Quantum scattering with time-decaying harmonic potentials

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Abstract

By controlling coefficients and decaying order of time-decaying harmonic potentials, the velocity of a quantum particle is decelerated by the effect of harmonic potentials but the particle is non-trapping. Deceleration makes the threshold of decaying order between the short-range class of potentials and long range class of potentials $1/(1 - \lambda)$ for some $0 < \lambda < 1/2$, where $\lambda$ is determined by the mass of the particle and coefficients of harmonic potentials. In this paper, we consider the quantum system with controlled harmonic potentials. By defining the wave operators for this system and suitable range of these, we can prove the existence and completeness of wave operators with respect to the short-range potentials $V(t, x)$ satisfying $|V(t, x)| = o((1 + |x|)^{-1/(1-\lambda)})$.

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Scattering theory, time-dependent Hamiltonian, Harmonic oscillator, time-dependent magnetic fields.

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1 Introduction

We consider a dynamics of a quantum particle under the influence of time-dependent harmonic potentials. Define the time-dependent harmonic potentials by $k(t)x^2/2$, where $x = (x_1, \cdots, x_n)$, $n \in \mathbb{N}$ is the position of the particle and $k \in L^\infty(\mathbb{R})$. Then the free time-dependent Hamiltonian under consideration is described by

$$H_0(t) = \frac{p^2}{2m} + k(t)x^2/2,$$

(1.1)

where $p = (p_1, \cdots, p_n) = -i(\partial_1, \cdots, \partial_n)$ and $m > 0$ are the momentum and the mass of the particle, respectively. Define $U_0(t, s)$ by the propagator for $H_0(t)$, that is, a family of unitary operators $\{U_0(t, s)\}_{(t, s) \in \mathbb{R}^2}$ in $L^2(\mathbb{R}^n)$ and each components satisfying

$$i\partial_t U_0(t, s) = H_0(t)U_0(t, s), \quad i\partial_s U_0(t, s) = -U_0(t, s)H_0(s),$$

$$U_0(t, \theta)U_0(\theta, s) = U_0(t, s), \quad U_0(s, s) = \text{Id}. $$

Here let us investigate the asymptotic behavior of the particle governed by the $H_0(t)$. Define the classical trajectory of the particle as

$$x_0(t, s) = U_0(t, s)^*xU_0(t, s),$$
and then commutator calculation shows that

\[ x_0''(t, s) + \left(\frac{k(t)}{m}\right)x_0(t, s) = 0, \quad x_0(s, s) = x, \quad x'_0(s, s) = p/m \]

holds, where \(x'_0(t, s) = (\partial_t x_0)(t, s)\) and \(x''_0(t, s) = (\partial^2_t x_0)(t, s)\). Define the fundamental solutions \(\zeta_1(t, s)\) and \(\zeta_2(t, s)\) as solutions to

\[
\begin{align*}
\zeta_j''(t, s) + \left(\frac{k(t)}{m}\right)\zeta_j(t, s) &= 0, \\
\zeta_1(s, s) &= 1, \\
\zeta_1'(s, s) &= 0, \\
\zeta_2(s, s) &= 0, \\
\zeta_2'(s, s) &= 1,
\end{align*}
\]

and then we have \(x_0(t, s) = \zeta_1(t, s)x + \zeta_2(t, s)p/m\). For the case where the \(k(t)\) is a periodic function in \(t\), the equations (1.2) are sometimes called Hill’s equation and asymptotic behavior of the solutions of (1.2) are well known. By using the results of the study of Hill’s equation, spectral theory, scattering theory and related issues were considered by Korotyaev [17], Hagedorn-Loss-Slawny [10], Huang [13], Adachi-Kawamoto [2] and Kawamoto [15].

As the other study of (1.2), the case where \(k(t)\) is decaying on time is also considered. In the case of \(k(t) = c_0t^{-2-c_1}\) for \(c_0 \neq 0\) and \(c_1 > 0\), it is well known that the solution of (1.2) satisfies \(\zeta_1(t, s)/t \sim c_2\) and \(\zeta_2(t, s)/t \sim c_3\) for some \(c_2, c_3 \in \mathbb{R}\), see e.g. Willett [20] and Naito [19]. Hence the classical trajectory is very similar to the case of \(k(t) \equiv 0\). Although this system is very similar to a system considered by Yafaev [21] or Kitada-Yajima [16], the scattering theory for this case has not been considered yet, as far as we know.

On the other hand, in the case of \(k(t) = c_4t^{-2+c_5}\), \(c_4 \neq 0\) and \(c_5 > 0\), the solution of (1.2) behaves like \(\zeta_1(t, s) \sim t^{c_6}\cos(\theta_1(t))\) and \(\zeta_2(t, s) \sim t^{c_6}\sin(\theta_2(t))\), asymptotically in \(t\), with \(0 < c_6 < 1\) and some functions \(\theta_1(t)\) and \(\theta_2(t)\) satisfying \(|\theta_j(t)| \to \infty\) as \(t \to \infty\). These facts can be easily proven by applying the same approach of Hochstadt [14]. In this case, the quantum particle will move out the any compact regions but it also will come back to the origin infinity times. This movement makes difficult even to prove the existence of wave operators and hence there are no works associate with scattering theory for this model.

Above all, the scattering theory for the harmonic potentials with time-decaying coefficients have not been considered yet.

Here we remarked the case of \(k(t) = O(t^{-2})\). In this case, Geluk-Marić-Tomić [7] proved that for \(k(t)\) which satisfies \(\lim_{t \to \infty} t^2k(t) = k\) with \(0 < k < m/4\), solutions of (1.2) satisfy

\[
\lim_{t \to \infty} \zeta_j(t, s)/t^{1-\lambda} = \tilde{c}_j(s),
\]

for \(j \in \{1, 2\}\), some constants \(\tilde{c}_1(s)\) and \(\tilde{c}_2(s)\), and \(\lambda = (1 - \sqrt{1 - 4k/m})/2\). Therefore, it seems that the scattering theory is easier to consider and hence the purpose of this paper is to consider the scattering theory for this \(k(t)\). However, general \(k(t)\) makes difficult to construct \(\mathcal{M}(t)\) (see, p12) and makes complex to deduce auxiliary estimates and propagation estimates (see, §2 and §3). Hence, unfortunately, we will consider the simplified case \(k(t) = kt^{-2}\) with \(0 < k < m/4\). For this \(k(t)\), the equations (1.2) can not be defined on \(t = 0\) and hence let us state the following assumption on \(k(t)\).
Assumption 1.1. Assume that the coefficient $k(t)$ in (1.1) satisfies

$$k(t) = \begin{cases} \frac{k_C(t)}{t^2}, & 0 \leq |t| \leq r_0, \\ k_C(t), & |t| > r_0, \end{cases}$$

(1.3)

where $k_C(t) \in L^\infty([-r_0, r_0])$, $r_0 > 0$ is a given constant and $0 < k < m/4$. Moreover, assume that for any fixed $s \in \mathbb{R}$, both of solutions of (1.2) with respect to (1.3) are included in $C^1(\mathbb{R})$ and are twice differentiable functions.

An example of $k(t)$ satisfying the assumption 1.1 can be seen in Appendix §A.

Let us define $\lambda$, $0 < \lambda < 1/2$ as the small one of the solution to $\lambda(\lambda - 1) + k/m = 0$, i.e.,

$$\lambda = \frac{1 - \sqrt{1 - 4k/m}}{2}.$$  

Under Assumption 1.1 for $|t| > r_0$, we have that $|t|^\lambda$ and $|t|^{1 - \lambda}$ are linearly independent solutions to $f''(t) + kt^{-2}f(t)/m = 0$, respectively. Hence solutions of (1.2) can be represented by

$$\zeta_1(t, s) = c_{1, \pm}(s)|t|^{1 - \lambda} + c_{2, \pm}(s)|t|^\lambda, \quad (1.4)$$

$$\zeta_2(t, s) = c_{3, \pm}(s)|t|^{1 - \lambda} + c_{4, \pm}(s)|t|^\lambda, \quad (1.5)$$

for $|t| > r_0$. Then, the classical trajectory also be calculated that

$$x(t) = \left(c_{1, \pm}(s)x + \frac{c_{3, \pm}(s)}{m}p\right)|t|^{1 - \lambda} + \left(c_{2, \pm}(s)x + \frac{c_{4, \pm}(s)}{m}p\right)|t|^\lambda.$$

Hence if $(c_{1, \pm}(s)x + c_{3, \pm}(s)p/m)\phi(s) \neq 0$ holds for all $\phi(s) \in \mathcal{D}$, $\mathcal{D}$ is some dense set in $L^2(\mathbb{R}^n) \cap \mathcal{D}(x) \cap \mathcal{D}(p)$, then we have

$$C_0|t|^{1 - \lambda} \leq |(xU_0(t, s)\phi(s), U_0(t, s)\phi(s))_{L^2(\mathbb{R}^n)}| \leq C_1|t|^{1 - \lambda}$$

(1.6)

holds, where $0 < C_0 < C_1$. Since $x'(t)\phi(s) = \mathcal{O}(t^{-\lambda})$ holds, the velocity of the classical trajectory of the charged particle is decelerated by the harmonic potentials $k(t)x^2/2$ but the particle is non-trapping. On the other hand, by (1.2) we have

$$\zeta_1(t, s)\zeta_2'(t, s) - \zeta_1'(t, s)\zeta_2(t, s) = 1$$

for all $t$, and which induces

$$c_{1, \pm}(s)c_{4, \pm}(s) - c_{2, \pm}(s)c_{3, \pm}(s) = \mp 1/(1 - 2\lambda).$$

(1.7)

By (1.7), we notice that, at least, either $c_{1, \pm}(s)$ or $c_{4, \pm}(s)$ is not 0. Thus there always exists the set $\mathcal{D} \subset L^2(\mathbb{R}^n) \cap \mathcal{D}(x) \cap \mathcal{D}(p)$, dense in $L^2(\mathbb{R}^n)$, such that for all $\phi(s) \in \mathcal{D}$, (1.6) holds. (see §3.4)

Next, we define $H(t) = H_0(t) + V(t)$, where $V(t)$ satisfies the following Assumption 1.2.
Assumption 1.2. $V(t)$ is a multiplication operator with respect to $V(t, x)$ and $V(t, x) \in L^\infty(R; C^1(R^n))$ satisfies that for some $\rho_{S, \lambda} > 1/(1 - \lambda)$,

$$|V(t, x)| \leq C_{S, 0} \langle x \rangle^{-\rho_{S, \lambda}}, \quad |\nabla V(t, x)| \leq C_{S, 1} \langle x \rangle^{-\rho_{S, \lambda} - 1}, \quad \langle x \rangle = \sqrt{1 + x^2}$$

holds, where $C_{S, 0}$ and $C_{S, 1}$ are some positive constants.

