Abstract

A solution to the dark matter problem is set forth in the framework of reductive semi-classical gravity, i.e., semiclassical gravity involving quantum state reduction. In that theory, the Einstein equation includes the energy-momentum tensor originating from pseudomatter and partially compensating for quantum jumps of the matter energy-momentum tensor. The compensation ensures the continuity of metric and of its first time derivative. Pseudomatter is actualized as pseudodust and perceived as a dark matter. The necessity of compensating for quantum jumps makes pseudomatter, i.e., dark matter of such a form, an indispensable rather than ad hoc element of the theory. Applications: The Schwarzschild solution with pseudomatter, pseudomatter halo, collapse involving pseudomatter, pseudomatter in the FLRW universe.

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**Introduction**

It is commonly known that astrophysics and cosmology are being confronted with the problem of dark matter [1–6]. In the framework of general relativity, which is a totally classical theory, the problem is a riddle: dark matter appears to be an entirely arbitrary element—what does it exist for?

Meanwhile, reductive semiclassical gravity, i.e., semiclassical gravity incorporating quantum state reduction involves an indispensable element, namely, pseudomatter, which is actualized as pseudodust and perceived as a dark matter [7].

In reductive semiclassical gravity, the Einstein equation includes the energy-momentum tensor originating from pseudomatter. That tensor partially compensates for jumps of the matter energy-momentum tensor resulting from state quantum jumps. The compensation ensures the continuity of metric and of its first time derivative, which is necessary for the Einstein equation to be fulfilled. It cannot be too highly stressed that in reductive semiclassical gravity dark matter in the form of pseudomatter is an indispensable rather than ad hoc element of the theory.

The idea of introducing a nonmaterial field into the Einstein equation had been advanced in [8]. The idea was realized in the form of pseudomatter in [9] and further developed in [10,7].

In this paper, the concept of pseudomatter is elaborated in detail.

Some applications of the concept to astrophysics and cosmology are given: the Schwarzschild solution with a realistic matter state equation due to pseudomatter, pseudomatter halo (an important result: the velocity $v = \text{const}$), collapse involving pseudomatter, pseudomatter in the FLRW universe (an essential result: the pseudomatter density $\varepsilon \propto 1/R$ where $R$ is the radius of the universe).

### 1 Fundamentals: Reductive semiclassical gravity

#### 1.1 Semiclassical gravity

Semiclassical gravity is based on the semiclassical Einstein equation and the Schrödinger equation.

For what follows, it is convenient to introduce the tensor

$$E_{\mu\nu} = G_{\mu\nu} - \Lambda g_{\mu\nu} - 8\pi \kappa T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3$$

(1.1.1)

where $G$ is the Einstein tensor, $\Lambda$ is the cosmological constant, $\kappa$ is the gravitational constant, $g$ is metric, $T$ is the energy-momentum tensor; $c = 1$, $\hbar = 1$, so that $\kappa = T_P^2$ ($T_P$ is the Planck time). $E$ may be called the dynamical Einstein tensor. Next,

$$T_{\mu\nu} = (\Psi, \hat{T}_{\mu\nu} \Psi)$$

(1.1.2)

where $\hat{T}_{\mu\nu}$ is the energy-momentum tensor operator and $\Psi$ is a state vector of quantized matter, i.e., quantum fields $\hat{\phi}$. Thus

$$G_{\mu\nu} = G_{\mu\nu}[g]$$

(1.1.3)

$$\hat{T}_{\mu\nu} = \hat{T}_{\mu\nu}[g, \hat{\phi}]$$

(1.1.4)
The semiclassical Einstein equation is

$$E = 0 \quad E_{\mu \nu} = 0 \quad (1.1.5)$$

The Schrödinger equation is

$$\frac{d\Psi}{dt} = -i\hat{H}\Psi \quad (1.1.6)$$

where $\hat{H}$ is the Hamiltonian and $t = x^0$ is a global time coordinate.

1.2 Quantum state reduction and violation of the Einstein equation

Quantum state reduction is a jump of the state vector:

$$\Psi_{\text{before jump}} =: \Psi^{<}\text{jump} \rightarrow \Psi^{>}: = \Psi_{\text{after jump}} \quad (1.2.1)$$

The jump of $\Psi$ results in that of $T$:

$$\Delta T = (\Psi^{>}, \hat{T}\Psi^{>}) - (\Psi^{<}, \hat{T}\Psi^{<}) \quad (1.2.2)$$

under the assumption that $\hat{T}$ is continuous. The discontinuity of $T$ causes a violation of the semiclassical Einstein equation (1.1.5). Indeed, the latter may be written as

$$G_{ij} - \Lambda g_{ij} - 8\pi \kappa T_{ij} = 0, \quad i, j = 1, 2, 3 \quad (6 \text{ equations}) \quad (1.2.3)$$

$$G_{0\mu} - \Lambda g_{0\mu} - 8\pi \kappa T_{0\mu} = 0, \quad \mu = 0, 1, 2, 3 \quad (4 \text{ equations}) \quad (1.2.4)$$

The space components of the Einstein tensor, $G_{ij}$, involve the second time derivatives $\ddot{g}_{kl}$ of the metric tensor components $g_{kl}$ [11,12]. It is reasonable to assume that quantum jumps of $T_{ij}$ result in those of $\ddot{g}_{kl}$. That is quite conceivable from the physical point of view: A jump of force ($T_{ij}$) causes a jump of acceleration ($\ddot{g}_{kl}$). On the other hand, the time/time-space components $G_{0\mu}$ involve only the first time derivatives $\dot{g}_{kl}$ [11,12], so that jumps of $T_{0\mu}$ do harm: a jump of $\dot{g}_{kl}$ results in the $\delta$-function in $\ddot{g}_{kl}$, which violates (1.2.3). (A jump of $g$ would result in the $\delta$-function in $\dot{g}$.)

