SOLITONS IN POLYMERIC CHAINS
WITH PERIODIC INTERACTIONS

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Abstract

In this paper we follow the lines of recent works to investigate systems of two coupled real scalar fields defined by potentials that describe periodic interactions between the scalar fields. We work with polymeric chains containing periodic interactions between the coupled fields, and we investigate the topological sectors to obtain explicit soliton solutions and their corresponding energy. In particular, we offer an example that considers deoxyribonucleic acid (DNA) as a system of coupled fields, and we present the main steps to describe DNA as a polymeric chain belonging to the class of systems of two coupled real scalar fields.

PACS Numbers: 03.50.Kk; 11.10.Lm; 87.15.By

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I. INTRODUCTION

This paper deals with solitons in nonlinear systems of two coupled real scalar fields, described by periodic interactions between them in bidimensional spacetime. This is important in nonlinear science in general, since there are many systems which are described by periodic interactions among their relevant degrees of freedom. The present work is a continuation of former papers [1–3], in which we have set the basic ideas we are now going to enlarge to include some specific systems that are described by periodic interactions. Our aim here is to investigate the presence of soliton solutions in systems belonging to a general class of systems of coupled real scalar fields, and containing periodic interactions. Evidently, we believe that such systems can engender interesting physical contents, and can be used to map relevant degrees of freedom in quasi-one-dimensional periodic chains usually considered to model actual systems in condensed matter, organic chemistry and biological science. Within this context, in this work we use the term polymeric chains to name these systems generically.

For simplicity, we consider the case of two fields, namely \( \phi \) and \( \chi \). Systems of this kind are usually described by a Lagrangian density that contains a potential \( U = U(\phi, \chi) \), in general a nonlinear function of the real scalar fields. It is this nonlinearity that enlarges the scope of the problem, since it can be mapped to many interesting systems in nonlinear science. In this paper, in particular, we shall investigate periodic interactions that can be of great interest in applications to nonlinear science due to their topological properties, which we are also going to explore below. In this case, as an illustration we shall consider the deoxyribonucleic acid (DNA) as a system of coupled fields. In this first attempt to describe DNA as a system of coupled fields belonging to the class already introduced in [3], the main motivation is to shed some light on the way this can be done. Since the basic motivation of the present work is to introduce generic polymeric chains and to deal mainly with their mathematical intricacies, we postpone to a future work a more specific investigation of DNA as a system of two coupled real scalar fields. Here we recall another recent work [4], in which
an alternate route to solitons in hydrogen bonded chains is introduced.

The paper is organized as follows. In the next Sec. II we briefly review the method of searching for soliton solutions we have already introduced in [1–3]. There we introduce a general Lagrangian density and set the conditions the potential has to obey, such that the static solution of the second-order equations of motion are obtained by solving first-order differential equations. In Sec. III we examine examples of periodic systems. These examples illustrate how to search for soliton solutions in systems described by periodic interactions between two coupled scalar fields. In Sec. IV we consider DNA as a two-field system, and we show how to describe it via the class of system here considered. In this case, we start from the model proposed by Watson and Crick, in which DNA is formed by two polynucleotide strands that constitute the double helix held together by hydrogen bonds. We end the paper in Sec. V, and there we present some comments and conclusions.

II. A GENERAL CLASS OF SYSTEMS

A general Lagrangian density describing a relativistic system of two coupled real scalar fields in bidimensional spacetime is given by

\[ \mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi - U(\phi, \chi), \]  

(1)

where \( U = U(\phi, \chi) \) is the potential, which specifies the particular system one is interested in. Our notation is usual: we are using natural units, in which \( \hbar = c = 1 \), and the metric tensor \( g^{\alpha\beta} \) is diagonal, with \( g^{00} = -g^{11} = 1 \). In this case the fields are dimensionless, and time \( (t) \) and space coordinate \( (x) \) have dimension inverse of energy. The potential will be specified with two kinds of parameters: \( (a, b, ...) \) as real and positive dimensionless parameters, and \( (\lambda, \mu, ...) \) as real parameters having dimension of energy.

