On trisecant lines to White surfaces

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Abstract

Inspired by an argument of Gambier, we show that the only White surface of \( \mathbb{P}^5 \) having a 4-dimensional trisecant locus is the Segre polygonal surface. This allows us to deduce that the generic point of the principal component of the subvariety \( W_{18}[5] \) of 18-tuples special in degree 5 of the Hilbert scheme of 18 points of the plane corresponds to a smooth 18-tuple of points in uniform position, not lying on any quartic. This refines, in this particular case, the general bound due to Coppo. We also give the number of trisecant lines, counted with multiplicity, which pass through the generic point of a White surface of non-Segre type.

Key words: Trisecant lines, linear systems of plane curves

1 Introduction

Let \( S \) be a smooth, non-degenerate surface of \( \mathbb{P}^5 \). In order to obtain properties of \( S \), it is classical to study the generic projection of \( S \) to \( \mathbb{P}^3 \), i.e., the image \( \overline{S} \) of \( S \) under the projection from a generic line \( L \) of \( \mathbb{P}^5 \). Let \( x \) be a point of \( \overline{S} \), we denote by \( \pi^{-1}(x) \) the scheme theoretical fiber of \( \pi \) over \( x \). Since \( \pi \) is a generic projection, \( \pi^{-1} \) is 0-dimensional. So, the length of the fibers of \( \pi \) provides a natural stratification of \( \overline{S} \):

\[
\cdots \overline{S}_{k+1} \subseteq \overline{S}_k \subseteq \cdots \overline{S}_1 = \overline{S},
\]

where, for all integers \( i \geq 1 \),

\[
\overline{S}_k := \{ x \in \overline{S} | \text{length}(\pi^{-1}(x)) \geq k \}
\]

is the set of \( k \)-tuple points of the projection \( \pi \). The following well-known result gives a stratification theorem for surfaces: the hypersurface \( \overline{S} \) contains a curve of double points, a finite number of triple points and no quadruple points ([17]...
p. 611-618 for this and further properties of the singularities of $\mathcal{S}$). Since the projection is generic, over each triple point of $\mathcal{S}$ lie three distinct points of $S$. Unfortunately, the classical proof of this result, as for instance in Griffiths and Harris’ book [17] p. 611-618 (or [3] chapter 9 in arbitrary dimension), is false. Indeed, as a corollary of the proof those three points must be in general position (see Dobler [9]). Equivalently, the dimension of the trisecant lines locus of $S$ is at most 3. From the classical proof of the stratification theorem, one can also deduce the following corollary ([17] top of page 613): “among 3-planes of $\mathbb{P}^5$ meeting $S$ at points not in general position, the generic one contains 4 points of $S$ spanning a 2-plane, and not three colinear points”.

Since there exist surfaces of $\mathbb{P}^5$ having a 4-dimensional trisecant line locus, e.g. smooth special Enriques surfaces (Conte and Verra [4], Dolgachev and Reider [10]) or Segre polygonal surfaces (Segre [28], Dobler [9]), the classical proof of the stratification theorem contains a mistake; we refer to Dobler’s thesis [9] for a detailed discussion of this matter. Following Dobler, we wish to point out, that surfaces of $\mathbb{P}^5$ with a 4-dimensional trisecant lines locus give counterexamples to both corollaries of the classical proof of the stratification theorem for surfaces. The stratification theorem holds nonetheless; indeed, the results of Mather ([23] and [24]) provide a stratification theorem for generic projections to hypersurfaces up to dimension 14, without any indication of the postulation of points in the fibers over multiple points, though.

We should point out that the existence of surfaces with a 4-dimension trisecant lines locus and, more generally, of embedded projective $n$-folds with fibers of unexpected postulation for a generic projection to $\mathbb{P}^{n+1}$ is the source of great difficulties in applying the general projection method to establish good Castelnuovo-Mumford regularity upper-bounds. For a discussion of the general projection method, see Kwak’s article [21]. This problem was raised by Greenberg [16], who showed how to bypass this issue for smooth surfaces, in view of establishing Castelnuovo’s regularity bound (Pinkham [26], Lazarsfeld [22]).

A dimension count suggests that the trisecant lines locus of a surface in $\mathbb{P}^5$ should be 3-dimensional; so, surfaces in $\mathbb{P}^5$ with a 4-dimensional trisecant lines locus are said to have an excess of trisecant lines.

Very few examples of surfaces with an excess of trisecant lines are known and no classification has been established so far. The first example of a surface with an excess of trisecant lines, known as the Segre polygonal surface in the literature, was constructed by Segre [28] in 1924. It belongs to the family of White surfaces, which was constructed by White and Gambier, also in 1924 [13,30]. A modern reference is the paper of Gimigliano [15].

In his thesis [9], Dobler shows that the Segre polygonal surface is the only
polygonal surface with a 4-dimensional trisecant lines locus. Moreover, he shows that the Segre polygonal surface is a degeneration of smooth special Enriques surfaces in $\mathbb{P}^5$. These surfaces are the only smooth surfaces in $\mathbb{P}^5$ with an excess of trisecant lines (Conte and Verra [4], Dolgatchev and Reider [10]).

Since polygonal surfaces belong to the family of White surfaces, it is natural to ask whether there exist other White surfaces with an excess of trisecant lines.

The purpose of this work is to show that the only White surfaces in $\mathbb{P}^5$ with an excess of trisecant lines are the Segre polygonal surfaces. To achieve this, we use a remarkable argument by Gambier [13, p. 184-186 and p. 253-256]. Most of this work is intended to give modern rigor to his approach. Furthermore, this result allows us to deduce information on the generic point of the principal component of $W_{18}\mathbb{P}^5$, the subvariety of the Hilbert scheme of 18-tuples of $\mathbb{P}^2$ special in degree 5. We prove the following results.

**Theorem 1** Let $S_5$ be a possibly singular White surface in $\mathbb{P}^5$ and $p$ a generic point on $S_5$.

(i) Through the point $p$ there passes at least one 3-secant line.
(ii) The surface $S_5$ has an excess of trisecant lines, if and only if $S_5$ is a Segre polygonal surface.
(iii) If $S_5$ doesn’t have an excess of trisecant lines, then exactly 6 trisecant lines, counted with multiplicity, pass through $p$.

As a corollary of (i) we obtain

**Corollary 2** The generic point of the principal component of $W_{18}\mathbb{P}^5$ is a smooth uniform 18-tuple of points in the plane not lying on any curve of degree 4.

2 Notations and basic facts

2.1 Linear systems of plane curves of given degree with assigned base points

Let $S$ denote either $\mathbb{P}^2$ or a surface obtained from $\mathbb{P}^2$ by a finite succession of blow-up at a single point, called $\sigma$-process. Let $S \xrightarrow{\sigma} \mathbb{P}^2$ denote the composition of these $\sigma$-processes. If $S = \mathbb{P}^2$, we set $\phi = id_{\mathbb{P}^2}$. Let $L$ denote the linear equivalence class of a line in $\mathbb{P}^2$. A curve on $S$ is of degree $d$ if it is linearly equivalent to $d\phi^*L$. Let us recall briefly the facts regarding linear systems of
curves of degree $d$ with assigned base points and multiplicities on $S$.

**Definition 3 (n-tuples of points on $S$)** An $n$-tuple of points $P$ on $S$ is the data of

1. a sequence of distinct points on $S$, $\{p_1, \cdots, p_k\}$ (These points are the **base points** of $P$.)
2. and a sequence of non-negative integers $\{m_1, \cdots, m_k\}$ (These integers are the **multiplicities** of $P$).

such that $\sum_{i=1}^{k} m_i = n$.

If $n = k$, we say that $P$ consists of distinct points. Let $P$ be a $n$-tuple of points on $S$. Let $\tilde{S} \twoheadrightarrow S$, denote the blow-up of $S$ at the base points of $P$. For $i \in \{p_1, \cdots, p_k\}$, let $E_i$ denote the component of the exceptional locus of $\pi$ contracted to $p_i$ by $\pi$. The curves on $S$ whose strict transform by $\pi$ are linearly equivalent to $d(\phi \circ \pi)^*(L) - \sum_{i=1}^{k} m_i E_i$, form a linear system: the **linear system of curves of degree** $d$ on $S$ passing through $P$. The plane curves of this system are the vanishing locus of a polynomial in $H^0(\mathbb{P}^2, \mathcal{I}_P|_{\mathbb{P}^2}(d))$; thus, they are parameterized by points of the projective space $|\mathcal{I}_P(d)| := \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{I}_P|_{\mathbb{P}^2}(d)))$.

If the linear system $|d(\phi \circ \pi)^*(L) - \sum_{i=1}^{k} m_i E_i|$ is base-point free, we say that the linear system $|\mathcal{I}_P(d)|$ is **complete**.

