Aggregation functions on \( n \)-dimensional ordered vectors equipped with an admissible order and an application in multi-criteria group decision-making

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Abstract

\( n \)-Dimensional fuzzy sets are a fuzzy set extension where the membership values are \( n \)-tuples of real numbers in the unit interval \([0, 1]\) increasingly ordered, called \( n \)-dimensional intervals. The set of \( n \)-dimensional intervals is denoted by \( L_n([0, 1]) \). This paper aims to investigate semi-vector spaces over a weak semifield and aggregation functions concerning an admissible order on the set of \( n \)-dimensional intervals and the construction of aggregation functions on \( L_n([0, 1]) \) based on the operations of the semi-vector spaces. In particular, extensions of the family of OWA and weighted average aggregation functions are investigated. Finally, we develop a multi-criteria
group decision-making method based on \( n \)-dimensional aggregation functions with respect to an admissible order and give an illustrative example.

**Keywords:** \( n \)-dimensional fuzzy sets, ordered semi-vector spaces, admissible orders, aggregation function, ordered weighted averaging, decision making.

## 1 Introduction

A mathematical function is a rule, which takes one or several inputs and returns an output such that for each input has a unique output associated to it, average aggregation functions are functions \( A : [0, 1]^n \to [0, 1] \) with special properties that takes real inputs from a closed interval \([0, 1]\) and aggregates them in a single real value which, in some sense, represents all the other values. One example of aggregation function is the ordered weighted averaging (OWA) introduced by Yager in [45]. Aggregation functions are widely used in physics and natural sciences such as pure and applied mathematics, computer science, engineering, economics, among others. An extensive study on average aggregation functions can be found in [8, 13].

In recent years the interest of researchers in generalizations and extensions of fuzzy set theory [46] has grown, considering both the different ways of interpreting the uncertainty of phenomena and the scope of the problem. In [10] it is given a historical and hierarchical analysis of the most important of such extensions, among which we can mention the interval-valued fuzzy sets [36, 39], hesitant fuzzy sets [38, 43], \( L \)-fuzzy sets [25] and \( n \)-dimensional fuzzy sets [4, 42, 47].

In particular, this last extension considers as membership degrees \( n \)-dimensional ordered vectors of real numbers in \([0, 1] \), i.e. elements of \( L_n([0, 1]) = \{ (x_1, \ldots, x_n) \in [0, 1]^n \mid x_1 \leq \cdots \leq x_n \} \). This kind of extension arises, for example, in situations where \( n \) experts (human or not) provide their membership degrees of each element of the discourse universe to the same fuzzy set, and their identity is irrelevant (and therefore all they have the same weight) or yet necessary. Later, Bedregal et al. in [4] have investigated the \( m \)-ary \( n \)-dimensional aggregation functions with respect to the natural partial order (also known as product order) on the set \( L_n([0, 1]) \) and in particular the \( n \)-dimensional t-norms.

Admissible orders are orders which refine the “natural” order on a set of membership degrees in a fuzzy extensions. Originally they were introduced in the context of interval-valued fuzzy sets by H. Bustince et al. in [11] and since then they have been widely used [12, 14, 40, 48]. Lately, such notion was studied in other types of fuzzy sets, such as interval-valued intuitionistic fuzzy sets [18, 20, 21], hesitant fuzzy sets [31, 32], multidimensional fuzzy sets [19] and \( n \)-dimensional fuzzy sets [22].

The main contribution of this paper is to introduce and study the notion of \( n \)-dimensional aggregation functions with respect to an admissible order on \( L_n([0, 1]) \) of the \( n \)-dimensional ordered vectors on \([0, 1] \). In particular, in order to extend the
OWA and the weighted average aggregation function to this context, we must consider addition and product by an scalar (the weights) in \([0, 1]\). Nevertheless, in order to preserve the most basic properties of the OWA and many other OWA-like operators, it is necessary some kind of compatibility between the admissible order and both operations. With this in mind, it was introduced the notion of ordered semi-vector spaces over a weak semifield, which is generalization of ordered vector spaces \([15]\). Finally, \(n\)-dimensional aggregation functions with respect to an admissible order were used to generate a score in \(L_n([0, 1])\), where \(n\) corresponds to the number of experts, for each alternative in a multiple criteria group decision making problem and thereby a ranking of the alternatives based on these scores and the admissible order.

To do so, the paper is organized as follows: some important concepts related to aggregation functions and \(n\)-dimensional fuzzy sets are recalled in Section 2. Next section is devoted to introducing the notion of ordered semi-vector spaces over weak semifields in general and for the particular case of the \(L_n([0, 1])\) equipped with an admissible order. In Section 4 \(n\)-dimensional aggregation functions over admissible orders are considered and it is proposed an OWA like operator for the admissibly ordered semi-vector space with \(L_n([0, 1])\) as the underlying set studied in previous section. Section 5 brings a multiple criteria and multiple expert decision making method based on \(n\)-dimensional aggregation functions over admissible orders joint with an illustrative example. Finally, Section 6 presents the final remarks and discusses further works.

2 Preliminary Concepts

In this section we recall some important concepts for our proposal in this work. For a detailed overview on these topics, we recommend \([4, 8, 13, 42]\).

In the whole text we are considering \(\mathbb{N}^*\) as the set of natural elements without the zero and \(N_m^n = \{m, m + 1, m + 2, \ldots, n\}\) for each \(m, n \in \mathbb{N}^*\) such that \(m \leq n\).

In addition, \(\omega = (w_1, \ldots, w_n) \in [0, 1]^n\) such that \(\sum_{i=1}^n w_i = 1\) is called a weighting vector. \(\omega\) is strictly positive if \(w_i > 0\) for each \(i \in \mathbb{N}^*_n\).

2.1 \(n\)-Dimensional fuzzy sets

In this subsection, we make a review of \(n\)-dimensional fuzzy sets over a nonempty referential set \(X\). These sets arise as a natural generalization of the interval-valued fuzzy sets \([56, 39]\), three-dimensional fuzzy sets \([50]\) and interval-valued Atanassov intuitionistic fuzzy sets \([2]\) which mathematically are equivalent to four-dimensional fuzzy sets. This generalization is useful in situations where \(n\) experts provide an evaluation of how much an alternative or element satisfies some criteria or condition and the identity of the expert (i.e. the order in which the evaluations are provided) is not relevant.
Definition 1. [42] Let $X$ be a non-empty set and $n \in \mathbb{N}^*$. According to [42], a set

$$A = \{ (x, \mu_A^1(x), \ldots, \mu_A^n(x)) \mid x \in X \}$$

is called an $n$-dimensional fuzzy set on $X$, given that for all $i \in N^n_1$ the membership functions $\mu_A^i : X \to [0, 1]$ satisfy $\mu_A^1(x) \leq \cdots \leq \mu_A^n(x)$.

Bedregal et al. in [3] have defined the set

$$L_n([0, 1]) = \{ (x_1, \ldots, x_n) \in [0, 1]^n \mid x_1 \leq x_2 \leq \cdots \leq x_n \}.$$  \hspace{1cm} (1)

Since $L_1([0, 1]) = [0, 1]$ and $L_2([0, 1]) = L([0, 1])$ (the closed subintervals of the unit interval $[0, 1]$), the elements of $L_n([0, 1])$ are called $n$-dimensional intervals (see [4]).

Remark 1. The $i$-th projection on $L_n([0, 1])$, for $i \in N^n_1$, is the function $\pi_i : L_n([0, 1]) \to [0, 1]$ given by $\pi_i(x_1, \ldots, x_n) = x_i$. For each $x \in [0, 1]$ the element $(x, \ldots, x) \in L_n([0, 1])$ is denoted by $|x|$ and it is called a degenerate element. The set of all degenerate elements in $L_n([0, 1])$ is denoted by $D_n$.

Shang, Yuan and Lee, in [42], considered the following partial order on $L_n([0, 1])$:

$$x \preceq^p_n y \iff \pi_i(x) \leq \pi_i(y) \text{ for each } i \in N^n_1,$$ \hspace{1cm} (2)

called the product order.

The poset $(L_n([0, 1]), \preceq^p_n)$ is a continuous lattice [4] and hence a complete distributive lattice, having $/0/$ and $/1/$ as its smallest and greatest elements, respectively (see [24, 41]). In particular, the infimum and supremum of a set $S \subseteq L_n([0, 1])$ in this lattice, denoted by $\land S$ and $\lor S$, are given by

$$\land S = (\min_{x \in S} \pi_1(x), \ldots, \min_{x \in S} \pi_n(x))$$

and

$$\lor S = (\max_{x \in S} \pi_1(x), \ldots, \max_{x \in S} \pi_n(x))$$

In [22], De Miguel et al. have introduced the concept of an admissible order on $L_n([0, 1])$ with respect to the product order $\preceq^p_n$.

Definition 2. A total order $\preceq$ on $L_n([0, 1])$ is called admissible if for each $x, y \in L_n([0, 1])$ we have that $x \preceq y$ whenever $x \preceq^p_n y$.

Example 1. Let $x, y \in L_n([0, 1])$ and $\tau : N^n_1 \to N^n_1$ be a bijection.

1. [22] Example 2] The reflexive closure of

$$x \prec \tau y \iff \exists k \in N^n_1 \text{ s.t. } \pi_{\tau(k)}(x) < \pi_{\tau(k)}(y) \text{ and } \forall i, 1 \leq i < k, \pi_{\tau(i)}(x) = \pi_{\tau(i)}(y)$$

is an admissible order. In particular, if $\tau$ is the identity then $\preceq_\tau$ corresponds to the lexicographical order and if $\tau(i) = n - i + 1$ then $\preceq_\tau$ is the anti-lexicographical order.
2. Let \( \omega = (w_1, \ldots, w_n) \) be a weighting vector and
\[
F_\omega(x, y) = \sum_{i=1}^{n} w_i \max(0, \pi_i(x) - \pi_i(y)).
\]
Then
\[
x \preceq_\omega y \iff F_\omega(x, y) < F_\omega(y, x) \text{ or } (F_\omega(x, y) = F_\omega(y, x) \text{ and } x \preceq_\tau y),
\]
is an admissible order.

