Asymptotic normality and analysis of variance of log-likelihood ratios in spiked random matrix models

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Abstract

The present manuscript studies signal detection by likelihood ratio tests in a number of spiked random matrix models, including but not limited to Gaussian mixtures and spiked Wishart covariance matrices. We work directly with multi-spiked cases in these models and with flexible priors on the signal component that allow dependence across spikes. We derive asymptotic normality for the log-likelihood ratios when the signal-to-noise ratios are below certain thresholds. In addition, we show that the variances of the log-likelihood ratios can be asymptotically decomposed as the sums of those of a collection of statistics which we call bipartite signed cycles.

Keywords: Contiguity; Finite rank deformation; Principal Component Analysis; Random graphs; Signal detection.

1 Introduction

An important class of signal detection problems share the following hypothesis testing framework. Under the null hypothesis, the observed data matrix consists of pure noise. Under the alternative, it has a “signal + noise” structure, where the signal component is of low rank and certain knowledge can be encoded as a prior distribution on the signal.

We consider two different cases of the problem.

- In the unnormalized case, we assume that the observed data matrix $X$ is in $\mathbb{R}^{n \times p}$. Let $Z = (Z_{ij}) \in \mathbb{R}^{n \times p}$ with $Z_{ij} \overset{iid}{\sim} N(0, 1)$. We aim to test

$$H_0 : X = Z, \quad \text{vs.} \quad H_1 : X = \frac{1}{\sqrt{p}} \Theta U' + Z,$$

where $\Theta \in \mathbb{R}^{n \times \kappa}$ follows some prior distribution $\pi_\Theta$ and $U \in \mathbb{R}^{p \times \kappa}$ follows some prior distribution $\pi_U$ and $\Theta, U$ and $Z$ are mutually independent. Throughout, we assume that under $\pi_\Theta$ (and $\pi_U$, resp.), the rows of $\Theta$ (and $U$, resp.) are i.i.d. random vectors with $E[\Theta_{i*}] = 0$ and $\text{Cov}(\Theta_{i1}^*, \Theta_{i2}) = \Sigma_{\Theta}$ (with $E[U_{i*}] = 0$ and $\text{Cov}(U_{i1}' U_{i2}) = \Sigma_{U}$, resp.). In other words, we assume that $X_{ij} \overset{iid}{\sim} N(0, 1)$ under $H_0$, and under $H_1$, $X_{ij} | (\Theta, U) \overset{ind}{\sim} N(\frac{1}{\sqrt{p}} \sum_{l=1}^{\kappa} \Theta_{il} U_{jl}, 1)$ where $E[\Theta_{i*}] = 0$, $E[U_{i*}] = 0$, $E[\Theta_{il} \Theta_{il}] = \Sigma_{\Theta}$ and $E[U_{il} U_{il}'] = \Sigma_{U}$.
\[ \Sigma_\Theta(l_1, l_2) \quad \text{and} \quad \mathbb{E}[U_{j_1}U_{j_2}] = \Sigma_U(l_1, l_2) \quad \text{for} \quad l, l_1, l_2 \in [\kappa]. \] Here and after, for any matrix \( A \in \mathbb{R}^{n \times p} \), \( A' \) stands for its transposition, \( A_{is} \in \mathbb{R}^{1 \times p} \) its \( i \)-th row, and \( A_{sj} \in \mathbb{R}^{n \times 1} \) its \( j \)-th column. We use \( A_{ij} \) and \( A(i, j) \) exchangeably to denote its \((i, j)\)-th entry. For any positive integer \( l \), \([l] = \{1, 2, \ldots, l\} \). Under \( H_1 \), if the distribution of \( \Theta_{is} \) is discrete and takes a finite number of values, then conditioning on \( U \) the rows of \( X \) in (1) can be viewed as i.i.d. observations from a Gaussian mixture distribution.

- In the normalized case, we test

\[ H_0 : X = Z, \quad \text{vs.} \quad H_1 : X = \Theta V' + Z, \tag{2} \]

where \( V = U(U'U)^{-1/2} \) and \( U \) is defined as is in the previous case. In other words, \( V \) is a self-normalized version of \( U \) such that \( V'V = I_\kappa \) and so \( V \in O(p, \kappa) \), i.e., the Stiefel manifold consisting of all \( \kappa \)-frames in \( \mathbb{R}^p \). Under \( H_1 \), if \( \Theta_{is} \stackrel{iid}{\sim} N(0, \Sigma_\Theta) \) with \( \Sigma_\Theta = H = \text{diag}(h_1, \ldots, h_\kappa) \) with \( h_1 \geq \cdots \geq h_\kappa > 0 \), then conditioning on \( U \) the rows of \( X \) are i.i.d. observations from a \( p \)-dimensional normal distribution with mean \( 0 \) and multi-spiked covariance matrix \( VHV' + I_p \). Here \( I_p \) is the \( p \)-dimensional identity matrix. In this case, (2) reduces to the high dimensional sphericity test against multi-spiked alternative.

In either case, we aim to detect a spiked random matrix model against the null. Moreover, we deal with simple vs. simple hypothesis testing since we put some prior distribution on the signal component under the alternative. Therefore, the Neyman–Pearson lemma dictates that the likelihood ratio test is optimal.

In this manuscript, we are concerned with the asymptotic behavior of likelihood ratios in the aforementioned testing problems. In particular, let \( p = pn \) scale with \( n \). We are interested in the asymptotic regime where

\[ \frac{p}{n} \rightarrow \gamma \in (0, \infty) \quad \text{as} \quad n \rightarrow \infty. \tag{3} \]

Let \( \mathbb{P}_{0,n} \) be the null distribution and \( \mathbb{P}_{1,n} \) the alternative. Let \( L_n = \frac{d\mathbb{P}_{1,n}}{d\mathbb{P}_{0,n}} \) denote the likelihood ratio, and we call \( \log(L_n) \) the log-likelihood ratio.

A series of papers have discovered the following general phenomenon in these testing problems. Depending on the signal-to-noise ratio (SNR), there are two different types of asymptotic behavior of the likelihood ratio. If the SNR is below certain threshold, then \( L_n \) has a nontrivial weak limit, and the null and alternative distributions are mutually contiguous. When the SNR is sufficiently large, the likelihood ratio converges to zero under null and diverges to infinity under alternative in probability as \( n \) tends to infinity. In this case, the two distributions are asymptotically singular. Banks et al. [5] focused on locating the boundary between asymptotically contiguous and singular regimes in both model (1) and its symmetric counterpart known as the Gaussian spiked Wigner model. Perry et al. [19] investigated the same issue in three single-spiked models: Gaussian spiked Wigner model, non-Gaussian spiked Wigner model and spiked Wishart model. In addition, they determined when spectral method, i.e., PCA, is optimal or sub-optimal in these models. The single-spiked Wishart model they considered is a special case of model (1) with \( \kappa = 1 \). In a slightly different line of research, Onatski et al. [17, 18] (see also [12]) derived asymptotic normality of the likelihood ratio under both single and multiple spiked Wishart models for all contiguous
alternative distributions, provided that the prior on the leading eigenvectors is the uniform probability measure on the corresponding Stiefel manifold. The scenario they considered can be viewed as model (2) with the entries of $U$ sampled independently and identically from a standard normal distribution. Furthermore, El Alaoui et al. [8] and El Alaoui and Jordan [7], among other results, derived asymptotic normality results for log-likelihood ratios for single spiked Wigner and Wishart models, which allowed for priors with uniformly bounded support size on each entry of the leading eigenvector or the leading singular vector pair. Their approach is borrowed from spin glass literature and uses cavity method. For single spiked Wishart models, their result is complementary to that in [17] as it required a bounded support condition on the prior and hence excluded the Gaussian prior that underpins the result in [17].

1.1 Main contributions

The main contributions of the present manuscript are two-folded:

(1) For both models (1) and (2) and for a large collection of priors on $\Theta$ and $U$, we show that when the SNR is below certain threshold that depends only on $\gamma$ and second moments and sub-Gaussianity parameters of the priors, the null and the alternative distributions are asymptotically mutually contiguous and that the log-likelihood ratio has normal limits under both null and alternative as $n \to \infty$ and $p/n \to \gamma \in (0, \infty)$. The limiting normal distributions have different means but the same variance, both of which depend only on $\gamma$ and second moments of the priors. We allow any prior on $\Theta$ that assigns independent sub-Gaussian row vectors, and any such prior on $U$. This allows rows of $\Theta$ (and $U$) to be i.i.d. according to any multivariate normal distribution or any multivariate discrete/continuous distribution with bounded support, among other possibilities. To the best of our limited knowledge, the present manuscript is the first to give such results for these multi-spiked signal detection problems beyond the case of uniform priors.

(2) In either model, when the SNR is below the threshold under which we have asymptotic normality for the log-likelihood ratio, we show that under either null or alternative, the log-likelihood ratio can be decomposed as the weighted sum of a collection of statistics, defined later as bipartite signed cycles, which are asymptotically independently and normally distributed. This provides an analysis of variance for the asymptotic log-likelihood ratio. Such a result sheds light on the source of randomness in the asymptotic likelihood ratio. In addition, it serves as a first step toward designing computationally efficient tests that can achieve the exact optimal power of the likelihood ratio test for the testing problems of interest. See, for instance, the effort made in [4] along this direction for testing Erdős–Rényi random graphs against planted partition models.

The approach we shall take in the present manuscript is inspired by a parallel line of research on contiguity and signal detection for Erdős–Rényi random graphs and the planted partition model (i.e., symmetric two block stochastic block models). Janson [11] introduced a second moment argument to study asymptotically contiguous random graph models with respect to random $d$-regular graphs where the degree parameter $d$ remains finite as the graph size tends to infinity. In addition, he showed that the asymptotic likelihood ratios between these sparse random graph models are determined by counts of cycles on graphs. Mossel
et al. [16] established a comparable set of results when studying the detection of planted partition model against Erdős–Rényi random graphs in the asymptotic regime where the average degree of nodes remain finite when the graph size tends to infinity. They determined the exact boundary between asymptotically contiguous and singular regimes and showed that within the contiguous regime, the asymptotic log-likelihood ratio is determined by counts of cycles and has a Poisson mixture limit. Banerjee [3] studied the same Erdős–Rényi model vs. planted partition model detection problem in a different asymptotic regime where average degree and graph size tend to infinity together. Similar to [16], he determined the exact boundary between contiguity and singularity while he also showed that in the contiguous regime, the asymptotic likelihood ratio is determined by a series of graph statistics called signed cycles as opposed to actual counts of cycles on graph. The major tool in [3] is a Gaussian version of the second moment method in [11]. This second moment method also serves as the backbone of arguments in the present manuscript, while the major difficulty here lies in constructing and analyzing the collection of statistics that determine the asymptotic likelihood ratios for models (1) and (2). As we shall show later, the bipartite signed cycles will serve this purpose.

1.2 Organization

The rest of this manuscript is organized as the following. In Section 2, we introduce the second moment method, define bipartite signed cycles, and state the main theorems. Section 3 establishes the asymptotic normality of bipartite signed cycles and Section 4 proves the main theorems. The appendix presents the details of the second moment method.

2 Main results

2.1 Preliminaries

Contiguity  For two sequences of probability measures $\mathbb{P}_n$ and $\mathbb{Q}_n$ defined on $\sigma$-fields $(\Omega_n, \mathcal{F}_n)$, we say that $\mathbb{Q}_n$ is contiguous with respect to $\mathbb{P}_n$, denoted by $\mathbb{Q}_n \ll \mathbb{P}_n$, if for any event sequence $A_n$, $\mathbb{P}_n(A_n) \to 0$ implies $\mathbb{Q}_n(A_n) \to 0$. We say that they are (asymptotically) mutually contiguous, denoted by $\mathbb{P}_n \ll \mathbb{Q}_n$, if both $\mathbb{Q}_n \ll \mathbb{P}_n$ and $\mathbb{P}_n \ll \mathbb{Q}_n$ hold. We refer interested readers to Le Cam [13] and Le Cam and Yang [14] for general discussions on contiguity.

To establish our main results, we rely on the following proposition for establishing contiguity and asymptotic normality of log-likelihood ratios. For any two probability measure $\mathbb{P}$ and $\mathbb{Q}$ on the same probability space, we write $\mathbb{Q} \ll \mathbb{P}$ when $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$.

**Proposition 1** (Janson’s second moment method). Let $\mathbb{P}_n$ and $\mathbb{Q}_n$ be two sequences of probability measures such that for each $n$, both are defined on the common $\sigma$-algebra $(\Omega_n, \mathcal{F}_n)$. Suppose that for each $i \geq 1$, $W_{n,i}$ are random variables defined on $(\Omega_n, \mathcal{F}_n)$. Then the probability measures $\mathbb{P}_n$ and $\mathbb{Q}_n$ are asymptotically mutually contiguous if the following conditions hold simultaneously:

(i) $\mathbb{Q}_n$ is absolutely continuous with respect to $\mathbb{P}_n$ for each $n$;

(ii) For any fixed $k \geq 1$, one has $(W_{n,1}, \ldots, W_{n,k}) \mid \mathbb{P}_n \xrightarrow{d} (Z_1, \ldots, Z_k)$ and $(W_{n,1}, \ldots, W_{n,k}) \mid \mathbb{Q}_n \xrightarrow{d} (Z_1', \ldots, Z_k')$. 

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(iii) \( Z_i \sim N(0, \sigma_i^2) \) and \( Z_i' \sim N(\mu_i, \sigma_i^2) \) are sequences of independent random variables.

