Equivalent Inequalities

Abstract: Equivalencies of many basic elementary inequalities are given.

1 Introduction
It is an amazing act that almost all elementary inequalities can be derived from a few extremely elementary results and are in fact equivalent to these results. Much of this is known and is mentioned in passing in most basic books on inequalities, see for instance [3 pp 212–213]. Other equivalencies arise from the ability for equivalent inequalities to hide beneath almost impenetrable disguises. In this note we collect all these various equivalencies and disguised equivalencies so as to make them rapidly available to anyone who is interested. The inequalities discussed here are for the most part those found in the references [3,8,9]. Of course there is a vast field beyond this, see for instance [1,6], and many others beyond the competence of the authors to discuss. In this way the present paper can be regarded as a permanent work in progress as others with this competence add to the list of equivalent inequalities.

The question of equivalence can cause other problems. While we are all agree that inequality $I$ is equivalent to inequality $J$ if and only if a proof of of $J$ can be found under the assumption that $I$ holds and conversely is not always that simple. Some equivalent inequalities are really almost equivalent typographically: that is $I: A \geq B$ is equivalent to $J: B \leq A$. Some arise from not very subtle changes of notation $I: \phi(x) \geq 0, x \geq 0$, $J: \phi(x^2) \geq 0, x \in \mathbb{R}$ while in others the change of variables is so intricate as to make the one an impenetrable disguise of the other and only the most perverse of mathematicians would quote the disguise as the original. In addition a simple inequality may imply a more general one of

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which it is special case and so should perhaps not be considered as equivalent to the general form. Finally there is the problem that many inequalities are thought of without specifying exactly all of the logical operators that are implicit in their formulations.

We do not consider integral inequalities although it might be noted that where they exist they are usually seen very easily to be equivalent to their discrete analogues; see for instance [3 pp.368–384]. However the cases of equality and the extra complications are avoided here by restricting the discussion to the discrete case.

Notations: \( \mathbb{N} = \{0, 1, 2, \ldots \}; \mathbb{N}^* = \{1, 2, \ldots \}; \mathbb{N}^{**} = \mathbb{N}^* \setminus \{1\} = \{2, 3, \ldots \}; \mathbb{R} \) is the set of all real numbers; \( \mathbb{R}_+ = \{x; x \geq 0\}; \mathbb{P} = \{x; x > 0\}; \mathbb{P}^n = (\mathbb{R}_+^*)^n, n \in \mathbb{N}^* \).

If \( a, b \in \mathbb{R} \) then \( a \leq b \) then the open and closed intervals with these endpoints are \([a, b], [a, b) \) respectively.

If \( n \in \mathbb{N}^* \) then \( a = (a_1, \ldots, a_n), w = (w_1, \ldots, w_n) \) are \( n \)-tuples of positive numbers, that is \( a, w \in \mathbb{P}^n \); if \( a_1 = \cdots = a_n \) we say the the \( n \) tuple \( a \) is constant; if \( 1 \leq k \leq n \) then \( W_k = \sum_{i=1}^{k} w_i \).

If \( f: \mathbb{R}_+^n \to \mathbb{R}_+^n \) then \( f(a) \) denotes the \( n \)-tuple \( (f(a_1), \ldots, f(a_n)) \);

and \( ab \) will denote the \( n \)-tuple \((a_1 b_1, \ldots, a_n b_n)\).

\[ \mathfrak{G}_n(a; w) = (\prod_{i=1}^{n} a_i^{w_i})^{1/W_n}; \]

\[ \mathfrak{M}_n^r(a; w) = \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i^r \right)^{1/r}, r \in \mathbb{R}^*, \]

\[ = \mathfrak{G}_n(a; w), r = 0. \]

\[ = \max a, r = \infty. \]

\[ = \min a, r = -\infty. \]

\[ \mathfrak{M}_n^1(a; w) = \mathfrak{G}_n(a; w) \quad \mathfrak{M}_n^{-1}(a; w) = \mathfrak{G}_n(a; w) \]

\[ \mathfrak{M}_n^r(a; w) = \mathfrak{G}_n(a; w) \quad \mathfrak{M}_n^r(a; w) = \mathfrak{M}_n^r(a_1, \ldots, a_n; w_1, \ldots, w_n) \quad \text{etc.} \]

If \( w \) is a constant it is omitted from these notations; \( \mathfrak{M}_n^r(a) \) etc.

If \( m \in \mathbb{N}^*, m < n \) and \( a, w \in \mathbb{P}^n \) then \( \mathfrak{M}_m(a; w), \text{ etc. }, \) is taken to mean \( \mathfrak{M}_m(a_1, \ldots, a_m; w_1, \ldots, w_m), \text{ etc.} \)

In the statement of an inequality \( V \) denotes its set of validity and \( E \) denotes the set of equality; clearly \( E \subset V \) and the equality is strict on \( V \setminus E \). So formally an inequality is a triple \((V, I, E)\).

If \( I \) and \( J \) are two inequalities then \( I \equiv J \) says that they are equivalent.

\( \sim \) \( I \) is the inequality \( I \) with inequality signs reversed, the reverse inequality.
2 Equivalent Inequalities

2.1 Basic Notations  Given \( t \in \mathbb{N}^* \), \( A \subseteq \mathbb{R}^t \) and \( F: A \mapsto \mathbb{R} \) then \( A = A_+ \times A_0 \times A_- \) where

\[
A_+ = \{ x; x \in A \land F(x) \geq 0 \}, \quad A_0 = \{ x; x \in A \land F(x) = 0 \}, \quad A_- = \{ x; x \in A \land F(x) \leq 0 \},
\]

An inequality \( \mathcal{I} \) is a triple \( \{ V, E, F(x) \geq 0 \} \) where: (a) \( V \) is the set of validity of \( \mathcal{I} \), \( V \subseteq A_+ \); (b) \( E \) is is the set of equality of \( \mathcal{I} \), \( E \subseteq A_0 \cap V \); (c) the formula \( F(x) \geq 0 \), often just called, by an abuse of language, the inequality. Alternatively we could write \( \{ V, E, F(x) \leq 0 \} \), \( V \subseteq A_- \), but this form is the same as the standard one if \( F \) is replaced by \(-F\) and we will not elaborate on this trivial point. Another variant is when \( F = F_1 - F_2 \) and \( \mathcal{I} \) is written \( \{ V, E, F_1(x) \geq F_2(x) \} \), or alternatively \( \{ V, E, F_1(x) \leq F_2(x) \} \).

It is important to note that in general \( V \subset A_+ \). The \( t \) variables occurring in \( F \) are of two kinds; the basic variables, \( t_1 \) in number say and the parameters \( t_2 = t - t_1 \) in number. Then we have the notation: \( \mathbb{R}^t \cong \mathbb{R}^{t_1} \times \mathbb{R}^{t_2} \), \( A = A_1 \times A_2 \) and \( A_1 \subseteq \mathbb{R}^{t_1}, A_2 \subseteq \mathbb{R}^{t_2} \) etc. It usual to require the parameters to be such that the formula holds for all values of the variables.

2.1.1 Example  Consider \( F(x, y, z, s, t) = (1-s-t)x+sy+tz : \mathbb{P}^3 \times \mathbb{R}^2 \mapsto \mathbb{R} \); here \( t_1 = 3 \) and \( t_2 = 2 \). The basic geometric-arithmetic mean inequality has \( V = \mathbb{P}^3 \times T \) where \( T \) is the triangle \( \{(s, t); s \geq 0, t \geq 0, s + t \leq 1 \} \). However \( F \) is positive on \( T \) except at the corners so if, as above, \( A_+ = B \times C \) then \( T \subset C \) but whereas the inequality holds on \( T \) for all values of the variables the set \( C \) depends on the variables.

In many cases the exact value of either or both of \( V \) and \( E \) may be unknown.

2.1.2 Example  If \( I \subset \mathbb{R} \) is an interval then to say that \( f : I \mapsto \mathbb{R} \) is convex is equivalent to the inequality \( \mathcal{I} \) defined as follows. Let \( F(x, y, \lambda) = (1-\lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y) : I^2 \times [0, 1] \mapsto \mathbb{R} \) then \( \mathcal{I} = \{ V, E, F(x, y, \lambda) \geq 0 \} \) where \( V = I^2 \times [0, 1] \) but in general \( E \) is not known except for the obvious \( E_1 = I^2 \times \{ 0 \} \cup \{ 1 \} \subseteq E \). If \( E = \{(x, y); (x, y) \in I^2 \land x = y\} \cup E_1 \) then \( f \) is said to be strictly convex.

