Thermodynamic Features of Black Holes Dressed with Quantum Fields

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Abstract

The thermal properties of black holes in the presence of quantum fields can be revealed through solutions of the semi-classical Einstein equation. We present a brief but self-contained review of the main features of the semi-classical back reaction problem for a black hole in the microcanonical ensemble. The solutions, obtained for conformal scalars, massless spinors and U(1) gauge bosons, are used to calculate the $O(\hbar)$ corrections to the temperature and thermodynamical entropy of a Schwarzschild black hole. In each spin case considered, the entropy corrections $\Delta S(r)$, are positive definite and monotone increasing with increasing distance $r$ from the hole, and are of the same order as the naive flat space radiation entropy.

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1 Introduction

The physics of black holes provides a fertile ground in which the confluence of gravitation, quantum mechanics and thermodynamics takes place. Progress in our understanding of the thermal features of black holes demands a deeper understanding of the relationships among the state functions of black holes in thermodynamic equilibrium with quantized matter. A black hole can exist in thermodynamical equilibrium provided it is surrounded by radiation with a suitable distribution of stress-energy. In effect, one studies the consequences of coupling the black hole with its own Hawking radiation. In the semi-classical approach, such radiation is characterized by the expectation value of a stress energy tensor obtained by renormalization of a quantum field on the classical spacetime geometry of a black hole. Using such a stress tensor as a source in the semi-classical Einstein equation defines the back reaction problem.

In this talk, we shall review briefly the steps involved in the 1-loop back reaction problem for black holes placing special emphasis on the need to impose boundary conditions and the issue of the perturbative validity of the modified black hole metric. When then go on to use the solutions so obtained to calculate the corrections to the temperature at spatial infinity, \( T_\infty \), and to the entropy \( \Delta S \) by which quantum fields of spin 0, 1/2, 1 augment the usual Bekenstein-Hawking entropy. One finds that \( T_\infty \neq T_H = \frac{\hbar}{8\pi M} \), while \( \Delta S \geq 0 \) has a global minimum at the renormalized event horizon and is monotonically increasing for increasing distance from the black hole. The topics sketched here can be found in more detail in \[1, 2, 3, 4\].

2 Mechanics of the 1-loop Back-reaction

We begin by choosing the background spacetime. We therefore suppose we have solved the classical vacuum Einstein equation

\[
\hat{G}_{\mu\nu}(\hat{g}) = 0,
\]

for \( \hat{g}_{\mu\nu} \), the classical metric. We now inject, or superimpose, a collection of quantum fields on this classical background and solve the semi-classical field equation

\[
G_{\mu\nu}(\hat{g} + \delta g) = 8\pi <T_{\mu\nu}(\phi, \psi, A_\mu, \cdots) >_{\hat{g}},
\]

where \( <T_{\mu\nu}>_{\hat{g}} \) represents the 1-loop quantum stress-energy tensor for the fields \( \phi, \psi, A_\mu, \cdots \) (scalars, spinors, vector bosons, etc.) renormalized over the background spacetime whose metric is \( \hat{g} \). The effect, or back-reaction, of the quantum fields on the geometry described by the metric \( \hat{g} \) induces a modification of this metric, \( \delta g \), so that

\[
\hat{g} \rightarrow \hat{g} + \delta g
\]

when we “turn on” the sources. These quantum stress tensors obey the background conservation law

\[
\hat{\nabla}_\mu <T^{\mu\nu} > = 0,
\]
with respect to the background covariant derivative: $\hat{\nabla} \sim \partial + \hat{\Gamma}(\hat{g})$. As a consequence, it turns out we may only solve for the modified metric to order $O(\bar{h})$. This we can demonstrate easily by expanding the Einstein tensor $G$ in powers of the Planck constant and using (1) and (3), thus leading to

$$\hat{\nabla}_\mu (\hat{G} + \delta G + \Delta G)^{\mu\nu} = 8\pi \hat{\nabla}_\mu <T^{\mu\nu}> = 0,$$

(4)

so that $\hat{\nabla}_\mu (\delta G + \Delta G)^{\mu\nu} = 0$, or to order $\bar{h}$, $\hat{\nabla}_\mu \delta G^{\mu\nu} = 0$. Here, $\delta G$ and $\Delta G$ denote the order $\bar{h}$ and higher-order contributions, respectively. Since the operator $\hat{\nabla}$ is $O(1)$ while the stress tensor is $O(\bar{h})$, the back-reaction equation (2) reduces to

$$\delta G^{\mu\nu}(\hat{g} + \delta g) = 8\pi <T^{\mu\nu}>.$$

(5)

The components of the stress tensors are treated as input. To solve (5), one inserts the most general metric ansatz compatible with the symmetries and coordinate dependence of $<T_{\mu\nu}>$ into the left hand side.

