On a classification of ideals of local rings for irreducible curve singularities

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Abstract
We consider a classification problem of ideals by codimension in case rings are the local rings of irreducible curve singularities. In this paper, we introduce a systematic method to solve this problem.

1 Introduction

Let \( k \) be an algebraically closed field of characteristic zero. In the present paper, we consider a local integral noetherian \( k \)-algebra \( \mathcal{O} \) of dimension one. We also assume that \( \mathcal{O} \) is complete. Our aim is to present an effective method for a classification of ideals of \( \mathcal{O} \) with fixed codimension. Such classification is important for the study of punctual Hilbert schemes for irreducible curve singularities. See \cite{2} about punctual Hilbert schemes for curve singularities. In \cite{3} and \cite{4}, the results in this paper were used implicitly. However, they were not described at all. So we explain the methods precisely in this paper.

In Section \( \text{2} \), we fix the notations and prove some lemmas needed later. We define the order set of an ideal. We also show that the set of all ideals with fixed codimension admits finitely many order sets. In Section \( \text{3} \), we consider generators of an ideal whose order set is a given \( \Gamma \)-module. We first consider normal forms of these. In general, we need to determine the coefficients in the normal forms to attain the given order set. The algorithm to determine the coefficients is also introduced. In Section \( \text{4} \), we first introduce two algorithms. One is to compute the minimal generating set for a given \( \Gamma \)-module and the other is to compute all order sets for ideals with codimension \( i \) from those for ideals with codimension \( i - 1 \). These algorithms allow us to construct a systematic way to determine the all order sets for ideals with a given codimension. Finally, we prove that Theorem \( \text{8} \) which gives a computational method to determine all ideals with a given codimension. In Section \( \text{5} \), we consider two examples, the cases for the singularities of \( A_6 \) and \( E_6 \) types.

2 Preliminaries

We first fix notations. For a local ring \( \mathcal{O} \) as in the previous section, its maximal ideal is expressed by \( \mathfrak{m}_\mathcal{O} \). We denote by \( \mathcal{O}_\text{normal} \) the normalization of \( \mathcal{O} \). In our
case, we have $\overline{O} = k[[t]]$. We call the set $\Gamma := \{\text{ord}(f) \mid f \in O\}$ the semigroup of $O$. The set $G := \mathbb{N} \setminus \Gamma$ is called the gap sequence of $\Gamma$. The conductor of $\Gamma$ is to be $c := \max\{G\} + 1$. A subset $S$ of $\Gamma$ is called a $\Gamma$-module, if the following two conditions hold: (i) $s_1 + s_2 \in S$ for $\forall s_1, s_2 \in S$, (ii) $s + \gamma \in S$ for $\forall s \in S$, $\forall \gamma \in \Gamma$. If a $\Gamma$-module $S$ is generated by a set $A = \{\alpha_1, \ldots, \alpha_m\}$ (i.e. $S = \langle \gamma + \sum_{i=1}^{m} \gamma \alpha_i \mid \gamma \in \Gamma \rangle$), then we express it by $S = \langle A \rangle = \langle \alpha_1, \ldots, \alpha_m \rangle$. We refer $A$ as a generating set of $S$. A generating set $A$ of $S$ is called minimal, if $A \setminus \{\alpha\}$ for $\forall \alpha \in A$ does not generate $S$. Let $I$ be a nonzero ideal in $O$. We call $\tau := \tau(I) := \dim_k O/I$ the codimension of $I$ and denote by $I_r$ the set of all ideals with $\tau(I) = r$. The set $\Gamma(I) := \{\text{ord}(f) \mid f \in I\}$ is called the order set of $I$. Note that $\Gamma(I)$ is a $\Gamma$-module. Put $G(I) := \Gamma \setminus \Gamma(I)$. We define the conductor of $\Gamma(I)$ by $c(I) := \max\{G(I)\} + 1$. The following lemma shows that the codimension of $I$ depends on $G(I)$. Namely, it is determined by $\Gamma(I)$.

