Massive Gauge Field Theory Without Higgs Mechanism
I. Ward-Takahashi Identities and Proof of Unitarity

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In our previously published papers, it was argued that a massive non-Abelian gauge field theory in which all gauge fields have the same mass can well be set up on the gauge-invariance principle. The quantization of the fields was performed by different methods. In this paper, it is proved that the quantum theory is invariant with respect to a kind of BRST-transformations. From the BRST-invariance of the theory, the Ward-Takahashi identities satisfied by the generating functionals of full Green’s functions, connected Green’s functions and proper vertex functions are successively derived. As an application of the above Ward-Takahashi identity, the Ward-Takahashi identity obeyed by the massive gauge boson propagator is derived and the renormalization of the propagator is discussed. Furthermore, based on the Ward-Takahashi identity, it is exactly proved that the S-matrix elements given by the quantum theory are gauge-independent and hence unitary.

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1. INTRODUCTION

In our previous papers\cite{1-4}, it has been shown that some massive gauge field theory in which the masses of all gauge fields are the same may really be set up on the gauge-invariance principle without the help of the Higgs mechanism. The essential points of achieving this conclusion are the following. (1) The massive gauge field must be viewed as a constrained system in the whole space of vector potential. Therefore, the Lorentz condition, as a necessary constraint, must be introduced from the onset and imposed on the massive Yang-Mills Lagrangian so as to restrict the unphysical degrees of freedom involved in the Lagrangian; (2) The gauge-invariance of gauge field dynamics should be more generally required to the action of the field other than the Lagrangian because the action is of more fundamental dynamical meaning. In particular, the gauge-invariance for the constrained system should be required to the action written in the physical subspace defined by the Lorentz condition in which the fields exist and move only; (3) In the physical subspace, only the infinitesimal gauge transformations are possible to exist and necessary to be considered in inspection of whether the theory is gauge-invariant or not; (4) To construct a correct gauge field theory, the residual gauge degrees of freedom existing in the physical subspace must be eliminated by the constraint condition on the gauge group. This constraint condition may be determined by requiring the action to be gauge-invariant. Thus, the theory is set up from beginning to end on the gauge-invariance principle. These points are important to build up a correct quantum massive non-Abelian gauge field theory. Such a theory, as will be proved, is renormalizable and unitary.

In Refs. \cite{1-3}, the quantum theory of the massive non-Abelian gauge fields without Higgs mechanism was established by different methods of quantization. In this paper, it will be shown that the quantum theory has an important property that the effective action appearing in the generating functional of Green’s functions is invariant with respect to a kind of BRST-transformations\cite{5}. From the BRST-symmetry, we will derive various Ward-Takahashi (W-T) identities\cite{6-12} satisfied by the generating functionals of Green’s functions and proper vertices. These W-T identities are of special importance in proofs of unitarity and renormalizability of the theory. In this paper, we confine ourselves to prove that the S-matrix elements evaluated from the theory is independent of the gauge parameter, that is to say, the gauge-dependent unphysical poles appearing in the gauge boson propagator and the ghost particle propagator do not contribute to the S-matrix elements. Therefore, the unitarity of the theory is ensured.
The arrangement of this paper is as follows. In section 2, we will derive the BRST-transformations under which the effective action of the massive non-Abelian gauge field theory is invariant. In doing this, we extend our discussion by including fermions. In section 3, we will derive the W-T identities satisfied by various generating functionals. In section 4, to illustrate applications of the above W-T identity, the W-T identity obeyed by the massive gluon propagator will be derived and the renormalization of the propagator will be discussed. In section 5, by virtue of the W-T identity, we will prove that the S-matrix elements calculated from the massive gauge field theory are gauge-independent and hence unitary. The last section is used to make some remarks on the nilpotency problem of the BRST-transformations and the BRST-invariance of the external source terms introduced to the generating functional. In Appendix, the W-T identities used to prove the unitarity will be given by an alternative derivation.

2. BRST-TRANSFORMATION

In the previous paper, we mainly discussed the gauge fields themselves without concerning fermion fields. For the gauge fields, in order to guarantee the mass term in the action to be gauge-invariant, the mass of all gauge fields are taken to be the same. If fermions are included, as pointed out in Refs. [1-3], the QCD with massive gluons fulfils this requirement because all the gluons can be considered to have the same mass \( m \). The SU(3)-symmetric action of the QCD with massive gluons is given by the following Lagrangian [1-4]

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu - igT^a A^a_\mu)) - M\psi - \frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \frac{1}{2}m^2 A^{a\mu}A^a_\mu
\]  

(2.1)

where \( \psi(x) \) denotes the quark field function, \( \bar{\psi}(x) \) is its Dirac-conjugate, \( T^a = \lambda^a/2 \) are the color matrices and \( M \) is the quark mass. The above Lagrangian is constrained by the Lorentz condition

\[
\partial^\mu A^a_\mu = 0
\]  

(2.2)

Under this condition, as was proved in Refs. [1-3], the action given by the Lagrangian in Eq. (2.1) is invariant with respect to the following gauge transformations:

\[
\delta A^a_\mu = \xi D^{ab}_\mu C^b
\]

\[
\delta \bar{\psi}(x) = igT^a C^a(x)\psi(x)
\]

\[
\delta \psi(x) = ig\xi(x)T^a C^a(x)
\]  

(2.3)

where we have set the parametric functions of the gauge group \( \theta^a(x) = \xi C^a(x) \) in which \( \xi \) is an infinitesimal Grassmann number and \( C^a(x) \) are the ghost field functions. According to the result given in paper I, the quantum theory built up from the Lagrangian in Eq. (2.1) and the constraint condition in Eq. (2.2) is described by the following generating functional of Green’s functions [5-9]

\[
Z[J^\mu_a, K^a, \bar{K}^a, \eta, \bar{\eta}] = \frac{1}{\mu^2} \int D(A^a_\mu, \bar{C}^a, C^a, \bar{\psi}, \psi) \exp\{iS + i \int d^4x (J^a_\mu A^a_\mu + \bar{K}^a C^a + \bar{C}^a K^a + \bar{\psi}\eta + \bar{\eta}\psi)\}
\]  

(2.4)

where \( J^a_\mu, \bar{K}^a, K^a, \eta \) and \( \bar{\eta} \) are the external sources and

\[
S = \int d^4x \{\bar{\psi}(i\gamma^\mu(\partial_\mu - igT^a A^a_\mu)) - M\psi - \frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \frac{1}{2}m^2 A^{a\mu}A^a_\mu - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 + \bar{C}a^\mu(\partial^\mu D^{ab}_\mu C^b)\}
\]  

(2.5)

is the effective action in which

\[
D^{ab}_\mu(x) = \frac{\mu^2}{\Box_x} \partial^\mu + D^{ab}_\mu(x)
\]  

(2.6)

here \( \mu^2 = \alpha m^2 \) and

\[
D^{ab}_\mu = \delta^{ab} \partial_\mu - gf^{abc} A^c_\mu
\]  

(2.7)
is the covariant derivative. Similar to the massless gauge theory, for the massive gauge theory, there are a set of BRST-transformations including the infinitesimal gauge transformations shown in Eq. (2.3) and the transformations for the ghost fields under which the effective action is invariant. The transformations for the ghost fields may be found from the stationary condition of the effective action under the BRST-transformations. By applying the transformations in Eq. (2.3) to the action in Eq. (2.5), one can derive

$$\delta S = \int d^4x \{[\delta \bar{C}^a - \frac{\xi}{\alpha} \partial^\mu A_\mu^a] \partial^\mu(D^a_{\mu} C^b) + \bar{C}^a \partial^\mu \delta(D^a_{\mu} C^b)\} = 0$$

(2.8)

This expression suggests that if we set

$$\delta \bar{C}^a = \frac{\xi}{\alpha} \partial^\nu A_\nu^a$$

(2.9)

and

$$\partial^\mu \delta(D^a_{\mu} C^b) = 0$$

(2.10)

The action will be invariant. Eq. (2.9) gives the transformation law of the ghost field variable \(\bar{C}^a(x)\) which is the same as the one in the massless gauge field theory. From Eq. (2.10), we may derive a transformation law of the ghost variables \(C^a(x)\). Noticing the relation in Eq. (2.6), we can write

$$\delta(D^a_{\mu}(x)C^b(x)) = \frac{\mu^2}{\Box_x} \partial^\mu \delta C^a(x) + \delta(D^a_{\mu}(x)C^b(x))$$

(2.11)

In the massless gauge theory, it has been proved that

$$\delta(D^a_{\mu}(x)C^b(x)) = D^a_{\mu}(x)[\delta C^b(x) + \frac{\xi}{2} \epsilon_{abcd} C^c(x)C^d(x)]$$

(2.12)

With this result, Eq. (2.11) can be written as

$$\delta(D^a_{\mu}(x)C^b(x)) = D^a_{\mu}(x)\delta C^b(x) - D^a_{\mu}(x)\delta C^b_0(x)$$

(2.13)

where

$$\delta C^a_0(x) = -\frac{\xi \epsilon}{2} \epsilon_{abcd} C^b(x)C^c(x)$$

(2.14)

