Fourier series for the three-dimensional random flight

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Abstract

The probability density function of the random flight with isotropic initial conditions is obtained by an expansion in the number of collisions and the in the spatial harmonics of the solution, as in a Fourier series. The method holds for any dimension and is worked out in detail for the three dimensional case. In this case the probability density functions conditional to 1 and 2 collisions are also found using a different method, which yields them in terms of elementary functions and the polylogarithm function \( \text{Li}_2 \). The latter method is exact in the sense that one does not have to truncate a series, as in the first method. This provides a reference to decide where to truncate the series. A link is provided to a web page where the reader may download the series truncated at 132 collisions; for times larger than 100 times the average inter-collision time, the Gaussian approximations is used. The case in which the initial condition is a particle moving along a fixed direction is briefly considered.

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1 Introduction

In the random flight a particle moves with a constant speed until, at times uniformly distributed with density $\lambda$, it takes a uniformly distributed direction, while maintaining the same speed. To summarize it colorfully, “a photon in a box of mirrors” [1]. There are, of course, generalizations of this definition (see page xix of [2]). The random flight is the simplest case of “continuous time random walk” (CTRW) ([3], p. 177). The words “random flight” may also be used when the changes of direction do not take place at uniformly distributed times but at equally spaced times. This usage is due to Rayleigh [4] (reprinted in [5]) as quoted in page 55 of [6]. See references [6] (chapters 2 and 5) and [6, 7, 8, 2, 9] and references therein for a broader introduction to the subject. The random flight is also the limit of a random Lorentz gas when the size of the obstacles goes to zero (for two recent references on the subject see [10, 11]). This introductory paragraph relies heavily on a similar paragraph written by this author in reference [9].

The random flight with isotropic initial conditions was solved in 1 dimension by Goldstein in 1950 [12], in 2 dimensions in 1987 by Stadje [13] and in 1993 by Masoliver et alii
The main result of this article is that the probability density function (henceforth, pdf) of the random flight with isotropic initial conditions can be obtained by an expansion in the number of collisions and the spatial frequencies of the solution, as in a Fourier series. How to do this is explained in sections 7 through 13 and Appendix II. This method is applicable to any dimension, including non-integer dimensions.

This main method is applied to the three dimensional case, conspicuously missing from the list of solutions given in the second paragraph of this article. It is also a case with numerous applications, in particular to the scattering of photons [17, 18, 19, 20, 21, 22] and seismic waves [23, 24]. In the three dimensional case the main method is numerically not that good when used to find the probability density function conditional to 1 and 2 collisions. To find these two pdf’s a second result is obtained (sections 5, 6 and Appendix I) which is obtained by developing a method presented in reference [25]. These two pdf’s also provide a standard to test the main method.

The case of the random flight having a particle moving with a fixed direction and velocity as initial condition (as opposed to isotropic initial conditions) has not been treated often [26, 27, 28, 25]. It is, however, necessary to furnish the solution to a problem with arbitrary initial conditions. We have not been able to find an analytic solution to this problem, but it may be solved by an integral which gives the solution in terms of the solution to the isotropic case (see section 14).

We have written a program which computes the pdf for the three-dimensional random flight with isotropic initial conditions using the methods presented in this article. It is written in the programming language Mathematica and can be downloaded from the site [29]. The said pdf is denoted Ro[r,t] in the program and values for the speed v of the particle and the scattering rate \( \lambda \) need to be provided by the user. See section 13 for further detail on this.

For the reader who has read reference [9], we note that there is a subtle difference between the random flight in one dimension and the random flight in more than one dimension. In one dimension if a particle is scattered there is a probability of 1/2 that it keeps its direction, while in the latter case this probability is 0. Therefore in more than 1 dimension the number of scatterings is equal to the number of changes of directions with probability 1, and the distinction made between the two models shown in section 2 of [9] disappears.

2 Definitions

The pdf for the 3-dimensional random flight with isotropic initial conditions is the sum of an expanding spherical shell of norm \( e^{-\lambda t} \) and a pdf \((1 - e^{-\lambda t})\rho(\vec{r}, t)\) of norm \(1 - e^{-\lambda t}\), that is,

\[
e^{-\lambda t} \frac{\delta(\vec{r} - vt)}{4\pi r^2} + (1 - e^{-\lambda t})\rho(\vec{r}, t),
\]

with \( \rho \) normalized to 1. Henceforth, we deal exclusively with \( \rho \) and its relatives, which we introduce right now:
\[\rho(t, \omega) : B_{(\mathbf{0}, vt)} \to \mathbb{R}^+\]
\[\rho_r(t) : [0, vt] \to \mathbb{R}^+\]
\[\rho_I(t) : [0, vt] \to \mathbb{R}^+\]
\[\rho_{rs}(t) : [-vt, +vt] \to \mathbb{R}^+\]
\[\rho_{Is}(t) : [-vt, +vt] \to \mathbb{R}^+\]
\[\rho_{rsp}(t) : \mathbb{R} \to \mathbb{R}^+\]
\[\rho_{Isp}(t) : \mathbb{R} \to \mathbb{R}^+\]
\[\rho_{proj}(t) : [-vt, +vt] \to \mathbb{R}^+\]
\[\rho_{rs}(t, \omega) \equiv \frac{1}{2} \rho_r(|\mathbf{r}|, t) = \begin{cases} 
\frac{1}{2} 4\pi x^2 \rho_I(x, t) , & x \geq 0 \\
\frac{1}{2} 4\pi x^2 \rho_I(-x, t) , & x < 0 
\end{cases}\]  \( (5) \)

\[\rho_{Is}(x, t) \equiv \frac{1}{2} \rho_I(|x|, t).\]  \( (6) \)

where \( B_{(\mathbf{0}, vt)} \) is the ball centered at the origin and of radius \( vt \). The function that we want to find is \( \rho(t, \omega) \), although the radial distribution, \( \rho_r(t) \), may also be of interest. They are related as follows:

\[\rho_r(r, t) \equiv 4\pi r^2 \rho(\tilde{r}, t),\]  \( (3) \)

where \( \tilde{r} \) is any point such that \(|\tilde{r}| = r\). The pdf \( \rho_I(t) \) is simply \( \rho(t) \) but regarded as a 1-dimensional function, that is,

\[\rho_I(r, t) \equiv \rho(\tilde{r}, t),\]  \( (4) \)

where \( \tilde{r} \) is any point such that \(|\tilde{r}| = r\). The pdf’s \( \rho_{rs}(t) \) and \( \rho_{Is}(t) \) are almost \( \rho_r(t) \) and \( \rho_I(t) \), respectively, but they are more convenient for technical reasons, because they are symmetric about the origin and thus their Fourier series are just cosine series. Their definitions are

\[\rho_{rs}(x, t) \equiv \frac{1}{2} \rho_r(|x|, t) = \begin{cases} 
\frac{1}{2} 4\pi x^2 \rho_I(x, t) , & x \geq 0 \\
\frac{1}{2} 4\pi x^2 \rho_I(-x, t) , & x < 0 
\end{cases}\]  \( (5) \)

and

\[\rho_{Is}(x, t) \equiv \frac{1}{2} \rho_I(|x|, t).\]  \( (6) \)

With the goal of using Fourier series as in section 3 of [9] we define periodic functions which are simply the repetition of \( \rho_{rs} \) and \( \rho_{Is} \) over the support \([-vt, +vt]\) to the left and to the right:

\[\rho_{rsp}(x, t) \equiv \rho_{rs}(((x + vt) \mod 2vt) - vt, t)\]
\[\rho_{Isp}(x, t) \equiv \rho_{Is}(((x + vt) \mod 2vt) - vt, t)\]  \( (7) \)

Note that on \([-vt, +vt]\), \( \rho_{rsp}(x, t) = \rho_{rs}(x, t) \) and \( \rho_{Isp}(x, t) = \rho_{Is}(x, t) \).

The pdf \( \rho_{proj}(t) \) is the projection of \( \rho(t) \) onto a diameter. Its connection with \( \rho(t) \) is more complicated than the above relations and will be presented in section 5.

3 Collision expansion and scaling relations

First, a word on notation. If we convolute two real functions of the real variable \( x \), say functions \( f \) and \( g \), we use the notation \( f \otimes g \). But when these depend also on other
arguments, say $x_f$ and $x_g$, different from $x$, then we use a different notation, which we explain by the following example:

$$\int_{-\infty}^{+\infty} dx' f(x_f, x') \otimes g(x_g, x - x').$$

(8)

The collision expansion (used in reference [9] and called Born expansion in references [30, 7]) for the pdf $\rho$ is

$$\rho(\vec{r}, t) = e^{-\lambda t} \sum_{c=1}^{\infty} \frac{(\lambda t)^c}{c!} \rho_c(\vec{r}, t),$$

(9)

where $\rho_c$ is the one-dimensional pdf conditional to $c$ collisions. The expansion follows from the fact that the collision times are uniformly distributed, so that the number of collisions in the time interval $[0, vt]$ is Poisson distributed. Note that the sum starts at $c = 1$ because we are not including the expanding spherical shell, which is

$$\rho_0(r, t) = \frac{\delta(r - vt)}{4\pi r^2}.$$  

(10)

It follows from expansion (15) in reference [30] or from expression (24) in [7] that

$$\rho_c(\vec{r}, t) = \frac{c!}{t^c} \otimes_{c+1} \rho_0(\vec{r}, t),$$

(11)

where $\otimes$ is a convolution on all the variables of $\rho_0$, that is, a convolution on space and time. For example,

$$\otimes^3 \rho_0(\vec{r}, t) = \int_0^{l_2} dt_1 \int_0^t dt_2 (\rho_0(\vec{r}, t_1) \otimes \rho_0(\vec{r}, t_2 - t_1) \otimes \rho_0(\vec{r}, t - t_2)(\vec{r})$$

(12)

where the inner $\otimes$’s are convolutions on the space coordinates only, as implied by the notation explained at the beginning of the section. The prefactor $\frac{c!}{t^c}$ in equality (11) ensures the normalization of $\rho_c$.

We denote by $\rho_{rs,c}$ the one-dimensional pdf $\rho_{rs}$ conditional to $c$ collisions. The collision expansion reads

$$\rho_{rs}(x, t) = e^{-\lambda t} \sum_{c=1}^{\infty} \frac{(\lambda t)^c}{c!} \rho_{rs,c}(x, t).$$

(13)

Similar expansions hold for the other pdf’s defined at the beginning of this section.

