Multiparticle d-level GHZ bases associated with generalized braid matrices

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Abstract – We investigate the generalized braid relation for an arbitrary multipartite d-level system and its application to quantum entanglement. By means of finite-dimensional representations of quantum plane algebra, a set of \(d^N \times d^N\) unitary matrix representations satisfying the generalized braid relation can be constructed. Such generalized braid matrices can entangle N-partite d-level quantum states. Applying the generalized braid matrices on the standard basis of product states, one can obtain a set of maximally entangled bases. Further study shows that such entangled states can be viewed as the N-partite d-level Greenberger-Horne-Zeilinger (GHZ) states.

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Introduction. – One of the most prominent properties of quantum states is quantum entanglement, which has motivated much of work in quantum information theory [1]. For instance, quantum entangled channels are essential in teleporting an unknown quantum state [2] and in sharing a secret key for cryptography [3]. For qubits, multipartite entangled states have been studied intensively in quantum information processing including quantum nonlocality [4], one-way quantum computation [5] and quantum error correcting codes [6–8], etc. Recently, investigations on these fields have been generalized to high-dimensional Hilbert spaces due to the enhanced security offered in quantum cryptography [8]. One of the most important multipartite d-level (qudit) entangled states is the GHZ state [9–12], since it is the maximally entangled multipartite state.

In mathematical physics, the Yang-Baxter equation (YBE) is a fundamental tool to deal with quantum integrable models and statistical models [13,14]. As is well known, the parameter-dependent solution to YBE denoted by \(\hat{R}(x)\) satisfies
\[
\hat{R}_{12}(x)\hat{R}_{23}(xy)\hat{R}_{12}(y) = \hat{R}_{23}(y)\hat{R}_{12}(xy)\hat{R}_{23}(x),
\]
where \(\hat{R}_{12}(x) \equiv \hat{R}(x) \otimes I\) and \(\hat{R}_{23}(x) \equiv I \otimes \hat{R}(x)\) with \(I\) being the identity matrix. The parameter-independent asymptotic form of \(\hat{R}(x)\) denoted by \(S\) obeys the braid relation
\[
S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23},
\]
where \(S_{12} \equiv S \otimes I\) and \(S_{23} \equiv I \otimes S\). Recently, the unitary \(\hat{R}(\theta)\) matrix as well as the unitary braided matrix \(S\) have been investigated in quantum entanglement theory. Applying the braid operator \(S\) on the bipartite product state \(|k, l\rangle \equiv |k\rangle \otimes |l\rangle\), one gets a set of well-known maximal entangled states (i.e., the Bell basis) [15–17]. This provides a novel way to study quantum entanglement based on the theory of braiding operators, as well as YBE [18–24]. These studies include the qubit case [15] and the qutrit case [23,24]. In refs. [25,26], the authors generalized the concept of braid group relation and YBE to multi-qubit systems, which are termed generalized YBE (gYBE). Very recently, Hastings et al. studied the Gaussian representation based on the \(p\)-group with odd prime \(p\) [27] and applied the Gaussian representation to describe the braiding
statistical behavior of metaplectic anyons [28,29]. Thus, it is natural to ask whether the YBE approach can be generalized to an arbitrary multipartite $d$-level case, or the so-called $N$-qubit case, and whether one can get useful $N$-partite $d$-level entangled states via such approach.

Following refs. [25,26], the aim of this paper is to generalize the Yang-Baxter approach to multipartite $d$-level systems. We also explore the interesting applications of the generalization in the field of quantum entanglement. To achieve this, we first briefly review the theory of finite-dimensional representations of quantum plane algebra (QPA) [30,31] and then construct $N$-body $d$-level braid matrices by means of QPA, which provides the main mathematical tools for this work. We show that the $d$-level GHZ states for a $N$-body system can be obtained via unitary braid transformations, and the cases of $d = 2$ and $d = 3$ are discussed in detail as examples.

