Exit Decision of Venture Capital Based on Linear Contract in Continuous Time: IPO or M&A

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Based on the assumption that the long-term value of a venture capital satisfies the algebraic Brownian motion, we develop a continuous-time exit model of venture capital under different exit modes, namely, initial public offering (IPO) and mergers and acquisitions (M&A). The employee incentive problem is analyzed jointly with the exit decision of the firm in terms of the exit timing and the exit mode. Further, the problem of capital exit is considered from two perspectives, namely, optimal venture capital and social welfare maximization, and the differences between these exit decisions are compared. Our model predicts that the timing of an IPO, the purpose of which is to maximize the utility of the capitalists, lags behind the exit timing, whose purpose is to maximize social welfare. Using a numerical analysis, this paper also proves that increasing the production efficiency, lowering the interest rates, and improving risk management can make the exit decision of venture capitalists converge with that of maximizing social welfare.

1. Introduction

1.1. Motivation. In this paper, we will focus on a few practical problems: (1) Will the improvement in the production efficiency make the venture capitalist (VC) prefer going public or acquisition? (2) Will the improvement in the production efficiency delay the exit timing of the capital? (3) What would be the influence of risk management on the exit timing and exit mode? (4) What would be the influence of the discount rate on the exit timing and exit mode? (5) What would be the conflict and reconciliation between optimizing the investors’ utility and maximizing social welfare in a capital exit.

The common exit mechanisms of venture capital include initial public offering (IPO) and mergers and acquisitions (M&A). According to the data from http://www.pedata.cn, in 2017, 63.14% of Chinese VCs chose IPO and 25.16% chose M&A as their exit mode. In 2018, IPO accounted for 60.34% and M&A accounted for 25.07%, whereas in 2019, IPO accounted for 74.76% and M&A accounted for 19.31% as the choice of their exit mode. In the existing research studies, the choice of the exit mode of VC also focuses on the decision-making between IPO and M&A [1–4]. Thus, this paper limits the research scope to these two methods.

In the existing literature, the research on the exit mechanism of venture capitals mainly adopts constructing a static model [1, 3], which is highly refined. Thus, it is convenient to directly analyze the choice between an IPO and M&A. However, in practice, venture capital is a long-term process, from the beginning of the investment to the exit of the project [5], which may last for decades. In the long run, the environment of a risky project is dynamic. The present work considers exit decisions in the dynamic environment and tries to expand the adaptability of the model.

We assume that the value of a company is a function of time and satisfies the algebraic Brownian motion. The problem of employee incentive is modeled jointly with the exit decision of a firm in terms of the exit timing and the exit mode. Following this, a capital exit model under continuous time is set up. A venture capitalist, as the principal, has a right to make an exit decision. However, the implementation of the venture project depends on the efforts of the entrepreneur (EN). Therefore, a principal-agent relationship is formed between the VC and the EN. The VC encourages the
EN to make efforts by giving them an equity contract, which ultimately affects the exit decision regarding the capital.

The first step is to solve the optimal contract. Owing to the closed-end fund management mode adopted by a VC [6], the long-term value of venture capital can only be realized at the termination date, \( T \). Therefore, an optimal contract adopts the following structure: a VC gives an EN a continuous fixed salary between \([0, T]\) before the capital exits. The EN makes continuous efforts, but the final equity incentive can only be allocated at the time of exit. According to a study by He and Li [1], when a venture capital exits through M&A or IPO, the long-term value function of a project is different, and thus we can design an optimal equity incentive contract under M&A and IPO, respectively.

Subsequently, the best exit timing is estimated. By determining the optimal incentive contract, we can separately solve the utility obtained by the VC at the time of exit in an IPO or an acquisition. These two utilities are functions of the exit timing, \( T \), and exhibit an inverted U shape (see Figure 1). Thus, they both have the optimal exit timing. Since the expressions of these two utility functions are already too complex, we will calculate the optimal exit timing in both cases by a numerical method described in Section 5 of the paper.

What follows is to make a decision about the optimal exit method. We will consider it from the perspective of the VC income maximization as well as social welfare maximization. After solving for the optimal exit timing, by comparing the final utility gained from an IPO or M&A, investors can choose the exit method and the exit timing that are most beneficial to them. At the same time, we can also examine the benefits of the ENs and repeat the above two steps to find the optimal exit timing and exit method that maximizes social welfare. The abovementioned results are summarized in Tables 1 and 2. By comparison, we found that the exit decision of a VC is different from the exit decision for maximizing the overall social welfare.

Finally, in the numerical analysis, we test some common government policies used for stimulating a venture capital to investigate if they can help in converging the decision of the VC with the decision on maximizing social welfare. The results show that the three management factors, namely, reducing the interest rate, increasing the production capacity, and controlling risk, can address the two mentioned contradictions. The detailed results are given in Table 3.

The main contributions of the present work are as follows.

First, by building a capital exit model under continuous time, the problem of the optimal exit timing is combined with that of the optimal exit mode. Most of the earlier studies have considered the optimal exit mode [1, 3, 7], while others have focused on the timing of IPO [8–10]. We incorporate these two problems in our model.

Second, in our model, the fact that the VC’s exit decisions are not just determined by investors but are also affected by the consideration of social welfare, such as government-sponsored venture capital [11–15], has been considered. The goal of the government’s participation in venture capital is different from that of a VC. The government is more inclined towards achieving maximization of the overall profit of a project, including the total sales [12], the number of employees [13], the growth potential [14], and the IPO exit [15] rather than only maximizing the benefits of the VC. Even if the government does not participate in the management and operation of risky projects, it provides low-interest loans, infrastructure construction, and support to new talents. All of the above methods have an indirect impact on the exit mode and timing of risky projects.

1.2. Related Literature. On the basis of a large quantity of empirical data and case studies, Chemmanur et al. [16] have classified the exit modes into the following five categories: IPO, acquisition, second sale, buy-back, and write-off. Qian and Zhang [17] have divided the exit methods into three types, namely, IPO, M&A, and liquidation bankruptcy. Since liquidation bankruptcy is not a subjective choice but a compulsory procedure, most of the studies in the literature only focus on the exit choice between M&A and IPO. Some works have tested the choice between M&A and IPO via empirical methods [2, 17, 18], while others have used a mathematical model to analyze the choice between these two modes [1, 3, 7]. Therefore, this paper also focuses on the choice of IPO versus M&A.

Based on the abovementioned classification of the exit mechanisms, the existing literature shows that the exit mode is mainly affected by regional factors and industry characteristics. By employing empirical methods, other studies have discussed the preference of the venture capital in different regions for the different exit modes. Brau et al. [2] found that IPO is preferred as the exit mode in the US, but Jeng and Wells [4] proved that in the more conservative European markets, investors prefer to exit via M&A. Even in the US, which has the world’s most active IPO market, the number of firms going public is far less than the ones that get
Building theoretical models, the current methods includetheuse of the real option theory to calculate the optimalmerging time [8–10, 21] and the use of the game theory tobuilding the theoretical models, the current methods includetheuse of the real option theory to calculate the optimalmerging time [8–10, 21] and the use of the game theory toBuilding theoretical models, the current methods includetheuse of the real option theory to calculate the optimalmerging time [8–10, 21] and the use of the game theory to

| $k$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $T_{\text{IPO(all)}}^*$ | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| $E[\Pi_{\text{IPO(all)}}]$ | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| $T_{\text{AC(all)}}^*$ | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| $E[\Pi_{\text{AC(all)}}]$ | 19.08 | 19.08 | 19.08 | 19.08 | 19.08 | 19.08 | 19.08 | 19.08 | 19.08 | 19.08 |
| Overall decision | M&A | M&A | M&A | M&A | M&A | M&A | M&A | M&A | M&A | M&A |

Table 1: Numerical simulation for exit decision.

merging [1]. Furthermore, Qian and Zhang [17] showed thatin China, the return on investment (ROI) from an IPO is not higher than that from other exit modes. In addition to theexternal market environment, certain factors related tocorporate governance also affect the way a VC exits. Basedon the principal-agent theory, He and Li [1] proved thatfirms having high human capital and high managementquality are more likely to go public and use high-poweredstock-based compensation. On the basis of the competitionin the product market, Bayar and Chemmanur [3] built a
game among a VC, an EN, and external investors for der-
viving a number of testable implications regarding the exitdecision. An explanation about the IPO valuation premiumpuzzle was also given in this work. In terms of the life cycle,they predicted that the later stage companies having businessmodels that are more viable against market competition aremore likely to go public. Other investigations in the liter-ature have used empirical methods and studied the influenceof certain special factors, such as control right and repu-
tation, on the exit decision [19, 20].

