The KTY formalism and nonadiabatic contributions to the neutrino oscillation probability

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Abstract
It is shown how to obtain the analytical expression for the effective mixing angle in matter using the formalism which was developed by Kimura, Takamura and Yokomakura. If the baseline of the neutrino path is long enough so that averaging over rapid oscillations is a good approximation, then with the help of Landau’s method, the nonadiabatic contribution to the oscillation probability can be expressed analytically by this formalism. We give two examples in which the present method becomes useful.

Keywords: neutrino oscillation, nonadiabatic transition, the KTY formalism

1. Introduction
Neutrino oscillation is a quantum mechanical interference effect which sometimes has complex behaviors, particularly in matter. To discuss the behaviors of neutrino oscillation intuitively, it is important to have analytical formulae for the oscillation probability. Unfortunately, an analytical formula in the three flavor mixing scheme in matter is quite complicated. In 2002 Kimura, Takamura and Yokomakura (KTY) discovered a compact formula \[1, 2\] for the neutrino oscillation probability in matter with constant density. Subsequently the KTY framework was generalized to more general cases. Ref. [3] discussed the four neutrino mixing scheme in matter with constant density. Ref. [4] discussed the case with unitarity violation. Ref. [5] discussed two cases of neutrino oscillation in the adiabatic approximation, the one with non-standard interactions where the matter potential has non-diagonal elements in the flavor basis, or the other with large neutrino magnetic moments in a magnetic field.

In general, however, adiabatic approximation may not be good, and in this talk I discuss nonadiabatic contributions to the oscillation probability. When there are more than two neutrino mass eigenstates, there can be more than one level crossing. It is believed\[4\] that the nonadiabatic contributions to the transition phenomena in a problem with three or more eigenstates can be treated approximately well by applying the method for two state problems \[8, 9\] at each level crossing, if the two resonances are sufficiently far apart. Throughout this talk I discuss the case in which the baseline of the neutrino path is long enough so that averaging over rapid oscillations is a good approximation, as in the case of the solar neutrino deficit phenomena.

2. The oscillation probability

2.1. The oscillation probability in the adiabatic approximation

The equation of motion for neutrinos propagating in matter with general potential is given by

\[
\frac{d\Psi}{dt} = \left(UEU^{-1} + A(t)\right)\Psi,
\]

1See, e.g., Ref. [8] and references therein. See also Ref. [7] for a discussion on the condition to justify such a treatment.
where
\[ \mathcal{E}_0 \equiv \text{diag}(E_1, E_2, E_3), \]
\[ \mathcal{A}(t) \equiv \begin{pmatrix} A_{\alpha\gamma}(t) & A_{\bar{\alpha}\bar{\gamma}}(t) & A_{\alpha\bar{\gamma}}(t) \\ A_{\bar{\alpha}\gamma}(t) & A_{\bar{\alpha}\bar{\gamma}}(t) & A_{\bar{\alpha}\bar{\gamma}}(t) \\ A_{\alpha\bar{\gamma}}(t) & A_{\bar{\alpha}\bar{\gamma}}(t) & A_{\alpha\bar{\gamma}}(t) \end{pmatrix}. \]

Since the matrix which is proportional to identity gives contribution only to the phase of the probability amplitude, instead of \( \mathcal{E}_0 \) itself, we use the following quantity:
\[ \mathcal{E} \equiv \mathcal{E}_0 - E_1 \mathbb{1} = \text{diag}(0, \Delta E_2, \Delta E_3), \]
where \( \Delta E_{jk} \equiv E_j - E_k \approx (m_j^2 - m_k^2)/2|\vec{p}|. \) The \( 3 \times 3 \) matrix on the right hand side of Eq. (1) can be formally diagonalized as:
\[ U E U^{-1} + \mathcal{A}(t) = \bar{U}(t) \tilde{\mathcal{E}}(t) \bar{U}^{-1}(t), \]
where \( \bar{U}(t) \equiv \text{diag}(\bar{E}_1(t), \bar{E}_2(t), \bar{E}_3(t)) \) is a diagonal matrix with the energy eigenvalues \( \bar{E}_i(t) \) in the presence of the matter effect. Substituting the diagonalized form of the Hamiltonian into the Dirac equation (1), we have
\[ i \frac{d\Psi}{dt} = \left[ \tilde{\mathcal{E}} - i \bar{U}^{-1} \left( \frac{d}{dt} \bar{U} \right) \right] \Psi, \]
where \( \Psi \) is the effective energy eigenstate defined by
\[ \Psi(t) \equiv \begin{pmatrix} \tilde{\psi}_1(t) \\ \tilde{\psi}_2(t) \\ \tilde{\psi}_3(t) \end{pmatrix} \equiv \bar{U}^{-1}(t)\Psi(t). \]

