Approximated solution algorithms for Urysohn-type equations

V Belozub, M Kozlova and V Lukianenko
Faculty of mathematics and computer science, V.I. Vernadsky Crimean Federal University, Simferopol, 295007, Russia
E-mail: art-inf@mail.ru

Abstract. Problems of remote sensing and gravity, magnetic, seismic, geological prospecting use systems of indirect data measurement, which can be modeled with non-linear Urysohn equations. The paper presents equations of Urysohn type and their operator analogues. A number of algorithms for their solution has been developed based on the availability of a priori information, asymptotic properties of integral operators, and specific features of a model. It has been formulated a theorem on constructing a solution of the original equation based on the neighboring one with an error estimate. Both the original and neighboring equations are taken as the regularized equations. The proposed approach allows for a variety of algorithms, depending on the type of regularization and iteration schemes, in particular, a modified version of the Levenberg-Marquardt algorithm. Additionally, the algorithm for searching characteristic points of a given function based on the asymptotics of an integral Urysohn operator is provided.

1. Introduction
The paper explores the problem of reconstructing solutions for the first kind Urysohn equations given that a priori information on a solution and additional information on a solution of neighboring equations are known. The operators and right-hand sides of such equations are known with a certain error. The right-hand side is obtained based on real or experimental measurements. Experimental data are processed to maximize the amount of reliable information on the real properties of the functions to be recovered. Only integral properties are observed. The integral properties take into account only the total effect produced by all points belonging to the observed object. Such properties are stable in that they disregard even large changes in values describing the object, provided these changes compensate each other. The direct problem is to measure the object’s integral properties, which are the right-hand sides of the first kind Urysohn equations $Az = u$ according to the given initial dependencies $z$ characterizing the object. The inverse problem is to solve the first kind Urysohn equation.

"Neighboring” equations are equations which have a solution close in some sense to a solution of an original equation. So, closeness can be considered in regard to the norm. The precise (or approximate) solution of a close ("neighboring") equation, having a simpler structure, is taken as an approximate solution of the initial equation.

Solving a non-linear integral Urysohn equation of the first kind is an ill-posed problem. The theory of ill-posed problems, methods and algorithms of their solution are among the most important directions of research. The publications by Tikhonov, Lavrentiev and Ivanov were first to introduce the regularization methods as applied to solving ill-posed problems. In the works by
Bakushinsky, Goncharsky, Leonov, Polyak, Yagola, there have been formulated the principles of iterative regularization, construction of algorithms in Banach spaces, regularization of variational inequalities with monotone operators, as well as proposed the iterative regularization techniques for the methods of Newton-Kantorovich, Gauss-Newton, etc.

The idea of using additional information on the properties of original functions is very important for constructing regularizing algorithms in applied problems. For equations with nonlinear monotone operators, Vasin developed the nonlinear modified process analogues. He also proposed an approach to solving ill-posed problems with a priori information based on the use of strongly Fejer mappings for constructing iterative algorithms using a priori constraints in the form of convex inequalities. The present work relies on the research by Vasin on the strong convergence of the Levenberg-Marquardt method and its modifications for solving Tikhonov-regularized nonlinear equations.

The aim of the present work is to construct the regularizing iterative algorithms for solving Urysohn-type equations of the first kind, based on the use of information on a solution, precedent information on a neighboring equation, and asymptotic approximations. In Sections 2.1 and 3, the authors overview a group of general Urysohn-type models and consider their specific solution aspects. In Section 2.2, novel solution algorithms are proposed. They involve usage of solutions of neighboring equations, for which it is possible to obtain effective solutions in advance. For this purpose, authors take, in particular, equations having a solution based on the Levenberg-Marquardt-type algorithms, studied by Vasin and his research school. In Section 3, there is considered a group of equations for which asymptotic solution methods (delta-shaped kernels) are admissible.

2. Materials and methods

2.1. Discrete-continuous Uryson-type models of the first kind

Discrete, discrete-continuous, integral nonlinear Urysohn-type equations are applicable to a broad class of problems such as indirect measurement models, remote sensing, gravity, magnetic, seismic, geological prospecting, etc. For example, let us consider a one-dimensional case of determining the shape of a geological anomaly and its parameters (density, conductivity, etc.) based on the measurements done on the Earth’s surface. It is required to determine density \( \rho(x) \) of the anomalous area located at depth \( H \), constrained by surface \( h(x) \), \( a \leq x \leq b \), by changes in its vertical component \( u(x) \) – the gravity force on the surface at point \( (x, H) \), with \( \gamma \) being its gravitational constant.

