TOWARDS HOMOLOGICAL PROJECTIVE DUALITY FOR $S^2\mathbb{P}^3$ AND $S^2\mathbb{P}^4$

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Abstract. We provide homological foundations to establish conjectural homological projective dualities between 1) $S^2\mathbb{P}^3$ and the double cover of the projective 9-space branched along the symmetric determinantal quartic, and 2) $S^2\mathbb{P}^4$ and the double cover of the symmetric determinantal quintic in $\mathbb{P}^{14}$ branched along the symmetric determinantal locus of rank at most 3.

1. Introduction

Throughout this paper, we work over $\mathbb{C}$, the complex number field. We fix a vector space $V$ of dimension $n+1$ and denote by $V^*$ the dual vector space to $V$.

Let us consider the projective space $\mathbb{P}(S^2V^*)$ which we identify with the space of quadrics in $\mathbb{P}(V)$. In a separate paper [14], we have considered the locus $S^r \subset \mathbb{P}(S^2V^*)$ which represents quadrics in $\mathbb{P}(V)$ of rank at most $r$. $S^r$ is a determinantal variety which is defined by $(r+1) \times (r+1)$ minors of the generic $(n+1) \times (n+1)$ symmetric matrix, and we have called it symmetric determinantal locus of rank at most $r$. As studied in [ibid.], when $r$ is even, $S^r$ has a double cover $T^r$ branched along $S^r_{r-1}$. $T^r$ is called double symmetric determinantal locus of rank at most $r$.

These definitions apply in the same way using the dual projective spaces, i.e., $S^r_*$ and its double cover $T^*_r$ (when $r$ is even) are defined using the dual projective space $\mathbb{P}(S^2V)$.

In [ibid.], we have studied algebro-geometric properties of $S^r$ and $T^r$ in detail motivated by the so-called homological projective duality (HPD) due to Kuznetsov [17]. HPD is a powerful framework to describe the derived category of a projective variety with its dual variety. Several interesting examples such as Pfaffian varieties (i.e., determinantal loci of anti-symmetric matrices) [18] and the second Veronese variety $S^*_2$ [21] as well as the linear duality in general [17, §8] have been studied.

The purpose of this paper is to lay homological foundations to establish the HPDs for $S^2_2$ with $n = 3, 4$. Indeed this paper is an extended version of the second part of [13] (the paper [13] contains generalizations of the first part of [ibid.]). It is useful to note that $S^*_2$ may be identified with $S^2\mathbb{P}(V)$ in a similar way to the relation of $S^*_1$ and the second Veronese variety $v_2(\mathbb{P}(V))$. These conjectural HPDs are special cases of two different types of plausible HPDs, which have naturally arisen from our algebro-geometric study on $S^r$ and $T^r$ in [14] §3.5, 3.6].

The first one is the duality between $S^*_n,_{n+2-r}$ and $T^r$ for each even $r$. We suspect that their suitable non-commutative resolutions are HPD to each other with respect to certain (dual) Lefschetz collections because there exist orthogonal linear sections of $S^*_n,_{n+2-r}$ and $T^r$ such that they are Calabi-Yau varieties of the same dimension (see [13] Prop. 3.2 and 3.6]). In this paper, we consider the case where $n = r = 4$,
namely, we study $S^*_2$ and $T_4$ for $n = 4$. In this case, there exist orthogonal linear sections $X$ of $S^*_2$ and $Y$ of $T_4$ such that they are smooth Calabi-Yau threefolds. $X$ is so called a Calabi-Yau threefold of Reye congruence, and we call $Y$ a double quintic symmetroid. In [13], we constructed certain resolutions $\tilde{S}^*_2$ and $\tilde{T}_4$ of $S^*_2$ and $T_4$, respectively. In this paper, we construct (dual) Lefschetz collections in the derived categories of the resolutions $\tilde{S}^*_2$ and $\tilde{T}_4$ (Corollaries 3.5 and 5.13). We remark that the (dual) Lefschetz collections have been originally read off from a locally free resolution of certain ideal sheaf on $\tilde{T}_4 \times \tilde{S}^*_2$ (see [12]). Based on these (dual) Lefschetz collections, we have shown in [12] that $X$ and $Y$ are derived equivalent (see also [10] [11]). Moreover, we show that the dual Lefschetz collection in $D^b(\tilde{S}^*_2)$ gives a dual Lefschetz decomposition of a categorical resolution of $D^b(S^*_2)$ [4] defined by Kuznetsov (Theorem 3.7). These should be strong evidence for HPD between $S^*_2$ and $T_4$.

The second plausible duality is between $S^*_{n+1-\frac{r}{2}}$ and $T_r$ for each even $r$. This duality may be observed in the resolution $\tilde{S}^*_{n+1-\frac{r}{2}}$ of $S^*_{n+1-\frac{r}{2}}$ and the fiber space $U_r$ of $T_r$, which have been constructed by using certain projective bundles over the Grassmannian $G(n+1-\frac{r}{2}, V)$ [14]. It turns out that $\tilde{S}^*_{n+1-\frac{r}{2}}$ and $U_r$ are given as certain incident varieties in $\mathbb{P}(S^2V) \times G(n+1-\frac{r}{2}, V)$ and $\mathbb{P}(S^2V^*) \times G(n+1-\frac{r}{2}, V)$, respectively, and are orthogonal to each other with respect to the dual pairing between $S^2V$ and $S^2V^*$. We will see that $\tilde{S}^*_{n+1-\frac{r}{2}}$ and $U_r$ are precisely in the setting of the linear duality established by Kuznetsov [17], §8 and hence HPD to each other. This duality between $\tilde{S}^*_{n+1-\frac{r}{2}}$ and $U_r$ indicates certain relationship between the derived categories of $S^*_{n+1-\frac{r}{2}}$ and $T_r$. In this paper, to provide a supporting evidence, we consider the case of $n = 3$ and $r = 4$, i.e., we study $S^*_3$ and $T_4$ for $n = 3$. In this case, we have an Enriques surface of Reye congruence and an Artin-Mumford double solid as an orthogonal linear sections of $S^*_2$ and $T_4$, respectively. We construct (dual) Lefschetz collections in the derived categories of the resolutions $\tilde{S}^*_2$ and $\tilde{T}_4$ (Corollaries 3.5 and 5.11). Based on these (dual) Lefschetz collections, we will show in [15] that there exists a close relationship between the derived categories of the two linear sections. Moreover, we show that the dual Lefschetz collection in $D^b(\tilde{S}^*_2)$ gives a dual Lefschetz decomposition of a categorical resolution of $D^b(S^*_2)$ [4] defined by Kuznetsov (Theorem 3.7). These should be strong evidences for HPD between $S^*_2$ and $T_4$.

Acknowledgement. This paper is supported in part by Grant-in Aid Scientific Research (S 24224001, B 23340010 S.H.) and Grant-in Aid for Young Scientists (B 20740005, H.T.). They thank Nicolas Addington and Sergey Galkin for useful communications. They also thank Jorgen Vold Rennemo for letting us know about his Ph.D. thesis.

2. Basic results

2.1. Borel-Weil-Bott Theorem. We frequently use the following Borel-Weil-Bott Theorem.

1) Similar results have been obtained in [23] using the category of matrix factorizations and the variation of GIT method [10] [8] [6].
For a locally free sheaf $\mathcal{E}$ of rank $r$ on a variety and a nonincreasing sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ of integers, we denote by $\Sigma^\beta \mathcal{E}$ the associated locally free sheaf with the Schur functor $\Sigma^\beta$.

**Theorem 2.1.** Let $\pi: G(r, \mathcal{A}) \to X$ be a Grassmann bundle for a locally free sheaf $\mathcal{A}$ on a variety $X$ of rank $n$ and $0 \to \mathcal{S} \to \mathcal{A} \to \mathcal{Q} \to 0$ the universal exact sequence. For $\beta = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ $(\alpha_1 \geq \cdots \geq \alpha_r)$ and $\gamma = (\alpha_{r+1}, \ldots, \alpha_n) \in \mathbb{Z}^{n-r}$ $(\alpha_{r+1} \geq \cdots \geq \alpha_n)$, we set $\alpha := (\beta, \gamma)$ and $\mathcal{V}(\alpha) := \Sigma^\beta \mathcal{S}^* \otimes \Sigma^\gamma \mathcal{Q}^*$. Define $\rho := (n, \ldots, 1)$ and for an element $\sigma$ of the $n$-th symmetric group $\mathfrak{S}_n$, we set $\sigma \cdot (\alpha) := \sigma(\alpha + \rho) - \rho$. Then the followings hold:

1. If $\sigma(\alpha + \rho)$ contains two equal integers, then $R^i \pi_* \mathcal{V}(\alpha) = 0$ for any $i \geq 0$.
2. If there exists an element $\sigma \in \mathfrak{S}_n$ such that $\sigma(\alpha + \rho)$ is strictly decreasing, then $R^i \pi_* \mathcal{V}(\alpha) = 0$ for any $i \geq 0$ except $R^l(\sigma) \pi_* \mathcal{V}(\alpha) = \Sigma^{\sigma(\alpha)} \mathcal{A}^*$, where $l(\sigma)$ represents the length of $\sigma \in \mathfrak{S}_n$.

**Proof.** See [1].

### 2.2. Basic definitions for triangulated categories

We recall some basic definitions from the theory of triangulated categories (cf. [2, 3]).

**Definition 2.2.** An object $\mathcal{E}$ in a triangulated category $\mathcal{D}$ is called an exceptional object if $\text{Hom}(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}$ and $\text{Hom}^\bullet(\mathcal{E}, \mathcal{E}) = 0$ for $\bullet \neq 0$.

**Definition 2.3.** A triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is called admissible if there are right and left adjoint functors for the inclusion functor $i_*: \mathcal{D}' \to \mathcal{D}$.

**Definition 2.4.** A sequence $\mathcal{D}_1, \ldots, \mathcal{D}_m$ of admissible triangulated subcategories in a triangulated category $\mathcal{D}$ is called a semiorthogonal collection if $\text{Hom}_\mathcal{D}(\mathcal{D}_i, \mathcal{D}_j) = 0$ for any $i > j$. Moreover, if $\mathcal{D}_1, \ldots, \mathcal{D}_m$ generates $\mathcal{D}$, then it is called a semiorthogonal decomposition.

A semiorthogonal collection of exceptional objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is called an exceptional collection. Moreover, if $\text{Hom}^\bullet(\mathcal{E}_i, \mathcal{E}_j) = 0$ holds for any $i, j$ and $\bullet \neq 0$, then it is called a strongly exceptional collection.

Hereafter, in this article, we restrict our attention to the cases of the derived categories of bounded complexes of coherent sheaves on a variety. In such cases, a special type of semiorthogonal collection plays an important role (cf. [17, 18]).

**Definition 2.5.** For a variety $X$, a Lefschetz collection of $\mathcal{D}^b(X)$ is a semiorthogonal collection of the following form:

$$\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{m-1}(m-1),$$

where $0 \subset \mathcal{D}_{m-1} \subset \mathcal{D}_{m-2} \subset \cdots \subset \mathcal{D}_0 \subset \mathcal{D}^b(X)$ and $(k)$ means the twist by $L^\otimes k$ with a fixed invertible sheaf $L$. Moreover, if $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{m-1}(m-1)$ generate $\mathcal{D}^b(X)$, then it is called a Lefschetz decomposition.

Similarly, a dual Lefschetz collection of $\mathcal{D}^b(X)$ is a semiorthogonal collection of the following form:

$$\mathcal{D}_{m-1}(-(m-1)), \mathcal{D}_{m-2}(-(m-2)), \ldots, \mathcal{D}_0,$$

where it holds that $0 \subset \mathcal{D}_{m-1} \subset \mathcal{D}_{m-2} \subset \cdots \subset \mathcal{D}_0 \subset \mathcal{D}^b(X)$. Moreover, if $\mathcal{D}_{m-1}(-(m-1)), \mathcal{D}_{m-2}(-(m-2)), \ldots, \mathcal{D}_0$ generate $\mathcal{D}^b(X)$, then it is called a dual Lefschetz decomposition.
3. Dual Lefschetz collection in $\mathcal{D}^b(\mathcal{X})$

3.1. Symmetric determinantal loci $S^*_2$ and its Springer resolution $\tilde{S}^*_2$. We remark that $S^*_2 \subset \mathbb{P}(S^2V)$ is the locus of quadrics in $\mathbb{P}(V^*)$ of rank at most two. The Springer resolution of $S^*_2$ as in [13], (2.1) is $\tilde{S}^*_2 := \mathbb{P}(S^2\mathcal{F})$, where $\mathcal{F}$ is the universal subbundle on $G(2, V)$. As in [13], Subsect. 3.2, we may identify $S^*_2$ with the Chow variety of length two 0-cycles in $\mathbb{P}(V)$. With this identification, we may interpret [13], Prop. 2.1 (2)] in this situation as follows:

$\tilde{S}^*_2 := \{([V_2], [\eta]) \mid \text{Supp } \eta \subset \mathbb{P}(V_2) \subset G(2, V) \times S^*_2,$

where we consider $\eta$ is a length two 0-cycle in $\mathbb{P}(V)$. From this description of $\tilde{S}^*_2$, it is easy to see that $\tilde{S}^*_2$ is the Hilbert scheme of length two 0-dimensional subschemes of $\mathbb{P}(V)$ and the Springer resolution $\tilde{S}^*_2 \to S^*_2$ is the Hilbert-Chow morphism. Note that $S^*_2 \to S^*_2$ is isomorphic over $[\eta] \in S^*_2$ if the support of $\eta$ consists of two distinct points since then $\eta$ determines the line $\mathbb{P}(V_2)$ such that $\eta \subset \mathbb{P}(V_2)$ uniquely. If the support of $\eta$ consists of one point, say $x$, then the fiber over $[\eta]$ parameterizes lines through $x$. Therefore the exceptional locus $E_{\mathcal{F}}$ of $\tilde{S}^*_2 \to S^*_2$ is a prime divisor isomorphic to $\mathbb{P}(\mathcal{F})$. In particular, $E_{\mathcal{F}}$ has a $\mathbb{P}^{n-1}$-bundle structure over the rank one locus $S^*_2 \simeq \mathbb{P}(V)$.

3.2. Homological properties of certain locally free sheaves on $\mathcal{X}$. For brevity of notation, we set

$\mathcal{X} := S^*_2$, $\tilde{\mathcal{X}} := \tilde{S}^*_2$, $f : \tilde{\mathcal{X}} \to \mathcal{X}$, $g : \mathcal{X} \to G(2, V)$.

We also denote by $H_{\tilde{\mathcal{X}}}$ and $L_{\tilde{\mathcal{X}}}$ the pull-back of $O_{\mathbb{P}(S^2V)}(1)$ and $O_{G(2, V)}(1)$. For brevity of notation, we often omit the subscripts $\tilde{\mathcal{X}}$ from $H_{\tilde{\mathcal{X}}}$ and $L_{\tilde{\mathcal{X}}}$, and $g^*$ for the pull-back of coherent sheaves to $\tilde{\mathcal{X}}$ of $G(2, V)$.

We consider the Euler sequence

$0 \to O_{\tilde{\mathcal{X}}}(-H) \to S^2\mathcal{F} \to T_{\tilde{\mathcal{X}}/G(2, V)}(-H) \to 0$

associated to $\mathbb{P}(S^2\mathcal{F}) \to G(2, V)$. Twisting this by $L$ we obtain

(3.1) $0 \to O_{\tilde{\mathcal{X}}}(-H + L) \to S^2\mathcal{F}(L) \to T_{\tilde{\mathcal{X}}/G(2, V)}(-H + L) \to 0.$

Theorem 3.1. Suppose $n = 3, 4$.

1. The ordered sequence $O_{\tilde{\mathcal{X}}}(-H)$, $O_{\tilde{\mathcal{X}}}(-L)$, $\mathcal{F}$, or $T_{\tilde{\mathcal{X}}/G(2, V)}(-H + L)$ is semi-orthogonal.

