ON THE BIRATIONAL GEOMETRY
OF THE UNIVERSAL PICARD VARIETY

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Abstract. We compute the Kodaira dimension of the univer-
sal Picard variety $P_{d,g}$ parameterizing line bundles of degree $d$ on
curves of genus $g$ under the assumption that $(d - g + 1, 2g - 2) = 1$.
We also give partial results for arbitrary degrees $d$ and we investi-
gate for which degrees the universal Picard varieties are birational.

1. Introduction

The study of the birational geometry of the moduli spaces has be-
come a very active research area after the unexpected result of Harris-
Mumford-Eisenbud ([HM82], [EH87]) that the moduli space $M_g$ of
curves of genus $g$ is a variety of general type for $g \geq 24$, contradicting
a long-standing conjecture of Severi on the unirationality of moduli
of curves. More recently, also the birational geometry of other mod-
uli spaces has been widely investigated: the moduli space of pointed
curves ([Log03], [CF07]), the moduli space of Prym varieties ([FL10]),
the moduli space of spin curves ([Lud10], [Far10], [FV10a]), to men-
tion at least some contributions in this area.

The aim of the present paper is to investigate the birational geometry
of the universal Picard variety

$$P_{d,g} \rightarrow M_g,$$

parameterizing smooth curves of genus $g$ together with a line bundle of
degree $d$. The following result is due to Verra (see [Ver05, Thm. 1.2]).

Theorem 1.1. ([Ver05]) The variety $P_{d,g}$ is unirational for $g \leq 9$ and
any $d$.

Our main result is the computation of the Kodaira dimension of $P_{d,g}$
with $g \geq 10$ under a technical assumption on the degree $d$. Recall that,
since $P_{d,g}$ is singular and not projective, the Kodaira dimension of $P_{d,g}$, which we denote by $\kappa(P_{d,g})$, is defined as the Kodaira dimension of any smooth projective model of it (see [Laz04, Example 2.1.5]). The previous result of Verra implies that $\kappa(P_{d,g}) = -\infty$ for $g \leq 9$ and any $d$.

**Theorem 1.2.** Assume that $(d - g + 1, 2g - 2) = 1$ and $g \geq 10$. The Kodaira dimension of $P_{d,g}$ is equal to

$$
\kappa(P_{d,g}) = \begin{cases} 
0 & \text{if } g = 10, \\
19 & \text{if } g = 11, \\
3g - 3 & \text{if } g \geq 12.
\end{cases}
$$

In Propositions 6.5 and 6.3, we also determine the Iitaka fibration (see [Laz04, Def. 1.3.6]) of $P_{d,g}$ in the non-trivial cases, namely for $g \geq 11$. Without any assumption on the degree $d$, we obtain the following partial result:

**Theorem 1.3.** The Kodaira dimension of $P_{d,g}$ (for $g \geq 10$) satisfies the following inequalities

$$
\kappa(P_{d,g}) \leq \begin{cases} 
0 & \text{if } g = 10, \\
19 & \text{if } g = 11, \\
3g - 3 & \text{if } g \geq 12.
\end{cases}
$$

Moreover, $\kappa(P_{d,g}) = 3g - 3$ if $\kappa(M_g) \geq 0$ (and in particular for $g \geq 22$).

Let us now explain the strategy that we use to prove the above results. The main tool we use is the GIT compactification constructed by Caporaso (see [Cap94])

$$
\phi_d : \overline{P}_{d,g} \rightarrow M_g
$$

of $P_{d,g}$ over the Deligne-Mumford moduli space $M_g$ of stable curves of genus $g$. The projective normal variety $\overline{P}_{d,g}$ is a good moduli space for the stack $\overline{\mathcal{P}}_{ic,d,g}$ (see [Cap08] and [Mel09]), whose section over a scheme $S$ is the groupoid $\overline{\mathcal{P}}_{ic,d,g}(S)$ of families of quasistable curves of genus $g$

$$
f : (\mathcal{C}, \mathcal{L}) \rightarrow S
$$

endowed with a balanced line bundle $\mathcal{L}$ of degree $d$ (see [2.1] for details). Furthermore, $\overline{P}_{d,g}$ is a coarse moduli scheme for the stack $\overline{\mathcal{P}}_{ic,d,g}$ if and only if $(d - g + 1, 2g - 2) = 1$, which is precisely the numerical hypothesis on the degree $d$ in Theorem 1.2.

Albeit $\overline{P}_{d,g}$ is singular, we can prove (under the same assumption on the degree) that $\overline{P}_{d,g}$ has canonical singularities and therefore pluricanonical forms on the smooth locus lift to any desingularization:

**Theorem 1.4.** Assume that $(d - g + 1, 2g - 2) = 1$ and that $g \geq 4$. Then $\overline{P}_{d,g}$ has canonical singularities. In particular, if $\overline{P}_{d,g}$ is a resolution
of singularities of $\overline{P}_{d,g}$, then every pluricanonical form defined on the smooth locus $\overline{P}_{d,g}^{\text{reg}}$ of $\overline{P}_{d,g}$ extends holomorphically to $\overline{P}_{d,g}$, that is, for all integers $m$ we have

$$h^0(\overline{P}_{d,g}^{\text{reg}}, mK_{\overline{P}_{d,g}^{\text{reg}}}) = h^0(\overline{P}_{d,g}, mK_{\overline{P}_{d,g}}).$$

The proof of this theorem is given in Section 4. The restriction on the degree $d$ comes from the fact that $\overline{P}_{d,g}$ has finite quotient singularities if and only if $(d - g + 1, 2g - 2) = 1$; hence only for such degrees $d$ we can apply the Reid–Tai criterion for the canonicity of finite quotient singularities (see e.g. [HMS82, p. 27] or [Lud07, Thm. 4.1.11]). Indeed, we establish in Theorem 4.8 a similar statement without any restriction on $d$ for the open subset $P_{d,g}^{\text{st}} \subset P_{d,g}$ of GIT-stable points of $P_{d,g}$, which coincides with $P_{d,g}$ if and only if $(d - g + 1, 2g - 2) = 1$. In proving Theorem 4.8 (from which Theorem 1.4 follows), we determine the non-smooth locus of $P_{d,g}^{\text{st}}$ in Proposition 4.7. Note that a proof of Theorem 1.4 for all degrees $d$ would imply the validity of Theorem 1.2 without any assumptions on the degree $d$.

The above Theorem 1.4 is crucial for our purposes because it allows us to compute the Kodaira dimension of $\overline{P}_{d,g}$ as the Iitaka dimension (see [Laz04, Def. 2.1.3]) of the canonical divisor $K_{\overline{P}_{d,g}}$ on the modular variety $\overline{P}_{d,g}$, instead of working on some (a priori non modular) desingularization of $\overline{P}_{d,g}$. The class of $K_{\overline{P}_{d,g}}$ is given by the following

**Theorem 1.5.** For any $g \geq 4$, we have

$$K_{\overline{P}_{d,g}} = \phi_d^*(14\lambda - 2\delta),$$

where $\lambda, \delta$ denote the Hodge and the boundary class on $\overline{M}_g$, respectively.

The proof of this theorem is given in Section 5. We first compute in Theorems 5.1 the canonical class of $\overline{P}_{d,g}$ through a careful application of the Grothendieck-Riemann-Roch theorem to the universal family over $\overline{P}_{d,g}$. Then we show that the pull-back of $K_{\overline{P}_{d,g}}$ via the canonical map $p : \overline{P}_{d,g} \to \overline{P}_{d,g}$ is equal to $K_{\overline{P}_{d,g}}$. Note that this is in contrast with what happens for $\overline{M}_g$ (or for the moduli space of Prym or spin curves), where the pull-back of the canonical class of the coarse moduli space is equal to the canonical class of the moduli stack plus some (small) corrections at the boundary.

Theorem 1.5 allows us to compute the Iitaka dimension of $K_{\overline{P}_{d,g}}$ as the Iitaka dimension of the divisor $14\lambda - 2\delta$ on $\overline{M}_g$ (because $\phi_d$ is a regular fibration). By exploiting the rich available knowledge on the birational geometry of $\overline{M}_g$, we prove the following
Theorem 1.6. The Iitaka dimension of $K_{P_{d,g}}$ on $P_{d,g}$ is equal to

$$\kappa(K_{P_{d,g}}) = \kappa(14\lambda - 2\delta) = \begin{cases} 
-\infty & \text{if } g \leq 9, \\
0 & \text{if } g = 10, \\
19 & \text{if } g = 11, \\
3g - 3 & \text{if } g \geq 12.
\end{cases}$$

The proof of the above theorem is given in Section 6 by combining Propositions 6.1, 6.3, 6.4, 6.5.

With the above results it is now easy to prove Theorems 1.2 and 1.3. Indeed, note that we have always the inequality

$$(1) \quad \kappa(P_{d,g}) \leq \kappa(K_{P_{d,g}}),$$

with equality if $(d - g + 1, 2g - 2) = 1$ by Theorem 1.4. From (1) and Theorem 1.6, we deduce Theorem 1.2 and the first part of Theorem 1.3.

The second part of Theorem 1.3 follows from Proposition 3.2, which is proved in Section 3 via a careful analysis of the regular fibration $\phi_d : P_{d,g} \to \mathcal{M}_g$.

In the final Section 7, inspired by Lemma 8.1 in [Cap94], we investigate for which values of $d$ and $d'$ the varieties $P_{d,g}$ and $P_{d',g}$ are birational. We prove the following

Theorem 1.7. Assume that $g \geq 22$ or $g \geq 12$ and $(d - g + 1, 2g - 2) = 1$. Then $P_{d,g}$ is birational to $P_{d',g}$ if and only if $d' \equiv \pm d \mod (2g - 2)$. In this case, $P_{d,g}$ is isomorphic to $P_{d',g}$.

This follows from Theorem 1.3, where we also determine the possible birational maps between the varieties $P_{d,g}$ for $g$ big enough. From the same result, we obtain a description the group of birational self-maps of $P_{d,g}$ (see Corollary 7.3) and we deduce that the boundary of $P_{d,g}$ is preserved by any automorphism of $P_{d,g}$ (see Corollary 7.4).

While this work was being written down, Farkas and Verra posted on the arXiv the preprint [FV10b], where they determine, among other things, the Kodaira dimension of $P_{g,g}$ (note that the degree $g$ satisfies the assumptions of our Theorem 1.2, so that their result is a particular case of our main theorem). However, their strategy is different from ours and it seems to apply only in the special case $d = g$. Indeed, the authors of loc. cit. consider the global Abel-Jacobi map $A_{g,d} : M_{g,d}/S_d \to P_{d,g}$ obtained by sending a curve $C$ together with a collection of unordered points $\{p_1, \ldots, p_d\}$ into the pair $(C, \mathcal{O}_C(p_1 + \cdots + p_d))$. It is well-known that the map $A_{g,d}$ is a birational isomorphism in degree $d = g$ (and only in this case). Using this fact, Farkas and Verra determine the Kodaira dimension of $P_{g,g}$ by studying the pluricanonical forms on the Deligne-Mumford-Knudsen compactification $\overline{M}_{g,g}/S_g$ (instead of the Caporaso compactification $\overline{P}_{g,g}$, as we do in this paper).
Throughout this paper, we work over the complex field $\mathbb{C}$. Moreover, we fix two integers $g \geq 2$ and $d$.

2. Preliminaries

2.1. The stack $\overline{\text{Pic}}_{d,g}$ and the scheme $\overline{P}_{d,g}$

In this subsection, we recall the definition of the stack $\overline{\text{Pic}}_{d,g}$ and its good moduli space $\overline{P}_{d,g}$, and collect some of their properties to be used later on.

Let $\text{Pic}_{d,g}$ be the universal Picard stack over the moduli stack $M_g$ of smooth curves of genus $g$. The fiber $\text{Pic}_{d,g}(S)$ of $\text{Pic}_{d,g}$ over a scheme $S$ is the groupoid whose objects are families of smooth curves $C \to S$ endowed with a line bundle $L$ over $C$ of relative degree $d$ over $S$ and whose arrows are the obvious isomorphisms. $\text{Pic}_{d,g}$ is a smooth irreducible (Artin) algebraic stack of dimension $4g - 4$ endowed with a natural forgetful map $\Phi_d : \overline{\text{Pic}}_{d,g} \to M_g$. The stack $\overline{\text{Pic}}_{d,g}$ admits a good moduli scheme $P_{d,g}$ of dimension $4g - 3$ which has a natural forgetful map $\phi_d : P_{d,g} \to M_g$ onto the coarse moduli scheme of smooth curves of genus $g$. We have the following commutative diagram

\[
\begin{array}{ccc}
\text{Pic}_{d,g} & \longrightarrow & P_{d,g} \\
\Phi_d & \downarrow & \phi_d \\
M_g & \longrightarrow & M_g.
\end{array}
\]

Warning 2.2. The fact that $\text{Pic}_{d,g}$ has dimension $4g - 4$ (and not $4g - 3$ as $P_{d,g}$) is due to the fact that on each object $(C \to S, L)$ of $\text{Pic}_{d,g}(S)$ there is an action of the multiplicative group $\mathbb{G}_m$ via scalar multiplication on $L$. Therefore the map $\Phi_d$ factors as

$\Phi_d : \text{Pic}_{d,g} \to \text{Pic}^{\mathbb{G}_m}_{d,g} \to P_{d,g}$,

where $\text{Pic}^{\mathbb{G}_m}_{d,g}$ (which is denoted by $\text{Pic}_{d,g}^{\mathbb{G}_m}$ by some authors) is the $\mathbb{G}_m$-rigidification of $\text{Pic}_{d,g}$ along the subgroup $\mathbb{G}_m$. Note that $\text{Pic}^{\mathbb{G}_m}_{d,g}$ is a Deligne-Mumford stack of dimension $4g - 4$ while $\text{Pic}_{d,g}$ is an Artin stack of dimension $4g - 4$ which is not Deligne-Mumford. However, we will never need the rigidified stack $\text{Pic}^{\mathbb{G}_m}_{d,g}$ in this work so that we refer to [Mel09, Sec. 4] for more details (note that in loc. cit. our stack $\text{Pic}_{d,g}$ is denoted by $G_{d,g}$ while its rigidification $\text{Pic}^{\mathbb{G}_m}_{d,g}$ is denoted by $P_{d,g}$).

