Vector bundles and Arithmetical Groups I. The higher Bruhat-Tits tree *

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Recently author has proposed a generalization of the Bruhat-Tits buildings for the $n$-dimensional local fields [11]. It was shown also that there is a connection of this construction with classification of vector bundles on algebraic surfaces. In this paper we give the proofs of a part of results from [11], concerning with the construction of Bruhat-Tits building for the group $PGL(2)$ over two-dimensional local field. Some results will be given in a more generality, for the groups $PGL(n)$ or for local fields of arbitrary dimension. The applications to vector bundles will be considered in another paper. We refer to [11] for the detailed motivation of our construction and we restrict ourselves only by short remarks in this introduction.

Usually local fields (fields of dimension 1 in this language) appear from formal neighbourhoods of the points on an algebraic curve (or on an arithmetical curve). This can be generalized to higher dimensional schemes where the points will be replaced by chains of irreducible subvarieties having strictly decreasing codimension. And there is a way to put all them together in a single group called the adele group of the variety (or the scheme). Many parts of the classical adelic theory can be generalized to this situation. In particular this is done for the cohomology theory of coherent sheaves and for class field theory (see [4] for a survey of the existing theory).

Here we would like to apply these notions to the theory of buildings. The main object of the theory is a simplicial complex attached to any reductive algebraic group $G$ defined over a field $K$. There are two parallel theories for the cases when $K$ has no additional structure (being a local field of dimension 0) and when $K$ is a local field (of dimension 1). They are known

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as the spherical and euclidean buildings correspondingly (see [2,3] for original papers and [1,12,15] for the surveys and text-books. There exists also a system of axioms which cover both of the cases and which is known as theory of $BN$-pairs or the Tits systems [1]. As was shown in [11] these two theories (for the groups $PGL$) are the special cases of the general construction for the groups over local fields of an arbitrary dimension.

The simplest case which still exhibits the main features of the theory is the case when $G$ is of rank 1 and more precisely is a group $PGL(V)$ of projective linear transformations of a vector space $V$ of dimension 2 over a field $K$. Here we restrict ourselves by this case because it is quite sufficient for the study of vector bundles (of rank 2) over algebraic surfaces (and supplies also a necessary background for the corresponding theory on arithmetical surfaces).

The paper contains four sections. In the first we remind the main notions and results on local fields of dimension 2. All the algebraic constructions, like the Weyl group or the Bruhat decomposition are collected in section 2. The known results for the Bruhat-Tits tree are shortly discussed in section 3. The last section contains the main results of the paper – construction and the properties of the Bruhat-Tits tree over two-dimensional local field.

A substantial part of this work was done during my visits to the "Sonderforschungsbereich 170" of the Mathematical Institute of the Göttingen University. I am very much grateful to the members of the SFB for the possibility to work there and my special thanks to Hans Opolka and Samuel Patterson for the hospitality.

Author has tried to generalize the Bruhat-Tits theory starting from the middle 80’s. The talks with L. Breen, B. Seifert, J. Tits and T. A. Springer were very useful for me at the beginning of this work.

I am greatly indebted to T. Fimmel who taught me how to work with TeX.

1 Local Fields

We begin with the main definition of the higher adelic theory.

**Definition 1.** Let $K$ and $k$ be fields. We say that $K$ is a $n$-dimensional local field with $k$ as last residue field if the field $K$ has the following structure. Either $n = 0$ or $K$ is the quotient field of a (complete) discrete valuation ring.
$\mathcal{O}_K$ whose residue field is a local field of dimension $n - 1$ with last residue field $k$. If $K', K''$, ... are the intermediate residue fields from the definition then we will write $K/K'/K''/.../k$ for the structure. The first residue field will be denoted mostly as $\bar{K}$.

In the sequel we restrict ourselves by the case of $n = 2$ (the general case was considered in [11]).

A typical example (which is quite sufficient if we have in mind the applications to algebraic surfaces) is the field of iterated power series

$$K = k((u))((t))$$

with an obvious inductive local structure on it

$$\mathcal{O}_K = k((u))[[t]], \bar{K} = k((u)).$$

(see [7, ch.2]) for other examples and classification theorem for complete local fields). Let us mention that the choice of local parameters $t, u$ in our example does not follow from the local structure.

For technical reasons we do not assume as usual that the discrete valuation rings which enter in our definition are the complete rings. We have reduction map $p : \mathcal{O}_K \to \bar{K}$ and we denote by $\wp$ and $m$ the maximal ideals of the local rings $\mathcal{O}_K$ and $\mathcal{O}_{\bar{K}}$ correspondingly. Also we denote by $t, u$ the generators of these ideals. Let

$$\mathcal{O}'_K = p^{-1}(\mathcal{O}_{\bar{K}})$$

be a subring in $K$.

Then we have a tower of valuation rings for the valuations $\nu^{(i)}$ of rank $i = 0, 1, 2$:

$$\mathcal{O}_{(0)} \supset \mathcal{O}_{(1)} \supset \mathcal{O}_{(2)},$$

where $\mathcal{O}_{(0)} = K$, $\mathcal{O}_{(1)} = \mathcal{O}_K$, $\mathcal{O}_{(2)} = \mathcal{O}'$.

For valuation groups

$$\Gamma^{(i)}_K = K^*/(\mathcal{O}_{(i)})^*, \quad i = 1, 2, \quad \Gamma_K = \Gamma^{(2)}_K$$

there is a filtration

$$\Gamma^{(2)}_K \to \Gamma^{(1)}_K,$$

which will be reduced to one homomorphism in our case. We denote it by $\pi$. 3
This filtration defines on $\Gamma_K$ a structure of ordered group. If we need to show the local structure we will write $\Gamma_{K/.../k}$ instead of $\Gamma_K$. If we chose local parameters $t, u$ of the field $K$ then the order becomes the lexicographical order. Inside the group $\Gamma_K$ there is a subset $\Gamma_K^+$ of non-negative elements.

In our situation we have two valuations (of ranks 1 and 2). They will be denoted by $\nu$ and $\nu'$ correspondingly.

If $K \supset O$ is a fraction field of a subring $O$ we call $O$-submodules $a \subset K$ fractional $O$-ideals (or simply fractional ideals).

**Theorem 1.** The local rings $O_{(i)}$ $i = 0, 1, 2$ have the following properties:

i) $O'/m = k$, $K^* = \{t\} \{u\}(O')^*$, $(O')^* = k^*(1 + m)$;

ii) every finitely generated fractional $O'$-ideal $a$ is a principal one and

$$a = m_{i,n} = (u^it^n), \ i, n \in \mathbb{Z};$$

iii) every infinitely generated fractional $O'$-ideal $a$ is equal to

$$a = \varphi_n = (u^it^n \mid \text{for all } i \in \mathbb{Z}), \ n \in \mathbb{Z};$$

iv) if $\varphi_{(i,j)} = \varphi$ for $(i,j) = (2,1)$ and $\varphi_{(i,j)} = (0)$ for $(i,j) = (i,0)$,

$$O_{(i)} \supset \varphi_{(i,i-1)} \supset \ldots \supset \varphi_{(i,1)} \supset \varphi_{(i,0)};$$

then

$$\mathrm{Hom}_{O'}(O_{(i)}, O_{(j)}) = \left\{ \begin{array}{ll} O_{(j)}, & i \geq j; \\ \varphi_{(j,i)}, & i < j. \end{array} \right.$$ 

**Proof.** The multiplicative structure of the field $K$ can be deduced immediately from the corresponding results for the fields of dimension 1 (see, for example, [13]). We get also that $\nu' : K^* \to \Gamma_K$ is a valuation, if we introduce on $\Gamma_K$ the lexicographical order. Thus for any $x, y \in K^*$ we have

$$x = ay \text{ with } a \in O' \iff \nu'(x) \geq \nu'(y). \quad (1)$$

Let now $a = (x_1, ..., x_n)$ be a finitely generated $O'$-module. If $x \in a$ then $x = \sum a_i x_i$, $a_i \in O'$. From here we see that $\min_{x \in a \setminus \{0\}} \nu'(x)$ exists and it can be achieved for some $x_0 \in a$. This shows that $a = (x_0)$. 

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If $b \subset K$ is an infinitely generated module then for some $i$ the group $b \otimes \mathcal{O}_{(i)}$ will be a $\mathcal{O}_{(i)}$-module with one generator. Let $i$ be the largest index with this property. Then $\nu^{(i)}$ has its minimum on $b$ and $\min \nu^{(j)}$ equals to infinity for $j > i$. This gives our claim.

The last property must be checked only for $i < j$ (otherwise it is obvious). Explicitly it means that

$$\text{Hom}_{\mathcal{O}'}(K, \mathcal{O}) = \text{Hom}_{\mathcal{O}'}(K, \mathcal{O}') = (0)$$

and

$$\text{Hom}_{\mathcal{O}'}(\mathcal{O}, \mathcal{O}') = \varnothing.$$

Both the equalities followed from the results already proved on structure of ideals in the ring $\mathcal{O}'$.

