Order and quantum phase transitions in the cuprate superconductors

Subir Sachdev

Department of Physics, Yale University, P.O. Box 208120, New Haven CT 06520-8120

(Dated: October 28, 2002)

It is now widely accepted that the cuprate superconductors are characterized by the same long-range order as that present in the Bardeen-Cooper-Schrieffer (BCS) theory: that associated with the condensation of Cooper pairs. The author argues that many physical properties of the cuprates require interplay with additional order parameters associated with a proximate Mott insulator. A classification of Mott insulators in two dimensions is proposed. Experimental evidence so far shows that the class appropriate to the cuprates has collinear spin correlations, bond order, and confinement of neutral, spin $S = 1/2$ excitations. Proximity to second-order quantum phase transitions associated with these orders, and with the pairing order of BCS, has led to systematic predictions for many physical properties. In this context the author reviews the results of recent neutron scattering, fluxoid detection, nuclear magnetic resonance, and scanning tunnelling microscopy experiments.

Contents

1. INTRODUCTION 1
2. BCS THEORY 2
3. MOTT INSULATORS 3
   A. Magnetically ordered states 4
      1. Collinear spins, $N_1 \times N_2 = 1$ 5
      2. Non-collinear spins, $N_1 \times N_2 \neq 0$ 5
   B. Paramagnetic states 6
      1. Bond-ordered states: confined spinons 7
      2. Topological order: free spinons 8
   C. Connections between magnetically ordered and paramagnetic states 9
      1. Collinear spins and bond order 10
      2. Non-collinear spins and topological order 10
4. ORDER IN STATES PROXIMATE TO MOTT INSULATORS 11
   A. Tuning order by means of a magnetic field 12
   B. Detecting topological order 14
   C. Non-magnetic impurities 15
   D. STM studies of the vortex lattice 15
5. A PHASE DIAGRAM WITH COLLINER SPINS, BOND ORDER, AND SUPERCONDUCTIVITY 16
6. OUTLOOK 18
Acknowledgments 19
References 19

I. INTRODUCTION

The discovery of high temperature superconductivity in the cuprate series of compounds by Bednorz and Müller [1986] has strongly influenced the development of condensed matter physics. It stimulated a great deal of experimental work on the synthesis and characterization of a variety of related intermetallic compounds. It also reinvigorated theoretical study of electronic systems with strong correlations. Technological applications of these materials have also appeared, and could become more widespread.

Prior to this discovery, it was widely assumed that all known superconductors, or superfluids of neutral fermions such as $^3$He, were described by the theory of Bardeen, Cooper and Schrieffer (BCS) [Bardeen et al. 1957]. Certainly, the quantitative successes of BCS theory in describing an impressive range of phenomena in the lower temperature superconductors make it one of the most successful physical theories ever proposed. Soon after the discovery of the high temperature superconductors, it became clear that many of their properties, and especially those at temperatures ($T$) above the superconducting critical temperature ($T_c$), could not be quantitatively described by the BCS theory. Overcoming this failure has been an important motivation for theoretical work in the past decade.

One of the purposes of this article is to present an updated assessment of the applicability of the BCS theory to the cuprate superconductors. We will restrict our attention to physics at very low temperature associated with the nature of the ground state and its elementary excitations. This will allow us to focus on sharp, qualitative distinctions. In particular, we will avoid the regime of temperatures above $T_c$, where it is at least possible that any failure of the BCS theory is a symptom of our inability to make accurate quantitative predictions in a strong coupling regime, rather than our having missed a qualitatively new type of order. Also, while this article will present a unified view of the important physics of the cuprate superconductors, it is not a comprehensive review, and it does not attempt to reflect the state of the field by representing the variety of viewpoints that have been taken elsewhere in the literature.

The primary assertions of this article are as follows. At the lowest energy scales, the longest length scales, in
the absence of strong external perturbations, and at ‘optimal’ carrier concentrations and above, all experimental indications are that the cuprate superconductors can indeed be described in the framework of the BCS theory: the theory correctly describes the primary order parameter of the superconducting state, and the quantum numbers of its elementary excitations. However, many experiments at lower doping concentrations and at shorter length scales require one or more additional order parameters, either conventional (i.e. associated with the breaking of a symmetry of the Hamiltonian) or ‘topological’ (see Section II.B.2 below). These order parameters are best understood and classified in terms of the physics of ‘Mott insulators,” a topic which will be discussed in greater detail below. The importance of the Mott insulator was stressed by Anderson [1987]. Our understanding of Mott insulators, and of their classification into categories with distinct physical properties has advanced greatly in the last decade, and a sharper question of experimental relevance is: which class of Mott insulators has its ‘order’ present in the cuprate superconductors? As we shall discuss below, the evidence so far supports a class quite distinct from that implied in Anderson’s proposal [Sachdev and Read, 1991].

How can the postulated additional order parameters be detected experimentally? In the simplest case, there could be long-range correlations in the new order in the ground state: this is apparently the case in La$_{2-x}$Sr$_x$CuO$_4$ at low carrier concentrations, and we will describe recent experiments which have studied the interplay between the new order and superconductivity. However, the more common situation is that there are no long range correlations in any additional order parameter, but the ‘fluctuating’ order is nevertheless important in interpreting certain experiments. A powerful theoretical approach for obtaining semi-quantitative predictions in this regime of fluctuating order is provided by the theory of quantum phase transitions: imagine that we are free to tune parameters so that ultimately the new order does acquire long range correlations somewhere in a theoretical phase diagram. A quantum critical point will separate the phases with and without long-range order: identify this critical point and expand away from it towards the phase with fluctuating order, which is the regime of experimental interest [Chubukov et al., 1994a; Sachdev and Ye, 1992]; see Fig 1. An illuminating discussion of fluctuating order near quantum critical points (along with a thorough analysis of many recent experiments which has some overlap with our discussion here) has been provided recently by Kivelson et al. [2002].

An especially important class of experiments involve perturbations which destroy the superconducting order of the BCS state locally (on the scale a few atomic spacings). Under such situations the theory outlined above predicts that the order of the Mott insulator is revealed in a halo surrounding the perturbation, and can, in principle, be directly detected in experiments. Perturbations of this type are Zn impurities substituting on the Cu sites, and

![Figure 1](image)

**FIG. 1** Our theoretical strategy for describing the influence of a new order parameter in a BCS superconductor. Here $g$ is some convenient coupling constant in the Hamiltonian, and we imagine that the superconductor of physical interest is a BCS superconductor with $g > g_c$. Theoretically, it is useful to imagine that we can tune $g$ to a value smaller than $g_c$, where there are long-range correlations in a new order parameter. Having identified and understood the quantum phase transition at $g = g_c$, we can expand away from it back towards the BCS superconductor (as indicated by the thick arrow) to understand the influence of quantum fluctuations of the new order parameter. This approach is most effective when the transition at $g = g_c$ is second order, and this will usually be assumed in our discussion. Note that the horizontal axis need not be the concentration of mobile carriers, and it may well be that the superconductor of physical interest does not exhibit the $g < g_c$ state at any carrier concentration.

the vortices induced by an applied magnetic field. We shall discuss their physics below.

To set the stage for confrontation between theory and experiment, we review some essential features of the BCS theory in Section III, and introduce key concepts and order parameters in the theory of Mott insulators in Section IV. We will combine these considerations in our discussion of doped Mott insulators in Section V, which will also include a survey of some experiments. A theoretical phase diagram which encapsulates much of the physics discussed here appears in Section IV, while Section V concludes with a discussion on possible directions for future work.

**II. BCS THEORY**

In BCS theory, superconductivity arises as an instability of a metallic Fermi liquid. The latter state is an adiabatic continuation of the free electron model of a metal, in which all single particle states, labeled by the Bloch crystal momentum $\mathbf{k}$, inside the $\mathbf{k}$-space Fermi surface are occupied by electrons, while those outside remain empty. With $c_{\mathbf{k}\sigma}^\dagger$ the creation operator for an electron with momentum $\mathbf{k}$ and spin projection $\sigma = \uparrow, \downarrow$, a reasonable description of the Fermi liquid is provided by the free electron Hamiltonian

$$H_0 = \sum_{\mathbf{k}\sigma} (\varepsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma},$$

where $\varepsilon_{\mathbf{k}}$ is the energy-momentum dispersion of the single-particle Bloch states and $\mu$ is the chemical potential; the locus of points with $\varepsilon_{\mathbf{k}} = \mu$ defines the Fermi
surface. Changes in electron occupation numbers near the Fermi surface allow low energy processes which are responsible for the conduction properties of metals.

BCS realized that an arbitrarily weak attractive interaction between the electrons would induce the electrons near the Fermi surface to lower their energy by binding into pairs (known as Cooper pairs) \(|\langle c_{k\uparrow} c_{-k\downarrow} + c_{-k\uparrow} c_{k\downarrow} \rangle \propto \Delta_0 \equiv \Delta_0 (\cos k_x - \cos k_y) . \) (2)

The functional form of (2) in spin and \( \vec{k} \)-space carries information on the internal wavefunction of the two electrons forming a Cooper pair: we have displayed a singlet pair with a \( d \)-wave orbital wavefunction on the square lattice, as is believed to be the case in the cuprates \cite{Scalapino, 1995; Tsuei and Kirtley, 2000}.

Along with (3) as the key characterization of the ground state, BCS theory also predicts the elementary excitations. These can be separated into two types: those associated with the motion of center of mass, \( \vec{R} \), of the Cooper pairs, and those in which a pair is broken. The center of mass motion (or superflow) of the Cooper pair is associated with a slow variation in the phase of the pairing condensate \( \Delta_0 \rightarrow \Delta_0 e^{i\phi(\vec{R})} \); the superconducting ground state has \( \phi(\vec{R}) = \text{a constant independent of } \vec{R} \) (and thus long-range order in this phase variable), while a slow variation leads to an excitation with superflow. A vortex excitation is one in which this phase has a non-trivial winding, while the superflow has a non-zero circulation:

\[ \int_C d\vec{R} \cdot \nabla \phi = 2\pi n_v, \] (3)

where \( n_v \) is the integer-valued vorticity, and \( C \) is a contour enclosing the vortex core. A standard gauge invariance argument shows that each such vortex must carry a total magnetic flux of \( n_v \hbar c/(2e) \), where the \( 2e \) in the denominator represents the quantum of charge carried by the “bosons” in the condensate. Excitations which break Cooper pairs consist of multiple \( S = 1/2 \) fermionic quasiparticles with dispersion

\[ E_{\vec{k}} = \sqrt{(\epsilon_{\vec{k}} - \mu)^2 + |\Delta_{\vec{k}}|^2}, \] (4)

and these reduce to the particle and hole excitations around the Fermi surface when \( \Delta_0 \rightarrow 0 \).

