ON SJÖLIN-SORIA-ANTONOV TYPE EXTRAPOLATION FOR LOCALLY COMPACT GROUPS

GIORGİ ONIANI

Abstract. Sjölin-Soria-Antonov type extrapolation theorem for locally compact σ-compact non-discrete groups is proved. As an application of this result it is shown that the Fourier series with respect to the Vilenkin orthonormal systems on the Vilenkin groups of bounded type converge almost everywhere for functions from the class $L \log^+ L \log^+ \log^+ \log^+ \log^+ L$.

Let $(X, \mu)$ be a measure space. Denote by:

- $L^0(X, \mu)$ the class of all measurable functions $f : X \to [-\infty, \infty]$;
- $\Phi$ the set of all increasing continuous functions $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\liminf_{u \to \infty} \varphi(u)/u > 0$;
- $\varphi(L)(X, \mu)$ the class of all measurable functions $f : X \to [-\infty, \infty]$ for which $\int_X \varphi(|f|)d\mu < \infty$;
- $\chi_E$ the characteristic function of a set $E \subset X$.

We say that an operator $T : L(X, \mu) \to L^0(X, \mu)$ is:

- of restricted weak type $\varphi$, where $\varphi \in \Phi$, if there is a number $C > 0$ such that $\mu(\{|T\chi_E| > \lambda\}) \leq C\varphi(1/\lambda)\mu(E)$ for every $E \subset X$ with finite measure and every $\lambda > 0$;
- of restricted weak type $p$, where $1 \leq p < \infty$, if $T$ is of weak type $\varphi$ for the case of the function $\varphi(u) = u^p$;
- of generalized weak type $\varphi$, where $\varphi \in \Phi$, if for every subset $Y \subset X$ with finite measure and every $\varepsilon > 0$ there is a number $C_{Y, \varepsilon} > 0$ such that $\mu(\{|Tf| > \lambda\} \cap Y) \leq \varepsilon + C_{Y, \varepsilon} \int_X \varphi(|f|/\lambda)d\mu$ for every $f \in L(X, \mu)$ and every $\lambda > 0$.

Denote by:

- $H_m(X, \mu)$ ($m > 0$) the class of all sub-linear operators $T : L(X, \mu) \to L^0(X, \mu)$ for which there are $r = r(T) > 1$ and $C = C(T) > 0$ such that $T$ is of restricted weak type $p$ for every $p \in (1, r)$ with constant $C_p$ satisfying the estimation $C_p \leq \left( \frac{C}{p-1} \right)^{mp}$.

2010 Mathematics Subject Classification. 42B25, 42C10.
Key words and phrases. extrapolation, maximal operator, locally compact group, almost everywhere convergence, Vilenkin system.
• $A_m(X, \mu) (m > 0)$ the class of all sub-linear operators $T : L(X, \mu) \to L^0(X, \mu)$ which are of generalized weak type $\varphi$, where $\varphi(u) = u(1 + \log^+ u)^m (1 + \log^+ \log^+ \log^+ u)$;

For a family $T = (T_j)$ of operators acting from $L(X, \mu)$ to $L^0(X, \mu)$ the maximal operator $M_T$ is defined as follows:

$$M_T f(x) = \sup_j |T_j f(x)| \quad (f \in L(X, \mu), x \in X).$$

**Remark 1.** If a sequence $(T_j f)$ converge a.e. for each function $f$ from some "dense" subclass of $\varphi(L)$, then the generalized weak type $\varphi$ estimation for $M_T$ by standard technique (see, e.g., Lemma 2 below) makes it possible to establish the a.e. convergence for each function $f$ from $\varphi(L)$.

Let $X$ be a locally compact group and $\mu$ be a left-invariant Haar measure in $X$. For a function $k \in L(X, \mu)$ denote by $T_k$ the convolution operator generated by $k$, i.e., $T_k f = f * k (f \in L(X, \mu))$. Denote by $M(X, \mu)$ the class of all maximal operators $M_T$, where $T$ is a sequence of convolution operators $(T_{kj})$ for some functions $k_j \in L(X, \mu)$ ($j \in \mathbb{N}$).

