DAVIES TYPE ESTIMATE AND THE HEAT KERNEL BOUND UNDER THE RICCI FLOW

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Abstract. We prove a Davies type double integral estimate for the heat kernel $H(y, t; x, l)$ under the Ricci flow. As a result, we give an affirmative answer to a question proposed by Chow et al. Moreover, we apply the Davies type estimate to provide a new proof of the Gaussian upper and lower bounds of $H(y, t; x, l)$ which were first shown in 2011 by Chan, Tam, and Yu.

1. Introduction

On a complete Riemannian manifold $(M^n, g_{ij})$, the heat kernel $H(x, y, t)$, is the smallest positive fundamental solution to the heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$ (1.1)

The heat kernel estimate is of great importance and interest due to its relation to many other features of the manifolds, such as Harnack estimate, Sobolev inequality, Log Sobolev inequality, Faber-Krahn type inequality, and Nash type inequality (see e.g. [22], [9], [12], [4], [19], [5]). Since the work of Nash [20] and Aronson [1], many methods have been discovered for deriving Gaussian upper and lower bounds of $H(x, y, t)$; see, e.g., [7], [19], [10], [13], [11], [18]. One of the methods was developed by Li-Wang in [18]. They obtained a Gaussian upper bound for $H(x, y, t)$ based on the parabolic mean value inequality and the following double integral upper estimate of the heat kernel proved by E. B. Davies [11]:

**Theorem 1.1.** Let $(M, g)$ be a complete Riemannian manifold. For any two bounded subsets $U_1$ and $U_2$ of $M$, one has:

$$\int_{U_1} \int_{U_2} H(x, y, t) d\mu(x) d\mu(y) \leq \text{Vol}^{\frac{1}{2}}(U_1) \text{Vol}^{\frac{1}{2}}(U_2) e^{-\frac{d^2(U_1, U_2)}{4t}},$$ (1.2)

where $d(U_1, U_2)$ is the distance between $U_1$ and $U_2$.

In this paper, we consider the heat kernel of the time-evolving heat equation under the Ricci flow on a complete manifold $M^n$, i.e.,

$$\frac{\partial u}{\partial t} = \Delta_t u,$$ (1.3)
where $\Delta_t$ is the Laplacian with respect to a complete solution $g_{ij}(t)$, $t \in [0, T)$ and $T < \infty$, of the following Ricci flow equation:

\[
\begin{cases}
\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}, \\
g(0) = g_0.
\end{cases}
\]

on $M$.

The existence and uniqueness of the heat kernel $H(y, t; x, l)$ to (1.3) were proved in [14] and [8]. When $M$ is compact, C. Guenther [14] obtained a Gaussian lower bound of $H(y, t; x, l)$ for type I ancient $\kappa$-solutions of the Ricci flow. When $M$ is compact, C. Guenther [14] obtained a Gaussian lower bound of $H(y, t; x, l)$ assuming uniformly bounded curvature and non-negative Ricci curvature along the Ricci flow. Q. Zhang [24] proved both Gaussian upper and lower bounds of $H(y, t; x, l)$ for type I ancient $\kappa$-solutions of the Ricci flow.

Remark 1.3. When the pair $(x_0, l_0)$ is fixed, $H(y, t; x_0, l_0)$ defines a measure $d\nu_t$ on $M$. In [16], H.-J. Hein and A. Naber proved a nice Gaussian upper bound for $\nu_t(A)\nu_t(B)$, for any subsets $A$ and $B$ of $M$.

Using Theorem 1.2, we are able to provide a new proof of the Gaussian upper bound and lower bounds of $H(y, t; x, l)$ which was first shown by Chau-Tam-Yu [6]. For the Gaussian upper bound, we have the following theorem.
Theorem 1.4. Under the Ricci flow, assume that $Rc \geq -K_1$ on $M \times [0, T)$ for some nonnegative constant $K_1$ and $T < \infty$, and that $\Lambda = \int_0^T \sup_M |Rc|(t)dt < \infty$. We have the following upper bound:

$$H(y,t;x,l) \leq C_1 e^{C_2e^{C_3\Lambda + C_4K_1T}} \min \left\{ \exp \left( -\frac{d^2(x,y)}{8e^{C_3\Lambda + C_4K_1T}} (t-l) \right), \exp \left( -\frac{d^2(x,y)}{8e^{C_3\Lambda + C_4K_1T}} (t-l) \right) \right\},$$

where constants $C_1$, $C_2$, $C_3$ and $C_4$ depend only on $n$.

Next, following a method of Li-Tam-Wang \[17\], we can use a gradient estimate of Q. Zhang in \[26\] (see also \[25\] and \[2\]) to show a Gaussian lower bound of $H(y,t;x,l)$.

Theorem 1.5. Let $(M^n, g_{ij}(t))$, $t \in [0, T)$ and $T < \infty$, be a complete solution to the Ricci flow \[1,4\]. Suppose that $Rc \geq -K_1$ on $M \times [0, T)$ and $\Lambda = \int_0^T \sup_M |Rc|(t)dt < \infty$. Then we have the following lower bound:

$$H(y,t;x,l) \geq \frac{C_5 e^{-C_6e^{C_7\Lambda + C_8K_1T}} \exp \left( -\frac{4d^2(x,y)}{t-l} \right)}{\text{Vol}(B_t(x, \sqrt{\frac{l-t}{8}}))},$$

where $C_5$, $C_6$, $C_7$ and $C_8$ are positive constants only depending on $n$.

This paper is organized as follows. In section 2, we review some known facts about the fundamental solutions of heat-type equations with the metric evolving under a group of more general equations than the Ricci flow, including the existence, uniqueness and the mean value inequality. In section 3, we prove Theorem \[1,2\]. The main idea is similar to that in the fixed metric case. However, since the heat kernel is not self-symmetric in this case, we use the symmetry between the heat kernel and the adjoint heat kernel instead. In section 4, we first use the method in \[18\] to prove a slightly different version of the Gaussian upper estimate from the one in Theorem \[1,4\]. Then we show that Theorem \[1,4\] can be derived from this upper estimate and an $L^1$ bound of $H$. In section 5, we finish the proof of Theorem \[1,3\].

2. Preliminaries

In this section, we present some basic results regarding the heat kernel and adjoint heat kernel of time evolving heat-type equations. The readers can refer to \[8\] for more detail.

Let $M^n$ be a Riemannian manifold, and $g_{ij}(t)$, $t \in [0, T)$ and $T < \infty$, a complete solution to the following equation:

$$\begin{cases}
\frac{\partial g_{ij}(t)}{\partial t} = -2A_{ij}, \\
g(0) = g_0,
\end{cases}$$

(2.1)

where $A_{ij}(t)$ is a time-dependent symmetric 2-tensor. Consider the following heat operator with potential:

$$L = \frac{\partial}{\partial t} - \Delta_t + Q,$$

where $\Delta_t$ is the Laplacian with respect to $g_{ij}(t)$ and $Q : M \times [0, T) \to \mathbb{R}$ is a $C^\infty$ function.
Definition 2.1. Let $\mathbb{R}^2_T = \{(t, l) \in \mathbb{R}^2 | 0 \leq l < t < T\}$. A fundamental solution for the operator $L$ is a function $$H : M \times M \times \mathbb{R}^2_T \to \mathbb{R}$$ that satisfies:

1. $H$ is continuous, with $C^2$ in the first two space variables and $C^1$ in the last two time variables,
2. $LH = (\frac{\partial}{\partial t} - \Delta_{t,y} + Q)H(y, t; x, l) = 0$,
3. $\lim_{t \searrow l} H(y, t; x, l) = \delta_x$,

where $H(y, t; x, l) = H(y, x, (t, l))$.

