Stationary quantum vortex street in a driven-dissipative quantum fluid of light

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We investigate the formation of a new class of density-phase defects in a resonantly driven 2D quantum fluid of light. The system bistability allows the formation of low density regions containing density-phase singularities confined between high density regions. We show that in 1D channels, an odd (1-3) or even (2-4) number of dark solitons form parallel to the channel axis in order to accommodate the phase constraint induced by the pumps in the barriers. These soliton molecules are typically unstable and evolve toward stationary symmetric or anti-symmetric arrays of vortex streets straightforwardly observable in cw experiments. The flexibility of this photonic platform allows implementing more complicated potentials such as maze-like channels, with the vortex streets connecting the entrances and thus solving the maze.

A fluid is called quantum when it exhibits quantum-mechanical effects at a macroscopic scale. For bosons, it typically occurs when many particles can be described by a single-particle wave function. This collective behaviour can arise spontaneously when particles undergo a phase transition towards a quantum-coherent state such as a superconducting state, a superfluid, or a Bose Einstein condensate (BEC). Cavity exciton-polaritons (polari-

tons) are 2D photonic modes interacting via their excitonic parts. Their quantum coherence can spontaneously occur through a BEC process, but a unique feature of this photonic system is that such coherence can also be imprinted by a resonant laser and be preserved for times being substantially longer than their lifetime.

The high control of the injected flow combined with a direct optical access to the full wave function (amplitude, phase), both in space and time make this platform very attractive to study quantum fluid physics. A typical example which has revealed the potential of this system is the observation of oblique dark solitons, which form when a supersonic quantum fluid hits a defect (initially proposed in 2006 in the context of atomic BECs). The 2D solitons forming behind the defect remain stable because the transverse "snake instability" makes 2D solitons normally unstable, is carried away by the supersonic flow making the soliton effectively 1D. However, such supersonic flow is energetically unstable. It turns out that polariton flows can be efficiently decoupled from thermal relaxation, which made possible the observation of oblique dark solitons.

An interesting regime occurs if the fluid velocity is decreased just below the speed of sound. In such a case, the subsonic flow still interacts with the defect and exhibits a local acceleration in its vicinity due to the conservation of flow. This leads to the formation of quantum vortex streets composed of vortex-antivortex pairs. In the absence of stabilization by the supersonic flow, this quantum version of von Karman vortex streets can be understood as the decay of the oblique solitons via the snake instability. The creation of vortex-antivortex pairs has been reported in time-resolved pulsed experiments, both in polaritons and atomic quantum fluids. However, the study of the snake instability dynamics leading to quantum vortex street requires both cw excitation and time resolution, and it remained elusive so far. In a recent theoretical work, it was proposed to improve this scheme by sustaining the propagating flow against radiative decay with a support laser covering the whole sample. Interestingly, this configuration demonstrates very original density-phase defects showing key differences but also similarities with the conservative and quasi-conservative cases considered previously.

This pump-support scheme is not limited to the study of the flow scattering on defects, but can be used in a much more general frame to create, at will, a large variety of topological defects and study their behavior. Topological defects in the driven-dissipative case can be stationary only when the support laser intensity falls in the bistability loop of the non-linear system where the density can be either low, or high, depending on the laser absorption. Stationary phase defects exist in low-density regions where the phase is not fixed because most particles are not injected by the laser but diffuse from higher-density regions. The control of the spatial distribution of intensity and phase allows to realize various confining potential for density-phase defects, such as 1D channels, 0D traps, or circuits made by the combination of both.

In this work, we show that the creation of narrow 1D low density channels surrounded by high-density regions leads to the formation of stationary dark soliton molecules. These molecules exhibit snake instabilities
along the translationally invariant dimension. It leads to the formation of coupled chains of vortex-antivortex pairs, which can be viewed as stationary vortex streets being stabilized by the confining potential. We finally show that the pump distribution can be used to create a maze. Vortex streets in the dead ends are unstable and disappear, while the entrance and the exit of the maze remain connected by the stationary vortex streets, implementing an efficient analog all-optical maze solving algorithm.