The stack $\text{Pic}_{d,g}$ and the scheme $P_{d,g}$ have been compactified in a modular way in [Cap94], [Cap08] and [Mel09]. To describe these compactifications, we need to recall some definitions.

Definition 2.3. A connected, projective nodal curve $C$ is said to be quasistable if it is (Deligne-Mumford) semistable and the exceptional components of $C$ do not meet.
Given a quasi-stable curve $C$, we will denote by $C_{\text{exc}}$ the subcurve of $C$ (called exceptional subcurve) given by the union of all the exceptional components of $C$; by $\widetilde{C} := C \setminus C_{\text{exc}}$ its complementary subcurve (called non-exceptional subcurve) and by $C^{\text{st}}$ the stabilization of $C$. Moreover, we will denote by $\gamma(\widetilde{C})$ the number of connected components of $\widetilde{C}$.

**Definition 2.4.** Let $C$ be a quasistable curve of genus $g \geq 2$ and $L$ a degree $d$ line bundle on $C$.

(i) We say that $L$ is balanced if

- for every subcurve $Z$ of $C$ the following (“Basic Inequality”) holds

\[
\frac{d \deg_Z(\omega_C|_Z)}{2g-2} - \frac{k_Z}{2} \leq \deg_Z L \leq \frac{d \deg_Z(\omega_C|_Z)}{2g-2} + \frac{k_Z}{2},
\]

where $k_Z$ is the number of intersection points of $Z$ with the complementary subcurve $Z^c := \overline{C \setminus Z}$.

- $\deg_E L = 1$ for every exceptional component $E$ of $C$.

(ii) We say that $L$ is strictly balanced if it is balanced and if for each proper subcurve $Z$ of $C$ for which one of the two inequalities in (3) is not strict, then the intersection $Z \cap Z^c$ is contained $C_{\text{exc}}$.

(iii) We say that $L$ is stably balanced if it is balanced and if for each proper subcurve $Z$ of $C$ for which one of the two inequalities in (3) is not strict, then either $Z$ or $Z^c$ is entirely contained in $C_{\text{exc}}$.

The above Definitions 2.4(i) and 2.4(iii) are taken from [CCC07, Def. 5.1.1] (see also [Cap08, Def. 4.6]) and they are equivalent, respectively, to the definitions of semistable in [Cap94, Sec. 5.5] and G-stable in [Cap94, Sec. 6.2]. The Definition 2.4(ii) is taken from [Cap10, Sec. 4.1] and it is equivalent to the definition of extremal in [Cap94, Sec. 5.2].

There is an equivalence relation of the set of balanced line bundles on a quasi-stable curve $C$.

**Definition 2.5.** Given two balanced line bundles $L$ and $L'$ on a quasi-stable curve $C$, we say that $L$ and $L'$ are equivalent, and we write $(C, L) \equiv (C, L')$, if $L_{\widetilde{C}} \cong L'_{\widetilde{C}}$. The equivalence class of a pair $(C, L)$ is denoted by $[(C, L)]$.

Note that the above equivalence relation $\equiv$ clearly preserves the multidegree of the line bundles, hence it preserves the condition of being strictly balanced or stably balanced.

**Remark 2.6.** In the GIT construction of $\overline{P}_{d,g}$ given in [Cap94], the equivalence classes $[(C, L)]$ such that $C$ is quasi-stable and $L$ is balanced (resp. strictly balanced, resp. stably balanced) correspond to the GIT-semistable (resp. GIT-polystable, resp. GIT-stable) orbits (see [Cap94, Prop. 6.1, Lemma 6.1] and also [CCC07, Thm. 5.1.6]).
The relationship between stably balanced and strictly balanced line bundles is given by the following

**Lemma 2.7.** A line bundle \(L\) on a quasi-stable curve \(C\) is stably balanced if and only if it is strictly balanced and \(\widetilde{C}\) is connected.

**Proof.** Assume first that \(L\) is strictly balanced and that \(\widetilde{C}\) is connected. Let \(Z\) be a proper subcurve of \(C\) such that one of the two inequalities in (3) is not strict. Then \(Z \cap Z^c \subset C_{exc}\) because \(L\) is strictly balanced by hypothesis. Therefore the non-exceptional subcurve \(\widetilde{C}\) can be written as a disjoint union of the two subcurves \(Z \cap \widetilde{C}\) and \(Z^c \cap \widetilde{C}\). Since \(\widetilde{C}\) is connected by hypothesis, we must have that either \(Z \cap \widetilde{C} = \emptyset\) or \(Z^c \cap \widetilde{C} = \emptyset\), which implies that either \(Z \subset C_{exc}\) or \(Z^c \subset C_{exc}\), respectively. This shows that \(L\) is stably balanced.

Conversely, assume that \(L\) is stably balanced. Clearly this implies that \(L\) is strictly balanced. Assume, by contradiction, that \(\widetilde{C}\) is not connected. Then we can find two proper disjoint subcurves \(D_1\) and \(D_2\) of \(C\) that are not contained in \(C_{exc}\) and such that \(E := (D_1 \cup D_2)^c\) is the union of \(r \geq 1\) exceptional components of \(C\). It is easily checked that

\[
\begin{align*}
\left\{ \begin{array}{l}
\deg_{D_1 \cup E}(\omega_C) = \deg_D(\omega_C), \\
k_{D_1 \cup E} = k_{D_1} = r, \\
\deg_{D_1 \cup E} L = \deg_{D_1} L + r.
\end{array} \right.
\tag{*}
\end{align*}
\]

Applying the inequality (3) to the subcurves \(D_1\) and \(D_1 \cup E\), we get

\[
\frac{r}{2} = \frac{k_{D_1}}{2} \leq \deg_{D_1} L - d \frac{\deg_D(\omega_C)}{2g - 1} = \deg_{D_1 \cup E} L - r - d \frac{\deg_{D_1 \cup E}(\omega_C)}{2g - 2} \leq \frac{k_{D_1 \cup E}}{2} - r = \frac{r}{2}.
\]

Therefore one of the inequalities (3) is strict fo the subcurve \(D_1\) and this contradicts the fact that \(L\) is strictly balanced since \(\emptyset \neq D_1 \subset C_{exc}\) by construction. \(\square\)

Let \(\overline{Pic}_{d,g}\) be the category whose objects are families of quasistable curves \(\mathcal{C} \to S\) endowed with a line bundle \(\mathcal{L}\) of relative degree \(d\) whose restriction to each geometric fiber is balanced and whose arrows are Cartesian diagrams of such families. Cleary \(\overline{Pic}_{d,g}\) is a category fibered in groupoids over the category of schemes. The following theorem summarizes some of the properties of \(\overline{Pic}_{d,g}\) and of its good moduli space \(\overline{P}_{d,g}\) known thanks to Caporaso and Melo (note that our stacks \(\overline{Pic}_{d,g}\) and \(\overline{P}_{d,g}\) are called \(G_{d,g}\) and \(\mathcal{T}_{d,g}\) in [Mel09]).

**Theorem 2.8.** ([Cap94], [Cap08], [Mel09])

1. \(\overline{Pic}_{d,g}\) is an irreducible, smooth and universally closed Artin stack of finite type over \(\mathbb{C}\) and of dimension \(4g - 4\). It contains the stack \(\overline{Pic}_{d,g}\) as a dense open substack.
(2) $\mathcal{P}_{d,g}$ admits a good moduli space $\overline{P}_{d,g}$, that is a normal irreducible projective variety of dimension $4g - 3$. The geometric points of $\overline{P}_{d,g}$ correspond bijectively to the equivalence classes of pairs $(C, L)$ where $C$ is a quasi-stable curve of genus $g$ and $L$ is a strictly balanced line bundle of degree $d$.

(3) $\overline{P}_{d,g}$ is a coarse moduli scheme for $\mathcal{P}_{d,g}$ if and only if $(d + 1 - g, 2g - 2) = 1$. In this case $\overline{P}_{d,g}$ has only finite quotient singularities.

The construction of the scheme $\overline{P}_{d,g}$ as a GIT-quotient is due to Caporaso (see [Cap94]); the construction of the stack $\mathcal{P}_{d,g}$ is due to Caporaso (see [Cap98]) in the case $(d + 1 - g, 2g - 2) = 1$ and to Melo (see [Mel09]) in the general case. Note that we have a natural commutative diagram compactifying the diagram (2):

$$
\begin{array}{ccc}
\mathcal{P}_{d,g} & \to & P_{d,g} \\
\Phi_c & \searrow & \downarrow \phi_d \\
M_g & \to & M_g.
\end{array}
$$

**Notation 2.9.** From now on, for the ease of notation, whenever we write $(C, L) \in \mathcal{P}_{d,g}$ we mean that $L$ is a strictly balanced line bundle on the quasi-stable curve $C$, considered up to the equivalence relation of Definition 2.5.

Next we introduce an open subset of $\mathcal{P}_{d,g}$ that will play a special role in the sequel.

**Definition 2.10.** We denote by $\mathcal{P}_{d,g}^{st}$ the open subset of $\mathcal{P}_{d,g}$ consisting of pairs $(C, L) \in \mathcal{P}_{d,g}$ where $L$ is stably balanced.

By the above Remark 2.8, $\mathcal{P}_{d,g}^{st}$ is the open subset of $\mathcal{P}_{d,g}$ where the GIT quotient is geometric. In [Cap94] Lemma 2.2, it is proved that the semistable locus (called $H_d$ in loc. cit.) inside the Hilbert scheme whose GIT quotient gives $\mathcal{P}_{d,g}$ is smooth. From this, it follows that $\mathcal{P}_{d,g}^{st}$ has finite quotient singularities (see (12) for an explicit local description). Moreover, $\mathcal{P}_{d,g}^{st} = \mathcal{P}_{d,g}$ if and only if $(d + 1 - g, 2g - 2) = 1$ by [Cap94] Prop. 6.2.

Albeit $\mathcal{P}_{d,g}$ has not necessarily finite quotient singularities, we have the following useful result (see the proof of [Fon05] Cor. 1)):

**Theorem 2.11.** ([Fon05]) $\mathcal{P}_{d,g}$ is a $\mathbb{Q}$-factorial variety.

In view of the above result, we will identify throughout this paper $\mathbb{Q}$-Weil divisors and $\mathbb{Q}$-Cartier divisors on $\mathcal{P}_{d,g}$.

**2.12. The automorphism group $\text{Aut}(C, L)$**

For later use, we describe the automorphism group of a pair $(C, L)$ consisting of a quasi-stable curve $C$ and a balanced line bundle $L$ on
An automorphism of \((C, L)\) is given by a pair \((\sigma, \psi)\) such that \(\sigma \in \text{Aut}(C)\) and \(\psi\) is an isomorphism between the line bundles \(L\) and \(\sigma^*(L)\). The group of automorphisms of \((C, L)\) is denoted by \(\text{Aut}(C, L)\).

We get a natural forgetful homomorphism
\[
F : \text{Aut}(C, L) \rightarrow \text{Aut}(C)
\]
\[
(\sigma, \psi) \mapsto \sigma
\]
whose kernel is the multiplicative group \(\mathbb{G}_m\), acting as fiberwise multiplication on \(L\), and whose image is the subgroup of \(\sigma \in \text{Aut}(C)\) such that \(\sigma^* (L) \cong L\). The quotient \(\text{Aut}(C, L)/\mathbb{G}_m\) is denoted by \(\text{Aut}_{\text{red}}(C, L)\). Note that \(\text{Aut}(C, L)\) depends only on the equivalence class \([\langle L, C \rangle]\) (see Def. 2.5).

By composing the above homomorphism \(F\) of (5) with the natural homomorphism \(\text{Aut}(C) \rightarrow \text{Aut}(C^s)\) induced by the stabilization map \(C \rightarrow C^s\), we get a homomorphism
\[
G : \text{Aut}(C, L) \rightarrow \text{Aut}(C^s),
\]
whose kernel is described in the next Lemma.

**Lemma 2.13.** We have a commutative diagram with exact rows
\[
\begin{array}{cccccccccc}
0 & \rightarrow & \mathbb{G}_m^{\gamma(C)} & \rightarrow & \text{Aut}(C, L) & \rightarrow & \text{Aut}(C^s) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{G}_m^{\gamma(C)}/\mathbb{G}_m & \rightarrow & \text{Aut}(C, L)/\mathbb{G}_m & \rightarrow & \text{Aut}(C^s) & \rightarrow & 0
\end{array}
\]
where \(\mathbb{G}_m \subseteq \mathbb{G}_m^{\gamma(C)}\) is the diagonal embedding.

