SPACES OF MULTIPLICATIVE MAPS BETWEEN HIGHLY STRUCTURED RING SPECTRA

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Abstract. We uncover a somewhat surprising connection between spaces of multiplicative maps between $A_{\infty}$-ring spectra and topological Hochschild cohomology. As a consequence we show that such spaces become infinite loop spaces after looping only once. We also prove that any multiplicative cohomology operation in complex cobordisms theory $MU$ canonically lifts to an $A_{\infty}$-map $MU \to MU$. This implies, in particular, that the Brown-Peterson spectrum $BP$ splits off $MU$ as an $A_{\infty}$-ring spectrum.

1. Introduction

The main purpose of the present work is to provide a workable method for computing the homotopy type of spaces of $A_{\infty}$-maps between $A_{\infty}$-ring spectra (or $S$-algebras in the terminology of [8]). We make substantial use of the previous results by the author in [10] and for the reader’s convenience a brief summary of these is given in Section 2. In Section 3 we collect miscellaneous technical results concerning function spectra and topological Hochschild cohomology. Some of these results are surely known to experts but never have been written down. The formula (3) (base change) deserves special mention. While easy to prove it is extremely convenient when computing with various spectral sequences.

Our first main result (Theorem 4.3) essentially states that the mapping space between two $S$-algebras $A$ and $B$ is determined after taking based loops by the spectrum of topological derivations $\text{Der}(A, B)$. Therefore the problem of computing higher homotopy groups of this mapping space is a problem of stable homotopy which turns out to be quite amenable, particularly because in many cases $\text{Der}(A, B)$ can be reduced to $\text{THH}(A, B)$, the topological Hochschild cohomology of $A$ with values in $B$.

The computation of the zeroth homotopy group of the mapping space is, of course, a completely different story. We give a simple answer in the special case when $A$ is a connective $S$-algebra while $B$ is coconnective (that is, with vanishing homotopy in positive dimensions). This is Theorem 4.8.

Even though the problem of computing homotopy classes of $S$-algebra maps $A \to B$ is essentially unstable it does lend itself to analysis by methods of obstruction theory developed in [10]. Our second main result (Theorem 5.4) demonstrates that any multiplicative cohomology operation in complex cobordisms theory $MU$ canonically (even uniquely in an appropriate sense) lifts to an $S$-algebra self-map of $MU$. This is used to show that for an $S$-algebra $E$ belonging to a fairly large class of complex-oriented theories any multiplicative operation $MU \to E$ lifts to an $S$-algebra map. Another corollary is that the Brown-Peterson spectrum $BP$ splits off $MU$ localized at $p$ as an $S$-algebra.

The paper is written in the language of $S$-modules of [8] and we routinely use the results and terminology of the cited reference.

Notations. In Sections 2 and 3 we work in the category of modules or algebras over a fixed $q$-cofibrant commutative $S$-algebra $R$, the smash product $\wedge$ and the function spectrum $F(-, -)$ are always understood as $\wedge_R$ and $F_R(\cdot, \cdot)$. In Section 4 we specialize to $R = S$. The free $R$-algebra on an $R$-module $M$ is denoted by $T(M)$. The space of maps between two $R$-algebras $A$ and $B$ is denoted by $F_{\text{alg}}(A, B)$. If $A$ and $B$ are commutative $R$-algebras then $[A, B]_c$ denotes the set of homotopy classes of commutative algebra maps from $A$ to $B$. If $A$ and $B$ are associative $R$-algebras then $[A, B]_s$ stands for homotopy classes of associative algebra maps. For an associative $R$-algebra $A$ and two $A$-modules $M$ and $N$ we denote by $[M, N]_{A-\text{mod}}$ homotopy classes of $A$-module maps from $M$ to $N$. Similarly $[M, N]_{A-\text{bimod}}$ stands for homotopy classes of bimodule maps. Finally we denote by $\text{Mult}(E, F)$ the set of multiplicative (up to homotopy) maps between ring spectra $E$ and $F$. To distinguish between strict isomorphisms and weak equivalences we will use, as a rule, the symbol $\cong$ for the former and $\simeq$ for the latter.
2. Topological derivations and topological singular extension of $S$-algebras

In this short section we give an overview of some of the author’s results from [10] which will be needed later on.

Let $A$ be a $q$-cofibrant $R$-algebra and $M$ a $q$-cofibrant $A$-bimodule. Then the $R$-module $A \vee M$ has the obvious structure of an $R$-algebra (‘square-zero extension’ of $A$). Consider the set $[A, A \vee M]_{a/A}$ of homotopy classes of $R$-algebra maps from $A$ to $A \vee M$ in the category of $R$-algebras over $A$, that is the $R$-algebras supplied with an $R$-algebra map into $A$.

**Theorem 2.1.** There exists an $A$-bimodule $\Omega_A$ and a natural in $M$ isomorphism

$$[A, A \vee M]_{a/A} \cong [\Omega_A, M]_{A-bimod}$$

where the right hand side denotes the homotopy classes of maps in the category of $A$-bimodules.

**Remark 2.2.** Sometimes we will need a refinement of the above theorem which is formulated as follows. Let $B$ be an $R$-algebra over $A$, i.e. there exists a fixed $R$-algebra map $B \to A$. Then there is a natural isomorphism

$$[A, B \vee M]_{a/B} \cong [\Omega_B, B \wedge_B A, M]_{A-bimod}.$$  

Furthermore an $A$-bimodule $M$ can be considered as a $B$-bimodule and we have

$$[B, B \vee M]_{a/B} \cong [\Omega_B, B \wedge_B A, M]_{A-bimod} \cong [B, A \vee M]_{a/A}.$$  

The isomorphism $[B, B \vee M]_{a/B} \cong [B, B \vee M]_{a/B}$ will be used without explicit mention later on in this paper.

**Definition 2.3.** The topological derivations $R$-module of $A$ with values in $M$ is the function $R$-module $F_{A\wedge A^{op}}(\Omega_A, M)$. We denote it by $\text{Der}_R(A, M)$ and its $i$th homotopy group by $\text{Der}_R^i(A, M)$.

The $A$-bimodule $\Omega_A$ is constructed as the $q$-cofibrant approximation of the homotopy fibre of the multiplication map $A \wedge A \to A$. There exists the following homotopy fibre sequence of $R$-modules:

$$\text{THH}_R(A, M) \to M \to \text{Der}_R(A, M)$$  

(1)

Here $\text{THH}_R(A, M)$ is the topological Hochschild cohomology spectrum of $A$ with values in $M$:

$$\text{THH}_R(A, M) := F_{A\wedge A^{op}}(\hat{A}, M)$$

where $\hat{A}$ is the $q$-cofibrant replacement of $A$ as an $A$-bimodule.

We will also have a chance to use topological Hochschild homology spectrum $\text{THH}^R(A, M) := A\wedge A^{op}M$.

If the $R$-algebra $A$ is commutative and the left and right $A$-module structures on $M$ agree then both $\text{THH}^R(A, M)$ and $\text{THH}_R(A, M)$ are $A$-modules and there is a weak equivalence of $A$-modules

$$\text{THH}_R(A, M) \cong F_A(\text{THH}^R(A, M), A).$$

Furthermore in this case the sequence (1) splits giving a canonical weak equivalence $\text{THH}_R(A, M) \cong \Sigma^{-1}\text{Der}_R(A, M) \wedge M$.

Suppose we are given a topological derivation $d : A \to A \vee M$. Consider the following homotopy pullback diagram of $R$-algebras

$$\begin{array}{ccc}
X & \rightarrow & A \\
\downarrow & & \downarrow \\
A & \leftarrow & A \vee M
\end{array}$$

where the rightmost downward arrow is the canonical inclusion of a retract. Then we have the following homotopy fibre sequence of $R$-modules:

$$\Sigma^{-1}M \longrightarrow X \longrightarrow A$$  

(2)

**Definition 2.4.** The homotopy fibre sequence (2) is called the topological singular extension associated with the derivation $d : A \to A \vee M$. 

Theorem 2.5. Let $\Sigma^{-1}M \to X \to A$ be a singular extension of $R$-algebras associated with a derivation $d : A \to A \wedge M$ and $f : B \to A$ a map of $R$-algebras. Then $f$ lifts to an $R$-algebra map $B \to X$ iff a certain element in $\text{Der}_R^1(B,M)$ is zero. Assuming that a lifting exists the homotopy fibre of the map

$$F_{\text{der}}(B,X) \to F_{\text{der}}(B,A)$$

over the point $f \in F_{\text{der}}(B,A)$ is weakly equivalent to $\Omega^{\infty} \text{Der}_R(B,\Sigma^{-1}M)$, the 0th space of the spectrum $\text{Der}_R(B,\Sigma^{-1}M)$.

Theorem 2.6. Assume that $R$ is connective and $A$ is a connective $R$-algebra. Then the Postnikov tower of $A$

$$A_0 = H\pi_0A \xleftarrow{} A_1 \xrightarrow{} \cdots \xleftarrow{} A_n \xrightarrow{} A_{n+1} \xrightarrow{} \cdots$$

is a tower of $R$-algebras. Moreover the homotopy fibre sequences

$$A_n \xleftarrow{} A_{n+1} \xrightarrow{} \Sigma^{n+1}H\pi_{n+1}A$$

are topological singular extensions.

