Abstract

We review localization techniques for functional integrals which have recently been used to perform calculations in and gain insight into the structure of certain topological field theories and low-dimensional gauge theories. These are the functional integral counterparts of the Mathai-Quillen formalism, the Duistermaat-Heckman theorem, and the Weyl integral formula respectively. In each case, we first introduce the necessary mathematical background (Euler classes of vector bundles, equivariant cohomology, topology of Lie groups), and describe the finite dimensional integration formulae. We then discuss some applications to path integrals and give an overview of the relevant literature. The applications
we deal with include supersymmetric quantum mechanics, cohomological
field theories, phase space path integrals, and two-dimensional Yang-Mills
theory.
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1 Introduction

In recent years, the functional integral has become a very popular tool in a branch of physics lying on the interface between string theory, conformal field theory and topological field theory on the one hand and topology and algebraic geometry on the other. Not only has it become popular but it has also, because of the consistent reliability of the results the functional integral can produce when handled with due care, acquired a certain degree of respectability among mathematicians.

Here, however, our focus will not be primarily on the results or predictions obtained by these methods, because an appreciation of these results would require a rather detailed understanding of the mathematics and physics involved. Rather, we want to explain some of the general features and properties the functional integrals appearing in this context have in common. Foremost among these is the fact that, due to a large number of (super-)symmetries, these functional integrals essentially represent finite-dimensional integrals.

The transition between the functional and finite dimensional integrals can then naturally be regarded as a (rather drastic) localization of the original infinite dimensional integral. The purpose of this article is to give an introduction to and an overview of some functional integral tricks and techniques which have turned out to be useful in understanding these properties and which thus provide insight into the structure of topological field theories and some of their close relatives in general.

More specifically, we focus on three techniques which are extensions to functional integrals of finite dimensional integration and localization formulae which are quite interesting in their own right, namely

1. the Mathai-Quillen formalism [1], dealing with integral representations of Euler classes of vector bundles;
2. the Duistermaat-Heckman theorem [2] on the exactness of the stationary phase approximation for certain phase space path integrals, and its generalizations [3, 4];
3. the classical Weyl integral formula, relating integrals over a compact Lie
group or Lie algebra to integrals over a maximal torus or a Cartan subalgebra.

We will deal with these three techniques in sections 2-4 of this article respectively. In each case, we will first try to provide the necessary mathematical background (Euler classes of vector bundles, equivariant cohomology, topology of Lie groups). We then explain the integration formulae in their finite dimensional setting before we move on boldly to apply these techniques to some specific infinite dimensional examples like supersymmetric quantum mechanics or two-dimensional Yang-Mills theory.

Of course, the functional integrals we deal with in this article are very special, corresponding to theories with no field theoretic degrees of freedom. Moreover, our treatment of functional integrals is completely formal as regards its functional analytic aspects. So lest we lose the reader interested primarily in honest quantum mechanics or field theory functional integrals at this point, we should perhaps explain why we believe that it is, nevertheless, worthwhile to look at these examples and techniques.

First of all, precisely because the integrals we deal with are essentially finite dimensional integrals, and the path integral manipulations are frequently known to produce the correct (meaning e.g. topologically correct) results, any definition of, or approach to, the functional integral worth its salt should be able to reproduce these results and to incorporate these techniques in some way. This applies in particular to the infinite dimensional analogue of the Mathai-Quillen formalism and to the Weyl integral formula.

Secondly, the kinds of theories we deal with here allow one to study kinematical (i.e. geometrical and topological) aspects of the path integral in isolation from their dynamical aspects. In this sense, these theories are complementary to, say, simple interacting theories. While the latter are typically kinematically linear but dynamically non-linear, the former are usually dynamically linear (free field theories) but kinematically highly non-linear (and the entire non-triviality of the theories resides in this kinematic non-linearity). This is a feature shared by all the three techniques we discuss.

Thirdly, in principle the techniques we review here are also applicable to theories with field theoretic degrees of freedom, at least in the sense that they provide alternative approximation techniques to the usual perturbative expansion. Here we have in mind primarily the Weyl integral formula and the generalized WKB approximation techniques based on the Duistermaat-Heckman formula.

Finally, we want to draw attention to the fact that many topological field theories can be interpreted as ‘twisted’ versions of ordinary supersymmetric field theories, and that this relation has already led to a dramatic improvement of our understanding of both types of theories, see e.g. [5]-[9] for some recent developments.
Hoping to have persuaded the reader to stay with us, we will now sketch briefly the organization of this paper. As mentioned above, we will deal with the three different techniques in three separate sections, each one of them beginning with an introduction to the mathematical background. The article is written in such a way that these three sections can be read independently of each other. We have also tried to set things up in such a way that the reader primarily interested in the path integral applications should be able to skip the mathematical introduction upon first reading and to move directly to the relevant section, going back to the more formal considerations with a solid example at hand. Furthermore, ample references are given so that the interested reader should be able to track down most of the applications of the techniques that we describe.

Originally, we had intended to include a fifth section explaining the inter-relations among the three seemingly rather different techniques we discuss in this article. These relations exist. A nice example to keep in mind is two-dimensional Yang-Mills theory which can be solved either via a version of the Duistermaat-Heckman formula or using the Weyl integral formula, but which is also equivalent to a certain two-dimensional cohomological field theory which provides a field theoretic realization of the Mathai-Quillen formalism. Unfortunately, we haven’t been able to come up with some satisfactory general criteria for when this can occur and we have therefore just added some isolated remarks on this in appropriate places in the body of the text.

Note Added for the Proceedings

This is a (significantly) extended version of lectures given by M.B. at the Cargèse Summer School. During the lectures I focussed on two-dimensional Yang-Mills theory in order to complement the more quantum mechanical aspects and applications of path integrals that prevailed in most of the other lectures.

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The list of references is already quite long and detailed and I have not attempted to add to it articles that have appeared in the meantime. However, I want to draw attention to one article:

Richard Szabo has written an excellent and extensive review of equivariant localization techniques, partially overlapping with but also significantly extending the material covered in section III of these lectures. It is available from the preprint archives as hep-th/9608068.

I want to thank the organizers of the School, Pierre Cartier, Cecile DeWitt-
Morette, Antoine Folacci, and Chris Isham, for inviting me to lecture in Cargèse and for all their efforts that made this meeting a truly memorable one.

2 Localization via the Mathai-Quillen Formalism

The sort of localization we are interested in in this section is familiar from classical differential geometry and topology. Namely, let us recall that classically there exist two quite different prescriptions for calculating the Euler number \( \chi(X) \) (number of vertices minus number of edges plus \ldots) of some manifold \( X \). The first is topological in nature and instructs one to choose a vector field \( V \) on \( X \) with isolated zeros and to count these zeros with signs (this is the Hopf theorem). The second is differential geometric and represents \( \chi(X) \) as the integral over \( X \) of a density (top form) \( e^\nabla \) constructed from the curvature of some connection \( \nabla \) on \( X \) (the Gauss-Bonnet theorem).

The integral over \( X \) of \( e^\nabla \) can hence be localized to (in the sense of: evaluated in terms of contribution from) the zero locus of \( V \). This localization can be made quite explicit by using a more general formula for \( \chi(X) \), obtained by Mathai and Quillen [1], which interpolates between these two classical prescriptions. It relies on the construction of a form \( e_{V,\nabla}(X) \) which depends on both a vector field \( V \) and a connection \( \nabla \). Its integral over \( X \) represents \( \chi(X) \) for all \( V \) and \( \nabla \) and can be shown to reduce to an integral over the zero locus \( X_V \) of \( V \) in the form

\[
\chi(X) = \int_X e_{V,\nabla}(X) = \int_{X_V} e^\nabla(X_V) = \chi(X_V) .
\]

Here \( \nabla' \) is the induced connection on \( X_V \).

What makes this construction potentially interesting for functional integrals is the following observation of Atiyah and Jeffrey [10]. What they pointed out was that, although \( e^\nabla \) and \( \int_X e^\nabla \) do not make sense for infinite dimensional \( E \) and \( X \), the Mathai-Quillen form \( e_{V,\nabla} \) can be used to formally define regularized Euler numbers \( \chi_V(X) \) of such bundles by

\[
\chi_V(X) := \int_X e_{V,\nabla}(X)
\]

for certain choices of \( V \) (e.g. such that their zero locus is finite dimensional so that the integral on the right hand side of (2.1) makes sense). Although not independent of \( V \), these numbers \( \chi_V(X) \) are naturally associated with \( X \) for natural choices of \( V \) and are therefore likely to be of topological interest. Furthermore, this construction (and its generalization to Euler classes \( e(E) \) of arbitrary real vector bundles over \( X \)) provides one with an interesting class of functional integrals which are expected to localize to, and hence be equivalent.
to, ordinary finite dimensional integrals, precisely the phenomenon we want to study in these notes.

We should add that it is precisely such a representation of characteristic classes or numbers by functional integrals which is the characteristic property of topological field theories. This suggests, that certain topological field theories can be interpreted or obtained in this way which is indeed the case. We will come back to this in the examples at the end of this section.

A familiar theory in which this scenario is realized, and which preceded and inspired both the Mathai-Quillen formalism and topological field theory is supersymmetric quantum mechanics, an example we will discuss in some detail in section 2.2.

### 2.1 The Mathai-Quillen Formalism

The Euler class of a finite dimensional vector bundle

Consider a real vector bundle \( \pi : E \to X \) over a manifold \( X \). We will assume that \( E \) and \( X \) are orientable, \( X \) is compact without boundary, and that the rank (fibre dimension) of \( E \) is even and satisfies \( \text{rk}(E) = 2m \leq \dim(X) = n \).

The Euler class of \( E \) is an integral cohomology class \( e(E) \in H^{2m}(X, \mathbb{R}) \equiv H^{2m}(X) \) and there are two well-known and useful ways of thinking about \( e(E) \).

The first of these is in terms of sections of \( E \). In general, a twisted vector bundle will have no nowhere-vanishing non-singular sections and one defines the Euler class to be the homology class of the zero locus of a generic section of \( E \). Its Poincaré dual is then a cohomology class \( e(E) \in H^{2m}(X) \).

The second makes use of the Chern-Weil theory of curvatures and characteristic classes and produces an explicit representative \( e_\varphi(E) \) of \( e(E) \) in terms of the curvature \( \Omega_\varphi \) of a connection \( \nabla \) on \( E \). Thinking of \( \Omega_\varphi \) as a matrix of two-forms one has

\[
e_\varphi(E) = (2\pi)^m \text{Pf}(\Omega_\varphi)
\]

where \( \text{Pf}(A) \) denotes the Pfaffian of the real antisymmetric matrix \( A \). Standard arguments show that the cohomology class of \( e_\varphi \) is independent of the choice of \( \nabla \). For later use we note here that Pfaffians can be written as fermionic (Berezin) integrals. More precisely, if we introduce real Grassmann odd variables \( \chi^a \), then

\[
e_\varphi(E) = (2\pi)^m \int d\chi^a \frac{1}{2} \chi^a \Omega_\varphi^{ab} \chi_b .
\]

If the rank of \( E \) is equal to the dimension of \( X \) (e.g. if \( E = TX \), the tangent bundle of \( X \)) then \( H^{2m}(X) = H^n(X) = \mathbb{R} \) and nothing is lost by considering,
instead of $e(E)$, its evaluation on (the fundamental class $[X]$ of) $X$, the Euler number $\chi(E) = e(E)[X]$. In terms of the two descriptions of $e(E)$ given above, this number can be obtained either as the number of zeros of a generic section $s$ of $E$ (which are now isolated) counted with signs,

$$\chi(E) = \sum_{x_k: s(x_k) = 0} (\pm 1) ,$$

(2.5)

or as the integral

$$\chi(E) = \int_X e_\nabla(E) .$$

(2.6)

If $2m < n$, then one cannot evaluate $e(E)$ on $[X]$ as above. One can, however, evaluate it on homology $2m$-cycles or (equivalently) take the product of $e(E)$ with elements of $H^n 2m(X)$ and evaluate this on $X$. In this way one obtains intersection numbers of $X$ associated with the vector bundle $E$ or, looked at differently, intersection numbers of the zero locus of a generic section. The Mathai–Quillen form to be introduced below also takes care of the situation when one chooses a non-generic section $s$. In this case, the Euler number or class of $E$ can be expressed as the Euler number or class of a vector bundle over the zero locus of $s$ (see [11] for an argument to that effect).

For $E = TX$, the Euler number can be defined as the alternating sum of the Betti numbers of $X$,

$$\chi(X) = \sum_{k=0}^n (-1)^k b_k(X) , \quad b_k(X) = \dim H^k(X, \mathbb{R}) .$$

(2.7)

In this context, equations (2.5) and (2.6) are the content of the Poincaré-Hopf theorem and the Gauss-Bonnet theorem respectively. For $E = TX$ there is also an interesting generalization of (2.5) involving a possibly non-generic vector field $V$, i.e. with a zero locus $X_V$ which is not necessarily zero-dimensional, namely

$$\chi(X) = \chi(X_V) .$$

(2.8)

This reduces to (2.5) when $X_V$ consists of isolated points and is an identity when $V$ is the zero vector field.

The Mathai-Quillen representative of the Euler class

One of the beauties of the Mathai-Quillen formalism is that it provides a corresponding generalization of (2.6), i.e. an explicit differential form representative $e_{s, \nabla}$ of $e(E)$ depending on both a section $s$ of $E$ and a connection $\nabla$ on $E$ such that

$$e(E) = [e_{s, \nabla}(E)] ,$$

(2.9)
for any \( s \) and \( \nabla \). The construction of \( e_{s,\nabla} \) involves what is known as the Thom class \( \Phi(E) \) of \( E \) \([12]\) or - more precisely - the explicit differential form representative \( \Phi_\nabla(E) \) of the Thom class found by Mathai and Quillen \([1]\). The construction of \( \Phi_\nabla(E) \) is best understood within the framework of equivariant cohomology. What this means is that one does not work directly on \( E \), but that one realizes \( E \) as a vector bundle associated to some principal \( G \)-bundle \( P \) as \( E = P \times_G V \) (\( V \) the standard fibre of \( E \)) and works \( G \)-equivariantly on \( P \times V \).

We will not go into the details here and refer to \([10, 13, 14, 15]\) for discussions of various aspects of that construction. Given \( \Phi_\nabla(E) \), \( e_{s,\nabla} \) is obtained by pulling back \( \Phi_\nabla(E) \) to \( X \) via a section \( s : X \to E \) of \( E \), \( e_{s,\nabla}(E) = s^*\Phi_\nabla(E) \). It is again most conveniently represented as a Grassmann integral, as in \((2.4)\), and is explicitly given by

\[
e_{s,\nabla}(E) = (2\pi)^m \int_X e^{-\frac{1}{2} |s|^2 + \frac{1}{2} \chi s + i \nabla s \chi} \ . \quad (2.11)
\]

Here the norm \( |s|^2 \) refers to a fixed fibre metric on \( E \) and \( \nabla \) is a compatible connection. Integrating out \( \chi \) one sees that \( e_{s,\nabla}(E) \) is a \( 2m \)-form on \( X \). That \( e_{s,\nabla}(E) \) is closed is reflected in the fact that the exponent in \((2.11)\) is invariant under the transformation

\[
\delta s = \nabla s \ , \quad \delta \chi = is \ . \quad (2.12)
\]

To make contact with the notation commonly used in the physics literature it will be useful to introduce one more set of anti-commuting variables, \( \psi^\mu \), corresponding to the one-forms \( dx^\mu \) of a local coordinate basis. Given any form \( \omega \) on \( X \), we denote by \( \omega(\psi) \) the object that one obtains by replacing \( dx^\mu \) by \( \psi^\mu \),

\[
\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p} \to \omega(\psi) = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} \psi^{\mu_1} \ldots \psi^{\mu_p} \ . \quad (2.13)
\]

With the usual rules of Berezin integration, the integral of a top-form \( \omega^{(n)} \) can then be written as

\[
\int_X \omega^{(n)} = \int_X dx \int d\psi \omega^{(n)}(\psi) \ . \quad (2.14)
\]

As the measures \( dx \) and \( d\psi \) transform inversely to each other, the right hand side is coordinate independent (and this is the physicist’s way of saying that integration of differential forms has that property). Anyway, with this trick, the exponent of \((2.11)\) and the transformations \((2.12)\) (supplemented by \( \delta x^\mu = \psi^\mu \)) then take the form of a supersymmetric ‘action’ and its supersymmetry.

We now take a brief look at \( e_{s,\nabla}(E) \) for various choices of \( s \). The first thing to note is that for \( s \) the zero section of \( E \) \((2.11)\) reduces to \((2.4)\), \( e_{s=0,\nabla}(E) = \)
$e_V(E)$. As $\Phi_V(E)$ is closed, standard arguments now imply that $e_{s, V}(E)$ will be cohomologous to $e_V(E)$ for any choice of section $s$. If $n = 2m$ and $s$ is a generic section of $E$ transversal to the zero section, we can calculate $\int_X e_{s, V}(E)$ by replacing $s$ by $\gamma s$ for $\gamma \in \mathbb{R}$ and evaluating the integral in the limit $\gamma \to \infty$.

In that limit the curvature term in (2.11) will not contribute and one can use the stationary phase approximation (see section 3) to reduce the integral to a sum of contributions from the zeros of $s$, reproducing equation (2.5). The calculation is straightforward and entirely analogous to similar calculations in supersymmetric quantum mechanics (see e.g. [16]) and we will not repeat it here. The important thing to remember, though, is that one can use the $s$- (and hence $\gamma$-)independence of $[e_{s, V}]$ to evaluate (2.10) in terms of local data associated with the zero locus of some conveniently chosen section $s$.

Finally, if $E = TX$ and $V$ is a non-generic vector field on $X$ with zero locus $X_V$, the situation is a little bit more complicated. In this case, $\int_X e_{V, V}$ for $\gamma \to \infty$ can be expressed in terms of the Riemann curvature $\Omega_{V'}$ of $X_V$ with respect to the induced connection $\nabla'$. Here $\Omega_{V'}$ arises from the data $\Omega_{V}$ and $V$ entering $e_{V, V}$ via the classical Gauss-Codazzi equations. Then equation (2.8) is reproduced in the present setting in the form (cf. (2.2))

$$\chi(X) = \int_X e_{V, V} = (2\pi)^{\dim(X_V)/2} \int_{X_V} \text{Pf}(\Omega_{V'}) .$$

(2.15)

Again the manipulations required to arrive at (2.15) are exactly as in supersymmetric quantum mechanics, the Gauss-Codazzi version of which has been introduced and discussed in detail in [17].