The resonantly pumped microcavity is modelled by the standard driven-dissipative Gross-Pitaevskii equation, formally equivalent to the Lugiato-Lefever equation [24]. We neglect the polarization degree of freedom, the non-parabolicity of the polariton dispersion and any thermal effects [26,27]. The equation reads:

\[i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} - i\Gamma + g |\psi|^2 \right] \psi + (S + P)e^{-i\omega_0 t},\]

where \(\Gamma = \hbar/(2\tau)\) is the polariton decay rate (\(\tau = 15\) ps), \(m = 8 \times 10^{-5}m_0\) is the polariton mass (\(m_0\) is the free electron mass), \(g = 5\) µeV/µm² is the polariton-polariton interaction constant. The detuning between the ground state and the pump laser is \(\omega_0 = 0.14\) meV/\(\hbar\). The support \(S\) and the pump \(P(r)\) are at normal incidence (zero wave vector).

We start by considering the system under spatially homogeneous pumping (support only). In such case, the system is bistable, showing stable homogeneous wave function profile. A typical bistability loop obtained for our parameters is shown in Fig. 1(a). Next, we add to the support a half-space pump \((x < 0)\) switching the system to the higher branch of the bistability loop, whereas the other half-space remains on the lower branch. The use of a spatially homogeneous support \(S\) and an extra pump \(P(r)\) allows to control the pumping intensity in the high- and low-density regions independently.

Figure 1(b) shows the two regions with the domain wall separating them (the black dashed line is the pump boundary, and the support is present everywhere): the solid black line is a 1D cut of the density profile, and the 2D false color map shows the phase. As expected, the region under the pump shows a large intensity and a fixed phase. The intensity at the border of the pumped region decays typically within one healing length \(\xi\) and then exhibits small periodic oscillations of both intensity and phase. The domain wall is stable against the development of instabilities along \(y\), but can propagate along \(x\). Such type of domain wall propagation has been previously considered for polaritons[28,29] and in general for switching waves in optics [30–32] and beyond. The velocity of the wall computed numerically as a function of support intensity is shown as a black line in Fig. 1(a). As expected for this class of differential equations, the propagation velocity \(v\) is linear in \(S - S_c\), where the critical value of the support \(S_c\) is given by the Maxwell construction [33,34]

\[S_c \approx \frac{2(\hbar\omega_0)^{3/2}}{3\sqrt{3}g^{1/2}}\]

When the support is larger than \(S_c\), the wall propagates to the right with velocity

\[v \sim \frac{2}{S - S_c} \frac{\xi}{S_c} \frac{\xi}{\tau},\]

where \(\xi = \hbar/\sqrt{2\hbar\omega_0} \approx 1.8\) µm is the healing length, and the high intensity region expands to the whole space. On the contrary, for support values below \(S_c\) the high-density region shrinks and the wall stops at the boundary of the pump \((x = 0)\). Around \(S_c\), spatially localized solutions of the Gross-Pitaevskii (or Lugiato-Lefever) equation with homogeneous pumping bifurcate under the form of dark solitons multiplets [33].

Next, we go beyond the homogeneous pumping case by considering a second high-density region with its boundary parallel to the first one, defining an all-optically controlled 1D confining potential. We start by considering high density regions with the same phase and a fixed channel width 23 µm (about 13\(\xi\) of the high density region). Fig. 2 is computed at \(S = 0.25S_c\) for 2 different values of walls pumping. The first column presents the stationary intensity distribution with 2 dark solitons in the corridor. The system is effectively one-dimensional, since it is translationally invariant along \(Y\) (with periodic boundary conditions). Dark solitons are anti-symmetric states with a \(\pi\) phase shift. The phase constraints imposed by the high-density regions therefore only allows an even number of solitons in the channel. This situation is however unstable with respect to the development of instability along the \(y\)-direction for a large range of parameters. The second column of Fig. 2 shows the imaginary part of the energy of the weak excitations of the condensate versus their longitudinal wave vector \(k_y\),
obtained from the Bogoliubov-de Gennes equations:

\[
L(r)u(r) + g\psi(r)^2v(r) = \hbar \omega u(r), \quad (4)
\]
\[
L(r)v(r) + g(\psi^*(r))^2u(r) = -\hbar \omega v(r),
\]

