Reduced Chern–Simons quiver theories and cohomological 3-algebra models

Joshua DeBellis$^{1,2,3,*}$ and Richard J. Szabo$^{1,2,3,*}$

$^1$Department of Mathematics, Heriot-Watt University, Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, UK
$^2$Maxwell Institute for Mathematical Sciences, Edinburgh, UK
$^3$The Tait Institute, Edinburgh, UK
*E-mail: jd111@hw.ac.uk, R.J.Szabo@hw.ac.uk

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We study the BPS spectrum and vacuum moduli spaces in dimensional reductions of Chern–Simons-matter theories with $\mathcal{N} \geq 2$ supersymmetry to zero dimensions. Our main example is a matrix model version of the ABJM theory that we relate explicitly to certain reduced 3-algebra models. We find the explicit maps from Chern–Simons quiver matrix models to dual IKKT matrix models. We address the problem of topologically twisting the ABJM matrix model, and along the way construct a new twist of the IKKT model. We construct a cohomological matrix model whose partition function localizes onto a moduli space specified by 3-algebra relations that live in the double of the conifold quiver. It computes an equivariant index enumerating framed BPS states with specified R-charges that can be expressed as a combinatorial sum over certain filtered pyramid partitions.

1. Introduction and summary

In this paper we study certain supersymmetric matrix models that are related to Chern–Simons-matter theories in three dimensions with $\mathcal{N} \geq 2$ supersymmetry. Our motivation comes from attempting to understand the spectrum of Bogolmony-Prasad-Sommerfeld (BPS) states and the geometry of their moduli spaces. A powerful tool for the enumeration of supersymmetric vacua is provided by the Witten index since it is invariant under deformations of the continuous parameters of the field theory. However, supersymmetric gauge theories have much richer structures that are only partially captured by the Witten index; to extract more information about the field theory, one needs to exploit its symmetries. In three dimensions, a generalization of the Witten index is constructed using not only the dilatation operator $H$, but also the $SO(2)$ angular momentum $J$ and the generators $R_i$ of the Cartan subalgebra of the R-symmetry group; schematically this refined index is given by

$$\mathcal{I}(x, y, t) = \text{Tr}_{\mathcal{H}_{\text{BPS}}} (-1)^F x^H y^{2J} \prod_i t_i^{R_i}$$

(1.1)

where the fugacities $x, y, t$ are inserted to resolve degeneracies. Like the Witten index, it can be interpreted as a Feynman path integral with euclidean action by compactifying the time direction on a circle $S^1$ with supersymmetric twisted boundary conditions involving the $SO(2)$ rotation $J$ and the global R-symmetry twists $R_i$; then $H$ is the generator of translations along $S^1$ and $\mathcal{H}_{\text{BPS}}$ is the Hilbert space of the theory with $\mathbb{R}^2$ regarded as the spatial slice. In the weak coupling limit $x \to 0$
where the circle decompactifies, this theory reduces to a supersymmetric quantum mechanics on the moduli space of BPS solutions; in this paper we aim to study an effective theory of these sorts of reduced models.

As these field theories are expected to flow to fixed points with superconformal symmetry, superconformal field theories have found a distinctive place in the study of supersymmetric gauge theories as well as in the AdS/CFT correspondence. The pertinent index is then the superconformal index that coincides with the partition function of the three-dimensional superconformal field theory in radial quantization on \( S^1 \times S^3 \). A large class of \( \mathcal{N} = 2 \) superconformal field theories in three dimensions arises as the low-energy effective theory of multiple M2-branes probing a Calabi–Yau fourfold singularity. Using AdS\(_4\)/CFT\(_3\) duality, they have a holographic dual description as M-theory on the product of four-dimensional anti-de Sitter space AdS\(_4\) and a seven-dimensional Sasaki–Einstein manifold \( Y_7 \), which is the base of the probed Calabi–Yau fourfold cone. The simplest example is provided by the theory of M2-branes at the orbifold singularity \( \mathbb{C}^4/\mathbb{Z}_k \), which in the low-energy limit is a Chern–Simons quiver gauge theory at level \( k \) with \( \mathcal{N} = 6 \) supersymmetry [1,2]; the holographic dual theory is M-theory on AdS\(_4\) \( \times \) \((S^7/\mathbb{Z}_k)\). For \( k = 1, 2 \), the \( \mathcal{N} = 6 \) supersymmetry is enhanced to \( \mathcal{N} = 8 \), where the additional SO(8) R-symmetry generators are monopole operators [1,3,4]. By regarding \( S^7 \) as a circle bundle over \( \mathbb{C}P^3 \) via the Hopf fibration with the orbifold action on the \( S^1 \) fiber, in the Type IIA frame the dual theory is supergravity on AdS\(_4\) \( \times \) \( \mathbb{C}P^3 \). This theory is the celebrated Aharony-Bergman-Jafferis-Maldacena (ABJM) theory; in this paper we mostly work with the more general ABJ theory [5] with gauge group \( U(N_L) \times U(N_R) \), which we still refer to as “ABJM models” for simplicity, though we refer to the specialization \( N_L = N_R = N \) as the “ABJM limit”. The superconformal index for ABJM theory is computed in Refs. [6,7], where it is matched with that of the dual Type IIA theory and also M-theory in the large \( N \) limit (see Ref. [8] for a review of superconformal indices in three dimensions).

In this work we study in detail zero-dimensional reductions of three-dimensional Chern–Simons quiver theories. This analysis is partly inspired by the observation that the partition function of any \( \mathcal{N} \geq 2 \) Chern–Simons-matter theory on \( S^3 \) with no anomalous dimensions is given by a matrix integral over the Cartan subalgebra of the gauge group [9]; although our supersymmetric matrix models are structurally rather different, we shall find that the matrix integrals reduce in the same way. The result of Ref. [9] can be used to confirm that the free energy of the infrared \( \mathcal{N} = 6 \) superconformal field theory on \( N \) M2-branes has the expected \( N^{3/2} \) scaling behavior for large \( N \), and it provides various precise checks of the AdS\(_4\)/CFT\(_3\) correspondence (see Ref. [10] for a review); more precisely, in the \( 't \) Hooft limit \( N, k \to \infty \) with \( \lambda := \frac{N}{k} \) fixed, the corresponding free energy grows like \( (N k)^{3/2} \) times the volume of the \( S^7/\mathbb{Z}_k \) orbifold. More generally, for M-theory on AdS\(_4\) \( \times Y_7 \), where \( Y_7 \) is a seven-dimensional Sasaki–Einstein space threaded by \( N \) units of flux, the gravitational free energy to leading order as \( N \to \infty \) is

\[
\mathcal{F} = N^{3/2} \sqrt{\frac{2\pi^6}{27 \text{Vol}(Y_7)}},
\]

where \( \text{Vol}(Y_7) \) is the riemannian volume of \( Y_7 \) with respect to its Sasaki–Einstein metric; by the AdS/CFT correspondence, this result can be compared with computations in the dual Chern–Simons-matter field theories that flow to superconformal fixed points. The partition functions of three-dimensional \( \mathcal{N} = 2 \) superconformal field theories are similarly computed in Refs. [11–13]; their magnitudes are extremized at the superconformal R-charges of the infrared conformal field.
theory. Here we take the point of view that supersymmetry allows one to relate the superconformal index to a matrix integral, with action given by the reduction to zero dimensions of the three-dimensional superconformal field theory. Supersymmetric matrix models analogous to ours, obtained by dimensional reduction of $\mathcal{N} \geq 2$ Chern–Simons quiver gauge theories on $S^3$ to zero dimensions, are considered in Refs. [14,15]; one-dimensional reductions of the ABJM theory are studied in Ref. [16].

Although much of our discussion will apply to reductions of generic Chern–Simons quiver gauge theories, in the present paper we focus mostly on the partition functions of Chern–Simons-matter theories with the highest amount $\mathcal{N} \geq 6$ of supersymmetry, i.e. the ABJM theory. In addition to the fact that they are important for the understanding of various aspects of M-theory, it is these theories whose moduli spaces can be described as geometric invariant theory (GIT) quotients of bracket relations in certain 3-algebras. The Van Raamsdonk formulation of the ABJM model [17] is a matrix field theory describing stacks of M2-branes, and with a gauge group isomorphic to $SU(2) \times SU(2)$ it is an $\mathcal{N} = 8$ field theory that is equivalent to the Bagger-Lambert-Gustavsson (BLG) theory [18,19]; the BLG model is a three-dimensional $\mathcal{N} = 8$ superconformal Chern–Simons theory whose matter fields take values in a 3-Lie algebra with inner product compatible with the 3-bracket. A general $\mathcal{N} = 6$ theory of multiple M2-branes was formulated in Ref. [20] by relaxing the requirement of total antisymmetry of the 3-bracket; then one may also reformulate the ABJM theory with matter fields taking values in a particular hermitian 3-algebra. In this way one may study the novel geometry of the supersymmetric gauge theory moduli spaces using 3-algebra structures. This program was initiated and studied in Ref. [21], where dimensional reduction of the BLG model was introduced and its BPS solutions were related to the quantization of Nambu–Poisson manifolds, in the sense of Ref. [22], which arise as worldvolume geometries of M2-branes and M5-branes. In this paper we explore the relationship between our Chern–Simons quiver matrix models and reduced 3-algebra models, and exploit it in our studies of the vacuum moduli spaces.

Another reason for focusing on these particular classes of reductions of Chern–Simons quiver theories is that these are the ones that bear an intimate relationship to the better understood and behaved Yang–Mills matrix models; in this paper we detail these matrix model relationships and use them as guidance in our computations of partition functions. The ABJM theory for $k = 1$ is believed to have $\mathcal{N} = 8$ supersymmetry and to be isomorphic to the infrared fixed point of maximally supersymmetric Yang–Mills theory in three dimensions. The partition functions for the (mass deformed) theories on $S^3$ are shown by Ref. [23] to agree, while their superconformal indices are matched in Ref. [24] and also with those of supergravity. The ABJM model is also dual under mirror symmetry to $\mathcal{N} = 4 U(N)$ supersymmetric Yang–Mills theory coupled to a single fundamental hypermultiplet and a single adjoint hypermultiplet. Structural similarities between observables in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and ABJM theory are also pointed out in Ref. [25,26]. In this paper we shall exploit such equivalences as a duality between our reduced ABJM model and the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model in ten dimensions [27]. Reduced Yang–Mills theories play a central role in the nonperturbative definitions of M-theory and superstring theory [27,28]. Yang–Mills matrix models also provide a nonperturbative framework for emergent spacetime geometry and noncommutative gauge theories [29–32] (see Refs. [33,34] for reviews); in particular, the quantum geometries that arise as BPS solutions in the reduced 3-Lie algebra model of Ref. [21] are dual to those of the IKKT model in ten dimensions. Investigations of the ABJM model as a new kind of matrix model for M-theory, which is perhaps more fundamental than the BFSS matrix theory [28], were carried out in Ref. [35].
The final simplifying aspect of this paper is that we wish to formulate and solve topologically twisted models that properly localize the dynamics onto the BPS moduli spaces; these are theories that have a fermionic scalar symmetry that is a twisted version of the supersymmetry of the original physical theory. While the IKKT Yang–Mills matrix models can be deformed and solved using the powerful formalism of cohomological Yang–Mills theory [36–38], we have not succeeded in finding analogous twists for our reduced Chern–Simons quiver models. Nevertheless, by using the duality with the IKKT model we are able to construct a cohomological matrix model with \( \mathcal{N} = 2 \) supersymmetry “by hand”, which possesses the desired properties and should capture the salient features of the index we are after. The major difference in our treatment is that we regard the F-term relations for the bifundamental component scalars as relations in the double of the quiver characterizing the Chern–Simons-matter theory, as opposed to the actual relations of the quiver, which are the F-term relations of the supersymmetric gauge theory among chiral superfields derived from the superpotential; this means that our moduli space is built on the cotangent bundle of the representation variety of the original quiver with fixed dimension vector. For the reduced ABJM theory, this quiver is the well studied conifold quiver and we are able to take this calculation through to the end. We explicitly compute an equivariant index that enumerates \( \text{SU}(4) \) R-charge assignments of framed BPS states; the equivariant index receives contributions from only those BPS states that are fixed by the action of the maximal torus of the R-symmetry group, and its explicit combinatorial formula is given as a sum over filtered pyramid partitions related to length two empty room configurations. This index can also be used to compute a regularized volume of the non-compact vacuum moduli space. We suggest that this quantity can be interpreted as a character in a dimensional reduction of M-theory to one dimension; however, a complete physical interpretation of our equivariant index is not clear at the moment, and in particular if it can be interpreted in some way as a “superconformal index”.

The structure of the remainder of this paper is as follows. In Sect. 2 we review some ternary algebra structures, in particular 3-Lie algebras, hermitian 3-algebras, and some specific examples we use later in the paper. Section 3 is dedicated to constructing the various reduced Chern–Simons quiver models we study. In Sect. 4 we explain how these quiver models are related to the reduced 3-algebra model introduced in Ref. [21] by taking various scaling limits, or by making particular choices of 3-algebra. The duality between the Chern–Simons quiver matrix models and the IKKT matrix model using the Mukhi–Papageorgakis map [39] is explained in Sect. 5 In Sect. 6 we apply the Mukhi–Papageorgakis map to a particular topological twist of the BLG theory, thus yielding a topologically twisted version of \( \mathcal{N} = 8 \) supersymmetric Yang–Mills theory in three dimensions; we dimensionally reduce this theory to find a new topologically twisted version of the IKKT matrix model, and explain why it is not possible to lift this new twist to the reduced ABJM theory. Finally, in Sect. 7 we construct a cohomological matrix model related to the reduced ABJM theory and compute an equivariant index using localization methods.

2. Ternary algebras

In this section we review the definitions and properties of the 3-algebras that we will encounter in this paper, primarily to set up our conventions and notation.

2.1. Metric 3-Lie algebras

A metric 3-Lie algebra is a finite-dimensional real vector space \( \mathcal{A} \) equipped with a positive-definite symmetric bilinear form \((-,-): \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R} \) and a totally antisymmetric trilinear map \([−,−,−]:\):
\( \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \). The 3-bracket satisfies the fundamental identity

\[
[X, Y, [Z_1, Z_2, Z_3]] = [[X, Y, Z_1], Z_2, Z_3] + [Z_1, [X, Y, Z_2], Z_3] + [Z_1, Z_2, [X, Y, Z_3]] \tag{2.1}
\]

for \( X, Y, Z_1, Z_2, Z_3 \in \mathcal{A} \), and it is compatible with the metric in the sense that

\[
([Z_1, Z_2, X], Y) + (X, [Z_1, Z_2, Y]) = 0. \tag{2.2}
\]

Given a basis \( \{\tau_a\} \) of generators for \( \mathcal{A} \), the 3-bracket

\[
[\tau_a, \tau_b, \tau_c] = f_{abcd} \tau_d \tag{2.3}
\]

gives the totally antisymmetric structure constants \( f_{abcd} \) of \( \mathcal{A} \).

Every metric 3-Lie algebra admits an associated Lie algebra, denoted \( g_{\mathcal{A}} \). Given \( (Y_1, Y_2) \in \mathcal{A} \wedge \mathcal{A} \), we define an operator \( D_{(Y_1, Y_2)} \in \operatorname{End} (\mathcal{A}) \) by the 3-Lie bracket

\[
D_{(Y_1, Y_2)}(X) = [Y_1, Y_2, X]. \tag{2.4}
\]

The fundamental identity (2.1) implies that \( D_{(Y_1, Y_2)} \) is an inner derivation of \( \mathcal{A} \). The linear span of the collection of operators \( \{D_{(Y_1, Y_2)}\} \) forms the Lie algebra \( g_{\mathcal{A}} \), which is a Lie subalgebra of \( \mathfrak{so}(\mathcal{A}) \). The fundamental identity guarantees closure of the commutator bracket in \( \operatorname{End} (\mathcal{A}) \), which is expressed in terms of the 3-Lie bracket as

\[
[D_{(Y_1, Y_2)}, D_{(Z_1, Z_2)}](X) = [[Y_1, Y_2, Z_1], Z_2, X] + [Z_1, [Y_1, Y_2, Z_2], X]. \tag{2.5}
\]

The Jacobi identity for \( g_{\mathcal{A}} \) also follows from the fundamental identity (2.1), while the compatibility condition (2.2) implies that the metric of \( \mathcal{A} \) is \( g_{\mathcal{A}} \)-invariant; moreover, the metric on \( \mathcal{A} \) induces an ad-invariant symmetric bilinear form \( \operatorname{Tr}_{g_{\mathcal{A}}} \) on \( g_{\mathcal{A}} \) given by

\[
\operatorname{Tr}_{g_{\mathcal{A}}}(D_{(Y_1, Y_2)} D_{(Z_1, Z_2)}) := ([Y_1, Y_2, Z_1], Z_2), \tag{2.6}
\]

for which every element is null, i.e. \( \operatorname{Tr}_{g_{\mathcal{A}}}(D_{(Y_1, Y_2)}^2) = 0 \). The generators of \( g_{\mathcal{A}} \) are the operators \( D_{ab} \in \operatorname{End} (\mathcal{A}) \) given by

\[
D_{ab}(\tau_c) = f_{abcd} \tau_d. \tag{2.7}
\]

Together with a real finite-dimensional orthogonal representation, the subalgebra \( g_{\mathcal{A}} \) can be used to reconstruct the 3-Lie algebra \( \mathcal{A} \) via the Faulkner construction [40].

One can also reduce a 3-Lie algebra \( \mathcal{A} \) to a Lie algebra \( \mathcal{A}' \), which is generally different from \( g_{\mathcal{A}} \) [41]. One chooses a fixed element \( Z_0 \in \mathcal{A} \) and identifies the vector space \( \mathcal{A}' \) with \( \mathcal{A} \). The Lie bracket on \( \mathcal{A}' \) is defined as

\[
[X, Y] = [X, Y, Z_0] \tag{2.8}
\]

for \( X, Y \in \mathcal{A}' \), and the corresponding Jacobi identity for \( \mathcal{A}' \) likewise follows from (2.1).

All 3-Lie algebras with metric of euclidean signature are direct sums of the four-dimensional 3-algebra \( \mathcal{A} = A_4 \). It is defined by generators \( \tau_a, a = 1, 2, 3, 4 \), obeying the relations \( [\tau_a, \tau_b, \tau_c] = \epsilon_{abcd} \tau_d \) and \( (\tau_a, \tau_b) = \delta_{ab} \). The associated Lie algebra is \( g_{A_4} = \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \), while the reduced Lie algebra for any fixed \( Z_0 = \tau_a \) is \( A_4' = \mathfrak{so}(3) = \mathfrak{su}(2) \). On the other hand, applying the Faulkner construction to the pair \( (\mathfrak{so}(4), \mathbb{R}^4) \) with inner product on \( \mathfrak{so}(4) \) given by the Cartan–Killing form makes the fundamental representation \( \mathbb{R}^4 \) into a Lie triple system [40].
2.2. Hermitian 3-algebras

We will now relax the requirement of total antisymmetry of the 3-bracket; these 3-algebras are generally called 3-Leibniz algebras. Here we are interested in the special class of 3-Leibniz algebras called hermitian 3-algebras. They comprise a complex metric 3-algebra that is a finite-dimensional complex vector space \( \mathcal{A} \) equipped with a hermitian inner product \((-,-)\) and a trilinear map \([-,-;-] : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}\). We require that the 3-bracket is antisymmetric in its first two entries only, and that it is complex linear in its first two arguments and complex antilinear in its third argument. It satisfies a version of the fundamental identity given by

\[
[[Z_1, Z_2; Z_3], X; Y] = [[Z_1, X; Y], Z_2; Z_3] + [Z_1, [Z_2, X; Y], Z_3] - [Z_1, Z_2; [Z_3, Y; X]],
\]  

(2.9)

and also the metric compatibility conditions

\[
([Z_1, Z_2; X], Y) - ([Z_2, Z_1; Y], X) = 0,
\]

\[
([X, Z_1; Z_2], Y) - (X, [Y, Z_2; Z_1]) = 0.
\]  

(2.10)

On generators \(\{\tau_a\}\) for \(\mathcal{A}\) the 3-bracket

\[
[\tau_a, \tau_b; \tau_c] = f_{abcd} \tau_d
\]  

(2.11)

defines the structure constants \(f_{abcd}\) of \(\mathcal{A}\); they are antisymmetric in their first two indices and have the additional symmetry properties \(f_{abcd} = f_{cdab} = f_{bacd}\).