2.2 Complementary and Complete inequalities  With each inequality \( \mathcal{I} \) is associated a complementary inequality, \( \sim \mathcal{I} \) being the triple \( \{ \sim V, \sim E, F(x) \leq 0 \} \) where \( \sim V \subseteq A_- \) and \( \sim E \subseteq A_0 \cap \sim V \).

The pair of inequalities \((\mathcal{I}, \sim \mathcal{I})\) is called a complete inequality, or more precisely the complete \( \mathcal{I} \)-inequality, \( \tilde{\mathcal{I}} \).
2.2.1 Example If $F(x,y,t) : \mathbb{P}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ then the geometric-arithmetic mean inequality, (GA), has: $V = \mathbb{P}^2 \times [0,1], E = \{(x,y); x = y\} \cup \{t = 0\} \cup \{t = 1\}$ and the formula $F(x,y,t) \geq 0$. The complementary inequality, $\sim$ (GA), has formula $F(x,y,t) \leq 0, \sim V = \mathbb{P}^2 \times ]-\infty,0] \cup [1,\infty[, \sim E = E$. The complete geometric-arithmetic mean inequality, $(\sim$GA), is then the pair $(GA, \sim(GA))$.

2.3 Equivalent Inequalities Given two inequalities $I, J$ we say that they are equivalent, $I \equiv J$, if $I$ can be deduced from $J$ without the use of any other inequality and vice versa. Two complete inequalities $\tilde{I}, \tilde{J}$ are equivalent, $\tilde{I} \equiv \tilde{J}$, if both $I$ is equivalent to $J$ and $\sim I$ is equivalent to $\sim J$.

There is a minor difficulty with this definition in that certain inequalities are so basic that they must be allowed in any reasonable mathematical argument. These are the primitive inequalities:

(a) inequalities between real numbers;
(b) the signs of the power functions, $x^n, n \in \mathbb{Z}, x \in \mathbb{R}$ or $\mathbb{R}^*.$

2.3.1 Examples In all the following cases $I \equiv J$,

(a) $I = \{V,E,F \geq 0\}; J = \{V,E,-F \leq 0\}$.
(b) $I = \{V,E,F_1 \geq F_2\}; J = \{V,E, aF_1 + b \geq aF_2 + b\}, a \in \mathbb{R}^*, b \in \mathbb{R}$.
(c) $I = \{V,E,F_1 \geq F_2\}; J = \{V,E, aF_1 + b \leq aF_2 + b\}, -a \in \mathbb{R}^*, b \in \mathbb{R}$.

Clearly this concept defines an equivalence relation in that if $I,J,K$ are inequalities then: (a) $I \equiv I$, (b) $I \equiv J \iff J \equiv I$, (c) $I \equiv J \land J \equiv K \implies I \equiv K$.

As result if we have proved $I \equiv J$ and $J \equiv K$ we will not always then state that $I \equiv K$.

3 The Geometric-Arithmetic Mean Inequality

3.1 The Equal Weight Case Perhaps the simplest inequality is the classical one that goes back at least to the time of Euclid\textsuperscript{1}.

$$(GA_{2,e}) \quad \sqrt{xy} \leq \frac{x+y}{2} \quad \text{or} \quad \mathcal{G}_2(x,y) \leq \mathcal{A}_2(x,y): E = \{(x,y) : x = y\}. \quad (1)$$

However this simple inequality is equivalent to a much more general inequality, the equal weight geometric-arithmetic mean inequality of order $n$, where $n \in \mathbb{N}^{*}\text{ii}$.

$$V = \{(n,a); n \in \mathbb{N}^{*} \land a \in \mathbb{P}^n\}.$$  

\textsuperscript{1}Euclid (fl:c.300 BC).
\textsuperscript{ii}We could of course take $n \in \mathbb{N}^*$ but then $n=1$ must be added to $E$; this simple observation is often valid but will not be repeated.
3.1.1 Theorem If \( n, m \in \mathbb{N}^* \) then
\[
(\mathcal{G}A_{n,e}) \equiv (\mathcal{G}A_{m,e}).
\]
□ This is immediate from known results:
(i) \( \forall n \in \mathbb{N}^* : (\mathcal{G}A_{2,e}) \Rightarrow (\mathcal{G}A_{2^n,e}) \); see [3 pp.85-86];
(ii) \( \forall n, n' \in \mathbb{N}, n, n' \geq 2, n \geq n' : (\mathcal{G}A_{n,e}) \Rightarrow (\mathcal{G}A_{n',e}) \); see [3 p.81]. □

3.1.1.1 Remark The results used in the above theorem are due to Cauchy\(^iii\), and were published in 1821.

There are several different looking inequalities that are equivalent to the equal weight geometric-arithmetic mean inequality of order \( n \); see [3 pp.82–84].

\( V = \{(n, a); n \in \mathbb{N}^* \land a \in \mathbb{P}^n \land \prod_{i=1}^{n} a_i = 1\} \),

\( \mathcal{I}_n \) \[ n \leq \sum_{i=1}^{n} a_i : \]

\( \mathcal{J}_n \) \[ \prod_{i=1}^{n} a_i \leq (1/n)^n; \]

\( E = \{a; a \in \mathbb{P}^n \land a \text{ is constant}\} \).

3.1.2 Theorem If \( n \in \mathbb{N}^* \), then
\[
(\mathcal{I}_n) \equiv (\mathcal{J}_n) \equiv (\mathcal{G}A_{n,e}).
\]
□ The implications \((\mathcal{G}A_{n,e}) \Rightarrow (\mathcal{I}_n)\) and \((\mathcal{G}A_{n,e}) \Rightarrow (\mathcal{J}_n)\) are immediate.

If now \( a \in \mathbb{P}^n \) and \( P = \prod_{i=1}^{n} a_i \) the implication \((\mathcal{I}_n) \Rightarrow (\mathcal{G}A_{n,e})\) follows by applying \((\mathcal{I}_n)\) to the \( n \)-tuple \( ((a_1/P^{1/n}), \ldots, (a_n/P^{1/n}))\).

A similar argument gives the remaining implication \((\mathcal{J}_n) \Rightarrow (\mathcal{G}A_{n,e})\) □

\(^iii\) Augustin Louis Cauchy,(1789-1857), a French mathematician who worked in Paris; of all mathematicians he is the one most often mentioned.
3.1.3 Theorem  (a) Given that the logarithmic function is continuous and strictly increasing we have:

\((G_Ae) \equiv \log\) is strictly concave.

(a) Given that the exponential function is continuous and strictly increasing we have:

\((G_Ae) \equiv \exp\) is strictly convex.

□ This follows from a very simple proof in [3 pp. 77, 92]. □

3.2 The General Case  After \((G_Ae)\) the next simplest inequality involving the geometric and arithmetic means is:

\[V = V_1 \times V_2, \quad V_1 = \{(x, y) : x, y \in \mathbb{R}^*_+\}, \quad V_2 = \{(u, v) : u, v \in \mathbb{R}^*_+\},\]

\[(GA_2) \quad (x^u y^v)^{1/u+v} \leq \frac{ux + vy}{u + v} \quad \text{or} \quad G_2(x, y; u, v) \leq A_2(x, y; u, v) : (3)\]

\[E = \{(x, y); x = y\}.\]

Some obviously equivalent forms of this are given in the following lemma.

