On quantum operations as quantum states

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We formalize the correspondence between quantum states and quantum operations isometrically, and harness its consequences. This correspondence was already implicit in the various proofs of the operator sum representation of Completely Positive-preserving linear maps; we go further and show that all of the important theorems concerning quantum operations can be derived directly from those concerning quantum states. As we do so the discussion first provides an elegant and original review of the main features of quantum operations. Next (in the second half of the paper) we find more results stemming from our formulation of the correspondence. Thus we provide a factorizability condition for quantum operations, and give two novel Schmidt-type decompositions of bipartite pure states. By translating the composition law of quantum operations, we define a group structure upon the set of totally entangled states. The question whether the correspondence is merely mathematical or can be given a physical interpretation is addressed throughout the text: we provide formulae which suggest quantum states inherently define a quantum operation between two of their subsystems, and which turn out to have applications in quantum cryptography.

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This article is concerned with the properties of positive matrices (quantum states) and the linear maps between these, i.e. Positive-preserving linear maps and Completely Positive-preserving linear maps (quantum operations), as provided by the density matrix formalism of finite dimensional quantum theory. The analysis we carry out is formal and mathematical, and although it focuses on some quantum information theoretical issues, it should have applications in other domains. The driving line of the article is in its method: formalizing and exploiting systematically an isomorphism from hermitian matrices to Hermitian-preserving linear maps and quantum states to quantum operations. To our knowledge, this isomorphism was first used by Sudarshan et al. in the quantum theoretical context, and was later popularized by Jamiołkowski and Choi. The operator sum representation theorem has been independently derived by Kraus (see also) – with a proof valid in infinite dimensions. Our investigation shows that the isomorphism between states and operations has a much wider range of implications, whether to simplify the proofs of well-known results or to point out novel properties, both technical and geometrical. The presentation is rigorous and self-contained, we give all the necessary background for someone to enter the subject.

In section we after setting our conventions, we relate vectors to matrices, and matrices to superoperators, the idea being to map an $m \times n$ matrix to a linear operator from $n \times n$ matrices to $m \times m$ matrices. These isomorphisms are often viewed pragmatically as rearrangements of the coordinates of vectors or matrices, but we formalize them more abstractly as norm-preserving bijections between tensor product spaces. We derive original formulae relating to these isomorphisms which we use throughout the article. One of them will simplify those numerous mathematical problems in quantum cryptography which require a careful optimization of the fidelities induced by a quantum operation. This formal setting leads in subsection to the state-operator equivalence, inherently present in the works many, but rarely exploited as such: non-normalized quantum states of an $mn$-dimensional system are equivalent to quantum operations from an $n$-dimensional system to an $m$-dimensional one. We use this correspondence in subsection to rederive all the main properties of quantum operations from those of quantum states: the operator sum decomposition and its unitary degree of freedom stem from the spectral decomposition and Hughston-Josza-Wooters theorems; the factorizability of quantum operations up to a trace-out corresponds to the purification of quantum states; and the polar decomposition of matrices is equivalent to the Schmidt decomposition of pure states. Next, in subsection we consider properties of states (or operations) whose translation in terms of operators (or states) was unknown to us previously. Mainly we give a factorizability condition for quantum operations, i.e. a criteria for an operator to be single operator in the operator sum representation; and we find two original triangular decompositions of pure states of a bipartite system. Throughout the section the normalization of density matrices is unimportant. Yet for completeness the reader is reminded of the well known Trace-preserving conditions in subsection (both in terms of states and operators). Moreover we
highlight the fact that maximally entangled pure states of
a bipartite system go hand in hand with isometric maps
from one subsystem to the other (unitary maps in case
both systems have the same dimension). Choi’s extremal
Trace-preserving condition is also presented and recasted
in terms of the rank of an easily constructed matrix.

Section III is devoted to geometrical structures of quan-
tum states. We exploit the composition law on Com-
pletely Positive-preserving maps to define a semi-group
structure on the states of n-dim-dimensional quantum sys-
tems, and show that the subset of totally entangled pure
states is isomorphic to the group of invertible n × n ma-
trices defined up to phase (with maximally entangled
pure states corresponding to unitary transforms as in
). These group isomorphisms have profound structural
meaning, and are useful in finding nice coordinate charts
on such spaces. We also give an exotic composition law
on operators stemming from the Schur product on states.
In subsection III we make use of the dual mapping be-
tween states and positive functionals, and readily show
that the space of Positive-preserving maps is dual to that
of separable states of a bipartite system. This yields a
simple result which is in fact equivalent to Peres’ sepa-
rbility criterion. More generally the notion of duality
seems to help provide possible physical interpretations
of the state-operator correspondence formulae, notably as
we show that the effect of any quantum operation can be
viewed as the trace out of a particular local single oper-
ation on its corresponding state.

We conclude in section IV and give a table summarizing
the main results.

I. THE SETTING

We denote by M_d(C) the set of d × d matrices of com-
plex numbers, and by Herm^n_d(C) its hermitian subset.
Amongst the latter we will denote by Herm^n+_d(C) the set
of positive matrices, and also refer to it as the set of (nor-
normalized) states of a d-dimensional quantum system.
An important subset of Herm^n+_{mn}(C) is the set of separable
states, i.e. those which can be written in the form

$$\rho = \sum \lambda_x \rho^x_1 \otimes \rho^x_2$$

where $\lambda_x \geq 0$ and $\rho^x_1$ and $\rho^x_2$ belong to Herm^n+_{m}(C)
and Herm^n+_{n}(C) respectively. Later we shall denote this
set by Herm^S_{mn}(C).

Throughout the dagger operation † will be somewhat
overloaded, in a manner which has now become quite
standard: as usual a ket $A = \sum A_i |i\rangle$ will be taken into
a bra $A^\dagger = \sum A_i^* \langle i|$, while a matrix $A = \sum A_{ij} |i\rangle \langle j|$ will
be mapped into its conjugate transpose $A^\dagger = \sum A_{ij}^* \langle j|\langle i|$. In
other words, † takes kets into bras using the canonical
complex scalar product for vectors, i.e. $B^\dagger \equiv [A \mapsto
(B, A) = \sum B_i^* A_i \equiv B^\dagger A]$; but for linear maps on vec-
tors it denotes the usual adjoint operation defined with
respect to the same scalar product. We also make fre-
cquent use of the conjugation operation * which is de-
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A. Isomorphisms

Next we relate vectors of $\mathbb{C}^m \otimes \mathbb{C}^n$ to endomorphisms from $\mathbb{C}^m$ to $\mathbb{C}^n$. The tensor split of $\mathbb{C}^{mn}$ into $\mathbb{C}^m \otimes \mathbb{C}^n$ is considered fixed, as will be all tensor splits throughout the article unless specified otherwise. (Notions of entanglement will refer to a particular tensor product of spaces, given a priori.) Let $\{|i\rangle\}$ and $\{|j\rangle\}$ be orthonormal bases of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, which we will refer to as canonical.

**Isomorphism 1** The following linear map

$$\hat{\cdot} : \mathbb{C}^m \otimes \mathbb{C}^n \rightarrow \text{End}(\mathbb{C}^n \rightarrow \mathbb{C}^m)$$

$$A \mapsto \hat{A} \quad \sum_{ij} A_{ij} |i\rangle\langle j| \rightarrow \sum_{ij} A_{ij} |i\rangle\langle j|$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$, is an isomorphism taking vectors $A$ into $m \times n$ matrices $\hat{A}$. It is isometric in the sense that:

$$\forall A, B \in \mathbb{C}^m \otimes \mathbb{C}^n, \quad B^\dagger A = \text{Tr}(\hat{B}^\dagger \hat{A}) \quad (1)$$

**Proof.** This is trivial, but note that the definition of this isomorphism is basis dependent. $\square$

Following a very convenient notation introduced by Sudarshan we will often use a semicolon ';' to separate output indices (on the left) from input indices (on the right), together with the repeated indices summation convention. For instance the matrix $\hat{A} : \mathbb{C}^n \rightarrow \mathbb{C}^m$ will be denoted $A_{ij}$, so that $w = \hat{A}v$ is simply written as $w_i = A_{ij}v_j$. Thus the 'hat' operation acts as follows:

$$\text{if } A \equiv A_{ij} \text{ then } \hat{A} \equiv \hat{A}_{ij} \text{ with } \hat{A}_{ij} = A_{ij} \quad (2)$$

Another useful interpretation of this operation is provided in 18, by considering the canonical maximally entangled state of $\mathbb{C}^n \otimes \mathbb{C}^n$, $|\beta\rangle = \sum |j\rangle|j\rangle$. Indeed we have:

$$A = (\hat{A} \otimes I_d_n)|\beta\rangle \quad (3)$$

$$\hat{A} = (I_d_m \otimes \langle \beta|)(A \otimes I_d_n)$$

We now use the previous isomorphism to relate elements of $M_{mn}(\mathbb{C})$ to linear maps from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$. This formalizes some of the key steps in 6,2,1.

**Isomorphism 2** The following linear map:

$$\hat{\cdot} : \mathbb{C}^{mn} \otimes (\mathbb{C}^{mn})^\dagger \rightarrow \text{End}(M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}))$$

$$\$ \mapsto \hat{\$} : \rho \mapsto \hat{\$}(\rho)$$

such that $AB^\dagger \rightarrow [\rho \mapsto \hat{A}\rho\hat{B}^\dagger]$ i.e.

$$\sum_{ijkl} A_{ij} B_{kl}^\dagger |i\rangle\langle k| |j\rangle\langle l| \rightarrow [\rho \mapsto \sum_{ijkl} A_{ij} B_{kl}^\dagger |i\rangle\langle j| \rho |l\rangle\langle k|]$$

where $i, k = 1, \ldots, m$ and $j, l = 1, \ldots, n$, is an isomorphism. It is isometric in the sense that:

$$\forall \$, \$ \in M_{mn}(\mathbb{C}), \quad \text{Tr}(\$^\dagger \$) = \sum_{jl} \text{Tr}(\hat{\$}(E_{jl})^\dagger \hat{\$}(E_{jl})) \quad (4)$$

where $\{E_{jl} = |j\rangle\langle l|\}$ is the canonical basis of $M_n(\mathbb{C})$.