The important feature of these two conditional probability density functions, $\rho_c$ and $\rho_{rs,c}$, is that their dependence on time is just a scaling. Indeed, the pdf $\rho_c(\vec{r}, vt)$ is a function which maps the sphere of radius $vt$ into $\mathbb{R}^+$. The quantity $\rho_c(\vec{r}, t)d^3r$ is the proportion of broken lines of length $vt$, with $c$ uniformly distributed corners and with uniformly distributed angles at the corners, which, starting at the origin of coordinates, end in a neighborhood $d^3r$ around $\vec{r}$. A similar definition holds for the quantity $\rho_c(\vec{r}, t')d^3r'$. There is a trivial one-to-one correspondence between the two sets of broken lines, related by a similarity transformation of ratio $t'/t$. Clearly (see Fig. 1),
\[ \rho_c(\vec{r}, t) d^3r = \rho_c(\vec{r}', t') d^3r' = \rho_c(\left\langle \vec{r}, t \right\rangle \left(\frac{t'}{t}\right)^3 d^3r, \]
from which the following scaling relation between \( \rho_c(\vec{r}, t) \) and \( \rho_c(\vec{r}', t') \) follows:

\[
\rho_c(\vec{r}, t) = \left(\frac{t'}{t}\right)^3 \rho_c\left(\frac{t'}{t} \vec{r}, t'\right) \quad \Rightarrow \quad \rho_c(\vec{r}, t) = \frac{1}{(vt)^3} \rho_c\left(\vec{r}, \frac{1}{vt}\right). \tag{14}
\]

This scaling property is inherited by \( \rho_{I,c} \), \( \rho_{Is,c} \) and \( \rho_{Is,c} \). It follows from its definition (3) that \( \rho_{r,c} \) scales as

\[
\rho_{r,c}(x, t) = \frac{t'}{t} \rho_{r,c}\left(\frac{t'}{t} x, t'\right) \quad \Rightarrow \quad \rho_{r,c}(x, t) = \frac{1}{vt} \rho_{r,c}\left(x, \frac{1}{vt}\right). \tag{15}
\]

This scaling property is inherited by \( \rho_{rs,c} \) and \( \rho_{rsp,c} \). Likewise, it follows from its definition (20) that \( \rho_{proj} \) scales as

\[
\rho_{proj,c}(x, t) = \frac{t'}{t} \rho_{proj,c}\left(\frac{t'}{t} x, t'\right) \quad \Rightarrow \quad \rho_{proj,c}(x, t) = \frac{1}{vt} \rho_{proj,c}\left(x, \frac{1}{vt}\right). \tag{16}
\]

Figure 1: To the left, a broken line of length \( vt \) starting at the origin and finishing at the element of volume \( d^3r \) is shown. To the right, the corresponding broken line of length \( vt' \) starting at the origin and finishing at the element of volume \( d^3r' \) is shown. In both cases \( c = 2 \).

This is very good news because it shows that the collision expansions give functions of two variables, \( (x, t) \) or \( (r, t) \), as a linear combination of effectively one dimensional functions. This is so because the time dependence is just a scaling and the spatial dependence reduces to a dependence on just one variable: the radius. As we shall see, \( \rho_{rs,c} \) can be found analytically for \( c = 0, 1, 2 \). As for \( c > 2 \), we shall give analytic expansions for \( \rho_{rs,c} \) (essentially the expansion (45)) and use the scaling relations (14) and (15) to speed their computation.

Equations (11) and (10) imply that all the \( \rho_{...,c}(\vec{r}, t) \) are actually functions \( \rho_{...,c}(\vec{r}, vt) \). If we define \( \Lambda = \lambda/v \) as the mean number of collisions per unit length and \( \ell \equiv vt \) as the radius of the expanding front, then the collision expansion (9) may be written as
\[ \rho(\vec{r}, \ell) = e^{-\Lambda} \sum_{c=1}^{\infty} \frac{(\Lambda\ell)^c}{c!} \rho_c(\vec{r}, \ell). \]

The advantage of this expression is that in expression (9) \( \rho(\vec{r}, t) \) is actually a function of \( v \), which has disappeared in expression (9). One can take the last step along this line and choose units of length in which \( \Lambda = 1 \). Then the collision expansion becomes

\[ \rho(\vec{r}, \ell) = e^{-\ell} \sum_{c=1}^{\infty} \frac{\ell^c}{c!} \rho_c(\vec{r}, \ell), \]

where \( \ell \) is now a dimensionless length. While these redefinition of variables is more convenient from a mathematical and computational point of view, in the remainder of this article we stick to the \( v, t, \lambda \) variables, which are a more intuitive set.

4 General plan

There are two distinct lines of attack in the present article, which are summarized in the diagram which follows. It is just a summary. In particular, it glosses over the fact that the supports of each of the three pdf’s shown in the diagram are not the same. Indeed, the supports of \( \rho_{rs} \) and \( \rho_{proj} \) are \([-vt, +vt]\), but the support of \( \rho \) is the sphere of radius \( vt \). In the approach presented in section 5 we start with an exact solution for \( \rho_{proj} \) and use the generalized inverse Abel transform to undo the projection. This is the approach shown on the right side of the diagram, and it is based on the results of reference [25]. It is analytically feasible only for 1 and 2 collisions. On the left hand side, the Fourier series approach, which was presented in [9], and which is feasible for any number of collisions. The reason why we do not use exclusively the method of the Fourier series, which is good for any number of collisions, is that it is numerically not as good as the other method for 1 and 2 collisions.

\[ \rho_{rs} \xrightarrow{\frac{1}{2\pi x^2}} \xrightarrow{\frac{1}{2\pi r^2}} \rho \xrightarrow{\text{Generalized inverse Abel transform}} \rho_{proj} \]

Fourier series

Exact solution from space-time convolution

5 Exact solution for up to two collisions

In reference [25] the solution to the Pearson problem [31] for odd dimensions was found as follows. Since the convolution and the projection commute, the problem was projected onto one dimension, it was solved there, and the solution was projected back onto three dimensions. “Projecting back from one onto three dimensions” is something which has
probably struck the reader as strange or plain wrong. Bear in mind, though, that the distributions involved in this problem are isotropic, and thus the inversion of the projection is possible. It can be achieved by a generalization of the Abel inverse transform. The projection and the generalized inverse Abel transform are given by formulae (17) and (18), respectively, of reference [25]. These formulae, written for the problem at hand, are:

\[ \rho_{\text{proj}}(x, t) = 2\pi \int_x^\infty dx' \rho_I(x', t) x' \]  

which is the projection onto the diameter (not used in the current article, it is given here just for completeness) and

\[
\begin{align*}
\rho_I(x, t) &= -\frac{1}{2\pi} \frac{d}{dx} \rho_{\text{proj}}(x, t) \\
\rho_{rs}(x, t) &= -x \frac{d}{dx} \rho_{\text{proj}}(x, t)
\end{align*}
\]

which are the generalized inverse Abel transforms for the case at hand.

We try now to make the preceding paragraph more concrete in the hope that the reader might understand this article without reading reference [25]. The pdf after collisions which take place at equally spaced times is obtained by convoluting \( c + 1 \) expanding spherical waves, that is, \( \rho_c(\ldots, t) = \rho_0\left(-\frac{t}{c+1}\right)^{\otimes c+1} \), where \( \rho_0(r, t) = \frac{\delta(r-vt)}{4\pi r^2} \). We define the projection operator \( P : \mathbb{R}^3 \to \mathbb{R} \) on isotropic functions and its inverse \( P^{-1} \) by equations (20) and (21), respectively. Then using the fact that the convolution and the projection commute,

\[
\rho_c(\ldots, t) = P^{-1} P \left( \rho_0\left(-\frac{t}{c+1}\right)^{\otimes c+1} \right) = P^{-1} \left( \rho_{\text{proj},0}\left(-\frac{t}{c+1}\right)^{\otimes c+1} \right).
\]

The key advantage provided by this procedure is that while the convolution in the second term takes place in 3 dimensions, the convolution in the third term takes place in 1 dimension. The projection of the spherical shell \( \frac{\delta(r-vt)}{4\pi r^2} \) onto a diameter is [25]:

\[
\rho_{\text{proj},0}(x, t) = \begin{cases} 
0, & x \notin [-vt, +vt] \\
\frac{1}{2vt}, & x \in [-vt, +vt]
\end{cases}
\]

where the subindex 0 stands for 0 collisions. These are the sort of rectangle functions that need to be convoluted in one dimension.

So far we have summarized the procedure used in reference [25] to solve the Pearson random walk and formula (22) is valid only when collisions happen at regular intervals. But in the problem at hand collisions happen at uniformly distributed times. The pdf \( \rho_c \) (defined by the expansion (9)) can be found with the methods of reference [25] supplemented with two steps: first, we let the \( c \) collisions happen at arbitrary times \( t_1, \ldots, t_c \); second, we average over \( t_1, \ldots, t_c \), which amounts to a time convolution.

For \( c = 0 \) the pdf \( \rho_c \) is trivial, it is just the expanding spherical shell \( \rho_0(r, t) = \frac{\delta(r-vt)}{4\pi r^2} \). In the next two sections we find \( \rho_c \) for \( c = 1 \). A lot of work is required to find the \( c = 2 \) solution and this is done in Appendix I.
6 Exact solution for one collision

6.1 Convolution of two rectangle functions

We define the rectangle function:

\[
\text{rect}(w, x) \equiv \begin{cases} 
0, & x \in (-\infty, -w/2) \\
\frac{1}{w}, & x \in [-w/2, +w/2] \\
0, & x \in (+w/2, +\infty)
\end{cases},
\]

(24)

where \( w \) is the width of the rectangle. This function is a pdf because it is positive and normalized and

\[
\rho_{\text{proj},0}(x, t) = \text{rect}(2vt, x),
\]

(25)

which is why we are interested in the rectangle function. The convolution of a function \( f \) with a rectangle function is

\[
(\text{rect}(w, \_ \otimes f))(x) = \int_{-\infty}^{+\infty} dx' f(x')\text{rect}(w, x - x') =
\]

\[
\int_{x-w/2}^{x+w/2} dx' \frac{f(x')}{w} = \frac{1}{w} \left( F(x + \frac{w}{2}) - F(x - \frac{w}{2}) \right),
\]

(26)

where \( F \) is some primitive of \( f \).

As a primitive of the rectangle function we choose this continuous function

\[
\text{Rect}(w, x) \equiv \begin{cases} 
-\frac{1}{2}, & x \in (-\infty, -w/2) \\
\frac{x}{w}, & x \in [-w/2, +w/2] \\
+\frac{1}{2}, & x \in (+w/2, +\infty)
\end{cases}.
\]

(27)

Then,

\[
(\text{rect}(w_1, \_ \otimes \text{rect}(w_2, \_))(x) = \frac{1}{w_2} \left( \text{Rect}(w_1, x + \frac{w_2}{2}) - \text{Rect}(w_1, x - \frac{w_2}{2}) \right) =
\]

\[
\left\{ \begin{array}{ll}
-\frac{1}{2w_2}, & x + \frac{w_2}{2} \in (-\infty, -w_1/2) \\
\frac{x + \frac{w_2}{2}}{w_1w_2}, & x + \frac{w_2}{2} \in [-w_1/2, +w_1/2] \\
+\frac{1}{2w_2}, & x + \frac{w_2}{2} \in (+w_1/2, +\infty)
\end{array} \right\} - \left\{ \begin{array}{ll}
-\frac{1}{2w_2}, & x - \frac{w_2}{2} \in (-\infty, -w_1/2) \\
\frac{x - \frac{w_2}{2}}{w_1w_2}, & x - \frac{w_2}{2} \in [-w_1/2, +w_1/2] \\
+\frac{1}{2w_2}, & x - \frac{w_2}{2} \in (+w_1/2, +\infty)
\end{array} \right\} =
\]

\[
\left\{ \begin{array}{ll}
-\frac{1}{2w_2}, & x \in (-\infty, \frac{-w_1-w_2}{2}) \\
\frac{x}{w_1w_2}, & x \in \left[ \frac{-w_1-w_2}{2}, \frac{w_1-w_2}{2} \right] \\
+\frac{1}{2w_2}, & x \in \left( \frac{w_1-w_2}{2}, +\infty \right)
\end{array} \right\} - \left\{ \begin{array}{ll}
-\frac{1}{2w_2}, & x \in (-\infty, \frac{-w_1+w_2}{2}) \\
\frac{x}{w_1w_2}, & x \in \left[ \frac{-w_1+w_2}{2}, \frac{w_1+w_2}{2} \right] \\
+\frac{1}{2w_2}, & x \in \left( \frac{w_1+w_2}{2}, +\infty \right)
\end{array} \right\}.
\]

(28)

If we choose \( w_1 < w_2 \), then
\(-\infty < \frac{-w_1 - w_2}{2} < \frac{w_1 - w_2}{2} < \frac{-w_1 + w_2}{2} < \frac{+w_1 + w_2}{2} < \infty\)

and

\[
(rect(w_1, -) \otimes rect(w_2, -))(x) =
\begin{cases}
0, & x \in (-\infty, -\frac{w_1 - w_2}{2}) \\
\frac{1}{2} \left( \frac{1}{w_1} + \frac{1}{w_2} \right) + \frac{x}{w_1 w_2}, & x \in [-\frac{w_1 - w_2}{2}, \frac{w_1 - w_2}{2}] \\
\frac{1}{w_2}, & x \in (\frac{w_1 - w_2}{2}, -\frac{w_1 + w_2}{2}) \\
\frac{1}{2} \left( \frac{1}{w_1} + \frac{1}{w_2} \right) - \frac{x}{w_1 w_2}, & x \in [-\frac{w_1 + w_2}{2}, \frac{w_1 + w_2}{2}] \\
0, & x \in (\frac{w_1 + w_2}{2}, +\infty)
\end{cases}.
\] (29)

The graph of \((rect(w_1, -) \otimes rect(w_2, -))\) has the shape of a trapezoid whose base has length \(w_1 + w_2\) and whose top side has length \(w_2 - w_1\).

