Expected value of letters of permutations with a given number of $k$-cycles

Peter Kagey

2021/12/13

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6 Acknowledgments

In this paper, we study permutations $\pi \in S_n$ with exactly $m$ transpositions. In particular, we are interested in the expected value of $\pi(1)$ when such permutations are chosen uniformly at random. When $n$ is even, this expected value is approximated closely by $(n+1)/2$, with an error term that is related to the number isometries of the $(n/2 - m)$-dimensional hypercube that move every face. Furthermore, when $k \mid n$, this construction generalizes to allow us to
compute the expected value of $\pi(1)$ for permutations with exactly $m$ $k$-cycles. In this case, the expected value has an error term which is related instead to the number derangements of the generalized symmetric group $S(k, n/k - m)$.

When $k$ does not divide $n$, the expected value of $\pi(1)$ is precisely $(n + 1)/2$. Indirectly, this suggests the existence of a reversible algorithm to insert a letter into a permutation which preserves the number of $k$-cycles, which we construct.

1 Background

In 2010, Mark Conger [3] proved that a permutation with $k$ descents has an expected first letter of $\pi(1) = k + 1$, independent of $n$. This paper has the same premise, but with a different permutation statistic: the number of $k$-cycles of a permutation.

This section, (Section 1) provides an overview of where we’re headed, and includes an critical example that will hopefully spark the reader’s curiosity and motivate the remainder of the paper.

Section 2 establishes some recurrence relations for the number of permutations in $S_n$ with a given number of $k$-cycles. It also contains a theorem that gives an explicit way to compute the expected value of the first letter based on these counts.

Section 3 describes an explicit correspondence between $k$-cycles of permutations in $S_{kn}$ and fixed points of elements of the generalized symmetric group $(\mathbb{Z}/k\mathbb{Z}) \wr S_n$. Using generating functions and results from the previous section, this shows that the expected value of $\pi(1)$ of a permutation with a given number of $k$-cycles is intimately connected to the number of derangements of a generalized symmetric group.

While Section 3 emphasizes the case of $S_{kn}$, Section 4 looks at $S_N$ where $k \nmid N$. Here, the expected value of $\pi(1)$ is simply $(N + 1)/2$, which agrees with the expected value of the first letter of a uniformly chosen $N$-letter permutation with no additional restrictions. This fact together with the main theorem from Section 2 implies the existence of a bijection $\varphi_k: S_{N-1} \times \{N\} \to S_N$ that preserves the number of $k$-cycles whenever $k \nmid N$. Section 4 constructs such a bijection explicitly, and proves that it has the desired properties.

1.1 Motivating Examples

In support of the first examples, we start by defining the first bit of notation.

**Definition 1.1.** Let $C_k(n, m)$ denote the number of permutations $\pi \in S_n$ such that $\pi$ has exactly $m$ $k$-cycles.

These theorems—and many of the following lemmas—were discovered by looking at examples such as the following, written in both one-line and cycle notation:
Example 1.2. There are $C_2(4, 0) = 15$ permutations in $S_4$ with no $2$-cycles:

\[
\begin{align*}
1234 &= (1)(2)(3)(4) & 2314 &= (31)(2)(4) & 3124 &= (321)(4) & 4123 &= (4321) \\
1342 &= (1)(4)(23) & 2341 &= (4123) & 3142 &= (2)(413) & 4213 &= (2)(431) \\
1432 &= (1)(3)(42) & 2431 &= (3)(412) & 3421 &= (4132) & 4321 &= (4231)
\end{align*}
\]

There are $C_2(4, 1) = 6$ permutations in $S_4$ with exactly one $2$-cycle:

\[
\begin{align*}
1243 &= (1)(2)(43) & 2134 &= (21)(3)(4) \\
1324 &= (1)(32)(4) & 3214 &= (2)(31)(4) \\
1432 &= (1)(3)(42) & 4231 &= (2)(3)(41)
\end{align*}
\]

And there are $C_2(4, 2) = 3$ permutations in $S_4$ with exactly two $2$-cycles,

\[
\begin{align*}
2143 &= (21)(43) & 3412 &= (31)(42) & 4321 &= (32)(41)
\end{align*}
\]

By averaging the first letter over these examples, we can compute that

\[
\begin{align*}
E[\pi(1) | \pi \in S_4 \text{ has no } 2\text{-cycles}] &= \frac{3(1) + 4(2 + 3 + 4)}{15} = \frac{13}{5}, \\
E[\pi(1) | \pi \in S_4 \text{ has exactly } 1 \text{ } 2\text{-cycle}] &= \frac{3(1) + (2 + 3 + 4)}{6} = 2, \text{ and} \\
E[\pi(1) | \pi \in S_4 \text{ has exactly } 2 \text{ } 2\text{-cycles}] &= \frac{2 + 3 + 4}{3} = 3.
\end{align*}
\]

The table in Figure 1 gives the expected value of $\pi(1)$ given that $\pi \in S_n$ and has exactly $m$ $2$-cycles in its cycle decomposition. Notice that when $i$ is odd, row $i$ has a constant value of $(i + 1)/2$. Also notice that the number in position $(i, j)$ has the same denominator as the number in position $(i + 2, j + 1)$, and that these denominators increase with $n$. The sequence of denominators begins

\[
1, 5, 29, 233, 2329, 27949, \ldots, \quad (1.1)
\]

which agrees with the type B derangement numbers, sequence A000354 in the On-Line Encyclopedia of Integer Sequences (OEIS) [6]. In other words, the denominators in the table appear to be related to the symmetries of the hypercube that move every facet.

2 Structure of permutations with $m$ $k$-cycles

This section is about connecting the number of permutations with a given number of $k$-cycles to the expected value of the first letter. Saying this, it is appropriate to start with a 1944 theorem of Goncharov that, by the principle of inclusion/exclusion, gives an explicit formula that counts the number of such permutations.
Figure 1: A table of the expected value of the first letter of π ∈ Sn with exactly m 2-cycles, E[π(1) | π ∈ Sn has exactly m 2-cycles].

