Wave Superposition Method Applied to Calculate the Radiation Sound Power from a Thin Plate

Yong-yong Zhu\(^1\), Jing-hui Peng\(^1\) and Bao Liu\(^1\)*

\(^1\) Naval Engineering University, Wuhan, Hubei, 430033, China

*Corresponding author’s e-mail: 909450046@qq.com

Abstract. A method for calculating the radiation sound power from a thin plate is described using the principle of wave superposition. After getting the velocity of the surface, the method can be used to calculate the strength of radiation source, which can be used to compute the sound power of the corresponding plate without computing the surface pressure. The paper gives an example of a rectangular baffled plate with simply supported boundary. And through comparing the computation result with the analytic result, it is shown that the mention method can reduce the unit and node number while it can ensure high precision without calculating the radiation resistance. Thus the method reduce computing time and improve the efficiency.

1. Introduction

The radiated noise of the structure is a widely existing acoustic problem in engineering, and it is very important to master the noise characteristics of the structure to control the noise. To study the structure of the vibration and acoustic radiation, the essence is to determine the acoustic radiation field of the medium according to the vibration response of the radiator. Numerical methods are usually adopted to analyse the vibration characteristics of the radiators to attain the approaching surface vibration velocity and perform numerical computations to the radiated sound field with the use of finite element method or boundary element method (FEM / BEM) [1-4], but the methods are insufficient. For examples, there are inherent problems such as surface singular integrals and the non-uniqueness in characteristic frequency when BEM is used. In order to overcome the shortcomings of the methods, Koopmann proposed the principle of wave superposition [5,6] in 1989. Through disposing virtual sound sources inside the structure to simulate equivalently structural radiated sound field, sound radiation problems are described as the superposition of virtual sound source functions satisfying wave equation. The surface normal velocity of radiators is used to obtain the strength of virtual sound sources and furthermore to obtain the characteristic parameters such as structural acoustic radiation power. Due to the non-coincidence of the plane of virtual sources and the structural surface, thus the singularity problem is avoided and the calculation is simple and carried out easily. Using wave superposition principle, Xiang-yang and Koopmann obtained surface pressure of piston source in rigid spherical surface and pulsating sphere source and further discussed the effect of unit, the node number and the shape to the efficiency and accuracy of the algorithms under the condition of surface vibration velocity obtained from the experiment [7,8]. As an example of a simply supported rectangular plate, the paper introduces wave superposition principle based on the match of unit volume velocity, gives the method to calculate the average sound radiation power using the principle of wave superposition and discusses the sensitivity of discrete element numbers using wave superposition principle.
2. Surface Velocity of the Plate

In order to illuminate the calculation progress using the principle of wave superposition, this paper takes the rectangular simply supported plate as an example to illustrate the method. The density of the plate is \( \rho_s \), the length of the \( x \) and \( y \) axial direction is \( a \) and \( b \) separately, the thickness is \( h \), and is located in the plane \( z = 0 \). The half space \( z \geq 0 \) is filled with light medium of density \( \rho_f \), and the plate vibrates due to external force and radiates the sound power in the half space, \( z < 0 \) half space is vacuum state, as shown in Fig.1. Because the medium is light, the sound pressure generated by the media is ignored.

![Figure 1 Schematic view of the thin plate](image)

For a simply supported rectangular plate, the vibration velocity of the plate can be expressed as a linear superposition of the vibration modes

\[
\mathbf{w}(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \psi_{mn}(x, y)
\]

It can be expressed as a form of a matrix

\[
\mathbf{\dot{w}} = \mathbf{\Psi} \mathbf{A}
\]

where \( A_{mn} \) is the undetermined coefficient of vibration mode. \( \psi_{mn}(x, y) = \sin(m \pi x / a) \sin(n \pi y / b) \) is the mode shape.

