs, continuous-time transmitting power, SNR, bit rate, and BER. Four different cases of tradeoff have been represented. In the table, value $x$ denotes that the value of the corresponding quantity is multiplied by $x$ as a result of the transition. The approximate changes shown for the BER are obtained by a comparison between Figs. 10 and 11 for relatively large values of SNR.

Finally, Fig. 12 shows the SER and BER performances of Alamouti’s code with BPSK constituent signals for Rayleigh fading. The figure exhibits both results based on the analysis presented in Section IV-B and simulation results. In the analytical approach, integral (36) has been evaluated numerically. Note that the simulation results are in excellent agreement with the analytical results.
Fig. 12. The SER and BER of Alamouti’s code with constituent BPSK signals for the Rayleigh fading channel.

VI. CONCLUSION

Based on the analysis of the distance properties of OSTBCs, an equivalent model for MIMO communication system with OSTBCs was proposed and a class of Euclidean codes equivalent to OSTBCs was introduced. Examples of distance spectra, signal constellations, and signal
coordinate diagrams of Euclidean codes equivalent to some simplest OSTBCs were given. A new asymptotic upper bound on the SER of OSTBCs, which is based on the distance spectra of the introduced equivalent Euclidean codes, was derived. Also, new general design criteria for the signal constellations of optimal OSTBCs were proposed for two asymptotic cases, (i) large SNR and (ii) large SNR and a large number of antennas. Exploiting the connection between OSTBCs and spherical codes, some bounds which link the distance properties, dimensionality, and cardinality of equal energy OSTBC signals were given. Then, two new optimal signal constellations with cardinalities $M = 8$ and $M = 16$ for Alamouti’s code were designed as an example of using the connection between OSTBCs and spherical codes. Finally, using the model introduced for MIMO communication systems with OSTBCs, a general methodology for performance analysis of OSTBCs was formulated. As an application example of this methodology, a new expression for the SER of Alamouti’s code with BPSK signals was derived. This result is the first example of exact SER analysis of OSTBCs.

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