Recall that the **virtual dimension** of $|\mathcal{I}_P(d)|$ is given by

$$v(\mathcal{I}_P(d)) = \frac{d(d + 3)}{2} - \frac{1}{2} \sum_{i=1}^{k} m_i(m_i + 1).$$

We denote by $s$ the irregularity of the linear system $|\mathcal{I}_P(d)|$

$$s := \dim H^1(\mathbb{P}^2, \mathcal{I}_P(d)) = \dim(|\mathcal{I}_P(d)|) - \text{Max}(v(\mathcal{I}_P(d)), -1).$$

The linear system $|\mathcal{I}_P(d)|$ defines a rational surface $X$ in $\mathbb{P}^N = |\mathcal{I}_P(d)|^\vee$, image of $S$ by the rational map

$$\Phi : S \to \mathbb{P}^N$$

$$q \mapsto |\mathcal{I}_{P \cup q}(d)|^\vee$$

(1)

The surface $X$ is of degree $\mathcal{I}_P(d)^2$, the intersection number of two curves of the linear system $|\mathcal{I}_P(d)|$, and of sectional genus $\pi(X) = g_{\mathcal{I}_P(d)}$, the arithmetic genus of a curve in $|\mathcal{I}_P(d)|$.

Following Dobler, we use a non-standard definition of a trisecant line; it is not defined as a line meeting $S$ along a 0-dimensional scheme of degree at least 3.
Definition 4 (Trisecant lines) A line \( l \) of \( \mathbb{P}^N \) is a trisecant line to \( X \) if \( l \) does not lie on \( X \) and cuts \( X \) at (at least) three distinct points.

The trisecant lines locus of \( X \) is then the closure in \( \mathbb{P}^N \) of the union of all the trisecant lines to \( X \).

A generic trisecant line to \( X \) passing through a generic point \( \Phi(q) \) of \( X \) can be directly seen on the linear system \( |\mathcal{I}_P(d)| \). Indeed, such a line corresponds to a pair of distinct points \( \Pi := (\pi_1, \pi_2) \) in \( S \) such that the sublinear system of \( |\mathcal{I}_P(d)| \) of curves containing \( P \cup \Pi \cup \{q\} \) is 1-irregular. We call such a pair \( \Pi \) an associated pair to \( |\mathcal{I}_P(d)| \) at \( q \). Note that, with our special definition of trisecant lines, an associated pair to \( |\mathcal{I}_P(d)| \) at \( q \) defines a trisecant line to \( X \) at \( \Phi(q) \), if and only if the three points \( \Phi(q), \Phi(\pi_1), \Phi(\pi_2) \) are distinct on \( X \).

2.2 Linear systems of plane curves of given degree through fixed intersection cycles

Let \( D_1 \) and \( D_2 \) be two irreducible plane curves. Consider \( Q := D_1 \cap D_2 \) the 0-scheme of intersection of these two curves. Let \( \{p_1, \ldots, p_m\} \) be the support of \( Q \). For every point \( p_i \in \text{Supp} Q \), let \( n_i := \mu_{p_i}(D_1, D_2) \) denote the multiplicity of intersection of \( D_1 \) and \( D_2 \) at the point \( p_i \). Then \( Q \) gives rise to the 0-cycle of the plane \( \sum_{i=1}^m n_ip_i \), that we still denote by \( Q \). We say that \( Q \) consists of distinct points if \( n_i = 1 \) for all \( i = 1, \ldots, m \). The linear system of plane curves of degree \( d \) passing through the intersection cycle \( Q \) is \( \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{I}_Q(\mathbb{P}^2(d)))) \), where \( \mathcal{I}_Q(\mathbb{P}^2) \) is the ideal sheaf defining the 0-scheme \( Q \) in \( \mathbb{P}^2 \). If \( Q \) consists of distinct points, this linear system is just the linear system of curves of degree \( d \) passing through the \( p_i \)'s.

Assume that \( Q \) does not consist of distinct points; we may assume that \( n_i \geq 2 \) for \( i = 1, \ldots, k \) and \( n_i = 1 \) otherwise. Suppose, for simplicity, that both curves \( D_1 \) and \( D_2 \) are smooth. A plane curve \( C \) of degree \( d \) belongs to \( \mathcal{L} \) if and only if it contains the 0-scheme \( Q = D_1 \cap D_2 \). Let \( S^{(1)} \xrightarrow{\phi_1} \mathbb{P}^2 \) denote the blow-up of \( \mathbb{P}^2 \) at \( \{p_1, \ldots, p_k\} \). For \( i = 1, \ldots, k \), we define \( E_i^{(1)} \) to be \( \phi_1^{-1}(p_i) \). Denote by \( Q^{(1)} \) (resp. \( D_1^{(1)} \) and \( D_2^{(1)} \)) the strict transform of the scheme \( Q \) (resp. \( D_1 \) and \( D_2 \)) by \( \phi_1 \); then, \( Q^{(1)} \) is the 0-scheme of intersection on \( S^{(1)} \) of the curves \( D_1^{(1)} \) and \( D_2^{(1)} \). We have, moreover

\[
\phi_1^{-1}(D_1) \cap \phi_1^{-1}(D_2) = Q^{(1)} \cup E_1^{(1)} \cup \cdots \cup E_k^{(1)}
\]

Thus, the strict transform \( \mathcal{L}^{(1)} \) by \( \phi_1 \) of the linear system \( \mathcal{L} \) is the system of curves of \( S^{1} \) linearly equivalent to

\[
\phi_1^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^k E_i^{(1)}
\]
and containing the 0-scheme \( Q^{(1)} \).

Recall that the multiplicity at a point \( p \) of an irreducible curve \( C \) lying on a surface is the sum of the multiplicities of infinitely near points of \( C \) of order 1 over \( p \) (see [2, p.33]). Moreover, there is a point \( q \in \text{Supp}(Q^{(1)}) \) such that \( \phi_1(q) = p_i \) if and only if \( n_i \geq 1 \). Since we have assumed that both \( D_1 \) and \( D_2 \) are smooth, such a point \( q \) is unique, if it exists; we denote it by \( p_1^{(1)} \). The point \( p_1^{(1)} \) is an infinitely near point of order 1 over \( p_i \) for both \( D_1 \) and \( D_2 \). Let \( n_i^{(1)} := \mu_{p_i^{(1)}}(D_1^{(1)}, D_2^{(1)}) \). If \( Q^{(1)} \) does not consist of distinct points, repeat the process of blowing-up the multiple points of the support of \( Q^{(1)} \). This process stops after a finite number of steps (see [25, theorem 4.2.5]). We get a sequence of blow-up maps

\[
S^{(l)} \xrightarrow{\phi_l} S^{(l-1)} \xrightarrow{\phi_{l-1}} \ldots S^{(1)} \xrightarrow{\phi_1} \mathbb{P}^2,
\]

such that the intersection cycle \( Q^{(l)} \) consists of distinct points and the strict transform by \( \phi := \phi_1 \circ \ldots \circ \phi_l \) of the linear system \( \mathcal{L} \) is the linear system of curves in

\[
|\phi^*(\mathcal{O}_{\mathbb{P}^2}(d))| - \sum_{i=1}^k \sum_{j=1}^{n_i-1} E_i^{(j)}|
\]

passing through the distinct points of \( Q^{(l)} \). Indeed, there are only \( n_i - 1 \) cycles among \( Q^{(1)}, \ldots, Q^{(l)} \) whose support contains a point over \( p \), for we have [2, p.33]

\[
\mu_{p_i}(D_1, D_2) = \mu_{p_i}(D_1)\mu_{p_i}(D_2) + \sum_{x \in \text{A}(p_i)} \mu_x(D_1)\mu_x(D_2),
\]

where \( \text{A}(p_i) \) is the set of infinitely near points of both \( D_1 \) and \( D_2 \) over \( p_i \). We can formally replace the cycle \( Q \) by the cycle of “distinct or infinitely near points” \( \sum_{i=1}^k (p_i + \sum_{j=1}^{n_i-1} p_i^{(j)}) + \sum_{k=1}^m p_k \). The linear system \( \mathcal{L} \) is thus the linear system of curves of degree \( d \) passing through the proper distinct points \( p_1, \ldots, p_m \) and the infinitely near points \( p_1^{(1)}, \ldots, p_i^{(n_i-1)}, \ldots, p_k^{(1)}, \ldots, p_k^{(n_k-1)} \).

Of course, a similar analysis can be made to understand better the linear system \( \mathcal{L} \) in case \( D_1 \) or \( D_2 \) is singular.

### 2.3 Standard irregularity estimation techniques

Let \( S = \mathbb{P}^2 \). Recall the standard geometric interpretation of the irregularity of pencils. Let \( |\mathcal{I}_P(d)| \) be as in section 2.2. Suppose that \( |\mathcal{I}_P(d)| \) contains a pencil, i.e.\( \dim \mathcal{H}^0(\mathcal{I}_P(d)) \geq 2 \). Suppose, moreover, that the linear system \( |\mathcal{I}_P(d)| \) is complete and that all the multiplicities of \( P \) are equal to one.