3.2.1 Lemma  \((GA_2)\) is equivalent to either of the following statements:

(a) \[V = V_1 \times V_2, \quad V_1 = \{(x, y) : x, y \in \mathbb{R}^*_+\}, \quad V_2 = \{\alpha; \alpha \in \mathbb{R} \land 0 < \alpha < 1\},\]

\[(GA_2) \quad x^{1-\alpha}y^\alpha \leq (1-\alpha)x + \alpha y : (4)\]

\[E = \{(x, y); x = y\}.\]

(b) \[V = \{(x, y, p, q); (x, y, p, q) \in \mathbb{P}^4 \land \frac{1}{p} + \frac{1}{q} = 1\},\]

\[(Y) \quad xy \leq \frac{x^p}{p} + \frac{y^q}{q} : (5)\]

\[E = \{(x, y); x^p = y^q\}.\]

□ The first statement is an obvious rewriting of \((GA_2)\) as the inequality is just \(G_2(x, y; 1-\alpha, \alpha) \leq A_2(x, y; 1-\alpha, \alpha)\)

The second is a rewriting of the first putting \(1-\alpha = 1/p, \alpha = 1/q\) and then replacing \(x\) by \(x^p\) and \(y\) by \(y^q\). □

Inequality \((Y)\) is sometimes called Young’s inequality\(^{iv}\) although it is really a very special case of that result, see [5 pp. 48–49].

\(^{iv}\)William Henry Young (1863 –1942) was one half of perhaps the most famous mathematical couple, his wife being Grace Chisholm Young (1868–1944); they had a son who was also a famous mathematician Laurence Chisholm Young (1905–2000).
3.2.2 Theorem

\[(\mathcal{G}A_2) \equiv \forall n \in \mathbb{N}^*, (\mathcal{G}A_{n,e}).\]

\[\square\] The one equivalence is trivial since \((\mathcal{G}A_{2,e})\) is a special case of \((\mathcal{G}A_2)\) and implies \((\mathcal{G}A_{n,e}), \forall n \in \mathbb{N}^*\) by 2.1.1 Theorem.

The other equivalence needs more work but follows from known results; see \[3\ pp.80–81\].

(a) if \(w_1, w_2\) are rational then for a suitable \(m \in \mathbb{N}^*\) and \(m\)-tuple \(b\) we can write \(\mathcal{A}_2(a; w)\) as \(\mathcal{A}_m(b)\), and similarly for the geometric mean. So that given \((\mathcal{G}A_{n,e}), \forall n \in \mathbb{N}^*\) we can deduce \((\mathcal{G}A_2)\) when the weights are rational and get the right set \(E\) since \(b\) is constant exactly when \(a\) is constant.

(b) if \(w_1, w_2\) are real the result follows by taking the limit of the rational case except possibly for the set \(E\).

(c) Finally if \(a\) is not constant write \(w_i = q_i + r_i, i = 1, 2\), where \(q_i, i = 1, 2\), is a non-zero rational. Now

\[\mathcal{A}_2(a; w) = \frac{Q_2}{W_2} \mathcal{A}_2(a; q) + \frac{R_2}{W_2} \mathcal{A}_2(a; r)\]

\[> \frac{Q_2}{W_2} \mathcal{G}_2(a; q) + \frac{R_2}{W_2} \mathcal{G}_2(a; r), \text{ by (a)}.\]

\[\geq \frac{Q_2}{W_2} \mathcal{G}_2(a; q) + \frac{R_2}{W_2} \mathcal{G}_2(a; r), \text{ by (b)}.\]

\[\geq (\mathcal{G}_2(a; q))^{\frac{Q_2}{W_2}} (\mathcal{G}_2(a; r))^{\frac{Q_2}{W_2}}, \text{ by (b)}.\]

\[= \mathcal{G}_2(a; w).\]

\[\square\]

3.2.2.1 Remark  The all important last part of the proof seems to be due to Hardy, Littlewood and Pólya\(^v\) the trio of famous mathematicians of the first half of the twentieth century who almost single handedly founded the theory of inequalities with their book \([6]\).

We now generalize 3.1.1 Theorem and agree to write:

\[V = \{(n, a, w); n \in \mathbb{N}^*, a, w \in \mathcal{P}^n\},\]

\[(\mathcal{G}A_n) \quad \mathcal{G}_n(a; w) \leq \mathcal{A}_n(a; w) : (6)\]

\[E = \{a; a \in \mathcal{P}^n\text{and a is a constant}\},\]

the geometric-arithmetic mean inequality of order \(n\).

\(^v\)Geoffrey Harold Hardy (1877–1947) and John Edensor Littlewood (1885–1977) were English and George Pólya (1887–1985) was born in Hungary.
3.2.3 Theorem  If $n, m \in \mathbb{N}^*$ then

$$(GA_n) \equiv (GA_m).$$

□  The proof is essentially the same as that of 3.2.1 Theorem but considerably easier.

(i) $\forall n \in \mathbb{N}^*: (GA_n) \implies (GA_{n+1})$; see [3 pp.90–91].

(i) $\forall n \in \mathbb{N}^*: (GA_{n+1}) \implies (GA_n)$; see [3, p.81]. This is just the Cauchy reverse induction used in the earlier theorem As the proof in the reference is garbled it given here in full and in a slightly more general form. Assume that $n, m \in \mathbb{N}^*, m < n$, and let $b$ be the $n$-tuple defined as

$$b_i = \begin{cases} a_i, & \text{if } 1 \leq i \leq m, \\ A_m(a; w), & \text{if } m < i \leq n. \end{cases}$$

Then by $(GA_n)$, $G_n b; w \leq A_n b; w = A_m a; w$, or

$$(G_m a; w)^{W_m/W_n} (A_m a; w)^{(1-w_m)/W_n} \leq A_m(a; w),$$

that is $\mathcal{G}_m(a; w) \leq \mathcal{A}_m(a; w)$. The set of equality is readily checked. That is we have deduced $(GA_m)$. □

3.2.4 The Inequalities of Rado and Popoviciu  The Rado and Popoviciu inequalities of order $n$ are\(^{vi}\):

$$V = \{(n, a, w); n \in \mathbb{N}^* \land a, w \in \mathbb{P}^n\},$$

$$W_n \mathcal{A}_n(a; w) - \mathcal{G}_n(a; w) \geq W_{n-1} \mathcal{A}_{n-1}(a; w) - \mathcal{G}_{n-1}(a; w) : \quad (7)$$

$$E = \{a; a \in \mathbb{P}^n \land a_n = \mathcal{G}_{n-1}(a; w)\}.$$ (R$_n$)

$$E = \{(n, a, w); n \in \mathbb{N}^* \land a, w \in \mathbb{P}^n\},$$

$$\mathcal{A}_n(a; w)^{1/W_n} \geq \mathcal{A}_{n-1}(a; w)^{1/W_{n-1}} : \quad (8)$$

$$E = \{a; a \in \mathbb{P}^n \land a_n = \mathcal{A}_{n-1}(a; w)\}.$$ (P$_n$)

The inequality (R$_n$) was given as an exercise in [5 p.64] but has been rediscovered many times and variants have been much studied; the multiplicative analogue (P$_n$) was given a little later by Popoviciu. While $(GA_n)$, $n \in \mathbb{N}^*$, says that if $a_1 \neq a_2$ the sequence $W_n \mathcal{A}_n(a; w) - \mathcal{G}_n(a; w))$, $n \in \mathbb{N}^*$, is positive the inequalities (R$_n$), $n \in \mathbb{N}^*$, together with $(GA_2)$, say that this sequence is positive and

\(^{vi}\) Richard Rado (1906–1989; Tiberiu Popoviciu (1906–1975).
increasing. Although this is an apparently stronger statement the inequalities are essentially equivalent.

### 3.2.4.1 Theorem

If $n \in \mathbb{N}^{**}$ then:

$$(R_n) \equiv (GA_2), \quad and \quad (P_n) \equiv (GA_2).$$

□

Again this is a consequence of known results see [3 p.26]. For instance

$$W_n(\mathbb{A}_n(a;w) - \mathbb{G}_n(a;w)) \geq W_{n-1}(\mathbb{A}_{n-1}(a;w) - \mathbb{G}_{n-1}(a;w))$$

can be rewritten as

$$\frac{w_n}{W_n}a_n + \frac{W_{n-1}}{W_n}\mathbb{G}_{n-1}(a;w) \geq \mathbb{G}_n(a;w).$$

If this is valid putting $a_n = x$ and $a_i = y, 2 \leq i \leq n$, gives $(GA_2)$. On the other hand using $(GA_2)$ proves this last inequality.

