Projective tensor product of protoquantum spaces

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Abstract

A proto-quantum space is a (general) matricially normed space in the sense of Effros and Ruan presented in a ‘matrix-free’ language. We show that these spaces have a special (projective) tensor product possessing the universal property with respect to completely bounded bilinear operators. We study some general properties of this tensor product (among them a kind of adjoint associativity), and compute it for some tensor factors, notably for $L_1$ spaces. In particular, we obtain what could be called the proto-quantum version of the Grothendieck theorem about classical projective tensor products by $L_1$ spaces. At the end, we compare the new tensor product with the known projective tensor product of operator spaces, and show that the standard construction of the latter is not fit for general proto-quantum spaces.

1. Introduction

In their paper [7], Effros and Ruan introduced and investigated the important notion of a matricially normed space. Very soon, after the discovery of Ruan Representation Theorem [21], the great majority of papers and monographs was dedicated only to the outstanding special class of these structures, the $\mathcal{L}^\infty$–matricially normed spaces. Now the latter are called abstract operator spaces (or just operator spaces), and sometimes quantum...
spaces. The theory of operator spaces is very rich and well-developed. It is presented in widely known textbooks [11, 19, 20, 2).

On the other hand, already in [7, 21] it was demonstrated that matricially normed spaces are a subject of considerable interest even outside the class of operator spaces, that is without the assuming of the second axiom of Ruan. In this paper we return to general matricially normed spaces; however, presented in the equivalent ‘non-coordinate’, or ‘matrix-free’ language. (The latter seems to be more convenient for us in this circle of questions). We hope that our observations, apart from some results in the cited papers, also show that general matricially normed spaces deserve an independent interest. Moreover, in their study we sometimes come to things that look very different from what we know about operator spaces.

Our main point is that the general matricially normed spaces, called in this paper proto-quantum spaces, have a tensor product, possessing the universal property relative to the class of completely bounded bilinear operators. In the context of operator spaces, such a tensor product was discussed in [11 II.7.1] and, in the non-coordinate language, in [14, Ch.7.2]. Therefore, following the terminology of these textbooks, we call this tensor product projective. (Note that another kind of tensor product, the Haagerup tensor product, was already discussed in [7, 21]).

The contents of the paper are as follows.

The second and third sections contain initial definitions, notably of a proto-quantum space, of a completely bounded operator and of a completely bounded bilinear operator. Also the simplest examples are presented, in particular, the maximal and minimal proto-quantization of a given normed space. Among several observations, we show that the maximal proto-quantization always gives rise to an \( L^1 \)-space, and that every bounded functional with the domain an \( L^p \)-space and with the range \( \mathbb{C} \) that was made an \( L^q \)-space, is ‘automatically’ completely bounded, provided \( p \leq q \). (As to the notation ‘ \( L^q \) ’, here and thereafter we mean the non-coordinate counterpart of the notion of a matricially normed \( L^p \)-space, initially introduced in [7]).

In Section 4 we consider several further examples of proto-quantum not quantum spaces. In particular, we introduce what we call the standard proto-quantization of the space \( L_p(X, E) \), where \( E \) is a proto-quantum space. (Among them one can find the \( L^p \)-space \( L_p(X, E) \), where \( E \) is an \( L^p \)-space). Also we show that some bilinear operators, related to these spaces,
are completely contractive; this will be used in subsequent sections.

In Section 5 we define the non-completed projective tensor product of proto-quantum spaces, denoted by ‘$\otimes_{\text{pop}}$’, and its ‘completed’ version, denoted by ‘$\hat{\otimes}_{\text{pop}}$’. We prove the respective existence theorems by displaying relevant explicit constructions.

In Section 6 we present some examples of the computation of the introduced tensor product. It turns out that, just as in the case of the classical projective tensor product of normed spaces, the especially nice tensor factors are $L_1$-spaces. As the base of most applications, we show that for all proto-quantum spaces $E$ and $F$ we have

$$L_1(X, E) \hat{\otimes}_{\text{pop}} L_1(Y, F) \simeq L_1(X \times Y, E \hat{\otimes}_{\text{pop}} F).$$

(Here and thereafter ‘$\simeq$’ means a completely isometric isomorphism). Another frequently used fact is that, under some assumptions on $E$ and, $F$ we have

$$E \hat{\otimes}_{\text{pop}} F \simeq E \hat{\otimes}_{\text{pr}} F,$$

where on the right is the classical projective tensor product of our spaces, made a proto-quantum space according to a certain recipe in Section 4. Combining these two theorems, we obtain that for a $p$-convex proto-quantum space $E$ (in particular, an $L^p$-space) and the complex plane, considered as an $L^p$-space, we have

$$L_1(X, \mathbb{C}) \hat{\otimes}_{\text{pop}} E \simeq L_1(X, E).$$

This result can be considered as a version, for proto-quantum spaces, of the Grothendieck Theorem on tensoring by $L_1$-spaces (cf., e.g., [12, §2, n°2]).

At the beginning of the next chapter we extend to general proto-quantum spaces the method of the quantization of a given space of completely bounded operators, first suggested in papers [8, p.140], [3], [9]; see also the textbooks [11, I.3.2] or [14, Ch.8.7]. Then we establish the suitable form of the so-called law of adjoint associativity, connecting spaces of operators with tensor products. (The form of that law in the context of the classical functional analysis is presented, e.g., in [14, Ch.6.1]). Namely, for proto-quantum spaces $E, F, G$, the space $\mathcal{CB}(E \hat{\otimes}_{\text{pop}} F, G)$ is, in a natural way, (completely) isometrically isomorphic to $\mathcal{CB}(F, \mathcal{CB}(E, G))$ and to $\mathcal{CB}(E, \mathcal{CB}(F, G))$.

(We recall that the mentioned method essentially differs from the initial approach to what to call a dual matricially normed space. This approach was
considered in [7]), and, as it was shown there, has some advantages. However, its essential drawback is that it does not lead to the adjoint associativity).

In the concluding Section 7 we compare the introduced tensor product ‘$\otimes_{\text{pop}}$’ with what could be called its prototype. By this we mean the well-known projective tensor product of operator spaces, denoted here by ‘$\otimes_{\text{op}}$’, that was independently discovered by Blecher/Paulsen [3] and Effros/Ruan [9]. For operator spaces (i.e. when the second axiom of Ruan is fulfilled) both tensor products coincide. However, for general proto-quantum spaces the standard formulae for the respective norms give different numbers: in the case of ‘$\otimes_{\text{op}}$’ they are essentially greater than in the case of ‘$\otimes_{\text{pop}}$’. As an example, we consider the projective tensor square of a certain proto-quantum space, and for every $n$ we display an element of its amplification, for which the first number is $n^2$, whereas the second is $n$.

2. Proto-quantum spaces and their first examples

As it was said, we use in this paper the so-called non-coordinate approach to the structures in question, and not the more widespread ‘matrix’ approach, as in the textbooks [11, 19, 20, 2]. Some of our terms and notation are contained in [14], where practically only (abstract) operator spaces, called there quantum spaces, were considered. For the convenience of the reader, we shall briefly repeat some of the most needed initial definitions.

To begin with, we choose an arbitrary separable infinite-dimensional Hilbert space, denote it by $L$ and fix it throughout the whole paper. We write $\mathcal{B}$ instead of $\mathcal{B}(L)$, the Banach algebra of all bounded operators on $L$ with the operator norm, usually denoted just by $\|\cdot\|$. The symbol $\otimes$ is used for the (algebraic) tensor product of linear spaces and for elementary tensors. The symbols $\otimes_{\text{pr}}$ and $\otimes_{\text{in}}$ denote the non-completed projective and injective tensor product of normed spaces, respectively. The complex-conjugate space of a linear space $E$ is denoted by $E^{\text{cc}}$. The identity operator on a linear space $E$ is denoted by $1_E$, and we write $1$ instead of $1_L$.

For $\xi, \eta \in L$ we denote by $\xi \circ \eta$ the rank 1 operator on $L$, taking $\zeta$ to $\langle \zeta, \eta \rangle \xi$. Recall that $\|x \circ y\| = \|x\| \|y\|$.

Denote by $\mathcal{F}$ the (non-closed) two-sided ideal of $\mathcal{B}$, consisting of finite rank bounded operators. Recall that there is a linear isomorphism $L \otimes L^{\text{cc}} \to$
\( \mathcal{F} \), well defined by taking \( \xi \otimes \eta \) to \( \xi \circ \eta \). For \( p \in [1, \infty] \) we denote by \( \|\cdot\|_p \) the norm of the \( p \)-th Schatten class on \( \mathcal{F} \), and write \( \mathcal{F}_p := (\mathcal{F}, \|\cdot\|_p) \); in particular, \( \mathcal{F}_\infty \) is \( \mathcal{F} \) with the operator norm.

In what follows we need the triple notion of the so-called amplification. First, we amplify linear spaces, then linear operators and finally bilinear operators.

The amplification of a given linear space \( E \) is the tensor product \( \mathcal{F} \otimes E \). Usually we briefly denote it by \( \mathcal{F}E \), and an elementary tensor \( a \otimes x; a \in \mathcal{F}, x \in E \), by \( ax \). Note that \( \mathcal{F}E \) is a bimodule over the algebra \( \mathcal{B} \) with the outer multiplications, denoted by ‘ \( \cdot \) ’ and well defined by \( a \cdot (bx) := (ab)x \) and \( (ax) \cdot b := (ab)x \).

**Definition 2.1.** A semi-norm on \( \mathcal{F}E \) is called proto-quantum semi-norm, or briefly, \( \text{PQ} \)-semi-norm on \( E \), if the \( \mathcal{B} \)-bimodule \( \mathcal{F}E \) is contractive, that is we always have the estimate \( \|a \cdot u \cdot b\| \leq \|a\|\|u\|\|b\| \). A \( \text{PQ} \)-semi-norm on \( E \) is called quantum semi-norm, or briefly, \( \text{Q} \)-semi-norm on \( E \), if for \( u, v \in \mathcal{F}E \) and (ortho)projections \( P, Q \in \mathcal{B}, PQ = 0 \) we always have \( \|P \cdot u \cdot P + Q \cdot v \cdot Q\| = \max\{\|P \cdot u \cdot P\|, \|Q \cdot v \cdot Q\|\} \).

The space \( E \), endowed with a \( \text{PQ} \)-semi-norm, is called semi-normed proto-quantum space, or briefly, semi-normed \( \text{PQ} \)-space. In the case of a normed \( \text{PQ} \)-space we usually omit the word ‘normed’.

In a similar way we use the terms semi-normed \( \text{Q} \)-space and (just) \( \text{Q} \)-space.

**Remark 2.2.** By their definition, \( \text{Q} \)-spaces can be treated as a special case of the so-called Ruan bimodules, considered in [15] and [23].

**Remark 2.3.** Let us recall, for the convenience of the reader, the way of the translation from the ‘matrix language’ to the ‘non-coordinate language’. Let \( E \) be a matricially normed space in the sense of [7], and we are given \( u \in \mathcal{F}E \). Clearly, there exists a finite rank projection \( P \) such that \( u \) has the form \( \sum_{k=1}^n a_kx_k; a_k = P \cdot a_k \cdot P, x_k \in E \). We choose an arbitrary orthonormal basis in \( P(L) \) and denote by \( (a_{ij}^k) \) the matrix, in this basis, of the restriction \( a_k \) to \( P(L) \). Then we take the matrix \( (u_{ij} := \sum_k a_{ij}^k x_k) \) with entries in \( E \) and set \( \|u\| := \|(u_{ij})\| \). It is easy to show that \( \|u\| \) does not depend on the choice of \( P \) and of a basis in \( P(L) \), and that the function \( u \mapsto \|u\| \) is a \( \text{PQ} \)-norm on \( E \).

A semi-normed \( \text{PQ} \)-space \( E \) becomes a semi-normed space (in the usual sense), if for \( x \in E \) we set \( \|x\| := \|Qx\| \), where \( Q \) is an arbitrary rank 1
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operator of norm 1. Obviously, the resulting semi-norm does not depend on the particular choice of \( Q \). The obtained semi-normed space is called underlying space of a given \( PQ \)–space, and the latter is called a proto-quantization (briefly \( P \)–quantization) or, if we deal with a \( Q \)–space, a quantization of the former. (The term ‘quantization’ ascends to the seminal Effros’ lecture [6]. Indeed, in the space \( E = \mathbb{C} \otimes E \) commutative scalars from \( \mathbb{C} \) are replaced by the ‘non-commutative scalars’ from \( \mathcal{F} \); a typical device of ‘quantum mathematics’).

**Proposition 2.4.** Let \( E \) be a semi-normed \( PQ \)–space with a normed underlying space. Then the semi-norm on \( \mathcal{F}E \) is a norm.

The proof, given in [14, Prop. 1.2.2] for \( Q \)–spaces is valid, without any modification, for \( PQ \)–spaces too.

**Example 2.5.** Every non-zero normed space, say \( E \), has a lot of \( P \)–quantizations. Among them we distinguish the so-called maximal and minimal, denoted by \( E_{\text{max}} \) and \( E_{\text{min}} \), respectively. The first space is obtained by the endowing \( \mathcal{F}E \) with the norm of \( L^{\otimes_p}L^\infty \otimes_{pr}E = \mathcal{F}_1 \otimes_{pr}E \), and the second with the norm of \( L^{\otimes_{in}}L^\infty \otimes_{in}E = \mathcal{F}_\infty \otimes_{in}E \). (Evidently, the first of these \( PQ \)–norms is never a \( Q \)–norm whereas it is not difficult to show that the second one is always a \( Q \)–norm.)

