Abstract—This paper proposes a novel unified interval-valued observer synthesis approach for locally Lipschitz nonlinear continuous-time (CT) and discrete-time (DT) systems with nonlinear observations. A key feature of our proposed observer, which is derived using mixed-monotone decompositions, is that it is correct by construction (i.e., the true state trajectory of the system is framed by the states of the observer) without the need for imposing additional constraints and assumptions such as global Lipschitz continuity or contraction, as is done in existing approaches in the literature. Furthermore, we derive sufficient conditions for designing stabilizing observer gains in the form of Linear Matrix Inequalities (LMIs). Finally, we compare the performance of our observer design with some benchmark CT and DT observers in the literature.

I. INTRODUCTION

Knowledge of system states is essential in almost all engineering applications, including fault detection, system identification and monitoring. However, in many realistic cases, system states are not fully measurable/measured and further, the sensor measurements may be limited or inaccurate. Thus, state observers have been designed to estimate system’s states based on system dynamics and noisy/uncertain observations. When dealing with systems with set-valued (i.e., distribution-free) uncertainties, interval observers have become increasingly popular due to their simple principles and computational efficiency [1], [2].

The design of interval observers (a particular form of set-valued observers) has been extensively investigated in the literature for various classes of dynamical systems such as linear time-invariant (LTI) systems [3], linear parameter varying (LPV)/quasi-LPV systems [1], [4], monotone/cooperative dynamics [5], [6], Metzler systems [2] and mixed-monotone dynamics [7]–[9]. To obtain cooperative observer error dynamics, the design of interval observers has either directly relied on monotone systems theory [10], or relatively restrictive assumptions about the existence of certain system properties were imposed to guarantee the applicability of the proposed approaches. However, even for linear systems, it is not easy nor guaranteed to synthesize the frame gain to satisfy correctness and stability at the same time [11]. This difficulty to obtain such properties was relaxed for certain classes of systems, by applying time-invariant/varying state transformations [3], [7], transformation to a positive system before designing an observer [12] (only applicable to linear systems) or leveraging interval arithmetic or Müller’s theorem-based approaches [13].

In the context of nonlinear systems, an interval observer design has been proposed in [14] for a class of continuous-time nonlinear systems by leveraging bounding functions, but no necessary and/or sufficient conditions for the existence of bounding functions or how to compute them have been discussed. Moreover, to conclude stability, restrictive assumptions on the nonlinear dynamics have been imposed. On the other hand, the authors in [7] applied bounding/mixed-monotone decomposition functions to design interval state estimation for nonlinear discrete-time dynamics, where to guarantee positivity of the error dynamics (i.e., the correctness property), conservative additive terms were added to the error dynamics. Moreover, to best of our understanding, the required conditions to guarantee that the computed bounding functions are decomposition functions were not included in the resulting Linear Matrix Inequalities (LMIs). On the other hand, our previous work [8], [9] designed interval observer for globally Lipschitz mixed-monotone nonlinear discrete-time systems, where the stability of the proposed observer relied on some sufficient structural system properties.

In this paper, we introduce a novel method for synthesizing interval observers using mixed-monotone decompositions for locally Lipschitz nonlinear CT and DT systems with nonlinear observation functions. The main feature and advantage of our proposed observer is that it is correct by construction. In particular, by leveraging remainder-form mixed-monotone decomposition functions, we show that the true state trajectory of the system is guaranteed to frame the true states of the observer by construction. In other words, the observer error system is by design positive (for DT systems) or cooperative (for CT systems) without the need for additional assumptions, e.g., global Lipschitz continuity and contraction. Moreover, we derive sufficient conditions in the form of LMIs that ensure that our proposed correct-by-construction interval observer is also stable, and they can be utilized to design stabilizing observer gains via semi-definite programming. Finally, our unified framework is the first to address the problem of synthesizing interval observers for a very broad range of locally Lipschitz CT and DT systems that are correct by construction.

