A SHORT PROOF OF A CONJECTURE OF MATSUSHITA

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Abstract. In this note we build on the arguments of van Geemen and Voisin [17] to prove a conjecture of Matsushita that a Lagrangian fibration of an irreducible hyperkähler manifold is either isotrivial or of maximal variation. We also complete a partial result of Voisin [19] regarding the density of torsion points of sections of Lagrangian fibrations.

Let $X$ be an irreducible compact hyperkähler manifold, that is, a simply-connected compact Kähler manifold $X$ for which $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$ for a nowhere-degenerate holomorphic two-form $\sigma$. A Lagrangian fibration of $X$ is a proper morphism $f : X \to B$ to a normal compact analytic variety $B$ whose generic fiber is smooth, connected, and Lagrangian (see [10] for a recent survey). It follows that every smooth fiber is an abelian variety. We let $B^\circ \subset B$ be a dense Zariski open smooth subset over which the restriction $f^\circ : X^\circ \to B^\circ$ is smooth. By the period map of $f$ we mean the period map $\varphi : B^\circ \to S$ to an appropriate moduli space $S$ of polarized abelian varieties associated to the natural variation of (polarized) weight one integral Hodge structures on $B^\circ$ with underlying local system $R^1f^\circ_*\mathbb{Z}_X^\circ$. We say $f$ is isotrivial if the period map is trivial (equivalently if $R^1f^\circ_*\mathbb{Z}_X^\circ$ has finite monodromy) and of maximal variation if the period map is generically finite.

Our main result is to resolve a conjecture of Matsushita:

**Theorem 1.** Let $X$ be an irreducible hyperkähler manifold (or more generally a primitive symplectic variety in the sense of [2]). Then any Lagrangian fibration $f : X \to B$ is either isotrivial or of maximal variation.

Both possibilities in Theorem 1 occur, even for K3 surfaces—see for example [9, Chapter 11]. Primitive symplectic varieties are the natural singular analog (as far as deformation theory is concerned) of irreducible hyperkähler manifolds; see below for the definition and the precise meaning of a Lagrangian fibration in this context. Let $T_0 \subset H^2(X, \mathbb{Q})$ be the rational transcendental lattice, namely, the smallest rational Hodge substructure containing $[\sigma] \in H^{2,0}(X)$. Theorem 1 was proven by van Geemen and Voisin [17] Theorem 5] assuming $X$ is smooth and projective, that $T_0$ has generic (special) Mumford–Tate group (namely $\text{SO}(T_0, q_X)$, where $q_X$ is the Beauville–Bogomolov–Fujiki form), and that $\text{rk} T_0 \geq 5$, by showing that under these conditions any fiber of a Lagrangian fibration that is not of maximal variation must be a factor of the Kuga–Satake variety of $T_0$. Their result in particular applies to the generic projective deformation of $f : X \to B$, at least for $X$ of a known deformation type.

We will instead prove Theorem 1 by considering the complex variation of Hodge structures on $R^1f^\circ_*\mathbb{C}_X^\circ$. We first recall the basic properties of complex variations. A complex variation of Hodge structures on a Zariski open subset of a compact analytic variety (see for example [3]) consists of a $\mathbb{C}$-local system $V$ and a holomorphic (resp. antiholomorphic) descending filtration $F^\bullet$ (resp. $\overline{F}^\bullet$) such that we have a splitting of the sheaf of $C^\infty$ sections $A^0(V) = \bigoplus_p A^0(V^p)$ where $V^p = F^p \cap \overline{F}^{-p}$ and the flat connection maps $A^0(V^p)$ to $A^{1,0}(V^{p-1}) \oplus A^1(V^p) \oplus A^{0,1}(V^{p+1})$. We refer to the
grading $V^p$ as the Hodge grading and we say the level of the variation is the difference $p_{\text{max}} - p_{\text{min}}$ where $p_{\text{max}}$ (resp. $p_{\text{min}}$) is the maximum (resp. minimum) Hodge degree $p$ for which $V^p \neq 0$. Observe that the level of a tensor product $V \otimes W$ is the sum of the levels of $V$ and $W$. A polarization of the variation is a flat hermitian form $h$ for which the splitting is orthogonal and $(-1)^p h$ is positive definite on $V^p$. In this case $F^{-p} = (F^{p+1})^\perp$. A variation which admits a polarization is said to be polarizable. We define $\mathbb{C}(-d)$ to be the polarizable complex Hodge structure on $V = \mathbb{C}$ with $V^d = V$.