3. Let \( A \) be an \( n \)-ary aggregation function. Then
\[
x \preceq_\tau y \iff A(x) < A(y) \text{ or } (A(x) = A(y) \text{ and } x \preceq_\tau y)
\]
is an admissible order.

Since each admissible order \( \preceq \) is total then any finite set \( S \subseteq L_n([0,1]) \) has a minimum and a maximum with respect to the order. We will denote such elements as \( \wedge S \) and \( \vee S \), respectively, and as \( \wedge_{j=1}^{m} x_j \) and \( \vee_{j=1}^{m} x_j \) when \( S = \{x_1, \ldots, x_m\} \).

3 Ordered Semi-Vector Spaces

Semi-vector spaces were introduced in [37] by Prem Prakash and Murat Sertel as a set of elements, called vectors, that is closed under finite vector addition and scalar multiplication satisfying some conditions. In this definition, it is considered as scalar set just the set of non-negative real numbers with their usual addition and product and the property of existence of a neutral element for the vector addition is not considered. In [27], Vasantha Kandasamy introduced a more general and complete definition for semi-vector spaces (SVS) and in this section we introduce definitions of SVS and ordered SVS which are related to a weak semifield instead of a semifield.

3.1 Ordered semi-vector spaces over a weak semifield

A semifield is an algebraic structure together with two binary operations, addition and multiplication. It is similar to a field but with some axioms removed or relaxed. There are different non-equivalent definitions of semifield (see for example [26, 27, 29]). In particular, the given in [27] does not encompass the notion of field. In this paper we will adopt a weak notion of semifield in order to guarantee that each field is also a semifield.

**Definition 3.** A weak semifield is an algebra \( \mathcal{F} = (F, +, \cdot) \) satisfying the following axioms for each \( r, s, t \in F \):

1. \( r + (s + t) = (r + s) + t \) and \( r \cdot (s \cdot t) = (r \cdot s) \cdot t \);
2. \( r \dagger s = s \dagger r \) and \( r \cdot s = s \cdot r \);
3. \( r \cdot (s \dagger t) = (r \cdot s) \dagger (r \cdot t) \);
4. \( \exists 0 \in F \) such that \( r \dagger 0 = r \);
5. \( \exists 1 \in F \) such that \( 1 \cdot r = r \).

\( F^+ \) we will denote \( F - \{0\} \).

Observe that weak semifields are fields where the inverses of the addition and multiplication are dropped, or in other words, commutative semirings (in the sense of [28]) with a multiplicative identity element.

**Definition 4.** A semi-vector space \( V \) over a weak semifield \( F = \langle F, \bar{\cdot}, \cdot \rangle \) is a nonempty set \( V \), equipped with two operations, \( + : V \times V \to V \) and \( \star : F \times V \to V \), fulfilling the following axioms, for each \( x, y, z \in V \) and \( r, s \in F \):

(SV1) \( x + (y + z) = (x + y) + z \);
(SV2) \( x + y = y + x \);
(SV3) \( r \star (s \star x) = (r \cdot s) \star x \);
(SV4) \( 1 \star x = x \);
(SV5) \( r \star (x + y) = (r \star x) + (r \star y) \);
(SV6) \( (r \dagger s) \star x = (r \star x) + (s \star x) \);
(SV7) \( \exists 0 \in V \) such that \( 0 + x = x \) for each \( x \in V \).

In a semi-vector space, \( V = \langle V, +, \star \rangle \) the elements of \( V \) are called vectors, the operation \( \star \) is called scalar multiplication and \( + \) is the vector addition.

In a semi-vector space the vector \( 0 \) in axiom (SV7) is called zero vector or additive identity vector. The zero vector is unique.

In mathematics, an ordered vector space is a vector space equipped with a partial order that is compatible with the vector space operations [15]. Clearly, this notion can be extended for semi-vector spaces. In [23], the notion of an ordered semi-vector space was introduced, but only considering the semifield of non-negative real numbers. Here we extend it for an arbitrary weak semifield.

**Definition 5.** Let \( V = \langle V, +, \star \rangle \) be a semi-vector space over a weak semifield \( F = \langle F, \bar{\cdot} \rangle \) and \( \leq \) be a partial order on \( V \). We say that \( \leq \) is compatible with the scalar product of \( V \) if

(SV8) \( r \star x \leq r \star y \) for each \( x, y \in V \) such that \( x \leq y \) and \( r \in F \),

and is compatible with the vector addition of \( V \) if

\[ 6 \]
In addition, the pair \( \langle \mathcal{V}, \leq \rangle \) is an ordered semi-vector space over \( F \) or, equivalently, \( \leq \) is compatible with \( \mathcal{V} \), if it is compatible with both, the scalar product and the vector addition of \( \mathcal{V} \).

Notice that for any semi-vector space \( \mathcal{V} \) over an arbitrary weak semifield, the pair \( \langle \mathcal{V}, \leq \rangle \) is an ordered semi-vector space, which is called trivial.

The following proposition guarantees that each semi-vector space determines a natural partial order which is compatible with the semi-vector space.

**Proposition 1.** Let \( \mathcal{V} = \langle V, +, \cdot \rangle \) be a semi-vector space and consider the binary relation defined for each \( x, y \in V \) as

\[
x \leq_V y \iff \exists z \in V, x + z = y.
\] (3)

If \( \leq_V \) is antisymmetric, then \( \langle \mathcal{V}, \leq_V \rangle \) is an ordered semi-vector space.

**Proof.** First of all, observe that by (SV7) and (SV2), \( \leq_V \) is reflexive. If \( x \leq_V y \) and \( y \leq_V z \), then by Eq. (3), \( x + u = y \) and \( y + v = z \) for some \( u, v \in V \). So, \( x + (u + v) = z \) and therefore, \( x \leq_V z \), i.e. \( \leq_V \) is transitive. Hence, \( \leq_V \) is a partial order on \( V \) whenever \( \leq_V \) is antisymmetric. Let \( \mathcal{V} = \langle V, +, \cdot \rangle \) be a semi-vector space , \( x, y \in V \). Then, we just need to prove (SV8) and (SV9). So,

\[
x \leq_V y \iff \exists z \in V, x + z = y \quad \text{by Eq. (3)}
\]

\[
\iff \exists z \in V, \forall r \in F, r \cdot (x + z) = r \cdot y
\]

\[
\iff \exists z \in V, \forall r \in F, (r \cdot x) + (r \cdot z) = r \cdot y \quad \text{by (SV5)}
\]

\[
\iff \forall r \in F, r \cdot x \leq_V r \cdot y \quad \text{by Eq. (3)}.
\]

Now, let \( x, y, w \in V \). Then

\[
x \leq_V y \iff \exists z \in V, x + z = y \quad \text{by Eq. (3)}
\]

\[
\iff \exists z \in V, \forall w \in V, (x + z) + w = y + w
\]

\[
\iff \exists z \in V, \forall w \in V, (x + w) + z = y + w \quad \text{by (SV1) and (SV2)}
\]

\[
\iff \forall w \in V, x + w \leq_V y + w \quad \text{by Eq. (3)}.
\]

Therefore, \( \langle \mathcal{V}, \leq_V \rangle \) is an ordered semi-vector space.
3.2 Aggregation Functions

In this subsection, we will define and give some examples of $n$-ary aggregation functions, which are functions that map values in $[0, 1]$ into a single value in $[0, 1]$ which in some sense represents all of them.

**Definition 6.** [8] Let $n \in \mathbb{N}$ with $n \geq 2$. A function $A : [0, 1]^n \rightarrow [0, 1]$ is an $n$-ary aggregation function if:

i) $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$;

ii) If $x_i \leq y_i$ for each $i \in \{1, \ldots, n\}$, then $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$.

In addition,

iii) $A$ is strict if for each $i \in \{1, \ldots, n\}$ and $x_1, \ldots, x_n, y \in [0, 1]$, $A(x_1, \ldots, x_n) < A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$ whenever $x_i < y$;

iv) If there is a positive real number $k$ such that for each $x_1, \ldots, x_n, \lambda \in [0, 1]$, $A(\lambda x_1, \ldots, \lambda x_n) = \lambda^k A(x_1, \ldots, x_n)$ then $A$ is said to be homogeneous of order $k$; and

v) $A$ is called internal if $A(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for each $x_1, \ldots, x_n \in [0, 1]$.

An $m$-ary aggregation function $B$ dominates another $m$-ary aggregation function $A$, denoted by $A \leq B$, if $A(x_1, \ldots, x_m) \leq B(x_1, \ldots, x_m)$ for each $x_1, \ldots, x_m \in [0, 1]$.

**Example 2.** Let $\omega = (w_1, \ldots, w_n)$ be an arbitrary weighting vector, $\bar{e} = (e_1, \ldots, e_n)$ be an $n$-dimensional vector of positive real numbers and $r > 0$ be a real number. Then the following functions are examples of $n$-ary aggregation functions.

(a) $P^r : [0, 1]^n \rightarrow [0, 1]$ defined by

$$P^r(x_1, \ldots, x_n) = \frac{\prod_{i=1}^{n} (x_i^r + 1) - 1}{2^n - 1}$$

(b) $\min_\omega (x_1, \ldots, x_n) = \arg \min (w_i x_i)$ (Weighted min)

(c) $\max_\omega (x_1, \ldots, x_n) = \arg \max (w_i x_i)$ (Weighted max)

(d) $M_\omega (x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$ (Weighted average)

(e) $G_\omega (x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{w_i}$ (Geometric mean)

(f) $\max_e (x_1, \ldots, x_n) = \max (x_1^{e_1}, \ldots, x_n^{e_n})$ (Exponentially distorted maximum)
(g) \( OW_A(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i \) where \( (x_1), \ldots, x_n \) is a permutation of \( (x_1, \ldots, x_n) \) such that \( x_{(i)} \geq x_{(i+1)} \) for each \( i \in N_{1}^{n-1} \) (Ordered weighted averaging)

In particular, \( P^p \) is strict, \( \min_{\omega} \), \( \max_{\omega} \) and \( \max_{\mathbf{e}} \) are not strict and if \( \omega \) is strictly positive then \( M_{\omega} \) and \( G_{\omega} \) are strict. In addition, \( P^p \), \( G_{\omega} \) and \( \max_{\mathbf{e}} \) (with some \( \mathbf{e}_i \not\in \{0,1\} \) are not homogeneous of any order whenever \( \mathbf{e}_i \neq \mathbf{e}_j \) for some \( i, j \in \{1, \ldots, n\} \), but \( \min_{\omega} \), \( \max_{\omega} \) and \( M_{\omega} \) are homogeneous of order 1. Moreover, we have that \( \min_{\omega} \leq G_{\omega} \leq M_{\omega} \leq \max_{\omega} \).

