REGULARITY CRITERIA AND UNIFORM ESTIMATES FOR THE BOUSSINESQ SYSTEM WITH TEMPERATURE-DEPENDENT VISCOSITY AND THERMAL DIFFUSIVITY

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ABSTRACT. In this paper we establish some regularity criteria for the 3D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity. We also obtain some uniform estimates for the corresponding 2D case when the fluid viscosity coefficient is a positive constant.

1. INTRODUCTION

In this paper we establish some regularity criteria for the 3D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity ([14, 33]):

\begin{align}
\partial_t u + u \cdot \nabla u - \text{div} (\mu(\theta)(\nabla u + \nabla u^T)) + \nabla \pi = \theta e_3, \\
\partial_t \theta + u \cdot \nabla \theta - \text{div} (\kappa(\theta) \nabla \theta) = 0, \\
\text{div} u = 0, \\
(u, \theta)(x, 0) = (u_0, \theta_0)(x), \quad x \in \mathbb{R}^3.
\end{align}

(1.1) (1.2) (1.3) (1.4)

Here the unknowns \( u, \pi, \) and \( \theta \) denote the velocity, pressure and the temperature of the fluid, respectively. \( e_3 := (0, 0, 1)^T \) is the unit vector in the vertical \( x_3 \)-direction. The kinematic viscosity \( \mu(\theta) \) and the thermal conductivity \( \kappa(\theta) \) are generally depending on the temperature \( \theta \). Throughout this paper, we assume that \( \mu(\theta) \) and \( \kappa(\theta) \) are smooth functions and satisfy

\begin{equation}
0 < C_1^{-1} \leq \mu(\theta), \kappa(\theta) \leq C_1 < \infty \quad \text{when} \quad |\theta| \leq C_2
\end{equation}

(1.5)
for some positive constant $C_1 > 1$ and $C_2 > 0$. Boussinseq type system (1.1)-(1.2) are used to model geophysical flows such as atmospheric fronts and ocean circulations (see [31,35]).

Due to its physical importance and mathematical complexity, there are a lot of research papers concerned on the Boussinseq system when both $\kappa(\theta)$ and $\mu(\theta)$ are independent of $\theta$, see [2–4,6–11,15,16,19–22,24,27,28,32] and the references cited therein. When $\kappa(\theta)$ or $\mu(\theta)$ depends on $\theta$, only a few results are available, see [1,12,13,29,30,37]. In [29,30], Lorca and Boldrini obtained the global existence of weak solution with small initial data and the local existence of strong solution for general data to the system (1.1)-(1.3), see also [12,13]. In [1], Abidi obtained the global existence and uniqueness result to the system (1.1)-(1.3) with $k = 0$ in some critical spaces when the initial temperature is small. Recently, Wang and Zhang [37] obtain the global existence of smooth solutions to the problem (1.1)-(1.4) with general initial data in $\mathbb{R}^2$. However, whether or not the smooth solution to the 3D boussinesq system (1.1)-(1.3) with general initial data blows up in finite time is still a big open problem. In this paper we provide some regularity criteria related to this topic. We will prove

\textbf{Theorem 1.1.} Let $\kappa(\theta) \equiv 1$ and $\mu(\theta)$ satisfy (1.5) and $u_0, \theta_0 \in H^2(\mathbb{R}^3)$ with $\text{div} \, u_0 = 0$ in $\mathbb{R}^3$. Suppose that there exists a $T > 0$ such that one of the following two conditions holds:

\begin{equation}
\begin{aligned}
&u \in L^p(0,T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty, \\
&\nabla u \in L^p(0,T; L^q(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 2, \quad 3/2 < q \leq \infty.
\end{aligned}
\end{equation}

Then the solution $(u, \theta)$ of the problem (1.1)-(1.4) satisfies

\begin{equation}
u, \theta \in L^\infty(0,T; H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3)).
\end{equation}

\textbf{Theorem 1.2.} Let $\mu(\theta)$ and $\kappa(\theta)$ satisfy (1.5). Let $u_0, \theta_0 \in H^2(\mathbb{R}^3)$ with $\text{div} \, u_0 = 0$ in $\mathbb{R}^3$. Assume that (1.6) holds with $3 < q < \infty$, then the solution $(u, \theta)$ of the problem (1.1)-(1.4) satisfies (1.8).
Remark 1.1. We can also obtain the regularity criteria (1.6) or (1.7) for the Cahn-Hilliard-Navier-Stokes system [18, 23]:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nabla \cdot \mu(\phi)(\nabla u + \nabla u^T) + \nabla \pi &= \eta \nabla \phi, \quad \text{div } u = 0, \\
\partial_t \phi + u \cdot \nabla \phi - \Delta \eta &= 0, \\
\eta &= -\Delta \phi + f'(\phi), \quad f(\phi) = \frac{1}{4}(\phi^2 - 1)^2, \\
(u, \phi)(x, 0) &= (u_0, \phi_0)(x), \quad x \in \mathbb{R}^3.
\end{align*}
\]

Since the proof is similar to that of Theorem 1.1, we omit the details here.

Because the local-in-time well-posedness can be obtained in a standard way (see Propositions 2.1 and 2.2 below), to prove Theorems 1.1 and 1.2, it suffices to obtain a priori estimates under the conditions (1.6) or (1.7). Due to the equations (1.1) and (1.2) are strongly coupled, it is difficult to obtain the \(L^p\) estimates of \(\nabla \theta\) and the Hölder continuity of \(\theta\), we shall use some results obtained in [5, 37], Sobolev imbedding, and the ingenious applications of Hölder and Gagliardo-Nirenberg inequalities to obtain the desired estimates.