Any two distinct curves \( D \) and \( D' \) of the linear system \( |\mathcal{I}_P(d)| \) meet in a 0-dimensional scheme. Let \( Q \) denote the cycle of intersection \( D \cdot D' \setminus P \). As \( D' \) varies in \( |\mathcal{I}_P(d)| \), the cycles \( Q \) fit into a linear series \( \chi(D, |\mathcal{I}_P(d)|) \) on \( D \),
called the characteristic series cut on $D$. A member of this series is said to be residual to $P$ on $D$.

**Proposition 5 (Duality Theorem)** Let $|I_P(d)|$ be as above. Let $D$ be a generic curve in $|I_P(d)|$ and $Y \in \chi(D, |I_P(d)|)$. Denote by $s$ the irregularity of $|I_P(d)|$. Then, $s = \dim H^0(\mathbb{P}^2, I_Y|_{\mathbb{P}^2}(d-3))$, where $I_Y|_{\mathbb{P}^2}$ is the ideal sheaf defining $Y$ in $\mathbb{P}^2$.

For a proof of this proposition see Griffiths and Harris’ book [17, p.713-716].

In order to estimate the irregularity of $|I_P(d)|$, we need to control the configuration of the group of points residual to $P$ with respect to the linear system. This is taken care of by the classical residuation theorem, which is a direct rewriting of Noether "AF+BG" theorem. We require the stronger version of it, which allows singular points in the intersection cycles. It can be found, for instance, in Walker’s book on algebraic curves [29, theorem 7.2].

**Proposition 6 (residuation theorem)** Let $C_n$ and $C_m$ be plane curves of degree $n$ and $m$. Using cycle notation, we write $C_n \cdot C_m = P + Q$. Suppose that there are two integers $n_1$ and $n_2$ such that $n_1 + n_2 \geq n$ and there exist two curves of degree $n_1$ and $n_2$, respectively, such that:

$$C_{n_1} \cdot C_m = P + P'$$

and

$$C_{n_2} \cdot C_m = Q + Q'.$$

Moreover assume that

1. either $C_m$ is smooth at any point of the support of $P + Q + Q' + P'$,
2. or, if $n_2 = 0$ (classical residuation theorem), that $C_m$ is smooth at the points of $C_m \cdot C_n$.

Then there exists a curve of degree $n_1 + n_2 - n$ such that

$$C_m \cdot C_{n_1 + n_2 - n} = P' + Q'.$$

2.4 White and polygonal surfaces

Let $d \geq 5$ be an integer.

**Definition 7 (White surfaces)** Pick $\binom{d+1}{2}$ distinct points $P$ in the plane, not lying on any curve of degree $d-1$; so the linear system $|I_P(d)|$ is complete and regular. The non-degenerate rational surface $S_d$ it defines in $\mathbb{P}^d$, is called a White surface (Gimigliano [15]).
We should stress that Gimigliano studies much more general White surfaces than the classical ones we are interested in.

**Definition 8 (polygonal surfaces)** A simple way to construct $\binom{d+1}{2}$ points of $\mathbb{P}^2$ not lying on any curve of degree $(d-1)$ is to take the points of intersection of $(d+1)$ lines general enough. White surfaces obtained this way are called polygonal surfaces.

**Definition 9 (Segre polygonal surfaces)** A Segre polygonal surface is a polygonal surface obtained by imposing that the $(d+1)$ lines defining the polygonal surface are tangent to a fixed conic.

We refer the reader to Gimigliano’s paper [15] for general properties of White surfaces. We gather here the properties we need further on.

**Proposition 10 (Gimigliano [15])** Let $S_d$ be a White surface and $P$ the points in $\mathbb{P}^2$ used to construct $S_d$. Let $\overline{P}$ be the blow-up of the plane along $P$ and $\phi$ the map $\overline{P} \to \mathbb{P}^d$ defined by $|I_P(d)|$.

(i) If no $d$ points of $P$ are aligned, the map $\phi$ is an embedding.
(ii) Under the map $\phi$, a line $l$ containing $d$ of the points $P$ is contracted to a $(d-1)$-fold point. Any two such lines have to meet in a point of $P$.
(iii) All the singularities of $S_d$ are obtained by contraction of lines containing $d$ base points.

For a proof, see Gimigliano [15] and Dobler [9].

Dobler deduces the following result on trisecant lines to smooth White surfaces from Bauer’s results on inner projections [1].

**Proposition 11 (Dobler, [9] proposition 2.7 )** Let $S_5$ be a smooth White surface in $\mathbb{P}^5$ and $q$ a generic point of $S_5$. Then, there exists a trisecant line to $S_5$ passing through $q$.

Dobler also proves the following results concerning the trisecant locus of polygonal surfaces.

**Proposition 12 (Dobler, [9] proposition 3.17)** Among polygonal surfaces in $\mathbb{P}^d$, only the Segre polygonal surfaces have a 4-dimensional trisecant locus. In fact, for any non-Segre polygonal surface, there are at most finitely many trisecant lines to $S_d$. For a general polygonal surface there are none.
2.5 The variety parameterizing $N$-tuples of points special in degree $d$

In this section we briefly sketch the results regarding the (generally non-
irreducible) varieties $W_N[d]$ that we will use later on; for further details, we
refer the reader to Coppo’s paper [7] and Ellia and Peskine’s paper [11].

Following Ellia and Peskine, we call group of points a 0-dimensional sche-
me of $\mathbb{P}^2$. A group of points $Z$ is special in degree $d$ if $H^1(\mathbb{P}^2, I_Z|\mathbb{P}^2(d)) \neq 0$. The
groups of points of degree $N$ (called $N$-tuples of points in Coppo’s termi-
nology) special in degree $d$ are parameterized by a possibly non-irreducible
subvariety, $W_N[d]$, of the Hilbert scheme $Hilb^N(\mathbb{P}^2)$ of groups of points of
length $N$.

Let $Z$ be such a group of points. Choose a generic line $l$ in $\mathbb{P}^2$, so that it
avoids the support of $Z$. We may assume its equation is given by $z = 0$, where
$x, y, z$ are projective coordinates of the plane. Let us recall that the coordinate
ring $A_Z$ of a 0-dimensional scheme $Z$ in $\mathbb{P}^2$ has a graded $A=\mathbb{C}[x,y]$-module
structure, which has a $A$-module resolution of the form

$$0 \to \bigoplus_{i=0}^{s-1} A[-n_i] \to \bigoplus_{i=0}^{s-1} A[-i] \to A_Z \to 0$$

with $n_0 \geq n_1 \cdots \geq n_{s-1} \geq s$. These positive integers $n_i$ give a numerical
invariant for $Z$ introduced by Gruson and Peskine [18], the numerical character
$\chi(Z) = (n_0, \cdots, n_{s-1})$. The integer $s$, called the height of $\chi(Z)$, is the minimal
degree of a plane curve containing $Z$. The index of specialty of $Z$, $n_0 - 2$, is
the greatest integer $n$ such that $H^1(I_Z(n)) \neq 0$. For the moment, assume that
the linear system $|I_Z(d)|$ is complete.

If $Z$ is generic among the $N$-tuples of points of character $\chi(Z)$, the super-
abundance of $|I_Z(d)|$ can be computed directly from $\chi(Z)$ by the following
formula [7, §1.1]

$$h^1(I_Z(d)) = \sum_{i=0}^{s-1} (n_i - d - 1)^+ - (i - d - 1)^+,$$

where $(n)^+$ is zero, if the integer $n$ is negative, and $n$ otherwise.

Of course, the degree of $Z$ can be recovered from $\chi(Z)$:

$$\deg(Z) = \deg(\chi) := (\sum_{i=0}^{s-1} n_i) - s - 1.$$

The following proposition is useful to compute $\chi$ from the geometry of the
group of points $Z$. 
Proposition 13 (Davis [8], Ellia-Peskine [11]) Let \( \chi = (n_0, \ldots, n_{s-1}) \) be the numerical character of a group of points \( Z \). Suppose that for some index \( 1 \leq t \leq s - 1 \) we have \( n_{t-1} > n_t + 1 \). Then there exists a curve \( T \) of degree \( t \) such that, if

1. \( Z' \) is the group of points \( Z \cap T \),
2. \( f = 0 \) is the local equation of \( T \) in \( A \) and
3. \( Z'' \) is the residual group of points of \( Z' \) in \( Z \) with respect to \( T \),

we have \( A_{Z''} = f A \) and \( \chi(Z'') = (m_i) \) with \( m_i = n_{t+i} - t \).

