On the Fundamental Theorem of \((p, q)\)-Calculus and Some \((p, q)\)-Taylor Formulas

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Abstract. In this paper, the \((p, q)\)-derivative and the \((p, q)\)-integration are investigated. Two suitable polynomial bases for the \((p, q)\)-derivative are provided and various properties of these bases are given. As application, two \((p, q)\)-Taylor formulas for polynomials are given, the fundamental theorem of \((p, q)\)-calculus is included and the formula of \((p, q)\)-integration by part is proved.

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1. Introduction

The important new discoveries during the last two decades were focussed on the \(q\)-calculus based on deformation of quantum spaces. Regarding their importance in various fields of mathematics, there is a considerable list of publications [4–10, 12, 13, 16] and the references therein. The \(q\)-differential operators having the so-called \(q\)-classical polynomials as eigenfunctions fulfils many interesting properties. This is based on orthogonal polynomial whose sequences of the corresponding \(q\)-derivatives or Jackson-derivative is also orthogonal. The \((p, q)\)-deformed differential calculus plays an important role in physics and its applications [4–6, 13] and their differential calculus can be given.
The Taylor formula for polynomials $f(x)$ evaluates the coefficients $f_k$ in the expansion
\[ f(x) = \sum_{k=0}^{\infty} f_k(x - c)^k, \quad f_k = \frac{f^{(k)}(c)}{k!}. \]
(1)

It is possible to generalize (1) by considering other polynomial bases and suitable operators.

The fundamental theorem of calculus can be stated as follows.

**Theorem 1.** If $f$ is a continuous function on an interval $(a; b)$, then $f$ has an antiderivative on $(a; b)$. Moreover, if $F$ is any antiderivative of $f$ on $(a; b)$, then
\[ \int_a^b f(x) \, dx = F(b) - F(a). \]
(2)

The $q$ version of this theorem was stated in [14] as follows.

**Theorem 2.** If $F(x)$ is an antiderivative of $f(x)$ and if $F(x)$ is continuous at $x = 0$, then
\[ \int_a^b f(x) \, dq_x = F(b) - F(a), \quad 0 \leq a \leq b \leq \infty. \]
(3)

Here the $q$-integral is defined by
\[ \int_0^a f(x) \, dq_x = (1 - q)a \sum_{k=0}^{\infty} q^k f(aq^k). \]
(4)

The aims of this paper is to present the differential analysis and notation for $(p, q)$-calculus, which leads to a new method for computations and classifications of $(p, q)$-special functions. We derived many formulas of integration and differentiation which enable to understand the algebra properties of deformed quantum algebra. Two generalizations of (1) are given and a generalization of (3) is stated. The paper is organised as follows.

- In Sect. 2, we introduce and give relevant properties of the $(p, q)$-derivative. The $(p, q)$-power basis is given and main of its properties are provided. The properties of the $(p, q)$-derivative combined with those of the $(p, q)$-power basis enable to state two $(p, q)$-Taylors for polynomials. It then follows connection formulas between the canonical basis and the $(p, q)$-power basis.
- In Sect. 3, we recall two $(p, q)$-extension of the exponential functions, give some of their properties and state some representations of their series expansions in any $(p, q)$-power basis.
- In Sect. 4, the $(p, q)$-antiderivative, the $(p, q)$-integral are introduced and sufficient condition for their convergence are investigated. Finally the fundamental theorem of $(p, q)$-calculus is proved and the formula of $(p, q)$-integration by part is derived.
The results of this paper are the improved version of the work saved on Arxiv https://arxiv.org/abs/1309.3934 by the same author in 2013. Some applications of this result can be found in [1–3].

2. The \((p, q)\)-Derivative and the \((p, q)\)-Power Basis

In this section, we introduce the \((p, q)\)-derivative, the \((p, q)\)-power and provide some of their relevant properties. Two \((p, q)\)-Taylor formulas for polynomials are stated and some consequences are investigated.

2.1. The \((p, q)\)-Derivative

Let \(f\) be a function defined on the set of the complex numbers.

**Definition 1.** The \((p, q)\)-derivative of the function \(f\) is defined as (see e.g. \([4,13]\))

\[
D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0, \quad (5)
\]

and \((D_{p,q}f)(0) = f'(0)\), provided that \(f\) is differentiable at 0. The so-called \((p, q)\)-bracket or twin-basic number is defined as

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}. \quad (6)
\]

It happens clearly that \(D_{p,q}x^n = [n]_{p,q}x^{n-1}\). Note also that for \(p = 1\), the \((p, q)\)-derivative reduces to the Hahn derivative given by

\[
D_qf(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0.
\]

As with ordinary derivative, the action of the \((p, q)\)-derivative on a function is a linear operator. More precisely, for any constants \(a\) and \(b\),

\[
D_{p,q}(af(x) + bg(x)) = aD_{p,q}f(x) + bD_{p,q}g(x).
\]

The twin-basic number is a natural generalization of the \(q\)-number, that is

\[
\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1}{1-q^n}, \quad q \neq 1. \quad (7)
\]

The \((p, q)\)-factorial is defined by

\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1. \quad (8)
\]

Let us introduce also the so-called \((p, q)\)-binomial coefficient

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n. \quad (9)
\]

Note that as \(p \to 1\), the \((p, q)\)-binomial coefficients reduce to the \(q\)-binomial coefficients.
Proposition 1. The \((p,q)\)-derivative fulfills the following product rules
\[
D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \\
D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).
\]

Proof. From the definition of the \((p,q)\)-derivative, we have
\[
D_{p,q}(f(x)g(x)) = \frac{f(px)g(px) - f(qx)g(qx)}{(p-q)x} = \frac{f(px)[g(px) - g(qx)] + g(qx)[f(px) - f(qx)]}{(p-q)x} = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x).
\]
This proves (10). Equation (11) is obtained by symmetry. \(\Box\)

Proposition 2. The \((p,q)\)-derivative fulfills the following quotient rules
\[
D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \\
D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.
\]

Proof. The proof of this statements can be deduced using (10). \(\Box\)

2.2. The \((p,q)\)-Power Basis
Here, we introduce the so-called \((p,q)\)-power
\[
(x \ominus a)^n_{p,q} = (x - a)(px - aq) \cdots (px^{n-1} - aq^{n-1})
\]
and investigate some of its relevant properties. These polynomials will be useful to state our Taylor formulas.