**Lemma 1.** An ideal $I$ of $O$ has $\tau(I) = r$ if and only if we have $\sharp G(I) = r$.

**Proof.** An ideal $I$ belongs to $I_r$ if and only if

$$O/I = \{a_0 + a_1 t^{d_1} + \cdots + a_{r-1} t^{d_{r-1}} \mid a_i \in k, d_1 < \cdots < d_{r-1}\}$$

(1)

holds. The last condition implies $\sharp G(I) = r$. \hfill $\square$

Let $S$ be a $\Gamma$-module. We denote by $I(S)$ the set of all ideals of $O$ with $\Gamma(I) = S$. We have the following proposition.

**Proposition 2.** There exist a finite number of distinct $\Gamma$-modules $S_1, \ldots, S_l$ such that the set $I_r$ is decomposed as

$$I_r = \bigcup_{i=1}^{l} I(S_i)$$

(2)

where $I(S_i) \cap I(S_j)$ for $i \neq j$.

**Proof.** Fix a codimension $r$. Let $I$ be an element of $I_r$. By Lemma 1 we have $\sharp G(I) = r$. Write $G(I) = \{0, d_1, \ldots, d_{r-1}\}$. Then it must satisfy the condition $\sharp$. It is clear that the number of such $d_1, \ldots, d_{r-1}$ is finite. Hence we have only finite number of $\Gamma$-module of the form $\Gamma \setminus \{0, d_1, \ldots, d_{r-1}\}$. \hfill $\square$

For the decomposition (2) of $I_r$, set $\mathcal{S}_r := \{S_1, \ldots, S_l\}$. Letting $A_i$ be the minimal generating set of $S_i$, define $A_r := \{A_1, \ldots, A_l\}$. Let $I$ be a nonzero ideal of $O$ such that $I \neq (1)$. Set $G_1(I) := G(I) \setminus \{0\}$. We define $b_1 := \min\{G_1(I)\}$. For $b_1$, put $B_1(I) := G_1(I) \cap (b_1 + \Gamma)$. If $G_1(I) \neq B_1(I)$, then put $b_2 := \min\{G_1(I) \setminus B_1(I)\}$ and $B_2(I) := \left(G_1(I) \setminus (b_1 + \Gamma)\right) \cap (b_2 + \Gamma)$. We continue this process successively. Namely, if $G_1(I) \neq \cup_{j=1}^{n-1} B_i(I)$, then we set $b_j := \min\{G_1(I) \setminus \cup_{i=1}^{n-1} B_i(I)\}$ and $B_j(I) := \left(G_1(I) \setminus \cup_{i=1}^{n-1} B_i(I)\right) \cap (b_j + \Gamma)$. Since $\sharp G(I) < \infty$, there exist a positive integer $n$ such that $G_1(I) = \cup_{i=1}^{n} B_i(I)$. For each $i$, we define $d_i := \max\{B_i(I)\}$.
3 Determination of ideals

From this section, we freely use the notations introduced in the previous section. Let \(O\) be the local ring of an irreducible curve singularity with a semigroup \(\Gamma\). For an element \(f\) of \(O\), we denote by \(\operatorname{LC}(f)\) the leading coefficient of \(f\). For a given \(\Gamma\)-module \(S\), we determine generators of the ideals in \(I(S)\). Let \(I\) be an element of \(I(S)\). For each \(\gamma \in \Gamma(I) = S\), we consider the following element of \(I\):

\[
 f_\gamma := t^\gamma + \sum_{j \in G, j > \gamma} a_{\gamma,j} t^j.
\]

**Lemma 3.** Let \(S\) be a \(\Gamma\)-module. For an ideal \(I\) in \(I(S)\), we can take

\[
 F := \{ f_\gamma \in I | \gamma \in S, \min(S) \leq \gamma \leq \min(S) + c - 1 \}
\]

as the set of generators of \(I\).

