1. Introduction.

Let $S$ be an algebraic space, $A$ an $S$-abelian algebraic space, $X$ an $A$-torsor on $S$ for the étale topology and $L$ a finite étale $S$-equivalence relation on $X$. Thus, $X$ is representable by an $S$-algebraic space and its quotient, in the sense of étale topology, by the finite étale $S$-equivalence relation $L$, is representable by a proper and smooth $S$-algebraic space $Y$ with geometrically irreducible fibers. In the following we define the property for $L$ to be a basic finite étale equivalence relation and show that this property is remarkably rigid and our first goal is to prove:

**Theorem 1.1.** — Let $f : Y \to S$ be a proper smooth morphism of algebraic spaces with $S$ connected. Assume that for one geometric point $t$ of $S$ there is a finite étale $t$-morphism from a $t$-abelian variety $A'_t$ onto $Y_t = f^{-1}(t)$ with degree $[A'_t : Y_t]$ prime to the residue characteristics of $S$.

Then:

a) There exist an $S$-abelian algebraic space $A$, an $A$-torsor $X$ on $S$ for the étale topology and a basic finite étale $S$-equivalence relation $L$ on $X$ such that $Y$ is $S$-isomorphic to the quotient $X/L$.

b) Let $(A, X, L)$ be as in a). Then $X$ is a $f^*G$-torsor on $Y$ for the étale topology where the $S$-group $G = f_*\text{Aut}_Y(X)$ is a finite étale $S$-algebraic space. The functor

$$(A', X', L') \mapsto P' = f_*\text{Isom}_Y(X, X')$$

induces a bijection from the set of all triples as in a) up to isomorphisms onto $H^1(S, G)$.

The proof is in §9.

When $f$ degenerates to $\overline{f} : \overline{Y} \to \overline{S}$ where $\overline{f}$ is separated of finite type and universally open with $\overline{S}$ locally noetherian, then a geometric fiber $\overline{f}^{-1}(\overline{s})$, if it is non-empty and does not have uniruled irreducible components, is irreducible; see §10, which, as a continuation of §4 and our main objective, contains further support for the Principle: Non-uniruled abelian degenerations are almost non-degenerate.

2. Definition.
Definition 2.1. — Let $S$ be an algebraic space, $A$ an $S$-abelian algebraic space, $X$ an $A$-torsor on $S$ for the étale topology and $L$ a finite étale $S$-equivalence relation on $X$. We say that $L$ is basic at a geometric point $s$ of $S$ provided the following condition holds:

If $E$ is a finite étale $s$-subgroup of $A_s$ such that the morphism

$$E \times_s X_s \to X_s \times_s X_s, \ (e, x) \mapsto (e \cdot x, x)$$

factors through the graph of $L_s$, then $E = 0$.

We say that $L$ is basic if it is basic at every geometric point of $S$.

Thus, put in words, $L$ is basic at $s$ if and only if no sub-equivalence relation of $L(s)$ other than the identity is generated by a translation. Let $Y$ denote the quotient $X/L$. The graph of $L$ is then identical to $X \times_Y X$. With the above notations, saying that the morphism

$$E \times_s X_s \to X_s \times_s X_s, \ (e, x) \mapsto (e \cdot x, x)$$

factors through the graph of $L_s$ amounts to saying that the translations of $X_s$ defined by the elements of $E(s)$ are $Y_s$-automorphisms.

2.2. Let $s' \to s$ be a morphism of geometric points of $S$. The base change

$$L_s \mapsto L_s \times_s s'$$

establishes a bijection between the collection of finite étale $s$-equivalence relations on $X_s$ which are basic at $s$ and the collection of finite étale $s'$-equivalence relations on $X_{s'}$ which are basic at $s'$. Indeed, for every proper $s$-algebraic space $Z$, the functor of base change by $s' \to s$ is an equivalence from the category of finite étale $Z$-algebraic spaces onto the category of finite étale $Z_{s'}$-algebraic spaces (SGA 4 XII 5.4). This same base change provides also an equivalence between the category of finite étale $s$-subgroups of $A_s$ and the category of finite étale $s'$-subgroups of $A_{s'}$. In particular, a given finite étale $S$-equivalence relation $L$ on $X$ is basic at $s$ if and only if it is basic at $s'$. The collection of basic finite étale equivalence relations on $X_s$ and on $X_{s'}$ are in canonical bijective correspondence. One deduces also that if $L$ is basic, so is $L_{s'}$ for every base change $S' \to S$. When $S' \to S$ is surjective, the condition that $L$ be basic is then equivalent to that $L_{s'}$ be basic.

3. Automorphisms I.

We study now the $Y$-automorphisms of $X$.

Lemma 3.1. — Keep the notations of (2.1). With each $S$-morphism $q : X \to X$ there is associated a unique $S$-group homomorphism $p : A \to A$ such that $q$ is $p$-equivariant.
Proof. One may assume the torsor $X$ to be trivial, as the existence of a unique homomorphism $p$ is an étale local question on $S$.

Each $S$-morphism $q : A \to A$ is the unique composite of a translation $(x \mapsto x + q(0))$ and an $S$-group homomorphism $p : A \to A$ ("Geometric Invariant Theory" 6.4). It is clear that $q$ is $p$-equivariant: for every two local $S$-sections $a, x$ of $A$, one has

$$q(a + x) = p(a) + q(x).$$

\[\square\]

**Lemma 3.2.** — Keep the notations of (2.1). Assume that $L$ is basic. Then every $Y$-endomorphism of $X$ is a $Y$-automorphism. If $X$ admits furthermore an $S$-section $x$, the only $Y$-automorphism of $X$ preserving $x$ is the identity morphism.

Proof. Every $Y$-morphism $X \to X$ is finite étale, as $X$ is finite étale over $Y$. Let $q$ be such a $Y$-endomorphism of $X$. By (3.1), $q$ is equivariant relative to a unique $S$-group homomorphism $p : A \to A$. And, $p$ is finite étale, since it is étale locally on $S$ isomorphic to $q$. In particular, $E = \text{Ker}(p)$ is a finite étale $S$-subgroup of $A$. Observe that the morphism

$$E \times_S X \to X \times_S X, \ (e, x) \mapsto (e + x, x)$$

factors through the graph of $L$. That is to say, for every local $S$-section $e$ (resp. $x$) of $E$ (resp. $X$), $(e + x, x)$ is a local $S$-section of $X \times_Y X$. Indeed, as $q$ is a $Y$-endomorphism of $X$, the sections $x$ and $q(x)$ (resp. $e + x$ and $q(e + x) = p(e) + q(x) = q(x)$) have the same image in $Y$.

So $E = 0$, as $L$ is basic. So $p$, hence $q$ as well, is an isomorphism.