Next, we consider the Boussinesq system (1.1)-(1.2) in two-dimensional spatial space \(\mathbb{R}^2\) and let the kinematic viscosity \(\mu(\theta)\) be a positive constant, i.e., \(\mu(\theta) \equiv \epsilon > 0\). In this case the system reads

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \epsilon \Delta u + \nabla \pi &= \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta - \text{div} (\kappa(\theta) \nabla \theta) &= 0, \\
\text{div } u &= 0, \\
(u, \theta)(x, 0) &= (u_0, \theta_0)(x), \quad x \in \mathbb{R}^2.
\end{align*}
\]

Here \(e_2 := (0, 1)^T\). As before, the thermal conductivity \(\kappa(\theta)\) is assumed to be smooth and satisfy

\[
0 < C_1^{-1} \leq \kappa(\theta) \leq C_1 < \infty \quad \text{when} \quad |\theta| \leq C_2
\]

for some positive constant \(C_1 > 1\) and \(C_2 > 0\).
Notice that in [37] Wang and Zhang established the global existence of classical solutions for the problem (1.9)-(1.12) with general initial data and fixed $\epsilon > 0$. In this paper we give some uniform-in-$\epsilon$ estimates for smooth solution to the problem (1.9)-(1.12). Our result reads

**Theorem 1.3.** Let $\kappa(\theta)$ satisfy (1.13) and $\epsilon \in (0, 1)$. Let $\theta_0 \in H^2(\mathbb{R}^2)$ and $u_0 \in H^3(\mathbb{R}^2)$ with $\text{div} u_0 = 0$ in $\mathbb{R}^2$. Then, for any given $T > 0$, the solution $(u, \theta)$ to the problem (1.9)-(1.12) satisfies

$$
\|u\|_{L^{\infty}(0,T;H^3(\mathbb{R}^2))} + \|\theta\|_{L^{\infty}(0,T;H^2(\mathbb{R}^2))} + \|\theta\|_{L^2(0,T;H^3(\mathbb{R}^2))} \leq C, \quad (1.14)
$$

where $C$ is a positive constant independent of $\epsilon$.

Finally, considering the following 2D Boussinesq system

$$
\begin{align*}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi &= \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta - \epsilon \text{div} (\kappa(\theta) \nabla \theta) &= 0, \\
\text{div} u &= 0, \\
(u, \theta)(x,0) &= (u_0, \theta_0)(x), \quad x \in \mathbb{R}^2,
\end{align*}
\quad (1.15)-(1.18)
$$

we can obtain a similar result on uniform-in-$\epsilon$ estimates for smooth solution to the problem (1.15)-(1.18).

**Theorem 1.4.** Let $\kappa(\theta)$ satisfy (1.13) and $\epsilon \in (0, 1)$. Let $u_0, \theta_0 \in H^2(\mathbb{R}^2)$ and $\text{div} u_0 = 0$ in $\mathbb{R}^2$. Then, for any given $T > 0$, the solution $(u, \theta)$ to the problem (1.15)-(1.18) satisfies

$$
\|u\|_{L^{\infty}(0,T;H^2(\mathbb{R}^2))} + \|\theta\|_{L^{2}(0,T;H^3(\mathbb{R}^2))} + \|\theta\|_{L^{\infty}(0,T;H^2(\mathbb{R}^2))} \leq C, \quad (1.19)
$$

where $C$ is a positive constant independent of $\epsilon$.

**Remark 1.2.** In [7], Chae has proved similar results to Theorems 1.3 and 1.4 when $\kappa(\theta) \equiv 1$, it seems impossible to apply his method directly to our case. In fact, if we follow his arguments we shall encounter some unpleasant terms involved $\kappa(\theta)$ which are out of control.
We give some comments on the proofs of Theorems 1.3 and 1.4. Because the global-in-time well-posedness results (see Propositions 2.3 and 2.4 below), to complete our proofs, we only need to prove the \textit{a priori} estimates (1.14) and (1.19). We shall employ an elaborate nonlinear energy method to obtain these desired bounds. More precisely, we first use maximum principle to obtain $L^\infty$ estimates of $\theta$. Next, we derive an energy estimate based on $L^2$ energy. Finally, we use Amann’s $L^p$ estimates on uniform parabolic equations, Sobolev imbedding, bilinear commutator estimates, logarithmic Sobolev inequality, and Gagliardo-Nirenberg inequalities to obtain the desired higher order spatial estimates on $\theta$ and $u$.

This paper is organized as follows. In Section 2, we recall some basic inequalities and state the local/global existence results on the problems (1.1)-(1.4), (1.9)-(1.12), and (1.15)-(1.18). The proofs of Theorems 1.1, 1.2, 1.3, and 1.4 are presented in the subsequent four sections.

2. Preliminary

In this section, we first recall some basic inequalities which shall be used frequently.

\textbf{Lemma 2.1} (Logarithmic Sobolev inequality [26]). For all $u \in H^s(\mathbb{R}^d)$ with $s > 1 + d/2$, there exists a constant $C$ such that the following estimate holds

$$\|\nabla v\|_{L^\infty} \leq C(1 + \|\text{div} v\|_{L^\infty} + \|\text{curl} v\|_{L^\infty})(1 + \log(e + \|v\|_{H^s})).$$  \hfill (2.1)

\textbf{Lemma 2.2} (Gagliardo-Nirenberg inequality [17,34]). Let $v \in W^{k,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $1 \leq q, r \leq \infty$. Then the following inequalities hold

$$\|D^i v\|_{L^p} \leq M_0 \|D^k v\|_{L^r}^{\frac{1}{k} - \alpha} \|v\|_{L^q}^{1 - \frac{1}{q} \alpha}, \quad \forall \ 0 \leq i < k,$$  \hfill (2.2)

where

$$\frac{1}{p} = \frac{i}{N} + \alpha \left( \frac{1}{r} - \frac{k}{d} \right) + (1 - \alpha) \frac{1}{q},$$

for all $\alpha$ in the interval

$$\frac{i}{k} \leq \alpha \leq 1.$$
The constant $M_0$ depending only on $d, m, j, q, r$ and $\alpha$, with the following exceptional case:

1. If $i = 0$, $rk < d, q = \infty$ then we make the additional assumption that either $v$ tends to zero at infinity or $v \in \dot{L}^{\tilde{q}}(\mathbb{R}^d)$ for some finite $\tilde{q} > 0$.

2. If $1 < r < \infty$, and $k - i - d/r$ is a non negative integer then (2.2) holds only for $\alpha$ satisfying $i/k \leq \alpha < 1$.

