A DIOPHANTINE PROBLEM CONCERNING
THIRD ORDER MATRICES

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Abstract. In this paper we find a third order unimodular matrix, none
of whose entries is 1 or −1, such that when each entry of the matrix is
replaced by its cube, the resulting matrix is also unimodular. Further,
we find third order square integer matrices (a_{ij}), none of the integers
a_{ij} being 1 or −1, such that det (a_{ij}) = k and det (a_{ij}^3) = k^3, where k
is a nonzero integer.

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1. Introduction

This paper is concerned with the problem of finding a 3 × 3 integer matrix
(a_{ij}), with no a_{ij} = ±1, such that det (a_{ij}) = 1, and further, when each entry
of the matrix A is replaced by its cube, then also the determinant is 1, that
is, det (a_{ij}^3) = 1. We also consider the more general problem of finding a
matrix (a_{ij}), none of the a_{ij} being 0 or ±1, such that det (a_{ij}) = k and
det (a_{ij}^3) = k^3, where k is a nonzero integer.

It is pertinent to recall that Molnar [7] had posed the problem of finding
an n × n integer matrix (a_{ij}), with no a_{ij} = ±1, such that det (a_{ij}) = 1
and also det (a_{ij}^2) = 1. Several authors found solutions of the problem when
n = 3 [3, 4, 5]. In fact, Dănescu, Văjăitu, and Zaharescu [2] solved Molnar’s
problem for matrices of arbitrary order. Guy restricted the problem to 3 × 3
matrices in his book, “unsolved problems in number theory” [6, problem
F28, pp. 265–266], but imposed the additional condition that all the entries
a_{ij} should also be nonzero. He concluded his discussion by asking, “Will
the problem extend to cubes?” This question has, until now, remained
completely unanswered.

If A = (a_{ij}) is any n × n matrix, we will write A(3) to denote the matrix
(a_{ij}^3) obtained by replacing each entry of the matrix A by its cube. We
obtain in this paper a 3 × 3 matrix A = (a_{ij}) whose entries are univariate
polynomials with integer coefficients, with no a_{ij} = ±1, and such that both
det A and det (A(3)) are equal to 1. We also obtain a parametric solution
of the more general problem of finding a 3 × 3 integer matrix (a_{ij}), none of
the a_{ij} being 0 or ±1, and such that det (a_{ij}) = k and det (a_{ij}^3) = k^3, where
k is a nonzero integer.

1This notation is adapted from the notation used by Dănescu et al. [2].
2. Unimodular matrices that remain unimodular when each entry is replaced by its cube

If \( M \) is any \( n \times n \) matrix such that both \( \det M = 1 \) and \( \det(M^{(3)}) = 1 \), then several other matrices satisfying these conditions can readily be derived from the matrix \( M \). In Section 2.1 we give a lemma that lists out such matrices, and in Section 2.2 we obtain third order matrices satisfying such conditions.

2.1. A general lemma. We will denote the transpose of a matrix \( M \) by \( M^T \). Further, we will write \( E_{ij} \) to denote the elementary matrix obtained by interchanging the \( i \)-th and \( j \)-th rows of the identity matrix, and \( E_i(\alpha) \) to denote the elementary matrix obtained by multiplying the \( i \)-the the row of the identity matrix by \( \alpha \).

**Lemma 2.1.** If \( M \) is any \( n \times n \) integer matrix with the property that both \( \det M = 1 \) and \( \det(M^{(3)}) = 1 \), then the following integer matrices, derived from the matrix \( M \), also have this property:

(i) the matrix \( M^T \);
(ii) the matrices \( M_1 = E_{i_1}(-1)E_{i_2}(-1)M \) and \( M_2 = ME_{i_1}(-1)E_{i_2}(-1) \) where \( i_1, i_2 \in \{1, \ldots, n\} \) such that \( i_1 \neq i_2 \);
(iii) the matrices \( M_3 = E_{i_1i_2}E_{j_1j_2}M, M_4 = ME_{i_1i_2}E_{j_1j_2} \) and \( M_5 = E_{i_1i_2}ME_{j_1j_2} \) where \( i_1 \neq i_2, j_1 \neq j_2 \) and \( i_1, i_2, j_1, j_2 \in \{1, \ldots, n\} \);
(iv) the matrix \( M([i, j], \alpha) = E_i(\alpha)ME_j(\alpha^{-1}) \) where \( i, j \in \{1, \ldots, n\} \) and \( \alpha \) is a nonzero rational number so chosen that the entries of the matrix \( M([i, j], \alpha) \) are all integers.

