ASYMPTOTIC CONTROL THEORY FOR A SYSTEM OF LINEAR OSCILLATORS

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Abstract. We present asymptotical control theory for a system of an arbitrary number of linear oscillators under common bounded control. We suggest a method for a design of a feedback control for the system. We prove by using the DiPerna–Lions theory of singular ODE that the suggested control law correctly defines a motion of the system. The obtained control is asymptotically optimal: the ratio of motion time to zero with this control to the minimum one is close to 1, if the initial energy of the system is large. Some of the results are based on a new lemma about observability of perturbed autonomous linear systems.

Keywords maximum principle, reachable sets, linear system
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1. Introduction

The problem of time-optimal steering of a given initial state to a given manifold is typical for the optimal control theory. One of the classical achievements in this area is the explicit construction, based on the Pontryagin maximum principle, of the minimum time damping of a single linear oscillator [1]. This system is described via the equation

$$
\ddot{x} + x = u, \quad |u| \leq 1,
$$

where $x$ is the position, $u$ is the control. Here, the oscillator frequency $\omega$ is assumed to be 1 without loss of generality. The optimal control is of bang-bang type, i.e. it takes values $u = \pm 1$, and the switching curve which separates the domain of the phase plane, where $u = -1$ from the domain $u = +1$ consists of unit semicircles centered at points of the form $(2k+1, 0)$, where $k \in \mathbb{Z}$ is an integer. When seen from afar, and this is our primary point of view in what follows, the switching curve looks like the $x$-axis, and the optimal control looks like the dry friction $u(x, \dot{x}) = -\text{sign} \dot{x}$.

1.1. Problem statement. Our work is devoted to a more general, and next in complexity problem of minimum time steering of a system of $N$ linear oscillators with eigenfrequencies $\omega_i$ under a common bounded control $u$

$$
\begin{align*}
\dot{x}_i &= y_i, \\
\dot{y}_i &= -\omega_i^2 x_i + u, \quad |u| \leq 1, \quad i = 1, \ldots, N.
\end{align*}
$$

It is probably impossible in principle to get an explicit formula for the optimal control in this case. Even a numerical solution is hardly achievable.

In this paper we deal with the feedback control and try to make the duration of the steering as small as possible. We assume that the steering is possible in principle which means, according to the Kalman controllability condition [2], that the eigenfrequencies of all oscillators are different. Our main result is a design of an asymptotically optimal and implementable numerically feedback control for system (1) in the non-resonant case, when there are no nontrivial relations between
eigenfrequencies of the form

\( \sum_{i=1}^{N} m_i \omega_i = 0, \) where \( 0 \neq m = (m_1, \ldots, m_N) \in \mathbb{Z}^N. \)

Here the adjective asymptotical refers to the large initial energy

\[ E = \frac{1}{2} \sum_{i=1}^{N} (\dot{x}_i^2 + \omega_i^2 x_i^2) \]

of the system (1). Our control works as well in the resonant case, then, however, it is not asymptotically optimal. Still, the ratio of the steering time to the minimum one is uniformity bounded.

System (1) can be interpreted in mechanical terms at least in two ways. In the first interpretation, the components \( x_i \) of the state vector are vertical deviations of pendulums attached to a cart moving under bounded acceleration \( u \). In the second interpretation, the components \( x_i \) are the displacements of masses attached to springs which, in turn, are attached to the same cart.

Our paper is organized as follows. In subsection 1.2 we explain basic difficulties of the minimum time problem. Subsection 1.3 describes our strategy of control. In Section 2 we discuss basic properties of our control within high and medium energy zones. A nontrivial issue of the nature of motion under the control is discussed in Section 4. Asymptotic optimality of the control within high energy zone is studied in Section 5. An essentially new technique for the study of efficiency of the control is developed in Subsection 6. Section 7 is devoted to the singular arcs of our control. Section 8 is devoted to the final stage of the control: the motion in a close vicinity of the target. In Section 9 we perform matching of controls defined within different zones. Our main result on asymptotic optimality is presented in Section 10. In Section 11 we illustrate our strategy in the case of a single oscillator.

A summary of our results is presented in [3].

1.2. Minimum time problem. Suppose we need to bring system (1) to the equilibrium in minimum time. This problem is a particular case of the minimum time problem for a linear control system

\[ \dot{x} = Ax + Bu, \quad x = (x_1, y_1, \ldots, x_N, y_N)^* \in \mathbb{V} = \mathbb{R}^{2N}, \quad u \in \mathbb{R}, \quad |u| \leq 1, \]

where the matrix \( A \) and the vector \( B \) are

\[
A = \begin{pmatrix}
0 & 1 \\
-\omega_1^2 & 0 \\
& & \ddots \\
0 & 1 \\
-\omega_N^2 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
1 \\
\vdots \\
0 \\
1
\end{pmatrix}.
\]

As it is well known, the problem is equivalent to the boundary value problem for the Pontryagin maximum principle corresponding to the Hamiltonian

\[ h(x, p) = \langle Ax, p \rangle + |\langle B, p \rangle| - 1 = \max_{|u| \leq 1} \{ \langle Ax, p \rangle + \langle Bu, p \rangle - 1 \}. \]

Here, angular brackets stand for the standard scalar product in \( \mathbb{R}^{2N} \), \( |\cdot| \) stands for the Euclidean norm, and the maximum is taken over the interval \( \{ u \in \mathbb{R} : |u| \leq 1 \} \).

The problem takes the form

\[
\dot{x} = Ax + Bu, \quad \dot{p} = -A^* p, \\
u = \text{sign} \langle B, p \rangle, \quad x(0) = x_0, \quad x(T) = 0, \quad h(x, p) = 0.
\]
Note, that system (5) is Hamiltonian with $2N$ degrees of freedom and $N+1$ integrals of motion: namely, $h$ and $I_k = \frac{1}{2}p_{2k-1}^2 + \frac{1}{2}\omega_k^2 p_{2k}^2$, $k = 1,\ldots,N$. These integrals are Poisson commuting. The fact, that the Poisson brackets $\{I_k, I_l\}$ are zero is obvious, while the identity $\{I_k, h\} = 0$ results from an easy computation. In the case $N = 1$ the number of degrees of freedoms coincides with the number of commuting integrals. This is the basic reason for the existence of an explicit optimal solution. The same equality is the basic assumption of the Liouville–Arnold theorem on complete integrability of a Hamiltonian system [4].

In general we deal with a nonlinear boundary value problem of dimension $4N$. If the vector $p(0)$ is known, then the control $u$ is also known, and $x(T)$ can be easily found via solution of the Cauchy problem. Therefore, the boundary problem reduces to a solution of $2N+1$ transcendental equations $x(T) = 0$, $h(x, p) = 0$ for the $2N+1$-dimensional vector with components $p(0)$, $T$. This is complicated to such an extent that forces us looking for approximate methods.

1.3. Proposed strategy. We present a method based on asymptotical behavior of a reachable set of the system. We consider both asymptotic $T \to \infty$ and $T \to 0$ theories of the reachable sets.

In this way, we suggest using a quasi-optimal feedback control based on combination of three strategies. We divide the entire phase space of the system into three zones: high energy zone, middle energy zone, and a close vicinity of the equilibrium, aka low energy zone. In the high energy zone we use a control based on an asymptotic $T \to \infty$ formula for the support function of the reachable set [5]-[6] for system (1). The control can be in principle applied as well in other zones, but then its quasi-optimal properties are lost. Moreover, the control act the system like a dry friction, so that in some states, where the energy is not too high, it does not allow moving at all. More generally, the control might force moving in a vicinity of a limit set (attractor) not containing the target, i.e. the equilibrium state. In other words, there arise basins of attractors; the greater is the upper bound for controls, the larger are the basins.

To prevent getting into an attractor, we use in the middle energy zone a scaled version of the high energy control, with a reduced amplitude. This makes the basins of attractors located in a close vicinity of the target, so that the sinking into an attractor cannot happen within high and middle energy zones. The strategy allows the system reach a close vicinity of the equilibrium, where a terminal control scenario is in force.

In the third, terminal stage we consider asymptotic $T \to 0$ behavior of reachable set. We use important properties of shapes of the reachable sets: applying gauge transformations and adding linear feedback do not change shapes of the reachable sets. By using these properties we reduce problem of feedback control design for system (3) to design of feedback control for canonical system in Brunovsky form [7]. Thus at final state we apply a method of control, based on common Lyapunov functions [8]-[10].

We stress, that since our main goal is asymptotic efficiency, the detailed construction of the control within a finite distance to the equilibrium is of a secondary importance.

2. Basic control: High energy zone

A well-known geometric interpretation of the maximum principle says that the momentum (adjoint vector) $p$ at point $x$ is the inner normal to the reachable set $\mathcal{D}(T(x))$ [11].

Here the reachable set $\mathcal{D}(T)$ is the set of ends at time instant $T$ of all admissible trajectories of system (3)-[11] starting at the origin at zero time.
2.1. Asymptotic $T \to \infty$ theory of the reachable sets. We would like to use as momentums the normals to an approximate reachable set. This is possible thanks to the asymptotic theory of reachable sets for linear systems as developed in \[5\].

One of the basic results of \[5\] (see Appendix I), applicable to the system of $N$ oscillators is this: The reachable set $\mathcal{D}(T)$ equals asymptotically as $T \to \infty$ to the set $T\Omega$, where $\Omega$ is a fixed convex body. More precisely:

**Theorem 1.** \[5\] Suppose that the momentum $p$ is written in the form $p = (p_i)$, where $p_i = (\xi_i, \eta_i)$, $i = 1, \ldots, N$, $\xi_i$ is the dual variable for $x_i$, $\eta_i$ is the dual variable for $y_i$, and $z_i = (\eta_i^2 + \omega_i^{-2} \xi_i^2)^{1/2}$. Suppose, that system (3)-(4) is non-resonant, i.e., satisfies \[2\]. Then, the support function $H_T$ of the reachable set $\mathcal{D}(T)$ has as $T \to \infty$ the asymptotic form

$$H_T(p) = T \int_\mathcal{T} \left| \sum_{i=1}^N z_i \cos \varphi_i \right| \, d\varphi + o(T),$$

where $d\varphi$ is the canonical volume element $\frac{1}{(2\pi)^n} d\varphi_1 \wedge \cdots \wedge d\varphi_N$ on torus $\mathcal{T} = (\mathbb{R}/2\pi\mathbb{Z})^N$.

Recall, that the support function of any subset $M \subset \mathbb{R}^n$ is defined as $H_M(\xi) = \sup_{x \in M} \langle \xi, x \rangle$ and defines the closed convex hull of $M$ uniquely \[13\]. In particular, the support function of the convex body $\Omega$ is given by the main term in \[6\]:

$$H_\Omega(p) = H_T(z) = \int_\mathcal{T} \left| \sum_{i=1}^N z_i \cos \varphi_i \right| \, d\varphi,$$

where the vector $z = (z_1, \ldots, z_N) \in \mathbb{R}^N$ has components $z_i = (\eta_i^2 + \omega_i^{-2} \xi_i^2)^{1/2}$.

If $N = 1$ we get $H_T(z) = \frac{4}{\pi} |z|$, if $N = 2$ the function

$$\mathcal{S}_T(z) = \int (|z_1 \cos \varphi_1| + |z_2 \cos \varphi_2|) \, d\varphi$$

can be expressed via elliptic integrals (see, Appendix III)

$$\mathcal{S}_T(z_1, z_2) = \frac{1}{\pi^2} \int_0^{2\pi} \frac{(z_2^2 - z_1^2) \, d\varphi}{\sqrt{z_2^2 - z_1^2 \cos^2 \varphi}}, \quad \text{if } |z_1| \leq |z_2|.$$

In general, by substitution $t_i = \cos \varphi_i$ we reduce integral \[7\] to an Euler type integral

$$\mathcal{S}_T(z) = \frac{1}{(2\pi)^N} \int_{\{ |t_i| \leq 1 \}} \left| \sum_{i=1}^N z_i t_i \right| \prod_{i=1}^N (1 - t_i^2)^{-1/2} \, dt_1 \cdots dt_N,$$

defining a hypergeometric function in the sense of I. M. Gelfand \[12\]. Function $\mathcal{S}_T(z)$ has an integral representation via the Bessel functions (see, Appendix IV).

Note, that equation \[6\] makes sense even in the resonant case, when there are nontrivial relations between eigenfrequencies. In this case, however, equation \[6\] does not give an asymptotic formula for the support function of the reachable set $\mathcal{D}(T)$.

The basic idea of our feedback control is to substitute the set $T\Omega$ for $\mathcal{D}(T)$. The idea works even in the resonant case, when $T\Omega$ is not an asymptotic approximation of $\mathcal{D}(T)$. Note that a phase vector $x \in \mathcal{V} = \mathbb{R}^{2N}$ belongs to the boundary of $T\Omega$ if and only if

$$T^{-1} x = \frac{\partial H_\Omega}{\partial p}(p)$$

for a momentum $p = p(x)$. We notice that the support function $H_\Omega$ is differentiable, and equation \[9\] has a unique up to scaling $p \mapsto \lambda p$, $\lambda > 0$ solution, because the
boundary of Ω is smooth [6]. The unique solvability of equation (11) is also proved below in subsection 2.2. We discuss the issue of efficient solution of equation (11) in the next section. Our basic control in the high energy zone is given by

\[ u(x) = -\text{sign}(B,p(x)), \]

and it depends on the direction of the vector \( p(x) \) only, so that the scaling \( p \mapsto \lambda p \), where \( \lambda > 0 \), does not affect the control. The minus sign in (11) is due to the fact that \( p(x) \) is the outer normal to \( T\Omega \) at the point \( x \).

2.2. Efficient computation of the control. In coordinates \( x_i, y_i \), equation (7) takes the form

\[ T^{-1}(x_i, y_i) = z_i^{-1} \left( \frac{\partial \tilde{\mathcal{H}}}{\partial z_i} \left( \frac{\xi_i}{\omega_i^2}, \eta_i \right) \right), \quad i = 1, \ldots, N, \]

where \( z_i = (\eta_i^2 + \omega_i^2 \xi_i^2)^{1/2}, \) and \( \tilde{\mathcal{H}}(z) \) is given by integral (7). To solve (11), we should find first the point \( 3 \) of the sphere \( S^{N-1} \) with positive-homogeneous coordinates \( (z_1 : \cdots : z_N) \). Here, the sphere \( S^{N-1} \) is regarded as the set of direction of non-zero vectors in \( \mathbb{R}^N \). To this end we define the “energetic” vector \( e = (e_i) \in \mathbb{R}^N \), where \( e_i = (\omega_i^2 x_i^2 + y_i^2)^{1/2} \), and obtain from (11) that

\[ T^{-1}e_i = \frac{\partial \tilde{\mathcal{H}}}{\partial z_i}(3), \quad i = 1, \ldots, N. \]

Solution of equation (11) gives an inversion of a map from one 2N-dimensional manifold to another, while the solution of (12) reduces to inversion of a map of \( N-1 \) dimensional manifolds. Still the solution of (11) reduces easily to the solution of (12). Similarly to master equation (9), equation (12) has, according to [6], a unique solution, which, however, is not a very easy find. Anyway, we obtain that \( T \) is a function of the “energetic” vector.