**Proof.** The exactness of the first row is proved using an argument similar to the one used in the proof of [CCC07, Lemma 2.3.2]. We sketch the argument for the sake of completeness. Let \(E_1, \ldots, E_m\) be the exceptional components of \(C\) and let \(X_1, \ldots, X_{\gamma(C)}\) be the connected components of \(\tilde{C}\). We identify each \(E_i\) to a copy of \(\mathbb{P}^1\) attached to the rest of the curve at the points 0 and \(\infty\). An element \((\sigma, \psi) \in \text{Aut}(C, L)\) belongs to the kernel of the map \(\text{Aut}(C, L) \rightarrow \text{Aut}(C^s)\) if and only if \(\sigma_{| \tilde{C}} = \text{id}_{\tilde{C}}\) and \(\sigma\) acts as multiplication by \(m_i \in \mathbb{G}_m(k) = k^*\) on the exceptional component \(E_i\). If we restrict the isomorphism \(\psi : L \rightarrow \sigma^*(L)\) to \(\tilde{C}\), we get that \(\psi\) is the fiberwise multiplication by \(l_j \in \mathbb{G}_m(k) = k^*\) on each line bundle \(L_{X_j}\). The scalars \(m_i\) are uniquely determined by the scalars \(l_j\): if \(0 \in E_i\) lies on the component \(X_j\) and \(\infty \in E_i\) lies on the component \(X_h\) (possibly with \(j = h\)), then by the compatibility between \(\sigma\) and \(\psi\) we get that \(m_i = l_j/l_h\) (see the proof of [CCC07, Lemma 2.3.2]). Therefore the element \((\sigma, \psi)\) is uniquely determined by the scalars \(l_1, \ldots, l_{\gamma(C)}\) and we are done.
From the above proof, it is clear that the homomorphisms corresponding to the diagonal embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^{(C)}$ are exactly the fiberwise automorphisms on $L$, hence the exactness of the second row follows. □

**Corollary 2.14.** If $(C, L)$ is stably balanced then $\text{Aut}(C, L)$ is a subgroup of $\text{Aut}(C^{st})$. In particular, $\text{Aut}(C, L)$ is a finite group.

**Proof.** The first assertion follows from the last row of the diagram in Lemma 2.13 together with the fact that if $L$ is stably balanced on $C$ then $\tilde{C}$ must be connected by Lemma 2.7. The last assertion follows from the first one together with the well-known fact that the automorphism group of a stable curve is finite. □

**2.15. The local structure of $\mathcal{P}_{d,g}$**

The complete local ring $\hat{O}_{\mathcal{P}_{d,g}}(C, L)$ of $\mathcal{P}_{d,g}$ at a point $(C, L)$ can be described using the deformation theory of pairs $(C, L)$, which can be found in [Ser06, Sec. 3.3.3] for $C$ smooth and has been extended to the singular case in [Wan10].

Let us recall the results of [Wan10]. We denote by $\mathcal{P}_C^1(L)$ the sheaf of one jets (or sheaf of principle parts) of $L$ on $C$ ([Wan10, Sec. 2]). The sheaf $\mathcal{P}_C^1(L)$ fits into an exact sequence (see [Wan10, Eq. (2.1)])

$$
0 \to \Omega_C^1 \to \mathcal{P}_C^1(L) \otimes L^{-1} \to \mathcal{O}_C \to 0.
$$

Explicitly, the above extension can be described as follows (see [Ser06, p. 145]). Let $\mathcal{O}_C^\ast \to \Omega_C^1$ be the homomorphism of sheaves given by sending $u \in \Gamma(U, \mathcal{O}_C^\ast)$ into $\frac{du}{u} \in \Gamma(U, \Omega_C^1)$ for any open subset $U \subseteq C$. By passing to cohomology, we get a group homomorphism $\theta_C : \text{Pic}(C) = H^1(C, \mathcal{O}_C^\ast) \to H^1(C, \Omega_C^1)$. By using the identification $H^1(C, \Omega_C^1) \cong \text{Ext}^1(\mathcal{O}_C, \Omega_C^1)$, the map $\theta_C$ sends the line bundle $L$ to the class of the extension.

Wang in [Wan10] proves that the sheaf $\mathcal{P}_C^1(L)$ controls the tangent and obstruction theory of the pair $(C, L)$. Let us denote by $\text{Def}_{(C,L)}$ the functor of infinitesimal deformations of the pair $(C, L)$ (see [Ser06, p. 146]) and by $T_{\text{Def}_{(C,L)}}$ the tangent space to $\text{Def}_{(C,L)}$ (in the sense of [Ser06, Lemma 2.2.1]).

**Theorem 2.16.** ([Wan10])

(i) We have that $T_{\text{Def}_{(C,L)}} = \text{Ext}^1(\mathcal{P}_C^1(L), L)$.

(ii) An obstruction space for $\text{Def}_{(C,L)}$ is given by $\text{Ext}^2(\mathcal{P}_C^1(L), L)$.

**Proof.** Part (i) is [Wan10, Thm. 3.1(1)]; Part (ii) is [Wan10, Thm. 4.6(a)]. □

Moreover, the infinitesimal automorphisms of the pair $(C, L)$ are governed by $\text{Ext}^0(\mathcal{P}_C^1(L), L)$, as shown by the following Lemma.
Lemma 2.17. The tangent space of Aut(C, L) at the identity is equal to Ext^0(\mathcal{P}^1_C(L), L).

Proof. The Lemma is certainly well-known to the experts, at least in the case where C is smooth. However, we include a proof for the lack of a suitable reference.

According to the discussion in Proposition 2.12, we have an exact sequence of groups

\[ 0 \to \mathbb{G}_m \to \text{Aut}(C, L) \to \text{Stab}_L(\text{Aut}(C)) \to 0, \]

where \text{Stab}_L(\text{Aut}(C)) is the stabilizer of L in Aut(C), i.e. the subgroup of Aut(C) consisting of all the automorphisms \sigma of C such that \sigma^*(L) \cong L. In other words, \text{Stab}_L(\text{Aut}(C)) is the image of Aut(C, L) via the map F of Equation (5). By passing to the tangent spaces at the origin, we get

\[ (7) \quad 0 \to T_0 \mathbb{G}_m = k \to T_0 \text{Aut}(C, L) \to T_0 \text{Stab}_L(\text{Aut}(C)) \to 0, \]

where we have denoted by 0 the identity element in each of the above groups.

On the other hand, by dualizing Equation (3), we get the short exact sequence

\[ (8) \quad 0 \to \mathcal{O}_C \to \mathcal{P}^1(L) \lor \otimes L \to T_C \to 0. \]

Passing to cohomology, we get the exact sequence

\[ (9) \quad 0 \to H^0(C, \mathcal{O}_C) = k \to H^0(C, \mathcal{P}^1(L) \lor \otimes L) \xrightarrow{\partial} H^0(C, T_C). \]

Compare now the exact sequences (7) and (9). Since \text{Stab}_L(\text{Aut}(C)) is the image of Aut(C, L) via the map F of Equation (5), by passing to the tangent spaces at the origin, we get

\[ (10) \quad T_0 \text{Stab}_L(\text{Aut}(C)) = \text{Im}(p). \]

Let \mathcal{U} = \{U_\alpha\} be an affine open covering of C trivializing L and let \eta_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{O}_C^*) the transition functions of L with respect to \mathcal{U}, where as usual \text{U}_{\alpha\beta} := \text{U}_\alpha \cap \text{U}_\beta. Then \theta_\alpha(C) \in H^1(C, \Omega^1_C) is represented by the Čech 1-cocycle \((\frac{\text{d} \eta_{\alpha\beta}}{\text{d} \eta_{\beta\alpha}})_{\alpha\beta}\) \in Z^1(\mathcal{U}, \Omega^1_C) (see [Ser06, p. 145]). From the exact sequence [5], it follows that the sheaf \((\mathcal{P}^1(L) \lor \otimes L)|_{U_0}\) is isomorphic to \((\mathcal{O}_C)|_{U_0} \lor (T_C)|_{U_0}\) and an element of Ext^0(\mathcal{P}^1(L), L) = H^0(C, \mathcal{P}^1(L) \lor \otimes L) is represented by a Čech 0-cochain

\[ (k_\alpha, d_\alpha) \in C^0(\mathcal{U}, \mathcal{P}^1(L) \lor \otimes L) = C^0(\mathcal{U}, \mathcal{O}_C) \lor C^0(\mathcal{U}, T_C), \]

which satisfies the cocycle conditions: \(d_\alpha = d_\beta\) and \(k_\beta - k_\alpha = \frac{d_\alpha(f_{\alpha\beta})}{f_{\alpha\beta}}\) on \text{U}_{\alpha\beta} = \text{U}_\alpha \cap \text{U}_\beta (see [Ser06, p. 145]). Since the \(f_{\alpha\beta}\)'s are the transition functions of L, we conclude that the image of p consists of all the Čech 0-cocycles \((d_\alpha) \in Z^0(\mathcal{U}, T_C)\) corresponding to the infinitesimal automorphisms of C which preserve the line bundle L. In other words, Equation (10) is satisfied and we are done.

\[ \square \]

We can now compute the dimension of the vector spaces Ext^1(\mathcal{P}^1_C(L), L).
Lemma 2.18. We have that
\[
\begin{align*}
\dim \text{Ext}^i(P_C^1(L), L) &= 0 \text{ for } i \geq 2, \\
\dim \text{Ext}^1(P_C^1(L), L) &= 4g - 4 + \gamma(C), \\
\dim \text{Ext}^0(P_C^1(L), L) &= \gamma(C).
\end{align*}
\]

Proof. By applying the functor Hom(−, O_C) to the exact sequence (6) and using that \(\text{Ext}^{\geq 2}(O_C, O_C) = H^{\geq 2}(C, O_C) = 0\) since \(C\) is a curve and that \(\text{Ext}^{\geq 2}(\Omega^1_C, O_C) = 0\) by [DM69, Lemma 1.3], we get the vanishing \(\text{Ext}^{\geq 2}(P_C^1(L), L) = 0\). The fact that \(\text{Ext}^0(P_C^1(L), L) = \gamma(C)\) follows from Lemma 2.13 and 2.17. Finally, from the exact sequence (6), we get
\[
\begin{align*}
\dim \text{Ext}^0(P_C^1(L), L) - \dim \text{Ext}^1(P_C^1(L), L) &= \\
\dim \text{Ext}^0(O_C, O_C) - \dim \text{Ext}^1(O_C, O_C) = \\
\dim \text{Ext}^0(\Omega^1_C, O_C) - \dim \text{Ext}^1(\Omega^1_C, O_C) &= -(g - 1) - (3g - 3) = -(4g - 4),
\end{align*}
\]
from which we conclude. \(\square\)

We can now prove that the functor \(\text{Def}_{(C, L)}\) has a semiuniversal formal element (in the sense of [Ser06, Def. 2.2.6]).

Proposition 2.19.

(i) The functor \(\text{Def}_{(C, L)}\) has a semiuniversal formal element \(\text{Def}_{(C, L)}\).

(ii) \(\text{Def}_{(C, L)}\) is equal to the formal spectrum of \(k[[x_1, \ldots, x_{4g - 4 + \gamma(C)}]]\).

Proof. Part (i) is proved in [Ser06, Thm 3.3.11(i)] in the case where \(C\) is smooth. The proof of loc. cit. consists in showing that Schlessinger’s conditions are satisfied and this extends to the case where \(C\) is nodal: the crucial point of the proof is showing that \(T_{\text{Def}_{(C, L)}}\) is finite dimensional, and this follows in our case from Theorem 2.16(i).

From Theorem 2.16 and Lemma 2.18 it follows that \(\text{Def}_{(C, L)}\) is formally smooth and that the dimension of the tangent space at its unique closed point is \(4g - 4 + \gamma(C)\), from which part (ii) follows. \(\square\)

Now we can describe the complete local ring \(\hat{\mathcal{O}}_{T_{d,g}(C, L)}\) of \(T_{d,g}\) at a point \((C, L)\). Note that the automorphism group \(\text{Aut}(C, L)\) acts on \(\text{Def}_{(C, L)}\) (hence on \(\mathbb{C}[[x_1, \ldots, x_{4g - 4 + \gamma(C)}]]\)) by the semiuniversality of \(\text{Def}_{(C, L)}\). By a standard argument based on Luna’s étale slice theorem (see [Lun73, p. 97] and also [Las96, Sec. II], [Dre04, Sec. 7.4]), the formal spectrum (which we denote by Spf) of the complete local ring \(\hat{\mathcal{O}}_{T_{d,g}(C, L)}\) of \(T_{d,g}\) at the point \((C, L) \in T_{d,g}\) is given by
\[
\text{Spf}\hat{\mathcal{O}}_{T_{d,g}(C, L)} = \frac{\text{Def}_{(C, L)}}{\text{Aut}(C, L)},
\]
where \(\text{Def}_{(C, L)}/\text{Aut}(C, L)\) is the quotient of \(\text{Def}_{(C, L)}\) with respect to the natural action of \(\text{Aut}(C, L)\). In other words, \(\text{Def}_{(C, L)}/\text{Aut}(C, L)\)
is equal to the formal spectrum of the ring of invariants \( \mathbb{C}[[x_1, \ldots, x_{d+4-g(C)}]]^{\text{Aut}(C, L)} \) (see Proposition 2.19).

Clearly, the scalar automorphisms \( \mathbb{G}_m \subset \text{Aut}(C, L) \) act trivially on \( \text{Def}_{(C, L)} \) and thus we get the alternative description:

\[
\text{Spf} \hat{O}_{\mathcal{P}_{d,g}, (C, L)} = \text{Def}_{(C, L)}/ \text{Aut}(C, L).
\]

Note that, from the above description and Lemma 2.13 it follows that \( \mathcal{P}_{d,g}^{\text{st}} \) is the open subset of \( \mathcal{P}_{d,g} \) consisting of pairs \((C, L) \in \mathcal{P}_{d,g}\) such that \( \tilde{C} \) is connected.

We can similarly describe the morphism \( \phi_d : \mathcal{P}_{d,g} \to \mathcal{M}_g \) locally at \((C, L) \in \mathcal{P}_{d,g}\). Denote by \( \text{Def}_C \) (resp. \( \text{Def}_{C^{\text{st}}} \)) the semiuniversal formal element associated to the infinitesimal deformation functor \( \text{Def}_C \) (resp. \( \text{Def}_{C^{\text{st}}} \)) of \( C \) (resp. \( C^{\text{st}} \)), see [Ser10, Cor. 2.4.2].