**Proof.** Clearly, \( \det(M^T) = 1 \), and \( (M^T)^{(3)} = (M^{(3)})^T \), hence \( \det((M^T)^{(3)}) = \det((M^{(3)})^T) = \det(M^{(3)}) = 1 \), which proves the first part of the lemma. To prove (ii), we note that \( \det(E_{i_1}(-1)) = \det(E_{i_2}(-1)) = -1 \), hence \( \det M_1 = \det M \), and by the definition of \( M_1^{(3)} \), it follows that \( M_1^{(3)} = E_{i_1}(-1)E_{i_2}(-1)M^{(3)} \), hence \( \det(M_1^{(3)}) = \det(M^{(3)}) = 1 \). This proves the result for the matrix \( M_1 \). The proofs for the other matrices listed at (ii) and (iii) above are similar and are accordingly omitted.

Finally, regarding the last matrix \( M([i, j], \alpha) \), it is readily seen that \( \det(M([i, j], \alpha)) = \det M = 1 \). Further, on multiplying the entries of the \( i \)th row of the matrix \( M^{(3)} \) by \( \alpha^3 \) and then multiplying the entries of the \( j \)th column by \( \alpha^{-3} \), we get the matrix \( (M([i, j], \alpha))^{(3)} \). It follows that \( \det(M([i, j], \alpha))^{(3)} = \det(M^{(3)}) = 1 \).

2.2. Third order unimodular matrices. We will now obtain third order square integer matrices \( A = (a_{ij}) \), with no \( a_{ij} = \pm 1 \), such that both \( \det(a_{ij}) \) and \( \det(a_{ij}^3) \) are equal to 1. We have to solve two simultaneous equations in nine independent variables. A fair amount of computer search yielded essentially only one such matrix, namely,

\[
(2.1) \quad A_1 = \begin{bmatrix} 7 & 11 & 2 \\ 13 & 20 & 3 \\ 2 & 3 & 0 \end{bmatrix},
\]
which satisfies the conditions \( \det A_1 = 1 \) and \( \det (A_1^{(3)}) = 1 \).

We now give a theorem that gives a parametric solution of the problem.

**Theorem 2.2.** The matrix \( A \) defined by
\[
A = \begin{bmatrix}
(16t + 1)(2592t^2 + 288 + 7) & (18t + 1)(24t + 1)(144t + 11) & 2 \\
(12t + 1)(5184t^2 + 540 + 13) & (72t + 5)(1296t^2 + 153t + 4) & 3 \\
2 & 3 & 0
\end{bmatrix},
\]
where \( t \) is an arbitrary parameter, satisfies the conditions \( \det A = 1 \) and \( \det (A^{(3)}) = 1 \).

**Proof.** We begin with the \( 3 \times 3 \) matrix \( B = (b_{ij}) \) where we take
\[
b_{13} = b_{23} = b_{31} = b_{32} = 1, \quad b_{33} = 0,
\]
so that the matrix \( B \) may be written as follows:
\[
B = \begin{bmatrix}
b_{11} & b_{12} & 1 \\
b_{21} & b_{22} & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

We then get,
\[
\det B = -b_{11} + b_{12} + b_{21} - b_{22},
\]
\[
\det (B^{(3)}) = -b_{11}^3 + b_{12}^3 + b_{21}^3 - b_{22}^3.
\]

We note that a parametric solution of the simultaneous diophantine equations,
\[
x_1 + x_2 + x_3 + x_4 + x_5 = 0,
\]
\[
x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0,
\]
given by Choudhry [1, p. 316], is as follows:
\[
x_1 = pq(r^2 - s^2) + q^2r^2,
\]
\[
x_2 = -(p^2s(r + s) - q^2rs),
\]
\[
x_3 = p^2r(r + s) + pqr^2 - q^2rs,
\]
\[
x_4 = -(p^2r(r + s) + pq(r^2 - s^2)),
\]
\[
x_5 = p^2s(r + s) - pq(r^2 - s^2),
\]
where \( p, q, r, \) and \( s \) are arbitrary parameters.

With the values of \( x_i, i = 1, \ldots, 5 \), defined by (2.7), we take
\[
b_{11} = x_2, \quad b_{12} = -x_3, \quad b_{21} = -x_4, \quad b_{22} = x_5,
\]
when we get,
\[
\det B = x_1
\]
\[
\det (B^{(3)}) = x_1^3.
\]

We now choose the parameters \( p, q, r, s \), as follows:
\[
p = 36t + 3, \quad q = -1, \quad r = 144t + 11, \quad s = -144t - 9,
\]
where \( t \) is an arbitrary parameter, when we get \( x_1 = 1 \). The entries of the matrix \( B \) may now be written in terms of the parameter \( t \). We rename this matrix as \( C \), and write it explicitly as follows:

\[
C = \begin{bmatrix}
9(16t + 1)(2592t^2 + 288t + 7) & 6(18t + 1)(24t + 1)(144t + 11) & 1 \\
6(12t + 1)(5184t^2 + 540t + 13) & 4(72t + 5)(1296t^2 + 153t + 4) & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Since \( x_1 = 1 \), it follows from (2.9) that the matrix \( C \) satisfies the conditions \( \det C = 1 \) and \( \det (C^{(3)}) = 1 \).