If the term \( \bar{U}^{-1}d\bar{U}/dt \) in (3) is negligible compared with \( \tilde{\mathcal{E}} \), i.e., if adiabatic approximation is good, then the oscillation probability is given by
\[ P(\nu_\alpha \rightarrow \nu_\beta) = \sum_{jk} \bar{U}_{\beta j}(L) \bar{U}_{\bar{\alpha} k}(L) \bar{U}_{\bar{\alpha} j}(0) \bar{U}_{k \beta}(0) \times \exp \left[ -i \int_0^L \Delta \bar{E}_{jk}(t) dt \right], \]
where we have defined \( \Delta \bar{E}_{jk}(t) \equiv \bar{E}_j(t) - \bar{E}_k(t) \). The bi-linear quantity
\[ \tilde{X}_i^{\alpha\beta}(t) \equiv \bar{U}_{\alpha j}(t) \bar{U}^*_{\bar{\beta} k}(t) \] can be expressed analytically as
\[ \begin{pmatrix} \tilde{X}_1^{\alpha\beta}(t) \\ \tilde{X}_2^{\alpha\beta}(t) \\ \tilde{X}_3^{\alpha\beta}(t) \end{pmatrix} = \frac{\Delta \bar{E}_{21}}{\Delta \bar{E}_{21} - \frac{1}{2}} \begin{pmatrix} -1 \\ \Delta \bar{E}_{21} \Delta \bar{E}_{31} \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{E}_2 - \bar{E}_3 \\ \bar{E}_3 - \bar{E}_1 \\ \bar{E}_1 - \bar{E}_2 \end{pmatrix}, \]
where
\[ x^{\alpha\beta} = \begin{pmatrix} \delta^{\alpha\beta} \\ 0 \end{pmatrix}. \]

where the \( t \)-dependence of the quantities \( \bar{E}_j, \Delta \bar{E}_{jk} \) is suppressed for simplicity in Eq. (5).

2.2. The nonadiabatic correction to the oscillation probability

If the adiabatic approximation is not good, on the other hand, then Eq. (4) should be modified by taking non-adiabatic contributions into account.

In the three flavor case, there are at most two level crossings for neutrinos as in the case of a supersnova [11]. Let us assume that there are two level crossings. Let \( t = t_H \) and \( t = t_L \) be the level crossing points, and we assume that the energy levels \( 1 \) (1) and \( 2 \) (3) cross at \( t = t_L \). \( (t = t_H) \) respectively (See Fig. 1).\]

Except the two regions near \( t = t_H \) and \( t = t_L \), we can integrate (5) in the adiabatic approximation. Near the two level crossings \( t = t_H \) and \( t = t_L \), the energy eigenstates at \( t = t_H \pm \epsilon \) and \( t = t_L \pm \epsilon \) are related by the matrices \( W_H \) and \( W_L \) as in the case of the two level problem [12]:
\[ \Psi(t_H + \epsilon) = W_H \Psi(t_H - \epsilon) \]
\[ \Psi(t_L + \epsilon) = W_L \Psi(t_L - \epsilon) \]

