Covariation inequality in Grand Lebesgue Spaces.

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Abstract

We represent in this preprint the exact estimate for covariation between two random variables (r.v.), which are measurable relative the corresponding sigma-algebras through anyhow mixing coefficients.

We associate a solution of this problem with fundamental function for correspondent rearrangement invariant spaces.

Key words and phrases: Probability space, sigma-algebra, covariation, distance, mixing coefficient, Young - Fenchel, or Legendre transform; uniform (Rosenblatt) and strong (Ibragimov’s) mixing, exact estimate, Lebesgue - Riesz and Grand Lebesgue Spaces (GLS), generating function, fundamental functions, factorization, rearrangement invariant (r.i.) space, exponential Orlicz spaces, natural function, Central Limit Theorem.

1 Definitions. Notations. Previous results. Statement of problem.

Let \((\Omega, \mathcal{B}, P)\) be probability space with correspondent expectation \(E\), variance \(\text{Var}\) and covariation \(\text{Cov}\):

\[
\text{Cov}(\xi, \eta) := E\xi\eta - E\xi E\eta.
\]

Denotation 1.1.

We denote for arbitrary sub-sigma algebra (field) \(F \subset \mathcal{B}\) and for arbitrary numerical valued random variable \(\xi\) the symbol

\[
\xi \in F
\]

iff the r.v. \(\xi\) is measurable relative the sigma-field \(F\).

Let \(F\) and \(G\) be two sub-sigma algebras of source sigma field \(\mathcal{B}\). We define as ordinary the so-called uniform mixing coefficient, or equally Rosenblatt’s coefficient \(\alpha(F, G)\) by the formula

\[
\alpha(F, G) := \sup_{\xi, \eta \in F \cap G} \frac{|\text{Cov}(\xi, \eta)|}{\text{Var}(\xi) \text{Var}(\eta)}.
\]
\[ \alpha = \alpha(F, G) \overset{\text{def}}{=} \sup_{A \in F, B \in G} |\mathbf{P}(AB) - \mathbf{P}(A) \mathbf{P}(B)|. \quad (1.1) \]

The strong mixing coefficient, on the other words, Ibragimov’s coefficient, \( \beta(F, G) \) is defined by the formula

\[ \beta = \beta(F, G) \overset{\text{def}}{=} \sup_{A \in F, B \in G, \mathbf{P}(A) > 0} |\mathbf{P}(B/A) - \mathbf{P}(B)|. \quad (1.2) \]

Denote as usually here and in the sequel by \( |\xi|_p \) the Lebesgue - Riesz \( L(p) \) norm of the r.v. \( \xi \):

\[ |\xi|_p = [\mathbf{E}|\xi|^p]^{1/p} := \left[ \int_{\Omega} |\xi(\omega)|^{p} \mathbf{P}(d\omega) \right]^{1/p}, \quad 1 \leq p < \infty; \]

\[ |\xi|_\infty := \text{vraisup}_{\omega \in \Omega} |\xi(\omega)|. \]

Let \( \xi \in F, \xi \in L(p), \eta \in G, \eta \in L(q), p, q \in [1, \infty]. \) Yu.A.Davydov in [6] proved the following important inequality

\[ |\text{Cov}(\xi, \eta)| \leq 12 \alpha^{1-1/p-1/q} |\xi|_p |\eta|_q, \quad \frac{1}{p} + \frac{1}{q} < 1. \quad (1.3) \]

The similar inequality for strong mixing coefficient \( \beta = \beta(F, G) \):

\[ |\text{Cov}(\xi, \eta)| \leq 2 \beta^{1/p} |\xi|_p |\eta|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad (1.4) \]

or equally

\[ q = q(p) = \frac{p}{p-1} = p', \quad p > 1; \quad q(\infty) = 1, \]

may be found in the famous monograph of I.A.Ibragimov and Yu.A.Linnik [9]; see also the recent survey [19] and the article [12].

The following estimate based only on the H’older’s inequality may be considered as trivial:

\[ |\text{Cov}(\xi, \eta)| \leq 2 |\xi|_p |\eta|_q, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (1.4a) \]

It is natural to expect that if the considered r.v. \( \xi, \eta \) have more light tails, for instance, satisfy the Kramer’s condition, then both the estimates (1.3) and (1.4) can be essentially strengthened.

Both the inequalities (1.3) and (1.4) are essentially, i.e. up to multiplicative constants, non-refinable for all the admissible values \( p, q \); see e.g. [6], for instance, on the classical probability space \([0, 1]\) equipped with Lebesgue measure there exist two sigma-fields \( F, G \) and two non-zero symmetrical distributed r.v. \( \xi, \eta \) which

\[ |\text{Cov}(\xi, \eta)| \geq \alpha(F, G)^{1-1/p-1/q} |\xi|_p |\eta|_q. \]
It is sufficient to take $F = G$.

The inequalities (1.3) and (1.4) are used in the investigation of the CLT for the dependent r.v., for the obtaining of the non-asymptotical estimation for sums of these r.v., in the statistics and in the Monte-Carlo method, see e.g. [3], [4], [12], [13] etc.

Our target in this short report is extension of the estimates (1.3), (1.4) into the r.v. belonging to the so-called Grand Lebesgue Spaces (GLS), in particular, into the exponential Orlicz spaces.

