A Generalization of the Savage-Dickey Density Ratio for Testing Equality and Order Constrained Hypotheses

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Abstract

The Savage-Dickey density ratio is a specific expression of the Bayes factor when testing a precise (equality constrained) hypothesis against an unrestricted alternative. The expression greatly simplifies the computation of the Bayes factor at the cost of assuming a specific form of the prior under the precise hypothesis as a function of the unrestricted prior. A generalization was proposed by Verdinelli and Wasserman (1995) such that the priors can be freely specified under both hypotheses while keeping the computational advantage. This paper presents an extension of this generalization when the hypothesis has equality as well as order constraints on the parameters of interest. The methodology is used for a constrained multivariate t test using the JZS Bayes factor and a constrained hypothesis test under the multinomial model.

Keywords: Bayes factors, constrained hypotheses, constrained multivariate Bayesian t test, constrained multinomial models.

1 Introduction

In the social and behavioral sciences, and related fields, researchers often formulate their expectations using specific equality and order constraints on the parameters of interest. Equality constraints are used to formulate precise hypotheses where a set of parameters are expected to be fixed to a constant, e.g., in a normal distribution it may be expected that the population mean
equals zero (e.g., Rouder et al., 2009). Order constraints are used to formulate order hypotheses (Hoijtink, 2011) which assume a specific ordering of the parameters of interest, e.g., in a regression model it may be expected that the effects of the predictor variables on the outcome variable follows a specific order (Bracken et al., 2015). Hypotheses may also be formulated with combinations of equality and order constraints on the parameter of interest (de Jong et al., 2017). The literature contains a variety of applications where constrained hypotheses are tested using the Bayes factor (Jeffreys, 1961; Kass & Raftery, 1995), such as repeated measurements (Mulder et al., 2009), multilevel models (Mulder & Fox, 2013, 2019), or structural equation models (Gu et al., 2019). For a thorough overview of the merits of the Bayes factor for testing scientific expectations, see, for example, Wagenmakers et al. (2018).

Two important challenges need to be overcome to compute the Bayes factor. First, priors need to be formulated for the free parameters under the hypotheses under investigation. Prior distributions quantify the plausibility of the values of the free parameters before observing the data. Priors need to be carefully specified as the Bayes factor is sensitive to the choice of the prior (Vanpaemel, 2010). Second, the marginal likelihood, which quantifies the prior predictive probability of the data, often needs to be computed numerically which can be computationally expensive (Kass & Raftery, 1995).

When testing an equality constrained hypothesis where \( \theta \) is equal to a vector of constants \( r^E \), \( H_1 : \theta = r^E \), with nuisance parameters denoted by \( \phi \), against an unconstrained alternative, \( H_u : \theta \) unconstrained, the Savage-Dickey density ratio is a special form of the Bayes factor which partly solves both challenges. The Savage-Dickey density ratio is defined as the ratio of the unconstrained marginal posterior evaluated at the null value (implied by the equality constraints) and the unconstrained marginal prior evaluated at the null value, i.e.,

\[
B_{1u} = \frac{\pi_u(\theta^E = r^E|Y)}{\pi_u(\theta^E = r^E)}. \tag{1}
\]

Thus this specific expression avoids the need for computing marginal likelihoods. The expression only holds when the prior for the nuisance parameters under \( H_1 \) is equal to the conditional prior for the nuisance parameters under the unconstrained alternative, i.e., \( \pi_1(\phi) = \pi_u(\phi|\theta^E = r^E) \). Thus, prior specification is simplified as one only needs to formulate the unconstrained prior under \( H_u \); the prior under the equality constrained hypothesis \( H_1 \) is
implicitly specified from the unconstrained prior. The increasing popularity of the Savage-Dickey density ratio may be due to its intuitive expression as the height of the unconstrained posterior at the null value relative to the height of the unconstrained prior at the null value provides a reasonable quantification of the evidence in the data (i.e., the change from prior to posterior, see also Lavine & Chervish, 1999, for a related discussion) between the hypotheses (Wagenmakers et al., 2010). There are also critiques on the use of the Savage-Dickey expression from a measure theoretical point of view as the unconstrained prior and posterior are evaluated at a point, $\theta^E = r^E$, which has zero probability (Marin & Robert, 2010). Here we simply use the Savage-Dickey expression as a computational tool to obtain Bayes factors.