1.3 Compensatory field: Pseudomatter

In order to compensate for the jumps of $G_{0\mu}$, it is necessary to extend (1.1.5). Put

$$E_{\mu \nu} = 8\pi \kappa T_{ps \mu \nu} \quad (1.3.1)$$

where $ps$ stands for pseudomatter, so that

$$G_{\mu \nu} - \Lambda g_{\mu \nu} = 8\pi \kappa (T_{\mu \nu} + T_{ps \mu \nu}) \quad (1.3.2)$$

The components $G_{0\mu}$ must be continuous, therefore the components $T_{ps 0\mu}$ have to compensate for the jumps of the four components $T_{0\mu}$. Thus the compensatory tensor $T_{ps \mu \nu}$ should involve four functions on the spacetime manifold $M^4$. 

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1.4 Pseudomatter as pseudodust. The extended Einstein equation

The most appropriate way of introducing four functions is to treat pseudomatter as pseudodust:

\[ T_{\text{ps}} = \varepsilon v \otimes v \quad T_{\text{ps} \mu \nu} = \varepsilon v_\mu v_\nu \]  \hspace{1cm} (1.4.1)

with

\[ v^\mu v_\mu = s(v) \in \{1, -1, 0\} \]  \hspace{1cm} (1.4.2)

It is pertinent to note that the vector \( v \) is not a four-velocity: for the latter \( s(v) = 1 \).

Thus the extended Einstein equation is of the form

\[ E_{\mu \nu} = 8\pi \kappa \varepsilon v_\mu v_\nu \]  \hspace{1cm} (1.4.3)

or

\[ G_{\mu \nu} - \Lambda g_{\mu \nu} = 8\pi \kappa (T_{\mu \nu} + \varepsilon v_\mu v_\nu) \]  \hspace{1cm} (1.4.4)

It is obvious that pseudomatter is perceived as a dark matter: it interacts with matter only via gravity.

There is no intrinsic dynamics of pseudomatter, and

\[ T^{\mu \nu}_{\text{ps} \mu \nu} = 0 \]  \hspace{1cm} (1.4.5)

so that the equation

\[ (T + T_{\text{ps}})^{\mu \nu}_{\mu \nu} = 0 \]  \hspace{1cm} (1.4.6)

holds only as a trivial consequence of (1.3.2) and

\[ (G - \Lambda g)^{\mu \nu}_{\mu \nu} \equiv 0 \]  \hspace{1cm} (1.4.7)

Therefore, there are 10 dynamical equations (1.4.3): the equations

\[ E_{0 \mu} = 8\pi \kappa \varepsilon v_0 v_\mu \]  \hspace{1cm} (1.4.8)

are dynamical as well as the equations

\[ E_{ij} = 8\pi \kappa \varepsilon v_i v_j \]  \hspace{1cm} (1.4.9)

1.5 The reduced Einstein equation

The pseudomatter can be easily eliminated from the extended Einstein equation. We have

\[ E_{00} E_{\mu \nu} = (8\pi \kappa)^2 v_0 v_\mu v_\nu = E_{0\mu} E_{0\nu} \]  \hspace{1cm} (1.5.1)

Introduce the metrodynamical tensor

\[ M_{\mu \nu} = E_{00} E_{\mu \nu} - E_{0\mu} E_{0\nu} \quad M_{ij} = E_{00} E_{ij} - E_{0i} E_{0j} \quad M_{0\mu} \equiv 0 \]  \hspace{1cm} (1.5.2)

Thus we obtain the metrodynamical, or the reduced Einstein equation

\[ M_{ij} = 0 \quad i, j = 1, 2, 3 \]  \hspace{1cm} (1.5.3)
In the six equations (1.5.3), jumps of $T_{\mu\nu}$ are balanced by those of $\delta_{kl}$. The pseudomatter variables are expressed in terms of $E_{\mu\nu}$. We have

$$\sum_{\mu} E_{\mu\mu} = 8\pi \kappa \varepsilon \sum_{\mu} (v_\mu)^2 \quad (1.5.4)$$

and

$$E^\mu_{\mu} = 8\pi \kappa \varepsilon s(v) \quad (1.5.5)$$

Let

$$\varepsilon \nu \otimes \nu \neq 0 \quad \varepsilon \neq 0 \quad \nu \neq 0 \quad (1.5.6)$$

then

$$\sum_{\mu} (v_\mu)^2 > 0 \quad \sum_{\mu} E_{\mu\mu} \neq 0 \quad (1.5.7)$$

and

$$\text{sign} \; \varepsilon = \text{sign} \left( \sum_{\mu} E_{\mu\mu} \right) = \pm 1 \quad (1.5.8)$$

Next, $s(v) = (\text{sign} \; \varepsilon) \text{sign} E^\mu_{\mu} \quad \text{sign} E^\mu_{\mu} = 1, -1, 0 \quad (1.5.9)$

If

$$E^\mu_{\mu} \neq 0 \quad (1.5.10)$$

then

$$\varepsilon = \frac{1}{8\pi \kappa} E^\mu_{\mu} s(v) \quad \frac{1}{8\pi \kappa} (\text{sign} \; \varepsilon) E^\mu_{\mu} \quad \frac{1}{8\pi \kappa} (\text{sign} \; \varepsilon) |E^\mu_{\mu}| \quad (1.5.11)$$

and

$$v_\mu v_\nu = \frac{1}{8\pi \kappa \varepsilon} E_{\mu\nu} = (\text{sign} \; \varepsilon) \frac{E_{\mu\nu}}{|E^\lambda_{\lambda}|} \quad (1.5.12)$$

