Finsler Geometry Modeling of an Orientation-Asymmetric Surface Model for Membranes

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Abstract. In this paper, a triangulated surface model is studied in the context of Finsler geometry (FG) modeling. This FG model is an extended version of a recently reported model for two-component membranes, and it is asymmetric under surface inversion. We show that the definition of the model is independent of how the Finsler length of a bond is defined. This leads us to understand that the canonical (or Euclidean) surface model is obtained from the FG model such that it is uniquely determined as a trivial model from the viewpoint of well definedness.

Keywords: Triangulated surface model, Non-Euclidean metric, Finsler geometry, Membranes, Surface inversion

1. Introduction
The Finsler geometry (FG) modeling technique expands the range of application of the surface model of Helfrich and Polyakov for membranes [1, 2]. Indeed, the so-called soft elasticity and elongation under temperature changes observed in liquid crystal elastomers are understood via the 3D FG modeling technique [3]. It has been shown that the J-shaped stress-strain curve of biological materials, such as human skin and muscle, can be calculated using a 2D FG model [4]. Moreover, the origin of the line tension energy, which plays an important role in the model for two-component membranes [5], is explained by another 2D FG model.

In Ref. [6], it was mentioned that this FG model is independent of how the Finsler length of a bond is defined. In this paper, we demonstrate that the FG model in Ref. [6] remains unchanged if the definition of the Finsler length of a bond is altered. The Finsler length of a bond is in general an essentially new and important ingredient in an FG model. In this paper, we discuss the details of this interesting property and its consequence.
2. Model
2.1. FG modeling
A surface model is defined by a mapping \( \mathbf{r} : M \ni (x_1, x_2) \mapsto \mathbf{r} \in \mathbb{R}^3 \), where \( M \) is a two-dimensional parameter space. For a discrete model, both \( M \) and \( \mathbf{r}(M) \) should be triangulated. We should note that the Finsler length of a bond refers to the edge length of triangles in \( M \). This edge length can be direction dependent in the FG modeling technique, in contrast with the standard modeling, where the bond length is always direction independent on the triangulated surfaces.

The discrete partition function and Hamiltonian are given by [6]

\[
Z(\zeta, \kappa) = \sum_{\chi} \sum_T \int_0^1 \prod_{i=1}^N d\mathbf{r}_i \exp \left[ -S(\mathbf{r}, \chi) \right], \quad S = S_1 + \kappa S_2 + \zeta S_3,
\]

\[
S_1 = \sum_{ij} \gamma_{ij} \ell_{ij}^2, \quad S_2 = \sum_{ij} \kappa_{ij} \left( 1 - \mathbf{n}^+ \cdot \mathbf{n}^- \right), \quad S_3 = \sum_{ij} \left( 1 - \chi^+ \cdot \chi^- \right), \quad \left( \chi^+ \in \{1,-1\} \right),
\]

\[
\gamma_{ij} = \frac{1}{4} \left( \gamma_{ij}^+ + \gamma_{ij}^- \right), \quad \kappa_{ij} = \frac{1}{4} \left( \kappa_{ij}^+ + \kappa_{ij}^- \right),
\]

\[
\gamma_{ij}^+ = \left\{ \begin{array}{ll}
\rho_i^+ + 1/\rho_j^+ & (\chi(\Delta^+) = 1) \\
1/\rho_i^+ + \rho_j^+ & (\chi(\Delta^+) = -1)
\end{array} \right., \quad \gamma_{ij}^- = \left\{ \begin{array}{ll}
\rho_j^- + 1/\rho_i^- & (\chi(\Delta^-) = 1) \\
1/\rho_j^- + \rho_i^- & (\chi(\Delta^-) = -1)
\end{array} \right.,
\]

\[
\kappa_{ij}^+ = \left\{ \begin{array}{ll}
\rho_i^+ + 1/\rho_j^+ & (\chi(\Delta^+) = 1) \\
1/\rho_i^+ + \rho_j^+ & (\chi(\Delta^+) = -1)
\end{array} \right., \quad \kappa_{ij}^- = \left\{ \begin{array}{ll}
\rho_j^- + 1/\rho_i^- & (\chi(\Delta^-) = 1) \\
1/\rho_j^- + \rho_i^- & (\chi(\Delta^-) = -1)
\end{array} \right.,
\]

where \( \mathbf{r}, T, \) and \( \chi \) denote the vertex position, the triangulation, and the variable corresponding to the orientation of triangle \( \Delta \), respectively (Fig. 1). \( S_1, S_2 \) and \( S_3 \) are the Gaussian bond potential, the bending energy, and the energy for the triangle orientation, respectively. The symbols \( \ell_{ij} \) and \( \mathbf{n}^\pm \) represent the length of bond \( ij \) and unit normal vectors of triangles + and −, respectively. The Finsler metric assumed here is

\[
g_{ab} = \begin{pmatrix} 1/\rho & 0 \\ 0 & \rho \end{pmatrix},
\]

where \( \rho \) is a function on triangle \( \Delta \).