**Remark 1.** These non-noetherian rings play an important role in the whole theory. Usually the higher local fields appear as fields attached to some chain of the subschemes of decreasing codimension [4, ch.4, 7] and many structures related with them can be interpreted in terms of simplicial stucture on the partially ordered set of such chains. It seems that the rings $\mathcal{O}_{(i)}$ cannot be described in these terms. It would be interesting to extend the simplicial language (as described in [4, ch.7]) to cover these rings also.

## 2 BN-pairs

Let $G = \text{SL}(n, K)$ where $K$ is a two-dimensional local field. We put

$$B = \begin{pmatrix} \mathcal{O}' & \mathcal{O}' & \ldots & \mathcal{O}' \\ m & \mathcal{O}' & \ldots & \mathcal{O}' \\ m & m & \ldots & \mathcal{O}' \end{pmatrix},$$

We denote in such way the subgroup of $G$ consisting of the matrices whose entries satisfy the written conditions. Also let $N$ be the subgroup of monomial matrices.

**Definition 2.** Let

$$T = B \cap N = \begin{pmatrix} (\mathcal{O}')^* & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & (\mathcal{O}')^* \end{pmatrix}$$
The group
\[ W_{K/\bar{K}/k} = N/T. \]
will be called the Weyl group.

We also introduce
\[ P = \left( \begin{array}{cccc} O' & \ldots & O' \\ \ldots & \ldots & \ldots \\ O' & \ldots & O' \end{array} \right) \cap G, \]
\[ A = \left\{ \begin{pmatrix} t^{i_k}u^{j_k} & \ldots & 0 \\ 0 & \ldots & 0 \\ & \ddots & \ddots \\ 0 & \ldots & t^{i_n}u^{j_n} \end{pmatrix} \right\}, \text{ for all } k, i_k, j_k \in \mathbb{Z}. \]

If the matrices in this definition satisfy the additional condition: the integer vectors \((i_1, j_1), \ldots, (i_n, j_n)\) are lexicographically ordered, then we get a subset \(A_+\). We set also
\[ U = \left( \begin{array}{ccc} 1 & \ldots & K \\ \ddots & \ddots & \ddots \\ 0 & \ldots & 1 \end{array} \right) \]

**Theorem 2** We have the following decompositions in the group \(G\):

i) the Bruhat decomposition
\[ G = \bigcup_{w \in W} BwB \]

ii) the Cartan decomposition
\[ G = \bigcup_{a \in A_+} PaP \]

iii) the Iwasawa decomposition
\[ G = \bigcup_{a \in A} PaU \]
where all the unions are disjoint ones.

**Proof** can be given as a generalization of the known proofs of these facts for local fields of dimension 1 (see, for example, [8, ch. VI] for the Cartan and Iwasawa decompositions and [9, theorem 3.15] for the Bruhat decomposition). We only outline the main steps here.

**Existence.** This can be done by standard application of elementary transformations to the rows and columns of matrices from the group $G$. Let $e_{i,j}(\lambda)$ be an elementary matrix with $\lambda$ on $(i,j)$-th place. Now let $g = (a_{k,i}) \in G$ and for some $k, i, j$ $\nu'(a_{k,i}) \leq (or < ) \nu'(a_{k,j})$. Then after a multiplication from the right by $e_{i,j}(\lambda)$, $\lambda = -a_{k,i}^{-1}a_{k,j}$ we get 0 on the $(k,j)$-th place. By (1), $\lambda \in O'$ (or $m$). The same fact is true for the multiplication by $e_{i,j}(\lambda)$ from the left.

Multiplying the given matrix from $G$ by $e_{i,j}(\lambda)$ with $\lambda \in O'$ for $i < j$ and $\lambda \in m$ for $i > j$, from the left and from the right we can get a monomial matrix after several steps. This gives the Bruhat decomposition. In other two cases we need also to multiply by permutation matrices (after a multiplication by a suitable matrix from $T$ they belong to $SL(n, O')$). Also we have to change $m$ on $O'$ in the second restriction on matrices $e_{i,j}(\lambda)$ given above.

**Uniqueness.** Let $L = O'e_1 \oplus \ldots \oplus O'e_n$ be a free $O'$-submodule of the space $V$. If $x \in V, g \in GL(V)$, then we put

$$\nu'(x) = \min_i \nu'(x_i), \nu'(g) = \min_{i,j} \nu'(a_{i,j}),$$

(2)

where $x = x_1 e_1 + \ldots + x_n e_n$.

**Lemma 1.** $\nu'(x) \in \Gamma_K \cup \infty$ and we have:

i) $\nu'(x) = \min_{x \in \lambda L} \nu'(\lambda)$,

ii) $\nu'(g) = \nu'(pq)$, if $p, q \in \text{Stab}(L) \cong GL(n, O')$,

iii) $\nu'(g) = \min_{\nu'(x) = 0} \nu'(g(x)) = \min_{x \in L} \nu'(g(x))$.

These properties can be checked precisely as in the case of discrete valuation rings. We denote $\nu'(x)$ by $\nu'_L(x)$ because it depends only on the submodule $L$.

We show how to get the uniqueness for the Bruhat decomposition.
Let $L_k = m_1 \oplus \ldots \oplus m_k \oplus \mathcal{O}'e_{k+1} \ldots \oplus \mathcal{O}'e_n$, $k = 0, \ldots, n - 1$ — free $\mathcal{O}'$-submodules in $V$. We put

$$\delta_{rkl}(g) = \min_{x \in \wedge^r L_k} \nu^r_{\wedge^r L_l}(\wedge^r g(x)),$$

where $r = 1, \ldots, n$ and $\wedge^r L_k$ is a $r$-th external power of the module $L_k$ in $\wedge^r V$. Then we can prove:

$$\delta_{rkl}(g') = \delta_{rkl}(g), \text{ if } g' \in BgB;$$

(since $\forall k, B(L_k) = L_k$) and

if $w, w' \in N$ and $\forall r, k, l \delta_{rkl}(w) = \delta_{rkl}(w')$, then $w'w^{-1} \in T$.

It gives our claim. The uniqueness for the Cartan decomposition can be proved along the same lines (with one module $L$ instead of all $L_k$). The uniqueness for the Iwasawa decomposition is a direct computation.

The proof is finished.

**Remark 2.** The same type decompositions also exist for the group $\text{GL}(n, K)$.

We also conjecture that the decompositions of the theorem (and the known decompositions for the parabolic(parahoric) subgroups in Tits theory [1, ch. IV, §2.5]) can be included in some general theorem formulated in an appropriate simplicial language using the rings $\mathcal{O}_{(i)}$.

Let us study the Weyl group $W$ more carefully. It contains the following elements of order two

$$s_i = \begin{pmatrix}
1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & & & & \\
0 & \ldots & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & \ldots & 0 \\
0 & \ldots & -1 & 0 & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ddots \\
0 & \ldots & 0 & 0 & \ldots & 1
\end{pmatrix}, \text{ } i = 1, \ldots, n - 1;$$
If $n = 2$ then we denote by $w_0$ the element $s_1$ of the group $G$. For general $n$ let $S$ be the constructed set of elements of the Weyl group. Then $\# S = n + 1$ (and $\text{rk}(G) + m$ for $m$-dimensional field) and we have

**Theorem 3.** The Weyl group $W$ has the following properties:

i) $W$ is generated by the set $S$ of its elements of order two,

ii) there exists an exact sequence

$$0 \to E(= \text{Ker} \, \Sigma) \to W_{K/\bar{K}/k} \to W_K \to 1,$$

where

$$\Sigma : \Gamma_K \oplus \ldots \oplus \Gamma_K \to \Gamma_K$$

is a summation map and $W_K$ is isomorphic to the symmetric group $\text{Symm}_n$ of $n$ elements,

iii) the elements $s_i$, $i = 1, \ldots, n - 1$ define a splitting of the exact sequence and the subgroup $\langle s_1, \ldots, s_{n-1} \rangle$ acts on $E$ by permutations,

iv) if $n = 2$ then the group $W$ has a presentation

$$W = \langle w_0, w_1, w_2/w_0^2 = w_1^2 = w_2^2 = e, (w_0w_1w_2)^2 = e \rangle,$$

v) the Weyl groups of the group $G$ (for $n = 2$) over the local fields $K, K/\bar{K}, K/\bar{K}/k$ can be related by the following diagram
Proof. The claims i) - iii) and v) follow from a direct computation. We have to use the multiplicative structure of the local field $K$ and the structure of its valuation rings (theorem 1 of the previous section).