Every complex metric 3-algebra satisfying the fundamental identity (2.9) admits an associated Lie algebra \(\mathfrak{g}_A\). The generators of \(\mathfrak{g}_A\) are defined to be operators \(D_{ab} \in \text{End}(\mathcal{A})\) expressed in terms of the 3-bracket as

\[
D_{ab}(\tau_c) := [\tau_c, \tau_a; \tau_b] = f_{cabd} \tau_d.
\]  

(2.12)

The fundamental identity guarantees closure of their commutator bracket

\[
[D_{ab}, D_{cd}](\tau_e) = [[\tau_e, \tau_c; \tau_d], \tau_a; \tau_b] - [[\tau_e, \tau_a; \tau_b], \tau_c; \tau_d],
\]  

(2.13)

as well as the Jacobi identity for \(\mathfrak{g}_A\).

In this paper we are primarily interested in the following hermitian 3-algebra. Consider the vector space \(\mathcal{A} = \text{Hom}(V_L, V_R)\) of linear maps \(X : V_L \rightarrow V_R\) between two complex inner product spaces \(V_L\) and \(V_R\). The 3-bracket defined by

\[
[X, Y; Z] = \lambda(X Z^\dagger Y - Y Z^\dagger X),
\]  

(2.14)

for an arbitrary constant \(\lambda \in \mathbb{C}\), satisfies the fundamental identity (2.9). The metric on \(\mathcal{A}\) given by the Schmidt inner product

\[
(X, Y) = \text{Tr}_{V_L}(X^\dagger Y)
\]  

(2.15)

then satisfies the compatibility conditions (2.10). This 3-algebra has associated Lie algebra \(\mathfrak{g}_A = \mathfrak{u}(V_L) \oplus \mathfrak{u}(V_R)\): An endomorphism \(\phi = (\phi_L, \phi_R) \in \mathfrak{g}_A\) acts on \(X \in \mathcal{A}\) as

\[
\phi X = X \phi_L - \phi_R X.
\]  

(2.16)
2.3. Lorentzian 3-Lie algebras

A large class of 3-Lie algebras \( A_h \) with compatible metric of lorentzian signature are described as the semisimple indecomposable lorentzian 3-Lie algebras of dimension \( d + 2 \), which are obtained by double extension from a semisimple Lie algebra \( h \) of dimension \( d \) [42]. Let \( \tau_a, a = 1, \ldots, d \), be a set of generators for \( h \) with antisymmetric structure constants \( f_{abc} \) defined by the Lie bracket \([\tau_a, \tau_b] = f_{abc} \tau_c\). The 3-Lie algebra \( A_h \) has generators \( \tau_0, \tau_a, \) and \( 1 \) with the 3-bracket relations

\[
[\tau_a, \tau_b, \tau_c] = f_{abc} \, 1, \quad [\tau_0, \tau_a, \tau_b] = f_{abc} \tau_c, \quad [1, \tau_a, \tau_b] = 0 = [1, \tau_a, \tau_0] \tag{2.17}
\]

and the inner product relations

\[
(1, 1) = 0, \quad (1, \tau_a) = 0, \quad (1, \tau_0) = -1, \quad (\tau_0, \tau_0) = \beta, \quad (\tau_0, \tau_a) = 0, \quad (\tau_a, \tau_b) = \delta_{ab}, \tag{2.18}
\]

where \( \beta \in \mathbb{R} \) is an arbitrary constant. Note that with \( Z_0 = \tau_0 \), the reduced bracket (2.8) coincides with the Lie bracket of \( h \) and \( A'_h = h \oplus \mathbb{R} \). On the other hand, the associated Lie algebra of \( A_h \) is the semi-direct sum

\[
\mathfrak{g}_{A_h} = u(1)^d \ltimes h. \tag{2.19}
\]

3. Dimensional reduction of Chern–Simons-matter theories

In this section we study the dimensional reduction of three-dimensional supersymmetric Chern–Simons-matter theories to zero dimensions. The resulting reduced models are supersymmetric quiver matrix models, which will be the focal points of our analysis in this paper. Depending on the choice of gauge group and superpotential, one is able to construct models with varying amounts of supersymmetry. From the point of view of the underlying quivers, the quiver theories we construct are quiver quantum mechanics describing BPS particles; the representations of the quiver then correspond to BPS bound states. Although the Chern–Simons level \( k \) must be an integer to ensure large gauge invariance in the field theory, our matrix integrals are well defined for non-integer \( k \) and hence we do not impose any quantization condition on the Chern–Simons coupling constant in what follows.

3.1. \( \mathcal{N} = 2 \) Chern–Simons quiver matrix models

The field content for the \( \mathcal{N} = 2 \) supersymmetric Chern–Simons gauge multiplet \( V \) in three-dimensional flat space \( \mathbb{R}^{1,2} \) consists of a gauge field \( A_{\mu}, \mu = 0, 1, 2 \), two auxiliary scalar fields \( D \) and \( \sigma \), and a two-component complex auxiliary fermion field \( \lambda \). The fields are valued in the Lie algebra \( \mathfrak{g} \) of a matrix gauge group \( G \). The action is given by

\[
S_g = \int d^3 x \, \kappa \, \text{Tr}_\mathfrak{g} \left( \epsilon^{\mu \nu \lambda} \left( A_\mu \partial_\nu A_\lambda + \frac{2i}{\lambda} A_\mu A_\nu A_\lambda \right) - \bar{\lambda} \lambda + 2D \sigma \right), \tag{3.1}
\]

where \( \kappa \in \mathbb{R} \) is a coupling constant and \( \text{Tr}_\mathfrak{g} \) is an invariant quadratic form on the Lie algebra \( \mathfrak{g} \). The generators of the Clifford algebra \( \mathcal{C}\ell(\mathbb{R}^{1,2}) \) are the gamma-matrices \( \gamma^\mu \) that satisfy the anticommutation relations

\[
\{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu \nu} \tag{3.2}
\]

and are taken to be Pauli spin matrices

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.3}
\]
while the spinor adjoint is
\[ \bar{\lambda} = \lambda^\dagger \gamma^0. \]  

We perform a dimensional reduction to zero dimensions in which the gauge fields \( A_\mu \) become a collection of \( g \)-valued scalar fields, and similarly for the other fields of \( V \). The reduced action is
\[ S_g = \kappa \text{Tr}_g (\sum_i \epsilon^{\mu_1 \nu_1} A_{\mu_1} A_{\nu_1} - \bar{\lambda} \lambda + 2D \sigma). \]

This action is invariant under the \( N = 2 \) supersymmetry transformations
\[
\begin{align*}
\delta A_\mu &= \frac{i}{2} (\bar{\eta} \gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \eta), \\
\delta \sigma &= \frac{i}{2} (\bar{\eta} \lambda - \bar{\lambda} \eta), \\
\delta D &= \frac{i}{2} (\bar{\eta} \gamma^{\mu \nu} [A_\mu, \lambda] + [A_\mu, \lambda] \gamma^{\mu \nu} \eta) + \frac{i}{2} (\bar{\eta} [\lambda, \sigma] + [\lambda, \sigma] \eta), \\
\delta \lambda &= -i \left( \frac{1}{2} \gamma^{\mu \nu} [A_\mu, A_\nu] + D + \gamma^{\mu} [A_\mu, \sigma] \right) \eta, \\
\delta \bar{\lambda} &= i \bar{\eta} \left( -\frac{1}{2} \gamma^{\mu \nu} [A_\mu, A_\nu] + D + \gamma^{\mu} [A_\mu, \sigma] \right),
\end{align*}
\]

where \( \eta \) and \( \epsilon \) are two independent Dirac spinors of \( \text{SO}(1, 2) \) and \( \gamma^{\mu \nu} := \frac{1}{2} [\gamma^\mu, \gamma^\nu] \). The two supersymmetry transformations generated by \( \eta \) or \( \epsilon \) alone commute. The commutator of an \( \eta \)-supersymmetry with an \( \epsilon \)-supersymmetry generates a sum of a gauge transformation, a Lorentz rotation, a dilatation, and an \( R \)-rotation.

This action can be extended to include supersymmetric matter fields. The matter content is a chiral multiplet \( \Phi \) with component fields \( \Phi = (Z, Z^\dagger, \psi, \bar{\psi}, F, F^\dagger) \), which are also valued in the Lie algebra \( g \). The field \( Z \) is a complex matter field, \( F \) is an auxiliary complex scalar field, and \( \psi \) is a two-component Dirac spinor field. The action reads
\[
S_m = \int d^3x \text{Tr}_g (\nabla_\mu Z^\dagger \nabla^{\mu \nu} Z - Z^\dagger \sigma Z + Z^\dagger D Z + F^\dagger F \\
+ i \bar{\psi} \gamma^{\mu \nu} \nabla_\mu \psi - \bar{\psi} \sigma \psi - i \bar{\psi} \lambda Z + i Z^\dagger \bar{\lambda} \psi),
\]

where the gauge covariant derivatives act as \( \nabla_\mu Z := \partial_\mu Z + i [A_\mu, Z] \). We perform a dimensional reduction as above, so that the reduced matter action reads as
\[
S_m = \text{Tr}_g \left( -[A_\mu, Z^\dagger] [A_\mu, Z] - Z^\dagger \sigma Z + Z^\dagger D Z + F^\dagger F \\
- \bar{\psi} \gamma^{\mu} [A_\mu, \psi] - \bar{\psi} \sigma \psi - i \bar{\psi} \lambda Z + i Z^\dagger \bar{\lambda} \psi. \right)
\]

The supersymmetry transformations are given by
\[
\begin{align*}
\delta Z &= \bar{\eta} \psi, \\
\delta Z^\dagger &= \bar{\psi} \epsilon, \\
\delta \psi &= i \left( \gamma^{\mu} [A_\mu, Z] - \sigma Z \right) \epsilon + F \epsilon^*, \\
\delta \bar{\psi} &= i \bar{\eta} \left( \gamma^{\mu} [A_\mu, Z^\dagger] + Z^\dagger \sigma \right), \\
\delta F &= \bar{\eta}^* \left( \gamma^{\mu} [A_\mu, \psi] + i \lambda Z + \sigma \psi \right).
\end{align*}
\]

The complete action of the reduced \( \mathcal{N} = 2 \) Chern–Simons-matter theory thus reads as
\[
S = \text{Tr}_g \left( \kappa \left( \sum_i \epsilon^{\mu_1 \nu_1} A_{\mu_1} A_{\nu_1} - \bar{\lambda} \lambda + 2D \sigma \right) - [A_\mu, Z^\dagger] [A_\mu, Z] - Z^\dagger \sigma Z + Z^\dagger D Z + F^\dagger F \\
- \bar{\psi} \gamma^{\mu} [A_\mu, \psi] - \bar{\psi} \sigma \psi - i \bar{\psi} \lambda Z + i Z^\dagger \bar{\lambda} \psi. \right)
\]
The BRST transformations imply that the supersymmetric configurations satisfy

\[ [A_\mu, A_\nu] = 0 = [A_\mu, \sigma], \quad [A_\mu, Z] = 0 = [A_\mu, Z^\dagger], \quad D = 0 = F. \quad (3.11) \]

When the gauge group is a product of unitary groups

\[ G = \prod_{a=1}^{r} U(N_a), \quad (3.12) \]

we decompose the reduced vector multiplet as \( V = \bigoplus_a V^a \) where \( V^a \in \text{End} (V_a) \) are regarded as linear transformations of complex inner product spaces \( V_a = \mathbb{C}^{N_a} \) for \( a = 1, \ldots, r \), while the reduced matter multiplet is decomposed as \( \Phi = \bigoplus_{a,b} \Phi^{ab} \) with \( \Phi^{ab} \in \text{Hom} (V_a, V_b) \) and \( \Phi^{ab\dagger} \in \text{Hom} (V_b, V_a) \) for \( a, b = 1, \ldots, r \); then \( \text{Tr}_g \) refers to the trace in the fundamental representation of \( G \) that is possibly graded over the factors of \( G \). In this case the supersymmetric Chern–Simons-matter theory reduces to a quiver matrix model, which defines a finite-dimensional representation of the double of the quiver with \( r \) nodes that carry the gauge degrees of freedom \( A_\mu^a \) (plus their superpartners and auxiliary fields) transforming in the adjoint representation of \( U(N_a) \), and with an arrow from node \( a \) to node \( b \) for every non-zero matter field \( Z_{ab} \) (plus their superpartners and auxiliary fields) transforming in the bifundamental representation of \( U(N_a) \times U(N_b) \), along with an arrow in the opposite direction for the adjoint \( Z_{ab}^\dagger \). The double quiver is further equipped with a set of relations among the arrows that follow from the BPS equations of the supersymmetric gauge theory, which define a system of static quiver vortices; geometrically, representations of the double quiver are cotangent to representations of the original quiver. In this paper we will use this double quiver to construct and study the vacuum moduli spaces of supersymmetric Chern–Simons-matter theories in three dimensions in terms of moduli spaces of quiver representations.

### 3.2. \( A_1 \) quiver matrix model

The simplest example of the above construction is with a product gauge group

\[ G = U(N_L) \times U(N_R). \quad (3.13) \]

The matter content \( \Phi \) of the theory provides a representation of the double of the \( A_1 \) quiver

\[ \bullet \longrightarrow \bullet \quad (3.14) \]

We place complex inner product spaces \( V_L = \mathbb{C}^{N_L} \) and \( V_R = \mathbb{C}^{N_R} \) at the left and right nodes of the quiver (3.14), respectively. The matter field is regarded as a linear map \( Z : V_L \rightarrow V_R \) representing the arrow of the quiver (3.14), with hermitian conjugate \( Z^\dagger : V_R \rightarrow V_L \). The matrices \( Z, F, \) and \( \psi \) are bifundamental fields, i.e. they transform in the fundamental representation of \( U(N_R) \) and in the anti-fundamental representation of \( U(N_L) \). The vector multiplet has field content \( V = (A_{L,R}^\mu, \sigma_{L,R}^\lambda, \lambda_{L,R}^\mu, \bar{\lambda}_{L,R}^\mu, D_{L,R}^\mu) \). The matrices \( A_{L,R}^\mu \in \text{End} (V_{L,R}) \) for \( \mu = 0, 1, 2 \) transform in the adjoint representation of \( U(N_{L,R}) \), \( \lambda_{L,R}^\mu \) are two-component complex fermionic matrices, while \( \sigma_{L,R}^\mu \) and \( D_{L,R}^\mu \) are auxiliary matrix fields. The invariant quadratic form is given by \( \text{Tr}_g = \ldots \)
Tr_{V_L} \oplus (-\text{Tr}_{V_R})$, and the action of the quiver matrix model takes the form

\[ S_{A_1} = \text{Tr}_V \left( \kappa \left( \frac{2i}{3} e^{i\mu\nu} \lambda (A_{\mu}^L A_{\nu}^L A_{\lambda}^L - A_{\mu}^R A_{\nu}^R A_{\lambda}^R) - \bar{\lambda}^L \lambda^L + \bar{\lambda}^R \lambda^R + 2D^L \sigma^L - 2D^R \sigma^R \right) \right. 
- \left. (A_{\mu}^L Z^\dagger - Z^\dagger A_{\mu}^L) (A_{\mu}^R Z - Z A_{\mu}^R) - \bar{\psi} \gamma^\mu (A_{\mu}^R \psi - \psi A_{\mu}^L) + F^\dagger F \right. 
+ \left. Z D^L Z^\dagger - Z^\dagger D^R Z - i \bar{\psi} Z \lambda^L + i \bar{\psi} \lambda^R Z + i \bar{\lambda}^L Z^\dagger \psi - i Z^\dagger \bar{\lambda}^R \psi \right. 
+ \left. Z^\dagger \sigma^2 L - Z^\dagger \sigma^2 R Z + 2Z^\dagger \sigma^R Z \sigma^L - \bar{\psi} \psi \sigma^L + \bar{\psi} \sigma^R \psi \right), \tag{3.15} \]

where the trace is taken over $V = V_L$ or $V = V_R$ where appropriate. The supersymmetry transformations of this model are given by

\[ \delta A_{\mu}^{L,R} = \frac{i}{2} \left( \bar{\eta} \gamma^\mu \lambda^{L,R} - \bar{\lambda}^{L,R} \gamma^\mu \eta \right), \]
\[ \delta \sigma^{L,R} = \frac{i}{2} \left( \bar{\eta} \lambda^{L,R} - \bar{\lambda}^{L,R} \eta \right), \]
\[ \delta D^{L,R} = \frac{i}{2} \left( \bar{\eta} \gamma^\mu \left[ A_{\mu}^{L,R}, \lambda^{L,R} \right] + \frac{1}{2} \left[ A_{\mu}^{L,R}, \bar{\lambda}^{L,R} \right] \gamma^\mu \right) + \frac{i}{2} \left( \eta \lambda^{L,R}, \sigma^{L,R} \right) + \frac{i}{2} \left( \bar{\lambda}^{L,R}, \sigma^{L,R} \right) \eta, \]
\[ \delta \lambda^{L,R} = i \left( \frac{1}{2} \gamma^{\mu\nu} \left[ A_{\mu}^{L,R}, A_{\nu}^{L,R} \right] - D^{L,R} - \gamma^{\mu} \left[ A_{\mu}^{L,R}, \sigma^{L,R} \right] \right) \eta, \]
\[ \delta Z = \bar{\eta} \psi, \]
\[ \delta Z^\dagger = \bar{\psi} \eta, \]
\[ \delta \psi = i \gamma^\mu \left( Z A_{\mu}^{L} - A_{\mu}^{R} Z \right) \eta - i \eta \left( Z \sigma^L - \sigma^R Z \right) + F \eta, \]
\[ \delta \bar{\psi} = i \bar{\eta} \gamma^\mu \left( Z^\dagger A_{\mu}^{L} - A_{\mu}^{R} Z^\dagger \right) + i \bar{\eta} \left( \sigma^L Z^\dagger - Z^\dagger \sigma^R \right), \]
\[ \delta F = -\bar{\eta} \gamma^{\mu} \left( \bar{\psi} A_{\mu}^{L} - A_{\mu}^{R} \bar{\psi} \right) + i \left( Z \lambda^L - \lambda^R Z \right) + \left( \psi \sigma^L - \sigma^R \psi \right). \tag{3.16} \]

It is straightforward to extend this construction to give a quiver matrix model based on the $A_{r-1}$ Dynkin diagram for all $r \geq 2$; the corresponding linear $A_{r-1}$ quiver is a chain of $r$ nodes with $r - 1$ arrows connecting nearest neighbor vertices.

