Homodyne tomography with homodyne-like detection

Stefano Olivares,1,2 Alessia Allevi,3,41 Giovanni Caiazzo,3 Matteo G. A. Paris,1,2 and Maria Bondani4

1Quantum Technology Lab, Department of Physics “Aldo Pontremoli”, University of Milan, via Celoria 16, I-20133 Milano (MI), Italy
2INFN, Sezione di Milano, via Celoria 16, I-20133 Milano (MI), Italy
3Department of Science and High Technology, University of Insubria, Via Valleggio 11, I-22100 Como (CO), Italy
4Institute for Photonics and Nanotechnologies, CNR, Via Valleggio 11, I-22100 Como (CO), Italy

We show that data from homodyne-like detection based on photon-number-resolving (PNR) detectors may be effectively exploited to reconstruct quantum states of light using the tomographic reconstruction techniques originally developed for homodyne detection based on photodiodes. Our results open new perspectives to quantum state reconstruction, and pave the way to the use of PNR-based homodyne-like detectors in quantum information science.

FIG. 1: (Left) HL detection scheme: the signal is mixed with a coherent state |β⟩ (LO) at a balanced BS. After the interference, the difference between the number of photons detected at the BS outputs is evaluated. (Right) Sketch of the experimental setup for the reconstruction of coherent and PHAV states.

A quantum tomographic scheme is a technique to evaluate the expectation value of any observables and, in turn, to reconstruct the density matrix or the Wigner function of the system, by processing the outcomes of a *quantum of observables*, measured on repeated preparations of the state under investigation [1–4]. In the continuous-variable domain, quantum states are usually reconstructed by means of optical homodyne tomography (OHT) [5]. From the technical point of view, OHT is based on an interferometric scheme (see the left panel of Fig. 1) in which a state ρ (the signal) is mixed at a balanced beam splitter (BS) with a high-intensity coherent state |β⟩ = |β|e^iφ (the local oscillator, LO). The two outputs of the interferometer are detected by two pin photodiodes, whose difference photocurrent is suitably amplified, rescaled by the LO amplitude |β|, and recorded as a function of the LO phase φ. By properly processing the data and applying advanced reconstruction algorithms, it is possible to achieve the complete knowledge of the state under investigation in terms of the density matrix [6,7] and to calculate the expectation values of given observables [4]. Among them, we mention the inversion algorithms, which can be applied if the effective quantum efficiency of the detection apparatus is above 50 %, and the maximum-likelihood reconstruction protocols, which can be applied also in the presence of strong losses [8,11].

In this Letter we consider the homodyne-like (HL) detection scheme shown in the left panel of Fig. 1. At variance with the standard homodyne detection scheme, in our apparatus the two photodiodes are replaced by photon-number-resolving detectors and the LO |β⟩ is a low-intensity (few tens of photons) coherent state. The physical information is obtained from the difference between the effective number of detected photons, rather than from the difference between two macroscopic photocurrents. Recently, we have demonstrated that HL detection may be successfully exploited for state-discrimination of coherent states [12,13]. Moreover, one has additional degrees of freedom available with HL data, and this may be used to outperform standard homodyne detection in quantum key distribution with continuous variables [14,15].

Motivated by the results achieved so far, here we address quantum tomography of CV systems by the HL detection scheme [15]. We prove that the HL scheme may be efficiently employed to reconstruct the density matrix of single-mode quantum states, and to evaluate the expectation values of the first moments of the quadrature operator. In particular, we show that robust and reliable results may be obtained with a relatively modest unbalancing between signal and LO. We also investigate the role played by quantum efficiency, which is a crucial parameter in the reconstruction of nonclassical states.

In OHT, the elements of the density matrix ρ of a quantum state can be reconstructed as follows [1–7]

\[
ρ_{nm} = \int_0^\pi dφ \int_{-\infty}^{+\infty} dx p(x, φ) F_{nm}(x, φ),
\]

where \( p(x, φ) = \langle x_φ | φ | x_φ \rangle \) are the homodyne probability distributions and |x_φ⟩ are the eigenstates of the quadratures \( \hat{x}_φ = (\hat{a} e^{-iφ} + \hat{a}^\dagger e^{iφ})/\sqrt{2}, \hat{a} \) being the annihilation operator of the field, \( [\hat{a}, \hat{a}^\dagger] = 1 \). In Eq. (1), \( F_{nm}(x, φ) \) are a set of suitable sampling functions (see Ref. [7] for details). In practice, the probabilities \( p(x, φ) \) are retrieved from the difference of the two macroscopic photocurrents exiting the homodyne detector [16,18], and the matrix elements are obtained by sampling \( F_{nm}(x, φ) \) over data.