It should be noted the following important relation between the \((p,q)\)-power basis and the \((p,q)\)-factorial
\[
[n]_{p,q} ! = \frac{(p \ominus q)^n_{p,q}}{(p-q)^n}.
\]

Proposition 3. The following assertion is valid
\[
D_{p,q}(x \ominus a)^n_{p,q} = [n]_{p,q}(px \ominus a)^{n-1}_{p,q}, \quad n \geq 1,
\]
and \(D_{p,q}(x \ominus a)^0_{p,q} = 0\).

Proof. The proof follows by a direct computation. \(\Box\)

Proposition 4. Let \(\gamma\) be a complex number and \(n \geq 1\) be an integer, then
\[
D_{p,q}(\gamma x \ominus a)^n_{p,q} = \gamma[n]_{p,q}(\gamma px \ominus a)^{n-1}_{p,q}.
\]
The following proposition generalizes (16).
Proposition 5. Let \( n \geq 1 \) be an integer, and \( 0 \leq k \leq n \), the following rule applies
\[
D^k_{p,q}(x \ominus a)^n_{p,q} = p^k \binom{n}{k} p^k x \ominus a)^{n-k}_{p,q}. \tag{18}
\]
Proof. The prove is done by induction with respect to \( k \). \( \square \)

Remark 1. For the classical derivative, it is known that for any complex number \( \alpha \), one has
\[
d_x \alpha = \alpha x^{\alpha - 1}.
\]

In what follows, we would like to state similar result for the \( D_{p,q} \) derivative as done for the \( D_q \) derivative in [14].

Proposition 6. Let \( m \) and \( n \) be two non negative integers. Then the following assertion is valid.
\[
(x \ominus a)^{m+n}_{p,q} = (x \ominus a)^m_{p,q} (p^m x \ominus q^m a)^n_{p,q}. \tag{19}
\]
In Proposition 6, if we take \( m = -n \), then we get the following extension of the \((p, q)\)-power basis.

Definition 2. Let \( n \) be a non negative integer, then we set the following definition
\[
(x \ominus a)^{-n}_{p,q} = \frac{1}{(p^{-n} x \ominus q^{-n} a)^n_{p,q}}. \tag{20}
\]

Proposition 7. For any two integers \( m \) and \( n \), (19) holds.
Proof. The case \( m > 0 \) and \( n > 0 \) has already been proved, and the case where one of \( m \) and \( n \) is zero is easy. Let us first consider the case \( m = -m' < 0 \) and \( n > 0 \). Then,
\[
(x \ominus a)^m_{p,q} (p^m x \ominus q^m a)^n_{p,q} = (x \ominus a)^{-m'}_{p,q} (p^{-m'} x \ominus q^{-m'} a)^n_{p,q}.
\]
by (20)
\[
= \frac{(p^{-m'} x \ominus q^{-m'} a)^n_{p,q}}{(p^{-m'} x \ominus q^{-m'} a)^{-m'}_{p,q}}
\]
by (19)
\[
= \begin{cases} 
(p^m (p^{-m} x) \ominus q^m (q^{-m} a))_{p,q}^{n-m'} & \text{if } n \geq m' \\
1 & \text{if } n < m'
\end{cases}
\]
by (20)
\[
= (x \ominus a)^{n-m'}_{p,q} = (x \ominus a)^{n+m}_{p,q}.
\]
If \( m \geq 0 \) and \( n = -n' < 0 \), then
\[
(x \ominus a)^m_{p,q} (p^m x \ominus q^m a)\]
\[
= (x \ominus a)^m_{p,q} (p^m x \ominus q^m a)^{-n'}_{p,q}
\]
\[
= \frac{(x \ominus a)^m_{p,q}}{(p^{-n'} x \ominus q^{-n'} a)^{n'}_{p,q}}
\]
Let Definition 3. as follows.

Before trying to answer this question, let us generalize the twin-basic number

The following relations are valid:

**Proposition 9.**

The proof follows by direct computations.

**Proof.**

Let **Proposition 10.**

Lastly, if \( m = -m' < 0 \) and \( n = -n' < 0 \),

\[
(x \ominus a)_{p,q}^m (p^m x \ominus q^m a)_{p,q}^n = (x \ominus a)_{p,q}^{-m'} (p^{-m'} x \ominus q^{-m'} a)_{p,q}^{-n'}
\]

\[
= \frac{1}{(p^{-m'} x \ominus q^{-m'} a)_{p,q}^{m'}} \frac{1}{(p^{-m'} x \ominus q^{-m'} a)_{p,q}^{-m'}}
\]

\[
= (x \ominus a)_{p,q}^{-m'} = (x \ominus a)_{p,q}^{m+n}.
\]

Therefore, (19) is true for any integers \( m \) and \( n \) as well. \( \Box \)

It is natural to ask ourselves if (16) is valid for any integer as well. But before trying to answer this question, let us generalize the twin-basic number as follows.

