Abstract

A Stern-Gerlach interferometer uses a magnetic field gradient to split particle wave functions into spatially separated wave-packets according to their respective spin projections. Over the years, quite a few proposals have been put forward by various groups to exploit this effect in order to create stable macroscopic spatial superpositions between micron-sized neutral test masses over appreciably long time scales. One such proposal put forward by Bose et al. and co-workers in 2017 uses this idea to show that two masses cannot be gravitationally entangled if not for the presence of a quantum coherent mediator (i.e., through spin correlation measurements between two quantum spins, each embedded in a test mass, they seek to demonstrate that gravity can act as a quantum coherent mediator [see [1]]. This primarily involves cooling the test mass to the ground state of a harmonic trap, thereby releasing it in a Stern-Gerlach interferometer. A key aspect of this approach involves the measure of the visibility of the SG-interferometer, a quantity that provides an estimate of the degree of spin coherence that is conserved over the total interferometric time after the wave-packets are combined in both, position and momentum space. A successful implementation of this idea however requires the knowledge of several experimental parameters, some of which include the temperature to which the test mass must be cooled initially, the admissible experimental errors in the measure of the phase-space observables (i.e., spatial and momentum splitting between the wave-packets with respect to the initial position and momentum uncertainties of the test mass.
in the ground state of the harmonic trap) and the total time-of-flight of the wave-packets in the interferometer. To this end, we present a rigorous mathematical analysis for the visibility in a general SG interferometer for non-squeezed and squeezed thermal coherent states of the Quantum harmonic oscillator. Additionally, by considering suitable experimental errors in the measure of the phase-space variables and subject to the desired accuracy in the measure of the visibility, we derive constraints on the temperature of the initially prepared wave-packet of the test mass for both, the non-squeezed and squeezed coherent states. We show that for wave-packet split sizes of the order of microns, masses of the order of $10^{-14} - 10^{-15}$ kg can be used to realize such a proposal in practice for time intervals as high as 0.5 seconds. Our results show that for the squeezed case, the temperatures required can be scaled up by several orders of magnitude (as opposed to the non-squeezed case) if one considers a squeezing in the momentum space of the initially prepared wave-packet. 

**Keywords:** Coherent states, Squeezing parameter, Temperature, Quantum harmonic oscillator, Visibility

1. **Introduction**

The Stern-Gerlach experiment has long been hailed as the first direct evidence of the quantum nature of spin of particles. One observes two distinct peaks corresponding to the spin-up and spin-down states of the emergent spin-1/2 particle beams on a detector placed at the far end of a Stern-Gerlach apparatus if one prepares the entrant particle beam in the $|\pm\rangle$ eigenstate of the spin operator $S_x$. A typical SG experiment involves the following: a particle beam in a pure spin state, say in the $|+\rangle$ eigenstate of $S_x$ enters the Stern-Gerlach magnet. The results of the measurement of the $S_z$ spin component of the emergent particle beam are consistent when one

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regards \( S_x \) as a coherent superposition over the spin eigenstates of \( S_z \). To estimate spin coherence, one generally measures the \( x \)-component of the spin to obtain \( |S_x\rangle \). As the quantum states of the split wave-packets evolve in time, their spatial components eventually become orthogonal to each other and spin coherence is completely lost. It must be noted that our analysis however, considers a micron-sized mesoscopic neutral test mass prepared initially in the ground state of a harmonic oscillator trap which is thereafter released from the trap and made to propagate through the Stern-Gerlach apparatus. We make use of some important results that appeared in the very early works of Schwinger et al. in our analysis.

We consider the typical setup of a Stern-Gerlach interferometer. An inherent assumption in our analysis is that we consider the magnetic field gradient to exist only along the \( z \)-direction. We work under the assumption that the field gradient applied is time dependent. As a brief introduction to the Stern-Gerlach theory [see [2]], we associate the familiar Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \) with the magnetic moment of the incoming particle beam. The magnetic moment \( \vec{\mu} \) can hence be expressed as \( \vec{\mu} = \mu\sigma \), where \( \sigma \) is the set of the familiar Pauli matrices. The force acting on the entrant particle beam can be expressed as the gradient of the interaction energy between the magnetic moment of the particles and the applied field gradient, as follows

\[
F(t) = \nabla(\vec{\mu}.\vec{B}(z,t)).
\]  

(1)

Maxwell’s equation \( \nabla.\vec{B}(z,t) = 0 \) dictates that if a finite force is to act on the incoming particle beam, in addition to a field gradient along the \( z \)-direction, it is essential to invoke the presence of a field gradient in the \( x-y \) plane as well. In our analysis however, we choose to ignore this effect by assuming that by some means, we are able to suppress the effect of the field gradient present in the \( x-y \) plane and the field gradient applied along the \( z \)-direction is dominant. For reasons listed out in [2], we consider a linear expansion of the field \( \vec{B}(z,t) \), as

\[
\vec{B}(z,t) \approx \vec{B}(t) + \frac{\partial B}{\partial z}(t)z.
\]  

(2)
The interaction energy in the SG setup is then \(-\mu \cdot \vec{B}(t) = -\mu \frac{\partial B}{\partial z}(t)\sigma_z z - \mu B(t)\sigma_z\) which serves as the potential energy term in the Stern-Gerlach Hamiltonian. The Hamiltonian for the Stern-Gerlach setup assumes the form

\[ H = \frac{p^2}{2m} - f(t)\sigma_z z - \mu B(t)\sigma_z, \]  

(3)

where \(f(t) = \mu \frac{\partial B}{\partial z}(t)\). We solve for the temporal evolution of the phase-space variables \(z(t)\) and \(p(t)\) by using the Heisenberg equation of motion, with the Hamiltonian given by Eq. (3). Note that \(z\) and \(p_z\) denote the position and momentum operators respectively. A trivial computation yields the following equations of motion for the split wave-packets and the temporal evolution of the Pauli spin operators \(\sigma_z\) and \(\sigma_+\) (note that \(\sigma_+ = \sigma_x + i\sigma_y\)) [see [2]]

\[ p(t) = p_0 + (\sigma_z)_0 \Delta p(t), \]  