In this case the flavor eigenstate is given by
\[ \Psi(L) = U(L) \exp \left[ -i \int_{t_H}^{t_L} \tilde{E}(t) dt \right] W_L \exp \left[ -i \int_{t_L}^{t_H} \tilde{E}(t) dt \right] \]
\[ \times W_H \exp \left[ -i \int_{t_H}^{t_L} \tilde{E}(t) dt \right] U(0)^{-1} \Psi(0). \]
Taking average over rapid oscillations, we have the probability
\[ P(y_a \to y_b) = \sum_{j,k,l} |U(L)_{jk}|^2 |W(L)_{kj}|^2 |W(H)_{kl}|^2 |U(O)_{bl}|^2 \]
\[ = \begin{pmatrix} |U_{11}|^2 & |U_{12}|^2 & |U_{13}|^2 \\ |U_{21}|^2 & |U_{22}|^2 & |U_{23}|^2 \\ |U_{31}|^2 & |U_{32}|^2 & |U_{33}|^2 \end{pmatrix} \begin{pmatrix} 1 - P_L & P_L & 0 \\ P_L & 1 - P_L & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |U_{11}(0)|^2 & |U_{12}(0)|^2 & |U_{13}(0)|^2 \\ |U_{21}(0)|^2 & |U_{22}(0)|^2 & |U_{23}(0)|^2 \\ |U_{31}(0)|^2 & |U_{32}(0)|^2 & |U_{33}(0)|^2 \end{pmatrix}. \]
\[ (6) \]

In (6) it was assumed that the nonadiabatic contributions to the transition phenomena in a problem with three eigenstates can be treated approximately well by applying the method for two state problems at each level-crossing, when the the two resonances are sufficiently far apart. Using the WKB approximation [8], the jumping factors in (6) are given by
\[ P_H = \exp \left[ -\text{Im} \int_C \Delta E_{31}(t) \, dt \right], \]
\[ P_L = \exp \left[ -\text{Im} \int_C \Delta E_{21}(t) \, dt \right]. \]
\[ (7) \]
\[ (8) \]

2.3. The effective mixing angles in matter

To evaluate the jumping factors \( P_H \) and \( P_L \) in the previous subsection, we assume here as in the two flavor case that the exponents in (7) and (8) are related by the ratio of the difference of the energy eigenvalues of the two levels to the derivative of the effective mixing angle at the level-crossing. For that purpose, it is necessary to know the effective mixing angles at the level crossings. In this subsection I show how to derive the expression for the effective mixing angle in the presence of the matter using the KTY formalism. Our strategy here is to start with effective matrix elements \( \tilde{X}^{ef} \) which are obtained by the KTY formalism and to determine the phase of each element by demanding that it be consistent with the standard parametrization of the mixing matrix element:
\[ \tilde{U} = e^{\theta_{23}\delta_{23}} \Gamma_{\delta}^{(13)} e^{\theta_{12}\delta_{12}} (\Gamma_{\delta}^{(13)})^{-1} e^{\theta_{12}\delta_{23}}, \]
\[ (9) \]
where \( \lambda_j (j = 2, 5, 7) \) are the Gell-Mann matrices defined by \( \lambda_2 \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( \lambda_5 \equiv \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \), and \( \lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \), and \( \Gamma_{\delta}^{(13)} \equiv \text{diag}(e^{-\delta/2}, 1, e^{\delta/2}). \)

From the identity \( \tilde{U}_{aj} = e^{i\epsilon \theta_{ij}} \tilde{X}_{ee} / \sqrt{\tilde{X}_{ee}} \), we can postulate the form for \( \tilde{U} \)
\[ \tilde{U} = e^{i\phi_{12}} e^{i\varphi_3} e^{i\varphi_3'} U_0 e^{i\varphi_{13}} e^{i\varphi_{13}'}, \]
\[ (10) \]
where \( (U_0)_{aj} = \tilde{X}_{ee}/\sqrt{\tilde{X}_{ee}} \), \( \lambda_3 \equiv \text{diag}(1, -1, 0), \lambda_9 \equiv \text{diag}(1, 0, -1) \), and choose the phases \( \varphi_0, \varphi_3, \varphi_3' \) so that the form (10) is consistent with (9) (See Ref. [13] for details). Comparing Eqs. (9) and (10), we find
\[ \cos 2\vartheta_{13} = \frac{\tilde{X}_{1e}^{ee} - \tilde{X}_{2e}^{ee}}{\tilde{X}_{1e}^{ee} + \tilde{X}_{2e}^{ee}}, \]
\[ (11) \]
\[ \cos 2\vartheta_{13} = 1 - 2\tilde{X}_{3e}^{ee}, \]
\[ (12) \]
\[ \cos 2\vartheta_{23} = \frac{\tilde{X}_{3e}^{ee} - \tilde{X}_{1e}^{ee}}{\tilde{X}_{3e}^{ee} + \tilde{X}_{1e}^{ee}}, \]
\[ (13) \]
\[ \tilde{d} = -\text{arg det} U_0 + \text{arg} \tilde{X}_{1e}^{ee} + \text{arg} \tilde{X}_{3e}^{ee} + \text{arg} \tilde{X}_{3e}^{ee}. \]
\[ (14) \]

