SPECTRAL SCALING LAWS IN MAGNETOHYDRODYNAMIC TURBULENCE SIMULATIONS AND IN THE SOLAR WIND

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ABSTRACT

The question is addressed as to what extent incompressible magnetohydrodynamics can describe random magnetic and velocity fluctuations measured in the solar wind. It is demonstrated that distributions of spectral indices for the velocity, magnetic field, and total energy obtained from high-resolution numerical simulations of magnetohydrodynamic turbulence are qualitatively and quantitatively similar to solar wind observations at 1 AU. Both simulations and observations show that in the inertial range the magnetic field spectrum $E_b$ is steeper than the velocity spectrum $E_v$ with $E_b \gtrsim E_v$ and that the magnitude of the residual energy $E_r = E_v - E_b$ decreases nearly following a $k^{-2}$ scaling.

Key words: magnetic fields – magnetohydrodynamics (MHD) – methods: statistical – plasmas – turbulence

Online-only material: color figures

1. INTRODUCTION

Plasma motions in astrophysical systems are usually magnetized and turbulent. At scales larger than characteristic plasma kinetic scales, one-fluid magnetohydrodynamics (MHD) provides a satisfactory framework for studying such systems (Biskamp 2003; Kulsrud 2005). MHD turbulence has long been invoked to explain the properties of the solar wind, where velocity and magnetic field fluctuations are measured in situ over a wide range of scales (e.g., Goldstein et al. 1995; Tu & Marsch 1995; Bruno & Carbone 2005). Recent high-resolution numerical simulations, however, reported intriguing contradictions with the observational data. The Fourier energy spectrum of MHD turbulence obtained from numerical simulations appears to have a different scaling compared to the scaling of magnetic field fluctuations inferred from observations. This raises some serious questions. Do the numerical simulations correctly represent the physics of solar wind fluctuations and, if so, then why does solar wind turbulence not seem to exhibit the same universal scaling found in three-dimensional MHD simulations?

To formulate the problem, we rewrite the incompressible MHD equations in terms of the Elsässer variables,

$$\left( \frac{\partial}{\partial t} + \mathbf{v}_A \cdot \nabla \right) \mathbf{z}^\pm + (\mathbf{z}^\top \cdot \nabla) \mathbf{z}^\pm = -\nabla P,$$

where the Elsässer variables are defined as $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$, $\mathbf{v}$ is the fluctuating plasma velocity, $\mathbf{b}$ is the fluctuating magnetic field normalized by $\sqrt{4\pi \rho_0}$, $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi \rho_0}$ is the Alfvén velocity corresponding to the uniform magnetic field $\mathbf{B}_0$, $P = (p/\rho_0 + b^2/2)$ includes the plasma pressure $p$ and the magnetic pressure, $\rho_0$ is the constant mass density, and we neglect driving and dissipation terms. It follows from these equations that for $\mathbf{z}^\pm(x, t) = 0$, an arbitrary function $\mathbf{z}^\sigma(x, t) = F^\pm(x \pm \mathbf{v}_A t)$ is an exact nonlinear solution that represents a non-dissipative Alfvén wave propagating along the direction $\pm \mathbf{v}_A$. Nonlinear interactions are thus the result of collisions between counterpropagating Alfvén wave packets.

Denote by $E^\pm = \langle |\mathbf{z}^\pm|^2 \rangle/4$ the energies associated with the $\pm$ waves. Those two quantities are independent integrals of motion of the ideal MHD system (1). They are related to the total energy and cross-helicity, $E = E^+ + E^-$ and $H_c = E^+ - E^-$, respectively. Cross-helicity provides a measure of imbalance between interacting Alfvén modes; when $H_c \neq 0$ the turbulence is called imbalanced, otherwise it is balanced. The solar wind is essentially imbalanced, as more Alfvén waves propagate away from the Sun than toward the Sun. In a turbulent state, when energy is supplied to the system at large scales, both $E^\pm$ cascade toward small scales where they are damped by viscosity and resistivity.