3 Projection of a surface from a multisecant line

Let \( H \) denote the class of hyperplanes in \( \mathbb{P}^5 \). Classically, we get geometric information on a surface \( X \) from the study of its double locus \( D \) by a generic projection to \( \mathbb{P}^3 \). If \( X \) is normal of degree \( d \) in \( \mathbb{P}^r \) and \( K_X \) is its canonical divisor, then a well-known consequence of the theory of subadjoint systems is that the class of \( D \) in \( \text{Pic}(X) \) is \( (d - 4)H - K_X \) ([27], [31]).

We need to establish a similar formula for the double locus of the projection of a smooth surface \( X \), if the center of projection is a multisecant line. This can be done by viewing this projection as a limit of regular projections, following a method that Franchetta ([12], [5]) used to prove his famous theorem on the irreducibility of the double locus of a generic projection of a codimension two surface.

Proposition 14 Let \( X \) be a smooth, non-degenerate, surface of degree \( d \) in \( \mathbb{P}^5 \). Assume, moreover, that \( l \) is a line in \( \mathbb{P}^5 \) cutting \( X \) along a 0-dimensional scheme of multiplicity \( \delta \). Let us consider the projection \( \pi_0 \) of \( X \) to \( \mathbb{P}^3 \) with center of projection \( l \). Suppose that any plane containing \( l \) intersects \( X \) at most in a finite number of points.

Then the class, in \( \text{Pic}(X) \), of the double locus \( D \) of \( \pi \) is given by

\[
D = (d - 4 - \delta)H - K_X.
\]  

We believe it is a classical result; although, we could not find any reference.

**Proof.** If the plane \( < q, l > \) meets \( X \) for all points \( q \) in \( \mathbb{P}^5 \setminus l \), then \( \pi_0(X) = \mathbb{P}^3 \). Therefore, we can find a plane \( \Pi \) containing \( l \) such that \( \Pi \cap X = l \cap X \). First, let us construct a family of regular projections degenerating at \( \pi_0 \).
Consider the pencil $\mathcal{P}$ of lines contained in $\Pi$ and passing through a given generic point $p_0$ on $l$. Let $0$ be a generic point on $l$. Let $\Delta$ be a generic line meeting $l$ at $0$; then, $\Delta$ parameterizes the lines $l_t$ of the pencil $\mathcal{P}$. Let us denote by $\{p_1, \ldots, p_s\}$ the support of $l \cap X$ and by $m_i$ the multiplicity of intersection of $l$ and $X$ at the point $p_i$.

For $t \neq 0$, the projection centers $l_t$ do not meet $X$, so the projections $\pi_t$ of $\mathbb{P}^5 \times \{t\}$ onto $\mathbb{P}^3 \times \{t\}$ define a rational map $\pi$ from the 3-fold $X \times \Delta$ to $\mathcal{F} = \pi(X \times \Delta)$ in $\mathbb{P}^3 \times \Delta$, whose indeterminacy locus is $\{(p_1, 0), \ldots, (p_s, 0)\}$.

Thus, the blow-up of $X \times \Delta$ at the points $(p_1, 0), \ldots, (p_s, 0)$, $\phi : \mathcal{X} \to X \times \Delta$, induces a regular map $g$ fibered over $\Delta$, which resolves the indeterminacy of $\pi$.

We have the following diagram:

\[
\begin{array}{ccc}
\mathcal{X} \\
\phi \downarrow \\
X \times \Delta- \pi \mathcal{F} \subset \mathbb{P}^3 \times \Delta \\
\downarrow h \\
\Delta
\end{array}
\]

where $h$ (resp. $\zeta$) denotes the projection to the second factor. For $1 \leq i \leq s$, let $E_i$ denote the exceptional divisor of $\mathcal{X}$ over $(p_i, 0)$.

Since $h \circ \phi$ is a morphism from $\mathcal{X}$ onto a 1-dimensional smooth base $\Delta$, it is a flat morphism [19, III 9.7 p.257].

Its special fiber $(h \circ \phi)^{-1}(0)$ is simply given by

$$
\mathcal{X}_0 = \tilde{X} \times \{0\} \cap \bigcap_{i=1}^s E_i
$$

where $\tilde{X}$ denotes the blow-up of $X$ at $p_1, \ldots, p_s$. Recall that the double locus $D$ of the map $g$ is defined on $\mathcal{F}$ by the ideal sheaf of the conductor $\mathcal{C} := (\mathcal{O}_\mathcal{F} : g_*(\mathcal{O}_\mathcal{X}))$.

Since $g$ is a finite morphism, $\mathcal{C}$ is also an ideal sheaf over $\mathcal{O}_\mathcal{X}$, defining the double locus $D' = g^{-1}(D)$ of $g$ on $\mathcal{X}$, see for instance [27]. The second projection from $\mathcal{F}$ onto $\Delta$ is a flat morphism, so the $\mathcal{O}_\mathcal{X}$-sheaf $\mathcal{O}_{\Delta,t}$ is locally free. If we tensor the previous exact sequence by this sheaf, we simply get the exact sequence defining the double locus $D_t$ of the map $g_t$, restriction of $g$ to the fiber $\mathcal{X}_t$.

Thus, in each fiber $\mathcal{X}_t$, the double locus $D$ of $g$ restricts to the double locus of $g_t$. Since $\mathcal{C}$ is an invertible $\mathcal{O}_\mathcal{X}$-module, $(D'_t)$ forms a flat family of divisors over $\Delta$. For $t \neq 0$, $\mathcal{X}_t = X \times \{t\}$ and $g_t = \pi_t$, the class of $D'_t$ in $Pic(\mathcal{X}_t)$ is
classically given by \[27, ?\]

\[D'_t = (d - 4)H'_t - K_{X_t}\]

where \(H'_t = \varphi^*((\zeta^*(H))_{|X_t})\), \(H\) being the class of hyperplanes in \(\mathbb{P}^3\). The divisor \(H'_t\) is naturally the restriction of \((\varphi)^*(\zeta^*(H))_{|X}\) in \(Pic(X)\).

**Claim 15** For \(t \neq 0\), we have \(K_{X_t} = K_X|_{X_t}\).

**PROOF.** Since \(X\) is a blow-up of \(X \times \Delta\), we have \(K_X = \phi^*(K_{X \times \Delta}) + \sum_{i=1}^{s} E_i\). Since, for all \(i = 1, \ldots, s\), we have \(X_i \cap E_i = \emptyset\), we find \(K_X|_{X_t} = \phi_t(K_{X \times \Delta}|_{X \times \{t_i\}})\). Recall that

\[Pic(X \times \Delta) \simeq \eta^*(Pic(X)) \oplus h^*(Pic(\Delta)) \simeq \eta^*(Pic(X)) \oplus h^*(H_{\Delta} \mathbb{Z})\]

where \(\eta\) is the projection from \(X \times \Delta\) onto the first factor and \(H_{\Delta}\) stands for the class of points in \(Pic(\Delta)\). From \(K_{X \times \Delta} = \eta^*(K_X) + h^*(-2H_{\Delta})\) we deduce that \(K_{X \times \Delta}|_{X \times \{t\}} = \eta^*(K_X)|_{X \times \{t\}}\). \(\square\)

By [19, proposition 9.7 p.258], the family \((D'_t)_{t \neq 0}\) has a unique flat limit, so the double locus \(D'\) of \(g\) on \(\tilde{X}\) has class \((d - 4)H^* - K_X\). Finally, we must determine the double locus divisor of \(g_0|_{\tilde{X}_0}\). Note that \(X \times \Delta\) is embedded in \(\mathbb{P}^5 \times \Delta\) by the linear system \(H' = h^*(H_X)\); so the linear system defining \(g\), is \(|H' - \sum_{i=1}^{s} m_i E_i|\) and \(g_0\) is defined by the restriction of this system to \(\tilde{X}_0\). Since \(K_{\tilde{X}_0} = \phi_0^*(K_X \times \{0\}) + \sum_{i=1}^{s} E_i\), we get \(D'_0 = (d - 4)g_0^*(H_0) - \phi_0^*(K_X \times \{0\}) - \sum_{i=1}^{s} E_i\). Moreover, the restriction of \(|H' - \sum m_i E_i|\) to \(X \times \{0\}\) embeds \(E_i\) as a plane in \(\mathbb{P}^3 \times \{0\}\). The curves \(e_i = \tilde{X} \cap E_i\) for \(1 \leq i \leq s\) sit in the double locus of \(g_0\) and are mapped by \(g_0\) to hyperplane sections of \(F_0\). Thus, the restriction of the double locus of \(g_0\) to \(\tilde{X}_0\) can only differ from the double locus of \(g_0|_{\tilde{X}}\) along the curves \(e_i\). Besides, the double locus of \(g_0|_{\tilde{X}}\) is not supported along these curves. Therefore, the class of the double locus of the projection of \(X\) from \(l\) is given by the formula

\[\phi_0^*((d - 4)g_0^*(H_0) - \phi_0^*(K_X \times \{0\}) - \sum_{i=1}^{s} e_i - g_0^*(\delta H)).\]

\(\square\)

**Remark 16** Let \(S\) be a singular surface of \(\mathbb{P}^5\), obtained as the image of \(\mathbb{P}^2\) by a complete linear system of plane curves with assigned base points. If \(S\) satisfies all the hypothesis of the theorem but smoothness, a slight modification of the argument shows that the same formula holds for the double locus in the Picard group of \(\tilde{P}\), the blow-up of \(\mathbb{P}^2\) at the base points.
4 Existence of a trisecant line through a generic point of $S_5$

Following Gambier [13, p. 184-186], we extend Dobler’s result on the existence of trisecant line to the case of singular White surfaces. We present Gambier’s beautiful construction in modern language and show how to fill a few gaps.