### 6.2 Space-time convolution of two rectangle functions

We define the function “restriction” by

\[
restr(a, x, b) \equiv \begin{cases}
0, & x < (-\infty, a) \\
1, & x \in [a, b] \\
0, & x \in (b, +\infty)
\end{cases}.
\] (30)

When \(w \leq vt\) it follows from equation (29) that

\[
(rect(w, -) \otimes rect(2vt - w, -))(x) =
\]

\[
restr(-\infty, x, -vt) \times 0 + restr(-vt, x, w - vt) \left( \frac{1}{2} \left( \frac{1}{w} + \frac{1}{2vt - w} \right) + \frac{x}{w(2vt - w)} \right) +
\]

\[
restr(w - vt, x, -w + vt) \frac{1}{2vt - w} + ..., \]

(31)

where the last two terms have been omitted but can be obtained from the first two by symmetry. The equal sign means that the lhs and the rhs are equal in probability, that is, integrals over any interval have the same value for both of them. But the lhs and the rhs might differ at the end points of the intervals. This is irrelevant for our purposes.

The average over the collision times mentioned at the end of section 5 is, for \(c = 1\),

\[
\rho_{proj,1}(x, t) = \frac{1}{vt} \int_0^t dw \, (\rho_{proj,0}(\cdot, w) \otimes \rho_{proj,0}(\cdot, 2vt - w))(x).
\] (32)

To do this average we use the equality (25) and distinguish two cases. If we take \(x\) to be negative, then
\[
\rho_{\text{proj,1}}(x, t) = \frac{1}{vt} \int_0^{vt} dw \left( \text{rect}(w, \omega) \otimes \text{rect}(2vt - w, \omega) \right)(x) =
\]
\[
\frac{1}{vt} \left[ 0 + \int_{x+vt}^{vt} dw \left( \frac{x}{w(2vt - w)} + \frac{1}{2} \frac{w}{2vt - w} + \frac{1}{2vt - w} \right) + \int_0^{x+vt} dw \frac{1}{2vt - w} \right] =
\]
\[
\frac{1}{vt} \left[ \frac{vt + x}{2vt} \ln \frac{vt - x}{vt + x} + \ln \frac{2vt}{vt - x} \right].
\] (33)

If we take \( x \) to be positive, then
\[
\rho_{\text{proj,1}}(x, t) = \frac{1}{vt} \int_0^{vt} dw \left( \text{rect}(w, \omega) \otimes \text{rect}(2vt - w, \omega) \right)(x) =
\]
\[
\frac{1}{vt} \left[ \int_0^{-x+vt} dw \frac{1}{2vt - w} + \int_{-x+vt}^{vt} dw \left( \frac{x}{w(2vt - w)} + \frac{1}{2} \frac{w}{2vt - w} - \frac{1}{2vt - w} \right) + 0 \right] =
\]
\[
\frac{1}{vt} \left[ \frac{vt + x}{2vt} \ln \frac{vt - x}{vt + x} + \ln \frac{2vt}{vt - x} \right],
\] (34)

also.

The three-dimensional pdf is found using the inverse Abel transform (21):
\[
\rho_1(\vec{r}, t) = -\frac{1}{2\pi r} \frac{d}{dr} \left[ \frac{1}{2\pi} \ln \frac{vt - r}{vt + r} + \ln \frac{2vt}{vt - r} \right] = \frac{1}{4\pi (vt)^2} \ln \frac{vt + r}{vt - r},
\] (35)

which is graphed in Fig. 4. Clearly, this function has a logarithmic divergence at \( r = vt \). This has a qualitative consequence on the shape of the graph, because this divergence shows up in \( \rho_I \) and \( \rho_r \). Indeed, the contribution of \( \rho_1 \) to the collision expansion (9) is never 0, even for large \( t \), so \( \rho_I \) and \( \rho_r \) have a vertical asymptote at \( r = vt \). It is still conceivable that other \( \rho_c \) might also have divergences at \( r = vt \) which cancel the one due to \( \rho_1 \), but it turns out that this is not so, as will be explained in section 13.

When \( \lambda t << 1 \) (very little scattering and/or very early times)
\[
\rho(\vec{r}, t) \approx \frac{1}{1 + \lambda t} \left( \frac{\delta(r - vt)}{4\pi r^2} + \lambda t \rho_1(\vec{r}, t) \right),
\] (36)

according to formula (57).
7 Solutions as Fourier series

7.1 Moments

To denote the moments we use the notation \( \langle \rangle \) for expected value, as in Quantum Mechanics. The relations between the even moments of \( \rho_{rs}(\mathbf{r}, t) \), \( \rho_{Is}(\mathbf{r}, t) \) and \( \rho(\mathbf{r}, t) \) are (the odd moments will not be used):

\[
\langle x^{2m} \rangle_{\rho_{rs}} = \langle r^{2m} \rangle_{\rho_c} = \langle r^{2m} \rangle_{\rho} \quad \text{for} \quad m = 0, 1, 2, \ldots \tag{37}
\]

and

\[
\langle x^{2m} \rangle_{\rho_{Is}} = \langle x^{2m} \rangle_{\rho} = \frac{1}{4\pi} \langle r^{2m-2} \rangle_{\rho} = \frac{1}{4\pi} \langle r^{2(m-1)} \rangle_{\rho} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{38}
\]

For \( m = 0 \) the last relation is still true, but \( \langle r^{-2} \rangle_{\rho} \) cannot be obtained in the way in which the positive even moments are obtained in reference [30].

Either from formulae (25) and (26) of reference [30] or from formulae (31) and (32) of reference [7] for the 3-dimensional case we obtain:

\[
\langle r^{2m} \rangle = (2m + 1)! \ e^{-\lambda t} \ (vt)^{2m} \sum_{c=0}^{\infty} \frac{(\lambda t)^c}{(2m + c)!} C(c, m), \tag{39}
\]

where

\[
C(c, m) \equiv \sum_{i_1, \ldots, i_{c+1} \in N \atop 1 + \ldots + i_{c+1} = m} \frac{1}{(2i_1 + 1) \cdots (2i_{c+1} + 1)} \tag{40}
\]

and \( c \) labels the number of collisions. See Appendix II for computation and some properties of the \( C(c, m) \) coefficients.

It follows from formula (39) and from the collision expansion (9) that

\[
\langle r^{2m} \rangle_{\rho_c} = (vt)^{2m} \frac{c!(2m + 1)!}{(2m + c)!} C(c, m) \quad \text{for} \quad m = 0, 1, 2, \ldots \tag{41}
\]

It follows from formulae (38) and (41) that

\[
\langle r^{2m} \rangle_{\rho_{Is,c}} = \frac{(vt)^{2m-2}}{4\pi} \frac{c!(2m - 1)!}{(2m - 2 + c)!} C(c, m - 1) \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{42}
\]

7.2 Fourier expansion in terms of the moments

The definition of Fourier transform that we use is

\[
\tilde{f}(\nu) \equiv \int dx \ f(x) \ e^{-i2\pi \nu x} = \langle e^{-i2\pi \nu x} \rangle \equiv \sum_{m=0}^{\infty} \frac{(-i2\pi)^m \langle x^m \rangle}{m!} \nu^m, \tag{43}
\]

\[12\]
where the last equality holds when the support of $f$ is bounded (30, Appendix C), as in the pdf’s of interest in this article. For such pdf’s the inverse Fourier transform may be written as follows

$$f(x) = \int d\nu \, \tilde{f}(\nu) \, e^{i2\pi \nu x} = \int d\nu \sum_{m=0}^{\infty} \frac{(-i2\pi)^m \nu^m \langle x^m \rangle}{m!} \, e^{i2\pi \nu x}. \quad (44)$$

For any periodic function $f_p$ of period $2vt$ (see the beginning of section 3 of reference [9]),

$$f_p(x) = \frac{1}{2vt} \sum_{h=-\infty}^{\infty} \tilde{f} \left( \frac{h}{2vt} \right) e^{i\frac{2\pi h}{2vt} x} = \frac{1}{2vt} \sum_{h=-\infty}^{\infty} \tilde{f} \left( \frac{h}{2vt} \right) e^{i\frac{\pi h}{vt} x}. \quad (45)$$

When $f_p$ is even,

$$f_p(x) = \frac{1}{2vt} \left( \langle x^0 \rangle + 2 \sum_{h=1}^{\infty} \left[ \sum_{m=0}^{\infty} \frac{(-4\pi)^m \langle x^{2m} \rangle}{(2m)!} \cdot \left( \frac{h}{2vt} \right)^{2m} \right] \cos \frac{\pi h}{vt} x \right). \quad (46)$$

A word of caution on the notation. We have chosen to work with the symmetrized pdf’s (the ones that have $s$ as subindex) in order to be able to use the above expansion, which has only even moments. So the index $m$ refers not to the $m$-th moment, but to the $2m$-th moment.

8 Fourier series of $\rho_{rs,c}$ and $\rho_{Is,c}$

We are going to find $\rho(x, t)$ in three dimensions by means of a Fourier series. The Fourier-Laplace transform of $\rho$ is known [32, 26], but it cannot be inverted analytically. But the moments of $\rho(x, t)$ are known [30], and from them a Fourier series for $\rho(x, t)$ can be obtained as in one dimension [9]. Indeed, substitution of expression (41) into formula (45) yields, for $x \in [-vt, +vt]$ (see definition (7)),

$$\rho_{rs,c}(x, t) = \rho_{rsp,c}(x, t) = \frac{1}{2vt} \left( \langle r^0 \rangle + 2 \sum_{h=1}^{\infty} \left[ \sum_{m=1}^{\infty} \frac{(-\pi^2)^m (2m+1)c!}{(2m+c)!} C(c, m) \cdot h^{2m} \right] \cos \frac{\pi h}{vt} x \right). \quad (46)$$

Likewise, substitution of expressions (38) (for $\langle r^0 \rangle_{\rho_{rs,c}}$) and (42) (for $\langle r^{2m} \rangle_{\rho_{rs,c}}$, $m = 1, 2, 3, \ldots$) into formula (45) yields, for $x \in [-vt, +vt]$ (see definition (7)),

$$\rho_{Is,c}(x, t) = \rho_{Isp,c}(x, t) =$$

$$\frac{1}{2vt} \left( \langle r^{-2} \rangle_{\rho_c} + \frac{1}{4\pi} \sum_{h=1}^{\infty} \left[ \langle r^{-2} \rangle_{\rho_c} + \sum_{m=1}^{\infty} \frac{(-\pi^2)^m c!}{2m(2m-2+c)!} \cdot h^{2m} \right] \cos \frac{\pi h}{vt} x \right). \quad (47)$$

Note that, unlike formula (46), in the previous formula the lower index of the summation sign over $m$ starts at 1, because the contribution of $\langle r^0 \rangle_{\rho_{Is,c}}$ has been pulled out of the
The Fourier series coefficients in expansion (47) when \( \rho_{rs,c}(x,t) \) and \( \rho_{I_{s,c}}(x,t) \) are, respectively, the ones stated in equations (14) and (15).