II. PRELIMINARIES

Notation. \( \mathbb{R}^n \), \( \mathbb{R}^{n \times p} \), \( \mathbb{D}_n \), \( \mathbb{N} \), \( \mathbb{N}_n \) denote the \( n \)-dimensional Euclidean space and the sets of \( n \) by \( p \) matrices, \( n \) by \( n \)
diagonal matrices, natural numbers and natural numbers up to \( n \), respectively, while \( M_n \) denotes the set of all \( n \) \( n \times n \) matrices. For \( M \in \mathbb{R}^{n \times p} \), \( M_{i,j} \) denotes \( M \)’s entry in the \( i \)’th row and the \( j \)’th column, \( M^+ = \max(M, 0_{n \times p}) \), \( M^- = M^+ - M \) and \( |M| \equiv M^+ + M^- \), where \( 0_{n \times p} \) is the zero matrix in \( \mathbb{R}^{n \times p} \), while \( \text{sgn}(M) \in \mathbb{R}^{n \times p} \) is the element-wise sign of \( M \) with \( \text{sgn}(M_{i,j}) = 1 \) if \( M_{i,j} \geq 0 \) and \( \text{sgn}(M_{i,j}) = -1 \), otherwise. Further, if \( p = n, M^d \) denotes a diagonal matrix whose diagonal coincides with the diagonal of \( M, M^{rd} \equiv M - M^d \) and \( M^{nd} \equiv M^d + |M^{rd}| \), while \( M > 0 \) and \( M < 0 \) (or \( M \geq 0 \) and \( M \leq 0 \)) denote that \( M \) is positive and negative (semi)-definite, respectively.

Next, we introduce some useful definitions and results.

**Definition 1** (Interval, Maximal and Minimal Elements, Interval Width). An (multi-dimensional) interval \( \mathcal{I} \equiv [\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^n \) is the set of all real vectors \( x \in \mathbb{R}^n \) that satisfies \( \mathbf{s} \leq x \leq \mathbf{t} \), where \( \mathbf{s} \) and \( |\mathbf{s} - \mathbf{t}| \equiv \max_{i \in \{1, \ldots, n\}} s_i \) are called minimal vector, maximal vector and interval width of \( \mathcal{I} \), respectively. An interval matrix can be defined similarly.

**Proposition 1**. [14, Lemma 1] Let \( A \in \mathbb{R}^{n \times p} \) and \( \mathbf{t} \leq x \leq \mathbf{s} \in \mathbb{R}^n \). Then, \( A^\top \mathbf{s} - A^\top \mathbf{t} \leq Ax \leq A^\top \mathbf{s} - A^\top \mathbf{t} \). As a corollary, if \( A \) is non-negative, \( A^\top \mathbf{s} \leq Ax \leq A^\top \mathbf{t} \).

**Definition 2** (Jacobi Sign-Stability). A mapping \( f : \mathcal{X} \subset \mathbb{R}^n \rightarrow \mathbb{R}^p \) is (generalized) Jacobian sign-stable (JSS), if its (generalized) Jacobian matrix entries do not change signs on its domain, i.e., if either of the following hold:

\[
\forall x \in \mathcal{X}, \forall i \in \mathbb{N}_p, \forall j \in \mathbb{N}_n, J_f(x)_{i,j} \geq 0 \quad \text{(positive JSS)}
\]

\[
\forall x \in \mathcal{X}, \forall i \in \mathbb{N}_p, \forall j \in \mathbb{N}_n, J_f(x)_{i,j} \leq 0 \quad \text{(negative JSS)},
\]

where \( J_f(x) \) denotes the Jacobian matrix of \( f \) at \( x \in \mathcal{X} \).

**Proposition 2** (Jacobi Sign-Stable Decomposition). Let \( J_f(x) \in \mathbb{R}^{p \times n} \) and suppose \( \forall x \in \mathcal{X}, \forall (i,j) \in \mathbb{N}_p \times \mathbb{N}_n, (J_f(x)_{i,j})_{i,j} \geq 0 \). Then, \( J_f(x) \) can be decomposed into a (remainder) affine mapping \( A(x) \) and a JSS mapping \( \mu(\cdot) \), in an additive form:

\[
\forall x \in \mathcal{X}, f(x) = \mu(x) + Ax,
\]

where \( A \) is a matrix in \( \mathbb{R}^{p \times n} \), that satisfies the following:

\[
\forall (i,j) \in \mathbb{N}_p \times \mathbb{N}_n, (J_f(x)_{i,j})_{i,j} = (\mu(x))_{i,j} \lor (\mu(x))_{i,j} = (J_f(x)_{i,j})_{i,j}.
\]

**Proof.** Let us define \( \mu(x) \equiv f(x) - Ax \), where \( H \) is given in (1). Then, it follows from (1) that \( \forall x \in \mathcal{X}, \forall (i,j) \in \mathbb{N}_p \times \mathbb{N}_n \), \( J_f(x)_{i,j} = (J_f(x)_{i,j})_{i,j} \lor (J_f(x)_{i,j})_{i,j} = (J_f(x)_{i,j})_{i,j} \).