where \( L(r) = -\hbar^2 \nabla^2 / 2m + 2g|\psi(r)|^2 - \hbar \omega_0 - i\Gamma. \) Comparing with Ref. [36], the chemical potential \( \mu \) for conservative system is replaced by the laser frequency in the driven-dissipative case [37, 38]. The system of Eqs. 4 is the eigenvalue problem for weak excitation frequency \( \omega \). Translational invariance along \( y \) allows to replace \(-\nabla^2 \rightarrow k^2_y - \partial^2 / \partial x^2 \), where \( k_y \) is the wave vector of perturbation. The period in space of the perturbations is thus \( 2\pi / k_y \). The positive imaginary part of the energy leads to the development of modulational instability (the snake instability, well known in conservative condensates [39]). The value of the maximal instability wave vector can be estimated [31] as \( k_y^* = 1 / \xi \sqrt{2} \). The lowest energy mode in the potential formed by the density dips is a symmetric bound state, whereas the highest mode is an anti-symmetric anti-bound state. However, one should keep in mind that each soliton is anti-symmetric by itself (\( \pi \) phase jump). Therefore, the modes with the highest real part of the energy (corresponding to the highest wave vector) lead to the development of symmetric patterns (Fig. 2 2nd line), whereas the mode with the lowest energy (wave vector) is associated with the development of an anti-symmetric pattern (Fig. 2 1st line). The development of the instabilities can be triggered by any noise or fluctuations in the numerical simulations. Here, we consider a weak Gaussian disorder potential with a correlation length of 2 \( \mu \)m and an amplitude of 0.01 meV. This noise breaks the translational symmetry along the \( y \)-axis and drives the system towards 2D, where the modulational instability can occur. The third and fourth column of Fig. 2 show the intensity and the phase of the stationary wave function after the development of the instability. The precise realization of the disorder potential determines the exact positioning of the pattern along \( y \) but it does not affect the shape, at least if the disorder amplitude is sufficiently small with respect to the other energy scales (essentially given by the detuning). We therefore expect these stationary patterns to be fully accessible experimentally. We see that in all cases, the solitons break into two vortex anti-vortex chains, which can be seen as stationary vortex streets. An interesting aspect is that these chains are really stable, as confirmed by the stability analysis of their 2D patterns. It means that the snake instability develops but is then frozen by the presence of the confining potential.
domain walls. This regime is qualitatively similar with the one which was used in the polariton neuron picture proposed a decade ago [28]. The dark grey region corresponds to the establishment of a non-stationary steady state (limit cycle), at least in the conditions of our simulations, namely without any energy relaxation and for a sufficiently low disorder. In this parameter range, we find a pair of breathing solitons oscillating in time as shown in a video [34]. The small light-grey domain corresponds to the narrow set of parameters where the lattice of four solitons is stable. This typically occurs for small $P$ and large $S$, meaning that the particles in the corridor region are mainly injected at $k = 0$ by the support rather than by the flow from the walls, which favours the stability of the soliton lattice. The next phase located at the bottom left corner of the phase diagram corresponds to the collapse of 4 solitons into a symmetric pair of vortex chains (see Fig. S1 [34]). The two next phases located above the blue line correspond to the collapse of 2 solitons into symmetric and anti-symmetric vortex chains respectively which corresponds to the situations shown in Fig. 2(a,b). A particularly tiny domain (lime-green) exhibits the collapse of 4 solitons into an anti-symmetric pair of vortex chains. The false color scale of the figure shows the maximal instability wave vector $k_y^*$ (except for the non-stationary and stable phases). It shows that the anti-symmetric solutions have a twice larger spatial period than the symmetric ones. One can also check that the $k_y^*$ gradient within a given phase is relatively small, which means that the possibility to observe experimentally these patterns will not be strongly affected by

FIG. 3. Phase diagram versus support $S$ and pump $P$. The color shows the wave vector $k_y$ for the excitation with maximal imaginary part. Green tones are for long period anti-symmetric excitations (snakes) and orange/red tones are for symmetric excitations with shorter period. Lower left corner separated by the blue curve corresponds to the 4-solitons initial state. Dark gray area is for oscillating in time solitons and violet is for high density in corridors without any solitons. Dots of various colors correspond to the panels in the boxes of the same color in Fig. 2. Blue dot is for maze pathfinding regime illustrated in Fig. 4. The insets show the transverse profiles of unperturbed density in the corridor.

pump/support intensity fluctuations.