Let $Z = \text{Ker}(q, \text{Id}_X)$, which is an open and closed sub-$Y$-algebraic space of $X$. If $q(x) = x$, that is, if $x \in Z(S)$, then $Z = X$ and $q = \text{Id}_X$, since $X$ has geometrically irreducible $S$-fibers.

\[\square\]

**Lemma 3.3.** — For $i = 1, 2$, let $X_i, L_i, Y_i = X_i/L_i$ be as in (2.1), with $L_i$ basic, let $x_i \in X_i(S)$ and let $y_i \in Y_i(S)$ be the image of $x_i$. Let $r : Y_1 \to Y_2$ be an $S$-isomorphism satisfying $r(y_1) = y_2$.

Then there exists a unique $r$-isomorphism $h : X_1 \to X_2$ satisfying $h(x_1) = x_2$.

Proof. Identify $Y_1$ with $Y_2$ by $r$. Write $Y = Y_1 = Y_2$ and $y = y_1 = y_2$. The uniqueness of $h$ follows from (3.2). Thus, the existence of $h$ is an étale local question on $S$.

One may assume that $S$ is strictly local with closed point $s$. For, if the lemma is proven under this assumption, then $h$ exists in an étale
neighborhood of $s$ in $S$ by a “passage à la limite projective” (EGA IV 8). As the closed immersion $Y_s \hookrightarrow Y$ induces an equivalence between the category of finite étale $Y$-algebraic spaces and the category of finite étale $Y_s$-algebraic spaces (SGA 4 XII 5.9 bis), one may further assume that $S = s$ is the spectrum of a separably closed field $k$.

Let $X$ be an algebraic space which is connected and finite étale Galois over $Y$ and which dominates $X_1$ and $X_2$, say by $q_i : X \to X_i$, $i = 1, 2$. By SGA 1 XI 2, $X$ is a (trivial) torsor under a $k$-abelian variety $A$ and $q_i$ is equivariant relative to a unique étale $k$-group homomorphism $p_i : A \to A_i$, $i = 1, 2$.

With each element $g \in \Gal(X/Y) = G$, there is associated a unique group automorphism $a(g)$ of the $k$-abelian variety $A$ such that $g$ is $a(g)$-equivariant. The map $g \mapsto a(g)$ is a homomorphism of groups and its kernel $a^{-1}(1) = E$ consists of translations by elements of $A(k)$.

By definition, $\Ker(p_1)$ and $\Ker(p_2)$ are $k$-subgroups of $E_k$. Hence, $X/E$ is dominated by $X_1$ and by $X_2$. So $E_k = \Ker(p_1) = \Ker(p_2)$, as $L_1$ and $L_2$ are basic.

There exist therefore a unique $Y$-isomorphism $q : X_1 \to X_2$ and a unique isomorphism of $k$-abelian varieties $p : A_1 \to A_2$ such that $q$ is $p$-equivariant and such that $qq_1 = q_2$, $pp_1 = p_2$.

Identify $X_1$ with $X_2$ by $q$ and identify $A_1$ with $A_2$ by $p$. Then, $X_1 = X_2 = X/E$ is Galois over $Y$ with Galois group $G/E$. Clearly, a unique element $h$ of $G/E$ transforms $x_1$ to $x_2$.

□

Proposition 3.4. — Keep the notations of (2.1). Assume that $L$ is basic. Let $f : Y \to S$ denote the structural morphism.

Then the $S$-group $G = f_*\Aut_Y(X)$ is a finite étale $S$-algebraic space and the canonical morphism

$$u : G \times_S X \to X \times_Y X, \ (g, x) \mapsto (g(x), x)$$

is an isomorphism.

Proof. The finite étale $Y$-group $N = \Aut_Y(X)$, considered as a proper smooth $S$-algebraic space, has Stein factorization $N \to f_*N \to S$ (SGA 1 X 1.2). By loc. cit., $f_*N \to S$ is finite étale and the formation of the Stein factorization commutes with every base change $S' \to S$. It remains only to verify that $u \times_S S$ is an isomorphism for each geometric point $s$ of $S$. One can thus assume that $S$ is the spectrum of a separably closed field. Then, by the proof of (3.3), $X \to Y$ is Galois and hence $u$ is an isomorphism.

□
Proposition 3.5. — For $i = 1, 2$, let $X_i, L_i, Y_i = X_i / L_i$ be as in (2.1) with $L_i$ basic, let $f_i : Y_i \to S$ be the structural morphism, let $G_i = f_i^* \text{Aut}_{Y_i}(X_i)$ and let $r : Y_1 \to Y_2$ be an $S$-isomorphism. Write $X'_i, L'_i, Y'_i$ for the base change of $X_i, L_i, Y_i, r$ by a morphism $S' \to S$. Then the sheaf on $(\text{Sch}/S)$ for the étale topology, $S' \mapsto \text{Isom}_r(X'_1, X'_2)$, is a $(G_2, G_1)$-bitorsor. The map
\[ \text{Isom}_r(X_1, X_2) \to \text{Isom}_r(X'_1, X'_2), \quad h \mapsto h \times_S S' \]
is a bijection if $S' \to S$ is 0-acyclic (SGA 4 XV 1.3).

Proof. Identify $Y_1$ with $Y_2$ by $r$, write $Y = Y_1 = Y_2$ and $f = f_1 = f_2$. Let $I = \text{Isom}_Y(X_1, X_2)$, which is a finite étale $Y$-algebraic space. By SGA 1 X I.2, $f_* I$ is a finite étale $S$-algebraic space and its formation commutes with every base change $S' \to S$ and it is a $(G_2, G_1)$-bitorsor by the same argument of (3.4). When $u : S' \to S$ is 0-acyclic, the adjunction morphism
\[ f_* I \to u_* u^* f_* I \]
is an isomorphism and in particular
\[ \Gamma(Y, I) = \Gamma(S, f_* I) \to \Gamma(Y', I') = \Gamma(S, u_* u^* f_* I) \]
is a bijection. □

4. Factorization over a separably closed field.

Proposition 4.1. — Over a separably closed field $k$, every finite étale surjective $k$-morphism from a $k$-abelian variety $A'$ to a $k$-algebraic space $Y$, $A' \to Y$, factors up to $k$-isomorphisms in a unique way as the composite of an étale isogeny of $k$-abelian varieties $A' \to A$ and a quotient $A \to A / L = Y$ by a basic finite étale $k$-equivalence relation $L$ on $A$.

Proof. Let $A' \to Y$ be dominated by a finite étale Galois $k$-morphism $A'' \to Y$ with $A''$ connected. By SGA 1 XI 2, $A''$ may be endowed with a $k$-abelian variety structure so that the projection $A'' \to A'$ is an étale $k$-group homomorphism. Let $E''$ denote the kernel of this homomorphism.

Let $E$ consist of the elements of $G = \text{Gal}(A''/Y)$ which are translations of $A''$, namely, of the form $x \mapsto x + a$, with $a \in A''(k)$. Evidently, $E$ is a subgroup of $G$, $E \supset E''(k)$ and the quotient $A = A'' / E$ inherits a $k$-abelian variety structure from that of $A''$.

