REGULARITY RESULTS OF NONLINEAR PERTURBED STABLE-LIKE OPERATORS

ANUP BISWAS AND MITESH MODASIYA

Abstract. We consider a class of fully nonlinear integro-differential operators where the nonlocal integral has two components: the non-degenerate one corresponds to the \( \alpha \)-stable operator and the second one (possibly degenerate) corresponds to a class of lower order\( \delta \) Lévy measures. Such operators do not have a global scaling property. We establish Hölder regularity, Harnack inequality and boundary Harnack property of solutions of these operators.

1. Introduction

In this article we are concerned with the regularity property of nonlinear integro-differential elliptic operators of the form

\[
Iu = \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u(x) = \int_{\mathbb{R}^d} (u(x + y) + u(x - y) - 2u(x)) \frac{k_{\alpha\beta}(y)}{|y|^d} \, dy,
\]

where \( k_{\alpha\beta} \) is symmetric and satisfies

\[
(2 - \alpha)\lambda \frac{1}{|y|^\alpha} \leq k_{\alpha\beta}(y) \leq \Lambda \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right), \quad 0 < \lambda \leq \Lambda,
\]

for some function \( \varphi : (0, \infty) \rightarrow (0, \infty) \) satisfying a weak upper scaling property with exponent \( \beta < \alpha \) (see Section 2 for a precise definition). Such operators are of great importance in control theory and were first considered by Pucci [15] to study the principal eigenvalue problem for local nonlinear elliptic operators. In a series of influential works \([5, 6, 7]\) Caffarelli and Silvestre develop a regularity theory of nonlinear stable-like integro-differential operator with symmetric kernels. Optimal boundary regularity for such operators are established by Serra and Ros-Oton in \([16, 17]\) whereas the boundary Harnack property is considered in \([18]\). Kriventsov in \([13]\) studies interior \( C^{1,\gamma} \) regularity for rough symmetric kernel and his result is further improved by Serra in \([20]\) who establishes interior \( C^{\alpha+\gamma} \) estimate with rough symmetric kernels.

There is also an extensive amount of work extending the results of Caffarelli and Silvestre \([5, 6, 7]\). In \([12]\) the authors generalized these results to fully nonlinear integro-differential operators with regularly varying kernels. Regularity results for nonsymmetric stable-like kernels are studied in \([8, 10]\). Recently, \([11]\) generalize these results for kernels with variable orders. These kernels are closely related to an important family of Lévy processes known as subordinate Brownian motions. Subordinate Brownian motions(sBM) are obtained by time-changing the Brownian motion by an independent subordinator (i.e., nondecreasing, non-negative Lévy process). In particular, when the subordinator is \( \alpha \)-stable we obtain a \( \alpha \)-stable process as sBM whose generator is given by the fractional Laplacian. Some further insightful discussion and comparison of \([11]\) with the current work are left for Section 2. Let us also mention \([1, 9]\) which also study regularity results for similar model. Our current work is closely related to a recent work of Mou \([14]\) where the author...
we study the interior Harnack inequality and boundary Harnack property for operators that are elliptic with respect to the family of linear operators having kernel function $k$ satisfying
\[ \lambda \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \leq k(y) \leq \Lambda \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right). \]

The rest of the article is organized as follows: In the next section we introduce the model and assumptions. Section 3 contains the proofs of ABP estimate (Theorem 3.1) and weak-Harnack inequality (Theorem 3.4). In Section 4 we study the interior Hölder regularity (Theorem 4.1) and the Harnack inequality (Theorem 5.1) is established in Section 5. Finally, in Section 6 we prove a boundary Harnack estimate (Theorem 6.1).

2. Our model and assumptions

Let $\varphi : (0, \infty) \to (0, \infty)$ be a locally bounded function satisfying a weak upper scaling property with exponent $\beta \in (0, 2)$ i.e.,
\[ \varphi(st) \leq \kappa_o s^\beta \varphi(t) \quad \text{for} \quad s \geq 1, t > 0, \tag{A1} \]
for some $\kappa_o > 0$. We also assume that
\[ \int_0^1 \frac{\varphi(y)}{y} dy < \infty. \tag{A2} \]

Note that (A1) and (A2) give us
\[ \int \rho \varphi(1/\rho^d) d\rho < \infty. \]

The ellipticity class is defined with respect to the set of nonlocal operators $\mathcal{L}$ containing operator $L$ of the form
\[ Lu(x) = \int \left( u(x + y) + u(x - y) - 2u(x) \right) \frac{k(y)}{|y|^d} dy, \tag{2.1} \]
where for some fixed $\lambda, \Lambda, 0 < \lambda \leq \Lambda$, it holds that $k(y) = k(-y)$ and
\[ (2 - \alpha)\lambda \frac{1}{|y|^\alpha} \leq k(y) \leq \Lambda \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \quad \text{for} \quad \alpha \in (\beta, 2). \tag{A3} \]

Then the extremal Pucci operators (with respect to $\mathcal{L}$) are defined to be $\mathcal{M}^+ u = \sup_{L \in \mathcal{L}} Lu$ and $\mathcal{M}^- u = \inf_{L \in \mathcal{L}} Lu$. Defining $\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x)$, we find from (A3) that
\[ \mathcal{M}^+ u(x) = \int \delta^+(u, x, y) \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) dy, \]
\[ \mathcal{M}^- u(x) = \int \delta^-(u, x, y) \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) dy. \]

In this article we would be interested in operators that are elliptic with respect to the class $\mathcal{L}$ (see [5, Definition 3.1]). Recall that a nonlinear operator $I$ is said to be elliptic with respect to the class $\mathcal{L}$ if it holds that
\[ \mathcal{M}^- (u - v) \leq I u - I v \leq \mathcal{M}^+ (u - v). \]

It should be observed that if an operator is elliptic with respect to a subset of $\mathcal{L}$ it is also elliptic with respect to $\mathcal{L}$. For instance, if we let $\varphi(r) = r^\beta, \beta \in (0, \alpha)$, and $\mathcal{L}_1 \subset \mathcal{L}$ be the collection of all kernel functions $k$ satisfying
\[ (2 - \alpha)\lambda \frac{1}{|y|^\alpha} \leq k(y) \leq \Lambda \left( \frac{2 - \alpha}{|y|^\alpha} + \mathbb{1}_{\mathcal{L}_1}(y) \frac{1}{|y|^\beta} \right), \]
then the results of this article hold for any operator that is elliptic with respect to \( \mathcal{L}_1 \). We remark that this class of operators are not covered by \([11, 12]\).

Also, by sub-solutions and super-solutions we shall mean viscosity sub and super-solutions, respectively.

**Definition 2.1.** A bounded function \( u : \mathbb{R}^d \to \mathbb{R} \) which is upper-semicontinuous (lower-semicontinuous) in \( \Omega \) is said to be a viscosity subsolution (supersolution) of \( Iu = f \) in \( \Omega \) and written as \( Iu \geq f \) (\( Iu \leq f \)) in \( \Omega \), if the following holds: if a \( C^2 \) function \( \psi \) touches \( u \) at \( x \in \Omega \) from above (below) in a small neighbourhood \( N_x \subset \Omega \), i.e., \( \psi \geq u \) in \( N_x \) and \( \psi(x) = u(x) \), then the function \( v \) defined by

\[
v(y) = \begin{cases} 
\psi(y) & \text{for } y \in N_x, \\
u(y) & \text{otherwise},
\end{cases}
\]

satisfies \( Iv(x) \geq f(x) \) (\( Iv(x) \leq f(x) \), resp.). A function \( u \) is said to be a viscosity solution if \( u \) is both a viscosity subsolution and a viscosity supersolution.

We refer to \([2, 3, 5]\) for more details on viscosity solutions. We also remark that the boundedness assumption of \( u \) assures integrability of \( u \) at infinity with respect to the jump kernel. This can be removed by assuming suitable integrability criterion and the results of this article will remain valid.

We also need scaled extremal operators which we introduce now. Define \( \varphi_i(|y|) = \frac{\kappa_i}{(2y)^{(\alpha - \beta)}} \varphi(|y|) \) for \( i \geq 0 \). The scaled extremal Pucci operators are defined to be

\[
\begin{align*}
\mathcal{M}_i^+ u(x) &= \int_{\mathbb{R}^d} \frac{\Lambda^+(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi_i(1/|y|) \right) - \frac{\lambda^+(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} \right) \, dy, \\
\mathcal{M}_i^- u(x) &= \int_{\mathbb{R}^d} \frac{\Lambda^-(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} \right) - \frac{\lambda^-(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi_i(1/|y|) \right) \, dy.
\end{align*}
\]

We conclude this section with a brief motivation for the above model. The works of Caffarelli and Silvestre \([5, 6, 7]\) are based on nonlinear generalization of the classical fractional Laplacian. These Pucci type operators appear in control theory when the underlying controlled dynamics is governed by stable-like process. Of course, one can consider nonlocal Pucci operators corresponding to other Lévy processes. For instance, \([11]\) considers a class of nonlinear integro-differential operator corresponding to a family of subordinate Brownian motion. Subordinate Brownian motion(sBM) forms an important family of Lévy process. For a large class of sBM, the jump kernel (or density of Lévy measure) is proportional to \( |y|^{-d} \Phi(1/|y|^2) \) where \( \Phi \) is a Bernstein function, in particular, increasing and concave. For more details we refer \([19]\). Note that \( \Phi(r) = r^{\alpha/2} \) corresponds to the stable kernel. The recent work \([11]\) deals with kernel of the form \( |y|^{-d} \Phi(1/|y|^2) \) where \( \Phi \) has both lower and upper weak scaling property. Our present model corresponds to a Lévy process which is obtained by adding two independent Lévy process: one is \( \alpha \)-stable process and the other one generated by the Lévy measure \( |y|^{-d} \varphi(1/|y|^2) \)dy. Since \( \varphi \) need not have a lower weak scaling property, the present model is not covered by \([11]\). Furthermore, for the results of Section 3 and Hölder regularity to hold we only allow \( \varphi \) in the upper bound of the kernel functions \( k \) (and not necessarily in the lower bound), as mentioned in (A3). So the above operator allows degeneracy in lower order operator.

