Stabilization of a Majorana Zero Mode through Quantum Frustration

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We analyze a system in which a topological Majorana zero mode (MZM) combines with a MZM produced by quantum frustration. At the boundary between the topological and non-topological states, a MZM does not appear. The system that we study combines two parts, a grounded topological superconducting wire that hosts two MZMs at its ends, and an on-resonant quantum dot connected to two dissipative leads. The quantum dot with dissipative leads creates an effective two-channel Kondo (2CK) state in which quantum frustration yields an isolated MZM at the dot. We find that coupling the dot to one of the wire Majoranas stabilizes the MZM at the other end of the wire. In addition to providing a route to achieving an unpaired Majorana zero mode, this scheme provides a clear signature of the presence of the 2CK Majorana.

Electron states with topological character and quantum frustration from competing interactions are two major themes in current condensed matter physics. In both contexts, fractionalized degrees of freedom on the boundary of a system occur. Perhaps the best known example is the possibility of Majorana zero modes (MZMs) at the ends of a one-dimensional (1D) system. MZMs are exotic self-conjugate edge states that can occur through either topology \cite{1–3} or fine-tuning of competing interactions \cite{4–7}. Here we study the interplay between these two types of MZMs in a nanoscale system of quantum dots and wires. We show that a MZM from quantum frustration can be used to stabilize a topological MZM.

Quantum frustration typically produces states of matter that are delicately balanced between competing options and which then show fractionalization \cite{8}. Quantum impurity models—an interacting quantum system coupled to non-interacting leads—provide several canonical examples. (Note that quantum impurity models are effectively 1D since the impurity couples to a limited set of states in the leads \cite{9}.) The two channel Kondo (2CK) model, for instance, has been extensively studied \cite{4, 7, 10–14}: an impurity spin is equally coupled to two metallic leads. It would be natural for the impurity spin to form a singlet with each lead, but it cannot because of entanglement exhaustion. This frustration in screening the impurity leads to a non-Fermi-liquid ground state in which there is a degeneracy of $\sqrt{2}$ at the impurity. This signals fractionalization and the existence of an unpaired MZM. MZM have also been discussed in the two impurity Kondo model \cite{6, 15} and the dissipative resonant level model \cite{16, 17}. Experimentally, several groups have investigated in detail nanoscale systems with an unpaired MZM of this type \cite{13, 14, 16, 18–20}—the fine tuning required appears to be easier to achieve than the creation of a topological state.

Topological MZMs, in addition to their inherent interest, have attracted attention because their non-Abelian statistics provide a possible route toward fault tolerant quantum computation \cite{1, 21}. To construct and observe such MZMs, researchers have proposed multiple systems \cite{2, 22–25}. One particularly promising 1D system consists of a semiconducting nanowire made of a material that has strong spin-orbit coupling which is placed in proximity to a $s$-wave superconductor and in a magnetic field \cite{2}. In this system, signatures of MZM through measurement of the conductance have been intensively pursued \cite{2}.

In contrast to the free elementary particle predicted by Majorana, the effective MZMs in condensed matter always appear in pairs in finite size systems \cite{1, 2}. Unfortunately, MZMs lose many of their interesting properties when they hybridize with their partners. The inter-MZM coupling decays as $\propto \exp(-L/\xi)$ \cite{1, 2}, with $L$ the distance and $\xi$ the superconducting correlation length in the nanowire. Experimentally, $L \sim 1 \mu m$ and $\xi \sim 300 \text{ nm}$ \cite{26}, so that the hybridization can not generally be ignored. Consequently, to see the full effect of a MZM, a method to stabilize the MZM against inter-MZM coupling is desirable.

In this paper, we stabilize a MZM against hybridization with its partner by coupling its partner to an unpaired Majorana fermion of a frustrated resonant level model. We show that the frustration induced Majorana fermion of the $R = R_Q$ dissipative resonant level model can be experimentally detected and is helpful in the stabilization of a MZM.

