Difference between Devaney chaos associated with two systems

Bingzhe Hou\textsuperscript{a,2} Xianfeng Ma\textsuperscript{b,c,*,1} Gongfu Liao\textsuperscript{a,2}

\textsuperscript{a}Institute of Mathematics, Jilin University, Changchun 130012, PR China
\textsuperscript{b}Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China
\textsuperscript{c}School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210097, PR China

Abstract

We discuss the relation between Devaney chaos in the base system and Devaney chaos in its induced hyperspace system. We show that the latter need not imply the former. We also argue that this implication is not true even in the strengthened condition. Additionally we give an equivalent condition for the periodically density in the hyperspace system.

1 Introduction

Let $(X, d)$ be a compact metric space with metric $d$ and $f : X \to X$ be continuous. Then $(X, f)$ is called a compact system. For every positive integer $n$, we define $f^n$ inductively by $f^n = f \circ f^{n-1}$, where $f^0$ is the identity map on $X$.

A compact system $(X, f)$ is called Devaney chaos \[1\] if it satisfies the following three conditions:

\* Corresponding author.

\textit{Email addresses: abellengend@163.com} (Bingzhe Hou),
\textit{xianfengma@gmail.com} (Xianfeng Ma), \textit{liaogf@email.jlu.edu.cn} (Gongfu Liao).

\textsuperscript{1} Supported by a grant from Postdoctoral Science Research Program of Jiangsu Province (No. 0701049C) and Specialized Research Fund for Outstanding Young Teachers of East China University of Science and Technology.

\textsuperscript{2} Supported by NSFC (No. 10771084).
(1) $f$ is transitive, i.e., for every pair $U$ and $V$ of non-empty open subsets of $X$ there is a non-negative integer $k$ such that $f^k(U) \cap V \neq \emptyset$;

(2) $f$ is periodically dense, i.e., the set of periodic points of $f$ is dense in $X$;

(3) $f$ is sensitive, i.e., there is a $\delta > 0$ such that, for any $x \in X$ and any neighborhood $V$ of $x$, there is a non-negative integer $n$ such that $d(f^n(x), f^n(y)) > \delta$.

It is worth noting that the conditions (1) and (2) imply that $f$ is sensitive if $X$ is infinite [3,8]. This means the condition (3) is redundant in the above definition.

In the works [6] Román-Flores investigated a certain hyperspace system $(\mathcal{K}(X), \overline{f})$ associated to the base system $(X, f)$, where $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is the nature extension of $f$ and $\mathcal{K}(X)$ is the family of all non-empty compact sets of $(X, d)$ endowed with the Hausdorff metric induced by $d$. He presented a fundamental question: Does the chaoticity of $(X, f)$ (individual chaos) imply that of $(\mathcal{K}(X), \overline{f})$ (collective chaos)? and conversely?

As a partial response to this question, Román-Flores [6] discussed the transitivity of the two systems, and showed that the transitivity of $\overline{f}$ implies that of $f$, but the converse is not true. Fedeli [7] showed that the periodically density of $f$ implies that of $\overline{f}$. Banks gave an example which has a dense set of periodic points in the hyperspace system but has none in the base system [8]. Gongfu Liao showed that there is an example on the interval which is Devaney chaos in the base system while is not in its induced hyperspace system [9].

In this paper, we show that $\overline{f}$ being Devaney chaos need not imply $f$ being Devaney chaos. Further, $f$ need not be Devaney chaos even if $\overline{f}$ is mixing and periodically dense. This answers the question posed by Román-Flores and Banks [8].

2 Preliminaries

$(X, f)$ is a compact system. If $Y \subset X$ and $Y$ is $f$-invariant, i.e. $f(Y) \subset Y$, then $(Y, f|_Y)$ is called the subsystem of $(X, f)$ or $f$, where $f|_Y$ is the restriction of $f$ on $Y$.

A subset $A \subset X$ is minimal if the closure of the orbit of any point $x$ of $A$ is $A$, i.e. $\overline{\text{Orb}(x)} = A$, for all $x \in A$. An equivalent notion is that $A$ has no proper $f$-invariant closed subset.

A point $x \in X$ is said to be an almost periodic point if for any $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that for any integer $q \geq 1$ exists an integer $r$ with property that $q \leq r < N + q$ and $f^r(x) \in B(x, \varepsilon)$, where $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. A
point $x \in X$ is an almost periodic point if and only if $\text{Orb}(x)$ is minimal \cite{5}.