**Definition 3.** Let \( \alpha \) be any number,

\[
[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p - q}.
\]  

**Proposition 8.** For any integer \( n \),

\[
D_{p,q}(x \ominus a)^n_{p,q} = [n]_{p,q}(px \ominus a)^{n-1}_{p,q}.
\]  

**Proof.** Note that \([0]_{p,q} = 0\). The result is already proved for \( n \geq 0 \). For \( n = -n' < 0 \), we use (12) and (20) to get the result. \( \Box \)

**Proposition 9.** The following relations are valid:

\[
D_{p,q}(x \ominus a)^n_{p,q} = \frac{-q[n]_{p,q}}{(qx \ominus a)^{n+1}_{p,q}};
\]  

\[
D_{p,q}(a \ominus x)^n_{p,q} = -[n]_{p,q}(a \ominus qx)^{n-1}_{p,q};
\]  

\[
D_{p,q}(a \ominus x)^n_{p,q} = \frac{p[n]_{p,q}}{(a \ominus px)^{n+1}_{p,q}}.
\]

**Proof.** The proof follows by direct computations. \( \Box \)

**Proposition 10.** Let \( n \geq 1 \) be an integer, and \( 0 \leq k \leq n \), we have the following

\[
D_{p,q}^k(a \ominus x)^n_{p,q} = (-1)^k q^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} (a \ominus q^k x)^{n-k}_{p,q}.
\]

**Proof.** The prove is done by induction with respect to \( k \). \( \Box \)
2.3. \((p, q)\)-Taylor Formulas for Polynomials

In this section, two Taylor formulas for polynomials are given and some of their consequences are investigated.

**Theorem 3.** For any polynomial \(f(x)\) of degree \(N\), and any number \(a\), we have the following \((p, q)\)-Taylor expansion:

\[
f(x) = \sum_{k=0}^{N} p^{-\binom{k}{2}} \frac{D^k_{p,q} f}{[k]_{p,q}!} (x \ominus a)^k_{p,q}.
\]

**Proof.** Let \(f\) be a polynomial of degree \(N\), then we have the expansion

\[
f(x) = \sum_{j=0}^{N} c_j (x \ominus a)^j_{p,q}.
\]

Let \(k\) be an integer such that \(0 \leq k \leq N\), then, applying \(D^k_{p,q}\) on both sides of (28) and using (18), we get

\[
(D^k_{p,q} f)(x) = \sum_{j=k}^{N} c_j \frac{[j]_{p,q}!}{[j-k]_{p,q}!} p^{\binom{j}{2}} (p^k x \ominus q)^{j-k}_{p,q}.
\]

Substituting \(x = ap^{-k}\), it follows that

\[
(D^k_{p,q} f)(ap^{-k}) = c_k [k]_{p,q}! p^{\binom{k}{2}},
\]

thus we get

\[
c_k = p^{-\binom{k}{2}} \frac{(D^k_{p,q} f)(ap^{-k})}{[k]_{p,q}!}.
\]

This proves the desired result. \(\square\)

**Corollary 1.** The following connection formula holds

\[
x^n = \sum_{k=0}^{n} p^{-\binom{k}{2}} \frac{n!}{[k]_{p,q}!} (ap^{-k})^{n-k} (x \ominus a)^k_{p,q}.
\]

**Theorem 4.** For any polynomial \(f(x)\) of degree \(N\), and any number \(a\), we have the following \((p, q)\)-Taylor expansion:

\[
f(x) = \sum_{k=0}^{N} (-1)^k q^{-\binom{k}{2}} \frac{D^k_{p,q} f}{[k]_{p,q}!} (a \ominus x)^k_{p,q}.
\]

**Proof.** Let \(f\) be a polynomial of degree \(N\), then we have the expansion

\[
f(x) = \sum_{j=0}^{N} c_j (a \ominus x)^j_{p,q}.
\]
Let \( k \) be an integer such that \( 0 \leq k \leq N \), then, applying \( D^k_{p,q} \) on both sides of (31) and using (26), we get
\[
(D^k_{p,q} f)(x) = \sum_{j=k}^{N} c_j (-1)^j \left[ \frac{[j]_{p,q}}{[j-k]_{p,q}} \right] q^{-\left( \frac{k}{2} \right)} (a \Theta q^k x)^{j-k}.
\]

Substituting \( x = aq^{-k} \), it follows that
\[
(D^k_{p,q} f)(aq^{-k}) = c_k (-1)^k [k]_{p,q}! q^{-\left( \frac{k}{2} \right)},
\]
thus we get
\[
c_k = (-1)^k q^{-\left( \frac{k}{2} \right)} \frac{(D^k_{p,q} f)(aq^{-k})}{[k]_{p,q}!}.
\]
This proves the desired result.

\[\square\]

**Corollary 2.** The following connection formula holds.
\[
x^n = \sum_{k=0}^{n} (-1)^k q^{-\left( \frac{k}{2} \right)} \left[ \frac{n}{k} \right]_{p,q} (aq^{-k})^{n-k} (a \Theta x)^k.
\] (32)

**Corollary 3.** The following connection formulas hold.
\[
(x \Theta b)^n_{p,q} = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{p,q} (a \Theta b)^{n-k} (x \Theta a)^k,
\] (33)
\[
(b \Theta x)^n_{p,q} = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{p,q} (b \Theta a)^{n-k} (a \Theta x)^k.
\] (34)

**Remark 2.** If one takes \( b = ab \) in (33), then one gets
\[
(x \Theta ab)^n_{p,q} = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{p,q} a^{n-k} (1 \Theta b)^{n-k} (x \Theta a)^k.
\]

Now, take \( x = 1 \) and \( p = 1 \), the following well known \( q \)-binomial theorem follows
\[
(ab; q)_n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{q} a^{n-k} (b; q)_{n-k} (a; q)_k.
\] (35)

Then, (33) is an obvious generalization of (35).