We define the operator $\Lambda := (-\Delta)^{1/2}$ via the Fourier transform

$$\hat{\Lambda}f(\xi) = |\xi|\hat{f}(\xi).$$

Generally, we define $\Lambda^s f$ for $s \in \mathbb{R}$ as

$$\hat{\Lambda}^s f(\xi) = |\xi|^s\hat{f}(\xi).$$

For $s \in \mathbb{R}$, we define

$$\|f\|_{\dot{H}^s} := \|\Lambda^s f\|_{L^2} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2}$$

and the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3) := \{ f \in L'(\mathbb{R}^3) : \|f\|_{\dot{H}^s} < \infty\}$. Similar, the Sobolev space $H^{s,p}(\mathbb{R}^3)$ is equipped with the norm

$$\|f\|_{H^{s,p}} := \|\Lambda^s f\|_{L^p}.$$

Now we recall the following bilinear commutator and the product estimates.

**Lemma 2.3** ([25]). Let $s > 0, 1 < p < \infty$. Assume that $f, g \in \dot{H}^{s,p}(\mathbb{R}^d)$, then there exists a constant $C$, independent of $f$ and $g$, such that,

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{n_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{n_2}}), \quad (2.3)$$

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{n_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{n_2}}), \quad (2.4)$$

where $p_1, p_2 \in (1, \infty)$ satisfies

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2},$$

and $q_1, q_2$ are defined in (2.2).
Lemma 2.4 ([36]). Let $s > 0, 1 < p < \infty$, and $f \in \dot{H}^{s,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Assume that $F(\cdot)$ is a smooth function on $\mathbb{R}$ with $F(0) = 0$. Then we have
\[
\|F(f)\|_{\dot{H}^{s,p}} \leq C(M)(1 + \|f\|_{L^\infty})^{[s]+1}\|f\|_{\dot{H}^{s,p}},
\]
where the constant $C(M)$ depends on $M := \sup_{k \leq [s]+2, |t| \leq \|f\|_{L^\infty}} \|F^{(k)}(t)\|_{L^\infty}$.

Finally, we state the local/global well-posedness results on the problems (1.1)-(1.4), (1.9)-(1.12), and (1.15)-(1.18).

Proposition 2.1. Let $\kappa(\theta) \equiv 1$ and $\mu(\theta)$ satisfy (1.5) and $u_0, \theta_0 \in H^2(\mathbb{R}^3)$ with $\text{div} u_0 = 0$ in $\mathbb{R}^3$. Then the problem (1.1)-(1.4) has a unique local strong solution $(u, \theta)$ satisfying
\[
(u, \theta) \in L^\infty(0, T; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3))
\]
for some positive constant $T > 0$.

Proof. We can prove it by the standard Galerkin method. Since the key step is to give a priori estimates (2.6), which are very similar to that of our proofs on the regularity criteria, and thus we omit the details here. \qed

Similarly, we can obtain the following proposition.

Proposition 2.2. Let $\mu(\theta)$ and $\kappa(\theta)$ satisfy (1.5). Let $u_0, \theta_0 \in H^2(\mathbb{R}^3)$ with $\text{div} u_0 = 0$ in $\mathbb{R}^3$. Then the problem (1.1)-(1.4) has a unique local strong solution $(u, \theta)$ satisfying (2.6) for some positive constant $T > 0$.

For the problem (1.9)-(1.12), we have

Proposition 2.3. Let $\kappa(\theta)$ satisfy (1.13) and $\epsilon \in (0, 1)$. Let $\theta_0 \in H^2(\mathbb{R}^2)$ and $u_0 \in H^3$ with $\text{div} u_0 = 0$ in $\mathbb{R}^2$. Then, for fixed $\epsilon > 0$, the problem (1.9)-(1.12) has a unique global solution $(u, \theta)$ satisfying
\[
\|u\|_{L^\infty(0, T; H^3(\mathbb{R}^2))} + \|\theta\|_{L^\infty(0, T; H^2(\mathbb{R}^2))} + \|\theta\|_{L^2(0, T; H^3(\mathbb{R}^2))} \leq C(\epsilon)
\]
for some positive constant $C(\epsilon)$. 

Proof. The local existence of solution can be proved by the standard Galerkin method. To obtain the global existence of solution, it suffices to obtain (2.7) and then apply continuity arguments. Noticing that our estimates (1.14) is stronger than (2.7), we only need to prove (1.14) which will be presented later. Hence we omit the details here. □

Similarly, for the problem (1.15)-(1.18), we have

**Proposition 2.4.** Let \( \kappa(\theta) \) satisfying (1.13) and \( \epsilon \in (0, 1) \). Let \( u_0, \theta_0 \in H^2 \) and \( \text{div} \, u_0 = 0 \) in \( \mathbb{R}^2 \). Then, for fixed \( \epsilon > 0 \), the problem (1.15)-(1.18) has a unique global solution \( (u, \theta) \) satisfying

\[
\|u\|_{L^\infty(0,T;H^2(\mathbb{R}^2))} + \|u\|_{L^2(0,T;H^3(\mathbb{R}^2))} + \|\theta\|_{L^\infty(0,T;H^2(\mathbb{R}^2))} \leq C(\epsilon)
\]

for some positive constant \( C(\epsilon) \).

In the subsequent sections, we use \( C \) (independent of \( \epsilon \) in Sections 5 and 6) to denote the positive constant which may change from line to line. We also omit the spatial domain \( \mathbb{R}^3 \) or \( \mathbb{R}^2 \) in the integrals below for simplicity.

### 3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, by Proposition 2.1, we only need to prove (1.8).

First, it follows from maximum principle and (1.2) and (1.4) that

\[
\|\theta\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \leq \|\theta_0\|_{L^\infty(\mathbb{R}^3)} \leq C.
\]

(3.1)

Multiplying (1.2) by \( \theta \), integrating the result over \( \mathbb{R}^3 \), and using (1.3), we see that

\[
\frac{1}{2} \frac{d}{dt} \int \theta^2 \, dx + \int |\nabla \theta|^2 \, dx = 0,
\]

which gives

\[
\|\theta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\theta\|_{L^2(0,T;H^1(\mathbb{R}^3))} \leq C.
\]

(3.2)
Multiplying (1.1) by \( u \), integrating the result over \( \mathbb{R}^3 \), and using (1.3), (3.1), (3.2) and (1.5), we find that

\[
\frac{1}{2} \frac{d}{dt} \int u^2 dx + \frac{1}{C} \int |\nabla u|^2 dx \leq \int \theta c_3 \cdot u dx \leq \| u \|_{L^2} \| \theta \|_{L^2} \leq C \| u \|_{L^2},
\]

which implies

\[
\| u \|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \| u \|_{L^2(0,T;H^1(\mathbb{R}^3))} \leq C. \tag{3.3}
\]

