Time-reversal Characteristics of Quantum Normal Diffusion

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(Dated: January 12, 2013)

This paper concerns with the time-reversal characteristics of intrinsic normal diffusion in quantum systems. Time-reversible properties are quantified by the time-reversal test; the system evolved in the forward direction for a certain period is time-reversed for the same period after applying a small perturbation at the reversal time, and the separation between the time-reversed perturbed and unperturbed states is measured as a function of perturbation strength, which characterizes sensitivity of the time reversed system to the perturbation and is called the time-reversal characteristic. Time-reversal characteristics are investigated for various quantum systems, namely, classically chaotic quantum systems and disordered systems including various stochastic diffusion systems. When the system is normally diffusive, there exists a fundamental quantum unit of perturbation, and all the models exhibit a universal scaling behavior in the time-reversal dynamics as well as in the time-reversal characteristics, which leads us to a basic understanding on the nature of quantum irreversibility.

PACS numbers: 05.30.-d, 05.45.Mt, 03.65.-w

I. INTRODUCTION

The origin of irreversibility in real world should not be attributed only to the huge amount of the degrees of freedom, and we can expect that it can be caused by the complex dynamical structure inside the system [1, 2]. Indeed, in homogeneously extended quantum systems defined in an infinite-dimensional Hilbert space, a stationary irreversible dynamics such as stationary diffusion and a uni-directional stationary energy flow can be self-organized in the quantum systems even if the number of degrees of freedom is small [2, 3].

However, so far, the nature of wavefunctions describing dissipative states has not been well understood, although it must have very complex structure. The purpose of the present work is to explore the nature of the quantum state related to the irreversibility by using simple quantum systems with a few degrees of freedom. To be more specific, quantum irreversibility of diffusive states in some quantum systems is investigated.

The normal diffusion realized in homogeneously extended quantum systems provides a typical irreversible phenomenon in quantum system [4]. Here, by the term normal diffusion we mean that the spread of wavefunction which is represented by the second-order moment $M(t)$ of the coordinate increases in proportion to time as, $M(t) \propto t^2$. The normal diffusion is not always attributed to a random source coming from an infinite number of degrees of freedom forming the heat reservoir. Indeed, the normal diffusion can be self-induced in the quantum systems with only a few degrees of freedom such as coupled kicked rotor [7] and three-dimensional disordered systems without any hidden degrees of freedom [4, 8, 12].

The purpose of our paper is to explore the time-reversibility of quantum system exhibiting normal diffusion. To quantify the time-reversibility of the dynamics we measure the effect of the external perturbation on the time-reversed evolution. It also represents the sensitivity of the time-reversed evolution process to the external perturbation. There are various types of spreading of wavepacket in quantum dynamics. One extreme limit is the localized state with $M(t) \propto t^0$, and the other limit is ballistic motion with $M(t) \propto t^2$. The normal diffusion characterized by $M(t) \propto t^2$ is just on the boundary of two extreme types of dynamics. When we consider stationarity of the quantum dynamics the localized states and ballistic motions are not time-stationary process. On the other hand, the quantum normal diffusion corresponds to one of the quantum stationary states.

In our preliminary report [13], we demonstrated the result concerning with quantum normal diffusion. In the present paper we give more details of the results together with discussions for it. To examine the difference of the time-reversibility among the different types of quantum dynamics is an another purpose of the present paper.

Unfortunately, we do not have so many quantum systems of small number degrees of freedom which genuinely exhibit normal diffusion without the help of any classical random noise source. However, the number of quan-
tum systems exhibiting normal diffusion increases if we include the stochastic quantum systems, namely, quantum systems driven by the classical stochastic force, into the object of our investigation. We will also investigate the time-reversibility of such quantum systems exhibiting noise-induced normal diffusion 11, 13.

Fidelity is frequently used to measure the sensitivity of the quantum dynamics to the perturbation 17, 24. However, the fidelity is not a good measure to characterize the complexity and irreversibility of quantum motion linking to classical integrability because the fidelity has no counterparts in classical dynamics 10.

In the present paper, instead of fidelity, we propose the following time-reversal test to measure the time-irreversibility of quantum dynamics. First, we evolve an initial quantum state $|\Psi_0\rangle$ forward in time by operating the unitary operator $U^T$, where $T$ denotes the time step of the evolution, which we hereafter call reversal time. Next, we perturb the evolved state by $P$, and evolve the perturbed state backward in time by operating the time-reversed unitary operator $U^{-T}$. At $t = 2T$ the difference between the perturbed time-reversed state $|\Psi'_0\rangle = U^{-T}P|\Psi_0\rangle$ and the initial state $|\Psi_0\rangle$ is measured by the following normalized deviation as a function of the perturbation strength $\eta(>0)$,

$$\delta Q = \frac{<\Psi'_0|\hat{Q}|\Psi'_0> - <\Psi_0|\hat{Q}|\Psi_0>}{<\Psi_0|\hat{Q}|\Psi_0>},$$

(1)

where $\hat{Q}$ is an appropriate observable related to the coordinate in which the diffusion takes place. The quantity $\delta Q$ observed as a function of the strength $\eta$ of the perturbation $P$ is called the time-reversal characteristic.

The above time-reversal test is immediately related to the fidelity of the quantum systems frequently estimated in Loschmidt echo experiments, which measures the decoherence 17, 19, 21 by the distance between the quantum states with perturbed and unperturbed time-reversed dynamics 23.

The above time-reversal characteristics were introduced by one of the authors in 1995 as a measure quantifying the sensitivity of quantum dynamics to perturbation 20. This has been motivated by a remarkable stability of quantum chaos systems compared with its classical counterpart, which was first demonstrated by Soviet-Italy group 27, 28. A quantum measurement-theoretic discussion about the significance of the time-reversal tests is given in Sect. II and appendix C.

The main results in the present paper are summarized as the following remarkable universal features of time-reversal characteristic.

**Result 1:** In the time reversal characteristic of quantum normal diffusion there always exists a universal characteristic strength of $P$ called the least quantum perturbation unit (LQPU), which is proportional to Planck constant 23. For the perturbation strength larger than the LQPU the normal diffusion is irreversible in the sense that it completely loses its ability to return to the initial state.

**Result 2:** In deterministic quantum maps the time reversal characteristic of quantum normal diffusion converges to a universal curve independent of the details of the systems in the large limit of the reversal time. Moreover, this universality and the stationarity of diffusion process are unified to a universal scaling of the time-reversed dynamics itself. On the other hand, the way of convergence toward the universal characteristics depends on the details of the systems.

**Result 3:** The universal features observed for the deterministic quantum maps exactly hold also for the normal diffusion process of stochastically perturbed quantum maps if the strength of the stochastic random force is large enough to realize a sufficiently rapid normal diffusion.

We note here that the presence of LQPU was first claimed by one of the authors and its existence was confirmed for the time-reversal characteristics of standard map Ref. 20. Later the existence of LQPU was verified within the limitation of lowest order perturbation theory 29.

The above universal features in the quantum region are first discussed for deterministic quantum systems, but it turns out that they are also exactly valid for stochastic quantum systems. It should be remarked that what we mean by the term "deterministic" quantum diffusion is the normal diffusion induced by the genuine quantum dynamics of the system which does not contain any explicit classical random force. We distinguish it from the noise-induced diffusion exhibited by quantum system driven by classical random force such as quantum Brownian motions. Further it should be noted that we refer to the diffusion as even if the normal diffusion $M(t) \propto t$ is maintained only within a limited time range. Indeed, some quantum systems such as one-dimensional quantum kicked rotor cannot show diffusion dynamics over an infinitely long time. However, if the diffusion dynamics has a well-defined time range on which a well-behaved $t$-linear dependence is maintained, we can use it as an example of quantum normal diffusion.

As mentioned above, we add the noise-driven quantum systems to our menu, and compare the time-reversal characteristics with those in the deterministic normal diffusion systems. It is shown that the time-reversal characteristics significantly deviate from the universal one when the strength of the random force is not large enough.

We further extend our investigation of time-reversal characteristics to quantum dynamics different from the normal diffusion, such as localization, subdiffusion and ballistic motion. We discuss a seemingly paradoxical result that the localized states are more sensitive to the perturbation than the normally diffusive states. On the other hand, the ballistic motion is entirely time-reversible in a sense that the time-reversal characteristics approaches to zero as the reversal time $T$ increases.

The outline of the present paper is as follows. In Sect. II model systems examined in the present work are introduced. In addition, we demonstrate numerically typical
examples of quantum normal diffusion and confirm its Markovian property. Section III is the core part of the present work in which the main results of time-reversal characteristics summarized as the Result 1 and Result 2 are clarified for deterministic quantum systems showing normal diffusion. In Sect. IV the main results mentioned above are examined for the stochastic quantum systems, which exhibit normal diffusion if it is driven by classical random force. The last section is devoted to summary and discussion.