**Corollary 4.** The following expansion holds
\[
\frac{1}{(1 \Theta x)^n_{p,q}} = 1 + \sum_{j=0}^{\infty} \left[ \frac{n+j-1}{j} \right]_{p,q} p^{j-\left( \frac{j}{2} \right)} x^j,
\] (36)
Note that (36) is the \((p, q)\)-analogue of the Taylor’s expansion of \(f(x) = \frac{1}{(1 - x)^n}\) in ordinary calculus. Note also that when \(p \to 1\), (36) becomes the well known Heine’s binomial formula.

3. Series Expansion of the \((p, q)\)-Exponential Functions

Here, we give a natural generalization of the \(q\)-hypergeometric series ([11])

\[
\begin{align*}
\sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(q, q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[ (-1)^n q(n) \right]^{1+s-r} z^n.
\end{align*}
\]

Definition 4. [15] The \((p, q)\)-hypergeometric series

\[
\Phi_r \left( a_1, a_2, \ldots, a_r \mid p, q; z \right) = \sum_{n=0}^{\infty} \frac{(a_1; p)_n \cdots (a_r; p)_n}{(p, q)_n (b_1; p)_n \cdots (b_s; q)_n} \left[ (-1)^n q(n) \right]^{1+s-r} z^n.
\]

\[\text{(37)}\]

(38)

\textbf{Theorem 5} (Compare to [13]). Let \(a, b\) be two complex numbers, then we have the following

\[
\Phi_r \left( a_1, a_2, \ldots, a_r \mid p, q; z \right) = \sum_{n=0}^{\infty} \frac{(a_1; p)_n \cdots (a_r; p)_n}{(p, q)_n (b_1; p)_n \cdots (b_s; q)_n} \left[ (-1)^n q(n) \right]^{1+s-r} z^n.
\]

\[\text{(37)}\]

\textbf{Proof.} We first note that

\[
\left( \frac{a \circ b}{p \circ q} \right)_{p, q} = \frac{\left( a, q \right)_n}{\left( p, q \right)_n} \left( \frac{a}{p} \right)^n.
\]

It follows from the \(q\)-binomial theorem (see [11]) that

\[
\sum_{n=0}^{\infty} \frac{(a \circ b)^n}{(p \circ q)^n} z^n = \sum_{n=0}^{\infty} \frac{\left( b \circ q \right)_n}{\left( p \circ q \right)_n} \left( \frac{a}{p} \right)^n = \frac{(p \circ b \circ z^n)}{(p \circ q)^n}.
\]

\[\text{(38)}\]

The following corollary also appears in [13].
Corollary 5 [15]. a, b and c are three complex numbers. Then
\[1 \Phi_0 \left( \begin{array}{c} a, b \\ p, q; z \end{array} \right) 1 \Phi_0 \left( \begin{array}{c} b, c \\ p, q; z \end{array} \right) = 1 \Phi_0 \left( \begin{array}{c} a, c \\ p, q; z \end{array} \right)\]

As in the q-case, there are many definitions of the \((p, q)\)-exponential function. The following two \((p, q)\)-analogues of the exponential function (see [13]) will be frequently used throughout this paper:
\[e_{p,q}(z) = 1 \Phi_0 \left( \begin{array}{c} (1, 0) \\ p, q; (p - q)z \end{array} \right) = \sum_{n=0}^{\infty} \frac{p^n(z)}{[n]_{p,q}} z^n, \quad (39)\]
\[E_{p,q}(z) = 1 \Phi_0 \left( \begin{array}{c} (0, 1) \\ p, q; (q - p)z \end{array} \right) = \sum_{n=0}^{\infty} \frac{q^n(z)}{[n]_{p,q}} z^n. \quad (40)\]

From the \((p, q)\)-binomial theorem (38) and the definitions (39) and (40) of the \((p, q)\)-exponential functions, it is easy to see that
\[e_{p,q}(x)E_{p,q}(-x) = 1. \quad (41)\]

The next two propositions give the \(n\)-th derivative of the \((p, q)\)-exponential functions. These formulas are very important for the computations the \((p, q)\)-Laplace transforms of some functions in the next sections.

Proposition 11 [15]. Let \(\lambda\) be a complex number, then the following relations hold
\[D_{p,q}e_{p,q}(\lambda x) = \lambda e_{p,q}(\lambda px),\]
\[D_{p,q}E_{p,q}(\lambda x) = \lambda E_{p,q}(\lambda qx).\]

Proof. The proof follows from the definitions of the \((p, q)\)-exponentials and the \((p, q)\)-derivative. \(\square\)

Proposition 12 [15]. Let \(n\) be a nonnegative integer, then the following equations hold
\[D_{p,q}^n e_{p,q}(\lambda x) = \lambda^n p^n(z) e_{p,q}(\lambda px), \quad (42)\]
\[D_{p,q}^n E_{p,q}(\lambda x) = \lambda^n q^n(z) E_{p,q}(\lambda qx). \quad (43)\]

Proof. The proof follows by induction from the definitions of the \((p, q)\)-exponentials and the \((p, q)\)-derivative. \(\square\)

Next, we give new expansions of the \((p, q)\)-exponential functions.