As a matter of fact, the first norm is maximal in the sense that it is the greatest of all norms of \( P \)–quantizations of \( E \). Indeed, we easily see that the norm on \( L \otimes L^p \otimes E \), corresponding to any given \( PQ \)–norm on \( E \), is a cross-norm. But among all cross-norms there is a greatest one, and it is exactly the norm on \( L \otimes_{pr}L^p \otimes_{pr}E \). In a similar sense the second norm is minimal, but this statement will be justified a little bit later.

As to \( E_{\text{max}} \), it is a member of the whole family of \( P \)–quantizations of \( E \), denoted by \( (p)E; 1 \leq p \leq \infty \); they are obtained by endowing \( \mathcal{F}E \) with the norm of \( \mathcal{F}_p \otimes_{pr}E \). Clearly, we have \( (1)E = E_{\text{max}} \). In particular, among various \( P \)–quantizations of \( \mathbb{C} \) we distinguish \( PG \)–spaces \( (p)\mathbb{C} \); we see that the amplification of such a space is identified with \( \mathcal{F}_p \). Moreover, there is only one \( P \)–quantization of \( \mathbb{C} \) which is a quantization, and this is \( (\infty)\mathbb{C} \).

In what follows, if numbers \( \lambda_k \geq 0; k = 1, ..., n \) are given, we shall understand the expression \( (\sum_{k=1}^{n} \lambda_k^p)^{1/p} \) as \( \max\{\lambda_1, ..., \lambda_n\} \) in the case \( p = \infty \).

We shall say that a projection \( P \in \mathcal{B} \) is a support of an element \( u \in \mathcal{F}E \), if we have \( P \cdot u \cdot P = u \).
For $p \in [1, \infty]$ we shall say that a PQ–space $E$ is an $L^p$–space, respectively, $p$–convex space and $p$–concave space, if for every $u_1, \ldots, u_n \in FE$ with pairwise orthogonal supports we have $\| \sum_{k=1}^n u_k \| = (\sum_{k=1}^n \| u_k \|_p^p)^{\frac{1}{p}}$, respectively, $\| \sum_{k=1}^n u_k \| \leq (\sum_{k=1}^n \| u_k \|_p^p)^{\frac{1}{p}}$, $\| \sum_{k=1}^n u_k \| \geq (\sum_{k=1}^n \| u_k \|_p^p)^{\frac{1}{p}}$. Obviously, it is sufficient to have similar relations for the case $n = 2$. We see that $\mathcal{L}^\infty$–space is just another name for a $Q$–space. Clearly, every PQ–space is 1–convex and $\infty$–concave. Moreover, $(p)_C$ is evidently an $L^p$–space.

Throughout the paper, for $a, b \in F$ we shall write $a \approx b$ provided we have $SaT = b$ for some unitary operators $S, T \in B$. Similarly, for $u, v \in FE$, where $E$ is a $PQ$–space, we shall write $u \approx v$ provided $S\cdot u\cdot T = v$ for $S, T$ as above. Clearly, $a \approx b$ implies $\|a\|_p = \|b\|_p$ for all $p \in [1, \infty]$, and $u \approx v$ implies $\|u\| = \|v\|$. It is well known (and easy to show) that for every $a \in F$ we have

$$a \approx h, \quad \text{where} \quad h = \sum_{k=1}^n s_k P_k$$

for some pairwise orthogonal rank one projections $P_k \in F$, and $s_k \geq 0$.

**Proposition 2.6.** If $E$ is a $p$–convex $PQ$–space, $p$–concave $PG$–space or an $L^p$–space, then, for all $a \in F, x \in E$, we have $\|ax\| \leq |a|_p \|x\|$, $\|ax\| \geq |a|_p \|x\|$ or $\|ax\| = |a|_p \|x\|$, respectively. In particular, for every $E$ we have $\|ax\| \leq \|a\|_1 \|x\|$.

**Proof.** Let $h$ be as in (2.1). Then we have $ax \approx hx$. But $hx = \sum_{k=1}^n s_k P_k x$, where the summands have pairwise orthogonal supports, namely $P_k$. Therefore in the ‘convex’ case we have $\|ax\| = \|hx\| \leq (\sum_{k=1}^n \|s_k P_k x\|_p^p)^{\frac{1}{p}} = (\sum_{k=1}^n s_k^p)^{\frac{1}{p}} \|x\|$, where, as we recall, $(\sum_{k=1}^n s_k^p)^{\frac{1}{p}}$ is just $\|h\|_p$, that is $\|a\|_p$. The remaining cases are treated in a similar way. \hfill \Box

**Example 2.7.** We shall show that the maximal $P$–quantization of a given normed space $E$ is an $L^1$–space. (And thus every normed space can be made an $L^1$–space).

Indeed, consider orthogonal projections $P, Q \in B$ and the subspaces $F_1^p := \{ PaP; a \in F \}, F_1^Q := \{ QaQ; a \in F \}$ and $F_1^{P,Q} = \{ PaP + QaQ; a \in F \}$ in $F_1$. Clearly, we have $F_1^{P,Q} = F_1^p \oplus_1 F_1^Q \in F_1$, where ‘$\oplus_1$’ is a sign of the $\ell_1$–sum of normed spaces.

It is well known (and easy to check) that the operator $j : F_1 \to F_1^{P,Q} : a \mapsto PaP + QaQ$ is contractive (in fact, it is a norm 1 projection). Consider the operators $j \otimes_{pr} 1_E : F_1 \otimes_{pr} E \to F_1^{P,Q} \otimes_{pr} E$, and also
i ⊗_{pr} 1_E : \mathcal{F}_1^{P,Q} ⊗_{pr} E \to \mathcal{F}_1 ⊗_{pr} E$, where $i$ is the respective natural embedding. Both of them, being projective tensor product of contractive operators, are contractive themselves. But their composition is evidently the identity operator on \( \mathcal{F}_1^{P,Q} ⊗_{pr} E \). It follows that \( i ⊗_{pr} 1_E \) is an isometry (whereas \( j ⊗_{pr} 1_E \) is a strict coisometry).

Now suppose that \( u, v \in \mathcal{F}_1 ⊗_{pr} E \) have \( P \) and \( Q \) as their respective supports. Observe that for every \( w \in \mathcal{F}_1 ⊗_{pr} E \) the equality \( w = P \cdot w \cdot P + Q \cdot w \cdot Q \) means exactly that \( w \in \mathcal{F}_1^{P,Q} \). Consequently, elements \( u, v \) and \( u + v \) have the same norms in \( \mathcal{F}_1^{P,Q} ⊗_{pr} E \) as in \( \mathcal{F}_1 ⊗_{pr} E \).

Finally, recall the known connection between the operations ‘ \( \oplus_1 \) ’ and ‘ \( ⊗_{pr} \) ’. In our situation we have the isometric isomorphism \( I : \mathcal{F}_1^{P,Q} ⊗_{pr} E \to (\mathcal{F}_1^P ⊗_{pr} E) \oplus_1 (\mathcal{F}_1^Q ⊗_{pr} E) \), well defined by taking \((a + b) \otimes x\) to \( a \otimes x \oplus b \otimes x\).

But for \( u, v \) as of elements of \( \mathcal{F}_1^{P,Q} ⊗_{pr} E \) we see that \( I(u) \in \mathcal{F}_1^P ⊗_{pr} E \) and \( I(v) \in \mathcal{F}_1^Q ⊗_{pr} E \). Therefore

\[
\|u + v\| = \|I(u + v)\| = \|I(u) + I(v)\| = \|I(u)\| + \|I(v)\| = \|u\| + \|v\|,
\]

and we are done.

Note, however, that the \( PG \)--space \((p)E\) for \( E \neq \mathbb{C} \) and \( p > 1 \) is, generally speaking, not an \( L^p \)--space.

**Remark 2.8.** Recall that numerous examples, all of them concerning \( Q \)--spaces, are presented in the cited textbooks. They include the example, which is the most important in the whole theory of \( Q \)--spaces, being, in a sense, universal \[21\] \[10\]. This is the so-called concrete quantization of a space, consisting of operators. But we do not need this material in the present paper.

### 3. Completely bounded linear and bilinear operators

Suppose that we are given an operator \( \varphi : E \to F \) between linear spaces. The *amplification* of \( \varphi \) is the operator \( \varphi_\infty : \mathcal{F}E \to \mathcal{FF} \), well defined on elementary tensors by \( ax \mapsto a \varphi(x) \). Clearly, \( \varphi_\infty \) is a morphism of \( B \)-bimodules.

**Definition 3.1.** An operator \( \varphi \), connecting semi-normed \( PQ \)--spaces, is called *completely bounded*, respectively, *completely contractive*, if its amplification is bounded, respectively, contractive in the usual sense. We set \( \|\varphi\|_{cb} := \|\varphi_\infty\| \).
In a similar way we define the notions of a \textit{completely isometric operator} and of a \textit{completely isometric isomorphism}.

If $\varphi$ is bounded, being considered in the context of the respective underlying semi-normed spaces, we say that it is (just) \textit{bounded} and denote its respective operator semi-norm, as usual, by $\|\varphi\|$. Every completely bounded linear operator is obviously bounded, and we have $\|\varphi\| \leq \|\varphi\|_{cb}$.

Denote by $\mathcal{CB}(E, F)$ the subspace in $\mathcal{B}(E, F)$, consisting of completely bounded linear operators. It is a normed space with respect to the norm $\|\cdot\|_{cb}$.

Some linear operators between $PQ$–spaces that are bounded, are ‘automatically’ completely bounded. Here is an observation of that kind.

\textbf{Proposition 3.2.} Let $E$ be an $\mathcal{L}^p$–space or, more general, $p$–concave $PQ$–space for some $p \in [1, \infty]$. Then every bounded functional $f : E \to (q)\mathbb{C}$, where $q \geq p$, is completely bounded, and $\|f\|_{cb} := \|f\|_q$.

\textbf{Proof.} Take an arbitrary $u \in F_E$. Setting in (2.1) $a := f_\infty(u)$ and $v := S \cdot u \cdot T$, where $S, T$ are relevant unitary operators, we see that for some pairwise orthogonal rank 1 projections $P_k$ and $s_k \geq 0$ we have

\begin{equation}
\sum_{k=1}^n s_k P_k, \quad \|u\| = \|v\| \quad \text{and} \quad \|f_\infty(u)\|_q = \|f_\infty(v)\|_q = \left(\sum_{k=1}^n s_k^q\right)^{\frac{1}{q}}.
\end{equation}

Therefore it suffices to prove that $\left(\sum_{k=1}^n s_k^q\right)^{\frac{1}{q}} \leq \|f\|\|v\|$.

Denote by $\zeta$ the prime $n$-th root of 1 and set, for $m = 1, \ldots, n$, $W_m := \sum_{k=1}^n \zeta^{-mk} P_k$ and $W'_m := \sum_{k=1}^n \zeta^{mk} P_k$. Then a routine calculation shows that for all $a \in \mathcal{B}$ we have $\sum_{m=1}^n W'_m a W_m = n (\sum_{k=1}^n P_k a P_k)$. Hence, representing $v$ as a sum of elementary tensors, we see that $\sum_{m=1}^n W'_m \cdot v \cdot W_m = n(\sum_{k=1}^n P_k \cdot v \cdot P_k)$. From this we have

\begin{equation}
\left\| \sum_{k=1}^n P_k \cdot v \cdot P_k \right\| \leq \frac{1}{n} \sum_{m=1}^n \|W'_m \cdot v \cdot W_m\| \leq \frac{1}{n} \sum_{m=1}^n \|W'_m\| \|v\| \|W_m\| \leq \|v\|.
\end{equation}

Since $P_k b P_k$ is proportional to $P_k$ for all $b \in \mathcal{B}$ and $k = 1, \ldots, n$, we easily see that for these $k$ we have $P_k \cdot v \cdot P_k = P_k x_k$ for some $x_k \in E$.

Therefore, by (3.1), we have for all $k$ that

$s_k P_k = P_k f_\infty(v) P_k = f_\infty(P_k \cdot v \cdot P_k) = f_\infty(P_k x_k) = f(x_k) P_k,$
hence \( f(x_k) = s_k \), and consequently

\[ s_k \leq \|f\| \|x_k\| = \|f\| \|P_k x_k\| = \|f\| \|P_k \cdot v \cdot P_k\|. \]

Therefore, taking into account that \( E \) is \( p \)-concave, we have

\[
\left( \sum_{k=1}^{n} s_k^q \right)^{\frac{1}{q}} \leq \|f\| \left( \sum_{k=1}^{n} \|P_k \cdot v \cdot P_k\|^q \right)^{\frac{1}{q}} \leq \|f\| \left( \sum_{k=1}^{n} \|P_k \cdot v \cdot P_k\|^p \right)^{\frac{1}{p}} \leq \|f\| \| \sum_{k=1}^{n} P_k \cdot v \cdot P_k\|.
\]

It remains to apply (3.2). \( \square \)

In particular (cf. [7]), for every \( PQ \)-space \( E \) every bounded functional \( f : E \to \ell^\infty C \) is completely bounded, and \( \|f\|_{cb} := \|f\| \).