The heat kernel for $L$ is defined to be the minimal positive fundamental solution.

Definition 2.2. The adjoint heat kernel is the minimal positive fundamental solution $G : M \times M \times \mathbb{R}^2_T \to \mathbb{R}$ for the operator $L^* = \frac{\partial}{\partial l} + \Delta_l - Q - A$, i.e., $G$ satisfies

1. $G$ is continuous, with $C^2$ in the first two space variables and $C^1$ in the last two time variables,
2. $L^* G = (\frac{\partial}{\partial l} + \Delta_l,x - Q - A)G(x, l; y, t) = 0$,
3. $\lim_{l \nearrow t} G(x, l; y, t) = \delta_y$,

where $G(x, l; y, t) = G(y, x, (t, l))$, and $A = tr_g(i)(A_{ij})$.

First of all, we have the following existence and uniqueness of the heat kernel and the adjoint heat kernel according to [14] and [8].

Theorem 2.3. Let $M^n$ be a complete manifold, and $g(t)$, $t \in [0, T)$ for some $T < \infty$, a smooth family of Riemannian metrics on $M$. If $\int_0^T \inf_M Q(t) dt < \infty$, then there exists a unique $C^\infty$ minimal positive fundamental solution $H(y, t; x, l)$ for the operator $L = \frac{\partial}{\partial t} - \Delta_{t,y} + Q$.

As in the fixed metric case, the heat kernel $H(y, t; x, l)$ for $L$ on $M$ is the limit of the Dirichlet heat kernels on a sequence of exhausting subsets in $M$.

Definition 2.4. Let $\Omega \subset M$ be a bounded subset. The Dirichlet heat kernel on $\Omega$ for $L$, denoted by $H_\Omega(y, t; x, l)$, is the fundamental solution to $\frac{\partial u}{\partial t} = \Delta_t u - Q u$ in $\Omega$ and satisfies

i) $\lim_{l \searrow t} H_\Omega = \delta_x$ for $x \in \text{Int}(\Omega)$;

ii) $H_\Omega(y, t; x, l) = 0$, for $y \in \partial\Omega$ and $x \in \text{Int}(\Omega)$.

Proposition 2.5 (See e.g. [8]). Let $\Omega_i \subset M$ be a sequence of exhausting bounded sets, and $H_{\Omega_i}(y, t; z, s)$ the Dirichlet heat kernel on $\Omega_i$. Then

$$\lim_{i \to \infty} H_{\Omega_i}(y, t; z, s) = H(y, t; z, s)$$

uniformly on any compact subset of $M \times M \times \mathbb{R}^2_T$.

Moreover, the heat kernel and the adjoint heat kernel satisfy the following important properties.
Proposition 2.6 (See e.g. [8]). With the assumptions above, we have

1. \( H_\Omega(y, t; x, l) = G_\Omega(x, l; y, t) \), for \( x, y \in \Omega \);
2. \( H(y, t; x, l) = G(x, l; y, t) \);
3. \( H_\Omega(y, t; x, l) = \int_\Omega H_\Omega(y, t; z, s) H_\Omega(z, s; x, l) d\mu_{g(s)}(z) \);
4. \( H(y, t; x, l) = \int_M H(y, t; z, s) H(z, s; x, l) d\mu_{g(s)}(z) \);

where \( \Omega \) is an open subset in \( M \) with compact closure, and \( G_\Omega(x, l; y, t) \) is the Dirichlet heat kernel for the adjoint operator \( \frac{\partial}{\partial t} + \Delta_l - Q - A \).

Let

\[ \Lambda = \int_0^T \sup_M |A_{ij}| g(t)(t) dt. \]

Assume that there exists a metric \( g' \) and a positive constant \( \hat{C} \geq 1 \) such that

\[ \hat{C}^{-1} g' \leq g(0) \leq \hat{C} g'. \]

Then we have

\[ \hat{C}^{-1} e^{-2\Lambda} g' \leq g(t) \leq \hat{C} e^{2\Lambda} g' \]

for all time \( t \in [0, T) \).

Suppose that \( u : M \times [0, T) \to \mathbb{R} \) is a positive subsolution to

\[ \frac{\partial u}{\partial t} \leq \Delta t u - Qu. \]

Define the parabolic cylinder:

\[ P_{g'}(x, \tau, r, -r^2) = B_{g'}(x, r) \times [\tau - r^2, \tau], \]

where \( B_{g'}(x, r) \) represents the geodesic ball of radius \( r \) centered at \( x \) in \( M \) with respect to the metric \( g' \).

By Moser iteration, Chau-Tam-Yu [6] got the following mean value inequality (see also [8]):

Theorem 2.7. In the above setting, assume that \( \text{Rc}(g') \geq -K_1 \) on \( M \) with some \( K_1 \geq 0 \). Then

\[ \sup_{P_{g'}(x_0, t_0, r_0, -(r_0)^2)} u \leq C_1 \hat{C}^{\frac{n(n+3)}{2}} e^{C_2 \Lambda + C_3 \sqrt{r_0}} r_0^{2n} \text{Vol}_{g'}(B_{g'}(x_0, r_0)) \int_{P_{g'}(x_0, t_0, 2r_0, -(2r_0)^2)} u(x, s) d\mu_{g'}(x) ds, \]

where \( C_1, C_2 \) and \( C_3 \) are constants only depending on \( n \), and

\[ \hat{C} = -\inf_{M \times [0, T]} \left\{ Q + \frac{1}{2} A \right\}. \]

3. **Davies type estimate for the heat kernel under the Ricci flow**

Assume that \((M^n, g_{ij}(t))\) is a complete solution to the Ricci flow \([14]\) for \( t \in [0, T) \) and \( T < \infty \).

Denote by \( H(y, t; x, l) > 0, 0 \leq l < t < T \), the heat kernel of \([13]\) under the Ricci flow, i.e.,

\[
\begin{cases}
\frac{\partial H}{\partial t} = \Delta_{t,y} H, \\
\lim_{t \searrow l} H = \delta_x.
\end{cases}
\]
Then $G(x, l; y, t) = H(y, t; x, l)$ is the adjoint heat kernel to the following conjugate heat equation:

\[
\begin{align*}
\frac{\partial G}{\partial t} &= -\Delta_{l,x} G + RG, \\
\lim_{t \searrow t} G &= \delta_y,
\end{align*}
\]

(3.2)

where $R$ is the scalar curvature of $M$.

In the rest of this paper, we will use the following notation: $B_t(x, r) := B_{g(t)}(x, r)$, $\text{Vol}_s(U) := \text{Vol}_{g(s)}(U)$, $d\mu_s := d\mu_{g(s)}$, where $U$ is a subset of $M$ and $d\mu_{g(s)}$ denotes the volume element of $g(s)$.

**Proof of Theorem 1.2** Let $\Omega_i \subset M$ be a sequence of exhausting bounded sets, and $H_{\Omega_i, i}(y; z, t, s)$ the Dirichlet heat kernel on $\Omega_i$. Since $U_1$ and $U_2$ are bounded, we may assume that $(U_1 \cup U_2) \subset \Omega_i$ for each $i$.

