Robust Online and Distributed Mean Estimation
Under Adversarial Data Corruption

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Abstract—We study robust mean estimation in an online and distributed scenario in the presence of adversarial data attacks. At each time step, each agent in a network receives a potentially corrupted data point, where the data points were originally independent and identically distributed samples of a random variable. We propose online and distributed algorithms for all agents to asymptotically estimate the mean. We provide the error-bound and the convergence properties of the estimates to the true mean under our algorithms. Based on the network topology, we further evaluate each agent’s trade-off in convergence rate between incorporating data from neighbors and learning with only local observations.

I. INTRODUCTION

Multi-agent cooperative learning scenarios, where agents in a network collect data and coordinate with each other to make inferences, have received significant attention in the research community. In the era of large-scale and decentralized systems, distributed learning has many applications in machine learning, data science, and decision making, e.g., multi-armed bandits [1], distributed correlation estimation [2], hypothesis testing [3], etc.

Due to the distributed nature of such systems, the data gathered by each agent may be corrupted in various ways (e.g., through adversarial attacks, or faulty sensor readings). Learning the true values and reaching consensus robustly and resiliently in the presence of corrupted data and misbehaving agents have been studied extensively in the literature. For example, [3] and [4] propose distributed learning algorithms that are robust against Byzantine attacks, [5] considers distributed heavy-tail stochastic bandit problems with robust mean estimators, and [6] analyzes resilient consensus in the presence of malicious agents.

Estimating the expected value is one of the most fundamental problems in statistics. Robust statistics address the problem of mean estimation under outliers and attacks, with classical estimators including median-of-means [7] and the trimmed mean [8], [9]. We refer the readers to the survey on robust mean estimation [10]. Early pioneering works on robust statistics attempt to estimate the mean given an outlier model [11], [12], and recent advancements consider stronger contamination models [13], [14].

In this paper, we study the problem of estimating the mean of a stream of corrupted data arriving at agents in the network. We design online and distributed algorithms to achieve robust estimation of the mean; these algorithms allow each agent to utilize its local data stream and information from neighbors to obtain accurate estimates of the true mean with high confidence, given a sequence of arbitrarily corrupted data under a strong contamination model.

Contributions: First, we provide two online algorithms that recursively utilize a modified trimmed mean estimator to estimate the mean from arbitrarily corrupted data (with an upper bound on the fraction of corrupted samples). Each of the two algorithms has different computational requirements and provides different performance guarantees. Second, we extend our online algorithm to a cooperative multi-agent setting, such that all the agents in the network can collaboratively estimate the mean (given a certain amount of arbitrarily corrupted data), and improve their estimates through communication with their neighbors. In all cases, we provide bounds on the estimation errors as a function of the number of samples and the fraction of corrupted samples.

II. PROBLEM FORMULATION

A. Data and Corruption Model

Let $X \in \mathbb{R}$ be a real-valued random variable that has finite variance $\sigma_X^2$. Let $\mu = \mathbb{E}[X]$ and define $\bar{X} = X - \mu$. The mean is assumed to be unknown, and we wish to estimate it through our algorithms. In this work, we assume that $X$ has an absolutely continuous distribution. At each time step $t \in \{1, 2, \ldots\}$, an independent and identically distributed copy of $X$ is collected. We denote the data point collected at time $t$ by $x_t$ and denote the set of data collected up to time $t$ as $\bar{X}_t = \{x_1, x_2, \ldots, x_t\}$.

We consider the existence of an adversary who can inspect all the samples and replace them with arbitrary values. We assume that due to a limited budget of the adversary, the total number of corrupted samples is at most $\eta t$ up to time $t$, where the corruption parameter $\eta \in (0, 1/16)$. We will discuss the range of the corruption in Section III. We say that a set of samples is $\eta$-corrupted if it is generated by the above process. We apply the tilde notation to any potentially corrupted data or dataset, e.g., $\tilde{x}_t$ is a potentially corrupted data and $\tilde{X}_t = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_t\}$ is a set of $\eta$-corrupted data up to time $t$.

B. Network Model

Consider a group of $m$ agents, $m \in \mathbb{N}$, with a known, unweighted, undirected, and connected communication graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, m\}$ is the set of vertices representing the agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. If an edge $(i, j) \in \mathcal{E}$, agent $i$ and $j$ can communicate with...
each other. The neighbors of agent $i \in \mathcal{V}$ and agent $i$ itself are represented by the set $\mathcal{N}_i = \{ j \in \mathcal{V} : (i,j) \in \mathcal{E} \cup \{ i \} \}$, which is the inclusive neighborhood of agent $i$.

