Ternary Codes and $\mathbb{Z}_3$-Orbifold Constructions of Conformal Field Theories*

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Abstract

We describe a pair of constructions of Eisenstein lattices from ternary codes, and a corresponding pair of constructions of conformal field theories from lattices which turn out to have a string theoretic interpretation. These are found to interconnect in a similar way to results for binary codes, which led to a generalisation of the triality structure relevant in the construction of the Monster module. We therefore make some comments regarding a series of constructions of $V^\natural$. In addition, we present a complete construction of the Niemeier lattices from ternary codes, which in view of the above analogies should prove to be of great importance in the problem of the classification of self-dual $c = 24$ conformal field theories. Other progress towards this problem is summarised, and some comments arise from this discussion regarding the uniqueness of the Monster conformal field theory.

1 Introduction

We will begin in section 2 by defining what shall be meant by a conformal field theory (cft). Essentially, the definition is that of the “vertex algebras” of Borcherds [1], who was inspired in part to write it down as an axiomatisation of the structure of the Monster module $V^\natural$ introduced by Frenkel, Lepowsky and Meurman (FLM) [10, 8, 9].

We shall then proceed to briefly describe the construction of the self-dual cft’s $\mathcal{H}(\Lambda)$ from an even self-dual lattice $\Lambda$. A similar construction of an even self-dual lattice $\Lambda_C$ from a doubly-even self-dual linear binary code $C$ will be described, which gives rise to various analogies between the three structures. These analogies then inspire us to write down a

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second construction of a self-dual cft $\tilde{H}(\Lambda)$ from $\Lambda$ from a second construction $\tilde{\Lambda}_C$ of an even self-dual lattice from $C$. This turns out to be what physicists would call a $\mathbb{Z}_2$-orbifold.

We shall explicitly describe the vertex operators involved, which is one of the main strengths of our work.

Initially, the connection with codes was regarded as just a useful analogy in that, as described above, one can use this “dictionary” of properties to translate statements in simple systems into conjectures in cft. However, we noticed that $\tilde{H}(\Lambda_C) \equiv \mathcal{H}(\tilde{\Lambda}_C)$, which shows a deep structure played by the code, and this leads to some understanding of FLM’s triality involution on $V^\natural$, which will be briefly discussed.

Section 3 should be regarded in some sense as an aside from the main stream of this work, as no direct application to cft’s and/or the Monster $\mathbb{M}$ has yet been found. In this section, we construct all of the Niemeier lattices by a pair of constructions from a self-dual ternary code of length 24, whereas the binary constructions provide only 12 of the 24 Niemeier lattices. Continuing our previous analogies, this would suggest that all cft’s of central charge 24 may be constructed from lattices by orbifolding. This has yet to be demonstrated. Though not directly relevant to $V^\natural$, there are some points of interest to the Monstrous community!

Inspired by the above, in section 3 $\mathbb{Z}_3$-orbifolds of the theories $\mathcal{H}(\Lambda)$ for $\Lambda$ Niemeier are investigated. The natural structure from which to begin, analogous to a ternary code, is a (complex) lattice over the ring of Eisenstein integers $\mathcal{E}$, which admits a natural third order no-fixed-point automorphism (NFPA). This is inherited by the cft $\mathcal{H}(\Lambda_R)$, where $\Lambda_R$ is the equivalent (real) $\mathbb{Z}$-lattice, and the orbifold we take is with respect to this automorphism. We find a similar picture to that in the binary case, though the correct cft analogy to the results of section 3 remains to be found.

We construct the vertex operators explicitly, using techniques valid for twists of orders 5, 7 and 13 also. This, in particular, gives us more constructions of $V^\natural$. This may help to further illuminate its unique structure.

Finally, in section 3, we summarise recent results on the classification of self-dual cft’s of central charge 24. This will be related to $V^\natural$ in the sense that all such cft’s should be contained in $V^\natural \otimes V_{1,1}$ (where $V_{1,1}$ is the two-dimensional Lorentzian theory), in an analogous way to the construction of the Niemeier lattices from $II_{25,1}$. Some comments regarding the uniqueness of the Monster module arise as a consequence.

## 2 $\mathbb{Z}_2$-Orbifolds and Binary Codes

### 2.1 Conformal field theories

We define a cft to consist of a Hilbert space $\mathcal{H}$ and a set $\mathcal{V} = \{V(\psi, z) : \psi \in \mathcal{H}, z \in \mathbb{C}\}$ of vertex operators, $V(\psi, z) : \mathcal{H} \to \mathcal{H}$ (we shall ignore any pretense at using formal variables) and a pair of states $|0\rangle, \psi_L \in \mathcal{H}$ such that

- $V(\psi, z)V(\phi, w) = V(\phi, w)V(\psi, z)$, the locality axiom. Note that the left hand side is strictly defined only for $|z| > |w|$, and similarly for the right hand side, and so we are to interpret this equality in the sense that on taking matrix elements of either side the resulting meromorphic functions are analytic continuations of one another.
This axiom is physically reasonable in that we are to interpret the vertex operator \( V(\psi, z) \) as representing the insertion of the state \( \psi \) on to the world sheet of a string at the point \( z \). For a bosonic string theory, the order of such operator insertions must clearly be irrelevant.

- \( V(\psi_L, z) = \sum_n L_n z^{-n-2} \), with
  \[
  [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n},
  \]
  for some scalar \( c \), known as the central charge.
- \( V(\psi, z)|0\rangle = e^{zL^{-1}}\psi \), the creation axiom.

The remaining axioms are technicalities, listed here only for the sake of completeness.

- \( x^{L_0} \) acts locally with respect to \( V \), i.e. \( x^{L_0}V(\psi, z)x^{-L_0} \) is local with respect to all the vertex operators in \( V \).
- The spectrum of \( L_0 \) is bounded below.
- \( |0\rangle \) is the only state annihilated by \( L_0, L_{\pm 1} \), i.e. the only su(1,1) invariant state in the theory.
- \( V\left(e^{\ast L_1}z^{\ast -2L_0}\psi, 1/z^\ast\right)^\dagger \) is local with respect to \( V \), i.e. the theory has a hermitian structure
  \[
  V\left(e^{\ast L_1}z^{\ast -2L_0}\psi, 1/z^\ast\right)^\dagger = V(\overline{\psi}, z)
  \]
  for some antilinear map \( \mathcal{H} \to \mathcal{H} \).

Physicists would call this a chiral bosonic meromorphic hermitian conformal field theory. Essentially, it is what Borcherds would refer to as a vertex algebra.