Appendixes are devoted to providing a lot of numerical data which supplements the main results from several points of view. Accordingly the busy readers can get the essential claims of the present work by reading only the main text. Those who are very interested in the topics of dissipation are recommended to consult the appendices.

II. MODEL SYSTEMS EXHIBITING QUANTUM DIFFUSION

A. Model systems

Typical examples of deterministic quantum systems with normal diffusion are seen in quantum chaos systems, quantum disordered systems and so on. In the present paper, we treat quantum systems defined on discrete-time, which are often called quantum maps, because it allows a very long-time evolution with numerical accuracy. We also deal noise-driven quantum maps, which exhibit normal diffusion, in comparison with the deterministic quantum maps. We will discuss the time-continuous quantum systems with normal diffusion in a separate paper [31].

As the quantum map, we use the following form of unitary operator,

\[ \hat{U} = e^{-i\frac{H_0(\hat{p})}{\hbar}} e^{-i\frac{V(\hat{q})}{\hbar}} e^{-i\frac{H_0(\hat{p})}{\hbar}}, \]  

where \( H_0(\hat{p}) \) and \( V(\hat{q}) \) represent translational-(or rotational-)kinetic energy and potential energy, respectively. Here \( \hat{p} \) and \( \hat{q} \) are momentum and positional operators, respectively.

1. Normally Diffusive deterministic quantum maps

Standard map (SM) is a typical deterministic quantum map which is given by

\[ H_0(p) = \frac{\hat{p}^2}{2}, \quad V(q) = K \cos \hat{q}. \]  

An important feature of SM is that it has the classical limit for \( \hbar \to 0 \). The position space is taken defined as \( 0 \leq q \leq 2\pi \), and a periodic boundary condition \( \Psi_t(q) = \Psi_t(q + 2\pi) \) is imposed on the wavefunction \( \Psi_t(q) = U^t\Psi_0(q) \) \((t \geq 0)\).

In the classical limit \( \hbar \to 0 \), its dynamics shows nearly integral motion on Kolmogorov-Arnold-Moser (KAM) torus for the nonlinear parameter \( K \ll 1 \), while it exhibits an unlimited normal diffusion for \( K \gg 1 \) in the momentum space \(-\infty < p < +\infty\), where the diffusion constant \( D = D_{cl} \sim K^2/2 \) for \( K \gg 1 \). The quantum counterpart exhibits the classical normal diffusion within a limited time range \( t \ll D/\hbar^2 \); the diffusion is suppressed and localized for \( t \gg D/\hbar^2 \). In the present paper, we investigate time-reversal characteristics of normal diffusion for \( t \ll D/\hbar^2 \), which can be made arbitrarily large by taking \( \hbar \ll 1 \) for \( K > 3 \).

We consider a quantum disordered system on one-dimensional discrete lattice \( q \in \mathbb{Z} \) as the next example. Unlike SM, it has no classical counterpart. The system is given by

\[ H_0(p) = \Delta \cos(q/\hbar) = \frac{e^{\beta\partial q} + e^{-\beta\partial q}}{2}, \]

\[ V(q) = \nu_q, \]

where \( H_0(p) \) describes hopping between nearest neighbour sites and the on-site potential \( \nu_q \) takes random value uniformly distributed over the range \([-W,W]\). The transfer operator \( \cos(p/\hbar) = (e^{\beta\partial q} + e^{-\beta\partial q})/2 \) evidently does not have a classical limit, and so the model has no classical counterpart as mentioned above. The unitary evolution operator given by Eq. (2) reduces to

\[ \hat{U}_{AM} = e^{-i(\cos(p/\hbar)+\nu_q)/\hbar}, \]

in the limit \( \Delta \to 0 \) with \( W/\Delta = \text{const.} \), where the Hamiltonian \( H_0 + \nu_q/W \) is nothing more than the one-dimensional Anderson model. Thus the model given by Eq. (4) is a map version of the one-dimensional Anderson model, and so we refer to it as the Anderson map (AM), and we take the special choice \( \Delta = 1 \) hereafter [31].

As commonly seen in the one-dimensional Anderson model, the initially localized wavepacket spreads in the \( q \)-space but is finally localized as time elapses. However, it was shown that the quantum motion exhibits a well-defined normal diffusion in \( q \)-space when we replace the static disordered potential \( V(q) \) with the following harmonic time-dependent one [31],

\[ V(q,t) = \nu_q \{1 + \sum_{i=1}^{M} \epsilon_i \cos \omega_i t\}, \]

where \( M \) is the number of the frequency component and the strength of the perturbation, respectively, and we call the model with potential given by Eq. (6) the perturbed Anderson map (PAM).

The external harmonic perturbation is equivalent to the coupling with quantum linear oscillators, and the model with Eq. (6) can be transformed into a one-dimensional discrete lattice system coupled with \( M \).
quantum linear oscillators [3]. In the following numerical calculation, we take \( W = 1.0 \) or 0.5 and \( \epsilon_i = \frac{1}{\sqrt{27}} \), for simplicity, and take incommensurate numbers \( \omega_i \sim O(1) \) as the frequency set.

2. Normally diffusive stochastic quantum maps

Unfortunately, we do not know many deterministic quantum system which shows rigorous normal diffusion without any stochastic perturbation. However, once we turn our attention to quantum systems perturbed by a stochastic random force varying at random from step to step, many one-dimensional quantum systems show normal diffusion. For example, if the potential in Eqs. [3] and [4] are perturbed by a stochastic random perturbation, i.e.,

\[
V(q,t) = V(q)(1 + \epsilon n_t), \quad < n_t n_{t'} > = \delta_{tt'}, \quad (7)
\]

then the corresponding SM and AM exhibit normal diffusion irrespective of the magnitude of \( \epsilon \) (if \( K \) is large enough).

Moreover, it is well-known that one-dimensional Hamiltonian \( H = \cos(\hat{p}/\hbar) + V(q,t) \) driven by spatio-temporal-uncorrelated noise written as

\[
V(q,t) = \epsilon n_{q,t}, \quad < n_{q,t} n_{q',t'} > = \delta_{qq'} \delta_{tt'}, \quad (8)
\]

shows normal diffusion. This model has been introduced by Haken and coworkers in the context of exciton migration in solid, and the diffusive behavior has been well-investigated. In addition to the stochastically driven SM and AM, we use the quantum map version of the Haken-Strobl model [14] as a prototype with normal diffusion. This model has been introduced by Haken and coworkers in the context of exciton migration in solid, and the diffusive behavior has been well-investigated. In addition to the stochastically driven SM and AM, we use the quantum map version of the Haken-Strobl model [14] as a prototype with normal diffusion. This model has been introduced by Haken and coworkers in the context of exciton migration in solid, and the diffusive behavior has been well-investigated.

3. Subdiffusive and localizing quantum maps, superdiffusive quantum maps

The quantum maps introduced above also exhibit subdiffusive behavior, \( M(t) \propto t^{\alpha}(0 < \alpha < 1) \), and localization (\( \alpha = 0 \)), depending on the value of the control parameters \( K, \hbar, \) and \( \epsilon_i \). Moreover, we can obtain the Bloch map by replacing the random potential with a periodic one in AM, in which the wavepacket shows a ballistic motion. We compare the time-reversal characteristics in the cases of localization, subdiffusion and ballistic motion, with those of normal diffusion exhibited by the models in the previous subsection.

B. Typical quantum diffusions

Let \( x \) be the coordinate of the space in which the diffusion takes place; namely \( x = p \) in SM and \( x = q \) in PAM and HM. We monitor time-dependence of mean square displacement (MSD) of wavepacket initially localized on a point at \( t = 0 \), i.e. \( < n | \Psi_0 > = \delta_{n,0} \).

\[
M(t) = < (\Delta x)^2 > = \sum_x P(x,t)(x- < x >)^2, \quad (10)
\]

where \( < ... > \) indicates quantum mechanical average for the quantum mechanical probability \( P(x,t) = |\psi(x,t)|^2 \), for example, \( < x >= \sum_x x P(x,t) \). In addition, in cases of AM and HM, we also take an ensemble average \( < ... >_Q \) over different on-site randomness and/or stochastic random force although it is not shown in equations to avoid complication of the symbols. In Fig.1 we show typical examples of the quantum normal diffusion observed for SM and PAM. The diffusion constant \( D \) increases with the increase of the control parameters: \( K \) in SM and \( \epsilon/W \) in PAM. In the realization of quantum normal diffusion, the first basic question is whether the square of quantum wave function can be identified with a classical probability distribution. First of all, we examine the Bayesian property of the quantum probability distribution by investigating whether or not the quantum transition probability \( P(t, x \rightarrow x') = | < x | U^t | x' > |^2 \) at time \( t \) satisfies Bayes theorem

\[
P_s(t, x \rightarrow x') = \sum_{x''} P(s, x \rightarrow x'') P(t-s, x'' \rightarrow x'). \quad (11)
\]