Note that the latter assertion immediately implies that for a normed space \( E \) the norm on \( FE \), given by \( \|u\| := \sup\{\|f_\infty(u)\|; f \in E^*; \|f\| \leq 1\} \), is the smallest among all norms of \( P \)-quantizations of \( E \). But this is exactly the norm on \( L_{\infty} \otimes in L^\infty \otimes in E \). This justifies the word ‘minimal’ for the latter norm (see above). Also we see that we got a \( Q \)-norm.

On the other hand, contrary to the situation with the space of all bounded operators, the space \( CB(E,F) \) can be very scanty. The following observation is taken from [21].

**Proposition 3.3.** Let \( E \) be \( p \)-convex, \( F \) be \( q \)-concave \( PQ \)-spaces, and \( p > q \). Then there is no non-zero completely bounded operators from \( E \) into \( F \).

**Proof.** Let \( \varphi : E \to F \) be an arbitrary non-zero operator; our task is to show that it is not completely bounded. Take \( x \in E \) with \( \varphi(x) \neq 0 \). Since \( p > q \), for every \( n \in \mathbb{N} \) there exist \( m \in \mathbb{N} \) such that \( m^{-\frac{1}{q}} > (n\|x\|/\|\varphi(x)\|)^{m^{-\frac{1}{q}}} \).

Take pairwise orthogonal rank 1 projections \( P_k; k = 1, ..., m \) and set \( u := \sum_k P_k x \in FE \); then \( \varphi_\infty(u) = \sum_k P_k \varphi(x) \in FF \). We see that elements \( P_k x \in FE \) as well as \( P_k \varphi(x) \in FF \) have pairwise orthogonal supports. Therefore we have

\[
\| \varphi_\infty(u) \| \geq \left( \sum_k \|P_k \varphi(x)\|^q \right)^{\frac{1}{q}} = \left( m \|\varphi(x)\|^q \right)^{\frac{1}{q}} = \|\varphi(x)\| m^{-\frac{1}{q}} >
\]
Since $n$ is arbitrary, this means that the operator $\varphi_\infty$ is not bounded. \hfill $\Box$

However, most of various known counter-examples (one of the earliest is due to Tomiyama \cite{22}) concern $Q$–spaces; see the textbooks cited above.

To amplify bilinear operators (in what follows, we shall say, for brevity, ‘bioperators’), we shall use a certain operation that imitate tensor product of operators on our Hilbert space $L$ but does not lead out of $L$. (Within the ‘matrix’ approach we would have to use the Kronecker product of matrices).

In what follows, the symbol $\otimes$ is used for the Hilbert tensor product of Hilbert spaces, as well as of bounded operators, acting on these spaces. By virtue of Riesz/Fisher Theorem, we can arbitrarily choose a unitary isomorphism $\iota : L \otimes L \to L$ and fix it throughout the whole paper. Following \cite{13}, for $\xi, \eta \in L$ we denote the vector $\iota(\xi \otimes \eta) \in L$ by $\xi \diamond \eta$, and for $a, b \in B$ we denote the operator $\iota(a \otimes b)\iota^{-1}$ on $L$ by $a \diamond b$; obviously, the latter is well defined by the equality $(a \diamond b)(\xi \diamond \eta) = a(\xi) \diamond b(\eta)$. Evidently, we have the identities

\begin{equation}
(a \diamond b)(c \diamond d) = ac \diamond bd, \quad \|\xi \diamond \eta\| = \|\xi\|\|\eta\| \quad \text{and} \quad \|a \diamond b\| = \|a\|\|b\|.
\end{equation}

Now suppose that we are given a bioperator $\mathcal{R} : E \times F \to G$ between linear spaces. Its \textit{amplification} is the bioperator $\mathcal{R}_\infty : \mathcal{F}E \times \mathcal{F}F \to \mathcal{FG}$, well defined on elementary tensors by $\mathcal{R}_\infty(ax, by) = (a \diamond b)\mathcal{R}(x, y)$.

\textbf{Remark 3.4.} We do not consider here another, different version of the amplification of a bioperator, that would lead us to the important notion of the Haagerup tensor product of $PQ$–spaces (cf. \cite{17} and, in the context of $Q$–spaces, \cite{11} \cite{3} and also the textbooks \cite{11} \cite{14}).

\textbf{Definition 3.5.} A bioperator $\mathcal{R}$, connecting semi-normed $PQ$–spaces, is called \textit{completely bounded}, respectively, \textit{completely contractive} if its amplification is bounded, respectively, contractive in the usual sense. We set $\|\mathcal{R}\|_{cb} := \|\mathcal{R}_\infty\|$. 

Note that, as it is easy to see, after restricting ourselves to $Q$–spaces and translating this definition back to the ‘matrix language’, we shall obtain
the standard definition of completely bounded (and completely contractive) bioperator between operator spaces (see [11, p.126]).

Here is another example of the ‘automatic complete boundedness’. If $E, F$ are $PQ$–spaces, and $f : E \to \mathbb{C}$, $g : F \to \mathbb{C}$ are bounded functionals, then the bilinear functional $f \times g : E \times F \to (\infty)\mathbb{C} : (x, y) \mapsto f(x)g(y)$ is completely bounded, and $\|f \times g\|_{cb} = \|f\|\|g\|$. This can be easily deduced from Proposition 3.2 with the help of the formula $(f \times g)_{\infty}(u, v) = f_{\infty}(u)\hat{\otimes}g_{\infty}(v), u \in FE, v \in FF$.

As a good exercise, we can mention the situation with the inner product bilinear functional $H \times H^{cc} \to (\infty)\mathbb{C} : (x, y) \mapsto \langle x, y \rangle$, where $H$ is a Hilbert space. It is completely contractive, if we endow both $H$ and $H^{cc}$ with the maximal $PQ$–norm (cf. Example 2.5), and it is not completely bounded, if we endow them with the minimal $Q$–norm.

4. Further examples of proto-quantum spaces and related bilinear operators

We introduce here several examples of $PQ$–spaces. Later some of them will show especially good behavior as tensor factors.

**Example 4.1.** Let $(X, \mu)$ be a measure space and $F$ be an arbitrary $PQ$–space. We want to endow the normed space $L^p(X, F); 1 \leq p \leq \infty$ of relevant $F$-valued measurable functions on $X$ with a $PQ$–norm.

As a preliminary step, consider the (non-completed) normed space $L^p(X, FF)$ and note that it is a $B$-bimodule with the outer multiplications defined by

$$[a \cdot \bar{x}](t) := a \cdot [\bar{x}(t)] \quad \text{and} \quad [\bar{x} \cdot b](t) := [\bar{x}(t)] \cdot b; \quad a, b \in B, \bar{x} \in L^p(X, FF), t \in X.$$

A routine calculation shows that this bimodule is contractive.

Now consider the operator $\alpha : \mathcal{F}(L^p(X, F)) \to L^p(X, FF)$, well defined on elementary tensors by taking $ax$ to the $FF$-valued function $\bar{x}(t) := a(x(t))$. Introduce the semi-norm on $\mathcal{F}(L^p(X, F))$ by setting $\|u\| := \|\alpha(u)\|$. Observe that $\alpha$ is a $B$-bimodule morphism: to show this, it is sufficient to consider respective elementary tensors.

Thus, there is an isometric morphism of the semi-normed bimodule $\mathcal{F}(L^p(X, F))$ into a contractive $B$-bimodule. It follows immediately that the former bimodule is itself contractive, hence the introduced semi-norm on
\( \mathcal{F}(L_p(X, F)) \) is a \( PQ \)-semi-norm on \( L_p(X, F) \). Further, for an arbitrary rank 1 operator \( Q \in \mathcal{F} \); \( \|Q\| = 1 \) and \( x \in L_p(X, F) \) we have \( \|Q[x(t)]\| = \|x(t)\| \) for all \( t \in X \). Therefore for \( Qx \in \mathcal{F}(L_p(X, F)) \) we easily have \( \|Qx\| = \|x\| \).

This means that the underlying semi-normed space of the constructed \( PQ \)-space is the ‘classical’ \( L_p(X, F) \). Consequently, Proposition 2.4 guarantees that the \( PQ \)-semi-norm on \( L_p(X, F) \) is actually a norm.

It is easy to verify that the \( PQ \)-space \( L_p(X, F) \) is \( p \)-convex or \( p \)-concave provided \( F \) has the same property. In particular, if \( F \) is an \( L_p \)-space, then \( L_p(X, F) \) is also an \( L_p \)-space. Note also that the \( PQ \)-space \( L_p(X, F) \) is not a \( Q \)-space whenever \( p < \infty \), and \( X \) is not a single atom.

**Example 4.2.** Now we want to introduce a \( P \)-quantization of the ‘classical’ tensor product \( E \otimes_{pr} F \) of normed spaces, when one of tensor factors, say, to be definite, \( F \), is a \( PQ \)-space.

Consider the linear isomorphism \( \beta : \mathcal{F}(E \otimes F) \rightarrow E \otimes_{pr}(\mathcal{F}F) \), well defined by taking \( a(x \otimes y) \) to \( x \otimes ay \), and introduce a norm on \( \mathcal{F}(E \otimes F) \) by setting \( \|U\| := \|\beta(U)\| \). The space \( E \otimes_{pr}(\mathcal{F}F) \), as a projective tensor product of a normed space and a contractive \( \mathcal{B} \)-bimodule, has itself a standard structure of a contractive \( \mathcal{B} \)-bimodule. The same, because \( \beta \) is obviously a \( \mathcal{B} \)-bimodule morphism, is true with \( \mathcal{F}(E \otimes F) \). Thus \( E \otimes F \) becomes a \( PQ \)-space, and we must show that its underlying normed space is exactly \( E \otimes_{pr} F \).

Denote the norm on \( E \otimes_{pr} F \) and on \( E \otimes_{pr}(\mathcal{F}F) \) by \( \|\cdot\|_{pr} \), and the introduced \( PQ \)-norm, as well as the norm of the respective underlying space, just by \( \|\cdot\| \).

Take an arbitrary \( u \in E \otimes F \). It is easy to check that the norm on the underlying space in question is a cross-norm, hence \( \|u\| \leq \|u\|_{pr} \). Therefore our task is to show that for a rank 1 projection \( P \in \mathcal{F} \) we have \( \|Pu\| \geq \|u\|_{pr} \).

Take an arbitrary representation of \( \beta(Pu) \) as \( \sum_{k=1}^{n} x_k \otimes w_k; x_k \in F, w_k \in FF \). Obviously, \( P \cdot w_k \cdot P = Py_k \) for some \( y_k \in F \); \( k = 1, \ldots, n \). Therefore \( \sum_{k=1}^{n} \|x_k\|\|w_k\| \geq \sum_{k=1}^{n} \|x_k\|\|P \cdot w_k \cdot P\| = \sum_{k=1}^{n} \|x_k\|\|y_k\|. \)

But we have \( \beta(Pu) = P \cdot \beta(Pu) \cdot P = \sum_{k=1}^{n} x_k \otimes P \cdot w_k \cdot P = \beta(P \sum_{k=1}^{n} x_k \otimes y_k) \).

It follows that \( u = \sum_{k=1}^{n} x_k \otimes y_k \) and consequently, \( \sum_{k=1}^{n} \|x_k\|\|w_k\| \geq \|u\|_{pr} \).

From this, by the definition of the norm on \( E \otimes_{pr}(\mathcal{F}F) \), we have \( \|Pu\| = \|\beta(Pu)\| \geq \|u\|_{pr} \).

The introduced \( PQ \)-spaces participate in some bioperators that we shall essentially use. Their study needs a certain extended version of the operation
‘\(\diamond\)’. Namely, if \(E\) is a linear space, \(a \in \mathcal{F}\) and \(u \in \mathcal{FE}\), then we introduce in \(\mathcal{FE}\) the elements, denoted by \(a\diamond u\) and \(u\diamond a\). They are well defined, if we set \(a\diamond(\sum_k b_k x_k) := \sum_k (a\diamond b_k)x\) and \((\sum_k b_k x_k)\diamond a := \sum_k (b_k\diamond a)x_k\). We shall use the following properties of such an operation that may have an independent interest.

As a preparatory step, for a given \(e \in L; \|e\| = 1\) we introduce the operator \(S\) on \(L\), acting as \(\zeta \mapsto e\diamond \zeta\); it is, of course, an isometry. It is easy to verify that for all \(b \in \mathcal{F}\) and \(P := e \circ e\) we have

\[
(4.1) \quad b = S^*(P\diamond b)S \quad \text{and} \quad P\diamond b = SbS^*.
\]

**Proposition 4.3.** Let \(E\) be a \(PQ\)-space, \(u \in \mathcal{FE}\). Then

(i) for every \(a \in \mathcal{F}\) we have \(\|a\diamond u\| = \|u\diamond a\|

(ii) for every \(Q \in \mathcal{F}\) of rank 1 we have \(\|Q\diamond u\| = \|Q\|\|u\|\).

(iii) for an arbitrary \(a \in \mathcal{F}\) we have \(\|a\diamond u\| \leq \|a\|_p\|u\|\) provided \(E\) is \(p\)-convex, \(\|a\diamond u\| \geq \|a\|_p\|u\|\) provided \(E\) is \(p\)-concave and, as a corollary, \(\|a\diamond u\| = \|a\|_p\|u\|\) provided \(E \in \mathcal{L}_p\). In particular, for all \(E\) we have \(\|a\diamond u\| \leq \|a\|_1\|u\|\).