2. Let $A$ or $B$ be one of the locally free sheaves $O_{\tilde{\mathcal{X}}}(-H)$, $O_{\tilde{\mathcal{X}}}(-L)$, $\mathcal{F}$, or $T_{\tilde{\mathcal{X}}/G(2, V)}(-H + L)$ on $\tilde{\mathcal{X}}$. Then

$H^*(A^* \otimes B(-t)) = 0$ for $\begin{cases} t = 1 : n = 3, \\ t = 2, 3 : n = 3 \text{ and } B \neq T_{\tilde{\mathcal{X}}/G(2, V)}(-H + L), \\ 1 \leq t \leq 4 : n = 4. \end{cases}$

We prepare the following lemma for our proof of the theorem in case $n = 4$. The lemma follows from [19] Prop.4.8], but for the reader’s convenience we present a proof.
Lemma 3.2. Suppose $n = 4$. For $C = A^* \otimes B$, it holds that $$H^\bullet(\mathcal{X}, C((-t))) \simeq H^{8-\bullet}(\mathcal{X}, C^{*}(t-5)) \text{ for any } t.$$ Proof. By [14, §2.1], we have $K_{\mathcal{X}} = -5H + E_f$ where $E_f$ is the $f$-exceptional divisor. By the Serre duality, we have $H^\bullet(\mathcal{X}, C(-t)) \simeq H^{8-\bullet}(\mathcal{X}, C^{*}(t-5)H + E_f))$ for any $t$. By the exact sequence $$0 \to C^*((t-5)) \to C^*((t-5)H + E_f) \to C^*|(t-5)H + E_f|_{E_f} \to 0,$$ we have only to show that $H^{8-\bullet}(E_f, C^*((t-5)H + E_f)|_{E_f}) = 0$. As we see in Subsection 3.1, the image of $E_f$ by $f$ is $S^*_1$ and $E_f \to S^*_1$ is a $\mathbb{P}^4$-bundle. Therefore it suffices to show the vanishing of cohomology groups of the restriction of $C^*((t-5)H + E_f)|_{E_f}$ to a fiber $\Gamma$ of $E_f \to S^*_2$. By [ibid.], $O_{\mathcal{X}}(E_f)|_{\Gamma} \simeq O_{\mathbb{P}^3}(-2)$ and $O_{\mathcal{X}}(H)|_{\Gamma} \simeq O_{\mathbb{P}^3}$. As we see in Subsection 3.1, $E_f$ parameterizes pairs of a point $x \in \mathbb{P}(V)$ and a line $t$ through $x$. Therefore $\Gamma \simeq \mathbb{P}^3$ parameterizes lines through a fixed point (i.e., $V_1 \subset V_2 \subset V$ for a fixed point $x = [V_1]$). This implies that $g^*\mathcal{F}|_{\Gamma} \simeq O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(1)$. Restricting the natural injection $O_{\mathcal{X}}(-H) \to g^*\mathcal{F}$ to $\Gamma$, we have an injection $$O_{\mathbb{P}^3} \to O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(-1) \oplus O_{\mathbb{P}^3}(-2).$$ Therefore, by the Euler sequence, we have $T_{\mathcal{X}/G(2,V)}(-H + L)|_{\Gamma} \simeq O_{\mathbb{P}^3} \oplus O_{\mathbb{P}^3}(-1)$ and $O_{\mathbb{P}^3}(-3)$ for any $C$ and $t$, hence all of its cohomology groups vanish. \hfill \square

Proof of Theorem 3.1. In any case, we can calculate the cohomology groups in a similar way. Thus we only give computations only for $n = 4$ and $A = \mathcal{F}$, $B = T_{\mathcal{X}/G(2,V)}(-H + L)$. Note that we may assume that $t = 0, 1, 2$ by Lemma 3.2 which simplifies the computations considerably. Twisted (3.1) with $\mathcal{F}^{*}(-tH)$, we obtain

(3.2) $$0 \to \mathcal{F}^{*}(-(t+1)H + L) \to \mathcal{F}^{*} \otimes \mathcal{F}^{*} \to \mathcal{F}^{*} \otimes T_{\mathcal{X}/G(2,V)}(-(t+1)H + L) \to 0.$$ We compute the cohomology groups of $\mathcal{F}^{*}(-(t+1)H + L)$ and $\mathcal{F}^{*} \otimes \mathcal{F}^{*}$. We see that $H^\bullet(\mathcal{F}^{*}(-(t+1)H + L)) = 0$ for $t = 0, 1$ since $\mathcal{F}^{*} \to G(2,V)$ is a $\mathbb{P}^2$-bundle. To compute $H^\bullet(\mathcal{F}^{*}(-(3H + L)))$, we take its Serre dual $H^{8-\bullet}(\mathcal{F}^{*}, \mathcal{F}^{*}(3L)) \simeq H^{8-\bullet}(G(2,V), \mathcal{F}^{*}(-4))$, which vanish by Theorem 2.1. We also see that $H^\bullet(\mathcal{F}^{*} \otimes S^2\mathcal{F}^{*}(-(t+1)H + L)) = 0$ for $t = 1, 2$ since $\mathcal{F}^{*} \to G(2,V)$ is a $\mathbb{P}^2$-bundle, and

$$H^\bullet(\mathcal{F}^{*} \otimes S^2\mathcal{F}(L)) \simeq H^\bullet(G(2,V), \mathcal{F}^{*} \otimes \{S^2\mathcal{F}\}(1)) \simeq H^\bullet(G(2,V), \Sigma^2\mathcal{F}^{*} \otimes \mathcal{F}^{*}),$$

which vanish except for $\bullet = 0$, and

$$H^0(G(2,V), \Sigma^2\mathcal{F}^{*} \otimes \mathcal{F}^{*}) \simeq H^0(G(2,V), \mathcal{F}^{*}) \simeq V^*$$
by Theorem 2.1. Therefore, by (3.2), we have $H^\bullet(\mathcal{F}^{*} \otimes T_{\mathcal{X}/G(2,V)}(-(t+1)H + L)) = 0$ except for $\bullet = 0$ and $t = 0$, and $H^0(\mathcal{F}^{*} \otimes T_{\mathcal{X}/G(2,V)}(-H + L)) \simeq V^*$. \hfill \square
3.3. Dual Lefschetz collection in $\mathcal{D}^b(\mathcal{X})$. It is straightforward to obtain the following result from Theorem 3.1.

**Corollary 3.3.** Suppose $n = 3$. Let $\Lambda := \{3, 2, 1_a, 1_b\}$ be an ordered set $(\Lambda, \prec)$. Define

\[(\mathcal{F}_\alpha)_{\alpha \in \Lambda} := (\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_{1_a}, \mathcal{F}_{1_b}) = (\Omega^1_{\mathcal{X}/G(2,V)}(-L + H), \mathcal{F}^*, \mathcal{O}_{\mathcal{X}}(L), \mathcal{O}_{\mathcal{X}}(H))\]

be an ordered collection of sheaves. We define the following triangulated subcategories of $\mathcal{D}^b(\mathcal{X})$:

- $\mathcal{D}^0_{\mathcal{X}} = \mathcal{D}^1_{\mathcal{X}} := (\mathcal{F}_{1_b}, \mathcal{F}_{1_a}, \mathcal{F}^*_2, \mathcal{F}^*_3)$,
- $\mathcal{D}^2_{\mathcal{X}} = \mathcal{D}^3_{\mathcal{X}} := (\mathcal{F}_{1_b}, \mathcal{F}_{1_a}, \mathcal{F}^*_2)$.

Then

$\mathcal{D}^3_{\mathcal{X}}(-3), \mathcal{D}^2_{\mathcal{X}}(-2), \mathcal{D}^1_{\mathcal{X}}(-1), \mathcal{D}^0_{\mathcal{X}}$

is a dual Lefschetz collection, where $(-t)$ represents the twist by the sheaf $\mathcal{O}_{\mathcal{X}}(-tH)$.

**Remark 3.4.** We can obtain the following results by a similar method to show Theorem 3.1.

1. We see that $\text{Ext}^4(\mathcal{F}^*_3, \mathcal{F}^*_3(-2)) \simeq \mathbb{C}$. This is the reason for the elimination of $\mathcal{F}^*_3$ from $\mathcal{D}^0_{\mathcal{X}}$.
2. $(\mathcal{F}_\alpha)_{\alpha \in \Lambda}$ is a strongly exceptional collection in $\mathcal{D}^b(\mathcal{X})$.
3. Hom’s of the sheaves in the above collection are given by the following diagram:

In case $n = 4$, the following dual Lefschetz collection is suitable for our purpose (see [12]).

**Corollary 3.5.** Suppose $n = 4$. Let $\Lambda := \{3, 2, 1_a, 1_b\}$ be an ordered set $(\Lambda, \prec)$. Define an ordered collection of sheaves on $\mathcal{X}$:

\[(\mathcal{F}_\alpha)_{\alpha \in \Lambda} := (\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_{1_a}, \mathcal{F}_{1_b}) = (\mathcal{O}_{\mathcal{X}}, \mathcal{F}^*, T_{\mathcal{X}/G(2,V)}(-H + 2L), \mathcal{O}_{\mathcal{X}}(L)).\]

Set $\mathcal{D}_{\mathcal{X}} := \langle \mathcal{F}_{1_b}^*, \mathcal{F}_{1_a}^*, \mathcal{F}_2^*, \mathcal{F}_3^* \rangle \subset \mathcal{D}^b(\mathcal{X})$. Then

$\mathcal{D}_{\mathcal{X}}(-4), \ldots, \mathcal{D}_{\mathcal{X}}(-1), \mathcal{D}_{\mathcal{X}}$

is a dual Lefschetz collection, where $(-t)$ represents the twist by the sheaf $\mathcal{O}_{\mathcal{X}}(-tH)$. 
Proof. By taking the dual of the sheaves of Theorem 3.1, we have the following dual Lefschetz collection:

\[ \Omega_{\mathcal{X}/G(2,V)}(-3H - L), \mathcal{F}^*(4H), \mathcal{O}_{\mathcal{X}}(-4H + L), \mathcal{O}_{\mathcal{X}}(-3H), \quad \ldots \]

\[ \Omega_{\mathcal{X}/G(2,V)}(H - L), \mathcal{F}^*, \mathcal{O}_{\mathcal{X}}(L), \mathcal{O}_{\mathcal{X}}(H). \]

Let \( \mathcal{C} \) be one of the sheaves in this collection except \( \mathcal{O}_{\mathcal{X}}(H) \). Then by Lemma 3.2, it holds that

\[ \text{Hom}^\bullet(\mathcal{O}_{\mathcal{X}}(H), \mathcal{C}) \simeq \text{Hom}^{8 - \bullet}(\mathcal{C}, \mathcal{O}_{\mathcal{X}}(-4H)). \]

Therefore we obtain the following dual Lefschetz collection.

\[ \mathcal{O}_{\mathcal{X}}(-4H), \Omega_{\mathcal{X}/G(2,V)}(-3H - L), \mathcal{F}^*(4H), \mathcal{O}_{\mathcal{X}}(-4H + L), \quad \ldots \]

\[ \mathcal{O}_{\mathcal{X}}, \Omega_{\mathcal{X}/G(2,V)}(H - L), \mathcal{F}^*, \mathcal{O}_{\mathcal{X}}(L). \]

Tensoring this with \( \mathcal{O}_{\mathcal{X}}(-L) \), we obtain the desired dual Lefschetz collection. \( \square \)

Remark 3.6. We may also obtain the following results by a similar method to show Theorem 3.1 (see [13]).

1. \( (\mathcal{F}_\alpha)_{\alpha \in \Lambda} \) is a strongly exceptional collection of \( \mathcal{D}^b(\mathcal{X}) \).
2. Hom’s of the sheaves in the above collection are given by the following diagram:

3.4. Categorical and noncommutative resolution of \( \mathcal{D}^b(\mathcal{X}) \). Since \( \mathcal{X} \rightarrow \mathcal{X} \) is a resolution of rational singularities whose exceptional locus is a prime divisor \( E_f \), we have a triangulated subcategory \( \mathcal{D} \subset \mathcal{D}^b(\mathcal{X}) \) called a categorical resolution of \( \mathcal{D}^b(\mathcal{X}) \) for every dual Lefschetz decomposition of \( \mathcal{D}^b(E_f) \) [13, Theorem 1]. There is a natural dual Lefschetz decomposition of \( \mathcal{D}(E_f) \) for the \( \mathbb{P}^{n-1} \)-bundle \( E_f \rightarrow S^1 = v_2(\mathbb{P}(V)) \) [24]:

1. \( n \) is even

\[ \mathcal{D}^b(E_f) = \langle C_{-1}(-n + 2), \ldots, C_1(-2), C_0 \rangle, \]

where \( C_0 = C_1 = \cdots = C_{-k} = \langle (f|_{E_f})^* \mathcal{D}^b(\mathbb{P}(V))(-1), (f|_{E_f})^* \mathcal{D}^b(\mathbb{P}(V)) \rangle \),

where \((-k)\) is the twist by \( \mathcal{O}_{\mathcal{X}}(-kL)|_{E_f} \).

2. \( n \) is odd

\[ \mathcal{D}^b(E_f) = \langle C_{-k}(-n + 1), \ldots, C_1(-2), C_0 \rangle, \]

where \( C_0 = C_1 = \cdots = C_{-k} = \langle (f|_{E_f})^* \mathcal{D}^b(\mathbb{P}(V))(-1), (f|_{E_f})^* \mathcal{D}^b(\mathbb{P}(V)) \rangle \), and

\[ C_{-k} = \langle (f|_{E_f})^* \mathcal{D}^b(\mathbb{P}(V)) \rangle. \]
Let $\mathcal{D}$ be the triangulated subcategory of $\mathcal{D}^b(\mathcal{X})$ which consists of objects $F$ such that $i^* F \in \mathcal{C}_0$, where $i$ is the natural closed embedding $E_f \to \mathcal{X}$. By [19], $\mathcal{D}^b(\mathcal{X})$ has the following semi-orthogonal decomposition:

$$
\mathcal{D}^b(\mathcal{X}) = \left\{ \begin{array}{ll}
\mathcal{D} \ni C_{-1}(-n+2), \ldots, C_1(-2), & n \text{ is even.} \\
\mathcal{D} \ni C_{-1}(-n+1), \ldots, C_1(-2), & n \text{ is odd.}
\end{array} \right.
$$

Recall that $\mathcal{X}$ is Gorenstein if and only if $n$ is even. [19] §2.1]. When $n$ is even, $\mathcal{D}$ is strongly crepant. Indeed, in this case, the conditions of [ibid., Prop. 4.7] holds:

- $C_0 = C_1 = \cdots = C_{n-1}$(the decomposition is called rectangular), and
- the discrepancy of $f$ is $n^2$ ([14] §2.1]), which is equal to the length of the decomposition of $\mathcal{D}^b(E_f)$.

The categorical resolution is also related to the noncommutative resolution by Van den Bergh ([19] Theorem 2). It is easy to see that

$$
\mathcal{R} := f_* \text{End}(O_{\mathcal{X}} \oplus O_{\mathcal{X}}(-L))
$$

satisfies the assumptions of [ibid.]; to check $\mathcal{C}_0$ is generated by $i^* \mathcal{E}$ is obvious, and to check $\mathcal{E}$ is tilting follows from the standard relative vanishing theorem. Thus

$$
\mathcal{D} \simeq \mathcal{D}(\mathcal{X}, \mathcal{R}).
$$

3.5. Dual Lefschetz decomposition of the categorical resolution for $n = 2, 3, 4$.

**Theorem 3.7.** For $n = 2, 3, 4$, we have the following dual Lefschetz decomposition of $\mathcal{D}$:

$$
\mathcal{D} \simeq \langle A_0(-n), \ldots, A_1(-1), A_0 \rangle,
$$

where

$$
\begin{align*}
A_0 &= A_1 = A_2 = \langle O_{\mathcal{X}}(-H), O_{\mathcal{X}}(-L), \mathcal{F} \rangle \text{ for } n = 2, \\
A_0 &= A_1 = \langle O_{\mathcal{X}}(-H), O_{\mathcal{X}}(-L), \mathcal{F}, T_{\mathcal{X}/G(2, V)}(-H+L) \rangle, \text{ and} \\
A_2 &= A_3 = \langle O_{\mathcal{X}}(-H), O_{\mathcal{X}}(-L), \mathcal{F}, T_{\mathcal{X}/G(2, V)}(-H+L) \rangle \text{ for } n = 3, \\
A_0 &= \cdots = A_4 = \langle O_{\mathcal{X}}(-H), O_{\mathcal{X}}(-L), \mathcal{F}, T_{\mathcal{X}/G(2, V)}(-H+L) \rangle \text{ for } n = 4.
\end{align*}
$$

We set

$$
\mathcal{I}_n := \mathbb{P}(S^2 \mathcal{F}), \quad \mathcal{X}_n := S^2 \mathbb{P}(V) \text{ for } \text{dim } V = n + 1.
$$

**Proof of Theorem 3.7 in case $n = 2$.**

Note that $\mathcal{D}$ is equivalent to $\mathcal{D}^b(\mathcal{X}_2)$ since $\mathcal{X}_2 \to \mathcal{X}_2$ is crepant. Since $\mathcal{X}_2$ is a $\mathbb{P}^2$-bundle over the projective plane $G(2, V) \simeq \mathbb{P}(V^*)$, the derived category $\mathcal{D}^b(\mathcal{X}_2)$ has the following standard semi-orthogonal decomposition by Beilinson’s and Orlov’s results:

$$
\mathcal{D}^b(\mathcal{X}_2) = \langle O_{\mathcal{X}_2}(-2H-L), \mathcal{F}(-2H), O_{\mathcal{X}_2}(-2H), \mathcal{F}(-H), O_{\mathcal{X}_2}(-H), \mathcal{F}, O_{\mathcal{X}_2}(-L), \mathcal{F} \rangle.
$$

Since $K_{\mathcal{X}_2} = -3H$, we obtain by mutating $O_{\mathcal{X}_2}$ to the left;

$$
\mathcal{D}^b(\mathcal{X}_2) = \langle O_{\mathcal{X}_2}(-3H), O_{\mathcal{X}_2}(-2H-L), \mathcal{F}(-2H), \mathcal{F}(-H), O_{\mathcal{X}_2}(-2H), O_{\mathcal{X}_2}(-H-L), \mathcal{F}(-H), \mathcal{F}(-H), O_{\mathcal{X}_2}(-H), O_{\mathcal{X}_2}(-L), \mathcal{F} \rangle.
$$

(3.4)
The rest of this subsection is occupied with our proof of Theorem 3.7 in case $n = 3, 4$.

We have already shown that $\mathcal{A}_n(-n), \ldots, \mathcal{A}_1(-1), \mathcal{A}_0$ is semi-orthogonal in Theorem 3.1. Besides, they are contained in the left orthogonal to

$$
\iota_!C_1(-2) := \begin{cases} 
\iota_!(f|_{E_j})^*\mathcal{D}^b(\mathbb{P}^3)(-2) : n = 3, \\
((f|_{E_j})^*\mathcal{D}^b(\mathbb{P}^4)(-3), (f|_{E_j})^*\mathcal{D}^b(\mathbb{P}^4)(-2)) : n = 4
\end{cases}
$$

since the restrictions of

$$
\mathcal{O}_{\mathcal{X}_n}(-H), \mathcal{O}_{\mathcal{X}_n}(-L), \mathcal{F}, T_{\mathcal{X}_n/G(2,V)}(-H + L)
$$

to a fiber $\mathbb{P}^{n-1}$ of $E_j \to \mathbb{P}(V)$ are direct sums of $\mathcal{O}_{\mathbb{P}^{n-1}}$ and $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ (cf. the proof of Lemma 3.2).

We set

$$
\begin{cases} 
\mathcal{A} := \mathcal{A}_0 = \mathcal{A}_1, \mathcal{A} := \mathcal{A}_2 = \mathcal{A}_3 : n = 3, \\
\mathcal{A} := \mathcal{A}_0 = \cdots = \mathcal{A}_4 : n = 4.
\end{cases}
$$

We define the following triangulated subcategory $\mathcal{C} \subset \mathcal{D}^b(\mathcal{X}_n)$:

$$
\mathcal{C} := \begin{cases} 
\perp (\iota_!C_1(-2), \mathcal{A}'(-3), \mathcal{A}'(-2), (\mathcal{A}(-1), \mathcal{A}) : n = 3, \\
\perp (\iota_!C_1(-2), \mathcal{A}(-4), \mathcal{A}(-3), \mathcal{A}(-2), (\mathcal{A}(-1), \mathcal{A}) : n = 4.
\end{cases}
$$

Then, by [2], we have the following semiorthogonal decomposition of $\mathcal{D}^b(\mathcal{X}_n)$:

(3.5)

$$
\mathcal{D}^b(\mathcal{X}_3) = \langle i_*C_1(-2), \mathcal{O}_{X_3}(-4H), \mathcal{O}_{X_3}(-3H - L), \mathcal{F}(-3H), \\
\mathcal{O}_{X_3}(-3H), \mathcal{O}_{X_3}(-2H - L), \mathcal{F}(-2H), \\
\mathcal{O}_{X_3}(-2H), \mathcal{O}_{X_3}(-H - L), \mathcal{F}(-H), \\
\mathcal{O}_{X_3}(-H), \mathcal{O}_{X_3}(-L), \mathcal{F}, T_{X_3/G(2,V)}(-2H + L), T_{X_3/G(2,V)}(-H + L) \rangle.
$$

Note that the restriction of the gray part of (3.5) corresponds to the r.h.s. of (3.4).