That the content of (2.11) is quite non-trivial even in finite dimensions where, as mentioned above, all the forms $e_{s, V}(E)$ are cohomologous, can already be seen in the following simple example, variants of which we will use throughout the paper to illustrate the integration formulae (see e.g. (3.4.3.13,3.47)).

Let us take $X = S^2$, $E = TX$, and equip $S^2$ with the standard constant curvature metric $g = d\theta^2 + \sin^2 \theta d\phi^2$ and its Levi-Civita connection $\nabla$. To obtain a representative of the Euler class of $S^2$ which is different from the Gauss-Bonnet representative, we pick some vector field $V$ on $S^2$. Let us e.g. choose $V$ to be the vector field $\partial_\phi$ generating rotations about an axis of $S^2$. Then the data entering (2.11) can be readily computed. In terms of an orthonormal frame $e^a = (d\theta, \sin \theta d\phi)$ one finds

$$|V|^2 = \sin^2 \theta , \quad \nabla V^a = (\sin \theta \cos \theta d\phi, -\cos \theta d\theta) ,$$

$$\Omega_{V}^{ab} = \sin \theta d\theta d\phi , \quad \frac{1}{2} \chi V^b \Omega_{V}^{ab} \chi b = \chi_1 \chi_2 \sin \theta d\theta d\phi .$$

(2.16)

To make life more interesting let us also replace $V$ by $\gamma V$. The $\chi$-integral in (2.11) is easily done and one obtains

$$e_{V, V}(TS^2) = (2\pi)^{1} e^{-\frac{\gamma^2}{2} \sin^2 \theta} (1 + \gamma^2 \cos^2 \theta) \sin \theta d\theta d\phi .$$

(2.17)
The quintessence of the above discussion, as applied to this example, is now that the integral of this form over \( S^2 \) is the Euler number of \( S^2 \) and hence in particular independent of \( \gamma \). We integrate over \( \phi \) (as nothing depends on it) and change variables from \( \theta \) to \( x = \sin \theta \). Then the integral becomes
\[
\chi(S^2) = \int_1^1 e^{\frac{x^2}{2}(x^2 - 1)}(1 + \gamma^2 x^2)dx.
\]  
(2.18)

For \( \gamma = 0 \), the integrand is simply \( dx \) and one obtains \( \chi(S^2) = 2 \) which is (reassuringly) the correct result. A priori, however, it is far from obvious that this integral is really independent of \( \gamma \). One way of making this manifest is to note that the entire integrand is a total derivative,
\[
d(e^{\frac{x^2}{2}(x^2 - 1)}x) = e^{\frac{x^2}{2}(x^2 - 1)}(1 + \gamma^2 x^2)dx.
\]  
(2.19)

This is essentially the statement that (2.17) differs from the standard representative \((1/2\pi)\sin \theta d\theta d\phi \) at \( \gamma = 0 \) by a globally defined total derivative or, equivalently, that the derivative of (2.17) with respect to \( \gamma \) is exact. This makes the \( \gamma \)-independence of (2.18) somewhat less mysterious. Nevertheless, already this simple example shows that the content of the Mathai-Quillen formula (2.11) is quite non-trivial.

### The Mathai-Quillen formalism for infinite dimensional vector bundles

Let us recapitulate briefly what we have achieved so far. We have constructed a family of differential forms \( e_s,\nabla(E) \) parametrized by a section \( s \) and a connection \( \nabla \), all representing the Euler class \( e(E) \in H^{2m}(X) \). In particular, for \( E = TX \), the equation \( \chi(X) = \int_X e_{V,\nabla}(X) \) interpolates between the classical Poincaré-Hopf and Gauss-Bonnet theorems. It should be borne in mind, however, that all the forms \( e_s,\nabla \) are cohomologous so that this construction, as nice as it is, is not very interesting from the cohomological point of view (although, as we have seen, it has its charm also in finite dimensions).

To be in a situation where the forms \( e_s,\nabla \) are not necessarily all cohomologous to \( e_\nabla \), and where the Mathai-Quillen formalism thus ‘comes into its own’ [10], we now consider infinite dimensional vector bundles where \( e_\nabla \) (an ‘infinite-form’) is not defined at all. In that case the added flexibility in the choice of \( s \) becomes crucial and opens up the possibility of obtaining well-defined, but \( s \)-dependent, ‘Euler classes’ of \( E \). To motivate the concept of a regularized Euler number of such a bundle, to be introduced below, recall equation (2.8) for the Euler number \( \chi(X) \) of a manifold \( X \). When \( X \) is finite dimensional this is an identity, while its left hand side is not defined when \( X \) is infinite dimensional. Assume, however, that we can find a vector field \( V \) on \( X \) whose zero locus is a finite dimensional submanifold of \( X \). Then the right hand side of (2.8) is well defined
and we can use it to tentatively define a *regularized Euler number* \( \chi_V(X) \) as

\[
\chi_V(X) := \chi(X) .
\]  

(2.20)

By (2.10) and the same localization arguments as used to arrive at (2.15), we expect this number to be given by the (functional) integral

\[
\chi_V(X) = \int_X e_V, \nabla(X) .
\]  

(2.21)

This equation can (formally) be confirmed by explicit calculation. A rigorous proof can presumably be obtained in some cases by probabilistic methods as used e.g. by Bismut [18, 19] in related contexts. We will, however, content ourselves with verifying (2.21) in some examples below.

More generally, we are now led to define the regularized Euler number \( \chi_s(E) \) of an infinite dimensional vector bundle \( E \) as

\[
\chi_s(E) := \int_X e_s, \nabla(E) .
\]  

(2.22)

Again, this expression turns out to make sense (for a physicist) when the zero locus of \( s \) is a finite dimensional manifold \( X_s \), in which case \( \chi_s(E) \) is the Euler number of some finite dimensional vector bundle over \( X_s \) (a quotient bundle of the restriction \( E|_{X_s} \), cf. [11, 20]).

Of course, there is no reason to expect \( \chi_s(E) \) to be independent of \( s \), even if one restricts one's attention to those sections \( s \) for which the integral (2.22) exists. However, if \( s \) is a section of \( E \) naturally associated with \( E \) (we will see examples of this below), then \( \chi_s(E) \) is also naturally associated with \( E \) and can be expected to carry interesting topological information. This is indeed the case.

2.2 The Regularized Euler Number of Loop Space, or: Supersymmetric Quantum Mechanics

Our first application and illustration of the Mathai-Quillen formalism for infinite dimensional vector bundles will be to the *loop space* \( X = LM \) of a finite dimensional manifold \( M \) and its tangent bundle \( E = T(LM) \). We will see that this is completely equivalent to supersymmetric quantum mechanics [21, 22] whose numerous attractive and interesting properties now find a neat explanation within the present framework. As the authors of [1] were certainly in part inspired by supersymmetric quantum mechanics, deriving the latter that way may appear to be somewhat circuitous. It is, nevertheless, instructive to do this because it illustrates all the essential features of the Mathai-Quillen formalism.
Geometry of loop space

We denote by \( M \) a smooth orientable Riemannian manifold with metric \( g \) and by \( LM = \text{Map}(S^1, M) \) the loop space of \( M \). Elements of \( LM \) are denoted by \( x = \{ x(t) \} \) or simply \( x^\mu(t) \), where \( t \in [0,1] \), \( x^\mu \) are (local) coordinates on \( M \) and \( x^\mu(0) = x^\mu(1) \). In supersymmetric quantum mechanics it is sometimes convenient to scale \( t \) such that \( t \in [0,T] \) and \( x^\mu(0) = x^\mu(T) \) for some \( T \in \mathbb{R} \), and to regard \( T \) as an additional parameter of the theory.

A tangent vector \( V(x) \in T_x(LM) \) at a loop \( x \in LM \) can be regarded as a deformation of the loop, i.e. as a section of the tangent bundle \( TM \) restricted to the loop \( x(t) \) such that \( V(x(t)) \in T_{x(t)}(M) \). In more fancy terms this means that

\[
T_x(LM) = \Gamma(x^*TM)
\]

is the infinite-dimensional space of sections of the pull-back of the tangent bundle \( TM \) to \( S^1 \) via the map \( x : S^1 \to M \).

There is a canonical vector field on \( LM \) generating rigid rotations \( x(t) \to x(t + \epsilon) \) of the loop, given by \( V(x(t)) = \dot{x}(t) \) (or \( V = \dot{x} \) for short). A metric \( g \) on \( M \) induces a metric \( \hat{g} \) on \( LM \) through

\[
\hat{g}_\epsilon(V_1, V_2) = \int_0^1 dt \ g_{\mu\nu}(x(t)) V_1^\mu(x(t)) V_2^\nu(x(t)) .
\]

Likewise, any differential form \( \alpha \) on \( M \) induces a differential form \( \hat{\alpha} \) on \( LM \) via

\[
\hat{\alpha}_\epsilon(V_1, \ldots, V_p) = \int_0^1 dt \ \alpha(x(t))(V_1(x(t)), \ldots, V_p(x(t))) .
\]

Because of the reparametrization invariance of these integrals, \( \hat{g} \) and \( \hat{\alpha} \) are invariant under the flow generated by \( V = \dot{x} \), \( L(V)\hat{g} = L(V)\hat{\alpha} = 0 \). We will make use of this observation in section 3.2.

Supersymmetric quantum mechanics

We will now apply the formalism developed in the previous section to the data \( X = LM \), \( E = T(LM) \), and \( V = \dot{x} \) (and later on some variant thereof). The zero locus \((LM)_V\) of \( V \) is just the space of constant loops, i.e. \( M \) itself. If we therefore define the regularized Euler number of \( LM \) via (2.20) by

\[
\chi_V(LM) := \chi((LM)_V) = \chi(M) ,
\]

we expect the functional integral \( \int_{LM} e^{-V} \chi(LM) \) to calculate the Euler number of \( LM \). Let us see that this is indeed the case.
The anticommuting variables $\chi_a$ parametrize the fibres of $T(LM)$ and we write them as $\chi_a = e^\mu_a \bar{\psi}_a$, where $e^\mu_a$ is the inverse vielbein corresponding to $g_{\mu\nu}$. This change of variables produces a Jacobian $\det e$ we will come back to below.

Remembering to substitute $dx^\mu(t)$ by $\psi^\mu(t)$, the exponent of the Mathai-Quillen form $e_{\nu,\nabla}(LM)$ (2.11) becomes

$$S_M = \int_0^T dt \left[ -\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{4} R^\alpha_{\mu\nu} \bar{\psi}_a \gamma^\mu \gamma^\nu \psi^a - i \bar{\psi}_a \nabla_t \psi^a \right], \quad (2.27)$$

where $\nabla_t$ is the covariant derivative along the loop $x(t)$ induced by $\nabla$. This is precisely the standard action of de Rham (or $N = 1$) supersymmetric quantum mechanics [21, 22, 23, 24, 25] (with the conventions as in [16]), the supersymmetry given by (2.12). In order to reduce the measure to the natural form $[dx][d\psi][d\bar{\psi}]$, one can introduce a multiplier field $B^\mu$ to write the bosonic part of the action as

$$\int_0^T dt \, g_{\mu\nu} (B^\mu \dot{x}^\nu + B^\nu \dot{x}^\mu) \approx \int_0^T dt \left[ -\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right], \quad (2.28)$$

because integration over $B$ will give rise to a determinant $\det[g]^{-1/2}$ that cancels the $\det[e]$ from above. With this understood, the integral we are interested in is the partition function of this theory,

$$\int_{LM} e_{\nu,\nabla}(LM) = Z(S_M) \quad (2.29)$$

Now it is well known that the latter indeed calculates $\chi(M)$ and this is our first non-trivial confirmation that (2.21) makes sense and calculates (2.20). The explicit calculation of $Z(S_M)$ is not difficult, and we will sketch it below.

It may be instructive, however, to first recall the standard a priori argument establishing $Z(S_M) = \chi(M)$. One starts with the definition (2.7) of $\chi(M)$. As there is a one-to-one correspondence between cohomology classes and harmonic forms on $M$, one can write $\chi(M)$ as a trace over the kernel $\text{Ker} \Delta$ of the Laplacian $\Delta = dd^* + d^*d$ on differential forms,

$$\chi(M) = \text{tr} \text{Ker} \Delta (-1)^F \quad (2.30)$$

(here $(-1)^F$ is $+1$ ($-1$) on even (odd) forms). As the operator $d + d^*$ commutes with $\Delta$ and maps even to odd forms and vice-versa, there is an exact pairing between ‘bosonic’ and ‘fermionic’ eigenvectors of $\Delta$ with non-zero eigenvalue. It is thus possible to extend the trace in (2.30) to a trace over the space of all differential forms,

$$\chi(M) = \text{tr} \Gamma_{\text{ev}(M)} (-1)^F e^{-T \Delta}. \quad (2.31)$$

As only the zero modes of $\Delta$ will contribute to the trace, it is evidently independent of the value of $T$. Once one has put $\chi(M)$ into this form of a statistical
mechanics partition function, one can use the Feynman-Kac formula to represent it as a supersymmetric path integral [24] with the action (2.27).

This Hamiltonian way of arriving at the action of supersymmetric quantum mechanics should be contrasted with the Mathai-Quillen approach. In the former one starts with the operator whose index one wishes to calculate (e.g. \( d + d^* \)), constructs a corresponding Hamiltonian, and then deduces the action. On the other hand, in the latter one begins with a finite dimensional topological invariant (e.g. \( \chi (M) \)) and represents that directly as an infinite dimensional integral, the partition function of a supersymmetric action.

What makes such a path integral representation of \( \chi (M) \) interesting is that one can now go ahead and try to somehow evaluate it directly, thus possibly obtaining alternative expressions for \( \chi (M) \). Indeed, one can obtain path integral ‘proofs’ of the Gauss-Bonnet and Poincaré-Hopf theorems in this way. This is just the infinite dimensional analogue of the fact discussed above that different choices of \( s \) in \( \int_X \epsilon_s \nabla (E) \) can lead to different expressions for \( \chi (E) \).

Let us, for example, replace the section \( V = \dot{x} \) in (2.27) by \( \dot{x} \gamma \) for some \( \gamma \in \mathbb{R} \). This has the effect of multiplying the first term in (2.27) by \( \gamma^2 \) and the third by \( \gamma \). In order to be able to take the limit \( \gamma \to \infty \), which would localize the functional integral to \( \dot{x} = 0 \), i.e. to \( M \), we proceed as follows. First of all, one expands all fields in Fourier modes. Then one scales all non-constant modes of \( x(t) \) by \( \gamma \) and all the non-constant modes of \( \psi (t) \) and \( \phi (t) \) by \( \gamma^{1/2} \). This leaves the measure invariant. Then in the limit \( \gamma \to \infty \) only the constant modes of the curvature term survive in the action, while the contributions from the non-constant modes cancel identically between the bosons and fermions because of supersymmetry. The net effect of this is that one is left with a finite dimensional integral of the form (2.4) leading to

\[
Z (S_M) = (2\pi)^{\dim (M)/2} \int_M \text{Pf}(\Omega_{\nabla} (M)) = \chi (M) . \tag{2.32}
\]

This provides a path integral derivation of the Gauss-Bonnet theorem (2.6). The usual supersymmetric quantum mechanics argument to this effect [23, 25] makes use of the \( T \)-independence of the partition function to suppress the non-constant modes as \( T \to 0 \). But the Mathai-Quillen formalism now provides an understanding and explanation of the mechanism by which the path integral (2.29) over \( LM \) localizes to the integral (2.32) over \( M \).

Other choices of sections are also possible, e.g. a vector field of the form

\[
V = \dot{x}^\mu + \gamma g^{\mu \nu} \partial_\nu W ,
\]

where \( W \) is some function on \( M \) and \( \gamma \in \mathbb{R} \) a parameter. This introduces a potential into the supersymmetric quantum mechanics action (2.27). It is easy to see that the zero locus of this vector field is the zero locus of the gradient vector field \( \partial_\mu W \) on \( M \) whose Euler number is the same as that of \( M \) by (2.8). Again this agrees with the explicit evaluation of the path integral of the corresponding supersymmetric quantum mechanics action which is, not
unexpectedly, most conveniently performed by considering the limit $\gamma \to \infty$. In that limit, because of the term $\gamma^2 g^{\mu\nu} \partial_\mu W \partial_\nu W$ in the action, the path integral localizes around the critical points of $W$. Let us assume that these are isolated. By supersymmetry, the fluctuations around the critical points will, up to a sign, cancel between the bosonic and fermionic contributions and one finds that the partition function is

$$\chi(M) = Z(S_M) = \sum_{x_k: dW(x_k) = 0} \text{sign} \left( \det H_{x_k}(W) \right), \quad (2.33)$$

where

$$H_{x_k}(W) = (\nabla_\mu \partial_\nu W)(x_k) \quad (2.34)$$

is the Hessian of $W$ at $x_k$. This is the Poincaré-Hopf theorem (2.5) (for a gradient vector field).

If the critical points of $W$ are not isolated then, by a combination of the above arguments, one recovers the generalization $\chi(M) = \chi(M_W)$ (2.8) of the Poincaré-Hopf theorem (for gradient vector fields) in the form (2.15).

Finally, by considering a section of the form $V = \dot{x} + v$, where $v$ is an arbitrary vector field on $M$, one can also, to complete the circle, rederive the general finite dimensional Mathai-Quillen form $e_v(M)$ (2.11) from supersymmetric quantum mechanics by proceeding exactly as in the derivation of (2.32) [26, 15].

This treatment of supersymmetric quantum mechanics has admittedly been somewhat sketchy and we should perhaps, summarizing this section, state clearly what are the important points to keep in mind as regards the Mathai-Quillen formalism and localization of functional integrals:

1. Explicit evaluation of the supersymmetric quantum mechanics path integrals obtained by formally applying the Mathai-Quillen construction to the loop space $LM$ confirms that we can indeed represent the regularized Euler number $\chi_V(LM)$, as defined by (2.20), by the functional integral (2.21).

2. In particular, it confirms that functional integrals arising from or related to the Mathai-Quillen formalism (extended to infinite dimensional bundles) localize to finite dimensional integrals.

3. This suggests that a large class of other quantum mechanics and field theory functional integrals can be constructed which also have this property. These integrals should be easier to understand in a mathematically rigorous fashion than generic functional integrals arising in field theory.
2.3 Other Examples and Applications - an Overview

The purpose of this section is to provide an overview of other applications and appearances of the Mathai-Quillen formalism in the physics literature. None too surprisingly, all of these are related to topological field theories or simple (non-topological) perturbations thereof.

It was first shown by Atiyah and Jeffrey [10] that the rather complicated looking action of Witten’s four-dimensional topological Yang-Mills theory [27] (Donaldson theory) had a neat explanation in this framework. Subsequently, this interpretation was also shown to be valid and useful in topological sigma models coupled to topological gravity [11]. In [26, 15] this strategy was turned around to construct topological field theories from scratch via the Mathai-Quillen formalism. A nice explanation of the formalism from a slightly different perspective can be found in the recent work of Vafa and Witten [9], and other recent applications include [28, 29].