Proof. First, by [17, corollary 2], the JSS function \( \mu(\cdot) \) admits a tight decomposition function that has the following form for any ordered \( x_1, x_2 \in \mathcal{X} \):

\[
\forall (i,j) \in \mathbb{N}_p \times \mathbb{N}_n, (J_f(x)_{i,j})_{i,j} = (\mu(x))_{i,j} \lor (\mu(x))_{i,j} = (J_f(x)_{i,j})_{i,j}
\]

where \( D_i \in \mathbb{R}_n \) is a binary diagonal matrix determined by which vertex of the interval \([x_{1,1}, x_{2,1}]\) or \([x_{1,2}, x_{2,2}]\) maximizes (if \( x_2 \leq x_1 \)) or minimizes (if \( x_2 > x_1 \)) the JSS function \( \mu(\cdot) \) and can be computed as follows:

\[
D_i = \text{diag}(\max(\text{sgn}(\mathbf{J}_{p,i}), 0_{1,n})).
\]
the case that \( i \) is positive JSS in dimension \( j \). Obviously, \( z^*_j = x_{1,j} \). On the other hand, \((J_{\mu})_{i,j} \geq 0\), and so, \(\max(\text{sgn}((J_{\mu})_{i,j}),0) = 1\). Hence, by (6), \((D_{\mu})_{i,j} = 1\), and therefore in (5), \((D_x x_{1} - (I_{n} - D_{\mu}) x_{2})_{j} = x_{1,j}\), which is consistent with what we obtained for the maximizer’s \( j \)’th entry, i.e., \( z_j^* = x_{1,j}\). Similar reasoning shows that such consistency also holds in the negative JSS case as well as when finding the minimizer of \( \mu(i) \) if \( x_2 > x_1 \).

Further, since \( \mu(\cdot) \) is JSS, \( J_{\mu,i} \) does not change signs and hence, \( \max(\text{sgn}(J_{\mu,i})),0) \) is well-defined and we can equivalently use \( \max(\text{sgn}(J_{\mu,i}),0) = \max(\text{sgn}(\overline{J}_{\mu,i}),0) \).

### III. PROBLEM FORMULATION

**System Assumptions.** Consider the following nonlinear continuous-time (CT) or discrete-time (DT) system:

\[
\begin{align*}
G: \begin{cases} 
    \dot{x}_t^+ = \hat{f}(x_t,u_t) \triangleq f(x_t), \\
    y_t = h(x_t,u_t) \triangleq h(x_t),
\end{cases}, \\
\text{for } x_t \in X, \quad t \in T, \tag{7}
\end{align*}
\]

where \( x_t^+ = x_t, T = \mathbb{R}_{\geq 0} \) if \( G \) is a CT and \( x_t^+ = x_{t+1}, T = \{0\} \cup \mathbb{N} \), if \( G \) is a DT system. Moreover, \( x_0 \in X \subset \mathbb{R}^n \), and \( y_t \in \mathbb{R}^l \) are continuous state, known (control) input and output (measurement) signals. Furthermore, \( \hat{f}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l \) are nonlinear state vector field and observation/constraint functions/mappings, respectively, from which, the functions/mappings \( f: \mathbb{R}^n \to \mathbb{R}^n \) and \( g: \mathbb{R}^n \to \mathbb{R}^l \) are well-defined since the input signal \( u_t \) is known. We are interested in estimating the trajectories of the plant \( G \) in (7), when they are initialized in a given interval \( X_0 \subset X \subset \mathbb{R}^n \).

**Assumption 1.** The initial state \( x_0 \) satisfies \( x_0 \in X_0 = [\underline{x}_0, \overline{x}_0] \), where \( \underline{x}_0 \) and \( \overline{x}_0 \) are known initial state bounds.

**Assumption 2.** The mappings \( f(\cdot) \) and \( h(\cdot) \) are known, differentiable, locally Lipschitz \( b \) and mixed-monotone in their domain with priori known upper and lower bounds for their Jacobian matrices, \( J_f, J_h \in \mathbb{R}^{n \times n} \) and \( J_h, J_h \in \mathbb{R}^{l \times n} \), respectively.

**Assumption 3.** The values of the input \( u_t \) and output/measurement \( y_t \) signals are known at all times.

Furthermore, we formally define the notions of framers, correctness and stability that are used throughout the paper.

**Definition 5 (Correct Interval Framers).** Suppose Assumptions (7)–(9) hold. Given the nonlinear plant (7), the mappings/signals \( \underline{x}, \overline{x}: T \to \mathbb{R}^n \) are called upper and lower framers for the states of system (7) if

\[
\forall t \in T, \quad \underline{x}_t \leq x_t \leq \overline{x}_t. \tag{8}
\]

In other words, starting from the initial interval \( \underline{x}_0 \leq x_0 \leq \overline{x}_0 \), the true state of the system in (7), \( x_t \), is guaranteed to evolve within the interval flow-pipe \( [\underline{x}_t, \overline{x}_t] \), for all \( t \in T \). Finally, any dynamical system whose states are correct framers for the states of the plant \( G \), i.e., any (tractable)

\[\text{algorithm that returns upper and lower framers for the states of plant } G \text{ is called a correct interval framer for system (7).} \]

**Definition 6 (Framer Error).** Given state framers \( \underline{x} \leq \overline{x} \), \( \varepsilon: T \to \mathbb{R}^n \), which denotes the interval width of \( [\underline{x}_t, \overline{x}_t] \) (cf. Definition 7), is called the framer error. It can be easily verified that correctness (cf. Definition 5) implies that \( \varepsilon_t \geq 0, \forall t \in T \).