**Theorem 17 (Gambier)** Let $P$ be a finite set of 15 distinct points in $\mathbb{P}^2$, such that $P$ is not contained in any curve of degree 4. Pick a generic point $q$ in $\mathbb{P}^2$. Then, there exists an associated pair $\Pi$ to $|\mathcal{I}_P(5)|$ at $q$.

**PROOF.** Let $D$ be a generic curve of $|\mathcal{I}_{P+q}(5)|$, so that $D$ is smooth. Fix a pencil $< D, D' >$ of curves in $|\mathcal{I}_{P+q}(5)|$. Let $Q$ denote the 0-scheme of length 9 residual to $P + q$ in the intersection cycle $D \cdot D'$. Pick a basis $D, D_1, \ldots, D_4$ of the linear system $|\mathcal{I}_{P+q}(5)|$ and choose $D'$ generic in the linear subsystem $< D_1, \ldots, D_4 >$, so that $D'$ is smooth.

An easy dimension count shows that the virtual dimension of the linear system of cubic curves passing through the cycle $Q$ is non negative.

**Lemma 18** Let $D$ and $D'$ be as above, then there is a unique cubic curve passing through $Q$.

**PROOF.** Suppose, to the contrary, that there is a pencil of cubic curves, $< C, C' >$, passing through $Q$. We have only two cases to consider:

(A) either those cubic curves are generically irreducible,
(B) or there is a fixed line or a fixed conic in the pencil $< C, C' >$.

Let us prove the lemma in case (A). First, assume that both cubic curves $C$ and $C'$ are irreducible. We claim that these curves are smooth along the supporting points of $Q$.

Let $r$ be a point of support of the cycle $Q$. We have, by construction,

$$\mu_r(C, C') \geq \mu_r(D, D'),$$

where $\mu_r(C, C')$ denotes the multiplicity of intersection of $C$ and $C'$ at $r$. Since the two cubic curves are assumed irreducible, by summation of the previous inequalities, we get

$$9 \geq \sum_{r \in \text{Supp}Q} \mu_r(C, C') \geq \sum_{r \in \text{Supp}Q} \mu_r(D, D') = 9,$$
so for every $r \in \text{Supp}Q$, we have
\[
\mu_r(C, C') = \mu_r(D, D') \geq \mu_r(C)\mu_r(C').
\]

If $r$ is not a multiple point of $Q$, the curves $C$ and $C'$ are therefore smooth at the point $r$.

Suppose, now, that $r$ is a multiple point of $Q$. We have $\mu_r(C, C') = \mu_r(D, D')$, so we get
\[
\mu_r(C)\mu_r(C') + \sum_{x \in A(r)} \mu_x(C)\mu_x(C') = \mu_r(D)\mu_r(D') + \sum_{x \in B(r)} \mu_x(D)\mu_x(D'),
\]
where $A(r)$ (resp. $B(r)$) is the set of infinitely near points of both $C$ and $C'$ (resp. $D$ and $D'$) over $r$. From the analysis made in section 2.2, we find $B(r) \subset A(r)$. Since $D$ and $D'$ are smooth, we find $\mu_r(D) = \mu_r(D') = 1$ and $\mu_x(D) = \mu_x(D') = 1$ for all $x \in B(r)$. Thus, from equation 3, we deduce that $A(r) = B(r)$, $\mu_r(C) = \mu_r(C') = 1$ and $\mu_x(C) = \mu_x(C') = 1$ for all $x \in A(r)$.

In particular, $C$ and $C'$ are smooth at $r$.

Since $C$ and $C'$ are smooth at the supporting points of $Q$, we may apply the residuation theorem for $n = m = 3$, $n_1 = 5$ and $n_2 = 0$. We find that the cycle of length 6, $V$, residual to $Q$ on $D \cdot C$, is the complete intersection of $C$ with some conic $C_2$. Applying, now, the residuation theorem to $V$ and $Q$ with $n_1 = 5$, $m = 5$, $n = 3$ and $n_2 = 2$, we deduce that $P + q$ lies on a quartic curve. This contradicts our assumption on $P$.

Only remains to prove the lemma in case (B). Either the pencil of cubics is the composite of a line $l$ with a pencil of conics or a fixed conic $C_0$ composed with a pencil of lines. Assume that we are in the first case and that the generic conic is irreducible, hence smooth. Thus, we can pick two smooth generators of the pencil: $C_2$ and $C_2'$. Applying the residuation theorem as before with $C_2 \cup l$ as cubic curve and $C_2'$ as conic, we get the same contradiction. The case of a fixed conic and a pencil of lines is similar. \(\square\)

The following proposition and lemma are a key ingredient used by Gambier without a proof.

**Proposition 19** The cubic curves $C$, constructed this way, vary in a linear system as the quintics $D'$ vary in $|\mathcal{I}_{P+q}(5)|$.

**PROOF.** It is enough to show that for any pencil $\mathcal{P}$ of quintic curves in $|\mathcal{I}_{P+q}(5)|$, such that $D \not\in \mathcal{P}$, the family of cubic curves passing through the residual groups of points to $|\mathcal{I}_{P+q}(5)|$ on $D$ vary in a rational pencil. \([31, p.\)
As $t$ varies in $\mathbb{P}^1$, the parameter space of the pencil $\mathcal{P}$, the curves $\mathcal{D}_t$ of the pencil vary in a rational family $\mathcal{D}$ over $\mathbb{P}^1$, sub-family of the trivial family $\mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1$. We have

$$\mathcal{D} \subset \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^2 \xrightarrow{g} \mathbb{P}^1$$

where $f$ (resp. $g$) is the projection onto the first factor (resp. the second factor) of $\mathbb{P}^2 \times \mathbb{P}^1$; the map $g$ is flat, since $g$ is surjective and $\mathbb{P}^1$ is a smooth 1-dimensional variety. Consider the family

$$\mathcal{Q} := \mathcal{D} \cap (D \times \mathbb{P}^1) \setminus (\{P + q\} \times \mathbb{P}^1) \to \mathbb{P}^1.$$

For $t$ generic in $\mathbb{P}^1$, both $\mathcal{D}_t$ and $D$ are smooth; thus, by the previous lemma, there is a unique cubic curve $\mathcal{C}_t$ passing through the intersection cycle $\mathcal{Q}_t$. A bigraded free resolution of $\mathcal{O}_\mathcal{Q}$ as $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}$-module is of the form

$$\cdots \to \bigoplus_{i=1}^k f^*(\mathcal{O}_{\mathbb{P}^2}(-n_i)) \otimes g^*(\mathcal{O}_{\mathbb{P}^1}(-m_i)) \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1} \to \mathcal{O}_\mathcal{Q} \to 0,$$

for some non-negative integers $k$, $m_i$ and $n_i$. Tensoring this resolution by $\mathcal{O}_{\mathbb{P}^2 \times \lbrace t \rbrace}$, we get a presentation of $\mathcal{I}_{\mathcal{Q}_t}$

$$\to \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^2}(-n_i) \to \mathcal{I}_{\mathcal{Q}_t} \to 0.$$

By lemma 18, (for $t$ generic) there is an index $i_0$ in $\{1, \ldots, k\}$ such that $n_{i_0} = 3$. Thus, there exist a non-negative integer $n$ and a non-zero polynomial $\sigma$ of bidegree $(3,n)$ whose vanishing locus contains the scheme $\mathcal{Q}$. For $t$ generic in $\mathbb{P}^1$, the curve $(\sigma(t) = 0)$ is the unique cubic curve passing through the 0-scheme $\mathcal{Q}_t$. Since the plane curves of the family $(\sigma = 0) \xrightarrow{g} \mathbb{P}^1$ have the same degree, hence the same Hilbert polynomial, $g$ is flat. Thus, $(\sigma = 0) \xrightarrow{g} \mathbb{P}^1$ is a rational pencil.