By Propositions 2.5 and 2.6, we have

\[
\int_{U_1} \int_{U_2} H(y, t; x, l)d\mu_t(y)d\mu_t(x)
\]

\[
= \lim_{i \to \infty} \int_{U_1} \int_{U_2} \int_{\Omega_i} H_{\Omega_i, i}(y, t; z, x, l)d\mu_{t \frac{t + l}{2}}(z)d\mu_t(y)d\mu_t(x)
\]

\[
= \lim_{i \to \infty} \int_{\Omega_i} \int_{U_1} \int_{U_2} H_{\Omega_i, i}(y, t; z, x, l)d\mu_{t \frac{t + l}{2}}(z)d\mu_t(y)d\mu_t(x)
\]

\[
= \lim_{i \to \infty} \int_{\Omega_i} u_i(z, l)u_i(z, l)d\mu_{t \frac{t + l}{2}}(z),
\]

where

\[
u_i(z, s) = \int_{U_1} H_{\Omega_i, i}(z, s; x, l)d\mu_t(x),
\]

and

\[
u_i(z, s) = \int_{U_2} H_{\Omega_i, i}(y, t; z, s)d\mu_t(y) = \int_{U_2} G_{\Omega_i, i}(z, s; y, t)d\mu_t(y).
\]

Here $G_{\Omega_i, i}(z, s; y, t)$ denotes the adjoint Dirichlet heat kernel on $\Omega_i$.

From $\bar{R}c \geq -K_1$, we have for any $s \in [0, T)$,

\[
g_{ij}(s) \leq e^{2K_1 T} g_{ij}(0),
\]

and

\[
d_s(x, y) \leq e^{K_1 T} d_0(x, y),
\]

where $d_s(x, y)$ is the distance function at time $s$.

Let $\xi(z, s) = \frac{d^2_s(z, U_1)}{2C_{K_1 T}(s - l)}$ for $C_{K_1 T} = e^{2K_1 T}$ and $s > l$. Since

\[
|\nabla d_0(z, U_1)|_{g(s)} \leq e^{K_1 T} |\nabla d_0(z, U_1)|_{g(0)} \leq e^{K_1 T},
\]

we have

\[
\frac{\partial \xi}{\partial s} + \frac{1}{2} |\nabla \xi|_{g(s)}^2 \leq -\frac{d^2_s(z, U_1)}{2C_{K_1 T}(s - l)^2} + \frac{d^2_s(z, U_1) \cdot e^{2K_1 T}}{2C_{K_1 T}^2(s - l)^2} = 0.
\]
We compute
\[
\frac{d}{ds} \int_{\Omega_i} u^2_i(z, s)e^{\xi(z, s)}d\mu_s(z) = \int_{\Omega_i} (2u_i \Delta_{s,z} u_i - Ru_i^2 + u_i^2 \cdot \frac{\partial \xi(z, s)}{\partial s})e^{\xi(z, s)}d\mu_s(z).
\]
Notice that
\[
2 \int_{\Omega_i} u_i \Delta_{s,z} u_i e^\xi d\mu_s(z) = -2 \int_{\Omega_i} (|\nabla u_i|^2 + u_i \nabla u_i \cdot \nabla \xi)e^\xi d\mu_s(z) + 2 \int_{\partial \Omega_i} u_i \partial_{\nu} u_i e^\xi dS \\
\leq \int_{\Omega_i} 1 \frac{u_i^2}{2} |\nabla \xi|^2 e^{\xi} d\mu_s(z),
\]
where we have used the Cauchy-Schwarz inequality for \(-u_i \nabla u_i \cdot \nabla \xi\) and the fact that \(u_i|_{\partial \Omega_i} = 0\).

Thus,
\[
\frac{d}{ds} \int_{\Omega_i} u^2_i(z, s)e^{\xi(z, s)}d\mu_s(z) \leq C_1 K_1 \int_{\Omega_i} u^2_i(z, s)e^{\xi(z, s)}d\mu_s(z),
\]
where \(C_1\) is a constant only depending on \(n\). Hence, we have
\[
\int_{\Omega_i} u^2_i(z, s)e^{\xi(z, s)}d\mu_s(z) \leq e^{C_1 K_1 (s-l)} \lim_{h \to l} \int_{\Omega_i} u^2_i(z, h)e^{\xi(z, h)}d\mu_h(z) \\
\leq e^{C_1 K_1 (s-l)} \text{Vol}_l(U_1).
\]

Let \(\eta(z, s) = \frac{d_0^2(z, U_2)}{2C_{K_1T}(t-s)^2}\) for \(C_{K_1T} = e^{2K_1T}\). Then
\[
\frac{\partial \eta}{\partial s} - \frac{1}{2} |\nabla \eta|^2 g(s) \geq \frac{d_0^2(z, U_2)}{2C_{K_1T}(t-s)^2} - \frac{d_0^2(z, U_2) \cdot e^{2K_1T}}{2C_{K_1T}(t-s)^2} = 0.
\]

Moreover,
\[
\frac{d}{ds} \int_{\Omega_i} v^2_i(z, s)e^{\eta(z, s)}d\mu_s(z) = \int_{\Omega_i} (-2u_i \Delta_{s,z} v_i + v_i^2 \cdot \frac{\partial \eta(z, s)}{\partial s})e^{\eta(z, s)}d\mu_s(z).
\]

Since,
\[
-2 \int_{\Omega_i} v_i \Delta_{s,z} v_i e^{\eta} d\mu_s(z) = 2 \int_{\Omega_i} (|\nabla v_i|^2 + v_i \nabla v_i \cdot \nabla \eta)e^{\eta} d\mu_s(z) - 2 \int_{\partial \Omega} v_i \partial_{\nu} v_i e^{\eta} dS \\
\geq - \frac{1}{2} \int_{\Omega_i} v_i^2 |\nabla \eta| e^{\eta} d\mu_s(z),
\]
we have
\[
\frac{d}{ds} \int_{\Omega_i} v_i^2(z, s)e^{\eta(z, s)}d\mu_s(z) \geq 0.
\]

It implies that, for large \(i\),
\[
\int_{\Omega_i} v_i^2(z, s)e^{\eta(z, s)}d\mu_s(z) \leq \lim_{h \to l} \int_{\Omega_i} v_i^2(z, h)e^{\eta(z, h)}d\mu_h(z) \\
= \text{Vol}_l(U_2).
\]
Now since
\[ \frac{1}{2} \xi(z, \frac{t + l}{2}) + \frac{1}{2} \eta(z, \frac{t + l}{2}) \geq \frac{d^2_{\tilde{g}}(U_1, U_2)}{4C_{K_1}T(t - l)}, \]
we have
\[
e^{-\frac{\tilde{d}^2_{\tilde{g}}(U_1, U_2)}{4C_{K_1}T(t - l)}} \int_{U_1} \int_{U_2} H(y, t; x, l) d\mu_t(y) d\mu_t(x)
= \lim_{i \to \infty} e^{-\frac{\tilde{d}^2_{\tilde{g}}(U_1, U_2)}{4C_{K_1}T(t - l)}} \int_{\Omega_i} u_i(z, \frac{t + l}{2}) v_i(z, \frac{t + l}{2}) d\mu_t(z)
\leq \lim_{i \to \infty} \int_{\Omega_i} u_i(z, \frac{t + l}{2}) e^{\frac{1}{2} \xi(z, \frac{t + l}{2})} v_i(z, \frac{t + l}{2}) e^{\frac{1}{2} \eta(z, \frac{t + l}{2})} d\mu_t(z)
\leq e^{-\frac{\tilde{d}^2_{\tilde{g}}(U_1, U_2)}{4C_{K_1}T(t - l)}} \Vol_t(U_1)^{1/2} \Vol_t(U_2)^{1/2},
\]
i.e.,
\[
\int_{U_1} \int_{U_2} H(y, t; x, l) d\mu_t(y) d\mu_t(x) \leq e^{-\frac{\tilde{d}^2_{\tilde{g}}(U_1, U_2)}{4C_{K_1}T(t - l)}} e^{-\frac{\tilde{d}^2_{\tilde{g}}(U_1, U_2)}{4C_{K_1}T(t - l)}} \Vol_t(U_1)^{1/2} \Vol_t(U_2)^{1/2}.
\]
The theorem follows from the fact that \( d_t(U_1, U_2) \leq e^{K_1T} d_0(U_1, U_2). \)

4. A Gaussian upper bound of \( H(y, t; x, l) \)

Recall that by the Volume Comparison Theorem, we have the following lemma (see e.g. [15]):

**Lemma 4.1.** Let \((M^n, g_{ij})\) be a complete Riemannian manifold. If \( Rc \geq -K_1 \), for some constant \( K_1 \geq 0 \), then for any point \( x \in M \) and any \( 0 < r \leq R \), we have
\[
\Vol(B(x, R)) \leq \left( \frac{R}{r} \right)^n e^{\sqrt{(n-1)K_1} R} \Vol(B(x, r)),
\]
where \( B(x, R) \) is the geodesic ball of radius \( R \) in \( M \) centered at \( x \).

