Approximate Kerr-Newman-like Metric with Quadrupole

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Abstract

The Kerr metric is known to present issues when trying to find an interior solution. In this work we continue in our efforts to construct a more realistic exterior metric for astrophysical objects. A new approximate metric representing the spacetime of a charged, rotating and slightly-deformed body is obtained by perturbing the Kerr-Newman metric to include the mass-quadrupole and quadrupole-quadrupole orders. It has a simple form, because is Kerr-Newman-like. Its post-linear form without charge coincides with post-linear quadrupole-quadrupole metrics already found.

1 Introduction

Since Kerr proposed his metric in 1963 \cite{1} multiple efforts have been directed to finding an interior solution of his spacetime \cite{2 3 4 5 6 7 8 9} or in generalizing the Kerr metric to a metric that allows a physical interior matching \cite{10 11 12 13}. Nonetheless, no physical interior solution exists. See \cite{14} for a relatively recent prespective in the issues present in the Kerr metric that complicate the interior matching.

Furthermore, from some of the attempts of constructing an interior metric of the Kerr metric we see a trend for preference of an oblate spheroid instead of a sphere \cite{15 16}. This is a mathematical motivation indicating the value of exploring metrics of deformed objects. However, the physical motivation is much more simple, real astrophysical objects are not perfect spheres, hence allowing for small deformations in a rotating object is meaningful.

Moreover, the interest in spacetimes capable of describing charged distributions has always been high. In the early 1916’s Reissner and Nordström \cite{17 18} found their metric, which described a static spherically symmetric charge distribution. Even though this metric does not handle rotation it has been used, for example, to do black hole lensing \cite{19} and study Hawking radiation from a Reissner-Nordström black hole \cite{20}. It is important to highlight that this kind of studies would benefit from a metric capable of describing charged objects but including rotation, and with the capabilities of allowing deformed objects, this last property rules out the Kerr-Newman metric \cite{21}.

In this article we continue our efforts in constructing a more realistic metric capable of representing a real astrophysical object. Here, we use a perturbative method which utilizes the Lewis metric \cite{22} in order to find spacetimes with quadrupole moment while using the Kerr spacetime as a seed metric. Basically, our technique consists in cleverly changing the potentials of the Lewis metric while maintaining the cross term (rotational term). We have already applied this technique and obtained other approximate metrics \cite{23, 24, 25}. In comparison to our previous efforts this work...
includes the addition of charge. Hence, our new metric is capable of representing a charged, rotating and slightly deformed massive object.

The usual question when computing new solutions of the Einstein Field Equations (EFE) is how to prove that a given metric would have physical meaning\(^1\). In order to confirm the physical legitimacy of a given metric one can expand it to its post-linear form and compare the result with the post-linear version of the Hartle-Thorne (HT) metric [10, 13]. Note that it is possible to find an inner solution to the HT metric [14], and hence if a metric has a similar form to the HT metric then an inner solution should exist. See [26, 27] for a more detailed discussion and some examples.

The rest of this paper is organized as follows: We start our perturbation method of the Kerr metric with the help of the Lewis potentials in section 2. In section 3, we obtained a new metric by means of our perturbative technique, this metric has rotation, quadrupole moments and charge. We checked that the metric is a solution of the EFE using a REDUCE program [28], this program is available upon request. We state our conclusions in section 4.

2 The Perturbing Method for the Kerr Metric

Here and in the following sections we will use the method developed by Frutos et al. in [23, 24, 25] to obtain a Kerr-Newman-like metric i.e. a spacetime capable of describing a slightly deformed rotating charged mass. We start by using the connection between the Lewis metric and the Kerr metric to obtain the Lewis potentials associated to our new metric. These potentials are later used in the perturbation method.

First of all, we start with the Lewis metric, which is given by [22]

\[ ds^2 = -V dt^2 + 2W dt d\phi + X d\rho^2 + Y dz^2 + Z d\phi^2, \]

where the chosen canonical coordinates are \( x^1 = \rho \) and \( x^2 = z \). The potentials \( V, W, Z, X = e^\mu \) and \( Y = e^\nu \) are functions of \( \rho \) and \( z \) with \( \rho^2 = VZ + W^2 \).

From [22] the transformation that leads to the Kerr metric is

\[ \rho = \sqrt{\Delta} \sin \theta \quad \text{and} \quad z = (r - M) \cos \theta, \]

where \( \Delta = r^2 - 2Mr + a^2 + e^2 \), \( a \) is the rotational parameter and \( e \) is the electric charge.