The abovementioned studies mainly focus on the exitmechanism. Due to the structure of the venture capitalindustry in which closed-end funds are often used, the exit
timing of the VC is critical [6]. There are various existingworks that have focused on the timing of IPOs. In terms ofbuilding theoretical models, the current methods includethe use of the real option theory to calculate the optimalstopping time [8–10, 21] and the use of the game theory to
Table 2: Numerical simulation for exit decision.

| k  | Overall decision | VC's decision |
|----|------------------|---------------|
| r  | $T^*_{(IPO,all)}$ | $T^*_{(IPO,all)}$ |
|    | $E_{(IPO,all)}$   | $E_{(IPO,all)}$   |
|    | $T^*_{(AC,all)}$  | $T^*_{(AC,all)}$  |
|    | $E_{(AC,all)}$    | $E_{(AC,all)}$    |
| 3  | IBO IPO IPO IPO IPO IPO IPO IPO IPO IPO | IPO IPO IPO IPO IPO IPO IPO IPO IPO IPO |
| 5  | IBO IPO IPO IPO IPO IPO IPO IPO IPO IPO | IPO IPO IPO IPO IPO IPO IPO IPO IPO IPO |
| 7  | IBO IPO IPO IPO IPO IPO IPO IPO IPO IPO | IPO IPO IPO IPO IPO IPO IPO IPO IPO IPO |
| 9  | IBO IPO IPO IPO IPO IPO IPO IPO IPO IPO | IPO IPO IPO IPO IPO IPO IPO IPO IPO IPO |

Table 3: Summary of numerical simulation results.

| Influence on exit method | Influence on exit timing |
|--------------------------|--------------------------|
| Improve the production efficiency (i) Make venture capital inclined towards choosing IPO as the exit method. (ii) Make the best exit mode of the VC converge towards the exit mode for maximizing social welfare. | (i) Shorten the IPO exit timing but extend the exit timing for M&A. (ii) Make the best exit time of the VC converge towards the exit time for maximizing social welfare. |
| Reduce the loan interest rate (i) Make venture capital inclined towards choosing IPO as the exit method. (ii) Make the best exit mode of the VC converge towards the exit mode for maximizing social welfare. | (i) Extend the exit timing of IPO and M&A. (ii) Cannot make the best exit timing of the VC converge towards the exit timing for maximizing social welfare. |
| Strengthen risk control The influence of risk control on the exit mode is not obvious. | (i) It shortens the IPO exit timing but has no obvious influence on the time of M&A. (ii) Make the VC’s IPO timing converge towards the IPO timing to maximize social welfare. |
analyze the optimal exit timing [6, 22, 23]. In addition, there is also a group of scholars who have tested the factors that affect the timing of exit via empirical methods [16, 24–26]. By treating the going-public decision as a real option, Draho [27] found that exercising this option must be viewed as the cost of an IPO. In the case of an unstable market, the VC hopes to exit as early as possible in order to avoid risks but tends to hold the project for a long time when the project provides very good returns. At the same time, the market environment will also affect the value of this option. Bus-tamante [10] found that in cold markets, firms with better investment prospects accelerate their IPO relative to their information benchmark to reveal the type to outside investors. However, in hot markets, all firms issue simultaneously; further, the issuing age is younger on average, and the IPO timing is uninformative. Using the real options theory to solve the optimal exit timing is often cumbersome, but using the game theory to directly calculate the exit timing greatly simplifies the analysis. By constructing a three-party game model of outside investors, ENs, and VCs, Chemmanur and Fulghieri [22] found that a VC having more information and bargaining power is not willing to invest continuously when firms require a huge investment or face great uncertainty. Thus, the firms will go public earlier, which is similar to the conclusion of Draho [27]. They also showed that an improvement in the production capacity can promote the acceleration of IPO. Reputation will also accelerate the IPO process. In order to get a foothold in the market as early as possible, underpricing serves as a device for young VCs to credibly invest in building up their reputation. The introduction of reputation into the game model also explains the “hot issue market behavior” problem satisfactorily [6]. Finally, a few works test the specific factors that affect the exit timing by empirical methods. Field and Hanka [24] showed that the length of the lock-in period would affect the exit timing of venture capital. Further, Giot and Schwienbacher [26] provide evidence on the impact of economic factors such as the size and composition of the syndicate, its geographical location, and value adding by a VC on the exit outcomes.

Some of the abovementioned studies focus on the optimal exit mode of the VC and the factors that affect these exit modes, whereas others focus on the optimal exit timing of venture capital, especially the timing of the IPOs. We hope to combine the two aspects. In terms of the mathematical methods, we need to build a continuous-time principal-agent model. However, unlike the work of Sannikov [28], in venture capital, the long-term returns of the participants are paid off in a lump sum at the time of exit. Sannikov [28] studied the optimal contract under continuous time but assumed that the principal offers the agent a contract that specifies a non-negative flow of consumption. The Hamilton–Jacobi–Bellman (H-J-B) equation is established to solve the optimal contract. This method can make the contract design very complicated and maximize the interest of the principal. However, this assumption does not apply to venture capital investment. The long-term growth of a risky project can only be realized when exiting [1]. At the time of exit, the VC and EN can either let the company go public and cash out the shares in their hands to obtain profits or sell the project to a third party to obtain the purchase price. Therefore, we assume that the equity incentives will be available at the termination date and use the variational method to solve the optimal contract.

Most of the abovementioned studies consider the capital exit from the perspective of maximizing the returns for the VC. However, there are still a large number of government-sponsored VCs in the capital market, which do not have the goal of maximizing the income from the principal.

Government-sponsored VCs have been popular for a long time, such as the small business investment company (SBIC) plan in the United States, the Yozma plan in Israel, and the innovation investment fund (IIF) plan in Australia. Taking Israel’s Yozma plan as an example, the evaluation system of Yozma includes the IPO/M&A ratio, number of exits, whether reputable investors invested in the VC company or not, total capital under management (critical mass effects), and demonstration effects [29]. In addition, the total sales [12], the number of employees [13], the growth potential [14], and the number of IPO exits [15] have been included in the evaluation system of risky projects. There are also various methods for the government to achieve these goals. In addition to the direct investment in risky projects, the government also influences venture capital through a variety of indirect methods, for example, through personnel appointment, supervision system, audit system, investment stimulation, and other methods [4, 30]. The Chinese government influences the development and exit of venture capital via talent introduction, low tax policies, low-interest loans, and infrastructure construction methods. Therefore, we attempt to know how the government can make the exit decision of the VC and social welfare maximization exit decision to converge by influencing factors such as the interest rate or production efficiency without directly participating in the exit decision.

Finally, by reviewing the literature in terms of the abovementioned three aspects, we will focus on the following two issues.

First, how to make a decision that jointly optimizes the exit method and the exit timing.

Second, whether the optimal exit decision of a VC and the exit decision of maximizing social welfare will conflict and how to make them converge.

Therefore, by introducing the time factor into the exit model, this work establishes a continuous time capital exit model and expands the theory of He and Li [1]. Our model can predict the exit timing and exit mode. Finally, we answer these two questions from the perspective of maximizing the income of the VC and maximizing the total project output.

The remainder of this paper is structured as follows: Section 2 gives the hypotheses of the model. Section 3 introduces the basic models. A comparative static analysis on the exit decision is described in Section 4. Section 5 describes the numerical analysis. Conclusions from this work are given in Section 6. All the proofs are given in the appendices.

2. Hypothesis of the Model

Owing to the lack of capital, start-ups need external finance to support their operation. However, as start-ups do not have “financial records” in the early stage, it is difficult
to obtain funds from banks. In this work, we assume that the ENs obtain investment from VCs, and VCs need to exit in a later stage of funding either through an IPO or M&A. However, in the context of information asymmetry, the investors need to give incentives to the ENs so that they will make efforts. The whole process is illustrated in Figure 2. In particular, due to the structure of the venture capital industry, closed-end funds are often used [6]. Therefore, long-term value can only be realized when exiting.

In order to facilitate modeling, the assumptions of this work are given first.