### 3.3. **ABJM matrix model**

The ABJM model \[1, 5, 17\] is based on adding arrows to the $A_1$ quiver to give the ABJM quiver

![ABJM quiver](image)

where there are now four complex bifundamental matter fields $\Phi^i = (Z^i, F^i, \psi^i)$ that further transform under the R-symmetry group $SU(4)$, which acts as rotations of the flavor index $i = 1, 2, 3, 4$. We deform the generic $\mathcal{N} = 2$ supersymmetric Chern–Simons quiver matrix model (3.10) by adding a suitable quartic superpotential of the chiral superfields $\Phi^i$ \[43, 44\], which reads as

\[ \mathcal{W}(\Phi) = \frac{\kappa}{4!} \varepsilon^{ijkl} \text{Tr}_V \left( \Phi_i \Phi_j^{\dagger} \Phi_k^{\dagger} \Phi_l^{\dagger} \right). \tag{3.18} \]
The extrema of the superpotential define the relations of the double quiver associated to the ABJM quiver (3.17). After integrating out the auxiliary fields, the action is then

\[
S_{\text{ABJM}} = \text{Tr} \left( \frac{2}{3} \kappa \epsilon^{\mu \nu \lambda} \left( A_\mu^L A_\nu^L A_\lambda^L - A_\mu^R A_\nu^R A_\lambda^R \right) 
- 2 A_\mu^L Z_\mu^i A_\mu^R Z^i + A_\mu^L Z_\mu^i A_\mu^L Z^i + A_\mu^R Z^i A_\mu^R - i \tilde{\psi}_i \gamma^\mu A_\mu^R \psi_i + i \tilde{\psi}_i \gamma^\mu \psi_i A_\mu^L 
+ \frac{1}{2\kappa} \left( Z_i^j \tilde{\psi}_j - \tilde{\psi}_j Z_i^j \right) + 2 \tilde{\psi}_j Z_i^j \psi_j - \epsilon^{ijkl} Z_i^j \psi_j Z_k^l \psi_l + \epsilon^{ijkl} Z_i^j \psi_j Z_k^l \psi_l 
+ \frac{1}{12\kappa} \left( Z_i^j Z^k Z^l Z^j Z^k Z^l - 6 Z_i^j Z^j Z_i^k Z^k \right) \right).
\]

(3.19)

The corresponding supersymmetry transformations are

\[
\delta Z_i^j = i \omega^{ij} \psi_j, \\
\delta Z_i^j = i \psi^i \omega_{ij}, \\
\delta \psi_i = -\gamma^\mu \omega_{ij} \left( Z_j^i A_\mu^L - A_\mu^R Z_j^i \right) - \frac{1}{2\kappa} \left( \omega_{ij} \left( Z^k Z_k^i Z_j^i Z_k^i - Z_j^i Z^i Z_k^k \right) - 2 \omega_{kl} Z^k Z_i^i Z^i \right), \\
\delta \tilde{\psi}_i = (Z_j^i A_\mu^L - A_\mu^R Z_j^i) \omega^{ij} \gamma^\mu - \frac{1}{2\kappa} \left( (Z_j^i Z^k Z_k^k Z_j^i) \omega^{ij} - 2 \omega_{kl} Z^k Z^i \omega^{i kl} \right), \\
\delta A_\mu^L = -\frac{1}{4\kappa} \left( \psi^i_j \gamma^i \omega_{ij} - \omega^{ij} \gamma^i \psi^i_j \right), \\
\delta A_\mu^R = -\frac{1}{4\kappa} \left( \psi^i_j \gamma^i \omega_{ij} - \omega^{ij} \gamma^i \psi^i_j \right).
\]

(3.20)

where \(\omega^{ij}\) are \(N = 6\) supersymmetry transformation parameters obeying \(\omega^{ij} = (\omega_{ij})^* = -\frac{1}{2} \epsilon^{ijkl} \omega_{kl}\).

The BPS equations of the ABJM theory were derived in Ref. [45]; here we present them for the dimensionally reduced model. They are determined by the quantities

\[
Z_{i}^{jk} := Z^{j} Z^{k} Z^{\dagger} - Z^{k} Z^{j} Z^{\dagger}
\]

(3.21)

for \(j < k\). We set the fermions equal to zero. The BPS equations for the supersymmetric solutions of the matrix model then follow from the fermionic supersymmetry variations in (3.20) using the independence of the gamma-matrices as a basis of the Clifford algebra, and are given by

\[
\begin{align*}
[A_\mu^L, A_\nu^L] &= 0 = [A_\mu^R, A_\nu^R], \\
A_1^L Z^1 - Z^1 A_1^L &= 0, \\
A_1^R Z^i - Z^i A_1^R &= 0 (i \neq 1, \mu = 1, 2), \\
A_0^L Z^2 - Z^2 A_0^R &= 0, \\
A_0^L Z^3 - Z^3 A_0^R &= 0, \\
A_0^L Z^4 - Z^4 A_0^R &= 0, \\
Z_{31}^{34} &= Z_{34}^{31} = Z_{31}^{41} = Z_{34}^{41}, \\
Z_{32}^{34} &= Z_{34}^{32} = Z_{32}^{42}, \\
Z_{i}^{4j} &= 0 (i \neq j \neq k).
\end{align*}
\]

(3.22)
The natural generalization of the ABJM model to a class of \( \mathcal{N} = 3 \) necklace \( A_{r-1} \) quiver theories is studied in Refs. [46–48]. The dual M-theory backgrounds for these models are \( \text{AdS}_4 \times Y_7 \), where \( Y_7 \) is a seven-dimensional tri-Sasaki–Einstein space, which is the base of a hyper-Kähler cone.

3.4. Gaiotto–Witten matrix model

A final example that we shall briefly make reference to in the following is the Gaiotto–Witten model [49], which is based on the quiver

\[
\bullet \overset{4}{\rightarrow} \overset{3}{\rightarrow} \overset{2}{\rightarrow} \overset{1}{\rightarrow} \bullet
\]  

(3.23)

We shall not spell out all details of this reduced model, which are completely analogous to the ABJM matrix model; indeed, we will regard the Gaiotto–Witten matrix model as a certain reduction of the matrix model of Sect. 3.3, which is schematically obtained by removing two of the arrows from the ABJM quiver (3.17). In this case there are two bifundamental matter fields and the R-symmetry is reduced from \( \text{SU}(4) = \text{SO}(6) \) to \( \text{SU}(2) \times \text{SU}(2) \). This model is also deformed by a quartic superpotential of the chiral superfields and possesses \( \mathcal{N} = 4 \) supersymmetry; its BPS equations are identical in form to those of the ABJM matrix model and are determined by the quantities \( Z_{i}^{12} \) from (3.21) with \( i = 1, 2 \).

4. 3-algebra models

In this section we describe the 3-algebra structures underlying the Chern–Simons quiver matrix models from Sect. 3, which we will use to analyze their vacuum structure. The various formulations of these models are related to each other, and it is possible to pass between them when certain constraints are placed on the relevant 3-algebras. For a particular 3-algebra, we show that it is possible to pass from a certain reduced 3-Lie algebra model to our ABJM matrix model. Furthermore, in a certain scaling limit, one can reach the 3-algebra model from the ABJM matrix model from Sect. 3.3, again for a particular 3-Lie algebra.

4.1. Reduced 3-Lie algebra model

Let us first recall the reduced 3-Lie algebra model derived in Ref. [21], which is the dimensional reduction of the BLG theory. It is a Chern–Simons-matter theory with matter fields taking values in a metric 3-Lie algebra \( \mathcal{A} \) and scalars \( A_{\mu} \) valued in the associated Lie algebra \( \mathfrak{g}_{\mathcal{A}} \). The matter fields consist of eight scalars \( X^{I} \), \( I = 1, \ldots, 8 \), which transform under the R-symmetry group \( \text{SO}(8) \), together with their superpartners, which can be combined into a Majorana spinor \( \Psi \) of \( \text{SO}(1, 10) \) satisfying \( \Gamma_{012} \Psi = - \Psi \); throughout we denote \( \Gamma_{M_{1} \ldots M_{k}} := \frac{1}{k!} \Gamma_{[M_{1} \ldots M_{k}]} \) where \( \Gamma^{I} \), together with the gamma-matrices \( \Gamma^{\mu} \), \( \mu = 0, 1, 2 \), form the generators of the Clifford algebra \( C\ell(\mathbb{R}^{1,10}) \). The Chern–Simons term is constructed using the alternative invariant form \( \text{Tr}_{\mathfrak{g}_{\mathcal{A}}} \) available on \( \mathfrak{g}_{\mathcal{A}} \) from (2.6). The action reads as

\[
S_{\text{BLG}} = \frac{1}{6} \epsilon^{\mu \nu \lambda} \text{Tr}_{\mathfrak{g}_{\mathcal{A}}} (A_{\mu} \llbracket A_{\nu}, A_{\lambda} \rrbracket) - \frac{1}{2} (A_{\mu} X^{I}, A^{\mu} X^{I}) + \frac{i}{2} (\bar{\Psi}, \Gamma^{\mu} A_{\mu} \Psi)
\]

\[
+ \frac{i}{2} (\bar{\Psi}, \Gamma_{IJ} [X^{I}, X^{J}, \Psi]) - \frac{1}{12} ([X^{I}, X^{J}, X^{K}], [X^{I}, X^{J}, X^{K}]).
\]  

(4.1)

This action is invariant under the \( \mathcal{N} = 8 \) supersymmetry transformations

\[
\delta X^{I} = i \bar{\epsilon} \Gamma^{I} \Psi,
\]

\[
\delta \Psi = A_{\mu} X^{I} \Gamma^{\mu} \Gamma_{I} \epsilon - \frac{1}{6} \llbracket X^{I}, X^{J}, X^{K} \rrbracket \Gamma_{IJK} \epsilon,
\]

\[
\delta A_{\mu} = i \bar{\epsilon} \Gamma_{\mu} \Gamma_{I} [X^{I}, \Psi, -].
\]  

(4.2)
It is also invariant under the gauge transformations generated by \( \Lambda \in \mathfrak{g}_A \) given as

\[
A_\mu \mapsto -[A_\mu, \Lambda], \quad X^I \mapsto \Lambda X^I, \quad \Psi \mapsto \Lambda \Psi.
\] (4.3)

The vacuum moduli space \( \mathfrak{M}_{BLG}^A \) of the 3-Lie algebra model is defined by setting \( A_\mu = 0 = \Psi \) and

\[
[X^I, X^J, X^K] = 0
\] (4.4)

in order to satisfy the BPS equations implied by (4.2). For the 3-Lie algebra \( A = A_4 \), the moduli space is given by [2]

\[
\mathfrak{M}_{BLG}^{A_4} = (\mathbb{R}^8/\mathbb{Z}_2) \times (\mathbb{R}^8/\mathbb{Z}_2).
\] (4.5)

### 4.2. Hermitian 3-algebra model

Alternative 3-algebra models can be written down if one relaxes the requirements of maximal supersymmetry and of total antisymmetry of the 3-bracket. We first break the SO(8) R-symmetry group of the maximally supersymmetric theory to SU(4) \( \times \) U(1). The supercharges transform under SU(4) \( \cong \) SO(6), whilst the U(1) factor provides an additional global symmetry. Introduce four complex 3-algebra valued scalar fields \( Z^i, i = 1, 2, 3, 4 \). Denote the corresponding four fermions by \( \bar{\psi}^i \); they are two-component Dirac spinors of SO(1, 2). We select a real set of gamma-matrices \( \gamma_\mu \), with \( \gamma_012 = 1 \). The Majorana condition is \( \bar{\psi} = \psi^\top \gamma_0 \). For a generic hermitian 3-algebra \( A \), the analog of our 3-Lie algebra model (4.1) is given by

\[
S_{BLG} = \frac{i}{6} \epsilon^{\mu\nu\lambda} \text{Tr}_{BA}(A_\mu [A_\nu, A_\lambda]) - (A_\mu Z^\dagger_i, A^\mu Z^i) + i (\bar{\psi}^i, \gamma^\mu A_\mu \psi^i) - \mathcal{V}(Z)
\]

\[
- (\bar{\psi}^i, [\psi_i, Z^i; Z_j]) + 2i (\bar{\psi}^i, [\psi_j, Z^i; Z_i]) + \frac{i}{2} \epsilon_{ijkl} (\bar{\psi}^i, [Z^j, Z^k; \psi^l])
\]

\[
- \frac{i}{2} \epsilon_{ijkl} (Z^i, [\bar{\psi}^j, \psi^k; Z^l]),
\] (4.6)

where the sextic potential is given by

\[
\mathcal{V}(Z) = \frac{2}{3} \left( \Upsilon^{jk}_i(Z), \Upsilon^j_i(Z) \right)
\] (4.7)

with

\[
\Upsilon^{jk}_i(Z) = [Z^j, Z^k; Z_i] - \frac{1}{2} \delta^j_i [Z^l, Z^k; Z_l] + \frac{1}{2} \delta^k_i [Z^l, Z^j; Z_l].
\] (4.8)

The supersymmetry transformations of this model read

\[
\delta Z^i = i \bar{\epsilon}^{ij} \psi_j,
\]

\[
\delta \bar{\psi}^i = -\gamma^\mu A_\mu Z_j \epsilon^{ij} + [Z_j, Z^k; Z_k] \epsilon^{ij} + [Z^k, Z^l; Z^l] \epsilon_{ijkl},
\]

\[
\delta A_\mu = -i [\cdot, Z_i; \psi_j] \gamma_\mu \epsilon^{ij} + i \bar{\epsilon}^{ij} \gamma_\mu [\cdot, \psi_j; Z_i].
\] (4.9)

We will now parallel the construction of Ref. [20] to demonstrate that our reduced model, for a particular choice of hermitian 3-algebra \( A \) and gauge group, yields the \( N = 6 \) ABJM matrix model. Let \( A = \text{Hom}(V_L, V_R) \) with 3-bracket (2.14) and inner product (2.15). The gauge group is the product \( U(N_L) \times U(N_R) \), corresponding to the associated Lie algebra \( \mathfrak{g}_A = u(N_L) \oplus u(N_R) \). With these
choices, the action (4.6) becomes
\[ S_{BLG} = \text{Tr}_V \left( A_\mu^L Z^i_1 Z^j_1 A_\mu^R + A_\mu^R Z^i_1 A_\mu^L - 2A_\mu^L Z^i_1 A_\mu^R Z^i_1 - i \bar{\psi}^i \gamma^\mu A_\mu^L \psi_i + i \bar{\psi}^i \gamma^\mu \psi_i A_\mu^L \right) + \frac{1}{6} \epsilon^\mu\nu\lambda (A_\mu^L [A_\nu^L, A_\lambda^L] - A_\nu^R [A_\lambda^R, A_\mu^R]) - \mathcal{V}(Z) - i \lambda \left( \bar{\psi}^i \psi_i Z_j^\dagger Z^i_j + \bar{\psi}^i \gamma^\mu \psi_i Z_j^\dagger Z^i_j + \bar{\psi}^i Z^i_j Z_j^\dagger \psi_j \right) + i \lambda \left( \epsilon_{ijkl} \bar{\psi}^i Z^k \bar{\psi}^j Z^l - \epsilon_{ijkl} Z_j^\dagger \psi_i Z_k^\dagger \psi_j \right). \] (4.10)

Our choice of 3-bracket is antisymmetric in the first two entries. This lets us rewrite the potential (4.7) as
\[ \mathcal{V}(Z) = \text{Tr}_V \left( - \frac{2}{3} [Z^i, Z^j] [Z_i^\dagger, Z_j^\dagger; Z_k] + \frac{1}{2} [Z^k, Z^i; Z_j^\dagger] [Z_k^\dagger, Z_j^\dagger; Z_i^\dagger] \right). \] (4.11)

The global minima of \( \mathcal{V}(Z) \) are described by the equations
\[ Z^{ijk} := [Z^j, Z^k; Z_i] = 0, \] (4.12)
which are just the BPS equations (3.22) with \( A_\mu = 0 \). These equations coincide with the extrema of the superpotential (3.18), and hence define the relations of the double of the ABJM quiver (3.17). We can evaluate the 3-brackets explicitly, and then the potential assumes the manifestly SU(4)-invariant form
\[ \mathcal{V}(Z) = -\frac{2}{3} \text{Tr}_V (2Z^k Z_j^\dagger Z_i^\dagger Z_k^\dagger Z_j^\dagger + 2Z^k Z_j^\dagger Z_i^\dagger Z_k^\dagger Z_j^\dagger + \frac{1}{2} Z^j Z_k^\dagger Z_i^\dagger + \frac{1}{2} Z^k Z_j^\dagger Z_i^\dagger Z_j^\dagger Z_k^\dagger Z^j). \] (4.13)

For the choice of constant \( \lambda = \frac{1}{2z} \), we recover the \( \mathcal{N} = 6 \) ABJM matrix model (3.19).

Note that the BPS equations (4.12) and their conjugates imply that the collection \( Z_i^\dagger Z^j \) of endomorphisms of \( V_L \) for \( i, j = 1, 2, 3, 4 \) form a mutually commuting set of \( N_L \times N_L \) matrices; similar \( Z^i Z_i^\dagger \) are a mutually commuting set of \( N_R \times N_R \) matrices. In the ABJM limit \( N_L = N_R = N \), the operators \( Z_i^\dagger Z^j \) and \( Z^i Z_i^\dagger \) moreover have the same spectra, and the vacuum moduli space \( \mathfrak{M}_N^{ABJM} \) is therefore given by the \( N \)th symmetric product orbifold
\[ \mathfrak{M}_N^{ABJM} = (\mathbb{C}^4)^N / \mathfrak{S}_N \] (4.14)
where \( \mathfrak{S}_N \) is the Weyl group of \( \text{U}(N) \) acting by permuting the components of \( N \)-vectors. As we will make use of this result later, let us derive it explicitly. For this, we note that the BPS equations in this case are solved by commuting matrices \( [Z^i, Z^j] = 0, i, j = 1, 2, 3, 4 \). Then \( Z^i \) can be put simultaneously into their Jordan normal forms, with \( k \) eigenvalues \( \zeta_1^i, \ldots, \zeta_k^i \) of each endomorphism \( Z^i \), i.e. for each fixed \( i \in \{1, 2, 3, 4\} \), each \( \zeta_l^i, l = 1, \ldots, k \), corresponds to a Jordan block; doing so breaks the \( \text{U}(N) \times \text{U}(N) \) gauge symmetry to a diagonal \( \text{U}(N) \) subgroup. To every Jordan block one associates its dimension \( \lambda_i \), independently of \( i \in \{1, 2, 3, 4\} \) because \( Z^i \) mutually commute. The collection \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of dimensions satisfies
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_{l=1}^k \lambda_l = N, \] (4.15)
and thus defines a linear partition of the rank \( N \) of length \( k \). Then the isomorphism (4.14) is generated by the map
\[ (Z^1, Z^2, Z^3, Z^4) \mapsto \sum_{l=1}^k \lambda_l z_l \in (\mathbb{C}^4)^N / \mathfrak{S}_N, \] (4.16)
where \( \{ \bar{z}_i = (\zeta_1^i, \zeta_2^i, \zeta_3^i, \zeta_4^i) \}_{i=1, \ldots, k} \) is a set of \( k \) points in \( \mathbb{C}^4 \) with multiplicities given by the linear partition \( \lambda \).

