On regular and irregular movement of cylinder colliding with a moving belt

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Abstract. Conveyors and moving belts are frequently used in the field of mechanical engineering. In many operating regimes they are subjected to impact loading, which can induce irregular motion and undesirable behaviour of the working machine. This paper focuses on the impacts between an impact body (the cylinder in our case) and a moving belt. Results of the simulations show that different combinations of the system parameters produce a high complexity of device motion. The bifurcation analysis together with outputs of the 0-1 test for chaos and sample entropy indicates different movement regions. The performed analysis elucidated more about the properties and behaviour of mechanical systems with strong nonlinearities such as impacts and dry friction.

1. Introduction
Moving belts are machine parts often used in the field of mechanical engineering. Mainly, they are used for the transport and processing of bodies and bulk materials (see e.g. [1–4] and references therein).

This paper focuses on the impacts of a cylinder constrained by a flexible rope on a moving belt. The upper end of the rope is excited in the vertical direction. The cylinder is not balanced, which arrives at eccentric collisions. Computational simulations were employed to carry out the research. The results show that the character (regular, chaotic) of the system vibration induced by the impacts, dry friction, and motion of the belt depends on the frequency of the excitation, its amplitude, and eccentricity of the cylinder. This paper is a natural continuation of a research of an analogous model introduced in [5] and further deeply developed in [6]. On contrary to this paper model (1) shown in Fig. 1, the colliding body was not assumed to be a cylinder but more complex body. More precisely, that body was constructed as a central cylindre and two end parts having 25 times higher moment of inertia.

This paper introduction continues with the development of the analyzed systems (Section 2), followed by the derivation of the mathematical model (Section 3). Fulfilling a goal, typical
movement cases are constructed (Section 4) using dynamics quantifiers and qualifiers varying the excitation frequency for balanced (Sub-section 4.1) and unbalanced case (Sub-section 4.2). Finally, concluding observations and remarks are made (Section 5).

2. The analysed system
The analyzed system is depicted in Fig. 1. It consists of slider 1, cylinder 2, flexible rope 3, conveyor belt 4, and rigid platform 5. The slider performs a reciprocating sliding motion in the vertical direction. The rope connects the slider with the centre of the cylinder. The belt moves in a horizontal direction at a constant velocity. The cylinder performs a general planar motion. It can slide in the vertical and horizontal directions, and can rotate about its axis. The rope transmits only the tensile force. All bodies of the system (except the rope) can be considered absolutely rigid. The flexibility of the rope makes it possible for the cylinder to collide with the moving belt. The material of the cylinder and the belt, in a small area near to the impact point and the direction normal to the contact surfaces, is considered to be linearly elastic, exhibiting some material damping. As the cylinder is unbalanced, the impacts are eccentric; the center of gravity of the cylinder is marked $T$ in Fig. 1. The friction in the contact area is considered to be of a Coulomb type and the connection of the rope with the cylinder and the slider as neutral (without resistances against the relative motion). The stiffness of the rope is linear, as is the damping of the cylinder caused by the environment. From the point of view of time history, vertical motion of the slider is harmonic with a short transient period at the beginning.

![Figure 1. Scheme of the investigated system.](image-url)
3. Mathematical model of the system
The equations of motion have been derived using the Lagrange equations of the second kind:

\begin{align*}
m\ddot{x} - me_T\dot{\phi}\sin(\phi + \psi_T) - me_T\dot{\phi}^2\cos(\phi + \psi_T) &= Q_x, \\
m\ddot{y} + me_T\dot{\phi}\cos(\phi + \psi_T) - me_T\dot{\phi}^2\sin(\phi + \psi_T) &= Q_y, \\
-me_T\ddot{x}\sin(\phi + \psi_T) + me_T\ddot{y}\cos(\phi + \psi_T) + (J_T + me_T^2)\ddot{\phi} &= Q_\phi
\end{align*}