An important limitation of the Savage-Dickey expression lies in the fact that it implicitly assumes a very special form of the prior for the nuisance parameters under the ('smaller') equality constrained hypothesis (Verdinelli & Wasserman, 1995; Heck, 2020), as was also mentioned above. Verdinelli & Wasserman (1995) therefore proposed a generalization of the Savage-Dickey density ratio which is given by

$$B_{1u} = \frac{\pi_u(\theta^E = r^E | Y)}{\pi_u(\theta^E = r^E)} \times \mathbb{E}\left\{ \frac{\pi_1(\phi)}{\pi_u(\phi | \theta^E = r^E)} \right\},$$

(2)

where the expectation is taken over the conditional posterior under the unconstrained model, $\pi_u(\phi | \theta^E = r^E, Y)$. This generalization allows one to freely specify the prior for the nuisance parameters under the restricted hypothesis, $\pi_1(\phi)$, while keeping the computational advantage of the Savage-Dickey density ratio.

The generalization in (2) is not designed for computing Bayes factors when the constrained hypothesis contains order constraints in addition to equality constraints. Given the increasing importance of order constraints for testing scientific expectations, the purpose of this paper is therefore to present an extension of the generalization of the Savage-Dickey density ratio for such constrained hypotheses with equality and order constraints. The generalization is presented in Section 2. Section 3 presents two different applications of the proposed method under different statistical models: A constrained multivariate Bayesian $t$ test for standardized effects using a novel extension of the JZS Bayes factor of Rouder et al. (2009), and a constrained hypothesis test on the cell probabilities under a multinomial model. The paper ends with some short concluding remarks in Section 4.
2 Extending the Savage-Dickey ratio

Lemma 1 presents our main result.

**Lemma 1** Consider an constrained statistical model, $H_1$, where the parameters $\theta^E$ are fixed to a constant, i.e., $\theta^E = r^E$, order (or one-sided) constraints are formulated on the parameters $\theta^O$, i.e., $\theta^O > r^O$, and the unconstrained nuisance parameters are contained in $\phi$, and an alternative unconstrained model $H_u$, where $(\theta^E, \theta^O, \phi)$ are unrestricted. If we denote the priors under $H_1$ and $H_u$ according to $\pi_1(\theta^O, \phi)$ and $\pi_u(\theta^E, \theta^O, \phi)$, respectively, then the Bayes factor of model $H_1$ against model $H_u$ given a data set $Y$ can be expressed as

$$B_{1u} = \frac{\pi_u(\theta^E = r^E \mid Y)}{\pi_u(\theta^E = r^E)} \times \mathbb{E}\left\{ \frac{\pi_{1,u}(\theta^O, \phi)}{\pi_u(\theta^O, \phi \mid \theta^E = r^E)} 1_{\{\theta^O > r^O\}}(\theta^O) \right\},$$

(3)

where the expectation is taken over the conditional posterior of $(\theta^O, \phi)$ given $\theta^E = r^E$ under $H_u$, i.e., $\pi_u(\theta^O, \phi \mid Y, \theta^E = r^E)$, and where $\pi_{1,u}(\theta^O, \phi)$ denotes the “completed” prior under $H_1$, i.e., the prior under $H_t$ where the equality constraints are omitted, $1_{\{\theta^O > r^O\}}(\theta^O)$ is the indicator function which equals 1 if $\theta^O > r^O$ holds, and 0 otherwise, and $Pr_{1,u}(\cdot)$ denotes the prior probability of $\theta^O > r^O$ under the completed prior under $H_t$.