All indications from experiments so far are that the cuprate superconductors do have a ground state characterized by (3), and the elementary excitations listed above. However, BCS theory does make numerous other predictions which have been successfully and thoroughly tested in the low temperature superconductors. In particular, an important prediction is that if an external perturbation succeeds in destroying superconductivity by sending \( |\Delta_0| \rightarrow 0 \), then the parent Fermi surface, which was swallowed up by the Cooper instability, would reappear. This prediction is quite different from the perspective discussed earlier, in which we argued for the appearance of a halo of order linked to the Mott insulator.

### III. MOTT INSULATORS

The Bloch theory of metals also specified conditions under which crystalline materials can be insulating: if, after filling the lowest energy bands with electrons, all bands are either fully occupied or completely empty, then there is no Fermi surface, and the system is an insulator. However, some materials are insulators even though these conditions are not satisfied, and one-electron theory would predict partially filled bands: these are Mott insulators. Correlations in the motion of the electrons induced by their Coulomb interactions are crucial in preventing metallic conduction.

One of the parent compounds of the cuprate superconductors, La\(_2\)CuO\(_4\), is a simple example of a Mott insulator. The lowest energy electronic excitations in this material reside on the Cu \( 3d_{x^2-y^2} \) orbitals, which are located on the vertices of a square lattice. The crystal has a layered structure of stacked square lattices, with only a weak amplitude for electron hopping between successive layers. (We shall neglect the interlayer coupling and focus on the physics of a single square lattice in the remainder of this article.) After accounting for the ionization states of the other ions in La\(_2\)CuO\(_4\), there turns out to be exactly one electron per unit cell available to occupy the Cu \( 3d_{x^2-y^2} \) band. With two available spin states, this
spins prefer opposite orientations. Classifying quantum $S = 1$ describes the spin dynamics then takes the form $J_{ij}$ where the antiferromagnetically aligned spins at the top and bottom in $(b)$ can access a high energy intermediate state (shown in the middle of $(b)$) and so undergo an exchange process.

band can accommodate two electrons per unit cell, and so is half-filled, and should have a metallic Fermi surface. Nevertheless, La$_2$CuO$_4$ is a very good insulator. The reason for this insulating behavior can be understood quite easily from a simple classical picture of electron motion in the presence of the Coulomb interactions. Classically, the ground state consists of one electron localized on each of the $3d_{x^2-y^2}$ orbitals: this state minimizes the repulsive Coulomb interaction energy. Any other state would have at least one orbital with two electrons, and one with no electrons: there is a large energetic penalty for placing two electrons so close to each other, and this prohibits motion of electrons across the lattice: hence the Mott insulator.

Let us now look at the quantum theory of the Mott insulator more carefully. While charge fluctuations on each site are expensive, it appears that the spin of the electron can be rotated freely and independently on each site. However, in the quantum theory virtual charge fluctuations do occur, and these lead to residual “super-exchange” interactions between the spins (Anderson, 1959). We represent the spin on the Cu site $j$ by the $S_{1/2}$ spin operator $S_j$; the effective Hamiltonian that describes the spin dynamics then takes the form

$$H = \sum_{i<j} J_{ij} S_i \cdot S_j + \ldots$$

where the $J_{ij}$ are short-ranged exchange couplings and the ellipses represent possible multiple spin couplings, all of which preserve full SU(2) spin rotation invariance. Because the Pauli principle completely prohibits charge fluctuations between two sites if they have parallel spin electrons, while they are only suppressed by the Coulomb repulsion if they have opposite spins (see Fig. 2), we expect an antiferromagnetic sign $J_{ij} > 0$, so that nearby spins prefer opposite orientations. Classifying quantum ground states of models like $(a)$ is a problem of considerable complexity, and has been the focus of extensive research in the last decade. We summarize the current understanding below.

In keeping with the spirit of this article, we characterize ground states of $H$ by a number of distinct order parameters. We only discuss states below which have long-range correlation in a single order parameter; in most cases, co-existence of multiple order parameters is also allowed (Balents et al., 1999; Senthil and Fisher, 2000), but we will ignore this complexity here. Our list of order parameters is not exhaustive, and we restrict our attention to the most plausible candidates (in the author’s opinion) for short-range $J_{ij}$.

Although our discussion below will refer mainly to Mott insulators, we will also mention ground states of non-insulating systems with mobile charge carriers: the order parameters we use to characterize Mott insulators can be applied more generally to other systems, and this will done in more detail in Section IV.

A. Magnetically ordered states

Such states are obtained by examining $H$ for the case of large spin $S$ on each site: in this limit, the $S_j$ can be taken as classical $c$-numbers, and these take a definite non-zero value in the ground state. More precisely, the SU(2) spin rotation symmetry of $H$ is spontaneously broken in the ground state by the non-zero values of $\langle S_j \rangle$, which are chosen to minimize the energy of $H$. We consider only states without a net ferromagnetic moment ($\sum_j \langle S_j \rangle = 0$), and this is expected because $J_{ij} > 0$. The pattern of non-zero $\langle S_j \rangle$ can survive down to $S = 1/2$, and this is often found to be the case, although quantum fluctuations do significantly reduce the magnitude of $\langle S_j \rangle$.

An especially important class of magnetically ordered states is characterized by a single ordering wavevector

\[ 1 \] An order that has been much discussed in the literature, which we do not discuss here, is that associated with the staggered flux state (Affleck and Marston, 1988), and the related algebraic spin liquid (Rantner and Wen, 2000; Wen, 2002). The low energy theory of these states includes a gapless $U(1)$ gauge field, and it has been argued (Sachdev and Park, 2000) that instantons, which are allowed because the underlying lattice scale theory has a compact gauge symmetry, always proliferate and render these unstable towards confining states (of the type discussed in Section III.B.1) in two spatial dimensions. However, states with a gapless $U(1)$ gauge field are allowed in the three spatial dimensions (Motrunich and Senthil, 2002; Wen, 2002).

\[ 2 \] Magnetically ordered states with the values of $\langle S_j \rangle$ non-coplanar (i.e. three dimensional spin textures) are not included in this simple classification. Their physical properties are expected to be similar to those of the non-collinear case discussed in Section III.A.2 in that quantum fluctuations of such a state lead to a paramagnet with topological order. However, this paramagnet is likely to have also a broken time-reversal symmetry.

![FIG. 2](image-url) Motion of the two ferromagnetically aligned spins in (a) is prohibited by the Pauli principle. In contrast, the antiferromagnetically aligned spins at the top and bottom in (b) can access a high energy intermediate state (shown in the middle of (b)) and so undergo an exchange process.
FIG. 3 States with collinear magnetic order on a square lattice with unit lattice spacing and wavevectors (a) $\vec{K} = (\pi, \pi)$, (b) and (c) $\vec{K} = (3\pi/4, \pi)$. Shown are the values of $\langle S_j \rangle$ on the square lattice sites $r_j$. A single unit cell is shown for the latter two states; they are crystallographically inequivalent and have different reflection planes: in (b) the reflection planes are on certain sites, while in (c) they are at the midpoint between two sites.

$\vec{K}$:

$$\langle S_j \rangle = N_1 \cos (\vec{K} \cdot \vec{r}_j) + N_2 \sin (\vec{K} \cdot \vec{r}_j)$$  \hspace{1cm} (6)

where $\vec{r}_j$ is the spatial location of the site $j$, and $N_{1,2}$ are two fixed vectors in spin space. We list two key subcategories of magnetically ordered Mott insulators which obey (6):

1. **Collinear spins**, $N_1 \times N_2 = 0$

   In this situation, the mean values of the spins in (6) on all sites $j$ are either parallel or anti-parallel to each other. The undoped insulator La$_2$CuO$_4$ is of this type$^3$ with $\vec{K} = (\pi, \pi)$; see Fig 3a. Insulating states with static holes appeared in Zaanen and Gunnarsson (1989), Machida (1989), Schulz (1989), and Poilblanc and Rice (1989) with ordering wavevectors which move continuously away from $(\pi, \pi)$. Another important illustrative example is the case $\vec{K} = (3\pi/4, \pi)$. Such a wavevector could be preferred in a Mott insulator by longer range $J_{ij}$ in (5), but in practice it is found in a non-insulating state obtained by doping La$_2$CuO$_4$ with a suitable density of mobile carriers (Kivelson and Emery, 1996; Martin et al., 2000; Scibold et al., 1998; Tranquada et al., 1995; Wakimoto et al., 2001, 1994; White and Scalapino, 1998a, 1999) — we can crudely view the mobile carriers as having induced an effective longer range exchange between the spins. Two examples of states with this value of $\vec{K}$ are shown in Fig 3b a site-centered state with $N_2 = 0$ in Fig 3b, and a bond-centered state with $N_2 = (\sqrt{2} - 1)N_1$ in Fig 3c. (The states have planes of reflection symmetry located on sites and the centers of bonds respectively, and so are crystallographically inequivalent. Also, these inequivalent classes are only present if the wavevector $K$ is commensurate with the underlying lattice.)

2. **Non-collinear spins**, $N_1 \times N_2 \neq 0$

   Now the spin expectation values in (6) lie in a plane in spin space, rather than along a single direction. For simplicity, we will only consider the simplest, and most common, case of non-collinearly ordered state, in which

   $$N_1 \cdot N_2 = 0 \ ; \ N_1^2 = N_2^2 \neq 0,$$  \hspace{1cm} (7)

   and then the values of $\langle S_j \rangle$ map out a circular spiral (Shraiman and Siggia, 1988, 1989), as illustrated in Fig 4.

B. **Paramagnetic states**

   The other major class of states comprises those having

   $$\langle S_j \rangle = 0,$$  \hspace{1cm} (8)

   and the ground state is a total spin singlet.$^4$ Loosely speaking each spin $S_j$ finds a partner, say $S_{j'}$, and the two pair up to form a singlet valence bond

   $$\frac{1}{\sqrt{2}} (| \uparrow_j \downarrow_{j'} \rangle - | \downarrow_j \uparrow_{j'} \rangle).$$  \hspace{1cm} (9)

---

$^3$ For this special value of $\vec{K}$ on the square lattice, and with the origin of $r$ co-ordinates on a lattice site, (6) is actually independent of $N_2$.

$^4$ In a finite system with an even number of spins, the magnetically ordered ground state also has total spin zero. However, to obtain a state which breaks spin rotation symmetry as in (6), it is necessary to mix in a large number of nearly degenerate states which carry non-zero total spin. The paramagnetic does not have such higher spin states available at low energy in a finite system.
FIG. 5 A crude variational wavefunction of a bond-ordered paramagnetic state. The true ground state will have fluctuations of the singlet bonds about the configuration shown here, but its pattern of lattice symmetry breaking will be retained. In other words, each bond represented by an ellipse above will have the same value of $\langle Q_a(\vec{r}_j) \rangle$, and this value will be distinct from that associated with all other bonds. This pattern of symmetry breaking is represented more abstractly in Fig. 6.