### 3. ABP estimates and weak Harnack inequality

In this section we obtain an Aleksandrov-Bakelman-Pucci (ABP) estimate which is the main ingredient for weak-Harnack inequality and point estimate. Let us begin by defining the concave envelope and contact set. Let \( u \) be a function that is non-positive outside \( \mathbb{B}_1 \) (unit ball around 0). The concave envelope \( \Gamma \) of \( u \) in \( \mathbb{B}_3 \) is defined as follows

\[
\Gamma(x) = \begin{cases} 
\inf \{ p(x) : p \text{ is a plane satisfying } p \geq u^+ \text{ in } \mathbb{B}_3 \} & \text{in } \mathbb{B}_3, \\
0 & \text{in } \mathbb{B}^c_3.
\end{cases}
\]
The contact set is defined to be \( \Sigma = \{ \Gamma = u \} \cap B_1 \). Here and in what follows we use the notation \( B_r \) to denote the ball of radius \( r \) around 0, and by \( B_r(x) \) we would denote the ball of radius \( r \) around \( x \). The following lemma follows by adapting [5, Lemma 8.1] in our setting.

**Lemma 3.1.** Let \( u \leq 0 \) in \( \mathbb{R}^d \setminus B_1 \) and \( \Gamma \) be its concave envelope in \( B_1 \). Assume \( M_i^+ u(x) \geq -f(x) \) in \( B_1 \) for some \( i \geq 0 \). Let \( \rho_0 = 1/16 \pi, r_k = \rho_0 2^{-\frac{i+2}{2} - \alpha} 2^{-k} \), and \( R_k(x) = B_{r_k}(x) \setminus B_{r_{k+1}}(x) \). Then there exists a constant \( C_0 \) independent of \( i \geq 0 \) and \( \alpha \) such that for any \( x \in \Sigma \) and any \( M > 0 \) there is a \( k \) satisfying

\[
|R_k(x) \cap \{ u(y) < u(x) + (y-x) \cdot \nabla \Gamma(x) - M r_k^2 \}| \leq C_0 \frac{f(x)}{M} |R_k(x)|.
\]

Furthermore, \( C_0 \) depends only on \((\lambda, d, \rho_0)\).

**Proof.** First we notice that \( M_i^0 u \geq M_i^+ u \) for all \( i \). Therefore \( M_i^+ u(x) \geq -f(x) \) implies that \( M_i^0 u(x) \geq -f(x) \). Hence it is enough to prove the lemma for the case \( i = 0 \).

Let \( x \in \Sigma \) and recall that \( \delta(u, x, y) = u(x + y) + u(x - y) - 2u(x) \). If both \( x + y \) and \( x - y \) belong to \( B_3 \) and \( x - y \in B_3 \) then \( \delta(u, x, y) \leq 0 \) since \( u(x) = \Gamma(x) = p(x) \) for some plane that remains above \( u \) in \( B_3 \). If either \( x + y \) or \( x - y \in B_3 \) then both \( x + y \) and \( x - y \) remain in \( B_3 \) and since \( u(x) = \Gamma(x) \geq 0 \) we have \( \delta(u, x, y) \leq 0 \). Thus, using (2.2), we find

\[
-f(x) \leq M_i^+ u(x) = (2 - \alpha) \int_{B_1} \frac{-\lambda \delta^-(u, x, y)}{|y|^d} \left( \frac{1}{|y|^{\alpha}} \right) dy
\]

\[
\leq (2 - \alpha) \int_{B_{r_0}} \frac{-\lambda \delta^-(u, x, y)}{|y|^d} \left( \frac{1}{|y|^{\alpha}} \right) dy,
\]

where \( r_0 = \rho_0 2^{-\frac{1}{2} - \alpha} \). Let

\[
E_k^\pm := \{ R_k \cap \{ u(x \pm y) < u(x) \pm y \cdot \nabla \Gamma(x) - M r_k^2 \} \}.
\]

Then on this set we will have \( \delta^{-}(u, x, y) \geq 2M r_k^2 \). Also \( |E_k^\pm| = |E_k(x)| \) where \( E_k(x) := \{ R_k(x) \cap \{ u(y) < u(x) + (y-x) \cdot \nabla \Gamma(x) - M r_k^2 \} \} \). Now suppose that the result does not hold for any \( C_0 \).

We will arrive at contradiction for large enough \( C_0 \). Using (3.1) we obtain that

\[
f(x) \geq (2 - \alpha) \lambda \sum_{k=0}^{\infty} \int_{R_k} \frac{\delta^-(u, x, y)}{|y|^{d+\alpha}} dy \geq (2 - \alpha) \lambda \sum_{k=0}^{\infty} \int_{E_k} \frac{2Mr_k^2}{|y|^{d+\alpha}} dy
\]

\[
\geq 2(2 - \alpha) \lambda \sum_{k=0}^{\infty} M \frac{r_k^2}{r_{k+1}^{d+\alpha}} |E_k|
\]

\[
\geq 2(2 - \alpha) \lambda \sum_{k=0}^{\infty} M \frac{r_k^2}{r_{k+1}^{d+\alpha}} \frac{C_0 f(x)}{M} |R_k|
\]

\[
\geq 2(2 - \alpha) \lambda \left[ \sum_{k=0}^{\infty} \frac{r_k^2}{r_{k+1}^{d+\alpha}} \omega_d (r_{k+1}^d - r_k^d) \right] C_0 f(x)
\]

\[
= 2(2 - \alpha) \lambda \omega_d \left[ \sum_{k=0}^{\infty} \frac{r_k^{2-\alpha}}{r_{k+1}^{d-\alpha}} \left( 1 - \left( \frac{1}{2} \right)^d \right) \right] C_0 f(x),
\]

since \( \frac{r_{k+1}}{r_k} = \frac{1}{2} \) for any \( k \), where \( \omega_d \) denotes volume of the unit ball. Now we notice that \( \sum_{k=0}^{\infty} \frac{r_k^{2-\alpha}}{r_{k+1}^{d-\alpha}} \) is a geometric series, and therefore,

\[
f(x) \geq 2(2 - \alpha) \lambda \omega_d \left( 1 - \left( \frac{1}{2} \right)^d \right) \left[ \frac{\rho_0^{2-\alpha}}{2} \left( \frac{1}{1 - 2(2-\alpha)} \right) \right] C_0 f(x).
\]
Take
\[ c = \lambda \omega_d \left( 1 - \left( \frac{1}{2} \right)^d \right) \left( \frac{\rho_0^2 (2 - \alpha)}{1 - 2(2 - \alpha)} \right), \]
and since \( \frac{(2-\alpha)}{1-2(2-\alpha)} \) remains bounded below for all \( \alpha \in (0, 2) \), we have \( c \) positive for any \( \alpha \in (0, 2) \). Thus
\[ f(x) \geq c C_0 f(x). \]
Choosing \( C_0 > c^{-1} \) leads to a contradiction. Hence the proof. \( \square \)

Using Lemma 3.1 and the arguments in [5, Section 8] we arrive at the following result. This is a mild extension to [5, Theorem 8.7].

**Theorem 3.1.** Let \( u \) and \( \Gamma \) be same as in Lemma 3.1. Then there is a finite family of open cubes \( Q_j \) with diameters \( d_j \) such that following hold.

(i) Any two cubes \( Q_i \) and \( Q_j \) in the family do not intersect.

(ii) \( \{ u = \Gamma \} \subset \bigcup_{j=1}^{\infty} Q_j \).

(iii) \( \{ u = \Gamma \} \cap Q_j \neq \emptyset \) for any \( Q_j \).

(iv) \( d_j \leq \rho_0 (2^{-1/2} - \alpha) \), where \( \rho_0 = 1/16\sqrt{d} \).

(v) \( |\nabla \Gamma(Q_j)| \leq C(\max_{Q_j} f(x))^{1/2} |Q_j| \).

(vi) \( \{|y \in 8\sqrt{d}Q_j : u(y) > \Gamma(y) - C(\max_{Q_j} f(x))d_j^2\} \geq \mu|Q_j| \).

The constant \( C > 0 \) and \( \mu > 0 \) depends only on \( (\lambda, d, \rho_0) \) but not on \( i \) and \( \alpha \).

Next we consider a special function which will play a key role in our analysis on point estimate and weak-Harnack inequality. Let \( p > 0 \) and \( \delta \) be small positive number. Define
\[ f(x) := \min\{\delta^{-p}, \max\{|x|^{-p}, (2\sqrt{n})^{-p}\}\}. \]

We claim that, for a given \( r \in (0, 1) \), we can choose \( p \) and \( \delta \) so that
\[ \mathcal{M}_r f(x) \geq 0 \quad \text{for } r < |x| \leq 2\sqrt{n}. \]  
(3.2)

For any \( 0 < r < 1 \), define
\[ \hat{f}(x) = \min \left\{ \left( \frac{\delta}{r} \right)^{-p}, \max \left\{ |x|^{-p}, \left( \frac{2\sqrt{n}}{r} \right)^{-p} \right\} \right\}. \]

Then clearly, \( f(rx) = r^{-p} \hat{f}(x) \) and for any \( |x| \geq r \) we have
\[
\int_{\mathbb{R}^d} \left( f(x + y) + f(x - y) - 2f(x) \right) \frac{k(y)}{|y|^d} dy = \int_{\mathbb{R}^d} \left( f(x + ry) + f(x - ry) - 2f(x) \right) \frac{k(ry)}{|y|^d} dy
\]
\[= r^{-p} \int_{\mathbb{R}^d} \delta(\hat{f}, x/r, y) \frac{k(ry)}{|y|^d} dy. \]