(iv) The likelihood ratio statistic \( Y_n = \frac{dQ_n}{dP_n} \) satisfies

\[
\limsup_{n \to \infty} \mathbb{E}_{P_n} [Y_n^2] \leq \exp \left\{ \sum_{i=1}^{\infty} \frac{\mu_i^2}{\sigma_i^2} \right\} < \infty. \tag{4}
\]

Furthermore, we have that under \( P_n \),

\[
Y_n \xrightarrow{d} \exp \left\{ \sum_{i=1}^{\infty} \frac{\mu_i Z_i - \frac{1}{2} \mu_i^2}{\sigma_i^2} \right\}. \tag{5}
\]

and that given any \( \epsilon, \delta > 0 \) there exists a natural number \( K = K(\delta, \epsilon) \) such that for any sequence \( n_l \) there is a further sub-sequence \( n_{l_m} \) such that

\[
\limsup_{m \to \infty} \mathbb{P}_{n_{l_m}} \left[ \left| \log(Y_{n_{l_m}}) - \left\{ \frac{2}{\sigma_k} \sum_{k=1}^{K} \left( \frac{2 \mu_k (W_{n_{l_m},k}) - \mu_k^2}{\sigma_k^2} \right) \right\} \right| \geq \epsilon \right] \leq \delta. \tag{6}
\]

Remark 1. The proposition can be viewed as a Gaussian version of Theorem 1 in Janson [11] which dealt with convergence to a Poisson mixture. In addition, it generalizes Proposition 3.4 in [3] where a more specific version of it appeared with an additional redundant condition (\( P_n \) being absolutely continuous with respect to \( Q_n \)) and without conclusion (6).

Bipartite signed cycles In view of Proposition 1, our proofs rely on finding out a class of random variables which are “asymptotically sufficient” for determining the likelihood ratio. To this end, we define the following set of statistics.

Definition 1 (Bipartite signed cycle of length 2k). For each \( k \in [n \wedge p] \), we define the bipartite signed cycle of length 2k as

\[
B_{n,k} = \frac{1}{n^k} \sum_{i_0, j_0, i_1, j_1, \ldots, i_{k-1}, j_{k-1} \in [n]} X_{i_0, j_0} X_{i_1, j_0} X_{i_1, j_1} \ldots X_{i_{k-1}, j_{k-1}} X_{i_0, j_{k-1}} \tag{7}
\]

where \( i_0, i_1, \ldots, i_{k-1} \in [n] \) are all distinct, and so are \( j_0, j_1, \ldots, j_{k-1} \in [p] \).

As we shall show later, for both models (1) and (2), the collection of bipartite signed cycles of increasing lengths determines the asymptotic likelihood ratio, at least for a large collection of prior distributions on \( \Theta \) and \( U \) which we now define.

Sub-Gaussian prior distributions We first define sub-Gaussian random vectors and their variance proxies.

Definition 2. Suppose \( X \) is a random vector of dimension \( d \). We say the random vector \( X \) is sub-Gaussian with variance proxy \( \tilde{\Sigma}_X \) if \( \mathbb{E}[X] = 0 \) and \( \mathbb{E}[\exp(t'X)] \leq \exp \left( \frac{1}{2} t' \tilde{\Sigma}_X t \right) \) for any \( t \in \mathbb{R}^d \). Here \( \tilde{\Sigma}_X \) is a non-negative definite matrix.
By definition, if $\Sigma_X$ is a variance proxy for $X$, then so is any matrix $\tilde{\Sigma}$ such that $\Sigma - \tilde{\Sigma}_X$ is non-negative definite. For any multivariate normal distribution the variance proxy matches with the true covariance matrix. If $X$ is a random vector with i.i.d. Rademacher entries then $X$ is sub-Gaussian with variance proxy $I_d$. Furthermore, if $X$ is sub-Gaussian with variance proxy $\tilde{\Sigma}_X$ then $AX$ is sub-Gaussian with variance proxy $\tilde{\Sigma}_X A$ for any $A$. Finally, if $X$ is a random variable taking values within $[a,b]$, then $X - E[X]$ is sub-Gaussian with variance proxy $\frac{1}{2} (a-b)^2$.

**Definition 3** (Sub-Gaussian prior). For any given number $\kappa < \min(n,p)$, let $P(n,\kappa,\Sigma_{\Theta},\tilde{\Sigma}_{\Theta})$ be the collection of all priors $\pi_{\Theta}$ on $\Theta$ such that under $\pi_{\Theta}$, the row vectors $\{\Theta_{i*} : i \in [n]\}$ are i.i.d. sub-Gaussian random vectors in $\mathbb{R}^p$ with mean zero, covariance matrix $\Sigma_{\Theta}$ and variance proxy $\tilde{\Sigma}_{\Theta}$. Let $P(p,\kappa,\Sigma_U,\tilde{\Sigma}_U)$ be defined analogously for $U$.

### 2.2 Statement of theorems

We first state the theorem for testing problem (1). For any matrix $A$, let $\|A\|_2$ be its spectral norm. For any event $E$, let $1_E$ be its indicator function.

**Theorem 1.** Consider the testing problem defined in (1) with the $\Theta$ prior $\pi_{\Theta} \in P(n,\kappa,\Sigma_{\Theta},\tilde{\Sigma}_{\Theta})$ and the $U$ prior $\pi_U \in P(p,\kappa,\Sigma_U,\tilde{\Sigma}_U)$. Denote the null distribution by $P_{0,n}$, the alternative distribution $P_{1,n}$ and the likelihood ratio $L_n = \frac{dP_{1,n}}{dP_{0,n}}$. Suppose as $n \to \infty$, $p/n \to \gamma \in (0,\infty)$ while $\kappa,\Sigma_{\Theta},\Sigma_U,\tilde{\Sigma}_{\Theta},\tilde{\Sigma}_U$ remain fixed. The following hold whenever $\|\tilde{\Sigma}_{\Theta} \tilde{\Sigma}_U\|_2 \|\Sigma_{\Theta} \Sigma_U\|_2 < \gamma$.

1. $P_{0,n}$ and $P_{1,n}$ are asymptotically mutually contiguous.

2. Under $H_0$,

$$L_n \xrightarrow{d} \exp \left\{ \sum_{k=1}^{\infty} \frac{2\mu_k Z_k - \mu_k^2}{4k\gamma^k} \right\} \quad (8)$$

where $Z_k$ are independent $N(0,2k\gamma^k)$ random variables and for any $k$, $\mu_k = \text{Tr} \left( (\Sigma_{\Theta} \Sigma_U)^k \right)$.

In other words, under $H_0$, $\log(L_n) \xrightarrow{d} N(-\frac{1}{2} \sigma_b^2, \sigma_b^2)$ with

$$\sigma_b^2 = \sum_{k=1}^{\infty} \frac{\mu_k^2}{2k\gamma^k} = \frac{1}{2} \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \log \left( 1 - \frac{h_i h_j}{\gamma} \right) \quad (9)$$

where $h_1 \geq \cdots \geq h_\kappa$ are the eigenvalues of $\Sigma_{\Theta} \Sigma_U$. Under $H_1$, we have (8) with $Z_k \sim N(\mu_k,2k\gamma^k)$ and $\log(L_n) \xrightarrow{d} N\left( \frac{1}{2} h_b^2, \sigma_b^2 \right)$.

3. Further under both null and alternative the log-likelihood ratio satisfies the following ANOVA type decomposition:

$$\log(L_n) - \sum_{k=1}^{m_n} \frac{2\mu_k (B_{n,k} - p1_{k=1}) - \mu_k^2}{4k\gamma^k} \xrightarrow{p} 0 \quad (10)$$

where $m_n$ is any sequence growing to $\infty$ at a rate $o(\sqrt{\log n})$.

Next we state the counterpart of Theorem 1 for the testing problem in (2).
Theorem 2. Consider the testing problem defined in (2). Denote the null distribution by \( P_{0,n} \), the alternative distribution \( P_{1,n} \) and the likelihood ratio \( L_n = \frac{\text{d}P_{1,n}}{\text{d}P_{0,n}} \). Under the condition of Theorem 1, whenever
\[
\|\Sigma_U^{-1/2} \Sigma_U - \Sigma_\Theta^{-1/2} \Sigma_\Theta\|_2 < \gamma,
\]
the three conclusions of Theorem 1 hold with \( \mu_k = \text{Tr}(\Sigma_k) \) and \( h_1 \geq \cdots \geq h_\kappa \) the eigenvalues of \( \Sigma_\Theta \).

It is not surprising that the results for the testing problems (1) and (2) are essentially the same. By law of large number, one has \( \|\sqrt{p}(UU')^{-1/2} - \Sigma_U^{-1/2}\|_2 \to P_0 0 \). Hence \( (UU')^{-1/2} \) is essentially same as \( \frac{1}{\sqrt{p}}\Sigma_U^{-1/2} \) for large values of \( p \). In addition, the distribution of \( X \) in (2) remains unchanged if we replace \( U \) with \( U\Sigma_U^{-1/2} \). Hence the testing problem is essentially the same as the unnormalized version (1) with \( \Sigma_U = I_\kappa \).

### Sphericity test against multi-spiked Wishart covariance matrix

Suppose \( \pi^0_\Theta \) assigns i.i.d. \( N(0, I_\kappa) \) rows vectors in \( \Theta \) and \( \pi^0_U \) does the same on \( U \). Then \( V = U(UU')^{-1/2} \) follows the uniform distribution on the Stiefel manifold \( O(p, \kappa) \), and (2) reduces to the high-dimensional sphericity testing problem considered in [18], since in this case, the full data likelihood ratio reduces to the likelihood ratio of the eigenvalues of the sample covariance matrix. Since \( \pi^0_\Theta \in \mathcal{P}(\kappa, I_\kappa, I_\kappa) \) and \( \pi^0_U \in \mathcal{P}(\kappa, I_\kappa, I_\kappa) \), we obtain the following corollary of Theorem 2 which reconstructs the key result in [18] for normal data. However, the proof approach we take is completely different from that used in [18].

**Corollary 1.** Let \( X_1, \ldots, X_n \) i.i.d. \( N_p(0, \Sigma) \). Consider testing \( H_0 : \Sigma = I_p \) vs. \( H_1 : \Sigma = I_p + V'HV \) where \( H = \text{diag}(h_1, \ldots, h_\kappa) \) with \( h_1 \geq \cdots \geq h_\kappa > 0 \) and \( V \) follows the uniform distribution on \( O(p, \kappa) \). Denote the null distribution by \( P_{0,n} \), the alternative distribution \( P_{1,n} \) and the likelihood ratio \( L_n = \frac{\text{d}P_{1,n}}{\text{d}P_{0,n}} \). Suppose as \( n \to \infty, p/n \to \gamma \in (0, \infty) \) while \( \kappa \) remains fixed. The following hold whenever \( h_1 < \sqrt{\gamma} \).

1. \( P_{0,n} \) and \( P_{1,n} \) are asymptotically mutually contiguous.

2. Under \( H_0 \), (8) holds where \( Z_k \) are independent \( N(0, 2k\gamma^k) \) random variables and for any \( k \), \( \mu_k = \text{Tr}(H^k) \). In other words, under \( H_0 \),
\[
\log(L_n) \overset{d}{\to} N\left( -\frac{1}{2}\sigma_b^2, \sigma_b^2 \right)
\]
with
\[
\sigma_b^2 = \frac{1}{2} \sum_{i=1}^\kappa \sum_{j=1}^\kappa \log \left( 1 - \frac{h_i h_j}{\gamma} \right).
\]

Under \( H_1 \), we have (8) with \( Z_k \overset{\text{ind}}{\sim} N(\mu_k, 2k\gamma^k) \) and \( \log(L_n) \overset{d}{\to} N\left( \frac{1}{2}\sigma_b^2, \sigma_b^2 \right) \).

3. Further under both null and alternative the log-likelihood ratio satisfies (10) where \( m_n \) is any sequence growing to \( \infty \) at a rate \( o(\sqrt{\log n}) \).

**Remark 2.** Since our main theorems depend on the priors only through covariance matrices and sub-Gaussian variance proxies, Corollary 1 holds for any prior \( \pi_U \in \mathcal{P}(p, \kappa, I_\kappa, I_\kappa) \), e.g., the prior that assigns entries of \( U \) with i.i.d. Rademacher random variables.
3 Asymptotic normality of bipartite signed cycles

We now give the limiting distribution of the statistics $B_{n,k}$’s defined in (7). These statistics serve the purpose of the $W_{n,i}$’s in Proposition 1 when it comes to proving Theorems 1 and 2.

**Proposition 2** (Bipartite signed cycles). Consider both testing problems (1) and (2). The following results hold:

(i) Under $H_0$, when $1 \leq k_1 < \ldots < k_l = o(\sqrt{\log n})$,
\[
\left( \frac{B_{n,k_1} - p_{1,k_1}}{\sqrt{2k_1\gamma^{k_1}}}, \ldots, \frac{B_{n,k_l}}{\sqrt{2k_l\gamma^{k_l}}} \right) \xrightarrow{d} N_l(0, I_l).
\] (11)

(ii) Under $H_1$, when $1 \leq k_1 < \ldots < k_l = o(\sqrt{\log n})$,
\[
\left( \frac{B_{n,k_1} - p_{1,k_1} - \mu_{k_1}}{\sqrt{2k_1\gamma^{k_1}}}, \ldots, \frac{B_{n,k_l} - \mu_{k_l}}{\sqrt{2k_l\gamma^{k_l}}} \right) \xrightarrow{d} N_l(0, I_l),
\] (12)

where for testing problem (1),
\[
\mu_k = \sum_{l_1,\ldots,l_{2k} \in \{1,\ldots,\kappa\}^{2k}} \Sigma_\Theta(l_1, l_{2k}) \Sigma_U(l_1, l_2) \Sigma_\Theta(l_2, l_3) \ldots \Sigma_U(l_{2k-1}, l_{2k})
\]
\[
= \text{Tr}((\Sigma_\Theta \Sigma_U)^k),
\] (13)

and for testing problem (2)
\[
\mu_k = \text{Tr}(\Sigma_\Theta^k).
\] (14)

3.1 Preliminaries

The proof of the foregoing proposition is inspired by the remarkable paper [1]. The fundamental idea is to prove the asymptotic normality by using the method of moments and showing that moments of the limiting distributions satisfy Wick’s formula. We first state the method of moments.