### 3.3 Admissible ordered semi-vector spaces on \( L_n([0,1]) \)

It is easy to note that the algebra \( U = ([0,1], +, \cdot) \) where \( \cdot \) is the usual multiplication and \( + \) is the bounded addition, i.e. \( r+s = \min(1, r+s) \) for each \( r, s \in [0,1] \), is a weak semifield. As usual we will omit the \( \cdot \) in expressions like \( r \cdot s \) when it is clear.

Given \( r \in [0,1] \) and \( x, y \in L_n([0,1]) \), we define:

- **Scalar product**: \( r \odot \mathbf{x} = (rx_1, \ldots, rx_n) \);
- **Bounded addition**: \( \mathbf{x} \oplus \mathbf{y} = (x_1+y_1, \ldots, x_n+y_n) \).

**Theorem 1.** \( L_n([0,1]) = \langle L_n([0,1]), +, \odot \rangle \) is a semi-vector space over the weak semifield \( U \). In addition, \( (L_n([0,1]), \leq_p) \) is an ordered semi-vector space over \( U \).

**Proof.** First note that the scalar product and the bounded addition are well defined, i.e. \( r \odot \mathbf{x}, \mathbf{x} \oplus \mathbf{y} \in L_n([0,1]) \) for each \( r \in [0,1] \) and \( x, y \in L_n([0,1]) \). Indeed, for each \( i,j \in N_1^n \) such that \( i \leq j \), \( rx_i \leq rx_j \) and \( \min(1, x_i+y_i) \leq \min(1, x_j+y_j) \).

Since, (SV1) to (SV4) are trivially satisfied by the scalar product and the bounded addition, we will only prove (SV5) - (SV9).

\[
\text{(SV5)} \quad r \odot (\mathbf{x} + \mathbf{y}) = (r(x_1+y_1), \ldots, r(x_n+y_n)) = (rx_1+ry_1, \ldots, rx_n+ry_n) = (rx_1, \ldots, rx_n) + (ry_1, \ldots, ry_n) = r \odot \mathbf{x} + r \odot \mathbf{y};
\]

\[
\text{(SV6)} \quad (r+s) \odot \mathbf{x} = ((r+s)x_1, \ldots, (r+s)x_n) = (rx_1+sx_1, \ldots, rx_n+sx_n) = (rx_1, \ldots, rx_n) + (sx_1, \ldots, sx_n) = r \odot \mathbf{x} + s \odot \mathbf{x};
\]

\[
\text{(SV7)} \quad \mathbf{x} \oplus 0 = (x_1, \ldots, x_n) = \mathbf{x};
\]

\[
\text{(SV8)} \quad \text{If } \mathbf{x} \leq_p \mathbf{y} \text{ then } x_i \leq y_i \text{ for each } i \in N_1^n \text{ and therefore, because } U \text{ is a weak semifield, } rx_i \leq ry_i \text{ for all } i \in N_1^n. \text{ So, } r \odot \mathbf{x} = (rx_1, \ldots, rx_n) \leq_p (ry_1, \ldots, ry_n) = r \odot \mathbf{y};
\]

\[
\text{(SV9)} \quad \text{If } \mathbf{x} \leq_p \mathbf{y} \text{ then } x_i \leq y_i \text{ for each } i \in N_1^n \text{ and therefore, because } U \text{ is a weak semifield, } x_i \odot z_i \leq y_i \odot z_i \text{ for all } i \in N_1^n. \text{ So, } \mathbf{x} \oplus \mathbf{z} = (x_1+z_1, \ldots, x_n+z_n) \leq_p (y_1+z_1, \ldots, y_n+z_n) = \mathbf{y} \oplus \mathbf{z}.
\]

Therefore, \( (L_n([0,1]), \leq_p) \) is an ordered semi-vector space over \( U \). \( \square \)
Observe that the natural preorder (Equation (3)) of the semi-vector space $L_n([0, 1])$ over the weak semifield $U$, i.e. $\leq_{L_n([0, 1])}$, is in fact, a partial order on $L_n([0, 1])$ and consequently, by Proposition 1, $(L_n([0, 1]), \leq_{L_n([0, 1])})$ is an ordered semifield over $U$. In addition, $\leq_{L_n([0, 1])}$ is refined by $\leq^P_n$ and therefore by any admissible order. Nevertheless, not for all admissible order $\leq$ the pair $(L_n([0, 1]), \leq)$ is an ordered semi-vector space over $U$. In fact, consider the aggregation function $A = \max_e$ of the Example 2 for $\bar{e} = (1, 2, 3, 4)$, $x = (0.4, 0.6, 0.7, 0.8)$ and $y = (0.2, 0.2, 0.2, 0.9)$. Then $\max_e(x) = 0.4096$ whereas $\max_e(y) = 0.6561$ and therefore, $x <^A y$. However, taking $r = 0.5$ we have that $\max_e(r \odot x) = \max_e(0.2, 0.3, 0.35, 0.4) = 0.2$ and $\max_e(r \odot y) = \max_e(0.1, 0.1, 0.1, 0.45) = 0.1$. So, $r \odot y <^A r \odot x$. Therefore, the pair $(L_n([0, 1]), \leq^A)$ does not satisfy (SV8) and consequently, it is not an ordered semi-vector space over $U$. This motivates the investigation of families of admissible orders which are compatible with the semi-vector space $L_n([0, 1])$.

**Proposition 2.** For any bijection $\tau : N_1^n \rightarrow N_1^n$, the pair $(L_n([0, 1]), \leq_{\tau})$ is an ordered semi-vector space over $U$.

**Proof.** From Theorem 1 we know that $L_n([0, 1]) = (L_n([0, 1]), +, \odot)$ is a semi-vector space over the weak semifield $U$. So it is necessary to prove (SV8) and (SV9).

(SV8) Let $x, y \in L_n([0, 1])$ and $r \in [0, 1]$ be such that $x \leq_{\tau} y$. If $r = 0$ or $x = y$ then, by Theorem 1, $r \odot x \leq^P_n r \odot y$ and since $\leq_{\tau}$ is admissible, then $r \odot x \leq_{\tau} r \odot y$. Now, consider the case $r > 0$ and $x \prec_{\tau} y$. Then $\exists k \in N_1^n$ such that $\pi_{\tau(k)}(x) < \pi_{\tau(k)}(y)$ and $\forall i \in N_1^{n-1}$, $\pi_{\tau(i)}(x) = \pi_{\tau(i)}(y)$. Therefore, $\pi_{\tau(k)}(r \odot x) = r \cdot \pi_{\tau(k)}(x) < r \cdot \pi_{\tau(k)}(y) = \pi_{\tau(k)}(r \odot y)$ and $\forall i \in N_1^{n-1}$, $\pi_{\tau(i)}(r \odot x) = r \cdot \pi_{\tau(i)}(x) = r \cdot \pi_{\tau(i)}(y)$ and $\pi_{\tau(i)}(r \odot y)$.

Hence, in both cases, $r \odot x \leq_{\tau} r \odot y$.

(SV9) Let $x, y, z \in L_n([0, 1])$ be such that $x \leq_{\tau} y$. If $x = y$ then trivially $x + z \leq_{\tau} y + z$. So, we just need to consider the case $x \prec_{\tau} y$. In this case, by definition, $\exists k \in N_1^n$ such that $\pi_{\tau(k)}(x) < \pi_{\tau(k)}(y)$ and $\forall i \in N_1^{n-1}$, $\pi_{\tau(i)}(x) = \pi_{\tau(i)}(y)$. Therefore, $\forall i \in N_1^{n-1}$, $\pi_{\tau(i)}(x + z) = \pi_{\tau(i)}(y + z)$ and $\pi_{\tau(k)}(x + z) = \pi_{\tau(k)}(y + z)$ and $\pi_{\tau(k)}(x + z) = \pi_{\tau(k)}(y + z)$.

If $\pi_{\tau(k)}(x + z) < \pi_{\tau(k)}(y + z)$ then $x + z <_{\tau} y + z$. On the other hand, if $\pi_{\tau(k)}(x + z) = \pi_{\tau(k)}(y + z)$ then $\pi_{\tau(l)}(x + z) = 1 = \pi_{\tau(l)}(y + z)$ for each $l \in N_1^n$ and consequently, $x + z = y + z$. 

\[\Box\]

**Proposition 3.** Let $n \geq 2$, $\tau : N_1^n \rightarrow N_1^n$ be a bijection and $\omega = (w_1, \ldots, w_n)$ be a weighting vector. Then $\leq_{\omega}$ is compatible with the scalar product $\odot$, but it is not compatible with the vector addition $+$.

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Proof. For each \( x, y \in L_n([0, 1]) \) and \( r \in [0, 1] \), we have that

\[
 r \cdot F_\omega(x, y) = r \cdot \sum_{i=1}^{n} w_i \max(0, \pi_i(x) - \pi_i(y))
 = \sum_{i=1}^{n} w_i \max(0, r \cdot (\pi_i(x) - \pi_i(y)))
 = \sum_{i=1}^{n} w_i \max(0, r \cdot (x \circ y))
 = F_\omega(r \circ x, r \circ y)
\]

Let \( x, y \in L_n([0, 1]) \) and \( r \in [0, 1] \) be such that \( x \preceq_\tau y \). If \( r = 0 \) or \( x = y \) then \( r \circ x \preceq_\omega r \circ y \) and since \( \preceq_\tau \) is admissible, it follows that \( r \circ x \preceq_\omega s \circ y \).