A completely analogous calculation shows that the moduli space of vacua \( \mathcal{M}^{GW}_N \) of the \( U(N) \times U(N) \) Gaiotto–Witten matrix model is the orbifold
\[
\mathcal{M}^{GW}_N = (\mathbb{C}^2)^N / \mathbb{S}_N.
\] (4.17)

In this case the eigenvalues of the commuting bilinear matrices \( Z^i Z^j, i, j = 1, 2 \), can be interpreted as points in \( \mathbb{R}^4 = \mathbb{C}^2 \).

For more general quiver matrix models, the moduli space of vacua typically contains the symmetric product \( X^N / \mathbb{S}_N \), where \( X \) is an affine Calabi–Yau fourfold; for a large class of such theories, \( X \) is topologically a cone over a compact Sasaki–Einstein seven-manifold \( Y_7 \) (see e.g. Ref. [50]).

4.3. Mapping to the lorentzian Lie algebra model

Following Ref. [51], we shall now demonstrate how a particular contraction relates the lorentzian version of the 3-Lie algebra model \((4.1)\) with the ABJM matrix model \((3.19)\). The first step is to construct the \textit{lorentzian Lie algebra model}. We fix a semisimple Lie algebra \( \mathfrak{g} \) and expand the fields of the reduced 3-Lie algebra model in terms of the generators of \( \mathfrak{g} \) satisfying the 3-bracket relations \((2.17)\) as
\[
X^I = X^I_c \mathbb{1} + X^I_0 \tau_0 + X^I_a \tau_a,
\]
\[
\Psi = \Psi_c \mathbb{1} + \Psi_0 \tau_0 + \Psi_a \tau_a,
\]
\[
A^\mu = A^\mu_{0a} D_{0a} + A^\mu_{ab} D_{ab}.
\] (4.18)

It is convenient to make the field definitions
\[
\hat{X}^I = X^I_c \tau_a,
\]
\[
\hat{\Psi} = \Psi_a \tau_a,
\]
\[
\hat{A}^\mu = A^\mu_{0a} \tau_a,
\]
\[
\hat{B}^\mu = f_{abc} A^\mu_{ab} \tau_c.
\] (4.19)

We insert these expansions into \((4.1)\), and denote the inner product \((2.18)\) by \( \text{Tr}_h \) here. Using \((2.8)\), 3-brackets involving the generator \( \tau_0 \) induce the Lie bracket of \( \mathcal{A}'_h \) through
\[
[X^I, X^J, \tau_0] = [\hat{X}^I, \hat{X}^J].
\] (4.20)

A similar reduction occurs for the brackets involving fermions. For the terms involving the gauge fields, we use \((2.18)\) to infer that terms proportional to the central element \( \mathbb{1} \) decouple from the gauge interactions, and in fact completely from the action. In this way we find the lorentzian Lie algebra model
\[
S_h = \text{Tr}_h \left( \frac{i}{2} \hat{A}^\mu \hat{X}^I + B^\mu X^I_0 \right)^2 + \frac{1}{4} (X^K_0)^2 [\hat{X}^I, \hat{X}^J]^2 - \frac{1}{2} (X^I_0 [\hat{X}^I, \hat{X}^J])^2
\]
\[+ \frac{i}{2} \hat{\Psi} \Gamma^\mu [\hat{A}_\mu, \hat{X}^I] - i \hat{\Psi}_0 \Gamma^\mu B_\mu \hat{\Psi} - \frac{i}{2} \hat{\Psi}_0 \hat{X}^I [\hat{X}^J, \Gamma_{IJ} \hat{\Psi}] + \frac{i}{2} \hat{\Psi} X^I_0 [\hat{X}^J, \Gamma_{IJ} \hat{\Psi}]
\]
\[+ \frac{i}{2} \epsilon^{\mu \nu \lambda} [\hat{A}_\mu, \hat{A}_\nu] B_\lambda \right).
\] (4.21)

It is invariant under the supersymmetry transformations
\[
\delta \hat{\Psi} = (\hat{[A}^\mu, \hat{X}^I] + B^\mu X^I_0) \Gamma_I \epsilon - \frac{1}{2} X^K_0 [\hat{X}^I, \hat{X}^J] \Gamma_{JK} \epsilon,
\]
\[
\delta \hat{X}^I = i \Gamma^I \hat{\Psi},
\]
\[
\delta X^I_0 = i \Gamma^I \hat{\Psi}_0,
\]
\[
\delta B^\mu = i \hat{\epsilon} \Gamma^\mu \Gamma_I [\hat{X}^I, \hat{\Psi}],
\]
\[
\delta \hat{A}^\mu = i \hat{\epsilon} \Gamma^\mu \Gamma_I \hat{X}^I \hat{\Psi} + i \hat{\epsilon} \Gamma^\mu \Gamma_I X^I_0 \hat{\Psi}.
\] (4.22)

In the following we show how this model is related to the ABJM matrix model \((3.19)\): For a particular choice of gauge symmetry breaking and scaling limit, we show that one can recover the lorentzian
Lie algebra model (4.21) from (3.19). As we will make use of similar reductions throughout this paper, we describe it here in detail.

For this, we consider the ABJM limit $N_L = N_R = N$. To take the scaling limit, we first make the gauge field redefinitions

$$A_\mu^L = A_\mu + i B_\mu, \quad A_\mu^R = A_\mu - i B_\mu, \quad (4.23)$$

which breaks the gauge symmetry to a diagonal $U(N)$ subgroup of $G = U(N) \times U(N)$. With this replacement, the Chern–Simons term from the first line of (3.19) reads

$$S_g = \kappa \epsilon^{\mu \nu \lambda} \text{Tr}_V \left(B_\mu \left[A_\nu, A_\lambda\right] - \frac{1}{3} B_\mu B_\nu B_\lambda\right). \quad (4.24)$$

We write the real and imaginary parts of the scalars and fermions as

$$Z^i = X^i + i X^{i+4}, \quad \psi^i = \chi^i + i \chi^{i+4} \quad (4.25)$$

for $i = 1, 2, 3, 4$. We decompose the scalars and fermions further into trace and traceless components as

$$Z^i = X^i_0 + i X^{i+4}_0 + X^i_a \tau_a + i X^{i+4}_a \tau_a, \quad \psi^i = \psi^i_0 + i \psi^{i+4}_0 + \psi^i_a \tau_a + i \psi^{i+4}_a \tau_a. \quad (4.26)$$

In this decomposition we have identified $\tau_0$ with the generator of $u(1)$, and $\tau_a, a = 1, \ldots, d = N^2 - 1$, are the generators of $su(N)$. We scale the fields as

$$B_\mu \rightarrow g B_\mu, \quad X^i_0 \rightarrow \frac{1}{g} X^i_0, \quad \psi^i_0 \rightarrow \frac{1}{g} \psi^i_0 \quad (4.27)$$

with all other fields unchanged, and the coupling constant as $\kappa \rightarrow \frac{1}{g} \kappa$. Taking the limit $g \rightarrow 0$ we find that the Chern–Simons term (4.24) reduces to

$$S_g = \kappa \epsilon^{\mu \nu \lambda} \text{Tr}_V \left(B_\mu \left[A_\nu, A_\lambda\right]\right), \quad (4.28)$$

while the second line of (3.19) becomes

$$S_k = -\text{Tr}_V \left(\left([A_\mu, X^I] + 2 B_\mu X^I_0\right)^2 + \bar{\psi} \gamma^\mu \left[A_\mu, \psi\right] - 2 \bar{\psi} \gamma^\mu B_\mu \psi_0 - 2 \bar{\psi}_0 \gamma^\mu B_\mu \psi\right). \quad (4.29)$$

In this reduction we have combined the indices $i$ and $i + 4$ for $i = 1, 2, 3, 4$ into an index $I = 1, \ldots, 8$, and the components of the spinors into a single Majorana fermion

$$\psi = (\chi^1, \ldots, \chi^8)^\top. \quad (4.30)$$

Now we consider the bosonic sextic potential. In the scaling limit, the surviving terms from the potential contain four trace components and eight real traceless components. Using $SU(4)$ R-symmetry we arrange them as

$$Z^i = \delta^{i+1} \left(X^i_0 + i X^{i+4}_0\right) \tau_0 + \left(X^i_a + i X^{i+4}_a\right) \tau_a. \quad (4.31)$$

If we combine the trace components as

$$X^I_0 = (X^1_0, 0, 0, 0, X^5_0, 0, 0, 0), \quad (4.32)$$

then the reduced bosonic potential reads

$$V_b(X) = -\frac{1}{2\kappa^2} \text{Tr}_V \left(\frac{1}{4} \left(X^K_0\right)^2 \left[X^I, X^J\right]^2 - \frac{1}{2} \left(X^K_0 \left[X^I, X^J\right]\right)^2\right). \quad (4.33)$$

We finally consider the quartic Yukawa potential. In this scaling limit, the surviving term of this potential has contributions from two bosonic trace components and two traceless bosonic components. We arrange them as in (4.31) and the spinor components into a Majorana fermion as in (4.30).
The resulting potential reads

$$V_f(X, \psi) = -\frac{1}{\kappa} \text{Tr}_V \left( \bar{\psi} X_0^I \left[ X^J, \gamma_{IJ} \psi \right] \right)$$

for suitable antisymmetrized products of $8 \times 8$ gamma-matrices $\gamma_{IJ}$ (see e.g. Ref. [51, App. A]).

The fully contracted theory thus reads

$$S_{\text{red}} = -\text{Tr}_V \left( \left( [A_\mu, X^I] + 2B_\mu X_0^I \right)^2 + i \bar{\psi} \gamma^\mu [A_\mu, \psi] - 2\bar{\psi} \gamma^\mu B_\mu \psi_0 - 2\bar{\psi}_0 \gamma^\mu B_\mu \psi 
- \frac{1}{2\kappa} \left( \frac{1}{4} (X_0^K)^2 [X^I, X^J]^2 - \frac{1}{2} X_0^J [X^I, X^J]^2 \right) - \frac{1}{8 \kappa} \bar{\psi} X_0^I [X^J, \gamma_{IJ} \psi] 
+ \kappa \epsilon^{\mu\nu\lambda} [A_\mu, A_\nu] B_\lambda \right).$$

This is just the original lorentzian Lie algebra model (4.21) with $\mathfrak{h} = \mathfrak{su}(N)$ and inner product (2.15).

The connection between the generic Gaiotto–Witten matrix models and reduced 3-algebra models is less clear. For the reduced 3-Lie algebra model of Sect. 4.1 based on $\mathcal{A} = A_4$, one can reduce the $\text{SO}(8)$ R-symmetry group to $\text{SO}(4)$ by keeping only half of the supersymmetries; then the matter content splits into a hypermultiplet plus a twisted hypermultiplet, and the reduced model can be regarded as the $\text{SU}(2) \times \text{SU}(2)$ Gaiotto–Witten model with an additional twisted hypermultiplet [52].

5. Dual IKKT matrix models

A main driving force in our subsequent analysis will be certain connections to the IKKT matrix model [27], whose vacuum moduli space is well understood and for which detailed localization techniques are available [37]. In this section we exploit the fact that it is also possible to reach gauge theories on D2-branes from M2-brane theories; this occurs in the limit away from the orbifold point of $\mathbb{R}^8/\mathbb{Z}_k$ where the orbifold geometry is $S^1 \times \mathbb{R}^7$. A field theory realization of this reduction is provided by the Higgs mechanism of Ref. [39], which relates the maximally supersymmetric BLG Chern–Simons-matter theory to $\mathcal{N} = 8$ supersymmetric Yang–Mills theory in three dimensions: One gives a vacuum expectation value $v$ to a single scalar field and takes the limit $\kappa, v \to \infty$ with $g^2 := \frac{v^2}{2\kappa}$ fixed; this reduction was exploited in Ref. [21] to relate the reduced 3-algebra model (4.1) and its classical solutions to those of the IKKT model in ten dimensions. In Ref. [53] it was shown how a certain variant of the lorentzian Lie algebra model (4.21) reduces to the ten-dimensional IKKT matrix model with $\mathfrak{h} = \mathfrak{su}(N)$ and gauge group $\text{SU}(N)$. It was shown in Ref. [54] that the Higgs mechanism proposed by Ref. [39] also reduces the ABJM theory to $\mathcal{N} = 8$ supersymmetric Yang–Mills theory in three dimensions. By considering dimensional reductions of generic $\mathcal{N} = 2$ Chern–Simons-matter theories, we can reduce via the Mukhi–Papageorgakis map to a variety of Yang–Mills matrix models.

We demonstrate this explicitly for a particular reduction of the $A_1$ quiver matrix model to the $\mathcal{N} = 1$ IKKT model in four dimensions. We then apply this map to our ABJM matrix model, and arrive at the $\mathcal{N} = 8$ ten-dimensional IKKT matrix model. In this section only we shall take the ABJM limit $N_L = N_R = N$ throughout.

5.1. Mapping of the $A_1$ quiver matrix model

We begin with the simplest example for which the reduction is relatively straightforward to construct. We consider the dimensionally reduced $\mathcal{N} = 2$ Chern–Simons-matter theory (3.15), and show that under the Mukhi–Papageorgakis map it reduces to the four-dimensional IKKT matrix model with $\mathcal{N} = 1$ supersymmetry and gauge group $\text{SU}(N)$. We work with the Clifford algebra $\mathcal{C}(\mathbb{R}^{1,2})$, and
use Dirac spinors. Our gamma-matrices are the Pauli spin matrices, and the Majorana conditions read
\[ \bar{\epsilon} \lambda = \bar{\lambda} \epsilon, \quad \bar{\epsilon} \gamma^\mu \lambda = -\bar{\lambda} \gamma^\mu \epsilon. \] (5.1)
As previously, we break the gauge symmetry to a $U(1)$ subgroup by making the field replacements $(4.23)$. We also restrict the matter field $Z$ to be hermitian. We decompose $Z$ into components $Z' \in su(N)$ and $Z_0 \in u(1)$, and expand it around a classical value proportional to the identity with a coupling constant $g$ as
\[ Z = g \mathbb{1} + Z_0 + Z'. \] (5.2)
Using global $U(1)$ symmetry, we may take $g \in \mathbb{R}$. For the gaugino and auxiliary fields, we take a diagonal limit in which
\[ \lambda^L = -\lambda^R =: \lambda, \quad D^L = D^R =: D, \quad \sigma^L = \sigma^R =: \sigma, \] (5.3)
and further couple the gauge and matter sectors of the model together by the requirements
\[ \lambda = -g \psi, \quad \sigma = g Z, \quad D = -g F. \] (5.4)

With these gauge field replacements, and diagonal limits of the gauginos and auxiliary fields, we find that the pure Chern–Simons component from the first line of the action $(3.15)$ reduces to $(4.24)$. For the remaining matter contributions in $(3.15)$, by inserting the field identifications above and expanding around the vacuum value we obtain
\[ S_m = \text{Tr}_V \left( -[A_\mu, Z']^2 - 4g^2 B_\mu B^\mu + i \bar{\psi} \gamma^\mu [A_\mu, \psi] - \bar{\psi} \gamma^\mu \{B_\mu, \psi\} + i g \bar{\psi} [Z', \psi] + F^2 \right). \] (5.5)
We now scale the fields appropriately and take the strong coupling limit $g \to \infty$. We can integrate out the auxiliary field $B_\mu$ using its equation of motion
\[ B_\mu = \frac{\kappa}{g^2} \epsilon_{\mu \nu \lambda} [A^\nu, A^\lambda]. \] (5.6)
In deriving this equation we have ignored cubic and higher order interactions involving $B_\mu$ that become suppressed in the strong coupling limit. Inserting $(5.6)$ into the pure Chern–Simons action $(4.24)$, we find
\[ S_g = -\frac{4g^2}{\kappa^2} \text{Tr}_V \left( [A_\mu, A_\nu]^2 \right). \] (5.7)
We scale the matter field $Z$ by the factor $\frac{1}{g}$, and similarly for the matter fermion (and its adjoint) and the auxiliary field $F$. Replacing $B_\mu$ by its equation of motion $(5.6)$, we find that the matter action $(5.5)$ reduces in the strong coupling limit to
\[ S_m = \text{Tr}_V \left( -\frac{1}{g^2} [A_\mu, Z']^2 - \frac{4g^2}{\kappa^2} [A_\mu, A_\nu]^2 + \frac{1}{g^2} \bar{\psi} \gamma^\mu [A_\mu, \psi] + \frac{1}{g^2} \bar{\psi} [Z', \psi] + \frac{1}{g^2} F^2 \right). \] (5.8)
We combine the scalar and gauge fields into a single field
\[ X^I = (X^\mu, X^3) = (A^\mu, Z') \] (5.9)
where $I = 0, 1, 2, 3$. Then with $\kappa = \frac{1}{g}$, the sum of $(5.7)$ with the first two terms of $(5.8)$ can be written as $-\frac{1}{2\kappa^2} \text{Tr}_V ([X^I, X^J]^2)$, which is the bosonic potential of the IKKT model. For the last three terms of $(5.8)$, we define a four-dimensional Majorana spinor of the Clifford algebra $C\ell(\mathbb{R}^{1,3})$ by
\[ \Psi = (\psi^1, \psi^2)^T. \] (5.10)
where each real component $\psi^1, \psi^2$ of the Dirac spinor $\psi$ is a two-component Majorana spinor. We then construct a set of four-dimensional gamma-matrices from our three-dimensional Pauli spin
matrices as
\[ \Gamma^\mu = i \begin{pmatrix} 0 & \gamma^\mu \\ -\gamma^\mu & 0 \end{pmatrix}, \quad \Gamma^3 = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \] (5.11)

For the chirality and charge conjugation matrices, we take
\[ \Gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad C = \begin{pmatrix} -i \gamma^2 & 0 \\ 0 & i \gamma^2 \end{pmatrix}. \] (5.12)

We can then combine the last three terms of (5.8) as \( \frac{1}{g^2} \text{Tr}_V \left( -\bar{\Psi} \Gamma^I [X_I, \Psi] + F^2 \right) \), which is the fermionic term of the IKKT model together with an auxiliary field.

Altogether the \( A_1 \) quiver matrix model action is reduced under the Mukhi–Papageorgakis map to the action of the four-dimensional IKKT model
\[ S_{IKKT} = \frac{1}{g^2} \text{Tr}_V \left( -\frac{1}{2} [X^I, X^J]^2 - \bar{\Psi} \Gamma^I [X_I, \Psi] + F^2 \right). \] (5.13)

5.2. Supersymmetry reduction

We will now show explicitly how the supersymmetry transformations of the \( A_1 \) quiver matrix model map to those of the IKKT model under the Mukhi–Papageorgakis map. The original matrix model has \( \mathcal{N} = 2 \) supersymmetry, while the IKKT model in four dimensions has \( \mathcal{N} = 1 \) supersymmetry. Hence the scaling limit must reduce the supersymmetry; we do this by identifying the infinitesimal supersymmetry generators in (3.16) so that \( \bar{\epsilon} = \eta \) are no longer independent. We demonstrate the reduction on each field transformation of (3.16) individually.