### 2.2 Construction of \( \mathcal{H}(\Lambda) \)

This brief description of a construction of the cft \( \mathcal{H}(\Lambda) \) from an even lattice \( \Lambda \), and also later its \( \mathbb{Z}_2 \)-orbifold \( \tilde{\mathcal{H}}(\Lambda) \), will give the essential flavour of the third order construction to be discussed later.

In this case\(^4\), the Hilbert space \( \mathcal{H} \) is taken to be the Fock space built up by the action of creation operators on momentum states \( |\lambda\rangle, \lambda \in \Lambda \) an even lattice of dimension \( d \). These satisfy

\[
\begin{align*}
[a^i_m, a^j_n] &= m\delta_{m,-n}\delta^{ij} \quad 1 \leq i, j \leq d \quad m, n \in \mathbb{Z} \\
[a^i_m |\lambda\rangle &= 0 \quad \forall \quad m > 0 \quad 1 \leq i \leq d \quad \lambda \in \Lambda \\
(p^i \equiv a^i_0 |\lambda\rangle &= \lambda^i |\lambda\rangle \quad 1 \leq i \leq d \quad \lambda \in \Lambda \quad (3) \\
a^i_n &= a^i_{-n} \\
e^{i\lambda q} |\mu\rangle &= |\lambda + \mu\rangle
\end{align*}
\]

\( (4) \)
and the vertex operator corresponding to the state

$$\psi = \prod_{a=1}^{M} a_{n_a}^i |\lambda\rangle$$

is

$$V(\psi, z) =: \left( \prod_{a=1}^{M} \frac{i}{(n_a - 1)!} \frac{d^{n_a}}{dz^{n_a}} X^i(z) \right) e^{i\lambda \cdot X(z)} :\sigma_\lambda,$$

where the normal ordering $:\cdots:\$ indicates that creation operators are written to the left of annihilation operators,

$$X^i(z) = q^i - ip^i \ln z + i \sum_{n \neq 0} a_n^i z^{-n}$$

is the string field (i.e. the coordinates of the string – this theory turns out to correspond to a bosonic string moving on the torus $\mathbb{R}^d/\Lambda$) and

$$\tilde{\sigma}_\lambda = e^{i\lambda \cdot q} \sigma_\lambda,$$

satisfying

$$\tilde{\sigma}_\lambda \tilde{\sigma}_\mu = (-1)^{\lambda \cdot \mu} \tilde{\sigma}_\mu \tilde{\sigma}_\lambda,$$

are cocycle operators which ensure that locality is satisfied.

For a cft $\mathcal{H}$, let

$$\chi_{\mathcal{H}}(\tau) = \text{tr}_{\mathcal{H}} q^{L_0 - c/24}; \quad q = e^{2\pi i \tau}.$$ (10)

This is the partition function, and is invariant under $T: \tau \rightarrow \tau + 1$ up to a phase of $e^{\pi i \tau}$, since we can show from our axioms that the spectrum of $L_0$ is integral.

We find that for $\Lambda$ self-dual $\chi_{\mathcal{H}(\Lambda)}(\tau)$ is also invariant under $S: \tau \rightarrow -1/\tau$, and so (for $c \in 24\mathbb{Z}$) under $\Gamma = \text{PSL}(2, \mathbb{Z})$, the modular group. [This is actually a physical requirement for the theory to be well-defined on the torus parameterised by $\tau$ in the usual sense.]

So, we define a cft $\mathcal{H}$ to be self-dual if $\chi_{\mathcal{H}}(\tau)$ is $S$-invariant.

This construction $\Lambda \mapsto \mathcal{H}(\Lambda)$ provides us with a “dictionary”, as mentioned in the introduction, between properties of lattices and cft’s [12].

### 2.3 Construction of $\Lambda_C$

We also have a similar dictionary between binary codes and lattices, provided by the construction of a lattice $\Lambda_C$ from a binary code $\mathcal{C}$ by

$$\Lambda_C = \frac{\mathcal{C}}{\sqrt{2}} + \sqrt{2} \mathbb{Z}^d,$$ (11)

e.g. $\Lambda_C$ is even for $\mathcal{C}$ doubly-even.
2.4 Construction of $\tilde{\Lambda}_C$ and $\tilde{\mathcal{H}}(\Lambda)$

A second, or “twisted”, construction

$$\tilde{\Lambda}_C = \Lambda_0(C) \cup \Lambda_3(C),$$

(12)

where

$$\begin{align*}
\Lambda_0(C) &= \frac{C}{\sqrt{2}} + \sqrt{2z^d_+} \\
\Lambda_1(C) &= \frac{C}{\sqrt{2}} + \sqrt{2z^d_-} \\
\Lambda_2(C) &= \frac{C}{\sqrt{2}} + \frac{1}{2\sqrt{2}} 1 + \sqrt{2z^d_{(\ldots)}_{n+1}} \\
\Lambda_3(C) &= \frac{C}{\sqrt{2}} + \frac{1}{2\sqrt{2}} 1 + \sqrt{2z^d_{(\ldots)}_n},
\end{align*}$$

(13)

of a lattice from a binary code for $d = 8n$ ($\mathbb{Z}^d_+ = \{x \in \mathbb{Z}^d : x^2 = 0 \mod 2\}, \mathbb{Z}^d_- = \{x \in \mathbb{Z}^d : x^2 = 1 \mod 2\}, \mathbb{1} = (1,1,\ldots,1)$) inspires a second (twisted) construction of a cft from a lattice (in $8n$ dimensions). This turns out to correspond to a string moving on the orbifold $(\mathbb{R}^d/\Lambda)/\mathbb{Z}_2$, and so we refer to it as an orbifold construction[4].

The basic idea is that we have a $\mathbb{Z}_2$ symmetry of the theory $\mathcal{H}(\Lambda)$, induced by the reflection symmetry of $\Lambda$, and we project out by this and add in a module for the resulting theory to restore self-duality. It turns out in fact that there are only two inequivalent such irreducible modules (a result still to be proven in general, but verified explicitly in a particular case by Dong [6, 7]). One clearly gives us back $\mathcal{H}(\Lambda)$, while the other gives us what we call $\tilde{\mathcal{H}}(\Lambda)$, the orbifold theory.

Explicitly, the vertex operators are

$$V \left( \left( \begin{array}{c} \psi \\ \chi \end{array} \right), z \right) = \left( \begin{array}{cc} V(\psi, z) & W(\chi, z) \\ W(\chi, z) & V_T(\psi, z) \end{array} \right)$$

(14)

acting on $H(\Lambda)_+ \oplus T$, where $H(\Lambda)_+$ is the projection of $H(\Lambda)$ under the $\mathbb{Z}_2$ symmetry and $T$ is an irreducible module for $H(\Lambda)_+$. The operators $V(\psi, z)$ are those of $H(\Lambda)$ restricted to $H(\Lambda)_+$, $V_T(\psi, z)$ form a representation of $H(\Lambda)_+$ acting on $T$, while the operators $W(\chi, z)$ and $\overline{W}(\chi, z)$ intertwine the two sectors.

