A HIGHER ORDER SYSTEM OF SOME COUPLED NONLINEAR
SCHRÖDINGER AND KORTEWEG-DE VRIES EQUATIONS

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Abstract. We prove existence and multiplicity of bound and ground state solutions, under appropriate conditions on the parameters, for a bi-harmonic stationary system of coupled nonlinear Schrödinger–Korteweg-de Vries equations.

1. Introduction

Recently in [8, 9] has been analyzed a system of coupled nonlinear Schrödinger–Korteweg-de Vries equations
\begin{equation}
\begin{align*}
if_x + f_{xx} + |f|^2 f + \beta fg &= 0, \\
g_t + g_{xxx} + gg_x + \frac{1}{2}\beta(|f|^2)_x &= 0,
\end{align*}
\end{equation}
with \(f = f(x,t) \in \mathbb{C}, g = g(x,t) \in \mathbb{R},\) and \(\beta \in \mathbb{R}\) a coupling parameter. This system appears in phenomena of interactions between short and long dispersive waves, arising in fluid mechanics, such as the interactions of capillary - gravity water waves [16]. Indeed, \(f\) represents the short-wave, while \(g\) stands for the long-wave; see references [2, 8, 9, 10, 14] for further details on similar system. Moreover, the interaction between long and short waves appears in magnetized plasma [15], [19] and in many physical phenomena as well, such that Bose-Einstein condensates [6].

The solutions studied in papers [8, 9] (see also [11, 12]) are taken as solitary traveling waves, i.e.,
\begin{equation}
(f(x,t), g(x,t)) = (e^{i\omega t} e^{i\frac{\lambda_1}{4} x} u(x - ct), v(x - ct)), \quad \text{where } u, v \text{ are real functions.}
\end{equation}
Choosing \(\lambda_1 = \omega + \frac{\lambda_2}{4}\) and \(\lambda_2 = c,\) then \(u, v\) are solutions of the following stationary system
\begin{equation}
\begin{align*}
-u'' + \lambda_1 u &= u^3 + \beta uv, \\
v'' + \lambda_2 v &= \frac{1}{2}v^2 + \frac{1}{2}\beta u^2.
\end{align*}
\end{equation}

In the present work we analyze the existence of solutions of a higher order system coming from [11]. More precisely, we consider the following system
\begin{equation}
\begin{align*}
i f_t - f_{xxxx} + |f|^2 f + \beta fg &= 0, \\
g_t - g_{xxxx} + \frac{1}{2}(|g|g)_x + \frac{1}{2}\beta(|f|^2)_x &= 0.
\end{align*}
\end{equation}
Looking for "standing-traveling" wave solutions of the form
\begin{equation}
(f(x,t), g(x,t)) = (e^{i\lambda_1 t} u(x), v(x - \lambda_2 t)), \quad \text{where } u, v \text{ are real functions,}
\end{equation}
then we arrive at the fourth-order stationary system
\begin{equation}
\begin{align*}
u^{(iv)} + \lambda_2 v &= \frac{1}{2}|v|^2 + \frac{1}{2}\beta u^2,
\end{align*}
\end{equation}

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1This is the first time, up to our knowledge, that the interaction of standing waves and traveling waves is analyzed in the mathematical literature.
Let us define the product Sobolev space $E$. We will take the inner product in $E$ case. Products:

which induces the following norm

$$\|u\|_j := \left\langle u, u_j \right\rangle := \int_{\mathbb{R}^N} \Delta u \cdot \Delta v \, dx + \lambda_j \int_{\mathbb{R}^N} uv \, dx,$$

where $u, v \in W^{2,2}(\mathbb{R}^N)$, $1 \leq N \leq 7$, $\lambda_j > 0$ with $j = 1, 2$ and $\beta > 0$ is the coupling parameter.

Recently, other similar fourth-order systems studying the interaction of coupled nonlinear Schrödinger equations have appeared; see [3], where the coupling terms have the same homogeneity as the nonlinear terms. Note that, as far as we know there is not any previous mathematical work analyzing a higher order system with the nonlinear and coupling terms considered here in the system (6).

Here we first analyze the dimensional case $2 \leq N \leq 7$ in the radial framework (see subsection [3.1]) by using the compactness described in Remark [3]. The one dimensional case is studied in subsection [3.2] where we use a measure Lemma due to P. L. Lions [18] to circumvent the lack of compactness.

System (6) has a non-negative semi-trivial solution, $v_2 = (0, V_2)$ defined in Remark [4]. Then in order to find non-negative bound or ground state solutions we need to check that they are different from $v_2$. To be more precise, we prove that there exists a positive critical value of the coupling parameter $\beta$, denoted by $\Lambda$ defined by (21), such that the associated functional constrained to the corresponding Nehari manifold possesses a positive global minimum, which is a critical point with energy below the energy of the semi-trivial solution under the following hypotheses: either $\beta > \Lambda$ or $\beta > 0$ and $\lambda_2 \gg 1$. Furthermore, we find a mountain pass critical point if $\beta < \Lambda$ and $\lambda_2 > 1$.