Therefore, to establish (3.2) it is enough to show that for all \( 1 \leq |x| \leq \frac{2\sqrt{n}}{r} \),
\[ r^{-p} \inf_k \int_{\mathbb{R}^d} \delta(\hat{f}, x, y) \frac{k(ry)}{|y|^d} dy \geq 0, \]  
(3.3)

where infimum is taken over all kernel \( k \) satisfying (A3). Note that \( \hat{f} \) is radially non-increasing function. Fix \( |x| \geq 1 \) and define \( \hat{f}(y) = |x|^p \hat{f}(|x|y) \). Then it implies that \( \hat{f}(y) \geq \hat{f}(y) \), for all \( y \in \mathbb{R}^d \) and \( \hat{f}(x/|x|) = \hat{f}(x) \). Thus we obtain
\[ \delta(\hat{f}, x, y) = \frac{1}{|x|^p} \left[ \hat{f}(\frac{x+y}{|x|}) + \hat{f}(\frac{x-y}{|x|}) - 2\hat{f}(\frac{x}{|x|}) \right] \geq \frac{1}{|x|^p} \left[ \hat{f}(\frac{x+y}{|x|}) + \hat{f}(\frac{x-y}{|x|}) - 2\hat{f}(\frac{x}{|x|}) \right]. \]
Without any loss of generality we may assume that $x/|x| = e_1 = (1, \ldots, 0)$. Then

$$
\int_{\mathbb{R}^d} \delta(f, x, y) \frac{k(r|y|)}{|y|^d} dy \geq \frac{1}{|x|^p} \int_{\mathbb{R}^d} \left[ \hat{f}(\frac{x+y}{|x|}) + \hat{f}(\frac{x-y}{|x|}) - 2\hat{f}(\frac{x}{|x|}) \right] \frac{k(r|y|)}{|y|^d} dy
$$

$$
= \frac{1}{|x|^p} \int_{\mathbb{R}^d} \left[ \hat{f}(\frac{x}{|x|}) + \hat{f}(\frac{x}{|x|}) - 2\hat{f}(\frac{x}{|x|}) \right] \frac{k(r|x||y|)}{|y|^d} dy
$$

$$
= \frac{1}{|x|^p} \int_{\mathbb{R}^d} \left[ \hat{f}(e_1 + y) + \hat{f}(e_1 - y) - 2\hat{f}(e_1) \right] \frac{k(r|x||y|)}{|y|^d} dy
$$

$$
\geq \frac{1}{|x|^p} \int_{\mathbb{R}^d} \delta(f, e_1, y) \frac{k(r|x||y|)}{|y|^d} dy.
$$

Hence, by (2.3), we get

$$
\inf_k \int_{\mathbb{R}^d} \delta(f, x, y) \frac{k(r|y|)}{|y|^d} dy \geq \frac{1}{|x|^p} \int_{\mathbb{R}^d} \lambda \delta^+(\hat{f}, e_1, y) \left( \frac{2 - \alpha}{(r|x|)^\alpha |y|^\alpha} \right)
$$

$$
- \frac{4\delta^-(\hat{f}, e_1, y)}{|y|^d} \left( \frac{2 - \alpha}{(r|x|)^\alpha |y|^\alpha} + \varphi_0(1/r|x||y|) \right) dy
$$

$$
:= I_1 - I_2.
$$

We now recall the following elementary relations that hold for any $a > b > 0$ and $q > 0$:

$$
(a + b)^{-q} \geq a^{-q} \left( 1 - q \frac{b}{a} \right),
$$

$$
(a + b)^{-q} + (a - b)^{-q} \geq 2a^{-q} + q(q + 1)b^2 a^{-q - 2}.
$$

Fixing $\delta < \frac{r}{2}$, we then see that for $|y| < 1/2$,

$$
\delta(f, e_1, y) = |e_1 + y|^{-p} + |e_1 - y|^{-p} - 2
$$

$$
= (1 + |y|^2 + 2y_1)^{-p/2} + (1 + |y|^2 - 2y_1)^{-p/2} - 2
$$

$$
\geq 2(1 + |y|^2)^{-p/2} + p(p + 2)2y_1^2(1 + |y|^2)^{-p/2 - 2} - 2
$$

$$
\geq p \left( -|y|^2 + (p + 2)y_1^2 - \frac{1}{2}(p + 2)(p + 4)y_1^2|y|^2 \right). \quad (3.4)
$$

Let us first calculate $I_2$. For any $|y| < \frac{1}{2}$ we have from (3.4) that

$$
\delta^-(\hat{f}, e_1, y) \leq p \left( 1 + \frac{1}{2}(p + 2)(p + 4) \right) |y|^2.
$$

Denote by $C_p = p \left( 1 + \frac{1}{2}(p + 2)(p + 4) \right)$. Then

$$
I_2 \leq C_p \Lambda \int_{|y| < \frac{1}{2}} \frac{|y|^2}{|y|^d} \left[ \frac{2 - \alpha}{(r|x|)^\alpha |y|^\alpha} + \varphi_0(1/r|x||y|) \right] dy
$$

$$
+ \Lambda \int_{|y| \geq \frac{1}{2}} \frac{2\hat{f}(e_1)}{|y|^d} \left[ \frac{2 - \alpha}{(r|x|)^\alpha |y|^\alpha} + \varphi_0(1/r|x||y|) \right] dy
$$

$$
= I_{21} + I_{22}.
$$

We observe that

$$
\frac{2 - \alpha}{(r|x|)^\alpha} \int_{|y| < \frac{1}{2}} \frac{|y|^2}{|y|^d} \frac{1}{|y|^\alpha} dy = \frac{\omega_d}{(r|x|)^\alpha} \left( \frac{1}{2} \right)^{2-\alpha},
$$

$$
\frac{2 - \alpha}{(r|x|)^\alpha} \int_{|y| \geq \frac{1}{2}} \frac{2}{|y|^d} \frac{1}{|y|^\alpha} dy = \frac{\omega_d}{(r|x|)^\alpha} 2^{-\alpha} \frac{2}{2^\alpha + 1}.
$$
On the other hand, for $r|x| \leq 2\sqrt{n}$,
\[
\int_{|y| < \frac{1}{2} r |y|} \frac{2}{|y|^d} \varphi_0(1/r |y|)dy \leq \kappa_0 \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \int_{|y| < \frac{1}{2} r |y|} \frac{y^2}{|y|^d} \beta \left( \frac{1}{2 \sqrt{n}} \right)dy
\]
\[
= \kappa_0 \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \beta \left( \frac{1}{2 \sqrt{n}} \right) \omega_d \left( 2 - \beta \right) \left( \frac{1}{2} \right)^{2-\beta},
\]
using the fact $|y| < \frac{1}{2}$ and (A1). Again using (A1)-(A2)
\[
\int_{|y| \geq \frac{1}{2} r |y|} \frac{2}{|y|^d} \varphi_0(1/r |y|)dy = 2\omega_d \int_{1/2}^{\infty} \frac{1}{t} \varphi(\frac{1}{r t})dt
\]
\[
\leq 2\omega_d \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \int_{1/2}^{\infty} \frac{1}{t} \varphi(\frac{1}{r t})dt
\]
\[
= 2\omega_d \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \int_{1/2}^{\infty} \frac{1}{t} \varphi(t)dt
\]
\[
= 2\omega_d \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \int_{0}^{1/\sqrt{n}} \frac{1}{t} \varphi(t)dt
\]
\[
\leq 2\omega_d \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \kappa_1,
\]
for some constant $\kappa_1$ depending only on $\varphi$. Thus combining we obtain for $1 \leq |x| \leq \frac{2\sqrt{n}}{r}$
\[
I_2 \leq C_p \Lambda \left[ \frac{\omega_d}{r |x|} \left( \frac{1}{2} \right)^{2-\alpha} \right] + \Lambda \left[ \frac{\omega_d}{r |x|} \left( \frac{2-\alpha}{\alpha} \right)^{2-\alpha+1} \right] + C_p \kappa_0 \Lambda \left[ \left( \frac{2\sqrt{n}}{r |x|} \right)^\beta \frac{1}{2 \sqrt{n}} \right] \omega_d \left( 2 - \beta \right) \left( \frac{1}{2} \right)^{2-\beta}
\]
\[
+ \kappa_0 \Lambda \left[ 2\omega_d \left( \frac{2 \sqrt{n}}{r |x|} \right)^\beta \kappa_1 \right].
\]
Next we calculate $I_1$. Notice that if $\delta < \frac{r}{10}$, then $\delta (\hat{f}, e_1, y) \geq (\delta/r)^{-p}$ for all $y \in B_{\delta/r}(e_1)$. Hence
\[
I_1 = \int_{\mathbb{R}^d} \frac{\lambda \delta^{+}(\hat{f}, e_1, y)}{|y|^d} \frac{2-\alpha}{(r |x|)^\alpha |y|^\alpha} dy \geq \frac{\lambda (2-\alpha)}{(r |x|)^\alpha} \kappa_2 (\delta/r)^{-p} \geq \frac{\lambda (2-\alpha)}{(2 \sqrt{n})^\alpha} \kappa_2 (\delta/r)^{-p},
\]
for some constant $\kappa_2$. Thus choosing $p > d$ and $\delta$ small enough (3.3) follows from (3.5). This leads to the following

**Lemma 3.2.** Given any $r, n > 0$ there are positive $p$ and $\delta$ such that the function
\[
f(x) = \min \{ \delta^{-p}, \max \{ |y|^{-p}, (2\sqrt{n})^{-p} \} \},
\]
is a solution to
\[
\mathcal{M}_0^- f(x) \geq 0,
\]
for every $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 2$ and $|x| > r$.

**Proof.** For any $|x| \geq 2\sqrt{n}$, by the definition of $f$, we get that $\delta(f, x, y) = f(x+y) + f(x-y) - 2f(x) \geq 0$ for all $y \in \mathbb{R}^d$. Therefore, $\mathcal{M}_0^- f(x) \geq 0$ for all $|x| \geq 2\sqrt{n}$. Hence the proof follows from (3.2). \(\square\)

Applying Lemma 3.2 we obtain the following corollary. The proof would be same as [5, Corollary 9.3] and thus omitted.