Of course, there are many other choices for the partner of spin $S_j$, and in the Feynman path integral picture we imagine that the pairing configuration fluctuates in quantum imaginary time; this is the ‘resonating valence bond’ picture of Pauling (1941), Fazekas and Anderson (1974), and Anderson (1987). However, there is a great of structure and information contained in the manner in which this fluctuation takes place, and research (Chubukov et al., 1994b,c; Read and Sachdev, 1991; Sachdev and Read, 1991) delineating this structure has led to the following classification of paramagnetic Mott insulators.

1. Bond-ordered states: confined spinons

This class of states can be easily understood by the caricature of its wavefunction shown in Fig. 5: here each spin has chosen its valence bond partner in a regular manner, so that there is a long-range ‘crystalline’ order in the arrangement of valence bonds. This ordering of bonds clearly breaks the square lattice space group symmetries under which the Hamiltonian is invariant. Of course, the actual wavefunction for any realistic Hamiltonian will have fluctuations in its valence bond configuration, but the pattern of lattice symmetry breaking implied by Fig. 5 will be retained in the true bond-ordered ground state. We can make this precise by examining observables which are insensitive to the electron spin direction: the simplest such observables we can construct from the low energy degrees of freedom of the Mott insulator are bond variables, which are a measure of the exchange energy between two spins:

$$Q_a(\vec{r}_j) = S_j \cdot S_{j+a}.$$  \hspace{1cm} (10)

Here $a$ denotes displacement by the spatial vector $\vec{r}_a$, and so the spins above are at the spatial locations $\vec{r}_j$ and $\vec{r}_j + \vec{r}_a$. We will mainly consider bond order with $\vec{r}_a \neq 0$, but note that the on-site variable $Q_0(\vec{r}_j)$, with $\vec{r}_a = 0$, is a measure of the charge density on site $\vec{r}_j$, and so this special case of (10) measures the “charge order.”

The state introduced in Fig. 5 can be characterized by the pattern of values of $\langle Q_a(\vec{r}_j) \rangle$ with $\vec{r}_a$ a nearest neighbor vector, in a number of paramagnetic states with $\langle S_j \rangle = 0$. For each state, the values of $\langle Q_a(\vec{r}_j) \rangle$ are equal on bonds represented by the same type of line, and unequal otherwise. The number of distinct values of $\langle Q_a(\vec{r}_j) \rangle$ are (a) 3, (b) 2, (c) 5, and (d) 5. The unit cells of the ground states have sizes (a) $2 \times 1$, (b) $2 \times 2$, (c) $4 \times 1$, and (d) $4 \times 4$.

FIG. 6 Pattern of the bond variables $\langle Q_a(\vec{r}_j) \rangle$, for $\vec{r}_a$ a nearest-neighbor vector, in a number of paramagnetic states with $\langle S_j \rangle = 0$. For each state, the values of $\langle Q_a(\vec{r}_j) \rangle$ are equal on bonds represented by the same type of line, and unequal otherwise. The number of distinct values of $\langle Q_a(\vec{r}_j) \rangle$ are (a) 3, (b) 2, (c) 5, and (d) 5. The unit cells of the ground states have sizes (a) $2 \times 1$, (b) $2 \times 2$, (c) $4 \times 1$, and (d) $4 \times 4$.

$^5$ By (10), $Q_0(\vec{r}_j) = S_j^2$. A site with a spin has $S_j^2 = 3/4$, while a site with a hole has $S_j^2 = 0$, and we assume that doubly occupied sites are very rare. Thus $S_j^2$, and hence $Q_0(\vec{r}_j)$ is seen to be linearly related to the charge density on site $j$. 

(a) 

(b) 

(c) 

(d)
that spontaneous bond order invariably appears in the
of the paramagnetic state. These computations show
"quantum dimer models" (Rokhsar and Kivelson, 1988)
(Fradkin and Kivelson, 1990; Read and Sachdev, 1990) of
a very different starting point—from duality mappings
markably, very closely related theories also appear from
semiclassical theory of quantum fluctuations near a mag-
states in Fig 6, as such modulations are not suppressed
associated with

Note, however, that the modulation in bond orders asso-
full Hilbert space of the model with charge fluctuations.

The physical mechanism inducing bond-ordered states
such as those in Fig 3 is illustrated in the cartoon pic-
tures in Fig 3. More detailed computations rely on a
semiclassical theory of quantum fluctuations near a mag-
etically ordered state (Read and Sachdev, 1990). Remark-
ably, very closely related theories also appear from a
very different starting point—from duality mappings
"quantum dimer models" (Rokhsar and Kivelson, 1988)
of the paramagnetic state. These computations show
that spontaneous bond order invariably appears in the
ground state in systems with collinear spin correlations in
two spatial dimensions (Read and Sachdev, 1990, Sachdev
and Park 2002). We will have more to say about this
connection between bond and collinear spin order in
Section II.C.1.

We also mention here the “nematic” states of Kivel-
son et al. (1998) in the doped Mott insulator. These
can also be characterized by the bond order variables in
(14). The symmetry of translations with respect to \( \vec{r}_j \)
is not broken in such states, but the values of \( \langle Q_a(\vec{r}_j) \rangle \)
for symmetry-related values of \( \vec{r}_a \) become unequal. For
example, \( \langle Q_a(\vec{r}_j) \rangle \) has distinct values for \( \vec{r}_a = (1, 0) \)
and \( (0, 1) \). Such states also appear in certain insulating anti-
ferromagnets (Read and Sachdev, 1989a,b, 1990).

It also interesting to note here that the bond order vari-
ables \( Q_a(\vec{r}_j) \) also have spatial modulations in some of the
magnetically ordered states considered in Section II.A
(Zachar et al. 1998). It is clear from (14) that any bro-
ked lattice symmetry in the spin-rotation invariant quan-
ty \( (S_j) \cdot (S_{j+a}) \) will generate a corresponding broken
symmetry in the bond variable \( \langle Q_a(\vec{r}_j) \rangle \). Evaluating
the former using (14) we can deduce the following: (i) the
\( \vec{K} = (\pi, \pi) \) state in Fig 3 and the spiral state in Fig 3
have \( \langle Q_a(\vec{r}_j) \rangle \) independent of \( \vec{r}_j \), and hence no bond or-
der; (ii) the bond-centered magnetically ordered state in
Fig 3c has precisely the same pattern of bond order as the
paramagnetic state in Fig 3a; (iii) the site-centered
magnetically ordered state in Fig 3b has bond order with
\( \langle Q_a(\vec{r}_j) \rangle \) \( \vec{r}_j \)-dependent, but with a pattern distinct from
any shown here—this pattern of bond order is in principle
also allowed for paramagnetic states, but has so far not
been found to be stable in various studies. Finally, note that in (ii) and (iii) the period of the bond order (four) is half that of the spin modulation (eight)—this is easily seen to be a general relationship following from the correspondence \( \langle Q_a(r_j) \rangle \sim \langle S_j \cdot (S_{j+a}) + \ldots \rangle \) in magnetically ordered states, which with (5) implies an \( r_j \)-dependent modulation of the bond order with wavevector \( 2\hat{K} \). It is worth reiterating here that this last relationship should not be taken to imply that there are no modulations in \( \langle Q_a(r_j) \rangle \) when \( \langle S_j \rangle = 0 \): there can indeed be bond modulations in a paramagnet, as discussed in the other paragraphs of this subsection, and as is already clear from the simple wavefunction in Fig 8—these will be important later for physical applications.

We continue our exposition of paramagnetic bond-ordered states by describing excitations with non-zero spin. These can be understood simply by the analog of a meson consisting of a quark and anti-quark as its quantum numbers and observable characteristics. A similar reasoning can be used to understand the influence of static spinless impurities i.e. the consequences of removing a \( S = 1/2 \) spin from a fixed site \( j \) in the crystal. Experimentally, this can be conveniently done by substituting a spinless \( \text{Zn}^{2+} \) ion in place of an \( S = 1/2 \) Cu ion. The main physical effect can be understood from the cartoon wavefunction in Fig 9: it is convenient to imagine placing 2 Zn impurities, and then moving them apart out to infinity to deduce the physics in the vicinity of a single impurity. As in our discussion above for spinons, note that there will initially be a defect line of valence bonds connecting the two Zn impurities, but it will eventually pay to annihilate this defect line by creating two spinons and binding each to a Zn impurity. Thus each Zn impurity confines a free \( S = 1/2 \) spinon in its vicinity, and this can be detected in experiments \[ Finkelson \text{ et al., 1990]}.\]

2. Topological order: free spinons

This type of paramagnet is the “resonating valence bond” (RVB) state \[ \text{Anderson, 1987; Baskaran and Anderson, 1988; Fazekas and Anderson, 1974; Kivelson et al., 1987; Moessner and Sondhi, 2001; Pauling, 1949} \] in which the singlet pairings fluctuate in a liquid-like configuration,\(^7\) in contrast to the crystalline arrangement in which the singlet pairings fluctuate in a liquid-like configuration.

\(^6\) In principle the Zn impurity could also bind an electron (with or without a spinon) but this is suppressed by the charge gap in a Mott insulator. Later, in Section IV.C when we consider Zn impurities in \( d \)-wave superconductors, a related phenomenon appears in the form of the Kondo effect.

\(^7\) In recent years, Anderson (2002) has extended the RVB concept to apply to doped Mott insulators at temperatures above \( T_C \). This extension is not in consonance with the classification of the present article. The topological order discussed in this subsection can only be defined at \( T = 0 \) in two spatial dimensions. The description at \( T > T_C \) requires solution of a problem of quantitative difficulty, and with incoherent excitations, but without sharp distinctions between different states.
FIG. 10 Topological order in a resonating valence bond state. Shown is one component of the wavefunction, with a particular pairing of the spins into local singlets: the actual wavefunction is a superposition over a very large number of such pairing configurations. The number of valence bonds cutting the dashed line is an invariant modulo 2 over these pairing configurations, as shown by the following simple argument. Any rearrangement of the valence bonds can be reached by repeated application of an elementary rearrangement between 4 spins: \((1, 2)(3, 4) \rightarrow (1, 3)(2, 4)\) (here \((i, j)\) denotes a singlet bond between \(S_i\) and \(S_j\)). So it is sufficient to check this conservation law for 4 spins: this is done easily by explicitly considering all different possibilities among spins 1, 2, 3, 4 residing to the left/right of the dashed line. If the system has periodic boundary conditions along the horizontal direction, then this conservation law is violated, but only by rearrangements associated with loops which circumnavigate the systems; these only occur with a probability which becomes exponentially small as the circumference of the system increases.

Despite the apparent ‘disorder’ in the valence bond configuration in the ground state, there is actually a subtle topological order parameter which characterizes this type of Mott insulator (Bonesteel, 1989; Kivelson, 1989; Read and Chakraborty, 1989; Read and Sachdev, 1991; Roikhar and Kivelson, 1988; Thouless, 1987; Wen, 1991), and which plays an important role in determining its excitation spectrum. The reader can see this in the context of the cartoon picture shown in Fig. 9. Count the number of singlet valence bonds cutting the dashed line in this figure: this number will clearly depend upon the particular valence bond configuration chosen from the many present in the ground state, and one such is shown in Fig. 10. However, as argued in the figure caption, the number of bonds cutting the dashed line is conserved modulo 2 between any two configurations which differ only local rearrangements of valence bonds: the quantum number associated with this conservation is the topological order in the ground state.