(3.6)

$$
\mathcal{D}^b(\mathcal{X}_4) = \langle i_*C_1(-2), \mathcal{O}_{X_4}(-5H), \mathcal{O}_{X_4}(-4H - L), \mathcal{F}(-4H), \\
\mathcal{O}_{X_4}(-4H), \mathcal{O}_{X_4}(-3H - L), \mathcal{F}(-3H), T_{X_4/G(2,V)}(-5H + L), T_{X_4/G(2,V)}(-4H + L) \\
\mathcal{O}_{X_4}(-3H), \mathcal{O}_{X_4}(-2H - L), \mathcal{F}(-2H), T_{X_4/G(2,V)}(-3H + L), T_{X_4/G(2,V)}(-2H + L), \\
\mathcal{O}_{X_4}(-2H), \mathcal{O}_{X_4}(-H - L), \mathcal{F}(-H), T_{X_4/G(2,V)}(-2H + L), T_{X_4/G(2,V)}(-H + L) \rangle.
$$

Note that the restriction of the gray part of (3.6) corresponds to a part of the r.h.s. of (3.5).

It remains to show the fullness of the collection $\iota_!C_1(-2), \mathcal{A}_n(-n), \ldots, \mathcal{A}_1(-1), \mathcal{A}_0$, equivalently, $\mathcal{C} = 0$. Following the inductive argument of Kuznetsov in the proof of [20] Thm. 4.1, we reduce the proof for $\mathcal{X}_3$ to the fullness of the collection for $\mathcal{X}_2$, and then the proof for $\mathcal{X}_4$ to the fullness of the collection for $\mathcal{X}_3$. For this, we take an $n$-dimensional vector subspace $V' \subset V$. Then we simply denote $\mathbb{P}(\mathbb{S}^2F)$ over $G(2,V')$ by $\mathcal{X}_{n-1}$ as a subvariety of $\mathcal{X}_n$. Let $j : \mathcal{X}_{n-1} \to \mathcal{X}_n$ be the natural closed immersion. It is well-known (see [20] Lem. 4.41) that the Koszul resolution of $j_*\mathcal{O}_{\mathcal{X}_{n-1}}$ is the following form:

(3.7)

$$
0 \to \mathcal{O}_{\mathcal{X}_n}(-L) \to \mathcal{F} \to \mathcal{O}_{\mathcal{X}_{n-1}} \to j_*\mathcal{O}_{\mathcal{X}_{n-1}} \to 0.
$$

Then, by a similar result to [20] Lem. 4.5, we have only to show $j^*\mathcal{E} = 0$ for any object $\mathcal{E}$ of $\mathcal{C}$ since we may choose $V'$ freely and we may assume the fullness of the
collection for $\mathcal{X}_{n-1}$. For this, arguing as in [20], p.165, we have only to show the following claim by [3.7):

**Lemma 3.8.** The following sheaves of the form $\mathcal{P} \otimes \mathcal{Q}$ are contained in $\mathcal{C}^\perp$:

- $(n=3)$ $\mathcal{P}$ is one of the sheaves in the gray part of $\{3.5\}$, namely, $\mathcal{P} = \mathcal{O}_{\mathcal{X}_3}(-iH) \ (1 \leq i \leq 3)$, $\mathcal{O}_{\mathcal{X}_3}(-iH - L) \ (0 \leq i \leq 2)$, $\mathcal{F}(-iH) \ (0 \leq i \leq 2)$, $\mathcal{Q}$ is one of the sheaves in the exact sequence $\{3.7\}$ except $j_* \mathcal{O}_{\mathcal{X}_2}$, namely, $\mathcal{Q} = \mathcal{O}_{\mathcal{X}_3}(-L)$, $\mathcal{F}$, $\mathcal{O}_{\mathcal{X}_3}$.

- $(n=4)$ $\mathcal{P}$ is one of the sheaves in the gray part of $\{3.6\}$, namely, $\mathcal{P} = \mathcal{O}_{\mathcal{X}_4}(-iH) \ (1 \leq i \leq 4)$, $\mathcal{O}_{\mathcal{X}_4}(-iH - L) \ (0 \leq i \leq 3)$, $\mathcal{F}(-iH) \ (0 \leq i \leq 3)$, $T_{\mathcal{X}_4/G(2,V)}(-iH + L) \ (i = 1, 2)$, $\mathcal{Q}$ is one of the sheaves in the exact sequence $\{3.7\}$ except $j_* \mathcal{O}_{\mathcal{X}_3}$, namely, $\mathcal{Q} = \mathcal{O}_{\mathcal{X}_4}(-L)$, $\mathcal{F}$, $\mathcal{O}_{\mathcal{X}_4}$.

To show Lemma 3.8 we prepare the following three results:

**Lemma 3.9.** We denote by $H_{\mathcal{P}(\mathcal{F})}$ the tautological divisor of $E_f \simeq \mathbb{P}(\mathcal{F})$. For any $k \in \mathbb{Z}$, there exist the following exact sequences:

1. $0 \to \mathcal{O}_{\mathcal{X}}(-2H - kL) \to \mathcal{O}_{\mathcal{X}}(-(2 + k)L) \to \mathcal{O}_{E_f}(-(2 + k)L) \to 0$, (3.8)
2. $0 \to \mathcal{F}(-H - kL) \to \mathcal{F}(-(k + 1)L) \to \mathcal{O}_{E_f}(H_{\mathcal{P}(\mathcal{F})} - (k + 2)L) \to 0$, (3.9)
3. $0 \to T_{\mathcal{X}/G(2,V)}(-2H - kL) \to T_{\mathcal{X}/G(2,V)}(-H - (k + 1)L) \to \mathcal{O}_{E_f}(2H_{\mathcal{P}(\mathcal{F})} - (k + 3)L) \to 0$, (3.10)
4. $0 \to \mathcal{O}_{\mathcal{X}}(-(H)) \to \mathcal{S}^2 \mathcal{F} \to T_{\mathcal{X}/G(2,V)}(-H) \to 0$, (3.11)
5. $0 \to T_{\mathcal{X}/G(2,V)}(-(H)) \to \mathcal{S}^2 \mathcal{F}(-H) \to \mathcal{O}_{\mathcal{X}}(2H - 3L) \to 0$, (3.12)

**Proof.** Noting that $E_f \sim 2(H - L)$ (14 §2.1]), (3.8) is obtained from the standard exact sequence $0 \to \mathcal{O}_{\mathcal{X}}(-(E_f)) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{E_f} \to 0$ by tensoring $\mathcal{O}_{\mathcal{X}}(-(k + 2)L)$. (3.10) is nothing but the Euler sequence. (3.11) is obtained by dualizing and twisting (3.11).

We will construct the exact sequence (3.9). Twisting $\mathcal{O}_{\mathcal{X}}(kL)$, it suffices to show the existence of the exact sequence

1. $0 \to \mathcal{F}(-(H)) \to \mathcal{F}(-(L)) \to \mathcal{O}_{E_f}(H_{\mathcal{P}(\mathcal{F})} - 2L) \to 0$, (3.13)

where we note that $\mathcal{O}_{\mathcal{X}}(H)|_{E_f} \simeq \mathcal{O}_{E_f}(2H_{\mathcal{P}(\mathcal{F})})$. The construction below is nothing but a relativization of the Kapranov’s construction of the spinor sheave $\mathcal{O}_{\mathbb{P}^1}(1)$ on a plane conic. By the Littlewood-Richardson rule, we have $\mathcal{S}^2 \mathcal{F} \otimes \mathcal{F} \simeq \mathcal{S}^3 \mathcal{P} \oplus \mathcal{F}(-1)$ on $G(2,V)$. Therefore we have a map $p: \mathcal{S}^2 \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}(-1)$ by the projection to the second factor. Let $[V_2] \in G(2,V)$ be any point. It is convenient to identify $\mathcal{S}^2 V_2$ with the spaces of binary $k$-forms with variables $x, y$. Then the map $p$ coincides up to constant with the SL($V_2$)-equivariant map $\delta: \mathcal{S}^2 V_2 \otimes V_2 \to V_2$ such that

$$\delta(f, g) = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$$
for \( f \in S^2V_2 \) and \( g \in V_2 \). Computing explicitly, we see that the composite
\[
S^2V_2 \otimes S^2V_2 \otimes V_2 \xrightarrow{1\otimes \delta} S^2V_2 \otimes V_2 \xrightarrow{\delta} V_2
\]
satisfies \((f, f, g) \mapsto (b^2 - 4ac)g\), where \( f = ax^2 + bxy + cy^2 \). Since \( S^2(S^2V_2) \cong S^4V_2 \oplus \mathbb{C} \) as an \( SL(V_2) \)-module, the map \( S^2V_2 \otimes S^2V_2 \to \mathbb{C} \) satisfying \((f, f) \mapsto b^2 - 4ac\) corresponding to the projection \( S^2(S^2V_2) \to \mathbb{C} \) to the second factor. Therefore the composite
\[
S^2\mathcal{F} \otimes S^2\mathcal{F} \otimes \mathcal{F} \xrightarrow{1\otimes \sigma} S^2\mathcal{F} \otimes \mathcal{F}(-1) \xrightarrow{\sigma} \mathcal{F}(-2)
\]
coincides with the map \( S^2\mathcal{F} \otimes S^2\mathcal{F} \otimes \mathcal{F} \to \mathcal{O}_{G(2,V)}(-2) \otimes \mathcal{F} \) induced from the projection
\[
S^2\mathcal{F} \otimes S^2\mathcal{F} \to S^2(S^2\mathcal{F}) \cong S^4\mathcal{F} \oplus \mathcal{O}_{G(2,V)}(-2) \to \mathcal{O}_{G(2,V)}(-2).
\]
Since the map \( S^2(S^2\mathcal{F}) \to \mathcal{O}_{G(2,V)}(-2) \) corresponds to the conic fibration \( E_f \to G(2,V) \), we see that \( p \) is the Clifford multiplication associated to \( E_f \to G(2,V) \). Therefore, by the construction of the spinor sheaf \( \mathcal{O}_{\mathbb{P}^1}(1) \) on a plane conic, we see that the map on \( \mathcal{F} \)
\[
\mathcal{O}_{\mathcal{F}}(-H) \otimes \mathcal{F} \hookrightarrow S^2\mathcal{F} \otimes \mathcal{F} \xrightarrow{p} \mathcal{F}(-L)
\]
is injective and the cokernel \( L \) is an invertible sheave on \( E_f \) of the form \( \mathcal{O}_{E_f}(H_{\mathbb{P}^1}(\mathcal{F}) + aL) \) with some \( a \in \mathbb{Z} \). Since the induced map \( \mathcal{F}(-L) \to L \) is \( SL(V) \)-equivariant, we see that \( a = -2 \). Therefore we have obtained the desired exact sequence (3.14).

The construction of (3.10) is similar to that of (3.9), so we only give a sketch below. Twisting \( \mathcal{O}_{\mathcal{F}}(kL) \), it suffices to show the existence of the exact sequence
\[
0 \to T_{\mathcal{F}/G(2,V)}(-2H) \to T_{\mathcal{F}/G(2,V)}(-H - L) \to \mathcal{O}_{E_f}(2H_{\mathbb{P}^1}(\mathcal{F}) - 3L) \to 0.
\]
By the Littlewood-Richardson rule, we have \( S^2\mathcal{F} \otimes S^2\mathcal{F} \simeq S^4\mathcal{F} \otimes \mathcal{O}_{G(2,V)}(-2) \) on \( G(2,V) \). Therefore we have a map \( q: S^2\mathcal{F} \otimes S^2\mathcal{F} \to S^2\mathcal{F}(-1) \) by the projection to the middle factor. We see that this map is locally defined as the map \( \delta \) above. Now we consider the composite of the maps on \( \mathcal{F} \)
\[
\mathcal{O}_{\mathcal{F}}(-H) \otimes S^2\mathcal{F} \hookrightarrow S^2\mathcal{F} \otimes S^2\mathcal{F} \xrightarrow{q} S^2\mathcal{F}(-L).
\]
Then, by local computations, we see that the restriction of this map to \( \mathcal{O}_{\mathcal{F}}(-H) \otimes \mathcal{O}_{\mathcal{F}}(-H) \) is zero. Therefore, noting \( S^2\mathcal{F}/\mathcal{O}_{\mathcal{F}}(-H) \simeq T_{\mathcal{F}/G(2,V)}(-H) \), we have the following map:
\[
\overline{q}: \mathcal{O}_{\mathcal{F}}(-H) \otimes T_{\mathcal{F}/G(2,V)}(-H) \to S^2\mathcal{F}(-L) \to T_{\mathcal{F}/G(2,V)}(-H - L).
\]
We consider the composite
\[
\mathcal{O}_{\mathcal{F}}(-H) \otimes \mathcal{O}_{\mathcal{F}}(-H) \otimes T_{\mathcal{F}/G(2,V)}(-H) \xrightarrow{1\otimes \overline{q}} \mathcal{O}_{\mathcal{F}}(-H) \otimes T_{\mathcal{F}/G(2,V)}(-H - L) \xrightarrow{\overline{q}} T_{\mathcal{F}/G(2,V)}(-H - 2L).
\]
Note that, by (3.13), we obtain a map
\[
\mathcal{O}_{\mathcal{F}}(-2H) \to \mathcal{O}_{\mathcal{F}}(-2L)
\]
 twisting the natural map \( \mathcal{O}_{\mathcal{F}}(-E_f) \to \mathcal{O}_{\mathcal{F}}(-2L) \). As in the case of (3.10), we see by local computations that (3.17) coincides with the map \( S^2\mathcal{F} \otimes T_{\mathcal{F}/G(2,V)}(-H) \). Therefore, \( \overline{q} \) is a Clifford multiplication. Again, by local computations, we see that \( \overline{q} \) is injective and the cokernel \( M \) is an invertible sheave on \( E_f \) of the form \( \mathcal{O}_{E_f}(2H_{\mathbb{P}^1}(\mathcal{F}) + bL) \) with some \( b \in \mathbb{Z} \). Since the induced map
\[
\mathcal{O}_{\mathcal{F}}(-2H) \to \mathcal{O}_{\mathcal{F}}(-2L)
\]
Proof of Lemma 3.8. First we assume that $\mathcal{P} = \mathcal{P}_3(\mathcal{F})$, then $\mathcal{P} \otimes \mathcal{F} = \mathcal{F}(-i\mathcal{F}) \in \mathcal{C}^\perp$ by Corollary 3.10 (1). If $\mathcal{P} = \mathcal{P}_3(\mathcal{F})$, then $\mathcal{P} \otimes \mathcal{F} = \mathcal{F}(-i\mathcal{F}) \in \mathcal{C}^\perp$ by Corollary 3.10 (3).

**Corollary 3.10.** For $n = 3, 4$, the following sheaves are contained in $\mathcal{C}^\perp$:

1. $\mathcal{O}_{\mathcal{X}_n}(-i\mathcal{F} - 2\mathcal{L})$ for $n = 3, 4$ and $-1 \leq i \leq n - 1$.
2. $\mathcal{O}_{\mathcal{X}_4}(-i\mathcal{F} - 3\mathcal{L})$ for $n = 4$ and $-2 \leq i \leq 2$.
3. $\mathcal{F}(-i\mathcal{F} - \mathcal{L})$ for $n = 3, 4$ and $-1 \leq i \leq n - 1$.
4. $\mathcal{F}(-i\mathcal{F} - 2\mathcal{L})$ for $n = 4$ and $-2 \leq i \leq 2$.
5. $T_{\mathcal{F}_n/\mathcal{G}(2, \mathcal{V})}(-i\mathcal{F})$ for $n = 3$ and $i = 0, 1$, and $n = 4$ and $0 \leq i \leq 4$.
6. $T_{\mathcal{F}_4/\mathcal{G}(2, \mathcal{V})}(-i\mathcal{F} - \mathcal{L})$ for $n = 4$ and $-1 \leq i \leq 3$.

**Proof.** All the assertions can be proved in a similar way by using the exact sequences in Lemma 3.9. Thus it should suffice to prove some of them.

To show (3), we consider $\mathcal{O}_{\mathcal{X}_4}(-i\mathcal{F}) \otimes \mathcal{F}$ for $k = 0$:

$$0 \to \mathcal{F}(-i + 1) \to \mathcal{F}(-i\mathcal{F} - \mathcal{L}) \to \mathcal{O}_{\mathcal{F}_4}((-2i + 1)\mathcal{F} - 2\mathcal{L}) \to 0.$$ 

Since $\mathcal{O}_{\mathcal{F}_4}((-2i + 1)\mathcal{F} - 2\mathcal{L}) \in i\mathcal{C}(\mathcal{F}) \subset \mathcal{C}^\perp$, and $\mathcal{F}(-i + 1) \mathcal{F} \in \mathcal{C}^\perp$ for $0 \leq i + 1 \leq n$ by Corollary 3.5, we have $\mathcal{F}(-i\mathcal{F} - \mathcal{L}) \in \mathcal{C}^\perp$ for $-1 \leq i \leq n - 1$.

To show (4), we consider $\mathcal{O}_{\mathcal{F}_4}(-i\mathcal{F}) \otimes \mathcal{F}$ for $k = 1$:

$$0 \to \mathcal{F}(-i + 1) \mathcal{F} - \mathcal{L} \to \mathcal{F}(-i\mathcal{F} - 2\mathcal{L}) \to \mathcal{O}_{\mathcal{F}_4}((-2i + 1)\mathcal{F} - 3\mathcal{L}) \to 0.$$ 

Since $\mathcal{O}_{\mathcal{F}_4}((-2i + 1)\mathcal{F} - 3\mathcal{L}) \in i\mathcal{C}(\mathcal{F}) \subset \mathcal{C}^\perp$ for $n = 4$, and $\mathcal{F}(-i + 1) \mathcal{F} - \mathcal{L} \in \mathcal{C}^\perp$ for $-2 \leq i \leq 3$ by (3) as we have proved, we have $\mathcal{F}(-i\mathcal{F} - 2\mathcal{L}) \in \mathcal{C}^\perp$ for $-2 \leq i \leq 2$.