Another interesting possibility offered by this driven-dissipative system is to tune the relative phase of the pump in the two walls, working at zero support. In such a case, an odd number of solitons forms [34]. They develop a snake instability completely similar to the previous case, with the formation of a corresponding number of chains of vortex-antivortex pairs. This tuning of the soliton number by varying the relative phase between the pumps is a generalization of [31], where a lattice of 1D solitons formed between two resonant pumps.

A crucial difference here is the 2D character of the system which allows instabilities to grow along $y$. In fact, modern optical techniques allow creating any shape of confining potential such as various 0D traps [24], or possibly lattices, connecting 0D islands with 1D channels. The geometry we address now is a maze of 1D channels [Fig. 4(a,b)]. For a proper choice of $(S, P)$, immediately after the jump of the walls on the upper branch the maze is filled with solitons. However, the dead ends represent a configuration different from that of Fig. 2: the heads of the vortex streets start to withdraw [34]. Fig. 4(a) shows the intensity distribution 20 ps after driving pump and support to maze solving regime (blue circle in Fig. 3 in the non-stationary phase). The heads are moving as symbolized by the arrows. Fig. 4(b) shows the final intensity distribution ($t = 1$ ns) where the street only connects the two exits of the maze. This configuration represents an optical maze solver (see also a supplementary movie for the dynamics [34]), belonging to a larger class of analog graph solving algorithms [31]. The solving time is determined by the velocity of the street’s head, which corresponds to the propagation velocity of a domain wall $v$ given by Eq. (20). In the worst case, the length of the dead end is $Nw$, where $N$ is the number of cells in the maze (or vertices in the graph) and $w$ is the width of a corridor. If $L$ is the overall system size, the maximal number of cells is $N = L^2/w^2$. The solving
analog implementation is the small value of the prefactor $\omega/\nu \sim 0.5$ ns: the high velocity $\nu$ reduces the solving time, allowing such an analog maze solver to outperform a modern PC with a clock frequency of several GHz, because tens or hundreds of clock ticks are required to check a single cell.

To conclude, we have shown that the modulational instability can be controlled and stabilized in a driven-dissipative polariton system allowing the on-demand formation of soliton molecules and vortex streets exhibiting a particularly rich phase diagram. Non-stationary regimes can be used for implementing fast analog maze solving.

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The behavior of bistable systems and, in particular, spatially inhomogeneous solutions present in such systems were a subject of studies for a long time. An overview on optical systems can be found in [30].

We start by looking for a stationary spatially inhomogeneous solution where a part of the system is at the upper bistability branch, whereas another part is at the lower bistability branch. These two parts are separated by a domain wall (or switching wall). This problem can be solved analytically for negligibly small dissipation $\Gamma$ (as compared with the laser detuning $\delta = \hbar \omega_0$), and then the solution can be generalized to non-zero $\Gamma$.

The stationary driven-dissipative Gross-Pitaevskii equation that needs to be satisfied by the wave function reads:
\begin{equation}
\left( -\hbar \omega_0 - \frac{\hbar^2}{2m} \Delta + g|\psi_0|^2 - i\Gamma_0 \right) \psi_0 + S = 0
\end{equation}
where we are going to neglect the term $\Gamma$ at first. Once it is neglected, the wave function $\psi$ and the pumping (support) $S$ can be assumed to take only real values without loss of generality (since all coefficients are real). The equation (6) can therefore be rewritten as
\begin{equation}
m_0 \frac{d^2x}{dt^2} = F(x)
\end{equation}
which is a Newton’s equation of motion for a material point $(x = \psi)$ with a mass $m_0 = \hbar^2/2m$ under the effect of a position-dependent “force”
\begin{equation}
F(x) = gx^3 - \delta x + S
\end{equation}
to which one may attribute a ”potential”
\begin{equation}
U(x) = -\int F(x) \, dx = -\frac{gx^4}{4} + \frac{\delta x^2}{2} - Sx
\end{equation}
The two maxima of this potential located at the coordinates \( x_1 \) and \( x_3 \) correspond to the two stable domains (high density and low density), while the minimum located at \( x_2 \) corresponds to the inaccessible part of the bistability curve. The system is stationary only if the values of the effective potentials at the two maxima are exactly the same:

\[
U(x_1) = U(x_3),
\]