The paper is organized as follows. In Section [2] we introduce the notation, establish the functional framework, define the Nehari manifold and study its properties. Section [3] is devoted to prove the main results of the paper. It is divided into two subsections, in the first one (Subsection [3.1]) we study the high-dimensional case ($2 \leq N \leq 7$), while the second one (Subsection [3.2]) deals with the one-dimensional case.

2. Functional setting, notation and Nehari manifold

Let $E$ be the Sobolev space $W^{2,2}(\mathbb{R}^N)$ then, we define the following equivalent norms and scalar products:

$$\left\langle u, v \right\rangle_j := \int_{\mathbb{R}^N} \Delta u \cdot \Delta v \, dx + \lambda_j \int_{\mathbb{R}^N} uv \, dx,$$

Let us define the product Sobolev space $\mathbb{E} := E \times E$ and denote its elements by $u = (u, v)$ with $0 = (0, 0)$. We will take the inner product in $\mathbb{E}$ as follow,

$$\langle u_1, u_2 \rangle := \langle u_1, u_2 \rangle_1 + \langle v_1, v_2 \rangle_2,$$

which induces the following norm

$$\|u\| := \sqrt{\|u\|_1^2 + \|v\|_2^2}.$$n

Moreover, for $u = (u, v) \in \mathbb{E}$, the notation $u \geq 0$, resp. $u > 0$, means that $u, v \geq 0$, resp. $u, v > 0$. We denote by $H$ the space of radially symmetric functions in $E$, and $\mathbb{H} := H \times H$. In addition, we define energy functionals associated to system (6) by

$$\Phi(u) = I_1(u) + I_2(v) - \frac{1}{2} \beta \int_{\mathbb{R}^N} u^2 v \, dx, \quad u \in \mathbb{E},$$

where

$$I_1(u) = \frac{1}{8} \|u\|_1^2 - \frac{1}{4} \int_{\mathbb{R}^N} u^4 \, dx, \quad I_2(v) = \frac{1}{8} \|v\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^N} |v|^4 \, dx, \quad u, v \in E,$$

are the energy functional associated to the uncoupled equations in (6).
Remark 1. We can easily see that the functional $\Phi$ is not bounded below on $\mathbb{E}$. Thus, we are going to work on the so called Nehari manifold which is a natural constraint for the functional $\Phi$, and even more the functional constrained to the Nehari manifold is bounded below.

We define

$$
\Psi(u) = \Phi'(u) | u| = \|u\|^2 - \int_{\mathbb{R}^N} u^4 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |v|^3 \, dx - \frac{3}{2} \beta \int_{\mathbb{R}^N} u^2 v \, dx. \quad (9)
$$

Using the previous definition, the Nehari manifold is given by

$$
\mathcal{N} = \{ u \in \mathbb{H} \setminus \{0\} : \Psi(u) = 0 \}. \quad (10)
$$

This manifold will be used in order to deal with the one dimensional case in subsection 3.2, in which there is no compactness, see Remark 3(ii).

In the dimensional case $2 \leq N \leq 7$, we restrict the Nehari Manifold to the radial setting, denoting it as

$$
\mathcal{N} = \{ u \in \mathbb{H} \setminus \{0\} : \Psi(u) = 0 \}. \quad (11)
$$

Furthermore, differentiating expression (9) yields

$$
\Psi'(u) | u| = 2\|u\|^2 - 4 \int_{\mathbb{R}^N} u^4 \, dx - \frac{3}{2} \int_{\mathbb{R}^N} |v|^3 \, dx - \frac{9}{2} \beta \int_{\mathbb{R}^N} u^2 v \, dx. \quad (12)
$$

Remark 2. All the properties we are going to prove in this section are satisfied for both $\mathcal{M}$ and $\mathcal{N}$, but the Palais-Smale condition, in Lemma 6, is only satisfied for $\mathcal{N}$ properties for the Nehari manifold

Using the fact that $\Phi$ is a smooth complete manifold, and there exists a constant $\rho > 0$ such that

$$
\Phi(u) = \frac{1}{\rho} \|u\|^2 - \frac{1}{4} \|u\|^4 \quad (13)
$$

This manifold will be used in order to deal with the one dimensional case in subsection 3.2, in which there is no compactness, see Remark 3(ii). To be short, we are going to demonstrate the following properties for the Nehari manifold $\mathcal{N}$.

Using the fact that $\Psi(u) = 0$ for any $u \in \mathcal{N}$, we have

$$
\Psi'(u) | u| = \Psi'(u) | u| - 3 \Psi(u) = -\|u\|^2 - \int_{\mathbb{R}^N} u^4 \, dx < 0, \quad \forall u \in \mathcal{N}. \quad (13)
$$

Then, $\mathcal{N}$ is a locally smooth manifold near any point $u \neq 0$ with $\Psi(u) = 0$. Taking the derivative of the functional $\Phi$, we find

$$
\Phi'(u) | h| = I'_1(u) | h_1| + I'_2(v) | h_2| - \beta \int_{\mathbb{R}^N} uwh_1 \, dx - \frac{1}{2} \beta \int_{\mathbb{R}^N} u^2 h_2 \, dx,
$$

