ON THE BOUNDARY BEHAVIOR OF MAPPINGS WITH A FINITE INTEGRAL OVER SPHERES

EVGENY SEVOST’YANOV

February 17, 2022

Abstract

We study mappings that satisfy the inverse Poletsky-type inequality in a domain of Euclidean space. It was proved that such mappings have a continuous extension to the boundary under some conditions on the geometry of the given and the mapped domains if the majorant responsible for the distortion of the families of paths has finite integrals over the set of spheres of positive linear measure. We have also studied the problem of discrete extension of such mappings to the boundary of a given domain. In particular, we have proved that mappings extended to the boundary are light under certain conditions, and also discrete under more stringent constraints.

2010 Mathematics Subject Classification: Primary 30C65; Secondary 31A15, 31A20, 30L10

1 Introduction

In our joint publication [SSD], we have obtained some results on the local and boundary behavior of maps with inverse Poletsky inequality. In particular, we obtained theorems on the equicontinuity of families of corresponding mappings and the possibility of their continuous extension to the boundary of a given domain under the condition that some function \( Q \) is integrable in the defining inequality. In this article, we will show a little more, namely that these results are met not only for integrable \( Q \), but also for those that have finite integrals on spheres centered at a fixed point on a set of radii some "not very small" measure. Let us point to examples of non-integrable functions that have these finite integrals by spheres and mappings that correspond to them (see, for example, [SevSkv3, Examples 1,2]).
Let us turn to the definitions. In what follows, $M_p(\Gamma)$ denotes the $p$-modulus of a family $\Gamma$ (see [Val] Section 6). We write $M(\Gamma)$ instead $M_n(\Gamma)$. Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0,r_1,r_2) = \{ y \in \mathbb{R}^n : r_1 < |y-y_0| < r_2 \}. \quad (1.1)$$

Given $x_0 \in \mathbb{R}^n$, we put

$$B(x_0,r) = \{ x \in \mathbb{R}^n : |x-x_0| < r \}, \quad \mathbb{B}^n = B(0,1),$$

$$S(x_0,r) = \{ x \in \mathbb{R}^n : |x-x_0| = r \}.$$ 

Given sets $E, F \subset \mathbb{R}^n$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma : [a,b] \to \mathbb{R}^n$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in [a,b]$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in f(D) \setminus \{ \infty \}$, and $0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0,r_1,r_2)$ a family of all paths $\gamma$ in $D$ such that $f(\gamma) \in \Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))$. Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. We say that $f$ satisfies the inverse Poletsky inequality at a point $y_0 \in f(D) \setminus \{ \infty \}$ if the relation

$$M(\Gamma_f(y_0,r_1,r_2)) \leq \int_{\Lambda(y_0,r_1,r_2) \cap f(D)} Q(y) \cdot \eta^n(|y-y_0|) \, dm(y) \quad (1.2)$$

holds for any Lebesgue measurable function $\eta : (r_1,r_2) \to [0,\infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \quad (1.3)$$

Using the inversion $\psi(y) = \frac{y}{|y|^2}$, we may also define the relation (1.2) at the point $y_0 = \infty$. A mapping $f : D \to \mathbb{R}^n$ is called discrete if the pre-image $\{ f^{-1}(y) \}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and open if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. A mapping $f$ of $D$ onto $D'$ is called closed if $f(E)$ is closed in $D'$ for any closed set $E \subset D$ (see, e.g., [Vu Chapter 3]). Let $h$ be a chordal metric in $\mathbb{R}^n$,

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|}} \quad h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2} \sqrt{1+|y|^2}} \quad x \neq \infty \neq y. \quad (1.4)$$

and let $h(E) := \sup_{x,y \in E} h(x,y)$ be a chordal diameter of a set $E \subset \mathbb{R}^n$ (see, e.g., [Val Definition 12.1]). Everywhere further the boundary $\partial A$ of the set $A$ and the closure $\overline{A}$ should be understood in the sense extended Euclidean space $\overline{\mathbb{R}}^n$. A continuous extension of the mapping $f : D \to \mathbb{R}^n$ also should be understood in terms of mapping with values in $\overline{\mathbb{R}}^n$ and relative to the metric $h$ in (1.4) (if a misunderstanding is impossible). Recall that a domain $D \subset \mathbb{R}^n$ is called locally connected at the point $x_0 \in \partial D$, if for any neighborhood $U$ of a
point \( x_0 \) there is a neighborhood \( V \subset U \) of \( x_0 \) such that \( V \cap D \) is connected. A domain \( D \) is locally connected at \( \partial D \), if \( D \) is locally connected at any point \( x_0 \in \partial D \). The boundary of the domain \( D \) is called weakly flat at the point \( x_0 \in \partial D \), if for any \( P > 0 \) and for any neighborhood \( U \) of a point \( x_0 \) there is a neighborhood \( V \subset U \) of the same point such that \( M(\Gamma(E,F,D)) > P \) for any continua \( E,F \subset D \), which intersect \( \partial U \) and \( \partial V \). The boundary of the domain \( D \) is called weakly flat if the corresponding property is fulfilled at any point of the boundary \( D \). The following result is correct.

**Theorem 1.1.** Let \( D \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain with a weakly flat boundary, and let \( D' \subset \mathbb{R}^n \) is locally connected at its boundary. Suppose that \( f \) is open discrete and closed mapping of \( D \) onto \( D' \) satisfying the relation \((1.2)\) at any point \( y_0 \in D' \). Suppose that, for each point \( y_0 \in D' \) and \( 0 < r_1 < r_2 < r_0 := \sup \{ y - y_0 \mid y \in D' \} \) there is a set \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}' \), while \( \overline{f}(D) = \overline{D}' \).

The following result contains a statement about the possibility of a continuous extension of the mapping to an isolated point of the boundary of a domain. Note that the formulation of this result does not contain any conditions on the mapped domain.

**Theorem 1.2.** Let \( D \) and \( D' \) be domains in \( \mathbb{R}^n \), \( n \geq 2 \), \( x_0 \in D \), and let \( f \) be an open discrete mapping of \( D \setminus \{ x_0 \} \) onto \( D' \) satisfying the relation \((1.2)\) at least at one finite point \( y_0 \in C(f,x_0) \). Let, in addition, \( C(f,x_0) \subset \partial D' \). Assume that, for any point \( y_0 \in D' \) and \( 0 < r_1 < r_2 < r_0 := \sup \{ y - y_0 \mid y \in D' \} \) there is a set of a positive linear Lebesgue measure \( E \subset [r_1, r_2] \) such that a function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then \( f \) has a continuous extension \( f : D \to D' \).

## 2 Proof of main results

**Proof of Theorem 1.1.** Put \( x_0 \in \partial D \). It is necessary to show the possibility of continuous extension of the mapping \( f \) to the point \( x_0 \). Using the Möbius transform \( \varphi : \infty \mapsto 0 \) and the invariance of the modulus \( M \) under a conformal mapping on the left side of the relation \((1.2)\) (see [V, Theorem 8.1]), we may assume that \( x_0 \neq \infty \).

Assume that the conclusion about the continuous extension of the mapping \( f \) to the point \( x_0 \) is not correct. Then there are sequences \( x_i, y_i \in D \), \( i = 1, 2, \ldots \), such that \( x_i, y_i \to x_0 \) as \( i \to \infty \), and

\[
h(f(x_i), f(y_i)) \geq a > 0 \tag{2.1}
\]

for some \( a > 0 \) and any \( i \in \mathbb{N} \), where \( h \) is a chordal metric. Since \( \mathbb{R}^n \) is a compact space, we may assume that the sequences \( f(x_i) \) and \( f(y_i) \) converge to \( z_1 \) and \( z_2 \) as \( i \to \infty \), respectively. We may assume also that \( z_1 \neq \infty \). Since \( f \) is closed, it is boundary preserving (see [Va, Theorem 3.3]). Thus, \( z_1, z_2 \in \partial D' \). Since \( D' \) is locally connected on the boundary, there are
neighborhoods $U_1$ and $U_2$ of points $z_1$ and $z_2$ such that $W_1 = D' \cap U_1$ and $W_2 = D' \cap U_2$ are connected. We may assume that $W_1$ and $W_2$ are locally connected, because $U_1$ and $U_2$ may be chosen open (see, e.g., [MRSY, Proposition 13.2]; see Figure 1). We may assume that

$$f(D) = D'$$

where $z_2 \in D'$ is some point sufficiently closed to $z_1$. Also we may assume that $f(x_i) \in W_1$ and $f(y_i) \in W_2$ for any $i = 1, 2, \ldots$. Join the points $f(x_i)$ and $f(x_1)$ by a path $\alpha_i : [0, 1] \to D'$, and points $f(y_i)$ and $f(y_1)$ by a path $\beta_i : [0, 1] \to D'$ such that $|\alpha_i| \subset W_1$ and $|\beta_i| \subset W_2$ as $i = 1, 2, \ldots$. Let $\tilde{\alpha}_i : [0, 1] \to D'$ and $\tilde{\beta}_i : [0, 1] \to D'$ be total liftings of paths $\alpha_i$ and $\beta_i$ starting at points $x_i$ and $y_i$, respectively (these liftings exist due to [Vu, Lemma 3.7]). Note that the points $f(x_1)$ and $f(y_1)$ may have no more than a finite number of pre-images under the mapping $f$ in the domain $D$, see [Vu, Lemma 3.2]. Then there exists $r_0 > 0$ such that $\tilde{\alpha}_i(1), \tilde{\beta}_i(1) \in D \setminus B(x_0, r_0)$ for any $i = 1, 2, \ldots$. Since the boundary of $D$ is weakly flat, for any $P > 0$ there is $i = i_P \geq 1$ such that

$$M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) > P \quad \forall i \geq i_P.$$  \hfill (2.3)