A convenient and powerful description of this topological order is provided by an effective model of the singlet sector formulated as \(Z_2\) gauge theory (Read and Sachdev, 1991; Sachdev and Read, 1991; Senthil and Fisher, 2000; Wen, 1991).⑧ We postpone a self-contained derivation of this \(Z_2\) gauge theory to Section III.C.2 (see especially Fig 11): here, we show that such a gauge theory has similar topological properties. In a system with periodic boundary conditions (with the topology of a torus), the \(Z_2\) gauge theory has different sectors depending upon whether there is a \(Z_2\) flux piercing any of the holes of the torus (following Senthil and Fisher, 2000), this \(Z_2\) flux is now commonly referred to as a ‘vison’). In the valence bond picture discussed in the previous paragraph, a vison changes the sign associated with every valence bond cutting a line traversing the system in the vison direction (the dashed line in Fig 9); in other words, the even and odd valence bond sectors mentioned above now have their relative signs in the wavefunction changed.

In addition to appearing in the holes of the torus, the vison can also appear as a singlet excitation within the bulk (Kivelson, 1989; Read and Chakraborty, 1989; Read and Sachdev, 1991; Senthil and Fisher, 2000). It is now a vortex excitation in the \(Z_2\) gauge theory, that requires a finite energy for its creation. We will see below in Section III.C.2 that there is an alternative, and physically revealing, interpretation of this vortex excitation in terms of the order parameters used earlier to characterize the magnetically ordered state, and that the topological order is intimately connected to the vison energy gap.

Finally, we can describe the spin-carrying excitations of this topologically ordered state using the crude, but instructive, methods used in Section III.B.1. As there is no particular bond order associated with the ground state, the spinons have no confining force between them, and are perfectly free to travel throughout the system as independent neutral \(S = 1/2\) quasiparticles. Similarly, there is no confining force between \(Z_n\) impurities and the spinons, and so it is not required that an \(S = 1/2\) moment be present near each \(Z_n\) impurity (Fendley et al., 2002; Sachdev and Voit, 2000).

C. Connections between magnetically ordered and paramagnetic states

A central ingredient in the reasoning of this article is the claim that there is an intimate connection between the magnetically ordered states in Section III.A and a corresponding paramagnetic state in Section III.B. In particular, the collinear states of Section III.A.2 are linked to the bond-ordered states in Section III.B.1 while ⑧ Readers not familiar with \(Z_2\) gauge theories may understand them by analogy to electromagnetism. The latter is a \(U(1)\) gauge theory in which the physics is invariant under the transformation \(z \rightarrow e^{i\eta} z\), \(A_\mu \rightarrow A_\mu - \partial_\mu \phi\) where \(z\) is some matter field, \(A_\mu\) is a gauge field, and \(\phi\) is an arbitrary spacetime-dependent field which generates the gauge transformation. Similarly, in a \(Z_2\) gauge theory, matter fields transform as \(z \rightarrow \eta z\), where \(\eta\) is a spacetime-dependent field which generates the gauge transformation, but is now allowed to take only the values \(\eta = \pm 1\). The \(Z_2\) gauge field \(\sigma_{ij}\) resides on the links of a lattice, and transforms as \(\sigma_{ij} \rightarrow \eta_i \sigma_{ij} \eta_j\).
the non-collinear states of Section III.A.2 are linked to
the topologically ordered states of Section III.B.2. The
reader will find a more technical discussion of the follow-
ing issues in a companion review article by the author
Sachdev, 2003).

Before describing these links in the following subsec-
tions, we discuss the meaning of the “connectedness” of
two states. The magnetically ordered phases are char-
2. Non-collinear spins and topological order

The first argument of Section II.C.1, when generalized
to non-collinear spins, leads quite simply to a surpris-
ingly subtle characterization of the associated paramag-
netic phase. Recall from Section II.A.2 that the non-collinear mag-
netic phase is characterized by two orthogonal, and equal
length, vectors \( \mathbf{N}_{1,2} \). It takes 6 real numbers to specify
two vectors, but the 2 constraints in (7) reduce the num-
ber of real parameters required to specify the ordered
state to 4. There is a useful parameterization (Chubukov
et al., 1994b) which explicitly solves the constraints (7)
by expressing \( \mathbf{N}_{1,2} \) in terms of 2 complex numbers \( z_{\uparrow}, z_{\downarrow} \)
which are equivalent to the required 4 real numbers:

\[
\mathbf{N}_1 + i \mathbf{N}_2 = \left( \begin{array}{c} z_{\uparrow}^2 - z_{\downarrow}^2 \\ i(z_{\uparrow}^2 + z_{\downarrow}^2) \end{array} \right) \quad (11)
\]

It can also be checked from (11) that \((z_{\uparrow}, z_{\downarrow})\) transforms
like an \( S = 1/2 \) spinor under spin rotations. So instead
of dealing with a constrained theory of \( \mathbf{N}_{1,2} \) fluctuations,
we can express the theory in terms of the complex spinor
\((z_{\uparrow}, z_{\downarrow})\), which is free of constraints. There is one cru-
"
vortex, we traverse a path in the order parameter space from \((z_1, z_1)\) to \((-z_1, -z_1)\), as shown in Fig. 11. As argued in the caption, a fundamental point is that such vortices can be defined as sensible excitations even in the paramagnetic phase, where \((z_1, z_1)\) is strongly fluctuating in quantum imaginary time: upon encircling the vortex, the path in order parameter space will also strongly fluctuate, but will always connect polar opposite points on \(S_3\). We identify these paramagnetic vortices with the visons of Section III.B, thus firmly establishing a connection between non-collinear magnetic order and the topologically ordered paramagnet.

Finally, we wish to consider a \(Z_2\) gauge theory in which magnetic order is lost continuously (Chubukov et al. 1994). Read and Sachdev 1991, and we obtain a paramagnetic phase in which the spinor \((z_1, z_1)\) fluctuates about 0. A pedagogical description of such a theory was provided by Lammert et al. (1993) and Lammert et al. (1995); in an entirely different context; they considered thermal phase transitions in a nematic liquid crystal, with order parameter \(S_2/Z_2\) in three spatial dimensions. However, their results can be transposed to the quantum phase transition in two spatial and one imaginary time dimension of interest here, with the primary change being in the order parameter space from \(S_2/Z_2\) to \(S_3/Z_2\): this change is only expected to modify uninteresting numerical factors in the phase diagram, as the global topologies of the two spaces are the same. As shown by Lammert et al. (1993) and Lammert et al. (1995), the magnetically ordered state (with states labeled by points in \(S_{2,3}/Z_2\)) does indeed undergo a continuous phase transition to a paramagnetic state in which spin rotation invariance is restored and a topological order is present. This topological order arises because the \(Z_2\) visons discussed in Fig. 11 do not proliferate in the paramagnetic state; in this sense, the topological order here is similar to the topological order in the low temperature phase of the classical XY model in 2 dimensions, where point vortices are suppressed below the Kosterlitz-Thouless transition (Thouless, 1973). We can also connect the nonproliferation of visons to our discussion in Section III.B.2, where we noted that there was an excitation gap towards the creation of \(Z_2\) visons (Senthil and Fisher 2000). Indeed, an explicit connection between the topological order being discussed here and the topological order noted in the caption to Fig 11 was established by Read and Sachdev (1991), Sachdev and Read 1991, and Chubukov et al. 1994.

Moreover, without the proliferation of visons in the ground state, the \((z_1, z_1)\) configurations can be defined as single-valued configurations throughout the sample. Normal-mode oscillations of \((z_1, z_1)\) about zero can now be identified as a neutral \(S = 1/2\) particle. This is clearly related to the spinon excitation of Section III.B.2: this is our final confirmation of the intimate connection between the non-collinear magnetic states of Section III.A.3 and the topologically ordered states of Section III.B.3.

This is a good point to mention, in passing, recent neutron scattering evidence for a RVB state in Cs$_2$CuCl$_4$ (Coldea et al. 2001): the measurements also show non-collinear spin correlations, consistent with the connections being drawn here.

IV. ORDER IN STATES PROXIMATE TO MOTT INSULATORS

We are now ready to discuss the central issue of order parameters characterizing the cuprate superconductors. These superconductors are obtained by introducing mobile charge carriers into the Mott insulator of the square lattice of Cu ions that was discussed at the beginning of Section III. The charge carriers are introduced by substitutional doping. For instance, in the compound La$_{2-x}$Sr$_x$CuO$_4$, each trivalent La$^{3+}$ ion replaced by a divalent Sr$^{2+}$ ion causes one hole to appear in the Mott insulator of Cu ions; the concentration of these holes is \(\delta\) per square lattice site.

For large enough \(\delta\), theory and experiment both indicate that such a doped Mott insulator is a \(d\)-wave superconductor characterized by the pairing amplitude \(\Delta\). The reader can gain an intuitive (but quite crude and incomplete) understanding of this by the similarity between the real-space, short-range pairing in \(\Delta\) and the momentum-space, long-range pairing in \(\Delta\). The undoped Mott insulator already has electrons paired into singlet valence bonds, as in \(\Delta\), but the repulsive Coulomb energy of the Mott insulator prevents motion of the charge associated with this pair of electrons. It should be clear from our discussion in Section III.B.3 that this singlet pairing is complete in the paramagnetic Mott...
insulators, but we can also expect a partial pairing in the magnetically ordered states. Upon introducing holes into the Mott insulator, it becomes possible to move charges around without any additional Coulomb energy cost, and so the static valence bond pairs in (1) transmute into the mobile Cooper pairs in (2); the condensation of these pairs leads to superconductivity. Note that this discussion is concerned with the nature of the ground state wavefunction, and we are not implying a “mechanism” for the formation of Cooper pairs.

The discussion in the previous sections has laid the groundwork for a more precise characterization of this superconductor using the correlations of various order parameters, and of their interplay with each other. The proximity of the Mott insulator indicates that the Cooper pairs should be considered descendants of the real-space, short range pairs in (1), and this clearly demands that all the magnetic, bond and topological order parameters discussed in Section II remain viable candidates for the doped Mott insulator. The motion of charge carriers allows for additional order parameters, and the most important of these is clearly the superconducting order of the BCS state noted below (2) in Section I. In principle, it is also possible to obtain new order parameters which are characteristic of neither the BCS state nor a Mott insulator, but we such order parameters shall not be discussed here (discussions of one such order may be found in Hsu \textit{et al.} (1991), Wen and Lee (1996), Lee and Sha (2003), Chakravarty \textit{et al.} (2001), and Schollwöck \textit{et al.} (2003)).

The arsenal of order parameters associated with Mott insulators and the BCS state permits a very wide variety of possible phases of doped Mott insulators, and of quantum phase transitions between them. Further progress requires experimental guidance, but we claim that valuable input is also obtained from the theoretical connections sketched in Section III.C.