A **Vilenkin group** is defined as the direct product of discrete cyclic groups $\mathbb{Z}_{m_j} = \{0, 1, \ldots, m_j - 1\}$ with $m_j \geq 2$ ($j \in \mathbb{N}$). A Vilenkin group is said to be of **bounded type** if $\sup_j m_j < \infty$. By $(\xi_j^X)$ we will denote the **Vilenkin orthonormal system** corresponding to a Vilenkin group $X$ (see, e.g., [1, App. (0.7)] or [2, §1.5] for the definition). Note that for the case $m_j = 2$ ($j \in \mathbb{N}$) the terms **dyadic group** and **Walsh orthonormal system** respectively are used.

Below $\mathbb{T}$ denotes the one-dimensional torus, i.e., $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Throughout the paper we will use the following convention: $\log n$ stands for $\log_2 n$.

### 1. Results

According to the famous theorem of Carleson [3] the Fourier series of every function $f \in L^2(\mathbb{T})$ converge almost everywhere. Hunt [4] extended this result to the spaces $L^p(\mathbb{T})$ ($p > 1$), furthermore, in [4] it was proved that the Carleson maximal operator (i.e. the operator $M_T$, where $T = (T_j)$ is the sequence of Fourier partial sums) belongs to the class $H_1(X, \mu)$. Using this estimation Antonov [5] have shown almost everywhere convergence of Fourier series of functions from the class $L \log^+ L \log^+ \log^+ L(\mathbb{T})$. Note that on the other hand, due to the result of Konyagin [6] in classes $\varphi(L)$ with $\varphi(u) = o(u \sqrt{\log u / \log \log u}) (u \to \infty)$ it is not guaranteed a.e. convergence of the Fourier series. Refining the method used in [5] Sjölin and Soria [7] proved the following extrapolation principle: Let $(X, \mu)$ be either $\mathbb{R}^n$ or $\mathbb{T}^n$.
with Lebesgue measure and \( m > 0 \). Then \( M(X, \mu) \cap H_m(X, \mu) \subset A_m(X, \mu) \).

The following extension of this result is true.

**Theorem 1.** Let \( X \) be a locally compact \( \sigma \)-compact non-discrete group, \( \mu \) be a left-invariant Haar measure in \( X \) and \( m > 0 \). Then
\[
M(X, \mu) \cap H_m(X, \mu) \subset A_m(X, \mu).
\]

**Remark 2.** Theorem 1 implies the validity of the inclusion (1) for the cases when \( X \) is either locally compact non-discrete group satisfying the second axiom of countability or compact non-discrete group.

Applying Theorem 1 to the a.e. convergence problem of Fourier series with respect to orthonormal systems of characters of compact groups we prove the following theorems.

**Theorem 2.** Let \( X \) be a compact Abelian group, \( \mu \) be a Haar measure in \( X \), \( (\xi_j) \) be a complete orthonormal system consisting of characters of \( X \), \( T = (T_j) \) be the sequence of partial sums of a Fourier series with respect to \( (\xi_j) \) and \( m > 0 \). If the maximal operator \( M_T \) belongs to the class \( H_m(X, \mu) \) then the Fourier series with respect to \( (\xi_j) \) of every function \( f \) from the class \( L(\log^+ L)^m \log^+ \log^+ \log^+ L(X, \mu) \) converge to \( f \) almost everywhere.

**Theorem 3.** Let \( X \) be a Vilenkin group of bounded type and \( \mu \) be a Haar measure in \( X \). Then the Fourier series with respect to the Vilenkin system \( (\xi^X_j) \) of every function \( f \) from the class \( L\log^+ L \log^+ \log^+ \log^+ L(X, \mu) \) converge to \( f \) almost everywhere.

In the setting of Theorem 3 a.e. convergence for functions from the classes \( L^p(X, \mu) \ (p > 1) \) was proved by Gosselin [8].

In [7] it was shown a.e. convergence of Walsh-Fourier series for functions from \( L\log^+ L \log^+ \log^+ \log^+ L([0, 1]) \). This result is equivalent to the conclusion of Theorem 3 for the case of the Walsh orthonormal system on the dyadic group. It must be mentioned here that by virtue of the result of Bochkarev [9] in classes \( \varphi(L) \) with \( \varphi(u) = o(u\sqrt{\log u}) \ (u \to \infty) \) it is not guaranteed a.e. convergence of the Walsh-Fourier series. Analogues theorem for the Vilenkin systems on Vilenkin groups of bounded type was established by Polyakov [10].