**Lemma 3.11.** (1) For $n = 3$, there exist the following exact sequences:

(3.19) $$0 \to S^2\mathcal{F} \to V \otimes \mathcal{F} \to \wedge^2 V \otimes \mathcal{O}_{\mathcal{X}_3} \to \mathcal{O}_{\mathcal{X}_3}(\mathcal{L}) \to 0,$$

(3.20) $$0 \to \mathcal{O}_{\mathcal{X}_3}(-3\mathcal{L}) \to \wedge^2 V^* \otimes \mathcal{O}_{\mathcal{X}_3}(-2\mathcal{L}) \to V^* \otimes \mathcal{F}(-\mathcal{L}) \to S^2\mathcal{F} \to 0.$$

(2) For $n = 4$, there exist the following exact sequences:

(3.21) $$0 \to S^2\mathcal{F} \to V \otimes \mathcal{F} \to \wedge^2 V \otimes \mathcal{O}_{\mathcal{X}_4} \to V^* \otimes \mathcal{O}_{\mathcal{X}_4}(\mathcal{L}) \to \mathcal{F}^*(\mathcal{L}) \to 0,$$

(3.22) $$0 \to \mathcal{O}_{\mathcal{X}_4}(-4\mathcal{L}) \to \wedge^3 V^* \otimes \mathcal{O}_{\mathcal{X}_4}(-3\mathcal{L}) \to \wedge^2 V^* \otimes \mathcal{F}(-2\mathcal{L}) \to V^* \otimes S^2\mathcal{F}(-\mathcal{L}) \to S^3\mathcal{F} \to 0.$$

**Proof.** See the proof of Lem. 4.3. □

**Proof of Lemma 3.8** First we assume that $n = 3$.

**Case** $\mathcal{Q} = \mathcal{O}_{\mathcal{X}_3}$. The assertion holds since $\mathcal{P} \in \mathcal{C}^\perp$.

**Case** $\mathcal{Q} = \mathcal{O}_{\mathcal{X}_3}(\mathcal{F})$. If $\mathcal{P} = \mathcal{O}_{\mathcal{X}_3}(-i\mathcal{F}) (1 \leq i \leq 3)$, then $\mathcal{P}(-\mathcal{F}) = \mathcal{O}_{\mathcal{X}_3}(-i\mathcal{F} - \mathcal{L}) \in \mathcal{C}^\perp$ by Corollary 3.11 (1). If $\mathcal{P} = \mathcal{O}_{\mathcal{X}_3}(-i\mathcal{F} - \mathcal{L}) (0 \leq i \leq 2)$, then $\mathcal{P}(-\mathcal{F}) = \mathcal{F}(-i\mathcal{F} - \mathcal{L}) \in \mathcal{C}^\perp$ by Corollary 3.11 (3).

**Case** $\mathcal{Q} = \mathcal{F}$. If $\mathcal{P} = \mathcal{O}_{\mathcal{X}_3}(-i\mathcal{F}) (1 \leq i \leq 3)$, then $\mathcal{F} \otimes \mathcal{P} = \mathcal{F}(-i\mathcal{F}) \in \mathcal{C}^\perp$ by Corollary 3.10 (3). Finally we assume that $\mathcal{P} = \mathcal{F}(-i\mathcal{F}) (0 \leq i \leq 2)$. We note that $\mathcal{F} \otimes \mathcal{F}(-i\mathcal{F}) \simeq$
$O_{\mathcal{F}_1}(-iH-L) \oplus S^2\mathcal{F}(-iH)$. Since $O_{\mathcal{F}_1}(-iH-L) \in \mathcal{C}^1$ by (3.5), it remains to show $S^2\mathcal{F}(-iH) \in \mathcal{C}^1$. For $S^2\mathcal{F}$, it suffices to show that $O_{\mathcal{F}_1}(-H), T_{\mathcal{F}_1/G(2,V)}(-H) \in \mathcal{C}^1$ by (5.11). By (5.5), $O_{\mathcal{F}_1}(-H) \in \mathcal{C}^1$. Moreover, by Corollary 3.10 (5), we see that $T_{\mathcal{F}_1/G(2,V)}(-H) \in \mathcal{C}^1$. The argument for $S^2\mathcal{F}(-iH)$ ($i = 1, 2$) is slightly involved.

We use also Lemma 3.11. By considering (3.19) $\otimes O_{\mathcal{F}_1}(-iH)$, we are reduced to show $\mathcal{F}(-iH), O_{\mathcal{F}_1}(-iH), O_{\mathcal{F}_1}(-iH + L) \in \mathcal{C}^1$. By (5.5), $\mathcal{F}(-iH), O_{\mathcal{F}_1}(-iH) \in \mathcal{C}^1$. For $O_{\mathcal{F}_1}(-iH + L)$, we consider (3.11) $\otimes O_{\mathcal{F}_1}(-(i-1)H + L)$. Then we are reduced to show that $S^2\mathcal{F}(-(i-1)H + L) \in \mathcal{C}^1$ since $T_{\mathcal{F}_1/G(2,V)}(-(i-1)H + L) \in \mathcal{C}^1$ by (5.4). For $S^2\mathcal{F}(-(i-1)H)$, we consider (3.20) $\otimes O_{\mathcal{F}_1}(-(i-1)H)$ + L). Then we are reduced to show that $O_{\mathcal{F}_1}(-(i-1)H) \in \mathcal{C}^1$ by Corollary 3.11 (1).

Next consider the case $n = 4$.

**Case** $Q = O_{\mathcal{F}_1}$. The assertion holds since $\mathcal{P} \in \mathcal{C}^1$.

**Case** $Q = O_{\mathcal{F}_1}(-iH)$. If $\mathcal{P} = O_{\mathcal{F}_1}(-iH)$ ($1 \leq i \leq 4$), then $\mathcal{P}(-L) = O_{\mathcal{F}_1}(-iH-L) \in \mathcal{C}^1$ by (3.6). If $\mathcal{P} = O_{\mathcal{F}_1}(-(i-1)H-L)$ ($0 \leq i \leq 3$), then $\mathcal{P}(-L) = O_{\mathcal{F}_1}(-(i-1)H-2L) \in \mathcal{C}^1$ by Corollary 3.10 (1). If $\mathcal{P} = \mathcal{F}(-iH)$ ($0 \leq i \leq 3$), then $\mathcal{P}(-L) = \mathcal{F}(-(i-1)H-L) \in \mathcal{C}^1$ by Corollary 3.10 (3). If $\mathcal{P} = T_{\mathcal{F}_1/G(2,V)}(-(i-1)H + L)$ ($i = 1, 2$), then $\mathcal{P}(-L) = T_{\mathcal{F}_1/G(2,V)}(-iH) \in \mathcal{C}^1$ by Corollary 3.10 (5).

The argument below is slightly involved. Considering (3.11) $\otimes \mathcal{F}(-(i-1)H + L)$, we are reduced to show that $S^2\mathcal{F} \otimes \mathcal{F}(-(i-2)H), \mathcal{F}(-(i-3)H + 2L) \in \mathcal{C}^1$. We have $\mathcal{F}(-(i-3)H + 2L) \in \mathcal{C}^1$ by Corollary 3.11 (4). As for $S^2\mathcal{F} \otimes \mathcal{F}(-(i-2)H)$, we note that the decomposition $S^2\mathcal{F} \otimes \mathcal{F}(-(i-2)H) \simeq S^3\mathcal{F}(-(i-2)H) \oplus \mathcal{F}(-(i-2)H)$ + L). We have $\mathcal{F}(-(i-2)H) \in \mathcal{C}^1$ by Corollary 3.10 (3). Now we show that $S^3\mathcal{F}(-(i-2)H) \in \mathcal{C}^1$. By (3.5), consider (3.22) $\otimes O_{\mathcal{F}_1}(-(i-2)H)$, we are reduced to show that $O_{\mathcal{F}_1}(-(i-2)H-4L), O_{\mathcal{F}_1}(-(i-2)H-3L), \mathcal{F}(-(i-2)H-2L), S^2\mathcal{F}(-(i-2)H-L) \in \mathcal{C}^1$. We have $O_{\mathcal{F}_1}(-(i-2)H-3L) \in \mathcal{C}^1$ by Corollary 3.11 (2) and (4) respectively. As for $S^2\mathcal{F}(-(i-2)H)$, considering (3.11) $\otimes O_{\mathcal{F}_1}(-(i-2)H-L)$, we are reduced to show that $O_{\mathcal{F}_1}(-(i-1)H-L) \in \mathcal{C}^1$, which follows from (3.6), and $T_{\mathcal{F}_1/G(2,V)}(-(i-1)H-L) \in \mathcal{C}^1$, which follows from Corollary 3.11 (6). To show that $O_{\mathcal{F}_1}(-(i-2)H-4L) \in \mathcal{C}^1$, we consider (3.12) $\otimes O_{\mathcal{F}_1}(-(i-2)H-L)$. Then we are reduced to show that $T_{\mathcal{F}_1/G(2,V)}(-(i-1)H-L) \in \mathcal{C}^1$, which follows from Corollary 3.11 (6), and $S^2\mathcal{F}(-(i-1)H - 2L) \in \mathcal{C}^1$. For the latter, we consider (3.11) $\otimes O_{\mathcal{F}_1}(-(i-1)H-2L)$. Then we are reduced to show that $\mathcal{F}(-(i-1)H-2L), O_{\mathcal{F}_1}(-(i-1)H-2L), O_{\mathcal{F}_1}(-(i-1)H-L), \mathcal{F}(-(i-1)H) \in \mathcal{C}^1$, which follow from Corollary 3.11 (4) and (1), and (3.6), respectively.

$\Box$
Now we have finished our proof of Theorem\[3.7\].

Remak 3.12. We believe that a similar method work for any $n$ as in [20] once we can find a suitable candidate of the dual Lefschetz collection of maximal length.

4. Locally free sheaves $\tilde{S}_L$, $\tilde{Q}$, $\tilde{\Omega}$ on $\tilde{Y}$

In this section, $n$ is any integer greater than or equal to 3.

4.1. Birational geometry of the double symmetric loci $T_4$. As in [14, §3.4], we set

$$H := S_4, \mathcal{U} := \tilde{S}_4, \mathcal{V} := T_4, \mathcal{Z} := U_4,$$

where $T_4$ is the double cover of $S_4$ branched along $S_3$. We quickly review the main result of [14], which describe the birational geometry of $\mathcal{V}$.

Let

$$\mathcal{Y}_3 := G(3, \wedge^2 \Omega)$$

with the universal quotient bundle $\Omega$ of $G(n - 3, V)$. In case $n = 3$, we consider $G(n - 3, V)$ is a point and $\Omega$ is the vector space $V$. We denote by $\mathcal{P}_\rho$ and $\mathcal{P}_\sigma$ the subvarieties of $\mathcal{Y}_3$ parameterizing $\rho$-planes and $\sigma$-planes respectively (we refer for the definitions of $\rho$-planes and $\sigma$-planes to [14, §4.1]). In [ibid., §4.5], we have seen that $\mathcal{P}_\rho \simeq F(n - 3, n - 2; V)$ and $\mathcal{P}_\sigma \simeq F(n - 3, n; V)$, where $F(a, b; V) := \{ (\mathbb{C}^a, \mathbb{C}^b) | \mathbb{C}^a \subset \mathbb{C}^b \subset V \}$ is the flag variety.

In [ibid.], we construct the following diagram:

\begin{equation}
\begin{array}{c}
\mathcal{Y}_3 \\
\downarrow \rho_{\sigma_3} \\
G(n - 3, V) \\
\downarrow \pi_{\sigma_3} \\
\mathcal{Y}_2 \\
\downarrow \rho_{\sigma_2} \\
\mathcal{Y}_1 \\
\downarrow \rho_{\sigma_1} \\
\mathcal{Y}_0 \\
\downarrow G_\rho \\
\mathcal{Y} \\
\end{array}
\end{equation}

where

- $\tilde{Y}$ is the normalization of the subvariety $Y$ of $G(3, \wedge^{n-1} V)$ parametrizing 3-planes annihilated by at least $n - 3$ linearly independent vectors in $V$ by the wedge product ([ibid., Prop. 4.8, 4.9]),
- $\mathcal{Y}_3 \to \tilde{Y}$ is a small contraction contracting $\mathcal{P}_\rho$ to $\tilde{Y}_\rho \simeq G(n - 2, V)$ (ibid., Prop. 4.11) with the isomorphic image $\tilde{Y}_\sigma$ of $\mathcal{P}_\sigma$,
- $\mathcal{Y}_3 \to \tilde{Y}$ is the (anti-) flip for the small contraction $\mathcal{Y}_3 \to \tilde{Y}$ (ibid., §4.4),
- $p_{\tilde{Y}} : \tilde{Y} \to \tilde{Y}$ is a small contraction contracting $G_\rho \simeq P(S^2 Q_{\rho})$ to $\tilde{Y}_\rho \simeq G(n - 2, V)$, where $Q_{\rho}$ is the universal quotient bundle on $G(n - 2, V)$ ([ibid., Prop. 4.15]),
• \(\rho_{\tilde{\mathcal{Y}}_2} : \tilde{\mathcal{Y}}_2 \to \mathcal{Y}_3\) is the blow-up along the subvariety \(\mathcal{P}_\rho\) ([ibid., §4.5 and §4.7]),

• \(\tilde{\rho}_{\mathcal{Y}_2} : \tilde{\mathcal{Y}}_2 \to \tilde{\mathcal{Y}}\) is the blow-up along the subvariety \(G_\rho\) of codimension \(n - 2\) ([ibid., Prop. 4.22 (1)]),

• \(\rho_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}} \to \mathcal{Y}\) is an extremal divisorial contraction with exceptional divisor \(\tilde{\mathcal{F}}_{\tilde{\mathcal{Y}}}\) ([ibid., §4.6, Prop. 4.22 (2), §5]),

• \(\mathcal{Y}_0 \to \tilde{\mathcal{Y}}\) is the blow-up along the strict transform of \(\mathcal{P}_\sigma\) ([ibid., §4.4, Rem. 4.23]).

**Remark 4.1.** In case \(n = 3\), we consider \(\mathcal{Y}_3 = \tilde{\mathcal{Y}} = G(3, \wedge^2 V)\) and \(\mathcal{Y}_2 \simeq \tilde{\mathcal{Y}}\).

In the subsequent subsections, we introduce locally free sheaves \(\tilde{\mathcal{S}}_L, \tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\) on \(\tilde{\mathcal{Y}}\), which will play central roles in our construction of the Lefschetz collection in \(\mathcal{D}^b(\tilde{\mathcal{Y}})\).

In Sections 4 and 5 and Appendix A we use the following convention for the invertible sheaves:

- \(L_{\Sigma}\): the pull back on a variety \(\Sigma\) of \(O_{G(n - 3, V)}(1)\) if there is a morphism \(\Sigma \to G(n - 3, V)\). In case \(n = 3\), we consider \(L_{\Sigma}\) as the trivial sheaf \(\wedge^3 V \otimes O_{\Sigma}\).
- \(M_{\Sigma}\): the pull back on a variety \(\Sigma\) of \(O_{S_4}(1)\) if there is a morphism \(\Sigma \to S_4\).

We often omit the subscripts \(\Sigma\) for \(L_{\Sigma}\) and \(M_{\Sigma}\) if no confusion is likely possible.

### 4.2. Locally free sheaves \(\tilde{\mathcal{S}}_L, \tilde{\mathcal{Q}}\) on \(\tilde{\mathcal{Y}}\).

Consider the following universal sequence of the Grassmann bundle \(\mathcal{Y}_3 = G(3, \wedge^2 \Omega)\) over \(G(n - 3, V)\) (cf. [14 p.434]):

\[
0 \to S \to \pi_3^* \wedge^2 \Omega \to \mathcal{Q} \to 0,
\]

where \(S\) is the relative universal subbundle of rank three and \(\mathcal{Q}\) is the relative universal quotient bundle of rank three. Similarly, we denote by \(\tilde{S}\) the universal subbundle of rank three of the Grassmannian \(G(3, \wedge^{n-1} V)\). Then we have

**Proposition 4.2.** \(S(-L_{\mathcal{Y}_3})\) is the pull-back of \(\tilde{S}\).

**Proof.** We may write a point of \(\mathcal{Y}_3\) by \(y = ([\tilde{U}], V_{n-3})\) with \([\tilde{U}] \in G(3, \wedge^2 (V/V_{n-3}))\), \([V_{n-3}] \in G(n - 3, V)\). \(y\) is mapped to \([U] = [\tilde{U} \wedge \wedge^{n-3} V_{n-3}] \in \tilde{\mathcal{Y}}\). Note that \(\wedge^{n-3} V_{n-3} = \pi_3^* O_{\tilde{\mathcal{Y}}}(1)|_{yg} = -L_{\mathcal{Y}_3}|_{yg}\). Therefore \(S(-L_{\mathcal{Y}_3})\) is the pull-back of \(\tilde{S}\).

Now we have the following proposition (and definition):

**Proposition 4.3.** There exist locally free sheaves \(\tilde{\mathcal{S}}_L\) and \(\tilde{\mathcal{Q}}\) on \(\tilde{\mathcal{Y}}\) which satisfy

\[
\rho_{\tilde{\mathcal{Y}}}^* S(-L_{\mathcal{Y}_3}) = \tilde{\rho}_{\tilde{\mathcal{Y}}}^* \tilde{\mathcal{S}}_L \text{ and } \rho_{\tilde{\mathcal{Y}}}^* \mathcal{Q} = \tilde{\rho}_{\tilde{\mathcal{Y}}}^* \tilde{\mathcal{Q}}.
\]

**Proof.** We define \(\tilde{\mathcal{S}}_L\) to be the pullback of \(\tilde{S}\) to \(\tilde{\mathcal{Y}}\), then the first claim is immediate by the commutativity of the morphisms in the diagram in Subsection [4.1.1. To see the existence of \(\tilde{\mathcal{Q}}\), consider the universal sequence ([4.11] on \(\mathcal{Y}_3\). Let \([V_{n-3}, V_{n-2}]\) be a point on the exceptional locus \(\mathcal{P}_\rho = F(n - 3, n - 2, V) \to \tilde{\mathcal{P}}_{\rho} = G(n - 2, V)\) of the small resolution \(\mathcal{Y}_3 \to \tilde{\mathcal{Y}}\). Since \(S|_{[V_{n-3}, V_{n-2}]} = (V/V_{n-2}) \wedge (V_{n-2}/V_{n-3}) \to \mathcal{Q}|_{[V_{n-3}, V_{n-2}]} \to 0\). Hence we have \(\mathcal{Q}|_{[V_{n-3}, V_{n-2}]} \simeq \wedge^2 (V/V_{n-2}),\) which implies \(\mathcal{Q}|_{\gamma} \simeq O_{\tilde{\mathcal{Y}}}^{\oplus 3}\) for the fiber \(\gamma = P^{n-3}\) of \(F(n - 3, n - 2, V) \to G(n - 2, V)\) over \([V_{n-2}]\). It also implies that \(\rho_{\tilde{\mathcal{Y}}}^* \mathcal{Q}\) is trivial on a fiber of \(\tilde{\rho}_{\tilde{\mathcal{Y}}} : \tilde{\mathcal{Y}}_2 \to \tilde{\mathcal{Y}}\) by [14 Prop. 4.22]. The last property
and Lemma 4.4 below ensure the existence of a locally free sheaf $\tilde{Q}$ on $\mathcal{F}$ such that $\rho_{\mathcal{F}}^\ast \mathcal{Q} = \rho_{\mathcal{F}}^\ast \tilde{Q}$. \hfill \Box

The following lemma should be well-known for experts but we give a proof for readers’ convenience:

**Lemma 4.4.** Let $Y$ be a smooth projective variety and $f: X \to Y$ the blow-up along a smooth subvariety $S \subset Y$. Let $\mathcal{E}$ be a locally free sheaf on $X$ of rank $r$ such that $\rho_\mathcal{E} \homotope \mathcal{E}$, holds that $\mathcal{E} |_{\gamma} \isom \mathcal{O}_{\gamma}$. Then there exists a locally free sheaf $\mathcal{E}$ on $Y$ such that $\mathcal{E} = f^\ast \mathcal{E}$.