Assumption 1. The employee decides his input \( a_t \) in the long-term growth. The monetary cost borne by the employee is hereafter denoted \( c(a_t) \). We assume a simple form for the cost function: \( dc(a_t) = b/2a_t^2 dt \), where \( b(b > 0) \) is a positive constant; the higher the management quality, the lower the coefficient \( b \) [31].

Assumption 2. A standard Brownian motion \( W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty \} \) on \( (\Omega, \mathcal{F}, P) \) drives the output process. The long-term value \( L(t) \) produced up to time \( t \) evolves according to

\[
dL(t) = ka_t dt + \sigma dW_t,
\]

where \( k \) represents the productivity of effort and \( \sigma \) is a constant, portraying the volatility of the company’s long-term gains. The long-term growth is valued at the terminal date \( T \). Therefore, this growth value of the company is \( L(T) = \int_0^T ka_t dt + \int_0^T \sigma dW_t \).

Assumption 3. When VC chooses IPO as the final exit mechanism, the final long-term profit is

\[
S^{IPO}(T) = L(T) - c,
\]

where \( c \) is the cost of IPO. When VC chooses acquisitions as the final exit mechanism, the final long-term profit is

\[
S^{A}(T) = \frac{L(T) + G + V_a(T)}{n},
\]

where \( V_a \) is the value of acquirer, \( G \) is the synergy value through acquisitions, \( n(n > 1) \) represents the share of the firm after M&A because we define the target firm’s (the venture capital project) share as 1. We can also follow Bayar and Chemmanur [3] and interpret \( V_a \) as the help from the acquirer.

Assumption 4. VC is risk neutral, but EN is risk averse with negative exponential utility function, and the risk aversion coefficient is \( \rho(\rho > 0) \). So, \( U_{EN}(T) = -\exp[-\rho\Pi_{EN}(T)] \).

Assumption 5. For simplicity, assume that both the principal and the agent discount the flow of profit and utility at a common rate \( r \).

3. The Basic Model

3.1. Venture Capital IPO Exit Model under Information Asymmetry. According to Holmstrom’s theory [32], due to information asymmetry, a VC cannot observe the degree of the efforts put by an EN. Thus, the model must meet the incentive compatibility (IC) constraints and participation constraint or individual rationality (IR). At the same time, the long-term value of venture capital can only be realized when exiting. Therefore, at the time of exiting, the payoff of the EN under an IPO is

\[
\Pi^{IPO}_{EN}(T) = \int_0^T a_t e^{-rt} dt + \int_0^T e^{-rt} \lambda(T)S^{IPO}(T) - \frac{b}{2} \int_0^T e^{-rt} a_t^2 dt.
\]

(4)

In a finite horizon problem, a VC provides a continuous fixed wage flow to an EN. Therefore, the fixed salary obtained by the EN is \( \int_0^T a_t e^{-rt} dt \). The equity incentive is \( \lambda(T)S^{IPO}(T) \), and the incentive coefficient is \( \lambda(T) \), which is a function of \( T \). In addition, equity incentive can only be realized by listing at the time of exit, and hence the discount rate is \( e^{-rt} \). Finally, before going public, the monetary cost of the EN’s efforts is \( \int_0^T e^{-rt} b/2a_t^2 dt \).

According to Assumption 2 and Assumption 3, we get

\[
\Pi^{IPO}_{EN}(T) = \int_0^T a_t e^{-rt} dt + \int_0^T e^{-rt} \left[ \lambda(T) \left( \int_0^T ka_t dt \right) + \int_0^T \sigma dW_t - c \right] - \frac{b}{2} \int_0^T e^{-rt} a_t^2 dt.
\]

(5)

According to Ito isometry and martingale properties of the Ito integral, we have

\[
E[\Pi^{IPO}_{EN}(T)] = \int_0^T a_t e^{-rt} dt + \int_0^T e^{-rt} \left[ \lambda(T) \left( \int_0^T ka_t dt - c \right) + \frac{b}{2} \int_0^T e^{-rt} a_t^2 dt, \right.
\]

(6)

\[
\text{Var}[\Pi^{IPO}_{EN}(T)] = e^{-2rT} \lambda^2 \sigma^2 T.
\]

(7)

According to Assumption 4, EN is risk averse with negative exponential utility function, so the certainty equivalent utility can be

\[
U^{IPO}_{EN}(T) = CE^{IPO}(T) = E[\Pi^{IPO}_{EN}(T)] - \frac{\rho}{2} \text{Var}[\Pi^{IPO}_{EN}(T)],
\]

(8)

and plugging (6) and (7) into (8), we have
At the same time, according to the hypothesis and model description, the payoff of VC is

\[ \Pi_{VC}^{IPO}(T) = -\int_0^T e^{-\gamma t} a_t dt + e^{-\gamma T} \left[ (1 - \lambda(T))\mathcal{I}^{IPO}(T) \right] - I. \] (10)

According to Assumption 3, we have

\[ \Pi_{VC}^{IPO}(T) = -\int_0^T e^{-\gamma t} a_t dt + e^{-\gamma T} \left[ \lambda(T) \int_0^T k\alpha_t dt + \int_0^T \sigma d\hat{W}_t - c \right] - I, \] (11)

where \( e^{-\gamma T} (1 - \lambda(T)) \left( \int_0^T k\alpha_t dt + \int_0^T \sigma d\hat{W}_t - c \right) \) is the VC’s dividend through IPO and \( \int_0^T a_t e^{-\gamma t} dt \) is the cost of paying entrepreneurs a fixed wage. For convenience, we assume that the total investment of VC in \([0, T]\) is \( I \).

Since VC is risk neutral, then

\[ U_{VC}^{IPO}(T) = E[\Pi_{VC}^{IPO}(T)]. \] (12)

Back to the agent problem, equation (9) satisfies the participation constraint, so we have

\[ U_{EN}^{IPO}(T) = \int_0^T e^{-\gamma t} a_t dt + e^{-\gamma T} \left[ \lambda(T) \left( \int_0^T k\alpha_t dt - c \right) \right] - \frac{b}{2} \int_0^T e^{-\gamma t} a_t^2 dt - e^{-2\gamma T} \frac{\lambda^2}{2} \sigma^2 T \geq 0. \] (13)

VC will adjust the salary to make the participation constraint be tight, so (13) becomes

\[ (IR)U_{EN}^{IPO}(T) = \int_0^T e^{-\gamma t} a_t dt + e^{-\gamma T} \left[ \lambda(T) \left( \int_0^T k\alpha_t dt - c \right) \right] - \frac{b}{2} \int_0^T e^{-\gamma t} a_t^2 dt - e^{-2\gamma T} \frac{\lambda^2}{2} \sigma^2 T = \mathbb{U}. \] (14)

Considering the information asymmetry, an employee can maximize his expect utility by following the contract, and we get the incentive compatibility constraint:

\[ (IC)a_t \in \arg \max_{a_t} \int_0^T e^{-\gamma t} a_t dt + e^{-\gamma T} \left[ \lambda(T) \left( \int_0^T k\alpha_t dt - c \right) \right] - \frac{b}{2} \int_0^T e^{-\gamma t} a_t^2 dt - e^{-2\gamma T} \frac{\lambda^2}{2} \sigma^2 T. \] (15)

Under the constraints of (IC) and (IR), VC optimizes the utility function (12) by designing the contract. According to the martingale property of the Itô integral, the objective function is

\[ \max_{a_t, \lambda(T)} U_{VC}^{IPO}(T) = -\int_0^T e^{-\gamma t} a_t dt + e^{-\gamma T} \left[ (1 - \lambda(T)) \left( \int_0^T k\alpha_t dt - c \right) \right] - I. \] (16)

To sum up, we get the principal-agent model (I) in the case of an IPO:
The following conclusions are obtained by solving model (I).

**Proposition 1.** If the venture capital adopts an IPO as the exit mode, the optimal incentive and optimal effort of EN are

\[
\begin{align*}
\lambda_{\text{IPO}}^* (T) &= \frac{k^2(e^{rt} - 1)}{k^2(e^{rt} - 1) + br T^2}, \\
a_{\text{IPO}}^* &= \frac{k^2(e^{rt} - 1)}{k^2(e^{rt} - 1) + br T^2} \frac{ke^{r(T-t)}}{b}.
\end{align*}
\]

**Proof.** See Appendix A.

From the analysis of the optimal incentive and the effort, given by Proposition 1, we obtain the following corollaries.