Consider the case \( r > 0 \) and \( x \prec_\tau y \). Then, either \( F_\omega(x, y) < F_\omega(y, x) \), or \( F_\omega(x, y) = F_\omega(y, x) \) and \( x \prec_\tau y \).

If \( F_\omega(x, y) < F_\omega(y, x) \) then \( r \cdot F_\omega(x, y) < r \cdot F_\omega(y, x) \) and therefore, by Equation (4), \( F_\omega(r \circ x, r \circ y) < F_\omega(r \circ y, r \circ x) \) and consequently, \( r \circ x \prec_\tau r \circ y \).

On the other hand, if \( F_\omega(x, y) = F_\omega(y, x) \) then \( r \cdot F_\omega(x, y) = r \cdot F_\omega(y, x) \) and therefore, by Equation (4).

\[
 F_\omega(r \circ x, r \circ y) = F_\omega(r \circ y, r \circ x).
\]

In addition, if \( x \prec_\tau y \) then, by Proposition 2

\[
 r \circ x \preceq_\tau r \circ y.
\]

Hence, from (5) and (6), we have that \( r \circ x \preceq_\omega r \circ y \) and consequently, the pair \( (L_n([0, 1]), \preceq_\omega) \) satisfies (SV8).

On the other hand, let \( x = (0.5, 0.6, 1, \ldots, 1) \), \( y = (0.3, 0.9, 1, \ldots, 1) \), \( z = (0.1, 0.3, 1, \ldots, 1) \in L_n([0, 1]) \) and \( \omega = (0.4, 0.6, 0, \ldots, 0) \). Then \( F_\omega(x, y) = 0.08 < 0.18 = F_\omega(y, x) \) and so \( x \prec_\omega y \). However, \( x \circ z = (0.6, 0.9, 1, \ldots, 1) \) and \( y \circ z = (0.4, 1, \ldots, 1) \). Therefore, \( F_\omega(y \circ z, x \circ z) = 0.06 < 0.08 = F_\omega(x \circ z, y \circ z) \), and consequently, \( y \circ z \prec_\omega x \circ z \). Hence, \( (L_n([0, 1]), \preceq_\omega) \) does not satisfy (SV9).

Therefore, as a direct consequence of Proposition 3 \( (L_n([0, 1]), \preceq_\omega) \) is not an ordered semi-vector space over \( U \).

**Proposition 4.** Let \( n \geq 2, \tau : N_1^n \to N_1^n \) be a bijection and \( A : [0, 1]^n \to [0, 1] \) be an aggregation function. If \( A \) is homogeneous of order \( k \) then \( \preceq_\tau \) is compatible
with the scalar product \( \odot \). In addition, \( \preceq_\tau^A \) is compatible with the vector addition \( \oplus \) if and only if \( \tau \) is the identity.

**Proof.** Let \( x, y \in L_n([0, 1]) \) and \( r \in [0, 1] \) such that \( x \preceq_\tau^A y \). If \( r = 0 \) or \( x = y \) then \( r \odot x \preceq_n^p r \odot y \) and since \( \preceq_\tau^A \) is an admissible order, then \( r \odot x \preceq_\tau^A r \odot y \).

Consider the case \( r > 0 \) and \( x \prec_\tau^A y \). Then, either \( A(x) < A(y) \) or, \( A(x) = A(y) \) and \( x \prec_\tau y \).

If \( A(x) < A(y) \) then \( A(r \odot x) = r^k \cdot A(x) < r^k \cdot A(y) = A(r \odot y) \) and therefore \( r \odot x \preceq_\tau^A r \odot y \).

If \( A(x) = A(y) \) and \( x \prec_\tau y \) then \( A(r \odot x) = r^k \cdot A(x) \leq r^k \cdot A(y) = A(r \odot y) \). If \( A(r \odot x) < A(r \odot y) \) then \( r \odot x \prec_\tau^A r \odot y \). If \( A(r \odot x) = A(r \odot y) \) then, by Proposition 2, \( r \odot x \prec_\tau r \odot y \). So, since \( \preceq_\tau^A \) is an admissible order, then \( r \odot x \prec_\tau^A r \odot y \). Hence, \( \preceq_\tau^A \) is compatible with the vector addition \( \oplus \).

In addition, if \( \tau \neq Id \) then there is \( m \in \mathbb{N}_1^{n-1} \) such that \( \tau(m) > \tau(m + 1) \). Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be such that \( y_{\tau(m)} > x_{\tau(m)} \) and \( x_{\tau(i)} = y_{\tau(i)} \) for all \( i < m \). Then, clearly, \( x \prec_\tau y \) and \( x_{\tau(i)} + (1 - x_{\tau(m)}) = y_{\tau(i)} + (1 - x_{\tau(m)}) \) for each \( i \leq m \) and \( x_{\tau(m+1)} + (1 - x_{\tau(m)}) > y_{\tau(m+1)} + (1 - x_{\tau(m)}) \). Therefore, \( y + z \preceq_\tau x + z \) for \( z = |1 - x_{\tau(m)}| \). Hence, \( \preceq_\tau^A \) is not compatible with the vector addition \( \oplus \). However, if \( \tau = Id \) and \( x, y, z \in L_n([0, 1]) \) are such that \( x \preceq_\tau^A y \) then there exists \( m \in \mathbb{N}_1^n \) such that \( x_m \leq y_m \) and \( x_i = y_i \) for each \( i < m \). Therefore, \( x_i + z_i = y_i + z_i \) for each \( i < m \) and if \( z_m < 1 - x_m \), \( x_k + z_k < y_k + z_k \) and therefore \( x + z \prec_\tau^A y + z \). If \( z_m \geq 1 - x_m \) then \( x + z = y + z \). So, \( \preceq_\tau^A \) is compatible with the vector addition \( \oplus \).

Therefore, as a direct consequence of Proposition 2 when \( A \) is homogeneous of some order, then \( (L_n([0, 1]), \preceq_\tau^A) \) is an ordered semi-vector space over \( U \) if and only if \( \tau = Id \).

## 4. \( n \)-Dimensional Aggregation Functions with respect to an Admissible Order

Bedregal et al. in 3 extended the notion of aggregation function for \( L_n([0, 1]) \) taking into account the order \( \preceq_n^p \). Here we consider an arbitrary admissible order on \( L_n([0, 1]) \).

**Definition 7.** Let \( \preceq \) be an admissible order on \( L_n([0, 1]) \). A function \( A : L_n([0, 1])^m \rightarrow L_n([0, 1]) \) is an \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq \) if:

\[
A(0/|, \ldots, 0/) = 0/ \text{ and } A(1/|, \ldots, 1/) = 1/; \text{ and}
\]

\[
A(x_1, \ldots, x_m) \preceq A(y_1, \ldots, y_m) \text{ whenever } x_j \preceq y_j \text{ for each } j \in \mathbb{N}_1^m.
\]

**Example 3.** Let \( \preceq \) be an admissible order on \( L_n([0, 1]) \). Then \( \bigwedge_{j=1}^m x_j \) and \( \bigvee_{j=1}^m x_j \) are \( m \)-ary \( n \)-dimensional internal aggregation functions with respect to \( \preceq \).
**Definition 8.** Let \( \preceq \) be an admissible order on \( L_n([0, 1]) \) and let \( A \) be an \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq \). Then \( A \)

1. is conjunctive if \( A(x_1, \ldots, x_m) \leq \bigwedge_{j=1}^m x_j \) for each \( x_j \in L_n([0, 1]) \);

2. is disjunctive if \( \forall \ x_j \preceq A(x_1, \ldots, x_m) \) for each \( x_j \in L_n([0, 1]) \);

3. is an average if \( \bigwedge_{j=1}^m x_j \preceq A(x_1, \ldots, x_m) \preceq \bigvee_{j=1}^m x_j \) for each \( x_j \in L_n([0, 1]) \);

4. is mixed if it is neither conjunctive, nor disjunctive nor average;

5. is idempotent if \( A(x, \ldots, x) = x \) for each \( x \in L_n([0, 1]) \);

6. is strict if for each \( x_1, \ldots, x_m, y \in L_n([0, 1]) \) and \( j \in N^m_1 \) such that \( x_j < y \) we have that \( A(x_1, \ldots, x_m) \preceq A(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m) \);

7. is internal if for each \( x_1, \ldots, x_m \in L_n([0, 1]) \) there exists \( j \in N^m_1 \) such that \( A(x_1, \ldots, x_m) = x_j \);

8. is symmetric if for each \( x_1, \ldots, x_m \in L_n([0, 1]) \) and for each bijection \( \tau : N^m_1 \to N^m_1 \) we have that \( A(x_1, \ldots, x_m) = A(x_{\tau(1)}, \ldots, x_{\tau(m)}) \).

**Proposition 5.** Let \( \preceq \) be an admissible order on \( L_n([0, 1]) \) and an \( m \)-ary \( n \)-dimensional aggregation function \( A \) with respect to \( \preceq \). Then \( A \) is an average if and only if \( A \) is idempotent, i.e. \( A(x, \ldots, x) = x \) for each \( x \in L_n([0, 1]) \).

**Proof.** (\( \Rightarrow \)) Since \( A \) is an average, then for each \( x \in L_n([0, 1]) \) we have that \( x = \bigwedge_{j=1}^m x \preceq A(x, \ldots, x) \preceq \bigvee_{j=1}^m x = x \) and therefore \( A \) is idempotent.