According to numerical simulations, statistics of MHD turbulence are anisotropic with respect to the local mean magnetic field. It has recently been argued that the field-perpendicular energy spectra of incompressible, homogeneous, strong MHD turbulence scale as $E^\pm(k) \propto k^{-3/2}$ in the inertial range, and that this scaling is consistent with analytic models (Boldyrev 2006; Perez & Boldyrev 2010). This picture seems however to contradict the observational data of the solar wind, which often find the spectrum of magnetic field fluctuations to be consistent with the Kolmogorov scaling $-5/3$ (e.g., Goldstein et al. 1995). At the same time, recent measurements of velocity fluctuations in the solar wind at 1 AU reveal an essentially shallower spectrum, closer to $E_v(k) \propto k^{-3/2}$ (e.g., Podesta et al. 2007; Tassein et al. 2009; Podesta & Borovsky 2010; Borovsky & Denton 2010; Chen et al. 2011). Several explanations have been proposed for the difference of the spectral exponents. One possibility is that the solar wind turbulence is in a transient state at 1 AU (Roberts 2010), and therefore it cannot be described by a steady-state model. Another possibility is that the solar wind contains magnetic field discontinuities or current sheets advected from the Sun and not generated by the turbulence, which make the magnetic energy spectrum steeper (Borovsky 2010; Li et al. 2011). Those explanations are interesting and may be plausible.

In the present work we propose a complementary explanation. We argue that the mismatch between kinetic and magnetic
energies is, in fact, a fundamental property of steadily driven MHD turbulence. We note that there is no strict requirement for the magnetic and velocity fluctuations to be in equipartition with each other in MHD turbulence. Indeed, even though analytic models often appeal to the picture of counterpropagating Alfvén modes, with \( v = \pm b \), such Alfvén modes are not statistically independent in strong turbulence. This means, in general, that \( \langle z^+ \cdot z^- \rangle = \langle v^+ \cdot v^- \rangle \neq 0 \). Besides, the nonlinear equations (1) are not symmetric with respect to the interchange \( v \leftrightarrow b \).

On a more quantitative level, we perform a high-resolution numerical study of both balanced and imbalanced MHD turbulence, and concentrate on individual magnetic and velocity spectra. We find that in both cases these spectra are generally not identical. We observe that the so-called residual energy, characterizing the mismatch of the spectra, \( E_r(k) = E_v(k) - E_b(k) \), is not universal in that its amplitude may depend on the details of the driving and the degree of imbalance. The scaling of the residual energy is, however, close to \( E(k) \propto k^{-3/2} \) in both balanced and imbalanced runs. In the balanced case this result was first obtained by Müller & Grappin (2005). While the total energy spectrum is close to \( E(k) \propto k^{-3/2} \), the presence of residual energy leads to a steeper magnetic spectrum and a shallower velocity spectrum in an inertial interval of limited extent. However, since the residual spectrum declines faster than the total spectrum, the universality of the turbulence should be restored asymptotically at large \( k \), and it can be observed if the inertial interval is large enough.

For a comparison with the solar wind measurements we then plot histograms of velocity and magnetic spectral indices measured for individual temporal snapshots in numerical simulations of MHD turbulence. Comparison of the results with analogous histograms obtained from individual solar wind measurements reveals good agreement. This indicates that an incompressible MHD framework may provide an adequate explanation for the breakdown of kinetic-magnetic equipartition observed in the solar wind, and it may serve as a good model for addressing the inertial-interval scaling of the solar wind turbulence.

2. NUMERICAL SIMULATIONS

The universal properties of MHD turbulence are accurately described by neglecting the parallel component of the fluctuating fields, associated with the pseudo-Alfvén mode (e.g., Galtier et al. 2002; Galtier & Chandran 2006; Perez & Boldyrev 2008). By setting \( z^+ = 0 \) in Equation (1), we obtain the closed system of equations

\[
\left( \frac{\partial}{\partial t} \mp v_A \cdot \nabla \right) z^\pm + (z^\mp \cdot \nabla_z) z^\pm = -\nabla_z P + f^\pm + v \nabla^2 z^\pm , \tag{2}
\]

