Scale invariant Euclidean field theory in any dimension

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Abstract
We discuss $D$-dimensional scalar field interacting with a scale invariant random metric which is either a Gaussian field or a square of a Gaussian field. The metric depends on $d$ dimensional coordinates (where $d < D$). By a projection to a lower dimensional subspace we obtain a scale invariant non-Gaussian model of Euclidean quantum field theory in $D - d$ or $d$ dimensions.

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1 Introduction

We consider a new method of a construction of Euclidean fields. A scalar field in $D$ dimensions is interacting with a metric depending on $d$ dimensional coordinates. An averaging over the metric and a projection of the scalar field to an $s$ dimensional subspace leads to a scalar field which is Euclidean invariant in $R^s$ (we consider $s = D - d$ and $s = d$). If the metric field is scale invariant with a scaling dimension $2\gamma$ then the scalar field is also scale invariant with a scaling dimension depending on $\gamma$. We discuss two models for the random metric. In the first model we consider a square of a Gaussian random field. We are unable to derive an upper bound for correlation functions in this model. Then, we
consider a metric which is Gaussian. We obtain scale invariant correlation functions with explicit upper and lower bounds. Our primary interest in this class of models \[1\]-\[3\] comes from quantum gravity. However, the method may be useful for a construction of relativistic quantum fields (although at the moment we are unable to prove the crucial Osterwalder-Schrader positivity \[3\]). The model can be interesting for statistical physics as a continuum version of spin glass models \[4\]. The lattice version of our model describes spins with a random coupling between them which is either Gaussian (then we have a mixing between ferromagnetic and antiferromagnetic couplings) or a square of the Gaussian field. A calculation of the average over the random coupling can be done explicitly. As a result we obtain models with many interacting spins in contradistinction to the conventional models based on bilinear spin-spin interactions.

2 \hspace{1em} D-dimensional scalar fields

We consider a complex scalar matter field $\Phi$ in $D$ dimensions interacting with gravitons varying only on a $d$-dimensional submanifold. We split the coordinates as $x = (X, \mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^d$. Without a self-interaction the $\Phi \Phi^\ast$ correlation function is equal to an average ($W(g)$ is the gravitational action)

$$\int \mathcal{D}g \exp \left( -\frac{1}{\hbar} W(g) \right) A^{-1}(x, y) \hspace{1em} (1)$$

over the gravitational field $g$ of the Green’s function $A^{-1}(x, y)$ of the operator

$$\mathcal{A} = \frac{1}{2} \sum_{\mu=0, \nu=0}^{D-d-1} g^{\mu\nu}(x) \partial_\mu \partial_\nu + \frac{1}{2} \sum_{k = D-d}^{D-1} \partial_k^2 \hspace{1em} (2)$$
In order to calculate the average (1) we repeat some steps of refs. [1]-[2]. We represent the Green’s function by means of the proper time method

\[ A^{-1}(x, y) = \int_0^\infty d\tau \, (\exp(\tau A))(x, y) \]  

(3)

For a calculation of \( (\exp(\tau A))(x, y) \) we apply the functional integral

\[ K_\tau(x, y) = (\exp(\tau A))(x, y) = \int Dx \exp\left(-\frac{1}{2} \int \frac{dx}{dt} \frac{dx}{dt} - \frac{1}{2} \int g^{\mu\nu}(x) \frac{dx_\mu}{dt} \frac{dx_\nu}{dt}\right) \delta(x(0) - x) \delta(x(\tau) - y) \]  

(4)

In the functional integral (4) we make a change of variables \( x \to b \) determined by Stratonovitch stochastic differential equations [5]

\[ dx^\Omega(s) = e^\Omega_A(x(s)) \, db_A(s) \]  

(5)

where for \( \Omega = 0, 1, \ldots, D - d - 1 \)

\[ e^\mu_a e^\nu_a = g^{\mu\nu} \]

and \( e^\Omega_A = \delta^\Omega_A \) if \( \Omega > D - d - 1 \).