Now, the Lewis potentials are chosen as follows

\[ V = V_K e^{-2\psi} = \frac{1}{\rho^2} [\Delta - a^2 \sin^2 \theta] e^{-2\psi} \]
\[ W = -\frac{2Jr}{\rho^2} \sin^2 \theta \]
\[ X = X_K e^{2\chi} = \frac{\rho^2 e^{2\chi}}{\Delta} \]
\[ Y = Y_K e^{2\chi} = \frac{\rho^2 e^{2\chi}}{\Delta} \]
\[ Z = Z_K e^{2\psi} = \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] e^{2\psi}, \]

where the potentials \( V_K, W_K, Y_K, Z_K \) are the Lewis potentials for the Kerr-Newman metric, and \( \rho^2 = r^2 + a^2 \cos^2 \theta \). Also, \( J = Ma \) is the angular momentum.

The cross term potential \( W \) is unaltered to preserve the following metric form

\(^1\)Here physical meaning stands for having an interior metric.
\[ ds^2 = -\Delta \frac{e^{-\psi} dt - ae^\psi \sin^2 \theta d\phi}{\rho^2} + \frac{\sin^2 \theta}{\rho^2} \left[(r^2 + a^2)e^\psi d\phi - ae^{-\psi} dt\right]^2 + \frac{\bar{\rho}^2 e^{2\chi}}{\Delta + d\theta^2}. \]  

These potentials guarantee that one gets the Kerr metric if \( \psi = \chi = 0 \). The function \( \psi \) and \( \chi \) will be found approximately from the EFE.

3 The Approximate Kerr-Newman Metric with Quadrupole

As was stated in the previous section our problem has been reduced to finding the functions \( \psi \) and \( \chi \). Here we proceed to find such functions by solving the EFE perturbatively. Also, we discuss the limiting cases of the new metric.

The EFE are given by

\[ G_{ij} = R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij}, \]

\[ \nabla_j F^{ij} = \frac{1}{\sqrt{-g}} \partial_j [\sqrt{-g} F^{ij}] = 0, \]

where \( G_{ij} \) \((i, j = 0, 1, 2, 3)\) are the Einstein tensor components, \( R_{ij} \) are the Ricci tensor components, \( R \) is the curvature scalar, \( \kappa = 8\pi G/c^4 \), \( g = \det(g_{ij}) \) and \( T_{ij} \) represent the energy-momentum tensor components, which are given by

\[ 4\pi T_{ij} = g^{kl} F_{il} F_{jk} - \frac{1}{4} F^{ab} F_{ab} g_{ij}, \]

where \( F_{ij} = \partial_j A_i - \partial_i A_j \) are the electromagnetic tensor components, and \( A_i \) is the vector potential components.

The 1-form for the vector potential \( \mathbf{A} \) can be written as follows

\[ \mathbf{A} = -\frac{e r}{\rho^2} \left[e^{-\psi} dt - ae^\psi \sin^2 \theta d\phi\right]. \]

In addition, the 2-form for the electromagnetic tensor \( F \) can be obtained as follows

\[ F = d\mathbf{A} = \frac{1}{2} F_{ij} dx^j \wedge dx^j. \]

From we can determine the energy-momentum tensor.

Let us highlight the terms that are neglected in our perturbative approach, which are

\[ W^2 \frac{\partial \psi}{\partial x^i} \sim 0, \]
\[ W \frac{\partial W}{\partial x^i} \frac{\partial \psi}{\partial x^i} \sim 0, \]
\[ W^2 \frac{\partial \chi}{\partial x^i} \sim 0, \]
\[ W \frac{\partial W}{\partial x^i} \frac{\partial \chi}{\partial x^i} \sim 0. \]
Moreover, eliminating the terms corresponding to the Kerr metric into the Ricci tensor components, we get the Ricci tensor component of the appendix of [24]. Note that $W$ plays the role of the rotation, since it is proportional to the angular momentum. However, the above expressions do not mean that factors of $J^2$ or $a^2$ vanish, what is being effectively restricted here are combinations of quadrupoles with the angular momentum or the rotation.

In order to obtain an expression for the Ricci tensor, we propose the following Ansatz

$$
\psi = \frac{q}{r^3}P_2 + \frac{Mq}{r^4}P_2, \\
\chi = \frac{qP_2}{r^3} + \frac{Mq}{r^4}(\beta_1 + \beta_2 P_2 + \beta_3 P_2^2) + \frac{q^2}{r^6}(\beta_4 + \beta_5 P_2 + \beta_6 P_2^2 + \beta_7 P_2^3),
$$

where $q$ represents the quadrupole parameter and $P_2$ is the usual Legendre polynomial, $P_2 = (3 \cos^2 \theta - 1)/2$. Furthermore, the $\alpha$ and $\beta$'s are constants to be determined.