(9)
otherwise the domain wall starts to propagate. Indeed, a material point should start its motion at one maximum and finish at the other maximum, and for this the two maxima have to be at the same potential height. The points \( x_1 \) and \( x_3 \), corresponding to the extrema of \( U(x) \), can be found analytically from the cubic equation \( F(x) = 0 \), and the condition \( \int_{x_1}^{x_2} F(x) \, dx = 0 \)

corresponding to the Maxwell construction in thermodynamics [33]. Solving this equation analytically for the unknown \( S \) gives finally:

\[
S_c = \frac{2 \delta^{3/2}}{3 \sqrt{3} g^{1/2}}
\]

(11)
and the ratio with respect to the pumping required for bistability \( S_{min} = \Gamma_0 \sqrt{\omega / g} \) is given by

\[
\frac{S_c}{S_{min}} = \frac{2 \delta}{3 \sqrt{3} \Gamma}
\]

(12)

For non-negligible \( \Gamma \), all terms in the equation become comparable. The real terms (the kinetic and the interaction energy, the detuning) are of the order of \( \delta = \hbar \omega_0 \), while the only imaginary term is \( \Gamma \) (imaginary part of kinetic energy is small with respect to \( \Gamma \)). Thus, the expression for the critical pumping \( S_c \) at \( \Gamma \) comparable with \( \hbar \omega_0 \) can be sought by replacing of the first three terms in \( \delta \) by \( \hbar \omega_0 \) with some coefficient. This coefficient can be found from numerical simulations by the small variation of parameters \( \omega_0 \to \omega_0 + \Delta \omega_0 \), \( \Gamma \to \Gamma + \Delta \Gamma \), and \( g \to g + \Delta g \), which allows obtaining the coefficients (\( c_\omega \approx 0.75, \ c_\Gamma \approx 0.75, \ c_g = -0.5 \)) in the Taylor expansion:

\[
\frac{S_c(\omega_0 + \Delta \omega_0, \Gamma + \Delta \Gamma, g + \Delta g)}{S_c(\omega_0, \Gamma, g)} = 1 + c_\omega \frac{\Delta \omega_0}{\omega_0} + c_\Gamma \frac{\Delta \Gamma}{\Gamma} + c_g \frac{\Delta g}{g}.
\]

(13)

As a net result, one obtains

\[
S_c = \sqrt{\frac{\hbar \omega_0}{g}} \sqrt{\left(\frac{3 \hbar \omega_0}{32}\right)^2 + \Gamma^2}
\]

(14)

One sees that this equation has the same Taylor expansion as Eq. (13) for actual values of system parameters.

For \( S \neq S_c \), one of the domains becomes more favorable than the other, and the domain wall starts to propagate. Assuming that the density changes linearly with position across the domain wall (which is valid in the vicinity of the inflexion point of this wall), we find the following expression for the speed of the wall:

\[
v = \frac{\partial n}{\partial t} \frac{\Delta x}{\Delta n}
\]

(15)

where \( \Delta x \) is the width of the domain wall and \( \Delta n \) is the difference of the densities in the high and low density regions.