After such a change of variables the functional integral in eq.(4) becomes Gaussian. In fact, this is the standard Wiener integral and \( b_A(t) \) for each \( A \) are independent Brownian motions

\[ E[b_A(t)b_C(s)] = \delta_{AC} \min(s, t) \]

The solution \( q_\tau \) of eq.(5) consists of two vectors \( (Q, q) \) where

\[ q(\tau, x) = x + b(\tau) \]  

(6)
and $Q$ has the components (for $\mu = 0, \ldots, D - d - 1$)

$$Q^\mu(\tau, X) = X^\mu + \int_0^\tau e_\mu^a(q(s, x)) dB^a(s)$$  \hfill (7)

The kernel is

$$K_\tau(x, y) = E[\delta(y - q_\tau(x))] = E[\delta(y - x - b(\tau)) \prod_\mu \delta(Y_\mu - Q_\mu(\tau, X))]$$

Using eq.(7) and the Fourier representation of the $\delta$-function we write the kernel $K_\tau$ in the form

$$K_\tau(x, y) = (2\pi)^{-D+d} \int dP \exp (i P \cdot (Y - X)) E[\delta(y - x - b(\tau)) \exp \left(-i \int P_\mu e_\mu^a(q(s, x)) dB^a(s)\right)]$$ \hfill (8)

We may choose a Gaussian field as a model for the tetrad (as we did in ref.[1])

$$\langle e_\mu^a(x) e_\nu^b(y) \rangle = \Gamma_{\mu\nu}^{ab}(x - y) = \alpha_{\mu\nu}^{ab} |x - y|^{-2\gamma}$$ \hfill (9)

where $\alpha$ is a scale invariant tensor. Then

$$\langle K_\tau(x, y) \rangle = (2\pi)^{-D+d} \int dP \exp (i P \cdot (Y - X)) E[\delta(y - x - \sqrt{\tau}b(1)) \exp \left(-\tau^{1-\gamma} P_\mu P_\nu \int_0^1 dB^a(s) \int_0^1 dB^b(s') \Gamma_{\mu\nu}^{ac}(b(s) - b(s'))\right)]$$ \hfill (10)

where we have changed the time $s \to \tau s$, used the equivalence $b(\tau s) = \sqrt{\tau}b(s)$, and the scale invariant form of the two-point function (9). Moreover, we renormalized the kernel $K_\tau$ removing from it the term (see [1][3])

$$\exp(-\frac{1}{2}\tau \Gamma_{\mu\nu}^{ab}(0)P_\mu P_\nu)$$

It can be seen that this procedure is equivalent to the normal ordering of the metric as a square of the tetrad

$$g^{\mu\nu}(x) = e_\mu^a(x)e_\nu^a(x) \rightarrow: e_\mu^a(x)e_\nu^a(x): = e_\mu^a(x)e_\nu^a(x) - \langle e_\mu^a(x)e_\nu^a(x) \rangle$$ \hfill (11)
We can prove that the double stochastic integral in eq.(10) is a finite square integrable random variable if $2\gamma < 1$. However, it remains unclear whether the momentum integral in eq.(10) is finite.

We can work without the stochastic integrals (8) if we explicitly integrate over $B$. The random variables $b$ and $B^a$ are independent. Hence, using the formula

$$E[\exp i \int f_a(q)dB^a] = E[\exp(-\frac{1}{2}\int f_a ds)]$$

we can rewrite eq.(8) solely in terms of the metric tensor

$$K_\tau(x, y) = (2\pi)^{-D+d} \int dP \exp(iP(Y - X))$$

$$E[\delta(y - x - b(\tau)) \exp(-\frac{1}{2}\int_0^{\tau} P_\mu g^{\mu\nu}(q(s, x)) P_\nu ds)]$$