3. BASE CHANGE AND TOPOLOGICAL DERIVATIONS OF SUPPLEMENTED $R$-ALGEBRAS

In this section we discuss topological Hochschild cohomology and topological derivations of supplemented $R$-algebras and the behaviour of the forgetful map $l$ in the hypercohomology spectral sequence. This material is largely parallel to [10], section 9 and so most of the proofs will be omitted.

We’ll start with some general lemmas.

Lemma 3.1. Let $A, B, C$ be $R$-algebras and $M, N, L$ be an $A \wedge B$-module, an $C \wedge A^{op}$-module and a $C \wedge B$-module respectively. Then there is a natural isomorphism of $R$-modules:

$$F_{A \wedge B}(M, F_C(N,L)) \cong F_{C \wedge A M, L}.$$  

Proof. Let us first check the above equivalences for $M = A \wedge B \wedge \tilde{M}$ and $N = C \wedge A^{op} \wedge \tilde{N}$. We have:

$$F_{A \wedge B}(M, F_C(N,L)) \cong F_{A \wedge B}(A \wedge B \wedge \tilde{M}, F_C(C \wedge A^{op} \wedge \tilde{N}, L))$$

$$\cong F(\tilde{M}, F(A^{op} \wedge \tilde{N}, L))$$

$$\cong F(\tilde{M} \wedge A^{op} \wedge \tilde{N}, L).$$

Likewise,

$$F_{C \wedge A M, L} \cong F_{C \wedge A^{op} \wedge \tilde{N} \wedge A} A \wedge B \wedge \tilde{M}, L$$

$$\cong F(A^{op} \wedge \tilde{N} \wedge \tilde{M}, L).$$

Observe that the above isomorphisms are natural in $M$ and $N$ that is, with respect to arbitrary maps of $A \wedge B$ modules $A \wedge B \wedge \tilde{M}_1 \to A \wedge B \wedge \tilde{M}_2$ and of $C \wedge A^{op}$-modules $C \wedge A^{op} \wedge \tilde{N}_1 \to C \wedge A^{op} \wedge \tilde{N}_2$ (not only those coming from $\tilde{M}_1 \to \tilde{M}_2$ and $\tilde{N}_1 \to \tilde{N}_2$). To obtain the general case it suffices to notice that for any $M$ and $N$ there exist standard split coequalizers of $R$-modules

$$(A \wedge B)^{\vee 2} \wedge M \xrightarrow{} A \wedge B \wedge M \xrightarrow{} M$$

and

$$(C \wedge A^{op})^{\vee 2} \wedge N \xrightarrow{} C \wedge A^{op} \wedge N \xrightarrow{} N.$$  

With this Lemma 3.1 is proved. \qed

Now let $C = N$ be an $A$-bimodule via an $R$-algebra map $f : A \to C$. Then the $C \wedge B$-module $L$ acquires a structure of an $A \wedge B$-module via the map

$$A \wedge B \xrightarrow{f \wedge \text{id}} C \wedge B.$$  

Furthermore simple diagram chase shows that the isomorphism $F_C(C,L) \cong L$ is in fact an isomorphism of $A \wedge B$-modules. This gives the following

Corollary 3.2. There exists the following natural isomorphism of $R$-modules:

$$F_{C \wedge A M, L} \cong F_{A \wedge B}(M, F_C(C,L)) \cong F_{A \wedge B}(M, L)$$  \hfill (3)
We will refer to the isomorphism (3) as base change. Related formulae are found in [8], III.6.

**Corollary 3.3.** If an $R$-algebra $B$ is an $A$-bimodule via a $q$-cofibration of $R$-algebras $A \to B$, then $\text{THH}_R(A, B) \simeq F_{B \wedge A^{op}}(\hat{B}, \hat{B})$ where $\hat{B}$ is the $q$-cofibrant approximation of the $B \wedge A^{op}$-module $B$. In particular, $\text{THH}_R(A, B)$ is an $R$-algebra under the composition product.

**Proof.** Denoting by $\hat{A}$ the $q$-cofibrant approximation of the $A$-bimodule $A$ we have the following isomorphisms of $R$-modules:

$$\text{THH}_R(A, B) \cong F_{A \wedge A^{op}}(\hat{A}, B) \cong F_{A \wedge A^{op}}(A, F_B(B, B)) \cong F_{B \wedge A^{op}}(B \wedge_A \hat{A}, B).$$

The $B \wedge A^{op}$-module $B \wedge_A \hat{A}$ is a $q$-cofibrant $B \wedge A^{op}$-module because the functor $? \to B \wedge A^{op} \wedge_A ?$ preserves $q$-cofibrant modules. Therefore $F_{B \wedge A^{op}}(B \wedge_A \hat{A}, B)$ represents derived function $B \wedge A^{op}$-module and is equivalent to $F_{B \wedge A^{op}}(\hat{B}, \hat{B})$. \hfill $\square$

We now discuss topological Hochschild cohomology and derivations of supplemented $R$-algebras. Let $A$ be a $q$-cofibrant $R$-algebra. We say that $A$ is supplemented if it is supplied with an $R$-algebra morphism $\epsilon : A \to B$ which we will assume to be a $q$-cofibration of $R$-algebras. Denote by $\Omega^B_A$ the homotopy fibre of the map $B \wedge A \to B$ that determines the structure of a right $A$-module on $B$. We will assume without loss of generality that $\Omega^B_A$ is a $q$-cofibrant right $A$-module. Recall that the module of differentials $\Omega_A$ for $A$ is defined from the homotopy fibre sequence

$$\Omega_A \to A \wedge A \to A$$

where the second arrow is the multiplication map. Smashing this fibre sequence on the left with $B$ over $A$ we get the fibre sequence

$$B \wedge_A \Omega_A \to B \wedge A \to B$$

That shows that $\Omega^B_A$ is weakly equivalent to $B \wedge_A \Omega_A$ as a $B \wedge A$-module. Further, base change gives a weak equivalence

$$F_{B \wedge A^{op}}(\Omega^B_A, B) \simeq F_{B \wedge A^{op}}(B \wedge_A \Omega_A, B) \simeq F_{A \wedge A^{op}}(\Omega_A, B) \cong \text{Der}_R(A, B).$$

Recall from [10] that there is a ‘universal derivation’ $d : A \to \Omega_A$ which is defined as the composite map

$$A \to A \vee \Omega_A \to \Omega_A$$

where the first map is the map of algebras over $A$ adjoint to the identity map $\Omega_A \to \Omega_A$ and the second map is the projection. The universal derivation allows one to define the forgetful map $l : \text{Der}_R(A, B) \to F(A, B)$ as the composite map

$$\text{Der}_R(A, B) \simeq F_{A \wedge A^{op}}(\Omega_A, B) \to F(\Omega_A, B) \to F(A, B)$$

where the last map is induced by $d$. In terms of $\Omega^B_A$ the forgetful map $l$ admits the following description. The fibre sequence $\Omega^B_A \to B \wedge A \to B$ splits via the map $B \cong B \wedge R \overset{id \wedge 1}{\to} B \wedge A$ so there is a weak equivalence of $R$-modules $B \wedge A \simeq B \vee \Omega^B_A$. Denote by $\overline{d} : A \to \Omega^B_A$ the following composite map

$$A \overset{\epsilon \wedge id}{\longrightarrow} B \wedge A \simeq B \vee \Omega^B_A \overset{id}{\longrightarrow} \Omega^B_A$$

the last arrow being the projection onto the wedge summand. Then $l$ coincides with the following composition:

$$\text{Der}_R(A, B) = F_{B \wedge A^{op}}(\Omega^B_A, B) \longrightarrow F(\Omega^B_A, B) \longrightarrow F(A, B)$$

the first arrow being the forgetful map and the second one is induced by $\overline{d}$.

Next we discuss the behaviour of the forgetful map $l$ in the hypercohomology spectral sequence. To do this we need to review algebraic Hochschild cohomology for supplemented algebras. The exposition will be somewhat sketchy since it is parallel to [10], section 9.
Definition 3.4. Let $A_*$ be a graded algebra over a graded commutative algebra $R_*$ supplied with an $R_*$-algebra map $e: A_* \rightarrow B_*$ (supplementation). Then algebraic Hochschild cohomology of $A_*$ with coefficients in $B_*$ is defined as
\[ HH^i_{R_*}(A_*, B_*) = Ext^i_{B_* \otimes_R A_*^{op}}(B_*, B_*) \]
where $\otimes_R^i$ denotes the derived tensor product.

Remark 3.5. If $A_*$ is flat as an $R_*$-module, then this definition is equivalent to the standard one found in, e.g., [6].