For the transformations of the gauge fields \( A_\mu \) in (3.16), we make the gauge field identifications, identify the supersymmetry generators with each other, and scale the spinor. In four dimensions we write the fermions as four-component Majorana spinors obeying (5.1), and along with the four-dimensional gamma-matrices (5.11) we can write
\[ \delta A_\mu = \bar{\epsilon} \Gamma_\mu \lambda. \] (5.14)

Following a similar process for the supersymmetry transformations of the matter field \( Z \), the requirement (5.3) reveals the Majorana spinor condition (5.1). After expanding \( Z \) around its classical value and scaling, we can combine its supersymmetry transformation with (5.14) to get
\[ \delta X^I = \bar{\epsilon} \Gamma^I \Psi. \] (5.15)

For the supersymmetry variation of the auxiliary field \( D \) in (3.16), we identify the supersymmetry generators with each other, scale the fields, and expand around the classical value, so that the resulting supersymmetry transformation reads as
\[ \delta D = \frac{i}{2} \left( \bar{\epsilon} \gamma^\mu [A_\mu, \lambda] + [A_\mu, \bar{\lambda}] \gamma^\mu \right) \epsilon + \frac{i}{2} \left( \bar{\epsilon} \gamma^2 [\lambda, Z'] + [\bar{\lambda}, Z'] \right) \epsilon. \] (5.16)

Identifying the four-dimensional gamma-matrices (5.11) and (5.12), applying the Majorana spinor identity (5.1), and combining \( Z' \) with \( A_\mu \) as in (5.9) results in the transformation
\[ \delta D = i \bar{\epsilon} \Gamma^5 \Gamma^I [X_I, \Psi]. \] (5.17)

A similar modification occurs for the supersymmetry transformation of the auxiliary field \( F \). We take an axial combination (4.23) of the gauge fields in (3.16), which reduces the interaction of the
fermion and the gauge field to a commutator, at the cost of introducing the field $B_\mu$, so that after making the field replacements (5.3) we arrive at

$$\delta F = \bar{\epsilon}^* \left( \gamma^\mu [A_\mu, \psi] + i \gamma^\mu \{ B_\mu, \psi \} + [\sigma, \psi] + i \{ \lambda, Z \} \right). \quad (5.18)$$

Expanding around the vacuum, and taking the appropriate scaling limit, the $B_\mu$ contribution decouples. After combining the gauge and matter fields, and rewriting the spinor and gamma-matrices, the reduction (5.18) coincides with (5.17).

Finally, we consider the spinor supersymmetry transformations. For the gaugino variation $\delta \lambda$ in (3.16), we make the usual field identifications and scalings, and combine the terms involving $A_\mu$ and $Z$ to get

$$\delta \Psi = -i \Gamma_{IJ} [X^I, X^J] \epsilon - F \epsilon. \quad (5.19)$$

For the matter fermions in (3.16), we take the axial limit of the gauge fields and make the field replacements to get

$$\delta \psi = i \gamma^\mu \epsilon \left( [A_\mu, Z] + i \{ B_\mu, Z \} \right) + F \epsilon^*. \quad (5.20)$$

Inserting the equation of motion (5.6) for $B_\mu$ and taking the scaling limit we find

$$\delta \psi = i \gamma^\mu \epsilon \left( [A_\mu, Z] + 2 i \kappa \epsilon_{\mu\nu\lambda} [A^\nu, A^\lambda] \right) + F \epsilon. \quad (5.21)$$

By setting $\kappa = \frac{1}{4}$ and using the Pauli spin matrix identity

$$\frac{i}{2} \epsilon^{\mu\nu\lambda} \gamma_\lambda = \gamma^{\mu\nu}, \quad (5.22)$$

we find that (5.21) coincides with (5.19).

### 5.3. Mapping of the ABJM matrix model

Let us now extend the Mukhi–Papageorgakis map to reduce the ABJM model (3.19). We break the product gauge group $G = U(N) \times U(N)$ to a diagonal $U(N)$ subgroup by taking an axial combination of the gauge fields (4.23). We write the real and imaginary parts of the scalars and the spinors as in (4.25). We further decompose the fields into the generators of the $U(N)$ gauge group exactly as in (4.26). We expand the scalar fields around a fixed vacuum configuration proportional to a coupling constant $g$. Using the SU(4) invariance of the matrix model, we can select the scalar field $Z^4$ to expand around so that

$$Z^i = i g \delta^{i,4} 1 + X^i_0 \tau_0 + X^i_a \tau_a + i X^{i+4}_0 \tau_0 + i X^{i+4}_a \tau_a. \quad (5.23)$$

We first investigate the effect of the scaling limit on the Chern–Simons matrix action from the first line of (3.19). The various terms of the action separate into $U(1)$ and $SU(N)$ components, and in the strong coupling limit $g \to \infty$ the $U(1)$ terms decouple so we will ignore them from now on. When we make the gauge field replacement (4.23), the Chern–Simons term reads as in (4.24), which in the scaling limit will reduce to (4.28).
The contributing terms to the reduction of the second line of (3.19) give

\[ S_k = \text{Tr}_V \left( -\sum_{i=1}^{4} \left( [A_\mu, X^i]^2 + [A_\mu, X^{i+4}]^2 \right) - 4g [A_\mu, X^8] B^\mu - 4g^2 B_\mu B^\mu \\
- i \sum_{i=1}^{4} (\bar{\chi}_i \gamma^\mu [A_\mu, X^i] + i \chi^{i+4}) \right) \] (5.24)

Combining (4.28) and (5.24), we can integrate out \( B_\mu \) using its equation of motion

\[ B_\mu = -\frac{1}{g} [A_\mu, X^8] + \frac{\kappa}{g^2} \epsilon_{\mu \nu \lambda} [A^\nu, A^\lambda]. \] (5.25)

This causes the scalar field \( X^8 \) to decouple from the action. We write the minimal spinor of \( \text{SO}(1, 2) \times \text{SO}(7) \) for the reduced theory as in (4.30), where each \( \chi^i \) is also a two-component Majorana spinor. Then the action (5.24) reduces to

\[ S_k = \text{Tr}_V \left( -\sum_{a=1}^{3} \left( [A_\mu, X^a]^2 + [A_\mu, X^{a+4}]^2 \right) - [A_\mu, X^4]^2 - 8\kappa^2 [A_\mu, A_\nu]^2 - i \bar{\psi} \gamma^\mu [A_\mu, \psi] \right) \] (5.26)

We now investigate the potential terms from the last four lines of (3.19). The surviving terms from the bosonic potential are of the form

\[ V_b(X) = -\frac{1}{8\kappa^2} \text{Tr}_V \left( \sum_{a,b=1}^{3} \left( [X^a, X^b]^2 + [X^{a+4}, X^{b+4}]^2 \right) + 2 \sum_{a=1}^{3} \left( [X^a, X^4]^2 + 2[X^{a+4}, X^4]^2 \right) \right) \] (5.27)

The fermions produce a potential that reads as

\[ V_f(X, \psi) = -\frac{i}{\kappa} \text{Tr}_V \left( \sum_{a=1}^{3} \left( \bar{\psi} \gamma_a [X^a, \psi] + \bar{\psi} \gamma_{a+4} [X^{a+4}, \psi] \right) + \bar{\psi} \gamma_4 [X^4, \psi] \right) \] (5.28)

for a suitable basis of \( \text{SO}(7) \) gamma-matrices \( \gamma^a, \gamma^4, \gamma^{a+4}, a = 1, 2, 3 \).

Finally, we rescale the fields as in (4.27), and then the full reduced action takes the form

\[ S_{\text{red}} = \frac{1}{g^2} \text{Tr}_V \left( -\sum_{a=1}^{3} \left( [A_\mu, X^a]^2 + [A_\mu, X^{a+4}]^2 \right) - [A_\mu, X^4]^2 - 8\kappa^2 [A_\mu, A_\nu]^2 \\
- \frac{1}{8\kappa^2} \sum_{a,b=1}^{3} \left( [X^a, X^b]^2 + [X^{a+4}, X^{b+4}]^2 \right) - \frac{1}{4\kappa^2} \sum_{a=1}^{3} \left( [X^a, X^4]^2 + [X^{a+4}, X^4]^2 \right) \\
- \frac{1}{\kappa} \sum_{a=1}^{3} \left( \bar{\psi} \gamma_a [X^a, \psi] + \bar{\psi} \gamma_{a+4} [X^{a+4}, \psi] \right) - \frac{1}{\kappa} \bar{\psi} \gamma_4 [X^4, \psi] - i \bar{\psi} \gamma^\mu [A_\mu, \psi] \right) \] (5.29)

We can combine the bosonic fields into a single field \( X^M = (2A_\mu, X^a, X^4, X^{a+4}) \) with \( M = 1, \ldots, 10 \). Then this action, along with the choice of Chern–Simons coupling constant \( \kappa = \frac{1}{2} \), produces the action of the ten-dimensional IKKT matrix model.
For later use, we note the similarity between the BPS equations of the ABJM and IKKT matrix models. In the case of the ABJM model the BPS equations are given by (3.22), while in the case of the IKKT model the BPS equations are determined by commuting matrices

\[ [X^M, X^N] = 0. \]

However, the 3-algebra form (4.12) of the ABJM equations does not map to the IKKT equations (5.30) under the scaling limit described here. This is due to the removal of the gauge fields from (3.22): In the axial limit (4.23) of the gauge fields, the field \( B_\mu \) causes a bosonic degree of freedom to decouple from the action in the scaling limit in order that one may combine the gauge fields with the scalars in the appropriate way.

A completely analogous calculation can be applied to the \( U(N) \times U(N) \) Gaiotto–Witten matrix models; the Mukhi–Papageorgakis map in this case identifies the dual theory as the \( \mathcal{N} = 4 \) six-dimensional IKKT matrix model (see Ref. [55]).

### 6. Cohomological 3-algebra models

In what follows we shall be interested in the exact computations of the partition functions of our Chern–Simons quiver matrix models using localization techniques. For this, we shall need to deform our matrix models in suitable ways in order to obtain theories with equivariant cohomological symmetries that will enable the localization procedure to be applied exactly. In this section we shall study cohomological versions of our quiver matrix models that are obtained by a topological twisting procedure, and point out various ensuing difficulties. The possible inequivalent twists of Chern–Simons-matter theories in three dimensions with \( \mathcal{N} \geq 4 \) supersymmetry were classified in Ref. [55].

In the case of an \( \mathcal{N} = 8 \) theory with R-symmetry group \( \text{SO}(8) \), restricting the supercharges to the vector representation does not generate any additional twists. However, letting the supercharges transform in the spinor representation via the triality of the R-symmetry group does allow for two additional twists. One of these new twists was constructed in Ref. [56]; in this section we investigate the effect of applying the Mukhi–Papageorgakis map to this topologically twisted theory. After dimensional reduction, the ensuing 3-algebra model can potentially induce a cohomological deformation of the ABJM matrix model under the mappings of Sect. 4, which is dual to a novel topological twisting of the ten-dimensional IKKT model.

#### 6.1. Topologically twisted BLG theory

We begin by briefly reviewing the topologically twisted theory constructed in Ref. [56]. In the conventions of Sect. 4.1, the BLG action in euclidean space reads

\[
S_{\text{BLG}} = \int d^3 x \left( \frac{1}{2} \epsilon^{\mu \nu \lambda} \text{Tr}_{\mathfrak{g}_A} \left( A_\mu \partial_\nu A_\lambda - \frac{1}{3} A_\mu [A_\nu, A_\lambda] \right) + \frac{1}{2} (\nabla_\mu X^I, \nabla^\mu X^I) - \frac{i}{2} (\bar{\Psi} \Gamma^\mu \nabla_\mu \Psi) + \frac{i}{4} (\bar{\Psi} \Gamma_{IJ} [X^I, X^J, \Psi]) + \frac{1}{12} ([X^I, X^J, X^K], [X^I, X^J, X^K]) \right).
\]

This action is invariant under the 16 supersymmetries generated by

\[
\delta X^I = i \bar{\epsilon} \Gamma^I \Psi,
\]

\[
\delta \Psi = \nabla_\mu X^I \Gamma^\mu \Gamma_I \bar{\epsilon} - \frac{i}{6} [X^I, X^J, X^K] \Gamma_{IJK} \bar{\epsilon},
\]

\[
\delta A_\mu = i \bar{\epsilon} \Gamma_\mu \Gamma_I [X^I, \Psi, -].
\]
The main difference from the split signature case is that the euclidean action involves only the holomorphic part of the spinor, so that we must make the definition

$$\bar{\Psi} := \Psi^\top C,$$

where $C$ is the charge conjugation matrix satisfying

$$C \Gamma^M C^{-1} = ( - \Gamma^M )^\top, \quad C^\top = - C,$$

and $M$ is the 11-dimensional vector index that decomposes into $\mu = 1, 2, 3$ and $I = 4, \ldots, 11$.

Consider now the rotational symmetry breaking $\text{Spin}(11) \rightarrow \text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(5)$, under which the corresponding gamma-matrices can be decomposed as

$$\Gamma^\mu = \gamma^\mu \otimes 1 \otimes 1 \otimes \gamma^3, \quad \Gamma^{\mu+3} = 1 \otimes \gamma^\mu \otimes 1 \otimes 1 \otimes \gamma^1, \quad \Gamma^{i+6} = 1 \otimes 1 \otimes \gamma^i \otimes \gamma^2$$

where $\gamma^\mu, \mu = 1, 2, 3$, are Pauli spin matrices and $\gamma^i, i = 1, \ldots, 5$, are $4 \times 4$ gamma-matrices in five euclidean dimensions. The charge conjugation matrix decomposes as

$$C = i \gamma^2 \otimes i \gamma^2 \otimes C \otimes 1,$$

where $C$ is the five-dimensional charge conjugation matrix. The $\text{SO}(8)$ chirality matrix is

$$\Gamma^{123} = -i \Gamma^{4 \ldots 11} = 1 \otimes 1 \otimes 1 \otimes i \gamma^3.$$ 

This means that the spinors have four indices: two for the $\text{SO}(3)$ factors, one for the $\text{SO}(5)$ factor, and one for $\text{SO}(8)$ chirality. The twist is constructed by replacing an $\text{SO}(3)$ factor with the diagonal subgroup of $\text{Spin}(3) \times \text{Spin}(3)$. Then we can expand the twisted spinors

$$\Psi = (\psi, \chi^\mu)$$

into an $\text{SO}(3)$ scalar and vector. We also decompose the bosons

$$X^I = (X^\mu, Y^i)$$

into an $\text{SO}(3)$ vector and five scalars.

The resulting twisted BLG action is the sum of a topological action

$$S_{\text{top}} = \int d^3 x \left( \frac{i}{2} \epsilon^{\mu \nu \lambda} \text{Tr}_{\text{BA}} (A^+_\mu \partial_\nu A^+_\lambda + \frac{1}{3} A^+_\mu [A^+_\mu, A^+_\lambda]) - \frac{i}{2} \epsilon^{\mu \nu \lambda} (\bar{\chi}_\mu \gamma^\nu \chi_\lambda - i \gamma_i [X^\nu, X^\lambda, Y^i]) \right)$$

plus a metric-dependent cohomological action

$$S_m = \int d^3 x \left( \frac{1}{4} (\nabla^\mu X^\nu - \nabla^\nu X^\mu, \nabla^\mu X^\nu - \nabla^\nu X^\mu) + \frac{1}{2} (\nabla^\mu Y^i, \nabla^\nu Y^i) ight. $$

$$+ \frac{1}{2} (\nabla^\mu X^\mu + \frac{i}{6} \epsilon_{\mu \nu \lambda} [X^\mu, X^\nu, X^\lambda], \nabla^\mu X^\mu + \frac{i}{6} \epsilon_{\mu \nu \lambda} [X^\mu, X^\nu, X^\lambda])$$

$$+ \frac{1}{2} ([Y^i, Y^j, Y^k], [Y^i, Y^j, Y^k]) + \frac{1}{2} ([X^\mu, Y^j, Y^k], [X^\mu, Y^j, Y^k])$$

$$+ (\bar{\psi}, \nabla^\mu \chi^\mu + i \gamma_i [Y^i, X^\mu, \chi_\mu] + \frac{i}{4} \gamma_{ij} [Y^i, Y^j, \psi]) + i \frac{1}{4} (\bar{\chi}^\mu \gamma_{ij} [Y^i, Y^j, \chi_\mu]).$$

(6.11)

where the gauge fields and covariant derivatives have been complexified so that

$$A^\pm_\mu := A_\mu \pm \frac{i}{2} \epsilon_{\mu \nu \lambda} [X^\nu, X^\lambda, -], \quad \nabla^\pm_\mu := \nabla_\mu \pm \frac{i}{2} \epsilon_{\mu \nu \lambda} [X^\nu, X^\lambda, -].$$

(6.12)
The total action $S_{\text{top}} + S_m$ is invariant under the supersymmetry transformations
\begin{align*}
\delta X^\mu &= \bar{\epsilon} \ X^\mu, \\
\delta Y^i &= \bar{\epsilon} \ Y^i, \\
\delta \psi &= -(\nabla_\mu X^\mu + \frac{i}{2} \epsilon_{\mu
u\lambda} [X^\mu, X^\nu, X^\lambda]) \epsilon, \\
\delta \chi_\mu &= \epsilon_{\mu
u\lambda} \nabla^\nu X^\lambda \epsilon + \nabla_\mu^+ Y^i \gamma_i \epsilon + \frac{i}{2} (Y_i, Y_j, X_\mu) \gamma_{ij} \epsilon, \\
\delta A_\mu &= i \bar{\epsilon} (\epsilon Y^i, \psi, \epsilon) + \epsilon_{\mu
u\lambda} [X^\nu, X^\lambda, -] + \gamma_i [Y^i, X_\mu, -]).
\end{align*}  

(6.13)

Setting the fermions equal to zero in (6.13), one finds that the corresponding BPS equations for the supersymmetric solutions of the field theory are
\begin{align*}
\nabla_\mu^+ X^\mu - i \epsilon_{\mu
u\lambda} [X^\mu, X^\nu, X^\lambda] &= 0 = \nabla_\mu^+ X^\mu - \nabla_\nu^+ X^\mu, \\
\nabla_\mu^+ Y^i &= 0 = F_{\mu
u}^+, \\
[Y^i, Y^j, Y^k] &= 0 = [Y^i, Y^j, X^\mu]
\end{align*}  

(6.14)

where the twisted field strength is defined by
\begin{equation}
F_{\mu
u}^+ := F_{\mu
u} - i \epsilon_{\nu\lambda\rho} [\nabla_\mu X^\lambda, X^\rho, -] + i \epsilon_{\mu\lambda\rho} [\nabla_\nu X^\lambda, X^\rho, -]
\end{equation}  

(6.15)

with $F_{\mu
u} = [\nabla_\mu, \nabla_\nu]$.