Let $\mathcal{J}$ be the characteristic function of $g^{\mu\nu}$

$$\mathcal{J}(h) = \langle \exp \left( -\frac{1}{2} g(h) \right) \rangle$$

Then, the mean value of the kernel (12) can be expressed in the form

$$\langle K_\tau(x, y) \rangle = (2\pi)^{-D+d} \int dP \exp(iP(Y - X))$$

$$E[\delta(y - x - b(\tau)) \mathcal{J}(h)]$$

where $g(h) = \int dz g^{\mu\nu}(z) h_{\mu\nu}(z)$ and

$$h_{\mu\nu}(z) = P_\mu P_\nu \int_0^\tau \delta(z - x - b(s)) ds$$

If $e^\mu_a$ is Gaussian then $\mathcal{J}$ can be calculated explicitly

$$\langle \exp \left( -\frac{1}{2} g(h) \right) \rangle = \det(1 + h\Gamma)^{-\frac{1}{2}}$$

where on the r.h.s. the renormalization of the determinant (through a multiplication by $\exp\left( \frac{1}{2}Tr(\Gamma h) \right)$ defining $\det_2$, see [3]) is equivalent to the normal ordering (11) (and subsequently to the renormalization of the kernel (10)).
We consider next another (simpler) model where the metric is Gaussian with two-point correlations

\[ \langle g^{\mu\nu}(x)g^{\sigma\rho}(y) \rangle = -D^{\mu\nu;\sigma\rho}(x-y) = -C^{\mu\nu;\sigma\rho}|y-x|^{-4}\gamma \]  

(16)

where \( C \) (a scale invariant operator) must be positive definite if the momentum integrals in the final formula are to exist. This requirement is not satisfied in a linearized Einstein gravity \( \text{[8]} \) (e.g., in the transverse-traceless gauge \( p_{\alpha}p_{\tau}g^{\alpha\tau}(x) \) would be zero in a covariant \( D \)-dimensional gravity; however our gravity is \( d \)-dimensional). The conformally flat metric \( C^{\mu\nu;\sigma\rho} = \delta^{\mu\nu}\delta^{\sigma\rho} \) would be a satisfactory model for our purposes.

The average over \( g \) in eq.(12) can be calculated

\[ \langle K_\tau(x, y) \rangle = (2\pi)^{-D+d} \int d\mathbf{p} \exp \left( i\mathbf{p} (\mathbf{Y} - \mathbf{X}) \right) \exp \left( \frac{1}{4}r^2 - 2\gamma \int_0^1 P_\mu P_\sigma P_\rho P_\nu D^{\mu\nu;\sigma\rho} (\mathbf{b}(s) - \mathbf{b}(s')) ds ds' \right) \]  

(17)

(as in eq.(10) we have changed the time \( s \rightarrow \tau s \)). By a scaling of momenta we can bring the propagator of eq.(3) to the form

\[ \langle A^{-1}(x, y) \rangle = \int_0^\infty d\tau \tau^{-\frac{1}{2}-(D-d)(1-\gamma)/2} F_2(\tau^{-\frac{1}{2}}(y-x), \tau^{-\frac{1}{2}+\frac{\gamma}{2}}(Y-X)) \]  

(18)

3 A projection to \( D - d \) dimensions

The two-point function (18) has a different scaling behaviour in \( x \) and \( X \) directions. We obtain a fixed scaling behaviour setting \( x = y = 0 \). Then, we have

\[ \langle A^{-1}(x, y) \rangle = R|\mathbf{X} - \mathbf{Y}|^{-D+2-(d-2)\gamma} \]  

(19)
where $R$ is a positive constant. Hence, if all the correlation functions are scale invariant then

$$
\Phi(0, X) \simeq \lambda^{d/2} \cdot \Phi(0, \lambda X)
$$

(20)

where the equivalence means that both sides have the same correlation functions.

In order to prove that $R$ is finite and not zero we need upper and lower bounds for the Gaussian model (17). We show first that the bilinear form $(f, \langle A^{-1} f \rangle)$ is finite and non-zero on a dense set of functions $f$. For this purpose we choose

$$
f_k(X) = (2\pi a^{-d})^{-d/2} \exp(-\frac{a}{2} X^2 + i k X)
$$

Then, (we keep $x \neq y$ in order to show that the model of sec.2 is non-trivial; for a scale invariant model of this section $x = y = 0$)

$$
(f_k, \langle A^{-1} f_k \rangle) = (2\pi)^{-d+\frac{d}{2}} \int_0^\infty \frac{d\tau}{\tau^{d/2}} \int dP

\left[ \delta\left(\tau^{-\frac{1}{2}} (y - x) - b(1)\right)