**Lemma 1.** Let $Y_{n,1}, \ldots, Y_{n,l}$ be a random vector of $l$ dimension. Then $(Y_{n,1}, \ldots, Y_{n,l}) \xrightarrow{d} (Z_1, \ldots, Z_l)$ if the following conditions are satisfied:

(i) $\lim_{n \to \infty} E[X_{n,1} \ldots X_{n,m}]$ exists for any fixed $m$ and $X_{n,i} \in \{Y_{n,1}, \ldots, Y_{n,l}\}$ for $1 \leq i \leq m$.

(ii) (Carleman’s Condition)[6]
\[
\sum_{h=1}^{\infty} \left( \lim_{n \to \infty} E[X_{n,i}^{2h}] \right)^{-\frac{1}{2h}} = \infty \quad \forall \ 1 \leq i \leq l.
\]

Further,
\[
\lim_{n \to \infty} E[X_{n,1} \ldots X_{n,m}] = E[X_1 \ldots X_m].
\]

Here $X_{n,i} \in \{Y_{n,1}, \ldots, Y_{n,l}\}$ for $1 \leq i \leq m$ and $X_i$ is the in distribution limit of $X_{n,i}$. In particular, if $X_{n,i} = X_{n,j}$ for some $i \neq j \in \{1, \ldots, l\}$ then $X_i = X_j$. 

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Now we state Wick’s formula for Gaussian random variables which was first proved by Isserlis [10] and later on introduced by Wick [21] in the physics literature.

**Lemma 2** (Wick’s formula [21]). Let \((Y_1, \ldots, Y_l)\) be a multivariate mean 0 random vector of dimension \(l\) with covariance matrix \(\Sigma\) (possibly singular). Then \((Y_1, \ldots, Y_l)\) is jointly Gaussian if and only if for any integer \(m\) and \(X_i \in \{Y_1, \ldots, Y_l\}\) for \(1 \leq i \leq m\)

\[
\mathbb{E}[X_1 \ldots X_m] = \left\{ \begin{array}{ll}
\sum_{\eta} \prod_{i=1}^m \mathbb{E}[X_{\eta(i,1)}X_{\eta(i,2)}] & \text{for } m \text{ even} \\
0 & \text{for } m \text{ odd}
\end{array} \right. \tag{15}
\]

Here \(\eta\) is a partition of \(\{1, \ldots, m\}\) into \(\frac{m}{2}\) blocks such that each block contains exactly 2 elements and \(\eta(i, j)\) denotes the \(j\)th element of the \(i\)th block of \(\eta\) for \(j = 1, 2\).

The proof of the aforesaid lemma is omitted. However, we note that the random variables \(Y_1, \ldots, Y_l\) may be the same. In particular, taking \(Y_1 = \cdots = Y_l\), Lemma 2 provides a description of the moments of multivariate Gaussian random variables.

### 3.2 Proof of Proposition 2

In this part, we focus on the proof of Proposition 2 for testing problem (1). In view of the discussion after the statement of Theorem 2, the modification of the formula for \(\mu_k\) for (2) is natural as we could in some sense treat this case similarly to (1) with \(\Sigma_U = I_k\). In particular, we could modify the proof below by considering a sequence of high probability events \(\Omega_n\) such that \(\|1_{\Omega_n}\sqrt{\mathbb{E}(UU')}^{-1/2} - \Sigma_U^{-1/2}\|_{\text{max}} \leq \delta_n \to 0\) and then establish all the weak limits on \(\Omega_n\). Here and after, for any matrix \(A\), \(\|A\|_{\text{max}} = \max_{i,j} |A_{ij}|\) is its vector \(\ell_\infty\) norm.

#### Additional notation and definition

Given a set \(S\), an \(S\) letter \(s\) is simply an element \(s \in S\). With two sets \(S_1\) and \(S_2\), a bi-word for \(S_1\) and \(S_2\) is defined as an alternating ordered sequence of letters where the letters at odd positions come from \(S_1\) and the letters at even positions come from \(S_2\). The final letter is required to come from \(S_1\). We call the letters from \(S_1\) type I and those from \(S_2\) type II. Given any bi-word \(w\), the \(i\)th type I letter is denoted by \(\alpha_i\) and the \(i\)th type II letter by \(\beta_i\). As a convention, we start the subscripts for letters in a bi-word with 0. Observe that any bi-word \(w\) starts from and ends with a type I letter and so the total number of letters in \(w\) is always odd. In particular, any bi-word \(w\) looks like \((\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_k)\). We use \(l(w) = 2k + 1\) to denote the length of \(w\). A bi-word is called closed if \(\alpha_0 = \alpha_k\).

Throughout the proof, we take \(S_1 = \{1, \ldots, p\}\) and \(S_2 = \{1, \ldots, n\}\). The bipartite graph induced by a bi-word \(w = (\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_k)\) is denoted by \(G_w\). It is defined as follows. One treats the letters \((\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_k)\) as nodes and one puts an edge between \(\alpha_i\) and \(\beta_j\) whenever \(|i - j| = 1\). Observe that for any closed bi-word \(w\), \(G_w\) is a cycle of even length\(^1\). Two bi-words \(w_1\) and \(w_2\) are called paired if the graphs \(G_{w_1}\) and \(G_{w_2}\) are the same. For a closed bi-word \(w = (\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_k)\), its mirror image is \(\tilde{w} = (\alpha_k, \beta_{k-1}, \alpha_{k-1}, \beta_{k-2}, \ldots, \alpha_0)\).

Furthermore, for a cyclic permutation \(\tau\) of the set \(\{0, 1, \ldots, k - 1\}\) and a closed bi-word \(w\), we define \(w^\tau := (\alpha_{\tau(0)}, \beta_{\tau(0)}, \alpha_{\tau(1)}, \beta_{\tau(1)}, \ldots, \beta_{\tau(k-1)}, \alpha_{\tau(k-1)})\). If two closed bi-words \(w_1\) and \(w_2\) are paired, then there exists a cyclic permutation \(\tau\) such that either \(w_1^\tau = w_2\) or \(\tilde{w}_1^\tau = w_2\).

---

\(^1\)Cycles of odd length in a bipartite graph do not exist.
Remark 3. These bi-words are not fundamentally different from the words defined in [1] and [2]. In particular, they form a restricted class of words where the alphabet set is taken to be $S_1 \cup S_2$. Hence all the properties of the words can be derived with minimal modifications of the proofs in [1] and [2].

We call an ordered tuple of $m$ words $(w_1, \ldots, w_m)$ a sentence. For any sentence $a = (w_1, \ldots, w_m)$, $G_a = (V_a, E_a)$ is the graph with $V_a = \bigcup_{i=1}^{m} V_{w_i}$ and $E_a = \bigcup_{i=1}^{m} E_{w_i}$. A sentence $a$ is called a weak CLT sentence if each edge in $G_a$ is traversed at least twice. By Lemma 4.10 of [1], the following lemma gives a bound on the number of weak CLT sentences. For any numbers $b$ and $c$, $b \lor c = \max(b, c)$ and $b \land c = \min(b, c)$.

**Lemma 3.** Let $A_t = A_t(l_1, \ldots, l_m)$ be the set of weak CLT sentences such that for each $a \in A_t$, it consists of $m$ words of lengths $l_1, \ldots, l_m$ respectively and $\#V_a = t$. Then

$$\#A_t \leq 2^{\sum_i l_i} \left( C_1 \sum_i l_i \right)^{C_{2m}} \left( \sum_i l_i \right)^{3(\sum_i l_i - 2t)} \, n^t \gamma \gamma 1^t. \quad (16)$$

**Proof.** The proof of this lemma is almost identical to the proof of Lemma 4.3 in [3]. The only difference is in the possible choices of vertices of $V_a$. Here this choice will be $n^t \gamma 2^2$ where $t_1 + t_2 = t$ and $t_1$ is the number of vertices which are from $S_1$ and $t_2$ is the number of vertices which are from $S_2$. It is easy to see in this case $n^t \gamma 2^2 = n^t \gamma \gamma 1^t$. \hfill \Box

**Proof of part (i)** We complete the proof of this part in two steps. In the first step we calculate the asymptotic variances of $(B_{n,k_1}, \ldots, B_{n,k_l})$. The second step is dedicated towards proving the asymptotic normality and independence of $(B_{n,k_1}, \ldots, B_{n,k_l})$.

**Step 1 (Calculation of variance):** Under $H_0$, the case $k_1 = 1$ is simple as it is a sum of i.i.d. random variables and hence its variance calculation is omitted. One important thing to note is that the case $k = 1$ depends on $E[X_{i,j}^4]$ (which is equal to 3 in the current case). This makes the asymptotic variance of $B_{n,1}$ equal to $2\gamma$, which is not the case in general.

In what follows, we focus on the case when $k_1 \geq 2$. Now we prove that $\text{Var}(B_{n,k}) = (1 + o(1))2k\gamma k$ for any finite $k$. Define for any bi-word $w = (\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{k-1}, \beta_{k-1}, \alpha_{k-1}, \beta_{k-1}) = 1_{k=1}$(17)

Now observe that

$$\text{Var}(B_{n,k}) = \left( \frac{1}{n} \right)^{2k} \mathbb{E} \left[ \left( \sum_w X_w \right)^2 \right] = \left( \frac{1}{n} \right)^{2k} \mathbb{E} \left[ \sum_{w_1,w_2} X_{w_1}X_{w_2} \right]. \quad (18)$$

Since both $X_{w_1}$ and $X_{w_2}$ are products of independent mean 0 random variables that appears exactly once with $X_{w_1}$ or $X_{w_2}$, $\mathbb{E}[X_{w_1}X_{w_2}] \neq 0$ if and only if all the edges in $G_{w_1}$ are repeated in $G_{w_2}$. This happens only if $w_1$ and $w_2$ are paired. Now there are $(1 + o(1))nk^k$ choices for $w_1$ and for each $w_1$ there are exactly $2k w_2$'s such that $w_1$ and $w_2$ are paired (images of cyclic permutations of $w_1$ and of $\tilde{w}_1$). As a consequence,

$$\text{Var}(B_{n,k}) = (1 + o(1))2k \frac{n^k p^k}{n^{2k}} = (1 + o(1))2k\gamma k.$$
Step 2 (Proof of asymptotic normality): In order to complete this step, it suffices to prove the following two limits:

$$\lim_{n \to \infty} \mathbb{E} \left[ (B_{n,k_1} - p1_{k_1=1}) B_{n,k_2} \right] \to 0$$  \hspace{1cm} (19)

whenever $k_1 < k_2$ and there exist random variables $Z_1, \ldots, Z_m$ such that for any fixed $m$

$$\lim_{n \to \infty} \mathbb{E}[X_{n,1} \ldots X_{n,m}] \to \begin{cases} \sum_{\eta} \prod_{i=1}^{m} \mathbb{E}[Z_{\eta(i,1)}Z_{\eta(i,2)}] & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd.} \end{cases}$$  \hspace{1cm} (20)

where $X_{n,i} \in \left\{ \frac{B_{n,k_1} - p1_{k_1=1}}{\sqrt{2k_1\gamma_{k_1}}}, \ldots, \frac{B_{n,k_l}}{\sqrt{2k_l\gamma_{k_l}}} \right\}$. To see this, observe that (20) simultaneously imply parts $i)$ and $ii)$ of Lemma 1. The implication of part $i)$ is obvious. For part $ii)$ one can take $X_{n,i}$’s to be all equal and from Wick’s formula (Lemma 2) the limiting distribution of $X_{n,i}$’s are normal and it is well known that normal random variables satisfy Carleman’s condition.

In addition, (20) also implies that the limiting distribution of $\left( \frac{B_{n,k_1} - p1_{k_1=1}}{\sqrt{2k_1\gamma_{k_1}}}, \ldots, \frac{B_{n,k_l}}{\sqrt{2k_l\gamma_{k_l}}} \right)$ is multivariate normal. Hence one gets the asymptotic independence by applying (19).

We first prove (19). Observe that

$$\mathbb{E} \left[ (B_{n,k_1} - p1_{k_1=1}) B_{n,k_2} \right] = \left( \frac{1}{n} \right)^{k_1+k_2} \mathbb{E} \left[ \sum_{w_1,w_2} X_{w_1}X_{w_2} \right].$$

However, here $l(w_1) \neq l(w_2)$. So $\mathbb{E} [X_{w_1}X_{w_2}] = 0$. As a consequence, (19) holds.

Now we prove (20). Let $l_i - 1$ be the length of the bipartite cycle corresponding to $X_{n,i}$ (so that $l_i$ is the length of the word corresponding to the bipartite cycle). Observe that $\frac{l_i-1}{2} \in \{k_1, \ldots, k_l\}$ for any $i$. At first we expand the left hand side of (20) as

$$\mathbb{E}[X_{n,1} \ldots X_{n,m}] = \left( \frac{1}{n} \right)^{\frac{1}{2}\sum_{i}(l_i-1)} \sum_{w_1,\ldots,w_m} \mathbb{E} [X_{w_1} \ldots X_{w_m}].$$  \hspace{1cm} (21)

Here each of the graphs $G_{w_1}, \ldots, G_{w_m}$ are cycles of length $l_1 - 1, \ldots, l_m - 1$ respectively.