$f$ is weakly mixing if the Cartesian product $f \times f$ is transitive, that is for any non-empty open set $U_1, U_2, V_1$ and $V_2$, there is a positive integer $n$ such that $f^n(U_i) \cap V_i \neq \emptyset$, $i = 1, 2$. $f$ is mixing if for any non-empty open sets $U$ and $V$, there is a positive integer $N$ such that for all $n \geq N$, $f^n(U) \cap V \neq \emptyset$.

Let $S = \{0, 1, \ldots, k - 1\}$, $\Sigma_k = \{x = x_0x_1 \cdots : x_i \in S, i = 0, 1, \ldots \}$. Define $\rho : \Sigma_k \times \Sigma_k \to R$ as follows: for any $x, y \in \Sigma_k$, $x = x_0x_1 \cdots$, $y = y_0y_1 \cdots$,

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

where $m = \min\{n \geq 1|x_n \neq y_n\}$. $\rho$ is a metric over $\Sigma_k$, and $(\Sigma_k, \rho)$ is a compact metric space and is called a symbol space.

Define $\sigma : \Sigma_k \to \Sigma_k$ as $\sigma(x) = x_1x_2 \cdots$, where $x = x_0x_1 \cdots \in \Sigma_k$. $\sigma$ is continuous and is called the shift map on $\Sigma_k$.

$A$ is called a block over $S$ if it is a finite permutation of some elements of $S$. For $A = a_0 \cdots a_{n-1}$, where $a_i \in S$, $0 \leq i \leq n - 1$, the length of $A$ is defined as $n$, i.e. $|A| = n$. Denote $[A] = \{x \in \Sigma_k : x_i = a_i, i = 0, 1, \ldots, n\}$, $[A]$ is called the cylinder set over $\Sigma_k$, it is an open and closed set. Let $B = b_0 \cdots b_{m-1}$ be another block, then the concatenation $C = AB = a_0 \cdots a_{n-1}b_0 \cdots b_{m-1}$ is a new block. $B$ is said to be admissible in $A$ if there exists $i \geq 0$ such that $b_j = a_{i+j}, j = 0, 1, \ldots, m - 1$. For $c \in S$, denote $c^{[n]}$ as a $n$ length permutation of $c$ (For example, $2^{[3]} = 222$), and $c^{[\infty]} = ccc \cdots$ as an infinite permutation of $c$.

**Lemma 1** $b_0b_1 \cdots b_n \cdots$ is an almost periodic point of a symbol space if and only if for any $j > 0$ exists $N > 0$ such that for any $i > 0$ symbol block $b_0b_1 \cdots b_j$ appears in $b_i b_{i+1} \cdots b_{i+N}$.

**Lemma 2** Let $(Y, \sigma)$ be a subsystem of a symbol system $(\Sigma_k, \sigma)$, then $(Y, \sigma)$ is mixing if and only if for any two admissible blocks $b_0b_1 \cdots b_s$ and $c_0c_1 \cdots c_t$ in $Y$, there exists $N > 0$ such that for any $n > N$ exists $n - 1$ length admissible block $w_1w_2 \cdots w_{n-1}$ in $Y$ with the property that $b_0b_1 \cdots b_sw_1w_2 \cdots w_{n-1}c_0c_1 \cdots c_t$ is an admissible block of $Y$.

Let $(X, \mathcal{J})$ be a topological space and $\mathcal{K}(X)$ be the set of all the non-empty compact sets of $X$. $\mathcal{K}(X)$ is called the hyperspace of $(X, \mathcal{J})$ when endowed with the Vietoris topology $\mathcal{J}_V$ whose base consists of sets of the form

$$\mathcal{B}(G_1, G_2, \ldots, G_n) = \{K \in \mathcal{K}(X) : K \subset \bigcup_{i=1}^{n} G_i \text{ and } K \cap G_i \neq \emptyset, 1 \leq i \leq n\},$$

where $G_1, G_2, \ldots, G_n$ are non-empty open subsets of $X$. 

3
(X,d) is a metric space, A ∈ K(X), “ε-dilatation of A” is defined as the set 
N(A,ε) = {x ∈ X : d(x,A) < ε}, where d(x,A) = inf_a∈A d(x,a). The Hausdorff 
separation ρ(A,B) is ρ(A,B) = inf{ε > 0 : A ⊂ N(B,ε)}. The Hausdorff 
metric on K(X) is H_d(A,B) = max{ρ(A,B),ρ(B,A)}. It is well known that 
the topology induced by the Hausdorff metric H_d on K(X) coincides with the 
Vietoris topology J_v[4].

The extension f of f to K(X) is defined as f(A) = {f(a) : a ∈ A}. It is 
easy to show that f being continuous implies f being continuous. If (X,f) 
is compact system, then (K(X),f) is also a compact system [4]. (K(X),f) is 
said to be the hyperspace system induced by the base system (X,f).