The factorization
\[ A' = A'' / E'' \to A = A'' / E \to Y = A'' / G \]
is then as desired. One argues as in (3.3) that such a factorization is unique up to $k$-isomorphisms.

\[ \square \]

5. Basicness is an open and closed property.

**Proposition 5.1.** — Keep the notations of (2.1). Let 

\[ F : (\text{Sch}/S)^{\circ} \to (\text{Sets}) \]

be the following sub-functor of the final functor:

For an $S$-scheme $S'$, $F(S') = \{\emptyset\}$, if $L \times_S S'$ is basic, and $F(S') = \emptyset$, otherwise.

Then $F$ is representable by an open and closed sub-algebraic space of $S$.

**Lemma 5.2.** — Keep the notations of (2.1). Assume that $S$ is the spectra of a discrete valuation ring with generic point $t$ and closed point $s$.

Then the following conditions are equivalent:

1) $L_s$ is basic.
2) $L_t$ is basic.

**Proof.** One may assume that $S$ is strictly henselian.

Assume 1). The finite étale morphism $X_s \to X_s/L_s$ is Galois (3.4). So by SGA 4 XII 5.9 bis, $X$ is Galois over $X/L = Y$, as $S$ is strictly henselien and $Y \to S$ is proper. Let $G = \text{Gal}(X/Y)$. Each element $g \in G$ is equivariant relative to a unique group automorphism $a(g)$ of the $S$-abelian algebraic space $A$. The homomorphism $g \mapsto a(g)_s$ is injective, as $L_s$ is basic. It follows that $g \mapsto a(g)_t$ is injective. For, the specialization homomorphism

\[ \text{Aut}_t(A_t) \hookrightarrow \text{Aut}_s(A) \to \text{Aut}_s(A_s) \]

is injective. So $L_t$ is basic.

Assume 2). Let $\overline{t}$ be the spectrum of a separable closure of $k(t)$. By (3.4), $X_{\overline{t}} \to Y_{\overline{t}}$ is Galois with Galois group, say $G$. Replacing if necessary $S$ by its normalization $S'$ in a finite sub-extension of $k(\overline{t})/k(t)$, $X$ by $X \times_S S'$, $Y$ by $Y \times_S S'$ and $L$ by $L \times_S S'$, one may assume that $X_t$ is a $G$-torsor on $Y_t$ and that $L_t$ is defined by the $G$-action on $X_t$. Now $X$ being the $S$-Néron model of $X_t$, each $t$-automorphism of $X_t$ uniquely extends to an $S$-automorphism of $X$. In particular, the $G$-action on $X_t$ extends to an action on $X$ with graph evidently equal to that of $L$. For each $g \in G$, let $a(g)$ be the unique $S$-automorphism of the $S$-abelian algebraic space $A$ relative to which $g : X \to X$ is equivariant. The
homomorphism $g \mapsto a(g)_t$ is injective, as $L_t$ is basic. Then $g \mapsto a(g)_s$ is injective, for the specialization homomorphism
\[
\text{Aut}_t(A_t) \hookrightarrow \text{Aut}_S(A) \rightarrow \text{Aut}_s(A_s)
\]
is injective. So $L_s$ is basic.

\[\square\]

**Lemma 5.3.** — Keep the notations of (2.1). Assume that $L$ is basic. Assume that $S$ is an affine scheme and that $(S_i)$ is a projective system of affine noetherian schemes, indexed by a co-directed set $I$, with limit $S$.

Then there exist an index $i \in I$, an $S_i$-abelian algebraic space $A_i$, an $A_i$-torsor $X_i$ on $S_i$ for the étale topology and a basic finite étale $S_i$-equivalence relation $L_i$ on $X_i$ such that $A = A_i \times_{S_i} S$, $X = X_i \times_{S_i} S$ and $L = L_i \times_{S_i} S$.

**Proof.** By the technique of “passage à la limite projective” (EGA IV 8, 9), there exist $i_o \in I$ and $A_{i_o}, X_{i_o}, L_{i_o}$ as desired except possibly the property of being basic. By (5.2), for each $j \geq i_o$, $L_{i_o} \times_{S_{i_o}} S_j$ is basic at precisely the points of an open and closed sub-scheme $S'_j$ of $S_j$. The projection $S \rightarrow S'_j$ factors through $S'_{j_o}$, as $L$ is basic. Now $(S_j - S'_{j_o})_{j \geq i_o}$ is a projective system of affine noetherian schemes with empty limit. So $S_i = S'_i$ for some $i \geq i_o$.

\[\square\]

5.4. Proof of (5.1).

The question being an étale local question on $S$, one may assume that $S$ is a scheme, then affine and by (5.3) noetherian. The claim is now immediate by (5.2).

\[\square\]

6. Automorphisms II.

We study the $S$-automorphisms of $Y$.

**Lemma 6.1.** — Keep the notations of (2.1). Assume that $L$ is basic. Let $p : X \rightarrow X/L = Y$ denote the projection. Let $r$ be an $S$-automorphism of $Y$.

Then there is at most one $S$-section $a$ of $A$ which verifies $rp = pT_a$, where $T_a : x \mapsto a + x$ is the $S$-automorphism of $X$, the translation by $a$.

**Proof.** If $rp = pT_a = pT_b$ holds for two $S$-sections $a, b$ of $A$, then $T_{a-b}$ acts on $X$ as a $Y$-automorphism. In particular, $T_{a-b}$ is locally on $S$ of finite order. So $T_{a-b} = \text{Id}_X$ and $a = b$, as $L$ is basic.
Lemma 6.2. — Keep the notations and assumptions of (6.1).

Let \( F : (\text{Sch}/S)^o \to (\text{Sets}) \) be the following sub-functor of the final functor:

For an \( S \)-scheme \( S' \), \( F(S') = \{\emptyset\} \), if \( r_{S'}p_{S'} = p_{S'}T_{a'} \) for an \( S' \)-section \( a' \) of \( A_{S'} \), and \( F(S') = \emptyset \), otherwise.

Then \( F \) is representable by an open and closed sub-algebraic space of \( S \).

Proof. The question being by (6.1) an étale local question on \( S \), one may assume \( S \) to be a scheme, then affine and by (5.3) noetherian.

— The functor \( F \) verifies the valuative criterion of properness:

Namely, given an \( S \)-scheme \( S' \) which is the spectra of a discrete valuation ring and which has generic point \( t' \), then \( F(S') = F(t') \). For, if \( r_{t'}p_{t'} = p_{t'}T_{a'} \) holds for some point \( a' \in A(t') \), then \( a' \) extends uniquely to an \( S' \)-section \( a' \) of \( A_{S'} \) and the equation \( r_{S'}p_{S'} = p_{S'}T_{a'} \) holds.

— The functor \( F \) is formally étale:

Namely, \( F(S') = F(S'') \) for every nilpotent \( S \)-immersion \( S'' \hookrightarrow S' \). Indeed, assume \( r_{S''}p_{S''} = p_{S''}T_{a'} \) holds for some section \( a'' \in A(S'') \). As both \( p \) and \( rp \) are étale, there is a unique \( S' \)-automorphism \( T' \) of \( X_{S'} \) such that \( T' \) restricts to \( T_{a''} \) on \( X_{S''} \) and such that \( r_{S''}p_{S''} = p_{S''}T' \) holds. This \( T' \) is equivariant with respect to a unique \( S' \)-group automorphism \( \varphi' \) of \( A_{S'} \). As the \( S \)-group \( \text{Aut}_S(A) \) is unramified over \( S \) and as \( \varphi' \) restricts to the identity automorphism of \( A_{S''} \), it follows that \( \varphi' = \text{Id}_{A_{S'}} \). So \( T' \) is of the form \( T_{a'} \) for an \( S' \)-section \( a' \in A(S') \).