Before we give a proof we shall reassert Sudarshan’s notation in this case. Suppose $\$ = $s_{ijkl}|j\rangle\langle k| |l\rangle\rangle$ so that we can write $\$ = $s_{ijkl}$. We then have:

$$\hat{\$} \equiv \hat{s}_{ikjl} \text{ with } \hat{s}_{ikjl} = s_{ijkl} \quad (5)$$

so that $\hat{\$} : \rho_{jl} \mapsto \hat{\$}(\rho)_{ik} = \hat{s}_{ikjl}\rho_{jl}$

This notation views $\text{End}(M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}))$ as $m^2 \times n^2$ matrices, or as superoperators, thus admitting the usual Hilbert-Schmidt inner-product:

$$\text{Tr}((\hat{\$}_{jl}^\dagger)(\hat{\$}'_{j'l'}) \rangle \langle \$_{jl} | \hat{\$}'_{j'l'} \rangle \langle \$_{jl}) \quad (6)$$

where $\hat{\$}_{jl}^\dagger$ is an $n^2 \times m^2$ matrix. The superoperator formalism simply consists of labelling a linear operator on matrices by a super-matrix, or more generally a linear map on tensors by a bigger tensor, and hence helps define operator norms. In fact it will turn out to be a corner stone of the state-operator correspondence. It has had many applications in physics, amongst them the super-scattering or “dollar” operator formalism introduced in Quantum Field Theory by Hawking [14], which, in contrast with the S-matrix formalism, allows non-unitary evolutions (hence our notation).

**Proof of Isomorphism** Elements of $\mathbb{C}^{mn} \otimes (\mathbb{C}^{mn})^\dagger$ are all of the form $\sum_x A_x B_x^\dagger$, and thus by linearity the map $\hat{\cdot}$ is fully determined by the above. The fact that it is an isomorphism is made obvious by Equation (5).

Now let $\$ = $s_{ijkl}$ and $\$ = $s_{ijkl}$. We now show that the notion of inner product given by (6) is precisely that of the RHS of Equation (4). Since $\hat{\$}_{ikjl} = \hat{\$}(E_{jl})_{ik}$ and $\hat{\$}'_{jl} = \hat{\$}'(E_{jl})_{ik}$ we have:

$$\text{Tr}((\hat{\$}_{jl}^\dagger)(\hat{\$}'_{j'l'}) \rangle \langle \$_{jl} | \hat{\$}'_{j'l'} \rangle \langle \$_{jl}) = \text{Tr}((\hat{\$}'_{jl}^\dagger)(\hat{\$}_{j'l'}) \rangle \langle \$_{jl} | \hat{\$}'_{j'l'} \rangle \langle \$_{jl})$$

Finally notice that $\hat{\$}_{jl} = \hat{s}_{ijkl}' = e_{ijkl}$, using (6).

Thus (6) is also equal to the LHS of (4):

$$\text{Tr}((\hat{\$}_j^\dagger)(\hat{\$}_l') \rangle \langle \$_j | \hat{\$}_l' \rangle \langle \$_j) = e_{ijkl}' s_{ijkl}' = e_{ijkl} s_{ijkl}$$

$$\text{Tr}(\$) \quad \square$$
In terms of the canonical maximally entangled state $|\beta\rangle$ of $\mathbb{C}^n \otimes \mathbb{C}^n$, using (4), we have that
\[ s = (\hat{s} \otimes I_d)(|\beta\rangle\langle\beta|) \]  
(7)

Note that $|\beta\rangle\langle\beta| = \sum E_{jl} \otimes E_{jl}$, so we get
\[ s = \sum_j \hat{s}(E_{jl}) \otimes E_{jl} \]  
(8)

This relation is quite handy when one seeks to visualize the isomorphism in terms of matrix manipulation. It is clear that the isomorphisms $\sim$ and $\hat{\sim}$ are biased towards interpreting states in $\mathbb{C}^{mn} = \mathbb{C}^m \otimes \mathbb{C}^n$ as linear operations from states in the second subspace $\mathbb{C}^n$ into states in the first subspace $\mathbb{C}^m$. This will be made explicit in the forthcoming theorems. Without difficulty we could do the contrary and view states in $\mathbb{C}^{mn}$ as operations from states in $\mathbb{C}^m$ to states in $\mathbb{C}^n$:

For $A = \sum_{ij} A_{ij} |i\rangle\langle j| \in \mathbb{C}^{mn}$, let $\hat{A} = \sum_{ij} A_{ij} |i\rangle\langle j|$, i.e. $\hat{A} = A_{ji} = \hat{A}$. For $s = AB^\dagger \in M_{mn}(\mathbb{C})$

let $\tilde{s} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$, $\rho \mapsto \sum_{ijkl} A_{ij} B_{kl}^\dagger |i\rangle\langle j| \otimes |k\rangle\langle l|$, which implies:
\[ s = \tilde{s}_{jl} \equiv s_{jli} = \tilde{s}_{ikjl} \text{, i.e. } \tilde{s}_{jl} = \tilde{s}_{ikjl}. \]  
(9)

In this case Equation (8) becomes:
\[ s = \sum_k E_{ik} \otimes \tilde{s}(E_{ik}) \]  
(10)

Note that with the usual tensor product convention of taking the right-hand-side matrix as the one to be plugged into each component of the left-hand-side matrix, Equation (10) is simply written $s = \tilde{s}(E_{ik})_{ik}$, which is precisely Choi’s formalism in [2]. Thus many view these two Isomorphisms as rearrangements of the coordinates of vectors or matrices. Although all would work equally well with $\hat{\sim}$, from now on we shall keep to our initial version of the isomorphisms, taking the second subspace into the first.

B. Useful Formulae

The following two lemmas are simple but useful results related to isomorphisms 1 and 2.

**Lemma 1** Let $A, B \in \mathbb{C}^m \otimes \mathbb{C}^n$, so that $AB^\dagger \in \mathbb{C}^{mn} \otimes (\mathbb{C}^{mn})^\dagger$, and let $T_1$ and $T_2$ denote the partial traces on $\mathbb{C}^{mn}$ and $\mathbb{C}^n$ respectively. Then we have:
\[ T_1(AB^\dagger) = (B^\dagger A)^\dagger \]  
(11)
\[ T_2(AB^\dagger) = \hat{A}\hat{B}^\dagger \]  
(12)

**Proof.** Let $A = A_{ij}$ and $B = B_{kl}$ with $i, k = 1, \ldots, m$ and $j, l = 1, \ldots, n$. $AB^\dagger = A_{ij} B_{kl}^\dagger |i\rangle\langle j| \otimes |k\rangle\langle l|$. Thus taking $T_1$ sets $i = k$ and taking $T_2$ sets $j = l$:
\[ T_1(AB^\dagger)_{jl} = A_{ij} B_{kl}^\dagger = B_{kl}^\dagger A_{ij} = (B^\dagger A)^\dagger_{jl} \]
\[ T_2(AB^\dagger)_{ik} = A_{ij} B_{kl}^\dagger = \hat{A}\hat{B}^\dagger_{ik} \]

**Lemma 2** Suppose $\hat{\sim}$ is defined for $n$ fixed and for all $d$ such that it takes any element of $\mathbb{C}^{dn} \otimes (\mathbb{C}^{dn})^\dagger$ to a linear map from $M_n(\mathbb{C})$ to $M_d(\mathbb{C})$:
\[ \forall d, \quad \hat{\sim} : \mathbb{C}^{dn} \otimes (\mathbb{C}^{dn})^\dagger \rightarrow \text{End}(M_n(\mathbb{C}) \rightarrow M_d(\mathbb{C})) \]

and let $T_1$ denote the partial trace on the first $r$-dimensional subsystem of any system. We then have:
\[ \forall s \in \mathbb{C}^{rnm} \otimes (\mathbb{C}^{rnm})^\dagger, \quad T_1(\hat{s}) = T_1 \circ \hat{s} \]

in other words $T_1$ and $\hat{\sim}$ commute.

**Proof:** In the following, $i, k = 1, \ldots, m$ and $j, l = 1, \ldots, n$ as usual, while $p, q = 1, \ldots, r$. Let $s = s_{ijkl} \in \mathbb{C}^{rnm} \otimes (\mathbb{C}^{rnm})^\dagger$, and $\rho = \tau_{ijkl}$.

Then $\tilde{s}(\rho)_{ijkl} = \tilde{s}_{ijkl} = s_{ijkl}$ is in $M_{rn}(\mathbb{C})$. Since $T_1$ sets $p = q$, $(T_1 \circ \tilde{s})(\rho)_{ijkl} = s_{ijkl}$, so on the other hand $T_1(\tilde{s})_{ijkl} = s_{ijkl}$ and $T_1(\tilde{s}) = T_1(s)_{ijkl} = s_{ijkl}$, thus $T_1(\tilde{s})(\rho)_{ijkl} = s_{ijkl}$.

Next we give a novel and powerful formula relating linear operations $s$ to trace outs of matrix multiplications involving $s$.

**Proposition 1** Let $\hat{s}$ a linear map from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$, $\sigma, \tau$ two elements of $M_n(\mathbb{C})$, $\kappa, \tau$ two elements of $M_m(\mathbb{C})$. Then we have:
\[ \kappa \hat{s}(\rho \sigma) \tau = T_2((\kappa \otimes \rho^\prime \sigma^\prime \tau) \circ \hat{s}) \]  
(13)

where $T_2$ denotes the partial trace over the second system $\mathbb{C}^n$ in $\mathbb{C}^m \otimes \mathbb{C}^n$. This implies that for all $\rho \in M_n(\mathbb{C})$ and $\kappa \in M_m(\mathbb{C})$,
\[ T_r(\kappa \hat{s}(\rho)) = T_r((\kappa \otimes \rho^\prime) \circ \hat{s}). \]  
(14)

**Proof:** Since $(\kappa \otimes \rho^\prime)_{ijkl} = \kappa_{ij} \rho^\prime_{jk}$, $(\tau \circ \sigma^\prime)_{ijkl} = \tau_{ij} \sigma^\prime_{jk}$, and tracing out $\mathbb{C}^n$ consists of setting $j = l$, we have
\[ (\kappa \otimes \rho^\prime) = (\kappa \otimes \rho^\prime)_{ijkl} = \kappa_{ij} \rho^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} \]  
\[ T_2((\kappa \otimes \rho^\prime) \circ \hat{s})_{ijkl} = \kappa_{ij} \rho^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} = \kappa_{ij} \rho^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} \]  
\[ = \kappa_{ij} \rho^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} \tau_{ij} \sigma^\prime_{jk} = \kappa \hat{s}(\rho \sigma) \tau. \]  

Equation (13) follows immediately by letting $\tau = Id_m$, $\sigma = Id_n$ and taking the total trace.

\[ \square \]
From Equation (13) one can also derive the following interesting formula: \( \forall \rho \in M_n(\mathbb{C}), \)

\[
\tilde{S}((\rho^\dagger \rho)^t) = \text{Tr}_2((Id_m \otimes \rho)(Id_m \otimes \rho^t)) = \text{Tr}_2((\rho^\dagger \otimes Id_n)(\rho \otimes Id_n)) \quad (15)
\]

We shall come back to Equation (15) in subsection III B, with a more physical point of view. For now note that the equation is slightly more general than the one given in \[15\] p4, and that its equivalent form for \( \tilde{\gamma} \) is clearly seen to define a map from the first subspace into the second:

\[
\tilde{\gamma}((\rho^\dagger \rho)^t) = \text{Tr}_1((\rho^\dagger \otimes Id_n)(\rho^t \otimes Id_n)).
\]

Moreover the original Equation (14) will have a wide range of applications in the field of quantum information theory. This is because many of the mathematical problems raised by quantum cryptography require a careful optimization of the fidelities induced by a linear operator \( \gamma \). By means of this formula such involved expressions can elegantly be brought to just the trace of the product of two matrices \( \gamma \).