Clearly, under Assumption 1.2, the uniqueness and existence of the unitary propagator for $H(t)$, $U(t, s)$, is guaranteed. Then the wave operators $W^\pm(s)$ for this model can be defined as

$$W^\pm(s) = s - \lim_{t \to \pm\infty} U(t, s)^* U_0(t, s)$$  \hspace{1cm} (1.8)

Theorem 1.3. Under Assumption 1.1 and 1.2, $W^\pm(s)$ exist.

1.1 Alternative energy

In the study of time depending systems, the energy is changing accordingly to the time $t$. Hence, by using decomposition formula given in Korotyaev [17], see Lemma 2.5. in [17], we introduce the following time-independent quantity $E(s)$:

$$E(s) = \zeta(t)^2 \left( (p - m\zeta'(t)x/\zeta(t))^2 U_0(t, s) \phi(s), U_0(t, s) \phi(s) \right)_{L^2(R^n)},$$

where $\phi(s) \in C_0^\infty(R^n)$ and $\zeta(t)$ is a twice differentiable function and which satisfies $\zeta''(t) + k(t)\zeta(t)/m = 0$. Indeed, by defining $\alpha(t) := \zeta(t)^2 (p - m\zeta'(t)x/\zeta(t))^2$, we have

$$\frac{d}{dt} \left( \alpha(t) U_0(t, s) \phi(s), U_0(t, s) \phi(s) \right)_{L^2(R^n)} = 0, \quad \phi(s) \in C_0^\infty(R^n).$$

In particular, $f(t) = |t|^\lambda$ is the solution to $f''(t) + k(t)f(t)/m = 0$ for $|t| > r_0$. Hence in the followings, we redefine

$$\alpha(t) = |t|^{2\lambda} \left( p \mp \frac{m\lambda x}{|t|} \right)^2, \quad \pm t > r_0$$

and call (alternative) energy.

Noting this energy, we characterize the range of wave operators as follows:

Definition 1.4. Let us define $\varphi_1 \in C_0^\infty([0, \infty))$ as follows

$$\varphi_1(\tau) = \begin{cases} 1 & 2\kappa_1 < \tau < R_1/2, \\ 0 & \tau < \kappa_1, \ R_1 < \tau, \end{cases} \hspace{1cm} (1.9)$$

where $0 < \kappa_1 \ll 1$ and $R_1 \gg 1$ are arbitrary constants. Moreover, we define $\varphi_2 \in C^\infty([0, \infty))$ as follows

$$\varphi_2'(\tau) \geq 0, \quad \begin{cases} \varphi_2(\tau) = 0 & \tau \in [0, \kappa_2], \\ \varphi_2(\tau) = 1 & \tau \in (2\kappa_2, \infty), \end{cases}$$

where $0 < \kappa_2 \ll 1$ is arbitrary constant.
**Definition 1.5.** For some $\kappa_1$, $R_1$ and $\kappa_2$, define $W^\pm_1(\lambda; s)$ and $W^\pm_2(\lambda; s)$ as ones satisfying

$$W^\pm_1(\lambda; s) = \left\{ \phi(s) \in L^2(\mathbb{R}^n) \mid \lim_{t \to \pm \infty} \| (1 - \varphi_1(\alpha(t))) U(t, s) \phi(s) \|^2_{L^2(\mathbb{R}^n)} = 0 \right\}$$

and

$$W^\pm_2(\lambda; s) = \left\{ \phi(s) \in L^2(\mathbb{R}^n) \mid \lim_{t \to \pm \infty} \| (1 - \varphi_2(x^2/t^{2\rho_\lambda})) U(t, s) \phi(s) \|^2_{L^2(\mathbb{R}^n)} = 0 \right\},$$

where

$$\rho_\lambda = 2\lambda(1 - \lambda).$$

Then we define $W^\pm(\lambda; s) := W^\pm_1(\lambda; s) \cap W^\pm_2(\lambda; s)$.

The result of the asymptotic completeness is the followings.

**Theorem 1.6.** Under Assumption 1.1 and 1.2, the wave operators are asymptotically complete, i.e.,

$$\text{Ran}(W^\pm(\lambda; s)) = \overline{W^\pm(\lambda; s)}$$

holds, where $\overline{W^\pm(\lambda; s)}$ is a closure of $W^\pm(\lambda; s)$ in the topology of $L^2(\mathbb{R}^n)$.

For some technical reasons, we need to take $\rho_\lambda$ as above. In the case $\lambda = 0$, the range $W^\pm(\lambda; s)$ is very similar to the one of the [16] and hence our proof will be another proof of the result of [16] for short-range potentials.

Before the proof of Theorem 1.6, we give the so-called propagation estimates for $U(t, s)$ (Proposition 3.1, 3.2 and 3.4) by using the Mourre type estimates and commutator calculations. This approach is well used in the proof of asymptotic completeness of wave operators not only quantum system but also relativistic equations, quantum fields theory and so on, (see e.g. Adachi [1], Adachi-Tamura [3], Bachelot [4], Daudé [5], Dereziński-Gérard, Gérard [8], Graf [9] and Herbst-Møller-Skibsted [11]). In particular, we referred to the approach of Graf [9]. However, since the Hamiltonian $H(t)$ never commutes with the propagator $U(t, s)$, dealing with the time-dependent energy cut-off such like $\varphi_1(H(t))$ is difficult. In order to get over this difficulty, we define the alternative energy $\alpha(t)$ instead of $H(t)$ and replace $\varphi_1(H(t))$ to $\varphi_1(\alpha(t))$ which commutes with $i\partial_t + H_0(t)$ (closely associated with the Floquet Hamiltonian, see, e.g. [2], [17], Yajima [22] and Yokoyama [23]). However, because of using the alternative energy, it appears the extra terms which disturbing to mimic the approach of Graf. In §2, we shall prove these disturbing terms are included in $L^1(t; dt)$ class on the suitable cut-off function and on the $W^\pm(\lambda; s)$. After, in §3, we shall prove the propagation estimates and existence of inverse wave operators by using the range of wave operators $W^\pm(\lambda; s)$, and at last, we will prove the completeness of wave operators.
2 Auxiliary estimates

In this section, we introduce some notations and estimations in order to simplify the proofs. From this section to the end of this paper, we only consider the case of $t > s > 0$. Other cases can be also proven by the same approach. Moreover, we also assume that $r_0 > s > 0$ since the case of $s > r_0$ is more simple.

2.1 Notations

We write $C$ as a constant which satisfies $0 < C < \infty$ and not depend on any other parameters under considering. We put state space of a particle as $L^2(\mathbb{R}^n)$ and denote a norm of $L^2(\mathbb{R}^n)$, $\| \cdot \|_{L^2(\mathbb{R}^n)}$, by $\| \cdot \|$, and we denote inner product on $L^2(\mathbb{R}^n)$ by $\langle \cdot, \cdot \rangle$. The set of bounded operators from $L^2(\mathbb{R}^n)$ to itself is described by $\mathcal{B}(L^2(\mathbb{R}^n))$, and we may denote components of $\mathcal{B}(L^2(\mathbb{R}^n))$ by (bdd). In addition to this, the operator norm on $\mathcal{B}(L^2(\mathbb{R}^n))$, $\| \|_{\mathcal{B}(L^2(\mathbb{R}^n))}$, is also described by $\| \|$. We will often use $O(\cdot)$. If a function $f(\sigma)$ or a $\sigma$-parametrized bounded operator $B(\sigma)$ satisfies that for $|\sigma| \to \infty$,
\[
|f(\sigma)| \leq C|\sigma|^\alpha, \quad \|B(\sigma)\| \leq C|\sigma|^\alpha
\]
for some constants $C$ and $\alpha$, we write $f(\sigma) = O(\sigma^\alpha)$ or $B(\sigma) = O(\sigma^\alpha)$. The commutator of two operators $A$ and $B$ is described by $[A, B]$ and it is defined by
\[
(i[A, B]\phi(s), \psi(s)) := i(B\phi(s), A^*\psi(s)) - i(A\phi(s), B^*\psi(s)),
\]
for $\phi(s), \psi(s) \in \mathcal{D}(A) \cap \mathcal{D}(B)$. At last, we sometimes use $(x) = \sqrt{1 + x^2}$.

