Weak asymptotic solution of the phase field system in the case of confluence of free boundaries in the Stefan problem with undercooling

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Abstract

We assume that the Stefan problem with undercooling has a classical solution until the moment of contact of free boundaries and the free boundaries have continuous velocities until the moment of contact. Under these assumptions, we construct a smooth approximation of the global solution of the Stefan problem with undercooling, which, until the contact, gives the classical solution mentioned above and, after the contact, gives a solution which is the solution of the heat equation.

1 Introduction

In this paper, we study the confluence of free boundaries in the Stefan problem with undercooling in the one-dimensional case. We at once note that an analysis of the one-dimensional problem is not the final goal but is only a necessary step in the study of the multidimensional problem.

In the present paper, we study the problem in the domain $Q = \Omega \times [0, t_1]$, where $\Omega = [l_1, l_2]$ is an interval. We assume that the interval $\Omega$ is divided into three parts $\Omega_{i+}(t)$ and $\Omega^-(t)$ as follows:

$$\Omega_{i+}(t) = [l_1, \varphi_1(t)], \quad \Omega^- (t) = [\varphi_1(t), \varphi_2(t)], \quad \Omega_{i+}^- (t) = [\varphi_2(t), l_2],$$

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where \( \varphi_i(t), i = 1, 2 \), are the free boundaries of phases "+" and "−". We assume that the phase "+" occupies the intervals \( \Omega^+_i(t) \) and the phase "−" occupies the interval \( \Omega^−(t) \).

We shall construct a smooth approximation of solutions of the Stefan problem with undercooling under the assumption that the motion of the free boundary is the motion of the front of a nonlinear wave and the confluence of free boundaries is interaction of solitary nonlinear waves. The possibility of this interpretation is given by the models of phase field \([2]\) proposed by G. Caginalp. In fact, the choice of the method for approximating the limit Stefan problems with undercooling is unessential for us, because we do not prove that the approximations thus constructed are close to the corresponding solutions of the phase field system.

For example, we could use the definition of the generalized solution of the limit problems including the order function (the nonlinear wave) \([10]\).

Here our considerations are based on the following simple fact. Suppose that there are two families of solutions (exact and approximate solutions) of some problem depending on a small parameter \( \varepsilon \). Suppose that both these families have the properties that permit passing to the limit as \( \varepsilon \to 0 \) in the weak sense. Suppose also that the family of approximate solutions satisfies a problem with a right-hand side small as \( \varepsilon \to 0 \) in the weak sense. Then the weak limits of both families are solutions of the same limit problem and if the latter has a unique solution, then the difference between the exact and approximate solutions tends to 0 as \( \varepsilon \to 0 \) at least in the weak sense.

We recall that the phase field system has the form

\[
L\theta = -\frac{\partial u}{\partial t}, \quad \varepsilon Lu - \frac{u - u^3}{\varepsilon} - \theta = 0, \tag{1}
\]

where

\[
L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \quad x \in [l_1, l_2], \quad t \in [0, t_1].
\]

The function \( \theta = \theta(x, t, \varepsilon) \) has the meaning of the temperature, and the function \( u = u(x, t, \varepsilon) \), which is called the order function, determined the phase state of the medium: \( u \simeq -1 \) corresponds to the phase "−" in the domain \( \Omega^−(t) \), and \( u \simeq 1 \) corresponds to the phase "+" in the domains \( \Omega^+_i(t) \).

Passing to the limit as \( \varepsilon \to 0 \) in \((1)\), we obtain the Stefan problem with undercooling.

This passage to the limit is possible, for example, in the case where the corresponding limit problems have classical solutions. In this case, the weak limits as \( \varepsilon \to 0 \) of solutions \([11]\) give these solutions \([2, 5, 13]\).
The smooth approximations of solutions of the Stefan problem with kinetic undercooling constructed in this paper are approximate (in the above sense) solutions of the phase field system and they admit a weak passage to the limit as $\varepsilon \to 0$. In this case, we obtain the limit problems (and their solutions) describing the process of confluence of free boundaries. To construct these approximations, we use the assumption that the classical sharp fronted solution of the Stefan problem with kinetic undercooling exists until the confluence of free boundaries begins. This is a natural assumption in our paper (otherwise, it is not clear the confluence of what is considered) and can be proved\(^1\).

We repeat that, under this assumption, we construct an approximation of solution of the limit problem, which is smooth for $\varepsilon > 0$ and uniformly bounded in $\varepsilon \geq 0$. As is well known, in a similar problem about the propagation of shock waves, the existence of such an approximation distinguishes a unique solution.

By $t^* \in [0, t_1)$ we denote the instant of confluence of free boundaries. Then, for any $t \leq t^* - \delta$ for all $\delta > 0$, we see that the limit problems have solutions. The asymptotic solution of system (1) has the form

$$\theta_{\varepsilon}^{as} = \bar{\theta}^-(x,t) + \left(\bar{\theta}^+(x,t) - \bar{\theta}^-(x,t)\right) \omega_1 \left(\frac{x - \hat{\varphi}_2(t)}{\varepsilon}\right) \omega_1 \left(\frac{x - \hat{\varphi}_1(t)}{\varepsilon}\right),$$

(2)

$$u_{\varepsilon}^{as} = 1 + \omega_0 \left(\frac{-x + \hat{\varphi}_1(t)}{\varepsilon}\right) + \omega_0 \left(\frac{x - \hat{\varphi}_2(t)}{\varepsilon}\right) + \varepsilon \left[\frac{\theta_{\varepsilon}^{as}}{2} + \omega \left(t, \frac{x - \hat{\varphi}_1(t)}{\varepsilon}, \frac{x - \hat{\varphi}_2(t)}{\varepsilon}\right)\right].$$

(3)

Here $\omega_1(z) \to 0, 1$ as $z \to \mp \infty$, $\omega_1^{(k)}(z) \in S(\mathbb{R}^1)$ for $k > 0$, $\hat{\varphi}_i(t), i = 1, 2$, are smooth functions, $\hat{\varphi}_1 \leq \hat{\varphi}_2$, $\omega_0(z) = \tanh(z)$, and $\omega(t, z_1, z_2) \in C^\infty([0, t^*]; S(\mathbb{R}^d))$. By $S(\mathbb{R}^n)$ we denote the Schwartz space of smooth rapidly decreasing functions. If the initial data for (1) has the form (2), (3) at $t = 0$, then, for $t \leq t^* - \delta$, we have the estimate

$$\|u - u_{\varepsilon}^{as}; C(0, T; L^2(\mathbb{R}^1))\| + \|\theta - \theta_{\varepsilon}^{as}; L^2(\Omega)\| \leq c\varepsilon^\mu,$$

where $(\theta, u)$ is a solution of system (1) (see [1, 5]). Here $Q = \Omega \times [0, t^* - \delta)$, and the constant $c$ is independent of $\varepsilon$.

The main obstacle to the construction of solutions of the form (2), (3), which could be used to describe the confluence of free boundaries, is the fact

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that, instead of an ordinary differential equation whose solution is the function $\omega_0(z) \[2, 5\]$, in the case of confluence of free boundaries, we must deal with a partial differential equation for which the explicit form of the exact solution is unknown.

In the present paper, we use the technique of the weak asymptotics method \[4, 8\], which allows us to avoid this problem. Let us explain several basic points.

**Definition 1.** A family of functions $f(x, t, \varepsilon) \in L^1(\Omega)$ integrable with respect to $x$ for all $t \in [0, t_1]$ and for $\varepsilon > 0$ admits the estimate $O(D'(\varepsilon^\nu))$ if, for any test function $\zeta(x) \in C_0^\infty$, we have the estimate

$$\left| \int_\Omega f(x, t, \varepsilon) \zeta(x) dx \right| \leq C_{t_1, \zeta} \varepsilon^\nu,$$  \hspace{1cm} (4)

where the constant $C_{t_1, \zeta}$ depends on $t_1$ and the test function $\zeta(x)$.

Generalizing (4), we shall say that the family of distributions $f(x, t, \varepsilon)$ depending on $t$ and $\varepsilon$ as on parameters admits the estimate $O(D'(\varepsilon^\nu))$ if, for any test function $\zeta(x)$, we have the estimate

$$\langle f(x, t, \varepsilon), \zeta(x) \rangle = O(\varepsilon^\nu), \quad 0 \leq t \leq t_1.$$

**Example 1.** Let $\omega(z) \in S(\mathbb{R}^1)$, $\int_{\mathbb{R}^1} \omega(z) dz = 1$, and let $x_0 \in [l_1, l_2]$. Then we have

$$\omega\left(\frac{x - x_0}{\varepsilon}\right) - \delta(x) = O(D'(\varepsilon)), \quad \omega\left(\frac{x}{\varepsilon}\right) = \delta(x - x_0) \int_{\mathbb{R}^1} \omega(z) dz + O(D'(\varepsilon)).$$

**Example 2.** Suppose that $\omega_i(z) \in C^\infty, (\omega_i)' \in S(\mathbb{R}^1)$, $\lim_{z \to -\infty} \omega_i(z) = 0$, and $\lim_{z \to +\infty} \omega_i(z) = 1$, $i = 1, 2$. Then we have

(a) $$\omega_i\left(\frac{x - x_i}{\varepsilon}\right) - H(x - x_i) = O(D'(\varepsilon)).$$

(b) $$\omega_1\left(\frac{x - x_1}{\varepsilon}\right) \omega_2\left(\frac{x - x_2}{\varepsilon}\right) = B_1\left(\frac{\Delta x}{\varepsilon}\right) H(x - x_1) + B_2\left(\frac{\Delta x}{\varepsilon}\right) H(x - x_2) + O(D'(\varepsilon)),$$

where $\Delta x = x_1 - x_2$, $B_1(\rho) \in C^\infty, B_1'(\rho) \in S(\mathbb{R}^1)$, $B_1(\rho) + B_2(\rho) = 1$, $B_1(+\infty) = 1$, and $B_1(-\infty) = 0$.  

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Example 3. The preceding relations easily imply the formula

\[ H(x - x_1)H(x - x_2) = B \left( \frac{\Delta x}{\varepsilon} \right) H(x - x_1) + (1 - B)H(x - x_2) + O_D'(\varepsilon), \]

where \( B(\rho) \in C^\infty \) is the function \( B_1 \) from the preceding example.

Example 4. The following relation is a corollary of Definition 1:

\[ \left( \frac{d}{dx} \right)^m O_D'(\varepsilon^\alpha) = O_D'(\varepsilon^\alpha) \]

for all \( m \in \mathbb{Z}_+ \) and \( \alpha > 0 \). Generally speaking, this is not true for the derivatives w.r.t. \( t \).

The relations in Examples 2(a), 2(b) and 4 are obvious, but, for the reader’s convenience, we prove some of these formulas in Section 5.

Here we only note that, in view of items (a) and (b), one can say that the product of approximations of the Heaviside functions or of the Heaviside functions themselves is a linear combination with accuracy up to small terms \( \sim O_D'(\varepsilon) \). The general nonlinear functions of linear combinations of approximations of the Heaviside functions can be linearized similarly. Precisely this property underlies the constructive study of the interaction between nonlinear waves with localized fast variations.

Definition 2. A pair of smooth functions \((\bar{u}, \bar{\theta})\) is a weak asymptotic solution of the phase field system (1) if, for any test functions \(\zeta(x), \xi(x) \in C^\infty_0(\Omega)\), the following relations hold:

\[
\int_\Omega (\bar{u}_t + \bar{\theta}_t) \zeta dx + \int_\Omega \bar{\theta}_x \zeta_x dx = O(\varepsilon),
\]

\[
\varepsilon \int_\Omega \bar{u}_x \xi dx + \frac{\varepsilon}{2} \int_\Omega \bar{u}_x^2 \xi_x dx - \frac{1}{\varepsilon} \int_\Omega \left( \frac{\bar{u}_x^4}{4} - \frac{\bar{u}_x^2}{2} + \frac{1}{4} \right) \xi_x dx + \kappa \int_\Omega \bar{u} \frac{\partial}{\partial x}(\bar{\theta}_x \xi) dx = O(\varepsilon^\mu), \quad \mu \in (0, 1/2).
\]

The left-hand side of Eq. (6) is obtained by multiplying the second equation in system (1) by \( \bar{u}_x \) and integrating by parts. The reminders \( O(\varepsilon^\alpha) \), \( \alpha = 1, \mu \), in the right-hand sides of (5) and (6) must be bounded locally in \( t \), i.e., for \( t \in [0, t_1] \), we have

\[ \max_{0 \leq t \leq t_1} |O(\varepsilon)| \leq C_{t_1} \varepsilon, \quad C_{t_1} = \text{const}. \]

This construction was introduced and analyzed in [6].
The fact that a noninteger exponent appears in the right-hand side of (6) is not directly related to the technique of the weak asymptotics method, see Examples 1–4. The source of this noninteger exponent is the non-smoothness of the function \( \theta \) (the temperature), which appears at the instant of confluence of the free boundaries.