**The system**—We consider a bare bones system, shown in Fig. 1, that has a pair of topological MZM, $\gamma_1$ and $\gamma_2$ (red dots), at its ends. The coupling between these two MZM is $\epsilon_M \neq 0$ and hence the Hamiltonian of the system is

$$H_{\text{sys.}} = i\epsilon_M \gamma_1 \gamma_2.$$

![Fig. 1. The structure of the system. Two MZMs $\gamma_1$ and $\gamma_2$ are realized at two ends of a nanowire on top of a grounded topological superconductor (TS). We calculate the conductance through the left quantum dot to detect the existence of the MZM $\gamma_1$. The right quantum dot couples to $\gamma_2$, thereby stabilizing $\gamma_1$.](image-url)
The goal is to have effectively $\epsilon_M = 0$ so that the topological MZM $\gamma_1$ can be manipulated for quantum computation.

In order to assess directly whether $\gamma_1$ is indeed an independent MZM, we incorporate a detector explicitly. As the presence of a MZM affects many physical properties, different types of detectors could be used. We choose to consider the conductance through a spinless quantum dot modeled as a resonant level [27], pictured on the left of Fig. 1. As there are different types of detectors could be used. We choose to consider the presence of a MZM affects many physical properties, different types of detectors could be used. We choose to consider the presence of a MZM. As the presence of a MZM affects many physical properties, different types of detectors could be used. We choose to consider the presence of a MZM affects many physical properties, different types of detectors could be used. We choose to consider the presence of a MZM.

The notation parallels that for the detector: $\epsilon_R$ is the dot energy level and $c_{kR\alpha}$ creates an electron in the lead labeled $\alpha$ (top or bottom). The dot is symmetrically coupled to the leads with amplitude $V_R$.

In modeling the dissipation, we follow a standard approach [30, 31]. The phase fluctuation operator $\phi_t$ becomes conjugate to the charge fluctuation operator for the capacitor between the dot and lead $\alpha$. Thus, the operator $e^{\pm i\phi_t}$ in Eq. (4) accounts for the change in charge upon tunneling. The current and voltage fluctuations caused by the electrons tunneling on and off the dot excite the ohmic environment of the leads. This environment is modeled as a bath of harmonic oscillators [30, 31, 33]. Because it is the charge moving across the dot that excites the environment, the difference $\varphi \equiv \phi_t - \phi_b$ couples to the harmonic oscillators. The bath causes the correlation function of these fluctuations to be $(e^{-i\varphi(t)}e^{i\varphi(0)}) \propto (1/t)^2 r$ where the exponent $r$ is related to the resistance of the environment by $r \equiv R/e^2/h \equiv R/R_Q$. With this correlation function, the conductance and scaling behavior can be found.

Because of the interactions introduced by dissipation [i.e. the second line of Eq. (4) is not quadratic], a RG treatment is natural [16, 17, 34–37]. For a generic set of parameters the dot-lead coupling scales to zero and so the conductance is zero. Physically, the dissipation suppresses the tunneling between the dot and lead, as one would expect. Effectively, the dot-lead barrier that is initially larger grows and cuts the system in two, while the the dot is incorporated into the other lead. However, there is a special value of the parameters at which this is not the case: for symmetric coupling and on resonance ($\epsilon_R = 0$), the RG flow is toward strong coupling. The dot becomes hybridized with both leads and the transmission becomes unity ($G_R = e^2/h$). Further analysis shows that this situation is analogous to the intermediate fixed point of the 2CK problem [16, 17, 37, 38]. The quantum frustration in the present context is the competition between incorporation of the dot into the left or right lead.

For the case $R = R_Q$ (i.e. $r = 1$), an explicit treatment using bosonization and refermionization is possible [16, 17, 38] following that for the 2CK model [12, 28] or a $g = 1/2$ Luttinger liquid resonant level model [7]. Two bosonic fields are introduced to treat the leads, one for the total density in the leads and one for the density difference. The density difference is related to the current through the system and so naturally combines with the dissipative phase fluctuations $\varphi$. For $r = 1$, the resulting entity can be expressed as a fermionic operator in its own right, which we shall denote $\psi_{k,\alpha}^\dagger$.
Meanwhile the bosonic field corresponding to the total density does not interact with the dissipation. Using a unitary transformation, one can remove it from the tunneling term at the cost of introducing an interaction between it and the density in the dot (i.e. \(d^\dagger_d d - 1/2\)). Under RG around the perfect-transmission fixed point, the magnitude of this interaction scales to zero (it is an irrelevant operator), and so this field can be treated as free \([17, 36, 38]\).