The second derivative of $\Phi$ is given by

$$
\Phi''(u) | h|^2 = ||h||^2 - 3 \int_{\mathbb{R}^N} u^2 h_1^2 \, dx - \int_{\mathbb{R}^N} |v|h_2^2 \, dx - \beta \int_{\mathbb{R}^N} v h_2^2 \, dx - 2 \beta \int_{\mathbb{R}^N} uh_1 h_2 \, dx.
$$

It satisfies

$$
\Phi''(0) | h|^2 = ||h||^2,
$$

which is positive definite, so that 0 is a strict minimum critical point for $\Phi$. As a consequence, we have that $\mathcal{N}$ is a smooth complete manifold, and there exists a constant $\rho > 0$ such that

$$
\|u\|^2 > \rho, \quad \forall u \in \mathcal{N}. \quad (14)
$$

Notice that by (13) and (14), [4, Proposition 6.7] proves that $\mathcal{N}$ is a Natural constraint of $\Phi$, i.e., $u \in \mathbb{H} \setminus \{0\}$ is a critical point of $\Phi$ if and only if $u$ is a critical point of $\Phi$ constrained on $\mathcal{N}$.

Remarks 3.

(i) The functional constrained on $\mathcal{N}$ takes the form

$$
\Phi_{|\mathcal{N}}(u) = \frac{1}{\rho} \|u\|^2 + \frac{1}{17} \int_{\mathbb{R}^N} u^4 \, dx. \quad (15)
$$

Even more, using (13) and (15),

$$
\Phi(u) > \frac{1}{\rho} \rho, \quad \forall u \in \mathcal{N}. \quad (16)
$$
Therefore, $\Phi$ is bounded from below on $\mathcal{N}$, so we can try to minimize it on the Nehari manifold. 

(ii) Let us define

$$2^* = \begin{cases} \frac{2N}{N-1} & \text{if } N > 4, \\ \infty & \text{if } 1 \leq N \leq 4. \end{cases}$$

One has the following Sobolev embedding

$$E \hookrightarrow L^p(\mathbb{R}^N), \quad \text{for } \begin{cases} 2 \leq p \leq 2^*, & \text{if } N \neq 4, \\ 2 \leq p < 2^*, & \text{if } N = 4, \end{cases}$$

see for instance, [17, 1].

In particular, this embeddings show that the functional $\Phi$ is well defined for every $1 \leq N \leq 7$.

Concerning the Palais-Smale condition for $2 \leq N \leq 7$, (see Lemma 7) we will use that if $N \geq 2$, replacing $E$ by the radial subspace $H$, we have the following compact embedding

$$H \hookrightarrow L^p(\mathbb{R}^N), \quad \text{for } 2 < p < 2^*.$$ 

The one dimensional case ($N = 1$) is analyzed in a different manner in Subsection 3.2 because of the lack of compactness.

**Remark 4.** System (8) only admits one kind of semi-trivial solutions of the form $(0,v)$. Indeed, if we suppose $v = 0$, the second equation in (8) gives us that $u = 0$ as well. Thus, let us take $v_2 = (0, V_2)$, where $V_2$ can be taken as a positive radially symmetric ground state solution of the equation $\Delta^2 v + \lambda_2 v = \frac{1}{2}|v|^p$. In particular, we can assume that $V_2$ is positive because in other case, taking $|V_2|$, it has the same energy. Moreover, if we denote by $V$ a positive radially symmetric ground state solution of the equation $\Delta^2 v + v = \frac{1}{2}|v|^p$, then, after some rescaling $V_2$ can be defined by

$$V_2(x) = \lambda_2 V(\sqrt{\lambda_2} x).$$ (17)

As a consequence, $v_2 = (0, V_2)$ is a non-negative semi-trivial solution of (8), independently of the value of $\beta$.

We define the Nehari manifold corresponding to the single second equation of (8) by

$$\mathcal{N}_2 = \{v \in H \setminus \{0\} : J_2(v) = 0\}$$

where

$$J_2(u) := I_2'(u)[u].$$

Let us define the tangent space to $\mathcal{N}$ on $v_2$ by

$$T_{v_2} \mathcal{N} := \{h \in E : \Psi'(v_2)[h] = 0\},$$

equivalently we define the tangent space to $\mathcal{N}_2$ on $V_2$ by

$$T_{V_2} \mathcal{N}_2 := \{h \in E : J_2'(V_2)[h] = 0\}.$$ 

We can see that the following equivalence holds:

$$h = (h_1, h_2) \in T_{v_2} \mathcal{N} \iff h_2 \in T_{V_2} \mathcal{N}_2, \quad (18)$$

in fact,

$$h \in T_{v_2} \mathcal{N} \iff \Psi'(v_2)[h] = 0 \iff 2\langle V_2, h_2 \rangle_2 - \frac{1}{2} \int_{\mathbb{R}^N} V_2^2 h_2 = 0 \iff J_2'(V_2)[h_2] = 0 \iff h_2 \in T_{V_2} \mathcal{N}_2.$$ 

If we denote by $D^2 \Phi_N$ the second derivative of $\Phi$ constrained on $\mathcal{N}$, using that $v_2$ is a critical point of $\Phi$, plainly we obtain that

$$D^2 \Phi_N(v_2)[h]^2 = \Phi''(v_2)[h]^2 \quad \forall \ h \in T_{v_2} \mathcal{N}. \quad (19)$$

In the following result we establish the character of $v_2$ in terms of the size of the coupling parameter.
Lemma 6. Let \( 2 < p < \infty \) from (15) Proposition 5. There exists \( u \) subsequence (denoted equals for short) \( u \parallel u \). To conclude this section we also prove that the functional \( \Phi \) satisfies the PS condition constrained to \( \mathcal{N} \). Therefore, \( \Phi \) is a saddle point of \( \Phi \) on \( \mathcal{N} \).

