On the functional Hodrick-Prescott filter with non-compact operators

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Abstract

We study a version of the functional Hodrick-Prescott filter where the associated operator is not necessarily compact, but merely closed and densely defined with closed range. We show that the associated optimal smoothing operator preserves the structure obtained in the compact case, when the underlying distribution of the data is Gaussian.

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1 Introduction

The study of functional data analysis is motivated by their applications in various fields of statistical estimations and statistical inverse problems (see Ramsay and Silverman (1997), Bosq (2000), Müller and Stadtmüller (2005) and references therein). One of the most common assumptions in the studies in statistical inverse problems is to deal with compact operators. This is due to their tractable spectral properties. The functional Hodrick-Prescott filter is often formulated as a statistical inverse problem that reconstructs an 'optimal smooth signal' \( y \) that solves an equation \( Ay = v \), corrupted by a noise \( v \) which is apriori unobservable, from observations \( x \) corrupted by a noise \( u \) which is also apriori unobservable:

\[
\begin{cases}
x = y + u, \\
Ay = v,
\end{cases}
\]

\[\text{(1)}\]
where \( A : H_1 \rightarrow H_2 \) is a compact operator between two appropriate Hilbert spaces \( H_1 \) and \( H_2 \).

By introducing a smoothing operator \( B \), the 'optimal smooth signal' \( y(B, x) \), associated with \( x \), is defined by

\[
y(B, x) := \arg \min_y \left\{ \|x - y\|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2} \right\},
\]

provided that

\[
\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1.
\]

In [6], the optimal smoothing operator is characterized as the minimizer of the difference between the optimal smoothing signal and the best predictor of the signal given the data \( x \), \( E[y|x] \), when the noise \( u \) and the signal \( v \) are independent Hilbert space-valued Gaussian random variables with zero means and covariance operators \( \Sigma_u \) and \( \Sigma_v \).

In this paper, we extend the functional Hodrick-Prescott filter to the case where the operator \( A \) is not necessarily compact. Moreover, we show that the optimal smoothing parameter preserves the structure obtained in [6], for the compact case.

An important class of non-compact operators to which we wish to extend the Hodrick-Prescott filter includes the Laplace operator \( \Delta = \frac{d^2}{dt^2} \), with Dirichlet boundary conditions, whose domain

\[ D(A) = \{ y \in H^2([0,1]), \quad y(0) = y(1) = 0 \}, \]

where, \( H^2([0,1]) \) is the Sobolev space of functions whose weak derivatives of order less than or equal to two belong to \( L^2([0,1]) \) (see e.g. [3] for further examples).

The paper is organized as follows. In Section 2, we generalize the functional Hodrick-Prescott filter under the assumption that the operator \( A \) is closed and densely defined with closed range. In Section 3, we prove that the optimal smoothing operator maintains same form when the covariance operators \( \Sigma_u \) and \( \Sigma_v \) are trace class operators. In Section 4, we illustrate this filter with two examples. In Section 5, we extend this characterization to the case where the covariance operators \( \Sigma_u \) and \( \Sigma_v \) are not trace class such e.g. white noise.

## 2 A Functional Hodrick-Prescott filter with closed operator

Let \( H_1 \) and \( H_2 \) be two separable Hilbert spaces, with norms \( \| \cdot \|_{H_i} \) and inner products \( (\cdot, \cdot)_{H_i}, \ i = 1, 2 \), and \( x \in H_1 \) be a functional time series of observables. Given a linear operator \( A : H_1 \rightarrow H_2 \), the Hodrick-Prescott filter extracts an 'optimal smooth signal' \( y \in H_1 \) that solves an equation \( Ay = v \), corrupted by a noise \( v \) which is apriori unobservable, from observations \( x \) corrupted by a noise \( u \) which is also apriori unobservable:

\[
\begin{cases}
x = y + u, \\
Ay = v.
\end{cases}
\]
Optimality of the extracted signal is achieved by the following Tikhonov-Phillips regularization of the system (3), by introducing a linear operator \( B : H_2 \to H_2 \) which acts as a smoothing parameter:

\[
y(B, x) := \arg \min_y \left\{ \| y - x \|_{H_1}^2 + \langle Ay, BAy \rangle_{H_2} \right\},
\]

provided that

\[
\langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1.
\]

As suggested in [6], the optimal smoothing operator minimizes the gap between the conditional expected value of \( y \) given \( x \), \( E[y|x] \), which is the best predictor of \( y \) given \( x \), and \( y(B, x) \):

\[
\hat{B} = \arg \min_B \| E[y|x] - y(B, x) \|_{H_1}^2.
\]

The main purpose of this work is to extend the characterization of the optimal smoothing operator obtained in [6] to the case where the linear operator \( A \) is not necessarily compact, and \( u \) and \( v \) are independent Hilbert space-valued generalized Gaussian random variables with zero means and covariance operators \( \Sigma_u \) and \( \Sigma_v \).