Recall that the category of polarizable complex variations of Hodge structures is semi-simple. The theorem of the fixed part \cite{16} says that for two polarizable complex variations $V, W$, the group Hom($V, W$) of morphisms of local systems has a natural complex Hodge structure whose degree zero part is exactly the morphisms of complex variations. We have the following further consequence due to Deligne:

**Theorem 2** (\cite{11} 1.13 Proposition). Suppose $V$ is a $\mathbb{C}$-local system underlying a polarizable complex variation of Hodge structures and that we have a splitting of $\mathbb{C}$-local systems

$$V = \bigoplus_i M_i \otimes A_i$$

where the $M_i$ are irreducible and pairwise non-isomorphic and the $A_i$ are nonzero complex vector spaces. Then

1. Each $M_i$ underlies a polarizable complex variation of Hodge structures, unique up to shifting the Hodge grading.
2. Each polarizable complex variation of Hodge structures with underlying local system $V$ arises from \cite{11} by equipping each $M_i$ with its unique polarizable complex variation of Hodge structures and each $A_i$ with a uniquely determined polarizable complex Hodge structure (up to shifting the Hodge grading), namely $A_i = \text{Hom}(M_i, V)$.

In particular, the theorem implies a polarizable complex variation is irreducible if and only if the underlying local system is.

Given an $\mathbb{R}$-local system $V$, a polarizable real variation of Hodge structures on $V$ in the usual sense naturally induces a polarizable complex variation on $V_{\mathbb{C}}$. Conversely, a polarizable complex variation on $V_{\mathbb{C}}$ comes from a polarizable real variation on $V$ if complex conjugation flips the Hodge grading, or more precisely if for some (hence any) polarization $h$ the isomorphism of local systems $V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee$ given by $y \mapsto h(-, \overline{y})$ induces an isomorphism of complex variations $V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee(-w)$ for some (uniquely determined) $w$. Indeed, if this is the case then $V^p \cong (V^{w-p})^\vee$ so $V^p = V^{w-p}$. Moreover, for even $w$ (resp. odd $w$) a real polarization is provided by the symmetric (resp. antisymmetric) real form $q(x, y) = h(x, \overline{y}) + h(y, \overline{x})$ (resp. $q(x, y) = i(h(x, \overline{y}) - h(y, \overline{x}))$, since $q(x, \overline{x}) = h(x, x) + h(\overline{x}, \overline{x})$ (resp. $-iq(x, \overline{x}) = h(x, x) - h(\overline{x}, \overline{x})$) is definite of alternating sign on $V^p$.

The category of polarizable real variations is also semi-simple. Observe that by Theorem 2 any isotypic component $W$ of a polarizable real variation $V$ is a real sub-variation, as the same is true over $\mathbb{C}$ and the isomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee(-w)$ coming from $h$ restricts to an isomorphism $W_{\mathbb{C}} \to W_{\mathbb{C}}^\vee(-w)$. If $V$ is a single isotypic factor, then $V_{\mathbb{C}}$ either has one self-conjugate irreducible factor $N$ or has two non-isomorphic conjugate irreducible factors $N, \overline{N}$. Note that $N^\vee \cong \overline{N}$ via the polarization, and that the level of $V$ is at least as large as the level of any of the irreducible factors of $V_{\mathbb{C}}$.

\footnote{Throughout, by a real variation we mean a pure real variation, unless otherwise specified.}
We say that a real or complex variation is isotrivial if the Hodge filtration is flat, or equivalently if the irreducible factors of the complexification are level zero. To summarize the above discussion:

**Lemma 3.** Let $V$ be an irreducible polarizable real variation of Hodge structures of level one. Then $V$ is either isotrivial, or every irreducible factor of $V_{\mathbb{R}}$ is level one.

Before turning to the proof of Theorem 1 we recall the definition of a primitive symplectic variety. Let $X$ be a symplectic variety in the sense of Beauville, that is, a compact Kähler variety with rational singularities and a nowhere degenerate 2-form $\omega$ on its regular locus $X_{reg}$. We say that $X$ is primitive symplectic if $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X_{reg}, \Omega_X^{\omega}_{X_{reg}}) = \mathbb{C}$. As the singularities are rational, for any resolution $\pi : Y \to X$ the form $\sigma$ extends to a two-form on $Y$ [11, Corollary 1.7]. Moreover, $\pi^* : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is injective, so the Hodge structure on $H^2(X, \mathbb{Q})$ is pure, and we have an induced isomorphism $\pi^* : H^{2,0}(X) \to H^{2,0}(Y)$ (see [2] for details). In particular we have a well defined class $[\sigma] \in H^{2,0}(X)$.