Considering a point force \( F \) on the \((x_0, y_0)\) location. It is known by the literature [9] that

\[
A_{mn} = \frac{\omega_m^2 F \psi_{mn}(x_0, y_0)}{\left[ \omega_m^2 (1 + i \eta) - \omega_c^2 \right] M_{mn}}
\]

where \( \eta \) is the damping coefficient, \( M_{mn} \) is mode mass, \( S \) is the surface of the plane, \( \omega_m \) is the resonance frequency of the mode \( m,n \) that is given by \( \omega_m = \left( B / \rho_s h^3 \right)^{1/2} \left( (m \pi / a)^2 + (n \pi / b)^2 \right) \), \( B \) is the rigidity modulus that is given by \( B = Eh^3 / 12(1 - \nu^2) \), \( E \) is elastic modulus, \( \nu \) is Poisson ratio, and \( \omega_c \) is angular frequency.

3. Lumped Parameter Model

The Green function of the second kind satisfying Neumann boundary conditions are used which is \( \nabla \cdot \mathbf{G}(r, r_s) \cdot \mathbf{n} = 0 \) to simplify the pressure[10] at any point \( r = (x, y, z) \) of the outside space described by Kirchhoff-Helmholtz equation.

\[
p(r) = -\frac{ik \rho_c}{4\pi} \iint_{S} \mathbf{G}(r, r_s) \mathbf{\dot{w}}(r_s) \cdot \mathbf{n} dS(r_s)
\]

Where \( k = \omega / c \) is acoustic wave number, \( c \) the sound velocity, \( \mathbf{\dot{w}}(r_s) \) is the velocity of \( r_s = (x_s, y_s, 0) \), \( \mathbf{n} \) is the outside normal to the surface \( S \), shown in figure 1. Equation (4) reflects the pressure outside the structure can be calculated through integrating the product of the velocity and the Green function under the condition that the surface velocity is known. Take the Green function in free space as the Green function in equation (4) which is

\[
\mathbf{G}(r, r_s) = e^{ik|\mathbf{r} - \mathbf{r_s}|}/|\mathbf{r} - \mathbf{r_s}|
\]
where \[ |r - r_s| = \sqrt{(x-x_s)^2 + (y-y_s)^2 + z^2}. \]

The surface of the thin plate is separated as \( N \) units with area \( S_\mu (1 \leq \mu \leq N) \), so the equation (4) can be expressed as

\[
p(r) = \frac{ik \rho \omega c}{4\pi} \sum_{r_s} \iint_{S_\mu} G(r, r_s) \tilde{w}(r_s) \cdot \mathbf{n}_s dS(r_s)
\]  

(6)

The volume velocity of the unit \( \mu \) is the sum of the normal velocity from the unit \( \mu \),

\[
u_\mu = \iint_{S_\mu} \tilde{w}(r_s) \cdot \mathbf{n}_s dS(r_s)
\]  

(7)

Substitution of Eq. (1) into Eq. (7) gives

\[
u_\mu = \sum_{\nu=1}^{N} \sum_{s=1}^{N} A_{\mu \nu} \iint_{S_\mu} \psi_{ss}(x, y) dS(r_s)
\]  

(8)

The equation above is expressed in the form of a matrix as

\[
u = \Phi A
\]  

(9)

The average pressure \( p_\mu \) of the unit \( \mu \) is

\[p_\mu = \frac{1}{S_\mu} \iint_{S_\mu} p(r) dS(r)
\]  

(10)

Substitution of Eq. (6) into Eq. (8) gives

\[p_\mu = \frac{ik \rho \omega c}{4\pi} \sum_{r_s} \iint_{S_\mu} G(r, r_s) \tilde{w}(r_s) \cdot \mathbf{n}_s dS(r_s) dS(r_s)
\]  

(11)

There are two possible circumstances under which the double surface integral is simplified. If the Green function was approximately constant over every surface element, then the function can be brought out from under the integral signs. However this approximation is not mathematically justifiable. The integral can be simplified by assuming that the normal surface velocity is constant over each element, as in the lowest order term of the series solution for the acoustic field. Although the approximation is somewhat coarse, it is certainly reasonable if computing the acoustic power of the vibrating structure. Thus it assumes that each of the elements vibrates as a piston, such that the surface velocity is constant over each element.