### 4 Bernoulli’s Inequality

*Convention: Where necessary we put $0^0 = 1$.*

#### 4.1 The Basic Bernoulli-Barrow Inequality

After $(GA_{2,e})$ perhaps the next simplest inequality is

$$V = V_1 \times V_2 \quad V_1 = \{x; x \in \mathbb{R}_+\}, \quad V_2 = \{\alpha; \alpha \in \mathbb{R} \land 0 \leq \alpha \leq 1\},$$

$$(B_1) \quad (1 + x)\alpha \leq 1 + \alpha x : \quad \quad (1)$$

$$E = \{(x, \alpha); x = 0 \lor \alpha = 0 \lor \alpha = 1\}.$$

Writing $F(x, \alpha)$ for the left-hand side of (1) and $A(x, \alpha)$ for the right-hand side and $V = [0, \infty] \times [0, 1]$ then we can express $(B_1)$ as:

$$\forall (x, \alpha) \in V : F(x, \alpha) \leq A(x, \alpha); \quad E = \partial V.$$

It is obvious that $F$ has a natural domain that is larger than $V$. The extension of the above inequality to this natural domain will be considered below.

The result estimates the binomial function $F$ by by the linear function $A$ and goes under the name of Bernoulli’s inequality. The original Bernoulli\textsuperscript{vii} inequality only considered the simplest case of the binomial function, namely $(1 + x)^n, n \in \mathbb{N}^{**}.

\textsuperscript{vii}Jacob Bernoulli (1654–1705), a Swiss mathematician who worked in Basel. The inequality was proved 20 years earlier by the British mathematician Isaac Barrow, (1630–1677). The inequality should be called the Barrow-Bernoulli inequality.
We now consider extending the inequality \((B_1)\) to the natural domain of the function \(F\).

First note an easy equivalent form of \((B_1)\):

\[
V = V_1 \times V_2, \quad V_1 = \{x; x \in \mathbb{R}^*_+\}, \quad V_2 = \{\alpha; \alpha \in \mathbb{R} \land 0 \leq \alpha \leq 1\},
\]

\[
(B_1) \quad (1 + x)^{1-\alpha} \leq 1 + (1 - \alpha)x : \quad (2)
\]

\[
E = \{(x, \alpha); x = 0 \lor \alpha = 0 \lor \alpha = 1\}.
\]

Now consider the rest of the natural domain for the variable \(x\).

\[
V = V_1 \times V_2, \quad V_1 = \{x; x \in \mathbb{R} \land 0 < x \leq 1\}, \quad V_2 = \{\alpha; \alpha \in \mathbb{R} \land 0 \leq \alpha \leq 1\},
\]

\[
(B_2) \quad (1 + x)^{\alpha} \leq 1 + \alpha x : \quad (1)
\]

\[
E = \{(x, \alpha); x = 0 \lor \alpha = 0 \lor \alpha = 1\}.
\]

4.1.1 Theorem \((B_1) \equiv (B_2)\). 

\(\square\)

(i) Consider the function \(\phi(x) = \frac{-x}{1 + x}\); it easy to see that if \(-1 \leq x \leq 0\) then \(\phi(x) \geq 0\), and obviously \(\phi(x) = 0\) if and only if \(x = 0\). Hence by (2)

\[
\left(1 - \frac{x}{1 + x}\right)^{1-\alpha} \leq 1 - (1 - \alpha)\frac{x}{1 + x}, \quad \text{or} \quad (1 + x)^{\alpha} \leq 1 + \alpha x,
\]

and the inequality is strict unless \(x = 0\), \(\alpha = 0\) or \(\alpha = 1\).

(ii) Now note that if \(-1 \leq \phi(x) \leq 0\) then \(x \geq 0\) and so we can reverse the preceding argument \(\square\)

Combining the two inequalities \((B_1)\) and \((B_2)\) gives

\[
V = V_1 \times V_2, \quad V_1 = \{x; x \in \mathbb{R} \land x > -1\}, \quad V_2 = \{\alpha; \alpha \in \mathbb{R} \land 0 \leq \alpha \leq 1\},
\]

\[
(B_3) \quad (1 + x)^{\alpha} \leq 1 + \alpha x : \quad (1)
\]

\[
E = \{(x, \alpha); x = 0 \lor \alpha = 0 \lor \alpha = 1\}.
\]

4.1.1.1 Corollary \((B_3) \equiv (B_1)\).

We finally consider the natural domain of \(\alpha\)

\[
V = V_1 \times V_2, \quad V_1 = \{x; x \in \mathbb{R}^*_+\}, \quad V_2 = \{\alpha; \alpha \in \mathbb{R} \land \alpha > 1\},
\]

\[
(B_4) \quad (1 + x)^{\alpha} \geq 1 + \alpha x : \quad (\sim 1)
\]

\[
E = \{(x, \alpha); x = 0\}.
\]
4.1.2 Theorem \((\mathcal{B}_1) \equiv (\mathcal{B}_4)\).

Since \(0 < 1/\alpha < 1\) we have by (1) that:

\[(1 + \alpha x)^{1/\alpha} < 1 + x,\]

which gives \((\sim 1)\). Reversing the argument and using \((\sim 1)\) gives (1).

4.1.3 Theorem \((\mathcal{B}_5) \equiv (\mathcal{B}_4)\).

Since \(1 - \alpha > 1\) and by 4.1.2 Theorem:

\[(1 + x)^{1-\alpha} > 1 + (1 - \alpha)x.\]

Using the function \(\phi(x)\) introduced in the proof of 3.1.2 Theorem,

\[\left(1 - \frac{x}{1+x}\right)^{1-\alpha} > 1 - (1 - \alpha)\frac{x}{1+x}, \text{ or } (1 + x)^\alpha > 1 + \alpha x,\]

This argument can be reversed and this completes the proof.

Now using the arguments that obtained 4.1.1 Theorem we can obtain inequalities equivalent to \(\mathcal{B}_4\) and \(\mathcal{B}_5\) but with the range of \(x\) as in \(\mathcal{B}_2\), that is \(-1 < x \leq 0\); we will neither state not prove these.

Now state the full Bernoulli-Barrow inequality:

\[V = V_1 \times V_2, \quad V_1 = \{x : x \in \mathbb{R}_+^x\}, \quad V_2 = \{\alpha : \alpha \in \mathbb{R} \land \alpha < 0\},\]

\[(\mathcal{B}_5) \quad (1 + x)^\alpha \geq 1 + \alpha x : \quad (\sim 1)\]

\[E = \{(x, \alpha); x = 0\}.\]

4.1.4 Theorem \((\mathcal{B}) \equiv (\mathcal{B}_1)\)

Immediate from the above discussion.
Also as above we can readily write an equivalent form of (B)

\[ V = \{(x, \alpha); x > -1 \wedge 0 \leq \alpha \leq 1\}, \]
\[ (1 + x)^{1-\alpha} \leq 1 + (1 - \alpha)x : \]
\[ V = \{(x, \alpha); x > -1 \wedge \alpha < 0 \lor \alpha > 1\}, \]
\[ (1 + x)^{1-\alpha} \geq 1 + (1 - \alpha)x : \]
\[ E = \{(x, \alpha); x = 0 \lor \alpha = 0 \lor \alpha = 1\}. \]

4.1.4.1 The inequality (2) can be rewritten as follows:

\[ (1 + x)^{\alpha} \geq \frac{1}{1 - \frac{\alpha x}{1 + x}} \]

There is a similar variant of (∼2) provided \(1 + (1 - \alpha)x > 0\); that is if \(\alpha > 1\) then \(-1 < x < 1/(\alpha - 1)\), and if \(\alpha < 0\) then \(x > 1/(\alpha - 1)\).

4.1.5 The Original Bernoulli Inequality Before leaving this topic it is worth noting that the historical Bernoulli inequality is equivalent to (B), as was pointed out in [5 pp. 40–41].

\[ V = V_1 \times V_2, \quad V_1 = \{x; x \in \mathbb{R}^*_+\}, \quad V_2 = \{n; n \in \mathbb{N}^{**}\} \]
\[ (1 + x)^n \geq 1 + nx : \]
\[ E = \{x = 1\}. \]

4.1.5.1 Theorem \((OB) \equiv (B)\).

\[ \square \quad \text{The one implication is trivial and the other is in the above reference.} \]

The proof of the one implication given by Hardy, Littlewood and Pólya is a little complicated but depends on the following interesting lemma.