The quantities \( \tilde{X}_{ij}^{ef} \) and det \( U_0 \) in Eqs. (11)–(14) are expressed in closed form by the known variables as is seen in Eq. (5) on the assumption that analytical expressions for all the eigenvalues are known [3].

The standard parametrization (9) is not the only one for \( 3 \times 3 \) unitary matrices, and other parametrizations are possible as is described in Appendix B of Ref. [13]. In the three flavor case, there can be at most two level crossings. Depending on which pair of the energy eigenvalues gets close at each level crossing, the relevant effective mixing angle varies. The appropriate parametrization is the one in which the orthogonal matrix, which mixes the two energy eigenstates, is located on the most right-hand side of the unitary matrix \( U \), because in such a parametrization the diagonalized matrix looks like \( \cdots O(\bar{\theta}_{12})\text{diag}(\cdots, \bar{E}_1, \cdots, \bar{E}_4, \cdots)O(\bar{\theta}_{12})^T \cdots \), and it becomes clear that \( \bar{\theta}_{12} \) in the orthogonal matrix \( O(\bar{\theta}_{12}) \) plays a role of the effective mixing angle which mixes the energy eigenstates with the energy \( \bar{E}_1 \) and \( \bar{E}_4 \). Furthermore in order for the effective mixing angle \( \bar{\theta}_{12} \) to be consistent with the two flavor description, \( \bar{\theta}_{12} \) should become maximal at the level crossing. Thus, while the effective mixing angle at \( t = t_L \) is \( \bar{\theta}_{12} \) defined by (11) for the standard parametrization (9), the one at \( t = t_H \) should be \( \varphi_{13} \) defined by
\[ \cos 2\varphi_{13} = \frac{\tilde{X}_{1e}^{ee} - \tilde{X}_{2e}^{ee}}{\tilde{X}_{1e}^{ee} + \tilde{X}_{2e}^{ee}}, \]
\[ (15) \]

Footnote 2: Here we note that the effective mixing angles were given in the standard three flavor case in Ref. [13], whereas Eqs. (11)–(14) are the results for the case of the general potential.
are given by
\[ \bar{U} = e^{i\phi_{31}} \Gamma^{(1)}_0 e^{i\phi_{12}} \Gamma^{(12)}_0^{-1} e^{i\phi_{21}}, \]
where \( \Gamma^{(12)}_0 \equiv \text{diag}(e^{-i\pi/2}, e^{-i\pi/2}, 1) \). From the analogy with the two flavor case, the jumping factors \( P_H \) and \( P_L \) are given by
\[ P_H = \exp \left( -\frac{\pi}{2} F \frac{\Delta E_{31}}{2|d\varphi_{13}/dt|_{t=\alpha}} \right), \]
\[ P_L = \exp \left( -\frac{\pi}{2} F \frac{\Delta E_{21}}{2|d\bar{\varphi}_{12}/dt|_{t=\beta}} \right), \]
where \( \varphi_{13} \) and \( \bar{\varphi}_{12} \) are defined by (15) and (11), and \( F \) is the factor which depends on the form of the potential \( A_{\phi\phi}(t) \), and \( F = 1 \) in the case of a linear potential.

3. Examples

Now I discuss two examples to demonstrate how the general discussions in the previous section are applied.