If

$$E^\mu_{\mu} = 0 \quad (1.5.13)$$

then

$$s(v) = 0 \quad (1.5.14)$$

Put

$$v^0 v_0 = 1 \quad v^i v_i = -1 \quad (1.5.15)$$

We obtain

$$\varepsilon = \frac{1}{8\pi \kappa} E^0_{\mu} \quad v_\mu v_\nu = \frac{E_{\mu\nu}}{E^0_0} \quad (1.5.16)$$
1.6 The reductive synchronous reference

Quantum jumps imply the existence of a universal (cosmic) time, so that the spacetime manifold has the structure of the direct product:

\[ M^4 = T \times S \]  \hspace{1cm} (1.6.1)

where \( T \) is cosmic time and \( S \) is cosmic space. The corresponding reference, or gauge, is reductive synchronous one with metric

\[ ds^2 = dt^2 + g_{ij}dx^i dx^j \quad T \ni t = x^0 \] \hspace{1cm} (1.6.2)

(Metric which admits of the global clock synchronization is a time-orthogonal one, i.e., with \( g_{0i} = 0 \); with \( g_{00} = 1 \), in addition, \( t \) represents the proper time at every point \( s \in S \) [12].)

Now there are 11 equations (1.4.3), (1.4.2) for 11 functions \( g_{ij}, \varepsilon, \upsilon^\mu \), or, alternatively, 6 equations (1.5.3) for 6 functions \( g_{ij} \); in the latter case, the pseudomatter variables are expressed in terms of \( E_{\mu \nu} \).

1.7 Static spacetime, reference, and metric

Static spacetime manifold is of the form

\[ M = T \times S \] \hspace{1cm} (1.7.1)

and there exists a static reference, or gauge, with metric

\[ ds^2 = g_{00}(s)dt^2 + g_{ij}(s)dx^i dx^j \quad dt = dx^0 \quad s \in S \] \hspace{1cm} (1.7.2)

In view of invariance with respect to \( dt \to -dt \), we have

\[ E_{0i} \equiv 0, \; i = 1,2,3 \] \hspace{1cm} (1.7.3)

so that

\[ \upsilon_0 \upsilon_i = 0 \] \hspace{1cm} (1.7.4)

**Stress pseudomatter:**

\[ \upsilon_0 = 0 \quad T_{ps0\mu} = 0 \quad T_{psij} \neq 0 \] \hspace{1cm} (1.7.5)

There are 8 equations:

\[ E_{00} = 0 \] \hspace{1cm} (1.7.6)

\[ E_{ij} = 8\pi \kappa \varepsilon \upsilon_i \upsilon_j \] \hspace{1cm} (1.7.7)

\[ \upsilon^i \upsilon_i = -1 \quad s(v) = -1 \] \hspace{1cm} (1.7.8)

for 11 functions: \( g_{00}, \; g_{ij}, \; \varepsilon, \; \upsilon_i \). We may preassign \( \varepsilon \) and \( \upsilon_i \) (under condition (1.7.8)), then there are 7 equations (1.7.6), (1.7.7) for \( g_{00}, \; g_{ij} \). Alternatively, if \( g \) and \( T \) are given, then

\[ \varepsilon = -\frac{1}{8\pi \kappa} E^l_l \] \hspace{1cm} (1.7.9)
\[ v_i v_j = - \frac{E_{ij}}{E_i^i} \]  

**(Energy ("cold") pseudomatter):**

\[ v_i = 0 \quad v^0 v_0 = 1 \quad s(v) = 1 \quad T_{\text{ps}ij\mu} = 0 \quad T_{00} \neq 0 \]  

There are 7 equations:

\[ E_0^0 = 8 \pi \kappa \epsilon \]  
\[ E_j^i = 0 \]  

for 8 functions: \( g_{00}, g_{ij}, \epsilon \). We may preassign \( \epsilon \); alternatively, with \( g \) and \( T \) given,

\[ \epsilon = \frac{1}{8 \pi \kappa} E_0^0 \]  

## 2  Spherical static objects

### 2.1  Spherical static reference frame

In spherical coordinates, \( x^1 = r \), \( x^2 = \theta \), \( x^3 = \varphi \), metric is of the form

\[ ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(\sin^2 \theta d\varphi^2) \quad \nu = \nu(r), \quad \lambda = \lambda(r) \]  

In view of spherical symmetry and invariance with respect to \( dx^2 \to -dx^2 \), \( dx^3 \to -dx^3 \) we have

\[ E_{k\mu} = E_{kk} \delta_{\mu k}, \quad k = 2, 3 \]  
\[ E_2^2 = E_3^3 \]  

so that

\[ v^2 = v^3 = 0 \quad v^0 = v^0(r) \quad v^1 = v^1(r) \]  

The equations are

\[ E_0^0 = 8 \pi \kappa \epsilon v^0 v_0 \]  
\[ E_1^1 = 8 \pi \kappa \epsilon v^1 v_1 \]  
\[ E_2^2 = 0 \]  
\[ v^0 v^1 = 0 \]  
\[ v^0 v_0 + v^1 v_1 = s(v) \]  

**Stress pseudomatter:**

\[ v^0 = 0 \quad v^1 v_1 = -1 \quad s(v) = -1 \]  

There are 3 equations:

\[ E_0^0 = 0 \]  
\[ E_1^1 = -8 \pi \kappa \epsilon \]  
\[ E_2^2 = 0 \]
for 3 functions: $e^\nu$, $e^\lambda$, $\varepsilon$.

Energy pseudomatter:

\[ v^1 = 0 \quad v^0v_0 = 1 \quad s(v) = 1 \]  

(2.1.14)

There are 3 equations:

\[ E^0_0 = 8\pi \kappa \varepsilon \]  

(2.1.15)

\[ E^1_1 = 0 \]  

(2.1.16)

\[ E^2_2 = 0 \]  

(2.1.17)

for 3 functions: $e^\nu$, $e^\lambda$, $\varepsilon$.