Let us to show that, the condition (2.3) contradicts the definition of $f$ in (1.2). Indeed, due to the relation (2.2) and by [Ku, Theorem 1.I.5.46] we obtain that

$$f(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) > \Gamma(S(z_s, R_0), S(z_s, 2R_0), A(z_s, R_0, 2R_0)).$$  \hfill (2.4)

It follows from (2.4) that

$$\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D) > \Gamma_f(z_s, R_0, 2R_0).$$  \hfill (2.5)

In addition, by (2.5) we obtain that

$$M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq M(\Gamma_f(z_s, R_0, 2R_0)) \leq \int_A Q(y) \cdot \eta^n(|y - z_s|) \, dm(y),$$  \hfill (2.6)
where \( A = A(z_*, R_0, 2R_0) \) and \( \eta \) is any Lebesgue measurable function satisfying the relation \( (1.3) \) for \( r_1 := R_0 \) and \( r_2 := 2R_0 \). Below we use the following conventions: \( a/\infty = 0 \) for \( a \neq \infty \), \( a/0 = \infty \) for \( a > 0 \) and \( 0 \cdot \infty = 0 \) (see, e.g., [Sal 3.1]). Put
\[
I = \int_{R_0}^{2R_0} \frac{dt}{tq_{z_*}^{1/(n-1)}(t)},
\]
where
\[
q_{y_0}(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int_{S(y_0,r)} Q(y) d\mathcal{H}^{n-1}(y),
\]
and \( \omega_{n-1} \) denotes the area of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \), and \( q_{z_*}(t) \) is defined in \( (2.8) \) for \( y_0 := z_* \). By the assumption, there is a set \( E \subset [R_0, 2R_0] \) of a positive measure such that \( q_{z_*}(t) \) is finite for all \( t \in E \). Thus \( I \neq 0 \) in \( (2.7) \). In this case, a function \( \eta_0(t) = \frac{1}{Iq_{z_*}^{1/(n-1)}(t)} \) satisfies the relation \( (1.3) \) for \( r_1 := R_1 \) and \( r_2 := 2R_0 \). Subsisting this function in the right-hand part of \( (2.6) \) and using the Fubini theorem, we obtain that
\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq \frac{\omega_{n-1}}{I_{n-1}} < \infty.
\]
The relation \( (2.9) \) contradicts with \( (2.3) \). The contradiction obtained above disproves the assumption made in \( (2.1) \).

The proof of the equality \( \overline{\mathcal{F}(\mathcal{D})} = \mathcal{D}' \) is similar to the second part of the proof of Theorem 3.1 in [SSD].

**Remark 2.1.** A slightly different formulation of Theorem \( 1.1 \) is also true.

Let \( D \subset \mathbb{R}^n, n \geq 2, \) be a domain which has a weakly flat boundary at a point \( x_0 \in \partial D \), and let \( D' \subset \mathbb{R}^n \) be a locally connected at any point \( z \in C(f, x_0) \). Assume that, \( f \) is an open discrete and closed mapping of \( D \) onto \( D' \) satisfying the relation \( (1.2) \) at any point \( y_0 \). Assume also that there exists at least one finite point \( z_1 \in C(f, x_0) \) for which there is \( 0 < r_1 = r_1(z_1) \) such that a function \( Q \) is integrable on \( S(z_1, r) \) for any \( r \in (0, r_1) \). Then \( f \) has a continuous extension \( \overline{f} : D \cup \{ x_0 \} \to \mathcal{D}' \).

The proof of this statement with minor modifications repeats the proof of the theorem \( 1.1 \).

We present this proof below.

Assume that the conclusion about the continuous extension of the mapping \( f \) to the point \( x_0 \) is not correct. Then there are at least two sequences \( x_i, y_i \in D, i = 1, 2, \ldots \), such that \( x_i, y_i \to x_0 \) as \( i \to \infty \), while the relation \( (2.1) \) holds for some \( a > 0 \) and all \( i \in \mathbb{N} \). Since \( z_1 \in C(f, x_0) \), we may choose the sequences \( f(x_i) \) and \( f(y_i) \) converge to \( z_1 \) and \( z_2 \) as \( i \to \infty \) respectively, where \( z_1 \in \partial D' \subset \mathbb{R}^n \). Since \( D' \) is locally connected at its boundary, there are disjoint neighborhoods of \( U_1 \) and \( U_2 \) of points \( z_1 \) and \( z_2 \) such that \( W_1 = D' \cap U_1 \) and \( W_2 = D' \cap U_2 \) are connected. We may assume that \( W_1 \) and \( W_2 \) are path connected. We may assume that
\[
U_1 \subset B(z_1, R_*) , \quad B(z_1, 2R_0) \cap U_2 = \varnothing, \quad 0 < R_* < 2R_0 < r_1.
\]
Join the points \( f(x_i) \) and \( f(x_1) \) by a path \( \alpha_i : [0, 1] \to \mathcal{D}' \), and points \( f(y_i) \) and \( f(y_1) \) by a path \( \beta_i : [0, 1] \to \mathcal{D}' \) such that \( |\alpha_i| \subset W_1 \) and \( |\beta_i| \subset W_2 \) for \( i = 1, 2, \ldots \). Let \( \tilde{\alpha}_i : [0, 1] \to \mathcal{D}' \) and \( \tilde{\beta}_i : [0, 1] \to \mathcal{D}' \) be whole liftings of \( \alpha_i \) and \( \beta_i \) starting at points \( x_i \) and \( y_i \), respectively (these liftings exist due to \([\text{V} u, \text{Lemma 3.7}]\)). Observe that, the points \( f(x_1) \) and \( f(y_1) \) have at least a finite number of pre-images in \( \mathcal{D} \) under \( f \), see \([\text{V} u, \text{Lemma 3.2}]\). Then there is \( r_0 > 0 \) such that \( \tilde{\alpha}_i(1), \tilde{\beta}_i(1) \in \mathcal{D} \setminus B(x_0, r_0) \) for any \( i = 1, 2, \ldots \). Since the boundary of \( \mathcal{D} \) is weakly flat, for any \( P > 0 \) there is \( i = i_P \geq 1 \) such that the relation (2.3) holds. On the other hand, due to \([Ku, \text{Theorem 1.I.5.46}]\)

\[
\int_I \frac{dt}{tq_{z_1}^{1/(n-1)}(t)} < \infty, \quad \int_0^{\delta(z_1)} \frac{dt}{tq_{z_1}^{1/(n-1)}(t)} = \infty \tag{2.15}
\]

for \( \varepsilon > 0 \) sufficiently small.

Indeed, literally repeating the proof of the statement given in Remark 2.1 we choose \( R_\ast \) so small that \( I \) in (2.14) is strictly positive (this is possible due to the conditions in (2.16)). The rest of the reasoning will not change. □

**Remark 2.2.** The statement given in Remark 2.1 remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that there is \( \delta(z_1) > 0 \) such that

\[
\int_{\varepsilon}^{\delta(z_1)} \frac{dt}{tq_{z_1}^{1/(n-1)}(t)} < \infty, \quad \int_0^{\delta(z_1)} \frac{dt}{tq_{z_1}^{1/(n-1)}(t)} = \infty \tag{2.16}
\]
Remark 2.3. The statement given in Remark 2.1 remains true, if in its formulation instead of the specified conditions on function \( Q \) to require that there is \( \varepsilon_0 = \varepsilon_0(z_1) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \to [0, \infty] \) such that

\[
I(\varepsilon, \varepsilon_0) := \int_\varepsilon^{\varepsilon_0} \psi(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0), \quad I(\varepsilon, \varepsilon_0) > 0 \quad \text{as} \quad \varepsilon \to 0, \quad (2.17)
\]

and, in addition,

\[
\int_{A(z_1, \varepsilon, \varepsilon_0)} Q(x) \cdot \psi^n(|x - x_0|) \, dm(x) \leq C_0 I^n(\varepsilon, \varepsilon_0), \quad (2.18)
\]
as \( \varepsilon \to 0 \), where \( C_0 \) is some constant, and \( A(x_0, \varepsilon, \varepsilon_0) \) is defined in (1.1).