The above system leads to the following set of equations of motion

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial U}{\partial \phi} = 0, \]

(2)

and
\[
\frac{\partial^2 \chi}{\partial t^2} - \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial U}{\partial \chi} = 0.
\]  
(3)

In the standard way \[4\] of searching for soliton solutions one consider static field configurations, and so \(\phi = \phi(x)\) and \(\chi = \chi(x)\). In this case, the equations of motion become

\[
\frac{d^2 \phi}{dx^2} = \frac{\partial U}{\partial \phi},
\]  
(4)

and

\[
\frac{d^2 \chi}{dx^2} = \frac{\partial U}{\partial \chi}.
\]  
(5)

As it was already stressed in \[5\], let us now consider potentials that can be written as

\[
U(\phi, \chi) = \frac{1}{2} \left( \frac{\partial H}{\partial \phi} \right)^2 + \frac{1}{2} \left( \frac{\partial H}{\partial \chi} \right)^2,
\]  
(6)

where the functions \(H = H(\phi, \chi)\) is a smooth but otherwise arbitrary functions of the fields \(\phi\) and \(\chi\). Here the equations of motion describing the static field configurations become

\[
\frac{d^2 \phi}{dx^2} = H_\phi H_{\phi\phi} + H_\chi H_{\phi\chi},
\]  
(7)

and

\[
\frac{d^2 \chi}{dx^2} = H_\phi H_{\phi\chi} + H_\chi H_{\chi\chi},
\]  
(8)

where we are using \(H_\phi = \partial H / \partial \phi\) etc.

At first glance, it seems that we rewrote equations (4) and (5) in a more complicated form, but this is not so, as it was already shown \[1\]–\[3\]. In this case the energy of the system can be cast to the form

\[
E = \int_{-\infty}^{\infty} dx \frac{dH}{dx} = H(\phi(\infty), \chi(\infty)) - H(\phi(-\infty), \chi(-\infty)),
\]  
(9)

and this is the minimum value for the energy, which is achieved when we impose the conditions

\[
\frac{d\phi}{dx} = H_\phi,
\]  
(10)
and
\[
\frac{d\chi}{dx} = H\chi. \tag{11}
\]

As we can easily see, solutions of the first-order equations (10) and (11) also satisfy the second-order equations of motion (7) and (8) since \(H(\phi, \chi)\) is smooth. Therefore, for the general system (1), when the potential has the specific form (5), the second-order differential equations of motion (7) and (8) can be replaced by the first-order differential equations (10) and (11).

We omit details here, but it was already shown [2,3] that every soliton solution this class of systems can comprise is classically or linearly stable. Furthermore, in the above class of systems the energy corresponding to static solutions is bounded from below, and gets to its minimum value given by (8). The function \(H(\phi, \chi)\) can also be used to define topological sectors, as shown in [3]: The topological charge, which is conserved, can be cast to the form
\[
Q_T = H(\phi(\infty), \chi(\infty)) - H(\phi(-\infty), \chi(-\infty)), \tag{12}
\]
and in this case it is identified with the energy of the corresponding static field configurations. However, since static field configurations should go to vacuum states, asymptotically, to maintain the energy finite, we see that the topological charge is nothing but the difference between two values of \(H(\phi, \chi)\), calculated at, say, \((\bar{\phi}, \bar{\chi})\) and \((\bar{\phi}', \bar{\chi}')\), which represent two neighbour vacuum states in the \((\phi, \chi)\) plane.

III. POLYMERIC CHAINS WITH PERIODIC INTERACTIONS

The investigations introduced in the former Sect. 1 are done on general grounds, and can be generalized to the case of three or more fields straightforwardly. Here, however, our motivation is to show how the general procedure we have already introduced works when investigating specific systems. In former papers [1,3] we have already introduced several examples, in which we presented explicit soliton solutions. There we have investigated
systems identified by potentials containing polynomial interactions between two coupled fields. In this section we focus our attention on periodic interactions, and consider potentials described by the sine and cosine functions.