In our scheme, we replace \( p(x, φ) \) with the HL probability distributions \( p_{HL}(Δ_φ) \), where \( Δ_φ = Δ/(√2|β|) \) and Δ is the (phase-dependent) difference between the number of photons detected at the two outputs of the homodyne detector [13]. We...
notice that, while the outcome $x$ of standard homodyne detection can assume any real value, the quantity $\Delta_\phi$ is intrinsically discrete. However, this discretization does not represent a limitation for the proper reconstruction of the states under examination, as shown hereafter. The information coming from the HL scheme is contained in the joint statistics $q(n, m)$ of the number of photons detected at the outputs of the BS. To find the theoretical expression of $q(n, m)$, it is useful to consider the Glauber-Sudarshan $P$ representation of the input state

$$P = \int_C d^2 z P(z) |z\rangle\langle z|,$$  \hspace{1cm} (2)

where $P(z)$ is the $P$-function associated with $\rho$. After the interference of the input state with the LO (the coherent state $|\beta\rangle$) at the balanced BS, the two-mode input state $R_{in} = \rho \otimes |\beta\rangle\langle \beta|$ becomes

$$R_{out} = \int_C d^2 z P(z) \left|\frac{\beta + z}{\sqrt{2}}\right\rangle\langle \frac{\beta + z}{\sqrt{2}}| \otimes \left|\frac{\beta - z}{\sqrt{2}}\right\rangle\langle \frac{\beta - z}{\sqrt{2}}|$$  \hspace{1cm} (3)

and the corresponding joint photon-number statistics can be written as $[19, 20]:$

$$q(n, m) = \int_C d^2 z P(z) e^{-\mu_c(z, \beta) - \mu_d(z, \beta)} [\mu_c(z, \beta)]^n [\mu_d(z, \beta)]^m n!m!$$  \hspace{1cm} (4)

where $\mu_c(z, \beta) = \frac{1}{2}|\beta + z|^2$ and $\mu_d(z, \beta) = \frac{1}{2}|\beta - z|^2$. We note that, thanks to this formalism, we can easily add the effect of quantum efficiency $\eta_k$ of the detector on the output modes $k = c, d$, by replacing $\mu_k(z, \beta)$ with $\eta_k \mu_k(z, \beta)$. Starting from the joint probability (4), it is possible to calculate the theoretical probability distribution of the quantity $\Delta = n - m$, namely

$$p_{HL}(\Delta) = \left\{ \begin{array}{ll} \sum_{k=0}^{\infty} q(\Delta + k, k) & \text{if } \Delta \geq 0, \\
   \sum_{k=1}^{\infty} q(k, k - \Delta) & \text{if } \Delta < 0, \end{array} \right.$$  \hspace{1cm} (5)

and the corresponding “rescaled” version $p_{HL}(\Delta_\phi)$, where $\Delta_\phi = \Delta/(\sqrt{2}|\beta|)$. In order to investigate the performance of the HL detector, we consider some paradigmatic optical states: the coherent state, which is phase-sensitive, its phase averaged counterpart (PHAV), which is of course phase insensitive, and the single-photon Fock state. While for the coherent and the PHAV states we provide an experimental demonstration, for the Fock state we performed Monte Carlo simulated experiments and we studied the effect of a non-unit quantum efficiency $\eta < 1$ with respect to the ideal case, namely $\eta = 1$.