| n   | 0     | 1     | 2     | 3     | 4     | 5     | 6     |
|-----|-------|-------|-------|-------|-------|-------|-------|
| 1   | 1/1   |       |       |       |       |       |       |
| 2   | 1/1   | 2/1   |       |       |       |       |       |
| 3   | 2/1   | 2/1   |       |       |       |       |       |
| 4   | 13/5  | 2/1   | 3/1   |       |       |       |       |
| 5   | 3/1   | 3/1   | 3/1   |       |       |       |       |
| 6   | 101/29| 18/5  | 3/1   | 4/1   |       |       |       |
| 7   | 4/1   | 4/1   | 4/1   | 4/1   |       |       |       |
| 8   | 1049/233 | 130/29 | 23/5 | 4/1  | 5/1   |       |       |
| 9   | 5/1   | 5/1   | 5/1   | 5/1   | 5/1   |       |       |
| 10  | 12809/2329 | 1282/233 | 159/29 | 28/5 | 5/1  | 6/1   |       |
| 11  | 6/1   | 6/1   | 6/1   | 6/1   | 6/1   | 6/1   |       |
| 12  | 181669/27949 | 15138/2329 | 1515/233 | 188/29 | 33/5 | 6/1  | 7/1   |
| 13  | 7/1   | 7/1   | 7/1   | 7/1   | 7/1   | 7/1   | 7/1   |

2.1 Counting permutations based on cycles

Theorem 2.1 ([5], [1]). The number of permutations in Sn with exactly m k-cycles is given by the following sum, via the principle inclusion/exclusion:

\[ C_k(n,m) = \frac{n!}{m!k^m} \sum_{i=0}^{\lfloor n/k \rfloor - m} \frac{(-1)^i}{i!k^i}. \]  \hspace{1cm} (2.1)

Corollary 2.2. For k \nmid n, there are exactly n times as many permutations in Sn with exactly m k-cycles than there are in Sn−1. When k \mid n, there is an explicit formula for the difference.

\[ C_k(n,m) - nC_k(n-1,m) = \begin{cases} 0 & k \nmid n \\ \frac{n!(-1)^{n-k} - m}{(n/k)!k^{n/k}} \binom{n/k}{m} & k \mid n \end{cases} \]  \hspace{1cm} (2.2a)

Proof. When k \nmid n, \[ \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{n-1}{k} \right\rfloor, \] so the bounds on the sums are identical and the result follows directly

\[ \frac{n!}{m!k^m} \sum_{i=0}^{\lfloor n/k \rfloor - m} \frac{(-1)^i}{i!k^i} - n \left( \frac{(n-1)!}{m!k^m} \sum_{i=0}^{\lfloor (n-1)/k \rfloor - m} \frac{(-1)^i}{i!k^i} \right) = 0. \]  \hspace{1cm} (2.3)
Otherwise, when \( k \mid n \), \( \left\lfloor \frac{n - 1}{k} \right\rfloor = \frac{n}{k} - 1 \), so

\[
\frac{n!}{m!k^m} \sum_{i=0}^{n/k-m} \frac{(-1)^i}{i!k^i} - n \left( \frac{(n - 1)!}{m!k^m} \sum_{i=0}^{n/k-1-m} \frac{(-1)^i}{i!k^i} \right)
\]

\[
= \frac{n!}{m!k^m} \left( \frac{(-1)^{n/k-m}}{(n/k - m)!k^{n/k-m}} \right)
\]

\[
= \frac{n!(-1)^{n/k-m}}{(n/k - m)!m!k^{n/k}}
\]

\[
= \frac{n!(-1)^{\frac{n}{k}-m}}{(n/k)!k^{n/k}} \binom{n}{m}. \tag{2.4}
\]

See Section 4 for a bijective proof of Equation 2.2a.

## 2.2 Permutations by first letter

In order to compute the expected value of the first letter of a permutation, it is useful to be able to compute the number of permutations that have a given number of \( k \)-cycles and a given first letter.

**Definition 2.3.** Let \( C^{(a)}_k(n, m) \) be the number of permutations \( \pi \in S_n \) that have exactly \( m \) \( k \)-cycles and \( \pi(1) = a \).

The expected value of \( \pi(1) \) with a given number of \( k \)-cycles is

\[
E[\pi(1) \mid \pi \in S_n \text{ has exactly } m \text{ \( k \)-cycles}] = \frac{1}{C_k(n, m)} \sum_{a=1}^{n} a C^{(a)}_k(n, m). \tag{2.5}
\]

The following three lemmas compute \( C^{(a)}_k(n, m) \) from \( C_k(n, m) \).

**Proposition 2.4.** For all \( k > 1 \), the number of permutations in \( S_n \) starting with 1 and having \( m \) \( k \)-cycles is equal to the number of permutations in \( S_{n-1} \) with \( m \) \( k \)-cycles:

\[
C^{(1)}_k(n, m) = C_k(n-1, m). \tag{2.6}
\]

**Proof.** The straightforward bijection from \( \{ \pi \in S_n : \pi(1) = 1 \} \) to \( S_{n-1} \) given by deleting 1 and relabeling preserves the number of \( k \)-cycles for \( k > 1 \).

**Proposition 2.5.** For all \( a, b \geq 2 \), the number of permutations having \( k \)-cycles and starting with \( a \) are the same as the number of those starting with \( b \):

\[
C^{(2)}_k(n, m) = \cdots = C^{(a)}_k(n, m) = \cdots = C^{(b)}_k(n, m) = \cdots = C^{(n)}_k(n, m). \tag{2.7}
\]
Proof. Since the permutations under consideration do not fix 1, conjugation by \((ab)\) is an isomorphism which takes all words starting with \(a\) to words starting with \(b\) without changing the cycle structure.

**Lemma 2.6.** For all \(2 \leq a \leq n\),

\[
C_k^{(a)}(n, m) = \frac{C_k(n, m) - C_k(n - 1, m)}{n - 1}. \tag{2.8}
\]

**Proof.** Since

\[
C_k(n, m) = C_k^{(1)}(n, m) + C_k^{(2)}(n, m) + \cdots + C_k^{(n)}(n, m), \tag{2.9}
\]

using Proposition 2.5 for the last \((n - 1)\) terms, this can be rewritten as

\[
C_k(n, m) = C_k^{(1)}(n, m) + (n - 1)C_k^{(a)}(n, m). \tag{2.10}
\]

Solving for \(C_k^{(a)}(n, m)\) and using the substitution from Proposition 2.4 gives the desired result.

Now, equipped with explicit formulas for \(C_k^{(a)}(n, m)\) and \(C_k(n, m)\), we can compute the expected value of \(\pi(1)\) for \(\pi \in S_n\) with exactly \(m\) \(k\)-cycles.