Indeed, literally repeating the proof of the statement given in Remark 2.1 to the ratio (2.13) inclusive, we put \[ \eta(t) = \begin{cases} \psi(t)/I(R_*, 2R_0), & t \in (R_*, 2R_0), \\ 0, & t \notin (R_*, 2R_0), \end{cases} \] where \( I(1/l, \varepsilon_0) = \int_{1/l}^{\varepsilon_0} \psi(t) \, dt \). Observe that \( \int_{1/l}^{\varepsilon_0} \eta(t) \, dt = 1 \). Now, by the definition of \( f \) in (1.2) and due to the relation (2.13) we obtain that

\[
M(\Gamma(|\tilde{\alpha}_i|, |\tilde{\beta}_i|, D)) \leq C_0 < \infty. \quad (2.19)
\]
The relation (2.19) contradicts with (2.3). The resulting contradiction indicates the falsity of the assumption made in (2.1). □

Proof of Theorem 1.2. Without loss of generality, we may assume that \( x_0 \neq \infty \). Due to the discreteness of \( f \) there is \( 0 < \varepsilon_0 < \text{dist}(x_0, \partial D) \) such that \( \infty \notin f(S(x_0, \varepsilon)) \) (if \( \partial D = \emptyset \), we choose any \( \varepsilon_0 > 0 \) with the condition mentioned above). Set

\[ g := f|_{B(x_0, \varepsilon_0) \setminus \{x_0\}}. \]
Suppose the opposite, namely, that the mapping \( f \) does not have a continuous boundary extension to the point \( x_0 \). Then also the mapping \( g \) does not have a continuous boundary extension to the same point. Since the space \( \mathbb{R}^n \) is compact, \( C(f, x_0) = C(g, x_0) \neq \emptyset \). Then there are \( y_1, y_2 \in C(f, x_0) \), \( y_1 \neq y_2 \), and at least two sequences \( x_m, x'_m \in B(x_0, \varepsilon_0) \setminus \{x_0\} \) such that \( x_m, x'_m \to x_0 \) as \( m \to \infty \), while \( z_m := g(x_m) \to y_1 \), \( z'_m = g(x'_m) \to y_2 \) as \( m \to \infty \). We may assume that \( y_1 \neq \infty \).

Let

\[ D_* := f(B(x_0, \varepsilon_0) \setminus \{x_0\}). \]
Let us to show that, there is \( \varepsilon_1 > 0 \) such that

\[ B(y_1, \varepsilon_1) \cap f(S(x_0, \varepsilon_0)) = \emptyset. \quad (2.20) \]
Observe that, \( y_1 \in \partial D_* \). Indeed, if \( y_1 \) is an inner point of \( D_* \), then \( y_1 \) is inner for \( D' \), because \( D_* \subset D' \). The latter contradicts with \( C(f, x_0) \subset \partial D' \). Since \( S(x_0, \varepsilon_0) \) is compact in \( D \), the set \( f(S(x_0, \varepsilon_0)) \) is a compactum in \( D' \). Thus

\[
 h(f(S(x_0, \varepsilon_0)), y_1) > \delta > 0.
\]

Therefore,

\[
 \text{dist}(y_1, f(S(x_0, \varepsilon_0))) > \delta_1 > 0, \quad (2.21)
\]

where \( \text{dist}(A, B) \) denotes the Euclidean distance between the sets \( A \) and \( B \) in \( \mathbb{R}^n \). Due to the relation (2.21), the inequality (2.20) holds for \( \varepsilon := \delta_1 \).

Now we will reason as follows. Let \( B_*(y_2, \varepsilon_2) = B(y_2, \varepsilon_2) \) for \( y_2 \neq \infty \) and \( B_*(y_2, \varepsilon_2) = \{ x \in \mathbb{R}^n : h(x, \infty) < \varepsilon_2 \} \) for \( y_2 = \infty \). Arguing similarly to the proof of the relation (2.20), we may show that there is \( \varepsilon_2 > 0 \) such that

\[
 B_*(y_2, \varepsilon_2) \cap f(S(x_0, \varepsilon_0)) = \emptyset. \quad (2.22)
\]

Without loss of generality, we may assume that \( B(y_1, \varepsilon_1) \cap \overline{B_*(y_2, \varepsilon_2)} = \emptyset \), in addition, \( z_m \in B(y_1, \varepsilon_1) \) and \( z'_m \in B_*(y_2, \varepsilon_2) \) for any \( m = 1, 2, \ldots \) (see Figure 2). Observe that, the set \( B(y_1, \varepsilon_1) \) is convex, and \( B_*(y_2, \varepsilon_2) \) is path connected. In this case, the points \( z_1 \) and \( y_1 \) may be joined by the segment \( I(t) = z_1 + t(y_1 - z_1), t \in (0, 1) \), which lies entirely in \( B(y_1, \varepsilon_1) \). Similarly, the points \( z'_1 \) and \( y_2 \) may be joined by a path \( J = J(t), t \in [0, 1] \), which belongs to the "ball" \( B_*(y_2, \varepsilon_2) \).

Observe that, by the construction \( |I| \cap \partial D_* \neq \emptyset \neq |J| \cap \partial D_* \). Denote

\[
 t_* := \sup_{t \in [0, 1]: I(t) \in D_*} t, \quad p_* := \sup_{t \in [0, 1]: J(t) \in D_*} t.
\]
Let
\[ C_1 := I_{[0,t_*)}, \quad C_2 := J_{[0,t_*)}. \]

By \[\text{MRV}_2\], Lemma 3.12 the paths \(C_1\) and \(C_2\) have maximal \(g\)-liftings \(C_1^* : [0,c_1) \to B(x_0,\varepsilon_0) \setminus \{x_0\}\) and \(C_2^* : [0,c_2) \to B(x_0,\varepsilon_0) \setminus \{x_0\}\) starting at the points \(x_1\) and \(x_1'\), respectively. Observe that, the case when \(C_1(t) \to z_0\) as \(t \to c_1 - 0\), where \(z_0 \in B(x_0,\varepsilon_0) \setminus \{x_0\}\), is impossible. Indeed, in this case, by \[\text{MRV}_2\], Lemma 3.12 we obtain that \(c_1 = t_*\) and \(I(t) \to f(z_0) \in D_0\), that contradicts with the definition of \(t_*\). Then, by \[\text{MRV}_2\], Lemma 3.12
\[
h(C_1^*(t), \partial(B(x_0,\varepsilon_0) \setminus \{x_0\})) \to 0, \quad t \to c_1 - 0. \tag{2.23}\]

Let us show that, the case \(h(C_1^*(t), S(x_0,\varepsilon_0)) \to 0\) as \(t \to c_1 - 0\) is impossible. Indeed, in the contrary case there is a sequence \(t_k \to c-0\) such that \(h(C_1^*(t_k), S(x_0,\varepsilon_0)) \to 0\) as \(k \to \infty\). Due to the compactness of the sphere \(S(x_0,\varepsilon_0)\), we may find a sequence \(w_k \in S(x_0,\varepsilon_0)\) such that \(h(C_1^*(t_k), S(x_0,\varepsilon_0)) = h(C_1^*(t_k), w_k)\). Since \(S(x_0,\varepsilon_0)\) is compact, we may assume that \(w_k \to w_0\) as \(k \to \infty\). Then \(C_1^*(t_k) \to w_0\) as \(k \to \infty\). Then, by the continuity of the mapping \(f\) in \(D\) we obtain that
\[
f(C_1^*(t_k)) = C_1(t_k) \to f(w_0) \in f(S(x_0,\varepsilon_0)) \tag{2.24}\]
as \(k \to \infty\). The latter contradicts with \(2.20\), because \(f(w_0) \in f(S(x_0,\varepsilon_0))\) and, simultaneously, \(f(w_0) \in |I| \subset B(y_1,\varepsilon_1)\). Now, it follows from \(2.23\) that
\[
h(C_1^*(t), x_0) \to 0, \quad t \to c_1 - 0. \tag{2.25}\]

Applying similar considerations to the path \(C_2^*(t)\), we may show that
\[
h(C_2^*(t), x_0) \to 0, \quad t \to c_2 - 0. \tag{2.26}\]

By \(2.25\) and \(2.26\), and due to \[\text{VA}\], Theorem 10.12 we obtain that
\[
M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0,\varepsilon_0) \setminus \{x_0\})) = \infty. \tag{2.27}\]

Let us show that \(2.27\) contradicts with the relation \(1.2\) at the point \(y_0 = y_1\). Since \(\overline{B(y_1,\varepsilon_1)} \cap B_*(y_2,\varepsilon_2) = \emptyset\), there is \(\varepsilon_1^* > \varepsilon_1\) such that \(\overline{B(y_1,\varepsilon_1^*)} \cap B_*(y_2,\varepsilon_2) = \emptyset\). Let \(\Gamma_* = \Gamma(|C_1|, |C_2|, D_0)\). Observe that
\[
\Gamma_* \supset \Gamma(S(y_1,\varepsilon_1^*), S(y_1,\varepsilon_1), A(y_1,\varepsilon_1,\varepsilon_1^*). \tag{2.28}\]