$\square$

Let us choose a basis $D, D_1, \ldots, D_4$ of $|\mathcal{I}_{P+q}(5)| \simeq \mathbb{P}^4$. As a consequence of the previous proposition, we get:

**Lemma 20** The characteristic series of $|\mathcal{I}_{P+q}(5)|$ defines an algebraic injective morphism $\Psi : \mathbb{P}^3 \xrightarrow{=} D_1, \ldots, D_4 \to \mathbb{P}^9 = |\mathcal{O}_{\mathbb{P}^2}(3)|$ which sends a curve $\mathcal{C}$ in $< D_1, \ldots, D_4 >$ to the unique curve of degree 3 passing through the cycle $(D \cdot C) - (P + q)$. 

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PROOF. Only remains to prove the injectivity of Ψ. Suppose, to the contrary, that there exist two quintics $C$ and $C'$ of the linear system $<D_1, \cdots, D_4>$, defining the same cubic $C_3$. We write

$$C \cdot D = P + q + R \text{ and } C' \cdot D = P + q + R'$$

$$D \cdot C_3 = R + V = R' + V'.$$

It follows that $R$ and $R'$ contain at least 3 common points.

Assume that $R$ and $R'$ contain exactly $3 + n$ common points $\Pi$, for $0 \leq n \leq 9$. The linear system $|\mathcal{I}_{P + q + \Pi}|$ is therefore special. Let us write $R = \Pi + W$ and $R' = \Pi + W'$; we have $V = W' + S$ and $V' = W + S$, where $S$ is a $(n - 3)$-tuple of points. Pick $\Pi_0$ in $\Pi$ such that $\Pi_0 + W$ has length 6. By the duality theorem, $W$ (resp. $W'$) lies on some conic $C_2$ (resp. $C'_2$). Finally, we apply the residual theorem with $n = 5$, $m = 3$, $n_1 = 5$ and $n_2 = 2$, to the curves $D$, $C_3$, $C$ and $C'_2$. Thus, $W + W' + 2\Pi_0$ lies on some conic $C'_2$. Thus, we have $C'_2 \cdot D \supset W + W' + 2\Pi_0$; this gives a contradiction, since $D$ is irreducible and $W + W' + 2\Pi_0$ has length 12. □

We are now able to prove the theorem. The locus in $\mathbb{P}^3 = |O_{\mathbb{P}^2}(3)|$ of this family of cubic curves is the image of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ by the Segre embedding. This is a variety of codimension 3 in $\mathbb{P}^9$. Since $\Psi$ is injective, we may choose $D'$ such that the unique cubic curve passing through $Q$, the group of points of $D$ residual to $P + q$ in $D \cdot D'$, is the union of 3 lines $l_1$, $l_2$ and $l_3$. Let $T$ denote the triangle $l_1 \cup l_2 \cup l_3$. Let $V^t$ denote the group of points on $D_t$ residual to $Q$ in $D_t \cdot T$, as $D_t$ varies in the pencil $<D, D'>$. We can find $t_0 \in \mathbb{P}^1$ such that 3 points of $V^{t_0}$ lie on a line, say $l_1$.

Since $D$ is irreducible, there are no more than 3 points of $V_{t_0}$, counted with multiplicity on $l_1$. Indeed otherwise, there would not be a unique cubic curve passing through $Q$. As $t$ tends to $t_0$, $D_t \cdot l_1$ tends to a length 5 cycle, $\Pi + (V_{t_0} \cdot l_1)$. Since $\Pi$ is a sub-cycle of $Q$, $\Pi$ belongs to the intersection cycle $D \cdot l_1$.

It is worth noticing that $\Pi$ is necessarily a cycle of length 2, i.e., $V_{t_0} \cdot l_1$ has length 3. Note also that $\Pi$ consists of 2 distinct points. Otherwise, $D$ is tangent to $l_1$ at $\Pi$, so this cycle $\Pi$ has to be supported at the point of intersection of $l_1$ with $l_2$ or $l_3$. But then, $Q$ does not lie on a unique cubic curve.

It follows from proposition 5 that $|\mathcal{I}_{P + q + \Pi}(5)|$ is 1-irregular, provided it is complete. This system clearly cannot contain any base curve, since $D$ is irreducible. The potential extra base points are in the cycle $R = Q - \Pi$. If $R$ is composed of 7 distinct points, it is clear that the irregularity of the system can be at most 2. Since the actual dimension of the system is at least 2, there can be at most 1 extra base point, $r$. Thus, the length 6 cycle $R \setminus r$ is supported on one of the lines $l_2$ or $l_3$. Since $D$ is irreducible, we get a contradiction. If $R$
is not reduced, it can at most have one non reduced point at \( l_2 \cap l_3 \). Since the curve \( D \) is irreducible, the linear system \( |\mathcal{I}_{P+q+\Pi}(5)| \) imposes also a tangency condition at \( r \) along the line \( l_2 \) (or \( l_3 \)). \( \square \)

**Remark 21** It follows from the previous proof, that \( S_5 \) has no quadrisecant lines, if it is smooth. For the same reason, if \( S_5 \) has excess of trisecant lines, a general trisecant line passing by a generic point is not a quadrisecant line.

5 Applications of the existence theorem

The existence theorem has an amusing consequence for the principal component of the subvariety \( W_{18}(5) \) of the Hilbert scheme \( \text{Hilb}_{18}(\mathbb{P}^2) \).

5.1 The geometry of the generic point of \( W_{18}[5] \)

The space of length 18 groups of points special in degree 5, \( W_{18}[5] \), is known to be irreducible of expected dimension 32 [6, theorem 3.2.1], so it coincides with its principal component. We shall show that its generic point has the same numerical character as the projection by a generic trisecant line of the generic White surface in \( \mathbb{P}^5 \).

**Corollary 22** The generic point of \( W_{18}[5] \) corresponds to a smooth uniform 18-tuple of points, of numerical character

\[ \chi = (7, 6, 5, 5); \]

therefore, it doesn’t lie on any curve of degree less or equal to 4.

**PROOF.** Let \( \chi^W = (5 + \epsilon_0, \ldots, 5 + \epsilon_4) \) be the character of the projection of a generic White surface \( S_5 \) by a generic 3-secant line. We have \( \sum \epsilon_i = 3 \). Since it corresponds to a 18-tuple of points special in degree 5, \( n_0 \geq 7 \). If \( n_0 > 7 \) or \( n_i > 6 \) for \( i > 0 \), we find \( h^1(\mathcal{I}_{18}(5)) \geq 2 \). Thus, we have \( \chi^W = (7, 6, 5, 5, 5) \). Since \( \chi^W \) is a uniform character, we find \( \text{dim}(\chi) = 32 = \text{dim}(W_{18}[5]) \). Furthermore, we have \( H_{\chi^W} \subset W_{18}[5] \). \( \square \)

Coppo’s bound [6] shows, in this particular case, that the generic point of \( W_{18}[5] \) corresponds to a smooth uniform 18-tuple of points not lying on any conic.
5.2 The number of 3-secant lines passing through a generic point of $S$

In 1882, H. Krey [20, p.505, for $n = 5$] showed, using techniques combining excess intersection theory and correspondence methods, that, in degree 5, there are 6 associated pairs to $Z$, for $Z$ a generic 16-tuple of points in the plane. In this section only, a trisecant line to $S_5$ through a generic point $p = \Phi(q)$ is a line $|I_P + q + \Pi(5)|$, where $\Pi$ is an associated pair to $I_P(5)$ at $q$. That is to say, we allow improper trisecant lines (i.e. $\Phi(q) + \Phi(\pi_1) + \Phi(\pi_2)$ does not consist of distinct points on $S_5$). We prove that Krey’s result still holds, if $Z$ corresponds to the generic projection of any White surface $S_5$ from a generic point on it, assuming that $S_5$ has a finite number of trisecant lines passing by that point.

**Remark 23** In the case of a general polygonal surface, this seems to be in apparent contradiction with T.Dobler’s result [9, proposition 3.17], which shows that there are no trisecant lines at all. But, the two results agree. Since a polygonal surface in $\mathbb{P}^5$ has 6 singular 4-fold points, the 6 trisecant lines through a generic point $q$ that we construct in the next theorem are simply the 6 bisection lines joining $q$ to a singular point of the surface. They are not counted by Dobler, for they are improper trisecant lines.

Notice that all trisecant lines to $S_5$ passing through a generic point are obtained by the construction of theorem 17.

**Theorem 24** Assume that, through a generic point $p := \Phi(q)$ of a White surface $S_5$ in $\mathbb{P}^5$, there passes only a finite number of trisecant lines. Then, through $p$, there pass exactly 6 trisecant lines counted with multiplicity.

**PROOF.** Let $D$ and $D'$ be two generic quintic curves in $|I_{P+q}(5)|$. Denote by $(D_\lambda)$ the pencil $<D, D'>$. From the construction of lemma 20, the cycle that $D_\lambda$ induces on $D$, $R := D \cdot D_\lambda$, lies on a unique cubic curve $E$. If $D'$ varies in $|I_{P+q}(5)|$, the cubic curves $E$ vary in a linear system of dimension at least 3, with no multiple base points. By Bertini’s theorem, for a generic choice of $D'$, $E$ is smooth.