**Solution to the generalized braid matrix.** We begin with parameter-independent YBE (i.e., braid relation). Let $S_1^{(d)},...,S_N^{(d)}$ be the $N$-body $d$-level braid matrix; the following so-called generalized braid relation [26] is satisfied:

$$S_1^{(d)}S_2^{(d)}...S_N^{(d)} = S_N^{(d)}S_{N-1}^{(d)}...S_2^{(d)}S_1^{(d)}.$$  

(3)

When $N = 2$, the generalized braid relation degenerates to its ordinary form (i.e., eq. (2)). When $N = 3$, the generalized $d$-level three-partite braid relation can be written as $S_{123}S_{234}S_{123} = S_{234}S_{123}S_{234}$. Actually, the unitary generalized braid matrix $S_{1,...,N}$ is a $N$-qubit quantum gate acting on the tensor product Hilbert space $V_{d_1}^{(d)} \otimes V_{d_2}^{(d)} \otimes \cdots \otimes V_{d_N}^{(d)}$, where $V_{d_i}$ is the $i$-th vector space of dimension $d$. We show in the following that the multipartite $d$-level braid matrix can be constructed by resorting to QPA.

**Matrix representations of QPA.** In this subsection, we review some basic results about the finite-dimensional representations of QPA which provides us a useful mathematical tool to describe a qudit system. Then the solutions to the $N$-body $d$-level braid relation can therefore be constructed. The so-called QPA generated by $\{q, X, Z\}$, is defined by the following relation:

$$XZ = qZX,$$  

(4)

here $q$ is a complex number. It is well known that the associative algebra generated by $X$ and $Z$ possesses a $d$-dimensional irreducible representation only when $q^d = 1$. In this paper, we take $q$ as a primitive $d$-th root of unity (i.e., $q \equiv q_{d} \equiv e^{2\pi i/d}$). Such special case has been studied by Weyl [32] and Schwinger [33]. Obviously, eq. (4) implies $X^nZ^n = q^{mn}Z^nX^n$ for all integers $m$ and $n$.

If one takes $\{|k\rangle; k = 0, 1, \ldots, d-1\}$ as an orthogonal basis for one qudit Hilbert space $H = V_{d}^{(d)}$, the operators $X$ and $Z$ possess the corresponding realizations $X = \sum_{k=0}^{d-1} |k \oplus 1\rangle\langle k|$ and $Z = \sum_{k=0}^{d-1} q^k|k\rangle\langle k|$ and the matrix forms read

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$  

and

$$Z = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & q & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{d-2} & 0 \\ 0 & 0 & \cdots & 0 & q^{d-1} \end{pmatrix}.$$  

(5)

Here and after we adopt a cyclic representation, which implies the relation $|k \oplus l\rangle \equiv |(k - l) \mod d\rangle$. Acting $X^i$ and $Z^i$ on the state $|k\rangle$, one obtains $X^i|k\rangle = |k \oplus i\rangle$, $Z^i|k\rangle = q^k|k\rangle$. The results imply that the operators $X$ and $Z$ satisfy the relation $X^d = Z^d = I$ (I is the identity matrix). Obviously, when $d = 2$, $q = -1$, the operators $X$ and $Z$ can be identified as Pauli matrices $\sigma_x$ and $\sigma_z$, respectively. From this viewpoint, the operators $X$, $Z$ and $F$ are often used in high-dimensional quantum error-correcting codes [35].

**Matrix representations of N-partite d-level braid algebra.** In this subsection, we present the $N$-partite $d$-level braid matrix based on QPA. From QPA, we first obtain the generalized $M$-algebra. As is well known, $M$-algebra (or extra-special 2-group [26]) obeying the relations $M^2 = -I$ and $M_{12}M_{23} = -M_{23}M_{12}$, plays an important role in the theory of YBE [15]. By means of $M$-algebra, the braid matrix and $R(z)$ can be constructed, and these matrix representations are applied to the studies of quantum entanglement and Berry phase. Now we show that the $M$-algebra can be generalized to the $N$-partite $d$-level system, and can be used to construct the generalized $M$-matrices (or extra-special d-group [27,28]).

