GRAPH REDUCTION TECHNIQUES AND
THE MULTIPlicity OF THE LAPLACIAN EIGENVALUES

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Abstract. Let $M = [m_{ij}]$ be an $n \times m$ real matrix, $\rho$ be a nonzero real number, and $A$ be a symmetric real matrix. We denote by $D(M)$ the $n \times n$ diagonal matrix $\text{diag} (\sum_{j=1}^{m} m_{1j}, \ldots, \sum_{j=1}^{m} m_{nj})$ and denote by $L^\rho_A$ the generalized Laplacian matrix $D(A) - \rho A$. A well-known result of Grone et al. states that by connecting one of the end-vertices of $P_3$ to an arbitrary vertex of a graph, does not change the multiplicity of Laplacian eigenvalue $1$. We extend this theorem and some other results for a given generalized Laplacian eigenvalue $\mu$. Furthermore, we give two proofs for a conjecture by Saito and Woei on the relation between the multiplicity of some Laplacian eigenvalues and pendant paths.

1. Introduction

Let $A$ be a real symmetric matrix. There exists a unique weighted graph $G$ such that the adjacency matrix of $G$, is $A$; i.e. for $i \neq j$, $A_{ij}$ is the weight of the edge $\{i, j\}$ and $A_{ii}$ is twice of the weight of the loop at the vertex $i$. In this paper, we look at real symmetric matrices in this point of view.

For a positive integer $n$, we denote by $\text{Sym}_n(\mathbb{R})$, the set of real symmetric matrices of order $n$ and denote by $[n]$ the set $\{1, \ldots, n\}$. The multiplicity of an eigenvalue $\lambda$ of $A$ is denoted by $m_A(\lambda)$ and a $\lambda$-eigenvector of $A$ is an eigenvector of $A$ corresponding to $\lambda$. For two positive integers $i$ and $j$, $J_n$ and $e_i$, denote the all $1$’s vector and the vector with a 1 in the $i^{th}$ coordinate and 0’s elsewhere, respectively, in $\mathbb{R}^n$. The restriction of a vector $x$ to any index set $I$ is denoted by $x_I$ and we denote the entry of $x$ corresponding to an index $u$, by $x(u)$. The identity matrix is denoted by $I_n$ or briefly $I$. The path, the cycle, and the star graph on $n$ vertices are denoted by $P_n$, $C_n$, and $S_n$, respectively.

Let $M = [m_{ij}]$ be an $n \times m$ real matrix. The transpose of $M$ is denoted by $MT$ and we denote by $D(M)$, the $n \times n$ diagonal matrix $\text{diag} (\sum_{j=1}^{m} m_{1j}, \ldots, \sum_{j=1}^{m} m_{nj})$. For $\rho \in \mathbb{R} - \{0\}$, we denote by $L^\rho_A$, the generalized Laplacian matrix $D(A) - \rho A$. If $\rho = 1$ ($\rho = -1$) and $A(G)$ is the adjacency matrix of a given graph $G$, then we have the Laplacian matrix $L^1_{A(G)} = L(G)$ (signless Laplacian matrix $L^{-1}_{A(G)} = Q(G)$, respectively).

Let $\mu$ be a Laplacian eigenvalue of $G$. We consider the results about the relation between $m_{L(G)}(\mu)$ and $m_{L(H)}(\mu)$, for a particular subgraph $H$ of $G$. We recall some of these results:

Theorem 1. [6] If $G'$ is a graph obtained from $G$ by connecting one of the end-vertices of $P_3$ to an arbitrary vertex of $G$, then we have $m_{L(G)}(1) = m_{L(G')}(1)$.

Theorem 2. [8] Let $G$ be any graph with a simple Laplacian eigenvalue $\mu$. Let $u$ be a vertex of $G$ such that an eigenvector corresponding to $\mu$ is nonzero on $u$. Let $H$ be any graph, and let $G'$ be the graph formed by joining an arbitrary vertex of $H$ to $u$. Then $m_{L(H)}(\mu) = m_{L(G')}(\mu)$.

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A connected sum of two graphs $G_1$ and $G_2$ is any graph $G$ where $V(G) = V(G_1) \cup V(G_2)$ and $E(G)$ differs from $E(G_1) \cup E(G_2)$ by the addition of a single edge joining some (arbitrary) vertex of $V(G_1)$ to some vertex of $V(G_2)$, and is denoted by $G = G_1 \# G_2$ [6].

**Theorem 3.** [6] Let $G$ be a nonempty graph on $n$ vertices. Let $H = G \# S_k$ be a connected sum of $G$ with the star on $k > 1$ vertices. Then $m_{L(G)}(k) = m_{L(H)}(k)$

A cluster of a graph $G$ is an independent set of two or more vertices of $G$, each of which has the same set of neighbours. The degree of a cluster is the cardinality of its shared set of neighbours, i.e., the common degree of each vertex in the cluster. A $d$-cluster is a cluster of degree $d$. The number of vertices in a $d$-cluster is its order. A collection of two or more $d$-clusters is independent if the sets of vertices comprising the $d$-clusters are pairwise disjoint [5].

**Theorem 4.** [5] Let $G$ be a graph with $k$ independent $d$-clusters of orders $r_1, \ldots, r_k$. Then $m_{L(G)}(d) \geq \sum_{i=1}^{k} r_i - k$.

Among other results, we generalize these results for weighted graphs and an arbitrary generalized Laplacian eigenvalue $\mu$.