2.3. Kuhn–Tucker theorem. For arbitrary \( N \) the search for solutions of (12) is equivalent, thanks to the Kuhn–Tucker theorem, to the optimization problem

\[ (e, z) \rightarrow \text{max}, \text{ provided that } \mathcal{H}(z) \leq 1, \]

and similar arguments can be applied to (9).

It is clear that the constraint \( \mathcal{H}(z) \leq 1 \) is equivalent to \( \mathcal{H}(z) = 1 \). The hypersurface \( \{ \mathcal{H}(z) = 1 \} \) is strictly convex, because of the obvious identity

\[ \left( \frac{\partial^2 \mathcal{H}}{\partial z^2}(z)\xi, \xi \right) = \int_{V(z)} \left( \sum_{i=1}^{N} \xi_i \cos \varphi_i \right)^2 d\sigma(\varphi), \]

where integration is over \( V(z) = \{ \varphi \in T : f(z, \varphi) = 0 \} \),

\[ f(z, \varphi) = \sum_{i=1}^{N} z_i \cos \varphi_i, \quad d\sigma(\varphi) = \frac{d\varphi_1 \wedge \cdots \wedge d\varphi_N}{(2\pi)^N} \]

is the canonical volume element on \( V(z) \). The identity implies that, if the vectors \( \xi \) and \( z \) are not collinear, then \( \frac{\partial^2 \mathcal{H}}{\partial z^2}(z)\xi, \xi \) is strictly positive. But, if the vector \( \xi \) is tangent to the hypersurface \( \{ \mathcal{H}(z) = 1 \} \) at \( z \), these two vectors cannot be collinear. Otherwise, we would obtain that \( \left( \frac{\partial \mathcal{H}}{\partial z}, z \right) = 0 \) which is impossible, since \( \left( \frac{\partial \mathcal{H}}{\partial z}, z \right) = \mathcal{H}(z) > 0 \) in view of the Euler identity. The proved strict convexity of \( \{ \mathcal{H}(z) = 1 \} \), as it is well-known, implies the uniqueness of solution of optimization problem (13). Indeed, it follows from the strict convexity of \( \{ \mathcal{H}(z) = 1 \} \) that the function \( f = \mathcal{H}^2 \) is strictly convex. At the same time, optimization problem (13) is equivalent to

\[ (e, z) \rightarrow \text{max}, \text{ provided that } f(z) \leq 1. \]
If \( z_1 \neq z_2 \) are solutions to (16), then \((e, z_1) = (e, z_2), f(z_1) = 1\). However, this implies \((e, \frac{z_1 + z_2}{2}) = (e, z_1), \text{ and } f\left(\frac{z_1 + z_2}{2}\right) < 1\) which contradicts optimality of \( z_1 \).

Thus, optimization problem (12) can be solved by well-developed efficient methods, which are still more difficult than the solution of a scalar transcendental equation, e.g., via Matlab Optimization Toolbox.

Now, we obtain from (12) the final formula for the momentum:

(16) \[
\langle \xi, \eta \rangle = \frac{z_i}{e_i} (\omega_i^2 x_1, y_i), i = 1, \ldots, N.
\]

Thus, if we know the point \( \mathbf{z} = (z_1 : \cdots : z_N) \in S^{N-1} \), then the direction of the momentum \( \rho(x) \) is defined by (16) uniquely. Control (10) depends on the direction of the momentum only. Therefore, it can be efficiently found in the form

(17) \[
u(x) = -\text{sign} \left( \sum_{i=1}^{N} e_i^{-1} z_i y_i \right).
\]

Here, the sign-function is understood as a multivalued map: \( \text{sign}(x) = \pm 1 \) if \( x \geq 0 \), and \( \text{sign}(0) \) might take any value from the interval \([-1, 1]\). We discuss the precise value of the control in the case of indefinite sign later, in section 7.2. Whatever the precise value is, the control \( u(x) \) is not a continuous function of \( x \). Therefore, to define the motion under the control we have to use solutions of ODE with a discontinuous right-hand side. This naturally requires a discussion of singular ODE which we start in Section 4. If \( N = 1 \) the control has the form of a dry friction \( u = -\text{sign} y_1 \). In what follows, we will also use a scaled control \( U u(x) = U u(x) \) with a smaller amplitude \(|U| \leq 1\).

3. Formal properties of the basic control

3.1. Polar-like coordinate system. We define a polar-like coordinate system, well suited for representation of the motion under the control \( u \). If \( N = 1 \) we get the proper polar coordinate system in a plane. To this end we take the boundary \( \omega = \partial \Omega \) of the set \( \Omega \) with support function (7) as a unit “sphere”. Every vector \( 0 \neq x \in \mathbb{R}^{2N} \) can be represented uniquely as \( x = \rho \phi \), where \( \rho = \rho(x) \) is a positive factor, and \( \phi \in \omega \). The pair \( \rho, \phi \) is the coordinate representation for \( x \), and \( \rho(\phi) = 1 \) is the equation of the “sphere” \( \omega \). It is important, that the set \( \omega \) is invariant under free (uncontrolled) motion of our system (3). This follows from the similar invariance of the support function \( H_\Omega(p) \) under evolution governed by \( \dot{p} = -A^* p \). The latter invariance is clear, because the support function depends only on variables \( z_i \), which are integrals of the motion. The invariance of \( \omega \) is equivalent to invariance of the homogeneous function \( \rho \), so that \( \langle \partial \rho/\partial x, Ax \rangle = 0 \). Therefore, under the control \( u \) the total (Lie) derivative of \( \rho \) takes the form

(18) \[
\dot{\rho} = \left\langle \frac{\partial \rho}{\partial x}, Ax + Bu \right\rangle = \left\langle \frac{\partial \rho}{\partial x}, Bu \right\rangle = -\left| \left\langle \frac{\partial \rho}{\partial x}, B \right\rangle \right|,
\]

where the last identity holds, because \( \partial \rho/\partial x \) is the outer normal to the set \( \rho \Omega \). Note, that the “radius” \( \rho \) is monotone nonincreasing. For any other admissible control we have

(19) \[
\dot{\rho} \geq -\left| \left\langle \frac{\partial \rho}{\partial x}, B \right\rangle \right|.
\]

The evolution of \( \phi \) by virtue of system (3) is described by

(20) \[
\dot{\phi} = A \phi + \frac{1}{\rho} (Bu - \phi \dot{\rho}) = A \phi + \frac{1}{\rho} \left( Bu + \phi \left| \left\langle \frac{\partial \rho}{\partial x}, B \right\rangle \right| \right).
\]
It is clear that if $\rho$ is large, then the second term in the right-hand side of (21) is $O(1/\rho)$ and affects the motion of $\phi$ over the “sphere” $\omega$ only slightly. The conclusion holds for any admissible control, not just for control (10).

We note that $p = \partial p/\partial x$ is a homogeneous function of degree 0, and, therefore, is a function of $\phi$. Geometrically, $p$ is the outer normal to the surface $\omega$ at $\phi$. It follows immediately from the Euler identity that

$$(21) \quad H_\Omega(p) = \langle p, \phi \rangle = \rho(\phi) = 1.$$ 

Thus, the function $\rho$ satisfies an eikonal-type equation which is “dual” to equation $\rho(\partial H/\partial p) = 1$ of the surface $\omega$. Here, $H$ stands for $H_\Omega$. We will use equation (21) in Section 5 for averaging the right-hand side of the identity (13) with respect to time.

3.2. Duality transform. Here, we discuss a general duality transformation related to equation (9). Toward this end we denote the function $H_\Omega$ just by $H = H(p)$, and the factor $T$ by $\rho(x)$. Then, the relation between $H$ and $\rho$ is similar to the Legendre transformation:

$$(22) \quad \langle x, p \rangle = \rho(x) H(p), \quad \rho(x) = \max_{H(p) \leq 1} \langle x, p \rangle, \quad H(p) = \max_{\rho(x) \leq 1} \langle x, p \rangle,$$

where the correspondence $x \mapsto p$ has the form

$$(23) \quad x = \rho(x) \partial H/\partial p(p), \quad p = H(p) \partial \rho/\partial x(x).$$

Here, $p$ and $x$ are the points where the maximums in (22) are attained. These relations make sense provided that $H$ and $\rho$ are norms, i.e., homogeneous of degree 1 convex bodies. These sublevels are mutually polar to each other. In other words, if $\Omega = \{\rho(x) \leq 1\}$, and $\Omega^* = \{H(p) \leq 1\}$, then $\Omega = \{x : \langle x, p \rangle \leq 1, p \in \Omega^*\}$ and vice versa. In the language of Banach spaces, the normed spaces $(\mathcal{V}, \rho)$ and $(\mathcal{V}^*, H)$ are dual to each other. The derivatives in (22) should be understood as subgradients. If the functions $H$ and $\rho$ are differentiable the equation (23) has the classical meaning. If one of the functions $H$ and $\rho$ is differentiable and strictly convex, then, the other one is also so.

We notice that besides the dual pair $H, \rho$ there is another related natural dual pair $\mathcal{H}, \mathcal{R}$, where $H(p) = \mathcal{H}(z(p)), \rho(x) = \mathcal{R}(e(x))$. Here, $z(p) = (z_1, \ldots, z_N)$ is the $N$-vector with components $z_i = (\eta_i + \omega_i^{-2} \xi_i^2)^{1/2}$, and $e(x) = (e_i) = ((\omega_i^2 x_i^2 + \eta_i^2)^{1/2})$.

By differentiating the defining relation (23) for $\rho$

$$(24) \quad x = \rho \partial H/\partial p \left( \partial \rho/\partial x \right)$$

we obtain a relation between second derivatives of the dual functions

$$(25) \quad 1 = \rho \partial^2 H/\partial p^2 \partial^2 x + \partial \rho/\partial x \otimes \partial H/\partial p,$$

which means in more detailed notation that for any constant vector $\zeta$

$$(26) \quad \zeta = \rho \partial^2 H/\partial p^2 \partial^2 x \hat{\zeta} + \left( \partial \rho/\partial x \right) \left( \partial H/\partial p \right),$$

3.3. The Hamiltonian structure. Here, we show that basic control (10) like the optimal one possesses a Hamiltonian structure, meaning that we can extend the corresponding dynamical system to a canonical one, similar to that of the maximal principle (5). This requires understanding the time-evolution of the momentum $p(x)$, involved in (10). We possess the expression $p = \partial \rho/\partial \phi$ for the momentum, where the point $\phi$ makes a controlled motion satisfying (20). It follows from the
identity $\left\langle \frac{\partial \rho}{\partial x}, A\phi \right\rangle = 0$, which expresses the invariance of the “radius” under free motion, that for any (constant) vector $\zeta$ we have

$$\left\langle \frac{\partial^2 \rho}{\partial x^2} \zeta, A\phi \right\rangle = -\left\langle \frac{\partial \rho}{\partial x}, A\zeta \right\rangle.$$ 

On the other hand, the total derivative $\left\langle \dot{\rho}, \zeta \right\rangle$ can be written in the form

$$\left\langle \frac{\partial^2 \rho}{\partial x^2} (A\phi + Bu), \zeta \right\rangle = \left\langle \frac{\partial \rho}{\partial x}, A\zeta \right\rangle - \left\langle \frac{\partial^2 \rho}{\partial x^2} \zeta, Bu \right\rangle,$$

which is equal to

$$-\left\langle \frac{\partial \rho}{\partial x}, A\zeta \right\rangle + \left\langle \frac{\partial^2 \rho}{\partial x^2} \zeta, Bu \right\rangle = -\left\langle A^*p, \zeta \right\rangle + \left\langle \frac{\partial^2 \rho}{\partial x^2} Bu, \zeta \right\rangle.$$

Therefore,

$$\dot{\rho} = -A^*p + \frac{\partial^2 \rho}{\partial x^2} Bu. \tag{27}$$

Now it is easy to check that the compound vector $(x, p)$, where $p = \frac{\partial \rho}{\partial x}$ satisfies the Hamiltonian system of a “maximum principle” different from the Pontryagin’s one

$$\dot{x} = Ax - B\text{sign}(B, p), \quad \dot{p} = -A^*p + \frac{\partial^2 \rho}{\partial x^2} B\text{sign} \left( B, \frac{\partial \rho}{\partial x} \right), \tag{28}$$

where the Hamiltonian is

$$H = \left\langle Ax, p \right\rangle + |\langle B, p \rangle| - \left| \left\langle B, \frac{\partial \rho}{\partial x} \right\rangle \right|. \tag{29}$$

We note, that $H = 0$ on admissible trajectories, because

$$|\langle B, p \rangle| = \left| \left\langle B, \frac{\partial \rho}{\partial x} \right\rangle \right| \quad \text{and} \quad \left\langle Ax, \frac{\partial \rho}{\partial x} \right\rangle = 0$$

in view of invariance of the function $\rho$ under free motion.

4. THE MOTION UNDER THE BASIC CONTROL

The control $u(x)$ is not everywhere uniquely defined, and is not a continuous function of $x$.

Nevertheless, a well known theorem of Filippov says that the Cauchy problem for the differential inclusion

$$\dot{x} = Ax - B\text{sign} \left( B, \frac{\partial \rho}{\partial x} \right), \tag{30}$$

is solvable for any initial condition $x(0)$, i.e., there exists a function $x(t)$ which is absolutely continuous, has a given value at zero, and satisfies (30) at points of differentiability $13$. This follows from the basic properties of the right-hand side $f(x)$:

- it grows linearly $|f(x)| \leq C(1 + |x|)$,
- its values are convex compacts
- it is semicontinuous as a multivalued map: if $y_n \in f(x_n)$ and $x_n \to x$, then $y \in f(x)$, where $y$ is any limit point of the sequence $y_n$.

However, the Filippov theorem does not guarantee the uniqueness of solution of the Cauchy problem. In particular, the theorem does not allow to define a motion $x \mapsto \phi_t(x)$ under control $u(x)$ in the phase space, because it is natural to expect that control defines all trajectories uniquely.
In this section we show that still the motion under the control can be defined uniquely in terms of the DiPerna–Lions theory [14]. First, a slight extension [15] of the DiPerna–Lions theory allows to define the motion under the singular Hamiltonian system (28) rigorously.