The simplest line of reasoning (Sachdev and Read, 1991) uses the fact that the undoped Mott insulator La$_2$CuO$_4$ has collinear magnetic order as sketched in Fig 3a. The arguments above and those in Section II.C then imply that the doped Mott insulator should be characterized by the collinear magnetic order of Section II.A, the bond order of Section III.B, along with the phase order of BCS theory. This still permits a large variety of phase diagrams, and some of these were explored in Sachdev and Read (1991), Vojta and Sachdev (1999), Vojta \textit{et al.} (2000), and Vojta (2002), with detailed results on the evolution of bond order and superconductivity with increasing doping. However, this reasoning excludes phases associated with the non-collinear magnetic order of Section II.A and the topological order of Section III.B.

Some support for this line of reasoning came from the breakthrough experiments of Tranquada \textit{et al.} (1995), Tranquada \textit{et al.} (1996), and Tranquada \textit{et al.} (1997) on La$_{2-y}$Ni$_y$Sr$_2$O$_4$ for hole concentrations near $\delta = 1/8$: they observed static, collinear magnetic order near the wavevectors $\vec{K} = (3\pi/4, \pi)$ shown in Figs 3b,c, which coexisted microscopically$^9$ with superconductivity for most $\delta$. They also observed modulations in the bond order $Q_0(\vec{r})$ (Eqn (13)) at the expected wavevector, $2\vec{K}$. The experimentalists interpreted their observations in terms of modulations of the site charge density—proportional to $Q_0(\vec{r})$—but the existing data actually do not discriminate between the different possible values of $\vec{r}$. As we noted earlier in Section III.B, the physical considerations of the present article suggest that the modulation may be stronger with $\vec{r} \neq 0$. (The existing data also cannot distinguish between the magnetic orders in Fig 3b (site-centered) and Fig 3c (bond-centered), or between the bond orders in Fig 3 (orthorhombic symmetry) and Fig 3 (tetragonal symmetry).) We also mention here the different physical considerations in the early theoretical work of Zaanen and Gunnarsson (1989), Machida (1989), Schulz (1989), and Poilblanc and Rice (1989) which led to insulating states with collinear magnetic order with wavevector $\vec{K} \neq (\pi, \pi)$ driven by a large site-charge density modulation in the domain walls of holes.

The following subsections discuss a number of recent experiments which explore the interplay between the order parameters we have introduced here. We argue that all of these experiments support the proposal that the cuprate superconductors are characterized by interplay between the collinear magnetic order of Section II.A, the bond order of Section III.B, (these are connected as discussed in Section III.C), and the superconducting order of BCS theory.

A. Tuning order by means of a magnetic field

In Section I, we identified a valuable theoretical tool for the study of systems with multiple order parameters: use a coupling $g$ to tune the relative weights of static or fluctuating order parameter correlations in the ground state. Is such a coupling available experimentally? One choice is the hole concentration, $\delta$, and we can assume here that $g$ increases monotonically with $\delta$. However, $\delta$ is often difficult to vary continuously, and it may be that sampling the phase diagram along this one-dimensional axis may not reveal the full range of physically relevant behavior. A second tuning parameter will be clearly valuable; here we argue that, under suitable conditions, this is provided by a magnetic field applied perpendicular to the two-dimensional layers.

Consider the case where both phases in Fig 1 are superconducting; the phase with $g < g_c$ then has co-existence of long-range order in superconductivity and a secondary order parameter. We also restrict attention to the case

\[ g < g_c \]
Evidence that the primary effect of a magnetic field is a spatially uniform modification of the magnetic order has appeared in recent muon spin resonance experiments (Sonier et al. 2003; Tomura 2003). where the transition at \( g = g_c\) is second order (related results apply also to first order transitions, but we do not discuss them here). Imposing a magnetic field, \( H\), on these states will induce an inhomogeneous state, consisting of a lattice of vortices surrounded by halos of superflow (we assume here that \( H > H_{c1}\), the lower critical field for flux penetration). In principle, we now need to study the secondary order parameter in this inhomogeneous background, which can be a problem of some complexity. However, it was argued by Demler et al. (2001) and Zhang et al. (2002) that the problem simplifies considerably near the phase boundary at \( g = g_c\). Because of the diverging correlation length associated with the secondary order parameter, we need only look at the spatially-averaged energy associated with the relevant order parameters. We know from the standard theory of the vortex lattice in a BCS superconductor (Parks, 1969) that the energy density of the superconducting order increases by the fraction \( 1969\) that the energy density of the superconducting order can also be characterized by a change in an effective coupling \( g_{\text{eff}}(H)\) (Chubukov et al., 1994a; Sachdev, 2000; Zhang, 1997). The phase boundary is determined by setting \( g_{\text{eff}}(H) = g_c\), which leads to a phase boundary at a critical field \( H \sim (g - g_c)/\ln(1/(g - g_c))\). We assume that \( g\) is a monotonically increasing function of \( \delta\). The collinear magnetic order of Figs 3 and 4 and \( \delta\) is the secondary order parameter investigated in recent neutron scattering experiments in doped \( \text{La}_2\text{CuO}_4\): the observations of Lake et al. (2001) are along the arrow A, and those of Katano et al. (2001), Lake et al. (2002), Khaykovich et al. (2002), and Khaykovich et al. (2003) are along the arrow B. The STM experiments of Hoffman et al. (2002), Hoffman et al. (2002b), Howald et al. (2002), Howald et al. (2003) are along arrow C, and will be discussed in Section V.D. The relationships (12) and (13) can be combined with Fig 1 to produce a phase diagram in the \((g, H)\) plane. This is shown in Fig 12. Notice that the phase boundary comes into the \( g = g_c\), \( H = 0\) point with vanishing slope. This implies that a relatively small field is needed in the \( g > g_c\) region to tune a BCS superconductor across a quantum phase transition into a state with long-range correlations in the secondary order parameter. There are also some interesting modifications to Fig 12 in the fully three-dimensional model which accounts for the coupling between adjacent \( \text{CuO}_2\) layers; these are discussed by Kivelson et al. (2002). A number of neutron scattering studies of the physics of Fig 12 in doped \( \text{La}_2\text{CuO}_4\) have recently appeared. The secondary order parameter here is the collinear magnetic order of Figs 3 and 4, which is also observed in \( \text{La}_{2-x}\text{Nd}_x\text{Sr}_2\text{O}_4\) as discussed above. Earlier, a series of beautiful experiments by Nakamoto et al. (1995), Lee et al. (1994), and Nakamoto et al. (2001) established that \( \text{La}_{2-x}\text{Sr}_2\text{CuO}_4\) has long-range, collinear magnetic order co-existing with superconductivity for a range of \( \delta\) values above \( \delta = 0.055\). Moreover, the anomalous frequency and temperature dependence of the dynamic spin structure factor (Chubukov et al., 1994a; Sachdev and Ye, 1992) in neutron scattering experiments by Aeppli et al. (1997) gave strong indications of a second-order quantum phase transition near \( \delta \approx 0.14\) at which the magnetic order vanished. We identify this transition with the point \( g = g_c, H = 0\) in Fig 12. Recent studies have explored the region with \( H > 0\): Lake et al. (2001) observed a softening of a collective spin excitation mode at \( \delta = 0.163\) in the presence of an applied magnetic field. We interpret this as a consequence of the low \( H\) approach to the phase boundary in Fig 12 in the \( g > g_c\) region, as indicated by the arrow labeled A. Notice that the field was not large.
enough to cross the phase boundary.

A separate set of experiments have examined the $H$ dependence of the static magnetic moment in the superconductor with $g < g_c$ in La$_2$−$_4$Sr$_3$CuO$_4$ (Katano et al., 2000; Lake et al., 2002) and La$_2$CuO$_4$+$\delta$ (Khaykovich et al., 2003, 2002), along the arrow indicated by $B$ in Fig 12. The theoretical prediction (Demler et al., 2001; Zhang et al., 2002) for these experiments is a simple consequence of (13) and (14). Let $I(H, \delta)$ be the observed intensity of the static magnetic moment associated with the order in Figs 3b,c at a field $H$ and doping $\delta$. If we assume that the dominant effect of the field can be absorbed by replacing $\delta$ by the effective $\delta_{\text{eff}}(H)$, we can write

$$I(H, \delta) \approx I(H = 0, \delta_{\text{eff}}(H)) \approx I(H = 0, \delta) + D \left( \frac{H}{H_c}\right) \ln \left( \frac{H_c^2}{H}\right),$$

(14)

where in the second expression we have used (13) and expanded in powers of the second argument of $I$. Reasoning as in the text below (13) for $C$, we use the experimental fact that a decrease in $\delta$ leads to an increase in the magnetic order, and hence $D > 0$. The results of recent experiments (Khaykovich et al., 2003, 2002; Lake et al., 2002) are in good agreement with the prediction (14), with a reasonable value for $D$ obtained by fitting (14) to the experimental data.

B. Detecting topological order

The magnetic and bond orders break simple symmetries of the Hamiltonian, and, at principle, these can be detected by measurement of the appropriate two-point correlation function in a scattering experiment. The topological order of Sections III.B.2 and III.C.2 is a far more subtle characterization of the electron wavefunction, and can only be observed indirectly through its consequences for the low energy excitations. We review here the rationale behind some recent experimental searches (Bonn et al., 2001; Wynn et al., 2001) for topological order.

The searches relied on a peculiar property of a superconductor proximate to a Mott insulator with topological order: there is a fundamental distinction in the internal structure of vortices in the superconducting order, specified by (3), which depends on whether the integer $n_v$ is even or odd. This difference was noted (Nagaosa and Lee, 1992; Sachdev, 1992) in the context of a simple mean-field theory of a superconductor near an insulating spin gap state. However, the significance and interpretation of the mean-field result, and in particular its connection with topological order, did not become apparent until the far-reaching work of Senthil and Fisher (2000), Senthil and Fisher (2001a), and Senthil and Fisher (2001b). The arguments behind the dependence on the parity of $n_v$ are subtle, and only an outline will be sketched here—the reader is referred to Senthil and Fisher (2001a) and Senthil and Fisher (2001b) for a complete exposition. Although the superconducting order of BCS theory in (2) and the topological order of the Mott insulator are quite distinct entities, there is an important connection between them in the superconducting state: each vortex with $n_v$ odd in (3) has a vison attached to it. The vison gap in the proximate Mott insulator then increases the energy required to create $n_v$ odd vortices, while this extra energy is not required for $n_v$ even.

