Second-order phase transition of Kehagias-Sfetsos black hole in deformed Hörava-Lifshitz gravity

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Abstract

We study the second-order phase transition (SOPT) for the spherically symmetric Kehagias-Sfetsos (KS) black hole in the deformed Hörava-Lifshitz gravity by applying the methods of equilibrium and non-equilibrium fluctuations. We find that, although the KS black hole has only one mass parameter as the usual Schwarzschild ones, the SOPT will take place if the mass of the KS black hole changes across the critical point $\sqrt[4]{\frac{5 + 3\sqrt{33}(1 - \sqrt{33})}{16\sqrt{2}}}$. The result show us that there is difference between the Hörava-Lifshitz gravity and the Einstein’s gravity theory.

Keywords: deformed Hörava-Lifshitz gravity, black hole, second-order phase transition

PACS numbers: 95.30.Tg, 04.70.-s, 97.60.Lf

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I. INTRODUCTION

It is well known that the non-renormalizable of gravity is a large challenge for Einstein’s gravity theory. Recently, Hörava [1–3] proposed a new class of quantum gravity which is non-relativistic and power-counting renormalizable. The key feature of this theory is that the space and time exhibit Lifshitz scale invariance $t \rightarrow b^z t$ and $x_i \rightarrow b^z x_i$ with $z > 1$. It is this anisotropic rescaling that makes Hörava’s theory power-counting renormalizable. In the IR region such a Hörava’s theory can reduce to the well-known Einstein gravity. Therefore, a lot of attention has been focused on this gravity theory [4–19].

In general, the IR vacuum in Hörava’s theory is anti de Sitter (AdS) spacetimes. In order to obtain a Minkowski vacuum in the IR sector, one can add the term “$\mu^4 R$” in the action and then take the $\Lambda_W \rightarrow 0$ limit. This does not change the UV properties of the theory, but it alters the IR properties. Making use of such a deformed action, Kehagias et al [10] obtain the asymptotic flat spherically symmetric vacuum black hole solution which has two event horizons. The heat capacity is positive for the small KS black hole and it is negative for the large one. It means that the small KS black holes are the stable in the Hörava’s theory. The result imply that there exists the distinct differences between the KS black hole and the Schwarzschild black hole.

The investigation of SOPT of black hole is helpful to explore the black hole’s property [20–22]. To the best of our knowledge, there is no report about investigation for SOPT of the KS black hole in the deformed Hörava-Lifshitz gravity. On the other hand, there is a paradox about where the SOPT is taken place for a long time. Some authors [23–26] argue that the SOPT is taken place when $C \rightarrow \infty$ by applying thermodynamical equilibrium fluctuations. The other [27] calculated the non-equilibrium fluctuation of mass and entropy and found that these fluctuations diverge when $r_+ \rightarrow r_-$ and they are finite when $C \rightarrow \infty$. So they put forwards that the SOPT of KS black hole takes place where $r_+ \rightarrow r_-$ rather than where $C \rightarrow \infty$. In this paper, we will address these question carefully by studying the SOPT of the KS black hole [10] in the deformed Hörava-Lifshitz gravity using the methods of equilibrium and non-equilibrium fluctuations.

The paper is organized as follows. In Sec. II, we give a brief description of solution in the deformed Hörava-Lifshitz black hole spacetime. In Sec. III, we calculate a SOPT point by using the method of equilibrium fluctuations. In Sec. IV, we make use of the non-
equilibrium thermodynamic fluctuations and find a SOPT point that is just the same point as we find in Sec. III. We present our conclusions and make some discussions in the last section. Throughout, we shall set $c = G = \hbar = k = 1$.

II. RIGOROUS SOLUTION IN DEFORMED HÓRAVA-LIFSHITZ GRAVITY

In the Hóra\v{v} theory, a deformed action of the non-relativistic renormalizable gravitational theory is given by \[10\]

$$S_{HL} = \int dtd^3x \left( \mathcal{L}_0 + \tilde{\mathcal{L}}_1 \right)$$

$$\mathcal{L}_0 = \sqrt{g}N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3 \Lambda_W^2)}{8(1 - 3 \lambda)} \right\}$$

$$\tilde{\mathcal{L}}_1 = \sqrt{g}N \left\{ \frac{\kappa^2 \mu^2 (1 - 4 \lambda)}{32(1 - 3 \lambda)} R^2 - \frac{\kappa^2}{2w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right) + \mu^4 R \right\},$$

where $\kappa^2$, $\lambda$, $\mu$, $w$ and $\Lambda_W$ are constant parameters, $K_{ij}$ is extrinsic curvature in the $(3 + 1)$-dimensional ADM formalism

$$K_{ij} = \frac{1}{2N} (g_{ij} - \nabla_i N_j - \nabla_j N_i),$$

and $C_{ij}$ is the Cotton tensor

$$C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R^{(3)j}_\ell - \frac{1}{4} R^{(3)} \delta^j_\ell \right).$$