As these applications thus range from topological gauge theory (intersection numbers of moduli spaces of connections) over topological sigma models (counting holomorphic curves) to topological gravity (intersection theory on some moduli space of metrics) and the string theory interpretation of 2d Yang-Mills theory, this section will not and cannot be self-contained. What we will do is to try to provide a basic understanding of why so-called cohomological topological field theories can generally be understood within and constructed from the Mathai-Quillen formalism.

The basic strategy - illustrated by Donaldson theory

On the basis of what has been done in the previous section, one possible approach would be to ask if topological field theories can in some sense be understood as supersymmetric quantum mechanics models with perhaps infinite-dimensional target spaces. For topological field theory models calculating the Euler number of some moduli space this has indeed been shown to be the case in [17]. In principle, it should also be possible to extend the supersymmetric quantum mechanics formalism of the previous section to general vector bundles $E$ and their Mathai-Quillen forms $e_s$, $\nu$. This generalization would then also cover other topological field theories.

This is, however, not the point of view we are going to adopt in the following. Rather, we will explain how the Mathai-Quillen formalism can be used to construct a field theory describing intersection theory on some given moduli space (of connections, metrics, maps, …) of interest.

Before embarking on this, we should perhaps point out that, from a purely pragmatic point of view, an approach to the construction of topological field theories based on, say, BRST quantization may occasionally be more efficient.
and straightforward. What one is gaining by the Mathai-Quillen approach is geometrical insight.

Assume then, that the finite dimensional moduli space on which one would like to base a topological field theory is given by the following data:

1. A space $\Phi$ of fields $\phi$ and (locally) a defining equation or set of equations $F(\phi) = 0$;
2. A group $\mathcal{G}$ of transformations acting on $\Phi$ leaving invariant the set $F(\phi) = 0$, such that the moduli space is given by
   $$\mathcal{M}_F = \{\phi \in \Phi : F(\phi) = 0\} / \mathcal{G} \subset \Phi / \mathcal{G} \equiv X .$$

This is not the most general set-up one could consider but will be sufficient for what follows.

The prototypical example to keep in mind is the moduli space of anti-self-dual connections (instantons), where $\Phi$ is the space $A$ of connections $A$ on a principal bundle $P$ on a Riemannian four-manifold $(M,g)$, $F(A) = (F_A)_{+}$ is the self-dual part of the curvature of $A$, and $\mathcal{G}$ is the group of gauge transformations (vertical automorphisms of $P$). In other cases either $F$ or $\mathcal{G}$ may be trivial - we will see examples of that below.

Now, to implement the Mathai-Quillen construction, one needs to be able to interpret the moduli space $\mathcal{M}_F$ as the zero locus of a section $s_F$ of a vector bundle $E$ over $X$. In practice, it is often more convenient to work 'upstairs', i.e. equivariantly, and to exhibit the set $\{F = 0\}$ as the zero locus of a $\mathcal{G}$-equivariant vector bundle over $\Phi$. This may seem to be rather abstract, but in practice there are usually moderately obvious candidates for $E$ and $s$. One also needs to choose some connection $\nabla$ on $E$.

Let us again see how this works for anti-self-dual connections. In that case, $F$ can be regarded as a map from $A$ to the space $\Omega^2_+ (M, \text{ad} P)$, i.e. roughly speaking the space of Lie algebra valued self-dual two-forms on $M$. The kernel of this map is the space of anti-self-dual connections. To mod out by the gauge group we proceed as follows. For appropriate choice of $A$ or $\mathcal{G}$, $A$ can be regarded as the total space of a principal $\mathcal{G}$-bundle over $X = A / \mathcal{G}$. As $\mathcal{G}$ acts on $\Omega^2_+ (M, \text{ad} P)$, one can form the associated infinite dimensional vector bundle

$$\mathcal{E}_+ = A \times _{\mathcal{G}} \Omega^2_+ (M, \text{ad} P) .$$

Clearly, the map $F$ descends to give a well-defined section $s_F$ of $\mathcal{E}_+$ whose zero-locus is precisely the finite dimensional moduli space of anti-self-dual connections and we can choose $E = \mathcal{E}_+$. This bundle also has a natural connection coming from declaring the horizontal subspaces in $A \to A / \mathcal{G}$ to be those orthogonal to the $\mathcal{G}$-orbits in $A$ with respect to the metric on $A$ induced by that on $M$. 
Once one has assembled the data \( X, E \) and \( s = s_F \) one can - upon choice of some connection \( \nabla \) on \( E \) - plug these into the formula (2.11) for the Mathai-Quillen form. In this way one obtains a functional integral which localizes onto the moduli space \( \mathcal{M}_F \). A few remarks may help to clarify the status and role of the ‘field theory action’ one obtains in this way.

1. \textit{A priori} this functional integral can be thought of as representing a regularized Euler number \( \chi_s(E) \) or Euler class of the infinite dimensional vector bundle \( E \). As mentioned in section 2.1 (and as we have seen explicitly in section 2.2), this can also be interpreted as the Euler number (class) of a vector bundle over \( \mathcal{M}_F \). If the rank of this vector bundle is strictly less than the dimension of \( \mathcal{M}_F \), then one needs to consider the pairing of this Euler class with cohomology classes on \( \mathcal{M}_F \).

From a mathematical point of view the necessity of this is clear while in physics parlance this amounts to inserting operators (observables) into the functional integral to ‘soak up the fermionic zero modes’.

2. Frequently, the connection and curvature of the bundle \( E \) are expressed in terms of Green’s functions on the underlying ‘space-time’ manifold. As such, the data entering the putative field theory action (the exponent of (2.11)) are non-local in space-time - an undesirable feature for a fundamental action. The ham-handed way of eliminating this non-locality is to introduce auxiliary fields. However, what this really amounts to is to working ‘upstairs’ and constructing an equivariant Mathai-Quillen form (cf. the remarks before (2.11)). In practice, therefore, it is often much simpler to start upstairs directly and to let Gaussian integrals do the calculation of the curvature and connection terms of \( E \) entering (2.11).

Let us come back once again to the moduli space of instantons. Note that, among all the possible sections of \( \mathcal{E}_+ \) the one we have chosen is really the only natural one. Different sections could of course be used in the Mathai-Quillen form. But in the infinite dimensional case there is no reason to believe that the result is independent of the choice of \( s \) and for other choices of \( s \) it would most likely not be of any mathematical interest. With the choice \( s_F \), however, the resulting action is interesting and, not unexpectedly, that of Donaldson theory [30, 27]. We refer to [16, pp. 198-247] for a partial collection of things that can and should be said about this theory.

A topological gauge theory of flat connections

With the Mathai-Quillen formalism at one’s disposal, it is now relatively easy to construct other examples. In the following we will sketch two of them, one for a moduli space of flat connections and the other related to holomorphic curves.
If one is, for instance, interested in the moduli space $\mathcal{M}^3$ of flat connections $F_A = 0$ on a principal $G$-bundle on a three-manifold $M$, one could go about constructing an action whose partition function localizes onto that moduli space as follows. The flatness condition expresses the vanishing of the two-form $F_A$ or - by duality - the one-form $*F_A \in \Omega^1(M, \text{ad} P)$. Now, as the space $A$ is an affine space modelled on $\Omega^1(M, \text{ad} P)$, we can think of $*F_A$ as a section of the tangent bundle of $A$, i.e. as a vector field on $A$. In fact, it is the gradient vector field of the Chern-Simons functional

$$S_{\text{CS}}(A) = \int_M \text{Tr} \ A dA + \frac{2}{3} A^3.$$  

(2.37)

This vector field passes down to a vector field on $A/G$ whose zero locus is precisely the moduli space $\mathcal{M}^3$ of flat connections and our data are therefore $X = A/G$, $E = TX$, and $V(A) = *F_A$ (in accordance with section 2.1 we denote sections of the tangent bundle by $V$). Following the steps outlined above to construct the action form the Mathai-Quillen form (2.11), one finds that it coincides with that constructed e.g. in [31, 32, 33, 26]. Again, as in supersymmetric quantum mechanics, one finds full agreement of

$$\chi_V(A/G) \equiv \chi((A/G)_V) = \chi(\mathcal{M}^3)$$  

(2.38)

with the partition function of the action which gives $\chi(\mathcal{M}^3)$ in the form (2.15), i.e. via the Gauss-Codazzi equations for the embedding $\mathcal{M}^3 \subset A/G$ [26]. We conclude this example with some remarks.

1. In [10] the partition function of this theory was first identified with a regularized Euler number of $A/G$. We can now identify it more specifically with the Euler number of $\mathcal{M}^3$.

2. In [34] it was shown that for certain three-manifolds (homology spheres) $\chi_V(A^3/G^3)$ is the Casson invariant. This has led us to propose the Euler number of $\mathcal{M}^3$ as a generalization of the Casson invariant for more general three-manifolds. See [26] for a preliminary investigation of this idea.

3. Note that the chosen moduli space $\mathcal{M}^3$ itself, via the defining equation $F_A = 0$, determined the bundle $E$ to be used in the Mathai-Quillen construction. Hence it also determined the fact that we ended up with a topological field theory calculating the Euler number of $\mathcal{M}^3$ rather than allowing us to model intersection theory on $\mathcal{M}^3$. While this may appear to be a shortcoming of this procedure, there are also other reasons to believe that intersection theory on $\mathcal{M}^3$ is not a particularly natural and meaningful thing to study.

4. Nevertheless, in two dimensions one can construct both a topological gauge theory corresponding to intersection theory on the moduli space $\mathcal{M}^2$ of
flat connections (this is just the 2d analogue of Donaldson theory) and a topological gauge theory calculating the Euler number of $M^2$. For the former, the relevant bundle is $E = A \times \mathcal{G} \Omega^0(M, \text{ad}P)$ (cf. (2.36)) with section $*F_A$. In the latter case, life turns out not be so simple. The base space $X$ cannot possibly be $A/\mathcal{G}$ as $*F_A$ does not define a section of its tangent bundle. Rather, the base space turns out to be $E$ with $E = TE$ and an appropriate section. Because the fibre directions of $E$ are topologically trivial, this indeed then calculates the Euler number of $M^2$ - see the final remarks in [15].

**Topological sigma models**

Finally, we will consider an example of a topological field theory which is not a gauge theory but rather, in a sense, the most obvious field theoretic generalization of supersymmetric quantum mechanics, namely the topological sigma model [35, 36]. In its simplest form this is a theory of maps from a Riemann surface $\Sigma$ to a Kähler manifold $M$ localizing either onto holomorphic maps (in the so-called $A$-model) or onto constant maps (in the $B$-model). As an obvious generalization of supersymmetric quantum mechanics, the interpretation of the $A$-model in terms of the Mathai-Quillen formalism is completely straightforward. It is already implicit in the Langevin equation approach to the construction of the model [37] and has recently been spelled out in detail in [28, 29]. We will review this construction below. The Mathai-Quillen interpretation of the $B$-model is slightly more subtle and we will discuss that elsewhere.

In terms of the notation introduced at the beginning of this section (cf. (2.35)), the space of fields is the space $\Phi = \text{Map}(\Sigma, M)$ of maps from a Riemann surface $\Sigma$ (with complex structure $j$) to a Kähler manifold $M$ (with complex structure $J$ and hermitian metric $g$). We will denote by $T^{(1,0)}\Sigma$ etc. the corresponding decomposition of the complexified tangent and cotangent bundles into their holomorphic and anti-holomorphic parts.

The defining equation is the condition of holomorphicity of $\phi \in \Phi$ with respect to both $j$ and $J$, $\Phi(\phi) = \bar{\partial}_j \phi = 0$. In terms of local complex coordinates on $\Sigma$ and $M$, $\phi$ can be represented as a map $(\phi^k(z, \bar{z}), \phi^\bar{k}(z, \bar{z}))$ and the holomorphicity condition can be written as

$$\partial_z \phi^k = \partial_{\bar{z}} \phi^\bar{k} = 0 \ .$$

Solutions to (2.39) are known as *holomorphic curves* in $M$. The moduli space of interest is just the space $\mathcal{M}_{\bar{J}_j}$ of holomorphic curves,

$$\mathcal{M}_{\bar{J}_j} = \{ \phi \in \text{Map}(\Sigma, M) : \bar{\partial}_j \phi = 0 \} \ ,$$

so that this is an example where the symmetry group $\mathcal{G}$ is trivial.
Let us now determine the bundle $E$ over $X = \text{Map}(\Sigma, M)$. In the case of supersymmetric quantum mechanics, the map $x(t) \rightarrow \dot{x}(t)$ could be regarded as a section of the tangent bundle $(2.23)$ of $LM = \text{Map}(S^1, M)$. Here the situation is, primarily notationally, a little bit more involved. First of all, as in supersymmetric quantum mechanics, the tangent space at a map $\phi$ is

$$T_\phi(\text{Map}(\Sigma, M)) = \Gamma(\phi^* TM) .$$

(2.41)

The map $\phi \rightarrow d\phi$ can hence be regarded as a section of a vector bundle $\mathcal{E}$ whose fibre at $\phi$ is $\Gamma(\phi^* TM \otimes T^* \Sigma)$. The map

$$s_\Sigma : \phi \rightarrow (\partial_\Sigma \phi^k, \partial_{\bar{z}} \phi^\bar{f})$$

(2.42)

we are interested in can then be regarded as a section of a subbundle $\mathcal{E}^{(0,1)}$ of that bundle spanned by $T^{(1,0)} M$-valued $(0,1)$-forms $\bar{\psi}^k_z$ and $T^{(1,1)} M$-valued $(1,0)$-forms $\bar{\psi}^\bar{f}_{z}$ on $\Sigma$. Evidently, the moduli space $M_{\Sigma}$ is the zero locus of the section $s_\Sigma(\phi) = \partial_\Sigma \phi$ of $E = \mathcal{E}^{(0,1)}$.

Since the bundle $\mathcal{E}^{(0,1)}$ is contracted from the tangent and cotangent bundles of $M$ and $\Sigma$, it inherits a connection from the Levi-Civita connections on $M$ and $\Sigma$. With the convention of replacing one-forms on $X$ by their fermionic counterparts (cf. (2.13,2.27)), the exterior covariant derivative of $s_\Sigma$ can then be written as

$$\nabla s_\Sigma(\psi) = (D_z \psi^k, D_{\bar{z}} \psi^\bar{f}) .$$

(2.43)

One can now obtain the action $S_{TSM}$ for the topological sigma model from the Mathai-Quillen form $S_{TSM}(\Phi) = \int_{\Sigma} d^2z g_{\Phi}(\frac{1}{2} \partial_\Phi \phi^k \partial_\Phi \phi^\bar{f} + i \bar{\psi}_{z}^k D_z \psi^k + i \bar{\psi}_{z}^\bar{f} D_{\bar{z}} \psi^\bar{f}) - R_{k\bar{l}m\bar{n}} \bar{\psi}_{z}^k \bar{\psi}_{\bar{n}z}^\bar{f} \psi_{\bar{l}z}^\bar{f} \psi_{\bar{l}z}^\bar{f} .

(2.44)

For $M$ a Ricci-flat Kähler manifold (i.e. a Calabi-Yau manifold), this action and its $B$-model partner have recently been studied intensely in relation with mirror symmetry, see e.g. [36, 20] and the articles in [38]. Usually, one adds a topological term $S_{TOP}$ to the action which is the pullback of the Kähler form $\omega$ of $M$,

$$S_{TOP} = \int_{\Sigma} \phi^* \omega = \int_{\Sigma} d^2z g_{\Phi}(\partial_\Phi \phi^k \partial_\Phi \phi^\bar{f} + \partial_\bar{z} \phi^k \partial_{\bar{z}} \phi^\bar{f}) .$$

(2.45)

This term keeps track of the instanton winding number. Its inclusion is also natural from the conformal field theory point of view as the ‘topological twisting’ of the underlying supersymmetric sigma model automatically gives rise to this term. We refer to [35] and [16] to background information on the topological sigma model in general and to [36, 20] for an introduction to the subject of mirror manifolds and holomorphic curves close in spirit to the point of view presented here.
The topological sigma model becomes even more interesting when it is coupled to topological gravity [39], as such a coupled model can be regarded as a topological string theory. A similar model, with both $\Sigma$ and $M$ Riemann surfaces, and with localization onto a rather complicated (Hurwitz) space of branched covers, has recently been studied as a candidate for a string theory realization of 2d Yang-Mills theory [28].

**BRST fixed points and localization**

This has in part been an extremely brief survey of some applications of the Mathai-Quillen formalism to topological field theory. The main purpose of this section, however, was to draw attention to the special properties the functional integrals of these theories enjoy. We have seen that it is the interpretation of the geometrical and topological features of the functional integral in terms of the Mathai-Quillen formalism which allows one to conclude that it is *a priori* designed to represent a finite dimensional integral.

In order to develop the subject further, and directly from a functional integral point of view, it may also be useful to know how physicists read off this property as a sort of fixed point theorem from the action or the 'measure' of the functional integral and its symmetries. The key point is that all these models have a Grassmann odd scalar symmetry $\delta$ of the kind familiar from BRST symmetry. This is essentially the field theory counterpart of (2.12), which we repeat here in the form

$$
\delta \chi = is\psi(x), \quad \delta s\psi(x) = \nabla s\psi(\psi).
$$

(2.46)

The 'essentially' above refers to the fact that this may be the $\delta$ symmetry of the complete action only modulo gauge transformations (equivariance) and equations of motion.

Now, roughly one can argue as follows (see [11, 36]). If the $\delta$ action were free, then one could introduce an anti-commuting collective coordinate $\theta$ for it. The path integral would then contain an integration $\int d\theta \ldots$. But $\delta$-invariance implies $\theta$-independence of the action, and hence the integral would be zero by the rules of Berezin integration. This implies that the path integral only receives contributions from some arbitrarily small $\delta$-invariant tubular neighbourhood of the fixed point set of $\delta$. While the integral over the fixed point set itself has to be calculated exactly, the integral over the ‘tranverse’ directions can then (for generic fixed points) be calculated in a stationary phase (or one-loop) approximation. If one wants to make something obvious look difficult, one can also apply this reasoning to the ordinary BRST symmetry of gauge fixing. In that case the above prescription says that one can set all the ghost and multiplier fields to zero provided that one takes into account the gauge fixing condition and the Faddeev-Popov determinant (arising from the quadratic ghost-fluctuation...
term - a truism.