**Theorem 2.** Suppose, we are given a (singular) linear Cauchy problem in $\mathbb{R}^n$
\begin{equation}
\frac{\partial v}{\partial t} = \sum b_i(x) \frac{\partial v}{\partial x_i}, \quad u(x, 0) = u(x)
\end{equation}
such that the extended DiPerna–Lions conditions are met:
\begin{equation}
\text{div} b \in L^\infty, \quad b \in W^{1,1}_{\text{loc}} = BV_{\text{loc}}, \quad \frac{b(x)}{1 + |x|} \in L^\infty + L^1,
\end{equation}
where $BV_{\text{loc}} = W^{1,1}_{\text{loc}}$ is the Sobolev space of locally integrable functions such that their first derivatives are locally finite measures, the remaining notations are standard. Then, there exists a measurable flow $x \mapsto x(t) = \phi_t(x)$ such that if $v(x)$ is a bounded measurable function, the function $v(x, t) = v(\phi_t(x))$ is the unique renormalized solution of Cauchy problem (27).

We remind that the renormalized solution of a Cauchy problem is a weak solution $v$ of the problem such that for any smooth function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ the function $\beta(v)$ is also a weak solution. Note that for a Hamiltonian system the divergence is identically zero. Other conditions (32) can also be easily checked for singular Hamiltonian system (28).

**Corollary 1.** The Cauchy problem for the transport equation, corresponding to Hamiltonian system (28) and a bounded initial condition $v(x, p)$ possesses a unique renormalized solution $v$. The solution has the form $v(x, p, t) = v(\phi_t(x, p))$, where $\phi_t : \mathbb{R}^{4N} \rightarrow \mathbb{R}^{4N}, \ t \in \mathbb{R}$ is a uniquely defined measurable flow. Each curve $t \mapsto (x(t), p(t)) = \phi_t(x, p)$ is absolutely continuous, and satisfies the system (28).

This theorem is general and good, but it does not define any flow in the phase space $\mathbb{R}^{2N}$ of system (28), because in the extended symplectic space $\mathbb{R}^{4N}$ the phase space has measure zero.

The following general result which can be easily deduced from the DiPerna–Lions theory along lines of [15] is better in this respect, meaning that it is applicable to system (28).

**Theorem 3.** Suppose, we are given a linear Cauchy problem in $\mathbb{R}^n$
\begin{equation}
\frac{\partial v}{\partial t} = -\sum b_i(x) \frac{\partial v}{\partial x_i}, \quad u(x, 0) = u(x)
\end{equation}
such that the extended DiPerna–Lions conditions are met:
\begin{equation}
\text{div} b \in L^\infty + P, \quad b \in BV_{\text{loc}}, \quad \frac{b(x)}{1 + |x|} \in L^\infty + L^1,
\end{equation}
where $P$ is the space of finite positive measures, other notations are the same as in Theorem 2. Then, there exists a measurable semiflow $x \mapsto x(t) = \phi_t(x)$, $t \geq 0$ such that if $v(x)$ is a bounded measurable function, the function $v(x, t) = v(\phi_t(x))$ is the unique renormalized solution of Cauchy problem (27).

The crucial difference of Theorem 3 with previous Theorem 2 is that in the latter we have a time-reversible dynamics, and a flow instead of semiflow. However, the flow is genuinely discontinuous.

**Corollary 2.** The Cauchy problem for the transport equation, corresponding to system (27), and a bounded initial condition $v(x)$ possesses a unique renormalized solution $v$. The solution has the form $v(x, t) = v(\phi_t(x))$, where $\phi_t : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N}$
for $t \geq 0$ is a uniquely defined measurable semiflow. Each curve $t \mapsto x(t) = \phi_t(x)$ is absolutely continuous, and satisfies the system (37).

The next important theorem says that the semiflow in Corollary 2 is continuous, and, therefore, uniquely defined everywhere in the phase space $\mathbb{R}^{2N}$ of system (30).

**Theorem 4.** There exists a continuous semiflow $x \mapsto x(t) = \phi_t(x)$, $t \geq 0$ such that if $v(x)$ is a bounded measurable function, the function $v(x, t) = v(\phi_t(x))$ is the unique renormalized solution of the Cauchy problem for the transport equation

$$
\frac{\partial v}{\partial t} = \left( Ax - B \text{sign} \left( B, \frac{\partial \rho}{\partial x}(x) \right) \right), \quad v(x, 0) = v(x).
$$

Moreover, each curve $t \mapsto x(t)$ is absolutely continuous, and

$$
\dot{x}(t) = Ax(t) - B \text{sign} \left( B, \frac{\partial \rho}{\partial x}(x(t)) \right), \quad x(0) = x,
$$

where the latter equation is to be understood as a differential inclusion, because the right-hand side is multivalued: $\text{sign}(0) = [-1, 1]$.

The main advantage of Theorem 4 over Corollary 2 is the continuity of the semiflow in Corollary 2, which is required to establish an a priori bound for the Lipschitz constant of $\dot{\phi}$. More precisely, the singular part of the right-hand side of (36) has the form $-\alpha(x) \frac{\partial \rho}{\partial x}$, where $\alpha$ is a smooth nonnegative symmetric matrix, while $f$ is a (non-smooth) convex function. Moreover, the quadratic form $(\alpha(x) \xi, \xi) + (x, \xi)^2$ is strictly positive, and the singular part of (30) is invariant under scaling $x \mapsto \lambda x$ of the phase space. Under these circumstances it is possible to deduce differential inequalities for $(\alpha(x) \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial x})$ and $(x, \frac{\partial \rho}{\partial x})^2$, which are sufficiently powerful to establish an a priori bound for the Lipschitz constant of $v$ in any domain of the form $\{ (t, x) \in \mathbb{R}^{2N+1} : \| \phi_t(x) \| \geq c \}$.

We provide details of the proof of Theorem 4 and will not dwell on proofs of Theorems 2, 3 because the latter are rather standard.

**Proof.** To get a solution of the Cauchy problem we use an approximation of our singular problem by a smooth one. In fact, we use two approximation scales: one is controlled by parameter $n \to \infty$ such that the smooth and even convex function $m_n : \mathbb{R} \to \mathbb{R}$ is a uniform approximation of the function $x \mapsto |x|$. Then, the derivative $s_n = m_n'$ approximates the sign-function in $L_1$. Note that $xs_n(x) \geq 0$ for any $x \in \mathbb{R}$. Another scale is controlled by the parameter $\delta \downarrow 0$, and picking up a particular value of $\delta$ means that we freeze the motion under system (30) within the $\delta$-neighborhood $U_\delta = \{ \rho(x) \leq \delta \}$ of zero w.r.t distance $\rho$. In other words, we approximate ODE (30) by the nonsingular equation

$$
\dot{x} = Ax - Bs_n \left( B, \frac{\partial \rho}{\partial x} \right)
$$

in the domain $V_\delta = \{ x \in \mathbb{R}^{2N} : \rho(x) \geq \delta \}$. It is important that all neighborhoods $U_\delta$ are invariant under the phase flow of (30) for positive times, because the radius-function $\rho$ is nonincreasing along the phase trajectories because of analogue of equation (18):

$$
\dot{\rho} = -s_n \left( \frac{\partial \rho}{\partial x}, B \right) \left( \frac{\partial \rho}{\partial x}, B \right) \leq 0.
$$
We rewrite equation (41) in the gradient form by using identity (25). It implies that

\[ B s_n \left( B \frac{\partial \rho}{\partial x} \right) = \rho \alpha(x) \frac{\partial}{\partial x} m_n \left( B \frac{\partial \rho}{\partial x} \right) + x s_n \left( B \frac{\partial \rho}{\partial x} \right) \left( B \frac{\partial \rho}{\partial x} \right) \]

which can be regarded as an approximation to

\[ B \text{sign} \left( B \frac{\partial \rho}{\partial x} \right) = \rho \alpha(x) \frac{\partial}{\partial x} \left| B \frac{\partial \rho}{\partial x} \right| + x \left| B \frac{\partial \rho}{\partial x} \right|, \]

where \( \alpha(x) = \frac{\partial^2 H}{\partial \rho \partial \rho}, H = H_\Omega \). In particular, the ODE takes the following form:

\[ \dot{x} = F(x) = f(x) - g(x) \frac{\partial}{\partial x} m_n (h(x)), \] if \( x \) is in the complement \( V_3 \) of \( U_3 \),

\[ \dot{x} = 0 \] if \( x \) is in \( U_3 \).

Here, the functions

\[ f(x) = Ax - x s_n \left( B \frac{\partial \rho}{\partial x} \right) \left( B \frac{\partial \rho}{\partial x} \right), \quad g = \rho \alpha, \quad h = \left| B \frac{\partial \rho}{\partial x} \right| \]

are rather smooth: they are locally Lipschitz outside zero. Equations (41), (42) form an approximation to (40) rewritten in the form

\[ \dot{x} = F(x) = f(x) - g(x) \frac{\partial}{\partial x} |h(x)|, \]

where \( f(x) = Ax - x \left( B \frac{\partial \rho}{\partial x} \right), \) while \( g, h \) are the same as above.

It is important that the matrix \( g = \rho \alpha \) is symmetric and nonnegative. To simplify we omit in what follows the index \( n \). The corresponding transport equation takes the form

\[ \frac{\partial v}{\partial t} = f_i v_i - g_i j h_i v_i s(h) = F_i v_i, \]

where \( v_i = \frac{\partial}{\partial x} v, h_i = \frac{\partial}{\partial x} h \), we use Einstein notations for summation, \( s(h) = \text{sign} h \), and \( F_i = f_i - g_i j h_i s(h) \).

By differentiation we get the following equation for vector-function \( V \) with components \( v_k \):

\[ \frac{\partial v_k}{\partial t} = F_i v_{k,i} + f_{i,k} v_i - g_{i,j,k} h_i v_i s(h) - g_{i,j} h_{j,k} v_i s(h) - g_{i,j} h_{j,k} v_i \delta(h), \]

where \( v_{k,i} = \frac{\partial v_k}{\partial x} \) \( v_{i,k} = \frac{\partial v_i}{\partial x} \), similarly \( h_{j,k} = \frac{\partial^2 h}{\partial x_j \partial x_k} \), \( g_{i,j,k} = \frac{\partial g_{i,j}}{\partial x_k} \), and \( \delta = \delta_n \) stands for \( m_n \).

Equation (44) is again a transport equation with extra terms \( f_{i,k} v_i - g_{i,j,k} h_i v_i s(h) - g_{i,j} h_{j,k} v_i s(h) - g_{i,j} h_{j,k} v_i \delta(h) \) in the right-hand side. Fortunately, the most “dangerous” and singular term \( g_{i,j,k} = g_{i,j} h_{j,k} v_i \delta(h) \) has a positivity property:

\[ g_{k,i} v_i \sigma_k = g_{k,i} h_k v_i g_{i,j} h_j v_i \delta(h) = \left( \sum g_{k,i} h_k v_i \right)^2 \delta(h) \]

is a positive measure.

All the other terms are linear functions of \( V \) with coefficients bounded outside any neighborhood of zero. This implies in fact that \( w = (gV, V) = g_{k,i} v_i v_k \) is a kind of quadratic Lyapunov function:

\[ \frac{\partial w}{\partial t} \leq F_i w_i + LW, \]

where \( L \) is a function uniformly bounded outside any neighborhood of zero, \( W = |V|^2 = \sum v_k^2 \). Since the matrix \( g = \rho \alpha \) is not strictly positive definite, \( W \) cannot be estimated via \( w \), and the Lyapunov equation (40) is insufficient for establishing an a priori bound for \( w \), not to mention \( W \). Still, we can use the estimate

\[ W = \sum v_k^2 \leq C \left( \left( \sum x_k v_k \right)^2 + (gV, V) \right), \]
where $C$ is a positive function bounded outside any neighborhood of zero. The bound holds because the kernel of the matrix $g(x)$ is the one-dimensional subspace of the phase space, generated by $x$. In view of bound (47) we have to find an estimate for $z = ∑ x_k v_k = E v$, where $E$ is the Euler operator $E v = ∑ x_k ∂v/∂x_k$. By applying the Euler operator to equation (44) we obtain:

$$\frac{∂z}{∂t} = F_i E v_i + (EF_i) v_i = F_i z_i - F_i v_i + (EF_i) v_i.$$  (48)

Here, we use commutation relation $\frac{∂}{∂x} E = E \frac{∂}{∂x} + \frac{∂}{∂x}$ which implies that $E v_i = z_i - v_i$. It is easy to compute $EF_i$: The function $F(x) = Ax - Bs \langle B, \frac{∂ρ}{∂x} \rangle$ is manifestly the sum of the homogeneous functions $Ax$ and $-Bs \langle B, \frac{∂ρ}{∂x} \rangle$ of degrees 1 and 0. Therefore, $EF_i$ is a locally bounded function. Relation (18) now implies that

$$\frac{∂y}{∂t} \leq F_i y_i + C' W,$$

where $y = z^2$, and $C'$ is a locally bounded function. Bound (47) says that $W \leq C(y + w)$. Therefore, by summing inequalities (46) and (49) we obtain that

$$\frac{∂Y}{∂t} \leq F_i Y_i + MY,$$

where $Y = w + y$ and the function $M$ is locally bounded outside zero uniformly wrt the scale $n$. This is the crucial estimate which allows to show that the flow $x \mapsto \Phi_t(x) = Φ_{n,t}(x)$ corresponding to equation (11) is locally Lipschitz. Most important is that the corresponding Lipschitz constant does not depend on the approximation scale $n$. Therefore, by passing to the limit $n → ∞$ we conclude that there exists the Lipschitz limit of $Φ_{n,t}$, which coincides with the measurable flow $φ_t(x)$ of Theorem 4 within $V_δ$. Since $δ$ is arbitrary, this proves, in particular, that the map $x \mapsto φ_t(x)$ is continuous if $x \neq 0$ and $φ_t(x) \neq 0$. It is in fact obvious that the map $x \mapsto φ_t(x)$ is continuous in zero, because the flow $φ$ maps any neighborhood $U_δ$ of zero into itself. It remains to consider the case $x \neq 0$, $φ_t(x) = 0$. Put $τ = \inf\{t > 0 : φ_t(x) = 0\}$. It suffices to show that $φ_τ(y)$ is close to $φ_τ(x)$ if $y$ is sufficiently close to $x$. We know already that for any $ε > 0$ the point $φ_{τ-ε}(x)$ depends on $x$ continuously. On the other hand, it is obvious that the map $t \mapsto φ_t(y)$ is uniformly Lipschitz for $y$ in a vicinity of $x$. Therefore, $||φ_τ(y) - φ_τ(x)|| ≤ C|ε| + |φ_{τ-ε}(y) - φ_{τ-ε}(x)|$. Since $ε$ is arbitrary, and $|φ_{τ-ε}(y) - φ_{τ-ε}(x)|$ is arbitrarily small if $y$ is sufficiently close to $x$ the continuity is proved.