\exp\left(-\frac{1}{2a} (P - k)^2 - \frac{1}{2\tau} (P - k')^2

- \frac{1}{4} \tau^{2-2\gamma} \int_0^1 \mu P \rho P \sigma P D^{\mu \sigma \rho} (b(s) - b(s')) ds ds' \right) \right]

(21)

In our estimates we apply Jensen inequalities in the form (for real functions $A$ and $f$)

$$
E[\exp A] \geq \exp E[A]
$$

(22)

and

$$
E[\exp \left(-\int_0^1 ds ds' f(s, s')\right)] \leq \int_0^1 ds ds' E[\exp(-f(s, s'))]
$$

(23)

An upper bound can be obtained by means of the Jensen inequality (23) ex-
pressed in the form
\[
(f_k, (A^{-1}) f_k) \leq 2 \int_0^\infty d\tau \int_0^\tau ds \int_0^s ds' \int du_1 du_2 d\mathbf{P} \\
\tau^{-\frac{d}{2}} \exp \left( -\frac{1}{2a} (\mathbf{P} - \mathbf{k})^2 - \frac{1}{2a} (\mathbf{P} - \mathbf{k}')^2 \right) p(s', u_1) p(s - s', u_2 - u_1)
\]
(24)
where \( p(s, u) = (2\pi s)^{-\frac{d}{2}} \exp(-u^2/2s) \). We can convince ourselves by means of explicit calculations (using a proper change of variables) that the integral on the r.h.s. of eq.(24) is finite. For the lower bound it will be useful to introduce the Brownian bridge \([9]\) starting from \(0\) and ending in \(u\) defined on the time interval \([0, 1] \)
\[
\mathbf{a}(u, s) = us + \mathbf{c}(s)
\]
where \( \mathbf{c} \) is the Gaussian process starting from \(0\) and ending in \(0\) with mean equal zero and the covariance
\[
E[c_j(s') c_k(s)] = \delta_{jk} s'(1 - s)
\]
for \( s' \leq s \). Then, the \( \delta \) function in eq.(21) determines the Brownian bridge and the Jensen inequality (22) takes the form
\[
(f_k, (A^{-1}) f_k) \geq (2\pi)^{-D+d} \int_0^{\infty} d\tau \tau^{-\frac{d}{2}} \int d\mathbf{P} \\
\exp \left( -\frac{1}{2a} (\mathbf{P} - \mathbf{k})^2 - \frac{1}{2a} (\mathbf{P} - \mathbf{k}')^2 \right) \\
-\frac{1}{4} \tau^{2-2s} \int_0^1 P_{\mu} P_{\sigma} P_{\nu} P_{\rho} E[D^{\mu\nu;\sigma\rho}] \left( \mathbf{a} \left( \tau^{-\frac{1}{2}} \mathbf{y} - \tau^{-\frac{1}{2}} \mathbf{x}, s \right) - \mathbf{a} \left( \tau^{-\frac{1}{2}} \mathbf{y} - \tau^{-\frac{1}{2}} \mathbf{x}, s' \right) \right) ds ds' \right)
\]
(25)
where the expectation value in the exponential on the r.h.s. of eq.(25) is equal to
\[
\int d\mathbf{u} \int ds \int_0^s ds' \left( 2\pi \omega(s, s') \right)^{-\frac{d}{2}} \exp \left( -\frac{1}{2\omega(s, s')} \mathbf{u}^2 \right) \\
\left| \mathbf{u} - \tau^{-\frac{1}{2}} s (\mathbf{y} - \mathbf{x}) + \tau^{-\frac{1}{2}} s' (\mathbf{y} - \mathbf{x}) \right|^{-4\gamma}
\]
(26)
where $\omega(s, s') = (s - s')(1 - s + s')$. It is finite if $\gamma < \frac{1}{2}$ (the form (16) of the graviton two-point function is assumed).