A pendant path of a graph $G$ is a path such that one of its end vertices has degree one and all the internal vertices have degree two and other end vertex has degree greater than two. $p_k(G)$ denotes the number of pendant paths of length $k$, and $q_k(G)$ the number of vertices of degree greater than three which are an end vertex of some pendant paths of length $k$. If $k = 1$, we have the well-known result of Faria [3] that $m_{L(G)}(1) \geq p_1(G) - q_1(G)$. Saito and Woei [9] conjectured that for any positive integer $k$, any graph $G$ has some Laplacian eigenvalue with multiplicity at least $p_k(G) - q_k(G)$ and proved it for $k = 2$. The following generalization of the conjecture has been proved in [4]. We give two proofs for this theorem in the next sections.

**Theorem 5.** [4] Let $G$ be a graph. Then $4 \cos^2(\frac{\pi i}{2k+1})$ for any $k \geq 1$ and $i = 1, \ldots, k$, is both a Laplacian and a signless Laplacian eigenvalue of $G$ with multiplicity at least $p_k(G) - q_k(G)$.

Let $A \in \text{Sym}_n(\mathbb{R})$ and $\lambda$ be an eigenvalue of $A$ of multiplicity $k$. A set $U \subseteq [n]$ is a star set for $\lambda$ (or $\lambda$-star set) of $A$ if $|U| = k$ and $\lambda$ is not an eigenvalue of the submatrix of $A$ obtained by removing rows and columns with index in $U$. It is known that for every eigenvalue $\lambda$ there exists a $\lambda$-star set [2].

We recall the following theorem about star sets that we use in the next sections.

**Theorem 6.** [2, Theorem 7.2.6] Let $U$ be a $\lambda$-star set of $A$. If $m_A(\lambda) = k$, then there exists a basis of eigenvectors $\{\alpha_s : s \in U\}$ such that $\alpha_s(t) = \delta_{st}$, whenever $s, t \in U$ and $\delta$ is the Kronecker delta function; its value is 1 if $s = t$, and 0 otherwise.

2. **Type I Reductions: Edge Deleting**

In this section, for a given eigenvalue $\mu$, we remove a particular subgraph corresponding to $\mu$ and consider the multiplicity of $\mu$ of remaining graph.

First, we state this following Edge Principle Theorem.

**Theorem 7.** [7] Let $\mu$ be a Laplacian eigenvalue of $G$ afforded by eigenvector $\mathbf{x}$. If $x_i = x_j$, then $\mu$ is an eigenvalue of $G'$ afforded by $\mathbf{x}$, where $G'$ is the graph obtained from $G$ by deleting or adding $e = \{i, j\}$ depending on whether or not it is an edge of $G$. 
Now, we state a weighted version of theorem above, for the Laplacian and the signless Laplacian of weighted graphs:

**Lemma 8.** Let \( n \in \mathbb{N}, \rho \in \{-1, 1\}, A \in \text{Sym}_n(\mathbb{R}) \), and \( \mu \) be an eigenvalue of \( L^\rho_A \) with a \( \mu \)-eigenvector \( x \). Suppose that \( a \in \mathbb{R} \) and \( x_i = \rho x_j \), for some \( i, j \in [n], i \neq j \). If \( A' \) is the matrix obtained from \( A \) by setting \( A'_{ij} = A_{ij} + a \), then \( x \) is a \( \mu \)-eigenvector of \( L^\rho_{A'} \).

**Proof.** We have \( L^\rho_A = L^\rho_{A'} - \frac{i}{j} (a - \rho a) \). So,

\[
\mu x = L^\rho_A x = L^\rho_{A'} x - \frac{i}{j} (a - \rho a) x = L^\rho_{A'} x - \frac{i}{j} (ax_i - \rho ax_j) = L^\rho_{A'} x.
\]

\( \square \)

The following theorem is the main theorem of this section.

**Theorem 9.** Let \( \mu \in \mathbb{R}, \rho \in \{-1, 1\} \), and \( H, L \) be real symmetric matrices with row and column indices \( I = I_1 \cup I_2 \cup I_3 \) and \( J = J_1 \cup J_2 \), respectively. Suppose that \( I_1 \cup I_2 \) is a \( \mu \)-star set of \( L^\rho_H \). If \( X, G, \) and \( E \) are matrices given below,

\[
X = \frac{1}{1} \begin{array}{cccc}
I_{11} & x_1 & 0 & \ldots & 0 \\
I_{12} & 0 & x_2 & \ldots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
I_{1|I|} & 0 & 0 & \ldots & x_{|I|}
\end{array},
\quad
G = \frac{1}{1} \begin{array}{cccc}
I_1 & I_2 & I_3 & I_1 \\
I_2 & H & 0 & 0 \\
I_3 & 0 & 0 & 0 \\
I_1 & X^T & 0 & 0 \\
J_2 & A^T & 0 & 0
\end{array},
\quad
E = \frac{1}{1} \begin{array}{cccc}
I_1 & I_2 & I_3 & 0 \\
I_2 & H & 0 & 0 \\
I_3 & 0 & 0 & 0 \\
J_1 & 0 & 0 & 0 \\
J_2 & A^T & 0 & 0
\end{array},
\]

for nowhere-zero vectors \( \{x_i\} \) and a matrix \( A \), where \( D(A) = 0 \) and \( A^T \alpha = 0 \), for every \( \mu \)-eigenvector \( \alpha \) of \( L^\rho_H \), then \( m_{L^\rho_G}(\mu) = m_{L^\rho_H}(\mu) + |I_1| \).