**Corollary 3.1.** Given any $\alpha \in [\alpha_0, \alpha_1]$ and $i \geq 0$ there is a function $\Phi$ such that

(i) $\Phi$ is continuous in $\mathbb{R}^d$,

(ii) $\Phi(x) = 0$ for $x$ outside $B_{2\sqrt{n}}$,
Theorem 3.1 and repeating the arguments of Lemma 3.3 we obtain the following.

Let \( \hat{\varepsilon} \) such that if \( \varepsilon \leq \varepsilon_0 \) for some universal constant \( \kappa, \varepsilon \) then |\( \{u \leq M\} \cap Q_1 | > \mu. \)

The above lemma is a key tool in obtaining weak-Harnack estimate. Combining a Calderón-Zygmund type argument with Lemma 3.3 we obtain the following.

**Theorem 3.2.** There exist constant \( \bar{\varepsilon} > 0 \), \( 0 < \bar{\mu} < 1 \), and \( \bar{M} > 1 \) (depending only on \( d, \lambda, \Lambda, \alpha, \varphi \)), such that if

(i) \( u \geq 0 \) in \( \mathbb{R}^d \),

(ii) \( \inf_{Q_1} u \leq 1 \), and

(iii) \( M^{-i} u \leq \varepsilon_0 \) in \( Q_{4\sqrt{d}} \) for some \( i \geq 0 \),

then \( |\{u \leq \mu\} \cap Q_1 | > \mu. \)

Thus we only need to show that

\[ |A| \leq (1 - \bar{\mu})|B|. \] (3.6)

Clearly, \( A \subset B \subset Q_1 \) and \( |A| \leq |\{u > \bar{M}\} \cap Q_1 | \leq 1 - \bar{\mu} \). We show that if \( Q \) is a dyadic cube such that

\[ |A \cap Q | > (1 - \bar{\mu})|Q|, \] (3.7)

then \( \tilde{Q} \subset B \), where \( \tilde{Q} \) is a predecessor of \( Q \). Then (3.6) follows from [4, Lemma 4.2]. Suppose that \( \tilde{Q} \notin B \). Take

\[ \tilde{x} \in \tilde{Q} \quad \text{such that} \quad u(\tilde{x}) \leq \bar{M}^{-1}. \]

We consider the function

\[ v(y) := \frac{u(x_0 + \frac{1}{2^n}y)}{M^{k-1}}. \]

Clearly, \( v \geq 0 \) in \( \mathbb{R}^d \) and \( \inf_{Q_3} v \leq 1 \). We claim that \( M^{-i} v \leq \varepsilon_0 \) in \( Q_{4\sqrt{d}} \) where \( \varepsilon_0 \) is given by Lemma 3.3. Let \( \hat{x} = x_0 + \frac{1}{2^n}x \) for some \( x \in Q_{4\sqrt{d}} \). Then simple calculation shows that

\[ \frac{1}{M^{k-1}} \delta(u, \hat{x}, \frac{y}{2^n}) = \delta(v, x, y), \]
and using (A1), we obtain
\[
\frac{1}{(2^i)^\alpha M_k^{-1}}\mathcal{M}^-u(\hat{x}) = \frac{1}{(2^i)^\alpha M_k^{-1}} \int_{\mathbb{R}^d} \left[ \lambda \delta^+ (u, \hat{x}, \frac{y}{\|y\|}) \left( \frac{(2 - \alpha)(2^i)^\alpha}{|y|^\alpha} \right)
- \Lambda \delta^-(u, \hat{x}, \frac{y}{\|y\|}) \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(2^i/|y|) \right) \right] dy
\geq \frac{1}{M_k^{-1}} \int_{\mathbb{R}^d} \lambda \delta^+ (u, \hat{x}, \frac{y}{\|y\|}) \left( \frac{2 - \alpha}{|y|^\alpha} \right)
- \Lambda \delta^-(u, \hat{x}, \frac{y}{\|y\|}) \left( \frac{2 - \alpha}{|y|^\alpha} + \frac{\kappa_o}{2^{i(\alpha - \beta)}} \varphi(\frac{1}{|y|}) \right) dy
\geq \mathcal{M}_i^- v(x).
\]

Thus we have the claim \( \mathcal{M}_i^- v \leq \varepsilon_0 \) in \( Q_{4\sqrt{\varepsilon}} \). Therefore, we can apply Lemma 3.3 to obtain
\[
\tilde{\mu} < |\{v(x) \leq M\} \cap Q_i| = 2^{id}|\{u(x) \leq M^k\} \cap Q|
\]
implying
\[
|\{u(x) \leq M^k\} \cap Q| > \tilde{\mu}|Q|.
\]
This gives us (3.7). This completes the proof. \( \square \)

Remark 3.1. Note that the constants \((\tilde{\varepsilon}, \tilde{M}, \tilde{\mu})\) in Theorem 3.2 also work if we replace \( \mathcal{M}^- \) by \( \mathcal{M}_i^- \) for all \( i \geq 0 \).

By a standard covering argument we obtain following result

**Theorem 3.3.** Let \( u \geq 0 \) in \( \mathbb{R}^d \), \( u(0) \leq 1 \), and \( \mathcal{M}_i^- u \leq \varepsilon_0 \) in \( B_2 \). Then
\[
|\{u \geq t\} \cap B_1| \leq C t^{-\varepsilon} \text{ for every } t > 0,
\]
where the constant \( C \) and \( \varepsilon \) depend only on \((d, \lambda, \Lambda, \alpha, \varphi)\).

We conclude the section by proving a weak-Harnack estimate.

**Theorem 3.4.** Let \( u \geq 0 \) in \( \mathbb{R}^d \) and \( \mathcal{M}^- u \leq C_0 \) in \( B_{2r}, 0 < r \leq 1 \). Then
\[
|\{u \geq t\} \cap B_r| \leq C r^d (u(0) + C_0 r^\alpha)^\varepsilon t^{-\varepsilon} \text{ for every } t > 0,
\]
for some constants \( C, \varepsilon \) as in Theorem 3.3. In particular,
\[
\|u\|_{L^{\varepsilon/(\varepsilon + 1)}(B_r)} \leq C (u(0) + C_0 r^\alpha).
\]

**Proof.** Choose \( k \in \mathbb{N} \cup \{0\} \) satisfying \( 2^{-k} < r \leq \frac{3}{2} 2^{-k} \). Let \( v(x) = u(\frac{1}{2^k}x) \). Then from the calculation of Theorem 3.2 it follows that
\[
\mathcal{M}_k^+ v(x) \leq \frac{1}{2^{\kappa_o}} \mathcal{M}^+ u(\frac{1}{2^k}x) \leq r^\alpha C_0 \text{ in } B_2.
\]
Multiplying \( v \) with \( \frac{\tilde{\varepsilon}_0}{v(0) + r^\alpha C_0} \), it follows from Theorem 3.3 (modifying the argument a bit)
\[
|\{v \geq t\} \cap B_{\frac{3}{2}}| \leq C t^{-\varepsilon} \quad t > 0.
\] (3.8)

Hence, by our choice of \( k \), we get
\[
|\{u \geq t\} \cap B_r| \leq C r^d (u(0) + C_0 r^\alpha)^\varepsilon t^{-\varepsilon}.
\]
The second conclusion follows by integrating both sides of (3.8) with respect to \( t \). \( \square \)
4. Hölder regularity

Using the results developed in Section 3, in this section we establish an interior Hölder regularity. The main theorem of this section is the following.

**Theorem 4.1.** Let \( u \) be a bounded continuous function defined on \( \mathbb{R}^d \) and satisfy
\[
\mathcal{M}^+ u \geq -C_0, \quad \mathcal{M}^- u \leq C_0 \quad \text{in } \mathcal{B}_1,
\]
for some constant \( C_0 \). Then \( u \in C^\gamma(\mathcal{B}_{\frac{1}{2}}) \) and
\[
\|u\|_{C^\gamma(\mathcal{B}_{\frac{1}{2}})} \leq C(\|u\|_{L^\infty(\mathbb{R}^d)} + C_0),
\]
where \( \gamma, C \) depend only on \( d, \lambda, \alpha, \varphi \).

We follow the approach of [5] to prove Theorem 4.1. Theorem 4.1 would follow from the following result.

**Lemma 4.1.** Let \( u \) be a continuous function satisfying
\[
-\frac{1}{2} \leq u \leq \frac{1}{2} \quad \text{in } \mathbb{R}^d, \quad \mathcal{M}^+ u \geq -\varepsilon, \quad \mathcal{M}^- u \leq \varepsilon \quad \text{in } \mathcal{B}_1.
\]
Then there is a \( \gamma > 0 \) (depending on \( \alpha, \varphi \)) such that \( u \in C^\gamma \) at the origin. In particular,
\[
|u(x) - u(0)| \leq C|x|^\gamma
\]
for some constant \( C \).

**Proof.** Following [5] We show that there exists sequences \( m_k \) and \( M_k \) satisfying \( m_k \leq u \leq M_k \) in \( \mathcal{B}_{8^{-k}} \) and
\[
M_k - m_k = 8^{-\gamma k}.
\]
Then the results follow by choosing \( C = 8^\gamma \).

For \( k = 0 \) we choose \( m_0 = -\frac{1}{2} \) and \( M_0 = \frac{1}{2} \). By assumption we have \( m_0 \leq u \leq M_0 \) in the whole space \( \mathbb{R}^d \). We proceed to construct the sequences \( M_k \) and \( m_k \) by induction. So by induction hypothesis we assume the construction of \( m_j, M_j \) for \( j = 0, \ldots, k \). We want to show that we can continue the sequences by finding \( m_{k+1} \) and \( M_{k+1} \).

