The Namioka property of $KC$-functions and Kempisty spaces

V.V. Mykhaylyuk

Chernivtsi National University, Department of Mathematical Analysis,
Kotsjubyn's’koho 2, Chernivtsi 58012, Ukraine

Abstract

A topological space $Y$ is called a Kempisty space if for any Baire space $X$ every function $f : X \times Y \to \mathbb{R}$, which is quasi-continuous in the first variable and continuous in the second variable has the Namioka property. Properties of compact Kempisty spaces are studied in this paper. In particular, it is shown that any Valdivia compact is a Kempisty space and the cartesian product of an arbitrary family of compact Kempisty spaces is a Kempisty space.

Key words: quasi-continuity, Namioka property, Kempisty space.
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1 Introduction

The notion of quasi-continuous mapping, which was introduced by Kempisty in [5], occupies an important place in investigations of the continuity point set of separately continuous mappings of several variables. Using it, the inductive pass on the quantity of variables can be reduced to the study of the continuity point set of two variables mappings which are quasi-continuous in one variable and continuous in the other variable.

Let $X$, $Y$ be topological spaces. A function $f : X \times Y \to \mathbb{R}$ is called a $KC$-function if it is quasi-continuous in the first variable and continuous in the second variable. The collection of all $KC$-functions $f : X \times Y \to \mathbb{R}$ we denote by $KC(X \times Y)$. A function $f : X \times Y \to \mathbb{R}$ is called a $\overline{KC}$-function ($\overline{KC}$-function) if it is continuous in the second variable and there exists a dense set (a dense $G_\delta$-set) $B \subseteq Y$ such that all functions $f_b : X \to \mathbb{R}$, $f_b(x) = f(x, b)$, where $b \in B$, are quasi-continuous on $X$. The collection of all $\overline{KC}$-functions ($\overline{KC}$-functions) on $X \times Y$ we denote by $\overline{KC}(X \times Y)$ ($\overline{KC}(X \times Y)$).
A mapping $f : X \times Y \to Z$ has the Namioka property if there exists a dense in $X \ G_{δ}$-set $A \subseteq X$ such that $f$ is jointly continuous at each point of $A \times Y$.

A topological space $Y$ is called a Kempisty space if for any Baire space $X$ every function $f \in KC(X \times Y)$ has the Namioka property. This notion was introduced by V. Maslyuchenko in $[9]$.

Spaces $Y$ such that for any Baire space $X$ every function $f \in \overline{KC}(X \times Y)$ has the Namioka property were studied in $[3,10,12]$. The most general result in this direction was obtained in $[8]$. It gives a possibility to replace the quasi-continuity by the continuity for those $Y$ all dense subsets of which are separable. The conditions of countability type (separability, second countability) on $Y$ are present in all these results.

The Namioka property of $KC$-functions was studied in $[6,7]$ in the case when $X$ is an arbitrary $α$-favorable space. In particular, it is shown that any $KC$-function on the product of a $α$-favorable space and a Valdivia compact has the Namioka property.

On the other hand a compact space $Y$ is called a co-Namioka space if for any Baire space $X$ every separately continuous function $f : X \times Y \to \mathbb{R}$ has the Namioka property. The name of these spaces is connected to the classical paper $[13]$ and was introduced by G. Debs in $[4]$. The class of co-Namioka compact spaces was intensively studied last time. In particular, it is shown in $[11]$ that any Valdivia compact and the product of an arbitrary family of co-Namioka spaces are co-Namioka spaces.

Thus the following questions arise naturally: is any Eberlein, Corson or Valdivia compact a Kempisty space and is the product of an arbitrary family of compact Kempisty spaces a Kempisty space?

First, using a dependence of a certain number of coordinates technique, we obtain a result which implies that a Valdivia compact is a Kempisty space. Thus, an investigation of the product of an arbitrary collection of compact Kempisty spaces we reduce to the case of two factors (these results are announced in $[11]$). Then, developing an approach to a study of separately continuous mappings from $[12]$ we show that: $(i)$ the product of two compact Kempisty spaces is a Kempisty space; $(ii)$ in the definition of a compact Kempisty space one can consider $KC$-functions which take values in metrizable spaces instead of real-valued functions, as it was done for co-Namioka spaces in $[1]$. 

2
2 Definitions and an auxiliary assertion

Let $X, Y$ be topological spaces. A mapping $f : X \to Y$ is called quasi-continuous at $x_0 \in X$ if for any neighborhood $U$ of $x_0$ in $X$ and any neighborhood $V$ of $y_0 = f(x_0)$ in $Y$ there exists a nonempty open in $X$ set $G \subseteq U$ such that $f(G) \subseteq V$. A mapping $f : X \to Y$ is called quasi-continuous if it is quasi-continuous at any point $x \in X$. It is easy to verify that a mapping $f : X \to Y$ is quasi-continuous if and only if $f(G) \subseteq f(A)$ for any nonempty open in $X$ set $G$ and any dense in $G$ set $A \subseteq X$, where $\overline{B}$ denotes the closure of $B$.