The well established the Bayesian property implies that the time-evolution of the wavepacket of the quantum mappings obeys a stationary Markov process. In Fig.2(a) and (b), instead of time-dependence of probability distribution, we compare the time-dependence of MSD computed using the probability for the actual time evolution,
i.e., $M(t) = \sum_{x'}(x' - x)^2P(t, x \rightarrow x')$, and MSD computed using the probability of Eq. (11), i.e.,

$$M_S(t) = \sum_{x'} P_s(t, x \rightarrow x')(x' - <x>_s)^2,$$  \hspace{1cm} (12)

where $<...>_s$ indicates quantum mechanical average for the probability $P_s(t, x \rightarrow x')$. Figure 2(c) shows an index $X(s)$ as a function of the intermediate time $s$ defined as,

$$X(s) = \frac{M_s(t)}{M(t)}.$$  \hspace{1cm} (13)

Figure 2(c) shows that $X(s)$ is equal to unity except for tiny fluctuation independently of the intermediate time $s$, which indicates that these pure quantum evolution processes can be well approximated by a stationary Markov process. We stress that in the present test we do not take the ensemble average over the random on-site potential. In an ordinary quantum process, Eq. (11) does not in general hold because of quantum coherence. Even if a measurement of $x$ is done at the intermediate time $s$ and so the wavefunction shrinks to one of the eigenstates of $x$, the Bayesian property do not in general hold for quantum probability. A surprising fact is that in the present cases the Bayesian property is valid without any measurement. This means that the conversion of a quantum wavefunction into a classical probability occurs spontaneously in the quantum evolution process.

On the other hand, such stationary Markovian property can not be observed if localization or subdiffusion takes place instead of normal diffusion. The details of the result are given in appendix D 4.

III. NORMALLY DIFFUSIVE DETERMINISTIC QUANTUM MAPS

This is the main section of this paper.

A. Time-reversal test

As seen in the last section, a normal diffusion obeying a stationary Markov process is realized in some simple quantum map systems. The presence of normal diffusion generally suggests that the underlying quantum dynamics is complex enough to lose the past memory, and further this complexity more or less reflects the instability of underlying dynamics. However, note that the quantum normal diffusion does not always correspond to instability in the quantum dynamics. Normal diffusion appears in the noise-driven (nearly) integrable systems without instability. We discuss this point in Sect [14].

We quantitatively characterize the instability of underlying quantum dynamics in terms of the sensitivity of time-reversed dynamics to the perturbation applied at a reversal time. This method provides a powerful tool for measuring the instability of quantum dynamics which have no corresponding classical orbit.

The time-reversal test is executed following the three steps mentioned below. First the initial state is evolved forward in time by operating the evolution operator $U^t$ until a reversal time $t = T$. At the reversal time $t = T$, a perturbation $\hat{P}(\eta)$ is applied, and next the perturbed state is evolved backward in time by operating the time-reversed evolution operator $U^{-T}$.

As introduced in Eq.(1), the relative irreversibility $(Q_\eta(2T) - Q_0(2T))/Q_0(T)$ as a function of the perturbation strength $\eta$ is considered as a measure of the sensitivity of the quantum state to the perturbation. Here $Q_\eta(2T)$ means physical quantity $Q$ when the perturbation with the strength $\eta$ is applied at the reversal time $T$, and $Q_0(2T)$ denotes the value of $Q$ in the unperturbed case, $\eta = 0$. There, however, are various ways to measure the distance between the quantum states. We adopt second moment $M(t) = \sum_x (x - <x>_s)^2|\Psi(x,t)|^2$ of the wavepacket as the quantity $Q$, because it captures the most characteristic feature of normal diffusion, and then we use the notation $R$ for the time-reversal characteris-
tics measured by MSD $M(t)$ as,
\[ R(\eta) = \frac{|M_{\eta}(2T) - M_0(2T)|}{M_0(T)}. \]  

As the perturbation $\hat{P}(\eta)$ applied at $T$, we mainly use the $\eta-$shift operator, which shifts the wavepacket by $\eta$ in the space $y$ canonically conjugate to the diffusion space $x$ (namely $y = q$ for $x = p$ as in SM, and $y = p$ for $x = q$ as in PAM and HM),
\[ \hat{P}_y(\eta) = \exp \{ i \eta \hat{y} / \hbar \} = \exp \{ i \eta \partial / \partial y \}. \]  

We call it the “perpendicular shift”. We also use the “parallel shift” $\hat{P}_x(\eta) = \exp \{ i \eta \hat{x} / \hbar \}$ which shifts the wavefunction by $\eta$ in the diffusion space $x$.

We give a brief comment on measure of time-irreversibility. Entropy can also be used to measure the probabilistic feature of a wavepacket. By using entropy in place of $M(t)$ we can also characterize the sensitivity of time-irreversibility to the perturbation. Indeed, the time-reversal characteristics based on the entropy give results similar to those based on the MSD. Indeed, as is discussed in appendix B a characterization by entropy has a clear measurement-theoretic significance [33, 32]. However, entropy is quantitatively less sensitive than MSD as the measure of the broadening of wavefunction, which is the significant feature of diffusion phenomenon, and we do not use it here. See appendix B for more details.

Before going on to quantum time-reversal characteristics, let us consider the behavior of the time-reversal characteristics $R_{cl}$ in the classical dynamics of Eq.(2), which is given by the mapping rule $(q, p) \rightarrow (q', p')$, namely,
\[ p' = p - V'(q + H_\eta(p)/2), \]
\[ q' = q + (H_\eta'(p) + H_\theta'(p'))/2, \]  

First, we discuss an integrable motion where the representation by action-angle variables $(\theta, I)$ is canonical, which is transformed into $(q, p)$-space by a canonical transformation $(q, p) = (Q(\theta, I), P(\theta, I))$, where $Q$ and $P$ are $2\pi$ periodic functions of $\theta$. In the integrable motion, the action is invariant and the angle exhibits a free motion at a constant angle velocity dependent only upon the action: $(\theta_t, I_t) = (\theta_0 + \omega(I_0)t, I_0)$, where $\omega(I_0) = \frac{\partial H_\eta}{\partial I}|_{I=I_0}$. At $t = T$ the applied perturbation in the $(q, p)$ plane makes the action-angle variables shift as $(\theta_T, I_T) \rightarrow (\theta_T + c_1 \eta, I_T + c_2 \eta)$, where $c_1, c_2$ are appropriate constants. Then, after time-reversed evolution for time $T$ the trajectory returns to recover the initial state
\[ \theta_{2T} = \theta_0 + c_1 \eta + T(\omega(I_0) - \omega(I_0 + c_2 \eta)), \]
\[ I_{2T} = I_0 + c_2 \eta. \]

The deviation is evaluated as
\[ |p_{2T} - p_0| \sim \left| \frac{\partial P(I_0, \theta_0)}{\partial \theta} \right| \frac{\partial \omega(I_0)}{\partial I} c_2 \eta T. \]  

Thus the time-reversal characteristics are
\[ R_{cl} = \frac{|M_{\eta}(2T) - M_0(2T)|}{M_0(T)} \sim \eta T. \]  

The difference linearly increases with the reversal time $T$. Therefore, we can control the accuracy of the system’s return to the initial state by controlling the magnitude of the perturbation strength as
\[ \eta \sim 1/T. \]  

See Fig.3(a) for an illustration of the $R_{cl}$.

On the other hand, in the case of chaotic motions, the perturbed orbit follows the unperturbed one only for a short period of time, and the deviation of the former from the latter grows exponentially as $d(\tau) \sim \eta e^{\lambda \tau}$, where $\tau \equiv t - T$ for $t > T$ and $\lambda$ is the Lyapunov exponent, up to the time $\tau > \tau_d$. $\tau_d$ is defined as time when $d(\tau)$ grows up to $O(1)$, namely $d(\tau_d) \sim O(1)(= C)$. We refer to $\tau_d$ as the delay-time hereafter, because it means the time required for the loss of memory in the dynamics, beyond which diffusion motion is recovered in backward time-evolution at the same diffusion constant as the forward time-evolution due to the time-reversal symmetry of the system. Accordingly, after the reversal-time the increment $M_{\eta}(t) - M_0(t)$ increases like $M_{\eta}(t) - M_0(t) = D(\tau - \tau_d)$ for $\tau > \tau_d$. Consequently, in the chaotic dynamics the time-reversal characteristics can be evaluated as
\[ R_{cl} \sim 2 - \frac{\tau_d(\eta)}{T}, \]  

where $\tau_d(\eta) = \frac{\text{log}(C/\eta)}{\lambda}$. Thus we have to keep $\eta$ exponentially as small as
\[ \eta \sim Ce^{-\lambda T}, \]  

if we would like to control the system to recover the time-reversibility. An illustration of the time-reversal characteristics $R_{cl}(\eta)$ for classical systems are shown in Fig.3(a).