### 6.2. Mapping to the Blau–Thompson model

Let us consider the metric 3-Lie algebra $\mathcal{A} = \mathfrak{a}_4$ and apply the Higgsing procedure to the twisted BLG theory. We proceed by letting the scalar fields $Y^i$ have classical values proportional to fixed 3-Lie algebra elements. Using $\mathrm{SO}(5)$ symmetry we can assume that only $Y^1$ acquires a vacuum expectation value, and by $\mathrm{SO}(4)$ invariance we can align this value in the 3-Lie algebra direction $\tau_4$. Hence we make the replacement
\begin{equation}
Y^1 \longrightarrow -g \tau_4 + Y^1,
\end{equation}  

(6.16)

where $g$ is a gauge coupling constant. The reduction of the gauge fields works in the usual way: With respect to the splitting $g_\mathcal{A} = \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, we make the replacements
\begin{equation}
A^\pm_\mu \longrightarrow A^\pm_\mu \pm \frac{1}{2} B_\mu,
\end{equation}  

(6.17)

where now we regard $A^\pm_\mu, B_\mu \in \mathfrak{so}(3)$. In the strong coupling limit $g \to \infty$, 3-brackets containing $Y^1$ reduce to the brackets $[X^\mu, X^\nu] := [X^\mu, X^\nu, -\tau_4]$ of the Lie algebra $\mathcal{A}' = \mathfrak{so}(3)';$ we denote the invariant form on either factors of $\mathfrak{so}(3) = \mathfrak{su}(2)$ by $\text{Tr}_{\mathcal{A}'}$, which coincides with the Cartan–Killing form. We also define a modified field strength
\begin{equation}
\tilde{F}_{\mu
u} := F_{\mu
u} - i \epsilon_{\nu\lambda\rho} [\nabla_\mu X^\lambda, X^\rho, -] + i \epsilon_{\mu\lambda\rho} [\nabla_\nu X^\lambda, X^\rho, -]
\end{equation}  

(6.18)

By inserting this combination of gauge fields into the total action $S_{\text{top}} + S_m$, we find that in the strong coupling limit the field $B_\mu$ only interacts with the Chern–Simons and scalar kinetic terms algebraically. Its equation of motion reads as
\begin{equation}
B_\mu = \frac{1}{2g} \epsilon_{\mu\nu\lambda} \tilde{F}^{\nu\lambda} - \frac{1}{2} \nabla_\mu Y^1,
\end{equation}  

(6.19)

where we keep only those terms that will remain in the strong coupling limit. Integrating out the field $B_\mu$, we find that the Chern–Simons term from the first line of (6.10) reduces to the modified...
Yang–Mills term $\int d^3 x \text{Tr}_A \left( \left( \widetilde{F}_{\mu \nu} \right)^2 \right)$. One further finds that the field $Y^1$ decouples from the remaining terms of the total action, so we introduce a new index $a = 1, 2, 3, 4$. Altogether, after suitable rescaling of the fields we thus find that the reduced action is given by

$$S_{\text{red}} = \int d^3 x \text{Tr}_A \left( \frac{1}{2} \left( \widetilde{F}_{\mu \nu} \right)^2 - \frac{i}{2} \epsilon^{\mu \nu \lambda} \bar{X}_\mu \left( \nabla_\nu X_\lambda + [X_\nu, X_\lambda] \right) \right)$$

$$+ \frac{1}{2} [X_\mu, X^\nu]^2 + [X_\mu, Y^\alpha]^2 + \frac{1}{2} [Y^\alpha, Y^\beta]^2$$

$$+ \frac{1}{2} \left( \nabla_\mu X^\nu \right)^2 - \nabla_\mu X^\nu \nabla_\nu X^\mu + \frac{1}{2} \left( \nabla_\mu X^\nu \right)^2 + \frac{1}{2} \nabla_\mu Y^\alpha \nabla_\mu Y^\alpha$$

$$+ \bar{\psi} \nabla_\mu X^\mu - i \bar{\psi} [X_\mu, \chi_\mu] - \frac{i}{2} \bar{\psi} \gamma_a [Y^a, \chi_\mu] - \frac{i}{2} \bar{\chi}^\mu \gamma_a [Y^a, \chi_\mu].$$

(6.20)

As with the original twisted BLG action, the reduced action is the sum of a topological term and a metric dependent cohomological action.

In Ref. [55] it was shown that the Mukhi–Papageorgakis map is compatible with the topological twisting procedure. We thus expect that the reduced model is some topological twist of $\mathcal{N} = 8$ supersymmetric Yang–Mills theory in three dimensions. The possible twists for this gauge theory were classified in Ref. [57]: One can either arrive at a twisted $\mathcal{N} = 2$ supersymmetric BF-theory, or a twisted $\mathcal{N} = 4$ equivariant extension of the Blau–Thompson model. Comparing our lagrangian (6.20) with those listed in Ref. [57], we find that we have obtained the on-shell formulation of the $\mathcal{N} = 4$ equivariant extension of the Blau–Thompson model; it can be realized as the world-volume gauge theory of D2-branes wrapping supersymmetric three-cycles in Type IIA string theory. Maximally supersymmetric Yang–Mills gauge theories on $S^3$ are also considered in Ref. [58].

### 6.3. Cohomological IKKT matrix model

As the equivariantly extended Blau–Thompson model is a twist of $\mathcal{N} = 8$ supersymmetric Yang–Mills theory, its dimensional reduction should yield some topological twist of the IKKT matrix model. The zero-dimensional reduction of the action (6.20) becomes

$$S_{\text{BT}} = \text{Tr}_A \left( \frac{1}{2} \left( [A_\mu, A_\nu] - i \epsilon_{\nu \lambda \rho} [A_\mu, X^\lambda], X^\rho \right) \right)$$

$$+ \frac{1}{2} [A_\mu, X^\nu]^2 - [A_\mu, X^\nu] [A_\nu, X^\mu] + \frac{1}{2} [A_\mu, X^\mu]$$

$$+ \frac{1}{2} [X_\mu, X^\nu] + [X_\mu, Y^\alpha]^2 + \frac{1}{2} [Y^\alpha, Y^\beta]^2 + \frac{1}{2} [A_\mu, Y^\alpha]^2$$

$$- i \epsilon^{\mu \nu \lambda} \bar{X}_\mu \left( [A_\nu, \chi_\alpha] + [X_\nu, \chi_\lambda] \right) - \frac{i}{2} \bar{X}_\mu \gamma_a [Y^a, \chi_\mu]$$

$$+ \bar{\psi} \left( [A_\mu, \chi_\mu] - i \bar{\psi} [X_\mu, \chi_\mu] - \frac{i}{2} \bar{\psi} \gamma_a [Y^a, \psi] \right).$$

(6.21)

This matrix model defines an $\mathcal{N} = 4$ equivariant extension of the usual IKKT matrix model in ten dimensions, which can be solved exactly by using localization techniques. It possesses a nilpotent $\mathcal{N} = 2$ topological symmetry that acts on the fields as

$$\delta A_\mu = \bar{\epsilon} \chi_\mu,$$

$$\delta X_\mu = i \bar{\epsilon} \chi_\mu,$$

$$\delta \chi_\mu = i \epsilon^{\mu \nu \lambda} [A_\mu, A_\nu] \bar{\epsilon},$$

$$\delta \bar{X}_\mu = - \gamma_a [A_\mu, Y^a] \epsilon,$$

$$\delta \psi = 0,$$

$$\delta \bar{\psi} = - [A_\mu, X^\mu] \bar{\epsilon} - i \gamma_{ab} [Y^a, Y^b] \bar{\psi},$$

$$\delta Y^a = - 2 i \bar{\epsilon} \gamma^a \psi.$$

(6.22)
In Sect. 4 we showed how the reduced ABJM and BLG models are related. One could thereby hope to lift the cohomological deformation (6.21) of the IKKT matrix model to obtain an analogous twist of the ABJM matrix model that would enable the exact computation of the deformed partition function using localization techniques. However, it was shown in Ref. [55] that for three-dimensional \( \mathcal{N} = 6 \) Chern–Simons-matter theories the only possible twists involve vector supercharges, and hence it is not possible to directly obtain such a cohomological deformation of the ABJM theory. In Sect. 7 we shall alleviate this problem by constructing a cohomological matrix model by hand that explicitly localizes onto the BPS equations of the ABJM matrix model; hence it computes an equivariant index for the model explicitly, and moreover possesses the same qualitative features as the matrix model (6.21) under the Mukhi–Papageorgakis map.

7. Equivariant 3-algebra models

In this final section we shall relate the computation of partition functions of our supersymmetric quiver matrix models to those of a particular cohomological matrix model. Cohomological matrix models comprise a certain type of topological field theory that are constructed by specifying a set of fields, a set of equations, and a set of symmetries; the correlation functions constructed from these data compute intersection numbers on the moduli space of solutions to the equations modulo the symmetries [59]. They have actions of the form

\[
S(\Phi) = Q \mathcal{V}(\Phi),
\]

where \( Q \) is the nilpotent BRST charge of the model acting on a gauge-invariant functional \( \mathcal{V}(\Phi) \) of the field content \( \Phi \). Matrix models of this type have appealing properties. For example, they are often exactly solvable by using localization methods. A prominent example of this type of theory is due to Moore, Nekrasov, and Shatashvili [37]: They computed the path integral for the Yang–Mills matrix model by constructing a related cohomological field theory, and then solving the cohomological deformation using localization techniques. This formalism was generalized to a large class of quiver matrix models in Ref. [60]. Since our dimensionally reduced Chern–Simons-matter theories and the IKKT matrix model are related via the Mukhi–Papageorgakis map, we could expect that the deformation approach of Ref. [37] can be lifted to our model; in this section we will apply this approach to the ABJM matrix model by constructing a related cohomological matrix model and then computing the path integral using localization methods. The deformation of the matrix integral is accomplished using the global \( \text{SU}(4) = \text{Spin}(6) \) R-symmetry of the model, and it preserves \( \mathcal{N} = 2 \) supersymmetry. It involves a choice of a generic element in the Cartan subalgebra of the R-symmetry group, which enables one to construct well defined matrix integrals. See Ref. [61, Sect. 5.1] for a relation between the cohomological deformation of the IKKT matrix model in four dimensions and the (non-supersymmetric) Chern–Simons matrix model obtained from dimensional reduction on \( S^3 \).

7.1. Equivariant localization

We begin by summarizing the main features involved in equivariant localization, in a form that we shall employ it. Localization is a technique used in supersymmetric quantum field theory by which a path integral over an infinite-dimensional field domain is reduced to a finite-dimensional integral; here we apply it to reduce the partition functions for our quiver matrix models to integrals over the critical point locus of some matrix functional. For this, we perturb the action \( S(\Phi) \) of our model and
consider the deformed partition function

\[ \mathcal{Z}_t = \int d\Phi \ e^{-S(\Phi) - t Q V(\Phi)}, \]  

(7.2)

where \( d\Phi \) is a suitably normalized, supersymmetry-invariant measure on field space and \( t \in \mathbb{R} \) parametrizes a continuous family of partition functions such that \( \mathcal{Z} := \mathcal{Z}_0 \) is the partition function of the original matrix model. Since the action \( S(\Phi) \) is supersymmetric, \( QS(\Phi) = 0 \), and the scalar supercharge \( Q \) is nilpotent on gauge-invariant operators, we have

\[ \frac{\partial \mathcal{Z}_t}{\partial t} = - \int d\Phi \ Q V(\Phi) e^{-S(\Phi) - t Q V(\Phi)} = - \int d\Phi \ Q \left( V(\Phi) e^{-S(\Phi) - t Q V(\Phi)} \right) = 0, \]  

(7.3)

where in the last step we have integrated by parts using the derivation property of the BRST operator \( Q \) with \( QS(\Phi) = 0 \), and used invariance of the measure \( d\Phi \) on field space under the BRST symmetry. This means that the original partition function \( \mathcal{Z} = \mathcal{Z}_0 \) is computed by (7.2) at any value of \( t \). In the limit \( t \to \infty \), the partition function often simplifies; in particular, if \( Q V(\Phi) \) is positive definite then the contributions to the integral in this limit come from the minima \( \Phi_0 \) in field space where \( Q V(\Phi_0) = 0 \). The partition function (7.2) can then be evaluated by applying the method of steepest descent. The differences between contributions from \( \Phi_0 \) and a generic point \( \Phi \) in field space are exponentially suppressed as \( t \to \infty \); the dominant contributions to this integral therefore come from points in a neighborhood \( \mathcal{N}(\Phi_0) \) of \( \Phi_0 \). Assuming that \( e^{-S(\Phi)} \) varies slowly with respect to \( e^{-t Q V(\Phi)} \), the partition function reduces to

\[ \mathcal{Z} = \int_{Q V(\Phi) = 0} d\Phi_0 e^{-S(\Phi_0)} \int_{\mathcal{N}(\Phi_0)} d\Phi' e^{-t Q V(\Phi')} \]  

(7.4)

with \( \Phi' \in \mathcal{N}(\Phi_0) \) denoting fluctuations around the minima \( \Phi_0 \); here we have dropped higher order terms using nilpotency of the supersymmetry variations. The \( t \)-dependence of the fluctuation integral in (7.4) cancels by supersymmetry of the measure \( d\Phi' \) when one performs the bosonic and fermionic integrations. Note that for cohomological matrix models with actions of the form (7.1), we can apply this argument directly to the integral \( \int d\Phi \ e^{-t S(\Phi)} \) itself, so that (7.4) is given by an integral over minima of the original action \( S(\Phi) \) with \( S(\Phi_0) = 0 \).

7.2 Localization of \( \mathcal{N} = 2 \) Chern–Simons quiver matrix models

Let us apply this formalism to the matrix models having actions (3.10) with a positive definite quadratic form \( \text{Tr} g \); to ensure convergence of the matrix integral, here we set \( A_0 = i A_3 \) with \( A_3 \) hermitian and \( \gamma^0 = i \gamma^3 \). For the cohomological deformation of this action we take

\[ Q V = Q \bar{\Phi} \text{Tr}_g \left( \frac{1}{2} \bar{\lambda} \lambda - 2 D \sigma \right), \]  

(7.5)

where the supercharge \( Q \) generates the nilpotent supersymmetry transformations (3.6) with \( \eta = \bar{\varepsilon} \) and the spinor normalization \( \bar{\varepsilon} \varepsilon = 1 \). The deformation term then reads explicitly as

\[ Q V = \text{Tr}_g \left( - \frac{1}{2} [A_\mu, A_\nu]^2 - [A_\mu, \sigma]^2 + D^2 + \frac{1}{2} \bar{\lambda} \gamma^\mu [A_\mu, \lambda] + i [\bar{\lambda}, \sigma] \lambda \right). \]  

(7.6)

Writing \( X^I = (A^\mu, \sigma \), \( \Psi = (\lambda^1, \lambda^2 \), \( \Gamma^I = (\gamma^\mu, i \mathbb{I} \), and \( F = D \), this is just the action of the four-dimensional Yang–Mills matrix model (5.13) (with \( g = 1 \)). The localization locus \( Q V = 0 \) is
given by

\[ [A_\mu, A_\nu] = 0 = [A_\mu, \sigma], \quad D = 0 = \lambda = \bar{\lambda} \]  

for the gauge sector, which coincides with the BPS equations (3.11). By noting that the matter part of the action (3.8) is itself a BRST-exact term

\[ S_m = Q \bar{Q} \text{Tr}(\bar{\psi} \psi - 2Z^\dagger \sigma Z), \]

we may choose the localization locus

\[ Z = F = 0 = \psi \]

for the matter interactions. Then the action (3.10) vanishes at the critical points. For gauge group \( G = U(N) \), the fixed point locus thus coincides with the moduli variety of quadruples \((A_\mu, \sigma)\) of commuting matrices; for \( G = U(N_L) \times U(N_R) \), it is a subvariety of the vacuum moduli space of the ABJM matrix model defined by the BPS equations (3.22). While the analogous localization procedure works nicely in the field theory setting to provide exact results for supersymmetric Chern–Simons-matter theories on \( S^3 \) [9,10] and their dimensional reductions to a point [14,15], in our dimensionally reduced model the result of the localization integral (7.4) comes out to involve terribly divergent integrals over the Cartan subalgebra of \( g \) that are beyond regularization; the partition function in our case is not well defined because the action lacks supersymmetric mass terms for the scalars. Below we shall cure this problem by constructing a cohomological matrix model whose fixed point locus provides a rigorous definition of the same moduli variety via a further equivariant deformation parametrized by the R-symmetry group of the matrix model. We follow the method of Ref. [60] to compute a supersymmetric equivariant index using localization techniques. Although the localization integral still formally diverges, the presence of twisted masses enables one to define it via a suitable prescription that we explain in detail.

### 7.3. Cohomological matrix model formalism

As only theories with \( N \geq 4 \) supersymmetry can be twisted to produce deformed scalar supercharges, we focus our attention henceforth on the \( N = 6 \) ABJM matrix model from Sect. 3.3 for definiteness; we construct a cohomological matrix model that localizes onto the BPS equations. In view of our discussion from Sect. 7.2, here we consider instead the localization locus with \( A^L_\mu, R = 0 \) as the gauge fields do not themselves transform under the R-symmetry; the BPS equations (3.22) then reduce to the relations (4.12) of the double of the ABJM quiver (3.17). Put differently, we localize the partition function of the matrix model onto the F-term constraints rather than the D-term constraints. We localize the matrix integral with respect to the equivariant BRST operator in the gauge group \( G = U(N_L) \times U(N_R) \), twisted by the toric action of the maximal torus \( \mathbb{T}^4 \) of the R-symmetry group \( SU(4) \) of the matrix model; this deforms the nilpotent BRST charge to a differential of \( SU(4) \)-equivariant cohomology. We denote the generators of this torus by \( \epsilon_i \in \mathbb{R}, i = 1, 2, 3, 4 \), and set \( t_i = e^{\epsilon_i} \) with the \( SU(4) \)-constraint

\[ \sum_{i=1}^4 \epsilon_i = 0 \]  

on the toric parameters. The full symmetry group of the equivariant model is thus \( U(N_L) \times U(N_R) \times \mathbb{T}^4 \). The transformation properties of the fields and equations of motion under the toric action of \( \mathbb{T}^4 \)
are given by

\[ Z^i \mapsto e^{-i\epsilon_i} Z^i, \quad Z^{jk}_i \mapsto e^{-i(\epsilon_j + \epsilon_k - \epsilon_i)} Z^{jk}_i. \]  

(7.11)

In order to construct a supersymmetric matrix model we assign superpartners to these fields to give multiplets \((Z^i, \psi^i)\) with BRST transformations

\[ QZ^i = \psi^i, \quad Q\psi^i = \phi_R Z^i - Z^i \phi_L - \epsilon_i Z^i, \]  

(7.12)

where the hermitian gauge parameters \(\phi_{L,R} \in \text{End}(V_{L,R})\) transform in the adjoint representation of the factors \(U(N_{L,R})\) of the gauge group. (There is no sum over \(i\) in the second equation.) We now add the Fermi multiplet of auxiliary fields \((\chi^{jk}_i, H^{jk}_i)\) related to the BPS equations, where the antighosts are defined as maps \(\chi^{jk}_i \in \text{Hom}(V_L, V_R)\) with transformations that read as

\[ QH^{jk}_i = \phi_R \chi^{jk}_i - \chi^{jk}_i \phi_L - (\epsilon_j + \epsilon_k - \epsilon_i) \chi^{jk}_i, \quad Q\chi^{jk}_i = H^{jk}_i. \]  

(7.13)

To these fields we include the gauge multiplet \((\phi_{L,R}, \tilde{\phi}_{L,R}, \eta_{L,R})\), which is necessary to close the BRST algebra off-shell; these fields have transformations

\[ Q\phi_{L,R} = 0, \quad Q\tilde{\phi}_{L,R} = \eta_{L,R}, \quad Q\eta_{L,R} = [\phi_{L,R}, \tilde{\phi}_{L,R}]. \]  

(7.14)

In order to obtain a localization onto a well defined moduli space of matrices that can be described as a non-singular quotient of a critical locus by the gauge group \(G\), we incorporate additional fields \(\varphi, I_{L,R}\) into the collection of bosonic fields, together with their superpartners \(\xi, \rho_{L,R}\) into the collection of fermions. The new field \(\varphi \in \text{Hom}(V_L, V_R)\) transforms in the bifundamental representation of the \(U(N_L) \times U(N_R)\) gauge group and in the determinant representation of the R-symmetry, and hence is invariant under the toric action of \(\mathbb{T}^4\) by (7.10). The fields \(I_{L,R} \in V_{L,R} = \mathbb{C}^{N_{L,R}}\) are also taken to be invariant under the action of the torus \(\mathbb{T}^4\) for simplicity, and they transform as vectors under the actions of the left and right gauge groups \(U(N_{L,R})\); in what follows we shall refer to the fundamental matter fields \(I_{L,R}\) as “framing vectors”. The equations of motion for these additional fields are given by

\[ \varphi I_L = 0 = \varphi^\dagger I_R \]  

(7.15)

and they ensure stability of the vacua of our quiver matrix model, as we discuss in detail later on. Their BRST transformations are

\[ Q\varphi = \xi, \quad QI_{L,R} = \rho_{L,R}, \quad Q\xi = \phi_R \varphi - \varphi \phi_L, \quad Q\rho_{L,R} = \phi_{L,R} I_{L,R}. \]  

(7.16)

We now add the corresponding antighost and auxiliary fields \(\xi_{L,R} \in V^*_{L,R}\) and \(h_{L,R}\) with the BRST transformations

\[ Q\xi_{L,R} = h_{L,R}, \quad Qh_{L,R} = -\xi_{L,R} \phi_{L,R}. \]  

(7.17)

The BRST symmetry \(Q\) squares to a gauge transformation twisted by a \(\mathbb{T}^4\) rotation of the fields.