We show that no points of $P + q$ lie on $E$, i.e., the pencil $D_\lambda$ induces a base point free $g_6^1$ on $E$. This follows from the fact that $P + q$ imposes independent conditions on quartics.

**Lemma 25** The pencil $(D_\lambda)$ cuts out on $E$ a base point free $g_6^1$.

**PROOF.** Indeed, suppose to the contrary, that the series $V_\lambda = (D_\lambda \cdot E) \setminus R$ has a base of length $l$. First notice that $l < 6$. Otherwise, applying the classical residuation theorem to $D \cdot E$ and $D \cdot D_\lambda$, we deduce that 10 of the points of
$P + q$ lie on the same conic. By genericity assumption, one can assume that 10 points of $P$ lie on the same conic; so, $P$ lies on a quartic, giving a contradiction.

Let $V_b$ denote the base of the series $V_{\lambda}$. Note that $V \setminus V_b$ lies on some conic $C_2^\lambda$. Let us apply the residuation theorem with $m = 5$, $n = 3$, $n_1 = 5$ and $n_2 = 2$; we get

$$D \cdot E = R + V_b + V' \text{ and } D \cdot D_{\lambda} = P + q + R = V_b + R + W$$

$$D \cdot C_2 = V' + T.$$

We find $W + T = D \cdot C_4$, for some quartic $C_4$. Suppose that $l = 1$; then, the system of quartics passing through $W$ is empty, since $P + q$ imposes independent conditions. Thus, $l \geq 2$. Notice that there is a pencil of conics $C_2^\xi$ passing through $V'$. This pencil induces a linear series $T_{\xi}$ on $D$ and a family of quartic curves $C_4^\xi$, resolving the residuation theorem.

Suppose now that $l = 2$; there is a unique quartic passing through $W$, so $T_{\xi}$ is fixed on $D$. Thus, the pencil of conics has 10 fixed points containing 9 aligned points. This line is then a fixed component of $D$, so $l \geq 3$.

Suppose that $l \geq 3$; we have, at least, a $(l - 1)$-dimensional system of conics, passing through $V'$ and cutting out the series $T_{\xi}$. Since $P + q$ imposes independent conditions on quartics, this series is cut out by the $(l - 2)$-dimensional system of quartic curves passing through $W$. This leads to a contradiction, since $D$ cannot be a component of any of those systems.

Thus, the pencil $(D_{\lambda})$ induces a base point free $g^1_6$ on $E$. $\square$

Let $L_0$ be the hyperplane section divisor on $E$ and $L$ the divisor associated to this $g^1_6$. We have an obvious morphism

$$H^0(E, \mathcal{O}_E(L_0)) \times H^0(E, \mathcal{O}_E(L_1)) \to H^0(E, \mathcal{O}_E(L)),$$

where $L_1 = L - L_0$. Its projectivization, $\mathbb{P}^2 \times \mathbb{P}^2 \overset{\phi}{\to} \mathbb{P}^5$, factors through the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ and a regular projection. Its image is an hypersurface $A$ of degree 6 in $\mathbb{P}^5$, which represents the divisors of $|L|$ containing an aligned subscheme of length 3. Thus, the intersection of $A$ with the line $D_{\lambda}$ parameterizes trisecant lines to $S_5$ passing through $q$. $\square$
6 The projection of $S_5$ by a generic trisecant

In this section, we present Gambier’s argument to show the finiteness of trisecant lines through a generic point of a White surface [13, p. 253-256]. Gambier assumes implicitly that the triple curve $\gamma$ is irreducible; we show how to fill this gap.

**Theorem 26 (Gambier)** The rational surface $\Sigma$, projection of $S_5$ from a generic trisecant line, has degree 7 and sectional genus 6. On $\Sigma$, the double locus of the projection is in fact a triple locus. Moreover, the curve $\gamma$ is a twisted cubic.

**PROOF.** Let $q$ be a generic point of $\mathbb{P}^2$ and $\Pi$ an associated pair to $|I_{\mathbb{P}^2}(5)|$ at $q$. Denote by $Z$ the cycle $P + q + \Pi$ and by $\tilde{\mathbb{P}} \xrightarrow{\pi} \mathbb{P}^2$ the blowing up of $\mathbb{P}^2$ at $P + q + \Pi$. We denote by $E$ the exceptional divisor of $\pi$.

Using Riemann-Roch’s theorem, we find $\deg(\Sigma) = 25 - 18 = 7$ and $\pi(\Sigma) = 6$. Only remains to prove that a generic hyperplane section of $\Sigma$ contains 3 triple points of the projection to $\mathbb{P}^3$.

A hyperplane section of $\Sigma$ has degree 7, arithmetical genus 15 and geometrical genus 6, so it has either 9 double points or 3 triple points. We shall construct these triple points. First, to construct a generic hyperplane section $h$ of $\Sigma$, pick a generic point $s$ in $\mathbb{P}^2$. The hyperplane section $h$ then corresponds to a curve $D \in |I_{Z+s}(5)|$. According to Bertini’s theorem, we may choose $D$ smooth, hence irreducible. We write $|I_{Z}(5)| = \langle D, D_1, \cdots, D_3 \rangle$. By assumption, any $Y \in \chi([D, |I_{Z}(5)|])$ lies on a unique conic curve. An argument similar to the proof of proposition 19 shows that the strict transforms by $\pi$ of these conics vary in a linear system $N$ on $\tilde{\mathbb{P}}$, as $Y$ varies in the linear series cut out by $D' \in \langle D_1, \cdots, D_3 \rangle$. We find, as before, $\dim(N) = \dim(\langle D_1, \cdots, D_3 \rangle) = 2$.

Denote by $\tilde{D}$ the strict transform of $D$ by $\pi$ and consider the series $N|_{\tilde{D}}$ cut on $\tilde{D}$ by $N$. Let $L$ denote the invertible sheaf of $O_{\tilde{\mathbb{P}}}$-modules $\pi^*(O_{\mathbb{P}^2}(5)) - E$. By construction, we have $\dim H^0(L|_{\tilde{D}}) = \dim(N|_{\tilde{D}}) + 1$. Tensoring by $L$ the exact sequence

$$0 \rightarrow L^{-1} \rightarrow O_{\tilde{\mathbb{P}}} \rightarrow O_{\tilde{D}} \rightarrow 0,$$

we find

$$\dim H^0(L|_{\tilde{D}}) = \dim H^0(L) - 1 = 3$$

$$\dim H^1(N|_{\tilde{D}}) = \dim H^1(L) = 1.$$

Thus, $\dim H^0(N|_{\tilde{D}}) = \dim H^0(L|_{\tilde{D}}) = 3$ and, by Riemann-Roch theorem, $L|_{\tilde{D}}$ is a series of degree 7. The free part of $L|_{\tilde{D}}$ is then a subsers of $N|_{\tilde{D}}$, of the same dimension, 2. The two series are therefore equal. For any conic $C_2$, we
have $D \cdot C_2 = 10$. So, $\mathcal{N}|_{\tilde{D}}$ has exactly 3 base points $A, B$ and $C$, distinct or infinitely near. Since $\mathcal{N}$ is 2-dimensional, the conics of this system cannot have a fixed line component, so the support of the points $A, B, C$ consist of at least 2 distinct points, say $A$ and $B$. For the same reason, the points $A, B, C$, if distinct, are not aligned.

We write $D \cdot L = A + B + \lambda + \lambda' + \lambda''$, where $\lambda, \lambda'$ and $\lambda''$ are possibly infinitely near.

**Lemma 27** Let $L$ be the line joining $A$ and $B$; the three points $\lambda, \lambda'$ and $\lambda''$ distinct or infinitely near are mapped by $|\mathcal{I}_Z(d)|$ to a triple point of $\Sigma$.

**PROOF.** We only have to show that $|\mathcal{I}_Z(\lambda)(5)| \subseteq |\mathcal{I}_Z(\lambda + \lambda' + \lambda''(5)|$. Let $r$ be any point of the plane. If $|\mathcal{I}_Z|$ contains a pencil, it is complete, since a linear space of conics with fixed points, is at most 1-dimensional.