□

Remark. One can prove the I. A. Bogachevskii Theorem [16] on continuous dependence of solutions to gradient differential equations $\dot{x} = -\frac{∂f}{∂x}$ on initial conditions, where $f$ is a convex function, in a similar but simpler way. The crucial differential inequality for solution $v$ of the corresponding transport equation looks like

$$\frac{∂w}{∂t} = - \left( \frac{∂w}{∂x} \cdot \frac{∂f}{∂x} \right) - 2 \left( \frac{∂^2 f}{∂x^2} \frac{∂v}{∂x^2} - \frac{∂v}{∂x} \right) \leq - \left( \frac{∂w}{∂x} \cdot \frac{∂f}{∂x} \right),$$

where $w = |\frac{∂v}{∂x}|^2$, since $\frac{∂^2 f}{∂x^2}$ is a measure with positive-definite matrix values.

5. Asymptotic optimality of the basic control

We commence with heuristic arguments. Assume that $ρ = ρ(x)$ is large, where $ρ$ is the radius-function defined in Section 3 and that there are no resonances. By neglecting the second term in the right-hand side of (20) we get the free motion of the vector $φ$, governed by $\dot{φ} = Aφ$. It follows from the invariance of the function $ρ$ under uncontrolled motion, that the motion of $p = \frac{∂ρ}{∂x}$ with the same accuracy is
governed by the Pontryagin equation for adjoint variables: \( \dot{p} = -A^*p \). This follows from the Lipschitz property of the function \( \frac{\partial \rho}{\partial x} \) which, in turn, follows from the boundedness of \( \frac{\partial^2 \rho}{\partial x^2} \) on the “sphere” \( \rho(x) = 1 \) (Appendix II). The averaging consists of computation of
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau ||(p(t), B)|| \, dt.
\]
But this average is, according to [5], the value \( H_\Omega(p) \) of the support function, where \( p \) is an arbitrary point of the curve \( p(t) \). By virtue of the eikonal equation (21) the last expression equals 1, and, therefore, “on the average” \( \dot{\rho} = -1 \). By using the same approximation, we obtain for any admissible control that \( \dot{\rho} \geq -1 \) in view of (19). The terminal condition for the controlled motion has the form \( \rho = 0 \). Thus, within the framework of the assumed approximation control (10) is optimal.

5.1. **Asymptotic optimality.** A precise statement on the asymptotic optimality of control (10) is as follows:

**Theorem 5.** Suppose, that there are no resonances: (2) holds. Consider evolution (18) of \( \rho \) under control (10). Put
\[
M = \min\{\rho(0), \rho(T), T\}.
\]
Then, as \( M \to +\infty \) we have
\[
\frac{(\rho(0) - \rho(T))}{T} = 1 + o(1).
\]
Under any other admissible control
\[
\frac{(\rho(0) - \rho(T))}{T} \leq 1 + o(1).
\]

**Proof.** Consider first the case, where the duration of the motion \( T \), although is large, but is much less than \( \rho(T) \), meaning that \( T/\rho(T) = o(1) \). Then, the controlled motion under (20) differs from the free one in the entire time interval \([0, T]\) by quantity of order \( T/\rho(T) = o(1) \). Therefore, the right-hand side of (19) differs from the similar quantity for the free motion by \( o(1) \). But we have already found out that for the free motion the average value of the right-hand side is \(-1 + o(1)\) as \( t \to \infty \). Thus, the average value of the right-hand side of (19) under control (10) is \(-1 + o(1)\) as \( M \to +\infty \). By integrating the right-hand side we arrive at (51). The statement (52) is proved similarly.

To prove the Theorem without the assumption of smallness of \( T/\rho(T) \) we divide the entire time interval \([0, T]\) into many pieces \([T_i, T_{i+1}]\), such that \( T_{i+1} - T_i \geq M \), and \((T_{i+1} - T_i)/\rho(T) = o(1)\), and apply to each piece the already proved special case of the Theorem. We obtain
\[
\rho(T_i) - \rho(T_{i+1}) = (T_{i+1} - T_i) + o(1)(T_{i+1} - T_i).
\]
Moreover, it follows from the previous arguments that the factor \( o(1) \) in the last identity is small uniformly with respect to \( i \). Summing up identities (53) over \( i \) we arrive at (51). Similarly, one can prove statement (52).

**Remark.** In what follows we will get a strengthening (Theorem 15) of Theorem 5 where only the initial point of the controlled motion is infinitely remote. Right now this is impossible, because if \( \rho(T) \) is not large, we can get under control (10) into a standstill zone. Then, \( (\rho(T)) \) does not depend on \( T \) for \( T \) large, and (51) does not hold.
5.2. Comparison with maximum principle. One can approach the issue of optimality of control (10) from another side: by comparison of the differential equations of the motion under the control with equations (5) of the Pontryagin maximum principle. The following informal statement holds true:

The maximum principle equation for the compound vector \((x, p)\), where \(p = \frac{\partial \rho}{\partial x}\), holds “on the average” with a small error as \(x\) is large.

Indeed, we obtain from the second equation of (28)

\[
\dot{p} = -A^* p + \tilde{B} u, \quad \tilde{B} = \frac{\partial^2 \rho}{\partial x^2} B.
\]

Note, that if the latter equation would not contain the second term \(\tilde{B} u\), then the equation for \(p\) would coincide with with the maximum principle equation for adjoint variables. However, the matrix \(\frac{\partial^2 \rho}{\partial x^2} B\) is a homogeneous function of \(x\) of degree \(-1\), and, according to Appendix II it is bounded on sphere \(|x| = 1\), therefore the said second term has order \(O\left(\frac{1}{|x|}\right)\) for \(x\) large, and, therefore, is small. We remark, that the maximum condition \(u = -\text{sign}\langle B, p \rangle\) holds for control (10). It remains to find out to what extent the condition \(h(x, p) = 0\) holds. We see, that the motion under control (10) is governed by the Hamiltonian \(H\), which is very much similar to the Pontryagin Hamiltonian \(h(x, \psi)\). The difference between the Hamiltonians is \(1 - |\langle B, \partial \rho / \partial x \rangle|\). The arguments of the previous section imply that the difference is zero “on the average” in the non-resonant case. Indeed, the average value of \(|\langle B, \partial \rho / \partial x \rangle|\) is close to 1 for \(x\) sufficiently large, as it is shown in the proof of Theorem 5.

6. Efficiency of basic control at finite distance from zero

We know already that asymptotically the time of motion from the level set \(\rho = M\) to the level set \(\rho = N\) under control (10) is \((M - N)(1 + o(1))\), if \(M, N\), and \(M - N\) are very large. Now we show that a nonasymptotical estimate holds: the time of motion \(T\) is \(O(M - N)\), if \(M, N\) and \(M - N\) are greater than a constant \(C(\omega)\), depending only on parameters \(\omega = (\omega_1, \ldots, \omega_N)\) of our system of oscillators. The equation (18) might be rewritten in notations of the previous section as

\[
\dot{p} = -|\langle p, B \rangle|,
\]

and this reduces the required estimate to the inequality

\[
\int_0^T |\langle p, B \rangle| dt \geq cT,
\]

where \(c = c(\omega)\) is a strictly positive constant. To prove (56) we use the perturbation theory of completely observable time-invariant linear systems (Appendix V).

Consider equation (54), regarded as a completely observable linear system, where the phase vector \(x = p\), matrices \(\alpha = -A^*\), \(\beta = B^*\), observation \(y = B^* p = \langle p, B \rangle\), and perturbation \(f = \tilde{B} u\). Assume, that in the entire time interval \(I\) of integer length \(T\) the motion of the state vector \(x\) takes place within the domain \(\rho(x) \geq C\). Then \(|f| = O(1/C)\) in the entire interval. Moreover, eikonal equation (21) holds for \(p\), and, therefore, \(1 \ll |p|\) and \(T \ll \int_I |p| dt\) (here \(\ll\) is the Vinogradov symbol, meaning \(O(\text{RHS})\)). The estimate of Theorem A.1 from Appendix V gives that

\[
T \ll \int_I |p| dt \ll \int_I |\langle p, B \rangle| dt + \frac{1}{C} T.
\]

By taking a sufficiently large constant \(C = C(A, B)\) we obtain, that

\[
T \ll \int_I |\langle p, B \rangle| dt.
\]
This is the inequality (56) in another notation. To be clear, we restate the result:

**Theorem 6.** Suppose that the motion from the level set $\rho = M$ to the level set $\rho = N$ under control (10) goes within the domain $\rho(x) \geq C(\omega)$, in the time integer of integer length $T$, where $C(\omega)$ is a (sufficiently large) constant, depending only on eigenfrequencies. Then, $T \leq c(M - N)$, where $c = c(\omega)$ is a strictly positive constant.

We emphasize that the Theorem holds both in the resonant and the non-resonant case; we need not worry about linear relation between eigenfrequencies, only the Kalman condition $\omega_i \neq \omega_j$ is relevant. One can find out easily what happens when we apply the scaled control

(57) $u_U(x) = Uu(x), |U| \leq 1$.

**Theorem 7.** Suppose that the motion from the level set $\rho = M$ to the level set $\rho = N$ under the control (10) goes within the domain $\rho(x) \geq UC(\omega)$, in the time integer of integer length $T$, where $C(\omega)$ is a (sufficiently large) constant from (6), depending on eigenfrequencies. Then, $T \leq c_U(M - N)$, where $c_U = c(\omega)$ is the constant from (6).

This statement follows from the previous theorem after uniform scaling $x \mapsto Ux$ of the phase space.

### 7. Singular trajectories

We know that if the system under control (10) goes sufficiently far from the target, the equilibrium, then, the control is efficient, meaning that we approach the target with a positive speed. However, within a zone more close to the equilibrium, might arise $\omega$-limit sets (attractors) so that by moving along them we do not approach the target. It is clear that the control should be changed before getting into an attractor. In fact, the attractors define the exact bound for the efficiency zone of the control.

#### 7.1. Standstill zone

The simplest attractor is a singleton, i.e., a fixed point. We call the set of these points the standstill zone. There is an obvious upper bound for standstill zones for any admissible control bounded by constant $U$: this is the interval

(58) $\{A^{-1}Bu, |u| \leq U\} = \{y_i = 0, \omega_i^2 x_i = \omega_j^2 x_j, |\omega_i^2 x_i| \leq U, i, j = 1, \ldots, N\}$.

#### 7.2. The motion along an attractor

More generally, consider the motion under control (10) along an attractor. It follows immediately from (18) and (20) that it is governed by the system

(59) $\dot{\rho} = 0, \quad \dot{\phi} = A\phi + \frac{1}{\rho}Bu,$

and the constraint $\langle \frac{\partial \rho}{\partial x}, B \rangle = 0$. Taking the relation $\langle \frac{\partial \rho}{\partial x}, A\phi \rangle = 0$ from the beginning of section (5) into account, we derive immediately the expression for the control:

(60) $u = u(\phi) = -\rho \frac{\langle \frac{\partial^2 \rho}{\partial x^2}A\phi, B \rangle}{\langle \frac{\partial^2 \rho}{\partial x^2}B, B \rangle}$,

where $\frac{\partial^2 \rho}{\partial x^2}$ is the Hessian of the function $\rho$.

We conclude that the motion along attractor is governed by equation

(61) $\dot{\rho} = 0, \quad \dot{\phi} = A\phi + B f(\phi)$.
More precisely, an integral curve of the latter system is contained in the attractor, if along the curve the inequality $|f| \leq 1/\rho$ holds. Note that the nontrivial existence issue of the integral curve is already solved by Theorem 4.

Thus, we get the following description of singular arcs of control (10). Consider the dynamic system on the manifold

$$
\sigma = \left\{ \rho = 1, \left( \frac{\partial \rho}{\partial x} B \right) = 0 \right\}
$$

of dimension $2N - 2$, described by equation

$$
\dot{\phi} = A\phi + Bf(\phi).
$$

Then, if the inequality $|f| \leq 1/\rho$ holds along an $\omega$-limit set $O$ of the system, the set $\rho O$ is an attractor for the motion under (10). Conversely, any attractor of the controlled motion can be obtained in the same way from dynamical system (63). In particular, we obtain the criterion for absence of nontrivial attractors in the form of the inequality for “radius”:

**Theorem 8.** The domain $\rho \geq \mu - 1$, where $\mu$ is the minimum over all attractors of system (63) of the maximum of $|f|$ over the attractor, is attractor free, i.e., does not contain nontrivial minimal $\omega$-limit sets of system (18), (20).

The value of the minimax $\mu$ is a primary characteristic of system (63). Its importance is due to the fact that it gives the exact bound for the efficiency zone for control (10).

The next theorem follows in a formal way from definitions and Theorem 6.

**Theorem 9.** Suppose $\epsilon > 0$ and the motion under control (10) in a sufficiently long time interval $[a, b]$ of length $T$ goes within the domain $\{ \rho \geq \mu - 1 + \epsilon \}$. Then, $\rho(a) - \rho(b) \geq c(\epsilon)T$, where $c(\epsilon)$ is a positive constant. On the other hand, there are infinitely long motions within $\{ \mu^{-1} - \epsilon \leq \rho \leq \mu^{-1} \}$, where $\rho(t)$ is a constant.

In notations of Theorem 6 this means that $C(\omega) = \mu^{-1} + \epsilon$, and stresses the importance of finding a lower estimate for $\mu$.

Note, that the manifold $\sigma$ is diffeomorphic to $2N - 2$-dimensional sphere. In particular, for the case of two oscillators the problem of the value of $\mu$ reduces to the classical problem of examination of a dynamical system on two-dimensional sphere.

It is convenient to study the dynamic system “dual” to (63), describing the motion of vector $p = \frac{\partial \rho}{\partial x}(\phi)$. By defining $\tilde{B} = \frac{\partial^2 \rho}{\partial x^2} B$, we obtain, according to formula (54), that

$$
\dot{p} = -A^* p + \tilde{B} u.
$$

The matrix $\frac{\partial^2 \rho}{\partial x^2}$ in the equation can be rewritten as a function of $p$. To do this, we use relation (26) between second derivatives of dual functions $H$ and $\rho$. In particular, for $\zeta = B$ we obtain, taking identities $\left( \frac{\partial \rho}{\partial x}, B \right) = 0$ and $\rho = 1$ into account, that

$$
\tilde{B} = \left( \frac{\partial^2 H}{\partial p^2} \right)^{-1} B.
$$

Moreover, in the motion along attractor the condition

$$
\langle p, \tilde{B} \rangle = \left( \left( \frac{\partial^2 \rho}{\partial x^2} \right)^{-1} p, \tilde{B} \right) = 0
$$
is fulfilled, which means that
\[ u = (p, AB) \left( \frac{\partial^2 \rho}{\partial x^2} B, B \right)^{-1}. \]

Note that the value of \( b = \left( \frac{\partial^2 \rho}{\partial x^2} (x) B, B \right) \) has, provided that \( \rho(x) = 1 \), a uniform upper estimate:
\[ b \leq C(A^2 B), \]
where \( C(A) \) is a positive constant depending only on the matrix \( A \) of the system studied. Therefore, in order to estimate \( \mu \) from below it suffices to estimate from below the minimum \( \tilde{\mu} \) over all attractors of system (64) of the maximum of the function \( \bar{f}(p) = |(p, AB)| \) on the attractor.