Relations (5)–(6) can be rewritten as

\[
\dot{\theta} + \ddot{\theta} = \varepsilon \frac{\partial^2 \theta}{\partial x^2} + \mathcal{O}_D(\varepsilon),
\]

\[
\varepsilon \ddot{\theta} = \varepsilon \frac{\partial^2 \ddot{\theta}}{\partial x^2} + \dot{\theta}(1 - \ddot{\theta}) + \mathcal{O}_D(\varepsilon^n).
\]

The paper is organized as follows.

In Section 2, we explain the structure of ansatzes of approximations of the temperature and the order function. These ansatzes are constructed under the assumption the classical sharp fronted solution of the Stefan problem exists. At the beginning of Section 2, we formulate what we exactly need.

Next, in Section 3, we substitute the constructed ansatzes in system (1) and derive equations for the unknown functions contained in the ansatzes.

The results of these calculations are summarized in Theorems 1–3 in Section 3. The final result of this paper is formulated in Theorem 4 in Section 3. It states that the assumption on the existence of the classical solution is sufficient for constructing formulas for the weak asymptotic solution of system (1) in the sense of Definition 2. Next, in Section 4, we analyze the constructed formulas and derive the following effects:

(a) the weak asymptotic solution is smooth for \( t > t^* \), the absolute values of the free boundaries velocities are equal to each other at the contact moment;

(b) the temperature has a negative jump at the instant and at the point of confluence of the free boundaries, and this jump is equal to the half-sum of the limits of the velocities of the free boundaries as \( t \to t^* - 0 \).

In particular, it follows from (a) and (b) that the velocities of the free boundaries have jumps at the point of contact.

These effects can also be discovered in the numerical analysis of the process of confluence of free boundaries. Some helpful technical results are given in Section 5. We note that the only example known to the authors, where the confluence of the free boundaries is studied, is given in [11].

In general, this paper turned out to be technically rather complicated and long in spite of the fact that the details in several justifications in Section 5 were omitted.
Although, as was shown above, the results obtained here are a necessary step in the study of the multidimensional problem, this paper shows that the following classification can be introduced:

1. problems in which the confluence of free boundaries leads to disappearance of one of the phases;
2. problems in which the domain occupied by one of the phases changes its connectivity, but the number of phases remains the same.

In this paper, we consider an example precisely from the first class of problems. As was mentioned above, we do not justify the asymptotics of the constructed solution. But this can be done based on our constructions. It is easy to see that the main role here is played by the (not proved) existence of the classical solution up to the moment of confluence of the free boundaries. This assumption reduces justifying the constructed weak asymptotic solution to estimating the solution of the heat equation with the right-hand side \( f_\varepsilon \) admitting the estimate

\[
f_\varepsilon = O_D(\varepsilon^\mu), \quad \mu \in (0, 1/2)
\]

and with zero initial and boundary conditions. An analysis of the structure of this right-hand side shows that \( f_\varepsilon \cdot \varepsilon^{-\mu} \) as \( \varepsilon \to 0 \) is a linear combination of functions \( \delta'(x - \varphi_i) \) and \( \delta(x - \varphi_i) \), \( i = 1, 2 \), with coefficients depending on \( t, \tau \), and these coefficients converge fast to zero as \( \tau \to \pm \infty \).

Hence we can conclude that for \( x \neq \varphi_i \) the solution of this heat equation belongs to \( C^\infty \) for \( \varepsilon \geq 0 \) and admits the estimate \( O(\varepsilon^\mu) \). In the whole domain \( \Omega \times [0, T] \), a rough analysis based on general theorems [14] shows that the solution belongs to \( W^{-\delta}_2, \delta > 0 \), and has the estimate \( O(\varepsilon^\mu) \) in the norm of this space. In particular, the weak limit of the constructed weak asymptotic solution is equal to the exact generalized global solution of the heat equation in the phase field system.

2 Ansatz of the approximation of the solution of the Stefan problem with kinetic undercooling

We recall that, in our case, the Stefan problem with kinetic undercooling has the form

\[
\frac{\partial \bar{\theta}}{\partial t} = \frac{\partial^2 \bar{\theta}}{\partial x^2}, \quad x \in [l_1, l_2], \quad x \neq \hat{\varphi}_i, \quad i = 1, 2
\]
Relations (7)–(9) are supplemented with (Dirichlet or Neumann) boundary conditions for \( x = l_i, \ i = 1, 2 \), and with a consistent initial condition. Obviously, problem (7)–(9) can be written as

\[
L \theta = 2 \hat{\phi}_{2 \Delta} \delta(x - \hat{\phi}_2) - 2 \hat{\phi}_{1 \Delta} \delta(x - \hat{\phi}_1),
\]

\[
\theta \big|_{x = \hat{\phi}_i} = (-1)^{i+1} 2^{-1} \hat{\phi}_i.
\]

We shall assume that the initial conditions are chosen so that the problem in question has the classical solution, \( \hat{\phi}_1(0) < \hat{\phi}_2(0) \), and there exists a \( t = t^* \in [0, t_1] \) such that \( \hat{\phi}_1(t^* - 0) = \hat{\phi}_2(t^* - 0) \). More precisely, we assume that

(i) the limits \( \lim_{t \to t^* - 0} \hat{\phi}_i(t) \) exist, \( i = 1, 2 \);

(ii)

\[
(0, t^*) \in C^1 \left[ \mathbb{R} \right] \cap C^1 \left( \mathbb{R}^+ \right) \cap C^1 \left( \mathbb{R}^- \right) \cap C^1 \left( \mathbb{R}^0 \right) \cap C^1 \left( \mathbb{R}^+ \right) \cap C^1 \left( \mathbb{R}^0 \right) \cap C^1 \left( \mathbb{R}^- \right)
\]

(here \( D \) denotes the interior of the domain \( D \)), cf. [3].

As will be shown below, these assumptions permit describing the interaction (the confluence of free boundaries) constructively.

We shall construct this approximation (ansatz) as a weak asymptotic solution of system (1). First, we introduce the ansatz of the order function. It has the form

\[
\bar{u} = \frac{1}{2} \left[ 1 + \omega_0 \left( \frac{\beta \Phi_1 - x}{\varepsilon} \right) + \omega_0 \left( \frac{\beta \Phi_2 - x}{\varepsilon} \right) \right.
\]

\[
\left. - \omega_0 \left( \frac{\beta \Phi_1 - x}{\varepsilon} \right) \omega_0 \left( \frac{\beta \Phi_2 - x}{\varepsilon} \right) \right].
\]

Here the unknowns are the functions \( \beta \) and \( \Phi_i = \varphi_i(t, \varepsilon), \ i = 1, 2 \). In what follows, we write them more precisely. Now we only note that if \( \beta > 0 \), then, for \( \varphi_1 < \varphi_2 \), the function \( \bar{u} \) coincides up to \( O(\varepsilon^N) \) with the sum of the first three terms in the right-hand side of (3), for \( \varphi_1 = \varphi_2 \), we have \( \bar{u} = 1 + O_{D'}(\varepsilon) \), and for \( \varphi_1 > \varphi_2 \), we have \( \bar{u} = 1 + O(\varepsilon^N) \). These relations can easily be verified directly. We note that, under the above assumptions, we can continue
the functions \( \hat{\varphi}_i(t) \) to the interval \([0, t_1]\) preserving the smoothness and the sign of the derivatives. We choose such continuations and denote them by \( \varphi_{i0}(t) \). Now, we can write the functions \( \varphi_i(t, \varepsilon) \) and \( \beta(t, \varepsilon) \) more precisely. Namely, we set

\[
\varphi_i(t, \varepsilon) = \varphi_{i0}(t) + \psi_0(t)\varphi_{i1}(\tau, t),
\]

where

\[
\psi_0(t) = \varphi_{20}(t) - \varphi_{10}(t), \quad \tau = \psi_0/\varepsilon.
\]

We note that

\[
\tau \to \infty, \quad t < t^*,
\]

\[
\tau \to -\infty, \quad t > t^*,
\]
as \( \varepsilon \to 0 \). Thus, the value of the variable \( \tau \) characterizes the process of confluence of free boundaries. Indeed, we could set \( \tau = (t^* - t)/\varepsilon \), but, for (13), the next formulas become simpler (this can be seen from the formulas in Example 2 if we put \( x_i = \varphi_{i0}(t) \)).

Similarly, we set

\[
\beta(\tau) = 1 + \beta_1(\tau).
\]

In addition, we assume that

\[
\beta_1^{(\alpha)}(\tau), \varphi_{i1}^{(\alpha)}(\tau) \to \mathcal{O}(\tau^{-1}), \quad \tau \to \infty, \quad \alpha = 0, 1,
\]

\[
0 < \delta_1 < \beta_1(\tau) < \delta_2, \quad \delta_1, \delta_2 = \text{const},
\]

\[
\varphi_{i1} \to \varphi_{i1}^- = \varphi_{i1} - \varphi_{i1}^- = \mathcal{O}(|\tau|^{-1}), \quad \tau \to -\infty, \quad \varphi_{i1}^- = \text{const}.
\]

Assumptions (13) for the functions \( \varphi_{i1} \) are the assumption that as \( \varepsilon \to 0 \) the functions \( \varphi_i(t, \varepsilon) \) approximate continuous functions with a possible discontinuity (jump) of the derivatives at \( t = t^* \).

More precisely, the functions \( \varphi_i(t, \varepsilon) \) determined in (12) are approximations of the functions

\[
\varphi_i(t, 0) = \varphi_{i0} + \psi_0\varphi_{i1}^-H(-\psi_0),
\]

where \( \varphi_{i1}^- = \lim_{\tau \to -\infty} \varphi_{i1}(\tau, t) \), \( \varphi_i(t, \varepsilon) - \varphi_i(t, 0) = \mathcal{O}(\varepsilon) \), and \( H(z) \) is the Heaviside function.

Now we describe the ansatz of the approximation of the temperature \( \bar{\theta}(x, t) \). Here we also must use the continuation procedure for matching the temperature for \( t < t^* \), when there are three subdomains of the domain \( Q \), and the temperature for \( t > t^* \), when there is only one phase. For this, we first introduce a "model" of the temperature whose continuation is reduced to
the problem of continuation of functions depending only on the time $t$. This model must have the same structure as the temperature $\bar{\theta}$. Obviously, the simplest function of this form is a function linear in $x$ in the domains $\Omega^+_{l,r}(t)$ and quadratic in the domain $\Omega^-(t)$.