A modern result in this direction is represented in the article of E.Rio [17]; we intend to give these covariation estimates in another terms.

2 Grand Lebesgue Spaces (GLS). Fundamental functions.

Let $Z = (\Omega, B, P)$ be again the source probability space with non-trivial normed measure $P$. Let also $\psi = \psi(p)$, $p \in [1, b)$, $b = \text{const} \in (1, \infty]$ (or $p \in [1, b]$) be certain bounded from below: $\inf \psi(p) > 0$ continuous inside the semi-open interval $p \in [1, b)$ numerical function. We can and will suppose $b = \sup\{p, \psi(p) < \infty\}$, so that $\text{supp } \psi = [1, b)$ or $\text{supp } \psi = [1, b]$. The set of all such functions will be denoted by $\Psi(b) = \{\psi(\cdot)\}$; $\Psi := \Psi(\infty)$.

We agree to extend the definition these functions. Indeed, we define for arbitrary $\psi(\cdot) \in \Psi(b)$ function in the case when $b < \infty$ for the values $p > b$ formally as follows:

$$\forall p > b \Rightarrow \psi(p) := \infty.$$ 

By definition, the (Banach) Grand Lebesgue Space (GLS) space $G\psi = G\psi(b)$ consists on all the (real or complex) numerical valued measurable functions (random variables) $\zeta$ defined on our probability space $\Omega$ and having a finite norm

$$||\zeta|| = ||\zeta||_{G\psi} \overset{\text{def}}{=} \sup_{p \in [1, b]} \left\{ \frac{||\zeta||_p}{\psi(p)} \right\}. \tag{2.0}$$

These spaces are Banach functional spaces, are complete, and rearrangement invariant in the classical sense, see [1], chapters 1, 2. They were investigated in particular in many works, see e.g. [5], [7], [8], [10], [11], [13], [14]. The function $\psi = \psi(p)$ is said to be the generating function for this space.

We refer here some used in the sequel facts about these spaces and supplement more.

It is known that if $\zeta \neq 0$, $\zeta \in G\psi$, then
where \( v(p) = v_\psi(p) := p \ln \psi(p) \) and \( v^*(\cdot) \) denotes the Young - Fenchel, or Legendre transform for the function \( v(\cdot) \):

\[
v^*(x) = \sup_{p \in \text{Dom}(v)} (px - v(p)).
\]

Conversely, the last inequality may be reversed in the following version: if

\[
\mathbf{P}(|\zeta| > y) \leq 2 \exp \left( -v^*_\psi(\ln(y/||\zeta||) \right), \quad y \geq e \cdot ||\zeta||,
\]

and if the function \( v_\psi(p), \ 1 \leq p < \infty \) is positive, finite for all the values \( p \in [1, \infty) \), continuous, convex and such that

\[
\lim_{p \to \infty} \ln \psi(p) = \infty,
\]

then \( \zeta \in G\psi \) and besides \( ||\zeta|| \leq C(\psi) \cdot K \):

\[
||\zeta||_{G\psi} \leq C_1||\zeta||_{L(M)} \leq C_2||\zeta||_{G\psi}, \ 0 < C_1 < C_2 < \infty. \tag{2.3}
\]

Moreover, let us introduce the following exponential Young - Orlicz function

\[
N(u) = N_\psi(u) := \exp \left( v^*_\psi(\ln |u|) \right), \quad |u| \geq e; \quad N_\psi(u) = Cu^2, \quad |u| < e.
\]

The Orlicz’s norm \( ||\zeta||_{L(N_\psi)} \) is quite equivalent under formulated above conditions on the function \( \psi(\cdot) \) to the GLS one:

\[
||\zeta||_{G\psi} \leq C_3||\zeta||_{L(N_\psi)} \leq C_4||\zeta||_{G\psi}, \ 0 < C_1 < C_2 < \infty.
\]

Furthermore, let now \( \eta = \eta(z), \ z \in W \) be arbitrary family of random variables defined on any set \( W \) such that

\[
\exists b = \text{const} \in (1, \infty), \ \forall p \in [1, b) \ \Rightarrow \psi_W(p) := \sup_{z \in W} |\eta(z)|_p < \infty. \tag{2.4}
\]

The function \( p \to \psi_W(p) \) is named as a natural function for the family of random variables \( W \). Obviously,

\[
\sup_{z \in W} ||\eta(z)||_{G\psi_W} = 1.
\]

The family \( W \) may consists on the unique r.v., say \( \Delta \):

\[
\psi_\Delta(p) := |\Delta|_p,
\]

if of course the last function is finite for some value \( p = p_0 > 1 \).

Note that the last condition is satisfied if for instance the r.v. \( \zeta \) satisfies the so - called Kramer’s condition; the inverse proposition is not true.
Let us bring two examples. Define as usually the tail function for arbitrary numerical valued random variable $\xi$

$$T_\xi(y) \overset{\text{def}}{=} \max(P(\xi \geq y), P(\xi \leq -y)), \ y \geq 0.$$ 

**Example 2.1.** Let $m = \text{const} > 0$; define the function

$$\psi_m(p) = p^{1/m}, \ p \in [1, \infty).$$

The tail inequality

$$T_\xi(y) \leq \exp(-Cy^m), \ y \geq 0$$

for some positive constant $C$ is quite equivalent to the inclusion $\xi \in G\psi_m$.