Theorem 6. Let \(a\) be a complex number. The following expansions hold.
\[e_{p,q}(\lambda x) = \sum_{n=0}^{\infty} \frac{(p - q)\lambda^n}{(p \ominus q)_{p,q}} (x \ominus a)_{p,q}^n = e_{p,q}(\lambda a) 1 \Phi_0 \left( \begin{array}{c} (x, a) \\ p, q; (p - q)\lambda \end{array} \right), \]
\[ E_{p,q}(\lambda x) = \sum_{n=0}^{\infty} \binom{n}{2} \frac{q^n E_{p,q}(\lambda a(q/p)^n)}{[n]_{p,q}!} (x \ominus a)^n_{p,q}. \]

**Proof.** Note that in (27), \( N \) can be taken to be \( \infty \) with the condition that the infinite series obtained is convergent. The formula becomes

\[ f(x) = \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{D_{p,q}^n f}{[n]_{p,q}!} (x \ominus a)^n_{p,q}. \]

For \( f(x) = e_{p,q}(x) \), and using the relations (15) and (42), it follows that

\[ e_{p,q}(\lambda x) = \sum_{n=0}^{\infty} (-1)^n p^{-\binom{n}{2}} \lambda^n p_{p,q}^{(2)} e_{p,q}(\lambda a) (x \ominus a)^n_{p,q}, \]

\[ = e_{p,q}(\lambda a) \sum_{n=0}^{\infty} \frac{\lambda^n}{[n]_{p,q}!} (x \ominus a)^n_{p,q}, \]

\[ = e_{p,q}(\lambda a) \sum_{n=0}^{\infty} \frac{(x \ominus a)^n_{p,q}}{(p \ominus q)^n_{p,q}} ((p - q)\lambda)^n \]

\[ = e_{p,q}(\lambda a) \Phi_0 \left( (x, a) - 1 \bigg| p, q, (p - q)\lambda \right). \]

For \( f(x) = E_{p,q}(x) \), and using the relations (15) and (43), it follows that

\[ E_{p,q}(\lambda x) = \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \lambda^n E_{p,q}(\lambda a(q/p)^n) (x \ominus a)^n_{p,q} \]

\[ = \sum_{n=0}^{\infty} \binom{n}{2} \frac{q^n E_{p,q}(\lambda a(q/p)^n)}{[n]_{p,q}!} (x \ominus a)^n_{p,q}. \]

\[ \square \]

**Theorem 7.** Let \( a \) be a complex number. The following expansions hold.

\[ e_{p,q}(x) = \sum_{n=0}^{\infty} \left( -\frac{p}{q} \right)^{\binom{n}{2}} \frac{\lambda^n e_{p,q}(\lambda a(p/q)^n)}{[n]_{p,q}!} (a \ominus x)^n_{p,q}, \]

\[ E_{p,q}(x) = E_{p,q}(\lambda a) \sum_{n=0}^{\infty} \frac{((q - p)\lambda)^n}{(p \ominus q)^n_{p,q}} (a \ominus x)^n_{p,q} \]

\[ = E_{p,q}(\lambda a) \Phi_0 \left( (a, x) - 1 \bigg| p, q, (q - p)\lambda \right). \]

**Proof.** Note that in (30), \( N \) can be taken to be \( \infty \) with the condition that the infinite series obtained is convergent. The formula becomes

\[ f(x) = \sum_{n=0}^{\infty} (-1)^n q^{-\binom{n}{2}} \frac{D_{p,q}^n f}{[n]_{p,q}!} (a \ominus x)^n_{p,q}. \]
For \( f(x) = e_{p,q}(x) \), and using the relations (15) and (42), it follows that
\[
e_{p,q}(\lambda x) = \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{\lambda^n p_{n}(\lambda a/p/q)^n}{[n]_{p,q}!} (a \ominus x)_p^n q^n e_{p,q}(\lambda a/p/q^n)(a \ominus x)_{p,q}.
\]

For \( f(x) = E_{p,q}(x) \), and using the relations (15) and (43), it follows that
\[
E_{p,q}(\lambda x) = \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{\lambda^n q^n E_{p,q}(\lambda a)}{[n]_{p,q}!} (a \ominus x)_p^n q^n e_{p,q}(\lambda a/p/q^n) (a \ominus x)_{p,q}.
\]

4. The \((p, q)\)-Antiderivative and the \((p, q)\)-Integral

4.1. The \((p, q)\)-Antiderivative

The function \( F(x) \) is a \((p, q)\)-antiderivative of \( f(x) \) if \( D_{p,q} F(x) = f(x) \). It is denoted by
\[
\int f(x) d_{p,q} x.
\]

Note that we say “a” \((p, q)\)-antiderivative instead of “the” \((p, q)\)-antiderivative, because, as in ordinary calculus, an antiderivative is not unique. In ordinary calculus, the uniqueness is up to a constant since the derivative of a function vanishes if and only if it is a constant. The situation in the twin basic quantum calculus is more subtle. \( D_{p,q} \varphi(x) = 0 \) if and only if \( \varphi(px) = \varphi(qx) \), which does not necessarily imply \( \varphi \) a constant. If we require \( \varphi \) to be a formal power series, the condition \( \varphi(px) = \varphi(qx) \) implies \( p^n c_n = q^n c_n \) for each \( n \), where \( c_n \) is the coefficient of \( x^n \). It is possible only when \( c_n = 0 \) for any \( n \geq 1 \), that is, \( \varphi \) is constant. Therefore, if
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]
is a formal power series, then among formal power series, \(f(x)\) has a unique \((p, q)\)-antiderivative up to a constant term, which is
\[
\int f(x)\,d_{p,q}x = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{[n+1]_{p,q}} + C. \tag{45}
\]

### 4.2. The \((p, q)\)-Integral

We define the inverse of the \((p, q)\)-differentiation called the \((p, q)\)-integration.

Let \(f(x)\) be an arbitrary function and \(F(x)\) be a function such that
\[
D_{p,q}F(x) = f(x),
\]
then
\[
F(px) - F(qx) = \varepsilon x f(x)
\]
where \(\varepsilon = (p - q)\). This relation leads to the formula
\[
F(p^{n+1}q^{-(n+1)}x) - F(p^n q^{-n}x) = \varepsilon p^n q^{-(n+1)} x \sum_{k=0}^{\infty} p^k q^{k+1} f \left( p^k q^{-(k+1)} x \right)
\]

By adding these formulas term by term, we obtain
\[
F(p^{n+1}q^{-(n+1)}x) - F(x) = (p - q) x \sum_{k=0}^{n} f \left( p^k q^{-(k+1)} x \right).
\]

Assuming \(\left| \frac{p}{q} \right| < 1\) and letting \(n \to \infty\), we have
\[
F(x) - F(0) = (q - p) x \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f \left( \frac{p^k}{q^{k+1}} x \right).
\]

Similarly, for \(\left| \frac{p}{q} \right| > 1\), we have
\[
F(x) - F(0) = (p - q) x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} x \right).
\]

Therefore, we give the following definition.