**Proof.** It is clear that \(I\) is generated by all \(f_\gamma\) with \(\gamma \in \Gamma(I)\). Note that we can take \(f_\gamma = t^\gamma\) for \(\gamma \geq c(I)\). Set \(a_1 := \min(\Gamma(I))\). It is clear that \(c(I) \leq a_1 + c\) holds. Since there exists \(t^j\) in \(O\) for \(\forall j \geq c\), for \(\gamma \in \Gamma\) with \(\gamma \geq a_1 + c\), we can reduce \(f_\gamma\) by \(f_{a_1}\) as

\[
 f_\gamma - \sum_{j \geq c} b_j t^j f_{a_1} = 0
\]

where \(b_j\)'s are suitable coefficients. Hence we can remove \(f_\gamma\) with \(\gamma \geq a_1 + c\) from the set of generators of \(I\). \(\Box\)

Let \(I\) be an element of \(I(S)\). In general, if (3) is the set of generators of \(I\), then the coefficients of its elements may satisfy some conditions to attain \(\Gamma(I) = S\). We denote by \(H\) the set of all such conditions. We must find \(H\) to determine the ideals in \(I(S)\) from the data \(S\) (see also Remark 10 in Section 5).

To determine \(H\) for a given \(S\), we introduce the reduction for two elements in \(O\). This is an analogue of S-polynomial for given two polynomials (cf. [1]). For \(h_1, h_2 \in O\), consider \(V := \{ (\gamma_1, \gamma_2) \in \Gamma \times \Gamma | \gamma_1 \cdot \operatorname{ord}(h_1) = \gamma_2 \cdot \operatorname{ord}(h_2) \}\). Let \((\alpha, \beta)\) be the element of \(V\) that makes the condition \(\gamma_1 \cdot \operatorname{ord}(h_1) = \gamma_2 \cdot \operatorname{ord}(h_2)\) minimal. In the similar manner of the definition of \(f_\gamma\), we consider the following elements of \(O\):

\[
 g_\gamma := t^\gamma + \sum_{j \in G, j > \gamma} b_{\gamma,j} t^j \quad (\gamma \in \Gamma)
\]

We define the reduction of \(h_1\) and \(h_2\) by

\[
 \operatorname{Red}(h_1, h_2) := \frac{g_\alpha}{\operatorname{LC}(h_1)} h_1 - \frac{g_\beta}{\operatorname{LC}(h_2)} h_2. \quad (4)
\]

**Proposition 4.** Let \(I\) be an ideal in \(I(S)\). We rewrite the set (3) as \(F = \{ f_1, \ldots, f_l \}\) where

\[
 f_i := t^{\gamma_i} + \sum_{j \in G, j > \gamma_i} a_{i,j} t^j \quad (i = 1, \ldots, l).
\]
The condition set $H$ of $F$ is obtained in a finite number of steps by the following algorithm:

Input: $F = \{f_1, \ldots, f_l\}$
Output: $H$

**DEFINE:** $H := \emptyset$

FOR each $i, j$ in $\{1, \ldots, l\}$ with $i \neq j$

$R := \text{Red}(f_i, f_j)$

WHILE $\deg(R) < c(I)$ DO

IF $\text{ord}(R) \notin \Gamma(I)$ THEN $H := H \cup \{\text{LC}(R) = 0\}$

ELSE $R := \text{Red} \left( R, \sum_{f_i \in L} b_i g_{\sigma_i} f_i \right)$ for $L = \{f_i \in F | \exists \sigma_i \in \Gamma \text{ s.t. } \gamma_i + \sigma_i = \text{ord}(R)\}$

**Proof.** For distinct elements $f_i$ and $f_j$ in $F$, we first compute $R_1 := \text{Red}(f_i, f_j)$. Note that $\text{LC}(R_1)$ is a polynomial with respect to the coefficients of $f_i$ and $f_j$. If $\text{ord}(R_1) \notin S$, then we must have $\text{LC}(R_1) = 0$. We add it to $H$ and put $R_2 := R_1$. On the other hand, if $\text{ord}(R_1) \in S$, then we can set $L_1 := \{f_i \in F | \exists \sigma_i \in \Gamma \text{ s.t. } \gamma_i + \sigma_i = \text{ord}(R_1)\}$ and reduce $R_1$ by the power series $\sum_{f_i \in L_1} b_i g_{\sigma_i} f_i$ ($b_i \in k$). Put $R_2 := \text{Red} \left( R_1, \sum_{f_i \in L_1} b_i g_{\sigma_i} f_i \right)$. Next we apply same arguments to $R_2$. If $\text{ord}(R_2) \notin S$, then the leading coefficient $\text{LC}(R_2)$ must be 0 for any value of $b_i$. This fact yields some conditions of the coefficients of $f_i$ and $f_j$. We add them to $H$ and rewrite $R_2$ by $R_3$. If $\text{ord}(R_2) \in S$, then we set $L_2 := \{f_i \in F | \exists \sigma_i \in \Gamma \text{ s.t. } \gamma_i + \sigma_i = \text{ord}(R_2)\}$ and reduce $R_2$ by $\sum_{f_i \in L_2} b_i g_{\sigma_i} f_i$. Put $R_3 := \text{Red} \left( R_2, \sum_{f_i \in L_2} b_i g_{\sigma_i} f_i \right)$. In this way, we have $\text{ord}(R_1) < \text{ord}(R_2) < \cdots$.