Consider the ball \( \mathcal{B}_{\frac{1}{8^{k+1}}} \). Then one of the following holds
\[
|\{u \geq \frac{M_k + m_k}{2}\} \cap \mathcal{B}_{\frac{1}{8^{k+1}}}| \geq \frac{1}{2}|\mathcal{B}_{\frac{1}{8^{k+1}}}|; \quad (4.1)
\]
\[
|\{u \leq \frac{M_k + m_k}{2}\} \cap \mathcal{B}_{\frac{1}{8^{k+1}}}| \geq \frac{1}{2}|\mathcal{B}_{\frac{1}{8^{k+1}}}|. \quad (4.2)
\]
Suppose that (4.1) holds. Define
\[
v(x) := \frac{u(8^{-k}x) - m_k}{(M_k - m_k)/2},
\]
Then that \( v(x) \geq 0 \) in \( \mathcal{B}_1 \) and \( |\{v \geq 1\} \cap \mathcal{B}_{1/8}| \geq |\mathcal{B}_{1/8}| / 2 \). Moreover, since \( \mathcal{M}^- u \leq \varepsilon \) in \( \mathcal{B}_1 \), we get from the calculation in Theorem 3.2
\[
\mathcal{M}_{3^k}^+ v \leq \frac{8^{-k\alpha \varepsilon}}{(M_k - m_k)/2} = 2\varepsilon 8^{-k(\alpha - \gamma)} \leq 2\varepsilon \quad \text{in } \mathcal{B}_{8^k},
\]
provided we set \( \gamma \leq \alpha \). From the induction hypothesis, for any \( j \geq 1 \), we have
\[
v \geq \frac{(m_{k-j} - m_k)}{(M_{k-j} - m_{k-j})/2} \geq 2(1 - 2\gamma^j) \quad \text{in } \mathcal{B}_{8^j}.
\]
Thus \( v(x) \geq \max\{-2(|8x|^\gamma - 1), -2(8^{(k+1)^\gamma} - 1)\} := -g(x) \) outside \( \mathcal{B}_1 \). Letting \( w(x) = \max(v, 0) \) we also see that
\[
\mathcal{M}_{3^k}^- w \leq \mathcal{M}_{3^k}^+ v + \mathcal{M}_{3^k}^+ v^-.
\]
We claim that $\mathcal{M}^+_3 v^- \leq 2\varepsilon$ in $\mathbb{B}_{3/4}$, for all $k$, if we choose $\gamma$ small enough. For $x \in \mathbb{B}_{3/4}$, since $v^-(x) = 0$, we have $\delta(v^-, x, y) = \delta^+(v^-, x, y) = v^-(x + y) + v^-(x - y)$ for all $y \in \mathbb{R}^d$, and by (2.2),

$$\mathcal{M}^+_3 v^-(x) = \int_{\mathbb{R}^d} \frac{\Lambda \delta^+(v^-, x, y)}{|y|^d} \left\{ \frac{2 - \alpha}{|y|^\alpha} + 8(\beta - \alpha)k \varphi_0(1/|y|) \right\} dy.$$ 

If $|y| < \frac{1}{2}$, then both $x + y$ and $x - y$ is in $\mathbb{B}_1$ so $v^-(x + y) = v^-(x - y) = 0$. This gives us

$$\mathcal{M}^+_0 v^- (x) = \Lambda \int_{\{|y| \geq \frac{1}{2}\}} \frac{v^-(x + y) + v^-(x - y)}{|y|^d} \left\{ \frac{2 - \alpha}{|y|^\alpha} + 8(\beta - \alpha)k \varphi_0(1/|y|) \right\} dy \leq \Lambda \int_{\{|y| \geq \frac{1}{2}\}} \frac{g^+(x + y) + g^+(x - y)}{|y|^d} \left\{ \frac{2 - \alpha}{|y|^\alpha} + 8(\beta - \alpha)k \varphi_0(1/|y|) \right\} dy \leq 2\Lambda \int_{\{|y| \geq \frac{1}{2}\}} \frac{g^+(x + y) - g^+(x - y)}{|y|^d} \left\{ \frac{2 - \alpha}{|y|^\alpha} + 8(\beta - \alpha)k \varphi_0(1/|y|) \right\} dy \leq 4\Lambda \int_{\{|y| \geq \frac{1}{2}\}} \frac{(32^\gamma|y| - 1)^+ + 2 - \alpha}{|y|^\alpha} |y|^d dy + 4\Lambda \int_{\{|y| \geq \frac{1}{2}\}} (8^\gamma(k + 1) - 1)8^{(\beta - \alpha)k} |y|^{\frac{1}{2}} \varphi_0(1/|y|)dy = I_1 + I_2.$$ 

Notice that the function $f_\gamma = (32^\gamma|y| - 1)^+ \mathbb{1}_{\{|y| \geq \frac{1}{2}\}}$ decreases to 0 as $\gamma \to 0$. Also, the function becomes integrable if we choose $\gamma < \alpha$. Thus for a small $\gamma$ we have $I_1 \leq \varepsilon$. So we calculate $I_2$. We fix $\gamma < \alpha - \beta$. Define the function $h(t) = \log[(8^\gamma t - 1)8^{(\beta - \alpha)t}]$ for $t > 0$. Note that

$$h'(t) = \log 8[-(\alpha - \beta) + \gamma \frac{8^\gamma t}{8^\gamma t - 1}] < 0,$$

for large $t$. So $h(t)$ attains its maximum and

$$h'(t) = 0 \Rightarrow 8^\gamma t - 1 = \gamma \frac{8^\gamma t}{\alpha - \beta}.$$ 

Thus,

$$\max_{t \geq 1} e^{h(t)} \leq \gamma \sup_{t \geq 1} \frac{8(\gamma - \alpha + \beta)t}{\alpha - \beta} \to 0,$$

as $\gamma \to 0$. Thus using (A2) we can choose $\gamma$ small enough to satisfy $I_2 \leq \varepsilon$, uniformly in $k$. This gives us the claim.

Using (4.3) we obtain $M^-_3 w \leq 4\varepsilon$ in $\mathbb{B}_{3/4}$, provided $\gamma$ is small enough. We also have

$$|\{w \geq 1\} \cap \mathbb{B}_{1/\gamma} | \geq \frac{|B_{1/\gamma}|}{2}.$$ 

Given any point $x \in \mathbb{B}_{1/\gamma}$, we can apply Theorem 3.4 in $B_{2/\gamma}(x)$ to obtain

$$C(w(x) + 4\varepsilon) \geq |\{w > 1\} \cap \mathbb{B}_{1/\gamma} (x) | \geq \frac{1}{2}|\mathbb{B}_{1/\gamma} |.$$ 

If we have chosen $\varepsilon$ small, this implies that $w > \theta$ in $\mathbb{B}_{1/\gamma}$ for some $\theta > 0$. Thus if we let $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta \frac{(M_k - m_k)}{2}$, we have $m_{k+1} \leq u \leq M_{k+1}$ in $B_{8^{-k+1}}$. Moreover, $M_{k+1} - m_{k+1} = (1 - \theta/2)8^{-\gamma k}$. So we must choose $\gamma$ and $\theta$ small and so that $(1 - \theta/2) = 8^{-\gamma}$ and we obtain $M_{k+1} - m_{k+1} = 8^{-\gamma(k+1)}$.

On the other hand, if (4.2) holds, we define

$$v(x) = \frac{M_k - u(8^{-k}x)}{(M_k - m_k)2}$$
and continue in the same way using that $\mathcal{M}^+ u \geq -\tilde{\varepsilon}$.

\[
\square
\]

5. Harnack Inequality

In this section and the next section we discuss Harnack’s inequality and boundary Harnack inequality. This will be done for a smaller class of operators. Let $\tilde{\mathcal{L}} \subset \mathcal{L}$ be the set of operators containing kernel function $k$ satisfying

\[
\lambda \left( \frac{2 - \alpha}{|y|^2} + \varphi(1/|y|) \right) \leq k(y) \leq \Lambda \left( \frac{2 - \alpha}{|y|^2} + \varphi(1/|y|) \right) \tag{A4}
\]

and $\varphi$ is non-decreasing. The associated extremal operators are denoted by $\tilde{\mathcal{M}}^\pm$. In particular,

\[
\tilde{\mathcal{M}}^+(x) = \int_{\mathbb{R}^d} \frac{\Lambda \delta^+(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^2} + \varphi(1/|y|) \right) - \frac{\lambda \delta^-(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^2} + \varphi(1/|y|) \right) \, dy,
\]

\[
\tilde{\mathcal{M}}^-(x) = \int_{\mathbb{R}^d} \frac{\lambda \delta^+(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^2} + \varphi(1/|y|) \right) - \frac{\Lambda \delta^-(u, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^2} + \varphi(1/|y|) \right) \, dy.
\]

It is also evident that $\mathcal{M}^+ u \leq \tilde{\mathcal{M}}^- u \leq \tilde{\mathcal{M}}^+ u \leq \mathcal{M}^+ u$. It should be observed that we do not require weak lower scaling property on $\varphi$ (compare with [11]). For instance, $\varphi(r) = \log(1 + r^\beta)$ does not satisfy weak lower scaling i.e. there is no $\mu > 0$ so that $\varphi(sr) \gtrsim s^\mu \varphi(r)$ for $s \geq 1, r > 0$. But it does satisfy a weak upper scaling property since for every $s \geq 1$,

\[
1 + s^\beta r^\beta \leq (1 + r^\beta)^s \Rightarrow \varphi(sr) \leq s^\beta \varphi(r).
\]

Our main result of this section is the following

**Theorem 5.1.** Let $u$ be a non-negative function satisfying

\[
\tilde{\mathcal{M}}^+ u \geq -C_0, \quad \text{and} \quad \tilde{\mathcal{M}}^- u \leq C_0 \quad \text{in } B_2.
\]

Then $u(x) \leq C(u(0) + C_0)$ for every $x \in B^{\frac{\gamma}{2}}_1$, for some constant $C$ dependent only on $\lambda, \Lambda, \alpha, \varphi$.

