Using GENERATINGFUNCTIONOLOGY to Enumerate Distinct-Multiplicity Partitions

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In fond memory of Guru Herbert Saul WILF (28 Sivan 5691- 12 Tevet 5772) zecher gaon l’bracha

Preamble

About a year ago, Herb Wilf posed, on-line, eight intriguing problems. I don’t know the answer to any of them, but I will say something about the sixth question.

Herb Wilf 6th Question: Let \( T(n) \) be the set of partitions of \( n \) for which the (nonzero) multiplicities of its parts are all different, and write \( f(n) = |T(n)| \). See Sloane’s sequence A098859 for a table of values. Find any interesting theorems about \( f(n) \) . . .

First, I will explain how to compute the first few terms of \( f(n) \). Shalosh can easily get the first 250 terms, but as \( n \) gets larger it gets harder and harder to compute, unlike its unrestricted cousin, \( p(n) \). I conjecture that the fastest algorithm takes exponential time, but I have no idea how to prove that claim. I am impressed that, according to Sloane, Maciej Ireneusz Wilczynsk computed 508 terms.

Recall that the generating function for the number of integer partitions of \( n \) whose largest part is \( \leq m \), \( p_m(n) \), is the very simple rational function

\[
\sum_{n=0}^{\infty} p_m(n) q^n = \frac{1}{(1-q)(1-q^2) \cdots (1-q^m)}.
\]

The main purpose of this note is to describe, using Generatingfunctionology, so vividly and lucidly preached in W’s classic book [W2], how to compute the generating function (that also turns out to be rational) for the number of partitions of \( n \) whose largest part is \( \leq m \) and all its (nonzero) multiplicities are distinct, let’s call it \( f_m(n) \). As \( m \) gets larger, the formulas get more and more complicated, but we sure do have an answer, in the sense of the classic article [W3], for any fixed \( m \), but of course not for a symbolic \( m \).

Even more is true! Because, like \( \frac{1}{(1-q)(1-q^2) \cdots (1-q^m)} \), the generating function of \( f_m(n) \), \( \sum_{n=0}^{\infty} f_m(n) q^n \), turns out (as we will see) to only have roots-of-unity poles, whose highest order is \( m \), it follows that \( f_m(n) \) is a quasi-polynomial of degree \( m - 1 \) in \( n \). Now that’s a very good answer! (in W’s sense, albeit only for a fixed \( m \)).

1 Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA. zeilberg at math dot rutgers dot edu , http://www.math.rutgers.edu/~zeilberg/ . Jan. 18, 2012. Accompanied by Maple package DMP downloadable from http://www.math.rutgers.edu/~zeilberg/tokhniot/DMP . Sample input and output files may be viewed in the front of this article: http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/dmp.html . Supported in part by the USA National Science Foundation.
How to Compute Many terms of $f(n)$?

$p_m(n)$ is very easy to compute. For example, one may use the recurrence

$$p_m(n) = p_{m-1}(n) + \sum_{i=1}^{\lfloor n/m \rfloor} p_{m-1}(n-mi)$$

together with the initial condition $p_1(n) = 1$, $p_m(0) = 1$.

How can we adapt this in order to compute $f_m(n)$? The contribution from the partitions counted by $f_m(n)$ where $m$ does not show up is $f_{m-1}(n)$, in analogy with the $p_{m-1}(n)$ term in the above recurrence. But if $m$ does show up, it does so with a certain multiplicity, $i$, say, where $1 \leq i \leq \lfloor n/m \rfloor$, and removing these $i$ copies of $m$ results in a partition counted by $f_{m-1}(n-mi)$-so all its multiplicities are different- and in addition none of these multiplicities may be $i$. Continuing, we are forced to introduce a much more general discrete function $f_m(n;S)$ whose arguments are $m$ and $n$ and a set of “forbidden multiplicities”, $S$.

So let’s define $f_m(n;S)$ to be the number of partitions of $n$ with parts $\leq m$, with all its multiplicities distinct and none of these multiplicities belonging to $S$. Our intermediate object of desire, $f_m(n)$, is simply $f_m(n;\emptyset)$, and the ultimate object, $f(n)$, is $f_n(n;\emptyset)$.

The recurrence for $f_m(n;S)$ is, naturally

$$f_m(n;S) = f_{m-1}(n;S) + \sum_{i=1, i \notin S}^{\lfloor n/m \rfloor} f_{m-1}(n-im;S \cup \{i\})$$

because once we decided on the number of times $m$ shows up, let’s call it $i$, where $i$ is between 1 and $\lfloor m/m \rfloor$ and $i \notin S$, the partition (of $n-im$) obtained by removing these $i$ copies of $m$ must forbid the set of multiplicities $S \cup \{i\}$.

In the Maple package DMP, procedure qnS(n,m,S) implements $f_m(n;S)$ and procedure qn(n) implements $f(n)$.