At each time step $t$, each agent $i \in \mathcal{V}$ collects a potentially corrupted data point $\tilde{x}_{i,t}$. In this work, we consider a scenario where at each time step $t$, every agent in the network $i \in \mathcal{V}$ has at most $\eta t$ corrupted data.

This paper aims to first design online algorithms such that given any $\eta$-corrupted data arriving in a stream, the algorithms efficiently output good estimates of the mean of $X$ with high probability. Secondly, we formulate a distributed algorithm for each agent $i$ to calculate and update the estimates of the mean of data in real-time. Given a sequence of corrupted observations of agent $i$, denoted by $\{ \tilde{x}_{i,1}, \tilde{x}_{i,2}, \tilde{x}_{i,3}, \ldots \}$, the objective of $i$ is to perform an online inference of the mean $\mu_{i,t}$ at time $t$ by incorporating its local data and information from its neighbors. In particular, we want all agents to reach consensus on their estimates asymptotically, i.e., as $t \to \infty$, the estimate of each agent converges asymptotically to the agreement: $\lim_{t \to \infty} \hat{\mu}_{1,t} = \lim_{t \to \infty} \hat{\mu}_{2,t} = \ldots = \lim_{t \to \infty} \hat{\mu}_{n,t} \approx \mu$.

### III. Trimmed Mean Estimator

In this section, we introduce a modified version of the trimmed mean estimator in [14] on which we build our online and distributed algorithms. To simplify notations, we describe the estimator for any agent $i \in \mathcal{V}$ and omit the subscript $i$.

At time step $2t$, each agent has $2t$ samples after an adversary corrupts up to $2\eta t$ points from the original independent and identically distributed (i.i.d) samples of the random variable $X$. The samples are arbitrarily divided into two sets, denoted by $\bar{X}_t = \{ \tilde{x}_1, \ldots, \tilde{x}_t \}$ and $\bar{Y}_t = \{ \tilde{y}_1, \ldots, \tilde{y}_t \}$. Given the corruption parameter $\eta \in (0, 1/16)$ and a desired confidence level $\delta \in (0, 1)$, define

$$
\epsilon_t = 8\eta + 12 \frac{\log(4/\delta)}{t}.
$$

Note that for sufficiently large $t$, $\epsilon_t < 0.5$ since $\eta < 1/16$.

Let $\bar{y}_1^t \leq \bar{y}_2^t \leq \ldots \leq \bar{y}_t^t$ be a non-decreasing rearrangement of $\bar{y}_1, \ldots, \bar{y}_t \in \mathbb{R}$. Define $\alpha_t = \bar{y}_{r_t}$ and $\beta_t = \bar{y}_{(1-r_t) t}$ to be trimming values. Note that $\alpha$ and $\beta$ are both agent specific and time specific but we omit the agent subscript $i$ and time subscript $t$ temporarily for convenience. For $\alpha \leq \beta$, and $x \in \mathbb{R}$, define the trim estimator

$$
\phi_{\alpha, \beta}(x) = \begin{cases} 
\beta & \text{ if } x > \beta, \\
\alpha & \text{ if } x < \alpha,
\end{cases}
$$

The estimated mean after trimming is given by

$$
\hat{\mu}_t = \frac{1}{t} \sum_{j=1}^{t} \phi_{\alpha_t, \beta_t}(\tilde{x}_j).
$$

For $0 < p < 1$, define the quantile $Q_p(X) = \sup\{ M \in \mathbb{R} : P(X \geq M) \geq 1-p \}$. We have $P(X \geq Q_p(X)) = 1 - p$ and from Chebyshev’s inequality,

$$
P(\bar{X} \geq Q_p(\bar{X})) = 1 - p \leq \frac{\sigma^2}{Q_p^2(\bar{X})}
$$

and as a result,

$$
|Q_p(\bar{X})| \leq \frac{\sigma_X}{\sqrt{1-p}}.
$$

The estimation error is defined in [14] to be

$$
E(\epsilon_t, X) := \max\{ E(\bar{X} | 1_{\tau_t \leq Q_{1-\epsilon_t}/2}(\bar{X}) \}, E(\bar{X} | 1_{\tau_t \geq Q_{1-\epsilon_t}/2}(\bar{X}) \},
$$

where $1$ is the indicator function and it is equal to one when the condition is satisfied, and zero otherwise.

From [14], we have that for every $X$,

$$
E(\epsilon_t, X) \leq \sigma_X \sqrt{8\epsilon_t}.
$$

The following result upper bounds the trimming values using quantiles.