3.1. The case with non-standard interactions

The first example is the oscillation probability in the presence of new physics in propagation [13, 16, 17]. In this case the mass matrix is given by
\[ U E U^{-1} + \mathcal{A}_{NP} \]
where \( \mathcal{A}_{NP} \equiv \sqrt{2} G_F N e \left( \begin{array}{ccc} 1 + \epsilon_{\tau} & \epsilon_{\mu} & \epsilon_{\tau} \\ \epsilon_{\mu}^* & \epsilon_{\mu}^* & \epsilon_{\mu} \\ \epsilon_{\tau}^* & \epsilon_{\tau}^* & \epsilon_{\tau} \end{array} \right) \). The dimensionless quantities \( \epsilon_{\phi\phi} \) stand for possible deviation from the standard matter effect. It is known [18] that the constraints on the parameters \( \epsilon_{\phi\phi}, \epsilon_{\mu\mu}, \epsilon_{\tau\tau} \) are strong \((|\epsilon_{\phi\phi}| \approx O(10^{-5}) \) \( \alpha = e, \mu, \tau \) while those on the parameters \( \epsilon_{\phi\phi}, \epsilon_{\mu\mu}, \epsilon_{\tau\tau} \) are weak \((|\epsilon_{\phi\phi}|, |\epsilon_{\mu\mu}|, |\epsilon_{\tau\tau}| \approx O(1)) \). In Ref. [19] it was found that large values \((O(1)) \) of the parameters \( \epsilon_{\phi\phi}, \epsilon_{\mu\mu}, \epsilon_{\tau\tau} \) are consistent with all the experimental data including those of the atmospheric neutrino data, provided that one of the eigenvalues of the matrix (17) at high energy limit becomes zero, and that such a constraint implies the relation \( \epsilon_{\tau\tau} \approx |\epsilon_{\mu\mu}|^2/(1 + \epsilon_{\mu\mu}) \). For simplicity, therefore, we consider the potential matrix
\[ \mathcal{A}_{NP} = A \left( \begin{array}{ccc} 1 + \epsilon_{\tau} & 0 & \epsilon_{\tau} \\ 0 & 0 & 0 \\ \epsilon_{\tau}^* & 0 & |\epsilon_{\tau\tau}|^2/(1 + \epsilon_{\tau}) \end{array} \right), \]
where \( \cap_{\phi\phi} = |\epsilon_{\tau\tau}|/(1 + \epsilon_{\mu}) \), \( \gamma' \equiv \text{arg}(\epsilon_{\tau\tau})/2 \), \( \lambda' = A(1 + \epsilon_{\mu})/\cos^2 \beta \). The mass matrix (17) can be written as
\[ \bar{U} E U^{-1} + \mathcal{A}_{NP} \]
\[ = e^{i\phi_{31}} e^{-i\phi_{12}} e^{-i\phi_{21}} \left( \begin{array}{ccc} U'' & e^{i\gamma'} & \mathcal{A}_{NP} U'' \end{array} \right) \]
\[ \times e^{i\bar{\phi}_{12}} e^{-i\bar{\phi}_{12}} e^{-i\gamma' \Lambda}, \]
(20)
where the phases \( \phi_{13}, \phi_{31}, \omega_1 \) and \( \omega_3 \) are defined in Appendix C of Ref. [13]. The expressions for the mixing angles \( \theta'' \) and the CP phase \( \theta'' \) in the standard parametrization of \( U'' \) are given in Ref. [13]. Among others the mixing angles \( \theta_{12} \) and \( \theta_{13} \) are given by
\[ \theta_{12}' = \tan^{-1} \left( \frac{\epsilon_{\mu\mu} e^{i\gamma} U_{e2} + s_\beta e^{i\gamma'} U_{e2}}{\epsilon_{\mu\mu} e^{i\gamma} U_{e1} + s_\beta e^{i\gamma'} U_{e1}} \right), \]
\[ \theta_{13}' = \sin^{-1} \left( \frac{|c_\beta e^{-i\gamma'} U_{e3} + s_\beta e^{i\gamma'} U_{e3}|}{|c_\beta e^{i\gamma'} U_{e3} + s_\beta e^{-i\gamma'} U_{e3}|} \right), \]
where \( c_\beta \equiv \cos \beta, s_\beta \equiv \sin \beta \). The inside of the square bracket in the mass matrix (20) has exactly the same form as that of the standard case with replacement \( \theta_{ij} \rightarrow \theta''_{ij}, \delta \rightarrow \theta'' \) and \( A \rightarrow A'' \). Furthermore, at the two-level-crossings specified by \( \Delta E_{21} \cos 2\theta''_{13} = A'' \) and \( \Delta E_{21} \cos 2\theta''_{12} = (\epsilon_{\mu\mu})^2 A'', \theta_{13}' \) and \( \theta_{13}' \) become \( \pi/4 \), respectively. Therefore, \( \theta_{13}' \) and \( \theta_{13}' \) can be regarded as the appropriate mixing angles to describe the nonadiabatic transition at the two-level-crossings. Hence we can deduce the jumping factors at the two-level-crossings (19).
3.2. The case with large magnetic moments and a magnetic field