**Corollary 1.**

\[\lim_{T \rightarrow \infty} \lambda_{\text{IPO}} (T) = \frac{r k^2}{r k^2 + br T^2}, \lim_{T \rightarrow \infty} a_{\text{IPO}}^* (T) = 1; \]

\[d \lambda_{\text{IPO}} (T)/dT > 0, \ d a_{\text{IPO}}^* (T)/dr > 0, \ d a_{\text{IPO}}^* (T)/dr > 0.\]

**Proof.** See Appendix B.

Corollary 1 shows that the optimal incentive factor, \(\lambda\), is a function of the exit timing, \(T\), and increases with \(T\) until it becomes 1. The financial intuition behind this conclusion is that the longer the risky project runs before exiting, the more is the incentive that the VC must give to the EN. Generally speaking, the longer the project develops, the higher is the value of the final listing, and hence a smart EN requires more incentives. When the exit timing of the project becomes infinite, the equity incentive becomes 1, and the VC will not get any dividend after listing. Therefore, we assert that the best timing, \(T_{\text{IPO,VC}}\), for the VC to make a decision is limited. Similarly, the increase in productivity efficiency, \(k\), will lead to more incentives for optimal contracts. When the parameter, \(b\), is smaller, the management quality of the EN is higher, and the incentive factor will also increase. Finally, the rise in the discount rate, \(r\), increases the incentive factors and will thus accelerate the process of IPO, which we will verify in Section 5.

**Corollary 2.** In the period \([0, T]\) before the exit of a project, an EN’s efforts increase.

On the basis of equation (18), Corollary 2 can easily be proved. Since the value of a project cannot be realized until it exits, the cost of effort is decreasing continuously (the effort cost is discounted continuously), and the EN must put additional personal efforts in the final stage. This phenomenon is common in corporate management. When the company is about to go public, the ENs will try their best to whitewash its financial statements and increase the publicity of the company.

**Corollary 3.** We define the expected output of the project as \(\Pi_{\text{IPO}}(T) = e^{-rT} \int_0^T k a_{\text{IPO}}^* \lambda_{\text{IPO}}^* dt\). Then,

\[
\lim_{T \rightarrow \infty} \Pi_{\text{IPO}} (T) = 0.
\]

**Proof.** See Appendix C.

Although, according to Corollary 2, in \([0, T]\), the efforts of the EN increase, the output generated by the project is not rising all the time. The intuition of Corollary 3 is that from the perspective of the whole project, the output of the project increases with the exit timing initially and then decreases. Thus, there must be the best exit timing, \(T_{\text{IPO,all}}\), for the project. In the discussion below, we also need to consider the effort cost of the ENs and finally determine the exit timing for maximizing social welfare.

### 3.2. Venture Capital M&A Exit Model under Information Asymmetry

In the case of M&A, the moral hazard problem similar to an IPO still exists. However, in this case, there is another participant, namely, the acquirer who will merge the venture project into his/her own company at a competitive price, \(P\). We follow the perfect competition hypothesis, according to which, in the same period of time, the acquirer will not buy at a loss (for example, how do we evaluate the cost of a dish? Let us say that we spend money and time to make the same dish as a reference to evaluate the price of this dish). The acquisition can be viewed as a project of zero net present value for the acquirer. Thus, the competitive price, \(P\), must include the long-term value, \(L(T)\), for the optimal incentive contract, and the synergy of the project. For simplicity, we assume that the variance of the stock price is the same in the case of an IPO as well as acquisition [1] (the stock price for M&A is not only affected by the long-term value, \(L(T)\), but also by the value, \(V_a(T)\), of the acquirer. For simplicity, we assume that the variance is the same for an IPO as well as acquisition. For further explanation, please refer to footnote in [1]).

We use a salary system similar to that used in the case of an IPO. According to equation (4), we get

\[
\Pi_{\text{EN}}^{AC} (T) = \int_0^T a_t e^{-rt} dt + e^{-rT} \lambda(T) S^{AC} (T) - \frac{b}{2} \int_0^T e^{-rt} a_t^2 dt.
\]

According to hypothesis 3, we have

\[
\Pi_{\text{EN}}^{AC} (T) = \int_0^T e^{-rt} a_t dt + e^{-rT} \left[ \frac{\lambda(T)}{n} \left( \int_0^T k a_t dt + \int_0^T \sigma dW_t + G + V_a(T) \right) \right] - \frac{b}{2} \int_0^T e^{-rt} a_t^2 dt,
\]

where \(\lambda(T)/n (\int_0^T k a_t dt + \int_0^T \sigma dW_t + G + V_a(T))\) represents the income of EN when the project exits through M&A, which can only be realized at the timing of exit, \(G\) is the synergy value through acquisition, and \(V_a(T)\) is the value of acquirer. Before going acquisition, the monetary cost of EN’s efforts is \(\int_0^T e^{-rt} (b/2) a_t^2 dt\).
According to Assumption 4, EN is risk averse with negative exponential utility function, so the certainty equivalent utility can be

\[ U_{EN}^{AC}(T) = CE^{AC}(T) = E[\Pi_{EN}^{AC}(T)] - \frac{\rho}{2} \text{Var}[\Pi_{EN}^{AC}(T)]. \]  

(22)

According to Ito isometry and martingale properties of the Ito integral, we have

\[
\begin{align*}
\text{(IR)} \int_0^T e^{-rt} a_i dt + e^{-rT} \lambda(T) \left[ \int_0^T k_0 dt + G + E(V_a(T)) \right] & - \frac{b}{2} \int_0^T e^{-rt} a_i^2 dt - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T = \mathcal{U}, \\
\text{(IC)} a_t \in \arg \max_{a_t} & \int_0^T e^{-rt} a_i dt + e^{-rT} \lambda(T) \left[ \int_0^T k_0 dt + G + E(V_a(T)) \right] & - \frac{b}{2} \int_0^T e^{-rt} a_i^2 dt - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T.
\end{align*}
\]

(24)

To calculate the acquisition price \( P \), the following equation should be satisfied on the condition of (IC) and (IR) above:

\[
E[V_a(0)] = \max_{\lambda(T),a_t} E \left\{ e^{-rT} \left[ n - \lambda(T) \left[ \int_0^T k_0 dt + G + V_a(T) \right] \right] \right\} - P - \int_0^T e^{-rt} a_i dt.
\]

(25)

The acquisition can be viewed as a project of zero net present value to the acquirer, where the expected acquirer value without the acquisition should be the same as the total value to be realized after acquisition in the same period.

**Proposition 2.** Let \( \Pi_a \triangleq E[V_a(T) - e^{rt}V_a(0)] \), and the acquisition price \( P \) is

\[
P(T) = \max_{\lambda(T)} \left\{ -e^{rt} \mathcal{U} + \int_0^T k_0 dt + G + \Pi_a \\
- e^{rt} \left[ \frac{b}{2} - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T \right]. \right\}
\]

(26)

**Proof.** See Appendix D.

Due to the assumption of a completely competitive market, the acquisition price, \( P \), given by the acquirer includes the synergy gain, \( G \), and the expected output of the risky project, \( \int_0^T k_0 dt \), monetary cost of effort, \( \int_0^T (b/2)e^{-rt} a_i^2 dt \), risk premium, \( -(\rho/2)e^{-rT} \lambda^2 \sigma^2 T \), and reservation utility \( \mathcal{U} \). In particular, it also includes the total profit of the acquirer, \( \Pi_a \), during \([0, T]\). Intuitively, the more profitable the company is, the higher is the purchase price that it will attain. Theoretically, \( \Pi_a \) is not always positive, and the acquirer may also have losses. In this case, the acquisition price will also be negatively affected.

Since VC is the direct beneficiary of acquisitions, we have the expected return of VC:

\[
U_{VC}^{AC}(T) = E[\pi_{VC}^{AC}(T)] = e^{-rt} P(T) - I.
\]

(27)

To sum up, we get the principal-agent model (II) in M&A as an exit way:
The remaining work is to solve the optimal incentive and effort in the case of M&A. This part is similar to the procedure in the previous section.

**Proposition 3.** In the case of acquisition, the optimal incentive and optimal effort are

$$\lambda^{*}_{AC}(T) = \frac{nk^2(e^T - 1)}{k^2(e^T - 1) + n^2brPT\sigma^2}$$

$$a^{*}_{AC} = \frac{k^2(e^T - 1)}{k^2(e^T - 1) + n^2brPT\sigma^2} b$$

**Proof.** See Appendix E. According to Proposition 3, we get the following inference.

**Corollary 4.** There is a threshold $n^*$, and if $n > n^*$, then $\partial\lambda^{*}_{AC}(T)/\partial n < 0$. Moreover, $\partial n^*/\partial k > 0; \partial n^*/\partial T > 0$.