(\( \Leftarrow \)) For each \( x_1, \ldots, x_n \in L_n([0, 1]) \) consider \( y = \bigwedge_{j=1}^m x_j \) and \( z = \bigvee_{j=1}^m x_j \). So, since \( A \) is idempotent and increasing with respect to \( \preceq \), then
\[
\bigwedge_{j=1}^m x_j = A(y, \ldots, y) \preceq A(x_1, \ldots, x_m) \preceq A(z, \ldots, z) = \bigvee_{j=1}^m x_j
\]
is an average. \( \square \)

**Proposition 6.** Let \( m \) and \( n \) be non-zero natural numbers, \( \tau : N^m_1 \to N^m_1 \) be a bijection and \( A_1, \ldots, A_n \) be \( m \)-ary aggregation functions such that \( A_1 \preceq \cdots \preceq A_n \) and they are strictly increasing when restricted to \( L_n([0, 1]) \). Then \( A_1 \cdots A_n : L_n([0, 1])^m \to L_n([0, 1]) \) defined by
\[
A_1 \cdots A_n(x_1, \ldots, x_m) = (A_1(\pi_1(x_1), \ldots, \pi_1(x_m)), \ldots, A_n(\pi_n(x_1), \ldots, \pi_n(x_m)))
\]
is an \( m \)-ary strict \( n \)-dimensional aggregation function with respect to \( \preceq_\tau \). In addition, if \( A_i \)'s are conjunctive (disjunctive, internal) \( m \)-ary aggregation functions for each \( i \in N^m_1 \) then \( A_1 \cdots A_n \) is a conjunctive (disjunctive, average) \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq_\tau \).
Remark 2. In Proposition 6, the condition that all \( A_1 \ldots A_n \) restricted to \( L_n([0,1]) \) must be strictly increasing is necessary because if some \( A_i \) does not satisfy this condition then \( A_1 \ldots A_n \) may not be a \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq \).

Proof. Clearly, \( A_1 \ldots A_n \) is well defined, \( A_1 \ldots A_n ([0,\ldots,0]) = [0,\ldots,0] \) and \( A_1 \ldots A_n ([1,\ldots,1]) = [1,\ldots,1] \). Let \( j \in \mathbb{N}_m \) and \( x_1, x_m, y \in L_n([0,1]) \) such that \( x_j \prec y \). Then, \( \exists k \in \mathbb{N}_1^m \) s.t. \( \pi_{\tau(k)}(x_j) < \pi_{\tau(k)}(y) \) and \( \forall i, 1 \leq i < k, \pi_{\tau(i)}(x_j) = \pi_{\tau(i)}(y) \). So, for each \( 1 \leq i < k \), we have that

\[
\pi_{\tau(i)}(A_{1} \ldots A_{n}(x_1, \ldots, x_m)) = A_{\tau(i)}(\pi_{\tau(i)}(x_1), \ldots, \pi_{\tau(i)}(x_m)) \\
= A_{\tau(i)}(\pi_{\tau(i)}(x_1), \ldots, \pi_{\tau(i)}(x_{j-1}), \pi_{\tau(i)}(y), \pi_{\tau(i)}(x_{j+1}), \ldots, \pi_{\tau(i)}(x_m)) \\
= \pi_{\tau(i)}(A_{1} \ldots A_{n}(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m))
\]

and, because \( A_{\tau(k)} \) when restricted to \( L_n([0,1]) \) is strictly increasing, we have that

\[
\pi_{\tau(k)}(A_{1} \ldots A_{n}(x_1, \ldots, x_m)) = A_{\tau(k)}(\pi_{\tau(k)}(x_1), \ldots, \pi_{\tau(k)}(x_m)) \\
< A_{\tau(k)}(\pi_{\tau(k)}(x_1), \ldots, \pi_{\tau(k)}(x_{j-1}), \pi_{\tau(k)}(y), \pi_{\tau(k)}(x_{j+1}), \ldots, \pi_{\tau(k)}(x_m)) \\
= \pi_{\tau(k)}(A_{1} \ldots A_{n}(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m))
\]

Therefore,

\[
A_{1} \ldots A_{n}(x_1, \ldots, x_m) \prec A_{1} \ldots A_{n}(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m).
\]

Consequently, \( A_1 \ldots A_n \) is an \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq \).

Now, suppose that all the \( m \)-dimensional aggregation functions \( A_i \) are conjunctive. Then \( A_i(\pi_i(x_1), \ldots, \pi_i(x_m)) \leq \min(\pi_i(x_1), \ldots, \pi_i(x_m)) \) for each \( i \in \mathbb{N}_1^n \). Therefore, \( A_1 \ldots A_n(x_1, \ldots, x_m) \leq \min \{ x_1, \ldots, x_m \} \) for each \( j \in \mathbb{N}_1^m \). So, because \( \preceq \) is an admissible order, \( A_1 \ldots A_n(x_1, \ldots, x_m) \preceq \min \{ x_1, \ldots, x_m \} \) in every \( j \), \( 1 \leq j \leq m \). Analogously, we can prove that \( \forall x_j \preceq \min \{ x_1, \ldots, x_m \} \preceq A_1 \ldots A_n(x_1, \ldots, x_m) \) whenever \( A_j \) is disjunctive.

If the \( A_j \)'s are internal aggregation functions then \( A_i(\pi_i(x_1), \ldots, \pi_i(x_m)) = \pi_i(x_{j_i}) \) for some \( j_i \in \mathbb{N}_1^m \). Let \( t = \min j_i \) and \( s = \max j_i \). Then \( \pi_i(x_t) \leq A_i(\pi_i(x_1), \ldots, \pi_i(x_m)) \leq \pi_i(x_s) \) and therefore \( x_t \leq \pi_i(x_1) \ldots x_m \leq x_s \) So, because \( \preceq \) is an admissible order,

\[
\bigwedge_{j=1}^m x_j \preceq x_t \preceq A_1 \ldots A_n(x_1, \ldots, x_m) \preceq x_s \preceq \bigvee_{j=1}^m x_j.
\]

Hence, \( A_1 \ldots A_n \) is an average \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq \). □

Remark 2. In Proposition 6 the condition that all \( A_i \) restricted to \( L_n([0,1]) \) must be strictly increasing is necessary because if some \( A_i \) does not satisfy this condition then \( A_1 \ldots A_n \) may not be a \( m \)-ary \( n \)-dimensional aggregation function with respect to \( \preceq \). Indeed, taking as \( \tau \) the identity, \( A_1(x_1, \ldots, x_m) = \ldots = A_{n-1} = \min(x_1, \ldots, x_m) \) and \( A_n(x_1, \ldots, x_m) = \frac{1}{m} \sum_{i=1}^m x_i \) then \( x = (0.5, \ldots, 0.5, 1) \).
\[(0.5, \ldots, 0.5, 0.6, 0.6) = y. \text{ However,}
\]
\[A_1 \cdots A_n(x, \ldots, x, y) = (0.5, \ldots, 0.5, 0.5 + \frac{0.1}{m}) \preceq (0.5, \ldots, 0.5, \frac{m + 1}{2m}) = A_1 \cdots A_n(x, \ldots, x).
\]

**Theorem 2.** Let \(\preceq\) be an admissible order on \(L_n([0, 1])\), \(\omega = (w_1, \ldots, w_m)\) be a weighting vector and \(L'^n_m([0, 1]) = \langle L_n([0, 1]), +, *, \preceq \rangle\) be an ordered semi-vector space over \(U\). Then the function \(\text{OW } A^{L'^n_m([0, 1])}_\omega : L_n([0, 1])^m \to L_n([0, 1])\) defined by
\[
\text{OW } A^{L'^n_m([0, 1])}_\omega(x_1, \ldots, x_m) = \sum_{j=1}^m w_j \cdot x(j)
\]
where the sum is with respect to \(+\), \(x_{(j+1)} \leq x(j)\) for each \(j \in N^{m-1}\) and \(\{(1), \ldots, (m)\} = \{1, \ldots, m\}\), is an idempotent \(m\)-ary \(n\)-dimensional aggregation function with respect to \(\preceq\) called \(m\)-ary \(n\)-dimensional ordered weighted aggregation function with respect to \(\preceq\).

**Proof.** For each \(x \in L_n([0, 1])\) we have that,
\[
\text{OW } A^{L'^n_m([0, 1])}_\omega(x, \ldots, x) = \sum_{j=1}^m w_j \cdot x, \text{ by Eq. (7)}
\]
\[
= \left(\sum_{j=1}^m w_j\right) \cdot x, \text{ by (SV6)}
\]
\[
= 1 \otimes x, \text{ by (SV4)}
\]
i.e. \(\text{OW } A^{L'^n_m([0, 1])}_\omega\) is idempotent and therefore, \(\text{OW } A^{L'^n_m([0, 1])}_\omega(0/\ldots, 0/\ldots) = 0/\ldots, 0/\ldots\) and \(\text{OW } A^{L'^n_m([0, 1])}_\omega(1/\ldots, 1/\ldots) = 1/\ldots, 1/\ldots\).

Let \(x_1, \ldots, x_m, y \in L_n([0, 1])\) and \(t \in N^n\). If \(x(t) \leq y\) then by (SV8) we have that \(w_t \cdot x(t) \leq w_t \cdot y\) and therefore,
\[
\text{OW } A^{L'^n_m([0, 1])}_\omega(x_1, \ldots, x_m) = \sum_{j=1}^m w_j \cdot x(j), \text{ by Eq. (7)}
\]
\[
= \left(\sum_{j=1}^m w_j \cdot x(j)\right) + (w_t \cdot x(t)), \text{ by (SV1) and (SV2)}
\]
\[
\leq \left(\sum_{j=1,j \neq t}^m w_j \cdot x(j)\right) + (w_t \cdot y), \text{ by (SV9)}
\]
\[
= \text{OW } A^{L'^n_m([0, 1])}_\omega(x_1, \ldots, x(t-1), y, x(t), \ldots, x_m), \text{ by Eq. (7)}
\]
Therefore, \(\text{OW } A^{L'^n_m([0, 1])}_\omega\) is increasing with respect to \(\preceq\) and consequently is an idempotent \(m\)-ary \(n\)-dimensional aggregation function with respect to \(\preceq\).

**Corollary 1.** Let \(\preceq\) be an admissible order on \(L_n([0, 1])\), \(\omega = (w_1, \ldots, w_m)\) be a weighting vector and \(L'^n_m([0, 1]) = \langle L_n([0, 1]), +, *, \preceq \rangle\) be an ordered semi-vector space over \(U\). Then \(\text{OW } A^{L'^n_m([0, 1])}_\omega\) is a \(m\)-ary \(n\)-dimensional average aggregation function with respect to \(\preceq\).
Proof. Straightforward from Proposition 5 and Theorem 2.

Corollary 2. Let $\tau : N_1^m \to N_1^m$ be a bijection and $\omega = (w_1, \ldots, w_m)$ be a weighting vector. Then $OW A^L_\omega([0,1])$ is an $n$-ary weighted average function with respect to $\leq$. 