Locally at \((C, L) \in \mathcal{P}_{d,g}\), the morphism \( \phi_d : \mathcal{P}_{d,g} \to \mathcal{M}_g \) is given by

\[
\text{Spf} \hat{O}_{\mathcal{P}_{d,g}, (C, L)} = \text{Def}_{(C, L)}/ \text{Aut}(C, L) \to \text{Spf} \hat{O}_{\mathcal{M}_g, \text{Cst}} = \text{Def}_{C^{\text{st}}}/ \text{Aut}(C^{\text{st}}),
\]

where the homomorphism of groups \( \text{Aut}(C, L) \to \text{Aut}(C^{\text{st}}) \) is the one given by Lemma 2.13 and the morphism \( \text{Def}_C \to \text{Def}_{C^{\text{st}}} \) is the composition of the forgetful morphism \( \text{Def}_C \to \text{Def}_f \) with the stabilization morphism \( \text{Def}_f \to \text{Def}_{C^{\text{st}}} \). The induced morphism at the level of tangent spaces

\[
T_{\text{Def}_C} = \text{Ext}^1(\mathcal{P}_C^{\text{st}}(L), L) \to T_{\text{Def}_{C^{\text{st}}}} = \text{Ext}^1(\Omega_C^{\text{st}}, \mathcal{O}_C^{\text{st}})
\]

is given by composing the morphism \( \text{Ext}^1(\mathcal{P}_C^{\text{st}}(L), L) \to \text{Ext}^1(\Omega_C^{\text{st}} \otimes L, L) = \text{Ext}^1(\Omega_C^{\text{st}}, \mathcal{O}_C) \) induced by the exact sequence \( \mathfrak{0} \) with the morphism \( \text{Ext}^1(\Omega_C^{\text{st}}, \mathcal{O}_C) \to \text{Ext}^1(\Omega_C^{\text{st}}, \mathcal{O}_C^{\text{st}}) \) induced by the stabilization map \( C \to C^{\text{st}} \). More precisely, let \( f : C \to C^{\text{st}} \) be the stabilization morphism and denote by \( Lf^* \) (resp. \( Rf_* \)) the left derived functor of \( f^* \) (resp. the right derived functor of \( f_* \)). We have a natural map

\[
\text{Ext}^1(\Omega_C^{\text{st}}, \mathcal{O}_C) \to \text{Ext}^1(Lf^* \Omega_C^{\text{st}}, \mathcal{O}_C) = \text{Ext}^1(\Omega_C^{\text{st}}, \mathcal{O}_C).
\]

The first map in \( \mathfrak{1} \) is induced by the composite map \( Lf^* \Omega_C^{\text{st}} \to \Omega_C^{\text{st}} = f^* \Omega_C^{\text{st}} \to \Omega_C^{\text{st}} \) in the derived category of coherent sheaves on \( C \). The equality in \( \mathfrak{1} \) follows from the adjointness of the functors \( Lf^* \) and \( Rf_* \) between the derived category of coherent sheaves on \( C \) and on \( C^{\text{st}} \), together with the fact that \( Rf_* \mathcal{O}_C \cong \mathcal{O}_C^{\text{st}} \) because \( f \) is a sequence of blow-ups with projective spaces as fibers.

3. The fibration \( \phi_d : \mathcal{P}_{d,g} \to \mathcal{M}_g \)

The aim of this section is to prove Proposition 3.2 below, which gives the second part of Theorem 1.3.

To this aim, we analyze the natural morphism \( \phi_d : \mathcal{P}_{d,g} \to \mathcal{M}_g \). Note that \( \phi_d \) is a regular fibration (i.e. a proper, surjective morphism with connected fibers), whose general fiber is the degree-\( d \) Jacobian Pic^d(C)
of a general \([C] \in \overline{M}_g\). Following Kawamata (see [Kaw85 Sec. 1] and [Kaw85 Cor. 7.3]), we define the variation \(\text{Var}(\phi_d)\) of \(\phi_d\) to be

\[
\text{Var}(\phi_d) = \dim \overline{M}_g - \dim \text{Ker}(\delta_C),
\]

where

\[
\delta_C : T_{M_g,C} = H^1(C, T_C) \to H^1(\text{Pic}^d(C), T_{\text{Pic}^d(C)})
\]

is the Kodaira-Spencer map associated to \(\phi_d\) at a general point \(C \in \overline{M}_g\).

**Lemma 3.1.** The algebraic fiber space \(\phi_d : \overline{P}_{d,g} \to \overline{M}_g\) has maximal variation, i.e. \(\text{Var}(\phi_d) = 3g - 3\).

**Proof.** By (16), we have to prove the injectivity of the Kodaira-Spencer map \(\delta_C\) for a general curve \(C \in \overline{M}_g\).

We will reinterpret the above Kodaira-Spencer map as composition of certain maps that were studied in [OS79 Sec. 2], in their analysis of the local Torelli problems for curves. We need to recall their setting, with the simplification that, since we are only interested in the general curve, we can work directly with the coarse moduli spaces \(M_g\) and \(A_g\) (where, as usual, \(A_g\) denotes the coarse moduli space of principally polarized abelian varieties of dimension \(g\)), without having to pass to their \(n\)-level covers. Consider the following commutative diagram (see [OS79 p. 169]):

\[
\begin{array}{ccc}
T_{M_g,C} & \xrightarrow{k_C} & H^1(C, T_C) \\
\downarrow dt_g & & \downarrow du \\
T_{A_g,\text{Pic}^d(C)} & \xrightarrow{k_{\text{Pic}^d(C)}} & \text{Pic}^d(C), T_{\text{Pic}^d(C)}
\end{array}
\]

where \(u : C \to \text{Pic}^d(C)\) is an Abel-Jacobi map (well-defined only up to translation), \(t_g : M_g \to A_g\) is the classical Torelli map, \(k_C\) is the Kodaira-Spencer map in \(C\) associated to the universal family over an open subset of \(M_g\) containing \(C\) and \(k_{\text{Pic}^d(C)}\) is the Kodaira-Spencer map in \(\text{Pic}^d(C)\) associated to the universal family over an open subset of \(A_g\) containing \(\text{Pic}^d(C)\). The map \(k_C\) is an isomorphism (see e.g. [OST99 Thm. 2.2]) and the map \(u^*\) is an isomorphism since \(T_{\text{Pic}^d(C)}\) is the trivial bundle of rank \(g\) and

\[
u^* : H^1(\text{Pic}^d(C), \mathcal{O}_{\text{Pic}^d(C)}) \xrightarrow{\cong} H^1(C, \mathcal{O}_C).
\]

It is easy to see that

\[
\delta_C = k_{\text{Pic}^d(C)} \circ dt_g = (u^*)^{-1} \circ du \circ k_C.
\]
Therefore, the injectivity of $\delta_C$ is equivalent to the injectivity of $du$. According to [OS79, Thm. 2.6], the map $du$ is the dual of the multiplication map

$$\mu : H^0(C, \omega_C) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C^2),$$

which is well-known (Noether’s theorem) to be surjective if $g = 2$ or if $g \geq 3$ and $C$ is not hyperelliptic. Since we assumed $C$ to be generic, we deduce the injectivity of $\delta_C$ and we are done. □

**Proposition 3.2.**

(1) We have that $\kappa(P_{d,g}) \leq 3g - 3$.

(2) If $\kappa(M_g) \geq 0$, then $\kappa(P_{d,g}) = 3g - 3$ and the map $\phi_d : P_{d,g} \to \overline{M}_g$ is the Iitaka fibration of $P_{d,g}$.

**Proof.** The subadditivity of the Kodaira dimension (see [Uen75, Thm. 6.12]) applied to the regular fibration $\phi_d$ gives that

$$\kappa(P_{d,g}) \leq \dim M_g + \kappa(\phi_d^{-1}(C)),$$

for a general $C \in \overline{M}_g$. Since, for a general $C \in \overline{M}_g$, the fiber $\phi_d^{-1}(C) = \text{Pic}^d(C)$ is an abelian variety, we have that $\kappa(\phi_d^{-1}(C)) = 0$, which proves part (1).

Assume now that $\kappa(M_g) \geq 0$. Observe that $\text{Pic}^d(C)$ is a good minimal model, since it is smooth and the canonical $K_{\text{Pic}^d(C)}$ is trivial and thus clearly semi-ample. Therefore, the Iitaka conjecture (in the stronger form of [Kaw85, p. 1]) does hold true by [Kaw85, Cor. 1.2] and gives that

$$\kappa(P_{d,g}) \geq \kappa(\phi_d^{-1}(C)) + \max\{\kappa(M_g), \text{Var}(\phi_d)\} = 3g - 3,$$

using the above Lemma 3.1. This, combined with part (1), proves that $\kappa(P_{d,g}) = 3g - 3$.

The last part follows from the birational characterization of the Iitaka fibration (see e.g. [Uen75, Thm. 6.11]) since $\phi_d : P_{d,g} \to \overline{M}_g$ is an algebraic fiber space such that $\dim \overline{M}_g = \kappa(P_{d,g})$ and the generic fiber $\phi_d^{-1}(C) = \text{Pic}^d(C)$ is smooth and irreducible of Kodaira dimension zero. □

4. The singularities of $P^{st}_{d,g}$

The purpose of this section is to study the singularities of $P_{d,g}$ with the aim of proving Theorem 1.4. More generally, we will prove a similar statement (see Theorem 4.8) for the open subvariety $P^{st}_{d,g} \subseteq P_{d,g}$ of (2.1) and for any degree $d$. Since $P^{st}_{d,g} = P^{st}_{d,g}$ if (and only if) $(d + 1 - g, 2g - 2) = 1$ (by [Cap94, Prop. 6.2]), Theorem 1.4 is a special case of Theorem 4.8 below. The need of restricting ourselves to the open subset $P^{st}_{d,g}$ is due to the fact that $P_{d,g}$ has finite quotient singularities and therefore we can apply the Reid–Tai criterion for the canonicity
of finite quotient singularities (see e.g. [HMS82, pp. 27-28] or [Lud07, Thm. 4.1.11]). For simplicity, we assume throughout this section that $g \geq 4$ in order to avoid problems with the hyperelliptic locus in $\overline{M}_g$.

In the analysis of the singularities of $\overline{P}^\text{st}_{d,g}$, the pairs $(C, L) \in \overline{P}^\text{st}_{d,g}$ such that $C$ contains an elliptic tail will play a special role. Let us give some definitions.

**Definition 4.1.** A connected subcurve $E$ of a quasi-stable curve $C$ is called an elliptic tail if it has arithmetic genus 1 and meets the rest of the curve in exactly one node $P$ which is called an elliptic tail node.

The following remark is straightforward.

**Remark 4.2.** If a quasi-stable curve $C$ has an elliptic tail $E \subseteq C$ then the image $E^\text{st} := \text{st}(E) \subseteq C^\text{st}$ of $E$ via the stabilization morphism $\text{st} : C \to C^\text{st}$ is an elliptic tail of $C^\text{st}$. Conversely, if $C^\text{st}$ has an elliptic tail $E' \subseteq C^\text{st}$, then $E := \text{st}^{-1}(E')$ is an elliptic tail of $C$ such that $E' = E^\text{st}$.

In the next Lemma, we describe the pairs $(C, L) \in \overline{P}^\text{st}_{d,g}$ such that $C$ has an elliptic tail $E \subseteq C$.

**Figure 1.** The possible elliptic tails of a quasi-stable curve $C$ such that $(C, L) \in \overline{P}^\text{st}_{d,g}$ for some line bundle $L$.

**Lemma 4.3.** Let $(C, L) \in \overline{P}^\text{st}_{d,g}$ such that $C$ has an elliptic tail $E \subseteq C$. Then we have:

(i) $d \not\equiv g - 1 \mod (2g - 2)$ and the degree $\deg_E(L)$ of $L$ on $E$ is the unique integer $d_E$ such that

$$\frac{1}{2} < d_E - \frac{d}{2g - 2} < \frac{1}{2}.$$
(ii) The elliptic tail node $P$ does not belong to $C_{\text{exc}}$.

(iii) $E$ is either smooth or it is a rational (irreducible) curve with one node $Q$ or it is formed by two smooth rational curves $R_1$ and $R_2$ meeting in two points $Q_1$ and $Q_2$, as depicted in Figure 1.

Proof. The equation (18) follows from the basic inequality (3) applied to the subcurve $E \subseteq C$ together with the fact that the inequalities must be strict since clearly $E \not\subseteq C_{\text{ex}}$ and $L$ is stably balanced by the hypothesis that $(C, L) \in \mathcal{P}^{st}_{d,g}$. The fact that $d \neq g - 1 \mod (2g - 2)$ follows from the fact that there exists an integer $d_E$ satisfying the strict inequalities in (18). This proves part (i).

Next we turn to Part (ii). By contradiction, if $P \in R$ where $R$ is an exceptional component of $C$, then $R^e$ is a disjoint union of two sub-curves of $C$ each of which contains some components of $C$. Therefore $C$ is disconnected and this contradicts the hypothesis that $(C, L) \in \mathcal{P}^{st}_{d,g}$ by Lemma 2.7.

Part (iii) follows from the fact that $C$ is quasi-stable together with Part (ii) and the fact that $E^t$ is either a smooth elliptic curve (which occurs for Type I) or a rational irreducible curve with one node $Q$ (which occurs for Types II and III). □

The elements $(C, L) \in \mathcal{P}^{st}_{d,g}$ such that $C$ has an elliptic tail $E$ have special automorphisms that will play a key role in the sequel.

Definition 4.4. Given an element $(C, L) \in \mathcal{P}^{st}_{d,g}$ such that $C$ has an elliptic tail $E$, an automorphism $\phi = (\sigma, \psi) \in \text{Aut}(C, L)$ (or its image in $\text{Aut}(C^t)$) is called an elliptic tail automorphism of $(C, L)$ of order $n \geq 1$ (with respect to the elliptic tail $E \subseteq C$) if $\sigma$ is the identity on $C \setminus E$ and $\sigma|_E$ has order $n$.

