Two null gravitational cones in the theory of GPS-intersatellite communications between two moving satellites. I. Physical and mathematical theory of the space-time interval and the geodesic distance on intersecting null cones.

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Abstract

Several space missions such as GRACE, GRAIL, ACES and others rely on intersatellite communications (ISC) between two satellites at a large distance one from another. The main goal of the theory is to formulate all the navigation observables within the General Relativity Theory (GRT). The same approach should be applied also to the intersatellite GPS-communications (in perspective also between the GPS, GLONASS and Galileo satellite constellations). In this paper a theoretical approach has been developed for ISC between two satellites moving on (one-plane) elliptical orbits, based on the introduction of two gravity null cones with origins at the emitting-signal and receiving-signal satellites. The two null cones account for the variable distance between the satellites during their uncorrelated motion. The intersection of the two null cones defines a distance, which can be found from a differential equation in full derivatives. This distance is the space-time interval in GRT. Applying some theorems from higher algebra, it was proved that this space-time distance can become zero, consequently it can be also negative and positive. But in order to represent the geodesic distance travelled by the signal, the space-time interval has to be "compatible" with the Euclidean distance. So this "compatibility condition", conditionally called "condition for ISC" is the most important consequence of the theory. The other important consequence is that the geodesic distance turns out to be the space-time interval, but with account also of the "condition for ISC". This interpretation enables the strict mathematical proof that the geodesic distance is greater than the Euclidean distance - a result, entirely based on the "two null cones approach" and moreover, without any use of the Shapiro delay formulae. The theory places also a restriction on the ellipticity of the orbit \( e \leq 0.816496580927726 \). For the typical GPS orbital parameters, the condition for ISC gives a value \( E = 45.00251 \) [deg], which is surprisingly close to the value for the true anomaly angle \( f = 45.54143 \) [deg] and also to the angle of disposition of the satellites in the GLONASS satellite constellation (the Russian analogue of the American GPS) - 8 satellites within one and the same plane equally spaced at 45 deg. Consistency between several other newly derived numerical parameters is noted. The paper is the first step towards constructing a new and consistent relativistic physical theory of ISC between moving (non-stationary) satellites on space-distributed Kepler orbits.

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I. INTRODUCTION

A. General idea about intersatellite communications

Currently GPS technologies have developed rapidly due to the wide implementation of atomic clocks [1] (particularly optical atomic clocks based on atomic optical transitions and precise frequency standards), which have found numerous applications in navigation, satellite communications, frequency and time transfer over optical fibres [2], time-variable gravity potential components induced by tides and non-tidal mass redistributions on the Earth [3].

The main stream of research in the last 20 years was concentrated mainly on the problem about the communications between satellites and ground stations. The central issue in this communication is the broadcast message [4], which contains information about the orbit of the satellite in the form of Kepler elements and also determines the accuracy of the navigation and point positioning. This message contains information about the perturbation - dependent deviation from the two-body ellipse.

In the past 10 years the problem about GPS satellite-ground station communications has been replaced by the problem about autonomous navigation and intersatellite communications (ISC) (links), which has been mentioned yet in 2005 in the monograph [12]. Autonomous navigation means that generations of satellite Block II F (replenishment) and Block III satellites have the capability to transmit data between them via intersatellite crosslink ranging and thus they will essentially position themselves without extensive ground tracking. Consequently, autonomous navigation is achieved by means of exchanging time signals and other information among the satellites through ISL (Inter Satellite Links) for ranging and calculating clock offsets [13]. In such way, navigation accuracy can be maintained for six months [14] without ground support and control. However, one of the serious problems is that the accuracy of the navigation message degrades over time such that the user range in satellite-to-satellite tracking is bounded by 10000 m after 180 days [15]. From a theoretical point of view ISC do not require the determination of the Earth geoid.

This paper will propose a new theoretical approach for intersatellite communications between satellites moving on one-plane Kepler elliptical orbits. In principle, the precision measurement of the propagation time of the signal (this is mostly performed for the signal
between station on the Earth and a satellite) is of key importance for Satellite Laser Ranging which measures relativistic effects on the light propagation between station and satellite with the accuracy of 1 micrometer (\(\sim 0.01\) ps - picoseconds) \(\text{[1]}\). But this accuracy (1 micrometer is diameter of the blood cell) can be attained also in measuring the distance between the two spacecrafts (about 220 km) in the GRACE (Gravity Recovery and Climate Experiment) space mission \(\text{[23]}\) on a low-Earth orbit by means of the microwave ranging (MWR) system \(\text{[24]}\).

It is important that next generation space missions will attain sub-millimeter precision of measuring distances beyond 10\(^6\) meters by means of ultrashort femtosecond pulse lasers. A new technology based on comparison of the phase of the laser pulses is proposed currently by the Jet Propulsion Laboratory (JPL). So the precision measurement of distances is the first important moment.

The second important moment is that the theoretical description of such measurements is inevitably related with General Relativity Theory (GRT). As an example one can point out the space mission GRAIL (Gravity Recovery and Interior Laboratory), also comprised of two spacecrafts, launched on September 10, 2011 and with data acquisition from March 1, 2012. This mission together with the one- and two-way Doppler observations from the NASA Deep Space Network (DSN) allows to recover the lunar gravitational field with the purpose to investigate the interior structure of the Moon from crust to core \(\text{[25]}\). The peculiar and essential fact in the theoretical formalism is that all the observables for the two radio links at the K- and Ka-bands (26 GHz and 32 GHz) for inter-satellite ranging, also for the second inter-spacecraft link at the S-band (\(\sim 2.3\) GHz) for the Time Transfer System (TTS) and the one-way X-band link should be formulated within the GRT.

## B. Intersatellite communications and the space experiments GRACE, GRAIL, ACES and the RadioAstron ground-space VLBRI project

The theory of intersatellite communications (ISC) is developed in the series of papers by S. Turyshev, V. Toth, M. Sazhin \(\text{[58, 59]}\) and S. Turyshev, N. Yu, V. Toth \(\text{[60]}\) and concerns the space missions GRAIL (Gravity Recovery and Interior Laboratory), GRACE- FOLLOW-ON (GRACE-FO - Gravity Recovery and Climate Experiment - Follow On) mission and the Atomic Clock Ensemble in Space (ACES) experiment \(\text{[61, 63]}\) on the International Space
Station (ISS). It should be stressed that the theory in these papers is developed for low-orbit satellites when in the theoretical description of the gravitational field the multipoles of the Earth as a massive celestial body should be taken into account. In the case currently investigated, GPS satellites are on a more distant orbit of 26560 km, so the gravitational field at such height will not be influenced by such multipoles. Nevertheless, many features of the theory may be applied also to the intersatellite GPS-communications theory. For example, a key property of the theory for the exchange of signals between two non-moving satellites is that if the first spacecraft A is sending $dn_{A_0}$ cycles (number of pulses), then they should be equal to the number of pulses $dn_{B_0}^B$ received by the spacecraft $B$ \[58\]. The question which arises in reference to the problem treated in this paper is: will this equality be preserved in the case when the satellites are moving?

There is also a concrete experimental situation, related to the RadioAstron interferometric project, where the baseline distance (which is in fact $R_{AB}$) is changing. RadioAstron is a ground-space interferometer \[33\], consisting of a space radio telescope (SRT) with a diameter 10 meters, launched into a highly elongated and perturbed orbit \[34\], and a ground radio telescope (GRT) with a diameter larger than 60 m. The baseline between the SRT and the GRT is changing its length due to the variable parameters of the orbit - the perigee varies from 7065 km to 81500 km, the apogee varies from 280 000 km to 353 000 km and for a period of 100 days the eccentricity of the orbit changes from 0.59 to 0.96. In this conjunction of SRT and a GRT, commonly called VLBRI (Very Large Baseline Radio Interferometry) \[35\], the SRT time turns out to be undefined due to the changing delay time, which is a difference between the time of the SRT and the time of the Terrestrial Station (TS). This might mean that the commonly accepted formulae for the Shapiro time delay might not account for the relative motion between the SRT and the GRT. This will be explained in the next section and represents one of the main motivation for the search for a formulae, which would account for a variable baseline distance between the SRT and the GRT.
C. Objectives of this paper

1. Two intersecting null gravitational cones and the algebraic geometry problem

The key objective of the paper is to construct a mathematical formalism for the exchange of signals in the gravitational field of the Earth between satellites, which are not stationary with time, but are moving on one-plane elliptic orbits. A new approach here is the introduction of two gravitational null cones with origins at the signal-emitting and signal receiving satellites. The two null cones signify a transition to the moving reference systems of the two satellites, parametrized by the Kepler parameters (the eccentricities, the semi-major axis and the eccentric anomaly angles) for the case of plane elliptic motions. The eccentricity angles can be found as solutions of the Kepler equation for a given value of the mean anomaly angles (determined for one full revolution along the Kepler orbit), but generally, they may be considered changing with time. For this general case, the distance between the satellites will be a variable quantity, which means that the distance will not be expressed by a number but may depend on the space coordinates. Thus, the known formulae for the Shapiro time delay cannot be applied because it presumes non-moving emitters and receivers in the gravitational field around a massive body. The most peculiar moment of the approach in this paper is that the Euclidean distance can be expressed from the two null cone equations - this variable distance will turn out to be a solution of a differential equation in full derivatives. The distance (note that again, it is a macroscopic quantity) will then be the distance between two points on the corresponding null cones. Consequently, since the null gravitational cones are an essential ingredient of General Relativity Theory (GRT), this distance will represent in fact the space-time interval, determined on the intersection of the two four-dimensional null cones. In other words, the initial function, denoting the Euclidean distance shall be expressed by another formulae, which will give the space-time interval.

2. The concept about the space-time interval of two intersecting gravitational null cones

In the first place, let us clarify first why in the present case the two gravitational null cones are intersecting. This is so because for a given (variable) Euclidean distance between two space points on the corresponding four-dimensional null cones, there will be a rela-
tion between the space-time coordinates on the two null cones. In other words, the two
four-dimensional null cones will be intersecting along some three-dimensional hypersurface,
which will be a function also of the variable Euclidean distance (depending on the space
coordinates). Therefore, the problem about finding the propagation times $T_1$ and $T_2$ in fact
is equivalent to the algebraic geometry problem about finding the intersection variety of
the two gravitational null cones, written in terms of the corresponding variables $dT_1$, $dx_1$,
$dy_1$, $dz_1$ and $dT_2$, $dx_2$, $dy_2$, $dz_2$ with the six-dimensional hyperplane in terms of the variables
$dx_1$, $dy_1$, $dz_1$, $dx_2$, $dy_2$, $dz_2$. For the investigated case of plane Keplerian motion, since there
will be no dependence on the $z_1$ and $z_2$ coordinates, the null cones will be three-dimensional
instead and the hyperplane - a four-dimensional one.

In the second place, following the GRT concepts, this space-time interval can be positive,
negative or zero. Further, the availability of all these options will be confirmed by the
concrete calculations. Let us clarify this unusual moment: in the standard literature no
proof is given whether the intersection of the two null cones will give again a space-time
distance with the property of being null, positive or negative. In this paper, this fact will
be proved for several partial cases (for example, equal eccentricities, semi-major axis but
different eccentric anomaly angles or the other case, when the eccentric anomaly angles are
also equal), but also for the general case when the space-time interval represents a fourth-
degree polynomial with respect to the square of the sine of the eccentric anomaly angle. By
applying the Shur theorem from higher algebra it was proved not only that the polynomial
has roots, but also the interval of values for the eccentric anomaly were found for which this
polynomial might have zeroes.

3. How does the concept about geodesic distance appear based on the notion of space-time
interval

Propagation of signals is a macroscopic process, because the signal (light or electromagnetic)
has to travel a certain macroscopic distance, which is of course positive. Therefore,
this space-time interval has to be compatible with the large-scale, Euclidean distance. Thus,
the equality between the space-time interval and the Euclidean distance will give the s.c.
"condition for intersatellite communications" (CISC) (further in the text - eq. (47)). Interest-
estingly, this condition can be obtained also (but only for a certain partial case) without
comparing with the Euclidean distance, only by means of setting up equal to zero the space-
time interval. Again, for coinciding points on the orbit (equal semi-major axis, eccentricities
and eccentric anomaly angles), this is a consistent result, since it does not change the physi-
cal essence about zero Euclidean distance and zero space-time interval for coinciding points.
If the CISC is substituted into the equation (45) for the space-time interval, the obtained
expression will be called the geodesic distance (equation (122)). This physical interpretation
will be correct, because by using some properties of the condition for intersatellite commu-
nications, it will be proved that the geodesic distance is greater than the Euclidean distance.
Again, this was established for certain partial cases, but it turned out that a simple proof
can be made also for the general case. This should be so, because from the Shapiro formulae
it follows that due to the action of the gravitational field, a signal travels a greater time (the
sum of the Euclidean time and the logarithmic correction). The curious and very interesting
fact in the present case is that this result has been confirmed without the use of the Shapiro
delay formulae and in the framework of the approach of the two gravitational null cones. The
other curious moment is that the algebraic treatment of the fourth-order algebraic equation
for the geodesic distance fully complies with its positivity. The Shur theorem applied to this
equation proves that it does not possess any roots. This is a substantial difference from the
previous case with the algebraic equation for the space-time interval. Therefore, in spite of
the relative motion between the satellites and the emission and reception of signals by the
moving satellites, some basic facts about the geodesic distance and the delay of the signal
in the gravitational field still remain.

In view of the above considerations, the paper will have the following major objectives:

1. Deriving the expression for the space-time interval as an intersection of the two null
cone equations, relating this interval also to the Euclidean distance. Proving that the space-
time interval for certain partial cases can be positive, negative and also zero. Presenting
a complicated mathematical proof (without solving the equation) that the fourth-order al-
gebraic equation for the space-time interval for the general case of different eccentricities,
semi-major axis and eccentric anomaly angles possesses roots (zeroes). The mathematical
proof is valid however for the case of small eccentricities, which is the case for GPS orbits.

2. Deriving the s.c. "condition for intersatellite communications" and by means of it,
clarifying the physical meaning of the space-time interval and the geodesic distance, by
considering also the limiting cases of equal eccentricities and semi-major axis, but different
eccentric anomaly angles and also another case - equal eccentricities, semi-major axis and eccentric anomaly angles.

3. A mathematical proof is given that the square of the geodesic distance is greater than the square of the Euclidean distance, based on the "two gravitational null cones approach" and not on the Shapiro delay formulae. Based on the proof that the geodesic distance is only positive and greater than the Euclidean distance, a new physical interpretation is proposed for the Euclidean distance as the positive space-time distance on the intersection of two gravitational null cones. However, the requirement for positive distance may not be taken into account, if the condition for intersatellite communications is taken into consideration, since it is derived from the equality of the space-time distance and the Euclidean distance. Then the distance measured on the intersection of the null cones will be positive.

4. Some numerical restrictions are found on the eccentricity of the orbit (valid for any eccentric anomaly angles and any semi-major axis) and on the eccentric anomaly angle (valid only for the typical eccentricity of the GPS orbit). It is very interesting to note that the first restriction is closely related to the fact that the geodesic distance is greater than the Euclidean one.

D. Organization of this paper

This paper is organized as follows:

In section I it has been pointed out that contemporary experiments such as GRACE, GRAIL, RadioAstron and others perform precision measurements of the distance between the satellites, which requires the formulation of all the observables for the radio links within the GRT. So the main prerequisite for this investigation comes from an experimental point of view and the necessity to establish the s.c. "intersatellite communications" between moving satellites (see Section I A and also Section I B). However, there is also a serious theoretical motivation for this, presented in section I C 1, which is based on the introduced new concept in this paper about the "intersecting null four-dimensional gravitational null cones". This concept, allowing to create a theory for exchange of signals between moving satellites, inevitably leads to two other concepts, the physical and mathematical aspects of which will be investigated in details in this paper: the concept in section I C 2 about the space-time distance on intersecting null cones and the other, closely related to the first one,
but yet different concept in section [C3] about the geodesic distance on intersecting null cones. For each concept, several partial cases will be worked out before investigating the general case since the partial cases will "suggest" the ideas about the space-time distance being positive, negative or null and the geodesic distance-being only positive.

But firstly, some initial and important notions will be reminded in section [IIA] namely the relation between the null cone equation and the world function, which has been defined in General Relativity by Synge. The next section [IIB] reminds the well known and currently widely used Shapiro time delay formulae for the theoretical modelling of the mentioned in section [I] experiments. It enables to find explicitly the propagation time of a signal, sent from one satellite to another but this formulae presupposes that the emitter and receiver of the signal are non-moving. But if they are moving, it cannot be expected that the distance travelled by the signal (this is in fact the geodesic distance) can be represented as a sum of the geometric distance and the logarithmic term in the Shapiro formulae multiplied by the velocity of light $c$. The formulae for the case of moving emitters and receivers will be another, and this will become evident from formulae (123) in section [VIIA] and formulae (130) in section [VIIIC].

Further in section [IID] the correspondence celestial time-eccentric anomaly angle is discussed. The reason is that the celestial time is proportional to the mean anomaly $M$, which is a numerical characteristics of the orbit. However, the really important characteristics for the elliptic motion is the eccentric anomaly angle $E$, which for known $M$ is found as solution of the Kepler equation. But at the same time, the null cone equation establishes a correspondence between the eccentric anomaly angle and the propagation time (see [IIE]). This means that the propagation time is a solution of the null cone equation (see section [IIG]), but the peculiar moment in the present investigation is that there are two propagation times - the propagation time $T_1$ of emission of the signal by the first satellite and a propagation time $T_2$ of reception of the signal, and these propagation times are the solutions of two null cone equations. In Section [IIF] a motivation is presented why a variable distance between an emitter and a receiver (or a variable baseline of a space interferometer such as RadioAstron) should be accounted by two gravitational null cones.

The next Section [III] has the purpose to find explicitly the formulae for the propagation time (the difference between the two propagation times) as a function of the eccentric anomaly angles of the two satellites. This is equivalent to solving the algebraic geometry
problem about the intersection of two null gravitational cones with a six-dimensional hyper-plane (Section III A), derived from the variable Euclidean distance. Section III D deals with the partial case of equal eccentric anomaly angles as characteristics of the orbits, after first clarifying what is the meaning of this notion.

The following two sections IV and VII are the most important contributions in this paper. Based on the initial physical concept about intersecting null cones, the two sections have the purpose to build up a detailed physical and mathematical theory of the space-time interval and of the geodesic distance. Since the geodesic distance is the distance travelled by light (or electromagnetic signal), it might seem that it should have a more important physical meaning in comparison with the space-time distance. But in fact, the space-time distance also has an important meaning, proposed for the first time in section IV I in this paper. Namely, the Euclidean distance may be considered as the positive distance measured on the intersection of two null four-dimensional null cones. This is a new and rather non-trivial moment, because a large-scale notion from celestial mechanics - the Euclidean distance, turns out to have another meaning and representation in terms of a notion with a ”broader” physical meaning - the space-time interval, which is related to General Relativity Theory. The explicite derivation of the expression for the space-time interval for the case of planar Keplerian (elliptic) motion has been performed in Section IV A. By ”broader” meaning it is meant that (positive) Euclidean distance is just one option for the space-time distance - besides positive, it can be also zero and negative (section IV B). The emergence of a negative (macroscopic) distance is not prohibited by geometry - these are the s.c. Lobachevsky geometries with a negative scalar curvature. On the other hand, the derivation of the formulae (45) for the space-time distance in Section IV C clearly suggests that there should be some compatibility between the space-time distance and the Euclidean one, especially when light or electromagnetic signals propagate a macroscopic distance. The mathematical expression of this compatibility is the ”condition for intersatellite communications”. It is important to stress that formulae (45) is fully legitimate and can be used independently from the compatibility condition (47). Then, for certain partial cases in Section IV E and in Section IV E, it can easily be established that the space-time interval can be of any signs. Particularly interesting is the simple proof in Section IV E that even for non-zero Euclidean distance, the space-time interval can also be negative. However, for the general case of non-equal eccentricities, semi-major axis and eccentric anomaly angles, it cannot become evident
whether or not the space-time distance can become zero, because the formulae represents a complicated polynomial of fourth degree. One of the main achievements of this paper is that even for such a complicated case and without solving the algebraic equation, it is possible to establish that the polynomial has roots. A general overview of the theorems from higher algebra is given in Section IV F and particularly in Section IV G. Among the several theorems from higher algebra, dedicated to polynomials with roots within the unit circle (see the monograph by Obreshkoff on higher algebra [18]), only two of the theorems are most appropriate to be applied - the Shur theorem in Section IV H and the so called "substitution theorem". Since the reader might not be familiar with these theorems, their mathematical proofs are given in Section X. The Shur theorem is applied to the fourth-order algebraic equation (63) for the space-time distance in Section XI and also the substitution theorem is applied to the same equation in Section XII. Both algebraic methods confirm that the space-time algebraic equation really does have roots within the unit circle, related to the chosen variable. So it is amazing that the results from the two higher algebra theorems are fully consistent with the simple algebraic analysis performed in Section IV E for the case of different eccentric anomaly angles, when the Euclidean distance is non-zero.

The next Section V gives the restriction on the eccentric anomaly angle for the assumed value of the eccentricity of the GPS orbit (see Section V A) and on the numerical value of the eccentricity of the orbit (see Section V B). The importance of these restrictions from a physical point of view will be discussed in the Discussion part of the paper.

Section VI does not propose a full solution to any problem and only discusses the perspectives for applying the mathematical formalism in this paper to the more complicated case of exchange of signals between satellites on different space-oriented orbits, which differ by the numerical values of the Keplerian orbital parameters. Surely there is a mathematical consistency for developing such an approach, but the main motivation outlined in Section VI A comes from the necessity for operational interaction between the satellites on different satellite constellations such as GPS, GLONASS and Galileo. As pointed out in the monograph [21] by Xu, a combined GNSS (Global Navigation Satellite System) of 75 satellites from the GPS, GLONASS and the Galileo constellations may increase greatly the visibility of the satellites, especially in critical areas such as urban canyons. However, the theoretical investigation (with account of General Relativity Theory) of the process of propagation of signals between satellites on different, space-oriented orbits contains a number of peculiar
moments, which are clarified in Sections VI D, VI E, VI F and VI G.

Further in Section VII the theory of the geodesic distance is exposed, again starting from the partial cases and afterwards treating the general case after applying a complicated higher algebra technique, which indeed confirms the main conclusion that the geodesic distance does not have the property to be negative or zero. In fact, confirming this important property entirely different from the space-time interval, is the key moment in this investigation. Firstly, the expression for the geodesic distance \(122\) is derived in Section VII A by means of substituting the derived compatibility ”condition for intersatellite communications” \(47\) in the space-time distance formulae \(45\). So the geodesic distance is a ”further step” in the theory after finding the space-time distance. In Sections VII B and VII C some subcases are investigated for the geodesic distance. Firstly, the subcase of equal eccentricities, semi major axis and eccentric anomaly angles is considered, when the geodesic distance is zero as it should be, because for zero Euclidean distance (coinciding points on the orbit), the geodesic distance should also be zero (i.e. no propagation of any signals between coinciding points). Although this case is trivial, it serves as a consistency check of the correctness of the calculations. The second case in Section VII C is more interesting and corresponds to the case of different eccentric anomaly angles (non-zero Euclidean distance). Most remarkable is formulae \(130\), showing that the geodesic distance is greater than the Euclidean distance. This is a result similar to formulae \(1\) for the Shapiro time delay, where the time travelled by the signal (consequently the distance) is greater than the geometric time (and also the ”geometric”, Euclidean distance). But since formulae \(130\) is derived in the framework of the formalism of ”two null intersecting gravitational cones”, the result can be interpreted as a clear evidence about the physical consistency of this formalism. It is interesting to note that the additional (second) term under the square root in this formulae is positive due to the found in Section VB restriction \(e \leq 0.816496580927726\) (formulae \(89\)), which follows from the condition for intersatellite communications. Therefore, this restriction on the ellipticity plays an important role for the positivity of the geodesic distance.

The greatness of the geodesic distance in comparison with the Euclidean one is proved also in Section VII D in the general case by substituting inequality \(137\) from the condition for intersatellite communications \(132\) into expression \(123\) for the differences between the squares of the geodesic distance and the Euclidean one. In Section VII E a numerical value for the lower bound of the eccentricity anomaly angle is obtained, but the condition for
intersatellite communications gives a higher bound.

In the general case of different eccentricities, semi-major axis and eccentric anomaly angles, the geodesic distance assumes the form of a fourth-degree algebraic equation (150), which was obtained in Section VII F and in Section VII G. The equation has been analyzed in Section XIII again by applying the Shur theorem and afterwards - the substitution theorem in Section XIV. Both higher algebra methods confirm that the fourth-order algebraic equation for the geodesic distance (309) does not have any roots, which is fully consistent with the previous considerations in Section VII C and in Section VII D about the positivity of the geodesic distance.

The results from the application of the higher algebra theorems to the space-time equation in Section IV and to the geodesic equation in Section VII are valid only under the assumption of the smallness of the eccentricity of the GPS orbit, but this fully corresponds to the real small value of the eccentricity.

In the Discussion section VIII some of the obtained results are summarized, but the emphasis is on the importance and consistency between the different numerical parameters, obtained as a result of the proposed new theoretical formalism.

II. PHYSICAL ARGUMENTATION FOR THE NEW APPROACH OF TWO GRAVITATIONAL NULL CONE EQUATIONS

A. World function in GPS theory and relation to the null-cone equation

The ”point positioning problem” in GPS theory means that if \( \{t_j, r_j\} \) \( (j = 1, 2, 3, 4) \) are respectively the time of the transmission events and the positions of the four satellites [5], then the position of the station on the ground and the time can be found from the s.c. ”navigation equations” (see also the review article by Neil Ashby [10]). All other issues of GPS satellite-ground station communications - Earths rotation, determination of the geoid, the gravitational frequency shifts and the second-order Doppler shifts are treated in the contemporary review articles [3–11].

For high accuracy measurements and determination of coordinate positions and time, performed over large distances (long baselines of at least 1000 kilometers) for the purposes of space-based interferometers, the navigation equations in the framework of the GRT are
modified and replaced by the two-point world function $\Omega(P_1, P_2)$ (initially determined in Synge monograph [26]), accounting for the delay of the electromagnetic signals due to the presence of the gravitational field. The physical meaning of the world function is that the flat-space null cones are replaced by the null geodesic equations [27]. In other words, from a mathematical point of view navigation in a curved space-time means that a set of four unique null geodesics connecting four emission events to one reception event should exist. This is an important theoretical fact meaning that each point of the orbit at which the satellite emits or intercepts a signal can be connected to a null geodesics. Note that this concerns the case when one null geodesics connects the signal-emitting and the signal-receiving points (i.e. satellites).