C. The correspondence

We proceed to give the well-known three fundamental theorems about isomorphism 2.

**Theorem 1** The linear operation \( \tilde{S} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is Hermitian-preserving if and only if \( S \) belongs to \( \text{Herm}_{mn}(\mathbb{C}) \).

**Proof.** \( \Rightarrow \) Suppose \( \tilde{S} \) is Hermitian-preserving, then by Remark II so is \( (\tilde{S} \otimes Id_n) \). Now since \( \beta \langle \beta \rangle \) is hermitian it must be the case that \( (\tilde{S} \otimes Id_n)(\langle \beta \rangle \langle \beta \rangle) = S \) is hermitian. We used Equation (17) for the last equality.

\( \Leftarrow \) Suppose \( S \) is Hermitian, so that \( S_{ij,kl} = S_{kl,ij}^* \). Let \( \rho_{ij} = \rho_{ij}^* \in \text{Herm}_n(\mathbb{C}) \). Using (15) we have

\[
\tilde{S}(\rho)_{ik,kl} = \tilde{S}_{ik,kl} \rho_{ij} = \delta_{ik,kl} \rho_{ij} = S_{kl,ij}^* \rho_{ij} = \tilde{S}(\rho_{ij}^*)_{ki,ij} = \tilde{S}(\rho)_{ki,ij}^*
\]

so that \( \tilde{S} \) is Hermitian-preserving. \( \Box \)

This result first appeared in \[13\]. In terms of components, \( \tilde{S} \) is Hermitian-preserving if and only if \( S_{ij,kl} = S_{kl,ij}^* \), or equivalently \( \tilde{S}_{ik,kl} = \tilde{S}_{kl,ij}^* \).

**Theorem 2** The linear operation \( \tilde{S} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is Positive-preserving if and only if \( S \) belongs to \( \text{Herm}_{mn}^+(\mathbb{C}) \) and is such that for all separable state \( \rho \) in \( \text{Herm}_{mn}^+(\mathbb{C}) \), \( \text{Tr}(\rho) \geq 0 \).

**Proof.** \( S \) is Hermitian by theorem II since \( \tilde{S} \) is Hermitian-preserving. Using Equation (14) in the following, with \( \rho, \rho_1 \in \text{Herm}_{mn}^+(\mathbb{C}) \) and \( \sigma, \rho_2 \in \text{Herm}_{mn}^+(\mathbb{C}) \), we have:

\[
\tilde{S} \text{ is Positive-preserving}
\]

\( \Leftrightarrow \forall \rho, \forall \sigma, \text{Tr}(\tilde{S}(\rho \otimes \rho_2)) \geq 0 \)

This result is shown for instance in \[16\], in a different manner. We shall come back to its geometrical consequences in section IIII.

**Theorem 3** The linear operation \( \tilde{S} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is Completely Positive-preserving if and only if \( S \) belongs to \( \text{Herm}_{mn}_{mn}^+(\mathbb{C}) \).

**Proof.** \( \Rightarrow \) Suppose \( \tilde{S} \) is Completely Positive-preserving. Since \( |\beta \rangle \langle \beta | \) is positive it must be the case that \( (\tilde{S} \otimes Id_n)(|\beta \rangle \langle \beta |) = S \) is positive. We used Equation (17) for the last equality.

\( \Leftarrow \) Suppose \( S \) is Positive. We want to show that for all \( \rho \) \( S \otimes Id_r : M_{mn}(\mathbb{C}) \rightarrow M_{mn}(\mathbb{C}) \) is Positive-preserving. Let \( C \in M_{mn}(\mathbb{C}) \) be such that:

\[
C = \tilde{S} \otimes Id_r.
\]

Explicitly, with \( s, t, u, v = 1, \ldots, r \), and \( i, k = 1, \ldots, m \) and \( j, l = 1, \ldots, n \) as usual,

\[
(V \otimes Id_r)(\langle is | (kt) \rangle \langle ju | lv \rangle) = \delta_{ss'} \delta_{tt'} S_{ij,kl}
\]

hence \( C \in \text{Herm}_{mn}^+(\mathbb{C}) \). Then, by theorem II \( \tilde{S} \otimes Id_r \) is Positive-preserving if for all \( \rho_1 \in \text{Herm}_{mn}^+(\mathbb{C}) \) and \( \rho_2 \in \text{Herm}_{mn}^+(\mathbb{C}) \), \( \text{Tr}(C(\rho_1 \otimes \rho_2)) \geq 0 \). This follows directly since \( \rho_1 \otimes \rho_2 \) and \( C \) are positive. \( \Box \)

This result first appears in \[16\], and later \[2\] with a different proof. The (possibly non-normalized) states of a \( mn \)-dimensional quantum system, or elements of \( \text{Herm}_{mn}^+(\mathbb{C}) \), are thus in one-to-one correspondence with the (possibly non Trace-preserving) quantum operations,
or Completely Positive-preserving maps, taking an $n$-dimensional system into an $m$-dimensional system. We claim that virtually all of the important, well-established results about quantum operations are in direct correspondence with those regarding quantum states, through the use of theorem $\text{8}$. In $\text{8}$, the operator sum representation for Completely Positive-preserving maps is derived in the proof of theorem $\text{8}$ but in our approach we will think of it as stemming directly from the properties of quantum states.

II. PROPERTIES OF QUANTUM STATES AND QUANTUM OPERATIONS

A. Properties rediscovered via the correspondence

Property 1 (Decomposition, degree of freedom.) A matrix $\rho$ is in $\operatorname{Herm}_d^+(\mathbb{C})$ if and only if it can be written as

$$\rho = \sum_x A_x A_x^\dagger$$

where each $A_x$ is a $d$-dimensional vector. Two decompositions $\{A_x\}$ and $\{B_y\}$ correspond to the same state $\rho$ if and only if there exists an isometric matrix $U$ (i.e. $U^\dagger U = \text{Id}$) such that $A_x = \sum U_{xy} B_y$. There is a decomposition $\{A_x\}$ with $\operatorname{rank}(\rho) \leq d$ non-zero elements and such that $A_x^\dagger A_x \propto \delta_{xx'}$.

Corollary 1 (Operator sum representation.) A linear map $\hat{\mathcal{S}} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is Completely Positive-preserving if and only if it can be written as

$$\hat{\mathcal{S}} : \rho \mapsto \sum_x \hat{A}_x \rho \hat{A}_x^\dagger$$

where each $\hat{A}_x$ is an $m \times n$ matrix. Two decompositions $\{\hat{A}_x\}$ and $\{\hat{B}_y\}$ correspond to the same $\hat{\mathcal{S}}$ if and only if there exists an isometric matrix $U$ (i.e. $U^\dagger U = \text{Id}$) such that $\hat{A}_x = \sum U_{xy} \hat{B}_y$. There is a decomposition $\{\hat{A}_x\}$ with $r \leq mn$ elements and such that $\operatorname{Tr}(\hat{A}_x^\dagger \hat{A}_x) \propto \delta_{xx'}$. $r$ will be referred to as the higher rank of $\hat{\mathcal{S}}$, as this is the decomposition having the least number of elements.

Proof of Property 1. This is the spectral decomposition theorem for positive matrices, together with the unitary degree of freedom theorem by Hugshot, Josza and Wootters $\text{11}$ p103. □

Proof of Corollary 1. Consider $\hat{\mathcal{S}}$ a Completely Positive-preserving linear operator. By theorem 8 $\hat{\mathcal{S}}$ is positive, and so Property 1 provides decompositions upon that state. One may translate back these decompositions in terms of quantum operations using Isomorphism 2: this yields nothing but Corollary 1. □

Notice that the higher rank of $\hat{\mathcal{S}}$ is equal to $\operatorname{rank}(\mathcal{S})$.

Property 2 (Purification.) A matrix $\rho$ is in $\operatorname{Herm}_d^+(\mathbb{C})$ if and only if it can be written as

$$\rho = \operatorname{Tr}_1(\rho_{\text{pure}}) \quad \text{with} \quad \rho_{\text{pure}} = V V^\dagger$$

where $V$ is an $r \times d$-dimensional vector and $\operatorname{Tr}_1$ traces out the first $r$-dimensional subsystem ($r$ can be chosen equal to $\operatorname{rank}(\rho) \leq d$).

Corollary 2 (Factorizable then trace representation.) A linear map $\hat{\mathcal{S}} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is Completely Positive-preserving if and only if it can be written as

$$\hat{\mathcal{S}} : \rho \mapsto \operatorname{Tr}_1(\hat{\rho}_{\text{pure}}(\rho)) \quad \text{with} \quad \hat{\rho}_{\text{pure}} : \rho \mapsto \hat{V} \rho \hat{V}^\dagger$$

where $\hat{V}$ is an $r \times n$ matrix and $\operatorname{Tr}_1$ traces out the first $r$-dimensional subsystem ($r$ may be chosen equal to $\operatorname{rank}(\hat{\mathcal{S}}) \leq mn$). Moreover if $\hat{\mathcal{S}}$ decomposes as $\{\hat{A}_x\}$ we have:

$$\hat{V}^\dagger \hat{V} = \sum_x \hat{A}_x^\dagger \hat{A}_x \tag{17}$$

Proof of Property 2. $[\Rightarrow]$ Suppose $\rho$ decomposes as $\{A_x\}$ and let $V = \sum |x\rangle A_x$, with $\{|x\rangle\}$ an orthonormal basis of an ancilla system.

$$\rho_{\text{pure}} = V V^\dagger = \sum_{xy} |x\rangle \langle y| \otimes A_x A_y^\dagger$$

$$\operatorname{Tr}_1(\rho_{\text{pure}}) = \sum_{xy} \langle y|x\rangle A_x A_y^\dagger = \sum_x A_x A_x^\dagger = \rho$$

If $\{A_x\}$ is a spectral decomposition of $\rho$ it counts $\operatorname{rank}(\rho)$ elements, and thus $r$ can be chosen to equal $\operatorname{rank}(\rho)$.