**Proof:** Appendix A.

**Remark 1** Note that in the special case where

$$\pi_t(\theta^O, \phi) = \pi_u(\theta^O, \phi \mid \theta^E = r^E) Pr_u(\theta^O > r^O \mid \theta^E = r^E) - 1_{\{\theta^O > r^O\}}(\theta^O)$$

then (3) results in the known generalization of the Savage-Dickey density ratio of the Bayes factor for an equality and order hypothesis against an unconstrained alternative,

$$B_{1u} = \frac{\pi_u(\theta^E = r^E \mid Y)}{\pi_u(\theta^E = r^E)} \times \frac{Pr_u(\theta^O > r^O \mid Y, \theta^E = r^E)}{Pr_u(\theta^O > r^O \mid \theta^E = r^E)}.$$  

(4)

This expression has been observed in, for example, Mulder & Gelissen (2018).

**Remark 2** In the special case with no order constraints, the parameters $\theta^O$ would be part of the nuisance parameters $\phi$, and thus (3) becomes equal to (2).
Remark 3 As the expectation in (3) is taken over the conditional posterior of \((\theta^E, \phi)\) given \(\theta^E = r^E\) under \(H_u\) while disregarding the order constraints, the computation is relatively cheap. Methods for efficient computation of such expectations have been discussed by Verdinelli & Wasserman (1995), among others.

Remark 4 The importance of the “completed” prior, i.e., the prior under the constrained hypothesis where the order constraints are omitted, was highlighted by Pericchi et al. (2008).

3 Applications

3.1 A multivariate \(t\) test using the JZS Bayes factor

The Cauchy prior for standardized effects is becoming increasingly popular for Bayes factor testing in the social and behavioral sciences (Rouder et al., 2009, 2012; Rouder & Morey, 2015). This Bayes factor is based on key contributions by Jeffreys (1961), Zellner & Siow (1980), and Liang et al. (2008), and is therefore also referred to as the JZS Bayes factor. Here we extend this Bayes factor to the multivariate normal model, and show how to compute the Bayes factor for testing a hypothesis with equality and order constraints on the standardized effects using Lemma 1.

We consider a multivariate dependent variable, \(y_i\), of \(p\) dimensions, which are assumed to follow a multivariate normal distribution, i.e., \(y_i \sim N(\mu, \Sigma)\), for \(i = 1, \ldots, n\). To explicitly model the standardized effects, we reparameterize the model according to

\[
y_i \sim N(L_{\Sigma} \delta, \Sigma),
\]

where \(\delta\) are the unknown standardized effects, and \(L_{\Sigma}\) is the lower triangular Cholesky factor of the unknown covariance matrix \(\Sigma\), such that \(L_{\Sigma} L'_{\Sigma} = \Sigma\).

Note that for the univariate case, i.e., \(p = 1\), the model can be written as \(y_i \sim N(\sigma \delta, \sigma^2)\), which was considered by Rouder et al. (2009).

As a motivating example we consider the bivariate data set \((p = 2)\) presented in Larocque & Labarre (2004), which contains the differences of CD45RA T and CD45RO T cell counts of 36 HIV-positive newborn infants.
We are interested in testing whether the standardized effects of the cell count differences is equal and positive, i.e.,
\begin{align*}
H_1 & : \delta_1 = \delta_2 > 0 \\
H_u & : (\delta_1, \delta_2) \in \mathbb{R}^2.
\end{align*}

The sample means the outcome variances were $\bar{y} = (86.94, 193.47)'$ and the estimated covariance matrix equalled $\hat{\Sigma} = [20197 23515; 23515 106350]$.

Extending the prior proposed by Rouder et al. (2009) to the multivariate case, we set an unconstrained Cauchy prior on $\delta$ under $H_u$ and the Jeffreys prior for the covariance matrix:
\begin{align*}
\pi_u(\delta, \Sigma) &= \pi_u(\delta) \times \pi_u(\Sigma) \\
&= \text{Cauchy}(\delta|S_u, 0) \times |\Sigma|^{-\frac{p+1}{2}}.
\end{align*}