**Proof.** See Appendix F.

The implication of Corollary 4 is that large enterprises pay small incentives to acquire small projects. However, small companies are often required to provide more incentives to acquire large projects, making it more difficult for them to enjoy the benefits brought by M&A. Higher project efficiency or longer project running time will obviously make risky projects bigger and stronger, thus raising their threshold. Small enterprises can only acquire better projects by increasing incentives.

**Corollary 5.** The purchase price will not increase indefinitely. If the reservation utility, $\mathcal{U}$, is zero, then

$$\lim_{T \to -\infty} P(T) = k^2/2br + G + \Pi_{a}.$$

**Proof.** See Appendix G.

Intuitively, if a project goes on longer, the value of the project will be higher and so will be its acquisition price. Corollary 5 proves that the acquisition price will not increase indefinitely and that there will be a limit. Thus, we can assert that there is a timing, $T^*$, such that when $T > T^*$, the discounted acquisition price decreases with $T$. Finally, when choosing M&A as an exit method, both the optimal exit timing $T^*_{AC,VC}$ for the VC and $T^*_{AC,all}$ for social welfare maximization are limited.

### 4. Choice of the Venture Capital Exit Mechanism

#### 4.1. Exit Decision of the VC

It is necessary to compare the utilities of a VC in each exit mechanism and conduct a comparative static analysis from different perspectives. We have already estimated the optimal incentives provided by the VC and the best efforts made by the EN in the case of an IPO as well as acquisition. Substituting the results of Properties 1 and 3 into equations (16) and (27), we have

$$U^{IPO}_{VC}(T) = -\mathcal{U} + \frac{k^4(e^T - 1)^2}{k^2(e^T - 1) + brPT\sigma^2} \cdot \frac{1}{2brPT\sigma^2} - \frac{G + \Pi_{a}}{e^T} - I,$$

$$U^{AC}_{VC}(T) = -\mathcal{U} + \frac{k^4(e^T - 1)^2}{k^2(e^T - 1) + n^2brPT\sigma^2} \cdot \frac{1}{2brPT\sigma^2} + \frac{G + \Pi_{a}}{e^T} - I.$$

By comparing $U^{IPO}_{VC}(T^*_{IPO,VC})$ and $U^{AC}_{VC}(T^*_{AC,VC})$, a VC makes a decision on the exit mode of the venture capital.

By comparing the utility of the investors for different exit modes, VCs decide to choose a specific exit timing and an exit mode to maximize their own utility. From the previous sections, it is understood that whether it is an M&A or an IPO, venture capital investors cannot always hold projects without capital exit. However, different exit modes have different exit timings. The best exit timing for an IPO is $T^*_{IPO,VC}$, whereas that for an acquisition is $T^*_{AC,VC}$. After calculating the best exit timing, we can obtain the best exit mode of a VC by comparing $U^{IPO}_{VC}(T^*_{IPO,VC})$ and $U^{AC}_{VC}(T^*_{AC,VC})$. 
Due to the complexity of the utility function, we use a numerical method to determine the optimal exit timing and exit mode. The examples of the same are given in Section 5.

4.2. Exit Decision for Maximizing Social Welfare. The above conclusion is from the perspective of the VC, in which the utility of the VC is maximized. However, if considered from the perspective of maximizing the overall income of the project, e.g., a risky project supported by the government, the best exit timing and exit mode of the project are different. In the case of capital exit, the government takes into account the benefits of the investors and the employees.

In the case of capital exit, the government takes into account the perspective of maximizing the overall income of the project, e.g., a risky project supported by the government, the best exit timing, and exit mode. In the following, we will select multiple parameters (the parameters from Figures 1, 3, and 4 are \( \alpha^2 = 16, k = 5, n = 2, G = c = \Pi_1 = 10, r = 0.05, \) and \( b = r = 1 \)) and the VC’s utility for different exit modes. It can be seen that the reservation utility and investment do not be observed that the best exit timing for an IPO lags behind that of M&A. Therefore, for the current parameters (the parameters from Figures 3, 4, and 5 are \( \alpha^2 = 16, k = 5, n = 2, G = c = \Pi_1 = 10, r = 0.05, \) and \( b = r = 1 \)), the VC will finally choose IPO as the best exit mode. In the following, we will select multiple sets of parameters to analyze the best exit timing, exit method, and final income.

By comparing \( E[\Pi_{\text{ipo}}(T_{\text{ipo}, \text{all}})] \) and \( E[\Pi_{\text{ac}}(T_{\text{ac}, \text{all}})] \), the best exit timing and mode for maximizing social welfare are chosen.

5. Numerical Experimental Analysis

In the previous sections, we have given the intuition behind the conclusions. Owing to the lack of data, conducting a numerical simulation is an effective way to test the conclusions drawn in the previous sections. In this section, we describe the numerical analysis carried out using MATLAB for assessing the conclusions drawn in the previous sections more intuitively and further explain these conclusions and corollaries. In particular, we will focus on the following practical problems: (1) Will the improvement in the production efficiency, \( k \), make a VC prefer IPO or M&A as the exit mode? (2) Will the improvement in the production efficiency, \( k \), delay the exit timing of the capital? (3) What is the influence of risk management on the exit timing and exit mode? (4) How does the adjustment of the discount rate, \( r \), affect the mode and timing of capital exit? (5) Can the government influence the discount rate and the long-term production efficiency to adopt the VC’s optimal exit choice as the overall optimal exit decision? For parameter selection, we refer to Sannikov [28], DeMarzo and Sannikov [33], Anderson et al. [34], and Williams [35].

5.1. Comparative Static Analysis of the Optimal Exit Timing.

From the previous sections, it can be seen that the optimal incentive coefficient, the optimal effort of an EN, the utility function of a VC, and the overall returns of the project are all related to the exit timing. Therefore, the first parameters that need to be solved are the optimal exit timings, \( T_{\text{ipo}, \text{vc}}, T_{\text{ac}, \text{vc}}, T_{\text{ipo}, \text{all}}, \) and \( T_{\text{ac}, \text{all}} \). From equations (30), (31), (32), and (33), it can be seen that the reservation utility and investment do not affect the decision of the optimal exit timing and the choice of the optimal exit mode and thus have all been set to 0.

Figure 1 describes the relationship between the exit timing, \( T \), and the VC’s utility for different exit modes. It can be observed that the best exit timing for an IPO lags behind that of M&A. However, the final returns from the IPO are higher than those from M&A. Therefore, for the current parameters (the parameters from Figures 1, 3, and 4 are \( \alpha^2 = 16, k = 5, n = 2, G = c = \Pi_1 = 10, r = 0.05, \) and \( b = r = 1 \)) and Figure 5 are \( \alpha^2 = 16, n = 2, G = c = \Pi_1 = 10, r = 0.05, \) and \( b = r = 1 \), the VC will finally choose IPO as the best exit mode. In the following, we will select multiple sets of parameters to analyze the best exit timing, exit method, and final income.

Figure 3 describes the relationship between the VC’s income, \( U_{\text{vc}}(T) \), the project’s overall income, \( \Pi_{\text{all}}(T) \), and the exit timing, \( T \), when IPO is chosen as the capital exit mode. Obviously, the IPO timing of the whole project is earlier than that of the VC, i.e., \( T_{\text{ipo}, \text{all}} < T_{\text{ipo}, \text{vc}} \). In other words, when choosing IPO as the capital exit mode, the VCs are willing to develop the project for a longer period from a personal perspective, like the Facebook founder Mark Zuckerberg (Mr. Zuckerberg told in a newspaper interview that the company would eventually have an IPO “because that is the contract that we have with our investors and our employees.” However, he added, “we are definitely in no rush.”—VASCHELARO, JESSICA, E. Facebook CEO in No Rush To “Friend” Wall Street. (cover story)[J], Wall Street Journal Eastern Edition, 2010). However, the delay in listing hampers the income of the employees. Therefore, how to guide a VC’s exit timing, \( T_{\text{ipo}, \text{vc}} \), to approach \( T_{\text{ipo}, \text{all}} \) will be one of the topics discussed below.