Let $X$ be a topological space, $Y$ be a metric space with a metric $d$, $f : X \to Y$ be a mapping, $A \subseteq X$ and $B \subseteq Y$ be nonempty sets. The oscillation $\sup_{x',x'' \in A} d(f(x'), f(x''))$ of $f$ on $A$ we denote by $\omega_f(A)$, the oscillation $\inf_{U \in \mathcal{U}} \omega_f(U)$ of $f$ at $x \in X$ we denote by $\omega_f(x)$, where $\mathcal{U}$ is the system of all neighborhoods of $x$ in $X$, and the diameter $\sup_{y',y'' \in B} d(y', y'')$ of $B$ we denote by $\text{diam} B$.

For a compact space $K$ by $C(K)$ we mean the Banach space of all continuous functions $x : K \to \mathbb{R}$ with the norm $\|x\| = \sup_{t \in K} |x(t)|$.

For a topological space $X = [0, 1]^T$ the space $\Sigma = \{x \in X : |\text{supp}x| \leq \aleph_0\}$, where $\text{supp}x = \{t \in T : x(t) \neq 0\}$, is called the $\Sigma$-subspace of $X$. A compact space $Y$ is called a Corson compact if it is embeded homeomorphically into the $\Sigma$-subspace of some $[0, 1]^T$ and a Valdivia compact if $Y$ is included homeomorphically into a $[0, 1]^T$ such that the $\Sigma$-subspace of $[0, 1]^T$ is dense in the image of $Y$. It is clear that any Corson compact is a Valdivia compact.

**Proposition 2.1** Let $T$ be a set, $(X_t : t \in T)$ be a family of compact spaces, $X \subseteq \prod_{t \in T} X_t$, be a compact space, $f : X \to \mathbb{R}$ be a continuous mapping and $\varepsilon > 0$. Then there exists a finite set $T_0 \subseteq T$ such that $|f(x') - f(x'')| < \varepsilon$ for any $x', x'' \in X$ with $x'|_{T_0} = x''|_{T_0}$.

**Proof.** Using the compactness of $X$ we pick a finite cover $(U_k : 1 \leq k \leq n)$ of $X$ by open basic sets $U_k$ such that $|f(x') - f(x'')| < \varepsilon$ if $x', x'' \in U_k$ for some $k \leq n$. For each $k \leq n$ there exists a finite set $T_k \subseteq T$ such that $x'' \in U_k$ if $x'' \in X$ and $x' \in U_k$ with $x'|_{T_k} = x''|_{T_k}$. It is easy to verify that the set $T_0 = \bigcup_{k=1}^n T_k$ is to be found. ∎
3 The Namioka property of KC-functions

Theorem 3.1 Let $X$ be a Baire space, $Y \subseteq [0,1]^T$ be a compact, $\Sigma$ be the $\Sigma$-subspace of a space $[0,1]^T$, $f : X \times Y \to \mathbb{R}$ be a continuous in the second variable mapping and $B \subseteq Y \cap \Sigma$ be a set with $Y = \overline{B}$ and the function $f_b : X \to \mathbb{R}$, $f_b(x) = f(x, b)$, is quasi-continuous for any $b \in B$. Then there exists a dense in $X$ $G_\delta$-set $A \subseteq X$ such that $f$ is jointly continuous at any point of $A \times Y$.

Proof. For any $\varepsilon > 0$ we put $A_\varepsilon = \{x \in X : \omega_f(x, y) \geq \varepsilon$ for some $y \in Y\}$. Now we prove that all sets $A_\varepsilon$ are nowhere dense in $X$.

Suppose that $\varepsilon > 0$ and let $U_0$ be an open in $X$ nonempty set with $U_0 \subseteq \overline{T}_\varepsilon$.

Lemma 3.2 For any open nonempty set $U \subseteq U_0$ and any set $S \subseteq T$ with $|S| \leq N_0$ there exist an open nonempty set $W \subseteq U$ and points $y_1, y_2 \in \Sigma \cap Y$ such that $y_1|_S = y_2|_S$ and $|f(x, y_1) - f(x, y_2)| \geq \frac{\varepsilon}{8}$ for each $x \in W$.

Proof of Lemma 3.2. Assume that it is false. Then there exist an open nonempty set $U \subseteq U_0$ and a set $S \subseteq T$ with $|S| \leq N_0$ such that for any $y_1, y_2 \in \Sigma \cap Y$ with $y_1|_S = y_2|_S$ the set $A(y_1, y_2) = \{x \in U : |f(x, y_1) - f(x, y_2)| < \frac{\varepsilon}{8}\}$ is dense in $U$.