in which force and dissipation terms have been added to address the case of steadily driven turbulence, and we assume that viscosity is equal to resistivity. This set of equations is known as the reduced MHD (RMHD) model (Kadomtsev & Pogutse 1974; Strauss 1976), which is appropriate for studying MHD turbulence with a strong guide field, \( v_A \gg v_{\text{rms}} \), and with the typical field-parallel gradients of the fluctuating fields ordered with respect to their field perpendicular ones according to \( v_{\text{rms}} \nabla \parallel \sim v_A \nabla \perp \). Numerical simulations of full MHD equations show that the universal regime of strong MHD turbulence is reproduced well for \( v_A/v_{\text{rms}} \geq 5 \) (e.g., Müller et al. 2003; Müller & Grappin 2005; Mason et al. 2008), which is properly captured by the RMHD system (2). RMHD allows one to reduce the number of fields by a factor of two and to speed up the numerical integration. We employ a fully dealiased Fourier pseudo-spectral method to solve Equation (2) in a rectangular periodic box, with field-parallel crossing section \( L_\parallel^2 = (2\pi)^2 \) and field-parallel box size \( L_\parallel = (v_A/v_{\text{rms}}) L_\perp \). The choice of a rectangular box, as discussed in Perez & Boldyrev (2009, 2010), allows for the excitation of elongated modes at large scales, necessary to avoid a long transition region between the forcing scale and the beginning of the inertial range.

The random forcing \( f^\pm \) is applied in Fourier space at wavenumbers \( 1 \leq k_\perp \leq 2, k_\parallel = 2\pi/L_\parallel \). The Fourier coefficients inside that range are Gaussian random numbers with amplitude chosen so that the resulting rms velocity fluctuations are of order unity. The individual random forces are refreshed independently for each mode, on average every \( \tau \). In the present simulations, we also introduce correlation between \( v_b \), which is achieved by taking \( f^\pm \) as uncorrelated Gaussian random forces with zero mean and variances \( \sigma^2 \). Denoting \( f_v = \frac{1}{2}(f^+ - f^-) \) and \( f_b = \frac{1}{2}(f^+ + f^-) \), we obtain that cross-helicity is controlled by \( \langle f_v \cdot f_b \rangle = \frac{1}{2}(\sigma^2 - \sigma^2_v) \). The results presented below have been conducted at numerical resolution of \( 1024^2 \times 256 \) points, which corresponds to the Reynolds

5 Note that independent driving of \( z^+ \) and \( z^- \) ensures that no residual energy is supplied to the system by the driving routine.
number of Re ≈ 5600. In the imbalanced run, the normalized cross-helicity is $H_c/E ≈ 0.6$. The cases were run for up to 200 large-scale eddy turnover times in order to get reliable statistics. More details on the numerical setup can be found in Perez & Boldyrev (2010).

The results of numerical simulations are presented in Figures 1 and 2. Two important observations should be made here. First, there is a tendency of magnetic energy to exceed the kinetic energy in the inertial range for both balanced and imbalanced cases. The presence of nonzero residual energy was noted in previous studies (e.g., Pouquet et al. 1976; Müller & Grappin 2005; Ng & Bhattacharjee 2007). Comparison of our results with previously available numerical data suggests that the level of residual energy is not universal; rather, it may be affected by the driving and the degree of imbalance. Second, the excess of magnetic energy persists in the whole inertial interval, however in quite a peculiar fashion. In both balanced and imbalanced cases, the residual energy spectrum has a power-law behavior close to $E_r(k_\perp) \propto k_\perp^{-2}$. In an inertial interval of limited extent, this leads to steepening of the magnetic spectrum and flattening of the velocity spectrum; however, the total energy scaling stays close to $-3/2$. Due to the relatively rapid spectral decline, the residual energy provides a subdominant contribution to both kinetic and magnetic energy spectra. We therefore propose that the mismatch between velocity and magnetic field energies becomes asymptotically irrelevant as the inertial range increases, in which case the universal scaling $-3/2$ is restored for both $E_v(k_\perp)$ and $E_b(k_\perp)$.