![Figure 1. The graphs of Theorem 9.](image)

In Theorem 9, by putting \( A = 0 \), we conclude the following corollary for the (signless) Laplacian matrix of simple graphs:

**Corollary 10.** Let \( \mu \in \mathbb{R}, \rho \in \{-1, 1\} \), and \( H \) be a graph and \( \{u_1, \ldots, u_t\} \) be a subset of a \( \mu \)-star set of \( L^\rho_H \). If \( L \) is an arbitrary graph disjoint from \( H \), and \( G \) is the graph formed by joining the vertex \( u_i \) to an arbitrary vertex \( v_i \) of \( L \) (not necessarily disjoint), \( i \in [t] \), then \( m_{L^\rho_G}(\mu) = m_{L^\rho_L}(\mu) + m_{L^\rho_H}(\mu) - t \).
Proof. Assume that \( L_G^\mu = u \begin{pmatrix} \alpha^T \\ M \end{pmatrix} \) and \( \alpha \) is a \( \mu \)-eigenvector of \( L_G^\mu \) such that \( \alpha(u) \neq 0 \). It is sufficient to show that \( \{u\} \) is a \( \mu \)-star set of \( L_G^\mu \). On the other hand, we show \( m_M(\mu) = 0 \). Suppose, by contradiction, \( M \) has a \( \mu \)-eigenvector \( \beta \). If \( \beta^T \beta = 0 \), then \( y \) is a \( \mu \)-eigenvector of \( L_G^\mu \), where \( y(v) = \begin{cases} \beta(v) & v \neq u, \\ 0 & v = u. \end{cases} \) Since \( \alpha(u) \neq 0 \), the vectors \( \alpha \) and \( y \) are independent and we have a contradiction with \( m_M(\mu) = 1 \). If \( \beta^T \beta \neq 0 \), then

\[
0 = (\mu I - L_G^\mu) \alpha = (\mu I - M) \alpha|_{V(G) - \{u\}} \Rightarrow \alpha(u) \beta^T x = \beta^T (\mu I - M) \alpha|_{V(G) - \{u\}} = 0 \Rightarrow \beta^T x \neq 0 \Rightarrow \alpha(u) = 0,
\]

and we have a contradiction. This completes the proof. \( \square \)

Remark 12. By Corollary 10, since \( m_L(P_3)(1) = 1 \) and the value of a 1-eigenvector is nonzero on every pendant vertex of \( P_3 \), we have Theorem 1. Also, \( m_L(S_k)(k) = 1 \) and every \( k \)-eigenvector of \( S_k \) is nowhere-zero, hence we have Theorem 3.

2.1. Edge Switching. In the following theorem, for a given eigenvalue \( \mu \), a particular subgraph, and given weights of the edges, we delete some edges and switch some weights from a section of graph to another section and give the relation between the multiplicity of \( \mu \) for two graphs.

Theorem 13. Let \( \mu \in \mathbb{R} \), \( \rho \in \mathbb{R} - \{0\} \) and \( \mathcal{H}, \mathcal{L} \) be real symmetric matrices with row and column indices \( I = I_1 \cup I_2 \cup I_3 \) and \( J = J_1 \cup J_2 \), respectively. Suppose that \( S \in \text{Sym}_{|I_1|}(\mathbb{R}) \) and \( I_1 \cup I_2 \) is a \( \mu \)-star set of \( L_H^\rho \). If \( \tilde{\mathcal{H}}, \tilde{\mathcal{L}}, \tilde{\mathcal{G}}, \) and \( \tilde{\mathcal{E}} \) are symmetric matrices given below,

\[
\tilde{\mathcal{H}} = \mathcal{H} + \begin{pmatrix} I_1 & I_2 \\ \rho \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & I_3 \end{pmatrix},
\tilde{\mathcal{L}} = \mathcal{L} - \begin{pmatrix} S' \\ 0 \end{pmatrix},
\tilde{\mathcal{G}} = \begin{pmatrix} I_1 & I_2 & I_3 \\ X & J_1 & J_2 \\ J_3 \end{pmatrix},
\tilde{\mathcal{E}} = \begin{pmatrix} I_1 & I_2 & I_3 \\ J_1 & J_2 \\ J_3 \end{pmatrix},
\]

for some matrices \( \mathcal{X} \) and \( X \), where \( D(\mathcal{A}) = 0 \), \( \mathcal{A}^T \alpha = 0 \), for every \( \mu \)-eigenvector \( \alpha \) of \( \mathcal{H} \), and \( S' \) is a solution of the equation \( L_S^\rho + D(X^T) = \rho^2 X^T (L_S^\rho + D(X))^{-1} X \), then \( m_{L_{\tilde{\mathcal{H}}}^\rho}(\mu) = m_{L_H^\rho}(\mu) + |I_1| \).

In particular, if \( L_S^\rho \) is invertible, then \( m_{L_{\tilde{\mathcal{H}}}^\rho}(\mu) = m_{L_H^\rho}(\mu) - |I_1| \).
For any $\rho \in \mathbb{R} \setminus \{0,1\}$ and $L \in \text{Sym}_n(\mathbb{R})$, it is easy to see that the equation $L = L_M^\rho$ has a unique solution $M \in \text{Sym}_n(\mathbb{R})$. Thus, for given $S$ ans $X$ such that there exists $(L_S^\rho + D(X))^{-1}$, the equation $L_S^\rho + D(X^T) = \rho^2X^T(L_S^\rho + D(X))^{-1}X$ has a solution for $S'$.

**Corollary 15.** Let $H = A(G)$, $L = A(L)$, $A = 0$, and consider the following two cases:

1. $X = I_{|I|}$ and $-S$ is a permutation matrix corresponding to an involution,
2. for a given $S$, suppose that $X$ is a solution of $X = L_S^\rho + D(X)$.

Then, for both cases, $S' = S$ is a solution and $m_{L_G}(\mu) + |I_1| = m_{L_H}(\mu) + m_{L_L}(\mu)$.

Since for a non-bipartite graph $H$, the signless Laplacian matrix $Q(H)$ is invertible, by a particular case of Theorem 13, if we set $S = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then we can conclude the next corollary.