2. Proofs

The Lemma 1 below extends the approximation principle proved in [7] (see Lemma 5) to the setting of Theorem 1. The rest part of the proof of Theorem 1 follows the scheme used in [7].

**Lemma 1.** Let \( X \) be a locally compact \( \sigma \)-compact non-discrete group and \( \mu \) be a left-invariant Haar measure in \( X \). Suppose \((k_j)_{j \geq 1}\) is a sequence of
functions from $L(X, \mu)$. Then for every $N \in \mathbb{N}$, a non-negative function $f \in L(X, \mu)$, increasing sequence $(a_n)_{n \geq 0}$ with $a_0 = 0$ and $\lim_{n \to \infty} a_n = \infty$, and $\varepsilon > 0$, there is a simple function $h$ of the form $h = \sum_{n=1}^{\nu} a_n \chi_{E_n}$ for some $\nu$ such that

- $E_n \subset \{a_n-1 < f \leq a_n\}$ and $\int_{\{a_n-1 < f \leq a_n\}} f \, d\mu = a_n \mu(E_n)$ for every $n = 1, \ldots, \nu$;
- $\int_X M_{(T_{k_1}, \ldots, T_{k_N})} (f - h) \, d\mu < \varepsilon$.

**Proof.** For the sake of simplicity we will consider the case when $X$ is Abelian.

Denote $f_t = f \chi_{\{t < t\}}$ and $f^t = f \chi_{\{t \geq 2\}}$ ($t > 0$). For every $t > 0$ we have

$$\int_X M_{(T_{k_1}, \ldots, T_{k_N})} (f^t) \, d\mu \leq \sum_{j=1}^{N} \int_X |f^t \ast k_j| \, d\mu \leq \sum_{j=1}^{N} \|f^t\|_{L(X, \mu)} \|k_j\|_{L(X, \mu)}.$$

Since $\lim_{t \to \infty} \|f^t\|_{L(X, \mu)} = 0$ then by the above estimation we can choose $\nu$ so big that

$$\int_X M_{(T_{k_1}, \ldots, T_{k_N})} (f^\nu) \, d\mu < \frac{\varepsilon}{2}. \quad (2)$$

For every function $f \in L(X, \mu)$ the mapping $X \ni s \mapsto f(\cdot - s) \in L(X, \mu)$ is uniformly continuous (see, e.g., [11, (20.4) Theorem]). Using this statement for functions $k_1, \ldots, k_N$ we find a neighbourhood $V$ of zero in $X$ such that

$$\int_X |k_j(x - s_2) - k_j(x - s_1)| \, d\mu(x) < \frac{\varepsilon}{4N(1 + \|f\|_{L(X, \mu)})} \quad (3)$$

for every $j = 1, \ldots, N$ and $s_1, s_2 \in X$ with $(s_2 - s_1) \in V$.

Let $X_m$ ($m \in \mathbb{N}$) be compact subsets of $X$ such that $X = \bigcup_{m=1}^{\infty} X_m$. For each $m$ we can find finite set $X_m^* \subset X_m$ for which $X_m \subset \bigcup_{x \in X_m^*} (x + V)$. Then the collection $\{x + V : x \in \bigcup_{m=1}^{\infty} X_m^*\}$ is at most countable. Without loss of generality let us assume that this collection is countable and let $V_1, V_2, \ldots$ be its members. Set $V^*_1 = V_1$ and $V^*_m = V_m \setminus \bigcup_{i=1}^{m-1} V_i$ when $m \geq 2$. Note that $V^*_m \subset V_m$ ($m \in \mathbb{N}$), $V^*_{m_1} \cap V^*_{m_2} = \emptyset$ ($m_1 \neq m_2$) and $X = \bigcup_{m=1}^{\infty} V^*_m$. Without loss of generality assume also that $\mu(V^*_m) > 0$ for each $m$.