Consider the continuous mapping $\varphi : Y \to [0,1]^S$, $\varphi(y) = y|_S$. It is clear that $\varphi(Y)$ is a metrizable compact. Since $B$ is dense in $Y$ then $\varphi(B)$ is dense in $\varphi(Y)$. Pick a set $\{b_1, b_2, \ldots, b_n, \ldots\} \subseteq B$ such that $\varphi(Y) = \{\varphi(b_n) : n \in \mathbb{N}\}$. Put $\tilde{Y} = \{b_n : n \in \mathbb{N}\}$, $\tilde{f} = f|_{X \times \tilde{Y}}$. Note that $\tilde{Y}$ is a metrizable compact, $\tilde{Y} \subseteq \Sigma$ and $\varphi(\tilde{Y}) = \varphi(Y)$. The mapping $\tilde{f}$ belongs to $\overline{KC}(X, \tilde{Y})$ and has the Namioka property by a theorem from [3]. Therefore there exists an open nonempty set $\tilde{U} \subseteq U$ such that $\text{diam}(\tilde{f}_b(\tilde{U})) < \frac{\varepsilon}{8}$ for each $\tilde{y} \in \tilde{Y}$ where $\tilde{f}_b : X \to \mathbb{R}$, $\tilde{f}_b(x) = \tilde{f}(x, \tilde{y})$.

We show that $\text{diam}(f_b(\tilde{U})) \leq \frac{3\varepsilon}{8}$ for each $b \in B$. Fix $b \in B$ and $\tilde{y} \in \tilde{Y}$ so that $\varphi(b) = \varphi(\tilde{y})$. Put $\tilde{A} = A(b, \tilde{y}) \cap \tilde{U}$. Then

$$|f(a', b) - f(a'', b)| \leq |f(a', b) - f(a', \tilde{y})| + |f(a', \tilde{y}) - f(a'', \tilde{y})| + |f(a'', \tilde{y}) - f(a'', b)| < \frac{\varepsilon}{8} + \omega f_b(\tilde{U}) + \frac{\varepsilon}{8} < \frac{3\varepsilon}{8}$$

for any $a', a'' \in \tilde{A}$. Thus $\text{diam}(f_b(\tilde{A})) \leq \frac{3\varepsilon}{8}$. By the assumption, $\tilde{A}$ is dense in $\tilde{U}$. Since $f_b$ is quasi-continuous then $f_b(\tilde{U}) \subseteq \tilde{f}_b(\tilde{A})$ and $\text{diam}(f_b(\tilde{U})) = \text{diam}(f_b(\tilde{A})) \leq \frac{3\varepsilon}{8}$ for any $b \in B$.

We prove that $\omega_f(x, y) < \varepsilon$ for any $x \in \tilde{U}$ and $y \in Y$. Fix $x_0 \in \tilde{U}$ and $y_0 \in Y$. Using the continuity of $f^{x_0}$, choose a neighborhood $V_0$ of $y_0$ in $Y$ such that
\(\omega_{f^0}(V_0) < \frac{\varepsilon}{10}\). Pick any points \((x_1, y_1), (x_2, y_2) \in \bar{U} \times V_0\). Since \(f\) is continuous in variable \(y\) and \(\bar{B} = Y\) then there exist points \(b_1, b_2 \in B \cap V_0\) such that 
\[
|f(x_1, y_1) - f(x_1, b_1)| < \frac{\varepsilon}{10} \text{ and } |f(x_2, y_2) - f(x_2, b_2)| < \frac{\varepsilon}{10}.
\]
Then 
\[
|f(x_1, y_1) - f(x_2, y_2)| \leq |f(x_1, y_1) - f(x_1, b_1)| + |f(x_1, b_1) - f(x_0, b_1)| + 
\]
\[
+ |f(x_0, b_1) - f(x_0, b_2)| + |f(x_0, b_2) - f(x_2, b_2)| + |f(x_2, b_2) - f(x_2, y_2)| < 
\]
\[
\frac{\varepsilon}{16} + \text{diam}(f_{b_1}(\bar{U})) + \omega_{f^0}(V_0) + \text{diam}(f_{b_2}(\bar{U})) + \frac{\varepsilon}{16} 
\]
\[
\leq \frac{\varepsilon}{16} + \frac{3\varepsilon}{8} + \frac{\varepsilon}{16} + \frac{3\varepsilon}{8} + \frac{\varepsilon}{16} = \frac{15\varepsilon}{16}.
\]
Therefore \(\omega_f(\bar{U} \times V_0) \leq \frac{15\varepsilon}{16}\) and \(\omega_f(x_0, y_0) < \varepsilon\). Thus \(A_\varepsilon \cap \bar{U} = \emptyset\) but it contradicts \(U_0 \subseteq \bar{U}_0\). Hence, Lemma 3.2 is proved.

For an arbitrary nonempty open set \(U \subseteq U_0\) and set \(S \subseteq T\) with \(|S| \leq \aleph_0\) pick a nonempty open set \(\tau(U, S)\) and points \(u(U, S), v(U, S) \in \Sigma \cap \bar{Y}\) which satisfy the conditions of Lemma 3.2.