2.2 Definitions

The cut-off function $F_\varepsilon \in C^\infty(\mathbb{R})$ is define as
\[
F_\varepsilon(s \leq \theta) = \begin{cases} 1, & s \leq \theta - \varepsilon, \\ 0, & s \geq \theta, \end{cases} \quad F_\varepsilon(s \geq \theta) = \begin{cases} 1, & s \geq \theta + \varepsilon, \\ 0, & s \leq \theta, \end{cases}
\]
where $\varepsilon > 0$ is a sufficiently small constant. For $\theta_1 < \theta_2$, we also define
\[
F_\varepsilon(\theta_1 \leq s \leq \theta_2) = F_\varepsilon(s \geq \theta_1)F_\varepsilon(s \leq \theta_2).
\]
Moreover, we define sets $C_{0,\delta}^\infty(\mathbb{R})$ and $\mathcal{B}_\delta^\infty(\mathbb{R})$ as follows:
\[
C_{0,\delta}^\infty(\mathbb{R}) = \{ \phi(s) \in C_0^\infty(\mathbb{R}) \mid \text{supp}(\phi(s)) = \{ x \in \mathbb{R} : |x| > \delta \} \}, \quad (2.2)
\]
\[
\mathcal{B}_\delta^\infty(\mathbb{R}) = \{ \phi(s) \in \mathcal{B}^\infty(\mathbb{R}) \mid \text{supp}(\phi(s)) = \{ x \in \mathbb{R} : |x| > \delta \} \}. \quad (2.3)
\]
We often use the notation $L^1(t; dt)$. A function $\beta(t)$ or $t-$parametrized operator $\beta(t)$ satisfies
\[
\int_{r_0}^\infty |\beta(t)|\, dt < \infty, \quad \text{or} \quad \int_{r_0}^\infty \|\beta(t)\|\, dt < \infty,
\]
then we denote $\beta(t) \in L^1(t;dt)$ and call $\beta(t)$ is integrable in $t$. For some linear operator $A$, we often denote

$$A + A^* \text{ by } (A + (\text{h.c.})).$$

For some $t$–parametrized linear operators $A(t)$ and $B(t)$, we define the Heisenberg derivative of $A(t)$ associated with $B(t)$ by

$$D_{B(t)}(A(t)) := i[B(t), A(t)] + i\frac{d}{dt} A(t).$$

### 2.3 Commutator estimate (Mourre estimates)

In this subsection, we will consider the commutator estimate which is closely related to the Mourre estimate (see, e.g. Mourre [18] and Yokoyama [23]). Let $\zeta(t) = t^\lambda$ (i.e., $\zeta(t)$ is the solution to $\zeta''(t) + k(t)\zeta(t)/m = 0$ for $t > r_0$). Define $\mathcal{A}(t)$ as follows

$$\mathcal{A}(t) = (x \cdot p + p \cdot x) - 2m\zeta'(t)x^2/\zeta(t) = \left(x \cdot \left(p - \frac{m\zeta'(t)x}{\zeta(t)}\right) + (\text{h.c.)}\right).$$

Straightforward calculation shows

$$\frac{d}{dt} \mathcal{A}(t) = -\frac{2m(\zeta(t)\zeta''(t) - (\zeta'(t))^2)}{\zeta(t)^2}x^2$$

and

$$i[H(t), \mathcal{A}(t)] = 2\left(\frac{p^2}{m} - \frac{\zeta'(t)}{\zeta(t)}(x \cdot p + p \cdot x) - k(t)x^2 - x \cdot \nabla V(t)\right).$$

Noting $-m\zeta''(t)/\zeta(t) - k(t) = 0$, for $t > r_0$, we have

$$\frac{d}{dt} \left(\mathcal{A}(t)U(t, s)\phi(s), U(t, s)\phi(s)\right)$$

$$= 2\left(\left(\zeta(t)^{-2}\alpha(t)/m - x \cdot \nabla V(t)\right)U(t, s)\phi(s), U(t, s)\phi(s)\right), \text{ for } t > r_0. \tag{2.4}$$

On the energy cut-off $\varphi_1(\alpha(t))$, it follows that

$$\left|\left(\zeta(t)^{-2}\alpha(t)\right)\varphi_1(\alpha(t))U(t, s)\phi(s), \varphi_1(\alpha(t))U(t, s)\phi(s)\right|$$

$$\geq \kappa_1\zeta(t)^{-2}\|\varphi_1(\alpha(t))U(t, s)\phi(s)\|^2 = \kappa_1t^{-2\lambda}\|\varphi_1(\alpha(t))U(t, s)\phi(s)\|^2.$$

Hence, it seems that the candidate of conjugate operator associated with $H(t)$ is $t^{2\lambda}\mathcal{A}(t)$. On the other hand, in order to deduce the positiveness of the term $\varphi_1(\alpha(t))(\alpha(t)/m - t^{2\lambda}x \cdot \nabla V(t))\varphi_1(\alpha(t))$, we assumed that $|t^{2\lambda}x \cdot \nabla V(t)\varphi_2(x^2/t^{\delta_1})| \leq C|t|^{-\delta_1}$ for some $\delta_1 > 0$, and hence we needed to assume specific assumption in the range of wave operators (in $W^2_2(\lambda; s)$).
2.4 Commutator estimates (potential estimates)

The energy $\alpha(t)$ satisfies $D_{H_0(t)}(\alpha(t)) = 0$ but it satisfies $D_{H(t)}(\alpha(t)) \neq 0$. In the proof of asymptotic completeness, the term $D_{H(t)}(\alpha(t)) = i[V(t), \alpha(t)]$ appears many times. Hence we give such a term is included in $L^1(t; dt)$ on the suitable cut-off function.

**Lemma 2.1.** For all $h(\cdot) \in \mathcal{B}_0^\infty(\mathbb{R})$, $\phi \in L^2(\mathbb{R}^n)$ and $\rho > \lambda$,

$$J(t) := h(|x|/t^\rho)[V(t), \varphi_1(\alpha(t))] \langle x \rangle^N \phi = \mathcal{O}(t^{-(\rho_s,\lambda+1)\rho+\lambda+N\rho})$$

(2.5)

holds, where $N \in \{0, 1\}$.

**Proof.** At first, we shall prove the case of $N = 0$. By the Helffer-Sj"{o}strand formula (see, Helffer-Sj"{o}strand [12]), for any fixed $t$,

$$\varphi_1(\alpha(t)) = \frac{1}{2\pi i} \int C \bar{\partial}_z \bar{\varphi}_1(z)(z - \alpha(t))^{-1} dzd\bar{z}$$

holds, where $\bar{\varphi}_1$ is called *almost analytic extension* of $\varphi_1$. By $\varphi_1 \in C_0^\infty(\mathbb{R})$, one can choose $\bar{\varphi}_1$ be such that $\bar{\varphi}_1 \in C_0^\infty(\mathbb{C})$ and

$$|\bar{\varphi}_1(z)| \leq C_{M_0}|\text{Im} z|^{M_0} (z)^{-M_0-1},$$

(2.6)

for all $M_0 \in \mathbb{N}$. Then

$$\|h(|x|/t^\rho)i[V(t), \varphi_1]\| \leq Ct^\lambda \int C |\bar{\partial}_z \bar{\varphi}_1(z)|$$

$$\times \left| h(|x|/t^\rho)(z - \alpha(t))^{-1} \left( t^\lambda \left( p - \frac{m\lambda x}{t} \right) \cdot \nabla V(t) + (\text{h.c.}) \right) (z - \alpha(t))^{-1} \right| dzd\bar{z}$$

holds. Here, for sufficiently large $M_1 > (\rho - \lambda)^{-1}$, we have

$$h(|x|/t^\rho)(z - \alpha(t))^{-1} = \sum_{M=0}^{M_1-1} C_M t^{(-\rho+\lambda)M} \left( t^\lambda \left( \frac{x}{|x|} \cdot \left( p - \frac{m\lambda x}{t} \right) \right) \right)^M$$

$$\times (z - \alpha(t))^{-M} h^{(M)}(|x|/t^\rho) + L^1(t; dt).$$

On the support of $h^{(M)}(|x|/t^\rho)$, there exists $\delta_2 > 0$ such that $|x| \geq \delta_2 t^\rho$ holds, which yields that

$$\|h(|x|/t^\rho)i[V(t), (z - \alpha(t))^{-1}]\|$$

$$\leq Ct^{-\rho(\rho_s,\lambda+1)+\lambda} \int C |\bar{\partial}_z \bar{\varphi}(z)| \langle z \rangle^{(M_1+1)/2} |\text{Im} z|^{-M_1-1} dzd\bar{z}.$$
2.5 Estimation for Remainder terms

In the proof of the asymptotic completeness, the following terms often appear:

\[ A_j(t_1, t_2) := \int_{t_1}^{t_2} (\Theta(t)h(|x|/t^{1-\lambda})(1 - \varphi_j)u(t, s), u(t, s)) \frac{dt}{mt^{1-\lambda}}, \quad (2.7) \]

where \( j \in \{1, 2\}, h \in C^\infty_{0, \delta}(\mathbb{R}), u(t, s) = U(t, s)\psi(s), \psi(s) \in \mathcal{W}^+(\lambda; s), \)

\[ \Theta(t) := \frac{x}{|x|} \cdot \left( p - \frac{m(1 - \lambda)x}{t} \right) \quad (2.8) \]

and

\[ \varphi_1 = \varphi_1(\alpha(t)), \quad \varphi_2 = \varphi_2(x^2/t^{2\rho_s}). \]

In the following Lemma, we shall prove \( A_j(r_0, \infty) \) exists.

**Lemma 2.2.** Let \( A_j(t_1, t_2) \) is the same one as in (2.7). Then

\[ \lim_{t_1, t_2 \to \infty} |A_j(t_1, t_2)| = 0 \]

holds.