Proof. We define \( \Lambda := \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_1^2}{\int_{\mathbb{R}^N} V_2 \varphi^2 \, dx} \). For \( h \in T_{v_2} \mathcal{N} \) one has that
\[
D^2 \Phi_N(v_2)[h]^2 = \Phi''(v_2)[h]^2 = \|h_1\|^2_1 + I''_2(V_2)[h_2]^2 - \beta \int_{\mathbb{R}^N} V_2 h_2^2 \, dx.
\]
By (18) \( h = (h_1, h_2) \in T_{v_2} \mathcal{N} \Leftrightarrow h_2 \in T_{v_2} \mathcal{N}_2 \). Then, using that \( V_2 \) is a minimum of \( J_2 \) on \( \mathcal{N}_2 \), there exists a constant \( c_2 > 0 \) such that
\[
I''_2(V_2)[h_2]^2 \geq c_2 \|h_2\|^2_2.
\]
Since \( \beta < \Lambda \), (21) and (22) there exists \( c_1 > 0 \) such that
\[
D^2 \Phi_N(v_2)[h]^2 \geq c_1 \|h_1\|^2_1 + c_2 \|h_2\|^2_2,
\]
proving that \( v_2 \) is a strict local minimum of \( \Phi \) on \( \mathcal{N} \).

(ii) Since \( \beta > \Lambda \), there exists \( \tilde{h} \in H \) such that
\[
\Lambda < \frac{\|\tilde{h}\|^2_1}{\int_{\mathbb{R}^N} V_2 \tilde{h}^2 \, dx} < \beta.
\]
Then, taking \( h_1 = (\tilde{h}, 0) \in T_{v_2} \mathcal{N} \) it yields
\[
D^2 \Phi_N(v_2)[h_1]^2 = \|\tilde{h}\|^2_1 - \beta \int_{\mathbb{R}^N} V_2 \tilde{h}^2 \, dx < 0,
\]
and taking \( h_2 \in T_{v_2} \mathcal{N}_2 \) not equal to zero, then \( h_2 = (0, h_2) \in T_{v_2} \mathcal{N}_2 \) and
\[
D^2 \Phi_N(v_2)[h_2]^2 = I''_2(V_2)[h_2]^2 \geq c_2 \|h_2\|^2_2 > 0.
\]
Therefore, \( v_2 \) is a saddle point of \( \Phi \) on \( \mathcal{N} \) and obviously inequality (20) holds.

To conclude this section we also prove that the functional \( \Phi \) satisfies the PS condition constrained to \( \mathcal{N} \) on the high-dimensional case.

Lemma 6. Assume that \( 2 \leq N \leq 7 \), then \( \Phi \) satisfies the PS condition constrained on \( \mathcal{N} \).

Proof. Let \( u_n = (u_n, v_n) \in \mathcal{N} \) be a PS sequence, i. e.,
\[
\Phi(u_n) \to c \quad \text{and} \quad \nabla_N \Phi(u_n) \to 0, \quad \text{as} \quad n \to \infty.
\]
From (16) and the first convergence in (24) it follows that \( u_n \) is bounded, then we have a weakly convergent subsequence (denoted equals for short) \( u_n \to u_0 \in \mathbb{H} \). Since \( H \) is compactly embedding into \( L^p(\mathbb{R}^N) \) for \( 2 < p < 4 + \frac{4}{N} \) and \( 2 \leq N \leq 7 \) (see Remark 3-(ii)), we infer that
\[
\int_{\mathbb{R}^N} u_n^4 \, dx \to \int_{\mathbb{R}^N} u_0^4 \, dx, \quad \int_{\mathbb{R}^N} |v_n|^3 \, dx \to \int_{\mathbb{R}^N} |v_0|^3 \, dx, \quad \int_{\mathbb{R}^N} u_n^2 v_n \, dx \to \int_{\mathbb{R}^N} u_0^2 v_0 \, dx.
\]
Moreover, using the fact that \( u_n \in \mathcal{N} \) and (14), we have
\[
\|u_n\|^2 = \int_{\mathbb{R}^N} u_n^4 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |v_n|^3 \, dx + \frac{1}{2} \beta \int_{\mathbb{R}^N} u_n^2 v_n \, dx \to \int_{\mathbb{R}^N} u_0^4 dx + \frac{1}{2} \int_{\mathbb{R}^N} |v_0|^3 dx + \frac{1}{2} \beta \int_{\mathbb{R}^N} u_0^2 v_0 \, dx \geq \rho.
\]
which implies that \( u_0 \neq 0 \). The constrained gradient satisfies
\[
\nabla \Phi(u_n) = \Phi'(u_n) - \lambda_n \nabla \Phi(u_n) \to 0,
\]
(26)
then, taking into account (13), (14), the fact that \( \Phi'(u_n)|_{u_n} = \Psi(u_n) = 0 \), and evaluating the identity of expression (26) at \( u_n \) we deduce that \( \lambda_n \to 0 \) as \( n \to \infty \). We also have that \( \|\Phi'(u_n)\| \) is bounded. Hence, from (13), jointly with the fact \( \lambda_n \to 0 \), we obtain
\[
\|\Phi'(u_n)\| \leq \|\nabla \Phi(u_n)\| + |\lambda_n|\|\Phi'(u_n)\| \to 0 \quad \text{as} \quad n \to \infty.
\]
To finish the proof, since \( \Phi'(u_n)|_{u_n} \to 0 \) as \( n \to \infty \), it follows that \( u_n \to u_0 \) strongly. \( \blacksquare \)