$[\Leftarrow]$ $\langle \psi|\rho|\psi\rangle = \sum_i \langle i|\langle \psi|VV^\dagger|i\rangle|\psi\rangle \geq 0 \quad \text{since}$

$$\forall i \quad \langle i|\langle \psi|VV^\dagger|i\rangle|\psi\rangle \geq 0 \quad \square$$

The second corollary is not traditionally thought of as a ‘quantum operation equivalent’ of quantum state purification. We now explicitly show how the result is again trivially obtained from Property 2 by virtue of Theorem 8.

Proof of Corollary 2. Consider $\hat{\mathcal{S}}$ a Completely Positive-preserving linear operator. By Theorem 8 $\hat{\mathcal{S}}$ is positive, and so Property 2 gives $\hat{\mathcal{S}} = \operatorname{Tr}_1(\hat{\rho}_{\text{pure}})$, $\hat{\rho}_{\text{pure}} = V V^\dagger$, where the ancilla system can be chosen to be of dimension $r = \operatorname{rank}(\hat{\mathcal{S}})$. As a consequence we can use Lemma 2 to retrieve $\hat{\mathcal{S}} = \operatorname{Tr}_1(\hat{\rho}_{\text{pure}})$, $\hat{\rho}_{\text{pure}} : \rho \mapsto \hat{V} \rho \hat{V}^\dagger$.

Moreover, denote by $\operatorname{Tr}_1$, the partial trace over the $m$-dimensional system. For $V = V_{\alpha\beta}|\alpha\rangle\langle\beta|$, let $\hat{V} = V_{\alpha\beta}|\alpha\rangle\langle\beta|$ the corresponding $r \times n$ matrix. Since
\[ \text{Tr}_1(\rho_{\text{pure}}) = \rho \] with \( \rho_{\text{pure}} = VV^\dagger \), and \( \rho = \sum_x A_x A_x^\dagger \), we get

\[ (\text{Tr}_1 \circ \text{Tr}_1)(VV^\dagger) = \sum_x \text{Tr}_1(A_x A_x^\dagger) \] implying

\[ \hat{V}^\dagger \hat{V} = \sum_x \hat{A}_x^\dagger \hat{A}_x \] by Equation \( \text{(11)} \).

Notice that whenever \( \hat{S} \) is Trace-preserving, then Equation \( \text{(11)} \) reads \( \hat{V}^\dagger \hat{V} = \text{Id}_n \), so that \( \hat{V} \) is isometric. Thus we have derived as a simple consequence of properties of state purification that any Trace-preserving quantum operation can arise as the trace-out of an isometric operation.

**Property 3 (Schmidt decomposition.)** Consider \( \rho = VV^\dagger \) a non-normalized pure state in \( \text{Herm}_{mn}^+(\mathbb{C}) \) with \( V = \sum V_i |i\rangle \langle j| \) in the canonical basis of \( \mathbb{C}^m \otimes \mathbb{C}^n \). Then there exists some positive reals \( \{\lambda_i\} \) and some orthogonal basis \( \{|\psi_i\rangle\} \) and \( \{|\phi_i\rangle\} \) of \( \mathbb{C}^m \) and \( \mathbb{C}^n \) respectively, such that

\[ V = \sum_{i=1}^r \lambda_i |\psi_i\rangle |\phi_i\rangle, \]

with \( r \leq m \) and \( r \leq n \). Moreover:

\[ \text{Tr}_1(\rho) = \sum_{i=1}^r \lambda_i^2 \langle \phi_i | \phi_i \rangle \quad (n \times n \text{ positive}) \]

\[ \text{Tr}_2(\rho) = \sum_{i=1}^r \lambda_i^2 \langle \psi_i | \psi_i \rangle \quad (m \times m \text{ positive}) \]

**Corollary 3 (Polar Decomposition.)** Consider \( \hat{S} : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \), \( \rho \mapsto \hat{V} \rho \hat{V}^\dagger \) a factorizable Completely Positive-preserving linear map, with \( \hat{V} = \sum V_i |i\rangle \langle j| \). Then there exists some positive reals \( \{\lambda_i\} \) and some orthogonal basis of \( \mathbb{C}^m \) and \( (\mathbb{C}^n)^\dagger \), namely \( \{|\psi_i\rangle\} \) and \( \{\phi_i^*\rangle\} \), such that

\[ \hat{V} = \sum_{i=1}^r \lambda_i |\psi_i\rangle \langle \phi_i | \]

with \( r \leq m \) and \( r \leq n \). In other words:

\[ \hat{V} = UJ = KU \] with

\[ J = \sqrt{\hat{V}^\dagger \hat{V}} = \sum_{i=1}^r \lambda_i |\phi_i^*\rangle \langle \phi_i | \] \( (n \times n \text{ positive}) \)

\[ K = \sqrt{VV^\dagger} = \sum_{i=1}^r \lambda_i |\psi_i\rangle \langle \psi_i | \] \( (m \times m \text{ positive}) \)

\[ U = \sum_{i=1}^n |\psi_i\rangle \langle \phi_i^* | \] \( (m \times n \text{ isometric, i.e. } U^\dagger U = \text{Id}_n) \)

**Proof of Property 3.** Let \( \rho = VV^\dagger \), \( V = \sum V_i |i\rangle \langle j| \), and \( \text{Tr}_2 \) the partial trace on the last \( n \)-dimensional system. Since \( \rho^A = \text{Tr}_2(\rho) \) is in \( \text{Herm}_{mn}^+(\mathbb{C}) \), we can write

\[ \rho^A = \sum_{i=1}^r \lambda_i^2 |\psi_i\rangle \langle \psi_i | \]

where \( \{\lambda_i\} \) are strictly positive reals, \( r \leq m \), and \( \{|\psi_i\rangle\} \) is an orthonormal family of vectors which we may complete into an orthonormal basis of \( \mathbb{C}^m \). By expressing the first subspace of \( V \) in this basis we can of course write:

\[ V = \sum_{i=1}^r |\psi_i\rangle |\phi_i\rangle \] with \( |\phi_i\rangle = (|\psi_i\rangle \otimes \text{Id}_n)V \)

We have:

\[ \langle \phi_i | \phi_j \rangle = \text{Tr}(|\phi_i\rangle \langle \phi_i |) \]

\[ = \text{Tr}((|\psi_i\rangle \otimes \text{Id})VV^\dagger(|\psi_i\rangle \otimes \text{Id})) \]

\[ = \text{Tr}(|\psi_j\rangle \langle \psi_j | \otimes \text{Id})VV^\dagger \]

\[ = \text{Tr}(|\psi_j\rangle \langle \psi_j | \rho^A) \]

\[ = \lambda_i^2 \delta_{ij} \]

Thus \( \{|\phi_i\rangle\} = \{|\phi_i\rangle / \lambda_i\} \) is an orthonormal family of vectors in \( \mathbb{C}^n \), which we may again complete into an orthonormal basis.

We now have \( V = \sum \lambda_i |\psi_i\rangle \langle \phi_i | \), from which it is straightforward to verify that

\[ \text{Tr}_1(\rho) = \sum_{i=1}^r \lambda_i^2 |\phi_i\rangle \langle \phi_i | \]

The well-known connection between the Schmidt decomposition and the polar decomposition (itself trivially equivalent to the singular value decomposition) is now shown to arise naturally using the state-operator correspondence.

**Proof of Corollary 3.** Consider \( \hat{S} : \rho \mapsto \hat{V} \rho \hat{V}^\dagger \). Using Isomorphism 2 the corresponding state in \( \text{Herm}_{mn}^+(\mathbb{C}) \) is \( \rho = VV^\dagger \). Applying the Schmidt decomposition theorem yields

\[ V = \sum_{i=1}^r \lambda_i |\psi_i\rangle \langle \phi_i | \] and thus

\[ \hat{V} = \sum_{i=1}^r \lambda_i |\psi_i\rangle \langle \phi_i^* | \]

with \( \{|\psi_i\rangle\} \) and \( \{\phi_i^*\rangle = \langle \phi_i | \} \) some orthogonal basis of \( \mathbb{C}^m \) and \( (\mathbb{C}^n)^\dagger \) respectively. Now if we call \( U \) the \( m \times n \) isometric (i.e. \( U^\dagger U = \text{Id}_n \)) matrix \( \sum_{i=1}^n |\psi_i\rangle \langle \phi_i^* | \), we have that \( \hat{V} = UJ = KU \), with

\[ K = \sum_{i=1}^r \lambda_i |\psi_i\rangle \langle \psi_i | = \sqrt{\text{Tr}_2(VV^\dagger)} = \sqrt{VV^\dagger} \]

\[ J = \sum_{i=1}^r \lambda_i |\phi_i^*\rangle \langle \phi_i^* | = \left( \sqrt{\text{Tr}_1(VV^\dagger)} \right)^t = \sqrt{\hat{V}^\dagger \hat{V}} \]
In the above \( K \) is \( m \times m \) whilst \( J \) is \( n \times n \), and the last equality of each line was derived from Equations (12) and (11). \( \square \)

Thus it seems that all the standard results about quantum operations are in correspondence with those concerning quantum states. Of course although we derived the properties of operators from those of states, we could equally have done the opposite. Next we seek to apply the same principle to derive new results, as we consider properties of states and operations which do not yet have any equivalent in terms of, respectively, operations and states.

B. Properties discovered via the correspondence

We first derive a factorizability condition on quantum operations by making use of the well-known property:

Property 4 (Purity condition.) Let \( \rho \) a matrix in \( \text{Herm}_n^+(\mathbb{C}) \). Then \( \rho \) is non-normalized pure, i.e. of the form \( \rho = VV^\dagger \), if and only if

\[
\text{Tr}(\rho)^2 - \text{Tr}(\rho^2) = 0
\]

Corollary 4 (Factorizability condition.) Let \( \hat{S} : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \) a Completely Positive-preserving linear operator. Then \( \hat{S} \) is of the form \( \hat{S} : \rho \mapsto V \rho V^\dagger \), i.e. it is factorizable, if and only if

\[
\left( \text{Tr}(\hat{S}(Id_n)) \right)^2 - \sum_{jl} \text{Tr}(\hat{S}(E_{jl}) \hat{S}(E_{jl})) = 0
\]

or equivalently in terms of indices

\[
(\hat{S}_{ii;jj})^2 - \hat{S}^*_{ik;jl} \hat{S}_{ik;jl} = 0.
\]

Proof of Property 4 \[ \Rightarrow \] is obvious since \( \rho \) pure has only got one non-zero eigenvalue.

\[ \Leftarrow \] Suppose \( \rho \) has eigenvalues \( \{\lambda_i\} \). The purity condition amounts to

\[
\sum_i \lambda_i^2 = \sum_i \lambda_i^2 \quad \text{implying} \quad \sum_{i<j} \lambda_i \lambda_j = 0.
\]

For the last relation to hold, since the \( \lambda_i \)'s are positive there can be at most one value of \( i \) such that \( \lambda_i \neq 0 \). \( \square \)

Proof of Corollary 4 \[ \Rightarrow \] \( \hat{S} \) is factorizable is equivalent to \( \hat{S} \) being pure, thus by Property 4 to

\[
\text{Tr}(Id_m \hat{S})^2 - \text{Tr}(\hat{S}^2) = 0.
\]

Using \( \hat{S}^\dagger = \hat{S} \) Equation (18) is a direct application of Equation (4) upon this last equation, as can be seen from

\[
Id_{mn} = \sum_{kl} |kl\rangle \langle kl|
\]

so that \( \hat{Id}_{mn} : \rho \mapsto \sum_{kl} E_{kl} \rho E_{kl}^\dagger \)

and \( \hat{Id}_{mn} : E_{jl} \mapsto \delta_{jl} Id_m \). \( \square \)

Next we give two new vector decompositions which stem from classical results on matrix decomposition.

