A REMARK ON QUANTUM HOCHSCHILD HOMOLOGY

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Abstract. Beliakova-Putyra-Wehrli studied various kinds of traces, in relation to annular Khovanov homology \cite{BPW19}. In particular, to a graded algebra and a graded bimodule over it, they associate a quantum Hochschild homology of the algebra with coefficients in the bimodule, and use this to obtain a deformation of the annular Khovanov homology of a link. A spectral refinement of the resulting invariant was recently given by Akhmechet-Krushkal-Willis \cite{AKW19}.

In this short note we observe that quantum Hochschild homology is a composition of two familiar operations, and give a short proof that it gives an invariant of annular links, in some generality. Much of this is implicit in \cite{BPW19}.

Definition 1. \cite[Section 3.8.5]{BPW19} Let $A$ be a graded ring, $M$ a chain complex of graded $A$-bimodules (so $M$ is bigraded), and $q \in A$ an invertible central element with grading 0. The quantum Hochschild complex of $A$ with coefficients in $M$ and parameter $q$ has $q\text{CH}_n(A; M) = M \otimes_Z A^\otimes_{zn}$ and differential

$$
\partial (m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n 
+ (-1)^n q^{-|a_n|} a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}.
$$

The homology of this complex is the quantum Hochschild homology $q\text{HH}_\bullet(A; M)$ of $A$ with coefficients in $M$ and parameter $q$.

The goal of this note is to reformulate this operation and deduce that it often leads to annular link invariants. The data of $A$ and $q$ specifies a ring homomorphism $f_q : A \to A$ defined on homogeneous elements $a$ of $A$ by

$$f_q(a) = q^{-|a|} a,$$

where $|a|$ denotes the grading of $a$. We can twist the left action of the $A$-bimodule $M$ by $f_q$ to obtain a new bimodule $f_q M$ which is equal to $M$ as a right $A$-module and has left action given by the composition $A \otimes_Z f_q M \xrightarrow{f_q \otimes id} A \otimes M \xrightarrow{m} M = f_q M$. This operation is a special case of tensor product:

$$f_q M \cong f_q A \otimes_A M.$$

Our first observation is:

Proposition 2. The quantum Hochschild homology of $A$ with coefficients in $M$ is isomorphic to the ordinary Hochschild homology of $A$ with coefficients in $f_q M$.

Proof. This is immediate from the definitions. \hfill $\square$
Call a chain complex of graded $A$-bimodules $M$ weakly central if for any graded $A$-bimodule $N$ there is a quasi-isomorphism $M \otimes_A^L N \simeq N \otimes_A^L M$.

**Lemma 3.** The bimodule $f_q A$ is weakly central.

**Proof.** The isomorphism $M \otimes_A^L f_q A \rightarrow f_q A \otimes M$ sends $m$ to $q^{-|m|} m$. □

We turn next to annular link invariants. Consider the category $\text{Tan}$ with one object for each even integer and $\text{Hom}(2m, 2n)$ given by the set of isotopy classes of $(2m, 2n)$-tangles (embedded in $D^2 \times [0, 1]$). Given a (graded) algebra $A$, a very weak action of $\text{Tan}$ on the category of $A$-modules is a choice of quasi-isomorphism class of chain complex of (graded) $A$-bimodules $C(T)$ for each $T \in \text{Hom}(2m, 2n)$ so that $C(T_2 \circ T_1)$ is quasi-isomorphic to $C(T_2) \otimes_A^L C(T_1)$. For example, if we take $A$ to be the direct sum of the Khovanov arc algebras [Kho02] then Khovanov defined a very weak action of $\text{Tan}$ on $A_{\text{Mod}}$, and if we define $A$ to be the direct sum of the Chen-Khovanov algebras [CK14] then Chen-Khovanov defined a very weak action of $\text{Tan}$ on $A_{\text{Mod}}$. (In fact, in both cases, they did more; cf. Remark 6.)

Any $(2n, 2n)$-tangle $T \subset D^2 \times [0, 1]$ has an annular closure in $D^2 \times S^1$.

**Proposition 4.** Fix a very weak action of $\text{Tan}$ on $A_{\text{Mod}}$ and a weakly central $A$-bimodule $P$. Then for any $(2n, 2n)$-tangle $T$, the isomorphism class of $\text{HH}^*_A(A; C(T) \otimes_A^L P)$ is an invariant of the annular closure of $T$.

**Proof.** This is immediate from the definitions and the trace property of Hochschild homology, i.e., that given $A$-bimodules $M$ and $N$,

$$\text{HH}^*_A(A; M \otimes_A^L N) \cong \text{HH}^*_A(A; N \otimes_A^L M).$$

**Corollary 5.** Up to isomorphism, the quantum Hochschild homology of the Chen-Khovanov bimodule associated to a $(2n, 2n)$-tangle $T$ is an invariant of the annular closure of $T$.

**Proof.** This is immediate from Lemma 3, Proposition 4, and the fact that the Chen-Khovanov bimodules induce a very weak action of $\text{Tan}$ [CK14]. □

**Remark 6.** To keep this note short, we have not discussed functoriality of these annular link invariants under annular cobordisms. To do so, one replaces $\text{Tan}$ by an appropriate 2-category of tangles and weak centrality by a notion keeping track of the isomorphisms. See [BPW19] for further discussion.

**References**

[AKW19] Rostislav Akhmechet, Vyacheslav Krushkal, and Michael Willis, *Stable homotopy refinement of quantum annular homology*, 2019, arXiv:2001.00077.

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