**Lemma 1 ([14]):** Consider the corruption-free sample $y_1, \ldots, y_t$. With probability at least $1 - 4e^{-c_t/12}$, the inequalities

$$
\{ i : y_i \geq \mu + Q_{1-2\epsilon_t}(\bar{X}) \} \geq 3/2 \epsilon_t t \\
\{ i : y_i \leq \mu - Q_{1-2\epsilon_t}(\bar{X}) \} \geq (1 - (3/4)\epsilon_t) t \\
\{ i : y_i \leq \mu + Q_{2\epsilon_t}(\bar{X}) \} \geq 3/2 \epsilon_t t \\
\{ i : y_i \geq \mu - Q_{2\epsilon_t}(\bar{X}) \} \geq (1 - (3/4)\epsilon_t) t
$$

hold simultaneously. We denote the event when the above four inequalities hold as event $A$. On event $A$, since corruption $\eta \leq \epsilon_t/8$, the following inequalities also hold

$$
Q_{1-2\epsilon_t}(\bar{X}) \leq \beta_t - \mu \leq Q_{1-\epsilon_t}(\bar{X}),
$$

$$
Q_{\epsilon_t/2}(\bar{X}) \leq \alpha_t - \mu \leq Q_{2\epsilon_t}(\bar{X}).
$$

The result below provides insight on the quality of the estimated mean at time $t$.

**Lemma 2 ([14]):** Let $\delta \in (0, 1)$ and $\delta \geq 4e^{-c_t}$. Using the trimmed mean estimator, with probability at least $1 - \delta$,

$$
|\hat{\mu}_t - \mu| \leq 3E(4\epsilon_t, X) + 2\sigma_X \sqrt{\log(4/\delta)}/t.
$$

### IV. Online Robust Mean Estimation

In the previous section, we provided an overview of the trimmed mean estimator for batch data. In this section, we provide update algorithms for each agent to estimate the mean from a sequence of data arriving in real-time.

**A. Fixed Trimming Thresholds**

We assume that each agent starts with a set of potentially $\eta$-corrupted data containing all the data available to the agent at time $t_0 \geq 1$, denoted as $\bar{X}_{i,t_0} = \{ \tilde{x}_{i,1}, \ldots, \tilde{x}_{i,t_0} \}$. At time $t = t_0$, each agent estimates the trimming thresholds $\alpha_{i,t_0}$ and $\beta_{i,t_0}$ following the procedures described in Section III. For all $t > t_0$, each agent starts the estimation process recursively using the previous estimates,

$$
\hat{\mu}_{i,t} = \frac{(t - t_0 - 1)\hat{\mu}_{i,t-1} + \phi_{\alpha_{i,t_0}, \beta_{i,t_0}}(\tilde{x}_{i,t})}{t - t_0}.
$$
Algorithm 1: Online Robust Mean Estimation - Fixed Trimming Thresholds

Initialization: $\hat{\mu}_{i,t_0} = 0$

Input: $t_0, \delta, \eta, \alpha, \beta$, a sequence of corrupted data $\tilde{x}_{i,t}$

for $t \leq t_0$ do
  Store data in $\hat{Y}_{i,t}$
  if $t = t_0$ then
    Compute $\epsilon_{t_0}$ as per (1)
    Sort data in $\hat{Y}_{i,t_0}$ and determine trimming thresholds $\alpha_{i,t_0}, \beta_{i,t_0}$
  end
  for $t > t_0$ do
    Update mean estimate as
    $$\hat{\mu}_{i,t} = \frac{(t-t_0-1)\hat{\mu}_{i,t-1} + \phi_{\alpha_{i,t_0}, \beta_{i,t_0}}(\tilde{x}_{i,t})}{t-t_0}$$
  end
end

We include the pseudo-code implementation for the online update in Algorithm 1.

We provide the following results that guarantee the quality of the estimates.

Theorem 1: Let $\delta \in (0, 1)$ be s.t. $\delta \geq 4e^{-(t-t_0)}, \forall t \geq 2t_0$. Following the procedures of Algorithm 1, with probability at least $1 - \delta$, the estimates of each agent $i \in \mathcal{V}$ satisfy

$$|\hat{\mu}_{i,t} - \mu| \leq 3\mathbb{E}(4\epsilon_{t_0}, X) + 2\sigma X \sqrt{\frac{\log(4/\delta)}{t-t_0}}, \forall t \geq 2t_0. \quad (10)$$

The proof of Theorem 1 is similar to Lemma 2 (batch algorithm). We include the proof in the Appendix of [15] for completeness.

Using the definition of $\epsilon_t$ in (1) and property (5) with Theorem 1, we obtain the result below on the convergence of the estimates.