In particular, letting \( r \to 0 \), we have
\[
\Vol(B(x, R)) \leq C R^n e^{\sqrt{(n-1)K_1} R}
\]
for some constant \( C \) only depending on \( n \).

Since we have obtained a Davies type estimate in Theorem 1.2, similarly to the method in [15], we can show a Gaussian upper bound of the heat kernel (see [6]) by using the mean value inequality in Theorem 2.1. In the following, unless otherwise stated, \( C, C_0, C_1, C_2, \cdots \), and \( \bar{C}, \bar{C}_0, \bar{C}_1, \bar{C}_2, \cdots \) all represent positive constants only depending on \( n \).

**Theorem 4.2.** Under the Ricci flow, assume that \( Rc \geq -K_1 \) on \( M \times [0, T] \) and \( \Lambda = \int_0^T \sup_M |Rc|(t) dt < \infty \). We have the following upper bound:
\[
H(y, t; x, l) \leq \frac{C_1 e^{C_2 \Lambda + C_3 K_1 T + C_4 \sqrt{K_1} T} \exp\left(-\frac{d^2_{\tilde{g}}(x, y)}{8e^{4K_1T(t - l)}}\right)}{\sqrt{\Vol_t(B_t(y, \sqrt{\frac{t - l}{8}}))} \sqrt{\Vol_t(B_t(x, \sqrt{\frac{t - l}{8}}))}}.
\]
Proof. We have by Theorem 2.7 that, for $0 \leq l < t_0 < T$ and $r_0$ with $t_0 - l_0 \geq 4r_0^2$,

\[
H(y_0, t_0; x_0, l_0) \leq \frac{C_0 e^{C_1 A + C_2 K_1 T + C_3 \sqrt{K_1} r_0}}{r_0^2 \text{Vol}_{l_0}(B_{l_0}(y_0, r_0))} \int_{t_0 - 4r_0^2}^{t_0} \int_{B_{l_0}(y_0, 2r_0)} H(y, s; x_0, l_0) d\mu_{l_0}(y) ds.
\]

Let $\tilde{l} = t_0 - l$, $\tilde{s} = t_0 - s$, and

\[
\tilde{G}(x, \tilde{l}; y, \tilde{s}) = G(x, t_0 - \tilde{l}; y, t_0 - \tilde{s}) = G(x, l; y, s) = H(y, s; x, l).
\]

Then, $\tilde{G}(x, \tilde{l}; y, \tilde{s})$ satisfies

\[
\frac{\partial \tilde{G}(x, \tilde{l}; y, \tilde{s})}{\partial \tilde{l}} = \tilde{\Delta}_l \tilde{G} - \tilde{R} \tilde{G}(x, \tilde{l}; y, \tilde{s}),
\]

where $\tilde{\Delta}_l$ is the Laplacian with respect to the solution $\tilde{g}(\tilde{l}) := g(t_0 - \tilde{l})$ of the backward Ricci flow:

\[
\frac{\partial \tilde{g}(\tilde{l})}{\partial \tilde{l}} = 2\tilde{R}_{ij}(\tilde{l}).
\]

Thus, according to Theorem 2.7 we have the following mean value inequality:

\[
\tilde{G}(x_0, \tilde{l}_0; y, \tilde{s}) \leq \frac{C_0 e^{C_1 A + C_2 K_1 T + C_3 \sqrt{K_1} r_0}}{r_0^2 \text{Vol}_{l_0}(B_{l_0}(y_0, r_0))} \int_{t_0 - 4r_0^2}^{t_0} \int_{B_{l_0}(y_0, 2r_0)} \tilde{G}(x, \tilde{l}; y, \tilde{s}) d\mu_{l_0}(y) d\tilde{l},
\]

where $4r_0^2 \leq t_0 - l_0 - 4r_0^2 \leq s - l_0$.

Hence formula (4.1) becomes

\[
H(y_0, t_0; x_0, l_0) \leq \frac{C_0 e^{C_1 A + C_2 K_1 T + C_3 \sqrt{K_1} r_0}}{r_0^2 \text{Vol}_{l_0}(B_{l_0}(y_0, r_0))} \int_{t_0 - 4r_0^2}^{t_0} \int_{B_{l_0}(y_0, 2r_0)} \tilde{G}(x, \tilde{l}; y, \tilde{s}) d\mu_{l_0}(y) d\tilde{l}.
\]

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Now let \( r_0 = r_1 = \sqrt{\frac{l_0 - l_0}{8}} \). We have by Theorem 1.2 that

\[
H(y_0, t_0; x_0, l_0) \leq \frac{C_4 e^{C_5 \Lambda + C_6 K_1 T + C_7 \sqrt{K_1 T}}}{(t_0 - l_0)^2 \Vol_0(B_{t_0}(y_0, \sqrt{\frac{t_0 - l_0}{8}})) \Vol_0(B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{8}}))}
\]

\[
\cdot \int_{t_0}^{t_0 + t_0} \int_{t_0}^{t_0 + t_0} \int_{B_{t_0}(y_0, \sqrt{\frac{t_0 - l_0}{8}})} \int_{B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{8}})} H(y, s; x, l) d\mu_t(x) d\mu_t(y) \, dl \, ds
\]

\[
\leq \frac{C_8 e^{C_9 \Lambda + C_{10} K_1 T + C_{11} \sqrt{K_1 T}} \sqrt{\Vol_0(B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{2}})) \Vol_0(B_{t_0}(y_0, \sqrt{\frac{t_0 - l_0}{2}}))}}{\Vol_0(B_{t_0}(y_0, \sqrt{\frac{t_0 - l_0}{8}})) \Vol_0(B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{8}}))}
\]

\[
\cdot \exp \left( -\frac{d^2_{t_0}(B_{t_0}(y_0, \sqrt{\frac{t_0 - l_0}{2}}), B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{2}}))}{4e^{4K_1 T}(t_0 - l_0)} \right)
\]

In the last step above, we have used Lemma 4.1 to get

\[
\Vol(B_t(z, \sqrt{\frac{t_0 - l_0}{2}})) \leq C_{16} e^{C_{17} \sqrt{K_1 T}} \Vol(B_t(z, \sqrt{\frac{t_0 - l_0}{8}}))
\]

for all \( t \in [0, T] \).