As we have already seen, the basic function we have to take under consideration is $H(\phi, \chi)$, and this is the understanding we shall now follow to investigate some explicit examples of systems that contain periodic interactions between the two scalar fields. The following polymeric chains are introduced with two main motivations: Firstly, to deal with mathematical intricacies that naturally appear in these systems and, secondly, as a preparation for possible applications to actual systems.

A. Polymeric Chain Number One

As a first polymeric chain containing periodic interactions, let us choose $H(\phi, \chi)$ in the form

$$H_1(\phi, \chi) = \mu \cos(\phi) + \nu \cos(a\chi) + \lambda \cos(\phi) \cos(a\chi).$$

(13)

Other forms similar to the above one can be obtained by just shifting $\phi \rightarrow \phi + \pi/2$ and/or $\chi \rightarrow \chi + \pi/2a$. In the present case, however, the first-order equations obtained from $H_1(\phi, \chi)$ are

$$\frac{d\phi}{dx} + \sin(\phi)[\mu + \lambda \cos(a\chi)] = 0,$$

(14)

and

$$\frac{d\chi}{dx} + a \sin(a\chi)[\nu + \lambda \cos(\phi)] = 0.$$  

(15)

After some algebraic manipulations, the potential can be cast to the following form:

$$U_1(\phi, \chi) = \frac{1}{8}[\chi^2(1 + a^2) + 2(\mu^2 + \nu^2 a^2)] +$$

$$+ \frac{1}{8}[\chi^2(a^2 - 1) - 2\mu^2] \cos(2\phi) +$$

$$+ \frac{1}{8}[\chi^2(1 - a^2) - 2\nu^2 a^2] \cos(2a\chi) +$$
\[
\begin{align*}
&+ \frac{1}{2} \lambda \nu a^2 \cos(\phi) + \frac{1}{2} \lambda \mu \cos(a\chi) - \\
&- \frac{1}{2} \lambda \mu \cos(2\phi) \cos(a\chi) - \frac{1}{2} \lambda \nu a^2 \cos(\phi) \cos(2a\chi) - \\
&- \frac{1}{8} \lambda^2 (1 + a^2) \cos(2\phi) \cos(2a\chi).
\end{align*}
\] (16)

Here we notice that in the above general function (13) the two first terms correspond to
decoupled self-interacting scalar field. The self-interacting fields have different amplitudes,
controlled by the parameters \( \mu \) and \( \nu \), and different periodicities, the difference being gov-
erned by the parameter \( a \). If we set \( \lambda = 0 \) in (13) the two first-order equations (14) and (15)
become

\[
\frac{d \phi}{dx} + \mu \sin(\phi) = 0,
\] (17)

and

\[
\frac{d \chi}{dx} + \nu a \sin(a\chi) = 0.
\] (18)

These equations can be integrated to give

\[
\phi(x) = 2 \arctan e^{-\mu x},
\] (19)

and

\[
\chi(x) = \frac{2}{a} \arctan e^{-\nu a^2 x},
\] (20)

which correspond to sine-Gordon systems [5].

The third term in (13) represents interactions between the two scalar fields, which is
controlled by the parameter \( \lambda \). The general system contains several parameters, and presents
the discrete \( Z_2 \) symmetry, and for \( a = 1 \) and \( \mu = \nu \) this symmetry becomes the \( Z_4 \) symmetry.
This last case seems to be interesting, since it describes two identical sine-Gordon systems
(represented by the \( \phi \) and \( \chi \) fields) interacting with each other. Another interesting case
is obtained by setting \( \mu = \nu = 0 \) in Eq. (13), and this will be investigated in the next
subsection.
If we set $a = 1$ and $\mu = \nu$ the potential becomes

$$U_1(\phi, \chi) = \frac{1}{4}[\chi^2 + 2\mu^2] - \frac{1}{4}\mu^2[\cos(2\phi) + \cos(2\chi)] + \frac{1}{2}\lambda\mu[\cos(\phi) + \cos(\chi)] - \frac{1}{2}\lambda\mu[\cos(2\phi) \cos(\chi) + \cos(\phi) \cos(2\chi)] - \frac{1}{4}\lambda^2 \cos(2\phi) \cos(2\chi).$$