Corollary 1: Let $\delta \in (0, 1)$ be s.t. $\delta \geq 4e^{-(t-t_0)}, \forall t \geq 2t_0$. Following the procedures of Algorithm 1, for any $\xi > 0$, there is a sufficiently large $t$, s.t. $\forall t \geq \xi$, the estimates of each agent $i \in \mathcal{V}$ satisfy the following inequality with probability at least $1 - \delta$,

$$|\hat{\mu}_{i,t} - \mu| \leq 24\sigma X \sqrt{4\eta + 6\frac{\log(4/\delta)}{t_0}} + \xi. \quad (11)$$

From the above results, we observe that the estimation error converges to the error introduced by initialization, i.e., $3\mathbb{E}(4\epsilon_{t_0}, X)$. Thus, it requires a relatively large $t_0$ to give reasonable estimates. However, Algorithm 1 is suitable when the memory of agent is limited, as the algorithm only requires $O(1)$ memory. If each agent has sufficient memory, maintaining a sorted list of data $\hat{Y}_t$ to update the trimming thresholds $\alpha_t$ and $\beta_t$ as new data arrives has advantages since the initial errors will converge to zero as data continuously stream in.

B. Updated Trimming Thresholds

In this subsection, we propose an alternative online algorithm that updates the trimming thresholds $\alpha_t$ and $\beta_t$ as new data point arrives, improving the performance of the trimming operator while sacrificing $O(t)$ memory and $O(\log t)$ computation. In comparison, the batch algorithm [14] considers static $\alpha$ and $\beta$ with a given dataset and the proposed online Algorithm 1 still uses static trimming values determined at $t_0$ but updates the mean recursively as data stream in.

We first describe the online algorithm and subsequently analyze the convergence properties and the trade-offs between the online and the batch algorithm.

At time step $2t$, the samples of each agent are arbitrarily divided into two sets, denoted $X_{i,t} = \{\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,t}\}$ and $\hat{Y}_{i,t} = \{\hat{y}_{i,1}, \ldots, \hat{y}_{i,t}\}$. In order to simplify the notation and discussion, we assume two data points $\tilde{x}_{i,t}$ and $\hat{y}_{i,t}$ arrive at each time step $t$.

Each agent will store $\hat{Y}_{i,t}$ as a binary search tree (BST), denoted $\text{BST}(\hat{Y}_{i,t})$. At each time step $t$, each agent updates $\epsilon_t$ following (1) and updates the trimming thresholds $\hat{\alpha}_{i,t} = \hat{y}_{i,t}^{\epsilon_t}$ and $\hat{\beta}_{i,t} = \hat{y}_{i,t}^{(1-\epsilon_t)}$ in $O(\ln t)$ complexity with the BST data structure. Subsequently, each agent performs a recursive local update using its previous estimates $\hat{\mu}_{i,t-1}$ and applying the new trimming thresholds $\hat{\alpha}_{i,t}, \hat{\beta}_{i,t}$ on data $\hat{x}_{i,t}$:

$$\hat{\mu}_{i,t} = \frac{(t-1)\hat{\mu}_{i,t-1} + \phi_{\hat{\alpha}_{i,t}, \hat{\beta}_{i,t}}(\tilde{x}_{i,t})}{t}. \quad (12)$$

The pseudo-code is provided in Algorithm 2 below.

Algorithm 2: Online Robust Mean Estimation - Updated Trimming Thresholds

Initialization: $\hat{\mu}_{i,0} = 0$

Input: $\delta, \eta$, a sequence of corrupted data $\hat{x}_{i,t}$ and $\hat{y}_{i,t}$

for $t \in \mathbb{N}$ do
  Update BST($\hat{Y}_{i,t}$) with $\hat{y}_{i,t}$
  Compute $\epsilon_t$ as per (1)
  Update $\hat{\alpha}_{i,t}$ and $\hat{\beta}_{i,t}$
  $$\hat{\mu}_{i,t} = \frac{(t-1)\hat{\mu}_{i,t-1} + \phi_{\hat{\alpha}_{i,t}, \hat{\beta}_{i,t}}(\tilde{x}_{i,t})}{t}$$
end

We provide the following results that guarantee the quality of the estimates of Algorithm 2.

Theorem 2: Following the procedures of Algorithm 2, for all sample paths in a set of measure 1, there exists a finite time $\tilde{t}$, such that for all $t \geq \tilde{t}$, the error between the estimated mean and the true mean of any agent $i \in \mathcal{V}$, denoted as $|\hat{\mu}_{i,t} - \mu|$, satisfies the error bound

$$|\hat{\mu}_{i,t} - \mu| \leq \frac{\tilde{t}}{t} |\hat{\mu}_{i,t} - \mu| + \frac{1}{t} \sum_{j=t+1}^{t} \frac{\sigma X}{4\eta + 6\log(4/\delta)} \quad (13)$$

Proof: To simplify notation, we omit the subscript $i$ denoting the agent in the proof. Let $\phi_t = \phi_{\hat{\alpha}_t, \hat{\beta}_t}$ denote the updated trimming operator at time step $t \in \mathbb{N}$.

