Self-calibration for self-consistent tomography

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Abstract. Self-calibrating states are introduced as quantum resources for the simultaneous reconstruction of a quantum state and measurement device. Simple criteria for a state being self-calibrating are formulated. Theory applies to realistic state estimation scenarios, where the detection model is nonlinear in unknown parameters of the measuring device. Examples discussed include time multiplexed detection with on/off detectors and quantum homodyne tomography. The thermally noised single-photon states and Gaussian states are self-calibrating for these schemes. For homodyne tomography, statistical tests applied to raw measured data are sufficient for judging whether or not the signal state is self-calibrating without any \textit{a priori} information.

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Quantum state and process tomographies are recognized as standard tools of quantum diagnostics [1]. Many applications of sophisticated detection techniques and, in particular, the successful implementation of quantum information protocols depend on our ability to verify the quantum state preparation, check the results of quantum computations or transmissions of quantum states through quantum communication channels and so on. For all those tasks, one takes for granted that the measuring apparatus of the tomography setup is well known and calibrated. This means that the measuring device is believed to give a priori known specific responses to known signals assuming that, for example, the efficiency, dark count rate of photodetectors, overall efficiency of the measurement setup, mismatch, etc are known. However, in practice, determination of those parameters is rather challenging and non-trivial, and in this sense tomographic schemes are not fully consistent. The problem of having efficient, precise and simple detector calibration methods, particularly in the realm of quantum physics, has been attracting increasing attention [2–7]. The problem becomes even more intriguing when parameters of the signal or the measurement apparatus are likely to change in the course of the experiment run, as a consequence of, for example, heating, instabilities, stochastic fluctuations, etc. Taking into account the calibration stage, it is always possible to consider tomography as the process of updating information about signal state and the measurement apparatus starting from some initial knowledge.

In other words, any generic tomography scheme deals simultaneously with some prior (initially available) information and information registered in the course of the measurement. Depending on the setup, the prior information might be distributed between the measured system and the measurement apparatus. The self-consistent reconstruction procedure enables one to reconstruct the system and to calibrate the experimental setup at the same time.

Here we will show that for any particular measurement setup, there is a certain class of ‘self-calibrating’ states allowing one to combine the calibration and reconstruction stages. In this sense, simultaneous tomography of the signal and the apparatus is possible. We demonstrate that for the widely used reconstruction schemes such as quantum homodyne tomography [8, 9] and on/off detection, where the detectors are able to distinguish between the presence and absence of the signal only, this is manifested by two possible outcomes: a ‘click’ or ‘no click’ [10, 12, 13]; Gaussian squeezed states may serve as self-calibration states.
2. The concept of self-calibration

The concept of self-calibration follows naturally from the description of quantum measurements. Let us denote by $\vec{\eta}$ a vector of unknown parameters describing the experimenter’s lack of knowledge about the measurement setup with $M$ outcomes, described by elements of the positive-values operator measure (POVM), $\{A_j(\vec{\eta})\}$, $j = 1, \ldots, M$. Similarly, let us denote by $\vec{\gamma}$ the vector of unknown parameters describing the experimenter’s lack of knowledge about the set of $N$ preparations $\{\rho_k(\vec{\gamma})\}$, $k = 1, \ldots, N$, of the system subject to the measurement. States $\{\rho_k(\vec{\gamma})\}$ will be called self-calibrating states for a given measurement $\{A_j(\vec{\eta})\}$ if the detection probabilities

$$p_{jk} = \text{Tr} \left[ A_j(\vec{\eta}) \rho_k(\vec{\gamma}) \right]$$

(1)

determine uniquely both the sets of parameters $\vec{\eta}$ and $\vec{\gamma}$. In this way, we obtain complete knowledge about the measurement setup, $\{A_j\}$, and system preparations, $\{\rho_k\}$. Note that standard state tomography and measurement tomography are just two special cases of the self-calibration scenario. Provided that complete knowledge about the measurement is available, $\text{dim}(\vec{\eta}) = 0$. So, the measurement happens to be informationally complete; any quantum state can be reconstructed. All quantum states are therefore self-calibrating according to this definition. At the opposite extreme consider the situation when the measurement has been made with a completely unknown apparatus $\text{dim}(\vec{\eta}) = M \text{dim}(A)^2$. In that case complete information about a set of preparations is required to reconstruct the measurement. Loosely speaking, in the self-calibrating setup, greater ignorance about the measurement, $\text{dim}(\vec{\eta})$, can be compensated for by less ignorance about the system, $\text{dim}(\vec{\gamma})$, and vice versa.