where

\begin{align*}
Q_x &= F_x + F_t - b\dot{x}, \\
Q_y &= F_y + F_c - mg - b\dot{y}, \\
Q_\phi &= F_tR - mge_T\cos(\phi + \psi_T) - b\dot{\phi}\ddot{\phi}
\end{align*}

and \(x\) and \(y\) denote the position of the cylinder centre, and \(\phi\) its angular rotation. \(F_x, F_y, F_t,\) and \(F_c\) are \(x\) and \(y\) components of the forces, by which the suspension acts on the cylinder, tangential component of the contact force (friction force), and normal component of the contact force acting on the cylinder, respectively. (\(\cdot\)) and (\(\ddot{}\)) stand for the first and second derivative with respect to time. The notations of the further quantities and their descriptions are shown in Table 1.

Extension of the cylinder suspension \(\Delta l\) and its rate can be expressed thusly:

\begin{align*}
\Delta l &= \sqrt{x^2 + (y_z + h_z - y)^2} - L_0, \\
\Delta \dot{l} &= 1/2 \left( x^2 + (y_z + h_z - y)^2 \right)^{-1/2} (2x\dot{x} + 2(y_z + h_z - y)(\dot{y}_z - \dot{y})).
\end{align*}

Now, denote

\[ F^* = k_l\Delta l + b_l\Delta \dot{l}. \]

The suspension of the cylinder can transmit only the tensile force it holds for its components. If \(\Delta l \leq 0\) or \(F^* \leq 0\) then \(F_x = 0\) and \(F_y = 0\). If \(\Delta l > 0\) or \(F^* > 0\) then

\begin{align*}
F_x &= -F^*\cos(\alpha), \\
F_y &= -F^*\sin(\alpha)
\end{align*}

where \(\alpha\) follows from the solution of the following equations:

\begin{align*}
\cos(\alpha) &= \frac{x}{\sqrt{x^2 + (y_z + h_z - y)^2}}, \\
\sin(\alpha) &= \frac{y_z + h_z - y}{\sqrt{x^2 + (y_z + h_z - y)^2}}.
\end{align*}

Next, put

\[ N^* = k_c(R - y) - b_c\dot{y}. \]

As the contact force acting between the cylindrical body and the belt can only be compressive, it holds for its normal components. If \(y > R\) or \(N^* < 0\) then

\[ F_c = 0. \]

In the opposite case,

\[ F_c = N^*. \]
For the friction force acting on the cylindrical body, it holds that

\[ F_t = -\frac{2F_c f}{\pi} \tan(a(\dot{x} + R\dot{\varphi} - v_p)). \]  

(8)

The vertical position of the driving body is given by

\[ y_z(t) = z_a(1 - e^{-\alpha t}) \sin(\omega t) \]  

(9)

where \( z_a \) is the amplitude of the driving body kinetic excitation, \( \alpha \) is the run up coefficient, and \( \omega \) stands for the excitation frequency. The time history of the position of the sliding body \( y_z \), which excites the system as shown in Fig. 2.

At the beginning, the system takes the rest position defined by the following initial conditions:

\[
\begin{align*}
    x(0) &= 0, \\
    \dot{x}(0) &= 0, \\
    y(0) &= h_z - L_0 - \frac{mg}{k_1}, \\
    \dot{y}(0) &= 0, \\
    \varphi(0) &= -\frac{\pi}{2} - \psi_T, \\
    \dot{\varphi}(0) &= 0.
\end{align*}
\]

Figure 2. Vertical simulation function \( y_z \) (9) for the parameters given in Table 1, \( \omega = 2.6 \text{ rad s}^{-1} \), and \( z_a = 7.5 \text{ mm} \).