Recall that the inequalities in Lemma 1 hold simultaneously with probability at least $1 - 4 \exp\{-\epsilon_t/12\}, \forall t \in \mathbb{N}$. Define a bad event at time $t$ to be the event that any inequalities in (6) or (7) do not hold. Define the random variable $B_t$, with $B_t = 1$ if the bad event occurs at the given time $t$, and 0 otherwise. Let $B = \sum_{t=1}^{T} B_t$ be the number of bad events up to time $t$. Summing up the probability of
bad events, we have
\[ \sum_{t=1}^{T} \mathbb{P}[B_j] \leq 4 \sum_{j=1}^{T} e^{-\epsilon_j t / 12}. \]  \tag{14} \]

As \( t \to \infty \), it can be shown that the series is convergent, i.e., \( \sum_{j=1}^{T} \mathbb{P}[B_j] < \infty \). From the Borel-Cantelli lemma, the probability of infinitely many bad events occurring is 0. Thus, for a set of sample paths of measure 1, there exists a sample path dependent finite time \( t \) such that no more bad events occur for \( t \geq \bar{t} \).

Rewriting the update for Algorithm 2 with finite time \( \bar{t} \), the deviation between the estimated mean and the true mean can be expressed as
\[ \left| \hat{\mu}_{\bar{t}} - \mu \right| = \left| \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} \phi_j(\tilde{x}_j) + \sum_{j=1}^{\bar{t}} \phi_j(\tilde{x}_j) - \mu \right| \leq \left| \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} \phi_j(\tilde{x}_j) - \mu \right| + \left| \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} \phi_j(\tilde{x}_j) - \frac{t - \bar{t}}{t} \mu \right|. \]  \tag{15} \]

Since (6) holds for all \( t \geq \bar{t} \), from (3) and (1), we have \( \phi_j(\tilde{x}_j) \leq \beta_t \leq Q_{1-\epsilon / 2} + \mu \leq -\frac{\sigma x}{\sqrt{\eta}} + \mu \leq -\frac{\sigma x}{\sqrt{\eta}} + \mu = -\frac{\sigma x}{\sqrt{\eta}} + \mu \). Let \( U_t = \frac{\sigma x}{\sqrt{4 \eta + 6 \epsilon (2 \eta / 12)}} \). We can see that \( U_{t-1} \leq U_t \leq \frac{\sigma x}{\sqrt{4 \eta}} \). Using \( U_t \), we can provide an upper bound
\[ \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} \phi_j(\tilde{x}_j) \leq \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} (U_j + \mu) = \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} U_j + \frac{t - \bar{t}}{t} \mu. \]

Similarly with the lower tail, \( \phi_j(\tilde{x}_j) \geq \alpha_t \geq Q_{1-\epsilon / 2} + \mu \geq -\frac{\sigma x}{\sqrt{\eta}} + \mu \). Thus, we have
\[ \frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} \phi_j(\tilde{x}_j) \geq -\frac{1}{\bar{t}} \sum_{j=1}^{\bar{t}} U_j + \frac{t - \bar{t}}{t} \mu. \]

The above result provides a closed-form upper bound on the quality of the estimates from Algorithms 2. We can observe from the below corollary, that deviation of the estimate and the true mean is indeed bounded as \( t \to \infty \).

**Corollary 2:** Following the procedures of Algorithm 2, for all sample paths in a set of measure 1, as \( t \to \infty \), the estimates of each agent satisfy
\[ \limsup \left| \hat{\mu}_{i,t} - \mu \right| \leq \frac{\sigma x}{2 \sqrt{\eta}}. \]  \tag{16} \]

In order to make inferences in real-time, additional costs in accuracy occur from Algorithm 2. As observed from (13), the error between the estimates of Algorithm 2 and the true mean has a term \( \frac{t}{\bar{t}} |\hat{\mu}_{i,t} - \mu| \), introduced by the accumulated error from trimming when \( t \) is relatively small. However, from Corollary 2, we see that asymptotically, the estimates are only influenced by the corruption rate \( \eta \). With the sacrifice of \( O(t) \) space and additional \( O(\ln t) \) computation complexity, Algorithm 2 eliminates the initialization error.