In the case investigated in this paper, when these points are moving and each one of them is related to its own null cone, the situation will be quite peculiar and a special condition will be derived. Since the world function is one-half the square of the space-time distance between the points $P_1$ and $P_2$, the fulfillment of the null-cone equation $ds^2 = 0$ is equivalent to a null cone value $\Omega(P_1, P_2) = 0$ of the world function.

**B. Shapiro time delay in VLBI radio interferometry, instantaneous and variable Euclidean distance**

The purpose of this section is to remind the basic assumptions, concerning the derivation of the Shapiro delay formulae. Since the formulae is valid for fixed (non-moving) space points of the emitter and the receiver, it will turn out that it is inappropriate to be used with respect to emitters and receivers on moving satellites.

Now for a moment we shall denote by $t=TCG$ the Geocentric Coordinate Time and we shall keep the notation $T$ for the propagation time of the signal between two space points. If the coordinates of the emitter on the first satellite and of the receiver on the second satellite are correspondingly $|x_A(t_A)| = r_A$ and $|x_B(t_B)| = r_B$, and $R_{AB} = |x_A(t_A) - x_B(t_B)|$ is the Euclidean distance between the signal-emitting satellite and the signal-receiving satellite, then from the null cone equation, the signal propagation time $T_{AB} = T_B - T_A$ can be expressed by the known formulae [28] (see also [29]) and also the review article [30] by
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\[ T_{AB} = \frac{R_{AB}}{c} + \frac{2GM_E}{c^3} \ln \left( \frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}} \right), \]  

(1)

where \( GM_E \) is the geocentric gravitational constant and \( M_E \) is the Earth mass. Note the important moment that \( R_{AB} \) is the Euclidean distance between the space point of emission at the emission time \( T_A \) and the point of reception of the signal at the reception time \( T_B \). In other words, this is a distance, depending on two different moments of time. The second term in formulae (1) is the Shapiro time delay, accounting for the signal delay due to the curved space-time. For low-orbit satellites, the Shapiro delay is of the order of few picoseconds. However, since in real experiments only the time of emission \( T_A \) is known instead of the time of reception \( T_B \), the use of the Euclidean distance \( R_{AB} \) in (1) is not very appropriate. Instead, as pointed out in the paper [28], one may use the s.c. "instantaneous distance" \( D_{AB} = x_A(T_A) - x_B(T_A) \) (defined at the moment \( T_A \) of emission of the signal) and the Taylor decomposition of \( x_B(t_B) \) around the moment of time \( t_A \). The resulting expression, however, does not possess the symmetry \( A \leftrightarrow B \), i.e. station \( A \) and satellite \( B \) cannot be interchanged [28] (see also the PhD thesis of Duchayne [31]). The lack of symmetry results in the fact that the relative motion between the emitter (the station) and the receiver cannot be accounted.

It can be concluded that the dependence of the propagation time \( T_{AB} \) on the distance \( R_{AB} \), determined at two different moments of time, leads to a loss of accuracy if \( R_{AB} \) is to be replaced by the instantaneous distance \( D_{AB} \). The complexity of the situation arises because in defining the Euclidean distance \( R_{AB} \), one has to keep account of the changes in the location of the space-time points and also of the correspondence between these space-points and the definite moments of time.

C. Coordinate parametrizations for moving satellites and the variable baseline distance \( R_{AB} \)

From the above point of view, it will be very convenient to find some new parametrization for the two space-time coordinates (defining the Euclidean distance), so that these new parametrization variables will be related in a prescribed way to a time variable, accounting for the motion of the satellites. This is the essence of the approach, developed in this paper.
for the case of two satellites moving along two elliptical orbits on one plane. For such a case of plane motion, the most convenient variable turns out to be the eccentric anomaly angle \( E \), which parametrizes the two \( x - y \) coordinates in the framework of the standard Kepler motion in celestial mechanics. In this approach, each satellite trajectory is parametrized by its own eccentric anomaly, so two eccentric anomaly angles \( E_1 \) and \( E_2 \) are used. Their values are taken at one and the same moment of time and since the eccentric anomaly angles change with time, the Euclidean distance \( R_{AB} \) also changes with time. However, since further it will be shown that the space-time interval, the condition for intersatellite communications and the geodesic distance do not depend on the propagation time explicitly, then the two eccentric anomaly angles \( E_1 \) and \( E_2 \) might be taken also at two different moments of time. In such a way, the variable Euclidean distance can be accounted, because there will be an initial and also a final value for the first eccentric anomaly angle, and an initial and a final value for the second angle.

This theoretical approach is unlike the standard approach for calculating the Shapiro delay in formulae (1), where \( R_{AB} \) has to be constant so that the first term \( \frac{R_{AB}}{c} \) will have the dimension of time.

### D. The correspondence celestial time - eccentric anomaly angle

The time coordinate is chosen to be the celestial time \( t_{cel} \), which is related to the eccentric anomaly through the Kepler equation

\[
E - e \sin E = n(t_{cel} - t_p) = M ,
\]

where \( e \) is the eccentricity of the orbit, \( n = \sqrt{\frac{GM}{a^3}} \) is the mean motion and \( M \) is the mean anomaly. The mean anomaly \( M \) is an angular variable, which increases uniformly with time and changes by \( 360^0 \) during one revolution. Let us remind also the geometrical meaning of the mean motion: this is the motion of the satellite along an elliptical orbit, projected onto an uniform motion along a circle with a diameter equal to the large axis of the ellipse. Usually \( M \) is defined with respect to some reference time - this is the time \( t_p \) of perigee passage, where the perigee is the point of minimal distance from the foci of the ellipse (the Earth is presumed to be at the foci). Since the eccentric anomaly \( E \) will play an important role in the further calculation, let us also remind how this notion is defined: if from a
point on the ellipse a perpendicular is drawn towards the large axis of the ellipse, then
this perpendicular intersects a circle with a centre \( O \) at the centre of the ellipse at a point
\( P \). Then the eccentric anomaly represents the angle between the joining line \( OP \) and the
semi-major axis (the line of perigee passage).

In \cite{32} \( E \) is determined as an auxiliary angular variable such that \( a - r = ae \cos E \),
which has the following geometrical meaning with respect to the ellipse: if \( r_p \) and \( r_a \) are
the corresponding radius-vectors at the perigee passage and at the apogee, then \( E = 0 \) for
\( r = r_p \) and \( E = \pi \) for \( r = r_a \).

All these notions can be found as well in the standard textbooks on celestial mechanics\
\cite{12,37,39,44,46,64} and in the books on theoretical geodesy \cite{14,48,49}. Extensive
knowledge about the most contemporary aspects of celestial mechanics can be found in the
recent textbook by Gurfil and Seidelmann \cite{50}. Note that the geometrical meaning of the
other three orbital parameters (\( \Omega, I, \omega \)) will also be outlined briefly in this paper, because
they are important for the space determination of the orbits. This will be necessary to be
done, when creating a theory of intersatellite communications between satellites on different
(space) Kepler orbits, characterized by the full set of six Kepler parameters (\( M, a, e, \Omega, I, \omega \)).
For example, this might be a theory of ISC between GPS, GLONASS and Galileo satellite
constellations, situated on different orbital planes.

Further, for concrete numerical values of the mean motion \( M \) in the framework of the
numerical characteristics of the GPS orbit, the numerical values for the eccentric anomaly \( E \)
will be calculated from the Kepler equation (2). The peculiar moment in such a consideration
is that the correspondence between \( E \) and the celestial time \( t_{cel} \) is not unique, because
\( E \) is a solution of the transcendental Kepler equation (2). This means that for a given
value of the celestial time, the solution of (2) with respect to \( E \) is given in terms of an
iterative procedure. The more iterations are performed, then the more exact will be the
correspondence \( E \iff t_{cel} \).

E. The correspondence propagation time - eccentric anomaly angle

The second correspondence, which will be established is between the propagation time \( T \)
and the eccentric anomaly angle \( E \). This is not a trivial correspondence because propagation
time is an intrinsic characteristic of the propagation of signals, which according to General
Relativity Theory takes place on the gravitational null cone \( ds^2 = 0 \) and the eccentric anomaly \( E \) is a notion from celestial mechanics, based on the Newton equation. But it turns out that these two characteristics are related - while moving along the (two-dimensional, one-plane) elliptical orbit parametrized by the equations

\[
x = a(\cos E - e) \quad , \quad y = a\sqrt{1 - e^2}\sin E
\]

the emitter of the first satellite emits a signal propagating on the gravitational null cone

\[
ds^2 = 0 = g_{00}c^2dT^2 + 2g_{0j}cdTdx^j + g_{ij}dx^idx^j
\]

For the null-cone equation (4), the solution of this quadratic algebraic equation with respect to the differential \( dT \) can be given as

\[
\frac{dT}{\pm} = \frac{1}{c} \frac{1}{\sqrt{-g_{00}}} \sqrt{\left( g_{ij} + \frac{g_{0i}g_{0j}}{-g_{00}} \right) dx^i dx^j + \frac{1}{c} \left( \frac{g_{0j}}{-g_{00}} \right) dx^j}
\]

where the metric tensor components are determined for an Earth Reference System with an origin at the centre of the Earth. If the space coordinates are parametrized in terms of the Keplerian (plane) elliptic orbital parameters (semi-major axis \( a \), eccentricity \( e \) and eccentric anomaly angle \( E \)), then formulae (5) is the mathematical expression of the correspondence eccentric anomaly angle \( E \rightarrow \) propagation time. After integrating, the propagation time can be found as

\[
T_1 = \pm \int_{E_0}^{E_1} \frac{1}{\sqrt{-g_{00}}} \sqrt{\left( g_{ij} + \frac{g_{0i}g_{0j}}{-g_{00}} \right) dx^i dx^j \overline{dE dE}} + \frac{1}{c} \int_{E_0}^{E_1} \frac{g_{0j}}{-g_{00}} \overline{dx^j dE} = \int_{E_0}^{E_1} M^{(1)}(e_1, a_1, g_{00}(x_1, y_1), g_0(x_1, y_1), g_{ij}(x_1, y_1))dE
\]

This is the time \( T_1 \) for propagation of the signal, while the eccentric anomaly angle of the (first) satellite changes from some initial value \( E = E_{init} = E_0 \) (for example - at the initial time of perigee passage \( t = t_{per} \)) to the final value \( E = E_1 \). More about the determination
of the propagation time $T_1$ as an initial moment of time of emission of the signal will be clarified in Section II G. The second propagation time $T_2$ can be written analogously, and the actual propagation time for the signal to travel from the emitter of the first satellite to the receiver of the second satellite is $T_2 - T_1$.

F. Two gravity null cones and the variable baseline distance $R_{AB}$

Note that the emission time $T$ is the time coordinate in this metric and the space coordinates are in fact the parametrization equations (3) for the (first) elliptic orbit. Further, the signal is intercepted by the receiver of the second satellite and this signal is propagating on a (second) null cone $ds^2_{(2)} = 0$, where the metric tensor components $g_{00}$, $g_{0j}$ and $g_{ij}$ are determined at a second space point $x_2$, $y_2$, parametrized again by the equations (3) in terms of new orbital parameters $a_2$, $e_2$ and $E_2$. The peculiar and very important feature of the newly proposed formalism in this paper is that we have two gravity null cones $ds^2_{(1)} = 0$ and $ds^2_{(2)} = 0$ for the emitted and the received signal (with time of emission $T_1$ and time of reception $T_2$) with cone origins at the points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. Also an equation about the differential of the square of the Euclidean distance $dR_{AB}^2$ is written, which now is a variable quantity. Thus, it can be noted that the two propagation times $T_1$ and $T_2$ are no longer treated in the framework of just one null cone equation (as is the case with the known equation (11) for the Shapiro time delay), but in the framework of two gravitational null cone equations. The derivation, the simultaneous solution of these three equations and some physical consequences of the found solution in terms of concrete numerical parameters for the GPS orbit are the main objectives of this paper. It should be stressed that according to the theory of propagation of electromagnetic signals in the gravitational field of (moving) bodies developed by S. Kopeikin [51–54], the interaction between the light and the gravitational field should be described in terms of two null cones [52] - the gravitational and the electromagnetic one. In the approach of Kopeikin, the null gravitational and light cones are situated at the light-deflecting body (Jupiter) and at the observer on the Earth. Since the light ray originates from a very distant quasar, there is no necessity of considering a second (moving) null gravitational or light cone with a centre coinciding with the quasar. In our case, the relative motion between the satellites is notable and is accounted by means of introducing two null (gravitational) cones.
G. Coordinate propagation time as a solution of the gravity null cone equation

The propagation times $T_1$ and $T_2$ are coordinate times, which by definition are determined for a region of space with a system of space-time coordinates chosen arbitrary. The coordinate time is an independent variable in the equations of motion for material bodies and in the equations for the propagation of electromagnetic waves. Examples of conventionally defined coordinate times are the Terrestrial Time ($TT$), the Geocentric Coordinate Time ($T_{CG}$), defined for the space around the Earth and also the Barycentric Coordinate Time ($T_{CB}$), defined for the region inside the Solar System. For the concrete case of the null-cone equation $ds^2 = 0$, it sets up a mathematical correspondence between the eccentric anomaly $E$ and the propagation time $T$ and thus the correspondence $T \iff E$ is realized. This is so, because the metric tensor components $g_{00}$, $g_{0i}$, $g_{ij}$ depend on the celestial coordinates (3), which in turn are expressed by the eccentric anomaly angle $E$. The propagation time will depend on the eccentric anomaly from some initial moment of time $t_0$ (of perigee passage) to some final moment of time when $E = E_{fin}$. After the initial moment of time, when $E > E_{init}$, the emitted electromagnetic signal from the satellite is completely decoupled from the motion of the satellite, but nevertheless it ”keeps track” of the position of the satellite along the elliptical orbit via the eccentric anomaly angle $E$. Let us note that the actual emission of the signal (when the signal decouples from the motion of the satellite) from the emitter of the first satellite is at the position on the orbit with an eccentric anomaly angle $E_1$, but in order to determine quantitatively this initial moment $T_1$ of emission, an additional assumption is made that formulae (6) gives the propagation time $T_1$ of the signal during that period of time while the satellite changes its position from the initial moment of perigee passage $t = t_{per}$ to the moment when the satellite will have an eccentric anomaly angle $E_1$. In such a way, together with the previously established correspondence $E \iff t_{cel}$, the correspondence between the celestial time and the propagation time is realized.

But if there is such a correspondence, it is natural to ask whether there is some standard (unit) for measuring both the celestial and the propagation times? The answer is affirmative, and this standard is the proper time, which is the actual reading of the atomic clock (the local time) in an inertial frame of reference at rest with respect to the reference frame, related to the coordinate time. In this paper we shall not deal with the proper time - this
shall be postponed elsewhere.

III. SIGNAL PROPAGATION TIMES FROM THE INTERSECTION OF THE TWO NULL GRAVITATIONAL CONES

A. General considerations about intersecting algebraic equations and the correspondence eccentric anomaly angles - propagation times

Having in mind the argumentation in the previous section, it is important to derive a formulae for the propagation time of a signal for the case, when both the emitter and the receiver are moving. We shall consider in this section only satellites moving along different elliptical orbits (different semi-major axis \(a_1, a_2\), eccentricity parameters \(e_1, e_2\) and eccentric anomalies \(E_1, E_2\)).

Let the gravitational null cone metric for the signal emitted by the first satellite at the space point \((x_1, y_1, z_1)\) is

\[
d s_1^2 = 0 = -(c^2 + 2V_1)(dT_1)^2 + (1 - \frac{2V_1}{c^2}) ((dx_1)^2 + (dy_1)^2 + (dz_1)^2) \tag{7}
\]

and the null cone metric for the second signal - receiving satellite is

\[
d s_2^2 = 0 = -(c^2 + 2V_2)(dT_2)^2 + (1 - \frac{2V_2}{c^2}) ((dx_2)^2 + (dy_2)^2 + (dz_2)^2) \tag{8}
\]

Note that often in the literature the square of the space distance is written as \(dx^2 + dy^2 + dz^2\) and not in the way as are written in the above formulae \(((dx_1)^2 + (dy_1)^2 + (dz_1)^2)\). In fact, the square of the space-time distance is \(ds^2 = g_{\mu\nu}dx^\mu dx^\nu \ (\mu, \nu = 0, 1, 2, 3)\), so this is an expression quadratic in the differentials while \(dx^2 + dy^2 + dz^2\) might be understood as \(2x dx + 2y dy + 2z dz\). For the use of the notation in (7) and (8), the interested reader may consult the monograph [67].

The first system of coordinates \(dT_1, dx_1, dy_1, dz_1\) for the null cone (7) are related to the emission time \(T_1\) of the first satellite and the differential \(dT_2\) in the second system of coordinates of the null cone (8) - to the time of reception of the signal by the second satellite. The space coordinates \(dx_2, dy_2, dz_2\) of the second satellite are again determined
at the initial moment of emission of the first satellite. However, the evolution of the space coordinates \( x_2, y_2, z_2 \) up to the moment of reception of the signal depends on the celestial motion of the satellite. Consequently, during the propagation of the signal from the first to the second satellite, the second satellite moves from the initial coordinates \( x_2, y_2, z_2 \) to the final coordinates \( x_2^{(fin)}, y_2^{(fin)}, z_2^{(fin)} \) of reception of the signal, and this evolution is governed by the Kepler equation \([2]\). But at the same time, these space coordinates enter the two gravitational null cone equations, which physically means that the propagation times \( dT_1 \) and \( dT_2 \), found from the intersection of these two null cones, also ”keep track” of the (constantly changing) positions of the two satellites. In other words, the space coordinates in the null cone equations simply are parametrized in terms of variables, related to the motion of the satellites. So this parametrization does not have relation with any ”mixing up” of the coordinates of the satellite and the space points of propagation of the signal.

It is important to mention that finding the differentials of the propagation times \( dT_1 \) and \( dT_2 \) turns out to be a complicated problem from algebraic geometry. In fact, the changing positions (Euclidean distance) between the satellites mean that the two four-dimensional null cones have to be additionally intersected with the six-dimensional hyperplane, which will depend also on the variable Euclidean distance (meaning that \( dR^2_{AB} \neq 0 \)). After finding the differentials \( dT_1 \) and \( dT_2 \) as solutions of the intersecting algebraic variety of algebraic equations, the solutions of the obtained complicated differential equations with respect to \( dT_1 \) and \( dT_2 \) will give the evolution of the propagation time as a function of the changing positions of both satellites (i.e. changing Euclidean distance). From the standard expression for the Euclidean distance between the points \( A \) and \( B \)

\[
R^2_{AB} = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2
\]  

(9)

one may obtain after differentiation the following hyperplane equation for the case of variable Euclidean distance (i.e. \( dR^2_{AB} \neq 0 \))

\[
dR^2_{AB} = 2(x_1 - x_2)d(x_1 - x_2) + \\
+ 2(y_1 - y_2)d(y_1 - y_2) + 2(z_1 - z_2)d(z_1 - z_2).
\]  

(10)

This is a 6–dimensional hyperplane in terms of the variables \( dx_1, dy_1, dz_1, dx_2, dy_2, dz_2 \), which intersects the two four-dimensional cones. Consequently, the notions of Euclidean distance \( R_{AB} \) and the propagation times \( T_1 \) and \( T_2 \) are closely related to the intersection
variety of the hyperplane with the two gravitational cones. In a subsequent paper, this complicated algebraic geometry approach \[36\] will be applied in the general case for the propagation of signals between satellites on different space orbits, characterized by the full set of 6 Keplerian parameters. This problem is important in view of the operational interaction (transmission of signals) between the satellites, belonging to different satellite constellations - GPS, GLONASS and Galileo. Further in this paper we shall deal only with the two-dimensional case of satellites on one and the same elliptical orbit or on one - plane non-intersecting orbits.

**B. Two types of differentials and the equation in full derivatives with respect to the square of the Euclidean distance**

Our further aim will be to find a relation between the Euclidean distance \( R_{AB} \) (a notion from Newtonian mechanics) and the variables in the null cone equations. As previously, the two plane elliptical orbits are parametrized by the following equations

\[
x_1 = a_1 (\cos E_1 - e_1) \quad , \quad y_1 = a_1 \sqrt{1 - e_1^2 \sin E_1} \quad ,
\]

\[
x_2 = a_2 (\cos E_2 - e_2) \quad , \quad y_2 = a_2 \sqrt{1 - e_2^2 \sin E_2} \quad .
\]

In order to relate the differential \( dR_{AB} \) to the space-coordinate differentials in the null cone equations (7) and (8), let us consider the differential

\[
d(x_1^2 + x_2^2) = d(x_1^2 + x_2^2 - 2x_1x_2 + 2x_1x_2) =
\]

\[
= d \left[ (x_1 - x_2)^2 + 2x_1x_2 \right] .
\]

Performing the same for the \( y_1, y_2 \) coordinates and summing up with (13), one can obtain

\[
d(x_1^2 + x_2^2 + y_1^2 + y_2^2) = d \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right] +
\]

\[
+ 2 \left[ x_2dx_1 + x_1dx_2 + y_2dy_1 + y_1dy_2 \right] .
\]

Substituting the coordinate expressions (11) and (12) and keeping in mind that \( R_{AB}^2 = [(x_1 - x_2)^2 + (y_1 - y_2)^2] \), one can derive

\[
d(x_1^2 + x_2^2 + y_1^2 + y_2^2) = dR_{AB}^2 +
\]

25
\[ S_1(E_1, E_2) := -2[a_1a_2 \sqrt{(1 - e_1^2)(1 - e_2^2)} \sin E_2 \cos E_1 + a_1a_2 \sin E_1 \cos E_2 - e_2a_1a_2 \sin E_1] \]  

The expression for \( S_2(E_1, E_2) \) is the same as the previous one, but with interchanged \( E_1 \leftrightarrow E_2 \), i.e. \( S_2(E_1, E_2) = S_1(E_2, E_1) \).

The next step is to compare the differentials \((dx_1)^2 + (dy_1)^2\) and \((dx_2)^2 + (dy_2)^2\), written in terms of the elliptical coordinates (11) and (12). Eliminating the term \([a_1^2(1 - e_1 \cos E_1)dE_1]\) in the expressions for the two differentials, one can obtain

\[ d(x_1^2 + y_1^2) = 2e_1 \sin E_1 \frac{[(dx_1)^2 + (dy_1)^2]}{dE_1} . \]  

Analogous expression can be obtained also for \((dx_2^2 + y_2^2)\). Now it can be noted that \([(dx_1)^2 + (dy_1)^2]\) and \([(dx_2)^2 + (dy_2)^2]\) can be expressed also from the gravitational null cone equations (7) and (8)

\[ (dx_1)^2 + (dy_1)^2 = \frac{c^2(c^2 + 2V_1)}{(c^2 - 2V_1)}(dT_1)^2 , \]  

and the second expression is the same but with the indice ”1” replaced by ”2”.

Substituting these expressions into (17) and into the analogous formulae for \((dx_2^2 + y_2^2)\), one can obtain a second representation for \((dx_1^2 + x_2^2 + y_1^2 + y_2^2)\)

\[ d(x_1^2 + x_2^2 + y_1^2 + y_2^2) = P_1(E_1) \frac{(dT_1)^2}{dE_1} + P_2(E_2) \frac{(dT_2)^2}{dE_2} , \]  

where \( P_1(E_1) \) is the expression

\[ P_1(E_1) := \frac{2e_1 \sin E_1}{(1 + e_1 \cos E_1)} \frac{c^2(c^2 + 2V_1)}{(c^2 - 2V_1)} . \]  

and the expression for \( P_2(E_2) \) is the same, but with the indice ”1” replaced by ”2”.

Setting up equal the two equivalent representations (15) and (19), one can find the following symmetrical relation

\[ dR_{AB}^2 + S_1(E_1, E_2)dE_1 + S_2(E_1, E_2)dE_2 = \]

\[ = P_1(E_1) \frac{(dT_1)^2}{dE_1} + P_2(E_2) \frac{(dT_2)^2}{dE_2} . \]  

(21)
C. The signal propagation times for the two satellites as two-point time transfer functions of the two eccentric anomaly angles

From the last equality $dT_2$ can be expressed if $dT_1$ is known. It follows also that if $T_1$ is a function only of the first eccentric anomaly angle, i.e. $T_1 = T_1(E_1)$, then the second propagation time $T_2$ for the process of signal propagation from the second satellite to the first will depend on both eccentric anomaly angles. Since the obtained relation is symmetrical with respect to the interchange of the indices 1 and 2, it is impossible to derive two expressions for the two propagation times $T_1$ and $T_2$.

If the expression for the square of the differential $(dT_2)^2$ in its standard form

$$(dT_2)^2 = \left( \frac{\partial T_2(E_1, E_2)}{\partial E_1} \right)^2 (dE_1)^2 +$$

$$+ 2 \frac{\partial T_2(E_1, E_2)}{\partial E_1} \frac{\partial T_2(E_1, E_2)}{\partial E_2} dE_1 dE_2 +$$

$$+ \left( \frac{\partial T_2(E_1, E_2)}{\partial E_2} \right)^2 (dE_2)^2$$

(22)

is substituted into (21) and both sides are divided by $(dE_1)^2$, then the following cubic polynomial with respect to $dE_2 dE_1$ is obtained

$$Q_1 \left( \frac{dE_2}{dE_1} \right)^3 + Q_2 \left( \frac{dE_2}{dE_1} \right)^2 + Q_3 \left( \frac{dE_2}{dE_1} \right) + Q_4 = 0 ,$$

(23)

where the functions $Q_1(E_1, E_2)$, $Q_2(E_1, E_2)$, $Q_3(E_1, E_2)$ and $Q_4(E_1, E_2)$ are given in Appendix A.

Equation (23) can be transformed into an equation depending on the derivatives $\frac{\partial T_1}{\partial E_1}$ and $\frac{\partial T_2}{\partial E_1}$

$$\left[ \frac{\partial T_2}{\partial E_1} + \frac{P_1(E_1)}{2P_2(E_2)} \frac{\partial T_1}{\partial E_1} \left( \frac{dE_2}{dE_1} \right) \right]^2 = K ,$$

(24)

where $K$ can be represented as

$$K := G_1(E_1, E_2) \left( \frac{dE_2}{dE_1} \right)^2 + G_2(E_1, E_2) \left( \frac{dE_2}{dE_1} \right) .$$

(25)

The functions $G_1(E_1, E_2)$ and $G_2(E_1, E_2)$ are given by expressions (164) and (165) in Appendix A. Equation (24) enables to express the unknown derivative $\frac{\partial T_2}{\partial E_1}$ as

$$\frac{\partial T_2}{\partial E_1} = - \frac{P_1}{2P_2} \frac{\partial T_1}{\partial E_1} \frac{dE_2}{dE_1} + \epsilon \sqrt{K} ,$$

(26)
where \( \epsilon = \pm 1 \).