Finally, we define a Majorana representation for the degree of freedom represented by \(d_R\): \(\chi_1 \equiv (d^\dagger_R + d_R)/\sqrt{2}\) and \(\chi_2 \equiv (d^\dagger_R - d_R)/\sqrt{2}\). Because of the unitary transformation mentioned in the last paragraph, this is no longer simply the dot level but rather a nonlinear mixture of the dot and the density in the two leads near the dot. Nevertheless, both resulting MZMs are highly localized near the quantum dot.

The result of these manipulations (specific for \(r = 1\)) is an effective Majorana Hamiltonian for the right-hand dot and leads:

\[
H_{\text{dissip}} = \sum_k \epsilon_k \psi^\dagger_k \psi_k + i\sqrt{2}V_R \left( \psi^\dagger_k \chi_2 + \psi_k \chi_2 + i\epsilon_R \chi_1 \chi_2 \right). \tag{5}
\]

The key point is that for \(\epsilon_R = 0\) only one of the Majoranas is coupled to the leads, leaving \(\chi_1\) as a true MZM. This is a Majorana obtained from quantum frustration by fine-tuning.

**Combining Majoranas produced by frustration and topology**—The key final ingredient in our problem is the connection between the right dot and the topological wire. This is simply tunneling, as for the left dot Eq. (3); for detailed discussions of tunneling between topological MZMs and those arising from Klein factors in bosonization see, e.g., Refs. [39–41]. However, now there is no freedom to choose the particular combination of \(d_R^\dagger\) and \(d_R\) that couples, since the Majorana representation is fixed to be \(\{\chi_1, \chi_2\}\) by the coupling to the dissipative leads. Thus the coupled Hamiltonian is

\[
H_{\text{sys.}} + H_{\text{sys.-dis.}} \equiv i\epsilon_M \gamma_1 \gamma_2 + i\epsilon_{RL} \gamma_1 \chi_1 + i\epsilon_{RR} \gamma_2 \chi_2, \tag{6}
\]

in which the topological MZM \(\gamma_2\) is coupled to both dot Majoranas with arbitrary coupling \(\epsilon_{RL}\) and \(\epsilon_{RR}\). In writing Eq. (6) we have for simplicity dropped factors involving the total density in the leads produced by the unitary transformation discussed above; including these factors does not change the results or arguments that follow [42].

The full Hamiltonian for our problem, \(H = H_{\text{sys.}} + H_{\text{detect.}} + H_{\text{dissip.}} + H_{\text{sys.-det.}} + H_{\text{sys.-dis.}}\), is quadratic and so can be solved through the equation of motion method. We calculate the conductance of the left quantum dot that probes the \(\gamma_1\) MZM. With symmetric coupling, its equilibrium conductance is related to the dot spectral function by

\[
G_L = -\Gamma_L \frac{e^2}{h} \int \frac{d\omega}{2\pi} \text{Im} \left\{ G^R(d_L^\dagger d_L)(\omega) \right\} \partial_\omega n_F(\omega), \tag{7}
\]

where \(G^R(d_L^\dagger d_L)(\omega)\) is the Fourier transform of the retarded Green function \(-i\theta(t)\{d_L(t), d^\dagger_L(0)\}\), \(n_F(\omega)\) is the Fermi distribution function, and \(\Gamma_L = \pi\hbar\Gamma L^2\) is the level broadening.

