Characterization of intermittency in renewal processes:
Application to earthquakes

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Abstract

We construct a one-dimensional piecewise linear intermittent map from the interevent time distribution for a given renewal process. Then, we characterize intermittency by the asymptotic behavior near the indifferent fixed point in the piecewise linear intermittent map. Thus, we provide a new framework to understand a unified characterization of intermittency, and also present the Lyapunov exponent of renewal processes. This method is applied to the occurrence of earthquakes using the Japan Meteorological Agency (JMA) catalog. We demonstrate that interevent times are not independent and identically distributed random variables by analyzing the return map of interevent times, but that there is a systematic change in conditional probability distribution functions of interevent times.

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I. INTRODUCTION

Recently, intermittent phenomena, characterized by a power law of the laminar state, have attracted interest in non-equilibrium statistical physics as well as biological and atomic physics. Examples of intermittent phenomena are the fluorescence of quantum dots and nanocrystals, and ion channel gating. Earthquakes can be recognized as an intermittent phenomenon. Actually, a $1/f^\delta$ spectrum is observed in the $P$-wave and $S$-wave velocity. Also, the residence time distribution of the laminar state, in which the number of earthquakes per unit time is lower than a threshold, obeys a power law, and the intermittency for the occurrence of earthquakes in Irpinia, Italy, has been quantified using the correlation codimension. Moreover, intermittency appears in the stick-slip model of earthquakes. In such a non-Poisson processes, a long time tail and aging have been clearly observed. Non-hyperbolic dynamical systems, which have at least one indifferent fixed point, also show intermittent behavior such as a long time tail and non-stationarity. By applying renewal theory to symbolic dynamics generated by the coarse graining of an orbit, it was also shown that non-hyperbolic dynamical systems generate a $1/f$ spectrum.

An outstanding problem in intermittent phenomena is the incompleteness of the usual statistical descriptors such as mean and variance because of the divergence of the mean interevent time. It is remarkable that infinite measure preserving dynamical systems, which are closely related to intermittent phenomena, exhibit intrinsic non-stationarity. Even when the mean interevent time is finite, the long tail of distribution makes it difficult to characterize the interevent time by the mean. Usually, intermittency is characterized by the exponent of a power law. However, this characterization does not include a long tail distribution heavier or not heavier than a power law. In the present paper, we characterize intermittency to estimate the difficulty of forecasting rare events. Moreover, the degree of activity of events is studied from the viewpoint of a non-hyperbolic dynamical system.

Renewal processes have been drawing attention not only in mathematics but also in physics, and are useful to analyze intermittent phenomena. In fact, intermittent phenomena can be rewritten as renewal processes by focusing attention on the residence time distribution of laminar state. In renewal processes, it is assumed that the interevent times between renewals are independent and identically distributed (i.i.d.) random variables. For
a given renewal process we construct a one-dimensional dynamical system to characterize intermittency in renewal processes. Then, we develop a concept of intermittency based on dynamical systems. One of our results is a unified characterization of intermittency in renewal processes, where we classify renewal processes into six different regimes according to difficulty to forecast the next event: (i) non-stationary essential singular intermittency, (ii) non-stationary very strong intermittency, (iii) stationary strong intermittency, (iv) stationary weak intermittency, (v) stationary very weak intermittency, and (vi) stationary non-intermittent chaos.

The paper is organized as follows. In Sec. II, we construct one-dimensional piecewise linear intermittent maps from renewal processes with any distribution, and then intermittency is characterized by the asymptotic behavior near the indifferent fixed point. Using the constructed map, we can calculate the Lyapunov exponent of a renewal process. In Sec. III, we study the occurrence of earthquakes. To apply our method to the occurrence of earthquakes in Japan, we verify whether the occurrence of earthquakes is a renewal process or not. Then, our method is applied to the occurrence of earthquakes by analyzing the conditional probability distribution functions of interevent times. Conclusions are given at the end of the paper.