(4.a)

\[ z(t) = z_0 + p_0 \frac{t}{m} + (\sigma_z)_0 \left( \Delta z(t) + \frac{t}{m} \Delta p(t) \right), \]  

(4.b)

\[ \sigma_z(t) = (\sigma_z)_0, \]  

(4.c)

and

\[ \sigma_+(t) = \exp \left( -i \left( \frac{2}{\hbar} \int_0^t \mu B(t') dt' + \frac{2\Delta p(t) z_0}{\hbar} - \frac{2\Delta z(t) p_0}{\hbar} \right) \right) (\sigma_+)_0. \]  

(4.d)

Here, \(z_0\) and \(p_0\) are initial conditions that we set to solve for the equations of motion, namely that the particle beam enters the SG interferometer at the point \(z_0\) with a non-zero momentum \(p_0\). The time dependent parameters in Eq. (4.a) and Eq. (4.b) denote the macroscopic displacements of the split wave-packets in phase-space. These are given as

\[ \Delta p(t) = \int_0^t f(t') dt', \]  

(5.a)
and
\[ \Delta z(t) = \int_0^t \frac{f(t')}{m} (t - t') dt'. \] (5.b)

Note that for a constant force \( f \), Eq. (5.a) and Eq. (5.b) assume the form
\[ \Delta p(t) = ft, \] (5.c)
and
\[ \Delta z(t) = \frac{ft^2}{2m}. \] (5.d)

Alternatively, from Eq. (5.a) and Eq. (5.b), we define the temporal evolution of the displacement of the split wave-packets in position space as
\[ \Delta \bar{z}(t) = - \int_0^t \frac{f(t')}{m} t' dt' = \Delta z(t) - \frac{t}{m} \Delta p(t) \] (5.e)

Schwinger et al. and co-workers pioneered early works on the realization of a full-loop Stern-Gerlach interferometer. Through a detailed analysis of the spin dynamics of particle beams in a Stern-Gerlach interferometer, they were able to arrive at a closed-form expression for the visibility in terms of the coordinate wave function of the \textit{initially prepared} spatial state (i.e., at time \( t = 0 \)), given as [3]
\[ \phi_{\text{coherence}} = \int_{-\infty}^{\infty} \psi_i^*(z - \Delta \bar{z}(t)) \psi_i(z + \Delta \bar{z}(t)) \times \exp \left( -\frac{2i \Delta p_z(t)}{\hbar} \right) \, dz, \] (6)

where \( z \) in Eq. (6) denotes the eigenvalue of the position operator \( z \). Note that \( \psi_i(z') \) denotes the coordinate wave function of the \textit{initially prepared} spatial state. For a \textit{stationary} Gaussian wave-packet (one that undergoes no temporal evolution), they arrived at a closed-form expression for the visibility in the SG interferometer as follows [see [3]]
\[ \phi_{\text{coherence}} = \exp \left( -\frac{1}{2} \left( \left( \frac{\Delta z}{\delta z} \right)^2 + \left( \frac{\Delta p_z}{\delta p_z} \right)^2 \right) \right), \] (7)
where $\delta z$ and $\delta p_z$ denote the initial position and momentum uncertainties in the Gaussian wave-packet respectively. It must be noted that the authors denote the visibility by $C$ in [3]. A key point to observe is that the visibility in the SG interferometer undergoes a Gaussian decay with increasing spatial and momentum splitting between the wave-packets [4]. This result however, suffers from a major drawback, due in part to the fact that it does not account for the temporal evolutions of the spatial and momentum splitting $\Delta z(t)$ and $\Delta p_z(t)$ between the split wave-packets respectively, namely that the consequences arising from Eq. (5.e) have been ignored. To this end, we perform a complete analysis of the visibility for the cases of the non-squeezed and squeezed thermal coherent states of the Quantum harmonic oscillator by taking into account the effect of Eq. (5.e). We note that it is necessary to keep the spatial and momentum splitting between the wave-packets as low as possible to 

\textit{maximize} the visibility, at least to the extent that the following conditions are satisfied

$$\Delta z \ll \delta z,$$  \hspace{1cm} (8.a)

and

$$\Delta p_z \ll \delta p_z.$$ \hspace{1cm} (8.b)

2. Analysis of the visibility for the case of a non-squeezed thermal coherent state of the Quantum harmonic oscillator

The explicit form of the coordinate wave function for a general non-squeezed coherent state $|\beta\rangle$ of the Quantum harmonic oscillator is given as (we consider a one-dimensional case) [refer appendix A for a detailed derivation of Eq. (9.a)]

$$\psi_{\beta}(z, t) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \exp\left(\frac{2m\omega}{\hbar} \Im[\beta(t)]z - \frac{m\omega}{2\hbar} \left(z - \sqrt{\frac{2\hbar}{m\omega}} \Re[\beta(t)]\right)^2\right), \hspace{1cm} (9.a)$$
where the quantities $\sqrt{\frac{2m\omega}{\hbar}} \Im[\beta(t)]$ and $\sqrt{\frac{2\hbar}{m\omega}} \Re[\beta(t)]$ are the expectation values of the momentum and position operators $p_z$ and $z$ respectively. Note that in Eq. (9.a), $\xi(t)$ is merely a time dependent phase factor. The time evolution operator for the coherent state of a Quantum harmonic oscillator assumes the form

$$\hat{U}(t) \equiv \exp(-i\omega t),$$

(9.b) for which the temporal evolution of the coherent state $|\beta\rangle$ is given as

$$\beta(t) = \beta(0) \exp(-i\omega t).$$

(10)