Namely, we set

$$
\frac{\partial \bar{\theta}}{\partial x} \bigg|_{x=\hat{\varphi}_2 \pm 0} = \pm \gamma^\pm_2,
$$

$$
\frac{\partial \bar{\theta}}{\partial x} \bigg|_{x=\hat{\varphi}_1 \pm 0} = \pm \gamma^\pm_1,
$$

We note that, by assumptions (i) and (ii), the functions $\gamma^\pm_j$, $j = 1, 2$, depend on $t$, are continuous for $t < t^*$, and have limits as $t \to t^* - 0$. Therefore, they can be continued to the interval $[0, t_1]$, $t_1 > t^*$, so that the properties of being continuous and the signs are preserved. We shall use the previous notation for the continued functions. Now we can introduce the temperature model

$$
\bar{T} = \gamma^-_1 (\varphi_1 - x) H(\varphi_1 - x) + \gamma^+_2 (x - \varphi_2) H(x - \varphi_2)
$$

$$
+ \gamma^-(x, t) \frac{(\varphi_1 - x)(x - \varphi_2)}{\psi} H(x - \varphi_1) H(\varphi_2 - x)
$$

$$
+ \hat{\gamma}(x, t) \frac{(\varphi_1 - x)(x - \varphi_2)}{\psi} H(\varphi_1 - x) H(x - \varphi_2) + I,
$$

where $\gamma^-$, $\hat{\gamma}$, and $I$ are linear functions of $x$, $\gamma^-|_{x=\varphi_i} = \gamma^-_i(t)$, $\hat{\gamma}|_{x=\varphi_i} = \hat{\gamma}_i(t)$, and $I|_{x=\varphi_i} = (-1)^{i+1} \varphi_i t$. In more detail, the functions $\hat{\gamma}(x, t)$ and $\gamma^-(x, t)$ can be written in the form

$$
\gamma^- = \frac{\gamma^+_1 - \gamma^-_2}{2} - (x - x^*) \frac{\gamma^+_1 + \gamma^-_2}{\psi},
$$

$$
\hat{\gamma} = \frac{\hat{\gamma}_1 + \hat{\gamma}_2}{2} - (x - x^*) \frac{\hat{\gamma}_1 - \hat{\gamma}_2}{\psi},
$$

$$
x^* = \frac{\varphi_1 + \varphi_2}{2}.
$$

We note that, by Lemma 3 in Section 5, for $t \leq t^*$, we have the relations

$$
\varphi_i(t, \varepsilon) = \hat{\varphi}_i(t) + \mathcal{O}(\varepsilon), \quad i = 1, 2.
$$
This implies that, for $t < t^*$,
\[
\left[ \frac{\partial \hat{T}}{\partial x} \right]_{x=\varphi_i} = \left[ \frac{\partial \theta}{\partial x} \right]_{x=\varphi_i},
\]
\[
\left[ \frac{\partial \hat{T}}{\partial x} \right] \delta(x - \varphi_1) = \left[ \frac{\partial \theta}{\partial x} \right] \delta(x - \varphi_1) + O_{D'}(\varepsilon).
\]

Using the formula in Example 3 above, we rewrite the product
\[
H(x - \varphi_1)H(\varphi_2 - x)
\]
as follows
\[
H(x - \varphi_1)H(\varphi_2 - x) = B(\rho)[H(x - \varphi_1) - H(x - \varphi_2)] + O_{D'}(\varepsilon).
\]
\[
H(\varphi_1 - x)H(x - \varphi_2) = (1 - B(\rho))[H(x - \varphi_2) - H(x - \varphi_1)] + O_{D'}(\varepsilon).
\]

Here $B(\rho) \in C^\infty$, $B(\rho) \to 1$, $\rho \to \infty$, $B(\rho) \to 0$, $\rho \to -\infty$, $B^{(\alpha)} = O(|\rho|^{-N})$, $|\rho| \to \infty$ for any $N > 0$, $\alpha > 0$.

In what follows, we write $B(\tau)$ instead of $B(\rho)$. The difference is that $B(\rho)$ is an unknown function ($\rho$ is unknown), while $B(\tau)$ is a known function. The replacement of $B(\rho)$ by $B(\tau)$ in (19) and (20) implies a correction of order $O_{D'}(\varepsilon)$ in the right-hand sides.

But we do not use this remark and, by definition, we set
\[
\hat{T} = \gamma_1^- (\varphi_1 - x)H(\varphi_1 - x) + \gamma_2^- (x - \varphi_2)H(x - \varphi_2)
\]
\[
+ \gamma^- (x, t) \frac{\varphi_1 - x}{\psi}(x - \varphi_2)B(\tau)[H(x - \varphi_1) - H(x - \varphi_2)] + \gamma^+ (x, t) \frac{\varphi_1 - x}{\psi}(x - \varphi_2)(1 - B(\tau))[H(x - \varphi_2) - H(x - \varphi_1)] + I.
\]

Now we can write the expression $\frac{\partial^2 \hat{T}}{\partial x^2}$ uniformly in $\psi = \varphi_2 - \varphi_1$. We have
\[
\frac{\partial^2 \hat{T}}{\partial x^2} = F_1(x, t, \tau)
\]
\[
+ \frac{\partial^2}{\partial x^2} \left( \gamma^- (\varphi_1 - x)(x - \varphi_2) \right)B(\tau)[H(x - \varphi_1) - H(x - \varphi_2)]
\]
\[
+ \frac{\partial^2}{\partial x^2} \left( \gamma^+ (\varphi_1 - x)(x - \varphi_2) \right)(1 - B(\tau))[H(x - \varphi_2) - H(x - \varphi_1)]
\]
\[
+ \left[ \gamma_1^+ + \gamma_1^- B(\tau) + \gamma_2(1 - B(\tau)) \right] \delta(x - \varphi_2)
\]
\[
+ \left[ \gamma_1^- + \gamma_1^+ B(\tau) - \gamma_1^+(1 - B(\tau)) \right] \delta(x - \varphi_1) + O_{D'}(\varepsilon),
\]
11
where $F_1(x,t,\tau)$ is a bounded piecewise continuous function. We set $\gamma_2^+ = -\hat{\gamma}_2$ and $\gamma_1^- = \hat{\gamma}_1$ and see that the coefficients for $\delta(x - \varphi_i)$ in (21) take the form

$$\left[\gamma_2^- + \gamma_2^+\right] B(\tau) \delta(x - \varphi_2) + \left[\gamma_1^- + \gamma_1^+\right] B(\tau) \delta(x - \varphi_1).$$

(22)

Now we note that the inequalities

$$[H(x - \varphi_1) - H(x - \varphi_2)]/\psi > 0, \quad [H(x - \varphi_2) - H(x - \varphi_1)]/\psi < 0,$$

hold for $\psi \neq 0$. We recall that, by (18) and the Stefan condition (9),

$$\varphi_{10t} = \gamma_1^+ + \gamma_1^-, \quad \varphi_{20t} = -\left(\gamma_2^+ + \gamma_2^-\right)$$

for $t < t^*$ and assume that the continuations of the functions contained in these relations are chosen so that these relations hold for $t \in [0, t_1]$.

The assumption that $\tilde{\theta}(x,t)$ is the classical solution of problem (7)–(9) implies that

$$\tilde{\theta} - T = \tilde{q} \in C([0, t^*); C^1(\Omega))$$

(23)

and the limit $\tilde{q}$ exists as $t \to t^* - 0$ and for $x \in \Omega$. Therefore, we can continue the function $\tilde{q}$ to the domain $\Omega \times [0, t_1]$, where $t_1 > t^*$ and the properties of smoothness are preserved, and moreover,

$$\tilde{q}|_{x = \hat{\varphi}_i} = 0, \quad i = 1, 2, \quad t < t^*.$$

We shall construct the global temperature $\tilde{\theta}$ in the form

$$\tilde{\theta} = e(x) \tilde{T} + \hat{q} + \tilde{q},$$

(24)

where $\hat{q}$ is the desired function, $e(x) \in C^\infty([l_1, l_2])$, and $e \equiv 1$ for $x \in [\hat{\varphi}_1(0), \hat{\varphi}_1(0)]$.

## 3 Construction of the weak asymptotic solution

We first consider the heat equation in system (11). Using the formulas of the weak asymptotics method (see Section 5) and taking (11) into account, we obtain

$$\frac{\partial \hat{q}}{\partial t} = \left[\frac{\varphi_{1t}(2 - B_{00})}{2} + \frac{\beta_\tau(\varphi_{20t} - \varphi_{10t})}{2\beta^2} B_{00}^2\right] \delta(x - \varphi_1)$$

(25)

$$+ \left[\frac{-\varphi_{2t}(2 - B_{00})}{2} + \frac{\beta_\tau(\varphi_{20t} - \varphi_{10t})}{2\beta^2} B_{00}^2\right] \delta(x - \varphi_2) + \mathcal{O}_D(\epsilon),$$

where $\mathcal{O}_D(\epsilon)$ is the remainder term.
where the estimate $O_{D^e}(\varepsilon)$ is uniform in $\tau$,

$$B_{00} = \int_{\mathbb{R}} \dot{\omega}_0(z)\omega_0(-\eta - z)\,dz, \quad B^z_{00} = \int_{\mathbb{R}} z\dot{\omega}_0(z)\omega_0(-\eta - z)\,dz,$$

and the function $\rho(\tau)$ is determined by the formula

$$\rho(\tau) = \frac{\varphi_2 - \varphi_1}{\varepsilon}. \quad (26)$$

We assume that function $\bar{q}$ and its continuation are chosen so that the function $\bar{q}$ satisfies the boundary conditions of the original problem. Then the boundary conditions for the function $\hat{q}$ are zero.

We substitute the function $\bar{\theta}$ determined by relation (24) and expression (25) for $\partial \bar{u}/\partial t$ into Eq. (5). With accuracy $O_{D^e}(\varepsilon)$, we obtain

$$L\bar{\theta} + \frac{\partial \bar{u}}{\partial t} = L\bar{e} + Lq + A_1\delta(x - \varphi_1) + A_2\delta(x - \varphi_2),$$

where

$$q = \bar{q} + \hat{q}, \quad A_i = (-1)^{i+1}\frac{\varphi_{i+1} - \varphi_i}{2}(2 - B_{00}) + \frac{\beta_i\psi^0_{i+1}}{2\beta^2}B^z_{00}, \quad i = 1, 2.$$

We let $\tau$ tend to $\infty$ (i.e., for $t < t^*$) and, in view of (10) and (18), obtain

$$L(e\bar{T} + q) = 2\varphi_2\delta(x - \varphi_2) - 2\varphi_1\delta(x - \varphi_1).$$

Since $e\bar{T} + q \to \bar{T}$ as $\tau \to \infty$, we see that $\hat{q} \to 0$ as $\tau \to \infty$. The total equation for the function $q = \bar{q} + \hat{q}$ has the form

$$L(q + e(x)I) = F(x, t, \tau) \quad (27)$$

where $F(x, t, \tau)$ is a piecewise continuous function containing of terms that are uniformly bounded in $\varepsilon$ on $\Omega \times [0, t_1]$ (they appear from $F_1(x, t, \tau)$ in (21) and the derivatives of the function $\bar{q}$).

Equation (27) was derived under the assumption that (see (22))

$$\sum_{i=1}^{2} \left\{ B[\gamma_i^+ + \gamma_i^-] - A_i \right\} \delta(x - \varphi_i) = O_{D^e}(\varepsilon). \quad (28)$$
Similarly to (31), with accuracy up to smooth functions, we can write the fundamental solution of the heat equation (in what follows, we shall consider only the first equation),

\[
\phi(t, \tau, x) = \sqrt{\frac{1}{\pi t}} \exp\left\{-\frac{(x - \tau)^2}{4t}\right\}.
\]

It is clear that the functions \(\hat{q}_i\) are uniformly bounded. It is also clear that \(\hat{q}_i \notin C^1(\Omega)\), but \(\hat{q}_i \in C^{1,2}(\Omega \times [0, t_1]) \setminus \{(x = \phi_1) \cup \{x - \phi_2\}\}. Therefore, to justify Eq. (27), we must verify that no \(\delta\)-functions arise in calculating the derivatives \(\frac{\partial \hat{q}_i}{\partial x^2}\) and \(\frac{\partial \hat{q}_i^*}{\partial t}\).