3. COMPARISON WITH SOLAR WIND DATA

Comparison of available numerical simulations with solar wind data is complicated by the fact that individual solar wind measurements typically last for a few correlation times ($\tau_c \sim 1$ hr), while in numerical simulations the spectra are averaged over tens or hundreds of turnover times to obtain good convergence. To make an appropriate comparison, we measure

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6 This tendency has been recently analytically explained in the case of weak MHD turbulence (Wang et al. 2011).

7 This may be related to the conservation of magnetic helicity that tends to accumulate at large scales in a turbulent regime (Biskamp 2003; Boldyrev & Perez 2009).

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Figure 2. Distributions of spectral indices for kinetic, magnetic, and total energies for individual snapshots in numerical simulations of MHD turbulence. Left plot: balanced turbulence, 80 snapshots; right plot: imbalanced turbulence, 196 snapshots. The average spectral indices are indicated by arrows. Normal distributions with the mean values and variances matching those of the data are also shown.

(A color version of this figure is available in the online journal.)

Figure 3. Histograms of measured spectral indices for the velocity spectrum (blue triangles), magnetic field spectrum (red squares), and total energy spectrum (black circles) in the solar wind using data from the ACE and Wind spacecraft. The average spectral indices are indicated by the arrows. Note the different horizontal scales in the two plots.

(A color version of this figure is available in the online journal.)
of 176 spectra studied in Podesta & Borovsky (2010) having the highest normalized cross-helicity $|\sigma_c| > 0.76$, that is, the greatest imbalance. The spectral indices are obtained from fits over the range of spacecraft frame frequencies from $10^{-3}$ to $3 \times 10^{-2}$ Hz for the Wind data (Podesta & Borovsky 2010) and from $1.8 \times 10^{-4}$ to $3.9 \times 10^{-3}$ Hz for the ACE data. It turns out that the scatter of individual indices in numerical simulations resembles the corresponding scatter in solar wind measurements, in that there is measurable deviation of the magnetic and kinetic spectra from the $-1.5$ scaling; however, the spectrum of the total energy stays remarkably close to $-1.5$. The mismatch of magnetic and kinetic energies is also similar to that seen in the solar wind where, in the study of Podesta & Borovsky (2010) for example, the power-law exponent of the residual energy takes typical values around $-1.75$. The solar wind also shows a tendency for the spectral indices of velocity and magnetic fields to be closer together when the normalized cross-helicity is high than when it is low (Borovsky & Denton 2010; Podesta & Borovsky 2010). A similar tendency is evident in the simulation results in Figure 2. We therefore propose that the mismatch between $E_v(k_{\perp})$ and $E_v(k_{\parallel})$ observed in the solar wind turbulence is neither the manifestation of non-universality of MHD turbulence nor does it indicate a breakdown of the applicability of incompressible MHD turbulence theory to the solar wind. Rather, it is a consequence of significant residual energy generated at large scales, in agreement with numerical simulations.

4. CONCLUSIONS

Velocity spectra, magnetic field spectra, and total energy spectra in high-resolution numerical simulations of three-dimensional incompressible MHD turbulence are shown to be in good agreement with solar wind observations at 1 AU where the respective spectral indices of $E_v$, $E_b$, and $E$ are approximately centered around $1.4$–$1.5$, $1.6$–$1.7$, and $1.5$–$1.6$ (see, e.g., Tessein et al. 2009; Podesta & Borovsky 2010). It is important to note that the large variability found in solar wind spectral indices is also observed in temporal snapshots of the numerical simulations. The unique scaling laws obtained from simulation data averaged over many eddy turnover times can also be obtained through a statistical averaging of the spectral indices of individual snapshots. This provides justification for the widely used statistical approach to the analysis of spectral indices in the solar wind, where averaging over many turnover times is not practical. Our results indicate that universal inertial range dynamics may be present in the solar wind in spite of the observed high variability of solar wind measurements, and that solar wind turbulence spectra are consistent with the characteristics of incompressible MHD turbulence.

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