(21)

Furthermore, the first-order equations get to the form

$$\frac{d\phi}{dx} + \sin(\phi)[\mu + \lambda \cos(\chi)] = 0,$$

(22)

and

$$\frac{d\chi}{dx} + \sin(\chi)[\mu + \lambda \cos(\phi)] = 0.$$

(23)

This system presents an infinity set of singular points. For $|\lambda| < |\mu|$ the singular points are at $(n\pi, m\pi)$, and alternate between stable and unstable points. For $|\lambda| > |\mu|$ there are more singular points, and they are located at $(\arccos(-\mu/\lambda), \arccos(\mu/\lambda))$. At the particular value $|\lambda| = |\mu|$ one gets to the picture that the system presents only stable or unstable points at the center of square cells with the four sides being continuous lines of singularities, as depicted in Fig. [1]. The following picture then emerges: for $|\lambda| < |\mu|$ ($|\lambda| > |\mu|$) the intra (inter) chain binding is stronger than the inter (intra) one; for $|\lambda| = |\mu|$ we have intra and inter chain binding equally stronge.

Fig. [1]. A cell of singular points in $(\phi, \chi)$ plane for $H_2(\phi, \chi)$ when $\mu = \nu = \lambda$ and $a = 1$. 

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We focus our attention at the specific case $|\mu| = |\lambda|$. To identify topological sectors in this system, we now use the topological current introduced at the very end of the former Sect. II. Here we verify that there is only one topological sector, with topological charge $Q_T = 4|\lambda|$. This topological sector represents any of the infinity possibilities of connecting the central point of the square cell to a point at its border, by a straight line. See Fig. [1]. A particularly simple soliton solution is obtained by setting $\chi = 2n\pi$ and $\lambda = \mu$. In this case we get, from the above first-order equations,

$$\frac{d\phi}{dx} + 2\lambda \sin(\phi) = 0,$$

which is the equation one gets in the sine-Gordon system, and this was already solved.

**B. Polymeric Chain Number Two**

As a second periodic system, we choose $H(\phi, \chi)$ in the following form

$$H_2(\phi, \chi) = \lambda \cos(\phi) \cos(a\chi).$$

(26)

This system is a particular case of the former system, and we treat it separately because it will be used in the next section. Here we also note that other similar forms are obtained by just shifting $\phi \to \phi + \pi/2$ and/or $\chi \to \chi + \pi/2a$. For instance, the function $H'_2 = \lambda \sin(\phi) \sin(a\chi)$ can be considered as an almost trivial case of the investigation we are now doing, using the above $H_2$.

In the present case, however, the first-order equations become

$$\frac{d\phi}{dx} + \lambda \sin(\phi) \cos(a\chi) = 0,$$

(27)

and

$$\frac{d\chi}{dx} + \lambda a \cos(\phi) \sin(a\chi) = 0.$$

(28)

The potential can be cast to the form
\[ U_2(\phi, \chi) = \frac{1}{8} \lambda^2 [(1 + a^2) + (a^2 - 1) \cos(2\phi) +
+ (1 - a^2) \cos(2a\chi) - (1 + a^2) \cos(2\phi) \cos(2a\chi)]. \] (29)

Note that the system presents \( \mathbb{Z}_2 \) symmetry, which becomes the \( \mathbb{Z}_4 \) symmetry for \( a = 1 \).

This system presents an infinity set of singular points in the \((\phi, \chi)\) plane. The singular points can be identified by: For \( n, m = 0, \pm 1, \pm 2, \ldots \), points at \((2n\pi, 2m\pi/a)\) and at \(((2n + 1)\pi, (2m + 1)\pi/a)\) alternate between unstable and stable respectively, and the points \(((2n + 1)\pi/2, (2m + 1)\pi/2a)\) are all saddle points. See Fig. [2].

![A cell of singular points in \((\phi, \chi)\) plane for \(H_1(\phi, \chi)\).](image)

Thin lines represent topological sectors.