The interchange of the indices \( 1 \leftrightarrow 2 \) allows to find

\[
\frac{\partial T_1}{\partial E_2} = -\frac{P_2}{2P_1} \frac{\partial T_2}{\partial E_2} \frac{dE_1}{dE_2} + \epsilon \sqrt{G_1(E_2, E_1) \left( \frac{dE_1}{dE_2} \right)^2 + G_2(E_2, E_1) \left( \frac{dE_1}{dE_2} \right)} ,
\]

(27)

where \( G_1(E_2, E_1) \) (expression (166) in Appendix A) and \( G_2(E_2, E_1) \) are the functions \( G_1(E_1, E_2) \) and \( G_2(E_1, E_2) \), but with interchanged indices 1 and 2. If the derivative \( \frac{\partial T_1}{\partial E_2} \) is known, then the dependence of the second propagation time on \( E_2 \) can be found as

\[
\frac{\partial T_2}{\partial E_2} = -\frac{1}{2} \left( \frac{dE_1}{dE_2} \right) \left( 1 + 4 \frac{\partial T_1}{\partial E_2} \right) + \epsilon \sqrt{N} ,
\]

(28)

where the function \( N \) is also given in Appendix A.

Thus, after integration of (26) and (28), it is possible to find the dependence of the second propagation time \( T_2 \) on the eccentric anomaly angles \( E_1 \) and \( E_2 \). The corresponding integrals however are very complicated and not possible to be solved analytically. It is important that both expressions depend on the derivatives of the square of the Euclidean distance \( R_{AB}^2 \), which according to the parametrization equations (11) and (12) is given by the formulae

\[
R_{AB}^2 = [(a_1 \cos E_1 - a_2 \cos E_2) + (a_2 e_2 - a_1 e_1)]^2 + \\
+ \left[ a_1 \sqrt{1 - e_1^2} \sin E_1 - a_2 \sqrt{1 - e_2^2} \sin E_2 \right]^2 .
\]

(29)

D. Signal propagation times for satellites moving on elliptical orbits with equal eccentric anomalies

For orbits with equal eccentric anomalies (this notion will be clarified further) there can be two cases.

First case: Equal eccentricities \( e_1 = e_2 = e \) and semi-major axis \( a_1 = a_2 = a \).

This is the case when two or more satellites move along one and the same orbit. This corresponds to the satellite dispositions for the GLONASS, GPS and Galileo constellations. For the GPS and GLONASS constellations, four satellites are selected per plane [15], while in the Galileo constellation nine satellites are equally spaced per plane.

Second case. Equal eccentric anomalies, but different eccentricities and semi-major axis.
If one considers the first case then
\[ P_1 = P_2, \quad S_1 = S_2, \quad \frac{dE_1}{dE_2} = 1, \quad \frac{\partial R^2_{AB}}{\partial E} = 0. \]  \hspace{1cm} (30)

Consequently, expression (26) for \( \frac{\partial T_2}{\partial E} \) can be rewritten as
\[ \frac{\partial T_2}{\partial E} = -\frac{1}{2} \frac{\partial T_1}{\partial E} + \varepsilon \sqrt{G_1(E, E) + G_2(E, E)} \]  \hspace{1cm} (31)

where expressions \( G_1(E, E) \) and \( G_2(E, E) \) can be found from formulae (164) and (165) in Appendix A
\[ G_1(E, E) + G_2(E, E) = \frac{1}{4} \left( \frac{\partial T_1}{\partial E} \right)^2 + \frac{1}{2} \left( \frac{\partial T_1}{\partial E} \right) - \frac{S}{P}. \]  \hspace{1cm} (32)

After performing the integration in (31), one can obtain
\[ T_2 = -\frac{1}{2} T_1 + \varepsilon \int dE \sqrt{\left( \frac{\partial T_1}{\partial E} \right)^2 + \frac{1}{2} \left( \frac{\partial T_1}{\partial E} \right) - \frac{S}{P}}. \]  \hspace{1cm} (33)

From (28) for \( E_1 = E_2 = E \) (however, it is not assumed that \( \frac{\partial T_1}{\partial E_2} = 0 \)) one can also derive
\[ \frac{\partial T_2}{\partial E} = -\frac{1}{2} \left( 1 + 4 \frac{\partial T_1}{\partial E} \right) + \varepsilon \sqrt{2 \left( \frac{\partial T_1}{\partial E} \right)^2 + 2 \left( \frac{\partial T_1}{\partial E} \right) + \frac{2S}{P} + \frac{1}{4}}. \]  \hspace{1cm} (34)

From the equality of the expressions (30) and (31) for \( \frac{\partial T_2}{\partial E} \), one can obtain the following quartic algebraic equation with respect to \( \frac{\partial T_1}{\partial E} \):
\[ \frac{5}{4} \left( \frac{\partial T_1}{\partial E} \right)^4 + 6 \left( \frac{\partial T_1}{\partial E} \right)^3 + \left[ \frac{7}{4} + \frac{6S}{P} \right] \left( \frac{\partial T_1}{\partial E} \right)^2 + \left( \frac{1}{2} - \frac{3S}{P} \right) \frac{\partial T_1}{\partial E} - \left( \frac{S}{P} + \frac{9S^2}{P^2} \right) = 0. \]  \hspace{1cm} (35)

This equation can be solved as an algebraic equation with respect to \( \frac{\partial T_1}{\partial E} \) and then, after integration, the function \( T_1 = T_1(E) \) can be found. However, if it is assumed that \( \frac{\partial T_1}{\partial E_2} = 0 \), then the function \( T_1(E) \) can be found from the following integral
\[ T_1 = \int dE \left[ \frac{\frac{1}{2} + \frac{3S}{P} - \sqrt{\frac{1}{4} + \frac{2S}{P}}}{\left[ 1 - \sqrt{\frac{1}{4} + \frac{2S}{P}} \right]} \right]. \]  \hspace{1cm} (36)

In both cases, the obtained integrals are rather complicated and not possible to be solved analytically. It is interesting also to see from (33) the asymmetry and inequality between the two propagation times \( T_1 \) and \( T_2 \). In the next section it will be shown how to determine the upper integration boundary in the integral (36) and in the integral derived from (35).
IV. PHYSICAL AND MATHEMATICAL THEORY OF THE SPACE - TIME INTERVAL ON INTERSECTING GRAVITATIONAL NULL CONES FOR THE CASE OF NON-SPACE ORIENTED ORBITS

A. Derivation of the formulae for the space - time distance after integrating a differential equation in full derivatives

Let us first express from the null cone equations (7) and (8) the square of the differentials of the propagation times $\left(\frac{dT_1}{2}\right)^2$ and $\left(\frac{dT_2}{2}\right)^2$. Combining these expressions with the relation (20) (also with the one for the indice 2) and making use of the parametrization equations (11) and (12) for the elliptical orbit, one can obtain the simple relation

$$
\left(\frac{dT_1}{2}\right)^2 P_1(E_1) = a_1^2 (1 - e_1 \cos E_1) 2e_1 \sin E_1 (dE_1)^2.
$$

If we substitute this relation and the analogous one for $\left(\frac{dT_2}{2}\right)^2 P_2(E_2)$ into (21), then the following equation in full differentials with respect to the (variable) square of the Euclidean distance is obtained

$$
dR_{AB}^2 = F_1(E_1, E_2)dE_1 + F_2(E_1, E_2)dE_2,
$$

where $F_1(E_1, E_2)$ and $F_2(E_1, E_2)$ are the expressions

$$
F_1(E_1, E_2) := 2e_1 a_1^2 (1 - e_1 \cos E_1) \sin E_1 - S_1(E_1, E_2),
$$

$$
F_2(E_1, E_2) = 2e_2 a_2^2 (1 - e_2 \cos E_2) \sin E_1 - S_2(E_1, E_2)
$$

and as previously, $S_1(E_1, E_2)$ is given by (16), $S_2(E_1, E_2)$ is the analogous expression, but with interchanged indices. The conditions (38) to be an equation in full differentials are (see any textbook on differential equations, for example [16])

$$
F_1(E_1, E_2) = \frac{\partial R_{AB}^2}{\partial E_1}, \quad F_2(E_1, E_2) = \frac{\partial R_{AB}^2}{\partial E_2}.
$$

If the first equation is integrated then

$$
R_{AB}^2 = \int F_1(E_1, E_2)dE_1 + \varphi(E_2) =
$$

$$
= -2e_1 a_1^2 \cos E_1 + \frac{1}{2} e_1^2 a_1^2 \cos(2E_1) +
$$
+2a_1a_2\sqrt{(1 - e_1^2)(1 - e_2^2)} \sin E_1 \sin E_2 - \\
- 2a_1a_2 \cos E_1 \cos E_2 + 2e_2a_1a_2 \cos E_1 + \varphi(E_2) \quad , \quad (43)

where \( \varphi(E_2) \) is a function, which has to be determined from the second equation in (41). If from (43) the derivative \( \frac{\partial \sqrt{R_{AB}^2}}{\partial E_2} \) is calculated and then is set equal to \( F_2(E_1, E_2) \) given by expression (40), the following simple differential equation for \( \varphi(E_2) \) can be obtained

\[
\frac{\partial \varphi(E_2)}{\partial E_2} = \left( 2e_2a_2^2 - 2e_1a_1a_2 \right) \sin E_2 - e_2^2a_2^2 \sin(2E_2) \quad . \quad (44)
\]

If the equation is integrated and the result is substituted into (43), then the final expression for \( R_{AB}^2 \) is obtained

\[
R_{AB}^2 = \left( -2e_1a_1^2 \cos E_1 - 2e_2a_2^2 \cos E_2 \right) + \\
+ (2e_2a_1a_2 \cos E_1 + 2e_1a_1a_2 \cos E_2) + \\
\frac{1}{2} \left( e_1^2a_1^2 \cos(2E_1) + e_2^2a_2^2 \cos(2E_2) \right) - 2a_1a_2 \cos E_1 \cos E_2 + \\
+ 2a_1a_2\sqrt{(1 - e_1^2)(1 - e_2^2)} \sin E_1 \sin E_2 \quad . \quad (45)
\]

Note that this expression is symmetrical and does not change under interchange of the indices 1 and 2, as it should be. This is in fact the second representation for the square of the Euclidean distance, based on the equation (37) in full differentials, when \( R_{AB}^2 \) is expressed according to (45).

B. General idea about the positive, negative or zero space - time distance from the intersection of the two null cones

Relation (45) has been found from the intersection of the two null cones (7) and (8) and consequently, represents a General Relativity Theory (GRT) notion. It is more correct to call it "a space-time interval", which according to GRT can be either positive, negative or equal to zero. In fact, an important clarification should be made: GRT clearly defines what is a "gravitational null cone" and also a "space-time interval". But as mentioned before, GRT and also Special Relativity Theory [17] do not give an answer to the problem: will the intersection of two gravitational null cones again possess the property of the space-time interval, i.e. can it be again positive, negative and zero? The investigation of the "intersecting space-time interval" (45) in some partial simplified cases, but also in the general
case will give an affirmative answer to this problem. In other words, it will become evident
that this "intersecting" interval will again preserve the property of being positive, negative
or null.

At the same time, the function $R_{AB}^2$ in (45) is the Euclidean distance $R_{AB}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ in the initial formulae (29), which can be presented in a more symmetrical way

$$R_{AB}^2 = a_1^2 + a_2^2 + (a_2 e_1 - a_1 e_2)^2 - 2 a_1 a_2 \cos E_1 \cos E_2 -$$

$$- 2 a_1 a_2 \sqrt{(1 - e_1^2) (1 - e_2^2) \sin E_1 \sin E_2 - a_1^2 e_1^2 \sin^2 E_1 -}$$

$$- a_2^2 e_2^2 \sin^2 E_2 + 2 a_1 a_2 (e_2 \cos E_1 + e_1 \cos E_2) -$$

$$- 2 e_1 a_1^2 \cos E_1 - 2 e_2 a_2^2 \cos E_2 .$$

(46)

C. The compatibility condition for intersatellite communications

It turns out that $R_{AB}^2$ has two equivalent representations - the first representation as a
space-time interval (45), found from the intersection of the null cones, which can be positive,
negative or zero, and the second representation as an Euclidean distance (46), which can
be only positive. There is nothing strange that the Euclidean distance can play also the
role of a space-time interval, since it is found also from another equations (the gravitational
null cone equations). But in any case, they denote one and the same function denoted as
$R_{AB}^2$. The only possibility for the compatibility of the two representations is they to be
equal both to zero or to be both positive. However, as further it shall be explained, it is
not obligatory to impose the requirement for the compatibility of the two representations
- the space-time interval (45) can be treated as an independent notion from the Euclidean
distance. But in the case of light or signal propagation, when the two distances have to be
compatible because the signal travels a macroscopic distance, the two representations (45)
and (46) have to be set up equal. Then from the equality of the two representations (45) and
(46) for $R_{AB}^2$, one can obtain the following simple relation between the eccentric anomalies,
semi-major axis and the eccentricities of the two orbits

$$4 a_1 a_2 \sqrt{(1 - e_1^2) (1 - e_2^2) \sin E_1 \sin E_2 =}$$

$$= a_1^2 + a_2^2 + (a_2 e_2 - a_1 e_1)^2 - \frac{1}{2} \left( e_1^2 a_1^2 + e_2^2 a_2^2 \right) .$$

(47)
This relation can be conditionally called "a condition for intersatellite communications between satellites on (one-plane) elliptical orbits". It is obtained as a compatibility condition between the large-scale, Euclidean distance (46) and the space-time interval (45).

D. Positivity and negativity of the space-time interval for the case of equal eccentric anomaly angles, eccentricities and semi-major axis - consistency check of the calculations

The best way to understand the difference between the physical meaning of the space-time interval and the Euclidean distance with account of the condition (47) is to prove that the space-time interval for some specific cases can be of any signs, while the situation will turn out to be different for the geodesic distance. It is remarkable that the positivity of the geodesic distance will become evident when performing a simple algebraic substitution of the condition (47) into formulae (45) and at the same time, this will be confirmed by the analysis of a complicated algebraic equation of fourth degree.

Let us first write again the space-time interval (45) for the case of equal eccentricities, semi-major axis and eccentric anomaly angles ($e_1 = e_2 = e$, $a_1 = a_2 = a$, $E_1 = E_2 = E$)

$$R_{AB}^2 = 4a^2 \sin^2 E (1 - e^2) + a^2 (e^2 - 2)$$.

(48)

Now it is interesting to note that this space-time interval is positive for

$$\sin^2 E \geq \frac{2 - e^2}{4(1 - e^2)}$$,

(49)

but for

$$\sin^2 E \leq \frac{2 - e^2}{4(1 - e^2)}$$

(50)

it can be also negative. This fact for the partial case suggests that it should be so for the general case of different one from another eccentricities, large semi-major axis and eccentricity anomaly angles. However, the general case will be much more complicated, because the space-time distance (45) will turn out to be a fourth-degree algebraic equation with respect to the variable $y = \sin E_2$. By means of theorems from higher algebra and without solving this complicated equation, it will further be proved that the space-time distance again can be zero, negative or positive.
The lower bound for which $R_{AB}^2 \geq 0$ for the case of a typical GPS orbit with eccentricity $e = 0.01323881349526$ (see the PhD thesis of Gulklett) is given by the limiting value $E_{\text{lim}}$ for the eccentric anomaly angle

$$ E_{\text{lim}} = \arcsin \left( \frac{1}{2} \sqrt{\frac{2 - e^2}{1 - e^2}} \right) = 45.002510943228 \, [\text{deg}] . \quad (51) $$

Respectively, the upper bound $\frac{2 - e^2}{4(1 - e^2)}$ in (50), for which $R_{AB}^2 \leq 0$, can be found from $E \leq E_{\text{lim}}$. It is curious to note that if the condition for intersatellite communications is taken into considerations, neither of the two inequalities is realized. The reason is that for $e_1 = e_2 = e$, $a_1 = a_2 = a$, $E_1 = E_2 = E$ this condition gives the relation

$$ 4a^2(1 - e^2) \sin^2 E = 2a^2 - e^2a^2 \implies \sin E = \frac{1}{2} \sqrt{\frac{2 - e^2}{1 - e^2}} , \quad (52) $$

which, if substituted into the space-time interval (48), gives $R_{AB}^2 = 0$. This should be expected and in fact is a consistency check of the calculations because for equal eccentricities, semi-major axis and eccentric anomaly angles, the Euclidean distance is equal to zero. Then the compatibility condition (51), when substituted in the formulae for the space-time interval, should give also zero. The obtained result is fully consistent with what should be expected. However, the equality to zero of the space-time distance (but only for this specific case investigated) is ensured only when the compatibility condition is applied, which justifies its name. Without the compatibility condition, the space-time interval is different from zero for equal eccentricities, semi-major axis and eccentric anomaly angles, while the Euclidean distance is equal to zero.

E. Positive and negative space-time interval from non-zero Euclidean distance - the case of different eccentric anomaly angles

Now we shall investigate the other case of non-zero Euclidean distance, when two points on the two corresponding orbits do not coincide. This will be the case of equal eccentricities and semi-major axis, but different eccentric anomaly angles ($E_1 \neq E_2$). Two important facts will be proved:

1. The space-time interval can be positive or negative.
2. It can be equal to zero even when the Euclidean distance is equal to zero. The space-time interval (45) can be written as

\[
R_{AB}^2 = e^2 a^2 - e^2 a^2 (\sin E_1 + \sin E_2)^2 - 2a^2 \cos(E_1 + E_2) .
\] (53)

The space-time interval will be positive (i.e. \( R_{AB}^2 > 0 \)), if the following inequality is satisfied

\[
e^2 - e^2 (\sin^2 E_1 + \sin^2 E_2) + 2(1 - e^2) \sin E_1 \sin E_2 > 2 \cos E_1 \cos E_2 .
\] (54)

If we take into account the standard inequalities for the \( \cos \)-function

\[
\cos E_1 \leq 1 , \text{ } \cos E_2 \leq 1 ,
\] (55)

then the first two terms on the first line of the above inequality can be written as

\[
e^2 - e^2 (\sin^2 E_1 + \sin^2 E_2) = e^2 \cos^2 E_1 - e^2 - e^2 \cos^2 E_2 \leq e^2 - e^2 + e^2 = e^2 .
\] (56)

Substituting into inequality (54), it can be derived

\[
2 \cos E_1 \cos E_2 < e^2 + 2 \sin E_1 \sin E_2 ,
\] (57)

which can be represented also as

\[
\cos(E_1 + E_2) < \frac{e^2}{2} \implies E_1 + E_2 > \arccos\left(\frac{e^2}{2}\right) .
\] (58)

For the typical value of the eccentricity of the GPS orbit, it can be obtained

\[
E_1 + E_2 > 89.994978993712 \text{ [deg]} .
\] (59)

Note that the sign is greater because \( \cos \) is a decreasing function with the increase of the angle. This is valid for the first and the second quadrant, for the third and the fourth quadrant \( \cos \) is an increasing function and the sign should be the reverse one. Also the sign (for the angle within the first quadrant) should be the reverse one to the sign in (59), if the space-time interval is negative, i.e. (i.e. \( R_{AB}^2 < 0 \)). For the moment, we shall investigate the
case when the eccentric anomaly angle is in the first and the second quadrant. If one sets up \( E_1 = E_2 = E \) in (57), then (let us take again the case of positive space-time interval)
\[
\sin^2 E > \frac{1}{2} \left( 1 - \frac{e^2}{2} \right) \quad \Rightarrow \quad E > \overline{E} = \arcsin \frac{\sqrt{2 - e^2}}{2},
\]
where the numerical result for \( \overline{E} \) is twice as smaller than \( \overline{E} = 44.997489496856 \, [\text{deg}] \). (60)

It should be clarified that this numerical value is a little lower that the limiting value \( 45.002510943228 \, [\text{deg}] \) in the preceding section, because in the case the property (55) of the trigonometric functions has been used. In the previous section, the limiting value has been obtained as an exact value. Comparison between these two values will be performed in the Discussion part of this paper.

**F. Zero space-time interval from non-zero Euclidean distance - analysis of fourth-order algebraic equations by means of higher algebra theorems**

Now it remains to see when the space-time interval can be zero for the case of non-zero Euclidean distance. For the purpose, expression (53) can be written as
\[
2 \sqrt{(1 - \sin^2 E_1)(1 - \sin^2 E_2)} = e^2 - e^2(\sin^2 E_1 + \sin^2 E_2) + 2(1 - e^2) \sin E_1 \sin E_2 .
\]
(62)

After some transformations and introducing the notation \( \sin^2 E_1 = y \), the above expression can be presented in the form of a quartic (fourth-degree) algebraic equation
\[
y^4 + a_1 y^3 + a_2 y^2 + a_3 y + a_4 = 0 .
\]
(63)

The coefficient functions of this equation will be given in Appendix C. Consequently, the problem about finding those values of the eccentric anomaly angle \( E_1 \) for which the space-time interval (53) is zero is equivalent to the algebraic problem of finding all the roots of the above quartic (fourth-order) algebraic equation, which are within the circle \( | y | = | \sin^2 E_1 | < 1 \) (we exclude the boundary points \( y = \sin^2 E_1 = 1 \)). It is well-known that an algebraic equation of fourth degree will always possess roots. The problem is that these roots should be within the circle \( | y | < 1 \).
G. General overview of some higher algebra theorems about the existence of roots within the unit circle

In order to prove that equation (63) has roots within the circle \(|y| < 1\), the following theorem of Enestrom-Kakeya from higher algebra [18] shall be used:

**Theorem 1** If for a \(n\)-th degree polynomial

\[ f(y) = a_0y^n + a_1y^{n-1} + a_2y^{n-2} + \ldots + a_n \]  

(64)

one has

\[ a_0 > a_1 > a_2 > a_3 > \ldots > a_n > 0, \]  

(65)

then the roots of the polynomial \(f(y)\) are situated within the circle \(|y| < 1\).

In the monograph of Prasolov [19] this theorem has a slightly different formulation:

**Theorem 2** If all the coefficients of the polynomial (64) are positive, then for every root \(\zeta\) of this polynomial the following estimate is valid

\[ \min_{1 \leq i \leq n} \left\{ \frac{a_i}{a_{i-1}} \right\} = \delta \leq |\zeta| \leq \gamma = \max_{1 \leq i \leq n} \left\{ \frac{a_i}{a_{i-1}} \right\}. \]  

(66)

Since the Enestrom-Kakeya theorem shall not be used in this paper due to reasons which are given below, its proof will not be given in Appendix B. In Appendix C the coefficient functions \(a_1, a_2, a_3, a_4\) of the polynomial (63) will be presented.

The theorem of Enestrom-Kakeya has two major shortcomings:

1. It requires the fulfillment of the "chain" of inequalities \(a_0 > a_1 > a_2 > a_3 > a_4 > 0\). The last means that all coefficient functions should be positive and moreover, beginning from the free term (the coefficient \(a_4\)), each subsequent coefficient should be greater than the preceding one. This is a serious restriction, since in the present case the coefficient functions \(a_0, a_1, a_2, a_3, a_4\) are not positive.

2. The theorem represents only a necessary condition. This means that the theorem is appropriate to be used if the condition (65) (the chain of inequalities) can be proved to be valid and thus it will follow that the roots of the polynomial \(f(y)\) are within the circle \(|y| < 1\). However, since the theorem does not represent a necessary and sufficient condition,
the non-fulfillment of the condition (65) does not guarantee that the polynomial \( f(y) \) will not have any roots. Some other condition may exist so that the polynomial still might possess roots.

Another approach may be proposed with the aim to eliminate these shortcomings by means of combining the theorem of Enestrom - Kakeya with some other theorems. For example, for the first case those coefficient functions which are negative shall be taken with a negative sign (so that they will become positive), and in this way a new chain of coefficients \( b_0, b_1, b_2, b_3, b_4 \) shall be obtained, and further they shall be arranged in a definite order

\[
\begin{align*}
    b_4 &> b_2 > b_3 > b_1 .
\end{align*}
\]

Introducing the new notations

\[
\begin{align*}
    b_0^* &> b_1^* > b_2^* > b_3^* > b_4^* ,
\end{align*}
\]

so that

\[
\begin{align*}
    b_0^* &= b_4 ,
    b_1^* &= b_2 ,
    b_2^* &= b_3 ,
\end{align*}
\]

the following polynomial can be constructed

\[
\begin{align*}
    B(y) := b_0^* y^4 + b_1^* y^3 + b_2^* y^2 + b_3^* y + b_4^* ,
\end{align*}
\]

where the coefficient function \( b_4^* \) shall be determined later from a special condition. The polynomial \( B(y) \) shall be defined also in another representation as

\[
\begin{align*}
    B(y) := \bar{b}_0 + \left( \begin{array}{c} 4 \\ 1 \end{array} \right) \bar{b}_1 y + \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \bar{b}_2 y^2 + \left( \begin{array}{c} 4 \\ 3 \end{array} \right) \bar{b}_3 y^3 + \left( \begin{array}{c} 4 \\ 4 \end{array} \right) \bar{b}_4 y^4 .
\end{align*}
\]

At the same time, the original polynomial (64) shall be represented in another form

\[
\begin{align*}
    f(y) = A(y) := \bar{a}_0 + \left( \begin{array}{c} 4 \\ 1 \end{array} \right) \bar{a}_1 y + \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \bar{a}_2 y^2 + \\
    + \left( \begin{array}{c} 4 \\ 3 \end{array} \right) \bar{a}_3 y^3 + \left( \begin{array}{c} 4 \\ 4 \end{array} \right) \bar{a}_4 y^4 ,
\end{align*}
\]

which requires the calculation of the coefficient functions with the "bar" sign above. Then the assertion of the s.c. Grace theorem (see again Obreshkoff monograph [18]) is
Theorem 3 If the roots of the polynomial $B(y) \ (71)$ are within the circle $|y| < 1$ (which again should be checked by means of the Enestrom-Kakeya theorem), then the polynomial $f(y) = A(y) \ (64)$ (or (72)) has at least one root within the circle $|y| < 1$, provided also that the following relation holds between the coefficient functions $\overline{a}_0, \overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4$ and $\overline{b}_0, \overline{b}_1, \overline{b}_2, \overline{b}_3, \overline{b}_4$ of the two polynomials

$$\overline{a}_0 \overline{b}_4 - 4 \overline{a}_1 \overline{b}_3 + \frac{4.3}{2} \overline{a}_2 \overline{b}_2 - \frac{4.3.2}{3.2} \overline{a}_3 \overline{b}_1 + \overline{a}_4 \overline{b}_0 = 0 \ . \quad (73)$$

From the last relation, the coefficient function $b_0 = \overline{b}_0$ of the newly constructed polynomial $B(y)$ may be expressed. In the mathematical literature [18] and [19], polynomials $B(y) \ (71)$ and $A(y) \ (72)$, satisfying the condition (73) are called apolar polynomials.