We now wish to compute analytically the visibility in a SG interferometer for the non-squeezed coherent state defined in Eq. (9.a). Note that for a total interferometric time $\tau$ (i.e., the time elapsed between the instant at which the wave-packets are initially split and the instant at which we begin to bring them together for recombination), from Eq. (5.e), we can express $\Delta \bar{z}(\tau)$ in terms of the final spatial and momentum splitting between the wave-packets as

$$\Delta \bar{z}(\tau) = \Delta z(\tau) - \frac{\tau}{m} \Delta p_z(\tau).$$

(11)

We are required to consider the form of the initially prepared spatial state for the analysis of the visibility in the SG interferometer (i.e., at time $t = 0$). To this end, we consider $\psi_\beta(z, t = 0)$ which from Eq. (9.a) can be written as

$$\psi_\beta(z, t = 0) = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \exp\left(i\xi(0) - \frac{m\omega}{2\hbar} \left(z - \sqrt{\frac{2\hbar}{m\omega}} \beta(0)\right)^2\right).$$

(12)

Note that the expectation value of the momentum operator $p_z$ vanishes at time $t = 0$. We now use the form of the wave function obtained in Eq. (12) (for the non-squeezed coherent state) to solve for the visibility parameter. We denote the visibility by $\phi_{\text{non-squeezed}}$. From Eq. (6), Eq. (11) and Eq. (12), we have

$$\phi_{\text{non-squeezed}} = \left(\frac{m\omega}{\pi \hbar}\right)^{1/4} \int_{-\infty}^{\infty} \exp\left(-\frac{m\omega}{2\hbar} \left(z - \Delta \bar{z}(\tau) - \sqrt{\frac{2\hbar}{m\omega}} \beta(0)\right)^2 + \left(z + \Delta \bar{z}(\tau) - \sqrt{\frac{2\hbar}{m\omega}} \beta(0)\right)^2\right).$$

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\[ \times \exp \left( -2i \frac{\Delta p_z(\tau)}{\hbar} \right) dz, \]  

(13.a)

which upon simplification (refer appendix B for the calculations involved herein) gives us for the visibility parameter

\[ \phi_{\text{non-squeezed}} = \exp \left( -\frac{m\omega}{\hbar} (\Delta z(\tau))^2 - \frac{(1 + \omega^2 \tau^2)}{m\hbar \omega} (\Delta p_z(\tau))^2 + \frac{2\Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar} \right) \]

\[ \times \exp \left( -2i \sqrt{\frac{2\hbar}{m\omega}} \beta(0) \frac{\Delta p_z(\tau)}{\hbar} \right). \]  

(13.b)

We note that the phase term in Eq. (13.b) is constant for a given set of experimental parameters and hence plays no role in the estimation of the visibility parameter. We are thus only concerned with the ‘amplitude’ part of \( \phi_{\text{non-squeezed}} \) which is given by the first exponential factor.

For the ground state of the Quantum harmonic oscillator, we define a characteristic length scale \( \sigma_0 \) and take the initial uncertainty in the measure of the position to be roughly equal to this length scale. For the ground state of the Quantum harmonic oscillator, we have

\[ \delta z \approx \sigma_0 = \sqrt{\frac{\hbar}{2m\omega}}. \]  

(14.a)

We know that the coherent state of the QHO is a minimum uncertainty state that saturates the uncertainty principle, for which we have \( \delta z \delta p_z = \hbar / 2 \). Using Eq. (14.a) and the uncertainty principle, Eq. (13.b) can be recast into the following form to account for the experimental errors in the measure of the phase-space variables (we consider only the ‘amplitude’ part of \( \phi_{\text{non-squeezed}} \))

\[ |\phi_{\text{non-squeezed}}| = \exp \left( -\frac{1}{2} \left( \left( \frac{\Delta z(\tau)}{\delta z} \right)^2 + (1 + \omega^2 \tau^2) \left( \frac{\Delta p_z(\tau)}{\delta p_z} \right)^2 + \frac{2\Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar} \right) \right) \]  

(14.b)

As outlined in Eq. (8.a) and Eq. (8.b), we note that similar conditions must be satisfied for the case of the non-squeezed coherent state of the QHO to maximize the visibility parameter.
We now seek to obtain an upper bound on the temperature at which the wave-packet of the neutral test mass must be initially cooled in the ground state of the harmonic oscillator trap. We note that in the classical approximation and at a finite temperature $T$, the equipartition theorem states that the average thermal energy of the QHO must equal $k_B T$, with one-half of the contribution coming from the kinetic energy term and one-half of the contribution coming from the potential energy term in the Hamiltonian of the Quantum harmonic oscillator (here $k_B$ is the Boltzmann’s constant). Cooling the neutral test mass to the ground state of the harmonic trap implies that the following relation must hold (note that the ground state energy of the QHO is given by $\frac{1}{2} \hbar \omega$)

$$\hbar \omega = k_B T$$  \hfill (15)

Suppose that we consider impose certain error tolerances in the measures of the phase-space variables, namely the accuracy to which we can maintain the ratios $\Delta z(\tau)/\delta z$ and $\Delta p_z(\tau)/\delta p_z$ over the total interferometric time $\tau$. We thus impose the following constraints

$$\left| \frac{\Delta z(\tau)}{\delta z} \right| \approx \eta_1, \hfill (16.a)$$

and,

$$\left| \frac{\Delta p_z(\tau)}{\delta p_z} \right| \approx \eta_2. \hfill (16.b)$$

Note that $\eta_1$ and $\eta_2$ are such that $0 < \eta_1, \eta_2 < 1$.