The second example is the case where there are three active neutrinos with magnetic moments and a large magnetic field. This is an example where the energy eigenvalues cannot be expressed as roots of a quadratic equation, and this case demonstrates the usefulness of the KTY formalism. Here we assume the magnetic interaction of Majorana type \( \mu_{\alpha\beta} \bar{\nu}_\alpha F_{\alpha\beta} \sigma^k \gamma^5 \nu_\beta + h.c. \), and in this case the magnetic moments \( \mu_{\alpha\beta} \) are real and antisymmetric in flavor indices: \( \mu_{\alpha\beta} = -\mu_{\beta\alpha} \).

\[
M \equiv \left( \begin{array}{cc} U E U^{-1} & B \\ B^\dagger U^* E(U^*)^{-1} \end{array} \right) \quad (24)
\]

with \( B_{\alpha\beta} \equiv B_{\mu\nu} = \text{the hermitian mass matrix for neutrinos and anti-neutrinos without the matter effect where neutrinos have the magnetic moments } \mu_{\alpha\beta} \text{ in the magnetic field } B. \)

For simplicity we consider the limit \( \theta_{13} \to 0 \) and \( \Delta m_{23}^2 \to 0 \), and we assume that all the CP phases vanish. Then the matrix \( (24) \) can be rewritten as

\[
M = \frac{1}{2} \left( \begin{array}{cc} 1 & i1 \\ i1 & 1 \end{array} \right) \left( \begin{array}{cc} U E U^{-1} + iB \\ 0 \\ U E U^{-1} - iB \end{array} \right) \times \left( \begin{array}{cc} 1 & -i1 \\ -i1 & 1 \end{array} \right),
\]

so the problem of diagonalizing the \( 6 \times 6 \) matrix \( (24) \) is reduced to diagonalizing the \( 3 \times 3 \) matrices \( U E U^{-1} \pm iB \).

Since we are assuming that all the CP phases vanish, all the matrix elements \( U_{\alpha\beta} \) and \( B_{\alpha\beta} = -B_{\beta\alpha} \) are real, \( U E U^{-1} \pm iB \) can be diagonalized by a unitary matrix and its complex conjugate:

\[
UEU^{-1} + iB = \hat{U} \hat{E} U^{-1},
\]

and the equation for motion is given by

\[
\frac{d}{dt} \left( \Psi(t) + i\Psi'(t) \right) = \left( \hat{U}(t) \hat{E}(t) \hat{U}^{-1}(t) \Psi(t) + i\Psi'(t) \right),
\]