Adopting the second method to simplify Eq. (11) gives

\[p_\mu = \frac{ik \rho \omega c}{4\pi} \sum_{r_s} \iint_{S_\mu} \tilde{w}(r_s) \cdot \mathbf{n}_s dS(r_s) \int_S \int_S G(r, r_s) dS(r_s) dS_\mu(r_s)
\]  

(12)

Substitution of Eq. (7) into Eq. (12) gives

\[p_\mu = \sum_{\nu=1}^{N} Z_{\mu \nu} \nu_\nu
\]  

(13)

where \[ Z_{\mu \nu} = \frac{-ik \rho \omega c}{4\pi S_\mu S_\nu} \iint_{S_\nu} \iint_{S_\mu} G(r, r_s) dS_\nu(r_s) dS_\mu(r)
\] is the radiation impedance of a piston source located on element \( \nu \) to the pressure field of a piston source located on element \( \mu \).

The amplitude of the piston vibration over \( \nu (1 \leq \nu \leq N) \) then becomes

\[\tilde{w}_{\nu, \text{piston}} = \frac{1}{S_\nu} \iint_{S_\nu} \tilde{w}(r_s) \cdot \mathbf{n}_s dS = \nu_\nu / S_\nu
\]  

(14)

The average power can be written in terms of the pressure and normal velocity on the surface of the plate as

\[\Pi_{\mu \nu} = \frac{1}{2} \iint_{S_\mu} \text{Re} \{ p(r) \tilde{w}^*(r) \cdot \mathbf{n}_s \} dS(r)
\]  

(15)

where \( \text{Re} \) and \( * \) express the real part and the conjugate of the plural respectively.

If it assumes that each element vibrates as a piston, the acoustic power can be approximated as
\[ \Pi_{\mu} = \frac{1}{2} \sum_{\nu=1}^{N} \text{Re} \left[ u_{\mu,\nu}^* \right] \int_{S_{\mu}} p(r) dS(r) \]  

(16)

Substitution of Eq. (14) then gives

\[ \Pi_{\mu} = \frac{1}{2} \sum_{\nu=1}^{N} \text{Re} \left[ u_{\mu,\nu}^* \right] \left[ \int_{S_{\mu}} p(r) dS(r) \right] = \frac{1}{2} \sum_{\nu=1}^{N} \text{Re} \left[ u_{\mu,\nu}^* p_{\nu} \right] \]  

(17)

Substitution of Eq. (13) into Eq. (17) considering the symmetry of \( Z_{\mu\nu} \) then gives

\[ \Pi_{\mu} = \frac{1}{2} \sum_{\nu=1}^{N} \text{Re} \left[ u_{\mu,\nu}^* Z_{\mu\nu} u_{\nu} \right] = \frac{1}{2} U^H \Phi U \]  

(18)

Where \( \Phi_{\mu\nu} = \frac{k \rho c}{4\pi S_{s} S_{r}} \int_{S_{s}} \int_{S_{r}} \text{Im} \{ G(r, r_{\nu}) \} dS_{s}(r_{\nu}) dS_{r}(r). \)

Through the analysis, the energy of the radiation of the structure is mainly derived from the piston-like component of the surface vibration, so the radiation acoustic power of the structure can be calculated accurately by the lumped parameter model.