### 2.2 The Einstein tensor. Matter distribution

The components of the Einstein tensor $G$ are [12]

\[ G^0_0 = e^{-\lambda} \left( \frac{\nu'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} \]  

(2.2.1)

\[ G^1_1 = -e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} \]  

(2.2.2)

\[ G^2_2 = -\frac{1}{2} e^{-\lambda} \left( \nu'' + \frac{\nu'^2}{2} + \frac{\nu'}{r} - \frac{\lambda'}{r} - \frac{\nu'\lambda'}{2} \right) \]  

(2.2.3)

where \( \frac{d}{dr} \) = \( \frac{d}{dr} \).

Let

\[ T = 0 \quad \text{for} \quad r > a \]  

(2.2.4)

It is convenient to put [13]

\[ e^{\lambda(r)} = \frac{1}{1 - 2m(r)/r} \quad m(0) = 0 \]  

(2.2.5)

We have

\[ \frac{dm}{dr} = \frac{r^2}{2} G^0_0 \]  

(2.2.6)

whence

\[ \int_0^a r^2 G^0_0 dr = 2m(a) = 2\kappa M =: r_S = r_{\text{Schwarzschild}} \]  

(2.2.7)

In this Section, we put

\[ \Lambda = 0 \]  

(2.2.8)

so that

\[ E = G - 8\pi \kappa T \]  

(2.2.9)

Now

\[ G^0_0 = 8\pi \kappa (T^0_0 + \varepsilon v^0 v_0) \]  

(2.2.10)
and we obtain
\[ 4\pi \int_0^a r^2 (T_0^0 + \varepsilon v^0 v_0) dr = M \quad (2.2.11) \]

Let matter be a perfect fluid:
\[ T_{\mu}^{\nu} = (\rho + p) v_\mu v_\nu - \delta_\mu^\nu p \quad v^\mu v_\mu = 1 \quad (2.2.12) \]

In the spherical static case
\[ v^i = 0 \quad v^0 v_0 = 1 \quad (2.2.13) \]
so that
\[ T_0^0 = \rho \quad T_1^1 = T_2^2 = T_3^3 = -p \quad (2.2.14) \]
Now
\[ 4\pi \int_0^a r^2 (\rho + \varepsilon v^0 v_0) dr = M \quad (2.2.15) \]

### 2.3 Dimensionless quantities

It is convenient to introduce dimensionless quantities:
\[
\begin{align*}
x &= \frac{r}{a} \\
' &= \frac{d}{dx} \\
\bar{r}_S &= \frac{r_S}{a} = \frac{2\kappa M}{a} \\
\bar{G} &= a^2 G \\
\bar{T} &= \kappa a^2 T \\
\bar{g} &= \kappa a^2 \rho \\
\bar{p} &= \kappa a^2 p \quad \bar{\varepsilon} = \kappa a^2 \varepsilon
\end{align*}
\]
Introduce
\[ \mu = \nu' \quad \nu(x) = \nu(1) - \int_x^1 \mu(x) dx \quad \nu'' = \mu' \quad (2.3.2) \]
Now
\[
\begin{align*}
\bar{G}_0 &= \frac{\mu e^{-\lambda}}{x} + \frac{1 - e^{-\lambda}}{x^2} \\
\bar{G}_1 &= -\frac{\mu e^{-\lambda}}{x} + \frac{1 - e^{-\lambda}}{x^2} \\
\bar{G}_2 &= \left[ \mu' + \left( \frac{1}{x} + \frac{\mu}{2} \right) (\mu - \lambda') \right] \quad (2.3.5)
\end{align*}
\]
(2.3.3)–(2.3.5) imply helpful formulas [14]:
\[
\begin{align*}
e^{-\lambda} &= 1 - \frac{1}{x} \int_0^x x^2 \bar{G}_0^0 dx \\
\mu &= e^\lambda x \left( \bar{G}_0^0 - \bar{G}_1^1 \right) - \lambda' \\
G_2^2 &= \frac{1}{2} x \left( \bar{G}_1^1 \right)' + \left( 1 + \frac{1}{4} x \mu \right) \bar{G}_1^1 - \frac{1}{4} x \mu \bar{G}_0^0
\end{align*}
\]
(2.2.7) amounts to
\[
\int_0^1 x^2 \bar{G}_0^0 \, dx = \bar{r}_S
\]  
(2.3.9)

The equations are:
\[
\begin{align*}
\bar{G}_0^0 &= 8\pi \{ \bar{T}_0^0 + \theta [s(\nu)] \bar{\varepsilon} \} \\
\bar{G}_1^1 &= 8\pi \{ \bar{T}_1^1 - \theta [-s(\nu)] \bar{\varepsilon} \} \\
\bar{G}_2^2 &= 8\pi \bar{T}_2^2
\end{align*}
\]  
(2.3.10)
(2.3.11)
(2.3.12)

where \( \theta \) is the step-function,
\[
s(\nu) = 1/ -1 \quad \text{for energy/stress pseudomatter}
\]  
(2.3.13)

For perfect fluid, the equations are:
\[
\begin{align*}
\bar{G}_0^0 &= 8\pi \{ \bar{\rho} + \theta [s(\nu)] \bar{\varepsilon} \} \\
\bar{G}_1^1 &= -8\pi \{ \bar{p} + \theta [-s(\nu)] \bar{\varepsilon} \} \\
\bar{G}_2^2 &= -8\pi \bar{p}
\end{align*}
\]  
(2.3.14)
(2.3.15)
(2.3.16)

2.4 Nonexistence of matterless pseudomatter static objects

Without matter, the equations are:
\[
\begin{align*}
\bar{G}_0^0 &= 8\pi \theta [s(\nu)] \bar{\varepsilon} \\
\bar{G}_1^1 &= -8\pi \theta [-s(\nu)] \bar{\varepsilon} \\
\bar{G}_2^2 &= 0
\end{align*}
\]  
(2.4.1)
(2.4.2)
(2.4.3)