Following the treatment of Sect. 7.2, we will now write down a cohomological Yang–Mills type matrix model that has this field content, equations of motion, and BRST transformations. It is given by the \(N' = 2\) action

\[
S_{\text{coh}} = Q \text{Tr} V \left( (\chi^{jk}_i \epsilon^2 H^{jk}_i - \imath[Z^j, Z^k, Z_i]) + \psi^j (\phi_L Z^i - Z^i \phi_R) + \eta [\phi_L, \tilde{\phi}_R] - \eta_R [\phi_R, \tilde{\phi}_L] 
+ \epsilon^L \otimes (g' h_L - I^L) \varphi - \xi_R \otimes (g' h_R - I^R) \psi^i
+ \phi_L \rho_L \otimes I^L - \tilde{\phi}_R \rho_R \otimes I^R 
+ (\phi_L \varphi^i - \phi^i \tilde{\phi}_R) \xi + \epsilon^{\dagger} (Z^i \psi^i + Z^i \psi^i) + \epsilon^{\dagger} (I_L \otimes \rho_L - I_R \otimes \rho_R) 
+ \frac{\epsilon^{\dagger}}{2} \left( \varphi^i \xi^j + \varphi^j \xi^i \right) \right)
\]  

(7.18)

where we used the canonical identifications \(\text{End}(V_{L,R}) = V_{L,R} \otimes V^*_{L,R}\). The deformation by the last three BRST-exact terms in (7.18) removes flat directions from the matrix integral for the partition
function (see Ref. [60] for details); the equivariant deformation further has the effect of generating mass terms for all bosonic fields, which as we will see yields a well defined matrix integral. Note that the relevant bosonic part of the action from the first line of (7.18) is \( \text{Tr}_V \left( \frac{g}{2} H_i^{jk} \right) \); integrating out \( H_i^{jk} \) gives the bosonic potential energy \( \frac{1}{2g} \text{Tr}_V \left( Z_i^{jk} \right) \) and supersymmetry, and thus the path integral of the matrix model localizes onto the configurations where \( Z_i^{jk} = 0 \), as desired.

Since this matrix model is cohomological, it is independent of the couplings \( g, g', g_1, g_2, g_3 \) in the action (7.18). We can compute the partition function by taking various limits of these couplings. The first step is to use the \( U(N_L) \times U(N_R) \) gauge symmetry to diagonalize the gauge generators \( \phi_{L,R} \); we denote their eigenvalues by \( \phi^a_L, a = 1, \ldots, N_L \), and \( \phi^b_R, b = 1, \ldots, N_R \). This change of variables produces Vandermonde determinants \( \prod_{a<b} (\phi^a_L - \phi^b_R)^2 \) and \( \prod_{a<b} (\phi^b_R - \phi^a_R)^2 \) in the path integral measure. Let us now take the limit \( g \to \infty \). The dominant part of the action is

\[
g_1 \prod_{i=1}^4 \text{Tr}_V \left( H_i^{jk} \phi R \phi_L \phi_R - \phi_L \phi_R - \phi_L \phi_R \right). 
\] (7.21)

Performing the matter integrations puts a term in the localized matrix integral of the form \( \prod_{a,b} \prod_i \prod_{j<k} (\phi^a_L - \phi^b_R + \epsilon_j + \epsilon_k - \epsilon_i) \). Now we take the limit \( g_1 \to \infty \). The relevant part of the action reads as

\[
g_1 \prod_{i=1}^4 \text{Tr}_V \left( \psi_i \psi_i \phi - \phi_L \phi_R \right). 
\] (7.22)

The fields \( h_{L,R} \) can be trivially integrated out, while performing the left and right fermionic integrations puts terms in the path integral of the form \( \prod_{a,b} \prod_i (\phi^a_L - \phi^b_R - \epsilon_i - i 0)^{-1} \), where we have added a small imaginary part to the generic real parameters \( \epsilon_i \) to ensure convergence of the gaussian integrations. Next we treat the stabilizing fields \( I_{L,R} \) and their superpartners. We first take the limit \( g' \to \infty \). The dominant part of the action is

\[
g' \text{Tr}_V \left( h^{\dagger}_{L} \otimes h_{L} - h^{\dagger}_{R} \otimes h_{R} - \xi^{\dagger}_{L} \otimes \xi_{L} + \xi^{\dagger}_{R} \otimes \xi_{R} \phi \right). 
\] (7.23)

As a Lebesgue integral, this expression formally diverges. Hence we define it via an analytic continuation to a suitable contour integral prescription in the complex plane that picks up the poles of the integrand; the precise choice of contour keeps track of the auxiliary multiplet of fields that have been
eliminated by taking the large coupling limits above. It is straightforward to see that the poles occur precisely on the supersymmetric solutions of the cohomological matrix model. For this, we consider the critical points of the action (7.18) where the fermions are set equal to zero. They are determined by the zeroes of the BRST charge. By (7.12) and (7.16) the fixed point equations are then

\[ Z^{i \alpha}_{ab} (\phi^b_L - \phi^a_R - \epsilon_i) = 0 = \varphi_{ab} (\phi^b_L - \phi^a_R) = 0, \quad I^a_R \phi^a_R = 0 = I^b_L \phi^b_L \]  

(7.24)

for each \( i = 1, 2, 3, 4, a = 1, \ldots, N_R, \) and \( b = 1, \ldots, N_L. \)

We can evaluate the integral (7.23) explicitly in dimensions \( N_L = N_R = 1. \) As its integrand depends only on the combination \( \phi := \phi_L - \phi_R \) in this case, it can be evaluated from the residue theorem by picking up the contributions from the simple poles at \( \phi = 0 \) and \( \phi = \epsilon_i, i = 1, 2, 3, 4, \) to get

\[ \mathcal{Z}^{ABJM}_{1,1} (\epsilon) = \prod_{i=1}^{4} \frac{1}{\epsilon_i} \prod_{j<k} (\epsilon_i - \epsilon_j - \epsilon_k) + \sum_{i=1}^{4} \frac{1}{\epsilon_i} \prod_{i \neq j} \frac{1}{\epsilon_i - \epsilon_j} \prod_{i'=1}^{4} \prod_{j<k} (\epsilon_i' + \epsilon_i - \epsilon_j - \epsilon_k). \]  

(7.25)

On the other hand, for \( N_R = 0 \) one finds that the contour integral vanishes for \( N_L \geq 5; \) more generally, the integral vanishes for \( |N_L - N_R| \) sufficiently large, in agreement with recent analysis of the ABJM theory through the partition function of the \( U(N_L) \times U(N_R) \) lens space matrix model [62]. However, for higher dimensions an explicit evaluation of (7.23) becomes increasingly intractable.

In the remainder of this section we shall develop an alternative local model for the fluctuation integrals in (7.4) through a geometric analysis of the neighborhoods \( \mathcal{N}(\Phi_0) \) around the fixed point subset of the critical point locus with respect to the action of the R-symmetry torus \( \mathbb{T}^d. \) In particular, we compute an equivariant index

\[ \mathcal{J}_{N_L, N_R} (t) = \text{Tr}_{H_{BPS}} (-1)^F \prod_{i=1}^{4} I_i^R \]  

(7.26)

whose infinitesimal limit \( \epsilon_i \to 0 \) explicitly evaluates the contour integrals (7.23); here \( H_{BPS} \) is the Hilbert space of framed BPS states of the cohomological field theory and \( R_i \) are the generators of the Cartan subalgebra of the global symmetry group \( SU(4) = SO(6). \) In writing (7.26) we have used the fact that the hamiltonian \( H \) vanishes in any cohomological field theory, and set the fugacity \( y = 1 \) for \( SO(2) \) rotations as we take \( A_\mu = 0. \)

Because the R-symmetry descends from the Lorentz group \( SO(1, 10) \) in 11 dimensions, the operators \( I_i^R \) in the equivariant index rotate the extra directions \( \mathbb{C}^4; \) this suggests that the index (7.26) could be interpreted as the partition function of M-theory compactified on the total space of an affine \( \mathbb{R}^{10} \)-bundle \( M \) over \( S^4, \) with locally flat metric, viewed as a fibration

\[ \mathbb{R}^2 \times \mathbb{C}^4 \longrightarrow M \]  

(7.27)

\[ \downarrow \]  

\[ S^4 \]

The affine bundle \( M \) is defined as the quotient of \( \mathbb{R}^{11} \) by the \( \mathbb{Z} \)-action given by

\[ (x^0, \bar{x}, z) \mapsto (x^0 + 2\pi n, \bar{x}, e^{2\pi i n} z) \]  

(7.28)
for \((x^0, x, z^i) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{C}^4\) and \(n \in \mathbb{Z}\). This background has a realization in 11-dimensional supergravity \([63]\) with the global metric

\[
dx^2_{11} = (dx^0)^2 + dx^2 + \sum_{i=1}^{4} |dz^i - i\epsilon_i z^i dx^0|^2.
\]

(7.29)

Since the \(N = 2\) supersymmetry of the cohomological field theory is not sufficient to fix the R-charges of gauge-invariant operators, the partition function depends on the R-charges of the matter fields.

### 7.4. Vacuum moduli space and fixed point analysis

The partition function (7.23) can be regarded as computing a regularized volume of the non-compact vacuum moduli space \(\mathcal{M}_{N_L,N_R}\) \([64]\), which we now define explicitly. For this, we recall the equations of motion (7.15), which imply that the vector \(I_L\) sits in the kernel and the vector \(I_R\) in the cokernel of \(\varphi\). The presence of the bifundamental field \(\varphi\) also implies that the quotient of the fixed point locus \(Z_i^{jk} = 0\) by the gauge group \(G\) is equivalent to a quotient by the action of the complexified gauge group \(G\). Then the moduli space can be represented as a quasi-projective variety

\[
\mathcal{M}_{N_L,N_R} = \{(Z_i^{jk})^{-1}(0) \} // \text{GL}(N_L, \mathbb{C}) \times \text{GL}(N_R, \mathbb{C}),
\]

(7.30)

where the GIT quotient on the right is taken by removing the points at which the action of \(G\) is not free. Such a quotient can be defined by imposing an additional stability condition on the data (\(Z_i^i, I_{L,R}, \varphi\)); a suitable notion of stability for our purposes can be given as follows: We say that a datum \((Z^i, I_{L,R}, \varphi)\) is stable if there are no non-trivial proper subspaces \(W_{L,R} \subset V_{L,R}\) that contain the vectors \(I_{L,R}\) and that are invariant under the bilinear commuting operators \(Z_i^j Z^i, Z_i^j Z_i^j\) for all \(i, j = 1, 2, 3, 4\), respectively. Let us demonstrate that the gauge group \(G\) acts freely on stable data. Suppose that \((Z^i, I_{L,R}, \varphi)\) is fixed by \((g_L, g_R) \in G\). Then \(g_R Z^i = Z^i g_L, g_L Z_j^j = Z_j^j g_R, g_{L,R} I_{L,R} = I_{L,R}\), which respectively imply that the subspaces \(W_{L,R} = \ker(1 - g_{L,R})\) have \(Z_i^j Z^i (W_{L,R}) \subset W_{L,R}, Z_i^j Z_i^j (W_{R}) \subset W_{R}\), and \(I_{L,R} \in W_{L,R}\). It follows by stability that \(g_{L,R} = 1\), and hence the \(G\)-action is free. The corresponding quotient (7.30) defines a suitable moduli space of solutions to the BPS equations (4.12) modulo gauge equivalence.

Let us now characterize the fixed points of this moduli space. A fixed point \(\Pi = (Z^i, I_{L,R}, \varphi) \in \mathcal{M}_{N_L,N_R}\) with respect to the action of \(\mathbb{T}^4 \subset \text{SU}(4)\) is characterized by the condition that an equivariant rotation is equivalent to a gauge transformation of the fields, so that

\[
g_R Z^i g_L^{-1} = t_i^{-1} Z^i, \quad g_{L,R} I_{L,R} = I_{L,R}, \quad g_R \varphi = \varphi g_L.
\]

(7.31)

Under the \(\mathbb{T}^4\)-action the vector spaces \(V_{L,R}\) admit the weight space decompositions

\[
V_{L,R} = \bigoplus_{\alpha \in \mathbb{Z}^4} V_{L,R}(\alpha)
\]

(7.32)

with

\[
V_{L,R}(\alpha) = \{ v \in V_{L,R} \mid g_{L,R}^{-1} v = t_1^{a_1} t_2^{a_2} t_3^{a_3} t_4^{a_4} v \}
\]

(7.33)

for \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^4\). It is a straightforward consequence of (7.31) that the nonvanishing components of the maps \((Z^i, I_{L,R}, \varphi)\) are given by

\[
Z^i : V_L(\alpha) \to V_R(\alpha - e_i), \quad I_{L,R} \in V_{L,R}(0), \quad \varphi : V_L(\alpha) \to V_R(\alpha),
\]

(7.34)

where \(e_i \in \mathbb{Z}^4, i = 1, 2, 3, 4\), is the vector with 1 in its \(i\)th component and 0 elsewhere. With the weight space decompositions (7.33) and (7.34), it is also easy to show that the solution of the fixed
point equations (7.24) is given by setting the eigenvalues of the gauge parameter matrices $\phi_{L,R}^\alpha$ in this basis equal to

$$
\phi_{L,R}^\alpha = \sum_{i=1}^4 \epsilon_i \alpha_i^{L,R},
$$

(7.35)

and $Z_i^j = 0 = I_{L,R}$ except for the components $Z_{\alpha-e_i,\alpha}^i$ and $I_{L,R}^0$. Moreover, the only non-trivial components of the BPS equations (3.22) are given by

$$
Z_j^{\alpha+e_i-e_j,\alpha} = Z_j^{\alpha+e_i-e_j,\alpha+e_k} \left( Z_i^{\alpha+e_i-e_k,\alpha} \right)^{\sum_{i=1}^4 \epsilon_i \alpha_i^{L,R}}
$$

and for the conjugates of these equations one has

$$
\left( Z_j^{\alpha+e_i-e_j,\alpha} \right)^{\sum_{i=1}^4 \epsilon_i \alpha_i^{L,R}} = \left( Z_j^{\alpha+e_i-e_j,\alpha+e_k} \right)^{\sum_{i=1}^4 \epsilon_i \alpha_i^{L,R}} \left( Z_i^{\alpha+e_i-e_k,\alpha} \right)^{\sum_{i=1}^4 \epsilon_i \alpha_i^{L,R}}.
$$

(7.36)

We can describe the graded components of the $T^4$-module decomposition (7.33) explicitly in terms of the fixed point maps as follows. Recalling the discussion at the end of Sect. 4.2, we unambiguously define subspaces of $V_{L,R}$ by

$$
W_L = \bigoplus_{n_{ij} \geq 0} \prod_{i,j=1}^4 \left( Z_i^j \right)^{n_{ij}} I_L, \quad W_R = \bigoplus_{n_{ij} \geq 0} \prod_{i,j=1}^4 \left( Z_j^i \right)^{n_{ij}} I_R.
$$

(7.37)

Clearly $I_{L,R} \in W_{L,R}$, the subspace $W_L$ is $Z_i^j$-invariant, and $W_R$ is $Z_j^i$-invariant for all $i, j$. Whence $W_{L,R} = V_{L,R}$ by stability, and hence

$$
V_L(\alpha) = \bigoplus_{\sum_j (n_{ij} - n_{ji}) = \alpha_i} \prod_{i,j=1}^4 \left( Z_i^j \right)^{n_{ij}} I_L, \quad V_R(\alpha) = \bigoplus_{\sum_j (n_{ij} - n_{ji}) = \alpha_i} \prod_{i,j=1}^4 \left( Z_j^i \right)^{n_{ij}} I_R.
$$

(7.39)

Note that the constraints on the sums in (7.39) imply that the weights must satisfy

$$
\sum_{i=1}^4 \alpha_i = 0.
$$

(7.40)

We define finite sets of lattice points $\Pi_{L,R} \subset \mathbb{Z}^4$ by

$$
\Pi_{L,R} = \left\{ \alpha \in \mathbb{Z}^4 \mid V_{L,R}(\alpha) \neq 0 \right\},
$$

(7.41)

with $|\Pi_{L,R}| = N_{L,R}$ nodes; the meaning of the restrictions $\Pi_{L,R} \subset \mathbb{Z}^3$ implied by (7.40) will be elucidated below. The vertices of these lattices are related by the actions of commuting matrices
through the commutative diagrams

\[
\begin{align*}
& V_L(\alpha) \quad \xrightarrow{Z^i_i Z^j_j} \quad V_L(\alpha + e_i - e_j) \\
& V_L(\alpha + e_k - e_l) \quad \xrightarrow{Z^i_i Z^j_j} \quad V_L(\alpha + e_i + e_k - e_j - e_l)
\end{align*}
\]

and

\[
\begin{align*}
& V_R(\alpha) \quad \xrightarrow{Z^i_i Z^j_j} \quad V_R(\alpha + e_i - e_j) \\
& V_R(\alpha + e_k - e_l) \quad \xrightarrow{Z^i_i Z^j_j} \quad V_R(\alpha + e_i + e_k - e_j - e_l)
\end{align*}
\]

\[(7.42)\]

We can gain a better combinatorial understanding of the sets (7.41) by employing some machinery from the theory of quiver representations (see e.g. Ref. [65]); in this setting we identify torus-invariant framed BPS states \( \Pi \) in the cotangent bundle of the moduli space of framed representations of the ABJM quiver (3.17) with fixed dimension vector \((N_L, N_R)\). In fact, many interesting features of BPS states in three-dimensional supersymmetric gauge theories find natural realizations within the quiver framework. For example, there is a conjectural Seiberg duality for Chern–Simons gauge theories with \( N \geq 2 \) supersymmetry (see e.g. Ref. [5]); in the present context this duality is realized as a mutation of quivers, which is a tilting procedure that therefore yields an equivalence of the corresponding derived categories of quiver representations [66].