The corresponding integral Urysohn-type equation has the form

\[
\gamma \int_a^b \frac{\rho(\xi)h(\xi)[H-h(\xi)]d\xi}{[(x-\xi)^2 + (H-h(\xi))^2]^{3/2}} = u(x), \quad a \leq x \leq b. \tag{1}
\]

Under an assumption that the shape of the anomaly located at depth \( H \) is of no interest to us, and the mass of the volume element in \( \Delta \xi \) is equal to \( \rho(\xi)\Delta \xi \), we obtain the equation

\[
\gamma H \int_a^b \frac{\rho(\xi)d\xi}{[(x-\xi)^2 + H^2]^{3/2}} = u(x), \quad a \leq x \leq b. \tag{2}
\]

The only unknown function here is density \( \rho(\xi) \) (and, probably, \( H,a,b \)), in contrast to equation (1), where unknown are two functions \( h(\xi) \) and \( \rho(\xi) \). A version of equation (1), when the mass of element \( \Delta \xi \) concentrated near point \( \xi \) is determined as \( \rho(\xi)\Delta \xi \) and does not depend on \( h(\xi) \), can be written in the form:

\[
Af \equiv \gamma \int_a^b \frac{\rho(\xi)[H-h(\xi)]d\xi}{[(x-\xi)^2 + (H-h(\xi))^2]^{3/2}} = u(x), \quad a \leq x \leq b, \quad f = (\rho,h). \tag{3}
\]
The problem of obtaining a solution for equations of the first kind (1)–(3) is ill-posed [2]. In works [5–8] different models are presented. The inverse problem of a logarithmic potential for an infinite by axis $OX$ contact surface $OY$ is described by the nonlinear integral equation [7]:

$$Az = \gamma \rho \int_{a}^{b} \ln \frac{(x - \xi)^2 + H^2}{(x - \xi)^2 + (H - z(\xi))^2} d\xi = u(x),$$

where $z(\xi)$ is the surface shape of a gravitating body; $H$ – depth under the surface; $\rho$ – density (if $\rho$ depends on $x$, then $\rho(\xi)$ is under the integral sign); $|z(\xi)| < H$.

The inverse problem of the Newtonian potential in a linear formulation for a body occupying area $V = \{x, y, z \mid a \leq x \leq b, c \leq y \leq d, -H \leq z \leq -H + \varphi(x, y)\}$, is described by equation [12]

$$A\varphi = \gamma \int_{a}^{b} \int_{c}^{d} \rho(\xi, \eta) \frac{H\varphi(\xi, \eta) d\xi d\eta}{((x - \xi)^2 + (y - \eta)^2 + H^2)^{3/2}} = u(x, y, 0).$$

In work [14], the inverse problem of gravimetry in a multilayer medium is considered. To solve the corresponding Urysohn equation of the first kind, relevant to the problem, the regularizing algorithms based on the Levenberg-Marquardt method with the use of weight factors are proposed. This is a component-wise method. Classical and component-wise methods are compared in terms of speed, convergence, relative error, and program execution time.

In work [12], theorems of convergence are formulated for a two-component algorithm based on the Lavrentiev regularization scheme and the modified Newton method. The results of a numerical solution of a gravimetry three-dimensional inverse problem on a model of a two-layer medium are provided. In the article [8] the authors use the homotopy analysis method to obtain an approximate solution for the inverse problems of logarithmic and Newtonian potentials, simulating the inverse problems of gravity and magnetic prospecting. For numerical algorithms, Bernstein polynomials are used.

For the class of considered problems, it is possible to present a hierarchy of mathematical models and many scenarios for measuring, collecting, and using information necessary for a correct reconstruction of original values.

