A projection formula for the ind-Grassmannian

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Abstract

Let $X = \bigcup_k X_k$ be the ind-Grassmannian of codimension $n$ subspaces of an infinite-dimensional torus representation. If $\mathcal{E}$ is a bundle on $X$, we expect that $\sum_j (-1)^j \Lambda^j(\mathcal{E})$ represents the $K$-theoretic fundamental class $[0_Y]$ of a subvariety $Y \subset X$ dual to $\mathcal{E}^*$. It is desirable to lift a $K$-theoretic “projection formula” from the finite-dimensional subvarieties $X_k$, but such a statement requires switching the order of the limits in $j$ and $k$. We find conditions in which this may be done, and consider examples in which $Y$ is the Hilbert scheme of points in the plane, the Hilbert scheme of an irreducible curve singularity, and the affine Grassmannian of $SL(2, \mathbb{C})$. In the last example, the projection formula becomes an instance of the Weyl-Kac character formula, which has long been recognized as the result of formally extending Borel-Weil theory and localization to $Y$ [49]. See also [10] for a proof of the MacDonald inner product formula of type $A_n$ along these lines.

1 Introduction

Let $X$ be a smooth complex projective variety, let

$$T \circlearrowleft X, \quad T = \mathbb{C}^* = \{z\},$$

be a one-dimensional complex torus action on $X$, and let $\mathcal{E}$ be an equivariant bundle on $X$. The $K$-theoretic Atiyah-Bott-Lefschetz localization formula describes the character of the derived push forward to a point, also known as the equivariant Euler characteristic,

$$\chi_X(\mathcal{E}) = \sum_i (-1)^i \text{ch} H^i_X(\mathcal{E}) = \sum_{F \subset X} \chi_F \left( \mathcal{E}_F \lambda(\mathcal{N}_{X/F})^{-1} \right), \quad (1)$$
Here $H^i$ is the Čech cohomology group, ch is the (Chern) character map, $F$ ranges over the fixed components of the torus action, $N_{X/F}$ is its normal bundle, and $\lambda$ is the usual operation on $K$-theory defined below. See [14] for a reference.

Suppose $Y \subset X$ is an invariant subvariety which is the zero set of an equivariant section of a bundle $\mathcal{E}^*$. Then the fundamental class $[\mathcal{O}_Y]$ is given by $\lambda(\mathcal{E}) \in \tilde{K}_T(X)$, and we have the projection formula

$$
\chi_Y(\gamma) = \chi_X(\gamma \lambda(\mathcal{E})).
$$

(2)

If we apply the localization formula to either side, the resulting identity is not mysterious. The fundamental class $\lambda(\mathcal{E})$ vanishes when restricted to a component $F \subset X$ that does not intersect with $Y$, and the two expressions are in fact equal termwise. However, if a formula is known describing $\chi_X$ as a Laurent polynomial rather than an unworkable rational function, then (2) produces such a formula for $Y$.

We will derive a version of (2) for several examples in which $Y$ is an interesting moduli space, and $X$ is the Grassmannian of codimension $n$ subspaces of an infinite-dimensional torus representation $Z$, defined as an ind-variety, i.e. a union of finite dimensional subvarieties,

$$
\cdots \subset X_{-1} \subset X_0 \subset X_1 \subset \cdots, \quad \bigcup_k X_k = X.
$$

We define the $K$-theory of this space as the inverse limit

$$
\tilde{K}_T(X) = \lim_{\leftarrow} K_T(X_k) \otimes \mathbb{C}[[z]], \quad \chi_X(\gamma) = \lim_{k \to \infty} \chi_X(\gamma_k) \in \mathbb{C}((z)).
$$

For instance, we have a class $[\mathcal{U}] \in \tilde{K}_T(X)$, where $\mathcal{U}_k$ is the tautological rank $n$ quotient bundle on $X_k$.

Consider the fairly general virtual bundle

$$
\mathcal{E} = A\mathcal{U} + B\mathcal{U}^* + C\mathcal{U}\mathcal{U}^* \in \tilde{K}_T(X),
$$

where $A, B, C$ are torus characters, i.e. elements of the equivariant $K$-theory of a point. We then define the class $\mathcal{Y} = \lambda(\mathcal{E})$ by its components

$$
\mathcal{Y}_k = \lim_{w \to 1} \sum_{j \geq 0} (-w)^j \lambda^j(\mathcal{E}) \in \tilde{K}_T(X_k),
$$

(3)

and define $\chi_{\mathcal{Y}}(\gamma) = \chi_X(\gamma \mathcal{Y})$. We will think of $\mathcal{Y}$ as the fundamental class of some subvariety $Y \subset X$, if formally we have $\chi_{\mathcal{Y}}(\gamma) = \chi_Y(i^*\gamma)$, even though $\mathcal{E}$ may not be an honest bundle, and $Y$ may be noncompact, infinite dimensional, or singular.