4.1.5.2 Lemma If \(y > 0\) and \(n \in \mathbb{N}^*\) then

\[ \frac{y^{n+1} - 1}{n+1} \geq \frac{y^n - 1}{n}, \]

with equality if and only if \(y = 1\)

\[ \square \quad \text{Inequality (3) is equivalent to} \]

\[ p(x) = ny^{n+1} - (n+1)y^n + 1 \geq 0, \quad y > 0. \]

Elementary arguments, see [6 pp.3–4], show that the polynomial \(p\) has two positive roots, a double root when \(y = 1\). This give (3) and the case of equality.
4.1.5.3 Corollary  If \( y > 0 \) and \( p, q \in \mathbb{N}^* \), \( p > q \), then
\[
\frac{y^p - 1}{p} \geq \frac{y^q - 1}{q},
\]  
(3)
with equality if and only if \( y = 1 \).

4.2 Variants of the Bernoulli Inequality

4.2.1 Changes of Variable  If \( I \) is a subinterval of \( \mathbb{R} \) and \( \phi: I \rightarrow] -1, \infty[ \) then \((B)\) gives

\[
V = \{(x, \alpha); \ x \in I \land 0 \leq \alpha \leq 1\}, \quad (I)
\]
\[
(1 + \phi(x))^\alpha \leq 1 + \alpha \phi(x) \quad : (I) \quad (4)
\]
\[
V = \{(x, \alpha); \ x \in I \land \alpha < 0 \lor \alpha > 1\}, \quad (I)
\]
\[
(1 + \phi(x))^\alpha \geq 1 + \alpha \phi(x) \quad : \sim (4)
\]
\[
E = \{(x, \alpha); \phi(x) = 0 \lor \alpha = 0 \lor \alpha = 1\}.
\]

Further if \( \phi \) is strictly monotonic applying \( \phi^{-1} \) we can reverse the argument and so in this case \((I) \equiv (B)\).

Such an argument has been used in the proofs of 4.1.1 Theorem and 4.1.2 Theorem. Applying this idea the following examples give various inequalities that are equivalent to \((B)\).

4.2.1.1 Example  \( I = ]-\infty, 1]\), \( \phi(x) = -x \); \( \phi \) is zero at \( x = 0 \). So from \((I)\):

\[
V = \{(x, \alpha); \ x < 1 \land 0 \leq \alpha \leq 1\}, \quad \text{(5)}
\]
\[
(1-x)^\alpha \leq 1 - \alpha x.
\]
\[
V = \{(x, \alpha); \ x < 1 \land \alpha \leq 0 \land \alpha \geq 1\}, \quad \text{(5)}
\]
\[
(1-x)^\alpha \geq 1 - \alpha x.
\]
\[
E = \{(x, \alpha); \ x = 0 \lor \alpha = 0 \lor \alpha = 1\}.
\]

4.2.1.2 Example  \( I = ]0, \infty[\), \( \phi(x) = x - 1 \); \( \phi \) is zero at \( x = 1 \). So from \((I)\):

\[
x^\alpha \leq (1-\alpha) + \alpha x \quad : (6)
\]
\[
V = \{(x, \alpha); \ x \in \mathbb{R}^*_+ \land 0 \leq \alpha \leq 1\}, \quad \text{(5)}
\]
\[
x^\alpha \geq (1-\alpha) + \alpha x \quad : \sim (6)
\]
\[
E = \{(x, \alpha); \ x = 1 \lor \alpha = 1 \lor \alpha = 1\}.
\]
4.2.2 Increasing the Number of Variables

(A) If $S$ is a set in $\mathbb{R}^n$ and $\phi : S \mapsto [-1, \infty]$ and $x = (x_1, \ldots, x_n)$ then (B) gives

$$V = \{(x, \alpha); x \in S \land 0 \leq \alpha \leq 1\},$$

$$(1 + \phi(x))^\alpha \leq 1 + \alpha \phi(x); \quad (7)$$

$$(\mathcal{J})$$

$$V = \{(x, \alpha); x \in S \land \alpha < 0 \land \alpha > 1\},$$

$$V = \{(x, \alpha); \phi(x) = 0 \lor \alpha = 0 \lor \alpha = 1\}.$$ 

If $\phi(x(t)) = t$ for some function $x$ then (6) implies (B) and so in this case $\mathcal{J} \equiv (B)$.

4.2.2.1 Example Let $S = \mathbb{R}^2$, $\phi(u, v) = v/u$ and apply the above idea in the situation of 4.2.1.2 Example. Then 4.2.1.2 (5) becomes

$$\left(\frac{v}{u}\right)^\alpha \leq (1 - \alpha) + \alpha \frac{v}{u} \implies u^{1-\alpha} v^\alpha \leq (1 - \alpha) u + \alpha v,$$

with equality when $\phi(u, v) = 1$, that is if $u = v$.

This however is, omitting the cases $\alpha = 0, 1$, just 4.2.1 (4) showing that this part of the Bernoulli inequality is essentially equivalent to $(GA_2)$.

4.2.2.2 Theorem $(B) \equiv (GA_2)$. \hfill \Box

4.2.2.3 Rüthing’s inequality\textsuperscript{viii};

\textsuperscript{viii} Dieter Rüthing; but the result appears earlier in [6 pp.39–42].
If $S = \mathbb{P}^2$, $\phi(a, b) = a/b - 1$; $\phi$ is zero when $a = b$; and $\phi(x, 1) = x$. From $(J)$ we get that

$$V = \{(a, b, \alpha); a, b \in \mathbb{R}^*_+ \land 0 \leq \alpha \leq 1\},$$

$$aa^\alpha - 1(b - a) \leq a^\alpha - b^\alpha \leq ab^\alpha - 1(a - b)$$

($(RU)$

$$V = \{(a, b, \alpha); a, b \in \mathbb{R}^*_+ \land 0 \leq \alpha \leq 1\},$$

$$aa^\alpha - 1(b - a) \geq a^\alpha - b^\alpha \geq ab^\alpha - 1(a - b);$$

$$E = \{(a, b); a = b \lor \alpha = 0 \lor \alpha = 1\};$$

4.2.2.4 Theorem $(RU) \equiv (B)$

□ Immediate from the general discussion. □

A rewriting of part of (7):

$$aa^\alpha + ba \geq aa^\alpha - 1b$$

is called Jacobsthal’s inequality and being equivalent to a part of $(B)$ is another inequality that is equivalent to $(B)$,

4.2.2.5 Remark It might be noted that $(RU)$ has a slightly different form for $V$ and $E$ in that in the left-hand inequalities $b = 0$ is allowed and in the right-hand inequalities $a = 0$ is allowed.

$(B)$ Alternatively if $S$ is a set in $\mathbb{R}^n$ and if $\psi : S \mapsto \mathbb{R}$ and $a = (a_1, \ldots, a_n)$ then $(B)$ gives

$$V = \{(x, a); x > -1 \land a \in S \land 0 \leq \psi(a) \leq 1\},$$

$$(1 + x)^{\psi(a)} \leq 1 + \psi(a)x :$$

$(J)$

$$V = \{(x, a); x > -1 \land a \in S \land \psi(a) < 0 \lor \psi(a) > 1\},$$

$$(1 + x)^{\psi(a)} \geq 1 + \psi(a)x :$$

$$E = \{(x, a); x = 0 \lor \psi(a) = 0 \lor \psi(a) = 1\}.$$
16 Equivalent Inequalities

\[ V = V_1 \cup V_2, \text{ where} \]
\[ V_1 = \{(x, p, q); x, p, q \in \mathbb{R} \land x \geq 0 \land 0 \leq p \leq q \leq -x \leq 0\}, \]
\[ V_2 = \{(x, p, q); x, p, q \in \mathbb{R} \land x \leq 0 \land 0 \leq -x < p \leq q \leq 0 \leq q < 0\}, \]
\[ (BU) \]
\[ (1 + x/p)^p \leq (1 + x/q)^q: \quad (8) \]
\[ (1 + x/p)^p \geq (1 + x/q)^q: \quad (\sim 8) \]
\[ E = \{(x, p, q); x = 0 \lor p = q\}. \]

Clearly (8) implies (B) and so is equivalent to (B). This result is stated in [2 p.8; 3 pp.36–37] but the expositions leave a lot to be desired. The result is a little complicated anyway but says: if the expressions in (8) have meaning and if \( p < q \) then (8) holds if \( p \) and \( q \) have the same sign, but (\sim 8) holds if \( p \) and \( q \) have opposite signs.iii