Let us consider a set of generalized Urysohn-type models of the first kind whose form has already been shown above. Equation (3) $Af = u$ with kernel

$$k(x, y) = \frac{\gamma y}{[x^2 + y^2]^{3/2}}$$

can be written in the form

$$Af \equiv \int_{a}^{b} k(x - \xi, H - h(\xi)) \rho(\xi) d\xi = u(x), \quad a \leq x \leq b. \quad (4)$$

If measurements are made at discrete points $x_i, i = 1, m$, then we get the discrete-continuous systems of convolution-type equations

$$(Af)_i = \int_{a}^{b} k(x_i - \xi, H - h(\xi)) \rho(\xi) d\xi = u(x_i) \equiv u_i, \quad a \leq x_i \leq b, \quad i = 1, m. \quad (5)$$

A discrete version of equation (5) represents a nonlinear algebraic system of equations:

$$\sum_{j=1}^{n} A_j k_{i-j}(h_j) \rho_j = u_i, \quad i = 1, m, k_{i-j}(h_j) = k(x_i - \xi_j, H - h(\xi_j)), \quad \rho_j = \rho(\xi_j), \quad u_i = u_i(x). \quad (6)$$
In many applied problems, including, in particular, abovementioned, the requirement of monotonicity for operator $A$ in the original equation is not met. The use of the finite-dimensional analogues may ease this requirement and justify the convergence of iterative processes [11–14]. Paper [13] considers a finite-dimensional operator $A$ for which the matrix corresponding to the derivative $A' u$ in a certain neighborhood of solutions has a spectrum consisting of various non-negative eigenvalues. Under certain additional assumptions, the results remain valid for the case with small negative eigenvalues of matrix $A' u$. The substitution of a derivative for operator

$$Az = \int_a^b m(s)n(t − z(s))ds$$

in the neighborhood of function $z_0$

$$(A'z_0)h = −\int_a^b m(s)n'(t − z_0(s))h(s)ds$$

with a discrete analogue leads to the system of equations

$$\sum_{j=1}^N a_{ij} h_j,$$

where

$$a_{ij} = m_j n_{ij}, \quad m_j = \Delta m(s_j), \quad n_{ij} = n'(t_i − z_0(j)), \quad z_0(j) = z_0(s_j), \quad j = 1,N, \quad i = 1,M,$$

$s_j$ are the points of splitting of interval of integration.

The problem of surface restoration from data of measuring a time period needed for a signal (reflected pulse) to reflect from surface $h(\xi)$, directed vertically downwards

$$\tau \equiv z(\xi) = \frac{2}{c} \left[ (x − \xi)^2 + (H − h(\xi))^2 \right]^{\frac{1}{2}},$$

$$Az(h) = \int_a^b m(s)n \left( t − \frac{2}{c} [(x − c)^2 + (H − h(s))^2]^{\frac{1}{2}} \right) ds,$$

where $c$ is the pulse propagation speed, and $\tau$ – twice the time period (to the surface and back) with the use of the antenna is reduced to Urysohn-type equations of the 1st kind.

As the reflected signal is integrated with the weighting function $k(x, \xi)$ (kernel, directional pattern, instrumental function of the measuring device), and the pulse has a certain shape $n(t)$, the resulting equation will be

$$Az \equiv \int_a^b m(s)k(x − s)n(t − z(s))ds = u(x, t), \quad a \leq x \leq b, \quad c \leq t \leq d, \quad (7)$$

where $m(s)$ indicates the surface reflection parameters, and $n(t)$ is the delta function. Function $m$ can depend both on $z$ and $z'$.

The discrete-continuous nonlinear operators of the following form

$$(Az)_i \equiv \int_a^b m(s)n(t_i − z(s))ds, \quad c \leq t_i \leq d, \quad i \in M \subset \mathbb{N},$$

$$(Az)_i \equiv \int_a^b k(x − s)n(t_i − z(s))ds, \quad c \leq t_i \leq d, \quad a \leq x \leq b, \quad (8)$$
are used as the model operators. The linearized equations involve Frechet derivatives on \( z \) in the form

\[
A'(z)h \equiv - \int_a^b m(s)n'(t_i - z(s))h(s)ds, \quad c \leq t_i \leq d,
\]

\[
A'(z)h \equiv - \int_a^b k(x - s)n'(t_i - z(s))h(s)ds, \quad c \leq t_i \leq d,
\]

\[
A'(z)h \equiv - \int_a^b m(s)k(x - s)n'(t_i - z(s))h(s)ds, \quad c \leq t_i \leq d.
\]

(9)

Note that it is impossible to reconstruct two functions \( h(\xi), \rho(\xi) \) from one equation based only upon the information about the right-hand side \( u(x) \). Hence, it is necessary to have a system of such equations or to aim at reconstructing only certain properties of given functions (extremum points, monotonicity areas, etc.) by using a priori, precedent or other kind of information on a solution of \( f = (h, \rho) \) and specific features of the model. One can pose the problem of finding one of two functions \( h \) or \( \rho \), under an assumption that another one is already given.