On substituting Eq. (2.13) into Eq. (2.10), we have

$$M^{ab}(x)\delta C^b(x) = M^{ab}(x)\delta C^b_0(x)$$

(2.15)

where we have defined

$$M^{ab}(x) \equiv \partial^\mu D^a_{\mu}(x) = \delta^{ab} (\Box_x + \mu^2) - g \epsilon^{abc} A_\mu^c(x) \partial^\mu$$

(2.16)

and

$$M^{ab}_0(x) \equiv \partial^\mu D^a_{\mu}(x) = M^{ab}(x) - \mu^2 \delta^{ab}$$

(2.17)

It is noted that the operator in Eq. (2.16) is just the operator appearing in the equation of motion for the ghost field \(C^a(x)\)

$$\partial^\mu (D^a_{\mu} C^b) = 0$$

(2.18)

(see Eq. (4.10) in paper I). Corresponding to this equation of motion, we may write an equation satisfied by the Green’s function \(\Delta^{ab}(x - y)\)

$$M^{ac}(x)\Delta^{ab}(x - y) = \delta^{ab} \delta^4(x - y)$$

(2.19)
The function $\Delta^{ab}(x-y)$ is nothing but the exact propagator of the ghost field which is inverse of the operator $M^{ab}(x)$. In the light of Eq. (2.19) and noticing Eq. (2.17), we may solve out the $\delta C^a(x)$ from Eq. (2.15)

$$\delta C^a(x) = (M^{-1}M_0\delta C_0)^a(x) = \{M^{-1}(M - \mu^2)\delta C_0\}^a(x)$$

$$= \delta C^a_0(x) - \mu^2 \int d^4y \Delta^{ab}(x-y)\delta C^b(y)$$

(2.20)

This just is the transformation law for the ghost variables $C^a(x)$. When the mass tends to zero, Eq. (2.20) immediately goes over to the corresponding transformation given in the massless gauge field theory. It is interesting that in the Landau gauge ($\alpha = 0$), due to $\mu = 0$, the above transformation also reduces to the form as given in the massless theory. This result is natural since in the Landau gauge, the gauge field mass term in the action is gauge-invariant. However, in general gauges, the mass term is no longer gauge-invariant. In this case, to maintain the action to be gauge-invariant, it is necessary to give the ghost field a mass $\mu$ so as to counteract the gauge-non-invariance of the gauge field mass term. As a result, in the transformation given in Eq. (2.20) appears a term proportional to $\mu^2$.

### 3. WARD-TAKAHASHI IDENTITIES

This section serves to derive the W-T identities for the quantum massive non-Abelian gauge field theory established in paper I and represented in Eqs. (2.4)-(2.7) on the basis of the BRST-symmetry of the theory. Since derivations of the W-T identities for the QCD with massive gluons are much similar to those for the QCD with massless gluons, we only need here to give a brief description of the derivations. When we make the BRST-transformations shown in Eqs. (2.3), (2.9) and (2.20) to the generating functional in Eq. (2.4) and consider the invariance of the generating functional, the action and the integration measure under the transformations (the invariance of the integration measure is easy to check), we obtain an identity such that

$$\frac{i}{\hbar} \int \mathcal{D}(A_\mu^a, \bar{C}^a, C^a, \bar{\psi}, \psi) \int d^4x \{J^{\mu a}(x)\delta A_\mu^a(x) + \delta \bar{C}^a(x)K^a(x) + \bar{K}^a(x)\delta C^a(x)$$

$$+ \bar{\eta}(x)\delta \bar{\psi}(x) + \delta \bar{\psi}(x)\eta(x)\} e^{iS + EST}$$

(3.1)

where $EST$ is an abbreviation of the external source terms appearing in Eq. (2.4). The Grassmann number $\xi$ contained in the BRST-transformations in Eq. (3.1) may be eliminated by performing a partial differentiation of Eq. (3.1) with respect to $\xi$. As a result, we get a W-T identity as follows

$$\frac{i}{\hbar} \int \mathcal{D}(A_\mu^a, \bar{C}^a, C^a, \bar{\psi}, \psi) \int d^4x \{J^{\mu a}(x)\Delta A_\mu^a(x) + \Delta \bar{C}^a(x)K^a(x) - \bar{K}^a(x)\Delta C^a(x)$$

$$- \bar{\eta}(x)\Delta \bar{\psi}(x) + \Delta \bar{\psi}(x)\eta(x)\} e^{iS + EST}$$

(3.2)

$$= 0$$

where

$$\Delta A_\mu^a(x) = D^{ab}_\mu(x)C^b(x)$$

$$\Delta C^a(x) = \frac{1}{\hbar}\partial^\mu A^a_\mu(x)$$

$$\Delta C^a_0(y) = -\frac{i}{\hbar}g \int \delta^{abcd}(y)C^d(y)$$

$$\Delta \bar{\psi}(x) = i\gamma^a T^a(x)\bar{\psi}(x)$$

$$\Delta \bar{\psi}(x) = i\gamma^a T^a(x)\bar{\psi}(x)$$

(3.3)

These functions defined above are finite. Each of them differs from the corresponding BRST-transformation written in Eqs. (2.3), (2.9) and (2.20) by an infinitesimal Grassmann parameter $\xi$.

In order to represent the composite field functions $\Delta A_\mu^a, \Delta C^a, \Delta \bar{\psi}$ and $\Delta \psi$ in Eq. (3.2) in terms of differentials of the functional $Z$ with respect to external sources, we may, as usual, construct a generalized generating functional by introducing new external sources (called BRST-sources later on) into the generating functional written in Eq. (2.4), as shown in the following[8-11]
\[ Z[J^n, \bar{K}^a, K^a, \bar{\eta}, \eta, u^{a\mu}, v^a, \bar{\zeta}, \zeta] = \frac{1}{\mathcal{N}} \int \mathcal{D}[A^n, C^n, \bar{C}^a, C^a, \psi, \bar{\psi}] \exp \{ i S + i \int d^4 x [u^{a\mu} \Delta A^n_a + v^a \Delta C^a + \Delta \bar{\psi} \zeta + \zeta \Delta \psi + J^{a\mu} A^n_a + \bar{K}^a + C^n K^a + \bar{\eta} \bar{\psi} + \eta \bar{\psi}] \} \]

where \( u^{a\mu}, v^a, \bar{\zeta} \) and \( \zeta \) are the sources which belong to the corresponding functions \( \Delta A^n_a, \Delta C^a, \Delta \bar{\psi} \) and \( \Delta \psi \) respectively. Obviously, the \( u^{a\mu} \) and \( \Delta A^n_a \) are anticommuting quantities, while, the \( v^a, \bar{\zeta}, \zeta, \Delta C^a, \Delta \bar{\psi} \) and \( \Delta \psi \) are commuting ones. We may start from the above generating functional to re-derive the W-T identity. In order that the identity thus derived is identical to that as given in Eq. (3.2), it is necessary to require the BRST-source terms \( u_i \Delta \Phi_i \) where \( u_i = u^{a\mu}, v^a, \bar{\zeta} \) or \( \zeta \) and \( \Delta \Phi_i = \Delta A^n_a, \Delta C^a, \Delta \bar{\psi} \) or \( \Delta \psi \) to be invariant under the BRST-transformations. How to ensure the BRST-invariance of the source terms? For illustration, let us introduce the source terms in such a fashion

\[ \int d^4 x [\bar{u}^{a\mu} \delta A^n_a + \bar{v}^a \delta C^a + \bar{\zeta} \delta \bar{\psi} + \zeta \delta \psi] \]

where

\[ u^{a\mu} = \bar{u}^{a\mu} \xi, \ v^a = \bar{v}^a \xi, \ \bar{\zeta} = \bar{\bar{\zeta}} \xi, \ \zeta = - \bar{\zeta} \xi \]

These external sources are defined by including the Grassmann number \( \xi \) and hence products of them with \( \xi \) vanish. This suggests that we may generally define the sources by the following condition

\[ u_i \xi = 0 \]

Considering that under the BRST-transformation, the variation of the composite field functions given in the general gauges can be represented in the form \( \delta \Delta \Phi_i = \xi \Phi_i \) where \( \Phi_i \) are functions without including the parameter \( \xi \), clearly, the definition in Eq. (3.7) for the sources would guarantee the BRST-invariance of the BRST-source terms. When the BRST-transformations in Eqs. (2.3), (2.9) and (2.20) are made to the generating functional in Eq. (3.4), due to the definition in Eq. (3.7) for the sources, we have \( u_i \xi \delta \Phi_i = 0 \) which means that the BRST-source terms give a vanishing contribution to the identity in Eq. (3.1). Therefore, we still obtain the identity as shown in Eq. (3.1) except that the external source terms is now extended to include the BRST-external source terms. This fact indicates that we may directly insert the BRST-source terms into the exponent in Eq. (3.1) without changing the identity itself. When performing a partial differentiation of the identity with respect to \( \xi \), we obtain a W-T identity which is the same as written in Eq. (3.2) except that the BRST-source terms are now included in the identity. Therefore, Eq. (3.2) may be expressed as

\[ \int d^4 x [J^{a\mu}(x) \delta_{\partial^{a\mu}(x)} - \bar{K}^a(x) \delta_{\bar{\partial}^{a\mu}(x)} - \bar{\eta}(x) \delta_{\bar{\partial}^\eta(x)} + \bar{\eta}(x) \delta_{\partial^{\bar{\eta}}(x)}] Z[J^n, \cdots, \zeta] = 0 \]

This is the W-T identity satisfied by the generating functional of full Green functions.