As follows from the definition, for any measurement, $\{A_j(\vec{\eta})\}$, a trivial class of self-calibrating states can always be found. Of course, one is mainly interested in those tomography schemes for which a wide non-trivial class of self-calibrating states exists and can be easily generated with the present-day technology. Relevant examples will be given below. To keep notation simple, in the following we consider measurements on a single state, $\rho(\vec{\gamma})$. Let us first illustrate the self-calibrating strategy by considering a single outcome of a qubit measurement. Using Pauli spin representation, the measured state is represented by Stokes parameters $\rho = \sum_i x_i \sigma_i$, where $\sigma_i$ are Pauli spin operators. Similarly, the measurement can be described in the same basis, $A = \sum_i y_i \sigma_i$. Since the detection probability

$$p = \sum_i x_i y_i$$

(2)

displays symmetry with respect to the state and measurement parameters, the preparation and measurement are equivalent. Assuming the experimenter’s ignorance about one of the measurement parameters, say $y_m$, prior knowledge about the corresponding state parameter $x_m$ can be used to calibrate $y_m$ and vice versa. Indeed, applying an imperfectly calibrated measurement $y_1, y_2, \ldots, y_m, \ldots$ to a partially known state $x_1, x_2, \ldots, x_m, \ldots$ is formally equivalent to carrying out a fully calibrated measurement $y_1, y_2, \ldots, x_m, \ldots$ of a completely unknown state $x_1, x_2, \ldots, y_m, \ldots$. This example can be easily extended to any measurement scheme with linear dependence of the detection probabilities on unknown parameters $\vec{\eta}$ and $\vec{\gamma}$. Typically, the set of linear equations (1) will have a unique solution once the number of unknown measurement parameters and the number of known state parameters match.
3. Realistic nonlinear self-calibration

The concept of the self-calibrating schemes outlined above would still be too restrictive, since many if not all state-of-art tomography schemes are highly nonlinear with respect to the relevant parameters. For example, both homodyne measurement and time-multiplexed measurement are highly nonlinear in the quantum efficiency of photo-detectors. Following the concept of self-calibration we propose to utilize robust properties of quantum states, such as Gaussianity, as prior information that amounts to imposing nonlinear conditions on the state preparation.

To our knowledge, the first advances toward simultaneous tomography of the signal and measurement setup were presented in [14], where the simultaneous inference of the signal state and the mismatch between the signal and reference states in the on/off reconstruction scheme was discussed. Later in [7], a protocol was proposed for performing the absolute calibration of a single-photon detector using an on/off detection scheme and squeezed noised signal states. Recently, the first experimental demonstration of a nonlinear self-calibration procedure was done with single qubits [15]. In that work, an unknown rotation angle of a unitary transformation induced by the measuring apparatus was determined simultaneously with the parameters of the signal state.

3.1. The general self-calibration procedure

Obviously, in nonlinear detection schemes, verification of the self-calibrating nature of any given class of quantum states is much trickier than in the trivial linear case discussed above. The existence of a unique solution to the nonlinear system equation (1) is not obvious. Nevertheless, we formulate a simple sufficient condition motivated by the statistical approach to quantum estimation, where model $p_j$ provides the best fitting of data $f_j$ with respect to some measure of success (distance) $D(p, f)$. For example, taking the Kullback–Leibler divergence as the measure,

$$D(\tilde{p}, \tilde{f}) \propto \ln (L) = \sum_j f_j \ln \left\{ p_j / P \right\}, \quad P = \sum_j p_j,$$

is equivalent to performing maximum-likelihood (ML) estimation [11], i.e. maximizing the likelihood $L$ of the model.

A generic self-calibration procedure can be divided into three steps: (i) initial values of the measurement parameters are set ($\tilde{\eta} = \tilde{\eta}_0$) and perturbed by small deviation $\delta \tilde{\eta}$; (ii) the state $\rho$ is estimated by minimizing the distance $D$; (iii) the resulting estimate is compared with the prior information about the state. If this check fails, these three steps are repeated with another choice of parameters, $\delta \tilde{\eta}$, until the prior information is reproduced to a sufficient accuracy. In this process, the prior information about the states is gradually transferred to measurement, and calibration of the measurement device is achieved.

Now we can formulate a simple criterion for self-calibration: for a given measurement scheme, $\{A_j(\tilde{\eta})\}$, $\rho(\tilde{y})$, to be self-calibrating, it is sufficient that the distance $D(\tilde{p}, \tilde{f})$ is strictly convex with respect to unknown parameters $\{\tilde{\eta}, \tilde{y}\}$. Indeed, convexity guarantees a unique best fit, which will coincide with the true values in the absence of noise.