To generalize the $M$-matrix, we introduce the matrices $A = ZX = \sum_{k=0}^{d-1} q^k|k \oplus 1\rangle\langle k|$ and $B = X$ which satisfy the algebraic relations $A^d = (-1)^{d-1}I$, $B^d = I$ and $AB = q^{-1}BA$. Then the generalized $M$-matrix can be obtained in terms of matrices $A$ and $B$ with the following relation:

$$M^{(d)} = A \otimes B \otimes \cdots \otimes B,$$  

(6)

where $M^{(d)}$ is the $d$-level $M$-matrix, which obeys the following algebraic relations:

$$[M^{(d)}]^d = (-1)^{d-1}I,$$

$$M_{1}^{(d)}M_{2}^{(d)}...M_{N+1}^{(d)} = qM_{2}^{(d)}...M_{N+1}^{(d)}M_{1}^{(d)},$$  

(7)

Here $M_{1}^{(d)}...N$ stands for $M^{(d)}$ acting on the tensor product Hilbert space $H = V_{d}^{(d)} \otimes V_{d}^{(d)} \otimes \cdots \otimes V_{d}^{(d)}$. 

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With this generalized M-matrix, one can construct a corresponding generalized braid matrix, which is denoted by $S^{(d)}$. Its specific form reads

$$S^{(d)} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{ik(\pi/d)} \left[ M^{(d)} \right]^{k},$$

with the parameter $\omega = e^{i\pi/d}$ with $d$ being even, and $\omega = e^{i2\pi/d}$ with $d$ being odd. The braid matrix $S^{(d)}$ is a $dN \times dN$ matrix acting on the tensor product space $\mathcal{H}_d^N$. Direct computation shows that the generalized braid matrix is unitary (i.e., $[S^{(d)}]^{\dagger} = [S^{(d)}]^{-1}$). Actually, some special cases for $d$ and $N$ have been discussed in detail in refs. [15,20,23,24,27]. When $d = N = 2$, one can calculate the so-called “eight-vertex” braid matrix, which can be viewed as localized representation of the Ising anyon (i.e. the Majorana fermion) [29] and can be used in the topological quantum computation theory [36].

For the case of odd prime $d \geq 3$ and $N = 2$, substituting $M^{(d)} = \omega^{-1}M^{(d)}$ into eq. (8), the braid matrix $S^{(d)}$ can be recast as $S^{(d)} = d^{-1/2} \sum_{k=0}^{d-1} \omega^k \left[ M^{(d)} \right]^{k}$. Such representation can be viewed as localization of the Gaussian representation, which is used in the metaplectic anyons theory [28,29,37]. It is worth mentioning that the N-body braid matrices in eq. (8) are different from the braid matrices in refs. [38,39] since such N-body braid matrices cannot be factorized. That is to say the braid matrix in eq. (8) cannot be rewritten as direct product of local operators.

**Construction of N-partite d-level GHZ bases.**

In this section, we demonstrate that the generalized GHZ states can be constructed by resorting to the generalized braid matrix (i.e., eq. (8)). Following Ge et al., one can view the generalized braid matrix as quantum gate acting on the tensor product Hilbert space $\mathcal{H}_d^N$. In other words, applying the generalized braid matrix on the standard basis, we are able to get a set of entangled states.

For instance, if $d = N = 2$, the Bell basis can be obtained. For the general case, one can achieve N-partite d-level entangled states.

The standard basis for the N-partite d-level tensor product space reads $\mathcal{B}_{sta} = \{ |k_1, k_2, \cdots, k_N \rangle; k_i \in \{0, 1, \cdots, d-1\} \}$. Acting $S^{(d)}$ on the standard basis state $|k_1, k_2, \cdots, k_N \rangle$, one easily gets

$$|\psi^{(d)}_{k_1, k_2, \cdots, k_N} \rangle = S^{(d)} |k_1, k_2, \cdots, k_N \rangle = \begin{cases} \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \alpha_i |k_1 \oplus i, k_2 \oplus i, \cdots, k_N \oplus i \rangle; & \text{odd } d, \\ \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \beta_i |k_1 \oplus i, k_2 \oplus i, \cdots, k_N \oplus i \rangle; & \text{even } d, \end{cases}$$