Now we return to the quantum problem and compare the quantum result with the classical one. Figure 3(c) and (d) show typical time-reversal tests executed by using classical and quantum SM in the normal diffusion region, respectively, and the classical time-reversal characteristics $R_{cl}$ and quantum one $R$ are shown in Fig.3(b). The quantum $R$ well coincides with classical $R_{cl}$ in large $\eta$ region, but the quantum $R$ abruptly decreases when $\eta$ becomes smaller than a certain threshold value, which we denote by $\eta_{th}$. It seems to approach algebraically to zero for $\eta \to 0$. This result imply that in the quantum dynamics the perturbation does not disturb the time-reversibility of the system for the perturbation strength smaller than the threshold, $\eta < \eta_{th}$. This is in sharp contrast to classical dynamics for which control with exponentially small perturbation strength is required $\eta$ in order to attain the time-reversibility. Quantum systems are more stable than the classical counterpart for $\eta < \eta_{th}$. In the next section, we give the qualitative evaluation of $\eta_{th}$. 
B. Time-reversal characteristics

In this section we examine the time-reversal test for quantum SM and PAM in the normal diffusion region to evaluate the time-reversal characteristics of the systems.

1. Case studies: standard map and perturbed Anderson map

We investigate time-reversal characteristics of diffusive quantum state by using SM, which has a classical limit, and PAM, which has no classical limit. As shown in Fig.3 in quantum SM the feature of time-reversal characteristics approaches classical one for relatively large \( \eta \), if the system is classically chaotic. Apparently there is a threshold \( \eta_{th} \) of the perturbation strength, below which the system shows an intrinsic quantum characteristic. It is expected quite naturally that the threshold \( \eta_{th} \) is related to Planck constant \( \hbar \).

In the case of SM, the wavepacket diffuses in the momentum space (i.e., \( x = p \)), and the wavepacket diffuses to cover the range of \( x \) with width \( \Delta x(T) = \sqrt{M(T)} \) at the reversal time. So the perpendicular perturbation Eq.(14) shifting the quantum state in the \( y (= q) \) space by \( \eta \) sweeps the phase space over the area \( A = \eta \Delta x(T) = \eta \sqrt{M(T)} \). Then, the shifted quantum state may be distinguished classically from the original state if the number of quantum states contained in the swept area \( A \) is larger than unity, and the separation of orbit from the shifted state deviates classically from the original state. Thereby, the threshold \( \eta_{th} \) of the perturbation strength is estimated as,

\[
\eta_{th} = \frac{2\pi\hbar}{\Delta x(T)}.
\]  

(23)

On the other hand, in the case of the parallel perturbation shifting the quantum state in the \( x (= p) \) space by \( \eta \), the wavepacket fills the full domain of definition of \( y (= q) \). Recall that it is defined by \( 0 \leq q \leq 2\pi \). Then the sweep area \( A = 2\pi \times \eta \), and the threshold perturbation strength is given by

\[
\eta_{th} = \hbar.
\]  

(24)

Hereafter, we mainly use the perpendicular \( \eta \) -shift as the perturbation, since \( \eta_{th} \) depends on both \( \hbar \) and \( T \) and can be varied in two ways. The results of the time-reversal characteristics for the parallel \( \eta \) -shift are given in appendix A. The above evaluation for \( \eta_{th} \) can obviously be extended to general quantum systems. On the other hand, Eq. (23) can be justified by semiclassical theory in case of SM because it has a classical limit. The time-reversal characteristics become classical in the condition that the difference among the magnitudes of action integrals associated with classical orbits contributing to the semiclassical wavepacket is large enough to avoid the quantum interference effect, which yields Eq. (23).

In Fig.4(a) both quantum and classical time-reversal characteristics of SM are illustrated for various reversal times \( T \). It suggests that the classical \( R_{cl} \) gradually increases with \( T \) and approaches \( R = 2 \) for \( T \rightarrow \infty \). The quantum \( R \) rapidly decreases with decrease of \( \eta \) in the quantum region, \( \eta < \eta_{th} \) as if it makes a hole in the vicinity of \( \eta \sim 0 \), which reflects the quantum stability against the perturbation. However, as \( \eta \) exceeds the threshold \( \eta_{th} \) the quantum \( R \) readily reaches the classical \( R_{cl} \). In particular, in the limit of \( T \gg T_{th} = \log(C/\hbar)/\lambda \), \( R \) reaches to 2 as \( \eta \) exceeds \( \eta_{th} \), which means that as \( \eta > \eta_{th} \) the system completely loses the memory to return to the initial state and changes to the backward diffusion. Thus we can see the significance of the threshold \( \eta_{th} \) as the least quantum perturbation unit (LQPU) above which the system become entirely irreversible. A relation between the quantum fidelity and the LQPU has been discussed by Sokolov et al[29].

It is reasonable to express the time-reversal characteristics as a function of scaled perturbation strength \( \eta/\eta_{th} \).
to eliminate the explicit dependence of $\eta_{th}$ upon $T$ and $\hbar$. Figure 4(b) shows the actual quantum and classical $R$ as a function of the scaled perturbation strength $\eta/\eta_{th}$ for various $T$’s in SM with $K = 6$.

The results clearly manifest the presence of the “quantum hole” for $\eta/\eta_{th} \leq 1$. In the following, we call the region $\eta > \eta_{th}$ post quantum region, in which we can observe classical behavior if the system has the classical limit. As is indicated by Fig.4(b) the $R$ is accompanied by some irregular fluctuation in the post-quantum region, which suggest the chaotic fluctuation inherent in the recovered diffusive motion.

Figure 5 shows $R$ for various parameter sets of SM. In the post quantum region, $\eta > \eta_{th}$, $R$ approaches the common line $R = 2$ in proportion to $1/T$ slowly. Also in the quantum region $\eta < \eta_{th}$, the results suggest that $R$ as a function of $\eta/\eta_{th}$ converges to a limit as $T \to \infty$. Moreover, it approaches a common curve if we take the limit $\hbar \to 0$ and/or $K \to \infty$ in which an ideal quantum normal diffusion is realized. The characteristics of convergence are given in appendix D. These facts strongly suggest that in SM the time-reversal characteristics represented by the scaled $\eta$ becomes the same common curve in the limit of normal diffusive motion.

We also numerically examined the curves of $R$ vs $\eta/\eta_{th}$ for SM with various different periodic potential $V(q)$ and confirmed that all curves coincide in the large limit of $T$ if the normal diffusion is maintained in the range $0 < t < 2T$.

In the following we use a well-converged curve of $R$ vs $\eta/\eta_{th}$ of SM as the reference curve to compare with $R$ of other systems. Thus we use $R$ taken at $T = 1200 \gg \tau_d(\eta_{th})$ for SM with $K = 12$ as the reference curve.

Figure 6(a) shows the time-reversal characteristics in the post quantum region for some cases with different Planck constant. It is shown that in the post quantum region the slope of the plots is insensitive to the change of the Planck constant, this is, it is determined only by the Lyapunov exponent once the quantum dynamics is classicalized.

Next, we investigate the time-reversal characteristics of PAM, which has no classical limit. We compare it with the time-reversal characteristics of the reference curve of SM. In PAM the normal diffusion occurs in the position space (namely $x = q$) quite differently from SM. In the case of PAM, $\eta_{th} = h$ for the parallel $\eta–$shift because $x = q$ and $y = p$. The physical origin of the normal diffusion in PAM is not chaotic dynamics like SM but destruction of Anderson localization by harmonic perturbations. As seen in Fig.6 the time-reversal characteristics of PAM also have the asymptotic limit for $T \to \infty$. The calculated $R$ for some parameter sets in PAM that generate
as is shown in Fig. 8(a) for SM, in the limit of \( T \to \infty \) the scaled curves form a common shape with no regard to the parameter \( K \) if the scaled perturbation strength \( \eta / \eta_h \) is taken at the same value. Finally, as shown in Fig. 8(c), the scaled difference \( \Delta M_\eta(T, \tau)/M_0(T) \) as a function of the scaled time \( \tau/T \) is on the common curve independent of the system in the limit \( T \to \infty \) if the scaled perturbation \( \eta / \eta_h \) is taken at the same value.

2. Universal similarity of time-reversal dynamics in asymptotic limit and non-universality in the way of convergence

The fact that the \( \mathcal{R} \) as the function of scaled \( \eta \) asymptotically converges to a common curve means that the time-reversed dynamics itself has an universal scaled feature in the limit of \( T \to \infty \), or more precisely \( T/T_{th} \gg 1 \). We begin with demonstrating some numerical results of time-reversed dynamics. Sensitivity of time-reversed dynamics to the perturbation is represented by the time-reversed characteristics.