Figure 4 describes the relationship between the VC’s income, \( U_{\text{vc}}(T) \), a project’s overall income, \( \Pi_{\text{all}}(T) \), and the exit timing, \( T \), when the acquisition is chosen as the capital exit mode. Similar to IPO, the best M&A timing of the whole project is earlier than that for the VC, i.e., \( T_{\text{ipo}, \text{all}} < T_{\text{ipo}, \text{vc}} \) or \( T_{\text{ac}, \text{all}} < T_{\text{ac}, \text{vc}} \). In addition, we hope to find a way to make \( T_{\text{ac}, \text{vc}} \) approach \( T_{\text{ac}, \text{all}} \) in the case of M&A.
Finally, on the basis of the results given in Table 1, the relationship between the exit timings, \( T_{\text{IPO,VC}}^*, T_{\text{AC,VC}}^*, T_{\text{IPO,all}}^*, \) and \( T_{\text{AC,all}}^* \) and the production efficiency factor, \( k \), is illustrated graphically in Figure 5. From the figure, we find that timings for the different modes of capital exit tend to be consistent with the increase in the long-term production efficiency of the project.

Numerical Conclusion 1. (1) The optimal IPO timings \( T_{\text{IPO,VC}}^* \) and \( T_{\text{IPO,all}}^* \) tend to decrease with the productivity \( k \). (2) The acquisition timings, \( T_{\text{AC,VC}}^* \) and \( T_{\text{AC,all}}^* \), tend to increase with the production productivity. (3) The optimal exit timings for the different exit modes tend to be consistent with the increase in production efficiency. (4) \( T_{\text{IPO,all}}^* < T_{\text{IPO,VC}}^* \).

As for the contradiction of inconsistent exit timing, i.e., \( T_{\text{IPO,all}}^* \neq T_{\text{IPO,VC}}^* \) and \( T_{\text{AC,all}}^* \neq T_{\text{AC,VC}}^* \), Numerical Conclusion 1 shows that increasing the production capacity can effectively alleviate this problem. Especially in the case of IPO, the first item in Numerical Conclusion 1 indicates that companies having a strong production capacity and good competitiveness will go public relatively early (note that this is not inconsistent with the previous section. A VC hopes to exit later than the overall optimal timing. However, from the horizontal analysis, more competitive enterprises are willing to go public as soon as possible). The second item in Numerical Conclusion 1 does not make much sense because we will show in the following that enterprises with a high production capacity will not choose M&A as the final exit method. As for the last item, the reader can obtain it by observing Tables 1 and 2. The intuition behind this is that the VCs are not in a hurry to go public.

5.2. Influence of the Production Efficiency, \( k \), on the Exit Decision. We set the value of the long-term production efficiency, \( k \), from 1 to 10, and the exit timing, \( T \), is limited to \([1, 100]\). The return on capital exit is calculated with an accuracy of 0.1. The numerical simulation is carried out in four situations: \( \sigma = 2, \sigma = 4, \sigma = 7, \) and \( \sigma = 9 \). The optimal exit timings \( T_{\text{IPO,VC}}, T_{\text{AC,VC}}, T_{\text{IPO,all}}, \) and \( T_{\text{AC,all}} \) and the corresponding optimal payoffs \( U_{\text{VC}}(T_{\text{IPO,VC}}), U_{\text{AC}}(T_{\text{AC,VC}}), \) are calculated using the MATLAB software, and the results obtained are given in Table 1 (the other parameters of Table 1 are \( G = c = \Pi_a = 10, r = 0.05, \) and \( a = \rho = 1 \)).

By comparing the benefits of the different exit modes, we simulate the results of the VC’s exit decision and the overall exit decision as a function of the change in \( k \). To facilitate comparison, the 0–1 variable is set, where 1 represents IPO and 0 represents M&A. On the basis of the choice of the exit mode for different volatility rates, Figures 6–8 are plotted. Since several studies in the literature have discussed the
optimal listing time of venture capital, we have summarized $T_{IPO,VC}^*$ with different variances, as shown in Figure 9.

By comparing Figures 6–8, we obtain Numerical Conclusion 2.

Numerical Conclusion 2. When the production efficiency is high (i.e., high $k$), the firm tends to prefer an IPO over acquisition.

Numerical Conclusion 2 shows that whether it is from the perspective of maximizing the utility of the VC or maximizing the overall value of the project, the improvement in the production efficiency, $k$, will encourage the VC to take IPO as the final exit mode. Therefore, improving production efficiency can alleviate the contradiction in the choice of the exit mode. For example, in Figure 7, when $k = 4$, a VC prefers to adopt M&A as the final exit method, while from the perspective of maximizing the social welfare of the project, IPO is the better option. However, when $k \geq 5$, the contradiction no longer exists. For low-efficiency projects, from the overall perspective of all the participants in the venture project as well as the personal perspective of the VC, M&A is chosen as the exit mode. However, we cannot conclude that low efficiency is conducive to resolving the contradiction of choosing an exit mode. The reasons for this are as follows: (i) $U_{AC}^*(T_{AC,VC}^*)$ and $E[\Pi_{AC}^*(T_{AC,all}^*)]$ will decrease with the decrease in the efficiency, (ii) IPO is generally considered as a better way to exit [5], and (iii) it is difficult for inefficient venture enterprises to find acquirers.

By combining the results of Figure 9 and Table 1, we obtain Numerical Conclusion 3.

Numerical Conclusion 3. (i) The IPO timings, $T_{IPO,VC}^*$, and $T_{IPO,all}^*$ increase with project volatility; (ii) the difference between $T_{IPO,VC}^*$ and $T_{IPO,all}^*$ increases with the project volatility; and (iii) the influence of project volatility on the exit mode is not obvious.

Numerical Conclusion 3 implies that risk control accelerates the listing of enterprises. However, its impact on the timing of M&A is not obvious. For example, from the values given in the fifth column for $k = 5$ in Table 1, it can be seen that the volatility gradually increases, $T_{AC,all}^*$: [12.9; 13.9; 12; 8.3], and from the results given in the sixth column, when $k = 6$, $T_{AC,VC}^*$: [13.7; 14.6; 12.5; 8.9], an opposite trend can be seen. Therefore, there is no obvious upward or downward trend. At the same time, the impact of volatility on the exit methods is also irregular. For example, from Figure 6, when $k \geq 6$, the VC chooses IPO; from Figure 7, when $k \geq 5$, the VC chooses IPO; and finally from Figure 8, when $k \geq 6$ once again, the VC chooses IPO as the exit mode.

At the beginning of this section, we raised the question, Will the control of the project volatility lead to the convergence of the exit mode decision of the VC and the overall exit mode decision of the project? By comparing Figures 6–9, we find that the answer to this is no.

5.3. Influence of the Discount Rate, $r$, on the Exit Decision

In this section, we focus on the influence of the discount rate on the exit decision. The government provides low-interest loans to stimulate venture capital, which affects the discount of cash flow. We wish to explore how changes in the discount rate would affect the VC’s exit decision. We set the discount rate uniformly from 0.01 to 0.1. The numerical simulation is carried out in four situations: $k = 3$, $k = 5$, $k = 7$, and $k = 9$.

The optimal exit timings $T_{IPO,VC}^*$, $T_{IPO,all}^*$, $T_{IPO,all}^*$, and $T_{AC,all}^*$ and the corresponding payoffs $U_{IPO}^*(T_{IPO,VC}^*)$, $U_{AC}^*(T_{AC,VC}^*)$, $E[\Pi_{AC}^*(T_{AC,all}^*)]$, and $E[\Pi_{AC}^*(T_{AC,all}^*)]$ are calculated using MATLAB, and the results obtained are presented in Table 2 (the other parameters of Table 2 are $G = c = \Pi_0 = 10$, $\sigma = 4$, and $\rho = 1$).

Similar to the calculations described in the previous section, we set the 0–1 variable, where 1 represents IPO and 0 represents M&A. Figures 10–12 show the choice of exit methods for different discount rates. At the same time, in
order to observe the change in the exit timing as a function of the discount rate, \( r \), Figure 13 is plotted by selecting the data of \( k = 5 \).

By comparing the results obtained from Figures 10–12, we obtain Numerical Conclusion 4.

Numerical Conclusion 4. The effects of reducing the discount rate are as follows: (i) the preference of the venture capital to exit via IPO increases; (ii) the overall returns of the project and the utility of the VC are improved; and (iii) the exit timing of the capital is delayed; however, the final returns are improved.

