I. INTRODUCTION

A quantum system may undergo unitary evolution in the simple case that the system is closed. However, many practical experiments involve systems for which this is not a good approximation. In these cases, the evolution is an open-system evolution where the external influences can be very important. When these systems are being considered for quantum information processing and/or transfer, understanding the external influence and modeling it is often a necessary part of the process of evaluating experiments.

The need for describing such evolution of a particular system using dynamical maps was identified by Sudarshan, Mathews, and Rau [1]. Their work provides a mapping from one density operator to another which can be used to describe a quite general open-system evolution. Some time later Kraus [2] restricted consideration to maps which are completely positive, a useful but not necessary assumption for open system evolution. The importance of the complete positivity assumption has recently been discussed in the literature [3–7]. In particular, it has recently been shown that vanishing quantum discord [8] is sufficient [9] (see also [4]) and necessary [7] for completely positive maps. Furthermore, the conditions an initial state must satisfy in order for a map to be positive were also found in Rev. [10].

In the case of a completely positive map, it is well-known that a unitary degree of freedom exists in the OSR. This freedom for completely positive maps is noted and used by Nielsen and Chuang [11] for error correction, modeling, and other applications to quantum information processing. Most notably perhaps is the application to quantum error correcting codes. In that case it was shown that the freedom can be used to model errors in a very useful way by choosing a complete basis for errors which acts as a basis for the operators in the operator-sum representation (OSR). Recently it was shown that complete positivity is not a necessary assumption for quantum error correcting codes [10].

In this article, we identify the pseudo-unitary degree of freedom in the OSR for maps that are not necessarily completely positive. We first provide some background and the origin of the OSR. We then review the unitary degree of freedom for completely positive maps. Finally, we show that the freedom for the more general case of maps which are not necessarily completely positive is pseudo-unitary.

II. BACKGROUND

In 1961, Sudarshan, Mathews, and Rau (SMR) described what they called “dynamical maps” [1]. These are maps from one density matrix (or density operator) to another with no other restrictions. The authors were able to arrive at a form for the map which is quite general and provides conditions for the map to be positive. We provide their argument here as a basis for what follows.

A. Dynamical Maps and the SMR Decomposition

Following the treatment of Sudarshan, Mathews, and Rau, let us consider a quite general mapping from one Hermitian matrix to another of the form

\[ \rho' = A \rho, \]

or more explicitly

\[ \rho'_{rs} = A_{r's',rs} \rho_{rs}. \]

It is apparent that this is a very general linear map \( A \) maps elements of \( \rho \) to elements of another operator \( \rho' \). For this reason it is sometimes referred to as a superoperator. Now, we recall that the density matrix is required to be Hermitian, positive semi-definite, and have trace one. Respectively, we write

\[ \rho = \rho^\dagger, \quad \rho \geq 0, \quad \text{Tr} \rho = 1. \]

These ensure a valid probability interpretation of the density matrix. One can show that, given the properties specified in Eq. (3), the mapping \( A \) must have the
B. Hermitian Preserving Maps

Here we consider the case where the map does not necessarily correspond to a trace-preserving, completely positive map. We will consider maps which are hermiticity-preserving, i.e. they take hermitian matrices to hermitian matrices. (They are also sometimes called Hermitian-preserving.) Thus a general map \( \Phi \) can be expressed as

\[
\Phi(\rho) = \sum_k \eta_k C_k \rho C_k^\dagger,
\]

and again, when the \( \eta_k \) are all positive, \( \Phi \) can be expressed as

\[
\Phi(\rho) = \sum_k A_k \rho A_k^\dagger.
\]

Note that if the \( \eta_k \) are not all positive, then we may take \( \eta_k = (\pm 1) \) and the square-roots of the positive magnitudes, \( \sqrt{|\eta_k|} \) may be absorbed into the \( C_k' \). (The \( \eta_k \) are real since \( B \) is Hermitian.) Therefore, we let \( C_k = \sqrt{|\eta_k|} C_k' \) and express Eq. (15) as

\[
\Phi(\rho) = \sum_{k=1}^{p+q} \eta_k C_k \rho C_k^\dagger,
\]

where there are a total of \( p+q \) terms in the sum with \( \eta_k = +1 \), for \( k = 1, \ldots, p \) and \( \eta_k = -1 \), for \( k = p+1, \ldots, p+q \). This is an expression of the fact that any Hermiticity-preserving map can be written as the difference between two completely postive maps (15).

III. UNITARY FREEDOM IN THE OSR

The description of the dynamical map is not unique. It can be represented by the set of \( C_k \) corresponding to the eigendecomposition of the map \( B \), but there are many other representations. In this section we find this freedom after reviewing the case for completely positive maps. For completely positive maps, we reiterate that a theorem describing the freedom, examples, and uses can be found in Ref. (11) although our presentation differs somewhat from theirs.