In Ref. [6], the Finsler lengths of bonds 12 and 21 are defined by (see Fig. 1)

\[
L_{12}(\Delta_1^+) = (1/2) \left( 1/\rho_i^+ + \rho_j^+ \right), \quad L_{21}(\Delta_1^-) = (1/2) \left( 1/\rho_2^- + \rho_i^- \right),
\]

\[
L_{12}(\Delta_2^-) = (1/2) \left( 1/\rho_2^- + \rho_j^- \right), \quad L_{21}(\Delta_2^+) = (1/2) \left( 1/\rho_i^+ + \rho_j^+ \right).
\]

(3)

In contrast, in this paper, we define the lengths of bonds 12 and 21 as follows:

\[
L_{12}(\Delta_1^+, \Delta_1^-) = (1/2) \left( 1/\rho_i^+ + \rho_j^- \right), \quad L_{21}(\Delta_2^+, \Delta_2^-) = (1/2) \left( 1/\rho_2^- + 1/\rho_i^- \right),
\]

\[
\bar{L}_{12}(\Delta_1^+, \Delta_2^-) = (1/2) \left( \rho_i^+ + 1/\rho_2^- \right), \quad \bar{L}_{21}(\Delta_2^+, \Delta_1^-) = (1/2) \left( 1/\rho_2^- + \rho_j^- \right).
\]

(4)

The expressions in Eq. (4) are different from those in Eq. (3). In this paper, we show that the canonical (Euclidean) model is obtained from the model defined in Eq. (1) as a well-defined model even when the lengths of bonds are given by Eq. (4). We should note that there are two other expressions for \( L_{12} \) and \( L_{21} \) for the combination of two neighboring triangles \( (\Delta_1^+, \Delta_2^-) \) and \( (\Delta_2^+, \Delta_1^-) \). For the expressions of \( L_{12} \) and \( L_{21} \), we use the inverted triangles \( (\Delta_1^+, \Delta_2^-) \) and \( (\Delta_2^+, \Delta_1^-) \). However, the following discussions remain unchanged although the expressions for several conditions become dependent on the definition of the bond length. In this paper, we confine ourselves to the lengths described in Eq. (4).
2.2. Discussions

In this subsection, we follow the arguments in Ref. [6]. We call a discrete surface model trivial (non-trivial) if the following conditions are (not) satisfied:

\[ \gamma_{ij} = \text{constant}, \quad \kappa_{ij} = \text{constant}. \]  (5)

It is well-known that the trivial models are equivalent to the canonical (or Euclidean) model, which is given by \( \gamma_{ij} = \kappa_{ij} = 1 \). This is understood from the scale invariance of the partition function [7].

A discrete surface model is called well defined (ill defined) if the following conditions are (not) satisfied:

(D1) The bond length is independent of its direction.
(D2) The bond length is independent of the surface orientation.

Note that the bond length in (D1) and (D2) is the bond length of triangles in the parameter space \( M \). Thus, from the expressions in Eq. (4), we have the following equations:

\[ 1/\rho_1^+ + \rho_1^- = 1/\rho_2^+ + \rho_2^- \quad (\Leftrightarrow \text{D1}), \]
\[ 1/\rho_1^+ + \rho_1^- = 1/\rho_1^+ + \rho_1^-, \quad 1/\rho_2^+ + \rho_2^- = 1/\rho_2^+ + \rho_2^- \quad (\Leftrightarrow \text{D2}). \]  (6)

These equations are different from those in Ref. [6] because these come from Eq. (4). We also have the following condition for the model to be orientation symmetric:

\[ 1/\rho_1^- + \rho_2^+ + 1/\rho_2^- + \rho_1^+ = 1/\rho_2^- + \rho_1^- + 1/\rho_1^+ + \rho_2^+. \]  (7)

This condition comes from the Hamiltonian in Eq. (1) and the following definition:

(D3) A discrete surface model is called orientation symmetric if the Hamiltonian is orientation symmetric.

We should note that the condition in Eq. (8) is identical to the one given in Ref. [6]. This is because the Hamiltonian is independent of how the length of a bond (in \( M \)) is defined.

The condition in Eq. (8) is also proved independently of the Hamiltonian. We make the following remarks:

(R1) Any well-defined model is orientation symmetric.
For any well-defined model, we have
\[ \frac{1}{\rho_i} + \rho_i = a (= \text{const}). \]  \hfill (9)

If Eq. (9) is satisfied, then the model is trivial. \hfill (R3)

(R1) is a direct consequence of Eqs. (6) and (7). (R2) is proved as follows: multiplying \( \rho_i^+ \rho_i^- \) by the first equation in Eq. (7), we have \( \rho_i^2 = \rho_i^- \). We also have \( \rho_2^+ = \rho_2^- \) from the second equation in Eq. (7). Then, we obtain \( \frac{1}{\rho_i} + \rho_i = 1/\rho_2 + \rho_2 \) from Eq. (6). This equality is satisfied for all bonds; therefore, we have Eq. (9). \hfill (R3)

We obtain from (R2) and (R3) the following theorem:

(Th) All well-defined models are trivial.

This means that all non-trivial models are ill defined. However, this ill definedness turns out to be well defined in the context of FG modeling because the Finsler length depends on the direction in general, where the bond length is called the Finsler length of a bond. The Finsler length is allowed to be dependent on the surface orientation in the FG model of Eq. (1). This is the main result reported in Ref. [6].

3. Summary

We have shown that all well-defined models are trivial. This indicates that a well-defined model is uniquely determined because the well-defined model, which is trivial, is determined independently of how the Finsler length of a bond is defined. This uniqueness is also supported by the physically understood fact that all trivial models are equivalent to the Euclidean model, which is given by \( \gamma_{ij} = 1/4 \left( \gamma_{ij}^+ + \gamma_{ij}^- \right) = a/2 \) and \( \kappa_{ij} = 1/4 \left( \kappa_{ij}^+ + \kappa_{ij}^- \right) = a/2 \).

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