In order that the resulting theory have a hermitian structure, as in (2), $\overline{W}$ is given in terms of $W$ by hermitian conjugation. Thus, we only need to define $W$ and $V_T$ to complete the definition of the theory.

Diagrammatically, it is easier to see the situation, as in figure [1].

$T$ is built up from a ground state (of dimension $2^{d/2}$) by the action of creation and annihilation operators

$$[c^i_r, c^j_s] = r \delta_{r-s} \delta^{ij} \quad r, s \in \mathbb{Z} + \frac{1}{2} \quad 1 \leq i, j \leq d$$

$$c^i_r \chi = 0 \quad \forall \quad r > 0 \quad 1 \leq i \leq d \quad \chi \text{ in ground state}$$

(15)

$$c^i_r = c^i_{-r}.$$  

(16)
The vertex operators $V_T$, forming a representation of $\mathcal{H}(\Lambda)_+$, can be written down by analogy with the operators $V$.

Define

$$C^i(z) = i \sum_r \frac{c^i_r}{r} z^{-r}, \quad (17)$$

by analogy with $X(z)$. Then, we guess

$$V_T^G(\psi, z) = \left( \prod_{\alpha=1}^M \frac{i}{(n_\alpha - 1)!} \frac{d^{n_\alpha}}{dz^{n_\alpha}} C^{\alpha}(z) \right) e^{i\lambda C(z)} : \gamma_\lambda : \gamma_{\lambda}, \quad (18)$$

where

$$\gamma_\lambda \gamma_\mu = (-1)^{\lambda_\mu + \gamma_\mu + \gamma_\lambda}. \quad (19)$$

(The ground state forms an irreducible representation of this algebra, introduced since we no longer have any momentum states on which to represent the cocycles.)

However, this is not quite correct, since in a cft it can be shown that we require

$$[L_{-1}, V(\psi, z)] = \frac{d}{dz} V(\psi, z). \quad (20)$$

We have that

$$L_n \equiv \frac{1}{2} : \sum_r c_r \cdot c_{n-r} : + \frac{d}{16} \delta_{n,0} \quad (21)$$

satisfy the Virasoro algebra with $c = d$ (and are the modes of $V_T(\psi_L, z)$ – which does need to be checked at the end). So, we can check

$$[L_{-1}, c^i_r] = -rc^i_{r-1}. \quad (22)$$

Thus, $c^i_{\frac{1}{2}} \rightarrow -c^i_{-\frac{1}{2}}$ under commutation with $L_{-1}$, and then normal ordering creates extra terms as the $c^i_{-\frac{1}{2}}$ is moved to the left past any $c^i_{\frac{1}{2}}$’s, for which we need to compensate.
Writing our vertex operators in an explicitly normal ordered form, i.e.
\[ V^G_T(\psi, z) = \sum_{\mu \in \Lambda} \langle \mu | e^{B_-(z)} e^{B_+(z)} | \psi \rangle \gamma_{\mu}, \]  
(23)
where
\[ B_{\pm}(z) = \sum_{n \geq 0, r > 0} B_{\pm}^{n r}(z) a_n \cdot c_{\pm r}, \]  
(24)
we see easily that
\[ V^T_T(\psi, z) = V^G_T \left( e^{A(z)} \psi, z \right) \]  
(25)
satisfies (22), with
\[ A(z) = \frac{1}{2} \sum_{n, m \geq 0} A_{nm}(z) a_n \cdot a_m \] 
defined uniquely up to a constant (of integration), which can be fixed by checking the representation property (i.e. locality of the \( V^T \)'s [4]). It is miraculous that these vertex operators turn out to be the correct ones. We have still to understand properly why a simple correction of the \( L_{-1} \) commutation relation should produce such far reaching consequences.

Finally, we find the intertwining operators \( W \) by a trick. The "skew-symmetry" relation
\[ V(\psi, z) \phi = e^{z L_{-1}} V(\phi, -z) \psi \]  
(26)
is essentially a consequence of locality and the creation axiom. In \( \tilde{H}(\Lambda) \), this requires
\[ W(\chi, z) \psi = e^{z L_{-1}} V_T(\psi, -z) \chi, \]  
(27)
fixing \( W \) (and hence \( W^T \)).

To check the axioms of a cft is difficult, but has been done in [4], and they are found to hold if (and only if [17]) \( \sqrt{2} \Lambda^* \) is even, a powerful constraint as it almost forces self-duality, i.e. the condition for consistency of the cft on a torus, whereas we are only working on the Riemann sphere.

2.5 Generalised triality

It was observed [5, 4] from the Kac-Moody algebras formed by the modes of the vertex operators corresponding to the states of conformal weight one that we appeared to have \( \tilde{H}(\Lambda_C) \cong H(\tilde{\Lambda}_C) \) in all cases considered. This was proved explicitly, and we obtained the picture shown in figure 2 (upward sloping lines represent the untwisted constructions and downward sloping lines the twisted constructions).

The codes in fact fit together into blocks of the form shown in figure 3, which are finite for 24 or more dimensions, but infinite in size for 8 and 16 dimensions.

Now, on \( H(\Lambda_C) \) there is an obvious triality structure, due to the existence of an affine \( su(2)^d \) algebra. The isomorphism noted above then allows us to lift this triality up to \( \tilde{H}(\Lambda_C) \) to give an \( S_3 \) group of triality automorphisms mixing straight and twisted sectors [5, 4].

For \( C \) the Golay code \( C_{24}, \tilde{\Lambda}_C \) is the Leech lattice \( \Lambda_{24} \). Now \( \tilde{H}(\Lambda) \) inherits an obvious group of automorphisms from \( Aut(\Lambda) \) (which do not mix straight and twisted sectors). In the case of the Leech lattice, these, together with a triality involution, generate the Monster group \( M \), which can in fact be shown to be the full automorphism group of \( V^2 = \tilde{H}(\tilde{\Lambda}_{C_{24}}) \) [10, 14, 25].

The beauty of this picture is that
1. it demonstrates that the codes play a much more fundamental role than previously thought in the cft structure, and
2. that the “unique” features of the Monster module can be seen in a much more general context.

3 Construction of Niemeier Lattices from Ternary Codes

Only 12 of the 24 Niemeier lattices are obtained by the two constructions from binary codes described in the previous section. Let us see if we can obtain more by a study of constructions from ternary codes [19].