**Proof.** We again follow the idea of [5]. Dividing by $u(0) + C_0$, it is enough to consider $u(0) \leq 1$ and $C_0 = 1$. Fix $\varepsilon > 0$ from Theorem 3.4 and let $\gamma = \frac{d}{2}$. Let

\[
t := \min \{ s : u(x) \leq h_s(x) := s(1 - |x|)^{-\gamma} \text{ for all } x \in B_1 \}.
\]

Let $x_0 \in B_1$ be such that $u(x_0) = h_t(x_0)$. Let $\eta = 1 - |x_0|$ be the distance of $x_0$ from $\partial B_1$. We show that $t < C$ for some universal $C$ which in turn, implies that $u(x) < C(1 - |x|)^{-\gamma}$. This would prove our result.

For $r = \frac{\eta}{2}$, we estimate the portion of the ball $B_r(x_0)$ covered by $\{ u < \frac{u(x_0)}{2} \}$ and $\{ u > \frac{u(x_0)}{2} \}$. Define $A := \{ u > \frac{u(x_0)}{2} \}$. Using Theorem 3.4 we then obtain

\[
|A \cap B_1| \leq C \left( \frac{2}{u(x_0)} \right)^\varepsilon \leq Ct^{-\varepsilon} \eta^d,
\]

whereas $|B_r| = \omega_d(\eta/2)^d$. In particular,

\[
\left| \left\{ u > \frac{u(x_0)}{2} \right\} \cap B_r(x_0) \right| \leq Ct^{-\varepsilon} |B_r|, \tag{5.1}
\]

So if $t$ is large, $A$ can cover only a small portion of $B_r(x_0)$. We shall show that for some $\delta > 0$, independent of $t$ we have

\[
|\{ u \leq \frac{u(x_0)}{2} \} \cap B_r(x_0) | \leq (1 - \delta)|B_r|
\]

which will provide an upper bound on $t$ completing the proof.
We start by estimating $|\{u \leq \frac{u(x_0)}{2}\} \cap B_{\theta r}(x_0)|$ for $\theta > 0$ small. For every $x \in B_{\theta r}(x_0)$ we have

$$ u(x) \leq h_t(x) \leq t \left( \frac{2\eta - \theta \eta}{2} \right)^{-\gamma} \leq u(x_0) \left( 1 - \frac{\theta}{2} \right)^{-\gamma}, $$

with $\left( 1 - \frac{\theta}{2} \right)$ close to 1. Define

$$ v(x) := \left( 1 - \frac{\theta}{2} \right)^{-\gamma} u(x_0) - u(x). $$

Then that $v \geq 0$ in $B_{\theta r}(x_0)$, and also $\hat{M}^{-} v \leq 1$ as $\hat{M}^+ u \geq 1$. We would like to apply Theorem 3.4 to $v$ but $v$ need not be non-negative in the whole space $\mathbb{R}^d$. Thus we consider $w = v^+$ and find an upper bound of $\hat{M}^{-} w$.

We already know that

$$ \hat{M}^{-} v(x) = \int_{\mathbb{R}^d} \frac{\lambda^+(v, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) - \frac{\Lambda \delta^-(v, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy \leq 1. $$

Therefore, for $x \in B_{\frac{\theta r}{4}}(x_0)$

$$ \hat{M}^{-} w(x) = \int_{\mathbb{R}^d \cap \{v(x+y) < 0\}} \frac{\lambda^+(w, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) - \frac{\Lambda \delta^-(w, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy 
\leq 1 + 2 \int_{\mathbb{R}^d \cap \{v(x+y) < 0\}} \frac{\lambda^+(w, x, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy $$

So to find an upper bound we must compute the second expression. Let us consider the largest value $\tau > 0$ such that $u(x) \geq g_{\tau} := \tau (1 - |4 x|^2)$. There must be a point $x_1 \in B_{\frac{1}{4}}$ such that $u(x_1) = \tau (1 - |4 x|^2)$. The value of $\tau$ cannot be larger than 1 since $u(0) \leq 1$. Also truncate $g_{\tau}$ and define $\hat{g}_{\tau} := g_{\tau} 1_{B_{\frac{1}{4}}}$, which implies $u(x) \geq \hat{g}_{\tau}(x) \geq g_{\tau}(x)$ for all $x \in \mathbb{R}^d$ and $u(x_1) = \hat{g}_{\tau}(x_1) = g_{\tau}(x_1)$.

Thus we have the upper bound

$$ \int_{\mathbb{R}^d} \frac{\delta^-(u, x_1, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy $$
\leq \int_{\mathbb{R}^d} \frac{\delta^-(\hat{g}_{\tau}, x_1, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy $$
\leq \int_{B_{1}} \frac{\delta^-(g_{\tau}, x_1, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy + \int_{\mathbb{R}^d \setminus B_{1}} \frac{32}{|y|^2} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy $$
\leq \tfrac{32}{|y|^2} \left( \frac{2 - \alpha}{|y|^\alpha} + \frac{\kappa_0}{|y|^\alpha} \varphi(1) \, dy \right) + C_1 \leq C_2,$n

for some constants $C_1, C_2$ dependent only on $(d, \alpha, \varphi)$, where we used following inequality

$$ \hat{g}_{\tau}(x_1 + y) + \hat{g}_{\tau}(x_1 - y) - 2 \hat{g}_{\tau}(x_1) \geq \tau (2 |4 x_1|^2 - |4(x_1 + y)|^2 - |4(x_1 - y)|^2) = 32 |y|^2, $$

for $y \in B_1$. Since $\hat{M}^{-} u(x_1) \leq 1$, we get using the above estimate that

$$ \int_{\mathbb{R}^d} \frac{\delta^+(u, x_1, y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy \leq C. $$

In particular, since $u(x_1) \leq 1$ and $u(x_1 - y) \geq 0$,

$$ \int_{\mathbb{R}^d} \frac{(u(x_1 + y) - 2)^+}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy \leq C. $$
We use this estimate to compute the RHS of (5.2). Without any loss of generality we may assume that $u(x_0) > 2$, since otherwise $t$ would not be large. We can use the inequality above to get following estimate

$$2(2 - \alpha) \int_{\mathbb{R}^d \setminus B_{1/2}(x_0 - x)} \frac{\Lambda(u(x + y) - (1 - \theta^2)^{-\gamma} u(x_0))^+}{|y|^{d+\alpha}} dy$$

$$\leq 2(2 - \alpha) \int_{\mathbb{R}^d \setminus B_{1/2}(x_0 - x)} \frac{\Lambda(u(x + x + y - x_1) - 2)^+}{|y + x - x_1|^{d+\alpha}} \frac{|y + x - x_1|^{d+\alpha}}{|y|^{d+\alpha}} dy$$

$$\leq C(\theta r)^{-d-\alpha},$$

here we used the fact that $y \notin B_{1/2}(x_0 - x)$ implies $y \notin B_{1/4}$. Again, a simple calculation gives

$$\frac{|y + x - x_1|}{|y|} \leq \frac{|y| + |x - x_1|}{|y|} \leq 12(\theta r)^{-1}$$

and using the monotonicity property of $\varphi$,

$$\frac{\varphi(1/|y|)}{\varphi(1/|y + x - x_1|)} \leq \frac{\varphi(1/|y + x - x_1|)}{\varphi(1/|y + x - x_1|)} \leq \kappa_0 \left(12(\theta r)^{-1}\right)^{\beta},$$

by (A1). This gives us

$$2\Lambda \int_{\mathbb{R}^d \setminus B_{1/2}(x_0 - x)} \frac{(u(x + y) - (1 - \theta^2)^{-\gamma} u(x_0))^+}{|y|^{d}} \varphi \left(\frac{1}{|y|}\right) dy$$

$$\leq 2\Lambda \int_{\mathbb{R}^d \setminus B_{1/2}(x_0 - x)} \frac{(u(x + x + y - x_1) - 2)^+}{|y + x - x_1|^{d}} \varphi \left(\frac{1}{|y + x - x_1|}\right)$$

$$\leq C \kappa_0(\theta r)^{-d-\beta} \leq C(\theta r)^{-d-\alpha}.$$
and hence,
\[
\left\{ u \leq \frac{u(x_0)}{2} \right\} \cap B_{\frac{\theta_0}{8}}(x_0) \leq \frac{1}{2} |B_{\frac{\theta_0}{8}}(x_0)|.
\]
This of course, implies that
\[
\left\{ u > \frac{u(x_0)}{2} \right\} \cap B_{\frac{\theta_0}{8}}(x_0) \geq C_2 |B_r|,
\]
but this is contradicting to (5.1). Therefore \( t \) cannot be large and we finish the proof. \( \square \)

Mimicking Theorem 5.1 we also obtain the following result which will be useful to establish a boundary Harnack property. The following also known as the half Harnack inequality for subsolutions.

**Theorem 5.2.** Let \( u \) be a function continuous in \( \overline{B}_1 \), and satisfy
\[
\int_{\mathbb{R}^d} \frac{|u(y)|}{1 + |y|^{d+\alpha}} \, dy + \int_{\mathbb{R}^d} \frac{|u(y)|}{1 + |y|^{d}(\varphi(1/|y|))^{-\frac{1}{2}}} \, dy \leq C_0,
\]
and
\[
\mathcal{N}^+ u \geq -C_0 \quad \text{in} \ B_1.
\]
Then
\[
u(x) \leq C \left. \frac{u(x)}{u(x_0)} \right|_{B_1},
\]
for every \( x \in B_{\frac{1}{2}} \), where the constant \( C > 0 \) depends only on \((d, \lambda, \Lambda, \alpha, \varphi)\).

**Proof.** We follow the approach of [7] and Theorem 5.1. Dividing \( u \) by \( C_0 \), it is enough to consider \( C_0 = 1 \). Also, without any loss of generality we may assume that \( u \) is positive somewhere in \( B_1 \). Otherwise, there is nothing to prove. As before we consider the minimum value of \( t \) such that
\[
u(x) \leq h_t(x) := t(1 - |x|)^{-d} \quad \text{for every} \ x \in B_1,
\]
and find \( x_0 \in B_1 \) with \( u(x_0) = h_t(x_0) \). Denote by \( \eta = 1 - |x_0|, r = \eta/2 \) and \( A = \{ u > u(x_0)/2 \} \).