We also have a generalization of the standard complex which computes Hochschild cohomology. Let $\tilde{A}_*$ be a differential graded supplemented $R_*$-algebra which is quasiisomorphic to $A_*$ and $R_*$-projective. Denote by $\tilde{\epsilon}: \tilde{A}_* \rightarrow B_*$ its supplementation. Consider the bar-resolution of the right $\tilde{A}_*$-module $B_*$. Here and later on $\otimes$ stands for $\otimes_R$.

\[ B_* \leftarrow B_* \otimes \tilde{A}_* \leftarrow B_* \otimes \tilde{\tilde{A}}_* \leftarrow \cdots \]

with the usual bar differential
\[ \partial(b \otimes a_1 \otimes \ldots \otimes a_n) = \pm b\tilde{\epsilon}(a_1) \otimes a_2 \ldots \otimes \tilde{\epsilon}(a_n) + \Sigma b \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n. \]

We don't specify the signs in this well-known formula, see e.g., [12], Chapter X. This is actually a bicomplex since $\tilde{A}_*$ is a differential graded algebra. Applying the functor $\text{Hom}_{B_* \otimes \tilde{A}_*^{op}}(?, B_*)$ to (4) we get the standard Hochschild cohomology (bi)complex
\[ C^{ij}(A_*, B_*) = \text{Hom}^i(\tilde{A}_*^\otimes j, B_*). \]

Now define the module of differentials $\Omega^{B_*}_{A_*}$ from the following short exact sequence
\[ 0 \rightarrow \Omega^{B_*}_{A_*} \rightarrow B_* \otimes \tilde{A}_* \rightarrow B_* \rightarrow 0. \]

Clearly $\Omega^{B_*}_{A_*}$ is quasiisomorphic as a complex of right $\tilde{A}_*$-modules to the truncated bar-resolution:
\[ B_* \otimes \tilde{\tilde{A}}_*^\otimes 2 \leftarrow B_* \otimes \tilde{\tilde{A}}_*^\otimes 3 \leftarrow \cdots \]

The universal derivation $d: \tilde{A}_* \rightarrow \Omega^{B_*}_{A_*}$ is induced by the map $\tilde{A}_* \rightarrow B_* \otimes \tilde{A}_*, \tilde{a} \rightarrow 1 \otimes \tilde{a} - \tilde{\epsilon}(\tilde{a}) \otimes 1$. If we take the complex (5) as a model for $\Omega^{B_*}_{A_*}$, then the universal derivation $d$ is a map of complexes
\[ d: \tilde{A}_* \rightarrow \{ B_* \otimes \tilde{\tilde{A}}_*^\otimes 2 \rightarrow B_* \otimes \tilde{\tilde{A}}_*^\otimes 3 \rightarrow \ldots \} \]

where $\tilde{A}_*$ is considered to be a complex concentrated in degree 0 and $d(\tilde{a}) = -1 \otimes \tilde{a} \otimes 1$.

Further define algebraic derivations of $A_*$ with coefficients in $B_*$ as
\[ \text{Der}^*_{R_*}(A_*, B_*) = Ext^*_{B_* \otimes \tilde{A}_*^{op}}(\Omega^{B_*}_{A_*}, B_*). \]

Then the (truncated) standard resolution (5) provides a (bi)complex for computing $\text{Der}^*_{R_*}(A_*, B_*)$:
\[ \overline{\text{C}}^*(A_*, B_*) : \text{Hom}(\tilde{A}_*, B_*) \rightarrow \text{Hom}(\tilde{\tilde{A}}_*^\otimes 2, B_*) \rightarrow \cdots \]

(This is indeed a bicomplex, the additional differential being induced from the internal differential in $\tilde{A}_*$).

As in the topological case the universal derivation determines the forgetful map
\[ l_{alg}: \text{Der}^*_{R_*}(A_*, B_*) = Ext^*_{B_* \otimes \tilde{A}_*^{op}}(\Omega^{B_*}_{A_*}, B_*) \rightarrow \text{Hom}^*(A_*, B_*). \]

Then we have the obvious

Proposition 3.6. The forgetful map
\[ \text{Der}^*_{R_*}(A_*, B_*) \rightarrow \text{Hom}^*(\tilde{A}_*, B_*) = Ext^*_{R_*}(A_*, B_*) \]

is induced by the projection $\overline{\text{C}}^*(A_*, B_*) \rightarrow \text{Hom}^*(\tilde{A}_*, B_*)$ times $(-1)$.

Returning to our topological situation we have the following result which is analogous to Proposition 9.3 in [10]:
**Proposition 3.7.** Let $A$, $B$ be $R$-algebras, $A \to B$ is an $R$-algebra map. Suppose that the Kunneth spectral sequence for $\pi_*(B \wedge A^{op})$ collapses and there is a ring isomorphism

$$\pi_*(B \wedge A^{op}) \cong B_* \otimes^L_{R_*} A_*^{op}$$

Then there are the following spectral sequences

$$E_2^{1} = \text{Der}_R^*(A_*, B_*) = \text{Ext}^*_{\text{Tor}_{op}(B_*, A^{op})}(\Omega^B_{A*}, R_*) \Rightarrow \text{Der}^*(A, B);$$

$$E_2^{2} = \text{Ext}^*_{R_*}(A_*, B_*) \Rightarrow [A, B]^*.$$ 

Furthermore, the forgetful map $l : \text{Der}^*(A, B) \to [A, B]^*$ induces a map of spectral sequences $E_2^{1} \to E_2^{2}$ which on the level of $E_2$-terms gives the forgetful map $l_{alg} := \text{Der}^*_{R_*}(A_*, B_*) \to \text{Ext}^*_{R_*}(A_*, B_*)$.

4. **Mappings spaces via derivations**

In this section we show that for two $R$-algebras $A$ and $B$ the higher homotopy groups of the space $F_{R-\text{alg}}(A, B)$ can be reduced to the computation of certain topological derivations. This is important because in many cases topological derivations can be further reduced to topological Hochschild cohomology which is an essentially stable object, so that one could apply standard methods of homological algebra for computation. As usual, we assume that $A$ is a $q$-cofibrant $R$-algebra.

Now consider the $R$-module $A \vee \Sigma^{-d} A$, $d > 0$. It can be supplied canonically with the structure of an $R$-algebra over $A$ so that $\Sigma^{-d} A$ is a 'square-zero ideal'. Denote this $R$-algebra by $A_d$.

Let us also introduce the algebra $A(d) := A^d$, the cotensor of $A$ and the $d$-sphere $S^d$. Then $A(d) \cong F(R\Sigma^{\infty}S^d_+, A)$ as an $R$-module (here $R$ stands for the free $R$-module functor. The structure of an $R$-algebra on $A(d)$ is induced by the $R$-algebra structure on $A$ and the topological diagonal $S^d \to S^d \times S^d$. The coefficient rings of $A(d)$ and $A_d$ are both isomorphic to the exterior algebra $\Lambda_{A_*, (y)}$ where $y$ has degree $-d$.

There is also a weak equivalence of $R$-modules:

$$A(d) \simeq A \vee \Sigma^{-d} A \cong A_d.$$ 

Notice that both $A(d)$ and $A_d$ are $R$-algebras over $A$, that is there exist maps of $R$-algebras $A(d) \to A$ and $A_d \to A$. (The first map is induced by choosing a base point in $S^d$, the second map is the canonical projection).

**Theorem 4.1.** The $R$-algebras $A(d)$ and $A_d$ are weakly equivalent in the category of $R$-algebras.

**Proof.** First consider the case $A = R$. Since $R$ is an $R(d)$-module it makes sense to consider self-maps of $R$ in the category of $R(d)$-modules. Notice that $R(d)$ is actually a commutative $R$-algebra so we need not distinguish between left and right $R(d)$-modules.

**Lemma 4.2.**

$$\pi_* F_{R(d)}(R, R) = R_* [[x]]$$

where the element $x$ has degree $d-1$.