In order to have $\mathbb{E} [X_{w_1} \ldots X_{w_m}] \neq 0$, we need each of the edges in $G_{w_1}, \ldots, G_{w_m}$ to be traversed more than once. This is true even for $l_i = 2$ for some $i$. In particular, in this case $G_{w_i}$ is a single edge and this edge is traversed twice. So one can think this as a cycle of length 2. Thus, $a = (w_{1_i}, \ldots, w_{p_i})$ is a weak CLT sentence. Given any weak CLT sentence $a$, we introduce a partition $\eta(a)$ of $\{1, \ldots, m\}$ in the following way: If $i$ and $j$ are in same block of the partition $\eta(a)$, then $G_{w_i}$ and $G_{w_j}$ have at least one edge in common. As a consequence, we can further expand the right hand side of (21) as

$$\left( \frac{1}{n} \right)^{\frac{1}{2}\sum_{i}(l_i-1)} \sum_{\eta} \sum_{w_1,\ldots,w_m: \eta(w_1,\ldots,w_m)=\eta} \mathbb{E} [X_{w_1} \ldots X_{w_m}].$$  \hspace{1cm} (22)

We now show that we only need to care about those $\eta$’s which have at most $\lfloor \frac{m}{2} \rfloor$ blocks when evaluating the expectation. For any number $b$, $[b]$ denotes the largest integer that is
the number of blocks in $\eta$ by the arguments similar to (2.1.32) in Anderson et al. [2]. Here we have also used the fact for any standard Gaussian random variable $m$ and the case of even $\eta$’s such that the number of blocks in them are exactly $\frac{m}{2}$ contribute to a non-vanishing asymptotic mean. Note that this necessarily requires $m$ to be even.

When $\eta(w_1, \ldots, w_m)$ have strictly less than $\frac{m}{2}$ blocks (including all cases of odd $m$ and the case of even $m$ when the number of blocks is strictly less than $\frac{m}{2}$), $G_a$ has strictly less than $\frac{m}{2}$ connected components. From Lemma 4.10 of [1] it follows that in this case $\# V_a < \sum_{i=1}^{m} \frac{l_i - 1}{2}$. Applying Lemma 3 and noting that the $a$’s are weak CLT sentences, we have

$$\left(\frac{1}{n}\right)^{\frac{1}{2}} \sum_{i} (l_i - 1) \sum_{\eta: \# V_a < \sum_{i=1}^{m} \frac{l_i - 1}{2}} \mathbb{E} [X_{w_1} \cdots X_{w_m}]$$

$$\leq \left(\frac{1}{n}\right)^{\frac{1}{2}} \sum_{t < \frac{1}{2}} \left( C_1 \sum_i l_i \right) C_2 m \left( \sum_i l_i \right)^{3(\sum_i l_i - 2t)} \left( n^t \gamma \vee 1 \right)^{\frac{1}{2}} \mathbb{E} \left[ X_{11} \right] \sum_{l_i} \sum_{t < \frac{1}{2}} \left( \sum_i l_i \right)^{3} \left( \frac{\left( \sum_i l_i \right)^{3}}{\sqrt{n}} \right)$$

$$\leq C_3 \sum_i l_i \left( \gamma \vee 1 \right)^{\frac{1}{2}} \sum_i l_i \cdot O \left( \frac{\left( \sum_i l_i \right)^{3}}{\sqrt{n}} \right)$$

(23)

Here we have also used the fact for any standard Gaussian random variable $\mathbb{E} \left[ |X|^t \right] \leq (Cd)^{Cdt}$. Observe that the rightmost side of (23) is $o(1)$ since for $l_1, \ldots, l_m = o(\sqrt{\log n})$, $(C_3 \sum_{i=1}^{m} l_i) C_4 \sum_{i=1}^{m} l_i / n^\alpha \to 0$ whenever $\alpha > 0$ and $m$ is finite.\footnote{The term $\mathbb{E} \left[ |X_{11}| \right]^{t/2}$ is not optimal. One can prove the CLT under the null up to $o(\log n)$ order by the arguments similar to (2.1.32) in Anderson et al. [2]. However for our purpose this suffices.}

Now the only remaining partitions are pair partitions which have exactly $\frac{m}{2}$ many blocks (and so naturally $m$ is even). We now fix a partition $\eta$ of this kind. Let for any $i \in \{1, \ldots, \frac{m}{2}\}$, $\eta(i, 1) < \eta(i, 2)$ be the elements in the $i$th block. Observe now that fixing a pair partition $\eta$ and $(w_1, \ldots, w_m)$ such that $\eta(w_1, \ldots, w_m) = \eta$, the random variables $X_{w_{\eta(i, 1)}}$ and $X_{w_{\eta(i, 2)}}$ are independent when ever $i_1 \neq i_2$ for any $j \in \{1, 2\}$. As a consequence, we now can rewrite (22) as

$$\left(\frac{1}{n}\right)^{\frac{1}{2}} \sum_{i} (l_i - 1) \sum_{\eta: \# \text{pair partition}} \mathbb{E} [X_{w_1} \cdots X_{w_m}]$$

$$= o(1) + \left(\frac{1}{n}\right)^{\frac{1}{2}} \sum_{\eta \text{ pair partition}} \sum_{\eta: \# \text{pair partition}} \prod_{i=1}^{\frac{m}{2}} \mathbb{E} \left[ X_{w_{\eta(i, 1)}} X_{w_{\eta(i, 2)}} \right] \quad (24)$$
Now observe that whenever $\prod_{i=1}^m \mathbb{E}[X_{w_{\eta(i,1)}}, X_{w_{\eta(i,2)}}] \neq 0$, we have $w_{\eta(i,1)}$ and $w_{\eta(i,2)}$ are paired. When $l(w_{\eta(i,1)}) = l(w_{\eta(i,2)}) \neq 3$, there are $(1 + o(1))(l_{\eta(i,1)} - 1)(n\sqrt{\gamma})^{l_{\eta(i,1)} - 1}$ many such choices of $(w_{\eta(i,1)}, w_{\eta(i,2)})$ for every $i$. Here $l_{\eta(i,1)} - 1$ equals the common length of the cycles induced by $w_{\eta(i,1)}$ and $w_{\eta(i,2)}$. In this case $\mathbb{E}[X_{w_{\eta(i,1)}}, X_{w_{\eta(i,2)}}] = 1$. On the other hand, when $l(w_{\eta(i,1)}) = 3$, there are $(1 + o(1))n^{l_{\eta(i,1)} - 1} \gamma$ many such choices of $(w_{\eta(i,1)}, w_{\eta(i,2)})$ for every $i$ and in this case $\mathbb{E}[X_{w_{\eta(i,1)}}, X_{w_{\eta(i,2)}}] = 2$. Hence, we get the following further reduction of the right side of (24):

$$o(1) + (1 + o(1)) \left( \frac{1}{n} \right)^\frac{1}{2} \sum_{\eta \text{ pair partition } i=1}^{\prod_{i=1}^n} \prod_{i=1}^n (l_{\eta(i,1)} - 1) 1_{l_{\eta(i,1)} = l_{\eta(i,2)}} (n\sqrt{\gamma})^{l_{\eta(i,1)} - 1} (25)$$

Recalling that $l_i = 2k_i + 1$ we complete the proof.

Proof of part (ii) We at first look at the case when $k = 1$. This is an exceptional case and needs to be handled differently. Then we deal with the general case of $k \geq 2$.

Analysis of $B_{n,1}$: Recall that $B_{n,1} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^p X_{i,j}^2$, we have

$$B_{n,1} | \Theta, U = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^p (Z_{i,j} + M_{i,j})^2$$

(26)

where for any $(i, j)$,

$$M_{i,j} = \frac{1}{\sqrt{p}} \sum_{l=1}^\kappa \Theta_{i,j} U_{j,l}$$

(27)

and $Z_{i,j} \sim N(0, 1)$. Observe that in this case one can apply the Lindeberg–Feller central limit theorem. So it suffices to calculate the limiting mean and variance of $B_{n,1} | \Theta, U$. Now

$$\mathbb{E} [X_{i,j}^2 | \Theta, U] = 1 + M_{i,j}^2$$

(28)

and

$$\text{Var} [X_{i,j}^2 | \Theta, U] = \text{Var} [Z_{i,j}^2 + 2Z_{i,j} M_{i,j} | \Theta, U]$$

$$= \text{Var} [Z_{i,j}^2] + 4\text{Var}[Z_{i,j}] M_{i,j}^2$$

$$= 2 + 4M_{i,j}^2$$

(29)

So it is enough to prove

$$\frac{1}{n} \sum_{i,j} M_{i,j}^2 \xrightarrow{\text{p}} \sum_{l_1, l_2} \sum_{\Theta(l_1, l_2)} \Sigma_U(l_1, l_2).$$

(30)

As a consequence,

$$\text{Var} [B_{n,1}] = \frac{1}{n^2} \left( 2np + \sum_{i,j} 4M_{i,j}^2 \right) \rightarrow 2\gamma.$$
To this end, note that

\[
\frac{1}{n} \sum_{i,j} M_{i,j}^2 = \frac{1}{n} \left[ \sum_{i,j} \sum_{i',j'} \frac{1}{p} \Theta_{i,i'} \Theta_{i,l} U_{j,j'} \right] \\
= \frac{\kappa}{\sum_{l=1}^{\kappa} \sum_{l'=1}^{\kappa} \left( \frac{1}{n} \sum_{i=1}^{n} \Theta_{i,l} \Theta_{i,l'} \right) \left( \frac{1}{p} \sum_{j=1}^{p} U_{j,l} U_{j,l'} \right)}.
\]

(31)

The weak law of large numbers then gives

\[
\frac{1}{n} \sum_{i=1}^{n} \Theta_{i,l} \Theta_{i,l'} \rightarrow \Sigma_{\Theta}(l,l') \quad \text{and} \quad \frac{1}{p} \sum_{j=1}^{p} U_{j,l} U_{j,l'} \rightarrow \Sigma_{U}(l,l').
\]

Since \( \kappa \) is fixed, we obtain (30).

**Analysis of \( B_{n,k} \) with \( k \geq 2 \):** We first write

\[
B_{n,k} = \frac{1}{n^k} \sum_{i_0,j_0,\ldots,i_{k-1},j_{k-1}} X_{i_0,j_0} \cdots X_{i_0,j_{k-1}}
\]

\[
= \frac{1}{n^k} \sum_{i_0,j_0,\ldots,i_{k-1},j_{k-1}} (Z_{i_0,j_0} + M_{i_0,j_0}) \cdots (Z_{i_0,j_{k-1}} + M_{i_0,j_{k-1}})
\]

(32)

\[
= \frac{1}{n^k} \sum_{i_0,j_0,\ldots,i_{k-1},j_{k-1}} Z_{i_0,j_0} \cdots Z_{i_0,j_{k-1}} + \mu_{n,k} + V_{n,k},
\]

where

\[
\mu_{n,k} := \frac{1}{n^k} \sum_{i_0,j_0,\ldots,i_{k-1},j_{k-1}} M_{i_0,j_0} \cdots M_{i_0,j_{k-1}},
\]

(33)

and \( V_{n,k} \) collects all the terms involving cross-products.

The proof of the asymptotic normality of \( \frac{1}{n^k} \sum_{i_0,j_0,\ldots,i_{k-1},j_{k-1}} Z_{i_0,j_0} \cdots Z_{i_0,j_{k-1}} \) is the same as the proof we have just finished for the null distribution. We shall prove later that \( \mu_k \) satisfies (13). Now we focus on \( V_{n,k} \). Observe that \( \mathbb{E}[V_{n,k} \mid \Theta, U] = 0 \) and hence \( \mathbb{E}[V_{n,k}] = 0 \).

So our goal is to prove \( \mathbb{E}[V_{n,k}^2] \rightarrow 0 \) which implies \( V_{n,k} \rightarrow 0 \).