In Fig. 8(b) for PAM, the functional form of \( \Delta M_\eta(T, \tau)/M_0(T) \) as a function of \( \tau/T \) converges to a well-defined limit in the limit \( T \to \infty \). Further, the scaled curves form a common shape with no regard to the parameter \( K \) if the scaled perturbation strength \( \eta / \eta_h \) is taken at the same value. Finally, as shown in Fig. 8(c), the scaled difference \( \Delta M_\eta(T, \tau)/M_0(T) \) as a function of the scaled \( \tau / T \) is on the common curve independent of the system in the limit \( T \to \infty \) if the scaled perturbation \( \eta / \eta_h \) is taken at the same value.
the time-reversed dynamics have no specific time-scale. Properties, we come to the result of Eq. (29) claiming that the memory of initial state is reversed time \( \tau \) as,

\[
\Delta M_\eta = G(\eta, \tau),
\]

where \( G \) is a function depending only on \( \eta \) and \( \tau \). Therefore, the difference \( \Delta M_\eta \) becomes,

\[
G(\eta, \tau) = D\tau F\left(\frac{\eta}{\eta_{th}(\tau)}\right),
\]

when we take \( T = \tau \) in Eq. (28). Accordingly, the relation immediately follows:

\[
\frac{M_\eta(T, T + \tau) - M_\eta(T, T + \tau)}{M_\eta(T, T)} = \frac{\tau}{T} F\left(\frac{\eta}{\eta_{th}(T)} \cdot \frac{\eta_{th}(\tau)}{\eta_{th}(T)}\right) = \frac{\tau}{T} F\left(\frac{\eta}{\eta_{th}(T)} \left(\frac{\tau}{T}\right)^{\chi}\right),
\]

where \( \eta_{th}(T) \propto T^{-\chi} \). The index \( \chi \) is determined by the type of perturbation as, \( \chi = 1/2 \) for perpendicular \( \eta \)-shift and \( \chi = 0 \) for parallel \( \eta \)-shift. (See Eqs. (20) and (24).) Thus \( \Delta M_\eta(T, T + \tau)/M_\eta(T) \) is determined only by the scaled perturbation strength \( \eta/\eta_{th} \) and the scaled time \( \tau/T \) for \( T(> T_{th}) \).

The above result is a natural consequence of the universal scaling of the time-reversibility and stationarity. The stationarity seems to mean that the past history up to the reversal time \( T \) does not influence further time evolution, which suggests that the memory of initial state is lost, while the scaling property of time-reversality, which is independent of \( T \), seems to suggest that the memory from \( t = 0 \) to \( T \) is maintained during the time evolved process. Unifying the above apparently contradictory properties, we come to the result of Eq. (29) claiming that the time-reversed dynamics have no specific time-scale.

As has been discussed, \( R = 2 \) in the post quantum region implies that an entire loss of memory is realized in the backward evolution and the backward diffusion of the wavepackets restored in the same way as the forward evolution without returning to the vicinity of the initial state. In the quantum diffusive systems the spread of the wavepacket reaches to a macroscopic level for \( T \to \infty \). The fact that \( R \) reaches 2 for \( \eta > \eta_{th} \) means that we have to control the external perturbation suffered by the wavepacket at the strength less than the LQPU in order to make the uncertainty of wavefunction extended in the macroscopic level to shrink to the initial level. Namely, we have to control the quantum unitary dynamics on the ultra small scale, i.e. \( \eta_{th} \) decided by the Planck constant \( h \). We can regard this extreme difficulty of recovering the time-reversibility of a quantum system as the appearance of quantum irreversibility.

The results discussed above and in the previous section suggest that the time-reversal dynamics leads to a limit of convergence which is insensitive to the details of the system in the limit \( T \to \infty \). However, the way of convergence with increase in \( T \) depends on the details of the system. This is related to the short time scale dynamics discarded in the above universality argument.

We investigate the short-time behavior of the time-reversed dynamics and the convergence properties focusing on the post quantum region. As discussed in Sect. III A, in the post quantum region of SM the time-reversed dynamics is governed by classical dynamics, and the convergence of \( R \) to 2 for \( \eta \gg \eta_{th} \) depends on the the parameter \( K \) through the Lyapunov exponent, which controls the sensitivity of the classical chaotic dynamics.

![FIG. 9](image-url) (Color online) The growth of the separation \( \Delta M_\eta(\tau) \) as a function of \( \tau = t - T \) for some perturbation strength \( \eta \). (a) SM with \( K = 6, h = 2\pi/121 \) and \( T = 40 \). (b) PAM with \( M = 3, \epsilon = 0.5 \) and \( T = 100 \). (c) Time-reversal characteristics of quantum SM and PAM, as functions of the scaled perturbation strength \( \eta/\eta_{th} \) at several reversal times \( T = 5, 10, 20, 40, 80, 160 \) from below for SM and from top for PAM, respectively. \( K = 6, h = 2\pi/121 \) for SM and \( W = 1.0, M = 3, \epsilon = 0.5 \) for PAM. (d) and (e) are plots of \( \Delta H_\eta(\tau) \) as a function of the time \( \tau \) at \( T = 40 \), which are correspond to data of (a) and (b), respectively. The dotted and bold lines represent cases of \( \eta < \eta_{th} \) and of \( \eta > \eta_{th} \), respectively.
Figure 9(a) and (b) show the time dependence of $\Delta M_\eta$ for various perturbation strength $\eta$. The delay time $\tau_d$ can then be decided by $\Delta M_\eta(T, \tau) \sim O(1)$. In case of SM (Fig. 9(a)), the difference $\Delta M_\eta$ shows a weak exponential growth with $\tau$ in the quantum region, while in the post quantum region it exponentially increases in the same way as the classical chaotic dynamics, namely $\Delta M_\eta(T, T + \tau) \sim e^{\eta T}$. See appendix D1 for the results obeying purely classical dynamics. However, as shown in Fig. 9(b), in case of PAM, $\Delta M_\eta$ increases very rapidly in the first stage, is not exponentially but algebraically. In the case of SM, the delay time obeys the classical result, $\tau_d = \log(C/\eta)/\lambda$, which is significantly large if $|\eta| \ll 1$ [37]. However, in the case of PAM, the increase of $\Delta M(T, \tau)$ in $\tau$ is so rapid that $\tau_d$ is less than 1 in the post quantum region. In the post quantum region of PAM the deviation of $R$ from the converged value 2 will be much less than that of SM when the reversal time $T$ takes a common value.

In Fig. 9(c) $R$ is displayed as functions of $\eta$ scaled by the LQPU for various $T$s. As seen in Fig. 9(b) the time-reversal characteristics of SM asymptotically approaches the universal curve from below when the reversal time increases. On the other hand, in the case of PAM we can not observe a significant $T$ dependence of the characteristics, and convergence to the asymptotic characteristics occurs much more rapidly than those in SM. As a result, in spite of the difference in the manner of convergence, the asymptotic limit exhibits a remarkable universality irrespective of the kind of the system.

We further introduce the following quantity $\Delta H_\eta$ in order to stress the difference of the sensitivity to the perturbation between SM and PAM,

$$\Delta H_\eta(\tau) = M(T + \tau) - M_\eta(T + \tau), \quad (30)$$

where $M(T + \tau)$ denotes MSD of the forward time-evolution without the time-reversal operation. In Figs. 9(d) and (e), the $\tau$ dependence of $\Delta H_\eta$ is shown for various perturbation strength $\eta$. $\Delta H_\eta$ linearly increases as $\Delta H_\eta(\tau) = 2D\tau$ for $\eta = 0$. It is expected that the existence of the quite different regions of time-reversal characteristics, namely, the stable quantum region and unstable post quantum region, is a general and common feature of quantum normal diffusion irrespective of details of the model. In addition, the difference between SM and PAM can clearly appear in the sensitivity of the $\eta$-dependence in the post quantum region. In the case of SM $\tau$-dependence of $\Delta H_\eta$ becomes flattened as the perturbation strength $\eta$ increases even in the post quantum region. On the other hand, in the case of PAM the $\tau$-dependence of $\Delta H_\eta$ drastically changes for the post quantum region, and it becomes insensitive to increase of $\eta$. We also give the other plots, $2 - R$ v.s. $\eta/\eta_{th}$, to express the difference between SM and PAM in appendix D3 (Fig. 22).

IV. NORMALLY DIFFUSIVE STOCHASTIC QUANTUM MAPS

Unfortunately, we do not know many quantum models which exhibit the rigorous normal diffusion and which allow us to examine a precise numerical simulation of normal diffusion. To our knowledge, the coupled standard map and three-dimensional Anderson model are typical models which can show a normal diffusion in a rigorous sense [35]. Even SM and PAM, which we take as typical examples showing definite normal diffusion, require some conditions in order to realize a definite normal diffusion. In fact, in case of SM the normal diffusion is observed only in a finite time range beyond which the diffusion is suppressed by the localization effect, while in the case of PAM an ideal diffusion is observed for sufficiently strong periodic perturbation force.

However, it is known that even quantum systems exhibiting incomplete diffusion can show a normal diffusion if they are perturbed by stochastically varying random force. (We remark that such noise-induced diffusion process is in general accompanied by a large fluctuation, and so an ensemble average over many noise processes is necessary to have reliable results.) In the present section we examine the time-reversal characteristics of noise-induced quantum normal diffusion. Here we emphasize that the term “time-reversal” means to reverse the whole time evolution rule including the externally applied random force.