### 2.3 Expected value of first letter

**Theorem 2.7.** For \(k > 1\), the expected value of the first letter of a permutation \(\pi \in S_n\) with \(m\) \(k\)-cycles is given by

\[
\mathbb{E}[\pi(1) \mid \pi \in S_n \text{ has exactly } m \text{ } k\text{-cycles}] = \frac{n}{2} \left(1 - \frac{C_k(n - 1, m)}{C_k(n, m)}\right) + 1. \tag{2.11}
\]

**Proof.** Using Proposition 2.5 we can consolidate all but the first term of the sum in Equation 2.5

\[
\sum_{a=1}^{n} aC_k^{(a)}(n, m) = C_k^{(1)}(n, m) + \sum_{a=2}^{n} aC_k^{(a)}(n, m) \tag{2.12}
\]

\[
= C_k^{(1)}(n, m) + \sum_{a=2}^{n} aC_k^{(a)}(n, m) \tag{2.13}
\]

\[
= C_k^{(1)}(n, m) + \frac{(n - 1)(n + 2)}{2} C_k^{(n)}(n, m) \tag{2.14}
\]

\[
= C_k(n - 1, m) + \frac{(n - 1)(n + 2)}{2} \left(\frac{C_k(n, m) - C_k(n - 1, m)}{n - 1}\right) \tag{2.15}
\]

\[
= \left(\frac{n}{2} + 1\right)C_k(n, m) - \frac{n}{2}C_k(n - 1, m). \tag{2.16}
\]

Dividing by \(C_k(n, m)\) yields the result.
Corollary 2.8. When $k \nmid n$, $C_k(n, m) = nC_k(n-1, m)$ by Equation 2.2a, so
\[
\mathbb{E}[\pi(1) \mid \pi \in S_n \text{ has exactly } m \text{ } k\text{-cycles}] = \frac{n}{2} \left(1 - \frac{1}{n}\right) + 1 = \frac{n+1}{2}. \quad (2.17)
\]
Together with Theorem 2.4, this theorem and its corollary provides our first formula for the expected value of $\pi(1)$ that performs exponentially better than brute force.

2.4 Identities for counting permutations with given cycle conditions

Both in practical terms (if computing the expected value of $\pi(1)$ by hand or optimizing an algorithm) and in a theoretical sense, the following recurrence is simple and useful.

Lemma 2.9. For $n < mk$ or $m < 0$, $C_k(n, m) = 0$. Otherwise, for all $k, m \geq 1$
\[
mC_k(n, m) = (k-1)! \left(\frac{n}{k}\right) C_k(n-k, m-1). \quad (2.18)
\]
While this can be proven directly by the algebraic manipulation of the identity in Theorem 2.4, a bijective proof has been included here because it is natural and may be of interest.

Proof. Let
\[
\mathcal{C}_k(n, m) = \{\pi \in S_n \mid \pi \text{ has exactly } m \text{ } k\text{-cycles}\}. \quad (2.19)
\]
Then consider the two sets, whose cardinalities match the left- and right-hand sides of the equation above:
\[
X_{n,m,k}^L = \{(\pi, c) \mid \pi \in \mathcal{C}_k(n, m), c \text{ a distinguished } k\text{-cycle of } \pi\}. \quad (2.20)
\]
\[
X_{n,m,k}^R = \{ (\sigma, d) \mid \pi \in \mathcal{C}_k(n-k, m-1), d \text{ an } n\text{-ary necklace of length } k\}. \quad (2.21)
\]
The first set, $X_{n,m,k}^L$, is constructed by taking a permutation in $\mathcal{C}_k(n, m)$ and choosing one of its $m$ $k$-cycles to be distinguished, so $\#X_{n,m,k}^L = mC_k(n, m)$.

In the second set, $X_{n,m,k}^R$, the two parts of the tuple are independent. There are $C_k(n-k, m-1)$ choices for the permutation $\sigma$ and $(k-1)! \binom{n}{k}$ choices for the necklace $d$. Thus $\#X_{n,m,k}^R = (k-1)! \binom{n}{k} C_k(n-k, m-1)$.

Now, consider the map $\varphi : X_{n,m,k}^L \to X_{n,m,k}^R$ which removes the distinguished $k$-cycle and relabels the remaining $n-k$ letters as $\{1, 2, \ldots, n-k\}$, preserving the relative order:
\[
(\pi_1 \pi_2 \cdots \pi_i, \pi_i) \overset{\varphi}{\mapsto} (\pi'_1 \pi'_2 \cdots \pi'_{i-1} \pi'_{i+1} \cdots \pi'_i, \pi_i) \quad (2.22)
\]
where $\pi'_i$ is $\pi_i$ after relabeling.

By construction, $\sigma$ has one fewer $k$-cycle and $k$ fewer letters than $\pi$. 

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The inverse map is similar. To recover $\pi$, increment the letters of $\sigma$ appropriately and add the necklace $d$ back in as the distinguished cycle. Thus $\varphi$ is a bijection and $\#X_{n,m,k}^L = \#X_{n,m,k}^R$.

**Example 2.10.** Suppose $\pi = (423)(61)(75)$ in cycle notation with $(61)$ distinguished. Then

$$\varphi((423)(61)(75), (61)) = ((312)(54), (61))$$

(2.23)

under the bijection $\varphi$, described in the proof of Lemma 2.9.

The recurrence in Lemma 2.9 suggests that understanding $C_k(n,m)$ is related to understanding $C_k(n - km, 0)$, the permutations of $S_{n-km}$ with no $k$-cycles. On the other hand, Corollary 2.2 suggests that the case where $k \mid n$ has some of the most intricate structure. We can, of course, combine these two observations and analyze the case of $C_k(kn, 0)$, which has a particularly simple generating function, which will show up again in a different guise.

**Lemma 2.11.** For $k \geq 2$,

$$\sum_{n=0}^{\infty} \frac{C_k(kn, 0)k^n}{(kn)!} x^n = \frac{\exp(-x)}{1 - kx}.$$  

(2.24)

**Proof.** By substitution of $C_k(kn, 0)$ via the identity in Theorem 2.1,

$$\sum_{n=0}^{\infty} \frac{C_k(kn, 0)k^n}{(kn)!} x^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^i k^i n^i x^n}{i!}$$

(2.25)

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-x)^i}{i!} (nx)^{n-i}$$

(2.26)

$$= \left( \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) \left( \sum_{n=0}^{\infty} (nx)^n \right)$$

(2.27)

$$= \frac{\exp(-x)}{1 - kx}.$$  

(2.28)

This section allowed for the practical computation of the expected value of $\pi(1)$ with a given number of $k$-cycles, but leaves the observation about Figure 1 unexplained. The following section will explain the connection between the expected values of $\pi(1)$ and the facet-derangements of the hypercube.

