Density of kinks after a sudden quench in the quantum Ising spin chain

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(Dated: October 21, 2009)

We investigate the time evolution of the density of kinks in the spin-1/2 quantum Ising spin chain after a sudden quench in the transverse field strength, and find that it relaxes to a value which depends on the initial and the final values of the transverse field, with an oscillating power-law decay. We provide analytical estimates of the long-time behavior and of the asymptotic value reached after complete relaxation, and discuss the role of quantum criticality in the quench dynamics. We show that, for a dynamics at the critical point, the residual density of kinks after the quench can be described by equilibrium statistical mechanics at a finite temperature dictated by the energy of the state after the quench. On the other hand, outside of criticality it does not exhibit thermalization.

PACS numbers: 75.40.Gb, 75.40.Cx, 75.10.Pq, 73.43.Nq

I. INTRODUCTION

The experimental developments in the physics of ultracold atomic gases that have been put forward in the last decade opened up the possibility to test some fundamental aspects of strongly interacting quantum many-body systems1,2. In particular, by loading cold atoms in optical lattices, the implementation of Hubbard-like Hamiltonians3,4 or artificial spin chain models5,6 has become possible. The key features of these setups are a detailed microscopic knowledge of the Hamiltonian, together with an extremely high accuracy in controlling the system parameters with tunable external fields. Moreover, due to the surprisingly long coherence times and to small thermal fluctuations, the dynamics can be monitored with very low dissipation and noise effects2. Recently emphasis has been put in the investigation of out-of-equilibrium properties of strongly correlated systems. We quote for example the spectacular observation of a superfluid-to-Mott insulator quantum phase transition in optical lattices8; the formation of topological defects during a quench of trapped atomic gases through a critical point7,8; the absence of thermalization in the non-equilibrium evolution of an integrable Bose gas9.

Such enormous experimental potentialities raised an increasing theoretical interest in the study of the effects induced by a sudden perturbation in such strongly correlated systems, which goes beyond a mere debate of statistical physics principles10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25. Integrability is believed to play a crucial role in the relaxation to the steady state, even if a comprehensive scenario is still lacking. The nonequilibrium dynamics of a nonintegrable system is expected to thermalize at the level of individual eigenstates17, following the standard statistical mechanics prescriptions15,16,17,28. On the contrary, for integrable models with nontrivial integrals of motion the asymptotic equilibrium states usually carry some memory of the initial conditions and are not the thermal ones11,12,13,14,18,19,20,21,22,23,24,25. In this case it is however possible to derive a statistical prediction for the steady state in terms of a generalized Gibbs ensemble21. Several works tested the relevance of such distribution for different classes of observables12,13,14,22,23, and found that it indeed works, provided the observables are sufficiently uncorrelated with respect to the constants of motion20,23.

In the present study we consider a quantum quench of the one-dimensional transverse field Ising chain; we investigate the time evolution of the density of kinks that are generated after a quench in the field strength, focusing on the thermalization properties of this observable and on the role of the quantum phase transition in the quench dynamics. Due to the complete integrability of the Ising model, observables are in general not expected to follow the prescriptions of statistical mechanics. As a matter of fact, the sensitivity to integrability can be traced back to the property of locality of operators with respect to the fermion quasiparticles that diagonalize the model in the continuum limit. The Ising chain possesses two sectors of operators: one that is local with respect to those particles, and another one that is non-local20. Observables belonging to the non-local sector, as the two-point correlation function of the order parameter, behave thermally26; on the other hand, local observables in general do not. Here indeed we find that the converged value of the density of kinks after the quench is not equal to a thermal expectation value, unless the system is quenched towards its critical point.

The paper is organized as follows. In Sec. III we describe our model and review some standard techniques that are used to approach the quenched dynamics of the Ising chain. In Sec. IIII we define the density of kinks, that is the quantity that we are going to analyze, and give the prescription to evaluate it within the free fermion formalism. We first focus on its behavior for the system at equilibrium, both at zero and at finite temperatures (Sec. IV), and then concentrate on the quenched case at...
zero temperature (Sec. [V]), where we analyze the asymptotic value as well as the finite-time transient. In Sec. [VI] we explicitly show that the density of kinks does not exhibit a thermal behavior outside criticality, while it thermalizes only at the critical point. Finally in Sec. [VII] we draw our conclusions. In the appendix we derive explicit analytic expressions for the density of kinks of the ground state (App. [A]) and after a quench (App. [B]).

II. MODEL

We consider the simplest nontrivial example of exactly solvable one-dimensional quantum many-body systems, exhibiting a quantum phase transition, that is the spin-1/2 quantum Ising chain with ferromagnetic interactions. This is characterized by the Hamiltonian

$$\mathcal{H}(\Gamma) = -J \sum_j (\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^z) \ ,$$

where $\sigma_j^\alpha$ ($\alpha = x, y, z$) are the Pauli matrices for the $j$th spin, while the parameters $J$ and $\Gamma$ respectively denote the nearest-neighbor antiferromagnetic exchange coupling and the transverse field strength (hereafter we set $J = 1$ as the system’s energy scale and take $\Gamma \geq 0$ without loss of generality; we also use units of $\hbar = k_B = 1$). In this paper we will assume periodic boundary conditions for the Ising chain, and then take the limit $L \to \infty$, in order to approach the thermodynamic limit. We will therefore suppose that the sum in Eq. (1) goes from 1 to $L$, with the rule $\sigma_{L+1}^\alpha \equiv \sigma_1^\alpha$ and with $L$ being the number of spins in the chain. We assume an even number for $L$. At zero temperature, the system is a quantum paramagnet for $\Gamma > 1$ and a ferromagnet with respect to the coupling direction for $\Gamma < 1$; these two phases are separated by a quantum critical point at $\Gamma_c = 1$.

We choose to drive the system out of equilibrium by performing a sudden quench of the Hamiltonian parameter $\Gamma$: we assume that the transverse magnetic field is suddenly quenched from $\Gamma_0$ to $\Gamma$ at time $t = 0$. The system is also supposed to be initialized at zero temperature (Sec. [V]), where we analyze the asymptotic value as well as the finite-time transient. In Sec. [VI] we explicitly show that the density of kinks does not exhibit a thermal behavior outside criticality, while it thermalizes only at the critical point. Finally in Sec. [VII] we draw our conclusions. In the appendix we derive explicit analytic expressions for the density of kinks of the ground state (App. [A]) and after a quench (App. [B]).