On substituting in Eq. (3.8) the relation

\[ Z = e^{iW} \]

where \( W \) denotes the generating functional of connected Green’s functions, one may obtain a W-T identity expressed by the functional \( W \)

\[ \int d^4 x [J^{a\mu}(x) \delta_{\partial^{a\mu}(x)} - \bar{K}^a(x) \delta_{\bar{\partial}^{a\mu}(x)} - \bar{\eta}(x) \delta_{\bar{\partial}^\eta(x)} + \bar{\eta}(x) \delta_{\partial^{\bar{\eta}}(x)}] W[J^n, \cdots, \zeta] = 0 \]

From this identity, one may get another W-T identity satisfied by the generating functional \( \Gamma \) of proper (one-particle-irreducible) vertex functions. The functional \( \Gamma \) is usually defined by the following Legendre transformation
\[ \Gamma[A^a, \bar{C}^a, C^a, \bar{\psi}, \psi; u^a, v^a, \zeta, \bar{\zeta}] = W[J^a_\mu, \bar{K}^a, K^a, \bar{\eta}, \eta; u^a, v^a, \zeta, \bar{\zeta}] - \int d^4x [J^a_\mu A^{a\mu} + K^a C^a + \bar{\eta} \psi + \psi \eta] \] (3.11)

where \( A^a_\mu, \bar{C}^a, C^a, \bar{\psi}, \psi \) are the field variables defined by the following functional derivatives [5-8]:

\[ A^a_\mu(x) = \frac{\delta W}{\delta j^a_\mu(x)}, \quad \bar{C}^a(x) = -\frac{\delta W}{\delta \bar{K}^a(x)}, \quad C^a(x) = \frac{\delta W}{\delta K^a(x)}, \] (3.12)

From Eq. (3.11), it is not difficult to get the inverse transformations [5-8]:

\[ J^{a\mu}(x) = -\frac{\delta \Gamma}{\delta \bar{A}^{a\mu}_\mu(x)}, \quad \bar{K}^a(x) = \frac{\delta \Gamma}{\delta \bar{\eta}}, \quad K^a(x) = -\frac{\delta \Gamma}{\delta \eta(x)}. \] (3.13)

It is obvious that

\[ \frac{\delta W}{\delta u^a} = \frac{\delta \Gamma}{\delta u^a}, \quad \frac{\delta W}{\delta v^a} = \frac{\delta \Gamma}{\delta v^a}, \quad \frac{\delta W}{\delta \zeta} = \frac{\delta \Gamma}{\delta \zeta}, \quad \frac{\delta W}{\delta \bar{\zeta}} = \frac{\delta \Gamma}{\delta \bar{\zeta}}. \] (3.14)

Employing Eqs. (3.13) and (3.14), the W-T identity in Eq. (3.10) will be written as

\[ \int d^4x \{ \frac{\delta \Gamma}{\delta A^a_\mu(x)} \frac{\delta \Gamma}{\delta u^a(x)} + \frac{\delta \Gamma}{\delta \bar{C}^a(x)} \frac{\delta \Gamma}{\delta \bar{v}^a(x)} + \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \frac{\delta \Gamma}{\delta \bar{\zeta}} \frac{\delta \Gamma}{\delta \zeta(x)} + m^2 \frac{\delta^2 \Gamma}{\delta \bar{A}^a_\mu(x) \delta \bar{A}^a_\mu(x)} \} = 0 \] (3.15)

This is the W-T identity satisfied by the generating functional of proper vertex functions.

The above identity may be represented in another form with the aid of the so-called ghost equation of motion. The ghost equation may easily be derived by firstly making the translation transformation: \( C^a \rightarrow \bar{C}^a + \lambda^a \) in Eq. (2.4) where \( \lambda^a \) is an arbitrary Grassmann variable, then differentiating Eq. (2.4) with respect to \( \lambda^a \) and finally setting \( \lambda^a = 0 \). The result is [8-11]:

\[ \frac{1}{N} \int D(A^a_\mu, \bar{C}^a, C^a, \bar{\psi}, \psi) \{ K^a(x) + \partial^a_\mu (D^{ab}_\mu(x)C^b(x)) \} e^{iS + EST} = 0 \] (3.16)

When we use the generating functional defined in Eq. (3.4) and notice the relation in Eq. (2.6), the above equation may be represented as [8-11]:

\[ [K^a(x) = i \partial^a_\mu \frac{\delta}{\delta u^a(x)} - i \mu^2 \frac{\delta}{\delta K^a(x)}] Z[J^a_\mu, \ldots, \zeta] = 0 \] (3.17)

On substituting the relation in Eq. (3.9) into the above equation, we may write a ghost equation satisfied by the functional \( W \) such that

\[ K^a(x) + \partial^a_\mu \frac{\delta W}{\delta u^a(x)} + \mu^2 \frac{\delta W}{\delta K^a(x)} = 0 \] (3.18)

From this equation, the ghost equation obeyed by the functional \( \Gamma \) is easy to be derived by virtue of Eqs. (3.12) - (3.14): [8-11]

\[ \frac{\delta \Gamma}{\delta C^a(x)} - \partial^a_\mu \frac{\delta \Gamma}{\delta u^a(x)} - \mu^2 C^a(x) = 0 \] (3.19)

Upon applying the above equation to the last term in Eq. (3.15), the identity in Eq. (3.15) will be rewritten as

\[ \int d^4x \{ \frac{\delta \Gamma}{\delta A^a_\mu(x)} \frac{\delta \Gamma}{\delta u^a(x)} + \frac{\delta \Gamma}{\delta \bar{C}^a(x)} \frac{\delta \Gamma}{\delta \bar{v}^a(x)} + \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \frac{\delta \Gamma}{\delta \bar{\zeta}} \frac{\delta \Gamma}{\delta \zeta(x)} + m^2 \frac{\delta^2 \Gamma}{\delta \bar{A}^a_\mu(x) \delta \bar{A}^a_\mu(x)} \} = 0 \] (3.20)
Now, let us define a new functional $\hat{\Gamma}$ in such a manner
\[
\hat{\Gamma} = \Gamma + \frac{1}{2\alpha} \int d^4x (\partial^\mu A^a_\mu)^2
\] (3.21)

From this definition, it follows that
\[
\frac{\delta \hat{\Gamma}}{\delta A^a_{\mu}} = \frac{\delta \Gamma}{\delta A^a_{\mu}} + \frac{1}{\alpha} \partial^\mu \partial^\nu A^a_\nu
\] (3.22)

When inserting Eq. (3.21) into Eq. (3.20) and considering the relation in Eq. (3.22), we arrive at
\[
\int d^4x \left\{ \frac{\delta \hat{\Gamma}}{\delta A^a_{\mu}} \frac{\delta \Gamma}{\delta u^{a\mu}(x)} + \frac{\delta \hat{\Gamma}}{\delta u^{a\mu}(x)} \frac{\delta \Gamma}{\delta \psi(x)} + \frac{\delta \hat{\Gamma}}{\delta C^a(x)} \frac{\delta \Gamma}{\delta \bar{\psi}(x)} + \frac{\delta \hat{\Gamma}}{\delta \bar{\psi}(x)} \frac{\delta \Gamma}{\delta \bar{\zeta}(x)} \right\} = 0
\] (3.23)

The ghost equation represented through the functional $\hat{\Gamma}$ is of the same form as Eq. (3.19)
\[
\frac{\delta \hat{\Gamma}}{\delta C^a(x)} - \frac{\partial^\mu}{\partial x^\mu} \frac{\delta \hat{\Gamma}}{\delta u^{a\mu}(x)} - \mu^2 C^a(x) = 0
\] (3.24)

In the Landau gauge, since $\mu = 0$ and $\partial^\nu A^a_\nu = 0$, Eqs. (3.23) and (3.24) respectively reduce to
\[
\int d^4x \left\{ \frac{\delta \hat{\Gamma}}{\delta A^a_{\mu}} \frac{\delta \Gamma}{\delta u^{a\mu}(x)} + \frac{\delta \hat{\Gamma}}{\delta u^{a\mu}(x)} \frac{\delta \Gamma}{\delta \psi(x)} + \frac{\delta \hat{\Gamma}}{\delta C^a(x)} \frac{\delta \Gamma}{\delta \bar{\psi}(x)} + \frac{\delta \hat{\Gamma}}{\delta \bar{\psi}(x)} \frac{\delta \Gamma}{\delta \bar{\zeta}(x)} \right\} = 0
\] (3.25)
and
\[
\frac{\delta \hat{\Gamma}}{\delta C^a(x)} - \frac{\partial^\mu}{\partial x^\mu} \frac{\delta \hat{\Gamma}}{\delta u^{a\mu}(x)} = 0
\] (3.26)

These equations formally are the same as those for the massless gauge field theory.