**Proof.** (i) Consider the unitary operator \(\triangle\) on \(L\), well defined by taking \(\xi\diamond \eta;\xi, \eta \in L\) to \(\eta\diamond \xi\). Obviously, for every \(a, b \in \mathcal{F}\) we have \(b\diamond a = \triangle(a\diamond b)\triangle\). From this we easily deduce that for every \(a \in \mathcal{F}\) we have \(\triangle(a\diamond u)\cdot \triangle = u\diamond a\). It remains to recall that the \(B\)-bimodule \(\mathcal{FE}\) is contractive.

(ii) We can assume that \(\|Q\| = 1\). Then \(Q = \xi\diamond \eta\) for some \(\xi, \eta \in L; \|\xi\| = \|\eta\| = 1\). Then, for \(e\) and \(P\) as above, the formulae (4.1), being combined with the equalities \(Q = R_1PR_2\) and \(P = R_1^*Q^R_2^*\), where \(R_1 := \xi \circ e\) and \(R_2 := e \circ \eta\), imply that

\[
Q\diamond b = (R_1\diamond 1)SbS^*(R_2\diamond 1) \quad \text{and} \quad b = S^*(R_1^*\diamond 1)(q\diamond b)(R_2^*\diamond 1)S.
\]

Therefore, representing \(u\) as a sum of elementary tensors, we obtain that

\[
Q\diamond u = [(R_1\diamond 1)S] \cdot u \cdot [S^*(R_2\diamond 1)] \quad \text{and} \quad u = [S^*(R_1^*\diamond 1)] \cdot (q\diamond u) \cdot [(R_2^*\diamond 1)S].
\]

But all operators, participating in these equalities, have norm 1, and the bimodule \(\mathcal{FE}\) is contractive. Consequently, we have the estimate \(\|Q\diamond u\| \leq \|u\|\) and its inverse.

(iii) By (2.1), for our \(a\) there exist \(h, P_k\) and \(s_k\) with mentioned properties. If \(S, T\) are relevant operators, we have \(a\diamond u = (S\diamond 1) \cdot (h\diamond u) \cdot (T\diamond 1)\),
hence \( \|a \triangle u\| = \|h \triangle u\| = \| \sum_k s_k P_k \triangle u\| \). Further, the elements \( P_k \triangle u \) have pairwise orthogonal supports, namely \( P_k \triangle 1 \). Combining this with (ii) and remembering, what is \( \|h\|_p \), we have, in ‘convex’ case, that \( \|a \triangle u\| \leq (\sum_{k=1}^n (s_k \|u\|^p)^{\frac{1}{p}})^\frac{1}{p} = \|a\|_p \|u\| \). Similarly, in the ‘concave’ case, we obtain the inverse estimate. \( \square \)

Here are several applications. In the following proposition \( p \in [1, \infty] \), and we consider \( L_p(X, F) \), where \( F \) is a given \( PQ \)-space, and also \( L_p(X, \langle \rangle) \) as \( PQ \)-spaces according to Example 4.1.

**Proposition 4.4.** Let \( F \) be \( p \)-convex. Then the bioperator \( \mathcal{R} : L_p(X, \langle \rangle) \times F \rightarrow L_p(X, F) \), taking a pair \((z, y)\) to the \( F \)-valued function \( t \mapsto z(t)y; t \in X \), is completely contractive.

**Proof.** Recall the isometric operator \( \alpha : \mathcal{F}(L_p(X, F)) \rightarrow L_p(X, \mathcal{FF}) \) and distinguish its particular case \( \alpha_0 : \mathcal{F}(L_p(X), \langle \rangle) \rightarrow L_p(X, \mathcal{F}p) \). Also introduce the bioperator \( \mathcal{S} : L_p(X, \mathcal{F}p) \times \mathcal{FF} \rightarrow L_p(X, \mathcal{FF}) \), taking a pair \((\omega, v)\) to the \( \mathcal{FF} \)-valued function \( t \mapsto \omega(t) \triangle v; t \in X \). Consider the diagram

\[
\begin{array}{ccc}
\mathcal{F}(L_p(X, \langle \rangle)) \times \mathcal{FF} & \xrightarrow{\mathcal{R}_\infty} & \mathcal{F}(L_p(X, F)) \\
\alpha_0 \times 1_{\mathcal{FF}} & \downarrow & \alpha \\
L_p(X, \mathcal{F}p) \times \mathcal{FF} & \xrightarrow{\mathcal{S}} & L_p(X, \mathcal{FF})
\end{array}
\]

It is commutative: this is easy to check on elementary tensors in the respective amplifications. Therefore, for \( w \in \mathcal{F}(L_p(X, \langle \rangle)) \) and \( v \in \mathcal{FE} \) we have

\[
(4.2) \quad \|\mathcal{R}_\infty(w, v)\| = \|\alpha(\mathcal{R}_\infty(w, v))\| = \|\mathcal{S}(\alpha_0(w), v)\|.
\]

But it follows from Proposition 4.3(iii) that for all \( \omega \in L_p(X, \mathcal{F}p), v \in \mathcal{FF} \) we have

\[
\|\mathcal{S}(\omega, v)\| = \left( \int_X (\|\omega(t) \triangle v\|^p dt) \right)^{\frac{1}{p}} \leq \left( \int_X \left( \|\omega(t)\|_p \|v\|\right)^p dt \right)^{\frac{1}{p}} = \|\omega\|_p \|v\|.
\]

Setting in (4.2) \( \omega := \alpha_0(w) \) and remembering that \( \alpha_0 \) is an isometry, we obtain that \( \|\mathcal{R}_\infty(w, v)\| \leq \|w\|_p \|v\| \). \( \square \)

In the following proposition \( E \) is a normed space, \( \langle \rangle \) \( p \) is its \( P \)-quantization from Example 2.2, \( F \) and \( E \otimes_{pr} F \) are \( PQ \)-spaces from Example 4.2.

**Proposition 4.5.** Let \( F \) be \( p \)-convex. Then the canonical bioperator \( \vartheta : \langle \rangle \times F \rightarrow E \otimes_{pr} F : (x, y) \mapsto x \otimes y, \) is completely contractive. In particular, the bioperator \( \mathcal{R} : \langle \rangle \mathcal{C} \times F \rightarrow F : (\lambda, x) \mapsto \lambda x \) is completely contractive.
Proof. Consider the trilinear operator $\mathcal{T} : E \times \mathcal{F} \times \mathcal{F} F \to E \otimes (\mathcal{F} F) : (x, a, v) \mapsto x \otimes (a \diamond v)$. It gives rise to the bioperator $\mathcal{S} : (E \otimes \mathcal{F}) \times \mathcal{F} F \to E \otimes \mathcal{F} F : (x \otimes a, v) \mapsto x \otimes (a \diamond v)$. Being considered with the domain $E \times \mathcal{F}_p \times \mathcal{F} F$ and the range $E \otimes_{pr} (\mathcal{F} F)$, $\mathcal{T}$ is contractive by virtue of Proposition 4.3(iii); therefore $\mathcal{S}$ is contractive, taken with the domain $(E \otimes_{pr} \mathcal{F}) \times \mathcal{F} F$ and the same range. Now recall the isometric operator $\beta : \mathcal{F}(E \otimes_{pr} F) \to E \otimes_{pr} \mathcal{F} F$ and distinguish its particular case, the “flip” $\beta_0 : \mathcal{F}(p) E \to E \otimes_{pr} \mathcal{F}_p$. Consider the diagram

$$
\begin{array}{ccc}
\mathcal{F}(p) E \times \mathcal{F} F & \xrightarrow{\vartheta_{\infty}} & \mathcal{F}(E \otimes_{pr} F), \\
\beta_0 \times 1_{\mathcal{F} F} \downarrow & & \downarrow \quad \gamma \\
(E \otimes_{pr} \mathcal{F}_p) \times \mathcal{F} F & \xrightarrow{\mathcal{S}} & E \otimes_{pr} \mathcal{F} F
\end{array}
$$

which is obviously commutative. Therefore for $w \in \mathcal{F}(p) E$ and $v \in \mathcal{F} F$ we have

$$
\|\vartheta_{\infty}(w, v)\| = \|\beta(\vartheta_{\infty}(w, v))\| = \|\mathcal{S}(\beta_0(w), v)\| \leq \|\beta_0(w)\| \|v\| = \|w\| \|v\|.
$$

Our third example of a completely contractive bioperator needs some preparatory observation which must be well known in its equivalent version for the ‘genuine’ Hilbert tensor product of operators.

**Proposition 4.6.** For $a, b \in \mathcal{F}_p$ we have $\|a \diamond b\|_p = \|a\|_p \|b\|_p$.

*Proof.* Take unitary operators $S, T, S', T' \in \mathcal{B}$ such that $SaT = \sum_{k=1}^n s_k P_k$ and $S'bT' = \sum_{t} t_l Q_k$, where $P_k; k = 1, ..., n$, as well as $Q_l; l = 1, ..., m$, is a family of pairwise orthogonal rank 1 projections. Then $\|a\|_p = (\sum_{k=1}^n s_k^p)^{\frac{1}{p}}$ and $\|b\|_p = (\sum_{l=1}^m t_l^p)^{\frac{1}{p}}$. Further, $(S \diamond S')(a \diamond b)(T \diamond T') = (\sum_{k=1}^n s_k P_k) \diamond (\sum_{l=1}^m t_l Q_k)$. Since $S \diamond S', T \diamond T'$ are unitary operators, this implies that

$$
\|a \diamond b\|_p = \|[(\sum_{k=1}^n s_k P_k) \diamond (\sum_{l=1}^m t_l Q_k)]\|_p = \|\sum_{k,l} s_k t_l P_k \diamond Q_l\|_p.
$$

But, since all $P_k \diamond Q_l$ are pairwise orthogonal rank 1 projections, the last number is $(\sum_{k,l} (s_k t_l)^p)^{\frac{1}{p}} = (\sum_{k=1}^n s_k^p)^{\frac{1}{p}} (\sum_{l=1}^m t_l^p)^{\frac{1}{p}} = \|a\|_p \|b\|_p$. \hfill \Box

**Proposition 4.7.** For every $p, q \in [1, \infty]$ and $r := \max\{p, q\}$ the bioperator $\mathcal{R} : (p) E \times (q) F \to (r)(E \otimes_{pr} F) : (x, y) \mapsto x \otimes y$ is completely contractive.
Proof. Take \( u \in \mathcal{F}((p)E), v \in \mathcal{F}((q)F) \) and choose \( \varepsilon > 0 \). By definition of \( \otimes_{pr} \), there exist representations of \( u \) as \( \sum_{k=1}^{n} a_k x_k \) and \( v \) as \( \sum_{l=1}^{m} b_l y_l \) such that \( \sum_{k=1}^{n} \|a_k\|_p \|x_k\| < \|u\|_{pr} + \varepsilon \) and \( \sum_{l=1}^{m} \|b_l\|_q \|y_l\| < \|v\|_{pr} + \varepsilon \). We have \( R_{\infty}(u, v) = \sum_{k,l} a_k \diamond b_l (x_k \otimes y_l) \); therefore, by the previous proposition,

\[
\| R(u, v) \| \leq \sum_{k,l} \|a_k \diamond b_l\|_r \|x_k\| \|y_l\| \leq \left( \sum_{k=1}^{n} \|a_k\|_r \|x_k\| \right) \left( \sum_{l=1}^{m} \|b_l\|_r \|y_l\| \right) \leq \left( \sum_{k=1}^{n} \|a_k\|_p \|x_k\| \right) \left( \sum_{l=1}^{m} \|b_l\|_q \|y_l\| \right) < (\|u\|_{pr} + \varepsilon)(\|v\|_{pr} + \varepsilon).
\]

Since \( \varepsilon \) is arbitrary, this implies that \( \| R_{\infty}(u, v) \| \leq \|u\| \|v\| \). \( \Box \)

5. The projective tensor product ‘ \( \otimes_{\text{pop}} \)’, its definition and the existence theorem

A widespread point of view, inherited from pure algebra, is that the raison d’etre of a ‘good’ tensor product is that it linearizes some respective ‘good’ class of bioperators (cf. [4, pp. 3-5]). As to the theory of PG- (= matricially normed) spaces, one could show that the Haagerup tensor product, introduced in [7], linearizes what was called in [11] multiplicatively bounded bioperators. But this is outside the scope of the present paper. Here we shall introduce another, ‘projective’ tensor product of PG–spaces that linearizes what was called in the cited textbook, as well as in this paper, completely bounded bioperators.

Fix, for a time, two normed PQ–spaces \( E \) and \( F \).

Definition 5.1. A pair \((\Theta, \theta)\), consisting of a normed PQ–space \( \Theta \) and a completely contractive bioperator \( \theta : E \times F \to \Theta \), is called non-completed proto-operator-projective tensor product of \( E \) and \( F \) or, for brevity, projective tensor product of \( E \) and \( F \) if, for every completely bounded bioperator \( R : E \times F \to G \), where \( G \) is a PQ–space, there exists a unique completely bounded operator \( R : \Theta \to G \) such that the diagram

\[
\begin{array}{ccc}
E \times F & \xrightarrow{\theta} & \Theta \\
\downarrow R & & \downarrow R \\
G
\end{array}
\]

is commutative, and moreover \( \|R\|_{cb} = \|R\|_{cb} \).
Uniqueness, in a proper sense, of such a pair is a particular case of the general-categorical observation, concerning the uniqueness of the initial object in a category (cf.\cite{18}). We shall prove the existence of such a pair, displaying its explicit construction.