It is now evident that \( F \to S \) is representable by an open and closed immersion.

□

Proposition 6.3. — Keep the notations of (2.1). Assume that \( L \) is basic. Let \( p : X \to X/L = Y \) denote the projection and \( f : Y \to S \) the structural morphism.

Then:

a) The \( S \)-subgroup \( R \) of \( \text{Aut}_S(Y) \) defined to consist of the local \( S \)-automorphisms \( r \) satisfying \( rp = pT_a \) for some local \( S \)-sections \( a \) of \( A \) is open and closed in \( \text{Aut}_S(Y) \).

b) The homomorphism \( r \mapsto a \) is a closed immersion of \( R \) into \( A \). In particular, \( R \) is commutative and representable by a proper \( S \)-algebraic space.
c) An $S$-section $a$ of $A$ lies in the image of $R$ if and only if $T_ag = gT_a$ for all local $S$-sections $g$ of $G = f_! \text{Aut}_{X}(X)$.

Proof. Note that $R \to A, r \mapsto a$, is a well-defined morphism by (6.1). This morphism is a monomorphism because $p$, being \'{e}tale surjective, is an epimorphism in the category of $S$-algebraic spaces.

The claim that the sub-$S$-group functor $R \subset \text{Aut}_S(Y)$ is open and closed is a rephrase of (6.2), hence $a$).

If an $S$-section $a$ of $A$ satisfies $T_ag = gT_a$ for every local $S$-section $g$ of $G$, then $T_a$, being $G$-equivariant, defines by passing to the $G$-quotient an $S$-automorphism $r$ of $X/G = Y$ (3.4). One has thus $rp = pT_a$ by construction. Conversely, the identity $rp = pT_a$ implies that, for every local $S$-section $g$ of $G$, one has $rp = rpg^{-1} = pgT_ag^{-1}$ and then by (6.1) $gT_ag^{-1} = T_a$. The characterization in c) therefore follows and it implies evidently that the monomorphism $R \to A, r \mapsto a$, is a closed immersion, hence b).

\[\square\]

7. Rigidity of torsors and albanese.

Lemma 7.1. — Let $S$ be the spectra of a discrete valuation ring and $t$ (resp. $s$) a geometric generic (resp. closed) point of $S$. Let $X$ be a proper smooth $S$-algebraic space.

Then the following conditions are equivalent :

1) $X \times_S s$ has an $s$-abelian variety structure.

2) $X \times_S t$ has a $t$-abelian variety structure.

Proof. One may assume that $S$ is strictly henselian. Then $X$ has $S$-sections by EGA IV 17.16.3. Fix an $S$-section $o$ of $X$. When 1) (resp. 2)) holds, after a translation of the origin, $X \times_S s$ (resp. $X \times_S t$) has an abelian variety structure over $s$ (resp. $t$) with $o_s$ (resp. $o_t$) as its zero section. Under either assumption, $X$ has geometrically irreducible $S$-fibers and admits a unique $S$-abelian algebraic space structure with $o$ being the zero section (“Geometric Invariant Theory” 6.14).

\[\square\]

Proposition 7.2. — Let $S$ be an algebraic space and $X$ a proper smooth $S$-algebraic space. Let $T : (\text{Sch}/S)^{\circ} \to (\text{Sets})$ be the following sub-functor of the final functor :

For an $S$-scheme $S'$, $T(S') = \{\emptyset\}$, if $X \times_S s'$ has an $s'$-abelian variety structure for each geometric point $s'$ of $S'$, and $T(S') = \emptyset$, otherwise.
Then $T$ is representable by an open and closed sub-algebraic space of $S$. And $X \times_S T = X_T$ is a $\text{Pic}^o_{P_T/T}$-torsor on $T$ for the étale topology where $P_T = \text{Pic}^o_{X_T/T}$ is a $T$-abelian algebraic space.

Proof. From (7.1) and by a “passage à la limite”, one deduces that $T \to S$ is representable by an open and closed immersion. Replacing $S$ by $T$, one may assume $T = S$. Let $S' \to S$ be a smooth morphism with geometrically irreducible fibers such that $X' = X \times_S S'$ admits an $S'$-section $e'$. One may for instance take $S' = X$ and $e'$ to be the diagonal section $\Delta_X/S$. Notice that in $S'' = S' \times_S S'$ the only open and closed neighborhood of the diagonal is $S''$. By “Geometric Invariant Theory” 6.14, on $X'$ there is a unique $S'$-abelian algebraic space structure with zero section $e'$. It follows that $P = \text{Pic}^o_{X/S}$, being representable by an $S$-algebraic space (Artin), is an $S$-abelian algebraic space and that $\text{Pic}^o_{X/S} = \text{Pic}^o_{X/S}$, as one verifies after the fppf base change $S' \to S$ on the $S'$-abelian algebraic space $X'$. Let the dual $S$-abelian algebraic space of $P$ be $A$. Let $p_1, p_2$ be the two projections of $S''$ onto $S'$ and $u : p_1^*X' \to p_2^*X'$ the descent datum on $X'$ relative to $S' \to S$ corresponding to $X$. By (3.1)+(7.3) $u$ is equivariant with respect to a unique $S''$-group automorphism $a$ of $A'' = A \times_S S''$. It suffices evidently to show that $a = 1$. Now the $S$-group $\text{Aut}_S(A)$ being unramified and separated over $S$, the relation “$a = 1$” is an open and closed relation on $S''$ and holds on the diagonal and so holds.

□

Lemma 7.3. — Let $S$ be an algebraic space, $A$ an $S$-abelian algebraic space and $X$ an $A$-torsor on $S$ for the étale topology.

Then there exists a canonical isomorphism

$$X A \times \text{Pic}^o_{A/S} \cong \text{Pic}^o_{X/S}$$

which induces isomorphisms

$$\text{Pic}^o_{A/S} = X A \times \text{Pic}^o_{A/S} \cong \text{Pic}^o_{X/S},$$

$$\text{NS}_{A/S} = X A \times \text{NS}_{A/S} \cong \text{NS}_{X/S}.$$

Proof. This is “Faisceaux amples sur les schémas en groupes et les espaces homogènes” XIII 1.1, by which it is justified to call $A$ the albanese of $X$.

□
8. Specialization and descent.

**Proposition 8.1.** — Let $S$ be the spectra of a discrete valuation ring and $t$ (resp. $s$) a geometric generic (resp. closed) point of $S$. Let $Y$ be a proper smooth $S$-algebraic space. Suppose that $Y_t$ is the quotient of a $t$-abelian variety $A_t$ by a basic finite étale $t$-equivalence relation such that $[A_t : Y_t]$ is prime to the characteristic of $k(s)$.