Suppose that $\lambda$ is an ordinary point of the plane. Let $D'$ be a generic curve of $|\mathcal{I}_Z(\lambda)|$. By the previous remark, $D'$ does not contain any of the points $A, B$ or $C$. The cycle $Y := (D' \cdot D) \setminus Z$ lies on a unique conic $C_2$. Since $C$ belongs to $\mathcal{N}$, the strict transform of this conic $C_2$ passes through $A, B$ and $C$. Thus, the conic curve $C_2$ is the union of two lines $L$ and $l$, whose strict transforms pass through $C$. By unicity of the conic $C_2$, two points of $Y$ must lie on $L$. Thus, $|\mathcal{I}_Z(\lambda)| \subseteq |\mathcal{I}_Z(\lambda + \lambda' + \lambda''(5)|$. $\blacksquare$

By taking for $L$ the associated line and by working in the blow-up of $\mathbb{P}^2$ at $B$, the same proof works, if $C$ is infinitely near to $B$. $\blacksquare$

**Remark 28**

(1) Using Riemann-Roch’s theorem on $\mathcal{N}|_{\tilde{D}}$, it is not hard to see that $A, B$ and $C$ are in fact the base points of $\mathcal{N}$.

(2) The surfaces of $\mathbb{P}^3$ of degree $2n + 3$, passing $n$ times through a given twisted cubic, have been studied by Gérard[14]. For $n = 2$, we obtain another geometric construction of White surfaces, or more precisely, of their projections to $\mathbb{P}^3$ from a trisecant line.

We have the following improvement of Dobler’s result:

**Theorem 29** The only White surface with a 4-dimensional trisecant line locus is a Segre polygonal White surface.

**PROOF.** Let $\tilde{S} \xrightarrow{\pi} \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at $Z$ for $Z = P + q + \Pi$. By remark 21, we may assume that a generic trisecant line is not a quadriseccant
line. The only hypothesis of proposition 14 that we still need to check, is that no plane curves are contracted by the projection from a trisecant line passing through a generic point of $S_5$.

**Lemma 30** If the White surface $S_5$ has an excess of trisecant lines, its projection to $\mathbb{P}^3$ from a generic trisecant line does not contract any plane curve.

**PROOF.** Let $\phi_{S_5}$ (resp. $\phi_{\Sigma}$) denote the rational map associated to the linear system $|I_{P}(5)|$ (resp. $|I_{Z}(5)|$). Let $l := < \phi_{S_5}(q), \phi_{S_5}(\pi_1), \phi_{S_5}(\pi_2) >$. Let $\pi_l$ denote the projection map from the trisecant line $l$, and $\Sigma$ the image of $S_5$ by $\pi_l$. The surface $\Sigma$ is the image of $\mathbb{P}^2$ by the rational map $\phi_{\Sigma}$.

Assume, to the contrary, that there exists $C$, a curve on $S_5$ which is contracted to a point $x$ of $\Sigma$ by the projection $\pi_l$. The curve $C$ lies in the 2-plane $< l, x >$.

**(A)** We may therefore assume that $C$ is the 1-dimensional part of the intersection $S_5 \cap < l, x >$.

We have $\deg(C) \leq \text{length}(l \cap S_5) = 3$. The rational map $\phi_{S_5}$ is birational; let $C$ denote the plane curve $\phi_{\Sigma}^{-1}(C)$. Then, the class of its strict transform $\tilde{C}$ by $\pi$, is given by

$$[\tilde{C}] \cdot (5\pi^*L - \sum_{z \in Z} E_z) = 0,$$

in $\text{Pic}(\tilde{S})$. Furthermore, if $r_1$ and $r_2$ are two points of the curve $C$, we have $|I_{Z \cup \{r_1\}}(5)| = |I_{Z \cup \{r_2\}}(5)|$, since the points $r_1$ and $r_2$ are both mapped to

$$x := \phi_{\Sigma}(C) = \pi_l(C) \in \Sigma.$$

So, the curve $C$ must be a fixed component of the linear system of quintic curves passing through $r_1$ and $Z$. From this follows that the degree of $C$ must be strictly less than 5.

We find $[\tilde{C}] = d\pi^*L - \sum_{z \in Z'} E_z$, where $d < 5$ and $Z'$ is a subset of $Z$.

From equation (4) we get that the cardinality of $Z'$ equals $5\deg(C)$. We denote by $P'$ the subset of $P$ contained in $Z'$. We get $\deg(C) = 5d - \text{length}(P')$. We conclude with a case by case study.

The case $d = 4$ cannot happen since $Z$ has only 18 points. If $d = 3$, then $C$ passes through at least 12 points of $P$. Any 11 points in $P'$ do not lie on any conic, for then $P$ would lie on a quartic curve. Hence, there is a unique cubic through those 11 points. We can assume that $q$ doesn’t belong to that cubic, since $q$ is a generic point of the plane. Therefore, $Z'$ contains at least 13 points of $P$, so that $P$ lies on a quartic curve.
Therefore we have $d \leq 2$.

If $d = 1$, then the line $C$ passes through 5 of the base points $Z$; so $P'$ consists of at least 2 points. By genericity assumption, we can assume that $q$ doesn’t lie on this line through $P'$. Thus, either $\overline{C}$ is a conic and $C$ is a line through 3 points of $P$ and the associated pair $\{\pi_1, \pi_2\}$ to $q$ or $\overline{C}$ is a line and $C$ is a line through 4 points of $P$ and only one point of the associated pair to $q$.

Suppose that $d = 2$; from equation (4), we deduce that $C$ is a conic passing through 10 points of $Z$, among which 7 at least belong to $P$. Since no 6 points of $P$ can be aligned, unless $P$ lies on a quartic curve, we deduce that this conic curve $C$ through $P'$ is unique. By genericity assumption, we may again assume that $q$ does not lie on $C$. Then, either $C$ is a conic and $C$ is a conic through 8 points of $P$ and the associated pair to $q$ or $C$ is a line and $C$ is a conic through 9 points of $P$ and a single point of the associated pair to $q$.

Therefore, $\overline{C}$ is either a conic or a line not passing through $\phi_{S_5}(q)$. Suppose that for $q$ generic in the plane and a generic trisecant line to $S_5$ through $q$, there is a conic $\overline{C}$ contracted by $l$ to a point on $\Sigma$. Then, $C$ is either a line through 3 points of $P$ or a conic through 8 points of $P$. Since there is a finite number of such curves, as $q$ varies on $S_5$ there is a finite number of conics $\overline{C}$. From assumption (A), $S_5$ is the union of a finite number of 1-dimensional schemes. This gives an obvious contradiction.

Suppose now that for $q$ generic in the plane, and $l$ a generic trisecant line to $S_5$, there is a 2-plane containing $l$ meeting $S_5$ along a 1-dimensional locus, $\overline{C}(q, l)$ and a finite number of points, such that $\overline{C}(q, l)$ is a line (not passing through $q$). The curve $\overline{C}(q, l)$, corresponds in the plane to a curve $C(q, l)$, which is either a line through 3 points of $P$ or a conic through 8 points of $P$. Since there is a finite number of such curves, as $q$ varies on $S_5$ there is a finite number of conics $\overline{C}$. From assumption (A), $S_5$ is the union of a finite number of 1-dimensional schemes. This gives an obvious contradiction.

The class of the triple locus of the projection in $Pic(\tilde{S})$ is therefore

$$9H^* - 2 \sum_{i=1}^{18} E_i,$$

where $H^*$ is the pull back to $\tilde{P}$ of the divisor of lines in $\mathbb{P}^2$ and $E_i$ is the exceptional divisor over the point $p_i$. The curve $\gamma$ is therefore the birational
transform of a plane curve of degree 9 passing twice through \((P + q + \Pi)\), which we still denote by \(\gamma\).

**Lemma 31** If \(S_5\) has a 4-dimensional trisecant locus, the curve \(\gamma\) contains a line joining the points of \(\Pi\), some associated pair to \(P + q\) in degree 5.

**Proof.** Suppose that \(S_5\) has an excess of trisecant lines. Consider two pairs, \(\Pi\) and \(\Pi'\), associated to \(|I_P(5)|\) at \(q\). The points of \(\Pi'\) are mapped to the same point of \(\Sigma\), so the linear system \(|I_{Z+\Pi'}(5)|\) is not complete. By construction, any extra base point, \(\omega\), of \(|I_{Z+\Pi'}(5)|\) lies on the line \(<\Pi'>\). Since we can exchange \(\Pi\) and \(\Pi'\), an extra base point, \(\omega\), must lie at the intersection of \(<\Pi>\) and \(<\Pi'>\). Thus, \(S_5\) has only a finite number of trisecant lines passing through our generic point, unless \(<\Pi>\) is contained in \(\gamma\).

Assume that by a generic point of \(S_5\) there pass infinitely many trisecant lines. Then this is true at any point of the surface. Since we can exchange the roles of \(q, \pi_1\) and \(\pi_2\), the curve \(\gamma\) is the union of 3 lines with a sextic curve, contracted to a single point on \(\Sigma\). Thus, \(P\), the double points of \(\gamma\) belong to this sextic, which is thus the union of 6 lines, meeting two by two at \(P\). Therefore, the White surface \(S_5\), we started with, was a polygonal surface. According to Dobler’s thesis [9, proposition 3.17], it is of Segre type. □

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