First, we need an additional version of the operation ‘\( \Diamond \)’, this time connecting elements of amplifications. Namely, for \( u \in \mathcal{F}E, v \in \mathcal{F}F \) we set
\[
u \Diamond \nu := \vartheta_{\infty}(u, v) \in \mathcal{F}(E \otimes F),
\]
where \( \vartheta : E \times F \to E \otimes F : (x, y) \mapsto x \otimes y \) is the canonical bilinear operator. In particular, for elementary tensors we have
\[
a \Diamond b = (a \cdot u \cdot c) \Diamond (b \cdot v \cdot d).
\]
(5.1)

One can immediately verify this formula on elementary tensors.

It is easy to show that every \( U \in \mathcal{F}(E \otimes F) \) can be represented as
\[
\sum_{k=1}^{n} a_k \cdot (u_k \Diamond v_k) \cdot b_k
\]
for some \( a_k, b_k \in \mathcal{B}, u_k \in \mathcal{F}E, v_k \in \mathcal{F}F, k = 1, \ldots, n \) (see details in \cite{14} Section 7.2)). This implies that the operator \( \mathcal{B} \otimes \mathcal{F}E \otimes \mathcal{F}F \otimes \mathcal{B} \to \mathcal{F}(E \otimes F) \), associated with the 4-linear operator \( (a, u, v, b) \mapsto a \cdot (u \Diamond v) \cdot b \), is surjective. Thus \( \mathcal{F}(E \otimes F) \) can be endowed with the semi-norm of the respective quotient space of \( \mathcal{B} \otimes \mathcal{F}E \otimes \mathcal{F}F \otimes \mathcal{B} \), denoted by \( \| \cdot \|_{\text{pop}} \). In other words, for \( U \in \mathcal{F}(E \otimes F) \) we have
\[
\| U \|_{\text{pop}} := \inf \left\{ \sum_{k=1}^{n} \| a_k \| \| u_k \| \| v_k \| \| b_k \| \right\},
\]
(5.3)

where the infimum is taken over all possible representations of \( U \) in the form given by (5.2).

Now observe that \( \mathcal{B} \otimes \mathcal{F}E \otimes \mathcal{F}F \otimes \mathcal{B} \) is a contractive \( \mathcal{B} \)-bimodule, being considered as a tensor product of the left \( \mathcal{B} \)-module \( \mathcal{B} \) with the linear space \( \mathcal{F}E \otimes \mathcal{F}F \) and the right \( \mathcal{B} \)-module \( \mathcal{B} \). Therefore \( \mathcal{F}(E \otimes F) \) is the image of a contractive \( \mathcal{B} \)-bimodule with respect to a quotient map of semi-normed spaces. Since the latter map is obviously a bimodule morphism, we easily obtain that the bimodule \( (\mathcal{F}(E \otimes F), \| \cdot \|_{\text{pop}}) \) is also contractive.

We see that \( \| \cdot \|_{\text{pop}} \) is a \( PQ \)-semi-norm on \( E \otimes F \). Denote the respective semi-normed \( PQ \)-space by \( E \otimes_{\text{pop}} F \).

Finally, note that if \( \mathcal{R} : E \times F \to G \) is a bioperator, and \( R : E \otimes F \to G \) is the associated linear operator, then we obviously have the formula
\[
R_{\infty}(u \Diamond v) = \mathcal{R}_{\infty}(u, v).
\]
(5.4)
**Theorem 5.2.** (Existence theorem). The pair \((E \otimes_{\text{pop}} F, \vartheta)\) is a non-completed projective tensor product of \(E\) and \(F\).

We prefer to give a self-contained proof of the theorem, despite some of its parts (not all) resemble with what was said in [14] under the assumption that we deal with quantum spaces.

**Proof.** First, for arbitrary \(u \in F, v \in F\) we have, of course, \(u \diamond v = 1 \cdot (u \diamond v) \cdot 1\). Therefore the bioperator \(\vartheta\), considered with values in \(E \otimes_{\text{pop}} F\), is completely contractive, or, equivalently, we have

\[
\|u \diamond v\|_{\text{pop}} \leq \|u\| \|v\|.
\]

(5.5)

Now let \(G\) be a \(PQ\)-space, \(\mathcal{R} : E \times F \to G\) a completely bounded bioperator, and \(R : E \otimes_{\text{pop}} F \to G\) the associated linear operator. We want to show that \(R\) is completely bounded and that \(\|R\|_{cb} = \|R\|_{cb}\).

Take \(U \in F(E \otimes_{\text{pop}} F)\) and represent it as in (5.2). Since \(R_{\infty}\) is a \(B\)-bimodule morphism, we have by (5.4) that \(R_{\infty}(U) = \sum_{k=1}^{n} a_k \mathcal{R}_{\infty}(u_k, v_k) b_k\), hence \(\|R_{\infty}(U)\| \leq \|\mathcal{R}\|_{cb} \sum_{k=1}^{n} \|a_k\| \|u_k\| \|v_k\| \|b_k\|\). Therefore the definition of \(\|\cdot\|_{\text{pop}}\) implies that \(R\) is completely bounded, together with \(\mathcal{R}\), and \(\|R\|_{cb} \leq \|\mathcal{R}\|_{cb}\). The inverse estimate follows from the inequality \(\|R_{\infty}(u, v)\| \leq \|R_{\infty}\| \|u\| \|v\|\), which, in its turn, immediately follows from (5.4) and (5.5).

Now consider the diagram from Definition 5.1 with \(E \otimes_{\text{pop}} F\) and \(\vartheta\) in the capacity of \(\Theta\) and \(\theta\), respectively. It is known from linear algebra, that \(R\) is the only linear operator, making the diagram commutative. Thus we see that the pair \((E \otimes_{\text{pop}} F, \vartheta)\) satisfies almost all requirements given in Definition 5.1. The only remaining thing is to show that the semi-norm \(\|\cdot\|_{\text{pop}}\) is actually a norm.

By Proposition 2.4, for this aim it is sufficient to show that, for every non-zero elementary tensor \(Qw\), where \(Q\) is a rank 1 operator of norm 1 and \(w \in E \otimes_{\text{pop}} F, w \neq 0\), we have \(\|Qw\|_{\text{pop}} \neq 0\). Since \(E\) and \(F\) are normed spaces, there exist bounded functionals \(f : E \to \mathbb{C}, g : F \to \mathbb{C}\) such that for \(f \otimes g : E \otimes F \to \mathbb{C}\) we have \((f \otimes g)w \neq 0\). As we know from the previous section, the bilinear functional \(\mathcal{R} := f \times g : E \times F \to \mathcal{C}\) is completely bounded. Therefore, choosing \(G := \mathcal{C}\), we see that the associated linear functional, that is \(f \otimes g : E \otimes_{\text{pop}} F \to \mathcal{C}\), is also completely bounded.

Hence, we have

\[
\|f \otimes g\| \cdot \|w\|_{\text{pop}} = \|Q[(f \otimes g)w]\| \leq \|f \otimes g\|_{cb} \|Qw\|_{\text{pop}}.
\]
Therefore \( \|Qw\|_{\text{pop}} \neq 0 \) since \((f \otimes g)w \neq 0\).

Note that in the underlying space of \(E \otimes_{\text{pop}} F\) we have

\[
(5.6) \quad \|x \otimes y\| \leq \|x\|\|y\| \quad \text{for all } x \in E, y \in F.
\]

Indeed, take two operators \(P, Q \in \mathcal{F}\) of rank 1 and of norm 1. We see that \(\|P \hat{\otimes} Q\| = 1\) and that \(P \hat{\otimes} Q\) has also rank 1. Therefore, \(\|x \otimes y\| = \|(P \hat{\otimes} Q)x \otimes y\|_{\text{pop}} = \|Px \hat{\otimes} Qy\|\). It remains to use (5.5).

(In fact, in (5.6), as well as in (5.5), we have the exact equality, but we shall not discuss it now).

So far, we spoke about general (normed) \(PQ\)-spaces. But their tensor product has a natural analogue in the context of complete or Banach \(PQ\)-spaces. The latter are, by definition, \(PQ\)-spaces with complete underlying normed spaces. As in the ‘classical’ context, for every \(PQ\)-space \(E\) there exists its completion, which is defined as a pair \((\overline{E}, i : E \to \overline{E})\), consisting of a complete \(PQ\)-space and a completely isometric operator, such that the same pair, considered for respective underlying spaces and operators, is the ‘classical’ completion of \(E\) as of a normed space. The proof of the respective existence theorem repeats word by word the simple argument given in [14, Chapter 4] for \(Q\)-spaces. We only recall that the norm on \(\mathcal{F}E\) is introduced with the help of the natural embedding of \(\mathcal{F}E\) into \(\overline{\mathcal{F}E}\), the ‘classical’ completion of \(\mathcal{F}E\).

It is easy to observe that the characteristic universal property of the ‘classical’ completion has its proto-quantum version (ibidem). Namely, if \((\overline{E}, i)\) is the completion of a \(PQ\)-space \(E, F\) a \(PQ\)-space and \(\varphi : E \to F\) is a completely bounded operator, then there exists a unique completely bounded operator \(\overline{\varphi} : \overline{E} \to \overline{F}\) that extends, in the obvious sense, \(\varphi\). Moreover, we have \(\|\overline{\varphi}\|_{\text{cb}} = \|\varphi\|_{\text{cb}}\).

Let us distinguish the following fact that will be useful. Its proof repeats word by word the argument in Proposition 4.8 in [14].

**Proposition 5.3.** Let \(\varphi : E \to F\) be a completely isometric isomorphism between \(PQ\)-spaces. Then its continuous extension \(\overline{\varphi} : \overline{E} \to \overline{F}\) is also a completely isometric isomorphism.

Now we can speak of the completed projective tensor product of two \(PQ\)-spaces. Its definition repeats Definition 5.1, but, what is essential, with the following difference: \(\Theta\) and \(G\) are supposed to be complete. Using the
universal property of the completion, we immediately see that the completed projective tensor product of $PQ$-spaces $E$ and $F$ exists: it is the pair $(\hat{E} \hat{\otimes}_{\text{pop}} F, \hat{\vartheta})$, where $\hat{E} \hat{\otimes}_{\text{pop}} F$ is the completion of the $PQ$-space $E \otimes_{\text{pop}} F$, and $\hat{\vartheta}$ acts as $\vartheta$, but with range $E \hat{\otimes}_{\text{pop}} F$.

6. Tensoring by $L_1(\cdot)$, and some other computation

In this section we show that for certain concrete tensor factors their projective tensor product also becomes something concrete and transparent. We shall see that the behavior of this tensor product resembles the behavior of the projective tensor product in the classical context.

Denote the completion of the $PQ$-space $E \otimes_{\text{pr}} F$ from Example 4.2 by $E \hat{\otimes}_{\text{pr}} F$. Clearly, it is a $P$-quantization of the ‘classical’ completed projective tensor product, denoted also by $E \hat{\otimes}_{\text{p}} F$; it will not create a confusion.

**Theorem 6.1.** Let $E$ be a normed space, $F$ a $PQ$-space, $p \in [1, \infty]$, $(p)E$ the $PQ$-space from Example 2.5, and $E \otimes_{\text{pr}} F$ the $PQ$-space from Example 4.2. Suppose that $F$ is $p$-convex. Then there exists a completely isometric isomorphism $I : (p)E \otimes_{\text{pop}} F \to E \otimes_{\text{pr}} F$, acting as the identity operator on the common underlying linear space of our $PQ$-spaces. As a corollary (see Proposition 5.3), there exists a completely isometric isomorphism $\hat{I} : (p)E \hat{\otimes}_{\text{pop}} F \to E \hat{\otimes}_{\text{pr}} F$, which is the extension by continuity of $I$.

**Proof.** Consider the canonical bioperator $\vartheta : (p)E \times F \to E \otimes_{\text{pr}} F$. Since the $PQ$-space is $p$-convex, it gives rise, by virtue of Proposition 4.5, to the completely contractive operator $I$, acting as in the formulation. Therefore it is sufficient to show that for every $U \in F(E \otimes F)$ its norm in $F[(p)E \otimes_{\text{pop}} F]$ is not greater than $\|I_\infty(U)\|$ or, equivalently, than the norm of $\beta(U)$ in $E \otimes_{\text{pr}} FF$.

Fix $U$ and choose $\varepsilon > 0$; then there exists a representation $\beta(U) = \sum_{k=1}^{n} x_k \otimes v_k \in E, v_k \in FF$ such that $\sum_{k=1}^{n} \|x_k\| \|v_k\| < \|\beta(U)\| + \varepsilon = \|I_\infty(U)\| + \varepsilon$.

Now choose an arbitrary rank one projection $P \in F$ and set $V := \sum_{k=1}^{n} Px_k \otimes v_k \in F(E \otimes F)$. By (4.1), there exists an isometry $S \in B$ such that $S^*(P \hat{\otimes} a)S = a$ for every $a \in F$. Representing every $v_k$ as a sum of elementary tensors, we easily see that $\beta(S^* \cdot V \cdot S) = \sum_{k=1}^{n} x_k \otimes v_k = \beta(U)$. Therefore $U = S^* \cdot V \cdot S$, hence $\|U\|_{\text{pop}} \leq \|V\|_{\text{pop}}$. But by (5.5) we have
\[\|V\|_{\text{pop}} \leq \sum_{k=1}^n \|P x_k\| \|v_k\| = \sum_{k=1}^n \|x_k\| \|v_k\|, \text{ and consequently } \|U\|_{\text{pop}} \leq \|I_\infty(U)\| + \varepsilon. \text{ Since such an estimate holds for every } \varepsilon > 0, \text{ we are done.} \]

**Corollary 6.2.** With \( p \) and \( F \) as above, there exists a completely isometric isomorphism \( I : \langle p \rangle \mathcal{C} \otimes_{\text{pop}} F \to F \), acting as \( \lambda \otimes x \mapsto \lambda x \).