Introducing the notations

\( B_{\alpha\beta} = B_{\mu\nu} \equiv \left( \begin{array}{ccc} 0 & -p_0 & -q_0 \\ p_0 & 0 & -r_0 \\ q_0 & r_0 & 0 \end{array} \right) \),

it can be shown [13] that Eq. (25) can be rewritten as

\[
e^{-i\omega t} e^{i\theta_{23} k} \left[ \text{diag}(0, 0, \Delta E_{31}) + \Lambda e^{i\theta_{12,13}} e^{-i\varphi_k} \right] \times e^{-i\omega t} e^{i\theta_{23} k},
\]

where \( \Lambda, \omega \) and \( \chi \) are defined by \( \Lambda \equiv \sqrt{\omega^2 + q^2 + r^2}, \omega \equiv \tan^{-1}(r/q), \chi \equiv \tan^{-1}(\sqrt{q^2 + r^2}/p). \) Since the we are mainly interested in the effective mixing angle which mixes the two energy eigenstates, the matrices \( e^{i\theta_{23} k} e^{i\varphi_k} \) on the left-hand side and \( e^{-i\varphi_k} e^{-i\theta_{23} k} \) on the right-hand side of the square bracket in Eq. (26) are irrelevant, so we discuss the following matrix:

\[
\Lambda \equiv \text{diag}(0, 0, \Delta E_{31}) = \frac{\Delta E_{31}}{3} \left[ 1 + \Lambda e^{i\theta_{12,13}} \right] e^{-i\varphi_k},
\]

where a matrix which is proportional to identity was subtracted for convenience in later calculations so that the trace of \( \Lambda \) vanishes. The eigenvalues \( \hat{E}_j \) of the matrix \( \Lambda \) are given by \( \hat{E}_j = 2 \sqrt{\Delta E_{31}^2 / 9 + \Lambda^2 / 3} \cos(\varphi + 2j\pi/3) \) \((j = 1, 2, 3)\), where \( \cos 3\varphi = \left( \Delta E_{31} / \Lambda^2 \Delta E_{31}/12 \right) / \left( \Delta E_{31}^2 / 9 + \Lambda^2 / 3 \right)^{3/2} \).

In Fig. 13 the three eigenvalues \( \hat{E}_j \) \((j = 1, 2, 3)\) which are normalized by \( 2 \sqrt{\Delta E_{31}^2 / 9 + \Lambda^2 / 3} \) are depicted as a function of \( \Lambda \equiv 3\Lambda^2 / \Delta E_{31}^2 \). If \( \chi \) is small, then the two of the three eigenvalues get close to each other, and \( \chi \) can be regarded as the vacuum mixing angle near the level-crossing in the present case. In this example, for a large value of \( \Lambda \gg \Delta E_{31} \), the energy eigenvalues are 0 and \( \pm \Lambda \), and we found that there is only one level-crossing for \( |\Delta E_{31}| \sim \Lambda \), unlike in the standard three flavor case. So in the following we discuss the contribution from one level-crossing only.

Furthermore, it can be shown [13] that the following relation holds:

\[
P(v_\alpha \to v_\beta) + P(v_\alpha \to \bar{v}_\beta) = \sum_{j,k} |U_{\beta j}|^2 |U_{\alpha k}|^2 |U_{\alpha k}(0)|^2
\]

\[
= \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 - P_H & P_H \\ 0 & P_H & 1 - P_H \end{array} \right)
\]

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4 The possibility that the magnetic moments of neutrinos in a large magnetic field affect the neutrino flavor transition caught a lot of attention after this idea was applied to the solar neutrino deficit in Refs. [33, 34, 35, 36].
where we have assumed that the level-crossing occurs between the energy eigenstates 2 and 3, and we have assumed that there is no magnetic field at the endpoint $t = L$, and $|\bar{U}_{\alpha j}(0)|^2$ in the transition probability can be obtained from the KTY formula \cite{24}.