### 3 Connection with the generalized symmetric group

This section explains the connection between the expected value of $\pi(1)$ given that $\pi$ has exactly $m$ 2-cycles and the facet-derangements of the hypercube,
by telling the more general story of derangements of the generalized symmetric group. Thus it is appropriate to start this section by defining both the generalized symmetric group and its derangements.

### 3.1 Derangements of the generalized symmetric group

**Definition 3.1.** The **generalized symmetric group** \( S(k, n) \) is the wreath product \((\mathbb{Z}/k\mathbb{Z}) \wr S_n\), which in turn is a semidirect product \((\mathbb{Z}/k\mathbb{Z})^n \rtimes S_n\).

A natural way of thinking about the symmetric group \( S_n \) is by considering how the elements act on length-\( n \) sequences by permuting the indices. Informally, we can think about the generalized symmetric group \( S(k, n) \) in an essentially similar way: each element consists of an ordered pair in \((\mathbb{Z}/k\mathbb{Z})^n \rtimes S_n\), where \((\mathbb{Z}/k\mathbb{Z})^n\) gives information about what to add componentwise, and \( S_n \) gives information about how to rearrange afterward.

**Example 3.2.** Consider the generalized permutation \(((1, 3, 0), (23)) \in S(4, 3)\). It acts on the sequence \((0, 1, 1) \in (\mathbb{Z}/2\mathbb{Z})^3\) first by adding element-wise, and then permuting:

\[
((1, 3, 0), (23)) \cdot (0, 1, 1) = (23) \cdot (1 + 0, 3 + 1, 0 + 1) = (23) \cdot (1, 0, 1) = (1, 1, 0).
\]

(3.1)

When \( k = 1 \), the sequence \((\mathbb{Z}/1\mathbb{Z})^n\) is trivially the zero sequence, so \( S(1, n) \cong S_n \). When \( k = 2 \), \( S(2, n) \) is the hyperoctahedral group that we brushed up against in Figure 1: the group of symmetries of the \( n \)-dimensional hypercube. When \( k \geq 3 \), \( S(k, n) \) does not have such an immediate geometric interpretation, but it is precisely the right analog for the expected value of \( \pi(1) \) when \( \pi \) has a given number of \( k \)-cycles.

**Definition 3.3.** A **derangement** or **fixed-point-free element** of the generalized symmetric group is an element \(((x_1, \ldots, x_n), \pi) \in S(k, n)\) such that for all \( i \), either \( \pi(i) \neq i \) or \( x_i \neq 0 \).

That is, when a derangement acts on a sequence in the manner described above, it changes the position or the value of every term in the sequence. When \( k = 1 \) and \( S(1, n) \cong S_n \), this recovers the usual sense of a derangement in \( S_n \): a permutation with no fixed points. In terms of the hyperoctahedral group, \( S(2, n) \), a derangement is a symmetry of the \( n \)-cube that moves each \((n-1)\)-dimensional face.

**Example 3.4.** The element \(((1, 3, 0), (23)) \in S(4, 3)\) is a derangement because it increments the first term and swaps the second and third terms—thus changing the position or value for each term.
The number of derangements of the generalized symmetric group can be described by an explicit sum via the principle of inclusion/exclusion, and it has a particularly elegant exponential generating function.

**Theorem 3.5** \((2)\). For \(k > 1\), the number of derangements of the generalized symmetric group \(S(k,n)\) is

\[
D(k, n) = k^n n! \sum_{i=0}^{n} \frac{(-1)^i}{k^i i!},
\]

which has exponential generating function

\[
\sum_{n=0}^{\infty} \frac{D(k, n)}{n!} x^n = \exp\left(-x\right) \frac{1}{1-kx}.
\]

Notice that this agrees identically with the generating function in Lemma \(2.11\) which is our first hint in explaining the connection between \(k\)-cycles in permutations and fixed points in elements of the generalized symmetric group.

### 3.2 Permutation cycles and derangements

**Lemma 3.6.** For \(k \geq 1\), the number of permutations with \(kn + km\) letters and \(m k\)-cycles is

\[
C_k(n + m, m) = \binom{kn + km}{kn} C_k(kn, 0) \frac{(km)!}{k^m m!}.
\]

**Algebraic proof.** This will proceed by induction on \(m\). The base case is clear when \(m = 0\), so suppose that the lemma is true up to \(m - 1\), that is

\[
C_k(n + m - 1, m - 1) = \frac{(km - k)!}{k^m - 1 (m - 1)!} \binom{kn + km - k}{kn} C_k(kn, 0).
\]

Rearranging Lemma \(2.9\)

\[
C_k(n + m, m) = \frac{(k - 1)!}{m} \binom{k(n + m)}{k} C_k(k(n + m - 1), m - 1)
\]

\[
= \frac{(kn + km)!}{km(kn + km - k)!} C_k(k(n + m - 1), m - 1).
\]

Now, notice there is a \((kn + km - k)!\) term in the numerator of Equation \(3.6\) and the denominator of Equation \(3.8\) so substituting and simplifying yields

\[
C_k(n + m, m) = \frac{(kn + km)!}{k^m m!(kn)!} C_k(kn, 0),
\]

as desired. \(\square\)
**Combinatorial proof.** This lemma lends itself to a combinatorial proof. The left hand side of the equation counts the number of permutations in $S_{kn+km}$ with exactly $m$ $k$-cycles. The right hand side of the equation says that this is the number of ways to choose $kn$ letters in the permutation that will not be in $k$-cycles, and for each of these, there are $C_k(kn,0)$ ways to arrange these such that they have no $k$-cycles. This leaves over $km$ letters, of which there are $(km)!/(k^m m!)$ ways to write them as products of $m$ disjoint $k$-cycles.

The following lemma uses the above identities to establish that the proportion of permutations in the symmetric group $S_{kn}$ with exactly $m$ $k$-cycles is equal to the proportion of elements in the generalized symmetric group $S(k,n)$ with exactly $m$ fixed points.