In its turn, this assertion, since \( (q)\mathcal{C} \) is \( p \)-convex provided \( q < p \), implies

**Corollary 6.3.** \( \langle p \rangle \mathcal{C} \otimes_{\text{pop}} \langle q \rangle \mathcal{C} = \langle r \rangle \mathcal{C}, \text{ where } r = \max\{p, q\}. \)

**Remark 6.4.** The projective tensor product of two \( \mathcal{L}^p \)-spaces is not bound to be again an \( \mathcal{L}^p \)-space. Indeed, consider the projective tensor square of the \( \mathcal{L}^2 \)-space \( \ell_2(\mathbb{C}^2) \) and the elements \( Pe_1, Qe_2 \in \mathcal{F}(\ell_2(\mathbb{C}^2)) \), where \( P, Q \) are orthogonal projections in \( \mathcal{F} \) (i.e. \( PQ = 0 \)), and \( e_1, e_2 \) are orts in \( \ell_2 \). Then it is not difficult to show that, despite our elements have orthogonal supports, we have that \( \|Pe_1 + Qe_2\|_{\text{pop}} = 2 \), whereas \( (\|Pe_1\|^2 + \|Qe_2\|^2)^{\frac{1}{2}} = \sqrt{2} \). Incidentally, it is shown in [16] that there exists a kind of projective tensor product in the class of the so-called \( p \)-convex \( p \)-multi-spaces (see [5] and also [17]), reflecting their special properties. Therefore one may suggest that \( p \)-convex \( PQ \)-spaces have their own projective tensor product, defined only within that class and not leading out of this class.

In the following proposition we deal, generally speaking, with \( PQ \)-spaces that are not \( p \)-convex.

**Theorem 6.5.** Let \( E \) and \( F \) be normed spaces, \( p \in [1, \infty] \). Then there exists a completely isometric isomorphism \( I : \langle p \rangle E \otimes_{\text{pop}} \langle p \rangle F \to \langle p \rangle (E \otimes_{pr} F) \), acting as the identity operator on the common underlying linear space of our \( PQ \)-spaces. As a corollary (see Proposition 5.3) there exists a completely isometric isomorphism \( I : \langle p \rangle E \otimes_{\text{pop}} \langle p \rangle F \to \langle p \rangle (E \otimes_{pr} F) \), which is the extension by continuity of \( I \).

**Proof.** Consider the canonical bioperator \( \vartheta : \langle p \rangle E \times \langle p \rangle F \to \langle p \rangle (E \otimes_{pr} F) \). It gives rise, by virtue of Proposition 4.7, to the completely contractive operator \( I \), acting as in the formulation. Therefore it is sufficient to show that for every \( U \in \mathcal{F}(\langle p \rangle E \otimes_{\text{pop}} \langle p \rangle F) \) we have \( \|U\|_{\text{pop}} \leq \|I_\infty(U)\| \).

Fix \( U \) and choose \( \varepsilon > 0 \). As a normed space, \( \mathcal{F}(\langle p \rangle E \otimes_{pr} F) \) is \( \mathcal{F} \otimes_{pr} E \otimes_{pr} F \). Consequently, in the linear space \( \mathcal{F}(E \otimes F) = \mathcal{F} \otimes E \otimes F \) there exists a representation of \( U \) or, which is the same, of \( I_\infty(U) \) as \( \sum_{k=1}^n a_k \otimes x_k \otimes y_k; a_k \in \mathcal{F}, x_k \in E, y_k \in F \) such that \( \sum_{k=1}^n \|a_k\| \|x_k\| \|y_k\| < \|I_\infty(U)\| + \varepsilon. \)
Take an arbitrary rank 1 projection \( P \in \mathcal{F} \) and introduce an element 
\( V := \sum_{k=1}^{n} P x_k \otimes (a_k y_k) = \sum_{k=1}^{n} (P \otimes a_k) x_k \otimes y_k \in \mathcal{F}(E \otimes F) \). As we know by (4.1), there exists an isometry \( S \in \mathcal{B} \) such that 
\( a_k = S^* (P \otimes a_k) S \) for all \( k \). It follows that 
\( S^* \cdot V \cdot S = \sum_{k=1}^{n} a_k (x_k \otimes y_k) = U \). Therefore, for \( U \) and \( V \) as of elements of \( \mathcal{F}((p) E \hat{\otimes}_{\text{pop}} (p) F) \), we have 
\( \| U \|_{\text{pop}} \leq \| V \|_{\text{pop}} \). But, by (5.5), we have 
\( \| V \|_{\text{pop}} \leq \sum_{k=1}^{n} \| P x_k \| \| a_k y_k \| \), where norms are taken in 
\( \mathcal{F}((p) E \otimes \mathcal{F}(F), \mathcal{F}(F) \otimes \mathcal{F}(E)) \), respectively.

Further, \( \| P x_k \| = \| x_k \| \) and, since 
\( \mathcal{F}(p) F = \mathcal{F}(p) E \hat{\otimes}_{\text{pop}} F \), we have 
\( \| a_k y_k \| = \| a_k \|_p \| y_k \| \). Consequently, we have 
\( \| U \|_{\text{pop}} \leq \sum_{k=1}^{n} \| a_k \|_p \| x_k \| \| y_k \| < \| I_\infty(U) \| + \varepsilon \). Since such an estimate holds for every 
\( \varepsilon > 0 \), we are done.

Setting in this theorem \( p := 1 \), we obtain

**Corollary 6.6.** For all normed spaces \( E \) and \( F \) we have, up to a completely isometric isomorphism, 
\( E_{\text{max}} \hat{\otimes}_{\text{pop}} F_{\text{max}} = (E \otimes_{\text{pr}} F)_{\text{max}} \) and
\( E_{\text{max}} \hat{\otimes}_{\text{pop}} F_{\text{max}} = (E \otimes_{\text{pr}} F)_{\text{max}} \).

As another particular case, we see that for for a Hilbert space \( H \) we have 
\( (p) H \hat{\otimes}_{\text{pop}} (p) H = (p) \mathcal{N}(H) \), where \( \mathcal{N}(H) \) is the Banach space of trace class operators on \( H \).

By virtue of Grothendieck Theorem, mentioned in Introduction, we may say that in the case of the classical projective tensor product of normed spaces, the especially nice tensor factors are \( L_1 \)-spaces. Now we would like to show that the same is true for the projective tensor product of \( PG \)-spaces.

At first we need some preparation.

**Proposition 6.7.** Let \((X, \mu), (Y, \nu)\) be measure spaces, and \( E, F \) be \( PQ \)-spaces, \( p \in [1, \infty] \). Then the bioperator \( \mathcal{R} : L_p(X, E) \times L_p(Y, F) \rightarrow L_p(X \times Y, E \otimes_{\text{pop}} F) \) : \((\bar{x}, \bar{y}) \mapsto \bar{z} ; \bar{z}(s, t) := x(s) \otimes y(t)\) is completely contractive.

(Here and thereafter, speaking about \( L_p(X, \cdot) \) and \( L_p(Y, \cdot) \), we mean \( X \) and \( Y \) with the given measures, and speaking about \( L_p(X \times Y, \cdot) \), we consider 
\( X \times Y \) with the cartesian product of these measures).

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
\mathcal{F}(L_p(X, E)) \times \mathcal{F}(L_p(Y, F)) & \xrightarrow{\mathcal{R}_\infty} & \mathcal{F}(L_p(X \times Y, E \otimes_{\text{pop}} F)) \\
\xrightarrow{\alpha_{X \times Y}} & & \xrightarrow{\alpha_{X \times Y}} \\
L_p(X, \mathcal{F}E) \times L_p(Y, \mathcal{F}F) & \xrightarrow{S} & L_p(X \times Y, \mathcal{F}E_{\text{pop}} F)
\end{array}
\]


where $\mathcal{S}$ takes a pair of vector–functions $(\bar{u}(s), \bar{v}(t))$ to the vector–function $\bar{w}(s, t) := \bar{u}(s)\Diamond \bar{v}(t); s \in X, t \in Y$, and $\alpha_X$ etc. are the respective specializations of $\alpha$ from Example 4.1. The diagram is evidently commutative, and $\alpha$ is an isometry. Therefore it suffices to show that $\mathcal{S}$ is contractive. Indeed, with the help of the estimate (5.5), we have

$$
\|\bar{w}\| = \left( \int_{X \times Y} \|\bar{u}(s)\Diamond \bar{v}(t)\|^p d(s, t) \right)^{1/p} \leq \left( \int_{X \times Y} \|\bar{u}(s)\|^p \|\bar{v}(t)\|^p d(s, t) \right)^{1/p} = \|\bar{u}\| \|\bar{v}\|.
$$

\[\square\]

**Theorem 6.8.** Let $(X, \mu), (Y, \nu)$ be measure spaces, $E, F$ PQ–spaces. Then there exists the complete isometry $I : L_1(X, E) \otimes_{pop} L_1(Y, F) \to L_1(X \times Y, E \otimes_{pop} F)$, well defined by $\bar{x} \otimes \bar{y} \mapsto \bar{z}$, where $\bar{z}(s, t) := \bar{x}(s) \otimes \bar{y}(t)$.

**Proof.** The bioperator $\mathcal{R}$ from Proposition 6.7, being considered for $p = 1$, gives rise to the completely contractive operator $R$, acting exactly as $I$ in the formulation. Therefore our task is to show that for every $U \in \mathcal{F}[L_1(X, E) \otimes_{pop} L_1(Y, F)]$ we have $\|U\|_{pop} \leq \|R_\infty(U)\|.

Obviously, $L_1(X, E)$ contains the dense subspace $L_0^0(X, E)$ consisting of vector-functions of the form $\sum_k \chi_k x_k$, where $\chi_k$ are characteristic functions of pairwise disjoint subsets of $X$ with finite measure, and $x_k \in E$. Similarly, $L_1(Y, F)$ contains the subspace $L_0^0(Y, F)$ with analogues properties. Therefore, thanks to the estimate (5.6) and the last estimate in Proposition 2.6, it is sufficient to prove that $R_\infty$ does not decrease norms of sums of elementary tensors of the form $a(x \otimes y)$, where $a \in \mathcal{F}, x \in L_0^0(X, E), y \in L_0^0(Y, F)$.

Let $U$ be such a sum. It is not difficult to show that it can be represented as

$$
U = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{kl} (\bar{x}_k \otimes \bar{y}_l),
$$

with $\bar{x}_k(s) := \chi_k(s) x_k; s \in X$, where $\chi_k(s)$ are characteristic functions of pairwise disjoint subsets $X_k; \mu(X_k) < \infty, x_k \in E$, and $\bar{y}_l(t) := \chi_l(t) y_l; t \in Y$, where $\chi_l(t)$ are characteristic functions of pairwise disjoint subsets $Y_l; \nu(Y_l) < \infty, y_l \in F$.

We obviously have $R_\infty(U) = \sum_{k,l} a_{kl} [\chi_k(s)\chi_l(t)x_k \otimes y_l]$. Therefore, by the recipe of Example 4.1, $\|R_\infty(U)\|$ is the norm of the function $\sum_{k,l} \chi_k(s) \chi_l(t)[a_{kl}(x_k \otimes y_l)] \in L_1(X \times Y), \mathcal{F}(E \otimes_{pop} F)$. Since we are in a space $L_1(\cdot)$, this implies that

$$
(6.1) \quad \|R_\infty(U)\| = \sum_{k=1}^{n} \sum_{l=1}^{m} \mu(X_k) \nu(Y_l) \|a_{kl}(x_k \otimes y_l)\|_{pop}.
$$
Fix, for a time, a pair $k, l$, and also $\varepsilon > 0$. By (5.3), we can represent $a_{kl}(x_k \otimes y_l) \in \mathcal{F}(E \otimes_{pop} F)$ in the form $\sum_i b^i_{kl}(u^i_{kl} \otimes v^i_{kl}) \cdot c^i_{kl}$, where $b^i_{kl}, c^i_{kl} \in \mathcal{B}, u^i_{kl} \in \mathcal{F}E, v^i_{kl} \in \mathcal{FF}$, such that

$$\sum_i \|b^i_{kl}\|\|u^i_{kl}\|\|v^i_{kl}\|\|c^i_{kl}\| < \|a_{kl}(x_k \otimes y_l)\|_{pop} + \varepsilon.$$  

(6.2)

(Here the number of summands, indexed by $i$, of course, depends on the pair $k, l$.)