Before investigating the time-reversal characteristics of randomly perturbed SM and AM, we take a quantum integrable map which shows a rigorous normal diffusion under the perturbation of randomly varying potential, and examine its time reversal characteristic. It provides a typical example which does not have the quantum time irreversibility discussed so far, although the system shows a rigorous normal diffusion and is seemingly time-reversible. Unlike SM, the quantum map has a linear kinetic energy:

$$H_0(p) = \omega p, \quad V(q) = \epsilon n_t \cos q. \quad (31)$$

Here $n_t$ is a time-dependent random variable with uncorrelated statistical average $\langle n_t n_{t'} \rangle = \delta_{t,t'}$. Without the noise (i.e., $n_t = \text{const.}$) the integrable map become equivalent to kicked harmonic oscillator, which is originally proposed as a model of kicked charges in a uniform magnetic field [39]. The equations of motion for the operators $(q, p)$ coincide with those of the classical map for the forward-backward evolution. By setting $(q_t, p_t) = (U^{-q} q_0 U^q, U^{-1} p_0 U^q)$, it immediately follows that

$$q_{t+1} = q_t \pm \omega, \quad p_{t+1} = p_t \pm \epsilon n_t \sin(q_t \pm \omega/2), \quad (32)$$

where + and − are taken in forward ($t \leq T - 1$) and backward ($t \geq T$) processes, respectively. Choosing $p_0 =$
\[ p_t = \sum_{j=0}^{t-1} \epsilon n_j \sin(\omega(j + 1/2) + q_0), \]  

(33)

immediately follows for \( t \leq T \), which rigorously results in normal diffusion \( M(t) = \langle (p_t - p_0)^2 \rangle = \epsilon^2 t/2 \) after averaging over the random force \( n_j \). After the shifting \( \langle q_T, p_T \rangle \to \langle q_T + \eta, p_T \rangle \), by taking the backward process into account with the time-reversal random variable \( n_j = n_{2T-j-1} \) \( (j \geq T) \), the time-reversal characteristics becomes

\[ \mathcal{R} = 4 \sin^2 \frac{\eta}{2}. \]  

(34)

\( \mathcal{R} \) is free from \( \hbar \) and increases from the reversal state \( \mathcal{R} = 0 \) with \( \eta \), however, it has no threshold related to \( h \) above which the time-reversal characteristics \( \mathcal{R} \) suddenly increase to order \( O(1) \), which means that the system is not irreversible in the sense argued in the previous section.

Our concern is whether or not the noise-induced normal diffusion may in general exhibit quantum irreversibility in the sense of the previous section. In the following we examine the time-reversal characteristics of stochastically perturbed SM and AM of Eq. (4) type, whose control parameters \( K \) and \( \epsilon \) are now taken small enough so that no spontaneous diffusion may take place. We compare them with those in self-induced normal diffusion of SM and PAM, which were discussed in the sections [11B.1 and 11B.2. The inverse limit of localization is the ballistic motion of Bloch electrons in the periodic lattice. Ballistic motion can be also converted into normal diffusion by applying external noise. A typical example of noise-induced diffusion in periodic lattice is seen in the Haken-Strobe model, which is described by Eq. (4), namely, a one-dimensional lattice with stochastic potential with no spatio-temporal correlation. In addition to the noise-driven SM and AM, we also investigate the time-reversal characteristics of HM as a typical example of noise-induced normal diffusion system in the periodic lattice.

It should be noted that in the noise-induced diffusion systems, the normal diffusion is in general realized irrespective of the noise strength, but in the following, we make the perturbation strength large enough such that a well-defined normal diffusion can be recognizable in a finite time observation.

In Fig. 10 we show the time-reversal characteristics of noise-induced diffusion observed for SM, AM, and HM in strong noise region. See also appendices D.6 and D.7 for the data of the time-reversal tests in the models. A remarkable feature in that the time-reversal characteristics agrees very well with the reference curve obtained for the self-induced diffusion in SM without the noise source. In particular, it is surprising that the HM, whose diffusion mechanism does not seems to contain any dynamical instability like SM and AM, also exactly traces the standard time-irreversibility. This fact implies that even the quantum system which is forced to diffuse normally by the externally applied noise has the same time-irreversibility as the quantum systems exhibiting the intrinsic normal diffusion. In appendix C we compare the time-irreversibility of the original systems such as localized systems and ballistic systems from which the noise source is removed with the irreversibility of the noise-induced diffusion systems discussed here.

In the relatively weak noise regime, the diffusion constant is small and the delay-time \( \tau_d \) is in general extremely long, and so the converges to the reference curve, which is attained for \( T \ll \tau_d \), is not yet very clear. It is still an open problem whether or not the time-reversal characteristics of weak-noise system convergence to the universal characteristics. It will be discussed elsewhere [30]. We will briefly discuss this in appendix C.2.

V. SUMMARY AND DISCUSSION

In the present paper, we investigated the time-reversal characteristics for typical quantum maps which exhibit rigorous normal diffusion. Time reversality is characterized by the difference between the forward time evolution and the backward time evolution which is a time-reversed process after an application of perturbation.

First, as examples of quantum deterministic maps, the standard map (SM), which is known to have a classical counterpart, and the perturbed Anderson map (PAM), having no classical counterpart, are examined. We con-
firmed that there always exists the least quantum perturbation unit (LQPU), first introduced in Ref.[26], which denotes the threshold of perturbation strength for the memory of backward evolution returning to the initial state to be lost. A remarkable fact is that, if the system exhibits normal diffusion, the time-reversal characteristics defined as a function of the scaled perturbation strength \( \eta/\eta_\text{th} \) is universal in the sense that all characteristics are plotted on the same curve in the limit of the reversal time \( T \rightarrow \infty \). In the quantum region where the perturbation strength \( \eta \) is less than the LQPU, the time-reversal characteristics decreases to zero smoothly as the scaled \( \eta \) goes to zero, and makes a "quantum hole" around \( \eta/\eta_\text{th} = 0 \). On the other hand, in the post quantum region where \( \eta \) is larger than LQPU, i.e. \( \eta/\eta_\text{th} > 1 \), it is accompanied by an intense fluctuation around the irreversible limit \( R = 2 \), which means that there is a sensitive dependence on the perturbation strength inherent in the quantum region. In particular, in SM the time-reversal characteristics in the post quantum region coincided very well with those expected from the classical chaotic instability even for finite \( T \).

The onset of normal diffusion seems to imply that the memory is lost steadily, and the temporal stationarity is attained. On the other hand, we have to accept the fact that the scaled time reversibility is universal in the sense mentioned above and the memory is conserved universally in the quantum region. If we accept these facts, we come to a conclusion that the time-reversed evolution process itself should obey a universal scaled dynamics if the perturbation strength is in the quantum region. This conspicuous consequence was indeed confirmed numerically for SM and PAM. These features correspond to the most unstable class of dynamics of the deterministic quantum systems.

We extend our time-reversal test to the class of quantum maps which exhibit normal diffusion under the influence of classical stochastic force. Stochastically driven standard map, stochastically driven Anderson map, and Haken map (HM) are tested, and the universal time-reversal characteristic examined. The universal time-reversed scaled dynamics confirmed for the deterministic quantum maps are observed also for all the stochastic quantum maps we examined, if the stochastic force is intense enough to induce a sufficiently rapid normal diffusion. This fact implies that even the stochastic quantum system genuinely possesses the same degree of instability as the typical deterministic unstable quantum systems.

However, whether or not the above universal time-reversal rules hold for a weakly perturbed stochastic map, which certainly exhibits normal diffusion but seems to share an "integrable" feature with the linear integrable map of Eq.(31), is still very unclear. There may be a threshold stochastic perturbation strength below which the universal time-reversal rules are violated. This is an interesting problem. All the universal features mentioned above hold only in the converged limit \( T \rightarrow \infty \), and the ways of convergence toward the universal time-reversible characteristics depend on the details of the system.

To compare with the ideal normal diffusion, we examined the time reversal characteristic of the localized motion, sub-diffusive motion and finally ballistic motion. In case of localization and subdiffusion the time-reversal characteristics deviates entirely from the universal curve of the normal diffusion, and it deviates upward from the universal curve. On the other hand, in the ballistic motion the time-reversal characteristics deviates downward from the universal curve, and the time-reversal characteristics asymptotically vanishes as the reversal time \( T \) increases. The behavior suggests that the ballistic state is essentially time-reversible as is the case in the integrable system. An application of external stochastic force transforms the localized and ballistic motions into a normal diffusion, and then the time-reversal characteristics approaches close to the universal curve from above in the localized case and from below in the ballistic cases, respectively.

Due to the recent development of laser techniques, the kicked rotor systems can now be implemented experimentally with ultracold atoms subjected to nearly resonant laser pulses. The time-reversal experiment proposed in the present paper can in principle be tested experimentally in atomic optic experiments [24]. However, the direct observation of the time-reversal characteristics seems to be still rather difficult because in order to observe ideal diffusion in the momentum space it requires extremely long spatio-temporal coherence of laser pulses.