A quiver representation is the same thing as a module for the path algebra \( A \) of the ABJM quiver (3.17) with relations (4.12). The path algebra \( A \) is generated by acting with arrows \( Z^i_i, Z^i_i \), \( i = 1, 2, 3, 4 \), on the framing vectors \( I_{L,R} \), as in (7.38); we refer to such quiver representations as cyclic modules. In this setting we replace our definition of stable points \( \Pi \) above with the more natural notion of \( \theta \)-stability appropriate to moduli spaces of quiver representations [67]. By regarding the conjugate fields \( Z^i_i \) as independent arrows, our quiver moduli problem is then formally equivalent to that of the conifold quiver whose path algebra is a noncommutative crepant resolution of the conifold singularity in six dimensions [68], except that we use multiple framings as in Ref. [69] in order to preserve the left/symmetry inherent in the original ABJM matrix model. This provides us with a concrete geometrical description of the vacuum moduli space; it would be interesting to investigate what low-energy brane dynamics this geometry could correspond to in this equivariant model.

The R-symmetry torus \( \mathbb{T}^4 \) acts on the arrows \( Z^i_i, i = 1, 2, 3, 4 \); hence it acts on the whole path algebra \( A \) and leaves the relations (4.12) invariant. The diagonal torus \( \mathbb{T}^2 \) of the gauge group \( G \) induces an action of \( \mathbb{T} = U(1) \) on the arrows via overall rescaling; this can be used to set e.g. \( \epsilon_4 = 0 \). Modding out by this gauge group action, the overall torus action is \( \mathbb{T}_Q \cong \mathbb{T}^3 \). We shall now argue that the \( \mathbb{T}_Q \)-fixed points are isolated and are parametrized by certain filtrations of the finite pyramid partitions of the conifold quiver. For this, we note that the \( \mathbb{T}_Q \)-fixed points in the moduli space of framed cyclic modules correspond bijectively to \( \mathbb{T}_Q \)-fixed ideals in the path algebra \( A \). There is a one-to-one correspondence between \( \mathbb{T}_Q \)-fixed modules of the path algebra \( A \) with relations and the \( \mathbb{T}_Q \)-fixed annihilator \( A \) of the framing vectors \( I_{L,R} \in V_{L,R} \) consisting of stabilizing bifundamental...
fields $\varphi$ that satisfy \eqref{7.15}; the finite-dimensional annihilator $\mathbb{A}$ is a left ideal of the path algebra and it is generated by linear combinations of elements of the same weight. We claim that $\mathbb{A}$ is generated by monomials of the path algebra, such that its class $[\mathbb{A}]$ is an isolated $\mathbb{T}_Q$-fixed point in the moduli space of cyclic representations with dimension vector $(N_L, N_R)$. For this, note that $\mathbb{A}$ is generated by linear combinations of path monomials of the same weights. Given a torus weight $t_1^{a_1} t_2^{a_2} t_3^{a_3}$, we can find finitely many monomial paths $p_l$ emanating from the nodes $V_{L,R}$. Elements of $\mathbb{A}$ with weight $t_1^{a_1} t_2^{a_2} t_3^{a_3}$ are most generally written as finite sums of paths $\sum_l \xi_l p_l$ for some $\xi_l \in \mathbb{C}$; if $\xi_l \neq 0$, then $p_l$ should be included as one of the monomial generators of the $\mathbb{T}_Q$-fixed annihilator $\mathbb{A}$, since each $p_l$ is a linear map from the framing vectors $I_{L,R}$ to different vector spaces. By exhausting all monomial generators in this way, we conclude that the torus fixed point $\mathbb{A}$ is generated by monomials and hence corresponds to an isolated point in the moduli space of quiver representations.

The problem of parametrizing finite-dimensional cyclic $\mathcal{A}$-modules (up to isomorphism) is now equivalent to the problem of parametrizing finite-codimensional ideals of $\mathcal{A}$ (up to $\mathcal{A}$-module isomorphism). Following Ref. \cite{69}, they are classified in terms of filtered pyramid partitions of length two empty room configurations. Recall \cite{68} that a pyramid partition consists of two types of layers of stones, labeled $L$ (colored white) and $R$ (colored black), which denote one-dimensional subspaces $V_{L,R}(\alpha)$ of given toric weights $\alpha$ from \eqref{7.39}. For $i \geq 0$, there are $(i + 1)^2$ $L$-type stones on layer $2i$, and $(i + 1)(i + 2)$ $R$-type stones on layer $2i + 1$. A finite subset $\Pi$ of this combinatorial arrangement is a pyramid partition if, for every stone of $\Pi$, the two stones immediately above it (of different color) are also in $\Pi$.

In the ABJM limit $N_L = N_R = N$, we can make this description of the vacuum moduli space somewhat more explicit. Then the stability condition implies that the moduli space is a resolution of the $N$th symmetric product orbifold \eqref{4.14} provided by the Hilbert scheme $\left( \mathbb{C}^4 \right)^{[N]}$ of $N$ points in $\mathbb{C}^4$, which parametrizes zero-dimensional subschemes of $\mathbb{C}^4$ of length $N$. The map $(Z^i, I) \mapsto \sum_l \lambda_l z_l$ from \eqref{4.16} gives the Hilbert–Chow map
\begin{equation}
\left( \mathbb{C}^4 \right)^{[N]} \longrightarrow \left( \mathbb{C}^4 \right)^N / \mathfrak{S}_N,
\end{equation}
which is constructed in detail in Ref. \cite{70}. Following the derivation in Ref. \cite{60}, the $\mathbb{T}^4$-fixed points in this case are parametrized by three-dimensional solid partitions \cite{71} of the positive integer $N$; they are specified by height functions $\Pi(i) \in \mathbb{Z}$ on a cubic lattice with sites $i \in \mathbb{Z}^3$, such that $\Pi(i) \geq 0$ are decreasing functions in each of the three lattice directions satisfying
\begin{equation}
\sum_{i \in \mathbb{Z}^3} \Pi(i) = N.
\end{equation}

7.5. Equivariant index for the ABJM quiver

The localization formula allows one to calculate the contribution to the partition function from each fixed point; as we have discussed, the sum over fixed points is captured by applying the residue theorem to write the contour integral \eqref{7.23} as a sum over simple poles at the critical points \eqref{7.35}. As the explicit form of the residue formula is difficult to handle, we generalize the technique of Ref. \cite{72} to extract the eigenvalues of the superdeterminants of the BRST operator $Q$, arising in the fluctuation integrals \eqref{7.4}, from the character of the tangent space to the moduli space at each critical point. Let $Q$ be the fundamental representation of $\mathbb{T}^4$ with weight $(1, 1, 1, 1)$; the dual module $Q^*$ has weight $(-1, -1, -1, -1)$. The local geometry of the moduli space of BPS solutions $\mathfrak{M}_{N_L,N_R}$
near a particular fixed point $\Pi = (Z', I_{L,R}, \varphi)$ can be described by the complex of vector spaces

\[
\text{End}(V_L) \oplus \frac{\text{d}^1_{\Pi}}{\text{im}(\text{d}^1_{\Pi})} V_L \oplus V_R \xrightarrow{\text{d}^1_{\Pi}} \text{Hom}(V_L, V_R) \oplus \frac{\text{d}^1_{\Pi}}{\text{im}(\text{d}^1_{\Pi})} V_L \oplus V_R
\]

\[
\text{Hom}(V_L, V_R) \oplus Q \oplus \text{Hom}(V_L, V_R) \oplus (Q^* \otimes \wedge^2 Q)
\]

(7.46)

where the map $\text{d}^1_{\Pi}$ is an infinitesimal gauge transformation

\[
\text{d}^1_{\Pi}(\phi_L, \phi_R) = \begin{pmatrix}
\phi_R Z^i - Z^i \phi_L \\
\phi_L I_L \\
\phi_R I_R \\
\phi_R \varphi - \varphi \phi_L
\end{pmatrix},
\]

(7.47)

while the map $\text{d}^2_{\Pi}$ is the differential of Eqs. (4.12) and (7.15) that define the vacuum moduli space so that

\[
\text{d}^2_{\Pi} \begin{pmatrix}
y^i \\
v_L \\
v_R \\
Y
\end{pmatrix} = \begin{pmatrix}
[Y^j, Z^k; Z_i] + [Z^j, Y^k; Z_i] + [Z^j, Z^k; Y_i] \\
\varphi^* v_R + Y^* I_R \\
\varphi v_L + Y I_L
\end{pmatrix}.
\]

(7.48)

The first cohomology $\ker(\text{d}^1_{\Pi})/\text{im}(\text{d}^1_{\Pi})$ parametrizes deformations and provides a local model for the tangent space $T_{\Pi}\mathcal{M}_{NL,N_R}$ at the fixed point $\Pi$. As supersymmetric ground states are in one-to-one correspondence with cohomology classes of $\mathcal{M}_{NL,N_R}$, the total cohomology of this complex is identified with the Hilbert space $\mathcal{H}_{BPS}$ of framed BPS states of the cohomological field theory.

The complex (7.46) has a natural meaning in the local geometry of the moduli space of representations of the framed ABJM quiver. Write $V$ for a given representation of the ABJM quiver (3.17) with fixed dimension vector $(N_L, N_R)$, and $\text{Ext}^p(-, -)$ for the extension groups in the abelian category of modules for the path algebra $\mathcal{A}$. Then the first term of (7.46) is the space $\text{Ext}^0(V, V) = \text{Hom}(V, V)$ of nodes of the quiver (3.17), the second term is the space $\text{Ext}^1(V, V)$ of arrows including the framing, and the third term is the vector space $\text{Ext}^2(V, V)$ of all relations; as there are no relations among the F-term relations (4.12), in our case $\text{Ext}^p(V, V) = 0$ for all $p \geq 3$ and the deformation complex contains only three terms. Note that since here the $\mathbb{T}^4$ action leaves invariant the F-term relations (4.12) but not the superpotential (3.18) itself, the deformation complex (7.46) is neither symmetric nor self-dual; as a consequence, the local weight of a fixed point $\Pi$ is not simply a sign $(-1)^{\dim T_{\Pi}\mathcal{M}_{NL,N_R}}$ but is rather a rational function of the equivariant deformation parameters $\epsilon_i$, $i = 1, 2, 3, 4$. In the following we compute the equivariant Euler character of the deformation complex (7.46) for the ABJM quiver. Via our deformation of the nilpotent BRST operator, the equivariant Euler character can still be interpreted as a Witten index in the topologically twisted supersymmetric quantum mechanics on the moduli space $\mathcal{M}_{NL,N_R}$ of supersymmetric vacua.

The equivariant character of the complex (7.46) can be calculated from its cohomology, which is given by an alternating sum of the weights of the various $\mathbb{T}^4$ representations. In the representation ring of the torus group $\mathbb{T}^4$, one has $Q = \sum_i t_i^{-1}$ and $\wedge^2 Q = \sum_{i < j} t_i t_j$, and we obtain the virtual
responding gravitational free energies for M-theory backgrounds AdS
sum
\[
\text{ch}^\mathbb{T}^4_{\Pi} = V_L^* \otimes V_L + V_R^* \otimes V_R - \left( (V_L^* \otimes V_R) \sum_{i=1}^4 t_i^{-1} + V_L + V_R + V_L^* \otimes V_R \right) + (V_L^* \otimes V_R) \sum_{i=1}^4 t_i \sum_{j<k} t_j^{-1} t_k^{-1} + V_L + V_R,
\]
where we use the weight decompositions of the vector spaces
\[
V_{L,R} = \sum_{a_{L,R} \in \Pi_{L,R}} \prod_{i=1}^3 t_i^{a_{L,R}} = \sum_{a_{L,R} \in \Pi_{L,R}} \prod_{i=1}^3 t_i^{a_{L,R}+a_1^{L,R}+a_2^{L,R}+a_3^{L,R}}
\]
as \mathbb{T}^4 representations, and the second equality here follows from the constraints (7.10) and (7.40); the dual involution acts on the weights as inversion \((t_i)^* = t_i^{-1}\). Inserting this decomposition into the character formula (7.49) and using the \(\text{SU}(4)\)-constraint \(t_1 t_2 t_3 t_4 = 1\) we find
\[
\text{ch}^\mathbb{T}^4_{\Pi} = \left( \sum_{j \neq k} t_j t_k^2 + 2 \sum_{j=1}^4 t_j^{-1} - 1 \right) \sum_{a_{L,R} \in \Pi_{L,R}} \prod_{i=1}^4 t_i^{a_{L,R}^+ - a_i^L} + \sum_{a_{L,R} \in \Pi_{L,R}} \prod_{i=1}^4 t_i^{a_{L,R}^+ - a_i^R} + \sum_{a_{L,R} \in \Pi_{L,R}} \prod_{i=1}^4 t_i^{a_{L,R}^+ - a_i^R}.
\]
The corresponding top form then gives the equivariant version of the fluctuation integral over the normal bundle \(\mathcal{N}(\Pi)\) in (7.4) at each fixed point \(\Pi\) of the vacuum moduli space \(\mathcal{M}_{NL, NR}\). As the second cohomology of the complex (7.46) is non-vanishing, there is a non-trivial obstruction theory for the moduli space and the localization formula computes the equivariant Euler character of the virtual tangent bundle on \(\mathcal{M}_{NL, NR}\), i.e. the difference in K-theory between the tangent and normal bundles at each fixed point of the moduli space. By summing over all fixed points \(\Pi\) we arrive at an explicit combinatorial expression for the contour integral (7.23) given by the finite sum
\[
\mathcal{Z}_{NL, NR}^{ABJM}(\epsilon) = \prod_{\Pi \in \mathcal{M}_{NL, NR}^{\mathbb{T}^4}} \prod_{a_{L,R} \in \Pi_{L,R}} \left( \sum_{i=1}^4 (a_i^L - a_i^R) \epsilon_i \right) \prod_{a_{L,R} \in \Pi_{L,R}} \left( \sum_{i=1}^4 (a_i^R - a_i^L) \epsilon_i \right)
\]
\[
\times \prod_{a_{L,R} \in \Pi_{L,R}} \prod_{j=1}^4 \left( (a_j^R - a_j^L - 1) \epsilon_j + \sum_{i \neq j} (a_i^R - a_i^L) \epsilon_i \right)^2 \prod_{j \neq k} \left( (a_j^R - a_j^L + 1) \epsilon_j + (a_k^R - a_k^L + 2) \epsilon_k + \sum_{i \neq j,k} (a_i^R - a_i^L) \epsilon_i \right).
\]
Consistently with the fact that it computes an equivariant index, the partition function \(\mathcal{Z}_{NL, NR}^{ABJM}(\epsilon)\) is a Laurent series in the deformation parameters \((\epsilon_1, \epsilon_2, \epsilon_3)\) with rational coefficients. The partition weights \(a_{L,R} \in \Pi_{L,R}\) in this formula are naturally interpreted as R-charges of framed BPS particles of the three-dimensional supersymmetric gauge theory.

It would be interesting to match these regularized volumes with those that would appear in corresponding gravitational free energies for M-theory backgrounds \(\text{AdS}_4 \times Y_7\) with metrics suitably
twisted by the $SU(4)$ rotations, as in (7.29); see e.g. Refs. [12,13,48] for some examples of field theory computations of (non-equivariant) moduli space volumes in this context as functions of R-charges. It would also be interesting to see if there are values of the equivariant deformation parameters for which our equivariant index compares with the superconformal indices computed for the full ABJM theory [6–8]. Remembering that the equivariant index of BPS states (7.26) is properly defined by compactifying the radial time direction on a circle, it is computed by the equivariant character of ABJM theory [6–8].

7.6. Equivariant index for the Gaiotto–Witten quiver

A completely analogous calculation of the equivariant index can be performed for the $\mathcal{N} = 4$ Gaiotto–Witten matrix model from Sect. 3.4 In order to demonstrate the generality of our formalism, and to justify the constructions of this section, let us briefly explain it. It can be obtained by formally reducing two of the flavor directions of the cohomological matrix model for the ABJM model above, which corresponds to restricting the indices $i$, $j$, $k = 1$, 2 that label the bifundamental matter fields $Z^i$. In this case we localize with respect to a two-torus $\mathbb{T}^2$, which is the maximal torus of the $R$-symmetry group $SU(2) \times SU(2)$; the corresponding deformation parameters $(\epsilon_1, \epsilon_2)$ are now independent. For the $U(N) \times U(N)$ model, the moduli space is the Hilbert scheme $(\mathbb{C}^2)^{[N]} \to \mathfrak{M}_N^{GW}$ whose $\mathbb{T}^2$-fixed points are parametrized by linear partitions $\lambda$ of the rank $N$. In the case $N_L \neq N_R$, the analogous restrictions on the vector space sums (7.39) imply that there is only one independent weight $\alpha = \alpha_1 = -\alpha_2$ in each of the left and right sectors of the character lattice $\mathbb{Z}^2$ of $\mathbb{T}^2$; whence there is a unique fixed point in each sector labeled by $\alpha_{L,R} \in \{1, \ldots, N_{L,R}\}$, which arises as the reduction of a filtered pyramid partition along two of its directions. As a consequence, the partition function will depend only on the combination $\epsilon_1 - \epsilon_2$.

The equivariant deformation complex for the Gaiotto–Witten quiver is formally the same as (7.46), where now $Q = t_1^{-1} + t_2^{-1}$ and $\wedge^2 Q = (t_1 t_2)^{-1}$ in the representation ring of $\mathbb{T}^2$, together with the $\mathbb{T}^2$ characters

$$V_{L,R} = \sum_{\alpha_{L,R}=1}^{N_{L,R}} (t_1 t_2)^{\alpha_{L,R}} = -\frac{t_1 t_2^{N_{L,R}}}{t_1 t_2^{N_{L,R}} - t_2^{N_{L,R}+1}}.
$$

The equivariant character is now simply

$$\text{ch}_{\mathbb{T}^2}^2(t) = V_L^* \otimes V_L + V_R^* \otimes V_R - V_L^* \otimes V_R
$$

(7.55)
and whence the equivariant index of BPS states (7.26) in this theory evaluates to

\[ \mathcal{G}_{N_L,N_R}^{\text{GW}}(t) = \frac{\prod_{\alpha_L,\beta_L=1}^{N_L} (1 - t^{\beta_L - \alpha_L}) \prod_{\alpha_R,\beta_R=1}^{N_R} (1 - t^{\beta_R - \alpha_R})}{\prod_{\alpha_L=1}^{N_L} \prod_{\alpha_R=1}^{N_R} (1 - t^{\alpha_L - \alpha_R})}. \]  

(7.56)

where we defined \( t := t_1 t_2^{-1} = e^{-i(e_1 - e_2)}. \) Note that in the diagonal limit \( t_1 = t_2 \) the index vanishes, thus demonstrating that our equivariant index indeed gives a more refined enumeration of degeneracies of BPS states.

The simplicity of the index here as compared to (7.53) follows from the topological twisting of this \( \mathcal{N} = 4 \) theory that is given in Ref. [55]; similarly to Sect. 6.1, the twist is constructed by replacing the Lorentz group \( \text{SO}(3) = \text{SU}(2) \) with the diagonal subgroup of its product with one of the \( \text{SU}(2) \) factors of the \( \text{SO}(4) = \text{SU}(2) \times \text{SU}(2) \) R-symmetry group. With the twisted BRST charge \( Q \), the resulting Gaiotto–Witten matrix model action in the absence of gauge fields is proportional to the \( Q \)-exact term \( Q \text{Tr}_V (\lambda Z_i^i) \) where \( \lambda \) is the \( \text{SO}(3) \) scalar part of the twisted spinors \( \psi^i \). By deforming this supercharge suitably as in (7.12), and adding the appropriate quartet of auxiliary fields as before, the corresponding cohomological matrix integral localizes onto a contour integral over the Cartan subalgebra of the gauge group \( U(N_L) \times U(N_R) \), which evaluates to (7.56).

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