**Definition 5.** Let \(f\) be an arbitrary function. We define the \((p, q)\)-integral of \(f\) as follows:
\[
\int f(x)\,d_{p,q}x = (p - q) x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} x \right). \tag{46}
\]
Remark 3. Note that this is a formal definition since the we do not care about the convergence of the right hand side of (46).

From this definition, one easily derives a more general formula

\[
\int f(x)D_{p,q}g(x)d_{p,q}x = (p - q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} x \right) D_{p,q}g \left( \frac{q^k}{p^{k+1}} x \right) \\
= (p - q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} x \right) \frac{g \left( \frac{q^k}{p^{k+1}} x \right) - g \left( \frac{q^{k+1}}{p^{k+1}} x \right)}{(p - q) \frac{q^k}{p^{k+1}} x} \\
= \sum_{k=0}^{\infty} f \left( \frac{q^k}{p^{k+1}} x \right) \left( g \left( \frac{q^k}{p^{k+1}} x \right) - g \left( \frac{q^{k+1}}{p^{k+1}} x \right) \right),
\]

or otherwise stated

\[
\int f(x)d_{p,q}g(x) = \sum_{k=0}^{\infty} f \left( \frac{q^k}{p^{k+1}} x \right) \left( g \left( \frac{q^k}{p^{k+1}} x \right) - g \left( \frac{q^{k+1}}{p^{k+1}} x \right) \right). \tag{47}
\]

We have merely derived (46) formally and have yet to examine under what conditions it really converges to a \((p, q)\)-antiderivative. The theorem below gives a sufficient condition for this.

**Theorem 8.** Suppose \(0 < \frac{q}{p} < 1\). If \(|f(x)x^{\alpha}|\) is bounded on the interval \((0, A]\) for some \(0 \leq \alpha < 1\), then the \((p, q)\)-integral (46) converges to a function \(F(x)\) on \((0, A]\), which is a \((p, q)\)-antiderivative of \(f(x)\). Moreover, \(F(x)\) is continuous at \(x = 0\) with \(F(0) = 0\).

**Proof.** Let us assume that \(|f(x)x^{\alpha}| \leq M\) on \((0, A]\). For any \(0 < x < A, j \geq 0\),

\[
\left| f \left( \frac{q^j}{p^{j+1}} x \right) \right| < M \left( \frac{q^j}{p^{j+1}} x \right)^{-\alpha}.
\]

Thus, for \(0 < x \leq A\), we have

\[
\left| \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} x \right) \right| < M \frac{q^j}{p^{j+1}} \left( \frac{q^j}{p^{j+1}} x \right)^{-\alpha} = Mp^{1-\alpha}x^{-\alpha} \left[ \left( \frac{q}{p} \right)^{1-\alpha} \right]^j. \tag{48}
\]

Since, \(1 - \alpha > 0\) and \(0 < \frac{q}{p} < 1\), we see that our series is bounded above by a convergent geometric series. Hence, the right-hand size of (46) converges point-wise to some function \(F(x)\). It follows directly from (46) that \(F(0) = 0\). The fact that \(F(x)\) is continuous at \(x = 0\), that is \(F(x)\) tends to zero as \(x \to 0\), is clear if we consider, using (48)

\[
\left| (p - q)x \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} x \right) \right| < \frac{M(p - q)x^{1-\alpha}}{p^{1-\alpha} - q^{1-\alpha}}, \quad 0 < x \leq A.
\]
In order to check that $F(x)$ is a $(p, q)$-antiderivative we $(p, q)$-differentiate it:

$$D_{p,q} F(x) = \frac{1}{(p-q)x} \left( (p-q)px \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} px \right) \right. $$

$$\left. - (p-q)qx \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} qx \right) \right)$$

$$= \sum_{k=0}^{\infty} \frac{q^k}{p^k} f \left( \frac{q^k}{p^k} x \right) - \sum_{k=0}^{\infty} \frac{q^{k+1}}{p^{k+1}} f \left( \frac{q^{k+1}}{p^{k+1}} x \right)$$

$$= \sum_{k=0}^{\infty} \frac{q^k}{p^k} f \left( \frac{q^k}{p^k} x \right) - \sum_{k=1}^{\infty} \frac{q^k}{p^k} f \left( \frac{q^k}{p^k} x \right)$$

$$= f(x).$$

Note that if $x \in (0, A]$ and $0 < \frac{q}{p} < 1$, then $\frac{q}{p} x \in (0, A]$, and the $(p, q)$-differentiation is valid.

**Remark 4.** Note that if the assumption of Theorem 8 is satisfied, the $(p, q)$-integral gives the unique $(p, q)$-antiderivative that is continuous at $x = 0$, up to a constant. On the other hand, if we know that $F(x)$ is a $(p, q)$-antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, $F(x)$ must be given, up to a constant, by (46), since a partial sum of the $(p, q)$-integral is

$$(p-q)x \sum_{j=0}^{N} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} x \right) = (p-q)x \sum_{j=0}^{N} \frac{q^j}{p^{j+1}} D_{p,q} F(t)\big|_{t=\frac{q^j}{p^{j+1}} x}$$

$$= (p-q)x \sum_{j=0}^{N} \frac{q^j}{p^{j+1}} \left( \frac{F \left( \frac{q^j}{p^j} x \right) - F \left( \frac{q^{j+1}}{p^{j+1}} x \right)}{(p-q)\frac{q^j}{p^{j+1}} x} \right)$$

$$= \sum_{j=0}^{N} \left( F \left( \frac{q^j}{p^j} x \right) - F \left( \frac{q^{j+1}}{p^{j+1}} x \right) \right)$$

$$= F(x) - F \left( \frac{q^{N+1}}{p^{N+1}} x \right)$$

which tends to $F(x) - F(0)$ as $N$ tends to $\infty$, by the continuity of $F(0)$ at $x = 0$.