In the present paper, we focused on the time-reversibility of the quantum dynamics only in the time-discrete systems (quantum maps). In a forthcoming separated paper we will compare the results with those of the time-continuous quantum systems like the Haken model and the perturbed Anderson model in which normal diffusion can takes places. The problems of weakly perturbed stochastic quantum maps mentioned above will be discussed there again.

Appendix A: A universality in parallel \( \eta \)-shift

In this appendix, the time-reversal characteristics for the parallel \( \eta \)-shift is discussed. (\( \eta \)-shift in \( q \)-space in SM, and \( \eta \)-shift in \( p \)-space in PAM, respectively.) It should be noted that in the direction of \( y \)-space perpendicular to the diffusion space \( x \), the periodic boundary conditions imposed on the system is \( \Delta q = 2\pi \) for SM and \( \Delta p = 2\pi \hbar \) for PAM, respectively. Therefore, according to the argument given in Eqs.(20) and (21), the LQPU’s are \( \eta_\text{th} = \hbar \) in SM and \( \eta_\text{th} = 1 \) in PAM, respectively. @ Figure 11(a) and (b) show the time reversal experiments with parallel \( \eta \)-shift for SM and PAM, respectively. It is found that the backward evolution after application of the perturbation at the reversal time shows almost linear \( \tau \)-dependence which is remarkably different from the case of perpendicular \( \eta \)-shift.

In Fig.11(c) and (d), it is shown that in the quantum
predicts the linear $\tau$–dependence of the backward evolution, which is just the manifestation for the universal feature of the scaled time-reversed dynamics in the case of parallel shift.

Appendix B: Time-reversal experiment and quantum measurement

An interpretation based on the measurement theory is possible for the time-reversal experiments. We select an arbitrary eigenstate $|x_0>$ of the coordinate $\hat{x} = \sum_x x|x_x><x|$, along which the diffusion takes place, and observe the unitary time evolution from it as $|\Psi(T) >= U(T)|x_0>$. Then we consider a measurement process of measuring an observable $\hat{X}$ for the wavepacket $|\Psi(T)>$. The best measurement which gives no statistical fluctuation is achieved if we choose the observable

$$\hat{X} = U(-T)\hat{x}U(T) = \sum_x U(-T)|x_x><x|U(T), \quad (B1)$$

Then, at time $T$, we always measure $X = x_0$ with the probability of 100%. This forms the “best” measurement with the least fluctuation which gives the least information entropy of the measurement

$$S = -\sum_x <X|\rho|X> \log <X|\rho|X>, \quad (B2)$$

region the time reversal characteristic are well scaled by a single common curve for SM and PAM. However, in the fully post quantum region $\eta/\eta_{th} \gg 1$ the results for SM and PAM does not always coincide with each other, and there seems to be a significant fluctuation.

We investigate more details of the time reversal characteristic of the parallel $\eta$–shift. Figure 12 shows the logarithmic plots of the time reversal characteristic for a wide range of $\eta/\eta_{th}$. In the case of SM, the slow convergence due to the $1/T$ dependence is observed in the post quantum region as in the case of perpendicular $\eta$–shift. In the case of PAM the convergence is so rapid that $\eta/\eta_{th}$ dependence in the limit $T \to \infty$. In addition, the presence of periodic oscillation around $\eta/\eta_{th} \sim 1$ is a notable feature probably due to the effect of the periodic boundary condition imposed for the coordinate perpendicular to the shift coordinate, which is different from the case of the perpendicular $\eta$–shift. It is surprising that the oscillating structure around the threshold $\eta \sim \eta_{th}$ also converges to a universal $\eta/\eta_{th}$ dependence in the limit $T \to \infty$.

Finally we claim that the remarkable $\tau$–linear dependence of the backward evolution depicted in Fig. 11 is a natural consequence of the scaled universality (Eq. (29)) of the time-reversed dynamics, which should hold also for the parallel shift. Indeed, by setting $\chi = 0$, Eq. (29)
where $|X>$ is the eigenstate of $\hat{X}$ and $\rho(T)$ is the density operator of the system, where $\rho(T) = U(T)|x_0><x_0|U(-T)$ in the present case. It is null and equals to the von Neumann entropy

$$S_{vN}(T) = -\text{Tr}\rho(T) \log \rho(T), \quad (B3)$$

which gives the lowest bound of the information entropy of the measurement process. Thus the measurement for Eq.(B3) provides an ideal measurement process making the information entropy minimum.

We are then interested in how a small perturbation $\hat{P}(|\eta)$ makes the unperturbed state $U(T)|x_0 >$ dirty, by observing the perturbed state by the best measurement process for the unperturbed state. Then the probability of measuring $X = x$ is $P_x(T) = |<x|U(-T)\hat{P}(|\eta)U(T)|x_0>|^2$, which defines the measurement entropy

$$S_{\eta}(T) = -\sum_x P_x(T) \log P_x(T). \quad (B4)$$

Since $U(-T)\hat{P}(|\eta)U(T)$ is nothing more than the time-reversal process, $P_x(T)$ is the probability of finding the system at $x$ when the time-reversal evolution is finished at $t = 2T$, and so $S_{\eta}$ is the information entropy of the time-reversed wavepacket which is measured by $\hat{x}$, and should be written as $S_{\eta}(2T)$ in the context of the time-reversal test. Inversely speaking, $M_\eta(2T) = \sum_x P_x(T)(x - x >)^2$, which we often used as the measure of deviation from the exactly time-reversed state, can be interpreted as the squared average of the relative fluctuation at the best measurement.

The time-reversal characteristics $R_x$ can be also defined by using the measurement entropy $S_{\eta}$ in Eq.(B4). Then the entropic time-reversal characteristics $R_x$ is also defined as

$$R_x(\eta, T) = \frac{S(\eta, 2T) - S(\eta = 0, 2T)}{S(\eta = 0, T)}, \quad (B5)$$

instead of time-reversal characteristics $R$ in the main text, where the normalizing entropy $S(\eta = 0, T)$ is the measurement entropy at the reversal time for the probability $P_x = |<x|U(T)|x_0>|^2$.

Figure 13 shows the entropic time-reversal characteristics as a function of the scaled perturbation strength $\eta/\eta_{th}$ for various quantum states in SM. The $R_x$ denotes the same tendency of $R$. In the quantum region $\eta < \eta_{th}$ the entropic “quantum hole” appears.

### Appendix C: Irreversibility of sub- and super-diffusion

In this appendix, we investigate the time-reversal characteristics of quantum states of the localized, subdiffusive and ballistic time evolution and compare the results with those of the normal diffusion.

FIG. 13: (Color online) The entropic time-reversal characteristics $R_x$ obtained by time evolution of entropy $S(T, \eta)$ for SM. The results for the quasi-integral($K = 0.5, T = 200$), diffusive($K = 12, T = 100$) and localized($K = 4, T = 800$) cases are plotted.

1. **Time-reversal Characteristics for Localization and Ballistic motion**

We employ here AM and weakly perturbed AM as the prototypes showing the localized motion and subdiffusion, respectively. The SM is also used as an example realizing localization if small $K$ and/or large $\hbar$ are taken. On the other hand, we use a binary periodic system $v_q = 1, -1, 1, -1, \ldots$ in Eq.(4) as a typical example showing the ballistic motion. Figure 14(a) and (b) show the time-reversal experiments for the localizing SM and AM. The quantum interference effect making the wavepacket localized is destroyed in part as the perturbation strength increases. As a result, for strong enough perturbation, the wavepacket transiently recovers the extending behavior and the MSD increases, attaining, at the most, about 2 times the MSD of the localized state at $t = 2T$, namely $M_\eta(2T) \sim 2M_{\eta_{th}}(T)$. Figure 14(c) shows the time-reversal experiments in the case with subdiffusive motion. A similar tendency to the localized states is readily seen. Figure 14(d) shows the time-reversal experiments examined for the ballistic evolution in a binary periodic system. The behavior is similar to those of the integrable systems.

In Fig.15 the time-reversal characteristics of the localized, subdiffusive and ballistic motion are summarized. In the localized state, the time-reversal characteristics $R(\eta/\eta_{th})$ of an individual system computed at several reversal times seem to be on a common curve peculiar to the individual system in the quantum region. However, the common curve deviates upward significantly from the universal characteristic curve of normal diffu-
FIG. 14: (Color online) Time dependence of MSD in the time-reversal experiments examined for typical systems which do not manifest normal diffusion. The time-reversal backward evolutions are plotted at two reversal times for some perturbation strength $\eta$. (a)SM with $K = 4$, $h = \frac{\pi}{2}$. (b)AM with $\epsilon = 0$. (c)PAM with $M = 3$, $\epsilon = 0.05$. (d)Case with periodic sequence for $v_n$.