Indeed, let \(\gamma \in \Gamma_*\), \(\gamma : [a,b] \to \mathbb{R}^n\). Since \(\gamma(a) \in |C_1| \subset B(y_1,\varepsilon_1)\) and \(\gamma(b) \in |C_2| \subset \mathbb{R}^n \setminus B(y_1,\varepsilon_1)\), by \[\text{Ku}\], Theorem 1.1.5.46 there is \(t_1 \in (a,b)\) such that \(\gamma(t_1) \in S(y_1,\varepsilon_1)\). Without loss of generality, we may assume that \(|\gamma(t) - y_1| > \varepsilon_1\) for \(t > t_1\). Since \(\gamma(t_1) \in B(y_1,\varepsilon_1^*)\) and \(\gamma(b) \in |C_2| \subset \mathbb{R}^n \setminus B(y_1,\varepsilon_1^*)\), by \[\text{Ku}\], Theorem 1.1.5.46 there is \(t_2 \in (t_1,b)\) such that \(\gamma(t_2) \in S(y_1,\varepsilon_1^*)\). We also may assume that \(|\gamma(t) - y_1| < \varepsilon_1^*\) for \(t_1 < t < t_2\). Thus, \(\gamma|_{[t_1,t_2]}\) is a subpath of \(\gamma\) which belongs to \(\Gamma(S(y_1,\varepsilon_1^*), S(y_1,\varepsilon_1), A(y_1,\varepsilon_1,\varepsilon_1^*))\). Therefore, the relation \(2.28\) is proved.
Let us to show now that
\[ \Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\}) > \Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*). \tag{2.29} \]

Indeed, if \( \gamma : [a, b] \rightarrow B(x_0, \varepsilon_0) \setminus \{x_0\} \) belongs \( \Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\}) \), then \( f(\gamma) \) belongs to \( D_\ast \), while \( f(\gamma(a)) \in |C_1| \) and \( f(\gamma(b)) \in |C_2| \), i.e. \( f(\gamma) \in \Gamma_\ast \). Then, by the proving above, and due to the relation (2.28) a path \( f(\gamma) \) has a subpath \( f(\gamma)^* := f(\gamma)|_{[t_1, t_2]} \), \( a \leq t_1 < t_2 \leq b \), in \( \Gamma(S(y_1, \varepsilon_1^*), S(y_1, \varepsilon_1), A(y_1, \varepsilon_1, \varepsilon_1^*)) \). Then \( \gamma^* := \gamma|_{[t_1, t_2]} \) is a subpath of \( \gamma \) which belongs to \( \Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*) \), as required.

By (2.29), we obtain that
\[ M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) \leq M(\Gamma_f(y_1, \varepsilon_1, \varepsilon_1^*)) \leq \int_A Q(y) \cdot \eta^n(|y - y_1|) \, dm(y), \tag{2.30} \]

where \( A = A(y_1, \varepsilon_1, \varepsilon_1^*) \) and \( \eta \) is arbitrary Lebesgue measurable function satisfying the relation (1.13) for \( r_1 := \varepsilon_1 \) and \( r_2 := \varepsilon_1^* \). As above, we use the standard conventions \( a/\infty = 0 \) as \( a \neq \infty \), \( a/0 = \infty \) as \( a > 0 \) and \( 0 \cdot \infty = 0 \) (see, e.g., [Sa, 3.1]). Put
\[ I = \int_{\varepsilon_1}^{\varepsilon_1^*} \frac{dt}{t q_{y_1}^{1/(n-1)}(t)}, \tag{2.31} \]

where \( \omega_{n-1} \) denotes the area of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \), and \( q_{y_1}(t) \) is defined in (2.8). By the assumption, there is a set \( E \subset [\varepsilon_1, \varepsilon_1^*] \) of the positive Lebesgue measure such that \( q_{y_1}(t) \) is finite for any \( t \in E \). Thus, \( I \neq 0 \) in (2.31). In this case, a function \( \eta_0(t) = \frac{1}{t q_{y_1}^{1/(n-1)}(t)} \) satisfies the relation (1.13) for \( r_1 := \varepsilon_1 \) and \( r_2 := \varepsilon_1^* \). Substituting this function in the right-hand part of (2.30) and using the Fubini theorem, we obtain that
\[ M(\Gamma(|C_1^*(t)|, |C_2^*(t)|, B(x_0, \varepsilon_0) \setminus \{x_0\})) \leq \frac{\omega_{n-1}}{I^{n-1}} < \infty. \tag{2.32} \]

The relation (2.32) contradicts with (2.27). The obtained contradiction completes the proof of the theorem. \( \Box \)

### 3 Examples

**Example 1.** First of all, let us use the construction given in Example 1 in [SevSkv3].

Consider the function \( \varphi : [0, 1] \rightarrow \mathbb{R} \), defined by equality
\[
\varphi(t) = \begin{cases} 
1, & t \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right], \ k = 1, 2, \ldots, \\
\frac{1}{n}, & t \in \left(\frac{1}{2k}, \frac{1}{2k-1}\right], \ k = 1, 2, \ldots, 
\end{cases}
\]

\[
Q(x) = \varphi(|x|), \quad Q : \mathbb{R}^n \setminus \{0\} \rightarrow [0, \infty). \tag{3.1}
\]
By the Fubini theorem and by the countable additivity of the Lebesgue integral, we obtain that
\[ \int_{\mathbb{B}^n} Q(x) \, dm(x) = \int_0^1 \int_{S(0,r)} Q(x) \, d\mathcal{H}^{n-1} dr = \]
\[ \omega_n^{-1} \int_0^1 r^{n-1} \varphi(r) \, dr \geq \omega_n^{-1} \sum_{k=1}^{\infty} \int_0^{1/(2k)} \frac{dr}{r} = \omega_n^{-1} \sum_{k=1}^{\infty} \ln \left( \frac{2k}{2k-1} \right). \]  
(3.2)

Note that the series on the right side of the relation (3.2) diverges. Indeed, according to Lagrange’s theorem on the mean we obtain that
\[ \ln \left( \frac{2k}{2k-1} \right) = \ln(2k) - \ln(2k-1) = \frac{1}{\theta(k)} \geq \frac{1}{2k}, \]
where \( \theta(k) \in [2k-1, 2k]. \) Since \( \sum_{k=1}^{\infty} \frac{1}{2k} = \infty, \) we obtain that \( \sum_{k=1}^{\infty} \ln \left( \frac{2k}{2k-1} \right) = \infty \) and thus
\[ \int_{\mathbb{B}^n} Q(x) \, dm(x) = \infty. \]

On the other hand,
\[ \int_0^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \geq \sum_{k=1}^{\infty} \int_{1/(2k+1)}^{1/(2k)} \frac{dr}{r} = \sum_{k=1}^{\infty} \ln \frac{2k + 1}{2k} = \infty. \]  
(3.3)

Set
\[ g(x) = \frac{x}{|x|} \rho(|x|), \quad g(0) := 0, \]
where
\[ \rho(r) = \exp \left\{- \int_r^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \right\}. \]  
(3.4)

Observe that, \( g \) is a homeomorphism of the unit ball \( \mathbb{B}^n \) onto itself. Let us establish that \( g \) satisfies the relation
\[ M(g(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leq \int_D Q(x) \cdot \eta^n(|x-x_0|) \, dm(x) \]  
(3.5)

for each nonnegative Lebesgue measurable function \( \eta, \) which satisfies the relation (1.3). Indeed, \( g \) belongs to the class \( ACL, \) and its Jacobian and the operator norm of the derivative are calculated by the formulae
\[ \|g'(x)\| = \frac{\exp \left\{- \int_{|x|}^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \right\}}{|x|}, \quad |J(x, g)| = \frac{\exp \left\{-n \int_{|x|}^1 \frac{dt}{t q_0^{1/(n-1)}(t)} \right\}}{|x|^n q_0^{1/(n-1)}(|x|)}, \]
see e.g. [IS, Proof of Theorem 5.2]. Thus, \( g \in W^{1,n}_{\text{loc}}(\mathbb{B}^n \setminus 0). \) Moreover, the so-called inner dilatation \( K_I(x, g) \) of the mapping \( g \) at \( x \) is calculated as follows: \( K_I(x, g) = q_0(|x|). \) In this
case, \( g \) satisfies the relation \([3.5]\) for \( Q = K_1(x, g) = q_0(|x|) \) (see, e.g., \[MRSY\] Corollary 8.5 and Theorem 8.6).

Therefore, the mapping \( f = g^{-1} \) satisfies the relation \([1.2]\) in \( \mathbb{B}^n \). Observe that, the function \( Q \), is extended by zero outside the unit ball, is integrable over almost all spheres with center at any point \( x_0 \in \mathbb{B}^n \), because this function is locally bounded in \( \mathbb{B}^n \setminus \{0\} \).

Observe also that, the corresponding integrable functions \( Q \) in \( \mathbb{B}^n \) do not exist. Indeed, otherwise we would have that \( K_1(x, g) \leq c_n Q(x) \) (see, e.g., \[SalSev\] Theorem 3.1]), but then the inner dilatation \( K_1(x, g) \) would also be integrable in \( \mathbb{B}^n \), which, due to the above, is not true.

Since \( g \) is a homeomorphism, \( g \) is open, discrete and closed. Obviously, the unit ball \( \mathbb{B}^n \) is locally connected on the boundary. In addition, \( \mathbb{B}^n \) has a weakly flat boundary (see, e.g., \[Va\] Theorem 17.12]). Thus, all of the conditions of Theorem \([1.1]\) are fulfilled.