Theoretically, we could find \( \rho_{I_{s,c}} \) from the equation

\[
\rho_{rs,c}(x,t) = 4\pi x^2 \rho_{I_{s,c}}(x,t)
\]  

and avoid using expansion (47), in which the quantity \( (r^{-2})_{\rho_c} \) is not provided by reference [30]. But in expression (46) the upper limits of the summation signs cannot actually be made infinity. Thus \( \rho_{rs,c}(x,t) \) is not known exactly and \( \frac{\rho_{rs,c}(x,t)}{4\pi x^2} \) is bound to be not well behaved near the origin, as shown in Fig. 6.

But there is an elegant and accurate way of finding \( (r^{-2})_{\rho} \) in expansion (47). In expression (47) we define the Fourier series coefficients

\[
FS(\rho_{I_{s,c}}(0)) = \frac{(r^{-2})_{\rho_c}}{8\pi}
\]  

These are the Fourier series coefficients in expansion (47) when \( vt = 1 \). When \( vt \neq 1 \) the pdf \( \rho_{I_{s,c}} \) may be computed using the scaling relations (14) and (15):

\[
\rho_{I_{s,c}}(x,t) = e^{-\lambda t} \sum_{c=1}^{\infty} \frac{(\lambda t)^c}{c!} \rho_{I_{s,c}}(x,t) = e^{-\lambda t} \sum_{c=1}^{\infty} \frac{(\lambda t)^c}{c!} \rho_{I_{s,c}}(\frac{x}{vt}, \frac{1}{v}) =
\]

\[
e^{-\lambda t} \frac{1}{(vt)^3} \sum_{c=1}^{\infty} \frac{(\lambda t)^c}{c!} \sum_{h=0}^{\infty} FS(\rho_{I_{s,c}}(h)) \cos \frac{\pi h}{vt} x.
\]  

We now show that \( \lim_{h \to \infty} FS(\rho_{I_{s,c}}(h)) = 0, \forall c \in \{1, 2, 3, \ldots\} \). Intuitively, if \( FS(\rho_{I_{s,c}}(h)) \) didn’t tend to zero as \( h \to \infty \), then there would be oscillations of unbounded frequency and \( \rho_{I_{s,c}} \) would then not be a continuous function. We now give a proof of the advertised limit.

**Proposition 1.** Let there be a natural number \( n \). Let there be a function \( f : \mathbb{R}^n \to \mathbb{R} \) which has minimum \( m \) and maximum \( M \). Let there be a pdf \( g : \mathbb{R}^n \to \mathbb{R}^+ \). Then the minimum \( m' \) and maximum \( M' \) of \( f \otimes g \) satisfy \( [m', M'] \subset [m, M] \).

**Proof.** The expected value of \( f \) with respect to any pdf, which we denote by \( \langle f \rangle \), certainly satisfies \( \langle f \rangle \in [m, M] \). The convolution \( f \otimes g \) at \( \vec{r} \in \mathbb{R}^n \) is

\[
(f \otimes g)(\vec{r}) = \int d^n r' f(\vec{r}') g(\vec{r} - \vec{r}') = \int d^n r' f(\vec{r}') g(-(\vec{r}' - \vec{r})),
\]

which is the expected value of \( f \) with respect to the pdf \( g(-(\vec{r}' - \vec{r})) = -T_{\vec{r}}(g) \), that is, the pdf \( g \) translated by the vector \( \vec{r} \) and then inverted. Therefore \( (f \otimes g)(\vec{r}) \in [m, M] \), which proves the proposition. QED.
Proposition 2. If \( \rho_c \) is bounded by \( M_c \in \mathbb{R}^+ \), then \( \rho_{c+1} \) is bounded by \( (c+1)M_c \).

Proof. It follows from formula (11) that

\[
\rho_{c+1}(\vec{r}, t) = \frac{c+1}{t} (\rho_c \otimes \rho_0)(\vec{r}, t) = \frac{c+1}{t} \int_0^t dt' \ (\rho_c(\omega, t') \otimes \rho_0(\omega - t'))(\vec{r}').
\]

(52)

Proposition 1 implies that the integrand in the last term is bounded by \( M_c \). The integrand is then averaged over \( t \) and multiplied by \( c+1 \), thus the final result is bounded by \( (c+1)M_c \).

QED.

Proposition 3. \( \lim_{h \to \infty} FS(\rho_{Is,c})(h) = 0 \), \( \forall c \in \{1, 2, 3, \ldots\} \).

Proof. Suppose that \( \lim_{h \to \infty} FS(\rho_{Is,c})(h) \neq 0 \). Then, by Parseval’s formula (see, e.g., reference [33]), the \( L^2 \) norm of \( \rho_{Is,c} \) is

\[
||\rho_{Is,c}||_2 = \frac{1}{(vt)^2} \sum_{h=0}^{\infty} FS(\rho_{Is,c})(h)^2 = \infty.
\]

(53)

But we can see that for \( c = 1, 2, 3, \ldots \), \( ||\rho_{Is,c}||_2 < \infty \). For \( \rho_{Is,1} = \frac{1}{4\pi (vt)^2} \ln \frac{vt+r}{vt-r} \) (formula (55)) this can be checked by direct computation. Since \( \rho_2 \) is bounded (see the graph of \( \rho_{Is,2}(r, t) = \frac{1}{2} \rho_2(\vec{r}, t) \) in Fig. 6), by applying Proposition 2 iteratively to \( \rho_2 \), \( \rho_3 \), \( \rho_4 \),..., it follows that the \( \rho_c \)’s are bounded for \( c \geq 2 \), which implies that their \( L^2 \) norm is finite.

QED.

We define a quantity related to \( FS(\rho_{Is,c})(h) \):

\[
FS_0(\rho_{Is,c})(h) \equiv \frac{1}{4\pi} \sum_{m=1}^{\infty} \frac{(-\pi^2)^m c! C(c, m-1)}{2m(2m-2+c)!} \cdot h^{2m} \quad \text{for} \quad h = 1, 2, 3, \ldots
\]

(54)

The result \( \lim_{h \to \infty} FS(\rho_{Is,c})(h) = 0 \), \( \forall c \in \{1, 2, 3, \ldots\} \) implies that

\[
\langle r^{-2} \rangle_{\rho_c} = -\lim_{h \to \infty} 4\pi \ FS_0(\rho_{Is,c})(h), \quad \forall c \in \{1, 2, 3, \ldots\}
\]

(55)

which renders formula (47) useful. An example of this is worked out in section 11.

9 Truncations

In order to compute \( \rho_r \) and \( \rho_I \), the summations in the collision expansion (9) and in the series (46) and (47) over the indices \( c \), \( h \) and \( m \) must be truncated. In this section we obtain upper bounds for these summations to compute \( \rho_r \) and \( \rho_I \) with reasonable accuracy. A more sophisticated, precise and extensive discussion of these issues than the one presented in this section is certainly possible. But the main emphasis of this article is not numerical.

We need to estimate which the upper limits of the summations over the harmonics \( h \) index), the moments \( m \) index) and the collisions \( c \) index) should be. The estimation of the upper limit of \( c \) is the only one which is easy.

The conditional pdf’s \( \rho_{Is,c} \) are all normalized. But the \( \rho \) given in the collision expansion (9) is just the non-singular part of the pdf, which is normalized to \( 1 - e^{-\lambda t} \). Therefore, if
we want to neglect less than \( \varepsilon_r \) of the norm of the non-singular part, we have to solve for \( c_{\text{max}} \) in the inequality
\[
\frac{e^{-\lambda t}}{1 - e^{-\lambda t}} \sum_{c = c_{\text{max}} + 1}^{\infty} \frac{(\lambda t)^c}{c!} \leq \varepsilon_r \rightarrow c_{\text{max}}(\lambda t, \varepsilon_r).
\]

Then we can be content with the approximation
\[
\frac{1 - e^{-\lambda t}}{\sum_{j=1}^{c_{\text{max}}(\lambda t, \varepsilon_r)} \frac{(\lambda t)^j}{j!}} \sum_{c=1}^{c_{\text{max}}(\lambda t, \varepsilon_r)} \frac{(\lambda t)^c}{c!} \rho_{rs,c}(x, t),
\]

where the prefactor ensures that the approximation is kept normalized to \( 1 - e^{-\lambda t} \). That the norm of a truncation in the number of collisions departs from 1 by less than, say, \( 10^{-3} \) does not guarantee that at any given point \( x \) the truncated pdf and the true one differ by less than 1 part in 1,000, but it is reasonable to expect that it cannot differ by a lot more.

The following data have been obtained numerically: \( c_{\text{max}}(0.1, 10^{-3}) = 2 \), \( c_{\text{max}}(1, 10^{-3}) = 5 \), \( c_{\text{max}}(10, 10^{-3}) = 21 \), \( c_{\text{max}}(50, 10^{-3}) = 73 \), \( c_{\text{max}}(100, 10^{-3}) = 132 \).

There is no easy theoretical way of finding at which moment or harmonic the sums over \( m \) or \( h \) have to be truncated. But we can actually test the accuracy of these truncations for \( \rho_{rs,0}, \rho_{rs,1} \) and \( \rho_{rs,2} \), for which exact solutions may be obtained. Similarly to (50) we define the Fourier series coefficient \( FS(\rho_{rs,c})(h) \) by
\[
FS(\rho_{rs,c})(h) \equiv \sum_{m=0}^{\infty} (-\pi^2)^m \frac{(2m + 1)c!}{(2m + c)!} C(c, m)(h^2)^m = \left\{ (-\pi^2)^m \frac{(2m + 1)c!}{(2m + c)!} C(c, m) \right\}^{\infty}_{m=0} \cdot \left\{ (h^2)^m \right\}^{\infty}_{m=0} \approx \left\{ (-\pi^2)^m \frac{(2m + 1)c!}{(2m + c)!} C(c, m) \right\}^{m_{\text{Max}}}_{m=0} \cdot \left\{ (h^2)^m \right\}^{m_{\text{Max}}}_{m=0} \quad \text{for} \quad h = 1, 2, 3, \ldots, \quad (58)
\]

which we have written in a manner that suggests a scalar product. The advantage of this is that, for each \( c \), \( \left\{ (-\pi^2)^m \frac{(2m + 1)c!}{(2m + c)!} C(c, m) \right\}^{m_{\text{Max}}}_{m=0} \) may be computed first for a large \( m_{\text{Max}} \). This sequence may be stored and then multiplied scalarly by the sequence \( \left\{ (h^2)^m \right\}^{m_{\text{Max}}}_{m=0} \), where \( m_{\text{Max}} \) might be smaller than \( m_{\text{Max}} \) if not much accuracy is needed. The trouble with the terms \( \left\{ (-\pi^2)^m \frac{(2m + 1)c!}{(2m + c)!} C(c, m)(h^2)^m \right\}^{\infty}_{m=0} \) is that \( m \) has to be large for the terms to start becoming small. For a given \( h_{\text{Max}} \), \( m_{\text{Max}} \) will be found by numerical experimentation.