From (2.46) one can read off that, in the case at hand, this fixed point set is precisely the desired moduli space described by the zero locus of $s_F$ and its tangents $\psi$ satisfying the linearized equation $\nabla_{s_F}(\psi) = 0$. In this way the functional integral reduces to an integral of differential forms on $M_F$.

This point of view has the virtue of being based on easily verifiable properties of the BRST like symmetry. It also brings out the analogy with the functional integral generalization of the Duistermaat-Heckman theorem, the common theme being a supersymmetry modelling an underlying geometric or topological structure which is ultimately responsible for localization.

3 Equivariant Localization and the Stationary Phase Approximation - the Duistermaat-Heckman Theorem

In this section we will discuss a localization formula which, roughly speaking, gives a criterion for the stationary phase approximation to an oscillatory integral to be exact.

To set the stage, we will first briefly recall the ordinary stationary phase approximation. Thus, let $X$ be a smooth compact manifold of dimension $n = 2l$, $f$ and $dx$ a smooth function and a smooth density on $X$. Let $t \in \mathbb{R}$ and consider the function (integral)

$$F(t) = \int_X e^{itf}dx.$$  \hspace{1cm} (3.1)

The stationary phase approximation expresses the fact that for large $t$ the main contributions to the integral come from neighbourhoods of the critical points of $f$. In particular, if the critical points of $f$ are isolated and non-degenerate (so that the determinant of the Hessian $H_f = \nabla df$ of $f$ is non-zero there), then for $t \to \infty$ the integral (3.1) can be approximated by

$$F(t) = \left(\frac{2\pi}{t}\right)^l \sum_{x_k, df(x_k) = 0} e^{it\sigma(H_f(x_k))} [\det(H_f(x_k))]^{1/2} e^{-itf(x_k)} + O(t^{-1}),$$  \hspace{1cm} (3.2)

where $\sigma(H_f)$ denotes the signature of $H_f$ (the number of positive minus the number of negative eigenvalues).

Duistermaat and Heckman [2] discovered a class of examples where this approximation gives the exact result and the error term in the above vanishes. To state these examples and the Duistermaat-Heckman formula in their simplest form, we take $X$ to be a symplectic manifold with symplectic form $\omega$ and Liouville measure $dx(\omega) = \omega^l / l$. We will also assume that the function $f$ generates
a circle action on $X$ via its Hamiltonian vector field $V_f$ defined by $i(V_f)\omega = df$, and denote by $L_f(x_k)$ the infinitesimal action induced by $V_f$ on the tangent space $T_{x_k}X$ (essentially, $L_f$ can be represented by the matrix $H_f$). Then the Duistermaat-Heckman formula reads

$$\int_X e^{itf} dx(\omega) = \left(\frac{2\pi i}{t}\right)^{\frac{1}{2}} \prod_{k: df(x_k) \neq 0} \det(L_f(x_k))^{1/2} e^{itf(x_k)}.$$  \hspace{1cm} (3.3)

Being careful with signs and the definition of the square root of the determinant, it can be seen that (3.3) expresses nothing other than the fact that the stationary phase approximation to $F(t)$ is exact for all $t$.

As a simple example consider a two-sphere of unit radius centered at the origin of $\mathbb{R}^3$, with $f(x, y, z) = z$ (or, more generally, $z + a$ for some constant $a$) the generator of rotations about the $z$-axis and $\omega$ the volume form. This integral can of course easily be done exactly, e.g. by converting to polar coordinates, and one finds

$$\int_{S^2} d\omega(x) e^{itf} = \int \sin \theta d\theta d\phi e^{it(\cos \theta + a)} = \left(\frac{2\pi i}{t}\right) \left(e^{-it(1 - a)} - e^{it(1 + a)}\right).$$  \hspace{1cm} (3.4)

We see that the result can be expressed as a sum of two terms, one from $\theta = \pi$ and the other form $\theta = 0$. This is just what one expects form the Duistermaat-Heckman formula, as these are just the north and south poles, i.e. the fixed points of the $U(1)$-action. The relative sign between the two contributions is due to the fact that $f$ has a maximum at the north pole and a minimum at the south pole. This example can be considered as the classical partition function of a spin system. Thinking of the two-sphere as $SU(2)/U(1)$, it has a nice generalization to homogeneous spaces of the form $G/T$, where $G$ is a compact Lie group and $T$ a maximal torus. The corresponding quantum theories are also all given exactly by their stationary phase (WKB) approximation and evaluate to the Weyl character formula for $G$. For an exposition of these ideas see the beautiful paper [40] by Stone.

It is now clear that the Duistermaat-Heckman formula (3.3) (and its generalization to functions with non-isolated critical points) can be regarded as a localization formula as it reduces the integral over $X$ to a sum (integral) over the critical point set. As the stationary phase approximation is very much like the semi-classical or WKB approximation investigated extensively in the physics (and, in particular, the path integral) literature, it is of obvious interest to inquire if or under which circumstances a functional integral analogue of the formula (3.3) can be expected to exist. In this context, (3.3) would now express something like the exactness of the one-loop approximation to the path integral.

The first encouraging evidence for the existence of such a generalization was pointed out by Atiyah and Witten [41]. They showed that a formal application of the Duistermaat-Heckman theorem to the infinite-dimensional loop space
$X = LM$ of a manifold $M$ and, more specifically, to the partition function of $N = \frac{1}{2}$ supersymmetric quantum mechanics representing the index of the Dirac operator on $M$, reproduced the known result correctly. This method of evaluating the quantum mechanics path integral has been analyzed by Bismut [19] within a mathematically rigorous framework. Subsequently, the work of Stone [40] brought the Duistermaat-Heckman theorem to the attention of a wider physicists audience. Then, in [42, 43], a general supersymmetric framework for investigating Duistermaat-Heckman- (or WKB-)like localization theorems for (non-supersymmetric) phase space path integrals was introduced, leading to a fair amount of activity in the field - see e.g. [44, 45, 46, 47, 48, 49, 50]. We will review some of these matters below.

While Duistermaat and Heckman originally discovered their formula within the context of symplectic geometry, it turns out to have its most natural explanation in the setting of equivariant cohomology and equivariant characteristic classes [51, 3, 4]. This point of view also suggests some generalizations of the Duistermaat-Heckman theorem, e.g. the Berline-Vergne formula [3] valid for Killing vectors on general compact Riemannian manifolds $X$. Strictly speaking, the example considered by Atiyah and Witten falls into this category as the free loop space of a manifold is not quite a symplectic manifold in general. We will discuss the interpretation of these formulae in terms of equivariant cohomology below. An excellent survey of these and many other interesting matters can be found in [4].

There is also a rather far-reaching generalization of the localization formula which is due to Witten [52]. It applies to non-Abelian group actions and to `actions' (the exponent on the lhs of (3.3)) which are polynomials in the `moment map' $f$ rather than just linear in the moment map as in (3.3). This generalization is potentially much more interesting for field-theoretic applications, and we give a brief sketch of the set-up in section 3.3, but as a detailed discussion of it would necessarily go far beyond the elementary and introductory scope of this paper we refer to [52, 53, 54] for further information.

### 3.1 Equivariant Cohomology and Localization

As mentioned above, the localization theorems we are interested in, in this section, are most conveniently understood in terms of equivariant cohomology and we will first introduce the necessary concepts. Actually, all we will need is a rather watered down version of this, valid for Abelian group actions. The following discussion is intended as a compromise between keeping things elementary in this simple setting and retaining the flavour and elegance of the general theory. We will see that the localization formulae of Berline-Vergne [3] and Duistermaat-Heckman [2] are then fairly immediate consequences of the
general formalism. Everything that is said in this section can be found in [4].

**Equivariant cohomology**

Let $X$ be a compact smooth $(n = 2l)$-dimensional manifold, and let $G$ be a compact group acting smoothly on $X$ (the restriction to even dimension is by no means necessary - it will just allow us to shorten one of the arguments below). We denote by $\mathfrak{g}$ the Lie algebra of $G$, by $\mathbb{C}[\mathfrak{g}]$ the algebra of complex valued polynomial functions on $\mathfrak{g}$, and by $\Omega^*(X)$ the algebra of complex valued differential forms on $X$. For any $V \in \mathfrak{g}$ there is an associated vector field on $X$ which we denote by $V_X$. We note that any metric $h$ on $X$ can be made $G$-invariant by averaging $h$ over $G$. Hence we can assume without loss of generality that $G$ acts on the Riemannian manifold $(X, h)$ by isometries and that all the vector fields $V_X$ are Killing vectors of $h$.

The $G$-equivariant cohomology of $X$, a useful generalization of the cohomology of $X/G$ when the latter is not a smooth manifold, can be defined by the following construction (known as the Cartan model of equivariant cohomology). We consider the tensor product $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X)$. As there is a natural (adjoint) action of $G$ on $\mathbb{C}[\mathfrak{g}]$ and the $G$-action on $X$ induces an action on $\Omega^*(X)$, one can consider the space $\Omega^*_G(X)$ of $G$-invariant elements of $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X)$. Elements of $\Omega^*_G(X)$ will be called equivariant differential forms. They can be regarded as $G$-equivariant maps $\mu : V \to \mu(V)$ from $\mathfrak{g}$ to $\Omega(X)$. The $\mathbb{Z}$-grading of $\Omega^*_G(X)$ is defined by assigning degree two to the elements of $\mathfrak{g}$. $G$-equivariance implies that the operator

$$d_G : \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X) \to \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(X)$$

$$(d_G \mu)(V) = d(\mu(V)) - i(V_X)(\mu(V)),$$  \hspace{1cm} (3.5)

which raises the degree by one, squares to zero on $\Omega^*_G(X)$. The $G$-equivariant cohomology $H^*_G(X)$ of $X$ is then defined to be the cohomology of $d_G$ acting on $\Omega^*_G(X)$. In analogy with the ordinary notions of cohomology an equivariant differential form $\mu$ is said to be equivariantly closed (respectively equivariantly exact) if $d_G \mu = 0$ ($\mu = d_G \lambda$ for some $\lambda \in \Omega^*_G(X)$).

It follows from these definitions that $H^*_G(X)$ coincides with the ordinary cohomology if $G$ is the trivial group, and that the $G$-equivariant cohomology of a point is the algebra of $G$-invariant polynomials on $\mathfrak{g}$, $H^*_G(\{\mathfrak{g}\}) = \mathbb{C}[\mathfrak{g}]^G$.

Integration of equivariant differential forms can be defined as a map from $\Omega^*_G(X)$ to $\mathbb{C}[\mathfrak{g}]^G$ by

$$\int_X \mu(V) = \int_X \mu(V),$$  \hspace{1cm} (3.6)

where it is understood that the integral on the right hand side picks out the top-form component $\mu(V)^{(n)}$ of $\mu(V)$ and the integral is defined to be zero if $\mu$ contains no term of form-degree $n$.  

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A final thing worth pointing out is that one can also generalize the standard notions of characteristic classes, Chern-Weil homomorphism etc. to the equivariant case. The equivariant Euler form of a vector bundle $E$ over $X$ to which the action of $G$ lifts, with $\nabla$ a $G$-invariant connection on $E$, will make a brief appearance later. It is defined in the standard way by (2.3), with the curvature $\Omega_\nabla$ replaced by the equivariant curvature

$$\Omega^E_{\nabla}(V) = \Omega_\nabla + (L^E(V) - \nabla_V)$$

(3.7)

of $E$. Here $L^E(V)$ denotes the infinitesimal $G$-action on $E$ and the term in brackets is also known as the equivariant moment of the action - in analogy with the moment map of symplectic geometry. It can be checked directly that the differential form one obtains in this way is equivariantly closed and the usual arguments carry over to this case to establish that its cohomology class does not depend on the choice of $G$-invariant connection $\nabla$.

For $E$ the tangent bundle of $X$ and $\nabla$ the Levi-Civita connection one finds that, as a consequence of $\nabla_V W - \nabla_W V = [V,W]$ (no torsion), the Riemannian moment map is simply $-(\nabla V)_{\mu\nu} = -h_{\mu\nu} V^\lambda$. Note that this matrix is antisymmetric because $V$ is Killing. Hence the Riemannian equivariant curvature is simply

$$\Omega^e_{\nabla}(V) = \Omega_\nabla - \nabla V.$$  

(3.8)

The equivariant $\hat{\chi}$-genus $\hat{\chi}(\Omega^e_{\nabla})$ of the tangent bundle appears in the localization theorem of [47] which we will discuss below.

This is as far as we will follow the general story. In our applications we will be interested in situations where we have a single Killing vector field $V_X \equiv V$ on $X$, perhaps corresponding to an action of $G = S^1$. This means that we will be considering equivariant differential forms $\mu(V)$ evaluated on a fixed $V \in \mathfrak{g}$. In those cases, nothing is gained by carrying around $\mathbb{C}[\mathfrak{g}]$ and we will simply be considering the operator $d(V) = d - i(V)$ on $\Omega^*(X)$. By the Cartan formula, the square of this operator is (minus) the Lie derivative $L(V)$ along $V$,

$$(d(V))^2 = (d - i(V))^2 = -(i(V)d + d(i(V))) = -L(V),$$  

(3.9)

so that it leaves invariant the space $\Omega^V_\nabla(X)$ of $V$-invariant forms on $X$ (the kernel of $L(V)$) and squares to zero there. Thus it makes sense to consider the cohomology of $d(V)$ on $\Omega^V_\nabla(X)$ and we will call forms on $X$ satisfying $d(V)\alpha = 0$ $d(V)$-closed or equivariantly closed etc. The final observation we will need is that if a differential form is $d(V)$-exact, $\alpha = d(V)\gamma$, then its top-form component is actually exact in the ordinary sense. This is obvious because the $i(V)$-part of $d(V)$ lowers the form-degree by one so there is no way that one can produce a
top-form by acting with $i(V)$.

The Berline-Vergne and Duistermaat-Heckman localization formulae

We now come to the localization formulae themselves. We will see that integrals over $X$ of $d(V)$-closed forms localize to the zero locus of the vector field $V$. There are two simple ways of establishing this localization and as each one has its merits we will present both of them. It requires a little bit more work to determine the precise contribution of each component of $X_V$ to the integral, and here we will only sketch the required calculations and quote the result.

The essence of the localization theorems is the fact that equivariant cohomology is determined by the fixed point locus of the $G$-action. The first argument for localization essentially boils down to an explicit proof of this fact at the level of differential forms. Namely, we will show that any equivariantly closed form $\alpha$, $d(V)\alpha = 0$, is equivariantly exact away from the zero locus $X_V = \{x \in X : V(x) = 0\}$ of $V$. To see this, we will construct explicitly a differential form $\beta$ on $X \setminus X_V$ satisfying $d(V)\beta = \alpha$. This implies that the top-form component of $\alpha$ is exact, it then follows from Stokes theorem that the integral $\int_X \alpha$ only receives contributions from an arbitrarily small neighbourhood of $X_V$ in $X$.

It is in the construction of $\beta$ that the condition enters that $V$ be a Killing vector. Using the invariant metric $h$ we can construct the metric dual one-form $h(V)$. It satisfies $d(V)h(V) = d(h(V)) - |V|^2$ and (since $V$ is a Killing vector) $L(V)h(V) = 0$. Away from $X_V$, the zero-form part of $d(V)h(V)$ is non-zero and hence $d(V)h(V)$ is invertible. Here the inverse of an inhomogeneous differential form with non-zero scalar term is defined by analogy with the formula $(1 + x)^{-1} = \sum_k (-x)^k$. We now define the (inhomogeneous) differential form $\Theta$ by

$$\Theta = h(V)(d(V)h(V))^{-1}. \quad (3.10)$$

It follows immediately that

$$d(V)\Theta = 1, \quad L(V)\Theta = 0. \quad (3.11)$$

Hence, 1 is equivariantly exact away from $X_V$ and we can choose $\beta = \Theta \alpha$,

$$d(V)\alpha = 0 \quad \Rightarrow \quad \alpha = (d(V)\Theta)\alpha = d(V)(\Theta \alpha) \quad \text{on} \quad X \setminus X_V \quad (3.12)$$

so that any equivariantly closed form is equivariantly exact away from $X_V$. In particular, the top-form component of an equivariantly closed form is exact away from $X_V$.

Let us see what this amounts to in the example (3.4) of the introduction to this section (cf. also (2.17,3.47)). Using the symplectic form $\omega = \sin \theta d\theta d\phi$, the Hamiltonian vector field corresponding to $\cos \theta$ is $V = \partial_\phi$. This vector field
generates an isometry of the standard metric \( d\theta^2 + \sin^2 \theta d\phi^2 \) on \( S^2 \), and the corresponding \( \Theta \) is \( \Theta = -d\phi \). This form is, as expected, ill defined at the two poles of \( S^2 \). Now the integral (3.4) can be written as

\[
\int_{S^2} \omega e^{itf} = (it)^{-1} \int_{S^2} d(e^{itf}d\phi) ,
\]
thus receiving contributions only from the critical points \( \theta = 0, \pi \), the endpoints of the integration range for \( \cos \theta \), in agreement with the explicit evaluation.

Alternatively, to establish localization, one can use the fact that the integral \( \int_X \alpha \) of a \( d(V) \)-closed form only depends on its cohomology class \([\alpha]\) to evaluate the integral using a particularly suitable representative of \([\alpha]\) making the localization manifest. Again with the help of an invariant metric, such a representative can be constructed. Consider the inhomogeneous form

\[
\alpha_t = \alpha e^{td(V)h(V)} , \quad t \in \mathbb{R} .
\]
Clearly, \( \alpha_t \) is cohomologous to \( \alpha \) for all \( t \). Equally clearly, as \( d(V)h(V) = -|V|^2 + d(h(V)) \), the form \( \alpha_t \) is increasingly sharply Gaussian peaked around \( X_V \subset X \) as \( t \to \infty \). Hence, evaluating the integral as

\[
\int_X \alpha = \lim_{t \to \infty} \int_X \alpha_t ,
\]
reestablishes the localization to \( X_V \).

Perhaps it is good to point out here that there is nothing particularly unique about the choice \( \alpha_t \) to localize the integral of \( \alpha \) over \( X \). Indeed, instead of \( h(V) \) one could choose some other \( L(V) \)-invariant form \( \beta \) in the exponent of (3.14) and try to evaluate \( \int_X \) as some limit of

\[
\int_X \alpha = \int_X \alpha e^{td(V)\beta} .
\]
This can potentially localize the integral to something other than \( X_V \) (we will see an example of this in the context of path integrals later on), and so lead to a seemingly rather different expression for the integral. In principle this argument for localization could also work without the assumption that \( V \) is a Killing vector, but it appears to be difficult to make any general statements in that case. However, as everything in sight is clearly \( d(V) \)-closed, it is possible to reduce the resulting expression further to \( X_V \) by applying the above localization arguments once more, now to the localized expression.