Note that \( V_{n,k} = \sum_{w} V_{n,k,w} \) where the summation is over all closed bi-words of length \( 2k + 1 \). Fix such a bi-word \( w \) and let \( \emptyset \subseteq E_f \subseteq E_w \) be any subset. Then

\[
V_{n,k,w} = \frac{1}{n^k} \sum_{\emptyset \subseteq E_f \subseteq E_w} \mu(E_f, w) \prod_{e \in E_w \setminus E_f} Z_e.
\]

(34)

Here

\[
\mu(E_f, w) = \prod_{e \in E_f} M_{\alpha_e, \beta_e},
\]

where for any edge \( e \), \( \alpha_e \) and \( \beta_e \) denote its two end points which belong to \( S_1 \) and \( S_2 \) respectively. Now

\[
\mathbb{E}[V_{n,k}^2 \mid \Theta, U] = \sum_{w_1, w_2} \mathbb{E}[V_{n,k,w_1} V_{n,k,w_2} \mid \Theta, U].
\]

(34)
We now give an upper bound to $\mathbb{E}[V_{n,k,w_1}V_{n,k,w_2}]$. At first fix any word $w_1$ and the set $\emptyset \subseteq E_f \subseteq E_{w_1}$ and consider all the words $w_2$ such that $E_{w_1} \cap E_{w_2} = E_{w_1 \setminus E_f}$. As every edge in $G_{w_1}$ and $G_{w_2}$ appear exactly once within $G_{w_1}$ and $G_{w_2}$,

$$\mathbb{E}[V_{n,k,w_1}V_{n,k,w_2} | \Theta, U] = \sum_{E_{w_1} \setminus E' \subseteq E_{w_1 \setminus E_f}} \left( \frac{1}{n} \right)^{2k} \left( \mu(E', w_1) \mu(E_{w_2} \setminus (E_{w_1 \setminus E_f}), w_2) \right) \mathbb{E} \prod_{e \in E' \setminus E'} (Z_e)^2 \quad (35)$$

$$= \sum_{E_{w_1} \setminus E' \subseteq E_{w_1 \setminus E_f}} \left( \frac{1}{n} \right)^{2k} \left( \mu(E', w_1) \mu(E_{w_2} \setminus (E_{w_1 \setminus E_f}), w_2) \right).$$

Now it is enough to prove

$$\mathbb{E} \left[ \left( \frac{1}{n} \right)^{2k} \sum_{w_1} \sum_{\emptyset \subseteq E_f \subseteq E_{w_1}} \sum_{E_f \subset E'} \sum_{\{w_2|E_{w_1} \cap E_{w_2} = E_{w_1 \setminus E_f}\}} \mathbb{E} \left[ \mu(E', w_1) \mu(E_{w_2} \setminus (E_{w_1 \setminus E_f}), w_2) \right] \right] \leq \left( \frac{1}{p} \right)^{\#E} \prod_{\emptyset \subseteq E_f \subseteq E_{w_1}} \mathbb{E} \left[ \sum_{l=1}^{\kappa} \Theta_{\alpha_{e,l}} U_{\beta_{e,l}} \right] \leq \left( \frac{1}{p} \right)^{\#E} \prod_{\emptyset \subseteq E_f \subseteq E_{w_1}} \mathbb{E} \left[ \sum_{l=1}^{\kappa} \Theta_{\alpha_{e,l}} U_{\beta_{e,l}} \right] \quad (36)$$

Now observe that for any $w$ in consideration and any subset $E$ of $E_w$,

$$|\mu(E, w)| = \left( \frac{1}{p} \right)^{\#E} \prod_{e \in E} \mathbb{E} \left[ \sum_{l=1}^{\kappa} \Theta_{\alpha_{e,l}} U_{\beta_{e,l}} \right].$$

Hence we have for any $E \subseteq E_{w_1}$ and $\bar{E} \subseteq E_{w_2}$ such that $\#E = \#\bar{E}$,

$$\mathbb{E} \left[ \mu(E, w_1) \mu(\bar{E}, w_2) \right] \leq \left( \frac{1}{p} \right)^{\#E} \prod_{e \in E} \mathbb{E} \left[ \sum_{l=1}^{\kappa} \Theta_{\alpha_{e,l}} U_{\beta_{e,l}} \right] \leq \left( \frac{1}{p} \right)^{\#E} (C_5 \#E)^{C_8 \#E} \leq \left( \frac{1}{p} \right)^{\#E} (C_7k)^{C_{8k}}. \quad (37)$$

The last step follows from the fact no matter what the value of $e$ is, $\sum_{l=1}^{\kappa} \Theta_{\alpha_{e,l}} U_{\beta_{e,l}}$ is sub-exponential with parameter $C$ for some constant $C$ that depends on $\kappa$, $\sum_{l=1}^{\kappa}$ and $\sum_{e \in E}$. Plugging the estimate obtained in (37) in (36), we have

$$\mathbb{E} \left[ \left( \frac{1}{n} \right)^{2k} \sum_{w_1} \sum_{\emptyset \subseteq E_f \subseteq E_{w_1}} \sum_{E_f \subset E'} \sum_{\{w_2|E_{w_1} \cap E_{w_2} = E_{w_1 \setminus E_f}\}} \mathbb{E} \left[ \mu(E', w_1) \mu(E_{w_2} \setminus (E_{w_1 \setminus E_f}), w_2) \right] \right] \leq \left( \frac{1}{n} \right)^{2k} \sum_{w_1} \sum_{\emptyset \subseteq E_f \subseteq E_{w_1}} \sum_{E_f \subset E'} \sum_{\{w_2|E_{w_1} \cap E_{w_2} = E_{w_1 \setminus E_f}\}} \left( \frac{1}{p} \right)^{\#E'} (C_7k)^{C_{8k}} \quad (38)$$

$$\leq \left( \frac{1}{n} \right)^{2k} (C_7k)^{C_{8k}} \sum_{w_1} \sum_{\emptyset \subseteq E_f \subseteq E_{w_1}} \left( \frac{1}{p} \right)^{\#E_f} \sum_{E_f \subset E'} \sum_{\{w_2|E_{w_1} \cap E_{w_2} = E_{w_1 \setminus E_f}\}} \sum_{\{w_2|E_{w_1} \cap E_{w_2} = E_{w_1 \setminus E_f}\}} 1.$$
Observe that the graph corresponding to the edges $E_{w_1} \setminus E_f$ is a disjoint collection of line segments. Let the number of such line segments be $\zeta$. Obviously $\zeta \leq \#(E_{w_1} \setminus E_f)$. The number of ways these $\zeta$ components can be placed in $w_2$ is bounded by $(2k)^\zeta \leq (2k)^{\#(E_{w_1} \setminus E_f)} \leq (2k)^{2k}$ and all other nodes in $w_2$ can be chosen freely. So there are at most $(1+o(1))[(\gamma \lor 1) n]^{2k - \#V_{E_w \setminus E_f}} (2k)^{2k}$ choices of such $w_2$. Here $V_{E_w \setminus E_f}$ is the set of vertices of the graph corresponding to $G_w$ with all edges in $E_f$ removed, i.e., $E_w \setminus E_f$. Observe that, whenever $2k = E_w > \#E_f > 0$, $E_w \setminus E_f$ is a forest and so $\#V_{E_w \setminus E_f} \geq \#(E_{w} \setminus E_f) + 1$ which is equivalent to

$$2k - \#V_{E_w \setminus E_f} \leq \#E_f - 1.$$ 

Also observe that there are no more than $2^{2k}$ many choices of $E_f$'s and for each $E_f$ there are no more than $2^{2k}$ many choices for $E_f$. Combining all these, we have the rightmost side of (38) is bounded by

$$\left(\frac{1}{n}\right)^{2k} (C_7 k)^{C_k \sum_{u_1} \sum_{\emptyset \subseteq E_f \subseteq E_w} \left(\frac{1}{p}\right)^{\#E_f} (2)^{2k} \times (2k)^{2k} [(\gamma \lor 1) n]^{\#E_f - 1} \right) \leq \frac{1}{p} (C_7 k)^{C_k (2k)^{2k} 2^{4k}} \gamma^{2k} \rightarrow 0. \quad (39)$$

Now our final task is to prove $\mu_n, k \xrightarrow{p} \mu_k$ defined in (13). First we expand $\mu_{n, k}$ in (33) as

$$\mu_{n, k} = \frac{1}{n^k} \frac{1}{p^k} \sum_{i_0, j_0, \ldots, k_{k-1}} \sum_{l_1, \ldots, l_{2k}} \Theta_{i_0, l_1} U_{j_0, l_1} \ldots \Theta_{i_0, l_{2k}} U_{j_{k-1}, l_{2k}}$$

$$= \sum_{l_1, \ldots, l_{2k}} \left(\frac{1}{n^k} \sum_{i_0, \ldots, i_{k-1}} \Theta_{i_0, l_1} \Theta_{i_0, l_{2k}} \Theta_{i_1, l_3} \ldots \Theta_{i_{k-1}, l_{2k-2}} \Theta_{i_{k-1}, l_{2k-1}} \right) \times \left(\frac{1}{p^k} \sum_{j_0, \ldots, j_{k-1}} U_{j_0, l_1} U_{j_0, l_{2k}} U_{j_1, l_3} U_{j_1, l_4} \ldots U_{j_{k-1}, l_{2k-1}} U_{j_{k-1}, l_{2k}} \right). \quad (40)$$

Now fix the values of $l_1, \ldots, l_{2k}$ and for this value of the group assignment we have

$$E \left[ \frac{1}{n^k} \sum_{i_0, \ldots, i_{k-1}} \Theta_{i_0, l_1} \Theta_{i_0, l_{2k}} \Theta_{i_1, l_3} \ldots \Theta_{i_{k-1}, l_{2k-2}} \Theta_{i_{k-1}, l_{2k-1}} \right] = m^{\Theta}_{l_1, \ldots, l_{2k}} = \mO(l_1, l_{2k}) \ldots \mO(l_{2k-2}, l_{2k-1}).$$

Now

$$\text{Var} \left[ \frac{1}{n^k} \sum_{i_0, \ldots, i_{k-1}} \Theta_{i_0, l_1} \Theta_{i_0, l_{2k}} \Theta_{i_1, l_3} \ldots \Theta_{i_{k-1}, l_{2k-2}} \Theta_{i_{k-1}, l_{2k-1}} \right] = \frac{1}{n^{2k}} \sum_{i_0^{(1)}, \ldots, i_{k-1}^{(1)}, i_0^{(2)}, \ldots, i_{k-1}^{(2)}} \sum_{l_1^{(1)}, \ldots, l_{2k}^{(1)}, l_1^{(2)}, \ldots, l_{2k}^{(2)}} \text{Var} \left[ \Theta_{i_0^{(1)}, l_1^{(1)}} \Theta_{i_0^{(1)}, l_{2k}^{(1)}} \Theta_{i_1^{(1)}, l_3^{(1)}} \ldots \Theta_{i_{k-1}^{(1)}, l_{2k-2}^{(1)}} \Theta_{i_{k-1}^{(1)}, l_{2k-1}^{(1)}} - m^{\Theta}_{l_1^{(1)}, \ldots, l_{2k}^{(1)}} \right] \times \left(\frac{1}{n^k} \sum_{i_0, \ldots, i_{k-1}} \Theta_{i_0, l_1} \Theta_{i_0, l_{2k}} \Theta_{i_1, l_3} \ldots \Theta_{i_{k-1}, l_{2k-2}} \Theta_{i_{k-1}, l_{2k-1}} - m^{\Theta}_{l_1, \ldots, l_{2k}} \right). \quad (41)$$
However, if the indices \((i_0^{(1)}, \ldots, i_{k-1}^{(1)})\) and \((i_0^{(2)}, \ldots, i_{k-1}^{(2)})\) are disjoint,

\[
\mathbb{E} \left[ \left( \Theta_{i_0^{(1)}, i_1^{(1)}, \ldots, i_{2k-2}^{(1)}}^{(1)} - m_{i_1^{(1)}, \ldots, i_{2k-1}^{(1)}}^{(1)} \right) \times \left( \Theta_{i_0^{(2)}, i_1^{(2)}, \ldots, i_{2k-2}^{(2)}}^{(2)} - m_{i_1^{(2)}, \ldots, i_{2k-1}^{(2)}}^{(2)} \right) \right] = 0.
\]

Now consider the indices

\[
A := \{(i_0^{(1)}, \ldots, i_{k-1}^{(1)}), (i_0^{(2)}, \ldots, i_{k-1}^{(2)}) \mid \#(\{i_0^{(1)}, \ldots, i_{k-1}^{(1)}\} \cap \{i_0^{(2)}, \ldots, i_{k-1}^{(2)}\}) \geq 1 \}.
\]

It is easy to see \(\#A \leq (c_1 k)^{c_2 k^2} n^{2k-1}\). Further from sub-Gaussianity and Hölder’s inequality we also have

\[
\mathbb{E} \left[ \left| \left( \Theta_{i_0^{(1)}, i_1^{(1)}, \ldots, i_{2k-2}^{(1)}}^{(1)} - m_{i_1^{(1)}, \ldots, i_{2k-1}^{(1)}}^{(1)} \right) \times \left( \Theta_{i_0^{(2)}, i_1^{(2)}, \ldots, i_{2k-2}^{(2)}}^{(2)} - m_{i_1^{(2)}, \ldots, i_{2k-1}^{(2)}}^{(2)} \right) \right| \right] = (c_3 k)^{c_4 k}
\]

uniformly over the indices. This gives us the final expression of (41) to be bounded by \(\frac{(c_1 c_3 k)^{c_2 + c_4 k}}{n} \to 0\). The proof for

\[
\frac{1}{p^k} \sum_{j_0, \ldots, j_{k-1}} U_{j_0, l_1} U_{j_0, l_2} U_{j_1, l_3} U_{j_1, l_4} \cdots U_{j_{k-1}, l_{2k-1}} U_{j_{k-1}, l_{2k}} \overset{p}{\to} \sum_{U(l_1, l_2)} \sum_{U(l_3, l_4)} \cdots \sum_{U(l_{2k-1}, l_{2k})}
\]

is analogous and so we omit the details. \(\square\)

## 4 Proof of main results

In this section, we focus on the proof of Theorem 1. The proof of Theorem 2 can be established analogously using the same strategy mentioned at the beginning of Section 3.2.

Throughout the proof, without further specification, all probability and expectation calculations are conducted with respect to \(\mathbb{P}_{p,n}\), i.e., under the null hypothesis. For any two matrices \(A = (a_{i,j}) \in \mathbb{R}^{m_1 \times m_2}\) and \(B = (b_{i,j}) \in \mathbb{R}^{n_1 \times n_2}\), we define their Kronecker product \(A \otimes B\) as

\[
A \otimes B = \begin{pmatrix}
a_{1,1}B & a_{1,2}B & \cdots & a_{1,m_2}B \\
a_{2,1}B & a_{2,2}B & \cdots & a_{2,m_2}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m_1,1}B & a_{m_1,2}B & \cdots & a_{m_1,m_2}B
\end{pmatrix}.
\]

In addition, \(\text{vec}(A) = (A_{1}^t, \ldots, A_{m_2}^t)^t \in \mathbb{R}^{m_1 m_2 \times 1}\) is the vector obtained from stacking all column vectors of \(A\) in order.