II. UNDERLYING DYNAMICAL SYSTEMS IN RENEWAL PROCESSES

A. Construction of one-dimensional maps

To characterize intermittency, we construct one-dimensional maps from discrete time renewal processes. Let \( f(m) \) be the probability distribution function of a random variable \( m \) \((m = 1, 2, \cdots)\), and \( F(m) = \sum_{k=1}^{m} f(k) \) and \( \mathcal{F}(m) = 1 - F(m) \), where \( F(0) = 0 \) and \( \mathcal{F}(0) = 1 \). Then, we can obtain the following relationship:

\[
f(m) = \mathcal{F}(m-1) - \mathcal{F}(m).
\]  

(1)

Using this relation, we can construct a one-dimensional map on \([0, c]\) in which the residence times in \([0, 1]\) are i.i.d. random variables with probability \( f(m) \). Concretely, the map is
FIG. 1: Piecewise linear map $T(x)$ for the Weibull distribution ($a = 0.5, \tau = 5, c = 2$). Circles indicate the end points of the straight line segments.

given by a piecewise linear map $T : [0, c] \rightarrow [0, c]$ defined by

$$
x_{n+1} = T(x_n) =
\begin{cases}
  a_{k-1} - a_k (x_n - a_k) + a_{k-1}, & x_n \in [a_{k+1}, a_k), \\
  \frac{x_n - 1}{c - 1}, & x_n \in [1, c],
\end{cases}
$$

(2)

where a sequence $a_k$ is given by $a_k = \mathcal{F}(k)$ and $\mathcal{F}(-1) = c$. Actually, a point in the interval $[a_m, a_{m-1}]$ is mapped into the interval $[1, c]$ by $m + 1$th iterations, and then the probability of the residence time $m$ in the interval $[0, 1]$ is given by $a_m - a_{m-1}$ (see Fig. 1).

Next, we focus on the constructed map near the fixed point ($x = 0$). The derivative of the map is given by

$$
T'(x)|_{x \in [a_n, a_{n-1})} = \frac{a_{n-1} - a_n}{a_n - a_{n+1}} = \frac{f(n)}{f(n+1)}.
$$

(3)

Considering the maps near the fixed point, i.e., $x \approx a_n = \mathcal{F}(n), a_{n-1} - a_n \approx 0$, we obtain the asymptotic form,

$$
T'(x) \sim \frac{f(\mathcal{F}^{-1}(x))}{f(\mathcal{F}^{-1}(x) + 1)} \text{ as } x \rightarrow 0.
$$

(4)

In particular, the asymptotic behavior for a power law distribution ($\mathcal{F}(m) \sim m^{-\beta}$) is given by

$$
T'(x) - 1 \propto x^{1/\beta} \text{ as } x \rightarrow 0.
$$

(5)

This map is the same as the Pomeau-Manneville map [16], which is a typical example of intermittent maps, and the piecewise linear version is also well studied as an intermittent
FIG. 2: Time series of piecewise linear intermittent maps. The map in the left figure is constructed using the power law distribution ($\mathcal{F}(m) = m^{-1}$). The map in the right figure is constructed using the Weibull distribution ($\mathcal{F}(m) = \exp(-(m/5)^{0.35})$).

The asymptotic behavior is given by

$$T'(x) \sim e^{1/\tau} \quad \text{as } x \to 0 \text{ for } \mathcal{F}(m) \sim e^{-m/\tau},$$

(6)

and

$$T'(x) - 1 \propto (-\log x)^{(a-1)/a} \quad \text{as } x \to 0$$

(7)

for the Weibull distribution ($\mathcal{F}(m) \sim e^{-(m/\tau)^a}$) ($a \neq 1$). In the case of the log-Weibull distribution ($\mathcal{F}(m) \sim e^{-(\log m/\tau)^b}$), the asymptotic behavior is given by

$$T'(x) - 1 \propto \exp(-\tau(-\log x)^{1/b}) \quad \text{as } x \to 0.$$  

(8)

We refer to Eqs. (7) and (8) as the Weibull map and the log-Weibull map, respectively. Note that the Weibull map with exponent $a < 1$ and the log-Weibull map have the indifferent fixed point ($T'(0) = 1$).