**Example 2.2.** Let $b = \text{const} > 1; \ \beta = \text{const} > 0$. Define the following tail function

$$T[b, \beta](y) := C y^{-b} (\ln y)^{\beta - 1}, \ y \geq e,$$

and the following $\Psi(b)$ function with finite support

$$\psi[b, \beta](p) = (b - p)^{-\beta}, \ p \in [1, b); \ \psi[b, \beta](p) = \infty, \ p \geq b.$$

The tail inequality of the form

$$T_\eta(y) \leq T[b, \beta](y), \ y \geq e$$

entails the inclusion $\eta \in G\psi[b, \beta]$.

Note that the inverse proposition is not true.

**Definition 2.1.** The fundamental function for GLS $G\psi_b$ $\phi[G\psi](\delta)$, $\delta \in (0, \infty)$ may be calculated in accordance by the general theory of rearrangement invariant spaces [1], chapters 1,2 by a formula

$$\phi[G\psi](\delta) := \sup_{p \in [1, b)} \left\{ \frac{\delta^{1/p}}{\psi(p)} \right\}. \quad (2.5)$$

This notion play a very important role in the Functional Analysis, theory of Fourier series, Operator Theory, Theory of Random Processes etc., see the classical monograph [1]. For the GLS this function was investigated in the preprint [15]. It is proved in particular that there exists a bilateral continuous interrelation between fundamental and generating function for these spaces.

**Definition 2.2.** (See [15].) The low truncated fundamental function for the GLS $G\psi_b$, namely, $\phi_s[G\psi](\delta)$, $\delta \in (0, \infty), 0 < s < b$ is defined by a formula

$$\phi_s[G\psi](\delta) := \sup_{p \in [s, b)} \left\{ \frac{\delta^{1/p}}{\psi(p)} \right\}, \ 1 \leq s < b. \quad (2.5a)$$
Definition 2.3. (See [15].) The upper truncated fundamental function for the GLS $G_{\psi_b}$, indeed:

$$\phi^s[G_{\psi}](\delta), \ \delta \in (0, \infty), 0 < s < b$$

is defined by a formula

$$\phi^s[G_{\psi}](\delta) := \sup_{p \in [s,b)} \left\{ \frac{\delta^{1/p}}{\psi(p)} \right\}, 1 \leq s < b. \quad (2.5b)$$

Example 2.1.a. Let $m = \text{const} > 0$; the fundamental function for the $G_{\psi_m}$ has a form

$$\phi[G_{\psi_m}](\delta) = (em)^{-1/m} |\ln \delta|^{-1/m}, \ \delta \in (0, 1/e). \quad (2.6)$$

Example 2.2.a. Define the following $\Psi$ - function with finite support

$$\tau_{b,\beta}(p) \overset{\text{def}}{=} (b - p)^{-\beta}, \ p \in [1, b). \quad (2.7)$$

Here $b = \text{const} \in (1, \infty), \ \beta = \text{const} \geq 0$. The fundamental function for these space has a form

$$\phi[G_{\tau_{b,\beta}}](\delta) = \frac{b^{2\beta - 1} \beta^{\beta} \delta^{1/b}}{|\ln \delta|^\beta} =: K(b, \beta) \delta^{1/b} \ |\ln \delta|^{-\beta}, \ \delta \in (0, 1/e]. \quad (2.8)$$

3 Main results. Strong mixing.

Suppose $\xi \in G_{\psi}, \ \eta \in G_{\nu}$ for certain $\Psi$ functions $\psi, \nu$, and that

$$\xi \in F, \ \eta \in G. \quad (3.0)$$

One can allow without loss of generality $||\xi||_{G_{\psi}} = ||\eta||_{G_{\nu}} = 1$. Then

$$|\xi|_p \leq \psi(p), \ |\eta|_{p/(p-1)} \leq \nu(p/(p-1)), \ p \in [1, b). \quad (3.1)$$

Define a new $\Psi$ function

$$\zeta(p) = \zeta[\psi, \nu](p) := \psi(p) \ \nu(p/(p-1)); \quad (3.2)$$

then we have using the estimate (1.4)

$$0.5 \ |\text{Cov}(\xi, \eta)| \leq \beta^{1/p}(F, G) \ \psi(p) \ \nu(p/(p-1)) = \beta^{1/p} \zeta[\psi, \nu](p),$$

therefore

$$0.5 \ |\text{Cov}(\xi, \eta)| \leq \inf_p \left[ \beta^{1/p} \zeta(p) \right] = \left\{ \sup_p \left[ \frac{\beta^{-1/p}}{\zeta[\psi, \nu](p/)} \right] \right\}^{-1} = \frac{1}{\phi[G_{\zeta}(1/\beta)].}$$
To summarize:

**Theorem 3.1.** We deduce under formulated above notations and conditions

\[ |\text{Cov}(\xi, \eta)|| \leq \frac{2 ||\xi||_{G\psi} ||\eta||_{G\nu}}{\phi[G\zeta(\psi, \nu)](1/\beta(F, G))}. \]  

(3.3)

Let us consider a particular case.

**Definition 3.1.** Let the function \( \psi = \psi(p) \) be from the set \( \Psi = \Psi(\infty) \) : \( \text{supp} \psi = [1, \infty) \). The function \( \hat{\psi} \) from this set is said to be *dual* to the function \( \psi(\cdot) \), iff

\[ \hat{\psi}(p/(p - 1)) = \psi(p); \iff \hat{\psi}(p) = \psi(p/(p - 1)). \]  

(3.4)

Evidently, \( \hat{\hat{\psi}} = \psi \).