**Example 2.** We may also specify an example of a mapping with the corresponding function \( Q \) in \([1.2]\), which has a singularity at the boundary of the unit sphere. For simplification, we limit ourselves to the case \( n = 2 \). We arbitrarily choose the point \( z_0 \in \partial \mathbb{B}^2 \) and \( z \in \mathbb{B}^2 \cap B(z_0, 1) \), and set

\[
g_1(z) = \frac{z - z_0}{|z - z_0|} \rho(|z - z_0|), \quad g_1(z_0) := 0,
\]

where the function \( \rho \) is still defined in \([3.4]\). For \( z \in \mathbb{B}^2 \setminus B(z_0, 1) \) we set \( f(z) = z \). Note that \( g_1 \) satisfies the relation \([3.5]\), where \( Q_1(z) = q_0(|z - z_0|) \) and \( q_0(z) \) is defined in \([3.1]\).

For the same reasons, \( Q_1(z) \) has finite integrals for almost all spheres with centers in \( \mathbb{B}^2 \), where, as usual, the function \( Q_1 \) vanishes outside \( \mathbb{B}^2 \). We show that the function \( Q_1 \) has infinite integrals over sufficient small balls \( B(z_0, \varepsilon_0) \). For this purpose, we introduce the polar coordinates \( z = (r, \varphi) \) centered at the point \( z_0 \), where \( r \) denotes the Euclidean distance from \( z_0 \) to \( z \), and \( \varphi \) is the angle between the radius vector \( z_0 - z \) and tangent to the disk \( \mathbb{B}^2 \), passing through the point \( z_0 \). Let

\[
\theta_1 = \inf_{z \in \mathbb{B}^2 \setminus S(z_0, \varepsilon)} \varphi, \quad \theta_2 = \sup_{z \in \mathbb{B}^2 \setminus S(z_0, \varepsilon)} \varphi.
\]

Using elementary methods of geometry, we will have that \( \sin \theta_1 = \varepsilon/2 \), \( \sin(\pi - \theta_2) = \varepsilon/2 \). Then, for \( \varepsilon \to 0 \), we obtain that \( \theta_1 \to 0 \), \( \theta_2 \to \pi \). From here it follows that the interval of change of angles \( \varphi \) is close to \( \pi \) for \( z \in \mathbb{B}^2 \cap B(z_0, \varepsilon_0) \). In particular, for some (rather small) \( \varepsilon_0 > 0 \) we have that \( \theta_2 - \theta_1 \geq 5\pi/6 \). Then, by the relation \([3.2]\), we obtain that

\[
\int_{\mathbb{B}^2 \cap B(z_0, \varepsilon_0)} Q_1(z) \, dm(z) = \int_0^{\varepsilon_0} \int_{S(z_0, r) \cap B(z_0, \varepsilon_0)} Q_1(z) \, |dz|dr = \\
\int_0^{\varepsilon_0} \int_{\theta_1}^{\theta_2} r \varphi(r) \, dr \geq \frac{5\pi}{6} \cdot \int_0^{\varepsilon_0} r \varphi(r) \, dr = \infty.
\]
Since $g_1(B^2)$ is simply connected, according to Riemann’s theorem on conformal mapping we may find a mapping $\varphi$ such that $(\varphi \circ g_1)(B^2) = B^2$. Put $f_1 := g_1^{-1} \circ \varphi^{-1}$. Then the mapping $f_1$ satisfies all the conditions of Theorem 1.1 in particular, the inequality (1.2) for $Q = Q_1(z)$.

**Example 3.** Finally, let us construct relevant examples of mappings with a branching. For this purpose, in the notations of Example 1 we put: $f_2(z) = (\varphi_1 \circ f)(z)$, where $\varphi_1(z) = z^2$. Observe that, a mapping $f_2$ is open, discrete and closed, in addition, it satisfies the relation (1.2) for $Q := 2K_f(x, g) = 2q_0(x)$ (see, e.g., [MRV, Theorem 3.2]).

4 On the discreteness of mappings with the inverse Poletsky inequality at the boundary of a domain

In [Vu], some issues related to the discreteness of a closed quasiregular map $f : B^n \to \mathbb{R}^n$ in $B^n$ are considered, see [Vu, Lemma 4.4, Corollary 4.5 and Theorem 4.7]. In this section we talk about the discreteness of mappings that satisfy the condition (1.2). Note that here we consider mappings $f$ with, generally speaking, an unbounded function $Q$ in (1.2), in addition, we consider the case of an arbitrary domain $D$ with some additional conditions on its boundary.

Here are some definitions. Let $p \geq 1$. Consider a more general definition compared to (1.2). We will say that $f$ satisfies the inverse Poletsky inequality at a point $y_0 \in f(D) \setminus \{\infty\}$ relative to $p$-modulus, if the relation

$$M_p(\Gamma f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^p(|y - y_0|) \, dm(y) \tag{4.1}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \tag{4.2}$$

Using the inversion $\psi(y) = |y|^2$, we also may defined the relation (4.1) at the point $y_0 = \infty$.

Following [NP, Section 2.4], we say that a domain $D \subset \mathbb{R}^n$, $n \geq 2$, is uniform with respect to $p$-modulus, if for any $r > 0$ there is $\delta > 0$ such that the inequality

$$M_p(\Gamma(F^*, F, D)) \geq \delta \tag{4.3}$$

holds for any continua $F, F^* \subset D$ with $h(F) \geq r$ and $h(F^*) \geq r$. When $p = n$, the prefix "relative to $p$-modulus" is omitted. Note that this is the definition slightly different from the "classical" given in [NP, Chapter 2.4], where the sets $F$ and $F^* \subset D$ are assumed to be arbitrary connected. We prove the following statement (see its analogue for quasiregular mappings of the unit ball in [Vu, Lemma 4.4]).
Lemma 4.1. Let \( n \geq 2 \), \( n - 1 < p \leq n \), and let \( D \) be a domain which is uniform with respect to \( p \)-modulus. Let \( f : D \to \mathbb{R}^n \) be an open discrete and closed mapping in \( D \), for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \), equals to zero outside of \( f(D) \), such that the relations (4.1)–(4.2) hold for some \( y_0 \in f(D) \). Assume that, there is \( \varepsilon_0 = \varepsilon_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \to [0, \infty] \) such that

\[
I(\varepsilon, \varepsilon_0) := \int_\varepsilon^{\varepsilon_0} \psi(t) \, dt < \infty \quad \forall \, \varepsilon \in (0, \varepsilon_0), \quad I(\varepsilon, \varepsilon_0) \to \infty \quad \text{as} \quad \varepsilon \to 0, \tag{4.4}
\]

and, in addition,

\[
\int_{A(y_0, \varepsilon, \varepsilon_0)} Q(y) \cdot \psi^p(|y - y_0|) \, dm(y) = o(I^p(\varepsilon, \varepsilon_0)), \tag{4.5}
\]

as \( \varepsilon \to 0 \), where \( A(y_0, \varepsilon, \varepsilon_0) \) is defined in (4.1). Let \( C_j, j = 1, 2, \ldots, \) be a sequence of continua such that \( h(C_j) \geq \delta > 0 \) for some \( \delta > 0 \) and any \( j \in \mathbb{N} \) and, in addition, \( h(f(C_j)) \to 0 \) as \( j \to \infty \). Then \( h(f(C_j), y_0) \geq \delta > 0 \) for any \( j \in \mathbb{N} \) and some \( \delta_1 > 0 \).

Proof. We may assume that \( y_0 \neq \infty \). Suppose the opposite, namely, let \( h(f(C_{j_k}), y_0) \to 0 \) as \( k \to \infty \) for some increasing sequence of numbers \( j_k, k = 1, 2, \ldots, \). Let \( F \subset D \) be any continuum in \( D \) such that \( y_0 \notin f(F) \). Let \( \Gamma_k := \Gamma(F, C_{j_k}, D) \). Then, due to the definition of the uniformity of the domain with respect to \( p \)-modulus, we obtain that

\[
M_p(\Gamma_k) \geq \delta_2 > 0 \tag{4.6}
\]

for any \( k \in \mathbb{N} \) and some \( \delta_2 > 0 \). On the other hand, let us to consider the family of paths \( f(\Gamma_k) \).