We assume that the linear operator \( A : H_1 \to H_2 \) is

**Assumption (1)**

(1a) Closed and defined on a dense subspace \( \mathcal{D}(A) \) of \( H_1 \),

(1b) Its range, \( \text{Ran}(A) \), is closed.

Its Moore-Penrose generalized inverse \( A^\dagger \) is defined on

\[ \mathcal{D}(A^\dagger) = \text{Ran}(A) + \text{Ran}(A)^\perp, \]

i.e. \( \text{Ker}(A^\dagger) = \text{Ran}(A)^\perp \).

Assumption (1) is equivalent to the fact that \( A^\dagger \) is bounded (see [8], [12]).

For all \( v \in \mathcal{D}(A^\dagger) \), the set of all solutions of the equation

\[ Ay = v, \quad y \in \mathcal{D}(A), \]

is given by

\[ \{ y^\dagger + y_0; \quad y_0 \in \text{Ker}(A) \}, \]

where \( y^\dagger \) is the unique minimal-norm solution given by \( y^\dagger = A^\dagger v \).

Hence, for arbitrary \( y_0 \in \text{Ker}(A) \), we have

\[ y = y_0 + A^\dagger v, \]

and, in view of (3),

\[ x = y_0 + A^\dagger v + u. \]

Let \( \Pi := A^\dagger A \). By the Moore-Penrose equations, we have \( \Pi A^\dagger = A^\dagger \), \( \Pi^2 = \Pi \) and \( \Pi^* = \Pi \) (self-adjoint). Therefore, \( \Pi \) an orthogonal projector. It is easily checked that, for every \( \xi \in H_1 \), the elements \( \Pi\xi \) and \( (I_{H_1} - \Pi)\xi \) are orthogonal:

\[ < \Pi\xi, (I_{H_1} - \Pi)\xi > = 0, \]
and

\[ \text{Ker}(A) = \text{Ker}(\Pi) = \text{Ran}(I_{H_1} - \Pi). \]  

(10)

Moreover, we have \((I_{H_1} - \Pi)y = y_0\) and \(A^\dagger v = \Pi y\).

In the next proposition we show that the problem (4) has a unique solution for a class of linear smoothing operators \(B\) satisfying (5).

**Proposition 1.** Let \(A : H_1 \rightarrow H_2\) be a closed, linear operator and its domain is dense in \(H_1\). Assume further the smoothing operator \(B : H_2 \rightarrow H_2\) is closed, densely defined and satisfies

\[ \langle Ah, BAh \rangle_{H_2} \geq 0, \quad h \in H_1. \]  

(11)

Then, there exists a unique \(y(B, x) \in H_1\) which minimizes the functional

\[ J_B(y) = \|x - y\|^2_{H_1} + \langle Ay, BAy \rangle_{H_2}. \]

This minimizer is given by the formula

\[ y(B, x) = (I_{H_1} + A^*BA)^{-1}x. \]  

(12)

**Proof.** It is immediate to check that the minimizer of the functional \(\|x - y\|^2_{H_1} + \langle Ay, BAy \rangle_{H_2}\) is \((I_H + A^*BA)^{-1}x\) provided that the function \((I_H + A^*BA)^{-1}\) exists everywhere. But, since the operator \(D := \sqrt{BA}\) is closed and densely defined, thanks to a result by Neumann (see [20], Sec. 118, Chap. VII), \((I_H + A^*BA)^{-1} = (I + D^*D)^{-1}\) exists everywhere and is bounded. This finishes the proof of the proposition.

3 **HP filter associated with trace class covariance operators**

In this section we prove that the optimal smoothing operator which solves (6) has the same structure as in [6], when \(u\) and \(v\) are independent Gaussian random variables with zero mean and trace class covariance operators \(\Sigma_u\) and \(\Sigma_v\).