Suppose now that \( c = 1 \). Since \( C(1, m) < 1 \), in order to find when the terms become small it is enough to think about \( \sum_{m=m_{\text{max}}+1}^{\infty} \frac{\pi^{2m}h^{2m}}{(2m)!} \approx \sum_{m=m_{\text{max}}+1}^{\infty} \frac{(\pi e)^{2m}h^{2m}}{(2m)!} \). The rhs is a geometric series whose ratio becomes \( \leq 1/2 \) when \( m \geq 9h \). This result gives us an idea of where to start in order to find the \( m_{\text{Max}} \) necessary for a given \( h_{\text{Max}} \). When \( c > 1 \) we still use the previous estimate, because \( (\pi^2)^m \frac{(2m + 1)c!}{(2m + c)!} C(c, m)(h^2)^m \) decays faster.
than $\frac{\pi^2 m^2 h^{2m}}{(2m)!}$. Indeed, we have found in the examples which follow that a $m_{Max}$ of 5-10 times $h_{max}$ is necessary.

It remains now to decide how many harmonics to use. For each $c$ we have computed enough Fourier coefficients $FS(\rho_{ls,c})(h)$ so that the ratio of the smallest (which is $FS(\rho_{ls,c})(h_{max})$ or $FS(\rho_{ls,c})(h_{max} - 1)$) to the largest (which is $FS(\rho_{ls,c})(1)$) is at most $10^{-3}$. We expect, then, the relative error of the series to be at most $10^{-3}$ in the region around the origin where the amplitudes, except for the ones which correspond to large frequencies, mostly add up. In the tail, however, the argument in the preceding sentence no longer holds, and the relative error can be larger than $10^{-3}$.

To compute the Fourier series coefficients we have used a high-level language and a rational approximation for $\pi$, $\pi \approx \frac{314159265358979323846}{100000000000000000000}$. In this way all calculations are done with rational numbers keeping all digits. Only when the calculation is done we ask the computer to convert the result to a decimal number.

\section{Computation of $\rho_{rs,c}$}

In this section we use the expansion (46) to find and comment several $\rho_{rs,0}$. For completeness we give the radial pdf conditional to 0 collisions, even though it is not in the collision expansion (9). Since $C(0, m) = \frac{1}{2m+1}$,

$$
\rho_{rs,0}(x, t) = \frac{1}{2vt} \sum_{h=-\infty}^{+\infty} \sum_{m=0}^{\infty} (-1)^m (\pi h)^{2m} (2m)! e^{\frac{\pi h}{vt} x} = \frac{1}{2vt} \sum_{h=-\infty}^{+\infty} \cos(\pi h) e^{\frac{\pi h}{vt} x} = \sum_{h=-\infty}^{+\infty} \frac{1}{2vt} \cos(\pi h) e^{\frac{\pi h}{vt} x},
$$

which the reader will recognize as the Fourier series expansion of a comb function whose spikes lie a distance $2vt$ apart and which has been shifted a distance $vt$ from the origin. Its restriction to the $[-vt, +vt]$ interval is one half Dirac delta function at each of its end points.

The radial pdf conditional to 1 collision is

$$
\rho_{rs,1}(x, t) = \frac{1}{2vt} \sum_{h=-\infty}^{+\infty} \sum_{m=0}^{\infty} (-1)^m (\pi h)^{2m} (2m)! \left( \sum_{i=0}^{m} \frac{1}{(2i+1)(2(m-i)+1)} \right) \cos(\pi h) e^{\frac{\pi h}{vt} x},
$$

\[(60)\]

to be compared with (see eq. (35))

$$
\rho_{rs,1}(x, t) = \frac{1}{2} 4\pi x^2 \rho_{I,1}(x, t) = \frac{x}{2(vt)^2} \ln \frac{vt + x}{vt - x}
$$

\[(61)\]
in the range $x \in [0, vt]$.

We have set $m_{Max} = 500$ and found that the Fourier series coefficients start to diverge when $h = 115$. To be conservative we sum the expression (60) up to the 100-th harmonic. See the comparison between the exact $\rho_{rs,1}$ and the sum of expression (60) up to the 50-th harmonic in Fig. 2.
Figure 2: Comparison between the Fourier series $\rho_{rs,1}(x,t)$ summed up to the 50-th harmonic and the exact expression for $vt = 1$. When the Fourier series is summed up to the 100-th harmonic the graph becomes barely distinguishable from the exact expression.

The radial pdf conditional to 2 collisions is

$$
\rho_{rs,2}(x,t) = \frac{1}{2vt} \sum_{h=-\infty}^{+\infty} \left( \sum_{m=0}^{\infty} (-\pi)^m \frac{(2m+1)}{(2m+2)!} C(2,m)(h^2)^m e^{\frac{i\pi h}{vt} x} \right) - \pi \frac{h}{vt} x.
$$

(62)

This expression is to be compared with the $\rho_{rs,2}(x,t)$ obtained from $\rho_{proj,2}(x,t)$ (see the end of Appendix I). This function, according to formula (21), is $-x\frac{d}{dx}\rho_{proj,2}(x,t)$. We compare this expression summed up to the 25-th harmonic with the exact expression in Fig. 3.

Figure 3: Comparison between the Fourier series $\rho_{rs,2}(x,t)$ summed up to the 25-th harmonic ($mMax = 500$) and the exact $\rho_{rs,2}(x,t)$ expression for $vt = 1$. When the Fourier series is summed up to the 50-th harmonic the graph becomes barely distinguishable from the exact expression.

The examination of the general form of the Fourier series coefficients (58) of $\rho_{rs,c}$ suggests that for larger $c$ they fall off faster as $h$ increases, and thus, as implied by
the caption of Fig. 3, 50 terms or less should give a good fit for all \( c > 2 \). Indeed, this is confirmed when the Fourier series coefficients for \( \rho_{rs,1} \), \( \rho_{rs,2} \), \( \rho_{rs,3} \) and \( \rho_{rs,10} \) are compared. See Fig. 4 for the first three. As of \( \rho_{rs,10} \), its first 12 coefficients are 
\[
\{0.329598, -0.468924, -0.340769, -0.028119, 0.010381, -0.00274265, 0.000735024, -0.000204368, 0.000056844, -0.0000143126, 2.1067 \times 10^{-6}, 1.0685 \times 10^{-6}\},
\]
they decay a lot faster than the ones of \( \rho_{rs,1} \), \( \rho_{rs,2} \) and \( \rho_{rs,3} \).

![Figure 4](image)

Figure 4: To the left, the first 50 Fourier series coefficients for \( \rho_{rs,1} \) when \( vt = 1/2 \). To the right, the first 50 Fourier series coefficients for \( \rho_{rs,2} \) (the outer points) and \( \rho_{rs,3} \) (the inner points) when \( vt = 1/2 \). Notice the different scales in each graph.

When \( c > 2 \) we don’t have exact \( \rho_{rs,c} \) to compare with, but, as we have just seen, we can obtain good approximations for \( \rho_{rs,c} \) with fewer Fourier series coefficients. A test to see how good the approximations to \( \rho_{rs,3} \) (with 100 Fourier series coefficients) and \( \rho_{rs,10} \) (with 25 Fourier series coefficients) are is to see how they behave at \( x = 0, vt \). They should be 0 but (setting \( v = 1 \)) \( \rho_{rs,3}(0, 1) = 0.00041, \rho_{rs,3}(1, 1) = 0.013, \rho_{rs,10}(0, 1) = -1.75 \times 10^{-8} \) and \( \rho_{rs,10}(1, 1) = -1.4 \times 10^{-7} \).

![Figure 5](image)

Figure 5: Graphs of \( \rho_{rs,3}(x, 1/v) \) and \( \rho_{rs,10}(x, 1/v) \) (the taller curve) with 100 and 25 Fourier series coefficients, and \( mMax = 500 \) and \( mMax = 125 \), respectively.

11 Computation of \( \rho_{I,c} \)

So far everything has seemed to be going rather well with the pdf’s \( \rho_{rs,c} \). The match between the exact solutions and the truncated Fourier series was good and the required
number of harmonics diminished with \( c \). However, the pdf that we are after is usually not \( \rho_r \) but \( \rho \). If, with equation (48) in mind, we divide the \( \rho_{rs,c} \) functions by \( 2\pi r^2 \) we can see in Fig. 6 that in the Fourier series oscillations appear in the vicinity of 0 which will not go away by increasing the frequency cut off.

Figure 6: To the left, the exact \( \rho_{Is,1}(x, 1/v) \) (one half of expression (35)) and the Fourier series (46) for \( \rho_{rs,1}(x, 1/v) \) with 50 harmonics divided by \( 4\pi x^2 \). To the right, the exact \( \rho_{Is,2}(x, 1/v) \) (one half of expression (92)) and the Fourier series (46) for \( \rho_{rs,2}(x, 1/v) \) with 50 harmonics divided by \( 4\pi x^2 \).

Figure 7: The graph of \( \rho_{rs,3}(x, 1/v) \) with 25 Fourier series coefficients and \( mMax = 125 \) divided by \( 4\pi x^2 \) (the curve which goes from \((0.01, 0.68)\) to \((1, 0)\)) misbehaves between \( x = 0 \) and about \( x = 0.1vt \). The Fourier series for \( \rho_{Is,3}(x, 1/v) \) with 25 harmonics and coefficients \( F\mathcal{S}_0(\rho_{Is,3})(h) \) oscillates wildly. But \( F\mathcal{S}(\rho_{Is,3})(h) \to F\mathcal{S}_0(\rho_{Is,3})(h) - \lim_{h \to \infty} F\mathcal{S}_0(\rho_{Is,3})(h) \) eliminates the oscillations.

Note that the problem of oscillations is less serious for \( c = 2 \) than for \( c = 1 \). Indeed, for \( c = 3 \) Fig. 7 shows that the problem is mild enough to be solved by various numerical methods. But a better method is the one given in the second half of section 8. It follows from expansion (51) that
\[
\rho_{Is,3}(x, t) = \frac{1}{(vt)^3} \sum_{h=0}^{\infty} FS(\rho_{Is,3})(h) \cos \frac{\pi h}{vt} x. \tag{63}
\]

When \(\langle r^{-2} \rangle_{\rho_{3}} = 0\) the series (63) oscillates wildly (see Fig. 7) and, from expression (50), for \(h = 1, 2, 3, \ldots\),

\[
FS(\rho_{Is,3})(h) = FS_0(\rho_{Is,3})(h) \equiv \frac{1}{4\pi} \sum_{m=1}^{\infty} \frac{(-\pi^2)^m 3! C(3, m - 1)}{2m(2m - 2 + 3)!} \cdot h^{2m}.
\]

Figure 8: Plot of the coefficients \(FS_0(\rho_{Is})(h) = 2FS_0(\rho_{Is,3})(h)\) from \(h = 30\) to \(h = 100\).

The plot of these coefficients in Fig. 8 shows that

\[
\langle r^{-2} \rangle_{\rho_{3}} = -\lim_{h \to \infty} 8\pi FS_0(\rho_{Is,3})(h) \approx 0.432965 \times 4\pi. \tag{64}
\]

As explained at the end of section 8 we use this \(\langle r^{-2} \rangle_{\rho_{3}}\) to redefine

\[
FS(\rho_{Is,3})(h) = FS_0(\rho_{Is,3})(h) - \lim_{h \to \infty} FS_0(\rho_{Is,3})(h)
\]

and the corresponding \(\rho_{Is,3}(x, t)\) given by expression (63) is now free of oscillations, as shown in Fig. 7.