If $\text{ord}(R_i) \geq c(I)$, then further reduction yields no element of $H$, because any integer $a$ with $a \geq c(I)$ belongs to $\Gamma(I) = S$. So these procedures terminate in finite many steps. Finally, we obtain the set $H$ of conditions. \hfill \Box

### 4 Computational algorithms

In this section, we give a computational method to determine $\mathcal{I}_r$.

**Proposition 5.** For a $\Gamma$-module $S$, we obtain the set $A$ of minimal generators for $S$ in a finite number of steps by the following algorithm:

**Input:** $S$
**Output:** $A$

**DEFINE:** $A := \emptyset$, $S := S$

WHILE $S \neq \emptyset$ DO

$A := A \cup \{\min\{S\}\}$

$S := S \setminus \{\min\{S\} + \gamma | \gamma \in \Gamma\}$

**Proof.** For $\alpha_1 := \min\{S\}$, consider the set $\alpha_1 + \Gamma$. If $S \neq \alpha_1 + \Gamma$, then we put $\alpha_2 := \min\{S \setminus (\alpha_1 + \Gamma)\}$. We repeat this process as follows: If $S \neq \bigcup_{i=1}^{j-1} (\alpha_i + \Gamma)$,
then set $\alpha_j := \min \{ S \setminus \cup_{i=1}^{j-1} (\alpha_i + \Gamma) \}$. Then we obtain the descending sequence

$$S \supseteq S \setminus (\alpha_1 + \Gamma) \supseteq \cdots \supseteq S \setminus \cup_{i=1}^{j} (\alpha_i + \Gamma) \supseteq \cdots .$$

Since the infinite set $\{ a \in \mathbb{N} \mid a \geq \alpha_1 + c \}$ is contained in both $S$ and $\alpha_1 + \Gamma$, we have $\sharp (S \setminus (\alpha_1 + \Gamma)) < \infty$. Thus there exists a positive integer $n$ such that $S = \cup_{i=1}^{n} (\alpha_i + \Gamma)$. This fact guarantees the termination of the algorithm. It is clear that the set $A$ obtained by this algorithm is the minimal generating set of $S$.

**Lemma 6.** If there exists an ideal $I$ in $\mathcal{I}_{r-1}$ with $\Gamma(I) = \langle \alpha_1, \ldots, \alpha_m \rangle$, then so does an ideal $I'$ in $\mathcal{I}_r$ with $G(I') = G(I) \cup \{ \alpha_i \}$ for each $i$. Conversely, if there exists an ideal $J$ in $\mathcal{I}_r$ with $G(J) = \cup_{i=1}^{m} B_i(J)$, then so does an ideal $J'$ in $\mathcal{I}_{r-1}$ with $G(J') = G(J) \setminus \{ d_i \}$ for each $i$.