**Proof.** Define \( h(\cdot) \in C^\infty_{0, \delta}(\mathbb{R}) \) and \( \hat{h}(t) = \int_{-\infty}^{t} h(\tau)d\tau \). Here we note \( \hat{h} \in B^\infty_{\delta}(\mathbb{R}) \) and define

\[ B_j(t) := \left( \hat{h}(|x|/t^{1-\lambda})(1 - \varphi_j)u(t, s), u(t, s) \right). \]

At first, we shall prove \( A_1(t_1, t_2) \to 0 \) as \( t_1, t_2 \to \infty \). Noting that

\[ D_{H(t)}(\hat{h}(|x|/t^{1-\lambda})) = \frac{\Theta(t)}{mt^{1-\lambda}}h(|x|/t^{1-\lambda}) + L^1(t; dt) \]

and \( D_{H_0(t)}(\alpha(t)) = 0 \) for \( t > r_0 \), we have

\[ \frac{d}{dt} B_1(t) = \frac{1}{mt^{1-\lambda}} (\Theta(t)h \left( |x|/t^{1-\lambda} \right) (1 - \varphi_1)u(t, s), u(t, s)) \]

\[ + \left( \hat{h} \left( |x|/t^{1-\lambda} \right) i[V(t), -\varphi_1] u(t, s), u(t, s) \right) + L^1(t; dt). \]

Consequently, we have

\[ |B_1(t_1) - B_1(t_2)| = \left| \int_{t_1}^{t_2} B'_1(t)dt \right| \]

\[ \geq \left| \int_{t_1}^{t_2} (\Theta(t)h \left( |x|/t^{1-\lambda} \right) (1 - \varphi_1)u(t, s), u(t, s)) \frac{dt}{mt^{1-\lambda}} \right| - \int_{t_1}^{t_2} L^1(t; dt)dt, \]
where we use Lemma 2.1 with $\rho = 1 - \lambda$ and $N = 0$. By the definition of $\varphi_1$ and $W^+(\lambda; s)$, we have $|B_1(t_1) - B_1(t_2)| \to 0$, and this yields $A_1(t_1, t_2) \to 0$.

Next we shall prove $|A_2(t_1, t_2)| \to 0$ as $t_1, t_2 \to \infty$. By using the same approach in the proof of $A_1(t_1, t_2) \to 0$, we notice that it is enough to prove

$$C_2(t) := \left( \hat{h}(|x|/t^{1-\lambda})D_{H(t)}(\varphi_2(x^2/t^{2\rho_\lambda}))u(t, s), u(t, s) \right) \in L^1(t; dt).$$

(2.9)

On the other hand,

$$\hat{h}(|x|/t^{1-\lambda})D_{H(t)}(\varphi_2) = \hat{h}(|x|/t^{1-\lambda}) \left( \frac{1}{t^{2\rho_\lambda}} \left( \varphi'_2 x \cdot \left( p - \frac{m\lambda x}{t} \right) \right) + (\text{h.c.}) \right) + \frac{2(\lambda - \rho_\lambda)\hat{h}(|x|/t^{1-\lambda})\varphi'_2(x^2/t^{2\rho_\lambda})x^2}{t^{1+2\rho_\lambda}}$$

holds. Since $|x| \leq 2\kappa_2 t^{\rho_\lambda}$ holds on the support of $\varphi'_2$ and $\rho_\lambda < 1/2 < 1 - \lambda$, we have that for enough large $t$ and for all $r > 0$,

$$|\hat{h}(|x|/t^{1-\lambda})\varphi'_2| \leq Ct^{-r(1-\lambda)}|x|^r |\varphi'_2| \leq Ct^{-r(1-\lambda)-\rho_\lambda}.$$ 

It implies

$$\frac{\hat{h}(|x|/t^{1-\lambda})}{t^{2\rho_\lambda}} \left( \varphi'_2 x \cdot \left( p - \frac{m\lambda x}{t} \right) \right) + (\text{h.c.}) \varphi_1 \quad \text{and} \quad \frac{2(\lambda - \rho_\lambda)\hat{h}\varphi'_2(x^2/t^{2\rho_\lambda})x^2}{t^{1+2\rho_\lambda}}$$

are in $L^1(t; dt)$. Hence $C_2(t)$ can be divided into

$$(\Xi(|x|/t^{1-\lambda})\Theta(t)(1 - \varphi_1)u(t, s), u(t, s)) \frac{1}{mt^{1-\lambda}} + L^1(t; dt),$$

$$\Xi(\tau) := mt^{-2\rho_\lambda + 2 - 2\lambda - \rho_\lambda} \hat{h}(\tau)\varphi'_2(t^{2-2\lambda-2\rho_\lambda} \tau^2).$$

Since $\Xi(\tau) \in C_{0,\delta}^{\infty}(\mathbb{R}_+)$, we have $C_2(t)$ is integrable in $t$ by using (2.7) with $j = 1$. □

**Corollary 2.3.** For all $j, l \in \{1, 2\}$ with $j \neq l$ and $l_1, l_2 \in \{0, 1\}$, define

$$A_{j,l_1,l_2}(t_1, t_2) := \int_{t_1}^{t_2} (\Theta(t)h(|x|/t^{1-\lambda})(1 - \varphi_j)\varphi_{l_1}^l(1 - \varphi_l)^{l_2}u(t, s), u(t, s)) \frac{dt}{mt^{1-\lambda}},$$

and then $A_{j,l_1,l_2}(r_0, \infty)$ exists.

### 3 Asymptotic completeness

At last, we will prove the asymptotic completeness of wave operators. Key for the proof is deducing so-called minimal velocity estimates by using Mourre estimate. This approach was
considered by Graf [9] in order to establish the many-body scattering theory. The approach of Graf for the two-body case can be seen e.g., in Deérezinski-Gérard [6], and hence we shall prove the similar estimates of Proposition 4.4.3, Proposition 4.4.4 and Proposition 4.4.7 of [6] for the Hamiltonian with time-decaying harmonic potentials.

In the following to the end of this section, for all \( \phi(s) \in L^2(\mathbb{R}^n) \), we write
\[
v(t, s) = U(t, s)\phi(s).
\]

3.1 Propagation estimates

As a first step of deducing minimal velocity estimate, we show the following so-called large velocity estimate.

**Proposition 3.1.** Let \( \varphi_1 \in C_0^\infty(\mathbb{R}) \) is the same one as in Definition 1.4, and then for all \( \eta_0 > 0 \) and \( \phi(s) \in L^2(\mathbb{R}^n) \), there exists \( C > 0 \) such that
\[
\int_{r_0}^{\infty} \left\| F_\varepsilon(\theta \leq |x|/t^{1-\lambda} \leq \theta + \eta_0)\varphi_1(\alpha(t))v(t, s) \right\|^2 \frac{dt}{t} < C \|\phi(s)\|^2 \quad (3.1)
\]
holds, where \( \theta = 2\sqrt{R_1}/(m(1 - 2\lambda)) \).

**Proof.** For simplicity, we will denote
\[
F^L = F_\varepsilon(\theta \leq |x|/t^{1-\lambda} \leq \theta + \eta_0), \quad \varphi_1(\alpha(t)) = \varphi_1
\]
Define \( G_\varepsilon(t) = \int_{-\infty}^{t} F_\varepsilon(\theta \leq s \leq \theta + \eta_0)^2 ds, \theta = 2\sqrt{R_1}/(m(1 - 2\lambda)) \). Then straightforward calculation shows that
\[
D_{H(t)}(\varphi_1 G_\varepsilon(|x|/t^{1-\lambda}) \varphi_1) = \frac{\varphi_1 F^L}{mt^{1-\lambda}} \Theta(t)F^L \varphi_1 + L^1(t; dt) \quad (3.2)
\]
holds, where \( \Theta(t) \) is the same one defined in (2.8) and we use \( G_\varepsilon(\cdot) \in B_2^\infty(\mathbb{R}) \) and Lemma 2.1 with \( \rho = (1 - \lambda) \) and \( N = 0 \). By noting the support of \( F^L \) and \( \varphi_1 \), one has that (3.2) is smaller than
\[
-\frac{1}{t}\varphi_1 F^L \left( \theta(1 - 2\lambda) - m^{-1}\sqrt{R_1} \right) F^L \varphi_1 + L^1(t; dt).
\]
Hence by using the same scheme of Proposition 4.4.3 and Lemma B.4.1 of [6], Proposition 3.1 can be proven. \( \square \)

As a second step, we will prove so-called middle velocity estimate.

**Proposition 3.2.** For all \( \phi(s) \in L^2(\mathbb{R}^n) \), \( \varepsilon_2 > \varepsilon \) and \( \varepsilon_3 > 2\varepsilon \), there exists \( C > 0 \) such that
\[
\int_{r_0}^{\infty} \left\| \left( p - \frac{m(1 - \lambda)}{t} \right) F_\varepsilon \left( \varepsilon_2 \leq |x|/t^{1-\lambda} \leq \theta + \varepsilon_3 \right) \varphi_1(\alpha(t))v(t, s) \right\|^2 \frac{dt}{t^{1-2\lambda}} \leq C \|\phi(s)\|^2 \quad (3.3)
\]
holds, where \( \theta \) is the same one as in Proposition 3.1.
Remark 3.3. We note that
\[
\left( p - \frac{m(1-\lambda)x}{t} \right) F_x \left( \varepsilon_2 \leq \frac{|x|}{t^{1-\lambda}} \leq \theta + \varepsilon_3 \right) \phi_1(\alpha(t)) = O(t^{-\lambda})
\]
holds.