To find the derivative \( \partial n / \partial t \), we define \( n_0 \) as the density at the inflexion point which corresponds to \( S_c \). The only contribution into \( \partial \psi / \partial t \) at \( S \neq S_c \) can come from the difference \( S - S_c \) (other terms in the time-dependent driven-dissipative Gross-Pitaevskii equation cancel):

\[
\frac{\partial \psi}{\partial t} = \frac{P - P_0}{i \hbar}
\]

(16)

which allows to write the solution

\[
\psi(t) = \frac{P - P_0}{i \hbar} t + \sqrt{n_0} e^{i \phi_0}
\]

(17)

This expression strongly depends on the phase \( \phi_0 \) of the wave function at the inflexion point (with respect to the phase of the pump). In a homogeneous system, the phase is given by \( \tan \phi = -1/(\omega_0 - \hbar \omega_0) \). In the low-density region, \( \phi \to 0 \), whereas in the high-density region \( \phi \to \pi/2 \) just above the threshold density \( gn \approx \hbar \omega_0 \). We assume that \( \phi_0 \) takes an intermediate value \( \phi_0 = \pi/4 \), and we can also assume that it changes linearly with \( S - S_c \) for small deviations from \( S_c \), in which case the time derivative of the density can be found as:

\[
\frac{\partial n}{\partial t} = \frac{\partial |\psi(t)|^2}{\partial t} \approx 2 \sqrt{n_0} S - S_c \frac{1}{2} \left( 1 - \frac{S - S_c}{S_c} \right)
\]

(18)

For \( S \approx S_c \), the second-order correction can be neglected, but it starts to become important for larger deviation of the pumping, creating an important difference between the speed of the wall in the case of shrinking or expanding high-density domain.

In the simplest case, the speed can be found as

\[
v = \sqrt{2} \sqrt{n_0} \frac{S - S_c}{\hbar} \frac{\Delta X}{\Delta N}
\]

(19)

By estimating \( \sqrt{\Delta N} \approx \sqrt{n_0} \approx S_c / \Gamma \) and taking \( \Delta X = \sqrt{2} \xi \) one rewrites Eq. (20) as

\[
v \sim 2 \frac{S - S_c}{S_c} \frac{\xi}{\tau} \frac{\Delta X}{S_c}
\]

(20)

The expression (20) for the values used in numerical simulations gives \( v \approx 12 \mu \text{m/ns} \), very close to the numerical value of 30 \( \mu \text{m/ns} \) for \( \frac{S - S_c}{S_c} = 0.05 \).
Stability of driven-dissipative solitons in 2D

We analyze the soliton stability in the 2D driven-dissipative configuration using the Bogoliubov-de Gennes equations for weak excitations of the condensate. Let ψ be the non-trivial solitonic solution of the driven-dissipative Gross-Pitaevskii equation written on the polaron basis

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + g|\psi|^2 \psi - i\Gamma \psi + P(x, y)e^{-i\omega_0 t} \] (21)

We can write \( \psi = \psi_0(x, y)e^{-i\omega_0 t} \), and the perturbed solution is \( \psi_0(x, y) + A(x)e^{i(k_y y - \omega t)} + B^*(x)e^{-i(k_y y - \omega t)} \), where \( \omega \) is the perturbation frequency relative to the laser frequency \( \omega_0 \) and \( \psi_0(x, y) \) is the solution of the stationary equation

\[ \hbar\omega_0 \psi_0 = -\frac{\hbar^2}{2m} \Delta \psi_0 + g|\psi_0|^2 \psi_0 - i\Gamma \psi_0 + P \] (22)

Linearizing (21), we obtain the Bogoliubov-de Gennes equations for \( A \) and \( B \):

\[
\begin{pmatrix}
-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} - k_y^2 \right) + 2g|\psi_0|^2 - i\Gamma - \hbar \omega_0 - \hbar \omega & g\psi_0^* \\
-g\psi_0 & -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} - k_y^2 \right) + 2g|\psi_0|^2 - i\Gamma - \hbar \omega_0 + \hbar \omega
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix} \tag{23}
\]

where we are interested in the bogolon states confined along the \( x \) direction. For a single bogolon state quantized in a dark soliton, its energy can be estimated as \([43]\):