Taking the $\Lambda_W \rightarrow 0$ limit and letting $\lambda = 1$, it was found that the speed of light, the Newton constant are described by the following relations \[10\]

$$c^2 = \frac{\kappa^2 \mu^4}{2}, \quad G = \frac{\kappa^2}{32 \pi c}$$

A static and asymptotically flat KS black hole was found in \[10\] which has the following form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

with

$$f(r) = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}$$

where $\omega = \frac{16\mu^2}{\kappa^2}$ and $M$ is an integration constant related to the mass of the KS black hole.
The outer (inner) horizons are given by

\[ r_\pm = M \pm \sqrt{M^2 - \frac{1}{2\omega}} \]  

(8)

The corresponding Hawking temperature and heat capacity are

\[ T_H = \frac{1}{4\pi} \partial_r f(r) \big|_{r=r_+} = \frac{\omega(r_+ - M)}{2\pi(1 + \omega r_+^2)} = \frac{2\omega r_+^2 - 1}{8\pi(\omega r_+^3 + r_+)} \]  

(9)

\[ C = \frac{\partial M}{\partial T} = -\frac{2\pi}{\omega} \frac{(1 + \omega r_+^2)^2(2\omega r_+^2 - 1)}{2\omega^2 r_+^4 - 5\omega r_+^4 - 1} \]  

(10)

III. EQUILIBRIUM FLUCTUATION FOR KS BLACK HOLE IN DEFORMED GRAVITY

In thermodynamical fluctuational theory, we always express fluctuations of thermodynamical quantities in an equilibrium system as some mean square fluctuations. If the mean square fluctuations are divergent at a point, we call the point as a phase transition point for the thermodynamical system. If the entropy of the system is continuous at the point, we name the phase transition as the SOPT. Therefore, we can look for the SOPT point in a thermodynamical system according to singularity of fluctuations of the thermodynamical quantities and continuity of the entropy.

We suppose that the KS black hole \([10]\) is initially at temperature \(T_0\) (the point \(O\) is arbitrary). The variation of temperature due to thermal fluctuation of the mass is \(\Delta\epsilon T_0\), where \(|\Delta\epsilon| \leq B\) and \(B\) being a sufficiently small constant. From the first law of the thermodynamics we have

\[ \Delta M = T_0(1 + \Delta\epsilon)\Delta S \approx T_0\Delta S \]  

(11)

where \(T_0\) is determined by (9).

The fluctuation probability \(p\) is

\[ p \propto \exp \Delta S \sim \exp \left( \frac{\Delta M}{T_0} \right). \]  

(12)

We know from the definition of heat capacity that

\[ \Delta M = CT_0\Delta\epsilon. \]  

(13)
where heat capacity $C$ is determined by (10).

Because $|\Delta \epsilon| \leq B$, the variation of the mass $\Delta M$ may be expressed as

$$|\Delta M| \leq |C| T_o B.$$ \tag{14}

With Eqs. \(12\), \(14\) and the condition of normalized

$$\int_{-B|C|T_o}^{B|C|T_o} p \ d(\Delta M) = 1,$$ \tag{15}

we find that the fluctuation probability $p$ can be rewritten as

$$p = \frac{1}{2T_o \sinh (B \ |C|)} \exp \left( \frac{\Delta M}{T_o} \right).$$ \tag{16}