This approach shall not be developed in this paper, because it requires rather tedious calculations, which at the end will not result in an equality or nonequality, giving the opportunity to make the conclusion whether the given polynomial has roots or not within the circle $|y| < 1$. The more serious reason is again in the lack of a necessary and sufficient condition in the formulation of the Grace theorem - this means that if the inequalities (67) $b_4 > b_2 > b_3 > b_1$ of the Enestrom-Kakeya theorem or the relation (73) are not fulfilled, then this by itself does not guarantee that the polynomial $A(y)$ will not have a root within the unit circle. In other words, the absence of a sufficient condition means that some other necessary condition instead of the inequalities (67) $b_4 > b_2 > b_3 > b_1$ may exist, so that the polynomial will have again roots within the unit circle. The formulation of the Shur theorem confirms this conclusion.

H. Shur theorem as a basic mathematical instrument for proving the existence of roots within the unit circle for the space-time algebraic equation

In this paper a preference is given to a theorem, which has a necessary and sufficient condition. This is the Shur theorem [18], which for the general $n-$dimensional case has the following formulation:

Theorem 4 (Shur) The necessary and sufficient conditions for the polynomial of $n-$th degree

$$f(y) = a_0 y^n + a_1 y^{n-1} + \ldots + a_{n-2} y^2 + a_{n-1} y + a_n \quad (74)$$
to have roots only in the circle $|y| < 1$ are the following ones:

1. The fulfillment of the inequality

$$|a_0| > |a_n|.$$  \hfill (75)

2. The roots of the polynomial of $(n-1)$ degree

$$f_1(y) = \frac{1}{y} [a_0 f(y) - a_n f^*(y)]$$  \hfill (76)

should be contained in the circle $|y| < 1$, where $f^*(y)$ is the s.c. "inverse polynomial", defined as

$$f^*(y) = y^a f\left(\frac{1}{y}\right) = a_n y^n + a_{n-1} y^{n-1} + \ldots + a_2 y^2 + a_1 y + a_0.$$  \hfill (77)

In case of fulfillment of the inverse inequality

$$|a_0| < |a_n|$$  \hfill (78)

the $(n-1)$ degree polynomial $f_1(y)$ (again with the requirement the roots to remain within the circle $|y| < 1$) is given by the expression

$$f_1(y) = a_n f(y) - a_0 f^*(y).$$  \hfill (79)

The proof of this theorem, taken from the Obreshkoff monograph \[18\], will be presented in Appendix B. Concerning the necessary conditions, the Shur theorem has one another advantage - if the condition \hfill (76) (or \hfill (79)) about the roots of the polynomial $f_1(y)$ is not fulfilled, then the polynomial $f(y)$ will not have any roots within the circle. This allows to apply the theorem not only with respect to the space-time interval algebraic equation (which shall be proved to have roots in Appendix C), but also with respect to the geodesic equation, which should not have any roots within the circle $|y| < 1$ (this shall be proved in Appendix E). The last fact shall be confirmed by independent calculations, since it shall be proved in the next sections that the geodesic distance is greater than the Euclidean distance, so it cannot become zero. This is fully consistent from a physical point of view, since it is not occasional that light or signal propagation is related to the geodesic distance and not to the space-time interval, which can also be equal to zero or even become negative. In this aspect, it is really amazing how the physical interpretation is consistent with the mathematical results about these two algebraic equations. It is important to mention that
these conclusions are valid in view of the fact that the eccentricity \( e \) is very small (in celestial mechanics, it is of the order of 0.01), and on the base of this it is possible to compare terms with inverse powers in \( e \) in the corresponding inequalities - the higher inverse powers in \( e \) will lead to a larger number. For example, a term of the order of \( \frac{1}{e^2} \) will give a number of the order of 10000, but as it will be shown, there will be terms proportional to \( \frac{1}{e^{10}} \), \( \frac{1}{e^{12}} \) and even \( \frac{1}{e^{14}} \), which are extremely large numbers. It is important that terms which differ by two orders in inverse powers of \( e \) will have greatly different numerical values.

On the base of such analysis, the Shur theorem gives the opportunity not only to prove the existence of roots for the space-time interval equation (without solving this equation), but also to predict the numerical interval for the eccentric anomaly angle \( E_2 \), where the space-time interval can become zero. This interval is

\[
15.64 \text{ [deg]} < E_2 < 56.88 \text{ [deg]} \quad (80)
\]

In the Discussion part it will be explained that the restriction (from the properties of trigonometric functions) on compatibility condition for intersatellite communications will give a higher lower bound for the above inequality, thus confirming the difference between the space-time interval and the geodesic distance, which will be derived by means of the compatibility condition. Since all the expressions are symmetric with respect to the two angles \( E_1 \) and \( E_2 \), the same interval is valid also for \( E_1 \).

It is important to mention one peculiarity of the Shur theorem which made possible the derivation of the above result. This is the fact that the polynomial of \((n-1)\) degree \((76)\) is a sufficient condition for the existence of roots within the unit circle of the initial polynomial of \(n\)-th degree. But then, if a new polynomial of \((n-2)\) degree is constructed according to formulae \((76)\) or \((79)\), then this polynomial can become a sufficient condition for the roots of the \((n-1)\) degree polynomial. In such a way, a chain of lower-degree polynomials is constructed - each polynomial represents a necessary and at the same time a sufficient condition for the construction of a lower degree polynomial. The last constructed polynomial will be of first order, and from it the condition for the roots to be contained in the unit circle can easily be found. Note the important role of the necessary and sufficient condition - if from the linear polynomial the condition for the roots is found, then it will be a sufficient condition for the second-order polynomial, further this polynomial will be a necessary and sufficient condition for the third-order polynomial and etc. In such a way, the first-order polynomial
will turn out to be a sufficient condition for the roots to remain within the unit circle with respect to the initial $n$-th degree polynomial, provided also that for each polynomial the corresponding inequalities between the coefficient functions are fulfilled. It can be claimed that this "chain" of lower-degree polynomials, together with the corresponding inequalities between the coefficient functions, represents a modified version of the Shur theorem. So from the point of view of pure mathematics, such a modified version without any doubt is interesting, the peculiar moment is that the physical information (availability of roots with respect to the space-time equation and absence of any roots with respect to the geodesic equation) is very important for the proof of such a modified version of the theorem. Of course, the proof is limited for the investigated case of polynomials of fourth degree.

I. New definition of the Euclidean distance by means of intersecting null cones - geometrical importance of the new result

The main result in this paper concerns the intersection of two four-dimensional gravitational null cones (7) and (8) and also with the hyperplane equation (10), which in fact defines a variable distance (45) on the intersection of these null cones. From Special Relativity Theory it is known that a distance on the null cone can be either positive, negative or null. So the main nontrivial result in this paper is that on the intersecting variety of these null cones, the distance again preserves this important characteristics and can be positive, negative or null.

The fact that this distance can be positive (see (49)) or negative (see (50)) for the partial case of equal semi-major axis, eccentricities and eccentric anomaly angles

1. confirms the correctness of the interpretation of the formulae (48) as the square of the space-time interval, which can be either positive, negative or zero.

2. raises up the important question whether this is a result only for the partial case or also for the more general case of different semi-major axis, eccentricities and eccentric anomaly angles. This is the case of equation (53) and (62), which is a fourth-order algebraic equation with respect to the variable $y = \sin^2 E_1$. The implementation of the Shur theorem proves that this equation has roots within the circle $|y| < 1$, which means that the space-time distance (45) can become zero, meaning also that it can be also positive or negative.

Negative distances are not prohibited by geometry - these are the s.c. hyperbolic ge-
ometries, known also as Lobachevsky geometries with negative scalar curvature. So these three-dimensional hyperbolic geometries are obtained as an intersection of four-dimensional null cones - this is an interesting fact from mathematical point of view, not studied yet in the literature. In fact, because of the assumption for plane orbital motion, the hyperbolic geometries will be two-dimensional ones.

It should be remembered also that the starting point for the calculations of the space-time distance $R_{AB}$ was the definition (29) of this function as the Euclidean distance. That is why, the Euclidean distance can be affirmed to represent a partial case of a more general case, related to the space-time distance. Thus, one can define the Euclidean distance as a positive space-time distance, measured along the intersection of two null four-dimensional gravitational null cones attached to two moving observers (on the emitting - signal satellite and on the receiving - signal satellite). Up to now the proof was given for the case of planar orbits. It will be interesting to see whether such a physical interpretation will be valid also for the more general case of non-planar satellite orbits.

V. IMPORTANT PHYSICAL CONSEQUENCES FROM THE CONDITION FOR INTERSATELLITE COMMUNICATIONS

A. Satellites on one orbit and the restriction on the GPS - orbit eccentric anomaly angle $E$

Let us begin with one important physical consequence from the condition for intersatellite communications (47), which gives restriction on the parameters of the orbit. More physical and numerical consequences shall be given in the second (forthcoming) part of the paper.

Let us calculate the found relation (47) for the case of elliptical orbits with equal semi-major axis $a_1 = a_2 = a$, equal eccentricities $e_1 = e_2 = e$ and equal eccentric anomalies $E_1 = E_2 = E$. In order to understand properly the meaning of ”equal eccentric anomalies”, let us remember the definition for the eccentric anomaly angle. Let us denote by $O$ the center of the ellipse and from the position of the satellite on the elliptical orbit, a perpendicular is drawn towards the large semi-major axis. If this perpendicular at the point $M$ intersects the circle with a radius equal to the semi-major axis, then the angle between the semi-major axis and the line $OM$ is called the eccentric anomaly angle $E$. Since the equal eccentricities and
semi-major axis correspond to the case of several satellites on one orbit, the notion ”equal eccentric anomalies” means, that for a fixed interval of time (counted from the moment of perigee passage), the satellites encircle a distance along the orbit corresponding to equal eccentric anomaly angles. However, when distances between satellites on one orbit are calculated, the eccentric anomaly angles of the two satellites should be different depending on their different, non-coinciding positions on the orbit.

Let us assume again the value $e = 0.01323881349526$ for the eccentricity of the orbit for a GPS satellite, which is taken from the PhD thesis [6] of Gulklett. From (47) it follows that the sine of the angle $E$ does not depend on the semi-major axis

$$4a^2(1 - e^2)\sin^2 E = 2a^2 - e^2a^2 \implies \sin E = \frac{1}{2\sqrt{(2 - e^2)/(1 - e^2)}}.$$  \hspace{1cm} (81)

For the given eccentricity, the eccentric anomaly angle can be found to be $E = 45.002510943228$ [deg] or in radians $E = 0.785441987624$ [rad].

**B. The restriction on the ellipticity of the orbit**

Let us compare this value with the one obtained as an iterative solution of the Kepler equation (2). For the purpose, the initial (zero) approximation $E_0$ is taken from the dissertation [6] to be equal to the mean anomaly $M$

$$E_0 = M = -0.3134513508155 \hspace{1cm} [rad].$$  \hspace{1cm} (82)

However, since the mean anomaly $M$ is related to a projected uniform motion along a circle, the more realistic angular characteristics is the eccentric anomaly $E$, which can be found as an iterative solution of the transcendental Kepler equation (2). The iterative solution, described for example in the monograph [43], is performed according to the formulae

$$E_{i+1} = M + e \sin E_i \hspace{1cm} , \hspace{1cm} i = 0, 1, 2, .....$$  \hspace{1cm} (83)

Consequently, the first three iterative solutions are given according to the following formulae:

$$E_1 = M + e \sin M,$$  \hspace{1cm} (84)

$$E_2 = M + e \sin E_1 = M + e \sin(M + e \sin M).$$  \hspace{1cm} (85)
\[ E_3 = M + e \sin E_2 = M + e \sin[M + e \sin(M + e \sin M)] \quad . \] (86)

The third iteration gives the value

\[ E_3 = M + e \sin E_2 = -0.31758547588467897473 \quad [\text{rad}] \quad , \] (87)

The above value for \( E \) is considerably lower than the calculated according to (81) value, which might only mean that this (initial) eccentric anomaly angle is not very favourable for intersatellite communications.

Since \( \sin E \leq 1 \), one should have also

\[ \sin E = \frac{1}{2} \sqrt{\frac{(2 - e^2)}{(1 - e^2)}} \leq 1 \quad , \] (88)

which is fulfilled for

\[ e^2 \leq \frac{2}{3} \quad \text{or} \quad e \leq 0.816496580927726 \quad . \] (89)

Surprisingly, highly eccentric orbits (i.e. with the ratio \( e = \frac{\sqrt{a^2 - b^2}}{a} \) tending to one, where \( a \) and \( b \) are the great and small axis of the ellipse), are not favourable for intersatellite communications. For GPS satellites, which have very low eccentricity orbits (of the order 0.01) and for communication satellites on circular orbits (\( e = 0 \)), intersatellite communications between moving satellites can be practically achieved. For the first time the above calculation shows that such communications depend on the eccentricity of the orbit.

As mentioned in the Introduction, the RadioAstron space mission with a large semi-major axis of \( a \approx 2 \times 10^8 \) m has a variable orbital eccentricity ranging from \( e = 0.59 \) to the large value \( e = 0.966 \), which is higher than the value 0.816496580927726. So for eccentricity in the interval \( 0.59 < e < 0.816 \), intersatellite communications of RadioAstron with another satellites on the same orbit will be possible, but this will not be possible for eccentricities in the interval \( 0.816 < e < 0.966 \).
VI. POSSIBLE EXTENSIONS OF THE NEW RESULTS FOR THE SPACE-
TIME INTERVAL TO THE CASE OF NON-PLANAR (SPATIALLY-ORIENTED)
ORBITS

A. GPS, GLONASS and Galileo satellite constellations and exchange of signals
between satellites on space-oriented orbits

This section has the aim only to point out a number of research topics related to the
problem about the generalization of the developed approach for planar orbits to the case
of non-planar, space-oriented orbits. This case is much more complicated and shall not be
investigated in this paper. Nevertheless, the purpose of the section will be to outline the
basic principles and equations to be used further for the construction of such a more general
theory. In particular, newly derived will be equations (118) and (120) in the subsequent
sections, where the modified version of the Kepler equation will be presented.

One of the main motivations for the idea for constructing an extension of the theory
for propagation of signals between moving satellites is the requirement that the Global
Navigation Satellite System (GNSS), consisting of 30 satellites and orbiting the Earth
at a height of 23616 km, should be interoperable with the other two navigational systems
GPS and GLONASS [20]. This means that satellites on different orbital planes should be
able to exchange signals between each other. The construction of such a theory is possible,
and the main prerequisite for this is the knowledge of the full set of six Keplerian elements
($M, a, e, \Omega, I, \omega$). For example, the satellites of the Galileo constellation are situated on three
orbital planes with nine-equally spaced operational satellites in each plane. The Galileo
satellites are in nearly circular orbits with semi-major axis of 29600 km and a period of
about 14 hours [21] and an inclination of the orbital planes 56 degrees. For comparison, the
Russian Global Navigation Satellite System GLONASS, managed by the Russian Space
Forces and launched in 1982, consists of 21 satellites in three orbital planes (with three non-
orbit spares). Each satellite operates in nearly circular orbits with semi-major axis of 25510
km, and the satellites within the same orbital plane are equally spaced by 45 degrees. Each
orbital plane has an inclination angle of 64.8 degrees, which is more than the inclination
angle 56 degrees of the orbital planes of the Galileo satellites. Moreover, a GLONASS
satellite completes an orbit in approximately 11 hours 16 minutes - less than the period of
14 hours for the Galileo satellite. Consequently, the three characteristic angles of rotation - the eccentric anomaly $E$, the mean anomaly $M = n(\tau - t)$ and the true anomaly $f$ should be different for the two satellites.

Different from GLONASS orbital parameters have also the satellites of the GPS satellite constellation, consisting of 24 operational satellites, deployed in six evenly spaced planes (A to F) with 4 satellites per plane and an inclination of the orbit 55 degrees [12].

**B. The non-planar (spatially oriented) orbits and their orbital characteristics**

The more interesting and complicated case is the one for non-planar orbits (space-oriented Keplerian orbits), when the orbit is parametrized by the full set of six Keplerian elements $(M, a, e, \Omega, I, \omega)$, where the first three ones are characteristics of the planar motion and have been previously defined. The next three Keplerian elements $(\Omega, I, \omega)$ characterize the spatial orientation of the orbit and are defined in the framework of the geocentric equatorial coordinate system, in which the $x-$axis is aligned with the vernal equinox ($\Upsilon$), the $z-$axis points to the north pole and the origin of the system is at the center of the Earth [39]. The vernal equinox describes the direction of the Sun as seen from the Earth at the beginning of the spring season, which is equivalent to considering the intersection of the equatorial plane with the Earth’s orbital plane. The direction of the $z-$axis clearly shows that the precession and nutation of the Earth are neglected. Therefore, the polar motion is not taken into account and a mean pole of the Earth rotation is chosen, representing the average of all the changes in the direction of the true rotational axis of the Earth [46]. Such a mean pole is called also Conventional International Origin (CIO).

In order to define the inclination of the orbit, first one should define the line of nodes. This is the line of intersection between the orbital plane and the equatorial plane (for Earth satellites and for celestial bodies in the Solar system, this reference plane will be the ecliptic) [47]. The line of intersection consists of two ending points - the ascending node and the descending node. Besides the inclination, the second measure to orient the orbital plane is the right ascension of the ascending node $\Omega$. It denotes the point where the satellite moves from the southern hemisphere of the Earth to the northern hemisphere [47], [64]. The angle $\Omega$ of the longitude of the ascending node is the angle between the ascending node and the $x-$axis (oriented towards the vernal equinox). The argument of perigee (periapsis) $\omega$ is the
angle within the orbital plane from the ascending node to perigee in the direction of the satellite motion \((0 \leq \omega \leq 360^\circ)\).

Very often, another variable is used - the argument of latitude \(u = \omega + f\), being defined as the sum of the argument of perigee \(\omega\) and the true anomaly \(f\) and geometrically representing the angle between the line of nodes and the position vector \(r\). The argument of latitude will appear further in the calculation of the propagation time in terms of the celestial coordinates. A similar additive angular variable is the eccentric longitude \(F = E + \omega + \Omega\) (equinoctial orbital characteristic), representing the sum of the eccentric anomaly \(E\), the right ascension of the ascending node \(\Omega\) and the argument of the perigee \(\omega\).

### C. Intersecting null cones of observers on spatially oriented orbits - the possible generalization of the approach

If a space-time interval is obtained after the intersection of null cones for the case of planar orbits, then it is reasonable to ask whether the intersection of null cones of observers on spatially oriented orbits (parametrized by \((M, a, e, \Omega, I, \omega)\)) will again produce a space-time structure with positive, negative and null distance? This general case will require the coordinate transformation from the orbital coordinates \((M, a, e, \Omega, I, \omega)\) to the cartesian coordinates \((x, y, z)\) of the geocentric equatorial coordinate system (see the monograph of Brauer, Clemence 38)

\[
x = r \cos^2 \frac{I}{2} \cos(f + \Omega + \omega) + r \sin^2 \frac{I}{2} \cos(f + \omega - \Omega) , \tag{90}
\]

\[
y = r \cos^2 \frac{I}{2} \sin(f + \Omega + \omega) - r \sin^2 \frac{I}{2} \sin(f + \omega - \Omega) , \tag{91}
\]

\[
z = r \sin I \sin(f + \omega) , \tag{92}
\]

where the true anomaly \(f\) is expressed through the eccentric anomaly \(E\) and the eccentricity parameter \(e\) by means of the formulae

\[
\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \implies f = 2 \arctan \left[ \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right] . \tag{93}
\]

In other words, if instead of the plane parametrization of the orbit (3) the parametrization (90)-(92) is applied, then an analogous formulae to (45) for the space-time distance can be
obtained. However, it can be expected that the corresponding equation will not be an algebraic one, since the radius-vector \( r \) is in the orbital plane and is expressed by the true anomaly by means of the standard formulae
\[
r = \frac{a(1 - e^2)}{1 + e \cos f}.
\] (94)

Consequently, the resemblance between the plane transformation \( \mathbf{3} \) in matrix notations
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= \begin{pmatrix}
  -ae \\
  0
\end{pmatrix}
+ \begin{pmatrix}
  a & 0 \\
  0 & a\sqrt{1 - e^2}
\end{pmatrix}
\begin{pmatrix}
  \cos E \\
  \sin E
\end{pmatrix}
\] (95)

and the non-planar transformations \( (90)-(92) \) represented as
\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= R_z(-\Omega)R_x(-I)R_z(-\omega)
\begin{pmatrix}
  r \cos f \\
  r \sin f \\
  0
\end{pmatrix}
\] , (96)

is only at first glance. The meaning of the above formulae is that expressions \( (90) - (92) \) can be obtained after performing three successive rotations \( R_z(-\omega), R_x(-I) \) and \( R_z(-\Omega) \) with respect to the orbital vector \( (r \cos f, r \cos f, 0)^T \) (the transposed vector to the vector-column in \( (96) \)), where \( R_z(-\omega) \) is the matrix of rotation at an angle \( (-\omega) \) in the counterclockwise direction around the \( z \) axis, \( R_x(-I) \) is the matrix of rotation at an angle \( (-I) \) around the \( x \)-axis, \( R_z(-\Omega) \) is the matrix of rotation at an angle \( (-\Omega) \) around the \( z \)-axis \( [39], [40] \). So the corresponding (non-algebraic and nonlinear) equation for the space-time distance shall be derived after substituting the transformations \( (96) \) (with the corresponding indices 1 and 2) in the null cone equations \( ds^2_{(1)} = 0 \) and \( ds^2_{(2)} = 0 \) for the emitted and the received signal. The cone origins will be at the space points \( (x_1, y_1, z_1) \) and \( (x_2, y_2, z_2) \).

It should be stressed that the transformations \( (90) - (92) \) for the general case do not depend explicitly on the eccentricity \( e \) because the true anomaly \( f \) depends on both the eccentricity \( e \) and on the eccentric anomaly angle \( E \). This is the substantial difference from the "planar orbit transformations" \( (95) \) used in this paper, which depended directly on the eccentricity, and further the coefficient functions of the fourth - order algebraic equations \( (213) \) for the space-time distance and \( (309) \) for the geodesic distance exhibited dependence on the eccentricity. The algebraic proof that first equation \( (213) \) has the property of the space-time distance was based substantially on the smallness of the eccentricity.
However, if in the general case there will be no dependence of the coefficient functions of the algebraic equation on the eccentricity parameter $e$, then an interesting problem arises: will the algebraic proof of the space-time distance property (of being positive, negative or zero) be again possible? In other words, is the smallness of the eccentricity $e$ a very important ingredient of the mathematical proof? The smallness of $e$ can be taken into account after a series decomposition of (93). In this aspect, the following important problem arises: is it accidental that the eccentricities of the orbits of the celestial bodies in the Solar system are very small? For example, for Venus the eccentricity of the orbit is 0.01, for the Earth - 0.02, for Mars - 0.09, for Jupiter - 0.05, for Saturn - 0.06, for Uranus - 0.05, for Neptune - 0.01. For artificial body such as the RadioAstron SRT (Space Radio Telescope), the range of the changing eccentricity of the orbit is between 0.59 and 0.966, but evidently such large eccentricities are not favoured by Nature. But this is valid inside our Solar system, outside the Solar system some distant stars can have relatively large eccentricities of their orbits of the order $0.4 - 0.6$.