Describe an strategy \(\sigma\) of the second player \(\beta\) in the topological Choquet game on the Baire space \(X\). The nonempty open set \(U_0\) is the first move of the player \(\beta\). Fix any nonempty set \(S_0 \subseteq S\) with \(|S_0| \leq \aleph_0\) and for every nonempty open in \(X\) set \(V_1 \subseteq U_0\) put \(U_1 = \sigma(U_0, V_1) = \tau(V_1, S_0)\). Choose any nonempty open in \(X\) set \(V_2 \subseteq U_1\). Since the points \(u_1 = u(V_1, S_0)\) and \(v_1 = v(V_1, S_0)\) belong to the set \(\Sigma\) then \(|T_1| \leq \aleph_0\) where \(T_1 = (\text{supp } u_1) \cup (\text{supp } v_1)\). Denote \(S_1 = S_0 \cup T_1\) and put \(U_2 = \sigma(U_0, V_1, U_1, V_2) = \tau(V_2, S_1)\).

For any open in \(X\) nonempty sets \(V_1, V_2, \ldots, V_{n-1}\) such that \(U_0 \supseteq V_1 \supseteq U_1 \supseteq \cdots \supseteq U_{n-1}\) where \(U_k = \sigma(U_0, V_1, \ldots, U_{k-1}, V_k) = \tau(V_k, S_{k-1}), u_k = u(V_k, S_{k-1}), v_k = v(V_k, S_{k-1}), T_k = (\text{supp } u_k) \cup (\text{supp } v_k), S_k = S_{k-1} \cup T_k\) by \(k = 1, 2, \ldots, n - 1\) and an arbitrary nonempty open in \(X\) set \(V_n \subseteq U_{n-1}\) we put \(U_n = \sigma(U_0, V_1, \ldots, U_{n-1}, V_n) = \tau(V_n, S_{n-1}), u_n = u(V_n, S_{n-1}), v_n = v(V_n, S_{n-1}), T_n = (\text{supp } u_n) \cup (\text{supp } v_n), S_n = S_{n-1} \cup T_n\).

Since \(X\) is a Baire space then the strategy \(\sigma\) is not winning for the player \(\beta\). Therefore there exists a sequence of nonempty open sets \(U_0 \supseteq V_1 \supseteq U_1 \supseteq \cdots \supseteq U_{n-1} \supseteq V_n \supseteq \cdots\) where \(U_n = \sigma(U_0, V_1, \ldots, U_{n-1}, V_n)\) such that \(\bigcap_{n=0}^{\infty} U_n \neq \emptyset\).

Choose a point \(x_0 \in \bigcap_{n=0}^{\infty} U_n\). The function \(f^{x_0}\) is continuous on the compact \(Y \subseteq [0, 1]^T\). Therefore by Proposition 2.1, there exists a finite set \(T_0 \subseteq T\) such that 
\[
|f(x_0, u) - f(x_0, v)| < \frac{\varepsilon}{8}\]
for any \(u, v \in Y\) with \(u|_{T_0} = v|_{T_0}\). Since \(x_0 \in U_n = \tau(V_n, S_{n-1}), u_n = u(V_n, S_{n-1}), v_n = v(V_n, S_{n-1})\) then 
\[
|f(x_0, u_n) - f(x_0, v_n)| \geq \frac{\varepsilon}{8}.
\]
Therefore \(u_n|_{T_0} \neq v_n|_{T_0}\) that is for any \(n \in \mathbb{N}\) there exists a point \(t_n \in T_0\) such that \(u_n(t_n) \neq v_n(t_n)\). Note that \(t_n \in (\text{supp } u_n) \cup (\text{supp } v_n) = T_n\), thus \(t_n \in S_{m-1}\) for all \(m > n\). Since \(u_m|_{S_{m-1}} = v_m|_{S_{m-1}}\) then \(u_m(t_n) = v_m(t_n)\) for all \(m > n\). Hence \(t_n \neq t_m\) for all \(n \neq m\) and all points \(t_k\) are distinct but this contradicts the finiteness of the set \(T_0\).
Corollary 3.3 Let $X$ be a Baire space, $Y$ be a Corson compact and $f \in KC(X \times Y)$. Then $f$ has the Namioka property.

Corollary 3.4 Let $X$ be a Baire space, $Y$ be a Valdivia compact and $f \in KC(X \times Y)$. Then $f$ has the Namioka property.

Proof. Since $Y$ is a Valdivia compact then without loss of the generality we can assume that $Y \subseteq [0, 1]^I \cap \Sigma$ where $\Sigma$ is the $\Sigma$-subspace of $[0, 1]^I$. Choose a dense in $Y$ $G_\delta$-set $B_1 \subseteq Y$ such that all functions $f_b : X \to \mathbb{R}$, $b \in B_1$, are quasi-continuous. Put $B = B_1 \cap \Sigma$. Since the space $B_2 = Y \cap \Sigma$ is countably compact then $B_2$ is a dense Baire subspace of the Baire space $Y$. Therefore the set $B = B_1 \cap B_2$ is dense in $Y$ and by Theorem 3.1, $f$ has the Namioka property. $\diamond$

Corollary 3.5 Any Valdivia compact is a Kempisty space.

4 The products of Kempisty spaces

The following result reduces a study of the products to the case of two factors.