For this, we note that, for any test function \(\zeta(x)\) up to functions smooth in \(\Omega\), omitting the number factor, we have the relation

\[
\int_0^1 \left(2\gamma_1^+ + \gamma_2^-\right)B(\gamma)\frac{f_i(t \gamma - \tau)}{\sqrt{t - \tau}} d\gamma.
\]

With accuracy up to functions smooth in \(\Omega\), we can calculate them using the fundamental solution of the heat equation (in what follows, we shall consider only the first equation),

\[
\hat{q}_1 = -\frac{1}{2\sqrt{2\pi}} \int_0^t \left(\gamma_1^+(t') + \gamma_2^-(t')\right)B(\tau) \exp\left\{-\frac{(x - \xi)^2}{(t - t')}\right\} d\xi d\tau.
\]

It is clear that the functions \(\hat{q}_i\) are uniformly bounded. It is also clear that \(\hat{q}_i \notin C^1(\Omega)\), but \(\hat{q}_i \in C^{1,2}(\Omega \times [0, t_1]) \setminus \{(x = \phi_1) \cup \{x - \phi_2\}\}. Therefore, to justify Eq. (27), we must verify that no \(\delta\)-functions arise in calculating the derivatives \(\frac{\partial \hat{q}_i}{\partial x^2}\) and \(\frac{\partial \hat{q}_i^*}{\partial t}\).

For this, we note that, for any test function \(\zeta(x)\) up to functions smooth in \(\Omega\), omitting the number factor, we have the relation

\[
\langle \hat{q}_i, \zeta(x) \rangle = \int_0^t \left(\gamma_1^+(t') + \gamma_2^-(t')\right)B(\tau) \exp\left\{-\frac{(x - \xi)^2}{(t - t')}\right\} d\tau, \quad \langle \zeta(x), f_i(t \gamma) \rangle = \int_0^t \left(\gamma_1^+(t') + \gamma_2^-(t')\right)B(\tau) \exp\left\{-\frac{(x - \xi)^2}{(t - t')}\right\} d\tau.
\]

where \(f(x, t) = \int_{\mathbb{R}^2} \zeta(x) \exp\left\{-\frac{(x - \xi)^2}{2\sqrt{2\pi} (t - t')}\right\} d\xi\) is the solution of the heat equation \(f_t - f_{\xi\xi} = \zeta(x)\) at the point \(t - t'\). It is clear that \(f(x, t) \in C^\infty(\Omega)\) for all \(t\).

Calculating the derivative \(\frac{\partial \hat{q}_i/\partial x^2}\), we obtain

\[
\langle \frac{\partial^2 \hat{q}_i}{\partial x^2}, \zeta(x) \rangle = \int_0^t \left(\gamma_1^+(t') \right)B(\tau) \exp\left\{-\frac{(x - \xi)^2}{(t - t')}\right\} d\xi d\tau.
\]

where \(f_1(\xi, t)\) is a solution of the equation \(f_t - f_{\xi\xi} = \zeta''(x)\). It is clear that the relation

\[
\langle \frac{\partial^2 \hat{q}_i}{\partial x^2}, \zeta(x) \rangle \neq \sum G_i \zeta(\phi_i) + \mathcal{O}(\varepsilon)
\]

cannot hold for any coefficients \(G_i\). The same is true for \(\partial q_i/\partial t, \partial q_i^*/\partial t\).

Similarly to (31), with accuracy up to smooth functions, we can write the
functions $\hat{q}^*_i$, $i = 1, 2$. They have the same properties as $\hat{q}_i$ with the only additional condition

$$\hat{q}^*_i|_{x=x^*} = 0.$$  

This easily follows from the explicit formula of the type of (31) and the fact that, for $x = x^*$, the integral with respect to $\xi$ is an integral of an odd function over a symmetric interval. Therefore, condition (28) is necessary for deriving Eq. (27). This implies that the function $\hat{q} + eI$ is uniformly bounded in $\Omega \times [0, t_1]$ and belongs to $C^{1,0}$ for $x \neq \varphi_i$, $i = 1, 2$. So if the function $\hat{q} + eI$ satisfies Eq. (27) with zero initial conditions and the boundary conditions that follow from the fact that the function $\hat{q} + eI$ must satisfy the boundary conditions of the original problem, while the functions $\hat{T}$ and $\hat{q}$ are known (more precisely, they will be determined after the functions $\varphi_i$ are found), then we have the following assertion.

**Theorem 1.** Suppose that the function $\hat{\vartheta}$ is determined by relation (28), and the function $\hat{q}$ is a solution of Eq. (27) with zero initial condition and zero boundary conditions. Suppose that relation (28) holds.

Then the pair of functions $\hat{\vartheta}$ and $\hat{u}$ is a weak asymptotic solution of the heat equation in the phase field system, i.e., relation (5) holds.

**Remark 1.** Expression (28) may seem to be strange, because we did not take into account the well-known fact that the Dirac functions are linearly independent. But this fact is taken into account in the framework of the weak asymptotics method (see Lemma 2, Section 5). Here we do not want to mix the substitution of the ansatz into the equation and the analysis of the results of this substitution (see Section 4).

Now we consider the second equation of phase field system, i.e., the Allen–Cahn equation. By Definition 2 (see formula (6)), we must calculate the weak asymptotics of the expression

$$\mathcal{F} \overset{\text{def}}{=} \frac{\partial \hat{u}}{\partial x} \left[ \varepsilon \mathcal{L} \hat{u} - \frac{\hat{u} - \hat{u}^3}{\varepsilon} - \varepsilon \hat{\vartheta} \right],$$

where the function $\hat{\vartheta}$ is determined by the relation (20). Calculating similarly as in Example 2 above (see Section 5 for details), we obtain

$$\mathcal{F} = V_1^1 \delta(x - \varphi_1) + V_2^1 \delta(x - \varphi_2)$$

$$+ V_1^2 \delta'(x - \varphi_1) + V_2^2 \delta'(x - \varphi_2) + O_D(\varepsilon),$$

where $V_j^i$, $i, j = 1, 2$, are linear combinations of several convolutions. Their expressions will be given below. Here we note the following. It is rather clear
that the expression
\[ \frac{\varepsilon}{2} \int_{\mathbb{R}^1} (\tilde{u}_x)^2 \zeta_x \, dx \]
leads in the singular part to functions of the form \( \delta'(x - \varphi_i) \).

Indeed, according to our construction, the expression \( \varepsilon \tilde{u}_x \) is approximately
the function \( \text{sign}_\varepsilon z \)\( ^\prime \), where \( \text{sign}_\varepsilon \) is the regularization of the function \( \text{sign} \), i.e., a soliton-type function. It does not change its structure when squared, while its derivative divided by \( \varepsilon \) becomes the derivative of \( \delta \), i.e., \( \delta' \).

Similarly, the term
\[ \int_{\mathbb{R}^1} \frac{\partial \tilde{u}}{\partial x} (\tilde{u} - \tilde{u}^3) \zeta \, dx \]
is transformed as
\[ \int_{\mathbb{R}^1} \frac{\partial \tilde{u}}{\partial x} (\tilde{u} - \tilde{u}^3) \zeta \, dx = \int_{\mathbb{R}^1} \frac{\partial}{\partial x} F(\tilde{u}) \zeta \, dx = - \int_{\mathbb{R}^1} F(\tilde{u}) \zeta_x \, dx. \]

Similar arguments show that the singular expression itself in the asymptotics of the last integral is also of the form \( \delta' \).

We denote
\[ \Omega(z, \eta) = \frac{1}{2} \{1 + \omega_0(z) + \omega_0(-z - \eta) - \omega_0(z) \omega_0(-z - \eta)\} . \tag{33} \]

We have the following estimates:
\[ \Omega(z, \eta) = 1 + f(z, \eta)e^{2\eta}, \quad \eta \to -\infty, \tag{34} \]
\[ \Omega(z, \eta) = \omega_0(z) + f_1(z, \eta)e^{-2\eta}, \quad \eta \to \infty, \tag{35} \]
where
\[ \int |f(z, \eta)| \, dz \leq \text{const}, \quad \int |f_1(z, \eta)| \, dz \leq \text{const}. \]

These relations readily follow from the explicit form of the function \( \omega_0(z) \),
\[ \omega_0(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}. \]

In fact, relations (34) and (35) express the above-described properties of the ansatz \( \tilde{u} \) (11) in different terms.

Using the technique of the weak asymptotics method (see Lemma 7 in Section 5), we can write the weak asymptotics of the expression \( \int_{\mathbb{R}^1} \zeta \tilde{u}_t \tilde{u}_x \, dx \) in the form
\[ \int_{\mathbb{R}^1} \zeta \tilde{u}_t \tilde{u}_x \, dx = \sum_{i=1}^{2} \varphi_{it} \int \Omega'_z (\Omega'_z - \Omega'_\eta) 
\frac{\beta_x}{\beta_z} \int \Omega'_z (\Omega'_z - \Omega'_\eta) 
\frac{1}{\beta_z} \int \Omega'_z (\Omega'_z - \Omega'_\eta) + \mathcal{O}(\varepsilon). \tag{36} \]
Similarly, we obtain (see Lemmas 5 and 7, Section 5):

\[
\int_{\mathbb{R}^1} \theta \tilde{u}_x \zeta \, dx = \frac{1}{2} \left( \varphi_{1t} + (\dot{q} + \ddot{q}) \big|_{x=\varphi_1} \right) \int_{\mathbb{R}^1} \hat{\Omega}_\eta(z, \eta) \zeta(\varphi_1) \, dz \\
- \frac{1}{2} \left( -\varphi_{2t} + (\dot{q} + \ddot{q}) \big|_{x=\varphi_2} \right) \int_{\mathbb{R}^1} \hat{\Omega}_\eta(z, \eta) \zeta(\varphi_2) \, dz + \mathcal{O}(\varepsilon^\mu), \quad \mu \in (0, 1/2).
\]

(37)

Thus, adding expressions (36) and (37), we see that the coefficients of the \( \delta \)-functions in formula (32) have the form

\[
V_i^1 = -\left[ \varphi_{it}B_\Omega + \frac{\beta_\tau \psi_0}{\beta^2}B^z_\Omega \right] + (-1)^i(\varphi_{it} + (eI + q)|_{x=\varphi_i})C_\Omega, \quad i = 1, 2, \quad (38)
\]

where

\[
B_\Omega = \int \Omega'_z(\Omega'_z - \Omega'_\eta) \, dz,
\]

\[
B^z_\Omega = \int [z(\Omega'_z - \Omega'_\eta) - (z + \eta)\Omega'_\eta](\Omega'_z - \Omega'_\eta) \, dz, \quad (39)
\]

\[
C_\Omega = \int (\Omega'_z - \Omega'_\eta) \, dz, \quad q = \dot{q} + \ddot{q}.
\]

Similarly, we obtain (see Lemma 6, Section 5):

\[
V_i^2 = \beta \hat{C} - \frac{1}{\beta} \hat{D}, \quad (40)
\]

where

\[
\hat{C} = \frac{1}{4} \int_{\mathbb{R}^1} (\Omega'_z)^2 \, dz \quad (41)
\]

\[
\hat{D} = \frac{1}{2} \int_{\mathbb{R}^1} F(\Omega) \, dz \quad (42)
\]

Thus, to obtain \( \beta = \beta(\eta) \) we have the equation (obviously, this is a necessary condition for the relation \( \mathcal{F} = \mathcal{O}(\varepsilon) \) to hold, see (32)):

\[
\beta^2 = \frac{\hat{D}}{\hat{C}} \quad (43)
\]

According to (41) and (42), the right-hand side of the relation (42) is positive.

So we have proved the following assertion.
Theorem 2. Suppose that the assumptions of Theorem 1 and (43) are satisfied and
\[\sum_{i=1}^{2} V_1^i \delta(x - \varphi_1) + V_2^1 \delta(x - \varphi_2) = O_D(\varepsilon).\] (44)

Then the pair of functions \(\tilde{\theta}, \tilde{u}\) is a weak asymptotic solution of the phase field system \(\mathbf{(1)}\) in the sense of Definition 2.

Thus, under the assumption that the classical solution of the phase field system exists (see (i) and (ii) above), relations (27), (28), (43), and (44) are sufficient conditions for constructing a weak asymptotic solution of system (1). In what follows, we prove that the relations mentioned above are equations for determining the functions \(\beta = \beta(\tau)\) and \(\varphi_i(\tau), i = 1, 2\). We present an algorithm for solving these equations.