Now, we would like to give another form of the W-T identity. The ghost equation (3.16) suggests that the first external source term in Eq.(3.5) which appearing in the generating functional in Eq.(3.4) may be replaced by
\[
\int d^4x \tilde{u}^{a\mu}(x) \delta A^a_{\mu}(x) = \int d^4x u^{a\mu}(x) \Delta A^a_{\mu}(x)
\] (3.27)
where
\[
\delta A^a_{\mu}(x) = \xi \mathcal{D}^{ab}_\mu(x) C^b(x) = \xi \Delta A^a_{\mu}(x)
\] (3.28)
with
\[
\Delta A^a_\mu(x) = \Delta A^a_{\mu} + \frac{\mu^2}{\alpha} \frac{\partial^\mu}{\partial x^\mu} C^a(x)
\] (3.29)

In this case, from the above relations, we see, the W-T identity in Eq. (3.8) will be rewritten as
\[
\int d^4x \left\{ \frac{\delta K^a_\mu(x)}{\delta u^{a\mu}(x)} - \frac{\delta K^a_\mu(x)}{\delta u^{a\mu}(x)} - \frac{\delta K^a_\mu(x)}{\delta u^{a\mu}(x)} + \frac{\delta K^a_\mu(x)}{\delta u^{a\mu}(x)} \right\} Z[J^a_\mu, \cdots, \zeta] = 0
\] (3.30)

and the ghost equation in Eq. (3.17) becomes
\[
\{K^a_\mu(x) - i \frac{\partial^\mu}{\partial u^{a\mu}(x)} \} Z[J^a_\mu, \cdots, \zeta] = 0
\] (3.31)
Repeating the derivations described in Eqs. (3.9)-(3.15) and (3.17)-(3.24), one may obtain from Eqs. (3.30) and (3.31) the identities expressed by the functional $\hat{\Gamma}$

$$\int d^4 x \{ \frac{\delta \hat{\Gamma}}{\delta A^\alpha_a(x)} \frac{\delta Z}{\delta u^\mu(x)} + \frac{\delta \hat{\Gamma}}{\delta \bar{C}^\alpha_a(x)} \frac{\delta Z}{\delta u^\mu(x)} + \frac{\delta \hat{\Gamma}}{\delta \gamma(x)} \frac{\delta Z}{\delta u^\mu(x)} \}
+ \frac{\delta \hat{\Gamma}}{\delta \gamma^\mu(x)} \frac{\delta Z}{\delta u^\nu(x)} - \frac{\delta \hat{\Gamma}}{\delta \gamma^\mu(x)} \frac{\delta Z}{\delta u^\nu(x)} = 0 \quad (3.32)$$

$$\frac{\delta \hat{\Gamma}}{\delta C^\alpha_a(x)} - \frac{\partial \mu}{\partial u^\mu(x)} = 0 \quad (3.33)$$

In comparison of Eqs. (3.32) and (3.33) with Eqs. (3.23) and (3.24), we see, the advantage of using $\delta A^\alpha_a$ to define the external source is that the ghost equation (3.33) becomes homogeneous. However, the price paying for this advantage is the increase of an inhomogeneous term (the fifth term) in Eq. (3.32). In the Landau gauge and in the zero-mass limit, Eqs. (3.32) and (3.33) still reduce to the homogeneous equations (3.25) and (3.26).

From the W-T identities formulated above, we may derive various W-T identities obeyed by Green’s functions and vertices, as will be illustrated later on. Particularly, these identities provide a firm basis for the proof of renormalizability and unitarity problems of the quantum massive gauge field theory as will be discussed in this paper and the next paper.

4. PROPAGATORS

In this section, as an application of the W-T identities derived in the preceding section, we have a particular interest in deriving the W-T identities satisfied by the massive gluon and ghost particle propagators and then discussing their renormalization. To derive the mentioned W-T identities, it is appropriate to start from the W-T identity in Eq. (3.30) and the ghost particle propagators and then discussing their renormalization. To derive the mentioned W-T identities in Eqs. (3.8) and (17) with respect to the external sources $K^a(x)$ and $K^a(y)$ respectively and then set all the sources except for the source $J^a_{\mu}(x)$ to be zero. In this way, we obtain the following identities

$$\frac{1}{\alpha^2} \frac{\partial Z[J]}{\partial u^\mu(x)} + \int d^4 y J^{\nu(y)} \frac{\delta^2 Z[J, K, u]}{\delta K^a(x) \delta u^\nu(y)} |_{K=0} = 0 \quad (4.1)$$

and

$$i \frac{\partial}{\partial u^\mu} \frac{\delta Z[J, K, u]}{\delta u^\nu(y)} |_{K=0} = i \mu^2 \frac{\delta^2 Z[J, K, u]}{\delta K^a(x) \delta K^b(y)} |_{K=0} + \delta^{ab} \delta^4(x-y) Z[J] = 0 \quad (4.2)$$

Furthermore, on differentiating Eq. (4.1) with respect to $J^a_{\mu}(y)$ and then letting the source $J$ vanish, we may get an identity which is, in operator representation, of the form $[8-11]$

$$\frac{1}{\alpha^2} < 0^+ | T [ \hat{A}^\mu_a(x) \hat{A}^\nu_b(y) ] | 0^- > = < 0^+ | T^* [ \hat{C}^\alpha_a(x) \hat{D}^{\nu \mu}_b(y) \hat{C}^\mu_d(y) ] | 0^- > \quad (4.3)$$

where $\hat{A}^\mu_a(x)$, $\hat{C}^\alpha_a(x)$ and $\hat{C}^{\alpha}_a(x)$ stand for the gauge field and ghost field operators and $T^*$ symbolizes the covariant time-ordering product. When the source $J$ is set to vanish, Eq. (4.2) will give such an equation $[8-11]$

$$i \frac{\partial}{\partial u^\mu} < 0^+ | T^* [ \hat{C}^\alpha_a(x) \hat{D}^{\nu \mu}_b(y) \hat{C}^\mu_d(y) ] | 0^- > + i \mu^2 < 0^+ | T [ \hat{C}^\alpha_a(x) \hat{C}^\mu_b(y) ] | 0^- > = \delta^{ab} \delta^4(x-y) \quad (4.4)$$

Upon inserting Eq. (4.3) into Eq. (4.4), we have

$$\frac{\partial}{\partial u^\mu} D^{ab}_{\mu \nu}(x-y) - \alpha \mu^2 \Delta^{ab}(x-y) = -\alpha \delta^{ab} \delta^4(x-y) \quad (4.5)$$
where

\[ iD_{\mu\nu}^{ab}(x-y) = \langle 0^+ | T \{ \hat{A}^a_{\mu}(x) \hat{A}^b_{\nu}(y) \} | 0^- \rangle \]  

(4.6)

which is the familiar full gluon propagator and

\[ i\Delta^{ab}(x-y) = \langle 0^+ | T \{ \hat{C}^a(x) \hat{C}^b(y) \} | 0^- \rangle \]  

(4.7)

which is the full ghost particle propagator. Eq. (4.5) just is the W-T identity respected by the gluon propagator which establishes a relation between the longitudinal part of gluon propagator and the ghost particle propagator. Particularly, in the Landau gauge, Eq. (4.5) reduces to the form which exhibits the transversity of the gluon propagator. By the Fourier transformation, Eq. (4.5) will be converted to the form given in the momentum space as follows

\[ k^\mu k^\nu D_{\mu\nu}^{ab}(k) - \alpha \mu^2 \Delta^{ab}(k) = -\alpha \delta^{ab} \]  

(4.8)

The ghost particle propagator may be determined by the ghost equation shown in Eq. (4.4). However, we would rather here to derive its expression from the Dyson-Schwinger equation\cite{13} satisfied by the propagator which may be established by the perturbation method.

\[ \Delta^{ab}(k) = \Delta_0^{ab}(k) + \Delta_0^{a'b'}(k)\Omega^{a'b'}(k) \Delta^{b'}(k) \]  

(4.9)

where

\[ i\Delta_0^{ab}(k) = i\delta^{ab} \Delta_0(k) = \frac{-i\delta^{ab}}{k^2 - \mu^2 + i\varepsilon} \]  

(4.10)

is the free ghost particle propagator obtained in paper I and \(-i\Omega^{ab}(k) = -i\delta^{ab}\Omega(k)\) denotes the proper self-energy operator of ghost particle. From Eq. (4.9), it is easy to solve that

\[ i\Delta^{ab}(k) = \frac{-i\delta^{ab}}{k^2[1 + \hat{\Omega}(k^2)] - \mu^2 + i\varepsilon} \]  

(4.11)

where the self-energy has properly been expressed as

\[ \Omega(k) = k^2\hat{\Omega}(k^2) \]  

(4.12)

Similarly, we may write a Dyson-Schwinger equation for the gluon propagator by the perturbation procedure\cite{13}

\[ D_{\mu\nu}(k) = D_{\mu\nu}^0(k) + D_{\mu\lambda}(k)\Pi^{\lambda\nu}(k)D_{\rho\nu}(k) \]  