Theorem 4.1 Let $(Y_i : i \in I)$ be a family of compact spaces $Y_i$ such that for any finite set $I_0 \subseteq I$ the product $\prod_{i \in I_0} Y_i$ is a Kempisty space. Then the product $Y = \prod_{i \in I} Y_i$ is a Kempisty space.

Proof. Consider a Baire space $X$ and $f \in KC(X \times Y)$. We shall reason similarly as in the proof of Theorem 3.1. Assume there exist $\varepsilon > 0$ and nonempty open set $U_0$ in $X$ such that $U_0 \subseteq A_\varepsilon$ where $A_\varepsilon = \{x \in X : (\exists y \in Y)(\omega_f(x, y) \geq \varepsilon)\}$. Fix $a \in Y$ and put $\Sigma = \{y \in Y : |\{t \in T : a(t) \neq y(t)\}| < \aleph_0\}$ (this set will play the same role as the $\Sigma$-subspace in the proof of Theorem 3.1).

Lemma 4.2 For any open nonempty set $U \subseteq U_0$ and a finite set $S \subseteq T$ there exist an open nonempty set $W \subseteq U$ and points $y_1, y_2 \in \Sigma$ such that $y_1|_S = y_2|_S$ and $|f(x, y_1) - f(x, y_2)| \geq \frac{\varepsilon}{8}$ for every $x \in W$.

Proof of Lemma 4.2. Suppose to the contrary that there exist an open nonempty set $U \subseteq U_0$ and a finite set $S \subseteq T$ such that for any $y_1, y_2 \in \Sigma$ with $y_1|_S = y_2|_S$ the set $A(y_1, y_2) = \{x \in U : |f(x, y_1) - f(x, y_2)| < \frac{\varepsilon}{8}\}$ is dense in $U$. Denote $\tilde{Y} = \prod_{t \in S} Y_t \times \prod_{t \in T \setminus S} \{a(t)\}$ and $\tilde{f} = f|_{X \times \tilde{Y}}$. It is clear that $\tilde{Y}$ is homeomorphic to $\prod_{t \in S} Y_t$, thus it is a Kempisty space and $\tilde{f} \in KC(X \times \tilde{Y})$. Hence $\tilde{f}$ has the Namioka property and there exists a nonempty open set $\tilde{U} \subseteq U$ such that $\text{diam}(\tilde{f}_g(\tilde{U})) < \frac{\varepsilon}{8}$ for every $\tilde{y} \in \tilde{Y}$. Then we obtain (analogously
as in the proof of Theorem 3.1) that \( \text{diam}(f_y(\bar{U})) < \frac{3\varepsilon}{8} \) for every \( y \in \Sigma \). Now using the continuity of \( f \) in the second variable and the density of \( \Sigma \) in \( Y \) we obtain that \( \omega_f(x, y) < \varepsilon \) for every \( x \in \bar{U} \) and \( y \in Y \), which is impossible and this completes the proof of the Lemma 4.2. \( \Diamond \)

For any finite set \( S \subseteq T \) and a nonempty open set \( U \subseteq U_0 \) a nonempty open in \( U \) set \( W \) and points \( y_1, y_2 \in \Sigma \) which satisfy the conditions of Lemma 4.2 we denote by \( \tau(U, S), u(U, S) \) and \( v(U, S) \) respectively. We construct (similarly as in the proof of Theorem 3.1) a winning strategy for \( \beta \) in the Choquet game on the Baire space \( X \) (here finite sets \( T_k \) are defined by \( T_k = \{ t \in T : a(t) \neq u_k(t) \text{ or } a(t) \neq v_k(t) \} \)). It is impossible because \( X \) is a Baire space. \( \Diamond \)

Now we pass to the study of the products of two compact Kempisty spaces. The following proposition plays a central role in these investigations. It constitutes a certain development of a somewhat stronger property of separately continuous functions which has been used in \cite{2} at similar investigations of co-Namioka spaces.

Note that for any \( \delta > 0 \) and a metric compact \( K \) with a metric \( d \) there exists an integer \( m \in \mathbb{N} \) such that every set \( T \subseteq K \) with \( |T| \geq m \) has two distinct elements \( t_1, t_2 \in T \) for which \( d(t_1, t_2) \leq \delta \) (it is sufficient to consider a finite \( \frac{\delta}{2} \)-net in \( K \)). The least of such numbers \( m \) we shall call the \( \delta \)-size of \( K \).

**Proposition 4.3** Let \( X \) be a Baire space, \( Y, Z \) be compact spaces, \( P = Y \times Z \), \( f : X \times P \to \mathbb{R} \) be a function which is continuous in the second variable. Let for any \( y \in Y \) the function \( f_y : X \times Z \to \mathbb{R}, f_y(x, z) = f(x, y, z), \) has the Namioka property. Then for any \( \varepsilon > 0 \) and open in \( X \) nonempty set \( U \) there exist functions \( b_1, b_2, \ldots, b_n \in C(Z) \) and an open in \( X \) nonempty set \( U_0 \subseteq U \) such that for every \( y \in Y \) there exists a dense in \( U_0 \) set \( A \) such that for an arbitrary \( x \in A \) there is a number \( k \in \{ 1, 2, \ldots, n \} \) with \( |f(x, y, z) - b_k(z)| \leq \varepsilon \) for each \( z \in Z \).