Turning now to the case at hand, we consider the black hole background, whose metric is

$$\hat{g}_{\mu\nu} = \text{diag}\left(-\left(1 - \frac{2M}{r}\right), (1 - \frac{2M}{r})^{-1}, r^2, r^2 \sin^2 \theta\right),$$

(6)

and $M$ is the mass of the black hole. As shown in [1, 3], the modified metric, which is static and spherically symmetric, can be put into the following form:

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) (1 + 2\epsilon \bar{\rho}(r)) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

(7)

where $m(r), \bar{\rho}(r)$ are two functions depending on $r$ and $\epsilon = (M_{\text{Planck}}/M_{\text{BH}})^2 < 1$ is the expansion parameter. Note that $\epsilon = h/M^2$ in units where $G = c = k_B = 1$. This parametrization of the new equilibrium metric reflects the fact that the back reaction induces static, spherically symmetric metric perturbations. Indeed, the stress tensors renormalized over $\hat{g}$ are static and depend only on the radial coordinate. $d\Omega^2$ is the standard metric of a normal round unit sphere. The mass function has the form $m(r) = M[1 + \epsilon(\mu(r) + CK^{-1})]$ where $C$ is a constant of integration which serves to renormalize the bare Schwarzschild mass $M$. Indeed, to the order we are working, we may write

$$m(r) = M \left(1 + \epsilon CK^{-1}\right) [1 + \epsilon \mu(r)] \equiv M_{\text{ren}}[1 + \epsilon \mu(r)].$$

(8)

We henceforth write $M$ for the black hole mass in what follows, with the understanding that this represents the physical black hole mass. We note from (11) that $M_{\text{rad}} = \epsilon M\mu(r)$ is the usual expression for the effective mass of a spherical source. The metric in (7) is completed by the determination of $\bar{\rho}$, where

$$\bar{\rho}(r) = \rho(r) + kK^{-1},$$

(9)
with \( k \) another constant of integration; \( K = 3840\pi \).

The corresponding semiclassical field equations, valid to \( O(\epsilon) \) are (\( w = 2M/r \))

\[
\frac{\epsilon}{w^4} \frac{d \rho}{dw} = - \frac{16\pi M^2}{w^3} (1 - w)^{-1} (T_r^r - T_t^t) \\
\frac{\epsilon}{w^4} \frac{d \mu}{dw} = \frac{32\pi M^2}{w^4} (T_t^t).
\]

These may be obtained substituting (7) into (5) keeping only the \( O(\epsilon) \) terms. Naturally, indices are raised/lowered with the background metric.

Once the indicated components of the renormalized stress-energy tensors are known, the solution of the semiclassical back-reaction equation (5) follows immediately from two simple integrations:

\[
\mu(r) = \frac{1}{\epsilon M} \int_{2M}^{r} (-T_t^t > 4\pi\tilde{r}^2 \tilde{r} \, d\tilde{r},
\]

\[
\rho(r) = \frac{1}{\epsilon} \int_{2M}^{r} (T_r^r - T_t^t > 4\pi\tilde{r}^2 \tilde{r} \, d\tilde{r}).
\]

Actually, the back reaction problem as it stands has no definite solution unless boundary conditions are specified \[1\]. There are a number of reasons for why this must be so. In the first instance, the constant of integration \( k \), appearing in \( \rho \) remains undetermined unless a boundary condition is invoked (asymptotic flatness does not fix this constant \[1\]). More importantly, the renormalized stress tensors employed are asymptotically \textit{constant}, thus the radiation in a sufficiently large spatial region would collapse onto the black hole thereby producing a larger one. It is therefore necessary to implant the system consisting of the black hole plus radiation in a finite cavity with wall radius \( r_o > 2M \). As discussed in \[3\], a very important consequence of considering black holes in spatially bounded regions, quite independent of the back reaction problem, is that the cavity stabilizes the black hole in the thermodynamic sense and yields a \textit{positive} heat capacity for the hole. There are at least two distinct types of physically relevant boundary conditions one may choose to impose. In the case of canonical boundary conditions, we specify the temperature of the cavity wall \( T(r_o) \) and immerse the cavity containing the black hole and radiation in an external heat reservoir whose temperature \( T = T(r_o) \). To obtain microcanonical boundary conditions, we specify the total energy at the cavity wall \( E(r_o) \) and match on an external Schwarzschild spacetime with effective mass \( m(r_o) \). In the former case, the integration constant is absorbed by a redefinition of the time coordinate. This is possible as coordinate time has no special meaning unless the metric is asymptotically constant. We can choose the timelike Killing vector to be \( \frac{\partial}{\partial \eta} = \lambda \frac{\partial}{\partial t} \) for \( \lambda \) a constant. The choice \( \lambda = (1 - \epsilon k K^{-1}) \) removes \( k \) from expressions for the physical quantities. In the latter case, we fix \( k \) by requiring continuity of the metric across the cavity wall: \( k = -\rho(r_o)K \).
How does one determine the size of the cavity? The answer to this question is tied up with the perturbative validity of the solutions, which may be maintained provided the cavity wall radius satisfies a certain inequality, to which we now turn. We recall that all the stress tensors renormalized on the black hole background satisfy