Denote $G_n = \{a_{n-1} < f \leq a_n\}$ ($n \in \mathbb{N}$). For every $n = 1, \ldots, \nu$ and $m \in \mathbb{N}$ we have

$$\int_{G_n \cap V^*_m} f \, d\mu \leq a_n \mu(G_n \cap V^*_m).$$

Since the group $X$ is non-discrete then the measure $\mu$ is non-atomic (see, e.g., [12, §58]). Consequently, by virtue of Liapounoff’s theorem (see, e.g.,
[13, Theorem 5.5]) there is a set $E_{n,m} \subset G_n \cap V_m^*$ such that
\[
\int_{G_n \cap V_m^*} f \, d\mu = a_n \mu(E_{n,m}).
\] (4)

Denote $E_n = \bigcup_{m=1}^{\infty} E_{n,m}$ $(n = 1, \ldots, \nu)$ and $h = \sum_{n=1}^{\nu} a_n \chi_{E_n}$. Then
\[
\int_X M_{(T_{k_1}, \ldots, T_{k_N})}(f_{a_{\nu}} - h) \, d\mu \leq \sum_{j=1}^{N} \int_X |(f - h) * k_j| \, d\mu \leq
\]
\[
\sum_{j=1}^{N} \sum_{n=1}^{\nu} \sum_{m=1}^{\infty} \int_X \left| \int_{G_n \cap V_m^*} k_j(x - y)(f(y) - a_n \chi_{E_{n,m}}(y)) \, d\mu(y) \right| d\mu(x). (5)
\]

By virtue of (4), for every $n = 1, \ldots, \nu$ and $m \in \mathbb{N}$ we have
\[
\int_{G_n \cap V_m^*} (f(y) - a_n \chi_{E_{n,m}}(y)) \, d\mu(y) = 0. (6)
\]

Let for every $m \in \mathbb{N}$, $x_m$ be the point for which $V_m = x_m + V$. Now using (3–6) we obtain that
\[
\int_X M_{(T_{k_1}, \ldots, T_{k_N})}(f_{a_{\nu}} - h) \, d\mu \leq
\]
\[
\sum_{j, n, m} \int_X \left| \int_{G_n \cap V_m^*} (k_j(x - y) - k_j(x - x_m))(f(y) - a_n \chi_{E_{n,m}}(y)) \, d\mu(y) \right| d\mu(x) \leq
\]
\[
\sum_{j, n, m} \int_{G_n \cap V_m^*} (f(y) + a_n \chi_{E_{n,m}}(y)) \left( \int_X |k_j(x - y) - k_j(x - x_m)| \, d\mu(x) \right) d\mu(y)
\]
\[
\leq \sum_{j=1}^{N} \sum_{n=1}^{\nu} \sum_{m=1}^{\infty} \frac{\varepsilon}{4N(1 + \|f\|_{L(X, \mu)})} \int_{G_n \cap V_m^*} (f(y) + a_n \chi_{E_{n,m}}(y)) \, d\mu(y) =
\]
\[
\frac{\varepsilon}{4(1 + \|f\|_{L(X, \mu)})} \sum_{j=1}^{N} \int_{G_n} f(y) \, d\mu(y) \leq \frac{\varepsilon\|f\|_{L(X, \mu)}}{2(1 + \|f\|_{L(X, \mu)})} < \frac{\varepsilon}{2}. (7)
\]

Combining the estimations (2) and (7) and using sub-linearity of the operator $M_{(T_{k_1}, \ldots, T_{k_N})}$ we conclude the validity of the lemma.

Let $(X, \mu)$ be a measure space with finite measure and $\varphi \in \Phi$. We say that a class of measurable functions $\Delta$ is $\varphi$-dense in $\varphi(L)(X, \mu)$ if for every $f \in \varphi(L)(X, \mu)$ and $\varepsilon > 0$ there is a function $h \in \Delta$ such that $\int_X \varphi(|f - h|) \, d\mu < \varepsilon$. Note that using the dominated convergence theorem it is easy to check $\varphi$-density in $\varphi(L)(X, \mu)$ of the class of all bounded measurable functions.

A function $\varphi : [0, \infty) \to [0, \infty)$ is said to satisfy $\Delta_2$-condition if there are positive numbers $c$ and $u_0$ such that $\varphi(2u) \leq c\varphi(u)$ when $u \geq u_0$. 
**Lemma 2.** Let \((X, \mu)\) be a measure space with finite measure, \(T = (T_j)\) be a sequence of linear operators acting from \(L(X, \mu)\) to \(L^0(X, \mu)\), \(\Pi\) be the class of all functions \(f \in L(X, \mu)\) for which \((T_j f)\) converge to \(f\) almost everywhere, and \(\varphi \in \Phi\) be a function satisfying the \(\Delta_2\) condition. If \(\Pi\) is \(\varphi\)-dense in \(\varphi(L)(X, \mu)\) and \(M_T\) is of generalized weak type \(\varphi\) then \(\varphi(L)(X, \mu) \subset \Pi\).