Let
\[
s(\nu) = 1
\]  
(2.4.4)

then \( \bar{G}_1^1 = 0 \), and from (2.3.8) follows
\[
\mu \bar{G}_0^0 = 0
\]  
(2.4.5)

We have
\[
\mu = 0 \Rightarrow [\text{by (2.3.4)}] \lambda = 0 \Rightarrow [\text{by (2.3.3)}] \bar{G}_0^0 = 0
\]  
(2.4.6)

Thus \( \bar{G}_0^0 = 0 \), so that, by (2.4.1),
\[
\bar{\varepsilon} = 0
\]  
(2.4.7)

Now let
\[
s(\nu) = -1
\]  
(2.4.8)

then
\[
\bar{G}_0^0 = 0 \Rightarrow [\text{by (2.3.6)}] \lambda = 0 \Rightarrow [\text{by (2.3.4)}] \mu = -x \bar{G}_1^1
\]  
(2.4.9)

and (2.3.8) amounts to
\[
\frac{1}{2} x (G_1^1)' + G_1^1 - \frac{1}{4} x^2 (G_1^1)^2 = 0
\]  
(2.4.10)
The solution is
\[ G_1^1(x) = \frac{1}{x^2[1/G_1^1(1) - (1/2) \ln x]} \] (2.4.11)
If \( G_1^1(1) \neq 0 \), then \( G_1^1(x) \) is singular at \( x = 0 \). Thus \( G_1^1(1) = 0 \), so that, by (2.4.2),
\[ \varepsilon = 0 \] (2.4.12)

With matter present, energy pseudomatter may only exist where \( T \neq 0 \).
It appears that, in view of (1.7.4), the results of this Subsection are valid in an arbitrary static case.

### 2.5 The Schwarzschild solution with energy pseudomatter

Let there be a perfect fluid and energy pseudomatter for \( 0 \leq x \leq 1 \) (interior) and vacuum for \( x > 1 \) (exterior). The interior equations (2.3.14)–(2.3.16) are
\[ G_0^0 = 8\pi \bar{\rho}_{\text{eff}} \] (2.5.1)
\[ G_1^1 = -8\pi \bar{p} \] (2.5.2)
\[ G_2^2 = -8\pi \bar{p} \] (2.5.3)
where
\[ \bar{\rho}_{\text{effective}} =: \bar{\rho}_{\text{eff}} = \bar{\rho} + \varepsilon \] (2.5.4)
and
\[ 8\pi \int_0^1 x^2 \bar{\rho}_{\text{eff}} dx = \bar{r}_S \] (2.5.5)

The exterior equations are (2.5.1)–(2.5.3) with \( \bar{\rho} = \bar{\varepsilon} = 0 \), \( \bar{p} = 0 \).
Let us consider the Schwarzschild solution to the above equations. We have [11,13,14]:
\[ \bar{\rho}_{\text{eff}} = \text{const} = \frac{3}{8\pi} \bar{r}_S \] (2.5.6)
\[ \bar{r}_S < \frac{8}{9} \] (2.5.7)
\[ \bar{p} = \bar{\rho}_{\text{eff}} \frac{(1 - \bar{r}_S x^2)^{1/2} - (1 - \bar{r}_S)^{1/2}}{3(1 - \bar{r}_S)^{1/2} - (1 - \bar{r}_S x^2)^{1/2}} \] (2.5.8)

Let \( p \) satisfy the standard condition:
\[ \bar{p} \leq \frac{1}{3} \bar{\rho}_{\text{eff}} \] (2.5.9)

If
\[ \bar{p} > \frac{1}{3} \bar{\rho}_{\text{eff}} \] (2.5.10)
then
\[ \frac{1}{3}(\bar{\rho} + \varepsilon) < \bar{p} \leq \frac{1}{3} \bar{\rho} \] (2.5.11)
whence
\[ \varepsilon < 0 \quad (2.5.12) \]
From \((2.5.10), (2.5.8)\) follows
\[ \bar{r}_S > \frac{5}{9} \text{ and } x < \frac{3}{2\bar{r}_S^{1/2}} \left(\bar{r}_S - \frac{5}{9}\right)^{1/2} \Rightarrow \varepsilon < 0 \quad (2.5.13) \]
Specifically,
\[ \text{for } \bar{r}_S \rightarrow \frac{8}{9}, \quad x < \left(\frac{27}{32}\right)^{1/2} \quad (2.5.14) \]
Thus the realistic condition \((2.5.3)\) implies the existence of energy pseudomatter with negative density.

### 2.6 Energy pseudomatter halo
Let
\[ \bar{r}_S \ll 1 \quad \bar{\rho}_{\text{eff}} = \frac{3}{8\pi} \bar{r}_S \ll 1 \quad (2.6.1) \]
then
\[ \bar{p} \approx \frac{1}{4\bar{\rho}_{\text{eff}}} \bar{r}_S(1 - x^2) = \frac{2\pi}{3} \bar{\rho}_{\text{eff}}^2 (1 - x^2) \ll \bar{\rho}_{\text{eff}} \quad (2.6.2) \]
Introduce
\[ 0 < x_m \ll 1 \quad x_m := x_{\text{matter}} \quad (2.6.3) \]
and put
\[ \text{for } 0 \leq x \lesssim x_m \quad \bar{\rho} \approx \bar{\rho}_{\text{eff}} \gg \varepsilon \approx 0 \quad (2.6.4) \]
\[ \text{for } x_m \lesssim x \leq 1 \quad \bar{\rho} \approx 3\bar{\rho} = 2\pi \bar{\rho}_{\text{eff}}^2 (1 - x^2) \ll \varepsilon \approx \bar{\rho}_{\text{eff}} \quad (2.6.5) \]
This is a material object in the small interior \((2.6.4)\) and an energy pseudomatter halo with little matter in the large exterior \((2.6.5)\).