Making use of Eqs. \(14\) and \(16\), we can get the mean square fluctuation of $M$

$$\langle (\Delta M)^2 \rangle = \frac{1}{2T_o \sinh (B \ |C|)} \int_{-B|C|T_o}^{B|C|T_o} (\Delta M)^2 \exp \left( \frac{\Delta M}{T_o} \right) d(\Delta M)$$

$$= T_o^2 \left( 2 + B^2 C^2 - 2B \ |C| \coth(B \ |C|) \right).$$ \tag{17}

In the same way, we obtain

$$\langle (\Delta S)^2 \rangle = \frac{1}{T_o^2} \langle (\Delta M)^2 \rangle = 2 + B^2 C^2 - 2B \ |C| \coth(B \ |C|) \rangle,$$ \tag{18}

$$\langle (\Delta T)^2 \rangle = \frac{1}{|C|^2} \langle (\Delta M)^2 \rangle = T_o^2 \left( 2 + B^2 C^2 - 2B \ |C| \coth(B \ |C|) \right).$$ \tag{19}

We observe from Eqs. \(9\), \(10\), \(17\) and \(18\) that $\langle (\Delta S)^2 \rangle \to \infty$ and $\langle (\Delta M)^2 \rangle \to \infty$ when $C \to \infty$. In other words, at the point $C \to \infty$, even if we add the small perturbation of $M$ to black hole, the fluctuations of mass and entropy of the KS black hole will become so large that we can’t describe it with a normal equilibrium state. Thus, when the heat capacity changes from positive to negative, we think that the KS black hole changes from a phase into another phase and the critical point lies at $\sqrt{5+\sqrt{33}}(\sqrt{33}-1) \frac{1}{16\sqrt{3}}$. It is a SOPT point because the entropy of the KS black hole is continuous at this critical point. On the other hand, when $r_+ \to r_-$, Eqs. \(17\) and \(18\) lead to $\langle (\Delta S)^2 \rangle \to 0$, $\langle (\Delta M)^2 \rangle \to 0$. It means that when the KS black hole has a small deviation from equilibrium state, all thermodynamic fluctuations goes to zero under the condition that $r_+ \to r_-$, and then the KS black hole will come back to the initial equilibrium state. So we think the SOPT of the KS black hole in the deformed Hörava-Lifshitz gravity doesn’t occur as $r_+ \to r_-$ but $C \to \infty$. 

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IV. NON-EQUILIBRIUM FLUCTUATION FOR KS BLACK HOLE IN DEFORMED GRAVITY

In Landau and Lifshitz fluctuational theory \[29\], thermodynamical parameters and entropy in an arbitrary thermodynamical system will change as variation of time. In order to describe the variation of the entropy, we introduce a new physical quantity called the rate of entropy production which can be expressed as

\[
\dot{S} = \sum_i X_i \dot{\chi}_i,
\]  

(20)

where \(\dot{\chi}_i\) is flux of a given thermodynamic quantity \(\chi_i\), and \(X_i\) is thermodynamical force which is the conjugate quantity of the thermodynamical flux \(\chi_i\). In the KS black hole, its mass parameter \(M\), will be changed because of radiation of the black hole in the real physic background. The rates \(\dot{M}\) for all emission processes can be written as

\[
\dot{M} = -f M^{-2},
\]  

(21)

where \(f\) is dimensionless coefficient and can be calculated numerically.

The rate of entropy production of the KS black hole \[10\] is written as specifically

\[
\dot{S} = X_M \dot{M}.
\]  

(22)

with

\[
X_M = \frac{\partial S}{\partial M} = \frac{1}{T},
\]  

(23)

In a general fluctuation-dissipative process, the flux \(\dot{\chi}_i\) is given by

\[
\dot{\chi}_i = \sum_k \Gamma_{ik} X_k,
\]  

(24)

where the quantity \(\Gamma_{ik}\) is the phenomenological transport coefficient which is defined as

\[
\Gamma_{ik} = \frac{\partial \dot{\chi}_i}{\partial X_k}.
\]  

(25)

In non-equilibrium thermodynamic theory, we always make use of a second moment to describe the fluctuations of a thermodynamic quantity in a system. The second moments in the fluctuations of the flux obey

\[
\langle \delta \dot{\chi}_i \delta \dot{\chi}_k \rangle = (\Gamma_{ik} + \Gamma_{ki}),
\]  

(26)
where angular bracket denotes the mean value with respect to the steady state, and the fluctuation \( \delta \chi_i \) is spontaneous deviation from the steady-state value \( \langle \chi_i \rangle \). If the second moment tends to infinite at a certain point, we think a phase transition occurs in the system.