The role of convexity can be illustrated with the reconstruction of the optical signal with an on/off detector of unknown quantum efficiency. On/off detections combined with controlled attenuation have been shown to be sufficient for the reconstruction of diagonal elements in the
Figure 1. (a, c) Examples of initial states taken for the ML reconstruction from on/off measurements equation (4). Prior knowledge \( \rho_{11} = \rho_{22} \) is available. Light gray, gray and black bars correspond to the true states, true states with 10% noise added and states with flat photon-number distributions. (b, d) Deviations from the constraint as calculated for the ML state are shown for different initial guesses of the detector efficiency \( \eta \) and different initial states. In panel (b), solid and dash-dotted lines are for initial true states with 1 and 10% noise added; in panel (d), solid and dashed lines correspond to the initial true states with 10% noise added and the initial state with equal density elements. Left and right panels correspond to truncations \( N = 4 \) and \( N = 2 \), respectively, in equation (5). Measurements on \( 10^5 \) copies were simulated for each of the 100 absorber transmittivities equidistantly distributed within the interval [0.1, 0.9].

The true value of the detector efficiency was \( \eta = 0.75 \).

Fock state basis [10–13],

\[
p(T_k) = \sum_{n=0}^{\infty} (1 - T_k \eta)^n \rho_{nn},
\]

where \( p(T_k) \) is the ‘off’ probability, \( \eta \) is the unknown efficiency of the detector, \( T_k \) are attenuating ratios and \( \rho_{nn} \) are the diagonal elements of the density matrix in truncated space to be reconstructed,

\[
\rho = \sum_{n=0}^{N} \rho_{nn} |n\rangle\langle n|.
\]

Let us assume for our ‘toy’ example a special kind of prior information about the state in the form of \( \rho_{22} = \rho_{11} \). The goal is to estimate the photon number distribution \( \rho_{nn}, n = 0, \ldots, N \) and, at the same time, perform absolute calibration of the detector using the ML technique. Note that this problem is highly nonlinear in \( \eta \) parameter.

Typical results are shown in figure 1. Right (left) panels correspond to the convex (non-convex) distance measures. Correct results with a non-convex likelihood are obtained only as
long as the initial guess is close to the true configuration. As a result, the prior knowledge, \( \rho_{11} = \rho_{22} \), gives rise to two-photon self-calibrating states, but fails to make four-photon states so. This example may serve as a motivation on how to construct practical, useful and feasible self-calibration schemes. As will be shown below, the detection of the noised single-photon state adopting on/off detection and the detection of Gaussian states using homodyne detection fall in this category. At the same time, these states and detection schemes play an important role in quantum information processing. The concept of self-calibration and these two examples represent the key result of this paper.

3.2. On/off detection with noised single-photon states

The single-photon state makes a perfect calibration tool for single-photon detectors. This idea was exploited in the ‘absolute’ calibration scheme based on parametric downconversion [2]. Here we demonstrate that even the thermally noised single-photon state maintains this perfect calibrating property. Indeed, let us assume a mixture of the single-photon state and the thermal state with an unknown average number of photons \( n_T \) subject to on/off detection, \( \gamma \equiv n_T \). Then equation (4) takes the following form [16, 17]:

\[
p(T_k) = \frac{1 + \eta T_k (n_T - 1)}{(1 + n_T T_k \eta)^2}.
\]

As seen from equation (6), with just two different settings of \( T_k \), it is possible to find both \( \eta \) and \( n_T \). This is a direct consequence of the strict convexity of the relevant functional (3) with respect to these parameters as shown in figure 2. It is intriguing that even strongly noised single-photon states, such as those used for the simulation, can serve as self-calibrating states. In contrast, coherent states with \( p(T_k) = \exp(\alpha^2 T_k \eta) \), where \( \alpha \) is the coherent state amplitude or thermal states alone with \( p(T_k) = 1/(1 + n_T T_k \eta) \), fail to be self-calibrating states for the on/off scheme. In these cases it is not possible to find a unique solution for the detector efficiency and state parameters. Another set of self-calibrating states according to our definition is provided by the family of squeezed thermal states [7]. However, it appears that the presence of squeezing in a Gaussian state is sufficient to make the state a self-calibrating one for both the on/off detection schemes and quantum homodyne tomography. In a recent work [7], it was shown that the thermal squeezed state described by the density matrix

\[
\rho = U(v) \rho_{\text{term}} U^\dagger(v)
\]

can serve as the self-calibrating state for the on/off measurement scheme. The squeezing operator is \( U(v) = \exp\{v(a^\dagger)^2 - v^*a^2\} \), and \( a, a^\dagger \) are photon annihilation and creation operators; \( \rho_{\text{term}} \) is the density matrix of the thermal state with the average number of photons \( n_T \).