**Proof.** Assume first that $R = S$, the sphere spectrum. We need this special case because the connectiveness of $S$ will be used. If $R$ is connective this step could be skipped. Consider the spectral sequence

$$E_2^{**} = \text{Ext}^{**}_{S(d)}(S_*, S_*)$$

$$= \text{Ext}^{**}_{\Lambda_{y, (y)}}(S_*, S_*)$$

$$= S_* [[x]] \Rightarrow \pi_* F_{S(d)}(S, S).$$

Here the element $x$ has degree $d-1$. This spectral sequence collapses for dimensional reasons. By Boardman’s criterion $[3]$ it converges strongly to its target which is complete with respect to the (cobar) filtration. Since this filtration coincides with the $x$-adic filtration on the associated graded $S_*$-module we conclude that

$$\pi_* F_{S(d)}(S, S) = S_* [[x]].$$

Notice that the fact that $S$ is connective was used to show the collapse of our spectral sequence. For instance if $d = 1$ the elements $x^k$ are located along the line of slope 1 and the whole spectral sequence $E_2^{**}$ lies above it.
Now let $R$ be an arbitrary commutative $S$-algebra. Consider the spectral sequence
\[ E_2^{**} = \text{Ext}^{**}_{R(d)}(R_s, R_s) = \text{Ext}^{**}_{R(d)}(R_s, R_s) = R_s[[x]] \to \pi_* F_{R(d)}(R, R) \]
and notice that the unit map $S \to R$ determines the map of spectral sequences $E_2^{**} \to E_2^{**}$ taking $x$ to $x$. It follows that $E_2^{**}$ collapses proving our claim. \hfill \Box

Let us now return to the proof of the theorem; recall that we are still handling the special case $A = R$. Consider the set of maps $R(d) \to R_d$ in the homotopy category of $R$-algebras over $R$. By Theorem 2.1 this set is an abelian group of topological derivations of $R(d)$ with values in $\Sigma^{-d}R$. Since $R(d)$ is a commutative $R$-algebra there is a canonical splitting
\[ T HH^*_R(R(d), R) \simeq R \vee \text{Der}^{* - 1}_R(R(d), R) \]
So the computation of $\text{Der}^*_R(R(d), R)$ reduces to the computation of topological Hochschild cohomology $T HH^*_R(R(d), R)$. Further Corollary 3.3 provides an isomorphism
\[ T HH^*_R(R(d), R) \cong F_{R(d)}(R, R). \]
It follows from Lemma 4.2 that the spectral sequence
\[ E_2^j = \text{Der}_{R(d)}(R(d)_* R_s) = \text{Ext}^*_{R(d)}(\Omega^R_{R(d)_*}, R_s) \Rightarrow \text{Der}^*_R(R(d), R) \]
collapses and
\[ \text{Der}^{* - 1}_R(R(d), R) = R_*[[x]]/R_* \]
Further,
\[ [R(d), R]^* = R_* \oplus \Sigma^d R_* = \Lambda_{R_*}(z) \]
where the symbol $z$ has degree $d$ (of course, we do not claim the existence of any multiplicative structure). The element $z$ maps the wedge summand $\Sigma^{-d}R$ of the $R$-module $R(d)$ isomorphically to $\Sigma^{-d}R$, and the other wedge summand maps to zero. It follows from Proposition 3.7 that the image of $x \in \text{Der}^*(R(d), R)$ in $[R(d), R]^*$ under the forgetful map is $z$ (up to an invertible factor).

In other words we proved that there exists a topological derivation of $R(d)$ with values in $R$, that is a map in the homotopy category of $R$-algebras over $R$
\[ R(d) \to R_d \tag{6} \]
such that the wedge summand $\Sigma^{-d}R$ of $R(d)$ maps isomorphically onto the corresponding wedge summand of $R_d$. Therefore the map (6) is a weak equivalence of $R$-algebras and our theorem is proved (in the special case $A = R$). To get the general case consider the canonical map
\[ R(d) \wedge A = F(R \Sigma^\infty S^d_+, R) \wedge A \to F(R \Sigma^\infty S^d_+, A) = A(d) \tag{7} \]
Since $A$ is a $q$-cofibrant $R$-algebra the point-set level smash product $F(R \Sigma^\infty S^d_+, R) \wedge A$ represents the derived smash product. Further (7) is a weak equivalence since $R \Sigma^\infty S^d_+$ is a finite cell $R$-module and diagram chase shows that this is an $R$-algebra map. So we have the following equivalences of $R$-algebras:
\[ A_d \cong R_d \wedge A \cong R(d) \wedge A \cong A(d) \]
With this Theorem 4.1 is proved. \hfill \Box

Now suppose that we have another $R$-algebra $B$ and a map $f : A \to B$ of $R$-algebras. Then the pair $(F_{R-alg}(A, B), f)$ is a pointed topological space. This space turns out to be closely related to $\text{Der}_R(A, B)$. We have the following theorem:

**Theorem 4.3.** For a $q$-cofibrant algebra $A$ and a map of $R$-algebras $f : A \to B$ the space of $d$-fold loops $\Omega^d(F_{R-alg}(A, B), f)$ is weakly equivalent to the space $\Omega^\infty \text{Der}_R(A, \Sigma^{-d}B)$. 


Proof. For two topological spaces $X$ and $Y$ we will denote the space of maps between them by $\mathcal{T}(X,Y)$ ($\mathcal{T}_0(X,Y)$ in the pointed case). Then we have the following commutative diagram of spaces where both rows are homotopy fibre sequences:

$$
\begin{array}{c}
\mathcal{T}_0(S^d, F_{R-alg}(A, B)) \longrightarrow \mathcal{T}(S^d, F_{R-alg}(A, B)) \longrightarrow F_{R-alg}(A, B) \\
\downarrow \quad \downarrow \quad \downarrow \\
? \longrightarrow F_{R-alg}(A, B^{S^d}) \longrightarrow F_{R-alg}(A, B)
\end{array}
$$

Here the horizontal rightmost arrows are both induced by the inclusion of the base point into $S^d$. Since the right and the middle vertical arrows are weak equivalences (even isomorphisms) it follows that the map $\mathcal{T}_0(S^d, Map(A, B)) \to ?$ is a weak equivalence. But Theorem 4.1 tells us that the $R$-algebra $B^{S^d}$ is weakly equivalent as an $R$-algebra to $B \vee \Sigma^{-d}B$. In other words the term $?$ is weakly equivalent to the topological space of maps $A \to B \vee \Sigma^{-d}B$ which commute with the projection onto $A$. Therefore $?$ is weakly equivalent to $\Omega^\infty Der_R(A, \Sigma^{-d}B)$ and our theorem is proved. $\square$

**Corollary 4.4.** For a $q$-cofibrant algebra $A$ and a map of $R$-algebras $f : A \to B$ there is a bijection between sets $\pi_d(F_{R-alg}(A, B), f)$ and $Der^{-d}_R(A, B)$ for $d \geq 1$. If $d \geq 2$ then this bijection is an isomorphism of abelian groups.

**Remark 4.5.** One might wonder whether Theorem 4.3 remains true in the context of commutative $S$-algebras. The answer is no. The crucial point is the weak equivalence of $S$-algebras $S \vee S^{-1}$ and $S^{S^1}$. It is clear that $\pi_0 S \wedge_{S^{S^1}} S$ is the divided power ring. However N.Kuhn and M.Mandell proved that $\pi_0 S \wedge_{S^{S^1}} S$ is the ring of numeric polynomials. Therefore $S \vee S^{-1}$ and $S^{S^1}$ cannot be weakly equivalent as commutative $S$-algebras.

We see, that the space $F_{R-alg}(A, B)$ when looped only once becomes an infinite loop space. This is somewhat surprising since $F_{R-alg}(A, B)$ is hardly ever an infinite loop space itself. In particular the set of connected components of $F_{R-alg}(A, B)$ does not have to be a group, let alone an abelian group. Therefore the connection between $\pi_0 F_{R-alg}(A, B)$ and $Der^{-1}_R(A, B)$ (provided the latter is defined) may be rather weak. For instance the set of homotopy classes of $A_{\infty}$ self-maps of the $p$-completed $K$-theory spectrum is the multiplicative group of $p$-adic integers whereas the corresponding topological derivations spectrum can be proved to be contractible. A generalization of this example is discussed in author’s work [11]. However there is some evidence for the following

**Conjecture 4.6.** For an $R$-algebra map $f : A \to B$ the connected component of $f$ in $F_{R-alg}(A, B)$ is weakly equivalent to the connected component of $\Omega^\infty Der_R(A, B)$. In particular it is an infinite loop space.

To see why this conjecture has a chance of being true notice that the Whitehead products in the homotopy groups of $F_{R-alg}(A, B)$ determine via Theorem 4.3 various brackets in $Der^*_R(A, B)$ and, for commutative $A$ and $B$ - also in $THH^*_R(A, B)$. No such brackets have been recorded so far and it seems likely that they should all vanish. This suggests that the connected component of $f$ in $F_{R-alg}(A, B)$ is an $H$-space.

There is another interesting question raised by Theorem 4.3. In recent work [12] J.McClure and J.Smith introduced the Gerstenhaber bracket on $THH_R(A, A)$. Their work probably implies the existence of the bracket on $Der_R(A, A)$. This is surely the case if $A$ is commutative since then $Der_R(A, A)$ splits off $THH_R(A, A)$ as a wedge summand. Then via Theorem 4.3 a Poisson bracket is defined on $\pi_* F_{R-alg}(A, A)$ for $* > 0$.

**Conjecture 4.7.** The bracket described above agrees with the Whitehead product on $BF_{R-alg}(A, A)$, the classifying space of the monoid $F_{R-alg}(A, A)$.

We see that the problem of computing $\pi_0 F_{R-alg}(A, B)$ differs sharply from computing higher homotopy groups. This problem is usually much harder, being essentially nonabelian. However there is one case when it is possible to give a complete general answer.

**Theorem 4.8.** Assuming that $R$ is connective let $A$ be a connective $q$-cofibrant $R$-algebra, and $B$ a coconnective $R$-algebra (i.e. $\pi_i B = 0$ for $i > 0$). Then any $\pi_0 R$-algebra map $\pi_0 A \to \pi_0 B$ lifts to a unique
R-algebra map $A \to B$ so that the forgetful map $[A, B]_a \to \text{Hom}_{\pi_0 R-\text{alg}}(\pi_0 A, \pi_0 B)$ is bijective. Moreover the topological space $F_{R-\text{alg}}(A, B)$ is homotopically discrete, i.e. $\pi_i F_{R-\text{alg}}(A, B) = 0$ for $i > 0$.