**Proposition 3.1.** Suppose in addition to the conditions of theorem 3.1 that in (3.2) \( \nu = \hat{\psi} \); then \( \zeta(p) = \psi^2(p) \) and following

\[ |\text{Cov}(\xi, \eta)|| \leq 2 \left[ \phi[G\psi] \left( \beta^{-1/2} \right) \right]^{-2} \cdot ||\xi||_{G\psi} \cdot ||\eta||_{G\hat{\psi}}. \]  

(3.5)

**Remark 3.1.** Let \( \nu(\cdot) \in \Psi(b), b = \text{const} \in (1, \infty] \). We assert that the Grand Lebesgue Space \( G\hat{\nu} \) consists only on the essentially bounded variables:

\[ ||\zeta||_{G\hat{\nu}} \simeq ||\zeta||_{\infty}. \]  

(3.6)

**Proof.** The inclusion \( L_{\infty} \subset G\hat{\nu} \) is evident; we must ground an inverse inclusion.

So, let \( \zeta \in G\hat{\nu}, \text{ supp}(\nu) = [1, b) \). We have taking into account the continuity of the function \( \nu(\cdot) \) at the point \( 1 + 0 \):

\[ \lim_{p \to \infty} \nu(p) = \lim_{p \to \infty} \nu \left( \frac{p}{p - 1} \right) = \nu(1) < \infty, \]

therefore

\[ \lim_{p \to \infty} |\zeta|_p < \infty \iff \text{vraisup}_{\omega \in \Omega} |\zeta(\omega)| < \infty. \]

4 Main results. Uniform (Rosenbatt) mixing.

Let as before \( \psi, \nu \) be two \( \Psi \) functions and let \( \alpha, \beta = \text{const} \in [0, 1] \).

Denote by \( T \) the domain in the positive quarter plane
\[ T = \{ p, q : p, q \geq 1, 1/p + 1/q < 1 \}; \]

"T" implies a triangle for the inverse values \( x = 1/p, y = 1/q \). Introduce the following functions

\[ \Phi[\psi, \nu](\alpha, \beta) \overset{\text{def}}{=} \sup_{(p,q) \in T} \left[ \frac{\alpha^{1/p} \beta^{1/q}}{\psi(p) \nu(q)} \right], \quad (4.1) \]

\[ \theta[\nu]_{\beta}(p) := \frac{\psi(p)}{\phi(p)}[G\nu](\beta), \quad (4.2) \]

We have

\[ \Phi[\psi, \nu](\alpha, \beta) = \sup_p \left\{ \alpha^{1/p} \sup_{q \geq q'} \beta^{1/q} \psi(p) \nu(q) \right\} = \sup_p \left[ \alpha^{1/p} \theta \nu \beta(p) \right] = \phi[G\theta_{\beta}](\alpha). \quad (4.3) \]

**Theorem 4.1.**

\[ |\text{Cov}(\xi, \eta)| \leq \frac{12 \alpha ||\xi||G\psi ||\eta||G\nu}{\Phi[\psi, \nu](\alpha, \beta)} = \frac{12 \alpha ||\xi||G\psi ||\eta||G\nu}{\phi[G\theta_{\alpha}](\alpha)}. \quad (4.4) \]

Here \( \alpha = \alpha(F, G) \).

**Proof.** Assume \( ||\xi||G\psi = ||\eta||G\nu = 1 \). Then

\[ |\xi|_p \leq \psi(p), \quad |\eta|_q \leq \nu(q), \quad (p, q) \in T. \]

One can apply the Davydov’s inequality (1.3):

\[ (12\alpha)^{-1}|\text{Cov}(\xi, \eta)| \leq \alpha^{-1/p} \alpha^{-1/q} \psi(p) \nu(q), \]

therefore

\[ (12\alpha)^{-1}|\text{Cov}(\xi, \eta)| \leq \inf_{(p,q) \in D} \left[ \alpha^{-1/p} \alpha^{-1/q} \psi(p) \nu(q) \right] = \frac{1}{\phi[G\theta_{\alpha}](\alpha)} = \frac{||\xi||G\psi ||\eta||G\nu}{\phi[G\theta_{\alpha}](\alpha)}. \quad (4.5) \]

Q.E.D.

Let’s turn again our attention on the function \( \Phi[\psi, \nu](\alpha, \beta) \) from the definition (4.1). In all the considered examples it allows a factorization

\[ \Phi[\psi, \nu](\alpha, \beta) = \phi[G\psi](\alpha) \cdot \phi[G\nu](\beta) \quad (4.6) \]

for all sufficiently small values \( \alpha, \beta \). We intend further to investigate the possibility of this relation (4.6).
We need first of all to return to the investigation of the fundamental function for GLS.