7.3. A bound for the attractor free domain. According to Theorems 8 and 9 any lower bound for the constant \( \mu \) gives a lower bound for the attractor-free domain.

**Theorem 10.** The number \( \mu \), defined as the minimum over trajectories of (63) of the maximum of the function \( |f| \) on a trajectory, is strictly positive.

**Proof.** To prove this, we need, according to Theorem 8, to indicate a lower bound for the constants \( \mu \) or \( \tilde{\mu} \). One can easily approach the problem on the base of perturbation theory of observable systems (Theorem A.1). Indeed, if the maximum of the function \( \bar{f}(p) = |(p, AB)| \) on an attractor is \( \leq c \), then, in particular, the vector \( p \), solution of system (64), satisfies the equation \( \dot{p} = -A^* p + f \), where \( |f| \ll c \) in a time interval of arbitrary length. Consider the observable coordinate \( (p, B) \) which is identically zero in the manifold
\[ \sigma^- = \{ p \in \mathbb{R}^{2N} : H(p) = 1, (p, B) = 0 \}, \]
where the motion takes place. The a priori bound of Theorem A.1 as applied to an interval of unit length, shows, that
\[ 1 \leq \int |p| dt \ll c, \]
and gives required bound for \( c \).

8. The feedback near the terminal point

8.1. Asymptotic \( T \to 0 \) theory of the reachable sets. The design of the basic control in the high energy zone is based upon the knowledge of the asymptotic behavior of reachable sets \( D(T) \) as \( T \to \infty \). A natural approach to feedback control design near the equilibrium point is consideration of the asymptotic behavior of the reachable set \( D(T) \) of system (3) as \( T \to 0 \). This problem was studied in detail for linear systems in [17]. The moral of this investigation is that basics of the asymptotic behavior of the reachable set \( D(T) \) are the same for all linear systems, so that it suffices to study only a single canonical system.

Remind that the the Banach-Mazur distance \( d \) between two convex bodies \( \Omega_1, \Omega_2 \) in a finite dimensional space \( V \) is
\[ d(\Omega_1, \Omega_2) = \log(t(\Omega_1, \Omega_2) t(\Omega_2, \Omega_1)), \quad t(\Omega_1, \Omega_2) = \inf \{ t \geq 1 : t \Omega_1 \supset \Omega_2 \}. \]

The main result of [17] can be restated as follows:

**Theorem 11.** Suppose that system (3) in space \( V \) is controllable. Then, there are matrices \( \Delta(T) \) and a fixed convex body \( \Omega \subset V \) such that the asymptotic equivalence
\[ D(T) \sim \Delta(T) \Omega \]
holds. The equivalence means that the Banach-Mazur distance between the right-hand and the left-hand sides of the asymptotic equality tends to 0 as \( T \to 0 \). Moreover, the Banach-Mazur distance \( d(D(T), \Delta(T) \Omega) \) is \( O(T) \).
The idea of our approach is to devise a control by using instead of the reachable sets $D(T)$ a family of ellipsoids $E(T)$ with a similar basic property $E'(T) = \Delta(T) E$, where $E$ is a fixed (time-invariant) ellipsoid. It turns out that the quadratic function defining the crucial ellipsoid $E$ is a common Lyapunov function for two explicitly constructed linear systems.

8.2. Common Lyapunov functions. The design of our local feedback control goes back to [5]. It uses a preliminary reduction of system (3)-(4) to a canonical form by means of transformations

\[
A \mapsto A + BC, \quad u \mapsto u - Cx, \quad A \mapsto D^{-1}AD, \quad B \mapsto D^{-1}B,
\]

corresponding to adding a linear feedback control, and to coordinate changes (gauge transformations). We state the result as follows:

**Lemma 1.** By transformations (68) system (3)-(4) can be reduced to the canonical form

\[
\dot{x} = Ax + Bu,
\]

\[
A = \begin{bmatrix}
0 & 0 & & & \\
-1 & 0 & & & \\
& -2 & 0 & & \\
& & & \ddots & \\
& & & & -2N + 1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]

To do this, the matrix of the linear feedback should be chosen in the form

\[
C = (c_1 \ 0 \ c_2 \ 0 \ \ldots \ c_N \ 0), \quad c_k = (-1)^{N+1} \omega_k^{2N} \prod_{i \neq k} (\omega_i^2 - \omega_k^2)^{-1}.
\]

The gauge matrix $D$ transforms the standard basis $e_i = (\delta_{ij})$ of $\mathbb{R}^{2N}$ into the basis

\[
e_i = \frac{(-1)^{i-1}}{(i-1)!} (A + BC)^{i-1} B, \quad i = 1, \ldots, 2N,
\]

and has the following form: Define $2 \times 2$ matrices

\[
d_{ij} = (-1)^{j-1} \lambda_i^{-1} \begin{bmatrix}
0 & -\frac{i}{(2i-1)!} \\
\frac{i}{(2i-1)!} & 0
\end{bmatrix}, \quad \text{where} \ \lambda_i = \sum_{\substack{i \neq k \ \omega_k^2}}.
\]

Then,

\[
D \text{ is the } N \times N \text{ matrix } (d_{ij}) \text{ of } 2 \times 2 \text{ blocks } d_{ij}.
\]

When regarded as an existence theorem of a canonical form, without explicit formulas for matrices $C$ and $D$, Lemma 1 is a particular case of the Brunovsky theorem [7]. We have placed a proof of the lemma in Appendix VI. By following [10], introduce a matrix function of time, related to system (70):

\[
\delta(T) = \text{diag}(\delta_1^1, \delta_2^2, \ldots, \delta_{2N}^{2N})^{-1}.
\]

In what follows the parameter $T$ will be a function $T(x)$ of the phase vector. Define, in accordance with [10] [6], the matrices

\[
q = (q_{ij}), \quad q_{ij} = \int_0^1 x^{i+j-2} (1 - x)dx = [i+j](i+j-1)^{-1},
\]

\[
\Omega = q^{-1}, \quad \mathcal{E} = -\frac{1}{2} \mathfrak{B}^* \Omega, \quad \mathfrak{M} = \text{diag}(1, 2, \ldots, 2N)
\]

and the feedback control by

\[
u(x) = \mathcal{C} \delta(T(x)) x,
\]
where the function $\mathcal{I} = \mathcal{I}(y)$ is given implicitly by
\begin{equation}
(82) \quad \langle \Omega \delta(\mathcal{I})\gamma, \delta'(\mathcal{I})\gamma \rangle = \kappa^2.
\end{equation}
The value of the positive constant $\kappa$ will be fixed below. The basic result on steering the canonical system \((70)\) to zero is as follows:

**Theorem 12.**

A) The matrix $\Omega$ defines a common quadratic Lyapunov’s function for matrices \(-M\) and $A + BC$.

B) Equation \((75)\) defines $\mathcal{I} = \mathcal{I}(y)$ uniquely.

C) Control \((77)\) is bounded: $|u| \leq \frac{\kappa}{2} \sqrt{\Sigma_{11}}$.

D) Control \((77)\) brings the point $y$ to $0$ in time $\mathcal{I}(y)$.

**Proof.** The statement A amounts to the matrix inequalities
\begin{equation}
(79) \quad \{M, q\} > 0, \{A, q\} - \frac{1}{2}\{B, B^*\} < 0,
\end{equation}
where we use the “Jordan brackets” $\{\alpha, \beta\} = \alpha \beta + \beta^* \alpha^*$. Indeed, if $Q(x, y) = (Qx, y)$ is a quadratic Lyapunov function for a stable matrix $A$, this means the matrix inequality $\{Q, A^*\} < 0$, or, the same thing, $\{A, Q^{-1}\} = Q^{-1} \{Q, A^*\} Q^{-1} < 0$. Moreover, the matrix $\frac{1}{2}\{A, Q^{-1}\}$ corresponds to the negative quadratic form $Q^{-1}(x, A^* x)$. A trivial computation reveals that $\{B, q, B^*\} = -\frac{1}{2}\{B, B^*\}$. We implement the phase space $\mathbb{R}^{2N}$ as the space of polynomials $f$ of degree < $2N$ of variable $x$. The canonical basis $\xi_k$ of $\mathbb{R}^{2N}$ is represented then by monomials $m_k(x) = x^{k-1}$. Note, that the matrix $A^*$ is represented by differentiation operator $f \mapsto -\frac{\partial}{\partial x} f$, while the matrix $M^* = M$ is represented by operator $f \mapsto \frac{\partial}{\partial x} x f$. The dual vector $B^* = (1, 0, \ldots, 0)$ is represented by the functional $f \mapsto f(0)$. Consider relations \((79)\) in the functional model. The quadratic form $q(f, f)$, related to matrix $q$, takes the form $\int_0^1 f^2(x)(1-x)dx$. It is a positive form. The matrices $\{M, q\}, \{A, q\}, \{B, B^*\}$ are represented by the following quadratic forms in the functional model:
\begin{equation}
\begin{split}
\mu(f) &= q(f, M^* f) = 2 \int \left( \frac{\partial}{\partial x} x f \right) f(x)(1-x)dx, \\
\alpha(f) &= q(f, A^* f) = -2 \int \left( \frac{\partial}{\partial x} f(x) \right) f(x)(1-x)dx, \quad \beta(f) = 2 f(0)^2,
\end{split}
\end{equation}
where integration is over the interval $[0, 1]$. By partial integration we obtain
\begin{equation}
\begin{split}
\alpha(f) &= -\int \frac{\partial}{\partial x} f^2(x)(1-x)dx = -\int f^2(x)dx + f^2(0) \\
\mu(f) &= 2 \int f^2(x)(1-x)dx - \int f^2(x)((1-x)x' dx = \\
&= 2 \int f^2(x)(1-x) + \frac{1}{2}(2x-1) dx = \int f^2(x)dx.
\end{split}
\end{equation}
Therefore, $\alpha(f) - \frac{1}{2} \beta(f) = -\mu(f)$, and both sides of the latter equality coincide with the negative quadratic form $-\int f^2(x)dx$. This proves the inequalities \((79)\), and the statement A of Theorem. Moreover, we have shown that
\begin{equation}
(82) \quad -\{M, q\} = \{A, q\} + \{B, B^*\}.
\end{equation}
The latter relation is equivalent to equality of quadratic forms
\begin{equation}
(83) \quad \langle \Omega y, [A + BC] y \rangle = -\langle \Omega y, M y \rangle.
\end{equation}
We note, that above arguments can be easily generalized to the case, when the matrix $q$ is represented by a quadratic form
\[ \int_0^\infty f^2(x)q(x)dx, \]
where the nonnegative function $q$ is monotone nonincreasing ($q' \leq 0$), decreases at infinity faster than any power of $x$, and $q(0) = 1$. Indeed, the matrices 
\{\mathcal{M}, q\}, \{\mathfrak{A}, q\}, \{\mathcal{B}, \mathcal{B}^\ast\}

 correspond in the functional model to the following quadratic forms:

$$
\mu(f) = q(f, \mathcal{M}^\ast f) = 2 \int \left( \frac{\partial}{\partial x} f \right) q dx,
$$

(84)

$$
\alpha(f) = q(f, \mathfrak{A}^\ast f) = -2 \int \left( \frac{\partial}{\partial x} f \right) q dx, \quad \beta(f) = 2f(0)^2,
$$

where integration is over the ray $[0, +\infty)$. By partial integration we obtain

$$
\alpha(f) = - \int \left( \frac{\partial}{\partial x} f^2 \right) q dx = \int f^2 q' dx + f^2(0)q(0)
$$

(85)

$$
\mu(f) = -2 \int xf(f'q + f q') dx = - \int \left( \frac{\partial}{\partial x} f^2 \right) xq + 2f^2 xq' \right) dx
$$

$$
= \int f^2(q - xq') dx.
$$

Thus, inequalities (72) hold true.

The statement $\mathbf{B}$ is a corollary of the strict monotonicity of the function $\mathfrak{T} \mapsto (\Omega \delta(\mathfrak{T}) \Re, \delta(\mathfrak{T}) \Im)$ which, in turn, follows immediately from the first inequality (70).

The statement $\mathbf{C}$ follows from the Cauchy inequality. Indeed, $u = -\frac{1}{2} (\Omega \mathcal{B}, y)$, where $y = \delta(\mathfrak{T}) \Re$ and $(\Omega y, y) = \kappa^2$. Therefore

$$
|u| \leq \frac{1}{2} (\Omega y, y)^{1/2} (\Omega \mathcal{B}, \mathcal{B})^{1/2} \leq \frac{\kappa}{2} (\Omega \mathcal{B}, \mathcal{B})^{1/2} = \frac{\kappa \sqrt{\Omega_{11}}}{2}.
$$

The statement $\mathbf{D}$ follows from computation of the total derivative $\mathfrak{T}$. Put $\delta = \delta(\mathfrak{T})$, then we have

$$
\delta \mathfrak{A} \delta^{-1} = \mathfrak{T}^{-1} A, \quad \delta \mathfrak{B} = \mathfrak{T}^{-1} \mathfrak{B}, \quad \frac{d}{d\mathfrak{T}} \delta = -\mathfrak{T}^{-1} \mathfrak{M} \delta,
$$

which immediately imply for $y = \delta(\mathfrak{T}) \Im$ the equation

$$
\dot{y} = \mathfrak{T}^{-1} \left( \mathfrak{A} y + \mathfrak{B} u - \mathfrak{T} \mathfrak{M} y \right).
$$

Then, it follows from relations (77), (78) that

$$
\left( \Omega y, [\mathfrak{A} + \mathfrak{B} \mathcal{C}] y - \mathfrak{T} \mathfrak{M} y \right) = 0,
$$

but, in view of (83), this implies $\mathfrak{T} = -1$.

Note, that in a more general situation, where the matrix $q$ is related to a quadratic form $\int_0^\infty f^2(x)q(x) dx$, the statement D is valid iff $q' = -(q - xq')$. This implies easily that $q = (1 - x)_+$, so that the statement D characterizes the matrix $q$ of this kind essentially uniquely.

\begin{remark}
Suppose, that $\tau(\mathfrak{T})$ is the minimum time for steering a state $\mathfrak{T}$ of canonical system (65) to zero by using any admissible control $v$, $|v| \leq 1$. Then, $\mathfrak{T}(\tau)$ and $\mathfrak{T}(\tau)$ are comparable, meaning that $1 \leq \mathfrak{T}(\tau)/\tau(\mathfrak{T}) \leq C$, where $C$ is a constant. This follows from equation (78) and the fact, that the matrix $\delta(\mathfrak{T})^{-1}$ brings the reachable set $D(1)$ of canonical system (65) in the unit time to $D(\mathfrak{T})$: $\delta(\mathfrak{T})D(\mathfrak{T}) = D(1)$.