By a Lagrangian fibration of a primitive symplectic variety we still mean a proper morphism $f : X \to B$ to a normal compact analytic variety $B$ whose generic fiber is smooth, connected, and Lagrangian. Each smooth fiber will still be an abelian variety. Moreover, $B$ is in fact Kähler and Moishezon (and in particular an algebraic space) since $f$ is equidimensional as in [10, Lemma 1.17], now using functorial pullback of reflexive forms [11, Theorem 1.11] and the fact that $R\pi_*\omega_Y \cong \omega_X \cong \mathcal{O}_X$ by the rationality of the singularities of $X$ [12, §5.1].

We use the same notation as above: $B^0 \subset B$ is a dense Zariski open smooth subset over which the restriction $f^0 : X^0 \to B^0$ is smooth and $\varphi : B^0 \to S$ is the period map associated to the variation of (polarized) weight one integral Hodge structures on $B^0$ with underlying local system $R^1 f^0_\ast \mathbb{Z}_{X^0}$.

**Proof of Theorem 1.** Let $V_{\mathbb{Z}} := R^1 f^\circ_\ast \mathbb{Z}_{X^0}$. We start with the following result of Voisin, whose proof we give for convenience (and to extend it slightly).

**Lemma 4 ([19, Lemma 5.5]).** $V_{\mathbb{R}}$ is irreducible as a polarizable real variation of Hodge structures.

**Proof.** First assume $X$ is smooth. By a result of Matsushita [13, Lemma 2.2] the restriction map $H^2(X, \mathbb{Q}) \to H^2(X_b, \mathbb{Q})$ to a generic fiber of $f^0$ is rank one and by Deligne’s global invariant cycles theorem $H^2(X, \mathbb{Q}) \to H^0(B^0, R^2 f^\circ_\ast \mathbb{Q}_{X^0})$ is surjective [6]. If $V_{\mathbb{R}}$ splits as a variation then the polarizations of the factors would yield a larger than one-dimensional space of sections of $R^2 f^\circ_\ast \mathbb{R}_{X^0} = \lambda^2 V_{\mathbb{R}}$, which is a contradiction.

Now if $X$ is a primitive symplectic variety, one easily checks using the results of [2] that Matsushita’s proof carries through verbatim and that $H^2(X, \mathbb{Q}) \to H^0(B^0, R^2 f^\circ_\ast \mathbb{Q}_{X^0})$ is still surjective, since the cokernel of $\pi^* : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is generated by exceptional divisors for a log resolution $\pi : Y \to X$ since $X$ has rational singularities. □

Suppose now that $f$ is not of maximal variation. Define the real transcendental lattice $T \subset H^2(X, \mathbb{R})$ to be the polarizable real Hodge substructure spanned by $[\sigma]$ and $[\overline{\sigma}]$. We next claim that the polarizable real variation of Hodge structures $V_{\mathbb{R}} \otimes T^\vee$ has a nontrivial subvariation of level at most one after a finite base-change; the argument below is that of [17], with some mild modifications.

Let $\nu : B^\circ \to B^0$ be a finite Galois étale cover so that the base-change $V_{\mathbb{Z}}^\nu := \nu^* V_{\mathbb{Z}}$ is pulled back along its period map $\varphi^\nu : B^\circ \to S'$, where $S'$ is a level cover of $S$. Note

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2 Or equivalently, if the monodromy is unitary (by Theorem 2); since there may not be an integral structure, this does not necessarily mean the monodromy is finite.