Let us emphasize on an example where the $(p, q)$-derivative fails. Consider $f(x) = \frac{1}{x}$. Since
\[ D_{p,q} \ln x = \frac{\ln px - \ln qx}{(p - q)x} = \frac{\ln p - \ln q}{p - q} x, \]  
(49)

we have

\[ \int \frac{1}{x} d_{p,q} x = \frac{p - q}{\ln p - \ln q} \ln x. \]  
(50)

However, the formula (46) gives

\[ \int \frac{1}{x} d_{p,q} x = (p - q) \sum_{j=0}^{\infty} 1 = \infty. \]

The formula fails because \( f(x)^{\alpha} \) is not bounded for any \( 0 \leq \alpha < 1 \). Note that \( \ln x \) is not continuous at \( x = 0 \).

We now apply formula (46) to define the definite \((p, q)\)-integral.

**Definition 6.** Let \( f \) be an arbitrary function and \( a \) be a real number, we set

\[ \int_{0}^{a} f(x) d_{p,q} x = (q - p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f \left( \frac{p^k}{q^{k+1}} a \right) \quad \text{if} \quad \left| \frac{p}{q} \right| < 1 \]  
(51)

\[ \int_{0}^{a} f(x) d_{p,q} x = (p - q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} a \right) \quad \text{if} \quad \left| \frac{p}{q} \right| > 1. \]  
(52)

**Remark 5.** Note that for \( p = 1 \), the definition (52) reduces to the well known Jackson integral (see [14, p. 67])

\[ \int f(x) d_q x = (1 - q)x \sum_{k=0}^{\infty} q^k f(q^k x). \]

For \( p = r^{1/2} \), \( q = s^{-1/2} \),

\[ \left| \frac{p}{q} \right| < 1 \quad \iff \quad |rs| < 1, \]

and the formula (51) reads

\[ \int_{0}^{a} f(x) d_{p,q} x = (s^{-1/2} - r^{1/2})a \sum_{k=0}^{\infty} r^{k/2} s^{(k+1)/2} f \left( r^{k/2} s^{(k+1)/2} a \right), \]

which is the formula (11) given in [5]. Once more, for \( p = r^{1/2} \), \( q = s^{-1/2} \),

\[ \left| \frac{p}{q} \right| > 1 \quad \iff \quad |rs| > 1, \]

and the formula (52) reads

\[ \int_{0}^{a} f(x) d_{p,q} x = (r^{1/2} - s^{-1/2})a \sum_{k=0}^{\infty} s^{-k/2} r^{-(k+1)/2} f \left( s^{-k/2} r^{-(k+1)/2} a \right), \]

which is the formula (10) given in [5].
Definition 7. Let $f$ be an arbitrary function $a$ and $b$ be two nonnegative numbers such that $a < b$, then we set
\[
\int_a^b f(x) d_{p,q} x = \int_0^b f(x) d_{p,q} x - \int_0^a f(x) d_{p,q} x. \tag{53}
\]
We cannot obtain a good definition of the improper integral by simply letting $a \to \infty$ in (52). Instead, since
\[
\int_{q_j/p_j}^{q_j/p_j+1} f(x) d_{p,q} x = \int_0^{q_j/p_j+1} f(x) d_{p,q} x - \int_0^{q_j/p_j} f(x) d_{p,q} x
\]
\[= (p - q) \left\{ \sum_{k=0}^{\infty} \frac{q^{k+j} f \left( \frac{q^{k+j}}{p^{k+1+j}} \right)}{p^{k+1+j}} - \sum_{k=0}^{\infty} \frac{q^{k+j+1} f \left( \frac{q^{k+j+1}}{p^{k+j+2}} \right)}{p^{k+j+2}} \right\}
\]
\[= (p - q) \frac{q^2}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} \right),
\]
it is natural to define the improper $(p, q)$-integral as follows.

Definition 8. The improper $(p, q)$-integral of $f(x)$ on $[0; \infty)$ is defined to be
\[
\int_0^{\infty} f(x) d_{p,q} x = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x) d_{p,q} x
\]
\[= (p - q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^j} f \left( \frac{q^j}{p^{j+1}} \right) \tag{54}
\]
if $0 < \frac{q}{p} < 1$ or
\[
\int_0^{\infty} f(x) d_{p,q} x = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x) d_{p,q} x \tag{55}
\]
if $\frac{q}{p} > 1$ where the formula is used.

Proposition 13. Suppose that $0 < \frac{q}{p} < 1$. The improper $(p, q)$-integral defined above converges if $x^\alpha f(x)$ is bounded in a neighbourhood of $x = 0$ with $\alpha < 1$ and for sufficiently large $x$ with some $\alpha > 1$. 
Proof. By (54) we have
\[
\int_0^\infty f(x) d_{p,q} x = (p-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} \right)
\]

\[
= (p-q) \left\{ \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} \right) + \sum_{j=1}^{\infty} \frac{q^{-j}}{p^{-j+1}} f \left( \frac{q^{-j}}{p^{-j+1}} \right) \right\}
\]

The convergence of the first sum is proved by Theorem 8. For the second sum, suppose for \( x \) large we have \( |x^\alpha f(x)| < M \) where \( \alpha > 1 \) and \( M > 0 \). Then, we have for sufficiently large \( j \),
\[
\left| \frac{q^{-j}}{p^{-j+1}} f \left( \frac{q^{-j}}{p^{-j+1}} \right) \right| = p^{\alpha-1} \left( \frac{q}{p} \right)^{j(\alpha-1)} \left| \left( \frac{q^{-j}}{p^{-j+1}} \right)^\alpha f \left( \frac{q^{-j}}{p^{-j+1}} \right) \right| < M p^{\alpha-1} \left( \frac{q}{p} \right)^{j(\alpha-1)}.
\]
Therefore, the second sum is also bounded above by a convergent geometric series, and thus converges. \( \square \)

Note that similar proposition can be stated when \( \frac{q}{p} > 1 \).