**Corollary 14 (Edge Switching).** Let $\rho^2 = 1$ and $L, H$ be two disjoint graphs. With the notations of Theorem 13, put $\mathcal{H} = A(H)$, $\mathcal{L} = A(L)$, $A = 0$, and consider the following two cases:

1. $X = I_{|I|}$ and $-S$ is a permutation matrix corresponding to an involution,
2. for a given $S$, suppose that $X$ is a solution of $X = L_S^\rho + D(X)$.

Then, for both cases, $S' = S$ is a solution and $m_{L_G}(\mu) + |I_1| = m_{L_H}(\mu) + m_{L_L}(\mu)$.

**Example 17.** In Corollary 16:
\begin{itemize}
  \item $n = 3$ and $k = 1$: A path with three vertices of valency 2 in a graph $G$ can be replaced by an edge, without changing the Laplacian multiplicity of 3 and the signless Laplacian multiplicity of 1.
  \item $n = 4$ and $k = 1$: A path with four vertices of valency 2 in a graph $G$ can be replaced by an edge, without changing the (signless) Laplacian multiplicity of 2.
\end{itemize}

3. Type II Reductions: Deleting Subgraphs

In this section we generalize Theorem 4 for a given generalized Laplacian eigenvalue $\mu$.

**Theorem 18.** Let $\mu \in \mathbb{R}, \rho \in \mathbb{R} - \{0\}, r \in \mathbb{N}$, and $\mathcal{H}, \{\mathcal{H}_i\}_{i=1}^{r-1}, \mathcal{L}, \{\mathcal{L}_i\}_{i=1}^{r-1}, \mathcal{K}$ be symmetric matrices. Suppose that $L^\rho_{\mathcal{E}_i}$ has a $\mu$-eigenvector $\gamma^i$ such that $\gamma_i^i \neq 0$ and $\gamma_i^i \neq 0, i \in \{r - 1\}$, and $L^\rho_{\mathcal{E}}$ has independent $\mu$-eigenvectors $\beta^i$ such that $\beta_i^i = 0, i \in \{s\}$. Then $m_{L^\rho_{\mathcal{E}}} (\mu) \geq s + r - 1$, where $m_{L^\rho_{\mathcal{E}}} (\mu) = \sum_{i=1}^r (r_i - 1) = \sum_{i=1}^k r_i - k$.

**Corollary 19.** [5] Let $G$ be a graph with $k$ independent $d$-clusters of orders $r_1, \ldots, r_k$. Then $m_{L(G)} (d) \geq \sum_{i=1}^k r_i - k$.

**Proof.** With the notations of Theorem 18, we put $\rho = 1, \mathcal{H} = \mathcal{H}_i = [0]_{1 \times 1}, \mathcal{L} = [0]_{d \times d},$ and $\mathcal{A} = B^T = j_d$, then $m_{L^\rho_{\mathcal{E}_i}} (d) = 1$, for $d \neq 2$, and $m_{L^\rho_{\mathcal{E}_i}} (d) = 2$, for $d = 2$, and $\gamma_i^i = (1, 0, \ldots, 0, -1)^T$ is a $d$-eigenvector. Since $d$-clusters are independent, by using Theorem 18, $k$ times, we have $m_{L^\rho_{\mathcal{E}}} (d) \geq \sum_{i=1}^k (r_i - 1) = \sum_{i=1}^k r_i - k$. $\square$

Now, we give our first proof for Theorem 5. First, we need the following lemma on eigenvectors of the path graph.

**Lemma 20.** [10] Let $n$ be a positive integer. Then $4 \cos^2 \left( \frac{l \pi}{2n} \right)$ for $j \in [n]$ ($4 \sin^2 \left( \frac{l \pi}{2n} \right)$ for $0 \leq l \leq n - 1$) is a Laplacian eigenvalue of $P_n$ with the corresponding eigenvector $v_j$, where $v_j (u) = \cos \left( \frac{u-j \pi}{2n} \right)$, for $u \in [n]$.

Since the signless Laplacian matrix and the Laplacian matrix of a path are similar, it is easy to see that $w_j$ is a signless Laplacian eigenvector corresponding to $4 \cos^2 \left( \frac{l \pi}{2n} \right)$, where $w_j (u) = (-1)^u v_j (u)$, for $j, u \in [n]$. 
First proof of Theorem 5: By Lemma 20, if \( n = 2k + 1, j = 2t, \) and \( u = k + 1, \) then \( v_j(u) = 0. \) With
the notations of Theorem 18, we put \( \mu = 4 \cos^2\left(\frac{2\pi}{2k+1}\right), \rho = 1, \mathcal{H} = \mathcal{H}_i = A(P_k), \mathcal{L}_i = [0]_{1 \times 1}, A = e_k, \) and
\( B_i^T = e_1, \) then we have \( E_i = A(P_{2k+1}). \) If we put \( \gamma^i = v_j, \) then by using Theorem 18, \( q_k(G) \) times, we have
\( m_{L_i}^e(4 \cos^2\left(\frac{2\pi}{2k+1}\right)) \geq p_k(G) - q_k(G), \) \( t \in [k]. \) By similar proof, we have the statement is true for \( \rho = -1. \)

4. TYPE III REDUCTIONS: SPLITTING VERTICES

In this section, we state a splitting method to simplify graphs for a generalized Laplacian eigenvalue \( \mu \) and
a particular subgraph corresponding to it.