Let us to prove that, for any \( l \in \mathbb{N} \) there is a number \( k = k_l \) such that

\[
f(C_{j_k}) \subset B(y_0, 1/l), \quad k \geq k_l. \tag{4.7}
\]

Suppose the opposite. Then there is \( l_0 \in \mathbb{N} \) such that

\[
f(C_{j_{m_l}}) \cap (\mathbb{R}^n \setminus B(y_0, 1/l_0)) \neq \emptyset \tag{4.8}
\]

for some increasing sequence of numbers \( m_l, l = 1, 2, \ldots \). In this case, there is a sequence \( x_{m_l} \in f(C_{j_{m_l}} \cap (\mathbb{R}^n \setminus B(y_0, 1/l_0)), l \in \mathbb{N} \). Since by the assumption \( h(f(C_{j_k}, y_0) \to 0 \) for some sequence of numbers \( j_k, k = 1, 2, \ldots, \) we obtain that to зокрема

\[
h(f(C_{j_{m_l}}), y_0) \to 0 \quad \text{as} \quad l \to \infty. \tag{4.9}
\]

Since \( h(f(C_{j_{m_l}}), y_0) = \inf_{y \in f(C_{j_{m_l}})} \psi(y, y_0) \) and \( f(C_{j_{m_l}}) \) is a compact as a continuous image of the compact set \( C_{j_{m_l}} \) under the mapping \( f \), it follows that \( h(f(C_{j_{m_l}}), y_0) = h(y_l, y_0) \), where \( y_l \in f(C_{j_{m_l}}) \). Due to the relation (4.9) we obtain that \( y_l \to y_0 \) as \( l \to \infty \). Since by the assumption
\[ h(f(C_j)) = \sup_{y,z \in f(C_j)} h(y, z) \to 0 \] as \( j \to \infty \), we have that \( h(y_l, x_{m_l}) \leq h(f(C_{j_{m_l}})) \to 0 \) as \( l \to \infty \). Now, by the triangle inequality, we obtain that
\[ h(x_{m_l}, y_0) \leq h(x_{m_l}, y_l) + h(y_l, y_0) \to 0 \quad \text{as} \quad l \to \infty . \]
The latter contradicts with (4.8). The contradiction obtained above proves (4.7).

The following considerations are similar to the second part of the proof of Lemma 2.1 in [Sev]. Without loss of generality we may consider that the number \( l_0 \in \mathbb{N} \) is such that \( 1/l < \varepsilon_0 \) for any \( l \geq l_0 \), and
\[ f(F) \subset \mathbb{R}^n \setminus B(y_0, 1/l_0) . \tag{4.10} \]
In this case, we observe that
\[ f(\Gamma_{k_l}) > \Gamma(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0)) . \tag{4.11} \]
Indeed, let \( \tilde{\gamma} \in f(\Gamma_{k_l}) \). Then \( \tilde{\gamma}(t) = f(\gamma(t)) \), where \( \gamma \in \Gamma_{k_l} \), \( \gamma : [0, 1] \to D \), \( \gamma(0) \in F \), \( \gamma(1) \in C_{j_{k_l}} \). Due to the relation \(4.10\), we obtain that \( f(\gamma(0)) \in f(F) \subset \mathbb{R}^n \setminus B(y_0, 1/l_0) \). In addition, by \(4.7\) we have that \( \gamma(1) \in C_{j_{k_l}} \subset B(y_0, 1/l_0) \). Thus, \( |f(\gamma(t))| \cap B(y_0, 1/l_0) \neq \emptyset \neq |f(\gamma(t))| \cap (\mathbb{R}^n \setminus B(y_0, 1/l_0)) \). Now, by [Ku Theorem 1.1.5.46] we obtain that, there is \( 0 < t_1 < 1 \) such that \( f(\gamma(t_1)) \in S(y_0, 1/l_0) \). Set \( \gamma_1 := \gamma|_{[t_1, 1]} \). We may consider that \( f(\gamma(t)) \in B(y_0, \varepsilon_0) \) for any \( t \geq t_1 \). Arguing similarly, we obtain \( t_2 \in [t_1, 1] \) such that \( f(\gamma(t_2)) \in S(y_0, 1/l) \). Put \( \gamma_2 := \gamma|_{[t_1, t_2]} \). We may consider that \( f(\gamma(t)) \in B(y_0, 1/l) \) for any \( t \in [t_1, t_2] \). Now, a path \( f(\gamma_2) \) is a subpath of \( f(\gamma) = \tilde{\gamma} \), which belongs to \( \Gamma(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0)) \). The relation \(4.11\) is established.

It follows from \(4.11\) that
\[ \Gamma_{k_l} > \Gamma_f(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0)) . \tag{4.12} \]
Set
\[ \eta_l(t) = \begin{cases} \psi(t)/I(1/l, \varepsilon_0), & t \in (1/l, \varepsilon_0), \\ 0, & t \notin (1/l, \varepsilon_0), \end{cases} \]
where \( I(1/l, \varepsilon_0) = \int_{1/l}^{\varepsilon_0} \psi(t) \, dt \). Observe that \( \int_{1/l}^{\varepsilon_0} \eta_l(t) \, dt = 1 \). Now, by the relations \(4.5\) and \(4.12\), and due to the definition of \( f \) in \(4.1\), we obtain that будемо мати:
\[ M_p(\Gamma_{k_l}) \leq M_p(\Gamma_f(S(y_0, 1/l), S(y_0, \varepsilon_0), A(y_0, 1/l, \varepsilon_0))) \leq \]
\[ \leq \frac{1}{I_p(1/l, \varepsilon_0)} \int_{A(y_0, 1/l, \varepsilon_0)} Q(y) \cdot \psi^p(|y - y_0|) \, dm(y) \to 0 \quad \text{as} \quad l \to \infty . \tag{4.13} \]
The latter contradicts with \(4.6\). The contradiction obtained above proves the lemma. □

Let \( X \) and \( Y \) be metric spaces. Recall that, a mapping \( f : X \to Y \) is called light, if for any point \( y \in Y \), the set \( f^{-1}(y) \) does not contain any nondegenerate continuum \( K \subset X \). The
following lemma generalizes Corollary 4.5 in [Vn] for the case of mappings with unbounded characteristic.

**Lemma 4.2.** Let \( n \geq 2, n - 1 < p \leq n \) and let \( D \) be a domain which is uniform with a respect to \( p \)-modulus. Let \( f : D \to \mathbb{R}^n \) be an open discrete and closed mapping of \( D \) for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \) equals to zero outside \( f(D) \) such that the conditions (4.1)–(4.2) hold for any point \( y_0 \in \partial f(D) \). Assume that, there is \( \varepsilon_0 = \varepsilon_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \to [0, \infty] \) such that the relations (4.3)–(4.5) hold, where \( A(y_0, \varepsilon, \varepsilon_0) \) is defined in (4.4). Assume also that, a domain \( D \) is locally connected on its boundary, and that \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \mathbb{R}^n \). Then \( \overline{f} \) is light.

**Proof.** Assume the contrary, namely, let \( y_0 \in \partial f(D) \) be some point such that \( f^{-1}(y_0) \supset K_0 \), where \( K_0 \subset \partial D \) is some nondegenerate continuum. Then, in particular, \( f(K_0) = y_0 \).

Since \( \overline{D} \) is a compactum in \( \mathbb{R}^n \) and, in addition, \( \overline{f} \) is continuous in \( \overline{D} \), the mapping \( \overline{f} \) is uniformly continuous in \( \overline{D} \). In this case, for any \( j \in \mathbb{N} \) there is \( \delta_j < 1/j \) such that

\[
h(\overline{f}(x), \overline{f}(x_0)) = h(\overline{f}(x), y_0) < 1/j \quad \forall \ x, x_0 \in \overline{D}, \ h(x, x_0) < \delta_j \, , \quad \delta_j < 1/j .
\]

(4.14)

Denote by \( B_h(x_0, r) = \{ x \in \mathbb{R}^n : h(x, x_0) < r \} \). Then, given \( j \in \mathbb{N} \), we set

\[
B_j := \bigcup_{x_0 \in K_0} B_h(x_0, \delta_j) , \quad j \in \mathbb{N} .
\]

Since the set \( B_j \) is a neighborhood of \( K_0 \), by [HK] Lemma 2.2 there is a neighborhood \( U_j \) of the set \( K_0 \) such that \( U_j \subset B_j \) and the set \( U_j \cap D \) is connected. Without loss of generality, we may assume that \( U_j \) is open. Then the set \( U_j \cap D \) is path connected, as well (see [MRSY] Proposition 13.1]). Since \( K_0 \) is a compact set, there are \( z_0, w_0 \in K_0 \) such that \( h(K_0) = h(z_0, w_0) \). It follows from this, that there are \( z_j \in U_j \cap D \) and \( w_j \in U_j \cap D \) such that \( z_j \to z_0 \) and \( w_j \to w_0 \) as \( j \to \infty \). We may assume that

\[
h(z_j, w_j) > h(K_0)/2 \quad \forall \ j \in \mathbb{N} .
\]

(4.15)

Since the set \( U_j \cap D \) is path connected, we may join points \( z_j \) and \( w_j \) by some path \( \gamma_j \in U_j \cap D \). Set \( C_j := |\gamma_j| \).