Case A. Infinite support.
Let \( \psi(\cdot) \in \Psi(\infty) = \Psi \). Denote for an arbitrary \( \Psi \) function the following transform
\[
g(x) = g[\psi](x) := -\ln \psi(1/x), \ x \in (0, 1),
\]
(4.7)
so that \( x_0 = x_0(\delta) = 1/p_0(\delta) \) and
\[
\frac{\delta^{1/p_0}}{\psi(p_0)} = \phi[G\psi](\delta).
\]
(4.8)

Lemma A. Suppose that the derivative \( g'(x) = g[\psi]'(x) \) there exists, is continuous in the open interval \((0,1)\), is strictly decreasing and such that
\[
\lim_{x \to 0} g'[\psi](x) = \infty.
\]
(4.9)
Then
\[
\lim_{\delta \to 0^+} p_0(\delta) = \infty.
\]
(4.10)

Proof follows immediately from the equation
\[
g'[\psi](1/p_0(\delta)) = g'(x_0) = g[\psi]'(x_0) = \ln(1/\delta), \ \delta \in (0, 1).
\]
(4.11)

Case B. Finite support.
Let now \( \psi(\cdot) \in \Psi(b) \), where \( 1 < b = \text{const} < \infty \). Denote as before
\[
g(x) = g[\psi](x) := -\ln \psi(1/x), \ x \in (0, 1/b),
\]
(4.12)
so that
\[
q_0 = q_0(\delta) := \arg\max_{q \in [1,b]} \left( \frac{\delta^{1/q}}{\psi(q)} \right).
\]
(4.13)

Lemma B. Suppose that the derivative \( g'(x) = g[\psi]'(x) \) there exists, is continuous in the open interval \((0, 1/b)\), is strictly increasing and such that
\[
\lim_{x \to 1/b} g'(x) = \infty.
\]
Then as \( \delta \downarrow 0^+ \)
\[ q_0(\delta) \uparrow 1/b. \]  

**Proof** follows immediately likewise before from the equation (4.11).

Let us return to the factorization equality (4.6). Introduce the following “rectangle”

\[ R := \{(p, q) : 1 \leq p, q < \infty\}. \]

Obviously,

\[ \Phi[\psi, \nu](\alpha, \beta) = \sup_{(p,q) \in T} \left[ \frac{\alpha^{1/p} \beta^{1/q}}{\psi(p) \nu(q)} \right] \leq \sup_{(p,q) \in R} \left[ \frac{\alpha^{1/p} \beta^{1/q}}{\psi(p) \nu(q)} \right] = \phi[G\psi](\alpha) \cdot \phi[G\nu](\beta). \]  

(4.14)

It remains to ground the opposite inequality, of course, for sufficiently smallest values \( \alpha \) and \( \beta \).

We consider consequently three cases.

**Case 4.1. Infinite supports.**

Suppose the two functions \( \psi = \psi(p) \), \( \nu = \nu(p) \) belonging to the set \( \Psi(\infty) = \Psi \) be a given and both these functions satisfy to the conditions of lemma A, in particular, the relation (4.9):

\[ \lim_{x \to 0} g'(\psi)(x) = \lim_{x \to 0} g'(\nu)(x) = \infty. \]

We have using Lemma A:

\[ \phi[G\psi](\alpha) \cdot \phi[G\nu](\beta) = \frac{\alpha^{1/p_0(\alpha)}}{\psi_m(p_0(\alpha))} \cdot \frac{\beta^{1/p_0(\beta)}}{\psi_n(p_0(\beta))}, \]  

(4.15)

as long as

\[ \lim_{\alpha \to 0^+} p_0(\alpha) = \lim_{\beta \to 0^+} p_0(\beta) = \infty. \]

We deduce therefore for sufficiently smallest values \( \alpha \) and \( \beta \)

\[ \frac{1}{p_0(\alpha)} + \frac{1}{p_0(\beta)} < 1, \]

on the other words the optimal pair \( (p_0(\alpha), p_0(\beta)) \) belongs to the set \( T \).

To be more specifically, note that if for instance

\[ \alpha_0 := \exp(-g'[(\psi)(1/e)]) \]

and correspondingly
\[ \beta_0 := \exp \left( -g'[\nu](1/e) \right), \]
we deduce under assumptions of lemma A that when \( \alpha \leq \alpha_0, \beta \leq \beta_0 \)
\[ p_0(\alpha) \geq e, \quad p_0(\beta) \geq e \]
whence
\[ \frac{1}{p_0(\alpha)} + \frac{1}{p_0(\beta)} < 1. \]

**Case 4.2. Finite supports.**

Let two functions \( \psi = \psi(p) \) and \( \nu = \nu(p) \) be a given; suppose that these functions belongs correspondingly to the sets
\[ \psi(\cdot) \in G\psi(b_1), \quad \nu(\cdot) \in G\psi(b_2), \quad b_1, b_2 > 1. \]

Assume that the derivatives \( g[\psi]'(x) \) and \( g[\nu]'(x) \) there exist, are continuous in the open intervals \((0, 1/b_1)\), are strictly increasing and such that
\[ \lim_{x \to 1/b_1} g'[\psi](x) = \infty = \lim_{x \to 1/b_2} g'[\nu](x). \]

We conclude using Lemma B that the relation (4.6) there holds if
\[ \frac{1}{b_1} + \frac{1}{b_2} < 1. \quad (4.16) \]

To be more concrete, suppose estimate (4.16) be satisfied. Define the following values
\[ \beta_1 = \exp \left( -g'[\psi] \left( \frac{b_1 + 1}{3b_1} \right) \right), \]
\[ \beta_2 = \exp \left( -g'[\nu] \left( \frac{b_2 + 1}{3b_2} \right) \right), \]
or equally
\[ q(\beta_j) = \frac{3b_j}{b_j + 1}, \quad j = 1, 2. \]