To identify topological sectors in this system, we now use the topological current introduced at the very end of the former Sect. [1]. Here we use (26) to verify that there are two distinct topological sectors: The first one connects adjacent unstable-stable points, which we name the parallel sector, since it joins points by a straight line parallel to the \( \phi \) or \( \chi \) axis, with the corresponding topological charge given by \( Q^p_T = 2|\lambda| \); the second sector connects adjacent saddle-stable or unstable-saddle points, which we name the transversal sector, in which we have \( Q^t_T = |\lambda| \). Here we notice that the topological charge (and so the energy) does not depend on the parameter \( a \); also, since \( Q^p_T = 2Q^t_T \), we see that energy considerations favor the presence of solitons at the transversal sector.

To find explicit soliton solutions, let us first investigate the parallel sector. In this case one sets \( \chi = 2n\pi/a \), and so the above set of first-order equations (27) and (28) reduces to the single equation

\[ \cdots \]

Fig. [2]. A cell of singular points in \((\phi, \chi)\) plane for \(H_1(\phi, \chi)\).
\[
\frac{d\phi}{dx} + (-1)^n \lambda \sin(\phi) = 0. \tag{31}
\]

This is the sine-Gordon equation that was already studied in the former subsection.

To investigate the transversal sector, we consider the simplest case: \(a = 1\). Here, we can use \(\phi = \chi + 2n\pi\), and the first-order equations (27) and (28) reduce to

\[
\frac{d\phi}{dx} + \frac{1}{2} (-1)^n \lambda \sin(2\phi) = 0. \tag{32}
\]

See the former subsection for more details. It is interesting to point out that the transversal sector is connected by a straight line if and only if \(a = 1\). We have being unable to find any explicit analytical solution in this transversal sector for \(a \neq 1\).

IV. THE DEOXYRIBONUCLEIC ACID

In this section we shall consider deoxyribonucleic acid (DNA) as a system of the form we have been treating in the former sections. Before doing this, however, we recall that the DNA model proposed by Watson and Crick is formed by two polynucleotide strands that form the double helix, which are held together by hydrogen bonds connecting the adenine-thymine (A-T, with two hydrogen bonds) and guanine-cytosine (G-C, with three hydrogen bonds) bases. Two-field mechanical models for DNA consider each one of the two field describing bases in each one of the two strands, with the two strands being formed by harmonic coupling responding to the stacking of the DNA chain. Since the intrastrand distance between bases is at least twice as larger as the corresponding interstrand distance, such a mechanical view appears to be interesting, although one still has to account for interactions between interstrand pair of bases governed by hydrogen bonds.

This mechanical picture for DNA is usual, and the complete model in general differs only on the way one describes interactions between interstrand pair of bases, as interestingly considered in [6–8], and in references therein. Since we are here interested in finding soliton solutions, we shall now follow the line of reasoning presented in [6] to model hydrogen bonds in conjugated pair of bases. Before going on, let us recall that the above mechanical picture
of DNA neglects inhomogeneities due to the base sequence in each strand, and asymmetries of the two strands. Within this context, although the continuum-limit approximation appears to be somehow severe for DNA [9], when one discards inhomogeneities in each strand and asymmetry of the two strands, we realize that the continuum-limit approximation appears to be not so severe anymore.

If we now examine the steps introduced in [7] to get to the equation of motion, we recognize that the harmonic interaction present in each one of the two strands give rise, in the continuum approximation, to the second-order derivative along the DNA chain. Because of this, in the potential there considered, we are just left with several terms, but all of them related to interactions between conjugate pair of bases that connects the two strands. This is all we need to jump from this mechanical model to our system of two coupled fields.

Here we consider the fields $\phi$ and $\chi$ as describing bases in each one of the two strands in DNA. Moreover, since we are dealing with relativistic systems, we already have space and time coordinates entering with second order in derivative in the equations of motion. Then, the function $H = H(\phi, \chi)$ we have already introduced is to be thought of as generating the potential to describe interactions between conjugate pair of bases connecting the two strands, in a way such that if we set to zero those interactions, we should end up with two independent strands, none of them containing self-interactions. We then recognize that the polymeric chain number two, already investigated in the former section, is the field system that appears to be more adequate to map the DNA model introduced in Ref. [7].