Our main theorem is the following:
Theorem A. Under certain conditions on $E, X, \gamma$, we have the following analog of (2):

$$
\chi_Y(\gamma) = \sum_{j \geq 0} (-1)^j \chi_X(\gamma \lambda^j(E)).
$$

Essentially, this theorem gives conditions under which we may switch the limits in $j$ and $k$ in (3), which turns out to imply the formula. The essential part of the argument is to calculate the rational function in $C(z, w)$ whose expansion in $w$ is the contribution of the higher Cech cohomology groups to (3), and show that it vanishes to high degree at $z = 0$.

We include the following examples:

1. $Y$ is the Hilbert scheme of $n$ points in the complex plane, and $X = G_{n,R}$, where $R$ is the total space of $\mathbb{C}[x, y]$. The embedding is the map which associates to a subscheme of $\mathbb{C}^2$ the total space its ideal. The projection formula becomes a power series expansion with integer coefficients for the Euler characteristic of a subbundle of $U \otimes m$, where $U$ is the tautological rank $n$ bundle on $Y$. This example may also be extended to the moduli space $M_{r,n}$ of higher rank sheaves (instantons), see [39] for a definition.

2. $Y$ is the Hilbert scheme of a plane curve singularity $y^2 = x^3$, and $X = G_{n,R}$ with

$$
R = \mathbb{C}[x, y]/(y^2 - x^3) \cong \mathbb{C}[u^2, u^3].
$$

Again the projection formula produces a power series formula form the euler characteristic. Since $Y$ is singular, $\chi_Y$ must be defined using virtual localization.

3. $Y$ is the affine Grassmannian of the loop group of $SL(2, \mathbb{C})$, and $X$ is the Sato Grassmannian of half infinite-dimensional subspaces of a faithful representation of $L\mathbb{C}^2$. The projection formula produces an instance of the Weyl-Kac character formula. This circumvents the technical ideas behind an idea that was discussed by Segal in [40], and has been studied by several authors, including generalizations to the analogous flag varieties [28, 51, 52].

In [10], the author also gives a proof of the MacDonald inner product formula of type $A_n$ in this way, which is too involved to reproduce here. From this point of view, the factorization of the inner product is explained as coming from a localization sum concentrated at a
single fixed point point, together with the form of the Pieri rules for MacDonald polynomials.

The motivations for this paper have do to with a fascinating and well studied interplay between the Hilbert scheme of points on a surface, representation theory, and modular forms. In many different studies, geometric correspondences between the Hilbert schemes of different points induce an action of various infinite-dimensional Lie algebras on $H^*$, the direct sum of the cohomology groups of $\text{Hilb}_n S$ over all $n$, see [4, 11, 19, 32, 30, 37] to name a few. There is a related story in $K$-theory which in many cases is based on Haiman’s character theory of the Bridgeland King and Reid isomorphism which identifies $K(\text{Hilb}_n \mathbb{C}^2)$ as an inner-product space with the ring of symmetric polynomials in infinitely many variables [6, 12, 43, 48]. In some cases, the resulting character theory leads to functional properties of the generating function of cohomological or $K$-theoretic constants in a variable $q$, over the number of points $n$ [8, 11, 27, 55].

These phenomena are closely related to a physical conjecture known as AGT (Alday, Gaiotto, Tachikawa) [1], which connects correlation functions in four dimensional gauge theory with a certain Liouville theory. In fact, there are two current mathematical proofs of this conjecture that proceed along these lines [36, 54]. It would be very desirable mathematically and physically to discover integrals on a larger moduli space which restrict to both sides of this dictionary under different specializations of the equivariant parameters. The motivation in extending the projection formula is that interesting integrands on a Grassmannian manifold are simply easier to construct than interesting moduli spaces. Haiman’s theory makes sense when the moduli space is the Hilbert scheme, whereas the structures on the cohomology and $K$-theory of $\mathcal{M}_{r,n}$ also lead to interesting character theory. A fundamental example is the action of the Kač-Moody algebra $\widehat{sl}_r \mathbb{C}$ on $H^*(\mathcal{M}_{r,n})$ [32, 42, 43, 44], which prompted the Kač-Moody example.

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2 Plethysm

Let $K_T(X)$ denote the complex equivariant $K$-theory of a smooth complex projective variety with an action of a torus $T = (\mathbb{C}^*)^d$. Let
\(\lambda^i\) denote the usual operation defined on bundles by
\[
\lambda^i([E]) = [\Lambda^i(E)].
\]
The total operation is defined by
\[
\lambda(\mathcal{E}) = \lim_{w \to 1} \lambda^i(w\mathcal{E}) = \lim_{w \to 1} \sum_j (-1)^j w^j \lambda^j(\mathcal{E})
\] (4)
where the limit is the analytic continuation to \(w = 1\) of the rational function defined by the right hand side for \(w\) near zero. The limit exists if \(\gamma = [\mathcal{E} - \mathcal{F}]\) for honest bundles \(\mathcal{E}, \mathcal{F}\), with \(\lambda(\mathcal{F})\) invertible, and equals \(\lambda(\mathcal{E})\lambda(\mathcal{F})^{-1}\).