Let \( \mathcal{C} \) be a self-dual ternary code of length \( d \). The form of the weight enumerator, given by a theorem of Gleason [4] as

\[
W_{\mathcal{C}}(x, y) \in \mathbb{C}[\psi_4, \psi_{12}]
\]

\[
\psi_4 \equiv x(x^3 + 8y^3)
\]

(28)
\[
\psi_{12} \equiv (8x^6 - 160x^3y^3 - 64y^6)^2,
\]
is such that we cannot require it to be the equivalent of doubly-even, \textit{i.e.} \( \hat{c}^2 \in 3\mathbb{Z} \forall \hat{c} \in \hat{C} \) but we cannot require \( \hat{c}^2 \in 6\mathbb{Z} \forall \hat{c} \in \hat{C} \).

So, we define
\[
\hat{C}_+ = \{ \hat{c} \in \hat{C} : \text{wt}(\hat{c}) \equiv 0 \mod 6 \}
\]
\[
\hat{C}_- = \{ \hat{c} \in \hat{C} : \text{wt}(\hat{c}) \equiv 3 \mod 6 \}.
\]

Set
\[
\Lambda_0(\hat{C}) = \left( \hat{C}_+ + \sqrt{3}\mathbb{Z}_d^+ \right) \cup \left( \hat{C}_- + \sqrt{3}\mathbb{Z}_d^- \right).
\]
This is even, but not self-dual. We add to it a sector shifted by some vector, \textit{i.e.} set
\[
\Lambda_\pm(\hat{C}) = \left( \hat{C}_+ + \frac{3\mathbb{1}}{\sqrt{3}} + \sqrt{3}\mathbb{Z}_d^+ \right) \cup \left( \hat{C}_- + \frac{3\mathbb{1}}{\sqrt{3}} + \sqrt{3}\mathbb{Z}_d^- \right).
\]

Then \( \Lambda_\pm = \Lambda_0(\hat{C}) \cup \Lambda_\pm(\hat{C}) \) is an even self-dual lattice, provided \( 1 \in \hat{C} \) and \( d \in 24\mathbb{Z} \). Note that \( \Lambda_\pm \) can be regarded as the two possible results of applying just one of the constructions to the set of codes equivalent to \( \hat{C} \).

Note that we already have all the even self-dual lattices from binary codes in 8 and 16 dimensions, so the restriction to \( d \in 24\mathbb{Z} \) is not too disastrous. Let us now consider the case \( d = 24 \) exclusively.

From the results of Venkov [28], we may identify the resulting Niemeier lattice uniquely by the number of vectors of squared length two and the number of orthogonal components into which the set of such vectors splits.

In fact, we find
\[
|\Lambda_\pm(2)| = 3n_3 + n_6 + n_{24}^\pm,
\]
where \( n_m \) is the number of codewords of weight \( m \), and \( n_m^\pm \) is the number of codewords of weight \( m \) in which the number of entries equal to 1 is (an) even or odd (multiple of 3) respectively.

However, for the purposes of computer calculation, this naive approach results in considerably too much computation. So, we use a theorem of Mallows, Pless and Sloane [16] on the form of the complete weight enumerator of a self-dual ternary code (similar to Gleason’s result for the weight enumerator), and we find that there are \( |\Lambda_\pm^+(2)| / 24 - 2 \) codewords of weight 6 with 6 1’s in the code, with a similar result for \( |\Lambda_\pm^-(2)| \).

We now need some codes to work on. However, the self-dual ternary codes have only been classified up to and including length 20. (There are 24 of these!) We do not need a full classification however, since it is really only the resulting lattices in which we are interested.

We proceed in two steps. Firstly, we show that we may construct a self-dual length 24 code with \( n_3 \geq 4 \) from a self-dual length 20 code (see [15] for full details). The construction
involves a choice of codeword of weight 18 in the length 20 code. The support of this codeword is all that is relevant, since sign changes in the coordinates trivially give rise to equivalent codes of length 24. As a bonus, we obtain an upper bound on the number of inequivalent length 24 self-dual ternary codes, though quite a bit larger (of order 1000) than the lower bound of 40 found by Leon, Pless and Sloane [15].

One can show, for \( n_3 \geq 2 \), that \( \Lambda_+^\pm \cong \Lambda_-^\pm \), and we find, by our previously described calculational technique, the corresponding lattices, 19 of them in total (11 of the 12 not produced by the binary constructions – the exception being \( A_{24} \)). This suggests that we should try to find all of the Niemeier lattices by these constructions, not merely those which cannot be obtained by the binary constructions.

[As an aside, we find (distinguishing them by the lattice they produce as well as their complete weight enumerators) at least 34 inequivalent indecomposable codes, c.f. the lower bound of 13 given by Leon, Pless and Sloane.]

There are some known length 24 codes containing 1 with \( n_3 < 4 \). Using these, we get 23 of the 24 Niemeier lattices. In particular, we get the Leech lattice from at least two codes \( (Q_{24} \text{ and } P_{24}) \), which is surprising as we would normally expect some sort of uniqueness to be associated to this structure.

Finally, we resort to a random basis generation technique on a computer, and find the \( A_{24} \) Niemeier lattice.

So, we obtain all Niemeier lattices from the constructions \( \Lambda_+^\pm, 23 \text{ of them from } \Lambda_+^\pm \) alone \( (|\Lambda_+^\pm(2)| \geq 48 \text{ by the known form of the complete weight enumerator, and so the Leech lattice cannot be obtained}) \) and at least 22 from \( \Lambda_-^\pm \).

As yet, there is no analogous cft construction. It would seem to have to be a \( \mathbb{Z}_2 \)-orbifold of a \( \mathbb{Z}_2 \)-orbifold, and we only know explicitly how to orbifold the theories \( \mathcal{H}(\Lambda) \) so far, though one can make arguments about orbifolds of other theories without knowledge of their construction by looking at the modular transformations of the generalised Thompson series [18]. This is a way of tackling the classification problem for \( c = 24 \) theories which will be discussed further in section 5.

4 \( \mathbb{Z}_3 \)-Orbifolds and Ternary Codes

In the absence of any inspiration for an analogue of section 3 for cft’s, let us bring in ternary codes into the picture in another way. The following in fact applies to any automorphism of prime order \( p \) with \( (p - 1)|24 \), i.e. \( p = 3, 5, 7 \) and 13. We shall restrict to \( p = 3 \) simply for ease of exposition, and consider a \( \mathbb{Z}_3 \)-NFPA (no-fixed-point automorphism) orbifold of \( \mathcal{H}(\Lambda) \).