We conclude for all the values \( \theta_j \in (0, \beta_j) \)
\[ \frac{1}{q(\theta_1)} + \frac{1}{q(\theta_2)} \leq \frac{1}{q(\beta_1)} + \frac{1}{q(\beta_2)} = \]
\[ \frac{b_1 + 1}{3b_1} + \frac{b_2 + 1}{3b_2} = \frac{2}{3} + \frac{1}{3b_1} + \frac{1}{3b_2} < 1. \]
Case 4.3. “Mixed “ case.
Assume \( \psi(\cdot) \in \Psi(\infty), \nu(\cdot) \in G_{\psi} b \), \( b = \text{const} > 1 \), and
\[
\lim_{x \to 0} g'[\psi](x) = \infty = \lim_{x \to 1/b} g'[\nu](x).
\]
We conclude likewise foregoing propositions that then the equality (4.6) there holds still in this case, since
\[
\frac{1}{\infty} + \frac{1}{b} = \frac{1}{b} < 1.
\]
To be more precisely, it is sufficient to pick the (positive) values \( \alpha_0 \) and \( \beta_0 \) as follows: \( \alpha_0, \beta_0 \in (0, 1) \) and
\[
g'[\psi] \left( \frac{b - 1}{3b} \right) = |\ln \alpha_0|,
\]
\[
g'[\nu] \left( \frac{b + 1}{2b} \right) = |\ln \beta_0|.
\]
Then \( p_0 := \frac{3b}{b-1} \) and \( q_0 := \frac{2b}{b+1} \), so that \( p_0 > 1 \),
\[
1 < q_0 = \frac{2b}{b+1} < b,
\]
and
\[
\frac{1}{p_0} + \frac{1}{q_0} = \frac{b - 1}{3b} + \frac{b + 1}{2b} < \frac{b - 1}{2b} + \frac{b + 1}{2b} = 1,
\]
as long as \( b > 1 \).

5 Main results. The case of identical spaces. Examples.

We suppose in addition to the conditions (and notations) of the last section that the \( \Psi \) functions \( \psi(\cdot), \nu(\cdot) \) coinsides: \( \psi(\cdot) = \nu(\cdot) \).

In detail: assume \( \xi \in F, \xi \in G_{\psi}, \eta \in G, \eta \in G_{\psi} \).

Theorem 5.1.
\[
|\text{Cov}(\xi, \eta)| \leq 12 \alpha(F, G) \cdot \frac{||\xi||G_{\psi} ||\eta||G_{\psi}}{\phi^2[G_{\psi}](\alpha(F, G))}.
\]  \hspace{1cm} (5.1)

Proof. Suppose for simplicity \( ||\xi||G_{\psi} = 1 = ||\eta||G_{\psi} \); then
\[
||\xi||_p \leq \psi(p), \ ||\eta||_p \leq \psi(p), \ p \in [1, b), \ b = \text{const} \in (1, \infty].
\]
We can apply again the Davydov’s inequality (1.3):

\[(12\alpha)^{-1}\left|\text{Cov}(\xi, \eta)\right| \leq \alpha^{-2/p} \psi^2(p),\]
therefore

\[\left(12\alpha\right)^{-1}\left|\text{Cov}(\xi, \eta)\right| \leq \inf_p \left[\alpha^{-2/p} \psi^2(p)\right] = \left[\sup_p \frac{\alpha^{1/p}}{\psi(p)}\right]^{-2} = \left[\frac{\phi[G\psi](\alpha)}{\phi^2[G\psi](\alpha)}\right]^{-2},\]

(5.2)

Q.E.D.

Example 5.1. Infinite supports.
Define the following \(\Psi\) functions

\[\psi_m(p) = p^{1/m}, \psi_n(p) = p^{1/n}, \quad p \in [1, \infty), \quad m, n = \text{const} > 0.\]

If \(\xi \in F, \eta \in G\), and \(\alpha = \alpha(G, F) \leq e^{-1}\), then

\[|\text{Cov}(\xi, \eta)| \leq 12 e^{1/m+1/n} \alpha \left|\ln \alpha\right|^{1/m+1/n} \left|\xi\right| \left|\eta\right| G_{\psi_m} \left|G_{\psi_n}\right|.\]

(5.3)

Example 5.2. Finite supports.
Recall the definition of the following \(\Psi\) function

\[\tau_{b,\beta}(p) = (b - p)^{-\beta}, \quad p \in [1, b).\]

(5.4)

Here \(b = \text{const} \in (1, \infty), \quad \beta = \text{const} \geq 0.\) As we knew, the fundamental function for these space has a form

\[\phi[G\tau_{b,\beta}](\delta) = \frac{b^{\beta - 1} \beta^\delta b^{1/b}}{\ln \delta^{b/\beta}} =: K(b, \beta) \delta^{1/b} \ln \delta^{-\beta}, \quad \delta \to 0 + .\]

(5.5)

This relation allows us to calculate the required covariation. Namely, if \(\xi \in F, \eta \in G\tau_{b_1,\beta_1}, \eta \in G, \eta \in G\tau_{b_2,\beta_2}, \quad b_{1,2} = \text{const} \in (1, \infty), \quad \beta_{1,2} = \text{const} \geq 0\) and \(\alpha = \alpha(G, F) \leq e^{-1}\), then

\[|\text{Cov}(\xi, \eta)| \leq 12 K(b_1, \beta_1) K(b_2, \beta_2) \alpha^{1-1/b_1-1/b_2} \left|\ln \alpha\right|^{\beta_1+\beta_2} \times \left|\xi\right| \left|\eta\right| G_{\tau_{b_1,\beta_1}} \left|G_{\tau_{b_2,\beta_2}}\right|.\]

(5.6)

if of course \(1/b_1 + 1/b_2 < 1\).