4.2.3 Inductive Variants It is often possible to disguise an inequality such as the ones we are discussing by an apparent generalization involving induction. Thus if \( x \geq -1, y \geq -1, \alpha \geq 0, \beta \geq 0, 0 \leq \alpha + \beta \leq 1, \) and with no loss in generality assume that \( y \leq x \), also assume that not both \( \alpha \) and \( \beta \) are zero, that is \( \alpha + \beta \neq 0:\)

\[
(1 + x)^{\alpha} (1 + y)^{\beta} = \left(1 + \frac{x - y}{1 + y}\right)^{\alpha/(\alpha+\beta)} (1 + y)^{\alpha+\beta} \\
\leq \left(1 + \frac{x - y}{\alpha + \beta}ight)^{\alpha+\beta} (1 + y)^{\alpha+\beta}, \text{ by (B)}, \\
= \left(1 + \frac{x}{\alpha + \beta} + \frac{\beta}{1 + y}\right)^{\alpha+\beta} \\
\leq (1 + \alpha x + \beta y) \quad \text{by (B)},
\]

the inequality is strict unless either (i) \( \alpha + \beta = 0 \), or (ii) \( \alpha + \beta = 1, x = y \), or (iii) \( 0 < \alpha + \beta < 1, x = y = 0 \). Since this inequality reduces to (B) if \( x = y \) it is equivalent to the more elementary result. Further this inequality is the first step in an induction to an even more general looking equivalent result, due to Pečarić, that is equivalent to (B).

\[ V = \{(n, a, w); n \in \mathbb{N}^+ \land a_i > -1, 1 \leq i \leq n \land a \in \mathbb{R}^n, 0 \leq W_n \leq 1\} \]

iii A different proof of part of (8) can be found in [7 p.365]. It is easy to get this result by a direct study of the function \( f(r) = (1 + x/r)^r \); \( f \) is strictly increasing on the two intervals of its domain; the domain is the complement of the interval \([(-x)^-), (-x)^+\]; the infimum of \( f \) in the left interval of the domain is the same as the supremum in the right interval, being \( e^r \); see below 4.3.1 Theorem.
Equivalent Inequalities

\( (E_n) \quad \prod_{i=1}^{n}(1+a_i)^{w_i} \leq 1 + \sum_{i=1}^{n} w_ia_i \) : \( B_n \) \( \frac{1}{1 + \sum_{i=1}^{n} w_i a_i} \)

\( E = \{ W_n = 0 \lor W_n = 1 \land a \text{ is constant} \lor 0 < W_n < 1 \land a = 0 \} \).

The proof of this inequality, by Pečarić, is not readily available so will be included for completeness.

If \( n = 1, 2 \) then (10) is just \( (B_1) \) and (9) respectively, so assume that \( n \geq 3 \) and \( (B_{n-1}) \).

\[
\prod_{i=1}^{n}(1+a_i)^{w_i} = \left( \prod_{i=1}^{n-1}(1+a_i)^{w_i/W_{n-1}} \right) W_{n-1}^{W_{n-1} - 1}(1+a_n)^{w_n}
\]

\[
\leq \left( 1 + \frac{1}{W_{n-1}} \sum_{i=1}^{n-1} w_i a_i \right)^{W_{n-1} - 1}(1+a_n)^{w_n}, \text{by the induction hypothesis,}
\]

\[
\leq 1 + \sum_{i=1}^{n} w_i a_i, \text{by (9), noting that } \frac{1}{W_{n-1}} \sum_{i=1}^{n-1} w_i a_i > -1.
\]

A similar argument can be used to prove that \( (\sim 10) \) holds if we assume that either \( w_i \leq 0, 1 \leq i \leq n \) or \( w_i \geq 1, 1 \leq i \leq n \).

4.2.3.1 Theorem \hspace{1em} \( \forall n \in \mathbb{N}^* \quad (E_n) \equiv (B) \). More generally:

\( \forall n, m \in \mathbb{N}^* \quad (E_n) \equiv (E_m) \).

Suppose than we have \((E_n)\) for a particular \( n \) then by taking \( x_1 = \cdots = x_n \) we get \((B)\) so these inequalities are equivalent. More generally in this way we see that \((E_{n_1})\) holds if and only if \((E_{n_2})\) holds.

4.3 Properties of Functions

The inequality \((B)\) can be used to obtain the properties of certain functions, that in turn are equivalent to the inequality. We see an example of this in 4.2.2.6 (7) which can be rephrased as saying that if \( x > 0 \) then the function \( f(r) = (1 + x/r)^r \) : \( \mathbb{R}^+ \to \mathbb{R} \) is strictly increasing; this is elaborated n the following theorem.

4.3.1 Theorem \hspace{1em} If \( a \in \mathbb{R} \), \( S = ]\infty, (-a)^- [ \cup ](-a)^+, \infty] \) then:

(a) \( f(x) = (1 + a/x)^x : S \to \mathbb{R} \) is strictly increasing on each of the intervals of \( S \); further if \( x < (-a)^- \), \( y > (-a)^+ \) then \( f(x) > f(y) \);

(b) \( g(x) = (1 + a/x)^x+ : S \to \mathbb{R} \) is strictly decreasing on each of the intervals of \( S \); further if \( x < (-a)^- \), \( y > (-a)^+ \) then \( g(x) < g(y) \).

(a) This is just \((B_0)\), see 4.2.2.6 note (viii).
(b) Similarly if \( x < y \) both in one of the intervals of \( S \) then \(-(y-a) < -(x-a)\) and both of these quantities are also in the same interval of \( S \) and so:

\[
(1 + \frac{a}{y})^{(y+a)} = (1 + \frac{a}{y+a})^{-(y+a)} < (1 + \frac{a}{x+a})^{-(x+a)}, \text{ by (a)}
\]

\[
= (1 + \frac{a}{x})^{(x+a)}.
\]

A similar argument can be used to complete the proof of (b).

\[\square\]

4.3.1.1 Corollary  
(a) The monotonicity property of either of the functions \( f, g \) implies that of the other.
(b) The monotonicity property of each of the functions \( f, g \) implies (B).
(c) All of the limits \( \lim_{x \to \pm \infty} f(x) \) and \( \lim_{x \to \pm \infty} g(x) \) exist and are equal to \( e^a \).

\[\square\]  
(a) This fact for the two functions \( f \) and \( g \) follows from the proof of part (c) of the theorem. The rest follows from (b).

(b) Simple changes in variable prove this.

(c) Since \( 0 < f < g \) it follows from the theorem that both of the limits exist. Further, since \( 0 < g(x)/f(x) = (1 + a/x)^a \), the two limits are the same. The value of the limit is a well known property of the exponential function.

\[\square\]

5 Mean Inequalities

An important part of inequalities is the area of inequalities between means. As can be seen from the literature, see for instance [3, 8, 9], there are many types of means but in this section we will concentrate on the power means.

The fundamental inequality between these means is the following result, called the power mean inequality\(^{\text{ix}}\)

\[
V = \{(n, a, w, r, s) : n \in \mathbb{N}^*; a, w \in \mathbb{P}^n; r, s \in \mathbb{R} \land r < s\},
\]

\[
(\mathcal{R}; \mathcal{S}_n) \quad \mathfrak{M}_{n}^{[r]}(a; w) \leq \mathfrak{M}_{n}^{[s]}(a; w) : (r, s) \in E = \{a \in \mathbb{P}^n, a \text{ is constant}\}.
\]

There are several proofs of this inequality and a discussion of its history in most books on inequalities; see [3 pp.202–207].

\(^{\text{ix}}\) In the equal weight case this inequality is due to Oscar Xavier Schlömich, (1823-1901), a French-born mathematician. He worked in Jena and Dresden. Cauchy’s techniques in analysis became well known in Germany through his textbook. The general case was given in 1879 by Davide Besso (1845-1906), an Italian mathematician who worked in Roma and Modena
Equivalent Inequalities

Special cases of this result are the geometric-arithmetic mean inequality and the harmonic-geometric mean inequality:

\[ G_n(a; w) \leq A_n(a; w), \quad (0, 1) \]
\[ H_n(a; w) \leq G_n(a; w); \quad (-1, 0) \]

and \((R; S_n)\) in these cases is referred to as \((GA_n)\) and \((HG_n)\) respectively.