**Proof.** For each \( x \in X \) denote by \( \varphi_x \) the continuous mapping \( \varphi_x : Y \to C(Z), \varphi_x(y)(z) = f(x, y, z) \). It is clear that \( \varphi_x(Y) \) is a compact in \( C(Z) \).

Fix \( \varepsilon > 0 \) and an open in \( X \) nonempty set \( U \). For every \( k \in \mathbb{N} \) denote by \( A_k \) the set of all \( x \in U \) such that \( \frac{\varepsilon}{2} \)-size of \( \varphi_x(Y) \) is not greater than \( k \). Since \( \bigcup_{k=1}^\infty A_k = U \) and \( X \) is a Baire space then there exist an open in \( X \) nonempty set \( U' \subseteq U \) and a number \( n \) such that \( A_n \) is dense in \( U' \) that is \( U' \subseteq \overline{A_n} \).

Suppose that for a fixed \( \varepsilon \) and an \( U \) the conclusion of Proposition 4.3 is false. In particular, for any open in \( X \) nonempty set \( U'' \subseteq U' \) and a finite set \( B \subseteq C(Z) \) there exists an \( y \in Y \) such that the set \( \bigcup_{b \in B} \{ x \in U'' : \|\varphi_x(y) - b\| \leq \varepsilon \} \) is not dense in \( U'' \).
We pick an arbitrary point \( y_1 \in Y \). Since \( f_{y_1} \) has the Namioka property then there exists an open in \( X \) nonempty set \( U_1 \subseteq U' \) such that \( \| \varphi_{x'}(y_1) - \varphi_{x''}(y_1) \| \leq \frac{\varepsilon}{2} \) for any \( x', x'' \in U_1 \). Fix \( x_1 \in U_1 \) and put \( b_1 = \varphi_{x_1}(y_1) \). By the assumption there exists \( y_2 \in Y \) such that the set \( \{ x \in U_1 : \| \varphi_x(y_2) - b_1 \| \leq \varepsilon \} \) is not dense in \( U_1 \). Therefore there exists an open in \( X \) nonempty set \( U'_1 \subseteq U_1 \) such that \( \| \varphi_x(y_2) - b_1 \| > \varepsilon \) for each \( x \in U'_1 \). Using the Namioka property of \( f_{y_2} \) we find an open in \( X \) nonempty set \( U_2 \subseteq U'_1 \) such that \( \| \varphi_{x'}(y_2) - \varphi_{x''}(y_2) \| \leq \frac{\varepsilon}{2} \) for any \( x', x'' \in U_2 \).

Given \( x_2 \in U_2 \) we put \( b_2 = \varphi_{x_2}(y_2) \). Again using the assumption we find a point \( y_3 \in Y \) and an open in \( X \) nonempty set \( U'_2 \subseteq U_2 \) such that \( \| \varphi_{x}(y_3) - b_2 \| > \varepsilon \) and \( \| \varphi_x(y_3) - b_2 \| > \varepsilon \) for every \( x \in U'_2 \). Now using the Namioka property of \( f_{y_3} \) chose an open in \( X \) nonempty set \( U_3 \subseteq U'_2 \) such that \( \| \varphi_{x'}(y_3) - \varphi_{x''}(y_3) \| \leq \frac{\varepsilon}{2} \) for any \( x', x'' \in U_3 \).

Applying the same arguments \( n \) times we obtain a decreasing finite sequence \( (U_k)_{k=1}^n \) of open in \( X \) nonempty sets \( U_k \) and finite sequences \( (x_k)_{k=1}^n \) and \( (y_k)_{k=1}^n \) of points \( x_k \in U_k \) and \( y_k \in Y \) such that \( \| \varphi_{x'}(y_k) - \varphi_{x''}(y_k) \| \leq \frac{\varepsilon}{2} \) for any \( x', x'' \in U_k \) and \( \| \varphi_{x}(y_k) - b_i \| > \varepsilon \) for arbitrary \( x \in U_k \) and \( i \in \{ 1, 2, \ldots, k-1 \} \) where \( b_i = \varphi_{x_i}(y_i) \).

Since \( U_n \subseteq U' \) then \( A_n \cap U_n \neq \emptyset \). Pick an arbitrary point \( x \in A_n \cap U_n \) and put \( u_k = \varphi_{x}(y_k) \) for \( k = 1, 2, \ldots, n \). Then for \( 1 \leq i < j \leq n \) we have

\[
\| u_j - u_i \| = \| \varphi_{x}(y_j) - \varphi_{x}(y_i) \| \geq \| \varphi_{x}(y_j) - b_i \| - \| \varphi_{x}(y_i) - \varphi_{x}(y_i) \| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.
\]

But this contradicts \( x \in A_n \).