Inclusion-Exclusion

Let $P_m(n)$ be the set of partitions of $n$ whose parts are all $\leq m$, in other words, the set that $p_m(n)$ is counting. Consider the set of all partitions whose largest part is $\leq m$, where we write a partition in frequency notation:

$$P_m := \{1^{a_1}2^{a_2} \ldots m^{a_m} \mid a_1, \ldots, a_m \geq 0\}$$

For example $1^32^54^2$ is the partition of twenty-one usually written as 4422222111. Introducing symbols $x_1, x_2, \ldots, x_m$, we define the Weight of a partition to be

$$\text{Weight}(1^{a_1}2^{a_2} \ldots m^{a_m}) := x_1^{a_1}x_2^{a_2} \cdots x_m^{a_m}$$
The weight-enumerator of $\mathcal{P}_m$ is, by ordinary-generating-functionology

$$
\text{Weight}(\mathcal{P}_m) = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_m)},
$$

since we make $m$ independent decisions:

- how many copies of 1? (Weight enumerator = $1 + x_1 + x_1^2 + \ldots = (1 - x_1)^{-1}$),
- how many copies of 2? (Weight enumerator = $1 + x_2 + x_2^2 + \ldots = (1 - x_2)^{-1}$),

... 
- how many copies of $m$? (Weight enumerator = $1 + x_m + x_m^2 + \ldots = (1 - x_m)^{-1}$).

But we want to find the weight-enumerator of the much-harder-to-weight-count set

$$
\mathcal{F}_m := \{1^{a_1}2^{a_2}\ldots m^{a_m} | a_1, \ldots, a_m \geq 0; a_i \neq a_j \text{ (if } a_i > 0, a_j > 0)\}.
$$

Calling the members of $\mathcal{F}_m$ good, we see that a member of $\mathcal{P}_m$ is good if it does not belong to any of the following $(\begin{pmatrix} m 
2 \end{pmatrix})$ sets, $S_{ij} \ 1 \leq i < j \leq m$:

$$
S_{ij} := \{1^{a_1}2^{a_2}\ldots m^{a_m} \in \mathcal{P}_m | a_i = a_j > 0\}.
$$

By inclusion-exclusion, the weight-enumerator of $\mathcal{F}_m$ is

$$
\sum_G (-1)^{|G|} \text{Weight} \left( \bigcap_{ij \in G} S_{ij} \right),
$$

where the summation ranges over all $2^{m(m-1)/2}$ subsets of $\{(i, j) | 1 \leq i < j \leq m\}$.

But the $G$'s can be naturally viewed as labeled graphs on $m$ vertices. Such a graph has several connected components, and together they naturally induce a set partition $\{C_1, C_2, \ldots, C_r\}$ of $\{1, 2, \ldots, m\}$. We have:

$$
\text{Weight} \left( \bigcap_{ij \in G} S_{ij} \right) = \prod_{i=1}^r \text{weight}(C_i),
$$

where if $|S| = 1$, $S = \{s\}$, say, then $\text{weight}(S) = \frac{1}{1-x_s}$, and if $|S| = d > 1$, $S = \{s_1, s_2, \ldots, s_d\}$, say, then

$$
\text{weight}(S) = \frac{x_{s_1}x_{s_2}\cdots x_{s_d}}{1-x_{s_1}x_{s_2}\cdots x_{s_d}}.
$$

To justify the latter, note that if vertices $s_1, s_2, \ldots, s_d$ all belong to the same connected component of our graph then, by transitivity, we have that all $a_{s_1} = a_{s_2} = \ldots = a_{s_d} > 0$, and the weight-enumerator is the infinite geometric series

$$
\sum_{\alpha=1}^{\infty} (x_{s_1} \cdots x_{s_d})^\alpha = \frac{x_{s_1}x_{s_2}\cdots x_{s_d}}{1-x_{s_1}x_{s_2}\cdots x_{s_d}}.
$$
But quite a few graphs correspond to any one set-partition. To find out the coefficients in front, for any set-partition \( \{C_1, C_2, \ldots, C_r\} \) of \( \{1, \ldots, m\} \) we must find
\[
\sum_G (-1)^{|G|},
\]
summed over all the graphs that gives rise to the above set partition. But this is the product of the analogous sums where one focuses on one connected component at a time, and then multiplies everything together.

Let’s digress and figure out \( \sum_G (-1)^{|G|} \) over all connected labeled graphs on \( n \) vertices. For the sake of clarity, let’s, more generally, figure out \( \sum_G y^{|G|} \) with a general variable \( y \).

By exponential-generatingfunctionology\([W2]\) (see also \([Z]\)), this sum is nothing but the coefficient of \( t^n/n! \) in
\[
\log \left( \sum_{i=0}^{\infty} (1 + y) \frac{t^i}{i!} \right). 
\]

Going back to \( y = -1 \), we see that we need the coefficient of \( t^n/n! \) in
\[
\log \left( \sum_{i=0}^{\infty} (1 - 1) \frac{t^i}{i!} \right) = \log \left( \sum_{i=0}^{\infty} 0 \frac{t^i}{i!} \right) = \log(1 + t)
\]
\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)! t^n}{n!}.
\]
So the desired sum is \( (-1)^{n-1}(n-1)! \).