Let us consider a relevant algorithm.

Let the approximations of function \( h(\xi) \) be given in the form of a sequence of functions \( h_n(\xi) \), \( n = 0, 1, 2, ... \) where \( h_0(\xi) = 0 \) then the \( n \)th approximation for \( \rho(\xi) \) will be found by solving an integral equation belonging to the type (4) (or a system of discrete-continuous equations (5)):

\[
\int_a^b k(x - \xi, H - h_n(\xi))\rho(\xi)d\xi = u(x), \quad a \leq x \leq b.
\]

(10)

The approximated solution of \( \rho(\xi) \) is used for constructing a new approximation for \( h(\xi) \):

\[
\int_a^b k(x - \xi, H - h(\xi))\rho_n(\xi)d\xi = u(x), \quad a \leq x \leq b.
\]

The resulting solution is \( f_n = (h_n(\xi), \rho_n(\xi)) \).

Obviously, the solution is obtained as a result of using regularizing algorithms. For \( h_n(\xi) \) the rationale for a discrete analogue comes from the works by Vasin (for the Levenberg-Marquardt algorithm see [9–13]).

2.2. Discrete-continuous Urysohn-type models of the first kind

If we know that the solution of \( z(s) \) is to be monotone, then the problem of solving the nonlinear Urysohn equation of the first kind is reduced to a linear convolution-type equation. Relevant results can be found in work [17]. Developing an intellectualized data processing system is needed when it is required to take into account diverse information [16].

Note that the approach accounting for a variety of information may demand different scenarios for the search of required parameters based on the results of indirect measurements. It seems reasonable to exploit the knowledge about a solution of a neighboring equation. Such precedent information is often used in different applied problems.

To justify the reconstruction of a solution for given equations upon the use of two equations, in particular, let us consider a simple scheme for solving equation \( Az = u \) based on the solution
of a neighboring equation $\tilde{A}\tilde{z} = \tilde{u}$. The problem is now reduced to a set of two neighboring extremal problems

$$M^\alpha[z] = \alpha\|z\|_{W_2}^2 + \|Az - u\|_{L_2}^2,$$

$$\tilde{M}^\alpha[\tilde{z}] = \tilde{\alpha}\|\tilde{z}\|_{W_2}^2 + \|\tilde{A}\tilde{z} - \tilde{u}\|_{L_2}^2.$$  

They have equivalent equations

$$Bz \equiv \alpha(-z'' + z) + A^*Az = A^*u \equiv g,$$
$$\tilde{B}\tilde{z} \equiv \tilde{\alpha}(-\tilde{z}'' + \tilde{z}) + \tilde{A}^*\tilde{A}\tilde{z} = \tilde{A}^*\tilde{u} \equiv \tilde{g}$$  

with regularization parameters $\alpha, \tilde{\alpha}$.

The following theorem is valid for linear equations $Bz = g$ and $\tilde{B}\tilde{z} = \tilde{g}$

**Theorem 1**

Let the following conditions be satisfied:

1. $\exists Y_0 \subset Y, g \in Y, \tilde{g} \in Y, \text{ then } (g - \tilde{g}) \in Y_0$;
2. Equation $\tilde{B}z_0 = \tilde{g}$ has a single solution;
3. $B - \tilde{B} : X \to Y_0$;
4. $\exists B^{-1} : Y_0 \to X_0$;
5. Condition $\|B^{-1}(B - \tilde{B})\| < 1$ is met.

Then, the iteration process $z_{n+1} = z_n \tilde{B}^{-1}[Bz_n - g]$ converges to $X_0$. All elements $z_n$ have the property $z^* - z_n \subset X_0$. The formula for the error is valid:

$$\|z_n - z^*\| \leq \frac{\|\tilde{B}^{-1}(B - \tilde{B})\|n}{1 - \|B^{-1}(B - \tilde{B})\|} \|\tilde{B}^{-1}g\|_{X_0}.$$ 

Here $X = Y = L_2, X_0 = W_2^1$.