**Definition 7 (Stability and Interval Observer).** An interval framer is stable, if the framer error (cf. Definition 6) asymptotically converges to zero, i.e., \( \lim_{t \to \infty} \| \varepsilon_t \| = 0 \). A stable interval framer is called an interval observer.

The observer design problem can be stated as follows:

**Problem 1.** Given the nonlinear system in (7), as well as Assumptions 1–3, synthesize an interval observer, i.e., a correct and stable framer (cf. Definitions 5 and 7).

### IV. PROPOSED INTERVAL OBSERVER

**A. Interval Observer Design**

Given the nonlinear plant \( G \), in order to address Problem 1, we propose an interval observer (cf. Definition 5) for \( G \) through the following dynamical system:

\[
\begin{align*}
    
    \dot{\overline{x}}_t &= (A-LC)\overline{x}_t - (A-LC)\underline{x}_t + Ly_t + \phi_d(\overline{x}_t,\underline{x}_t), \\
    \dot{\underline{x}}_t &= (A-LC)\underline{x}_t - (A-LC)\overline{x}_t + Ly_t + \phi_d(\overline{x}_t,\underline{x}_t),
\end{align*}
\]

(9)

where if \( G \) is a CT system, then

\[
\overline{x}_t \triangleq \overline{x}_{t+1}, (A-LC)^t \triangleq (A-LC)^{t+1}, (A-LC)^d \triangleq (A-LC)^{d+1},
\]

and if \( G \) is a DT system, then

\[
\overline{x}_t^+ \triangleq \overline{x}_{t+1}, (A-LC)^t \triangleq (A-LC)^{t+1}, (A-LC)^d \triangleq (A-LC)^{d+1}.
\]

Moreover, \( A \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^{n \times l} \) are chosen such that the following decompositions hold (cf. Definition 8 and Proposition 2):

\[
\forall x \in X: \begin{cases} 
    f(x) = Ax + \phi(x), \\
    h(x) = Cx + \psi(x),
\end{cases} \quad \text{s.t. } \phi, \psi \text{ are JSS in } X. \tag{12}
\]

Furthermore, \( \phi \) and \( \psi \) are tight mixed-monotone decomposition functions of \( \phi \) and \( \psi \), respectively (cf. Definition 3 and Propositions 3 and 4).

Finally, \( L \in \mathbb{R}^{n \times l} \) is the observer gain matrix, designed via Theorem 2 such that the proposed observer \( G \) possesses the desired properties discussed in the following subsections.

**B. Observer Correctness (Framer Property)**

Our strategy is to design a correct by construction interval observer for plant \( G \). To accomplish this goal, first, note that from (9) and (12) we have \( y_t - C x_t - \psi(x_t) = 0 \), and so \( L(y_t - C x_t - \psi(x_t)) = 0 \), for any \( L \in \mathbb{R}^{n \times l} \). Adding this “zero” term to the right hand side of (7) and applying (12) yield the following equivalent system to \( G \):

\[
x_t^+ = (A-LC)x_t + Ly_t + \phi(x_t) - L\psi(x_t). \tag{13}
\]
From now on, we are interested in computing embedding systems, in the sense of Definition 4, for the system in (13), so that by Proposition 2, the state trajectories of (13) are “framed” by the state trajectories of the computed embedding system. To do so, we split the right hand side of (13) (except for \( L\dot{y} \) that is independent of the states) into two constituent systems: the linear constituent \((A - LC)x_1 \) and the nonlinear constituent, \( \phi(x_1) - L\psi(x_1) \). Then, we compute embedding systems for each constituent, separately. Finally, we add the computed embedding systems to construct an embedding system for (13). We start with framing the linear constituent through the following lemma.

**Lemma 1** (Linear Embedding). Consider a dynamical system \( \mathcal{G}_t \) in the form of (13), with domain \( \mathcal{X} \) and state equation \( f_t(x_t) = (A - LC)x_t \). Then, a tight decomposition function (cf. Definition 3) for \( \mathcal{G}_t \) can be computed as follows:

\[
\tilde{f}_{td}(x_1, x_2) = (A - LC)^+ x_1 - (A - LC)^- x_2, \quad (14)
\]

where \((A - LC)^+ \) and \((A - LC)^- \) are given in (10) and (11) for CT and DT systems, respectively.