B. Characterization of intermittency

In intermittent chaos, an orbit stagnates near indifferent fixed points for an extremely long time (laminar state), and then irregular chaotic motion occurs (see Fig. 2). The residence time distribution of the laminar state is determined by the structure of a map near the indifferent fixed points. Here, we characterize intermittency from the asymptotic behavior of the derivative at the indifferent fixed point ($x = 0$):

$$T'(x) - 1 \propto A(x)x^\alpha \quad \text{as } x \to 0,$$

(9)

where $T(x)$ is the one-dimensional map constructed by a renewal process and $A(x) = o(1)$ and $x^\alpha = o(A(x))$. The degree of intermittency is classified into six types:
(i) \( \alpha = \infty \), i.e., \( T'(x) - 1 \propto e^{-x^{\alpha'}} \) as \( x \to 0 \) (\( \alpha' > 0 \)): non-stationary essential singular intermittency,

(ii) \( \alpha \geq 1 \): non-stationary very strong intermittency,

(iii) \( 0 < \alpha < 1 \): stationary strong intermittency,

(iv) \( \alpha = 0 \) and \( A(x) = o(L(x)) \) as \( x \to 0 \): stationary weak intermittency,

(v) \( \alpha = 0 \) and \( A(x) \sim L(x) \) as \( x \to 0 \): stationary very weak intermittency,

(vi) \( T'(x) - 1 > 0 \) as \( x \to 0 \): stationary non-intermittent chaos,

where the function \( L(x) \) is slowly varying at 0 \([19]\). The intensity of intermittency can be quantified by the exponent \( \alpha \). The larger \( \alpha \) is, the more difficult to forecast the next event becomes. This is because slightly different reinjections near the fixed point make the residence time in \([0,1]\) completely different in the case of large \( \alpha \). With regard to type (iv), to be more precise,

\[
T'(x) - 1 = O(\exp(-(-\log x)\gamma)) \quad \text{as} \quad x \to 0,
\]

we can quantify the intensity of intermittency by the exponent \( \gamma \). For type (v),

\[
T'(x) - 1 = O\left(\frac{1}{(-\log x)^\eta}\right) \quad \text{as} \quad x \to 0,
\]

so that the intensity of intermittency is determined by the exponent \( \eta \). Exponents \( \alpha, \gamma \) and \( \eta \) represent the degree of the difficulty to forecast events in renewal processes because the sensitive dependence of the interevent time on reinjection points is determined by these exponents.

Note that the renewal function does not increase linearly with time; that is, the occurrence of renewals is not stationary, when the mean of the interevent times is not finite \([20]\). Accordingly, the occurrence of renewals becomes non-stationary in the case of \( \gamma \geq 1 \). For \( \gamma > 1 \), intermittency is classified into the non-stationary essential singular intermittent regime.
C. Lyapunov exponent

We can calculate the Lyapunov exponent in renewal processes because we constructed one-dimensional maps from renewal processes. In general, the Lyapunov exponent $\lambda$ is defined by

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln |T'(x_k)|,$$

where $T(x)$ is a one-dimensional map constructed from a renewal process. The slope of the piecewise linear map on $[a_1, a_0]$ and $[1, c]$ is given by $(c-1)/(1-a_1)$ and $1/(c-1)$, respectively. Therefore, the Lyapunov exponent does not depend on $c$ because $\ln(c-1)/(1-a_1) + \ln 1/(c-1) = -\ln(1-a_1)$. However, the Lyapunov exponent strongly depends upon the unit of time in renewal processes. Physical meaning of the Lyapunov exponent is the degree of activity of events. In other words, a large Lyapunov exponent implies high activity of an underlying dynamical system.

III. APPLICATION TO THE OCCURRENCE OF EARTHQUAKES

We apply this method to the occurrence of earthquakes using the JMA catalog in the area enclosed within 25°-50°N latitude and 125°-150°E longitude with magnitude $M \geq 2$. We are interested in the interevent time distribution for a tail part to construct a one-dimensional piecewise linear map. Similar to previous studies, we consider earthquakes with magnitude above a certain threshold $M_c$. In other words, we study the interevent time statistics in which magnitude is greater than $M_c$. Here, we use the earthquake data from January 1, 2001 to October 31, 2007.