The a diagonal prior scale matrix is set with diagonal elements of $\frac{5}{2}$. This prior implies that standardized effects of about 0.5 are likely under $H_u$. Under the constrained hypothesis $H_1$ the free parameters are the common standardized effect, say, $\delta = \delta_1 = \delta_2$, and the error covariance matrix, $\Sigma$. We set a univariate Cauchy prior for $\delta$ with scale $s_1^2$ truncated in $\delta > 0$, and the Jeffreys prior for $\Sigma$, i.e.,
\begin{align*}
\pi_1(\delta, \Sigma) &= \pi_1(\delta) \times \pi_1(\Sigma) \\
&= 2 \times \text{Cauchy}(\delta|s_1^2) \times 1(\delta > 0) \times |\Sigma|^{-\frac{p+1}{2}}.
\end{align*}

The 2 comes from the reciprocal of the prior probability that $\delta > 0$ in order for the prior to integrate to one. As $\delta$ has a similar interpretation as $\delta_1$ and $\delta_2$ under $H_u$, the prior scale is also set to $s_1^2 = \frac{5}{2}$.

By applying the following linear transformation on the standardized effects,
\begin{align*}
\theta &= \begin{bmatrix} \theta_E \\ \theta_O \end{bmatrix} = \begin{bmatrix} \delta_1 - \delta_2 \\ \delta_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = T\delta,
\end{align*}

the model can be written as $y_i \sim N(LT^{-1}\theta, \Sigma)$, and the hypotheses can be written as
\begin{align*}
H_1 & : \theta_E = 0, \theta_O > 0 \\
H_u & : (\theta_E, \theta_O) \in \mathbb{R}^2.
\end{align*}
Note here that $\theta_O$ corresponds to the standardized effect $\delta$ under $H_1$. Further note that the prior for $(\theta_E, \theta_O)$ under $H_u$ then follows a bivariate Cauchy distribution with scale matrix $T S_{u,0} T' = [0.5 \quad -0.25; -0.25 \quad 0.25]$.

Thus, if one would be testing the hypotheses with the Savage Dickey density ratio in (4), it is easy to show that the implied prior for $\delta$ under $H_1$ (i.e., the conditional unconstrained prior for $\theta_O$ given $\theta_E = 0$ under $H_u$) follows a Student $t$ distribution with 2 degrees of freedom with a scale parameter of $0.25^2 = 0.125$; thus assuming that standardized effects of 0.25 are likely under $H_1$. There is no logical reason however that the standardized effect should be smaller under the more restricted model under $H_1$.

The JSZ Bayes factor for this constrained testing problem using Lemma 1 based on the actual Cauchy priors for the standardized effects can be computed using MCMC output from a sampler under $H_u$, which is described in Appendix B. The four key quantifies in (3) then follow automatically:

- As the unconstrained marginal prior for $\theta_E$ follows a Cauchy(0.25) distribution (Figure 1, left panel, dashed line), the prior density equals $\pi_u(\theta_E = 0 | Y) = 2/\pi$.

- As the complete prior for $\delta$ under $H_1$ follows a Cauchy(0.25) distribution that is centered at zero, the prior probability equals $Pr_1(\delta > 0) = 0.5$.

- The estimated marginal posterior for $\theta_E$ under $H_u$ follows from MCMC output. The estimated posterior for $\theta_E$ is plotted in Figure 1 (left panel, solid line). This yields $\hat{\pi}_u(\theta_E = 0 | Y) = 1.000$.

- As the priors for the covariance matrices cancel out in the fraction, the expected value can be written as $E \left\{ \frac{\text{Cauchy}(\theta_O|0.25)}{\text{Cauchy}(\theta_O|0.125)} 1_{\{\theta_O > 0\}}(\theta^O) \right\}$ under the conditional posterior for $\theta_O$ given $\theta_E = 0$ under $H_u$. Appendix B also shows how to get posterior draws from $\theta_O$ under $H_u$ given $\theta_E = 0$. The estimated posterior is displayed in Figure 1 (right panel). A Monte Carlo estimate can then be used to compute the expectation, which yields 1.146.