**Case I: Assume that (1.6) holds.**

Applying the operator \( \partial_i \) to (1.2), multiplying the result by \((\partial_i \theta)^3\), integrating over \( \mathbb{R}^3 \), summing over \( i \), and using (1.3), the Hölder and Gagliardo-Nirenberg inequalities, we have

\[
\frac{1}{4} \frac{d}{dt} \int |\nabla \theta|^4 dx + \frac{3}{4} \int |\nabla |\nabla \theta|^2|^2 dx \\
\leq C \int |u| |\nabla \theta|^2 |\nabla |\nabla \theta|^2| dx \\
\leq C \| u \|_{L^4} \| |\nabla \theta|^2 \|_{L^{\frac{6}{4}}} \| |\nabla |\nabla \theta|^2|_{L^2} \\
\leq C \| u \|_{L^4} \| |\nabla \theta|^2 \|_{L^2}^{1-\alpha_1} \| |\nabla |\nabla \theta|^2|_{L^2}^{1+\alpha_1} \\
\leq \frac{1}{2} \int |\nabla |\nabla \theta|^2|^2_{L^2} + C \| u \|_{L^4} \| |\nabla \theta|^2 \|_{L^2}^2 \quad \left( p = \frac{2}{1-\alpha_1} \right).
\]

Hence it holds

\[
\| \nabla \theta \|_{L^\infty(0,T;L^4(\mathbb{R}^3))} \leq C. \tag{3.4}
\]

Multiplying (1.1) by \( -\Delta u \), integrating the result over \( \mathbb{R}^3 \), and using (1.3), (3.4), the Hölder and Gagliardo-Nirenberg inequalities, we derive that

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \frac{1}{C} \int |\Delta u|^2 dx \\
\leq \int (u \cdot \nabla u) \cdot \Delta u dx + C \int |\nabla \theta| |\nabla u| |\Delta u| dx \\
\leq \| u \|_{L^6} \| |\nabla u| \|_{L^{\frac{6}{4}}} \| \Delta u \|_{L^2} + C \| \nabla \theta \|_{L^4} \| |\nabla u| \|_{L^4} \| \Delta u \|_{L^2} \\
\leq C \| u \|_{L^4} \| |\nabla u| \|_{L^2}^{1-\alpha_1} \| \Delta u \|_{L^2}^{1+\alpha_1} + C \| u \|_{L^4} \| \Delta u \|_{L^2} \\
\leq \frac{1}{2C} \| \Delta u \|_{L^2}^2 + C \left\{ \| u \|_{L^4}^2 \| |\nabla u| \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right\},
\]

which gives

\[
\| u \|_{L^\infty(0,T;H^1(\mathbb{R}^3))} + \| u \|_{L^2(0,T;H^2(\mathbb{R}^3))} \leq C. \tag{3.5}
\]
Applying the operator $\Delta$ to (1.2), multiplying the result by $\Delta \theta$, integrating over $\mathbb{R}^3$, and using (1.3), (3.5), the Hölder and Gagliardo-Nirenberg inequalities, we obtain that

$$\frac{1}{2} \frac{d}{dt} \int |\Delta \theta|^2 dx + \int |\nabla \Delta \theta|^2 dx = \sum_{i=1}^{3} \int \partial_i (u \cdot \nabla \theta) \cdot \partial_i \Delta \theta dx$$

$$\leq (\|u\|_{L^6} \|\Delta \theta\|_{L^3} + \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^\infty}) \|\nabla \Delta \theta\|_{L^2}$$

$$\leq C(\|\Delta \theta\|_{L^3} + \|\nabla \theta\|_{L^\infty}) \|\nabla \Delta \theta\|_{L^2}$$

$$\leq C \|\Delta \theta\|_{L^2}^{1/2} \|\nabla \Delta \theta\|_{L^2}^{3/2} \leq \frac{1}{2} \|\nabla \Delta \theta\|_{L^2}^2 + C \|\Delta \theta\|_{L^2}^2,$$

which leads to

$$\|\theta\|_{L^\infty(0,T;H^2(\mathbb{R}^3))} + \|\theta\|_{L^2(0,T;H^3(\mathbb{R}^3))} \leq C. \quad (3.6)$$

Applying the operator $\Delta$ to (1.1), multiplying the result by $\Delta u$, integrating over $\mathbb{R}^3$, and using (1.3), (1.5), (3.1), (2.5), and (3.5), we arrive at

$$\frac{1}{2} \frac{d}{dt} \int |\Delta u|^2 dx + \frac{1}{C} \int |\nabla \Delta u|^2 dx$$

$$\leq \sum_{i=1}^{3} \int \partial_i (u \cdot \nabla u) \cdot \partial_i \Delta u dx + C \int (|\Delta \mu(\theta)||\nabla u| + |\nabla \mu(\theta)||\nabla^2 u|) |\nabla \Delta u| dx$$

$$\leq (\|u\|_{L^6} \|\Delta u\|_{L^3} + \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^\infty}) \|\nabla \Delta u\|_{L^2}$$

$$+ C(\|\Delta \theta\|_{L^3} \|\nabla u\|_{L^6} + \|\nabla \theta\|_{L^\infty} \|\Delta u\|_{L^2}) \|\nabla \Delta u\|_{L^2}$$

$$\leq C(\|\Delta u\|_{L^3} + \|\nabla u\|_{L^\infty}) \|\nabla \Delta u\|_{L^2}$$

$$+ C(\|\Delta \theta\|_{L^3} + \|\nabla \theta\|_{L^\infty}) \|\Delta u\|_{L^2} \|\nabla \Delta u\|_{L^2}$$

$$\leq \frac{1}{2} \|\nabla \Delta u\|_{L^2}^2 + C \|\Delta u\|_{L^2}^2 + C(\|\Delta \theta\|_{L^3}^2 + \|\nabla \theta\|_{L^\infty}^2) \|\Delta u\|_{L^2}^2,$$

which implies that

$$\|u\|_{L^\infty(0,T;H^2(\mathbb{R}^3))} + \|u\|_{L^2(0,T;H^3(\mathbb{R}^3))} \leq C. \quad (3.7)$$

Thus (1.8) holds.