The following two theorems can be found in [8,9,10,11]

Theorem 3 Let (K(X),f) be the hyperspace system induced by the compact 
system (X,f), then the following are equivalent.
(i) f is weakly mixing,
(ii) f is weakly mixing,
(iii) f is transitive.

Theorem 4 Let (K(X),f) be the hyperspace system induced by the compact 
system (X,f), then f being mixing is equivalent to f being mixing.

3 Periodically dense

Theorem 5 If (K(X),f) is the hyperspace system induced by the compact 
system (X,d), then K(X) has a dense set of periodic points if and only if for 
any non-empty open subset U of X there exists a compact subset K ⊂ U and 
an integer n > 0 such that f^n(K) = K.

Proof

⇒ Since K(X) has a dense set of periodic points, every non-empty open 
subset of K(X) has at least one periodic point. Further, every basic element 
has at least one periodic point. Let U be an arbitrary open subset of X. Then 
by the definition of the Vietoris topology B(U) is a basic element of K(X). 
According to the condition, there exists K ∈ K(X) and an integer n > 0 such 
that f^n(K) = K. Since f^n = f^n, then f^n(K) = K. Hence, there is a K ⊂ U 
in X such that f^n(K) = K.

⇐ Let U_1,U_2,...,U_n be the open subsets of X, then 
B(U_1,U_2,...,U_n) = {W ∈ K(X) : W ⊂ \bigcup_{i=1}^n U_i \text{ and } W \cap U_i \neq \emptyset, 1 \leq i \leq n}
is a basic element of $\mathcal{K}(X)$. According to the condition, there exists $K_i \subset U_i, m_i > 0$ such that $f^{m_i}(K_i) = K_i, \forall 1 \leq i \leq n$. Let $m$ be the least common multiple of $\{m_i\}$ and $K = \bigcup_{i=1}^{n} K_i$. Then $f^m(K) = K$, and $K \subset \bigcup_{i=1}^{n} U_i, K \cap U_i \neq \emptyset, 1 \leq i \leq n$. Therefore, $K \in \mathcal{B}(U_1, U_2, \ldots, U_n)$.

This means that every element of $\mathcal{K}(X)$ has a $m$-invariant set of $X$ in it. Further, every open set of $\mathcal{K}(X)$ has a periodic point in it. Then $\mathcal{K}(X)$ has a dense set of periodic points.

**Remark 6** If $X$ is a topology space, and $\mathcal{P}(X)$ is the family of all non-empty subset of it, then with the similar argument, it is not hard to see the above result is also hold for the Vietoris topology.

4 Devaney chaos

If $a$ is an almost periodic point of $\Sigma_2$ and $X = \text{orb}(a)$, then $(X, \sigma)$ is usually a mixing non-trivial minimal subsystem of $(\Sigma_2, \sigma)$, which depends on the selection of the point $a$. There are many such examples [12]. From these examples we take an arbitrary one as our research object and this do not affect the proof and our results.

We construct a subsystem $(\tilde{X}, \sigma)$ of $(\Sigma_3, \sigma)$ associated to the selected $(X, \sigma)$ as follows.

For $b = b_0b_1 \cdots b_n \cdots \in X$, denote $X_b = \{x = x_0x_1 \cdots \in \Sigma_3 : \exists n_i \to \infty \text{ such that } x_{n_i} = b_i, \text{ and } x_n = 2 \text{ if } n \neq n_i\}$, then $X_b$ is called the extension of $b$ in $\Sigma_3$. Denote $\tilde{X} = \bigcup_{b \in X} X_{b}$. It is easy to see that $\tilde{X}$ is $\sigma$-invariant set of $\Sigma_3$. Therefore, $(\tilde{X}, \sigma)$ is a subsystem of $(\Sigma_3, \sigma)$.

**Proposition 7** $(\tilde{X}, \sigma)$ is mixing.

**Proof** Let $U = [x_0x_1 \cdots x_{n_1}]$ and $V = [y_0y_1 \cdots y_{n_2}]$ be the cylinder sets of $\tilde{X}$. They are two basic elements of $\tilde{X}$ and both are closed sets.

By the construction of $\tilde{X}$, there exists two integer sequences $\{i_l\}, \{j_m\}, i_l, j_m \geq -1, l, m \geq -1$ such that $x_{i_l} = b_l, y_{j_m} = c_m$, and $U_1 = [b_0b_1 \cdots b_l \cdots b_s], V_1 = [c_0c_1 \cdots c_m \cdots b_t], s \leq n_1, t \leq n_2$, are cylinder sets of $X$ (We assign $U = [2^{n_1}]$ for $i_l = -1$, and $V = [2^{n_2}]$ for $j_m = -1$). So $u = b_0b_1 \cdots b_l \cdots b_s, v = c_0c_1 \cdots c_m \cdots b_t$ are the admissible blocks of $X$.