Then $Y_s$ is the quotient of an $s$-abelian variety by a basic finite étale $s$-equivalence relation.

**Proof.** One may assume that $S$ is strictly henselian with closed point $s$. Recall that by $(3.4)$ $A_t$ is Galois over $Y_t$. As $Y$ is proper smooth over $S$, $S$ strictly henselian and $G = \text{Gal}(A_t/Y_t)$ of order prime to the characteristic of $k(s)$, the specialization morphism

$$H^1(Y, G) \to H^1(Y_t, G)$$

is by SGA 4 XVI 2.2 a bijection. One finds thus a $G$-torsor on $Y$ for the étale topology, $A \to Y$, which specializes to $A_t \to Y_t$ at $t$. In particular, $A$ is proper smooth over $S$ and has $S$-sections by EGA IV 17.16.3 and has by (7.2) an $S$-abelian algebraic space structure. The finite étale $S$-equivalence relation on $A$ of graph $A \times_Y A$ is basic because it is basic at $t$ (5.1).

□

**Proposition 8.2.** — Let $S$ be an algebraic space. Let $Y$ be an algebraic space which is proper flat of finite presentation over $S$.

Let $U : (\text{Sch}/S)^\circ \to \text{(Sets)}$ be the following sub-functor of the final functor:

For an $S$-scheme $S'$, $U(S') = \{\emptyset\}$, if $Y \times_S S'$ is the quotient of $s'$-abelian variety by a finite étale $s'$-equivalence relation for each geometric point $s'$ of $S'$, and $U(S') = \emptyset$, otherwise.

Then $U \to S$ is representable by an open immersion of finite presentation. There exist a $U$-abelian algebraic space $A$, an $A$-torsor $X$ on $U$ for the étale topology and a basic finite étale $U$-equivalence relation $L$ on $X$ such that $Y \times_S U$ is $U$-isomorphic to the quotient $X/L$.

**Proof.** Observe that, by (4.1), for a given geometric point $s$ of $S$, $U(s) = \{\emptyset\}$ if and only if $Y_s$ is the quotient of an $s$-abelian variety by a basic finite étale $s$-equivalence relation.

— Reduction to the case where $Y$ is $S$-smooth:

Consider the functor $V : (\text{Sch}/S)^\circ \to \text{(Sets)}$:

For an $S$-scheme $S'$, $V(S') = \{\emptyset\}$, if $Y \times_S S'$ is smooth over $s'$ for each geometric point $s'$ of $S'$, and $V(S') = \emptyset$, otherwise.
It is evident that $V \to S$ is representable by an open immersion of finite presentation, that $Y \times_S V$ is proper and smooth over $V$ and that $V$ contains $U$ as a sub-functor. Restricting to $V$, one may assume that $S = V$, namely, that $Y$ is smooth over $S$.

— Assume that $Y$ is $S$-smooth. Then $U \to S$ is representable by an open immersion of finite presentation. There exist an étale surjective morphism $U' \to U$, a $U'$-abelian algebraic space $A'$, an $A'$-torsor $X'$ on $U'$ for the étale topology and a basic finite étale $U'$-equivalence relation $L'$ on $X'$ such that $Y \times_S U'$ is $U'$-isomorphic to the quotient $X'/L'$.

The question being an étale local question on $S$, one may assume that $S$ is a scheme, then affine, then by (5.3) noetherian and strictly local with closed point $s$ and finally that $U(s) = \{ \emptyset \}$, i.e., that $Y_s$ is the quotient of an $s$-abelian variety $A_s$ by a basic finite étale $s$-equivalence relation $L_s$. By SGA 4 XII 5.9 bis, there is a finite étale $S$-morphism $A \to Y$ which specializes to $A_s \to A_s/L_s = Y_s$ at $s$. This algebraic space $A$ is in particular proper smooth over $S$ and has closed fiber the abelian variety $A_s$. As $S$ is strictly local, $A$ admits $S$-sections (EGA IV 17.16.3). So by (7.2) $A$ has an $S$-abelian algebraic space structure. The finite étale $S$-equivalence relation $L$ on $A$ of graph $A \times_Y A$, basic at $s$, is basic (5.1).

— Assume that $Y$ is $S$-smooth. Then up to an étale localization on $U'$ there exists a descent datum on $(A', X', L')$ relative to $U' \to U$:

Restricting to the open sub-algebraic space $U$, one may assume that $U = S$. Write $S'$ for $U'$ and $Y'$ for $Y \times_S S'$. It suffices to prove the existence of a finite étale $Y'$-algebraic space $X$ such that the finite étale $Y'$-algebraic space $\text{Isom}_{Y'}(X \times_Y Y', X')$ is surjective over $Y'$, for then $X$ is an $A$-torsor on $S$ for the étale topology (7.2) where the $S$-abelian algebraic space $A$ satisfies $\text{Pic}^0_{A/S} = \text{Pic}^0_{X/S}$ (7.3) and $X \times_Y X$ is the graph of a basic finite étale $S$-equivalence relation $L$ on $X$.

i) Case where $Y$ has an $S$-section $y$:

By an étale localization on $S'$, one may assume that there is an $S'$-section $x'$ of $X'$ which is mapped to $y' = y \times_S S'$ by $X' \to Y'$. Let $S'' = S' \times_S S'$ and $p_1, p_2$ the two projections of $S''$ onto $S'$. By (3.3), there exists a unique $(Y \times_S S'')$-isomorphism $h : p_1^*X' \to p_2^*X'$ which transforms $p_1^*(x')$ to $p_2^*(x')$. That is, $h$ is a gluing datum on $(X', x')$ relative to $S' \to S$. By (3.3) again, $h$ is a descent datum. This descent provides a finite étale $Y$-algebraic space $X$ which verifies $X \times_Y Y' = X'$ and which is equipped with an $S$-section $x$ having image $y$ in $Y(S)$.

ii) General case:
By the second projection, \( Y \times_S Y = Y_1 \) has a \( Y \)-algebraic space structure which admits the \( Y \)-section \( \Delta_{Y/S} \). One finds by i) a finite étale \( Y_1 \)-algebraic space \( X_1 \) such that the finite étale \( Y_1' \)-algebraic space 

\[
\text{Isom}_{Y_1'}(X_1 \times_{Y_1} Y_1', X' \times_{Y'} Y_1')
\]

is surjective over \( Y_1' \), where \( Y_1' := Y \times_S Y' = Y_1 \times_Y Y' \). Let the Stein factorization of the proper smooth \( Y \)-algebraic space \( X_1 \) be 

\[
X_1 \to X \to Y
\]

and let \( c \) be the canonical morphism 

\[
c : X_1 \to X \times_Y Y_1.
\]

