Behavior of the solution of the Cauchy problem for an inhomogeneous hyperbolic equation with periodic coefficients

Hovik A. Matevossian$^1$, Anatoly V. Vestyak$^2$

$^1$ Federal Research Center "Informatics and Control", Dorodnitsyn Computing Center of the Russian Academy of Sciences, Vavilov str., 40, Moscow 119333 RUSSIA; and Department of Mathematical Modeling, Moscow Aviation Institute (National Research University), Volokolomske shosse, 4, Moscow 125993 RUSSIA

$^2$ Department of Mathematical Modeling, Moscow Aviation Institute (National Research University), Volokolomske shosse, 4, Moscow 125993 RUSSIA

E-mail: hmatevossian@graduate.org; v.a.vestyak@mail.ru

Abstract. We study the asymptotic behavior of the solution $u(x,t)$ of the Cauchy problem for a one-dimensional second-order hyperbolic equation with periodic coefficients (as $t \to \infty$) in which the initial data are zero, and the right-hand side of the equation has the form $f(x)e^{-i\omega t}$, where $\omega$ is real.

1. Introduction

Let $u(x,t)$ be the solution of the Cauchy problem

$$u_{tt}(x,t) - (p(x)u_x(x,t))_x + q(x)u(x,t) = f(x)e^{-i\omega t}, \quad (x,t) \in \mathbb{R}^1 \times \{ t > 0 \},$$

with the initial conditions

$$u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = 0, \quad x \in \mathbb{R}^1$$

where the functions $p(x)$ and $q(x)$ are 1-periodic,

$$p(x+1) = p(x) \geq \text{const} > 0, \quad q(x+1) = q(x) \geq 0.$$ 

Moreover, the functions $p(x)$ and $q(x)$ are continuous or have finitely many discontinuities of the first kind on the period. Eq. (1) describes the oscillation caused by a periodic force in $t$.

This paper is a continuation of research conducted in [10] and presented at the International Conference on "Mathematical Modeling in Physical Sciences (5th IC–MSQUARE)" [6], where for the solution $u(x,t)$ of the following Cauchy problem

$$u_{tt}(x,t) - (p(x)u_x(x,t))_x + q(x)u(x,t) = 0, \quad (x,t) \in \mathbb{R}^1 \times \{ t > 0 \},$$

$$u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = \psi(x), \quad x \in \mathbb{R}^1$$

1

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd.
the asymptotic expansion (as \( t \to \infty \)) was obtained, and, as above, the functions \( p(x) \) and \( q(x) \) are 1-periodic, \( p(x + 1) = p(x) \geq const > 0, q(x + 1) = q(x) \geq 0 \), the functions \( p(x) \) and \( q(x) \) are continuous or have finitely many discontinuities of the first kind on the period, \( \psi \in C_0^\infty(\mathbb{R}^1) \), \( \text{supp} \, \psi \subset [0, 1] \).

The behavior (as \( t \to \infty \)) of solutions of problems similar to problem (1), (2) (and also (3), (4)) and of the corresponding multi-dimensional problems under the condition that the potential differs from a constant by a finite function or tends to a constant sufficiently fast at infinity has been studied in many papers (see, e.g., [5],[7],[9],[10],[12]).

An asymptotic expansion as \( t \to \infty \) of the solution \( u(x, t) \) of the Cauchy problem

\[
\begin{align*}
&u_{tt}(x, t) - u_{xx}(x, t) + (\alpha_0 + q(x))u = 0, \quad (x, t) \in \mathbb{R}^1 \times \{ t > 0 \}, \\
&u(x, t)|_{t=0} = \varphi(x), \quad u_t(x, t)|_{t=0} = \psi(x), \quad x \in \mathbb{R}^1,
\end{align*}
\]

with finite initial functions, \( \varphi(x) \in C^2(\mathbb{R}^1) \) and \( \psi(x) \in C^1(\mathbb{R}^1) \) and under weaker restrictions on the potential \( \alpha_0 + q(x) \), where \( \alpha_0 = const \) and \( q(x) \) satisfies, for some \( k \geq 1 \), the condition

\[
\int_{-\infty}^{+\infty} |x|^k |q(x)| < \infty,
\]