The Theorem 12 was obtained in [9] in a less precise form. Our proof is about ten times shorter. Moreover, the method applied allows us to indicate a large class of common quadratic Lyapunov functions for matrices $-\mathfrak{M}$ and $\mathfrak{A} + \mathfrak{B} \mathcal{C}$. Another curious result is not directly related to control problems:

**Theorem 13.** The matrix $\Omega$ is even integer: $\Omega \in 2M_{2N}Z$.

The idea of using orthogonal polynomials in the presented proof goes back at least to Hilbert [20]. A strengthening of the above result is related to computation of the matrix element $\Omega_{11}$: 
Theorem 14. The matrix element $\Omega_{11} = 2N(2N + 1)$.

We place proofs of these theorems in Appendices 13–14.

Corollary 3. Control (77) is bounded by $\frac{\kappa}{2} \sqrt{2N(2N + 1)}$.

The corollary is obvious. Numerical experiments suggest the following:

1. $\Omega_{11}$ is a divisor of all elements of the matrix $\Omega$: $\Omega \in \Omega_{11} M_{2N}(\mathbb{Z})$.

The explicit form of $\Omega = q^{-1}$ in the 4-dimensional case (note, that $\Omega_{11} = 20$) is the following:

$$
\begin{pmatrix}
1 & -9 & 21 & -14 \\
-9 & 111 & -294 & 210 \\
21 & -294 & 840 & -630 \\
-14 & 210 & -630 & 490
\end{pmatrix}
$$

This gives equation (88) in explicit form:

$$
\begin{array}{l}
\mathcal{T}^2 - 20T_1^2T_5^2 + 360T_1T_2T_5T_3^2 - (2220T_2^2 + 840T_1T_3)T_5^4 + (11760T_2T_3^2 + 560T_1T_4)T_5^3 \\
- (840T_3T_4 + 16800T_2^2)T_5^2 + 25200T_3T_4T_5 - 9800T_4^2 = 0.
\end{array}
$$

The precise bound for absolute value of control (77) is

$$
\frac{\kappa}{2} \sqrt{2N(2N + 1)} = \kappa \sqrt{5},
$$

where $\kappa$ is the constant from (78). If we want that $|u| \leq 1/2$, we put $\kappa = \frac{1}{2\sqrt{5}}$. This is the bound we use at the terminal stage of the control.

9. The control matching

In section 5 we designed a local feedback control, which works in a vicinity of zero. The switching to this control should happen at the boundary of an invariant domain with respect to the phase flow such that the local feedback control can be applied within the interior. We confine ourselves with the invariant domains of the form

$$
G_\Theta = \{ x : \mathcal{T}(x) \leq \Theta \} = \{ x : (\Omega(\Theta)x, \delta(\Theta)x) \leq 1 \}.
$$

The invariant domain $G_\Theta$ should satisfy two conditions:

A: The domain $G_\Theta$ contain the inefficiency domain $\{ \rho(x) \leq UC(\omega) \}$ of the preceding control;

B: The domain $G_\Theta$ is contained in the strip $\{ |Cx| \leq 1/2 \}$, where $C$ is the matrix (77).

The condition B allows to use at the terminal stage controls $u$ which are less than $1/2$ in absolute value. Therefore, the constant $\kappa^2$ in (77) should be equal to $\frac{1}{2N(2N + 1)}$. If we applied at the preceding stage the control (57), the condition A says that the set $UC(\omega)\Omega$ is contained in $G_\Theta$. Here, $C(\omega)$ is the estimate found in section 7.3 for the “radius” of the attractor free domain. In other words, the inequality should be fulfilled for the support functions

$$
UC(\omega)H_\Omega(D^*p) \leq (\delta(\Theta)^{-1}q\delta(\Theta)^{-1}p, p)^{1/2},
$$

where $D$ is the matrix (74). It is clear that the inequality holds, provided that $U$ is sufficiently small.

The condition B says precisely, that the value of the support function of ellipsoid $G_\Theta$ at the vector $D^{-1}C$ does not exceed $1/2$ in absolute value. In other words,

$$
\left( \delta(\Theta)^{-1}q\delta(\Theta)^{-1}D^{-1}C, D^{-1}C \right)^{1/2} \leq 1/2.
$$

Certainly, the inequality holds for sufficiently small $\Theta$. After choice of $\Theta$ we have to choose the bound $U$ for the control at the second stage in accordance with the inequality (88). Then, conditions A and B are met. The switching to the third,
terminal stage should happen upon arriving at the boundary \(\{(Q\delta(\Theta)x, \delta(\Theta)x) = 1\}\) of \(G_{\Theta}\). Here, the vector \(x\) is related to the phase vector \(x\) by \(x = Dx\), \(D\) is matrix (74).

The switching to the second stage of control, when the bound for admissible controls drops from 1 to \(U\) should happen before getting into the inefficiency zone of the initial control. Therefore, the switching should happen upon reaching the value \(C(\omega)\) of the “radius”.

10. The final asymptotic result

Now we can state the final asymptotic theorem:

**Theorem 15.** Assume that system (3)-(4) of oscillators is non-resonant. Let \(T = T(x)\) be the motion time from the initial point \(x\) to the equilibrium under our three-stage control, and let \(\tau = \tau(x)\) be the minimum time. Then, as \(\rho(x) \to +\infty\) we have asymptotic equalities

\[
(90) \quad \rho(x)/T(x) = 1 + o(1), \quad \tau(x)/T(x) = 1 + o(1).
\]

In the resonant case, we have non-asymptotic inequalities

\[
(91) \quad C(\omega) \geq \rho(x)/T(x) \geq c(\omega), \quad 1 \geq \tau(x)/T(x) \geq c(\omega)
\]

for \(\rho(x) \geq 1\), where \(C(\omega), c(\omega)\) are strictly positive constants, depending on eigenfrequencies of the system.

**Proof.** The proof consists in summing up the already proved results. Consider first the controlled motion from the value \(\rho(x)\) of the “radius” to the value \(\sqrt{\rho(x)}\). It follows from Theorem 5 that in the non-resonant case the duration under control (10) is asymptotically equivalent to \(\rho(x) - \sqrt{\rho(x)} \sim \rho(x)\) as \(\rho(x) \to +\infty\), while for any other control, including the time-optimal one, is no less asymptotically. Then we move to the boundary of the inefficiency zone. It is clear, in view of Theorem 6, that the motion time under control (11) is \(O(\sqrt{\rho(x)})\), which is negligible compared to \(\rho(x)\). The remaining two stages of the motion to zero, according to Theorems 7 and 12 take a (uniform over all initial conditions) finite time. Therefore, they are negligible and the total duration is asymptotically \(\rho(x)\), while the optimal time is asymptotically the same.

To prove inequalities (91) one could argue in the same way, by appealing to Theorem 6 instead of Theorem 5. \(\square\)

11. Toy model: \(N = 1\)

We illustrate our previous constructions in the simplest case of a single oscillator. For a further simplification we assume that it has the unit frequency, so that the control system is

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + u, \quad |u| \leq 1.
\end{align*}
\]

We divide the entire phase space \(\mathbb{R}^2\) into three domains. The “basic” one is the exterior of the disk \(B_2\) of radius 2, wherein we apply the “dry-friction” control \(u = -\text{sign}(y)\). We can, in principle, use a disk \(B_r\) of any radius \(r > 1\). A substantially different control \(u(x, y) = x + 6 \Xi^{-2}x + 3 \Xi^{-1}y\) is applied in a vicinity of zero. Here \(\Xi\) is the function of \((x, y)\), defined by equation (83), where \(\kappa^2 = 1/\Omega_{11} = 1/6\). In this case it takes the form

\[
\Xi^{-2}6y^2 - \Xi^{-3}24xy + \Xi^{-4}36x^2 = 1/6.
\]

The neighborhood \(G_{\Theta}\) of zero, where this control is used, is the interior of the ellipse \(\Theta^{-2}6y^2 - \Theta^{-3}24xy + \Theta^{-4}36x^2 = 1\), where the parameter \(\Theta = 3^{1/4}\) is found from...
The ellipse contains the disk $B_\lambda$ of radius $\lambda = 0.1378446 \ldots$ The complete description of control is as follows: in $\mathbb{R}^2 \setminus (B_2 \cup G_G)$ we apply the control $u = -\text{sign}(y)$, in $B_2 \setminus G_G$ the control $u = -U\text{sign}(y)$, where $U = \lambda/2$, finally, in $G_G$ we apply the control $u(x, y) = x + 6T^{-2}x - 3T^{-1}y$, where $T$ satisfies (93). If we would use a disk $B_r$, $r > 1$, instead of $B_2$ at the first stage, the parameter $U$ would be $\lambda/r$.

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APPENDIX I. ASYMPTOTICS OF THE SUPPORT FUNCTION $H_{D(T)}$

We present here a sketch of the proof of Theorem 1. By definition, $H_{D(T)}(p) = \sup(x(T), p)$, where sup is taken over admissible controls, and $x(T)$ is the state at time $T$ of the control system (3)–(4) such that $x(0) = 0$. In view of the Cauchy formula

$$\langle x(T), p \rangle = \int_0^T (e^{A(T-t)} Bu(t), p)dt = \int_0^T u(t)B^*e^{A^*(T-t)}pdt,$$

and after taking supremum under the integral sign and change of variables $t \mapsto T-t$ we obtain

$$H_{D(T)}(p) = \int_0^T \sup_{|u(t)| \leq 1} u(t)B^*e^{A^*(T-t)}pdt = \int_0^T |B^*e^{A^*t}p|dt.$$

In coordinates $\xi, \eta$, the latter formula takes the form

$$H_{D(T)}(p) = \int_0^T \sum_{i=1}^N \eta_i \cos \omega_i t + \omega_i^{-1} \xi_i \sin \omega_i t \bigg| dt.$$

The latter value is the integral of the function $f(\varphi) = \sum_{i=1}^N \eta_i \cos \varphi_i + \omega_i^{-1} \xi_i \sin \varphi_i$ taken over the rectilinear winding $\varphi(t) = \omega_i t$ of the torus $T = (\mathbb{R}/2\pi \mathbb{Z})^N$ with angular coordinates $\varphi_i$. Suppose that the system of oscillators is nonresonant, i.e., condition (2) is fulfilled. Then (4), the time average $\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi(t))dt$ coincides with the space average $\int_T f(\varphi)d\varphi$. In order to prove Theorem 1 we note that $\eta_i \cos \varphi_i + \omega_i^{-1} \xi_i \sin \varphi_i = z_i \cos (\varphi_i + \alpha_i)$, where $\alpha$ is a constant point of the torus. Therefore

$$\int_T f(\varphi)d\varphi = \int_T f(\varphi - \alpha)d\varphi = \int_T \sum_{i=1}^N z_i \cos \varphi_i \bigg| d\varphi.$$

Thus,

$$\lim_{T \to \infty} \frac{1}{T} H_{D(T)}(p) = \int_T \sum_{i=1}^N z_i \cos \varphi_i \bigg| d\varphi,$$

which is the statement of Theorem 1.

APPENDIX II. DIFFERENTIABILITY PROPERTIES OF FUNCTIONS $\mathcal{F}, H, \mathcal{R}$ AND $\rho$

Here we study basic analytic properties of the integral

$$(A.1) \quad \mathcal{F}(z) = \int_T \sum_{i=1}^N z_i \cos \varphi_i \bigg| d\varphi$$

as a function of $z \in \mathbb{R}^N$ and derive from this study differentiability properties of functions $H, \rho, \mathcal{R}$.

First, it is clear that $\mathcal{F}(z)$ is of class $C^1$ outside zero, and

$$(A.2) \quad \frac{\partial \mathcal{F}}{\partial z_i} = \int_T \text{sign} \left( \sum_{i=1}^N z_i \cos \varphi_i \right) \cos \varphi_i \, d\varphi,$$

because integrand in (A.2) is bounded and continuous with respect to $z$ outside the analytic hypersurface

$$(A.3) \quad V(z) = \{ \varphi \in T : f(z, \varphi) = 0 \}, \quad f(z, \varphi) = \sum_{i=1}^N z_i \cos \varphi_i$$
of \(d\varphi\)-measure zero (cf. \[19\] §3.1). As to the second derivatives, we again have the integral formula

\[
(A.4) \quad \left(\frac{\partial^2 \tilde{\gamma}}{\partial z^2}(z, \xi, \xi)\right) = \int_{V(z)} \left(\sum_{i=1}^{N} \xi_i \cos \varphi_i \right)^2 \, d\sigma(\varphi),
\]

where

\[
d\sigma(\varphi) = \frac{d\varphi_1 \wedge \cdots \wedge d\varphi_N}{(2\pi)^N df}
\]
is the canonical volume element on \(V(z)\). The problem is that the positive measure \(d\sigma(\varphi)\) is not necessarily finite: there are exceptional vectors \(z\) such that the integral \((A.4)\) is \(+\infty\) for all vectors \(\xi\) not collinear with \(z\). We proceed to determination of the exceptional locus. It is convenient to make the substitution \(t = \cos \varphi_i\) and assume without loss of generality that \(z_N \neq 0\). The measure \(d\sigma\) can be rewritten as \(f(t) dt_1 \ldots dt_{N-1}\), where

\[
(A.5) \quad f(t) = z_N^{-1}(2\pi)^{-N} \prod_{i=1}^{N-1} (1 - t_i^2)^{-1/2} \left(1 - \frac{1}{z_N} \sum_{i=1}^{N-1} z_i t_i\right)^2 \left(\sum_{i=1}^{N-1} z_i t_i\right)^{-1/2}
\]
on the polytope defined by conditions

\[
(A.6) \quad |t_i| \leq 1, i = 1, \ldots, N-1, \quad \sum_{i=1}^{N-1} z_i t_i \leq z_N.
\]

If the linear forms \(1 \pm t_i, i = 1, \ldots, N-1\) and \(1 \pm \frac{1}{z_N} \left(\sum_{i=1}^{N-1} z_i t_i\right)\) are all different, then, \(f\) is Lebesgue-integrable. The opposite happens exactly when

\[
(A.7) \quad z_i = \pm z_j, i \neq j, \text{ and } z_k = 0 \text{ for } k \neq i, j \text{ and } i, j, k = 1, \ldots, N.
\]

Then, the singularity takes the nonintegrable form \((1 \pm t_i)^{-1}\). Thus, condition \((A.7)\) determines the exceptional locus \(\text{sing}(\mathcal{R})\), where the quadratic form \(\frac{\partial^2 \tilde{\gamma}}{\partial z^2}(z) = +\infty\) on the factorspace \(\mathbb{R}^N / \mathbb{R}z\). The corresponding locus \(\text{sing}(\mathcal{R})\) for the dual function \(\mathcal{R}\) can be obtained from the set \((A.7)\) by the gradient map \(\psi(z) = \frac{\partial \mathcal{R}}{\partial z}(z)\).