From Eqs. (21), (23) and (25), we can get the phenomenological transport coefficients

\[
\Gamma_{MM} = -\frac{2fT^2}{M^3} \left( \frac{\partial M}{\partial T} \right) = -\frac{2fT^2C}{M^3},
\]

(27)

Substituting (27) into (26), we find the second moment for mass \( M \)

\[
\langle \delta \dot{M} \delta \dot{M} \rangle = 2\Gamma_{MM} = -\frac{4fT^2C}{M^3}.
\]

(28)

The second moments of entropy \( S \) and temperature \( T \) can be expressed as

\[
\langle \delta \dot{S} \delta \dot{S} \rangle = \frac{\langle \delta \dot{M} \delta \dot{M} \rangle}{T^2} = -\frac{4fC}{M^3},
\]

(29)

\[
\langle \delta \dot{T} \delta \dot{T} \rangle = \frac{\langle \delta \dot{M} \delta \dot{M} \rangle}{C^2} = -\frac{4fT^2}{M^3C},
\]

(30)

\[
\langle \delta \dot{S} \delta \dot{T} \rangle = \frac{\langle \delta \dot{M} \delta \dot{M} \rangle}{TC} = -\frac{4fT}{M^3}.
\]

(31)

As \( r_+ \to r_- \), we find from Eqs. (28), (29), (30) and (31) that

\[
\langle \delta \dot{M} \delta \dot{M} \rangle \to 0, \quad \langle \delta \dot{S} \delta \dot{S} \rangle \to 0, \quad \langle \delta \dot{T} \delta \dot{T} \rangle \to 0, \quad \langle \delta \dot{S} \delta \dot{T} \rangle \to 0.
\]

(32)

These equations show that all second moments of the KS black hole tend to zero when \( r_+ \to r_- \), i.e., all thermodynamic fluctuations vanish under this condition. Thus, the phase transition does not occur in the KS black hole as \( r_+ \to r_- \).

On the other hand, when the heat capacity \( C = \frac{\partial M}{\partial T} \to \infty \), according to Eqs. (28) and (29), we have

\[
\langle \delta \dot{M} \delta \dot{M} \rangle \to \infty,
\]

\[
\langle \delta \dot{S} \delta \dot{S} \rangle \to \infty.
\]

(33)

It is obvious that these second moments tend to infinite when heat capacity is discontinuous. Thus, the thermal non-equilibrium fluctuations of some thermodynamical quantities are divergent. According to the theory of thermal non-equilibrium fluctuation, we can draw a conclusion that a phase transition of the KS black hole occurs when \( C \to \infty \). The phase transition is SOPT since the entropy of the thermodynamical system is continuous.
V. SUMMARY AND DISCUSSIONS

In this paper, we investigate the SOPT of the KS black hole [10] in the deformed Hörava-Lifshitz gravity by using the thermal equilibrium and the thermal non-equilibrium fluctuation methods, respectively. We find that as the heat capacity goes to infinity the mean square fluctuations of mass and entropy in equilibrium fluctuation diverges, so does the second moments of mass and entropy in non-equilibrium fluctuation. However, under the condition $r_+ \to r_-$, all of fluctuations of thermodynamic quantities in the both equilibrium and non-equilibrium fluctuations vanish. Comparing to Schwarzschild black hole, although both of them have one mass parameter, there are two phases (one is $\frac{1}{\sqrt{2\omega}} \leq M < \frac{\sqrt{5+\sqrt{33}(\sqrt{33}-1)}}{16\sqrt{\omega}}$ and the other one is $M > \frac{\sqrt{5+\sqrt{33}(\sqrt{33}-1)}}{16\sqrt{\omega}}$) and the SOPT can taken place for the KS black hole in the deformed Hörava-Lifshitz gravity. In this sense, there is difference between the Hörava-Lifshitz gravity and the usual general relativity.

Acknowledgements

This work was supported by the National Natural Science Foundation of China under Grant No 10875040; a key project of the National Natural Science Foundation of China under Grant No 10935013; the National Basic Research of China under Grant No. 2010CB833004, the Hunan Provincial Natural Science Foundation of China under Grant No. 08JJ3010, PCSIRT under Grant No. IRT0964, and the Construct Program of the National Key Discipline. S. B. Chen’s work was partially supported by the National Natural Science Foundation of China under Grant No.10875041; the Scientific Research Fund of Hunan Provincial Education Department Grant No.07B043.

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