In view of (7) and (8), a stochastic model for \((x, y)\) being determined by models for \(y_0\) and \((u, v)\), we assume

**Assumption (2)** \(y_0\) deterministic.

**Assumption (3)** \(u\) and \(v\) are independent Gaussian random variables with zero mean and covariance operators \(\Sigma_u\) and \(\Sigma_v\) respectively.

Assumption (2) is made to ease the analysis. The independence between \(u\) and \(v\) imposed in Assumption (3) is natural because a priori there should not be any dependence between the ‘residual’ \(u\) which is due to the noisy observation \(x\) and the required degree of smoothness of the signal \(y\).

Assumption (3) implies that \(\Pi y = A^\dagger v\) and \(u\) are also independent. Thus, with regard to the following decomposition of \(x\),

\[ x = y_0 + \Pi y + \Pi u + (I_{H_1} - \Pi)u, \]

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it is natural to assume that even the orthogonal random variables $\Pi u$ and $(I_{H_1} - \Pi) u$ independent. This would mean that the input $x$ is decomposed into three independent random variables. This is actually the case for the classical HP filter. Also, as we will show below, thanks to this property the optimal smoothing operator has the form of a ‘noise to signal ratio’ in line with the classical HP filter.

**Assumption (4)** The orthogonal (in $H_1$) random variables $\Pi u$ and $(I_{H_1} - \Pi) u$ are independent:

$$\Pi \Sigma = \Sigma u \Pi.$$  

We note that (13) is equivalent to

$$\Pi \Sigma u \Pi = \Pi \Sigma u.$$  

Given Assumptions (2) and (3), by (7) and (8), it holds that $(x, y)$ is Gaussian with mean $(E[x], E[y]) = (y_0, y_0)$, and covariance operator

$$\Sigma = \begin{pmatrix} \Sigma_u + Q_v & Q_v \\ Q_v & Q_v \end{pmatrix},$$  

(15)

where,

$$Q_v := A^\dagger \Sigma v (A^\dagger)^*.$$  

(16)

**Lemma 2.** The linear operator $Q_v$ is trace class.

Moreover, the linear operator

$$T := Q_v [\Sigma_u + Q_v]^{-1/2}$$  

is Hilbert-Schmidt.

**Proof.** Since the covariance operator $\Sigma_v$ is a trace class operator, $\Sigma_v^{1/2}$ is Hilbert-Schmidt. Therefore, $A^\dagger \Sigma_v^{1/2}$ is Hilbert-Schmidt, since, by Assumption (1), $A^\dagger$ is bounded. Hence,

$$Q_v = A^\dagger \Sigma_v^{1/2} (A^\dagger)^*$$

as a product of two Hilbert-Schmidt operators, is a trace class operator. Furthermore, since $\Sigma_u + Q_v$ is injective and trace-class, the operator $[\Sigma_u + Q_v]^{-1/2}$ is Hilbert-Schmidt. Hence, $T := Q_v [\Sigma_u + Q_v]^{-1/2}$ is Hilbert-Schmidt.

We may apply Theorem 2 in [17] to obtain the conditional expectation of the signal $y$ given the functional data $x$:

$$E[y|x] = y_0 + Q_v [\Sigma_u + Q_v]^{-1} (x - y_0).$$  

(17)

The following theorem is a generalization of Theorem 4 in [6].

**Theorem 3.** Under Assumptions (1), (2) (3) and (4), the smoothing operator

$$\hat{B} := (A^\dagger)^* \Sigma_u A^* \Sigma_v^{-1}$$  

(18)

is the unique operator which satisfies

$$\hat{B} = \arg \min_B \|E[y|x] - y(B, x)\|_{H_1},$$

where the minimum is taken with respect to all linear closed and densely defined operators which satisfy the positivity condition (11).

**Proof.** The proof is similar to that of Theorem 4 in [6].
4 Examples

In this section, we apply Theorem 3 to two examples for which the operators are densely defined with closed range.

Example 1. Inspired by an example discussed in [13], we consider the operator

\[ A(x_1, x_2, x_3, \ldots, x_n, \ldots) = (0, 2x_2, 3x_3, \ldots, nx_n, \ldots), \]

with domain

\[ D(A) = \{ x := (x_1, x_2, x_3, \ldots, x_n, \ldots) \in l^2 : \sum_{j=1}^{\infty} |jx_j|^2 < \infty \}. \]

This operator is self-adjoint, unbounded, closed and densely defined with \( D(A) = l^2 \).