More precisely, \(\text{sing}(\mathcal{R})\) is the set of points \(\rho \frac{\partial \mathcal{R}}{\partial z}(z)\), where \(z \in \text{sing}(\mathcal{S})\) and \(\rho\) is an arbitrary positive factor. Formula \((A.2)\) immediately implies that \(\psi\) maps the exceptional locus \((A.7)\) into itself. Luckily, it turns out that \(\frac{\partial^2 \mathcal{R}}{\partial z^2}(\epsilon)\) is continuous everywhere outside zero so that there is no exceptional set for the dual function. The reason is simple: at singular points \(\frac{\partial^2 \mathcal{S}}{\partial z^2}(z) = +\infty\) which means that \(\frac{\partial^2 \mathcal{R}}{\partial z^2}(\epsilon) = 0\) at the corresponding point \(\epsilon\). Indeed, this follows from the general duality relation

\[
(A.8) \quad 1 = \mathcal{R} \frac{\partial^2 \mathcal{S}}{\partial z^2} \frac{\partial^2 \mathcal{R}}{\partial \xi^2} \frac{\partial \mathcal{R}}{\partial \xi} \otimes \frac{\partial \mathcal{S}}{\partial \eta} \frac{\partial \mathcal{R}}{\partial \eta}.
\]

An important observation is this: Consider the canonical map \(\pi\) from the space of quadratic forms on \(\mathbb{R}^N / \mathbb{R}z\) of dimension \(N(N-1)/2\) to the corresponding sphere \(S^{N(N-1)/2-1}\) of rays. In other words, by the map \(\pi\) we put into correspondence a quadratic form and the all its multiples by a positive factor. Then, the map

\[
(A.9) \quad z \mapsto \frac{\partial^2 \tilde{\gamma}}{\partial z^2}(z)
\]
is continuous. Indeed, the at singular locus the nonintegrability of the measure \(d\sigma\) affects the right-hand side of identity \((A.4)\) like multiplication by an infinite positive scalar factor. This means, in particular, that the ratio \(\langle \eta, \frac{\partial^2 \mathcal{S}}{\partial z^2}(z)\rangle : \langle \zeta, \frac{\partial^2 \mathcal{S}}{\partial z^2}(z)\rangle\) is a continuous function of \(z\). Here, \(\eta, \zeta\) are continuous vector fields in \(\mathbb{R}^N\) and \(\zeta(z)\) is not collinear with \(z\).
The duality relation (A.8) allows to get a similar conclusion for the quadratic form ∂^2 H/∂z^2.

Now we turn to singularities of second derivatives of the dual pair of functions H(p) = δ(z(p)) and ρ(x) = R(e(x)). The corresponding singular locus sing(H) might include singular points of the mapping z : R^{2N} → R^N besides the preimage z^{-1}(sing(δ)). A direct computation gives that

(A.10) \[ \frac{\partial^2 H}{\partial z^2} = \frac{\partial^2 z}{\partial p} \delta_{\partial^2 z} \frac{\partial z}{\partial p} + \frac{\partial \delta}{\partial z} \frac{\partial^2 z}{\partial p^2}, \]

and each component \[ z_i = (Q_i p, p)^{1/2} \] for a nonnegative symmetric matrix \[ Q_i , \]
which implies that

(A.11) \[ \frac{\partial^2 z_i}{\partial p^2} = \frac{1}{z_i} \left( Q_i - \frac{Q_i p \otimes Q_i p}{z_i^2} \right). \]

The expression is surely singular as \[ z_i \to 0 \] but \[ \frac{\partial^2 z_i}{\partial p^2} \] is bounded (and nonnegative). The matrix \[ \frac{\partial z}{\partial p} \] is everywhere bounded. Thus, in order to find singularities of \[ \frac{\partial^2 z_i}{\partial p^2} \]
we have to find

(A.12) \[ \lim_{z_i \to 0} \frac{1}{z_i} \frac{\partial \delta}{\partial z_i}(z). \]

It is clear from (A.2) that \[ \frac{\partial \delta}{\partial z_i}(z) = 0 \] if the component \[ z_i = 0 \], because one-dimensional integral \[ \int_0^{2\pi} \cos \varphi_i d\varphi_i = 0 \]. Therefore, expression (A.12) equals \[ \frac{\partial^2 \delta}{\partial z^2} \]
and

(A.13) \[ \frac{\partial \delta_i}{\partial z_i} \frac{\partial^2 z}{\partial p^2} = \sum_{i=1}^N \frac{1}{z_i} \frac{\partial \delta_i}{\partial z_i} \left( Q_i - \frac{Q_i p \otimes Q_i p}{z_i^2} \right) \]
tends to

\[ \sum_{i=1}^N \frac{\partial \delta_i}{\partial z_i} \left( Q_i - \frac{Q_i p \otimes Q_i p}{z_i^2} \right). \]

The latter expression is a nonnegative symmetric matrix because of inequality \[ \frac{\partial^2 \delta}{\partial z^2} \geq 0 \], implied by the convexity of \[ \delta_i \], and the Cauchy inequality. The term \[ \frac{\partial^2 \delta_i}{\partial z_i} \] from (A.10) defines a strictly positive quadratic form on \[ \mathbb{R}^{2N}/\mathbb{R}p \]. Therefore, outside the preimage \[ z^{-1}(\text{sing}(\delta)) \] the symmetric matrix \[ \frac{\partial^2 \delta}{\partial z^2}(z) \] remains locally bounded and strictly positive, although it is not continuous at points \[ p \], where a component \[ z_i(p) = 0 \]. In view of duality relation

(A.14) \[ 1 = \rho \frac{\partial^2 H}{\partial p^2} \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial \rho}{\partial x} \otimes \frac{\partial H}{\partial p}, \]

we conclude that the symmetric matrix \[ \frac{\partial^2 \rho}{\partial x^2}(x) \] is bounded on the “sphere”

\[ \omega = \{ x \in \mathbb{R}^{2N} : \rho(x) = 1 \}, \]

but it is discontinuous at points \[ x \] such that a component \[ e_i(x) = 0 \].

APPENDIX III. Elliptic integrals

Here, we study our basic function (A.1) in the case \[ N = 2 \], when it belongs to the realm of elliptic functions. In this case

\[ \frac{\partial \delta_i}{\partial z_i} = \frac{1}{(2\pi)^2} \int \int \cos \varphi_1 \text{sign}(z_1 \cos \varphi_1 + z_2 \cos \varphi_2) d\varphi_1 d\varphi_2. \]
To fix ideas, consider the index \( i = 1 \), and perform the inner integration over \( \varphi_2 \). Taking positivity of \( z_2 \) into account, we have to compute the integral

\[
\frac{1}{2\pi} \int_0^{2\pi} \text{sign}(C + \cos \varphi_2) d\varphi_2 = \frac{2}{\pi} \arccos C - 1, \quad \text{where } |C| \leq 1,
\]

where \( C = k \cos \phi_1, \ k = -z_1/z_2 \). One can assume, by making transposition of indices if necessary, that \(|k| \leq 1\). Note, that the latter assumption makes a “disparity” between \( z_1 \) and \( z_2 \). From (A.15) we obtain, that if \(|k| \leq 1\)

\[
\frac{\partial \delta}{\partial z_1} = \frac{1}{\pi^2} \int_0^{2\pi} \cos \varphi_1 \arccos(k \cos \varphi_1) d\varphi_1,
\]

since \( \int_0^{2\pi} \cos \varphi_1 d\varphi_1 = 0 \). Integral (A.16) after partial integration can be rewritten in an “elliptic” form:

\[
\int_0^{2\pi} \cos \varphi \arccos(k \cos \varphi_1) d\varphi = \int_0^{2\pi} \frac{k \sin^2 \varphi}{\sqrt{1 - k^2 \cos^2 \varphi}} d\varphi.
\]

This gives for the derivative of the support function the final formula

\[
\frac{\partial \delta}{\partial z_1} = \frac{1}{\pi^2} \int_0^{2\pi} \frac{k \sin^2 \varphi}{\sqrt{1 - k^2 \cos^2 \varphi}} d\varphi, \quad \text{where } k = -z_1/z_2,
\]

valid for \(|k| \leq 1\). To compute \( \frac{\partial \delta}{\partial z_2} \) we need the inner integral

\[
\frac{1}{2\pi} \int_0^{2\pi} \cos \varphi_2 \text{sign}(C + \cos \varphi_2) d\varphi_2 = \frac{2}{\pi} \sin \arccos C, \quad \text{if } |C| \leq 1,
\]

where from we obtain

\[
\frac{\partial \delta}{\partial z_2} = \frac{1}{\pi^2} \int_0^{2\pi} \sqrt{1 - k^2 \cos^2 \varphi} d\varphi.
\]

Note, that the apparent asymmetry between integral formulas (A.18) and (A.19) is illusive: the change of variables \( z_1 \leftrightarrow z_2 \) makes the change of parameters \( k \leftrightarrow k^{-1} \). Under this change, the integrals

\[
I_1(k) = \int_0^{2\pi} \frac{k \sin^2 \varphi}{\sqrt{1 - k^2 \cos^2 \varphi}} d\varphi \quad \text{and} \quad I_2(k) = \int_0^{2\pi} \sqrt{1 - k^2 \cos^2 \varphi} d\varphi,
\]

regarded as (multivalued) meromorphic functions of \( k \), are transposed: \( I_1(k^{-1}) = I_2(k) \). Functions \( I_1 \) are integrals of meromorphic differential form \( \alpha = \frac{(1 - k^2 x^2) dx}{y} \) on the elliptic curve \( \mathcal{E} = \{y^2 = (1 - x^2)(1 - k^2 x^2)\} \), taken over some paths \( \gamma_i \), where \( \gamma_1 \) goes from \((-1,0)\) to \((1,0)\) and gets back, while \( \gamma_2 \) goes from \((-k^{-1},0)\) to \((k^{-1},0)\) and gets back. The form \( \alpha \) has a second order pole at infinity, so that it is a differential of the second kind. Key equation (17) defining control (17), has the form of equation for \( k = -z_1/z_2 \):

\[
\frac{e_2}{e_1} = \frac{I_2}{I_1}(k) = \int_{\gamma_2} \alpha / \int_{\gamma_1} \alpha.
\]

Note, that the support function itself has the form

\[
\delta(z_1, z_2) = \frac{1}{\pi^2} \int_0^{2\pi} \frac{(z_2^2 - z_1^2) d\varphi}{\sqrt{z_2^2 - z_1^2 \cos^2 \varphi}} \quad \text{if } |z_1| \leq |z_2|,
\]

and is expressed via a period of the holomorphic form \( \frac{dz}{y} \) on \( \mathcal{E} \).
APPENDIX IV. Another representation of the function \( H(z) \)

Besides definition \((7)\) there is another useful presentation \([5]\) of the hypergeometric function \( H(z) \). Namely,

\[
H(z) = \frac{1}{\pi} \int_0^\infty \left( 1 - \prod_{i=1}^N J_0(z_i \lambda) \right) \frac{d\lambda}{\lambda^2},
\]

where \( J_0(x) = \frac{1}{\pi} \int_0^\pi e^{ix \cos \phi} \, d\phi = \sum_{k=0}^\infty \frac{(-1)^k}{k!^2} \left( \frac{z \lambda}{2} \right)^k \) is the Bessel function of order zero. Indeed, for any real \( x \)

\[
|z| = \frac{1}{2\pi} \int_\mathbb{R} 1 + i \lambda x - e^{i\lambda x} \frac{d\lambda}{\lambda^2},
\]

since the right-hand side \( I(x) \) has the property \( I(\mu x) = |\mu| I(x) \) for any real \( x \).

This argument proves \((A.23)\) up to a constant factor. To determine this factor we consider the second (distributional) derivative of the right- and left-hand sides of \((A.23)\). This reduces the problem to the identity

\[
\delta(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\lambda x} \, d\lambda,
\]

or, equivalently,

\[
\phi(0) = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \phi(x) e^{i\lambda x} \, dx \, d\lambda,
\]

where \( \phi \) is a Schwartz function, which is a well known formula for the inverse Fourier transform. Therefore,

\[
\int_\mathbb{R} \sum_{k=1}^N z_k \cos \varphi_k \, d\varphi = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \frac{e^{i\lambda \sum_{k=1}^N z_k \cos \varphi_k} - i \lambda \sum_{k=1}^N z_k \cos \varphi_k - 1}{\lambda^2} \, d\lambda \, d\varphi = \frac{1}{2\pi} \int_\mathbb{R} \int_\mathbb{R} \frac{1 - \prod_{k=1}^N z_k \cos \varphi_k}{\lambda^2} \, d\lambda \, d\varphi = \frac{1}{\pi} \int_0^\infty \left( 1 - \prod_{i=1}^N J_0(z_i \lambda) \right) \frac{d\lambda}{\lambda^2}.
\]

APPENDIX V. Perturbation theory of observable linear systems

The subject of the Kalman observability theory is a linear time-invariant system \( \dot{x} = \alpha x \), which is observed, so that the vector \( y = \beta x \) is the observation result. Here, \( \alpha \) and \( \beta \) are constant matrices. The system is said to be completely observable, if the knowledge of the curve \( y(t) \) in an open time interval allows to recover \( x(t) \) uniquely. We consider a perturbed situation where the observed vector has the same structure, but the vector \( x \) satisfies the perturbed equation \( \dot{x} = \alpha x + f \). Then, it is impossible to recover \( x \) from knowledge of \( y \) precisely, but, if the perturbation \( f \) is small, we can do this with a small error.

More quantitatively, the error size is described by the following:

**Theorem A.1.** Suppose \( \dot{x} = \alpha x, \ y = \beta x \) is a completely observable time-invariant linear system. Then, for a solution \( z \) of \( \dot{z} = \alpha z + f \) in the interval \( I \) of integer length the a priori estimate

\[
\int_I |z| \, dt \leq C \left( \int_I |\beta z| \, dt + \int_I |f| \, dt \right)
\]

holds, where the constant \( C \) does not depend on the interval \( I \).

It is clear by summation over adjacent intervals of length 1 that it suffices to consider the case \( I = [0, 1] \). We present two proofs of the theorem. Both are based on the following lemma:
Lemma A.1. Under assumptions of theorem [A.1] consider the map
\[ \Phi : z \mapsto [y, f] = [\beta z, \dot{z} - \alpha z] \]
from \( W^{1,1} \otimes \mathbb{R}^n \) to \( V = L_1 \otimes \mathbb{R}^m \oplus L_1 \otimes \mathbb{R}^n \) and its image \( L \). Then, there exists a pair of linear ordinary differential operators \( P = P \left( \frac{\partial}{\partial t} \right) \) and \( Q = Q \left( \frac{\partial}{\partial t} \right) \) with constant coefficients such that \( (y, f) \in L \) iff \( Py = Qf \). Moreover, the degrees of polynomials \( P, Q \) are \( \leq n \).