Let’s define for any set of positive integers, \( S \),
\[
mishkal(S) = \begin{cases} 
1/(1 - x_s), & \text{if } |S| = 1 \text{ where } S = \{s\}; \\
(-1)^{d-1}(d-1)!/(1-x_{s_1}\cdots x_{s_d}),(1-x_{s_1}\cdots x_{s_d}), & \text{if } |S| = d > 1 \text{ where } S = \{s_1,\ldots,s_d\}.
\end{cases}
\]

For any set partition \( C = \{C_1,\ldots,C_r\} \) let’s define
\[
Mishkal(C) = mishkal(C_1)\cdots mishkal(C_r) .
\]

It follows that the weight-enumerator of \( \mathcal{F}_m \) according to \( \text{Weight}(1^{a_1}2^{a_2}\ldots m^{a_m}) := x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m} \) is
\[
\sum_C Mishkal(C) ,
\]
where the sum has \( B_m \) terms (\( B_m \) being the Bell numbers), one for each set-partition of \( \{1,\ldots,m\} \).

Finally, to get an “explicit” formula (as a sum of \( B_m \) terms, each a simple rational function of \( q \)), for the generating function \( \sum_{n=0}^{\infty} f_m(n)q^n \), all we need is replace \( x_i \) by \( q^i \), for \( i = 1 \ldots m \), getting
\[
\sum_{n=0}^{\infty} f_m(n)q^n = \sum_C Poids(C) ,
\]
where the sum has \( B_m \) terms (\( B_m \) being the Bell numbers), one for each set-partition of \( \{1,\ldots,m\} \).
where for a set partition \( C = \{C_1, \ldots, C_r\} \)

\[
Poids(C) = poids(C_1) \cdots poids(C_r) ,
\]

and where for an individual set \( S \):

\[
poids(S) = \begin{cases} 
1/(1-q^s), & \text{if } |S| = 1 \text{ where } S = \{s\}; \\
(-1)^{d-1}(d-1)!q^{s_1+\ldots+s_d}/(1-q^{s_1+\ldots+s_d}), & \text{if } |S| = d > 1 \text{ where } S = \{s_1, \ldots s_d\}.
\end{cases}
\]

It follows that indeed \( f_m(n) \) is a quasi-polynomial of degree \( m - 1 \) in \( n \). Furthermore, since the only pole that has multiplicity \( m \) is \( q = 1 \), it follows that the leading term (of degree \( m - 1 \)) is a pure polynomial.

The generating function, \( \sum_{n=0}^{\infty} f_m(n)q^n \), for any desired positive integer \( m \), is implemented in procedure \( \text{GFmq}(m,q) \) in the Maple package \( \text{DMP} \). For the Weight-enumerator (or rather with \( x_i \) replaced by \( q^ix_i \), for \( i = 1,\ldots,m \)), see \( \text{GFmxq}(m,x,q) \). Since the Bell numbers grow very fast, the formulas get complicated rather fast, but in principle we do have a very nice answer for any specific \( m \), but in practice, for large \( m \) it is only “nice” in principle. Of course it is anything but nice when viewed also as function of \( m \), and that’s why \( f(n) = f_n(n) \) is probably very hard to compute for larger \( n \).

To see the outputs of \( \text{GFmq}(m,q) \) for \( 1 \leq m \leq 8 \) see: http://www.math.rutgers.edu/~zeilberg/tokhniot/oDMP3.

**Asymptotics**

Recall that Hardy and Ramanujan tell us that as \( n \) goes to infinity, \( p(n) \) is asymptotic to \( \frac{1}{4n\sqrt{3}}exp(C\sqrt{n}) \) where \( C = \sqrt{2/3\pi} = 2.565099661 \ldots \), and hence \( \log p(n)/\sqrt{n} \) converges to \( C \). By looking at the sequence \( \log f(n)/\sqrt{n} \) for \( 1 \leq n \leq 508 \), it seems that this too converges to a limit, that appears to be a bit larger than 1.517 (but of course way less than 2.565099661 \ldots \). Let’s call that constant the Wilf constant.

The numerical evidence is here: http://www.math.rutgers.edu/~zeilberg/tokhniot/oDMP4.

Let me conclude with two challenges.

- Prove that the Wilf constant exists.
- Determine the exact value of the Wilf constant (if it exists) in terms of \( \pi \) or other famous constants. Failing this, find non-trivial rigorous lower and upper bounds.
References

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[Z] Doron Zeilberger, *Enumerative and Algebraic Combinatorics*, in: “Princeton Companion to Mathematics” , (Timothy Gowers, ed.), Princeton University Press, 550-561. Available from: http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/enu.pdf