Property 5 (One-sided triangular decomposition.) Let \( \rho = VV^\dagger \) a non-normalized pure state in \( \text{Herm}_{mn}^+(\mathbb{C}) \), with \( V = \sum V_{ij} |i\rangle |j\rangle \) in the canonical basis, and suppose \( m \geq n \). Then there exists some orthogonal basis of \( \mathbb{C}^m \), namely \( \{|\psi_i\rangle\} \) such that

\[
V = \sum_{i<j} \mu_{ij} |\psi_i\rangle |j\rangle
\]

Proof. According to the QR decomposition theorem \[ \text{E} \] the \( m \times n \) matrix \( \hat{V} \) can be decomposed as \( \hat{V} = QR \), where \( Q \) is \( m \times n \) and verifies \( Q^\dagger Q = Id_n \) whilst \( R \) is \( n \times n \) upper triangular. Thus we have:

\[
\hat{V} = Q \sum_{i<j} \mu_{ij} |i\rangle |j\rangle
\]

\[
\hat{V} = \sum_{i<j} \mu_{ij} |\psi_i\rangle |j\rangle
\]

\[
V = \sum_{i<j} \mu_{ij} |\psi_i\rangle |j\rangle
\]

Since \( Q \) is isometric, the \( \{|\psi_i\rangle = Q|i\rangle\} \) are orthonormal and can be extended to form a basis of \( \mathbb{C}^m \). \( \square \)

On the one hand Property 5 is less powerful than the Schmidt decomposition, in the sense that it yields ‘upper triangular’ coefficients \( \mu_{ij} \) instead of the neat diagonal form \( V = \sum \lambda_i |\psi_i\rangle |\phi_i\rangle \). On the other hand however our Property requires a change of basis for the first subsytem only. Such a distinction is perfectly analogous to what separates the polar decomposition (or more expressively its singular value decomposition corollary) from the QR decomposition when speaking about matrices. Just like the QR decomposition the one-sided state triangularization is easily computed.

Schur’s triangularization theorem can also be given a quantum state equivalent, as we now explain. This seems of a lesser interest however, since the procedure involves two changes of basis, one for each subsystem - a case which seems better covered by the Schmidt decomposistion (though here the two basis are simply related).

Property 6 (Two-sided triangular decomposition.) Let \( \rho = VV^\dagger \) a non-normalized pure state in \( \text{Herm}_{mn}^+(\mathbb{C}) \), with \( V = \sum V_{ij} |i\rangle |j\rangle \) in the canonical basis. Then there exists some orthogonal basis of \( \mathbb{C}^m \), namely \( \{|\psi_i\rangle\} \) such that

\[
V = \sum_{i<j} \mu_{ij} |\psi_i\rangle |\psi_j^*\rangle
\]

where \( * \) denotes complex conjugation of the coordinates of a vector in the canonical basis. Moreover the set \( \{\mu_{ii}\} \) is the set of the Schmidt coefficients \( \{\lambda_i\} \) of \( V \) (as defined in Property 4).
Moreover we say that $\hat{V}$ is maximally mixed if and only if one of the following six equivalent conditions is satisfied:

(i) $\sum A_i^x A_x = I_{d_n}$,
(ii) $\hat{\delta}_{kk;jl} = \delta_{jl}$.

In terms of $\hat{\delta}$ this is

(iii) $\sum A_x A_x^\dagger = I_{d_n}$,
(iv) $\hat{\delta}(I_{d_m}) = I_{d_n}$,

In terms of the state $\hat{\delta}$ this is

(v) $\text{Tr}(\hat{\delta}) = I_{d_n}$,
(vi) $\hat{\delta}_{kk;jl} = \delta_{jl}$.

Proof. We have $\hat{\delta}(\rho) = \sum \hat{A}_i \rho \hat{A}_i^\dagger$ or using components $\hat{\delta}(\rho)_{ij} = \hat{\delta}_{ik;jl} \rho_{kl}$, so that $\text{Tr}(\hat{\delta}(\rho)) = \text{Tr}(\sum \hat{A}_i^\dagger \hat{A}_i \rho) = \hat{\delta}_{kk;jl} \rho_{kl}$. Thus (i) and (ii) follow immediately. Using that $\hat{A} = \hat{A}^\dagger$ and $\hat{\delta}_{jl;kk} = \hat{\delta}(I_{d_m})_{jl} = \hat{\delta}_{jl;kk}$ from (9), we get (iii) and (iv). (v) and (vi) follow from (i) and (ii) using (11) and (5) respectively. $\square$

C. Trace-preserving Quantum Operations

The results of subsection 11A, although extremely useful in quantum theory (quantum information theory in particular), are in fact general results on positive matrices and Completely Positive-preserving linear maps. The same is true of subsection 11B and this is the reason why we have barely mentioned the unit trace condition on density matrices so far. Yet in quantum theory the states must have trace one (unless we start to consider the trace as encoding some overall probability of occurrence), and quantum operation must be Trace-preserving (so that they may always occur). We now give an account of the main known results related to these restrictions, augmented with some results stemming from the state-operator correspondence.

Definition 4 A linear map $\Omega : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is Trace-preserving if and only if for all $\rho \in M_n(\mathbb{C})$, $\text{Tr}(\Omega(\rho)) = \text{Tr}(\rho)$.

Definition 5 The state $1/d\text{Id}_d \in \text{Herm}^+_n(\mathbb{C})$ is called the maximally mixed state of $\mathbb{C}^d$. Moreover we say that $\hat{\delta} \in \text{Herm}^+_m(\mathbb{C})$ is a maximally entangled state of $\mathbb{C}^m \otimes \mathbb{C}^n$ if and only if $\hat{\delta}$ is pure and verifies either of

\[
\begin{align*}
(n \leq m) & \quad \text{Tr}_1(\hat{\delta}) = \text{Id}_{d_n} \\
(m \leq n) & \quad \text{Tr}_2(\hat{\delta}) = \text{Id}_{d_m}
\end{align*}
\]

depending on the integers $m$ and $n$ (if $m = n$ the two conditions are indifferent).

Lemma 3 (Trace-preserving linear maps.) A Completely positive-preserving linear map $\hat{\delta} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ with decomposition \{\hat{A}_x\} is Trace-preserving if and only if one of the following six equivalent conditions is satisfied:

(i) $\sum A_i^x A_x = I_{d_n}$,
(ii) $\hat{\delta}_{kk;jl} = \delta_{jl}$.

In terms of $\hat{\delta}$ this is

(iii) $\sum A_x A_x^\dagger = I_{d_n}$,
(iv) $\hat{\delta}(I_{d_m}) = I_{d_n}$,

In terms of the state $\hat{\delta}$ this is

(v) $\text{Tr}(\hat{\delta}) = I_{d_n}$,
(vi) $\hat{\delta}_{kk;jl} = \delta_{jl}$.

Proof. We have $\hat{\delta}(\rho) = \sum \hat{A}_i \rho \hat{A}_i^\dagger$ or using components $\hat{\delta}(\rho)_{ij} = \hat{\delta}_{ik;jl} \rho_{kl}$, so that $\text{Tr}(\hat{\delta}(\rho)) = \text{Tr}(\sum \hat{A}_i^\dagger \hat{A}_i \rho) = \hat{\delta}_{kk;jl} \rho_{kl}$. Thus (i) and (ii) follow immediately. Using that $\hat{A} = \hat{A}^\dagger$ and $\hat{\delta}_{jl;kk} = \hat{\delta}(I_{d_m})_{jl} = \hat{\delta}_{jl;kk}$ from (9), we get (iii) and (iv). (v) and (vi) follow from (i) and (ii) using (11) and (5) respectively. $\square$

Note that these conditions imply, but are not equivalent to, $(1/n)\hat{\delta}$ having unit trace. This is because $\hat{\delta}$ has unit trace' reads:

\[
\text{Tr}(\hat{\delta}) = \text{Tr}(\hat{\delta}(I_{d_n})) = \text{Tr}(\hat{\delta}(I_{d_m})) = 1
\]

or $\hat{\delta}_{kk;kl} = \hat{\delta}_{kk;jl} = \hat{\delta}_{ll;kk} = 1$.

Thus we have shown that Trace-preserving quantum operations $\hat{\delta} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ go hand in hand with unit trace states $(1/n)\hat{\delta} \in \text{Herm}^+_m(\mathbb{C})$ whose partial trace on the first subsystem yields the maximally mixed state: $\text{Tr}_1((1/n)\hat{\delta}) = (1/n)\text{Id}_{d_n}$. We immediately obtain the following, which is a generalization of a result in 8 and 13:

Lemma 4 (Unitary maps.) Let $\hat{\delta} : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ a Completely Positive-preserving map. Then $\hat{\delta}$ is isometric (i.e it can be written as $\hat{\delta} : \rho \mapsto \hat{U} \rho \hat{U}^\dagger$ with $\hat{U} \hat{U}^\dagger = \text{Id}_{d_n}$) if and only if $n \leq m$ and the corresponding state $\hat{\delta}$ is maximally entangled (i.e. pure with $\text{Tr}_1(\hat{\delta}) = \text{Id}_{d_n}$). Equivalently, in terms of indices, $\hat{\delta}$ must verify $\hat{\delta}_{kk;jl} = \delta_{jl}$ and

\[
\sum_{jl} \text{Tr}(\hat{\delta}(E_{jl})) \hat{\delta}(E_{jl})) = n^2
\]

Remark 2 (Bistochastic maps.) $\hat{\delta}$ is bistochastic, i.e. it is Trace-preserving and satisfies $\hat{\delta}(I_{d_n}) = I_{d_m}$, if and only if the state $\hat{\delta}$ satisfies $\text{Tr}_1(\hat{\delta}) = \text{Id}_{d_n}$ and $\text{Tr}_2(\hat{\delta}) = \text{Id}_{d_m}$. Thus bistochastic maps cannot be factorizable whenever $m \neq n$.