Let us deduce the presentation iv). It is easy to see that the elements $w_0, w_1, w_2$ satisfy the conditions of the theorem. In order to show that there are no other relations we observe that according to the claim ii) of the theorem, the group $W$ can be presented by some generators $a, b$ (free generators of the subgroup $E$), $w_0$, and defining relations:

$$w_0^2 = e, \quad w_0aw_0 = a^{-1}, \quad w_0bw_0 = b^{-1}, \quad ab = ba.$$ 

We may assume that $a = w_0w_1$ and $b = w_0w_2$. It is enough to show that these relations are equivalent to the relations of the claim iv). Indeed, we have

$$w_0aw_0 = w_0w_0w_1w_0 = (w_0w_1)^{-1}$$

and similarly for $b$. Then

$$e = (w_0w_1w_2)^2 = w_0w_1w_2w_0w_1w_0w_0w_2 = ab^{-1}a^{-1}b, \text{ i.e. } ab = ba.$$ 

The same formulas will also give the equivalence between our defining relations in the opposite direction also.

The theorem is proved.

Corollary. The pair $(W, S)$ is not a Coxeter group and furthermore there is no subset $S$ of involutions in $W$ such that $(W, S)$ will be a Coxeter group.

Proof. We prove the second claim at once. Assume that the opposite is true and consider the map of $S$ into the quotient-group $W_K$. Choose an involution $s$ from the image of this set. Then the set $S'$ of elements $S$ mapping to $s$ will generate the Coxeter group $W'$ [1, ch. IV, §1.8]. By the theorem, it will be an extension of the free abelian group $E$ of rank $> 1$ by a group of order 2. We see simultaneously that there is a subgroup of $E$ which has rank $> 1$ and on which the quotient-group acts as a multiplication by $-1$.

We show that this is impossible for a Coxeter group. Let us consider the Coxeter diagram of the pair $(W', S')$(for its definition and the properties we need see [1, ch. IV, §1.9]). It is clear that $\#S' > 1$ and the diagram contains
at least two vertices. If they are not connected by an edge then the group
has to contain a subgroup \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), which is obviously wrong. If the edge
connecting the two vertices is marked by some finite number \( m > 2 \), then our
group has to contain a subgroup \( \mathbb{Z}/m \) which is also impossible. It remains
that the diagram of our group is connected and all the edges are marked by
the symbol \( \infty \). Now if we have only two vertices then \( W' \) cannot include a
free abelian subgroup of rank \( > 1 \). If the number of vertices \( > 2 \) then \( W' \)
must contain a free subgroup of rank \( > 1 \) and this is also impossible.

**Remark 3.** If we consider the Weyl group for the \( \text{SL}(2, K) \) defined over
\( n \)-dimensional local field then it will have \( n + 1 \) generators \( w_0, \ldots, w_n \) and
the defining relations will be \( w_i^2 = 1 \), \( (w_0 w_i w_j)^2 = 1 \) for all \( i, j \). It is not a
Coxeter group also.

**Remark 4.** Here we see the first basic difference between the Tits theory
and ours. The formalism of the \( BN \)-pairs cannot be applied in our situation,
at least, without some substantial modifications. Nevertheless some corollaries
of the Tits axioms are valid, for example the Bruhat decomposition (see
theorem 2 above)

In our situation there exists still some weaker form of the Tits axioms
(from [\[ ch. IV.2\]]. More precisely they will be true only partially and for
\( n = 2 \) we can replace them by the following formula. Let

\[
    w = \begin{pmatrix} 0 & x^{-1} \\ -x & 0 \end{pmatrix}, \quad v = v'(x), \quad w(y) = \begin{pmatrix} 0 & y^{-1} \\ -y & 0 \end{pmatrix}
\]

If \( s = w_1 \) then there are three possibilities:

\[
\begin{align*}
    v \geq 0 & \quad (BwB)(BsB) = BwsB \\
    (0, -1) < v < 0 & \quad (BwB)(BsB) = BwsB \cup_{(1, -1) + v \leq (0, 1)} Bw(y)B \\
    v \leq (0, -1) & \quad (BwB)(BsB) = BwsB \cup_{(1, -1) + v \leq (0, 1) + v} Bw(y)B
\end{align*}
\]

We have the same expression for the diagonal elements \( w \in W \). And if
\( s = w_0, w_2 \) then the Tits axiom T3

\[
    BwBsB \subset BwB \cup BwsB
\]

will be valid. These expressions can be deduced by straightforward but
rather lengthy computations using the elementary transformations from the
proof of theorem 2.
Problem 1. To generalize the notion of $BN$-pair in order to include both the Tits axioms and the infinite decompositions for non-Coxeter groups which appear here.

For the $BN$-pairs attached to the algebraic groups in the Bruhat-Tits theory we also know some finiteness property for the double classes $BwB$. Namely, they are the finite unions of the cosets $B_\eta$. This property is important for the definition of the Hecke rings (see [10]). It is easy to see that this property is not preserved in the higher dimensions. Thus the usual construction cannot be done in our case.

Problem 2. To define an analog of the Hecke ring for the groups over $n$-dimensional local fields for $n > 1$.

3   Bruhat-Tits tree over local field of dimension 1

The complex $\Delta(G, K)$. First we assume that the field $K$ has no additional structure. Then the spherical building of $G = PGL(V)$ over $K$ is a complex $\Delta(G, K)$ whose vertices are lines $l$ in the space $V$. All simplices of higher dimension are degenerate and thus its dimension equals zero. The group $G(K)$ of rational points over $K$ acts on $\Delta(G, K)$ in a transitive way.

Let $B$ be a Borel subgroup of $G$,

$$B = \begin{pmatrix} K^* & K \\ 0 & K^* \end{pmatrix},$$

Then $B$ is the stabilizer of a line in $V$ and thus $B$ is a stabilizer of a vertex of $\Delta(G, K)$. This gives us a one to one correspondence between the Borel subgroups and the stabilizers of the vertices.

The next important object inside $\Delta(G, K)$ is an apartment $\Sigma$. To specify it we need to choose a maximal torus $T$ of $G$

$$T = \begin{pmatrix} K^* & 0 \\ 0 & K^* \end{pmatrix}$$

or equivalently a splitting $V = l_1 \oplus l_2$. This means that the torus $T$ will fix the pair of vertices corresponding to the lines $l_1$ and $l_2$. And this pair is
called an apartment. Its stabilizer is the normalizer $N$ of the torus $T$, 

$$N = \left( \begin{array}{cc} K^* & 0 \\ 0 & K^* \end{array} \right) \cup \left( \begin{array}{cc} 0 & K^* \\ K^* & 0 \end{array} \right).$$

The group $W = N/T$ is called the *Weyl group*. In our case it is of order two and has as a generator an involution

$$w_0 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

We see that the apartments are precisely the orbits of the Weyl group $W$.

**The Complex $\Delta(G,K/k)$**. Now we turn to the case when the field $K$ is a local field (of dimension 1) with residue field $k$. Denote by $\mathcal{O}$ the valuation ring $\mathcal{O}_K$, by $u$ a generator of the maximal ideal $m$ and by $\nu : K \rightarrow \Gamma_K \cong \mathbb{Z}$ the valuation homomorphism. Again $G = PGL(V)$.

We define the euclidean building $\Delta(G,K/k)$ as a one-dimensional complex constructed from classes of lattices in $V$. A lattice $L$ is an $\mathcal{O}_K$-submodule in $V$ which is free and of rank 2. A class $< L >$ of lattices is the set of all lattices $aL$ for $a \in K^*$. We say that two classes $< L >$ and $< L' >$ are connected as the vertices by an edge iff for some choice of $L$ and $L'$ we have an exact sequence

$$0 \rightarrow L' \rightarrow L \rightarrow k \rightarrow 0.$$ 

This is equivalent to existence of a maximal totally ordered chain of $\mathcal{O}$-submodules in $V$ which is invariant under multiplication on $K^*$ and contain $L$ and $L'$ [3]. Then from the combinatorial point of view $\Delta(G,K/k)$ is a homogeneous tree [14].

We denote by $\Delta_i(G,K/k)$ the set of $i$-dimensional simplices. From the construction we deduce easily the following property which we will use in the sequel:

**Link property**

Let $P \in \Delta_0(K/k)$ be represented by a lattice $L$. Then the set of edges going from $P$ is in one to one canonical correspondence with the set of lines $\mathbb{P}(V_P)$ in the vector space $V_P = L/mL$ of dimension 2 over $k$. The last set does not depend on the choice of $L$ (or better to say that there are canonical isomorphisms between these $\mathbb{P}(V_P)$ for different $L$’s in the same class $< L >$).
In particularly, if \( k = \mathbb{F}_q \) is a finite field then \( \Delta(G, K/k) \) is locally finite.

The group \( SL(V) \) acts on \( \Delta(G, K/k) \) in the following way. It is transitive on the edges \( \in \Delta(G, K/k) \) and has two orbits on the vertices \( \in \Delta(K/k) \).