Similarly if A and B are both commutative R-algebras, where B is cocommutative and A is q-cofibrant and connective then the forgetful map $[A, B]_c \to \text{Hom}_{\pi_0 R-\text{alg}}(\pi_0 A, \pi_0 B)$ is bijective and the space of commutative R-algebra maps from A to B is homotopically discrete.

Proof. We will deal only with the associative case, the commutative one being completely analogous. Picking a system of generators and relations for the $\pi_0 R$-algebra $\pi_0 A$ we construct the following pushout diagram in the category of $R$-algebras:

$$
T(\coprod_I R) \to R \quad \downarrow \quad \downarrow
$$

$$
T(\coprod_I R) \to A^0 \quad (8)
$$

Here the sets I and J run respectively through the systems of generators and relations in $\pi_0 A$. There is a canonical $R$-algebra map from $A^0$ to A that induces an isomorphism on zeroth homotopy group. The $R$-algebra $A^0$ is the zeroth skeleton of $A$ in the category of $R$-algebras and (the CW-approximation of) A is obtained from $A^0$ by attaching $R$-algebra cells in higher dimensions. Then induction up the CW-filtration of A shows that the map $A^0 \to A$ induces a weak equivalence $F_{R-\text{alg}}(A, B) \simeq F_{R-\text{alg}}(A^0, B)$.

Further applying the functor $F_{R-\text{alg}}(?, B)$ to the diagram (8) we get the following homotopy pullback of topological spaces:

$$
\coprod_I \pi_0 B \leftarrow \quad pt \quad \leftarrow \quad \coprod_I \pi_0 B \leftarrow F_{R-\text{alg}}(A^0, B)
$$

It follows that the space $F_{R-\text{alg}}(A^0, B)$ is homotopically discrete with the set of connected components being equal to $\text{Hom}_{\pi_0 R-\text{alg}}(\pi_0 A, \pi_0 B)$. \qed

Now let $k$ be an associative ring. Recall that according to [8], Proposition IV.3.1 the Eilenberg-MacLane spectrum $Hk$ admits a structure of an $S$-algebra or a commutative $S$-algebra if $k$ is commutative. Theorem 4.8 shows that this structure is unique up to a weak equivalence of $S$-algebras or commutative $S$-algebras. We also have the following evident corollary which will be used in the next section.

**Corollary 4.9.** Let $A$ be a connective q-cofibrant $S$-algebra or commutative $S$-algebra. Then the topological space of $S$-algebra maps (or commutative $S$-algebra maps) from $A$ to $H\pi_0 A$ is homotopically discrete and

$$
\pi_0 F_{S-\text{alg}}(A, H\pi_0 A) = \text{End}_{\text{rings}}(\pi_0 A).
$$

5. **Spaces of multiplicative self-maps of MU**

In this section we study the homotopy groups of $A_\mathbb{Q}$-maps from the complex cobordism spectrum $MU$ into itself. Our main result here is that any homotopy multiplicative operation $MU \to MU$ lifts canonically to an $S$-algebra map. We also calculate completely higher homotopy groups of $S$-algebra maps out of $MU$ into an arbitrary $MU$-algebra $E$. In this section we work with various homotopy categories and so smash products and function spectra are understood in the derived sense.

Before we state our main theorem we need to introduce the notion of $\mathbb{Q}$-commutative $S$-algebras and $\mathbb{Q}$-preferred $S$-algebra maps.

**Definition 5.1.** Let $A$ be an $S$-algebra and denote by $A_\mathbb{Q}$ its rationalization. We say that $A$ is $\mathbb{Q}$-commutative if the $A_\mathbb{Q}$ is weakly equivalent as an $S$-algebra to a commutative $S$-algebra.

**Remark 5.2.** Later on all $\mathbb{Q}$-commutative $S$-algebras which we encounter will in fact be commutative. Notice, however, that it is not always the case. Denote by $S[x_i]$ the free $S$-algebra on the $S$-module $S^{2i}_S$, the cell approximation of the $2i$-dimensional sphere. Then clearly $S_\mathbb{Q}[x_i]$ is weakly equivalent to the free commutative $S$-algebra on $S^{2i}_\mathbb{Q}$. Therefore $S[x_i]$ is a $\mathbb{Q}$-commutative $S$-algebra which is not commutative unless $i = 0$.

Consider two $\mathbb{Q}$-commutative $S$-algebras $A_\mathbb{Q}$ and $B_\mathbb{Q}$. We have the following maps:

$$
k : [A_\mathbb{Q}, B_\mathbb{Q}]_c \longrightarrow [A_\mathbb{Q}, B_\mathbb{Q}]_a \longrightarrow [A, B]_a : q.
$$

Here $k$ is the forgetful map and $q$ is induced by rationalization.
Definition 5.3. A map \( f \in [A,B]_a \) is called \( \mathbb{Q} \)-preferred if \( q(f) \in [A_\mathbb{Q},B_\mathbb{Q}]_a \) is in the image of \( k \). Similarly for an \( A \)-bimodule \( M \) which is \( \mathbb{Q} \)-symmetric (that is, the square-zero extension \( A \vee M \) is \( \mathbb{Q} \)-commutative) an \( S \)-algebra derivation \( d : A \rightarrow A \vee M \) is called \( \mathbb{Q} \)-preferred if \( d \) is \( \mathbb{Q} \)-preferred as an \( S \)-algebra map.

In other words a map of \( S \)-algebras (or a topological derivation) is \( \mathbb{Q} \)-preferred if it lifts to a map (to a derivation) of commutative \( S \)-algebras after rationalization.

Theorem 5.4. The forgetful map of monoids

\[
[MU,MU]_a \longrightarrow \text{Mult}(MU,MU)
\]

admits a unique section whose image consists of \( \mathbb{Q} \)-preferred \( S \)-algebra maps.

The proof will be given below after a succession of lemmas.

Remark 5.5. The set \( \text{Mult}(MU,MU) \) is relatively well understood. One can describe it for example as the set of all \( MU_* \)-algebra maps

\[
MU_*MU = MU_*[t_1,t_2,\ldots] \longrightarrow MU_*.
\]

Our next result is the computation of topological Hochschild cohomology of \( MU \) with coefficients in an \( MU \)-algebra \( E \). Since there is a canonical splitting of spectra

\[
\text{THH}_S(MU,E) \cong E \vee \Sigma^{-1}\text{Der}_S(MU,E)
\]

the combination of this result with Corollary 4.4 gives a complete calculation of higher homotopy groups of the based space \( F_{\text{alg}}(MU,E) \).

Proposition 5.6. For an \( MU \)-algebra \( E \) considered as an \( MU \)-bimodule the following isomorphism holds

\[
\text{THH}_S^S(MU,E) \cong \hat{\Lambda}_{E_*}(y_1,y_2,\ldots)
\]

where the hat denotes the completed exterior algebra and the exterior generator \( y_i \) has cohomological degree \( 2i-1 \).

Proof. Consider the topological Hochschild homology \( S \)-module of \( MU \) with coefficients in \( E \),

\[
\text{THH}_S^S(MU,E) := MU \wedge_{MU \wedge MU} E \cong MU \wedge_{MU \wedge MU} MU \wedge_{MU} E.
\]

We have the spectral sequence of \( MU_* \)-algebras

\[
E_2 = \text{Tor}_{MU_*}^{MU}(MU_* , MU_*) \cong MU_* \otimes \Lambda(\hat{y}_1,\hat{y}_2,\ldots) \Rightarrow \pi_* \text{THH}_S^S(MU,MU).
\]

Since the differentials applied to the exterior generators \( \hat{y}_i \) are trivial for dimensional reasons we conclude that it collapses. It follows that

\[
\pi_* \text{THH}_S^S(MU,E) = \pi_* \text{THH}_S^S(MU,MU) \otimes_{MU_*} E_*
\]

\[
= E_* \otimes \Lambda(\hat{y}_1,\hat{y}_2,\ldots).
\]

Now the result for topological Hochschild cohomology follows by virtue of the universal coefficients formula and the isomorphism

\[
\text{THH}_S^S(MU,E) \cong F_E(\text{THH}_S^S(MU,E),E).
\]

Proposition (5.6) is proved.

Recall that we are using the notation \( S_\mathbb{Q}[x_i] \) for the free commutative \( S \)-algebra on the \( S \)-module \( S_\mathbb{Q}^{2i} \), the rationalized \( 2i \)-sphere \( S \)-module. The coefficient ring of \( S_\mathbb{Q}[x_i] \) is isomorphic to \( \mathbb{Q}[x_i] \) where the polynomial generator \( x_i \) has degree \( 2i \). Further denote the infinite smash power \( S_\mathbb{Q}[x_i]^{\wedge \infty} \) by \( S_\mathbb{Q}[x_1,x_2,\ldots] \).