The retarded Green function of the left dot from the equation of motion method [43] is

\[
G^R(d_L^\dagger d_L)(\omega) = \frac{1}{\omega + i\Gamma_L - \epsilon_L - \Sigma(\omega)}, \tag{8}
\]

where the self-energy \(\Sigma(\omega)\) incorporates the effects of the coupling between the dot \(d_L\) and \(\gamma_1\) as well as the three couplings in Eq. (6) [see the supplementary material [42] for an explicit expression for \(\Sigma(\omega)\)]. One finds that the conductance through the left dot when on resonance \((\epsilon_L = \epsilon_R = 0)\) is

\[
G_L = \frac{1}{2} \frac{e^2}{\hbar}, \tag{9}
\]

independent of the values of any parameters (such as \(\epsilon_{RL}\) or \(\epsilon_M\)). This striking independence implies, for instance, that fine tuning of the coupling between \(\gamma_2\) and the right dot is not needed, a significant experimental simplification. This result holds only within the validity of our model, of course: one should have \(\epsilon_M \ll \Delta\) in order to have a topological MZM (\(\Delta\) is the superconducting gap in the proximitized nanowire) and \(\epsilon_M \ll \Gamma_R\) in order to have a MZM from frustration (\(\Gamma_R\) is the level broadening of the right resonant level model).

The conductance Eq. (9) is one of our main results. It indicates that the introduction of the \(R = R_Q\) dissipative quantum dot stabilizes \(\gamma_1\).

Fine-tuning of the energy level of the quantum dot is required only for the right-hand dot, \(\epsilon_R = 0\). We do not need a fine-tuned left dot since \(\epsilon_L\) is irrelevant at the non-trivial fixed point: at this point, \((d^\dagger_L + d_L)/\sqrt{2}\) couples with \(\gamma_1\) into a singlet, thus reducing its connection to \((d^\dagger_R - d_R)/i\sqrt{2}\).

Experimentally, the irrelevance of \(\epsilon_L\) is a signature of the MZM: instead of the usual Lorentzian shape from the resonant model, the zero temperature conductance of the MZM-coupled resonant level model is expected to be flat as a function of \(\epsilon_L\). For non-zero temperature, the conductance will be constant as long as \(\epsilon_L(T) < \Gamma_L(T)\), both of which may vary with temperature because of renormalization effects.

**Dissipation-free right dot**—To highlight the role of dissipation, we now consider what happens if there is no dissipation in the right-hand leads, \(r = 0\). Thus the full-transmission fixed point Hamiltonian Eq. (5) is replaced by a second copy of the resonant level Hamiltonian Eq. (2). Since the Hamiltonian remains quadratic, we again use the equation of motion method to find the retarded Green function of the left dot; the explicit form of the self-energy to use in Eq. (8) is given in the supplementary material [42]. In the absence of dissipation, the two MZMs on the right dot become equivalent, allowing us to freely choose \(\gamma_2\) to couple to any linear combination of them with the coupling strength \(t_R\). The remaining MZM will be hybridized by the leads.

The conductance of the left quantum dot in this case is

\[
G_L = \frac{e^2}{4\hbar} \frac{2t_R^2 + \epsilon_M^2 \Gamma L \Gamma R}{2\epsilon_M^2 \Gamma L \Gamma R + \epsilon_M^2 \Gamma L \Gamma R}, \tag{10}
\]

in which one of the Majoranas is coupled to the leads, leaving \(\chi_1\) as a true MZM. This is a Majorana obtained from quantum frustration by fine-tuning.
where $\Gamma_R = \pi \rho_0 V_R^2$ is the broadening of the right resonant level. Eq. (10) displays an interesting feature: the equilibrium zero temperature conductance varies between the trivial \((e^2/h)\) and the non-trivial \((e^2/2h)\) values, depending on the details of the system. This crossover originates from the competition between the dot-MZM couplings and the hybridization of the quantum dots by the leads.

**Summary**— We have shown that the conductance through the MZM-coupled left quantum dot is strongly influenced by the nature of the system on the right. (i) In the absence of a right-hand system, the two MZMs $\gamma_1$ and $\gamma_2$ couple into a trivial state for which $G_L/(e^2/h) = 1$. (ii) When the right-hand system is present but without dissipation, the conductance through the left dot is between the trivial and nontrivial values, depending on the details of the system. (iii) Finally, the nontrivial state is stabilized when the right-hand system becomes dissipative $R = R_Q$, leading to the zero temperature conductance $G_L/(e^2/h) = 1/2$.