One caveat to keep in mind is that the above statements require some qualifications when the manifold \( X \) is not compact. In that case, \( t \)-independence of the rhs of (3.16) is only ensured if the asymptotic behaviour of \( \alpha \) is not changed.
by the replacement $\alpha \to \alpha \exp d(V)$. For a version of the Duistermaat-Heckman formula for non-compact manifolds see [55]. The extension to non-compact groups is also not immediate. For example [4], consider the nowhere vanishing vector field
\[ V = (1 + \frac{1}{2} \sin x) \partial_y \]  
(3.17)
on the two-torus $(x, y \in \mathbb{R}/2\pi\mathbb{Z})$. Then one can easily check that
\[ \alpha(V) = -\frac{1}{2} (7 \cos x + \sin 2x) + (1 - 4 \sin x) dx dy \]  
(3.18)
is equivariantly closed. Its integral over the torus, on the other hand, is equal to $(2\pi)^2$, so that the two-form component of $\alpha(V)$ cannot possibly be exact.

Another ambiguity can arise when $\alpha$ is equivariantly closed with respect to two vector fields $V$ and $W$, $d(V)\alpha = d(W)\alpha = 0$. In this case one can get seemingly different expressions by localizing to either $X_V$ or $X_W$ or $X_V \cap X_W$. Of course, in whichever way one chooses to evaluate the integral, the results are guaranteed to agree, even if not manifestly so.

Having established that the integral will indeed localize, to obtain a localization formula one needs to determine the contributions to the integral from the connected components of $X_V$. This is most readily done when $X_V$ consists of isolated points $x_k$, $V(x_k) = 0$. In that case the Lie derivative $L_V$ induces invertible linear transformations $L_V(x_k)$ on the tangent spaces $T_{x_k}X$. One can introduce local coordinates in a $G$-invariant neighbourhood of these points such that the localized integrals become essentially Gaussian. Then one finds that the contribution of $x_k$ to the integral is just $(-2\pi)^d \det (L_V(x_k))^{1/2}$. Restricting the equivariantly closed form $\alpha$ to $X_V$, only the scalar part $\alpha^{(0)}(x_k)$ survives and one finds the Berline-Vergne localization formula
\[ d(V)\alpha = 0 \quad \Rightarrow \quad \int_X \alpha = (-2\pi)^d \sum_{x_k \in X_V} \det (L_V(x_k))^{1/2} \alpha^{(0)}(x_k). \]  
(3.19)
If $X_V$ has components of non-zero dimension, things become a little bit more complicated. In that case, in the above discussion the tangent space $T_{x_k}X$ has to be replaced by the normal bundle $N$ to $X_V$ in $X$ and $L_V(x_k)$ by its equivariant curvature (3.7). As $V = 0$ on $X_V$, the scalar part of (3.7) reduces to $L^N(V)$, thought of as an endomorphism of $N$. The localization formula one obtains in this case is then
\[ d(V)\alpha = 0 \quad \Rightarrow \quad \int_X \alpha = \int_{X_V} \det(L^N(V) + \Omega_V)^{1/2} \alpha|_{X_V}, \]  
(3.20)
(a sum over the components of $X_V$ being understood). As pointed out before, the form appearing in the denominator of this formula can be regarded as the equivariant Euler form of the normal bundle $N$. We refer to [4] and [56] for details and some of the beautiful applications of this formula.
We will now consider a particular case of the Berline-Vergne formula. Namely, let \((X, \omega)\) be a symplectic manifold and assume that there is a Hamiltonian action of \(G\) on \(X\). This means that the vector fields \(V_X\) generating the action of \(G\) on \(X\) are Hamiltonian. That is, there is a moment map \(\mu : X \to g^*\) such that \(\langle \mu, V \rangle \equiv \mu(V) \in C^\infty(X)\) generates the action of \(G\) on \(X\) via the Hamiltonian vector fields \(V_X\). Since \(\omega\) is closed, the defining equation

\[
i(V_X)\omega = d(\mu(V))
\]

implies that \(\mu + \omega\) is an equivariantly closed form,

\[
d_\theta(\mu + \omega)(V) = d(\mu(V)) - i(V_X)\omega = 0 .
\]

In fact, finding an equivariantly closed extension of \(\omega\) is equivalent to finding a moment map for the \(G\)-action above. Indeed in geometric quantization it is well known that the ‘prequantum operator’ \(L_E(V)\) can be realized as \(L_E(V) = \nabla_V + \mu(V)\).

To produce from this an equivariantly closed form on \(X\) which we can integrate (which has a top-form component), we consider the form \(\exp \, it(\mu(V) + \omega)\) which is annihilated by \(d(V)\). Its integral is

\[
\int_X e^{it(\mu(V) + \omega)} = \int_X \frac{(it\omega)^l}{l!} e^{it\mu(V)}
\]

\[
= \langle it \rangle^l \int_X dx(\omega) e^{it\mu(V)} . \tag{3.23}
\]

Assuming that the zeros of \(V_X\) are isolated, we can apply the Berline-Vergne formula (3.19) to obtain

\[
\int_X dx(\omega) e^{it\mu(V)} = \langle it \rangle^l (-2\pi)^l \sum_{x_k \in X_V} \det(L_V(x_k)) \frac{1}{l!} e^{it\mu(V)}(x_k) . \tag{3.24}
\]

As the critical points of \(\mu(V)\) are precisely the zeros of \(V_X\) (\(\omega\) is non-degenerate) this is the Duistermaat-Heckman formula (3.3) if \(V\) is a generator of a circle action for some \(U(1) \subset G\). For an example we refer back to the classical spin system discussed in the introduction to this section. Of course there is a corresponding generalization of the Duistermaat-Heckman formula for the case that the zeros of \(V_X\) are not isolated which follows from applying (3.20) to (3.23).

### 3.2 Localization Formulae for Phase Space Path Integrals

In quantum mechanics there are not too many path integrals that can be evaluated explicitly and exactly, while the semi-classical approximation can usually
be obtained quite readily. It is therefore of obvious interest to investigate if there is some path integral analogue of the Duistermaat-Heckman and Berline-Vergne formulae.

One class of quantum mechanics models for which the Duistermaat-Heckman formula clearly seems to make sense is $N = \frac{1}{2}$ supersymmetric quantum mechanics. We will come back to this in section 3.3. What one would really like, however, is to have some version of the equivariant localization formulae available which can be applied to non-supersymmetric models and when the partition functions cannot be calculated directly (or only with difficulty) by some other means. One non-trivial field theoretic example in which (a non-Abelian version of) the Duistermaat-Heckman theorem has been applied successfully is two-dimensional Yang-Mills theory [52]. This theory has a natural symplectic interpretation because the space of gauge fields in two dimensions is symplectic and the Yang-Mills action is just the square of the moment map generating gauge transformations of the gauge fields. Hence one is in principle in the right framework to apply equivariant localization. We will sketch an Abelian version of the localization formula for the square of the moment map in section 3.3.

A large class of examples where one also has an underlying equivariant cohomology which could be responsible for localization is provided by phase space path integrals, i.e. the direct loop space analogues of the lhs of (3.3). Now phase space path integrals are of course notoriously awkward objects, and it is for this reason that the localization formulae we will obtain in this way should not be regarded as definite predictions but rather as suggestions for what kind of results to expect. Because of the lack of rigour that goes into the derivation of these localization formulae it is perhaps surprising that nevertheless some of the results that have been obtained are not only conceptually interesting but also physically reasonable.

**Equivariant cohomology for phase space path integrals**

In this section we choose our manifold $X$ to be the loop space $X = LM$ (see section 2.2) of a finite-dimensional symplectic manifold $(M, \omega)$, the phase space of a classical system. $LM$ is also a symplectic manifold in the sense that $\omega$ (cf. (2.25)) defines a closed and non-degenerate two-form on $LM$. We denote by $S = S[x]$ the phase space action and by $H$ its Hamiltonian. Thus, if we denote by $\theta$ a local symplectic potential for $\omega$, $d\theta = \omega$, then the action is essentially of the form

$$S[x] = \int_0^T dt \ (\theta_\mu(x(t))dx^\mu(t) - H(x(t))).$$

(3.25)

Of course, when $\omega$ is not globally exact this has to be suitably defined by regarding the first term as a Wess-Zumino like term for $\omega$, but this is a standard procedure which will be tacitly adopted in the following.
Consider now the phase space path integral

$$Z(T) = \int_{LM} [dx(\hat{\omega})] e^{\frac{i}{\hbar} S[x]} .$$

(3.26)

As in (3.23), we will lift the symplectic form into the exponent which will then take the form \( S + \hat{\omega} \). Clearly, therefore, the integrand is equivariantly closed with respect to the flow generated on the loop space by the Hamiltonian vector field \( V_S \) of the action \( S \),

$$d(V_S)(S + \hat{\omega}) = 0 .$$

(3.27)

The zeros of this vector field are precisely the critical points of the action, i.e. the classical solutions. This is the reason why this set-up has the potential to produce WKB-like localization formulae \cite{42, 43}.

Let us make the above formulæ a little bit more explicit. First of all, if we denote by \( H \) the Hamiltonian corresponding to \( S \), then the components of \( V_S \) can be written as

$$V_S^\mu(x(t)) = \dot{x}^\mu(t) - \omega^{\mu\nu}(x(t))\partial_\nu H(x(t)) ,$$

(3.28)

so that clearly \( V_S(x) = 0 \) when \( x(t) \) satisfies the classical equations of motion. The first term is just the canonical vector field \( V = \dot{x} \) on \( LM \), while the second is the Hamiltonian vector field \( V_H \) of \( H \) on \( M \), regarded as a vector field on \( LM \).

Using also the trick of section 2 (2.14) to replace the loop space one-forms \( dx(t) \) by anticommuting variables \( \psi(t) \), we can write the partition function (3.26) as

$$Z(T) = \int_{LM} [dx][d\psi] e^{\frac{i}{\hbar}(S + \hat{\omega}(\psi))} .$$

(3.29)

Then the fact (3.27) that \( S + \hat{\omega} \) is equivariantly closed translates into the statement that the augmented action \( S[x, \psi] \) appearing in (3.29) is invariant under the supersymmetry

$$\delta x^\mu = \psi^\mu , \quad \delta \psi^\mu = -V_S^\mu \quad \Rightarrow \quad \delta(S + \hat{\omega}(\psi)) = 0 .$$

(3.30)

We will mostly set \( \hbar = 1 \) in the following.

### Localization formulæ

The above is of course not yet sufficient to establish that (formally) the path integral localizes. As we know from the previous section, at the very least we need a metric on \( LM \) with respect to which \( V_S \) is Killing. To investigate when this is the case, we choose a metric \( \hat{g} \) on \( LM \) of the ultra-local form (2.24). As any such metric is invariant under \( V = \dot{x} \), \( L(V)\hat{g} = 0 \), the condition \( L(V_S)\hat{g} = 0 \)
reduces to the condition that the Hamiltonian vector field $V_H$ of $H$ on $M$ be a Killing vector of the metric $g$,

$$ L(V_S)g = 0 \iff L(V_H)g = 0. \quad (3.31) $$

This is exactly the same condition we encountered in the finite-dimensional case. For these examples, then, for which there are localization formulae for the classical partition function, can we hope to find analogous formulae for the quantum partition function as well.

We now proceed as in the finite-dimensional case and localize the integral by adding some appropriate $d(V_S)$-exact term $d(V_S)\beta$ with $L(V_S)\beta = 0$ to the action. First of all we need to show that this does not change the value of the path integral. In the finite dimensional case this (or the $t$-independence of integrals like (3.15,3.16)) relied, at least implicitly, on Stokes' theorem which is not directly available for integrals over $LM$. However, instead of that one has a kind of Stokes' theorem in the form of a Ward identity associated with the supersymmetry $\delta$ (or, in more elementary terms, a change of variables argument [43]) to reach the desired conclusion provided that the supersymmetry is non-anomalous. We will assume this and proceed with fingers crossed.

The obvious candidate for $\beta$ is, as in the finite dimensional case, the metric dual one-form $\tilde{g}(V_S)$ of the Hamiltonian vector field $V_S$ itself [43] which, by our assumption $L(V_H)g = 0$, satisfies $L(V_S)\tilde{g}(V_S) = 0$. This choice will formally lead to a localization onto the critical points of $S$ and hence onto classical trajectories $x_c(t)$. If these are isolated and non-degenerate this will lead to the exact analogue of the Duistermaat-Heckman formula (3.3), namely

$$ Z(T) \sim \sum_{x_c(t)} \det[L_S(x_c)]^{1/2} e^{i\frac{\pi}{4} S[x_c]} . \quad (3.32) $$

Here the determinant is essentially the functional determinant of the Jacobi operator (the Hessian of the action). This formula can be interpreted as saying that the WKB approximation to the path integral is exact, the only difference to the usual WKB approximation being that here one is to sum over all critical points of the action and not just the local minima.

The validity of (3.32) has been well investigated in the path integral literature (although more for configuration space path integrals), and we have little to add to that discussion. We only want to point out that in those examples we know of where the semi-classical approximation is known to be correct, the assumptions that went into the derivation of (3.32) here, like the existence of an invariant phase space metric, are indeed satisfied (see e.g. [58, 59, 44] for the propagator of a particle moving on a group manifold).

Nevertheless, (3.32) should be taken with a grain of salt. First of all, as (3.32) is a sum over classical periodic trajectories with period $T$, these will typically...
either occur only at the critical points of the Hamiltonian (in which case (3.32) resembles even more closely the classical Duistermaat-Heckman formula) or on a non-zero dimensional submanifold of $M$ (in which case one should really use the degenerate version of the Duistermaat-Heckman formula). Moreover, more care has to be exercised (already in the finite dimensional case) when one is dealing with non-compact phase spaces, which is the rule in systems of physical interest.

However, even if this formula or its obvious modifications are not correct as they stand, it would be interesting to uncover the reason for that. At the very least this could then provide one with a systematic geometric method for analyzing corrections to the WKB approximation. We are not aware of any work that has been done in this direction yet.

One of the strengths of the present setting is the ability to produce other, different, localization formulae, with perhaps different ranges of validity, by exploiting the flexibility in the choice of $\beta$. For example, one can easily convince oneself that the two terms of $\hat{g}(V_S)$ (3.28), namely $\hat{g}(V = \dot{x})$ and $\hat{g}(V_H)$, are separately invariant under $V_S$ so that one can choose $\beta$ to be a general linear combination of them [45, 46],

$$\beta = s\hat{g}(x) - r\hat{g}(V_H).$$

(3.33)

For $s = r$ this reduces to what we did above. But, by choosing e.g. $r = 0$ and $s \to \infty$ one can localize the path integral (3.29) to an integral over the classical phase space $M$. Alternatively, by choosing $s = 0$ and $r \to \infty$, one can obtain an expression for $Z(T)$ in terms of the critical points of the Hamiltonian. We will only consider the first of these possibilities here, as it leads to a nice expression in terms of equivariant characteristic classes. The action one obtains in this way is

$$S^s[x, \psi] = S[x] + \hat{\omega}(\psi) + s\hat{d}(V_S)\hat{g}(x)$$

$$= S[x] + \hat{\omega}(\psi) + s\hat{d}(\hat{g}(x))(\psi) - s\hat{g}(\dot{x}, \dot{x}) + s\hat{g}(\dot{x}, V_H)$$

$$= S[x] + \hat{\omega}(\psi) + s\int_0^T dt \psi(t)\nabla_1\psi(t) + \hat{g}(\dot{x}(t), \dot{x}(t)) - g(\dot{x}(t), V_H(\dot{x}(t)))$$

As in (2.27), $\nabla_1$ denotes the covariant derivative along the loop $x(t)$ induced by the Riemannian connection $\nabla$ on $M$. This action can be evaluated as in supersymmetric quantum mechanics. One possibility is to expand all fields in Fourier modes and to scale the non-constant modes by $s^{-1/2}$ so as to eliminate positive powers of $s$ from the action. Then the limit $s \to \infty$ can be taken with impunity, the integration over the non-constant modes gives rise to a ratio of determinants and in the exponent one is left with the contribution from the
constant modes, namely $T(\omega(\psi) - H)$. The result one obtains is

$$Z(T) = \int_M dx d\psi \det \left[ \frac{\Omega^2}{\sinh[\frac{\Omega^2}{2}(\Omega^2 + \nabla^2 V_H)]} \right]^{1/2} e^{iT(\omega(\psi) - H)}$$

(see (3.42) and (3.43) in section 3.3 for a brief explanation of the appearance of this particular determinant in supersymmetric path integrals). This is the Nemi-Tirkkonen localization formula [47]. In this expression we recognize the $d(-V_H)$-equivariant Riemannian curvature (3.8) and its Â-genus (the occurrence of minus signs at awkward places is unavoidable in symplectic geometry ...). The exponent can also be interpreted in equivariant terms. Namely, as we have seen in the discussion of the Duistermaat-Heckman theorem, as the $(-V_H)$-equivariant extension of the symplectic form $\omega$. Roughly speaking (modulo the fact that $\omega$ need not be integral) the exponential can then be regarded as an equivariant Chern character and the result can then be written more succinctly and elegantly in the notation of (3.6) as

$$Z(T) = \left( \int_M \hat{A}(T\Omega_{V_H}^{\omega^{ec}(n)}) \mathrm{Ch}(T\omega^{ec(n)}) \right) (-V_H) .$$

This differs from the classical partition function for the dynamical system described by $H$ by the Â-term which can be thought of as encoding the information due to the quantum fluctuations.

As the integrand is clearly $d(-V_H)$-closed, this integral can be localized further to an integral over the critical points of $H$ provided that the zero locus of $V_H$ is non-degenerate. The possible advantage of the present formula is that no such assumption appears to have been necessary in the derivation of (3.35) which thus potentially applies to examples where the ordinary WKB approximation breaks down (e.g. when classical paths coalesce).

This possibility has been analyzed in [48] for some simple one-dimensional quantum mechanics examples, the harmonic oscillator and the 'hydrogen atom', i.e. a particle moving in a $1/|x|$-potential. The latter example, in particular, is interesting because there the classical paths are known to coalesce so that the traditional WKB formula is not applicable. The results obtained in [48] as well as some further considerations in [49] illustrate the potential usefulness of these generalized localization formulae. However, in order to establish to what extent formulae like the above are trustworthy and can be turned into reliable calculational tools, other (higher dimensional) examples will need to be worked out.

### 3.3 Other Examples and Applications - an Overview

In this section we shall sketch some of the other applications of equivariant localization and, in particular, the Duistermaat-Heckman formula. Most prominent
among these are applications to supersymmetric quantum mechanics and index
theorems.