4. Main results

The movement character of investigated system (1) depends on 20 parameters whose values are summarized in Table 1. To solve the main task of this paper the system was simulated varying two parameters, the excitation frequency \( \omega \) and eccentricity \( e_T \), where \( \omega \) was chosen from the interval \([3, 6] \text{ rad s}^{-1}\) and \( e_T \in \{0, 0.02\} \text{ m}\) to show how the system performs. The second parameter selection corresponds to the balanced and unbalanced case, respectively, discussed separately in the following two subsections.

In the following, all simulations were performed using Runge–Kutta explicit method with the final time \( 4 \times 10^4 \text{ s} \) with recorded \( 4 \times 10^6 \) values and all characteristics of this research were done on the last 25% of each simulation to skip the start up of the (1) system.

For the characterization of the movement patterns it was utilized: bifurcation diagrams (constructed as projection of Poincaré sections), the 0-1 test for chaos, and sample entropy underlined by phase portraits and FFT.

The 0-1 test for chaos (denoted \( K \)) as a qualifier of regularity movement returns binary output: 0 for regular case (period or quasi-period) and 1 for irregular situation (chaos). This method was introduced by [7] (see also [8]) applicable to real data [9] as well as to time series generated by continuous [10–12] or discrete models [13].

For the dynamical qualifier, sample entropy \( E_{\text{Samp}} \) is used, which was introduced by [14], and subsequently applied in wide range of scientific disciplines; [15], [16], and [17]. \( E_{\text{Samp}} \) returns a value that measures the complexity of the system; as \( E_{\text{Samp}} \) increases, the complexity increases, see also [18], [19] and references therein.

The tested vector \( \phi \) that is applied to the \( K \) and \( E_{\text{Samp}} \) test is constructed as Euclidean norm of state variables:

\[ \phi(j) = \sqrt{x(j)^2 + y(j)^2}. \]
Table 1. Parameters of the system (1).

| quantity | value         | description                                                                 |
|----------|---------------|------------------------------------------------------------------------------|
| $m$      | 10 kg         | mass of the cylinder                                                        |
| $J_T$    | 0.008 kg m$^2$ | moment of inertia of the cylinder referred to its centre of gravity         |
| $\psi_T$| 0.4 rad       | angular phase shift of the center of gravity                                |
| $L_0$    | 0.35 m        | length of the unloaded suspension of the cylinder                           |
| $h_z$    | 0.89 m        | the mean height of the excitation body above the belt                        |
| $R$      | 0.04 m        | radius of the cylinder                                                       |
| $k_c$    | $1 \times 10^7$ Nm$^{-1}$ | contact stiffness (cylinder – belt)                                  |
| $b_c$    | 100 Nsm$^{-1}$ | the coefficient of contact damping (cylinder – belt)                        |
| $k_l$    | 200 N m$^{-1}$ | stiffness of the cylinder suspension                                         |
| $b_l$    | 5 Nsm$^{-1}$  | the damping coefficient of the cylinder suspension                           |
| $g$      | 9.80665 m s$^{-2}$ | the gravity acceleration                                                   |
| $a_\alpha$ | 0.004 m      | the amplitude of the driving body kinetic excitation                         |
| $\alpha$ | 1 s$^{-1}$    | the run up coefficient                                                      |
| $v_p$    | 0.5 ms$^{-1}$ | velocity of the belt                                                        |
| $b$      | 0.01 Ns m$^{-1}$ | the outer damping coefficient of the cylinder (sliding motion)             |
| $b_\varphi$ | 0.001 Nms rad$^{-1}$ | the outer damping coefficient of the cylinder (rotational motion)        |
| $f$      | 0.2           | coefficient of friction                                                     |
| $a$      | 100 s m$^{-1}$ | mathematical constant defining the shape of the friction characteristic     |

Figure 3. Investigated system’s (1) phase diagram for $e_T = 0$ m; (left) for $\omega = 3.15$ rad s$^{-1}$, (middle) $\omega = 4.035$ rad s$^{-1}$, and (right) $\omega = 3.38$ rad s$^{-1}$.