4 An analysis of the relations obtained

We begin with relation (28) and, for a while, forget everything said about the notion of linear independence. Then, for this relation to hold, it suffices to have the two relations
\[B[\gamma_i^+ + \gamma_i^-] = A_i, \quad i = 1, 2.\] (45)

In view of (18), we have \(\gamma_i^+ + \gamma_i^- = \left[\frac{\partial \tilde{\theta}}{\partial x}\right]_{x = \hat{\varphi}_i(t)}\). Taking this into account and adding relations (45), we obtain
\[\psi_0' \left(\rho_t - \frac{1}{2} \int_{\mathbb{R}^1} \dot{\omega}(z) \omega_0(-z - \eta) \left(\rho_t - 2\frac{2\beta}{\beta}z\right) dz\right) = B(\tau) \left\{\left[\frac{\partial \tilde{\theta}}{\partial x}\right]_{x = \hat{\varphi}_1(t)} + \left[\frac{\partial \tilde{\theta}}{\partial x}\right]_{x = \hat{\varphi}_2(t)}\right\} = 2B\psi_0'.\] (46)

Here we used the relation
\[\psi_t = \varphi_{21} - \varphi_{11} = \psi_0' \rho_t + \psi_0(\varphi_{21} - \varphi_{11})t,\]
which follows from the definition of the function \(\rho\), see (25).

We set the last term to be zero, since we show below that \((\varphi_{21} - \varphi_{11}) = O(|\tau|^{-1})\). In view of Lemma 3 in Section 5, in this case we have \((\varphi_{21} - \varphi_{11})t\psi_0 = O(\varepsilon)\). Moreover, in view of the Stefan conditions and the choice of the continuation of the functions \(\gamma_i^\pm\) and \(\varphi_0(t)\), we have
\[\gamma_1^+ + \gamma_2^+ + \gamma_1^- + \gamma_2^- = \left[\frac{\partial \tilde{\theta}}{\partial x}\right]_{x = \hat{\varphi}_1(t)} + \left[\frac{\partial \tilde{\theta}}{\partial x}\right]_{x = \hat{\varphi}_2(t)} = 2\psi_0'.\]
We transform the left-hand side of relation (02):

\[
I \overset{\text{def}}{=} \psi_0 t \left[ \rho_r - \frac{1}{2} \int_{\mathbb{R}^1} \dot{\omega}_0(z) \omega_0(-z - \eta) \left( \rho_r - \frac{2 \beta_r}{\beta^2} z \right) dz \right]
\]

\[
= \psi_0 t \left[ \rho_r + \frac{1}{2} \int_{\mathbb{R}^1} \dot{\omega}_0(z) \omega_0(-z - \eta) \frac{\beta_r}{\beta^2} z dz - \frac{1}{2} \int_{\mathbb{R}^1} \dot{\omega}_0(z) \omega_0(-z - \eta) \left( \rho_r + \frac{\beta_r}{\beta^2} (z - \eta) + \frac{\beta_r}{\beta^2} \eta \right) \right].
\]

Next, we change the variables in the last integral

\[
\frac{\beta_r}{\beta^2} \int_{\mathbb{R}^1} (-z - \eta) \dot{\omega}_0(z) \omega_0(-z - \eta) dz = -\frac{\beta_r}{\beta^2} \int_{\mathbb{R}^1} z \dot{\omega}_0(-z - \eta) \omega_0(z) dz
\]

and note that

\[
\rho_r - \frac{\beta_r}{\beta^2} \eta = \frac{1}{\beta} \frac{\partial}{\partial \eta} \eta.
\]

Finally, we get

\[
I = \psi_0 t \left[ \rho_r - \frac{1}{2} \int_{\mathbb{R}^1} \left\{ z \dot{\omega}_0(z) \omega_0(-z - \eta) + z \dot{\omega}_0(-z - \eta) \omega_0(z) \right\} dz - \frac{1}{2 \beta} \frac{\partial}{\partial \eta} \int_{\mathbb{R}^1} \dot{\omega}_0(z) \omega_0(-z - \eta) dz \right].
\]

Since the function \( \omega_0(z) \) is odd, we see that the expression in braces in the first integral in the right-hand side is

\[
-\frac{\partial}{\partial z} (1 - \omega_0(z) \omega_0(z + \eta)).
\]

Hence, integrating by parts, we obtain

\[
I = \psi_0 t \left[ \rho_r - \frac{\beta_r}{2 \beta^2} \int_{\mathbb{R}^1} (1 - \omega_0(z) \omega_0(z + \eta)) dz - \frac{1}{2 \beta} \frac{\partial}{\partial \eta} \int_{\mathbb{R}^1} \dot{\omega}_0(z + \eta) \omega_0(z) dz \right].
\]

Or, finally,

\[
I = \psi_0 t \left[ \rho_r + \frac{1}{\beta} \frac{\partial}{\partial \eta} \tilde{B} \right],
\]

where

\[
\tilde{B}(\eta) = \int_{\mathbb{R}^1} (1 - \omega_0(z + \eta) \omega_0(z)) dz = 2 \eta \tanh \eta.
\]

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So, in view of (46), we obtain the equation for $\rho$:

$$\frac{\partial}{\partial \tau} \left( \rho + \frac{1}{2\beta} \tilde{B}(\eta) \right) = (\tau). \quad (48)$$

Or

$$\frac{\partial}{\partial \tau} (\beta^{-1}(\eta + \eta \tanh \eta)) = 2B. \quad (49)$$

After integration, we obtain

$$\eta(1 + \tanh \eta) = 2\beta \int_{-\infty}^{\tau} B(\tau') d\tau'. \quad (50)$$

Since $B(\tau) \to 1$ as $\tau \to \infty$, we have $\eta^{-1} \to 1$ as $\tau \to \infty$.

Since $B(\tau) = O(|\tau|^{-N})$ for any $N$ as $\tau \to -\infty$, the integral in the left-hand side of (49) converges and hence the right-hand side tends to the limit

$$\lim_{\tau \to -\infty} \eta(1 + \tanh \eta) = \lim_{\tau \to -\infty} 2\beta \int_{-\infty}^{\tau} B(\tau') d\tau' = 0.$$

Moreover, in view of the inequality $B \geq 0$, we have $\eta \geq 0$ for $\tau \in \mathbb{R}^2$, which implies that

$$\eta \to 0 \quad \text{as} \quad \tau \to -\infty.$$

Here we took into account the inequality $\beta > 0$, which follows from (43).

Substituting $\beta = (\hat{C} \hat{D}^{-1}(\eta))^{1/2}$ into (49), we obtain the following equation for the function $\eta$:

$$\eta(1 + \tanh \eta) = (\hat{C} \hat{D}^{-1})^{1/2} \int_{0}^{\tau} B(\tau') d\tau'. \quad (51)$$

Since the functions contained in (50) are monotone and the limits exist as $\tau \to \pm \infty$, this equation is obviously solvable.

Next, we have

$$\beta(\tau) = \sqrt{\frac{\hat{C}(\eta)}{\hat{D}(\eta)}} \to \beta^- = \text{const}, \quad \tau \to -\infty.$$

Moreover, as readily follows from the exponential rate of convergence to the limit of the functions $\hat{C}/\hat{D}, 1 - B$, and $1 + \tanh \eta$ as $\tau \to \pm \infty$, the derivatives satisfy the estimate

$$\frac{\partial^\alpha \beta}{\partial \tau^\alpha} = O(|\tau|^{-N}), \quad |\tau| \to \infty. \quad (51)$$
Now we calculate the limits of the expressions

$$J_i = B(\gamma_i^+ + \gamma_i^-) - A_i, \quad i = 1, 2,$$

as $\tau \to \pm \infty$. As $\tau \to \infty$, we have

$$\rho \to \infty \quad (\rho \sim \tau), \quad B \to 1, \quad A_i \to (-1)^{i+1}\varphi_{\Omega i}.$$

Therefore, we have

$$\lim_{\tau \to \infty} J_i = 0$$

in view of the Stefan conditions (9), and $J_i = \mathcal{O}(\tau^{-1})$, $i = 1, 2$.

As $\tau \to -\infty$, we proved that $\eta \to 0$ and hence

$$\varphi_1 - \varphi_2 = \mathcal{O}(\varepsilon), \quad \tau \to -\infty. \quad (52)$$

**Theorem 3.** Relation (50) is a sufficient condition for relation (28) to hold.

*Proof.* In view of the corollary of Lemma 2, Section 5, the estimate for the coefficients of the $\delta$-functions in (28) as $\tau \to \infty$ and (52) are sufficient conditions for relation (28) to follow from (46). In turn, relation (50) follows from (46).

Now we analyze the relations $V_i^2 = 0$ and (44). We use the explicit form of the function $F(z)$ and the formula

$$\omega_0 = \frac{e^z - e^{-z}}{e^z + e^{-z}}. \quad (53)$$

As $\eta \to \infty$, we have

$$\Omega(z, \eta) = \omega_0(z) + \mathcal{O}(e^{-2\eta}) \quad (54)$$

and hence $\beta \to 1$ as $\eta \to \infty$.

From relation (33) we easily obtain

$$\Omega'_z(z, \eta) = \mathcal{O}(e^{-|z|}), \quad |z| \to \infty,$$

$$\Omega'_\eta(z, \eta) = \mathcal{O}(e^{-|z|}), \quad |z| \to \infty,$$

$$C_\Omega = \int_{\mathbb{R}^1} \dot{\omega}_0(z)(1 - \omega_0(-z - \eta)) \, dz \geq 0.$$
Similarly to (54), using (45), we can verify the relations
\[
\beta_r = O(e^{-2|\eta|}), \quad \eta \to \infty,
\]
\[
B_{\Omega} = 1 + O(e^{-2|\eta|}), \quad \eta \to \infty,
\]
\[
C_{\Omega} = 4 + O(e^{-2\eta}), \quad \eta \to \infty,
\]
\[
B_{\tilde{\Omega}} = 0, \quad \eta \to \infty.
\]

We rewrite the expressions for the coefficients \(V^1_i\) in more detail:
\[
V^1_1 = -\varphi_1t(B_{\Omega} + C_{\Omega}) + \frac{\beta_r \psi_0'}{\beta^2} B^z_{\Omega} - q|_{x=\varphi_1} C_{\Omega},
\]
\[
V^2_1 = -\varphi_2t(B_{\Omega} + C_{\Omega}) + \frac{\beta_r \psi_0'}{\beta^2} B^z_{\Omega} + q|_{x=\varphi_2} C_{\Omega}.
\]

Form formulas (57), the definition of the functions \(\varphi_i\), and the kinetic overcooling conditions (9), we obtain
\[
\lim_{\tau \to \infty} V^1_i = 0,
\]
because \(\hat{q}|_{x=\varphi_i} = 0\), while \(\hat{q} \equiv 0\) for \(t < t^*\), i.e., because of the fact that the model function \(\bar{T} + \epsilon I\) is constructed so that conditions (8) be satisfied. Thus, in view of the corollary of Lemma 2, Section 5, for the relation
\[
V^1_1 \delta(x - \varphi_1) + V^2_1 \delta(x - \varphi) = O_{D'}(\epsilon)
\]
to hold, it is sufficient to have \(V^1_1 + V^2_2 = 0\) or, in more detail,
\[
(\varphi_{1t} + \varphi_{2t})(B_{\Omega} + C_{\Omega}) - \frac{\beta_r \psi_0'}{\beta^2} B^z_{\Omega} + (q|_{x=\varphi_1} - q|_{x=\varphi_2}) C_{\Omega} = 0. \tag{56}
\]
Thus, we have proved the following assertion.

**Theorem 4.** The conditions of Theorem 1 and relations (49), (55), (56) are sufficient for the functions \(\bar{\theta}, \bar{u}\) to be a weak asymptotic solution of the phase field system.