The proof follows from theorems 1, 2 by Cherskiy [1]. The multitude of algorithms is obtained by using different methods of regularization (in (11) Tikhonov’s regularization method is applied [2]). Theorem 1 is applicable to linearized equations. For Lavrentiev’s regularization method $F(z) \equiv Az - u + \alpha(z - z_m)$, the iterative procedure of constructing $z_{n+1} = z_n + h$ approximation leads to the algorithm of searching function $h$ from a linear equation $[\alpha I + (A'z_m)]h - F(z_n) = 0$ or

$$z_{n+1} = z_n - \gamma[\alpha I + (A'z_m)]^{-1}[Az_n - u + \alpha(z_n - z_m)].$$  

The following equations should be considered as neighboring linear equations:

$$\mathbb{K}h \equiv [\alpha I + (A'z_m)]h = Az_n - u + \alpha(z_n - z_m) \equiv g,$$
$$\tilde{\mathbb{K}}h \equiv [\tilde{\alpha}I + (\tilde{A}'\tilde{z}_m)]h = \tilde{A}\tilde{z}_n - \tilde{u} + \tilde{\alpha}(\tilde{z}_n - \tilde{z}_m) \equiv \tilde{g}$$

Provided $\delta = ||\tilde{\mathbb{K}}^{-1}(\mathbb{K} - \tilde{\mathbb{K}})|| < 1$, it is possible to find a needed solution via a solution of a neighboring equation with simpler structure, or a solution already found for different error levels (taking into account precedent information).

The follow theorem is valid for linear equations (13): operators $\mathbb{K}, \tilde{\mathbb{K}} : H \to G, H_0 \subset H, H_0 - Banach$ spaces.

**Theorem 2**

Let the following conditions be satisfied:

1. $\exists G_0 \subset G, g \in G, \tilde{g} \in G, \text{ then } (g - \tilde{g}) \in G_0$;
   
   $g - \tilde{g} = \tilde{A}z_k - \tilde{u} + \tilde{\alpha}(\tilde{z}_k - \tilde{z}_m) - [Az_k - u + \alpha(z_k - z_m)] \in G_0$;

2. Equation $\tilde{\mathbb{K}}h_0 = \tilde{g}$ has a single solution;
3. Operator \( \mathcal{K} - \tilde{\mathcal{K}} : H \to G_0; \)
\[
(\mathcal{K} - \tilde{\mathcal{K}}) h = [(\alpha - \tilde{\alpha}) I + (A'z_k - \tilde{A}'z_k)] h = y \in G_0 \subset G;
\]
4. \( \exists \mathcal{K}^{-1} : G_0 \to H_0, \quad \tilde{\mathcal{K}}g^{-1} = (\tilde{\alpha}I + \tilde{A}'z_k)^{-1} \tilde{g} = \tilde{h} \in H_0; \)
5. Condition \( ||\tilde{\mathcal{K}}^{-1}(\mathcal{K} - \tilde{\mathcal{K}})|| = ||(\tilde{\alpha}I + \tilde{A}'z_k)^{-1}[(\alpha - \tilde{\alpha}) I + A'z_k - \tilde{A}'z_k]|| < 1 \) is met.

Then equation \( \mathcal{K}h = g \) has single solution
\[
h = \tilde{h} + [I + \mathcal{K}^{-1}(\mathcal{K} - \tilde{\mathcal{K}})]^{-1} \mathcal{K}^{-1}(g - \mathcal{K}\tilde{h}),
\]
\( h - \tilde{h} \in H_0 \subset H, \)
\( \mathcal{K}h - g \in G_0, z_{k+1} = z_k + h. \)

The formula for the error is valid:
\[
||h - \tilde{h}||_{H_0} \lesssim \frac{||\mathcal{K}^{-1}(g - \mathcal{K}\tilde{h})||_{H_0}}{1 - ||\mathcal{K}^{-1}(\mathcal{K} - \tilde{\mathcal{K}})||}
\]