A. Statics

The Ising Hamiltonian can be exactly diagonalized by means of a Jordan-Wigner transformation (JWT), followed by a Bogoliubov rotation. First one represents the spins in terms of JW fermions, which are defined by:

$$c_l \equiv \sigma_l^- \exp \left( i \pi \sum_{j=1}^{l-1} \sigma_j^+ \sigma_j^- \right) ,$$

with $\sigma_l^\pm = (\sigma_l^x \pm i \sigma_l^y)/2$ raising and lowering spin operators. This leads to a Hamiltonian that is quadratic in the c-fermions. Since the ground state has an even number of c-fermions, and their parity is conserved during the evolution, we will assume that the actual system state $|\psi(t)\rangle$ always lies in the even c-fermionic Hilbert space sector. This requires imposing antiperiodic boundary conditions for the fermions.

The Hamiltonian can then be easily handled by switching to momentum representation: $c_k = \frac{1}{\sqrt{L}} \sum_j e^{-ijkl} c_j$, where the possible values of $k$ are fixed by the antiperiodic boundary conditions, and are given by $k = \pm \frac{\pi (2n+1)}{L}$, with $n = 0, 1, \ldots, \frac{L}{2} - 1$. Indeed, $\mathcal{H}$ decouples into a sum of independent terms, each of them acting on the subspace $(-k, +k)$. This is finally diagonalized through a Bogoliubov transformation, by introducing the new fermionic operators

$$\begin{align*}
\gamma_k &= (\bar{u}_k)^* c_k + (\bar{v}_k)^* c_{-k}^\dagger,
\gamma_{-k}^\dagger &= -\bar{u}_k^* c_k + \bar{v}_k^* c_{-k}.
\end{align*}$$

where the coefficients

$$\begin{align*}
\bar{u}_k &= \frac{\epsilon_k^\Gamma + a_k^\Gamma}{\sqrt{2\epsilon_k^\Gamma (\epsilon_k^\Gamma + a_k^\Gamma)}} ,
\bar{v}_k &= \frac{ib_k}{\sqrt{2\epsilon_k^\Gamma (\epsilon_k^\Gamma + a_k^\Gamma)}} ,
\end{align*}$$

with $a_k^\Gamma = -2(\cos \Gamma + \Gamma)$ and $b_k = 2 \sin k$, while their dispersion is characterized by

$$\epsilon_k^\Gamma \equiv \sqrt{(a_k^\Gamma)^2 + (b_k)^2} = 2\sqrt{1 + \Gamma^2 + 2\Gamma \cos k} .$$

The Ising model is then recasted into a free fermionic system, where the Hamiltonian can be written as:

$$\mathcal{H}(\Gamma) = \sum_{0<k<\pi} \epsilon_k^\Gamma (\gamma_k^\dagger \gamma_k^\dagger + \gamma_k^\dagger \gamma_k - 1) .$$

The ground state $|\Psi(\Gamma)\rangle$ of $\mathcal{H}(\Gamma)$ is therefore the vacuum state for the fermionic Bogoliubov quasiparticles $\gamma_k$:

$$|\Psi(\Gamma)\rangle = \prod_{0<k<\pi} \gamma_k \gamma_{-k} |0\rangle ,$$

where $|0\rangle$ is the vacuum state of c-fermions.
B. Dynamics

The dynamics of the Ising chain is conveniently described within the Heisenberg representation. The Heisenberg representation of an operator $O(t)$ is defined by $O^H(t) = U^\dagger(t) O(t) U(t)$, and it evolves in time according to the equation of motion:

$$i \frac{d}{dt} O^H(t) = U^\dagger(t) \left( [O(t), H(t)] U(t) + i \frac{d}{dt} O(t) \right) U(t),$$

where $[\cdot, \cdot]$ denotes the commutator of two operators.

Since the Ising Hamiltonian is quadratic in the $c$-fermions, the Heisenberg equations of motion for the operators $c^H_k(t)$ are linear and can be solved with a standard Bogoliubov-de Gennes approach. Their solution is written according to

$$c^H_k(t) = u_k(t) \gamma^0_k - v^*_k(t) \gamma^0_{-k}^\dagger,$$

$$c^{H\dagger}_k(t) = v_k(t) \gamma^0_k + u^*_k(t) \gamma^0_{-k}^\dagger,$$

where $\gamma^0_k$ are the Bogoliubov operators that diagonalize $H(\Gamma_0)$ at the initial time, and the coefficients $u_k(t), v_k(t)$ obey the time-dependent Bogoliubov-de Gennes equations and are given by

$$\begin{pmatrix} u_k(t) \\ v_k(t) \end{pmatrix} = U^\Gamma_k \begin{pmatrix} e^{-i\epsilon_k t} & 0 \\ 0 & e^{i\epsilon_k t} \end{pmatrix} U^\dagger \begin{pmatrix} u_k(0) \\ v_k(0) \end{pmatrix}$$

with initial conditions $u_k(0) = \bar{u}_k^{\Gamma_0}$, $v_k(0) = \bar{v}_k^{\Gamma_0}$, and

$$U^\Gamma_k = \begin{pmatrix} \bar{u}_k^{\Gamma_0} & \bar{v}_k^{\Gamma_0} \\ \bar{v}_k^{\Gamma_0} & -\bar{u}_k^{\Gamma_0} \end{pmatrix}.$$  

III. DENSITY OF KINKS

The quantity we will analyze throughout this paper is the density of defects (or kinks), defined by:

$$\mathcal{N} = \frac{1}{L} \sum_j \frac{1 - \sigma^z_j \sigma^x_{j+1}}{2},$$

where the sum goes from $j = 1$ to $N$, for periodic boundaries. Switching to the Heisenberg representation, one can compute the expectation value of the density of kinks $\rho(t)$ at a certain time $t$. In the thermodynamic limit, this is defined by:

$$\rho(t) = \lim_{L \to \infty} \langle \Psi(\Gamma_0) | \mathcal{N}^H(t) | \Psi(\Gamma_0) \rangle.$$  

In order to evaluate this expectation value, one has first to express $\mathcal{N}^H(t)$ in terms of $c$-fermions, and then write them as combinations of quasiparticles $\gamma^0_k$ that diagonalize the Hamiltonian before the quench, using Eqs. (18).