4.1 Reformulation of \( \mathcal{H}(\Lambda) \)

We begin with the theory \( \mathcal{H}(\Lambda) \) for \( \Lambda \) an even self-dual lattice of dimension \( 2d \). As remarked before, any automorphism \( R \) of \( \Lambda \) extends to an automorphism of the cft, i.e.

\[
\begin{align*}
    a_n^i & \mapsto R^i_j a_n^j \\
    |\lambda\rangle & \mapsto (-1)^{\lambda \cdot \mu} |R\lambda\rangle, \quad \mu \in \Lambda/2\Lambda.
\end{align*}
\]
[In fact, it has been shown that any finite order automorphism of $H(\Lambda)$ is conjugate to such an automorphism \cite{20}.]

Now, suppose that $\Lambda$ admits a third order NFPA. It is then easily shown that $\Lambda$ corresponds to a $d$-dimensional complex lattice $\hat{\Lambda}$ over the Eisenstein ring $E$ of cyclotomic integers $\mathbb{Z}[\omega] = \{ m + n\omega : m, n \in \mathbb{Z} \}, \omega = e^{\frac{2\pi i}{3}}$. Let us rewrite the theory $H(\Lambda)$ in terms of oscillators $b_i^\dagger, \overline{b}_i^\dagger, 1 \leq i \leq d$, such that

$$[b_i, b_j^\dagger] = m \delta_{ij}, \quad [b_i, b_j] = [\overline{b}_i, \overline{b}_j] = 0$$

$$b_i^{\dagger} = \overline{b}_i^n$$

$$\langle p \equiv b_0^\dagger | \lambda \rangle = \frac{\lambda}{\alpha} | \lambda \rangle, \quad \lambda \in \hat{\Lambda}$$

$$\langle \overline{p} \equiv \overline{b}_0^\dagger | \lambda \rangle = \frac{\lambda}{\alpha} | \lambda \rangle$$

$$b_i^\dagger | \lambda \rangle = \overline{b}_i^n | \lambda \rangle = 0, \quad n > 0,$$

for some scale factor $\alpha$, fixed later by locality.

For

$$\psi = \prod_{a=1}^M b_i^{i_{a_{-m_a}}} \prod_{b=1}^N \overline{b}_i^{b_{-n_b}} | \lambda \rangle,$$

we have

$$V(\psi, z) =: \prod_{a=1}^M \frac{i}{(m_a - 1)!} d^{m_a} X_+^{i_{a_1}}(z) \prod_{b=1}^N \frac{i}{(n_b - 1)!} d^{n_b} X_-^{b_{-1}}(z) e^{i\frac{\pi}{\alpha} X_+(z)} e^{i\frac{\pi}{\alpha} X_-(z)} : \sigma_\lambda,$$

where

$$X_+(z) = q - ip \ln z + i \sum_{n \neq 0} b_n \frac{z^{-n}}{n}$$

and

$$X_-(z) = \overline{q} - i \overline{p} \ln z + i \sum_{n \neq 0} \overline{b}_n \frac{z^{-n}}{n},$$

with

$$e^{i\frac{\pi}{\alpha} q(z)} e^{i\frac{\pi}{\alpha} \overline{q}} | \mu \rangle = | \lambda + \mu \rangle.$$  

The relation

$$\hat{\sigma}_\lambda (\equiv e^{i\frac{\pi}{\alpha} q(z)} e^{i\frac{\pi}{\alpha} \overline{q}}) \hat{\sigma}_\mu = (-1)^{(\lambda, \mu)/\alpha^2} \hat{\sigma}_\mu \hat{\sigma}_\lambda$$

ensures locality.

The third order automorphism $\theta$ in this picture is simply given by

$$\theta b_i^\dagger \theta^{-1} = \omega b_i^\dagger$$

$$\theta \overline{b}_i^\dagger \theta^{-1} = \overline{\omega} \overline{b}_i^\dagger$$

$$\theta | \lambda \rangle = | \overline{\omega} \lambda \rangle,$$

$$\theta \overline{b}_i^\dagger = \overline{\omega} b_i^\dagger.$$
4.2 Construction of $\tilde{\mathcal{H}}(\Lambda)$

We now need to define the other vertex operators in the orbifold theory. We shall have three sectors (one for each conjugacy class of the discrete symmetry group $[22]$). In diagrammatic form we have figure 4.

![Diagram](image)

Figure 4:

Sector 0 is the $\theta = 1$ projection of $\mathcal{H}(\Lambda)$, while sectors 1 and 2 form irreducible meromorphic representations of sector 0. So, we write down vertex operators $V_1(\psi, z)$ acting on sector 1, $V_2(\psi, z)$ acting on sector 2, for $\psi$ a state in sector 0, by analogy with $V_0(\psi, z)$, just as in the $\mathbb{Z}_2$ case.

Introduce oscillators $c^i_r$, $1 \leq i \leq d$, $r \in \mathbb{Z} \pm \frac{1}{3}$, such that

$$[c^i_r, c^j_s] = r \delta_{r-s} \delta^{ij}, \quad c^i_r \dagger = c^{-i}_{-r}, \quad (43)$$

and similarly operators $\overline{c}^i_r$. All other commutation relations vanish. Sector 1 is then built up by the action of the $c$'s on some ground state, and sector 2 similarly by the $\overline{c}$'s.

The operators

$$L_n = \sum_{r \in \mathbb{Z} + \frac{1}{3}} : c_r \cdot c_{n-r} : + \frac{d}{9} \delta_{n0} \quad (44)$$

satisfy the $c = 2d$ Virasoro algebra (and turn out, as necessary for consistency, to be the modes of $V_1(\psi_L, z)$). The conformal weight of

$$\chi = \prod_{a=1}^M c^{-j_a}_{-r_a} \chi_0, \quad (45)$$

for $\chi_0$ in the ground state, is then $\frac{d}{2} + \sum_{a=1}^M r_a$. Hence, for a meromorphic representation, we require $d \in 3\mathbb{Z}$ and we must project onto states with $\theta = 1$. where

$$\theta c^i_r \theta^{-1} = e^{2\pi ir} c^i_r, \quad \theta \chi_0 = e^{-2\pi i d/9} \chi_0 \quad (46)$$
extends the automorphism from sector 0 to sector 1 (provided we define $V_1$ suitably).