Iteration process (4) can be modified in the form
\[
z_{n+1} = z_n - \gamma (\tilde{\alpha}I + (\tilde{A}'z_n))^{-1}[Az_n - u + \alpha(z_n - z_m)].
\]

Here the initial approximation \( \xi = z_m \) is included in the condition of the approximate source representability of the initial residual
\[
z^* - \xi = F^{\text{app}}(z^*)v + w, \quad v \in Y, \quad w \in X, \quad ||w|| \leq \Delta, \quad F : X \to Y
\]
and is widely used when studying Tikhonov’s convergence methods and its iterative variants [3,4]. Parameter \( \alpha \) is chosen to be close to known \( \tilde{\alpha} \), to provide an estimate of \( \delta < 1 \). Function \( \tilde{z}_n \) is fixed in operator \([\tilde{\alpha}I + (\tilde{A}'z_n)]\), which allows for a significant decrease in computation time when using an iterative formula. For a modified version of the Levenberg-Marquardt method [9–11], the solution of nonlinear equations, in a particular case, has the form
\[
z_{n+1} = z_n - \gamma (\tilde{\alpha}'(z_n)\tilde{A}'(z_n) + \alpha I)^{-1}[\tilde{A}'(z_n)(Az_n - u_\delta) + \alpha(z_n - \tilde{z}_m)].
\]

Here, \( \tilde{z}_m \) is an initial approximation for a needed solution.

The justification for the convergence of the proposed algorithms can be obtained by analogy with the works [12,13]. In work [12], it is given an overview of Newtonian and gradient type iterative processes used for a stable approximation of solutions of nonlinear irregular operator equations in Hilbert spaces. A modified Newton’s method is considered in a form similar to (12), in which operator \( B_n = (\alpha I + A'z_n) \) is replaced with operator \( B_0 = (\beta I + A'z_0) \) where \( z_0 \) is the initial approximation. It is proved a theorem that provides sufficient conditions for converging the process (12) to a solution of an equation assumed to be solvable [12].

If we select such an equation as a neighboring one, then, according to Theorem 1, we can obtain a solution of the original equation with an error estimate that takes into account the error in the neighboring equation solution. Thus, the availability of neighboring equations, that can be effectively solved, allows one to construct algorithms for the original equations. In particular, asymptotic solution methods can be used for these purposes.

3. Results and discussion

In model equations (7) function \( n(t) \) is delta-shaped, in particular. The example are
\[
n(t) = \frac{1}{\pi} \frac{\beta}{\beta^2 + t^2}, \quad n(t) = \frac{\beta}{\pi} e^{-\beta^2 t^2}, \quad n(t) = \frac{\beta}{2} e^{-\beta |t|}
\]
which for $\beta \to \infty$ converge to the Dirac delta function $\delta(t)$ (the sense of the theory of generalized function [1]). For big values of parameter $\beta$, function $n(t)$ can be replaced with function $\delta(t)$, but it does not simplify the equation (in contrast to the linear case). The equation is simplified in the monotonic case:

$$Az \equiv \int_{\mathbb{R}} m(s)\delta(t - z(s))ds = u(t), \ t \in \mathbb{R}$$

with substitute $\tau = z(s)$, $s = \psi(\tau)$ it converges to differential equation $z'(\psi(\tau)) = m(\psi(\tau))u(t)$. The delta-shape of the kernel and the dependency on the parameter $\beta$ make it possible to pose the problem of searching for the characteristic points of the solution ("shiny", stationary). Based on the determined points, a solution can be obtained. So, for example, it’s possible to reconstruct a solution in the monotonic areas. Such approximate solutions are used as an initial iteration in iterative algorithms.

Here is the procedure for constructing characteristic points. When solving inverse problems (7)–(10), asymptotic methods are used to reconstruct the characteristic points of a solution for nonlinear integral equations based on the Laplace method and the use of generalized functions. Let us consider a model system of the nonlinear integro-differential Urysohn-type equations of the first kind in the form

$$\int_{a}^{b} f_k(x, h(x), h'(x))e^{-\lambda(t - \frac{2}{\lambda} R_k(h))} dx = u_k(t), \ k = 1, 2,$$

where

$$R_k(h) = \left( (H - h(x))^2 + (x - a_k)^2 \right)^{\frac{1}{2}},$$

$\lambda \gg 1$ is a fixed parameter, $c, H, a_1, a_2$ – fixed values, and $H \gg 1$, $R_k(h) \neq 0$.