Example 5.3. “Mixed “ case.

Assume \(\xi \in F, \xi \in G_{\psi_m}, m = \text{const} > 0; \eta \in G, \eta \in G_{\tau_{b,\beta}}, b = \text{const} > 1, \quad \beta = \text{const} > 0\), and denote as before \(\alpha = \alpha(F, G)\). We obtain after some calculations
\[ |\text{Cov}(\xi, \eta)| \leq 12 \ (\text{em})^{1/m} K(b, \beta) \alpha^{1-1/b} |\ln \alpha|^{\beta+1/m} \times \]
\[ ||\xi||G_{\psi b} ||\eta||G_{\tau b, \beta}. \] (5.7)

**Remark 5.1.** For the “greatest” values \( \alpha \) and \( \beta \), say \( \alpha \geq 1/e \), one can use the trivial estimate 1.4a.

**Example 5.4.** “Combined” event.

Suppose here that \( \xi \in G_{\psi b} \) for some \( b = \text{const} > 1 \) and that \( \eta \in L(q(0)) \), where \( q(0)' < b \). We derive consequently \( |\xi|_p \leq ||\xi||G_{\psi} \psi(p), \ 1 \leq p < b; \)
\[ (12\alpha)^{-1} \alpha^{1/q(0)} (|\text{Cov}(\xi, \eta)|||\xi||G_{\psi} |\eta|_{q(0)}) \leq \alpha^{-1/p} \psi(p), \ p \geq q'(0), \]
whence
\[ (12\alpha)^{-1} \alpha^{1/q(0)} (||\text{Cov}(\xi, \eta)||)/(|\xi||G_{\psi} |\eta|_{q(0)}) \leq= \]
\[ \inf_{p \in [q'(0), b]} \alpha^{-1/p} \psi(p) = \frac{1}{\phi_{q(0)}[G_{\psi}](\alpha)}. \]

Thus, we deduce in the considered case
\[ |\text{Cov}(\xi, \eta)| \leq 12 \alpha^{1-1/q(0)} \frac{||\xi||G_{\psi} \cdot |\eta|_{q(0)}}{\phi_{q(0)}[G_{\psi}](\alpha)}. \]

6 **Application to the classical CLT.**

Let \( \gamma(i), \ i = 0, \pm 1, \pm 2 \ldots \) be a centered strictly stationary sequence of r.v. A new denotations:
\[ S(n) := n^{-1/2} \sum_{i=1}^{n} \gamma(i), \ n = 1, 2, \ldots ; \] (6.0)
\[ \Sigma(n) := \text{Var}(S(n)), \ \Sigma := \lim_{n \to \infty} \Sigma(n); \] (6.1)
\[ F_0 := \sigma\{\gamma(i), \ i \leq 0\}; \ F_k := \sigma\{\gamma(j), \ j \geq k\}; \]
\[ \alpha(k) := \alpha \left(F_0, F_k\right); \ \beta(k) := \beta \left(F_0, F_k\right). \] (6.2)

Introduce also the following \( \Psi \) function as a natural function for the sequence \( \{\gamma(i)\} : \)
\[ \psi[\gamma](p) := |\gamma(0)|_p; \] (6.3)
if of course there exists for some value \( p \) greatest than one.

Recall that the sequence \( \gamma(\cdot) \) satisfies the CLT, iff \( \exists \Sigma \in (0, \infty) \) and the sequence \( \{S(n)\} \) converges in distribution as \( n \to \infty \) to the centered normal (Gaussian) law with variance \( \Sigma \).

The methods of obtaining CLT for the random sequences satisfying some mixing conditions are well known, see e.g. [9], [16], [18], [19] etc. The essential moment for this proof is the following variation estimate, which follows immediately from the foregoing covariation estimates.

**Theorem 6.1.** Assume that the function \( \psi[\gamma](p) \) is not trivial: \( \exists p_0 > 1 \Rightarrow \psi[\gamma](p_0) < \infty \). Define also a numerical positive sequence

\[
y(k) = y[\gamma](k) := \frac{\alpha(k)}{\phi^2[G\psi[\gamma]](\alpha(k))}, \quad k = 2, 2, \ldots.
\]

(6.4)

If

\[
\sum_{k=2}^{\infty} y[\gamma](k) < \infty,
\]

then the value \( \Sigma \) there exists and is finite: \( \Sigma \in [0, \infty) \).

Define now a new \( \Psi \) function

\[
\zeta[\psi](p) := \psi(p) \psi(p/(p - 1)).
\]

(6.5)

and the correspondent numerical sequence, also positive

\[
z(k) = z[\gamma](k) := \frac{1}{\phi[G\zeta[\psi]](1/\beta(k))}.
\]

(6.6)

**Theorem 6.2.** We deduce under formulated above notations and conditions that if

\[
\sum_{k=2}^{\infty} z[\gamma](k) < \infty,
\]

then the value \( \Sigma \) there exists and is finite: \( \Sigma \in [0, \infty) \).