Since

\[
d_{t_0}(B_{t_0}(y_0, e^{K_1 T} \sqrt{\frac{t_0 - l_0}{2}}), B_{t_0}(x_0, e^{K_1 T} \sqrt{\frac{t_0 - l_0}{2}})) = \begin{cases} 0, & \text{if } d_{t_0}(x_0, y_0) \leq e^{K_1 T} \sqrt{2(t_0 - l_0)} \\ d_{t_0}(x_0, y_0) - e^{K_1 T} \sqrt{2(t_0 - l_0)}, & \text{if } d_{t_0}(x_0, y_0) > e^{K_1 T} \sqrt{2(t_0 - l_0)} \end{cases}
\]

it follows that when \( d_{t_0}(x_0, y_0) \leq e^{K_1 T} \sqrt{2(t_0 - l_0)} \),

\[
\exp \left( -\frac{d^2_{t_0}(B_{t_0}(y_0, e^{K_1 T} \sqrt{\frac{t_0 - l_0}{2}}), B_{t_0}(x_0, e^{K_1 T} \sqrt{\frac{t_0 - l_0}{2}}))}{4e^{4K_1 T}(t_0 - l_0)} \right)
\]

\[
= 1
\]

\[
\leq e^{\frac{1}{4e^{4K_1 T}(t_0 - l_0)}} \exp \left( -\frac{d^2_{t_0}(x_0, y_0)}{8e^{4K_1 T}(t_0 - l_0)} \right).
\]
When \( d_{t_0}(x_0, y_0) > e^{K_1 T} \sqrt{2(t_0 - l_0)} \), we have
\[
\exp \left( -\frac{d_{t_0}^2(B_{t_0}(y_0, e^{K_1 T} \sqrt{\frac{t_0 - l_0}{2}}), B_{t_0}(x_0, e^{K_1 T} \sqrt{\frac{t_0 - l_0}{2}}))}{4e^{4K_1 T} (t_0 - l_0)} \right) = \exp \left( -\frac{(d_{t_0}(x_0, y_0) - e^{K_1 T} \sqrt{2(t_0 - l_0)})^2}{4e^{4K_1 T} (t_0 - l_0)} \right) \]
\[
\leq \exp \left( -\frac{1/2 \cdot d_{t_0}^2(x_0, y_0) + 2e^{2K_1 T} (t_0 - l_0)}{4e^{4K_1 T} (t_0 - l_0)} \right) = e^{2e^{2K_1 T}} \cdot \exp \left( -\frac{d_{t_0}^2(x_0, y_0)}{8e^{4K_1 T} (t_0 - l_0)} \right).
\]

Therefore, we get
\[
H(y_0, t_0; x_0, l_0) \leq \frac{C_{18} \exp \left( C_{19} T + C_{20} K T + C_{21} \sqrt{K_1 T} \right) \exp \left( -\frac{d_{t_0}^2(x_0, y_0)}{8e^{4K_1 T} (t_0 - l_0)} \right)}{\sqrt{\text{Vol}_{t_0}(B_{t_0}(y_0, \sqrt{\frac{t_0 - l_0}{8}}))} \sqrt{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{8}}))}}.
\]

□

The rest of the proof of Theorem [1.4] follows [6]. We include them here for the purpose of completeness. The following lemma shows an \( L^1 \) bound of the Dirichlet heat kernel.

**Lemma 4.3.** Let \( \Omega \) be a compact manifold with nonempty boundary \( \partial \Omega \), and \( g(t), t \in [0, T) \), a solution to the Ricci flow [1.4] on \( \Omega \). Denote by \( H_{\Omega}(y, t; x, l) \) and \( G_{\Omega}(x, l; y, t) \) the Dirichlet heat kernel for \( \frac{\partial}{\partial t} - \Delta_t \) and \( \frac{\partial}{\partial t} + \Delta_t - R(x, l) \) on \( \Omega \), respectively. Then we have
\[
ev_t^\kappa \sup_{M} R(t) dt \leq \int_{\Omega} H_{\Omega}(y, t; x, l) d\mu_t(y) \leq e^{-\int_t^\kappa \inf_{M} R(t) dt},
\]
and
\[
\int_{\Omega} G_{\Omega}(x, l; y, t) d\mu_t(x) \equiv 1
\]
for any \( x, y \in \text{int}(\Omega) \).

**Proof.** Since
\[
\frac{d}{dt} \int_{\Omega} H_{\Omega}(y, t; x, l) d\mu_t(y) = \int_{\Omega} (\Delta_t H_{\Omega}(y, t; x, l) - R \cdot H_{\Omega}(y, t; x, l)) d\mu_t(y)
\]
\[
= \int_{\partial \Omega} \nu_z(H_{\Omega}(z, t; x, l)) dS - \int_{\Omega} R \cdot H_{\Omega}(y, t; x, l) d\mu_t(y)
\]
\[
= - \int_{\Omega} R \cdot H_{\Omega}(y, t; x, l) d\mu_t(y)
\]
and
\[
d\frac{d}{dt} \int_{\Omega} G_\Omega(x, l; y, t) d\mu_l(x) = \int_{\Omega} -\Delta_{l,x} G_\Omega(x, l; y, t) d\mu_l(x)
= - \int_{\partial\Omega} \nu_z (G_\Omega(z, l; y, t)) dS
= 0,
\]
the lemma follows immediately. \hfill \Box

Since the heat kernel on a complete manifold is the limit of the Dirichlet heat kernels on a family of exhausting open subsets of the manifold, Lemma 4.3 implies that

**Corollary 4.4.** Let \((M^n, g_t), t \in [0, T],\) be a complete solution to the Ricci flow \([1.4].\) Then we have the following estimates for the heat kernel \(H(y, t; x, l)\) and the adjoint heat kernel \(G(x, l; y, t)\):

\[
e^{-\int_t^1 \sup_M R(t) dt} \leq \int_M H(y, t; x, l) d\mu_l(y) \leq e^{-\int_t^1 \inf_M R(t) dt},
\]
and

\[
\int_M G(x, l; y, t) d\mu_l(x) \equiv 1.
\]

From Corollary 4.4 and the mean value inequality, we can also show the following rough \(C^0\) bound of \(H\).

**Lemma 4.5.** Let \((M^n, g(t)), t \in [0, T)\) and \(T < \infty,\) be a complete solution to the Ricci flow. Assume that \(Rc \geq -K_1\) for all time \(t,\) and that \(\Lambda = \int_0^T \sup_M |Rc(t)| dt < \infty.\) Then there exist constants \(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3\) and \(\tilde{C}_4\) such that

\[
H(y, t; x, l) \leq \min \left\{ \tilde{C}_1e^{\tilde{C}_2\Lambda + \tilde{C}_5K_1T + \tilde{C}_4\sqrt{K_1T}} \frac{\Vol_l(B_t(y, \sqrt{\frac{t-1}{8}}))}{\Vol_l(B_t(x, \sqrt{\frac{t-1}{8}}))}, \tilde{C}_4e^{\tilde{C}_3\Lambda + \tilde{C}_5K_1T + \tilde{C}_4\sqrt{K_1T}} \right\}.
\]

**Proof.** In Theorem 2.7 by choosing the parabolic cylinder \(P_{g(t)}(y, t, r_0, -(r_0)^2)\) with \(r_0 = \sqrt{\frac{t-1}{8}},\) we have

\[
H(y, t; x, l) \leq \sup_{P_{g(t)}(y, t, r_0, -(r_0)^2)} H(\cdot, \cdot; x, l)
\leq \frac{C_1e^{C_3\Lambda + C_5K_1T + C_4\sqrt{K_1T}}}{(t - l)\Vol_l(B_t(y, r_0))} \int_{\frac{t-1}{2}}^t \int_{B_t(y, \sqrt{\frac{t-1}{2}})} H(z, s; x, l) d\mu_l(z) ds.
\]