Now for every $u \in \mathcal{F}E$ we denote, for brevity, by $\bar{u} \in \mathcal{FL}_1(X, E)$ the element $B_{\infty}(u) \in \mathcal{F}(L_1(X, E))$, where $B : E \rightarrow L_1(X, E)$ takes $x$ to the vector-function $\bar{x}(s) := \chi_k(s) x$. Similarly, for $v \in L_1(Y, F)$ we set $\bar{v} := B'_{\infty}(v) \in \mathcal{F}(L_1(Y, F))$, where $B' : F \rightarrow L_1(Y, F) : y \mapsto \bar{y} := \chi'(t) y$. Evidently we have

$$\|\bar{u}\| = \mu(X_k)\|u\| \quad \text{and} \quad \|\bar{v}\| = \nu(Y_k)\|v\|.$$  

(6.3)

Further, since for $x \in E, y \in F$ we have $R(\bar{x} \otimes \bar{y}) = \chi_k(s) \chi'(t) x \otimes y$, it easily follows that for $u \in \mathcal{F}E, v \in \mathcal{FF}$ we have $R_{\infty}(\bar{u} \otimes \bar{v}) = \chi_k(s) \chi'(t) u \otimes v$. Consequently, taking in account (6.3) and (6.2), we have

$$R_{\infty}\left(\sum_i b^i_{kl}(\bar{u}^i_{kl} \otimes \bar{v}^i_{kl}) \cdot c^i_{kl}\right) = \chi_k(s) \chi'_i(t) \sum_i b^i_{kl}(u^i_{kl} \otimes v^i_{kl}) \cdot c^i_{kl} =$$

$$\chi_k(s) \chi'_i(t) a_{kl}(x_k \otimes y_l) = R_{\infty}[a_{kl}(\bar{x}_k \otimes \bar{y}_l)].$$

But $R$ is obviously injective and, of course, the same is true for $R_{\infty}$. It follows that $\sum_i b^i_{kl}(\bar{u}^i_{kl} \otimes \bar{v}^i_{kl}) \cdot c^i_{kl} = a_{kl}(\bar{x}_k \otimes \bar{y}_l)$, and consequently $U = \sum_{k,l} [\sum_i b^i_{kl}(\bar{u}^i_{kl} \otimes \bar{v}^i_{kl}) \cdot c^i_{kl}]$. This, with the help of (6.3), implies that

$$\|U\|_{pop} \leq \sum_{k,l} \sum_i \|b^i_{kl}\|\|\bar{u}^i_{kl}\|\|\bar{v}^i_{kl}\|\|c^i_{kl}\| = \sum_{k,l} \mu(X_k)\nu(Y_l) \sum_i \|b^i_{kl}\|\|u^i_{kl}\|\|v^i_{kl}\|\|c^i_{kl}\|.$$  

From this, by virtue of (6.2), we obtain that

$$\|U\|_{pop} \leq \sum_{k,l} \mu(X_k)\nu(Y_l)(\|a_{kl}(x_k \otimes y_l)\|_{pop} + \varepsilon).$$

Since $\varepsilon$ is arbitrary, it follows that $\|U\|_{pop} \leq \sum_{k,l} \mu(X_k)\nu(Y_l)\|a_{kl}(x_k \otimes y_l)\|_{pop}$, that is, by (6.1), $\|U\|_{pop} \leq \|R_{\infty}(U)\|$. \hfill \Box

Since $L_1(X, E) \otimes_{pop} L_1(Y, F)$ is dense in $L_1(X, E) \otimes_{pop} L_1(Y, F)$, and the image of $I$ is obviously dense in $L_1(X \times Y, E \otimes_{pop} F)$, we have, as an immediate corollary,
Theorem 6.9. Let \((X, \mu), (Y, \nu), E, F\) be as before. Then there exists a complete isometric isomorphism \(I : L_1(X, E) \hat{\otimes}_{pop} L_1(Y, F) \to L_1(X \times Y, E \hat{\otimes}_{pop} F)\), well defined by \(\bar{x} \otimes \bar{y} \mapsto \bar{z}\), where \(\bar{z}(s, t) := \bar{x}(s) \otimes \bar{y}(t)\).

Combining Theorem 6.8 or 6.9 with the previous results in this section, one can obtain various corollaries. For example, taking the one-point \(Y\) and using Corollary 6.2, we get the assertion that can be considered as a ‘\(PQ\)-version’ of the Grothendieck Theorem in its usual formulation:

Corollary 6.10. Let \(p \in [1, \infty], X\) be a measure space, and \(F\) be a complete \(p\)-convex \(PQ\)-space. Then we have, up to a completely isometric isomorphism, \(L_1(X, (p)\mathbb{C}) \hat{\otimes}_{pop} F = L_1(X, F)\).

Note that the same assertion could be obtained without using Theorem 6.8, by combining easier Proposition 4.4 with Corollary 6.2.

Also, combining Theorem 6.9 (or Proposition 4.4) with Corollary 6.3, one can get the completely isometric isomorphism \(L_1(X, (p)\mathbb{C}) \hat{\otimes}_{pop} L_1(Y, (q)\mathbb{C}) \simeq L_1(X \times Y, (r)\mathbb{C})\) with \(r := \max\{p, q\}\), and so on.

7. Quantum duality and adjoint associativity

We proceed to show that the projective tensor product of \(PQ\)-spaces satisfies the law of adjoint associativity ( = exponential law), connecting it with the proper \(P\)-quantization of the space of completely bounded operators. Such a \(P\)-quantization extends what was well known in the context of quantum (operator) spaces. In that context the relevant construction cropped up in [8, p.140], but was fully realized and put in proper place independently and simultaneously in [3] and [9]. In the matrix-free language, again only for \(Q\)-spaces, it was presented in [14, 8.1.8.] Here we give all needed details for general \(PQ\)-spaces.

Let \(E, G\) be two \(PQ\)-spaces. Our task is to endow the normed space \(\mathcal{CB}(E, G)\) with a \(P\)-quantization. For this aim we consider the evaluation bioperator \(\mathcal{E} : E \times \mathcal{CB}(E, G) \to G : (x, \varphi) \mapsto \varphi(x)\) and its amplification \(\mathcal{E}_\infty : \mathcal{F}E \times \mathcal{F}[\mathcal{CB}(E, G)] \to \mathcal{F}G\); the latter, as we remember, is well defined by \((ax, b\varphi) \mapsto (a \otimes b)\varphi(x)\). Set, for \(\Phi \in \mathcal{F}[\mathcal{CB}(E, G)]\),

\[
(7.1) \quad \|\Phi\| := \sup\{\|\mathcal{E}_\infty(u, \Phi)\| : u \in \mathcal{F}E; \|u\| \leq 1\}.
\]
(We see that such a definition closely imitates the definition of the ‘classical’ operator norm: indeed, \( \| \varphi \| \) is \( \sup \{ \| \mathcal{E}(x, \varphi) \| ; x \in E; \| x \| \leq 1 \} \), where \( \mathcal{E} : E \times \mathcal{B}(E, G) \to G \) is the obvious ‘classical’ evaluation operator).

**Proposition 7.1.** The function \( \Phi \mapsto \| \Phi \| \) is a PQ–norm on \( \mathcal{CB}(E, G) \), and the resulting PQ–space is a P–quantization of \( \mathcal{CB}(E, G) \) as of a normed space.

**Proof.** For every \( b \in \mathcal{F} \), \( u \in \mathcal{FE} \) and \( \varphi \in \mathcal{CB}(E, G) \) we obviously have the equality

\[ \mathcal{E}_\infty(u, b \varphi) = \varphi_\infty(u) \hat{\circ} b. \]  

Therefore, by the Proposition 4.3(iii), we have \( \| \mathcal{E}_\infty(u, b \varphi) \| \leq \| b \|_1 \| \varphi_\infty(u) \| \). It follows that the number \( \| b \varphi \| \) is well-defined. Consequently, proceeding from elementary tensors to their (finite) sums, that the number \( \| \Phi \| \) is well-defined for all \( \Phi \in \mathcal{F}\mathcal{CB}(E, G) \).

Further, for all \( a \in \mathcal{F} \), \( u \in \mathcal{FE} \) and \( \Phi \in \mathcal{F}[\mathcal{CB}(E, G)] \) we have the equalities

\[ \mathcal{E}_\infty(u, a \cdot \Phi) = (1 \hat{\circ} a \cdot \mathcal{E}_\infty(u, \Phi)) \quad \text{and} \quad \mathcal{E}_\infty(u, \Phi \cdot a) = \mathcal{E}_\infty(u, \Phi) \hat{\circ} (1 \hat{\circ} a) \]

that can be immediately checked on elementary tensors. Consequently,

\[ \| \mathcal{E}_\infty(u, a \cdot \Phi) \|, \| \mathcal{E}_\infty(u, \Phi \cdot a) \| \leq \| 1 \hat{\circ} a \| \| \mathcal{E}_\infty(u, \Phi) \| = \| a \| \| \mathcal{E}_\infty(u, \Phi) \|. \]

It follows that \( \| a \cdot \Phi \| \leq \| a \| \| \Phi \| \), and similarly \( \| \Phi \cdot a \| \leq \| a \| \| \Phi \| \). Therefore the introduced seminorm on \( \mathcal{F}[\mathcal{CB}(E, G)] \) is a PQ–seminorm on \( \mathcal{F}[\mathcal{CB}(E, G)] \).

Finally, take a rank one projection \( P \in \mathcal{F} \). Then, considering \( \mathcal{CB}(E, G) \) as the underlying semi-normed space of the introduced PQ–space, we have, by virtue of (7.2) and Proposition 4.3(ii), that

\[ \| \varphi \| = \sup \{ \mathcal{E}_\infty(u, P \varphi) ; \| u \| \leq 1 \} = \sup \{ \| \varphi_\infty(u) \hat{\circ} P \| ; \| u \| \leq 1 \} = \sup \{ \| \varphi_\infty(u) \| ; \| u \| \leq 1 \} = \| \varphi \|_{cb}. \]

Consequently, our underlying space is just \( \mathcal{CB}(E, G) \) with its \( cb \)–norm. Therefore, by Proposition 2.4, the seminorm on \( \mathcal{F}[\mathcal{CB}(E, G)] \) is actually a norm, and the respected PQ–space is a P–quantization of the given space of completely bounded operators. \( \square \)

If a a given normed space \( E \) is endowed with a quantization (not just a P–quantization), then there is a well known standard way to make its dual space \( E^* \) again a Q–space. Namely, if we identify the normed spaces \( E^* \) and \( \mathcal{CB}(E, (\infty)\mathbb{C}) \) (that is, if we consider \( \mathbb{C} \) with its unique quantization), then
the recipe above provides the $PQ$-norm on $E^*$ which is in fact a $Q$-norm (see, e.g., \cite[Ch.8.2]{14}). However, if we shall consider other $P$-quantizations of $\mathbb{C}$, the normed space $\mathcal{CB}(E, \mathbb{C})$ is not bound to be the dual of $E$. Actually, we already know this: by Proposition 3.2 and 3.3, for $E := (p, q)\mathbb{C}$ the space $\mathcal{CB}(E, (q, q)\mathbb{C})$ is $E^*$, that is just $\mathbb{C}$, if, and only if $p \leq q$; otherwise, it is 0. In the first case the respected $P$-quantization is as follows.

**Proposition 7.2.** If $p \leq q$, then the $PQ$-space $\mathcal{CB}((p, q)\mathbb{C}, (q, q)\mathbb{C})$, after the identification of its underlying space with $\mathbb{C}$, is $(q, q)\mathbb{C}$.

**Proof.** In our case every $u \in F\mathbb{C}$ has a unique presentation as $a1; a \in F, 1 \in \mathbb{C}$ whereas every $\Phi \in F[\mathcal{CB}((p, q)\mathbb{C}, (q, q)\mathbb{C})]$, after the mentioned identification, has a unique presentation as $b1; b \in F, 1 \in \mathbb{C}$. Consequently, the bioperator $E_\infty$ can be considered as taking $(a1, b1)$ to $(a\diamond b)1$. Therefore, by virtue of Proposition 4.6, the $PQ$-norm of a given $\Phi$, presented as $b1$, is

$$
\|\Phi\| := \sup\{\|a\diamond b\|_q : a \in F_p, \|a\|_p \leq 1\} = \sup\{\|a\|_q \|b\|_q : a \in F_p, \|a\|_p \leq 1\} = \|b\|_q \sup\{\|a\|_q : a \in F_p, \|a\|_p \leq 1\}
$$

But, since $p \leq q$, the last supremum is obviously 1.

Denote the space of completely bounded bioperators from $E \times F$ into $G$ by $\mathcal{CB}(E \times F, G)$. Obviously, it is the normed space with respect to $\|\cdot\|_{cb}$.

**Theorem 7.3.** There exists the isometric isomorphism (of normed spaces) $I_F : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(F, \mathcal{CB}(E, G))$, well defined by taking (exactly as in the ‘classical’ context) the bioperator $R$ to the operator $\mathcal{R}^F : F \to \mathcal{CB}(E, G) : y \mapsto \mathcal{R}^y$, where $\mathcal{R}^y : E \to G$ takes $x$ to $R(x, y)$.

To put it in more detailed form,

(i) for every $R \in \mathcal{CB}(E \times F, G)$ and $y \in F$ the operator $\mathcal{R}^y : E \to G$ is completely bounded.

(ii) The operator $\mathcal{R}^F : F \to \mathcal{CB}(E, G) : y \mapsto \mathcal{R}^y$, which is well defined because of (i), is completely bounded with respect to the $PQ$-norm on $\mathcal{CB}(E, G)$, defined above.

(iii) The operator $I_F : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(F, \mathcal{CB}(E, G))$, which is well defined because of (ii), is an isometric isomorphism.