By SGA 1 X 1.2, the morphism \( X \to Y \) is finite étale and the formation of the Stein factorization commutes with every base change \( T \to S \). It suffices to show that \( c \) is an isomorphism, for then 

\[
\text{Isom}_{Y_1'}(X_1 \times_{Y_1} Y_1', X' \times_{Y'} Y_1') = \text{Isom}_{Y_1'}(X \times_Y Y', X') \times_{Y'} Y_1'
\]

and \( X \) is the sought after \( Y \)-algebraic space. This amounts to showing that \( c \times_S s \) is an isomorphism for each geometric point \( s \) of \( S \). One may thus assume that \( S \) is the spectrum of a separably closed field. Then \( S' \), being étale surjective over \( S \), is a non-empty disjoint union of \( S \). Index these components of \( S' \) as \( S_i \), \( i \in \pi_0(S') = \Pi \), write \( X' = \sum_{i \in \Pi} X_i \), fix a point \( y \in Y(S) \) and choose a point \( x_i \in X_i(S_i) \) above \( y \) for each \( i \in \Pi \). These \( (X_i, x_i) \)'s are all mutually \( Y \)-isomorphic by (3.3). Clearly, \( c \) is an isomorphism.

\[\square\]

9. Proof of Theorem 1.1.

a) Let the open sub-algebraic space \( U \) of \( S \), the \( U \)-abelian algebraic space \( A \), the \( A \)-torsor \( X \) on \( U \) for the étale topology be as in (8.2) so that \( Y \times_S U = Y_U \) is the quotient of \( X \) by a basic finite étale \( U \)-equivalence relation \( L \). It suffices to show that \( U = S \).

Factor \( A'_t \to Y_t \) as \( A'_t \to X_t \to Y_t \) (4.1). The degree \( [X_t : Y_t] = d \), which divides \( [A'_t : Y_t] \), is prime to the residue characteristics of \( S \). By (3.4), there is a maximal open and closed sub-algebraic space \( U' \) of \( U \) such that \( X_{U'} \) is of constant degree \( d \) over \( Y_{U'} \). By (8.1), \( U' \) is closed in \( S \). So \( U' = U = S \), as \( S \) is connected.

b) The assertion is immediate by (3.5)+(7.3). And the map which with \( P' \in H^1(S, G) \) associates

\[
X' = P' \wedge X
\]

provides the inverse.
10. Irreducibility of non-uniruled degenerate fibers. Almost non-degeneration.

The irreducibility of non-uniruled degenerate fibers of an abelian fibration is shown in [4], hence:

**Proposition 10.1.** — Keep the notations of (1.1). Assume that S is open in an algebraic space \( \overline{S} \) and that \( Y \) is open dense in a separated finitely presented \( \overline{S} \)-algebraic space \( \overline{Y} \) with structural morphism \( \overline{f} \). Let \( \overline{s} \) be a geometric point of \( \overline{S} \) with values in an algebraically closed field.

Then \( \overline{f}^{-1}(\overline{s}) \) is irreducible if it is non-empty and does not have uniruled irreducible components and if one of the following two conditions holds:

a) \( \overline{f} \) is flat at all maximal points of \( \overline{f}^{-1}(\overline{s}) \).

b) \( \overline{S} \) is locally noetherian and \( \overline{f} \) is universally open at all maximal points of \( \overline{f}^{-1}(\overline{s}) \).

**Proof.** By standard arguments one may assume that \( \overline{S} \) is the spectra of a complete discrete valuation ring with \( \overline{S} - S = \{ \overline{s} \} \), that \( \overline{f} \) is flat and that \( \overline{f}^{-1}(\overline{s}) \) has no imbedded components. Let \( X \) be as in (1.1) and \( X_\overline{s} \) the normalization of \( Y \) in \( X \). Then \( X_\overline{s} \) is non-empty and does not have uniruled irreducible components and hence by [4] 4.1 is irreducible. So \( \overline{f}^{-1}(\overline{s}) \) is irreducible.

**Theorem 10.2.** — Let \( S \) be the spectra of a discrete valuation ring and \( \overline{t} \) (resp. \( \overline{s} \)) a geometric generic (resp. closed) point of \( S \) with values in an algebraically closed field. Let \( Y \) be an \( S \)-algebraic space with structural morphism \( f \). Assume that \( f \) is separated of finite type, that \( Y \) is normal integral and at each of its geometric codimension \( \geq 2 \) points either regular or pure geometrically para-factorial of equal characteristic, that \( f^{-1}(\overline{t}) \) is the quotient of a \( \overline{t} \)-abelian variety \( A_{\overline{t}} \) by a finite étale \( \overline{t} \)-equivalence relation with degree \( [A_{\overline{t}} : f^{-1}(\overline{t})] \) prime to the characteristic of \( k(\overline{s}) \) and that \( f^{-1}(\overline{s}) \) is non-empty, proper, of total multiplicity prime to the characteristic of \( k(\overline{s}) \) and does not have uniruled irreducible components.

Then \( f \) factors canonically as \( Y \to E \to S \) with \( Y \to E \) proper and smooth, where \( E \) is a finite flat \( S \)-algebraic stack, regular, tame over \( S \) and satisfies \( E \times_S \overline{t} = \overline{t} \).

**Proof.** Notice that \( f \) is faithfully flat and, being separated of finite type with geometrically irreducible (10.1) and proper fibers, that \( f \) is also
proper (EGA IV 15.7.10). Let the total multiplicity of \( f^{-1}(\mathfrak{s}) \), that is by definition, the greatest common divisor of the lengths of the local rings of \( f^{-1}(\mathfrak{s}) \) at its maximal points, be \( \delta \), which by hypothesis is prime to the characteristic of \( k(\mathfrak{s}) \). Thus, the 1-codimensional cycle on \( Y \) with rational coefficients, \( \Delta = f^* \text{Cyc}_{S}(\pi)/\delta \), where \( \pi \) is a uniformizer of \( S \), has integral coefficients and is a prime cycle and is locally principal, for \( Y \), being normal and geometrically para-factorial at all its geometric codimension \( \geq 2 \) points, has geometrically factorial local rings (EGA IV 21.13.11). With \( \Delta \) one associates a canonical \( \mu_{\delta} \)-torsor on \( Y \) for the étale topology, \( Y' \to Y \). Let \( S' = \text{Spec} \Gamma(Y', \mathcal{O}_{Y'}) \). There is by quotient by \( \mu_{\delta} \) an \( S \)-morphism
\[
Y = [Y'/\mu_{\delta}] \to [S'/\mu_{\delta}] = E.
\]

It suffices evidently to show that \( Y' \to S' \) is smooth, for then
\[
Y \to E \to S
\]
is the desired factorization of \( f \). Replacing \( f \) by \( Y' \to S' \), we assume from now on that \( \delta = 1 \), namely, that \( f^{-1}(\mathfrak{s}) \) is integral. And, replacing \( f \) by \( f \times_{S} S(\mathfrak{s}) \), where \( S(\mathfrak{s}) \) is the strict henselization of \( S \) at \( \mathfrak{s} \), we assume that \( S \) is strictly henselian.