**V. DISTRIBUTED ROBUST MEAN ESTIMATION**

Previously, we provide algorithms that allow each agent in the network to update their estimates of the mean in an online manner, using only local data. In this section, we propose a multi-agent distributed algorithm that enables the agents to estimate the mean collaboratively (i.e., by exchanging information with each other). First, we describe the proposed algorithm. Subsequently, we analyze the convergence properties and demonstrate the improvement in the learning rate.

To simplify analysis, an agent \( o \in \mathcal{V} \) is randomly selected initially, whose trimming values \( \alpha_o \) and \( \beta_o \) are eventually transmitted to all agents in the network, i.e., \( \alpha_i = \alpha_o \) and \( \beta_i = \beta_o \), \( \forall i \in \mathcal{V} \). Depending on the algorithm used, \( \alpha_o \) and \( \beta_o \) can be fixed or updated at each time step.

At the beginning of each time step \( t \), upon receiving a new data point \( \tilde{x}_{i,t} \), agent \( i \) updates its local estimate \( \hat{\mu}_{i,t} \) using either Algorithm 1 or Algorithm 2, determined by the system designer according to hardware conditions and applications. After the local update, each agent transmits its estimated mean \( \hat{\mu}_{i,t} = \hat{\mu}_{i,t}^0 \) to its neighbors. Here, we use superscript \( k \) in \( \hat{\mu}_{i,t}^k \) to denote the rounds of communications. At each time step, each agent can communicate \( K \in \mathbb{N} \) times with its neighbors, updating its local estimates following
\[ \hat{\mu}_{i,t}^{k+1} = \sum_{j \in \mathcal{N}_i} \frac{\hat{\mu}_{j,t}^k}{|\mathcal{N}_i|}, \quad k = 0, \ldots, K - 1. \]  \tag{17} \]

We provide the pseudo-code in Algorithm 3. We only include the online local update from Algorithm 1 as the distributed algorithm can be modified for Algorithm 2.

**Algorithm 3:** Distributed Robust Mean Estimation of Agent \( i \)

**Initialization:** \( \hat{\mu}_{i,t_0}^0 = 0 \)

**Input:** \( t_0, \delta, \eta, K \), a sequence of corrupted data \( \tilde{x}_{i,t} \)

for \( t \in \{ t_0, t_0 + 1, \ldots \} \) do

if \( t < t_0 \) then

Store data in \( \hat{Y}_{i,t} \)

if \( t = t_0 \) then

if \( i = 0 \) then

Compute \( \epsilon_o \) as per (1)

Sort data in \( \hat{Y}_{i,t_0} \), determine and transmit trimming thresholds \( \alpha_{o,t_0}, \beta_{o,t_0} \)

else

Receive and transmit \( \alpha_{o,t_0}, \beta_{o,t_0} \)

if \( t > t_0 \) then

Update mean as

\[ \hat{\mu}_{i,t}^0 = \left( t - t_0 - 1 \right) \hat{\mu}_{i,t-1}^K + \phi_{\alpha_{o,t_0},\beta_{o,t_0}}(\tilde{x}_{i,t}) \]

for \( k = 1 : K \) do

Transmit \( \hat{\mu}_{i,t}^{k-1} \) to neighbors

Receive \( \hat{\mu}_{k-1}^{k-1} \) from neighbors

Update mean estimate \( \hat{\mu}_{i,t}^k = \frac{\sum_{j \in \mathcal{N}_i} \hat{\mu}_{j,t}^{k-1}}{|\mathcal{N}_i|} \)

end for

end if

end if

end if

end for

end do

end Algorithm 3

We analyze the theoretical properties of the proposed distributed algorithm. First, we rewrite the update (17) as
\[ \hat{\mu}_{i,t}^{k+1} = (I + D)^{-1}(I + A)\hat{\mu}_{i,t}^k = P\hat{\mu}_{i,t}^k, \]  \tag{19} \]
where \( \hat{\mu}_k^t \in \mathbb{R}^m \) is a vector of all agents’ estimates at communication round \( k \) of time step \( t \), \( I \in \mathbb{R}^{m \times m} \) is the identity matrix, \( D \) is the degree matrix, \( A \) is the adjacency matrix of the communication network, and finally, \( P = (I + D)^{-1}(I + A) \) is referred to as the Perron matrix [16]. At each fixed time step \( t \), the group decision value for all agents, denoted \( \hat{\mu}_t \), is the average of all initial mean estimates \( \hat{\mu}_0 \) at time \( t \). It can be shown that for Algorithm 1, the consensus value at time \( t \) is

\[
\hat{\mu}_t = \frac{1}{m(t-t_0)} \sum_{i=1}^{m} \sum_{j=t_0+1}^{t} \phi_{\alpha_{o_i},\beta_{o_j}}(\hat{x}_{i,j}). \tag{20}
\]

If all agents apply Algorithm 2, the consensus value at time \( t \) can be expressed as

\[
\hat{\mu}_t = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=t_0+1}^{t} \phi_{\alpha_{o_i},\beta_{o_j}}(\tilde{x}_{i,j}). \tag{21}
\]

We provide the following lemmas that bound the error of the group decision mean value to the true mean. The proofs are included in the Appendix of [15].