We compute now higher order correlation functions in the Gaussian model

$$\langle \Phi(x)\Phi(x')\Phi^*(y)\Phi^*(y') \rangle = \langle A^{-1}(x, y) A^{-1}(x', y') \rangle + (x \to x')$$

$$= (2\pi)^{-2D+2d} \int d\tau_1 d\tau_2 \int dP dP' \exp \left\{ iP (Y - X) + iP' (Y' - X') \right\}$$

$$E[\delta(y - x - b(\tau_1)) \delta(y' - x' - b'(\tau_2))]$$

$$\exp \left\{ -\frac{1}{4} \int_0^{\tau_1} \int_0^{\tau_2} P_{\mu} P_{\nu} P_{\rho} P_{\sigma} D^{\mu\nu\rho\sigma} (b(s) - b(s')) ds ds' \right\} + (x \to x')$$

(27)

where $(x \to x')$ means the same expression in which $x$ is exchanged with $x'$.

The fourlinear form (27) calculated on the basis $f$ reads

$$\langle \Phi(f_k)\Phi(f_{k'})\Phi^*(f_{k''})\Phi^*(f_{k'''}) \rangle$$

$$= (2\pi)^{-2D+2d} \int d\tau_1 d\tau_2 \int dP dP' \exp \left\{ iP (Y - X) + iP' (Y' - X') \right\}$$

$$E[\delta(y - x - b(\tau_1)) \delta(y' - x' - b'(\tau_2))]$$

$$\exp \left\{ -\frac{1}{4} \int_0^{\tau_1} \int_0^{\tau_2} P_{\mu} P_{\nu} P_{\rho} P_{\sigma} D^{\mu\nu\rho\sigma} (b(s) - b(s')) ds ds' \right\} + (1, 2 \to 3, 4)$$

(28)

where the last term means the same expression with exchanged wave numbers.

We introduce the spherical coordinates on the $(\tau_1, \tau_2)$-plane $\tau_1 = r \cos \theta$ and $\tau_2 = r \sin \theta$. Let us rescale the momenta $k = p\sqrt{\tau}$, $k' = p'\sqrt{\tau}$, $K = P^{\frac{1}{2} - \frac{\tau}{2}}$ and $K' = P'^{\frac{1}{2} - \frac{\tau}{2}}$. Then, we can see that the four-point function (27) takes the form

$$\langle \Phi(x)\Phi(x')\Phi^*(y)\Phi^*(y') \rangle$$

$$= \int d\theta drrr' d^{-(D-\gamma)} F_4(\theta, r^{\frac{1}{2}}(x - y), r^{\frac{1}{2}}(x' - y'),$$

$$r^{\frac{1}{2}}(x' - x), r^{\frac{1}{2}}(x' - y), r^{\frac{1}{2}}(y' - x),$$

$$r^{\frac{1}{2}}(y' - x'))$$

(29)

It follows just by scaling of coordinates (the $r$-integral scales as twice the $\tau$-
integral in eq.(18)) that at $x = x' = y = y' = 0$ the correlations are scale invariant with the same scaling dimension as in eq.(20).

It is clear from eq.(28) that in the same way as we did it in eqs.(24)-(25) we can obtain finite upper and lower bounds on the correlation functions (28) by means of the Jensen inequalities.

We could continue with higher order correlation functions. Again through an introduction of spherical coordinates in the $(\tau_1, ..., \tau_3)$ space we can show that

$$\langle \Phi(x_1) \ldots \Phi(x_3) \Phi^*(y_1) \ldots \Phi^*(y_3) \rangle$$

scales with the same dimension as in eq.(20). The scaling of higher order correlation functions is now evident. We introduce the spherical coordinates for the $\tau$-integrals. The resulting scaling is a consequence of the fact that the $\tau$-volume and $P$ integrals have the scaling dimensions proportional to the order of the correlation function.