In terms of the coefficients $u_k(t)$ and $v_k(t)$, the expectation value is written as

$$\rho(t) = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - \langle |v_k(t)|^2 - |u_k(t)|^2 \rangle \cos k - i \langle u_k(t) v^*_k(t) - u^*_k(t) v_k(t) \rangle \sin k \right].$$

Substituting the solution of the Bogoliubov equations for $u_k(t)$ and $v_k(t)$, Eq. (19), one obtains the following integral expression for the density of kinks at zero temperature:

$$\rho(t) = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - 2 \sum_{\mu, \nu} \frac{1 + \Gamma_{\mu, \nu}}{\epsilon^\Gamma_{\mu, \nu}} \cos k \right].$$

The generalization to finite temperatures $T$ is quite straightforward; however, concerning the case $T > 0$, we are only interested in the equilibrium situation, to which we restrict ourselves. If the average is taken on the thermal state at temperature $T$, the density of kinks at equilibrium is defined by

$$\rho^{eq}_T = \lim_{L \to \infty} \frac{1}{Z} \text{Tr} [e^{-\beta H(\Gamma)} \mathcal{N}],$$

where $e^{-\beta H(\Gamma)}/Z$ is the canonical ensemble of the system at a given temperature $T = 1/\beta$, and $Z = \text{Tr} [e^{-\beta H(\Gamma)}]$ is the partition function. At finite temperatures, one has to take into account the possibility of having thermally excited quasiparticles, according to the Fermi distribution function

$$n_\mu(T) = \frac{1}{\exp(\beta \epsilon^\Gamma_\mu) + 1},$$

where $\epsilon^\Gamma_\mu$ is the energy of the $\gamma_\mu$ quasiparticle. The quasiparticle averages on the thermal state can thus be evaluated in the same way as on the ground state, using $\langle \gamma_\mu^\dagger \gamma_\nu \rangle_T = \langle \gamma_\mu \gamma_\nu \rangle_T = 0$ and $\langle \gamma_\mu^\dagger \gamma_\nu \rangle_T = (1 - \langle \gamma_\mu \gamma_\nu \rangle_T) \delta_{\mu, \nu}$, where $\langle \cdot \rangle_T$ denotes the thermal average and $\delta_{\mu, \nu}$ is the Kronecker delta. Proceeding in a way analogous to the zero-temperature case, one obtains this expression for $\rho^{eq}_T$:

$$\rho^{eq}_T = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - 2 \sum_{\mu} \left( \frac{1}{\exp(\beta \epsilon^\Gamma_\mu) + 1} \right) \cos k - i \left( 1 - 2 n_k(T) \right) \left( |\bar{u}^\dagger_k(\epsilon^\dagger_k)|^2 - |\bar{u}_k^\dagger(\epsilon_k)|^2 \right) \sin k \right].$$

In the following, we will analyze in detail Eq. (19) and Eq. (19).

IV. BEHAVIOR AT EQUILIBRIUM

Let us now start focusing on the expectation value of the density of kinks for the case in which system is time-independent, i.e. at equilibrium. In this situation the density of kinks is a time-independent quantity, and is given by the expression in Eq. (19).
A. Zero temperature

At zero temperature the system is frozen in the ground state $|\Psi(\Gamma)\rangle$, that is the vacuum of $\gamma_k$ particles, Eq. [5], therefore the Fermi distribution function reduces to zero and the only non zero expectation value is $\langle \gamma^+_\mu \gamma^-_{\mu} \rangle = 1$. In this case, the density of kinks can be straightforwardly obtained by putting $\Gamma_0 = \Gamma$ in Eq. [16], and is given by:

$$\rho_0 = \rho^{eq}_{\Gamma = 0} = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - \frac{1 + \Gamma \cos k}{\sqrt{1 + \Gamma^2 + 2\Gamma \cos k}} \right] .$$

(20)

This expression can be computed analytically, and is written in terms of complete elliptic integrals $K$ which are defined as follows:

$$K(\lambda) \equiv \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-\lambda u^2)}} ,$$

(21)

$$\Pi(c,\lambda) \equiv \int_0^1 \frac{du}{(1-cu^2)(1-\lambda u^2)} .$$

(22)

Following Appendix A one gets:

$$\rho_0 = \frac{1}{2} \left\{ \begin{array}{ll} \frac{1 - \Gamma^2}{\pi} \Pi(\Gamma^2,\Gamma^2) & \text{for } \Gamma < 1 \\ \frac{1}{\pi} & \text{for } \Gamma = 1 \\ \frac{\Gamma^2 - 1}{\pi\Gamma} \left\{ \Pi\left(\frac{1}{\Gamma^2},\frac{1}{\Gamma^2}\right) - K\left(\frac{1}{\Gamma^2}\right) \right\} & \text{for } \Gamma > 1 . \end{array} \right.$$  

(23)

The density of kinks at equilibrium and at zero temperature, as a function of the field strength $\Gamma$, is plotted in Fig. 1 with black solid curve. As one can see, $\rho_0$ is a monotonic increasing function with $\Gamma$, and its first derivative diverges at the critical point $\Gamma_c = 1$.

B. Finite temperature

The density of kinks at equilibrium at finite temperatures is given by Eq. [19], and can be reexpressed in the following way:

$$\rho^{eq}_T = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - \frac{1 + \Gamma \cos k}{\sqrt{1 + \Gamma^2 + 2\Gamma \cos k}} (1 - 2n_k(T)) \right] .$$

(24)

Unlike the zero temperature case, it is not possible to derive a simple analytic expression for $\rho^{eq}_T$, and one has to evaluate it numerically. Results are shown in Fig. 1, where we plot $\rho^{eq}_T$ as a function of the transverse field strength $\Gamma$. As for the $T = 0$ case, $\rho^{eq}_T$ increases monotonically with $\Gamma$, towards the maximum value 1/2, that is reached in the limit $\Gamma \rightarrow \infty$. The singularity in its first derivative at $\Gamma_c$ disappears for $T > 0$ and is progressively smoothed out, as $T$ increases.

V. BEHAVIOR AFTER A QUENCH

We now discuss in details the behavior of the density of kinks at $T = 0$ after a quantum quench in the transverse field. As explained in Sec. II, the system is supposed to be initially in the ground state at $\Gamma_0$, and then evolved according to the quenched Hamiltonian, following Eq. [2]. Therefore, concerning the non-equilibrium situation, we specialize to the zero-temperature case, and denote with $\rho_Q(t)$ the density of kinks. This is explicitly given by Eq. [16], and can be written as the sum of a time-independent part and a time-dependent term, which decays in time and vanishes for asymptotically long times:

$$\rho_Q(t) \equiv \rho_Q^{(0)} + \rho_Q^{(1)}(t)$$

(25)

with $\rho_Q^{(1)}(t) \xrightarrow{t \rightarrow \infty} 0$. In the rest of this section we will separately discuss the behaviors of $\rho_Q^{(0)}$ and $\rho_Q^{(1)}(t)$.