**Theorem 21.** Let \( \mu \in \mathbb{R}, \rho \in \mathbb{R} - \{0\} \) and \( \mathcal{H}, \mathcal{L} \) be real symmetric matrices. Suppose that \( m_{L_i}^e + D(\mathbf{x}) = 1. \)
If \( \mathbf{x}^T \alpha \neq 0, \) for a \( \mu \)-eigenvector \( \alpha \) of \( L_i^p + D(\mathbf{x}), \) then \( m_{L_i}^e(\mu) = m_{L_i}^e(\mu), \) where

\[
\mathcal{G} = \begin{pmatrix}
I & \mathcal{H} \\
\mathcal{L} & 0
\end{pmatrix} \quad \mathbf{y} = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_{|J|}
\end{pmatrix}
\]

**Figure 6. The splitting method: Theorem 21.**

**Lemma 22.** Let \( n \in \mathbb{N}, a, \mu \in \mathbb{R}, \rho \in \mathbb{R} - \{0\}, \mathcal{H} \in \text{Sym}_n(\mathbb{R}), \) and \( \mathbf{x} \in \mathbb{R}^n. \) The following statements are equivalent:

(i) \( m_{L_i}^e + D(\mathbf{x}) = 1 \) and \( \mathbf{x}^T \alpha \neq 0, \) for a \( \mu \)-eigenvector \( \alpha \) of \( L_i^p + D(\mathbf{x}); \)
(ii) \( m_{L_i}^e(\mu) = 1 \) and \( \beta(\mathbf{v}) = 0, \) for a \( \mu \)-eigenvector \( \beta \) of \( L_i^p. \)

Furthermore, (i) and (ii) imply \( m_{L_i}^e(\mu) = 0, \) where

\[
\mathcal{K} = \begin{pmatrix}
I & \mathcal{H} \\
\mathcal{L} & 0
\end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{H}} = \begin{pmatrix}
I & \mathcal{H} \\
\mathcal{L} & 0
\end{pmatrix}.
\]

**Figure 7. The graphs of Lemma 22.**
Proof. Suppose that $\alpha, \beta, \gamma$ are $\mu$-eigenvectors of $L^\rho_H + D(x)$, $L^\rho_K$, and $L^\rho_H$, respectively.

(i) $\Rightarrow$ (ii): For the index $I$ of $L^\rho_K$, we have

\[ (\mu I - L^\rho_H - D(x))\beta_{11} = -\rho \left( \begin{array}{c} x \\ 0 \end{array} \right) \left( \frac{\beta(v)}{\beta_{11}} \right). \]

Multiplying relation (1) by $\alpha^T$ from the left, we obtain

\[ 0 = \alpha^T (\mu I - L^\rho_H - D(x))\beta_{11} = -\rho \alpha^T \alpha \beta(v). \]

Hence, $\beta(v) = 0$ and $\beta_{11} = a_1 \alpha$ and similarly $\beta_{11'} = a_2 \alpha$, for some $a_1, a_2 \in \mathbb{R}$. For the index $v$ of $L^\rho_K$, we have

\[ (\mu - a + \rho a - 2D(x^T))\beta(v) = \beta_{11} \beta(v) = -\rho (z^T \beta_{11} + z^T \beta_{11'}) = -\rho (a_1 + a_2) z^T \alpha. \]

Hence $a_2 = -a_1$ and the proof is done.

(ii) $\Rightarrow$ (i): It follows by the relations above in a similar manner.

Now, we show that $m_{L^\rho_K}(\mu) = 0$. We have

\[ (\mu I - L^\rho_H - D(x))\gamma_{11} = -\rho \alpha \gamma(v). \]

Multiplying relation (2) by $\alpha^T$ from the left, we obtain $\gamma(v) = 0$ and $\gamma_{11} = b \alpha$, for $b \in \mathbb{R}$. For the index $v$ of $L^\rho_H$, we have

\[ (\mu - a + \rho a - D(x^T))\gamma(v) = -\rho \alpha^T \gamma_{11} = -\rho \alpha^T \alpha. \]

Hence $b = 0$ and $m_{L^\rho_H}(\mu) = 0$. \qed

The following corollary is a straightforward consequence of Theorem 21 and Lemma 22.

Corollary 23. Let $\mu \in \mathbb{R}, \rho \in \mathbb{R} - \{0\}$ and $H, L$ be two disjoint graphs and $u \in V(H)$ and $v \in V(L)$. Suppose that $H_1, \ldots, H_t$ are $t$ copies of $H$ and $E, K, G$ are graphs as shown below (see Figure 8). If $m_{L^\rho_K}(\mu) = 1$ and $\beta(v') = 0$, for a $\mu$-eigenvector $\beta$ of $L^\rho_K$, then $m_{L^\rho_E}(\mu) = m_{L^\rho_H}(\mu) + t - 1$.

\[ \text{Figure 8. The graphs of Corollary 23.} \]

Now, we give the second proof for Theorem 5.

**Second proof of Theorem 5:** For brevity, we set $q_k(G) = q$, $p_k(G) = p$. With the notations of Corollary 23, we put $\mu = 4 \cos^2 \left( \frac{2\pi}{2k+1} \right)$, $\rho = \pm 1$, $H = P_k$, then by using the splitting method of Corollary 23, for $q$ vertices of $G$, $q$ times, we have $m_{L^\rho_E}(4 \cos^2 \left( \frac{2\pi}{2k+1} \right)) = m_{L^\rho_E}(4 \cos^2 \left( \frac{2\pi}{2k+1} \right)) + p - q \geq p - q$, for $t \in [k]$.

Example 24. $\mu = 4 \cos^2 \left( \frac{2\pi}{2k+1} \right)$: Suppose that $k \in \mathbb{N}$ and $G, E$ are the graphs as shown below (see Figure 9). Then $m_{L(G)}(\mu) = m_{L(E)}(\mu)$ and $m_{Q(G)}(\mu) = m_{Q(E)}(\mu)$.
\[ \mu = 4 \cos^2 \left( \frac{\pi}{2m+1} \right); \quad t \in [k]. \]

Figure 9. The graphs of Example 24.