**Theorem 4.4** Let \( X \) be a Baire space, \( Y, Z \) be a compact spaces, \( P = Y \times Z \), \( f : X \times P \to \mathbb{R} \) be a function which is quasi-continuous in the first variable and continuous in the second variable and for any \( y \in Y \) and \( z \in Z \) the functions \( f_y : X \times Z \to \mathbb{R} \) and \( f^z : X \times Y \to \mathbb{R} \), \( f_y(x, z) = f^z(x, y) = f(x, y, z) \), have the Namioka property. Then \( f \) has the Namioka property.

**Proof.** It is sufficient to prove that for any open in \( X \) nonempty set \( U \) and an \( \varepsilon > 0 \) there exists an open in \( X \) nonempty set \( U_0 \subseteq U \) such that \( \| f(x', y, z) - f(x'', y, z) \| < \varepsilon \) for arbitrary \( x', x'' \in U_0 \), \( y \in Y \) and \( z \in Z \).

Fix an open in \( X \) nonempty set \( U \) and \( \varepsilon > 0 \). Proposition 4.3 implies the existence of functions \( b_1, b_2, \ldots, b_n \in C(Z) \) and an open in \( X \) nonempty set \( U_1 \subseteq U \) such that for any \( y \in Y \) there exists a dense in \( U_1 \) set \( \bar{A}_y \) such that for every \( x \in \bar{A}_y \) there is a number \( k \in \{ 1, 2, \ldots, n \} \) with \( \| f(x, y, z) - b_k(z) \| \leq \frac{\varepsilon}{8} \) for \( z \in Z \).

For continuous on \( Z \) functions \( b_1, b_2, \ldots, b_n \) we find a finite open covering \( W \) of \( Z \) by nonempty sets \( W \) such that \( \omega_{b_k}(W) < \frac{\varepsilon}{8} \) for any \( k \in \{ 1, 2, \ldots, n \} \)
and $W \in \mathcal{W}$. For every $W \in \mathcal{W}$ choose a point $z_W \in W$. Since all functions $f^{z_W}$ have the Namioka property then there exists an open in $X$ nonempty set $U_0 \subseteq U_1$ such that $|f(x', y, z_W) - f(x'', y, z_W)| < \frac{\varepsilon}{8}$ for arbitrary $x', x'' \in U_0$, $y \in Y$ and $W \in \mathcal{W}$. Show that $U_0$ is to be found.

For every $y \in Y$ put $A_y = U_0 \cap \overline{A_y}$. It is clear that $U_0 \subseteq A_y$ for each $y \in Y$. Fix $y \in Y$ and $W \in \mathcal{W}$. Recall that for every $a \in A_y$ there exists a number $k \in \{1, 2, \ldots, n\}$ such that $|f(a, y, z) - b_k(z)| \leq \frac{\varepsilon}{8}$ for all $z \in W$. Since $\omega_{b_k}(W) < \frac{\varepsilon}{8}$ then for arbitrary $z', z'' \in W$ the following inequality holds

$$|f(a, y, z') - f(a, y, z'')| \leq |f(a, y, z') - b_k(z')| + |b_k(z') - b_k(z'')| + |f(a, y, z'') - f(a, y, z)| \leq \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{7\varepsilon}{8}.$$

Now for arbitrary $a', a'' \in A_y$ and $z \in W$ we have

$$|f(a', y, z) - f(a'', y, z)| \leq |f(a', y, z) - f(a', y, z_W)| + |f(a', y, z_W) - f(a'', y, z_W)| + |f(a'', y, z_W) - f(a'', y, z)| \leq \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} = \frac{7\varepsilon}{8}.$$

Since $f$ is quasi-continuous in the first variable and the set $A_y$ is dense in the open set $U_0$ then $|f(x', y, z) - f(x'', y, z)| < \frac{7\varepsilon}{8} < \varepsilon$ for any $x', x'' \in U_0$.\)

**Corollary 4.5** The product of two compact Kempisty spaces is a Kempisty space.

Theorem 4.1 and Corollary 4.5 imply the following property of compact Kempisty spaces.

**Theorem 4.6** The product of an arbitrary family of compact Kempisty spaces is a Kempisty space.

### 5 KC-mappings with values in metrizable spaces

In this section we carry over the corresponding result from [1] for separately continuous mappings on the case of $KC$-mappings which take values in metrizable spaces.

The following statement is an analog of Proposition 4.3.

**Proposition 5.1** Let $X$ be a Baire space, $Y$ be a compact space, $Z$ be a metric space with a metric $d$ and $f : X \times Y \to Z$ be a mapping which is quasi-continuous in the first variable and continuous in the second variable. Then for any $\varepsilon > 0$ and an open in $X$ nonempty set $U$ there exist a set...
\{z_1, z_2, \ldots, z_n\} \subseteq Z and an open in X nonempty set \( U_0 \subseteq U \) such that for each \( x \in U_0 \) and each \( y \in Y \) there exists a number \( k \in \{1, 2, \ldots, n\} \) such that \( d(f(x, y), z_k) \leq \varepsilon \).