4.1. Balanced case

As a first research situation $e_T = 0$ m was set – that is the balanced case is considered. The phase portraits in Fig. 3 show that the movement can act in periodic, quasi-periodic, and also chaotic regime. In this figure, the black point indicates the initial position of the cylinder, the green curve stands for the full trajectory, and the red one for the basin of attraction. The corresponding FFT analysis in Fig. 4 clearly detect all three previously described movements. This three cases are summarized in Table 2.

The evolution of the movement character was done using bifurcation diagrams and the dynamics, as it was mentioned above, by the $K$ and $E_{Samp}$ test was observed. Firstly, in Fig. 5 bifurcation diagram depending on $\omega \in [3,6]$ rad s$^{-1}$ for $x$ and $y$ variable shows wide variety of behaving motion regimes. In this figure, by brown box, restriction of focussing interval is
Figure 4. Investigated system’s (1) FFT(y) for $e_T = 0$ m; (left) for $\omega = 3.15$ rad s$^{-1}$, (middle) $\omega = 4.035$ rad s$^{-1}$, and (right) $\omega = 3.38$ rad s$^{-1}$.

Table 2. Movement character of three study cases of (1) with $e_T = 0$ m.

| $\omega$ [rad s$^{-1}$] | movement character | $K$ | $E_{Samp}$ | contact | Fig. |
|-------------------------|-------------------|-----|-----------|---------|------|
| 3.15                    | period            | -0.0003 | 0.051 | no      | 3 and 4 (left) |
| 4.035                   | quasi-period      | 0.0017 | 0.085 | yes     | 3 and 4 (middle) |
| 3.385                   | chaos             | 0.612  | 0.321 | yes     | 3 and 4 (right) |

denoted. The magnified part’s $\omega \in [3.35, 3.5]$ rad s$^{-1}$ bifurcation diagram is shown in 6. For the two mentioned intervals $K$ and $E_{Samp}$ in Figs 7, 8 respectively are given. Comparing outputs of all three detection methods one can easily observe that the system often remains in regular regimes (period, quasi-period) and rarely in irregular ones. Moreover, it the contact appearance is visible from Figs. 5 and 6. More precisely, maximum and minimum values of $x$ and $y$ trajectories in the steady state are marked in green and magenta respectively. The contact is detectable if minimum value of $y$ is 0.04 m.

Figure 5. Bifurcation diagrams varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0$ m.

4.2. Unbalanced case

For the second research situation $e_T = 0.02$ m was set – that is the unbalanced case is considered. Again as in the balanced case, the pase portraits in Fig. 9 show that the movement can act in periodic, quasi-periodic, and also chaotic regime. The corresponding FFT analysis in Fig. 4 clearly detect all three previously described movements. This three cases are summarized in Table 3.
Figure 6. Magnified bifurcation diagrams varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0$ m.

Figure 7. Output of the 0-1 test for chaos varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0$ m; (right) magnified part of (left).

Figure 8. Output of the sample entropy $E_{Samp}$ varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0$ m; (right) magnified part of (left).

| $\omega$ [rad s$^{-1}$] | movement character | $K$ | $E_{Samp}$ | contact | Fig. |
|-------------------------|--------------------|-----|------------|---------|------|
| 3.12                    | period             | -0.001 | 0.051    | no      | 9 and 10 (left) |
| 5.215                   | quasi-period       | 0.004  | 0.043     | yes     | 9 and 10 (middle) |
| 5.263                   | chaos              | 0.9177 | 0.3       | yes     | 9 and 10 (right) |

As in the foregoing, balanced, researched case, bifurcation diagrams and the $K$ and $E_{Samp}$ test were utilized. Firstly, in Fig. 11 bifurcation diagram depending on $\omega \in [3, 6]$ rad s$^{-1}$ for
Figure 9. Investigated system’s (1) phase diagram for $e_T = 0.02$ m; (left) for $\omega = 3.12$ rad s$^{-1}$, (middle) $\omega = 5.263$ rad s$^{-1}$, and (right) $\omega = 5.1$ rad s$^{-1}$.