Now consider the Hodrick-Prescott filter associated the operator \( A \). Under the assumption that \( u \) and \( v \) are independent Gaussian random variables with zero means and covariance operators of trace class of the form

\[ \Sigma_u x = (\sigma_u^1 x_1, \sigma_u^2 x_2, \sigma_u^3 x_3, \ldots, \sigma_u^n x_n, \ldots), \]

and

\[ \Sigma_v x = (\sigma_v^1 x_1, \sigma_v^2 x_2, \sigma_v^3 x_3, \ldots, \sigma_v^n x_n, \ldots), \]

respectively.

In view of the form of the operator \( A \), an appropriate class of smoothing operator \( B \) is

\[ B(x_1, x_2, x_3, \ldots, x_n, \ldots) = (b_1 x_1, b_2 x_2, b_3 x_3, \ldots, b_n x_n, \ldots), \]

where, the coefficients \( \frac{\sigma_u^j}{\sigma_v^j}, j = 1, 2, \ldots \) are chosen so that the operator \( B \) is closed, densely defined and satisfies the positivity condition (11). In view of Theorem 3, the optimal smooth operator \( \hat{B} \) given by (18) reads

\[ \hat{B} x = A^{-1} \Sigma_u A^{-1} x = (0, \frac{\sigma_u^2}{\sigma_v^2} x_2, \frac{\sigma_u^3}{\sigma_v^3} x_3, \ldots, \frac{\sigma_u^n}{\sigma_v^n} x_n, \ldots), \quad x \in l^2. \]

Moreover, the corresponding optimal signal given by (12) is

\[ y(\hat{B}, x) = (I_{H^1} + A^* \hat{B} A)^{-1} x = (x_1, \frac{1}{4b_2 + 1} x_2, \frac{1}{9b_3 + 1} x_3, \ldots, \frac{1}{n^2 b_n + 1} x_n, \ldots), \]

where, \( \hat{b}_j := \frac{\sigma_u^j}{\sigma_v^j}, \quad j = 1, 2, \ldots \).

Example 2. Consider the Laplace operator \( A = -\frac{d^2}{dx^2} \), with Dirichlet boundary conditions, whose domain is

\[ D(A) = \{ y \in H^2([0, 1]), \quad y(0) = y(1) = 0 \}, \]

where, \( H^2([0, 1]) \) is the Sobolev space of functions whose weak derivatives of order less than or equal to two belong to \( L^2([0, 1]) \).
The signal process \( y \) corrupted by \( v \) satisfies

\[
Ay(t) = -\frac{d^2y(t)}{dt^2} = v(t), \quad y \in D(A),
\] (20)

The Laplacian \( A \) is one-to-one, non-negative, self-adjoint, closed, unbounded operator, with domain \( D(A) \) dense in \( L^2([0,1]) \) (see e.g. [3]). The eigenvalues and eigenvectors of \( A \) satisfy:

\[
\begin{cases}
\lambda_n = n^2\pi^2, & n \geq 1 \quad \text{and} \quad n \in \mathbb{N} \\
e_n(t) = \sqrt{2}\sin n\pi t.
\end{cases}
\]

The inverse of the operator \( A \) is a self-adjoint Hilbert-Schmidt operator and given by

\[
(A^{-1} x)(t) = \int_0^1 G(t,s)x(s)ds
\]

where the Green function \( G : [0,1] \times [0,1] \rightarrow [0,1] \) is given by

\[
G(t,s) := \begin{cases}
(1-t)s, & 0 \leq s \leq t, \\
t(1-s), & t \leq s \leq 1
\end{cases}
\]

Hence, the operator \( A \) can be written

\[
Ay(t) = \sum_{n=1}^{\infty} n^2\pi^2\langle y, e_n \rangle e_n(t),
\]

and the solution of equation (20) is given in terms of eigenvalues and eigenvectors by

\[
y(t) = A^{-1} v(t) = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \langle v, e_n \rangle e_n(t).
\]