**Proof.** Define \( \mathcal{R} \in C^\infty(\mathbb{R}^n) \) as
\[
\mathcal{R}(s) = \begin{cases} 
 s^2, & |s| \geq \varepsilon_2, \\
 0, & |s| \leq \varepsilon_2 - \varepsilon, \\
 \Delta \mathcal{R} \geq 0,
\end{cases}
\]
where \( \varepsilon < \varepsilon_2 \), and \( \mathcal{M}(t) \) as
\[
\mathcal{M}(t) = t^\lambda \left( m \left( p - \frac{m(1-\lambda)}{t}x \right) \cdot \nabla \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) + \text{(h.c.)} \right) + 2m^2(1-2\lambda)\mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right).
\]

One has
\[
\mathcal{M}'(t) = \left( m \lambda t^{\lambda-1} \left( p - \frac{m(1-\lambda)}{t}x \right) \cdot \nabla \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) + \text{(h.c.)} \right)
+ t^{-2+\lambda} \left( 2m^2(1-\lambda) - 2m^2(1-\lambda)(1-2\lambda) \right) x \cdot \nabla \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right)
- \left( \frac{1}{t^{1-2\lambda}} \left( p - \frac{m(1-\lambda)}{t}x \right) \Delta \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) \cdot \frac{m(1-\lambda)x}{t} + \text{(h.c.)} \right)
\]
and
\[
i[p^2/(2m), \mathcal{M}(t)] = \frac{1}{t^{1-2\lambda}} \left( p - \frac{m(1-\lambda)}{t}x \right) \Delta \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) \cdot p + \text{(h.c.)}
+ \frac{1}{t^{1-\lambda}} \left( -m(1-\lambda) + m(1-2\lambda) \right) \left( p \cdot \nabla \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) + \text{(h.c.)} \right)
+ it^{-2+3\lambda} \left( \left( p - \frac{m(1-\lambda)}{t}x \right) \cdot (\nabla \Delta \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) + \text{(h.c.)} \right),
\]
\[
i[k(t)x^2/2, \mathcal{M}(t)] = -2mt^\lambda k(t)x \cdot \nabla \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right)
= 2m^2\lambda(\lambda - 1)t^{-2+\lambda}x \cdot \nabla \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right)
\]
for \( t > r_0 \). Sum up all results, one also has
\[
\mathcal{D}_{H(t)}(\mathcal{M}(t)) = \frac{1}{t^{1-2\lambda}} \left( p - \frac{m(1-\lambda)}{t}x \right) \Delta \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) \left( p - \frac{m(1-\lambda)}{t}x \right)
+ it^{-2+3\lambda} \left( \left( p - \frac{m(1-\lambda)}{t}x \right) \cdot (\nabla \Delta \mathcal{R} \left( \frac{x}{t^{1-\lambda}} \right) + \text{(h.c.)} \right)
+ \Theta_V(t; x)
\]
(3.5)
where \( \Theta_V(t; x) := i[V(t; x), \mathcal{M}(t)] \in L^1(t; dt) \). Here we note the middle term of the above equation is clearly integrable on \( \varphi(\alpha(t)) \). Define \( \mathcal{L}(t) \) as follows

\[
\mathcal{L}(t) = \varphi_1 F_\varepsilon \left( \frac{|x|}{t^{1-\lambda}} \leq \theta + \varepsilon_4 \right) \mathcal{M}(t) F_\varepsilon \left( \frac{|x|}{t^{1-\lambda}} \leq \theta + \varepsilon_4 \right) \varphi_1,
\]

with \( \varepsilon_4 = \varepsilon_3 + \varepsilon > 0 \). For simplicity, we denote

\[
F_\varepsilon \left( \frac{|x|}{t^{1-\lambda}} \leq \theta + \varepsilon_4 \right) = F^{MS}, \quad F_\varepsilon \left( \varepsilon_2 \leq t^{-1+\lambda} |x| \leq \theta + \varepsilon_3 \right) = F^M.
\]

Here we note that, on the support of \( M(t) \), \( |x| > (\varepsilon_2 - \varepsilon) t^{1-\lambda} \) holds and which yields \( i[V(t, \varphi_1)] F^{MS} M(t) F^{MS} \varphi_1 \in L^1(t; dt) \) by Lemma 2.1 with \( \rho = (1 - \lambda) \) and \( N = 0 \).

By the virtue of (3.5), for \( \phi(s) \in L^2(\mathbb{R}^n) \),

\[
\frac{d}{dt} (\mathcal{L}(t)v(t, s), v(t, s)) = \left( \varphi_1 \mathcal{D} H(t)(F^{MS}) M(t) F^{MS} \varphi_1 + (\text{h.c.}) \right) v(t, s), v(t, s)
\]

\[
+ \left\| \Delta \nabla \left( \frac{x}{t^{1-\lambda}} \right) \left( \frac{p - m(1 - \lambda)}{t} \right) F^{MS} \varphi_1 v(t, s) \right\|^2 \frac{1}{t^{1+2\lambda}} + L^1(t; dt)
\]

holds. Here by the definition of \( \mathcal{R}(\cdot) \), one can see that

\[
F^M \sqrt{\Delta \nabla \left( \frac{x}{t^{1-\lambda}} \right)} F^{MS} = \sqrt{2n} F^M
\]

holds. It implies that

\[
\left\| \Delta \nabla \left( \frac{x}{t^{1-\lambda}} \right) \left( \frac{p - m(1 - \lambda)}{t} \right) F^{MS} \varphi_1 v(t, s) \right\|^2 \frac{1}{t^{1+2\lambda}}
\]

\[
\geq \left\| F^M \left( \frac{p - m(1 - \lambda)}{t} \right) \varphi_1 v(t, s) \right\|^2 \frac{1}{t^{1+2\lambda}} - L^1(t; dt)
\]

holds. Thus Proposition 3.2 is proven if the term

\[
\left| \left( M(t) F^{MS} \varphi_1 v(t, s), \mathcal{D} H(t)(F^{MS})^* \varphi_1 v(t, s) \right) \right| \quad (3.6)
\]

is integrable in \( t \). Here we define \( F_{MM} = F_\varepsilon (\theta + \varepsilon_4 - 2 \varepsilon \leq |x| t^{-1+\lambda} \leq \theta + \varepsilon_4 + \varepsilon) \) and suppose that \( \varepsilon_4 > 3 \varepsilon \). Straightforward calculation shows that

\[
\left\| (F^{MS})' (1 - F_{MM}^2) \right\| = 0
\]

holds since \( t^{-1+\lambda} |x| \leq \theta + \varepsilon_4 - \varepsilon \) or \( \theta + \varepsilon_4 \leq t^{-1+\lambda} |x| \) holds on the support of \( 1 - F_{MM}^2 \). On the other hand, we put \( \eta_1 \geq \theta + \varepsilon_4 + 2 \varepsilon \). Then by \( \varepsilon_4 > 3 \varepsilon \) and \( \eta_1 \geq \theta + \varepsilon_4 + 2 \varepsilon \), we can see \( F_{MM} F_\varepsilon (\theta \leq |x| t^{1-\lambda} \leq \eta_1) = F_{MM} \). Thus, by putting \( \tilde{\varphi}_1 = \varphi_1(\alpha(t)) \) as an operator be such that \( \tilde{\varphi}_1 \varphi_1 = \varphi_1 \) and \( \tilde{\varphi}_1 \in C^\infty_0 (\mathbb{R}, \alpha(t)) \), (3.6) is smaller than

\[
C t^{-1+\lambda} \left\| \Theta(t) (F^{MS})' \right\| \left\| M(t) F^{MS} \tilde{\varphi}_1 \right\| \left\| F_{MM} \tilde{\varphi}_1 v(t, s) \right\|^2 + L^1(t; dt)
\]

\[
\leq C t^{-1} \left\| F_\varepsilon (\theta \leq |x| t^{1-\lambda} \leq \eta_1) \varphi_1 v(t, s) \right\|^2 + L^1(t; dt).
\]
By using the Proposition 3.1, (3.6) is integrable in \( t \). Thus we obtain
\[
\frac{d}{dt} \mathcal{L}(t) + L^1(t; dt) \\
\geq \left\| F_\varepsilon \left( \varepsilon_2 \leq \frac{|x|}{t^{1-\lambda}} \leq \theta + \varepsilon_3 \right) \left( p - \frac{m(1-\lambda)}{t} x \right) \varphi_1 v(t, s) \right\|^2 \frac{1}{t^{1-2\lambda}}.
\]
By the same argument in Proposition 3.1, (3.3) is obtained. \( \square \)

### 3.2 Minimal velocity estimate

We define selfadjoint operators \( \mathcal{A}(t) \) and \( \mathcal{C}(t) \) as follows
\[
\mathcal{A}(t) := x \cdot \left( p - \frac{m\lambda x}{t} \right),
\]
\[
\mathcal{C}(t) := \varphi_2(x^2/t^{2\rho\lambda}) \varphi_1(\alpha(t)) \mathcal{A}(t) \varphi_1(\alpha(t)) \varphi_2(x^2/t^{2\rho\lambda})
\]
(3.7)

where \( \varphi_2 \) is the same one as in Definition 1.4 and we also define
\[
\mathcal{M}_2(t) = t^\lambda \left( \frac{m x}{|x|} \cdot \left( p - \frac{m(1-\lambda)}{t} x \right) \left( F^S \right)' + (\text{h.c.}) \right) + 2m^2(1-2\lambda)F^S
\]
\[
F^S = F_\varepsilon(|x|/t^{1-\lambda} \leq \varepsilon_5), \quad \left( F^S \right)' = F_\varepsilon'(|x|/t^{1-\lambda} \leq \varepsilon_5)
\]
and
\[
\mathcal{L}_2(t) = \mathcal{M}_2(t) \mathcal{C}(t) t^{1-2\lambda} \mathcal{M}_2(t)
\]

Here we assume that \( 3\varepsilon + \varepsilon_2 < \varepsilon_5 < \kappa_1/(m(1-2\lambda)\sqrt{R_1}) \). Under Assumption 1.2, we obtain the following proposition:

**Proposition 3.4.** For all \( \phi(s) \in L^2(\mathbb{R}^n) \), there exists \( C > 0 \) such that
\[
\int_{t_0}^\infty \left\| F_\varepsilon(|x|/t^{1-\lambda} \leq \varepsilon_5) \varphi_2(x^2/t^{2\rho\lambda}) \varphi_1(\alpha(t)) v(t, s) \right\|^2 \frac{dt}{t} \leq C \| \phi(s) \|^2
\]
holds. Moreover, for all \( \psi(s) \in \mathcal{W}^+(\lambda; s) \),
\[
\lim_{t \to \infty} \left\| F_\varepsilon(|x|/t^{1-\lambda} \leq \varepsilon_5) U(t, s) \psi(s) \right\| = 0
\]
(3.8)
holds.