Proof. Analogous to the proof of Theorem 2.

Theorem 3. Let $\leq$ be an admissible order on $L_n([0,1])$, $\omega = (w_1, \ldots, w_m)$ be a weighting vector and $L'_n([0,1]) = \langle L_n([0,1]), +, \ast, \leq \rangle$ an ordered semi-vector space over $U$. Then the function $M^L_\omega([0,1]) : L_n([0,1])^m \to L_n([0,1])$ defined by

$$M^L_\omega([0,1]) (x_1, \ldots, x_m) = \sum_{j=1}^m w_j \ast x_j$$

where the sum is with respect to $+$, is an idempotent $m$-ary $n$-dimensional aggregation function with respect to $\leq$ called of $m$-ary $n$-dimensional weighted average with respect to $\leq$.

Proof. Straightforward from Proposition 5 and Theorem 3.

Corollary 3. Let $\leq$ be an admissible order on $L_n([0,1])$, $\omega = (w_1, \ldots, w_m)$ be a weighting vector and $L'_n([0,1]) = \langle L_n([0,1]), +, \ast, \leq \rangle$ an ordered semi-vector space over $U$. Then $M^L_\omega([0,1])$ is a $m$-ary $n$-dimensional average aggregation function with respect to $\leq$.

Proof. Straightforward from Proposition 5 and Theorem 3.

Corollary 4. Let $\tau : N_1^m \to N_1^m$ be a bijection and $\omega = (w_1, \ldots, w_m)$ be a weighting vector. Then $M^L_\omega([0,1])$ is a $m$-ary $n$-dimensional idempotent and average aggregation function with respect to $\leq$. 

Proof. Straightforward from Propositions 2 and 5 and Theorem 2.

In the following, we will explore some properties of $M^L_\omega([0,1])$.

Lemma 1. Let $\tau : N_1^m \to N_1^m$ be a bijection, $\lambda \in (0,1)$ and $x, y, z \in L_n([0,1])$ such that $x \prec \tau y$ and $\pi_n(z) \leq 1 - \lambda$. Then $z \ast \lambda \circ x \prec \tau z \ast \lambda \circ y$.

Proof. Since $x \prec \tau y$ then there exists $k \in N_1^m$ such that $\pi_k(x) < \pi_k(y)$ and $\pi_i(x) = \pi_i(y)$ for each $i < k$. So, because $\lambda > 0$, then $\lambda \pi_k(x) < \lambda \pi_k(y)$ and $\lambda \pi_i(x) = \lambda \pi_i(y)$ for each $i < k$, and therefore, $\lambda \circ x \prec \tau \lambda \circ y$. So, once $\lambda < 1$ and $\pi_k(z) \leq 1 - \lambda$, we have that

$$\pi_k(z) \ast \lambda \circ x = \pi_k(z) + \lambda \pi_k(x)$$

$$< \pi_k(z) + \lambda \pi_k(y) = \pi_k(z) \ast \lambda \pi_k(y) = \pi_k(z) \ast \lambda \circ y$$
Thus, since clearly, \( \pi_i(z) + \pi_i(\lambda \odot x) = \pi_i(z) + \pi_i(\lambda \odot y) \) for each \( i < k \) then \( z + \lambda \odot x <_r z + \lambda \odot y \).

\( \square \)

**Proposition 7.** Let \( \tau : N_1^m \to N_1^m \) be a bijection and \( \omega = (w_1, \ldots, w_m) \) be a weighting vector. Then \( M_{\omega}^{c_n}([0,1]) \) is

1. **strict if and only if** \( w_j > 0 \) for each \( j \in N_1^m \);
2. **symmetric if and only if** \( w_j = \frac{1}{m} \) for each \( j \in N_1^m \);
3. **additive with respect to** \( \odot \), i.e. \( M_{\omega}^{c_n}([0,1]) (x_1 + y_1, \ldots, x_m + y_m) = M_{\omega}^{c_n}([0,1]) (x_1, \ldots, x_m) + M_{\omega}^{c_n}([0,1]) (y_1, \ldots, y_m) \) for each \( x_1, y_1, \ldots, x_m, y_m \in L_n([0,1]) \);
4. **homogeneous, i.e.** \( M_{\omega}^{c_n}([0,1]) (\lambda \odot x_1, \ldots, \lambda \odot x_m) = \lambda \odot M_{\omega}^{c_n}([0,1]) (x_1, \ldots, x_m) \) for each \( \lambda \in [0,1] \) and \( x_1, \ldots, x_m \in L_n([0,1]) \).

**Proof.** (\( \Rightarrow \)) For each \( j \in N_1^m \),

\[
\begin{align*}
\leq & \ M_{\omega}^{c_n}([0,1]) (0, \ldots, 0) \\
< & \ M_{\omega}^{c_n}([0,1]) (0, \ldots, 1, 0, \ldots, 0) \\
& \ (j-1)-times \quad \quad \quad \ (m-j)-times \\
= & \ w_j \odot 1 \\
= & \ w_j
\end{align*}
\]

and therefore \( 0 < w_j \) for each \( j \in N_1^m \).

(\( \Leftarrow \)) Let \( x_1, \ldots, x_m, y \in L_n([0,1]) \). Since \( 0 < w_j \) for each \( j \in N_1^m \) then for each \( x, y \in L_n([0,1]) \) we have that \( w_j \odot x <_r w_j \odot y \) whenever \( x <_r y \). So, for each \( t \in N_1^m \) such that \( x_t \prec y \) we have that

\[
M_{\omega}^{c_n}([0,1]) (x_1, \ldots, x_m) = \sum_{j=1}^{m} w_j \odot x_j
= \left( \sum_{j=1, j \neq t}^{m} w_j \odot x_j \right) + w_t \odot x_t
\]

(\( \Leftarrow \)) Let \( j \in N_1^m \) then

\[
\begin{align*}
|w_j| & = M_{\omega}^{c_n}([0,1]) (0, \ldots, 0, 1, \ldots, 0) \\
& = M_{\omega}^{c_n}([0,1]) (0, \ldots, 1, 0, \ldots, 0, 0, \ldots, 0) \\
& = \frac{|w_{\tau(j)}|}{m}.
\end{align*}
\]
Therefore, \( w_j = w_{\tau(j)} \) for each bijection \( \tau \) and, consequently, \( w_j = \frac{1}{m} \) for each \( j \in N_n^m \).

(\( \Leftarrow \)) Straightforward.

3. Let \( x_1, \ldots, x_m, y_1, \ldots, y_m \in L_{n}([0, 1]) \). Then

\[
M_{\omega}^{L_n}([0,1]) (x_1 + y_1, \ldots, x_m + y_m) = \sum_{j=1}^{m} w_j \odot (x_j + y_j)
\]

\[
= \sum_{j=1}^{m} (w_j (\pi_1(x_j) + \pi_1(y_j)), \ldots, w_j (\pi_n(x_j) + \pi_n(y_j)))
\]

\[
= \sum_{j=1}^{m} (w_j \pi_1(x_j) + w_j \pi_1(y_j), \ldots, w_j \pi_n(x_j) + w_j \pi_n(y_j))
\]

\[
= \sum_{j=1}^{m} ((w_j \pi_1(x_j), \ldots, w_j \pi_n(x_j)) + (w_j \pi_1(y_j), \ldots, w_j \pi_n(y_j)))
\]

\[
= \left( \sum_{j=1}^{m} w_j \odot x_j \right) + \left( \sum_{j=1}^{m} w_j \odot y_j \right)
\]

\[
= M_{\omega}^{L_n}([0,1]) (x_1, \ldots, x_m) + M_{\omega}^{L_n}([0,1]) (y_1, \ldots, y_m)
\]

4. Let \( x_1, \ldots, x_m \in L_{n}([0, 1]) \) and \( \lambda \in [0, 1] \). Then

\[
M_{\omega}^{L_n}([0,1]) (\lambda \odot x_1, \ldots, \lambda \odot x_m) = \sum_{j=1}^{m} w_j \odot (\lambda \odot x_j)
\]

\[
= \sum_{j=1}^{m} \lambda \odot (w_j \odot x_j) \quad \text{(by (SV3) and (WF2))}
\]

\[
= \lambda \odot \sum_{j=1}^{m} w_j \odot x_j \quad \text{(by (SV5))}
\]

\[
= \lambda \odot M_{\omega}^{L_n}([0,1]) (x_1, \ldots, x_m)
\]

\[\square\]

5  A Decision-Making Method Based on \( n \)-dimensional Aggregation Functions

Multiple criteria group decision making (MCGDM) methods are procedures to rank a set of alternatives, by considering some criteria and the opinion of some experts on how much the alternatives satisfy each one of the criteria. In general, the criteria may be in conflict to each other, which makes more difficult the generation of the ranking in a reasonable way.

Let \( A = \{a_1, \ldots, a_p\} \) \( (p \geq 2) \) be a set of alternatives, \( E = \{e_1, \ldots, e_n\} \) \( (n \geq 2) \) be a group of experts, and \( C = \{c_1, \ldots, c_m\} \) \( (m \geq 2) \) be a set of attributes or criteria. From the opinion of each expert \( e_k \), we generate a decision matrix \( R^k = \{R^k_{ij}\}_{p \times m} \) such that \( R^k_{ij} \) is the evaluation of \( e_k \) of how much the alternative \( a_i \) meets the criterion \( c_j \), which is expressed by a value in \([0, 1]\).
We will consider the immersion \( \sigma : [0, 1]^n \to L_n([0, 1]) \), i.e. the function

\[
\sigma(x_1, \ldots, x_n) = (x_{(1)}, \ldots, x_{(n)})
\]

such that \((x_{(1)}, \ldots, x_{(n)})\) is the element of \(L_n([0, 1])\) which is a permutation of \((x_1, \ldots, x_n)\). Observe that \((x_{(1)}, \ldots, x_{(n)})\) always exists and is unique. For example, \(\sigma(0.2, 0.5, 0.2, 0.4, 0.5, 0.7, 0.5) = (0.2, 0.2, 0.4, 0.5, 0.5, 0.5, 0.7)\). However, as this example shows, the corresponding permutation may not be unique.