4. The Principle of Wave Superposition

As is known from the above analysis, calculating the radiated sound power does not need to obtain the exact value of the surface pressure, if only the average velocity in each unit of the surface is accurately computed. The volume velocity matching scheme requires an assumed form for the pressure field in terms of a series of basic functions. So the approximate solution for the acoustic pressure field can be written as Refs. [11]

\[ p(r) = \sum_{i=1}^{N} s_{i} \left\{ \alpha_{i} G(r, r_{i}) + \beta_{i} \left[ \nabla_{s} G(r, r_{i}) \cdot n_{s} \right]_{s_{i}} \right\} \]  

(19)

where the \( \alpha_{i} \) and \( \beta_{i} \) are known constants, as listed in Tab. 1 based on Refs. [7]. \( r_{\nu} \) is the location of source \( s_{\nu} \), \( r_{s} \) is the geometric center of element \( s_{\nu} \), \( s_{v} \) is the strength of virtual source \( s_{v} \).

Table. 1 Values of the constants \( \alpha_{s} \) and \( \beta_{s} \)

| \( \alpha_{s} \) | \( \beta_{s} \) | Source type | Description of the surface element |
|----------------|----------------|-------------|-----------------------------------|
| 1              | 0              | Simple      | In the plane of an infinite baffle |
| 0              | \( i/k \)      | Dipole      | Part of a surface enclosing no volume |
| 1              | \( i/k \)      | Tripole     | Part of a surface enclosing a finite volume |

Euler’s equation can be written as

\[ \nabla p(r) = ik \rho c v(r) \]  

(20)

Using Euler’s equation, Eq. (19) can be expressed as

\[ v(r) = \frac{1}{ik \rho c} \sum_{i=1}^{N} s_{i} \nabla \left\{ \alpha_{i} G(r, r_{i}) + \beta_{i} \left[ \nabla_{s} G(r, r_{i}) \cdot n_{s} \right]_{s_{i}} \right\} \cdot n_{s} \]  

(21)

The volume velocity over element \( \mu \) is determined by integrating Eq. (21) over the element surface as

\[ u_{\mu} = \frac{1}{ik \rho c} \sum_{i=1}^{N} s_{i} \int_{S_{\mu}} \nabla \left\{ \alpha_{i} G(r, r_{i}) + \beta_{i} \left[ \nabla_{s} G(r, r_{i}) \cdot n_{s} \right]_{s_{i}} \right\} \cdot n_{s} dS(r) \]  

(22)

Eq. (22) can be written in matrix form as

\[ \mathbf{u} = \mathbf{U} \mathbf{s} \]  

(23)

where the individual terms of the matrix \( \mathbf{U} \) becomes
\[ U_{ij} = \frac{1}{ikpc} \iiint_{V_s} \nabla \left\{ \alpha_i G(r_i, r_j) + \beta_j \left[ \nabla_s G(r_i, r_j) \cdot \mathbf{n}_i \right] \right\} \cdot \mathbf{n}_j dS(r) \] (24)

So \( s \) can be expressed as
\[ s = U^\dagger \mathbf{u} \] (25)

Substitution of Eq. (9) into Eq. (25) gives
\[ s = U^\dagger \Phi \mathbf{A} \] (26)

5. The Solution for the Power Output

The partial differential equation governing the acoustic field of a number of sources is written as
\[ \nabla^2 p(r) + k^2 p(r) = -4\pi \sum_{i=1}^{N} s_i \cdot \left\{ \alpha_i \delta(r-r_i) + \beta_i \left[ \nabla_s \delta(r-r_i) \right] \right\} \cdot \mathbf{n}_i \] (27)

Noting \( \nabla \left[ \left( p^*(r) \nabla p(r) \right) \right] = \nabla p^*(r) \nabla p(r) + p^*(r) \nabla^2 p(r) \),
multiplying by \( p(r) \) gives
\[ \nabla \left[ \left( p^*(r) \nabla p(r) \right) \right] - [\nabla p(r)]^2 + k^2 |p(r)|^2 = -4\pi \sum_{i=1}^{N} s_i \cdot \left\{ \alpha_i \delta(r-r_i) + \beta_i \left[ \nabla_s \delta(r-r_i) \right] \right\} \cdot \mathbf{n}_i \] (28)

