Gravitational Zero Point Energy and the
Induced Cosmological Constant

Remo Garattini

Abstract We discuss how to extract information about the cosmological constant from the Wheeler-DeWitt equation, considered as an eigenvalue of a Sturm-Liouville problem in a generic spherically symmetric background. The equation is approximated to one loop with the help of a variational approach with Gaussian trial wave functionals. A canonical decomposition of modes is used to separate transverse-traceless tensors (graviton) from ghosts and scalar. We show that no ghosts appear in the final evaluation of the cosmological constant. A zeta function regularization and a ultra violet cutoff are used to handle with divergences. A renormalization procedure is introduced to remove the infinities. We compare the result with the one obtained in the context of noncommutative geometries.

1 Introduction

One of the biggest challenges of our century is the explanation of why the observed cosmological constant is so small when compared to the one estimated by Zero Point Energy (ZPE) computations in Quantum Field Theory. Indeed there exists a difference of 120 orders of magnitude between them. However, it appears that a definitive answer is still lacking. One possible approach to this problem comes from the Wheeler-DeWitt equation (WDW)[1], which is described by

\[ \mathcal{H} \Psi = \left( 2\kappa G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (\hat{3} R - 2\Lambda) \right) \Psi = 0, \]  

(1)

where \( \kappa = 8\pi G \), \( G_{ijkl} \) is the super-metric and \( \hat{3} R \) is the scalar curvature in three dimensions. The main reason to use such an equation is that its most general formu-
lation intrinsically includes a cosmological term. Moreover, if we formally re-write the WDW equation as\footnote{See also Ref.\cite{3} for an application of the method to a $f(R)$ theory.\[2\]}

\begin{equation}
\frac{1}{V} \int \frac{\mathcal{D}[g_{ij}] \Psi^* [g_{ij}] \int_{\Sigma} d^3 x \hat{\Lambda}_{\Sigma} \Psi [g_{ij}]}{\int \mathcal{D}[g_{ij}] \Psi^* [g_{ij}] \Psi [g_{ij}]} = \frac{1}{V} \frac{\langle \Psi^* \int_{\Sigma} d^3 x \hat{\Lambda}_{\Sigma} \Psi \rangle}{\langle \Psi^* \Psi \rangle} = \frac{-\Lambda}{\kappa},
\end{equation}

where

\begin{equation}
V = \int_{\Sigma} d^3 x \sqrt{g}
\end{equation}

is the volume of the hypersurface $\Sigma$ and

\begin{equation}
\hat{\Lambda}_{\Sigma} = (2\kappa) G_{ijkl} \pi^{ijkl} - \sqrt{g}^3 R / (2\kappa),
\end{equation}

we recognize that the WDW equation can be represented by an expectation value. In particular, (2) represents the Sturm-Liouville problem associated with the cosmological constant. In this form the ratio $\Lambda_c / \kappa$ represents the expectation value of $\hat{\Lambda}_\Sigma$ without matter fields. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that we would like to explain the cosmological constant ($\Lambda_c / \kappa$) as a ZPE effect. To fix ideas, we will work with the following form of the metric

\begin{equation}
ds^2 = -N^2 (r) dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\end{equation}

where $b(r)$ is subject to the only condition $b(r_i) = r_i$. As a first step, we begin to decompose the gravitational perturbation in such a way to obtain the graviton contribution enclosed in (2).

\section{2 Extracting the graviton contribution}

We can gain more information if we consider $g_{ij} = \bar{g}_{ij} + h_{ij}$, where $\bar{g}_{ij}$ is the background metric and $h_{ij}$ is a quantum fluctuation around the background. Thus (2) can be expanded in terms of $h_{ij}$. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3 x \sqrt{g}^3 R$ up to the quadratic order. However, to proceed with the computation, we also need an orthogonal decomposition on the tangent space of 3-metric deformations\cite{4, 5}:

\begin{equation}
h_{ij} = \frac{1}{3} \left( \sigma + 2 \nabla \cdot \xi \right) g_{ij} + \left( L \xi \right)_{ij} + h_{ij}^\perp.
\end{equation}

The operator $L$ maps $\xi_i$ into symmetric tracefree tensors.