was obtained in [5]. For the equation of string oscillations, the Cauchy problem

\[
\begin{align*}
&u_{tt}(x, t) = (a(x) u_x(x, t))_x, \quad 0 < a_0 \leq a(x) \leq A_0 < +\infty, \quad (x, t) \in \mathbb{R}^1 \times \{ t > 0 \}, \\
&u(x, t)|_{t=0} = \varphi(x), \quad u_t(x, t)|_{t=0} = 0, \quad x \in \mathbb{R}^1,
\end{align*}
\]

was considered in [7]. Sufficient conditions for the stabilization (as \( t \to +\infty \)) of the solution \( u(x, t) \) uniformly in \( x \) on any compact set and necessary and sufficient conditions for the stabilization of the solution \( u(x, t) \) in the mean where obtained under some assumptions about the rate of tension \( a(x) \).

In the case of periodic \( p(x) \) and \( q(x) \), the asymptotic behavior (as \( t \to \infty \)) of the solution of the Cauchy problem for a hyperbolic equation was first studied by the authors [10], [11]. This note contains the results announced in [11].

The Bloch principle for elliptic equations with periodic coefficients was obtained in [12].

We introduce the notation: \( C_0^\infty(\Omega) \) is the space of the infinitely differentiable functions on the domain \( \Omega \) with compact support in \( \Omega \); \( L_2(\Omega) \) is the space of measurable functions in \( \Omega \) such that

\[
||u(x); L_2(\Omega)|| = \left( \int_{\Omega} |u(x)|^2 dx \right)^{1/2} < \infty.
\]

The Sobolev space \( H^1(\Omega) \) in \( \Omega \) is defined as follows:

\[
H^1(\Omega) = \{ u : u \in L_2(\Omega), \frac{\partial u}{\partial x_i} \in L_2(\Omega), i = 1, \ldots, n \},
\]

equipped with the norm

\[
||u(x); H^1(\Omega)|| = \left( \int_{\Omega} \left| u(x) \right|^2 + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{1/2}.
\]
2. Definitions and auxiliary statements

**Definition 1** A function \( u \in C^2(\mathbb{R}^1 \times \{ t \geq 0 \}) \) is called a periodic (antiperiodic) solution of the Cauchy problem (1), (2), if it is satisfies the relation

\[
u(x+1,t) = (-1)^j u(x,t)\]

for any \( x \in \mathbb{R}^1, t \geq 0 \), and \( j = 0 \) \((j = 1)\) in the case of periodic (antiperiodic) solutions, respectively.

By \( H_0 \) we denote the Hill operator

\[
H_0 = -\frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x)
\]

in the space \( L_2(\mathbb{R}^1) \). The Hill operator has only a continuous spectrum which is located on the real axis and is left semibounded \[8\].

Let \( y_n(x,\lambda_n) \) be an eigenfunction, normed by the condition \( ||y_n; L_2([0,1])|| = 1 \), of the periodic Sturm-Liouville problem:

\[-(p(x)y')' + q(x)y = \lambda_n y, \quad x \in [0,1], \quad y(0) = y(1), \quad y'(0) = y'(1),\]

and let \( y_n(x,\mu_n) \) be an eigenfunction, normed in \( L_2([0,1]) \), of the antiperiodic Sturm-Liouville problem:

\[-(p(x)y')' + q(x)y = \mu_n y, \quad x \in [0,1], \quad y(0) = -y(1), \quad y'(0) = -y'(1),\]

where \( \lambda_n (\mu_n), n = 0,1,2,... \), are eigenvalues of the corresponding problems; these eigenvalues are numbered in ascending order with the multiplicity taken into account.

By \( v_n(x,\lambda_n) \) \((v_n(x,\mu_n))\) we denote a periodic (antiperiodic) function in \( \mathbb{R}^1 \) whose restriction on the segment \([0,1]\) coincides with \( y_n(x,\lambda_n) \) \((y_n(x,\mu_n))\).

Let

\[
h(x,\omega) = \begin{cases} 
(H_0 - \omega^2)^{-1}(f), & \text{if } \omega^2 \notin \sigma(H_0), \\
\lim_{\omega \to 0} \frac{1}{\omega} (H_0 - \omega^2)^{-1}(f), & \text{if } \omega^2 \in \sigma(H_0), \end{cases}
\]

where the limit is understood in the uniform metric in each interval in \( \mathbb{R}^1 \). The function \( h(x,\omega) \) is a solution of the equation

\[(p(x)y')' + (\omega^2 - q(x))y = f(x),\]

and belongs to \( L_2(\mathbb{R}^1) \) if the point \( \omega^2 \) does not belong to the spectrum \( \sigma(H_0) \) of the operator \( H_0 \). Let \( C > 0 \) be an arbitrary fixed constant.