Let us mention that the on/off detection scheme is not limited to the reconstruction of diagonal elements of the density matrix. A complete reconstruction would be possible with a minor modification of the scheme (e.g. adding coherent shifts to the signal state) [13]. So, one should expect that the class of self-calibration states for this scheme is far wider than the set of examples given above. In addition, the detector efficiency is not the only setup parameter that can be reconstructed by the self-calibration technique. A number of other parameters (such as mismatch, phase of the reference coherent state, intensity of the thermal noise, etc) can be inferred provided that the condition of self-calibration is satisfied.
3.3. Homodyne tomography with Gaussian states

Gaussian states form an important class of self-calibrating states for homodyne tomography with unknown overall quantum efficiency. Taking Gaussianity as the prior information is equivalent to imposing nonlinear constraints on the measured state, since the density matrix elements appear to be highly nonlinear in the state parameters (such as squeezing, temperature, etc). Gaussianity is a robust property that is carried over to the statistics of measured quadrature distributions [18], given for the thermal squeezed state (7) as

\[ p(\theta, x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x^2/2\sigma^2), \]  

(8)

\[ \theta \] being the quadrature angle, \( x \) being the quadrature value and \( \sigma^2 \) being the variance of the squeezed thermal state,

\[ \sigma^2 = \eta \left[ \frac{2n_T + 1}{2Q} (Q^2 \cos^2 \theta + \sin^2 \theta) + \delta \right], \]  

(9)

where \( Q = \exp(2\nu) \), \( \delta = (1 - \eta)/2\eta \) and the squeezing parameter \( \nu \) is taken to be real valued. Let us emphasize that, in this particular case, prior information is not necessary for deciding whether or not the measured state belongs to the class of self-calibrating states. Indeed, well-known normality tests [19, 20] can be applied directly to raw data to confirm/refuse Gaussianity of all measured statistics. Consequently, no prior information about the signal state is required and data alone are sufficient for this purpose.
Figure 3. Log-likelihood (arbitrary units) of squeezed thermal states subject to homodyne measurement equation (8). (a) 2D slice for a fixed squeezing $Q = 2$; (b) 2D slice for a fixed thermal state intensity $n_T = 0.25$. True values: $\eta^{\text{true}} = 0.75$, $n_T^{\text{true}} = 0.25$ and $Q^{\text{true}} = 2$. The simulated measurement consisted of six quadratures equidistantly distributed in the interval $\Theta \in [0, \pi]$ with $2 \times 10^5$ detections per quadrature.

Slices of the log-likelihood (3) for the homodyne tomography with self-calibrating Gaussian states are shown in figure 3. Using the procedure described above, sufficiently accurate estimates of the state parameters $Q$ and $n_T$ together with the absolute calibration of the setup efficiency $\eta$ can be obtained under realistic assumptions. Also, a possible coherent shift can be easily incorporated into the scheme.

In order to reach the accuracy of the standard tomography, a larger number of measurements would be required by the self-calibration approach. However, as was demonstrated in [7, 14], this increase is not critical and for moderately sized systems remains well within the limits of current practical feasibility. For example, the number of measurements is approximately quadrupled in the simple scheme considered in [7]. This disadvantage is more than compensated for by avoiding the possible reconstruction bias resulting from incorrectly set parameters of the standard scheme. Also note that in all the cases considered here, the total number of unknown parameters is rather small and the amount of data processing required to perform the self-calibration is not a limiting factor. Should it be otherwise, such as in analyzing large quantum systems, other methods of dealing with the experimenter’s uncertainties might be considered; see, e.g., maximum-entropy methods in this context [21].

4. Conclusion

We have proposed the concept of self-calibrating states, which can be utilized for performing tomography of a quantum state and measurement simultaneously. A simple criterion based on
the convexity of the relevant fitting measure was formulated for verifying the self-calibration status of a given tomography scheme. Two pertinent examples of nonlinear self-calibrating schemes were discussed: the on/off detection with thermally noised single-photon states and Gaussian states, and homodyne detection with Gaussian states, where the detector efficiency can be estimated together with the state. In that case, statistical data analysis alone without any prior information about the state is sufficient for achieving this goal.

Of course, the set of self-calibrating states for the nonlinear reconstruction schemes considered is far from being exhausted by the given examples. Establishing a general class of self-calibrating states for a given measurement scheme and finding non-Gaussian self-calibrating states for homodyne tomography, in particular, seems to be a rather challenging task. This will be addressed in our future work on the subject.

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