**Proof.** In this proof, we denote that
\[
\varphi_1(p^2) = \varphi_1, \quad \varphi_2(x^2/t^{2\rho\lambda}) = \varphi_2, \quad \Theta(t) = p \cdot \frac{x}{|x|} - \frac{m(1-\lambda)}{t} |x|
\]
(3.9)
moreover define

$$Q(t) = (L_2(t)v(t,s), v(t,s))$$

for the sake of simplicity. In the following, we will calculate $dQ(t)/(dt)$. By the same calculation in the proof of Proposition 3.2, one has

$$D_{H(t)}(M_2(t)) = t^{-1+2\lambda} \Theta(t)(F^S)'' \Theta(t) + \theta_V(t;x) + L^1(t;dt),$$

$$\theta_V(t;x) = i[V(t,x), M_2(t)] \in L^1(t;dt).$$

Here we define $F^S_M = F_\epsilon(\delta_5 - 2\epsilon \leq |x|/t^{1-\lambda} \leq \delta_5 + \epsilon)$ and obtain $(F^S_M)^2(F^S)'' = (F^S)'$. Noting

$$\Theta(t)(F^S_M) \varphi_1 = O(t^{-\lambda}), \quad \varphi(t) \varphi_1 F^S_M = O(t^{1-2\lambda}), \quad M_2(t) \varphi_1 = O(1)$$

and

$$\varphi_1[\Theta(t), M_2(t)] = O(t^{-1+\lambda}), \quad [F^S_M, \Theta(t)] = O(t^{-1+\lambda}), \quad [F^S_M, M_2(t)] = O(t^{-1+\lambda}),$$

we have

$$t^{-1+2\lambda} \left| D_{H(t)}(M_2(t)) \mathcal{C}(t), M_2(t)v(t,s), v(t,s) \right| \leq t^{-2+4\lambda} \left| (\Theta(t) F^S_M(F^S)'F^S_M\varphi(t) \varphi_2 \varphi_1 \varphi_2 M_2(t) v(t,s), v(t,s)) \right| + L^1(t;dt)$$

$$\leq Ct^{-1+2\lambda} \left| \Theta(t) F^S_M \varphi_1 v(t,s) \right|^2 + L^1(t;dt), \quad (3.10)$$

where we use $|x| \leq t^{1-2\lambda}$ on the support of $F^S_M$ and $\varphi_2 F^S_M = F^S_M$ for $t \gg 1$. By Proposition 3.2 with $\delta_5 - 2\epsilon > \delta_2 + \epsilon$, (3.10) is in $L^1(t;dt)$.

Now we calculate about the term

$$D_{H(t)}(t^{-1+2\lambda} \mathcal{C}(t)).$$

By the straightforward calculation, we have

$$D_{H(t)}(t^{-1+2\lambda} \mathcal{C}(t)) = \sum_{j=1}^{6} J_j(t),$$

where

$$J_1(t) = -2\rho \lambda t^{-2+2\lambda-2\rho} x^2 \varphi_2 \varphi_1 \mathcal{A}(t) \varphi_1 \varphi_2 + (h.c.),$$

$$J_2(t) = -(1 - 2\lambda) t^{-2+2\lambda} \mathcal{C}(t),$$

$$J_3(t) = (t^{-1-2\rho+2\lambda}/m) ((x \cdot p) \varphi_2 + (h.c.) \varphi_1 \mathcal{A}(t) \varphi_1 \varphi_2 + (h.c.)), \quad J_4(t) = (2/(mt^{1-2\lambda})) \frac{\varphi_2 \varphi_1}{p - m \lambda x t^{-1}} \varphi_1 \varphi_2, \quad J_5(t) = t^{-1+2\lambda} \varphi_2 i[V(t), \varphi_1(\alpha(t))] \varphi(t) \varphi_1 \varphi_2 + (h.c.),$$

$$J_6(t) = -2t^{-1+2\lambda} \varphi_2 \varphi_1 (x \cdot \nabla V(t)) \varphi_1 \varphi_2.$$
By the definition of \( \varphi_2 \), we notice that
\[
\| x |\varphi_2'(x^2/t^{2\rho_\lambda}) \| \leq Ct^{\rho_\lambda},
\]
holds. Hence, we have
\[
\mathcal{J}_1(t) = \mathcal{O}(t^{-2+\lambda+\rho_\lambda}) \in L^1(t; dt),
\]
(3.11)
where we use \( \lambda < 1/2 \) and \( \rho_\lambda = 2\lambda(1-\lambda) < 1/2 \). Moreover,
\[
| (\mathcal{J}_0(t)\phi, \phi) | \leq Ct^{-1+2\lambda} \| \langle x \rangle^{-\rho_\delta} \varphi_1 \varphi_2 \phi \| = \mathcal{O}(t^{-1+2\lambda-\rho_\delta \kappa_2}) \in L^1(t; dt)
\]
(3.12)
holds for all \( \phi \in L^2(\mathbb{R}^n) \), on the support of \( \varphi_2(x^2/t^{2\kappa_2}) \). By
\[
\| \mathcal{J}_5(t) \| \leq Ct^{-1+2\lambda} \| \varphi_2[V(t), \varphi_1] \langle x \rangle \| \| \langle x \rangle^{-1} \mathcal{A}(t) \varphi_1 \|
\]
and Lemma 2.1 with \( \rho = \rho_\lambda \) and \( N = 1 \), we have
\[
\mathcal{J}_5(t) = \mathcal{O}(t^{-1+2\lambda-(\rho_\delta,\lambda+1)\rho_\lambda+\lambda+\rho_\lambda-\lambda}) \in L^1(t; dt).
\]
(3.13)
Equations
\[
\varphi'_2 x \cdot p + (\text{h.c.}) = (\varphi'_2 x \cdot (p - m\lambda x/t) + (\text{h.c.})) + 2m\lambda \varphi'_2 x^2/t
\]
and
\[
\varphi'_2 \varphi_1 \mathcal{A}(t) = \mathcal{O}(t^{-\lambda+\rho_\lambda}), \quad x^2 \varphi'_2/t = \mathcal{O}(t^{-1+2\rho_\lambda})
\]
yield
\[
\mathcal{J}_3(t) = L^1(t; dt) + (mt^{1+2\rho_\lambda})^{-1} \sqrt{\varphi'_2 \varphi_2 \varphi_1 \mathcal{A}(t)^2 \varphi_1} \geq L^1(t; dt).
\]
(3.14)
Next we calculate \( \mathcal{J}_2(t) + \mathcal{J}_4(t) \). Inequalities
\[
\| \mathcal{A}(t) \varphi_1 L^S \| \leq 2\varepsilon_5 \sqrt{R_1} t^{1-2\lambda}, \quad \mathcal{J}_4(t) \geq (2\kappa_1/(mt)) \varphi_2 \varphi_1^2 \varphi_2,
\]
yield
\[
((\mathcal{J}_2(t) + \mathcal{J}_4(t)) \mathcal{M}_2(t)v(t, s), \mathcal{M}_2(t)v(t, s),)
\geq t^{-1} \left( 2\kappa_1/m - 2(1-2\lambda)\varepsilon_5 \sqrt{R_1} \right) \| \mathcal{M}_2(t) \varphi_2 \varphi_1 v(t, s) \|^2 - L^1(t; dt).
\]
Here we take \( \varepsilon_5 > 0 \) sufficiently small be such that \( 2\kappa_1/m - 2(1-2\lambda)\varepsilon_5 \sqrt{R_1} > 0 \), divide
\( \mathcal{M}_2(t) = \mathcal{M}_3(t) + \mathcal{M}_4(t) \) with \( \mathcal{M}_4(t) = 2m^2(1-2\lambda)F_S \) and \( \mathcal{M}_3(t) = \mathcal{M}_2(t) - \mathcal{M}_4(t) \)
and define \( 0 < \kappa_0 \ll 1 \) as a sufficiently small constant. Then we can see \( |\mathcal{M}_2(t)|^2 \geq \)
\[(1 - \kappa_0^2)\mathcal{M}_4(t)^2 + (1 - 4\kappa_0^{-2})\mathcal{M}_3(t)^2.\]

By using this, it can be deduced that there exist \(\delta_3 > 0\) and \(\delta_4 > 0\) such that
\[
((J_2(t) + J_3(t)) M_2(t)v(t, s), M_2(t)v(t, s)) \\
\geq \delta_3 \left\| F^S \varphi_2 \varphi_1 v(t, s) \right\|^2 \frac{1}{t} - \delta_4 \left\| (F^S)'(t) \varphi_2 \varphi_1 v(t, s) \right\|^2 \frac{1}{t^{1-2\lambda}} + L^1(t; dt)
\]
holds. By \((3.11), (3.12), (3.13), (3.14)\) and \((3.15)\), we obtain
\[
((D_{\mathcal{H}(t)}(\mathcal{S}(t)/t^{1-2\lambda})) M_2(t)v(t, s), M_2(t)v(t, s)) \\
\geq \delta_3 \left\| F^S \varphi_2 \varphi_1 v(t, s) \right\|^2 \frac{1}{t} - \delta_4 \left\| (F^S)'(t) \varphi_2 \varphi_1 v(t, s) \right\|^2 \frac{1}{t^{1-2\lambda}} + L^1(t; dt)
\]
(3.16)

By using \((3.10)\) and \((3.16)\) with Proposition 3.2, we obtain
\[
\frac{d}{dt} Q(t) \geq \delta_3 \left\| F^S \varphi_2 \varphi_1 v(t, s) \right\|^2 \frac{1}{t} - L^1(t; dt)
\]
holds. Thus, as the same reason in the proof of Proposition 3.1, we obtain
\[
\int_{t_0}^\infty \left\| F^S \varphi_2 \varphi_1 v(t, s) \right\|^2 \frac{dt}{t} \leq C \left\| \phi(s) \right\|^2.
\]
(3.17)