D. The true anomaly $f$ and the Runge-Lentz-Laplace vector

The dependence of the true anomaly $f$ on both the eccentricity and the eccentric anomaly angle $E$ is a more peculiar feature of the orbital characteristics. The true anomaly is defined as the geometric angle in the plane of the ellipse between periapsis (the closest approach to the central body) and the position of the orbiting satellite at any given time [64]. There is also another more ”mathematical” definition, which makes use of the s.c. ”Runge-Lentz” (or Laplace) vector $A_L$ [39], which lies in the orbital plane and thus is orthogonal to the angular-momentum vector $J = r \times \dot{r}$ (called also areal velocity $h = r \times \dot{r} = \text{const} = J$)

$$A_L := r \times (r \times \dot{r}) - G_M \frac{r^2}{r} = r \times J - G_M \frac{r^2}{r}.$$  

(97)

Thus, the true anomaly $f$ is the angle between the Runge-Lentz vector $A_L$ and the position vector $r$. The vector $A_L$ appears as an additive constant after integrating the equation

$$h \times \dot{r} = -G_M \frac{d}{dt} \left( \frac{r}{r^2} \right),$$  

(98)

and $\Delta A = \frac{1}{2} | r \times \dot{r} \Delta t | = \frac{1}{2} | h | \Delta t$ is the area, swept by radius-vector $r$ during the time $\Delta t$. The square of the Runge-Lentz-Laplace vector can be calculated to be [39]

$$A_L^2 = G_M^2 + 2J^2 \left( \frac{1}{2} r^2 - \frac{G_M}{r} \right) = G_M^2 + 2J^2F,$$  

(99)
where $\mathcal{E}$ is the conserved energy per unit mass. Thus, since the magnitude and direction of the vector $A_L$ are conserved, the number of the independent integrals of motion of the reduced two-body problem is increased by one. It can be calculated that

$$r^2 = \frac{n^2 a^2}{1-e^2} [1 + e^2 (1 - \frac{1}{2} \sin(2\Omega) \sin(2\omega)(1 - \cos i)) +$$

$$+ 2e (\sin \omega \sin(\omega + f) + \cos \omega \cos^2 i \cos(\omega + f) +$$

$$+ \cos \omega \cos(\omega + f) \sin i)] . \quad (100)$$

Consequently, after the decomposition of the prefactor $\frac{n^2 a^2}{1-e^2}$ into an infinite sum of the small eccentricity parameter $e$ and combining (99) and (100), it can be seen that the conserved energy of unit mass $\mathcal{E}$ and the orbital parameters $\omega, f, i$ depend on the geocentric gravitational constant $G_M = 3986205.266 \times 10^8 \left[ \frac{m^3}{sec^2} \right]$. This numerical value however is not strictly determined - for example, the value for $G_M$ obtained from the analysis of laser distance measurements of artificial Earth satellites is

$$G_M = (3986004.405 \pm 1) \times 10^8 \left[ \frac{m^3}{sec^2} \right] . \quad (101)$$

In the review papers [40], [41] by P. Mohr, B. Taylor and D. B. Newell, where the value of $G$ (experimentally determined by means of different experiments, performed by different groups) ranges from

$$G_M = 3986056.75236 \times 10^8 \left[ \frac{m^3}{sec^2} \right] , \quad (102)$$

to

$$G_M = 3987999.07898 \times 10^8 \left[ \frac{m^3}{sec^2} \right] . \quad (103)$$

### E. Generalized Kepler equation for space-oriented orbits

The eccentric anomaly angle $E$ and the true anomaly $f$ are by definition plane characteristics of the orbit, related to one another by means of the differential relation

$$dE = \frac{\sqrt{1-e^2}}{1+e \cos f} df \quad \text{or} \quad ndt = \frac{(1-e^2)^{\frac{3}{2}}}{(1+e \cos f)^{\frac{3}{2}}} df . \quad (104)$$

However, for space-oriented orbits the radius vector can be determined by the formulae [39], [42]

$$r = a(1 - k \cos F - h \sin F) , \quad (105)$$
which is a generalization of the usual plane-orbit expression \( x = a(\cos E - e) \), \( y = a\sqrt{1 - e^2}\sin E \). Correspondingly, instead of the Kepler equation (2) for the eccentric anomaly \( E \), the evolution of the eccentric longitude \( F = E + \omega + \Omega \) for the case of spatial orbit is governed by the modified Kepler equation \( F - k\sin F + h\cos F = l \),

\[
F - k\sin F + h\cos F = l, \quad (106)
\]

where \( l = M + \omega + \Omega \) is the mean longitude and \( k \) and \( h \) are the trigonometric functions

\[
k = e\cos(\omega + \Omega) \quad \text{,} \quad h = e\sin(\omega + \Omega) \quad . \quad (107)
\]

In fact, the functions \( k \) and \( h \) specify the orientation of the orbital plane after a rotation at an angle \( \omega + \Omega \). It is seen also that for the case \( \omega = 0 \) and \( \Omega = 0 \) the modified Kepler equation (106) transforms in the usual Kepler equation. The problem is: if the above two orbital parameters are kept constant, then will the Kepler equation also preserve its form for such a case? At first glance, it might seem that this will happen. However, one should bear in mind that although the eccentric anomaly \( E \) and the true anomaly \( f \) are plane characteristics, their value is being accounted by means of the radius-vector. In the case of space orbit with non-zero, but constant \( \omega \) and \( \Omega \), the radius vector may not lie in the orbital plane so there will be an angle between the orbital plane and the radius vector. So it might be expected that the standard Kepler equation will be modified.

One more comparison may be performed between the plane-orbit and the space-orbit cases. Previously, it was mentioned that the Kepler equation establishes a correspondence between the eccentric anomaly \( E \) and the celestial time \( t_{cel} \), i.e. \( E \rightarrow t_{cel} \). In fact, if the celestial time \( t_{cel} \) is known then the iterative solution of the Kepler equation establishes an approximate correspondence \( t_{cel} \rightarrow E \). In the case of space orbits, due to the complicated integral

\[
t_{cel} = \frac{1}{n} \int \frac{(1 - e^2)^{\frac{3}{2}}}{(1 + e\cos f)^2} df
\]

resulting from the differential relation (104) for the true anomaly \( f \) and the celestial time \( t_{cel} \), it might seem that a correspondence between the true anomaly \( f \) and the celestial time \( f \rightarrow t_{cel} \) is not possible. In fact, this can be proved to be not true since the above integral can be exactly calculated

\[
t_{cel} = \frac{\sqrt{1 - e^2}}{n} \left[ -e \frac{\sin f}{(1 + e\cos f)} \right] +
\]
\[
+ \frac{2}{\sin \delta} \arctan \left( \cot \frac{\delta}{2} \tan \frac{f}{2} \right),
\]

(109)

where \( \delta \) is the following numerical parameter

\[
\delta = \arccos e .
\]

(110)

However, an approximate correspondence \( t_{cel} \rightarrow f \) in this case cannot be established since \( f \) cannot be expressed from (109) if \( t_{cel} \) is known. It should be noted that in most monographs on celestial mechanics only approximate solutions of the integral (108) are given. An integral of the kind (109) will appear also in another problems, related to the change of the proper time of an atomic clock, when transported along a given orbit. The analytical techniques for finding the exact value of this integral will be presented in another paper.

F. Modified Kepler equation only in terms of the eccentric anomaly angle

Now it shall be proved that such a modification of the Kepler equation will really take place but most strangely, this modification will include terms with the eccentric anomaly \( E \) only.

For constant \( \omega \) and \( \Omega \), from (106) it can be found

\[
(-k \cos F - h \sin F)dE = Edt .
\]

(111)

But on the other hand, making use of the second formulae in (104), the last formulae can be rewritten as

\[
\frac{dE}{df} = \frac{(1 - e^2)^{\frac{3}{2}}}{n(1 + e \cos f)} \cdot \frac{E}{(-k \cos F - h \sin F)} .
\]

(112)

Since this determination of \( E \) and \( f \) should be compatible with the plane-orbit relation (104), both relations (104) and (112) should be fulfilled. This compatibility gives

\[
1 = \frac{(1 - e^2)^{\frac{1}{2}}}{n(1 + e \cos f)} \cdot \frac{E}{(-k \cos F - h \sin F)} .
\]

(113)

The second expression in the denominator can be written as

\[
-k \cos(E + \omega + \Omega) - h \sin(E + \omega + \Omega) =
\]

(114)

\[
= -e \cos(\omega + \Omega) [\cos E \cos(\omega + \Omega) - \sin E \sin(\omega + \Omega)] -
\]
\[-e \sin(\omega + \Omega)\sin E \cos(\omega + \Omega) + \\
+ \cos E \sin(\omega + \Omega) = -e \cos E \ .\] (115)

Substitution of this expression into (113) gives a formulae for the eccentric anomaly \(E\) not dependent on the orbital parameters \(\omega\) and \(\Omega\)

\[E = -\frac{ne(e + \cos f)}{1 - e^2} \ .\] (116)

Note that this expression is not identical with the relation between \(E\) and \(f\) for the case of planar orbits

\[\cos E = \frac{e + \cos f}{1 + e \cos f} \ .\] (117)

Denoting \(\tilde{q} = \cos f\) and combining the last two expressions, the following transcendental equation with respect to \(\tilde{q}\) can be obtained

\[\cos \left[ \frac{ne(e + \tilde{q})}{1 - e^2} \right] = \frac{e + \tilde{q}}{1 + e \tilde{q}} \ .\] (118)

In terms of the eccentric anomaly \(E\), this equation can be written also as

\[\cos X = \cos E \ , \ X = \frac{ne \cos E}{1 - e \cos E} \ ,\] (119)

or, taking into account that \(X = E + 2k\pi\), the following modified version of the Kepler equation can be obtained

\[E - eE \cos E = ne \cos E + \\
+ 2k\pi - 2k\pi e \cos E \ .\] (120)

This equation is second order in \(E\), unlike the standard Kepler equation.

G. Space orbits and the nontrivial problem for small eccentricities

For the case of space orbits, the smallness of the eccentricity of the GPS orbit creates an additional problem since, as mentioned in the monograph [39], for small \(e\) and nearly circular orbits, the argument of perigee \(\omega\) is not a well-defined orbital element. The reason is that small changes of the orbit may change the perigee location significantly. For such a case, instead of the usual full set of Kepler parameters \(a_\alpha = (a, e, I, M, \omega, \Omega)\), another set of parameters \(p_\alpha = (a, l, h, k, \tilde{p}, \tilde{q})\) had been implemented in the papers by Broucke, Cefola
\[ \text{[42] and Deprit, Rom [43], where } l \text{ is the mean longitude, } h \text{ and } k \text{ are given by (107) and } \tilde{p} \text{ and } \tilde{q} \text{ are the expressions} \]

\[ \tilde{p} = \tan \frac{I}{2} \sin \Omega, \quad \tilde{q} = \tan \frac{I}{2} \cos \Omega . \]  

(121)

All these definitions enable the determination of the derivatives \( \frac{\partial x}{\partial \tilde{p}} \) and \( \frac{\partial x}{\partial \tilde{q}} \) in such a way, so that they are consistent even when \( e = 0 \).

VII. PHYSICAL AND MATHEMATICAL THEORY OF THE GEODESIC DISTANCE FOR THE CASE OF NON-SPACE ORIENTED ORBITS

A. Geodesic distance as a result of the compatibility between the condition for intersatellite communications and the space-time interval

We shall begin with a simple explanation, concerning how the formulae for the geodesic distance is obtained. If (47) is substituted into expression (45) for \( R_{AB}^2 \) and the simple formulae \( \cos(2E) = 1 - 2\sin^2 E \) is used, then expression (45) can be written as

\[ \tilde{R}_{AB}^2 = \frac{1}{2}(a_1^2 + a_2^2) + \frac{1}{2}(a_2e_2 - a_1e_1)^2 + \frac{1}{4}(a_1^2e_1^2 + a_2^2e_2^2) - \\
(2e_1a_1^2\cos E_1 + 2e_2a_2^2\cos E_2) - \\
- (e_1^2a_1^2\sin^2 E_1 + e_2^2a_2^2\sin^2 E_2) - 2a_1a_2\cos E_1\cos E_2 + \\
+ 2a_1a_2(e_2\cos E_1 + e_1\cos E_2) . \]  

(122)

The square \( \tilde{R}_{AB}^2 \) of the Euclidean distance, when it is in the form of the condition for intersatellite communications (transmission of signals) is a two-point function, depending on the semi-major axis \( a_1, a_2 \), eccentricities \( e_1, e_2 \) and eccentric anomaly angles \( E_1, E_2 \) of the two satellites. It is denoted with the tilda sign \( \tilde{R}_{AB}^2 \) in order to distinguish it from the usual expression (46) for the Euclidean distance \( R_{AB}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \), expressed in the orbital elliptic coordinates. In the paper [60], when the propagation of signals from one satellite to another is investigated, the distance travelled by light is called "geodesic distance". So in analogy with this paper, we shall make a distinction between the Euclidean distance \( R_{AB} \) and the geodesic distance \( \tilde{R}_{AB}^2 \). The difference between the two distances can be found by substracting (122) from (46)

\[ R_{AB}^2 - \tilde{R}_{AB}^2 = \frac{1}{2}(a_1^2 + a_2^2) - e_1e_2a_1a_2 + \]
\[ + \frac{1}{4}(a_1^2 e_1^2 + a_2^2 e_2^2) - 2a_1 a_2 \sqrt{(1 - e_1^2)(1 - e_2^2)} \]  

(123)

**B. Consistency of the calculations - geodesic and Euclidean distances for equal eccentricities, semi-major axis and eccentric anomaly angles**

Note also another consistency of the calculations - from (47) for \( e_1 = e_2 = e, \) \( a_1 = a_2 = a \) and \( E_1 = E_2 = E \) it can be derived that \( \sin^2 E = \frac{2 - e^2}{4(1 - e^2)} \). If substituted into expressions (46) for \( R_{AB}^2 \) and (122) for \( \tilde{R}_{AB}^2 \), it can be obtained that both these expressions equal to zero. This should be so - for coinciding positions of the satellites \( (R_{AB}^2 = 0) \), the geodesic distance should also equal zero, i.e. \( \tilde{R}_{AB}^2 = 0 \). For equal eccentricities, semi-major axis and eccentric anomaly angles and without taking into account the compatibility condition (47), the geodesic distance (45) is different from zero. However, since the geodesic distance is derived by using the compatibility condition, it should always be taken into account.

Now let us demonstrate that the boundary value \( E_{\text{lim}} = 45.002510943228 \) [deg] (51) is essential for the consistency between the space-time interval, the geodesic distance and the Euclidean distance for this particular case, but is irrelevent for the definition of the geodesic distance itself. For the purpose, let us write formulae (122) for the geodesic distance again for the case of equal eccentricities, equal semi-major axis and equal eccentric anomaly angles

\[ \tilde{R}_{AB}^2 = -a_2^2 + \frac{1}{2} a^2 e^2 + 2a_2^2 (1 - e^2) \sin^2 E \]  

(124)

Then for

\[ \sin^2 E \geq \sin^2 E_{\text{lim}} = \frac{1}{4} \frac{(2 - e^2)}{(1 - e^2)} \]  

(125)

substituting the above inequality in the expression (124), it can be derived

\[ \tilde{R}_{AB}^2 \geq -a_2^2 + \frac{1}{2} a^2 e^2 + 2a_2^2 \frac{1}{4}(2 - e^2) = 0 \]  

(126)

However, this cannot be considered as a proof of the positivity of the geodesic distance, since from the condition for intersatellite communications (47) it follows that the equality sign in (126) should be fulfilled, i.e. \( \sin^2 E = \sin^2 E_{\text{lim}} \). Consequently, from (126) \( \tilde{R}_{AB}^2 = 0 \). For the value (125) of \( E \), taken to be equal to \( E_{\text{lim}} \) and also identical with the value (81), both the space-time interval and the geodesic distance are equal to zero. Therefore, this supports the consistency of their defining formulae (45) in Section IV and (122) in Section VII A.
Now it can also be understood why the choice $E_1 = E = E_{\text{lim}}$ for the inverse inequality to (59), when the space-time interval will be negative in Section IV E, will not be acceptable. Let us put

\[ E_1 = E_2 = E = E_{\text{lim}} = 45.002510943228 \text{ [deg]} \]  \hspace{1cm} (127)

in the inverse inequality

\[ E_1 + E_2 < 89.994978993712 \text{ [deg]} . \]  \hspace{1cm} (128)

Then it will follow

\[ E < 44.997489496 \text{ [deg]} < 45.002510943228 \text{ [deg]} . \]  \hspace{1cm} (129)

However, the value (127) for the space-time interval has to be compatible with the geodesic distance, which means that the geodesic distance $\tilde{R}_{AB}^2$ according to (126) should be equal to zero. This is not possible for the inequality sign in (129), consequently the choice $E = 45.002510943228 \text{ [deg]}$ in (127) is incompatible with the inequality (128).

C. Compatibility condition and positive geodesic distance - the case of different eccentric anomaly angles but equal eccentricities and semi-major axis

It is instructive to investigate the case for non-zero Euclidean distance (given by formulae (46)) and to compare it with the geodesic distance (122). The Euclidean distance will be non-zero when the eccentric anomaly angles $E_1$ and $E_2$ are different. Note however that the difference $R_{AB}^2 - \tilde{R}_{AB}^2$ in (123) does not depend on the eccentric anomaly angles. So for $e_1 = e_2 = e$ and for $a_1 = a_2 = a$ one can represent (123) as

\[ \tilde{R}_{AB} = \sqrt{R_{AB}^2 + a^2(1 - \frac{3}{2}e^2)} . \]  \hspace{1cm} (130)

Taking into account the restriction (59) $e^2 \leq \frac{2}{3}$ on the value of the ellipticity of the orbit, the second term under the square root in (130) is positive. Due to this

\[ \tilde{R}_{AB} \geq R_{AB} , \]  \hspace{1cm} (131)

which means that the geodesic distance, travelled by the signal is greater than the Euclidean distance. This simple result, obtained by applying the formalism of two intersecting null cones is a formal proof of the validity of the Shapiro time delay formulae for the case of
moving emitters and receptors of the signals. Due to the larger geodesic distance, any signal in the presence of a gravitational field will travel a greater distance and thus will be additionally delayed.

From a formal point of view, an equality sign in (131) is possible when $e^2 = \frac{2}{3}$. This may take place if $\sin E = 1$. This would mean that $E = \frac{\pi}{2} + 2k\pi$, which will contradict the initial assumption about arbitrary values of the eccentric anomaly angle. Therefore, the geodesic distance should be considered strictly greater than the Euclidean one.

D. Positivity of the geodesic distance in the general case of different plane orbital elements

It is natural to expect that the geodesic distance will be greater than the Euclidean one also in the general case of different eccentricities of the two orbits, different semi-major axis and eccentric anomaly angles.

Let us first write the condition for intersatellite communications (17) as

$$\sin E_1 \sin E_2 = p,$$

where $p$ is the introduced notation for

$$p = \frac{P_1(e_1, a_1; e_2, a_2)}{Q_1(e_1, a_1; e_2, a_2)},$$

$P_1(e_1, a_1; e_2, a_2)$ and $Q_1(e_1, a_1; e_2, a_2)$ for given values of the two eccentricities and the semi-major axis are the numerical parameters

$$P_1(e_1, a_1; e_2, a_2) := a_1^2 + a_2^2 + (a_2 e_2 - a_1 e_1)^2 - \frac{1}{2}(e_1^2 a_1^2 + e_2^2 a_2^2),$$

$$Q_1(e_1, a_1; e_2, a_2) := 4a_1 a_2 \sqrt{(1 - e_1^2)(1 - e_2^2)}.$$

Since

$$\sin E_1 \sin E_2 \leq 1,$$

from the preceding relations it can be obtained

$$-\frac{1}{2}(a_1^2 + a_2^2) + a_1 a_2 e_1 e_2 \geq$$

$$\geq -\frac{1}{4}(e_1^2 a_1^2 + e_2^2 a_2^2) + \frac{1}{2}(e_1^2 a_1^2 + e_2^2 a_2^2) -$$
Substituting the terms in the left-hand side of the above inequality in the expression \( (123) \) for \( \tilde{R}_{AB}^2 - R_{AB}^2 \), it can be obtained

\[
\tilde{R}_{AB}^2 - R_{AB}^2 \geq -\frac{1}{4}(e_1^2a_1^2 + e_2^2a_2^2) + \frac{1}{2}(e_1^2a_1^2 + e_2^2a_2^2) - \frac{1}{4}(e_1^2a_1^2 + e_2^2a_2^2) + 2a_1a_2\sqrt{(1 - e_1^2)(1 - e_2^2)} .
\]  

(138)

All the terms in the right-hand side of the above inequality cancel, so one obtains

\[
\tilde{R}_{AB}^2 \geq R_{AB}^2 .
\]  

(139)

Note the interesting fact that for different eccentricities and semi-major axis the equality sign is fully legitimate. So the Euclidean distance becomes equal to the geodesic distance when

\[
\sin E_1 = \sin E_2 = 1 .
\]  

(140)

Since then

\[
E_1 = E_2 = \frac{\pi}{2} + 2k\pi
\]  

(141)
in a coordinate system, in which the small axis of the ellipses coincides with the \( y \)-axis, the two satellites should be situated one above another (of course, remaining on the plane orbit). From the relation \( (47) \) or \( (123) \), it can be found that this may happen, when the ratio \( m = \frac{a_1}{a_2} \) of the two semi-major axis satisfies the quadratic algebraic equation

\[
\frac{1}{4}(1 + 2e_1^2)m + \frac{1}{4}(1 + 2e_2^2)\frac{1}{m} - e_1e_2 = 2\sqrt{(1 - e_1^2)(1 - e_2^2)} .
\]  

(142)

E. Geodesic distance in terms only of the first eccentric anomaly angle and for equal eccentricities and equal semi-major axis

Let us substitute \( \sin E_2 \) from the condition for intersatellite communications \( \sin E_1 \sin E_2 = p \) \( (132) \) in expression \( (53) \) for the square of the geodesic distance, which can be rewritten as

\[
\tilde{R}_{AB}^2 = 2pa^2(1 - e^2) + e^2a^2(\cos^2 E_1 -
\]
\[- \frac{p^2}{\sin^2 E_1} - 2a^2 \cos E_1 \sqrt{1 - \frac{p^2}{\sin^2 E_1}} \]  \hspace{1cm} (143)

Note that expression (53) was obtained after applying (132) to the expression (46) for the Euclidean distance. Consequently, the above formulae is obtained after a subsequent application of (132) with respect to (53), assuming again the equality of the two eccentricities and of the two semi-major axis. Since (143) depends only on the first eccentric anomaly \(E_1\), it is not fully equivalent to (122) but only in the sense that (122) depends on both eccentric anomaly angles.

Remembering the numerical value for \(p = \frac{2 - e^2}{4(1 - e^2)}\), one can write

\[2pa^2(1 - e^2) + e^2a^2 = a^2 + \frac{3}{2}e^2a^2 \]  \hspace{1cm} (144)

By means of this expression and considering the geodesic distance \(\tilde{R}_{AB}\) to be positive, from (143) one can obtain the inequality

\[2 \sqrt{(1 - \sin^2 E_1) \left(1 - \frac{p^2}{\sin^2 E_1}\right)} < \left(1 + \frac{3}{2}e^2\right) - e^2 \left(\sin^2 E_1 + \frac{p^2}{\sin^2 E_1}\right) \]  \hspace{1cm} (145)

**F. Restrictions on the lower bound of the eccentric anomaly angle**

Since the left-hand side of the above inequality is positive, the right-hand side should also be positive. It can be written as

\[\frac{\sin^4 E_1 + p^2}{\sin^2 E_1} < \frac{3}{2} + \frac{1}{e^2} \]  \hspace{1cm} (146)

For small eccentricities \(e \sim 0.01\), the number \(\frac{1}{e^2} \sim 10000\) in the right-hand side is much greater than the numerical value in the left-hand side. So the inequality is fulfilled, and since it is a consequence of the previous one, based on the inequality (131) \(\tilde{R}_{AB} \geq R_{AB}\), it should be considered as another indirect confirmation of the positiveness of the geodesic distance. In a rough approximation, the above inequality will be fulfilled, if \(\sin E_1 > e\), which for the typical eccentricity of the GPS orbit \(e = 0.01323881349526\) gives

\[E_1 > \arcsin e = 0.013238426779 \text{ [rad]} = \]
So the eccentric anomaly angle should not be smaller than the above value, which is an extremely small number. Note also that up to the sixth digit after the decimal dot the simple relation \( \sin E \approx E \) (\( \arcsin E \approx E \)) is fulfilled. This is evident if the angle \( E_1 \) is expressed in radians.

However, the lower bound on \( E_1 \) should be more stringent because from the condition (132) and from the requirement to define properly the second eccentric anomaly angle \( E_2 \), it can be obtained

\[
\sin E_2 = \frac{p}{\sin E_1} < 1 .
\]

From here and for the numerical value \( p = 0.50004382422659548 \) for the case of equal eccentricities and semi-major axis it follows that

\[
E_1 > \arcsin p = 30.00289942985 \, [\text{deg}] =
= 0.523649380196 \, [\text{rad}] .
\]

Of course, this value for \( p \) is just for one specific case, it might be in principle another for other cases of different eccentricities and semi-major axis, but in any case the eccentric anomaly angles should not be small. The value (149) turns out to be higher than the lower bound (80) (from formulae (293) in Appendix C) for the space-time interval. But since the geodesic distance is positive, while the space-time interval can be of any signs, it is natural to expect that the range of values for the eccentric anomaly angle, related to the definition of the geodesic distance will be more restrictive in comparison with the range of values of this angle, related to the space-time interval. This conclusion is confirmed by the numerical analysis.

G. Fourth-order algebraic equation for the geodesic distance without any roots

Taking the square of inequality (145), it can be written as

\[
(1 + \frac{9}{4}e^4 + 3e^2 + 2e^4p^2 - 4 - 4p^2) +
+e^4(\sin^4 E_1 + \frac{p^4}{\sin^4 E_1}) - e^2(2 + 3e^2)\sin^2 E_1 =
\]

\[
= 0.000231060882 \, [\text{deg}] .
\]
\[- \frac{e^2 p^2}{\sin^2 E_1} (2 + 3e^2) + 4\left( \frac{p^2}{\sin^2 E_1} + \sin^2 E_1 \right) > 0 \quad .
\]

The most significant contributions will be given by the terms with the smallest powers in \( E \). Now one can realize the importance of the fact that the angle \( E_1 \) should not be a small one - a term with \( \sin E_1 \) (\( \sin^2 E_1 \) or \( \sin^4 E_1 \)) in the denominator will give a very large value. But in view of the lower bound (149) in the previous section, the numerical value of the eccentric anomaly angle cannot be a small number, since the geodesic distance is ultimately related to the restrictions imposed by the condition for intersatellite communications.

That is why, another choice is made in this paper. By denoting \( y = \sin^2 E_1 \), the above expression has been presented in the form of a fourth-degree algebraic equation \( g(y) = \pi_0 y^4 + \pi_1 y^3 + \pi_2 y^2 + \pi_3 y + \pi_4 \) with coefficient functions, some of which depend on the inverse powers of the eccentricity parameter \( e \). The equation (309) is investigated in Appendix E again by applying the Shur theorem. Since the final polynomial from the chain of polynomials will not satisfy the sufficient condition of the theorem for small eccentricities, it will follow that the necessary condition of the theorem for the roots to be within the unit circle \( |y| < 1 \) will not be fulfilled. Thus, the polynomial will not have any roots and so it cannot be equal to zero. This is consistent with the previous proof about the positivity of the geodesic distance.

\section*{VIII. DISCUSSION}

In this paper we have presented a theoretical model of intersatellite communications based on two gravitational null cones, the origin of each one of them situated at the emitter and at the receiver of the corresponding satellites. The satellites are assumed to move on (one-plane) elliptic orbits with different eccentricities \( e_1, e_2 \) and semi-major axis \( a_1, a_2 \). The standard formulae (11) for the Shapiro time delay, used in VLBI radio interferometry \( [30] \) (see also the textbook \( [68] \) on GR theory) presumes that the coordinates \( r_A = |x_A(t_A)| \) and \( r_B = |x_B(t_B)| \) are varying in such a way so that the Euclidean distance \( R_{AB} = |x_A(t_A) - x_B(t_B)| \) is not changing. In such a formalism, it is natural to derive the signal propagation time from one light cone equation. It should be noted also that the coordinates \( r_A \) and \( r_B \) enter formulae (11) in a symmetrical manner.