The assumption that $(C, L)$ belongs to $\mathcal{P}^{st}_{d,g}$ puts some constraints on the possible elliptic tail automorphisms that can occur. Indeed, under this assumption, using that the map $G : \text{Aut}(C, L) \to \text{Aut}(C^t)$ is injective (see Corollary 2.14), we deduce immediately the following

Remark 4.5. Given $(C, L) \in \mathcal{P}^{st}_{d,g}$, an element $\phi \in \text{Aut}(C, L)$ is an elliptic tail automorphism of order $n \geq 1$ with respect to the elliptic tail $E \subseteq C$ if and only if $G(\phi) \in \text{Aut}(C^t)$ (see the notation of Lemma 2.13) is an elliptic tail automorphism of order $n$ of $C^t$ with respect to the elliptic tail $E^t \subseteq C^t$, i.e. $G(\phi)$ is the identity on $C^t \setminus E^t$ and $G(\phi)|_{E^t}$ has order $n$.

Using this Remark, we can give a complete description of the possible elliptic tail automorphisms of elements $(C, L) \in \mathcal{P}^{st}_{d,g}$.

Lemma 4.6. Assume that $(C, L) \in \mathcal{P}^{st}_{d,g}$ and that $C$ has an elliptic tail $E \subseteq C$. For an elliptic tail automorphism $\phi = (\sigma, \psi) \in \text{Aut}(C, L)$
of \((C, L)\) of order \(n > 1\) with respect to the elliptic tail \(E \subseteq C\), the restriction \(\sigma_E\) of \(\sigma\) to \(E\) must satisfy the following conditions (according to whether the elliptic tail \(E\) is of Type I, II or III as in Lemma 4.3(iii)):

(i) Type I: \(\sigma_E\) is an automorphism of \(E\) fixing \(P\) and \(n = 2\) or \(n = 4\) (which can occur if and only if \(E\) has \(j\)-invariant equal to 1728) or \(n = 3, 6\) (which can occur if and only if \(E\) has \(j\)-invariant equal to 0).

(ii) Type II: \(\sigma_E\) is an automorphism of order \(n = 2\) fixing \(P\) and \(Q\).

If we call \(\nu : E^\nu \to E\) the normalization map and identify \(E^\nu\) with \(P_1\) in such a way that \(\nu^{-1}(P) = \infty\) and \(\nu^{-1}(Q) = \{1, -1\}\), then the automorphism \(\sigma_E\) is induced by the automorphism \(x \mapsto -x\) on \(P_1\).

(iii) Type III: \(\sigma_E\) is an automorphism of order \(n = 2\) such that, if we identify \(R_i\) (for \(i = 1, 2\)) with \(P_1\) in such a way that \(Q_1\) and \(Q_2\) get identified with 1 and \(-1\) (on both copies of \(P_1\)) and \(P \in R_1\) gets identified with \(\infty\), then \(\sigma_{E^\nu}\) (for \(i = 1, 2\)) is equal to the automorphism \(x \mapsto -x\) on \(P_1\). In particular, \(\sigma_E\) fixes \(P\) and exchanges \(Q_1\) with \(Q_2\).

Proof. Parts (i) and (ii) follow easily from Remark 4.5 together with the fact that \(E^\ast \cong E\) for Type I and II and the well-known description of the elliptic tail automorphisms of stable curves (see e.g. [Lud07, Rmk. 4.2.2]).

In order to prove Part (iii), observe that in this case \(E^\ast \subseteq C^\ast\) is a rational curve with one node. Therefore, there exists a unique elliptic tail automorphism of \(C^\ast\) with respect to \(E^\ast\), namely the automorphism \(\sigma\) whose restriction \(\sigma_E\) is described in Part (i). We conclude by Remark 4.5 together with the fact the elliptic tail automorphism of \((C, L)\) described in Part (iii) is the unique (by Corollary 2.14) lift to \(\operatorname{Aut}(C, L)\) of the elliptic tail automorphism of \(C^\ast\) with respect to \(E^\ast\) described in Part (i). \(\square\)

We can now determine the singular locus of \(T_{d,g}^\ast\).

**Proposition 4.7.** The singular locus of \(T_{d,g}^\ast\) (for \(g \geq 4\)) is exactly the locus of pairs \((C, L)\) such that \(\operatorname{Aut}(C, L)\) is not trivial.

Proof. Near a point \((C, L) \in T_{d,g}^\ast\), using the local description (12) and Corollary 2.14, the scheme \(T_{d,g}^\ast\) is isomorphic to the finite quotient

\[T_{\operatorname{Def}(C,L)}/\operatorname{Aut}(C,L),\]

where \(T_{\operatorname{Def}(C,L)}\) is a \(\mathbb{C}\)-vector space of dimension \(4g - 3\) (by Proposition 2.13(iii)) and \(\operatorname{Aut}(C,L)\) can be naturally identified with a finite subgroup of \(\operatorname{GL}(T_{\operatorname{Def}(C,L)})\).

By a well-known result of Prill (see [Pri67]), it is enough to prove that \(\operatorname{Aut}(C, L) \subseteq \operatorname{GL}(T_{\operatorname{Def}(C,L)})\) does not contain quasi-reflections, i.e.
elements \( \phi \) such that 1 is an eigenvalue of \( \phi \) with multiplicity equal to \( 4g - 4 \) or, equivalently, such that the fixed locus \( \text{Fix}(\phi) \) of \( \phi \) is a divisor inside \( T_{\text{Def}(C,L)} \).

Consider the morphism \( \phi_d : \mathcal{P}_{d,g} \to \overline{M}_g \) which, according to (14), locally looks like

\[
T_{\text{Def}(C,L)}/\text{Aut}(C,L) \to T_{\text{Def}_{\text{cost}}} / \text{Aut}(C^{\text{st}}),
\]

where \( T_{\text{Def}(C,L)} \to T_{\text{Def}_{\text{cost}}} \) is surjective with kernel \( V \) of dimension \( g \) and \( \text{Aut}(C^{\text{st}}) \) can be naturally identified with a finite subgroup of \( \text{GL}(T_{\text{Def}_{\text{cost}}} C^{\text{st}}) \).

Assume, by contradiction, that \( \phi \in \text{Aut}(C,L) \subseteq \text{GL}(T_{\text{Def}(C,L)}) \) is a quasi-reflection. By the above local description of the morphism \( \phi_d \), there are two possibilities for the image \( G(\phi) \) of \( \phi \) in \( \text{Aut}(C^{\text{st}}) \subseteq \text{GL}(T_{\text{Def}_{\text{cost}}} C^{\text{st}}) \) via the homomorphism \( G \) of Lemma 2.13:

(i) 1 is an eigenvalue of multiplicity \( 3g - 3 \) for \( G(\phi) \), i.e. \( G(\phi) = \text{id} \in \text{Aut}(C^{\text{st}}) \);
(ii) 1 is an eigenvalue of multiplicity \( 3g - 4 \) for \( G(\phi) \), i.e. \( G(\phi) \) is a quasi-reflection for \( \text{Aut}(C^{\text{st}}) \subseteq \text{GL}(T_{\text{Def}_{\text{cost}}} C^{\text{st}}) \).

In case (i), we conclude that \( \phi = \text{id} \in \text{Aut}(C,L) \) since \( G \) is injective for an element \( (C,L) \in \mathcal{P}_{d,g}^{\text{st}} \) by Corollary 2.14. This contradicts the fact that \( \phi \) is a quasi-reflection.

In case (ii), it is well-known (see e.g. [Lud07, Cor. 4.2.6]) that \( C^{\text{st}} \) must have an elliptic tail \( E \) and \( G(\phi) \) must be equal to the elliptic tail automorphism \( i \) of \( C^{\text{st}} \) of order 2 with respect to \( E \) (see Lemma 4.6). Since \( i = G(\phi) \) admits a lifting to \( \overline{\text{Aut}}(C,L) \), namely \( \phi \), the line bundle \( L \) on \( C \) must be such that the restriction \( L|_E \) to \( E \) is a suitable translate of a 2-torsion point of \( \text{Pic}^0(E) \) (using some identification \( \text{Pic}^{dE}(E) \cong \text{Pic}^0(E) \) and the fact that \( i \) acts on \( \text{Pic}^0(E) \) sending \( \eta \) into \( \eta^{-1} \)). Therefore, the fixed locus \( \text{Fix}(\phi) \) of \( \phi \) inside \( T_{\text{Def}(C,L)} \) has codimension at least two, hence \( \phi \) is not a quasi-reflection.

By applying the Reid–Tai criterion for canonical singularities, we can prove the following result.

**Theorem 4.8.** Assume \( g \geq 4 \). Then the stable locus \( \overline{\mathcal{P}}_{d,g}^{\text{st}} \) has canonical singularities. In particular, if \( \overline{\mathcal{P}}_{d,g}^{\text{st}} \) is a resolution of singularities of \( \overline{\mathcal{P}}_{d,g}^{\text{st}} \), then every pluricanonical form defined on the smooth locus \( (\overline{\mathcal{P}}_{d,g}^{\text{st}})^{\text{reg}} \) of \( \overline{\mathcal{P}}_{d,g}^{\text{st}} \) extends holomorphically to \( \overline{\mathcal{P}}_{d,g}^{\text{st}} \), that is, for all integers \( m \) we have

\[
h^0\left( (\overline{\mathcal{P}}_{d,g}^{\text{st}})^{\text{reg}}, mK_{\overline{\mathcal{P}}_{d,g}^{\text{st}}^{\text{reg}}} \right) = h^0\left( \overline{\mathcal{P}}_{d,g}^{\text{st}}, mK_{\overline{\mathcal{P}}_{d,g}^{\text{st}}} \right).
\]

**Proof.** We use the notation introduced in the proof of Proposition 4.7.
Given an element $\phi \in \overline{\text{Aut}(C, L)} \subseteq \text{GL}(T_{\text{Def}(C, L)})$ of order $n$, we can choose suitable coordinates of $T_{\text{Def}(C, L)}$ and a primitive $n$-th root of unity $\zeta$, such that the action of $\phi$ on $T_{\text{Def}(C, L)}$ is given by the sum $M(\phi) \oplus N(\phi)$ of two matrices (with $0 \leq a_i < n$ for $1 \leq i \leq 4g - 3$):

$$M(\phi) = \begin{pmatrix} \zeta^{a_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \zeta^{a_{4g-3}} \end{pmatrix} \quad \text{and} \quad N(\phi) = \begin{pmatrix} \zeta^{a_{3g-2}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \zeta^{a_{4g-3}} \end{pmatrix}$$

in such a way that the action of $\phi$ on $V$ is given by $N(\phi)$ and the action of $\overline{G(\phi)}$ on $T_{\text{Def} \, \text{cur}}$ is given by $M(\phi)$.

Recall that, according to the Reid–Tai criterion for the canonicity of finite quotient singularities (see e.g. [HM82, pp. 27–28] or [Lud07, Thm. 4.1.11]), a point $(C, L)$ is a canonical singularity if and only if for every $\phi \in \overline{\text{Aut}(C, L)}$ of some order $n$ and every $n$-th root of unity $\zeta$ we have

$$(19) \quad \sum_{i=1}^{4g-3} a_i \frac{n}{n} \geq 1.$$ 

Note that this is true because $\overline{\text{Aut}(C, L)}$ does not contain quasi-reflections (see the proof of Proposition 4.7).

Denote, as usual, by $\Delta_1$ the divisor of $\overline{M_g}$ consisting of curves having an elliptic tail. If $C^{\text{st}} \not\in \Delta_1$ or $C^{\text{st}} \in \Delta_1$ but $\overline{G(\phi)}$ is not an elliptic tail automorphism (or equivalently, by Remark 4.5 $\phi$ is not an elliptic tail automorphism) then by [HM82 Thm. 2] we get

$$\sum_{i=1}^{4g-3} a_i \frac{n}{n} \geq \sum_{i=1}^{3g-3} a_i \frac{n}{n} \geq 1,$$

and we are done in this case.

If $C^{\text{st}} \in \Delta_1$ and $\overline{G(\phi)}$ is an elliptic tail automorphism with respect to the elliptic tail $E^{\text{st}} \subset C^{\text{st}}$ (where $E^{\text{st}}$ is equal to the image via $\text{st} : C \to C^{\text{st}}$ of the elliptic tail $E \subset C$ as in Remark 4.2) then we choose, as in [Lud07 Prop. 4.2.5], the first two coordinates $t_1$ and $t_2$ of $T_{\text{Def} \, \text{cur}}$ in such a way that (in the notation of Lemma 4.3): $t_1$ corresponds to the elliptic tail node $P$ and $t_2$ correspond to $Q$ if $E^{\text{st}}$ is singular and is a coordinate for $T_{(E, P)}(M_{1,1})$ if $E$ is smooth. In [Lud07 Prop. 4.2.5], it is proved that the matrix $M(\phi)$ is given by (depending
on the choice of the primitive $n$-th root of unity $\zeta$):

$$M(\phi) = \begin{cases} 
\begin{pmatrix} 
\zeta^1 \\
\zeta^0 \end{pmatrix} & \text{if } n = 2, \\
\begin{pmatrix} 
\zeta^1 \\
\zeta^2 \end{pmatrix} & \text{or } \begin{pmatrix} 
\zeta^3 \\
\zeta^2 \end{pmatrix} \text{ if } n = 4, \\
\begin{pmatrix} 
\zeta^1 \\
\zeta^2 \end{pmatrix} & \text{or } \begin{pmatrix} 
\zeta^1 \\
\zeta^1 \end{pmatrix} \text{ if } n = 3, \\
\begin{pmatrix} 
\zeta^5 \\
\zeta^4 \end{pmatrix} & \text{or } \begin{pmatrix} 
\zeta^1 \\
\zeta^2 \end{pmatrix} \text{ if } n = 6,
\end{cases}$$

where $I$ is the suitable unit matrix.