**Lemma 3.7.** For $k \geq 2$,

$$\frac{C_k(kn,m)}{(kn)!} = \binom{n}{m} \frac{D(k,n-m)}{k^m n!}.$$  

\[(3.10)\]

**Proof.** By solving for $D(k,n-m)$ on the right hand side and substituting $n+m$ for $n$, it is enough to show that the exponential generating function for $D(k,n)$ (as shown in Theorem 3.5) is also the exponential generating function for

$$C_k(kn + km, m) \frac{m! n! k^{n+m}}{(kn+km)!}.$$  

\[(3.11)\]

By the identity in Lemma 3.6

$$\sum_{n=0}^{\infty} C_k(kn + km, m) \frac{m! n! k^{n+m}}{(kn+km)!} \frac{x^n}{n!}$$

\[(3.12)\]

$$= \sum_{n=0}^{\infty} \frac{(km)!}{m! k^m} \binom{kn}{km} \frac{C_k(kn,0)}{(kn+km)!} \frac{m! n! k^{n+m}}{(kn+km)!} \frac{x^n}{n!}$$

\[(3.13)\]

$$= \sum_{n=0}^{\infty} C_k(kn,0) \frac{k^n x^n}{(kn)!}$$

\[(3.14)\]

$$= \frac{\exp(-x)}{1 - kx},$$

\[(3.15)\]

with the final equality being the identity in Lemma 2.11.

**3.3 Expected value of letters of permutations**

We now have the ingredients we need to prove the pattern that we observed in Figure 1 that purported to show a relationship between permutations given number of 2-cycles and derangements of the hyperoctahedral group. These ingredients come together in the following theorem, which establishes the more general relationship between permutations with a given number of $k$-cycles and derangements of the generalized symmetric group, $S(k,n)$. 
Theorem 3.8. The expected value of the first letter of a permutation \( \pi \in S_{kn} \) with exactly \( m \) \( k \)-cycles, where \( k > 1 \) and \( 0 \leq m \leq n \), is

\[
E[\pi(1) \mid \pi \in S_{kn} \text{ has exactly } m \text{ } k\text{-cycles}] = \frac{kn + 1}{2} + \frac{(-1)^{n-m}}{2D(k, n-m)} \tag{3.16}
\]

where \( D(k, n) \) is the number of derangements of the generalized symmetric group \( S(k, n) = (\mathbb{Z}/m\mathbb{Z}) \wr S_n \).

Proof. Inverting the identity in Lemma 3.7 yields

\[
\frac{(kn)!}{n!} \frac{\binom{n}{m}}{C_k(kn,m)} = \frac{1}{D(k, n-m)}.
\tag{3.17}
\]

Multiplying through by \((-1)^{n-m}\) to match the right hand side of Equation 3.16 together with some small manipulations yields

\[
1 - \frac{C_k(kn, m) - (-1)^{n-m} \binom{kn}{m}}{C_k(kn,m)} = \frac{(-1)^{n-m}}{D(k, n-m)}.
\tag{3.18}
\]

Now adding \( kn + 1 \) and dividing by 2 yields

\[
\frac{kn}{2} \left( 1 - \frac{C_k(kn, m) - (-1)^{n-m} \binom{kn}{m}}{knC_k(kn,m)} \right) + 1
\]

\[
= \frac{kn + 1}{2} + \frac{(-1)^{n-m}}{2D(k, n-m)},
\tag{3.19}
\]

which gives the right hand side as desired. Since the numerator on the left hand side is equal to \( knC_k(kn - 1, m) \) by Equation 2.2, the proof then follows from by Theorem 2.7. \( \square \)

With the expected value of the first letter found, we can generalize this one more step to find the expected value of the \( i \)-th letter of these permutations.

Corollary 3.9. The expected value of the \( i \)-th letter of a permutation in \( S_{kn} \) with exactly \( m \) \( k \)-cycles, where \( n \in \mathbb{N} \), \( k > 1 \), \( 1 \leq i \leq kn \), and \( 0 \leq m \leq n \), is

\[
E[\pi(i) \mid \pi \in S_{kn} \text{ has exactly } m \text{ } k\text{-cycles}] = \frac{kn + 1}{2} + \frac{(-1)^{n-m}}{2D(k, n-m)} \frac{kn + 1 - 2i}{kn - 1}.
\]

Proof. Denote by \( N \) the number of permutations in \( S_{kn} \) with \( m \) \( k \)-cycles where 1 is a fixed point; denote by \( M \) the number of permutations in \( S_{kn} \) with \( m \) \( k \)-cycles where \( \pi(1) = a \neq 1 \). Note that while \( N \) and \( M \) implicitly depend on \( m \), \( n \), and \( k \), \( M \) does not depend on \( a \) by Proposition 2.8.

Thus

\[
E[\pi(1) \mid \pi \in S_{kn} \text{ has exactly } m \text{ } k\text{-cycles}] \]

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\[ \frac{1}{N + (kn - 1)M} \left( N + \sum_{a=2}^{kn} aM \right) \]
\[ = \frac{1}{N + (kn - 1)M} \left( N + \left( \frac{kn(kn + 1)}{2} - 1 \right) M \right). \quad (3.20) \]

More generally, if we conjugate with \( (1i) \) then
\[ \mathbb{E}[\pi(i) \mid \pi \in S_{kn} \text{ has exactly } m \text{ } k\text{-cycles}] \]
\[ = \frac{1}{N + (kn - 1)M} \left( N + \sum_{a\neq i} aM \right) \]
\[ = \frac{1}{N + (kn - 1)M} \left( iN + \left( \frac{kn(kn + 1)}{2} - i \right) M \right). \quad (3.21) \]

We can extend the function \( \mathbb{E}[\pi(i) \mid \pi \in S_{kn} \text{ has exactly } m \text{ } k\text{-cycles}] \) to a function \( f(n,k,m,i) \) where \( i \in \mathbb{Q} \) is not necessarily an integer. As can be seen in Equation 3.21, \( f \) is affine function in \( i \). By Theorem 3.8 when \( i = 1 \),
\[ f(n,k,m,1) = \frac{kn + 1}{2} + \frac{(-1)^{n-m}}{2D(k,n-m)}. \]

When \( i = (kn + 1)/2 \) yields
\[ f(n,k,m,(kn + 1)/2) = \frac{kn + 1}{2}. \]

Because \( f(n,k,m,i) \) is affine in \( i \), it is enough to use linear interpolation and extrapolation to compute \( f \) for arbitrary \( i \). This can be done by scaling the \( \frac{(-1)^{n-m}}{2D(k,n-m)} \) term by an affine function of \( i \) which is 1 when \( i = 1 \) and which vanishes when \( i = (kn + 1)/2 \), namely \( \frac{kn + 1 - 2i}{kn - 1} \), as desired. \( \square \)

**Example 3.10.** For \( n = 2 \), \( k = 2 \), and \( m = 0 \) the expected value of the first letter in a permutation in \( S_{2k} = S_4 \) with no \( k = 2\)-cycles is \( \frac{13}{9} \), as shown in Example 1.2. This agrees with Theorem 3.8
\[ \frac{kn + 1}{2} + \frac{(-1)^{n-m}}{2D(k,n-m)} = \frac{4 + 1}{2} + \frac{(-1)^{2-0}}{2D(2,2-0)} = \frac{5}{2} + \frac{1}{10} = \frac{13}{5}, \quad (3.22) \]

since \( D(2,2) = 5 \) as illustrated in Figure 6.