For a coherent state $\rho = |\alpha\rangle\langle \alpha|$, the theoretical joint photon-number statistics, and the HL distribution, can be obtained by setting $P(z) = \delta^{(2)}(z - \alpha)$, where $\delta^{(2)}(\zeta)$ is the complex Dirac’s delta function. The experimental setup is sketched in the right panel of Fig. 1. The second harmonic pulses ($\sim 5$ ps pulse duration) at 523 nm of a mode-locked Nd:YLF laser regeneratively amplified at 500 Hz were divided at a Mach-Zehnder interferometer into two parts in order to yield the signal and the LO. The length of one arm of the interferometer was changed in steps by means of a piezoelectric movement ($Pz$) in order to vary the relative phase $\phi$ between the two arms in the interval $[0, \pi]$. Two variable neutral density filter wheels (ND) were used to change the balancing between the two beams, which were then recombined in a balanced BS. At the BS outputs two multi-mode 600-$\mu$m-core fibers (MF) delivered the light to a pair of photon-number-resolving detectors. In detail, we employed two hybrid photodetectors (HPDs, mod. R10467U-40, Hamamatsu Photonics), whose outputs were amplified, synchronously integrated and digitized. HPDs are commercial detectors endowed with partial photon-number-resolving capability and a good linearity up to 100 photons. As already demonstrated elsewhere (see for instance [13, 21, 22]), HPD response can be characterized in a self-consistent way with the same light under examination. Here we just remark that from the experimental data it is possible to calculate the gain of the detection apparatus in order to recover the distribution of detected photons at each BS output. In addition, this kind of detector allows us to obtain information on the relative phase between the two arms of the interferometer by simply monitoring the mean number of detected photons measured at each BS output as a function of the piezoelectric movement [13, 23]. In the present case, $3 \times 10^6$ data corresponding to subsequent laser pulses were recorded and used to reconstruct the density matrix of the coherent state and of the PHAV state, which is a diagonal state obtained by randomizing the phase of a coherent state $|\alpha\rangle = |\alpha\rangle e^{i\phi}$. \hspace{1cm} (6)

In order to process the data corresponding to the coherent state, once obtained the number of detected photons at each BS output, we calculated the shot-by-shot photon-number difference, $\Delta$. The corresponding phase value $\phi$ was determined by acquiring a suitable data sample $(5 \times 10^4$ pulses) for each one of the 60 piezo positions. As the piezo moves in regular steps, the measured mean number of detected photons follows a sinusoidal trend due to the interference at the beam splitter, from which the effective value of $\phi$ for each piezo position can be evaluated. The $5 \times 10^4$ data corresponding to the same $\phi$ were uniformly distributed around that value with a step of $1/(5 \times 10^4)$.

The typical behavior of $\Delta_\phi$ as a function of $\phi$ is shown in Fig. 2(a) ($3 \times 10^5$ data). Note that, due to irregularities in the piezoelectric movement, the data are not uniformly distributed in $\phi$. The reconstruction procedure was applied to 10 sets of $3 \times 10^5$ data drawn from the whole sample: the modulus of the average density matrix is shown in Fig. 2(b). The corresponding mean number of photons is $\langle n \rangle = \sum_n n \rho_{n,n} = 1.06$. The fidelity [24] between the reconstructed coherent state and a coherent state with the same amplitude is $F = 99.4 \%$.

As a matter of fact, the fidelity may not fully assess the quality of a reconstruction scheme [25] and therefore, in or-
FIG. 2: Experimental reconstruction of a coherent state with measured amplitude $\alpha = 1.03$. The LO has an amplitude $|\beta| = 3.82$. (a) $\Delta_\phi$ as a function of LO phase, $\phi$. The experimental data (green dots) are shown together with the mean value (solid line) and the standard deviation (dashed lines) of the quadrature, as obtained through the pattern function tomography. (b) Reconstructed density matrix. The fidelity with the expected state is $F = 99.4\%$.

FIG. 3: Experimental reconstruction of a PHA V state with amplitude $|\alpha| = 1.08$. The LO has an amplitude $|\beta| = 3.82$. (a) $\Delta_\phi$ as a function of $\phi$. The experimental data (green dots) are shown together with the mean value (solid line) and the standard deviation (dashed lines) of the quadrature as obtained through the pattern function tomography. (b) Reconstructed density matrix. The fidelity with the expected state is $F = 99.9\%$.