If \(\gamma = \sum_I a_I x^I\) for \(a_I \in \mathbb{Z}\), and \(x^I\) are monomials in some set of indeterminants, called plethystic variables, we define
\[
\lambda(\gamma) = \prod_I (1 - x^I)^{a_I}, \quad \gamma^* = \sum_I x^{-I},
\]
\[
\dim(\gamma) = \sum_I a_i, \quad \det(\gamma) = \prod_I x^{a_i I}.\tag{5}\]
Given a symmetric polynomial \(f \in \Lambda_n\), we also have a homomorphism defined in the elementary symmetric function basis by
\[
\gamma \mapsto f(\gamma), \quad e_{i_1} \cdots e_{i_k}(\gamma) = \lambda^{i_1}(\gamma) \cdots \lambda^{i_k}(\gamma).
\]
We may think of \(\gamma\) as an element of \(K_T(pt)\), when the plethystic variables are the torus variables \(z_i\). In this paper, every variable will be considered plethystic, meaning it counts as an indeterminant for the purposes of (5). Furthermore, we will often identify a torus representation and its character in \(\mathbb{Z}_{\geq 0}[z_i^{\pm 1}]\), denoting both by the same letter.

## 3 The Grassmannian

Suppose \(Z\) is a representation of \(T\) of dimension \(d\), and consider the Grassmannian variety of codimension \(n\) subspaces of \(Z\),
\[
X = G_{n,Z} = \{ V \subset Z \mid \text{codim}(V) = n \}.
\]
There is a tautological bundle \(\mathcal{V}\) whose fiber over \(V \subset Z\) is \(V\) itself, and a rank \(n\) quotient bundle \(\mathcal{U} = \mathcal{Z}/\mathcal{V}\), where \(\mathcal{Z} = G_{n,Z} \times Z\). The action
of $T$ on $Z$ induces an action on the Grassmannian, and on the above bundles. We may consider the characters of the Cech cohomology groups
\[ \chi^i(E) = \text{ch} H^i_X(E), \quad \chi = \sum_i (-1)^i \chi^i. \]

Only $\chi_0$ however, descends to a map on $K$-theory.

Let $P = G_{1,Z}$, and let us define a linear map $\xi: x^m \mapsto \chi^0_x(U^m) \in \mathbb{C}[z_1^\pm 1], \quad \epsilon = 0, 1, \ldots$

where $\chi^0 = \chi$. The answer is well known to be
\[ \xi_x^0 f(x) = [x^0]^j \mathcal{P} x^{-1} f(x) \chi_x, \quad \xi_x^d f(x) = [x^0]^j \mathcal{P} f(x) \chi_x, \]
\[ \chi_x = \lambda(Z x^{-1})^{-1}, \quad \xi_x^i f(x) = 0, \quad i \notin \{0, d\}. \] (6)

where $[x^i]$ denotes the coefficient of $x^i$, and
\[ \mathcal{P} x^\pm 1 : \mathbb{k}(x) \hookrightarrow \mathbb{k}((x^{\pm 1})) \subset \mathbb{k}[[x^{\pm 1}]] \]
is the map that sends a rational function over $\mathbb{k}$ to its Laurent series about $x = 0$ or $\infty$ respectively.

Equivariant localization gives a second expression for the Euler characteristic,
\[ \xi_x f(x) = \sum_j \xi_{x,j} f(x), \quad \xi_{x,j} f(x) = \chi \left( \lambda(N_{F_j/F_j})^{-1} \iota_{x,j} f(x) \right), \]
\[ \iota_{x,j} : x^m \mapsto \iota_{x,j}^* (U^m) \in K(F_j) \otimes \mathbb{C}(z), \] (7)
where $F_j = G_{1,Z_j}$ are the torus fixed components of $P$, and $Z_j$ is the invariant subspace of $Z$ with character $z^j$. We have that
\[ K(F_j) \cong \mathbb{C}[y]/(y^c), \quad c = \dim F_j. \]
The restriction map is given by
\[ \iota_{x,j}^* (x^m) = z^{jm} p_y^j (1 + y)^m \in \mathbb{C}[[y]] \rightarrow \mathbb{C}[[y]]/((y^c)). \] (8)
The pushforward map is
\[ \chi : K(F_j) \rightarrow \mathbb{C}, \quad y^i \mapsto \left( \frac{c - 1}{i} \right). \] (9)
There is an elegant expression for the Euler characteristic in terms of residues, which follows easily from the formulas above:

\[ \xi_0^0(f(x)) = -\text{Res}_{x=\infty} g(x), \quad \xi_d^d(f(x)) = -(-1)^d \text{Res}_{x=0} g(x), \]

\[ \xi_{x,j}(f(x)) = \text{Res}_{x=z_j} g(x), \quad g(x) = x^{-1} f(x)X_x, \]  \hspace{1cm} (10)

where

\[ \text{Res}_{x=c} f(x) = [x^{-1}] f(x + c) \]

is the algebraic residue operation. Then

\[ \xi_x^0 + (-1)^d \xi_x^d = \xi_x = \sum_j \xi_{x,j} \]  \hspace{1cm} (11)

corresponds to the fact that the sum of the residues of a meromorphic function on \( \mathbb{CP}^1 \) equals zero.