A. Asymptotic value

Similar to the density of kinks at zero temperature, $\rho_0$, the asymptotic value $\rho_Q^{(0)}$ can be expressed in terms of complete elliptic integrals $K(\lambda)$ and $\Pi(c,\lambda)$, Eqs. (21) and (22). We introduce the following notation for convenience:

$$\varphi_1 \equiv \Gamma^2 - 2\Gamma_0\Gamma + 1 , \quad \varphi_2 \equiv \Gamma_0\Gamma^2 + \Gamma_0 - 2\Gamma .$$

(26)

The expression of $\rho_Q^{(0)}$ is then given separately for the following three cases with respect to the value of $\Gamma_0$, as

FIG. 1: (color online). Density of kinks $\rho^{eq}_T$ for the thermal state at temperature $T$, as a function of the transverse field $\Gamma$. The black solid line denotes $\rho_0$, that is the equilibrium density of kinks in the ground state $|\Psi(\Gamma)\rangle$. Inset: first derivative of $\rho^{eq}_T$ with respect to $\Gamma$. 

explained in Appendix B. 
(i) Case $\Gamma_0 > 1$:

$$
\rho_Q^{(0)} = \frac{1}{2} - \frac{1}{2\pi} \left[ \frac{2(\Gamma_0^2 - 1)(1 - \Gamma_0^2)}{\Gamma_0^2 \varphi_1} K\left( \frac{1}{\Gamma_0^2} \right) \right] + \left( \Gamma_0 + \Gamma \right) \frac{(1 - \Gamma_0^2)}{\Gamma_0^2 \varphi_2} \Pi\left( \frac{1}{\Gamma_0^2}, \frac{1}{\Gamma_0^2} \right) + \frac{(\Gamma_0 - \Gamma)(1 - \Gamma_0^2)(1 - \Gamma_0^2)}{\Gamma_0^2 \varphi_1 \varphi_2} \Pi\left( \frac{(\varphi_1)^2}{\varphi_2}, \frac{1}{\Gamma_0^2} \right). 
$$

(ii) Case $\Gamma_0 < 1$:

$$
\rho_Q^{(0)} = \frac{1}{2} - \frac{1}{2\pi} \left[ - \frac{2\Gamma(\Gamma_0 - \Gamma)(1 - \Gamma_0^2)}{\Gamma_0^2 \varphi_2} K\left( \Gamma_0^2 \right) \right] + \left( \Gamma_0 + \Gamma \right) \frac{(1 - \Gamma_0^2)}{\Gamma_0^2 \varphi_2} \Pi\left( \Gamma_0^2, \Gamma_0^2 \right) + \frac{(\Gamma_0 - \Gamma)(1 - \Gamma_0^2)(1 - \Gamma_0^2)}{\varphi_1 \varphi_2} \Pi\left( \frac{(\varphi_2^2)^2}{\varphi_1^2}, \Gamma_0^2 \right). 
$$

(iii) Case $\Gamma_0 = 1$:

$$
\rho_Q^{(0)} = \frac{1}{2} + \frac{\Gamma - \Gamma_0}{2\pi} \ln \left[ 1 + \frac{\Gamma + 2\sqrt{\Gamma}}{1 - \Gamma - 2\sqrt{\Gamma}} \right] .
$$

In Fig. 2 we show the asymptotic value of the density of kinks after the quench, as a function of the transverse field $\Gamma$. We notice that, when the system after the quench is a paramagnet ($\Gamma > 1$), the density of kinks always decreases with decreasing $\Gamma$, for any value of the initial field $\Gamma_0 > 0$. In that case, the shape of the curve is qualitatively the same as that for the ground state value $\rho_0$, even if it has a smaller gradient. The situation is subtler for quenches ending in the ferromagnet, where the behavior of $\rho_Q^{(0)}$ becomes non monotonic. As a matter of fact, decreasing $\Gamma$, $\rho_Q^{(0)}$ has a minimum at a certain value $\Gamma < 1$ and then it increases abruptly.

B. Time dependent transient

The relaxation of the density of kinks toward its asymptotic value can be unveiled by an asymptotic expansion of the time-dependent part of Eq. (10), $\rho_Q^{(1)}(t)$, for large $t$. As explained in Appendix B an explicit expression for the asymptotic expansion can be given separately for the following cases, up to order $O(t^{-2})$ in time.

(i) Case $\Gamma_0 \neq 1$ and $\Gamma \neq 1$:

$$
\rho_Q^{(1)}(t) = \frac{\Gamma - \Gamma_0}{16\sqrt{2\pi} \Gamma t^{3/2}} \times \left\{ \frac{1}{\sqrt{|1 - \Gamma_0| |1 - \Gamma|}} \cos \left( 4|1 - \Gamma|t - \frac{\pi}{4} \right) + \frac{1}{\sqrt{1 + \Gamma_0}} \cos \left( 4(1 + \Gamma_0)t + \frac{\pi}{4} \right) \right\} + O(t^{-2}).
$$

(ii) Case $\Gamma_0 \neq 1$ and $\Gamma = 1$:

$$
\rho_Q^{(1)}(t) = - \frac{1}{8\pi} \left\{ \frac{1}{t} \sin \left( 4(1 - \Gamma) t \right) \right. 
+ \left. \frac{\sqrt{\pi}(1 - \Gamma)}{4\sqrt{2\Gamma(1 + \Gamma) t^{3/2}}} \cos \left( 4(1 + \Gamma) t + \frac{\pi}{4} \right) \right\} + O(t^{-2}).
$$

The decay of $\rho_Q^{(1)}(t)$ for large times is a power-law with oscillations. This feature does not depend on whether the quench is across the critical point or not. If the system before the quench is not critical, $\Gamma_0 \neq 1$, the leading term decays as $t^{-3/2}$. The oscillatory part of the leading term consists of two frequencies, $4(1 + \Gamma)$ and $4(1 - \Gamma)$; the second vanishes if the system is quenched to the critical point. On the other hand, if the system before the quench is critical, $\Gamma_0 = 1$, the leading term decays as $t^{-1}$ with an oscillating term of frequency $4|1 - \Gamma|$. These frequencies of the oscillation come from the modes with zero group velocity, defined by $\partial \epsilon_k^f/\partial k = 0$. Indeed, such modes are found from Eq. (15) to be $k = 0$ and $k = \pi$, the latter of which disappears when $\Gamma = 1$. As it is seen from Eq. (16), the frequency corresponding to the mode $k$ is given by $2\epsilon_k^f$. This leads to the frequencies $4(1 + \Gamma)$ for $k = 0$ and $4|1 - \Gamma|$ for $k = \pi$.