Substituting of Eq. (20) into Eq. (28) gives
\[ ik\rho c \nabla \left[ \left( p^*(r) \nabla p(r) \right) \right] - [\nabla p(r)]^2 + k^2 |p(r)|^2 = -4\pi \cdot p^*(r) \sum_{i=1}^{N} s_i \cdot \left\{ \alpha_i \delta(r-r_i) + \beta_i \left[ \nabla_s \delta(r-r_i) \right] \right\} \cdot \mathbf{n}_i \] (29)

Dividing Eq. (29) by \( ik\rho c \) and taking the real part of the results then yields
\[ \text{Re} \left\{ \nabla \left[ \left( p^*(r) \nabla p(r) \right) \right] \right\} = \frac{4\pi}{kpc} \sum_{i=1}^{N} \text{Re} \left[ ip^*(r) s_i \cdot \left\{ \alpha_i \delta(r-r_i) + \beta_i \left[ \nabla_s \delta(r-r_i) \right] \right\} \cdot \mathbf{n}_i \right] \] (30)

To convert the left hand side of Eq. (30) to the time-averaged acoustic power output of the sources, integrating over any volume \( V \) enclosing all the sources and apply Gauss’ theorem to the result gives
\[ \text{Re} \iiint_{V} \nabla \left[ \left( p^*(r) \nabla p(r) \right) \right] dV(r) = \iint_{S} \text{Re} \left[ p^*(r) \nabla p(r) \cdot \mathbf{n} \right] dS(r) \] (31)

As is shown above, the formula (31) is two times of the average sound radiation power of the plate. Integrating over \( V \) and use the property of the Dirac delta function gives
\[ \frac{4\pi}{kpc} \sum_{i=1}^{N} \iint_{S} \text{Re} \left[ ip^*(r) s_i \cdot \left\{ \alpha_i \delta(r-r_i) + \beta_i \left[ \nabla_s \delta(r-r_i) \right] \right\} \cdot \mathbf{n}_i \right] dV(r) = \frac{4\pi}{kpc} \sum_{i=1}^{N} \text{Re} \left\{ ix_i \alpha_i p^*(r_i) + ix_i \beta_i \left[ \nabla_s p^*(r_i) \right] \right\} \cdot \mathbf{n}_i \] (32)

So the acoustic power can be written as
\[ \Pi_m = \frac{2\pi}{kpc} \sum_{i=1}^{N} \text{Re} \left\{ ix_i \alpha_i p^*(r_i) + ix_i \beta_i \left[ \nabla_s p^*(r_i) \right] \right\} \cdot \mathbf{n}_i \] (33)

Since a simple supported rectangular barrier plate is taken as the research object, it is known from Tab.1 that \( \alpha_i = 1, \beta_i = 0 \). Substituting these into Eq. (32) gives
\[ \Pi_m = \frac{2\pi}{kpc} \sum_{i=1}^{N} \text{Re} \left\{ ix_i p^*(r_i) \right\} \] (34)

\( p^*(r) \) can be expressed by Eq. (19), then Eq. (34) can be written as
\[ \Pi_m = \frac{2\pi}{kpc} \sum_{i=1}^{N} \sum_{\mu=1}^{N} \text{Re} \left\{ ix_i G(r_{\mu}, r_i) \alpha_{\mu}^* \right\} \] (35)

Further, it can be simplified as
\[ \Pi_{av} = \frac{2\pi}{\rho f c} \sum_{n=1}^{N} \sum_{m=1}^{N} s_{n} s_{m} j_{0}(kR_{n,m}) R_{n,m} \]

where \( j_{0}(kR_{n,m}) = \sin(kR_{n,m})/(kR_{n,m}) \) is the 0th order spherical Bessel function of the first kind, \( R_{n,m} = |r_{n} - r_{m}| \) is the distance between sound source \( n \) and \( m \).

Eq. (33) can be written in matrix form as

\[ \Pi_{av} = \frac{2\pi}{\rho f c} \mathbf{s}^{H} \mathbf{j} \]

where the superscript \( \mathbf{H} \) denotes the Hermitian transpose.