When a truncated Fourier series is used to approximate a bounded function whose support is an interval, the truncated Fourier series has to have a horizontal tangent at the ends of the said interval, as seen in Fig. 3 and the right figure of Fig. 6. Based on these figures we believe that the value of the truncated Fourier series at \(\pm vt\) is an upper bound of the error. With the value for \(\langle r^{-2} \rangle_{\rho_{Is,3}}\) given by the equality (64) and \(h_{\text{max}} = 25\), the value of the Fourier series (63) for \((x, t) = (1, 1)\) is about 0.001, and when \(h_{\text{max}} = 100\), the value for \((x, t) = (1, 1)\) is about 0.0001. We remind the reader that \(\rho_{Is,3}(x, 1)\) is plotted in Fig. 7.

We have obtained \(\rho_{Is,10}\) in the same way as \(\rho_{Is,3}\) and plotted them together in Fig. 9. The Fourier series for the \(\rho_{Is,c}\) seem to work at least as well as the Fourier series for the \(\rho_{rs,c}\). At the end of the last section we saw that, setting \(v = 1\) and using 25 harmonics, \(\rho_{rs,10}(0, 1) = -1.75 \times 10^{-8}\) and \(\rho_{rs,10}(1, 1) = -1.4 \times 10^{-7}\). For \(c = 10\) and \(h_{\text{max}} = 12\), the value of the the Fourier series (47) for \((x, t) = (1, 1)\) (setting \(v = 1\)) is about \(-10^{-6}\), and when \(h_{\text{max}} = 25\), the value for \((x, t) = (1, 1)\) is about \(-10^{-8}\). All the 4 preceding values should be 0, so the values obtained give us an idea of the absolute error.
Figure 9: Graphs of $\rho_{Is,3}$ and $\rho_{Is,10}$ (big, bell shaped). They are the graphs of Fig. 5 divided by $4\pi x^2$. While the curves in Fig. 5 had to cover the same area, this needs not to be so now.

12 Asymptotics

It has been shown [34, 18, 35, 28, 7] that as $t \to \infty$, $\rho$ tends to a Gaussian. Two parameters are needed to specify a Gaussian: its norm and its variance, $\langle r^0 \rangle$ and $\langle r^2 \rangle$, respectively. In three dimensions a Gaussian with given norm and variance is:

$$\langle r^0 \rangle \left( \frac{2}{3\pi \langle r^2 \rangle} \right)^{3/2} \exp \left( -\frac{3}{2} \frac{r^2}{\langle r^2 \rangle} \right). \tag{66}$$

In our case the norm of the limiting Gaussian is

$$\langle r^0 \rangle(t) = 1 - e^{-\lambda t}. \tag{67}$$

From formula (29) of reference [30] we know that the second moment of the limiting Gaussian is

$$\langle r^2 \rangle(t) = 2 \left( \frac{\nu}{\lambda} \right)^2 (e^{-\lambda t} - 1 + \lambda t) - e^{-\lambda t}(vt)^2, \tag{68}$$

where we have subtracted the contribution due to the expanding spherical shell from the said formula. Therefore

$$\lim_{t \to \infty} \rho(r, t) =$$

$$\frac{1 - e^{-\lambda t}}{\left( \frac{2}{3\pi \frac{2(\nu)^2(e^{-\lambda t} - 1 + \lambda t) - e^{-\lambda t}(vt)^2)}{1-e^{-\lambda t}} \right)^{3/2} \exp \left( -\frac{3}{2} \frac{2(\nu)^2(e^{-\lambda t} - 1 + \lambda t) - e^{-\lambda t}(vt)^2}{1-e^{-\lambda t}} \right) \cdot \frac{r^2}{\langle r^2 \rangle}}. \tag{69}$$

At $(r, \lambda t) = (0, 100)$ the ratio of the Gaussian approximation to the Fourier series with 132 harmonics is 1.0088. Since we have taken 132 harmonics, it follows from the examples of $c_{max}$ given in section 9 and from the discussion in the same section, that the relative error of the series (47) at $(r, \lambda t) = (0, 100)$ is at most $10^{-3}$ in the region around the $r = 0$, which is almost an order of magnitude less than 0.0088. We may then take the expansion (47) with 132 collisions as exact relative to the Gaussian approximation in the said region.
and conclude that if, for $\lambda t > 100$, we use the Gaussian approximation, then the relative error in the central region will always be less than 1%. This is a drawback from the $10^{-3}$ relative error for $\lambda t > 100$, but pushing the transition to the Gaussian approximation to a larger time where the Gaussian approximation became significantly better would require the computation of a couple of hundred more collections of Fourier coefficients, up to 300 collisions or more. Such a numerical effort is beyond the scope of this paper.

13 Graphs

In [29] there are four Mathematica files which produce a function called $R_0[r, t]$, which is $(1 - e^{-\lambda t})\rho_I(r, t)$. They have to be compiled in the order 1,2,3,4. The function $R_0[r, t]$ interpolates linearly between $(1 - e^{-\lambda t})\rho_I(r, t)$ computed with the series (51) for $\lambda t < 100$ and $(1 - e^{-\lambda t})\rho_I(r, t)$ for $\lambda t > 105$ approximated by the Gaussian presented in this section. The reader who does not have the program Mathematica will not be missing any understanding, just the chance of producing graphs by him-/herself. In Fig. 13 we show graphs of $(1 - e^{-\lambda t})\rho_I(r, t)$ for $\lambda t = 0.1, 1$ and 10.

Graphical representations of $(1 - e^{-\lambda t})\rho_I(r, t) = 4\pi r^2(1 - e^{-\lambda t})\rho_I(r, t)$, which is the probability enclosed in a spherical shell of radius $r$ divided by the width of the shell, are shown in Fig. 13.
Figure 12: Graphs of \((1 - e^{-\lambda t}) \rho_r(r, t)\) for \(\lambda = 1, v = 1\) and (from left to right) \(t = 0.1, 1, 10\). Notice the different scales in each graph.

The reader will probably wonder whether the divergences which appear at \(r = vt\) for \(t = 0.1, 1\) are still there for \(t = 10\). They are, as was argued at the end of section 6. There, the possibility that divergences in \(\rho_c\), \(c \geq 2\) cancel the divergence due to \(\rho_1\) was raised. But we saw in Proposition 3 of section 8 that the \(\rho_c\)'s are bounded for \(c \geq 2\). Therefore \(\rho_I\) and \(\rho_r\) have a logarithmic divergence at \(r = vt\) which is the divergence due to the pdf \(\rho_1\) (35). In Fig. 13 the said divergences are made visible.

Figure 13: Divergences at \(r = vt\) of \((1 - e^{-\lambda t}) \rho_I(r, t)\) and \((1 - e^{-\lambda t}) \rho_r(r, t)\) for \(\lambda = 1, v = 1\) and \(t = 10\).

14 The bullet initial condition

The topic of initial conditions in the random flight has been discussed seldom [26, 27, 28, 25]. Here we follow the approach of section 5 of reference [7]. As usual in random walks in more than one dimension, we have restricted the discussion to the isotropic initial condition, that is, the probability density of the yet unscattered particles is an expanding spherical shell of norm \(e^{-\lambda t}\). But this is not enough to give the solution of a general case. This is accomplished if the solution to the bullet initial condition [7] is known. In the bullet initial condition the probability density of the yet unscattered particles is \(e^{-\lambda t} \delta(x - vt, y, z)\), where we have taken as the positive direction of the \(x\)-axis the direction along which the unscattered particles travel. The solution \(\rho_b\) to the bullet initial condition can be put in terms of the solution to the isotropic initial condition as follows [27]:

\[
\rho_b(\vec{r}, t) = e^{-\lambda t} \delta(\vec{0}) \delta(y) \delta(z) + \lambda \int_0^t dt' e^{-\lambda t'} \rho_s(\vec{r} - vt', t - t').
\] (70)
The above integral has been done analytically in the two-dimensional case \[27, 26\], but this seems impossible or quite difficult in 3 dimensions, where it would be best to do the integral numerically.
Appendix I: Exact solution for two collisions

15.1 Convolution of three rectangle functions

In order to obtain the convolution of three rectangle functions we are going to decompose the trapezoid which is the convolution of two rectangle functions into a central rectangle and one triangle on each side. Let \( a, b \) and \( c \) be positive real numbers such that \( a+b+c = 2vt \) and \( 0 < a < b < c < 2vt \).

Since \( 2vt - b - c \leq b \leq c \),

\[
\text{rect}(2vt - b - c,_{\bot}) \otimes (\text{rect}(b,_{\bot}) \otimes \text{rect}(c,_{\bot}))(x) =
\]

\[
\text{rect}(2vt - b - c,_{\bot}) \otimes \left( \text{restr}\left(\frac{-b - c}{2}, \frac{-c + b}{2}\right) \left( \frac{1}{2} \left( \frac{1}{b} + \frac{1}{c} \right) + \frac{v}{bc} \right) + \right.
\]

\[
\left. \text{restr}\left(\frac{-c + b}{2}, \frac{-c - b}{2}\right) \frac{1}{c} + \text{restr}\left(\frac{c - b}{2}, \frac{-b + c}{2}\right) \left( \frac{1}{2} \left( \frac{1}{b} + \frac{1}{c} \right) - \frac{v}{bc} \right) \right) (x). \quad (71)
\]

All of the terms which appear when the preceding expression is developed are proportional to the already seen convolution of two rectangle functions except for the terms \( \text{rect}(2vt - b - c,_{\bot}) \otimes \left( \text{restr}\left(\frac{-b - c}{2}, \frac{-c + b}{2}\right) \right) (x) \) and \( -\text{rect}(2vt - b - c,_{\bot}) \otimes \left( \text{restr}\left(\frac{c - b}{2}, \frac{-b + c}{2}\right) \right) (x) \).

According to formula (26), the first of the new terms is

\[
\text{rect}(2vt - b - c,_{\bot}) \otimes \left( \text{restr}\left(\frac{-b - c}{2}, \frac{-c + b}{2}\right) \right) (x) =
\]

\[
\frac{1}{2vt - b - c} \int_{x - vt + \frac{b - c}{2}}^{x + vt - \frac{b + c}{2}} dx' \text{ restra}\left(\frac{-b - c}{2}, \frac{-c + b}{2}\right) \frac{x'}{bc} =
\]

\[
\frac{\text{restr}(-vt, x, vt - c)}{2bc(2vt - b - c)} \left( \min \left( x + vt - \frac{b + c}{2}, \frac{-c + b}{2} \right)^2 - \max \left( x - vt + \frac{b + c}{2}, \frac{-b - c}{2} \right)^2 \right) =
\]

\[
\frac{1}{2bc(2vt - b - c)} \left[ \text{restr}(-vt, x, vt - b - c) \left( \left( x + vt - \frac{b + c}{2} \right)^2 - \left( \frac{-b - c}{2} \right)^2 \right) \right.
\]

\[
\left. + \right. \text{restr}(vt - b - c, x, -vt + b) \left( \left( x + vt - \frac{b + c}{2} \right)^2 - \left( x - vt + \frac{b + c}{2} \right)^2 \right) +
\]

\[
\text{restr}(-vt + b, x, vt - c) \left( \left( \frac{-c + b}{2} \right)^2 - \left( x - vt + \frac{b + c}{2} \right)^2 \right) \right] =
\]

\[
- \frac{1}{2bc(2vt - b - c)} \text{restr}(-vt, x, vt - b - c) \left( b + c - vt - x \right) \left( vt + x \right) +
\]
\[\text{restr}(vt - b - c, x, -vt + b) (2(b + c - 2vt)x) + \]
\[\text{restr}(-vt + b, x, vt - c) ((b - vt + x)(c - vt + x))\].