3. Existence results

This section is divided into two subsections depending on the dimension of problem (10).

3.1. High-dimensional case, \( 2 \leq N \leq 7 \).

In this subsection we will see that the infimum of \( \Phi \) constrained on the radial Nehari manifold, \( \mathcal{N} \), is attained under appropriate parameter conditions. We also prove the existence of a mountain pass critical point.

**Theorem 7.** Suppose \( \beta > \Lambda \) and \( 2 \leq N \leq 7 \). The infimum of \( \Phi \) on \( \mathcal{N} \) is attained at some point \( \tilde{u} \geq 0 \) with \( \Phi(\tilde{u}) < \Phi(v_2) \) and both components \( \tilde{u}, \tilde{v} \neq 0 \).

**Proof.** By the Ekeland’s Variational Principle (see [13] for further details) there exists a minimizing PS sequence \( u_n \in \mathcal{N} \), i.e.,
\[
\Phi(u_n) \to c := \inf_{\mathcal{N}} \Phi \quad \text{and} \quad \nabla \Phi(u_n) \to 0.
\]
Due to the Lemma [5] there exists \( \bar{u} \in \mathcal{N} \) such that
\[
u_n \to \bar{u} \quad \text{strongly as} \quad n \to \infty,
\]
hence \( \bar{u} \) is a minimum point of \( \Phi \) on \( \mathcal{N} \). Moreover, taking into account Proposition [5](ii), we have:
\[
\Phi(\bar{u}) = c < \Phi(v_2).
\]
Note that the second component \( \tilde{v} \) can not be zero, because if that occur then \( \bar{u} \equiv 0 \) due to the form of the second equation of (6), and zero is not in \( \mathcal{N} \). On the other hand, if we suppose that the first component \( \bar{u} \equiv 0 \), then
\[
I_2(\bar{v}) = \Phi(\bar{u}) < \Phi(v_2) = I_2(V_2),
\]
and this is a contradiction with the fact that \( V_2 \) is a ground state of the equation \( \Delta^2 v + \lambda_1 v = \frac{1}{4} |v|^4 v \).

In general we can not ensure that both components of \( \bar{u} \) are non-negative, thus, in order to obtain this fact we take \( t|\bar{u}| \in \mathcal{N} \), and we will show that
\[
\Phi(t|\bar{u}|) \leq \Phi(\bar{u}).
\]
Note that by (16) we have that
\[
\Phi(t|\bar{u}|) = \frac{1}{6} t^2 \|\bar{u}\|^2 + \frac{1}{12} t^4 \int_{\mathbb{R}^N} |\bar{u}|^4 \, dx, \quad \Phi(\bar{u}) = \frac{1}{6} \|\bar{u}\|^2 + \frac{1}{12} \int_{\mathbb{R}^N} |\bar{u}|^4 \, dx.
\]
Hence, to prove \( \Phi(t|\bar{u}|) \leq \Phi(\bar{u}) \) is equivalent to show that \( t \leq 1 \). Taking into account that \( \Psi(t|\bar{u}|) = 0 \), we find:
\[
0 = \Psi(t|\bar{u}|) = t^2 \|\bar{u}\|^2 - t^4 \int_{\mathbb{R}^N} |\bar{u}|^4 \, dx - \frac{1}{2} t^3 \int_{\mathbb{R}^N} |\bar{v}|^3 \, dx - 4 t^2 \beta \int_{\mathbb{R}^N} \bar{u}^2 |\bar{v}| \, dx,
\]
which is equivalent to
\[
0 = \|\bar{u}\|^2 - t^2 \int_{\mathbb{R}^N} |\bar{u}|^4 \, dx - \frac{1}{2} t^3 \int_{\mathbb{R}^N} |\bar{v}|^3 \, dx - 4 t^2 \beta \int_{\mathbb{R}^N} \bar{u}^2 |\bar{v}| \, dx.
\]
(28)
Furthermore, since \( \bar{u} \in \mathcal{N} \) we also have,

\[
0 = \Psi(\bar{u}) = ||\bar{u}||^2 - \int_{\mathbb{R}^N} \bar{u}^4 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |\bar{u}|^3 \, dx - \frac{3}{2} \beta \int_{\mathbb{R}^N} \bar{u}^2 \bar{v} \, dx.
\]  