$$< T_{\mu}^{\nu} > \rightarrow (\text{spin-dependent const.}) \times \text{diag}(-3,1,1,1)^{\mu}_{\nu},$$

(13)
as $r \rightarrow \infty$, which results in asymptotically unbounded metric perturbations $\delta g_{\mu\nu} = (g_{\mu\nu} - \hat{g}_{\mu\nu})$. In fact, one can show [1, 2, 3, 4] that the relative corrections diverge quadratically

$$|\frac{\delta g}{\hat{g}}| \rightarrow = \epsilon \alpha_s \left( \frac{2}{3K} \right) \left( \frac{r}{2M} \right)^2,$$

(14)

($tt$ and $rr$ components only, since $\delta g_{\theta\theta} = \delta g_{\phi\phi} = 0$) with $\alpha_s$ a spin-dependent constant: $\alpha_s = (1/2, 7/8, 1)$ for the spins $s = (0, 1/2, 1)$, respectively. So, we can obtain solutions that are uniformly small over the entire range $2M < r < r_{\text{domain}}$, taking $|\frac{\delta g}{\hat{g}}| < \delta < 1$, which from (14), and by saturating the previous inequality, defines the radius of the domain of perturbative validity via

$$\left( \frac{r_{\text{domain}}}{2M} \right)^2 = \frac{3K}{2\alpha_s} \left( \frac{\delta}{\epsilon} \right).$$

(15)

So, we should take the cavity radius $r_o < r_{\text{domain}}$. By way of illustration, setting $\epsilon = \delta < 1$ results in rather large perturbatively valid domains, indeed $r_{\text{domain}} \approx (380M, 286M, 268M)$ for the spins $s = (0, 1/2, 1)$, respectively.

3 The Quantum Stress-Energy Tensors

These have been obtained in exact form for the conformal scalar and $U(1)$ gauge boson, and in an approximate form for the massless spin 1/2 fermion. For the former two cases, they are expressed as

$$< T_{\nu}^{\mu} >_{\text{renormalized}} = < T_{\nu}^{\mu} >_{\text{analytic}} + \left( \frac{\hbar}{\pi^2(4M)^4} \right) \Delta^{\mu}_{\nu},$$

(16)

where the analytic piece, in the case of the conformal scalar, was first given by Page [10]. The term $\Delta^{\mu}_{\nu}$ is obtained from a numerical mode sum. As this term is small in comparison to the analytic piece, we do not include it here. This affects none of our qualitative results; both pieces seperately obey the the required regularity and consistency conditions. The analytic part has the exact trace anomaly in both cases. We display only the analytic part for the $U(1)$ case and direct the interested reader to the original literature for the other cases. Dropping the angular brackets, we have ($w = 2M/r$)

$$T^{t}_{t} = -\frac{1}{3} a T_{H}^{4} \left( 3 + 6w + 9w^{2} + 12w^{3} - 315w^{4} + 78w^{5} - 249w^{6} \right),$$
\[
T_r^r = \frac{1}{3} a T_H^4 \left( 1 + 2w + 3w^2 - 76w^3 + 295w^4 - 54w^5 + 285w^6 \right),
\]
\[
T_\theta^\theta = \frac{1}{3} a T_H^4 \left( 1 + 2w + 3w^2 + 44w^3 - 305w^4 + 66w^5 - 579w^6 \right),
\]

where \( a = \left( \pi^2/15h^3 \right) \) and \( T_H = \hbar/8\pi M \) is the (uncorrected) Hawking temperature.