Figure 10. Investigated system’s (1) FFT($y$) for $e_T = 0.02$ m; (left) for $\omega = 3.12$ rad s$^{-1}$, (middle) $\omega = 5.251$ rad s$^{-1}$, and (right) $\omega = 5.263$ rad s$^{-1}$.

$x$ and $y$ variable shows wide variety of behaving motion regimes. In this figure, restriction of focussing interval is denoted by orange box. The magnified part’s $\omega \in [5.05, 5.356]$ rad s$^{-1}$ bifurcation diagram is shown in 12. For the two mentioned intervals $K$ and $E_{Samp}$ in Figs 13, 14 respectively are given. Comparing outputs of all three detection methods one can easily observe that the system often remains in regular regimes (period, quasi-period) and rarely in irregular ones.

Figure 11. Bifurcation diagrams varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0.02$ m.

Closing the above given results, the effect of contact of the cylindre with the moving belt plays the crucial role here. Hence, by computer graphics analysis studying the phase diagrams together with FFT underlined by their dynamic characteristics $K$ and $E_{Samp}$ (this dynamic behavior coincides ) it can be proven:
Figure 12. Magnified bifurcation diagrams varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0.02$ m.

Figure 13. Output of the 0-1 test for chaos varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0.02$ m; (right) magnified part of (left).

Figure 14. Output of the sample entropy $E_{Samp}$ varying $\omega$ of (1) for the parameters given in Table 1 and $e_T = 0.02$ m; (right) magnified part of (left).

Property 1  Let $e_T \in \{0, 0.02\}$ and $\omega \in [3, 6]$ rad s$^{-1}$. Then if chaotic movement appears the system’s (1) cylinder (body 2, Fig. 1) contacts the belt (body 4, Fig. 1).

5. Conclusions
In this paper, a newly constructed mechanical system consisting of a cylinder constrained by a flexible rope and a moving belt is introduced; through extensive numerical simulation, its very rich dynamics are revealed.

To achieve this, a model of three degrees of freedom was constructed and then simulated in Matlab [20] using the Runge-Kutta method as an in-function ode45 adaptive solver.

As a main aim, the research was divided into two cases balanced (for $e_T = 0$ m in Section 4.1) and unbalanced (for $e_T = 0.02$ m in Section 4.2). In both cases, firstly phase portraits (Figs 3
and 9) with FFT (Figs 4 and 10) were given for selected cases to show the full variety of motion patterns. Next, varying the excitation frequency $\omega$ bifurcation diagrams (Figs 5 and 11) with its magnified parts (Figs 6 and 12) and the outputs of $K$ (Figs 7 and 13) and $E_{\text{Samp}}$ (Figs 8 and 14) test were given.

Comparing the balanced and unbalanced cases it can be concluded that in both cases regular and irregular are observable, while the influence of unbalance drives the system in the direction of higher complexity that can be derived from $E_{\text{Samp}}$ (Figs 8 and 14).

To the end, as a final observation, the effect of contact of the cylinder with the moving belt plays the crucial role here. Consequently, it can be concluded Property 1 that the necessary condition for chaos is the contact.

Acknowledgments
Acknowledgments This work was supported by The Ministry of Education, Youth and Sports from the National Programme of Sustainability (NPU II) project “IT4Innovations excellence in science – LQ1602”; by The Ministry of Education, Youth and Sports from the Large Infrastructures for Research, Experimental Development and Innovations project “IT4Innovations National Supercomputing Center – LM2015070”; by SGC grant No. SP2020/137 “Dynamic system theory and its application in engineering”, VSB - Technical University of Ostrava, Czech Republic, Grant of SGS No. SP2020/114, VSB - Technical University of Ostrava, Czech Republic, and by grant project of the Czech Science Foundation 19-06666S.

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