Suppose that we desire a certain accuracy in the measure of the visibility parameter. We set the exponential factor in Eq. (14.b) to be less or than equal to a desired value $\eta$, subject to which we estimate a bound on the temperature $T$ required (note that $0 < \eta < 1$). We thus have

$$\frac{1}{2} \left( \left( \frac{\Delta z(\tau)}{\delta z} \right)^2 + (1 + \omega^2 \tau^2) \left( \frac{\Delta p_z(\tau)}{\delta p_z} \right)^2 \right) - \frac{2 \Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar} \leq \eta$$  \hfill (17.a)

This is a quadratic inequality in $\omega$ which we can solve for by using the quadratic formula. From
Eq. (15), Eq. (16.a), Eq. (16.b) and Eq. (17.a), we have for the temperature $T$ (note that $T > 0$)

$$T \leq \frac{\hbar}{k_B \eta^2 \tau^2} \left[ \frac{2 \Delta z(\tau) \Delta p_z(\tau) \tau}{\hbar} + \sqrt{\left( \frac{2 \Delta z(\tau) \Delta p_z(\tau) \tau}{\hbar} \right)^2 - 2 \eta^2 \tau^2 \left( \frac{\eta_1^2 + \eta_2^2}{2} - \eta \right)} \right].$$  

(17.b)

We see that the temperature required primarily depends on the admissible experimental errors, the total interferometric time and the final spatial and momentum splitting between the wave-packets before they are brought together for recombination.

3. Analysis of the visibility for the case of a squeezed thermal coherent state of the Quantum harmonic oscillator

Squeezed coherent states are often encountered in the study of quantum optics. A generic version of the wave function of a squeezed coherent state of the Quantum harmonic oscillator assumes the form

$$\psi_\beta(z,t) = \frac{1}{\sqrt{s}} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \exp \left( i \xi(t) + i \sqrt{\frac{2 m \omega}{\hbar}} \Im[\beta(t)] z - \frac{m \omega}{2 \hbar s^2} \left( z - \sqrt{\frac{2 \hbar}{m \omega}} \Re[\beta(t)] \right)^2 \right),$$

(18)

where $s$ is a dimensionless free parameter, referred to as the squeezing parameter of the squeezed state. Effectively, one can squeeze the coherent state in either the position or the momentum quadrature which would lead to a modified scaling in the phase space parameters for the harmonic oscillator state. The characteristic length scale that we define for the squeezed state of the QHO now reads

$$\delta z \approx \sigma_0(s) = \sqrt{\frac{\hbar}{2 m \omega} s}.$$  

(19)

We note that this is now an explicit function of the squeezing parameter $s$. If $0 < s < 1$, it refers to a squeezing in the position quadrature and if $s > 1$, it refers to a squeezing in the momentum quadrature. In general, the initial uncertainties in the measure of the position and momentum of the initially prepared wave-packet (one that corresponds to the ground state of a harmonic trap)
get modified as follows

\[ \delta z_{squeezed} = \delta z_s, \tag{20.a} \]

and

\[ \delta p_{z_squeezed} = \frac{\delta p_z}{s}, \tag{20.b} \]

where \( \delta z \) and \( \delta p_z \) denote the initial uncertainties in the measure of the position and momentum of the initially prepared wave-packet in the non-squeezed case. We assume a similar approach as outlined in section II to compute the visibility parameter in a general Stern-Gerlach interferometer for the case of a squeezed coherent state of the QHO. We consider the form of the initially prepared spatial state (given by Eq. (18)) at time \( t = 0 \) to solve for the visibility parameter. The visibility parameter for the squeezed case is denoted by \( \phi_{squeezed} \) and we denote the initial uncertainties in the measure of the position and momentum of the initially prepared wave-packet by \( \delta z' \) and \( \delta p'_z \) respectively. From Eq. (6), Eq. (11) and Eq. (18), we obtain for the visibility parameter \( \phi_{squeezed} \) (refer appendix C for the calculations involved herein)

\[ |\phi_{squeezed}| = \exp \left( -\frac{1}{2} \left( \left( \frac{\Delta z(\tau)}{\delta z'} \right)^2 + \left( 1 + \frac{\omega^2 \tau^2}{s^4} \right) \left( \frac{\Delta p_z(\tau)}{\delta p'_z} \right)^2 \right) + \frac{2 \Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar s^2} \right). \tag{21} \]

Note that we have ignored the phase term that arises in the computation of the visibility parameter since being a fixed quantity for a given squeezing parameter \( s \), it plays no role in the estimation of \( \phi_{squeezed} \). We have also used Eq. (18) and the uncertainty principle to obtain \( \phi_{squeezed} \) in terms of the initial uncertainties in the measure of the position and momentum of the initially prepared wave-packet. We note that the conditions outlined in Eq. (8.a) and Eq. (8.b) must be met in order to maximize the visibility parameter in the SG interferometer.

To estimate the temperature to which the neutral test mass must be cooled to the ground state of the harmonic trap (now squeezed), we consider certain error tolerances in the measure of the
phase-space observables as before, namely that we have

\[ \left| \frac{\Delta z(\tau)}{\delta z'} \right| \approx \eta_1, \quad (22.a) \]

and

\[ \left| \frac{\Delta p_z(\tau)}{\delta p'_z} \right| \approx \eta_2. \quad (22.b) \]

For a given squeezing \( s \) in either the position or the momentum uncertainties, we seek to obtain a constraint on the temperature required for the initially prepared wave-packet, subject to a certain desired accuracy \( \eta \) in the measure of the visibility parameter \( \phi_{\text{squeezed}} \). As in section II, we consider cooling the neutral test mass to the ground state of the harmonic trap (now scaled due to squeezing), for which Eq. (15) holds true. We now have

\[
\frac{1}{2} \left( \left( \frac{\Delta z(\tau)}{\delta z'} \right)^2 + \left( 1 + \frac{\omega^2 \tau^2}{s^4} \right) \left( \frac{\Delta p_z(\tau)}{\delta p'_z} \right)^2 \right) - \frac{2 \Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar s^2} \leq \eta, \quad (23.a)
\]

where we now consider error tolerances \( \eta_1 \) and \( \eta_2 \) in the measure of \( \Delta z(\tau) / \delta z' \) and \( \Delta p_z(\tau) / \delta p'_z \) respectively. From Eq. (15) and Eq. (23.a), we get for the temperature \( T \) (note that \( T > 0 \))

\[
T(s) \leq \frac{\hbar s^4}{k_B \eta_1^2 \tau^2} \left[ \frac{2 \Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar s^2} + \sqrt{\left( \frac{2 \Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar s^2} \right)^2 - \frac{2 \eta_2^2 \tau^2}{s^4} \left( \frac{\eta_1^2 + \eta_2^2}{2} - \eta \right)} \right]. \quad (23.b)
\]

We see that besides other experimental parameters, the temperature \( T \) strongly depends on the squeezing parameter \( s \). An intuitive observation that one make in Eq. (23.b) is that the temperature can be scaled up considerably if one considers the squeezing parameter \( s \) to be relatively large.