The situation changes when two satellites are moving with respect to one another and
with respect to the origin of the chosen reference system. For such a case, two different 
parametrizations of the space coordinates are used, corresponding to the two different (un-
correlated) motions of the satellites. This means that if in terms of the first parametrization 
the null cone gravitational equation $ds^2 = 0$ is fulfilled, then it might not be fulfilled with 
respect to the other parametrization. Consequently, this is the motivation for making use 
of the two gravitational null cone equations (7) and (8). The positions of the satellites 
are characterized by the two eccentric anomaly angles $E_1$ and $E_2$ and by the semi-major 
axis $a_1$, $a_2$ and eccentricities $e_1$, $e_2$, and all these parameters satisfy the s.c. "condition for 
intersatellite communications” (47), derived for the first time in this paper. On the base 
of this expression the subsequent formulae (122) for the difference between the squares of 
the geodesic distance $\tilde{R}_{AB}^2$ and the Euclidean distance $R_{AB}^2$ is obtained. Since light and 
radio signals travel along null geodesics, the geodesic distance will be different from the Eu-
clidean distance, and this is exactly proved mathematically by formulae (123) for the case 
$a_1 = a_2 = a$ and $e_1 = e_2 = e$. One of the main purposes of this paper is to build up a con-
sistent physical and mathematical theory of the two notions which are closely related one to 
another - the space-time interval and the geodesic distance. From a conceptual point of view, 
the most significant contribution in this paper is the conclusion that the intersection of two 
space-time intervals can be related to the macroscopic Euclidean distance. There is nothing 
strange in this conception since General Relativity approaches, for example the motion of a 
body in a spherically-symmetric gravitational field of a Schwarzschild metric, turns out to be 
the key for understanding a celestial-mechanics effect - the precession of the perihelion of 
an orbit (see the monograph by T.Padmanabhan [69]). Moreover, the perihelion shift of the 
orbit of Mercury could not be found by the methods of celestial mechanics [68]. Likewise, 
the space-time distance and the geodesic distance in this paper provide interesting and new 
information about some celestial mechanics parameters.

Now let us summarize the physical consequences of the condition for intersatellite com-
munications (47), which constitutes one of the most important results in this paper. The 
first important consequence is that unlike formulae (1), where the symmetry between the 
coordinates $r_A$ and $r_B$ results in the "reversibility" of the propagation time difference, i.e. 
$T_2 - T_1 = -(T_1 - T_2)$, there is no time reversibility for the case of moving gravitational null 
cones. This can be seen from expression (33) $T_2 = -\frac{1}{2}T_1 + \epsilon \int d\epsilon \sqrt{\left(\frac{\partial T_1}{\partial \epsilon}\right)^2 + \frac{1}{2} \left(\frac{\partial T_1}{\partial \epsilon}\right)^2} - \frac{s}{T}$ 
(for the case $E_1 = E_2 = E$) where $T_2$ and $T_1$ are not symmetrical since $T_2$ depends not
only on $T_1$, but also on $\frac{\partial T_1}{\partial E}$. In view of the motion of the satellites while transmitting the signals this means that the "forward" and "backward" optical paths will be different, i.e. $\tilde{R}_{A_1B_1} \neq \tilde{R}_{B_1A_2}$, where the optical paths will be in fact the geodesic distances $\tilde{R}_{A_1B_1}$ and $\tilde{R}_{B_1A_2}$, given by formulae (122). The idea that the optical paths for the case of moving satellites should not be equal was proposed by Klioner in [70]. The difference of the optical paths however has nothing to do with the "reversibility" of the numeration of points - one may choose the space point 2 as the "initial" point from where the signal is being sent in the direction of the second receiving-signal satellite, situated at the space-point 1. Then formulae (33) should be rewritten as (the indices 1 and 2 are interchanged)

$$T_1 = -\frac{1}{2} T_2 + \epsilon \int dE \sqrt{\frac{(\partial T_2)}{\partial E}^2 + \frac{1}{2} \left( \frac{\partial T_2}{\partial E} \right)^2 - \frac{S}{P}}. \quad (151)$$

The equations (33) and (151) give a solution for the propagation time $T_2$ in the form of a complicated integro-differential equation.

The second consequence concerns the derived relation (81) $\sin E = \frac{\cos E - e}{1 - e \cos E}$, which does not depend on the semi-major axis of the orbit and is derived from the condition for intersatellite communications (47) for the case of equal semi-major axis, eccentricities and eccentric anomaly angles (the case of zero Euclidean distance). For the typical eccentricity of the GPS-orbit $e = 0.01323881349526$, the obtained value for the eccentric anomaly angle from equality (81) is $E = 45.00251094 \text{ [deg]}$, which is surprisingly close to the value $f = 45.541436900412 \text{ [deg]}$ for the true anomaly angle $f$ calculated according to formulae $\cos f = \frac{\cos E - e}{1 - e \cos E}$ [44], [45]. In fact, these numerical values are the only ones for the case, when the satellites may move along one and the same orbit situated equidistantly one from another (without colliding with each other). The value $E = 45.00251094 \text{ [deg]}$ was proved to be the same as the limiting value (51) $E_{\lim} = 45.00251094 \text{ [deg]}$, above which the space-time interval (48) is positive and below which this interval is negative. So this fact suggests the interpretation that the condition for intersatellite communication represents a boundary value above which the space-time interval is positive or below which this interval is negative.

In (81) the condition for intersatellite communications was written for a partial case but in (47) it was represented in the general case. In deriving the formulae (150) (taken with the positive sign) in Section VII G, the $\sin E_2$ function was expressed from the condition for intersatellite communications (132) and was substituted into the geodesic equation (122).
Consequently, the positivity of the geodesic distance was established with respect to equation (150) and not with respect to the initial equation (122). Nevertheless, the obtained result about the absence of any roots of this equation is mathematically correct. Note also that in deriving both the algebraic equations for the space-time distance and the geodesic distance, for simplicity the case of equal semi-major axis and eccentricities was considered. However, the algebraic structure of these equations depends on the eccentric anomaly angles and not on the eccentricities and semi-major axis because the important fact is that different eccentric anomaly angles give different Euclidean distances. Concretely for the algebraic treatment, the other important fact is the smallness of the eccentricities of the GPS orbits - the proofs based on higher algebra theorems are valid only for such a case.

Some facts from experimental point of view may be pointed out which might be related to the obtained in this paper value $E = 45.00251094$ [deg] for the eccentric anomaly angle. For example, in the GLONASS constellation the satellites within one and the same plane are equally spaced at 45 degrees. Eight satellites can be situated in this way. The eccentricity of the orbit for the GLONASS constellation is $e = 0.02$ (close to the eccentricity of the GPS orbit), so the value for the eccentric anomaly angle according to (81) is obtained to be $E = 45.00573$ [deg]. This is surprisingly close to the angle of equal spacing for the GPS satellites within one plane. For the Galileo constellation, the satellites are 9 per one plane, thus equally spaced at 40 degrees. Of course, from a formal point of view the coincidence between the angle of equal spacing with the eccentric anomaly angle from the formalism of two null gravitational null cones might seem to be accidental but yet the question remains: what is the role of the angle 45 [deg] in the GPS (or GLONASS) intersatellite communications?

Now some evidence shall be presented for the mutual consistency of the obtained numerical results. When considering the case of non-zero Euclidean distance and the positive or negative space-time distance, the numerical inequality (59) $E_1 + E_2 > 89.994978993712$ [deg] was derived for the lower bound of the sum of the two eccentric anomaly angles. But since this case is more general, the derived formulae (59) should be valid also for the partial case of equal eccentric anomaly angles when the Euclidean and the space-time distances are equal to zero - as a consequence formulae (81) was derived and also the equivalent formulae (51) for the limiting value. This means that inequality (59) should be fulfilled for the partial case of equal eccentric anomaly angles, given by $E = E_{\text{lim}} = 45.00251094$ [deg] according to
formulae (81) and (51). Indeed, twice the value of $E = E_{\lim}$ is greater than the number 89.994978993712 [deg] in the right-hand side of inequality (59).

Another evidence for a consistent result is the derived formulae (80) 15.64 [deg] < $E_2$ < 56.88 [deg] in Section D3 of Appendix C for the numerical interval for the angle $E_2$, when the space-time interval can become zero. The last means that the four possible roots of the space-time algebraic equation (213) in Section C of Appendix C are expected to be found in this numerical interval. Since in all formulae there is a symmetry with respect to the angles $E_1$ and $E_2$, the same inequality as (80) should be valid with respect to $E_1$, i.e.

$$15.64 \text{ [deg]} < E_1 < 56.88 \text{ [deg]} \quad .$$

If (80) and (152) are summed up, then one can obtain

$$31.28 \text{ [deg]} < E_1 + E_2 < 113.76 \text{ [deg]} \quad .$$

(153)

Since in deriving the formulae (152) in Appendix C one of the basic assumptions was about the smallness of the eccentricity $e$, the lower bound is in a ”broader range” (i.e. considerably smaller) in comparison with the lower bound 89.994978993712 [deg] from inequality (59), derived under the assumption that trigonometric functions are less or equal to 1, i.e. $\cos E_1 \leq 1, \cos E_2 \leq 1$. So in fact (153) should be written as

$$89.994978993712 \text{ [deg]} < E_1 + E_2 < 113.76 \text{ [deg]} \quad .$$

(154)

It is really amazing that twice the value of $E = E_{\lim}$ (the partial case for equal eccentric anomaly angles) from (81) and (51) remains within this interval! Moreover, the value $E = E_{\lim}$ is obtained for a partial case, while the upper bound 113.76 [deg] in (154) is a result for the general case of different eccentric anomaly angles, derived after the application of the Shur theorem.

There is one more curious and interesting fact. The lower bound 15.64 [deg] in (152) is related to the numerical interval for the eccentric anomaly angle, when the space-time distance can become zero. However, if the restriction (138) $\sin E_2 = \frac{p}{\sin E_1} < 1$ from the condition for intersatellite communications (132) is taken into account, then the allowed lower bound of $E_1$ (or $E_2$) $E_1 > \arcsin p = 30.002899$ [deg] (149) turns out to be higher than the lower bound in 15.64 [deg] < $E_1$ < 56.88 [deg] (152). Consequently, there is an interval

$$15.64 \text{ [deg]} < E_1 < 30.002899 \text{ [deg]} \quad .$$

(155)
where the space-time interval can exist (and can have zeroes) but the condition for intersatellite communications and the resulting from it geodesic distance equation (122) cannot be defined. This confirms the conclusion that the space-time distance is a more broader notion and has a more general meaning in comparison with the Euclidean distance and the geodesic distance. This also means that the notion of space-time distance can be defined independently from the geodesic distance. In fact, this was the logical consequence of derivation of these equations - first the space-time distance equation (45) was derived, then after setting up equal (45) with the Euclidean distance (46), the condition for intersatellite connections (47) was derived and finally - formulae (46) for the geodesic distance. It should be clear also that the space-time interval is defined outside the interval (155) as well, where the space-time algebraic equation does not have roots (zeroes in terms of the chosen variable $y = \sin^2 E_2$), but can have either positive or negative values. Whether the sign of the space-time algebraic polynomial outside the interval (292) will be positive or negative depends on the values of the polynomial $f(y)$ at the endpoints $y = 0$ and $y = 1$. However, in order to ensure the fulfillment of the substitution theorem for the availability of an even or odd number of roots inside the given interval, at these endpoints the values of the polynomial should be determined by the conditions (299) and (301), defined by the inequalities $f(y = 0) < 0$ and $f(y = 1) < 0$ respectively. These conditions, investigated in details in Appendix D again under the realistic assumption about smallness of the eccentricity of the orbit, guarantee that there will be an even number of roots (two or four) of the investigated fourth-degree space-time algebraic polynomial. The second inequality $f(y = 1) < 0$ is proved to be fulfilled if the inequality (307) $E_2 < 2 \arctan(-2) = -126.869897645844$ [deg] is valid, which in terms of the variable $y = \sin^2 E_2$ and the fact that $\sin(...)$ in the third quadrant is a decreasing function, can be rewritten as

$$y > \sin^2(-126.869897645844) = 0.64 = y_0 .$$

(156)

Now it is interesting to compare this result from the application of the substitution theorem with the final inequality (292), derived after the application of the Shur theorem in Appendix C. In terms of the chosen variable $y$ the inequality assumes the form

$$\sin^2(15.64) \leq y \leq \sin^2(56.88)$$

(157)

or, taking into account the numerical values, it can be written as

$$0.07267993 \leq y \leq 0.701453 .$$

(158)
It is very interesting to note that the value $y_0 = 0.64$ from (156) falls within the interval (158), which is an evidence about the consistency between the two theorems. On the base of this consistency, it can be asserted that in the interval

$$0.64 < y \leq 0.701453$$

(159)

there should be at least two (i.e. two or four) roots of the space-time algebraic equation, and in the other interval $0.07267993 \leq y < 0.64$ - either two roots, or no roots at all (in case if all the roots fall within the interval (159)).

Note also that the numerical boundaries of the interval (152) are determined by the inequality (262) after comparing the terms $\frac{2\sin^2 E_2}{e^2}$ in (257) and $\frac{16\sin E_2}{e^4} - \frac{16}{e^4} \sin E_2 \cos E_2$ in (260), which represent the highest inverse powers of the eccentricity $e$. Since the next inverse powers of $e$ are proportional to $\sim \frac{1}{e^{12}}$, they will be 10000 times smaller than the leading terms and therefore will not give any substantial contributions to inequality (262), from where the interval (152) is obtained. Consequently, this numerical estimate (although approximate) can be trusted.

The third consequence follows from relation (88) and since $\sin E \leq 1$, it places a restriction on the value of the eccentricity of the orbit. Hence, it can be derived that $e \leq 0.816496580927726$ ($e^2 \leq \frac{7}{2}$). Since this value is too high and GPS-orbits have a very low eccentricity, the above restriction is of no importance for the GPS- intersatellite communications. It is of importance for the RadioAstron space project where the eccentricity of the orbit varies in a wide range. In this paper it is established that intersatellite communications of RadioAstron with another satellites on the same orbit will be possible in the range $0.59 < e < 0.816$, but this will not be possible for eccentricities in the interval $0.816 < e < 0.966$. In other words, the formalism of the two gravitational null cones will not be valid in this range.

However, the eccentricity restriction plays an important role for the greatness of the geodesic distance $\tilde{R}_{AB}$ in comparison with the Euclidean distance $R_{AB}$ in formulae (130) $\tilde{R}_{AB} = \sqrt{R_{AB}^2 + a^2(1 - \frac{3}{2}e^2)}$. This again confirms the fact about the mutual consistency between the different numerical values obtained in the framework of the formalism.
IX. APPENDIX A: SOME COEFFICIENT FUNCTIONS IN EQUATIONS (23) AND (24)

The coefficient functions $Q_1(E_1, E_2)$, $Q_2(E_1, E_2)$, $Q_3(E_1, E_2)$ and $Q_4(E_1, E_2)$ in equation (23) have the following form

\[
Q_1(E_1, E_2) := \frac{P_1(E_1) \partial T_1(E_1)}{P_2(E_2)} \partial E_2 , \quad (160)
\]

\[
Q_2(E_1, E_2) := \left( \frac{\partial T_2(E_1, E_2)}{\partial E_2} \right)^2 + 2\frac{P_1(E_1) \partial T_1(E_1) \partial T_2(E_1, E_2)}{P_2(E_2) \partial E_1 \partial E_2} - \frac{1}{P_2(E_2)} \frac{\partial R^2_{AB}}{\partial E_2} - \frac{S_2(E_1, E_2)}{P_2(E_2)} , \quad (161)
\]

\[
Q_3(E_1, E_2) := \frac{P_1(E_1)}{P_2(E_2)} \left( \frac{\partial T_1(E_1)}{\partial E_1} \right)^2 - \frac{1}{P_2(E_2)} \frac{\partial R^2_{AB}}{\partial E_1} - \frac{S_1(E_1, E_2)}{P_2(E_2)} , \quad (162)
\]

\[
Q_4(E_1, E_2) := \left( \frac{\partial T_2(E_1, E_2)}{\partial E_1} \right)^2 . \quad (163)
\]

The coefficient functions $G_1(E_1, E_2)$ and $G_2(E_1, E_2)$ in (24) have the following form

\[
G_1(E_1, E_2) := \frac{1}{2} \frac{S_2(E_1, E_2)}{P_2(E_2)} + \left( \frac{P_1(E_1)}{2P_2(E_2)} \right)^2 \left( \frac{\partial T_1(E_1)}{\partial E_1} \right)^2 - \frac{P_1(E_1)}{2P_2(E_2)} \frac{\partial T_1(E_1)}{\partial E_1} , \quad (164)
\]

\[
G_2(E_1, E_2) := \frac{1}{2} \frac{S_1(E_1, E_2)}{P_2(E_2)} + \frac{1}{P_2(E_2)} \frac{\partial R^2_{AB}}{\partial E_1} - \frac{P_1(E_1)}{2P_2(E_2)} \left( \frac{\partial T_1(E_1)}{\partial E_1} \right)^2 . \quad (165)
\]

The function $\overline{G}_1(E_1, E_2)$ in (27) is obtained from (164) by interchanging the indices 1 and 2

\[
\overline{G}_1(E_1, E_2) := \frac{1}{2} \frac{S_1(E_1, E_2)}{P_1(E_1)} + \left( \frac{P_2(E_2)}{2P_1(E_1)} \right)^2 \left( \frac{\partial T_2(E_1, E_2)}{\partial E_2} \right)^2 - \frac{P_2(E_2)}{2P_1(E_1)} \frac{\partial T_2(E_1, E_2)}{\partial E_2} . \quad (166)
\]
The function $G_2(E_1, E_2)$ can be obtained in an analogous way from (165).

The function $N$ in (28) can be written as follows

$$N := \frac{1}{4} \left( \frac{dE_1}{dE_2} \right)^2 \left( 1 + 4 \left( \frac{\partial T_1}{\partial E_2} \right)^2 \right) + \frac{S_1(E_1, E_2)}{P_2(E_2)} \left( \frac{dE_1}{dE_2} \right)^2 + \frac{S_2(E_1, E_2)}{P_2(E_2)} + \frac{1}{P_2(E_2)} \frac{\partial R_{AB}^2}{\partial E_2} - \frac{2P_1(E_1)}{P_2(E_2)} \frac{dE_1}{dE_2} \left( \frac{\partial T_1(E_1)}{\partial E_1} \right)^2 . \quad (167)$$

X. APPENDIX B: THREE THEOREMS FROM HIGHER ALGEBRA

This appendix does not present new material, but contains the proofs of three theorems from higher algebra, which shall be extensively used for proving the existence of roots (within the unit circle) for the space-time algebraic equation and for the nonexistence of such roots for the geodesic algebraic equation. These theorems are: the substitution theorem, the Rouche theorem, and the Shur theorem. All the proofs are taken from the Obreshkoff monograph [18].

**Theorem 5**  If for two numbers $a$ and $b$ the polynomial $f(y)$ of arbitrary degree has equal signs, then $f(y)$ has an even number of roots (zeroes) in the interval $(a, b)$. If the signs of $f(y)$ at the endpoints $a$ and $b$ are different, then the polynomial $f(y)$ has an odd number of roots.

**Proof:** Let the roots of the polynomial in the interval $(a, b)$ are

$$a \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m < b . \quad (168)$$

Since the polynomial has also roots outside the interval $(a, b)$, it can be decomposed as

$$f(y) = (y - \alpha_1)(y - \alpha_2)\ldots(y - \alpha_n)\varphi(y) , \quad (169)$$

where $\varphi(y)$ contains binomial multipliers of the kind $y - \beta$ ($\beta$ is a number outside the interval $(a, b)$), and also quadratic multipliers of the kind $(y - \mu)^2 + \nu^2$, responsible for the imaginary roots. Since $\beta > b > a$, one can write also

$$\frac{a - \beta}{b - \beta} > 0 \quad \Rightarrow \quad \frac{\varphi(a)}{\varphi(b)} > 0 . \quad (170)$$
But for each root $\alpha_k$ ($1 \leq k \leq m$) inside the interval $(a, b)$, the sign of $\frac{a - \alpha_k}{b - \alpha_k}$ is negative. Consequently, if the decomposition (169) is used, then the following equality can be written

$$\frac{f(a)}{f(b)} = \left[ \frac{(a - \alpha_1)(a - \alpha_2) \ldots (a - \alpha_m)}{(b - \alpha_1)(b - \alpha_2) \ldots (b - \alpha_m)} \right] \phi(a) \phi(b).$$

(171)

Due to the positivity of $\frac{\phi(a)}{\phi(b)}$, the sign of $\frac{f(a)}{f(b)}$ will be determined by the $m$-multipliers in the square bracket. Since each one is with a negative sign, the overall sign of $\frac{f(a)}{f(b)}$ will be given by $(-1)^m$. So for $m$ even, one has $\frac{f(a)}{f(b)} > 0$ and for $m$ odd it can be obtained $\frac{f(a)}{f(b)} < 0$. This proves the theorem.

Next, the Rouche theorem has the following formulation [18]:

**Theorem 6**  Let $f(x)$ and $\phi(x)$ are two polynomials and $C$ is a closed curve on which these polynomials are defined. If on $C$ the following inequality is defined

$$|f(x)| > |\phi(x)|,$$

(172)

then the two equations

$$f(x) = 0, \quad f(x) + \phi(x) = 0$$

(173)

have an equal number of roots inside $C$.

Proof: The theorem is valid in principle for the case of complex roots $\alpha_k$ of the polynomial $f(x)$, when for each root $\alpha_k$ within the curve $C$ it can be written

$$x - \alpha_p = r_p \cos \varphi_p + i \sin \varphi_p, \quad p = 1, 2, \ldots, k,$$

(174)

and for each root $\beta_s$ outside the curve $C$, it can also be written

$$x - \beta_s = r_s \cos \Psi_s + i \sin \Psi_s, \quad s = 1, 2, \ldots, m.$$

(175)

Then the function $f(x)$ can be represented as

$$f(x) = R \cos \Phi + i \sin \Phi,$$

(176)

where $R$ and $\Phi$ are correspondingly the module and the argument of $f$. When encircling along the curve, the argument $\Phi$ will change in one or another direction, but one full encircling along the curve $C$ will correspond to a change of the argument by $2k\pi$. It should be noted that the argument $\Psi_s$ for the outside roots can increase or decrease within certain limits, but $\Psi_s$ cannot change by $2\pi$.  

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Further, the following function is constructed

\[ F(x) = f(x) + \varphi(x) = f(x) \left[ 1 + \frac{\varphi(x)}{f(x)} \right]. \] (177)

Let the first equation \( f(x) \) has \( p \) roots inside \( C \). Then upon one full encircling along \( C \), the argument of \( f(x) \) will increase by \( 2p\pi \). Since \( |\varphi(x)| < |f(x)| \), the point \( u = \frac{\varphi(x)}{f(x)} \) will remain within a circle with a radius smaller than 1. Thus the point

\[ 1 + u = 1 + \frac{\varphi(x)}{f(x)} \] (178)

will be within a circle, centered at the point equal to 1. Therefore, the argument of \( f(x) \) will return to its initial value upon one full encircling along \( C \). Since the argument of \( F(x) \) is a sum of the arguments of the functions \( f(x) \) and \( (1 + u) \), this argument will have to increase by \( 2p\pi \). This means that the function \( F(x) \) has \( p \) roots, which precludes the proof.

By means of the Rouche theorem, let us prove the Shur theorem, which further will be the basic mathematical tool for investigation of the space-time equation and the geodesic equation.

**Theorem 7** [18] The necessary and sufficient conditions for the equation

\[ f(y) = a_0y^n + a_1y^{n-1} + \ldots + a_n = 0, \quad a_0 \neq 0 \] (179)

to have roots within the circle \( |y| < 1 \) are:

1. The inequality

\[ |a_0| > |a_n| \] (180)

should be fulfilled.

2. The polynomial of \((n-1)\)-degree

\[ f_1(y) = \frac{1}{y} [\overline{a}_0 f(y) - \overline{a}_n f^*(y)] \] (181)

should have roots only within the circle \( |y| < 1 \). In (181) \( \overline{a}_0, \overline{a}_n \) are the complex conjugated quantities of the coefficients \( a_0, a_n \) and \( f^*(y) \) is the s.c. “inverse” polynomial, defined as

\[ f^*(y) = y^n \overline{f\left(\frac{1}{y}\right)} = \overline{a}_n y^n + \overline{a}_{n-1} y^{n-1} + \ldots + \overline{a}_0, \] (182)

where \( \overline{a}_n, \overline{a}_{n-1}, \ldots, \overline{a}_0 \) are the complex conjugated coefficients (complex coefficient functions), related to the coefficients (complex coefficient functions) \( a_n, a_{n-1}, \ldots, a_0 \). For the present
case, all the coefficient functions will be real, so there will be no complex conjugated quantities and no "barred" coefficients, i.e.

\[ f^*(y) = y^n f\left(\frac{1}{y}\right) = a_n y^n + a_{n-1} y^{n-1} + \ldots + a_0. \]  

(183)

Proof: Let us prove first the necessary condition, assuming that the roots of the polynomial \( f(y) = 0 \) are in the unit circle \( |y| < 1 \). Since according to the Wiet formulae the multiplication of all the roots \( y_1, y_2, \ldots, y_{n-1}, y_n \) gives

\[ (-1)^n \frac{a_n}{a_0} = y_1 y_2 \ldots y_n \]  

(184)

and also \( |y| < 1 \), it follows that

\[ \left| \frac{a_n}{a_0} \right| < 1 \implies |a_0| > |a_n|. \]  

(185)

Consequently, for \( |y| = 1 \) one can write also

\[ |a_0 f(y)| > |a_n f^*(y)|. \]  

(186)

Then from the Rouche theorem it follows that the polynomial \( f(y) \) and the polynomial of \((n - 1)\)-degree

\[ y f_1(y) = a_0 f(y) - a_n f^*(y) \]  

(187)

have \( n \) roots within the circle \( |y| < 1 \).

XI. APPENDIX C: THE SHUR THEOREM AND THE PROOF THAT THE FOURTH-ORDER ALGEBRAIC EQUATION FOR THE SPACE-TIME INTERVAL HAS ROOTS WITHIN THE UNIT CIRCLE

A. The general strategy for constructing a "chain" of lower-degree polynomials

Since the Shur theorem is based on the construction of the \((n - 1)\) degree polynomial \( f_1(y) \) with roots inside the circle \( |y| < 1 \), it is important to check whether the condition \( |a_0| > |a_4| \) (for the case of the four-dimensional polynomial (63)) is fulfilled. Then the condition the polynomial \( f_1(y) \) to have roots within the unit circle will be equivalent to the condition for the original polynomial \( f(y) \) to have roots within the same circle.
Following the above mentioned algorithm and also formulae (181), another polynomial of \((n - 2)\) degree may be constructed. It will have roots inside the circle \(|y| < 1\) only if condition (180) is valid with respect to the coefficients \(b_0, b_1, b_2, \ldots, b_{n-1}\) of the polynomial of \((n - 1)\) degree, i.e. the inequality \(|b_0| > |b_{n-1}|\) should be satisfied (for the 4-dimensional case it will be \(|b_0| > |b_3|\)). Further, if the coefficients of the \((n - 2)\) degree polynomial are \(c_0, c_1, \ldots, c_{n-2}\) and \(|c_0| > |c_{n-2}|\) is fulfilled, then another polynomial of \((n - 3)\) degree can be constructed with roots within the circle \(|y| < 1\). In such a way, a chain of polynomials of diminishing degrees can be constructed, the final polynomial being a first order (linear) equation. It can easily be found when its root by module is smaller than 1. But then, since Shur theorem is a necessary and sufficient condition, all the preceding polynomials of second, third, \((n - 1)\), \(n\)-th degree will have also roots within the unit circle, provided that the chain of coefficient inequalities

\[
|a_0| > |a_n|, \quad |b_0| > |b_{n-1}|, \quad |c_0| > |c_{n-2}| \quad \ldots\ldots
\]

(188)

is fulfilled. In the following subsections, the algorithm will be developed in details for the fourth order algebraic equation for the space-time interval.