There is a type of the vertices \( P \) which has two values. To understand this consider an exact sequence

\[
0 \to PGL^+(V) \to PGL(V) \to \mathbb{Z}/2\mathbb{Z} \to 0
\]

where the last homomorphism is \( \nu(det(.)) \mod 2 \). The group \( PGL(V) \) acts on \( \Delta_0(G, K/k) \) in a transitive way and the group \( PGL^+(V) \) has the same two orbits as it’s subgroup \( SL(V) \).

Now let

\[
B = \begin{pmatrix}
\mathcal{O} & \mathcal{O} \\
m & \mathcal{O}
\end{pmatrix}
\]

be the subgroup of \( SL(V) \) consisting of the matrices whose entries satisfy the written conditions. Then \( B \) is a stabilizer of an edge of the \( \Delta(G, K/k) \) and all the stabilizers look like this in an appropriate coordinate system of \( V \). The stabilizers of the boundary vertices of the edge are

\[
P_0 = \begin{pmatrix}
\mathcal{O} & \mathcal{O} \\
\mathcal{O} & \mathcal{O}
\end{pmatrix},
P_1 = \begin{pmatrix}
\mathcal{O} & m^{-1} \mathcal{O} \\
m & \mathcal{O}
\end{pmatrix}
\]

We define the subgroup \( N \) as above (so it does not reflect the local structure on \( K \)). Instead of the maximal torus we take

\[
T = \begin{pmatrix}
\mathcal{O}^* & 0 \\
0 & \mathcal{O}^*
\end{pmatrix}
\]

and the Weyl group \( W = N/T \) is an extension

\[
0 \to \mathbb{Z} \to W \to \mathbb{Z}/2\mathbb{Z} \to 0
\]

Here we can identify the \( \mathbb{Z} \) with the valuation group \( \Gamma_K \) and the \( \mathbb{Z}/2\mathbb{Z} \) with the previous Weyl group of \( G \) over the field \( K \) without local structure.

We have a new involution

\[
w_1 = \begin{pmatrix}
0 & u \\
-u^{-1} & 0
\end{pmatrix}
\]
and the group \( W \) is generated by \( w_0 \) and \( w_1 \).

The appartments \( \Sigma \) are now the infinite lines:

\[
\ldots \quad x_{n-1} \quad x_n \quad x_{n+1} \quad \ldots
\]

The group \( T \) acts trivially on \( \Sigma \) and it’s stabilizer coincides with \( N \). Thus appartments are the orbits of \( W \). The vertices of \( \Sigma \) can be represented by lattices

\[
x_n = < L_n >, \quad L_n = \mathcal{O} \oplus m^n, \quad -\infty < n < \infty
\]  

(4)

The action of \( W \) on \( \Sigma \) can now be easily described. If \( w \in \mathbb{Z} \) then \( w \) acts by a translation of even length, and if \( w \notin \mathbb{Z} \) then \( w \) acts as an involution with a unique fixed point \( x_{n_0} \):

\[
w(x_{n_0} + n) = x_{n_0} - n
\]

It can be proved that all the appartments look like this and thus they are in one to one correspondence with the splittings \( V = l_1 \oplus l_2 \) of the space \( V \).

Relations between \( \Delta(G, K) \) and \( \Delta(G, K/k) \). With the local field \( K \) of dimension 1 we can connect two local fields, namely \( K \) itself and \( k \). They are local fields of dimension 0. Thus we have three buildings attached to \( G : \Delta(G, K/k), \Delta(G, K) \) and \( \Delta(G, k) \).

The remark made above (the Link property) shows that the link of a point \( P \in \Delta(G, K/k) \) (= the boundary of the Star\((P)\)) is isomorphic to \( \Delta(G, k) \). The group \( G(k) \) acts on the last building, \( P_0 \) acts on the link of \( P \) and the isomorphism between the buildings is an equivariant respective canonical homomorphism from \( P_0 \) onto \( G(k) \) (reduction map mod \( m \)).

To formalize the connection with \( \Delta(G, K) \) we define a boundary point of a tree as a class of half-lines such that intersection of any two half-lines from the class is a half-line in both of them. We have now an isomorphism of \( G(K) \)-sets between the set of boundary points and \( \Delta(G, K) \). If the half-line is represented by \( L_n = \mathcal{O} \oplus m^n, n > 0 \) then the corresponding vertex in \( \Delta(K) \) is the line \( K \oplus (0) \) in \( V \).

It seems reasonable to consider the complexes \( \Delta(G, K) \) and \( \Delta(G, K/k) \) together.

Denote by \( \Delta.[1](G, K/k) \) the complex of lattices introduced above. We define the tree of \( G \) as a union

\[
\Delta(G, K/k) = \Delta.[1](G, K/k) \cup \Delta.[0](G, K)
\]

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where $\Delta[1](G, K/k) = \Delta(G, K/k)$ and $\Delta[0](G, K/k)$ is a complex of classes of $\mathcal{O}$-submodules in $V$ isomorphic to $K \oplus \mathcal{O}$ and we will call the subcomplex $\Delta[0](G, K/k)$ the boundary of the tree. The definition of the boundary gives a topology on $\Delta_0(G, K/k)$ which is discrete on both subsets $\Delta_0[1]$ and $\Delta_0[0]$.

Let $P_n = \langle L_n \rangle$ and $L_n = \mathcal{O}e_1 + m^n e_2$. If $P \in \Delta[0]$ is represented by a line $l_1 = Ke_1$ then $P_n \to P$ since $\cap L_n = \mathcal{O}e_1$ belongs to a unique line, namely to $l_1$ (see [14, ch.II.1.1]). We can interpret the points from $\Delta[0]$ as classes of $\mathcal{O}$-submodules which are isomorphic to $K \oplus \mathcal{O}$ (see lemma 2 below). Then we have $P = \langle L \rangle$, $L = Ke_1 + \mathcal{O}e_2$ instead of $l_1$ and the definition of the convergence can be given as $\cup m^{-n} L_n = L$.

It is easy to extend it to 1-simplexes. In our case their limits at infinity will be the degenerate simplexes. Thus we have a structure of a simplicial topological space on the tree. It is simply a simplicial object in the category of topological spaces. This topology is stronger then the topology usually introduced to connect these complexes together (see [3]).

Now the connections between the buildings over local fields of dimension 0 and 1 can be summarized as follows.

For any $P \in \Delta[1](PGL(V), K/k)$, $\text{Link}(P) = \Delta(PGL(V), k)$

$$\Delta[0](PGL(V), K/k) = \Delta(PGL(V), K)$$

The last subset can be called a star (or link) at infinity. It will be interesting to define the last notion in purely simplicial terms (see remark 6 below).

4 Bruhat-Tits tree over local field of dimension 2

As above let $G = \text{PGL}(V)$ be projective linear group of a vector space $V$ of dimension 2 over a field $K$ and we assume now that $K$ is a two-dimensional local field.

**Definition 3.** The vertices of the Bruhat-Tits tree.

$$\Delta_0[2](G, K/\bar{K}/k) = \{\text{classes of } \mathcal{O}'\text{-submodules } L \subset V : L \cong \mathcal{O}' \oplus \mathcal{O}'\},$$

$$\Delta_0[1](G, K/\bar{K}/k) = \{\text{classes of } \mathcal{O}'\text{-submodules } L \subset V : L \cong \mathcal{O}' \oplus \mathcal{O}\},$$

$$\Delta_0[0](G, K/\bar{K}/k) = \{\text{classes of } \mathcal{O}'\text{-submodules } L \subset V : L \cong \mathcal{O}' \oplus K\}.$$
The two submodules $L$ and $L'$ belong to one class $< L >= < L ' >$, iff $L = aL'$, with $a \in K^*$. 

\[ \Delta_0(G, K/\bar{K}/k) = \Delta_0[2](G, K/\bar{K}/k) \cup \Delta_0[1](G, K/\bar{K}/k) \cup \Delta_0[0](G, K/\bar{K}/k) \]

We say that the points from $\Delta_0[2]$ are the inner points, the points from $\Delta_0[1]$ are the inner boundary points and the points from $\Delta_0[0]$ are the external boundary points.

Sometimes we will delete $G$ and $K/\bar{K}/k$ from our notation if this does not lead to a confusion.

We have defined the vertices only. For the simplices of higher dimension we have the following

**Definition 4.**

Let \( \{ L_\alpha, \alpha \in I \} \) be a set of $\mathcal{O}'$-submodules in $V$. We say that \( \{ L_\alpha, \alpha \in I \} \) is a chain iff:

i) for any $\alpha \in I$ and for any $a \in K^*$ there exists an $\alpha' \in I$ such that $aL_\alpha = L_{\alpha'}$,

ii) the set \( \{ L_\alpha, \alpha \in I \} \) is totally ordered by the inclusion.

\( \{ L_\alpha, \alpha \in I \} \) is a maximal chain iff it cannot be included in a strictly larger set satisfying the same conditions i) and ii).