Case II: Assume that (1.7) holds.
Applying the operator $\partial_i$ to (1.2), multiplying the result by $(\partial_i \theta)^3$, integrating over $\mathbb{R}^3$, summing over $i$, and using (1.3), the Hölder and Gagliardo-Nirenberg inequalities, we obtain that

$$
\frac{1}{4} \frac{d}{dt} \int |\nabla \theta|^4 dx + \frac{3}{4} \int |\nabla |\nabla \theta|^2|^2 dx 
\leq C \int |\nabla u||\nabla \theta|^2 : |\nabla \theta|^2 dx 
\leq C \|\nabla u\|_{L^4} \|\nabla \theta\|^2 L^{4} 
\leq C \|\nabla u\|_{L^4} \|\nabla \theta\|^2_{L^2} |\nabla \nabla \theta|^2_{L^2} 
\leq \frac{1}{2} \|\nabla |\nabla \theta|^2\|_{L^2}^2 + C \|\nabla u\|_{L^p} \|\nabla \theta\|^2_{L^2} \left( p = \frac{1}{1 - \alpha_3} \right),
$$

which proves (3.4).

Multiplying (1.1) by $-\Delta u$, integrating the result over $\mathbb{R}^3$, and using (1.3), (3.4), the Hölder and Gagliardo-Nirenberg inequalities, we get

$$
\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \frac{1}{C} \int |\Delta u|^2 dx 
\leq \sum_i \int \partial_i u \partial_i u \partial_j udx + C \int |\nabla \theta||\nabla u||\Delta u|dx 
\leq - \sum_{i=1}^3 \int \partial_j u \partial_i u \partial_j udx + C \|\nabla \theta\|_{L^4} \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} 
\leq C \|\nabla u\|_{L^4} \|\nabla u\|^2_{L^2} + C \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} 
\leq C \|\nabla u\|_{L^4} \|\nabla u\|^{2(1 - \alpha_2)}_{L^2} \|\Delta u\|^{2\alpha_2}_{L^2} + C \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} 
\leq \frac{1}{2C} \|\Delta u\|_{L^2}^2 + C \left\{ \|\nabla u\|_{L^4} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right\},
$$

which implies (3.5).

Noticing that the calculations for (3.6) and (3.7) still hold in this case, we hence complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2, by Proposition 2.2, we only need to prove (1.8).
First, by maximum principle, it is easy to prove that

\[ \| \theta \|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \leq C. \]  

(4.1)

Noticing the condition (1.5), we still have (3.2) and (3.3).

Similar to the Proposition 4.1 in [37], we can prove Proposition 4.1.

Let \( u \) satisfy (1.6) and \( \text{div} \ u = 0 \) in \( \mathbb{R}^3 \times (0,\infty) \). Assume that \( \theta \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \) is a weak solution to (1.2) and (1.4). Then there exists a \( \alpha \in (0,1) \) such that \( \theta \in C^\alpha([0,T] \times \mathbb{R}^3) \).

**Proof.** Most of the calculations are as same as that in [37], the only difference is the calculations of the following term

\[ \left| \int_{t_1}^{t_2} \theta_k^2 u \cdot \nabla \eta_t \eta dx dt \right| \]

\[ \leq \int_{t_1}^{t_2} \| u \|_{L^q} \| \eta \theta_k \|_{L^\frac{2q}{{p}_1}} \| \theta_k \nabla \eta \|_{L^2} dt \]

\[ \leq \int_{t_1}^{t_2} C \| u \|_{L^q} \| \eta \theta_k \|_{L^\frac{4(1-\alpha_1)}{3}} \| \nabla (\eta \theta_k) \|_{L^\frac{2}{3}} \| \theta_k \nabla \eta \|_{L^2} dt \]

\[ \leq \epsilon \int_{t_1}^{t_2} \| u \|_{L^q} \| \eta \theta_k \|_{L^\frac{4(1-\alpha_1)}{3}} \| \nabla (\eta \theta_k) \|_{L^\frac{2}{3}} dt + C \| \theta_k \nabla \eta \|_{L^2((t_1,t_2);L^2(\mathbb{R}^3))} \]

\[ \leq \epsilon \int_{t_1}^{t_2} \| u \|_{L^p} \| \eta \theta_k \|_{L^\infty((t_1,t_2];L^2(\mathbb{R}^3))}^2 dt + C \| \theta_k \nabla \eta \|_{L^2((t_1,t_2);L^2(\mathbb{R}^3))}^2 \]

\[ + C \epsilon \| \nabla (\eta \theta_k) \|_{L^2((t_1,t_2);L^2(\mathbb{R}^3))}^2 + C \| \theta_k \nabla \eta \|_{L^2((t_1,t_2);L^2(\mathbb{R}^3))}^2 \]

for any \( 0 < \epsilon < 1 \). This completes the proof. \( \square \)

Now, using an estimate of the gradient of solution to the following parabolic equation

\[ \partial_t \theta - \text{div} \ (\kappa(\theta) \nabla \theta) = -\text{div} \ (u \theta), \]

we have (see [5])

\[ \| \nabla \theta \|_{L^p(0,T;L^q(\mathbb{R}^3))} \leq C \| u \theta \|_{L^p(0,T;L^q(\mathbb{R}^3))} + C \]

\[ \leq C \| \theta \|_{L^\infty(0,T;L^\infty)} \| u \|_{L^p(0,T;L^q(\mathbb{R}^3))} + C \]

\[ \leq C \| u \|_{L^p(0,T;L^q(\mathbb{R}^3))} + C. \]  

(4.2)
Multiplying (1.1) by \(-\Delta u\), integrating the result over \(\mathbb{R}^3\), and using (1.3), (4.2), the Hölder and Gagliardo-Nirenberg inequalities, we have

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \frac{1}{C} \int |\Delta u|^2 dx \\
\leq \int (u \cdot \nabla) u \cdot \Delta u dx + C \int |\nabla \theta| |\nabla u| |\Delta u| dx \\
\leq \|u\|_{L^q} \|\nabla u\|_{L^{\frac{3q}{2}}} \|\Delta u\|_{L^2} + C \|\nabla \theta\|_{L^q} \|\nabla u\|_{L^{\frac{3q}{2}}} \|\Delta u\|_{L^2} \\
\leq C(\|u\|_{L^q} + \|\nabla \theta\|_{L^q}) \|\nabla u\|_2 \|\Delta u\|_2 \|\nabla \theta\|_{L^2} \|\Delta u\|_2 \|\nabla u\|_2 \|\Delta u\|_2 \|\nabla \theta\|_{L^2} \|\Delta u\|_2 \|\nabla \theta\|_{L^2},
\]

which gives (3.5).