5. Proofs of the Main Theorems

For an \( n \times m \) matrix \( M \) and \( I \subseteq [n], J \subseteq [m], \) let \( M[I|J] \) denote the submatrix of \( M \) formed by rows with index in \( I \) and columns with index in \( J \).

**Proof of Theorem 9.** Suppose that \( m_{L^E_H}(\mu) = k \) and \( \alpha^1, \ldots, \alpha^k \) are the eigenvectors of \( L^E_H \) corresponding to \( I_1 \cup I_2 \) by Theorem 6. We set

\[
E = \left( \begin{array}{c|c} \alpha^1 & \cdots & \alpha^k \end{array} \right)^T = \left( \begin{array}{c|c} I_k & * \end{array} \right)
\]

and extend \( \alpha^i \) to \( \hat{\alpha}^i = \frac{l_j}{\bar{\rho}} \left( \begin{array}{c} \alpha^i \\ 0 \end{array} \right) \) for \( i \in [k] \). It is easy to check that \( \hat{\alpha}^1, \ldots, \hat{\alpha}^k \) are \( k \mu \)-eigenvectors of \( L^E \).

Suppose that \( \beta \) is a \( \mu \)-eigenvector of \( L^E_G \). We show that \( \beta|_{I_1} = \rho \beta(v_j)j, j \in [J_1] \). We have

\[
(\mu I - \mu L^E_G)|_{I_1} \beta|_{I_1} = -\rho \left( \begin{array}{c|c} X & 0 \\ 0 & A \end{array} \right) \beta|_{I_1}
\]

and

\[
(\mu I - \mu L^E_H - D \left( \begin{array}{c|c} X & 0 \\ 0 & A \end{array} \right) \beta|_{I_1} = -\rho \left( \begin{array}{c|c} X & 0 \\ 0 & A \end{array} \right) \beta|_{I_1}.
\]

Multiplying relation (3) by \( E \) from the left, we obtain

\[
0 = E(\mu I - \mu L^E_H)\beta|_{I_1} = \left( \begin{array}{c|c} D(X)^T \beta|_{I_1} & 0 \\ 0 & 0 \end{array} \right) - \rho E(A \beta|_{I_1} = \left( \begin{array}{c|c} D(X)^T \beta|_{I_1} - \rho \beta|_{I_1} X & 0 \\ 0 & 0 \end{array} \right).
\]

Since, \( x_j \) is nowhere-zero, hence, we have

\[
\beta|_{I_1} = \rho \beta(v_j)j, j \in [I_1].
\]

Thus \( \beta \) is a \( \mu \)-eigenvector of \( L^E \) by Lemma 8.

Now, we show that \( m_{L^E}(\mu) \geq m_{L^E_G}(\mu) + |I_1| \). Assume that \( m_{L^E_G}(\mu) = r \) and \( \beta^1, \ldots, \beta^r \) are independent eigenvectors of \( L^E_G \). We show that \( \beta^1, \ldots, \beta^r, \alpha^1, \ldots, \alpha^{|I_1|} \) are independent. Suppose that for some \( c_1, \ldots, c_r, d_1, \ldots, d_{|I_1|} \in \mathbb{R} \),

\[
\sum_{i=1}^{r} c_i \beta^i = \sum_{i=1}^{|I_1|} d_i \alpha^i.
\]

Hence \( \sum_{i=1}^{r} c_i \beta^i = \sum_{i=1}^{|I_1|} d_i \alpha^i = (d_1, \ldots, d_{|I_1|}, *, *, \ldots, *)^T \).

Thus \( (\mu I - \mu L^E_G)(\sum_{i=1}^{|I_1|} d_i \alpha^i) = \mathbf{0} \). From relation (4), \( (d_1, \ldots, d_{|I_1|})^T = \mathbf{0} \) and so \( c_1 = \cdots = c_r = 0 \) and \( m_{L^E}(\mu) \geq m_{L^E_G}(\mu) + |I_1| \).

Next, we show that \( m_{L^E}(\mu) \leq m_{L^E_G}(\mu) + |I_1| \).
\( \alpha^1, \ldots, \alpha^{|I_1|} \) are \(|I_1| \) \( \mu \)-eigenvectors of \( L^p_E \). Suppose that \( m_{L^p_E}(\mu) = s + |I_1| \) and \( \alpha^1, \ldots, \alpha^{|I_1|}, \gamma^1, \ldots, \gamma^s \) are independent \( \mu \)-eigenvectors of \( L^p_E \). For \( i \in [s] \), we define \( \gamma^i \) as below,

\[
\gamma^i = \gamma^i + \frac{I}{J_2} \left( \frac{\sum_{j \in |I_1|} E^T[I | I_1](\mu \gamma^j | I_1, i) - \gamma^i | I_1, i)}{0} \right)
\]

We show that \( \gamma^1, \ldots, \gamma^s \) are \( s \) independent \( \mu \)-eigenvectors of \( L^p_E \). We have

\[
L^p_E \gamma^i = \sum_{j \in |I_1|} D(X) \gamma^j | I_1, i) + \mu \gamma^i | I_1, i) - \gamma^i | I_1, i) + L^p_E \gamma^i | I_1, i) = 0
\]

Now, suppose that

\[
0 = \sum_{i=1}^s c_i \gamma^i = \sum_{i=1}^s c_i \gamma^i + \frac{\sum_{j \in |I_1|} E^T[I | I_1](\mu \gamma^j | I_1, i) - \gamma^j | I_1, i)}{0}
\]

for some \( c_1, \ldots, c_s \in \mathbb{R} \). So, \( \sum_{i=1}^s c_i \gamma^i = \sum_{i \in |I_1|} \gamma^i | I_1, i) \), and hence \( m_{L^p_E}(\mu) \leq m_{L^p_E}(\mu) + |I_1| \).