Let us now turn to the matrix $N(\phi)$. We choose the first coordinate $s_1$ on $V$ so that it is a coordinate for $T_{L/E}(\text{Pic}^{d_E}(E))$, where $d_E$ is defined in Lemma 1.3.3. In order to compute the action of $\phi$ on $s_1$, we distinguish three cases according to whether the elliptic tail $E \subset C$ is of Type I, II or III (see Lemma 1.3.3 and Figure 1).

If $E$ is of Type I, i.e. $E$ is smooth, then we can identify $E$ with $\text{Pic}^{d_E}(E)$ sending $q \in E$ into $O_E(q + (d_E - 1)P) \in \text{Pic}^{d_E}(E)$. Since $\phi$ acts on $\text{Pic}^{d_E}(E)$ via pull-back, if the action of $G(\phi)$ on $T_P(E)$ is given by the multiplication by a root of unity $\zeta$, then the action of $\phi$ on $T_{L/E}(\text{Pic}^{d_E}(E))$ is given by the multiplication by $\zeta^{-1}$. Therefore the matrix $N(\phi)$ is equal to (with respect to the same choice of the primitive $n$-th root of unity $\zeta$ as in the above matrix $M(\phi)$):

$$N(\phi) = \begin{cases} 
\begin{pmatrix} 
\zeta^1 \\
I \end{pmatrix} & \text{if } n = 2, \\
\begin{pmatrix} 
\zeta^3 \\
I \end{pmatrix} & \text{or } \begin{pmatrix} 
\zeta^1 \\
I \end{pmatrix} \text{ if } n = 4, \\
\begin{pmatrix} 
\zeta^2 \\
I \end{pmatrix} & \text{or } \begin{pmatrix} 
\zeta^1 \\
I \end{pmatrix} \text{ if } n = 3, \\
\begin{pmatrix} 
\zeta^1 \\
I \end{pmatrix} & \text{or } \begin{pmatrix} 
\zeta^5 \\
I \end{pmatrix} \text{ if } n = 6,
\end{cases}$$

If $E$ is of type II, i.e. $E$ is an irreducible rational curve with one node $Q$ (as in Figure 1), then $\text{Pic}^{d_E}(E) \cong \mathbb{G}_m$. Explicitly, if we consider the normalization morphism $\nu : E^\nu \cong \mathbb{P}^1 \to E$ and let $\nu^{-1}(Q) = \{u, v\}$, then any $\lambda \in \mathbb{G}_m(k)$ determines a unique line bundle $L_\lambda \in \text{Pic}^{d_E}(E)$ whose local sections are the local sections $s$ of $O_{\mathbb{P}^1}(d_E)$ such that $s(u) = \lambda$.
\[ \lambda s(v) \]. Since \( \phi_{|E} \) is induced by an involution of \( E' \) that exchanges \( u \) and \( v \) (by Lemma 4.6(ii)), then clearly \( \phi \) will send \( L_\lambda \) into \( L_{\lambda^{-1}} \). This implies that the action of \( \phi \) on \( T_{L_{E'}(\text{Pic}^{d_E}(E))} \) is given by multiplication by \(-1\), hence the matrix \( N(\phi) \) is also in this case given by (21) with \( n = 2 \).

If \( E \) is of type III, i.e. \( E \) is made of two irreducible rational components \( R_1 \) and \( R_2 \) meeting in two points \( Q_1 \) and \( Q_2 \) (as in Figure 1), then again \( \text{Pic}^{d_E}(E) \cong \mathbb{G}_m \). Explicitly, if we consider the normalization morphism \( \nu : E' = R_1 \coprod R_2 \to E \) and let \( \nu^{-1}(Q_i) = \{ u_i, v_i \} \) with \( u_i \in R_1 \) and \( v_i \in R_2 \) (for \( i = 1, 2 \)), then any \( \lambda \in \mathbb{G}_m(k) \) determines a unique line bundle \( L_\lambda \in \text{Pic}^{d_E}(E) \) whose local sections are pairs of local sections \( (s_1, s_2) \) of \( (\mathcal{O}_{R_1}(d_E - 1), \mathcal{O}_{R_1}(1)) \) such that

\[
\frac{s_1(u_1)}{s_1(u_2)} = \frac{s_2(v_1)}{s_2(v_2)}.
\]

Since \( \phi_{|E} \) is induced by an involution of \( E' \) that exchanges \( u_1 \) with \( u_2 \) and \( v_1 \) with \( v_2 \) (by Lemma 4.6(ii)), then clearly \( \phi \) will send \( L_\lambda \) into \( L_{\lambda^{-1}} \). This implies that the action of \( \phi \) on \( T_{L_{E'}(\text{Pic}^{d_E}(E))} \) is given by multiplication by \(-1\), hence the matrix \( N(\phi) \) is also in this case given by (21) with \( n = 2 \).

An easy inspection of the matrices \( M(\phi) \) in (20) and \( N(\phi) \) in (21) reveals that the condition \([19]\) is always satisfied, which shows that \( \mathcal{P}_{d,g} \) has canonical singularities.

The last assertion of the theorem follows from the well-known fact that canonical singularities do not impose adjoint conditions on the pluricanonical forms.

\[ \square \]

5. The canonical class of \( \mathcal{P}_{\text{ic},d,g} \) and of \( \mathcal{P}_{d,g} \).

The aim of this section is to prove Theorem 1.5. To achieve that, we first determine the canonical class of the stack \( \mathcal{P}_{\text{ic},d,g} \).

**Theorem 5.1.** The canonical class of \( \mathcal{P}_{\text{ic},d,g} \) is equal to

\[ K_{\mathcal{P}_{\text{ic},d,g}} = \Phi_d' (14 \lambda - 2 \delta), \]

where \( \lambda \) and \( \delta \) are the Hodge and total boundary class on \( \mathcal{M}_g \).

**Proof.** Let \( \pi : \mathcal{P}_{\text{ic},d,g,1} \to \mathcal{P}_{d,g} \) the universal family over \( \mathcal{P}_{d,g} \) and \( \mathcal{L}_d \) the universal line bundle over \( \mathcal{P}_{\text{ic},d,g,1} \) (see [Mel11] for a modular description of \( \mathcal{P}_{\text{ic},d,g,1} \)). Denote by \( \Omega_\pi \) and \( \omega_\pi \) the sheaf of relative Kähler differentials and the relative dualizing sheaf, respectively. Let \( d : \Omega_{\mathcal{P}_{\text{ic},d,g,1}} \to \Omega_\pi \) be the universal derivation and consider the map induced in cohomology \( \theta : \text{Pic}(\mathcal{P}_{\text{ic},d,g,1}) \to H^1(\mathcal{P}_{\text{ic},d,g,1}, \Omega_\pi) \). Since \( H^1(\mathcal{P}_{\text{ic},d,g,1}, \Omega_\pi) \cong \text{Ext}^1(\mathcal{O}_{\mathcal{P}_{\text{ic},d,g,1}}, \Omega_\pi) \), the map \( \theta \) sends the line bundle \( \mathcal{L}_d \) on \( \mathcal{P}_{\text{ic},d,g,1} \) into the class of an extension

\[
(22) \quad 0 \to \Omega_\pi \to \mathcal{E} \to \mathcal{O}_{\mathcal{P}_{\text{ic},d,g,1}} \to 0.
\]
The restriction of the above extension (23) to a geometric fiber \((C, L)\) of \(\pi\) is the extension (22) as it follows from the discussion in Section 2.15.

From this and the analysis of the deformation theory of the pair \((C, L)\) carried out in 2.15, it follows that the tangent space of \(P_{ic}^{d,g}\) at a geometric point \((C, L)\) is equal to \(\text{Ext}^1(E|_C, O_C) - \text{Ext}^0(E|_C, O_C)\). Therefore, using relative duality for \(\pi\), it follows that the canonical class \(K_{P_{ic}^{d,g}}\) of \(P_{ic}^{d,g}\) is equal to

\[
K_{P_{ic}^{d,g}} = c_1(\pi_!(\mathcal{E} \otimes \omega_{\pi})).
\]

To compute this class, we apply the Grothendieck-Riemann-Roch Theorem for quotient stacks \([EG00]\) relative to the morphism \(\pi\):

\[
\text{ch}(\pi_!(\omega_\pi \otimes \mathcal{E})) = \pi^*(\text{ch}(\omega_\pi \otimes \mathcal{E}) \cdot \text{Td}(\Omega_\pi)^{-1}).
\]

Let us compute the degree one part of the right hand side of (23). We set \(\tilde{K} := c_1(\omega_\pi)\) and \(\tilde{\eta} := c_2(\Omega_\pi)\). Note that, as remarked in \([HM98,\text{p. 158}]\), we have \(\tilde{K} = c_1(\Omega_\pi)\).

The first three terms of inverse of the Todd class of \(\Omega_\pi\) are equal to

\[
\frac{\tilde{K}^2}{2} + \frac{\tilde{K}^2}{2} = 1 - \frac{c_2(\Omega_\pi)}{12} = 1 - \frac{c_2(\Omega_\pi)}{12} + \ldots
\]

Using (22), we get

\[
\text{ch}(\mathcal{E} \otimes \omega_\pi) = \text{ch}(\mathcal{E}) \cdot \text{ch}(\omega_\pi) = (\text{ch}(\mathcal{E}) + \text{ch}(O_{P_{ic}^{d,g,1}})) \cdot \text{ch}(\omega_\pi) = (\text{ch}(\Omega_\pi) + 1) \cdot \text{ch}(\omega_\pi).
\]

Moreover we have

\[
\text{ch}(\omega_\pi) = 1 + c_1(\omega_\pi) + \frac{c_1(\omega_\pi)}{2} + \ldots = 1 + \tilde{K} + \frac{\tilde{K}^2}{2} + \ldots
\]

\[
\text{ch}(\Omega_\pi) = 1 + c_1(\Omega_\pi) + \frac{c_1(\Omega_\pi)}{2} - c_2(\Omega_\pi) + \ldots = 1 + \tilde{K} + \frac{\tilde{K}^2}{2} - \tilde{\eta} + \ldots
\]

Substituting into (25), we arrive at

\[
\text{ch}(\mathcal{E} \otimes \omega_\pi) = 2 + 3\tilde{K} + \frac{5}{2}\tilde{K}^2 - \tilde{\eta} + \ldots
\]

Combining (24) and (26), we get

\[
[\text{ch}(\omega_\pi \otimes \mathcal{E}) \cdot \text{Td}(\Omega_\pi)^{-1}]_2 = \frac{\tilde{K}^2 + \tilde{\eta}}{6} - \frac{3}{2}\tilde{K}^2 + \frac{5}{2}\tilde{K}^2 - \tilde{\eta} = \frac{7}{6}\tilde{K}^2 + \frac{5}{6}\tilde{\eta},
\]

hence, from (23), we deduce

\[
K_{P_{ic}^{d,g}} = \frac{7}{6}\pi^*(\tilde{K}^2) - \frac{5}{6}\pi^*(\tilde{\eta}).
\]
Let us now apply the Grothendieck-Riemann-Roch theorem to the sheaf $\omega_\pi$. Since $R^1\pi_*\omega_\pi = \mathcal{O}_{\overline{Pic}_{d,g}}$ by relative duality, we get

$$c_1(\pi_*\omega_\pi) = \pi_*\left(\frac{\tilde{K}^2 + \tilde{\eta}}{12} - \frac{1}{2}\tilde{K}^2 + \frac{1}{2}\tilde{K}^2\right) = \frac{1}{12}\pi_*\left(\tilde{K}^2\right) + \frac{1}{2}\pi_*\left(\tilde{\eta}\right).$$

If we set $\tilde{\lambda} := c_1(\pi_*\omega_\pi)$ and $\tilde{\delta} := \pi_*\left(\tilde{\eta}\right)$, then the previous relation becomes $12\tilde{\lambda} = \pi_*\left(\tilde{K}^2\right) + \tilde{\delta}$. Substituting into (27), we obtain

$$K_{\overline{Pic}_{d,g}} = 14\tilde{\lambda} - 2\tilde{\delta}.$$

The Lemma below completes the proof.

**Lemma 5.2.** With the notation of Theorem 5.1, we have

$$\tilde{\lambda} = \Phi^*_g(\lambda) \quad \text{and} \quad \tilde{\delta} = \Phi^*_g(\delta).$$

**Proof.** Consider the diagram

$$\begin{array}{ccc}
\overline{Pic}_{d,g,1} & \xrightarrow{\Phi_{d,1}} & M_{g,1} \\
\pi \downarrow & & \downarrow \pi' \\
\overline{Pic}_{d,g} & \xrightarrow{\Phi_d} & M_g
\end{array}$$

Recall that the classes $\lambda$ and $\delta$ on $\overline{M}_g$ are defined as

$$\lambda := c_1(\pi_*\omega_\pi) \quad \text{and} \quad \delta := \pi_*\left(c_2(\Omega_\pi)\right),$$

where $\Omega_\pi$ and $\omega_\pi$ are the sheaf of relative Kähler differentials and the relative dualizing sheaf of $\pi$, respectively.

The map $\Phi_d$ sends an element $(C \to S, \mathcal{L}) \in \overline{Pic}_{d,g}(S)$ into the stabilization $C^{st} \to S \in \overline{M}_g(S)$. Recall that for every quasi-stable (or more generally semistable) curve $C$ with stabilization morphism $\psi : C \to C^{st}$, the pull-back via $\psi$ induces a natural isomorphism $\psi^* : H^0(C^{st}, \omega_{C^{st}}) \xrightarrow{\sim} H^0(C, \omega_C)$. Therefore we have $\Phi^*_d(\pi_*\omega_\pi) = \pi_*\omega_\pi$ and, by taking the first Chern classes, we get $\tilde{\lambda} = \Phi^*_g(\lambda)$.