For $i_l \geq 0, j_m \geq 0, \forall l, m$, from Lemma [2] there exists $N \in \mathbb{N}$ such that for any $n \geq N$ there exists an admissible block $w = d_1d_2 \cdots d_{n-1}$ of $X$ with the
property that \( uwv \) is an admissible block of \( X \). Since \( U, V \) can be written as 
\[
U = [x_0x_1 \cdots x_i 2^{[r]}], \quad r = n_1 - i_s \\
V = [2^{[j_0]} y_{j_0} y_{j_0+1} \cdots y_{n_2}],
\]
according to the construction of \( \bar{X} \), obviously,
\[
x_0x_1 \cdots x_i 2^{[r]} w^{[j_0]} y_{j_0} y_{j_0+1} \cdots y_{n_2}
\]
is an admissible block of \( \bar{X} \). Then
\[
[x_0x_1 \cdots x_i 2^{[r]} w^{[j_0]} y_{j_0} y_{j_0+1} \cdots y_{n_2}] \subset \bar{X}
\]
is the basic element of \( \bar{X} \). This means that \( \sigma^n(U) \cap V \neq \emptyset \), \( \forall n \geq N \).

For \( i_l \) or \( j_m \) is \(-1\), we just need to add enough symbols \( 2 \) in the admissible blocks \( U \) and \( V \).

Therefore, \( (\bar{X}, \sigma) \) is mixing.

Corollary 8 \( (\mathcal{K}(\bar{X}), \sigma) \) is mixing.

This is a direct result of Proposition 7 and Theorem 4.

Proposition 9 \( (\bar{X}, \sigma) \) is not minimal and has only one periodic point. The periodic point is the unique fixed point of \( \bar{X} \).

Proof From the construction, it is easy to see that \( X \subset \bar{X} \) and \( X \) is minimal proper set of \( \Sigma_3 \). By the definition of minimal, \( \bar{X} \) is not minimal. We now prove \( \bar{X} \) has only one fixed point \( 2^{[\infty]} \). It is the unique periodic point of \( \bar{X} \).

Suppose that there exists \( x \in \bar{X} \) and \( n > 0 \) such that \( \sigma^n(x) = x \), then \( x \) can be written as an infinite permutation of block \( x_0x_1 \cdots x_{n-1} \),
\[
x = x_0x_1 \cdots x_{n-1} \cdots x_0x_1 \cdots x_{n-1} \cdots \n
\]
If \( x_i = 2 \) for all \( i, 0 \leq i \leq n - 1 \), then \( x = 2^{[\infty]} \). Obviously, \( x \) is a fixed point of \( \bar{X} \).

If there exists a finite integer sequence \( \{i_l\} \), \( 0 \leq i_l \leq n - 1 \), \( 0 \leq l \leq m \), \( 0 \leq m \leq n - 1 \) such that \( x_{i_l} \neq 2 \), then \( x_{i_0}x_{i_1} \cdots x_{i_m} \) is an admissible block of \( X \), we can write it for convenience as \( a_0a_1 \cdots a_m \). Accordingly, \( a = a_0a_1 \cdots a_m \cdots a_0a_1 \cdots a_m \cdots \) is an infinite permutation of \( a_0a_1 \cdots a_m \), then \( a \) is a \((m + 1)\)-periodic point.

Since \( X \) is a non-trivial minimal set, so \( a \not\in X \), and then \( x \not\in X_a \). Further, it is not hard to see that \( x \not\in \bar{X} \). This is a contradiction.
Remark 10 The result implies that $\tilde{X}$ is not periodically dense.

Proposition 11 The hyperspace system $(\mathcal{K}(\tilde{X}), \sigma)$ induced by $(\tilde{X}, \sigma)$ is periodically dense.

Proof According to Theorem 5, we just need to prove that every open set $U$ of $X$ has a $m$-invariant compact subset $K$, i.e. $\sigma^m(K) = K$. Further, it is enough to prove that every basic element of $X$ has a $m$-invariant compact subset.

Let $U = [a_0a_1 \ldots a_{n-1}]$ be a cylinder set of $\tilde{X}$.

If $x_i = 2$ for all $i$, $0 \leq i \leq n - 1$, then the single set $\{2^{[\infty]}\}$ is a $\sigma$-invariant closed set of $U$. It is a 1-invariant set of $\sigma$ and satisfies the condition.