From Figures 10–12, we find that when the discount rate drops, the entire project and the VC are more inclined to choose IPO as the best exit mode of capital. This result is consistent with the fact that the government hopes to stimulate more venture capital listing through low-interest loans. From the result given in Table 2, we find that \( U_{\text{VC}}^{\text{IPO}}(T_{\text{IPO,VC}}^*), U_{\text{VC}}^{\text{AC}}(T_{\text{AC,VC}}^*), E[\Pi_{\text{all}}^{\text{IPO}}(T_{\text{IPO,all}}^*)] \), and \( E[\Pi_{\text{all}}^{\text{AC}}(T_{\text{AC,all}}^*)] \) increase as \( r \) decreases. Finally, from Figure 13, it can be seen that with the decrease in the discount \( r \), the exit timings \( T_{\text{IPO,VC}}^*, T_{\text{AC,VC}}^*, T_{\text{IPO,all}}^* \), and \( T_{\text{AC,all}}^* \) increase. However, when \( r = 0.02 \), the difference
between the various exit timings is minimal. Therefore, it is not advisable to promote convergence of different exit timings by simply lowering the interest rates.

6. Conclusion

In the management of a risky project, a VC needs to decide the exit mechanism and timing of the project in advance.

Based on the fact that the cash flow of a risky project satisfies the algebraic Brownian motion, we have introduced the time factor into the exit model of the venture capital and have attempted to estimate the optimal exit mode and the corresponding optimal exit timing. In this paper, we first build the principal-agent model in which the investor chooses IPO as the exit method. Further, we establish the principal-agent model in which the investor chooses M&A as the exit method. The optimal incentive contracts under the two exit modes are solved. As a result, the problem of choosing the exit mode and the exit timing can be solved. Finally, we also determine the best exit decision with the goal of maximizing social welfare. It provides a reference for the government to guide the exit of venture capital. This work shows that

1. The incentive of the optimal contract increases with the exit timing, $T$. In a firm with long exit timing, the efforts taken by the employee in the growth of the firm will have a longer time to be realized. Such a firm is therefore expected to use a high-power contract to motivate the employee to contribute to the long-term growth.

2. The optimal effort of the EN increases in $[0,T]$. The closer the project is to the exit, the more willing will be the EN to put in the effort. In other words, there is a phenomenon of “cramming temporarily.”

3. VCs will make an exit decision according to the production efficiency, volatility, and external interest factors of the project. Generally speaking, improving the efficiency and reducing the loan interest rate can make the venture capital exit via an IPO rather than an acquisition. In the case of the life cycle, our model predicts that efficient firms and firms with a higher risk management quality will go public sooner. However, companies with lower lending rates may delay their IPO.

4. An explanation of the government’s policy on venture capital is given. Exit decisions by capitalists are often not arbitrary. For example, risky projects supported by the government tend to pay more attention to the social welfare of the projects. Therefore, there will be two contradictions: (i) with regard to the exit timing, e.g., from the perspective of maximizing social welfare, the exit of a VC should occur at $T_1$, but the capitalists wish to exit at $T_2$; (ii) with regard to the exit methods, e.g., from the perspective of maximizing social welfare, an IPO should be chosen to withdraw the VC; however, the capitalists want to choose M&A. Therefore, we carry out simulation to investigate whether increasing the production capacity, lowering the interest rate, and improving risk management can alleviate the abovementioned contradictions. A summary of the results obtained in this work is given in Table 3. Finally, it is worth mentioning that these three management methods can alleviate the above two contradictions in certain aspects.

Appendix.

A. Proof of Proposition 1

According to (9), we have
\[ U_{\text{EN}}^{\text{IPO}}(T) = \int_0^T a_t e^{-rT} dt + e^{-rT} \left[ \lambda(T) \left( \int_0^T k a_t dt - c \right) \right] - \frac{b}{2} \int_0^T e^{-rT} a_t^2 dt - \frac{b^2}{2\sigma^2} T \]

\[ = \int_0^T a_t e^{-rT} dt + \left[ \int_0^T e^{-rT} \lambda(T) k a_t \right] \frac{b}{2} \int_0^T e^{-rT} a_t^2 dt - \lambda(T)e^{-rT}c - \frac{\lambda^2 \sigma^2 T}{2}. \]  

\[(A.1)\]

Let \( F = e^{-rT} \lambda(T) k a_t - (b/2)e^{-rT} a_t^2 \), and according to the Euler equation, we have

\[ F_a = \frac{dF}{dr} = 0. \]  

\[(A.2)\]

Therefore, we get

\[ e^{-rT} \lambda(T) k - be^{-rT} a_t = 0. \]  

\[(A.3)\]

The best effort of entrepreneurs in an IPO is

\[ a_t^* = \frac{\lambda(T) k}{b} e^{-r(T-t)}. \]  

\[(A.4)\]

According to the IR constraint, we have

\[ \int_0^T a_t e^{-rT} dt = U - \left[ \int_0^T e^{-rT} \lambda(T) k a_t \frac{b}{2} \int_0^T e^{-rT} a_t^2 dt \right], \]

\[ -\lambda(T)e^{-rT}c + e^{-2\gamma T} \alpha^2 T \]  

\[(A.5)\]

and substituting (A.4) into (16),

\[ U_{\text{VC}}^{\text{IPO}}(T) = -\int_0^T a_t e^{-rT} dt + e^{-rT} \left[ (1 - \lambda(T)) \left( \int_0^T k a_t dt - c \right) - I \right] = -\bar{U} + e^{-rT} \left( \int_0^T k a_t dt - c \right) - \frac{b}{2} \int_0^T e^{-rT} a_t^2 dt - \frac{b^2}{2\sigma^2} T - I \]

\[ = -\bar{U} + e^{-2rT} \lambda(T) \left[ \lambda(T) k^2 \right] \frac{1}{br} (e^{rT} - 1) - ce^{-rT} - \frac{1}{2} \lambda^2 k^2 e^{-2rT} (e^{rT} - 1) - e^{-2\gamma T} \alpha^2 T - I. \]  

\[(A.6)\]

VC optimizes personal income by controlling incentive factors \( \lambda(T) \), so there are

\[ \frac{\partial U_{\text{VC}}^{\text{IPO}}(T)}{\partial \lambda(T)} = e^{-2rT} \frac{k^2}{br} (e^{rT} - 1) - \frac{\lambda(T) k^2}{br} e^{-2rT} (e^{rT} - 1) - 2e^{-2rT} \alpha^2 T = 0, \]  

\[ \frac{\partial U_{\text{VC}}^{\text{IPO}}(T)}{\partial \lambda(T)} = - \frac{k^2 (e^{rT} - 1)}{k^2 (e^{rT} - 1) + br \alpha^2 T}, \]  

\[ \frac{dU_{\text{VC}}^{\text{IPO}}(T)}{dr} = - \frac{k^2 l^2 (e^{rT} - 1) + br \alpha^2 T}{k^2 (e^{rT} - 1) + br \alpha^2 T} \]

\[ = - \frac{k^2 l^2 (e^{rT} - 1) + br \alpha^2 T}{k^2 (e^{rT} - 1) + br \alpha^2 T} \]

\[ \frac{dU_{\text{VC}}^{\text{IPO}}(T)}{dr} = - \frac{k^2 l^2 (e^{rT} - 1) + br \alpha^2 T}{k^2 (e^{rT} - 1) + br \alpha^2 T} \]

\[ (A.7)\]

and \( \frac{\partial U_{\text{VC}}^{\text{IPO}}(T)}{\partial \lambda(T)} < 0. \) After calculation, we get the optimal incentive and effort in the case of an IPO:

\[ \lambda_{\text{IPO}}^*(T) = \frac{k^2 (e^{rT} - 1)}{k^2 (e^{rT} - 1) + br \alpha^2 T}, \]

\[ a_{\text{IPO}}^* = \frac{k^2 (e^{rT} - 1) + br \alpha^2 T}{k^2 (e^{rT} - 1) + br \alpha^2 T} \]  

\[(A.8)\]

**B. Proof of Corollary 1**

According to Proposition 1, we have

\[ \frac{d\lambda_{\text{IPO}}(T)}{dT} = \frac{rk^2 e^{rT} [k^2 (e^{rT} - 1) + br \alpha^2 T] - k^2 (e^{rT} - 1) [rk^2 e^{rT} + br \alpha^2 T]}{[k^2 (e^{rT} - 1) + br \alpha^2 T]^2} = \frac{brk^2 \alpha^2 (re^{rT} - e^{rT} + 1)}{[k^2 (e^{rT} - 1) + br \alpha^2 T]^2}. \]  

\[(B.1)\]

Because \( (d(\lambda_{\text{IPO}}(T))/dT) \) is \( r^2 e^{rT} \) and when \( T = 0 \) the incentive increases with the increase of exit timing \( T \). And, it is easy to prove \( (\lambda_{\text{IPO}}(T))/dT \geq 0 \).