As discussed in the main text, the complete memory loss is characterized by $R = 2$ in the post quantum region. For the localized and subdiffusive states the complete memory loss does not take place and quantum interference effect stays in the time evolution because the time-reversal characteristics deviates from $R = 2$.

On the other hand, in the ballistic motion the time-reversal characteristics deviates markedly downward from the universal curve, and $R$ asymptotically vanishes as the reversal time $T$ increases. The behavior suggests that the ballistic state is essentially time-reversible as is the case in the integrable system observed in Sect.III B. In general, the quantum states which show the normal diffusion, ballistic motion (and probably also superdiffusion) are all recognized as “extended states” because all cases show continuous spectrum. However, the normal diffusion is markedly different from ballistic motion (and probably also from superdiffusion) in terms of the time-reversal characteristics.

2. Time-reversal characteristics of stochastically driven quantum maps

As shown in Sect.11 the localized motion is destroyed as it is driven by stochastic noise and a transformation into a delocalized motion showing a normal diffusion in general occurs. The external stochastic noise, on the other hand, transforms the ballistic motion into a normal diffusion as is typically exemplified by the Haken map. Here we investigate how the time-reversal characteristic changes with increasing strength $\epsilon_n$ of the external noise.

The reversal time $T$ is here taken finite, and we emphasize the strong possibility that all the characteristic converges to a common universal limit even for the small $\epsilon_n$ if the limit $T \to \infty$ is taken first time.

The time-reversal experiments and the characteristics of the stochastically driven quantum maps are given in Figs.23, 24 and 25, respectively. It can be seen that, by applying the stochastic noise, the localized and the ballistic motions transform into a diffusive motion in which the time dependence of MSD increases in proportion to $t$, and with increase of the strength $\epsilon$ and the reversal time $T$ the time-reversal characteristics approach to the universal curve from above in the localized case and from below in the ballistic case, respectively.

It can be expected that the stochastic perturbation introduced by the coupling with the external degrees of freedom can establish irreversibility even in the quantum systems which are not genuinely unstable (typical example is the Ballistic map). Such a coupling makes the time-reversal characteristics indistinguishable from the universal curve of deterministic unstable systems such as SM and PAM, which show a normal diffusion without stochastic perturbation.
Appendix D: Complementary Data Sets

In this appendix, we give some data which supplements the main text.

1. Classical counterpart of the separation $\Delta M_\eta(T, \tau)$

In SM whose classical counterpart has chaotic phase space, the separation exponentially increases for adequately small $\tau$ region as,

$$\Delta M_\eta^cl(T, \tau) = M_\eta^cl(T, T + \tau) - M_\eta^cl(T, T + \tau)(D1) \sim \eta e^{\lambda \tau}, \quad (D2)$$

where $\lambda$ is the Lyapunov exponent. Here $M_\eta^cl(T, T + \tau)$ means MSD of classical SM at time $\tau$ after shift-perturbation at the reversal time $T$. Figure 16 shows the time dependence of the actual separation in SM for various perturbation strengths. It is found that $\Delta M_\eta$ exponentially increases independent of the perturbation strength.

![Figure 16](image)

**FIG. 16:** (Color online) Growth of the separation $\Delta M_\eta^cl(T, T + \tau)$ of classical SM with $K = 6$ for some $\eta$'s as a function of the time $\tau(= t - T)$, where $T = 40$. Plots using logarithmic scale are given in (b). The growth asymptotically approaches $\Delta M_\eta^cl(T, \tau) \sim 2D_\eta(t - \tau_d)$, where $D_\eta$ is classical diffusion constant and $\tau_d$ is delay time, as the perturbation strength $\eta$ increases.

2. Bayesian property for localized states

Figure 17 shows the time-evolution of MSD in the localization regime, where the panels (a) and (b) are AM and SM, respectively. Here, as is the case of Fig. 18, a measurement of $\dot{x}$ is taken at several intermediate values of time $s$ taken as several values. Hence these are the localization version of Fig. 2.

The measurement process operated at $t = s$ destroys the accumulated quantum interference effect leading to the localization, and the evolution process is reset. In Fig. 17(c) it is shown that the index $X(s)$ asymptotically approaches $X(s) = 1$ from above as $s \rightarrow T_f$. On the other hand, for the ballistic motion the index $X(s)$ asymptotically approaches $X(s) = 1$ from below when $s \rightarrow T_f$, as shown in Fig. 17(c). (The MSD is not given.)

![Figure 17](image)

**FIG. 17:** (Color online) Time-dependence of MSD with observation at several intermediate times. $s = 10, 20, 30$ for SM, and $s = 30, 60$ for PAM. (a)SM with $K = 6$, $\hbar = 2\pi305$. (b)AM with localized behavior $\epsilon = 0$. (c)Index $X(s) = M_\eta(T)/M(T)$ as a function of $s/T$ for the cases. The curve for $s = T$ shows time-dependence of MSD without intermediate observation as a reference. The result for the ballistic motion is also plotted. In this case we take one sample for PAM.

3. Time-reversal test of SM

Figure 18 shows the time-reversal experiments in SM with various values of the parameter sets, $\hbar$, $K$ and $T$. We investigate the convergence property of the time-reversal characteristics to the universal curve.

![Figure 18](image)
of Lyapunov exponent.

4. Time-reversal test for PAM

Most readers will be, however, unfamiliar with the perturbed Anderson map (PAM), so we first depict how the normal diffusion is attained with increase in the perturbation strength $\epsilon$. As shown in Fig. 20 in the Anderson map (AM) the increase of MSD soon saturates without the perturbation and a typical localization behavior takes place, but the nature of time evolution changes as localization $\rightarrow$ subdiffusion $\rightarrow$ normal diffusion with increase in the strength $\epsilon$. We here consider the normal diffusion regime.

Figure 21 shows the time dependence of MSD for several values of the parameter sets of PAM exhibiting a normal diffusion. The time-reversal characteristics for the normal diffusion shows the universal feature in the quantum region independent of the diffusion coefficient, as shown in Fig. 7 in the main text.

5. $2 - R$ in SM and PAM

Supposing that Eq. (21) can be used in the post quantum region in general, the deviation of $R$ from converged value 2 in the post quantum region of PAM will be much
less than that of SM, as is depicted in Fig. 22(a) and (b), if \( T \) takes a common value. Thus the sensitivity of the time-reversed dynamics to the perturbation is responsible for the convergence of the time-reversal characteristics.

We finally remark that the time-reversal characteristics plotted in Fig. 22(a) and (b) manifests the presence of intense fluctuation inherent in the post quantum region, which contrasts with the quantum region in which \( 2 - R \) decreases smoothly as the function of \( \eta \).

### FIG. 22: (Color online) Plots of \( 2 - R \) as a function of the scaled perturbation strength \( \eta/\eta_{th} \) in double logarithmic scale. (a) SM with \( K = 6 \), \( h = \frac{2\pi 121}{24} \) and \( T = 320, 640 \). (b) PAM with \( M = 3, 2, \epsilon = 0.5 \) and \( T = 400 \).

6. Strongly noise-induced normal diffusion

Figure 23 shows the time dependence of MSD for normal diffusion with large diffusion coefficient in noise-induced SM, PAM and HM, which is driven by stochastic forces with large strength \( \epsilon_n \). The normal diffusion by the backward evolution asymptotically approaches to the normal diffusion by the forward evolution, and the time-reversal characteristics approaches to the universal one as the perturbation strength \( \eta \) increases.

7. Weakly noise-induced normal diffusion

In this subsection, the data of the time-reversal experiments are given for normal diffusion realized by the weakly noise driven SM, PAM and HM. Figure 24 shows the time dependence of MSD. Note that in weakly noise driven cases the time dependence of MSD deviates from the typical \( t \)-linear dependence of the normal diffusion, at least in the the early stage of the time-evolution, although it shows the exact normal diffusion with the \( t \)-linear dependence on a longer time-scale. However, whether the time-reversal characteristics of the weakly noise driven system approaches to the universal curve is still an open question.

Figure 25 shows the actual time-reversal characteristics for the weakly noise driven maps. In the Haken map the time-reversal characteristics deviates downward from the reference curve due to the ballistic-like growth.
FIG. 25: (Color online) Time-reversal characteristics of some stochastic models in pre-normal diffusive behavior due to weak stochastic perturbation. Results for SM with $K = 6$, $\hbar = \frac{2\pi}{1001}$, $\epsilon_n = 0.05$, AM with $\epsilon_n = 0.05$ and HM with $\epsilon_n = 0.05$, $0.1$ at several reversal-times are plotted. The reference curve is also shown.

in the early stage, and it asymptotically approaches the reference curve. On the other hand, in noise driven SM and AM the time-reversal characteristics deviates upward from the universal curve. The behavior is similar to that in the localized state.

Acknowledgments

This work is partly supported by Japanese people’s tax via MEXT, and the authors would like to acknowledge them. They are also very grateful to Dr. T.Tsuji and Koike memorial house for using the facilities during this study.

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