We say that $< L_0 >, < L_1 >, ..., < L_m >$ belong to a simplex of dimension $m$ iff the $L_i, i = 0, 1, ..., m$ belong to a maximal chain of $\mathcal{O}'$-submodules in $V$. The faces and the degeneracies can be defined in a standard way (as a deletion or a repetition of some vertex).

Thus the set $\Delta(G, K/\bar{K}/k)$ becomes a simplicial set. The group $G = \text{PGL}(V)$ acts on $\mathcal{O}'$-modules. This gives a simplicial action on $\Delta(G, K/\bar{K}/k)$.

**Proposition 1.** The set of all maximal chains of $\mathcal{O}'$-submodules in the space $V$ will be exhausted by the following three possibilities:

i) $\ldots \supset m_{i,n}L \supset m_{i,n}L' \supset m_{i+1,n}L \supset m_{i+1,n}L' \supset \ldots \supset m_{i,n+1}L \supset m_{i+1,n+1}L \supset \ldots$, $i, n \in \mathbb{Z}$,

where $< L >= < L ' > \in \Delta_0(G, K/\bar{K}/k)[2]$ and $L \cong \mathcal{O}' \oplus \mathcal{O}'$, $L' \cong m \oplus \mathcal{O}'$.

ii) $\ldots \supset m_{i,n}L \supset m_{i+1,n}L \supset m_{i+2,n} \supset \ldots \supset m_{i,n}L' \supset m_{i+1,n}L' \supset \ldots \supset m_{i,n+1}L \supset m_{i+1,n+1}L \supset \ldots$, $i, n \in \mathbb{Z}$,
where \( < L >, < L' > \in \Delta_0(G, K/\bar{K}/k)[1] \) and \( L \cong O' \oplus O, L' \cong \wp \oplus O' \).

where \( m_{i,n}L \supset m_{i+1,n}L \supset m_{i+1,n+1}L \supset \ldots, i, n \in \mathbb{Z} \)

\( \text{iii) } \ldots \supset m_{i,n}L \supset m_{i+1,n}L \supset \ldots \supset m_{i+1,n+1}L \supset \ldots, i, n \in \mathbb{Z} \)

\( \text{where } < L > \in \Delta_0(G, K/\bar{K}/k)[0] \).

\[ \text{Proof.} \]

The chains in the claim of our proposition can be completed by the \( O' \)-modules which are isomorphic to \( O \oplus O \) and thus do not belong to the modules from definition 3. Then a part (with \( n = 0 \)) of the chain of first type will look as follows:

\[ \ldots \supset O \supset \ldots \supset L \supset mL \supset \ldots \supset \wp L \supset \ldots, \]  

where \( O = O' \cong O \oplus O \) and \( \wp = \wp' \cong \wp \oplus \wp \). There is an isomorphism \( O/L \cong \bar{K} \oplus \bar{K} \). For the same part of the chain of second type we have:

\[ \ldots \supset O \supset \ldots \supset L \supset mL \supset \ldots \supset O' \supset \ldots \]

\[ \ldots \supset L' \supset mL' \supset \ldots \supset \wp L \supset \ldots, \]  

where \( O \cong O \oplus O, O' \cong \wp \oplus O \) and \( \wp \cong \wp \oplus \wp \). Again there exist isomorphisms \( O/L \cong \bar{K}, O'/\wp L \cong \bar{K} \). The last chain has the following structure:

\[ \ldots \supset O \supset \ldots \supset L \supset mL \supset \ldots \supset \wp L \supset \ldots, \]  

where \( L \cong K \oplus O' \) and \( O/L \cong \bar{K} \).

Let us go to the proof of our proposition. It is easy to see that the modules which we have inserted into our chains are the unions (intersections) of those \( L_\alpha \) which are just to the right (left) of them. Furthermore, if \( L_{\alpha'} \) is the module, located after \( L_\alpha \), then \( L_\alpha /mL_\alpha \cong k \) (theorem 1). It follows that all the chains from i) – iii) are maximal ones. Now let \( L_\alpha \) be an arbitrary maximal chain satisfying to the definition 3. We consider three cases:

1) For some \( \alpha L_\alpha \cong K \oplus O' \). Then all modules \( m_{i,n}L_\alpha \) enter into our chain, i.e. it will coincide with the chain from iii).

2) For some \( \alpha L_\alpha \cong O' \oplus O' \). Again all modules \( m_{i,n}L_\alpha \) belong to the chain, but now \( L_\alpha /mL_\alpha \) has dimension 2 over \( k \) and since our chain is supposed to be a maximal one there exists a module \( L' \) between these two. All \( m_{i,n}L' \) are in the chain, which should coincide with the chain from i).

3) Now if \( L_\alpha \cong O \oplus O' \), then the chain contain subchains \( \ldots \supset m_{i,n}L_\alpha \supset m_{i+1,n}L_\alpha \supset \ldots, \) lying in between the modules \( \wp L_\alpha \) and \( \wp_{n+1}L_\alpha \). Choose some \( n \). The intersection of all \( m_{i,n}L_\alpha \) for varying \( i \) gives us a module \( L'' \supset \ldots \)
φ_{n+1}L_α. An "empty" place which we have to the right of $L''$ can be filled out if we set $L' = \varphi^{-1}(\mathcal{O}_K)$, where $\varphi : L'' \to L''/\varphi L \cong \bar{K}$. Then all "multiples" $m_i,nL'$ should be presented in the chain because of its maximality. We see that the chain constructed in such way is equal to the chain from ii).

The proposition is proved.

**Corollary.** The simplicial set $\Delta$ is a disconnected union of its subsets $\Delta[m]$, $m = 0, 1, 2$. The dimension of the subset $\Delta[m]$ equals to 0 for $m = 0$ and 1 for $m = 1, 2$.

This is obvious. We need only note that all vertices of any simplex can be represented by the modules of the same type and that in the case of subset $\Delta[0]$ the chains of the type iii) contain only one class of modules.

**Definition 5.** The projection map.

For any $\mathcal{O}'$-module $L$ we have an $\mathcal{O}$-module $M = L \otimes_{\mathcal{O}'} \mathcal{O}$. This gives a map

$$
\pi : \Delta(K/\bar{K}/k) \rightarrow \Delta(K/\bar{K})
$$

in the tree of the same group $G$ over the field $K$, which we consider as a local field of dimension 1 over $\bar{K}$.

**Proposition 2.** The map $\pi$ has the following properties:

i) $\pi$ is a simplicial $G$-equivariant surjective map,

ii) $\pi$ induces a bijective map of the set $\Delta(G, K/\bar{K}/k)[0]$ onto the set $\Delta(G, K/\bar{K})[0]$,

iii) if $\sigma = < L > \in \Delta_0(G, K/\bar{K})[1]$, then there exist simplicial and $\text{Stab}(< L >)$-equivariant isomorphisms

$$
\pi^{-1}(\sigma) \bigcap \Delta(G, K/\bar{K}/k)[2] \cong \Delta(PGL(L/\varphi L), \bar{K}/k)[1],
$$

$$
\pi^{-1}(\sigma) \bigcap \Delta(G, K/\bar{K}/k)[1] \cong \Delta(PGL(L/\varphi L), \bar{K}/k)[0],
$$

where $L/\varphi L$ is a vector space of dimension 2 over $\bar{K}$. Also we have

$$
\pi^{-1}(\sigma) \bigcap \Delta(G, K/\bar{K}/k)[0] = \emptyset,
$$

iv) if two vertices from $\Delta_0(G, K/\bar{K}/k)[2]$ are connected by an edge then they belong to the same fiber of the map $\pi$,

v) the image of any edge $\sigma \in \Delta_1(K/\bar{K}/k)[1]$ will be (non-degenerate) edge in $\Delta(K/\bar{K})$. 

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The property i) is obvious. Let \( < L > \in \Delta_0(K/\bar{K})[0] \) and let \( l \subset L \) be the set of elements from \( L \), divisible in \( L \) by all \( a \in K^* \).

**Lemma 2.** The correspondence \( < L > \mapsto l \subset V \) is a bijection between \( \Delta_0[0] \) and \( P(V) \).

**Proof.** If \( L = Ke_1 \oplus \mathcal{O}e_2 \), then \( l = Ke_1 \) and depends only on class \( < L > \). We can get all the lines in such way. Now let \( L = Ke_1 \oplus \mathcal{O}e_2 \), \( M = Ke_1 \oplus \mathcal{O}e_2' \). If \( e_2' = ae_1 + be_2 \), \( a, b \in K \), then \( M = Ke_1 \oplus \mathcal{O}be_2 = bL \), i.e. \( < L > = < M > \).

This is also true for \( \Delta_0(K/\bar{K})[0] \). The claim ii) follows since the projection commutes with the constructed correspondence.