3. Main results

**Theorem 1** If the Hill operator \( H_0 \) is positive, \( p(x) \geq \text{const} > 0, q(x) \geq 0 \), and the point \( \omega^2 \) does not lie on the boundary of the spectrum \( \sigma(H_0) \) of the operator \( H_0 \), such that, for \( |x| < C \) and \( t > 0 \), the solution of the problem (1), (2) has the form

\[
u(x,t) = -ie^{-i\omega t}h(x,\omega) + \frac{1}{\sqrt{t}}(u_1(x,t) + u_2(x,t)) + v(x,t),\]
where \( u_1(x, t) \) is the periodic solution of the Cauchy problem

\[
u_1(x, t) = \sum_{n=0}^{\infty} a_{\lambda_n} \psi_{\lambda_n} v_n(x, \lambda_n) \sin(\sqrt{\lambda_n} t + (-1)^n \frac{\pi}{4} + b_{\lambda_n}),
\]

\( u_2(x, t) \) is the antiperiodic solution of the Cauchy problem

\[
u_2(x, t) = \sum_{n=0}^{\infty} a_{\mu_n} \psi_{\mu_n} v_n(x, \mu_n) \sin(\sqrt{\mu_n} t + (-1)^{n+1} \frac{\pi}{4} + b_{\mu_n}),
\]

and the function \( v(x, t) \) satisfies the estimate

\[
|v(x, t)| \leq \frac{C_0}{t} ||f; L_2(\mathbb{R}^1)||, \quad C_0 = \text{const}(C),
\]

for \( |x| < C \) and \( t > 0 \), where \( \psi_{\lambda_n} (\psi_{\mu_n}) \) are coefficients of the expansion of the function \( \psi(x) \) in the Fourier series with respect to the system \( \{y_n(x, \lambda_n)\} \) \( \{y_n(x, \mu_n)\} \) and \( a_{\lambda_n} (a_{\mu_n}) \), \( b_{\lambda_n} (b_{\mu_n}) \) are certain constants of order \( o(1/n^2) \) as \( n \to \infty \).

Here the sums are taken only over the \( n \) for which \( \lambda_n (\mu_n) \) are simple eigenvalues of the periodic (antiperiodic) Sturm-Liouville problem.

Let \( \varepsilon > 0 \) be a sufficiently small number and the point \( \omega^2 \) lies on the boundary of the spectrum \( \sigma(H_0) \) of the operator \( H_0 \). Then in the domain

\[
\Omega_k = \{k : \text{Im}k > 0, |k - \omega| < \varepsilon\}
\]

the equality holds

\[
(H_0 - k^2)^{-1}(f) = h_1(x, \omega)(k - \omega)^{-1/2} + h_2(x, \omega) + h_3(x, \omega)(k - \omega)^{1/2} + O(k - \omega).
\]

**Theorem 2** If the Hill operator \( H_0 \) is positive, \( p(x) \geq \text{const} > 0, q(x) \geq 0 \), and the point \( \omega^2 \) lies on the boundary of the spectrum \( \sigma(H_0) \) of the operator \( H_0 \), then, for \( |x| < C \) and \( t > 0 \), the solution of the problem (1), (2) has the form

\[
u(x, t) = C_1\sqrt{e^{-i\omega t}}h_1(x, \omega) - C_2e^{-i\omega t}h_2(x, \omega) + \frac{1}{4!}\{u_1(x, t) + u_2(x, t) + C_3e^{-i\omega t}h_3(x, \omega)\} + v(x, t),
\]

where the functions \( u_1(x, t) \) and \( u_2(x, t) \) have the same form as in Theorem 1, only with the difference that in the expansions for \( u_1(x, t) \) and \( u_2(x, t) \) do not include a term corresponding to the \( \omega^2 \), and for the function \( v(x, t) \), for \( |x| < C \) and \( t > 0 \), as in Theorem 1, the following estimate is holds

\[
|v(x, t)| \leq \frac{C_0}{t} ||f; L_2(\mathbb{R}^1)||, \quad C_i = \text{const}, i = 0, 1, 2, 3.
\]

It is well known [8] that the left end of the spectrum \( \sigma(H_0) \) of the operator \( H_0 \) coincides with \( \lambda_0 \). Assume that \( \lambda_0 \leq 0 \).