From Eq. (20.b), we note that this corresponds to squeezing the initially prepared wave-packet in momentum space.

4. Discussion

We now consider a few sample cases for both, the non-squeezed and squeezed cases to get a feel of the numbers involved. Suppose that we prepare the neutral test mass in a non-squeezed
coherent state of the harmonic trap and cool it to its motional ground state at a finite temperature $T$. We consider an error tolerance of $10^{-1}$ in the measure of $\Delta z(\tau)/\delta z$ to be maintained over the course of the total interferometric time (i.e., with respect to a scale set by the initial uncertainty in the measure of the position of the test mass in the trap) and an error tolerance of say, $10^{-3}$ in the measure of $\Delta p_z(\tau)/\delta p_z$, also to be maintained over the course of the total interferometric time $\tau$. As mentioned in section II, we denote these error tolerances by $\eta_1$ and $\eta_2$ respectively. We consider a total interferometric time of $\tau = 0.5$ seconds and we seek to estimate the temperature required for obtaining a visibility of a desired value, say $\eta$ in the SG interferometer. Here, we take $\eta$ to be $10^{-1}$. We consider a maximum spatial split size of 10 microns between the wave-packets in the SG interferometer, corresponding to which we require a maximum momentum splitting of approximately $5.275 \times 10^{-34}$ kg-m/sec between the wave-packets. We consider a mesoscopic test mass, of the order of $10^{-14}$ kg. Using Eq. (17.b), we obtain an upper bound of $6.4835$ nK on the temperature to which the test mass must be cooled in the ground state of the harmonic trap. Given these parameters, we obtain a visibility of about 94.89% in the SG interferometer (using Eq. (14.b)). For the same set of experimental parameters, we set $\eta$ equal to $10^{-2}$. The upper bound on the temperature that we obtain in this case is $3.4016$ nK, with the visibility being about 99.25%.

In this context however, we note that a squeezing in the momentum quadrature can help scale up the temperatures required considerably (i.e., when $s$ is relatively large, as can be seen in Eq. (20.b)). We now consider a maximum spatial split size between the wave-packets of about 20 microns. For $\eta_1 = 10^{-1}, \eta_2 = 10^{-3}, \eta = 10^{-1}, m \approx 10^{-14}$ kg and a maximum momentum split size of about $2.6375 \times 10^{-34}$ kg-m/sec, we observe that a squeezing in the initial momentum uncertainty to about a tenth of its initial value (i.e., $s = 10$) yields an upper bound of $3.129 \mu K$ on the temperature of the initially prepared wave-packet and a visibility of roughly 94.87% in the SG interferometer (using Eq. (21)). For a larger value of $s$, say $s = 50$, we observe that for the same set of parameters as considered previously (i.e., the case for $s = 10$), we obtain an upper bound of
78.22 \, \mu K \) on the temperature of the initially prepared wave-packet, with the visibility in the SG interferometer being about 94.88%.

This clearly demonstrates that a squeezing of the initially prepared wave-packet in momentum space is both, desirable and effective. It is worth commenting here that squeezing the wave-packet in position space yields temperatures in the nanokelvins range (i.e., for \( 0 < s < 1 \)) which is far lower than what one would require when the initially prepared wave-packet is squeezed in momentum space, as has been demonstrated in the above cases. We also note that our results allow for a lot of experimental flexibility, in the sense that the visibility in the SG interferometer and the temperatures required for the initially prepared harmonic trap are primarily dependent on the desired experimental errors in the measure of the phase-space variables and the total time-of-flight of the split wave-packets in the SG interferometer, which the experimenter has good control over. Hence, the interplay of parameters external to our Stern-Gerlach analysis that may or may not have to be assumed is non-existent.

5. Summary

In this work, we have derived the closed-form expressions for the visibility in a general full-loop Stern-Gerlach interferometer for the cases of non-squeezed and squeezed thermal coherent states of the Quantum harmonic oscillator. In an effort to maximize the visibility obtained in the SG interferometer, we have analytically obtained constraints on the required temperatures of the initially prepared harmonic traps for both, the non-squeezed and squeezed coherent state cases in terms of the experimental errors that one must account for in the measure of the phase-space variables, the total time-of-flight of the wave-packets inside the SG interferometer and the desired accuracy in the measure of the visibility in the SG interferometer. We have shown that masses of the order of \( 10^{-14} \) kg and spatial split sizes of the order of microns (with the inclusion of suitable experimental errors) can be used to obtain relatively high visibilities in the SG interferometer over
time scales as high as 0.5 seconds, thus confirming that a proposal of the kind put forward in [1] can be realized in principle. We have demonstrated that for the case of the squeezed coherent state, a squeezing in the initial momentum uncertainty of the prepared spatial state would prove far more effective, since given the dependence of the temperature $T$ on the squeezing parameter $s$, temperatures of higher values (those that can be easily achieved for say, in conventional magneto-optical traps) can be implemented in a typical experimental setup. There however, remain a few issues that need to be addressed. For instance, we have chosen to ignore the fluctuations in the magnetic field gradient in our analysis. This of course, requires a full and rigorous QFT treatment in the context of the problem at hand. We also assume that the Pauli spin operator $\sigma_z$ is a constant of the motion, which would not be the case for an arbitrary configuration of the field gradient in the SG interferometer setup. We work under the assumption that the $z$-component of the field gradient greatly suppresses the effects that arise due to the presence of the field gradient in the $x$-$y$ plane (the presence of which is required due to Eq. (1)), in which case the assumption that $\sigma_z$ is a constant of the motion holds approximately good. We wish to circle back to these issues in some future works.