In the halo,
\[ \bar{M}(x) = 4\pi \int_0^x x^2 \bar{\rho}_{\text{eff}} dx \propto x^3 \quad (2.6.6) \]
so that we obtain for the velocity \(v\):
\[ \frac{v^2}{x} \propto \frac{\bar{M}(x)}{x^2} \propto x \quad v \propto x \quad (2.6.7) \]

### 2.7 Stress pseudomatter halo
Let us consider the case when in the exterior, \(x > 1\), there is only stress pseudomatter. The exterior equations are:
\[ \bar{G}^0_0 = 0 \quad (2.7.1) \]
\[ \bar{G}^1_1 = -8\pi \varepsilon \quad (2.7.2) \]
\( \tilde{G}_2^2 = 0 \) (2.7.3)

In the exterior, we obtain from (2.3.8), (2.3.4)

\[
(\tilde{G}_1)' + \frac{2}{x} \left[ 1 + \frac{1}{4} (e^\lambda - 1) \right] \tilde{G}_1 - \frac{1}{2} x e^\lambda (\tilde{G}_1)^2 = 0
\] (2.7.4)

and from (2.3.6), (2.3.9)

\[
e^\lambda = \frac{1}{1 - \tilde{r}_S / x}
\] (2.7.5)

Thus we obtain the equation for \( \tilde{G}_1^1 \):

\[
(\tilde{G}_1)' + \frac{2}{x} \left[ 1 + \frac{1}{4} \frac{\tilde{r}_S}{x - \tilde{r}_S} \right] \tilde{G}_1^1 - \frac{1}{2} \frac{x^2}{x - \tilde{r}_S} (\tilde{G}_1^1)^2 = 0
\] (2.7.6)

Put

\[
\tilde{G}_1 = \frac{1}{y}
\] (2.7.7)

which gives

\[
y' - \frac{2}{x} \left[ 1 + \frac{1}{4} \frac{\tilde{r}_S}{x - \tilde{r}_S} \right] y = -\frac{1}{2} \frac{x^2}{x - \tilde{r}_S}
\] (2.7.8)

whence

\[
y = x^{3/2} \left( \frac{x - \tilde{r}_S}{1 - \tilde{r}_S} \right)^{1/2}
\times \left\{ y(1) - \frac{1}{2} (1 - \tilde{r}_S)^{1/2} \left[ \ln x + 2 \ln \frac{1 + \sqrt{1 - \tilde{r}_S / x}}{1 + \sqrt{1 - \tilde{r}_S}} + 2 \left( \frac{1}{\sqrt{1 - \tilde{r}_S}} - \frac{1}{\sqrt{1 - \tilde{r}_S / x}} \right) \right] \right\}
\] (2.7.9)

To obtain metric (2.1.1), we have to find \( e^\nu \). From (2.3.2) follows

\[
e^\nu = [e^\nu](1) \exp \left\{ \int_1^x \mu \, dx \right\}
\] (2.7.10)

and from (2.3.4)

\[
\mu = \frac{e^\lambda - 1}{x} - x e^\lambda \tilde{G}_1^1 = \frac{\tilde{r}_S}{x(x - \tilde{r}_S)} - \frac{x^2}{(x - \tilde{r}_S)y}
\] (2.7.11)

so that

\[
e^\nu = [e^\nu](1) \frac{1}{1 - \tilde{r}_S} \left( 1 - \frac{\tilde{r}_S}{x} \right) \exp \left\{ -\int_1^x \frac{x^2}{(x - \tilde{r}_S)y} \, dx \right\}
\] (2.7.12)

\([e^\nu](1) \) depends on the choice of \( t \) in (2.1.1). In view of (2.7.5), we introduce the condition

\[
\left[ \frac{d}{dx} e^\nu \right](1) = \left[ \frac{d}{dx} \left( c + 1 - \frac{\tilde{r}_S}{x} \right) \right](1)
\] (2.7.13)
i.e.,
\[
\left[ \frac{d}{dx} e^\nu \right] (1) = \bar{r}_S \tag{2.7.14}
\]

In what follows in this Subsection, we consider the case
\[
\bar{r}_S \ll 1 \tag{2.7.15}
\]

From (2.7.9) follows
\[
y \approx x^2 \left[ y(1) - \frac{1}{2} \ln x \right] \tag{2.7.16}
\]

so that to avoid a singularity, we put
\[
y(1) = -b \quad b > 0 \tag{2.7.17}
\]

Thus
\[
e^\nu = \left[ e^\nu \right] (1) \frac{1}{1 - \bar{r}_S} \left( 1 - \frac{\bar{r}_S}{x} \right) \left( 1 + \frac{1}{2b} \ln x \right)^2
\approx \left[ e^\nu \right] (1) \frac{1}{1 - \bar{r}_S} \left[ 1 - \frac{\bar{r}_S}{x} + \frac{\ln x}{b} + \frac{(\ln x)^2}{4b^2} \right] \tag{2.7.18}
\]

Now put
\[
e^\nu = g_{00} = 1 + 2\varphi \quad \varphi \ll 1 \tag{2.7.19}
\]

where \( \varphi \) is the Newtonian potential \([12,15]\), and
\[
\left[ e^\nu \right] (1) \frac{1}{1 - \bar{r}_S} = 1 + \beta \quad |\beta| \ll 1 \tag{2.7.20}
\]

Then
\[
\varphi = \frac{\beta}{2} + \frac{1 + \beta}{2} \left[ -\frac{\bar{r}_S}{x} + \frac{\ln x}{b} + \frac{(\ln x)^2}{4b^2} \right] \tag{2.7.21}
\]

and
\[
\varphi' = \frac{1 + \beta}{2} \left[ \frac{\bar{r}_S}{x^2} + \frac{1}{bx} + \frac{\ln x}{2b^2x} \right] \tag{2.7.22}
\]