Lemma 5.7. There is a weak equivalence of commutative \( S \)-algebras

\[
S_\mathbb{Q}[x_1,x_2,\ldots] \longrightarrow MU_\mathbb{Q}.
\]

Proof. The polynomial generators \( x_i \) of the ring \( MU_\mathbb{Q} = \mathbb{Q}[x_1,x_2,\ldots] \) determine a collection of maps \( S_\mathbb{Q}^{2i} \rightarrow MU_\mathbb{Q} \) and therefore a map of commutative algebras \( S_\mathbb{Q}[x_i]^{\wedge \infty} \rightarrow MU_\mathbb{Q} \) which is clearly a weak equivalence.
Definition 5.8. Let $E$ be a ring spectrum (in the traditional up to homotopy sense) with multiplication $m : E \wedge E \to E$ and $M$ an $E$-bimodule spectrum with the left action $m_1 : E \wedge M \to M$ and the right action $m_r : M \wedge E \to M$. We say that a map $f : E \to M$ is a primitive operation if $f \circ m$ and $m_i \circ (f \circ id) + m_j \circ (id \wedge f)$ are homotopic as maps from $E \wedge E$ to $M$. The set of all primitive operation from $E$ to $M$ is denoted by $\text{Prim}(E, M)$.

Remark 5.9. Perhaps it is more natural to use the term ‘derivation’ instead of ‘primitive operation’ but this term is already overworked in this paper.

The next lemma provides a description of topological derivations of $MU$ with coefficients in $HZ$, the integral Eilenberg-MacLane spectrum.

Lemma 5.10. There is the following isomorphism of graded abelian groups:

$$\text{Der}^*_S(MU, HZ) \cong \Lambda^{*-1}(y_1, y_2 \ldots)/\mathbb{Z}.$$ 

Under the forgetful map

$$l : \text{Der}^*_S(MU, HZ) \to [MU, HZ]^* = \text{Hom}(\mathbb{Z}[t_1, t_2, \ldots], \mathbb{Z})$$

the elements $y_i \in \text{Der}^{2*-2}_S(MU, HZ)$ correspond to the derivations $\partial_{i^*}$ evaluated at 0. Moreover the elements $y_i$ are $\mathbb{Q}$-preferred topological derivations.

Proof. We have the spectral sequence

$$\text{Der}^*_S(HZ, MU, \mathbb{Z}) = \Lambda^{*-1}(y_1, y_2 \ldots)/\mathbb{Z}$$

$$\Rightarrow \text{Der}^*_S(HZ \wedge MU, HZ) = \text{Der}^*_S(MU, HZ).$$

This spectral sequence clearly collapses. Next using Proposition 3.7 we see that the image of the element $y_i$ under the forgetful map $l$ in the group

$$[HZ \wedge MU, HZ]_{HZ, \text{mod}} = [MU, HZ]^* = \text{Hom}^*(\mathbb{Z}[t_1, t_2, \ldots], \mathbb{Z})$$

is precisely the algebraic derivation $\partial_{i^*}$ evaluated at 0 (up to elements of higher filtration). Since this image is contained in the subgroup of primitive operations $MU \to HZ$ none of these elements of higher filtration are present.

To see that $y_i$ are $\mathbb{Q}$-preferred derivations let us introduce the notation $\text{CDer}^*(MU_\mathbb{Q}, H\mathbb{Q})$ to denote topological derivations of $MU_\mathbb{Q}$ with values in $H\mathbb{Q} \cong S_\mathbb{Q}$ in the category of commutative $S$-algebras. (These commutative derivations are also known as topological Andre-Quillen cohomology, cf.[2]). Then since $MU_\mathbb{Q}$ is a free commutative $S$-algebra we see immediately that

$$\text{CDer}^*(MU_\mathbb{Q}, H\mathbb{Q}) = \text{Der}^*_S(MU_\mathbb{Q}_*, \mathbb{Q}) = \mathbb{Q} < \tilde{\partial}_{x_1}, \tilde{\partial}_{x_2}, \ldots>$$

the right hand side being the set of derivations (in the usual algebraic sense) of the algebra $MU_\mathbb{Q}_*$ with values in the rational numbers. Here we denoted by $\tilde{\partial}_{x_i}$ the standard derivation $\partial_{x_i}$ of the ring $MU_\mathbb{Q} = \mathbb{Q}[x_1, x_2, \ldots]$ composed with evaluation at zero.

On the other hand $\text{Der}^*_S(MU_\mathbb{Q}, H\mathbb{Q}) \cong \Lambda^{*-1}(y_1, y_2 \ldots)/\mathbb{Q}$. We need to prove therefore that the forgetful map

$$\text{CDer}^*(MU_\mathbb{Q}, H\mathbb{Q}) \to \text{Der}^*_S(MU_\mathbb{Q}, H\mathbb{Q})$$

sends the elements $\tilde{\partial}_{x_i}$ to $y_i$.

Since the commutative $S$-algebra $H\mathbb{Q}[x_i] \cong S_\mathbb{Q}[x_i]$ is free as a commutative $S$-algebra as well as an (associative) $S$-algebra it follows that $\text{CDer}^*(H\mathbb{Q}[x_i], H\mathbb{Q}) = \text{Der}^*(\mathbb{Q}[x_i], \mathbb{Q}) = \mathbb{Q} < \tilde{\partial}_{x_i}, >$.

There is a unique map of commutative $S$-algebras $MU_\mathbb{Q} \to H\mathbb{Q}[x_i]$ which corresponds to quotienting out the ideal $(x_1, x_2, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1} \ldots)$ in the coefficient ring of $MU_\mathbb{Q}$. We have the following commutative diagram:

$$\text{CDer}^*(H\mathbb{Q}[x_i], H\mathbb{Q}) \xrightarrow{\cong} \text{Der}^*(\mathbb{Q}[x_i], \mathbb{Q})$$

$$\text{CDer}^*(MU_\mathbb{Q}, H\mathbb{Q}) \to \text{Der}^*_S(MU_\mathbb{Q}, H\mathbb{Q})$$
from which it is clear that the image of $\partial_x$ in $\text{Der}^*(MU_\mathbb{Q}, H\mathbb{Q})$ is $y_1$ and the lemma is proved.

\begin{corollary}
The set of $\mathbb{Q}$-preferred derivations of $MU$ with values in $HZ$ maps bijectively onto the set of primitive cohomology operations $MU \to HZ$ under the forgetful map $\text{Der}^*_S(MU, HZ) \to [MU, HZ]^*$.
\end{corollary}

\begin{lemma}
Let $A$ be an $S$-algebra and $B$ a commutative $S$-algebra. Suppose that $B$ has a structure of an $A$-bimodule via a map of $S$-algebras $A \to B$. Then $\text{THH}_S(A, B)$ has a structure of a $B$-bimodule and there is a canonical splitting of $B$-bimodules $\text{THH}_S(A, B) \simeq B \vee \Sigma^{-1} \text{Der}_S(A, B)$.
\end{lemma}

\begin{proof}
Consider the following sequence of $S$-algebra maps:
\[
B \to \text{THH}_S(B, B) \cong F_{BA}(B, B) \to \text{THH}_S(A, B).
\]

The first map exists because $B$ is commutative, the middle map is induced by the $S$-algebra map $A \to B$ and the last equivalence is Corollary 3.3. The composite map $B \to \text{THH}_S(A, B)$ supplies $\text{THH}_S(A, B)$ with a structure of a $B$-bimodule and splits the canonical map $\text{THH}_S(A, B) \to B$.

Let us introduce the notation $MU_n$ for the $n$th Postnikov stage of $MU$. Then $MU_n$ is an $S$-algebra (even a commutative $S$-algebra).

\begin{lemma}
There is the following weak equivalence of $MU_n$-modules:
\[
\text{THH}_S(MU_n, MU_n) \cong \Lambda_{MU_n}(y_1, y_2, \ldots)
\]

where $MU_n$ is considered as an $MU$-bimodule via any (not necessarily central) map of $S$-algebras $f : MU \to MU_n$.
\end{lemma}

\begin{proof}
Since $MU_n$ is a commutative $S$-algebra the multiplication map
\[
MU_n \wedge MU_n \xrightarrow{m} MU_n
\]
is an $S$-algebra map. Therefore the composition
\[
MU \wedge MU_n \xrightarrow{f \wedge \text{id}} MU_n \wedge MU_n \xrightarrow{m} MU_n
\]
is also an $S$-algebra map. This gives the following weak equivalence of $S$-modules:
\[\text{THH}_S(MU_n, MU_n) \cong F_{MU_n}(MU_n, MU_n) \cong F_{MU_n}(MU_n \wedge MU_n, MU_n, MU_n).\]

Therefore it is enough to show that
\[\pi_* MU_n \wedge_{MU_n} MU_n = \Lambda_{MU_n}(y_1, y_2, \ldots).\]
(The exterior generators $y_i \in \text{THH}_S(MU, MU_n)$ will be dual to $\tilde{y}_i$).