7  Concluding remarks.

A. Offered here results may be easily generalized onto another types of mixing, as well as onto others r.i. spaces: Lorentz, Marcinkiewicz etc. instead GLS. All we need - the source \( L(p), L(q) \) estimate of the form

\[
|\text{Cov}(\xi, \eta)| \leq h(p, q) |\xi|_p |\eta|_q, \quad (p, q) \in D,
\]

(7.1)
where $D$ is arbitrary domain in the positive quarter plane. Suppose $\xi \in G\psi$, $\eta \in G\nu$. Then

$$|\text{Cov}(\xi, \eta)| \leq \inf_{(p,q) \in D} \left[ h(p,q) \psi(p) \nu(q) \right] \|\xi\|_{G\psi} \|\eta\|_{G\nu}. \quad (7.2)$$

B. It is interest by our opinion to derive the lower bound for considered covariance for different Banach spaces.

C. Define the following $\psi(r) = \psi(r)(p)$, $r = \text{const} > 1$, $-\text{ function as follows:}$

$$b(\psi(r)) = r \quad \text{and} \quad \psi(r)(p) = 1, \; 1 \leq p \leq r.$$ 

One can define formally $\psi(r)(p) = \infty$, $p > r$. Then

$$\|f\|_{G\psi(r)} = |f|_{L^r}.$$ 

Thus, the classical Lebesgue - Riesz spaces $L^r$ are a particular, more precisely, extremal cases of the Grand Lebesgue ones.

As long as the fundamental function for $L^r(\Omega)$ space built by atomless measure $P$ is equal

$$\phi[L^r](\delta) = \delta^{1/r}, \; \delta \in [0,1],$$ 

we derive the Davydov’s estimate (1.3) in turn from the proposition of theorem 4.1; as well as the inequality (1.4) follows from theorem (3.1).

As a slight consequence: both the assertions of theorems (3.1) and (4.1) are in general case essentially non-improvable.

D. The case of the so-called $\rho$ - mixing is very simple for covariation estimation, see e.g. (20).

References.

1. Bennet C., Sharpley R. Interpolation of operators. Orlando, Academic Press Inc., (1988).

2. Billingsley P. Convergence of probability measures. New York: Wiley 1968.

3. Bradley, R.C. (2007). Introduction to strong mixing conditions. Vol. 1,2,3. Kendrick Press.

4. Bryc, W. and Dembo, A. Large deviations and strong mixing. Ann. Inst. Henri Poincar’e, 32. (1996), pp. 549569.
5. Buldygin V.V., Kozachenko Yu.V. *Metric Characterization of Random Variables and Random Processes.* 1998, Translations of Mathematics Monograph, AMS, v.188.

6. Davydov Yu.A. *Convergence of distributions generated by stationary stochastic processes.* Theory Probab. Appli. 13, 691-696.

7. A. Fiorenza. *Duality and reflexivity in grand Lebesgue spaces.* Collect. Math. 51, (2000), 131 - 148.

8. A. Fiorenza and G.E. Karadzhov. *Grand and small Lebesgue spaces and their analogs.* Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico 272/03, (2005).

9. Ibragimov I.A., Linnik Yu.A. *Independent and stationary sequences of random variables.* Groningen Wolters-Noordhoff, 1971

10. T. Iwaniec and C. Sbordone. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat.Mech. Anal., 119, (1992), 129-143.

11. Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian Type.* Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43-57.

12. Florence Merlev’ede, Magda Peligrad, Emmanuel Rio. *Bernstein inequality and moderate deviations under strong mixing conditions.* Edited by Christian Houdr’e, Vladimir Koltchinskii, David M. Mason and Magda Peligrad. High dimensional probability V : the 5th International Conference (HDP V), May 2008, Luminy, France. Institute of Mathematical Statistics, Beachwood, OH, pp.273-292, 2009. ¡inria-00360856¿

13. Ostrovsky E.I. (1999). *Exponential estimations for Random Fields and its applications,* (in Russian). Moscow-Obninsk, OINPE.

14. Ostrovsky E. and Sirota L. *Entropy and Grand Lebesgue Spaces approach for the problem of Prokhorov - Skorokhod continuity of discontinuous random fields.* arXiv:1512.01909v1 [math.Pr] 7 Dec 2015

15. Ostrovsky E. and Sirota L. *Fundamental function for Grand Lebesgue
16. Magda Peligrad and Sergey Utev. *Central Limit Theorem for linear processes*. The Annals of Probability, 1997, Vol 25, N 1, 443 - 456.

17. Rio E. *Covariance inequalities for strongly mixing processes*. 1993, Ann. Inst. of H. Poincare, 29, 589 - 597.

18. S. Utev, M. Peligrad. *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*. Journal of Theoretical Probability, 16 (2003). 101 - 115.

19. K. Yoshishara. *Moment inequalities for mixing sequences*. Kodai Math. J., 1, (1978), 316-328.

20. Q.Y. Wu, Y.Y. Jiang. *Some strong limit theorems for mixing sequences of random variables*. Statistics and Probability Letters, 78, (2008), 1017 - 1023.