Application of Lemma 1 then yields a Bayes factor for $H_1$ against $H_u$ of $B_{1u} = 3.6$. Thus there is 3.6 time more evidence in the data for equal and positive standardized count differences than for the unconstrained alternative hypothesis. Assuming equal prior probabilities for $H_1$ and $H_u$ this would
Figure 1: Left panel. Marginal posterior (solid line) and prior (dashed line) for $\theta_E$. The dotted lines indicate the estimated density values at $\theta_E = 0$. Right panel. Estimated conditional posterior for $\theta_O$ given $\theta_E = 0$ under $H_u$. The grey area displays the region where $\theta_O$ (i.e., the common $\delta$ under $H_0$) is positive, $\theta_O > 0$.

yield posterior probabilities of $Pr(H_1|Y) = .783$ and $Pr(H_u|Y) = .217$. Thus there is very mild evidence for $H_1$ relative to $H_u$. In order to draw clearer conclusions more data would need to be collected.

3.2 Constrained hypothesis testing under the multinomial model

When analyzing categorical data using a multinomial model, researchers are often interested in testing the relationships between the probabilities of the different cells (Robertson, 1978; Klugkist et al., 2010; Heck & Davis-Stober, 2019). As an example we consider an experiment for testing the Mendelian inheritance discussed in Robertson (1978). A total of 556 peas coming from crosses of plants from round yellow seeds and plants from wrinkled green seeds were divided in four categories. The cell probabilities for these categories are contained in the vector $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, where $\gamma_1$ denotes the probability that a pea resulting from such a mating is round and yellow; $\gamma_2$ denotes the probability that it is wrinkled and yellow; $\gamma_3$ denotes the probability that it is round and green; and $\gamma_4$ denotes the probability that it is wrinkled and green. The Mendelian theory states that $\gamma_1$ is largest, followed by $\gamma_2$ and $\gamma_3$
which are assumed to be equal, and \( \gamma_4 \) is expected to be smallest. This can be summarized as \( H_1 : \gamma_1 > \gamma_2 = \gamma_3 > \gamma_4 \). In particular the theory dictates that the four probabilities are proportional to 9, 3, 3, and 1, respectively. Therefore we consider a conjugate Dirichlet prior on the three free parameters under \( H_1 \) given by \( \pi_1(\gamma_1, \gamma_2, \gamma_4) = \text{Dirichlet}(9, 3, 1) \), such that the prior means for \((\gamma_1, \gamma_2, \gamma_4)\) are equal to \(\left( \frac{9}{15}, \frac{3}{15}, \frac{1}{15} \right)\), which correspond to the anticipated probabilities according to Mendelian theory. This Mendelian hypothesis will be tested against an unconstrained alternative which does not make any assumptions about the relationships between the cell probabilities. A uniform prior on the simplex will be used under the alternative, i.e., \( \pi_u(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \text{Dirichlet}(1, 1, 1, 1) \). The observed frequencies in the four respective categories were equal to 315, 101, 108, and 32.

In order to use Lemma 1, the Mendelian hypothesis will equivalently be formulated on the transformed parameters \((\theta^E, \theta^O_1, \theta^O_2, \phi) = (\gamma_2 - \gamma_3, \gamma_1 - \gamma_2, \gamma_2 - \gamma_4, \gamma_2)\) so that \( H_1 : \theta^E = 0, \theta^O_1 > 0, \theta^O_2 > 0 \). The four quantities in (3) can then be computed as follows.

- The unconstrained marginal prior density at \( \theta^E = 0 \) can be estimated from a sample of \( \theta^E = \gamma_2 - \gamma_3 \) where \( \gamma \) is sampled from the unconstrained \( \text{Dirichlet}(1, 1, 1, 1) \) prior, resulting in \( \hat{\pi}_u(\theta^E = 0) = 1.483 \).

- Similarly, the unconstrained marginal posterior density at \( \theta^E = 0 \) can be obtained by sampling \( \gamma \) from the unconstrained \( \text{Dirichlet}(316, 102, 109, 33) \) posterior, resulting in \( \hat{\pi}_u(\theta^E = 0 | \mathbf{Y}) = 13.72 \).