**Proof.** We start with the DT case, where \((A - LC)^+ \triangleq (A - LC)^+\), \((A - LC)^- \triangleq (A - LC)^-\). It is easy to verify that \( \tilde{f}_{td} \) is increasing in \( x_1 \) since \((A - LC)^+ \geq 0\), is decreasing in \( x_2 \) since \((- (A - LC)^-) \leq 0\), and \( \tilde{f}_{td}(x, x) = ((A - LC)^+ - (A - LC)^-)x = (A - LC)x = f_t(x) \). Hence, \( \tilde{f}_{td} \) is a DT decomposition function of \( f_t \). The proof for tightness goes through similar lines of the proof of [18, Lemma 1]. As for the CT case, where \((A - LC)^+ \triangleq (A - LC)^+\) \((A - LC)^- \triangleq (A - LC)^-\), the proof is similar to the one for the DT case, with the slight difference that in the CT case, we need increasing monotonicity of \( \tilde{f}_{td} \) only in off-diagonal elements of \( x_1 \) (cf. Definition 3), which is guaranteed by non-negativity of \((A - LC)^{-}\). 

Next, we compute an embedding system for the nonlinear constituent system in (13), i.e., \( \phi(x_t) - L\psi(x_t) \), as follows.

**Lemma 2** (Nonlinear Embedding). Consider a dynamical system \( \mathcal{G}_t \) in the form of (13), with domain \( \mathcal{X} \) and state equation \( f_t(x_t) = \phi(x_t) - L\psi(x_t) \). Then, a decomposition function (cf. Definition 3) for \( \mathcal{G}_t \) can be computed as follows:

\[
f_{vd}(x_1, x_2) = \phi_d(x_1, x_2) - L^+ \psi_d(x_2, x_1) + L^- \psi_d(x_1, x_2), \quad (15)
\]

where \( \phi_d(\cdot, \cdot), \psi_d(\cdot, \cdot) \) are tight decomposition functions for the JSS mapping \( \phi(\cdot, \cdot), \psi(\cdot, \cdot) \), computed via Proposition 4.

**Proof.** \( f_{vd} \) is increasing in \( x_1 \) since it is a summation of three increasing mappings in \( x_1 \), including \( \phi_d(x_1, x_2) \) (a decomposition function that by construction is increasing in \( x_1 \)), \(- L^+ \psi_d(x_2, x_1) \) (a multiplication of the non-positive matrix \(- L^+ \) and the decomposition function \( \psi_d(x_2, x_1) \) which is decreasing on \( x_1 \) by construction) and \( L^- \psi_d(x_1, x_2) \) (a multiplication of the non-negative matrix \( L^- \) and the decomposition function \( \psi_d(x_1, x_2) \) which is itself increasing on \( x_1 \) by construction). Similar reasoning shows that \( g_{vd} \) is decreasing in \( x_2 \). Finally, \( f_{vd}(x, x) = \phi_d(x, x) - L^+ \psi_d(x, x) + L^- \psi_d(x, x) = \phi(x) - L\psi(x) = f_{\mu}(x) \).

We conclude this subsection by combining the results in Lemmas 1 and 2 as well as Proposition 3 that results in the following theorem on correctness of the proposed observer.

**Theorem 1** (Correct Interval Framer). Consider the non-linear plant \( \mathcal{G} \) in (7) and suppose Assumptions 4 and 5 hold. Then, the dynamical system \( \mathcal{G'} \) given as (9) constructs a correct interval framer for the nonlinear plant \( \mathcal{G} \). In other words, \forall \epsilon \in \mathbb{T}, x_t \leq x_t_1 \leq \mathcal{F}_{1}, \) where \( x_t \) and \( \mathcal{F}_{1} \) are the state vectors in \( \mathcal{G} \) and \( \mathcal{G'} \) at time \( t \in \mathbb{T} \), respectively.

**Proof.** It is straightforward to show that the summation of decomposition functions of constituent systems, is a decomposition function of the summation of the constituent systems. Combining this with Lemmas 1 and 2 implies that \( f_d(x_1, x_2) \triangleq g_{vd}(x_1, x_2) + g_{vd}(x_1, x_2) = (A - LC)^+ x_1 - (A - LC)^- x_2 + \phi_d(x_1, x_2) + \psi_d(x_2, x_1) + L^- \psi_d(x_1, x_2) \) is a decomposition function for the system in (13), and equivalently, for System (7). Consequently, the 2n-dimensional system \( \left( \mathcal{F}^{\epsilon}_{1} \right) \triangleq \left( \mathcal{F}^{\epsilon}_{1} \right)^+ \) of (7) with initial condition \( \mathcal{F}^{\epsilon}_{1} = \left( \mathcal{F}^{\epsilon}_{1} \right)^+ \), is an embedding system for (7) (cf. Definition 4). So, \( x_t \leq x_t_1 \leq \mathcal{F}_{1} \), by Proposition 3.