The connection between $n_v$ odd vortices and visons is most transparent for the case where the spinons in the Mott insulator obey fermionic statistics. We considered bosonic spinons $z_\sigma$ in Section II.C.2, but they can transmute into fermions by binding with a vison (Demler et al., 2002; Kivelson, 1989; Read and Chakraborty, 1989): we represent the fermionic spinon by $f_{j\sigma}$. In the doped Mott insulator, each electron annihilation operator, $c_{j\sigma}$, must create at least one neutral $S = 1/2$ spinon excitation, along with a charge $e$ hole (Kivelson et al., 1987), and we can represent this schematically by the operator relation

$$c_{j\sigma} = b^\dagger_{j\uparrow} f_{j\sigma},$$

(15)

where $b^\dagger$ creates a bosonic spinless hole. In this picture of the doped Mott insulator, the presence of superconductivity as in (3) requires both the condensation of the $b_{j\uparrow}$, along with the condensation of “Cooper pairs” of the spinons $f_{j\sigma}$. We can deduce this relationship from (3) and (13) which imply, schematically

$$\Delta_0 = \Delta f b^2,$$

(16)

where we have ignored spatial dependence associated with the internal wavefunction of the Cooper pair (hence there are no site subscripts $j$ in (16)), and $\Delta f \sim \langle f_{j\uparrow}, f_{j\downarrow}\rangle$ is the spinon pairing amplitude. From (16) we see if the phase of $b_{j\uparrow}$ winds by $2\pi$ upon encircling some defect site, then phase of $\Delta_0$ will wind by $4\pi$, and this corresponds to a vortex in the superconducting order with $n_v = 2$ in (3). Indeed, the only way (16) can lead to an elementary vortex with $n_v = 1$ is if the phase of the spinon pair amplitude, $\Delta_f$, winds by $2\pi$ upon encircling the vortex: the latter is another description of a vison (Senthil and Fisher, 2000). This argument is easily extended to show that every odd $n_v$ vortex must be associated with at least an elementary vortex in the phase of $\Delta_f$, thus establishing our claimed connection.

Sufficiently close to the Mott insulator, and near a second-order superconductor-insulator transition, the energy required to create a vison raises the energy of $n_v = 1$ vortices, and the lowest energy vortex lattice state in an applied magnetic field turns out to have vortices with flux $hc/e$, which is twice the elementary flux (Sachdev, 1992). This should be easily detectable, but such searches have not been successful so far (Wynn et al., 2001).

More recently Senthil and Fisher (2001a) have proposed an ingenious test for the presence of visons, also relying on the binding of a vison to a vortex with flux
undertaken (Bonn et al. 2001). An experimental test for this “flux memory effect” has also been undertaken (Bonn et al. 2001), but no such effect has yet been found.

So despite some innovative and valuable experimental tests, no topological order has been detected so far in the cuprate superconductors.

C. Non magnetic impurities

We noted in Section II.B.1 that one of the key consequences of the confinement of spinons in the bond-ordered paramagnet was that each non-magnetic impurity would bind a free $S = 1/2$ moment. In contrast, in the topologically ordered RVB states of Section II.B.2, such a moment is not generically expected, and it is more likely that the “liquid” of valence bonds would readjust itself to screen away the offending impurity without releasing any free spins.

Moving to the doped Mott insulator, we then expect no free $S = 1/2$ moment for the topologically ordered case. The remaining discussion here is for the confining case; in this situation the $S = 1/2$ moment may well survive over a finite range of doping, beyond that required for the onset of superconductivity. Eventually, at large enough hole concentrations, the low energy fermionic excitations in the $d$-wave superconductor will screen the moment (by the Kondo effect) at the lowest temperatures. However, unlike the case of a Fermi liquid, the linearly vanishing density of fermionic states at the Fermi level implies that the Kondo temperature can be strictly zero for a finite range of parameters (Gonzalez-Buxtorf and Ingersent 1998; Vojta and Bulla 2002; Withoff and Frahm 1990). So we expect each non-magnetic impurity to create a free $S = 1/2$ moment that survives down to $T = 0$ for a finite range of doping in a $d$-wave superconductor proximate to a confining Mott insulator. The collinear magnetic or bond order in the latter insulator may also survive into the superconducting state, but there is no fundamental reason for the disappearance of these long-range orders (bulk quantum phase transitions) to coincide with the zero temperature quenching of the moment (an impurity quantum phase transition).

A very large number of experimental studies of non-magnetic Zn and Li impurities have been carried out. Early on, in electron paramagnetic resonance experiments (Finkelstein et al. 1990) observed the trapping of an $S = 1/2$ moment near a Zn impurity above the superconducting critical temperature, and also noted the implication of their observations for the confinement of spinons, in the spirit of our discussion above. Subsequent specific heat and nuclear magnetic resonance experiments (Alloul et al. 1991; Bobroff et al. 2001; Julien et al. 2001; Sisson et al. 2001) have also explored low temperatures in the superconducting state, and find evidence of spin moments, which are eventually quenched by the Kondo effect in the large doping regime. Especially notable is the recent nuclear magnetic resonance evidence (Bobroff et al. 2001) for a transition from a $T = 0$ free moment state at low doping, to a Kondo quenched state at high doping.

We interpret these results as strong evidence for the presence of an $S = 1/2$ moment near non-magnetic impurities in the lightly doped cuprates. We have also argued here, and elsewhere (Sachdev and Vojta 2000), that the physics of this moment formation is most naturally understood in terms of the physics of a proximate Mott insulator with spinon confinement.

The creation of a free magnetic moment (with a local magnetic susceptibility which diverges as $1/T$ as $T \to 0$) near a single impurity implies that the cuprate superconductors are exceptionally sensitive to disorder. Other defects, such as vacancies, dislocations, and grain boundaries, which are invariably present even in the best crystals, should also have similar strong effects. We speculate that this tendency to produce free moments (and local spin order) which will be induced in their vicinity which is responsible for the frequent recent observation of magnetic moments in the lightly doped cuprates (Sidis et al. 2001; Sonier et al. 2001).

D. STM studies of the vortex lattice

Section IV.A discussed the tuning of collinear magnetic order by means of an applied magnetic field, and its detection in neutron scattering experiments in doped $La_2CuO_4$. This naturally raises the question of whether it may also be possible to detect the bond order of Section III.B.2 somewhere in the phase diagram of Fig 12. Clearly the state with co-existing collinear magnetic and superconducting order (explored by experiments along the arrow B) should, by the arguments of Section II.C.1, also have co-existing bond order. However, more interesting is the possibility that the BCS superconductor itself has local regions of bond order for $H \neq 0$ (Park and Sachdev 2001). As we have argued, increasing $H$ increases the weight of the Mott insulator order parameter correlations in the superconducting ground state. The appearance of static magnetic order requires breaking of spin rotation invariance (in the plane perpendicular to the applied field), and this cannot happen until there is a bulk phase transition indicated by the phase boundary in Fig 12. In contrast, bond order only breaks translational symmetry, but this is already broken by the vortex lattice induced by a non-zero $H$. The small vortex
cores can pin the translational degree of freedom of the bond order, and a halo of static bond order should appear around each vortex core (Demler et al., 2001; Park and Sachdev, 2001; Polkovnikov et al., 2002a, 2002b; Zhang et al., 2002). Notice that this bond order has appeared in the state which has only superconducting order at $H = 0$, and so should be visible along the arrow labelled C in Fig. 12. Recall also our discussion in Section II.B.1 that site charge order is a special case of bond order (with $\vec{r}_{\alpha} = 0$ in the bond order parameter $Q_{\alpha}(\vec{r})$).

Many other proposals have also been made for additional order parameters within the vortex core. The earliest of these involved dynamic antiferromagnetism (Nagaosa and Lee, 1992; Sachdev, 1992), and were discussed in Section IV.B in the context of topological order. Others (Andersen et al., 2003; Arovas et al., 1997; Chen et al., 2002; Chen and Ting, 2002; Franz et al., 2002; Ghosal et al., 2002; Ichioka and Machida, 2002; Zhang, 1997; Zhu et al., 2002) involve static magnetism within each vortex core in the superconductor. This appears unlikely from the perspective of the physics of Fig 12 in which static magnetism only appears after there is a cooperative bulk transition to long-range magnetic order, in the region above the phase boundary; below the phase boundary there are no static “spins in vortices,” but there is bond order as discussed above (Park and Sachdev, 2001; Zhang et al., 2002). (Static spins do appear in the three space dimensional model with spin anisotropy and inter-planar couplings considered in Kivelson et al. (2002b).) A separate proposal involving staggered current loops in the vortex core (Kishine et al., 2001; Lee and Sha, 2003; Lee and Wen, 2001) has also been made.

Nanoscale studies looking for signals of bond order along the arrow C in Fig 12 would clearly be helpful. Scanning tunnelling microscopy (STM) is the ideal tool, but requires atomically clean surfaces of the cuprate crystal. The detection of collinear magnetic order in doped La$_2$CuO$_4$ makes such materials ideal candidates for bond order, but they have not been amenable to STM studies so far. Crystals of Bi$_2$Sr$_2$CaCu$_2$O$_8+\delta$ have been the focus of numerous STM studies, but there is little indication of magnetic order in neutron scattering studies of this superconductor. Nevertheless, by the reasoning in Fig 12, and using the reasonable hypothesis that a common picture of competing superconducting, bond, and collinear magnetic order applies to all the cuprates, it is plausible that static bond order should appear in Bi$_2$Sr$_2$CaCu$_2$O$_8+\delta$ for large enough $H$ along the arrow C in Fig 12.

A number of atomic resolution STM studies of Bi$_2$Sr$_2$CaCu$_2$O$_8+\delta$ surfaces have appeared recently (Hoffman et al., 2002a, 2002b; Howald et al., 2003, 2002). Hoffman et al. (2002a) observed a clear signal of modulations in the local density of electronic states, with a period of 4 lattice spacings, in a halo around each vortex core. There was no corresponding modulation in the surface topography, implying there is little modulation in the charge density. However, a bond order modulation, such as those in Figs 13 and 14, could naturally lead to the required modulation in the local density of states. Other studies (Hoffman et al., 2002a, 2002b; Howald et al., 2003, 2002) have focused on the $H = 0$ region: here the modulations appear to have significant contributions (Byers et al., 1993; Wang and Lee, 2003) from scattering of the fermionic $S = 1/2$ quasiparticles of the superconductor (Section II), but there are also signals (Howald et al., 2003, 2002) of a weak residual periodic modulation in the density of states, similar to those found at $H \neq 0$.

Theoretically (Howald et al., 2003; Kivelson et al., 2002a; Polkovnikov et al., 2003), it is quite natural that these quasiparticle and order parameter modulations co-exist. Howald et al. (2002) and Howald et al. (2003) also presented results for the energy dependence of this periodic modulation, and these appear to be best modelled by modulations in microscopic bond, rather than site, variables (Podolsky et al., 2003; Vojta, 2002; Zhang, 2002). This is a rapidly evolving field of investigation, and future experiments should help settle the interpretation of the density of states modulations both at $H = 0$ and $H \neq 0$. It should be noted that because translational symmetry is broken by the vortices or the pinning centers, there is no fundamental symmetry distinction between the quasiparticle and the pinned-fluctuating-order contributions; nevertheless, their separate spectral and spatial features should allow us to distinguish them.