While Theorem 2.7 gave us our first way to efficiently compute the expected value of the first letter of a permutation on \( kn \) letters with a given number of \( k\)-cycles, we can also compute this efficiently with Theorem 3.8 by using the formulas for \( D(k,n) \) in Theorem 3.9. But this is not the only reason that
Figure 2: The $2^22! = 8$ symmetries of a square with fixed sides circled. The square (2-dimensional hypercube) has symmetry group $S(2,2) = (\mathbb{Z}/2\mathbb{Z}) \wr S_2$ and $D(2,2) = 5$ of these symmetries are derangements, meaning that they do not fix any sides.

Theorem 3.8 is of interest; because of the structure of the formula it provides, this theorem suggests other quantitative and qualitative insights.

Recall that when there are no restrictions on a permutation $\pi \in S_{kn}$, the first letter is equally likely to take on any value, so $E[\pi(1) \mid \pi \in S_{kn}] = (kn + 1)/2$. The first insight given by Theorem 3.8 is that the expected value of $\pi(1)$ given some number of $k$ cycles differs from $(kn + 1)/2$ by at most $1/2$, because $D(k,N) \geq 1$ for $k \geq 2$. Secondly, since $D(k,N)$ increases as a function of $N$, the expected value gets closer to $(kn + 1)/2$ as the number of $k$-cycles decreases. Lastly, the numerator of $(-1)^{n-m}$ in the second summand of Equation 3.16 shows that the expected value of the first letter is larger than $(kn + 1)/2$ if and only if $n$ and $m$ have the same parity.

4 A $k$-cycle preserving bijection

Motivated by Equation 2.2a, this section describes a family of bijections, $\phi_k : S_{n-1} \times [n] \to S_n$,

each of which preserves the number of $k$-cycles when $k \nmid n$. Of course, there is no map that preserves the number of $k$-cycles when $k \mid n$. For example, a permutation in $S_n$ consisting entirely of $k$-cycles contains $n/k$ $k$-cycles, while a permutation in $S_{n-1}$ can contain at most $n/k - 1$ $k$-cycles by the pigeonhole principle.

Informally, these maps are defined by writing down a permutation $\sigma \in S_{n-1}$ in canonical cycle notation, incrementing all letters in $\sigma$ that are greater than or equal to $x \in [n]$, inserting $x$ into the rightmost cycle, and then recursively
moving letters into or out of subsequent cycles, whenever a $k$-cycle is turned into a $(k + 1)$-cycle or a $(k - 1)$-cycle is turned into a $k$-cycle.

### 4.1 Example of recursive structure

The definition of the map can look complicated, so it’s worthwhile to start with an example to give some sense of the overarching idea.

**Example 4.1.** This example illustrates how the map $\phi_3$ inserts $I$ into the permutation $(D76)(E)(F32)(G91C)(K54)(LJ8)(MB)(NAH)$ while preserving the number of 3-cycles. The maps $\phi_k$ and $\psi_k$ are the result of moving letters according to the arrows and are applied from right-to-left. (This example uses the convention that $1 < 2 < \cdots < 9 < A < B < \cdots < N$.)

\[
\phi_3((D76)(E3)(F29)(G1)(KC5)(L4J)(M8BA)(NHI),I) = (D76)(E3)(F29)(G1)(KC5)(L4J)(M8BA)(NHI)
\]

\[
\psi_3((D76)(E3)(F29)(G1)(KC5)(L4J)(M8BA)(NHI),I) = ((D76)(E)(F32)(G91C)(K54)(LJ8)(MB)(NAH),I)
\]

Again, it is worth reemphasizing that the following definitions will follow the convention that permutations are written in canonical cycle notation,

\[
\pi = (\underbrace{c_1^{(1)} \cdots c_{\ell_1}^{(1)}}_{c_1^{(1)}}, \ldots \underbrace{c_1^{(\ell_1)} \cdots c_{\ell_1}^{(\ell_1)}}_{c_{\ell_1}^{(1)}}),
\]

where cycle $c_{\ell_i}^{(i)} = (c_{1}^{(i)} \cdots c_{\ell_i}^{(i)})$ has $\ell_i$ letters. This means that the first letter in each cycle, $c_1^{(i)}$, is the largest letter in that cycle, and that the cycles are ordered in increasing order by first letter when read from right-to-left: $c_1^{(i+1)} < c_1^{(i)}$ for all $i$.

### 4.2 Formal definition and properties

**Definition 4.2.** Define $\phi_k : S_{n-1} \times [n] \mapsto S_n$ recursively as follows:

\[
\phi_k(\emptyset, 1) = (1), \quad (4.1)
\]
and for \( n > 1, \pi \in S_{n-1}, \) and \( x \in [n], \)

\[
\phi_k(\pi, x) = \begin{cases} 
\ell_1 = k & \text{if } k \mid n \text{ and for } x > c_1^{(1)} \\
\phi_k(c^{(1)} \cdots c^{(2)} c_1^{(1)} \cdots c_k^{(1)} x) & \ell_1 = k - 1, t > 1 \\
\pi'(c_1^{(1)} x' c_2^{(1)} \cdots c_{k-1}^{(1)} x) & \text{otherwise.}
\end{cases}
\]

Here, \( \phi_k \) depends on the auxiliary function \( \psi_k : S_n \mapsto S_{n-1} \times [n], \)

\[
\psi_k(\pi) = \begin{cases} 
\left( c^{(1)} \cdots c^{(2)}, c_1^{(1)} \right) & \ell_1 = 1 \\
\left( \phi_k(c^{(1)} \cdots c^{(2)}, c_1^{(1)} \cdots c_k^{(1)}), c_1^{(1)} \right) & \ell_1 = k + 1 \\
\left( \pi'(c_1^{(1)} x' c_2^{(1)} \cdots c_{k-1}^{(1)}), c_k^{(1)} \right) & \ell_1 = k, t > 1 \\
\left( c^{(1)} \cdots c^{(2)} c_1^{(1)} \cdots c_{\ell_1-1}^{(1)}, c_{\ell_1}^{(1)} \right) & \text{otherwise,}
\end{cases}
\]

and in both functions, \( (\pi', x') = \psi(c^{(1)} \cdots c^{(2)}). \)

**Note 4.3.** Strictly speaking, \( \phi_k \) and \( \psi_k \) have an additional implicit parameter \( n, \) which indicates the size of permutation that these functions act on. Since the construction of these functions do not depend on \( n, \) this is suppressed in the notation.