Proof. The Lemma follows immediately from Lemma 8 and Corollary 13. The remark follows from Lemma 8 and the fact that $\text{Tr}_2(\hat{\delta}) = \hat{\delta}(I_{d_m})$. $\square$

The set of states $\hat{\delta} \in \text{Herm}^+_m(\mathbb{C})$ satisfying $\text{Tr}_1(\hat{\delta}) = \text{Id}_{d_n}$ is convex, hence its extremal points correspond to

\[
\begin{align*}
(n \leq m) & \quad \text{Tr}_1(\hat{\delta}) = \text{Id}_{d_n} \\
(m \leq n) & \quad \text{Tr}_2(\hat{\delta}) = \text{Id}_{d_m}
\end{align*}
\]
extremal Trace-preserving quantum operations. Recall that the extremal elements of a convex set \( S \) are those which cannot be written as sums of two distinct elements of \( S \). Extremal elements are important since they generate \( S \), and so now we restate Choi’s well-known theorem about extremal Trace-preserving maps (without reproducing the proof).

**Theorem 4 (Extremal Trace-preserving.)** Let \( \hat{\mathcal{S}} : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \) a Trace-preserving Completely Positive-preserving linear map with decomposition \( \{ A_x \} \) and higher rank \( r \) (i.e. \( r = \text{rank}(\mathcal{S}) \)). Then \( \hat{\mathcal{S}} \) is extremal in the set of Trace-preserving Completely Positive-preserving maps if and only if one of the following equivalent conditions is satisfied:

(i) the span of the set \( \{ A_x^\dagger A_y \} \) in \( M_n(\mathbb{C}) \) is \( r \)-dimensional;

(ii) the span of the set \( \{ A_x A_y^\dagger \} \) in \( M_n(\mathbb{C}) \) is \( r \)-dimensional;

(iii) the span of the set \( \{ \text{Tr}_1(A_x A_y^\dagger) \} \) in \( M_n(\mathbb{C}) \) is \( r \)-dimensional.

Notice that this is a slightly different formulation from the one given in [2], where the \( \{ A_x^\dagger A_y \} \) have to form a linearly independent set. This implies that the \( \{ A_x \} \) are automatically linearly independent themselves, and hence there must be \( r \) of them. Since different decompositions can give the same operation, we thought it better to express the extremality conditions in terms of any decomposition, and not just a minimal one.

**Proof.** We just prove the equivalence with Choi’s formulation. Let \( \{ \hat{V}_\alpha \} \) a minimal decomposition of \( \hat{\mathcal{S}} \). Then \( \text{Span}(\{ A_x \}) = \text{Span}(\{ V_\alpha \}) \) since both are equal to the support (the image space) of \( \mathcal{S} \) (see Corollary 1), and trivially \( \text{Span}(\{ A_x \}) = \text{Span}(\{ \hat{V}_\beta \}) \) implies \( \text{Span}(\{ A_x^\dagger A_y \}) = \text{Span}(\{ V_\alpha^\dagger \hat{V}_\beta \}) \). \( \square \)

**Remark 3** An extreme map \( \hat{\mathcal{S}} \) has higher rank \( r \leq n \) since it must satisfy \( r^2 \leq n^2 \), but this condition is not sufficient.

**Proof.** Suppose \( U_1 \neq U_2 \) unitary and \( \hat{\mathcal{S}} : \rho \mapsto (1/2)U_1 \rho U_1^\dagger + (1/2)U_2 \rho U_2^\dagger \). Clearly \( U_1^\dagger U_1 = U_2^\dagger U_2 = I_{n^2} \), and thus this Trace-preserving Completely Positive-preserving map cannot be extremal Trace-preserving. Yet it has higher rank 2 regardless of a choice for \( n \). \( \square \)

By pushing the consequences of Choi’s theorem further we obtain the following original criteria for extremal Trace-preserving linear maps:

**Proposition 2 (Extremal Trace-preserving.)** Let \( \hat{\mathcal{S}} : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \) a Trace-preserving Completely Positive-preserving linear map of Choi rank \( r \) (i.e. \( r = \text{rank}(\mathcal{S}) \)) with \( \mathcal{S} \) its corresponding state, and \( \hat{\mathcal{S}}^\dagger : M_m(\mathbb{C}) \to M_n(\mathbb{C}) \) its adjoint map (i.e. \( \hat{\mathcal{S}} \equiv \hat{\mathcal{S}}_{jl;ik} \)).

Then \( \hat{\mathcal{S}} \) is extremal if and only if one of the following equivalent conditions is satisfied:

(i) The higher rank of \( \hat{\mathcal{S}}^\dagger \circ \mathcal{S} \) is equal to \( r^2 \),

(ii) \( \mathcal{S} \) is such that the state in \( \text{Herm}^+_{n^2}(\mathbb{C}) \) defined by

\[
\mathcal{E}_{jj';ll'} = S_{ij;kl}^* S_{ij';kl'}
\]

has rank \( r^2 \).

**Proof.** If \( \hat{\mathcal{S}} \) has operator sum decomposition \( \{ A_x \} \) i.e. \( \mathcal{S}(\rho) = \sum A_x \rho A_x^\dagger \), then we get that \( \hat{\mathcal{S}}^\dagger \) has decomposition \( \{ A_x^\dagger \} \) i.e. \( \hat{\mathcal{S}}^\dagger (\sigma) = \sum A_x^\dagger \sigma A_x \). This can be seen using \( \hat{\mathcal{S}}_{jl;ik} = \hat{\mathcal{S}}_{ik;jl}^\dagger \) for example. Thus \( \hat{\mathcal{S}}^\dagger \circ \hat{\mathcal{S}} \) has decomposition \( \{ A_x^\dagger A_y \} \), and (i), using Corollary 1 is equivalent to (i) in Theorem 4.

Next we restate (i) using indices and Equation 5.

\[
(\mathcal{S}^\dagger \circ \mathcal{S})_{jj';ll'} = S_{jl;ik}^* S_{ik;jj'}^\dagger = S_{ik;jl}^* S_{ik;jj'} = S_{ij;kl}^* S_{ij';kl'} = \mathcal{E}_{jj';ll'} \circ \mathcal{E}_{jj';ll'}.
\]

Since \( \hat{\mathcal{S}}^\dagger \circ \hat{\mathcal{S}} \) is a Completely Positive-preserving map from \( M_n(\mathbb{C}) \) to \( M_m(\mathbb{C}) \), \( \mathcal{E} \) is in \( \text{Herm}^+_{n^2}(\mathbb{C}) \) by Theorem 5. We see that (i) is equivalent to (ii). \( \square \)

The relation between condition (i) and (ii) suggests that the composition law on quantum operations could yield, through Isomorphism 2, an interesting structure upon states. We pursue this idea in the following section.

### III. INDUCED GEOMETRICAL STRUCTURE

The beginning of this section is maybe aimed at a mathematically-minded reader. We investigate simple algebraic and geometric properties stemming from the operator state correspondence. These yield a nice group theoretic description of totally entangled states of a bipartite system (Proposition 4), and a description of Positive-preserving maps as dual to separable states (Theorem 5, restated). Proposition 6 however unravels a possible physical interpretation of the correspondence.

#### A. Composition laws

We make use of some elementary facts about operators or positive matrices to define new composition laws on the spaces of operators or positive matrices.

First, the set of Completely Positive-preserving linear maps from \( M_n(\mathbb{C}) \) into itself is stable under composition. This induces the following semi-group structure for states (recall that semi-group elements do not need to have an inverse):

**Proposition 3** If \( \mathcal{S} \) and \( \mathcal{E} \) are in \( \text{Herm}^+_{n^2}(\mathbb{C}) \), then so is

\[
\mathcal{S} \circ \mathcal{E} \equiv (\mathcal{S} \circ \mathcal{E})_{ij;kl} = S_{iv;kk} \mathcal{E}_{ij';kl'}
\]

(19)
where all the indices run from 1 to $n$. $(\text{Herm}^+_n(\mathbb{C}), \circ)$ is a semi-group with identity element the canonical maximally entangled state $|\beta\rangle\langle\beta| = \delta_{ij}\delta_{kl}$.

The set of non-normalized pure states, the set of entangled states and the set of separable states (together with $|\beta\rangle\langle\beta|$), are sub-semi-groups of $(\text{Herm}^+_n(\mathbb{C}), \circ)$. More precisely,

$$(AA^\dagger) \circ (BB^\dagger) = VV^\dagger \text{ where } V = \hat{A}\hat{B} \quad (20)$$

$$(\mu_1 \otimes \mu_2) \circ (\sigma_1 \otimes \sigma_2) = \text{Tr}(\mu_2^\dagger\sigma_1)\mu_1 \otimes \sigma_2$$

**Proof.** The composition law is just the transcription of $(\hat{S} \circ \hat{E})_{ik;jl} = \hat{S}_{ik;\ell k'}\hat{E}_{\ell'k';jl}$ using [9], and the identity element is clearly $\delta_{ij}\delta_{kl}$. Next, the composition of two factorizable operations is factorizable and trivially yields [20]. Let $\hat{E} = \sigma_1 \otimes \sigma_2$ and $\hat{s} = \mu_1 \otimes \mu_2$ two unentangled states. We have using Equation (15):

$$\hat{E}(\rho) = \text{Tr}_2((Id \otimes \rho')(\sigma_1 \otimes \sigma_2)) = (\text{Tr}(\sigma_2^\dagger\rho))\sigma_1,$$

hence $\hat{s} \circ \hat{E}(\rho) = \text{Tr}(\mu_2^\dagger(\text{Tr}(\sigma_2^\dagger\rho)\sigma_1))\mu_1$

$$= \text{Tr}(\mu_2^\dagger\sigma_1)\text{Tr}(\sigma_2^\dagger\rho)\mu_1$$

and the last equation follows immediately.

Since the composition law is bilinear, the space of separable states of $\text{Herm}^+_n(\mathbb{C})$ is also a sub-semi-group of $(\text{Herm}^+_n(\mathbb{C}), \circ)$. \qed

It seems natural at this point to look for subgroups of $(\text{Herm}^+_n(\mathbb{C}), \circ)$. Clearly the largest subgroup corresponds to the set of invertible quantum operations $\hat{S}$, of which it is difficult to give a physical description in terms of the states $\hat{S}$; we just require $\hat{s}_{ik;jl}$ to be invertible. Since unentangled states yield projections (as was illustrated in the proof above), they are not in this group; yet mixtures of them (separable states) may well yield invertible operations.