Now consider the Hodrick-Prescott filter associated the operator \( A \). Under the assumption that \( u \) and \( v \) are independent Gaussian random variables with zero means and covariance operators of trace class of the form

\[
\Sigma_u h(t) = \sum_{n=1}^{\infty} \sigma_n^u(h, e_n) e_n(t),
\]

and

\[
\Sigma_v h(t) = \sum_{n=1}^{\infty} \sigma_n^v(h, e_n) e_n(t),
\]

respectively, where the sums converge in the operator norm. The smoothing operator \( B \) is defined as

\[
Bh(t) = \sum_{n=1}^{\infty} \beta_n(h, e_n) e_n(t),
\]
where, the coefficients \( \{\beta_n, n = 1, 2, \ldots\} \) are chosen so that the operator \( B \) is closed, densely defined and satisfies the positivity condition (11). By Theorem 3, the optimal smooth operator \( B \) given by (18) reads

\[
\hat{B} h(t) = A^{-1} \Sigma_u A \Sigma_v^{-1} h(t) = \sum_{n=1}^{\infty} \frac{\sigma_u^2}{\sigma_n^2} \langle h, e_n \rangle e_n(t).
\]

The corresponding optimal signal given by (12) is

\[
y(\hat{B}, x) = (I_{L^2(0,1)} + A^* \hat{B} A)^{-1} x = \sum_{n=1}^{\infty} \left(1 + n^4 \pi^4 \frac{\sigma_u^2}{\sigma_n^2}\right)^{-1} \langle x, e_n \rangle e_n(t).
\]

5 Extension to non-trace class covariance operators

In this section we show that the characterization (18) of the optimal smoothing operator is preserved even when the covariance operators of \( u \) and \( v \) are not necessarily trace class operators.

Assuming that \( u \sim N(0, \Sigma_u) \) and \( v \sim N(0, \Sigma_v) \) where \( \Sigma_u \) and \( \Sigma_v \) are self-adjoint positive-definite bounded but not trace class operators on \( H_1 \) and \( H_2 \), respectively. One important case of this extension is where \( u \) and \( v \) are white noise with covariance operators of the form \( \Sigma_u = \sigma_u^2 I_{H_1} \) and \( \Sigma_v = \sigma_v^2 I_{H_2} \), respectively, for some constants \( \sigma_u \) and \( \sigma_v \).

Following Rozanov (1968) (see also Lehtinen et al. (1989)), we consider these Gaussian variables as generalized random variables on an appropriate Hilbert scale (or nuclear countable Hilbert space), where the covariance operators can be maximally extended to self-adjoint positive-definite, bounded and trace class operators on an appropriate domain.

We first construct the Hilbert scale appropriate to our setting. This is performed using the linear operator \( A \) as follows (see Engle et al. (1996) for further details).

In view of Assumption (1), the operator \( \hat{A}^\dagger : H_2 \rightarrow H_1 \) is linear and bounded operator. Put \( H_3 := \text{Ran}(A) \), \( H_3 \) is a Hilbert space, since it is a closed subspace of Hilbert space \( H_2 \). Let \( \hat{A}^\dagger \) be the restriction of \( \hat{A}^\dagger \) on \( H_3 \) i.e. \( \hat{A}^\dagger : H_3 \rightarrow H_1 \). Hence \( \hat{A}^\dagger \) is injective bounded linear operator.

**Remark 4.** In view of Hodrick-Prescott Filter (3), \( v \in \text{Ran}(A) = H_3 \) i.e. it can be seen as \( H_3 \)-random variable with covariance operator \( \Sigma_v : H_3 \rightarrow H_3 \).

Set

\[
K_1 := (\hat{A}^\dagger (\hat{A}^\dagger)^*)^{-1} : H_1 \rightarrow H_1.
\]

We can define the fractional power of the operator \( K_1 \) by

\[
K_1^s h = (\hat{A}^\dagger (\hat{A}^\dagger)^*)^{-s} h, \quad h \in H_1, \quad s \geq 0,
\]

and define its domain by

\[
\mathcal{D}(K_1^s) := \{ h \in H_1; \quad (\hat{A}^\dagger (\hat{A}^\dagger)^*)^{-s} h \in H_1 \}.
\]

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Let \( \mathcal{M} \) be the set of all elements \( x \) for which all the powers of \( K_1 \) are defined i.e.

\[
\mathcal{M} := \bigcap_{n=0}^{\infty} \mathcal{D}(K_1^n).
\]

For \( s \geq 0 \), let \( H_1^s \) be the completion of \( \mathcal{M} \) with respect to the Hilbert space norm induced by the inner product

\[
(x,y)_{H_1} := (K_1^nx, K_1^ny)_{H_1}, \quad x, y \in \mathcal{M},
\]

and let \( H_1^{-s} := (H_1^s)^\ast \) denote the dual of \( H_1^s \) equipped with the following inner product:

\[
(x,y)_{H_1^{-s}} := (K_1^{-s}x, K_1^{-s}y)_{H_1}, \quad x, y \in \mathcal{M}.
\]

Then, \( (H_1^s)_{s \in \mathbb{R}} \) is the Hilbert scale induced by the operator \( K_1 \).