**Proof.** We give a proof by using Mori theory. Let $\pi: \mathbb{P}(\mathcal{E}) \to X$ be the natural projection and $H$ the tautological divisor. We denote by $E$ the $f$-exceptional divisor and set $F := \pi^{-1}(E)$. Since $\mathcal{E}|_{\gamma} \isom \mathcal{O}_{\gamma}^\oplus r$, we have $\pi^{-1}(\gamma) \isom \mathbb{P}^{r-1} \times \gamma$ and $H |_{\pi^{-1}(\gamma)}$ is a divisor of type $(1,0)$. Moreover, since $K_{\mathbb{P}(\mathcal{E})} = -rH + \pi^\ast(\det \mathcal{E} + K_X)$, any curve in a fiber of $\pi^{-1}(\gamma) \to \mathbb{P}^{r-1}$ is negative for $K_{\mathbb{P}(\mathcal{E})}$. Therefore any curve in a fiber of $\pi^{-1}(\gamma) \to \mathbb{P}^{r-1}$ spans a $K_{\mathbb{P}(\mathcal{E})}$-negative extremal ray in $\mathcal{N}_E(\mathbb{P}(\mathcal{E})/Y)$ and a sufficient multiple of $H$ defines a birational morphism $p: \mathbb{P}(\mathcal{E}) \to \mathbb{P}$ over $Y$ contracting fibers of $\pi^{-1}(\gamma) \to \mathbb{P}^{r-1}$. Let $q: \mathbb{P} \to Y$ be the induced morphism. It is easy to see that $\mathbb{P}$ is smooth and $p$ is the blow-up along $p(F)$. Moreover, $H$ is the pull-back of a Cartier divisor $\mathcal{H}$ on $\mathbb{P}$ and the restriction of $\mathcal{H}$ to any fiber $\simeq \mathbb{P}^{r-1}$ of $\mathbb{P} \to Y$ is $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Therefore $\mathcal{E} := q_\ast \mathcal{O}_\mathcal{H}(\mathcal{H})$ is a locally free sheaf of rank $r$ on $Y$ and it holds that $\mathcal{E} \isom f^\ast \mathcal{E}$ by construction. \hfill \Box

### 4.3 Locally free sheaf $\tilde{Q}$ on $\mathcal{F}$.

Let us focus on the local geometry of the blow-up $\mathcal{F} \to \mathcal{F}$ which is described by $\mathcal{F} = F(n - 3, n - 2, V) \to \mathcal{F} = G(n - 2, V)$. We denote the universal sub-bundles of the partial flag variety $F(n - 3, n - 2, V)$ by $\mathcal{R}_{n-3} \subset \mathcal{R}_{n-2} \subset \mathcal{R}$. We set $\mathcal{R} = V \otimes \mathcal{O}_{F(n-3, n-2)}$ and rank $\mathcal{R} = k$. There is an exact sequence

$$0 \to \mathcal{R}_{n-2}/\mathcal{R}_{n-3} \to \mathcal{R}_V/\mathcal{R}_{n-3} \to \mathcal{R}_V/\mathcal{R}_{n-2} \to 0. \tag{4.2}$$

It is also useful to identify $\mathcal{F}$ with the projective bundle $\mathbb{P}(\mathcal{Q})$ over $G(n - 3, V)$. Then the exact sequence above is nothing but the relative Euler sequence of the projective bundle $\pi_\mathcal{F}: \mathbb{P}(\mathcal{Q}) \to G(n - 3, V)$ with

$$\mathcal{R}_{n-2}/\mathcal{R}_{n-3} = \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(-1), \mathcal{R}_V/\mathcal{R}_{n-3} = \pi_\mathcal{F}^\ast \mathcal{Q},$$

and $\mathcal{R}_V/\mathcal{R}_{n-2}$ is the twisted relative tangent bundle $T_{\mathcal{F}/G(n-2,V)}(-1)$.

**Proposition 4.5.** Let $\pi_{\mathcal{Q}} = \pi_{\mathcal{F}} \circ \rho_{\mathcal{F}}$ be the composite of $\rho_{\mathcal{F}}: \mathcal{F} \to \mathcal{F}$ with $\pi_{\mathcal{Q}}: \mathcal{F} \to G(n - 3, V)$. Denote by $i: F_3 \hookrightarrow \mathcal{F}$ the inclusion of the exceptional divisor, and by $\rho_\mathcal{F}: F_3 \to \mathcal{F}$ the restriction $\rho_\mathcal{F} := \rho_{\mathcal{F}}|_{F_3}$. Then the kernel

$$\mathcal{R} := \text{Ker} \left\{ \pi_{\mathcal{F}}^\ast \mathcal{Q}^* \to i_\ast \circ \rho_\mathcal{F}^\ast \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(1) \right\}$$

is a locally free sheaf on $\mathcal{F}$.

**Proof.** From (1.2), we have a surjection $\rho_\mathcal{F}^\ast (\mathcal{R}_V/\mathcal{R}_{n-3})^* \to \rho_\mathcal{F}^\ast (\mathcal{R}_{n-2}/\mathcal{R}_{n-3})^* \to 0$ and also $i_\ast \circ \rho_\mathcal{F}^\ast (\mathcal{R}_V/\mathcal{R}_{n-3})^* \to i_\ast \circ \rho_\mathcal{F}^\ast (\mathcal{R}_{n-2}/\mathcal{R}_{n-3})^* \to 0$. Let $\mathcal{R}$ be the kernel of the composite of the latter and the natural surjection $\pi_{\mathcal{F}}^\ast \mathcal{Q}^* \to i_\ast \circ \rho_\mathcal{F}^\ast \mathcal{Q}^* = i_\ast \circ \rho_\mathcal{F}^\ast (\mathcal{R}_V/\mathcal{R}_{n-3})^*$. Then we obtain the exact sequence

$$0 \to \mathcal{R} \to \pi_{\mathcal{F}}^\ast \mathcal{Q}^* \to i_\ast \circ \rho_\mathcal{F}^\ast \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(1) \to 0. \tag{4.3}$$
By taking $\mathcal{E}xt(-, \mathcal{O}_{\mathcal{Y}_2})$ of this sequence, we see that $\mathcal{R}$ is a locally free sheaf on $\mathcal{Y}_2$ (see [3] III, Ex. 6.6).

**Lemma 4.6.** $\mathcal{R}|_{\delta} = \mathcal{O}_{\mathcal{Y}_2}^{\mathbb{P}^4}$ for each fiber $\delta \simeq \mathbb{P}^{n-3}$ of $F_\rho \to G_\rho$.

**Proof.** Each fiber $\delta$ of $F_\rho \to G_\rho$ projects isomorphically to a fiber of $F(n - 3, n - 2, V) \to G(n - 2, V)$, and further to a copy of $\mathbb{P}^{n-3}$ in $G(n - 3, V)$. Therefore $\pi_{\mathcal{Y}_2}^* \mathcal{O}^*|_{\delta} \simeq \mathcal{O}_{\mathcal{F}_n}^{\mathbb{P}^3} \oplus \mathcal{O}_{\mathcal{F}_{n-3}}(-1)$ and also $\rho_{\mathcal{F}}^* \mathcal{O}_{\mathcal{F}_n}(1)|_{\delta} \simeq \mathcal{O}_{\mathcal{F}_{n-3}}(-1)$. By restricting the exact sequence (4.3), we obtain

$$\mathcal{R}|_{\delta} \to \mathcal{O}_{\mathcal{F}_n}^{\mathbb{P}^3} \oplus \mathcal{O}_{\mathcal{F}_{n-3}}(-1) \to \mathcal{O}_{\mathcal{F}_{n-3}}(-1) \to 0,$$

which shows that there is a surjection $\mathcal{R}|_{\delta} \to \mathcal{O}_{\mathcal{F}_n}^{\mathbb{P}^3}$ with its kernel being an invertible sheaf $\mathcal{L}$. Note that $\det \mathcal{R} \simeq \mathcal{O}_{\mathcal{F}_n}(-L_{\mathcal{Y}_2} - F_\rho)$ from (4.3). Now, since $L_{\mathcal{Y}_2}|_{\delta} = \mathcal{O}_\delta(1)$ by definition and also $F_\rho|_{\delta} = \mathcal{O}_{\mathcal{F}_{n-3}}(-1)$, we have $\det \mathcal{R}|_{\delta} \simeq \mathcal{O}_\delta$ and $\mathcal{L} \simeq \mathcal{O}_\delta$. Hence $\mathcal{R}|_{\delta} \simeq \mathcal{O}_{\mathcal{Y}_2}^{\mathbb{P}^4}$. □

Now we define $\Omega_2 := \mathcal{R}_\mathcal{Y}_2^*$. From Lemmas 4.6 and 4.7, we have the following proposition (and definition):

**Proposition 4.7.** There exists a locally free sheaf $\tilde{\Omega}$ on $\mathcal{Y}$ such that

$$\tilde{\Omega}_2 = \mathcal{R}_\mathcal{Y}_2^* \tilde{\Omega}.$$

The following exact sequence will be used in our later calculations:

**Proposition 4.8.** There exists the following exact sequence:

$$0 \to \pi_{\mathcal{Y}_2}^* \Omega \to \tilde{\Omega}_2 \to i_\ast \circ \rho_{\mathcal{F}}^* \mathcal{O}_{\mathcal{F}_n}(1)(F_\rho|_{F_\rho}) \to 0.$$

**Proof.** By taking $\mathcal{H}om(-, \mathcal{O}_{\mathcal{Y}_2})$ of (4.3), we obtain:

$$0 \to \pi_{\mathcal{Y}_2}^* \Omega \to \tilde{\Omega}_2 \to \mathcal{E}xt^1_{\mathcal{O}_{\mathcal{Y}_2}}(i_\ast \circ \rho_{\mathcal{F}}^* \mathcal{O}_{\mathcal{F}_n}(1), \mathcal{O}_{\mathcal{Y}_2}) \to 0.$$

The claim follows by evaluating $\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{Y}_2}}$ as

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{Y}_2}}(i_\ast \circ \rho_{\mathcal{F}}^* \mathcal{O}_{\mathcal{F}_n}(1), \mathcal{O}_{\mathcal{Y}_2}) \simeq i_\ast \mathcal{E}xt^0_{\mathcal{F}_\rho}(\rho_{\mathcal{F}}^* \mathcal{O}_{\mathcal{F}_n}(1), i^* \mathcal{O}_{\mathcal{Y}_2} \otimes \omega_{F_\rho/\mathcal{Y}_2}|[-1] = \mathcal{O}_{\mathcal{F}_\rho}(F_\rho|_{F_\rho}|[-1]).$$

□

4.4. **Properties of $S^*$, $Q$ restricted on $\mathcal{P}_\rho$ and $\mathcal{P}_\sigma$.** As in the last subsection, we identify $\mathcal{P}_\rho = F(n - 3, n - 2, V)$ with the projective bundle $\mathbb{P}(\mathcal{Q})$ with $\mathcal{P}_\rho : \mathbb{P}(\mathcal{Q}) \to G(n - 3, V)$. We introduce two divisors on $\mathcal{P}_\rho$:

$$H_{\mathcal{P}_\rho} = \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(1) \quad \text{and} \quad L_{\mathcal{P}_\rho} := \pi_{\mathcal{P}_\rho}^* \mathcal{O}_{G(n-3,V)}(1).$$

**Proposition 4.9.** $Q|_{\mathcal{P}_\rho} \simeq S^*(L_{\mathcal{F}_\rho})|_{\mathcal{P}_\rho}$ and $Q|_{\mathcal{P}_\rho} \simeq S^*(L_{\mathcal{F}_\sigma})|_{\mathcal{P}_\rho}$.

**Proof.** Proofs of the both relations are similar, so we only prove the former. Take a point $[\mathcal{P}_{V_{n-3}/V_{n-3}}]$ of $\mathcal{P}_\rho \subset G(3, \wedge^2 \mathcal{Q})$. Let $W_1$ and $W_2$ be the fiber of $S^*$ and $Q$ at $[\mathcal{P}_{V_{n-3}/V_{n-3}}]$, respectively. We compare the restrictions of the universal exact sequence (4.1) and its dual:

$$0 \to W_1 \to \wedge^2 V/V_{n-3} \to W_2 \to 0,$$

$$0 \to (W_2)^* \to (\wedge^2 V/V_{n-3})^* \to (W_1)^* \to 0.$$
Let \( P(W_1) = P_{V_{n-2}/V_{n-3}} \). Choose a basis \( \{ e_1, e_2, e_4 \} \) of \( V_{n-3} \) so that \( V_{n-2}/V_{n-3} = \langle e_1 \rangle \). Then \( W_1 = \langle e_1 \cdot e_2, e_1 \cdot e_3, e_1 \cdot e_4 \rangle \). By the non-degenerate pairing \( \wedge^2 V_{n-3} \times \wedge^2 V_{n-3} \to \wedge^4 V_{n-3} \cong \mathbb{C} \), we may identify \( \wedge^2 V_{n-3} \) and \( \langle \wedge^2 V_{n-3} \rangle \). Under this identification, we see from the explicit basis of the rest of this section is devoted to our proof of Theorem 5.1, where we compute the rest of this section is devoted to our proof of Theorem 5.1, where we compute the cohomology groups \( H^\bullet(\wedge^t(-t)) \) for \( n = 3 \) and \( n = 4 \) and \( 1 \leq t \leq 9 \). Our strategy is to reduce the computations of cohomology groups on \( \mathcal{Y} \) to those on \( \mathcal{B}_3 \) and use Theorem 2.1 for the \( G(3, 6) \)-bundle \( \mathcal{B}_3 \to G(n-3, V) \). Let \( \mathcal{F}_2 \) be a locally free sheaf on \( \mathcal{Y} \) and \( \mathcal{F}_2 := \hat{\rho}_{\mathcal{Y}_2}^* \mathcal{F}_2 \). Since \( \hat{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \to \mathcal{Y} \) is a blow-up of a smooth variety, it holds that

\[
(5.1) \quad H^\bullet(\mathcal{Y}, \mathcal{F}_2(-t)) \cong H^\bullet(\mathcal{Y}_2, \mathcal{F}_2(-t)),
\]

where \( (-t) \) on the right hand side represents the twist by \( O_{\mathcal{Y}_2}(-t M_{\mathcal{Y}_2}) \). Therefore it suffices to compute \( H^\bullet(\mathcal{A}_2^* \otimes \mathcal{B}_2(-t)) \).
5.2. **Divisors on \( \mathcal{M}_2 \).** Recall the universal sequence (1.1) of the Grassmann bundle \( \mathcal{M}_2 = G(3, \Lambda^2 \Omega) \). Taking the determinant and using \( \wedge^6(\Lambda^2 \Omega) = \mathcal{O}_{\mathcal{M}_2}(3) \), we have
\[
\det Q = \det S^* + 3L_{\mathcal{M}_2} = \det \{S^*(L_{\mathcal{M}_2})\}. 
\] (5.2)
Also, since \( T_{\mathcal{M}_2/\mathbb{P}(V)} = S^* \otimes Q \) (see [4] p.435), we have
\[
K_{\mathcal{M}_2} = -\det(Q \otimes S^*) - (n+1)L_{\mathcal{M}_2} = -3(\det Q + \det S^*) - (n+1)L_{\mathcal{M}_2} = -6 \det Q + (8-n)L_{\mathcal{M}_2},
\]
where we note \( \text{rank } S = \text{rank } Q = 3 \) and we use (5.2) in the last equality.

We now see some relations among divisors on \( \mathcal{M}_2 \). Note that
\[
K_{\mathcal{M}_2} = \rho_{\mathcal{M}_2}^* K_{\mathcal{M}_2} + 5F_{\rho}
\]
(5.4)
since \( \rho_{\mathcal{M}_2} \) is the blow-up along a smooth subvariety of codimension 6. By this and (5.3), we have
\[
K_{\mathcal{M}_2} = -6\rho_{\mathcal{M}_2}^* \det Q + (8-n)L_{\mathcal{M}_2} + 5F_{\rho}.
\] (5.5)

**Proposition 5.2.** The pull-back \( M_{\mathcal{M}_2} \) of \( \mathcal{O}_{\mathcal{M}}(1) \) is given by
\[
M_{\mathcal{M}_2} = \rho_{\mathcal{M}_2}^*(\det Q) - L_{\mathcal{M}_2} - F_{\rho}.
\]
**Proof.** Note that \( \pi_{\mathcal{M}_2}^* \mathcal{O}_{\mathcal{M}_2}(\Lambda^3(\Lambda^2 \Omega))^\vee(1) = \Lambda^3(\Lambda^2 \Omega)^*, \) and \( \pi_{\mathcal{M}_2}^* \mathcal{O}_{\mathcal{M}_2}(\Lambda^2 \Omega)^* \cdot (1) = \mathcal{S}^2 \Omega. \) Therefore, by the decomposition (1.1) (4.5) and the construction of \( \mathcal{M}_2 \rightarrow \mathcal{M} \) as a relative linear projection, we have
\[
\rho_{\mathcal{M}_2}^*(\det S^*) - F_{\rho} = q_{\mathcal{M}_2}^*(M_{\mathcal{M}} - 2L_{\mathcal{M}}) = M_{\mathcal{M}_2} - 2L_{\mathcal{M}_2}.
\]
Then we have the assertion by (5.2).