**Lemma 3.** Let \( L \) be a \( \mathcal{O}' \)-submodule in \( V \) and \( L \cong \mathcal{O}' \oplus \mathcal{O}' \). Then for any point \( P \in \Delta_0[2] \) there exists a unique module \( L' \) such that \( < L' > = P \), \( L' \subset L \) and one of the following equivalent conditions are true:

- i) \( L' \not\subset mL \),
- ii) \( L/L' \cong \mathcal{O}'/a \), where \( a \) is a principal ideal,
- iii) \( L/L' \) is a module of rank 1.

**Proof.** Take some module \( L'' \) in the class of the vertex \( P \). By the Cartan decomposition (theorem 2) there exists a basis \( e_{1,2} \) in \( V \) such that \( L = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \), \( L'' = a_1e_1 \oplus a_2e_2 \), \( a_1, a_2 \) are principal ideals. The standard arguments (see [14], ch. II, §1.1) give the claim of the lemma.

We now prove property iii). Fix a module \( L_0 \cong \mathcal{O}' \oplus \mathcal{O}' \), \( L_0 \otimes_{\mathcal{O}'} \mathcal{O} = L \), i.e. \( < L_0 > \in \pi^{-1}(\sigma) \cap \Delta_0[2] \). If \( P \in \pi^{-1}(\sigma) \), then according to lemma the vertex \( P \) can be represented as \( < L' > \). Then \( L' = m_{i,n}e_1 + \mathcal{O}'e_2 \), \( L_0 = \mathcal{O}'e_1 + \mathcal{O}'e_2 \) and from the equality \( \pi < L_0 > = \pi < L' > = < L > \) we get that \( n = 0 \). It follows that \( L' \supset \mathcal{O}L_0 \) and \( L' \) defines a free \( \mathcal{O}_K \)-module \( L'/\mathcal{O}L_0 \subset L_0/\mathcal{O}L_0 \subset L/\mathcal{O}L \) of rank 2 in space \( L/\mathcal{O}L \).

This correspondence gives the first bijection from iii). It is easy to see that it preserves the simplicial structure of both sets and it is equivariant under the stabilizer of the vertex \( < L > \).
To construct the second bijection from ii) we take $P \in \pi^{-1}(\sigma) \cap \Delta_0[1]$. If $P = < L' >$ then $< L' \otimes O > = < L >$. Changing the module $L'$ to an equivalent one we can assume that $L' \otimes O = L$ and $L' \subset L$. All such modules $L'$ can be transformed into one by a multiplication by some $a \in O^*$. We have a map $L' \to L/\varphi L$. The image $\text{Im} \ L'$ will be a $O_K$-module in $L/\varphi L$ isomorphic to $\bar{K} \oplus O_K$. As we saw the class $< \text{Im} \ L' >$ will be defined in a unique way. It defines a point in $\Delta_0(\text{PGL}(L/\varphi L), \bar{K}/k)[1]$. The constructed correspondence will be a bijection with the properties we need.

The last claim from iii) follows from the property ii) proved above.

To get iv) it is enough to apply lemma 2 and proposition 1, i).

The property v) can be seen from the description of the chain (6) which represents the edge $\sigma$. We need only to take it's quotient by the ideal $\varphi$.

We check now the last property from the proposition. Let $P = < L >$, $Q = < L' >$ be two vertices of the tree $\Delta(K/\bar{K})$ connected by an edge $\sigma$. It is possible to choose a basis in $V$ such that $L = Oe_1 + Oe_2$, $L' = \varphi e_1 + Oe_2$. Then $O'$-modules $M = O'e_1 + O'e_2$ and $M' = \varphi e_1 + O'e_2$ will represent the boundary points of the fibers $\pi^{-1}(P)$ and $\pi^{-1}(Q)$ correspondingly. By the proposition 1, ii) they are connected by an edge which is mapped onto an edge $\sigma$. Thus the set $\pi^{-1}(\sigma)$ is not empty.

It consists of only one edge. To make this clear we denote by $M, M'$ the modules which represent the vertices of the edge lying over $\sigma$. By the proposition 1, ii) they belong to a chain as in (6). Now we remark that the set of lines of the space $L/\varphi L$ is bijective to the following sets of simplices of our trees:

- the set of the edges from $\Delta(K/\bar{K})$ going out from the vertex $P$ (link property, see section 3).
- the set $\pi^{-1}(P) \cap \Delta_1[1]$ of the boundary points of the fiber $\pi^{-1}(Q)$ (the bijection constructed above).

From the definition of the bijection we conclude that the line corresponding to the vertex $< M >$, coincides with the line corresponding to the edge $\sigma$, i. e. the vertex $< M >$ will be defined uniquely. As this is true also for the vertex $< M' >$ we get that the edge connecting them will be also defined in an unique way.

The proposition is proved.
Corollary 1. Any vertex \( P \in \Delta_0[1] \) belongs to precisely one edge.

Corollary 2. If \( P \in \Delta_0(K/K)[1] \) then the stabilizer \( G_P \subset G \) of the vertex \( P \) acts on the fiber \( \pi^{-1}(P) \) by the reduction map 
\[
G_P \cong \text{SL}(2, \mathcal{O}_K) \to \text{SL}(2, \bar{K}).
\]

Here we have fixed the modules \( L \) with \( < L > = P \) and \( L_0 \) with \( < L_0 > \in \pi^{-1}(P) \cap \Delta_0[2] \).

We see that "inside" our construction there are five trees coming from the dimensions \( \leq 1 \), namely
\[
\Delta(K/K), \Delta(K), \Delta(\bar{K}/k), \Delta(\bar{K}), \Delta(k).
\]
The first one is the tree which is the target of the projection map \( \pi \), the second one is the external boundary and the three last trees will occur infinitely many times.

Thus the constructed simplicial set will be a disconnected union of it’s connected components. The \( \Delta[2] \)-piece of our tree is an infinite disconnected union of the usual Bruhat-Tits trees = fibers of the map \( \pi \). The \( \Delta[1] \)-piece will be an infinite disconnected union of the edges.

In order to change this and to have a possibility to pass from one fiber to another one has to use some topology which will be a generalization of the topology we have introduced in section 3.

Definition 6. We say that a sequence \( P_n \in \Delta_0[2] \) converges to \( Q \in \Delta_0[1] \) iff there is a basis of \( V \) and a sequence \( i(n) \) of integers such that \( i(n) \to \infty \) as \( n \to \infty \) and for large \( n \) \( P_n \) and \( Q \) can be represented by the following modules
\[
P_n = < \mathcal{O}' \oplus m^{i(n)} >, \quad Q = < \mathcal{O} \oplus \mathcal{O}' >
\]
Also a sequence \( Q_n \) from \( \Delta_0[1] \) converges to a point \( R \) from \( \Delta_0[0] \) iff in some basis and for some sequence \( i(n) \) as above
\[
Q_n = < \mathcal{O}' \oplus \mathcal{O}^{i(n)} >, \quad R = < K \oplus \mathcal{O}' >
\]
Combining these two definitions we can get also a condition for a sequence of points from \( \Delta_0[2] \) to converge to a point from \( \Delta_0[0] \).

We introduce a topology on \( \Delta_0(G, K/K/k) \) as a discrete one on any of the sets \( \Delta_0[m] \) and for which the sequences introduced above are the only convergent sequences on the whole set. \( \Delta_0 \). The convergence on the set of simplices \( \Delta_1 \) can be defined as convergence of their vertices.
Theorem 4. $\Delta(G, K/\bar{K}/k)$ is a simplicial topological space. Let $|\Delta|$ be its geometrical realization. Then

i) $|\Delta|$ is a connected contractible topological space of dimension 1 having a cell structure,

ii) if $x \in |\Delta|$ then $x$ has a neighbourhood homeomorphic to an interval, if $x \notin \Delta_0[2]$, and to a bouquet of (finite number if $k = F_q$) intervals otherwise.

iii) the group $G$ acts on $|\Delta|$ by homeomorphisms

iv) $|\pi|$ is a continuous map

v) if $\sigma = <L> \in \Delta_0(G, K/\bar{K})[1]$ then the fiber $\pi^{-1}(\sigma)$ is isomorphic to $\Delta(PGL(L/\wp L), K/\bar{K})$ as a simplicial topological space.

We refer to [6] for the notions of simplicial topological space and its geometrical realization.

Proof can be given by a direct check with an application of the proposition 2 and of the corresponding facts for the trees $\Delta(K/\bar{K})$ and $\Delta(\bar{K}/k)$ related to the local fields of dimension 1.

Remark 5. $|\Delta|$ is not a CW-complex even if $n = 1$ and $k = F_q$ but it is a closure finite complex. Also we note that $|\Delta(K/\bar{K}/k)|$ is not a compact space just as in the case of local fields of dimension 1.

We can make the results proved more transparent by drawing all that in the following picture where the dots of different kinds belong to the different $\Delta[m]$-pieces of the tree:
Usually the buildings are defined as combinatorial complexes having a system of subcomplexes called appartments (see, for example, [12, 15]). We show how to introduce them in our case.