**Lemma 3:** Let \( \delta \in (0,1) \) be s.t. \( \delta \geq 4e^{-(t-t_0)} \), \( \forall t \geq 2t_0 \). Following Algorithm 1, the consensus value of the entire network \( \hat{\mu}_t \) satisfies the below condition with probability at least \( 1 - \delta \),

\[
|\hat{\mu}_t - \mu| \leq 3E(4\epsilon_{o,t}, X) + 2\sigma X \sqrt{\frac{\log(4/\delta)}{m(t-t_0)}} \quad \forall t \geq 2t_0. \tag{22}
\]

From the results above, we see that the convergence rate of the final estimates of each agent to the true mean is increased by a factor of \( \sqrt{1/m} \). In Algorithm 2, since the upper bound is loose, we do not observe a rate improvement from the following result.

**Lemma 4:** Following Algorithm 2, for all sample paths in a set of measure 1, there exists a finite time \( \tilde{t} \), such that for all \( t \geq \tilde{t} \), the consensus value of the entire network \( \tilde{\mu}_t \) satisfies condition

\[
|\tilde{\mu}_t - \mu| \leq \frac{\tilde{t}}{t} |\hat{\mu}_t - \mu| + \frac{1}{\tilde{t}} \sum_{j=t+1}^{t} \frac{\sigma X}{\sqrt{4\eta + 6\log(4/\delta)}}. \tag{23}
\]

In order to analyze the convergence properties of the multi-agent algorithm, we will require the classical result below.

**Lemma 5 ([16]):** Following (19), a discrete consensus is globally exponentially reached with a speed that is faster or equal to \( \lambda \) for a connected undirected network, where \( 0 < \lambda < 1 \) is the second largest eigenvalue of the matrix \( P \).

The error of the estimates is given by

\[
|\hat{\mu}_t^K - \hat{\mu}_t| \leq c\lambda^K |\hat{\mu}_0^K - \hat{\mu}_t|, \tag{24}
\]

where \( c \) is some constant.

Below, we present the bound of error for the proposed distributed algorithms and include the proof in the Appendix of [15].

**Theorem 3:** For all \( t \geq 2t_0, \forall k \in \{1,2,\ldots,K\} \), and \( \forall i \in \mathcal{V} \), let \( \delta \in (0,1) \) be s.t. \( \delta \geq 4e^{-(t-t_0)} \). When agents follow Algorithm 1, the estimated mean of each agent satisfies the following condition with probability at least \( 1 - \delta \),

\[
|\hat{\mu}_{i,t}^K - \mu| \leq (c\lambda^K)^{t-t_0}|\hat{\mu}_{i,t_0}^K - \hat{\mu}_{t_0}|
+ \sum_{j=t_0+1}^{t} (c\lambda^K)^{t-j+1} \frac{2\max(|\alpha_{o_{i,j}}|,|\beta_{o_{i,j}}|)}{j-t_0}
+ 3E(4\epsilon_{o,t}, X) + 2\sigma X \sqrt{\frac{\log(4/\delta)}{m(t-t_0)}}. \tag{25}
\]

**Corollary 3:** Let \( \delta \in (0,1) \) be s.t. \( \delta \geq 4e^{-(t-t_0)} \), \( \forall t \geq 2t_0 \). For all \( k \in \{1,2,\ldots,K\} \) and \( \forall K \in \mathbb{N} \) s.t. \( c\lambda^K < 1 \), for any \( \xi > 0 \), there is a sufficiently large \( \tilde{t} \), \( \forall t \geq \tilde{t} \), when agents \( i \in \mathcal{V} \) follow Algorithm 1, the estimated mean of each agent satisfies the following condition with probability at least \( 1 - \delta \),

\[
|\hat{\mu}_{i,t}^K - \mu| \leq 24\sigma X \sqrt{4\eta + 6\log(4/\delta)/t_0} + \xi. \tag{26}
\]

**Proof:** From (25), as \( t \to \infty \), the first term goes to zero. The second term is an instance of Lemma 6 (see Appendix of [15]), where \( \zeta = c\lambda^K < 1 \) and \( \gamma = \max\{\alpha_{o_{i,j}},\beta_{o_{i,j}}\} \to 0 \). As \( t \to \infty \), the fourth term also converges to zero. Using the previous results and technique of proofs, we arrive at the convergence properties if all agents follow Algorithm 2.