**Proof.** For \(f \in L(X, \mu)\) and \(\lambda > 0\) denote

\[
E(f, \lambda) = \{\limsup_{j \to \infty} |T_j f - f| > \lambda\}.
\]

Note that if \(f \in \Pi\) then \(\mu(E(f, \lambda)) = 0\) for every \(\lambda > 0\).

Let \(f, h \in L(X, \mu)\). Since for every \(x \in X\),

\[
\limsup_{j \to \infty} |T_j f(x) - f(x)| \leq \limsup_{j \to \infty} |T_j f(x) - T_j h(x)| + \limsup_{j \to \infty} |T_j h(x) - h(x)| + |f(x) - h(x)|
\]

then for every \(\lambda > 0\) we have

\[
E(f, \lambda) \subseteq \{\limsup_{j \to \infty} |T_j f - T_j h| > \lambda/3\} \cup E(h, \lambda/3) \cup \{|f - h| > \lambda/3\} \subseteq \{M_T (f - h) > \lambda/3\} \cup E(h, \lambda/3) \cup \{|f - h| > \lambda/3\}. \tag{8}
\]

Suppose, \(f \in \varphi(L)(X, \mu)\). Let us show that for every \(\lambda \in (0, 1)\) the set \(E(f, \lambda)\) has zero measure. It will imply the needed conclusion. Let \(\varepsilon\) be an arbitrary positive number. For every \(h \in L(X, \mu)\) we have

\[
\mu(\{M_T (f - h) > \lambda/3\}) \leq \varepsilon + C_{X, \varepsilon} \int_X \varphi \left(\frac{3|f - h|}{\lambda}\right) d\mu, \tag{9}
\]

and

\[
\mu(\{|f - h| > \lambda/3\}) \leq \frac{1}{\varphi(\lambda/3)} \int_X \varphi(|f - h|) d\mu. \tag{10}
\]

Since \(\varphi\) satisfy the \(\Delta_2\) condition then \(3f/\lambda \in \varphi(L)(X, \mu)\). Consequently, we can choose a function \(g \in \Pi\) so that

\[
\int_X \varphi \left(\left|\frac{3f}{\lambda} - g\right|\right) d\mu < \frac{\varepsilon}{C_{X, \varepsilon} + 1/\varphi(\lambda/3)}. \tag{11}
\]

Set \(h = \lambda g/3\). Note that the class \(\Pi\) is positively homogeneous. Therefore \(h \in \Pi\) and consequently,

\[
\mu(E(h, \lambda/3)) = 0. \tag{12}
\]

On the other hand we have

\[
\int_X \varphi \left(\left|\frac{3f - h}{\lambda}\right|\right) d\mu = \int_X \varphi \left(\left|\frac{3f}{\lambda} - g\right|\right) d\mu \tag{13}
\]
and
\[ \int_X \varphi(|f-h|)d\mu = \int_X \varphi\left(\frac{\lambda}{3}\left|\frac{3f}{\lambda} - g\right|\right)d\mu \leq \int_X \varphi\left(\frac{|3f}{\lambda} - g|\right)d\mu. \]  
(14)

From (8 – 14) we obtain the estimation \( \mu(E(f, \lambda)) \leq 3\varepsilon. \) Consequently, by arbitrariness of \( \varepsilon > 0 \) we have \( \mu(E(f, \lambda)) = 0. \) The lemma is proved.