In the approximation of the small mixing angle $\chi$, the jumping factor $P_H$ can be calculated as \cite{13}

$$P_H \approx \exp\left(-\frac{\pi}{d\Lambda|d\tau_{\text{diab}}|} \Delta E_{31}^2 \chi^2 \right). \tag{29}$$

It can be also shown that the exponent of the jumping factor $P_H$ coincides with $-\pi/2$ times the $\chi$ factor in the case of a linear potential ($F = 1$):

$$\gamma = \frac{\Delta E_{32}}{2|d\phi_{23}/dt|_{\text{diab}}} \approx \frac{2\Delta E_{31}^2 \chi^2}{|d\Lambda|/d\tau_{\text{diab}}} = -\frac{\log P_H}{\pi/2}. \tag{30}$$

4. Conclusions

Using the formalism which was developed by Kimura, Takamura and Yokomakura to express analytically the combination $X_{ij}^{\alpha j} \equiv \bar{U}_{\alpha j}^\dagger \bar{U}_{\alpha i}^\dagger$ of the mixing matrix elements in matter with constant density, we have shown that the effective mixing angle can be analytically expressed in terms of the mixing matrix elements in vacuum and the energy eigenvalues. The analytical expression for the effective mixing angle enables us to evaluate the nonadiabatic contribution to the transition probability based on the two assumptions: (i) The nonadiabatic transitions in the case with more than two energy eigenstates can be separately treated as a two state problem at each level crossing. (ii) The exponent of the probability obtained by the WKB method is proportional to the factor $\gamma$ which is the ratio of the energy difference of the two eigenstates to the derivative of the effective mixing angle at the level crossing. We have given two examples: one with flavor dependent non-standard interactions in neutrino propagation and the other with magnetic moments in a large magnetic field.

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References

[1] K. Kimura, A. Takamura and H. Yokomakura, Phys. Lett. B 253, 869 (2002) [arXiv:hep-ph/0203009].
[2] K. Kimura, A. Takamura and H. Yokomakura, Phys. Rev. D 66, 073005 (2002) [arXiv:hep-ph/0203295].
[3] H. Zhang, Mod. Phys. Lett. A 22, 1341 (2007) [arXiv:hep-ph/0606040].
[4] E. Fernandez-Martinez, M. B. Gavela, J. Lopez-Pavon and O. Yasuda, Phys. Lett. B 649, 427 (2007) [hep-ph/0703098].
[5] O. Yasuda, [arXiv:0704.1531 [hep-ph]].
[6] T. K. Kuo and J. T. Pantaleone, Rev. Mod. Phys. 61, 937 (1989).
[7] K. Yamamoto, [arXiv:1003.2853 [hep-ph]].
[8] L. Landau, Phys. Z. Sowj. 2, 46 (1932).
[9] C. Zener, Proc. Roy. Soc. Lond. A 137, 696 (1932).
[10] Z. x. Xing and H. Zhang, Phys. Lett. B 618, 131 (2005) [arXiv:hep-ph/0503118].
[11] A. S. Dighe and A. Y. Smirnov, Phys. Rev. D 62, 033007 (2000) [hep-ph/9907243].
[12] S. J. Parke, Phys. Rev. Lett. 57, 1275 (1986).
[13] O. Yasuda, Phys. Rev. D 89 (2014) 093023 [arXiv:1402.5569 [hep-ph]].
[14] H. W. Zaglauer and K. H. Schwarzer, Z. Phys. C 40, 273 (1988).
[15] L. Wolfenstein, Phys. Rev. D 17, 2369 (1978).
[16] M. M. Guzzo, A. Masiero and S. T. Petcov, Phys. Lett. B 260, 154 (1991).
[17] E. Roulet, Phys. Rev. D 44, 935 (1991).
[18] S. Davidson, C. Pena-Garay, N. Rius and A. Santamaria, JHEP 0303, 011 (2003) [arXiv:hep-ph/0302093].
[19] A. Friedland and C. Lunardini, Phys. Rev. D 72, 053009 (2005) [arXiv:hep-ph/0506143].
[20] A. Friedland, C. Lunardini and C. Pena-Garay, Phys. Lett. B 594, 347 (2004) [hep-ph/0402266].
[21] A. Cisneros, Astroparticle Physics 10, 87 (1997).
[22] L. B. Okun, M. B. Voloshin and M. I. Vysotsky, Sov. Phys. JETP 64, 446 (1986) [Zh. Eksp. Teor. Fiz. 91, 754 (1986)].
[23] C. S. Lim and W. J. Marciano, Phys. Rev. D 37, 1368 (1988).
[24] E. K. Akhmmedov, Phys. Lett. B 213, 64 (1988).