Consider the spectral sequence
\[\text{Tor}_{**}^{MU}(MU_n, MU_n) = \Lambda_{MU_n}(y_1, y_2, \ldots) \Rightarrow \pi_* MU_n \wedge_{MU_n} MU_n\] (9)

This spectral sequence is not multiplicative since the map $f : MU \to MU_n$ may not be central. However it is a spectral sequence of $MU_n$-modules.

Let us introduce another spectral sequence
\[\text{Tor}_{**}^{MU}(MU_n, MU_n) = \Lambda_{MU_n}(y_1, y_2, \ldots) \Rightarrow \pi_* MU \wedge_{MU} MU\] (10)

Then the map $f : MU \to MU_n$ induces a map of spectral sequences (10)$\to$ (9). Further the spectral sequence (10) is multiplicative and collapses for that reason. Therefore in (9) all elements of the form $y_1 \wedge y_2 \wedge \ldots y_k$ are permanent cycles and it follows that (9) collapses. Lemma 5.13 is proved.

\end{proof}

Suppose as before that we have an $MU$-bimodule structure on $MU_{n+1}$ via some $S$-algebra map $f : MU \to MU_{n+1}$. Composing $f$ with the canonical map in the Postnikov tower $p_n : MU_{n+1} \to MU_n$ we get an $MU$-bimodule structure on $MU_n$ also. Then we have the following

\begin{corollary}
The induced map $\text{Der}_S^*(MU, MU_{n+1}) \to \text{Der}_S^*(MU, MU_n)$ is onto.
\end{corollary}
Proof. Indeed, the map
\[ THH^*_S(MU, MU_{n+1}) = \Lambda_{MU_{n+1}}(y_1, y_2, \ldots) \]
\[ \rightarrow \Lambda_{MU_n}(y_1, y_2, \ldots) = THH^*_S(MU, MU_n) \]
is clearly onto and our claim follows from Lemma 5.12. □

Proof of Theorem 5.4. We start by outlining the strategy of the proof. Take a multiplicative operation \( f \in Mult(MU, MU) \). Define \( f_n : MU \to MU_n \) as the composition
\[ MU \xrightarrow{f} MU \xrightarrow{p_n} MU_n. \]
(Recall that we denoted by \( p_n \) the canonical projection onto the \( n \)th Postnikov stage.) Then by Corollary 4.9 the map \( f_0 : MU \to MU_0 = HZ \) is homotopic to a unique \( S \)-algebra map which we will denote by \( \tilde{f}_0 \).
Proceeding by induction assume that there exists a unique \( \mathbb{Q} \)-preferred \( S \)-algebra map \( \tilde{f}_n : MU \to MU_n \) which is homotopic to \( f_n \) when considered as a map of \( S \)-modules. We will see that
- \( \tilde{f}_n \) admits a \( \mathbb{Q} \)-preferred lifting to an \( S \)-algebra map \( MU \to MU_{n+1} \) and
- there is a one-to-one correspondence between such liftings and the set of liftings of \( \tilde{f}_n \) to a homotopy multiplicative map \( MU \to MU_{n+1} \).

In particular \( f_{n+1} \) being one of such liftings can be realized as a \( \mathbb{Q} \)-preferred \( S \)-algebra lifting in a unique fashion.

We now proceed to realize the above program in detail. The first thing is to show that there exists a lifting of \( \tilde{f}_n \) in the category of \( S \)-algebras. The homotopy fibre sequence
\[ \Sigma^{n+1} \pi_{n+1} MU \longrightarrow MU_{n+1} \longrightarrow MU_n \tag{11} \]
is a topological singular extension by Theorem 2.6. Then Theorem 2.5 tells us that the obstruction to an \( S \)-algebra lifting \( \tilde{f}_n \) to \( MU_{n+1} \) is a certain element \( \sigma \in Der^0_S(MU, \Sigma H\pi_{n+1} MU) \). More precisely, the extension (11) is associated with a derivation
\[ d : MU_n \longrightarrow MU_n \vee \Sigma^{n+2} H\pi_{n+1} MU \]
and \( \sigma : MU \to MU_n \vee \Sigma^{n+2} H\pi_{n+1} MU \) is the composition of \( d \) with \( p_n : MU \to MU_n \).

Furthermore, notice that the set of \( S \)-algebra maps \( MU_\mathbb{Q} \to MU_{\mathbb{Q}^{n+1}} \) is in bijective correspondence with the set of ring maps \( MU_\mathbb{Q} \to MU_{\mathbb{Q}^{n+1}} \). Since \( MU_\mathbb{Q} \) is a polynomial algebra we see that a lift of \( \tilde{f}_n \) does exists after rationalization. Therefore the image of \( \sigma \) in \( Der^0_S(MU, \Sigma^{n+2} H\pi_{n+1} MU \mathbb{Q}) \) is zero. But \( H\pi_{n+1} MU \) is a wedge of suspensions of \( H\mathbb{Z} \) and according to Lemma 5.10 the abelian group \( Der^0_S(MU, H\pi_{n+1} MU) \) has no torsion. It follows that \( \sigma = 0 \) as an element in the group \( Der^0_S(MU, \Sigma H\pi_{n+1} MU) \) and a lift of \( \tilde{f}_n \) exists integrally (though not necessarily \( \mathbb{Q} \)-preferred). By Theorem 2.5 the homotopy fibre of the map
\[ FS_{-alg}(MU, MU_{n+1}) \longrightarrow FS_{-alg}(MU, MU_n) \]
taken over the point \( \tilde{f}_n \in FS_{-alg}(MU, MU_n) \) is weakly equivalent to the zeroth space of the spectrum \( Der_S(MU, \Sigma^{n+1} H\pi_{n+1} MU) \).

Therefore denoting by \( [MU, MU_{n+1}]_{a}^{hfg} \subset [MU, MU_{n+1}]_{a} \) the set of homotopy classes of \( S \)-algebra maps \( MU \to MU_{n+1} \) lifting \( \tilde{f}_n \) we have the following long exact sequence:
\[ \ldots \longrightarrow Der^{1}_S(MU, \Sigma^{n+1} H\pi_{n+1} MU) \longrightarrow \pi_1 FS_{-alg}(MU, MU_{n+1}) \]
\[ \longrightarrow \pi_1 FS_{-alg}(MU, MU_n) \longrightarrow Der^0_S(MU, \Sigma^{n+1} H\pi_{n+1} MU) \]
\[ \longrightarrow [MU, MU_{n+1}]_{a}^{hfg} \longrightarrow pt \]
which is the same (by Theorem 4.3) as the long exact sequence

\[ \ldots \rightarrow \text{Der}_S^{-1}(MU, \Sigma^{n+1}H\pi_{n+1}MU) \rightarrow \text{Der}_S^{-1}(MU, MU_{n+1}) \rightarrow \text{Der}_S^{-1}(MU, MU_n) \rightarrow \text{Der}_S^0(MU, \Sigma^{n+1}H\pi_{n+1}MU) \rightarrow [MU, MU_{n+1}]^{\text{lf}} \rightarrow pt. \]

By Corollary 5.14 the map

\[ \text{Der}_S^{-1}(MU, MU_{n+1}) \rightarrow \text{Der}_S^{-1}(MU, MU_n) \]

is onto and we conclude that the map

\[ \text{Der}_S^0(MU, \Sigma^{n+1}H\pi_{n+1}MU) \rightarrow [MU, MU_{n+1}]^{\text{lf}} \]

is bijective. So the set of all possible lifts of \( \tilde{f}_k \) is in one-to-one correspondence with elements in the group \( \text{Der}_S^0(MU, \Sigma^{n+1}H\pi_{n+1}MU) \). Clearly the set of all Q-preferred lifts corresponds under this isomorphism to the set of Q-preferred topological derivations of \( MU \) with values in \( \Sigma^{n+1}H\pi_{n+1}MU \). By Corollary 5.11 these Q-preferred derivations are identified with the set of primitive operations from \( MU \) to \( \Sigma^{n+1}H\pi_{n+1}MU \). So we established a one-to-one correspondence between the set of Q-preferred lifts of \( \tilde{f}_n \) and \( \text{Prim}(MU, \Sigma^{n+1}H\pi_{n+1}MU) \).

Now we examine the question of lifting the map \( \tilde{f}_n \) up to homotopy to a homotopy multiplicative map \( MU \rightarrow MU_{n+1} \). Clearly the homotopy class of any map of \( S \)-modules \( MU \rightarrow MU_k \) is determined by its rationalization, i.e., the rationalization map \( [MU, MU_k] \rightarrow [MU, MU_{Qk}] \) is injective. It follows that the map

\[ \text{Mult}(MU, MU_k) \rightarrow \text{Mult}(MU, MU_{Qk}) \]

is also injective.