**Proof.** First, distinguish the formula $\mathcal{R}^\infty(u)\diamond b = \mathcal{R}^\infty(u, by); u \in FE, b \in F$, easily verified on elementary tensors. If our $b$ is a rank 1 projection, it implies, by virtue of Proposition 4.3(ii), that $\|\mathcal{R}^y(u)\| = \|\mathcal{R}^\infty(u, by)\| \leq \|R\|\infty\|u\|\|by\| = \|R\|\infty\|u\|\|y\|$. This gives (i).
Now we may speak about the operator $R^F_\infty : \mathcal{F} F \to \mathcal{F} [\mathcal{CB}(E, G)]$. This time we shall use the formula
\[
E_\infty (u, R^F_\infty (v)) = R_\infty (u, v); u \in \mathcal{F} E, v \in \mathcal{F} F,
\]
also easily verified on elementary tensors in respective amplifications. Together with (7.1), it implies that for $v \in \mathcal{F} F$ we have
\[
\|R^F_\infty (v)\| = \sup\{\|E_\infty (u, R^F_\infty (v))\| : u \in \mathcal{F} E; \|u\| \leq 1\} = \\
\sup\{\|R_\infty (u, v)\| : u \in \mathcal{F} E; \|u\| \leq 1\}.
\]

Consequently, $R^F_\infty$ is a bounded operator, and we obviously have $\|R^F_\infty \| = \|R_\infty \|$. This gives (ii), and also the equality $\|R^F\|_{cb} = \|R\|_{cb}$.

Thus, the operator $I_F : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(F, \mathcal{CB}(E, G))$ is well defined and isometric. To conclude the proof of the assertion (iii) we shall show that it is surjective.

Take $S \in \mathcal{CB}(F, \mathcal{CB}(E, G))$ and set $R : E \times F \to G : (x, y) \mapsto [S(y)](x)$. Clearly, $R$ is bounded, and $R^F = S$. Therefore our task is to verify that $R$ is completely bounded. But the formula (7.3) is obviously valid, if we replace $R^F$ by $S$, hence $\|R_\infty (u, v)\| = \|E_\infty (u, S_\infty (v))\|$. Finally, it follows from (7.1) that $\|E_\infty (u, S_\infty (v))\| \leq \|u\|\|S_\infty (v)\| \leq \|S\|_\infty \|u\|\|v\|$, and we are done. \hfill \Box

A similar, up to obvious modifications, argument provides the ‘twin’ isometric isomorphism $I_E : \mathcal{CB}(E \times F, G) \to \mathcal{CB}(E, \mathcal{CB}(F, G))$, well defined by taking (again exactly as in the “classical” context) the bioperator $R$ to the operator $R^E : x \mapsto R^z$, where $R^z : F \to G$ acts as $y \mapsto R(x, y)$.

Now recall that, by virtue of the universal property of the projective tensor product of $PQ$–spaces, we can identify the spaces $\mathcal{CB}(E \times F, G)$ and $\mathcal{CB}(E \otimes_{pop} F, G)$ by means of the isometric isomorphism, taking a bioperator to its linearization. Therefore, as an immediate corollary of the previous proposition, we obtain the following ‘proto-quantum’ version of the so-called law of adjoint associativity in classical functional analysis. (As to the ‘classical’ formulation, see, e.g., [14, Ch.6.1]).

**Theorem 7.4.** There exists an isometric isomorphism $I_F : \mathcal{CB}(E \otimes_{pop} F, G) \to \mathcal{CB}(F, \mathcal{CB}(E, G))$, uniquely determined by the equality
\[
([I_F(\varphi)]y)(x) = \varphi(x \otimes y).
\]
A similar argument provides an isometric isomorphism $\mathcal{I}_E : \mathcal{CB}(E \otimes_{\text{pop}} F, G) \to \mathcal{CB}(E, \mathcal{CB}(F, G))$ by means of the equality $(\mathcal{I}_F(\varphi)(x)(y) = \varphi(x \otimes y)$.

**Remark 7.5.** In fact, the operators $\mathcal{I}_F$ and $\mathcal{I}_F$ are complete isometric isomorphisms. As to $\mathcal{I}_F$, one can prove this in the following way. First, we identify, up to a natural complete isometric isomorphism, $\mathcal{F}[\mathcal{CB}(E \otimes_{\text{pop}} F, G)]$ with $\mathcal{CB}(E \otimes_{\text{pop}} F, F G)$ and $\mathcal{F}[\mathcal{CB}(F, \mathcal{CB}(E, G))]$ with $\mathcal{CB}(F, \mathcal{CB}(E, F G))$, and then apply Theorem 7.4 to the triple $E, F, F G$. Here $F G$ is equipped with a $P Q$-norm by means of the embedding of $F [F G]$ into $F G$, well defined by taking $a[bz]$ to $(a \diamond b)z; a, b \in F, z \in G$. The details are given in [14, Ch.8.8] in the context of $Q$-spaces, but the argument is valid, up to some minor modifications, for $P Q$-spaces as well.

### 8. Comparison of proto-operator-projective and operator-projective tensor products

In conclusion, we consider the relationship between the introduced tensor product and the well-known operator-projective tensor product of operator spaces. We recall that the latter linearizes completely bounded bilinear operators within the class of $Q$-spaces which is essentially narrower than the class of $P Q$-spaces. Its initial definition was given in terms of an explicit construction (cf., e.g., the textbook [11, p. 124]), which after the translation from the ‘matrix’ to the ‘non-coordinate’ language sounds as follows.

Let $E, F$ be linear spaces. It is not difficult to show that every $U \in \mathcal{F}(E \otimes F)$ can be expressed with the help of a ‘single diamond’, namely as $a \cdot (u \diamond v) \cdot b; a, b \in F, u \in \mathcal{F}E, v \in \mathcal{F}F$. Thus, we can introduce the number

\[ (8.1) \quad \|U\|_{op} = \inf \{ \|a\|\|u\|\|v\|\|b\| \}. \]

where the infimum is taken over all possible representations of $U$ in the form $a \cdot (u \diamond v) \cdot b$.

If $E, F$ are $Q$-spaces, then $\|\cdot\|_{op}$ is an $Q$-norm on $E \otimes F$, and the $Q$-space $E \otimes_{op} F := (E \otimes F, \|\cdot\|_{op})$, together with $\vartheta$, possess the universal property, characteristic for an operator-projective tensor product (see, e.g., [14, Ch.7.2]).

It is known that within the class of quantum spaces the norm $\|U\|_{op}$ coincides with the norm $\|U\|_{\text{pop}}$, given by the formula (5.3) (*ibidem*). The difference lies in another corner: outside this class the former number is,
generally speaking, essentially greater than the latter number. This can be demonstrated by the following example.

Set $E := F := \ell_1$, where by $\ell_1$ we denote, for brevity, the $PQ$–space $L_1(\mathbb{N}, (\infty)\mathbb{C})$ with the counting measure on $\mathbb{N}$, which is a particular case of the $PQ$–spaces $L_p(\cdot)$ from Section 3.

Denote by $e_k \in \ell_1; k = 1, 2, \ldots$ the sequence $(..., 0, 1, 0, ...)$ with 1 as the $k$–th coordinate, fix $n \in \mathbb{N}$ and choose arbitrary pairwise orthogonal rank 1 projections $P_k \in F; k = 1, ..., n$, in $F$. Finally, take in $F(\ell_1 \otimes_{op} \ell_1)$ the element

$$V_n := \sum_{k=1}^{n} P_k (e_k \otimes e_k).$$

**Proposition 8.1.** We have $\|V_n\|_{op} = n$, whereas $\|V_n\|_{op} = n^2$.

*Proof.* To show the first equality, we use Theorem 6.9. As a particular case, it provides a completely isometric isomorphism $I : \ell_1 \otimes_{op} \ell_1 \rightarrow L_1(\mathbb{N} \times \mathbb{N}, (\infty)\mathbb{C}) \otimes_{op} (\infty)\mathbb{C})$. Clearly, the latter $PQ$–space can be identified with $L_1(\mathbb{N} \times \mathbb{N}, (\infty)\mathbb{C})$. Thus, we can say that $I_{\infty}(V_n) = \sum_{k=1}^{n} P_k \tilde{e}_k$, where $\tilde{e}_k$ is the function ( = double sequence) taking $(k, k)$ to 1 and taking other pairs in $\mathbb{N} \times \mathbb{N}$ to 0. Therefore the definition of $PQ$–norm on $L_1(\mathbb{N} \times \mathbb{N}, (\infty)\mathbb{C})$ implies that $\|V_n\|_{op}$ is the norm of the $F$-valued function in $L_1(\mathbb{N} \times \mathbb{N}, F)$, taking $(k, k)$ to $P_k$ in the case, when $1 \leq k \leq n$ and taking other pairs in $\mathbb{N} \times \mathbb{N}$ to 0. This norm is, of course, $\sum_{k=1}^{n} \|P_k\| = n$.

Turn to the second equality. To begin with, we shall display the representation of $V_n$ in the form (8.1), such that $\|a\| \|u\| \|v\| \|b\| = n^2$.

Take $\tilde{e}_k \in L$ such that for every $k$ we have $P_k = \tilde{e}_k \otimes \tilde{e}_k$. Further, take $u = v = \sum_{k=1}^{n} P_k e_k$, $a = \sum_{k=1}^{n} \tilde{e}_k \circ (\tilde{e}_k \otimes \tilde{e}_k)$ and $b = \sum_{k=1}^{n} (\tilde{e}_k \otimes \tilde{e}_k) \circ \tilde{e}_k$.

A simple calculation shows that indeed $a \cdot (u \otimes v) \cdot b = V_n$, and it remains to observe that $\|a\| = \|b\| = 1$ whereas $\|u\| = \|v\| = n$.

Now we must show that for every representation of $V_n$ as $a \cdot (u \otimes v) \cdot b$ we have $\|a\| \|u\| \|v\| \|b\| \geq n^2$.

Since $u, v \in F\ell_1$, it is easy to observe that $u$ can be represented as $u = \sum_{k=1}^{n} a_k e_n + \sum_{l=1}^{m_1} a'_l f_l$ for some $m_1$, and $v$ can be represented as $v = \sum_{k=1}^{n} b_k e_n + \sum_{l=1}^{m_2} b'_l g_l$ for some $m_2$, where $a_k, a'_l, b_k, b'_l \in F$, the sequences $f_l, g_l \in \ell_1$ begin with $n$ zeroes and the systems $f_l; l = 1, ..., m_1$ and $g_l; l = 1, ..., m_2$ are linearly independent. Consequently, we have $V_n = a \cdot W \cdot b$, where $W$ is

$$\sum_{k=1}^{n} \sum_{l=1}^{n} (a_k \otimes b_l)(e_k \otimes e_l) + \sum_{k=1}^{n} \sum_{i=1}^{m_2} (a_k \otimes b'_i)(e_k \otimes g_i) +$$
\[
\sum_{k=1}^{n} \sum_{j=1}^{m_1} (a'_j \otimes e_k)(f_j \otimes e_k) + \sum_{j=1}^{m_1} \sum_{i=1}^{m_2} (a'_j \otimes b'_i)(f_j \otimes g_i).
\]

But at the same time \(V_n = \sum_{k=1}^{n} P_k(e_k \otimes e_k)\) and the system of elements in \(\ell_1 \otimes \ell_1\), consisting of all \(e_k \otimes e_i, e_k \otimes g_i, f_j \otimes e_k\) is obviously linearly independent. Therefore, comparing both representations of \(V_n\), we see that \(a(a_k \otimes b_k)b = P_k; k = 1, \ldots, n\), and all operators \(a(a_k \otimes b_l)b\), where \(k \neq l\), as well as all \(a(a_k \otimes b'_l)b, a(a'_j \otimes e_k)b, a(a'_j \otimes b'_j)b\), are zeroes. In particular, we have

\[
1 = \|P_k\| = \|a(a_k \otimes b_k)b\| \leq \|a\|\|a_k \otimes b_k\|\|b\| = \|a\|\|a_k\|\|b_k\|\|b\|
\]

for every \(k\).

Embedding \(F\ell_1\) into \(\ell_1(F)\) by the recipe in Section 4, we see that the indicated forms of \(u\) and \(v\) imply that \(\|u\| \geq \sum_{k=1}^{n} \|a_k\|\) and \(\|v\| \geq \sum_{k=1}^{n} \|b_k\|\). Set \(\lambda_k := \|a\|\|a_k\|\) and note that \(\|b_k\|\|b\| \geq \lambda^{-1}\). Therefore we have

\[
\|a\|\|u\|\|v\|\|b\| \geq \left(\sum_{k=1}^{n} \lambda_k\right) \left(\sum_{k=1}^{n} \lambda_k^{-1}\right) = \sum_{k,l=1}^{n} \lambda_k \lambda_l^{-1},
\]

the obvious estimate \(\lambda_k \lambda_l^{-1} + \lambda_k^{-1} \lambda_l \geq 2\) implies that the latter sum is \(\geq n^2\).

This proposition shows, in particular, that outside the class of \(Q\)-spaces the function \(U \mapsto \|U\|_{op}; U \in F(E \otimes F)\) is not bound to be a norm. Indeed, let \(m\) be a natural number such that \(1 \leq m < n\). Set \(V := V_m, W := V_n - V_m\). Then practically the same argument shows that \(\|W\|_{op} = (n - m)^2\), hence contrary to the triangle inequality, we have

\[
\|V + W\|_{op} = \|V_n\|_{op} = n^2 > m^2 + (n - m)^2 = \|V\|_{op} + \|W\|_{op}.
\]

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