Observe that, \( h(f(C_j)) \to 0 \) as \( j \to \infty \). Indeed, since \( f(C_j) \) is a continuum in \( \mathbb{R}^n \), there are points \( y_j, y_j' \in f(C_j) \) such that \( h(f(C_j)) = h(y_j, y_j') \). Then there are \( x_j, x_j' \in C_j \) such that \( y_j = f(x_j) \) and \( y_j' = f(x_j') \). Then points \( x_j \) and \( x_j' \) belong to \( U_j \subset B_j \). Therefore, there are \( x_j^1 \) and \( x_j^2 \in K_0 \) such that \( x_j \in B(x_j^1, \delta_j) \) and \( x_j' \in B(x_j^2, \delta_j) \). In this case, by the relation (4.14) and due to the triangle inequality we obtain that

\[
h(f(C_j)) = h(y_j, y_j') = h(f(x_j), f(x_j')) \leq
\]

\[
\leq h(f(x_j), f(x_j^1)) + h(f(x_j^1), f(x_j')) + h(f(x_j'), f(x_j')) < 2/j \to 0 \quad \text{as} \quad j \to \infty .
\]

(4.16)
It follows from (4.15) and (4.16) that, the continua $C_j$, $j = 1, 2, \ldots$, satisfy the conditions of Lemma 4.1. By this lemma we may obtain that $h(f(C_j), y_0) \geq \delta_1 > 0$ for any $j \in \mathbb{N}$. On the other hand, by the proving above $x_j \in B(x^j_1, \delta_j)$. Now, by the relation (4.14) we obtain that $h(f(x_j), y_0) < 1/j$, $j = 1, 2, \ldots$. The resulting contradiction indicates the incorrectness of the assumption that $f$ is not light in $\partial D$. Lemma is proved. $\blacksquare$

**Corollary 4.1.** The statements of Lemmas 4.1 and 4.2 are fulfilled if we put $D = \mathbb{B}^n$.

**Proof.** Obviously, the domain $D = \mathbb{B}^n$ is locally connected at its boundary. We prove that this domain is uniform with respect to the $p$-modulus for $p \in (n-1, n)$. Indeed, since $\mathbb{B}^n$ is a Loewner space (see [He, Example 8.24(a)]), the set $\mathbb{B}^n$ is Ahlfors regular with respect to the Euclidean metric $d$ and Lebesgue measure in $\mathbb{R}^n$ (see [He, Proposition 8.19]). In addition, in $\mathbb{B}^n$, $(1; p)$-Poincaré inequality holds for any $p \geq 1$ (see e.g. [HaK, Theorem 10.5]). Now, by [AS, Proposition 4.7] we obtain that the relation

$$M_p(\Gamma(E, F, \mathbb{B}^n)) \geq \frac{1}{C} \min\{\text{diam } E, \text{diam } F\}, \quad (4.17)$$

holds for any $n-1 < p \leq n$ and for any continua $E, F \subset \mathbb{B}^n$, where $C > 0$ is some constant, and diam denotes the Euclidean diameter. Since the Euclidean distance is equivalent to the chordal distance on bounded sets, the uniformity of the domain $D = \mathbb{B}^n$ with respect to the $p$-modulus follows directly from (4.17). $\blacksquare$

We need the following statement (see [Na, Theorem 4.2]).

**Proposition 4.1.** Let $\mathcal{F}$ be a family of connected sets in $D$ such that $\inf_{F \subset \mathcal{F}} h(F) > 0$, and let $\inf_{F \subset \mathcal{F}} M(\Gamma(F, A, D)) > 0$ for some continuum $A \subset D$. Then

$$\inf_{F, F^* \subset \mathcal{F}} M(\Gamma(F, F^*, D)) > 0.$$  

Let $p \geq 1$. Due to [MRSY, Section 3] we say that a boundary $D$ is called strongly accessible with respect to $p$-modulus at $x_0 \in \partial D$, if for any neighborhood $U$ of the point $x_0 \in \partial D$ there is a neighborhood $V \subset U$ of this point, a compactum $F \subset D$ and a number $\delta > 0$ such that $M_p(\Gamma(E, F, D)) \geq \delta$ for any continua $E \subset D$ such that $E \cap \partial U \neq \emptyset \neq E \cap \partial V$. The boundary of a domain $D$ is called strongly accessible with respect to $p$-modulus, if this is true for any $x_0 \in \partial D$. When $p = n$, prefix "relative to $p$-modulus" is omitted. The following lemma is valid (see the statement similar in content to [Na, Theorem 6.2]).

**Lemma 4.3.** A domain $D \subset \mathbb{R}^n$ has a strongly accessible boundary if and only if $D$ is uniform.

**Proof.** The fact that uniform domains have strongly accessible boundaries has been proved in [SevSky1, Remark 1]. It remains to prove that domains with strongly accessible boundaries are uniform.
We will prove this statement from the opposite. Let \( D \) be a domain which has a strongly accessible boundary, but it is not uniform. Then there is \( r > 0 \) such that, for any \( k \in \mathbb{N} \) there are continua \( F_k \) and \( F^*_k \subset D \) such that \( h(F_k) \geq r, h(F^*_k) \geq r \), however,

\[
M(\Gamma(F_k, F^*_k, D)) < 1/k.
\]

Let \( x_k \in F_k \). Since \( \overline{D} \) is compact in \( \mathbb{R}^n \), we may assume that \( x_k \to x_0 \in \overline{D} \). Note that the strongly accessibility of the domain \( D \) at the boundary points is assumed to be, and at the inner points it is even weakly flat, which is the result of Väisälä’s lemma (see e.g. \[Va\] Sect. 10.12, cf. \[SevSkv2\] Lemma 2.2). Let \( U \) be a neighborhood of the point \( x_0 \) such that \( h(x_0, \partial U) \leq r/2 \). Then there is a neighborhood \( V \subset U \), a compactum \( F \subset D \) and a number \( \delta > 0 \) such that the relation \( M(\Gamma(E, F, D)) \geq \delta \) holds for any continuum \( E \subset D \) such that \( E \cap \partial U \neq \emptyset \neq E \cap \partial V \). By the choice of the neighborhood \( U \), we obtain that \( F_k \cap U \neq \emptyset \neq F_k \cap (D \setminus U) \) for sufficiently large \( k \in \mathbb{N} \). Observe that, for the same \( k \in \mathbb{N} \), the condition \( F_k \cap V \neq \emptyset \neq F_k \cap (D \setminus V) \) holds. Then, by \[Ku\] Theorem 1.I.5.46 we obtain that \( F_k \cap \partial U \neq \emptyset \neq F_k \cap \partial V \). Observe that, a compactum \( F \) can be imbedded in some continuum \( A \subset D \) (see \[Sm\] Lemma 1]). Then the inequality \( M(\Gamma(E, A, D)) \geq \delta \) will only increase. Given the above, we obtain that

\[
M(\Gamma(F_k, A, D)) \geq \delta \quad \forall k \geq k_0
\]

for some. Taking inf over all \( k \geq k_0 \) in (4.19), we obtain that

\[
\inf_{k \geq k_0} M(\Gamma(F_k, A, D)) \geq \delta.
\]

Set \( \mathfrak{I} := \{F_k\}_{k=k_0}^\infty \). Now, by the condition (4.20) and by Proposition 4.1, we obtain that \( \inf_{k \geq k_0} M(\Gamma(F_k, F^*_k, D)) > 0 \), that contradicts the assumption made in (4.18). The resulting contradiction completes the proof of the lemma. \( \square \)

Obviously, weakly flat boundaries are strongly accessible. Now, by Lemma 4.3 we obtain the following.

**Corollary 4.2.** If \( D \subset \mathbb{R}^n \) has a weakly flat boundary, then \( D \) is uniform.

Given a mapping \( f : D \to \mathbb{R}^n \), a set \( E \subset D \) and \( y \in \mathbb{R}^n \), we define the *multiplicity function* \( N(y, f, E) \) as a number of preimages of the point \( y \) in a set \( E \), i.e.

\[
N(y, f, E) = \text{card} \{ x \in E : f(x) = y \},
\]

\[
N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E).
\]

Note that, the concept of a multiplicity function may also be extended to sets belonging to the closure of a given domain. Finally, we formulate and prove a key statement about the discreteness of mapping (see \[Vu\] Theorem 4.7]).
Lemma 4.4. Suppose that, under the conditions of Lemma 4.2, \( p = n \), the domain \( D \) is weakly flat and the domain \( f(D) \) is locally connected at its boundary. Then the mapping \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \mathbb{R}^n \) such that \( N(f, D) = N(f, \overline{D}) < \infty \). In particular, \( \overline{f} \) is discrete in \( \overline{D} \).

Proof. First of all, the possibility of continuous extension of \( f \) to a mapping \( \overline{f} : \overline{D} \to \mathbb{R}^n \) follows by Remark 2.3. Note also that \( N(f, D) < \infty \), see [MS, Theorem 2.8]. Let us to prove that \( N(f, D) = N(f, \overline{D}) \). Next we will reason using the scheme proof of Theorem 4.7 in [Yu]. Assume the contrary. Then there are points \( y_0 \in \partial f(D) \) and \( x_1, x_2, \ldots, x_k, x_{k+1} \in \partial D \) such that \( f(x_i) = y_0, i = 1, 2, \ldots, k + 1 \) and \( k := N(f, D) \). We may assume that \( y_0 \neq \infty \). Since by the assumption \( f(D) \) is locally connected at any point of its boundary, for any \( p \in \mathbb{N} \) there is a neighborhood \( U_p \subset B(y_0, 1/p) \) such that the set \( \overline{U}_p \cap f(D) = \overline{U}_p \) is connected.