(4.13)

where the color indices are suppressed for simplicity and

\[ iD_{\mu\nu}^{(0)ab}(k) = i\delta^{ab} D_{\mu\nu}^{(0)}(k) = -i\delta^{ab} \left[ \frac{g_{\mu\nu} - k_{\mu}k_{\nu}/k^2}{k^2 - m^2 + i\varepsilon} + \frac{\alpha k_{\mu}k_{\nu}/k^2}{k^2 - \mu^2 + i\varepsilon} \right] \]  

(4.14)

is the free gluon propagator as derived in paper I and \(-i\Pi_{\mu\nu}^{ab}(k) = -i\delta^{ab}\Pi_{\mu\nu}(k)\) stands for the gluon proper self-energy operator. Let us decompose the propagator and the self-energy operator into transverse and longitudinal parts:

\[ D^{\mu\nu}(k) = D_T^{\mu\nu}(k) + D_L^{\mu\nu}(k), \Pi^{\mu\nu}(k) = \Pi_T^{\mu\nu}(k) + \Pi_L^{\mu\nu}(k) \]  

(4.15)

where

\[ D_T^{\mu\nu}(k) = (g^{\mu\nu} - k^\mu k^\nu/k^2)D_T(k^2), \quad D_L^{\mu\nu}(k) = k^\mu k^\nu/k^2 D_L(k^2), \]

\[ \Pi_T^{\mu\nu}(k) = (g^{\mu\nu} - k^\mu k^\nu/k^2)\Pi_T(k^2), \quad \Pi_L^{\mu\nu}(k) = k^\mu k^\nu/k^2 \Pi_L(k^2) \]  

(4.16)

Considering these decompositions and the orthogonality between the transverse and longitudinal parts, Eq. (4.13) will be split into two equations.
Solving the equations (4.17) and (4.18), one can get

$$D_{T\mu\nu}(k) = D_{T\mu\nu}^0(k) + D_{T\mu\nu}^0(k)\Pi_T^0(k)D_{T\mu\nu}(k)$$  \hspace{1cm} (4.17)$$

and

$$D_{L\mu\nu}(k) = D_{L\mu\nu}^0(k) + D_{L\mu\nu}^0(k)\Pi_L^0(k)D_{L\mu\nu}(k)$$  \hspace{1cm} (4.18)$$

Solving the equations (4.17) and (4.18), one can get

$$iD_{\mu\nu}^{ab}(k) = -i\delta^{ab}\left\{\frac{g_{\mu\nu} - k_{\mu}k_{\nu}/k^2}{k^2 + \Pi_T(k^2) - m^2 + i\varepsilon} + \frac{\alpha k_{\mu}k_{\nu}/k^2}{k^2 + \alpha\Pi_L(k^2) - \mu^2 + i\varepsilon}\right\}.$$  \hspace{1cm} (4.19)$$

With setting

$$\Pi_T(k^2) = k^2\Pi_1(k^2) + m^2\Pi_2(k^2)$$  \hspace{1cm} (4.20)$$

which follows from thr Lorentz-covariance of the operator $\Pi_T(k^2)$ and

$$\alpha\Pi_L(k^2) = k^2\tilde{\Pi}_L(k^2),$$  \hspace{1cm} (4.21)$$

Eq. (4.19) will be rewritten as

$$iD_{\mu\nu}^{ab}(k) = -i\delta^{ab}\left\{\frac{g_{\mu\nu} - k_{\mu}k_{\nu}/k^2}{k^2[1 + \Pi_1(k^2)] - m^2[1 - \Pi_2(k^2)] + i\varepsilon} + \frac{\alpha k_{\mu}k_{\nu}/k^2}{k^2[1 + \Pi_L(k^2)] - \mu^2 + i\varepsilon}\right\}.$$  \hspace{1cm} (4.22)$$

We would like to note that the expressions given in Eqs. (4.12), (4.20) and (4.21) can be verified by practical calculations and are important for renormalizations of the propogators and the gluon mass.

Substitution of Eqs. (4.11) and (4.22) into Eq. (4.8) yields

$$\tilde{\Pi}_L(k^2) = \frac{\mu^2\tilde{\Omega}(k^2)}{k^2[1 + \Omega(k^2)]}$$  \hspace{1cm} (4.23)$$

From this relation, we see, either in the Landau gauge or in the zero-mass limit, the $\tilde{\Pi}_L(k^2)$ vanishes.

Now let us discuss the renormalization. The function $\tilde{\Omega}(k^2)$ in Eq. (4.11), the functions $\Pi_1(k^2)$, $\Pi_2(k^2)$ and $\tilde{\Pi}_L(k^2)$ in Eq. (4.22) are generally divergent in higher order perturbative calculations. According to the conventional procedure of renormalization, the divergences included in the functions $\Omega(k^2)$, $\Pi_1(k^2)$, $\Pi_2(k^2)$ and $\tilde{\Pi}_L(k^2)$ may be subtracted at a renormalization point, say, $k^2 = \nu^2$. Thus, we can write$^{[5-9]}$

$$\tilde{\Omega}(k^2) = \tilde{\Omega}(\nu^2) + \tilde{\Omega}'(k^2), \quad \Pi_1(k^2) = \Pi_1(\nu^2) + \Pi_1'(k^2),$$

$$\Pi_2(k^2) = \Pi_2(\nu^2) + \Pi_2'(k^2), \quad \tilde{\Pi}_L(k^2) = \tilde{\Pi}_L(\nu^2) + \tilde{\Pi}_L'(k^2)$$  \hspace{1cm} (4.24)$$

where $\tilde{\Omega}(\nu^2)$, $\Pi_1(\nu^2)$, $\Pi_2(\nu^2)$ and $\Omega'(k^2)$, $\Pi_1'(k^2)$, $\Pi_2'(k^2)$, $\tilde{\Pi}_L'(k^2)$ are respectively the divergent parts and the finite parts of the functions $\Omega(k^2)$, $\Pi_1(k^2)$, $\Pi_2(k^2)$ and $\tilde{\Pi}_L(k^2)$. The divergent parts can be absorbed in the following renormalization constants defined by$^{[5-9]}$

$$Z_3^{-1} = 1 + \tilde{\Omega}(\nu^2), \quad Z_3^{-1} = 1 + \Pi_1(\nu^2), \quad Z_3' = 1 + \tilde{\Pi}_L(\nu^2),$$

$$Z_m^{-1} = \sqrt{Z_3[1 - \Pi_2(\nu^2)]} = \sqrt{[1 - \Pi_1(\nu^2)][1 - \Pi_2(\nu^2)]}$$  \hspace{1cm} (4.25)$$

where $Z_3$ and $\tilde{Z}_3$ are the renormalization constants of gluon and ghost particle propagators respectively, $Z_3'$ is the additional renormalization constant of the longitudinal part of gluon propagator and $Z_m$ is the renormalization constant of gluon mass. With the above definitions of the renormalization constants, on inserting Eq. (4.24) into Eqs. (4.11) and (4.22), the ghost particle propagator and gluon propagator can be renormalized, respectively, in such a manner

$$i\Delta^{ab}(k) = \tilde{Z}_3 i\Delta^{ab}_R(k)$$  \hspace{1cm} (4.26)$$
\[ iD_{\mu\nu}^{ab}(k) = Z_3 iD_{R\mu\nu}^{ab}(k) \] (4.27)

where
\[ i\Delta_R^{ab}(k) = \frac{-i\delta^{ab}}{k^2[1 + \Omega_R(k^2)] - \mu_R^2 + i\varepsilon} \] (4.28)

and
\[ iD_{R\mu\nu}^{ab}(k) = -i\delta^{ab}\left(\frac{g_{\mu\nu} - k_{\mu}k_{\nu}/k^2}{k^2 - m_R^2 + \Pi_L(k^2) + i\varepsilon} + \frac{Z_3\alpha_R k_{\mu}k_{\nu}/k^2}{k^2[1 + \Pi_L(k^2)] - \mu_R^2 + i\varepsilon}\right) \] (4.29)

are the renormalized propagators in which \( m_R, \bar{m}_R \) and \( \mu_R \) are the renormalized masses, \( \alpha_R \) is the renormalized gauge parameter, \( \Omega_R(k^2), \Pi_L(k^2) \) and \( \Pi_{L\alpha}(k^2) \) denote the finite corrections coming from the loop diagrams. They are defined as

\[ m_R = Z_m^{-1}m, \quad \alpha_R = Z_3^{-1}\alpha, \quad \bar{m}_R = \sqrt{Z_3}\mu, \quad \mu_R = \sqrt{Z_3}\mu, \]
\[ \Omega_R(k^2) = 3\bar{\Pi}r(k^2), \quad \Pi_L(k^2) = Z_3[k^2\Pi_L(k^2) + m^2\bar{\Pi}_L(k^2)], \quad \Pi_{L\alpha}(k^2) = Z_{3\alpha}\bar{\Pi}_{L\alpha}(k^2). \] (4.30)