It is assumed that there is solution $h(x)$, for example, in the class of sufficiently smooth functions. The goal is to determine the characteristic points ("shiny", extremum points) of a system solution for given $u_k(t), k=1, 2$.

To solve the problem, the Laplace method is used, determining the behavior of integral

$$\int_{a}^{b} f(x)e^{-\lambda(t - S(x))^2} dx = q(t),$$

under condition that $\lambda \gg 1$. It is assumed that:

1. For $t$: $\min (t - S(x))^2 \neq 0$, $q(t) = O(\lambda^{-\infty})$.

2. For fixed $t$, there are the roots $(x_1, ..., x_p)$ of equation $S(x) = t$, each of them is simple, and none of these points is boundary; then the following asymptotic representation is valid

$$q(t) = \lambda^{-\frac{1}{2}} \sqrt{\pi} \left( \sum_{k=1}^{p} \frac{f(x_k)}{|S'(x_k)|} + O(\lambda^{-1}) \right).$$

3. If for some $t$ there is a single simple root $\hat{x}$ of multiplicity 1 for equation $S(\hat{x}) = t$, such that $S'(\hat{x}) = 0$, $S''(\hat{x}) \neq 0$, then

$$q(t) = \lambda^{-\frac{1}{4}} \left( \frac{f(\hat{x})}{|S''(\hat{x})|^\frac{1}{2}} \cdot \frac{1}{\Gamma \left( \frac{1}{4} \right)} + O(\lambda^{-\frac{1}{2}}) \right),$$

where $\Gamma(x)$ is the gamma function. Provided that $f(x)$ is a positive, continuous, slowly varying function, then, based on the asymptotic representations, it follows that function $q(t)$ takes its
maximum value at the points lying in the vicinity \( \lambda^{-\frac{1}{2}} \) of points \( \hat{t} \), for which there is a stationary point \( \hat{x} \) such that \( S(\hat{x}) = \hat{t} \).

These representations allow constructing an algorithm for searching the characteristic points of the original equation. It is assumed that the variation range for \( t \) contains the range \([m_k, M_k]\) where

\[
m_k = \min_{[a,b]} \frac{2}{c} R_k(h),
M_k = \max_{[a,b]} \frac{2}{c} R_k(h), \quad k = 1, 2.
\]

Under the formulated conditions for \( H \) stationary points \( h(x) \) are close to stationary points \( R_k(h) \). Let \( h_1, h_2, ..., h_m \) be a set of function \( h(x) \) values in its stationary points \( x_1, x_2, ..., x_m \) ordered descending. The distance from \( a_1, a_2 \) to \([a,b]\) has the order \( \ll H \) hence the sequence

\[
R_k(h_i) = \frac{2}{c} \left( (H - h_i) + (x_i - a_k)^2 \right)^{\frac{1}{2}}
\]

is ordered ascending. In such a case, the stationary points \( h(x) \) can be found with the right-hand side using the following algorithm.

---

**Algorithm for constructing stationary points of Urysohn equation**

1° Determine the points of maximum in ascending order \( u_k(t) : t^{k}_1, t^{k}_2, ..., t^{k}_p, \quad k = 1, 2 \).

2° For each \( i = 1, p \) in respect to \( h, x \), solve the system of equations

\[
R_k(h, x) = t^{k}_i, \quad k = 1, 2.
\]

Obtain \( (h_i, x_i) \).

3° Assume \( h(x_i) = h_i, \quad i = 1, p \).

---

In the second step, practical calculations use regularization of solution and \( k = N > 2 \).

It has also been considered the solution of a system of nonlinear Uryson-type equations (15) in the class of piecewise constant functions and piecewise linear functions. Based on the performed numerical experiments, the following results were obtained.

1. High accuracy in determining the waveforms on the right-hand sides of a system of integro-differential Uryson-type equations provides sufficient accuracy in determining stationary points.

2. In the case of inaccurate calculation of the sharp maximum points (bursts) on the right-hand sides, the method is stable for determining the height \( h \) and less stable for determining the \( x \) coordinate of the stationary point \( h(x) \).