3 This definition is equivalent to Beauville’s original one.
that up to replacing $B^{\omega}$ with a further finite cover, we may assume $\varphi'$ can be embedded in a proper map $\bar{\varphi}' : \overline{B}^{\omega} \to S'$ [8]. Denote by $Z \subset S'$ the image of $\varphi'$, by $\psi : B^{\omega} \to Z$ the resulting map, and by $V_Z^{\omega}$ the variation on $Z$ so that $V_Z^{\omega} = \psi^*V_Z^{\omega}$. The map $\bar{\varphi}'$ and its image $Z$ are in fact algebraic [4, Theorem 3.1]. We shrink $Z$ (and $B^{\omega}, B^{\omega}, X^{\omega}$) so that it is smooth and so that $R^1\psi_*\mathcal{R}_{B^{\omega}}$ is a local system, naturally underlying a graded polarizable real variation of mixed Hodge structures whose only nonzero Hodge components are $(0,0), (1,0), (0,1), (1,1)$ (for example using Saito’s theory of mixed Hodge modules [14, 15]). Let $f^{\omega} : X^{\omega} \to B^{\omega}$ be the base-change of $f$. The natural map $H^2(X, \mathbb{C}) \to H^0(B^{\omega}, R^2f^{\omega}_*\mathcal{C}_{X^{\omega}})$ sends $[\sigma]$ and hence $T_\mathbb{C}$ to zero, since the fibers of $f$ are Lagrangian. By the Leray spectral sequence we have a natural morphism $\mathrm{pt}_Z^*T \to R^1\psi_*V_Z^{\omega} \cong V_Z^{\omega} \otimes R^1\psi_*\mathcal{R}_{B^{\omega}}$ in the category of real variations of mixed Hodge structures. This map is nonzero from the following geometric description as in [17].

Through a very general point $b \in B^{\omega}$, say above a point $z \in Z$, let $F$ be the positive-dimensional fiber of $\psi$ through $b$. The restricted family $X_F$ is isotrivial, so after replacing $F$ with a finite base-change we can trivialize the monodromy of $V_C'|_F$ and the following natural diagram commutes

$$
\begin{array}{c}
T \\
\downarrow \\
H^2(X_F, \mathbb{C}) \longrightarrow H^1(X_b, \mathbb{C}) \otimes H^1(F, \mathbb{C})
\end{array}
$$

where the bottom arrow comes from the degeneration of the Leray spectral sequence for $X_F \to F$. In the projective case we have $X_F \cong X_b \times F$ (possibly after a further base-change) and this map is just the Künneth projection. The image of $[\sigma]$ is nonzero in the bottom right corner since: (i) $\sigma$ is nonzero when restricted to $X_F$ since $\dim X_F > \frac{1}{2}\dim X$; (ii) $[\sigma]|_{X_F}$ extends to a smooth compactification since $\sigma$ extends to a smooth compactification of $X$, so $[\sigma] \neq 0 \in H^2(X_F, \mathbb{C})$; (iii) the image of $[\sigma]$ in $H^2(X_b, \mathbb{C}) \otimes H^0(F, \mathbb{C})$ vanishes and $[\sigma]$ is not in the image of $H^0(X_b, \mathbb{C}) \otimes H^2(F, \mathbb{C})$, as it is not pulled back from $F$.

Thus, there is a nonzero morphism of real variations

$$V_Z^{\omega} \otimes T^{\omega} \to \mathrm{gr}^{W}_{-1} \psi^*(R^1\psi_*\mathcal{R}_{B^{\omega}})^{\vee}.$$

As the category of polarizable real variations of (pure) Hodge structures is semi-simple, we therefore have a splitting

$$V_Z^{\omega} \otimes T^{\omega} = U \oplus W$$

of real variations, where $U \neq 0$ is the image of (2). In particular, $U$ has level at most one and weight -1.

Now by Lemma 4 the Galois group of $\nu$ acts transitively on the isotypic factors of $V_Z^{\omega}$. In particular, if $f$ (and therefore $V_{\mathbb{R}}$) is not isotrivial, no factor of $V_Z^{\omega}$ (as a variation) is isotrivial, or else its entire isotypic component would be, and so would $V_Z$. But then there can be no nonzero morphism of variations $V_Z^{\omega} \otimes T^{\omega} \to U$. Indeed, by Lemma 3 an irreducible factor $N$ of $V_Z$ has level one of degrees 0,1, and $N \otimes T_C^{\omega}$ can map nontrivially to an irreducible factor of $U_C$ of the form $N(1)$, while $\mathrm{Hom}(N \otimes T_C^{\omega}, N(1)) = T_C(1) \cong \mathbb{C}(-1) \oplus \mathbb{C}(1)$ has no degree 0 elements. Thus, $f$ must be isotrivial.