Let \( t \) (resp. \( s \)) be the generic (resp. closed) point of \( S \). Choose by (1.1) a triple \((A_t, X_t, L_t)\) so that, for a \( t \)-abelian variety \( A_t \), \( Y_t = f^{-1}(t) \) is the quotient of an \( A_t \)-torsor \( X_t \) by a basic finite étale \( t \)-equivalence relation \( L_t \). Let \( G_t = f_{ts} \text{Aut}_{A_t}(X_t) \). The degree \([X_t : f^{-1}(t)] = [G_t : t]\), which divides \([A'_t : f^{-1}(t)]\) (4.1), is prime to the characteristic of \( k(s) \).

Let \( X \) be the normalization of \( Y \) in \( X_t \). Then \( X_{\mathfrak{s}} \) is non-empty and does not have uniruled irreducible components and hence is irreducible (10.1). Let \( p : X \to Y \) be the projection and \( x \) (resp. \( y = p(x) \)) the generic point of \( X_{s} \) (resp. \( Y_{s} \)). Note that there is an open neighborhood of \( x \) (resp. \( y \)) in \( X \) (resp. \( Y \)) which is a scheme (2 3.3.2).

--- Reduction to the case where \( G_t \) is constant and cyclic ---

One has that \([X_t : Y_t] = [\mathcal{O}_{X,t} : \mathcal{O}_{Y,y}] = e[k(x) : k(y)]\) is prime to the characteristic of \( k(s) \), where \( e \) is the ramification index of \( \mathcal{O}_{X,t} \) over \( \mathcal{O}_{Y,y} \). One deduces that there is an open sub-scheme \( V \) of \( Y \) containing \( y \) such that \( V \) is \( S \)-smooth, that \((U_{s})_{\text{red}} \) is finite étale surjective over \( V_{s} \) of rank \([k(x) : k(y)]\), where \( U = p^{-1}(V) \), and that the ideal of \( U \) defining the closed sub-scheme \((U_{s})_{\text{red}} \) is generated by one section \( h \in \Gamma(U, \mathcal{O}_{U}) \). In particular, \( U \) is regular with \( h_{u} \) being part of a regular system of parameters at each point \( u \) of \( U_{s} \). Now \( S \) being strictly henselian, choose a point \( u \in U_{s}(s) \), let \( v = p(u) \), let \( n = \text{dim}_{h}(U_{s}) \) and choose \( h_{1}, \ldots, h_{n} \in \mathcal{O}_{U,u} \) so that \( \{h_{1} \text{ mod } h, \ldots, h_{n} \text{ mod } h\} \) is the
image of a regular system of parameters of $V_s$ at $v$. Then $h, h_1, \cdots, h_n$ form a regular system of parameters of $U$ at $u$. Let

$$R = \text{Spec}(\mathcal{O}_{U,u}/(h_1, \cdots, h_n)),$$

which is regular local of dimension 1 and finite flat tame along $s$ of rank $e$ over $S$. Let $r$ be the generic point of $R$. The closed image of $p(r)$ in $Y$ is an $S$-section lying in $V$ and one obtains the following commutative diagram of $S$-schemes:

$$\begin{array}{ccc}
R & \longrightarrow & U = p^{-1}(V) \\
\downarrow & & \downarrow p \\
S & \longrightarrow & V
\end{array}$$

The $G_t$-torsor structure on $p^{-1}(p(r))$, where one identifies the $t$-rational point $p(r)$ with $t$, induces an epimorphism

$$G_t \times_t r \to p^{-1}(p(r)), \quad (g, \lambda) \mapsto g.\lambda$$

and an isomorphism

$$G_t/\text{Norm}_{G_t}(r) \cong \pi_o(p^{-1}(p(r))).$$

Let $Z_t = X_t/N$ be the quotient of $X_t$ by $N = \text{Norm}_{G_t}(r)$ and let $Z$ be the normalization of $Y$ in $Z_t$. The fiber $Z_s$ is irreducible with generic point $z$ being the image of $x$. Observe that, on writing $w$ for the image of $u$ in $Z$, $Z$ is by construction étale over $Y$ at $w$ and a priori at $z$. So $Z \to Y$ is étale, as $Y$ by hypothesis is pure at all its geometric codimension $\geq 2$ points. Observe next that $N$ is constant and cyclic. Replacing $Y$ by $Z$ and $G_t$ by $N$, one may assume that $G_t \simeq \mathbb{Z}/e\mathbb{Z}$.

Assume $G_t = \mathbb{Z}/e\mathbb{Z}$. Reduction to the case where $p : X \to Y$ is étale:

Identify $\mathbb{Z}/e\mathbb{Z}$ with $\mu_e$. The $\mu_e$-torsor $X_t \to Y_t$ corresponds to a pair $(J_t, \alpha_t)$ which consists of an invertible module $J_t$ on $Y_t$ and of an isomorphism $\alpha_t : \mathcal{O}_{Y_t} \cong J_t^{\otimes e}$. There is an invertible $\mathcal{O}_Y$-module $J$ extending $J_t$, since $Y$ has geometrically factorial local rings. Since furthermore $f$ has geometrically integral fibers, $J^{\otimes e}$ is isomorphic to $\mathcal{O}_Y$, say by $\beta : \mathcal{O}_Y \cong J^{\otimes e}$. The difference in $H^1(Y_t, \mu_e)$ of the classes of $(J_t, \alpha_t)$ and $(J_t, \beta_t)$, that is, the class of $\mathcal{O}_{Y_t}/\beta_t^{-1}\alpha_t$, is contained in the image of the map

$$f^* : H^1(t, \mu_e) \to H^1(Y_t, \mu_e),$$

for one has $\Gamma(Y_t, \mathcal{O}_{Y_t}) = k(t)$. By (1.1) $b$ replacing $X_t$ by the $\mu_e$-torsor $X'_t \to Y_t$ defined by $(J_t, \beta_t)$, one may assume that $p : X \to Y$ is étale. It suffices to show that $X$ is smooth over $S$. This follows from (4) 4.3. \qed
Lemma 10.3. — Keep the notations of (1.1). Assume that $S$ is a noetherian local scheme with closed point $s$ such that one of the following two conditions holds:

a) $S$ is regular of dimension $> 0$.

b) $S$ is pure geometrically para-factorial along $s$ of equal characteristic.

Let $U = S - \{s\}$. Then each $U$-section of $f|U$ extends uniquely to an $S$-section of $f$.

Proof. In case a) one applies [4] 2.1 as the geometric fibers of $f$ do not contain rational curves. Assume b). One may assume $S$ strictly local. Choose $(A, X, L)$ as in (1.1), let $G = f_*\text{Aut}_Y(X)$, $p : X \to Y$ the projection and $y : U \to Y$ a section of $f|U$. The finite étale $S$-group $G$ is constant and the $G|U$-torsor $p^{-1}(y)$ is trivial, for $S$ is strictly local and pure along $s$. By loc.cit. 5.2+5.3 each $U$-section of $p^{-1}(y)$ extends uniquely to an $S$-section of $X$, hence the claim.