**Proof of Theorem 13.** Suppose that \( m_{L^p_E}(\mu) = k \) and \( \alpha^1, \ldots, \alpha^k \) are the eigenvectors of \( L^p_H \) corresponding to \( I_1 \cup I_2 \) by Theorem 6. We set \( E = \left( \begin{array}{ccc} \alpha^1 & \cdots & \alpha^k \end{array} \right)^T = \left( \begin{array}{c} I_1 \end{array} \right) \) and extend \( \alpha^i \) to \( \tilde{\alpha}^i = \frac{I}{J} \left( \alpha^i \right) \), for \( i \in [k] \). It is easy to check that \( \tilde{\alpha}^1, \ldots, \tilde{\alpha}^k \) are \( k \) \( \mu \)-eigenvectors of \( L^p_E \).

Suppose that \( \beta \) is a \( \mu \)-eigenvector of \( L^p_E \). We have

\[
0 = (\mu I - L^p_E) \beta | I_1 = - \rho \left( \begin{array}{c} X \\ 0 \\ 0 \end{array} \right) \beta | I_1
\]

Multiplying relation (5) by \( E \) from the left, we obtain

\[
0 = E (\mu I - L^p_H) \beta | I_1 = \left( \begin{array}{c} (L^p_E + D(X)) \beta | I_1 \\ 0 \\ 0 \end{array} \right) - \rho E \left( \begin{array}{c} X \beta | I_1 \\ 0 \\ 0 \end{array} \right) - \rho E A \beta | I_1 = \left( \begin{array}{c} (L^p_E + D(X)) \beta | I_1 + \rho X \beta | I_1 \end{array} \right).
\]

Hence,

\[
\beta | I_1 = \rho (L^p_E + D(X))^{-1} X \beta | I_1.
\]

So, \( \rho X^T \beta | I_1 = (L^p_E + D(X)) \beta | I_1 \) and it is easy to see that \( \beta \) is a \( \mu \)-eigenvector of \( L^p_E \).
Now, we show that $m_{L^0_\rho}(\mu) \geq m_{L^0_\rho}(\mu) + |I_1|$. Assume that $m_{L^0_\rho}(\mu) = r$ and $\beta^1, \ldots, \beta^r$ are independent eigenvectors of $L^0_\rho$. We show that $\beta^1, \ldots, \beta^r, \alpha^1, \ldots, \alpha^{|I_1|}$ are independent. Suppose that for some $c_1, \ldots, c_r, d_1, \ldots, d_{|I_1|} \in \mathbb{R}$,

$$\sum_{i=1}^r c_i \beta^i = \sum_{i=1}^{|I_1|} d_i \alpha^i = (d_1, \ldots, d_{|I_1|}, *, *, \ldots)^T.$$ 

Thus $(\mu I - L^0_\rho)(\sum_{i=1}^{|I_1|} d_i \alpha^i) = 0$. From relation (6), $(d_1, \ldots, d_{|I_1|})^T = 0$ and so $c_1 = \cdots = c_r = 0$. Hence, $m_{L^0_\rho}(\mu) \geq m_{L^0_\rho}(\mu) + |I_1|$.

Next, we show that $m_{L^0_\rho}(\mu) \leq m_{L^0_\rho}(\mu) + |I_1|$.

$\alpha^1, \ldots, \alpha^{|I_1|}$ are $|I_1|$ $\mu$-eigenvectors of $L^0_\rho$. Suppose that $m_{L^0_\rho}(\mu) = s + |I_1|$ and $\alpha^1, \ldots, \alpha^{|I_1|}$, $\gamma^1, \ldots, \gamma^s$ are independent $\mu$-eigenvectors of $L^0_\rho$. For $i \in [s]$, we define $\gamma^i$ as below,

$$\gamma^i = \gamma^i + \frac{I}{J} \left( E^T[I] I_1 (\rho L^0_\rho + D(X)^{-1} X \gamma^i|_{I_1} - \gamma^i|_{I_1}) \right).$$

We show that $\gamma^1, \ldots, \gamma^s$ are $s$ independent $\mu$-eigenvectors of $L^0_\rho$. We have

$$L^0_\rho \gamma^i = (L^0_\rho + D(X)) \gamma^i + \left( E^T[I] I_1 (\rho L^0_\rho + D(X)^{-1} X \gamma^i|_{I_1} - \gamma^i|_{I_1}) \right) + \left( \rho X \gamma^i|_{I_1} - (L^0_\rho + D(X)) \gamma^i|_{I_1} \right)$$

$$= \mu \gamma^i.$$

Now, suppose that

$$0 \in \sum_{i=1}^s c_i \gamma^i = \sum_{i=1}^s c_i \gamma^i + \left( E^T[I] I_1 (\rho L^0_\rho + D(X)^{-1} X \sum_{i=1}^s c_i \gamma^i|_{I_1} - \sum_{i=1}^s c_i \gamma^i|_{I_1}) \right).$$

for some $c_1, \ldots, c_s \in \mathbb{R}$. So, $\sum_{i=1}^s c_i \gamma^i = \sum_{i \in [s], i \in |I_1|} d_{ij} \alpha^i$, for some real numbers $d_{ij}$. Thus $c_1 = \cdots = c_s = 0$ and hence $m_{L^0_\rho}(\mu) \leq m_{L^0_\rho}(\mu) + |I_1|$. This inequality implies that $m_{L^0_\rho}(\mu) = m_{L^0_\rho}(\mu) + |I_1|$. Finally, to prove $m_{L^0_\rho}(\mu) = m_{L^0_\rho}(\mu) - |I_1|$, it suffices to put $X = 0$ and $L = 0$. \hfill \square