**C. Stability**

Besides the correctness property that we already obtained by construction, we are interested in studying the stability of the proposed framer. In other words, we wish to design the observer gain \( L \) such that the observer error, \( \epsilon_t \triangleq x_t - \mathcal{F}_{1} \), converges to zero asymptotically (cf. Definitions 6 and 7). Before stating our main results on observer stability, we derive some upper bounds for the interval widths of the JSS functions in terms of the interval widths of their domains, which will be helpful in deriving the stability conditions.

**Lemma 3** (JSS Function Interval Width Bounding). Let \( f : \mathcal{X} \subset \mathbb{R}^{n} \rightarrow \mathbb{R}^{p} \) be a mapping that satisfies the assumptions in Proposition 2 and hence, can be decomposed in the form of (10). Let \( \mu_{d,i} \triangleq \left[ \mu_{d,1} \ldots \mu_{d,p} \right] \mathcal{X} \rightarrow \mathbb{R}^{p} \) be the tight decomposition function for the JSS mapping \( \mu_{d} \), given in Proposition 4. Then, for any interval domain \( \mathcal{X} \), with \( x, x_{1} \in \mathcal{X} \), the following inequality holds:

\[
\Delta \mu_{d,i} \leq \mu_{d} \mathcal{X}, \text{ where } \left[ \mu_{d} \mathcal{X} \right] = \max(\mathcal{J} \mu - H, 0)_{p.n} - \mathcal{J} f + H. \quad (16)
\]

where \( \Delta \mu_{d,i} \triangleq \left[ \Delta \mu_{d,1} \ldots \Delta \mu_{d,p} \right] \triangleq \mu_{d}(\mathcal{X}) - \mu_{d}(\mathcal{X}) \) and \( \mathcal{X} \) is the interval width of \( \mathcal{X} \) (cf. Definition 7).

**Proof.** First, note that using Proposition 4, \forall i \in \mathbb{N}^m, \Delta \mu_{d,i} \triangleq \mu_{d,i}(\mathcal{X}) - \mu_{d,i}(\mathcal{X}) = i_{d,i}(D_{1}, I - D_{1})_{x} - \mu_{i,d}(D_{1}, I - D_{1})\mathcal{X}, \) where \( D_{1} \) is given in (6) with \( i_{d,i} \) defined below. Applying the mean value theorem, the last equality can be rewritten as \( \Delta \mu_{d,i} = \mu_{i,d}(\xi)(D_{1} - I)_{x} - \mu_{i,d}(D_{1} - I)\mathcal{X}, \) where \( \xi \in [\mathcal{X}, I] \), \( \mu_{i,d}(\xi) \in [i_{d,i} - H_{i} + H_{i}, i_{d,i} - H_{i} - H_{i}] \) and \( H_{i} \) is the \( i \)th row of \( H \). Combining this, Proposition 1 and the facts that \( D_{1} - I \geq 0 \) (since \( D_{1} \) is binary diagonal) and \( \epsilon \geq 0 \) (by the correctness property) yields

\[
\Delta \mu_{d,i} \leq \mathcal{J}_{i,d}, \text{ where } \mathcal{J}_{i,d} = \mathcal{J} f_{i,d} + H_{i}(I - D_{1}). \quad (17)
\]
On the other hand, from (6) we have 
\[ J_{f,i} D_i = J_{\mu,i} \max\{\text{sgn}(J_{\mu,i,e}), 0\} J_{\mu,i} \max\{\text{sgn}(J_{\mu,i,0}), 0\} = \max(J_{\mu,i}, 0_{1,n}) = \max(J_{f,i} - H_i, 0_{1,n}), \] (18)
where the second equality can be verified element-wise: for \( j \in N_n \), if \( J_{\mu,i,j} \) is positive sign-stable, then 
\[ \text{sgn}(J_{\mu,i,j}) = 1, \] hence \( \max(\text{sgn}(J_{\mu,i,j}), 0) = 1 \), and therefore \( J_{f,i} \max(\text{sgn}(J_{\mu,i,j}), 0) = J_{\mu,i,j} \). Moreover, the \( j \)’th entry of the row vector \( J_{\mu,i} \) also equals \( \max(J_{\mu,i,j}, 0) = J_{\mu,i,j} \) since \( J_{\mu,i,j} \geq 0 \) by positive sign-stability. The verification process can be done for the negative sign-stable case through similar reasoning. Next, combining (17) and (18) results in 
\[ F_{\mu,i} = 2 \max(J_{f,i} - H_i, 0_{1,n}) - J_{f,i} + H_i, \]
which when plugged into (17) and stacked for all \( i \in N_n \), returns the result in (16).