Supersymmetric quantum mechanics and index theorems

As mentioned in the introduction to this section, the first infinite dimensional
application of the Duistermaat-Heckman theorem is due to Atiyah and Witten
[41] (see also [19] and [60]) who applied it to the loop space $LM$ of a Riemannian
manifold and $N = \frac{1}{2}$ supersymmetric quantum mechanics. This is the model
which represents the index of the Dirac operator [23, 24, 25] in the same way
that the $N = 1$ model we discussed in section 2.2 represents the index of the de
Rham complex. Just like the Mathai-Quillen formalism provides the appropriate
framework for understanding the topological and localization properties of $N = 1$ (de Rham)
supersymmetric quantum mechanics, those of $N = \frac{1}{2}$ (Dirac)
supersymmetric quantum mechanics find their natural explanation within the
framework of loop space equivariant localization. And, just as in the case of the
Mathai-Quillen formalism, this way of looking at the $N = \frac{1}{2}$ models provides
some additional insights and flexibility in the evaluation of the path integrals
[46].

Roughly speaking, the action of the $N = \frac{1}{2}$ model can be obtained from that
of the $N = 1$ model (2.27) by setting $\psi = \bar{\psi}$. Then the four-fermi curvature term
drops out by the Bianchi identity $R^{\mu}_{\nu \rho \sigma} = 0$ and (with a suitable rescaling of
the fermi fields) one is left with

$$S_M[x, \psi] = \int_0^\beta dt \, g_{\mu\nu} \left( \dot{x}^\mu(t) \dot{x}^\nu(t) - \psi^\mu(t) \nabla t \psi^\nu(t) \right) . \quad (3.37)$$

This action has the supersymmetry

$$\delta \dot{x}^\mu(t) = \psi^\mu(t) , \quad \delta \psi^\mu(t) = -\dot{x}^\mu(t) \quad \Rightarrow \quad \delta S_M[x, \psi] = 0 , \quad (3.38)$$

which we can now recognize immediately as the action of the equivariant exterior
derivative $d(V = \dot{x}) = d - i(\dot{x})$ on $LM$. Hence the action of $N = \frac{1}{2}$ supersymmetric
quantum mechanics defines an equivariant differential form on $LM$ and
on the basis of the general arguments we expect its integral to localize to an
integral over $M$, the zero locus of $\dot{x}$. This is of course well known to be the case
[23, 24, 25].

One new feature of this model is that the action is actually equivariantly
exact,

$$S_M = d(V) \hat{\gamma}(V) \quad \Leftrightarrow \quad S_M[x, \psi] = \delta \int_0^\beta dt \, g_{\mu\nu} \dot{x}^\mu(t) \psi^\nu(t) , \quad (3.39)$$

so that the partition function will not depend on the coefficient of the action
(and thus manifestly localizes onto $\dot{x} = 0$). As one can think of this coefficient
as \( \hbar \), this is another way of seeing that the semi-classical approximation to this model is exact. Nevertheless one can of course not simply set the coefficient to zero as the integral is ill-defined in this case (infinity from the \( x \)-integral times zero from the \( \psi \)-integral).

There are now at least three different ways of calculating the partition function

\[
Z[M] = \int [d\tau][d\psi]e^{i \int_0^\beta dt \sum_{\mu} \left( g_{\mu\nu} (\dot{\psi}^\mu(t) \dot{\psi}^\nu(t) - \psi^\mu(t) \nabla^\nu \psi^\nu(t)) \right)}.
\]

The traditional method is to make use of the presumed \( \beta \)-independence of the theory to evaluate \( Z \) in the limit \( \beta \to 0 \) using a normal coordinate expansion [23, 24, 25]. One can also apply directly the Berline-Vergne formula (3.20), valid when the zero locus is not zero-dimensional. In that case one has to calculate the equivariant Euler form of the normal bundle \( N \) to \( M \) in \( LM \) (this is the bundle spanned by the non-constant modes) and evaluate the determinant appearing in (3.20) using e.g. a zeta-function prescription. This was the approach adopted in [41]. Finally, one can scale the action (3.39) by some parameter \( s \), scale the non-constant modes of \( x \) and \( \psi \) by \( s^{1/2} \) and take the limit \( s \to \infty \). The remaining integral is then Gaussian. In whichever way one proceeds one obtains the result

\[
Z[M] = \int_M \hat{A}(M),
\]

which is the index of the Dirac operator on \( M \) if \( M \) is a spin-manifold.

The reason for the ubiquitous appearance of the \( \hat{A} \)-character in supersymmetric quantum mechanics path integrals is, that it arises whenever one calculates the determinant of a first-order differential operator on the circle. Let \( D_a = \partial_t + a \) be such an operator. We can without loss of generality assume that \( a \) is constant as the non-constant modes of \( a \) could always be removed by conjugating the entire operator by \( \exp if(t) \) for some function \( f \), an operation that does not change the determinant of \( D_a \). Acting on periodic functions on \( S^1 \), the eigenvalues of \( D_a \) are \( a + 2\pi m \) for \( m \in \mathbb{Z} \) (and \( m \neq 0 \) if one excludes the constant mode). The determinant of \( D_a \) is now formally defined as the product of the eigenvalues. Of course, this requires some regularization and a suitable prescription is zeta-function regularization. Then the determinant \( \det D_a \) over the non-constant modes can be written as

\[
\det D_a = \prod_{n \neq 0} (2\pi in + a) = \prod_{n > 0} (a^2 + (2\pi n)^2) = \prod_{n > 0} (1 + a^2 / (2\pi n)^2) = \prod_{n > 0} (1 + a^2 / (2\pi n)^2)
\]

as formally the infinite prefactor is equal to one by zeta-function regularization. The function defined by the infinite product in the last line has zeros for \( a \in \]
$2\pi i Z, a \neq 0$, and is equal to one at $a = 0$. It is nothing other than the function $(\sinh a/2)/(a/2)$, so that we can write
\[
\det' D_a = \frac{\sinh a/2}{a/2} = \hat{\Lambda}(a)^1,
\]
where we define the function $\hat{\Lambda}(x) = (x/2)/\sinh(x/2)$.

This can of course be generalized in various ways to include the coupling of fermions to a gauge field on a vector bundle $E$, yielding the result
\[
Z[M, E] = \int_M \text{Ch}(E)\hat{\Lambda}(M).
\]

The actual calculations involved in these three different approaches are practically identical and it is really only in the interpretation of what one is doing that they differ. However, there seems to be one instance where method three appears to be superior to method one [61], namely when one is dealing with odd-dimensional open-space index theorems like that of Callias and Bott. In that case it is simply not true that the partition function is independent of $\beta$ (the index is obtained for $\beta \to \infty$) so that one cannot simplify matters by going to the $\beta \to 0$ limit. In [61] it was shown that the correct result can be obtained by introducing the parameter $s$ and calculating the partition function as $s \to \infty$.

A localization formula for the square of the moment map

In [52], Witten introduced a new non-Abelian localization formula for finite dimensional integrals and applied it to path integrals. He was able in this way to deduce the intersection numbers of the moduli space of flat connections on a two-surface $\Sigma$ from the solution of Yang-Mills theory on $\Sigma$. Subsequently, certain cases of this localization formula have been derived rigorously by Jeffrey and Kirwan [54] and Wu [53].

In its simplest version, this formula applies to integrals over symplectic manifolds of the form
\[
Z(\epsilon) = (2\pi\epsilon)^{m/2} \int_X e^{-I/2\epsilon + \omega},
\]
where $I = (\mu, \mu)$ is the square with respect to some invariant scalar product on $g^*$ of the moment map $\mu : X \to g^*$ of a Hamiltonian $G$-action on $M$ and $\dim G = m$. By introducing an auxiliary integral over $g$, (3.45) can be put into a form very similar to the Duistermaat-Heckman integral (3.3.3.24), namely
\[
Z(\epsilon) = (2\pi)^m \int_{g^*} d\phi e^{-\epsilon(\phi, \phi)/2} \int_M e^{\omega + i(\mu, \phi)}.
\]
Here we recognize again the equivariant extension of the symplectic form (the factor of $i$ is irrelevant). Hence, as the integrand is equivariantly closed, one can again attempt to localize it by adding a $dg$-exact term $d_g\beta$ to the exponent, where $\beta \in \Omega^*_G(X)$. A reasonably canonical choice for $\beta$ is $\beta = J(dI)/2$, where $J$ is some positive $G$-invariant almost complex structure on $X$. In that case it can be shown that the integral localizes to the critical points of $I$. These are either critical points of $\mu$, as in the Duistermaat-Heckman formula, or zeros of $\mu$. The latter are a new feature of this localization theorem and are interesting for a number of reasons. For one, as the absolute minima of $I$ they give the dominant contribution to the integral in the 'weak coupling' limit $\epsilon \to 0$, the contributions form the other points being roughly of order $\exp(-1/\epsilon)$. Furthermore, the contribution from $\mu = 0$ is, by $G$-equivariance, related to the Marsden-Weinstein reduced phase space (or symplectic quotient) $X/\!/G = \mu^{-1}(0)/G$. Hence, integrals like (3.45) can detect the cohomology of $X/\!/G$.

This is of interest in 2d Yang-Mills theory, where $X = A$ is the space of gauge potentials $A$, and $\mu(A) = F_A$ is the curvature of $A$. The Yang-Mills action is therefore just the square of the moment map and the symplectic quotient is the moduli space of flat connections.

In general, the contribution from the non-minimal critical points of $I$ can be quite complicated, even for $G$ Abelian. In that case the integral over $M$ in (3.46) can also be evaluated via the Duistermaat-Heckman formula and a comparison of the expressions one obtains by following either route has been performed in [53]. To illustrate the complicated structure that arises, the example of the spin system (3.4,3.13) will suffice. In that case we have to study the integral

$$Z(\epsilon) = (2\pi)^{1/2} \int_{\infty}^{\infty} d\phi e^{-\epsilon \phi^2 / 2} \int_X \omega + i\phi(\cos \theta + a)$$

$$= (2\pi/\epsilon)^{1/2} \int_{1}^{1} dx e^{-(x+a)^2 / 2\epsilon} . \quad (3.47)$$

For $|a| < 1$ this can be written as

$$Z(\epsilon) = 2\pi(1 - I_+ - I_-) , \quad (3.48)$$

where

$$I_{\pm} = \pm(2\pi\epsilon)^{1/2} \int_{\pm1}^{\pm\infty} dx e^{-(x+a)^2 / 2\epsilon} . \quad (3.49)$$

These terms correspond to the contributions from the three critical points of $I = \mu^2$, the absolute minimum at $\cos \theta = -a$ contributing the simple first term, and the other two contributions coming from the critical points of $\mu$ at $\theta = 0$ and $\theta = \pi$. The appearance of the error function in this example is in marked contrast with the elementary functions that appear as the contributions from the critical points in the Duistermaat-Heckman formula. The above integral should
be compared with the (closely related) integral (2.17) which we discussed in the context of the Mathai-Quillen formalism. There is much more that should be said about these localization formulae, but for this we refer to [52].

Character Formulae and other applications

In this section we will just mention some other applications of the Duistermaat-Heckman formula and other equivariant localization theorems in the physics literature. We have already mentioned the path integral derivation of the Weyl character formula by Stone [40] and the application of the Duistermaat-Heckman theorem to the particle on a group manifold by Picken [44] (recovering old results by Schulman [58] and Dowker [59] in this way). In a similar setting, the action invariant of Weinstein [62], an invariant probing the first cohomology group of the symplectomorphism group of a symplectic manifold, has been related to what is known as Chern-Simons quantum mechanics using the Duistermaat-Heckman formula in [42]. The derivation of Stone has been generalized by Perret [63] to the Weyl-Kac character formula for Kac-Moody algebras.

Because of their classical properties, coherent states are particularly well suited for studying semi-classical properties of quantum systems, and in [50] the predictions of the Duistermaat-Heckman theorem have been verified for coherent state path integrals associated with $SU(2)$ and $SU(1,1)$. A very careful analysis of the WKB approximation for these coherent state path integrals has recently been performed in [64].

It is a longstanding conjecture that the quantum theories of classically integrable systems are given approximately by their semi-classical approximation. An application of the finite-dimensional Duistermaat-Heckman formula to certain integrable models can be found in [65] and some suggestive formulae for the phase space path integrals of integrable models have been obtained in [66].

Finally, we have recently found [67] that a localization formula of Bismut [68] for equivariant Kähler geometry has a field theoretic realization in the $G/G$ gauged Wess-Zumino-Witten model and can be used to shed some light on and give an alternative derivation to [69] of the Verlinde formula from the $G/G$ model.

4 Gauge Invariance and Diagonalization - the Weyl Integral Formula

In this section we discuss a technique for solving or simplifying path integrals which is quite different in spirit to those we encountered in sections 2 and 3.
Both the Mathai-Quillen formalism and the Duistermaat-Heckman and Berline-Vergne localization theorems are fundamentally cohomological in nature and can be understood in terms of a supersymmetry allowing one to deform the integrand without changing the integral. The technique we will discuss here, on the other hand, requires not a supersymmetry but an ordinary non-Abelian symmetry of the integrand (as in non-Abelian gauge theories), the idea being to reduce such an integral to one with a (much more tractable) Abelian symmetry.

More precisely, the classical integration formula, which we wish to generalize to functional integrals, is what is known in group theory and harmonic analysis as the Weyl integral formula. To state this formula, we need some notation (and refer to the next section for details). We denote by $G$ a compact Lie group and by $T$ a maximal torus of $G$. As every element of $G$ is conjugate to some element of $T$ (in other words, every $g \in G$ can be 'diagonalized'), a conjugation invariant function $f$ on $G$, $f(g) = f(h^{-1}gh)$ for all $h \in G$, is determined by its restriction to $T$. In particular, therefore, the integral of $f$ over $G$ can be expressed as an integral over $T$, and the Weyl integral formula gives an expression for the integrand on $T$,

\[
\int_G dg f(g) = \int_T dt \det \Delta_W(t) f(t) .
\] (4.1)

Here $\Delta_W(t)$ is the Weyl determinant whose precise form we will give below. One can read this formula as expressing the fact that an integral of a function with a non-Abelian (conjugation) symmetry can be reduced to an integral of a function with an Abelian symmetry. In physics parlance one would say that one has integrated over the $G/T$ part of the gauge volume or partially fixed the gauge using an abelianizing gauge condition.

It is this formula that we wish to generalize to functional integrals, i.e. to integrals over spaces of maps $\text{Map}(M, G)$ from some manifold $M$ into $G$. What is interesting about this generalization is the fact that the naive extension of (4.1) to this case is definitely wrong for basic topological reasons. This is in marked contrast with the other path integral formulae we have discussed above which appear to correctly capture the topological aspects of the situation without any modifications. The correct formula in this case (correct in the sense that the topology comes out right) turns out to include a summation over isomorphism classes of non-trivial $T$-bundles on $M$ on the right hand side of (4.1), so that (very roughly) one has

\[
\text{“} \int_{\text{Map}(M, G)} [dg] F[g] = \sum_{\text{bundles}} \int_{\text{Map}(M, T)} [dt] \det \Delta_W(t) [F[t]] \text{”}
\] (4.2)

(see (4.21) for a less outrageous rendition of this formula). To see how this comes about, suffice it to note here that in order to apply (4.1) to spaces of maps, one first needs to establish that maps from $M$ to $G$ can be diagonalized. It turns out that there are topological obstructions to doing this globally (again, in physics
parlance, to choose this abelianizing gauge globally), and the summation over topological sectors reflects this fact.

The classical finite dimensional version of the Weyl integral formula (4.1) and its various ramifications have played an important role in physics in the context of matrix models for a long time [70], and in particular recently in view of the connections between matrix models and quantum gravity (see e.g. [71] for a review). Some of these integration formulae [72] can also be understood in terms of the Duistermaat-Heckman theorem applied to integrals over coadjoint orbits [40], which provides an intriguing link between these two types of localization.

The path integral version (4.2) of the Weyl integral formula was first used in [69], where we calculated the partition function and correlation functions of some low-dimensional gauge theories (Chern-Simons theory and the $G/G$ gauged Wess-Zumino-Witten model) from the path integral. Subsequently, we also applied it to 2d Yang-Mills theory [73], rederiving the results which had been previously obtained by other methods [74, 75, 76, 77]. The method has been applied by Witten [78] to solve the Grassmannian $N = 2$ sigma model, and the topological obstructions to diagonalization have been analysed in detail in [79].

In the following we will first recall the necessary background from the theory of Lie groups and Lie algebras and then discuss the topological aspects to the extent that the more precise version of (4.2) acquires some degree of plausibility. To keep the presentation as simple as possible, we will here only deal with simply connected groups. We then discuss 2d Yang-Mills theory as an example. For more details, the reader is referred to the review [69] and to [79].

### 4.1 The Weyl Integral Formula

Background from the theory of Lie groups

Let $G$ be a compact connected and simply-connected Lie group. We denote by $T$ a maximal torus of $G$, i.e. a maximal compact connected Abelian subgroup of $G$, and by $r$ the rank of $G$, $r = \text{rk}(G) = \dim T$. We also denote by $G_r$ the set of regular elements of $G$, i.e. those lying in one and only one maximal torus of $G$, and set $T_r = G_r \cap T$. The set of non-regular elements of $G$ is of codimension 3 and $\pi_1(G) = 0$ implies $\pi_1(G_r) = 0$.

The crucial information we need is that any element $g$ of $G$ lies in some maximal torus and that any two maximal tori are conjugate to each other. We will henceforth choose one maximal torus $T$ arbitrarily and fix it. It follows that any element $g$ of $G$ can be conjugated into $T$, $h^{-1}gh \in T$ for some $h \in G$. Such an $h$ is of course not unique. First of all, $h$ can be multiplied on the right
by any element of $T$, $h \rightarrow ht, t \in T$ as $T$ is Abelian. To specify the residual ambiguity in $h$, we need to introduce the Weyl group $W$. It can be defined as the quotient $W = N(T)/T$, where $N(T) = \{ g \in G : g^{-1}tg \in T \forall t \in T \}$ denotes the normalizer of $T$ in $G$. If $h^{-1}gh = t \in T$, then $(hn)^{-1}g(hn) = n^{-1}tn \in T$, $n \in N(T)$, is one of the finite number of images $w(t)$ of $t$ under the action of the Weyl group $W$. It follows that for regular elements $g \in G_r$, the complete ambiguity in $h$ is $h \rightarrow hn$. For non-regular elements this ambiguity is larger (e.g. $g$ the identity element $h$ is completely arbitrary).