**Definition 7.** Let us fix a basis $e_1, e_2 \in V$. The *apartment*, defined by this basis is the following set

$$\Sigma = \bigcup_{0 \leq m \leq 2} \Sigma[m],$$
where

\[ \Sigma_0[m] = \begin{cases} 
\langle L \rangle & | L = a_1 e_1 \oplus a_2 e_2, \\
& \text{where } a_1, a_2 \text{ are } \mathcal{O}'-\text{submodules in } K \\
& \text{and there exists a permutation } s, \\
& \text{such that } a_{s(1)} \cong \mathcal{O}_{(2)} = \mathcal{O}', \ a_{s(2)} \cong \mathcal{O}_{(m)} 
\end{cases} \]

\( \Sigma[m] \) is the minimal subcomplex having \( \Sigma_0[m] \) as vertices.

Let us denote the edge connecting the vertices \( P \) and \( Q \) by \( \sigma(P, Q) \).

**Proposition 3**. In some basis we have the following relations:

i) if

\[
\begin{align*}
\sigma_{\alpha} &= \langle m_{i,n} \oplus \mathcal{O}' \rangle = \langle \mathcal{O}' \oplus m_{i,-n} \rangle, \\
\sigma_{\beta} &= \langle m_{i,n} \oplus \mathcal{O} \rangle = \langle m_{j,n} \oplus \mathcal{O} \rangle = \langle \mathcal{O}' \oplus \mathcal{O}^{-n} \rangle, \\
\sigma_{\gamma} &= \langle \mathcal{O} \oplus m_{i,-n} \rangle = \langle \mathcal{O} \oplus m_{j,-n} \rangle = \langle \mathcal{O}' \oplus \mathcal{O}' \rangle,
\end{align*}
\]

then

\[
\begin{align*}
\Sigma_0[2] &= \{ x_{i,n} \ | i, n \in \mathbb{Z} \}, \quad \Sigma_1[2] = \{ \sigma(x_{i,n}, x_{i+1,n}) \ | i, n \in \mathbb{Z} \}, \\
\Sigma_0[1] &= \{ y_n, z_n \ | n \in \mathbb{Z} \}, \quad \Sigma_1[1] = \{ \sigma(y_n, z_n) \ | n \in \mathbb{Z} \}, \\
\Sigma[0] &= \{ x_0, x_\infty \}.
\end{align*}
\]

ii) let \( \text{Stab}(\sigma) \) be a stabilizer of a simplex \( \sigma \) in the subgroup \( \text{SL}(V) \). Then

\[
\begin{align*}
\text{Stab}(x_{i,n}) &= \begin{pmatrix} \mathcal{O}' & m_{i,n} \\ m_{i,-n} & \mathcal{O}' \end{pmatrix}, \quad \text{Stab}(\sigma(x_{i,n}, x_{i+1,n})) = \begin{pmatrix} \mathcal{O}' & m_{i+1,n} \\ m_{i,-n} & \mathcal{O}' \end{pmatrix}, \\
\text{Stab}(z_n) &= \begin{pmatrix} \mathcal{O} & \mathcal{O}' \\ \mathcal{O}' & \mathcal{O} \end{pmatrix}, \quad \text{Stab}(y_n) = \begin{pmatrix} \mathcal{O}' & \mathcal{O}' \\ \mathcal{O}' & \mathcal{O}' \end{pmatrix}, \\
\text{Stab}(\sigma(y_{n-1}, z_{n})) &= \begin{pmatrix} \mathcal{O}' & \mathcal{O}' \\ \mathcal{O}' & \mathcal{O}' \end{pmatrix}, \\
\text{Stab}(x_0) &= \begin{pmatrix} K^* & 0 \\ 0 & K^* \end{pmatrix}, \quad \text{Stab}(x_\infty) = \begin{pmatrix} K^* & K^* \\ K^* & K^* \end{pmatrix}.
\end{align*}
\]

The stabilizers in the \( \text{PGL}(V) \) are represented by the matrices from \( \text{GL}(V) \) satisfying the same conditions.
Proof. It is obvious that all the vertices from i) belong to Σ. It follows from the theorem 1 (section 1) that there are no other vertices. It is also clear that the simplicial complex described in i) is a minimal complex containing it’s vertices.

The formulas for the stabilizers (property ii) can be confirmed by direct computations.

Thus the simplicial structure of an apartment can be presented as the following triangulation of compactified line $\mathbb{R} \cup -\infty, \infty$:

```
x_0 \ y_{n-1} \ z_n \ x_{i,n} \ x_{i+1,n} \ y_n \ z_{n+1} \ x_{i,n+1} \ x_{\infty}
```

Pic. 2

Theorem 5 The appartments $\Sigma$. have the following properties:

i) any two simplices are contained in an apartment,

ii) for any two apppartments $\Sigma, \Sigma'$ there exists an isomorphism $i : \Sigma \to \Sigma'$ such that $i|_{\Sigma \cap \Sigma'} = \text{identity}$,

iii) for any apartment $\Sigma \subset \Delta(G, K/\bar{K})$ there exists a unique apartment $\Sigma \subset \Delta(G, K/\bar{K}/k)$ such that $\pi(\Sigma) = \bar{\Sigma}$,

iv) a geometrical realization $|\Sigma|$ of an apartment $\Sigma$, is homeomorphic to a closed interval,

v) $\Sigma = \{\sigma \in \Delta \mid \forall g \in T \ g(\sigma) = \sigma\}, \ N(\Sigma) \subset \Sigma$, and the Weyl group $W$ acts on $\Sigma$.

If $w \in W$ is an involution then it has a fixed point $x_{i_0,n_0} \in \Sigma_0[2]$ and $w$ is a reflection:

\[ w(x_{i,n}) = x_{2i-2n-n_0}, \]

\[ w(y_{n_0+n}) = z_{n_0-n}, \ w(z_{n_0+n}) = y_{n_0-n}, \ w(x_0) = x_{\infty}. \]

If $w \in \Gamma_K \cong \mathbb{Z} \oplus \mathbb{Z} \subset W$ then $w = (0, 1)$ acts as a shift of the whole structure to the right

\[ w(x_{i,n}) = x_{i,n+2}, \]

\[ w(y_n) = y_{n+2}, \ w(z_n) = z_{n+2}, \ w(x_0) = x_0, \ w(x_{\infty}) = x_{\infty}. \]
The element \( w = (1, 0) \) acts as a shift on the points \( x_{i,n} \) but leaves fixed the points in the inner boundary

\[
w(x_{i,n}) = x_{i+2,n}, \quad w(y_n) = y_n, \quad w(z_n) = z_n, \quad w(x_0) = x_0, \quad w(x_\infty) = x_\infty.
\]

Under the map \( W_{K/\bar{K}/k} \to W_{K/\bar{K}} \) this action goes to the action of Weyl group \( W_{K/\bar{K}} \) on an appartment of the tree \( \Delta(K/\bar{K}) \).

**Proof.** If we compare the modules belonging to an appartment according to proposition 3, ii) and the modules belonging to an appartment of the tree over a local field of dimension 1 (see (I) in section 3) we will see that they will go one to another under the projection map. Thus we can find an appartment \( \Sigma \) in \( \Delta(K/\bar{K}/k) \), projecting onto any given appartment of the tree \( \Delta(K/\bar{K}) \). Note that an appartment always contains an edge from \( \Delta_1[1] \). Then from proposition 2, vi) we get that \( \Sigma \) will be defined in an unique way. We have proved iii).

Let us prove the property i). For any two simplices there exists a subcomplex in \( \Delta(K/\bar{K}/k) \), having the same combinatorial and topological structure as the line from picture 2 and containing our simplices. This is obvious from the picture 1. The image of this complex will be an infinite chain, i.e. an appartment \( \bar{\Sigma} \) in the tree \( \Delta(K/\bar{K}) \) (see section 3). By the result proved above, all edges of our subcomplex which belong to \( \Delta[1] \) will also belong to an appartment \( \Sigma \) lying over \( \bar{\Sigma} \). Next we consider the trees which are the fibers of \( \pi \). Looking at them we see that the other simplices of our subcomplex also belong to \( \Sigma \) (if two appartments in the usual Bruhat-Tits tree have the same boundary points then they coincide).

To get ii) we remark that the intersection \( \Sigma \cap \Sigma' \) will be an “interval”, consisting of all the simplices lying between two extreme points. From picture 2 we see the existence of an isomorphism with the properties which we need.

The property iv) is obvious. In v) we check only the first claim. The other formulas can be deduced by direct computations. Let \( \sigma \notin \Sigma \), and let \( \sigma \) be a vertex. Connect this vertex with \( \Sigma \) by a minimal ”path” (= interval of an appartment). This path will enter into the appartment \( \Sigma \) at an inner point (corollary 1 of proposition 2). Let \( P \) be a vertex nearest to \( \Sigma \) belonging to this path. Then \( P \) belong to a usual Bruhat-Tits tree and there exists \( g \in G \) such that \( g(P) \neq P \). Thus \( g(\sigma) \neq \sigma \). For the usual tree this property
will follow from the link property (section 3). Namely, if $P_0$ is a point of an appartment then the group $T$ acts in a simply transitive way on all edges coming out from $P_0$ and not lying in the appartment.