By (4.4), we can see that

\[
\int_{\frac{t-1}{2}}^t \int_{B_t(y, \sqrt{\frac{t-1}{2}})} H(z, s; x, l) d\mu_l(z) ds \leq e^{\Lambda} \int_{\frac{t-1}{2}}^t \int_M H(z, s; x, l) d\mu_s(z) ds
\leq e^{\Lambda} \int_{\frac{t-1}{2}}^t e^{C_5K_1(s-l)} ds
= e^{\Lambda} \frac{e^{C_5K_1(t-l)} - e^{C_5K_1(t-\frac{l-1}{2})}}{C_5K_1}.
\]
Hence,
\[
H(y, t; x, l) \leq \frac{C_1 e^{C_2 A + C_3 K_1 T + C_4 \sqrt{K_1 T}}}{\text{Vol}_t(B(y, \sqrt{\frac{l-t}{5}}))} \cdot \frac{e^{C_5 K_1 (t-l)} - e^{C_6 K_1 (t-l)}}{C_5 K_1 (t-l)} 
\leq \frac{C_6 e^{C_7 A + C_8 K_1 T + C_9 \sqrt{K_1 T}}}{\text{Vol}_t(B(y, \sqrt{\frac{l-t}{5}}))}.
\]

Similarly, using (4.5), we can get
\[
H(y, t; x, l) \leq \frac{\hat{C}_1 e^{\hat{C}_2 A + \hat{C}_3 K_1 T + \hat{C}_4 \sqrt{K_1 T}}}{\text{Vol}_t(B(x, \sqrt{\frac{l-t}{5}}))}.
\]

Now we are ready to prove Theorem 1.4.

\begin{proof}[Proof of Theorem 1.4]
Let \( \sigma = \sqrt{\frac{t_0 - l_0}{\delta}} \) and \( r_0 = d_{t_0}(x_0, y_0) \). By Theorem 4.2, we have
\[
H(y_0, t_0; x_0, l_0) \leq \frac{C_0 e^{C_1 A + C_2 K_1 T + C_3 \sqrt{K_1 T}}}{\text{Vol}_t(B(y_0, \sqrt{\frac{t_0 - l_0}{2}}))} \cdot \frac{\text{Vol}_{t_0}(B(y_0, e^A(\sigma + r_0)))}{\text{Vol}_{t_0}(B(y_0, \sigma))}.
\]

Since \( B_{t_0}(x_0, \sigma) \subset B_{t_0}(y_0, \sigma + r_0) \subset B_{t_0}(y_0, e^A(\sigma + r_0)) \),
\[
\text{Vol}_{t_0}^{-1}(B_{t_0}(y_0, \sigma)) \leq \frac{\text{Vol}_{t_0}^{-1}(B_{t_0}(x_0, \sigma))}{\text{Vol}_{t_0}(B_{t_0}(y_0, \sigma))} \cdot \frac{\text{Vol}_{t_0}(B_{t_0}(y_0, e^A(\sigma + r_0)))}{\text{Vol}_{t_0}(B_{t_0}(y_0, \sigma))} \leq e^{nA} \text{Vol}_{t_0}^{-1}(B_{t_0}(x_0, \sigma)) \cdot \frac{\text{Vol}_{t_0}(B_{t_0}(y_0, e^A(\sigma + r_0)))}{\text{Vol}_{t_0}(B_{t_0}(y_0, \sigma))} \leq e^{nA} \text{Vol}_{t_0}^{-1}(B_{t_0}(x_0, \sigma)) \cdot e^{nA(1 + \frac{r_0}{\sigma})}.
\]

Let \( \hat{C}_1 = \frac{1}{64 e^{nA}} \), \( \hat{C}_2 = C_4 \sqrt{K_1 T} e^A \) and \( \eta = \frac{r_0}{\sigma} \). We have
\[
\exp \left( -\frac{d_{t_0}^2(x_0, y_0)}{8e^{4K_1 T}(t_0 - l_0)} \right) \cdot (1 + \frac{r_0}{\sigma})^n e^{C_4 \sqrt{K_1 T} e^A(1 + \frac{r_0}{\sigma})} = \exp \left( -\hat{C}_1 \eta^2 - \hat{C}_2 \eta \right) (1 + \eta)^n \leq C_5 e^{C_6 e^{C_7 A + C_8 K_1 T}}.
\]

Thus,
\[
H(y_0, t_0; x_0, l_0) \leq \frac{C_9 e^{C_{19} A + C_{13} K_1 T}}{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{2}}))} \cdot \exp \left( -\frac{d_{t_0}^2(x_0, y_0)}{8e^{4K_1 T}(t_0 - l_0)} \right).
\]
\end{proof}
Similarly, one can show that
\[
H(y, t_0; x, l_0) \leq C_1 e^{C_1 e^{C_{15} + C_{16} K_1 T}} \exp \left( - \frac{d^2_{t_0}(x, y)}{8 e^{4 K_1 T (t_0 - l_0)}} \right) \frac{\text{Vol}_{t_0}(B_{t_0}(y, \sqrt{\frac{t_0 - l_0}{8}}))}{\text{Vol}_l(B_l(y, \sqrt{t_0 - l_0}))},
\]
\[\square\]

Remark 4.6. If one assumes \( R_c \geq 0 \) on \([0, T]\), and \( \Lambda = \int_0^T \sup_M |Rc|(t) dt < \infty \), then the upper bound above can be improved to
\[
H(y, t; x, l) \leq C_1 e^{C_2 \Lambda} \exp \left( \frac{d^2_t(x, y)}{8(t - l)} \right) \min \left\{ \frac{1}{\text{Vol}_l(B_l(y, \sqrt{\frac{t - l}{8}}))}, \frac{1}{\text{Vol}_l(B_t(x, \sqrt{\frac{t - l}{8}}))} \right\}.
\]

5. A Gaussian lower bound of \( H(y, t; x, l) \)

In this section, we prove Theorem 1.5 following [3].

Proof of Theorem 1.5. Let \( W \) be a large constant to be determined later. By Theorem 1.4 and (4.4), we have
\[
\int_{B_{t_0}(x_0, \sqrt{W(t_0 - l_0)})} H^2(y, t_0; x_0, l_0) d\mu_{t_0}(y) \geq \frac{1}{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0 - l_0)}))} \left( \int_{B_{t_0}(x_0, \sqrt{W(t_0 - l_0)})} H(y, t_0; x_0, l_0) d\mu_{t_0}(y) \right)^2
\]
\[
= \frac{1}{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0 - l_0)}))} \left( \int_{M} H(y, t_0; x_0, l_0) d\mu_{t_0}(y) \right)^2
\]
\[
\geq \frac{1}{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0 - l_0)}))} \cdot \left( e^{-C_0 \Lambda} - \int_{M - B_{t_0}(x_0, \sqrt{W(t_0 - l_0)})} \tilde{C}_0 \hat{C}_0 \exp \left( - \frac{d^2_{t_0}(x, y)}{8 e^{4 K_1 T (t_0 - l_0)}} \right) \frac{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{\frac{t_0 - l_0}{8}}))}{\text{Vol}_l(B_l(x_0, \sqrt{\frac{t_0 - l_0}{8}}))} d\mu_{t_0}(y) \right)^2,
\]
where \( \tilde{C}_0 = e^{\tilde{C}_1 e^{\tilde{C}_2 \Lambda + \tilde{C}_3 K_1 T}}. \)
Notice that

\[
\int_{M-B_{t_0}(x_0,\sqrt{W(t_0-l_0)})} e^{\exp \left(-\frac{d_{t_0}^2(x_0,y)}{8e^{4K_1T(t_0-l_0)}}\right)} \frac{d\mu_{t_0}(y)}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))}
\]