- The prior probability of \( \theta^O > 0 \) under \( H_1 \) can be obtained by sampling \( (\gamma_1, \gamma_2, \gamma_4) \) from an unconstrained \( \text{Dirichlet}(9, 3, 1) \) distribution, and then compute the proportion of draws satisfying \( \gamma_1 - \gamma_2 > 0 \) and \( \gamma_2 - \gamma_4 > 0 \), resulting in \( \Pr_1(\gamma_1 > \gamma_2 > \gamma_3) = 0.842 \).

- The conditional posterior for \((\gamma_1, \gamma_2, \gamma_4)\) given \( \theta^E = \gamma_2 - \gamma_3 = 0 \) under \( H_u \) follows a \( \text{Dirichlet}(316, 210, 33) \) distribution. The expectation in (3) can then be computed as the arithmetic mean of \( \frac{\text{Dirichlet}(\gamma_1, \gamma_2, \gamma_4 | \alpha = (9, 3, 1))}{\text{Dirichlet}(\gamma_1, \gamma_2, \gamma_4 | \alpha = (1, 1, 1))} f(\gamma_1 > \gamma_2 > \gamma_4) \) based on a sufficiently large sample of \((\gamma_1, \gamma_2, \gamma_4)\) from a \( \text{Dirichlet}(316, 210, 33) \) distribution, resulting in an estimate of 4.426.

In sum the Bayes factor of the Mendelian hypothesis against the noninformative unconstrained alternative is equal to \( B_{1u} = \frac{13.72}{1.483} \times 0.842 = 48.6309 \). This can be interpreted as relatively strong evidence for the Mendelian hypothesis based on the observed data.
4 Concluding remarks

As Bayes factors are becoming increasingly popular to test hypotheses with equality as well as order constraints on the parameters of interest, simple and accurate estimation methods to acquire these Bayes factors are needed. The generalization of the Savage-Dickey density ratio that was presented in this paper will be a useful contribution for this purpose. The expression allows one to compute Bayes factors in a straightforward manner using MCMC output while being able to freely specify the priors for the free parameters under the competing hypotheses. The applicability of the proposed methodology was illustrated in a constrained multivariate $t$ test using a novel extension of the JSZ Bayes factor to the multivariate normal model and in a constrained hypothesis test under the multinomial model.

Acknowledgements

The authors would like to thank Florian Böing-Messing for helpful discussions at an early stage of the paper. The first author is supported by an ERC Starting Grant (758791).

A Proof of Lemma 1

As the constrained model $H_1$ is nested in the unconstrained model $H_u$, the likelihood under $H_1$ can be written as the truncation of the unconstrained likelihood, i.e., $p_1(\mathbf{Y}|\theta^O, \phi) = p_u(\mathbf{Y}|\theta^E = r^E, \theta^O, \phi)1_{\{\theta^O > r^O\}}(\theta^O)$. The re-
sult in Lemma 1 then follows via the following steps,

\[
B_{1u} = \frac{p_1(Y)}{p_u(Y)} = \frac{\int_{\theta^O \succ r^O} p_1(Y|\theta^O, \phi)\pi_1(\theta^O, \phi)d\theta^O d\phi}{\int \int_{\theta^E, \theta^O} p_u(Y|\theta^E, \theta^O, \phi)\pi_u(\theta^E, \theta^O, \phi)d\theta^E d\theta^O d\phi}
\]

\[
= \int_{\theta^O \succ r^O} p_u(Y|\theta^E = r^E, \theta^O, \phi)\pi_1(\theta^O, \phi)\pi_u(\theta^E = r^E|Y, \theta^E = r^E)d\theta^O d\phi
\]

\[
\times \pi_u(\theta^E = r^E|Y)
\]

\[
= \int_{\theta^O \succ r^O} \pi_1(\theta^O, \phi)\pi_u(\theta^O, \phi|Y, \theta^E = r^E)d\theta^O d\phi
\]

\[
\times \pi_u(\theta^E = r^E|Y)
\]

\[
= \int_{\theta^O \succ r^O} \frac{\pi_1(\theta^O, \phi)\pi_u(\theta^O, \phi|Y, \theta^E = r^E)}{\pi_u(\theta^E = r^E|Y)}\pi_u(\theta^O, \phi|Y, \theta^E = r^E)d\theta^O d\phi
\]