The finite corrections above are zero at the renormalization point \( \nu \). As we see from Eq. (4.29), the longitudinal part of the gluon propagator, except for in the Landau gauge, needs to be renormalized and has an extra renormalization constant \( Z_m^\prime \). This fact coincides with the general property of the massive vector boson propagator (see Ref. (8), Chap.V). From Eqs. (4.23)-(4.25), it is easy to find that the longitudinal part in Eq. (4.22) can be renormalized as

\[ \prod_{\mu\nu}^{\alpha} = \frac{\alpha}{k^2[1 + \Pi_L(k^2)] - \mu^2 + i\varepsilon} = Z_3\alpha_R[1 + \Omega_R(k^2)]\Delta_R(k^2) \] (4.31)

where
\[ \Delta_R(k^2) = \frac{1}{k^2[1 + \Omega_R(k^2)] - \mu_R^2 + i\varepsilon} \] (4.32)

which appears in Eq. (4.28) and the renormalization constant \( Z_3^\prime \) can be expressed as

\[ Z_3^\prime = [1 + \frac{\mu_R^2}{\mu^2}(1 - \bar{Z}_3)]^{-1} \] (4.33)

If choosing \( \nu = \mu_R \), we have
\[ Z_3^\prime = \bar{Z}_3 \] (4.34)

5. GAUGE-INDEPENDENCE AND UNITARITY

This section serves to prove that the S-matrix elements evaluated by the massive gauge field theory are independent of the gauge parameter. That is to say, the gauge-dependent spurious pole appearing in the ghost particle propagator and the longitudinal part of the gauge boson propagator as shown in Eqs. (4.10) and (4.14) would not contribute to the S-matrix elements. This fact just ensures the unitarity of the S-matrix. According to the reduction formula\(^\text{[11]}\), the S-matrix elements can be obtained from the corresponding Green’s functions. So, we first examine how the Green’s functions are dependent on the gauge parameter.

Let us start from the generating functional of Green’s functions given in Eqs. (2.4) and (2.5). Since the fermion field in the generating functional is not related to the gauge parameter, for simplicity of statement, we will omit the fermion field functions in the generating functional and rewrite the functional in the form

\[ Z[J, \bar{K}, K] = \frac{1}{\hbar} \int \mathcal{D}[A, \bar{C}, C] \exp\{iS + i \int d^4x [-\frac{1}{2}\left(\partial^\mu A^a_{\mu}\right)^2 + J^a_{\mu} A^a_{\mu} + \bar{K}^a C^a + \bar{C}^a K^a + i \int d^4xd^4y \bar{C}^a(x) M^{ab}(x, y) C^b(y)\} \] (5.1)
where

\[ S = \int d^4x \left[ -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} m^2 A^{\mu} A_{\mu} \right] \] (5.2)

and

\[ M^{ab}(x, y) = \partial^u [D^{ab}_u(x) \delta^4(x - y)] \] (5.3)

in which \( D^{ab}_u(x) \) was defined in Eqs. (2.6) and (2.7).

When we make the following translation transformations in Eq. (5.1)

\[ C^\alpha(x) \to C^\alpha(x) - \int d^4y (M^{-1})^{ab}(x, y) K^b(y) \]
\[ \bar{C}^\alpha(x) \to \bar{C}^\alpha(x) - \int d^4y K^b(y) (M^{-1})^{ba}(y, x) \] (5.4)

and complete the integration over the ghost field variables, Eq. (5.1) will be expressed as

\[ Z[J, \bar{K}, K] = e^{-i \int d^4x d^4y \bar{K}^a(x)(M^{-1})^{ab}(x, y, \delta/i\delta J) K^b(y) Z[J]} \] (5.5)

where \( Z[J] \) is the generating functional without the external sources of ghost fields\[8,12\].

\[ Z[J] = \frac{1}{N} \int \mathcal{D}(A) \Delta_F[A] \exp \left\{ iS + i \int d^4x \left[ -\frac{1}{2\alpha} (\partial^\mu A^\alpha_{\mu})^2 + J^{\alpha \mu} A^\alpha_{\mu}\right] \right\} \] (5.6)

in which

\[ \Delta_F[A] = det M[A] \] (5.7)

here the matrix \( M[A] \) was defined in Eq. (5.3). From Eq. (5.5), we may obtain the ghost particle propagator in the presence of the external source \( J \)

\[ i\Delta^{ab}[x, y, J] = \frac{\delta^2 Z[J, \bar{K}, K]}{\delta K^b(x) \delta \bar{K}^a(y) |_{\bar{K} = K = 0}} = i(M^{-1})^{ab}(x, y, \delta/i\delta J) Z[J] \] (5.8)

The above result allows us to rewrite the W-T identity in Eq. (4.1) in terms of the generating functional \( Z[J] \) when completing the derivative with respect to \( u^{\alpha \nu}(y) \) and setting \( K^\alpha(x) = \bar{K}^\alpha(y) = 0 \),

\[ \frac{1}{\alpha} \partial^\mu x \delta Z[J] + \int d^4y J^{\alpha \nu}(y) D^{bd}_\mu[y, \delta/i\delta J](M^{-1})^d_a(y, x, \delta/i\delta J) Z[J] = 0 \] (5.9)

where

\[ D^{bd}_\mu(y) = D^{bd}_\mu(y) - \frac{\mu^2}{\Box} \partial_\mu \delta^{bd} \] (5.10)

is the ordinary covariant derivative. On completing the differentiations with respect to the source \( J \), Eq. (5.9) reads

\[ \frac{1}{N} \int \mathcal{D}[A] \Delta_F[A] \exp \left\{ iS + i \int d^4x \left[ -\frac{1}{2\alpha} (\partial^\mu A^\alpha_{\mu})^2 + J^{\alpha \mu} A^\alpha_{\mu}\right] \right\} \times \left[ \int d^4y J^{\alpha \nu}(y) D^{bd}_\mu(y)(M^{-1})^d_a(y, x, \delta/i\delta J) \right] = 0 \] (5.11)

By making use of Eqs. (5.3), (5.8) and (5.10), the ghost equation shown in Eq. (4.2) may be written as

\[ M^{ac}[x, \delta/i\delta J](M^{-1})^{cb}[x, y, \delta/i\delta J] Z[J] = \delta^{ab} \delta^4(x - y) Z[J] \] (5.12)

When the source \( J \) is turned off, we get

\[ M^{ac}(x) \Delta^{cb}(x - y) = \delta^{ab} \delta^4(x - y) \] (5.13)
This equation only affirms the fact that the ghost particle propagator is the inverse of the matrix $M$.

Now we are in a position to describe the proof of the unitarity mentioned in the beginning of this section. To do this, it is suitable to use the generating functional written in Eq. (5.6) and the W-T identity shown in Eq. (5.11) because the S-matrix only has gluon external lines, without any ghost particle external line to appear. For simplifying statement of the proof, in the following, we use the matrix notation to represent the integrals. In this notation, Eqs. (5.6) and (5.11) are respectively represented as

$$Z[J] = \frac{1}{N} \int \mathcal{D}(A) \Delta F[A] e^{i[S[A] - \frac{1}{4} F^2 + J \cdot A]}$$

and

$$\frac{1}{N} \int \mathcal{D}(A) \Delta F[A] e^{i[S[A] - \frac{1}{4} F^2 + J \cdot A]} [J_b D_{bc}(M_F^{-1})]_{ca} - \frac{1}{\sqrt{\alpha}} F_a = 0$$

where we have defined

$$F_a \equiv \frac{1}{\sqrt{\alpha}} \partial^\mu A^a_\mu(x)$$

with $F$ corresponding to the gauge $\alpha$, the subscript $a, b$ or $c$ stands for the color and/or the Lorentz indices and the space-time variable, and the repeated indices imply summation and/or integration.

Let us consider the generating functional in the gauge $\alpha + \Delta \alpha$ where $\Delta \alpha$ is taken to be infinitesimal

$$Z[J]_{\alpha + \Delta \alpha} = \frac{1}{N} \int \mathcal{D}(A) \Delta F[A] e^{i[S[A] - \frac{1}{4} (F + \Delta F)^2 + J \cdot A]}$$

In the above,

$$e^{-\frac{i}{4} (F + \Delta F)^2} = e^{-\frac{i}{4} F^2} [1 + \frac{i \Delta \alpha}{2 \alpha} F^2]$$

$$\Delta F + \Delta F[A] = det M_F + \Delta F$$

According to the definitions given in Eqs. (5.3), (2.6) and (2.7), it is seen that

$$M_{F + \Delta F}^{ab} = M_F^{ab} + \delta^{ab} \Delta \alpha m^2$$

Therefore,

$$\Delta F + \Delta F[A] = det[M_F(1 + \Delta \alpha m^2 M_F^{-1})] = det M_F e^{Tr ln(1 + \Delta \alpha m^2 M_F^{-1})} = \Delta F[A][1 + \Delta \alpha m^2 Tr M_F^{-1}]$$

Upon substituting Eqs. (5.18) and (5.21) into Eq. (5.17), we obtain

$$Z_{F + \Delta F}[J] = \frac{1}{N} \int \mathcal{D}(A) \Delta F[A] e^{i[S[A] - \frac{1}{4} F^2 + J \cdot A]} \times \{1 + \frac{i \Delta \alpha}{2 \alpha} F^2 + \Delta \alpha m^2 Tr M_F^{-1}\}$$