This immediately implies the following

Corollary A.1. The image \( L \) of the map \( \Phi \) is closed in \( V \).

Indeed, the condition \( Py = Qf \) defines a closed subspace in the space of (pairs of vector valued) distributions.

Now it is easy to derive Theorem [A.1] from Corollary [A.1]. The map \( \Phi : W^{1,1} \otimes \mathbb{R}^n \rightarrow L \) is a continuous linear map.

Proof. By Corollary [A.1] the image \( L \) is closed in \( V \). The observability condition means that the kernel of the map \( \Phi \) is zero. Hence, one can apply the Banach inverse operator theorem and conclude that
\[ |z|_1 \leq C(|\beta z|_0 + |\dot{z} - \alpha z|_0). \]
Here, \( C \) is the norm of the inverse operator \( \Phi^{-1} \), and \( |z|_n = \sum_{k=0}^n \int_0^1 |\frac{\partial^k z}{\partial t^k}| \, dt \) is the standard Sobolev norm in \( W^{n,1}([0,1]) \). The conclusion of Theorem [A.1] is an obvious weakening of inequality (A.27).

Another proof of Theorem [A.1] is more constructive and lengthy. We present it only for the case of scalar observation (\( m = 1 \)), because this simplifies our arguments, and this is the only case needed in the main body of the paper. It starts with the following weakening of inequality (A.26).

Lemma A.2. Suppose \( \dot{x} = \alpha x, y = \beta x \), where \( \alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n, \beta : \mathbb{R}^n \rightarrow \mathbb{R} \), is a completely observable time-invariant linear system. Then, for any solution \( z \) of equation \( \dot{z} = \alpha z + f \) an a priori estimate
\[ |z|_0 \leq C(\beta z|_0 + |f|_n) \]
holds, where \( C \) is a positive constant.

In this case \( m = 1 \) the polynomial \( P \left( \frac{\partial}{\partial t} \right) \) is scalar and monomial of order \( n \). We rewrite equation \( Py = Qf \) of Lemma [A.1] in the form
\[ P \left( \frac{\partial}{\partial t} \right) y = g, \]
where \( |g|_0 \leq C|f|_n \), and this reduces the proof of Lemma [A.2] to a priori estimate (A.28).

\[ |y|_n \leq C(\beta y|_0 + |g|_0), \]
resembling basic estimates in the \( L_p \)-theory of elliptic equations, for any solution of (A.28). We prove the estimate by induction with respect to the order \( n \) of the operator \( P \). For \( n = 1 \) we have a scalar equation \( \frac{\partial y}{\partial t} + ay = g \), with constant \( a \), which implies \( \frac{\partial y}{\partial t} \ll |y|_0 + |g|_0 \). The latter inequality is equivalent to (A.29) for \( n = 1 \).

To perform the induction step we write \( P \) in the form \( \prod_{k=1}^n \left( \frac{\partial}{\partial t} + a_k \right) \), and define \( v = (\frac{\partial}{\partial t} + a_1) y \). By induction we obtain that \( |v|_{n-1} \ll |v|_0 + |g|_0 \). This immediately implies that
\[ |y|_n \ll |y|_1 + |g|_0. \]
In order to arrive at (A.29) we invoke the Kolmogorov-type inequality
\[
\int_0^1 \left| \frac{\partial y}{\partial t} \right| \, dt \ll \left( \int_0^1 \left| \frac{\partial^n y}{\partial t^n} \right| \, dt \right)^{1/n} \left( \int_0^1 |y| \, dt \right)^{(n-1)/n},
\]
proven in [22], which implies inequality of the form
\[
\int_0^1 \left| \frac{\partial y}{\partial t} \right| \, dt \leq \epsilon \int_0^1 \left| \frac{\partial^n y}{\partial t^n} \right| \, dt + \frac{C}{\epsilon} \left( \int_0^1 |y| \, dt \right),
\]
where \( C \) is a fixed constant, and \( \epsilon > 0 \) is arbitrary. This last inequality follows also from compactness of the Sobolev spaces imbedding \( W^{n,1} \hookrightarrow W^{1,1} \) for \( n > 1 \). Thus, \( |y| \leq \epsilon |y|_n + \frac{C}{\epsilon} |y|_0 \), and we obtain (A.29) from (A.30) provided that the chosen \( \epsilon \) is sufficiently small. This proves the induction step and Lemma A.2.

In order to get from the estimate of Lemma A.2 to (A.26) we denote by \( z_g \), where \( g \in L_1 \), the solution \( z_g = \alpha z_g + g \) with the initial condition \( z_g(0) = z(0) \). It is clear, e.g., from the Cauchy formula, that \( |z - z_g(0) \ll |f - g|_0 \), and, therefore, \( |\beta z - \beta z_g|_0 \ll |f - g|_0 \). Thus, the following strengthening of the Lemma A.2 holds:

**Lemma A.3.** Suppose that \( \dot{x} = \alpha x, y = \beta x \) is a completely observable time-invariant linear system. Then, for a solution \( z \) of \( \dot{z} = \alpha z + f \) in the interval \( [0,1] \) the a priori estimate \( |z|_0 \ll |\beta z|_0 + \inf \{|f - g|_0 + |g|_n\} \) take place, where inf is taken over \( g \in W^{n,1} \), and \( n \) is the dimension of the phase space.

Since \( \inf \{|f - g|_0 + |g|_n\} \leq |f|_0 \) we arrive at estimate (A.29).

Now it remains to prove Lemma A.1.

**Proof.** Note that the change of parameters
\[
\alpha \mapsto \alpha + \gamma \beta, \quad \beta \mapsto \beta, \quad \alpha \mapsto \delta \alpha \delta^{-1}, \quad \beta \mapsto \beta \delta,
\]
where \( \gamma \) is an arbitrary matrix, and \( \delta \) is an arbitrary invertible matrix does not affect the validity of Lemma A.1 because it corresponds to substitutions \( f \mapsto f + \gamma y \), and \( z \mapsto \delta z \). In view of the Brunovsky normal form (see [7] and Lemma I of Section 5), and the Kalman duality between controllability and observability we may assume that the observable system takes the form of direct sum of systems with a scalar observation of the form
\[
\begin{align*}
\dot{z}_1 &= z_2 + f_1 \\
\vdots & \\
\dot{z}_{n-1} &= z_n + f_{n-1}, \quad \text{and } y = z_1. \\
\dot{z}_n &= f_n.
\end{align*}
\]
(A.32)

This allows to prove by induction that the defining relation of Lemma A.1 holds in the form of direct sum of equations of the form
\[
\frac{\partial^n}{\partial t^n} y = \sum_{k=0}^{n} \frac{\partial^k}{\partial t^k} f_{n-k}.
\]

**APPENDIX VI. A PROOF OF LEMMA 11**

**Proof.** We begin with identity [21]. The feedback matrix \( C \) is to be found from nilpotency of matrix \( A + BC \). In other words, we need that the characteristic polynomial \( P(s) = \det(s - (A + BC)) \) is equal to \( s^{2N} \). We rewrite \( P(s) \) in the form
\[
\det \left( (s - A)(1 - (s - A)^{-1}BC) \right)
\]
and use the general property of determinants [21]:
\[
\det(1_n - \alpha \beta) = \det(1_n - \beta \alpha)
\]
(A.33)
for any pair $\alpha, \beta$ of matrices of size $n \times m, m \times n$. By applying \( (A.34) \) to the pair
\[
\alpha = (s - A)^{-1}B, \quad \beta = C
\]
we obtain that
\[
(A.34) \quad P(s) = \det(s - A) - CF(s, A)B,
\]
where $F(s, A) = [\det(s - A)](s - A)^{-1}$. Note, that elements of $F(s, A)$ are polynomials in $s$ of degree $< 2N$, for they are cofactors to some elements of the matrix $(s - A)$, and $CF(s, A)B$ is a scalar polynomial with the same bound for degree. If the matrix $C$ is given by \( (A.35) \), then $CF(s, A)B$ has the form $\sum c_k \prod_{i \neq k} (s^2 + \omega_i^2)$, and $\det(s - A) = \prod_{i=1}^{N} (s^2 + \omega_i^2)$. Therefore, the equation $P(s) = s^{2N}$ is equivalent to identity
\[
(A.35) \quad \prod_{i=1}^{N} (s^2 + \omega_i^2) - s^{2N} = \sum c_k \prod_{i \neq k} (s^2 + \omega_i^2).
\]
This is the Lagrange interpolation formula for the polynomial $f(\lambda) = \prod_{i=1}^{N} (\omega_i^2 + \lambda) - \lambda^N$ of degree $N - 1$ with nodes $\lambda = -\omega_i^2$, $i = 1, \ldots, N$, which implies \( (A.36) \).

We prove statements \( (E.10) \) and \( (E.2) \) simultaneously. We know already that the matrix $\tilde{A} = A + BC$ is nilpotent: $\tilde{A}^{2N} = 0$. Define a new basis by formula \( (E.2) \):
\[
\xi_i = \frac{(-1)^{i-1}}{(i-1)!} A^{-1}B \quad \text{for} \quad i = 1, \ldots, 2N.
\]
The fact that the vectors $\xi_i$ form a basis follows from the complete controllability of system \( (E.3)-(E.4) \). It is clear that $\xi_1 = B$, and $A\xi_i = -it_{i+1}$ for $i < 2N$. If $i = 2N$ it follows from the nilpotency of $\tilde{A}$ that $A\xi_{2N} = \frac{(-1)^{2N-1}}{(2N-1)!} \tilde{A}^{2N} B = 0$. This shows that the matrix $\tilde{A}$ has canonical form $\tilde{A}(0)$ in basis \( (E.2) \).

We show now that the matrix $D$ can be represented as block-matrix \( (E.3) \). The vectors $\xi_i$ are, by definition, the columns of $D$. Denote by $\lambda$ and $\omega$ the diagonal matrices
\[
\lambda = \text{diag}(\lambda_1, \lambda_1, \ldots, \lambda_N, \lambda_N), \quad \omega^2 = \text{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_N^2, \omega_N^2),
\]
where the scalar $\lambda_k$ is defined in \( (E.3) \). It is obvious that $CB = 0$. Denote $\tilde{A}B = AB$ by $B'$. It is clear that $\xi_1 = B$, and $\xi_2 = -B'$. We compute $CB' = \sum_{i=1}^{N} c_i \xi_i$, where $c_i$ is defined in \( (E.11) \). We show that $\sum_{i=1}^{N} c_i = \sum_{i=1}^{N} \omega_i^2$. To do this, we divide both sides of \( (A.35) \) by $s^{2N-2}$, and pass to the limit $s \to \infty$. We get $\sum_{i=1}^{N} c_i$ in the right-hand side, and $\sum_{i=1}^{N} \omega_i^2$ in the left-hand side. Now we can compute $\tilde{A}B' = A^2 B + BCB' = -\omega^2 B + \left( \sum_{i=1}^{N} \omega_i^2 \right)^2 B = \lambda B$ and $\tilde{A}B' = \tilde{A}\lambda B = \lambda\tilde{A}B = B'$. Therefore, we conclude by induction that
\[
\varepsilon_{2k-1} = \frac{(-1)^{k-1}}{(2k-1)!} \lambda^{k-1} B \quad \text{and} \quad \varepsilon_{2k} = \frac{(-1)^{k-1}}{(2k)!} \lambda^{k-1} B',
\]
which is equivalent to representation $D$ as block-matrix \( (E.3) \). \( \square \)

**APPENDIX VII. A PROOF OF THEOREM 13**

**Proof.** This follows from consideration of orthogonal polynomials (shifted Jacobi polynomials) with respect to the measure $d\mu = (1-x)dx$ in the interval $[0, 1]$. The required polynomials $P_n$ are given by the Rodrigues formula
\[
(A.36) \quad P_n(x) = \frac{1}{n!(1-x)^n} \frac{d^n}{dx^n} \left[ (1-x)(x-x^2)^n \right],
\]
The latter formula obviously implies that $Q$ following reason. It follows from Rodrigues formula (A.36) that
definition
where the partial integration, and identity $\partial^n x^m = 0$ are used. Therefore, the polynomials $P_n$ and $P_m$ are orthogonal if $n \neq m$. One can easily compute the higher order coefficient $c_n$ of $P_n$. It is the same as the higher order coefficient of the polynomial $\pi_n(x) = \frac{(-1)^n}{n!(n+1)!} \partial^n [x^{2n+1}]$, which obviously equals $\frac{(-1)^n(2n+1)!}{n(n+1)!}$. The square norm of the polynomial $P_n$ is
\[
\int P_n^2 d\mu = c_n \int P_n(x)x^n(1-x)dx = \frac{(-1)^n(2n+1)!}{n!(n+1)!} \frac{(-1)^n}{n}(1-x)^{n+1} \int P_n^2 d\mu = \frac{(2n+1)!}{n!(n+1)!} B(n+2, n+1),
\]
where $B(\alpha, \beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$ is the Euler $B$-function. Finally we have
\[
\int P_n^2 d\mu = \frac{(2n+1)!}{n!(n+1)!} \frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(n+3)} = \frac{(2n+1)!}{n!(n+1)!} \frac{(n+1)!n!}{(2n+2)!} = \frac{1}{2(n+1)}. \tag{A.37}
\]
It follows immediately from Rodrigues formula (A.36) that $P_n \in \mathbb{Z}[x]$, because operator $\frac{1}{n!} \partial^n$ maps $\mathbb{Z}[x]$ into itself. This fact can be rewritten in the form $P_{n-1} = \sum a_{ij}m_j$, where $m_j = x^{j-1}$ elements of the standard monomial basis, and $A = (a_{ij})$ is an integer (triangular) matrix of coefficients of the Jacobi polynomials. The above formulas for the scalar product can be rewritten in the form $A \Omega A^* = \text{diag}(\Omega)$, or, what is the same thing,
\[
\Omega = A^* \text{diag}(2k) A.
\]
The latter formula obviously imply that $\Omega$ is an even integer matrix. \hfill \Box

**APPENDIX VIII. A PROOF OF THEOREM 14**

**Proof.** We obtain from (A.37) that $\Omega_{11} = \sum_{k=1}^{2N} 2k\alpha_k^2$, where $\alpha_k = P_{k-1}(0)$ is the free term of the Jacobi polynomial of degree $k-1$. This term is always 1 by the following reason. It follows from Rodrigues formula (A.36) that
\[
P_n(0) = \frac{1}{n!} \partial^n [(x-x)^n] |_{x=0}.
\]
But $\partial^n [(x-x)^n] |_{x=0} = (1-x)^n \partial^n [x^n] |_{x=0} = n!$. Therefore, $\alpha_k = 1$ for all $k$, and $\Omega_{11} = \sum_{k=1}^{2N} 2k = 2N(2N+1)$. \hfill \Box

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