\[
\leq e^{-\frac{W}{16e^{4K_1T}}} \int_{M-B_{t_0}(x_0,\sqrt{W(t_0-l_0)})} e^{\exp \left(-\frac{d_{t_0}^2(x_0,y)}{16e^{4K_1T}(t_0-l_0)}\right)} \frac{d\mu_{t_0}(y)}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))}
\]

\[
\leq \frac{e^{-\frac{W}{16e^{4K_1T}}}}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))} \int_{\sqrt{W(t_0-l_0)}}^\infty e^{-\frac{\rho^2}{16e^{4K_1T}(t_0-l_0)}} \frac{d\text{Vol}_{t_0}(B_{t_0}(x_0,\rho))}{d\rho}
\]

\[
= \frac{e^{-\frac{W}{16e^{4K_1T}}}}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))} \left( -\int_{\sqrt{W(t_0-l_0)}}^\infty \text{Vol}_{t_0}(B_{t_0}(x_0,\rho)) \cdot \frac{d(e^{-\frac{16e^{4K_1T}\rho^2}{(t_0-l_0)}})}{d\rho} \right) + \text{Vol}_{t_0}(B_{t_0}(x_0,\rho))e^{-\frac{16e^{4K_1T}\rho^2}{(t_0-l_0)}} \left( -\int_{\sqrt{W(t_0-l_0)}}^\infty \right) \frac{d\rho}{d\rho}
\]

\[
\leq e^{-\frac{W}{16e^{4K_1T}}} \int_{\sqrt{W(t_0-l_0)}}^\infty \text{Vol}_{t_0}(B_{t_0}(x_0,\rho)) e^{-\frac{\rho^2}{16e^{4K_1T}(t_0-l_0)}} \frac{\rho}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))} \frac{d\rho}{d\rho}
\]

\[
\leq e^{-\frac{W}{16e^{4K_1T}}} \int_{\sqrt{W(t_0-l_0)}}^\infty \left( \frac{\rho}{\sqrt{t_0-l_0}} \right)^{n-1} e^{(n-1)\sqrt{K_1T}\rho} e^{-\frac{\rho^2}{16e^{4K_1T}(t_0-l_0)}} \frac{\rho}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))} \frac{d\rho}{d\rho}
\]

Set \( \eta = \frac{\rho}{\sqrt{t_0-l_0}} \). If we choose \( W = C_1 e^{C_2\Lambda + C_3K_1T} \) large enough so that

\[
\int_{\sqrt{W}}^\infty \eta^n e^{(n-1)\sqrt{K_1T}\eta} e^{-\frac{\eta^2}{16e^{4K_1T}}} \frac{\eta}{8e^{4K_1T}} d\eta \leq \frac{1}{2C_0C_1} e^{\frac{W}{32e^{4K_1T}}},
\]

and

\[
e^{-\frac{W}{32e^{4K_1T}}} \leq e^{-C_0\Lambda},
\]

then

\[
\int_{M-B_{t_0}(x_0,\sqrt{W(t_0-l_0)})} \tilde{C}_0 \hat{C}_0 e^{\exp \left(-\frac{d_{t_0}^2(x_0,y)}{8e^{4K_1T}(t_0-l_0)}\right)} \frac{d\mu_{t_0}(y)}{\text{Vol}_{t_0}(B_{t_0}(x_0,\sqrt{\frac{t_0-l_0}{8}}))}
\]

\[
\leq \tilde{C}_0 \hat{C}_0 e^{-\frac{W}{16e^{4K_1T}}} \int_{\sqrt{W}}^\infty \eta^n e^{(n-1)\sqrt{K_1T}\eta} e^{-\frac{\eta^2}{32e^{4K_1T}}} \frac{\eta}{8e^{4K_1T}} d\eta
\]

\[
\leq \frac{1}{2} e^{-\frac{W}{32e^{4K_1T}}}.
\]
Thus, we can get
\[
\int_{B_{t_0}(x_0, \sqrt{W(t_0-l_0)})} H^2(y, t_0; x_0, l_0) d\mu_{t_0}(y)
\geq \frac{1}{\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0-l_0)}))} \left( e^{-C_0 \Lambda} - \frac{1}{2} e^{-\frac{16}{3} \sqrt{W(t_0-l_0)}} \right)^2
\geq \frac{1}{4\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0-l_0)}))} e^{-2C_0 \Lambda}.
\]

By Corollary 4.4, there exists a point \( y_1 \in B_{t_0}(x_0, \sqrt{W(t_0-l_0)}) \) such that
\[
H(y_1, t_0; x_0, l_0) \geq \frac{1}{4\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0-l_0)}))} e^{-3C_0 \Lambda}.
\]

From Theorem 3.3 in [26] (see also Theorem 6.5.1 in [25] and Theorem 5.1 in [2]), we know
\[
(5.1) \quad H(y_1, t_0; x_0, l_0) \leq C_3 H^{1+\frac{\delta}{2}}(y_0, t_0; x_0, l_0) K^{\frac{\delta}{2}} e^{2d^2_{t_0}(y_1, y_0)},
\]
where \( \delta \) is any positive number, and \( K = \max_{M \times [\frac{t_0+t_0}{2}, t_0]} \{ H(y, t; x_0, l_0) \} \).

Since for \( t \in [\frac{t_0+t_0}{2}, t_0] \), we have
\[
\text{Vol}_t(B_t(x_0, \sqrt{\frac{t-l_0}{8}})) \geq \text{Vol}_t(B_t(x_0, \sqrt{\frac{t_0-l_0}{16}})) 
\geq e^{-n\Lambda} \text{Vol}_{t_0}(B_{t_0}(x_0, e^{-\Lambda} \sqrt{\frac{t_0-l_0}{16}})),
\]
according to Lemma 4.5, we have for any \( (y, t) \in M \times [\frac{t_0+t_0}{2}, t_0] \),
\[
H(y, t; x_0, l_0) \leq \frac{\tilde{C}_0 e^{c_1 A} + \tilde{C}_2 K_{1T} + \tilde{C}_3 \sqrt{K_{1T}}}{\text{Vol}_t(B_t(x_0, \sqrt{\frac{t-l_0}{8}}))} \leq \frac{\tilde{C}_0 e^{C_4 A + \tilde{C}_2 K_{1T} + \tilde{C}_3 \sqrt{K_{1T}}}}{\text{Vol}_{t_0}(B_{t_0}(x_0, e^{-\Lambda} \sqrt{\frac{t_0-l_0}{16}}))},
\]
i.e.,
\[
K \leq \frac{\tilde{C}_0 e^{C_4 A + \tilde{C}_2 K_{1T} + \tilde{C}_3 \sqrt{K_{1T}}}}{\text{Vol}_{t_0}(B_{t_0}(x_0, e^{-\Lambda} \sqrt{\frac{t_0-l_0}{16}}))}.
\]

In (5.1), letting \( \delta = 1 \) and noticing that
\[
d^2_{t_0}(y_1, y_0) \leq 2(d^2_{t_0}(y_1, x_0) + d^2_{t_0}(x_0, y_0)) \leq 2W(t_0 - l_0) + 2d^2_{t_0}(x_0, y_0),
\]
we get
\[
H(y_0, t_0; x_0, l_0) \geq \frac{C_5 e^{-C_4 A - C_7 K_{1T} - C_8 \sqrt{K_{1T}}} e^{-2W} \text{Vol}_{t_0}(B_{t_0}(x_0, e^{-\Lambda} \sqrt{\frac{t_0-l_0}{16}})) e^{-4d^2_{t_0}(y_1, y_0)}}{\text{Vol}_{t_0}^2(B_{t_0}(x_0, \sqrt{W(t_0-l_0)}))} e^{-\frac{4d^2_{t_0}(y_1, y_0)}{t_0-t_0}}.
\]