Now we prove \((3.3)\). Let \(\psi(s) \in \mathcal{W}^+(\lambda; s)\) and \(u(t, s) = U(t, s)\psi(s)\).
\[
\left\| F^S u(t_1, s) \right\|^2 - \left\| F^S u(t_2, s) \right\|^2 \\
= 2 \int_{t_2}^{t_1} \text{Re} \left( (F^S)'(u(t, s), F^S u(t, s)) \right) \frac{dt}{mt^{1-\lambda}} + \int_{t_2}^{t_1} L^1(t; dt) dt \\
\leq C(I_1)^{1/2} \times (I_2)^{1/2} + C(I_3 + I_4) + \int_{t_2}^{t_1} L^1(t; dt) dt,
\]
where
\[
I_1 = \int_{t_2}^{t_1} \left\| (F^S)' \varphi_1 u(t, s) \right\|^2 \frac{dt}{t^{1-2\lambda}}, \quad I_2 = \int_{t_2}^{t_1} \left\| F^S \varphi_2 \varphi_2 u(t, s) \right\|^2 \frac{dt}{t^{1}},
\]
\[
I_3 = \int_{t_2}^{t_1} (F^S \varphi_1(1 - \varphi_1) u(t, s), F^S u(t, s)) \frac{dt}{mt^{1-\lambda}},
\]
\[
I_4 = \int_{t_2}^{t_1} (F^S \varphi_2 u(t, s), F^S \Phi_1 u(t, s)) \frac{dt}{mt^{1-\lambda}},
\]
\[
\Phi_1 = (1 - \varphi_1) \varphi_2 + \varphi_1 (1 - \varphi_2) + (1 - \varphi_1)(1 - \varphi_2)
\]

Here, by using Proposition 3.2, \(I_1 \to 0\) holds, and by using \((3.17)\), \(I_2 \to 0\) holds. Moreover, by Lemma 2.2 and Corollary 2.3, \(I_3, I_4 \to 0\) holds, and hence the limit \(\lim_{t \to \infty} F^S u(t, s)\) exists. Here note that
\[
F^S(1 - \varphi_1) u(t, s) \to 0, \quad F^S(1 - \varphi_2) u(t, s) \to 0, \quad \text{for, } t \to \infty
\]
hold, and then we can see \(\lim_{t \to \infty} F^S \varphi_2 \varphi_1 u(t, s)\) exists. Combining this result and \((3.17)\), we can also obtain \(\lim_{t \to \infty} F^S \varphi_2 \varphi_1 u(t, s) = 0\), which implies \((3.8)\) holds.
3.3 Existence of inverse wave operators

In order to prove the asymptotic completeness of wave operators. At first, we denote that $\mathcal{P}^\pm(s)$ is a projection onto $\mathcal{W}^\pm(\lambda; s)$ from $L^2(\mathbb{R}^n)$ and prove that

$$W_{\text{in}}^\pm(s) := \lim_{t \to \pm \infty} U_0(t, s)^* U(t, s) \mathcal{P}^\pm(s)$$

exist.

By the uniformly boundedness of $U_0(t, s)^* U(t, s) \mathcal{P}^\pm(s)$, it can be proven that

$$U_0(t, s)(1 - \varphi_1^2) U(t, s) \mathcal{P}^\pm(s) \to 0$$

holds. By Proposition 3.4, we obtain

$$s - \lim_{t \to \pm \infty} U_0(t, s)^* \varphi_1^2 U(t, s) \mathcal{P}^\pm(s)$$

$$= s - \lim_{t \to \pm \infty} U_0(t, s)^* \varphi_1^2 (1 - F^S) U(t, s) \mathcal{P}^\pm(s) + 0.$$

Here $\|V(t, x)(1 - F^S)\| \in L^1(t; dt)$ holds and hence one can obtain the existence of

$$\lim_{t \to \pm \infty} U_0(t, s)^* U(t, s) \mathcal{P}^\pm(s)$$

by proving

$$U_0(t, s)^* \varphi_1^2 \mathcal{D}_{H_0(t)}(1 - F^S) U(t, s) \mathcal{P}^\pm(s) \in L^1(t; dt).$$

On the other hand, by the same calculation in the proof of Proposition 3.2, we obtain

$$\|U_0(t, s)^* \varphi_1^2 \mathcal{D}_{H_0(t)}(1 - F^S) U(t, s) \mathcal{P}^\pm(s) u(s)\|$$

$$\leq \frac{C}{|t|^{1 - \lambda}} \left( \sup_{|l| = 1} \left| \Theta(t) \varphi_1(F^S)' U(t, s) \mathcal{P}^\pm(s) u(s), \varphi_1 U_0(t, s) v(s) \right| \right) + L^1(t; dt),$$

where $\Theta(t)$ is the same one as in (3.9). Taking $\tilde{F}_S = \tilde{F}_C(\varepsilon - 2 \varepsilon \leq |x|/t \leq \varepsilon + \varepsilon)$ and noting $\tilde{F}_S(F^S)' = (F^S)',$ we obtain

$$\pm \int_{|t| = 1}^{\pm \infty} \|U_0(t, s)^* \varphi_1^2 \mathcal{D}_{H_0(t)}(1 - F^S) U(t, s) \mathcal{P}^\pm(s) u(s)\| dt$$

$$\leq C \left( \pm \int_{|t| = 1}^{\pm \infty} \|\Theta(t)(F^S)' \varphi_1 U(t, s) \mathcal{P}^\pm(s) u(s)\|^2 dt \right)^{1/2}$$

$$\times \left( \sup_{|l| = 1} \int_{|t| = 1}^{\pm \infty} \|\tilde{F}_S \varphi_1 \varphi_2 U_0(t, s) v(s)\|^2 \frac{dt}{|t|^{1 - 2 \lambda}} \right)^{1/2} + \int_{|t| = 1}^{\pm \infty} L^1(t; dt) dt,$$

where we use $\tilde{F}_S \varphi_1(1 - \varphi_2) \in L^1(t; dt).$ By Proposition 3.4 and (3.17) with $V \equiv 0,$ we can also prove all terms in the integration of the above inequality are integrable in $t$ and which implies that the all terms of above inequality are integrable in $t.$ By Cook-Kuroda method, we can conclude that the inverse wave operators $W_{\text{in}}^\pm(s)$ exist.
3.4 Asymptotic completeness

At last, we will prove Theorem 1.6. Suppose that \( \overline{\phi}_\pm(s) \in \overline{\mathcal{W}^\pm(\lambda; s)} \) and for any \( \varepsilon_0 > 0 \) take \( \phi_\pm(s) \in \mathcal{W}^\pm(\lambda; s) \) be such that \( \| \phi_\pm(s) - \overline{\phi}_\pm(s) \| \leq \varepsilon_0 \) holds. Then one can denote \( \phi_\pm(s) = \mathcal{P}_\pm(s)\phi_\pm(s) \) and notice that there exists \( u_\pm(s) \in L^2(\mathbb{R}^n) \) such that

\[
0 = \lim_{t \to \pm \infty} \left\| u_\pm(s) - U(t, s)^*U(t, s)\mathcal{P}_\pm(s)\phi_\pm(s) \right\| \\
= \left\| W^\pm(s)u_\pm(s) - \phi_\pm(s) + \overline{\phi}_\pm(s) - \overline{\phi}_\pm(s) \right\| \geq \left\| W^\pm(s)u_\pm(s) - \phi_\pm(s) \right\| - \varepsilon_0
\]

holds. It implies \( \overline{\phi}_\pm(s) \in \text{Ran}(W^\pm(s)) \), i.e., \( \overline{\mathcal{W}_\pm(\lambda; s)} \subset \text{Ran}(W^\pm(s)) \). Next we will prove \( \text{Ran}(W^\pm(s)) \subset \overline{\mathcal{W}_\pm(\lambda; s)} \). For all \( \psi_\pm(s) \in L^2(\mathbb{R}^n) \), define \( \phi_\pm(s) \in \text{Ran}(W^\pm(s)) \) by \( \phi_\pm(s) = W^\pm(s)\psi_\pm(s) \), then

\[
\| (1 - \varphi_j)U(t, s)\phi_\pm(s) \| \\
\leq \| (1 - \varphi_j)(U(t, s)\phi_\pm(s) - U_0(t, s)\psi_\pm(s)) \| + \| (1 - \varphi_j)U_0(t, s)\psi_\pm(s) \| \\
\leq C \| \phi_\pm(s) - U(t, s)^*U_0(t, s)\psi_\pm(s) \| + \| (1 - \varphi_j)U_0(t, s)\psi_\pm(s) \| .
\]

Clearly,

\[
\| \phi_\pm(s) - U(t, s)^*U_0(t, s)\psi_\pm(s) \| \to 0, \quad \text{as} \quad |t| \to \infty
\]

and hence we shall prove for all \( \hat{\varepsilon} > 0 \), \( \| (1 - \varphi_j)U_0(t, s)\psi_\pm(s) \| \leq \hat{\varepsilon} \) holds as \( |t| \to \infty \). Energy \( \alpha(t) \) satisfies, for \( \pm t > r_0 \),

\[
U_0(t, s)^*\alpha(t)U_0(t, s) = U_0(\pm r_0, s)^*\alpha(\pm r_0)U_0(\pm r_0, s) \\
= r_0^{2\lambda}U_0(\pm r_0, s)^* \left( p + \frac{m\lambda x}{r_0} \right) U_0(\pm r_0, s) \\
= r_0^{2\lambda} \left( m\zeta_1'(\pm r_0, s)x + \zeta_2(\pm r_0, s)p \right. \\
+ \frac{m\lambda}{r_0} \left. \left( \zeta_1(\pm r_0, s)x + \frac{\zeta_2(\pm r_0, s)}{m} p \right) \right)^2 \\
= r_0^{2\lambda - 2} \left( m(r_0\zeta_1'(\pm r_0, s) \mp \lambda(\pm r_0, s))x + (r_0\zeta_2(\pm r_0, s) \mp \lambda(\pm r_0, s))p \right)^2 ,
\]

\( \equiv \Xi_\pm(s) \).