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*Conflict of interest:* The author declares no conflict of interest with any third party.
References

[1] Bose S, Mazumdar A, Morley GW, Ulbricht H, Toroš M, Paternostro M, et al. Spin Entanglement Witness for Quantum Gravity. Physical Review Letters. 2017;119(24):1-7.

[2] Englert BG, Schwinger J, Scully MO. Is spin coherence like Humpty-Dumpty? I. Simplified treatment. Foundations of Physics. 1988;18(10):1045-1056.

[3] Schwinger J, Scully MO, Englert BG. Is spin coherence like Humpty-Dumpty? II. General theory. Zeitschrift für Physik D Atoms, Molecules and Clusters. 1988;10(2-3):135-144.

[4] Keil M, Machluf S, Margalit Y, Zhou Z, Amit O, Dobrowski O, et al. Stern-gerlach interferometry with the atom chip. arXiv. 2020;(September).

Appendix A. Derivation of the coordinate representation of a coherent state of the QHO

The Hamiltonian for the one-dimensional Quantum harmonic oscillator assumes the form

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 z^2, \]  

(A.1)

where we consider \( m \) to be the mass of the Quantum harmonic oscillator and \( \omega \) to be its natural oscillation frequency. The eigenvalue equation corresponding to the number states \( |n\rangle \) of the QHO assumes the form

\[ H|n\rangle = E_n|n\rangle, \]  

(A.2)

where the energy eigenvalues \( E_n \) of the QHO are given by

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega. \]  

(A.3)
for \( n \in I^+ \). We introduce the familiar creation and annihilation operators for the QHO \( \hat{a}^\dagger \) and \( \hat{a} \) respectively, the properties of which are

\[
\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (A.4)
\]

and

\[
\hat{a} |n\rangle = \sqrt{n} |n-1\rangle. \quad (A.5)
\]

The number states \(|n\rangle\) of the QHO form an orthonormal basis, for which we have \( \langle n_1 | n_2 \rangle = \delta_{n_1 n_2} \) for any two eigenstates \(|n_1\rangle\) and \(|n_2\rangle\). We know that the coherent states of the QHO are eigenstates of the annihilation operator \( \hat{a} \). For a general coherent state \(|\beta\rangle\) that belongs to the orthonormal basis, we have the following eigenvalue equation for the operator \( \hat{a} \)

\[
\hat{a} |\beta\rangle = \beta |\beta\rangle. \quad (A.6)
\]

We note that the number states \(|n\rangle\) form a complete set of orthonormal basis eigenkets for the system. The coherent state \(|\beta\rangle\) can hence be expressed as a linear superposition over the number states \(|n\rangle\), as

\[
|\beta\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad (A.7)
\]

where the coefficients \( c_n \) are generally complex. From Eq. (A.6) and Eq. (A.7), we have

\[
\sum_{n=0}^{\infty} c_n \hat{a} |n\rangle = \beta \sum_{n=0}^{\infty} c_n |n\rangle, \quad (A.8)
\]

which after using the property outlined in Eq. (A.5) simplifies this to

\[
\sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle = \beta \sum_{n=0}^{\infty} c_n |n\rangle. \quad (A.9)
\]

On expanding the summation on both sides of Eq. (A.9) and comparing like terms, we get for the coefficients \( c_n \)

\[
c_n = \frac{\beta^n}{\sqrt{n!}} c_0. \quad (A.10)
\]
Thus, from Eq. (A.7) and Eq. (A.10), we have

\[ | \beta \rangle = c_0 \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} | n \rangle. \]  

(A.11)

Taking the adjoint on both sides of Eq. (A.11), we get

\[ \langle \beta | = c_0^* \sum_{n=0}^{\infty} \frac{(\beta^*)^n}{\sqrt{n!}} \langle n |. \]  

(A.12)

We are required to satisfy the normalization condition, viz. \( \langle \beta | \beta \rangle = 1 \) which from Eq. (A.11) and Eq. (A.12) gives us

\[ |c_0|^2 \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \frac{\beta^n (\beta^*)^p}{\sqrt{p!n!}} \langle p | n \rangle = 1, \]  

(A.13)

which on simplification yields

\[ |c_0|^2 \sum_{n=1}^{\infty} \frac{\beta^{2n}}{n!} = 1. \]  

(A.14)

Using the Taylor series expansion for \( e^x \) in Eq. (A.14), we arrive at a closed-form expression for \( c_0 \), as

\[ |c_0| = \exp \left( -\frac{1}{2} |\beta|^2 \right). \]  

(A.15)

From Eq. (A.11) and Eq. (A.15), we get

\[ | \beta \rangle = \exp \left( -\frac{1}{2} |\beta|^2 \right) \times \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} | n \rangle. \]  

(A.16)

To establish the unitary time evolution of coherent states, we consider the matrix elements of \( \hat{a}(t) \) between any two eigenstates \( | m \rangle \) and \( | n \rangle \) of the Hamiltonian. From Eq. (A.2) and Eq. (A.3), we have

\[ H |n\rangle = \left( n + \frac{1}{2} \right) \hbar \omega |n\rangle. \]  

(A.17)
For this, we now consider the quantity $\langle m|\hat{a}(t)|n \rangle$. This gives us

$$
\langle m|\hat{a}(t)|n \rangle = \exp \left( i \left( n + \frac{1}{2} \frac{\hbar \omega}{\hbar} \right) \right) \times \langle m|\hat{a}(t)|n \rangle \times \exp \left( -i \left( n + \frac{1}{2} \frac{\hbar \omega}{\hbar} \right) \right),
$$

which on further simplification yields

$$
\langle m|\hat{a}(t)|n \rangle = \langle n-1|\hat{a}(t)|n \rangle \exp(-i\omega t).
$$