### 4 Solutions

The explicit solutions to the \( O(\epsilon) \) back reaction (5) obtained using the above stress-tensors may be summarized compactly as follows. Denoting with the subscripts \( S, f, V \) the conformal scalar, massless fermion and vector boson respectively, the metric functions in (11,12) turn out to be

\[
K_\mu S = \frac{1}{2} \left[ \frac{2}{3} w^{-3} + 2w^{-2} + 6w^{-1} - 8 \ln(w) - 10w - 6w^2 + 22w^3 - \frac{44}{3} \right],
\]
\[
K_\rho S = \frac{1}{2} \left[ \frac{2}{3} w^{-2} + 4w^{-1} - 8 \ln(w) - \frac{40}{3} w - 10w^2 - \frac{28}{3} w^3 + \frac{84}{3} \right],
\]

for the conformal scalar,

\[
K_\mu f = \frac{7}{8} \left[ \frac{2}{3} w^{-3} + 2w^{-2} + 6w^{-1} - 8 \ln(w) - \frac{90}{7} w - \frac{62}{7} w^2 + \frac{46}{3} w^3 - \frac{16}{7} \right],
\]
\[
K_\rho f = \frac{7}{8} \left[ \frac{2}{3} w^{-2} + 4w^{-1} - 8 \ln(w) - \frac{200}{21} w - \frac{50}{7} w^2 - \frac{52}{7} w^3 + \frac{136}{7} \right],
\]

for the massless spinor, and

\[
K_\mu V = \frac{2}{3} w^{-3} + 2w^{-2} + 6w^{-1} - 8 \ln(w) + 210w - 26w^2 + \frac{166}{3} w^3 - 248,
\]
\[
K_\rho V = \frac{2}{3} w^{-2} + 4w^{-1} - 8 \ln(w) + \frac{40}{3} w + 10w^2 + 4w^3 - 32,
\]

for the U(1) gauge boson. The various spin cases are distinguished by the spin dependence of the numerical coefficients.

### 5 Corrected Black Hole Temperature

As is well known, a black hole in empty space radiates quanta possessing a temperature characterized by the mass \( M \) of the hole. At large distances from the hole \( (r >> M) \), the temperature of the radiation approaches \( T_\infty = \hbar/8\pi M \equiv T_H \). The presence of this radiation however, leads to modifications of this temperature when the back reaction is properly taken into account. For microcanonical boundary conditions, the corrected temperature at spatial infinity takes the form

\[
T_\infty = \frac{\hbar}{8\pi M} \left( 1 + \epsilon f(r, s, M) \right),
\]
where \( f \) is a calculable function of the cavity radius \( r_o \), the spin \( s \) of the quantum radiation and the renormalized mass \( M \) of the hole. Note that generally, \( T_\infty \) is not equal to the Hawking temperature.

To find the form of \( f \) we recall that

\[
T_\infty = \frac{\hbar \kappa_H}{2\pi},
\]

where \( \kappa_H \) is the surface gravity of the event horizon; for a “free” black hole, \( \kappa_H = (4M)^{-1} \). In the case of back reaction, the surface gravity is given by

\[
kappa_H = \frac{1}{4M} \left( 1 + \epsilon(\bar{\rho} - \mu) + 8\pi r^2 < T^t_t > \right)_{r=2M}.
\]

(22)

With the microcanonical boundary conditions, one then obtains \[1, 3\]

\[
T^{(s)}(t) = \frac{\hbar}{8\pi M} \left( 1 - \epsilon\rho_s(r_o) + \epsilon n_s K^{-1} \right),
\]

(23)

where the constant \( n_s = (12, -4, 304) \) for the spin \( s = (0, 1/2, 1) \). The local temperature, \( T_{loc} \), is obtained by blueshifting back from infinity to a finite value of \( r \) (by means of the Tolman factor \[7\]):

\[
T_{loc}(r) = T_\infty \left[ -g_{tt}(r) \right]^{-1/2}.
\]

(24)

This yields the local temperature valid for all \( r > 2M \) (dropping the spin labels)

\[
T_{loc}(r) = T_H (1 - w)^{-1/2} \left[ 1 + \epsilon(\rho(r) - nK^{-1} - \frac{w}{2}(1 - w)^{-1} \mu(w)) \right].
\]

(25)

6 Thermodynamical Entropy

One way to calculate the entropy is as follows. Consider first the case of the free black hole. From the 1st law of thermodynamics applied to slightly differing equilibrium systems

\[
dE = dQ + dW,
\]

(26)

and as no work is done on the system, \( dW = 0 \), so that

\[
dS = \frac{dQ}{T} = \frac{dE}{T} = \frac{dM}{T_\infty},
\]

(27)

where \( M \) is the black hole mass and \( T_\infty \) the temperature at spatial infinity. Integrating, we have that the entropy

\[
S = \int \frac{dM}{T_\infty} = \int \left( \frac{8\pi M}{\hbar} \right) dM = \frac{4\pi M^2}{\hbar} + \text{const.}
\]