On the other hand, since the class $\tilde{\delta}$ is the total boundary class of $\overline{Pic}_{d,g}$ and $\delta$ is the total boundary class of $\overline{M}_g$ (see [HMS2, pp. 49–50]), it is clear that $\tilde{\delta} = \Phi^*_g(\delta)$. □

**Proof of Theorem 1.5.** Let $p : \overline{Pic}_{d,g} \to \overline{P}_{d,g}$ the natural map from the stack $\overline{Pic}_{d,g}$ to its good moduli space. In view of Theorem 5.1 it is enough to show that $p^*(K_{\overline{Pic}_{d,g}}) = K_{\overline{P}_{d,g}}$.

Clearly, the two classes agree on the interior $\overline{Pic}_{d,g}$, since for $C$ varying in an open subset of $M_g$ whose complement has codimension at least two (since $g \geq 4$), by Lemma 2.13 we have that $\text{Aut}(C, \mathcal{L}) \subseteq \text{Aut}(C) = \{\text{id}\}$.

Let us now look at the boundary of $\overline{P}_{d,g}$. By [Pon05, Prop. 4], the boundary of $\overline{P}_{d,g}$ is the union of the irreducible divisors $D_i := \phi_{d,i}^{-1}(\Delta_i)$,
ON THE BIRATIONAL GEOMETRY OF \( P_{d,g} \)

for \( i \in \{0, \ldots, \lfloor g/2 \rfloor \} \). Let \( k_{d,g} := (2g - 2, d - g + 1) \). A general element of \((C, L) \in D_i\) looks as follows (see e.g. [Cap94] Ex. 7.1, 7.2 and [Mel09] Prop. 2):

1. If \( i = 0 \) then \( C \) is a general irreducible nodal curve with one node and \( L \) is a general line bundle of degree \( d \) on \( C \);
2. If \( i > 0 \) and \( 2g - 2 \) does not divide \((2i - 1) \cdot k_{d,g}\), then \( C \) is a stable curve consisting of two general smooth curves \( C_1 \) and \( C_2 \) of genera, respectively, \( i \) and \( g - i \) meeting in one point and \( L \) is a general line bundle of multidegree \((\deg_{C_1}(L), \deg_{C_2}(L)) = (d_1, d_2 = d - d_1)\) where \( d_1 \) is the unique integer such that
   \[
   \left| d_1 - \frac{d(2i - 1)}{2g - 2} \right| < \frac{1}{2}.
   \]
3. If \( i > 0 \) and \( 2g - 2 \) divides \((2i - 1) \cdot k_{d,g}\), then \( C \) is a quasi-stable curve consisting of two general smooth curves \( C_1 \) and \( C_2 \) of genera, respectively, \( i \) and \( g - i \) joined by a rational curve \( R \) and \( L \) is a general line bundle whose multidegree is such that \( \deg_R L = 1 \) and
   \[
   \begin{cases}
   d_1 := \deg_{C_1} L = \frac{d(2i - 1)}{2g - 2} - \frac{1}{2},
   \\
   d_2 := \deg_{C_2} L = \frac{d(2g - 2i - 1)}{2g - 2} - \frac{1}{2}.
   \end{cases}
   \]

We claim that the automorphism group of a general point \((C, L) \in D_i\) is equal to

\[
(29) \quad \text{Aut}(C, L) = \begin{cases}
\mathbb{G}_m & \text{in cases 1 and 2} \\
\mathbb{G}_m^2 & \text{in case 3}
\end{cases}
\]

Indeed, by the explicit description above, \( \gamma(\tilde{C}) = 1 \) in cases 1 and 2, and \( \gamma(\tilde{C}) = 2 \) in case 3. Therefore, the claim will follow from Lemma 2.13 if we show that the image of \( \text{Aut}(C, L) \to \text{Aut}(C^{\text{st}}) \) is trivial. This is trivially true if \( i \neq 1 \) since in this case \( C^{\text{st}} \) is a general curve in \( \Delta_i \), hence \( \text{Aut}(C^{\text{st}}) = \{\text{id}\} \). If \( i = 1 \) then \( \text{Aut}(C^{\text{st}}) = \mathbb{Z}/2\mathbb{Z} \) generated by the elliptic tail involution \( \sigma \) with respect to the elliptic tail \( C_1 \) (see Remark 4.5 and the notation there). However, in this case, \( \sigma \) comes from an automorphism of the pair \((C, L)\) if and only if \( L_{|C_1}(-d_1 \cdot P) \) is a 2-torsion point of \( \text{Pic}^0(C_1) \), where \( P = C_1 \cap R \) is the elliptic tail node of \( C_1 \). Clearly, this is not the case for a general strictly balanced line bundle \( L \) on \( C \).

In cases 1 and 2, \( \text{Def}_{(C,L)} \) has dimension \( 4g - 3 \) and \( \text{Aut}(C, L) = \mathbb{G}_m \) acts trivially on it (see 2.15). Therefore, the morphism \( p \) looks locally at \((C, L)\) as \( \tilde{p} : [\text{Def}_{(C,L)}/\mathbb{G}_m] = \text{Def}_{(C,L)} \times B\mathbb{G}_m \to \text{Def}_{(C,L)}. \) It is clear that in this case \( \tilde{p}^*(K_{\text{Def}_{(C,L)}}) = K_{\text{Def}_{(C,L)} \times B\mathbb{G}_m}. \)
In case (3), Def\(_{(C,L)}\) has dimension \(4g - 2\) (see 2.13). If we choose the first two coordinates \(x\) and \(y\) of Def\(_{(C,L)}\) in such a way that they correspond to the local deformations of the two nodes \(P_1 := C_1 \cap R\) and \(P_2 := C_2 \cap R\), then the action of \((\mu, \nu) \in \text{Aut}(C,L) = G_m^2\) on the first two coordinates of Def\(_{(C,L)}\) is given by

\[
(\mu, \nu) \cdot (x, y) = (\mu \nu^{-1} x, \mu^{-1} \nu y),
\]

while it is trivial on the other coordinates. Therefore, neglecting the trivial coordinates, at \((C,L)\) the morphism \(p\) looks locally as

\[
X := \text{Spf}\mathbb{C}[[x,y]]/G_m^2 \rightarrow \text{Spf}\mathbb{C}[[x,y]]/G_m^2 := X.
\]

Since the ring of invariants for the action (30) of \(G_m^2\) on \(C[[x,y]]\) is generated by \(xy\), the quotient \(X\) is isomorphic to \(\text{Spf}\mathbb{C}[[xy]]\). The quotient map \(Y := \text{Spf}\mathbb{C}[[x,y]]/\mathfrak{g} \rightarrow X = \text{Spf}\mathbb{C}[[xy]]\) induces a pull-back map

\[
q^* : \langle d(xy) \rangle = \Omega^1_X \rightarrow \Omega^1_Y = \langle dx, dy \rangle,
\]

\[
d(xy) \mapsto xdy + ydx.
\]

On the other hand, the cotangent complex of \(X\) is equal to the \(G_m^2\)-equivariant cotangent complex of \(Y := \text{Spf}\mathbb{C}[[x,y]]\) (see e.g. [Moc09, p. 37]), which in our case looks like:

\[
L : \langle dx, dy \rangle = \Omega^1_Y \rightarrow \mathcal{O}_Y \otimes \text{Lie}(G_m^2)^* = \mathcal{O}_Y \otimes \left\langle \frac{d\lambda}{\lambda}, \frac{d\mu}{\mu} \right\rangle.
\]

The map \(f\) is the dual of the infinitesimal action of \(\text{Lie}(G_m^2)\) on \(Y\), hence, by the explicit action (30), we compute

\[
\begin{align*}
  f(dx) &= x \frac{d\lambda}{\lambda} - \frac{d\mu}{\mu}, \\
  f(dy) &= y \frac{d\mu}{\mu} - \frac{d\lambda}{\lambda}.
\end{align*}
\]

Since the image of \(q^*\) is equal to the kernel of \(f\), we deduce that \(\tilde{p}^*(K_X) = K_X\), which concludes the proof.

6. The Iitaka Dimension of \(K_{\mathcal{M}_{d,g}}\)

The aim of this section is to prove Theorem 1.6. Since \(K_{\mathcal{M}_{d,g}} = \phi_d'(14\lambda - 2\delta)\) (by Theorem 1.5) and \(\phi_d\) has connected fibers, we have

\[
\kappa(K_{\mathcal{M}_{d,g}}) = \kappa(14\lambda - 2\delta).
\]

Therefore, we are reduced to study the Iitaka dimension of the divisor \(14\lambda - 2\delta\) on \(\mathcal{M}_g\). Notice that the slope of \(14\lambda - 2\delta\) is equal to 7. Hence, if the slope \(s(\mathcal{M}_g)\) of \(\mathcal{M}_g\) (in the sense of Harris-Morrison [HM90]) is strictly less than 7 then we conclude that \(14\lambda - 2\delta\) is big, i.e. that \(\kappa(14\lambda - 2\delta) = 3g - 3\); while if \(s(\mathcal{M}_g) > 7\) then \(14\lambda - 2\delta\) is not pseudo-effective and \(\kappa(14\lambda - 2\delta) = -\infty\) (see the discussion at the beginning of [HM90]).
Proposition 6.1. If \(g \leq 9\) then \(\kappa(K_{\mathcal{P}_{d,g}}) = -\infty\).

Proof. This follows by the fact that \(s(M_g) > 7\) for \(g \leq 9\) (see [Tan98]). □

Remark 6.2. By combining the above Proposition 6.1 with the inequality (1), we obtain another proof of the fact that \(\kappa(P_{d,g}) = -\infty\) for \(g \leq 9\) and any \(d\) (which of course follows from the stronger Theorem 1.1).

For \(g \geq 12\) we can prove the following

Proposition 6.3. If \(g \geq 12\) then \(\kappa(K_{\mathcal{P}_{d,g}}) = 3g - 3\) and the fibration \(\phi_d : \mathcal{P}_{d,g} \rightarrow M_g\) is the Iitaka fibration of \(K_{\mathcal{P}_{d,g}}\).

Proof. We have already observed, in the proof of Proposition 3.2, that if \(\kappa(K_{\mathcal{P}_{d,g}}) = 3g - 3\) then \(\phi_d : \mathcal{P}_{d,g} \rightarrow M_g\) is the Iitaka fibration (see [Laz04, Def. 2.1.34]) of \(K_{\mathcal{P}_{d,g}}\). Therefore, it is enough to prove the first assertion. As pointed out before, this will follow if we show that \(s(M_g) < 7\) for \(g \geq 12\).

By computing the class of the Brill-Noether divisor \(D_{1+g+1}^{g+1}\), Harris and Mumford proved in [HMS82] that

\[
s(M_g) \leq 6 + \frac{12}{g+1} \quad \text{if } g \text{ is odd.}
\]

Since \(6 + \frac{12}{g+1} < 7\) if and only if \(g > 11\), we get that

(32) \(s(M_g) < 7\) if \(g\) is odd and \(g \geq 13\).

By computing the class of the Petri divisor \(E_{1+g+1}^{g+1}\), Eisenbud and Harris in [EH87, Thm. 2] proved that

\[
s(M_g) \leq 6 + \frac{14g + 4}{g(g+2)} \quad \text{if } g \text{ is even.}
\]

Since \(6 + \frac{14g + 4}{g(g+2)} < 7\) if and only if \(g > 13\), we get that

(33) \(s(M_g) < 7\) if \(g\) is even and \(g \geq 14\).

By computing the slope of some effective divisors on \(M_g\) associated to curves equipped with secant-exceptional linear series, Cotterill in [Cor10, §6.2] showed in particular that

(34) \(s(M_{12}) \leq 6,979... < 7\).

Equations (32), (33) and (34) together imply the result. □

The cases \(g = 10\) and \(g = 11\) requires a special care since it is known that in this case \(s(M_g) = 7\) (see [Tan98] and [FP05, Cor. 1.3]). We start with the case \(g = 10\).

Proposition 6.4. If \(g = 10\) then \(\kappa(K\mathcal{P}_{d,g}) = 0\).
Proof. Farkas and Popa proved in [FP05, Thm. 1.6] that the effective irreducible divisor $F$ (which is denoted by $\overline{K}$ in loc. cit.) given by the closure of the locus of smooth curves of genus 10 lying on a $K3$ surface has class equal to

$$
(35) \quad F = 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - B_5\delta_5,
$$

with $B_5 \geq 6$. Since it is easily checked that $14\lambda - 2\delta$ is the sum of $2F$ and an effective boundary divisor, we get, using (31), that $\kappa(\overline{K}_{d,\lambda}) = \kappa(14\lambda - 2\delta) \geq \kappa(2F) = \kappa(F) \geq 0$.

It remains to prove that $h^0(M_{10}, m(14\lambda - 2\delta)) = 1$ for any $m$ sufficiently divisible.

Claim 1: If $m$ is sufficiently divisible then $2mF$ is contained in the base locus of $|m(14\lambda - 2\delta)|$.

Take $D \in |m(14\lambda - 2\delta)|$ and let $r$ be the multiplicity of $F$ inside $D$. Consider a Lefschetz pencil of curves of genus 10 lying on a general $K3$ surface of degree 18 in $\mathbb{P}^{10}$. This gives rise to an irreducible curve $B$ in the moduli space $M_{10}$. Such pencils $B$ fill the divisor $F$, by results of Mukai (Muk92). Therefore $B$ is not contained in the support of $D - rF$, hence $(D - rF) \cdot B \geq 0$. Using the well-known formulas $\lambda \cdot B = g + 1 = 11$, $\delta_0 \cdot B = 6(g + 3) = 78$, $\delta_i \cdot B = 0$ for $i \geq 1$ (see e.g. [FP05, Lemma 2.1]), together with the expression (35), we get that

$$
0 \leq (D - rF) \cdot B = (2m - r)(7\lambda - \delta_0) \cdot B = r - 2m,
$$

which concludes the proof of the Claim.