(A.19)

where we have used the fact that the only non-zero matrix element of $\hat{a}(t)$ exists for $m = n - 1$. Since the energy eigenstates of the QHO form a complete basis, from Eq. (A.19), we can identify the unitary time evolution operator to be

$$
\hat{U}(t) = \exp(-i\omega t),
$$

(A.20)

which is the same as Eq. (9.b). We now operate with $|\beta\rangle$ on both sides of Eq. (A.16) to obtain

$$
\langle \hat{z}|\beta \rangle = \exp \left( -\frac{1}{2} \beta^2 \right) \times \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} \langle \hat{z}|n \rangle.
$$

(A.21)

Using the general form of the normalized wavefunction for a Quantum harmonic oscillator in its $n^{th}$ energy eigenstate, we explicitly write out the term $\langle \hat{z}|n \rangle$ present inside the infinite sum on the RHS of Eq. (A.21) to get

$$
\langle \hat{z}|\beta \rangle = \exp \left( -\frac{1}{2} \beta^2 \right) \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \times \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} H_n \left( \sqrt{\frac{m \omega}{\hbar}} z \right) \frac{1}{n!} ...
$$

$$
\times \exp \left( -\frac{m \omega z^2}{2\hbar} \right),
$$

(A.22)

where in Eq. (A.22), $H_n(z')$ denotes the $n^{th}$ order Hermite polynomial. We note that the infinite sum on the RHS of Eq. (A.22) is the generating function with parameter $\beta/\sqrt{2}$ (i.e., $G(z, \beta/\sqrt{2})$).

After much simplification, we arrive at the coordinate representation of a coherent state of the QHO, as outlined in Eq. (9.a). We note that $\xi(t)$ in Eq. (9.a) is simply a time dependent phase
factor. Since it appears in the exponential of a complex phase, it plays no role in the estimation of the visibility parameter. For the sake of completeness however, we state here that one can solve for $\xi(t)$ by taking note of the fact that the wavefunction in Eq. (9.a) satisfies the time dependent Schrödinger equation.

**Appendix B. Computation of the visibility parameter in the SG interferometer for a non-squeezed coherent state of the QHO**

For the sake of notational simplicity, we define the following parameters

$$a \equiv \sqrt{\frac{2\hbar}{m\omega}} \mathbb{R}[\beta(t)], \quad (B.1)$$

and

$$b \equiv \sqrt{\frac{2m\omega}{\hbar}} \mathbb{I}[\beta(t)]. \quad (B.2)$$

To compute the visibility parameter (over a total interferometric time $\tau$), we are required to consider the initially prepared spatial state (i.e., at time $t = 0$). From Eq. (6) and Eq. (12), we have (note that we state here a simplified version of Eq. (B.3))

$$\phi_{\text{non-squeezed}} = \left( \frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{m\omega}{2\hbar} \left( (z - \Delta \bar{z}(\tau) - a)^2 + (z + \Delta \bar{z}(\tau) - a)^2 \right) \right) \ldots \times \exp \left( -2i \frac{z \Delta p_z(\tau)}{\hbar} \right) dz, \quad (B.3)$$

where we note that the expectation value of $p_z$ (i.e., $b$, as defined in Eq. (B.2)) vanishes at time $t = 0$ and the time dependent phase term $\xi(t)$ cancels out. Simplifying Eq. (B.3) gives us

$$\phi_{\text{non-squeezed}} = \left( \frac{m\omega}{\pi\hbar} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{m\omega}{\hbar} \left( (z - a)^2 + (\Delta \bar{z}(\tau))^2 \right) \right) \times \exp \left( -2i \frac{z \Delta p_z(\tau)}{\hbar} \right) dz, \quad (B.4)$$

where we note that $\Delta \bar{z}(\tau)$ is independent of $z$ and thus can be factored out of the integral to give us

$$\phi_{\text{non-squeezed}} = \left( \frac{m\omega}{\pi\hbar} \right)^{1/2} \exp \left( -\frac{m\omega}{\hbar} (\Delta \bar{z}(\tau))^2 \right) \int_{-\infty}^{\infty} \exp \left( -\frac{m\omega}{\hbar} (z - a)^2 \right) \ldots$$
\[ \exp \left( -2i \frac{z \Delta p_z(\tau)}{\hbar} \right) dz. \]  

(B.5)

We note that the integral in Eq. (B.5) is the standard form of a Gaussian integral, simplifying which we obtain (note that \( a \) is independent of \( z \), which allows us to simplify Eq. (B.5) further)

\[ \phi_{\text{non-squeezed}} = \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \sqrt{\frac{\pi \hbar}{m \omega}} \exp \left( -\frac{m \omega}{h} (\Delta \bar{z}(\tau))^2 \right) \times \exp \left( \frac{\hbar}{m \omega} \left( \frac{m \omega}{h} - \frac{2i \Delta p_z(\tau)}{h} \right)^2 \right), \]  

(B.6)

which on further simplification gives us

\[ \phi_{\text{non-squeezed}} = \exp \left( -\frac{m \omega}{h} (\Delta \bar{z}(\tau))^2 - \frac{1}{m \hbar \omega} (\Delta p_z(\tau))^2 \right) \times \exp \left( -2i (a. \Delta p_z(\tau)) \right). \]  

(B.7)

Note that the phase term in Eq. (B.7) is constant for a given set of parameters, given which it plays no role in the estimation of the visibility parameter. To account for the temporal evolution of the spatial split between the wave-packets in the SG interferometer, we use Eq. (11) to simplify Eq. (B.7) further. This gives us

\[ \phi_{\text{non-squeezed}} = \exp \left( -\frac{m \omega}{h} \left( \Delta z(\tau) - \frac{\tau}{m} \Delta p_z(\tau) \right)^2 - \frac{1}{m \hbar \omega} (\Delta p_z(\tau))^2 \right), \]  