The operator \( K_2 := ((\bar{A}^\dagger)^\ast \bar{A}^\dagger)^{-1} : H_3 \rightarrow H_3 \) has the same properties as \( K_1 \). Repeating the same procedure as before we get that \( (H_3^s)_{s \in \mathbb{R}} \) is the Hilbert scale induced by the operator \( K_2 \), where the norm in \( H_3^n \) is given by \( \|h\|_{H_3} = \|K_2^nh\|_{H_3}, \quad h \in H_3^n \).

Noting that

\[
H_1^{-n} = \text{Im} \left( (\bar{A}^\dagger(\bar{A}^\dagger)^\ast)^n \right) = (\bar{A}^\dagger(\bar{A}^\dagger)^\ast)^n(H_1),
\]

\[
H_3^{-n} = \text{Im} \left( ((\bar{A}^\dagger)^\ast \bar{A}^\dagger)^n \right) = ((\bar{A}^\dagger)^\ast \bar{A}^\dagger)^n(H_3)
\]

with \( \ker((\bar{A}^\dagger(\bar{A}^\dagger)^\ast)^n) = \ker(\bar{A}^\dagger) = \{0\} \) and \( \ker(((\bar{A}^\dagger)^\ast \bar{A}^\dagger)^n) = \ker((\bar{A}^\dagger)^\ast) = \ker(\bar{A}) = \ker(A) \), it follows that the operator \( \bar{A}^\dagger \) extends to a continuous operator from \( H_3^{-n} \) into \( H_1^{-n} \), and the operator \( \bar{A} = A \) extends to a continuous operator from \( H_1^{-n} \) into \( H_3^{-n} \), and the operators \( \bar{A}^\dagger(\bar{A}^\dagger)^\ast \) and \( (\bar{A}^\dagger)^\ast \bar{A}^\dagger \) extend as well to a continuous operator onto \( H_1^{-n} \) and \( H_3^{-n} \) respectively.

Extending the HP filter to the larger Hilbert spaces \( H_1^{-n} \) and \( H_3^{-n} \) due to the flexibility offered by the Hilbert scale, where \( n \) is chosen so that the second moments \( E[\|x\|^2_{H_1^{-n}}], E[\|y\|^2_{H_1^{-n}}], E[\|u\|^2_{H_1^{-n}}] \) and \( E[\|v\|^2_{H_1^{-n}}] \) of the Gaussian random variables \( x, y, u \) and \( v \) in \( H_1^{-n} \) and \( H_3^{-n} \) respectively, are finite. This amounts to make their respective covariance operators

\[
\bar{\Sigma}_u = (\bar{A}^\dagger(\bar{A}^\dagger)^\ast)^n\Sigma_u(\bar{A}^\dagger(\bar{A}^\dagger)^\ast)^n, \quad \bar{\Sigma}_v = ((\bar{A}^\dagger)^\ast \bar{A}^\dagger)^n\Sigma_v((\bar{A}^\dagger)^\ast \bar{A}^\dagger)^n
\]

and

\[
\bar{\Sigma} = \begin{pmatrix}
\bar{\Sigma}_u + \bar{Q}_v & \bar{Q}_v \\
\bar{Q}_v & \bar{Q}_v
\end{pmatrix},
\]

where

\[
\bar{Q}_v := \bar{A}^\dagger\bar{\Sigma}_v(\bar{A}^\dagger)^\ast.
\]

trace class. We make the following assumption:

**Assumption (5).** There is \( n_0 > 0 \) such that the covariance operators \( \bar{\Sigma}_u, \bar{\Sigma} \) and \( \bar{\Sigma}_v \) are trace class on the Hilbert spaces \( H_1^{-n} \) and \( H_3^{-n} \), respectively.
It is worth noting that since \( y_0 \in \ker(A) = \ker((\bar{A}^\dagger)^n) \) then \( \|y_0\|_{\mathcal{H}^{-n}} = \|(\bar{A}^\dagger)^n y_0\|_{\mathcal{H}_1} = 0 \). Hence, the \( \mathcal{H}_1^{-n} \times \mathcal{H}_1^{-n} \)-valued random vector \((x, y)\) has mean \((E[x], E[y]) = (0, 0)\).