where $\alpha_i = \omega^{(2k_1+i+1)/2}$ and $\beta_i = \omega^{2k_1i}$. Obviously, the state $|\psi^{(d)}_{k_1, k_2, \cdots, k_N} \rangle$ cannot be written as a product state $|\psi^{(d)}_1 \rangle \otimes |\psi^{(d)}_2 \rangle \otimes \cdots \otimes |\psi^{(d)}_N \rangle$, thus we can say that the new basis $\mathcal{B}_{ent} = \{ |\psi^{(d)}_{k_1, k_2, \cdots, k_N} \rangle; k_i \in \{0, 1, \cdots, d-1\} \}$ is an entangled basis. We next utilize an entanglement measure to test the degree of the entangled states. There are lots of entanglement measures for the multipartite system presented, such as genuine-multipartite-entanglement (GME) concurrence [40], n-tangle [41] and so on. Here we adopt Scott’s Q-measure [42]. Such entanglement measure was introduced to determine quantum entanglement for a multi-qudit state $|\psi \rangle$, which is a vector in the tensor product space $\mathcal{H}_d^N$. The so-called Q-measure reads

$$Q^m_n(\psi) \equiv \frac{d^m}{d^{m-1} - 1} \left( 1 - \frac{n!(N-m)!}{N!} \sum_{|s| = m} Tr \rho^s_s \right),$$

where $m = 1, 2, \cdots, [N/2]$, $s \subset \{1, 2, \cdots, N\}$ and $\rho_s = Tr_s |\psi \rangle \langle \psi |$. The so-called Q-measure can be expressed as $Q^m_n(\psi^{(d)}_{k_1, k_2, \cdots, k_N})$ as is follows:

$$Q^m_n(\psi^{(d)}_{k_1, k_2, \cdots, k_N}) = 1 - \frac{d^m - 1}{d^m - 1}.$$
If we let $N = 2$, the specific form of the braid matrix $S^{(2)}$ is of the form

$$S^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$ 

The above braid matrix is the so-called “eight-vertex” braid matrix, which connects the standard basis $\{|i,j\rangle; i, j = 0, 1\}$ with the Bell basis [15,17]. Acting the $N$-partite two-level braid matrix $S^{(2)}$ on the standard basis, we can get a set of entangled states (i.e. GHZ basis) having the same degree of entanglement. For example, acting $S^{(2)}$ on the product state $|00\cdots 0\rangle$, the standard GHZ state for the $N$-qubit system is thus found,

$$|\psi^{(2)}_{00\cdots 0}\rangle = S^{(2)}|00\cdots 0\rangle = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle).$$

Our result for this special case is consistent with the ones obtained in ref. [25] by Ge et al. and in ref. [26] by Rowell et al.

The case of $N$ qutrits. When $d = 3$, according to eq. (5), the generators $X$ and $Z$ for $d = 3$ can be obtained as follows:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \quad (13)$$

By virtue of $X$ and $Z$, it is not difficult to find matrices $A$ and $B$, which satisfy the relations $A^{3} = B^{3} = I$ and $BA = \omega AB$. The action of $A$ and $B$ on the standard basis of a qutrit system gives $A|i\rangle = \omega^{-1}|i \oplus 1\rangle$ and $B|j\rangle = |j \oplus 1\rangle$. The $N$-level $M$-matrix is therefore $M^{(3)} = A \otimes B \otimes \cdots \otimes B$, which fulfills the following algebraic relations:

$$[M^{(3)}]^{3} = 1, \quad M^{(3)}_{1 \rightarrow N}M^{(3)}_{2 \rightarrow N+1} = \omega M^{(3)}_{2 \rightarrow N+1}M^{(3)}_{1 \rightarrow N}.$$ 