\[
\times \frac{\pi_u(\theta^E = r^E)}{\pi_u(\theta^E = r^E|Y)}
\]

\[
= \int_{\theta^O \succ r^O} \frac{\pi_{1,u}(\theta^O, \phi)1_{\{\theta^O > r^O\}}(\theta^O)}{\pi_u(\theta^O, \phi|\theta^E = r^E)}\pi_{1,u}(\theta^O, \phi|Y, \theta^E = r^E)d\theta^O d\phi
\]

\[
\times \frac{\pi_u(\theta^E = r^E)}{\pi_u(\theta^E = r^E|Y)}
\]

\[
= \int_{\theta^O \succ r^O} \frac{\pi_{1,u}(\theta^O, \phi)1_{\{\theta^O > r^O\}}(\theta^O)}{\pi_u(\theta^O, \phi|\theta^E = r^E)}\pi_{1,u}(\theta^O, \phi|Y, \theta^E = r^E)d\theta^O d\phi
\]

\[
\times \text{Pr}_1(\theta^O > r^O)^{-1} \times \frac{\pi_u(\theta^E = r^E|Y)}{\pi_u(\theta^E = r^E)},
\]

which completes the proof. Note that in the third step the indicator function, \(1_{\{\theta^O > r^O\}}(\theta^O)\), could be omitted as the integrand is integrated over the subspace where \(\theta^O > r^O\).
B MCMC sampler for the multivariate Student \( t \) test

1. Drawing the standardized effects \( \delta \). It is well-known that a multivariate Cauchy prior of \( p \) dimensions can be written as a Multivariate normal distribution with an inverse Wishart distribution on the normal covariance matrix with \( p \) degrees of freedom, i.e.,

\[
\pi_u(\delta) = \text{Cauchy}(\delta|\mathbf{S}_0) = \int N(\delta|0, \Phi) \times IW(\Phi|p, \mathbf{S}_0) d\Phi.
\]

Thus the conditional prior for \( \delta \) given the auxiliary parameter matrix \( \Phi \) follows a \( N(0, \Phi) \) distribution. Consequently, as \( \mathbf{z}_{\Sigma,i} = \mathbf{L}_\Sigma^{-1} \mathbf{y}_i \sim N(\delta, \mathbf{I}_p) \), the conditional posterior of \( \delta \) follows a multivariate normal posterior,

\[
\delta|\Phi, \Sigma, \mathbf{Y} \sim N(n(\Phi^{-1} + n\mathbf{I}_p)^{-1}\bar{\mathbf{z}}_\Sigma; (\Phi^{-1} + n\mathbf{I}_p)^{-1}),
\]

where \( \bar{\mathbf{z}}_\Sigma \) are the sample means of \( \mathbf{z}_{\Sigma,i} \), for \( i = 1, \ldots, n \).

2. Drawing the auxiliary covariance matrix \( \Phi \). The conditional posterior for \( \Phi \) only depends on the standardized effects and it follows an inverse Wishart distribution,

\[
\Phi|\delta \sim IW(p + 1, \mathbf{S}_0 + \delta\delta').
\]

3. Drawing the error covariance matrix \( \Sigma \). The conditional posterior for the covariance matrix does not follow a known distribution. For this reason we use a random walk (e.g., [Gelman et al., 2004]) for sampling the separate elements of \( \Sigma \).

The sampler under the unconstrained model while restricting \( \delta_1 = \delta_2 (= \delta) \) is very similar except that the prior for \( \delta \) is now univariate Cauchy(\( \delta|0.125 \)) and \( \Phi = [\phi^2] \) is a scalar, and thus the conditional posterior for \( \delta \) is univariate normal \( N(2n(\phi^{-2} + 2n)^{-1}\bar{z}_\Sigma, (\phi^{-2} + n)^{-1}) \), where \( \bar{z}_\Sigma \) is the mean of \( \bar{z}_\Sigma \). Also note that the inverse Wishart distribution in Step 2 now is an inverse gamma distribution.
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