**Proof.** Let $I$ be an ideal in $\mathcal{I}_{r-1}$ with $\Gamma(I) = \langle \alpha_1, \ldots, \alpha_m \rangle$. For an element $\alpha_i \in \Gamma(I)$, put $S = \{ \Gamma(I) \setminus \{ \alpha_i \} \} \cup \{ \alpha_i + \gamma \mid \gamma \in \Gamma \setminus \{ 0 \} \}$. It is easy to check that $S$ is a $\Gamma$-module. Consider an ideal $I'$ generated by $f_\gamma$ with $\min \{ S \} \leq \gamma \leq \min \{ S \} + c - 1$. We see that $\Gamma(I') = S$ and $G(I') = G(I) \setminus \{ \alpha_i \}$. Since $I \in \mathcal{I}_{r-1}$, we have $\sharp G(I) = r - 1$ by Lemma 4. So $\sharp G(I') = r$. We also have $I' \in \mathcal{I}_r$ by Lemma 4. Conversely, let $J$ be an element of $\mathcal{I}_r$ with $G(J) = \cup_{i=1}^{m} B_i(J)$. Note that, for each $i$, the set $\Gamma(J) \cup \{ d_i \}$ is a $\Gamma$-module. Indeed, by definition of $d_i$, we see that $d_i + \gamma \in \Gamma(J)$ for any $\gamma \in \Gamma \setminus \{ 0 \}$. So the set $\Gamma(J) \cup \{ d_i \}$ satisfies the definition of $\Gamma$-module. In the same manner as above, we take an ideal $J'$ generated by $f_\gamma$ with $\min \{ \Gamma(J) \cup \{ d_i \} \} \leq \gamma \leq \min \{ \Gamma(J) \cup \{ d_i \} \} + c - 1$. It is an element of $\mathcal{I}_{r-1}(\Gamma(J) \cup \{ d_i \})$.

It follows from Lemma 4 that the following proposition.

**Proposition 7.** We obtain the set $\mathcal{G}_r$ from $\mathcal{G}_{r-1}$ in a finite number of steps by the following algorithm:

**Input:** $\Gamma$ and $\mathcal{G}_{r-1} = \{ S_1 = \langle A_1 \rangle, \ldots, S_l = \langle A_l \rangle \}$ where $A_i = \{ \alpha_{i,1}, \ldots, \alpha_{i,m(i)} \}$

**Output:** $\mathcal{G}_r$

**DEFINE:** $\mathcal{G}_r := \emptyset$

FOR each $i$ in $\{1, \ldots, l\}$ and each $j$ in $\{1, \ldots, m(i)\}$ DO

$S := [S_i \setminus \{ \alpha_{i,j} \}] \cup \{ \alpha_{i,j} + \gamma \mid \gamma \in \Gamma \setminus \{ 0 \} \}$

IF $S \notin \mathcal{G}_r$ THEN $\mathcal{G}_r := \mathcal{G}_r \cup \{ S \}$ ELSE do nothing

**Proof.** By the argument in the proof of Lemma 4, the set $[S_i \setminus \{ \alpha_{i,j} \}] \cup \{ \alpha_{i,j} + \gamma \mid \gamma \in \Gamma \setminus \{ 0 \} \}$ is an order set for some ideal in $\mathcal{I}_r$. Since we have $\sharp \mathcal{G}_r < \infty$ by Proposition 7 and a $\Gamma$-module is generated by finely many generators by Proposition 8, this algorithm terminates and yields $\mathcal{G}_r$.

Using Proposition 4, 5 and 7 we obtain the following theorem:

**Theorem 8** (Computational steps for $\mathcal{I}_r$). For a given codimension $r$, we obtain $\mathcal{I}_r$ in the following $r$ steps.

**Step 1:** Set $\mathcal{G}_1 = \{ \Gamma(m_0) \}$ and find generators of $\Gamma(m_0)$ by Proposition 5.
Step i \((i = 2, \ldots, r)\): First compute \(\mathcal{G}_i\) by Proposition 7 Next determine \(\mathfrak{A_i}\) by applying Proposition 5 to each elements in \(\mathcal{G}_i\).

Step \(r + 1\): For each \(S \in \mathcal{G}_r\), determine in the coefficients in the generators of \(I(S)\) by Proposition 4.

**Proof.** It is clear that \(I_1 = \{m_\mathcal{O}\}\). So we have \(\mathcal{G}_1 = \{\Gamma(m_\mathcal{O})\}\). By using Proposition 5 we can obtain the minimal generating set \(A\) of \(\Gamma(m_\mathcal{O})\). Set \(\mathfrak{A}_1 := \{A\}\).