Applying the operator \(\Delta\) to (1.2), multiplying the result by \(\Delta \theta\), integrating over \(\mathbb{R}^3\), and using (1.3), (3.5), (2.5), (4.2), the Hölder and Gagliardo-Nirenberg inequalities, we have

\[
\frac{1}{2} \frac{d}{dt} \int |\Delta \theta|^2 dx + \frac{2}{C} \int |\nabla \Delta \theta|^2 dx \\
\leq \sum_{i=1}^3 \int \partial_i (u \cdot \nabla \theta) \cdot \partial_i \Delta \theta dx + C \|\nabla \theta\|_{L^q} \|\Delta \theta\|_2 \|\nabla \theta\|_2 \|\nabla \Delta \theta\|_2 \\
\leq (\|u\|_{L^q} \|\Delta \theta\|_2 + \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^\infty}) \|\nabla \Delta \theta\|_2 + C \|\nabla \theta\|_{L^q} \|\Delta \theta\|_2 \|\nabla \Delta \theta\|_2 \|\nabla \theta\|_{L^2} \|\nabla \Delta \theta\|_2 \\
\leq C(\|\Delta \theta\|_{L^2} + \|\nabla \theta\|_{L^\infty}) \|\nabla \Delta \theta\|_2 + C \|\nabla \theta\|_{L^q} \|\Delta \theta\|_2 \|\nabla \theta\|_{L^2}^2 + \frac{1}{4C} \|\nabla \Delta \theta\|_2^2 \\
\leq \frac{1}{2C} \|\nabla \Delta \theta\|_2^2 + C \{ \|\Delta \theta\|_2^2 + \|\nabla \theta\|_{L^q} \|\Delta \theta\|_2 \} \|\nabla \theta\|_{L^2},
\]

which implies (3.6).

Noticing the calculations for (3.7) still hold here, we hence complete the proof of Theorem 1.2.

5. Proof of Theorem 1.3

To complete our proof of Theorem 1.3, by Proposition 2.3, we only need to prove a priori estimates (1.14). By maximum principle, it follows from (1.10) and (1.12) that

\[
\|\theta\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \leq \|\theta_0\|_{L^\infty(\mathbb{R}^2)} \leq C.
\]
Multiplying (1.10) by \( \theta \), integrating the result over \( \mathbb{R}^2 \), and using (1.11), (1.13), and (5.1), we see that
\[
\frac{1}{2} \frac{d}{dt} \int \theta^2 dx + \int \kappa(\theta)|\nabla \theta|^2 dx = 0,
\]
which gives
\[
\| \theta \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \| \theta \|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq C. \tag{5.2}
\]

Multiplying (1.9) by \( u \), integrating the result over \( \mathbb{R}^2 \), and using (1.11) and (5.2), we find that
\[
\frac{1}{2} \frac{d}{dt} \int u^2 dx + \epsilon \int |\nabla u|^2 dx = \int \theta u^2 dx \leq \| \theta \|_{L^2} \| u \|_{L^2} \leq C \| u \|_{L^2},
\]
which yields
\[
\| u \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \sqrt{\epsilon} \| u \|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq C. \tag{5.3}
\]

Multiplying (1.9) by \( -\Delta u \), integrating the result over \( \mathbb{R}^2 \), and using (1.11), (5.2) and the fact that
\[
\int (u \cdot \nabla) u \cdot \Delta u dx = 0,
\]
we get
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \epsilon \int |\Delta u|^2 dx \leq \int |\nabla \theta| |\nabla u| dx \leq \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2},
\]
which leads to
\[
\| u \|_{L^\infty(0,T;H^1(\mathbb{R}^2))} + \sqrt{\epsilon} \| u \|_{L^2(0,T;H^2(\mathbb{R}^2))} \leq C. \tag{5.4}
\]

Then by taking the same calculations as those in [37], we obtain that
\[
\| \theta \|_{C^\alpha ([0,T])} \leq C \tag{5.5}
\]
for some \( \alpha \in (0,1) \) independent of \( \epsilon > 0 \).

Now, using an estimate of the gradient of solution of the parabolic equation
\[
\partial_t \theta - \text{div} (\kappa(\theta) \nabla \theta) = -\text{div} (u \theta),
\]
we obtain (see [5])
\[
\| \nabla \theta \|_{L^4(0,T;L^4(\mathbb{R}^2))} \leq \| u \theta \|_{L^4(0,T;L^4(\mathbb{R}^2))} + C \leq \| u \|_{L^4(0,T;L^4(\mathbb{R}^2))} \| \theta \|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} + C \leq C. \tag{5.6}
\]
Applying the operator $\Delta$ to (1.9), multiplying the result by $\Delta \theta$, integrating over $\mathbb{R}^2$, and using (1.11), (5.4), the Hölder and Gagliardo-Nirenberg inequalities, we derive that

\[
\frac{1}{2} \frac{d}{dt} \int |\Delta \theta|^2 dx + \frac{1}{C} \int |\nabla \Delta \theta|^2 dx \\
\leq \sum_{i=1}^{2} \int \partial_i (u \cdot \nabla \theta) \partial_i \Delta \theta dx + C \|\nabla \theta\|_{L^4} \|\Delta \theta\|_{L^4} \|\nabla \Delta \theta\|_{L^2} \\
\leq C \left( \|u\|_{L^4} \|\Delta \theta\|_{L^4} + \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^\infty} \right) \|\nabla \Delta \theta\|_{L^2} + C \|\nabla \theta\|_{L^4} \|\Delta \theta\|_{L^4} \|\nabla \Delta \theta\|_{L^2} \\
\leq C \left| |\Delta \theta|^{1/2} \|\nabla \Delta \theta\|_{L^2}^{1/2} + \|\nabla \theta\|_{L^4}^{2/3} \|\nabla \Delta \theta\|_{L^2}^{1/3} \|\nabla \theta\|_{L^2} \right| \\
+ C \|\nabla \theta\|_{L^4} \|\Delta \theta\|_{L^4} \|\nabla \Delta \theta\|_{L^2}^{3/2} \\
\leq \frac{1}{2C} \|\nabla \Delta \theta\|_{L^2}^2 + C \left\{ |\Delta \theta|_{L^2}^2 + \|\nabla \theta\|_{L^4}^2 + \|\nabla \theta\|_{L^4} \|\Delta \theta\|_{L^2}^2 \right\}.
\]

which implies that

\[
\|\theta\|_{L^\infty(0,T;H^2(\mathbb{R}^2))} + \|\theta\|_{L^2(0,T;H^3(\mathbb{R}^2))} \leq C. \quad (5.7)
\]