4 A projection to $d$ dimensions

There is still another option that we let all $X = Y = 0$. In such a case

$$\langle K_\tau(x, y) \rangle = (2\pi)^{-D+d} \int dP$$

$$E[\delta(y - x - \sqrt{7}b(1)) \exp \left(-\frac{1}{4} \tau^2 - 2\gamma \int_0^1 P_\mu P_\nu P_\rho D^{\mu:\nu:\rho}(b(s) - b(s')) dsds' \right)]$$

(31)

By a scaling of momenta we can bring the propagator of eq.(3) to the form

$$\langle A^{-1}(x, y) \rangle = \int_0^\infty d\tau \tau^{-\frac{D}{2}-(D-d)(1-\gamma)/2} F_2(\tau^{-\frac{1}{2}}(y - x))$$

(32)
Hence
\[ \langle A^{-1}(x, y) \rangle = R|x - y|^{-d+2-(D-d)(1-\gamma)} \] (33)
where \( R \) is a positive constant. Hence, if all the correlation functions are scale invariant then
\[ \Phi(x, 0) \simeq \lambda^{\frac{2\gamma}{2-D+2}} \Phi(\lambda x, 0) \] (34)
We can prove all the inequalities of sec.3 in this model. So, the upper bound for the two-point function reads
\[
\begin{align*}
|\langle A^{-1}(x, y) \rangle| &= 2 \int_0^\infty d\tau \int_0^1 ds \int_0^s ds' \int d\mathbf{u}_1 d\mathbf{u}_2 d\mathbf{P} \tau^{-\frac{D}{2}} p(s', \mathbf{u}_1) p(s - s', \mathbf{u}_2 - \mathbf{u}_1) \exp \left( -\frac{2\gamma}{4} \mathbf{P}_\mu \mathbf{P}_\sigma \mathbf{P}_\nu \mathbf{P}_\rho D_{\mu\nu,\rho\sigma} (\mathbf{u}_1 - \mathbf{u}_2) \right) \\
\end{align*}
\] (35)
The lower bound takes the form
\[
\begin{align*}
|\langle A^{-1}(x, y) \rangle| &\geq (2\pi)^{-D+d} \int_0^\infty d\tau \tau^{-\frac{D}{2}} \int d\mathbf{P} \exp \left( -\frac{1}{4} \tau^{2-2\gamma} \int_0^1 \mathbf{P}_\mu \mathbf{P}_\sigma \mathbf{P}_\nu \mathbf{P}_\rho E_{D_{\mu\nu,\rho\sigma}} (\mathbf{a} \left( \tau^{-\frac{1}{2}} y - \tau^{-\frac{1}{2}} x, s \right) - \mathbf{a} \left( \tau^{-\frac{1}{2}} y - \tau^{-\frac{1}{2}} x, s' \right) ) ds ds' \right) \\
\end{align*}
\] (36)
where the expectation value in the exponential on the r.h.s. of eq.(36) is equal to
\[
\int d\mathbf{u} \int ds \int_0^s ds' (2\pi \omega(s, s'))^{-\frac{D}{2}} \exp \left( -\frac{1}{4\omega(s, s')} \mathbf{u}^2 \right) |\mathbf{u} - \tau^{-\frac{1}{2}} s (y - x) + \tau^{-\frac{1}{2}} s' (y - x)|^{-\gamma} \] (37)
where \( \omega(s, s') = (s - s')(1 - s + s') \). It is finite if \( \gamma < \frac{1}{2} \). The bounds (35)-(36) in fact have the form
\[
R_1 \leq |x - y|^{d-2+\gamma(D-d)(1-\gamma)} \langle A^{-1}(x, y) \rangle \leq R_2 \] (38)
with certain positive \( R_1 \) and \( R_2 \).
The inequalities (38) can be proved from the inequalities (35)-(36) just by rescaling of variables. It is more tedious to show that the constants $R_1$ and $R_2$ are finite and not zero (but the estimates reduce to finite dimensional integrals and are straightforward).

We can project now to $R^d$ higher order correlation functions

$$
\langle \Phi(x) \Phi(x') \Phi^*(y) \Phi^*(y') \rangle
= \langle A^{-1}(x,y) A^{-1}(x',y') \rangle + (x \to x')
= (2\pi)^{-2D+2d} \int d\tau_1 d\tau_2 \int dP dP' \ E[\delta(y - x - b(\tau_1)) \delta(y' - x' - b'(\tau_2))]
\exp \left( -\frac{1}{2} \int_{\tau_1}^{\tau_2} P_\mu P_\sigma P_\nu P_\rho D^{\mu \nu \sigma \rho} (b(s) - b(s)) ds ds' \right)
\exp \left( -\frac{1}{2} \int_{\tau_1}^{\tau_2} P_\mu P_\nu P_\rho P_\sigma D^{\mu \nu \sigma \rho} (x - x' + b(s) - b'(s')) ds ds' \right)
+ (x \to x')
$$