Lemma 10.4. — Let $S$ be a noetherian normal local scheme of equal characteristic zero pure geometrically para-factorial along its closed point $s$. Let $U = S - \{s\}$. Let $E$ be the fiber category on the category of $S$-algebraic spaces whose fiber over each $S$-algebraic space $S'$ is the full sub-category of the category of $S'$-algebraic spaces consisting of the $S'$-algebraic spaces $Y'$ which, for an $S'$-abelian algebraic space $A'$, an $A'$-torsor $X'$ on $S'$ for the étale topology of finite order and a basic finite étale $S'$-equivalence relation $L'$ on $X'$, is representable as the quotient $X'/L'$.

Then the restriction functor $E(S) \to E(U)$ is an equivalence of categories.

Proof. This restriction functor is fully faithful by (10.3), since every $S$-smooth algebraic space is pure geometrically para-factorial along its closed $S$-fiber.

This functor is essentially surjective. For, $U$-abelian algebraic spaces extend to $S$-abelian algebraic spaces ([4] 5.1). And, as $S$ is pure along $s$, if $n$ is an integer $\geq 1$, $A$ an $S$-abelian algebraic space and $\pi A = \text{Ker}(n.\text{Id}_A)$, $nA|U$-torsors on $U$ for the étale topology extend to $nA$-torsors on $S$ for the étale topology. Thus, every object of $E(U)$ is a quotient $(X|U)/(G|U)$ where, for an $S$-abelian algebraic space $A$, $X$ is an $A$-torsor on $S$ for the étale topology of finite order and $G$ is a finite étale $S$-group such that $G|U$ acts and defines a basic finite étale $U$-equivalence relation on $X|U$. Each such action $G|U \times_U X|U \to X|U$
extends by \[\mu : G \times_S X \to X\], since \(G \times_S X\) is geometrically para-factorial along its closed \(S\)-fiber. Clearly, \(\mu\) represents a \(G\)-action and defines a basic finite étale \(S\)-equivalence relation on \(X\). And \(X/G\) is the desired extension of \((X|U)/(G|U)\).

\[
\square
\]

**Theorem 10.5.** — Let \(S\) be an integral scheme with generic point \(t\). Let \(F\) be the fiber category on the category of \(S\)-algebraic spaces whose fiber over each \(S\)-algebraic space \(S'\) is the full sub-category of the category of \(S'\)-algebraic spaces consisting of the \(S'\)-algebraic spaces \(Y'\) which, for an \(S'\)-abelian algebraic space \(A'\), an \(A'\)-torsor \(X'\) on \(S'\) for the étale topology and a basic finite étale \(S'\)-equivalence relation \(L'\) on \(X'\), is representable as the quotient \(X'/L'\). Let \(Y\) be an \(S\)-algebraic space with structural morphism \(f\). Assume that \(Y\) is locally noetherian normal integral of residue characteristics zero and at all its geometric codimension ≥ 2 points pure and geometrically para-factorial. Assume furthermore that \(f^{-1}(t)\) is an object of \(F(t)\) and that, for each geometric codimension 1 point \(\overline{\gamma}\) of \(Y\), \(f \times_S S_{[\overline{\gamma}]}\) is separated of finite type and flat at \(\overline{\gamma}\) and the geometric fiber \(f^{-1}(\overline{s})\) is proper and does not have uniruled irreducible components, where \(S_{[\overline{s}]}\) denotes the strict localization of \(S\) at \(s = f(\overline{s})\).

Then up to unique isomorphisms there exists a unique groupoid in the category of \(S\)-algebraic spaces whose nerve \((Y, d, s)\) satisfies the following conditions:

a) \(Y = Y_o\).

b) The \(Y\)-algebraic space with structural morphism \(d_1 : Y_1 \to Y_o\) is an object of \(F(Y)\).

c) Over \(t\), \(Y_t = \cosq_o(f^{-1}(t)/t)\).

**Proof.** Notice that for each geometric codimension 1 point \(\overline{\gamma}\) of \(Y\) the localization of \(S\) at the image \(s\) of \(\overline{\gamma}\) is noetherian regular of dimension ≤ 1, since \(f\) is by hypothesis flat at \(\overline{\gamma}\). If \(S\) is local of dimension 1 with closed point \(s\), then with the notations of (10.2) the asserted \(S\)-groupoid is \(\cosq_o(Y/E)\) which, as \(Y\) is regular (10.2), is unique up to unique isomorphisms (10.3). By a “passage à la limite” and by gluing, one obtains in the general case an \(S\)-groupoid \(U\) satisfying the following conditions:

a) \(U_o\) is open in \(Y\) such that \(\text{codim}(Y - U_o, Y) \geq 2\).

b) The \(U_o\)-algebraic space with structural morphism \(d_1 : U_1 \to U_o\) is an object of \(F(U_o)\).

c) Over \(t\), \(U_t = \cosq_o(f^{-1}(t)/t)\).
As \( d_1 : U_1 \to U_o \) has the canonical section \( s_o : U_o \to U_1 \), one may by the proof of (8.2) write \( d_1 \) as a quotient by a basic finite étale \( U_o \)-equivalence relation on a \( U_o \)-abelian algebraic space with \( s_o \) being the image of the zero section. Hence, by (10.4) there is a cartesian diagram of \( S \)-algebraic spaces

\[
\begin{array}{ccc}
U_1 & \longrightarrow & Y_1 \\
\downarrow d_1 & & \downarrow d_1 \\
U_o & \longrightarrow & Y_o
\end{array}
\]

where \( Y_o = Y \) whose vertical arrow on the right is an object of \( F(Y_o) \). By (10.3) this diagram is unique up to unique isomorphisms and there is a unique extension of \( s_o : U_o \to U_1 \) to a section \( s_o : Y_o \to Y_1 \) of \( d_1 : Y_1 \to Y_o \). By again (10.3) and arguing as in [4] 5.7 one finds a unique \( S \)-groupoid \( Y \) with \( d_1 : Y_1 \to Y_o \) being a face.

\[
\square
\]

Similarly as [4] 3.1, we call every groupoid \( Y \) satisfying the conditions (10.5) \( a)+b \) an almost non-degenerate fibration structure on \( f : Y \to S \) with \([Y]\) being called the ramification \( S \)-stack. Each such fibration has again a tautological factorization

\[
f : Y \to [Y] \to S.
\]

We say that this structure is non-degenerate if the groupoid \( Y \) is simply connected with \( \text{Coker}(d_o, d_1) = S \), that is, if \( f : Y \to S \) is an object of \( F(S) \) of (10.5). Note that \( Y \to [Y] \) is then non-degenerate in this sense as in loc.cit.

**Proposition 10.6.** — Keep the notations of (10.5). Consider the following conditions:

1) \( f \) is proper, \( S \) is excellent regular.

2) \( f \) is proper, \( S \) is locally noetherian normal and at each of its points satisfies the condition (W) (EGA IV 21.12.8).

Then, if 1) (resp. 2)) holds, \( S \) is the cokernel of \((d_o, d_1)\) in the full sub-category of the category of \( S \)-algebraic spaces consisting of the \( S \)-algebraic spaces (resp. \( S \)-schemes) which are \( S \)-separated and locally of finite type over \( S \).

**Proof.** One argues as in [4] 5.11.

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\square
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**Department of Mathematics, University of Toronto**

*E-mail address: zongying@math.utoronto.ca*