It should be kept in mind that the above coefficient inequalities might not be fulfilled. For example, instead of \(|a_0| > |a_n|\), the inverse inequality \(|a_0| < |a_n|\) might be fulfilled. For such a case, instead of the formulae (187), the following formulae for obtaining the \((n - 1)\)-degree polynomial should be used

\[
an f(y) - a_0 f^*(y) = f_1(y) .
\]

(189)

Analogously, if for example another inverse inequality \(|b_0| < |b_{n-1}|\) is fulfilled, then the next \((n - 2)\) degree polynomial \(f_2(y)\) will be given by

\[
b_{n-1} f_1(y) - b_0 f_1^*(y) = f_2(y) .
\]

(190)

So some of the inequalities (188) might be fulfilled, but the remaining ones might be the inverse ones. Then both the formulae of the type (189) and (190) should be applied.

B. Calculation of the coefficient functions for the chain of polynomials of diminishing degrees according to the Shur theorem

In order to derive the first-degree polynomial and to impose the requirement \(|y| < 1\), one has to calculate the coefficient functions of all the polynomials of \(n\)-th, \((n - 1)\), \((n - 2)\), \ldots \ldots
degree.

Let us first calculate the coefficient functions for the case when equalities (188) hold. For the case of the inequalities (188), and applying formulae (187), one can obtain for the general case of the \( n \)-th degree polynomial

\[
f(y) = a_0 y^n + a_1 y^{n-1} + \ldots + a_{n-2} y^2 + a_{n-1} y + a_n
\]  

the following \((n - 1)\) degree polynomial

\[
f_1(y) = \frac{1}{y} [a_0 (a_0 y^n + a_1 y^{n-1} + \ldots + a_{n-2} y^2 + a_{n-1} y + a_n) -
\]

\[
- a_n (a_0 y^n + a_1 y^{n-1} + \ldots + a_1 y + a_0)] = b_0 y^{n-1} + b_1 y^{n-2} + \ldots + b_{n-2} y + b_{n-1}
\]

where the coefficients \( b_0, b_1, \ldots, b_{n-2}, b_{n-1} \) are obtained to be

\[
b_0 = a_0^2 - a_n^2 , \quad b_1 = a_0 a_1 - a_{n-1} a_n , \quad b_2 = a_0 a_2 - a_{n-2} a_n , \quad b_{k-1} = a_0 a_{k-1} - a_{n-k+1} a_n \quad ,\quad b_{n-1} = a_0 a_{n-1} - a_1 a_n \quad .
\]

In the same way, the \((n - 2)\)-degree polynomial

\[
f_2(y) = \frac{1}{y} [b_0 f_1(y) - b_{n-1} f_1^*(y)] =
\]

\[
= c_0 y^{n-2} + c_1 y^{n-3} + c_2 y^{n-4} + \ldots + c_{n-3} y^2 + c_{n-2}
\]

has the coefficient functions

\[
c_0 = b_0^2 - b_{n-2}^2 = (a_0^2 - a_n^2)^2 - (a_0 a_{n-1} - a_1 a_n)^2 ,
\]

\[
c_1 = b_0 b_1 - b_{n-3} b_{n-1} , \quad c_2 = b_0 b_2 - b_{n-3} b_{n-1} ,
\]

\[
c_{n-3} = b_0 b_{n-3} - b_{2} b_{n-1} , \quad c_{n-2} = b_0 b_{n-2} - b_1 b_{n-1} ,
\]

\[
c_{k-2} = b_0 b_{k-2} - b_{n-k+1} b_{n-1} .
\]

According to the Shur theorem, the polynomial \( f_2(y) \) of \((n - 2)\) degree has roots within the circle \(| y | < 1\) if and only if the inequality \(| b_0 | > | b_{n-1} |\) for the coefficient functions of the
polynomial $f_1(y)$ (193) is fulfilled. Taking into account (194) for $b_0$ and (195) for $b_{n-1}$, one can rewrite this inequality as

$$| a_0^2 - a_n^2 | > | a_0 a_{n-1} - a_1 a_n | .$$  

(202)

In the next subsection it will be shown that the non-fulfillment of the initial inequality $| a_0 | > | a_n |$ will lead to the non-fulfillment of the above equality. Yet, a general rule cannot be formulated.

Next, from the polynomial $f_2(y)$ (197) of $(n - 2)$ degree one can construct the $(n - 3)$ degree polynomial

$$f_3(y) = \frac{1}{y} [c_0 f_2(y) - c_{n-2} f_2^*(y)] .$$  

(203)

For the initial polynomial (63) of fourth degree, $f_2(y)$ will be the second-order polynomial

$$f_2(y) = c_0 y^2 + c_1 y + c_2 , \quad f_2^*(y) = c_2 y^2 + c_1 y + c_0$$  

(204)

and the polynomial $f_3(y)$ will be the linear polynomial

$$f_3(y) = d_0 y + d_1 = (c_0^2 - c_2^2) y + (c_0 c_1 - c_1 c_2) .$$  

(205)

Taking into account expressions (194), (195) for $b_0$, ..., $b_{n-1}$ and (198) - (200) for $c_0$, ..., $c_{n-2}$, one can represent the coefficient functions $c_0$, $c_1$, $c_2$ as

$$c_0 = b_0^2 - b_3^2 = (a_0^2 - a_1^2)^2 - (a_0 a_3 - a_1 a_4)^2 ,$$  

(206)

$$c_1 = b_0 b_1 - b_2 b_3 = (a_0^2 - a_1^2)(a_0 a_1 - a_3 a_4) -$$

$$- (a_0 a_2 - a_2 a_4)(a_0 a_3 - a_1 a_4) ,$$  

(207)

$$c_2 = b_0 b_2 - b_1 b_3 = (a_0^2 - a_1^2)(a_0 a_2 - a_2 a_4) -$$

$$- (a_0 a_1 - a_3 a_4)(a_0 a_3 - a_1 a_4) .$$  

(208)

The following expressions for the $b_0$, $b_1$, $b_2$, $b_3$ coefficient functions have been used

$$b_0 = a_0^2 - a_4^2 , \quad b_1 = a_0 a_1 - a_3 a_4 ,$$  

(209)

$$b_2 = a_0 a_2 - a_2 a_4 , \quad b_3 = a_0 a_3 - a_1 a_4 .$$  

(210)

From the linear equation (205) one can find when the root is modulo less than 1

$$| y | = | - \frac{d_1}{d_0} | = \left| \frac{c_0 c_1 - c_1 c_2}{c_0^2 - c_2^2} \right| = \left| \frac{c_1}{c_0 + c_2} \right| < 1 .$$  

(211)
Taking into account the preceding expressions (206) - (210), the inequality can be rewritten as 

\[
| (a_0a_2 - a_2a_4)(a_0a_3 - a_1a_4) - (a_0^2 - a_4^2)(a_0a_1 - a_3a_4) | < | (a_0^2 - a_4^2) + (a_0 - a_4) [a_2(a_0^2 - a_4^2) - (a_0a_3 - a_1a_4)(a_1 + a_3)] | .
\] 

(212) 

A convincing argument, demonstrating the validity of the Shur theorem for the chain of algebraic equations with diminishing degrees is that the inequality (211), derived from the linear equation (205) can also be obtained from the quadratic equation (204) \( f_2(y) = c_0 y^2 + c_1 y + c_2 \) after finding its roots and imposing the restriction \( |y| < 1 \). This simple calculation shall be performed in the following subsections.

The above calculational scheme shall not be applied with respect to the space-time interval algebraic equation since the basic calculational inequalities (188) will not be fulfilled. However, some of these inequalities will be fulfilled with respect to the other fourth-order algebraic equation called in this paper ”the geodesic equation”. Again, by means of the Shur theorem it will be proved that this equation will have no roots in the circle \( |y| < 1 \). From an algebraic point of view, it will be interesting to see how the inequality (212) changes when the coefficient inequalities (188) are the inverse ones - all of them or some of them.

\[ \text{C. Coefficient functions and inequalities for the chain of polynomials derived from the space-time interval algebraic equation} \]

In this subsection the analogue of the inequality (212) for the case of the space-time interval algebraic equation

\[ f(y) = a_0y^4 + a_1y^3 + a_2y^2 + a_3y + a_4 = 0 \] 

(213)

will be derived. This equation has the following coefficient functions

\[ a_0 = 1 , \quad a_1 = -\frac{4}{e^4} \left( 1 - e^2 \right) \sin E_2 , \] 

(214)

\[ a_2 = \frac{1}{e^4} \left[ 4(1 - e^4) - 6e^2(1 - e^2) \sin^2 E_2 - 2e^2 \sin^2 E_2 \right] , \] 

(215)

\[ a_3 = \frac{4}{e^4} (1 - e^2) \sin E_2 \cos E_2 , \] 

(216)

\[ a_4 = 1 - \frac{4}{e^4} - 2 \sin^2 E_2 + \sin^4 E_2 . \] 

(217)
1. The first coefficient inequality

Let us check first whether the inequality $|a_0| > |a_4|$ is fulfilled, which can be written as

$$1 > |1 - \frac{4}{e^4} - 2 \sin^2 E_2 + \sin^4 E_2| .$$  \hspace{1cm} (218)

Since $e$ is the eccentricity of the orbit, which is approximately 0.01, its inverse powers will be very great numbers. It should be stressed that the proof that the space-time interval equation (213) has roots is based on the smallness of the eccentricity number $e$. Some physical implication of this fact have been pointed out in the Discussion part.

The term $\left(-\frac{4}{e^4}\right)$ in the right-hand side of (218) is a very large number equal to $-4.10^8$, which is predominant over the other ones. Since $|x| = -x$, when $x < 0$, (218) should be written as

$$1 > -1 + \frac{4}{e^4} + 2 \sin^2 E_2 - \sin^4 E_2 .$$ \hspace{1cm} (219)

This inequality cannot be fulfilled because it is impossible for a large positive number $\frac{4}{e^4}$ to be smaller than the number 1. Thus it is proved that the inverse inequality $|a_0| < |a_4|$ holds.

Therefore, the $(n - 1)$ degree polynomial $\tilde{f}_1(y)$ should be calculated according to formulae (189). For the fourth-degree algebraic equation (213), the polynomial $\tilde{f}_1(y)$ is given by the formulae

$$\tilde{f}_1(y) = a_4 f(y) - f^*(y) = b_0^*y^3 + b_1^*y^2 + b_2^*y + b_3^* ,$$ \hspace{1cm} (220)

where the star "*" subscripts denote coefficient functions derived for the case, when the inverse inequalities between the coefficient functions are fulfilled. The coefficient functions $b_0^*, b_1^*, b_2^*, b_3^*$ can be expressed as follows

$$b_0^* = -b_3 , \quad b_1^* = -b_2 , \quad b_2^* = -b_1 , \quad b_3^* = -b_0 ,$$ \hspace{1cm} (221)

where the coefficients $b_0, b_1, b_2, b_3$ are given according to formulae (209) - (210).

2. The second coefficient inequality

Let us check whether the inequality $|b_0^*| > |b_3^*|$ between the coefficient functions of the polynomial $\tilde{f}_1(y)$ (220) is fulfilled. It can be rewritten as

$$|a_1a_4 - a_0a_3| > |a_4^2 - 1| ,$$ \hspace{1cm} (222)
which is in fact the inverse inequality of \((202)\) for \(a_0 = 1\). Taking into account the expressions for the coefficient functions, it can be rewritten as (keeping only the highest inverse powers of \(e\), which constitute the predominant contribution)

\[
\left| \frac{16 \sin E_2}{e^6} - \frac{16 \sin E_2}{e^4} + \frac{l_1}{e^2} + \ldots + l_2 \right| > \left| \frac{16}{e^8} - \frac{[8(1 + \sin^4 E_2)]}{e^4} + \ldots + l_3 \right| ,
\]

where \(l_1, l_2, l_3\) are expressions, containing trigonometric functions

\[
l_1 := -4 \sin E_2 + 8 \sin^3 E_2 - 4 \sin E_2 - 2 \sin(2E_2) ,
\]

\[
l_2 := 8 \sin E_2 - 8 \sin^3 E_2 + 2 \sin(2E_2) ,
\]

\[
l_3 := \sin^8 E_2 - 4 \sin^6 E_2 + 6 \sin^4 E_2 - 4 \sin^2 E_2 .
\]

The right-hand side of \((223)\) contains a term, inversely proportional to the eight power of the eccentricity \(e\) of the orbit, while the left-hand side contains a term inversely proportional to the sixth power of the eccentricity. Therefore, the right-hand side is nearly 10000 times greater than the left-hand side, due to which inequality \((223)\) cannot be fulfilled. Moreover, the trigonometric terms in \((224)-(226)\) influence insignificantly both sides of the inequality.

So since the inverse inequality \(| b^*_0 | < | b^*_3 |\) is valid, the second degree polynomial \(\tilde{f}_2(y)\) should be given by an analogous to \((189)\) formulae

\[
\tilde{f}_2(y) = b^*_3 \tilde{f}_1(y) - b^*_0 \tilde{f}_1^*(y) = c^*_0 y^2 + c^*_1 y + c^*_2 ,
\]

where \(\tilde{f}_1^*(y)\) is the inverse polynomial to \((220)\)

\[
\tilde{f}_1^*(y) = b^*_3 y^3 + b^*_2 y^2 + b^*_1 y + b^*_0 .
\]

The coefficient functions \(c^*_0, c^*_1, c^*_2\) are given by the expressions

\[
c^*_0 = b^*_1 b^*_3 - b^*_0 b^*_2 = b_2 b_0 - b_3 b_1 = c_2 ,
\]

\[
c^*_1 = b^*_2 b^*_3 - b^*_0 b^*_1 = b_0 b_1 - b_2 b_3 = c_1 ,
\]

\[
c^*_2 = b^*_3^2 - b^*_0^2 = b_0^2 - b_3^2 = c_0 .
\]

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3. The third coefficient inequality

It remains to check whether the inequality $|c^*_0| > |c^*_2|$ is fulfilled. It can be rewritten as

$$|c_2| > |c_0| \quad \Longrightarrow \quad |b_0b_2 - b_1b_3| > |b_0^2 - b_3^2| ,$$

(232)

or, in terms of the $a_0, a_1, a_2, a_3, a_4$ coefficient functions, as

$$| (1 - a_4^2)(a_0a_2 - a_2a_4) - (a_0a_1 - a_3a_4)(a_0a_3 - a_1a_4) | > | (1 - a_4^2)^2 - (a_0a_3 - a_1a_4)^2 | .$$

(233)

Now let us write the two highest inverse powers of $\epsilon$ for the left-hand side of the above inequality, which give the predominant contributions:

$$\begin{align*}
(1 - a_4^2)(a_0a_2 - a_2a_4) & \simeq \\
& \simeq - \frac{16^2}{\epsilon^{16}} + \frac{2.16^2 \sin^2 E_2}{\epsilon^{14}} + \ldots ,
\end{align*}$$

(234)

$$\begin{align*}
-(a_0a_1 - a_3a_4)(a_0a_3 - a_1a_4) & \simeq \\
& \simeq \frac{16^2 \sin^2 E_2 \cos E_2}{\epsilon^{12}} - \frac{2.16^2 \sin^2 E_2 \cos E_2}{\epsilon^{10}} + \ldots .
\end{align*}$$

(235)

Because of the presence of the large negative term $- \frac{16^2}{\epsilon^{16}}$ in (234) (10000 times larger than the next term $\simeq + \frac{2.16^2 \sin^2 E_2}{\epsilon^{14}}$) and $10^8$ and $10^{12}$ times larger than the corresponding terms in (235), the sum of the two terms (234) and (235) is negative. Thus with account of (232), it is proved that $c_2 < 0$.

The largest terms on the right-hand side of (233) are

$$\begin{align*}
(1 - a_4^2)^2 & \simeq \frac{16^2}{\epsilon^{16}} - \frac{16^2(1 - k_1)}{\epsilon^{12}} + \frac{32f}{\epsilon^{8}} + \ldots ,
\end{align*}$$

(236)

$$\begin{align*}
-(a_0a_3 - a_1a_4)^2 & \simeq - \frac{16^2 \sin^2 E_2}{\epsilon^{12}} + \frac{2.16^2 \sin^2 E_2}{\epsilon^{10}} - \frac{m_3}{\epsilon^{8}} ,
\end{align*}$$

(237)

where $f, k_1$ and $m_3$ are combinations of trigonometric functions. Both the left-hand and the right-hand side have terms $\sim \frac{16^2}{\epsilon^{16}}$, but in the left-hand side of (234) a term $\sim \frac{1}{\epsilon^{14}}$ is contained which is not present in the right-hand sides of expressions (236) and (237). Evidently the left-hand side of inequality (233) is indeed greater that the right-hand side due to which this inequality is fulfilled. It can be seen also that because of the large positive term $\frac{16^2}{\epsilon^{16}}$ in (236)
which is absent in expression (237) for \((a_0a_3 - a_1a_4)^2\), the calculated according to (206) coefficient function \(c_0\) is positive, i.e.

\[
c_0 = b_0^2 - b_3^2 = (1 - a_1^2)^2 - (a_0a_3 - a_1a_4)^2 > 0 .
\]

(238)

From (234) - (237) it can easily be seen that

\[
c_0 + c_2 > 0 ,
\]

(239)

which is fully consistent with the inequality \(|c_2| > |c_0|\) because of the chain of inequalities

\[
c_0 > -c_2 \implies c_2 < -c_0 \implies -c_2 < -c_0 \implies \]

\[
\implies c_2 > c_0 \implies |c_2| > |c_0| . \quad (240)
\]

In (240) \(c_2\) is the positive part of \(c_2\) because \(c_2\) is negative.

4. Positive and negative coefficient function \(c_1\)

We shall prove that the coefficient \(c_1\) can be positive or negative, depending on the value of the eccentricity angle \(E_2\). This is important in view of the fact that the restriction \(|y| < 1\) can be proved either from the linear equation or equivalently, from the quadratic equation (227).

The two parts of the expression (207) can be calculated by using expressions (214) - (217) for the coefficient functions \(a_0, a_1, a_2, a_3\), taking into account only the highest inverse powers of the eccentricity

\[
(a_0^2 - a_1^2)(a_0a_1 - a_3a_4) \simeq -\frac{16^2}{e^{14}} \sin E_2 \cos E_2 +
\]

\[
+ \frac{16^2}{e^{12}} + \frac{64}{e^{10}} [(1 - k_1) \sin(2E_2) + n_2] + .... , \quad (241)
\]

\[-(a_0a_2 - a_2a_4)(a_0a_3 - a_1a_4) \simeq \frac{16^2}{e^{14}} \sin E_2 -
\]

\[-\frac{16^2}{e^{14}} \sin E_2 \cos E_2 - \frac{2.16^2 \sin^3 E_2}{e^{12}} + .... . \quad (242)
\]

Again, the terms \(\sim \frac{16^2}{e^{14}}\) are 10000 times larger than the terms \(\sim \frac{1}{e^{12}}\), so the sum only of the terms \(\sim \frac{1}{e^{12}}\) in the above approximate equalities gives

\[
c_1 = \frac{16^2}{e^{14}} \sin(2E_2) \left[2 \sin E_2 \sin(2E_2) - \frac{1}{2}\right] .
\]

(243)
If \( \sin(2E_2) > 0 \), i.e. \( E_2 \subset \left[ 0, \frac{\pi}{2} \right] \) and also if
\[
2 \sin E_2 \sin(2E_2) - \frac{1}{2} > 0 ,
\]
then the coefficient function \( c_1 \) is positive. In terms of the notation \( \sin^2 E_2 = y \), the preceding inequality can be written as
\[
y^3 - y^2 + \frac{1}{64} < 0 .
\]
Conversely, if \( \sin(2E_2) < 0 \) (i.e. \( E_2 \subset \left[ \frac{\pi}{2}, \frac{3\pi}{4} \right] \cup \left[ \frac{3\pi}{4}, \pi \right] \)) then the inequality (245) has to be with a reversed sign so that \( c_1 \) is again positive. In the same way, the case \( c_1 < 0 \) can be investigated. In this paper, the above inequality will not be investigated because the final inequality (quite similar to (245)) for the localization of the roots of the initial polynomial within the unit circle will be valid for both positive and negative \( c_1 \).

D. Finding the condition (in its general form) for localization of the roots of the fourth-order space-time interval equation within the unit circle

Let us construct the linear equation
\[
\tilde{f}_3(y) = d_0^* y + d_1^* = c_2^* \tilde{f}_2(y) - c_0^* \tilde{f}_2^*(y)
\]
after applying the chain of higher-degree algebraic equations and inequalities from the preceding subsections. Note also that the first-order equation is derived on the base of the modified formulae (227) and not (203).

The coefficient functions \( d_0^* \) and \( d_1^* \) are given by the following expressions
\[
d_0^* = c_1^* c_2^* - c_0^* c_1^* = c_1 (c_0 - c_2) ,
\]
\[
d_1^* = c_2^* - c_0^* = c_0^* - c_2^* .
\]
Consequently, the condition for the roots to remain within the unit circle is
\[
|y| = | - \frac{d_1^*}{d_0^*} | = | \frac{c_2^* - c_0^*}{c_1^*} | = | \frac{c_0 + c_2}{c_1^*} | < 1 .
\]
From the preceding subsection it follows that the following case is fulfilled
\[
c_2 < 0 , \quad c_0 > 0 , \quad c_0 + c_2 > 0 , \quad c_1 > 0 \text{ or } c_1 < 0 .
\]
From inequality \((249)\) it can be written also that

\[
c_0 + c_2 < c_1 \quad .
\] (251)

This in fact represents the condition for the roots of the linear equation to remain within the unit circle. In view of the necessary and sufficient conditions of the Shur theorem, it will represent also the condition for the roots of all other higher-order algebraic equations (second, third and the final one - the fourth order) to remain within the circle \(|y| < 1\).

Let us demonstrate that the theorem is fulfilled for the first- and second-order algebraic equations. This means that the found relation \((251)\) \(c_0 + c_2 < c_1\) from the first-order equation should be derived also from the second-order equation \((227)\), which in view of \((229) - (231)\) should be written as (see also \((227)\))

\[
\tilde{f}_2(y) = c_0^* y^2 + c_1^* y + c_2^* = c_2 y^2 + c_1 y + c_0 \quad .
\] (252)

From the roots of this quadratic equation and the condition \(|y| < 1\), it follows

\[
|y| = |{-c_1 \over 2c_2} \pm \sqrt{{c_1^2 \over (2c_2)^2} - {c_0 \over c_2}}| < 1 \quad .
\] (253)

Let \(c_1 > 0\) and let us assume a minus sign in front of the square root. The expression inside the module will be negative (then \(|x| = -x\) if \(x < 0\)) and if \(\overline{c}_2\) is the positive part of the negative coefficient function \(\overline{c}_2\), then it follows

\[
\sqrt{{c_1 \over 2c_2} - {c_0 \over c_2}} < 1 + {c_1 \over 2\overline{c}_2} \quad ,
\] (254)

from where

\[
{c_0 \over \overline{c}_2} < 1 + {c_1 \over \overline{c}_2} \implies {c_1 + \overline{c}_2 - c_0 \over \overline{c}_2} > 0 \implies c_0 - \overline{c}_2 < c_1 \quad ,
\] (255)

which is the same as \((251)\). This condition with account of the expressions \((206) - (208)\) for \(c_0, c_1, c_2\) can be represented in the following way

\[
| (a_0^2 - a_4^2)^2 - (a_0 a_3 - a_1 a_4)^2 +
+(a_0^2 - a_4^2)a_2(a_0 - a_4) -
-(a_0 a_1 - a_3 a_4)(a_0 a_3 - a_1 a_4) | <
< | (a_0^2 - a_4^2)(a_0 a_1 - a_3 a_4) |
\]
The purpose further will be to derive the exact expression, using the coefficient functions \((214) - (217)\) and then to prove that the right-hand side contains higher inverse powers of the eccentricity of the orbit. Since it is a small number and the inverse powers represent very large numbers, this would mean that the right-hand side will be greater than the left-hand side and consequently, the inequality will be fulfilled.

1. Final proof of the validity of the Shur theorem with respect to the algebraic equation for the space-time interval

A lengthy calculation shows that the first three terms with highest inverse powers of \(e\) in the left-hand side of the preceding inequality [(256)] can be represented as

\[
\frac{16^2}{e^{16}} - \frac{16^2}{e^{16}} + \frac{2 \cdot 16^2 \sin^2 E_2}{e^{14}} + \frac{p_1}{e^{12}},
\]  

where

\[
p_1 = 16^2 \sin^2 E_2 \cos E_2 + 64(2 - 3k_1 - k_2) - 16^2 \sin^2 E_2 - 16^2(1 - k_1),
\]

where \(k_1\) and \(k_2\) are the trigonometric expressions

\[
k_1 := 2 \sin^2 E_2 - \sin^4 E_2, \quad k_2 = 6 \sin^2 E_2 - 4.
\]

Since terms \(\sim \frac{16^2}{e^{16}}\) cancel, the highest term is \(\frac{2 \cdot 16^2 \sin^2 E_2}{e^{14}}\). The next term in \(\frac{p_1}{e^{12}}\) is 10000 times smaller than the first one and moreover, two of the constituent terms in \(p_1\) are with a positive sign and two - with a negative sign. So the term \(\frac{p_1}{e^{12}}\) is really much smaller than the preceding term.