To see this more explicitly, let us consider the function

$$H_3(\phi, \chi) = \mu \cos \phi \cos \chi + \nu \sin \phi \sin \chi,$$

which represents the addition of two functions $H(\phi, \chi)$ belonging to the polymeric chain number two already investigated in the former section. In this case we get the first-order equations

$$\frac{d\phi}{dx} + \mu \sin \phi \cos \chi - \nu \cos \phi \sin \chi = 0,$$
and
\[
\frac{d\chi}{dx} + \mu \cos \phi \sin \chi - \nu \sin \phi \cos \chi = 0. \tag{35}
\]

This system has an infinity set of singular points, and this is illustrated in Fig. [3]. In this Fig. [3] the central point is stable (unstable), and the other four points are unstable (stable). Here we see that there is just one soliton sector, which connects the central point to one of the other four points. The energy corresponding to this single soliton sector is given by

\[
E_M = |\mu - \nu|. \tag{36}
\]

\[
\begin{array}{c}
\text{Fig. [3]. A cell of singular points in } (\phi, \chi) \text{ plane for } H_3(\phi, \chi).
\end{array}
\]

The thin line represent the topological sector.

We deal with the first-order equations (34) and (35) to write

\[
\frac{d(\phi + \chi)}{dx} = (-\mu + \nu) \sin(\phi + \chi), \tag{38}
\]

which is solved to give

\[
\phi + \chi = 2 \arctan e^{(-\mu + \nu)x}. \tag{39}
\]

Furthermore, we can also write

\[
\frac{d(\phi - \chi)}{dx} = -(\mu + \nu) \sin(\phi - \chi), \tag{40}
\]

and now we have

\[
\phi - \chi = 2 \arctan e^{-(\mu + \nu)x}. \tag{41}
\]
The pair of solutions are

$$\phi = \arctan e^{(-\mu+\nu)x} + \arctan e^{-(\mu+\nu)x}, \quad (42)$$

and

$$\chi = \arctan e^{(-\mu+\nu)x} - \arctan e^{-(\mu+\nu)x}. \quad (43)$$

It is interesting to see that in this case the potential has the form

$$U_3(\phi, \chi) = \frac{1}{4} \mu^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \left( \mu^2 + \nu^2 \right) \cos 2\phi \cos 2\chi - \frac{1}{2} \mu \nu \sin 2\phi \sin 2\chi. \quad (44)$$

On the other hand, in [7] an interesting model to map DNA is considered. There the author uses the following potential

$$U_z(\varphi, \varphi') = (-B - 2\beta) \cos \varphi \cos \varphi' + (-B + \beta) \sin \varphi \sin \varphi' - \lambda \cos \varphi - \lambda \cos \varphi', \quad (45)$$

where $B$ is the parameter associated with the hydrogen bond energy, $\beta$ is the parameter associated with the dipole-dipole interaction energy, and $\lambda$ is the coupling constant associated with the dipole–induced-dipole interaction energy. These are the parameters that appear in the DNA model considered in Ref. [7].