In order to test the quality of HL reconstruction, we also consider the HL evaluation of the expectation values of some observables. To this aim, we remind that given a generic observable, we may obtain its expectation value $\langle \hat{A} \rangle = \text{Tr}[\rho \hat{A}]$ from the distribution of homodyne data as [4]:

$$\langle \hat{A} \rangle = \frac{1}{\pi} \int_0^\pi d\phi \int dx p(x, \phi) \mathcal{R}[\hat{A}](x, \phi)$$

(7)

where $\mathcal{R}[\hat{A}](x, \phi)$ is a kernel, or pattern function, associated with operator $\hat{A}$. For instance, in the following we will use the first two moments of the quadrature operator, which may be obtained from the kernels [4]:

$$\mathcal{R}[\hat{x}_\theta](x, \phi) = 2x \cos(\phi - \theta),$$

(8)

$$\mathcal{R}[\hat{x}_\theta^2](x, \phi) = \left(4x^2 - \frac{1}{\eta^2}\right) \left[4 \cos^2(\phi - \theta) - 1\right] \frac{1}{4} + \frac{1}{4}.$$  

(9)

By using the HL probabilities $p_{\text{HL}}(\Delta_\phi)$ instead of $p(x, \phi)$ and setting $\theta = 0$ from Eq. (7) we got: $\langle \hat{x}_\theta \rangle = 1.451 \pm 0.003$ and $\text{var}[\hat{x}_\theta] = \langle \hat{x}_\theta^2 \rangle - \langle \hat{x}_\theta \rangle^2 = 0.688 \pm 0.015$ for the case under investigation. These values must be compared to those expected in the limit of a HL scheme, namely $\langle \hat{x}_\theta \rangle_{\text{theo}} = \sqrt{2} \alpha = \sqrt{2} 1.03 = 1.456$ and $\text{var}[\hat{x}_\theta]_{\text{theo}} = \frac{1}{2} + \frac{1}{2} |\alpha|^2 / |\beta|^2 = 0.536$. The last result explicitly shows the contribution to the variance due to the low-intensity LO, which becomes negligible as $|\beta| \gg |\alpha|$. Notice that since we are dealing with classical states, the quantum efficiency just rescales the energy of the detected states, and we can safely set $\eta = 1$: the results we obtain are thus referred to the detected states. We see that while the mean value of the quadrature is well reconstructed by the experimental data, the variance is larger than expected. Such discrepancy is likely due to phase fluctuations occurred during the measurements session. In order to check whether this interpretation holds, we consider the PHA V state, which is phase insensitive and thus should not be influenced by the presence of possible phase fluctuations. As in the case of the coherent state, we saved 10 sets of $3 \times 10^5$ data by calculating the shot-by-shot photon-number difference between the two BS outputs. Since the PHA V state is phase-insensitive, we randomly assigned a phase value to each experimental value of $d$. A typical trace of $d$ vs. $\phi$ is shown in Fig. 3(a), whereas in panel (b) the modulus of the reconstructed density matrix is presented. As expected, the off-diagonal elements are absent. In order to compare the reconstructed matrix to the theoretical prediction in Eq. (6) with $|\alpha| = 1.08$, we calculated the fidelity and obtained $F = 99.9\%$. For what concerns the first and second moment of the quadrature, we obtained $\langle \hat{x}_\theta \rangle = 0.004 \pm 0.003$ and $\text{var}[\hat{x}_\theta] = 1.725 \pm 0.013$, $\forall \theta$. In this case, the expected values are $\langle \hat{x}_\theta \rangle_{\text{theo}} = 0$ and $\text{var}[\hat{x}_\theta]_{\text{theo}} = \frac{1}{2} + |\alpha|^2 / |\beta|^2 = 1.706$, respectively. The very good agreement between theory and experiment for this phase-insensitive state confirms our considerations about the variance of the reconstructed coherent state.