(29)

Now, if we suppose that \( t > 1 \) it follows that

\[
t^2 \int_{\mathbb{R}^N} \bar{u}^4 \, dx + \frac{1}{2} t \int_{\mathbb{R}^N} |\bar{u}|^3 \, dx + \frac{3}{2} t^2 \beta \int_{\mathbb{R}^N} \bar{u}^2 \bar{v} \, dx > \int_{\mathbb{R}^N} \bar{u}^4 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\bar{u}|^3 \, dx + \frac{3}{2} \beta \int_{\mathbb{R}^N} \bar{u}^2 \bar{v} \, dx.
\]

Then, thanks to (28) we obtain

\[
0 < ||\bar{u}||^2 - \frac{1}{2} \int_{\mathbb{R}^N} |\bar{u}|^3 \, dx - \frac{3}{2} \beta \int_{\mathbb{R}^N} \bar{u}^2 \bar{v} \, dx.
\]  

(30)

Combining (29) with (30) we arrive at

\[
0 < \frac{3}{2} \beta \int_{\mathbb{R}^N} \bar{u}^2 (\bar{v} - |\bar{v}|) \, dx,
\]

which is a contradiction. Consequently, \( t \leq 1 \) and therefore \( \Phi(|\bar{u}|) \leq \Phi(\bar{u}) \). On the other hand, we know that \( \Phi \) attains its infimum at \( \bar{u} \) on \( \mathcal{N} \) and, therefore, the last inequality can not be strict. Moreover, due to (27) it can not happen that \( t < 1 \) and, hence, \( t = 1 \) and

\[
\Phi(|\bar{u}|) = \Phi(\bar{u}).
\]

Redefining \( \bar{u} \) as \( |\bar{u}| \) we finally have that the minimum on the Nehari manifold is attained at \( \bar{u} \geq 0 \) with non-trivial components.

**Theorem 8.** Assume \( 2 \leq N \leq 7, \beta > 0 \). There exists a positive constant \( \Lambda_2 \) such that, if \( \lambda_2 > \Lambda_2 \), the functional \( \Phi \) attains its infimum on \( \mathcal{N} \) at some \( \bar{u} \geq 0 \) with \( \Phi(\bar{u}) < \Phi(v_2) \) and both \( \bar{u}, \bar{v} \neq 0 \).

**Proof.** Using the same argument as above in the Theorem 7 we prove that the infimum is attained at some point \( \bar{u} \in \mathcal{N} \), but to show that \( \bar{u}, \bar{v} \neq 0 \) we need to ensure that \( \Phi(\bar{u}) < \Phi(v_2) \). In Theorem 7 this fact was proved for the case \( \beta > \Lambda \) and here we need to prove it for \( 0 < \beta \leq \Lambda \). In this case the point \( v_2 \) is a strict local minima and this does not guarantee that \( \bar{u} \neq v_2 \).

Then, to see \( \Phi(\bar{u}) < \Phi(v_2) \) we will use a similar procedure to the one applied in [8] showing that there exists an element of the form

\[
w = t(V_2, V_2) \in \mathcal{N} \quad \text{with} \quad \Phi(w) < \Phi(v_2),
\]

for \( \lambda_2 \) big enough.

Notice that, thanks to the equation \( \Psi(w) = 0 \) we have that any \( t > 0 \) satisfies the following condition

\[
t^2 \|(V_2, V_2)\|^2 - t^4 \int_{\mathbb{R}^N} V_2^4 \, dx - \frac{1}{2} t^3 (1 + 3 \beta) \int_{\mathbb{R}^N} V_2^3 \, dx = 0,
\]  

(31)

and by definition we also have

\[
\|(V_2, V_2)\|^2 = 2 \|V_2\|^2 + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^N} V_2^2 \, dx.
\]

(32)

Moreover, since \( V_2 \in \mathcal{N}_2 \), we have

\[
\|V_2\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V_2^2 \, dx = 0.
\]

(33)

Substituting (32) and (33) in (31) it follows

\[
t^2 \left( \int_{\mathbb{R}^N} V_2^2 \, dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^N} V_2^2 \, dx \right) - t^4 \int_{\mathbb{R}^N} V_2^4 \, dx - \frac{1}{2} t^3 (1 + 3 \beta) \int_{\mathbb{R}^N} V_2^3 \, dx = 0.
\]

(34)

Hence, applying the rescaling (17) yields

\[
\int_{\mathbb{R}^N} V_2^p \, dx = \lambda_2^{p-4} \int_{\mathbb{R}^N} V^p \, dx.
\]