**Remark 3.** Lemma 2 remains true without assumption of \( \varphi \) to satisfy the \( \Delta_2 \)-condition. In this setting the proof becomes a bit longer because there appears an additional step. Namely, first we have to approximate a function \( f \in \varphi(L)(X, \mu) \) by its lower truncation \( f_t = f\chi_{\{f<t\}} \) and then \( f_t \) by a function \( h \in \Pi. \)

**Proof of Theorem 2.** Note that the partial sum operators \( T_j \) of a Fourier series with respect to the system of characters \( (\xi_j) \) is represented by the convolution (in \( X \)) with Dirichlet kernels corresponding to \( (\xi_j) \), i.e. \( T_j(f) = f \ast D_j \), where \( D_j = \sum_{i=0}^{j-1} \xi_i \). Consequently, by virtue of Theorem 1 the maximal operator \( M_T = M_{(T_j)} \) is of generalized weak type \( \varphi \), where
\[ \varphi(u) = (1 + \log^+ u)^m(1 + \log^+ \log^+ \log^+ u). \]

Due to the interpolation theorem of Stein and Weiss (see, e.g., [14, Corollary 1.4.21]), \( M_T \) is of strong type \( p \) for every \( p \in (1, r(M_T)) \). Consequently, assuming \( p \in (1, \min\{r(M_T), 2\}) \), and, taking into account density in \( L^2(X, \mu) \) of the class of all polynomials with respect to the system \( (\xi_j) \), we conclude that the Fourier series with respect to \( (\xi_j) \) of every function \( f \) from \( L^p(X, \mu) \) converge almost everywhere. Now using \( \varphi \)-density in \( \varphi(L)(X, \mu) \) of the class of bounded measurable functions, on the basis of Lemma 2 we obtain the needed conclusion.

**Proof of Theorem 3.** Let \( T = (T_j) \) be the sequence of partial sum operators of a Fourier series with respect to the system \( (\xi_j^X) \). Gosselin [8] proved that for every \( q \in (1, 2) \) there is a constant \( C_q > 0 \) for which
\[ \mu\{|M_T^X E > \lambda\} \leq \frac{C_q}{X^q} \mu(E), \]
for every measurable set \( E \subset X \) and \( \lambda > 0 \). Perusing the proof of this result one can observe that (see the estimation (84) for \( \mu(E_1) \), the estimation for \( \mu(E_2) \) after the relation (93) and the estimation for \( |S_n f(x)| \) at the end of the proof of the basic result in [8]) for the constants \( C_q \) the estimation
\[ C_q \leq \left(\frac{C}{q-1}\right)^q \ (q \in (1, 2)) \]
is valid, where \( C > 0 \) is an absolute constant. After it by virtue of Theorem 2 we finish the proof.

**References**

[1] F. Schipp, W. R. Wade and P. Simon, *Walsh series: An introduction to dyadic harmonic analysis*, Adam Hilger, Bristol, New York, 1990.
[2] B. Golubov, A. Efimov, V. Skvortsov, *Walsh Series and Transforms: Theory and Applications*, Springer, 1991.
[3] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116, No.1-2 (1966), 135-167.
[4] R.A. Hunt, *On the convergence of Fourier series*. Orthogonal Expansions and their Continuous Analogues, Proc. Conf. Edwardsville, III. (1967), Southern Illinois Univ. Press, Carbondale III. (1968), 235-255.
[5] N.Yu. Antonov, *Convergence of Fourier series*, East J. Approx. 2 (1996), 187-196.
[6] S. V. Konyagin, *On everywhere divergence of trigonometric Fourier series*, Sb. Math. 191 (2000), 97-120.
[7] P. Sjölín and F. Soria, *Remarks on a theorem by N. Yu. Antonov*, Studia Math. 158, No. 1 (2003), 79-97.
[8] J. Gosselin, *Almost everywhere convergence of Vilenkin-Fourier series*, Trans. Amer. Math. Soc. 185 (1973), 345-370.
[9] S.V. Bochkarev, *Everywhere divergent Fourier series with respect to the Walsh system and with respect to multiplicative systems*, Russian Math. Surveys, 2004, 59 (2004), 103-124.
[10] I.V. Polyakov, *Example of a divergent Fourier series in the Vilenkin system*, Math. Notes, 89, No.5 (2011), 734-740.
[11] E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis*, Springer, 1970.
[12] P.R. Halmos, *Measure Theory*, Springer, 1950.
[13] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.
[14] L. Grafakos, *Classical Fourier Analysis*, Springer, 2008.

Department of Mathematics, Akaki Tsereteli State University, 59 Tamar Mepe St., Kutaisi 4600, Georgia

*E-mail address*: oniani@atsu.edu.ge