A useful way of summarizing the above result for regular elements, one which will allow us to pose the question of diagonalizability of $G_r$-valued maps in a form amenable to topological considerations, is to say that the conjugation map

$$ q : G/T \times T_r \rightarrow G_r \\
([h], t) \mapsto hth^{-1} $$

is a $|W|$-fold covering onto $G_r$ (see e.g. [80, 81]). As $G_r$ is simply-connected, this covering is trivial and we can think of $G/T \times T_r$ as the total space of a trivial $W$-bundle over $G_r$. Thus any element of $G_r$ can be lifted to $G/T \times T_r$ (with a $|W|$-fold ambiguity). As coverings induce isomorphisms on the higher homotopy groups, it also follows that $\pi_2(G_r) = \pi_2(G/T) = \pi_1(T) = \mathbb{Z}$, to be contrasted with $\pi_2(G) = 0$.

All of the above results are also true for Lie algebras of compact Lie groups. In particular, if we denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, there is a trivial $W$-fibration

$$ q : G/T \times \mathfrak{t}_r \rightarrow \mathfrak{g}_r \\
([h], t) \mapsto h\tau h^{-1} $$

In order to state the Weyl integral formula, which can be thought of as a formula relating an integral over $G_r$ to an integral over $G/T \times T_r$ via (4.3), we will need some more information. First of all, we choose an invariant metric $(\cdot, \cdot)$ on $\mathfrak{g}$ and will use it to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ whenever convenient. This metric also induces natural Haar measures $dg$ and $dt$ on $G$ and $T$, normalized to $\int_{\mathfrak{g}} dg = \int_T dt = 1$.

For the purpose of integration over $G$ we may restrict ourselves to $G_r$ and we can thus use (4.3) to pull back the measure $dg$ to $G/T \times T_r$. To calculate the Jacobian of $q$, we need to know the infinitesimal conjugation action of $T$ on $G/T$. Corresponding to a choice of $T$ we have an orthogonal direct sum decomposition of the Lie algebra $\mathfrak{g}$ of $G$, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}$. $G$ acts on $\mathfrak{g}$ via the adjoint representation $Ad$. This induces an action of $T$ which acts trivially on $\mathfrak{t}$ and leaves $\mathfrak{t}$ invariant (the isotropy representation $Ad_t$ of $T$ on $\mathfrak{t}$, the tangent space to $G/T$). Therefore the Jacobian matrix is

$$ \Delta_W(t) = 1 - Ad_t(t) $$
and one finds the Weyl integral formula
\[ \int_G dg \, f(g) = \frac{1}{|W|} \int_T dt \, \det \Delta_W(t) \int_{G/T} dh \, f(hth^{-1}) . \] (4.6)

In particular, if \( f \) is conjugation invariant, this reduces to
\[ \int_G dg \, f(g) = \frac{1}{|W|} \int_T dt \, \det \Delta_W(t)f(t) , \] (4.7)
which is the version of the Weyl integral formula which we will make use of later on. The infinitesimal version of (4.5) is the determinant of \( \Delta_W(\tau) = \text{ad}_k(\tau) \), where \( \tau \in \mathfrak{t} \). It appears in the corresponding formulae for integration over Lie algebras which are otherwise the exact analogues of (4.6,4.7). A more explicit version of (4.7) for \( G = SU(2) \) is given in (4.25).

The determinant can be calculated by using the Cartan decomposition of \( \mathfrak{g} \). The complexified Lie algebra \( \mathfrak{g}_C \) splits into \( \mathfrak{t}_C \) and the one-dimensional eigenspaces \( \mathfrak{g}_\alpha \) of the isotropy representation, labelled by the roots \( \alpha \). It follows that the Jacobian (Weyl determinant) can be written as
\[ \det \Delta_W(t) = \prod_\alpha (1 - e^{\alpha}(t)) , \]
\[ \det \Delta_W(\tau) = \prod_\alpha (\alpha, \tau) . \] (4.8)

For later use we note that \( \Delta_W(t) \) vanishes precisely on the non-regular elements of \( T \), this being the mechanism by which non-regular maps (i.e. maps taking values also in the non-regular elements of \( G \)) should be suppressed in the path integral.

Topological obstructions to diagonalization

We have seen above that the crucial ingredient in the Weyl integral formula is the fact that any element of \( G \) is conjugate to some element of \( T \). In [79] we investigated to which extent this property continues to hold for spaces of (smooth) maps \( \text{Map}(M,G) \) from a manifold \( M \) to a compact Lie group \( G \). Here we will summarize the relevant results for simply-connected \( G \), as they will be essential for the generalization of the Weyl integral formula to integrals over \( \text{Map}(M,G) \) and \( \text{Map}(M,\mathfrak{g}) \).

Thus the question we need to address is if a smooth map \( g \in \text{Map}(M,G) \) can be written as
\[ g(x) = h(x)t(x)h(x)^{-1} \] (4.9)
for \( t \in \text{Map}(M,T) \) and \( h \in \text{Map}(M,G) \). Of course there is no problem with doing this pointwise, so the question is really if it can be done consistently.
in such a way that the resulting maps are smooth. Intuitively it is clear that problems can potentially arise from the ambiguities in $h$ and $t$. And it is indeed easy to see (by examples) that (4.9) cannot be achieved globally and smoothly on $M$ in general. In fact, the situation turns out to be manageable only for regular maps, i.e. for maps taking values in $G_r$. The reason for this is that non-regular maps may not be smoothly diagonalizable in any open neighbourhood of a preimage of a non-regular element even if the map takes on non-regular values only at isolated points (we will see an example of this below). Clearly then, methods of (differential) topology do not suffice to deal with the problems posed by non-regular maps. Henceforth we will deal almost exclusively with maps taking values in the dense set $G_r$ of regular elements of $G$. For these regular maps, the problem can be solved completely and for simply-connected groups the results can be summarized as follows:

1. Conjugation into $T$ can always be achieved locally, i.e. in open contractible neighbourhoods of $M$.

2. The diagonalized map $t$ can always be chosen to be smooth globally.

3. Non-trivial $T$-bundles on $M$ are the obstructions to finding smooth functions $h$ which accomplish (4.9) globally. In particular, when $M$ and $G$ are such that there are no non-trivial $G$ bundles on $M$ (e.g. for $G$ simply-connected and $M$ two-dimensional), all isomorphism classes of torus bundles appear as obstructions.

We actually want to go a little bit further than that. Namely, as we want to use (4.9) as a change of variables (gauge transformation) in the path integral, we need to inquire what the effect of the occurrence of these obstructions will be on the other fields in the theory. In particular, when gauge fields are present, a transformation $g \rightarrow h^{-1}gh$ has to be accompanied by a gauge transformation $A \rightarrow A^h = h^{-1}Ah + h^{-1}dh$. E.g. in [69] it was found that the Weyl integral formula leads to the correct results for the gauge theories investigated there only when the gauge field integral includes a sum over $T$-connections on all isomorphism classes of $T$-bundles on $M$ (a two-manifold in those examples), even though the original bundle was trivial. One would therefore like to know if the above obstructions to diagonalization can account for this. It turns out that the results concerning the behaviour of gauge fields are indeed in complete agreement with what one expects on the basis of the results of [69, 73], namely:

4. If $P_T$ is the (non-trivial) principal $T$-bundle which is the obstruction to the global smoothness of $h$, then the $t$-component of $A^h$ defines a connection on $P_T$.

5. Hence the Weyl integral formula, when applied to gauge theories, should contain a sum over all those isomorphism classes of $T$-bundles on $M$ which arise as obstructions to diagonalization.
Rather than present proofs of all these statements (which can be found in [79]), we will illustrate the above in two different ways. On the one hand we will present a very hands-on example which allows one to see explicitly the obstruction and how it is related to connections on non-trivial $T$-bundles. On the other hand we will set up the general problem in such a way that the reader with a background in topology will be able to read off the answers to at least the questions (1)-(3) almost immediately.

Our example will be a particular map from $S^2$ to $SU(2)$. We parametrize elements of $SU(2)$ as

\[
x_4 1 + \sum_{k=1}^{3} x_k \sigma_k = \begin{pmatrix} x_4 + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & x_4 - ix_3 \end{pmatrix},
\]

subject to $\sum_{k=1}^{3} (x_k)^2 = 1$, and consider the identity map $g(x) = \sum_{k=1}^{3} x_k \sigma_k$ from the two-sphere to the equator $x_4 = 0$ of $SU(2)$. As the only non-regular elements of $SU(2)$ are $\pm 1$, $g$ is a smooth regular map. To detect a possible obstruction to diagonalizing $g$ smoothly, we proceed as follows. To any map $f$ from $S^2$ to $S^2$ we can assign an integer, its winding number $n(f)$. It can be realized as the integral of the pull-back of the normalized volume-form $\omega$ on the target-$S^2$, $n(f) = \int_{S^2} f^* \omega$. This winding number is invariant under smooth and continuous homotopies of $f$. Clearly for the identity map $g$ we have $n(g) = 1$. Representing (as above) $f$ as $f = \sum_k f_k \sigma_k$ with $\sum_k (f_k)^2 = 1$, a more explicit realisation of $n(f)$ is

\[
n(f) = -\frac{1}{32\pi} \int_{S^2} \text{tr} \, [df, df].
\]

Here and in the following expressions like $[df, df] = df_k df_l [\sigma_k, \sigma_l]$ denote the wedge product of forms combined with the commutator in the Lie algebra. Now suppose that one can smoothly conjugate the map $g$ into $U(1)$ via some $h$, $h^{-1} gh = t$. As the space of maps from $S^2$ to $SU(2)$ is connected (since $\pi_2(G) = 0$), $g$ is homotopic to $t$ and one has $n(g) = n(t)$. But, since $g^2 = -1$, $t$ is a constant map (in fact, $t = \pm \sigma_3$, the two being related by a Weyl transformation) so that $n(t) = 0$, a contradiction. This shows that there can be no smooth or continuous $h$ satisfying $h^{-1} gh = t$. As both $g$ and its diagonalization $\pm \sigma_3$ may just as well be regarded as Lie algebra valued maps, this example establishes that obstructions to diagonalization will also arise in the (seemingly topologically trivial) case of Lie algebra valued maps.

To see how connections on non-trivial bundles arise, we consider a slight generalisation $n(f, A)$ of $n(f)$,

\[
n(f, A) = -\frac{1}{32\pi} \int_{S^2} \text{tr} \, [df, df] - \frac{1}{2\pi} \int_{S^2} \text{tr} \, [d(fA)],
\]
depending on both $f$ and an $SU(2)$-connection $A$. Because the second term is a total derivative, (4.12) obviously coincides with (4.11) when $A$ is globally defined. The crucial property of $n(f, A)$ is that it is gauge invariant, i.e. invariant under simultaneous transformation of $f$ and $A$,

$$n(h^{-1}fh, A^h) = n(f, A),$$

(4.13)
even for discontinous $h$ (the key point being that no integration by parts is required in establishing (4.13). The advantage of a formula with such an invariance is that it allows one to relate maps which are not necessarily homotopic. In particular, let us now choose some (possibly discontinous) $h$ such that it conjugates $g$ into $U(1)$, say $g = h\sigma h^{-1}$ (since this can be done pointwise, some such $h$ will exist). Using (4.13) we find

$$n(g, A) = 1 = \frac{1}{2\pi} \int_{S^2} \text{tr} \sigma d(A^h).$$

(4.14)

In particular, if we introduce the Abelian gauge field $a = -\text{tr} \sigma A$, we obtain

$$n(g, A) = 1 = \frac{1}{2\pi} \int_{S^2} da.$$}

(4.15)

We now see the price of conjugating into the torus. The first Chern class of the $U(1)$ component of the gauge field $A^h$ is equal to the winding number of the original map! We have picked up the sought for non-trivial torus bundles. In this case it is just the pull-back of the $U(1)$-bundle $SU(2) \to SU(2)/U(1) \sim S^2$ via $g$ and this turns out to be more or less what happens in general.

Finally, we will show that a certain non-regular extension of this map provides us with an example of a map which cannot be smoothly diagonalized in any open neighbourhood of a non-regular point. As a preparation, we make the obvious observation that if two maps from $M$ to $G_r$ are regularly homotopic, i.e. homotopic in $\text{Map}(M, G_r)$, then if there are no obstructions to diagonalizing one of them, there are also none for the other. Incidentally, this also shows why winding numbers play a role at all in this discussion although the space of maps from $S^2$ to $SU(2)$ is connected. As we have seen in our discussion of $G_r$, $\pi_2(G_r) = \mathbb{Z}^r$, so that the space of regular maps is not connected but decomposes into disjoint sectors labelled by an $r$-tuple of winding numbers.

Consider now the extension $\tilde{g}$ of $g$ to the identity map from the three-sphere to $SU(2)$, $\tilde{g}(x) = x_41 + \sum_3 x_4\sigma_k$. This map takes on non-regular values only at $x_4 = \pm 1$. There is clearly no smooth diagonalization of the restriction of this map to any open set containing the north-pole $\{x_4 = 1\}$. If there were, this would in particular imply the existence of a global smooth diagonalization of a map from $S^2$ to $SU(2)$, which is regularly homotopic to $g$, as any neighbourhood of the north pole contains a surrounding two-sphere - a contradiction.
We will now return to the general situation and briefly describe how to set up the problem concerning obstructions to diagonalization in such a way that it can be solved by standard topological arguments.

Being able to (locally) conjugate smoothly into the maximal torus is the statement that one can (locally) find smooth maps $h$ and $t$ such that $gu = hth^{-1}$. In other words, one is looking for a (local) lift of the map $g \in \text{Map}(M, G_r)$ to a map $(h, t) \in \text{Map}(M, G) \times \text{Map}(M, T_r)$. It will be convenient to break this problem up into parts and to establish the (local) existence of this lift in a two-step procedure, as indicated in the diagram below.

\[
\begin{array}{c}
\text{M} \\
\downarrow g \\
\text{G} \\
\downarrow p \times 1 \\
\text{G} \times T_r \\
\downarrow (h, t) \\
\text{F} \\
\downarrow q \\
\text{G} / T \times T_r \\
\end{array}
\]

Here, on the right hand side of this diagram we recognize the conjugation map (4.3) and its trivial $W$-fibration. The top arrow, on the other hand, is essentially the non-trivial fibration $G \to G/T$. In the first step one attempts to lift $g$ along the diagonal, i.e. to construct a pair $(f, t)$, where $f \in \text{Map}(M, G/T)$, which projects down to $g$ via the projection $q$. By a standard argument the obstruction to doing this globally is the possible non-triviality of the $W$-bundle $g^*(G/T \times T_r)$ on $M$. Happily, when $G$ is simply connected, (4.3) is trivial and this obstruction is absent. Thus we have just established the existence of a globally smooth diagonalization $t$ of $g$.

In the second step, dealing with the upper triangle, one needs to lift $f$ locally to $\text{Map}(M, G)$. The obstruction to doing this globally is the possible non-triviality of the principal $T$-bundle $f^*(G)$ over $M$ (as in the above example). If this bundle is trivial (e.g. when $H^2(M, \mathbb{Z}) = 0$, so that there are no non-trivial torus bundles at all on $M$), a smooth $h$ accomplishing the diagonalization will exist globally. Locally, a lift $h$ can of course always be found.

The principal $T$-bundles which appear as obstructions are precisely those that can be obtained from the trivial principal $G$-bundle $P_G$ by restriction of the structure group from $G$ to $T$ (or, more informally, that sit inside $P_G$). Under such a restriction, the $t$-part of a connection on $P_G$ becomes a connection on $P_T$, while the $t$-part becomes a one-form with values in the bundle associated to $P_T$ via the isotropy representation of $T$ on $\mathfrak{t}$.

Because of the similarity of (4.3) and (4.4), the situation for Lie algebra
valued maps is exactly the same as above (differences only occur for non-simply connected groups).

The Weyl integral formula for path integrals

We have now collected all the results we need to state the Weyl integral formula for integrals over Map(M,G) which correctly takes into account the topological considerations of the previous section.

For concreteness, consider a local functional $S[g;A]$ (the ‘action’) of maps $g \in \text{Map}(M,G)$ and gauge fields $A$ on a trivial principal $G$-bundle $P_G$ on $M$ (a dependence on other fields could of course be included as well). Assume that $\exp iS[g,A]$ is gauge invariant,

$$e^{iS[g,A]} = e^{iS[h^{-1}gh, A^h]} \forall h \in \text{Map}(M,G),$$

(4.17)

at least for smooth $h$. If, e.g., a partial integration is involved in establishing the gauge invariance (as in Chern-Simons theory), this may fail for non-smooth $h$'s and more care has to be exercised when such a gauge transformation is performed. Then the functional $F[g]$ obtained by integrating $\exp iS[g,A]$ over $A$,

$$F[g] := \int [dA] e^{iS[g,A]},$$

(4.18)

is conjugation invariant,

$$F[h^{-1}gh] = F[g].$$

(4.19)

It is then tempting to use a formal analogue of (4.7) to reduce the remaining integral over $g$ to an integral over maps taking values in the Abelian group $T$. In field theory language this amounts to using the gauge invariance (4.17) to impose the ‘gauge condition’ $g(x) \in T$. The first modification of (4.7) will then be the replacement of the Weyl determinant $\det \Delta_W(t)$ by a functional determinant $\det \Delta_W[t]$ of the same form which needs to be regularized appropriately (see the Appendix of [73]).

However, the crucial point is, of course, that this is not the whole story. We already know that this ‘gauge condition’ cannot necessarily be achieved smoothly and globally. Insisting on achieving this ‘gauge’ nevertheless, albeit via non-continuous field transformations, turns the $t$-component $A^t$ of the transformed gauge field $A^h$ into a gauge field on a possibly non-trivial $T$ bundle $P_T$ (and the $t$-component turns out to transform as a section of the associated bundle $P_T \times_T \mathbb{T}$). Moreover, we know that all those $T$ bundles will contribute which arise as restrictions of the (trivial) bundle $P_G$. Let us denote the set of isomorphism classes of these $T$ bundles by $[P_T;P_G]$. Hence the ‘correct’ (meaning correct modulo the analytical difficulties inherent in making any field theory functional integral rigorous) version of the Weyl integral formula, capturing the
topological aspects of the situation, is one which includes a sum over the contributions from the connections on all the isomorphism classes of bundles in \([P_T; P_G]\).