Then, taking the derivative with respect to \( r \), there are

\[ \frac{d\lambda_{\text{IPO}}(T)}{dr} = \frac{T k^2 e^{rT} [k^2 (e^{rT} - 1) + br \alpha^2 T] - k^2 (e^{rT} - 1) [Tk^2 e^{rT} + br \alpha^2 T]}{[k^2 (e^{rT} - 1) + br \alpha^2 T]^2} \]

\[ = \frac{k^2 br \alpha^2 (rT e^{rT} - e^{rT} + 1)}{[k^2 (e^{rT} - 1) + br \alpha^2 T]^2}. \]  

\[(B.2)\]

Defining the function \( F = rT e^{rT} - e^{rT} + 1 \), let \( rT = x, x \in [0, +\infty) \). Then, \( F(x) = xe^x - e^x + 1 \). It is easy to prove that the minimum value is \( F(0) = 0 \), so \( (\lambda_{\text{IPO}}(T))/dr \geq 0 \).
\[
\lim_{t \to \infty} k^2(e^T - 1) / k^2(e^T - 1) + br \rho \sigma^2 T = \lim_{t \to \infty} r k e^T / r k^2 e^T + br \rho \sigma^2 = r k e^T / r k^2 + br \rho \sigma. 
\]
(B.3)

C. Proof of Corollary 3

According to Proposition 1, we have

\[
\lim_{t \to \infty} e^{-rT} \int_0^T \frac{k^2(e^T - 1)}{k^2(e^T - 1) + br \rho \sigma^2 T} \cdot \frac{ke^{-r(t-t)}}{b} dt = \lim_{t \to \infty} \frac{k^4}{br k^2 e^T(e^T - 1) + br \rho \sigma^2 T} \left( \frac{e^T - 1}{e^T} \right)^2 = 0. 
\]
(C.1)

D. Proof of Proposition 2

According to the IR restriction under M&A, we have

\[
- \mathcal{U} + e^{-rT} \lambda(T) \left[ \int_0^T k_\alpha dt + G + E(V_a(T)) \right] 
\]
\[
- \int_0^T \frac{b}{2} e^{-rT} d\alpha^2 dt - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T = - \int_0^T e^{-rT} d\alpha dt. 
\]
(D.1)

\[
E[V_a(0)] = \max_{\lambda(T)} \left\{ e^{-rT} \left( \left[ n - \lambda(T) \right] \int_0^T k_\alpha dt + G + V_a(T) \right) - p \right\} - \mathcal{U} 
\]
\[
+ e^{-rT} \lambda(T) \left[ \int_0^T k_\alpha dt + G + V_a(T) \right] - \int_0^T \frac{b}{2} e^{-rT} d\alpha^2 dt - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T. 
\]
(D.2)

Since \( \Pi_a \triangleq E[V_a(T) - e^{-rT} V_a(0)] \), then

\[
P = \max_{\lambda(T)} \left\{-e^{-rT} \mathcal{U} + \int_0^T k_\alpha dt + G + \Pi_a \right\} 
\]
\[
- e^{-rT} \int_0^T \frac{b}{2} e^{-rT} d\alpha^2 dt - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T. 
\]
(D.3)

E. Proof of Proposition 3

According to the IC restriction under M&A, we have

\[
\max_a \int_0^T e^{-rT} k_\alpha dt + e^{-rT} \lambda(T) \int_0^T k_\alpha dt + G + E(V_a(T)) 
\]
\[
- \int_0^T \frac{b}{2} e^{-rT} d\alpha^2 dt - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T 
\]
\[
= \max_a \int_0^T e^{-rT} k_\alpha dt + \int_0^T \frac{\lambda(T) e^{-rT}}{n} k_\alpha - \frac{b}{2} e^{-rT} d\alpha^2 dt 
\]
\[
+ e^{-rT} \left[ G + E(V_a(T)) \right] - \frac{\rho}{2} e^{-2rT} \lambda^2 \sigma^2 T. 
\]
(E.1)

Let \( F = (\lambda(T)/n)e^{-rT} k_\alpha - b/2 e^{-rT} d\alpha^2 \), and according to the Euler equation, we have
\[ F_{\alpha} = \frac{dF_{\alpha}}{dt} = 0. \]  
(E.2)

So, the best effort of entrepreneurs is

\[ a^*_{\alpha, AC} = \frac{\lambda(T)k}{nb}e^{-r(T-t)}. \]  
(E.3)

Plugging (E.3) into (27), we have

\[
\max_{\lambda(T)} \left\{ -\mathcal{U} + e^{-rt} \left( \int_0^T k\alpha_i dt + G + \Pi_a \right) - \int_0^T b e^{-rt} a^*_i dt \right\}
= \max_{\lambda(T)} \left\{ -\mathcal{U} + e^{-rt} \left( \int_0^T \frac{\lambda(T)k}{nb}e^{-r(T-t)} dt + G + \Pi_a \right) - \int_0^T b e^{-rt} \frac{\lambda(T)k}{nb}e^{-r(T-t)} e^{-rT} - 1 \right\} - \frac{\rho e^{-2rt} \lambda(T)\sigma^2 T - 1}{2}
= \max_{\lambda(T)} e^{-rT} \left[ \frac{\lambda(T)k^2}{nbr} (1 - e^{-rt}) + G + \Pi_a \right] - \frac{\lambda(T)k^2}{2nbr} e^{-2rt} (e^{-T}) - \frac{\rho e^{-2rt} \lambda(T)\sigma^2 T - 1}{2}.
\]  
(E.4)

Using the first-order condition to find the optimal incentive coefficient,

\[
\frac{\partial \mathcal{U}_{\alpha, AC}}{\partial \lambda(T)} = e^{-rT} \frac{k^2}{nbr} (1 - e^{-rt}) - \frac{\lambda(T)k^2}{nbr} e^{-2rt} (e^{-T} - 1) - \rho e^{-2rt} \lambda(T)\sigma^2 T = 0.
\]  
(E.5)

Therefore, the optimal incentive and effort under M&A are

\[
\lambda^*_{AC}(T) = \frac{nk^2(e^{-T} - 1)}{k^2(e^{-T} - 1) + n^2 k^2 r T \sigma^2 T}
\]  
(F.1)

\[
\frac{a^*_{\alpha, AC}}{b} = \frac{k^2(e^{-T} - 1)}{k^2(e^{-T} - 1) + n^2 k^2 r T \sigma^2 T} \frac{ke^{-r(T-t)}}{b}.
\]  
(F.1)

\[
\frac{\partial n^*}{\partial k} = \left( k^2(e^{-T} - 1) \right)^{(1/2)} k^2(e^{-T} - 1) > 0,
\]

\[
\frac{\partial n^*}{\partial T} = \frac{k^2(e^{-T} - 1)}{br \sigma^2 T} \frac{T e^{-T} - e^{-T} + 1}{T^2} > 0.
\]  
(F.2)

**G. Proof of Corollary 5**

\[
\lim_{T \to \infty} \int_0^T k \alpha_i dt + G + \Pi_a = \int_0^T b e^{-rt} a^*_i dt - \frac{\rho e^{-rT} \lambda(T)\sigma^2 T - 1}{2}
= \lim_{T \to \infty} \int_0^T k \alpha_i dt + G + \Pi_a = \int_0^T b e^{-rt} a^*_i dt - \frac{\rho e^{-rT} \lambda(T)\sigma^2 T - 1}{2}.
\]

\[
\frac{k^2(e^{-T} - 1)}{k^2(e^{-T} - 1) + n^2 br \sigma^2 T} \frac{ke^{-r(T-t)}}{b} dr + G + \Pi_a
= \lim_{T \to \infty} \frac{k^2(e^{-T} - 1)}{k^2(e^{-T} - 1) + n^2 br \sigma^2 T} \frac{1 - e^{-rT}}{br} + G + \Pi_a
= \frac{b}{2br} + G + \Pi_a.
\]  
(G.1)

**Data Availability**

Data are provided in the manuscript.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

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