Definition 9. Let \( f \) be an arbitrary function and \( a \) be a nonnegative real number, then we put
\[
\int_0^\infty f(x) d_{p,q} x = (p-q) a \sum_{k=0}^{\infty} \frac{p^{-k}}{q^{-(k+1)}} f \left( \frac{p^{-k}}{q^{-(k+1)}} a \right) \quad \text{if} \quad \left| \frac{p}{q} \right| < 1 \quad (56)
\]
\[
\int_0^\infty f(x) d_{p,q} x = (p-q) a \sum_{k=0}^{\infty} \frac{q^{-k}}{p^{-(k+1)}} f \left( \frac{q^{-k}}{p^{-(k+1)}} a \right) \quad \text{if} \quad \left| \frac{p}{q} \right| > 1. \quad (57)
\]

Remark 6. Combining (51) with (56) and (52) with (57) we have for \( a = 1 \)
\[
\int_0^\infty f(x) d_{p,q} x = (q-p) \sum_{k=-\infty}^{\infty} \frac{p^k}{q^{k+1}} f \left( \frac{p^k}{q^{k+1}} \right) \quad \text{if} \quad \left| \frac{p}{q} \right| < 1 \quad (58)
\]
\[
\int_0^\infty f(x) d_{p,q} x = (p-q) \sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{q^k}{p^{k+1}} \right) \quad \text{if} \quad \left| \frac{p}{q} \right| > 1. \quad (59)
\]

4.3. The Fundamental Theorem of \((p, q)\)-Calculus

In ordinary calculus, a derivative is defined as the limit of a ratio, and a definite integral is defined as the limit of an infinite sum. Their subtle and surprising relation is given by the Newton-Leibniz formula, also called the fundamental theorem of calculus. Following the work done in \(q\)-calculus, where the introduction of the definite integral (see \([14]\)) has been motivated by an antiderivative, the relation between the \((p, q)\)-derivative and the \((p, q)\)-integral
is more obvious. Similarly to the ordinary and the \( q \) cases, we have the following fundamental theorem, or \((p, q)\)-Newton-Leibniz formula.

**Theorem 9** (Fundamental theorem of \((p, q)\)-calculus). If \( F(x) \) is a \((p, q)\)-antiderivative of \( f(x) \) and \( F(x) \) is continuous at \( x = 0 \), we have

\[
\int_a^b f(x) d_{p,q}x = F(b) - F(a),
\]

where \( 0 \leq a < b \leq \infty \).

**Proof.** Since \( F(x) \) is continuous at \( x = 0 \), \( F(x) \) is given by the formula

\[
F(x) = (p - q)x \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} x \right) + F(0).
\]

Since by definition,

\[
\int_0^a f(x) d_{p,q}x = (p - q) a \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f \left( \frac{q^j}{p^{j+1}} a \right),
\]

we have

\[
\int_0^a f(x) d_{p,q}x = F(a) - F(0).
\]

Similarly, we have, for a finite \( b \),

\[
\int_0^b f(x) d_{p,q}x = F(b) - F(0),
\]

and thus

\[
\int_a^b f(x) d_{p,q}x = \int_0^b f(x) d_{p,q}x - \int_0^a f(x) d_{p,q}x = F(b) - F(a).
\]

Putting \( a = \frac{q^{j+1}}{p^{j+1}} \) and \( b = \frac{q^j}{p^j} \) and considering the definition of the improper \((p, q)\)-integral (54), we see that (60) is true for \( b = \infty \).

**Corollary 6.** If \( f'(x) \) exists in a neighbourhood of \( x = 0 \) and is continuous at \( x = 0 \), where \( f'(x) \) denotes the ordinary derivative of \( f(x) \), we have

\[
\int_a^b D_{p,q} f(x) d_{p,q}x = f(b) - f(a).
\]

**Proof.** Using L’Hospital’s rule, we get

\[
\lim_{x \to 0} D_{p,q} f(x) = \lim_{x \to 0} \frac{f(px) - f(qx)}{(p - q)x} = \lim_{x \to 0} \frac{pf'(px) - qf'(qx)}{p - q} = f'(0).
\]

Hence \( D_{p,q} f(x) \) can be made continuous at \( x = 0 \) if we define \((D_{p,q} f)(0) = f'(0)\), and (61) follows from the theorem.
As the $q$-integral, an important difference between the $(p, q)$-integral and its ordinary counterpart is that even if we are integrating a function on an interval like $[1; 2]$, we have to care about the behaviour at $x = 0$. This has to do with the definition of the definite $(p, q)$-integral and the condition for the convergence of the $(p, q)$-integral.

Now suppose that $f(x)$ and $g(x)$ are two functions whose ordinary derivatives exist in a neighbourhood of $x = 0$. Using the product rule (10), we have

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x).$$

Since the product of differentiable functions is also differentiable in ordinary calculus, we can apply Corollary 6 to obtain

$$f(b)g(b) - f(a)g(a) = \int_a^b f(px)(D_{p,q}g(x))d_{p,q}x + \int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x,$$

or

$$\int_a^b f(px)(D_{p,q}g(x))d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x,$$

which is the formula of $(p, q)$-integration by part. Note that $b = \infty$ is allowed.

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