Now on starting with the matrix \( C \), and using the last matrix listed in Lemma 2.1 four times, in succession, we obtain four matrices \( C_i, i = 1, \ldots, 4 \), as follows:

\[
C_1 = C(\lfloor 1, 3 \rfloor, 1/3), \quad C_2 = C(\lfloor 2, 3 \rfloor, 1/2), \\
C_3 = C(\lfloor 3, 1 \rfloor, 3), \quad C_4 = C(\lfloor 3, 2 \rfloor, 2).
\]

In view of Lemma 2.1, each of the matrices \( C_i, i = 1, \ldots, 4 \), satisfies the conditions \( \det C_i = 1 \) and \( \det (C_i^{(3)}) = 1 \). In fact, the matrix \( C_4 \) is the matrix \( A \) mentioned in the theorem. It follows that \( \det A = 1 \) and \( \det (A^{(3)}) = 1 \).

When \( t = 0 \), the matrix \( A \), defined by (2.2), reduces to the matrix \( A_1 \) given by (2.1). As a second numerical example, when \( t = 1 \), we get the matrix

\[
A_2 = \begin{bmatrix}
49079 & 73625 & 2 \\
74581 & 111881 & 3 \\
2 & 3 & 0
\end{bmatrix},
\]

which satisfies the conditions \( \det A_2 = 1 \) and \( \det (A_2^{(3)}) = 1 \).

We note that one of the entries of the matrix \( A \) given by Theorem 2.2 is always zero. While it would be interesting to find a \( 3 \times 3 \) integer matrix \( A \), none of whose entries is 0 or \( \pm 1 \), such that both \( \det A \) and \( \det (A^{(3)}) \) are equal to 1, we could not find such an example.

3. A more general problem

We will now find third order square integer matrices \( A \) such that \( \det A = k \) and \( \det (A^{(3)}) = k^3 \), where \( k \neq 1 \) is a nonzero integer.

In fact, in Section 2.2 we have already obtained a solution to this problem in terms of four arbitrary parameters \( p, q, r, s \), with \( k = pq(r^2 - s^2) + q^2r^2 \), since the matrix \( B \), whose entries are defined by (2.3) and (2.8), satisfies the conditions (2.9) where the value of \( x_1 \) is given by (2.7). We note, however, that one entry of the matrix \( B \) is always 0.

A computer search for \( 3 \times 3 \) integer matrices, none of the entries being 0 or \( \pm 1 \), such that \( \det A = k \) and \( \det (A^{(3)}) = k^3 \), where \( k \) is an integer \(< 10 \), yielded just one such example, namely the matrix

\[
M = \begin{bmatrix}
-5 & 4 & 10 \\
5 & 3 & 11 \\
3 & 2 & 7
\end{bmatrix},
\]
such that \( \text{det} \ M = 7 \) and \( \text{det} \ M^{(3)} = 7^3 \). The following theorem gives a more general solution of the problem with the entries of the matrix \( A \) being given in terms of polynomials in six arbitrary integer parameters.

**Theorem 3.1.** If the polynomial \( \phi(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \) is defined by

\[
\phi(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = -(\alpha_2 \beta_3 + \alpha_3 \beta_2 + \alpha_1 \alpha_2 \alpha_3) \alpha_1 \alpha_2 \alpha_3 \beta_2 \beta_3 - (\alpha_2 \beta_3 + \alpha_3 \beta_2 + \alpha_1 \alpha_2 \alpha_3) \\
+ \frac{\alpha_2 \alpha_2 \beta_2 \beta_2}{\alpha_2} - \alpha_3 \alpha_3 \beta_2 \beta_3 + (\alpha_3 \beta_2) (\alpha_2 \beta_3 + \alpha_3 \beta_2)^2 \alpha_1 \beta_1 \beta_2 \beta_3 - (\alpha_2 \beta_3 + \alpha_3 \beta_2) \\
\times (\alpha_2 \beta_2 + \alpha_2 \beta_2 \beta_2 + \alpha_3 \beta_2) \alpha_1 \alpha_2 \alpha_3 \beta_2 \beta_3 - 2 \alpha_1 \alpha_2 \alpha_3 \beta_2 \beta_3 \beta_3 \\
-(\alpha_2 \beta_3 + \alpha_3 \beta_2 + \alpha_3 \beta_2) (\alpha_2 \beta_3 + \alpha_3 \beta_2 + \alpha_3 \beta_2)^2 - 2 \alpha_2 \alpha_3 \beta_2 \beta_3 + \alpha_4 \beta_4 \alpha_4 \beta_4 \\
+ 2 (\alpha_2 \beta_3 + \alpha_2 \beta_3) \alpha_1 \alpha_2 \beta_2 \beta_3 + (\alpha_2 \beta_3 + \alpha_3 \beta_2) (\alpha_2 \beta_3 + \alpha_3 \beta_2) \\
\times \alpha_2 \alpha_3 \beta_2 \beta_3 + (\alpha_2 \beta_3 + \alpha_3 \beta_2) \alpha_1 \alpha_2 \alpha_3 \beta_2 \beta_3 \beta_3,
\]