With these datum, we compute \(\mathcal{G}_2\) by using Proposition 7. Applying Proposition 5 to each element in \(\mathcal{G}_2\), we get \(\mathfrak{A}_2\). Continuing this process successively, we obtain \(\mathcal{G}_r\) from \(\mathcal{G}_{r-1}\) for a codimension \(r\). Finally, for each \(S\) in \(\mathcal{G}_r\), consider the generators of \(I(S)\) obtained by Lemma 8 Their coefficients are determined by Proposition 4. \(\square\)

5 Examples

We consider two examples in this section. If \(\mathcal{G}_r\) consists of \(l\) elements, then we write \(S_{r,i} (i = 1, \ldots, l)\) for these elements. We also use the notation \(I_{r,i}\) instead of \(I(S_{r,i})\). We first consider the \(A_{2d}\) type singularity (i.e. the curve singularity whose local ring \(\mathcal{O}\) is isomorphic to \(k[[t^2, t^{2d+1}]]\)). Similar to Lemma 8, we can prove the following lemma for the \(A_{2d}\) type singularity:

**Lemma 9.** Let \(I\) be an ideal of \(k[[t^2, t^{2d+1}]]\). If we have \(\Gamma(I) = \langle \alpha_1, \alpha_2 \rangle\) where \(\alpha_1 < \alpha_2\), then \(I\) is generated by

\[
f_{\alpha_1} = t^{\alpha_1} + \sum_{j \in G(I), j > \alpha_1} a_j t^j, \quad f_{\alpha_2} = t^{\alpha_2}.
\]  

(5)

On the other hand, if \(I\) has \(\Gamma(I) = \langle \alpha_1 \rangle\), then \(I\) is generated by \(f_{\alpha_1}\) above.

**Remark 10.** Lemma 7 implies that, for the \(A_{2d}\) type singularity, the coefficients in \((5)\) are completely determined by \(\Gamma(I)\). Namely, \(H = 0\). So we can omit the use of Proposition 4 in Step \(r + 1\) in Theorem 8.

Now consider the \(A_6\) singularity. Its local ring \(\mathcal{O}\) is isomorphic to \(k[[t^2, t^7]]\). We classify the elements of \(I_{r}\) for \(1 \leq \tau \leq 6\). We observe that \(\Gamma = \{0, 2, 4, 6, 7, 8, \ldots\}\) where \(c = 6\). Carrying out Theorem 8 the order sets \(S_{r,1}\) and the components of \(I_{r}\) are determined as in the following two tables:

| \(\tau\) | order sets |
|--------|---------|
| 1 \(S_{1,1}\) | \(\langle 2, 7 \rangle\) |
| 2 \(S_{2,1}\) | \(\langle 4, 7 \rangle\), \(S_{2,2} = \langle 2 \rangle\) |
| 3 \(S_{3,1}\) | \(\langle 6, 7 \rangle\), \(S_{3,2} = \langle 4, 9 \rangle\) |
| 4 \(S_{4,1}\) | \(\langle 7, 8 \rangle\), \(S_{4,2} = \langle 6, 9 \rangle\), \(S_{4,3} = \langle 4 \rangle\) |
| 5 \(S_{5,1}\) | \(\langle 8, 9 \rangle\), \(S_{5,2} = \langle 7, 10 \rangle\), \(S_{5,3} = \langle 6, 11 \rangle\) |
| 6 \(S_{6,1}\) | \(\langle 9, 10 \rangle\), \(S_{6,2} = \langle 8, 11 \rangle\), \(S_{6,3} = \langle 7, 12 \rangle\), \(S_{6,4} = \langle 6 \rangle\) |
In the above table, we obtain the following two tables:

| τ | components of \( \mathcal{I}_\tau \) |
|---|---|
| 1 | \( I_{1,1} = (t^2, t^7) \) |
| 2 | \( I_{2,1} = (t^4, t^7), I_{2,2} = (t^3 + at^7) \) |
| 3 | \( I_{3,1} = (t^6, t^7), I_{3,2} = (t^4 + at^7, t^9) \) |
| 4 | \( I_{4,1} = (t^7, t^8), I_{4,2} = (t^6 + at^7, t^9), I_{4,3} = (t^4 + at^7 + bt^9) \) |
| 5 | \( I_{5,1} = (t^8, t^9), I_{5,2} = (t^7 + at^8, t^{10}), I_{5,3} = (t^6 + at^7 + bt^9, t^{11}) \) |
| 6 | \( I_{6,1} = (t^9, t^{10}), I_{6,2} = (t^8 + at^9, t^{11}), I_{6,3} = (t^7 + at^8 + bt^9, t^{12}), I_{6,4} = (t^6 + at^7 + bt^9 + ct^{11}) \) |

In the above table, \( a, b, c \) are elements of \( k \).