The theorem is proved.

We note that the transformations from the Weyl group will be continuous but not necessarily smooth maps of compactified line $\mathbb{R} \cup -\infty, \infty$ into itself.

If $P, Q \in \Sigma_0$, then the subcomplex in $\Sigma_0$ containing all the simplices lying between $P$ and $Q$ will be called a path from $P$ to $Q$ (see picture 2).

**Corollary.** For any two vertices $P, Q \in \Delta_0$, $P \neq Q$ there exists a unique path $PQ$ between them.

As in the usual theory of the Bruhat-Tits tree we can introduce some intrinsically defined metric over our tree.

If $< L >, < L' > \in \Delta_0(G)[2]$ then by Cartan decomposition (theorem 2 of section 2) there exists a basis $e_1, e_2$ in $V$ such that

$$L = \mathcal{O}e_1 \oplus \mathcal{O}e_2, \quad L' = a_1e_1 \oplus a_2e_2,$$

where $a_1, a_2$ are some fractional $\mathcal{O}'$-ideals and $\nu'(a_1) \geq \nu'(a_2)$.

**Definition 8** $d(< L >, < L' >) = \nu'(a_1) - \nu'(a_2)$, where $< L >, < L' >$ are two vertices from $\Delta_0[2]$.

**Theorem 6.** The function $d(., .)$ is a correctly defined metric on the set $\Delta_0[2]$ having non-archimedean values in $\Gamma_K^+$. It has the following properties

i) $d(., .)$ is invariant under the action of $G$.

ii) the projection map $\pi$ is a distance-decreasing map, precisely

$$d(\pi(x), \pi(y)) = \pi(d(x, y))$$

iii) for any appartment $\Sigma$ there exists a simplicial map $\rho : \Delta \rightarrow \Sigma$, which is a retraction onto $\Sigma$ and which is a distance-decreasing map on subset $\Delta[2]$.

iv) let $u, t$ be local parameters of the field $K$ and $P, Q \in \Delta_0[2]$. Then $d(P, Q) = (m, n)$ and we have

$$n = d(\pi P, \pi Q) \text{ in } \Delta(K/\bar{K})[1],$$

$$m = d(Q, Q') \text{ in } \Delta(\bar{K}/k)[1] \cong \pi^{-1}(Q) \cap \Delta(K/\bar{K}/k)[2].$$
where $Q' = \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} P \in \pi^{-1}(Q)$

v) if $R \in PQ$, then $d(P, R) + d(R, Q) = d(P, Q)$,

vi) for $P, Q, P', Q' \in \Delta_0[2]$ there exist $g \in G$ such that $gP = P', gQ = Q'$ if and only if $d(P, Q) = d(P', Q')$.

**Proof.** If we change a module inside it's class then the same number will be added to the $\nu'(a_1)$ and $\nu'(a_2)$. Consequently, their difference will be unchanged. The properties i) and ii) follows directly from the definition. Let us show how to construct the retracting map.

Take an edge $\sigma$ of the appartment $\Sigma$. Then $\Sigma - \sigma$ can be decomposed into two pieces $\Sigma_+$ and $\Sigma_-$. Let $0$ and $\infty$ be the points from the external boundary of the appartment. We assume that $0$ ($\infty$) are the limit points for $\Sigma_+$ ($\Sigma_-$).

For any point $P \in \Delta_0[0]$ which does not belong to the appartment there is a unique shortest path which connects $P$ with some point $Q(P)$ of the appartment.

Thus the whole external boundary $\Delta[0]$ can be divided into two pieces $\Delta[0]_+$ and $\Delta[0]_-$. The first piece $\Delta[0]_+$ will contain $0$ and all the points which are connected with $\Sigma_+$. All the other points will belong to $\Delta[0]_-$. We start to construct $\rho$ from the external boundary:

$$\rho(\Delta[0]_+) = 0, \quad \rho(\Delta[0]_-) = \infty.$$ 

Then if $P \in \Delta[0]_+, P \neq 0$ there are two paths connecting the point $Q(P)$ with external boundary: the path between $P$ and $Q(P)$, and the path between $0$ and $Q(P)$ (a part of $\Sigma_+$). There exists a unique simplicial bijection $s_P$ of one path onto another one. Let us put

$$\rho(\sigma) = s_P(\sigma), \quad \text{if } \sigma \text{ lies on the path between } P \text{ and } Q(P).$$

The same definition works for $\Delta[0]_-$. It is straightforward that the constructed map is correctly defined on the whole tree and satisfies all the conditions from iii).

Properties iv) and v) follows from ii) and direct computations (compare with theorem 5, v) ). To get vi) we first observe that we can assume $P' = P$ (since $G$ is transitive on the tree) and $\pi(Q) = \pi(Q')$ (apply the same property for the tree $\Delta(K/\bar{K})$). Now let $n = d(P, Q) = d(P, Q')$ and we assume
Let $R$ be a common point of the paths $PQ$ and $PQ'$ such that the intersection of the paths $RQ$ and $RQ'$ is $R$. Let us denote by $R_0$ the inner boundary point of the path $RP$ which is closest to $R$. Then $R, R_0$ belong to the same fiber as $Q$ and $Q'$ and the equality $d(P,Q) = d(P,Q')$ is equivalent to $d(R,Q) = d(R,Q')$ in the tree $\Delta(\bar{K}/k)[1] \cong \Delta' = \pi^{-1}(\pi(Q)) \cap \Delta(\bar{K}/k)[2]$. By proposition 3, ii) the stabilizer $G'$ of the points $R_0$ and $P$ has the matrix form

$$\begin{pmatrix} O' & O \\ m_{i,n} & O' \end{pmatrix}$$

for some $i$. By the corollary 2 of proposition 2 $G'$ acts on $\Delta'$ as a group of upper triangular matrices

$$\begin{pmatrix} \mathcal{O}_K^* & \bar{K} \\ 0 & \mathcal{O}_K^* \end{pmatrix}.$$ 

This group acts transitively on the boundary of $\Delta'$ outside $R_0$ and thus under our distance condition it will move $Q$ to $Q'$.

The theorem is proved.

The last general notion which will be mentioned here is the type of the vertices and also of the simplices. Let us consider an exact sequence

$$0 \to PGL^+(V) \to PGL(V) \to \Gamma_K/2\Gamma_K \to 0$$

where the right hand map is $\nu'(\det(\cdot)) \mod 2$. As we know

$$\Gamma_K/2\Gamma_K \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

It can be shown that the stabilizers of the vertices belong to the subgroup $PGL^+(V)$ and thus we have a canonical map

$$\Delta_0[2] \to \Gamma_K/2\Gamma_K$$

which assign to the vertices four possible values, their type. The type of a simplex will be then a subset of $\Gamma_K/2\Gamma_K$. The type is invariant under the action of $SL(V)$ and the fundamental domain of this action is a disjoint union of two edges which are mapped by the projection map on the adjacent vertices of an edge in the tree $\Delta(\bar{K}/\bar{K})$.

The integer points of the lattice $\Gamma_K$ can be located on an real plane and it seems more reasonable to have a structure of dimension 2 on our simplicial set.

In the case of local field $K$ of dimension $n$ the number of types equals to $2^n$ and the building of $G$ (see [1]) could have a dimension depending on $n$. 

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But if we would like to preserve one of the most important features of the Tits theory - the geometrical structure of reflections, walls, chambers and so on then we are forced to introduce the simplicial structure as we did above. The reason is that the involutions from the Weyl group have very small fixed point set on the lattice $\Gamma_K$ (see theorem 3 above). In particularly, in dimension two they have the points as the fixed points but not the lines as would be the case if the dimension of our building were two.

Remark 6. Our use of the topology was rather artificial. It seems there should exist a purely simplicial construction which binds the $\Delta[m]$-pieces of the tree together. We can define $\Delta (G) = \Delta [2] \ast \Delta [1] \ast \Delta [0]$, where $\ast$ is a join of the simplicial complexes. Then the group $G \times G \times G$ will act on the whole $\Delta_{\text{max}}(G)$ in a transitive way and we will have a one to one correspondence between subgroups of this larger group and the simplexes of the new complex which has a dimension 4. The same remark is true for the groups of higher rank over arbitrary local fields \([11]\).

We also add the following problem.

Problem 3. It is well known that the buildings of the group $\operatorname{PGL}(V)$ (and in particularly the Bruhat-Tits tree) can be defined as classes of norms on the space $V$ \([3] [14] \text{II, 1.1}\). There is no doubt that this approach can be developed also for the higher buildings of this group also. But this should give directly a geometrical realization of the simplicial set $\Delta (G)$ which was defined in \([11]\).

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