**Proof of Theorem 18.** We extend $\beta^i$ to $\hat{\beta}^i$ and $\gamma^i$ to $\hat{\gamma}^i$, where

$$\hat{\beta}^i = \begin{bmatrix} I & \beta^i |_{I_1} \\ I_1 & 0 \\ \vdots & \vdots \\ J_{i-1} & 0 \\ J_i & \beta^i |_{I_1} \end{bmatrix}, \quad i \in [s], \quad \hat{\gamma}^i = \begin{bmatrix} I & \gamma^i |_{I_1} \\ I_1 & 0 \\ \vdots & \vdots \\ J_{i-1} & 0 \\ J_i & \gamma^i |_{I_1} \end{bmatrix}, \quad i \in [r - 1].$$

It is easy to check that $\{\hat{\beta}^i\}_{i=1}^s$ and $\{\hat{\gamma}^i\}_{i=1}^{r-1}$ are $\mu$-eigenvectors of $L^0_\rho$. By the definitions, the independence of these eigenvectors is obvious. \hfill \square
Proof of Theorem 21. Suppose that $\alpha$ is the $\mu$-eigenvector of $L^p_\mu + D(\mathbf{x})$ such that $\mathbf{x}^T \alpha = 1$ and $\beta$ is a $\mu$-eigenvector of $L^p_\alpha$. We have

$$
(\mu I - L^p_\alpha) \beta_{ij} = -\rho \left( \mathbf{x} \mid 0 \right) \left( \frac{\beta(v)}{\beta_{ij}} \right)
$$

(7)

$$
(\mu I - L^p_\mu - D \left( \mathbf{x} \mid 0 \right)) \beta_{ij} = -\rho \left( \mathbf{x} \mid 0 \right) \left( \frac{\beta(v)}{\beta_{ij}} \right).
$$

Multiplying relation (7) by $\alpha^T$ from the left, we obtain

$$
0 = \alpha^T (\mu I - L^p_\alpha - D(\mathbf{x})) \beta_{ij} = -\rho \alpha^T \mathbf{z} \beta(v).
$$

Hence, $\beta(v) = 0$ and $\beta_{ij} = \alpha \beta$ for an $a \in \mathbb{R}$. For the index $v$ of $L^p_\alpha$, we have

$$
(\mu - a + \rho a - D(\mathbf{x}^T) - D(\mathbf{y}^T)) \beta(v) = -\rho(\mathbf{x}^T \beta_{ij} + \mathbf{y}^T \beta_{ij}).
$$

Hence,

$$
\mathbf{x}^T \beta_{ij} = -\mathbf{y}^T \beta_{ij}, \quad a = -\frac{\mathbf{y}^T \beta_{ij}}{\mathbf{x}^T \alpha} = -\mathbf{y}^T \beta_{ij}.
$$

By similar method, if $\gamma$ is a $\mu$-eigenvector of $L^p_\gamma$, then

$$
\gamma(v_j) = 0, \quad \gamma_{ij} = a_j \alpha, \quad \text{and} \quad a_j = \mathbf{x}^T \gamma_{ij} = -y_J \gamma(j), \quad \text{for} \ j \in [\lceil J \rceil].
$$

Now, we show that $m_{L^p_\gamma}(\mu) \geq m_{L^p_\alpha}(\mu)$. Assume that $m_{L^p_\alpha}(\mu) = r$ and $\beta^1, \ldots, \beta^r$ are independent $\mu$-eigenvectors of $L^p_\alpha$. Put

$$
\hat{\beta}^i = \begin{pmatrix}
I_1 \\
v_1 \\
\vdots \\
v_{i-1} \\
I_{\lceil J \rceil} \\
v_{\lceil J \rceil} \\
J \\
\beta_{ij}^i
\end{pmatrix}, \quad i \in [r].
$$

It is easy to see that $\hat{\beta}^i$ is a $\mu$-eigenvector of $L^p_\gamma$. We show that $\hat{\beta}^1, \ldots, \hat{\beta}^r$ are independent. From the relation (8), we can conclude that $\beta^1, \ldots, \beta^r$ are independent, if and only if $\beta^1_{ij}, \ldots, \beta^r_{ij}$ are independent. So, $\hat{\beta}^1, \ldots, \hat{\beta}^r$ are independent and hence, $m_{L^p_\gamma}(\mu) \geq m_{L^p_\alpha}(\mu)$.

Next, we show that $m_{L^p_\gamma}(\mu) \leq m_{L^p_\alpha}(\mu)$. Suppose that $m_{L^p_\alpha}(\mu) = s$ and $\gamma^1, \ldots, \gamma^s$ are independent $\mu$-eigenvectors of $L^p_\alpha$. For $i \in [s]$, put

$$
\hat{\gamma}^i = \begin{pmatrix}
I \\
v \\
J \\
\gamma^i_{ij}
\end{pmatrix}
$$

It is easy to see that $\hat{\gamma}^i$ is a $\mu$-eigenvector of $L^p_\alpha$. From the relation (9), $\gamma^1, \ldots, \gamma^s$ are independent, if and only if $\gamma^1_{ij}, \ldots, \gamma^s_{ij}$ are independent. So, $\hat{\gamma}^1, \ldots, \hat{\gamma}^s$ are independent and hence $m_{L^p_\gamma}(\mu) \leq m_{L^p_\alpha}(\mu)$. This inequality implies that $m_{L^p_\gamma}(\mu) = m_{L^p_\alpha}(\mu)$. \qed

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