Further we have the following bijection (for any \( k \))

\[ \text{Mult}(MU_{Q}, MU_{Qk}) \cong \text{Hom}_{\text{rings}}(MU_{Q*}, MU_{Qk*}) \]

Therefore there is a short exact sequence

\[ 0 \rightarrow \text{Prim}(MU_{Q}, \Sigma^{n+1}H\pi_{n+1}MU_{Q}) = \text{Der}(MU_{Q*}, \pi_{n+1}MU_{Q}) \rightarrow \text{Mult}(MU, MU_{Qn}) \rightarrow \text{Mult}(MU, MU_{Qn+1}) \rightarrow pt \quad (12) \]

Of course the last three terms are only sets. The exactness here means that \( \text{Mult}(MU_{Q}, MU_{Qn+1}) \) has a faith-ful action of the group \( \text{Prim}(MU_{Q}, \Sigma^{n+1}H\pi_{n+1}MU_{Q}) \) so that the quotient is isomorphic to \( \text{Mult}(MU, MU_{Qn+1}) \).

Consider the diagram of fibre sequences

\[
\begin{array}{ccc}
\Sigma^{n+1}H\pi_{n+1}MU & \rightarrow & MU_{n+1} \\
\downarrow & & \downarrow \\
\Sigma^{n+1}H\pi_{n+1}MU_{Qn+1} & \rightarrow & MU_{Qn+1}
\end{array}
\]

Taking into account the fact that \( MU^*_{k+1}MU \) surjects onto \( MU^*_kMU \) for any \( k \) we obtain a map of short exact sequences

\[
\begin{array}{ccc}
0 & \rightarrow & [MU, \Sigma^{n+1}H\pi_{n+1}MU] \\
\downarrow & & \downarrow \\
0 & \rightarrow & [MU_{Q}, \Sigma^{n+1}H\pi_{n+1}MU_{Q}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & [MU, MU_{n+1}] \\
\downarrow & & \downarrow \\
0 & \rightarrow & [MU_{Q}, MU_{Qn+1}]
\end{array}
\]

\[
\begin{array}{ccc}
[MU, MU_{n}] & \rightarrow & 0 \\
\downarrow & & \downarrow \\
[MU_{Q}, MU_{Qn}] & \rightarrow & 0
\end{array}
\]
Notice that all downward maps are injections. Combining this with (12) we find that there is a short exact sequence
\[ 0 \to \text{Prim}(\text{MU}, \Sigma^{n+1}H\pi_{n+1}\text{MU}) \]
\[ \to \text{Mult}(\text{MU}, \text{MU}_{n+1}) \to \text{Mult}(\text{MU}, \text{MU}_n) \to 0 \]

That is the indeterminacy in lifting the map \( \tilde{f}_n : \text{MU} \to \text{MU}_n \) is precisely the set of primitive cohomology operations \( \text{Prim}(\text{MU}, \Sigma^{n+1}H\pi_{n+1}\text{MU}) \). We see that the set of lifts of the map \( f_n \) to a \( \mathbb{Q} \)-preferred \( S \)-algebra lifts. This completes the inductive step and shows that the original homotopy multiplicative map \( f : \text{MU} \to \text{MU} \) can be improved in a unique way to a \( \mathbb{Q} \)-preferred \( S \)-algebra map.

So we succeeded in finding a section \( i : \text{Mult}(\text{MU}, \text{MU}) \to [\text{MU}, \text{MU}]_a \) of the forgetful map \( j : [\text{MU}, \text{MU}]_a \to \text{Mult}(\text{MU}, \text{MU}) \) so that the image of \( i \) consists of \( \mathbb{Q} \)-preferred \( S \)-algebra self-maps of \( \text{MU} \). To see that \( i \) respects composition notice that for \( f, g \in \text{Mult}(\text{MU}, \text{MU}) \) the \( S \)-algebra map \( i(f) \circ i(g) \) is \( \mathbb{Q} \)-preferred and \( j(i(f) \circ i(g)) = f \circ g \). Since there is a unique \( \mathbb{Q} \)-preferred \( S \)-algebra self-map whose image under \( j \) is \( f \circ g \) we conclude that \( i(f) \circ i(g) = i(f \circ g) \). With this the proof of Theorem 5.4 is completed.

**Remark 5.15.** Using the Bousfield-Kan mapping space spectral sequence (cf. [5]) it is possible to calculate the set of all \( S \)-algebra self-maps of \( \text{MU} \). However this approach leads to the identification of \([\text{MU}, \text{MU}]_a \) only as a set, not as a monoid. It seems that the monoid structure on \([\text{MU}, \text{MU}]_a \) should be related to the Gerstenhaber bracket on \( \text{THH}_S^*(\text{MU}, \text{MU}) \).

Now consider an \( S \)-algebra \( E \) with a fixed map of \( S \)-algebras \( f : \text{MU} \to E \). Suppose that \( E \) satisfies the following condition:

\( (S) \) The unit map \( f : \text{MU}_s \to E_s \) is surjective.

**Remark 5.16.** In [7] it was proved that a rather broad class of \( C \)-oriented spectra (namely those which are obtained by killing any regular ideal in the ring \( \text{MU} \)) can be supplied with \( \text{MU} \)-algebra structures. For this class of spectra the condition \( (S) \) is obviously satisfied.

**Corollary 5.17.** For an \( S \)-algebra \( E \) satisfying the condition \( (S) \) any multiplicative operation \( \text{MU} \to E \) can be lifted (perhaps in a non-unique way) to an \( S \)-algebra map.

**Proof.** The condition \( (S) \) guarantees that the map
\[ \text{Mult}(\text{MU}, \text{MU}) \to \text{Mult}(\text{MU}, E) \]
induced by the given map \( f : \text{MU} \to E \) is surjective. In other words any multiplicative operation \( g : \text{MU} \to E \) can be represented as a composition \( h \circ f \) where \( h \in \text{Mult}(\text{MU}, \text{MU}) \). Since \( h \) can be lifted to an \( S \)-algebra self-map of \( \text{MU} \) our claim follows.

As another consequence of Theorem 5.4 we will show that the \( p \)-local Brown-Peterson spectrum \( BP \) is an \( A_{\infty} \)-retract of \( \text{MU}_{(p)} \), the spectrum \( \text{MU} \) localized at \( p \). Recall from e.g. [13], 4.1 that there exists a multiplicative cohomology operation \( g : \text{MU}_{(p)} \to \text{MU}_{(p)} \) which is idempotent and whose image is the \( p \)-local spectrum \( BP \).

**Theorem 5.18.** There exists an \( S \)-algebra map \( f : \text{MU}_{(p)} \to BP \) which has a right inverse \( S \)-algebra map \( h : BP \to \text{MU}_{(p)} \).

**Proof.** According to Theorem 5.4 the multiplicative operation \( g : \text{MU}_{(p)} \to \text{MU}_{(p)} \) determines a map of \( S \)-algebras which we will denote by the same letter. Without loss of generality we can assume \( g \) to be a \( q \)-cofibration of \( S \)-algebras. Consider the diagram in the category of \( S \)-algebras:

\[
\begin{array}{ccc}
\text{MU}_{(p)} & \xrightarrow{g} & \text{MU}_{(p)} \\
\downarrow{g} & & \downarrow{g} \\
\text{MU}_{(p)} & \xrightarrow{id} & \text{MU}_{(p)}
\end{array}
\]

Each square in this diagram is commutative since the operation \( g \) is idempotent. The colimit of the upper row taken in the category of \( S \)-algebras coincides with the colimit taken in the category of spectra by Cofibration.
Hypothesis ([8], VII.4) and both are equivalent to $BP$. (That shows that $BP$ is an $S$-algebra). Now the map $f : MU(p) \to BP$ is just the canonical map to the colimit. Next the colimit of the lower row is obviously $MU(p)$ and therefore there exists an $S$-algebra map $h : BP \to MU(p)$. It follows that $f \circ h : BP \to BP$ is homotopic to the identity and Theorem 5.18 is proved.

**Remark 5.19.** It can be shown (cf. [10], [1], [7]) that $BP$ actually supports a structure of an $MU$-algebra.

**Remark 5.20.** It seems natural to conjecture that any $Q$-preferred $S$-algebra self-map of $MU$ lifts to a commutative $S$-algebra self-map. This conjecture, if true, would imply the existence of a canonical $E_\infty$ ring structure on $BP$, a long-standing problem posed by P.May. The first (to author’s knowledge) serious attack on this problem was undertaken by I. Kriz in his 1993 preprint [9]. This paper inspired much activity in the area, however it is still regarded as a program for further work rather than a definitive solution.

Even though we don’t know whether $BP$ is a commutative $S$-algebra we can use Theorem 5.18 to compute homotopy classes of $A_\infty$-maps out of $BP$.

**Corollary 5.21.** For an $S$-algebra $E$ satisfying the condition $(S)$ every multiplicative operation $BP \to E$ lifts to an $A_\infty$ ring map $BP \to E$ (perhaps in a non-unique way).

**Proof.** The composition of the multiplicative operation $BP \to E$ with the canonical projection $MU \to BP$ determines a multiplicative operation $MU \to E$. This operation lifts to an $S$-algebra map. Composing this $S$-algebra map with the splitting map $BP \to MU$ (which we know is an $S$-algebra map by Theorem 5.18) we find the desired $S$-algebra map $BP \to E$.

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