The \(x\) which appears in the limits of the integral in the second line cannot take any values, because \(x\) has to satisfy \(x - vt + \frac{b+c}{2} \leq \frac{b-c}{2}\) and \(x + vt - \frac{b+c}{2} \geq -\frac{b+c}{2}\). The \(\text{restr}(-vt, x, vt - c)\) prefactor in the third line is equivalent to enforcing the satisfaction of both inequalities.

Since \(\text{rect}(w, x) = \frac{1}{w} \text{restr}\left(-\frac{w}{2}, x, +\frac{w}{2}\right)\), similarly to formula (26) we obtain
\[(\text{restr}(a, \_, b) \otimes f)(x) = F(x - a) - F(x - b),\]
where \(F\) is some primitive of \(f\), and similarly to (29) we obtain
\[(\text{restr}(x_1, \_, x_2) \otimes \text{restr}(x_3, \_, x_4))(x) = \]
\[
\left\{ \begin{array}{ll}
0, & x \in (-\infty, x_3 + x_1) \\
+x - x_3 - x_1, & x \in \left[x_3 + x_1, x_3 + x_2\right] \\
x_2 - x_1, & x \in \left(x_3 + x_2, x_4 + x_1\right) \\
-x + x_2 + x_4, & x \in \left[x_4 + x_1, x_4 + x_2\right] \\
0, & x \in \left(x_4 + x_2, +\infty\right)
\end{array} \right.,
\]
provided \(x_2 - x_1 \leq x_4 - x_3 \Leftrightarrow x_3 + x_2 \leq x_4 + x_1\).

The last three formulae (72,73) serve to expand the convolution (71) of three rectangle functions. Since \(c\) is the side of the longest rectangle, \(\frac{2vt}{3} \leq c\) is always satisfied. When \(\frac{2vt}{3} \leq c \leq vt:\)
\[f_1(vt, c, b, x) \equiv \text{rect}(2vt - b - c, \_, \_) \otimes (\text{rect}(b, \_, \_) \otimes \text{rect}(c, \_, \_))(x) = \]
\[
\frac{1}{2vt - b - c} \left[ 0 \times \text{restr}(-\infty, x, -vt) + \frac{(vt + x)^2}{2bc} \text{restr}(-vt, x, -b - c + vt) + \right.
\]
\[
\left. \frac{(b + c + 2x)(2vt - b - c)}{2bc} \text{restr}(-b - c + vt, x, b - vt) + \right.
\]
\[
\left. \frac{-2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc} \text{restr}(b - vt, x, c - vt) + \right.
\]
\[
\left. \frac{-b^2 + b(c - 2vt) + (c - vt)^2 + x^2}{bc} \text{restr}(c - vt, x, -c + vt) + \right.\]
\[-\frac{2b^2 + 2b(c - 2vt) + (c - vt - x)^2}{2bc}\text{restr}(-c + vt, x, -b + vt) +
\frac{(b + c - 2x)(2vt - b - c)}{2bc}\text{restr}(-b + vt, x, b + c - vt) +
\frac{(vt - x)^2}{2bc}\text{restr}(b + c - vt, x, vt) + 0 \times \text{restr}(vt, x, \infty)\]. 
\tag{75}

Likewise, when \(vt \leq c \leq 2vt:\)
\[f_2(vt, c, b, x) \equiv \text{rect}(2vt - b - c, \ldots) \otimes (\text{rect}(b, \ldots) \otimes \text{rect}(c, \ldots))(x) =
\frac{1}{2vt - b - c}\left[0 \times \text{restr}(-\infty, x, -vt) + \frac{(vt + x)^2}{2bc}\text{restr}(-vt, x, -b + c + vt) +
\frac{(b + c + 2x)(2vt - b - c)}{2bc}\text{restr}(-b - c + vt, x, b - vt) +
\frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc}\text{restr}(b - vt, x, -c + vt) +
\frac{2vt - b - c}{c}\text{restr}(-c + vt, x, c - vt) +
\frac{2b^2 + 2b(c - 2vt) + (c - vt - x)^2}{2bc}\text{restr}(c - vt, x, -b + vt) +
\frac{(b + c - 2x)(2vt - b - c)}{2bc}\text{restr}(-b + vt, x, b + c - vt) +
\frac{(vt - x)^2}{2bc}\text{restr}(b + c - vt, x, vt) + 0 \times \text{restr}(vt, x, \infty)\right]. \tag{76}

For later use in figures we assign notation to the different analytic forms which appear in the convolution of three rectangle functions:
\[
\begin{align*}
    h_1(vt, c, b, x) & \equiv \frac{(vt + x)^2}{2bc(2vt - b - c)} \\
    h_2(vt, c, b, x) & \equiv \frac{(b + c + 2x)}{2bc} \\
    h_3(vt, c, b, x) & \equiv -\frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc(2vt - b - c)} \\
    h_4(vt, c, b, x) & \equiv -\frac{b^2 + b(c - 2vt) + (c - vt)^2 + x^2}{bc(2vt - b - c)} \\
    h_4'(vt, c, b, x) & \equiv \frac{1}{c}
\end{align*}
\tag{77}
15.2 Space-time convolution of three rectangle functions

We define

\[ f(vt, c, b, x) \equiv \begin{cases} f_1(vt, c, b, x), & \frac{2vt}{3} \leq c \leq vt \\ f_2(vt, c, b, x), & vt \leq c \leq 2vt \end{cases} \]  

(78)

Then, since

\[ \int_{\frac{vt}{3}}^{vt} dc \int_{\frac{2vt-c}{2}}^{c} db 1 + \int_{vt}^{2vt} dc \int_{\frac{2vt-c}{2}}^{\min(c,2vt-c)} db 1 = \frac{v^2t^2}{3}, \]

the wanted space-time convolution is the average

\[ \frac{3\sqrt{2}t^2}{v^2t^2} \left( \int_{\frac{vt}{3}}^{vt} dc \int_{\frac{2vt-c}{2}}^{c} db \ f_1(vt, c, b, x) + \int_{vt}^{2vt} dc \int_{\frac{2vt-c}{2}}^{2vt-c} db \ f_2(vt, c, b, x) \right). \]  

(79)

The above integrals have a different structure depending on whether \(|x| \in [0, vt/3]\), \(|x| \in [vt/3, vt/2]\) or \(|x| \in [vt/2, vt]\). These structures will be developed now. For simplicity, the prefactor \(\frac{3\sqrt{2}t^2}{v^2t^2}\) will be omitted.

In the next three figures the isosceles right triangle is depicted for which \(b + c < 2vt\). This right triangle is divided into 6 triangles. The region of integration is the upper right triangle. This triangle is the set of points for which \(b < c\) and \(\frac{2vt-c}{2} < b\), that is, the set of points for which \(a < b < c\). For each of the three cases that follow, there are 4 different integration regions within the said triangle, separated by thin, continuous lines.
Figure 14: The above tiling showing the different integrands and regions of integration in the \((b, c)\) plane is valid for \(-vt \leq x \leq -vt/2\).

\[
\text{restr}( -vt, x, -vt/2) \\
\left( \int_{2vt - x}^{vt - x} dc \int_{2vt - x}^{c} db \frac{(vt + x)^2}{2bc(2vt - b - c)} + \int_{2vt - x}^{-2x} dc \int_{2vt - x}^{vt - c - x} db \frac{(vt + x)^2}{2bc(2vt - b - c)} \right) = \\
\frac{(vt + x)^2}{8vt} \text{restr}( -vt, x, -vt/2) \\
\left( \pi^2 - 3(ln 2)^2 - \ln 3 \ln \frac{243}{63} + (\ln vt)^2 - \text{Li}_2 \left( \frac{1}{9} \right) - 6 \text{Li}_2 \left( \frac{2}{3} \right) + (\ln(vt-x))^2 + \right)
\]
\[
2 \left[ \left( \text{arg tanh} \frac{vt}{2vt + x} - \text{arg tanh} \frac{x}{vt} \right) \ln 4 + \ln \frac{vt + x}{vt - x} \ln(3vt + x) - \ln(vt + x) \ln vt \right. \\
\left. - \text{Li}_2 \left( \frac{vt - x}{4vt} \right) + \text{Li}_2 \left( \frac{vt - x}{2vt} \right) + \text{Li}_2 \left( \frac{-vt + x}{2vt} \right) + \text{Li}_2 \left( \frac{vt + x}{vt - x} \right) - \Re \left( \text{Li}_2 \left( \frac{1}{2} + \frac{vt}{vt + x} \right) \right) \right] 
\]

(80)

\[
\text{restr}(vt, x, -vt/2) \left( \int_{vt-x}^{vt} dc \int_{vt-c-x}^{c} db \frac{b + c + 2x}{2bc} + \int_{vt}^{-2x} dc \int_{vt-c-x}^{2vt-c} db \frac{b + c + 2x}{2bc} + \right. \\
\left. \int_{-2x}^{vt-x} dc \int_{vt+x}^{2vt-c} db \frac{b + c + 2x}{2bc} \right) = \\
\text{restr}(vt, x, -vt/2) \\
\frac{vt}{2} 
\]

\[
-vt-x-(\ln 2)^2 - x \ln vt - x \ln 4 \ln vt - x (\ln vt)^2 + vt \ln(vt-x) - x \ln(vt-x) + x \ln 4 \ln(vt-x) +
\]

\[
x(\ln(vt-x))^2 + x \ln(-vt/x) + x \ln(-8x) + x \ln(-2x) + x \ln 4 \ln(-2x) + 2x \ln vt \ln(-2x) -
\]

\[
2x \ln(vt-x) \ln(-2x) + x \ln(-x) - vt \ln(vt+x) - x \ln(4(vt+x)) + 2x \ln(-2x) \ln \frac{vt + x}{2vt}
\]

\[
- 2x \ln(vt-x) \ln \frac{vt + x}{vt} - 2x \text{Li}_2 \left( \frac{vt - x}{2vt} \right) + 2x \text{Li}_2 \left( \frac{-2x}{vt - x} \right) \right]. \quad (81)
\]

\[
\text{restr}(vt, x, -vt/2) \int_{-2x}^{vt-x} dc \int_{vt-x}^{vt+x} db \frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc(b - c - 2vt)} =
\]

\[
\text{restr}(vt, x, -vt/2) \left( vt(vt+x) - \frac{\pi^2}{48} (vt+x)^2 + vt x \ln \frac{vt - x}{-2x} - \frac{1}{4} (vt-x)^2 \text{Li}_2 \frac{vt + x}{vt - x} \right). \quad (82)
\]

\[
\text{restr}(vt, x, -vt/2) \int_{vt-x}^{2vt} dc \int_{vt-x}^{2vt-c} db \frac{1}{c} =
\]

\[
\text{restr}(vt, x, -vt/2) \left( - \frac{vt + x}{2} + vt \ln \frac{2vt}{vt - x} \right). \quad (83)
\]
Figure 15: The above tiling showing the different integrands and regions of integration in the \((b, c)\) plane is valid for \(-\frac{vt}{2} \leq x \leq -\frac{vt}{3}\).