Representing \( \zeta_1(t, s) \) and \( \zeta_2(t, s) \) as in (1.4) and (1.5), respectively, then the condition \( c_{1,\pm}(s)c_{4,\pm}(s) - c_{2,\pm}(s)c_{3,\pm}(s) = \mp 1/(1 - 2\lambda) \) yields that either or both of the following terms

\[
r_0\zeta_1'(\pm r_0, s) \mp \lambda(\pm r_0, s) = \pm c_{1,\pm}(s)(1 - 2\lambda)r_0^{1-\lambda} ,
\]

\[
r_0\zeta_2'(\pm r_0, s) \mp \lambda(\pm r_0, s) = \pm c_{3,\pm}(s)(1 - 2\lambda)r_0^{1-\lambda}
\]

will not be 0. Hence a space

\[
\Omega^\pm_R(s) := \left\{ \phi(s) \in \mathcal{S}(\mathbb{R}^n) \mid \delta^2 \| \phi(s) \|^2 \leq \| \Xi_\pm(s)\phi(s) \|^2 \leq R^2 \| \phi(s) \|^2 \right\}
\]

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is dense in $L^2(\mathbb{R}^n)$. Take $2\kappa_1 < \hat{\delta} < 3\kappa_1$ and $R_1/2 < \hat{R} < R_1$, where $\kappa_1$ and $R_1$ are the same ones as in (1.9). Since $\Omega_{\pm}^{\hat{R}}(s)$ is dense in $L^2(\mathbb{R}^n)$, for all $\hat{\varepsilon} = \varepsilon_{\hat{\delta}, \hat{R}} > 0$, there exists $\psi_{\pm}^{\hat{R}}(s) \in \Omega_{\pm}^{\hat{R}}(s)$ such that

$$\left\| \psi_{\pm}^{\hat{R}}(s) - \psi_{\pm}(s) \right\| \leq \varepsilon_{\hat{\delta}, \hat{R}}/3 = \hat{\varepsilon}/3$$

(3.18) and

$$\left\| (1 - \varphi_1)U_0(t, s)\psi_{\pm}^{\hat{R}}(s) \right\| = \left\| (1 - \varphi_1(\Xi(s)))\psi_{\pm}^{\hat{R}}(s) \right\| = 0$$

hold. Thus we have $\left\| (1 - \varphi_1)U_0(t, s)\psi_{\pm}(s) \right\| \leq \hat{\varepsilon}/3$. Next, we shall prove $\left\| (1 - \varphi_2)U_0(t, s)\psi_{\pm}(s) \right\| \leq \hat{\varepsilon}$ as $|t| \to \infty$. Recall that for all $\psi_{\pm}^{\hat{R}}(s) \in \Omega_{\pm}^{\hat{R}}(s)$, $(1 - \varphi_1)U_0(t, s)\psi_{\pm}(s) = 0$ holds. By replacing $\mathcal{C}(t)$ in (3.7) with $\varphi_1\mathcal{A}(t)\varphi_1$, one can prove

$$F_{\varepsilon}(|x|/|t|^{1-\lambda} \leq \varepsilon_{\chi})U_0(t, s)\psi_{\pm}^{\hat{R}}(s) \to 0 \quad (\leq \hat{\varepsilon}/3),$$

(3.19)

by mimicking the same approach of the proof of Proposition 3.4. Since

$$\left\| (1 - F_{\varepsilon}(|x|/|t|^{1-\lambda})\right\| \leq C|t|^{-N(\kappa_1-1-\lambda)} |x|^N \left\| (1 - \varphi_2) \right\| \leq C|t|^{-N(\kappa_1-1-\lambda)} \leq \hat{\varepsilon}/3, \text{ as } t \to \pm\infty$$

(3.20)

holds, we have

$$\left\| (1 - \varphi_2)U_0(t, s)\psi_{\pm}(s) \right\| \leq (\text{l.h.s. of } (3.18)) + (\text{l.h.s. of } (3.19)) + (\text{l.h.s. of } (3.20)) \leq \hat{\varepsilon}$$

holds. Thus we have $\text{Ran}(W^\pm(s)) \subset W^\pm(\lambda; s) \subset \overline{W^\pm(\lambda; s)}$. Consequently, we have $\text{Ran}(W^\pm(s)) = \overline{W^\pm(\lambda; s)}$.

### A Example of $k(t)$

In this section, we introduce an example of $k(t)$ satisfying the assumption $[\lambda 1]$. Let us define

$$k(t) = \begin{cases} 
\frac{t_0}{|t|} & |t| \leq r_0, \\
\frac{k_t^2}{|t|} & |t| > r_0, \\
\frac{k_0}{m} =: \omega_0.
\end{cases}$$

Then for $|t| \leq r_0$, we have

$$\zeta_1(t, s) = \cos(\omega_0(t - s)), \quad \zeta_2(t, s) = \omega_0^{-1} \sin(\omega_0(t - s))$$

and for $|t| > r_0$, we have

$$\zeta_1(t, s) = c_{1, \pm}(s)|t|^{1-\lambda} + c_{2, \pm}(s)|t|^\lambda, \quad \zeta_2(t, s) = c_{3, \pm}(s)|t|^{1-\lambda} + c_{4, \pm}|t|^\lambda.$$
Hence, one can easily deduce the conditions of $c_{1,\pm}(s)$, $c_{2,\pm}(s)$, $c_{3,\pm}(s)$ and $c_{4,\pm}(s)$ so that $\zeta_j(t,s) \in C^1(\mathbb{R})$ and be the twice differentiable functions. Indeed, by the simple calculations, we have

$$r_0^{1-\lambda}(1-2\lambda)c_{1,\pm}(s) = -\lambda \cos(\omega_0(r_0 \mp s)) \mp r_0\omega_0 \sin(\omega_0(r_0 \mp s)),$$

$$r_0^{\lambda}(1-2\lambda)c_{2,\pm}(s) = (1-\lambda) \cos(\omega_0(r_0 \mp s)) \pm r_0\omega_0 \sin(\omega_0(r_0 \mp s)),$$

$$\omega_0r_0^{1-\lambda}(1-2\lambda)c_{3,\pm}(s) = \omega_0r_0 \cos(\omega_0(r_0 \mp s)) \mp \lambda \sin(\omega_0(r_0 \mp s)),$$

$$\omega_0r_0^{\lambda}(1-2\lambda)c_{4,\pm}(s) = -\omega_0r_0 \cos(\omega_0(r_0 \mp s)) \pm (1-\lambda) \sin(\omega_0(r_0 \mp s)).$$

For all given $\omega_0$, $s$, $k_0$ and $k$, by taking $c_{j,\pm}(s)$, $j = 1, 2, 3, 4$, as above, $\zeta_j(t,s)$, the solution of (1.2) satisfy Assumption 1.1. Here we remark that one can admit $k_0 = 0$ since $\omega_0^{-1} \sin(\omega_0(r_0 \mp s)) \to r_0 \mp s$ as $\omega_0 \to 0$.

References

[1] Adachi, T.: Propagation estimates for N-body Stark Hamiltonians. Ann. del'I.H.P. 62, 409-428, (1995).

[2] Adachi, T., Kawamoto, M.: Quantum scattering in a periodically pulsed magnetic field, Ann. H. Poincaré. 17, 2409-2438, (2016).

[3] Adachi, T., Tamura, H.: Asymptotic completeness for long-range many-particle systems with Stark effect. J. Math. Soc. Univ. Tokyo, 2, 77-116, (1995).

[4] Bachelot, A.: Gravitational scattering of electromagnetic field by Schwarzschild black-hole. Ann. del'I. H. P. 54, 261-320, (1991).

[5] Daudé T.: Propagation estimates for Dirac operators and applications to scattering theory. Ann. Inst. Fourier Grenoble 54, 2021-2083, (2004).

[6] Dereziński, J., Gérard, C.: Scattering theory of classical and quantum N-particle systems, Text Monographs. Phys., Springer, Berlin, (1997).

[7] Geluk, J. L., Marić, V., Tomić, M.: On regularly varying solutions of second order linear differential equations, Differential and Integral Equ., 6, 329-336, (1993).

[8] Gerard, C.: Scattering theory for Klein-Gordon equations with non-positive energy. Ann. Henri. Poincaré, 13, 883-941, (2012).

[9] Graf, G.-M.: Asymptotic completeness for N-body short-range quantum system: A new proof, Comm. Math. Phys. 132, 73-101, (1990).

[10] Hagedorn, G. A., Loss, M., and Slawny, J.: Nonstochasticity of time-dependent quadratic Hamiltonians and the spectra of canonical transformations. J. Phys. A 19, 521-531 (1986).
[11] Herbst, I., Möller, J.S., Skibsted, E.: Asymptotic completeness for N-body Stark Hamiltonians, Comm. Math. Phys., 174, 509-535, (1996).

[12] Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et equation de Harper, Springer Lecture Notes in Physics, 345, 118-197, (1989).

[13] Huang, M. J.: On stability for time-periodic perturbations of harmonic oscillators. Ann. Inst. H. Poincaré, Phys. Théor. 50, 229-238 (1989).

[14] Hochstadt, H.: Functiontheoretic properties of the discriminant of Hill’s equation, Math. Z. 82, 237-242, (1963).

[15] Kawamoto, M.: Mourre theory for time-periodic magnetic fields, (preprint, 2015).

[16] Kitada, H., Yajima, K.: A scattering theory for time-dependent long-range potentials, Duke Math. J., 49, 341-376, (1982).

[17] Korotyaev, E. L.: On scattering in an external, homogeneous, time-periodic magnetic field, Math. USSR-Sb., 66, 499-522, (1990).

[18] Mourre, E.: Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys., 91, 391-408, (1981).

[19] Naito, M.: Asymptotic behavior of solutions of second order differential equations with integrable coefficients, Trans. A.M.S., 282, 577-588, (1984).

[20] Willett, D.: On the oscillatory behavior of the solutions of second order linear differential equations, Ann. Polon. Math., 21, 175-194, (1969).

[21] Yafaev, D. R.: Scattering subspaces and asymptotic completeness for the time-dependent Schrödinger equation, Mat. Sb., 118 (160), 262-279, (1982).

[22] Yajima, K.: Scattering theory for Schrödinger equations with potentials periodic in time. J. Math. Soc. Jpn. 29, 729-743, (1977).

[23] Yokoyama, K.: Mourre theory for time-periodic systems, Nagoya Math. J., 149, 193-210, (1998).