(28)
This yields the usual Bekenstein-Hawking expression for the black hole entropy after setting the constant of integration to zero: 

\[ S = S_{BH} \]

Now turn on the back reaction and compute \( S \) from (28). We must of course use the corrected asymptotic temperature (23) and the effective mass of the combined system of black hole plus radiation: 

\[ m(r_o) = M[1 + \epsilon \mu(r_o)] \]

in (28). With (remembering that \( \epsilon = \epsilon(M) \))

\[
dm(r_o) = \left[ 1 + \epsilon \left( w \frac{\partial \mu}{\partial w} - \mu(w) \right) \right] r_o \ dM, \tag{29}
\]

we obtain \((dr_o = 0)\) that

\[ S = S_{BH} + \Delta S(r_o) + \text{constant}. \tag{30} \]

The constant of integration is fixed by requiring that \( \Delta S(r_o = 2M) = 0 \), that is, with no “room” for the fields to contribute anything further, one should obtain just the Bekenstein-Hawking entropy \( \frac{1}{4} A_H/\hbar \), as would be expected. With this choice, we find \([4]\) for the conformal scalar field

\[
\Delta S_S = \frac{8\pi}{K} \left( \frac{1}{2} \right) \left( \frac{8}{9} w^{-3} + \frac{8}{3} w^{-2} + 8w^{-1} + \frac{32}{3} \ln(w) \right) - \frac{40}{3} w - 8w^2 \\
+ \frac{104}{9} w^3 - \frac{16}{9}, \tag{31}
\]

while for the massless spin-\( \frac{1}{2} \) fermion

\[
\Delta S_f = \frac{8\pi}{K} \left( \frac{7}{8} \right) \left( \frac{8}{9} w^{-3} + \frac{8}{3} w^{-2} + 8w^{-1} + \frac{128}{7} \ln(w) \right) - \frac{200}{21} w - 8w^2 \\
+ \frac{488}{63} w^3 - \frac{16}{9}, \tag{32}
\]

and

\[
\Delta S_V = \frac{8\pi}{K} \left( \frac{8}{9} w^{-3} + \frac{8}{3} w^{-2} + 8w^{-1} - 96 \ln(w) \right) + \frac{40}{3} w - 8w^2 \\
+ \frac{344}{9} w^3 - \frac{496}{9}, \tag{33}
\]

for the \( U(1) \) gauge field. Among the features enjoyed by these entropy corrections, it is important to note that they all are positive definite, \( \Delta S \geq 0 \), and are monotone increasing functions of \( r_o \). They are thus amenable to arguments relating thermodynamical to statistical entropy. Moreover, the corrections are of order \( O(1) \) in \( \hbar \) and so are of the same order as the naive flat space radiation entropy: \( S_{\text{flat}} = \frac{4}{3} a T_H^3 V \). These corrections are therefore important.

### 7 Concluding Remarks

We should like to comment on the following points.
Lowest order solutions of the semi-classical back reaction problem have been used to calculate the $O(\hbar)$ corrections to the temperature and the order $O(1)$ correction to the entropy of a Schwarzschild black hole. It should be emphasized that this is a perturbative calculation. It would be of interest to extend the results to higher order, although a non-perturbative treatment would clearly be preferable. At higher orders, one needs to be careful, however, because the semi-classical field equation may well receive corrections arising from the stress tensor renormalization:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + aH^{(1)}_{\mu\nu} + bH^{(2)}_{\mu\nu} = 8\pi G <T_{\mu\nu}>, \quad (34)$$

where $\Lambda, a, b$ are constants and the tensors $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$ are linear combinations of quadratic curvature terms [12, 13].

Perhaps of greater importance is the fact that the contribution of quantum metric fluctuations has not been taken into account. Provided the semi-classical back reaction program leads qualitatively in the right direction (that is, towards a correct quantum gravity), one should include the spin-2 graviton contribution to the effective stress-energy tensor. One in fact expects the effects of linear gravitons to contribute a term of the same order to the stress tensor as those coming from ordinary matter and radiation fields. The corresponding calculations could be carried out once a renormalized effective energy-momentum tensor for quantized linear metric perturbations over a black hole background is worked out. However, at present, certain technical obstructions appear to make this a difficult task [11].

Lastly, one can use the results of the back reaction problem to explore the nature of the modified black hole spacetime by means of the effective potential [4], which determines the motion of test particles in the vicinity of the perturbed black hole, and to estimate the black hole free energy, of particular importance for assessing under what conditions the nucleation phase transition of black holes from hot flat space is likely to occur [14].

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