Consequently, the largest term \(\frac{2 \cdot 16^2 \sin^2 E_2}{e^{14}}\) from the left-hand side of (257) has to be compared with the highest inverse degree terms with the eccentricity parameter in the right-hand side of (256). These terms are

\[
\frac{16^2 \sin E_2}{e^{14}} - \frac{16^2 \sin E_2 \cos E_2}{e^{14}},
\]

and they will be greater than the term in the left-hand side of (257), if the following inequality is fulfilled

\[
\sin E_2 \sin^2(2E_2) > \sin^2 E_2.
\]
Introducing the notation $\overline{y} = \sin E_2$, this inequality can be rewritten as

$$4 \left[ \overline{y}^3 - \overline{y} + \frac{1}{4} \right] > 0 .$$

(262)

Further a distinction should be made between the notations $\overline{y} = \sin E_2$ and the initially introduced notation $y = \sin^2 E_1$.

2. Finding the solutions of the algebraic inequality with respect to the function $\overline{y} = \sin E_2$

Suppose that the cubic equation

$$\overline{y}^3 - \overline{y} + \frac{1}{4} = 0$$

(263)

has three roots $\overline{y}_1$, $\overline{y}_2$ and $\overline{y}_3$. It will be positive if the following inequalities are fulfilled

$$\overline{y}_1 \leq \overline{y} \leq \overline{y}_2 \ , \ \overline{y} \geq \overline{y}_3 .$$

(264)

However, one additional requirement should be taken into account - $|\overline{y}| < 1$. In view of the simplicity of the cubic equation, the algebraic criteria for the localization of roots from the Shur and the substitution theorems shall not be applied, but instead, the roots of the cubic polynomial (263) shall be found directly. It is known from higher algebra that the roots of a cubic polynomial of the kind

$$\overline{y}^3 + p\overline{y} + q = 0$$

(265)

are found from the Kardano formulae

$$\overline{y} = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} ,$$

(266)

where

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}$$

(267)

is the discriminant of the cubic polynomial. In the present case $p = -1$, $q = \frac{1}{4}$, so the discriminant is

$$\Delta = \frac{164}{64} - \frac{1}{27} = \frac{27 - 64}{64.27} < 0 .$$

(268)

Thus the Kardano formulae becomes

$$\overline{y} = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} ,$$

(269)
where $\tilde{\Delta}$ is the positive part of $\Delta$. Denoting the complex expression under the cubic root in (269) as

$$-\frac{q}{2} + \sqrt[3]{\Delta} = \rho (\cos \varphi + i \sin \varphi)$$

and making equal the corresponding real and imaginary parts on both sides, one can find

$$\rho = \sqrt[3]{\frac{q^2}{4} - \Delta} = \sqrt[3]{-\frac{p^3}{27}} , \quad \cos \varphi = -\frac{q}{2\sqrt[3]{\frac{p^3}{27}}} = -\frac{q}{2\rho} .$$

For the numerical values of $p$ and $q$, the numerical values for $\rho$ and $\cos \varphi$ are

$$\rho = \sqrt[3]{\frac{1}{27}} = 0.192450089 ,$$

$$\cos \varphi = -\frac{1}{8\rho} = -0.649519052 ,$$

$$\varphi = \arccos(-0.649519052) =$$

$$= 130.505350 \ [\text{deg}] .$$

Further, the two parts of the solution (269) can be represented as

$$\overline{y} = x + z = \sqrt[3]{\rho} (\cos \varphi + i \sin \varphi) + \sqrt[3]{\rho} (\cos \varphi - i \sin \varphi) =$$

$$= \sqrt[3]{\rho} \left( \cos \frac{\varphi + 2k_1 \pi}{3} + i \sin \frac{\varphi + 2k_1 \pi}{3} \right) +$$

$$+ \sqrt[3]{\rho} \left( \cos \frac{\varphi + 2k_2 \pi}{3} + i \sin \frac{\varphi + 2k_2 \pi}{3} \right) .$$

On the other hand, substituting (275) $\overline{y} = x + z$ in the cubic equation (265), one can derive

$$\overline{y}^3 - 3zx\overline{y} - (x^3 + z^3) = 0 ,$$

from where it follows

$$xz = -\frac{p}{3} , \quad x^3 + z^3 = q .$$

Making use of (275) and (276), one can obtain from the first relation in (278)

$$\rho^{\frac{2}{3}} \left( \cos \frac{2(k_1 - k_2)\pi}{3} + i \sin \frac{2(k_1 - k_2)\pi}{3} \right) = -\frac{p}{3} ,$$

from where

$$k_1 = k_2 , \quad \rho = \left( i^2 \frac{p}{3} \right)^{\frac{3}{2}} .$$
Consequently, the three roots of the cubic equation \( y^3 - y + \frac{1}{4} = 0 \) can be written as

\[
\bar{y}_m = \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2k\pi}{3}, \quad k = 0, 1, 2; \quad m = 1, 2, 3
\]  \hspace{1cm} (281)

Now let us assume that \( 0 \leq E_2 \leq \frac{\pi}{2} \), i.e. the eccentric anomaly angle is in the first quadrant. Then in view of (281) and the inequalities (264), it follows that

\[
\frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \leq \sin E_2 \leq \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3}, \quad (282)
\]

\[
\sin E_2 \geq \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3}. \quad (283)
\]

For the first quadrant one can write the obvious relation

\[
\sin E_2 = \cos \left( \frac{\pi}{2} - E_2 \right) \quad (284)
\]

and since \( \cos \) is a decreasing function, the substitution into (282) will give the inequality

\[
\frac{\pi}{2} + \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) \leq E_2 \leq \\
\leq \frac{\pi}{2} + \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3} \right) \quad . \hspace{1cm} (285)
\]

3. Finding the interval for the numerical values of the eccentric anomaly angle, where the space-time interval can become zero

Note that we have taken into account two important facts: with the increase of the angle \( \varphi \), \( \cos \varphi \) is a decreasing function in the first quadrant, but since \( \arccos(...) \) is also a decreasing function, \( \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) \) will be an increasing function. However, (285) gives values for \( E_2 \) greater than \( \frac{\pi}{2} \), in contradiction with the initial assumption. The correct approach will be to take into account that \( \cos \left( \frac{\pi}{2} - E_2 \right) = \cos \left( E_2 - \frac{\pi}{2} \right) \), then equality (282) can be rewritten as

\[
\frac{\pi}{2} - \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3} \right) \leq E_2 \leq \\
\leq \frac{\pi}{2} - \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) \quad . \hspace{1cm} (286)
\]

Now the concrete numerical value (274) for \( \varphi = 130.505 \) [deg] should be used in order to prove that the left-hand side of inequality (286) is undetermined, since it can be calculated

\[
\frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3} = -1.107159 \quad . \hspace{1cm} (287)
\]
Consequently, the second root ($m = 2, k = 1$ in (281)) is undefined with respect to the angle $E_2$ (since $| \sin E_2 | = | y | \leq 1$) and because of that, the function $\arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3} \right)$ has an invalid argument and the root $y_2$ is outside the circle $| y | < 1$.

It is important however that for the value $\varphi = 130.505$ [deg] the expression in the right-hand side of (286) can be exactly calculated. So we have

$$\arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) = 33.116078469887 \text{ [deg]}$$

and from (286)

$$E_2 \leq 56.883921530113 \text{ [deg]} \ .$$

Let us now investigate the second inequality (283). It can be found that

$$\frac{2}{\sqrt{3}} \cos \frac{\varphi + 4\pi}{3} = 0.2695944364053715 \ .$$

Since in the first quadrant $\arcsin(...)$ is an increasing function, from (283) one can obtain

$$E_2 \geq \arcsin \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi + 4\pi}{3} \right) = 15.640134887147 \text{ [deg]} \ .$$

Therefore, from the above analysis two important conclusions can be made:

1. The roots of the cubic equation (263) (one of them is undetermined) are not related to the number of roots of the original algebraic equation (213).

2. The two roots of the equation (263) allow to determine the possible range of values for the eccentric anomaly angle $E_2$, for which the space-time interval can become zero. This is the range

$$15.64 \text{ [deg]} \leq E_2 \leq 56.88 \text{ [deg]} \ .$$

In fact, the lower bound should not be 15.64 [deg], because earlier in (149) it was established that $E_1 > \arcsin p = 30.00289942985$ [deg]. Since the eccentric anomaly angles $E_1$ and $E_2$ enter all formulaes symmetrically, the same bound should be valid also for the angle $E_2$. So if one would like to define properly both the space-time interval and the geodesic distance, one should write the admissible numerical interval for the eccentric anomaly angle as

$$30.002 \text{ [deg]} \leq E_2 \leq 56.88 \text{ [deg]} \ .$$

In the Discussion part of this paper it was however clarified that since the space-time interval is a notion, which can be defined independently from the condition for intersatellite
communications and from the geodesic distance, then the numerical estimate (292) can also be accounted as denoting the boundaries of the interval, where the space-time interval can become zero (in the first quadrant). It should be mentioned that the upper limit 56.88 [deg] in the above equality may be obtained directly from (282), taking into account that \( \arcsin \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) \) in the first quadrant is a decreasing function. Therefore, it can be written

\[
\arcsin \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3} \right) \leq E_2 \leq \arcsin \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) .
\]  

(294)

Direct calculation confirms that

\[
\arcsin \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) = 56.883921530113 \text{ [deg]}
\]  

(295)

and consequently

\[
\frac{\pi}{2} - \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) = \arcsin \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) .
\]  

(296)

In the same way, a similar analysis may be performed for the other three quadrants. For example, for the second quadrant \( E_2 \in \left[ \frac{\pi}{2}, 2\pi \right] \), where \( \arccos \left( \frac{2}{\sqrt{3}} \cos \frac{\varphi}{3} \right) \) is an increasing function, it can be received from inequality (283) for the third root

\[
- \cos E_2 = \sin \left( \frac{\pi}{2} + E_2 \right) \geq \frac{2}{\sqrt{3}} \cos \left( \frac{\varphi + 4\pi}{3} \right) ,
\]  

(297)

from where

\[
E_2 \leq \arccos \left( -\frac{2}{\sqrt{3}} \cos \frac{\varphi + 4\pi}{3} \right) =
\]

\[
= 105.640134887147 \text{ [deg]}
\]  

(298)

Therefore, \( E_2 \) is in the range \( E_2 \in \left[ \frac{\pi}{2}, 105.64 \text{ [deg]} \right] \).

XII. APPENDIX D: ROOTS OF THE SPACE-TIME INTERVAL ALGEBRAIC EQUATION FROM THE SUBSTITUTION THEOREM

In order to prove that an even or an odd number of roots of a polynomial remains within a given interval, according to the substitution theorem one has to compute the signs of the polynomial at the two endpoints of the interval. Thus, in order to check whether the
fourth-degree polynomial \( f(y) = a_0 y^4 + a_1 y^3 + a_2 y^2 + a_3 y + a_4 = 0 \) has roots within the circle \(| y |< 1\), one has to determine the sign of the polynomial \( f(y) \) at the endpoints 0 and 1.

Let us compute \( f(0) \), keeping in mind that the coefficient functions of \( f(y) \) are given by the expressions (214)-(217). We have

\[
f(0) = 1 - \frac{4}{e^4} - 2 \sin^2 E_2 - \sin^4 E_2 < 0 \tag{299}
\]

This is an expression with a negative sign because of the large negative term \(-\frac{4}{e^4} = -4.10^{-8}\). The value of \( f \) at \( y = 1 \) is

\[
f(1) = a_0 + a_1 + a_2 + a_3 + a_4 = 1 - \frac{4}{e^2} (1 - e^2) \sin E_2 + \frac{1}{e^4} [4(1 - e^4) - 6e^2(1 - e^2) \sin^2 E_2 - 2e^2 \sin^2 E_2] + \frac{4}{e^2} (1 - e^2) \sin E_2 \cos E_2 + 1 - \frac{4}{e^4} - 2 \sin^2 E_2 - \sin^4 E_2 \tag{300}
\]

Note that terms with \( \frac{4}{e^4} \) cancel. According to the substitution theorem, if at the two endpoints \( f(0) < 0 \) and \( f(1) < 0 \) (i.e. the polynomial has equal signs), then the polynomial will have an even number of roots. In view of the fact that the polynomial is of fourth degree, the even number of roots might be two or four. The last seems more probable, so the condition \( f(1) \) to be negative can be written as

\[
\frac{1}{e^4} \left[ -4 \sin E_2 - 8 \sin^2 E_2 + 2 \sin(2E_2) \right] < 0 \tag{301}
\]

taking into account only the terms proportional to \( \frac{1}{e^4} \), which have the predominant contribution. Let us now find what is the consequence from inequality (301), which can be rewritten as

\[
\sin(2E_2) - 2 \sin E_2 - 4 \sin^2 E_2 < 0 \tag{302}
\]
or in an equivalent form

\[
8 \sin^2 \frac{E_2}{2} \cos^2 \frac{E_2}{2} \left( \tan \frac{E_2}{2} + 2 \right) > 0 \tag{303}
\]

The inequality is fulfilled if

\[
E_2 > 2 \arctan(-2) \quad \text{or}\quad E_2 < 2 \arctan(-2) \tag{304}
\]
The first case will take place if \( \arctan() \) is an increasing function and the second case - if \( \arctan() \) is a decreasing function. Since

\[
2 \arctan(-2) = -126.869897645844 \text{ [deg]} ,
\]

the choice of the first or the second option in (304) will depend on whether the function \( \arctan(...) \) in the third quadrant will be increasing or decreasing. It should be clarified first that a minus sign of the degrees is counted from the positive \( x \)-axis by rotation in the anticlockwise direction. Thus \(-126.86 \text{ [deg]} \) is within the third quadrant \( E_2 \subset \left[ \pi, \frac{3\pi}{2} \right] \), where the function \( \tan(...) \) is a decreasing one. This can be established by computing two arbitrary values, for example

\[
\tan(-120) = 1.73 , \quad \tan(-150) = 0.57 \quad .
\]

Consequently, since the function \( \arctan(...) \) is also a decreasing one, one should choose the second case in (304), i.e.

\[
E_2 < 2 \arctan(-2) = -126.869897645844 \text{ [deg]} ,
\]

which means that

\[
E_2 \subset (-126.86, -\frac{\pi}{2}) \cup \left[ -\frac{\pi}{2}, 0 \right) .
\]

XIII. APPENDIX E: PROOF BY MEANS OF THE SHUR THEOREM THAT THE FOURTH DEGREE ALGEBRAIC EQUATION FOR THE GEODESIC DISTANCE DOES NOT HAVE ANY ROOTS WITHIN THE UNIT CIRCLE

A. The geodesic algebraic equation and its coefficient functions

In this Section we shall follow again the algorithm, based on the Shur theorem for construction of a chain of lower-degree polynomials, beginning from the initial fourth-degree geodesic equation

\[
g(y) = \pi_0 y^4 + \pi_1 y^3 + \pi_2 y^2 + \pi_3 y + \pi_4 = 0 \quad .
\]

Then it shall be proved that the above equation will not result in an equality similar to (256) or (212), which is supposed to be fulfilled for small eccentricities. This would mean that the polynomial shall have no roots within \( |y| < 1 \), as it should be expected for the geodesic
distance. It should be remembered also that by means of the condition for intersatellite communications, the geodesic distance was proved to be greater than the Euclidean distance. The coefficient functions in the above polynomial are the following ones

\[
\begin{align*}
\bar{a}_0 &= 1 \quad , \quad \bar{a}_1 = -\frac{(2 + 3e^2)}{e^2} \quad , \quad \bar{a}_2 = \frac{\tilde{M}}{e^4} \\
\bar{a}_3 &= -\frac{p^2(2 + 3e^2)}{e^2} \quad , \quad \bar{a}_4 = p^4
\end{align*}
\]  

(310)

(311)

where \( p \) and \( \tilde{M} \) are the numerical parameters

\[
p := \frac{2 - e^2}{4(1 - e^2)} \quad , \quad \tilde{M} := \frac{9}{4}e^4 + 3e^2 + 2e^4p^2 - 3 - 4p^2 .
\]

(312)

Earlier we have introduced the notation \( p \) in (133) for the case of different eccentricities \( e_1, e_2 \) and different semi-major axis \( a_1, a_2 \). Here we use the same notation because (312) is in fact expression (133) for the case \( e_1 = e_2 \) and \( a_1 = a_2 \).

B. Construction of the chain of lower-degree polynomials and the corresponding coefficient inequalities

In order to construct the first third-degree polynomial \( g_1(y) \) and its inverse one \( g_1^*(y) \)

\[
g_1(y) = \bar{b}_0y^3 + \bar{b}_1y^2 + \bar{b}_2y + \bar{b}_3 \quad , \quad g_1^*(y) = \bar{b}_3y^3 + \bar{b}_2y^2 + \bar{b}_1y + \bar{b}_0
\]

(313)

on the base of the defining polynomial (192)-(193) or on the base of the polynomial (220) with coefficient functions (221), one has to check whether the inequality \( |\bar{a}_0| > |\bar{a}_4| \) is fulfilled. In the present case this inequality has the simple form

\[
1 > p^4 
\]

(314)

and since \( p \) can be represented as

\[
p := \frac{2 - e^2}{4(1 - e^2)} = \frac{1}{4} + \frac{1}{4(1 - e^2)} < 1 ,
\]

(315)

the inequality is fulfilled. Moreover, for the typical GPS eccentricity \( e \), the parameters \( p \) and \( p^4 \) have the numerical values

\[
p = 0.50004382422651548 \implies p^4 = 0.062521 .
\]

(316)
Consequently, the coefficient functions \( \bar{b}_0, \bar{b}_1, \bar{b}_2, \bar{b}_3 \) of the third-degree polynomial \( g_1(y) \) (313) are given by the standard formulae (209) - (210), but now with the "bar" coefficients \( \bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4 \) in (310) and (311).

In order to write the next second-degree polynomial from the chain of polynomials, one has to check whether the inequality

\[ | \bar{b}_0 | > | \bar{b}_3 | \]  

is fulfilled where \( \bar{b}_0 \) and \( \bar{b}_3 \) are given by the expressions

\[ \bar{b}_0 = a_0^2 - a_4^2 \quad , \quad \bar{b}_3 = a_0 a_3 - a_1 a_4 \]  

(318)

The inequality (317) acquires the form

\[ | 1 - p^8 | > \frac{p^2(2 + 3e^2)}{e^2}(p^2 - 1) \]  

(319)

Denoting \( p^2 = \bar{p} < 1 \) and keeping in mind that \( | p^2 - 1 | = 1 - p^2 \) since \( p^2 - 1 < 0 \) (\( | x | = -x \) if \( x < 0 \)), inequality (319) can be rewritten as

\[ 1 + \bar{p}^2 + \bar{p}^3 - \bar{p}^2(1 + e^2) > 0 \]  

(320)

The first three terms are not large, but they are positive, while the fourth term \( -\bar{p}^2(1 + e^2) \) is a very large negative term due to the second inverse power of the eccentricity \( e \). Because of this, the term will be proportional to \(-10^4\). So the left-hand side of the above inequality cannot be positive and inequalities (319) and (317) cannot be fulfilled. Consequently, since instead of (317) one has the inequality \( | \bar{b}_0 | < | \bar{b}_3 | \), the second-order polynomial

\[ g_2(y) = \bar{c}_0 y^2 + \bar{c}_1 y + \bar{c}_2 = \bar{b}_3 g_1(y) - \bar{b}_0 g_1^*(y) \]  

(321)

is constructed by means of formulae (189) and (220). The "tilda" signs of the coefficients \( \bar{c}_0, \bar{c}_1, \bar{c}_2 \) mean that these coefficients are not derived from formulae (206)-(208), but from (229)-(231). In the present case, these expressions are

\[ \bar{c}_0 = \bar{b}_1 \bar{b}_3 - \bar{b}_0 \bar{b}_2 = -\bar{c}_2 \quad , \quad \bar{c}_2 = \bar{b}_3^2 - \bar{b}_0^2 = -\bar{c}_0 \quad , \]

(322)

\[ \bar{c}_1 = \bar{b}_2 \bar{b}_3 - \bar{b}_0 \bar{b}_1 = -\bar{c}_1 \quad . \]

(323)

Next, we have to check whether

\[ | \bar{c}_0 | > | \bar{c}_2 | \quad \text{i.e.} \quad | \bar{c}_2 | > | \bar{c}_0 | \]  

(324)
Making use of the standard expressions (206)-(208) (now with the "barred" coefficients $\bar{a}_0$, $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, $\bar{a}_4$ instead of the "unbarred" ones), the modulus $|\bar{c}_2|$ and $|\bar{c}_0|$ can be written as

$$
|\bar{c}_2| = \frac{(1-p^2)}{e^4} \cdot |\bar{M}(1-p)(1+p^2) - (2+3e^2)p^2(1-p^6)| .
$$

(325)

$$
|\bar{c}_0| = (1-p^2)^2 \cdot \left| (1+p^2)^2 - \frac{p^4(2+3e^2)^2}{e^4} \right| .
$$

(326)

Both sides of the inequality $|\bar{c}_2| > |\bar{c}_0|$ have denominators proportional to $\frac{1}{e^4}$. Let us rewrite the inequality in the following form

$$
|\bar{M}(1+p^2)^2(1+p^4) - (2+3e^2)p^2(1+p^2+p^4)| > e^4(1+p^2)^2 - p^4(2+3e^2)^2 | .
$$

(327)

Terms proportional to $p$ (see also (315)) and not to the eccentricity $e$ will be larger. So the largest terms in the left-hand side will be those having the smallest powers in $p$. This is the term $\bar{M}p$, and since $\bar{M}$ is given by (312), this will be the term $|-3p |$, numerically equal to 1.50013. In the right-hand side, the largest will be the term not multiplied by powers of the eccentricity $e$ - this will be the term $|-4p^4 |$, equal to 0.25008. Consequently, inequality $|\bar{c}_2| > |\bar{c}_0|$ is fulfilled.

C. The last linear polynomial and the proof that the final inequality is not satisfied

From the polynomial (321) and formulae (205) from the preceding appendix, the final condition for the roots of the linear polynomial

$$
g_3(y) = \vec{d}_0 y + \vec{d}_1 =
$$

$$
(\vec{c}_0^2 - \vec{c}_2^2)y + (\vec{c}_0 \vec{c}_1 - \vec{c}_1 \vec{c}_2)
$$

(328)

to be in the circle $|y| = |\frac{\vec{d}_1}{\vec{d}_0}| < 1$ can be expressed as

$$
|-(\vec{c}_0 \vec{c}_1 - \vec{c}_1 \vec{c}_2)| < |(\vec{c}_0^2 - \vec{c}_2^2)| \implies |\vec{c}_1| < |\vec{c}_2 + \vec{c}_0|
$$

(329)

In terms of the coefficient functions $\bar{a}_0$, $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, $\bar{a}_4$ (given by (310) - (311)), the above inequality can be written as

$$
|(a_0^2 - a_4^2)(a_0 a_1 - a_3 a_4)| -
$$

94
\[-(\bar{a}_0\bar{a}_2 - \bar{a}_2\bar{a}_4)(\bar{a}_0\bar{a}_3 - \bar{a}_1\bar{a}_4) \mid<\]

\[<\mid(\bar{a}_0^2 - \bar{a}_4^2)\bar{a}_2(\bar{a}_0 - \bar{a}_4) -
-(\bar{a}_0\bar{a}_1 - \bar{a}_3\bar{a}_4)(\bar{a}_0\bar{a}_3 - \bar{a}_1\bar{a}_4) +
+(\bar{a}_0^2 - \bar{a}_4^2)^2 - (\bar{a}_0\bar{a}_3 - \bar{a}_1\bar{a}_4)^2 \mid.\]

(330)

In a simple and compact form, this inequality can be represented as

\[
\frac{p^2(2 + 3e^2)}{e^2} \mid (1 + p^2)(1 + p^4) - \frac{\tilde{M}}{e^4} \mid<
\]

(332)

\[< (1 + p^2). \mid (1 + p^4)^2 + \frac{\tilde{M}(1 + p^4)}{e^4} -
- \frac{p^2(2 + 3e^2)^2}{e^4} \mid.
\]

(333)

and the numerical parameter \(\tilde{M}\) is given again by expression (312). Now the largest terms in both sides of the inequality have to be compared - in fact, these are the terms, proportional to the inverse powers of \(e\). In the left-hand side, this is the term

\[\mid - \frac{2p^2\tilde{M}}{e^6} \mid \sim \mid 6p^2 \mid .\]

This is a large number with 12 digits. On the other hand, in the right-hand side the largest term is

\[\sim \frac{(1 + p^2)4p^2}{e^4} .\]

This is a number with 8 digits. So, the right-hand side is impossible to be larger than the left-hand side. Consequently, the Shur theorem cannot be fulfilled, due to which the geodesic equation (309) cannot have any roots in the interval \((0, 1)\). This is in full accord with the fact that the geodesic distance cannot be zero, since it is greater than the Euclidean one.

**XIV. APPENDIX F: THE SUBSTITUTION THEOREM APPLIED TO THE GEODESIC EQUATION**

The application of the substitution theorem to the geodesic equation (309) \(g(y) = \bar{a}_0y^4 + \bar{a}_1y^3 + \bar{a}_2y^2 + \bar{a}_3y + \bar{a}_4 = 0\) presumes that the signs of the functions \(g(0)\) and \(g(1)\) have to be determined.
It can easily be found that
\[ g(0) = p^4 > 0 , \] \hspace{1cm} \text{(334)}
\[ g(1) = 1 - \frac{(2 + 3e^2)}{e^2} + \frac{\tilde{M}}{e^4} - \frac{p^2(2 + 3e^2)}{e^2} + p^4 . \] \hspace{1cm} \text{(335)}
It is seen that the largest term is \( \frac{\tilde{M}}{e^4} \). More exactly, since \( \tilde{M} \) is given by (312), the largest term will be this part of \( \tilde{M} \), which does not contain powers of \( e \) - they will cancel with \( e^4 \) in the denominator and thus smaller inverse powers of \( e \) will result in a smaller term \( \frac{\tilde{M}}{e^4} \).
So the part of \( \tilde{M} \), not containing \( e \) is \( -\frac{3 - 4p^2}{e^4} \), consequently the predominant contribution in \( f(1) \) is negative, i.e.
\[ g(1) = -\frac{(3 + 4p^2)}{e^4} < 0 . \] \hspace{1cm} \text{(336)}
Since at the both endpoints, equal to the numbers zero and one, the polynomial has equal signs (since \( g(0) < 0 \) and \( g(1) < 0 \), it should have an odd number of roots. The odd number could be one or three. However, the polynomial is of fourth degree, so it should have an even (four) number of roots. If these roots are determined as \( y = \sin^2 E_1 \), then all of them should be situated in the circle \( | y | < 1 \). Since this is not what follows from the substitution theorem, it can be concluded that there should be no roots at all in the interval \( (0, 1) \), which confirms the conclusion in the preceding appendix.

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