Then one can acquire the three-level generalized BGR, $S^{(3)} = \frac{1}{\sqrt{3}}(I + \omega^{2}M^{(3)} + [M^{(3)}]^{2})$ with the generalized braid relation satisfied, $S^{(3)}_{1 \rightarrow N}S^{(3)}_{2 \rightarrow N+1}S^{(3)}_{1 \rightarrow N} = S^{(3)}_{2 \rightarrow N+1}S^{(3)}_{1 \rightarrow N+1}S^{(3)}_{2 \rightarrow N+1}$. We would like to mention that one special example of the $N$-qutrit case has been studied in detail in refs. [23,24] with $N = 2$. Acting $S^{(3)}$ on the standard basis of $N$ qutrits, it is easy to obtain a set of entangled states for tensor product space $S^{N}_{3}$, which are the eigenvalues and eigenvectors of $S^{(3)}$. An example of this is to act the three-level $N$-body braid matrix ($S^{(3)}$) on the product state $|00\cdots 0\rangle$, and the following three-level GHZ-like state is found:

$$|\psi^{(3)}_{00\cdots 0}\rangle = \frac{1}{\sqrt{3}}(|00\cdots 0\rangle + |11\cdots 1\rangle + \omega|22\cdots 2\rangle). \quad (14)$$

And then by applying a local unitary operator $U = u \otimes u \otimes \cdots \otimes u$ with $u = |0\rangle\langle 0| + |1\rangle\langle 1| + \omega^{-1/2}|2\rangle\langle 2|$, one can verify that the phase factor $\omega$ in eq. (14) will vanish, and thus the standard $N$-partite $d$-level GHZ state can be obtained.

Summary and discussion. – In this paper, we have investigated the generalized braid relation ($N$-body $d$-level braid relation) and its applications to quantum entanglement. By means of finite-dimensional matrix representations of QPA, a set of unitary matrix representations for the generalized braid relation can be constructed. We have shown that the generalized braid matrices are unitary, and such braid matrices can be viewed as quantum gates for $N$-body $d$-level systems. The action of the generalized braid matrix on the standard basis results in a set of entangled basis. The detailed calculations show that all the quantum states are $N$-partite $d$-level GHZ-type states.

Let us make some discussions to end this paper. i) In fact, we can always associate a Hamiltonian with the generalized unitary braid matrix. The unitary generalized braid matrix $S^{(d)}$ is of the form $S^{(d)} = \sum_{k} e^{i\phi_{k}}|u_{k}\rangle\langle u_{k}|$, where $e^{i\phi_{k}}$’s and $|u_{k}\rangle$’s are the eigenvalues and eigenvectors of $S^{(d)}$. Then the corresponding Hamiltonian operator reads $H^{(d)} = -i \log S^{(d)} = \sum_{k} \phi_{k}|u_{k}\rangle\langle u_{k}|$. This allows us to study braid transformation in a special physical system (such as NMR system). ii) The generalized unitary braid matrices in this paper are spectral parameter independent and time independent. The braid matrices cannot be used to describe a parameter-dependent entangled basis and its dynamical properties. An exploration in the more complicated case with parameter dependence is still an open problem. As is known, the Yang-Baxter equation can be viewed as spectral parameter-dependent braid relation. Via the Yang-Baxterization approach, it is possible to generalize the parameter-independent braid matrix to the parameter-dependent $N$-partite $d$-level Yang-Baxter $\hat{R}$-matrix such that one can study the relation between $N$-partite $d$-level quantum entanglement and generalized YBE, and this is under investigation.

**REFERENCES**

[1] Nielsen M. A. and Chuang I. L., Quantum Computation and Quantum Information (Cambridge University Press) 2000.

[2] Bennett C. H., Brassard G., Crépeau C., Jozsa R., Peres A. and Wootters W. K., Phys. Rev. Lett., 70 (1993) 1895.

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[3] Bennett C. H., Brassard G. and Ekert A. K., Sci. Am., **267** (1992) 50.

[4] Greenberger D. M., Horne M. A., Shimony A. and Zeilinger A., *Am. J. Phys.*, **58** (1990) 1131.

[5] Raussendorf R. and Briegel H. J., *Phys. Rev. Lett.*, **86** (2001) 5188.

[6] Calderbank A. R. and Shor P. W., *Phys. Rev. A*, **54** (1996) 1098.

[7] Brauinstein S. L., *Phys. Rev. Lett.*, **80** (1998) 4084.

[8] Cerf N. J., Bourreannane M., Karlsson A. and Gisin N., *Phys. Rev. Lett.*, **88** (2002) 127902.

[9] Cabello A., *Phys. Rev. A*, **63** (2001) 022104.