Set

$$
V_1^G(\psi, z) =: \prod_{a=1}^{M} \frac{i}{(m_a - 1)!} \frac{d^{m_a}}{dz^{m_a}} C^a_+(z) \prod_{b=1}^{N} \frac{i}{(m_b - 1)!} \frac{d^{m_b}}{dz^{m_b}} C^b_-(z) e^{\frac{i}{\pi} \sum C^a_+(z) e^{i\lambda_a} C^a_-(z)} : \gamma_\lambda : ,
$$

where

$$
C^a_\pm(z) = i \sum_{r \in Z_{\pm 1}} \frac{c_r}{r} z^{-r}
$$

and the $\gamma_\lambda$’s are a set of cocycle operators (for which the ground state forms an irreducible representation, though we will not go into such details here). This definition is simply an analogy of that for $V_0$ in terms of $X_\pm$, c.f. the $Z_2$ case. (Note that $\theta V_1^G((\psi, z) \theta^{-1} = V_1^G(\theta \psi, z)$.)

Similarly, we define $V_2^G$ by replacing $C_\pm$ by $\overline{C}_\mp$, where these have the obvious definition.

However, as in the $Z_2$ case, there is a problem, in that the relation (20) is not satisfied by $V_1^G$, $V_2^G$ due to a similar normal ordering problem to that in the $Z_2$ case. We find, as before, that this can be corrected by setting

$$
V_1(\psi, z) = V_1^G \left( e^{A_1(-z)} \psi, z \right) ,
$$

where

$$
A_1(z) = \sum_{n,m \geq 0} A_{nm}^1(z) b_n \cdot \overline{b}_m
$$

and the $\mathbb{C}$-numbers $A_{nm}^1$ are determined up to one arbitrary constant (of integration) fixed later by locality requirements. A similar result holds for $V_2(\psi, z)$, i.e. the expression written down by analogy with sector 0 is “almost” correct. Again, we still have no proper understanding of why this procedure works.

So, we have defined the representations of sector 0. Now we need to define the intertwining operators mixing the three sectors. Also, of course, if we want to show we have a cft we must verify all the locality relations. For the sake of brevity, however, let us simply content ourselves with defining the operators $W_1$, $W_2$, $W_3$ and $\overline{W}_1$, $\overline{W}_2$, $\overline{W}_3$. Full details of the locality calculations may be found in [21].

In the original theory $\mathcal{H}(\Lambda)$ we have [2], and if we are to preserve such a relation in the orbifold theory we must have

$$
\overline{W}_i(\chi, z) = W_i \left( e^{z^* L_1 z^{* -2L_0}} \chi, 1/z^* \right) ,
$$

extending the conjugation map to interchange the two twisted sectors. This is just as in the $Z_2$ case.

$W_1$ and $W_2$ are defined easily by the same trick as before, i.e. we must have the locality relation

$$
W_i(\chi, z) V_0(\psi, w) = V_i(\psi, w) W_i(\chi, z) ,
$$

for $i = 1, 2$. Acting on the vacuum state and requiring the creation property $V(\rho, z)|0\rangle = e^{zL_{-1}}|\rho\rangle$ for the orbifold cft gives us

$$
W_i(\chi, z) e^{w L_{-1}}|\psi\rangle = V_i(\psi, w) e^{z L_{-1}}|\chi\rangle ,
$$

where

$$
C^a_\pm(z) = i \sum_{r \in Z_{\pm 1}} \frac{c_r}{r} z^{-r}
$$
or

\[ W_i(\chi, z - w)|\psi\rangle = e^{(z-w)L_{-1}}V_i(\psi, w - z)|\chi\rangle. \] (54)

Hence

\[ W_i(\chi, z)|\psi\rangle = e^{zL_{-1}}V_i(\psi, -z)|\chi\rangle, \] (55)

for \( i = 1, 2 \), fixes \( W_i \) uniquely. (Note that we here effectively derived the “skew-symmetry” relation which we quoted at the corresponding point in section 2, giving a flavour of the sort of manipulations involved.) Though it remains to be checked that this definition is consistent with locality, it is certainly forced upon us as a consequence of locality.

### 4.2.1 Definition of \( W_3 \)

Defining \( W_3 \) is more complicated. This is the point at which we must leave the analogy with the \( \mathbb{Z}_2 \) case. It maps from sector 1 to sector 2, and so we cannot use the above trick of allowing it to act on the vacuum. Instead, we use the fact that sectors 1 and 2 are irreducible representations of sector 0 to give an implicit definition.

Fix \( \chi_0 \in \text{sector 1} \). Then define

\[ W_3(V_1(\phi_1, w_1)\chi_0, z)V_1(\phi_2, w_2)\chi_0 = V_2(\phi_2, w_2)V_2(\phi_1, w_1 + z)W_3(\chi_0, z)\chi_0, \] (56)

which must be true if we are to have locality in the orbifold theory, i.e. \( W_3 \) is defined in terms of \( W_3(\chi_0, z)\chi_0 \equiv F_{\chi_0}(z) \) for some fixed state \( \chi_0 \). [We lose the advantage over previous work of having an explicit form for our vertex operators by such an approach, while also obscuring the symmetry between \( W_3 \) and \( \overline{W}_3 \) under \( c \leftrightarrow \bar{c} \), though this has been demonstrated [17].]

Take \( \chi_0 \) as a ground state for simplicity (and work with \( c \) a multiple of 72). Then let us conjecture

\[ F_{\chi_0}(z) = z^{-\Delta_0} \sum_\alpha e^{s \pi_+ \pi_+ \chi_0} e^{r \pi_- \pi_- \chi_0} e^{\sum_{r,s > 0} D_{r,s} \chi_0 \chi_0}, \] (57)

where \( \Delta_0 \) is the conformal weight of the twisted sector ground state, \( r \in \mathbb{Z} + \frac{1}{3}, s \in \mathbb{Z} - \frac{1}{3} \), \( \chi_0 \) is a ground state and the sum over \( \alpha \) is over some lattice (presumably either \( \Lambda \) or its dual).

Now, since \( \chi_0 \) is quasi-primary (annihilated by \( L_1 \)),

\[ e^{\frac{1}{2}L_{-1}}F_{\chi_0}(\zeta) = \left(1 - \frac{\zeta}{\bar{\zeta}}\right)^{-2\Delta_0} F_{\chi_0} \left(\frac{\zeta}{\bar{\zeta}}\right), \] (58)

by the standard Möbius transformation result (or hermitian conjugation of the \( L_{-1} \) commutation relation, if preferred). This gives us

\[ F_r = \frac{F}{r} \left(\frac{-2}{r - \frac{1}{3}}\right) (-1)^{r - \frac{1}{3}}, \] (59)

and similarly for \( G_s \) up to some constant \( G \), while

\[ (r + s)D_{r,s} = (r + 1)D_{r+1,s} + (s + 1)D_{r,s+1} + \frac{2}{9} D_{\frac{1}{3},s} D_{\frac{1}{3},s}. \] (60)
The constants $F$ and $G$ are easily determined by trying to check a suitable locality relation 
($G = 0$ in fact). Solving for the coefficients $D_{rs}$ is however more of a problem. We have 
one indeterminate at each level (level($D_{rs}$) $\equiv r + s$).