For further derivation, it is necessary to employ the W-T identity described in Eq. (5.15). Acting on Eq. (5.15) with the operator $\frac{1}{2} \Delta \alpha \alpha^{-\frac{i}{2}} F_a \{\frac{\delta}{\delta J}\}$ and noticing

$$i F_a \{\frac{\delta}{\delta J}\} J_b e^{i J_c A_c} = i F_a \{\frac{\delta}{\delta J}\} \frac{\delta}{\delta A_a} e^{i J_c A_c} = \frac{\delta}{\delta A_a} F_a \{\frac{\delta}{\delta J}\} e^{i J_c A_c}$$

$$= e^{i J \cdot A} \{i J_b F_a[A] + \frac{\delta F_a[A]}{\delta A_a}\}$$

we have
\[
\frac{1}{N} \int \mathcal{D}(A) \delta F \{ A \} e^{i[S[A] - \frac{1}{2} F^2 + J \cdot A]} \frac{\delta A}{\sqrt{\alpha}} \{ i J_a D_{bc} [A] (M_F^{-1})_{ca} F_a [A] \\
+ \frac{\delta F}{\delta A_b} D_{bc} [A] (M_F^{-1})_{ca} - \frac{i}{\sqrt{\alpha}} F^2 \} = 0
\] (5.24)

Adding Eq. (5.24) to Eq. (5.22) and considering where Eqs. (5.10) and (5.3) has been used, one may reach the following result

\[
Z_{F+\Delta F} [J] = \frac{1}{N} \int \mathcal{D}(A) \Delta F \{ A \} e^{i[S[A] - \frac{1}{2} F^2 + J \cdot A]} \{ 1 + i \Delta S + i J^a \left[ \frac{\Delta \alpha}{2 \sqrt{\alpha}} D_{ab} (M_F^{-1})_{bc} F_c \right] \} = \frac{1}{N} \int \mathcal{D}(A) \Delta F \{ A \} e^{i[S[A] + \Delta S - \frac{1}{2} F^2 + J \cdot A']}
\] (5.26)

where

\[
A'_a = A_a + \frac{\Delta \alpha}{2 \sqrt{\alpha}} D_{ab} (M_F^{-1})_{bc} F_c
\] (5.27)

\[
\Delta S = - \frac{i \Delta \alpha}{2 \alpha} Tr [1 + \mu^2 M_F^{-1}]
\] (5.28)

in which

\[
Tr M_F^{-1} = \int d^4 x \Delta^{\alpha a} (0) = \text{const.}
\] (5.29)

Since the $\Delta S$ is a constant (even though it is infinite), it may be taken out from the integral sign and put in the normalization constant $N$. Thus, Eq. (5.26) will finally be represented as

\[
Z_{F+\Delta F} [J] = \frac{1}{N} \int \mathcal{D}(A) \Delta F \{ A \} e^{i[S[A] - \frac{1}{2} F^2 + J \cdot A']}
\] (5.30)

In comparison of Eq. (5.30) with Eq. (5.14), it is clear to see that the difference between the both generating functionals merely comes from the vector potentials in the external source terms, while, the remaining terms belonging to the dynamical part in the both generating functionals are completely the same. This indicates that any change of the gauge parameter can only lead to different field functions in the source terms of the generating functional. According to the equivalence theorem, the different field functions in the source terms does not influence on the S-matrix elements, it can only affect the renormalization of external lines for the Green’s functions and wave functions[8–12]. This point will be explained more specifically in the following.

The n-point gluon Green’s functions computed from the generating functionals $Z_F [J]$ and $Z_{F+\Delta F} [J]$ are represented in the position space as

\[
G_F (x_1, x_2, \cdots, x_n) = \langle 0 | T \{ A(x_1) A(x_2) \cdots A(x_n) \} | 0 \rangle
\] (5.31)

\[
G_{F+\Delta F} (x_1, x_2, \cdots, x_n) = \langle 0 | T \{ A'(x_1) A'(x_2) \cdots A'(x_n) \} | 0 \rangle
\] (5.32)

where $A(x_i)$ and $A'(x_i)$ denote the field operators corresponding the gauges $F$ and $F + \Delta F$ and $x_i$ designates the space-time for the i-th particle. Here and afterward, we adopt the matrix notation to represent the field functions and Green’s functions, therefore, the Lorentz and color indices are suppressed. In light of the renormalization of the field operators[11]

\[
A^{a\mu} (x) = Z_F^\frac{1}{2} A_R^{a\mu} (x)
\] (5.33)

\[
A'^{a\mu} (x) = Z_{F+\Delta F}^\frac{1}{2} A_R^{a\mu} (x)
\] (5.34)
where $R$ marks the renormalized quantities and $Z_F$ and $Z_{F+\Delta F}$ are the renormalization constants given in the gauges $F$ and $F + \Delta F$ respectively, the renormalization of the above Green's functions, in the momentum space, can be written as

\[
G_F(k_1, k_2, ..., k_n) = Z_F^{\frac{1}{2}} G^R_F(k_1, k_2, ..., k_n)
\]  

(5.35)

\[
G_{F+\Delta F}(k_1, k_2, ..., k_n) = Z_{F+\Delta F}^{\frac{1}{2}} G^R_{F+\Delta F}(k_1, k_2, ..., k_n)
\]  

(5.36)

It is well-known that the Green's functions computed from the generating functionals $Z_F[J]$ and $Z_{F+\Delta F}[J]$ have different external lines, but the same internal structure. Therefore, by the equivalence theorem and noticing Eqs. (5.35) and (5.36), we have

\[
\prod_{i=1}^n \lim_{k_i^2 \to m_R^2} (k_i^2 - m_R^2) G_{F+\Delta F}(k_1, k_2, ..., k_n)
\]

(5.37)

\[
= Z_{F+\Delta F}^{\frac{1}{2}} / Z_F^{\frac{1}{2}} \prod_{i=1}^n \lim_{k_i^2 \to m_R^2} (k_i^2 - m_R^2) G_F(k_1, k_2, ..., k_n)
\]

The propagators given in the gauges $F$ and $F + \Delta F$ are respectively renormalized in such a manner

\[
D_F(k_i) = Z_F D_R(k_i) = D^T_F(k_i) \hat{T} + R_F(k_i)
\]  

(5.38)

\[
D_{F+\Delta F}(k_i) = Z_{F+\Delta F} D_R(k_i) = D^T_{F+\Delta F}(k_i) \hat{T} + R_{F+\Delta F}(k_i)
\]  

(5.39)

where $\hat{T}$ denotes the transverse projector, $D^T_F(k_i)$ and $D^T_{F+\Delta F}(k_i)$ come from the transverse parts of the propagators $D_F(k_i)$ and $D_{F+\Delta F}(k_i)$ and have a physical pole at $k_i^2 = m_R^2$

\[
D^T_F(k_i) = \frac{Z_F}{k_i^2 - m_R^2}
\]  

(5.40)

\[
D^T_{F+\Delta F}(k_i) = \frac{Z_{F+\Delta F}}{k_i^2 - m_R^2}
\]  

(5.41)

while, $R_F(k_i)$ and $R_{F+\Delta F}(k_i)$ represent the remaining parts of the propagators $D_F(k_i)$ and $D_{F+\Delta F}(k_i)$ which are regular at the physical pole, therefore,

\[
\lim_{k_i^2 \to m_R^2} (k_i^2 - m_R^2) R_F(k_i) = \lim_{k_i^2 \to m_R^2} (k_i^2 - m_R^2) R_{F+\Delta F}(k_i) = 0
\]  

(5.42)

According to the reduction formula, the S-matrix elements for multi-gluon scattering which are evaluated in the gauges $F$ and $F + \Delta F$ may be respectively represented as

\[
S_F(k_1, ..., k_n) = Z_F^{\frac{1}{2}} \prod_{i=1}^n \lim_{k_i^2 \to m_R^2} A_0(k_i)(k_i^2 - m_R^2) G_F(k_1, ..., k_n)
\]

(5.43)

\[
S_{F+\Delta F}(k_1, ..., k_n) = Z_{F+\Delta F}^{\frac{1}{2}} \prod_{i=1}^n \lim_{k_i^2 \to m_R^2} A_0(k_i)(k_i^2 - m_R^2) G_{F+\Delta F}(k_1, ..., k_n)
\]

(5.44)

where $A_0(k_i)$ is the free wave function of the $i$-th gluon which represents the state of transverse polarization and therefore is free of the gauge parameter. On substituting Eq. (5.37) into Eq. (5.44) and noticing Eq. (5.43), we find
which shows that the S-matrix elements are independent of the gauge parameter. The gauge-independence of the S-matrix elements implies nothing but the unitarity of the S-matrix because the gauge-dependent spurious pole which appears in the longitudinal part of the gluon propagator and the ghost particle propagator and represent the unphysical excitation of the massive gauge field in the intermediate states must eventually be cancelled out in the S-matrix elements. From the construction of the theory, the cancellation seems to be natural. In fact, in the massive Yang-Mills Lagrangian, all the unphysical degrees of freedom have been restricted by the constraint conditions imposed on the gauge field and the gauge group. When these constraint conditions are incorporated in the Lagrangian, the theoretical principle we based on would automatically guarantee the cancellation of the unphysical excitations. This conclusion drawn from the above general proof can be checked by practical perturbative calculations, as will be demonstrated in a subsequent paper.

