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A recovery of two determinantal representations for derangement numbers

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Abstract: In the paper, the authors recover, correct, and extend two representations for derangement numbers in terms of a tridiagonal determinant.

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1. Introduction

In combinatorics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. The number of derangements of a set of size \( n \) is called the derangement number and sometimes denoted by \( !n \). The problem of counting derangements was first considered in 1708 and solved in 1713 by Pierre Raymond de Montmort, as did Nicholas Bernoulli at about the same time. Derangement numbers \( !n \) arise naturally in many different contexts. More generally, the number of derangements in various families of transitive permutation groups has been studied extensively in recent years. For more information on \( !n \), please refer to Aigner (2007), Andreescu and Feng (2004), Wilf (1994, 2006) and plenty of references therein.

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The first ten derangement numbers \( D_n \) for \( 0 \leq n \leq 9 \) are
\[
0, 1, 2, 9, 44, 265, 1854, 14833, 133496.
\] (1)

In Kittappa (1993, p. 216, Example 2), it was given that
\[
\begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 & 0 \\
3 & 3 & -1 & \ldots & 0 & 0 & 0 \\
0 & 4 & 4 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n-1 & -1 & 0 \\
0 & 0 & 0 & \cdots & n & n & -1 \\
0 & 0 & 0 & \cdots & 0 & n+1 & n+1 \\
\end{pmatrix}, \quad n \in \mathbb{N}.
\] (2)

In Janjić (2012, p. 8, 5°) and Janjić (2011, p. 5, 5°), it was deduced that
\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 \\
-1 & 1 & 2 & 0 & \ldots & 0 \\
0 & -1 & 2 & 3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & r-1 & r \\
0 & 0 & 0 & \cdots & -1 & r \\
\end{pmatrix}, \quad r \in \mathbb{N}.
\] (3)

By the determinantal expression (3), we figure out that \( D_2 = 1, D_3 = 3, D_4 = 6 \), and \( D_5 = 24 \). It is clear that the latter two values \( 6 \) and \( 24 \) do not coincide with the numbers \( 9 \) and \( 44 \) in (1). Therefore, the expression (3) appeared in Janjić (2011, 2012) is slightly wrong.

It is known in Comtet (1974, p. 182, Theorem B) that derangement numbers \( D_n \) have an exponential generating function
\[
D(x) = \frac{e^{-x}}{1-x} = \sum_{n=0}^\infty \frac{D_n x^n}{n!}.
\] (4)

The aim of this paper is, by computing the \( n \)th derivative of the exponential generating function \( D(x) \), to recover, correct, and extend the above determinantal representations (2) and (3) for derangement numbers \( D_n \).

THEOREM 1  For \( n \in \{0\} \cup \mathbb{N} \), derangement numbers \( D_n \) can be represented by a tridiagonal \( (n+1) \times (n+1) \) determinant
\[
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & -2 & 2 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 3 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & n-3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & -n(n-2) & n-2 & 1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -(n-1) & n-1 \\
\end{pmatrix} = -|\mathbf{e}_1^{(n+1)\times(n+1)}|^2.
\] (5)
2. Lemma
In order to recover Theorem 1, we need the following lemma which was reformulated in Qi (2015, Section 2.2, p. 849), Qi and Chapman (2016, p. 94), and Wei and Qi (2015, Lemma 2.1) from Bourbaki (2004, p. 40, Exercise 5).

**Lemma 1** Let $u(x)$ and $v(x) \neq 0$ be differentiable functions, let $U_{(n+1)\times 1}(x)$ be an $(n+1) \times 1$ matrix whose elements $u_{k,1}(x) = (e^x)^{k-2} = e^{x} \rightarrow 1$ for $1 \leq k \leq n+1$ and $V_{(n+1)\times n}(x)$ be an $(n+1) \times n$ matrix whose elements $v_{i,j}(x) = (1+x)^{i-j}$ for $1 \leq i \leq n+1$ and $1 \leq j \leq n$, and let $W_{(n+1)\times (n+1)}(x)$ denote the determinant of the $(n+1) \times (n+1)$ matrix

$$W_{(n+1)\times (n+1)}(x) = \left[ \begin{array}{c|c} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{array} \right].$$

Then the $n$th derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{d^n}{dx^n} \left[ \frac{u(x)}{v(x)} \right] = (-1)^n \frac{W_{(n+1)\times (n+1)}(x)}{v^{n+1}(x)}$$

(6)

3. Proof of Theorem 1
Now we are in a position to prove Theorem 1.

Applying $u(x) = e^x$ and $v(x) = 1 + x$ in Lemma 1 gives

$u_{k,1} = (e^x)^{(k-1)} = e^x \rightarrow 1$

as $x \rightarrow 0$ for $1 \leq k \leq n+1$ and

$$v_{i,j} = \binom{i-1}{j-1}(1+x)^{i-j} = \begin{cases} \binom{i-1}{j-1}(1+x), & i-j = 0 \\ \binom{i-1}{j-1}, & i-j = 1 \\ 0, & i-j \neq 0,1 \end{cases}$$

$= \begin{cases} 1+x, & i-j = 0 \\ i-1, & i-j = 1 \\ 0, & i-j \neq 0,1 \end{cases}$

as $x \rightarrow 0$ for $1 \leq i \leq n+1$ and $1 \leq j \leq n$. Consequently, employing (6) reveals
From (4), it follows that

\[ \frac{d^n D(-x)}{dx^n} = \frac{(-1)^n}{(1 + x)^{n+1}} \]

is equal to the determinant

\[
\begin{vmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 1 & 0 & \ldots & 0 & 0 \\
1 & 0 & 2 & 1 & \ldots & 0 & 0 \\
1 & 0 & 0 & 3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 1 & 0 \\
1 & 0 & 0 & 0 & \ldots & n - 1 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & n \\
\end{vmatrix}
\]

as \( x \to 0 \). From (4), it follows that

\[ D(-x) = \frac{e^x}{1 + x} = \sum_{k=0}^{\infty} (-1)^n n! \frac{x^n}{n!}. \]

This implies that

\[
(-1)^n n! = \lim_{x \to 0} \frac{d^n D(-x)}{dx^n} = (-1)^n
\]

Subtracting the \( n \)th row from the \((n+1)\)th row, then the \((n-1)\)th row from the \( n \)th row, \ldots, then the 1st row from the 2nd row of the above determinant leads to

\[ !n = \begin{vmatrix}
1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 1 & 1 & \ldots & 0 & 0 \\
0 & 0 & -2 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & n - 2 & 1 \\
0 & 0 & 0 & 0 & \ldots & -(n-1) & n - 1 \\
\end{vmatrix}
\]

which can be readily rearranged as the formula (5). The proof of Theorem 1 is complete.

**Remark 1** On 10 May 2016, Dr Wiwat Wanicharpichat at Naresuan University in Thailand told the first author that the matrix
is known as the “population projection matrix”. See (Kirkland & Neumann, 2013, p. 48, Equation (4.1)).

Remark 2 In the paper (Qi, 2016), an alternative proof of Theorem 1 was given.

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