By Lemma 4.1, we have
\[
\text{Vol}_{t_0}(B_{t_0}(x_0, \sqrt{W(t_0-l_0)})) \leq C_9 e^{n\Lambda} W^{\tilde{c}_2} e^{C_{10} \sqrt{W} K_{1T}} \text{Vol}_{t_0}(B_{t_0}(x_0, e^{-\Lambda} \sqrt{\frac{t_0-l_0}{16}}))
\]
and
\[ \text{Vol}_t(B_t(x_0, \sqrt{W(t_0 - l_0)}) \leq C_{11} W^{\frac{3}{2}} \exp(C_{12} \sqrt{WK_1 T}) \text{Vol}_t(B_t(x_0, \sqrt{\frac{t_0 - l_0}{8}})). \]

Therefore,
\[
H(y_0, t_0; x_0, l_0) \geq C_{11} \exp(C_{14} \Lambda + C_{16} K_1 T) \exp(-\frac{4d_t^2(y_0, x_0)}{(t_0 - l_0)}).
\]

\[\square\]

**Remark 5.1.** If \( Rc \geq 0 \) on \([0, T)\) and \( \Lambda = \int_0^T \sup_M |Rc(t)| dt < \infty \), then we have
\[
H(y, t; x, l) \geq C_1 \exp(-C_2 \lambda \exp(-\frac{4d_t^2(x, y)}{(t - l)})) \text{Vol}_t(B_t(x, \sqrt{\frac{t - l}{8}})).
\]

**Acknowledgements**

The author would like to thank Professor Huai-Dong Cao for many valuable suggestions, and for his constant support and encouragement. The author also wants to thank Professors Qing Ding, Jixiang Fu, Jiaxing Hong, Hong-Quan Li, Jun Li, Jiaping Wang, Quanshui Wu, Guoyi Xu and Weiping Zhang for helpful conversations and encouragement, and Professors Luen-Fai Tam and Qi S. Zhang for their interest in this work. In addition, the author expresses appreciation to the Shanghai Center for Mathematical Sciences, where this work was carried out, for its hospitality and support. This research was partially supported by the China Postdoctoral Science Foundation Grant No. 2013M531105.

**References**

[1] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. 73 (1967), 890–896. MR0217444 (36 #534)

[2] Xiaodong Cao and Richard S. Hamilton, *Differential Harnack estimates for time-dependent heat equations with potentials*, Geom. Funct. Anal. 19 (2009), no. 4, 989–1000, DOI 10.1007/s00039-009-0024-4. MR2570311 (2010j:53124)

[3] Xiaodong Cao and Qi S. Zhang, *The conjugate heat equation and ancient solutions of the Ricci flow*, Adv. Math. 228 (2011), no. 5, 2891–2919, DOI 10.1016/j.aim.2011.07.022. MR2838064 (2012k:53127)

[4] E. A. Carlen, S. Kusuoka, and D. W. Stroock, *Upper bounds for symmetric Markov transition functions* (English, with French summary), Ann. Inst. H. Poincaré Probab. Statist. 23 (1987), no. 2, suppl., 245–287. MR898496 (88i:35066)

[5] Gilles Carron, *Inégalités isopérimétriques de Faber-Krahn et conséquences* (French, with English and French summaries), Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 205–232. MR1427759 (97m:58198)

[6] Albert Chau, Luen-Fai Tam, and Chengjie Yu, *Pseudolocality for the Ricci flow and applications*, Canad. J. Math. 63 (2011), no. 1, 55–85, DOI 10.4153/CJM-2010-076-2. MR2779131 (2012g:53132)

[7] Siu Yuen Cheng, Peter Li, and Shing Tung Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. 103 (1981), no. 5, 1021–1063, DOI 10.2307/2374257. MR630777 (83c:58083)

[8] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni, *The Ricci flow: techniques and applications. Part III. Geometric-analytic aspects*, Mathematical Surveys and Monographs, vol. 163, American Mathematical Society, Providence, RI, 2010. MR2604955 (2011g:53142)
[9] E. B. Davies, Explicit constants for Gaussian upper bounds on heat kernels, Amer. J. Math. 109 (1987), no. 2, 319–333, DOI 10.2307/2374577. MR882426 (88g:58174)
[10] E. B. Davies, Heat kernels and spectral theory, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. MR990239 (90e:35123)
[11] E. B. Davies, Heat kernel bounds, conservation of probability and the Feller property, J. Anal. Math. 58 (1992), 99–119, DOI 10.1007/BF02790359. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. MR1226938 (94e:58136)
[12] Alexander Grigor’yan, Heat kernel upper bounds on a complete non-compact manifold, Rev. Mat. Iberoamericana 10 (1994), no. 2, 395–452, DOI 10.4171/RMI/157. MR1286481 (96b:58107)
[13] Alexander Grigor’yan, Gaussian upper bounds for the heat kernel on arbitrary manifolds, J. Differential Geom. 45 (1997), no. 1, 33–52. MR1443330 (98g:58167)
[14] Christine M. Guenther, The fundamental solution on manifolds with time-dependent metrics, J. Geom. Anal. 12 (2002), no. 3, 425–436, DOI 10.1007/BF02922048. MR1901749 (2003a:58034)
[15] Emmanuel Hebey, Nonlinear analysis on manifolds: Sobolev spaces and inequalities, Courant Lecture Notes in Mathematics, vol. 5, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. MR1688256 (2000e:58011)
[16] Hans-Joachim Hein and Aaron Naber, New logarithmic Sobolev inequalities and an $\epsilon$-regularity theorem for the Ricci flow, Comm. Pure Appl. Math. 67 (2014), no. 9, 1543–1561, DOI 10.1002/cpa.21474. MR3245102
[17] Peter Li, Luen-Fai Tam, and Jiaping Wang, Sharp bounds for the Green’s function and the heat kernel, Math. Res. Lett. 4 (1997), no. 4, 589–602, DOI 10.4310/MRL.1997.v4.n4.a13. MR1470428 (98j:58110)
[18] Peter Li and Shing-Tung Yau, On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153–201, DOI 10.1007/BF02399203. MR834612 (87f:58156)
[19] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958), 931–954. MR0100158 (20 #6592)
[20] Lei Ni, Ricci flow and nonnegativity of sectional curvature, Math. Res. Lett. 11 (2004), no. 5-6, 883–904, DOI 10.4310/MRL.2004.v11.n6.a12. MR2106247 (2005m:53123)
[21] N. Th. Varopoulos, Hardy-Littlewood theory for semigroups, J. Funct. Anal. 63 (1985), no. 2, 240–260, DOI 10.1016/0022-1236(85)90087-4. MR803094 (87a:31011)
[22] Guoyi Xu, An equation linking $W$-entropy with reduced volume, arXiv:1211.6354, to appear in J. Reine Angew. Math.
[23] Qi S. Zhang, Heat kernel bounds, ancient $\kappa$ solutions and the Poincaré conjecture, J. Funct. Anal. 258 (2010), no. 4, 1225–1246, DOI 10.1016/j.jfa.2009.11.002. MR2565838 (2011e:53110)
[24] Qi S. Zhang, Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture, CRC Press, Boca Raton, FL, 2011. MR2676527 (2011m:53127)
[25] Qi S. Zhang, Some gradient estimates for the heat equation on domains and for an equation by Perelman, Int. Math. Res. Not., posted on 2006, Art. ID 92314, 39, DOI 10.1155/IMRN/2006/92314. MR2250008 (2007f:35116)

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