**Physical interpretation**— Through the g-theorem, we now provide a simple way to understand these results. As the counterpart of the famous c-theorem of two-dimensional conformal field theory [44, 45], the g-theorem treats boundary phase transitions and relates the stability of the fixed points to the impurity or boundary entropy. Specifically, if the bulk parameters remain invariant during the RG flows, the flow will bring the system toward the fixed point with a smaller impurity entropy [46]. We have calculated the ground state degeneracy of our system at the two fixed points in the three scenarios above. The results are compiled in Table I, and we now discuss each scenario in turn.

(i) If the right-hand system is absent ($t_R = 0$) and $\epsilon_M \neq 0$, the trivial fixed point is non-degenerate: the two MZMs $\gamma_1$ and $\gamma_2$ form into a singlet and the left quantum dot is completely hybridized with the leads. In contrast, the nontrivial fixed point has a decoupled MZM, namely $\gamma_2$, yielding a ground state degeneracy $\sqrt{2}$. The g-theorem then implies, in agreement with the conductance calculation above, that the nontrivial fixed point is unstable. Alternatively, we notice that the leading operator at the nontrivial fixed point is the hybridization between the leads and $(d_L^\dagger + d_L)/\sqrt{2}$, which has the scaling dimension $1/2$. Consequently, the hybridization operator is relevant and sabotages the nontrivial fixed point.

(ii) If the right-hand system is a dissipationless quantum dot, the ground states of both the trivial and nontrivial fixed points are non-degenerate. Consequently, the operator that connects these two fixed points is marginal, leading to a crossover between the trivial and nontrivial fixed points. This crossover is reflected in the intermediate value of the conductance, Eq. (10). From an RG point of view, because the parity of the superconducting island is conserved, tunneling happens simultaneously in the left and right quantum dots [42]. Thus the scaling dimension of the hybridization doubles compared to case (i), rendering it marginal.

(iii) Finally, when the right quantum dot is dissipative, at the nontrivial fixed point the isolated MZM $\chi_2$ couples to $\gamma_2$ in a singlet, thus leading to a non-degenerate ground state. In contrast, at the trivial fixed point, $\chi_2$ remains isolated, yielding degeneracy $\sqrt{2}$. Consequently, the g-theorem predicts that the RG flow brings the system toward the nontrivial fixed point. Alternatively, the lead-dot hybridization is suppressed by the dissipation and so has a larger scaling dimension than in case (ii). The hybridization thus becomes irrelevant (for any $R \neq 0$) and the Majorana feature is protected.

**Quasi-Majorana detector**— The frustration-induced Majorana fermion $\chi_1$ can be used to distinguish a quasi-Majorana mode from a topologically non-trivial one. Briefly, it has been shown [47] that a topologically trivial superconducting island will host a pair of quasi-Majorana at both boundaries if they are smooth enough (Fig. 2). Suppose two out of the four Majorana fermions couple to the quantum dots ($\gamma_1$ and $\gamma_4$ in Fig. 2, the more interesting case [48]), and take the overlap between the two boundaries to be zero in order to favor a topological MZM. Note that the non-trivial fixed point ground state has the degeneracy two, which is larger than that of the trivial fixed point ($\sqrt{2}$). Consequently, with the presence of two pairs of quasi-Majoranas, the system prefers the trivial ground state, which is in sharp contrast to when they are absent. Based on this fact, the measured conductance $G_L$ can be used to detect the existence of possible quasi-Majoranas.

**Outlook**— We show that it is possible in principle to combine a MZM arising from topology with one produced by fine-tuned quantum frustration. This is a highly unusual situation:

| $t_R = 0$ | $t_R \neq 0$ and $R = 0$ | $t_R \neq 0$ and $R = R_Q$ |
|---|---|---|
| \( g_{\text{trivial}} \) | 1 | 1 |
| \( g_{\text{nontrivial}} \) | $\sqrt{2}$ | 1 |
| \( G_L/(e^2/h) \) | $\frac{2d_L^d + d_R^d}{4t_R^d + d_R^\dagger d_L^\dagger}$ | 1/2 |

**FIG. 2.** The case where two pairs of quasi-Majorana pairs form at each end of the superconducting island, within the range of the superconducting coherence length. In the non-trivial case (a) the impurity entropy is larger than that of the trivial case (b), leading to a more stable trivial fixed point. Here MZMs connected by the black lines form into singlets.
we connect a topological system with one that is not topological and find no unusual feature, and in particular no MZM, at the boundary.