Now, equipped with the tools in Lemma 3, we derive sufficient LMI’s to synthesize the stabilizing observer gain \( L \) for both DT and CT systems through the following theorem.

**Theorem 2 (Stability).** Consider the nonlinear plant \( G \) and suppose all the assumptions in Lemma 7 hold. Then, the proposed correct interval framer \( \hat{G} \) is stable, and hence, is an interval observer in the sense of Definition 2 if there exist matrices \( R_n \times n \ni P > 0_{n,n}, X \in R_n x n \) and \( J \in \mathbb{R}^{l \times n} \), \( J \leq 0 \), such that

(i) (if \( G \) is a CT system)
\[
\begin{bmatrix}
\Omega & A \\
A^T & -\alpha(X + X^T)
\end{bmatrix} < 0, J^T C \in M_n, X \in D_n, (19)
\]
for all \( \alpha > 0 \), where \( \Omega \triangleq ((A^m + \bar{T}_\phi) X + X^T (A^m + \bar{T}_\phi)) + (C^T - \bar{T}_\psi) J + J(C - \bar{T}_\psi) \) and \( \Delta \triangleq P + \alpha((A^m + \bar{T}_\phi) X + \alpha(C - \bar{T}_\psi) J) \);

(ii) (if \( G \) is a DT system)
\[
\begin{bmatrix}
-P & \Gamma \\
\Gamma^T & P - X - X^T
\end{bmatrix} < 0, J^T C \leq 0, -X \in M_n, (20)
\]
where \( \Gamma \triangleq (|A| + \bar{T}_\phi) X - (|C| + \bar{T}_\psi)) J \).

Furthermore, in both cases, \( \bar{T}_\phi \) and \( \bar{T}_\psi \) are computed by applying Lemma 3 on the ISS functions \( \phi \) and \( \psi \), respectively. Finally, the corresponding stabilizing observer gain \( L \) can be obtained as \( L = -X^{(-1)} J^T \).

**Proof.** Starting from (9), we first derive the framer error \( (\varepsilon_t, x - \hat{x}) \) dynamics. Then, we show that the provided conditions in (19) and (20) are sufficient for stability of the error system in the CT and DT cases, respectively. To do so, define \( \Delta \mu = \mu - \hat{\mu} \) and \( \Delta \phi = \phi - \hat{\phi} \), \( \forall \mu \in \{\phi, \psi\} \) and note that the LMIs in (19) and (20) and Schur complements imply that \( X \) is positive definite and hence invertible (non-singular) in both CT and DT cases. Now, considering the CT case, from (9) and (10), we obtain the observer error dynamics:
\[
\dot{\epsilon}_t = ((A - LC)^d + |(A - LC)^{md}|)\varepsilon_t + \Delta \phi_{et} + |L| \Delta \psi_d \leq (A^d - (LC)^d + |A^md| + |(LC)^{md}| + \bar{T}_\phi + |L| \bar{T}_\psi)\varepsilon_t \leq (A^m + \bar{T}_\phi + |L| \bar{T}_\psi)\varepsilon_t, (22)
\]
where \( \forall \mu \in \{\phi, \psi\} \), \( \bar{T}_\mu \) is given in (16), the inequality holds by Lemma 3 Proposition 1 and the facts that \( \forall M, N \in R_{n \times n}, (M + N)^d = M^d + N^d, (M + N)^{md} = M^{md} + N^{md} \), \( |M + N| \leq |M| + |N| \) by triangle inequality and the fact that \( \varepsilon_t \geq 0 \) by the correctness property (Lemma 1). Now, note that by the Comparison Lemma [19, Lemma 3.4] and positivity of the systems in (21) and (22), stability of the system in (22) implies stability for the actual error system in (21). To show the former, we require the following:

i) \( J \) and \( X \) are non-positive and diagonal matrices, respectively: This forces \( X \) and its inverse to be diagonal matrices with strictly positive diagonal elements, and since \( J \) is forced to be non-positive, \( L = -(X^{-1}) J^T \) must be non-negative, and hence \( |L| = L \);

ii) \( J^T C \) is Metzler: This results in \( -LC = (X^{-1}) J^T C \) being Metzler, since it is a product of a diagonal and positive matrix \( (X^{-1}) \) and a Metzler matrix \( J^T C \), and it can be shown that their product is Metzler. Thus, \((LC)^m = -(LC)\).

By i) and ii), the system in (22) turns into the linear comparison system \( \dot{\varepsilon}_t \leq (A^m + \bar{T}_\phi + \bar{T}_\psi)|L|\varepsilon_t \), whose stability is guaranteed by the LMI in (19) by [20, (12)].