Here we have used the estimate (see (2.5)):

\[
\|\Delta \kappa(\theta)\|_{L^p} \leq C \|\Delta \theta\|_{L^p} \quad \text{with} \quad 1 < p < \infty. \quad (5.8)
\]

Denote $\omega = \text{curl} u = \partial_1 u_2 - \partial_2 u_1$ for $u = (u_1, u_2)^T$. Taking curl to (1.9) and using (1.11), we infer that

\[
\partial_t \omega + u \cdot \nabla \omega - \epsilon \Delta \omega = \partial_1 \theta. \quad (5.9)
\]

Multiplying (5.9) by $|\omega|^{q-2} \omega$, integrating the result over $\mathbb{R}^2$, and using (1.11) and (5.7), we have

\[
\frac{1}{q} \frac{d}{dt} \int |\omega|^q dx \leq \int \partial_1 \theta |\omega|^{q-2} \omega dx \leq \|\partial_1 \theta\|_{L^q} \|\omega\|_{L^q}^{q-1},
\]

which gives

\[
\frac{d}{dt} \|\omega\|_{L^q} \leq C \|\nabla \theta\|_{L^q},
\]

and thus

\[
\|\omega\|_{L^q} \leq \|\omega_0\|_{L^q} + C \int_0^T \|\nabla \theta\|_{L^q} dt. \quad (5.10)
\]
Taking $q \to +\infty$ in (5.10), we obtain that

$$\| \omega \|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \leq C. \tag{5.11}$$

Applying the operator $\Lambda^3$ to (1.9), multiplying the result by $\Lambda^3 u$, integrating over $\mathbb{R}^2$, and using (1.11), (5.7), (5.11), (2.3), and (2.1), we conclude that

$$\frac{1}{2} \frac{d}{dt} \int |\Lambda^3 u|^2 \, dx \leq -\int (\Lambda^3 (u \cdot \nabla u) - u \nabla \Lambda^3 u) \Lambda^3 u \, dx + \int \Lambda^3 (\theta e_2) \cdot \Lambda^3 u \, dx$$

$$\leq C \| \nabla u \|_{L^\infty} \| \Lambda^3 u \|^2_{L^2} + C \| \Lambda^3 \theta \|_{L^2} \| \Lambda^3 u \|_{L^2}$$

$$\leq C (1 + \log(e + \| \Lambda^3 u \|_{L^2})) \| \Lambda^3 u \|^2_{L^2} + C \| \Lambda^3 \theta \|_{L^2} \| \Lambda^3 u \|_{L^2},$$

which leads to

$$\| u \|_{L^\infty(0,T;H^3(\mathbb{R}^2))} \leq C. \tag{5.12}$$

This completes the proof of Theorem 1.3.

6. PROOF OF THEOREM 1.4

To complete our proof of Theorem 1.4, by Proposition 2.4, we only need to prove a priori estimates (1.19). First, the estimate (5.1) still holds. Similarly to (5.2) and (5.3), we have

$$\| \theta \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \sqrt{C} \| \theta \|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq C, \tag{6.1}$$

$$\| u \|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \| u \|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq C. \tag{6.2}$$

Multiplying (1.15) by $-\Delta u$, integrating the result over $\mathbb{R}^2$, and using (1.17), (6.1) and the fact that

$$\int (u \cdot \nabla) u \cdot \Delta u \, dx = 0,$$

we find that

$$\frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 \, dx + \int |\Delta u|^2 \, dx = -\int \theta e_2 \Delta u \, dx \leq \| \theta \|_{L^2} \| \Delta u \|_{L^2} \leq C \| \Delta u \|_{L^2},$$