**Remark 1** If the left end of the spectrum \( \sigma(H_0) \) of the Hill operator \( H_0 \) coincides with \( \lambda_0 = 0 \) and \( \omega > 0 \), then in Theorems 1 and 2 in the right-hand side in the expansions for the function \( u(x, t) \) we add a term of the form \( h_0(x) = a_0 \psi_0 v_0(x, \lambda_0) \), where \( a_0 \) is some constant.

**Remark 2** In the case of the finite-zone potential, the functions \( u_1(x, t) \) and \( u_2(x, t) \) can be represented as a finite sum of oscillating in \( t \), because the spectrum of the operator \( H_0 \) has a zone structure and the ends of the zones coincide with the simple eigenvalues \( \lambda_n \) and \( \mu_n \) [8].
The validity of the principle of limiting amplitude, that is, the output of the solution (as \( t \to \infty \)) to the periodic regime with respect to \( t \) of the driving force, depends on whether \( \omega^2 \) coincides with the spectrum boundary of the Hill operator.

The basic idea of the proof of theorems are the same as in the case of the finite function \( q(x) \) [9]. Problem (1), (2) can be solved by using the Fourier transform, and the contour of integration in the inverse Fourier transform is shifted to the "nonphysical leaf" of the Riemann surface of the spectral parameter. After this, the integral determining the solution is analyzed, i.e. the leading term of the asymptotics is distinguished, and then the remainder is estimated.

In the case, the integral in question and the methods for studying it differ strongly from those in the case of a finite function \( q(x) \). The study of this integral is significantly based on the properties of the resolvent of the Hill operator \( H_0 \) ([1]–[4]).

References
[1] Firsova N E 1984, "Resonances of a Hill operator, perturbed by an exponentially decreasing additive potential", Math. Notes, 36:5, 854–861
[2] Firsova N E 1987, "The direct and inverse scattering problems for the one-dimensional perturbed Hill operator", Math. USSR–Sbornik, 58:2, 351–388.
[3] Firsova N E 1989, "On solution of the Cauchy problem for the Korteweg-de Vries equation with initial data the sum of a periodic and a rapidly decreasing function", Math. USSR–Sbornik, 63:1, 257–265.
[4] Hochstadt H 1965, "On the determination of a Hill’s equation from its spectrum", Arch. Rational Mech. Analysis, 19:5, 353–362.
[5] Laptev S A 1975, "On the behavior for large values of the time of the solution of the Cauchy problem for the equation \( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \alpha(x)u = 0 \)", Math. USSR–Sbornik, 26:3, 403–426
[6] Matevossian Hovik A and Vestyak Anatoly V 2016, "On the behavior of solutions of the Cauchy problem for the hyperbolic equation with periodic coefficients for large time value", 5th IC–MSQUARE: International Conference on Mathematical Modeling in Physical Sciences. Athens, Greece, May 23–26, 1–5.
[7] Peržan A V 1978, "Stabilization of the solution of the Cauchy problem for a hyperbolic equation", (Russian) Differ. Uravnenija [Diff. Equations], 14:6, 1065–1075.
[8] Titchmarsh E C 1958, Eigenfunction expansions: associated with second-order differential equations, Vol. 2, Oxford: Clarendon Press, 404 pp.
[9] Vainberg B R 1982, Asymptotic methods in equations of mathematical physics, Moscow: Moscow University Press, 296 pp. (Russian); English transl.: New York: Gordon and Breach, 1989, 498 pp.
[10] Vestyak A V and Matevossian O A 2016, "On the behavior of the solution of the Cauchy problem for the hyperbolic equation with periodic coefficients", Math. Notes, 100:5, 126–129.
[11] Vestyak A V and Matevossian O A 2017, " On the behavior of the solution of the Cauchy problem for an inhomogeneous hyperbolic equation with periodic coefficients", Math. Notes, 102:3.
[12] Zhikov V V and Pastukhova S E 2016, "Bloch principle for elliptic differential equations with periodic coefficients", Russian J. Math. Phys., 23:2, 257–277.