As shown in Theorem 5.1, we need to find an upper bound of \( t \).

By assumption, we have \( u \in L^1(B_1) \) and thus
\[
|A \cap B_1| \leq C \frac{2}{u(x_0)} \leq C t^{-1} \eta^d,
\]
whereas \( |B_r| = C \eta^d \), so if \( t \) is large, \( A \) can cover only a small portion of \( B_r(x_0) \) at most. In particular,
\[
\left\{ u > \frac{u(x_0)}{2} \right\} \cap B_r(x_0) \leq C t^{-1} |B_r|.
\]

We define
\[
v(x) = \left( 1 - \frac{\theta}{2} \right)^{-d} u(x_0) - u(x)
\]
for small \( \theta > 0 \), and observe that \( v \geq 0 \) in \( B_{\theta r}(x_0) \). Let \( w = v^+ \). Repeating the arguments of Theorem 5.1 we find, for \( x \in B_{\frac{\theta r}{8}}(x_0) \), that
\[
\mathcal{M}^- w(x) \leq 1 + 2 \int_{\mathbb{R}^d \cap \{v(x+y) < 0\}} -\Lambda \frac{v(x+y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy,
\]
since \( v \geq 0 \) in \( B_{\theta r}(x_0) \) and \( x \in B_{\frac{\theta r}{8}}(x_0) \), we will have \( x+y \) and \( x-y \) both in \( B_{\theta r}(x_0) \) for all \( y \in B_{\theta r} \).

Now we need to estimate the integral on the RHS of the above. Note that \( u \) need to be non-negative
here and thus we can not apply the technique of cut-off function as done in Theorem 5.1. So we use the integral condition imposed on $u$.

\[
\mathcal{M}^- w(x) \leq 1 + 2 \int_{\mathbb{R}^d \setminus B_{\frac{r}{2}}(x)} \Lambda \frac{(u(x+y) - (1 - \frac{\theta}{2})^{-d}u(x_0))^+}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy
\]
\[
\leq 1 + 2\Lambda \int_{\mathbb{R}^d \setminus B_{\frac{r}{2}}(x)} \frac{u^+(x+y)}{|y|^d} \left( \frac{2 - \alpha}{|y|^\alpha} + \varphi(1/|y|) \right) \, dy
\]
\[
\leq 1 + 2\Lambda \int_{\mathbb{R}^d \setminus B_{\frac{r}{2}}(x)} \frac{|u(x)|}{|x-y|^d} \left( \frac{2 - \alpha}{x-y} + \varphi(1/|x-y|) \right) \, dy. \quad (5.4)
\]

Using $|x-y| \geq \frac{\theta r}{2}$ and $|x| < 1$ we obtain the following estimates

\[
\frac{1}{|x-y|^{d+\alpha}} = \frac{1}{1 + |y|^{d+\alpha}} \cdot \frac{1 + |y|^{d+\alpha}}{|x-y|^{d+\alpha}} \leq \frac{1}{1 + |y|^{d+\alpha}} \cdot \left( \frac{1}{|x-y|^{d+\alpha}} + \left( \frac{|x| + |x-y|}{|x-y|} \right)^{d+\alpha} \right)
\]
\[
\leq \frac{1}{1 + |y|^{d+\alpha}} \left( \frac{\theta r}{2} \right)^{-d-\alpha} \left[ 1 + 2^{d+\alpha} \right] \leq C(\theta r)^{-d-\alpha} \frac{1}{1 + |y|^{d+\alpha}},
\]

and, since $\frac{|y|}{|x-y|} \leq 1 + \frac{|x|}{|x-y|}$,

\[
\varphi(1/|x-y|) \leq \kappa_0 \left( 1 + \frac{|x|}{|x-y|} \right)^\beta \varphi(1/|y|), \quad \varphi(1/|x-y|) \leq \varphi(2/r\theta) \leq \kappa_0 (2/r\theta)^\beta \varphi(1),
\]
giving us

\[
\frac{1}{|x-y|^d(\varphi(1/|x-y|))^{d+\beta}} = \frac{1}{1 + |y|^d(\varphi(1/|y|))^{d+\beta}} \left[ \frac{\varphi(1/|x-y|)}{|x-y|^d} \varphi(1/|y|) + \frac{|y|^d}{|x-y|^d} \varphi(1/|y|) \right]
\]
\[
\leq \frac{1}{1 + |y|^d(\varphi(1/|y|))^{d+\beta}} \left[ \kappa_0 (2/r\theta)^\beta \varphi(1) + \kappa_0 \left( 1 + \frac{|x|}{|x-y|} \right)^{d+\beta} \right]
\]
\[
\leq C_1 \frac{1}{1 + |y|^d(\varphi(1/|y|))^{d+\beta}} \left( \theta r \right)^{-d-\beta}
\]
\[
\leq C_2 \frac{1}{1 + |y|^d(\varphi(1/|y|))^{d+\beta}} \left( \theta r \right)^{-d-\alpha},
\]

for some constant $C_1$ dependent on $d, \varphi$. Using these estimates in (5.4) we thus obtain

\[
\mathcal{M}^- w(x) \leq C_3 (\theta r)^{-d-\alpha} \quad \text{in } \mathcal{B}_{\frac{r}{4}}(x_0).
\]

Now we repeat the arguments of Theorem 5.1 and get a contradiction to (5.3) if $t$ is large. This completes the proof. \hfill \Box

6. Boundary Harnack estimate

We prove a boundary Harnack property in this section for operators in $\tilde{\mathcal{L}}$ introduced in Section 5. Being inspired from [18] we prove the following result

**Theorem 6.1.** Let $\Omega \subset \mathbb{R}^d$ be any open set. Assume that there is $x_0 \in \mathcal{B}_{\frac{1}{2}}$ and $\varrho > 0$ such that $\mathcal{B}_{2\varrho}(x_0) \subset \Omega \cap \mathcal{B}_{\frac{1}{2}}$. Then there exists $\delta > 0$, dependent only on $(d, \alpha, \varrho, \varphi, \lambda, \Lambda)$, such that the following statement holds.
Let \( u_1, u_2 \in C(\mathcal{B}_1) \) be viscosity solutions of
\[
\dot{\mathcal{M}}^+ (au_1 + bu_2) \geq -\delta (|a| + |b|) \quad \text{in} \quad \mathcal{B}_1 \cap \Omega,
\]
\[
u_1 = u_2 = 0 \quad \text{in} \quad \mathcal{B}_1 \setminus \Omega,
\]
for all \( a, b \in \mathbb{R} \), and such that
\[
u_i \geq 0 \quad \text{in} \quad \mathbb{R}^d \;
\int_{\mathbb{R}^d} \frac{u_i(y)}{1 + |y|^{d+\alpha}} dy + \int_{\mathbb{R}^d} \frac{|u_i(y)|}{1 + |y|^d (\varphi(1/|y|))^{-1}} dy = 1.
\]
Then
\[
C^{-1} u_2 \leq u_1 \leq C u_2 \quad \text{in} \quad \mathcal{B}_{1/2},
\]
where the constant \( C \) depends only on \( (d, \alpha, \rho, \varphi, \lambda, \Lambda) \).

Theorem 6.1 is bit stronger than the boundary Harnack principle. To see it suppose that for some \( L \in \mathcal{L} \) we have \( Lu_i = 0 \) in \( \mathcal{B}_1 \cap \Omega \), in viscosity sense, and \( u_i = 0 \) in \( \mathcal{B}_1 \setminus \Omega \). Then clearly (6.1) holds for all \( a, b \in \mathbb{R} \) (cf. [5, Theorem 5.9]). Furthermore, if (6.2) holds, then Theorem 6.1 gives us
\[
C^{-1} u_2 \leq u_1 \leq C u_2 \quad \text{in} \quad \mathcal{B}_{1/2}.
\]

To prove Theorem 6.1 we need Lemmas 6.1 and 6.2 below.

Lemma 6.1. Assume that \( u \in C(\mathcal{B}_1) \) and satisfies \( \dot{\mathcal{M}}^- u \leq C_0 \) in \( \mathcal{B}_1 \) in viscosity sense. In addition, assume that \( u \geq 0 \) in \( \mathbb{R}^d \). Then
\[
\int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d+\alpha}} dy + \int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^d (\varphi(1/|y|))^{-1}} dy \leq C \left( \inf_{\mathcal{B}_{1/2}} u + C_0 \right),
\]
where the constant \( C \) depends only on \( (d, \lambda, \Lambda, \alpha, \varphi) \).

Proof. We need few basic estimates. We show that there exists a constant \( \kappa > 0 \) such that for any \( x_0 \in \mathcal{B}_{3/4} \) and \( z \in \mathbb{R}^d \) we have
\[
|x_0 - z|^{d+\alpha} \leq \kappa (1 + |z|^{d+\alpha}),
\]
\[
|x_0 - z|^d (\varphi(1/|x_0 - z|))^{-1} \leq \kappa (1 + |z|^d (\varphi(1/|z|))^{-1}).
\]
(6.3) is trivial since \( |x_0 + z| \leq 1 + |z| \) implies \( |x_0 - z|^{d+\alpha} \leq 2^{d+\alpha} (1 + |z|^{d+\alpha}) \). On the other hand
\[
\frac{1}{|z|} \leq \left( \frac{1 + |z|}{|z|} \right) \frac{1}{|x_0 - z|},
\]
implies
\[
\varphi \left( \frac{1}{|z|} \right) \leq \varphi \left( \frac{1 + |z|}{|z|} \frac{1}{|x_0 - z|} \right) \leq \kappa_0 \left( \frac{1 + |z|}{|z|} \right)^\beta \varphi \left( \frac{1}{|x_0 - z|} \right).
\]
Thus
\[
(\varphi(1/|x_0 - z|))^{-1} \leq \kappa_0 \left( \frac{1 + |z|}{|z|} \right)^\beta (\varphi(1/|z|))^{-1}.
\]
Let \( |z| \leq 1 \). Then using (6.5) we get
\[
|x_0 - z|^d (\varphi(1/|x_0 - z|))^{-1} \leq (1 + |z|)^{d+\beta} (\varphi(1/|x_0 - z|))^{-1} \leq 2^{d+\beta} (1 + |z|^{d+\beta}) (\varphi(1/|x_0 - z|))^{-1} \leq 2^{d+\beta} \left((\varphi(1/|x_0 - z|))^{-1} + |z|^{d+\beta} (\varphi(1/|x_0 - z|))^{-1}\right)
\]
Lemma 6.1

\[\leq 2^{d+\beta} \left( \kappa + \left| z \right|^{d+\beta} (\varphi(1/|x_0-z|))^{-1} \right)\]

\[\leq 2^{d+\beta} \left( \kappa + \kappa_0 2^\beta |z|^{d}(\varphi(1/|z|))^{-1} \right),\]

where \(\kappa = \max \{ (\varphi(1/2))^{-1}, 1\}\). Here we use \((\varphi(1/|x_0-z|))^{-1} \leq (\varphi(1/2))^{-1}\), since \(|x_0-z| < 2\).