Figure 3 displays the behavior of $\rho_Q^{(1)}(t)$ for various types of quenches, obtained by numerical integration of Eq. (10) (solid curves), as well as by the analytic asymptotic expressions in Eqs. (30)-(32) (dotted curves). As it can be seen from the main panel, if the system before the quench is not critical, the power-law exponent of
the decay is $t^{-3/2}$; notice also that, in the case where the system is quenched outside criticality, the oscillations are superpositions of different frequencies; if it is quenched to the critical point, oscillations are more regular and consist of just one frequency. On the other hand, if the initial state is the ground state critical, the decay is $t^{-1}$, as shown in the inset.

VI. NON-THERMAL BEHAVIOR OUTSIDE CRITICALITY

We now concentrate on the asymptotic value $\rho_Q^{(0)}$ of the density of kinks after a quench, and discuss the possibility to track a behavior of this quantity in terms of an equilibrium situation at a finite temperature. In particular we would like to study if it is possible to define an effective temperature for the quenched system (out of equilibrium) in the most natural way so that, looking at $\rho_Q^{(0)}$, the system behaves as if it was at equilibrium, at the same effective temperature.

Due to the quench, the ground state $|\Psi(\Gamma_0)\rangle$ of a given Hamiltonian $\hat{H}(\Gamma_0)$ becomes an excited state for the quenched Hamiltonian $\hat{H}(\Gamma)$, therefore it has a positive energy $E_0 = \frac{1}{\Gamma} \langle \Psi(\Gamma_0) | \hat{H}(\Gamma) | \Psi(\Gamma_0) \rangle$, as compared to the ground energy (we take all the energies normalized per single site). An effective temperature for the system out of equilibrium can be defined by equating such energy $E_0$ to that of a fictitious thermal state for the quenched Hamiltonian:

$$\langle \hat{H}(\Gamma) \rangle_{T_{\text{eff}}} = \int_0^\infty \frac{dk}{2\pi} \epsilon_k \left[ n_k(T_{\text{eff}}) + n_{-k}(T_{\text{eff}}) - 1 \right]$$

where we used the representation of $\hat{H}(\Gamma)$ in terms of free fermionic Bogoliubov quasiparticles, Eq. (7), and $n_k(T)$ is given by Eq. (13). We therefore define the effective temperature by the implicit equation:

$$E_0 \equiv \langle \hat{H}(\Gamma) \rangle_{T_{\text{eff}}} .$$

For a given pair of values $(\Gamma_0, \Gamma)$, this always admits a solution; we point out that, for a fixed $\Gamma$, there are two values of $\Gamma_0$ corresponding to the same $T_{\text{eff}}$, one for $\Gamma_0 < \Gamma$ and the other for $\Gamma_0 > \Gamma$, except at $T_{\text{eff}} = 0$, which coincides with the static case $\Gamma_0 = \Gamma$.

We are now ready to perform a quantitative comparison between the values of $\rho_Q^{(0)}$ after a quench and $\rho_Q^{eq}$ at equilibrium, where $T_{\text{eff}}$ for the out-of-equilibrium system is obtained from Eq. (34). This is done in Fig. 4. As one can see, outside criticality the two quantities are evidently not related and behave in different ways. On the other hand, at the critical point $\Gamma_c = 1$ symbols (diamonds) perfectly follow the solid line (green data). We interpret this as a thermal behavior, in the sense that the density of kinks after a quench to the critical point is univocally determined by an effective temperature $T_{\text{eff}}$ that depends only on the initial state energy after the quench, according to Eq. (34). Remarkably such behavior is not found for a non-critical dynamics, where fine details of the initial condition seem to be important.
The thermal behavior at criticality can be recovered analytically. Indeed, in that case the density of kinks at equilibrium with an effective temperature $T_{\text{eff}}$ is given, by substituting $\Gamma = \Gamma_c = 1$ in Eq. (24):

$$\rho_{T_{\text{eff}}}^{\text{eq}, \Gamma_c} = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - \frac{1}{4} c_k^1 \left[ 1 - 2 n_k(T_{\text{eff}}) \right] \right]$$

$$= \frac{1}{2} + \langle H(\Gamma) \rangle_{T_{\text{eff}}} - \frac{1}{2} + \frac{E_0}{4}, \quad (35)$$

where $c_k^1 = 2\sqrt{2 + 2\cos k}$ is the energy of the $\gamma_k$-fermion at criticality ($\Gamma = \Gamma_c = 1$), the second equality follows from Eq. (33) with $n_k(T) = n_{-k}(T)$, and the third from the definition of $T_{\text{eff}}$ in Eq. (34). The energy $E_0$ of the state $\langle \Psi(\Gamma_0) | \rangle_{\Gamma_c}^n$ is obtained directly from Eq. (7), by evaluating $\langle \Psi(\Gamma_0) | \gamma_k^+ \gamma_k | \Psi(\Gamma_0) \rangle$, and is given by

$$E_0 = -\int_0^\pi \frac{dk}{2\pi} \frac{2(1 + \Gamma_0)(1 + \cos k)}{c_k^1}. \quad (36)$$

On the other hand, the value of $\rho_Q^{(0)}$ for a quench at criticality reads, from Eqs. (10) and (25),

$$\rho_Q^{(0)} \big|_{\text{cr}} = \int_0^\pi \frac{dk}{2\pi} \left[ 1 - 2 \frac{1 + \Gamma_0 \cos k}{c_k^1} - 8 \frac{(\Gamma_0 - 1) \sin^2 k}{(c_k^1)^2 c_k^2} \right]$$

which, after simple algebra, can be shown to reduce to $1/2 + E_0/4$, hence obeying the equaligy

$$\rho_Q^{(0)} \big|_{\text{cr}} = \rho_{T_{\text{eff}}}^{\text{eq}, \Gamma_c}. \quad (37)$$

VII. CONCLUSIONS

In this paper we have analyzed the behavior of the density of kinks in the quantum Ising chain, both at equilibrium and after a sudden quench in the transverse field. At equilibrium the density of kinks is monotonically increasing with the transverse field strength, and presents a divergence in its first derivative at the critical point. This singularity is smoothed out by finite temperature effects. In the quenched situation, it exhibits a temporal decay down to a residual value which depends on the initial and the final values of the field. The finite-time transient is characterized by an oscillating power-law decay, whose rate depends on whether the initial state of the system is critical or not.