**Remark 5.** We revisit the example from [17] §4. Let $p \geq 5$ be a prime and $\lambda$ a $p$th root of unity. Consider a family of abelian varieties $f : X \to B$ with a cyclic automorphism such that the induced automorphism $\alpha$ of $V_{\mathbb{R}} = R^1f_*\mathcal{R}_X$ has $\lambda$ as an eigenvalue on $V^{1,0}$ but not on $V^{0,1}$. Let $\alpha'$ be the automorphism of $T^{\omega}$ with eigenvalue $\lambda^{-1}$ on $(T^{\omega})^{-2,0}$ and eigenvalue $\lambda$ on $(T^{\omega})^{0,-2}$. Then $V_{\mathbb{R}} \otimes T^{\omega}$ has a level one factor, namely...
the 1 eigenspace \((V_\mathbb{R} \otimes T^')^1\) of \(\alpha \otimes \alpha'\). But the condition on the eigenvalues means the eigenspaces \((V_\mathbb{C})^\lambda\) and \((V_\mathbb{C})^{\lambda-1}\) are level zero, and the real variation \((V_\mathbb{C})^\lambda \oplus (V_\mathbb{C})^{\lambda-1}\) is an isotrivial real factor.

We also obtain the following:

**Corollary 6.** Let \(X\) be a primitive symplectic variety and \(f : X \to B\) a Lagrangian fibration. Let \(L\) be a line bundle whose restriction to the smooth fibers is topologically trivial. Then the set of points \(b \in B^0(\mathbb{C})\) for which \(L|_{X_b}\) is torsion is analytically dense in \(B\).

Corollary 6 was proven by Voisin [19, Theorem 1.3] assuming either \(f\) is of maximal variation and \(\dim X \leq 8\) or isotrivial with no restriction on the dimension. Some applications of Corollary 6 (and more generally Proposition 7 below) to the Chow group and the construction of constant cycle curves are discussed in [19, §1.2].

We deduce this corollary using the following result of Gao. Recall that for a projective family \(f : X \to B\) of \(g\)-dimensional abelian varieties equipped with a section \(s\) and letting \(\tilde{B} \to B^\text{an}\) be the universal cover, the Betti map \(\beta : \tilde{B} \to H_1(X_b, \mathbb{R})\) is the real analytic map obtained by taking the coordinates of the section \(s\) with respect to the flat real-analytic trivialization of \(f\). Observe that \(\beta^{-1}(H_1(X_b, \mathbb{Q}))\) is the set of points of \(\tilde{B}\) at which \(s\) is torsion.

**Proposition 7** ([7, Theorem 9.1]). Let \(f : X \to B\) be a projective family of \(g\)-dimensional abelian varieties with \(\dim B \geq g\) and \(s : B \to X\) a section. Assume \(f\) is of maximal variation, that \(s\) is non-torsion, and that the very general fiber of \(f\) has no nontrivial \(\mathbb{Q}\)-factor. Then the Betti map \(\beta : \tilde{B} \to H_1(X_b, \mathbb{R})\) associated to \(s\) is generically submersive.

Gao proves Proposition 7 as a simple application of the Ax–Schanuel theorem for universal families of abelian varieties [7, Theorem 1.1]. This generalizes the results of André–Corvaja–Zannier [1] which were used in [19].

**Proof of Corollary 6.** By Voisin’s result and Theorem 1, we may assume \(f\) is of maximal variation. Consider the family of abelian varieties \(h : \text{Pic}^0(X^\circ/B^\circ) \to B^\circ\) and the section \(s : b \mapsto L|_{X_b}\). Let \(\nu : B^{\text{an}} \to B^\circ\) be a Galois finite base-change for which the \(\mathbb{Q}\)-factors of the very general fiber of \(h\) are defined over \(B^{\text{an}}\). As the Galois group of \(\nu\) acts transitively on the factors by Lemma 4, the \(d\) factors all have the same dimension \(q'\), and the image of the period map of each factor must have dimension \(\geq q'\), or else the image of the period map of \(f\) would have dimension smaller than \(dg' = \dim(X^\circ/B^\circ) = \dim(B^\circ)\). The base-change of the section \(s\) is also Galois invariant, so it suffices to prove the density statement for its projection to a single factor \(Y^\circ \to B^{\text{an}}\). Applying Proposition 7 the Betti map \(\beta : \tilde{B} \to H_1(Y_b, \mathbb{R})\) is submersive, so \(\beta^{-1}(H_1(Y_b, \mathbb{Q}))\) is analytically dense in \(\tilde{B}\) as claimed.

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