We propose the following general multi-criteria and multi-expert decision-making method:

**Step 1.** For every expert \(e_k \in E\), generate the matrix \(R^k\) with his/her evaluations of how much each alternative meets each criterion.

**Step 2.** Generate the collective matrix \(R^*\) from the \(R^k\)'s with \(k = 1, \ldots, n\), by ordering it, so that \(R^*_{ij}\) belongs to \(L_n([0, 1])\) for each \(i \in N^p_1\) and \(j \in N^m_1\), i.e.,

\[
R^*_{ij} = \sigma(R^1_{ij}, \ldots, R^n_{ij}).
\]

**Step 3.** Select a strictly positive weighting vector \(\omega = (w_1, \ldots, w_m)\) which reflects the importance of each criterion \(c_j\) in the decision making process.

**Step 4.** Choose an admissible order \(\preceq\) on \(L_n([0, 1])\) and a commutative \(m\)-ary \(n\)-dimensional aggregation function \(A\), with respect to \(\preceq\).

**Step 5.** Apply \(A\) to each row of \(R^*\), considering **Step 3**, i.e. \(s_i = A(R^*_1, \ldots, R^*_m)\) is the \(L_n([0, 1])\)-score of the alternative \(a_i\).

**Step 6.** Determine the ranking of the alternatives in \(A\) from the \(L_n([0, 1])\)-scores obtained in the **Step 5**, taking into account the admissible order \(\preceq\) choice in the **Step 4**.

Many MCGDM methods had been proposed but most of them do not consider any study of their general properties and therefore, we have no guarantee or evidence that the resulting ranking is reasonable. For example, the first illustrative example in [33] presents 12 different rankings obtained from the proposed method when considering 12 different aggregation function, and in [16, 17] for that same illustrative example, 3 new different rankings were obtained. That is, at all, we get 15 different ranking for the same problem and we have no mathematical foundation to determine which of these ranking is best one. Nevertheless, it is possible to determine some good properties for decision making methods, as made in [9] for decision making methods based on preference relations and in the light of Arrow’s social choice theory (see [1, 35]).

In the following, we present two reasonable principles that any MCGDM methods based on decision matrices should satisfy, in our opinion:

**Increasingness:** If the value of some \(R^k_{ij}\) is increased then the ranking of \(a_i\) not can decrease.
**Domination:** If the alternative $a_i$ is better evaluated than $a_j$ in all the criteria and in the opinion of all the experts, then the ranking of $a_i$ must be better than the ranking of $a_j$.

**Insensibility to indexations:** The order (indexation) of the expert, alternatives and criteria is irrelevant, i.e. different orders of the experts must never modify the final ranking of the alternatives.

The first principle is guaranteed in our method. In fact, when $R^k_{ij}$ is increased to $R^k_{ij}'$, for some $i \in N^p_1$, $j \in N^m_1$ and $k \in N^n_1$, then $R^*_ij \leq R^*_ij'$, where $R^*$ and $R^*_i$ are the collective matrices generated in the step 2 before and after the increasing of $R^k_{ij}$. So, since $\preceq$ is an admissible order and $\mathcal{A}$ is an aggregation function with respect to this admissible order, we have that $s_i \preceq s'_i$, where $s_i$ and $s'_i$ are the $L_n([0,1])$-scores, computed in the step 5, of the alternative $a_i$ before and after this increasing, respectively, whereas the $L_n([0,1])$-scores of the other alternatives remain the same. Therefore, if $a_j$ is worse ranked than $a_i$ before of the change, then $s_j \preceq s_i \preceq s'_i$. Hence, $a_j$ will remain worse ranked than $a_i$ after of the change.

The domination condition is also satisfied by the proposed method. Indeed, suppose that an alternative $a_i$ is better evaluated than $a_j$ in all the criteria for all the experts, i.e. $R^k_{ij} \leq R^k_{il}$ for each $k \in N^p_1$ and $l \in N^n_1$. Then, $R^*_ij \leq R^*_il$ and therefore $R^*_ij' \leq R^*_il'$. So, because $\mathcal{A}$ is a $m$-ary aggregation function with respect to $\preceq$, we have that $\mathcal{A}(R^*_ij_1, \ldots, R^*_jm) \preceq \mathcal{A}(R^*_il_1, \ldots, R^*_im)$ and therefore, $s_j \preceq s_i$ which means that the ranking of the alternative $a_j$ is worse than the ranking of alternative $a_i$.

Finally, for each permutation $\rho_u$ of $N^u_1$ with $u \in \{m,n,p\}$, define $P^k_{ij} = R^k_{ij(\rho_u(i))\rho_u(j)}$ and $t_i$ the $L_n([0,1])$-score of the alternative $b_i = a_{\rho_u(i)}$ obtained by the method. Since $\sigma$ and $\mathcal{A}$ are commutative, then $t_i = s_{\rho_u(i)}$ for each $i \in N^p_1$ and therefore, the ranking of the alternative $a_{\rho_u(i)}$ is the same as the alternative $b_i$, but $b_i = a_{\rho_u(i)}$. Therefore, the ranking is not modified by the permutations and consequently, we can claim that the proposed method also satisfies the insensibility to indexations principle.

### 5.1 An energy police selection problem under a multidimensional fuzzy environment

As an illustrative example of the proposed method based on $n$-dimensional aggregation functions with respect to an admissible order, we present an energy policy problem adapted from [44]. Since the energy issue has major economic, social and environmental impacts, addressing the right ways to deal with this issue affects economic and environmental development in societies. Therefore, selecting the most appropriate energy policy is very significant. Suppose there are five energy projects, i.e., our set of alternatives is $\{a_1, a_2, a_3, a_4, a_5\}$. Such alternatives are evaluated according to the criteria: technology ($C_1$), environment
(C₂), socio-political (C₃) and economic (C₄). We will consider that five experts (e₁, e₂, e₃, e₄, e₅) provide their decision matrices, i.e. matrices $R^k$ (where $k$ denote that is the decision matrix of expert $e_k$) such that each position $(i,j)$ in $R^k$ contain a value $R_{ij}^k$ in $[0,1]$ corresponding to their belief or evaluation of how much the alternative $a_i$ attends the criteria $C_j$. The decision matrices provided by the five experts are shown in Tables 1-5 (Step 1).

| Table 1: Decision matrix of the expert e₁ | Table 2: Decision matrix of the expert e₂ | Table 3: Decision matrix of the expert e₃ |
|------------------------------------------|------------------------------------------|------------------------------------------|
| $R^1$ | $R^2$ | $R^3$ |
| C₁, C₂, C₃, C₄ | C₁, C₂, C₃, C₄ | C₁, C₂, C₃, C₄ |
| a₁ | a₁ | a₁ |
| 0.4 0.7 0.2 0.3 | 0.5 0.5 0.5 0.5 | 0.4 0.8 0.2 0.9 |
| a₂ | a₂ | a₂ |
| 0.5 0.9 0.1 0.4 | 0.5 0.5 0.1 0.4 | 0.3 0.7 0.6 0.7 |
| a₃ | a₃ | a₃ |
| 0.6 0.6 0.5 0.4 | 0.7 0.6 0.3 0.6 | 0.7 0.6 0.5 0.4 |
| a₄ | a₄ | a₄ |
| 0.8 0.7 0.8 0.6 | 0.7 0.2 0.8 0.8 | 0.4 0.4 0.1 0.8 |
| a₅ | a₅ | a₅ |
| 0.6 0.4 0.7 0.7 | 0.9 0.6 0.8 0.3 | 0.1 0.6 0.7 0.6 |

The main difference between our approach and that of Xu and Zhang [44] is the use of hesitant fuzzy sets rather than hesitant fuzzy sets (for more details about hesitant fuzzy sets one may see [38, 43] and [6, 7] for typical hesitant fuzzy sets) in decision making. The use of admissible orders which reduces the number of cases where alternatives with different initial evaluations lead to the same final ranking. Now, following the proposed method, we construct the $L_n([0,1])$-valued collective matrix $R^*$ in Table 7 (Step 2).

If we take together the values on the same positions of the $R^k$'s, we generate the Table used as starting point in [44]. Observe that some equal evaluations given by different experts collapsed in this hesitant fuzzy matrix. The main difference between our approach and that of Xu and Zhang [44] is the use of $n$-dimensional fuzzy sets rather than hesitant fuzzy sets (for more details about hesitant fuzzy sets one may see [38, 43] and [6, 7] for typical hesitant fuzzy sets) in the decision matrix and the use of admissible orders which reduces the number of cases where alternatives with different initial evaluations lead to the same final ranking. Now, following the proposed method, we construct the $L_n([0,1])$-valued collective matrix $R^*$ in Table 7 (Step 2).

| Table 4: Decision matrix of the expert e₄ | Table 5: Decision matrix of the expert e₅ |
|------------------------------------------|------------------------------------------|
| $R^4$ | $R^5$ |
| C₁, C₂, C₃, C₄ | C₁, C₂, C₃, C₄ |
| a₁ | a₁ |
| 0.3 0.9 0.4 0.3 | 0.5 0.1 0.5 0.6 |
| a₂ | a₂ |
| 0.3 0.2 0.8 0.3 | 0.3 0.6 0.5 0.7 |
| a₃ | a₃ |
| 0.7 0.9 0.3 0.6 | 0.6 0.6 0.7 0.6 |
| a₄ | a₄ |
| 0.8 0.4 0.8 0.9 | 0.3 0.7 0.1 0.6 |
| a₅ | a₅ |
| 0.3 0.8 0.9 0.9 | 0.7 0.7 0.8 0.6 |