where $(x \to x')$ means the same expression in which $x$ is exchanged with $x'$. We introduce the spherical coordinates on the $(\tau_1, \tau_2)$-plane $\tau_1 = r \cos \theta$, $\tau_2 = r \sin \theta$ and we rescale the momenta $k = p \sqrt{r}$, $k' = p' \sqrt{r}$, $K = P r^{\frac{d - 2}{2}}$. Then, we can see that the four-point function (39) takes the form

$$
\langle \Phi(x) \Phi(x') \Phi^*(y) \Phi^*(y') \rangle
= \int d\delta d\tau d\mu d\nu d\sigma d\rho \ F_4(\delta, r^{-\frac{d}{2}}(x - y), r^{-\frac{d}{2}}(x' - y'),
\tau^{-\frac{d}{2}}(x' - x), \tau^{-\frac{d}{2}}(x' - y), \tau^{-\frac{d}{2}}(y' - x))
$$

The upper bound now reads

$$
\left| \langle \Phi(x) \Phi(x') \Phi^*(y) \Phi^*(y') \rangle \right| \leq
2(2\pi)^{-2D+2d} \int d\tau_1 d\tau_2 \int ds ds' \int dP dP' E[\delta(y - x - \sqrt{\tau_1} b(1))]
\exp \left( -\frac{1}{2} \int_{\tau_1}^{\tau_2} P_\mu P_\sigma P_\nu P_\rho D^{\mu \nu \sigma \rho} (\sqrt{\tau_1} b(s) - \sqrt{\tau_1} b(s')) \right)
$$

\begin{align*}
&\exp \left( -\frac{1}{2} \int_{\tau_1}^{\tau_2} P_\mu P_\nu P_\rho P_\sigma D^{\mu \nu \sigma \rho} (\sqrt{\tau_2} b'(s) - \sqrt{\tau_2} b'(s')) \right)
&\exp \left( -\tau_1 \tau_2 P_\mu P_\nu P_\rho P_\sigma D^{\mu \nu \sigma \rho} (x - x' + \sqrt{\tau_1} b(s) - \sqrt{\tau_2} b'(s')) \right)
+ (x \to x')
\end{align*}

The expectation value (41) can be expressed by the transition functions (as usual for the Wiener process). The bound is scale invariant and the scale invariant
function on the r.h.s. could be calculated explicitly. The lower bound takes the form

\[
\langle \Phi(x)\Phi(x')\Phi^*(y)\Phi^*(y') \rangle \\
\geq (2\pi)^{-2D+2d} \int d\tau_1 d\tau_2 \int d\mathbf{P} d\mathbf{P}' \\
\exp \left( -E \frac{1}{4} \int_0^{\tau_1} \int_0^{\tau_2} P_\mu P_\nu P_\rho P_\sigma D^{\mu\nu;\sigma\rho} (a(s) - a(s')) ds ds' \right) \\
\frac{1}{4} \int_0^{\tau_1} \int_0^{\tau_2} P_\mu P_\nu P_\rho P_\sigma D^{\mu\nu;\sigma\rho} (a'(s) - a'(s')) ds ds' \\
\frac{1}{2} \int_0^{\tau_1} \int_0^{\tau_2} P_\mu P_\nu P_\rho P_\sigma D^{\mu\nu;\sigma\rho} (a(s) - a'(s')) ds ds' \\
+ (x \to x')
\]

Here

\[ a(s) = x + (y - x) \frac{s}{\tau_1} + \sqrt{\tau_1} c \frac{s}{\tau_1} \]

and

\[ a'(s) = x' + (y' - x') \frac{s}{\tau_2} + \sqrt{\tau_2} c' \frac{s}{\tau_2} \]

In the exponential of the formula (42) we have the expectation over three Gaussian processes. The first is with the mean \((y - x)(s - s')/\tau_1\) and the covariance \((s - s')(\tau_1 - s + s')/\tau_1\), the second has the mean \((y' - x')(s - s')/\tau_2\) and the covariance \((s - s')(\tau_2 - s + s')/\tau_2\), the third has the mean \(x - x' + (y - x)s/\tau_1 - (y' - x')s'/\tau_2\) and the covariance \(s(\tau_1 - s)/\tau_1 + s'(\tau_2 - s')/\tau_2\). The lower bound can be explicitly calculated and is given by a scale invariant function. It is clear how to calculate the higher order scale invariant functions and their scale invariant lower and upper bounds.