(B.8)

where, as before, we consider only the ‘amplitude’ part of \( \phi_{\text{non-squeezed}} \). On simplifying Eq. (B.8) further and expressing the visibility parameter in terms of the initial uncertainties in the measures of the position and momentum of the initially prepared spatial state, we obtain Eq. (14.b), which is our required result. Thus, we get

\[ \phi_{\text{non-squeezed}} = \exp \left( -\frac{1}{2} \left( \left( \frac{\Delta z(\tau)}{\delta z} \right)^2 + (1 + \omega^2 \tau^2) \left( \frac{\Delta p_z(\tau)}{\delta p_z} \right)^2 \right) + \frac{2 \Delta \bar{z}(\tau) \Delta p_z(\tau) \omega \tau}{h} \right). \]  

(B.9)

Using Eq. (B.9), we obtain constraints on the temperature required for the initially prepared harmonic trap, in terms of the admissible experimental errors in the measure of the phase-space variables, the total time-of-flight of the wave-packets inside the SG interferometer, the maximum spatial and momentum split size between the wave-packets and the desired accuracy in the measure of the visibility parameter, as has been demonstrated above.
Appendix C. Computation of the visibility parameter in the SG interferometer for a squeezed coherent state of the QHO

We adopt a similar approach (as we did in appendix B) to compute the visibility parameter in the SG interferometer for the squeezed case. We consider the initially prepared squeezed spatial state in our analysis (squeezed in either the position or the momentum quadrature). From Eq. (6) and Eq. (18) [with Eq. (18) evaluated at time \( t = 0 \)], we get

\[
\phi_{squeezed} = \frac{1}{s} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{m \omega}{2 \hbar s^2} \left( (z - \Delta \bar{z}(\tau) - a)^2 + (z + \Delta \bar{z}(\tau) - a)^2 \right) \right) \ldots 
\]

\[
\times \exp \left( -2i \frac{z \cdot \Delta p_z(\tau)}{\hbar} \right) dz,
\]

where we state a simplified version of Eq. (C.1), without going into the details of the intermediate steps involved, which are otherwise trivial. Here again, we note that the expectation value of \( p_z \) (i.e., \( b \), as defined in Eq. (B.2)) vanishes at time \( t = 0 \) and the time dependent phase term \( \xi(t) \) cancels out. Note that the parameters defined in Eq. (B.1) and Eq. (B.2) will be used here as well. Simplifying Eq. (C.1) gives us

\[
\phi_{squeezed} = \frac{1}{s} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{m \omega}{\hbar s^2} \left( (z - a)^2 + (\Delta \bar{z}(\tau))^2 \right) \right) \ldots 
\]

\[
\times \exp \left( -2i \frac{z \cdot \Delta p_z(\tau)}{\hbar} \right) dz,
\]

where again, we note that \( \Delta \bar{z}(\tau) \) is independent of \( z \) and thus can be factored out of the integral to give us

\[
\phi_{squeezed} = \frac{1}{s} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \exp \left( -\frac{m \omega}{\hbar s^2} (\Delta \bar{z}(\tau))^2 \right) \int_{-\infty}^{\infty} \exp \left( -\frac{m \omega}{\hbar s^2} (z - a)^2 \right) \ldots 
\]

\[
\times \exp \left( -2i \frac{z \cdot \Delta p_z(\tau)}{\hbar} \right) dz.
\]
We note that the integral in Eq. (C.3) is the standard form of a Gaussian integral, simplifying which we obtain (note that $a$ is independent of $z$, which allows us to simplify Eq. (C.3) further)

$$
\phi_{\text{squeezed}} = \frac{1}{s} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \sqrt{\frac{\pi \hbar s^2}{m \omega}} \exp\left(-\frac{m \omega}{\hbar s^2} (\Delta \bar{z}(\tau))^2 \right) \times \exp\left( \frac{\hbar s^2}{m \omega} \left( \frac{ma \omega}{\hbar s^2} - \frac{2i \Delta p_z(\tau)}{\hbar} \right)^2 \right),
$$

which on further simplification gives us

$$
\phi_{\text{squeezed}} = \exp\left(-\frac{m \omega}{\hbar s^2} (\Delta \bar{z}(\tau))^2 - \frac{s^2}{m \hbar \omega} (\Delta p_z(\tau))^2 \right) \times \exp\left( -2i \left( a \Delta p_z(\tau) \right) \right). \quad (C.5)
$$

Note that the phase term in Eq. (C.5) is constant for a given set of parameters, given which it plays no role in the estimation of the visibility parameter. To account for the temporal evolution of the spatial split between the wave-packets in the SG interferometer, we use Eq. (11) to simplify Eq. (C.5) further. This gives us

$$
\phi_{\text{squeezed}} = \exp\left( -\frac{m \omega}{\hbar s^2} \left( \Delta z(\tau) - \frac{\tau}{m} \Delta p_z(\tau) \right)^2 - \frac{s^2}{m \hbar \omega} (\Delta p_z(\tau))^2 \right), \quad (C.6)
$$

Finally, simplifying Eq. (C.6) and expressing the visibility parameter in terms of the measures of the initial position and momentum uncertainties of the spatial state, we arrive at Eq. (21), which for the sake of completeness, we state here

$$
\phi_{\text{squeezed}} = \exp\left( -\frac{1}{2} \left( \frac{\Delta z(\tau)}{\delta z'} \right)^2 + \left( 1 + \frac{\omega^2 \tau^2}{s^4} \right) \left( \frac{\Delta p_z(\tau)}{\delta p_z'} \right)^2 \right) + \frac{2 \Delta z(\tau) \Delta p_z(\tau) \omega \tau}{\hbar s^2}. \quad (C.7)
$$