with \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \), being arbitrary integer parameters, the matrix \( A \) defined by

\[
A = \begin{bmatrix}
\phi(p, q, r, u, v, w) & \phi(q, r, u, v, w) & \phi(r, p, q, w, u) \\
p & q & r \\
u & v & w
\end{bmatrix},
\]
satisfies the conditions,

\[
det A = k, \quad \text{and} \quad det (A^{(3)}) = k^3,
\]

where

\[
k = pqr (pv - qu) (pw - ru) (qw - rv) (p^2 v^2 + pqw + q^2 w^2) \\
\times (p^2 w^2 + pru + r^2 u^2) (q^2 w^2 + qru + r^2 v^2) (pqw + prv + qru).
\]

**Proof.** We begin with the matrix \( A \) defined by

\[
A = \begin{bmatrix}
x & y & z \\
p & q & r \\
u & v & w
\end{bmatrix},
\]
when Eqs. (3.4) may be written as follows:

\[
(qw - rv) x + (ru - pw) y + (pv - qu) z = k, \\
(q^3 w^3 - r^3 v^3) x^3 + (r^3 u^3 - p^3 w^3) y^3 + (p^3 v^3 - q^3 u^3) z^3 = k^3.
\]

On eliminating \( k \) from Eqs. (3.7) and (3.8), we get,

\[
(q^3 w^3 - r^3 v^3) x^3 + (r^3 u^3 - p^3 w^3) y^3 + (p^3 v^3 - q^3 u^3) z^3 \\
- ((qw - rv) x + (ru - pw) y + (pv - qu) z)^3 = 0.
\]

We note that when \( (x, y, z) = (p, q, r) \), both \( \text{det} A \) and \( \text{det} (A^{(3)}) \) vanish, and hence \( (x, y, z) = (p, q, r) \) is a solution of Eq. (3.9). Similarly, \( (x, y, z) = (u, v, w) \) is also a solution of Eq. (3.9).

Equation (3.9) is a homogeneous cubic equation in the variables \( x, y \) and \( z \), and accordingly, we may consider it as an elliptic curve in the projective plane \( \mathbb{P}^2 \) with two known points on the curve being \( P_1 = (p, q, r) \) and
$P_2 = (u, v, w)$. If we draw a line joining the points $P_1$ and $P_2$ to intersect the elliptic curve (3.9) in a third rational point, say $(x_1, y_1, z_1)$, and take $(x, y, z) = (x_1, y_1, z_1)$, the left-hand side of both Eqs. (3.7) and (3.8) becomes 0, and we do not get a nonzero value of $k$ as desired. Accordingly, we draw a tangent at the point $P_1$ to intersect the elliptic curve in a point $P_3$ whose coordinates are as follows:

$$x = \phi(p, q, r, u, v, w), \quad y = \phi(q, r, p, v, w, u), \quad z = \phi(r, p, q, w, u, v),$$

where $\phi(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ is defined by (3.2).

The values of $x, y, z$, given by (3.10) satisfy Eq. (3.9), and further, the value of $k$ obtained from Eq. (3.7) is given by (3.5). On substituting the values of $x, y, z$, given by (3.10) in (3.6), we get the matrix $A$, defined by (3.3), that satisfies the conditions (3.4) with the value of $k$ being given by (3.5) as stated in the theorem.

We note that in any numerical example of the matrix $A$, if the greatest common divisor $g$ of the entries of any row (or column) of the matrix $A$ is $> 1$, then $g$ is also a factor of $\det A$, and we can divide the entries of that row (or column) by $g$ to get a numerically smaller example of a matrix satisfying the specified conditions. For instance, when $(p, q, r, u, v, w) = (2, -3, 3, -2, 4)$, we get a matrix which, on factoring out the greatest common divisor of the entries of the first row, yields the matrix,

$$A = \begin{bmatrix}
-57797 & -109147 & -22789 \\
2 & -3 & 3 \\
3 & -2 & 4
\end{bmatrix},$$

for which $\det A = 123690$ and $\det (A^{(3)}) = 123690^3$.

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