One can find the undetermined constants by substituting this Ansatz into the Ricci tensor components. We get a set of linear equations for these constants $\alpha$, and $\beta_n$ $(n = 1, \ldots, 7)$. After solving these linear equations, the constants are found to be the same as in [27]

$$
\alpha = 3, \\
\beta_1 = -\frac{1}{3}, \\
\beta_2 = \beta_3 = \frac{5}{3}, \\
\beta_4 = \frac{2}{9}, \\
\beta_5 = -\frac{2}{3}, \\
\beta_6 = -\frac{7}{3}, \\
\beta_7 = \frac{25}{9}.
$$

Finally we have all the information we require to construct our Kerr-Newman-like metric. From (11), the metric components are given by

$$
\begin{align*}
  g_{tt} &= e^{-2\psi} \frac{e^{-2\psi}}{\rho^2} [a^2 \sin^2 \theta - \Delta], \\
  g_{t\phi} &= \frac{a}{\rho^2} \frac{e^{2\chi}}{\Delta} [\Delta - (r^2 + a^2)] \sin^2 \theta = \frac{\sin^2 \theta}{\rho^2} (aQ^2 - 2Jr), \\
  g_{rr} &= \rho^2 \frac{e^{2\chi}}{\Delta}, \\
  g_{\theta\theta} &= \rho^2 e^{2\chi}, \\
  g_{\phi\phi} &= \frac{e^{2\psi}}{\rho^2} [(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta] \sin^2 \theta,
\end{align*}
$$

We checked that (11) was valid up to order $O(aq^2, a^2q, Maq, Mq^2, M^2q, q^3)$ using a REDUCE program [28].

Now, we will focus in the limiting cases of the new metric. We summarized the limiting cases in table 1. First, note that if $e = 0$ in (11) one recovers the metric found by Frutos et al. in [24].
| Absent physical property | Small physical property | Limiting metric                  |
|--------------------------|-------------------------|----------------------------------|
| Charge                   | Quadrupole (linear)     | Metric found in [24]              |
| Charge                   | Quadrupole (quadratic)  | Metric found in [27]              |
| Quadrupole               |                         | Kerr-Neuman                      |
| Quadrupole and rotation  |                         | Reissner-Nordström               |
| Charge and quadrupole    |                         | Kerr                             |
| Charge and rotation      |                         | Erez-Rosen-like                  |
| Charge                   | Quadrupole and rotation | Metric found in [23]              |
| Charge, quadrupole and rotation |         | Schwarzschild                    |

Table 1: Limiting cases.

Therefore, following [24] the other interesting limiting cases are: The Kerr metric if \( e = q = 0 \), the metric found in [23] if \( e = a^2 = q^2 = 0 \), the Erez-Rosen-like metric described in [24] if \( a = e = 0 \), and the Schwarzschild metric if \( a = e = q = 0 \). Furthermore, one obtains the Kerr-Newman geometry if \( q = 0 \). Also, the Reissner-Nordström metric is obtained if \( a = q = 0 \). Thus, all the expected limiting cases can be obtained from this new metric.

Here we will not show the matching of (11) with the HT metric. This matching is vital because it guarantees that an interior solution exists. We will not show it since it should be trivial to follow [24] since our metric has the same form, or equivalently follow [26]. Moreover, in [27], it was shown that the multipole structure of this metric without charge is non-isometric with the Quevedo-Mashhoon [12, 13] and the Manko-Novikov [29] metrics. Then, this new metric should not be isometric with charged version of these metrics.

### 4 Conclusion

A metric with charge, deformations and rotation was obtained by solving the EFE pertubatively. The expected limiting cases of this new metric were explored and found, which is a positive sign. Moreover, these limiting cases confirm that our metric adequately describes a mass with charge and quadrupole under rotation. We successfully applied the perturbation method developed by Frutos et al in [23, 24, 25, 27] to obtain a new approximate metric. Notice that the main improvement of our work with respect to [24, 27] is the inclusion of charge. Ideally, this moves us closer to representing an actual astrophysical object. Further, one could expect that careful understanding of this procedure might eventually lead us towards a magnetized object.

Even though we did not explicitly showed that the new metric can be matched to the Hartle-Thorne metric the previous success in the the non-charged metrics [24, 27] is encouraging. Particularly, because of the similarity with that non-charged metric we do not expect any issues in the matching. This matching is crucial for the sake of showing that an interior metric for our new metric exists. Since this new metric can describe real charged astrophysical objects in a more realistic fashion than the Kerr-Newman or Reissner-Nordström metrics, then our metric should be attractive for astrophysical applications, like gravitational lensing and relativistic magnetohydrodynamics. Furthermore, computational implementation of this metric should not imply additional difficulties since it maintains a similar form to the Kerr metric.

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