Now, in the case of $W_1$ and $W_2$, the skew-symmetry relation fixed the operators uniquely. In this case, it gives

$$e^{zL^{-1}} F_{x_0}(-z) = F_{x_0}(z).$$  \hfill{(61)}

Let us also consider the locality relation

$$\mathbb{W}_2(\chi_0, z) F_{x_0}(\zeta) = \mathbb{W}_2(\chi_0, \zeta) F_{x_0}(z).$$  \hfill{(62)}

Replace $A^{-m}$ by $x^m$ and $a^{-n}$ by $y^n$ in these equations to obtain a functional relation. (This is actually also sufficient in proving locality, since we know that we only have exponentials of bilinears, whereas in general it would be impossible in reversing the argument to identify e.g. $x^2$ with $a^{-1} \cdot a^{-1}$ or $a^{-2}$ uniquely.) Set

$$g(a, b) \equiv \sum_{r,s>0} D_{rs} a^r b^s.$$  \hfill{(63)}

Then we find that these two relations, after much manipulation, reduce to

$$g(a, b) - g(1 - a, 1 - b) = -(1 - \omega) \ln \left( \frac{(1 - a)^{\frac{1}{3}} - \omega(1 - b)^{\frac{1}{3}}}{(-a)^{\frac{1}{3}} - \omega(-b)^{\frac{1}{3}}} \right) + (\omega \rightarrow \bar{\omega})$$ \hfill{(64)}

and

$$g(a, b) - g \left( \frac{a}{a - 1}, \frac{b}{b - 1} \right) = -(1 - \omega) \ln \left( \frac{(1 - \frac{1}{a})^{\frac{1}{3}} - \omega(1 - \frac{1}{b})^{\frac{1}{3}}}{(\frac{1}{b})^{\frac{1}{3}} - \omega(\frac{1}{a})^{\frac{1}{3}}} \right) + (\omega \rightarrow \bar{\omega}).$$ \hfill{(65)}

Now, we only need one relation at each level, so consider these for $a = b$. Also, consider

$$h(a) \equiv \frac{d}{da} \frac{d}{db} g(a, b) \bigg|_{a=b}$$  \hfill{(66)}

to remove the logarithms ($h$ arises in correlation functions, and so is a more natural object to consider in any case).

We find

$$h(a) - h(1 - a) = \frac{2a - 1}{9a^2(1 - a)^2}$$ \hfill{(67)}

and

$$h(a) - \frac{1}{(1 - a)^4} h \left( \frac{a}{a - 1} \right) = \frac{2 - a}{9a(1 - a)^2}.$$ \hfill{(68)}

Noting an obvious solution to these, write

$$h(a) = \frac{1}{9a(1 - a)^2} + k(a),$$ \hfill{(69)}
where
\[ k(a) = k(1 - a) = \frac{1}{(1 - a)^4} k\left(\frac{a}{a - 1}\right). \quad (70) \]

Now, \( h(a) \) has a simple pole at the origin, from its definition, of residue \( \frac{2}{9}D_{\frac{1}{2}} \). But the skew-symmetry relation \((61)\) gives us \( D_{\frac{1}{2}} = \frac{1}{2} \), and hence \( k(a) \) is regular at 0, and therefore at 1 and \( \infty \) by \((70)\). Since \( h \) is essentially a correlation function, these are the only possible poles. So Liouville’s theorem tells us that \( k \) is constant, and furthermore this constant value vanishes by \((70)\).

We can then evaluate \( D_{rs} \) at successive levels. Doing so for the first half dozen or so, one arrives at the (surprisingly asymmetric) conjecture
\[ D_{rs} = \frac{(-1)^{r+s}}{r + s} \left( \frac{1}{3s} - 1 \right) \left( \frac{-\frac{2}{3}}{r - \frac{1}{3}} \right) \left( \frac{-\frac{1}{3}}{s - \frac{2}{3}} \right), \quad (71) \]
which is found, by substitution, to be the solution.

Note however that full verification of the consistency of the \( \mathbb{Z}_3 \)-orbifold \( \text{cft} \) has yet to be completed.

### 4.3 Construction of Eisenstein lattices from ternary codes

Now, we may generalise a pair of constructions due to Sloane of \( \mathcal{E} \)-lattices from ternary codes, inspired by results such as those shown in figure 4, which strongly indicate, having seen the binary situation, the existence of some code structure to the left, presumably based on a ternary code [and also that the \( \text{cft}'s \) are consistent!].

Let \( \hat{\mathcal{C}} \) be a self-dual ternary code of length \( d \). Define the “straight” construction by
\[ \Lambda_\mathcal{E}(\hat{\mathcal{C}}) = \hat{\mathcal{C}} + (\omega - \overline{\omega})\mathcal{E}^d, \quad (72) \]
and for \( \underline{1} \in \hat{\mathcal{C}} \) define the “twisted” construction by
\[ \tilde{\Lambda}_\mathcal{E}(\hat{\mathcal{C}}) = \Lambda_0(\hat{\mathcal{C}}) \cup \Lambda_1(\hat{\mathcal{C}}) \cup \Lambda_2(\hat{\mathcal{C}}), \quad (73) \]

where
\begin{align*}
\lambda_0(\hat{\mathcal{C}}) &= \hat{\mathcal{C}} + (\omega - \overline{\omega})\mathcal{E}_0^d \\
\lambda_1(\hat{\mathcal{C}}) &= \hat{\mathcal{C}} + (\omega - \overline{\omega})\left(\mathcal{E}_D^d + \frac{1}{3}\underline{1}\right) \\
\lambda_2(\hat{\mathcal{C}}) &= \hat{\mathcal{C}} + (\omega - \overline{\omega})\left(\mathcal{E}_{-D}^d - \frac{1}{3}\underline{1}\right),
\end{align*}

with
\[ \mathcal{E}_\rho^d \equiv \left\{ x = (x_1, \ldots, x_d) \in \mathcal{E}^d : \sum_{i=1}^d x_i \equiv \rho \mod (\omega - \overline{\omega}) \right\} \quad (75) \]
and \( d = 12D \).
4.4 Results and relationships between the constructions

For $D = 1$, we have 3 self-dual codes of length 12. Two contain $\mathbb{Z}_4$, and so can be used for the twisted construction.