The following theorem motivates this map, and together with Lemma 4.7 it implies Equation 2.2a

**Theorem 4.4.** If \( k \mid n, \) the number of \( k \)-cycles of \( \pi \in S_{n-1} \) is equal to the number of \( k \)-cycles in \( \phi_k(\pi, x). \)

**Proof.** By construction, the maps \( \phi_k \) and \( \psi_k \) change the rightmost cycle into a (different) \( k \)-cycle if it was previously a \( k \)-cycle, and they change non-\( k \)-cycles into non-\( k \)-cycles, except for the case where there is one cycle remaining with length \( k - 1 \) (in the case of \( \phi \)) or length \( k \) (in the case of \( \psi \)). These cases can only be achieved when \( k \mid n, \) by the following lemma.

**Lemma 4.5.** The number of letters in \( \pi \) in (recursive) applications of \( \phi_k \) and \( \psi_k \) are of congruent to \( n - 1 \mod k \) and \( n \mod k \), respectively. Therefore, the only time that the input to \( \phi_k \) can be a single cycle of length \( k - 1 \) or the input to \( \psi_k \) can be a single cycle of length \( k \) is when \( n \equiv 0 \pmod{k}. \)
Proof. The proof proceeds by induction on the number of recursive iterations of \( \phi_k \) and \( \psi_k \). The base case is clear: on the first application of a map is always \( \phi_k : S_{n-1} \times [n] \to S_n \), and the input permutation has \( n-1 \) letters by definition.

Now, either we’re finished, or we recurse (Equations 4.2b, 4.2c, 4.3b, or 4.3c), which we look at case-by-case.

Case 1. In Equation 4.2b, the map \( \phi_k \) sets aside \( k \) letters from the input, so the number of letters in the recursive input to \( \phi_k \) is also congruent to \( n-1 \mod k \).

Case 2. In Equation 4.2c, the map \( \phi_k \) sets aside \( k-1 \) letters from the leftmost cycle of the input. Since the number of letters in the original permutation was congruent to \( n-1 \mod k \), the number of letters in the permutation being input to \( \psi_k \) is congruent to \( n \mod k \).

Case 3. In Equation 4.3b, the map \( \psi_k \) sets aside \( k+1 \) letters from the leftmost cycle of the input. Since the number of letters in the original permutation was congruent to \( n \mod k \), the number of letters in the permutation being input to \( \phi_k \) is congruent to \( n-1 \mod k \).

Case 4. In Equation 4.3c, the map \( \psi_k \) sets aside \( k \) letters from the input, so the number of letters in the recursive input to \( \psi_k \) is also congruent to \( n \mod k \).

The following lemma provides a certain “niceness” property of the map, which allows us to analyze it. In particular, all recursive inputs in both \( \phi_k \) and \( \psi_k \) are written in canonical cycle notation.

Lemma 4.6. The output of \( \phi_k \) is in canonical cycle notation.

Proof. Canonical cycle notation is preserved by construction. In particular, \( \phi_k \) moves the first letter in any cycle, and Equation 4.2a guards against inserting a number into a cycle that is bigger than the largest number already in the cycle. Similarly, \( \psi_k \) only moves the first letter in the case of Equation 4.3a, but in this case, the cycle only has one letter, so this is equivalent to deleting the cycle.

4.3 Inverting the bijection

Lemma 4.7. The maps \( \phi_k : S_{n-1} \times [n] \to S_n \) and \( \psi_k : S_n \to S_{n-1} \times [n] \) are inverse to one another.

Proof. To prove this lemma, it suffices to show that \( \psi_k \circ \phi_k = \id \) by induction on the number of cycles of \( \pi \). This will simultaneously prove that \( \phi_k \circ \psi_k = \id \), because \( S_{n-1} \times [n] \) and \( S_n \), both having \( n! \) elements, have the same cardinality.

When \( \pi \) has no cycles, the base case is clear: \( \psi_k(\phi_k(\emptyset, x)) = \psi_k((x)) = (\emptyset, x) \).

Now there are five remaining cases to check, corresponding to each of the cases in the definition of \( \phi_k(\pi, x) \)
Case 1. Assume \( x > c_1^{(1)} \), so that \( \phi_k(\pi, x) \) is evaluated via Equation 4.2a

\[
\psi_k(\phi_k(\pi, x)) = \psi_k(c^{(t)} \cdots c^{(1)}(x)) = (c^{(t)} \cdots c^{(1)}, x) = (\pi, x).
\]

Case 2. Assume \( \ell_1 = k \), so that \( \phi_k(\pi, x) \) is evaluated via Equation 4.2b

\[
\psi_k(\phi_k(\pi, x)) = \psi_k(\phi_k(c^{(t)} \cdots c^{(2)}, c^{(1)}_{2 \cdots (1)})_{\text{length } k}) = (\pi', x') = (\pi, x).
\]

Case 3. Assume \( \ell_1 = k - 1 \) and \( t > 1 \), so that \( \phi_k(\pi, x) \) is evaluated via Equation 4.2c

\[
\psi_k(\phi_k(\pi, x)) = \psi_k(\pi' c^{(1)}_{1 \cdots (1)} x'_{1 \cdots (k-1)})_{\text{length } k+1} (\pi', x') = (\pi, x).
\]

Case 4. Assume that \( x > c_1^{(1)} \) and \( \ell_1 \notin \{k - 1, k\} \), so that \( \phi_k(\pi, x) \) is evaluated via Equation 4.2d

\[
\psi_k(\phi_k(\pi, x)) = \psi_k(c^{(t)} \cdots c^{(2)}(c^{(1)}_{1 \cdots (1)}))_{\text{length } \ell_1} (\pi', x') = (\pi, x).
\]

Case 5. Assume that \( \ell_1 = k - 1 \) and \( t = 1 \), so that \( \phi_k(\pi, x) \) is evaluated via Equation 4.2d

\[
\psi_k(\phi_k(\pi, x)) = \psi_k((c^{(1)}_{1 \cdots (1)} \cdots c^{(1)}_{k-1})) = (c^{(1)}_{1 \cdots (1)}, x) = (\pi, x).
\]
In this section we constructed a recursively-defined map and its inverse to give a bijective proof that $C_k(n, m) = nC_k(n-1, m)$ when $k \nmid n$. This is a novel, reversible algorithm for inserting a letters into a permutation that preserves the number of $k$-cycles whenever possible.