There is a well known formula for the general case to in terms of the case \( n = 1 \):

\[ \chi^\epsilon_X(f(\mathbb{U})) = \frac{1}{n!} \xi_{x_1}^\epsilon \cdots \xi_{x_n}^\epsilon f(x_1, \ldots, x_n) \Delta_x, \]  \hspace{1cm} (12)

\[ \Delta_x = \lambda \left( \sum_{i \neq j} x_i x_j^{-1} \right), \quad \epsilon \in \{\emptyset, 0\}. \]

The \( \epsilon = \emptyset \) case is a simple instance of a theorem of Shaun Martin [35] for general symplectic quotients, see also the Jeffrey-Kirwan residue formula [25]. The \( \epsilon = 0 \) case is an application of Borel-Weil theory, we refer the reader to [13].

4 The fundamental class

Suppose now that \( T \) is one-dimensional with torus parameter \( z \), and define a finite-dimensional torus representation by its character

\[ Z = \sum_i d_i z^i \in \mathbb{Z}_{\geq 0}[z^{\pm 1}]. \]

Let \( k, k' \) respectively denote the largest and smallest \( i \) such that \( d_i \neq 0 \), and let \( d = \dim(Z) \). Fix a positive integer \( n \), and let \( X = G_{n,Z} \).

Now define an element \( X = X_\mathbb{U} \in K_T(X)[[w]] \) by

\[ X_\gamma = \sum_i (-w)^i \lambda^i (E_\gamma), \quad E_\gamma = A\gamma + B\gamma^* + C\gamma^*, \]  \hspace{1cm} (13)

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for some Laurent polynomials
\[ A = \sum_i a_i z^i, \quad B = \sum_i b_i z^i, \quad C = \sum_i c_i z^i \]
with integer coefficients. Then we have that \( X = P_w Y_w \) for some class
\[ Y_w \in K_T(X) \otimes \mathbb{C}(z, w), \]
which can easily be seen using localization
\[ K_T(X) \otimes \mathbb{C}(z, w) \cong \bigoplus_{F \subseteq X^T} K(F) \otimes \mathbb{C}(z, w). \]
If this class is well defined at \( w = 1 \), we define
\[ Y = Y_1 \in K_T(X) \otimes \mathbb{C}(z). \]
Let us define
\[ \chi_Y(\gamma) = \chi_X(\gamma Y) \in \mathbb{C}(z), \]
which should be thought of as the Euler characteristic of \( \gamma \) over a subvariety \( Y \subset X \) which is the intersection of a section of \( \mathcal{E}^* \) with the zero section, even though \( \mathcal{E} \) is not an honest bundle, and such a variety may not exist. Now suppose that \( \gamma = \gamma_{u,m} \), where
\[ \gamma_{A,m} = \det(A)^m f(A), \quad f \in \Lambda, \quad m \in \mathbb{Z}. \] (14)
It follows easily from (10) and (12) that there are rational functions satisfying
\[ f^\epsilon(z, w) \in \mathbb{C}(z, w), \quad P_w f^\epsilon(z, w) = \chi^\epsilon(\gamma X), \]
when \( \epsilon \) is zero or blank. For instance, \( f(z, w) = \chi_X(\gamma Y_w) \). Define a rational function whose power series in \( w \) measures the contribution from the higher cohomology groups,
\[ g(z, w) = f(z, w) - f^0(z, w). \]
The following lemmas study the expansion of \( g(z, w) \) in the \( z \) direction.

**Lemma 1.** Let \( x \) be a variable. We have
\[ \nu_z(\xi^0_x(x^m X_x)) \geq o_k', \quad \nu_z(\xi^d_x(x^m X_x)) \geq o_k, \quad \nu_z(\xi_{x,i}(x^m X_x)) \geq o_i, \]
\[ o_i = mi + \sum_{j \leq -i} a_j(i + j) + \sum_{j \leq i} b_j(j - i) - \sum_{j \leq i} d_j(j - i). \]
Proof. Let us prove the first bound. Consider the power series
\[ P_x(A) \big|_{x=\lambda x^{-1}} = \sum_i f_i(z, w) x^i, \quad A = X_{\lambda} A(Z x^{-1})^{-1}. \]
Using (10), it suffices to show that
\[ \nu_z(f_i(z, w)) \geq o_k. \]
To prove this we simply study each factor in \( A \) separately and use
\[ \nu_z(fg) \geq \nu_z(f) + \nu_z(g). \]
The others are similar.