We now turn the attention to the Fock state $|1\rangle$, for which we show the results obtained by using Monte Carlo simulated experiments setting $\beta = \sqrt{2}\alpha$. The state $|1\rangle$ has a highly singular $P$ function given by:

$$P(z) = \delta^{(2)}(z) + \frac{\partial^2}{\partial z \partial z^*} \delta^{(2)}(z),$$

(10)

and the joint probability in Eq. (4) reads:

$$q_{\eta}(n, m) = \frac{e^{-\eta|\beta|^2}}{n! m!} \left( \frac{\eta|\beta|^2}{2} \right)^{n+m} \left[ 1 + \frac{(n-m)^2 - \eta|\beta|^2}{|\beta|^2} \right],$$

(11)

where we assumed that both the detectors have the same quantum efficiency $\eta$. The corresponding $p_{\text{HL}}(\Delta)$ is quite clumsy and is not explicitly reported here.
thus the fidelity of the state coincides with that of the photon-number distribution. In particular, for the density matrix in Fig. 4(b) we have $F = 99.9\%$.

Up to this point, we have shown the reliability of the HL detection scheme and of the reconstruction strategy under ideal detection conditions. However, since photon-number-resolving detectors are real detectors, it is worth investigating whether the HL scheme also works in the presence of an overall quantum efficiency $\eta < 1$. When standard HD is adopted, it is well known that a non-unit quantum efficiency rescales the mean value of the reconstructed density matrix in the case of classical states of light, such as coherent and PHAV states, whereas it deeply modifies the properties of the reconstruction in the case of nonclassical states, such as Fock states [17]. In order to check if analogous results can be achieved by means of the HL scheme, we present the results obtained by simulated experiments for the Fock state $|1\rangle$. We assume the same value of LO adopted above, i.e. $\beta = \sqrt{20}$, but now we set $\eta = 0.4$, a realistic value for many kinds of photon-number-resolving detectors [21–26]. In Fig. 5 we plot the modulus of the reconstructed density matrix, panel (a), and the photon-number statistics, panel (b), for the Fock state $|1\rangle$. It is clear from the reconstructed density matrix that the effect of a non-unit quantum efficiency is to add a vacuum component to the state: in this case the expected density matrix is $\rho = \eta |1\rangle \langle 1| + (1 - \eta) |0\rangle \langle 0| [17]$ and its fidelity with respect to the reconstructed one is $F = 99.0\%$.

In conclusion, our numerical and experimental results demonstrate that the HL detection scheme may be used, instead of standard (pin-based) homodyne detectors, to reconstruct the density matrix of quantum states of light, as well as the first two moments of the quadrature operator. Our results open new perspectives to quantum state reconstruction, and pave the way to the use of PNR-based homodyne-like detectors in quantum information science.
a state-balancing detector, Phys. Rev. A 93, 043805 (2016).
[17] A. I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, and S. Schiller, State Reconstruction of the Single-Photon Fock State, Phys. Rev. Lett. 87, 050402 (2001).
[18] M. Esposito, F. Benatti, R. Floreanini, S. Olivares, F. Randi, K. Titimbo, M. Pividori, F. Novelli, F. Cilento, F. Parmigiani, and D. Fausti, Pulsed homodyne Gaussian quantum tomography with low detection efficiency, New. J. Phys. 16, 043004 (2014).
[19] S. L. Braunstein, Homodyne statistics, Phys. Rev. A 42, 474-481 (1990).
[20] M. Freyberger, K. Vogel and W. Schleich, From photon counts to quantum phase, Phys. Lett. A 176, 41-46 (1993).
[21] M. Bondani, A. Allevi, A. Agliati, and A. Andreoni, Self-consistent characterization of light statistics, J. Mod. Opt. 56, 226-231 (2009).
[22] A. Allevi, and M. Bondani, Nonlinear and quantum properties and applications of intense twin-beams, Adv. At. Mol. Opt. Phys. 66, 49-110 (2017).
[23] M. Bondani, A. Allevi, and A. Andreoni, Self-consistent phase determination for Wigner function reconstruction, J. Opt. Soc. Am. B 27, 333-337 (2010).
[24] R. Jozsa, Fidelity for Mixed Quantum States, J. Mod. Opt. 41, 23152323 (1994).
[25] M. Bina, A. Mandarino, S. Olivares, and M. G. A. Paris, Drawbacks of the use of fidelity to assess quantum resources, Phys. Rev. A 89, 012305 (2014).
[26] G. Chesi, L. Malinverno, A. Allevi, R. Santoro, A. Martemiyanov, M. Caccia, and M. Bondani, Optimizing Silicon photomultipliers for Quantum Optics, to be submitted.