## 5 Further directions

In the introduction, we mentioned Conger’s paper which analyzed how the number of descents of a permutation affects the expected value of the first letter of the permutation. And similarly in the following sections, we looked at how the number of $k$-cycles affects the expected value of the first letter of the permutation. This section will principally look at the obvious generalization: given some permutation statistic $\text{stat}: S_n \rightarrow \mathbb{Z}$, does the map

$$f(n, m) = \mathbb{E}[\pi(i) \mid \pi \in S_n, \text{stat}(\pi) = m]$$  \hspace{0.5cm} (5.1)

have any interesting structure?

But notice that the first letter of a permutation is itself a statistic, so we can play a more general game. Given pairs of statistics $(\text{stat}_1, \text{stat}_2)$, does the map

$$g(n, m) = \mathbb{E}[\text{stat}_1(\pi) \mid \pi \in S_n, \text{stat}_2(\pi) = m]$$  \hspace{0.5cm} (5.2)

have any interesting structure?

### 5.1 FindStat database

The result by Conger gives the expected value of $\pi(1)$ given $\text{des}(\pi)$, and this paper gave the expected value of $\pi(1)$ given the number of $k$-cycles of $\pi$. Of course, it would be interesting to do analogous analysis with other permutations. In particular, the FindStat permutation statistics database \[7\] contains over 370 different permutation statistics, and many of these appear to have some structure with respect to the expected value of the first letter of a permutation.

### 5.2 Mahonian statistics

In particular, the family of Mahonian statistics may be fruitful to investigate. Below, we have given conjectures about two: the major index and the inversion number. Mahonian statistics are maps $\text{mah}: S_n \rightarrow \mathbb{N}_{\geq 0}$ that are equidistributed with the inversion number.\[4\] That is,

$$\#\{w \in S_n : \text{mah}(w) = k\} = \#\{w \in S_n : \text{inv}(w) = k\}.$$ 

Naturally, all Mahonian statistics share the same generating function:

$$\sum_{\sigma \in S_n} x^{\text{mah}(\sigma)} = [n]_q! = \prod_{i=0}^{n-1} \sum_{j=0}^{i} (q^i).$$
Because the expected value of the first letter is given by the weighted sum of the permutations with \( \text{mah}(w) = k \) divided by the number of such permutations, \( \mathbb{E}[\pi(1) | \pi \in S_n, \text{mah}(\pi) = k] \) has a denominator that is (a factor of) \( M(n,k) \), the number of permutations of \( w \in S_n \) such that \( \text{inv}(w) = k \). For fixed \( k \), these satisfy a degree \( k \) polynomial for all \( n > k \). Notably, in the cases of the major index and the inversion number, the numerators appear to satisfy degree \( k \) and degree \( k - 1 \) polynomials respectively.

**Conjecture 5.1.** For fixed \( k \) and \( n > k \), the expected value of the first letter of a permutation with a given number of inversions satisfies a rational function in \( n \) given by

\[
\mathbb{E}[\pi(1) | \pi \in S_n, \text{inv}(\pi) = k] = \frac{M(n+1,k)}{M(n,k)},
\]

where \( M(n,k) \), as above, is the number of permutations \( w \in S_n \) such that \( \text{inv}(w) = k \).

**Conjecture 5.2.** For fixed \( k > 0 \) and \( n \geq k \), \( \mathbb{E}[\pi(1) | \pi \in S_n, \text{maj}(\pi) = k] \) satisfies a rational function in \( n \) that is \( 1/(k+1) \) times the quotient of a monic degree-\( (k+1) \) polynomial by a monic degree-\( k \) polynomial. Specifically,

\[
\mathbb{E}[\pi(1) | \pi \in S_n, \text{maj}(\pi) = 1] = \frac{1}{2} \left( \frac{n^2 + n - 2}{n - 1} \right), \quad (5.3)
\]

\[
\mathbb{E}[\pi(1) | \pi \in S_n, \text{maj}(\pi) = 2] = \frac{1}{3} \left( \frac{n^3 - n - 6}{n^2 - n - 2} \right), \quad (5.4)
\]

\[
\mathbb{E}[\pi(1) | \pi \in S_n, \text{maj}(\pi) = 3] = \frac{1}{4} \left( \frac{n^4 + 6n^3 - 13n^2 - 18n}{n^3 - 7n} \right), \text{ and} \quad (5.5)
\]

\[
\mathbb{E}[\pi(1) | \pi \in S_n, \text{maj}(\pi) = 4] = \frac{1}{5} \left( \frac{n^5 + 20n^4 - 45n^3 - 80n^2 - 16n}{n^4 + 2n^3 - 13n^2 - 14n} \right). \quad (5.6)
\]

Note that the denominator is given by an integer multiple of \( M(n,k) \), a degree \( k \) polynomial.

### 5.3 An elusive bijection

Let \( F_k(n,m) \) be the number of elements of the generalized symmetric group \( S(k,n) = (\mathbb{Z}/k\mathbb{Z}) \wr S_n \) with \( m \) fixed points, and recall that \( C_k(n,m) \) is the number of elements of \( S_{km} \) with \( m \) \( k \)-cycles. Then for each pair of nonnegative integers \((\alpha, \beta) \) with \( \alpha, \beta \leq n \), then as Lemma 3.7 suggests, there exists a bijection of sets

\[
C_k(n,\alpha) \times F_k(n,\beta) \rightarrow C_k(n,\beta) \times F_k(n,\alpha). \quad (5.7)
\]

This bijection has proven to be elusive to construct outside of the special cases where \( n = 1 \) or \( k = 1 \). Note that, the map cannot be a group automorphism of
$S_{kn} \times S(k, n)$, because the identity of this group is in $C_k(n, 0) \times F_k(n, n)$, so it cannot be preserved under this map.

It would be especially interesting if there’s a way to use the embedding of $(\mathbb{Z}/k\mathbb{Z}) \wr S_n$ into $S_{kn}$ as the centralizer of an element that is the product of $n$ disjoint $k$ cycles.

6 Acknowledgments

A special thanks to my advisor, Sami Assaf, for sharing the spark that she found for this questions in a remark by Jim Pitman, and for her patient guidance. This paper benefitted from the feedback from my colleague, Sam Armon and his generosity, kindness, and sharp eye. It is unlikely that this paper would have been written if not for the On-Line Encyclopedia of Integer Sequences, which gave a several crucial hints, especially around the pattern in Figure 1.

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