[10] Cheong Y. W., Lee S.-W., Lee J. and Lee H.-W., *Phys. Rev. A*, **76** (2007) 042314.

[11] Cerf N. J., Massar S. and Pironio S., *Phys. Rev. Lett.*, **89** (2002) 080402.

[12] Lee J., Lee S.-W. and Kim M. S., *Phys. Rev. A*, **73** (2006) 032316.

[13] Yang C. N., *Phys. Rev. Lett.*, **19** (1967) 1312.

[14] Baxter R. J., *Ann. Phys. (N.Y.)*, **70** (1972) 193.

[15] Chen J.-L., Xue K. and Ge M.-L., *Phys. Rev. A*, **76** (2007) 042324.

[16] Xue K. and Ge M.-L., *Int. J. Mod. Phys. B*, **26** (2012) 1243007.

[17] Kauffman L. H. and Lomonaco S. J., *New J. Phys.*, **6** (2004) 134.

[18] Chakraborti A., Chakraborti A. and Hidalgo E. G., *J. Math. Phys.*, **54** (2013) 013517.

[19] Ho C.-L., Solomon A. I. and On C.-H., *EPL*, **92** (2010) 30002.

[20] Galindo C. and Rowell E. C., *J. Math. Phys.*, **55** (2014) 061702.

[21] Wang Z., Franko J. M. and Rowell E. C., *J. Knot Theory Ramificat.*, **15** (2006) 413.

[22] Wang G., Xue K., Sun C. and Du G., *Quantum Inf. Process.*, **11** (2012) 1775.

[23] Wang G., Xue K., Sun C., Zhou C., Hu T. and Wang Q., *Quantum Inf. Process.*, **9** (2009) 699.

[24] Rowell E. and Wang Z., *Commun. Math. Phys.*, **311** (2012) 595.

[25] Zhan Y. and Ge M.-L., *Quantum Inf. Process.*, **6** (2007) 363.

[26] Rowell E. C., Zhang Y., Wu Y. S. and Ge M. L., *Quantum Inf. Comput.*, **10** (2010) 685.

[27] Jones V. F. R., *Commun. Math. Phys.*, **125** (1989) 459.

[28] Hastings M., Nayak C. and Wang Z., *Phys. Rev. B*, **87** (2013) 165421.

[29] Hastings Matthew B., Nayak Chetan and Wang Zhenghan, *Commun. Math. Phys.*, **330** (2014) 45.

[30] Zhou D., Zeng B., Xu Z. and Sun C., *Phys. Rev. A*, **68** (2003) 062303.

[31] Ge Mo-Lin, Liu Xu-Feng and Sun Chang-Pu, *J. Phys. A: Math. Gen.*, **25** (1992) 2907.

[32] Weyl H., *Theory of Groups and Quantum Mechanics* (E. P. Dutton Co., New York) 1932.

[33] Schwinger J., *Proc. Natl. Acad. Sci. U.S.A.*, **46** (1960) 570.

[34] Thas K., *EPL*, **86** (2009) 60005.

[35] Gottesman D., Kitaev A. and Preskill J., *Phys. Rev. A*, **64** (2001) 012310.

[36] Hu S.-W., Xue K. and Ge M.-L., *Phys. Rev. A*, **78** (2008) 022319.

[37] Cobanera E. and Ortiz G., *Phys. Rev. A*, **89** (2014) 012328.

[38] Yu L.-W., Zhao Q. and Ge M.-L., *Ann. Phys. (N.Y.)*, **348** (2013) 106.

[39] Sun C., Xue K., Wang G., Zhou C. and Du G., *Quantum Inf. Process.*, **11** (2012) 385.

[40] Chen Z.-H., Ma Z.-H., Chen J.-L. and Severini S., *Phys. Rev. A*, **85** (2012) 062320.

[41] Wong A. and Christensen N., *Phys. Rev. A*, **63** (2001) 044301.

[42] Scott A. J., *Phys. Rev. A*, **69** (2004) 052330.

[43] Klimek A., Sych D., Sanchez-Soto L. and Leuchs G., *Phys. Rev. A*, **79** (2009) 052101.