We consider relation (56). It follows from the above that \(C_{\Omega} = C_{\Omega}(\eta)\) decreases sufficiently fast as \(\eta \to \infty\), in any case \(|\eta| C_{\Omega}(\eta)| \leq \text{const}\). This fact, the estimate
\[
\hat{q}|_{x=\varphi_1} - \hat{q}|_{x=\varphi_2} = O(|\varphi_1 - \varphi_2|^{\mu}), \quad \mu \in (0, 1/2), \quad t \leq t^*.
\]
proved in Lemma 4 in Section 5, relation (52) implies that the obvious estimate
\[
||\varphi_1 - \varphi_2||^{\mu} C_{\Omega} \leq \epsilon \text{const} |\eta C_{\Omega}| = O(\epsilon^{\mu})
\]
for \( t \in [0, t_1] \).

For \( t \leq t^* \), by Lemma 6, we have

\[
q \big|_{x=\varphi_1} - q \big|_{x=\varphi_{10}} = \mathcal{O}(|\psi_0\psi_1|^\mu),
\]

where \( \psi_1 = \varphi_{21} - \varphi_{11} \). Hence, by (15), and a statement similar to Lemma 3 in Section 5 (see also (17)), we have

\[
2(\varphi_{10t} + \varphi_{20t}) - q|_{x=\varphi_1} + q|_{x=\varphi_2} = \mathcal{O}(\varepsilon\mu), \quad t \leq t^*.
\]

We introduce a function \( V(\tau) \in C^\infty \) such that \( V' \in S(\mathbb{R}^1) \), \( V(-\infty) = 0 \), \( V(\infty) = 1 \). Then, in view of considerations similar to those used in Lemma 3, we can show that the following estimate holds:

\[
(B_\Omega + C_\Omega)(V(\tau)(\varphi_{10t} + \varphi_{10t}) + (q|_{x=\varphi_2} - q|_{x=\varphi_1})C_\Omega = \mathcal{O}(\varepsilon\mu), \quad \mu \in (0, 1/2),
\]

for \( t \leq t^* \).

This implies that the left-hand side of (32) is estimated as \( \mathcal{O}_D(\varepsilon\mu) \) if

\[
(B_\Omega + C_\Omega)[-V(\varphi_{10t} + \varphi_{20t}) + \varphi_{1t} + \varphi_{2t}] - 2\frac{\beta't\psi'_{0t}}{\beta^2} B_\Omega^2 = 0.
\]

(57)

From this relation we obtain

\[
\frac{\partial}{\partial \tau}(\tau(\varphi_{11} + \varphi_{21})) = (V - 1)(\varphi_{10t} + \varphi_{20t}) + 2\frac{\beta't\psi'_{0t}}{\beta^2} \frac{B_\Omega^2}{B_\Omega + C_\Omega}.
\]

(58)

From this equation we determine the extensions of the functions \( \varphi_{11} \) and \( \varphi_{21} \). We note that our argument results in an equation that does not contain the temperature, namely, the assumption that the classical solution exists till \( t = t^* \) is sufficient for constructing a global smooth approximation of the solution.

Let us calculate the function \( B_\Omega = B_\Omega(\eta) \) in more detail. We have

\[
B_\Omega(\eta) = \int \Omega'_z(\Omega'_z - \Omega'_\eta) \, dz.
\]

In this integral, we make the change of variable \( z \rightarrow -z - \eta \). Then we obtain

\[
\Omega'_\eta \rightarrow -(\Omega'_z - \Omega'_\eta), \quad (\Omega'_z - \Omega'_\eta) \rightarrow -\Omega'_z.
\]

Hence we have

\[
B_\Omega(\eta) = \int \Omega'_z\Omega'_\eta \, dz.
\]

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Finally, we obtain

$$B_{\Omega} = \frac{1}{2} \int (\Omega'_{\Omega} - \Omega'_{\eta}) + \Omega'_{\Omega} \Omega'_{\eta} \, dz = \frac{1}{2} \int (\Omega'_{\eta})^2 \, dz.$$ 

Thus, for finite \(\eta\), the denominator \((B_{\Omega} + C_{\Omega})\) in the last term in the right-hand side of (58) does not vanish. Moreover, using the explicit form of the functions \(\omega_0(z), B_{\Omega}^z, B_{\Omega},\) and \(C_{\Omega}\), we can verify that

$$2 \frac{\beta' B_{\Omega}^z}{\beta^2} = \mathcal{O}(|\eta|^{-N}), \quad \tau \to \infty,$$

where \(N \gg 1\) is an arbitrary number.

Therefore, we have

$$\varphi_{11} + \varphi_{12} = \frac{\varphi_{10t} + \varphi_{20t}}{\tau} \int_{0}^{\tau} (V(\tau') - 1) \, d\tau' + \frac{2\psi_{0t}}{\tau} \int_{0}^{\tau} \frac{\beta' B_{\Omega}^z}{\beta^2 (B_{\Omega}^z + C_{\Omega})} \, d\tau'.$$

(59)

We note that

$$\psi_{0t}' \left( 1 + \frac{\partial}{\partial \tau} [\tau (\varphi_{21} - \varphi_{11})] \right) = \psi_{0t}' \rho_{r},$$

(60)

where \(\rho = \eta \beta^{-1}\) and the functions \(\eta\) and \(\beta\) are determined by Eqs. (50) and (43).

The system of Eqs. (61) and (62) allows one to find the functions \(\varphi_{i1}, \varphi_{i2}, i=1,2\), and thus completely determines the functions contained in the ansatz of the weak asymptotic solution of system (1).

It is easy to see that the solutions thus constructed satisfy conditions (15).

Next, using the explicit formulas for \(B_{\Omega}^z, B_{\Omega},\) and \(C_{\Omega}\), we can easily verify that \(B_{\Omega}^z|_{\eta=0} = 0\) (because \(\omega_0(z)\) is odd) and \((B_{\Omega} + C_{\Omega})|_{\eta=0} \neq 0\), because \(\dot{\omega}_0(z)\) is even. This means that

$$((\varphi_{1t} + \varphi_{2t})|_{\eta=0} = 0.$$ 

(61)

We note that the right-hand side of (27) is a piecewise smooth function for \(|\psi| \geq \text{const} > 0\) and smooth for \(x \neq \varphi_i, i = 1,2\). As \(\psi \to 0 (t \to t^*)\), the right-hand side of (27) becomes proportional to \(\delta(x - x^*)\), where \(x_0^* = x^*|_{\psi=0}\), and the proportionality coefficient is negative. Thus, if \(x\) and \(t\) vary in the respective neighborhoods of the points \(x_0^*\) and \(t^*\), then the \(\delta\)-function with a negative coefficient appears and disappears in the right-hand side of (27), which results in a negative soliton-like jump of the temperature in a neighborhood of the point \(t = t^*, x = x_0^*\).
Calculation of the temperature jump. To prove the statement in the Introduction concerning the temperature jump, we must calculate the quantity

\[ [eI + \hat{q} + \hat{q}]|_{t^*, x=x^*}. \]

Let us verify that \( \hat{q} + \hat{q} \) is a continuous function. This function is the sum of solutions of the heat equation with singular right-hand sides, but these singularities arise after the substitution of the continuous function \( e\bar{T} \) (which, of course, is not a solution) into the left-hand side. It remains to prove that the solution differs from the function \( e\bar{T} \) by a continuous function. For this, it suffices to verify that the right-hand sides, which arise after the substitution of \( e\bar{T} \) into the equation, do not generate any singularities in the solution of the heat equation in addition to those contained in the function \( e\bar{T} \).

Here we, in contrast to the preceding statements, use the fact that the singularities in the right-hand sides arise as the result of the substitution.

By \( \hat{q}_i, i = 1, 2 \), we denote the terms in \( \hat{q} + \hat{q} \) corresponding to the right-hand sides:

\[
\begin{align*}
  f_1 &= -\left( \frac{\partial^2}{\partial x^2} \gamma^-(\varphi_1 - x)(x - \varphi_2) \right) B(\tau)[H(x - \varphi_1) - H(x - \varphi_2)], \\
  f_2 &= -\left( \frac{\partial^2}{\partial x^2} \gamma^-(\varphi_1 - x)(x - \varphi_2) \right) (1 - B(\tau))[H(x - \varphi_2) - H(x - \varphi_1)],
\end{align*}
\]

The other terms in \( q \) are solutions of the heat equation with piecewise continuous right-hand side and hence are continuous.

The functions \( \hat{q}_i \) have a similar property if they are calculated up to \( O(\varepsilon) \). Indeed, we denote

\[ \Pi = \gamma^-(\varphi_1 - x)(x - \varphi_2) \]

and represent, for example, the function \( \hat{q}_1 \) in the form

\[ \hat{q}_1 = -\frac{1}{2\sqrt{2\pi}} \int_0^t B(\tau(t', \varepsilon)) \frac{\varphi_2}{\sqrt{t - t'}} \int_{\varphi_1}^{\varphi_2} \frac{\partial^2 \Pi}{\partial \xi^2} e^{-\frac{(x - \xi)^2}{4(t - t')}} d\xi dt'. \]

Omitting the number factor and integrating by parts in the integral over \( \xi \), we obtain

\[
\begin{align*}
  \hat{q}_1 &= -\int_0^t \frac{B}{\sqrt{t - t'}} \left[ e^{\frac{(x - \xi)^2}{4(t - t')}} \gamma_2^- \right] \varphi_2 |_{\varphi_1}^{\varphi_2} - \int_0^t \frac{B}{\sqrt{t - t'}} \frac{\partial^2 \Pi}{\partial \xi^2} e^{-\frac{(x - \xi)^2}{4(t - t')}} d\xi dt' \\
  &= -\int_0^t \frac{B}{\sqrt{t - t'}} \left[ e^{\frac{(x - \xi)^2}{4(t - t')}} \gamma_2^- - e^{\frac{(x - \xi)^2}{4(t - t')}} \gamma_1^+ \right] dt' \\
  &\quad + \int_0^t \frac{\partial}{\partial t'} \left( \int_{\varphi_1}^{\varphi_2} \frac{\Pi e^{\frac{(x - \xi)^2}{4(t - t')}}}{\sqrt{t - t'}} d\xi \right) dt' + \int_0^t \left( \int_{\varphi_1}^{\varphi_2} \frac{B}{\sqrt{t - t'}} \frac{\partial \Pi e^{\frac{(x - \xi)^2}{4(t - t')}}}{\partial t'} d\xi \right) dt'.
\end{align*}
\]
The last term is a solution of the heat equation with the piecewise continuous right-hand side

\[ B \frac{\partial \Pi}{\partial t} [H(x - \varphi_1) - H(x - \varphi_2)], \]

and hence it is continuous uniformly w.r.t \( \varepsilon \geq 0 \). The other terms are also continuous functions uniformly w.r.t \( \varepsilon \geq 0 \).

It is easy to see that in the formula for \( \bar{q}_1 \) there is no term containing the derivative \( \frac{\partial B}{\partial t} \), because this term would be of order \( O(\varepsilon) \).

Indeed

\[ \left| \frac{\partial B}{\partial t} \Pi[H(x - \varphi_1) - H(x - \varphi_2)] \right| \leq C |B'| \cdot \psi'_0 \cdot \rho, \]

since \( |\Pi| \leq C\psi \) for \( x \in [\varphi_1, \varphi_2] \) and \( \varepsilon^{-1}\psi = \rho \). The derivative \( B'_\rho \) decreases faster than any power of \( |\rho|^{-1} \), \( |\rho'_\tau| \leq \text{const}. \) This implies that

\[ \int_0^t \frac{\partial B}{\partial t} dt' = O(\varepsilon). \]

Hence we have

\[ [\bar{q} + \hat{q}]|_{x = x^*, t = t^*} = 0. \]

Let us calculate the function \( eI \). We have

\[ eI|_{x = x^*} = \frac{1}{2}(\varphi_{1t} - \varphi_{2t}) = \frac{\psi'_0}{2} \hat{\rho}_\tau. \]

It follows from (48) that

\[ \hat{\rho}_\tau \to 1, \quad \tau \to \infty, \]
\[ \hat{\rho}_\tau \to 0, \quad \tau \to -\infty \]

and hence

\[ [eI]|_{x = x^*, t = t^*} = -\frac{1}{2} \lim_{t \to -t^*} (\varphi_{10t} + \varphi_{20t}). \]

## 5 Technique of the weak asymptotics method

First, we recall the definition of \textit{regularization of the generalized function}. 