Suppose that $a_0a_1 \ldots a_{n-1}$ have symbols which are not 2 and the number of which is $j$, $1 \leq j \leq n$. On the basis of the definition of $\tilde{X}$, there exists a point $b = b_0b_1 \ldots$ in $X$ satisfied: the first $j$ symbols in order are just the symbols of $a_0a_1 \ldots a_{n-1}$ which are not 2.

Since $b$ is an almost periodic point, there exists $N > 0$ and $n_k \to \infty$ such that $j \leq n_{k+1} - n_k \leq N + j$ and the first $j$ symbols of $\sigma^{n_k}(b)$ and $b$ are completely same.

Let $m = n + N$ and define $\bar{b} = \bar{b}_0\bar{b}_1 \ldots \in \Sigma_3$ as follows. For any $s \geq 0$,

$$\bar{b}_s = \begin{cases} a_i, & \text{if } s = km + i, \text{ where } k \geq 0, \ 0 \leq i < n, \\ b_{n_k+j+i}, & \text{if } s = km + n + i, \text{ where } k \geq 0, \ 0 \leq i < n_{k+1} - n_k - j, \\ 2, & \text{other.} \end{cases}$$

It is not hard to see that $\bar{b} \in X_b \subset \tilde{X}$.

On the other hand, obviously, for any $k \geq 0$, $\sigma^{km}(\bar{b}) \in U$.

Let $K$ be the $\omega$-limit set of the point $\bar{b}$ under the action of $\sigma^m$, i.e. $K = \omega(\bar{b}, \sigma^m)$, then $K \subset U$ is a closed set and $\sigma^m(K) = K$.

The proof of Proposition 11 is completed.

Theorem 12 The hyperspace system being periodically dense need not imply the base system being periodically dense.

Proof It is a direct result of Proposition 11 and Remark 10.

Corollary 13 The hyperspace system being mixing and periodically dense need not imply the base system being periodically dense.
Proof We can get the result from Proposition\textsuperscript{[7, 9, 11]} and Corollary\textsuperscript{8}.

**Theorem 14** The hyperspace system being Devaney chaos need not imply the base system being Devaney chaos.

Proof Since mixing implies weakly mixing, and weakly mixing is equivalent to the transitivity by Theorem\textsuperscript{3} then the hyperspace system being mixing and periodically dense implies it being Devaney chaos. However, from Theorem\textsuperscript{12} the base system need not be periodically dense, then it need not be Devaney chaos.

5 Conclusion

In summary, we showed some difference between the chaoticity of a compact system and the chaoticity of its induced hyperspace system. Devaney chaos in the hyperspace system need not implies Devaney chaos in its base system. Even if the hyperspace system is mixing, the former need not implies the latter. This kind of investigation should be useful in many real problems, such as in ecological modeling, demographic sciences associated to migration phenomena and numerical simulation, etc.

References

[1] Devaney RL. An introduction to chaotic dynamical systems. Redwood City: Addison-Wesley; 1989.

[2] Banks J, Brooks J, Cairns G, Davis G, Stacey P. On Devaney’s definition of chaos. Amer Math Monthly 1992;99:332-4.

[3] Silverman S. On maps with dense orbits and the definition of chaos. Rocky Mountain J Math 1992;22:353-75.

[4] Klein E., Thompson A. C. Theory of Correspondences: Including Applications to Mathematical Economics. New York: John Wiley and Sons, 1984.

[5] Zhou Z. Symbolic Dynamics. Shanghai: Shanghai Scientific and Technological Education Publishing House, 1997, pp. 17-18

[6] Román-Flores H. A note on transitivity in set-valued discrete systems. Chaos Solitons and Fractals 2003;17:99-104.

[7] Fedeli A. On chaotic set-valued discrete dynamical systems. Chaos Solitons and Fractals 2005;23:1381-4.
[8] Banks J. Chaos for induced hyperspace maps. Chaos Solitons and Fractals 2005;25:681-5.

[9] Gongfu L, Lidong W, Yucheng Zh. Transitivity, mixing and chaos for a class of set-valued mappings. Science in China(Series A), 2006;49:1-8.

[10] Peris A. Set-valued chaos. Chaos Solitons and Fractals 2005;26:19-23.

[11] Rongbao G, Wenjing G. On mixing property in set-valued discrete systems. Chaos Solitons and Fractals 2006;28:747-54.

[12] Liao G, Fan Q. Minimal subshifts which display Schweizer-Šmítal chaos and have zero topological entropy. Science in China (Series A) 1998;41:33-8.