Lemma 2. Suppose condition (d) of theorem 1 below is satisfied. Then
\[ g(z, w) = \sum_{r \geq 1} (-1)^r d \binom{n}{r} f^{r,n-r}(z, w), \]
where
\[ f^{r,s}(z, w) = \sum_{j_1, \ldots, j_s} \text{Res}_{\{y_q = z^q\}} \text{Res}_{\{x_p = 0\}} \Omega_{r,s}, \]
\[ \Omega_{r,s} = \Omega_{x_1 + \cdots + x_r + y_1 + \cdots + y_s}, \quad \text{Res}_{\{e_1, \ldots, e_k\}} = \text{Res}_{e_1} \cdots \text{Res}_{e_k}, \]
\[ \Omega_A = \frac{1}{n!} \gamma_A \lambda(w \mathcal{E}_A) \lambda(Z A)^{-1} \Delta_A. \]
Proof. Using (10), (11), (12), we may write
\[ \mathcal{P}_w g(z, w) = \sum_{r \geq 1} (-1)^r d \binom{n}{r} \mathcal{P}_w f^{r,n-r}(z, w), \]
where \( f^{r,s}(z, w) \) is defined by
\[ \mathcal{P}_w f^{r,s}(z, w) = \xi_{x_1} \cdots \xi_{x_r} \xi_{y_1} \cdots \xi_{y_s} \mathcal{X}_{x_1 + \cdots + x_r + y_1 + \cdots + y_s} = \]
\[ \sum_{j_1, \ldots, j_s} \text{Res}_{\{y_q = z^q\}} \text{Res}_{\{x_p = 0\}} \mathcal{P}_w \Omega_{r,s}. \]
It suffices to prove that
\[ \left( \text{Res}_{\{y_q = z^q\}} \text{Res}_{\{x_p = 0\}} \mathcal{P}_w - \mathcal{P}_w \text{Res}_{\{y_q = z^q\}} \text{Res}_{\{x_p = 0\}} \right) \Omega_{r,s} = 0, \]
so that $f^{r,s}(z,w) = \tilde{f}^{r,s}(z,w)$.

We now claim the following commutation relations between $P_w$ and the residue

$$\left(\text{Res}_{x_{p}=0} P_w - P_w \text{Res}_{x_{p}=0}\right) \Omega_{r,s} = P_w \sum_{q,i} \text{Res}_{y_{q}=wz^{i}} \Omega_{r,s}, \quad (16)$$

$$\left(\text{Res}_{y_{q}=z^{j}} P_w - P_w \text{Res}_{y_{q}=z^{j}}\right) \text{Res}_{x_{p}=0} \Omega_{r,s} = 0. \quad (17)$$

Both are algebraic facts which may be described in terms of formal distributions, but it is simpler to imagine the residues about zero as contours about $|x_{p}| = \epsilon$. The first commutator may be thought of as the residues picked up from swapping the range $|w| \ll |x_{p}|$ for $|x_{p}| \ll |w|$. The second is also straightforward.

We have that $\text{Res}_{x_{p}=wz^{i}} \Omega_{r,s}$ vanishes to order

$$a_{-i-j} + b_{i+j} + c_{i} - d_{j} + 1$$

at $y_{q} = z^{j}$. By condition (11) this number is nonnegative, so that

$$\text{Res}_{y_{q}=z^{j}} \text{Res}_{x_{p}=wz^{i}} \Omega_{r,s} = 0. \quad (18)$$

Furthermore, taking additional residues at $x_{i_{p}} = 0$ can only increase the degree of vanishing. Applying this to (16) and combining with (17) establishes (15), proving the lemma.

\[\square\]

## 5 The main theorem

Now suppose that $Z$ is an infinite-dimensional representation, defined via its character

$$Z = \sum_{i} d_{i}z^{i} \in \mathbb{Z}_{\geq 0}(z).$$

Suppose furthermore that $A, B \in \mathbb{Z}(z)$, again with coefficients $a_{i}, b_{i}$, respectively. Let $X = G_{n,Z}$ be the ind-Grassmannian of codimension $n$ subspaces of $Z$, taken as a limit of subspaces

$$X = \bigcup_{k} X_{k}, \quad X_{k} = G_{n,Z_{\leq k}},$$

where $Z_{\leq k}$ is the direct sum of the subspaces with torus weight $j \leq k$. 

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We set 
\[ \tilde{K}_T(X) = \lim_{\leftarrow} \tilde{K}_T(X_k), \quad \tilde{K}_T(X_k) = K_T(X_k) \otimes \mathbb{C}[[z]]. \]

The above inverse system is determined by the pullback maps $i_{ab}^*$ where

\[ i_{ab} : X_a \to X_b, \quad V \mapsto \pi_b^{-1}(V), \quad a \leq b, \]

and $\pi_{ab} : Z_{\leq b} \to Z_{\leq a}$ is the projection map. Given an element $\gamma \in \tilde{K}_T(X)$, with components $\gamma_k$, let us define

\[ \chi_X(\gamma) = \lim_{k \to \infty} \chi_{X_k}(\gamma_k) \in \mathbb{C}((z)), \]

when that limit exists. One way to guarantee existence is to have that $i_{F}^* \gamma = 0$ for all but finitely many fixed components $F \in X^T$. If this is the case, we will say that $\chi_X(\gamma)$ is well defined.

If it exists, we define an element $Y \in \tilde{K}_T(X)$ by

\[ Y_k = \lim_{l \to \infty} \mathcal{P}_z Y_{k,l,1} \in \tilde{K}_T(X_k), \]

where $Y_{k,l,w}$ is the class obtained by replacing $A, B$ by the finite dimensional spaces $A_{\leq l}, B_{\leq l}$ in the definition of $Y_w$.