We showed this result by calculating the conductance through a detector quantum dot and then supported it using the RG method and the g-theorem. The combination of the two types of MZM originates from the preference of the system for a non-degenerate ground state, arrived at by coupling the topological MZM $\gamma_2$ and the quantum dot Majorana fermion $\chi_1$ into a singlet.

One implication of these results is that the quantum-frustration MZM can be used to stabilize a topological MZM against coupling to its partner. Another is that the long-ignored frustration generated Majorana fermion is in principle detectable.

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[42] In this Supplementary Material, we provide details about: (i) the parity conservation of the superconducting island when the right resonant level model is free from dissipation, (ii) the expression of the self energy without and with the presence of the stabilizer and (iii) the non-equilibrium current and shot noise calculated with the FCS (full counting statistics) technique.

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[48] In the “boring” case, both $\gamma_1$ and $\gamma_2$ couple to the left dot. The quasi-Majorana fermion pair is then naturally different from a real Majorana.
Supplemental Material for
“Stabilization of a Majorana Zero Mode through Quantum Frustration”
Gu Zhang and Harold U. Baranger
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In this Supplemental Material, we provide details on three topics: (i) in Sec. S1, the parity conservation of the superconducting island when the right resonant level model is free from dissipation, (ii) in Sec. S2 the expression of the self energy of the left quantum dot, and (iii) in Sec. S3, the effect of the unitary transformation on the impurity operators.

S1. Parity Conservation Induced Joined Tunneling

We start with the nontrivial fixed point \((\epsilon_M = 0)\) where \((d^d_{L,R} + d_{L,R})/\sqrt{2}\) and \(\gamma_{1,2}\) form into a singlet. For simplicity, we label this state as \(|0\rangle_L \otimes |0\rangle_R\). We further label the exited state as \(|1\rangle_{L,R}\).

Now we add the weak inter-MZM coupling \(\epsilon_M\), which alters the system ground state. In this case we calculate the system ground state of the Hamiltonian

\[
H'_{\text{Impurity}} = i t_L (d^d_L + d_L) \gamma_1 + i t_R (d^d_R + d_R) \gamma_2 + i \epsilon_M \gamma_1 \gamma_2
\]  

through exact diagonalization. Without loss of generality, we take \(t_L = t_R = t\) for simplification. The system ground state has the energy \(\sqrt{2} t - \sqrt{\epsilon_M^2 + 8t^2}/2\), with the eigenvector

\[
\frac{1}{\sqrt{\epsilon_M^2 + (-2\sqrt{2} t + \sqrt{\epsilon_M^2 + 8t^2})^2}} \left[ i(-2\sqrt{2} t + \sqrt{\epsilon_M^2 + 8t^2})|0\rangle_L \otimes |1\rangle_R + \epsilon_M |0\rangle_L \otimes |0\rangle_R \right],
\]  

which reduces to \(|0\rangle_L \otimes |0\rangle_R\) at the limit \(\epsilon_M \rightarrow 0\).

Eq. (S2) indicates that the system ground state only involves the dot states with even parity \(|0\rangle_L \otimes |0\rangle_R\) and \(|1\rangle_L \otimes |1\rangle_R\). Physically, this originates from the fact that the inter-MZM coupling operator \(\epsilon_M \gamma_1 \gamma_2\) changes the parity of the impurity states at both sides simultaneously. Consequently, if we turn on a weak tunneling at both resonant level models, the system only allows tunneling events that occur simultaneously at both sides, thus doubling the scaling dimension by a factor of two, as mentioned in the main text.

S2. Self Energy of the Left Dot

To better understand the left dot conductance, here we provide the expression of the self energy \(\Sigma(\omega)\) of Eq. (8) of the left dot in different cases. In this section we take \(\epsilon_L = 0\) for simplicity.

To begin with, when the stabilizer does not exist \((t_R = 0)\), the self energy becomes

\[
\Sigma^{-1}(\omega) = \frac{\omega}{t_L^2} - \frac{1}{\omega + \epsilon_L + i\Gamma_L} - \frac{\epsilon_M^2}{t_L^2 \omega}, \tag{S3}
\]

where the three terms correspond to the coupling of \((d^d_L + d^d_L)/\sqrt{2}\) to \(\gamma_1\), its hybridization to the lead, and the \(\gamma_1-\gamma_2\) coupling, respectively.