3. Increasing the number of observation points, i.e. an increase in the number of jointly considered nonlinear equations, improves the stability in determining stationary points \( h(x) \) relative to errors in determining the maximum points on the right-hand sides.

Similar results are valid for the two-dimensional case. Determining the stationary points of a studied function may be sufficient to solve the problem. Reconstructing the function in the monotonic areas can be reduced to solving linear integral convolution-type equations. For such equations, it can be applied the technique of obtaining a solution based on the use of iterative algorithms (12) and Fourier transforms [15–17].

For equations of the form (7), the Fourier transform simplifies the solution algorithm. Replacing continuous operators with their discrete analogues allows to use the results of research on the convergence of iterative methods obtained by Vasin.
Conclusion
The results of the work are as follows. There has been provided a set of the first kind Urysohn-type discrete-continuous models applicable to the problems of solution reconstruction. There have been obtained the algorithms with regularization based on different a priori information. There have been proposed new algorithms for solving Urysohn-type equations of the first kind, based on effective solutions of neighboring equations. An option of implementing asymptotic methods was demonstrated.

The procedure of replacing original equations with neighboring ones makes it possible to extend the applicability of iterative algorithms. Prospective development of the proposed method implies extending the class of iterative algorithms via the use of various regularization methods and conducting broad computational experiments.

For one and two-dimensional cases, asymptotic methods can simplify the use of relevant models. Prospective work implies further implementation of a composition of different iterative algorithms and asymptotic models.

References
[1] Gakhov F D and Chersky Yu I 1978 Equations of convolution type (Moscow: Nauka) (in Russian)
[2] Tikhonov A N, Goncharsky A V, Stepanov V V and Yagola A G 1983 Regularizing algorithms and a priori information (Moscow: Nauka) (in Russian)
[3] Bakushinsky A B and Kokurin M Yu 2002 Iterative methods for solving ill-posed operator equations with smooth operators (Moscow: Editorial URSS) (in Russian)
[4] Engl H, Kunisch K and Neubauer A 1996 Regularization of inverse problems (Dortrecht: Kluwey)
[5] Zhdanov M S 1984 Analogues of the Cauchy integral in the theory of geophysical fields (Moscow: Nauka) (in Russian)
[6] Zhdanov M S 2002 Geophysical Inverse Theory and Regularization Problems (Elsevier, Academic Press)
[7] Boykov I V 2004 Approximate methods for solving singular integral equations (Penza: Publishing house of Penza state University) (in Russian)
[8] Boykov I V and Boykova A I 2012 News of higher educational institutions. Volga region. Physical and mathematical Sciences 3 pp 17-28 (in Russian)
[9] Vasin V V and Eremin I I 2009 Operators and Iterative Processes of Fejer Type (Berlin : Walter de Gruyer GmbH&Co.KG)
[10] Vasin V V 2012 The Levenberg-Marquardt method for approximation of solutions of irregular operator equations Automation and Remote Control 73(3) pp 440-449
[11] Vasin V V and Perestoronina G Ya 2013 The Levenberg-Marquardt method and its modified versions for solving nonlinear equations with application to the inverse gravimetry problem Proceedings of the Steklov Institute of Mathematics 280 pp 174-182
[12] Vasin V V, Akimova E N and Miniaachmetova A F 2013 Bulletin of the South Ural state University. Series: Mathematical modeling and programming vol 6(3) pp 26-37 (in Russian)
[13] Vasin V V and Skurydina A F 2018 Proceedings of the Steklov Institute of Mathematics (Supplementary issues) 301 suppl 1 pp 173–190
[14] Skurydina A F 2017 Bulletin of the South Ural state University. Series: Computational mathematics and computer science vol 6(3) pp 5-15
[15] Lukyanenko V A, Kozlova M G and Hazova U A 2010 Proceedings of the International Conference Integral Equations (Lviv) pp 80-84
[16] Belozub V A and Kozlova M G 2016 International Conference 'XXVIII Crimean Autumn Mathematical School-Symposium on spectral and evolutionary problems' (Simferopol) pp 132-135 (in Russian)
[17] Belozub V A and Kozlova M G 2019 Marchuk scientific readings-2019: Proceedings of the International conference 'Actual problems of computational and applied mathematics' (Novosibirsk) pp 45-51 (in Russian)