From the previous Claim, it follows that

$$
h^0(M_{10}, m(14\lambda - 2\delta)) = h^0(M_{10}, m(14\lambda - 2\delta) - 2mF).
$$

Note that $m(14\lambda - 2\delta) - 2mF = m(\sum_{i \geq 1} a_i \delta_i)$ for some $a_i \geq 0$. Therefore the proof of the theorem is concluded by the following

Claim 2: Let $\Delta$ be an effective divisor in $M_g$ (for $g \geq 3$) whose class in Pic($M_g$)$_\mathbb{Q}$ is equal to $\sum_{i \geq 0} a_i \delta_i$, with $a_i \geq 0$. Then $h^0(M_g, m\Delta) = 1$ for any $m$ sufficiently divisible.

Take $E \in |m\Delta|$. We have to show that $E = m\Delta$.

If $E$ meets the interior $M_g$ of $M_g$, then, from the well-known result that Pic($M_g$)$_\mathbb{Q}$ is generated by $\lambda$ and $\lambda$ is ample on $M_g$, we get that the class of $E$ in Pic($M_g$)$_\mathbb{Q}$ is equal to $a\lambda + \sum_{i \geq 0} b_i \delta_i$ with $a > 0$ and $b_i \in \mathbb{Z}$. However the class of $E$ in Pic($M_g$)$_\mathbb{Q}$ is also equal to the class of $m\Delta$, which is $\sum_{i \geq 0} m a_i \delta_i$. This produces a non-trivial relation between $\lambda$ and the boundary classes $\delta_i$, which contradicts the well-known result that Pic($M_g$)$_\mathbb{Q}$ is freely generated by $\lambda$ and the boundary classes $\delta_i$ for $g \geq 3$ (see AC87).

Therefore, $E$ must be entirely contained in the boundary $M_g \setminus M_g = \cup_{i \geq 0} \Delta_i$ of $M_g$. This implies that $E = \sum_{i \geq 0} b_i \Delta_i$ for some $b_i \geq 0$. Looking at the classes of $\sum_{i \geq 0} b_i \Delta_i$ and $m\Delta$ in Pic($M_g$)$_\mathbb{Q}$ and using
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the independence of the boundary classes $\delta_i$ in $\text{Pic}(\overline{M}_g)_{\mathbb{Q}}$, we deduce that $E = m\Delta$, as required. □

We finally examine the case $g = 11$. As usual, denote by $F_g$ ($g \geq 3$) the moduli space of K3 surfaces endowed with a polarization of degree $2g - 2$. By work of Mukai [Muk96], there exists a fibration

$$\psi: \overline{M}_{11} \dashrightarrow F_{11},$$

sending a general curve $C$ of genus $g$ into $(S, \mathcal{O}_S(C))$, where $S$ is the unique K3 surface containing $C$.

**Proposition 6.5.** If $g = 11$ then $\kappa(K_{\mathcal{P}_{d,11}}) = 19$ and the Iitaka fibration of $K_{\mathcal{P}_{d,11}}$ is the composition

$$\mathcal{P}_{d,11} \xrightarrow{\phi_d} \overline{M}_{11} \xrightarrow{\psi} F_{11}.$$  

**Proof.** Farkas and Popa proved in [FP05, Prop. 6.2] that the Iitaka dimension of the divisor

$$E := 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 8\delta_3 - 7\delta_4 - 7\delta_5$$

is 19. Since it is easily checked that $14\lambda - 2\delta$ is the sum of $2E$ and an effective boundary divisor, we get, using (51), that $\kappa(K_{\mathcal{P}_{d,11}}) = \kappa(14\lambda - 2\delta) \geq \kappa(2E) = \kappa(E) = 19$.

Consider now a general point $(S, L) \in F_{11}$. The fiber of $\psi$ over $(S, L)$ is the open subset of the complete linear series $|L| \cong \mathbb{P}^{11}$ consisting of smooth connected curves. Pick a Lefschetz pencil on $S$ and consider the associated curve $B$ inside $\overline{M}_{11}$. It is well-known (see e.g. [FP05, Lemma 2.1]) that $\lambda \cdot B = 12$, $\delta_0 \cdot B = 84$ and $\delta_i \cdot B = 0$ for every $i > 1$. This easily implies that

$$(14\lambda - 2\delta) \cdot B = 0. \tag{36}$$

Consider now the Iitaka fibration of the divisor $K_{\mathcal{P}_{d,11}}$, which we denote by

$$i_{K_{\mathcal{P}_{d,11}}} : \mathcal{P}_{d,11} \dashrightarrow I(K_{\mathcal{P}_{d,11}}).$$

Since $K_{\mathcal{P}_{d,11}} = \phi_d^*(14\lambda - 2\delta)$, the Iitaka fibration $i_{14\lambda - 2\delta}$ of $K_{\mathcal{P}_{d,11}}$ is the composition of the Iitaka fibration of $14\lambda - 2\delta$ with $\phi_d$, i.e. we have a natural diagram (up to birationality)

$$\begin{align*}
\mathcal{P}_{d,11} \xrightarrow{i_{K_{\mathcal{P}_{d,11}}}} \overline{M}_{11} \\
\xrightarrow{\phi_d} I(K_{\mathcal{P}_{d,11}}) = I(14\lambda - 2\delta)
\end{align*}$$

Now, equation (36) implies that the Iitaka fibration $i_{14\lambda - 2\delta}$ contracts the general fiber $\psi^{-1}(S, L) \subset |L| \cong \mathbb{P}^{11}$. Therefore the Iitaka fibration...
$i_{14\lambda-2\delta}$ factors through the fibration $\psi$:

\[
M_{11} \xrightarrow{\psi} i_{14\lambda-2\delta} \xrightarrow{\rho} I(14\lambda-2\delta)
\]

Recall that $\dim F_{11} = 19$. On other hand, by the usual properties of the Iitaka fibration and what proved before, we have that $\dim I(14\lambda-2\delta) = \kappa(14\lambda-2\delta) \geq 19$. Since $\rho$ is dominant and has connected general fiber, this implies that $\rho$ is a birational isomorphism, hence we are done.

\[\square\]

7. Birationalities among different $P_{d,g}$’s

In this section, inspired by Lemma 8.1 in [Cap94], we investigate the following

**Question 7.1.** For what values of $d$ and $d'$ is $P_{d,g}$ birational to $P_{d',g}$? How do the birational maps among them look like?

Note that if $d' = d + n(2g-2)$ for some $n \in \mathbb{Z}$ then we have the isomorphism

\[
\psi_1^n : P_{d,g} \xrightarrow{\cong} P_{d',g},
\]

\[(C,L) \mapsto (C, L \otimes \omega_C^n),
\]

while if $d' = -d + n(2g-2)$ for some $n \in \mathbb{Z}$ we have the isomorphism

\[
\psi_2^n : P_{d,g} \xrightarrow{\cong} P_{d',g},
\]

\[(C,L) \mapsto (C, L^{-1} \otimes \omega_C^n).
\]

Clearly the maps $\psi_1^n$ and $\psi_2^n$ commute with the projections $\phi_d$ and $\phi_{d'}$ onto $M_g$. Indeed, the converse is also true, as it follows from an argument of Caporaso (see [Cap94, Lemma 8.1] and also, for further details, [Cap10, Prop. 3.2.2]).

**Theorem 7.2.** ([Cap94]) If $\eta : P_{d,g} \dashrightarrow P_{d',g}$ is a birational map over $M_g$, i.e. a map $\eta$ inducing a commutative diagram

\[
P_{d,g} \xrightarrow{\eta} P_{d',g}
\]

\[
\phi_d \downarrow \quad \phi_{d'}
\]

\[
M_g \xrightarrow{\text{id}} M_g
\]

then there exists $n \in \mathbb{Z}$ such that either $d' = d + n(2g-2)$ and $\eta = \psi_1^n$ or $d' = -d + n(2g-2)$ and $\eta = \psi_2^n$.

By using our results on the Kodaira dimension of $P_{d,g}$, we can improve Theorem 7.2 at least for genus big enough.
Theorem 7.3. Assume that \( g \geq 22 \) or \( g \geq 12 \) and \( (d-g+1,2g-2) = 1 \). Let \( \eta : P_{d,g} \dashrightarrow P_{d',g} \) be a birational map. Then there exists \( n \in \mathbb{Z} \) such that either \( d' = d + n(2g-2) \) and \( \eta = \psi^n_d \) or \( d' = -d + n(2g-2) \) and \( \eta = \psi^n_d \). In particular, \( \eta \) is an isomorphism.

Proof. By Theorems 1.2 and 1.3 the assumptions of the statement imply that \( \kappa(P_{d,g}) = 3g - 3 \), hence that \( \kappa(P_{d',g}) = 3g - 3 \). From the proof of Proposition 6.2 it follows that \( \phi_d : P_{d,g} \rightarrow M_g \) is the Iitaka fibration of \( P_{d,g} \) and similarly for \( P_{d',g} \). Since the Iitaka fibration is a birational invariant, the map \( \eta \) induces a birational map \( \xi : M_g \dashrightarrow M_g \) such that the following diagram commutes:

\[
\begin{array}{ccc}
P_{d,g} & \xrightarrow{\eta} & P_{d',g} \\
\phi_d \downarrow & & \phi_{d'} \downarrow \\
M_g & \xrightarrow{\xi} & M_g.
\end{array}
\]

The map \( \xi \) sends a very general curve \( C \in M_g \) to a very general curve \( C' \in M_g \) so that the restriction of \( \eta \) induces a birational map \( J(C) \rightarrow \text{Pic}^d(C) \rightarrow \text{Pic}^{d'}(C') \rightarrow J(C') \). By Lemma 7.4 below, we get that \( C \cong C' \), hence \( \xi = \text{id} \). Finally, we conclude by Theorem 7.2.

Lemma 7.4. If \( C \) and \( C' \) are very general curves in \( M_g \) such that \( J(C) \) is birational to \( J(C') \), then \( C \cong C' \).

Proof. Let \( \epsilon : J(C) \dashrightarrow J(C') \) be a birational map. Since \( J(C) \) and \( J(C') \) are abelian varieties, then it is well-known that \( \epsilon \) extends to an isomorphism \( \epsilon : J(C) \cong J(C') \). Since \( C \) (resp. \( C' \)) are very general curves in \( M_g \), we may assume (by [BL04, Cor. 17.5.2]) that \( \text{NS}(J(C)) = \mathbb{Z} \) (resp. \( \text{NS}(J(C')) = \mathbb{Z} \)) generated by the class of the theta divisor \( [\Theta_C] \) (resp. \( [\Theta_{C'}] \)). Therefore we must have \( \epsilon^*([\Theta_C]) = \pm [\Theta_{C'}] \). Moreover, since \( [\Theta_C] \) is ample and the pull-back morphism \( \epsilon^* \) preserves ampleness, we get that actually \( \epsilon^*([\Theta_{C'}]) = [\Theta_C] \). We conclude that \( C \cong C' \) by the classical Torelli theorem.

From the previous result, we can deduce two corollaries. The first one concerns the group of birational self maps \( \text{Bir}(P_{d,g}) \) and the group of automorphisms \( \text{Aut}(P_{d,g}) \) of \( P_{d,g} \).

Corollary 7.5. With the same assumptions as in Theorem 7.3, we have

\[
\text{Bir}(P_{d,g}) = \text{Aut}(P_{d,g}) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & \text{if } d(g-1) \text{ for some } n \in \mathbb{Z}, \\
\{\text{id}\} & \text{otherwise},
\end{cases}
\]

where in the first case the generator of the cyclic group \( \mathbb{Z}/2\mathbb{Z} \) is \( \psi^2_n \).

Proof. The claim follows from Theorem 7.3 since the only maps \( \psi^1_n \) and \( \psi^2_n \) having domain \( P_{d,g} \) and codomain \( P_{d,g} \) are \( \psi^1_n = \text{id} \) and \( \psi^2_n \) if \( d(g-1) \), and in this case \( (\psi^2_n) \circ (\psi^2_n) = \text{id} \).
The second corollary is analogous to [GKM02, Cor. (0.12)], which states that the boundary \( \partial \bar{M}_g \) of the moduli space \( \bar{M}_g \) of stable curves of genus \( g \geq 2 \) is preserved by any automorphism of \( \bar{M}_g \).

**Corollary 7.6.** Same assumptions as in Theorem 7.3. Then any automorphism \( \phi : \bar{P}_{d,g} \to \bar{P}_{d,g} \) preserves the boundary \( \partial \bar{P}_{d,g} := \bar{P}_{d,g} \setminus P_{d,g} \).

**Proof.** The restriction \( \eta := \phi |_{P_{d,g}} \) of \( \phi \) to \( P_{d,g} \) defines a birational self map of \( P_{d,g} \). By Corollary 7.5, \( \eta \) is an automorphism of \( P_{d,g} \). Therefore \( \phi \) maps \( P_{d,g} \) isomorphically onto \( P_{d,g} \), hence it preserves the boundary \( \partial \bar{P}_{d,g} \). \( \square \)

**Remark 7.7.** Under the same assumptions as in Theorem 7.3, Corollary 7.6 implies that we have a restriction map

\[
\text{res} : \text{Aut}(\bar{P}_{d,g}) \to \text{Aut}(P_{d,g}).
\]

The map \( \text{res} \) is injective since \( \bar{P}_{d,g} \) is separated. In [Cap94, Lemma 8.1], it is claimed that the map \( \psi_2^2 \) of (38) extends to a map \( \bar{P}_{d,g} \to \bar{P}_{d,g} \) (the analogous statement for \( \psi_1^1 \) is easy to prove). This fact, together with Corollary 7.5, would imply that \( \text{res} \) is an isomorphism.

Finally, note that if one could remove our technical assumption on the degree in Theorem 1.2, then Theorem 7.3 and Corollaries 7.5 and 7.6 would follow for \( g \geq 12 \) without any hypothesis on the degree.

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