It follows from this result that both of the limits \(\lim_{r \to \pm \infty} M_n[r](a; w)\) exist and it can be shown that:

\[ \lim_{r \to \infty} M_n[r](a; w) = \max a, \quad \lim_{r \to -\infty} M_n[r](a; w) = \min a. \]

In addition

\[ \lim_{r \to 0, r \neq 0} M_n[r](a; w) = G_n(a; w); \]

see [3 pp.175–177].

It turns out that many of the inequalities contained in the collection \((R; S_n)\) are equivalent and this is what we now discuss.

5.1 Theorem  Given \(n, a, w, r, s \in V\):

(a) if \(r < s < 0\) then \((R; S_n)(r, s) \equiv (R; S_n)(-s, -r)\);

(b) if \(r \leq 0 \leq s\) then \((R; S_n)(r, s) \equiv (GA_n)\);

(c) if \(0 < r < s\), then \((R; S_n)(r, s) \equiv (R; S_n)(1, s/r) \wedge (R; S_n)(r/s, 1)\).

(d) if \(1 < s\) then \((R; S_n)(1, s) \equiv (R; S_n)(1/s, 1)\).

\(\square\)  (a) This follows from the second part of the useful identity. If \(r, s \in \mathbb{R^+}, t = s/r, b = a^r, c = a^t\) then

\[ M_n[s](a; w) = (M_n[r](b; w))^{t/s} = (M_n[r](c; w))^{1/t} \quad (1) \]

(b) This follow from the identity obtained by taking \(r = s\) in the first identity in (1), that is:

\[ M_n[s](a; w) = (A_n(b; w))^{1/s}. \quad (2) \]

(c), (d) These follow by again using (2).

\(\square\)

We now turn to equivalencies amongst the simpler inequalities \((GA_n)\).

5.2 Theorem  (a) \((B) \equiv (GA_2)\).

(b) If \(m, n \in \mathbb{N^*}\) then \((GA_{m,n,e}) \equiv (GA_{n,e})\).

(c) If \(n \in \mathbb{N^*}\) then \((GA_{n,e}) \equiv (GA_n)\).

\(\square\)  (a) See 4.2.2.2 Theorem.
(b) This is implied by the famous Cauchy backward induction, see [3 pp.81–82].

(c) See [3 pp.81–82]. □

6 The Hölder and Minkowski Inequalities

Two other basic inequalities are the Hölder and Minkowski inequalities:

(a) \[ V = \{ (n, p, a, b); n \in \mathbb{N}^*; p \in \mathbb{R} \land p > 1; a, b \in \mathbb{F}^n \}, \]

(b) \[ (H_{n,p}) \quad \sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} b_i^{p-1} \right)^{1-1/p} : \]

(c) \[ E = \{ a, b \in \mathbb{F}^n \land \exists \lambda, \mu \in \mathbb{R} \text{ such that } \lambda a^p + \mu b \frac{1}{p} = 0; \}

The inequality \( (H_{n,2}) \) is usually called the Cauchy inequality denoted by \( (C_n) \)

and we have the following result.

6.1 Theorem  Given \( n \in \mathbb{N}^*; p \in \mathbb{R}; p > 1 \), \( (H_{n,p}) \equiv (M_{n,p}). \)

The inequality \( (H_{n,2}) \) is usually called the Cauchy inequality denoted by \( (C_n) \)

and we have the following result.

6.2 Theorem  If \( n \in \mathbb{N}^*; p \in \mathbb{R}; p > 1 \) then \( (H_{n,p}) \equiv (C_n). \)

6.3 Theorem  If \( n \in \mathbb{N}^*; p \in \mathbb{R}; p > 1 \) then \( (H_{n,p}) \equiv (R; S_n)(1, 1/p). \)

---

\( x \) Recently the name Hölder-Rogers inequality has been suggested as being more in keeping with the historical record; [7]. Otto Ludwig Hölder, (1859-1937), German mathematician who worked in Leipzig and proved the inequality in 1889. Leonard James Rogers, (1862-1933), an English mathematician who worked in Leeds and Oxford proved an equivalent inequality in 1888. Later Frigyes Riesz (1880-1956), a Hungarian mathematician who and worked in Budapest and gave a proof for both sums and for the integrals in 1910.

\( xii \) Also called the Cauchy-Schwarz-Buniakovskii inequality and also by any one of those three names, all namings being reasonable. Cauchy gave the inequality for finite sums in 1825. Victor Yakovlevich Buniakovski, B./Ya. Buniakovski, also transliterated as Bunyakovski, (1804-1889), a Russian mathematician who worked in St. Petersburg and proved the result for integrals in 1859. Hermann Amandus Schwarz (1843-1921), German mathematician who worked in Göttingen and Berlin and proved the result in 1885 for integrals. The inequality for spaces with inner product was proved by Hermann Günther Grassman (1809–1877) and later by Hermann Klaus Hugo Weyl (1809–1955) in 1918.
The inequality \((M_{n,2})\) is usually called the triangle inequality, written \((T_n)\).

**6.4 Theorem** If \(n \in \mathbb{N}^{**}, p \in \mathbb{R}, p > 1\) then \((M_{n,p}) \equiv (T_n)\).

In addition these inequalities can be proved by induction resulting in the following equivalencies.

**6.5 Theorem** Given \(p \in \mathbb{R}, p > 1\):

(a) \((H_{2,p}) \equiv \forall n \in \mathbb{N}^{**} (H_{n,p})\).

(b) \((M_{2,p}) \equiv \forall n \in \mathbb{N}^{**} (M_{n,p})\).

\(\square\) See [3 pp.183–185, 191–192].

The value of the parameter \(p\) in both of these inequalities can be extended to all of \(\mathbb{R}\) and the resulting inequality is as follows. First let us introduce the following notation: if \(p \in \mathbb{R} \setminus \{1\}\) then the conjugate index \(p'\) is defined by

\[(p - 1)(p' - 1) = 1:\text{ or: if } p \neq 0, \frac{1}{p} + \frac{1}{p'} = 1, \text{ and if } p = 0, p' = 0.\]

Note: \(p > 1 \implies p' > 1,\) \(0 < p < 1 \implies p' < 0;\) \(p < 0 \implies 0 < p' < 1.\)

\[V = \{(n,p,a,b); n \in \mathbb{N}^{**}; p \in \mathbb{R}^* \setminus \{1\}; a, b \in \mathbb{F}^n\},\]

\[(\tilde{H}_{n,p}) \quad \left(\sum_{i=1}^{n} a_i b_i\right)^{p} \leq \left(\sum_{i=1}^{n} a_i^{p}\right)^{p'} \left(\sum_{i=1}^{n} b_i^{p'}\right)^{p} :\]

\[E = \{a, b \in \mathbb{F}^n \land \exists \lambda, \mu \in \mathbb{R} \text{ such that } \lambda a^p + \mu b^{p'} = 0\}\]

If \(p > 1\) then (3) is just (1) while if \(p < 1\) then (3) is the same as \((\sim 1)\).

**6.6 Theorem** (a) If \(0 < p < 1\) then \((H_{n,p}) \equiv (H_{n,1/p})\).

(b) If \(p < 0\) then \((H_{n,p}) \equiv (H_{n,p'})\).

\(\square\) See [6 pp.24–25].