\[
\text{restr}\left(-\frac{vt}{2}, x, -\frac{vt}{3}\right)
\]

\[
\left( \int_{\frac{vt}{2}}^{c} dc \int_{\frac{vt-c}{2}}^{c} db \frac{(vt+x)^2}{2bc(2vt-b-c)} + \int_{\frac{vt-x}{2}}^{-2x} dc \int_{\frac{vt-x}{2}}^{vt-c-x} db \frac{(vt+x)^2}{2bc(2vt-b-c)} \right) =
\]

\[
\frac{(vt+x)^2}{8vt} \text{restr}\left(-\frac{vt}{2}, x, -\frac{vt}{3}\right)
\]
\[
\left( \pi^2 - 3(\ln 2)^2 - \ln 3 \ln \frac{243}{63} + (\ln vt)^2 - \text{Li}_2 \left( \frac{1}{9} \right) - 6 \text{Li}_2 \left( \frac{2}{3} \right) + (\ln(vt - x))^2 + \right.
\]
\[
2 \left[ \left( \text{arg tanh} \frac{vt}{2vt + x} - \text{arg tanh} \frac{x}{vt} \right) \ln 4 + \ln \frac{vt + x}{vt - x} \ln(3vt + x) - \ln(vt + x) \ln vt 
\right.
\]
\[
- \text{Li}_2 \left( \frac{vt - x}{4vt} \right) + \text{Li}_2 \left( \frac{vt - x}{2vt} \right) + \text{Li}_2 \left( \frac{-vt + x}{2vt} \right) + \text{Li}_2 \left( \frac{vt + x}{vt - x} \right) - \Re \left( \text{Li}_2 \left( \frac{1}{2} + \frac{vt}{vt + x} \right) \right) \right]
\right). \tag{84}
\]

\[
\text{restr}(-vt/2, x, -vt/3)
\]
\[
\left( \int_{\frac{vt}{2}}^{-2x} dc \int_{c-x}^{c} db \frac{b + c + 2x}{2bc} + \int_{-2x}^{vt} dc \int_{c}^{c-x} db \frac{b + c + 2x}{2bc} + \int_{vt}^{vt-x} dc \int_{vt+x}^{2vt-c} db \frac{b + c + 2x}{2bc} \right) = 
\]
\[
\text{restr}(-vt/2, x, -vt/3)
\]
\[
\left( \frac{1}{2} \left( -vt - x - x(\ln 2)^2 + (vt - x)(\ln(vt^2 - x^2) + x \ln(vt - x)^2 - x(\ln vt)^2) + \right. \right.
\]
\[
x \ln(-2x) \left( 1 + \ln \frac{vt + x}{vt - x} \right) - x \ln 2 \ln vt - vt \ln(vt + x) - x \ln(vt - x) \ln \frac{vt + x}{2vt} + 
\]
\[
x \text{Li}_2 \left( \frac{-2x}{vt - x} \right) - x \text{Li}_2 \left( \frac{vt - x}{2vt} \right) \right). \tag{85}
\]

\[
\text{restr}(-vt/2, x, -vt/3) \int_{-2x}^{vt-x} dc \int_{\frac{2vt-c}{4}}^{vt+c} db \frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc(b + c - 2vt)} = 
\]
\[
\text{restr}(-vt/2, x, -vt/3) \int_{vt-x}^{vt-x} dc \int_{\frac{2vt-c}{4}}^{2vt-c} db \frac{1}{c} = 
\]
\[
\text{restr}(-vt/2, x, -vt/3) \left( -\frac{vt + x}{2} + vt \ln \frac{2vt}{vt - x} \right). \tag{87}
\]

33
Figure 16: The above tiling showing the different integrands and regions of integration in the \((b, c)\) plane is valid for \(-vt/3 \leq x \leq 0\).

If \(-vt/3 < x < 0\), then \(vt - b - c < x\), therefore the first case doesn’t happen. The next question is whether \(x\) is smaller or greater than \(-vt + b\). When it is smaller the integrand is \(\frac{b + c + 2x}{2bc}\). When it is larger we have to distinguish the \(c \in [2vt/3, vt]\) and the \(c \in [vt, 2vt]\) cases. In the first case, when \(x \in [-vt + b, -vt + c]\) the integrand is \(\frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc(b + c - 2vt)}\), and when \(x \in [-vt + c, 0]\) the integrand is \(\frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2 + x}{2bc(b + c - 2vt)}\). In the second case, when \(x \in [-vt + b, vt - c]\) the integrand is \(\frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc(b + c - 2vt)}\), and when \(x \in [vt - c, 0]\) the integrand is \(1/c\).

\[ \text{restr}(-vt/3, x, 0) \]
\[
\left( \int_{vt+x}^{vt} dc \int_{vt+x}^{c} db \frac{b+c+2x}{2bc} + \int_{vt}^{vt-x} dc \int_{vt+x}^{2vt-c} db \frac{b+c+2x}{2bc} \right) = \\
\text{restr}(-vt/3, x, 0)
\]

\[
-x(\ln vt)^2 + \frac{1}{2} x \ln \frac{vt}{vt+x} \left( -1 + \ln \frac{vt}{vt+x} \right) + \frac{x}{2} + x \ln \left( \frac{vt^2 - x^2}{2} \right) \ln vt + \\
\frac{1}{12} \left( 12 + \pi^2 - 6(\ln 2)^2 \right) x - 6vt \ln (vt+x) + 6(\ln 4)x - 2x \ln (vt+x) \ln (vt-x) - x \text{Li}_2 \frac{vt-x}{2vt} \right).
\]

(88)

\[
\text{restr}(-vt/3, x, 0) \int_{vt+x}^{vt-x} dc \int_{vt+x}^{v+vt+x} db \frac{2b^2 + 2b(c - 2vt) + (c - vt + x)^2}{2bc(b + c - 2vt)} = \\
\frac{\text{restr}(-vt/3, x, 0)}{12vt} \left( -2vt(\pi^2 vt + 12x) + 3(5vt^2 + 2vtx + x^2)(\ln (vt-x))^2 - 8vtx \ln (vt+x) + \\
(vt-x)^2(\ln (vt+x))^2 - 2 \ln (vt-x)(-2vt + (3vt^2 + x^2) \ln (vt+x)) + 4vt \ln (-2x)(x + (vt + x) \ln \frac{vt + x}{vt-x}) + \\
12vt(vt + x) \text{Li}_2 \left( -\frac{2x}{vt-x} \right) + 3(5vt^2 + 2vtx + x^2) \text{Li}_2 \frac{vt + x}{vt-x} - 3(vt + x)^2 \Re \left( \text{Li}_2 \left( -1 + \frac{2vt}{vt+x} \right) \right) \right).
\]

(89)

\[
\text{restr}(-vt/3, x, 0) \int_{2vt/3}^{vt+x} dc \int_{2vt/3-c}^{c} db \frac{b^2 + b(c - 2vt) + (c - vt)^2 + x^2}{bc(b + c - 2vt)} = \\
-\frac{\text{restr}(-vt/3, x, 0)}{12vt} \left( -6vt^2 + (vt^2 + x^2)(\pi^2 - 3(\ln 3)^2) - 18vtx + 6(vt^2 + x^2) \ln \frac{4}{3} \ln 3 \\
-6(vt^2 + x^2) \ln 2 \ln 2vt + 3(vt^2 + x^2)(\ln 2vt)^2 + 6 \ln (-2x)(2vtx + (vt^2 + x^2) \ln \frac{vt-x}{vt(vt+x)}) \\
+ 3(vt^2 + x^2)(\ln (vt+x))^2 + 2 \ln (vt+x) \left( -2vt(x + (vt^2 + x^2) \ln \frac{2x}{vt-x} \right)
\]
\(-2(vt^2+x^2)\left(\text{Li}_2\left(-\frac{1}{3}\right) + 2\text{Li}_2\left(\frac{2}{3}\right)\right) + 6(vt^2+x^2)\left(\text{Li}_2\left(\frac{x}{vt}\right) - \text{Li}_2\left(\frac{vt+x}{2vt}\right) + \text{Li}_2\left(\frac{vt+x}{vt}\right)\right)\) \tag{90}

\[
\text{restr}(-vt/3, x, 0) \int_{vt}^{2vt} dc \int_{2vt-c}^{2vt} db \frac{1}{c} = \text{restr}(-vt/3, x, 0)\left( -\frac{vt+x}{2} - vt \ln \frac{vt-x}{2vt} \right). \tag{91}
\]

The addition of the last 12 results is \(\rho_{\text{proj},2}(x, t)\). From this pdf

\[
\rho_{I,2}(x, t) = -\frac{1}{2\pi x} \frac{d}{dx} \rho_{\text{proj},2}(x, t), \tag{92}
\]

according to (21). The pdf \(\rho_{I,2}\) is denoted as “ro[2]” in the file 4roc.nb in site.09 and it is graphed in Fig. 6.

16 Appendix II: the coefficients \(C(c, m)\)

The coefficients \(C(c, m)\) are defined by

\[
C(c, m) = \sum_{i_1, \ldots, i_{c+1} \in \mathbb{N}} \frac{1}{(2i_1 + 1) \cdots (2i_{c+1} + 1)}. \tag{93}
\]

They can be computed using nested summations, that is

\[
C(c, m) = \sum_{i_1=0}^{m} \frac{1}{2i_1+1} \sum_{i_2=0}^{m-i_1} \frac{1}{2i_2+1} \cdots \sum_{i_c=0}^{m-i_{c-1}} \frac{1}{2i_c+1} \frac{1}{2(m-i_1-\cdots-i_c)+1}. \tag{94}
\]

With symbolic software it might be more convenient to compute \(C(c, m)\) as follows. We remark that \(C(c, m)\) is the coefficient of \(x^m\) in \(\left(1 + \frac{x}{3} + \frac{x^2}{5} + \cdots\right)^{c+1}\). This series is reminiscent of the expansion of \(\ln(1 + x)\). Indeed,

\[
1 + \frac{x}{3} + \frac{x^2}{5} + \cdots = f(x) \equiv \frac{1}{2\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} \quad \text{(for } 0 < x < 1). \]

Thus,

\[
C(c, m) = \frac{1}{m!} \frac{d^m}{dx^m} \bigg|_{x=0} (f(x))^{c+1}. \tag{95}
\]

However, the computation is faster if one defines the \(m\)-th degree polynomial

\[
f(m; x) \equiv \sum_{i=0}^{m} \frac{x^i}{2i+1} \tag{96}
\]
and then
\[ C(c, m) = \frac{1}{m!} \frac{d^m}{dx^m} \bigg|_{x=0} (f(m; x))^{c+1}. \] (97)

It can be read off from the definition that
\[ C(0, m) = \frac{1}{2m + 1}. \] (98)

Bounds for \( C(c, m) \) can be obtained as follows. The number of terms in the sum
\[ C(c, m) \equiv \sum_{i_1, \ldots, i_{c+1} \in \mathbb{N}} \frac{1}{(2i_1+1) \cdots (2i_{c+1}+1)} \] is the number of ways in which \( c \) separators may be placed between \( m \) objects, which is \( \binom{m+c}{m} \) (see, e. g., [36], chapter 2, section 5). The largest term is \( \frac{1}{2m+1} \), and \( \left( \frac{1}{2(m/(c+1)+1)} \right)^{c+1} \) is a lower bound for all the terms. Therefore,
\[ \left( \frac{m+c}{m} \right) \left( \frac{1}{2(m/(c+1)+1)} \right)^{c+1} \leq C(c, m) \leq \left( \frac{m+c}{m} \right) \frac{1}{2m+1}. \] (99)

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