We identify the corresponding Niemeier lattices as before, and we obtain figure confirming that we do indeed have the correct analogy. (We identify the twisted cft’s by use of a theorem in [3], which states that when the rank of the Lie algebra generated by the zero modes of the vertex operators corresponding to the states of conformal weight one is equal to the central charge, the cft is isomorphic to a cft of the form $\mathcal{H}(\Lambda)$, for some even lattice $\Lambda$.)

In some sense, the constructions of $\mathbb{Z}$-lattices in the previous section are $d$-dimensional projections of the constructions of $2d$-dimensional $\mathbb{Z}$-lattices from ternary codes described here. So, we would guess that all $c = 24$ self-dual cft’s could be obtained by projecting out the $c = 48$ cft’s produced by the straight and $\mathbb{Z}_3$-twisted constructions on suitable 24-dimensional Eisenstein lattices. The status of this remark is as yet unclear.

In any case, we must verify that $\hat{\mathcal{H}}(\Lambda(\tilde{C}_12))$ is $V^\natural$. We provide evidence for this only. One point is that the ternary code $C_{12}$ enjoys similar unique properties to the binary Golay code $C_{24}$. Also, $\Lambda(\tilde{C}_12)$, the complex Leech lattice, has symmetry group 6Suz. Note that $F_{3-} = $Suz, c.f. $F_{2-} = \text{Aut} (\Lambda_{24})/\mathbb{Z}_2$, suggesting that then automorphism group is again $M$. (We will discuss the uniqueness of the Monster module in the next section.) An analogous analysis to that for the $\mathbb{Z}_2$ case should complete the Suzuki group to the Monster and also generalise to other codes. [We utilise the affine $su(3)^d$ algebra present in $\mathcal{H}(\Lambda(\tilde{C}))$.]
5 Classification of Self-Dual $c = 24$ Conformal Field Theories and Uniqueness of $V^\frac{1}{2}$

Finally, we will briefly summarize and discuss some results on the classification of self-dual cft’s with $c = 24$. Schellekens [23] has derived results for these analogous to those of Venkov [28] for the Niemeier lattices. In particular, he has shown that the Kac-Moody algebra generated by the modes of the states of conformal weight one is restricted to contain components whose central charges (under the Sugawara construction – see e.g. [13]) sum to 24 and have

$$\frac{g}{k} = \frac{N}{24} - 1,$$  \hspace{1cm} (76)

where $N$ is the number of weight one states in the cft, $g$ is the Coxeter number of the Kac-Moody component and $k$ is its level. [He has since derived stronger constraints [24], reducing to 71 (the largest prime divisor of the order of the Monster!) the number of possible Kac-Moody algebras, though only 39 have been found in constructed cft’s so far (those constructed in section 2). Also, there are no known examples as yet of distinct cft’s
with the same algebra, though some do have coincident partition functions.

Let us look at the consequences of this work for the uniqueness of a self-dual \( c = 24 \) theory \( \mathcal{H} \) with no weight one states.

Suppose there exists an involution \( g \) (say if \( \mathcal{H} \) were a Monster module, or an orbifold of \( V^2 \) with respect to some known automorphism as considered in recent work by Tuite [27]), and consider the orbifold \( \mathcal{H}_g \) constructed using it (which we shall assume to exist). We then have the partition function (using notation explained in e.g. [11])

\[
\chi_{\mathcal{H}_g}(\tau) = \frac{1}{2} \left( J(\tau) + T_g(\tau) + T_g(S(\tau)) + T_g(ST(\tau)) \right).
\] (77)

Now, we proceed along the lines of work done by Tuite [26]. Clearly \( T_g(\tau) \) is \( \Gamma_0(2) \) invariant. If \( \mathcal{H}_g \) has the correct twisted sector ground state energy (\( \geq 1 \)) then \( T_g(\tau) \) is a \( \Gamma_0(2) \) hauptmodul, and so is known, i.e.

\[
T_g(\tau) = \left( \frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} + 24.
\] (78)

This then gives

\[
\chi_{\mathcal{H}_g}(\tau) = J(\tau) + 24.
\] (79)

Schellekens’ partial classification then restricts us to an algebra \( U(1)^{24} \). Now, as mentioned previously, we have the theorem that when the rank of the corresponding Lie algebra is equal to the central charge, a cft \( \mathcal{H} \cong \mathcal{H}(\Lambda) \) for some even lattice \( \Lambda \). We thus deduce \( \mathcal{H}_g \cong \mathcal{H}(\Lambda_{24}) \) (the Leech lattice being the only lattice producing the appropriate partition function).

To proceed analogously to Venkov’s proof of the uniqueness of the Leech lattice, we would have to assume the existence of an orbifolding inverse to the original, i.e. such that \( (\mathcal{H}_g)^*_h \equiv \mathcal{H} \).

Instead, we write \( \mathcal{H}_g = \mathcal{H}_{g}^{0} \oplus U \), where \( \mathcal{H}_{g}^{0} \) is the subspace of \( \mathcal{H} \) invariant under the action of \( g \) and \( U \) forms an irreducible representation of \( \mathcal{H}_{g}^{0} \).

Now, \( a_{-1}^{i}0 \in U \), so \( a_{-m}^{i}a_{-n}^{j}|0\rangle \) lie in \( \mathcal{H}_{g}^{0} \), as \( U \times U \subset \mathcal{H}_{g}^{0} \). So \( \mathcal{H}_{g}^{0} \cong \mathcal{H}(\Lambda_{24})_+ \) (consistent with its known partition function). But there exist only 2 irreducible representations of \( \mathcal{H}(\Lambda)_+ \) for \( \Lambda \) self-dual. These are easily distinguished by the number of weight one states, and we deduce that \( \mathcal{H} \cong V^2 \), as required.

Note that we may also, of course, use a similar argument beginning with an automorphism of higher order, though some of the relevant results remain to be verified in those cases.

6 Conclusions

To conclude, we have shown how the connections between \( \mathbb{Z}_2 \)-orbifolds and binary codes lead to a generalization of the triality structure of FLM, and that similar links between
ternary codes and $\mathbb{Z}_3$-orbifolds appear to generalise another “triality” to a class of conformal field theories. The orbifold construction should follow through as in the $\mathbb{Z}_3$ case for higher prime ordered twists, providing a series of constructions of the Monster module. In addition, the construction of the Niemeier lattices from ternary codes holds out the hope that a conformal field theory analogue will produce a complete set of self-dual theories, at least at central charge 24. This is work which is currently in progress. The comments regarding the uniqueness of the Monster module (or rather a self-dual theory with no weight one states) are intimately connected with the ideas of Tuite. Much research is still to be done in this promising area on the boundary between fundamental physics and the hitherto abstract realms of groups and the theory of modular functions.

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