To verify the hypothesis that the occurrence of earthquakes is a renewal process, we analyze the return map of interevent times. As shown in Fig 3, the average of an interevent time will be small if the previous interevent time is small. Thus, an interevent time depends clearly on the one preceding it. However, the average of an interevent time is constant when the previous one is relatively large, which suggests that interevent times are i.i.d. random variables.

To analyze the dependence of the distribution of interevent times on the previous one in detail, we sorted data in order and divided it into ordered data sets. To avoid statistical error, the number of data in each ordered data set is fixed at $10^4$. Analyzing the conditional
TABLE I: The Weibull exponent $a$, the intensity of intermittency $\eta$, and the Lyapunov exponent $\lambda$.

| $M_c$ | $a$   | $\eta$ | $\lambda$        | number of earthquakes |
|-------|-------|--------|-------------------|-----------------------|
| 2.0   | 0.977 | 0.023  | $4.7 \times 10^{-3}$ | 130243                |
| 2.5   | 0.956 | 0.046  | $2.7 \times 10^{-3}$ | 67912                 |
| 3.0   | 0.937 | 0.067  | $1.4 \times 10^{-3}$ | 31857                 |

probability distribution functions of interevent times for each ordered data set, we find that they change systematically. Moreover, we find that the conditional probability distribution functions converge to the Weibull distribution according to increases of previous interevent times, where the Weibull distribution $F(t)$ is defined by

$$F(t) = 1 - \exp(-(t/\tau)^a),$$

(see Figs. 4 and 5). It is remarkable that all conditional probability distribution functions obey the Weibull distribution in a tail region. Therefore, the intermittency of the occurrence of earthquakes is stationary very weak intermittency because the conditional probability distribution functions of interevent times in the tail region are invariant and described by the Weibull distribution.

Analyzing the distribution for different $M_c$, we find that the conditional probability distribution functions obey the Weibull distribution when the previous interevent time is relatively large. The Weibull exponent $a$, the intensity of intermittency $\eta$, and the Lyapunov exponent $\lambda$ are summarized in Table I, where the time unit is set to be one second in the calculation of the Lyapunov exponent.

IV. CONCLUSIONS

Intermittency of renewal processes is studied by constructing one-dimensional piecewise linear maps from renewal processes. As a result, characterization of intermittent phenomena is extended to extremely heavy tail and stretched exponential relaxation phenomena, and we can also estimate the Lyapunov exponent of renewal processes, which measures the activity of events.
Analyzing the occurrence of earthquakes, we found that the occurrence of earthquakes is not a renewal process, which is in agreement with [24]. However, interevent times are i.i.d. random variables when the previous interevent time is a relatively large. Moreover, the conditional probability distribution functions are characterized by the Weibull distribution, which means that the occurrence of earthquakes is stationary very weak intermittency. The intensity of intermittency of earthquakes depends on the threshold \( M_c \). In particular, the intensity of intermittency increases monotonically with the threshold of magnitude, indicating the view that it is difficult to forecast the occurrence of large earthquakes.

To quantify intermittency in point processes, Bickel proposed a clear estimation of intermittency using the correlation codimension [25]. Although we assume that interevent times are i.i.d. random variables, which does not always hold in point processes, we can study underlying dynamics of intermittent phenomena with the aid of one-dimensional maps. If we
FIG. 4: (color online). Weibull plot of the distribution of interevent times of earthquakes in Japan ($M_c = 2.0$). Each legend is a time interval $[T_{\text{min}}^i, T_{\text{max}}^i]$, where $T_{\text{min}}^i$ and $T_{\text{max}}^i$ are the minimum and maximum in the $i$th sorted data set ($i = 1, \cdots, 13$), respectively.

Assume the integrate-and-fire model [26], the orbit $x_n$ in dynamical systems is considered to be the integrated value of a signal that is behind intermittent phenomena. In the occurrence of earthquakes, the integrated value would be the accumulated energy on the crust.

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FIG. 5: (color online). Weibull plot of the distribution of interevent times of earthquakes in Japan ($M_c = 3.0$). Each legend is a time interval $[T_{i\min}^i, T_{i\max}^i]$, where $T_{i\min}^i$ and $T_{i\max}^i$ are the minimum and maximum in the $i$th sorted data set ($i = 1, 2, 3$), respectively.

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