From (2.7.14) and (2.7.19) follows
\[
\varphi'(1) = \frac{\bar{r}_S}{2} \tag{2.7.23}
\]

and
\[
\beta = -\frac{1}{1 + \bar{r}_S b} \quad 1 + \beta = \frac{1}{1 + 1/\bar{r}_S b} \quad \bar{r}_S b \gg 1 \tag{2.7.24}
\]

Next,
\[
\ln x \ll b \tag{2.7.25}
\]

so that
\[
\varphi' = \frac{1}{2(1 + \bar{r}_S b)} \left( \frac{\bar{r}_S}{x^2} + \frac{1}{bx} \right) \tag{2.7.26}
\]
We have for the velocity \( v \)

\[
\frac{v^2}{x} = \varphi' \quad v \ll 1 \tag{2.7.27}
\]

whence

\[
v^2 = \frac{1}{2(1 + 1/\bar{r}_Sb)} \left( \frac{\bar{r}_S}{x} + \frac{1}{b} \right) \tag{2.7.28}
\]

Specifically,

\[
\text{for } 1 \ll \bar{r}_S b \ll x \ll e^b \quad v = \sqrt{\frac{1}{2b}} = \text{const} \tag{2.7.29}
\]

This result is in good agreement with galaxy-scale observations [6].

It might be worthwhile to point out that in the treatment in this Subsection, the localization of the matter and gravitational potential are spatially separated; this takes place in the bullet cluster [6].

3 Dynamical objects

3.1 Collapse with pseudomatter

In the treatment of a collapse, we will exploit the relevant results of [11]. In the spherical case, with \( x^1 = r \), \( x^2 = \theta \), \( x^3 = \varphi \), the metric is of the form

\[
ds^2 = dt^2 - U(r, t)dr^2 - V(r, t)(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{3.1.1}
\]

The quantities which are not identically zero are:

\[
\begin{align*}
G_{00} & \quad G_{01} & \quad G_{11} & \quad G_{22} & \quad G_3^3 = G_2^2 \\
E_{00} & \quad E_{01} & \quad E_{11} & \quad E_{22} & \quad E_3^3 = E_2^2
\end{align*} \tag{3.1.2}
\]

Now we use the metrodynamical equation (1.5.3) with \( E_{00} \neq 0 \):

\[
E_{ij} = \frac{E_{0i}E_{0j}}{E_{00}} \tag{3.1.3}
\]

whence

\[
E_{11} = \frac{(E_{01})^2}{E_{00}} \tag{3.1.4}
\]

\[
E_{22} = 0 \tag{3.1.5}
\]

We put

\[
\Lambda = 0 \quad E = G - 8\pi \kappa T \tag{3.1.6}
\]

so that

\[
G_{22} - 8\pi \kappa T_{22} = 0 \tag{3.1.7}
\]

\[
G_{11} - 8\pi \kappa T_{11} = \frac{(G_{01} - 8\pi \kappa T_{01})^2}{G_{00} - 8\pi \kappa T_{00}} \tag{3.1.8}
\]
Now we consider the simplest case: a solution to (3.1.7), (3.1.8) for which

\[ ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right] \] (3.1.9)

\[ G_{22} = 0 \quad G_{11} = 0 \quad G_{01} = 0 \quad G_{00} = 8\pi \kappa \sigma \quad \sigma = \frac{\sigma(0)}{R^3(t)} \] (3.1.10)

\[ k = \frac{8\pi \kappa}{3} \sigma(0) = \frac{2\kappa M}{a^3} \quad ka^2 < 1 \quad \frac{2\kappa M}{a} < 1 \] (3.1.11)

\( R(t) \) is given in [11]. Again,

\[ T_{22} = 0 \quad T_{2} = 0 \] (3.1.12)

\[ T_{11} = \frac{(T_{01})^2}{T_{00} - \sigma} - \frac{R^2}{1 - kr^2} T_{1}^1 = \frac{(T_{1}^0)^2}{T_{0}^0 - \sigma} \] (3.1.13)

Pseudomatter is determined by (1.4.3), (1.4.2):

\[ E_{\mu}^{\nu} = 8\pi \kappa \varepsilon v_{\mu} v^{\nu} \quad v_{\mu} v^{\mu} = s(v) \] (3.1.14)

From (3.1.5) follows

\[ v_2 = v_3 = 0 \] (3.1.15)

Thus

\[ v_0 v^0 + v_1 v^1 = s(v) \] (3.1.16)

\[ \varepsilon v_0 v^0 = \sigma - T_0^0 \quad \varepsilon v_1 v^1 = -T_1^1 \] (3.1.17)

whence

\[ \varepsilon s(v) = \sigma - T_0^0 - T_1^1 \] (3.1.18)

Next,

\[ T_0^0 = \rho \quad T_1^1 - p_r \] (3.1.19)

where \( p_r \) is the radial pressure. Thus

\[ \frac{R^2}{1 - kr^2} p_r = \frac{(T_1^0)^2}{\rho - \sigma} \] (3.1.20)

\[ \varepsilon v_0 v^0 = \sigma - \rho \quad \varepsilon v_1 v^1 = p_r \] (3.1.21)

\[ -\varepsilon s(v) = D \] (3.1.22)

where

\[ D = \rho - p_r - \sigma \] (3.1.23)

We have

\[ p_r > 0 \quad \rho - \sigma > 0 \] (3.1.24)

so that

\[ \varepsilon < 0 \] (3.1.25)