**6.6.1 Corollary** For \(p \in \mathbb{R}^* \setminus \{1\}\) \((\tilde{H}_{n,p}) \equiv \forall p \in \mathbb{R}, p > 1 (H_{n,p})\)

In a similar way we can extend Minkowski’s inequality.

\[V = \{(n,p,a,b); n \in \mathbb{N}^{**}; p \in \mathbb{R}^*; p \geq 1; a, b \in \mathbb{F}^n\},\]

\[\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^{p}\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^{p}\right)^{1/p}:\]

\[\left(\tilde{M}_{n,p}\right) \quad \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/p} \geq \left(\sum_{i=1}^{n} a_i^{p}\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^{p}\right)^{1/p}:\]

\[E = \{a, b \in \mathbb{F}^n \land p = 1 \lor p \neq 1 \lor \exists \lambda, \mu \in \mathbb{R} \text{ such that } \lambda a + \mu b = 0\}\]

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6.7 Theorem \( \forall p \in \mathbb{R}^*, (\tilde{M}_{n,p}) \equiv \forall p \in \mathbb{R}, p \geq 1 (M_{n,p}) \)

\[ \blacktriangleleft \text{See [6 pp.30–32].} \]\[ \blacktriangleleft \]

6.8 Theorem \( \forall r, s \in \mathbb{R}, (\mathcal{R}, S_n)(r, s) \equiv \forall p \in \mathbb{R}^*, (\tilde{H}_{n,p}) \)

\[ \blacktriangleleft \text{A well known proof of } (\tilde{H}_{n,p}) \text{ shows that } (GA_n) = \Rightarrow (H_{n,p}); \text{ see [3 pp.178–179].} \]

It is also known that \( (H_{n,p}) = \Rightarrow \forall s \geq 1, (\mathcal{R}, S_n)(1, s) \); see 6.3 Theorem.

The rest of the equivalences then follows from 5.1 Theorem. \[ \blacktriangleleft \]

6.9 Other Equivalencies It is a notorious fact that many well known inequalities are just a well-known inequality in what is often an almost impenetrable disguise. This disguise is brought about by the various changes of variables described in section 4.2 above. We now look at some of these “hidden” inequalities but first consider a couple of easy equivalences.

6.9.1 Weighted Inequalities First let us note a very elementary fact that the above inequalities are equivalent to weighted forms in which (1), (2) and (3) become:

\[
\sum_{i=1}^{n} w_i a_i b_i \leq \left( \sum_{i=1}^{n} w_i a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} w_i b_i^{1/p} \right)^{1-1/p}, \\
\left( \sum_{i=1}^{n} w_i(a_i + b_i)^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} w_i a_i^p \right)^{1/p} + \left( \sum_{i=1}^{n} w_i b_i^p \right)^{1/p}, \\
\left( \sum_{i=1}^{n} w_i a_i b_i \right)^{p'} \leq \left( \sum_{i=1}^{n} w_i a_i^p \right)^{p'} \left( \sum_{i=1}^{n} w_i b_i^{p'} \right)^{p},
\]

respectively, and where of course \( w \in \mathbb{P}^n \).

(1w) is just (1) applied to the \( n \)-tuples \( w^{1/p}a \) and \( w^{(1-1/p)b} \), and (1) is just (1w) where \( w \) is constant. A similar argument shows that (3w) and (3) are equivalent.

(2w) is just (2) applied to the \( n \)-tuples \( w^{1/p}a \) and \( w^{1/p}b \), and (2) is just (2w) where \( w \) is constant. \[ \blacktriangleleft \]

Of course there is an equivalent weighted inequality \( \sim 2w \).

6.9.2 Radon’s Inequality Another inequality that is equivalent to Hölder’s inequality is the following

\[
V = V_1 \times V_2, \quad V_1 = \{ (n, a, b); n \in \mathbb{N}^*, a, b \in \mathbb{P}^n \}, \quad V_2 = \{ s; s \in \mathbb{R} \land 0 < s < 1 \}.
\]
\sum_{i=1}^{n} a_i^s b_i^{1-s} \leq \left( \sum_{i=1}^{n} a_i \right)^s \left( \sum_{i=1}^{n} b_i \right)^{1-s} : (4)

V = V_1 \times V_3, \quad V_3 = \{s; s \in \mathbb{R} \land s < 0 \lor s > 1\}; \quad \left( \sum_{i=1}^{n} a_i \right)^s \left( \sum_{i=1}^{n} b_i \right)^{1-s} : (\sim 4)

E = \{a, b; a \sim b\}.

□ This is seen to be equivalent to \((\tilde{\mathcal{H}}_{n,p})\) by a simple change of variable; see \[3 pp.181–182\]. □

If (4) and \((\sim 4)\) are written as

\[
\sum_{i=1}^{n} \frac{a_i^s}{b_i^{1-s}} \leq \left( \sum_{i=1}^{n} a_i \right)^s \left( \sum_{i=1}^{n} b_i \right)^{s-1}
\]

\[
\sum_{i=1}^{n} \frac{a_i^s}{b_i^{1-s}} \geq \left( \sum_{i=1}^{n} a_i \right)^s \left( \sum_{i=1}^{n} b_i \right)^{s-1}
\]

the result is called Radon’s \[xixi\] inequality. In \[6 p.61\] it is set as an exercise and in \[3 pp.181–182\] it is Theorem 3(c) and it is not noticed to be a mere rewriting of Theorem 3(a).

6.9.3 Liapunov’s Inequality The inequality known as Liapunov’s \[xiv\] inequality is the following.

\[
V = V_1 \times V_2, \quad V_1 = \{(n, x, w); n \in \mathbb{N}^*; x, w \in \mathbb{P}^n\},
\]

\[
V_2 = \{(r, s, t); r, s, t \in \mathbb{R} \land t < s < r \lor r < t < s \lor s < r < t\},
\]

\[
\left( \sum_{i=1}^{n} w_i x_i^r \right)^{r-t} \leq \left( \sum_{i=1}^{n} w_i x_i^s \right)^{r-s} \left( \sum_{i=1}^{n} w_i x_i^t \right)^{s-t} : (5)
\]

\[
(L_{n,r,s,t})
\]

\[
V = V_1 \times V_3,
\]

\[
V_3 = \{(r, s, t); r, s, t \in \mathbb{R} \land t < r < s \lor s < t < r \lor r < s < t\},
\]

\[
\left( \sum_{i=1}^{n} w_i x_i^r \right)^{r-t} \geq \left( \sum_{i=1}^{n} w_i x_i^s \right)^{r-s} \left( \sum_{i=1}^{n} w_i x_i^t \right)^{s-t} : (\sim 5)
\]

\[E = \{x \in \mathbb{P}^n \land x \text{ is constant}\}.
\]

6.9.3.1 Theorem \[\forall r, s, t \in \mathbb{R} (L_{n,r,s,t}) \equiv \forall p \in \mathbb{R} (\tilde{\mathcal{H}}_{n,p})\]

\[xixi\] Johann Radon (1887-1956), Czech-born mathematician. He worked in Vienna and proved the inequality in 1913.

\[xiv\] Aleksandr Mikhailovič Liapunov (1857–1918), A. M. Liapunov: also transliterated as Liapunoff, Lyapunov a Russian mathematician who worked in Kharkov and St. Petersburg. He gave this inequality in 1901.
(\mathcal{L}_{n,r,s,t}) follows from the weighted (\widetilde{\mathcal{H}}_{n,p}) by an application of the change of variables:

\[ p = \frac{r - t}{r - s}, \quad a = x^{t/p}, \quad b = x^{r/p'}. \]

When:

\[ p' = \frac{r - t}{s - t}, \quad ab = x^s. \]

Now we see that the equal weighted (\widetilde{\mathcal{H}}_{n,p}) follows from (\mathcal{L}_{n;r,s,t}) by the change of variables:

\[ p = \frac{r - t}{r - s}, \quad p' = \frac{r - t}{s - t}, \quad a = x^{-1/(r-s)}b^{1/(s-t)}, \quad w = a^{r/(r-s)}b^{-t/(s-t)}. \]

When:

\[ wx^s = ab, \quad wx^t = a^p, \quad wx^t = b^{p'}, \quad \text{and} \quad a^p x^{r-t} = b^{p'}. \]

Since we have noted that the weighted form of (\widetilde{\mathcal{H}}_{n,p}) is equivalent to the equal weighted form the equivalency is proved since the cases of equality are easily discussed. \(\square\)

7 Some Remarks

This paper originated in an attempt to rewrite a joint paper with Yuan-Chuan Li and Cheh-Chih Yeh. This was not successful but the idea for the paper came from these two mathematicians and I recognise their efforts and inspiration. The historical remarks originate from a very interesting communication received from Lech Maligranda whose knowledge of the history of the various inequalities discussed far exceeds my own.

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Finally it should be remarked that this paper ought to be considered a work in progress. There are more inequalities that should be discussed in the field considered here and there are many fields with their own inequalities. The various equivalencies between the members of this vast array has yet to be fully determined, the present paper is but a very short introduction to the whole topic.

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