A. Unitary Freedom for Completely Positive Maps

A completely positive map can always be written in the form given in Eq. (16). If we then let a new set be given by \( A'_j = \sum_i u_{ji} A_i \), then

\[
\Phi'(\rho) = \sum_k A'_k \rho A'_k = \sum_{ijk} u_{jk} A_k \rho u_{ji} A_i^\dagger.
\]
along the diagonal are +1 and the next elements of the matrix of freedom. By this we mean the freedom is described by the operator-sum representation for maps which are not the same by appending zero operators to the shorter list and making the matrix \( U \) a square unitary matrix \( [11] \).

### B. Pseudo-unitary freedom for Hermiticity-preserving maps

Now let us consider the map \( \Phi(\rho) = \sum_{j} \eta_{j} C_{j} \rho C_{j}^{\dagger} \) and introduce a set of operators \( D_{j} \) corresponding to the map \( \Phi'(\rho) = \sum_{j} \eta_{j} D_{j} \rho D_{j}^{\dagger} \). As stated above, we may take \( \eta_{j} = \pm 1 \). We can choose the number of operators to be the same by appending zero operators to the shorter list. This enables the number of \(-1\) and \(+1\) to be chosen to be the same. Furthermore, we will order the set of \( \eta_{j} \) such that the first \( p \) are \(+1\) and the next \( q \) are \(-1\).

As stated in the title of this section, the freedom in the operator-sum representation for maps which are not necessarily completely positive is a pseudo-unitary degree of freedom. By this we mean the freedom is described by the group \( U(p, q) \). This group is often called a pseudo-unitary group due to its relation to the unitary group and it is a metric-preserving group with the signature of the metric determined by the integers \( p, q \). See for example \( [16] \), pages 45, 197), \( [17] \), page 392), \( [18] \), page 12), or \( [19] \), page 444).

Let \( \eta \) be an \( N \times N \) diagonal matrix with the first \( p \) entries \(+1\), the next \( q \) entries \(-1\), and \( N = p + q \). Then for all \( U \in U(p, q) \),

\[
U^{\dagger} \eta U = \eta U^{\dagger} - 1. \tag{20}
\]

We may express the matrix \( \eta \) as a diagonal matrix with the matrix elements being \( \eta_{k} \), \( \eta_{k} = +1 \), for \( k = 1, \ldots, p \) and \( \eta_{k} = -1 \), for \( k = p + 1, \ldots, p + q = N \). Alternatively, we may express the matrix \( \eta \) using elements \( (\eta)_{kl} = \eta_{kl} \delta_{kl} \). This is clearly a diagonal matrix since the elements are zero if \( k \neq l \). Furthermore, the first \( p \) entries along the diagonal are \(+1\) and the next \( q \) are \(-1\). Let the elements of the matrix \( U \) be given by \( u_{ij} \) and those of \( U^{\dagger} \) be \( u_{ij}^{\ast} \). Then the Eq. \( (20) \) can be written as \( U^{\dagger} \eta U = \eta \), or since \( \eta^{2} = I \), \( U \eta U^{\dagger} = \eta \). In components, this can be written as

\[
\sum_{jk} u_{ij} \eta_{j} \delta_{jk} u_{kl}^{\ast} = \eta_{i} \delta_{il}. \tag{21}
\]

Having established this property for elements of the group \( U(p, q) \), the following theorem may now be stated.

**Theorem 1** Pseudo-unitary freedom: Suppose \( \{C_{1}, C_{2}, \ldots, C_{n}\} \) and \( \{D_{1}, D_{2}, \ldots, D_{m}\} \), are operation elements giving rise to quantum operations (maps) \( \Phi \) and \( \Phi' \) respectively. Explicitly,

\[
\Phi = \sum_{i} \gamma_{i} C_{i} C_{i}^{\dagger} \tag{22}
\]

and

\[
\Phi' = \sum_{j} \mu_{j} D_{j} D_{j}^{\dagger}, \tag{23}
\]

where each \( \gamma_{i} \) and each \( \mu_{j} \) is \( \pm 1 \) and ordered as above, with all \(+1\) eigenvalues first. Furthermore, we can always take \( \gamma_{i} = \mu_{j} \) with zero-valued \( C_{i} \) or zero-valued \( D_{j} \) appended to the shorter list for the \(+1\) (\(-1\)) eigenvalue. Then \( \Phi = \Phi' \) if and only if

\[
D_{j} = \sum_{i} u_{ji} C_{i}, \tag{24}
\]

where the numbers \( u_{ij} \) form a \( p + q \) by \( p + q \) matrix in \( U(p, q) \).