**Definition 6** The positive definite matrices of $\text{Herm}^+_n(\mathbb{C})$ are sometimes called the totally mixed states of $\mathbb{C}^d$.

Moreover we say that $\hat{S} \in \text{Herm}^+_{mn}(\mathbb{C})$ is a totally entangled state of $\mathbb{C}^m \otimes \mathbb{C}^n$ if and only if $\hat{S}$ is pure and verifies either of

$$(n \leq m) \quad \text{Tr}_1(\hat{S}) \text{ is totally mixed}$$

$$(m \leq n) \quad \text{Tr}_2(\hat{S}) \text{ is totally mixed}$$

depending on the integers $m$ and $n$ (if $m = n$ the two conditions are indifferent).

Let $\text{GL}_n(\mathbb{C})$ denote the group of invertible $n \times n$ complex matrices, $U(1)$ its (normal) subgroup of matrices of the type $e^{i\theta}I_{dn}$, and $SU(n)$ the group of special unitary $n \times n$ matrices, i.e. matrices $U$ satisfying $U^\dagger U = UU^\dagger = I_{dn}$ and det $U = 1$. We have the following:

**Proposition 4** The set of totally entangled pure states in $\text{Herm}^+_{n^2}(\mathbb{C})$, equipped with the composition law $\circ$, is a group which is isomorphic to the group $\text{GL}_n(\mathbb{C})/U(1)$. Its subset of maximally entangled states is a subgroup isomorphic to $SU(n)$.

**Proof.** Let us denote by $T$ the set of totally entangled (pure) states in $\text{Herm}^+_{n^2}(\mathbb{C})$. Note that for any $\hat{A} \in M_n(\mathbb{C})$, $\hat{A}$ is invertible if and only if $\hat{A}A^\dagger$ is invertible, which by [12] is equivalent to $\text{Tr}_2(\hat{A}A^\dagger)$ invertible, in other words $AA^\dagger$ totally entangled. Thus $T = \{AA^\dagger \mid A \in \text{GL}_n(\mathbb{C})\}$, and from (20), $(T, \circ)$ is a group with identity element $|\beta\rangle\langle\beta|$. $\phi : \text{GL}_n(\mathbb{C}) \rightarrow T$

$$\hat{A} \mapsto AA^\dagger$$

is then trivially a group homomorphism, since $\phi(\hat{A}\hat{B}) = AA^\dagger \circ BB^\dagger$ by [20]. $\phi$ is clearly onto, and its kernel is $U(1)$. Thus $\text{GL}_n(\mathbb{C})/U(1)$ is isomorphic to $T$.

$\phi$ restricted to $U(n)$ maps onto the set of maximally entangled states by Lemma 4 so that $SU(n) = U(n)/U(1)$ is isomorphic to it. \qed

These results are useful when one seeks to parameterize certain pure states of an $n^2$-dimensional system. The description of pure states in $\text{Herm}^+_{n^2}(\mathbb{C})$ in terms of the homogeneous space $SU(n^2)/SU(n^2 - 1)$ is well-known, but yields a very complicated parameterization since one must mod out the $SU(n^2 - 1)$. We have shown that we can in fact parameterize the set of maximally entangled (pure) states of $\text{Herm}^+_{n^2}(\mathbb{C})$ in terms of (the Euler angles of) $SU(n)$, without having to mod out any redundancy. The parameterization could have potential applications in the study of entanglement, Bell states and EPR scenarios.

Next one can also define an original semi-group structure on the set of Completely Positive-preserving maps by using an exotic composition law (the Schur product $\triangleright$) on the set of states:

**Proposition 5** If $\hat{S}$ and $\hat{E}$ are Completely Positive-preserving maps from $M_m(\mathbb{C})$ to $M_m(\mathbb{C})$, then so is

$$\hat{S} \triangleright \hat{E} \equiv (\hat{S} \triangle \hat{E})_{ik;jl} = \hat{S}_{ik;jl}\hat{E}_{kl;jl}$$

where the summation convention is suspended, and $i, k = 1, \ldots, m$, and $j, l = 1, \ldots, n$. This composition law is obviously commutative, and the set of factorizable operations is stable under it.

**Proof.** This stems, via Theorem 3, from the stability of the set of positive matrices under of the Schur (or Hadamard) product $\hat{S} \triangle \hat{E}$. I.e. the fact that the component-wise product of two positive matrices is a positive matrix, when applied to $\hat{S}$ times $\hat{E}$, induces the corresponding result for $\hat{S} \times \hat{E}$. We use the same symbol $\triangle$ to denote all component-wise
products of matrices. If $\hat{A}$ and $\tilde{C}$ have decompositions $\{A_x\}$ and $\{B_y\}$ respectively, then $\hat{A} \triangle \tilde{C}$ has decomposition $\{A_x \triangle B_y\}$: this implies the stability of factorizable operations. $\square$

B. Duality: states and functionals

When relating operators and states of a physical theory notions of duality between vector spaces are often illuminating; operators sometimes induce functionals on the space of states, which can in turn be thought of as states. In finite-dimensional Quantum Mechanics, a given positive matrix can either represent a state or a positive functional, and we can switch from one to the other easily.

So far we have equipped the algebra of complex $d \times d$ matrices, $M_d(\mathbb{C})$, with the complex-bilinear form: $\langle \xi, \eta \rangle = \text{Tr}(\xi^* \eta)$. This non-degenerate form naturally defines a canonical pairing of $M_d(\mathbb{C})$ with $\hat{M}_d(\mathbb{C})$, the space of linear functionals on $M_d(\mathbb{C})$:

$$\sim : M_d(\mathbb{C}) \rightarrow \hat{M}_d(\mathbb{C})$$

$$\xi \mapsto \{ \tilde{C} : \xi \mapsto \text{Tr}(\xi^* \tilde{C}) \}$$

Since $\sim$ is an (anti-linear) isomorphism, any linear functional on $M_d(\mathbb{C})$ has a unique antecedent by $\sim$, thus is uniquely represented by an element of $\hat{M}_d(\mathbb{C})$. Let $\{E_{ij}\}_{1 \leq i, j \leq d}$ a canonical basis of $M_d(\mathbb{C})$ and $\{F_{kl}\}_{1 \leq k, l \leq d}$ its corresponding peered basis, i.e. $E_{ij} = \delta_{ik}\delta_{lj}$. Then the functional of $\xi$, namely $\tilde{C}_\xi$ is represented in the peered basis by $\xi^\ast$. Indeed, $\tilde{C}_\xi(E_{kl}) = \xi^\ast(\tilde{C}_\xi) = \xi^\ast(\xi) = \text{Tr}(\xi^\ast \xi)$.

When restricted to the real vector space of hermitian matrices $\text{Herm}_d(\mathbb{C})$, $\langle \xi, \eta \rangle \rightarrow \text{Tr}(\xi^\ast \eta)$ yields a real scalar product, and $\text{Herm}_d(\mathbb{C})$ is defined similarly. It then becomes possible to define the dual (sometimes called polar) of a subspace $S$ of $\text{Herm}_d(\mathbb{C})$ as follows:

$$S^* = \{ \tilde{\sigma} \in \text{Herm}_{mn}(\mathbb{C}) / \forall \rho \in S, \ \tilde{\sigma}(\rho) \geq 0 \}$$

(21)

The convex cone of hermitian positive matrices $\text{Herm}_d^+(\mathbb{C})$ is clearly self-dual under this pairing:

$$\xi \in \text{Herm}_d^+(\mathbb{C}) \iff \forall \xi \in \text{Herm}_d^+(\mathbb{C}), \ \text{Tr}(\xi^\ast \xi) \geq 0$$

$$\iff \forall \xi \in \text{Herm}_d^+(\mathbb{C}), \ \xi^\ast(\xi) \geq 0$$

$$\iff \tilde{C}_\xi \in \text{Herm}_d^+(\mathbb{C})^*$$

In the last line we have used the definition (21). Thus $\text{Herm}_d^+(\mathbb{C})^* = \text{Herm}_d^+(\mathbb{C})$, hence the set of non-normalized states is isomorphic to that of non-normalized linear probability distributions on states, i.e. functionals which are positive on $\text{Herm}_d^+(\mathbb{C})$. In this sense, if $\xi$ is an element of $\text{Herm}_d^+(\mathbb{C})$, then $\xi^\ast \equiv \tilde{C}_\xi$, represents its dual element, or associated linear probability distribution, and conversely. We shall now explain why this picture is illuminating.

1. Separable states and Positive-preserving maps

Now call $\text{Herm}^S_{mn}(\mathbb{C})$ the set (convex cone) of separable states of $\mathbb{C}^m \otimes \mathbb{C}^n$, and define its dual space by (21):

$$\text{Herm}^S_{mn}(\mathbb{C})^* = \{ \tilde{\sigma} \in \text{Herm}^S_{mn}(\mathbb{C}) / \forall \rho \in \text{Herm}^S_{mn}(\mathbb{C}), \ \tilde{\sigma}(\rho) \geq 0 \}$$

This is a convex cone too. The geometrical meaning of Theorem 4 is now clear in this formalism:

**Theorem 2 (restatement.)** A linear operation $\hat{\sigma} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ is Positive-preserving if and only if the linear functional of its associated state $\hat{\sigma}$, namely $\hat{\sigma}$, is in $\text{Herm}^S_{mn}(\mathbb{C})^*$. In other words, the convex cone of Positive-preserving maps is isomorphic to the dual of the convex cone of separable states.

Remember that inclusions are reversed by duality:

$$\text{Herm}^S_{mn}(\mathbb{C}) \subseteq \text{Herm}^+_{mn}(\mathbb{C})$$

$$\Leftrightarrow \text{Herm}^+_{mn}(\mathbb{C})^* \subseteq \text{Herm}^S_{mn}(\mathbb{C})^*.$$ 

Since not all states are separable, this confirms the fact that Positive-preserving maps are not necessarily Complete Positive-preserving.

**Remark 4** The set of $\hat{\sigma}$ in $\text{Herm}^S_{mn}(\mathbb{C})$ such that $\hat{\sigma}$ is Positive-preserving, i.e. such that $\hat{\sigma}$ belongs to $\text{Herm}^S_{mn}(\mathbb{C})^*$, is stable under the transposes $t_1$ on $\mathbb{C}^m$ and $t_2$ on $\mathbb{C}^n$.