(35)
Subsequently, substituting (3.4) for \( p = 2, 3, 4 \) into (25) and dividing by \( t^2 \lambda_2^{3-N} \) we have that
\[
\int_{\mathbb{R}^N} V^3 \, dx + \frac{1}{t} \int_{\mathbb{R}^N} V^2 \, dx - t^2 \lambda_2 \int_{\mathbb{R}^N} V^4 \, dx - \frac{1}{2} t^3(1 + 3 \beta) \int_{\mathbb{R}^N} V^3 \, dx = 0. \tag{36}
\]
Moreover, due to (3.4), (32) and (33) we find respectively the expressions
\[
\Phi(w) = \frac{1}{t^2} \left( \int_{\mathbb{R}^N} V_2^2 \, dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^N} V_2^2 \, dx \right) + \frac{1}{12} t^4 \int_{\mathbb{R}^N} V_2^4 \, dx,
\tag{37}
\]
\[
\Phi(v_2) = I_2(v_2) = \frac{1}{2} t \| V_2 \|_2^2 - \frac{1}{6} \int_{\mathbb{R}^N} V_2^3 \, dx = \frac{1}{12} \int_{\mathbb{R}^N} V_2^3 \, dx. \tag{38}
\]
Furthermore, we are looking for the inequality \( \Phi(w) < \Phi(v_2) \), or equivalently,
\[
\frac{1}{6} t^2 \left( \int_{\mathbb{R}^N} V_2^3 \, dx + (\lambda_1 - \lambda_2) \int_{\mathbb{R}^N} V_2^2 \, dx \right) + \frac{1}{12} t^4 \int_{\mathbb{R}^N} V_2^4 \, dx - \frac{1}{12} \int_{\mathbb{R}^N} V_2^3 \, dx < 0, \tag{39}
\]
and then, applying again (3.5) and multiplying (39) by \( 6 \lambda_2^{N-3} \), we actually have
\[
t^2 \left( \int_{\mathbb{R}^N} V_2^3 \, dx + \frac{\lambda_1 - \lambda_2}{\lambda_2} \int_{\mathbb{R}^N} V_2^2 \, dx \right) + \frac{1}{12} t^4 \lambda_2 \int_{\mathbb{R}^N} V_2^4 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} V_2^3 \, dx < 0. \tag{40}
\]
Solving (40) the corresponding will provide us (41) for \( \lambda_2 \) large enough.

Therefore, there exists a positive constant \( \Lambda_2 \) such that for \( \lambda_2 > \Lambda_2 \) inequality (41) holds, proving that
\[
\Phi(\tilde{u}) \leq \Phi(w) < \Phi(v_2).
\]

Finally, to show that \( \tilde{u} \geq 0 \) and \( \tilde{u}, \tilde{v} \neq 0 \) we can use the same argument as in Theorem 7.

In the following we will prove the existence of a MP critical point of \( \Phi \) on \( \mathcal{N} \).

**Theorem 9.** Assume \( 2 \leq N \leq 7 \) and \( \beta < \Lambda \). There exists a constant \( \Lambda_2 \) such that, if \( \lambda_2 > \Lambda_2 \), then \( \Phi \) constrained on \( \mathcal{N} \) has a Mountain-Pass critical point \( u^* \) with \( \Phi(u^*) > \Phi(v_2) \).

**Proof.** Due to Proposition (5), \( v_2 \) is a strict local minima of \( \Phi \) on \( \mathcal{N} \), and taking into account Theorem 3 we obtain \( \Lambda_2 \) such that, for \( \lambda_2 > \Lambda \), we have \( \Phi(\tilde{u}) < \Phi(v_2) \). Under those conditions we are able to apply the Mountain Pass Theorem (see [3] for further details) to \( \Phi \) on \( \mathcal{N} \), that provide us with a PS sequence \( v_n \in \mathcal{N} \) such that
\[
\Phi(v_n) \to m := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\gamma(t)),
\]
where
\[
\Gamma := \{ \gamma : [0, 1] \to \mathcal{N} \text{ continuous } | \gamma(0) = v_2, \gamma(1) = \tilde{u} \}.
\]
Furthermore, applying the Lemma 3 we are able to find a subsequence of \( v_n \) such that (relabelling) \( v_n \to u^* \) strongly in \( H \). Thus, \( u^* \) is a critical point of \( \Phi \) satisfying
\[
\Phi(u^*) > \Phi(v_2),
\]
which conclude the proof.

### 3.2. One-dimensional case, \( N = 1 \).

Here we must point out that we do not have the compact embedding even for \( H \). However, we will show that for a PS sequence we are able to find a subsequence for which its weak limit is a solution of (20) belonging to \( E \). Thus, in order to avoid the lack of compactness for \( N = 1 \) we will use the following result of measure theory that one can find in [18]; see also [7, 9] for an application of this procedure to a similar problem.
Lemma 10. If \( 2 < q < \infty \), there exists a constant \( C > 0 \) so that

\[
\int_\mathbb{R} |u|^q \, dx \leq C \left( \sup_{z \in \mathbb{R}} \int_{|z-x|<1} |u(x)|^2 \, dx \right)^{\frac{q-2}{2}} \|u\|^2_E, \quad \forall \, u \in E.
\]

The next result is analogous to Theorem 7 for the one-dimensional case and working on the full Nehari manifold \( \mathcal{M} \) defined by (11).

Theorem 11. Suppose \( N = 1 \) and \( \beta > \Lambda \). The infimum of \( \Phi \) on \( \mathcal{M} \) is attained at some \( \tilde{u} \geq 0 \) with both components \( \tilde{u}, \tilde{v} \neq 0 \). Moreover, \( \Phi(\tilde{u}) < \Phi(\nu_2) \).