**Proof:** We first consider whether the condition is necessary and use the notation \( C_{i} = |i\rangle \), \( D_{i} = |j\rangle \). Suppose that

\[
\Phi = \Phi'. \tag{25}
\]

(Or, if one would like to display the argument explicitly, \( \Phi(\rho) = \Phi'(\rho) \).) For a general map \( \Phi \), there exists a corresponding matrix (see Sec. \( [10] \) such that \( \Phi = B(\rho) \) (i.e. \( \Phi(\rho) = B \rho \)). \( B \) has an eigenvector decomposition \( B = \sum_{k} \lambda_{k} |k\rangle \langle k'| \) where the set of \( |k'\rangle \) are linearly independent since they are orthogonal. This follows from the fact that the eigenvectors can be chosen orthogonal. Now \( |k\rangle = \sqrt{|\lambda_{k}|} |k'\rangle \). These vectors are clearly also orthogonal and thus linearly independent if the \( |k'\rangle \) are.

Then \( B \) can be re-expressed as

\[
B = \sum_{k} \eta_{k} |k\rangle \langle k| = \sum_{k=1}^{p} |k\rangle \langle k| - \sum_{k=p+1}^{p+q} |k\rangle \langle k|, \tag{26}
\]

which is an eigenvector decomposition of the map \( \Phi \), Eq. \( (12) \). Now, let us consider another decomposition of \( B \) corresponding to the set of \( C_{i} \), \( B = \sum_{i} \gamma_{i} |i\rangle \langle i| \). Each \( |i\rangle \) can be written as a linear combination of the \( |k| \), \( |i\rangle = \sum_{w} w_{ik} |k\rangle \). (See for example Ref. \( [11] \), page 104.) Given \( \Phi = B \),

\[
\sum_{k} \eta_{k} |k\rangle \langle k| = \sum_{kl} \left( \sum_{i} \gamma_{i} w_{ik} w_{il}^{\ast} \right) |k\rangle \langle l|. \tag{27}
\]

Since the \( |k\rangle \) are linearly independent, it is clear that this can only happen if

\[
\sum_{i} \gamma_{i} w_{ik} w_{il}^{\ast} = \delta_{kl} \eta_{k}. \tag{28}
\]
We may always take $\eta_i = \gamma_i$ by appending the shorter list of vectors $\{|i\rangle\}$ or $\{|k\rangle\}$ with zero vectors. This will ensure that the matrices $\gamma$ with elements $\delta_{ij} \gamma_i$ and $\eta$ with elements $\delta_{nk} \eta_k$ are equal. Furthermore, $w$ can then be taken to be square with $|i\rangle = \sum_k w_{ik} |k\rangle$. The condition, Eq. (28), can then be written as

$$w^\dagger \eta w = \eta,$$

(29)

which is the condition for the matrix $w$ to be in $U(p,q)$. Now, we can use the same argument, with $B = \Phi'$ and $v_{jk}$ such that $|j\rangle = \sum_k v_{jk} |k\rangle$, to show

$$v^\dagger \eta v = \eta,$$

(30)

Since each of these two are related to the same expression for $B$ using elements of $U(p,q)$ which is a group, then the linear transformation which takes the $C_i$ to the $D_j$, $u = w^{-1}$ is in $U(p,q)$.

Next, we consider whether $u \in U(p,q)$ will imply that $\Phi = \Phi'$, i.e., if the condition is sufficient. This is straightforward algebra. Given Eq. (24)

$$\Phi'(\rho) = \sum_j \mu_j D_j \rho D_j^\dagger$$

$$= \sum_{lkj} \mu_{lkj} u_{lj}^* u_{jk}^* C_l \rho C_k^\dagger$$

$$= \sum_{lkj} \left( \sum_j \mu_{lj} u_{lj} u_{jk}^* \right) C_l \rho C_k^\dagger$$

$$= \sum_l \gamma_l \delta_{lk} C_l \rho C_k^\dagger$$

$$= \Phi(\rho),$$

which shows that the two sets of operators $C_j$ and $D_j$ related by a pseudo-unitary matrix $u$ will yield the same map. □

IV. CONCLUSIONS

The unitary degree of freedom in the operator-sum representation of quantum maps has multiple uses including applications in quantum error prevention schemes since it essentially provides the set of physical operators producing a given map. With the recent extensive discussions in the literature concerning maps which are not completely positive, and the extension of quantum error correction to maps which are not completely positive, we believe it important to have an extension of the unitary degree of freedom for completely positive maps to the cases when the map is not necessarily completely positive. We have done here by providing the pseudo-unitary freedom for any Hermitian-preserving map.

It is natural to ask which maps are genuinely different in the sense that they are not equivalent. In other words, if we consider the unitary degree of freedom to be a symmetry of the system, what is unique to two different maps? This can, in principle, be determined from the work here and Ref. [15] where the authors parameterized the space of positive maps. Also we see that the minimal decomposition provided by the spectral decomposition is very important. The map $B$ is unique [13] and provides an almost canonical form [20] which has a set number of positive and negative operators when written as the sum of two completely positive maps. These important issues should be addressed in future work.

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