**Proof.** We shall reason similarly like in the proof of Proposition 4.3.

Fix an open in \( X \) nonempty set \( U \) and \( \varepsilon > 0 \). Since \( f \) is continuous in the second variable then for any \( x \in X \) the set \( Z_x = \{ f(x, y) : y \in Y \} \) is a metric compact in \( Z \). Choose an open in \( X \) nonempty set \( U' \subseteq U \), a dense in \( U' \) set \( A \) and an integer \( n \in \mathbb{N} \) such that \( \frac{\varepsilon}{2} \)-size of \( Z_a \) is not greater than \( n \) for every \( a \in A \).

Suppose that for fixed \( \varepsilon \) and \( U \) the conclusion of this proposition is false. Choose arbitrary points \( x_1 \in U' \), \( y_1 \in Y \) and an open in \( X \) nonempty set \( U_1 \subseteq U' \) such that \( d(f(x, y_1), z_1) < \frac{\varepsilon}{2} \), where \( z_1 = f(x_1, y_1) \) for every \( x \in U_1 \). By the assumption, there exist points \( x_2 \in U_1 \) and \( y_2 \in Y \) such that \( d(f(x_2, y_2), z_1) > \varepsilon \). Put \( z_2 = f(x_2, y_2) \) and using the quasi-continuity of \( f \) in the first variable we find an open in \( X \) nonempty set \( U_2 \subseteq U_1 \) such that \( d(f(x, y_2), z_2) < \frac{\varepsilon}{4} \) for every \( x \in U_2 \).

Applying the same arguments \( n \) times we obtain a decreasing finite sequence \((U_k)_{k=1}^n \) of open in \( X \) nonempty sets \( U_k \) and finite sequences \((x_k)_{k=1}^n \) and \((y_k)_{k=1}^n \) of points \( x_k \in U_{k-1} \) for \( k = 2, \ldots, n \) and \( x_1 \in U' \) and \( y_k \in Y \) such that \( d(f(x, y_k), z_k) < \frac{\varepsilon}{4} \) for every \( x \in U_k \) and \( d(z_k, z_m) > \varepsilon \) for distinct \( k, m \in \{1, 2, \ldots, n\} \) where \( z_i = f(x_i, y_i) \) for \( i = 1, 2, \ldots, n \).

Consider an arbitrary point \( a \in A \cap U_n \). Then for \( 1 \leq k < m \leq n \) we have

\[
d(f(a, y_k), f(a, y_m)) \geq d(z_k, z_m) - d(f(a, y_k), z_k) - d(f(a, y_m), z_m) > \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\]

But this contradicts the choice of \( A \). \( \Diamond \)

**Theorem 5.2** For any Baire space \( X \) and compact space \( Y \) the following conditions are equivalent:

(i) every function \( f : X \times Y \to \mathbb{R} \) which is quasi-continuous in the first variable and continuous in the second variable has the Namioka property;

(ii) for any metrizable space \( Z \) every mapping \( f : X \times Y \to Z \) which is quasi-continuous in the first variable and continuous in the second variable has the Namioka property.

**Proof.** It is sufficient to prove the implication \((i) \implies (ii)\). Consider an arbitrary metrizable space \( Z \) and fix a metric \( d \) on \( Z \) which generates on \( Z \) its topology.
Fix $\varepsilon > 0$ and an open in $X$ nonempty set $U$. By Proposition 5.1 there exist an open in $X$ nonempty set $U_0 \subseteq U$ and a set $\{z_1, z_2, \ldots, z_n\} \subseteq Z$ such that for any $x \in U_0$ and $y \in Y$ there exists a number $k \in \{1, 2, \ldots, n\}$ with $d(f(x, y), z_k) < \frac{\varepsilon}{4}$.

For each $k \in \{1, 2, \ldots, n\}$ the function $f_k : X \times Y \to \mathbb{R}$ is defined by $f_k(x, y) = d(f(x, y), z_k)$. By (i) all functions $f_k$ have the Namioka property. Therefore there exists an open in $X$ nonempty set $U_1 \subseteq U_0$ such that $|f_k(x', y) - f_k(x'', y)| < \frac{\varepsilon}{4}$ for every $k \in \{1, 2, \ldots, n\}$, $x', x'' \in U_1$ and $y \in Y$.

Fix $y \in Y$ and $x', x'' \in U_1$. Choose $k \in \{1, 2, \ldots, n\}$ such that $d(f(x', y), z_k) < \frac{\varepsilon}{4}$. Then

$$d(f(x', y), f(x'', y)) \leq d(f(x', y), z_k) + d(f(x'', y), z_k) = f_k(x', y) + f_k(x'', y) \leq 2f_k(x', y) + |f_k(x', y) - f_k(x'', y)| < \frac{3\varepsilon}{4}.$$

Thus $\omega_f(x, y) < \varepsilon$ for every point $(x, y) \in U_1 \times Y$. Hence $f$ has the Namioka property. ♦

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V.V. Mykhaylyuk  
Chernivtsi National University  
Dept. of Mathematical Analysis  
Kotsyubyns’koho 2  
Chernivtsi 58012  
UKRAINE  

mathan@chnu.cv.ua  

telephone number: (380) (372) (584888)