We find

\[ s(v) = \text{sign} \, D \] (3.1.26)
If \( D \neq 0 \) \hfill (3.1.27)
then
\[
\varepsilon = -|D| \hfill (3.1.28)
\]
\[
v_0v^0 = \frac{\varrho - \sigma}{|D|} \quad v_1v^1 = -\frac{p_r}{|D|} \hfill (3.1.29)
\]

If \( D = 0 \) \hfill (3.1.30)
then
\[
s(v) = 0 \quad v_0v^0 + v_1v^1 = 0 \hfill (3.1.31)
\]
Put
\[
v_0v^0 = 1 \quad v_1v^1 = -1 \hfill (3.1.32)
\]
then
\[
\varepsilon = -(\varrho - \sigma) \hfill (3.1.33)
\]
Let
\[
R \to 0 \hfill (3.1.34)
\]
We have
\[
\varrho \propto \frac{1}{R^4} \quad p_r \propto \frac{1}{R^4} \quad T^0_1 \propto \frac{1}{R^3} \hfill (3.1.35)
\]
let \( p_r < \varrho \), then
\[
D > 0 \quad D \propto \frac{1}{R^4} \quad s(v) = 1 \quad v_0v^0 = \frac{\varrho}{\varrho - p_r} > 1 \quad v_1v^1 = -\frac{p_r}{\varrho - p_r} \hfill (3.1.36)
\]

### 3.2 The FLRW universe with energy ("cold") pseudomatter

The FLRW universe metric is of the form [16]
\[
ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad k = 1, 0, -1 \hfill (3.2.1)
\]
\( (t = x^0, \, r = x^1, \, \theta = x^2, \, \phi = x^3) \). The dynamical Einstein tensor (1.1.1) is
\[
E = G - \Lambda g - 8\pi \kappa T \hfill (3.2.2)
\]
and the metrodynamical equation (1.5.3) is
\[
E_{00}E_{ij} - E_{0i}E_{0j} = 0 \hfill (3.2.3)
\]
From symmetry considerations follows
\[
E_{0i} \equiv 0 \quad E_{ij} \equiv 0 \quad \text{for} \ j \neq i \hfill (3.2.4)
\]
Thus (3.2.3) reduces to
\[
E_{00}E_{ii} = 0 \hfill (3.2.5)
\]
Now

\[ E_{\mu\nu} = 8\pi \kappa v_\mu v_\nu \quad v_\mu v^\mu = s(v) \]  

(3.2.6)

and from symmetry considerations follows

\[ v_i = 0 \quad s(v) = 1 \quad (v_0)^2 = v_0 v^0 = 1 \quad \text{energy ("cold") pseudomatter} \]  

(3.2.7)

so that

\[ E_{ii} = 0 \quad E_{11} = 0 \Leftrightarrow E_{22} = 0 \Leftrightarrow E_{33} = 0 \]  

(3.2.8)

Thus we have two equations:

\[ E_1^1 = 0 \quad G_1^1 - \Lambda - 8\pi \kappa T_1^1 = 0 \]  

(3.2.9)

\[ E_0^0 = 8\pi \kappa \varepsilon G_0^0 - \Lambda - 8\pi \kappa (T_0^0 + \varepsilon) = 0 \]  

(3.2.10)

The components of the Einstein tensor \( G \) are [16]:

\[ G_0^0 = 3 \frac{\dot{R}^2 + k}{R^2} \quad G_1^1 = \frac{1}{R^2} (2R\ddot{R} + \dot{R}^2 + k) \]  

(3.2.11)

Now we regard matter as a perfect fluid (2.2.12) with \( v^i = 0 \) [12,16], so that

\[ T_0^0 = \rho \quad T_1^1 = -p \]  

(3.2.12)

and we obtain two equations:

\[ 2R\ddot{R} + \dot{R}^2 + k - \Lambda R^2 + 8\pi \kappa p R^2 = 0 \]  

(3.2.13)

\[ 3(\dot{R}^2 + k) - \Lambda R^2 - 8\pi \kappa (\rho + \varepsilon) R^2 = 0 \]  

(3.2.14)

for two functions: \( R, \varepsilon \) (\( p \) and \( \rho \) are determined by (1.1.2), (1.1.6)).

From (3.2.13), (3.2.14) the energy equation follows:

\[ \frac{d}{dt}[(\rho + \varepsilon)R^3] = -p \frac{dR^3}{dt} \]  

(3.2.15)

Specifically, in the case of \( k = 1 \) (closed universe), \( V = 2\pi^2 R^3 \), and

\[ \frac{d}{dt}[(\rho + \varepsilon)V] = -p \frac{dV}{dt} \]  

(3.2.16)

Now, the equation

\[ T_0^{\mu \mu} = 0 \]  

(3.2.17)

for perfect fluid results in

\[ \frac{d}{dt}[\rho R^3] = -p \frac{dR^3}{dt} \]  

(3.2.18)

Thus

\[ \frac{d}{dt}[\varepsilon R^3] = 0 \]  

(3.2.19)
and
\[ \varepsilon = \frac{B}{R^3}, \quad B = \text{const} \tag{3.2.20} \]

In the actual universe \( \varepsilon > 0 \), so that
\[ B > 0 \tag{3.2.21} \]

Let
\[ R \to +0 \tag{3.2.22} \]

then
\[ \varrho = \frac{A}{R^4} \quad p = \frac{1}{3} \varrho = \frac{A}{3R^4} \tag{3.2.23} \]

We have
\[ \frac{\varepsilon}{\varrho} = \frac{B}{A} R \quad \frac{d}{dR} \left( \frac{\varepsilon}{\varrho} \right) = \frac{B}{A} > 0 \tag{3.2.24} \]

Now,
\[ \text{for } R \to \infty \quad \varrho \propto \frac{1}{R^3} \quad \frac{\varepsilon}{\varrho} = \text{const} \tag{3.2.25} \]

Thus for small \( R \), \( \varepsilon/\varrho \) increases with time.

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