Definition 3. A family of functions $f(x, \varepsilon)$ smooth for $\varepsilon > 0$ and satisfying the condition

$$\lim_{\varepsilon \to 0} w f(x, \varepsilon) = f(x)$$

is called the regularization of the generalized function $f(x)$.

We note that, by definition, the last relation can be rewritten as

$$\lim_{\varepsilon \to 0} \langle f(x, \varepsilon), \zeta(x) \rangle = \langle f, \zeta \rangle$$

for any test function $\zeta(x)$ (from now on, $\langle \cdot, \cdot \rangle$ denotes the action of a generalized function on a test function).

Lemma 1. Let $\Gamma_t = \{ x - \varphi(t) = 0 \}, x \in \mathbb{R}$, where $\varphi(t)$ is a smooth function, let $\omega(z) \in \mathcal{S}$ ($\mathcal{S}$ is the Schwartz space), and let $\beta = \beta(t) > 0$. Then the following relation holds for any test function $\zeta(x)$:

$$\frac{1}{\varepsilon} \langle \omega \left( \frac{\beta x - \varphi(t)}{\varepsilon} \right), \zeta(x) \rangle = \frac{1}{\beta} A_\omega \zeta(\varphi) + \mathcal{O}(\varepsilon),$$

where $A_\omega = \int_{-\infty}^{\infty} \omega(z) dz$.

Proof. The expression in the right-hand side can be written as

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^1} \omega \left( \frac{\beta x - \varphi(t)}{\varepsilon} \right) \zeta(x) dx = \frac{\zeta(\varphi)}{\beta} \int_{\mathbb{R}^1} \omega(z) dz + \mathcal{O}(\varepsilon).$$

Here we perform the change of variables $z = \beta(x - \varphi)/\varepsilon$ and apply the Taylor formula to the integrand at the point $x = \varphi$. By definition, the last integral is the action of the generalized function $\beta^{-1} A_\omega \delta(x - \varphi)$ on the test function $\zeta$.

Proof of the formula in Example 1 in the Introduction. Let $\omega(z) \in C^\infty$, $\omega' \in \mathcal{S}(\mathbb{R})$, $\lim_{z \to +\infty} \omega(z) = 0$, and $\lim_{z \to -\infty} \omega(z) = 1$. We verify that

$$\omega \left( \frac{x - x_0}{\varepsilon} \right) - H(x - x_0) = \mathcal{O}(\varepsilon), \quad x_0 = \text{const},$$

where $H$ is the Heaviside function. By Definition we consider the expression

$$\int_{\mathbb{R}^1} \left[ \omega \left( \frac{x - x_0}{\varepsilon} \right) - H(x - x_0) \right] \zeta(x) dx = \varepsilon \zeta(x_0) \int_{\mathbb{R}^1} [\omega(z) - H(z)] \, dz + \mathcal{O}(\varepsilon^2).$$
Here we performed the change of variables \( z = (x - x_0)/\varepsilon \) and applied the Taylor formula to the functions \( \zeta(x) \) at the point \( x = x_0 \). In view of our assumptions and the properties of the function \( H \), the integral in the right-hand side of the last relation converges and hence the right-hand side of \( (63) \) is of order \( O(\varepsilon) \). We thus obtain estimate \( (62) \).

**Proof of the formula in Example 2 in the Introduction.** Let \( \omega_i(z) \in C^\infty, (\omega_i)' \in \mathcal{S}(\mathbb{R}^1), \lim_{z \to +\infty} \omega_i(z) = 0, \) and \( \lim_{z \to -\infty} \omega_i(z) = 1, i = 1, 2 \). We consider the integral

\[
J \overset{\text{def}}{=} \int_{\mathbb{R}^1} \omega_1 \left( \frac{x - x_1}{\varepsilon} \right) \omega_2 \left( \frac{x - x_2}{\varepsilon} \right) \zeta(x) \, dx
\]

Integrating by parts, we obtain

\[
J = - \int_{\mathbb{R}^1} \omega_1 \left( \frac{x - x_1}{\varepsilon} \right) \omega_2 \left( \frac{x - x_2}{\varepsilon} \right) \left( \int_{-\infty}^x \zeta(y) \, dy \right)' 
\]

We perform the change of variables \( z = (x - x_i)/\varepsilon \) with \( i = 1 \) in the first integral and with \( i = 2 \) in the second integral and apply the Taylor formula to the function \( F(x) = \int_{-\infty}^x \zeta(y) \, dy \) at the points \( x = x_i, i = 1, 2 \), respectively. Then we calculate the first and the second integral, we have

\[
J = B_1 \left( \frac{\Delta x}{\varepsilon} \right) H(x - x_1) + B_2 \left( \frac{\Delta x}{\varepsilon} \right) H(x - x_2) + O_{\mathcal{D}'}(\varepsilon), \quad \Delta x = x_1 - x_2,
\]

\[
B_1(\rho) = \int_{\mathbb{R}^1} \omega_1(z) \omega_1(-z - \rho) \, dz, \quad B_2(\rho) = \int_{\mathbb{R}^1} \omega_2(z) \omega_1(z - \rho) \, dz.
\]

To calculate the linear combination of generalized functions up to \( O_{\mathcal{D}'}(\varepsilon^\alpha) \), we must improve the classical definition of linear independence. This improvement plays the key role in the study of problems with interaction of nonlinear waves.

Indeed, let \( \phi_1 \neq \phi_2 \) be independent of \( x \). We consider the expression

\[
g_1 \delta(x - \phi_1) + g_2 \delta(x - \phi_2) = O_{\mathcal{D}'}(\varepsilon^\alpha), \quad \alpha > 0,
\]

where the functions \( g_i \) are independent of \( \varepsilon \). Clearly, for the last relation to be satisfied, it suffices to have

\[
g_i = O(\varepsilon^\alpha), \quad i = 1, 2,
\]

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or, with the properties of the functions $g_i$ taken into account,

$$g_i = 0, \quad i = 1, 2.$$  

But, if we assume that the coefficients $g_i$ depend on the parameter $\varepsilon$, then the above estimates do not work. Namely, let us consider the following specific case of this dependence:

$$g_i = A_i + S_i(\Delta \phi / \varepsilon), \quad i = 1, 2, \quad (65)$$

where $A_i$ are independent of $\varepsilon$, the functions $S_i(\sigma)$ decrease sufficiently fast as $|\sigma| \to \infty$, and $\Delta \phi = \phi_2 - \phi_1$.

**Lemma 2** (Linear independence of generalized functions). Suppose that the estimate

$$|\sigma S_i(\sigma)| \leq \text{const}, \quad i = 1, 2, \quad -\infty < \sigma < +\infty$$

holds. Then, for $\alpha = 1$, expression (64) implies the relations

$$A_1 = 0, \quad A_2 = 0, \quad S_1 + S_2 = 0. \quad (66)$$

**Proof.** Using the Taylor formula in (64) and taking (65) into account, we obtain

$$S_1 \zeta(\phi_1) + S_2 \zeta(\phi_2) = S_1 \zeta(\phi_1) + S_2 \zeta(\phi_1) + S_2(\phi_2 - \phi_1) \zeta'(\phi_1 + \mu \phi_2),$$

where $0 < \mu < 1$ Since the function $\sigma S_2(\sigma)$ is uniformly bounded in $\sigma \in \mathbb{R}^1$, we obtain

$$S_2(\Delta \phi / \varepsilon)(\phi_2 - \phi_1) = \{-\sigma S_2(\sigma)\}|_{\sigma = \Delta \phi / \varepsilon} \cdot \varepsilon = O(\varepsilon).$$

So expression (64) can be rewritten as

$$A_1 \zeta(\phi_1) + A_2 \zeta(\phi_2) + (S_1 + S_2) \zeta(\phi_1) = O(\varepsilon).$$

Thus, because the coefficients $A_i$ are independent of $\varepsilon$, we obtain the assertion of Lemma 2.

**Corollary 1.** Suppose that

$$|\sigma S_i(\sigma)| \leq \text{const}, \quad i = 1, 2, \quad \sigma \geq 0$$

and the functions $\phi_i = \phi_i(t, \varepsilon)$ are continuous and uniformly continuous and satisfy the condition that $\Delta \phi > 0$ for $t > 0$ and $\varepsilon \geq 0$ and $\Delta \phi = O(\varepsilon)$ for $t < 0$. Then, for $\alpha = 1$, expression (66) implies (64).
Proof. Following the above argument, we obtain

\[ S_1 \zeta(\phi_1) + S_2 \zeta(\phi_2) = (S_1 + S_2) \zeta(\phi_1) + S_2 (\phi_2 - \phi_1) \zeta'(\phi_1 + \mu \phi_2). \]

In view of our assumptions,

\[ S_2(\phi_2 - \phi_1) = O(\varepsilon) \]

uniformly in \( t \), which implies the desired assertion.

\[ \square \]

**Lemma 3.** Suppose that \( f(t) \in C^1 \), \( f(t_0) = 0 \), and \( f'(t_0) \neq 0 \). Suppose also that \( g(t, \tau) \) locally uniformly in \( t \) satisfies the condition

\[ |\tau g(t, \tau)| \leq \text{const}, \quad |\tau g'_t(t, \tau)| \leq \text{const}, \quad -\infty < \tau < \infty, \]

and \( g(t_0, \tau) = 0 \). Then the inequality

\[ |g(t, f(t)/\varepsilon)| \leq \varepsilon C, \]

where \( C_i = \text{const} \), holds in any interval \( 0 \leq t \leq \hat{t} \) that does not contain zeros of the function \( f(t) \) except for \( t_0 \).

**Proof.** The fraction \( \frac{f(t)}{t-t_0} \) is locally bounded in \( t \). The fraction \( \frac{\tau g(t, \tau)}{t-t_0} \) is also locally bounded. We have

\[ g(t, f(t)/\varepsilon) = \varepsilon \cdot \frac{g(t, f(t)/\varepsilon)}{(t-t_0)} \cdot \frac{f(t)}{\varepsilon} \cdot \frac{t-t_0}{f(t)}. \]

According to the assumptions of the lemma, the last factor in the right-hand side is bounded on the interval under study. The product of the first and second factors (without \( \varepsilon \)) is bounded in view of the properties of the function \( g(t, \tau) \).

\[ \square \]

**Corollary 2.** Suppose that the assumptions of Lemma 3 are satisfied for \( 0 \leq \tau < \infty \) \((-\infty < \tau \leq 0) \). Then the assertion of Lemma 3 holds on any half-interval \((t_0, \hat{t})\), which does not contain zeros of the function \( f(t) \), and \( \text{sign} \hat{t} = \text{sign} f(t) \), \( t \in (t_0, \hat{t}) \).

The proof of Corollary 2 is obvious.

**Lemma 4.** The following inequality holds:

\[ \hat{q}|_{x=\varphi_1} - \hat{q}|_{x=\varphi_2} = O(|\varphi_1 - \varphi_2|^\mu). \]
Proof. We consider only one of the the functions \( \hat{q} \) from (29), namely, the functions determined by relation (30), and apply the method developed in [14]. We consider teh difference of expressions (30) omitting the number factors:

\[
\hat{q}|_{x=\varphi_1} - \hat{q}|_{x=\varphi_2} = \int_0^t \int_{\varphi_1}^{\varphi_2} \frac{\exp\left(\frac{(\varphi_1 - \xi)^2}{4(t-t')} - \exp\left(-\frac{(\varphi_2 - \xi)^2}{4(t-t')}\right)\right)}{\psi \sqrt{t-t'}} \, d\xi \, d\tau
\]

\[
= \int_0^t \int_{\varphi_1}^{\varphi_2} \exp\left(-\frac{(\varphi_2 - \xi)^2}{4(t-t')}\right)\left[\exp\left(\frac{(\varphi_2 - \varphi_1)(\varphi_1 + \varphi_2 - 2\xi)}{4(t-t')} - 1\right)ight]
\]

\[
\times \frac{(t-t')^{\mu}}{(\varphi_2 - \varphi_1)^{\mu}} \frac{(\varphi_2 - \varphi_1)^{\mu}}{(t-t')^{\mu}} \, d\xi \, d\tau
\]

\[
= \int_0^t \int_{\varphi_1}^{\varphi_2} \exp\left(-\frac{(\varphi_1 - \xi)^2}{4(t-t')}\right)\left[\exp\left(\frac{(\varphi_1 - \varphi_2)(\varphi_1 + \varphi_2 - \xi)}{4(t-t')} - 1\right)ight]
\]

\[
\times \frac{(t-t')^{\mu}}{(\varphi_2 - \varphi_1)^{\mu}} \frac{(t-t')^{\mu}}{(t-t')^{\mu}} \, d\xi \, dt',
\]

where we choose \( \mu \in (0, 1/2) \).