Proof. Again, by the Ekeland’s variational principle there exists a PS sequence \( u_n \in \mathcal{M} \), i.e.,

\[
\Phi(u_n) \to c := \inf_{\mathcal{M}} \Phi \quad \text{and} \quad \nabla \Phi(u_n) \to 0,
\]

such that, \( u_n \) is bounded since (15). Also, we can assume that the sequence \( u_n \) possesses a subsequence such that (relabelling) it weakly converges \( u_n \to u \) in \( \mathbb{R} \), \( u_n \to u \) strongly in \( L^q_{\text{loc}}(\mathbb{R}) = L^q(\mathbb{R}) \times L^q(\mathbb{R}) \) for every \( 1 \leq q < \infty \) and \( u_k \to u \) a.e. in \( \mathbb{R} \). Moreover, arguing in the same way as in Lemma 6 we obtain \( \Phi(u_n) \to 0 \) as \( n \to \infty \).

Furthermore, using the idea performed in (8) we will prove that there is no loss of mass at infinity for \( \mu_n(x) := u_n^2(x) + v_n^2(x) \), where \( u_n = (u_n, v_n) \), i.e., there exist \( R, C > 0 \) such that

\[
\sup_{z \in \mathbb{R}} \int_{|z-x|<R} \mu_n(x) \, dx \geq C > 0, \quad \forall n \in \mathbb{N}.
\]

On the contrary, if we suppose

\[
\sup_{z \in \mathbb{R}} \int_{|z-x|<R} \mu_k(x) \, dx \to 0,
\]

and thanks to Lemma 10 applied in a similar way as in (7), we find that \( u_k \to 0 \) strongly in \( L^q(\mathbb{R}) \) for any \( 2 < q < \infty \). This is a contradiction since \( u_n \in \mathcal{N} \), and due to (10) jointly with the fact \( \Phi(u_n) \to c \) we have

\[
0 < \frac{1}{q} \rho < c + o_n(1) = \Phi(u_n), \quad \text{with} \quad o_n(1) \to 0 \quad \text{as} \quad n \to \infty,
\]

hence (12) is true and there is no loss of mass at infinity.

We observe that there is a sequence of points \( \{z_n\} \subset \mathbb{R} \) such that by (12), the translated sequence \( \overline{\mu}_n(x) = \mu_n(x + z_n) \) satisfies

\[
\liminf_{n \to \infty} \int_{B_R(0)} \overline{\mu}_n \, dx \geq C > 0.
\]

Taking into account that \( \overline{\mu}_n \to \overline{\mu} \) strongly in \( L^1_{\text{loc}}(\mathbb{R}) \), we obtain that \( \overline{\mu} \neq 0 \), thus, the weak limit of \( \overline{\mu}_n(x) := u_n(x + z_n) \), which we denote it by \( \overline{\mu} \), is non-trivial. Notice that \( \overline{\mu}_n, \overline{\mu} \in \mathcal{M} \) and \( \overline{\mu}_n \) is PS sequence of level \( c \) for \( \Phi \) on \( \mathcal{M} \). Moreover, if we set \( F = \Phi|_\mathcal{M} \) (similarly to (15)) and using Fatou’s lemma we obtain the following

\[
\Phi(\overline{\mu}) = F(\overline{\mu}) \leq \liminf_{n \to \infty} F(\overline{\mu}_n) = \liminf_{n \to \infty} \Phi(\overline{\mu}_n) = \liminf_{n \to \infty} \Phi(u_n) = c.
\]

Therefore, \( \overline{\mu} \) is a non-trivial critical point of \( \Phi \) constrained on \( \mathcal{M} \). Furthermore, it is not a semi-trivial solution because of \( \Phi(\overline{\mu}) < \Phi(\nu_2) \) from Proposition 5(ii). Finally, to show that \( \overline{\mu} \geq 0 \) and both components \( \overline{\mu}, \overline{\nu} \neq 0 \), we apply the same argument used in Theorem 7. \( \blacksquare \)

Theorem 8 can be extended to the one-dimensional case directly using the same idea as we have performed in the last proof, obtaining the following.

Corollary 12. Assume \( N = 1, \beta > 0 \). There exists a positive constant \( \Lambda_2 \) such that, if \( \lambda_2 > \Lambda_2 \), the functional \( \Phi \) attains its infimum on \( \mathcal{N} \) at some \( \tilde{u} \geq 0 \) with \( \Phi(\tilde{u}) < \Phi(\nu_2) \) and both \( \tilde{u}, \tilde{v} \neq 0 \).

To finish, for \( N = 1 \), Theorem 9 can be obtained in a similar manner, obtaining the following.
Corollary 13. Assume $N = 1$ and $\beta < \Lambda$. There exists a constant $\Lambda_2$ such that, if $\lambda_2 > \Lambda_2$, then $\Phi$ constrained on $\mathcal{N}$ has a Mountain-Pass critical point $u^*$ with $\Phi(u^*) > \Phi(v_2)$.

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