We have also shown that, if the system is quenched towards the critical point, the density of kinks in the out-of-equilibrium situation can be described by equilibrium statistical mechanics, provided an effective temperature is defined according to the energy of the state after the quench. We say that this observable exhibits a thermal behavior only for a critical dynamics, while it does not thermalize for quenches to non-critical phases. These results should be regarded in light of the discussion in Ref. 26, where it was conjectured that only non-local operators in the fermion quasiparticles that diagonalize the model in the continuum limit exhibit a general thermal behavior. Indeed, we find non-thermal behavior for our density of kinks, which is a local operator in terms of fermions. However, the surprising thermal behavior at criticality should be better understood in a more general framework.

Acknowledgments

We acknowledge fruitful discussions with R. Fazio, G. Mussardo and A. Silva. One of the authors (SS) is supported by Grant-in-Aid for Scientific Research (Grant No. 20740225) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

APPENDIX A: DENSITY OF KINKS IN THE GROUND STATE

In this appendix we derive an analytic expression for the expectation value of the $T = 0$ density of kinks $\rho_0$ with respect to the system ground state $|\Psi(\Gamma)\rangle$, in the thermodynamic limit.

As explained in Sec. IV A the density of kinks for the ground state is given by Eq. (20). Applying the variable transformation $x = \cos k$ we get

$$\rho_0 = \left[ \frac{1}{2} - \frac{1}{2\pi} \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} \frac{1 + \Gamma x}{\sqrt{1 + \Gamma^2 + 2\Gamma x}} \right]. \quad (A1)$$

In order to evaluate the integral, we consider the following three cases with respect to $\Gamma$:

i) Case $\Gamma > 1$. We perform the change of variable

$$x = -\frac{\Gamma u + 1}{u + \Gamma},$$

so that the integral in r.h.s. of Eq. (A1) becomes

$$\int_{-\infty}^{\infty} \frac{du}{\sqrt{(1 - u^2)(1 - u^2/\Gamma^2)}} \left(1 + \frac{1}{1 - u^2/\Gamma^2}\right) = 2\Gamma^2 - 1 - K\left(\frac{1}{\Gamma^2}\right) + \Pi\left(\frac{1}{\Gamma^2}, \frac{1}{\Gamma^2}\right),$$

where complete elliptic integrals are defined by Eqs. (21)- (22). Therefore we obtain:

$$\rho_0 = \left[ \frac{1}{2} - \frac{\Gamma^2 - 1}{\pi\Gamma} \left\{ \Pi\left(\frac{1}{\Gamma^2}, \frac{1}{\Gamma^2}\right) - K\left(\frac{1}{\Gamma^2}\right) \right\} \right]. \quad (A2)$$

ii) Case $\Gamma < 1$. From the change of variable

$$x = -\frac{u + \Gamma}{\Gamma u + 1},$$

...
it follows that:

\[
\int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} \frac{1 + \Gamma x}{\sqrt{1 + \Gamma^2 + 2\Gamma x}} = \int_{-1}^{1} \frac{du}{\sqrt{(1 - u^2)/(1 - \Gamma^2 u^2)}}\frac{1 - \Gamma^2}{1 - \Gamma^2 u^2} = 2(1 - \Gamma^2)\Pi(\Gamma^2, \Gamma^2).
\]

Therefore:

\[
\rho_0 = \frac{1}{2} - \frac{1 - \Gamma^2}{\pi} \Pi(\Gamma^2, \Gamma^2). \tag{A3}
\]

iii) Case \(\Gamma = 1\). The integral in Eq. (A1) reduces to

\[
\frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x}} = 2.
\]

Therefore:

\[
\rho_0 = \frac{1}{2} - \frac{1}{\pi}. \tag{A4}
\]

**APPENDIX B: ANALYTIC CALCULATION OF THE DENSITY OF KINKS AFTER A QUENCH**

We derive here analytic expressions for the density of kinks after a quench \(\rho_Q(t)\). We consider both the time-independent part \(\rho_Q^{(0)}(t)\) and the time-dependent transient \(\rho_Q^{(1)}(t)\).

1. **Asymptotic value**

The time-independent part \(\rho_Q^{(0)}\) of the density of kinks after a quench is written in integral form, starting from Eq. (10), as

\[
\rho_Q^{(0)} = \int_0^\infty \frac{dk}{2\pi} \left\{ 1 - 2 + \frac{\Gamma_0}{e_k} \cos k - \frac{8 (\Gamma_0 - \Gamma) \sin^2 k}{(e_k^i)^2 e_k^o} \right\}
= \frac{1}{2} + I. \tag{B1}
\]

Applying a variable transformation by \(x = \cos k\) and substituting \(e_k^i = 2\sqrt{1 + \Gamma^2 + 2\Gamma \cos k}\), the integral \(I\) is explicitly written as:

\[
I = \frac{1}{2\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} \left\{ \frac{1 + \Gamma_0 x}{\sqrt{1 + \Gamma_0^2 + 2\Gamma_0 x}} \right\}
+ \frac{\Gamma_0 - \Gamma)(1 - x^2)}{(1 + \Gamma^2 + 2\Gamma x)\sqrt{1 + \Gamma_0^2 + 2\Gamma_0 x}}. \tag{B2}
\]

In order to evaluate the integral, we separately consider three cases with respect to \(\Gamma_0\):

i) Case \(\Gamma_0 > 1\). We apply a variable transformation

\[
x = \frac{\Gamma_0 u + 1}{u + \Gamma_0}.
\]

Then Eq. (B2) is arranged into

\[
I = \frac{1}{2\pi} \int_{-1}^{1} \frac{du}{\sqrt{(1 - u^2)/(1 - \Gamma_0^2 u^2)}} \times \left\{ \frac{\Gamma_0 - \Gamma)(1 - \Gamma_0^2)}{\Gamma_0^2} \right\}
\]

where we remark the notation defined by Eq. (20). Using the definition of the elliptic integrals, Eqs. (B1) and (22), we reduce it to

\[
I = \frac{1}{2\pi} \left\{ \frac{2(\Gamma_0^2 - 1)(\Gamma_0 - 1)}{\Gamma_0^2} K \left( \frac{1}{\Gamma_0} \right) \right\}
+ \frac{\Gamma_0 + \Gamma}{\Gamma_0^2} \Pi \left( \frac{1}{\Gamma_0^2}, \frac{1}{\Gamma_0} \right)
+ \frac{(\Gamma_0 - \Gamma)(1 - \Gamma_0^2)(1 - \Gamma_0^2)}{\Gamma_0^2} \Pi \left( \frac{\varphi_1^2}{\varphi_2^2}, \frac{1}{\Gamma_0^2} \right) \right\}
\]