Next, using the inequality

\[
\left|\frac{e^{-\alpha x} - 1}{x^\mu}\right| \leq \text{const}
\]

for \( \alpha > 0, x \in [0, \infty) \), and taking into account the fact that the integral

\[
\int_0^t (t-t')^{-(1/2+\mu)} \, dt', \quad \mu \in (0, 1/2),
\]

converges, we obtain the statement of the lemma for \( \varphi_2 \geq \varphi_1 \). If \( \varphi_2 \leq \varphi_1 \) (this is true for \( t \geq t^* \)), then we must change the exponents outside the square brackets in the integrands.

\[\Box\]

**Lemma 5.** Suppose that \( \omega(z) \in C^\infty \) decreases faster than any power of \( |z|^{-1} \) as \( |z| \to \infty \). Then

\[
\varepsilon^{-1} \omega((x - \varphi_i)/\varepsilon) \hat{q}_j = \hat{q}_j \delta(x - \varphi_i) \int \omega(z) \, dz + O_{\mathcal{D}^*}(\varepsilon^\mu),
\]

where \( \hat{q}_j \) is one of the functions defined by the relations (29), \( i = 1, 2, \mu \in (0, 1/2) \).
Proof. We consider one of the functions \( \hat{q} \), namely, the function defined by the relations

\[
\hat{q} = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{dt'}{\sqrt{t-t'}} \int_{\phi_1}^{\phi_2} \exp\left(-\frac{(x-\xi)^2}{4(t-t')}\right) d\xi.
\]

The desired relation can be written as

\[
\varepsilon^{-1} \int \zeta(x)\omega\left(\frac{x-\varphi_i}{\varepsilon}\right) \hat{q} \, dx = \zeta(\varphi_i) \hat{q} \big|_{x=\varphi_i} \int \omega(z) \, dz + O(\varepsilon^\mu)
\]

or, omitting the number factors, in the form

\[
I \overset{\text{def}}{=} \varepsilon^{-1} \int \zeta(x)\omega\left(\frac{x-\varphi_i}{\varepsilon}\right) \hat{q} \, dx \tag{67}
\]

\[
\times \int_0^t \frac{dt'}{\sqrt{t-t'}} \int_{\phi_1}^{\phi_2} \exp\left(-\frac{(x-\xi)^2}{4(t-t')}\right) \, d\xi \, dx
\]

\[
= \zeta(\varphi_i) \int_0^t \frac{dt'}{\sqrt{t-t'}} \int_{\phi_1}^{\phi_2} \exp\left(-\frac{(\varphi_i-\xi)^2}{4(t-t')}\right) \, d\xi + O(\varepsilon^\mu).
\]

In the left-hand side, we change the variables \((x-\varphi_i)/\varepsilon = z\) and obtain

\[
I = \int \zeta(\varphi_i + \varepsilon z)\omega(z) \int_0^t \frac{dt'}{\sqrt{t-t'}} \int_{\phi_1}^{\phi_2} \exp\left(-\frac{(\varphi_i-\xi+\varepsilon z)^2}{4(t-t')}\right) \, dz \, d\xi.
\]

It is clear that to prove relation (67), it suffices to prove that, with accuracy up to small values, the term \(\varepsilon z\) can be omitted in the exponent.

We consider the expression

\[
J = \int \zeta(\varphi_i + \varepsilon z)\omega(z) \int_0^t \frac{dt'}{\sqrt{t-t'}} \int_{\phi_1}^{\phi_2} \left[ \exp\left(-\frac{(\varphi_i-\xi+\varepsilon z)^2}{4(t-t')}\right) - \exp\left(-\frac{(\varphi_i-\xi)^2}{4(t-t')}\right) \right] \, dz \, d\xi.
\]

Since the function \(\omega\) decreases fast, we can assume that \(|z| < \varepsilon^{-\delta}, \delta \in (0,1)\). Next it suffices to consider the integral over \(t'\) from 0 to \(t-\varepsilon^{1-\delta}\), because if \(t' \in [t-\varepsilon^{1-\delta}, t]\), then the obtained integral can be estimated as \(O(\varepsilon^{(1-\delta)/2})\).

In the remaining integrals, we can use the same method as in the proof of Lemma 4. Namely, we transform the difference of the exponential functions:

\[
\exp\left(-\frac{(\varphi_i-\xi+\varepsilon z)^2}{4(t-t')}\right) - \exp\left(-\frac{(\varphi_i-\xi)^2}{4(t-t')}\right)
\]

\[
= \left[ \exp\left(-\frac{2(\varphi_i-\xi)-\varepsilon z \varepsilon z}{4(t-t')}\right) - 1 \right] \left( \frac{\varepsilon z}{t-t'} \right)^{\gamma} \left( \frac{\varepsilon z}{t-t'} \right)^{-\gamma},
\]

\(\gamma \in (0,1/2)\),
and apply the estimate
\[
\left| \left( \exp \left( - \frac{(2(\varphi_i - x) - \varepsilon z)e^z}{4(t - t')} \right) - 1 \right) / \left( \frac{t - t'}{\varepsilon z} \right)^\gamma \right| < \text{const},
\]
which, obviously, is valid in the required range of variables. Hence, as in Lemma 4, we obtain the estimate
\[
J = O(\varepsilon^\mu), \quad \mu \in (0, 1/2),
\]
which proves relation (67).

Similarly to Lemmas 4 and 5, we prove the following statement.

**Lemma 6.** The following relation holds for any of the functions \( \hat{q}_i \) defined by relations (29):
\[
\hat{q}_i(\varphi_i, t) - \hat{q}_i(\hat{q}_i, t) = O(\varepsilon^\mu), \quad \mu \in (0, 1/2).
\]

**Lemma 7.** Relations (38), (39) hold.

*Proof.* In Section 3, we introduced the function \( \Omega(z, \eta) \). Of course, we can change the signs in the arguments using the fact that the function \( \omega_0(z) \) is odd, but the form presented above is convenient for calculations.

Calculating the derivatives \( u_t \) and \( u_x \), we obtain
\[
\hat{u}_t = \frac{1}{2\varepsilon} \left\{ \beta(x - \varphi_2) \omega_0 \left( \frac{x - \varphi_2}{\varepsilon} \right) + \beta(x - \varphi_2) \omega_0 \left( \frac{x - \varphi_1}{\varepsilon} \right) \right\},
\]
\[
\hat{u}_x = \frac{\beta}{2\varepsilon} \left\{ -\omega_0 \left( \frac{\varphi_1 - x}{\varepsilon} \right) + \omega_0 \left( \frac{x - \varphi_1}{\varepsilon} \right) \right\}. \tag{68}
\]

Now we group the terms in the subintegral expression so that one group contain terms with the factor \( \omega_0(\beta(x - \varphi_1)/\varepsilon) \), and the other group contain terms with the factor \( \omega_0(\beta(x - \varphi_2)/\varepsilon) \). Thus, we write the integral as the sum of two integrals. Then we perform the change of variables
\[
x \rightarrow \varphi_1 - \frac{\varepsilon z}{\beta} \tag{70}
\]
in the first integral and the change of variables

\[ x \rightarrow \varphi_2 + \frac{\varepsilon z}{\beta} \]  

(71)

in the second integral.

From (68) and (69), we obtain

\[
\int u_t u_x \zeta(x) \, dx = \frac{1}{2} \left[ \varphi_1 t + \frac{\beta_\tau \psi_0}{\beta^2} \right] \zeta(\varphi_1) \left( (1 + z)(\Omega'_\eta - \Omega'_0) + (1 - z - \eta)\Omega'_\eta [\Omega'_\eta - \Omega'_\xi] \right) \, dz 
\]

\[
- \frac{1}{2} \left[ \varphi_2 t + \frac{\beta_\tau \psi_0'}{\beta^2} \right] \zeta(\varphi_2) \left( (1 + z)\Omega'_{1\eta} + (1 - z - \eta)\Omega'_\eta [\Omega'_\eta - \Omega'_\xi] \right) \, dz 
\]

\[+ O(\varepsilon). \]

Here we used the formula for the derivative \( \beta_t = \varepsilon^{-1} \beta_\tau \psi_0' \) and the fact that the functions in the integrand in (72) are invariant under the change \( z \rightarrow -z - \eta \), since each bracket is multiplied by \( -1 \) in this change.

Now we calculate the other expressions contained in (6). We have

\[
\frac{\varepsilon}{2} \int \zeta'(x) (\bar{u}_x)^2 \, dx = \frac{\beta^2}{2 \varepsilon} \int \zeta'(x) \left[ - \dot{\omega}_0 \left( \frac{\beta \varphi_1 - x}{\varepsilon} \right) + \dot{\omega}_0 \left( \frac{\beta \varphi_2 - x}{\varepsilon} \right) \right] + \dot{\omega}_0 \left( \frac{\beta \varphi_1 - x}{\varepsilon} \right) \omega_0 \left( \frac{\beta \varphi_2 - x}{\varepsilon} \right) \left[ \dot{\omega}_0 \left( \frac{\beta \varphi_1 - x}{\varepsilon} \right) \right] \, dx 
\]

\[
= \frac{\beta}{8} \left( \zeta'(\varphi_1) + \zeta'(\varphi_2) \right) \int \left\{ \dot{\omega}_0(z)(1 - \omega_0(-z - \eta))^2 
\right.
\]

\[
- \dot{\omega}_0(z) \dot{\omega}_0(-z - \eta)(1 - \omega_0(z))(1 - \omega_0(-z - \eta)) \right\} \, dz + O(\varepsilon). 
\]

It is easy to see that the integrand in the right-hand side of (73) can be written as

\[
\int \left\{ \dot{\omega}_0(z)(1 - \omega_0(-z - \eta))^2 
\right.
\]

\[
- \dot{\omega}_0(z) \dot{\omega}_0(-z - \eta)(1 - \omega_0(z))(1 - \omega_0(-z - \eta)) \right\} \, dz = \frac{1}{4} \int (\Omega'_x)^2 \, dz. 
\]

Thus, we finally obtain

\[
\frac{\varepsilon}{2} \int \zeta'(\bar{u}_x)^2 \, dx = \frac{\beta}{4} \left( \zeta'(\varphi_1) + \zeta'(\varphi_2) \right) \int (\Omega'_x)^2 \, dz + O(\varepsilon). 
\]

Then the left-hand side of (73) is positive, and hence the integrand expression in the right-hand side is also positive (until the measure of the points at which \(|u_x| \geq \text{const} \) is positive).
The last term in (6) can be considered similarly:

$$\frac{1}{\varepsilon} \int F(\bar{u}) \zeta_{x} \, dx = \frac{\beta^{-1}}{2} (\zeta'(\varphi_1) + \zeta'(\varphi_2)) \int F(\Omega) \, dz + O(\varepsilon).$$

To prove this relation, we must successively perform the change of variables (71) in the integral in the left-hand side and then consider the half-sum of the integrals obtained, which, obviously, is equal to the original integral in the left-hand side. Moreover, we must take into account the estimates (34), (35), and

$$\Omega(z, \eta) = \begin{cases} 1 + O(\exp(-2z)), & z \to \infty, \\ \omega_0(-z - \eta) + O(\exp(2z)), & z \to -\infty, \end{cases}$$

which together with the explicit form of the function $F(u) = \frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{4}$ imply that the integrals

$$\int zF(\Omega) \, dz$$

converge. 

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