Again, for \(|z| > 1\), we use (6.5) to obtain

\[|x_0-z|^d (\varphi(1/|x_0-z|))^{-1} \leq (1 + |z|)^d \kappa_0 \left( 1 + \frac{|z|}{|z|} \right)^\beta (\varphi(1/|z|))^{-1}\]

\[\leq \kappa_0 2^\beta (1 + |z|)^d (\varphi(1/|z|))^{-1}\]

\[\leq \kappa_0 2^{\beta+1} |z|^{d}(\varphi(1/|z|))^{-1}\]

\[\leq \kappa_0 2^{\beta+1} (1 + |z|)^d (\varphi(1/|z|))^{-1}.\]

This gives us (6.4).

Let \(\chi \in C_c^\infty(\mathcal{B}_1)\) be such that \(0 \leq \chi \leq 1\) and \(\chi = 1\) in \(\mathcal{B}_1\). Let \(t > 0\) be the maximum value for which \(u \geq t\chi\). It is easily seen that \(t \leq \inf_{\mathcal{B}_1} u\). Since \(u\) and \(\chi\) are continuous in \(\mathcal{B}_1\) there exists \(x_0 \in \mathcal{B}_1\) such that \(u(x_0) = t\chi(x_0)\). We also get

\[\tilde{\mathcal{M}}(u - t\chi)(x_0) \leq \tilde{\mathcal{M}}(u(x_0) - t\chi(x_0) \leq C_0 + C t \quad \text{in } \mathcal{B}_1.\]

On the other hand, since \(u - t\chi \geq 0\) in \(\mathbb{R}^d\) and \((u - t\chi)(x_0) = 0\), we also obtain from (6.3)-(6.4)

\[\tilde{\mathcal{M}}^{-1}(u - t\chi)(x_0) = 2\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^d} \left( \frac{2 - \alpha}{|x_0 - z|^\alpha} + \varphi(1/|x_0-z|) \right) dz\]

\[\geq 2(2 - \alpha)\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^d} dz + 2\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^d} dz + 2\lambda \int_{\mathbb{R}^d} \frac{u(z) - t\chi(z)}{|x_0 - z|^d} dz\]

\[\geq C_1 \left( \int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^{d+\alpha}} dz + \int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^{d}(\varphi(1/|z|))^{-1}} dz \right) - C_2 t,\]

for some constants \(C_1, C_2\). Combining we get

\[(C + C_2) \inf_{\mathcal{B}_1} u \geq (C + C_2)t \geq -C_0 + C_1 \left( \int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^{d+\alpha}} dz + \int_{\mathbb{R}^d} \frac{u(z)}{1 + |z|^{d}(\varphi(1/|z|))^{-1}} dz \right),\]

and the result follows. 

Using Theorem 5.2 and Lemma 6.1 we obtain the following

Lemma 6.2. Let \(\Omega \subset \mathbb{R}^d\) be any open set. Suppose there exists \(x_0 \in \mathcal{B}_1\), and \(\varrho > 0\) such that \(\mathcal{B}_2(\varrho(x_0)) \subset \Omega \cap \mathcal{B}_1\). Denote by \(D = \mathcal{B}_2(\varrho(x_0))\). Let \(u \in C(B_1)\) be a viscosity solution of

\[\mathcal{M}^+ u \geq -C_0 \quad \text{and} \quad \tilde{\mathcal{M}}^- u \leq C_0 \quad \text{in } \mathcal{B}_1 \cap \Omega,\]

\[u = 0 \quad \text{in } \mathcal{B}_1 \setminus \Omega.\]

Assume in addition, that \(u \geq 0\) in \(\mathbb{R}^d\). Then

\[\sup_{\mathcal{B}_1} u \leq C \left( \inf_D u + C_0 \right),\]

where constant \(C\) depends only on \((d, \lambda, \Lambda, \alpha, \varphi, \varrho)\).
Proof. Since $u \geq 0$ in $\mathcal{B}_1$ and $\mathcal{M}^+u \geq -C_0$ in $\mathcal{B}_1 \cap \{u > 0\}$, we have $\mathcal{M}^+u \geq -C_0$ in all of $\mathcal{B}_1$. Thus, by Theorem 5.2 and a standard covering argument, we have

$$
\sup_{\mathcal{B}_{\frac{3}{4}}} u \leq C \left( \int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d+\alpha}} dy + \int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d(\varphi(1/|y|))-1}} dy + C_0 \right).
$$

Again, using Lemma 6.1 in the ball $\mathcal{B}_{2\rho}(x_0)$, we get

$$
\int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d+\alpha}} dy + \int_{\mathbb{R}^d} \frac{u(y)}{1 + |y|^{d(\varphi(1/|y|))-1}} dy \leq C \left( \inf_D u + C_0 \right),
$$

where $D = B_{\rho}(x_0)$. Combining the previous estimates, the result follows. □

Finally, we prove Theorem 6.1

Proof of Theorem 6.1. Proof follows by following the arguments of [18, Theorem 1.2] and using Lemmas 6.1 and 6.2. □

Acknowledgements

The research of Anup Biswas was supported in part by DST-SERB grants EMR/2016/004810 and MTR/2018/000028. Mitesh Modasiya is partially supported by CSIR PhD fellowship (File no. 09/936(0200)/2018-EMR-I ).

References

[1] J. Bae. Regularity for fully nonlinear equations driven by spatial-inhomogeneous nonlocal operators. Potential Anal. 43 (2015), no. 4, 611–624.
[2] G. Barles, E. Chasseigne, C. Imbert. On the Dirichlet Problem for Second-Order Elliptic Integro-Differential Equations, Indiana Univ. Math. J.,57(2008), no.1, 213–246.
[3] G. Barles, C. Imbert. Second-Order Elliptic Integro-Differential Equations: Viscosity Solutions Theory Revisited, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), no. 3, 567–585.
[4] L. A. Caffarelli, X. Cabré. Fully nonlinear elliptic equations. American Mathematical Society Colloqium Publications, 43. American Mathematical Society, Providence, R.I., 1995.
[5] L. Caffarelli, L. Silvestre. Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), 597–638.
[6] L. Caffarelli, L. Silvestre. Regularity results for nonlocal equations by approximation, Arch. Ration. Mech. Anal. 200(1) (2011), 59–88.
[7] L. Caffarelli, L. Silvestre. The Evans-Krylov theorem for nonlocal fully nonlinear equations, Ann. of Math. 174 (2011), 1163–1187.
[8] H. Chang Lara, G. Dávila. Regularity for solutions of nonlocal, nonsymmetric equations. Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 6, 833–859.
[9] M. Kassmann, A. Mimica. Intrinsic scaling properties for nonlocal operators. J. Eur. Math. Soc. (JEMS) 19 (2017), no. 4, 983–1011.
[10] Y.C. Kim and K.A. Lee. Regularity results for fully nonlinear integro-differential operators with nonsymmetric positive kernels, Manuscripta mathematica 139(2012), 291–319.
[11] M. Kim, K. A. Lee. Regularity for fully nonlinear integro-differential operators with kernels of variable orders, to appear in Nonlinear Analysis, arxiv.org/abs/1805.07955
[12] S. Kim, Y. C. Kim, K. A. Lee. Regularity for fully nonlinear integro-differential operators with regularly varying kernels. Potential Anal. 44 (2016), no. 4, 673–705.
[13] D. Kriventsov. $C^{1,\alpha}$ interior regularity for nonlinear nonlocal elliptic equations with rough kernels, Comm. Partial Differential Equations 38 (2013), 2081–2106.
[14] C. Mou. Existence of $C^\alpha$ solutions to integro-PDEs. Calc. Var. Partial Differential Equations 58 (2019), no. 4, Paper No. 143, 28 pp.
[15] C. Pucci, Maximum and minimum first eigenvalues for a class of elliptic operators, Proc. Amer. Math. Soc. 17 (1966), 788–795.
[16] X. Ros-Oton, J. Serra. Boundary regularity for fully nonlinear integro-differential equations, Duke Math. J. 165 (2016), 2079–2154.
[17] X. Ros-Oton, J. Serra. Boundary regularity estimates for nonlocal elliptic equations in $C^1$ and $C^{1,\alpha}$ domains. Ann. Mat. Pura Appl. (4) 196 (2017), no. 5, 1637–1668.
[18] X. Ros-Oton, J. Serra. The boundary Harnack principle for nonlocal elliptic operators in non-divergence form. Potential Anal. 51 (2019), no. 3, 315–331.
[19] R. Schilling, R. Song and Z. Vondraček. Bernstein Functions, Walter de Gruyter, 2010.
[20] J. Serra. Regularity for fully nonlinear nonlocal parabolic equations with rough kernels, Calc. Var. Partial Differential Equations 54 (2015), 615–629.