**Proposition 5.3.** \( \mathcal{M}_2 \) is a weak Fano manifold, namely, \( -K_{\mathcal{M}_2} \) is nef and big.
**Proof.** Note that \( \det Q \) is nef since \( Q \) is the image of the surjection from the globally generated bundle \( \pi_{\mathcal{M}_2}^* \Lambda^2 \Omega. \) By (5.3) and Proposition 5.2, we have \( -K_{\mathcal{M}_2} = 5M_{\mathcal{M}_2} + (n-3)L_{\mathcal{M}_2} + \rho_{\mathcal{M}_2}^* \det Q, \) which is clearly nef, and is also big since so is \( M_{\mathcal{M}_2}. \)

**Proposition 5.4.** Let \( F'_{\mathcal{M}} \) be the strict transform of \( F_{\mathcal{M}} \). It holds that \( F'_{\mathcal{M}} = 2M_{\mathcal{M}_2} - L_{\mathcal{M}_2} - F_{\rho}. \)
**Proof.** By [14] Prop. 2.5, Cor. 5.2], we have
\[
K_{\mathcal{M}} = \rho_{\mathcal{M}_2}^* K_{\mathcal{M}_2} + (n-2)F_{\mathcal{M}_2} = -2(n+1)M_{\mathcal{M}} + (n-2)F_{\mathcal{M}_2}.
\] (5.6)
By [14] Prop. 4.2], \( \rho_{\mathcal{M}_2} : \mathcal{M}_2 \rightarrow \mathcal{M} \) is the blow-up along \( G_{\rho}. \) Therefore
\[
K_{\mathcal{M}_2} = \rho_{\mathcal{M}_2}^* K_{\mathcal{M}_2} + (n-3)F_{\rho}.
\] (5.7)
Since \( G_{\rho} \) is not contained in \( F_{\mathcal{M}_2} \), we deduce from (5.6) and (5.7) that
\[
K_{\mathcal{M}_2} = -2(n+1)M_{\mathcal{M}_2} + (n-2)F'_{\mathcal{M}} + (n-3)F_{\rho}.
\]
Combining this with (5.6) and Proposition 5.2, we obtain
\[
(n-2)F'_{\mathcal{M}_2} = (2n-4)M_{\mathcal{M}_2} - (n-2)L_{\mathcal{M}_2} - (n-2)F_{\rho}.
\]
Since \( \mathcal{M}_2 \) is a weak Fano manifold, Pic \( \mathcal{M}_2 \) is torsion-free. Therefore we obtain the desired equality.

\[\Box\]
5.3. Case $\bar{A} = \bar{\Omega}$ or $\bar{B} = \bar{\Omega}$. Among $\mathcal{O}_{\bar{\mathcal{Y}}_2}$, $\Omega_2$, $\rho_{\bar{\mathcal{Y}}_2} S(-L)$, and $\rho_{\bar{\mathcal{Y}}_2} \mathcal{Q}$, only $\Omega_2$ is not the pull-back of a locally free sheaf on $\mathcal{Y}_2$. We will show that to compute $H^\bullet(\mathcal{Y}_2, \mathcal{A}_2^\bullet \otimes \mathcal{B}_2(-t))$ for $1 \leq t \leq 5$ and $\mathcal{A}_2 = \Omega_2$ or $\mathcal{B}_2 = \Omega_2$, we may replace $\Omega_2$ by $\rho_{\bar{\mathcal{Y}}_2} \mathcal{Q}$.

**Lemma 5.5.** With the notation as in Theorem 5.1, it holds that

\begin{align}
(5.8) & \quad H^\bullet(\Omega_2^\bullet \otimes \mathcal{B}_2(-t)) \simeq H^\bullet(\pi_{\bar{\mathcal{Y}}_2}^\bullet \mathcal{Q}^* \otimes \mathcal{B}_2(-t)), \\
(5.9) & \quad H^\bullet(\mathcal{A}_2^\bullet \otimes \Omega_2(-t)) \simeq H^\bullet(\mathcal{A}_2^\bullet \otimes \pi_{\bar{\mathcal{Y}}_2} \mathcal{Q}(-t)), \\
(5.10) & \quad H^\bullet(\Omega_2^\bullet \otimes \Omega_2(-t)) \simeq H^\bullet(\pi_{\bar{\mathcal{Y}}_2}^\bullet \mathcal{Q}^* \otimes \pi_{\bar{\mathcal{Y}}_2} \mathcal{Q}(-t))
\end{align}

for any $\bullet$ and $1 \leq t \leq 5$. Moreover, (5.8) holds also for $t = 0$ and $\mathcal{B}_2 = \mathcal{O}_{\bar{\mathcal{Y}}_2}$.

**Proof.**

**Proof of (5.8).** By the exact sequence (4.3), we have

\begin{equation}
0 \to \Omega_2^\bullet \otimes \mathcal{B}_2(-t) \to \pi_{\bar{\mathcal{Y}}_2}^\bullet \mathcal{Q}^* \otimes \mathcal{B}_2(-t) \to i_* \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \mathcal{B}_2(-t)|_{F_p} \to 0.
\end{equation}

There, it suffices to show

\begin{equation}
H^\bullet(F_p, \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \mathcal{B}_2(-t)|_{F_p}) = 0.
\end{equation}

Note that this will imply for $\mathcal{B}_2 = \Omega_2$ that

\begin{equation}
H^\bullet(\Omega_2^\bullet \otimes \Omega_2(-t)) \simeq H^\bullet(\pi_{\bar{\mathcal{Y}}_2}^\bullet \mathcal{Q}^* \otimes \Omega_2(-t))
\end{equation}

Case $n = 4$. In this case, this vanishing holds for any $t$ by the Leray spectral sequence for $\rho_{\bar{\mathcal{Y}}_2}|_{F_p} : F_p \to G_{\rho}$, since $\rho_{\bar{\mathcal{Y}}_2}|_{F_p}$ is a $\mathbb{P}^1$-bundle and the restriction of $\rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \mathcal{B}_2(-t)|_{F_p}$ to the fiber is a direct sum of $\mathcal{O}_{\mathbb{P}}(1)$ by the proof of Lemma 4.6. In particular, (5.8) holds also for $t = 0$ and $\mathcal{B}_2 = \mathcal{O}_{\bar{\mathcal{Y}}_2}$ in this case.

Case $n = 3$. If $\mathcal{B}_2 = \mathcal{O}_{\bar{\mathcal{Y}}_2}$, $\rho_{\bar{\mathcal{Y}}_2} S(-L)$, and $\rho_{\bar{\mathcal{Y}}_2} \mathcal{Q}$, then the vanishing (5.12) holds for $1 \leq t \leq 5$ by the Leray spectral sequence for $\rho_{\bar{\mathcal{Y}}_2}$: $F_p \to \mathcal{P}_\rho$ since $\rho_{\bar{\mathcal{Y}}_2}$ is a $\mathbb{P}^2$-bundle and the restriction of $\rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \mathcal{B}_2(-t)|_{F_p}$ to the fiber is a direct sum of $\mathcal{O}_{\mathbb{P}}(-t)$.

Suppose $t = 0$ and $\mathcal{B}_2 = \Omega_{\bar{\mathcal{Y}}_2}$. Then (5.8) holds by (5.11) since $H^\bullet(V^* \otimes \mathcal{O}_{\bar{\mathcal{Y}}_2})$ and $H^\bullet(\rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1))$ are isomorphic to each other for $\bullet = 0$ and are zero for $\bullet > 0$.

Suppose $\mathcal{B}_2 = \Omega_2$. First we calculate the restriction of $\Omega_2^\bullet$ on $F_p$. By restricting (4.3) to $F_p$, we obtain the exact sequence

\begin{equation}
\Omega_2^\bullet|_{F_p} \to V^* \otimes \mathcal{O}_{F_p} \to \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \to 0.
\end{equation}

Since the kernel of $V^* \otimes \mathcal{O}_{F_p} \to \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1)$ is isomorphic to $\rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1)$, we have the exact sequence

\begin{equation}
0 \to \mathcal{L} \to \Omega_2^\bullet|_{F_p} \to \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \to 0,
\end{equation}

where $\mathcal{L}$ is an invertible sheaf on $F_p$. Taking the determinants in (5.14), we have $\det \Omega_2^\bullet = \mathcal{O}_{\bar{\mathcal{Y}}}(F_p) \otimes \wedge^4 V^*$. Therefore $\mathcal{L} \simeq \mathcal{O}_{F_p}(F_p) \otimes \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \wedge^4 V^*$. Now dualizing (5.14), we obtain

\begin{equation}
0 \to \rho_{\bar{\mathcal{Y}}_2} T_{\bar{\mathcal{Y}}}(1) \to \Omega_2|_{F_p} \to \mathcal{O}_{F_p}(F_p) \otimes \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \wedge^4 V^* \to 0.
\end{equation}

Let $P$ be a fiber of $\rho_{\bar{\mathcal{Y}}_2}$: $F_p \to \mathcal{P}_{\rho} \simeq \mathbb{P}(V)$. We have the vanishing (5.12) for $1 \leq t \leq 4$ since the restriction to $P$ of $\Omega_2|_{F_p}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{\mathbb{P}}(-1)$ by (5.15). Finally we consider the case where $t = 5$. By the Serre duality, it suffices to show the vanishing of

\begin{equation}
H^{5-\bullet}(F_p, \rho_{\bar{\mathcal{Y}}_2} \mathcal{O}_{\bar{\mathcal{Y}}}(1) \otimes \Omega_2(-t))|_{F_p},
\end{equation}

where $\mathcal{A}_2 = \Omega_2$ or $\mathcal{B}_2 = \Omega_2$.
which can be seen by twisting \( [5.14] \) with \( \rho_2^*\mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1) \otimes \mathcal{O}_{\mathcal{F}_n}(-\det \mathcal{Q}_2 + F_\rho) \).

**Proof of (5.9) and (5.10).** By the exact sequence (4.4), we have

\[
0 \to \pi_{\mathbb{G}_2}^* \Omega \otimes \mathfrak{F}(t) \to \Omega_2 \otimes \mathfrak{F}(t) \to i_* \rho_2^* \mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1) \otimes \mathfrak{F}(-tM_{\mathbb{G}_2} + F_\rho)|_{\mathcal{F}_n} \to 0
\]

for a locally free sheaf \( \mathfrak{F} \) on \( \mathbb{G}_2 \). It suffices to show the vanishings of

\[
H^s(F_\rho, \rho_2^* \mathcal{O}_{\mathcal{P}(\mathcal{V})}(-1) \otimes \mathfrak{F}(-tM_{\mathbb{G}_2} + F_\rho)|_{\mathcal{F}_n}) \quad (1 \leq t \leq 5)
\]

with \( \mathfrak{F} = \mathcal{O}_{\mathbb{G}_2}, \rho_2^* \mathcal{Q}^*, \rho_2^* \mathcal{S}(-L_{\mathbb{G}_2}) \) for (5.9), and with \( \pi_{\mathbb{G}_2}^* \Omega^* \) for (5.10) in view of (5.11). Since \( \mathcal{Q}^*|_{\mathcal{P}_n} \simeq \mathcal{S}(-L_{\mathbb{G}_2})|_{\mathcal{P}_n} \) by Proposition 4.9, we have only to consider the cases \( \mathfrak{F} = \mathcal{O}_{\mathbb{G}_2}, \rho_2^* \mathcal{Q}^*, \pi_{\mathbb{G}_2}^* \Omega^* \). For \( 1 \leq t \leq 4 \), the vanishings of (5.16) follow from the Leray spectral sequence for \( \rho_2^* : F_\rho \to \mathcal{P}_n \) since \( \mathcal{P}_n \) is a \( \mathbb{P}^5 \)-bundle and the restriction of \( \rho_2^* \mathcal{O}_{\mathcal{P}(\mathcal{V})}(-1) \otimes \mathfrak{F}(-tM_{\mathbb{G}_2} + F_\rho)|_{\mathcal{F}_n} \) to the fiber is a direct sum of \( \mathcal{O}_3^*(-(t+1)) \) by Proposition 5.2. For \( t = 5 \), note that (5.16) is Serre dual to

\[
H^s(F_\rho, \rho_2^* \mathcal{O}_{\mathcal{P}(\mathcal{V})}(-1) \otimes \mathfrak{F}^*(\rho_2^* \mathcal{Q}(-\det \mathcal{Q} - L_{\mathbb{G}_2}))|_{\mathcal{F}_n})
\]

with \( s = 8 - \bullet \) for \( n = 3 \) and \( s' = 12 - \bullet \) for \( n = 4 \) by (5.5) and Proposition 5.2. Since \( \rho_2^* \mathcal{Q} \) is the blow-up of a smooth variety and \( \mathfrak{F} = \rho_2^* \mathfrak{F}_3 \) with a locally free sheaf \( \mathfrak{F}_3 \) on \( \mathbb{G}_3 \), each of (5.17) is isomorphic to

\[
H^s(\mathcal{P}_n, \rho_2^* \mathcal{O}_{\mathcal{P}(\mathcal{V})}(-1) \otimes \mathfrak{F}_3^*(-\det \mathcal{Q} - L_{\mathbb{G}_2})|_{\mathcal{P}_n}).
\]

Using Proposition 4.10 (1) and (2), we can write

\[
\mathcal{O}_{\mathcal{P}(\mathcal{V})}(-1) \otimes \mathfrak{F}_3^*(-\det \mathcal{Q} - L_{\mathbb{G}_2})|_{\mathcal{P}_n} = \begin{cases} 
\mathcal{O}_{\mathcal{P}_n}(-H_{\mathcal{P}_n} - 3L_{\mathcal{P}_n}) & \text{for } \mathfrak{F}_3 = \mathcal{O}_{\mathcal{G}_3} \\
(\mathcal{R}_V/\mathcal{R}_n)^*(-2L_{\mathcal{P}_n}) & \text{for } \mathfrak{F}_3 = \mathcal{Q}^* \\
\pi_{\mathcal{P}_n}^* \Omega(-H_{\mathcal{P}_n} - 3L_{\mathcal{P}_n}) & \text{for } \mathfrak{F}_3 = \pi_{\mathcal{G}_2}^* \Omega^*.
\end{cases}
\]

We see that all \( H^s(\mathcal{P}_n, -)'s \) of these sheaves vanish when restricted to fibers of \( \pi_{\mathcal{P}_n} : \mathcal{P}_n = \mathcal{P}(\mathcal{V}) \to \mathbb{G}(n - 3, V) \). Hence, by the Leray spectral sequence for \( \pi_{\mathcal{P}_n} \), all of (5.17) vanish, too. \( \square \)

**5.4. Case 0 \leq t \leq 5.** By using the following lemma, we can reduce our computations of cohomology groups to those on \( \mathbb{G}_3 \) in the case where \( 0 \leq t \leq 5 \):

**Lemma 5.6.**

1. \( R^t \rho_{\mathbb{G}_2}^* \mathcal{O}_{\mathbb{G}_2}(tF_\rho) = 0 \) for any \( t \leq 5 \) and \( q > 0 \).
2. \( \rho_{\mathbb{G}_2}^* \mathcal{O}_{\mathbb{G}_2}(tF_\rho) = \mathcal{O}_{\mathbb{G}_2} \) for \( t \geq 0 \).

**Proof.** (1) follows from the relative Kodaira vanishing theorem since \( tF_\rho - K_{\mathbb{G}_2} = \mathbb{G}_3 \) \((t - 5)F_\rho \) is \( \rho_{\mathbb{G}_2} \)-nef and \( \rho_{\mathbb{G}_2} \)-big if \( t \leq 5 \). (2) is a standard result. \( \square \)

In view of Lemma 5.5, we will replace \( \mathbb{G}_2 \) by \( \rho_{\mathbb{G}_2} \mathbb{G}_2 \) if \( \mathcal{A}_2 \) or \( \mathcal{B}_2 = \mathbb{G}_2 \) and write \( \mathcal{A}_2 \) or \( \mathcal{B}_2 = \rho_{\mathbb{G}_2} \mathbb{G}_2 \). Then, in any case, \( \mathcal{A}_2 \) and \( \mathcal{B}_2 \) are the pull-back of a locally free sheaf \( \mathcal{A}_3 \) and \( \mathcal{B}_3 \) on \( \mathbb{G}_3 \). Therefore, by the Leray spectral sequence for \( \rho_{\mathbb{G}_2} \), and Proposition 5.2 and Lemma 5.6 we have the following:

**Lemma 5.7.**

(5.19) \( H^s(\mathbb{G}_3, \mathcal{A}_3^* \otimes \mathcal{B}_3(-2t)) \simeq H^s(\mathbb{G}_3, \mathcal{A}_3^* \otimes \mathcal{B}_3(-t \det \mathcal{Q} + tL_{\mathbb{G}_2})) \) \((0 \leq t \leq 5)\).

**Proof of Theorem 5.1 for 0 \leq t \leq 5.** We have only to show the vanishing of \( H^s(\mathbb{G}_3, \mathcal{A}_3^* \otimes \mathcal{B}_3(-t \det \mathcal{Q} + tL_{\mathbb{G}_2})) \) for \( 1 \leq t \leq 5 \) and \( \mathcal{A}_3, \mathcal{B}_3 = \mathcal{O}_{\mathbb{G}_3}(\mathbb{G}_3, \mathcal{S}^*(L), \mathbb{Q} \), and for \( t = 0 \) and \((\mathcal{A}_3, \mathcal{B}_3) = (\mathbb{G}_3, \mathcal{O}_{\mathbb{G}_3}, \mathcal{S}^*(L), \mathbb{O}_{\mathbb{G}_3}, \mathcal{Q}, \mathcal{O}_{\mathbb{G}_3}, \mathcal{S}^*(L), \mathbb{Q}, \mathcal{Q}, \mathcal{O}_{\mathbb{G}_3}, \mathbb{Q}, \mathcal{S}^*(L)). \)
Let \( G \simeq G(3, 6) \) be a fiber of \( \pi_{\mathscr{G}} \). Noting \( L_{\mathscr{G}} \) and \( \mathcal{Q} \) are the pull-backs of locally free sheaves on \( \mathscr{G} \), we see that the restriction of \( \mathcal{A}_3^* \otimes \mathcal{B}_4(-(t \det \mathcal{Q} + tL_{\mathscr{G}})) \) to \( G \) is a direct sum of the following sheaves:

\[
\begin{align*}
\mathcal{O}_G(-t) \\
\mathcal{S}_G(-t), \\
\mathcal{S}_G^*(-t), \\
\mathcal{Q}(1)_G(-t), \\
\mathcal{Q}_G^*(-t), \\
\mathcal{S}_G \otimes \mathcal{Q}_G(-t), \\
\mathcal{S}_G^* \otimes \mathcal{Q}_G^*(-t), \\
\mathcal{Q}_G \otimes \mathcal{Q}_G(-t), \\
\mathcal{Q}_G^* \otimes \mathcal{Q}_G(-t)
\end{align*}
\]

(5.20)

By Theorem 2.1 all the cohomology groups of the sheaves in (5.20) vanish for \( 0 \leq t \leq 5 \) except for \( t = 0 \) and \( \mathcal{S}_G^* \otimes \mathcal{S}_G, \mathcal{S}_G^* \otimes \mathcal{Q}_G \), or \( \mathcal{Q}_G^* \otimes \mathcal{Q}_G \). Therefore we may show Theorem 5.1 by the Leray spectral sequence for \( \pi_{\mathscr{G}} \). \( \square \)

In the subsequent subsections, we will reduce the proof of Theorem in the remaining cases to the case where \( 1 \leq t \leq 4 \).