Let us to prove that, for any \( i = 1, 2, \ldots, k + 1 \) there is a component \( V^i_p \) of the set \( f^{-1}(U^i_p) \) such that \( x_i \in \overline{V}^i_p \). Fix \( i = 1, 2, \ldots, k + 1 \). By the continuity of \( \overline{f} \) in \( \overline{D} \), there is \( r_i = r_i(x_i) > 0 \) such that \( \overline{f}(B(x_i, r_i) \cap D) \subset \overline{U}_p \). By [MRSY, Lemma 3.15], a domain with a weakly flat boundary is locally connected on its boundary. Thus, we may find a neighborhood \( W_i \subset B(x_i, r_i) \) of the point \( x_i \) such that \( W_i \cap D \) is connected. Then \( W_i \cap D \) belongs to one and only one component \( V^i_p \) of the set \( f^{-1}(U^i_p) \), while \( x_i \in \overline{W}_i \cap D \subset \overline{V}_p \), as required.

Next we show that the sets \( \overline{V}^i_p \) are disjoint for any \( i = 1, 2, \ldots, k + 1 \) and large enough \( p \in \mathbb{N} \). In turn, we prove for this that \( h(\overline{V}^i_p) \to 0 \) as \( p \to \infty \) for each fixed \( i = 1, 2, \ldots, k + 1 \). Let us prove the opposite. Then there is \( 1 < i_0 \leq k + 1 \), a number \( r_0 > 0 \), \( r_0 < \frac{1}{2} \min_{1 \leq i,j \leq k+1,i \neq j} h(x_i, x_j) \) and an increasing sequence of numbers \( p_m, m = 1, 2, \ldots \), such that \( S_h(x_{i_0}, r_0) \cap \overline{V}^m_p \neq \emptyset \), where \( S_h(x_{i_0}, r) = \{ x \in \mathbb{R}^n : h(x, x_{i_0}) = r \} \). In this case, there are \( a_m, b_m \in V^m_p \) such that \( a_m \to x_{i_0} \) as \( m \to \infty \) and \( h(a_m, b_m) \geq r_0/2 \). Join the points \( a_m \) and \( b_m \) by a path \( C_m \), which entirely belongs to \( V^m_p \). Then \( h(|C_m|) \geq r_0/2 \) for \( m = 1, 2, \ldots \). On the other hand, since \( |C_m| \subset f(V^m_p) \subset B(y_0, 1/p_m) \), then simultaneously \( h(f(|C_m|)) \to 0 \) as \( m \to \infty \) and \( h(|C_m|, y_0) \to 0 \) as \( m \to \infty \), that contradicts with Lemma 4.1. The resulting contradiction indicates the incorrectness of the above assumption.

By [Yu] Lemma 3.6 \( f \) is a mapping of \( \overline{V}^i_p \) onto \( U^i_p \) for any \( i = 1, 2, \ldots, k, k + 1 \). Thus, \( N(f, D) \geq k + 1 \), which contradicts the definition of the number \( k \). The obtained contradiction refutes the assumption that \( N(f, \overline{D}) > N(f, D) \). The lemma is proved. \( \square \)

We say that a function \( \varphi : D \to \mathbb{R} \) has a finite mean oscillation at a point \( x_0 \in D \), write \( \varphi \in FMO(x_0) \), if

\[
\lim_{\varepsilon \to 0} \sup_{\Omega} \frac{1}{\Omega} \int_{B(x_0, \varepsilon)} |\varphi(x) - \bar{\varphi}_\varepsilon| \, dm(x) < \infty,
\]

where \( \bar{\varphi}_\varepsilon = \frac{1}{\Omega} \int_{B(x_0, \varepsilon)} \varphi(x) \, dm(x) \). We also say that a function \( \varphi : D \to \mathbb{R} \) has a finite mean oscillation at any point
Let us now turn to the main results of this section.

**Theorem 4.1.** Let \( n \geq 2 \), let \( D \) be a domain with a weakly flat boundary and let \( D' \) be a domain which is locally connected on its boundary. Let \( f \) be open discrete and closed mapping of \( D \) onto \( D' \) for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \), equal to zero outside \( D' \), such that the relations (1.2)-(1.3) hold at any point \( y_0 \in \partial D' \). Assume that, one of the following conditions hold:

1) \( Q \in \text{FMO}(\partial D') \);

2) for any \( y_0 \in \partial D' \) there is \( \delta(y_0) > 0 \) such that

\[
\int_\varepsilon^{\delta(y_0)} \frac{dt}{tu_0^{-\alpha}}(t) < \infty, \quad \int_0^{\delta(y_0)} \frac{dt}{tu_0^{-\alpha}}(t) = \infty \tag{4.22}
\]

for sufficiently small \( \varepsilon > 0 \).

Then \( f \) has a continuous extension \( \overline{f} : \overline{D} \to \mathbb{R}^n \) such that \( N(f, D) = N(f, \overline{D}) < \infty \). In particular, \( \overline{f} \) is discrete in \( \overline{D} \).

**Proof.** In the case 1), we choose \( \psi(t) = \frac{1}{t \log \frac{1}{t}} \), and in the case 2), we set

\[
\psi(t) = \begin{cases} 
1/[t^{\frac{n-1}{n-\alpha}}q_0^{-\frac{1}{\alpha}}(t)] , & t \in (\varepsilon, \varepsilon_0) , \\
0 , & t \notin (\varepsilon, \varepsilon_0) ,
\end{cases}
\]

Observe that, the relations (4.4)-(4.5) hold for these functions \( \psi \), where \( p = n \) (the proof of this facts may be found in [Sev, Proof of Theorem 1.1]). The desired conclusion follows from Lemma 4.4. \( \square \)

**References**

[AS] **Adamowicz, T. and N. Shanmugalingam:** Non-conformal Loewner type estimates for modulus of curve families. - Ann. Acad. Sci. Fenn. Math. 35, 2010, 609-626.

[HaK] **Hajlasz, P. and P. Koskela:** Sobolev met Poincare. - Mem. Amer. Math. Soc. 145:688, 2000, 1-101.

[He] **Heinonen, J.:** Lectures on Analysis on metric spaces. - Springer Science+Business Media, New York, 2001.

[HK] **Herron, J. and P. Koskela:** Quasiextremal distance domains and conformal mappings onto circle domains. - Compl. Var. Theor. Appl. 15, 1990, 167-179.

[IS] **Il'yutko, D.P. and E.A. Sevost’yanov:** Boundary behaviour of open discrete mappings on Riemannian manifolds. - Sbornik Mathematics 209:5, 2018, 605-651.
ON THE BOUNDARY BEHAVIOR...

[Ku] Kuratowski, K.: Topology, v. 2. – Academic Press, New York–London, 1968.

[MRV₁] Martio, O., S. Rickman and J. Väisälä: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 448, 1969, 1-40.

[MRV₂] Martio, O., S. Rickman and J. Väisälä: Topological and metric properties of quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 488, 1971, 1-31.

[MRSY] Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: Moduli in modern mapping theory. - Springer Science + Business Media, LLC, New York, 2009.

[MS] Martio, O., U. Srebro: Automorphic quasimeromorphic mappings in $\mathbb{R}^n$. - Acta Math. 135, 1975, 221-247.

[Na] Näkki, R: Extension of Loewner’s capacity theorem. - Trans. Amer. Math. Soc. 180, 1973, 229-236.

[NP] Näkki, R., and B. Palka: Uniform equicontinuity of quasiconformal mappings. - Proc. Amer. Math. Soc. 37:2, 1973, 427-433.

[Sa] Saks, S.: Theory of the Integral. - Dover, New York, 1964.

[SalSev] Sevost’yanov, E.A. and R.R. Salimov: On inner dilatations of the mappings with unbounded characteristic // J. Math. Sci. (N. Y.) 178:1, 2011, 97-107.

[Sev] Sevost’yanov, E.A.: On open and discrete mappings with a modulus condition. - Ann. Acad. Sci. Fenn. 41, 2016, 41–50.

[SevSkv₁] Sevost’yanov, E.A. and S.A. Skvortsov: On the equicontinuity of families of mappings in the case of variable domains. - Ukrainian Mathematical Journal 71:7, 2019, 1071-1086.

[SevSkv₂] Sevost’yanov, E.A. and S.A. Skvortsov: On mappings whose inverse satisfy the Poletsky inequality. - Ann. Acad. Scie. Fenn. Math. 45, 2020, 259-277.

[SevSkv₃] Sevost’yanov, E.A. and S.A. Skvortsov: Logarithmic Hölder continuous mappings and Beltrami equation. - Analysis and Mathematical Physics 11:3, 2021, 138.

[SSD] Sevost’yanov, E.A., S.O. Skvortsov and O.P. Dovhopiatyi: On non-homeomorphic mappings with inverse Poletsky inequality. - Journal of Mathematical Sciences 252:4, 2021, 541-557.

[Sm] Smolovaya, E.S.: Boundary behavior of ring $Q$-homeomorphisms in metric spaces. - Ukr. Math. Journ. 62:5, 2010, 785–793.

[Vu] Vuorinen, M.: Exceptional sets and boundary behavior of quasiregular mappings in $n$-space. - Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes 11, 1976, 1-44.

[Va] Väisälä, J.: Lectures on $n$-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin etc., 1971.
Evgeny Sevost’yanov  
1. Zhytomyr Ivan Franko State University,  
40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE  
2. Institute of Applied Mathematics and Mechanics  
of NAS of Ukraine,  
1 Dobrovol’skogo Str., 84 100 Slavyansk, UKRAINE  
esevostyanov2009@gmail.com