**Theorem 1.** Suppose $\gamma = \gamma_{U,m}$, and the following conditions are satisfied:

- a) $a_i \geq 0$ for $i \leq -k'$.
- b) $b_i \geq 0$ and $b_i = d_i$ for large enough $i$.
- c) $c_i = 0$ for $i \leq 0$, and $c_i = 0$ for large enough $i$.
- d) For any weights $i, j \in \mathbb{Z}$ with $d_j \neq 0$, $c_i < 0$, we have
  \[ a_{-i-j} + b_{i+j} + c_i - d_j + 1 \geq 0. \]
- e) Both $Y$ and $\chi_Y$ are defined in the sense described above.

Then for large enough $m$ we have the projection formula,

\[ \chi_Y(\gamma) = \sum_{j \geq 0} (-1)^j \chi_X(\gamma \lambda^j(\xi)). \quad (19) \]
Proof. Since \( f(U) \) is arbitrary, we may assume without loss of generality that \( A \in \mathbb{Z}_{\geq 0}[z^{\pm 1}] \). Let \( f_{k,l}(z, w) \) and \( g_{k,l}(z, w) \) denote the rational functions from the last section with \( Z \leq k, B \leq l \) in place of \( Z, B \), and let

\[
f_k(z, w) = \lim_{l \to \infty} f_{k,l}(z, w),
\]

pointwise. By condition \( \mathbf{c} \), \( f_k(z, w) \) is defined at \( w = 1 \). By definition, the limit over \( k \) of \( \mathcal{P}_z f_k(z, 1) \) agrees with the left hand side of (19).

We next claim that the limit of \( f_0(z, w) \) agrees with the right hand side of (19). Since the higher cohomology of \( f(U) \) vanishes for large enough dimension of \( Z \leq k \), we have

\[
\lim_{k \to \infty} \mathcal{P}_z f_k(z, w) = \lim_{k \to \infty} \mathcal{P}_z f_0(z, w),
\]

(20)

Using condition \( \mathbf{c} \), lemma 1, and (12), we can see that

\[
\lim_{i \to \infty} \nu_z(e_{k,i}(z)) = \infty, \quad \mathcal{P}_w f_0(z, w) = \sum_i e_{k,i}(z) w^i.
\]

(21)

This shows that the left side of (20) converges in \( \mathbb{C}((z)) \) at \( w = 1 \). It also shows that

\[
(\mathcal{P}_z \mathcal{P}_w - \mathcal{P}_w \mathcal{P}_z)f_k(z, w) \in \mathbb{C}((w))(z),
\]

whereas a priori, it is only an element of \( \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]] \). On the other hand, by multiplying by the denominator of \( f_0(z, w) \), we see that it is a zero divisor, and so must be zero. But the left side of (20) agrees with the right side of (19) at \( w = 1 \), proving the claim.

We now prove that they are equal. By lemma 1 we find that

\[
\nu_z(f_{r,s,k}(z, w)) \geq rmk + ak + b
\]

for some constants \( a, b \). The constants may be chosen independently of \( l \), by condition \( \mathbf{b} \), and because \( o_k \) depends only on the differences \( b_i - d_i \), for \( i \leq k \). By lemma 2, \( g_{k,l}(z, w) \) is a linear combination of \( f_{r,s,k}(z, w) \) for \( r \neq 0 \), so that

\[
\lim_{k \to \infty} \mathcal{P}_z f_k(z, w) - \mathcal{P}_z f_0(z, w) = \lim_{k \to \infty} \lim_{l \to \infty} \mathcal{P}_z g_{k,l}(z, w) = 0,
\]

including the value \( w = 1 \), as long as \( m > -a \).
6 Examples

6.1 The Hilbert scheme of points in the plane

Let $Y = \text{Hilb}_n \mathbb{C}^2$, the Hilbert scheme of $n$ points in the plane. There is a standard torus action on $Y$ induced by pullback of ideals from the action on the plane

$$(z_1, z_2) \cdot (x, y) = (z_1^{-1}x, z_2^{-1}y).$$

(22)

The fixed points of $Y$ are the monomial ideals indexed by Young diagrams $I_\mu = (x^{\mu_1}, x^{\mu_2}, ..., y^{\ell(\mu)}) \subset R = \mathbb{C}[x, y]$. The character of the cotangent space to this point is a polynomial in $z_i$ with nonnegative integer coefficients summing to $\dim(Y) = 2n$. By deformation theory and a standard Čech cohomology argument, it is given by

$$T_\mu Y = \chi(R, R) - \chi(I_\mu, I_\mu),$$

(23)

where $\chi$ is the Euler characteristic

$$\chi(F, G) = \sum_{i} (-1)^i \text{ch Ext}^i_R(F, G).$$

There is an interesting formula for this polynomial in terms of the arm and leg lengths of boxes in $\mu$, which we will not need. See [11, 38] for a reference on this calculation.

Now let $Z$ be the total space of $R$, so that

$$Z = \mathcal{P}_{z_1, z_2} M^{-1}, \quad M = (1 - z_1)(1 - z_2).$$

Let $X = G_{n, Z}$, and let

$$A = -z_1 z_2, \quad B = Z - 1, \quad C = M - 1,$$

with $\mathcal{E}$ and $\mathcal{Y}$ as in the last section. There is an injection $Y \hookrightarrow X$ determined by sending an ideal to its total space in $Z = R$, which comes up in the construction of $Y$. The images of the fixed points are

$$V_\mu = H^0(I_\mu) \subset Z, \quad U_\mu = Z/V_\mu = \sum_{(i, j) \in \mu} z_1^i z_2^j,$$

where $(i, j)$ are the coordinates of a box in $\mu$. 