5.5. **Case** \( n = 4 \) **and** \( 6 \leq t \leq 9 \). In this case, we have only to show the following:

**Proposition 5.8.** For \( \mathcal{C} = \mathcal{A}^* \otimes \mathcal{B} \) as above, it holds

\[
H^\bullet(\tilde{\mathcal{V}}, \mathcal{C}(-t)) \simeq H^{13-\bullet}(\tilde{\mathcal{V}}, \mathcal{C}^*(t - 10))
\]

for any integer \( t \).

Then the proof of Theorem 5.1 in the case where \( 6 \leq t \leq 9 \) is reduced immediately to the case where \( 1 \leq t \leq 4 \). For our proof of the above proposition, we note that each of the cohomology groups \( H^\bullet(\tilde{\mathcal{V}}, \mathcal{C}(-t)) \) is Serre dual to

\[
H^{13-\bullet}(\tilde{\mathcal{V}}, \mathcal{C}^*((t - 10)M_{\tilde{\mathcal{V}}} + 2F_{\tilde{\mathcal{V}}}))
\]

Then, from the exact sequence

\[
0 \to \mathcal{C}^*((t - 10)M_{\tilde{\mathcal{V}}} + (i - 1)F_{\tilde{\mathcal{V}}}) \to \mathcal{C}^*((t - 10)M_{\tilde{\mathcal{V}}} + iF_{\tilde{\mathcal{V}}}) \to \mathcal{C}^*((t - 10)M_{\tilde{\mathcal{V}}} + iF_{\tilde{\mathcal{V}}})|_{F_{\tilde{\mathcal{V}}}} \to 0,
\]

we see that it suffices to show the following vanishings:

\[
H^{13-\bullet}(F_{\tilde{\mathcal{V}}}, \mathcal{A}_i) = 0 \text{ for } i = 1, 2 \text{ and any } \bullet,
\]

where we set

\[
\mathcal{A}_i := \mathcal{C}^*((t - 10)M_{\tilde{\mathcal{V}}} + iF_{\tilde{\mathcal{V}}})|_{F_{\tilde{\mathcal{V}}}^\vee}.
\]

We evaluate the cohomologies (5.21) on the exceptional divisor \( F_{\tilde{\mathcal{V}}} \) by using the flattening \( F^{(3)} \to \hat{G}' \) of the contraction \( F_{\tilde{\mathcal{V}}} \to G_{\mathscr{G}} \), which we have constructed in \cite[Subsect. 5.5]{14}. We will use the notation there freely. Note that, since the morphism \( \hat{F} \to F_{\tilde{\mathcal{V}}} \) is finite, and \( \hat{F} \) has only rational singularities by its construction, the desired vanishings follow from the vanishings of the cohomology groups of the pull-backs of \( \mathcal{A}_i \) on \( F^{(3)} \). Also, since the morphism \( F^{(3)} \to \hat{G}' \) is flat, we have only to show the vanishing along its fibers. Then, by the upper semi-continuity of cohomology groups on fibers, it suffices to prove the vanishing on the fibers \( Fib^{(3)}(V_3, V_4, V_4) = A \cup B \) over the points \( ([V_3], [V_4], [V_4]) \in \hat{G}' \), where we refer \cite[Prop. 5.11]{14} for the notation. Note that the restriction of the pull-back of \( M_{\tilde{\mathcal{V}}} \) to the fibers is trivial. Therefore, it suffices to show

\[
H^\bullet(A \cup B, C_{A \cup B}^*(iF_{A \cup B})) = 0 \text{ for } i = 1, 2,
\]

(5.22)
where $C_{A \cup B}$ and $F_{A \cup B}$ are the pull-backs of $C$ and $F_{\bar{X}}$ to $A \cup B$, respectively. We recall $A = \mathbb{P}(O_{G(2,V)} \oplus U_{G(2,V)}(1)) |_{G(2,V)}$ and a natural morphism $A \rightarrow G(2,V) = \mathbb{P}(V^*)$. Also recall that $E_{\overline{A}B} = A \cap B$ is a divisor on $A$.

In Lemma A.1 in Appendix A we describe several pull-backs to $A$ and $B$ of locally free sheaves on $\mathcal{Y}$. Using this we can complete our proof of Proposition 5.8 as follows:

**Proof of Proposition 5.8.** It suffices to show the vanishings of (5.22). Tensoring $C_{A \cup B}(iF_{A \cup B})$ with the Mayor-Vietris sequence, we have

(5.23) \[ 0 \rightarrow C_{A \cup B}(iF_{A \cup B}) \rightarrow C_{A}(iF_{A}) \oplus C_{B}(iF_{B}) \rightarrow C_{A \cap B}(iF_{A \cap B}) \rightarrow 0, \]

where $C_{A}$, $C_{B}$, and $C_{A \cap B}$ are the restrictions of $C_{A \cup B}$ to $A$, $B$ and $A \cap B$ respectively, and $F_{A \cap B}$ is the restriction of $F_{A \cup B}$ to $A \cap B$. By Lemma A.1 (1), it is straightforward to verify the vanishings of $H^*(A, C_{A}(iF_{A}))$. Also, by Lemma A.1 (2), we see that the restriction maps $H^*(B, C_{B}(iF_{B})) \rightarrow H^*(A \cap B, C_{A \cap B}(iF_{A \cap B}))$ are isomorphisms. Then we have the desired vanishings of $H^*(A \cup B, C_{A \cup B}(iF_{A \cup B}))$.

\[ \square \]

5.6. **Case** $n = 3$ and $t = 6, 7$. For $n = 3$, it remains to show the vanishings of $H^*(\mathcal{Y}, B(-6))$ and $H^*(\mathcal{Y}, B(-7))$ for $B = \mathcal{O}_{\mathcal{Y}}, \mathcal{O}, \mathcal{S}_L^t$, or $\mathcal{Q}$. For this, we have only to show the following:

**Proposition 5.9.** It holds

$H^*(\mathcal{Y}, B(-t)) \simeq H^{n*t}(\mathcal{Y}, B^*(t - 8))$

for any integer $t$.

**Proof.** We can show the assertion in the same way to the proof of Proposition 5.8 using Lemma A.1. \[ \square \]

5.7. **Calculating** $H^0(\mathcal{Y}, \mathcal{Q}^* \otimes \mathcal{Q}(M))$. In the last part of the proof of [15 Thm. 8.3.2], we use the following lemma:

**Lemma 5.10.** Suppose $n = 3$. There exists a unique $\text{SL}(V)$-equivariant map $\mathcal{Q}^* \otimes \mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{Q}^*(M) \otimes \mathcal{O}_{\mathcal{Y}}(H)$ up to constant.

**Proof.** We compute

$H^0(\mathcal{Q}^* \otimes \mathcal{O}_{\mathcal{Y}}(H)) \simeq H^0(\mathcal{Y}, \mathcal{Q}^* \otimes \mathcal{Q}(M)) \otimes H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(H)).$

We have $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(H)) \simeq S^2V^*$. To compute $H^0(\mathcal{Y}, \mathcal{Q}^* \otimes \mathcal{Q}(M))$, we tensor $\mathcal{Q}^* \otimes \mathcal{Q}$ to the exact sequence

$0 \rightarrow \mathcal{O}_{\mathcal{Y}}(M) \rightarrow \text{det} \mathcal{Q} \otimes \wedge^4V^* \rightarrow \text{det} \mathcal{Q}|_{P_6} \otimes \wedge^4V^* \rightarrow 0$

induced from Proposition 5.2. By Theorem 2.1 and the Littlewood-Richardson rule, we have

$H^0(\mathcal{Y}, \mathcal{Q}^* \otimes \text{det} \mathcal{Q}) \otimes \wedge^4V^* \simeq H^0(\mathcal{Y}, \wedge^2\mathcal{Q} \otimes \mathcal{Q}) \otimes \wedge^4V^* \simeq (H^0(\mathcal{Y}, \wedge^2\mathcal{Q} \otimes \wedge^3\mathcal{Q}) \oplus H^0(\mathcal{Y}, \mathcal{Q}^{(2,1)}) \otimes \wedge^4V^*) \simeq (\wedge^3(\wedge^2V) \oplus \Sigma^{(2,1,0,0,0)} \otimes \wedge^2V) \otimes \wedge^4V^*.$

By the plethysm of the Schur functors

$\wedge^3(\wedge^2V) \simeq \Sigma^{(3,1,1,1)}V \oplus \Sigma^{(2,2,2)}V,$
By Proposition 4.10, we have
\[ \Sigma^{(2,1,0,0,0,0)} \land^2 V \simeq \Sigma^{(2,2,1,1)} V \oplus \Sigma^{(3,2,1,0)} V, \]
we obtain
\[ (5.24) \quad H^0(\mathcal{F}, \mathcal{Q}^* \otimes \mathcal{Q}(\det \mathcal{Q})) \otimes \land^4 V^* \simeq \Sigma^{(2,0,0,0)} V \oplus \Sigma^{(1,1,1,-1)} V \oplus \Sigma^{(1,1,0,0)} V \oplus \Sigma^{(2,1,0,-1)} V. \]

By Theorem 2.1 and the Littlewood-Richardson rule, we obtain
\[ \text{Corollary 5.11.} \]
\[ (5.25) \quad H^0(F_p, \mathcal{Q}^* \otimes \mathcal{Q}(\det \mathcal{Q})|_{F_p}) \otimes \land^4 V^* \simeq H^0(\mathcal{P}(V), (\Omega^1_{\mathcal{P}(V)}(1))^* \otimes \Omega^1_{\mathcal{P}(V)}(1) \otimes \mathcal{O}_{\mathcal{P}(V)}(2)) \otimes \land^4 V. \]

By Theorem 2.1 and the Littlewood-Richardson rule, we obtain
\[ (5.25) \quad H^0(F_p, \mathcal{Q}^* \otimes \mathcal{Q}(\det \mathcal{Q})|_{F_p}) \otimes \land^4 V^* \simeq \Sigma^{(1,1,1,-1)} V \oplus \Sigma^{(2,1,0,-1)} V. \]

Since the identity of Hom(\mathcal{Q}, \mathcal{Q}) induces that of Hom(\Omega^1_{\mathcal{P}(V)}(2), \Omega^1_{\mathcal{P}(V)}(2)), the component \( \Sigma^{(1,1,1,-1)} V \) in (5.24) is mapped isomorphically in (5.25). Therefore, \( H^0(\mathcal{F}, \mathcal{Q}^* \otimes \mathcal{Q}(M)) \) consists of at most \( \Sigma^{(2,0,0,0)} V \simeq S^2 V, \Sigma^{(1,1,0,0)} V, \) and \( \Sigma^{(2,1,0,-1)} V \) (it seems that \( H^0(\mathcal{F}, \mathcal{Q}^* \otimes \mathcal{Q}(M)) \simeq \Sigma^{(2,0,0,0)} V \oplus \Sigma^{(1,1,0,0)} V \) but we do not need to prove this). Hence SL(V)-invariant sections of \( H^0(\mathcal{F}, \mathcal{Q}^* \otimes \mathcal{Q}(M)) \otimes H^0(\mathcal{F}, \mathcal{O}_M(H)) \) come from the identity element of \( S^2 V \otimes S^2 V^* \) up to constant. \( \square \)

5.8. **Lefschetz collection in \( D^b(\mathcal{F}) \).** It is straightforward to obtain the following result from Theorem 5.1.

**Corollary 5.11.** Suppose \( n = 3 \). Let \( \Lambda := \{3, 2, 1_a, 1_b\} \) be an ordered set \( (\Lambda, \prec) \). Define \( (\mathcal{E}_\alpha)_{\alpha \in \Lambda} := (\mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b}) \) be an ordered collection of sheaves. Set \( D_{\mathcal{F}} := \langle \mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b} \rangle \subset D^b(\mathcal{F}) \).

Then \( D_{\mathcal{F}}, D_{\mathcal{F}}(1), \ldots, D_{\mathcal{F}}(5), \mathcal{O}_{\mathcal{F}}(6), \mathcal{O}_{\mathcal{F}}(7) \) is a Lefschetz collection, where \( (t) \) represents the twist by the sheaf \( \mathcal{O}_{\mathcal{F}}(tM) \).

**Remark 5.12.** By similar calculations to show Theorem 5.1, we obtain the following:

1. The ordered collection of sheaves \( (\mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b}) \) is a strongly exceptional collection in \( D^b(\mathcal{F}) \).

2. Hom’s of the sheaves in the above collection are given by the following diagram:

\[ \begin{tikzcd}
\mathcal{E}_3 \arrow[r] \arrow[d, Mapsto, \land^2 V] & \mathcal{E}_2 \arrow[d, Mapsto, \land^2 V] \\
\mathcal{E}_{1a} \arrow[r, Mapsto, V] & \mathcal{E}_{1b}
\end{tikzcd} \]

In case \( n = 4 \), the following Lefschetz collection is suitable for our purpose (see [12]).
Corollary 5.13. Suppose \( n = 4 \). Let \( \Lambda := \{3, 2, 1_a, 1_b\} \) be an ordered set (\( \Lambda, \prec \)). Define

\[
(E_\alpha)_{\alpha \in \Lambda} := (E_3, E_2, E_{1_a}, E_{1_b}) = (\tilde{S}_L, \tilde{Q}^* , \mathcal{O}_{\tilde{\mathcal{Y}}}, \tilde{Q}^*(M))
\]

be an ordered collection of sheaves on \( \tilde{\mathcal{Y}} \). Set

\[
D_{\tilde{\mathcal{Y}}} := \langle E_3, E_2, E_{1_a}, E_{1_b} \rangle \subset D^b(\tilde{\mathcal{Y}}).
\]

Then

\[
D_{\tilde{\mathcal{Y}}}, D_{\tilde{\mathcal{Y}}}(1), \ldots, D_{\tilde{\mathcal{Y}}}(9)
\]

is a Lefschetz collection, where \((t)\) represents the twist by the sheaf \( \mathcal{O}_{\tilde{\mathcal{Y}}}(tM) \).

Proof. Writing \( D_{\tilde{\mathcal{Y}}}, D_{\tilde{\mathcal{Y}}}(1), \ldots, D_{\tilde{\mathcal{Y}}}(9) \) explicitly, we obtain

\[
\begin{align*}
\tilde{S}_L, \tilde{Q}^*, \mathcal{O}_{\tilde{\mathcal{Y}}}, \\
\tilde{Q}^*(M), \tilde{S}_L(M), \tilde{Q}^*(M), \mathcal{O}_{\tilde{\mathcal{Y}}}(M), \\
\ldots, \\
\tilde{Q}^*(9M), \tilde{S}_L(9M), \tilde{Q}^*(9M), \mathcal{O}_{\tilde{\mathcal{Y}}}(9M), \\
\tilde{Q}^*(10M).
\end{align*}
\]

Let \( \mathcal{C} \) be any sheaf in this sequence except \( \tilde{Q}^*(10M) \). Then, by Proposition 5.8, we have \( \text{Hom}^\bullet(\tilde{Q}^*(10M), \mathcal{C}) \simeq \text{Hom}^\bullet(\mathcal{C}, \tilde{Q}^* ) \). Hence we may replace the sequence by \( \mathcal{D}', \mathcal{D}'(1), \ldots, \mathcal{D}'(9) \) with \( \mathcal{D}' = (\tilde{Q}^*, \tilde{S}_L, \tilde{Q}^*, \mathcal{O}_{\tilde{\mathcal{Y}}}) \). Now the assertion follows immediately from Theorem 5.1. \( \square \)

Remark 5.14. By similar calculations to show Theorem [5.1] we obtain the following (see [13]):

1. \( (E_\alpha)_{\alpha \in \Lambda} \) is a strongly exceptional collection.
2. Hom’s of the sheaves in the above collection are given by the following diagram:

\[
\begin{align*}
& \wedge^2 V \\
\Rightarrow & E_3 \quad V \\
& E_2 \quad V \\
& S^2 V \quad E_{1_b}
\end{align*}
\]

Remark 5.15. Since the singularity of the double symmetroid \( \mathcal{Y} \) is complicated, [13] Theorem 1] seems to be difficult to apply for the resolution \( \tilde{\mathcal{Y}} \to \mathcal{Y} \) to obtain a categorical resolution of \( D^b(\mathcal{Y}) \). However, we expect that the Lefschetz collection \( D_{\tilde{\mathcal{Y}}}, D_{\tilde{\mathcal{Y}}}(1), \ldots, D_{\tilde{\mathcal{Y}}}(9) \) gives a Lefschetz decomposition of a strongly crepant categorical resolution, if exists, of \( D^b(\mathcal{Y}) \).

Appendix A. Pull-backs of sheaves to the flattening of \( F_{\tilde{\mathcal{Y}}} \to G_{\mathcal{Y}} \)

In this section, we consider the situation as in [14 Subsect. 5.5] and use the notation there freely.

Here we fix \( V_{n-1} \) and \( V_n \), and consider sheaves on the fiber \( \text{Fib}^{(3)}(V_{n-1}, V_n, V_n) = A \cup B \) of \( F^{(3)} \to \hat{G}' \).
Lemma A.1. Denote by $H_A$ the pull-back on $A$ of $\mathcal{O}_{\mathbb{P}(V)}(1)$. We denote by $\mathcal{E}_A$ and $\mathcal{E}_B$ the pull-backs of a locally free sheaf $\mathcal{E}$ on $\mathbb{P}$ to $A$ and $B$, respectively. In particular, $F_A$ and $F_B$ stand for the pull-backs of $F$ to $A$ and $B$. Then we have the following isomorphisms:

(1) $F_A \sim -(E_{AB} + 2H_A)$, $(\tilde{S}_L^*)_A \sim \tilde{Q}_A \cong \mathcal{O}_A \oplus \mathcal{V}$, and $\tilde{Q}_A \cong \mathcal{O}^{B^2}_A \oplus \mathcal{V}$ with a locally free sheaf $\mathcal{V}$ given by a unique non-split extension

$0 \to \mathcal{O}_A(H_A + E_{AB}) \to \mathcal{V} \to \mathcal{O}_A(H_A) \to 0$.