V. A PHASE DIAGRAM WITH COLLINER SPINS, BOND ORDER, AND SUPERCONDUCTIVITY

We have already discussed two experimental possibilities for the coupling $g$ in Fig 1, which we used to tune the ground state of the doped Mott insulator between various distinct phases: the doping concentration, $\delta$, and the strength of a magnetic field, $H$, applied perpendicular to the layers. A simple phase diagram in the small $H$ region as a function of these parameters was presented in Fig 12, and its implications were compared with a number of experiments in Sections V.A and V.D. However, even though it is experimentally accessible, the field $H$ induces a large scale spatial modulation associated with the vortex lattice, and is consequently an inconvenient choice for microscopic theoretical calculations. Here we follow the strategy of introducing a third theoretical axis, which we denote schematically by $\tilde{g}$, to obtain a global view of the phase diagram. As we argue below, information on the phases present as a function of $\tilde{g}$ sheds considerable light on the physics as a function of $H$.

The crucial role of order parameters characterizing Mott insulators in our discussion suggests that we should work with a coupling, $\tilde{g}$, which allows exploration of
different ground states of Mott insulators already at $\delta = 0$. The range of this coupling should obviously include regimes where the Mott insulator has the magnetically ordered ground state of Fig 3a, found in $\text{La}_2\text{CuO}_4$. Now imagine adding further neighbor couplings in (5) which frustrate this magnetic order, and eventually lead to a phase transition to a paramagnetic state. As discussed in Section 11.C.1, it was argued (Read and Sachdev, 1989b; Sachdev and Park, 2003) that any paramagnetic state so obtained should have bond order, most likely in the patterns in Figs 4a and b.

It would clearly be useful to have numerical studies which tune a coupling $\tilde{g}$ acting in the manner described above. Large scale computer studies of this type have only appeared recently. The first results on a quantum antiferromagnet which has a spin of $S = 1/2$ per unit cell, whose Hamiltonian maintains full square lattice symmetry, and in which it is possible to tune a coupling to destroy the collinear magnetic order, were obtained recently by Sandvik et al. (2002). Their model extended (4) with a plaquette ring-exchange term, and had only a $U(1)$ spin rotation symmetry. Theoretical extensions to this case have also been discussed (Lannert et al., 2001; Park and Sachdev 2002). Along with the collinear magnetic state in the small ring-exchange region (small $\tilde{g}$), Sandvik et al. (2002) found the bond-ordered paramagnetic state of Fig 4a in the large ring-exchange region (large $\tilde{g}$).

A second large scale computer study of the destruction of collinear magnetic order on a model with $S = 1/2$ per unit cell was performed recently by Harada et al. (2003). They generalized the spin symmetry group from $\text{SU}(2)$ to $\text{SU}(N)$; in our language, they used the value of $N$ as an effective $\tilde{g}$. They also found the bond order of Fig 4a in the paramagnetic region.

These theoretical studies give us confidence in the theoretical phase diagram as a function of $\tilde{g}$ and $\delta$ sketched in Fig 13. Sachdev and Read (1991); Vojta 2002; Voja et al. and Sachdev (1999); Vojta et al. (2000a). Phase diagrams with related physical ingredients, but with significant differences, appear in the work of Kivelson et al. (1998) and Zaanen (1999).

Important input in sketching Fig 13 was provided by theoretical studies of the effects of doping the bond-ordered paramagnetic Mott insulator at large $\tilde{g}$. In this region without magnetic order, it was argued that a systematic and controlled study of the doped system was provided by a generalization of the $\text{SU}(2)$ spin symmetry to $\text{Sp}(2N)$, followed by an expansion in $1/N$. This approach directly gives (Sachdev and Read 1991) a stable bond-ordered state at $\delta = 0$, a stable $d$-wave superconductor at large $\delta$, and a region in which these two orders co-exist at small values of $\delta$; all of these phases are nicely in accord with the overall philosophy of the present article. This analysis of a model with purely short-range interactions also found a phase separation instability at small values of $\delta$ (Sachdev and Read 1991), whose importance had been emphasized by others (Bang et al., 1991; Emery et al., 1990) on different grounds. With long-range Coulomb interactions no macroscopic phase separation is possible, and we have to deal with the physics of frustrated phase separation (Emery et al. 1990). The interplay between bond order and $d$-wave superconduc-

---

12 We assume that there is no intermediate state with non-collinear magnetic order, as this is not supported by observations so far.

13 The group $\text{SU}(2)$ is identical to the symplectic group $\text{Sp}(2)$, but the group $\text{SU}(2N)$ is distinct from $\text{Sp}(2N)$ for $N > 1$. Consequently, distinct $1/N$ expansions are generated by models with $\text{SU}(2N)$ or $\text{Sp}(2N)$ symmetry. The $\text{Sp}(2N)$ choice better captures the physics discussed in this article, for reasons explained in Sachdev and Read (1991).
tivity has been studied in some detail in this region \cite{Vojta:2003a,Vojta:1999b,Vojta:2000a}: more complex bond ordered structures with large periods can appear, usually co-existing with superconductivity (as sketched in Fig 13). Predictions were made for the evolution of the ordering wavevector with \( \delta \), and the period 4 structures in Figs 13c and d were found to be especially stable over a wide regime of doping and parameter space.

The phase diagram of Fig 13 also includes a region at small \( g \), with collinear magnetic order, which is not directly covered by the above computations. “Stripe physics” \cite{Machida:1989,Poliblanc:1989,Schulz:1989,Zaanen:1989b}—the accumulation of holes on sites which are anti-phase domain walls between Néel ordered regions—is associated with this region. However, these stripe analyses treat the magnetic order in a static, classical manner, and this misses the physics of valence bond formation that has been emphasized in our discussion here. A related feature is that their domain walls are fully populated with holes and are insulating. Upon including quantum fluctuations accounting for valence bonds, it appears likely to us that the stripes will have partial filling \cite{Kivelson:1996b,Nayak:1997} acquire bond order, and co-exist with superconductivity, as has been assumed in our phase diagram in Fig 13. Indeed, as we have emphasized throughout, it may well be that the modulation in the site charge density—which is proportional to \( Q_0(\vec{r}) \) with \( \vec{r}_a = 0 \) in (10)—is quite small, and most of the modulation is for \( \vec{r}_a \neq 0 \).

The reader should now be able to use the perspective of the phase diagram in Fig 13 to illuminate our discussion of experiments in Section IV. The phase diagram in Fig 12, used to analyze neutron scattering experiments in Section IV.A and STM experiments in Section IV.D, has its horizontal axis along the line labeled \( \alpha \) in Fig 13: the phases that appear in Fig 12 as a function of increasing \( H \) should be related to those in Fig 13 as a function of increasing \( g \), although the detailed location of the phase boundaries is surely different.\textsuperscript{14} The absence of topological order in the experiments discussed in Section IV.B is seen in Fig 13 to be related to the absence of states with non-collinear spin correlations or topological order. The formation of \( S = 1/2 \) moments near non-magnetic impurities is understood by the proximity of confining, bond-ordered phases in Fig 13. The possible signals of bond order in a superconductor at \( H = 0 \) in the STM observations of Howald et al. \cite{Howald:2003a,Howald:2002a}, may be related to the B+SC phase along the line \( \beta \) in Fig 13, similarly, the observations of Hoffman et al. \cite{Hoffman:2001,Hoffman:2002a} at \( H \neq 0 \) can be interpreted by the proximity of the B+SC phase at \( H = 0 \).

VI. OUTLOOK

The main contention of this article is that cuprate superconductors are best understood in the context of a phase diagram containing states characterized by the pairing order of BCS theory, along with orders associated with Mott insulators; the evidence so far supports the class of Mott insulators with collinear spins and bond order. The interplay of these orders permits a rich variety of distinct phases, and the quantum critical points between them offer fertile ground for developing a controlled theory for intermediate regimes characterized by multiple competing orders. This approach has been used to analyze and predict the results of a number of recent neutron scattering, fluid detection, NMR, and STM experiments, as we have discussed in Sections IV.A, IV.B, IV.C, and IV.D. Further experimental tests have also been proposed, and there are bright prospects for a more detailed, and ultimately quantitative, confrontation between theory and experiment.

All of the experimental comparisons here have been restricted to very low temperatures. The theory of crossovers near quantum critical points also implies interesting anomalous dynamic properties at finite temperature \cite{Sachdev:1999,Sachdev:1992}, but these have not been discussed. However, we did note in Section IV.A that the transition involving loss of magnetic order in a background of superconductivity was a natural candidate for explaining the singular temperature and frequency dependence observed in the neutron scattering at \( \delta \approx 0.14 \) \cite{Appli:1997}.

There have also been several recent experimental proposals for a quantum critical point in the cuprates at \( \delta \approx 0.19 \), linked to anomalous quasiparticle damping \cite{Valla:1999}, thermodynamic \cite{Tallon:2001}, or magnetic \cite{Panagopoulos:2003,Panagopoulos:2002} properties. The study of Panagopoulos and collaborators presents evidence for a spin glass state below the critical doping, and this is expected in the presence of disorder at dopings lower than that of the point M in Fig 13.

Among theoretical proposals, a candidate for a quantum critical point \cite{Sachdev:2001,Zaanen:2002a,Zaanen:2002b,Zhang:2002} at large dopings is a novel topological transition which can occur even in systems with collinear spin correlations. While the topological order present in systems with non-collinear spin correlation leads to fractionalization of the electron (as discussed in Section IV.B), the collinear spin case exhibits a very different and much less disruptive transition in which the electron retains its integrity, but the spin and charge collective modes fractionalize into independent entities. Note that this fractionalization transition was not explicitly shown in Fig 13, and is associated with an additional intermediate state which may appear near the

\textsuperscript{14} More precisely, generalizing the arguments leading to (12) and (13), we can state that the system is characterized by an effective \( g \) which increases linearly with \( H \ln(1/H) \), and an effective \( \delta \) which decreases linearly with \( H \ln(1/H) \).
point $M$. Other theoretical proposals for quantum critical points are linked to the bond/charge order (Kivelson et al. 1998; Seibold et al. 1998) in Fig 3, to order associated with circulating current loops (Chakravarty et al. 2001; Varma 1997) which has not been discussed in this paper, and to a time-reversal symmetry breaking transition (Gliveshenko and Pasukas 2001; Laughlin 1998; Sangiovanni et al. 2001; Vojta et al. 2000) between $d_{x^2-y^2}$ and $d_{x^2-y^2} + id_{xy}$ superconductors. This last proposal offers a possible explanation of the quasiparticle damping measurements (Valla et al. 1999). Note that this transition does not involve any order associated with the Mott insulator. Indeed, the $d_{x^2-y^2} + id_{xy}$ order can be understood entirely within the framework of BCS theory, and experimental support for $d_{x^2-y^2} + id_{xy}$ superconductivity in recent tunnelling experiments (Dagan and Deutsch 2001) appears in the overdoped regime, well away from the Mott insulator.