The diagonal terms of the matrix are evaluated in the limit as \( kR_{n,n} \to 0 \). The diagonal components of \( S \) are finite, as they must be if the overall acoustic power is to be finite. The contribution from the simple sources is easily determined as

\[ \lim_{kR_{n,n} \to 0} j_{0}(kR_{n,n}) = \sin(kR_{n,n})/(kR_{n,n}) = 1 \]

Thus, in the limit as \( kR_{n,n} \to 0 \), the term in braces in Eq. (33) goes to 1.

6. The Example

The considered rectangular simply supported steel plate has a length of 1m, a width of 0.7m, and the thickness is 0.003m. The geometric center is subjected to a point force with an amplitude of 1N. The parameter of the plate is \( \rho = 7800 \text{kg/m}^3 \), \( \nu = 0.3 \), \( E = 2.16 \times 10^{11} \text{N/m}^2 \), \( \rho_f = 1.21 \text{kg/m}^3 \), \( c = 343 \text{m/s} \), \( \eta = 0.01 \), and reference sound power \( W_f = 10^{-12} W \). In this paper, the influence of the number of discrete elements on the calculation accuracy of the wave superposition method is emphatically discussed.

For the simple supported rectangular barrier plate, the analytical formula to calculate the average sound radiation power in the light medium is

\[ \Pi_{av} = \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{x=1}^{N} \left\{ A_{nn} A_{mm} \cdot \int_{0}^{2\pi} \left[ e^{ijkr} \frac{u_{m}(r)}{2\rho f c} \right] r^2 \sin \theta d\theta d\phi \right\} \]

where \( u_{m}(r) = jk\rho_c \frac{e^{-jkr}}{2\pi} \frac{ab}{n\pi} \left[ \begin{array}{c} (-1)^e e^{iax} - 1 \\ (-1)^f e^{ibx} - 1 \end{array} \right] \left[ \begin{array}{c} \alpha = ka \sin \theta \cos \phi \\ \beta = kb \sin \theta \sin \phi \end{array} \right] \).

The surface of the thin plate is divided into three kinds of different numbers of rectangular elements. Their numbers are respectively 5x5, 10x10, 15x15.

In this paper, the results obtained by the principle of wave superposition and analytic method is compared with the analytic results (dashed line: the results of wave superposition; real line: the results of analytic method).
Figure 2 The contrast between the results of $5 \times 5$ virtual sound sources and analytic method.

Figure 3 The contrast between the results of $10 \times 10$ virtual sound sources and analytic method.

Figure 4 The contrast between the results of $15 \times 15$ virtual sound sources and analytic method.
Table 2 The greatest relative errors using different numbers of virtual sound sources

| The number of virtual sound sources | 5×5 | 10×10 | 15×15 |
|------------------------------------|-----|-------|-------|
| The greatest relative errors       | 18.8% | 7.1% | 5.5% |

The diagram shows that when the frequency is low (\(ka \leq 7\)), the results using three different number of virtual sound sources are almost identical. It is shown from Tab.2, the relative error increases with the frequency increasing. With the number of elements increasing from 5×5 to 10×10, the relative error decreases rapidly from 18.8% to 7.1%. But with the number further increasing to 15×15, the relative error reduces from 7.1% to 5.5%, and the velocity of relative error decrease becoming slow mean the results using wave superposition tends to the analytic results.

7. Conclusion

The research results above shows that using the wave superposition method can estimate the structural acoustic radiation power well when obtaining the surface vibration velocity.

From the examples, the wave superposition method has the following advantages compared with the other algorithm. Compared with the boundary element method, it does not need to deal with the singularity problem, and the calculation is simplified; in solving structural acoustic radiation power, it does not need to calculation of radiation impedance, and the efficiency can be further improved; the method can be applied in wide frequency domain, and using a small number of virtual sound sources can achieve high accuracy in the low frequency range.

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