Now, if we use a dissipation-free resonant level model as the stabilizer, the self energy changes to be

\[
\Sigma(\omega)^{-1} = \frac{\omega}{t_L^2} - \frac{1}{\omega + \epsilon_L + i\Gamma_L} - \frac{\epsilon_M^2}{t_L^2} \left( \omega - \frac{t_R^2}{\omega + \epsilon_R + i\Gamma_R} - \frac{t_R^2}{\omega - \epsilon_R + i\Gamma_R} \right)^{-1}. \tag{S4}
\]
We can see that the third term of the self-energy now becomes more complicated: the extra terms correspond to the coupling between $\gamma_2$ and one of the impurity Majorana fermions of the stabilizer.

Finally, we take $R = R_Q$ for the right resonant level model, and the self energy becomes

\[
\Sigma(\omega)^{-1} = \frac{\omega}{t_L^2} - \frac{1}{\omega + \epsilon_L + i\Gamma_L} - \frac{\epsilon_M^2}{t_L^2} \left( \frac{\omega - t_{R_1}^2}{\omega + \epsilon_R + i\eta} - \frac{t_{R_1}^2}{\omega - \epsilon_R + i\eta} \right)^{-1} - \frac{\epsilon_M^2}{t_L^2} \left( \frac{\omega - t_{R_2}^2}{\omega + \epsilon_R + i\Gamma_R} - \frac{t_{R_2}^2}{\omega - \epsilon_R + i\Gamma_R} \right)^{-1},
\]

where $\eta$ is a positive infinitesimal. In comparison to that of Eq. (S4), Eq. (S5) becomes more complicated since now the two right impurity Majoranas become inequivalent, and we have to consider the coupling of $\gamma_2$ to both of them. Notice that the denominators of the second two functions in the second line have the positive infinitesimal $\eta$ instead of the level broadening $\Gamma_R$. This reflects the fact that $\chi_1$ is effectively decoupled from the lead.

**S3. IMPURITY OPERATOR AFTER THE UNITARY TRANSFORMATION**

In the main text, to get the Majorana Hamiltonian Eq. (5) we have taken an unitary transformation with the operator [S1]

\[
U = e^{i(d^\dagger d - \frac{1}{2})\phi_c(0)/\sqrt{2}}
\]

that decouples the field $\phi_c(0)$ from the tunneling Hamiltonian. Here $\phi_c(0)$ is proportional to the number of electrons in the leads.

However, such a transformation introduces two minor side effects. First of all, with such a transformation,

\[
d \rightarrow de^{iK\frac{\phi_c(0)}{\sqrt{2}}},
\]

where $K = 1$ initially. Since they are dressed by the bosonic field, the impurity operators $d$ and $d^\dagger$ of Eq. (5) are different from those in the bare Hamiltonian Eq. (2). Second, the unitary transformation introduces a quartic interaction [S1]

\[
H_{\text{extra}} = -\frac{v}{2\sqrt{2}} \left( d^\dagger d - \frac{1}{2} \right) \partial_x \phi_c(x) \bigg|_{x=0}
\]

that couples the $\phi_c$ field to the impurity occupation number, where $v$ is the Fermi velocity.

Strictly, the phase factor $\exp[i\phi_c(0)/\sqrt{2}]$ that attaches to $d$ as well as the interaction Eq. (S8) are quite important at high temperatures [S1]. However, it has been shown that $K$ decreases in the RG equation [S2] so that at low temperature $d \exp[iK\phi_c(0)/\sqrt{2}]$ becomes indistinguishable from the bare operator $d$. Meanwhile, the introduced extra interaction Eq. (S8) has the scaling dimension 3/2 (see, for instance, [S3] or [S4], where a similar term has been encountered) and is thus RG irrelevant.

Consequently, if we are only interested in the low temperature physics near the system ground state, we can safely ignore the phase attached to $d$ and $d^\dagger$, and write the Majorana Hamiltonian as in Eq. (5).

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