Summing up, by Assumption 5, for \( n \geq n_0 \), the vector \((x, y)\) is an \( \mathcal{H}_1^{-n} \times \mathcal{H}_1^{-n} \)-valued Gaussian vector with mean \((0, 0)\) and covariance operator \( \hat{\Sigma} \). Thus, by Theorem 2 in [17], we have
\[
E[|y|/x] = \tilde{Q}_v \left[ \tilde{\Sigma}_u + \tilde{Q}_v \right]^{-1} x, \quad \text{a.s. in } \mathcal{H}_1^{-n}.
\]
provided that the operator
\[
\hat{T} := \tilde{\Sigma}_{XY} \tilde{\Sigma}^{-\frac{1}{2}}_X
\]
is Hilbert-Schmidt. But, in view of Assumption (5) and Lemma [2], the operator \( T \) is Hilbert-Schmidt.

The deterministic optimal signal associated with \( x \) in \( \mathcal{H}_1^{-n} \), \( n \geq n_0 \), is given by the formula (cf. Proposition [1])
\[
y(B, x) = (I_{\mathcal{H}_1^{-n}} + A^* BA)^{-1} x,
\]
which is the unique minimizer of the functional
\[
J_B(y) = \|x - y\|^2_{\mathcal{H}_1^{-n}} + \langle Ay, BAy \rangle_{\mathcal{H}_1^{-n}},
\]
with a linear operator \( B : \mathcal{H}_3^{-n} \rightarrow \mathcal{H}_3^{-n} \) such that \( \langle Ah, BAh \rangle_{\mathcal{H}_3^{-n}} \geq 0 \) for all \( h \in \mathcal{H}_1^{-n} \).

The following theorem gives an explicit expression of the optimal smoothing operator \( \hat{B} \).

**Theorem 5.** Let assumption 5 hold. Then, the unique optimal smoothing operator associated with the HP filter associated with \( \mathcal{H}_1^{-n} \)-valued data \( x \) is given by:
\[
\hat{B}h := (\bar{A}^\dagger)^* \hat{\Sigma}_u A^* \hat{\Sigma}_v^{-1} h, \quad h \in \mathcal{H}_3^{-n}.
\]

The proof is similar to that of Theorem 6 in [4].

**5.1 The white noise case- Optimality of the noise-to-signal ratio**

In this section we show that the optimal smoothing operator \( \hat{B} \) given by [32] reduces to the noise-to-signal ratio where \( u \) and \( v \) are white noises. Assuming \( u \) and \( v \) independent and Gaussian random variables with zero means and covariance operators \( \Sigma_u = \sigma_u I_{\mathcal{H}_1} \) and \( \Sigma_v = \sigma_v I_{\mathcal{H}_3} \), where \( I_{\mathcal{H}_1} \) and \( I_{\mathcal{H}_3} \) denote the \( \mathcal{H}_1 \) and \( \mathcal{H}_3 \) identity operators, respectively and \( \sigma_u \) and \( \sigma_v \) are constant scalars. Assumption 5 reduces to

**Assumption 6.** There is an \( n_0 > 0 \) such that \((\bar{A}^\dagger(\bar{A}^\dagger)^*)^{2n} \) and \((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n} \) are trace class for all \( n \geq n_0 \).
Under this assumption, the associated covariance operators

\[ \tilde{\Sigma}_u = (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n \Sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^n = \sigma_u (\bar{A}^\dagger (\bar{A}^\dagger)^*)^{2n}, \]

\[ \tilde{\Sigma}_v = ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n \Sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^n = \sigma_v ((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n}, \]

and

\[ \tilde{Q}_v = \sigma_v A^\dagger ((\bar{A}^\dagger)^* \bar{A}^\dagger)^{2n} (\bar{A}^\dagger)^* = \sigma_v (A^\dagger (\bar{A}^\dagger)^*)^{2n+1} \]

are trace class, the expression (32) giving the optimal smoothing operator \( \hat{B} \) reduces to

\[ \hat{B} = (\bar{A}^\dagger)^* \tilde{\Sigma}_u A^* \tilde{\Sigma}_v^{-1} h = \frac{\sigma_u}{\sigma_v} I_{\mathbb{H}^{-n}}, \quad (33) \]

i.e. \( \hat{B} \) is the noise-to-signal ratio which is in the same pattern as in the classical HP filter.

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