Here we see that if one takes, for instance,

$$\mu = \sqrt{2} \sqrt{B + 2\beta + \sqrt{3\beta(2B + \beta)}}, \quad (46)$$

and

$$\nu = \sqrt{2} \sqrt{B + 2\beta - \sqrt{3\beta(2B + \beta)}}, \quad (47)$$

our potential (44) exactly reproduces the one considered in [7] in the case $\lambda = 0$. This result is very interesting, and shows that when the dipole–induced-dipole interaction is absent all the soliton solutions present minimum energy and are linearly or classically stable.
We now recall that the main motivation presented in [7] for studying soliton solutions in DNA is to perhaps find mechanisms to explain duplication in DNA. Within this context, if one takes the reasonable point of view that stable solitons will hardly lead to mechanisms to explain duplication in DNA, we can very naturally improve the model by just adding the dipole–induced-dipole interaction term to the potential. In this case we see that this new potential is not of the form required to belong to the class of systems already introduced in Sec. II, and so we may perhaps open the possibility of introducing instability for guiding us toward mechanisms to explain duplication in DNA. To work with this more realist case, we can proceed as follows: We add the dipole–induced-dipole interaction term as an extra, infinitesimal contribution to the potential; then, we follow the procedure introduced in [10] to calculate corrections to the energy of the soliton solutions even when we do not know the solutions explicitly. Here we recall that the above issue was also considered in Ref. [7], and there the author introduced another approach, alternative. On the other hand, when the dipole–induced-dipole interaction term is turned off we see that the energy \( E_M = |\mu - \nu| \) can be written as \( E_M = \sqrt{B(\Delta_+ - \Delta_-)} \), with \( \Delta_{\pm} \) given by

\[
\Delta_{\pm} = \sqrt{2 + 4\delta \pm 2\sqrt{3\delta(2 + \delta)}} ,
\]

(48)

where we have set \( \delta = \beta/B \). Furthermore, as we have already shown, in this case there is only one topological sector, and the corresponding energy is given by the above expression. Therefore, this value for the energy may be seen as a reference value, and can be used as a bound for energy considerations in investigating duplication in DNA. Furthermore, we could also consider the polymeric chain number one, and in this case we would introduce interactions in each one of the two strands, and these self-interactions would certainly improve the model considered in [7] to describe the DNA chain. Evidently, these issues are out of the scope of the present paper, and so we postpone to a future work the very specific investigation concerning the DNA polymeric chain.
V. COMMENTS AND CONCLUSIONS

In this paper we have investigated a general class of systems of coupled real scalar fields. This class of systems is defined by

\[ \mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi - \frac{1}{2} H_\phi^2 - \frac{1}{2} H_\chi^2, \]

where \( H = H(\phi, \chi) \) is a smooth but otherwise arbitrary function of the two fields \( \phi \) and \( \chi \). In this case, the second-order equations of motion corresponding to static field configurations are solved by the following first-order equations

\[ \frac{d\phi}{dx} = H_\phi, \]

and

\[ \frac{d\chi}{dx} = H_\chi. \]

The above set of first-order differential equations can be seen as a dynamical system, and so we can take advantage of all the mathematical tools available to dynamical systems to deal with it. We have also shown that static field configurations in this class of systems have minimum energy and are classically or linearly stable, and so the presence of soliton solutions will certainly play some important role in understanding physical properties of the system.

Our investigation is done on general grounds, but we have introduced some examples of polymeric chains in Sec. [III]. There we have investigated specific systems, described by periodic interactions, with the motivation of showing how to deal with issues that naturally appear in the general procedure. As we have shown explicitly, periodic interactions lead to potentials that can be made periodic too, and the topological behavior is easy to be identified and can be used to guide us toward finding soliton solutions.

This fact seems to be important in nonlinear science, since in this case there are many systems which can be described by periodic interactions among their degrees of freedom. For instance, as it was shown explicitly, we have found a model that maps the model used...
in [7] to describe DNA as a two-field system when the dipole–induced-dipole interaction term is neglected. This result seems to be interesting since it informs that without the dipole-induced-dipole interaction term, the continuum version of the model considered in [7] presents stable soliton solutions, and stable solitons can hardly be used to model open states in DNA. Within this context, the presence of the dipole-induced-dipole interaction seems to be a small correction that induces instability and so gives rise to a mechanism that may perhaps lead to open states, thus explaining duplication in DNA. We shall return to this issue in a future work, in which we deal specifically with solitons in DNA, owing to explain duplication via the presence of open states in the double helix model of Watson and Crick.

As an ending comment, we would like to add that we can perhaps find other applications for the models investigated in the present paper. For instance, we believe that the polymeric chains we have introduced in this work can be used to map other systems, in particular the model introduced in [11] to describe topological solitons in polyethylene. These and other related issues are presently under consideration.

ACKNOWLEDGMENTS

We would like to thank M. M. Santos for interesting discussions. DB, MTT, and EV also thank Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq, Brazil: DB and MTT for partial support and EV for a fellowship.
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