For the DT case, from (9) and (11) and by similar reasoning to the CT case, we obtain
\[
\varepsilon_{t+1} = |A - LC| \varepsilon_t + \Delta \phi_{et} + |L| \Delta \psi_d \leq (|A| + |LC| + \bar{T}_\phi + |L| \bar{T}_\psi)\varepsilon_t, (23)
\]
In addition, we enforce \(-X \) to be Metzler, as well as \( J \) and \( J^T C \) to be non-positive. Consequently, since \( X \) is positive definite, \( X \) becomes a non-singular M-matrix \( \frac{1}{2} \) and hence is inverse-positive [21, Theorem 1], i.e., \( X^{-1} \geq 0 \). Therefore, \( L = -(X^{-1}) J^T \geq 0 \) and because they are matrix products of non-negative matrices, \( (X^{-1}) \) and \( -(J^T C) \), respectively. Hence, \( |L| = L, |LC| = LC \), and so, the system in (22) turns into \( \varepsilon_{t+1} \leq (|A| + |LC| + \bar{T}_\phi + \bar{T}_\psi)\varepsilon_t \), which is stable if the LMI in (20) holds, by [20, (10)].

Finally, note that a coordinate transformation (cf. [22] and references therein) may also be helpful for making the LMIs in Theorem 2 feasible, as observed in Section V-A.

V. ILLUSTRATIVE EXAMPLES

The effectiveness of our interval observer design is illustrated for CT and DT systems (using YALMIP [23]).

A. CT System Example
Consider the CT system in [24, Section IV, Eq. (30)]:
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = b_1 x_3 - a_1 \sin(x_1) - a_2 x_2, \quad \dot{x}_3 = -a_2 a_3 x_3 + \frac{a_1}{b_1} (a_4 \sin(x_1) + \cos(x_1)) x_2 - a_3 x_2 - a_4 x_3,
\]
with output \( y = x_1 \). Without a coordinate transformation, the LMIs in (19) were infeasible, but with a coordinate transformation \( z = T x \) with \( T = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & 0.06 \\ 0 & 0.1 & -10 \end{bmatrix} \) and adding
\[
\begin{array}{c}
a_1 = 35.63, b_1 = 15, a_2 = 0.25, a_3 = 36, a_4 = 200, x_0 = \begin{bmatrix} 19.5 & 9 \end{bmatrix} \times [0.5, 1.5].
\end{array}
\]

An M-matrix is a square matrix whose negation is Metzler and whose eigenvalues have nonnegative real parts.
and subtracting 5\(y\) to the dynamics of \(\dot{x}_1\), we obtained the observer gain \(L = 10^{-6} \times [3.44 \ 0 \ 0.04]^\top\). As shown in Figure 1 (\(x_1, x_2\) omitted for brevity), the state framers returned by our approach, \(\mathcal{F}\), are tighter than the ones obtained by the interval observer in [24], \(\mathcal{F}_{DMN}\) (primarily because of outer-approximations of the initial framers \(X_0\) due to different coordinate transformations). Further, the framer error \(\epsilon_t = \mathcal{F}_t - \mathcal{F}_2\) is observed to tend to zero asymptotically.

**B. DT System Example**

Consider a variant of DT Hénon chaos system model [25]:

\[
x_{t+1} = Ax_t + r[1 - x_{t,1}^2], \quad y_t = x_{t,1},
\]

where \(A = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}\), \(r = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix} \) and \(X_0 = [-2, 2] \times [-1, 1]\). E

Employing YALMIP to solve the corresponding LMIs in (20), a stabilizing observer gain \(L = [0.0393 \ 0.0346]^\top\) was obtained. Figure 2 (\(x_1\) omitted for brevity) shows that our observer returns comparable estimates to those obtained from the approach in [7] and the framer error converges to zero.

**VI. CONCLUSION**

A novel unified interval observer synthesis approach was presented for locally Lipschitz nonlinear continuous-time (CT) and discrete-time (DT) systems with nonlinear observations. Leveraging mixed-monotone decompositions, the proposed observer satisfies the correctness property by construction, i.e., the true state trajectory of the system was shown to be framed by the states of the observer at all times, without needing restrictive assumptions such as global Lipschitz continuity or contraction. Moreover, by solving a semi-definite program based on some sufficient conditions with LMIs, a stabilizing observer gain was designed to ensure that the observer errors converge to zero asymptotically. Finally, the effectiveness of the proposed observer, when compared to some benchmark observers, was demonstrated using illustrative DT and CT system examples. In our future work, we will consider noise and uncertainties within our framework as well as hybrid system dynamics, and extend our approach to simultaneous input and state estimation.

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