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Lemma 3. Suppose $U \in X^T$ is an invariant subspace.

a) We have that $\lambda(E_U)$ vanishes unless $U = U_\mu$ for some $\mu$.

b) If $U = U_\mu$, then

$$T_\mu^* X - T_\mu^* Y = E_\mu, \quad E_\mu = E_{U_\mu}.$$ 

Proof. For part (a) it suffices to show that the constant term of $E_U$ is positive unless $U = U_\mu$, in which case it is zero. Consider the graph whose vertices are $\mathbb{Z}^2 \subset \mathbb{R}^2$, and whose edge set $E$ connects horizontal and vertical neighbors. Color each box with lower-left corner $(i, j)$ white if $z_1^i z_2^j$ is a weight of $V$, and black otherwise. Define subsets by

$$X_0 = \{ v \in \mathbb{Z}^2 : v(\nearrow) \text{ is black, } v(\swarrow) \text{ is white}\}$$

$$X_1 = \{ e \in E : e(\nearrow) \text{ is black, } e(\swarrow) \text{ is white}\}$$

Here $v(\nearrow)$ is the upper-right neighboring box to $v$, $e(\nearrow)$ is the upper or right neighboring box to the edge $e$ depending on whether $e$ is horizontal or vertical, and similarly for the southwest arrow.

Expanding $E$, we see that the constant term is

$$[z_1^0 z_2^0] E_U = x_1 - x_0, \quad x_i = |X_i|.$$ 

Now notice that every vertex in $X_0$ is the endpoint of exactly two edges in $X_1$, but each edge in $X_1$ always has at most two endpoints in $X_0$, proving that $x_1 - x_0 \geq 0$. If $U$ does not come from a Young diagram, then the set $X_1$ is nonempty, and there must be some edge in $X_1$ whose endpoints are not both in $X_0$, leading to strict inequality.

Part (b) may be deduced easily from (23), and

$$\chi(I_\mu, I_\nu) = z_1^{-1} z_2^{-1} MV_\mu^* V_\nu.$$ 

Now restrict to a one-dimensional torus $z_i = z^{a_i}$, where the $a_i$ are large enough that the fixed points of $Y$ are isolated. Lemma 3 combined with localization on each fixed component $F$ with respect to the two-dimensional torus proves that condition of the theorem is satisfied. Condition (d) holds because $Z$ has the same character as $R$, $B$ is the character of the total space of its maximal ideal $m$, and $xR$ and $yR$ are contained in $m$. The others are obvious.
By lemma 3, we see that \( \chi_{\text{Hilb}}(\gamma_U, m) = \chi_Y(\gamma_U, m) \), where \( U \) also denotes the tautological rank \( n \) bundle on \( Y = \text{Hilb}_n \), which is pulled back from \( X \). By theorem 1, we have

\[
\chi_{\text{Hilb}}(\gamma_U, m) = \sum_j (-1)^j \chi_X(\gamma \lambda^j(\mathcal{E})) \in \mathbb{Z}[[z]]
\] (24)

for large enough \( m \). Since the answer is a rational functions of \( z_i \), it is determined by its values on the restricted torus. We may therefore drop the assumption that \( z = z^a_i \), and have an equality of functions of two distinct torus variables \( z_i \). It may be checked that both sides are given by elements of \( \mathbb{C}(z_i)[z^m_i] \) for \( m \geq 0 \), and so (24) holds for all nonnegative \( m \). The point of this formula is the the right hand side is given explicitly by a sum of power series with integer coefficients.

### 6.2 The Hilbert scheme of a singular curve

Let \( C \) denote the singular curve \( y^2 = x^3 \), and consider the action

\[
T = \mathbb{C}^* \times C, \quad z \cdot (x, y) = (z^{-2}x, z^{-3}y).
\]

Let \( Y \) denote the Hilbert scheme of \( n \) points in this curve, whose points correspond to ideals in

\[
R = \mathbb{C}[x, y]/(y^2 - x^3) \cong \mathbb{C}[u^2, u^3],
\]

with \( \dim_{\mathbb{C}} R/I = n \). The torus fixed points of \( Y \) are those of the form

\[
I_S = \bigoplus_{i \in S} \mathbb{C} \cdot u^i \subset R,
\]

for \( S \) a sub-semigroup of \( \{0, 2, 3, 4, \ldots\} \).

There is an injection \( Y \hookrightarrow X \), where the data for \( X \) is given by

\[
Z = \text{ch} R = p_z(1 - z^6)M^{-1}, \quad M = (1 - z^2)(1 - z^3).
\]

Now let

\[
A = -z^5, \quad B = Z + z^6 - 1, \quad C = M - 1.
\]

We find that \( \gamma_U \) vanishes at all fixed points in \( X \) except those whose weights form a semigroup \( S \). We would find that \( T_S^*X - \mathcal{E}_S \) does not consist entirely of nonnegative weights, but that the signed dimension is always \( n \). This corresponds to the fact that the Hilbert scheme of
points on this curve only has a virtual tangent bundle of expected dimension $n$. Its character may be calculated by realizing $Y$ as an lci subvariety of the Hilbert scheme of $n$ points in the plane. For a reference, see [50].