### 4.1 Proof of parts 1 and 2

Recall that \(p = p_n\) is a sequence depending on \(n\). In this proof we shall use the following two sequences of \(\sigma\)-fields:

\[
\mathcal{G}_n = \sigma \left( \{(X_i)^n\}_{i=1}^n \right), \quad \mathcal{F}_n = \sigma \left( \{\Theta_{i^n}\}_{i=1}^n, \{U_{j^n}\}_{j=1}^p \right).
\]  \(\quad (42)\)

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It is straightforward to verify that
\[ L_n = \mathbb{E} [L_n^f | G_n] \] (43)
where the expectation is taken over \( \Theta \) and \( U \) and for \( M_{i,j} \) defined in (27),
\[ L_n^f := \exp \left\{ \sum_{i=1}^n \sum_{j=1}^p \left( X_{i,j} M_{i,j} - \frac{1}{2} M_{i,j}^2 \right) \right\} . \]

**Step 1.** We now consider any sequence of events \( \Omega_n \in \mathcal{F}_n \) such that \( \mathbb{P} [ \Omega_n^c ] \to 0 \) as \( n \to \infty \). An explicit description of the \( \Omega_n \)'s of our interest will be given in step 2. Now define
\[ \tilde{L}_n := \mathbb{E} [L_n^f 1_{\Omega_n} | G_n] . \]

In the rest of this step, we argue that it suffices to prove the desired results for \( \tilde{L}_n \). Since \( \tilde{L}_n \leq L_n \) almost surely under \( \mathbb{P}_{0,n} \), the measure \( \tilde{Q}_n \) on \( G_n \) defined as
\[ \tilde{Q}_n(A_n) = \frac{1}{\mathbb{P} [ \Omega_n]} \mathbb{E}_{\mathbb{P}_{0,n}} [\tilde{L}_n 1_{A_n}] , \quad \forall \ A_n \in G_n, \]
is a probability measure. By definition,
\[ 0 \leq \left| \mathbb{P}_{1,n}(A_n) - \tilde{Q}_n(A_n) \right| \]
\[ \leq \frac{1}{\mathbb{P} [ \Omega_n]} \mathbb{E}_{\mathbb{P}_{0,n}} [(L_n - \tilde{L}_n) 1_{A_n}] + \mathbb{P}_{1,n}(A_n) \frac{\mathbb{P}[\Omega_n^c]}{\mathbb{P}[\Omega_n]} \]
\[ \leq \frac{1}{\mathbb{P} [ \Omega_n]} \mathbb{E}_{\mathbb{P}_{0,n}} [L_n - \tilde{L}_n] + \frac{\mathbb{P}[\Omega_n^c]}{\mathbb{P}[\Omega_n]} = \frac{1}{\mathbb{P} [ \Omega_n]} \mathbb{E} [L_n^f 1_{\Omega_n}] + \frac{\mathbb{P}[\Omega_n^c]}{\mathbb{P}[\Omega_n]} \]
\[ = \frac{1}{\mathbb{P} [ \Omega_n]} \mathbb{E} [1_{\Omega_n} \mathbb{E} [L_n^f | \mathcal{F}_n]] + \frac{\mathbb{P}[\Omega_n^c]}{\mathbb{P}[\Omega_n]} = 2 \frac{\mathbb{P}[\Omega_n^c]}{\mathbb{P}[\Omega_n]} . \] (44)

In other words, the total variation distance between \( \mathbb{P}_{1,n} \) and \( \tilde{Q}_n \) converges to zero. As a consequence, for any fixed \( l \in \mathbb{N} \) and any \( 1 \leq k_1 < \cdots < k_l = o(\sqrt{\log(n)}) \), under \( \tilde{Q}_n \),
\[ \left( \frac{B_{n,k_1} - p 1_{k_1=1} - \mu_{k_1}}{\sqrt{2k_1^2}}, \ldots, \frac{B_{n,k_l} - \mu_{k_l}}{\sqrt{2k_l^2}} \right) \rightarrow_N (0, I_l). \]

Now if one can choose \( \Omega_n \) in such a way that
\[ \limsup_{n \to \infty} \mathbb{E}_{\mathbb{P}_{0,n}} \left[ \tilde{L}_n^2 \right] = \limsup_{n \to \infty} \mathbb{E}_{\mathbb{P}_{0,n}} \left[ \left( \frac{1}{\mathbb{P} [ \Omega_n]} \tilde{L}_n \right)^2 \right] \leq \exp \left\{ \sum_{k=1}^{\infty} \frac{\mu_k^2}{2k^2} \right\} , \] (45)
then one can use Proposition 1 to conclude that
\[ \frac{1}{\mathbb{P} [ \Omega_n]} \tilde{L}_n \mid \mathbb{P}_{0,n} \quad \rightarrow_D \exp \left\{ \sum_{k=1}^{\infty} \frac{2\mu_k Z_k - \mu_k^2}{4k^2} \right\} . \]
Hence, $\bar{L}_n \mid \mathbb{P}_{0,n}$ converges in distribution to the same limit. Then it remains to prove that

$$L_n - \bar{L}_n \mid \mathbb{P}_{0,n} \xrightarrow{P} 0.$$ 

Observe that $L_n \geq \bar{L}_n$ almost surely under $\mathbb{P}_{0,n}$. If the last display is not true, then there exist positive constants $c_1, c_2 > 0$ and a subsequence $n_k$ such that

$$\liminf_{n_k \to \infty} \mathbb{P}_{0,n_k} \left[ 1_{L_{n_k} - \bar{L}_{n_k} > c_1} \right] \geq c_2.$$

However

$$\liminf_{n_k \to \infty} \left[ \mathbb{P}_{1,n_k} \left( 1_{L_{n_k} - \bar{L}_{n_k} > c_1} \right) - \bar{Q}_{n_k} \left( 1_{L_{n_k} - \bar{L}_{n_k} > c_1} \right) \right] \geq \liminf_{n_k \to \infty} \left\{ c_1 \mathbb{P}_{0,n_k} \left( 1_{L_{n_k} - \bar{L}_{n_k} > c_1} \right) - \mathbb{P} \left[ \Omega_{n_k}^c \mid \bar{Q}_{n_k} \left( 1_{L_{n_k} - \bar{L}_{n_k} > c_1} \right) \right] \right\} \geq c_1 c_2. \tag{46}$$

This contradicts (44) since that bound is uniform over all $A_n \in \mathcal{G}_n$.

**Step 2.** Now we prove (45) by making appropriate choices of the $\Omega_n$’s. First observe that

$$E[\hat{L}^2_n] = E \left[ E \left[ L_n^2 \mid \mathcal{G}_n \right] \right]$$

$$= E \left[ E \left[ L_n^{(1)} L_n^{(2)} \mid \mathcal{G}_n \right] \right]$$

$$= E \left[ L_n^{(1)} L_n^{(2)} \mid \mathcal{G}_n \right]$$

$$= E \left[ 1_{\mathcal{G}_n} E \left[ L_n^{(1)} L_n^{(2)} \mid \mathcal{F}_n \right] \right]. \tag{47}$$

Here $L_n^{(1)}$ and $L_n^{(2)}$ are two independent copies of $L_n$ where the $X_i$’s are kept fixed but one takes two i.i.d. copies of the $\Theta$’s and $U$’s. This is feasible (only) under the null hypothesis when the $X_i$’s are independent of $\Theta$ and $U$. With slight abuse of notation, we use $\mathcal{F}_n$ to denote the $\sigma$-field generated by both copies. We call the corresponding random variables $\{\Theta^{(1)}, U^{(1)}\}$ and $\{\Theta^{(2)}, U^{(2)}\}$. Observe that

$$E \left[ L_n^{(1)} L_n^{(2)} \mid \mathcal{F}_n \right]$$

$$= \exp \left[ \sum_{i=1}^n \sum_{j=1}^p \left( \sum_{l=1}^\kappa \frac{1}{\sqrt{p}} \Theta_{i,l}^{(1)} U_{i,l}^{(1)} \right) \left( \sum_{l=1}^\kappa \frac{1}{\sqrt{p}} \Theta_{i,l}^{(2)} U_{i,l}^{(2)} \right) \right]$$

$$= \exp \left[ \sum_{l_1=1}^\kappa \sum_{l_2=1}^\kappa \frac{1}{p} \langle \Theta_{sl_1}^{(1)}, \Theta_{sl_2}^{(2)} \rangle \langle U_{sl_1}^{(1)}, U_{sl_2}^{(2)} \rangle \right]. \tag{48}$$

We denote $E[L_n^{(1)} L_n^{(2)} \mid \mathcal{F}_n] = \psi_n = \psi_n(\Theta^{(1)}, \Theta^{(2)}, U^{(1)}, U^{(2)})$ for conciseness.

Now define

$$\Omega_n^{(1)} := \left\{ \max_{1 \leq l_1, l_2 \leq \kappa} \left( \frac{1}{n} \langle \Theta_{sl_1}^{(1)}, \Theta_{sl_2}^{(2)} \rangle - \Sigma_{\Theta}(l_1, l_2) \right), \frac{1}{p} \langle U_{sl_1}^{(1)}, U_{sl_2}^{(2)} \rangle - \Sigma_{U}(l_1, l_2) \right\} \leq \delta \right\} \tag{49}$$
where $\delta_n \to 0$ and $\mathbb{P}((\Omega_n^{(1)})^c) \to 0$ as $n \to \infty$. Such a sequence of $\delta_n$ exists due to law of large numbers. Define $\Omega_n^{(2)}$ as an identical and independent copy of $\Omega_n^{(1)}$ that depends on $\Theta^{(2)}, U^{(2)}$. Conditioning on $\Theta^{(1)}, U^{(1)}$ and $U^{(2)}$, the exponent in (48) can be written as

$$
\sqrt{\frac{n}{p}} \langle Z, V \rangle
$$

where

$$
V = A \text{vec}(U^{(2)}) \in \mathbb{R}^{\kappa^2}
$$

for $A = \frac{1}{\sqrt{p}} I_\kappa \otimes (U^{(1)})' \in \mathbb{R}^{\kappa^2 \times \kappa p}$, \hspace{1cm} (51)

$$
Z = B \text{vec}(\Theta^{(2)}) \in \mathbb{R}^{\kappa^2}
$$

for $B = \frac{1}{\sqrt{n}} I_\kappa \otimes (\Theta^{(1)})' \in \mathbb{R}^{\kappa^2 \times \kappa n}$. \hspace{1cm} (52)

Our goal is to prove the random variables $\{\psi_n 1_{\Omega_n^{(1)}} 1_{\Omega_n^{(2)}}\}_{n \geq 1}$ are uniformly integrable. To this end, it suffices to show that $\mathbb{E}[\psi_n^{(1+\eta)} 1_{\Omega_n^{(1)}} 1_{\Omega_n^{(2)}}]$ is uniformly bounded for some $\eta > 0$. Now from assumption on the priors, we have for sufficiently large values of $n$,

$$
\mathbb{E} \left[ \psi_n^{(1+\eta)} 1_{\Omega_n^{(1)}} 1_{\Omega_n^{(2)}} \right] = \mathbb{E} \left[ \psi_n^{(1+\eta)} 1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} 1_{\tilde{\Omega}_n^{(2)}} \right]
$$

$$
\leq \mathbb{E} \left[ 1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} \exp \left( \frac{1}{2\gamma} (1 + 2\eta)^2 V'B\Theta B'V \right) \right].
$$

Here

$$
\tilde{\Omega}_n^{(2)} = \left\{ \max_{1 \leq i, j \leq \kappa} \left| \frac{1}{p} (U_{sl_i}^{(2)}, U_{sl_j}^{(2)}) - \Sigma_U(l_i, l_j) \right| \leq \delta_n \right\},
$$

$$
\tilde{\Omega}_n^{(2)} = \left\{ \max_{1 \leq i, j \leq \kappa} \left| \frac{1}{n} (\Theta_{sl_i}^{(2)}, \Theta_{sl_j}^{(2)}) - \Sigma_\Theta(l_i, l_j) \right| \leq \delta_n \right\}
$$

and

$$
D_\Theta = \tilde{\Sigma}_\Theta \otimes I_n \in \mathbb{R}^{\kappa^2 \times \kappa^2}.
$$

As a consequence, for $B$ defined in (52), we have

$$
BD_\Theta B' = \tilde{\Sigma}_\Theta \otimes \left[ \frac{1}{n} (\Theta^{(1)})' \Theta^{(1)} \right].
$$