which implies that

$$\| u \|_{L^\infty(0,T;H^1(\mathbb{R}^2))} + \| u \|_{L^2(0,T;H^2(\mathbb{R}^2))} \leq C. \tag{6.3}$$
By the Sobolev embedding theorem, it is easy to check that
\[ \partial_t u - \Delta u + \nabla \pi = f, \quad f := \theta e_2 - u \cdot \nabla u \in L^2(0, T; L^p(\mathbb{R}^2)), \quad (6.4) \]
for any \( p \in (2, \infty) \). Therefore, it follows from (6.4) and the regularity theory of Stokes equation that
\[ \| u \|_{L^2(0, T; W^{2,p}(\mathbb{R}^2))} \leq C \| f \|_{L^2(0, T; L^p(\mathbb{R}^2))} + C \]
(6.5)
And thus
\[ \| \nabla u \|_{L^2(0, T; L^\infty(\mathbb{R}^2))} \leq C \| u \|_{L^2(0, T; W^{2,p}(\mathbb{R}^2))} \leq C. \]
(6.6)
Let
\[ \begin{align*}
\tilde{\kappa}(\theta) := \int_0^\theta \kappa(\xi) d\xi.
\end{align*} \]
We deduce from (1.16) that
\[ \begin{align*}
\partial_t \tilde{\kappa}(\theta) + u \cdot \nabla \tilde{\kappa}(\theta) - \epsilon \kappa(\theta) \Delta \tilde{\kappa}(\theta) = 0.
\end{align*} \]
(6.7)
Multiplying (6.7) by \(- \Delta \tilde{\kappa}(\theta)\), integrating the result over \( \mathbb{R}^2 \), and using (1.17) and (6.6), we derive that
\[ \begin{align*}
\frac{1}{2} \frac{d}{dt} \int |\nabla \tilde{\kappa}(\theta)|^2 dx + \epsilon \int \kappa(\theta)|\Delta \tilde{\kappa}(\theta)|^2 dx
&= \int u \nabla \tilde{\kappa}(\theta) \Delta \tilde{\kappa}(\theta) dx
= - \sum_{i,j} \partial_j u_i \partial_i \tilde{\kappa}(\theta) \partial_j \tilde{\kappa}(\theta) dx
\leq \| \nabla u \|_{L^\infty} \| \nabla \tilde{\kappa}(\theta) \|^2_{L^2},
\end{align*} \]
which implies
\[ \| \theta \|_{L^\infty(0, T; H^1(\mathbb{R}^2))} + \sqrt{\epsilon} \| \Delta \tilde{\kappa}(\theta) \|_{L^2(0, T; L^2(\mathbb{R}^2))} \leq C. \]
(6.8)
Noting that (see (2.5))
\[ \begin{align*}
\sqrt{\epsilon} \| \Delta \theta \|_{L^2(0, T; L^2(\mathbb{R}^2))} &= \sqrt{\epsilon} \| \Delta (\tilde{\kappa}^{-1} \circ \tilde{\kappa}(\theta)) \|_{L^2(0, T; L^2(\mathbb{R}^2))}
\leq \sqrt{\epsilon} C \| \Delta \tilde{\kappa}(\theta) \|_{L^2(0, T; L^2(\mathbb{R}^2))} \leq C.
\end{align*} \]
(6.9)
By the Gagliardo-Nirenberg inequality, it follows from (6.8) and (6.9) that
\[ \begin{align*}
\epsilon \| \nabla \theta \|^4_{L^4(0, T; L^4(\mathbb{R}^2))} &= \epsilon \int_0^T \| \nabla \theta \|^4_{L^4} dt
\leq C \epsilon \int_0^T \| \nabla \theta \|^2_{L^2} \| \Delta \theta \|^2_{L^2} dt
\leq C \epsilon \int_0^T \| \Delta \theta \|^2_{L^2} dt
\leq C.
\end{align*} \]
(6.10)
 Applying the operator \( \partial_t \) to (1.16) gives
\[
\partial_t \partial_t \theta + u \nabla \partial_t \theta + \partial_t u \cdot \nabla \theta - \epsilon \text{div} \{ \kappa(\theta) \nabla \partial_t \theta + k'(\theta) \partial_t \theta \nabla \theta \} = 0.
\] (6.11)

Multiplying (6.11) by \( (\partial_t \theta)^3 \), integrating the result over \( \mathbb{R}^2 \), summing over \( i \), and using (1.17), (6.6) and (6.10), we obtain
\[
\frac{1}{4} \frac{d}{dt} \int |\nabla \theta|^4 \, dx + \frac{\epsilon}{C} \int |\nabla \nabla \theta|^2 \, dx 
\leq C \epsilon \int |\nabla \theta| \cdot |\nabla \theta|^2 |\nabla \theta|^2 \, dx + C \| \nabla u \|_{L^\infty} \int |\nabla \theta|^4 \, dx 
\leq C \epsilon \| \nabla \theta \|_{L^4}(\mathbb{R}^2) \| (\nabla \theta)^2 \|_{L^4} \| \nabla \theta \|_{L^2} + C \| \nabla u \|_{L^\infty} \int |\nabla \theta|^4 \, dx 
\leq \epsilon \frac{1}{2C} \| \nabla \nabla \theta \|_{L^2}^2 + C \epsilon \| \nabla \theta \|_{L^4}^4 \| \nabla \theta \|_{L^2}^2 + C \| \nabla u \|_{L^\infty} \int |\nabla \theta|^4 \, dx,
\]
which gives
\[
\| \nabla \theta \|_{L^\infty(0,T; L^4(\mathbb{R}^2))} \leq C.
\] (6.12)

Applying the operator \( \Delta \) to (1.16), multiplying the result by \( \Delta \theta \), integrating over \( \mathbb{R}^2 \), and using (2.5), (6.12), (5.8), (6.5), and (6.6), we conclude that
\[
\frac{1}{2} \frac{d}{dt} \int |\Delta \theta|^2 \, dx + \frac{\epsilon}{C} \int |\nabla \Delta \theta|^2 \, dx 
\leq - \int (\Delta(u \cdot \nabla \theta) - u \nabla \Delta \theta) \Delta \theta \, dx + C \epsilon \| \nabla \theta \|_{L^4} \| \Delta k \|_{L^4} \| \nabla \Delta \theta \|_{L^2} 
\leq C(\| \Delta u \|_{L^4} \| \nabla \theta \|_{L^4} + \| \nabla u \|_{L^\infty} \| \Delta \theta \|_{L^2}) \| \Delta \theta \|_{L^2} + C \epsilon \| \Delta \theta \|_{L^4} \| \nabla \Delta \theta \|_{L^2} 
\leq C(\| \Delta u \|_{L^4} + \| \nabla u \|_{L^\infty} \| \Delta \theta \|_{L^2}) \| \Delta \theta \|_{L^2} + \frac{\epsilon}{2C} \| \nabla \Delta \theta \|_{L^2}^2 + C \epsilon \| \Delta \theta \|_{L^2}^2,
\]
which leads to
\[
\| \theta \|_{L^\infty(0,T; H^2(\mathbb{R}^2))} \leq C.
\] (6.13)

Finally, applying the operator \( \Delta \) to (1.15), multiplying the result by \( \Delta u \), integrating over \( \mathbb{R}^2 \), and using (1.17), (6.13), and (6.6), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int |\Delta u|^2 \, dx + \int |\nabla \Delta u|^2 \, dx 
= \int \Delta \theta \epsilon_{2} \cdot \Delta u \, dx - \int \Delta(u \cdot \nabla u) \cdot \Delta u \, dx 
\leq \| \Delta \theta \|_{L^2} \| \Delta u \|_{L^2} + C \| \nabla u \|_{L^\infty} \| \Delta u \|_{L^2}^2
\]
\[ \leq C \| \Delta u \|_{L^2} + C \| \nabla u \|_{L^\infty} \| \Delta u \|_{L^2}^2, \]

which gives
\[ \| u \|_{L^\infty(0,T;H^2(\mathbb{R}^2))} + \| u \|_{L^2(0,T;H^3(\mathbb{R}^2))} \leq C. \]

This completes the proof of Theorem 1.4.

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