(2) $F_B \sim p_B^*\mathcal{O}_{\mathbb{P}(V)}(n-3,V_{n-1})(-1)$, $(\tilde{S}_L)_B \sim \tilde{Q}_B \cong \mathcal{O}_B \oplus p_B^*\mathcal{Q}_{V_{n-1}}$, and $\tilde{Q}_B \cong \mathcal{O}^{B^2}_B \oplus p_B^*\mathcal{Q}_{V_{n-1}}$, where $\mathcal{Q}_{V_{n-1}}$ is the universal quotient bundle on $\mathbb{P}(n-3,V_{n-1})$, and $p_B : B \to G(n-3,V_{n-1})$ is given in [14] Prop. 5.10 and 5.11.

Proof. Let $G$ be the exceptional divisor for $\tilde{A} \to A$ and $L_{\tilde{A}}$, $H_A$, and $E_{AB}$ the pull-backs on $\tilde{A}$ of $\mathcal{O}_{\mathbb{P}(n-3,V_{n-1})}(1)$, $H_A$ and $E_{AB}$, respectively.

Step 1. $\tilde{E}_{AB} + 2H_{\tilde{A}} - G = L_{\tilde{A}}$.

As in [14] Prop. 5.13], we denote by $s_A \subset A$ the locus of $\rho$-conics, which is a section associated to an injection $\mathcal{O}_{\tilde{A}} \to \mathcal{O}_{\mathbb{P}(V_{n-1})} \cong \mathcal{O}_{\mathbb{P}(V_{n-1})} \oplus T_{\mathbb{P}(V_{n-1})}$.

Let $\mathcal{I}$ be the ideal sheaf of $s_A$ in $A$, and consider the exact sequence

$0 \to \mathcal{O}_A(E_{AB} + 2H_A) \oplus \mathcal{I} \to \mathcal{O}_A(E_{AB} + 2H_A) \to \mathcal{O}_{s_A}(2H_A) \to 0$,

where the last term is obtained since $s_A \cap E_{AB} = \emptyset$. Let $\pi_A : A \to \mathbb{P}(V_{n-1})$ be the natural morphism. Then $\pi_A_*([\mathcal{O}(E_{AB} + 2H_A) \otimes \mathcal{I}]) \cong \mathcal{O}_{H_n}^{(2)} \cong \mathcal{O}_{H_n}^{(2)}$.

From this, we see that the natural map

$H^0(\mathcal{O}(E_{AB} + 2H_A) \otimes \mathcal{I}) \to \mathcal{O}_A(E_{AB} + 2H_A) \otimes \mathcal{I}$

is surjective and $H^0(\mathcal{O}(E_{AB} + 2H_A) \otimes \mathcal{I}) \cong H^0(\mathbb{P}(V_{n-1})) \cong \mathbb{P}(V_{n-1})$.

This is equivalent to that $|\tilde{E}_{AB} + 2H_{\tilde{A}} - G|$ is base point free and it defines a morphism $\Phi : \tilde{A} \to \mathbb{P}(\mathbb{P}(V_{n-1}))$. $\Phi$ factors through $A_{y_2}$ since it contracts $E_{AB}$. Let $\Phi' : A_{y_2} \to \mathbb{P}(\mathbb{P}(V_{n-1}))$ be the induced morphism. $\Phi'$ does not coincide with $A_{y_2} \to \tilde{A}$ since the latter contracts the image of $G$. Therefore $\Phi'$ induces the quadric fibration $A_{y_2} \to G(n-3,V_{n-1})$. In particular, we have $\tilde{E}_{AB} + 2H_{\tilde{A}} - G = L_{\tilde{A}}$ as desired.

Step 2. $\det(\tilde{S}_L) \sim \det(\tilde{Q}) = E_{AB} + 2H_A$.

By (5.2), we have only to determine $\det(\tilde{Q})$. Note that $G$ coincides with the pull-back of $F_{y_2}$, where $F_y$ is the exceptional divisor of $\mathbb{P}_2 \to \mathbb{P}$. Therefore, by Proposition 5.3, we have $\det(\tilde{Q}) = G + L_{\tilde{A}}$ since $M_{\tilde{y}}$ is trivial on a fiber of $F_{\tilde{y}} \to G$. Now the assertion follows from Step 1.

Step 3. $F_A = -(E_{AB} + 2H_A)$.

By Proposition 5.4, we have $F'_{\tilde{y}} = 2M_{y_2} - L_{y_2} - F_y$. Since $G$ coincides with the pull-back of $F_{y_2}$, $M_{\tilde{y}}$ is trivial on a fiber of $F_{\tilde{y}} \to G$, it holds that $F_{\tilde{A}} = -(L_{\tilde{Y}} + G)$. Therefore the assertion follows from Step 1.

Step 4. $(\tilde{S}_L)_A \sim \tilde{Q}_A \cong \mathcal{O}_A \oplus \mathcal{V}$.

We investigate the restriction of the universal exact sequence (5.1) on $A_{y_2}$. Let $S_{A_{y_2}}$ and $Q_{A_{y_2}}$ be the restrictions of $S$ and $Q$, respectively. Then we obtain

$(A.1) \quad 0 \to S_{A_{y_2}} \to \pi_{A_{y_2}}^*(\otimes^2 Q|_{G(n-3,V_{n-1})}) \to Q_{A_{y_2}} \to 0$. 

Therefore we have the following isomorphisms:

\[(A.2) \quad \bigwedge^2 (\mathcal{Q}|_{\mathbb{G}_m}) \cong \mathcal{O}_{\mathbb{G}_m}(1) \oplus \left( \mathcal{Q}|_{\mathbb{G}_m} \otimes (V/V_{n-1}) \right) \oplus \bigwedge^2 (V/V_{n-1}) \otimes \mathcal{O}_{\mathbb{G}_m}.
\]

Let \([V_n]_3\) be a point of \(G(n-3,V_{n-1})\). \((A.2)\) means fiberwise

\[(A.3) \quad \bigwedge^2 (V/V_{n-3}) \cong \bigwedge^2 (V_{n-1}/V_{n-3}) \oplus (V_{n-1}/V_{n-3}) \otimes (V/V_{n-1}) \oplus \bigwedge^2 (V/V_{n-1}).
\]

Now we recall [41 Rem. 5.14]. Let \(\Gamma\) be the fiber of \(A_{\mathbb{G}_m} \to G(n-3,V_{n-1})\) over \([V_n]_3\). The vertex of the quadric cone \(\Gamma\) corresponds to the \(\sigma\)-plane \(P_{V_n}/V_{n-3} = \{C^2 \subset V_n/V_{n-3}\}\). Points \([P_{V_{n-2}/V_{n-3}}]\) which correspond to \(\rho\)-planes and are contained in \(\Gamma\) satisfy \(V_{n-3} \subset V_{n-2}\). Since \(\Gamma\) is the cone over the Veronese curve \(v_2(P(V_{n-1}/V_{n-3}))\), it is swept out by lines joining \([P_{V_{n-2}/V_{n-3}}]\) and \([P_{V_{n-2}/V_{n-3}}]\) such that \(V_{n-3} \subset V_{n-2} \subset V_{n-1}\). A line in \(G(3,\bigwedge^2 (V/V_{n-3}))\) is of the form \(\{W_2 \subset C^3 \subset W_4\}\) with some \(W_i \simeq C^i\) \((i = 2, 4)\). We take a basis \(e_1, \ldots, e_4\) of \(V/V_{n-3}\) such that

\[V_{n-2}/V_{n-3} = \langle e_1 \rangle, V_{n-1}/V_{n-3} = \langle e_1, e_2 \rangle, \text{and } V_{n-3}/V_{n-3} = \langle e_1, e_2, e_3 \rangle.\]

For the line joining \([P_{V_{n-2}/V_{n-3}}]\) and \([P_{V_{n-2}/V_{n-3}}]\), it is easy to see that

\[W_2 = \langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle \cong \bigwedge^2 (V_{n-1}/V_{n-3}) \oplus (V_{n-2}/V_{n-3}) \otimes (V/V_{n-1}),
\]

\[W_4 = \langle e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3 \rangle \subset \bigwedge^2 (V_{n-1}/V_{n-3}) \oplus (V_{n-1}/V_{n-3}) \otimes (V/V_{n-1}),\]

and

\[\bigwedge^2 (V_{n-1}/V_{n-3}) \oplus (V_{n-1}/V_{n-3}) \otimes (V/V_{n-1})/W_4 \cong \langle e_2 \wedge e_4 \rangle \cong V_{n-1}/V_{n-2} \otimes V/V_{n-1}.
\]

As for \(S_{\hat{A}}\), these imply the following:

- We can see that \(S_{\hat{A}}\) contains the line bundle \(L_{\hat{A}}\) with fiber \(\bigwedge^2 (V_{n-1}/V_{n-3})\) as a direct summand. Hence, let us write \(S_{\hat{A}} = S'_{\hat{A}} \oplus L_{\hat{A}}\) with a locally free sheaf \(S'_{\hat{A}}\) of rank two on \(\hat{A}\).
- \(S'_{\hat{A}}\) contains a sub line bundle with fiber \((V_{n-1}/V_{n-3}) \otimes (V/V_{n-1})\), which is isomorphic to \(-H_{\hat{A}} + L_{\hat{A}}\).

Therefore, by Step 2, we obtain

\[0 \to \mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \to S'_{\hat{A}}(-L_{\hat{A}}) \to \mathcal{O}_{\hat{A}}(-H_{\hat{A}} - E_{AB}) \to 0.
\]

Since all the terms of the exact sequence are the pull-backs of locally free sheaves on \(A\), the dual of this exact sequence descends to

\[0 \to \mathcal{O}_A(H_A + E_{AB}) \to (S'_{L})_{\hat{A}} \to \mathcal{O}_A(H_A) \to 0,
\]

where \((S'_{L})_{\hat{A}}\) is the locally free sheaf on \(A\) such that its pull-back on \(\hat{A}\) is equal to \((S'_{L})^\vee (L_{\hat{A}})\). This sequence does not split since \((S'_{L})_{\hat{A}}\) comes from a locally free sheaf on \(A_{\mathbb{G}_m}\) while \(H_A\) does not. From

\[\text{Ext}^1(\mathcal{O}_A(H_A), \mathcal{O}_A(H_A + E_{AB})) \cong H^1(A, \mathcal{O}_A(E_{AB})) \cong H^1(P(V^*_{n-1}), \mathcal{O}_{P(V^*_{n-1})} \oplus \Omega^1(P(V^*_{n-1}))) \cong \mathbb{C},\]

such a nonsplit extension is unique, which we denote by \(\mathcal{V}\). Thus we obtain \((S'_{L})_{\hat{A}} \cong \mathcal{V} \oplus \mathcal{O}_{\hat{A}}\), with a locally free sheaf \(\mathcal{V}\) as described in (1).

Similarly, as for \(\mathcal{Q}_{\hat{A}}\), the above facts imply the following:
• We can see that $\mathcal{Q}_{A}$ contains the line bundle $\mathcal{O}_{A}$ with fiber $\wedge^2(V/V_{n-1})$ as a direct summand. Hence, let us write $\mathcal{Q}_{A} = \mathcal{Q}_{A}' \oplus \mathcal{O}_{A}$ with a locally free sheaf $\mathcal{Q}_{A}'$ of rank two on $\hat{A}$.

• $\mathcal{Q}_{A}'$ have a quotient line bundle with fiber $V_{n-1}/V_{n-2} \otimes V/V_n$, which is isomorphic to $H_{\hat{A}}$.

Therefore, by Step 2, we obtain

$$0 \to \mathcal{O}(H_{\hat{A}} + \hat{E}_{AB}) \to \mathcal{Q}_{A}' \to \mathcal{O}(H_{\hat{A}}) \to 0.$$ 

In a similar way to determine $(\hat{S}_{L}^{*})_{A}$, we may obtain $\mathcal{Q}_{A} \simeq \mathcal{V} \oplus \mathcal{O}_{A}$ as desired.

**Step 5.** $\mathcal{Q}_{A} \simeq \mathcal{O}_{A} \oplus \mathcal{V}$.

By [13] Rem. 5.14, $\mathcal{P}_{\rho} \cap \mathcal{A}_{\mathcal{G}_{3}} \simeq \mathbb{P}(\Omega_{V_{n-1}}) \simeq F(n-3, n-2, V_{n-1})$. Restricting [13] to $\mathcal{A}_{\mathcal{G}_{2}}$, we obtain

$$(A.4) \quad 0 \to \mathcal{R}_{A, \mathcal{G}_{2}} \to \pi_{A, \mathcal{G}_{2}}^{*} \Omega_{V_{n-1}}^{1} \oplus \mathcal{O}_{A, \mathcal{G}_{2}}^{\oplus 2} \to \iota_{*} \mathcal{O}_{\mathcal{P}(V_{n-1})}(1) \to 0,$$

where we set $\mathcal{R}_{A, \mathcal{G}_{2}} = \mathcal{R}_{2}|_{A, \mathcal{G}_{2}}$, $\pi_{A, \mathcal{G}_{2}} = \pi_{\mathcal{G}_{2}}|_{A, \mathcal{G}_{2}}$, $\iota : \mathbb{P}(\Omega_{V_{n-1}}) \hookrightarrow A_{\mathcal{G}_{2}}$ and use $\mathcal{R}_{2}/\mathcal{R}_{1} \simeq \mathcal{O}(\mathcal{P}(\mathcal{G}^{n-1}))(-1)$. Since $\text{Hom}(\mathcal{A}_{\mathcal{G}_{2}}, \mathcal{O}(\mathcal{P}(\Omega_{V_{n-1}}))(1)) = H^{0}(\Omega_{V_{n-1}}) = 0$, we have the decomposition $\mathcal{R}_{A, \mathcal{G}_{2}} \simeq \mathcal{O}_{A, \mathcal{G}_{2}}^{\oplus 2} \oplus \mathcal{V}'$ and also

$$(A.5) \quad 0 \to \mathcal{V}' \to \pi_{A, \mathcal{G}_{2}}^{*} \Omega_{V_{n-1}}^{1} \to \iota_{*} \mathcal{O}_{\mathcal{P}(V_{n-1})}(1) \to 0.$$ 

Note that $\pi_{A}$ is a $\mathbb{F}_{2}$-fibration and it decomposes as $\hat{A} \to \mathbb{P}(\Omega_{V_{n-1}}) \to G(n-3, V_{n-1})$. On $\mathbb{P}(\Omega_{V_{n-1}}) \simeq F(n-3, n-2, V_{n-1})$, we have a natural exact sequence

$$0 \to \mathcal{O}_{\mathcal{P}(V_{n-1})}(-H_{\hat{A}}) \to \pi_{A}^{*} \Omega_{V_{n-1}}^{1} \to \mathcal{O}_{\mathcal{P}(V_{n-1})}(1) \to 0.$$ 

Pulling back on $\hat{A}$, we obtain $0 \to \mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \to \pi_{A, \hat{A}}^{*} \Omega_{V_{n-1}}^{1} \to \mathcal{O}_{\hat{A}}(H_{\hat{A}} - L_{\hat{A}}) \to 0$ such that the composite of $\mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \to \pi_{A, \hat{A}}^{*} \Omega_{V_{n-1}}^{1}$ with $\pi_{A, \mathcal{G}_{2}}^{*} \Omega_{V_{n-1}}^{1}$ is a 0-map. Therefore, it induces an injection $\mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \to \mathcal{V}'$ with a locally free cokernel. Computing the determinants, we obtain

$$0 \to \mathcal{O}_{\mathcal{A}}(H_{\hat{A}} - L_{\hat{A}} - G) \to 0.$$ 

Since $H_{\hat{A}} - L_{\hat{A}} - G = -H_{\hat{A}} - \hat{E}_{AB}$ by Step 1, we obtain $\mathcal{Q}_{A} \simeq \mathcal{O}_{A} \oplus \mathcal{V}$ in a similar way to determine $(\hat{S}_{L}^{*})_{A}$.

**Step 6.** $F_{B}$, $\tilde{S}_{L}^{*}B$, $\tilde{Q}_{B}$ and $\tilde{Q}_{B}$. 

By [13] Prop. 5.10 and 5.11, the image of $B$ on $F_{\tilde{\mathcal{G}}}$ is the $G(n-3, V_{n-1})$ in $A_{\tilde{\mathcal{G}}}$. Therefore, $F_{B}$, $(\tilde{S}_{L}^{*})_{B}$, $\tilde{Q}_{B}$ and $\tilde{Q}_{B}$, respectively, are the pull-backs of the restrictions of $F_{\tilde{\mathcal{G}}}$, $\tilde{S}_{L}^{*}$, $\tilde{Q}$, and $\tilde{Q}$ to $G(n-3, V_{n-1})$. Since $F_{A}|_{E_{AB}} \simeq -(E_{AB} + 2H_{A})|_{E_{AB}}$ by Step 3, and this is the pull-back of $\mathcal{O}(G(n-3, V_{n-1})|_{-1})$, we have $F_{B} = p_{B}^{*} \mathcal{O}(G(n-3, V_{n-1})|_{-1})$. Also, since $\tilde{Q}_{A} \simeq \mathcal{O}_{A} \oplus (\tilde{S}_{L}^{*})_{A} \simeq \mathcal{O}_{A} \oplus \tilde{Q}_{A}$ as above, we have $\tilde{Q}_{B} \simeq \mathcal{O}_{B} \oplus (\tilde{S}_{L}^{*})_{B} \simeq \mathcal{O}_{B} \oplus \tilde{Q}_{B}$. Thus we have only to determine $\tilde{Q}_{B}$. Since $G(n-3, V_{n-1})$ is contained in the locus of $\sigma$-plane, it is disjoint from the locus $G_{\rho}$ of $\rho$-conics. Therefore, by [13], we have $\tilde{Q}_{B} \simeq p_{B}^{*} \mathcal{Q}(G(n-3, V_{n-1})) \simeq \mathcal{O}_{B}^{\oplus 2} \oplus p_{B}^{*} \mathcal{Q}(V_{n-1})$. □
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