Next we consider the \( E_6 \) singularity as second example. It is the irreducible curve singularity whose local ring \( O \) is \( k[[t^3, t^4]] \). We have \( \Gamma = \{3, 4, 6, 7, 8, \ldots\} \) where \( c = 6 \). We determine the elements of \( I_\tau \) for \( 1 \leq \tau \leq 6 \). Applying Theorem 8 we obtain the following two tables:

| τ | order sets |
|---|---|
| 1 | \( S_{1,1} = (3, 4) \) |
| 2 | \( S_{2,1} = (4, 6), S_{2,2} = (3, 8) \) |
| 3 | \( S_{3,1} = (6, 7, 8), S_{3,2} = (4, 9), S_{3,3} = (3) \) |
| 4 | \( S_{4,1} = (7, 8, 9), S_{4,2} = (6, 8), S_{4,3} = (6, 7), S_{4,4} = (4) \) |
| 5 | \( S_{5,1} = (8, 9, 10), S_{5,2} = (7, 9), S_{5,3} = (7, 8), S_{5,4} = (6, 11) \) |
| 6 | \( S_{6,1} = (9, 10, 11), S_{6,2} = (8, 10), S_{6,3} = (8, 9), S_{6,4} = (7, 12), S_{6,5} = (6) \) |

In the above table, \( a, b, c \in k \).

Here we explain how to determine \( I_{6,5} \) by Step 7 in Theorem 8. By Proposition 5 we see that \( S_{6,5} \) generated by 6. Let \( I \) be an ideal in \( I_{6,5} \). We have \( \Gamma(I) = S_{6,5} = \{6, 9, 10, 12, 13, \ldots\} \) where \( c(I) = 12 \) and \( G(I) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 11\} \). By Lemma 8 we can take \( F = \{f_\gamma \in I | \gamma \in \{6, 9, 10\} \cup \{\gamma | 12 \leq \gamma \leq 17\} \} \) as the set \( 3 \) of generators of \( I \). We simply write \( f_6 = t^6 + at^7 + bt^8 + ct^{11}, f_9 = t^9 + dt^{11}, f_{10} = t^{10} + et^{11} \) and \( f_7 = t^7 \) for \( 12 \leq \gamma \leq 17 \). We first apply Proposition 4 to \( f_6 \) and \( f_9 \). Then we have

\[
R_1 := \text{Red}(f_6, f_9) = t^3 f_6 - f_9 = at^{10} + (b - d) t^{11}.
\]

Note that \( 10 \in \Gamma(I) \). Since \( 10 = 4 + 6 \) \((4 \in \Gamma, 6 \in \Gamma(I))\) is the unique expression in \( S_{6,5} \), we set \( L_1 = \{f_6\} \). Taking \( t^4 \) as \( g_4 \), we reduce \( R_1 \) by \( pt^4 f_6 \) \((p \in k)\).

\[
R_2 := \text{Red}(R_1, pt^4 f_6) = \left(\frac{b - d - a^2}{a}\right) t^{11} - bt^{12} - ct^{15}
\]
Since 11 \not\in \Gamma(I), we must have \( b - d - a^2 = 0 \). Hence, we add \( d = b - a^2 \) to \( H \). Since \( \text{ord}(R_2) \) is 12, further reduction step yields no relations. Next we consider the reduction of \( f_6 \) and \( f_{10} \). In the same argument as above, we can see that \( e = a \). Furthermore, it is easy to check that \( f_{10} \) is expressed by \( f_6 \). We also can show that all \( f_\gamma \) with \( 12 \leq \gamma \leq 17 \) are expressed by \( f_6 \). Finally, we conclude that \( I_{6,5} = (t^6 + at^7 + bt^8 + ct^{11}, t^9 + (b - a^2)t^{11}) \).
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