Let us denote the space of connections on \(P_G\) and on a principal \(T\) bundle \(P_T\) representing an element \(l \in [P_T; P_G]\) by \(A\) and \(A[l]\) respectively and the space of one-forms with values in the sections of \(P_T^* \times_T \mathfrak{t}\) by \(B[l]\). Then, with

\[
Z[P_G] = \int [dA] \int [dg] e^{iS[g,A]},
\]

the Weyl integral formula for functional integrals reads

\[
Z[P_G] = \sum_{l \in [P_T; P_G]} \int_{[A[l]]} [dA^1] \int_{[B[l]]} [dA^4] \int [dt] \Delta_W[t] e^{iS[t,A^1,A^4]} (4.20)
\]

(modulo a normalization constant on the right hand side). The \(t\)-integrals carry no \(l\)-label as the spaces of sections of \(\text{Ad} P_T^*\) are all isomorphic to the space of maps into \(T\). There is an exactly analogous formula generalizing the Lie algebra version of the Weyl integral formula. Namely, assuming that we have a gauge invariant action \(S[\phi,A]\), where \(\phi \in \text{Map}(M,\mathfrak{g})\), we can use the Weyl integral formula to reduce its partition function to

\[
Z[P_G] = \sum_{l \in [P_T; P_G]} \int_{[A[l]]} [dA^1] \int_{[B[l]]} [dA^4] \int [d\tau] \Delta_W[\tau] e^{iS[\tau,A^1,A^4]} (4.21)
\]

In the examples considered in [69, 73], the fields \(A^4\) entered purely quadratically in the reduced action \(S[t,A^1,A^4]\) and could be integrated out directly, leaving behind an effective Abelian theory depending on the fields \(t\) and \(A^1\) (respectively \(\tau\) and \(A^4\)). The information on the non-Abelian origin of this theory is contained entirely in the measure determined by \(\Delta_W[t]\) and the (inverse) functional determinant coming from the \(A^4\)-integration. In all the examples considered so far, e.g. in [69, 73, 78], this procedure simplified the path integral to the extent that it could be calculated explicitly. One does of course not expect to be able to get that far in general. Nevertheless, the simplification brought about by replacing an interacting non-Abelian theory with a theory with only an Abelian gauge symmetry should allow one to obtain at least some new information from the path integral. We will see the formula (4.21) at work in the next section, where we will use it to solve 2d Yang-Mills theory.

4.2 Solving 2d Yang-Mills Theory via Abelianization

In this section we will apply the Weyl integral formula (4.21) to Yang-Mills theory on a two-dimensional closed surface \(\Sigma\) [73]. As mentioned before, this theory
has recently been solved by a number of other techniques as well [75, 74, 76, 77], but it appears to us that none of them is as simple and elementary as the one based on Abelianization. As the partition function of Yang-Mills theory for a gauge group $G$ can be expressed entirely in terms of the representation theory (more precisely the dimensions and the quadratic Casimirs of the representations) of $G$, we will first briefly recall the relevant formulae below. Then we will calculate the partition function. To keep the Lie algebra theory as simple as possible (as it is only of tangential interest here) we will assume that $G$ is not only simply-connected but also simply-laced. For the relation and application of these results to topological Yang-Mills theory see [52, 73].

**Some Lie algebra theory**

It turns out that the partition function of Yang-Mills theory for a group $G$ can be expressed in terms of the dimensions and quadratic Casimirs of the unitary irreducible representations of $G$. We will recall here the relevant formulae, see e.g. [80].

First of all, we choose a positive Weyl chamber $C^+$ and introduce a corresponding set of simple roots $\{\alpha_a, a = 1, \ldots, r\}$, a dual set of fundamental weights $\{\lambda^a, a = 1, \ldots, r\}$ and denote the weight lattice by $\Lambda = \mathbb{Z}[\lambda^a]$. Then the highest weights of the unitary irreducible representations can be identified with the elements of $\Lambda^+ = \Lambda \cap \mathbb{C}^+$, $\mathbb{C}^+$ denoting the closure of the Weyl chamber. For $\mu \in \Lambda^+$ we denote by $d(\mu)$ and $c(\mu)$ the dimension and the quadratic Casimir of the corresponding representation. The formulae for these are

$$d(\mu) = \prod_{\alpha > 0} \frac{\langle \alpha, \mu + \rho \rangle}{\langle \alpha, \rho \rangle}$$

$$c_\mu = (\mu + \rho, \mu + \rho) - (\rho, \rho) ,$$

where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is the Weyl vector.

Let us illustrate this and the formulae obtained in our discussion of the Weyl integral formula in the case $G = SU(2)$ and $T = U(1)$. We use the trace to identify the Lie algebra $t$ of $T$ with its dual. Then we can choose the positive root $\alpha$ and the fundamental weight $\lambda$ to be $\alpha = \text{diag}(1, -1)$ and $\lambda = \alpha / 2$, so that $\rho = \lambda$. We also parametrize elements of $T$ as $t = \exp i\phi = \text{diag}(\exp i\phi/2, \exp -i\phi/2)$. It follows that the Weyl determinant $\Delta_W(t)$ (4.5) is $\det \Delta_W(t) = 4 \sin^2 \phi / 2$. Hence the Weyl integral formula for class functions (conjugation invariant functions) is

$$\int_G dg f(g) = \frac{1}{2\pi} \int_0^{1\pi} d\phi \sin^2 (\phi / 2) f(\phi).$$

For the spin $j$ representation of $SU(2)$ (with highest weight $\mu(j) = 2j \lambda \in \Lambda^+$),
one finds \( d(\mu(j)) = 2j + 1 \) and \( c(\mu(j)) = 2j(j + 1) \).

2d Yang-Mills theory

The action of Yang-Mills theory on a two-dimensional surface \( \Sigma_g \) of genus \( g \) is

\[
S[A] = \frac{1}{2\epsilon} \int_{\Sigma_g} \text{tr} F_A \ast F_A .
\]  

(4.26)

Here \( F_A = dA + \frac{1}{2}[A, A] \) is the curvature of a gauge field \( A \) on a trivial principal \( G \)-bundle \( P_G \) on \( \Sigma_g \), \( \ast \) denotes the Hodge duality operator with respect to some metric on \( \Sigma_g \), \( \text{tr} \) refers to the trace in the fundamental representation for \( G = SU(n) \). \( \epsilon \) represents the coupling constant of the theory. A \( \text{tr} \) will henceforth be understood in integrals of Lie algebra valued forms. It is readily seen that, since \( \ast F_A \) is a scalar, the action does not depend on the details of the metric but only on the area \( A(\Sigma_g) = \int_{\Sigma_g} 1 \) of the surface. As a change in the metric can thus be compensated by a change in the coupling constant, we can without loss of generality choose a metric and a corresponding volume two-form \( \omega \) with unit area. \( \omega \) is a symplectic form on \( \Sigma_g \) representing the generator of \( H^2(\Sigma_g, \mathbb{Z}) \sim \mathbb{Z} \).

The action is invariant under gauge transformations \( A \to A^g \) and the path integral that we would like to compute is

\[
Z_{\Sigma_g}(\epsilon) = \int_A [dA] e^{\frac{1}{2\epsilon} \int_{\Sigma_g} F_A \ast F_A} .
\]  

(4.27)

It will be convenient to rewrite this in first order form by introducing an additional \( g \)-valued scalar field \( \phi \in \text{Map}(\Sigma_g, g) \) (cf. \((3.46)) \), so that the partition function becomes

\[
Z_{\Sigma_g}(\epsilon) = \int_A [dA] \int_{\text{Map}(\Sigma_g, g)} [d\phi] e^{\int_{\Sigma_g} i\phi F_A + \frac{i}{2} \phi \ast \phi} .
\]  

(4.28)

The gauge invariance of the action \( S[\phi, A] \) appearing in \((4.28)\) is

\[
S[\epsilon \phi g, A^g] = S[\phi, A] ,
\]  

(4.29)

and we are thus precisely in a situation where we can apply the considerations of the previous section. Before embarking on that, it will however be useful to recall some properties of the quantum field theory defined by the classical action \((4.26)) \).

Yang-Mills theory in two-dimensions is super-renormalisable and there are no ultraviolet infinities associated with diagrams involving external \( A \) or \( \phi \) fields. Furthermore, on a compact manifold \( \Sigma_g \) there are no infrared divergences associated with these diagrams either. However, because of the coupling of \( \phi \) to
the metric, there are diagrams which do not involve external $\phi$ or $A$ legs but do have external background graviton legs and which require regularisation. These terms arise in the determinants that are being calculated and they depend only on the area and topology of $\Sigma_g$, i.e. they have the form $\alpha_1(\epsilon)A(\Sigma_g) + \alpha_2(\epsilon)\chi(\Sigma_g)$, where $\chi(\Sigma_g) = 2 - 2g$ is the Euler number of $\Sigma_g$. These may be termed area and topological standard renormalisations. We wish to ensure that the scaling invariance that allowed us to move all of the metric dependence into $\epsilon$ is respected. Hence only those regularisation schemes which preserve this symmetry are to be considered. Then the dependence of $\alpha_1$ on $\epsilon$ is fixed to be $\alpha_1(\epsilon) = \epsilon\beta$ and $\alpha_2$ and $\beta$ are independent of $\epsilon$.

With these remarks in mind, we can now proceed by using the gauge freedom to conjugate $\phi$ into $\text{Map}(\Sigma_g, \mathfrak{t})$ (i.e. to impose $\phi^t = 0$). This simply amounts to plugging $S[\phi, A]$ into (4.22). Alternatively, one can of course implement this gauge condition by following the standard Faddeev-Popov procedure. The ghost determinant one obtains in this way is precisely the Weyl determinant $\det \Delta_W[\tau]$ appearing in (4.22).

We should perhaps stress once more that in choosing this gauge we are assuming that $\phi$ is regular. We have discussed this issue at some length in [69, 73] and have nothing to add to this here. We just want to point out that only constant modes of $\phi$ contribute to the path integral and that a non-regular constant $\phi$ would give a divergent contribution to the partition function from the gauge fields (and a zero from the Weyl determinant). Thus any way of regularizing this divergence will set the contribution from the non-regular points to zero (by the ghost contribution) and is therefore tantamount to discarding the non-regular maps which is what we will do.

Let us now determine the ingredients that go into (4.22). The Weyl determinant $\det \Delta_W[\tau]$ is the functional determinant of the adjoint action by $\tau$ on the space $\text{Map}(\Sigma_g, \mathfrak{t}) = \Omega^0(\Sigma_g, \mathfrak{t})$,

$$\det \Delta_W[\tau] = \det \text{ad}(\tau)|_{\Omega^0(\Sigma_g, \mathfrak{t})} .$$

This determinant of course requires some regularization. We will say more about it once we have combined this with another determinant that will arise below.

We now look at the reduced action $S[\tau, A^1, A^t]$. Because of the orthogonality of $\mathfrak{t}$ and $\mathfrak{t}$ with respect to the trace, the only terms that survive in the action are

$$S[\tau, A^1, A^t] = \int_{\Sigma_g} \tau dA^1 + \frac{1}{2}[\tau, A^1]A^t + \frac{e}{2}\tau \ast \tau .$$

(4.31)

Integrating out $A^t$, one obtains the inverse square root of the determinant of $\text{ad}(\tau)$, this time acting on the space $\Omega^1(\Sigma_g, \mathfrak{t})$ of $\mathfrak{t}$-valued one-forms,

$$\int [dA^t] \Rightarrow \det^{1/2} \text{ad}(\tau)|_{\Omega^1(\Sigma_g, \mathfrak{t})} .$$

(4.32)
It is clear that this almost cancels against the Weyl determinant as, modulo zero-modes, a one-form in 2d has as many degrees of freedom as two scalars. The zero mode surplus is one constant scalar mode minus \((2g/2) = g\) harmonic one-form modes so that the combined determinant would simply reduce to a finite dimensional determinant,

\[
\frac{\det \text{ad}(\tau)_{\mathfrak{g} \otimes \mathfrak{g}}}{\det^{1/2} \text{ad}(\tau)_{\mathfrak{g} \otimes \mathfrak{g}}} = (\det \text{ad}(\tau)_{\mathfrak{g}})^{\chi(\Sigma_g)/2},
\]

if \(\tau\) were constant. As we will see shortly that only the constant modes of \(\tau\) contribute to the path integral, we will just leave it at (4.33) and refer to [73] for a more careful discussion using a heat kernel regularization of the determinants involved.

Putting all the above together we see that we are left with an Abelian functional integral over \(\tau\) and \(A^1\) and a sum over the topological sectors,

\[
Z_{\Sigma_g}(\epsilon) = \sum_{l \in [P_T; P_G]} \int_{[A^1]} [dA^1] \int [d\tau] (\det \text{ad}(\tau)_{\mathfrak{g}})^{\chi(\Sigma_g)/2} e^{\int_{\Sigma_g} (i\tau dA^1 + \frac{1}{2} \tau \ast \tau)}. 
\]

(4.34)

To evaluate this, we will use one more trick, namely to change variables from \(A^1\) to \(F^1 = dA^1\). To see how to do that let us first note that the theory we have arrived at has still got local Abelian gauge invariance (as well as \(W\)-invariance as a remnant of the non-Abelian gauge invariance of the original action). Let us therefore choose some gauge fixing condition \(E(A^1) = 0\) and change variables from \(A^1\) to \((F^1, E(A^1))\). This change of variables is fine away from the gauge equivalence classes of flat connections (i.e. the moduli space \(T^2g\) of flat \(T\)-connections on \(\Sigma_g\)). As these flat connections do not appear in the action, they just give a finite volume factor (depending on the normalization of the metric on \(T\)) which we will not keep track of. On the remaining modes of the gauge field this change of variables is then well defined. What makes it so useful is the fact, demonstrated in [82], that the corresponding Jacobian cancels precisely against the Faddeev-Popov determinant arising from the gauge fixing.

As a consequence, after this change of variables and integration over the ghosts, the multiplier field and \(E(A^1)\) all derivatives disappear from (4.38) and the remaining path integral over \(\tau\) and \(F^1\) is completely elementary. The only thing to keep track of is that we are not integrating over arbitrary two-forms \(F^1\) but only over those which arise as the curvature two-form on some \(T\)-bundle over \(\Sigma\) which arises as the restriction of the trivial \(G\)-bundle \(P_G\). As there are no non-trivial \(G\)-bundles on \(\Sigma_g\), the sum over topological sectors \(l \in [P_T; P_G]\) will extend over all isomorphism classes of \(T\)-bundles on \(\Sigma_g\). These are classified by their first Chern class in \(H^2(\Sigma_g, \mathbb{Z}) \sim \mathbb{Z}^r\), and hence the summation over \(l\) in (4.22) can be thought of as a summation over \(r\)-tuples of integers. In particular, if we expand \(F^1\) in a basis of simple roots \(\{a_a\}\) of \(\mathfrak{g}\) as \(F^1 = i\alpha_a F^a\), then in the
topological sector characterized by \( l = (n^1, \ldots, n^r) \) we have

\[
\int_{\Sigma_S} F^a = 2\pi n^a .
\]  

(4.35)

As all topological sectors appear, one way to enforce (4.35) in the path integral is to impose a periodic delta-function constraint on the curvature \( F^1 \) of the torus gauge field, i.e. to insert

\[
\delta^\Lambda (\int F^1) = \prod_{a=1}^r \sum_{m_a \in \mathbb{Z}} e^{im_a \int_{\Sigma_S} F^a}.
\]

(4.36)

into the path integral. It will be convenient to write this as a sum over the weight lattice \( \Lambda \) dual to the lattice spanned by the simple roots (or Chern classes),

\[
\delta^\Lambda (\int F^1) = \sum_{\lambda \in \Lambda} e^{\int_{\Sigma_S} \lambda F^1}.
\]

(4.37)

Then the path integral to be evaluated reads

\[
Z_{\Sigma_S} (\epsilon) = \sum_{\lambda \in \Lambda} \int |F^1| \int [d\tau] (\text{det} \, \text{ad}(\tau))^{\chi(\Sigma_S)/2} e^{\int_{\Sigma_S} (i\tau + \lambda) F^1 + \frac{1}{2} \tau \ast \tau}.
\]

(4.38)

Let us do the \( F^1 \)-integral first. This imposes the delta-function constraint \( \tau = i\lambda \) on \( \tau \), so that in particular only the constant modes of \( \tau \) ever contribute to the partition function. We are then left with just the sum over the (regular) elements of \( \Lambda \). Eliminating the Weyl group invariance is then the same as summing only over \( \lambda \)'s in the interior of the positive Weyl chamber \( C^+ \). Writing \( \lambda = \mu + \rho \) one sees that the sum is now precisely over all the highest weights \( \mu \in \Lambda^+ = \Lambda \cap \tilde{C}^+ \) of the unitary irreducible representations of \( G \),

\[
Z_{\Sigma_S} (\epsilon) = \prod_{\mu \in \Lambda^+} a > 0 (\alpha, \mu + \rho)^{\chi(\Sigma_S)} e^{-\frac{i}{2} (\mu + \rho, \mu + \rho)}.
\]

(4.39)

Using (4.23) and (4.24), we can rewrite this as

\[
Z_{\Sigma_S} (\epsilon) = e^{-\frac{i}{2} (\rho, \rho)} \prod_{\alpha > 0} (\alpha, \rho)^{\chi(\Sigma_S)} \sum_{\mu \in \Lambda^+} d(\mu)^{\chi(\Sigma_S)} e^{-\frac{i}{2} c(\mu)}.
\]

(4.40)

This differs from the expression obtained in e.g. [74, 76, 75, 77] only by the first two terms which are precisely of the form of a standard area and topological renormalisation respectively. What is interesting, though, is that the form of the partition function obtained here (with \( (\mu + \rho, \mu + \rho) \) in the exponential instead of \( c_\mu \)) agrees with what one obtains from non-Abelian localization [52] and is
the correct one to use for determining the intersection numbers of the moduli space of flat $G$-connections on $\Sigma_g$.

The uses of the Weyl integral formula are of course not restricted to Yang-Mills theory (and its non-linear cousin, the $G/G$-model). For instance, in principle it can also be applied to gauge theories on manifolds of the form $M \times S^1$, where the ‘temporal gauge’ $\partial_t A_0 = 0$ can profitably be augmented by using time-independent gauge transformations to achieve the gauge condition $A^t = 0$ (see e.g. [69] for an application to Chern-Simons theory). In other words, the temporal gauge reduces $A_0$ to a map from $M$ to $\mathfrak{g}$ to which the considerations of this section regarding diagonalizability and the ensuing form of the path integral can be applied.

In practice, however, this method has its limits, not only because in higher dimensions the issue of non-regular maps raises its ugly head but also because simplifications brought about by a suitable choice of gauge alone will not be sufficient to make the theory solvable without recourse to more traditional techniques of quantum field theory as well. On a more optimistic note one may hope that the technique developed here can shed some light on the global aspects of the Abelian projection technique introduced in [83], currently popular in lattice gauge theories.

Finally, we want to mention that arguments as in section 4.1 permit one to also deduce a path integral analogue of the Weyl integral formula (4.6) valid for functions which are not conjugation invariant. This formula is then applicable to theories with less or no gauge symmetry, and can potentially help to disentangle the degrees of freedom of the theory. This appears to be the case e.g. in the ordinary (ungauged) Wess-Zumino-Witten model.

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