Hence, from Eq. (B1), we get the analytic expression for \(\rho_Q^{(0)}\) in Eq. (27).

ii) Case \(0 \leq \Gamma_0 < 1\). We perform a variable transformation by

\[
x = \frac{u + \Gamma_0}{\Gamma_0 u + 1}.
\]

Using the new variable \(u\), Eq. (B2) is written as

\[
I = \frac{1}{2\pi} \int_{-1}^{1} \frac{du}{\sqrt{(1 - u^2)/(1 - \Gamma_0^2 u^2)}} \times \left\{ \frac{-\Gamma_0 \Gamma_0^2}{\Gamma_0^2 \varphi_2^3} + \frac{\Gamma_0 + \Gamma}{2\Gamma_0^2} \right\}
+ \frac{(\Gamma_0 - \Gamma)(1 - \Gamma_0^2)(1 - \Gamma_0^2)}{\Gamma_0^2} \Pi \left( \frac{\varphi_1^2}{\varphi_2^2}, \frac{1}{\Gamma_0^2} \right) \right\}
\]

The integrals above are expressed in terms of complete elliptic integrals to yield

\[
I = \frac{1}{2\pi} \left\{ -\frac{2\Gamma_0 \Gamma_0^2}{\varphi_2^3} K \left( \Gamma_0^2 \right) \right\}
+ \frac{(\Gamma_0 + \Gamma)(1 - \Gamma_0^2)}{\Gamma_0^2 \varphi_2^3} \Pi \left( \frac{\varphi_1^2}{\varphi_2^2}, \frac{1}{\Gamma_0^2} \right) \right\}
\]

Therefore, taking Eq. (B1) into account, one obtains the analytic expression for \(\rho_Q^{(0)}\) in Eq. (28).

iii) Case \(\Gamma_0 = 1\). Eq. (B2) is simplified into

\[
I = \frac{1}{2\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x}} \left\{ \frac{1 + \Gamma_0}{\sqrt{2}} \right\}
\]

The first term in r.h.s. yields \(-1/\pi\). Regarding the second term, we apply a variable transformation: \(t =
\[ \sqrt{2\Gamma(1-x)}/4\sqrt{2(2\Gamma)^{3/2}} \{-4\sqrt{\Gamma} + (1+\Gamma) \ln \left[ 1 + \Gamma + 2\sqrt{\Gamma} \right] \}. \]

Summing the two terms, I is written as
\[ I = \frac{1+\Gamma}{2\pi} - \frac{1-\Gamma^2}{8\pi\Gamma} \ln \left[ 1 + \Gamma + 2\sqrt{\Gamma} \right]. \]

From Eq. (B3), one obtains the analytic expression for \( \rho_Q^{(0)} \) in Eq. (20).

2. Time-dependent transient

Following Eq. (16), the time-dependent part \( \rho_Q^{(1)}(t) \) of the density of kinks after a quench is written as
\[ \rho_Q^{(1)}(t) = 8 \int_0^\pi dk \frac{(\Gamma_0 - \Gamma)\sin^2 k}{2\pi} \theta_k^2 \cos(2\xi_k t). \]

Applying a variable transformation by \( u = \frac{\xi_k}{2(1 + \Gamma)} \), this equation is arranged into
\[ \rho_Q^{(1)}(t) = \frac{(\Gamma_0 - \Gamma)(1+\Gamma)}{4\pi\Gamma_0 t} \int_{u_0}^1 du \frac{\sqrt{(u^2 - u_0^2)(1-u^2)}}{\sqrt{u^2 - \psi}} \cos(t'u), \tag{B5} \]

where we defined \( u_0 = |1-\Gamma|/(1+\Gamma) \), \( t' = 4(1+\Gamma)t \), and \( \psi = (\Gamma_0 - \Gamma)(1-\Gamma_0)/(\Gamma_0(1+\Gamma)^2) \). Since \( \psi \) as a function of \( \Gamma_0 \) has a derivative \( \partial\psi/\partial\Gamma_0 = (1-\Gamma_0^2)/\Gamma_0^2 \), it is monotonically increasing with \( \Gamma_0 \) when \( 0 < \Gamma_0 < 1 \), while it decreases when \( \Gamma_0 > 1 \). The maximum of \( \psi \) for a given \( \Gamma \) is found at \( \Gamma_0 = 1 \) and its value is \( (1-\Gamma^2)/(1+\Gamma)^2 = u_0^2 \). The variable \( \psi \) can be either positive or negative, depending on \( \Gamma_0 \).

i) Case \( \Gamma_0 \neq 1 \) and \( \Gamma \neq 1 \). We introduce a number \( a \) that satisfies
\[ 0 < a < \min\{u_0 - \text{Re}(\sqrt{\psi}), 1-u_0\}, \]
where we remark that \( u_0 - \text{Re}(\sqrt{\psi}) \) is larger than zero, since \( 0 < u_0 < 1 \) and \( \psi < u_0^2 \). We define
\[ I_1(t') = \int_{u_0+a}^{u_0} du \frac{\sqrt{(u^2 - u_0^2)(1-u^2)}}{\sqrt{u^2 - \psi}} \cos(t'u), \tag{B6} \]
\[ I_2(t') = \int_{u_0}^{1-a} du \frac{\sqrt{(u^2 - u_0^2)(1-u^2)}}{\sqrt{u^2 - \psi}} \cos(t'u). \tag{B7} \]

We first consider the asymptotic behavior of \( I_1(t') \) for large \( t' \). Shifting the variable as \( u' = u - u_0 \), the integral \( I_1(t') \) is arranged into
\[ I_1(t') = \left[ \frac{2u_0(1-u_0^2)}{u_0^2(u_0^2-\psi)} \right]^{1/2} \int_0^a du' \cos \left[ t'(u' + u_0) \right] \times \left[ \frac{(1 + u'/u_0)(1 - u'/u_0)(1 + u'/u_0)}{(1 + u'/u_0)^2(1 + u'/u_0 + \sqrt{\psi})(1 + u'/u_0 - \sqrt{\psi})} \right]^{1/2}, \tag{B8} \]