**Proof:** For $\hat{\sigma}$ Positive-preserving, $\hat{\sigma} \circ t_2 \equiv \hat{\sigma}^t_2$ and $t_1 \circ \hat{\sigma} \equiv \hat{\sigma}^t_1$ are Positive-preserving too. From this simple observation we readily obtain that the set of the $\hat{\sigma}$ is stable under partial transpositions. $\square$

**Remark 5** Remark 4 is equivalent the Peres criterion [12] for separability, which states that the set of the separable states $\text{Herm}^S_{mn}(\mathbb{C})$ is stable under partial transposition.

**Proof:** [Peres $\Rightarrow$ Remark 4] If $\hat{\sigma}$ is such that $\hat{\sigma}$ belongs to $\text{Herm}^S_{mn}(\mathbb{C})^*$, then we have that

$$\forall \xi \in \text{Herm}^S_{mn}(\mathbb{C}), \text{Tr}(\xi^\ast \xi) \geq 0$$

$$\Rightarrow \forall \xi \in \text{Herm}^S_{mn}(\mathbb{C}), \text{Tr}(\xi^\ast \xi) \geq 0$$

by Peres

$$\Rightarrow \forall \xi \in \text{Herm}^S_{mn}(\mathbb{C}), \text{Tr}(\xi^\ast \xi) \geq 0$$

which means, by definition, that $\hat{\sigma}^t_2$ belongs to $\text{Herm}^S_{mn}(\mathbb{C})^*$. The same applies with $t_1$. 


Linear operator
Projection
Theorems on quantum operations
Hermitian
Operator Sum decomposition,
Factorizability condition
Bistochastic
Hermitian-preserving
Schur’s triangularization
Invertible
Positive-preserving
Formulae on quantum operations
Polar
Positive
Factorizable

Remark 4 [Peres]

Now let $\mathcal{S}$ belong to $\text{Herm}_{mn}^+(\mathbb{C})$. Since $\text{Herm}_{mn}^+(\mathbb{C})$ is a closed convex set containing 0 we have, by the bipolar theorem (see for instance [12]), that $\text{Herm}_{mn}^+(\mathbb{C}) = \text{Herm}_{mn}^+(\mathbb{C})^{**}$. Thus $\mathcal{S}$ belongs to $\text{Herm}_{mn}^+(\mathbb{C})^{**}$, and so

$$\forall \mathcal{S} \in \text{Herm}_{mn}^+(\mathbb{C})^{**}, \quad \text{Tr}(\mathcal{S} \mathcal{E}) \geq 0 \Rightarrow \forall \mathcal{S} \in \text{Herm}_{mn}^+(\mathbb{C})^{**}, \quad \text{Tr}(\mathcal{S} \mathcal{E}^2) \geq 0 \quad \text{by Remark 4}

\Rightarrow \forall \mathcal{S} \in \text{Herm}_{mn}^+(\mathbb{C})^{**}, \quad \text{Tr}(\mathcal{S} \mathcal{E}^2) \geq 0$$

which means, by definition, that $\mathcal{S}^{t2}$ belongs to $\text{Herm}_{mn}^+(\mathbb{C})^{**}$. The same applies with $^t$: we have recovered Peres' criterion.

That the Peres’ criterion corresponds to the simple fact that $\mathcal{S}$ Positive-preserving implies $\mathcal{S}^{t}$ Positive-preserving is a somewhat striking fact. This insight may well help to build tighter criterions: recently the Horodeckis [10] have been following this line of thought.

2. Physical interpretation of formulae

When attempting to characterize separability the notions of duality seem to play a simplifying role, as they help to clarify the correspondence induced by Isomorphism 2. Thus one may wonder if these concepts could facilitate the interpretation of other results in this article. We now give a formulation of quantum operations $\mathcal{S}$ in terms of single operations on its corresponding state $\mathcal{S}$.

**Proposition 6** Let $\mathcal{S}$ represent a non-normalized quantum state of a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^n \otimes \mathbb{C}^m$ shared by Alice and Bob. Suppose Bob performs on $\mathcal{S}$ a local generalized measurement \{Id$_m \otimes M_i^{(x)}\}$$_x$. Call $\text{Id}_m \otimes M$ the element whose outcome occurs and let $\rho_B \equiv (\text{Id}_M)I \in \text{Herm}^+_m(\mathbb{C})$. Then the unrescaled post-measurement state as viewed by Alice is precisely $\widehat{\mathcal{S}}(\rho_B)$. Thus the effect of any quantum operation $\mathcal{S}$ can be viewed as the trace out of a particular local single operation on its corresponding state $\mathcal{S}$.

**Proof:** The unrescaled post-measurement state is simply $\mathcal{S}_M = (\text{Id}_m \otimes \mathcal{S})(\text{Id}_m \otimes M_1)$. Using (13) this yields for Alice the state:

$$\text{Tr}_2(\mathcal{S}_M) = \text{Tr}_2((\text{Id}_m \otimes M)\mathcal{S}(\text{Id}_m \otimes M_1))$$

$$= \widehat{\mathcal{S}}\left((\text{Id}_M)I\right)$$

$$= \widehat{\mathcal{S}}(\rho_B) \quad \square$$

The fact that there is a transpose corroborates the idea of duality. Indeed, first $\text{Id}^tM$ is thought of as defining a functional $\sigma \mapsto \text{Tr}(\text{Id}^tM \sigma)$, but then as we think of a quantum operation as acting on states we act upon its transpose. The map $\text{Id}^tM \mapsto \widehat{\mathcal{S}}\left((\text{Id}_M)I\right)$, though it is Positive-preserving, is not Completely Positive-preserving since it can be written as $\mathcal{S} \circ \iota$. However the same map defined from states to states, i.e. $(\text{Id}^tM)^t \mapsto \widehat{\mathcal{S}}((\text{Id}_M)I)^t$, is Completely Positive-preserving. Proposition 4 suggests that quantum states in $\text{Herm}^+_m(\mathbb{C})$ inherently defines a quantum operation between their two subsystems.

| Table I: Summary |
|-------------------|
| **Matrix $\mathcal{S}$** | **Linear operator $\mathcal{S}$** |
| Hermitian | Hermitian-preserving |
| Dual to separable | Positive-preserving |
| Positive | Completely Positive-preserving |
| **Particular state $\mathcal{S}$** | **Particular quantum operation $\mathcal{S}$** |
| Pure | Factorizable |
| Unentangled $\sigma_1 \otimes \sigma_2$ | Projection $\rho \mapsto (\text{Tr}_2(\sigma'_2)\rho)\sigma_1$ |
| Separable | Sum of projections |
| $\text{Tr}_1(\mathcal{S}) = \text{Id}$ | Dual to Positive-preserving |
| $\text{Tr}_1(\mathcal{S}) = \text{Id}$ and $\text{Tr}_2(\mathcal{S}) = \text{Id}$ | Bistochastic |
| **Particular ket $A$** | **Particular evolution matrix $A$** |
| Maximally entangled | Unitary |
| Totally entangled | Invertible |
| $\sum_i |i\rangle |i\rangle$ | $\text{Id}$ |
| $\sum_i \lambda_i |i\rangle |i\rangle$ | $\text{Diag}\{\lambda_i\}$ |
| with $\forall i, \lambda_i \in \mathbb{R}$ | Hermitian |
| with $\forall i, \lambda_i \in \mathbb{R}^+$ | Positive |

**Theorems on states**

Spectral decomposition,
Unitary degree of freedom
Operator Sum decomposition,
Unitary degree of freedom
Purification
$\mathcal{S}(\rho) = \text{Tr}_1(U\rho U^\dagger)$
Bipartite decompositions:
Schmidt
Matrix decompositions:
Polar
One-sided triangular
QR
Two-sided triangular
Schur’s triangularization
Purity condition
Factorizability condition

| Formulae on states | Formulae on quantum operations |
|--------------------|-------------------------------|
| $\text{Tr}_{1/2}(AB^t)$ | $= (B^tA)/AB^t$ |
| $\text{Tr}_2(\kappa \otimes \rho)\mathcal{S}(\tau \otimes \sigma^t)$ | $= \kappa \mathcal{S}(\rho \sigma)\tau$ |
| $\text{Tr}_2(\text{Id} \otimes \rho)\mathcal{S}(\text{Id} \otimes \rho^t)$ | $= \mathcal{S}(\rho^t \rho^t)$ |
| $\text{Tr}(\mathcal{S}(\rho \otimes \rho^t))$ | $\text{Tr}(\mathcal{S}(\rho))$ |
| $\text{Tr}(\mathcal{E}(\mathcal{S}(\rho))$ | $\equiv \sum \text{Tr}(\mathcal{E}(E_{\left| i\right>}) \mathcal{S}(E_{\left| i\right>})$ |
IV. SUMMARY AND CONCLUDING REMARKS

In this article we make several new contributions, some technical, others more geometrical. Amongst the technical results we provide two triangular decompositions for pure states of a bipartite system, i.e. local changes of basis so that vectors in \( \mathbb{C}^m \otimes \mathbb{C}^n \) may be written with triangular coefficients only. We also give two original algebraic tests on Completely Positive-preserving maps: one regarding the factorizability or single operator decomposition, the other testing extremality in the set of Trace-preserving operations. These are particularly interesting in the sense that they do not depend on the operator sum decompositions of these maps. The formulae in Proposition 4 should yield simplifications in optimization of fidelities of quantum operations as encountered for instance in quantum cryptographic problems. On the more geometrical side we endow \( \text{Herm}^+_{n_2} (\mathbb{C}) \) with a semi-group structure stemming from the composition law on quantum operations. This in turn yields a group isomorphism between totally entangled (pure) states and \( GL_n (\mathbb{C})/U (1) \), and maximally entangled (pure) states and \( SU (n) \). This result sheds light on the geometry of entangled states as it suggests, for future work, simple parameterizations and bi-invariant metrics on the corresponding (group-)submanifolds of the set of pure states in \( \text{Herm}^+_{n_2} (\mathbb{C}) \). In addition we show that the set of quantum operations is stable under component-wise product. These contributions are interesting enough by themselves, but perhaps the most significant achievement of this article is to demonstrate the central, transversal role of the state-operator isomorphism as formalized in Isomorphism 2 and justified by Theorem 3. We have shown that virtually all the main results regarding states/operators can be elegantly brought as corollaries of their operator/state analogue, which makes this correspondence one of the most fruitful linear algebraic tool in the surroundings of quantum theory (see table I for summary). Even for more specialist issues of quantum information theory we find that the isomorphism has a role to play, as was illustrated by the problem of characterizing separable states. On this occasion we introduced notions of duality, which serve both to facilitate the interpretation of the state-operator correspondence and its related formulae, and to understand the underlying geometry from a slightly more abstract point of view. The formulae themselves should have numerous applications in quantum information theory, and maybe (as suggested in Proposition 6) provide a novel interpretation of states versus operations in open systems.

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