6. REMARKS

In the last section, we would like to make some remarks on the BRST-external source terms introduced in Eq. (3.4). Ordinarily, to guarantee the BRST-invariance of the source terms, the composite field functions $\Delta \Phi_i$ are required to have the nilpotency property $\delta \Delta \Phi_i = 0$ under the BRST-transformations [8–11]. For the massless gauge field theory, as one knows, the composite field functions are indeed nilpotent. This nilpotency property is still preserved for the massive gauge field theory established in the Landau gauge because in this gauge the BRST-transformations are the same as for the massless theory. However, for the massive gauge field theory set up in the general gauges, we find $\delta \Delta \Phi_i \neq 0$, the nilpotency loses, since in these gauges the ghost field acquires a spurious mass $\mu$. In this case, as pointed out in section 2, to ensure the BRST-invariance of the source terms, we may simply require the sources $u_i$ to satisfy the condition denoted in Eq. (3.7). The definition in Eq. (3.7) for the sources is reasonable. Why say so? Firstly, we note that the original W-T identity formulated in Eq. (3.2) does not involve the BRST-sources. This identity is suitable to use in practical applications. Introduction of the BRST source terms in the generating functional is only for the purpose of representing the identity in Eq. (3.2) in a convenient form, namely, to represent the composite field functions in the identity in terms of the differentials of the generating functional with respect to the corresponding sources. For this purpose, we may start from the generating functional defined in Eq. (3.4) to re-derive the identity in Eq. (3.2). In doing this, it is necessary to require the source terms $u_i \Delta \Phi_i$ to be BRST-invariant so as to make the derived identity coincide with that given in Eq. (3.2). How to ensure the source terms to be BRST-invariant? If the composite field functions $\Delta \Phi_i$ are nilpotent under the BRST-transformation, $\delta \Delta \Phi_i = 0$, the BRST-invariance of the source terms is certainly guaranteed. Nevertheless, the nilpotency of the functions $\Delta \Phi_i$ is not a uniquely necessary condition to ensure the BRST-invariance of the source terms, particularly, in the case where the functions $\Delta \Phi_i$ are not nilpotent. In the latter case, considering that under the BRST-transformations, the functions $\Delta \Phi_i$ can be, in general, expressed as $\delta \Delta \Phi_i = \xi \Phi_i$ where the $\Phi_i$ are some nonvanishing functions, we may alternatively require the sources $u_i$ to satisfy the condition shown in Eq. (3.7) so as to guarantee the source terms to be BRST-invariant. Actually, this is a general trick to make the source terms to be BRST-invariant in spite of whether the functions $\Delta \Phi_i$ are nilpotent or not. As mentioned before, the sources themselves have no physical meaning. They are, as a mathematical tool, introduced into the generating functional just for performing the differentiations. For this purpose, only a certain algebraic and analytical properties of the sources are necessarily required. Particularly, in the differentiations, only the infinitesimal property of the sources are concerned. Therefore, the sources defined in Eq. (3.7) are mathematically suitable for the purpose of introducing them. The reasonability of the arguments stated above for the source terms is substantiated by the correctness of the W-T identities derived in section 4. Even though the identities in Eqs. (4.1) and (4.2) are derived from the W-T identity in Eq. (3.8) which is represented in terms of the differentials with respect to the BRST-sources, they give rise to a correct relation between the gluon propagator and the ghost particle one as shown in Eq. (4.8). The correctness of the relation in Eq. (4.8) may easily be verified by the free propagators written in Eqs. (4.10)
and (4.14). These propagators were derived in paper I by employing the perturbation method, without concerning the BRST-source terms and the nilpotency of the BRST-transformations. A powerful argument of proving the correctness of the way of introducing the BRST-sources is that after completing the differentiations in Eq. (3.8) and setting the BRST-sources to vanish, we immediately obtain the W-T identity in Eq. (3.2) which is irrelevant to the BRST-sources. Therefore, all identities or relations derived from the W-T identity in Eq. (3.8) are completely the same as those derived from the identity in Eq. (3.2). An important example of showing this point will be presented in Appendix where the identity in Eq. (5.11) which was derived from the W-T identities in Eqs. (3.8) and used to prove the unitarity of the theory can equally be derived from the generating functional in Eq. (5.6) which does not involve the BRST-sources.

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8. APPENDIX

To confirm the correctness of the identity given in Eq. (5.11), we derive the identity newly by starting from the generating functional written in Eq. (5.6). The generating functional in Eq. (5.6) was directly derived from the massive Yang-Mills Lagrangian by the Faddeev-Popov method of quantization[12]. Let us make the ordinary gauge transformation

$$\delta A_{\mu}^{a} = D_{\mu}^{ab} \theta^{b}$$

(A1)

to the generating functional in Eq. (5.6). Considering the gauge-invariance of the functional integral, the integration measure and the functional $\Delta_{F}[A] = \det M[A]$, we get[8,12]

$$\delta Z[J] = \frac{1}{D} \int D(A) \Delta_{F}[A] \int d^{4}y \{ J^{\mu}_{\nu}(y) + m^{2} A^{\mu}(y) \}
- \frac{1}{\alpha} \partial^{\mu} A_{\nu}^{a}(y) \partial^{\nu} D_{\mu}^{bc}(y) \theta^{c}(y) \exp \{ iS + i \int d^{4}x [- \frac{1}{2\alpha} (\partial^{\mu} A_{\nu}^{a})^{2} + J^{\mu\nu} A_{\nu}^{a}] \} = 0$$

(A2)

According to the well-known procedure, the group parameter $\theta^{a}(x)$ in Eq. (A2) may be determined by the following equation[5,9]

$$M^{ab}(x) \theta^{b}(x) \equiv \partial_{\nu}^{\mu}(D_{\mu}^{ab}(x) \theta^{b}(x)) = \lambda^{a}(x)$$

(A3)

where $\lambda^{a}(x)$ is an arbitrary function. When setting $\lambda^{a}(x) = 0$, Eq. (A3) will be reduced to the constraint condition on the gauge group (the ghost equation) which is used to determine the $\theta^{a}(x)$ as a functional of the vector potential $A_{\nu}^{a}(x)$. However, when the constraint condition is incorporated into the action by the Lagrange undetermined multiplier method to give the ghost term in the generating functional, the $\theta^{a}(x)$ should be treated as arbitrary according to the spirit of Lagrange multiplier method. That is why we may use Eq. (A3) to determine the functions $\theta^{a}(x)$ in terms of the function $\lambda^{a}(x)$. From Eq. (A3), we solve

$$\theta^{a}(x) = \int d^{4}x (M^{-1})^{ab}(x - y) \lambda^{b}(y)$$

(A4)

Upon substituting the above expression into Eq. (A2) and then taking derivative of Eq. (A2) with respect to $\lambda^{a}(x)$, we obtain

$$\frac{1}{D} \int D(A) \Delta_{F}[A] \int d^{4}y \{ J^{\mu}_{\nu}(y) + m^{2} A^{\mu}(y) \}
- \frac{1}{\alpha} \partial^{\mu} A_{\nu}^{a}(y) \partial^{\nu} D_{\mu}^{bc}(y) (M^{-1})^{ab}(y - x) \exp \{ iS + i \int d^{4}x [- \frac{1}{2\alpha} (\partial^{\mu} A_{\nu}^{a})^{2} + J^{\mu\nu} A_{\nu}^{a}] \} = 0$$

(A5)

According to the expression denoted in Eq. (2.7) and the identity $f^{bcd} A^{c\mu} A_{\mu}^{d} = 0$, it is easy to see
\[ A^{\mu y} D_{\mu}^{bc} (y)(M^{-1})^{ca}(y - x) = A^{\mu y} \partial_{\mu}^{y}(M^{-1})^{ba}(y - x) \]  

(A6)

By making use of the relation in Eq. (5.10), the definition in Eq. (5.3) and the equation in Eq. (5.12), we deduce

\[ \frac{1}{\alpha} \partial_{\nu}^y A^\nu_x(y) \partial_{\mu}^y D_{\mu}^{bc}(y)(M^{-1})^{ca}(y - x) = \frac{1}{\alpha} \partial_{\nu}^y A^\nu_x(y) \delta^4(x - y) - m^2 \partial_{\nu}^y A^\nu_x(y)(M^{-1})^{ba}(y - x) \]  

(A7)

On inserting Eqs. (A6) and (A7) into Eq. (A5), we obtain an identity which is exactly identical to that given in Eq. (5.11) although in the above derivation, we started from the generating functional without containing the ghost field functions and the BRST-sources and, therefore, the derivation does not concern the nilpotency of the composite field functions appearing in the BRST-source terms. This fact indicates that the W-T identities derived in section 3 are correct and hence the procedure of introducing the BRST-invariant source terms into the generating functional is completely reasonable.

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