Since for \( 0 \leq u' \leq a \)
\[ \frac{u'}{2u_0} \begin{array}{c} < 1 - u_0 \quad 1 + u_0 \quad u' \quad u' \end{array} \begin{array}{c} u' \quad u' \quad \left| \frac{u'}{u_0 + \sqrt{\psi}} \right| \quad \left| \frac{u'}{u_0 - \sqrt{\psi}} \right| \end{array} \]
are non-negative and less than unity, one can expand the integrand term inside the square brackets into a power series of the type \( \left[ \cdot \right]^{1/2} = \sqrt{u'(1+a_1u'+a_2u'^2+\ldots)} \). Applying this power series expansion and using asymptotic expansions of the Fresnel’s type integral, one obtains:
\[ I_1(t) = \left[ \frac{2u_0(1-u_0^2)}{u_0^2(u_0^2-\psi)} \right]^{1/2} \left\{ \frac{\sin[t'(u_0 + a)]}{t'} \times \right. \frac{\sqrt{\pi}}{2\Gamma^{3/2}} \cos \left[ t' u_0 - \frac{\pi}{4} \right] + O(t'^{-2}) \]
Looking at Eqs. (B3) and (B10), one finds that the first terms in r.h.s. of both equations cancel in $I_1(t) + I_2(t)$. After expressing $u_0$, $\psi$ and $t'$ in terms of $\Gamma$, $\Gamma_0$ and $t$, from Eq. (B5) we arrive at the following expression:

$$\rho_Q^{(1)}(t) = \frac{\Gamma - \Gamma_0}{16\sqrt{2\pi} t^{3/2}} \times \left\{ \frac{1}{\sqrt{|1+\Gamma|} |1-\Gamma_0|} \cos \left[ 4(1+\Gamma) t + \frac{\pi}{4} \right] \right. $$

$$+ \frac{1}{\sqrt{1+\Gamma} (1+\Gamma_0)} \cos \left[ 4(1+\Gamma_0) t + \frac{\pi}{4} \right] \right\} + \mathcal{O}(t^{-2}).$$

ii) Case $\Gamma_0 \neq 1$ and $\Gamma = 1$. Notice that $u_0 = 0$ and $\psi = -((1-\Gamma_0)^2/4\Gamma_0 < 0$. We hereafter define $\psi' = -\psi$. $\rho_Q^{(1)}(t)$ is written as

$$\rho_Q^{(1)}(t) = \frac{\Gamma_0 - 1}{2\pi\sqrt{\Gamma_0}} I_3(t),$$

where we have defined

$$I_3(t) = \int_0^1 du \frac{1-u^2}{\sqrt{u^2 + \psi'}} \cos(8tu).$$

Changing the variable by $u' = 1-u$, the integral is arranged into

$$I_3(t) = \sqrt{\frac{2}{1+\psi'}} \int_0^{1-u'} du' \cos \left[ 8t(1-u') \right]$$

$$\times \left[ \frac{u'(1-\psi' / (1+\psi'))(1-\psi' / (1+\psi'))}{(1-u' / (1+\psi'))} \right]^{1/2}.$$  

For $0 \leq u' \leq 1$, one can expand the term under the square root in power series of $u'$: $[\cdot]^{1/2} = \sqrt{u'}(1 + c_1 u' + c_2 u'^2 + \cdots)$. This and the asymptotic expansions of Fresnel’s integrals lead to:

$$I_3(t) = -\sqrt{\frac{2}{1+\psi'}} \frac{\sqrt{\pi}}{3\sqrt{2t^{3/2}}} \cos \left[ 8t + \frac{\pi}{4} \right] + \mathcal{O}(t^{-2}).$$

Writing $\psi'$ in terms of $\Gamma_0$, Eq. (B12) is then evaluated as

$$\rho_Q^{(1)}(t) = \frac{1 - \Gamma_0}{32\sqrt{\pi}(1+\Gamma_0) t^{3/2}} \cos \left[ 8t + \frac{\pi}{4} \right] + \mathcal{O}(t^{-2}).$$

iii) Case $\Gamma_0 = 1$ and $\Gamma \neq 1$. Notice that $\psi = u_0^2$. Eq. (B5) is reduced to

$$\rho_Q^{(1)}(t) = \frac{1 - \Gamma^2}{4\sqrt{\pi} \sqrt{t}} I_4(t),$$

where we have defined

$$I_4(t) = \int_0^1 du \frac{\sqrt{1-u^2}}{u} \cos(t' u).$$

Changing the variable by $u' = 1-u$, the integral is arranged into

$$I_4(t) = \sqrt{\frac{2}{\pi}} \int_0^{1-u_0} du' \cos \left[ t'(1-u') \right] \times \left[ \frac{\sqrt{u'(1-u')}}{1-u'} \right].$$

For $0 \leq u' \leq 1-u_0$, we perform a power series expansion in $u'$ as follows: $[\cdot] = 1 + d_1 u' + d_2 u'^2 + \cdots$. Using this and the asymptotic expansion of Fresnel’s integrals,$^{35}$ $I_4(t)$ is evaluated as follows.

$$I_4(t) = \sqrt{\frac{2}{\pi}} \left\{ -\frac{1-u_0}{t} \sin(u_0 t') \right.$$ 

$$\times (1 + d_1 (1-u_0) + d_2 (1-u_0)^2 + \cdots) - \sqrt{\frac{\pi}{2t^{3/2}}} \cos \left[ t' + \frac{\pi}{4} \right] + \mathcal{O}(t^{-2}) \right\}$$

The power series of $(1-u_0)$ are summed to yield

$$I_4(t) = -\sqrt{\frac{1-u_0^2}{u_0 t'}} \sin(u_0 t') - \sqrt{\frac{\pi}{2t^{3/2}}} \cos \left[ t' + \frac{\pi}{4} \right] + \mathcal{O}(t^{-2}).$$

Writing $u_0$ in terms of $\Gamma$ and substituting $t' = 4(1+\Gamma)t$, Eq. (B15) is evaluated as

$$\rho_Q^{(1)}(t) = -\frac{1}{8\pi} \left\{ \frac{1}{t} \sin \left[ 4(1-\Gamma) t \right] \right.$$ 

$$+ \frac{\sqrt{\pi}(1-\Gamma)}{4\sqrt{2}(1+\Gamma)^{3/2}} \cos \left[ 4(1+\Gamma) t + \frac{\pi}{4} \right] \right\} + \mathcal{O}(t^{-2}).$$

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