**Theorem 4:** For all sample paths in a set of measure 1, there exists a finite time \( \tilde{t} \), such that \( \forall t \geq \tilde{t} \), \( \forall K \in \{1,2,\ldots,K\} \), if all agents \( i \in \mathcal{V} \) follow Algorithm 2, the estimated mean of each agent satisfies

\[
|\hat{\mu}_{i,t}^K - \mu| \leq (c\lambda^K)^{t-t_0}|\hat{\mu}_{i,t_0}^K - \hat{\mu}_{t_0}|
+ \sum_{j=t_0+1}^{t} (c\lambda^K)^{t-j+1} \frac{2\max(|\alpha_{o_{i,j}}|,|\beta_{o_{i,j}}|)}{j}
+ \frac{t}{\tilde{t}} |\bar{\mu}_t - \mu| + \frac{1}{\tilde{t}} \sum_{j=t+1}^{t} \frac{\sigma X}{\sqrt{4\eta + 6\log(4/\delta)}}. \tag{27}
\]

**Proof:** For all \( t \geq \tilde{t} + 1 \), we have \( |e_t| \leq 2\max\{\alpha_{o_{i,j}},\beta_{o_{i,j}}\} \) and \( |e_t| \to 0 \) as \( t \to \infty \). The remaining proof is similar to Theorem 3.

**Corollary 4:** For all \( k \in \{1,2,\ldots,K\} \), and \( \forall K \in \mathbb{N} \) such that \( c\lambda^K < 1 \), if all agents \( i \in \mathcal{V} \) follow Algorithm 2, for all sample paths in a set of measure 1, as \( t \to \infty \), the estimated mean of each agent satisfies

\[
\limsup_{t \to \infty} |\hat{\mu}_{i,t}^K - \mu| \leq \frac{\sigma X}{2\sqrt{\eta}}. \tag{28}
\]

From the above results, the asymptotic rate at which \( \hat{\mu}_{i,t}^K \) converges to the true mean is dominated by the convergence of the data and the amount of corruption. Also, compared to the centralized algorithm, increasing the total rounds of communication \( K \) of each time step will decrease the deviation of each agent’s estimates to the centralized estimates. As time \( t \to \infty \), the estimation errors converge to that of the centralized algorithms, with a rate improvement of \( \sqrt{1/m} \) for Algorithm 1.
VI. SIMULATIONS

In this section, we demonstrate the performance of our proposed algorithms via simulations. We generate data points from a Gaussian distribution \( G_1 \) with \( \mu_1 = 0, \sigma_1^2 = 1 \). We then let the dataset to be \( \eta = 0.02 \) corrupted with data sampled from another Gaussian distribution \( G_2 \) where \( \mu_2 = 100 \) and \( \sigma_2^2 = 1 \). We let \( \delta = 0.3 \).

In Fig. 1, we plot the estimated mean of Algorithm 1 and 2 over the number of data points used for the estimation \((t_0 = 100)\). Algorithm 2 obtains a better estimate asymptotically. The estimated mean of Algorithm 1 is 0.095 and the estimated mean of Algorithm 2 is \(-0.016\), given 1000 data points for estimations. Both algorithms achieve good results as without the trimmed operator, the estimated mean is 1.98.

We observe that Algorithm 1 converges to the error introduced by initialization while with additional computation and memory complexity, Algorithm 2 obtains better estimates.

![Fig. 1: Estimated mean of Algorithm 1 and 2](image)

We next demonstrate the distributed algorithm using Algorithm 1 with a network of \( m = 5 \) agents. In Fig. 2 and Fig. 3, we show the convergence of the absolute errors between the estimates and the true mean for each agent with different \( K \) values (the number of communication rounds per data point), where \( K = 0 \) means no communication. All agents’ estimates converge to the true mean with some error, and as \( K \) increases, the estimates converge to the estimates of the centralized algorithm at a faster rate.

VII. CONCLUSIONS AND FUTURE WORKS

In this work, we proposed online adaptations for robust mean estimation from arbitrarily corrupted data. We subsequently proposed distributed algorithms for agents to collaboratively estimate the mean. We analyzed the convergence properties of the proposed algorithms. However, we only considered a simple case when the corruption level of each agent is upper bounded by \( \eta \). In the future, we will consider \( \eta \)-corrupted global data sets, in which case the data of some agents can be entirely corrupted. Additionally, we would like to consider the multi-dimensional mean estimation problem, with the presence of partially observing agents that are receiving data from a subset of the dimensions.

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