Recall that for any matrix $A$, let $\|A\|_{\text{max}} = \max_{i,j} |A_{ij}|$ be the vector $\ell_\infty$-norm of $A$. On the event $\tilde{\Omega}_n^{(2)} \cap \Omega_n^{(1)}$, we have $\|BD_\Theta B' - \tilde{\Sigma}_\Theta \otimes \Sigma_\Theta\|_{\text{max}} < \|\tilde{\Sigma}_\Theta\|_{\text{max}} \delta_n$. Now we know that for any symmetric matrix $\Sigma$ of dimension $\kappa^2 \times \kappa^2$, $\|\Sigma\|_2 \leq \|\Sigma\|_F \leq \kappa^2 \|\Sigma\|_{\text{max}}$ where $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the spectral norm and Frobenius norm respectively. So

$$
1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} V'BD_\Theta B' V \leq 1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} V' \left( \tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \kappa^2 \|\Sigma_\Theta\|_{\text{max}} \delta_n I_n \right) V
$$

$$
\leq 1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} V' \left( \tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n I_n \right) V
$$
where $\delta'_n \to 0$ is a sequence depending only on $\kappa$, $\bar{\Sigma}_\theta$ and $\delta_n$. Therefore, we have

$$\mathbb{E} \left[1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} \exp \left(\frac{1}{2\gamma} (1 + 2\eta)^2 V'B D_\theta B'V \right) \right] \leq \mathbb{E} \left[1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} \exp \left(\frac{1}{2\gamma} (1 + 2\eta)^2 V'(\bar{\Sigma}_\theta \otimes \Sigma_\theta + \delta'_n I_{\kappa^2}) V \right) \right]$$

$$= \mathbb{E} \left[1_{\Omega_n^{(1)}} \mathbb{E} \left[1_{\tilde{\Omega}_n^{(2)}} \exp \left(\frac{1}{2\gamma} (1 + 2\eta)^2 V'(\bar{\Sigma}_\theta \otimes \Sigma_\theta + \delta'_n I_{\kappa^2}) V \right) \right] \right] \left[\Theta^{(1)}, U^{(1)} \right] \right] \right]$$

$$\leq \mathbb{E} \left[1_{\Omega_n^{(1)}} \exp \left(\frac{1}{2\gamma} (1 + 2\eta)^2 V'(\bar{\Sigma}_\theta \otimes \Sigma_\theta + \delta'_n I_{\kappa^2}) V \right) \right]$$

In Step 3 we prove that the sequence

$$\lim_{n \to \infty} \sup \mathbb{E} \left[1_{\Omega_n^{(1)}} \exp \left(\frac{1}{2\gamma} (1 + 2\eta)^2 V'(\bar{\Sigma}_\theta \otimes \Sigma_\theta + \delta'_n I_{\kappa^2}) V \right) \right] < \infty$$ for some $\eta > 0$. (57)

If we assume (57), the rest of the proof can be completed as follows. Observe that by central limit theorem

$$\psi_n \ 1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} \overset{d}{\to} \exp \left(\frac{1}{\sqrt{\gamma}} \sum_{i=1}^\kappa \sum_{l_1, l_2=1}^\kappa T_{l_1, l_2} Y_{l_1, l_2} \right)$$

where $\frac{1}{\sqrt{n}}(\theta_{l_1}^{(1)}, \theta_{l_2}^{(2)}) \overset{d}{\to} T_{l_1, l_2}$ and $\frac{1}{\sqrt{b}}(u_{l_1}^{(1)}, u_{l_2}^{(2)}) \overset{d}{\to} Y_{l_1, l_2}$. In addition, the collections $\{T_{l_1, l_2}\}$ and $\{Y_{l_1, l_2}\}$ are mutually independent. Furthermore, the random variables $T_{l_1, l_2}$ are jointly Gaussian with mean 0 and $\text{Cov}(T_{l_1, l_2}, T_{l_3, l_4}) = \Sigma_{\theta}(l_1, l_3) \Sigma_{\theta}(l_2, l_4)$ and analogous results hold for $\{Y_{l_1, l_2}\}$. Let $T = (T_{l_1, l_2})$ and $Y = (Y_{l_1, l_2})$ be $\kappa \times \kappa$ matrices. Then the foregoing discussion implies that $\text{vec}(T) \sim N_{\kappa^2}(0, \Sigma_{\theta} \otimes \Sigma_{\theta})$ and is independent of $\text{vec}(Y) \sim N_{\kappa^2}(0, \Sigma_U \otimes \Sigma_U)$. This, together with the uniform integrability of $\psi_n 1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}}$, implies that

$$\lim_{n \to \infty} \mathbb{E} \left[1_{\Omega_n^{(1)}} 1_{\tilde{\Omega}_n^{(2)}} \psi_n \right] = \mathbb{E} \left[\exp \left(\frac{1}{\sqrt{\gamma}} (\text{vec}(T), \text{vec}(Y)) \right) \right]$$

$$= \mathbb{E} \left[\exp \left(\frac{1}{2\gamma} \text{vec}(Y)'(\Sigma_{\theta} \otimes \Sigma_{\theta})\text{vec}(Y) \right) \right]$$

$$= \exp \left\{\frac{1}{2} \sum_{i=1}^\kappa \log \left(1 - \frac{\lambda_i}{\gamma} \right) \right\}$$

$$= \exp \left\{\sum_{k=1}^\infty \frac{\sum_{i=1}^\kappa \lambda_i^k}{2k\gamma^k} \right\}$$

Here $\lambda_{1\leq i \leq \kappa^2}$ are the eigenvalues of the matrix $(\Sigma_U \otimes \Sigma_U)^{1/2}(\Sigma_{\theta} \otimes \Sigma_{\theta})(\Sigma_U \otimes \Sigma_U)^{1/2}$. We complete the proof by noting that $\text{Tr}((\Sigma_{\theta} \Sigma_U)^k \otimes (\Sigma_{\theta} \Sigma_U)^k) = [\text{Tr}((\Sigma_{\theta} \Sigma_U)^k)]^2 = \mu_k^2$. 21
Step 3. In the final step of the proof, we verify (57). Recall (51) to observe that
\[
\exp\left(\frac{1}{2\gamma}(1 + 2\eta)^2V'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)V\right)
= \exp\left(\frac{1}{2\gamma}(1 + 2\eta)^2\text{vec}(U^{(2)})'A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)\text{vec}(U^{(2)})\right).
\]

Now write
\[\tilde{U}^{(2)} = D_{U}^{-1/2}U^{(2)}\]
where
\[D_{U} = \tilde{\Sigma}_U \otimes I_p \in \mathbb{R}^{\kappa p \times \kappa p}.\]

So we have
\[
\exp\left(\frac{1}{2\gamma}(1 + 2\eta)^2\left(\text{vec}(U^{(2)})'A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)\text{vec}(U^{(2)})\right)\right)
= \exp\left(\frac{1}{2\gamma}(1 + \eta)^2\text{vec}((\tilde{U}^{(2)})'D_{U}^{1/2}A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)AD_{U}^{1/2}\text{vec}(\tilde{U}^{(2)})\right)\).
\]

Theorem 1 from [9] implies for any non-random non-negative definite \(\tilde{\Sigma}\) and all \(t > 0\),
\[
P\left(\left|\text{vec}(\tilde{U}^{(2)})\right| > \text{Tr}(\Sigma) + \sqrt{\text{Tr}(\Sigma^2) t + 2\|\Sigma\|_{2} t}\right) \leq e^{-t}. \tag{62}
\]

In particular, the tail bound in (62) only depends on the nonzero eigenvalues of \(\tilde{\Sigma}\). Now the nonzero eigenvalues of
\[D_{U}^{1/2}A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)AD_{U}^{1/2}\]
are the same as those of
\[AD_{U}A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right).
\]

However on \(\Omega_{n}^{(1)}\), we have \(AD_{U}A' = \tilde{\Sigma}_U \otimes \Sigma_U + P\) where \(P\) is a perturbation matrix with \(\|P\|_{\text{max}} = O(\delta_n)\). As a consequence, Theorem 5.5.4 of [20] implies that the nonzero eigenvalues of \(AD_{U}A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)\) are the eigenvalues of
\[\tilde{\Sigma}_U \otimes \Sigma_U)(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta) + O(\delta_n).\]

Here the constant in the \(O(\delta_n)\) term depends on the eigenvalues of \((\tilde{\Sigma}_U \otimes \Sigma_U)(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta)\) and \(\gamma\), but not on \(n\) and \(p\).

For convenience, we define \(\tilde{\Sigma} := D_{U}^{1/2}A'\left(\tilde{\Sigma}_\Theta \otimes \Sigma_\Theta + \delta_n' I_{\kappa^2}\right)AD_{U}^{1/2}\). On \(\Omega_{n}^{(1)}\), \(\text{Tr}(\tilde{\Sigma})\) and \(\text{Tr}(\tilde{\Sigma}^2)\) are uniformly bounded. So given any \(\epsilon > 0\), there exists a sufficiently large \(t_0 > 0\) that is independent of \(n\) such that for all \(t \geq t_0\),
\[
\frac{\text{Tr}(\tilde{\Sigma}) + \sqrt{\text{Tr}(\tilde{\Sigma}^2) t}}{t} \leq \epsilon.
\]
So for all \( t > t_0 \) we have
\[
1_{\Omega_n^{(1)}} \mathbb{P} \left[ \frac{1}{\gamma} \text{vec}(\tilde{U}^{(2)})' \tilde{\Sigma} \text{vec}(\tilde{U}^{(2)}) > \left( \frac{2\|\tilde{\Sigma}\|_2}{\gamma} + \epsilon \right) t \left| \Theta^{(1)}, U^{(1)} \right| \right] \leq e^{-t},
\]
and hence
\[
1_{\Omega_n^{(1)}} \mathbb{P} \left[ \exp \left( \frac{1}{2\gamma} (1 + 2\eta)^2 \text{vec}(\tilde{U}^{(2)})' \tilde{\Sigma} \text{vec}(\tilde{U}^{(2)}) \right) > t \left| \Theta^{(1)}, U^{(1)} \right| \right] \leq \left( \frac{1}{t} \right)^{\frac{2}{(1 + 2\eta)^2\|\tilde{\Sigma}\|_2/\gamma + \epsilon}}
\]
Since \( \|\tilde{\Sigma} U \otimes \Sigma_U \tilde{\Sigma} \otimes \Sigma_U \Sigma_V \|_2 < 2 \), we can choose \( \epsilon \) and \( \eta \) small enough such that on \( \Omega_n^{(1)} \),
\[
\frac{2}{(1 + 2\eta)^2(2\|\tilde{\Sigma}\|_2/\gamma + \epsilon)} \geq \alpha_0 > 1.
\]
Hence we have the last expression in (63) is bounded from above by \( t^{-\alpha_0} \). As a consequence,
\[
E \left[ 1_{\Omega_n^{(1)}} \exp \left( \frac{1}{2\gamma} (1 + 2\eta)^2 \text{vec}(\tilde{U}^{(2)})' \tilde{\Sigma} \text{vec}(\tilde{U}^{(2)}) \right) \right]
\]
is uniformly bounded. This completes the proof.

### 4.2 Proof of part 3

We have from the proof of Proposition 1 that for any given \( \epsilon, \delta > 0 \) there exists \( K = K(\epsilon, \delta) \) and for any subsequence \( n_l \) there exists a further subsequence \( n_{l_q} \) such that
\[
\mathbb{P}_{n_{l_q}} \left[ \left| \log(L_{n_{l_q}}) - \sum_{k=1}^{K} \frac{2\mu_k(B_{n_{l_q},k} - p1_{k=1}) - \mu_k^2}{2\sigma_k^2} \right| \geq \frac{\epsilon}{2} \right] \leq \delta / 2. \quad (64)
\]
Now choose \( K' \geq K \) such that
\[
\sum_{k=K'+1}^{\infty} \frac{\mu_k^2}{\sigma_k^2} \leq \max_{\left\{ \epsilon \right\} \leq \frac{\delta}{100}, \frac{\epsilon}{100}}.
\]
Now observe that for any \( k_1 < k_2 < m_n = o(\sqrt{\log n}) \), \( \mathbb{E}_{\tilde{\Sigma}} \{ B_{n,k_1} \} = 0 \), \( \text{Cov}(B_{n,k_1}, B_{n,k_2}) = 0 \) and \( \text{Var}(B_{n,k_i}) = 2k_1 \gamma^{k_i} \left( 1 + O \left( \frac{k_i^2}{n} \right) \right) \) for \( i \in \{1, 2\} \). So
\[
\text{Var} \left[ \sum_{k=K'+1}^{m_{l_q}} \frac{2\mu_k B_{n_{l_q},k} - \mu_k^2}{2\sigma_k^2} \right] = \left( 1 + O \left( \frac{m_{n_{l_q}}}{n_{l_q}} \right) \right) \sum_{k=K'+1}^{m_{l_q}} \frac{\mu_k^2}{\sigma_k^2} \leq \frac{\delta \epsilon^2}{100},
\]
Now for large value of \( n_{l_q} \),
\[
\mathbb{P}_{n_{l_q}} \left[ \left| \sum_{k=K'+1}^{m_{l_q}} \frac{2\mu_k B_{n_{l_q},k}}{\sigma_k^2} \right| \geq \frac{\epsilon}{4} \right] \leq \frac{16\delta \epsilon^2}{100\epsilon^2}, \quad \text{and so}
\]
\[
\mathbb{P}_{n_{l_q}} \left[ \left| \sum_{k=K'+1}^{m_{l_q}} \frac{2\mu_k (B_{n_{l_q},k} - \mu_k^2)}{2\sigma_k^2} \right| \geq \frac{\epsilon}{4} + \frac{\epsilon}{100} \right] \leq \frac{16\delta \epsilon^2}{100\epsilon^2}.
\]

Plugging in the estimates of (64) and (65) we have for all large values of \( n \),

\[
\mathbb{P}_{n_{tq}} \left[ \log(L_{n_{tq}}) - \sum_{k=1}^{m_{n_{tq}}} \frac{2\mu_k(B_{n_{tq},k} - p1_{k=1}) - \mu_k^2}{2\sigma_k^2} \right] \geq \varepsilon \leq \delta. \tag{66}
\]

Since (66) occurs to any subsequence and any \((\varepsilon, \delta)\) pair, this completes the proof. \( \square \)

A Appendix: Proof of Proposition 1

At first we introduce the concept of Wasserstein’s metric which will be used in the proof of Proposition 1. Let \( F \) and \( G \) be two distribution functions with finite \( p \)-th moment. Then the Wasserstein distance \( W_p \) between \( F \) and \( G \) is defined to be

\[
W_p(F,G) = \inf_{X \sim F, Y \sim G} \mathbb{E}|X - Y|^p \right]^{1/p}.
\]

Here \( X \) and \( Y \) are random variables having distribution functions \( F \) and \( G \) respectively. The following result will be useful in our proof. See, for instance, Mallows [15] for its proof.