In a similar way to the last subsection, theorem 1 gives a power series formula for the Euler characteristic, but now for $m \geq 1$.

### 6.3 The affine Grassmannian

Let $G = SL(2, \mathbb{C})$, and consider the affine Grassmannian

$$Y = LG_{\mathbb{C}}/L^+G_{\mathbb{C}},$$

where $LG$ is the space of maps from the circle into $G$, and $L^+G$ are those maps which extend to a holomorphic function in the disc of radius 1.

In [49], Segal noted that there should be a proof of the Weyl-Kac character formula using this variety, which is analogous of the well-known geometric proof of the Weyl character formula using $K$-theoretic localization combined with Borel-Weil-Bott, see [14]. He also pointed out that there was a gap in the reasoning due to the fact that $Y$ is infinite-dimensional with singular closure, and the explanation that the higher cohomology groups vanish. This topic, and generalizations to the related flag varieties have been studied by several authors, including [28, 51, 52].

We now demonstrate how theorem 1 can be used to circumvent these two difficulties in the case of the Jacobi triple product formula, which corresponds to the basic representation for $G = SL(2, \mathbb{C})$. It would be interesting to see how far this approach generalizes. Let us ignore technicalities and simply motivate the choice of data for theorem 1. There is an action of a two-dimensional torus on $Y$ by

$$(g \cdot f)(x) = Ad \left( \begin{array}{c} z^{-1} \\ z \end{array} \right) \cdot f(qx), \quad g = (q, z).$$

Ignoring the infinite-dimensionality of $Y$, we can write down the character of the cotangent bundle to this space at a fixed point as follows. The cotangent bundle at the image of the identity in $Y$ is given by

$$T^*_1Y = (Lg_{\mathbb{C}}/L^+g_{\mathbb{C}})^* = \frac{q}{1-q}(z^2 + 1 + z^{-2}).$$

The character at a general fixed point can be extracted from by applying elements of the affine Weyl group.
Let $\mathcal{H} = L\mathbb{C}^2$, the Hilbert space of maps to $\mathbb{C}^2$. Then $LG_C$ acts in the obvious way on this space, and $L^+G_C$ is precisely the subgroup that preserves the subspace $V \subset \mathcal{H}$ of all maps which are holomorphic at the origin. The action on $\mathcal{H}$ induces an inclusion $Y \subset X$ in which 1 maps to $V$, where $X$ is the Sato Grassmannian of half-infinite dimensional subspaces of $\mathcal{H}$, by taking the orbit space of $V$. The character of the cotangent bundle at $V$ is given by

$$\text{ch} T^*_V X = \frac{q}{(1-q)^2} (z^2 + 2 + z^{-2}),$$

(26)

Now let us derive the Jacobi triple product. For each $n$, let

$$M = 1 - q, \quad W = q^{-n}(z + z^{-1}), \quad Z = q^{-n}(z + z^{-1}) \partial_z M^{-1},$$

$$A = 0, \quad B = Z - W, \quad C = M - 1, \quad X^{(n)} = G_{2n, Z},$$

so that $Z \subset \mathcal{H}$, and includes the whole space as $n$ becomes large. As in section 6.1, we find that the projection formula holds for the two-dimensional torus, and that $m \geq 0$ is sufficient.

We may check that $\lambda(\mathcal{E}^{(n)})$ vanishes at all fixed points of $X^{(n)}$ except those whose complementary subspace has character

$$U_k = \sum_{-n \leq i \leq k-1} zq^i + \sum_{-n \leq i \leq -k-1} z^{-1}q^i,$$

and the character at such a point satisfies

$$\lim_{n \to \infty} \left( T^*_k X^{(n)} - \mathcal{E}_k^{(n)} - T^*_k Y \right) = \frac{q}{1-q}.$$

Now taking the limit over $n$ of (19) gives

$$\sum_k (q; q)_\infty^{-1} \theta(z^2, q)^{-1} \left( z^{4k}q^{2k^2+k} - z^{4k-2}q^{2k^2-k} \right) =$$

$$\sum_j (-1)^j q^j \lim_{n \to \infty} \chi_X \left( \lambda^j(T^*_n X) \right) = (q; q)_\infty^{-1},$$

(27)

where

$$(x; q)_\infty = \prod_{i \geq 0} (1 - xq^i), \quad \theta(x; q) = (q; q)_\infty (xq; q)_\infty (x^{-1}; q)_\infty.$$

The second equality follows from

$$\chi_{\text{Gr}(k,n)} \left( \lambda^j(T^*_n) \right) = (-1)^j p(j),$$

for sufficiently large $k, n - k$, where $p(j)$ is the number of partitions of $j$, and the answer holds equivariantly for any group action on $\mathbb{C}^n$. 17
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