Acknowledgments

I have benefited from discussions and collaborations with many physicists: here I would like to especially thank Gabriel Aeppli, Henri Alloul, Robert Birgeneau, Andrey Chubukov, Seanus Davis, Eugene Demler, Matthew Fisher, Aharon Kapitulnik, Steve Kivelson, Christos Panagopoulos, Kwon Park, Anatoli Polkovnikov, T. Senthil, Matthias Vojta, Jan Zaanen, and Ying Zhang for valuable interactions in recent years. This article is based on the F. A. Matsen Endowed Regents Lecture on the Theories of Matter at the University of Texas at Austin, October 2002. This research was supported by US NSF Grant DMR 0098226.

References

Aeppli, G., T. E. Mason, S. M. Hayden, H. A. Mook, and J. Kulda, 1997, Science 278, 1432.
Affleck, I., and J. B. Marston, 1988, Phys. Rev. B 37, 3774.
Alloul, H., P. Mendels, H. Casalta, J.-F. Marucco, and J. Arabiski, 1991, Phys. Rev. Lett. 67, 3140.
Altman, E., and A. Auerbach, 2002, Phys. Rev. B 65, 104508.
Andersen, B. M., P. Hedegard, and H. Bruus, 2003, Phys. Rev. B 67, 134528.
Anderson, P. W., 1959, Phys. Rev. 115, 2.
Anderson, P. W., 1987, Science 235, 1196.
Anderson, P. W., 2002, eprint cond-mat/0201343.
Arovas, D. P., A. J. Berlinsky, C. Kallin, and S.-C. Zhang, 1997, Phys. Rev. Lett. 79, 2871.
Balents, L., M. P. A. Fisher, and C. Nayak, 1999, Phys. Rev. B 60, 1654.
Bang, Y., G. Kotliar, C. Castellani, M. Grilli, and R. Raimondi, 1991, Phys. Rev. B 43, 13724.
Bardeen, J., L. N. Cooper, and J. R. Schrieffer, 1957, Phys. Rev. 108, 1175.
Baskaran, G., and P. W. Anderson, 1988, Phys. Rev. B 37, 580.
Bednorz, J. G., and K. A. Müller, 1986, Z. Phys. B 64, 188.
Bobroff, J., H. Alloul, W. A. MacFarlane, P. Mendels, N. Blanchard, G. Collin, and J.-F. Marucco, 2001, Phys. Rev. Lett. 86, 4116.
Bonesteel, N. E., 1989, Phys. Rev. B 40, 8954.
Bonn, D. A., J. C. Wynn, B. W. Gardner, Y.-J. Lin, R. Liang, W. N. Hardy, J. R. Kirtley, and K. A. Moler, 2001, Nature 414, 887.
Byers, J. M., M. E. Flatté, and D. J. Scalapino, 1993, Phys. Rev. Lett. 71, 3363.
Chakravarty, S., B. I. Halperin, and D. R. Nelson, 1989, Phys. Rev. B 39, 2344.
Chakravarty, S., R. B. Laughlin, D. Morr, and C. Nayak, 2001, Phys. Rev. B 63, 94503.
Chen, H.-D., J.-P. Hu, S. Capponi, E. Arrigoni, and S.-C. Zhang, 2002, Phys. Rev. Lett. 89, 137004.
Chen, Y., and C. S. Ting, 2002, Phys. Rev. B 65, 180513.
Chubukov, A. V., S. Sachdev, and J. Ye, 1994a, Phys. Rev. B 49, 11919.
Chubukov, A. V., T. Senthil, and S. Sachdev, 1994b, Phys. Rev. Lett. 72, 2089.
Chubukov, A. V., T. Senthil, and S. Sachdev, 1994c, Nucl. Phys. B 426, 601.
Clay, R. T., S. Mazumdar, and D. K. Campbell, 2002, J. Phys. Soc. Japan 71, 1816.
Coldea, R., D. A. Tennant, A. M. Tsvelik, and Z. Tylczynski, 2001, Phys. Rev. Lett. 86, 1335.
Cooper, L. N., 1956, Phys. Rev. 104, 1189.
Dagan, Y., and G. Deutsch, 2001, Phys. Rev. Lett. 87, 177004.
Demler, E., C. Nayak, H.-Y. Kee, Y.-B. Kim, and T. Senthil, 2002, Phys. Rev. B 65, 155103.
Demler, E., S. Sachdev, and Y. Zhang, 2001, Phys. Rev. Lett. 87, 067202.
Dombre, T., and G. Kotliar, 1989, Phys. Rev. B 39, 885.
Emery, V. J., S. A. Kivelson, and H. Q. Lin, 1990, Phys. Rev. Lett. 64, 475.
Fazekas, P., and P. W. Anderson, 1974, Philos. Mag. 30, 23.
Fendley, P., R. Moessner, and S. L. Sondhi, 2002, Phys. Rev. B 66, 214513.
Finkelstein, A. M., V. E. Kataev, E. F. Kukovitskii, and G. B. Teitel’baum, 1990, Physica C 168, 370.
Fradkin, E., and S. A. Kivelson, 1990, Mod. Phys. Lett. B 4, 225.
Franz, M., D. E. Sheehy, and Z. Tesanovic, 2002, Phys. Rev. Lett. 88, 257005.
Ghosal, A., C. Kallin, and A. J. Berlinsky, 2002, Phys. Rev. B 66, 214502.
Gonzalez-Buxton, C., and K. Ingersent, 1998, Phys. Rev. B 57, 14254.
Harada, K., N. Kawashima, and M. Troyer, 2003, Phys. Rev. Lett. 90, 117203.
Hoffman, J. E., E. W. Hudson, K. M. Lang, V. Madhavan, H. Eisaki, S. Uchida, and J. C. Davis, 2002a, Science 295, 466.
Hoffman, J. E., K. McElroy, D.-H. Lee, K. M. Lang, H. Eisaki, S. Uchida, and J. C. Davis, 2002b, Science 297, 1148.
Howald, C., H. Eisaki, N. Kaneko, M. Greven, and A. Kapitulnik, 2003, Phys. Rev. B 67, 014533.
Howald, C., H. Eisaki, N. Kaneko, and A. Kapitulnik, 2002, eprint cond-mat/0201544.
Hsu, T., J. B. Marston, and I. Affleck, 1991, Phys. Rev. B 43, 2866.
Ichikawa, M., and K. Machida, 2002, J. Phys. Soc. Japan 71, 1836.
Bonn, W. N. Hardy, R. Liang, and R. H. Heffner, 2001, Science 292, 1692.
Sonier, J. E., K. F. Poon, G. M. Luke, P. Kyriakou, R. I. Miller, R. Liang, C. R. Wiebe, P. Fournier, and R. L. Greene, 2003, eprint cond-mat/0302223.
Sushkov, O. P., J. Oitmaa, and Z. Weihong, 2001, Phys. Rev. B 63, 104420.
Tallon, J. L., and J. W. Loram, 2001, Physica C 349, 53.
Thouless, D. J., 1987, Phys. Rev. B 36, 7187.
Thouless, D. J., 1998, Topological Quantum Numbers in Non-relativistic Physics (World Scientific, Singapore).
Tranquada, J. M., J. D. Axe, N. Ichikawa, A. R. Moodenbaugh, Y. Nakamura, and S. Uchida, 1997, Phys. Rev. Lett. 78, 338.
Tranquada, J. M., J. D. Axe, N. Ichikawa, Y. Nakamura, S. Uchida, and B. Nachumi, 1996, Phys. Rev. B 54, 7489.
Tranquada, J. M., B. J. Sternlieb, J. D. Axe, Y. Nakamura, and S. Uchida, 1995, Nature 375, 561.
Tsuei, C. C., and J. R. Kirtley, 2000, Rev. Mod. Phys. 72, 969.
Uemura, Y. J., 2003, in Talk at the 7th International Conference on Materials and Mechanisms of Superconductivity, Rio de Janeiro, May 25-30.
Valla, T., A. V. Fedorov, P. D. Johnson, B. O. Wells, S. L. Hulbert, Q. Li, G. D. Gu, and N. Koshizuka, 1999, Science 285, 2110.
Varma, C. M., 1997, Phys. Rev. B 55, 14554.
Vojta, M., 2002, Phys. Rev. B 66, 104505.
Vojta, M., and R. Bulla, 2002, Phys. Rev. B 65, 014511.
Vojta, M., and S. Sachdev, 1999, Phys. Rev. Lett. 83, 3916.
Vojta, M., Y. Zhang, and S. Sachdev, 2000a, Phys. Rev. B 62, 6721.
Vojta, M., Y. Zhang, and S. Sachdev, 2000b, Phys. Rev. Lett. 85, 4940.
Wakimoto, S., R. J. Birgeneau, Y. S. Lee, and G. Shirane, 2001, Phys. Rev. B 63, 172501.
Wakimoto, S., G. Shirane, Y. Endoh, K. Hirota, S. Ueki, K. Yamada, R. J. Birgeneau, M. A. Kastner, Y. S. Lee, P. M. Gehring, and S. H. Lee, 1999, Phys. Rev. B 60, 769.
Wang, Q.-H., and D.-H. Lee, 2003, Phys. Rev. B 67, 020511.
Wen, X.-G., 1991, Phys. Rev. B 44, 2664.
Wen, X.-G., 2002a, Phys. Rev. B 65, 165113.
Wen, X.-G., 2002b, Phys. Rev. Lett. 88, 011602.
Wen, X.-G., and P. A. Lee, 1996, Phys. Rev. Lett. 76, 503.
White, S. R., and D. J. Scalapino, 1998a, Phys. Rev. Lett. 80, 1272.
White, S. R., and D. J. Scalapino, 1998b, Phys. Rev. Lett. 81, 3227.
White, S. R., and D. J. Scalapino, 1999, Phys. Rev. B 60, 753.
Witchoff, D., and E. Fradkin, 1990, Phys. Rev. Lett. 64, 1835.
Wynn, J. C., D. A. Bonn, B. W. Gardner, Y.-J. Lin, R. Liang, W. N. Hardy, J. R. Kirtley, and K. A. Moler, 2001, Phys. Rev. Lett. 87, 197002.
Zaanen, J., 1999, Physica C 317, 217.
Zaanen, J., and O. Gunnarsson, 1989, Phys. Rev. B 40, 7391.
Zaanen, J., and Z. Nussinov, 2003, Phys. Stat. Sol. B 236, 332.
Zaanen, J., O. Y. Osman, H. V. Kruis, Z. Nussinov, and J. Tworzydlo, 2001, Phil. Mag. B 81, 1485.
Zachar, O., S. A. Kivelson, and V. J. Emery, 1998, Phys. Rev. B 57, 1422.
Zhang, D., 2002, Phys. Rev. B 66, 214515.
Zhang, S.-C., 1997, Science 275, 1089.
Zhang, Y., E. Demler, and S. Sachdev, 2002, Phys. Rev. B 66, 094501.
Zhu, J.-X., I. Martin, and A. R. Bishop, 2002, Phys. Rev. Lett. 89, 067003.