The selected weighting vector (Step 3) is $\omega = (0.2341, 0.2474, 0.3181, 0.2004)$ (as in [44]) and the admissible order on $L_n([0,1])$ chosen (Step 4) is $\preceq_\tau$ (as in Example 1) for $\tau(1) = 3, \tau(2) = 2, \tau(3) = 4, \tau(4) = 1$ and $\tau(5) = 5$ (notice that for this illustrative example the choice of $\tau$ is ad-hoc). The $m$-ary $n$-dimensional aggregation function with respect to $\preceq_\tau$ chosen in the step 4 is $M^{L_n([0,1])}_\omega$ defined in Equation 8 where the semi-vector space $L_n([0,1])$ is the given in Theorem 11.
Table 6: Hesitant fuzzy collective decision matrix in [44]

| R  | C₁     | C₂       | C₃       | C₄       |
|----|--------|----------|----------|----------|
| a₁ | (0.5,0.4,0.3) | (0.9,0.8,0.7,0.1) | (0.5,0.4,0.2) | (0.9,0.6,0.5,0.3) |
| a₂ | (0.5,0.3)      | (0.9,0.7,0.6,0.5,0.2) | (0.8,0.6,0.5,0.1) | (0.7,0.4,0.3) |
| a₃ | (0.7,0.6)      | (0.9,0.6)   | (0.7,0.5,0.3) | (0.6,0.4) |
| a₄ | (0.8,0.7,0.4,0.3) | (0.7,0.4,0.2) | (0.8,0.1) | (0.9,0.8,0.6) |
| a₅ | (0.9,0.7,0.6,0.3,0.1) | (0.8,0.7,0.6,0.4) | (0.9,0.8,0.7) | (0.9,0.7,0.6,0.3) |

Table 7: n-dimensional interval collective decision matrix

| R² | C₁     | C₂       | C₃       | C₄       |
|----|--------|----------|----------|----------|
| a₁ | (0.3,0,4,0.4,0.5,0.5) | (0.1,0.7,0.7,0.8,0.9) | (0.2,0.2,0.4,0.5,0.5) | (0.3,0,3,0.5,0.6,0.9) |
| a₂ | (0.3,0,3,0,3,0,5,0,5) | (0.2,0,5,0.6,0.7,0.9) | (0.1,0.1,0.5,0.6,0.8) | (0.3,0,4,0.4,0.7,0.7) |
| a₃ | (0.6,0,6,0.7,0.7,0.7) | (0.6,0,6,0,6,6,0.6,0.9) | (0.3,0.3,0.5,0.5,0.7) | (0.4,0,4,0,6,6,0.6) |
| a₄ | (0.3,0,4,0,7,0,8,0,8) | (0.2,0,4,0,4,0,7,0,7) | (0.1,0,1,0,8,0,8,0,8) | (0.6,0,6,0,8,0,8,0,9) |
| a₅ | (0.1,0,3,0,6,0,7,0,9) | (0.4,0,6,0,6,0,7,0,8) | (0.7,0,7,0,8,0,8,0,9) | (0.3,0,6,0,6,0,7,0,9) |

In the step 5, $M_{ω}^{L_n}([0,1])$ is applied to each row of the $R*$ resulting in the following $L_n([0,1])$-scores for each alternative:

\[
s₁ = M_{ω}^{L_n}([0,1]) (R_{11}^*, \ldots, R_{14}^*) \\
= 0.2341 \odot (0.3,0.4,0.4,0.5,0.5) \div 0.2474 \odot (0.1,0.7,0.7,0.8,0.9) \div 0.3181 \odot (0.2,0.2,0.4,0.5,0.5) \div 0.2004 \odot (0.3,0.3,0.5,0.6,0.9) \\
= (0.07023, 0.09364, 0.09364, 0.11705, 0.11705) \div (0.02474, 0.17318, 0.17318, 0.19792, 0.22266) \div (0.06362, 0.06362, 0.12724, 0.15905, 0.15905) \div (0.06012, 0.06012, 0.1002, 0.12024, 0.18036) \\
= (0.21871, 0.39056, 0.49426, 0.59426, 0.67912)
\]
Alternative is example, with the evaluation of the alternative \[ s_2 = \mathcal{M}_{\omega}^{L_{\alpha}}([0,1]) (R^*_{21}, \ldots, R^*_{24}) \]
= 0.2341 \oplus (0.3, 0.3, 0.3, 0.5, 0.5) \oplus 0.2474 \oplus (0.2, 0.5, 0.6, 0.7, 0.9) \oplus
0.3181 \oplus (0.1, 0.1, 0.5, 0.6, 0.8) \oplus 0.2004 \oplus (0.3, 0.4, 0.4, 0.7, 0.7)
= (0.07023, 0.07023, 0.07023, 0.11705, 0.11705) \oplus (0.04948, 0.1237, 0.19792, 0.17318, 0.22266) \oplus
(0.03181, 0.03181, 0.15905, 0.19086, 0.25448) \oplus (0.06012, 0.08016, 0.08016, 0.14028, 0.14028)
= (0.21164, 0.3059, 0.50736, 0.62137, 0.73447)

\[ s_3 = \mathcal{M}_{\omega}^{L_{\alpha}}([0,1]) (R^*_{31}, \ldots, R^*_{34}) \]
= 0.2341 \oplus (0.6, 0.6, 0.7, 0.7, 0.7) \oplus 0.2474 \oplus (0.6, 0.6, 0.6, 0.6, 0.9) \oplus
0.3181 \oplus (0.3, 0.3, 0.5, 0.5, 0.7) \oplus 0.2004 \oplus (0.4, 0.4, 0.6, 0.6, 0.6)
= (0.14046, 0.14046, 0.16387, 0.16387, 0.16387) \oplus (0.14844, 0.14844, 0.14844, 0.14844, 0.22266) \oplus
(0.09543, 0.09543, 0.15905, 0.15905, 0.22267) \oplus (0.08016, 0.08016, 0.12024, 0.12024, 0.12024)
= (0.46449, 0.46449, 0.5916, 0.5916, 0.72944)

\[ s_4 = \mathcal{M}_{\omega}^{L_{\alpha}}([0,1]) (R^*_{41}, \ldots, R^*_{44}) \]
= 0.2341 \oplus (0.3, 0.4, 0.7, 0.8, 0.8) \oplus 0.2474 \oplus (0.2, 0.4, 0.4, 0.7, 0.7) \oplus
0.3181 \oplus (0.1, 0.1, 0.8, 0.8, 0.8) \oplus 0.2004 \oplus (0.6, 0.6, 0.8, 0.8, 0.9)
= (0.07023, 0.09364, 0.16387, 0.18728, 0.18728) \oplus (0.03181, 0.03181, 0.09896, 0.17318, 0.17318) \oplus
(0.06362, 0.06362, 0.25448, 0.25448, 0.25448) \oplus (0.12024, 0.12024, 0.16032, 0.16032, 0.18036)
= (0.2859, 0.30931, 0.67763, 0.77526, 0.7953)

\[ s_5 = \mathcal{M}_{\omega}^{L_{\alpha}}([0,1]) (R^*_{51}, \ldots, R^*_{54}) \]
= 0.2341 \oplus (0.1, 0.3, 0.6, 0.7, 0.9) \oplus 0.2474 \oplus (0.4, 0.6, 0.6, 0.7, 0.8) \oplus
0.3181 \oplus (0.7, 0.7, 0.8, 0.8, 0.9) \oplus 0.2004 \oplus (0.3, 0.6, 0.6, 0.7, 0.9)
= (0.02341, 0.07023, 0.14046, 0.16387, 0.21069) \oplus (0.09896, 0.14844, 0.14844, 0.17318, 0.19792) \oplus
(0.22267, 0.22267, 0.25448, 0.25448, 0.28629) \oplus (0.06012, 0.12024, 0.12024, 0.14028, 0.18036)
= (0.40516, 0.56158, 0.66362, 0.73181, 0.87526)

Thereby, we obtain the following ranking of the alternatives: \( a_1 < a_2 < a_3 < a_5 < a_4 \) (Step 6), since \( s_1 \preceq_s s_2 \preceq_s s_3 \preceq_s s_5 \preceq_s s_4 \); and thus the most desirable alternative is \( a_4 \), which differs from the best alternative obtained in \[44\]. The problem with applying the method of Xu and Zhang, is that when two experts provide the same degree for the satisfaction of an alternative with respect to a criterion, they collapse in a single value in the hesitant decision matrix as, for example, with the evaluation of the alternative \( a_1 \) with respect to the criterion \( C_1 \) done by the experts \( e_1 \) and \( e_3 \) (see Tables \[1\] and \[3\]). Moreover, in the step 2 of
the Xu and Zhang method, the least value is repeated in order to obtain vectors on $[0, 1]$ of the same dimension. Thus, this process does not faithfully reflect the opinion of the experts, and therefore the ranking obtained will be biased. Indeed, if the expert $e_2$ changes the evaluation for the alternative $a_4$ with respect to $C_3$ to 0.1, this change will not have any effect over the hesitant decision matrix in Table 4 and therefore the final ranking will remain the same, whereas in the proposed method, the ranking will change.

6 Conclusion

In this paper, $m$-ary aggregation functions with respect to admissible orders on $n$-dimensional intervals, i.e. on $L_n([0, 1])$, were studied. Different classes of such functions have been considered, such as conjunctive, disjunctive, average, idempotent and symmetric $m$-ary $n$-dimensional aggregation functions with respect to an arbitrary admissible order. On the other hand, the notion of ordered semi-vector spaces on the semifield of non-negative real numbers was extended for arbitrary weak semifields. In particular, an ordered semi-vector space for $L_n([0, 1])$ where the order is admissible is given and OWA-like and weighted average operators for $L_n([0, 1])$ are defined. Then, we have investigated on admissible orders such that these operators are aggregation functions. Finally, we have developed an application in a multicriteria group decision making method based on $n$-dimensional aggregation functions with respect to an admissible order. Moreover, an illustrative example is considered where the $m$-ary $n$-dimensional weighted average aggregation function for a given admissible order is applied. Some minimal desirable properties that all multiple criteria group decision making method must satisfy and that are preserved by the proposed method are discussed.

As further work, we will investigate on new classes and on the way of constructing admissible order on $L_n([0, 1])$. We also intend to study $m$-ary $n$ dimensional aggregation functions with respect to such admissible orders. On the other hand, continuity is an important property in many applications, since it guarantees that small changes in the inputs of an aggregation function do not imply substantial changes of the results. So, in a future work a deeper study of continuous $n$-dimensional aggregation functions will be made, not only in the light of [5, 34], but also taking into account the admissible order which is being considered.

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