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 A}{\partial x^2} + g|\psi_0|^2 A - \hbar \omega_0 A \approx -\frac{\hbar \omega_0}{2} A \tag{24}
\]

which supposes that the system is above the bistability threshold and that the soliton is almost dark. The diagonalization of the matrix \([23]\) thus gives an equation

\[
\left( \frac{\hbar^2 k_y^2}{2m} + g|\psi_0|^2 - i\Gamma - \frac{\hbar \omega_0}{2} \right)^2 - g^2|\psi_0|^4 - \hbar^2 \omega^2 = 0 \tag{25}
\]

which allows to estimate the value of the imaginary part of \( \omega \). Indeed, if we suppose \( \Gamma \ll \hbar \omega_0 \), the solution for \( \omega \) becomes imaginary if the real part of \([25]\) is negative. Maximal imaginary part is achieved if the first square is zero and the second square is maximized, which is obtained at a point \( x_0 \) where \( g|\psi_0(x_0)|^2 = \hbar \omega_0/2 \), which determines the maximal possible positive imaginary part of \( \hbar \omega \) as

\[
\Gamma_{\text{max}} = \frac{\hbar \omega_0}{2} - \Gamma \tag{26}
\]

In this case, the soliton is clearly always unstable, because the negative contribution to the imaginary part is much smaller than the positive one, as required for the bistability. We can conclude that 2D solitons in the driven-dissipative configuration remain unstable with respect to small perturbations, at least if they are obtained at the upper bistability branch.

This also provides a criterion for the maximal wavevec-
A final remark is due on the phase diagram shown in the main text. In experiments, the patterns arising from the development of the instability will be fixed in space due to the pinning by the disorder [44]. It is also important to note that the precision of the phase diagram calculation is probably at the limit of the present experimental possibilities.

**Solitons under \( \pi \) phase shift**

In the main text, we show the results obtained when the phase of the pumping laser is homogeneous in space, and only the density profile is varying (allowing to obtain the high-density walls). In this subsection, we present additional results concerning the formation of solitons and their stability for a \( \pi \) phase difference between the pump at the walls. In this case, no support is used (otherwise it would exhibit different interference with the two pumping lasers of different phase).

Figure 6 shows the results obtained in this configuration, with top and bottom rows corresponding to two different distances between the walls. The first column shows the spatial density profile with 3 or 1 solitons, depending on the distance available for them. The second column shows the imaginary part of the energy obtained from the Bogoliubov-de Gennes analysis described above. Both curves exhibit a maximum with positive imaginary part, confirming the existence of modulational instability. The final stage of the development of this instability is shown in the 3rd column: it exhibits either 3 vortex chains or a single vortex chain. Finally, the phase distribution shown in the 4th column confirms the formation of vortices and anti-vortices evidenced by the density shown in the 3rd column.

**Solitons in a maze**

In Figs. 1, 2, 3 of the main text, we were considering a system infinite in the \( Y \) direction (implemented by periodic boundary conditions). As soon as we consider a dead end instead of an infinite corridor, the "head" of the pair of solitons comes into play. This head can also be considered as a domain wall along the \( Y \) direction, between a high-density region (the wall limiting the dead end) and a low-density region (soliton). The extra kinetic energy appearing because of the variation of the wavefunction in the \( Y \) direction, absent in the infinite system, changes the conditions of the local bistability loop and makes the low-density regime impossible for the same parameters for which it was possible in the infinite system. The domain wall starts to propagate, leading to the expansion of the high-density region erasing progressively the solitons in the channel.

One could imagine that this should lead to the switching of the whole maze structure to the high-density regime. However, this does not occur, because when the domain wall associated with a soliton edge disappearing from a dead end arrives to a T-junction where there is another soliton, the situation becomes fully equivalent to the \( Y \)-invariant case of the main text: the T-junction with one corridor in the high-density regime is just a corridor with homogeneous walls. We conclude that the solitons can only disappear from the dead ends, whereas solitons connecting the exits of the maze persist, thus allowing to solve the maze. Having solitons connected to a low density region is the same as having infinite solitons as presented in the first part of the article, and so independently of the geometry of the soliton path.

**Supplementary Movie**

The supplementary video file [https://youtu.be/8Yjrrr9ag8](https://youtu.be/8Yjrrr9ag8) shows an example of the solution of a large
maze with time. The image size is $1024 \mu m \times 1024 \mu m$. The maze solving time is 16 ns. We see that the vortex streets withdraw from the dead ends and only the vortex street connecting the two exits of the maze remains.

FIG. 6. Development of the instability of the initial configuration with $\pi$ phase shift.