5 Discussion

The model discussed in sec.3 is invariant under Euclidean rotations in \(D - d\) dimensions. Euclidean fields with the Osterwalder-Schrader positivity cannot be more regular than the free field (this follows from the Källen-Lehman rep-
presentation). In $D$ dimensions the short distance behaviour of the correlation functions (18) is more regular than the one of the free fields. However, after setting all $x = 0$ the behaviour is more singular than the canonical one in $D - d$ dimensions. We can suggest a lattice model whose formal continuum limit coincides with our scale invariant Euclidean field theory. The simplest possibility is to take the conformally flat metric placed on a sublattice just between the lattice sites of the scalar field (as the gauge fields in ref.[10]). It seems that the Gaussian model of sec.3 mixing the ferromagnetic and antiferromagnetic couplings would fail to be reflection positive (in any case this would not be easy to prove, see ref.[11]). The model (11) with the metric which is a square of a Gaussian field can be reflection positive (it is scale invariant with correlation functions expressed by the characteristic function $\mathcal{J}(h)$ (13)). The continuum limit and the subsequent analytic continuation to Minkowski space would give a model of relativistic quantum field theory satisfying all Wightman axioms. The Wick square of a Gaussian field is an example of an infinitely divisible field [12]. An infinitely divisible field can take non-negative values. Its characteristic function has an explicit integral representation. Such a field can be a good candidate for a random metric.

More interesting are the models in sec.4. The lattice version of the Lagrangian

$$L = g^{\mu\nu}(x) \frac{\partial}{\partial X^\mu} \Phi \frac{\partial}{\partial X^\nu} \Phi^* + \nabla_X \Phi \nabla_X \Phi^*$$

will have the form $-L = F + \theta F + M\theta M$ where $\theta$ is the reflection in the
plane perpendicular to one of the coordinates $x$ (which will be chosen as time). This representation holds true because the random metric does not mix the temporal coordinates in $\nabla_x \Phi \nabla_x \Phi^*$. Then, the reflection positivity results (see [11]). In the lattice approximation we have to replace the (formal) Gaussian measure with a negative definite covariance (16) by a non-Gaussian measure on the metrics which has a formal Gaussian limit (e.g., replacing $\frac{1}{2} x^2$ by $1 - \cos x$). There remains to be proven that such a lattice approximation is convergent to the continuum.

References

[1] Z. Brzezniak and Z. Haba, Journ.Phys. A34, L139(2001)

[2] Z. Haba, Phys.Lett. B528, 129(2002)

[3] K. Osterwalder and R. Schrader, Commun.Math.Phys. 31, 83(1973), 42, 281(1975)

[4] D. Chowdhury, Spin Glasses and Other Frustrated Systems, World Scientific, Singapore, 1987

[5] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North Holland, 1981

[6] Z. Haba, Regularizing effect of gravitons, in New Developments In Fundamental Interaction Theories, AIP Conference Proc., No.589, ed. by J. Lukierski and J. Rembielinski, AIP, 2001
[7] E. Seiler, Commun.Math.Phys.24,163(1975)
    B. Simon, Adv.Math.24,244(1977)

[8] S. Weinberg, Phys.Rev.138,B88(1965)

[9] B. Simon, Functional Integration and Quantum Physics, Academic,New York,1979

[10] K. Osterwalder and E. Seiler, Ann.Phys.(N.Y.)110,440(1978)

[11] J. Fröhlich, R. Israel, E.H. Lieb and B. Simon,
    Commun.Math.Phys.62,1(1978)

[12] K. Baumann and G.C. Hegerfeldt, Journ.Math.Phys.18,171(1977)