Proposition 3. Suppose \( F_n \) be a sequence of distribution functions and \( F \) be a distribution function. Then \( F_n \overset{d}{\to} F \) in distribution and \( \int x^2 dF_n(x) \to \int x^2 dF(x) \) if \( W_2(F_n,F) \to 0 \).

Proof of Proposition 1. We now prove the proposition.

Proof of mutual contiguity and (5): This proof is broken into two steps. We focus on proving (5). Given (5), mutual contiguity is a direct consequence of Le Cam’s first lemma [13].

Step 1. We first prove the random variable on the righthand side of (5) is almost surely positive and has mean 1. Let us define

\[
L := \exp \left\{ \sum_{i=1}^{\infty} \frac{2\mu_i Z_i - \mu_i^2}{2\sigma_i^2} \right\}, \quad L^{(m)} := \exp \left\{ \sum_{i=1}^{m} \frac{2\mu_i Z_i - \mu_i^2}{2\sigma_i^2} \right\}, \quad \forall m \in \mathbb{N}.
\]

As \( Z_i \sim N(0,\sigma_i^2) \), for any \( i \in \mathbb{N} \), and so

\[
\mathbb{E} \left[ \frac{2\mu_i Z_i - \mu_i^2}{2\sigma_i^2} \right] = 1.
\]

So \( \{L^{(m)}\}_{m=1}^{\infty} \) is a martingale sequence and

\[
\mathbb{E} \left[ \left( L^{(m)} \right)^2 \right] = \prod_{i=1}^{m} \exp \left\{ \frac{\mu_i^2}{\sigma_i^2} \right\} = \exp \left\{ \sum_{i=1}^{m} \frac{\mu_i^2}{\sigma_i^2} \right\}.
\]

Now by the righthand side of (4), \( L^{(m)} \) is a \( L^2 \) bounded martingale. Hence, \( L \) is a well defined random variable with

\[
\mathbb{E}[L] = 1, \quad \mathbb{E}[L^2] = \exp \left\{ \sum_{i=1}^{\infty} \frac{\mu_i^2}{\sigma_i^2} \right\}.
\]
On the other hand log(\(L\)) is a limit of Gaussian random variables, hence log(\(L\)) is Gaussian with
\[
\mathbb{E}[\log(L)] = -\frac{1}{2} \sum_{i=1}^{\infty} \frac{\mu_i^2}{\sigma_i^2}, \quad \text{Var}[\log(L)] = \sum_{i=1}^{\infty} \frac{\mu_i^2}{\sigma_i^2}.
\]
Hence \(\mathbb{P}[L = 0] = \mathbb{P}[\log(L) = -\infty] = 0\).

**Step 2.** Now we prove \(Y_n \xrightarrow{d} L\). Since
\[
\limsup_{n \to \infty} \mathbb{E}_{P_n} [Y_n^2] < \infty,
\]
condition (iv) implies that the sequence \(Y_n\) is tight. Prokhorov’s theorem further implies that there is a subsequence \(\{n_k\}\) such that \(Y_{n_k}\) converge in distribution to some random variable \(L(\{n_k\})\). In what follows, we prove that the distribution of \(L(\{n_k\})\) does not depend on the subsequence \(\{n_k\}\). In particular, \(L(\{n_k\}) \xrightarrow{d} L\). To start with, note that since \(Y_{n_k}\) converges in distribution to \(L(\{n_k\})\), for any further subsequence \(\{n_{k_l}\}\) of \(\{n_k\}\), \(Y_{n_{k_l}}\) also converges in distribution to \(L(\{n_k\})\).

Given any fixed \(\epsilon > 0\) take \(m\) large enough such that
\[
\exp \left(\sum_{i=1}^{\infty} \frac{\mu_i^2}{\sigma_i^2}\right) - \exp \left(\sum_{i=1}^{m} \frac{\mu_i^2}{\sigma_i^2}\right) < \epsilon.
\]
For this fixed number \(m\), consider the joint distribution of \((Y_{n_k}, W_{n_k,1}, \ldots, W_{n_k,m})\). This sequence of \(m + 1\) dimensional random vectors with respect to \(\mathbb{P}_{n_k}\) is tight by condition (ii). So it has a further subsequence such that
\[
(Y_{n_{k_l}}, W_{n_{k_l},1}, \ldots, W_{n_{k_l},m})|_{\mathbb{P}_{n_{k_l}}} \xrightarrow{d} (L(\{n_k\}), Z_3, \ldots, Z_m).
\]
We are to show that we can define the random variables \(L^{(m)}\) and \(L(\{n_k\})\) in such a way that there exist suitable \(\sigma\)-algebras \(\mathcal{F}_1 \subset \mathcal{F}_2\) such that \(L^{(m)} \in \mathcal{F}_1\), \(L(\{n_k\}) \in \mathcal{F}_2\), and \(\mathbb{E}[L(\{n_k\})| \mathcal{F}_1] = L^{(m)}\).

Since \(\limsup_{n \to \infty} \mathbb{E}_{P_n} [Y_n^2] < \infty\), the sequence \(Y_{n_{k_l}}\) is uniformly integrable. This, together with condition (i), leads to
\[
\mathbb{E}[L(\{n_k\})] = \lim_{l \to \infty} \mathbb{E}_{P_{n_{k_l}}}[Y_{n_{k_l}}] = 1. \tag{67}
\]
Now take any positive bounded continuous function \(f : \mathbb{R}^m \to \mathbb{R}\). By Fatou’s lemma
\[
\liminf_{l \to \infty} \mathbb{E}_{P_{n_{k_l}}} \left[ f(W_{n_{k_l},1}, \ldots, W_{n_{k_l},m} Y_{n_{k_l}}) \right] \geq \mathbb{E} \left[ f(Z_1, \ldots, Z_m) L(\{n_k\}) \right]. \tag{68}
\]
However for any constant \(\xi\), (67) implies \(\xi = \mathbb{E}_{P_{n_{k_l}}} [Y_{n_{k_l}}] \to \xi \mathbb{E}[L(\{n_k\})] = \xi\). So (68) holds for any bounded continuous function \(f\). On the other hand, replacing \(f\) by \(-f\) we have
\[
\lim_{l \to \infty} \mathbb{E}_{P_{n_{k_l}}} \left[ f(W_{n_{k_l},1}, \ldots, W_{n_{k_l},m} Y_{n_{k_l}}) \right] = \mathbb{E} \left[ f(Z_1, \ldots, Z_m) L(\{n_k\}) \right]. \tag{69}
\]
Now condition (ii) leads to
\[
\int f(W_{n_{k_l},1}, \ldots, W_{n_{k_l},m}) Y_{n_{k_l}} d\mathbb{P}_{n_{k_l}} = \int f(W_{n_{k_l},1}, \ldots, W_{n_{k_l},m}) d\mathbb{Q}_{n_{k_l}} \to \int f(Z_1', \ldots, Z_m') d\mathbb{Q}.
\]
Here $Q$ is the measure induced by $(Z'_1, \ldots, Z'_m)$. In particular, one can take the measure $Q$ such that $(Z_1, \ldots, Z_m)$ themselves are distributed as $(Z'_1, \ldots, Z'_m)$ under the measure $Q$. This is true since

$$\int f(Z'_1, \ldots, Z'_m) dQ = \mathbb{E}\left[ f(Z_1, \ldots, Z_m) L^{(m)} \right].$$

for any bounded continuous function $f$, and so $\int_A dQ = \mathbb{E}[1_A L^{(m)}]$ for any $A \in \sigma(Z_1, \ldots, Z_m)$. Now looking back into (69), we have for any $A \in \sigma(Z_1, \ldots, Z_m)$, $\mathbb{E}[1_A L^{(m)}] = \mathbb{E}[1_A L(\{n_k\})]$. Since $L^{(m)}$ is $\sigma(Z_1, \ldots, Z_m)$ measurable, we have

$$L^{(m)} = \mathbb{E}[L(\{n_k\}) | \sigma(Z_1, \ldots, Z_m)].$$

From Fatou’s lemma

$$\mathbb{E}[L(\{n_k\})^2] \leq \liminf_{n \to \infty} \mathbb{E}_{\mathbb{P}_n}[Y_n^2] = \exp \left\{ \sum_{i=1}^{\infty} \frac{\mu_i^2}{\sigma_i^2} \right\}.$$

As a consequence, we have

$$0 \leq \mathbb{E}[L(\{n_k\}) - L^{(m)}]^2 = \mathbb{E}[L(\{n_k\})]^2 - \mathbb{E}[L^{(m)}^2] < \varepsilon.$$

So $L_2(F^{L^{(m)}}, F^{L(\{n_k\})}) < \sqrt{\varepsilon}$. Here $F^{L^{(m)}}$ and $F^{L(\{n_k\})}$ denote the distribution functions corresponding to $L^{(m)}$ and $L(\{n_k\})$ respectively. As a consequence, $W_2(F^{L^{(m)}}, F^{L(\{n_k\})}) \to 0$ as $m \to \infty$. Hence by Proposition 3, $L^{(m)} \xrightarrow{d} L(\{n_k\})$. On the other hand, we have already proved $L^{(m)}$ converges to $L$ in $L^2$. So $L(\{n_k\}) \xrightarrow{d} L$.

**Proof of (6):** Consider any fixed pair of $(\varepsilon, \delta) \in (0, 1) \times (0, 1)$. First observe that the sequence $\log(Y_n)$ is tight from the proof of the previous part. For the given $\delta$, there exists a fixed number $M < \infty$ such that $\mathbb{P}_n [\bar{Z} = \log(Y_n) = E] \geq 1 - \frac{\varepsilon}{100} \delta$ for all $n$, implying $\mathbb{P}_n [e^{-M} \leq Y_n \leq e^{M}] \geq 1 - \frac{\varepsilon}{100} \delta$. Now consider $\tau \in (0, e^{-M})$. The function $\log(\cdot)$ is uniformly continuous on $[\tau, e^{M+1}]$. On this interval consider $\dot{\varepsilon}$ such that $|\log(x) - \log(y)| < \frac{\dot{\varepsilon}}{2}$ for all $x, y$ on this interval with $|x - y| < \dot{\varepsilon}$. Let $\varepsilon_1 = \min\{\dot{\varepsilon}, e^{-M} - \tau, e^{M+1} - e^{-M}\}$ and pick a sufficiently large $K \in \mathbb{N}$ such that

$$\exp \left\{ \sum_{k=1}^{\infty} \frac{\mu_k^2}{\sigma_k^2} \right\} - \exp \left\{ \sum_{k=1}^{K} \frac{\mu_k^2}{\sigma_k^2} \right\} \leq \frac{\delta \varepsilon_1^2}{100}. \tag{70}$$

From the proof of the previous part, we also know given any subsequence $n_l$ there exists a further subsequence $\{n_{l,m}\}$ so that under $\mathbb{P}_n$,

$$\left( Y_{n_{l,m}}, \exp \left\{ \sum_{k=1}^{K} \frac{2\mu_k (W_{n_{l,m}} - \mu_k^2)}{2\sigma_k^2} \right\} \right) \xrightarrow{d} \left( L, \exp \left\{ \sum_{k=1}^{K} \frac{2\mu_k (Z_k - \mu_k^2)}{2\sigma_k^2} \right\} \right)$$

and

$$\mathbb{E}\left[ \left( L - \exp \left\{ \sum_{k=1}^{K} \frac{2\mu_k (Z_k - \mu_k^2)}{2\sigma_k^2} \right\} \right)^2 \right] \leq \frac{\delta \varepsilon_1^2}{100}.$$
As a consequence,
\[
\limsup_{n_{lm} \to \infty} \mathbb{P}_{n_{lm}} \left[ \left| Y_{n_{lm}} - \exp \left\{ \sum_{k=1}^{K} \frac{2 \mu_k (W_{n_{lm}, k}) - \mu_k^2}{2 \sigma_k^2} \right\} \geq \frac{\epsilon_1}{2} \right| \right] \leq \mathbb{P} \left[ \left| (L - \exp \left\{ \sum_{k=1}^{K} \frac{2 \mu_k (Z_k) - \mu_k^2}{2 \sigma_k^2} \right\} \geq \frac{\epsilon_1}{2} \right| \right] \leq \frac{\delta}{25}.
\]  \tag{71}

As a consequence, for large values of \( n_{lm} \),
\[
\mathbb{P}_{n_{lm}} \left[ \left| Y_{n_{lm}} - \exp \left\{ \sum_{k=1}^{K} \frac{2 \mu_k (W_{n_{lm}, k}) - \mu_k^2}{2 \sigma_k^2} \right\} \geq \frac{\epsilon_1}{2} \right. \mbox{ and } Y_{n_{lm}} \notin [e^{-M}, e^M] \right] \leq \frac{\delta}{25} + \frac{\delta}{100} < \frac{\delta}{2}.
\]  \tag{72}

Therefore, for large values of \( n_{lm} \),
\[
\mathbb{P}_{n_{lm}} \left[ \left| \log(Y_{n_{lm}}) - \left\{ \sum_{k=1}^{K} \frac{2 \mu_k (W_{n_{lm}, k}) - \mu_k^2}{2 \sigma_k^2} \right\} \geq \frac{\epsilon}{2} \right| \right] \leq \frac{\delta}{2}.
\]

This completes the proof. \( \square \)

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