Stochastic electrodynamics and the interpretation of quantum theory

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Abstract

I propose that quantum mechanics is a stochastic theory and quantum phenomena derive from the existence of real vacuum stochastic fields filling space. I revisit stochastic electrodynamics (SED), a theory that studies classical systems of electrically charged particles immersed in an electromagnetic (zeropoint) radiation field with spectral density proportional to the cube of the frequency, Planck’s constant appearing as the parameter fixing the scale. Aside from briefly reviewing known results, I make a detailed comparison between SED and quantum mechanics. Both theories make the same predictions when the stochastic equations of motion are of first order in Planck constant, but not in general. I propose that SED provides a clue for a realistic interpretation of quantum theory.
1 Introduction

1.1 Charges immersed in a random (vacuum) radiation field

The basic assumption in this article is that the quantum vacuum fields are real stochastic fields. For the sake of clarity let us consider the best known vacuum field, the electromagnetic zeropoint radiation. The spectrum, that here I define as the energy per unit volume and unit frequency interval, is given by eq.(1) below. The parameter fixing the scale of the field is Planck constant. Therefore it is interesting to see whether the reality of the vacuum electromagnetic field, combined with classical physics, allows to explain some phenomena believed as typically quantal, thus providing a hint for the realistic interpretation of quantum theory. With that purpose I shall study a restricted domain of phenomena with a theory defined by:

1) Just one of the interactions of nature, that is electromagnetic, a choice that we expect should lead to an approximation of quantum electrodynamics (QED). Actually QED includes two (quantum) vacuum fields, namely electromagnetic and electron-positron.

2) Nonrelativistic energies. Thus we shall exclude positrons and study only electrons ignoring spin, or more generally charged particles without structure, in given electromagnetic fields and/or interacting with other charges.

3) Planck constant $\hbar$ appears exclusively in the vacuum electromagnetic radiation. Consequently for the evolution we shall use the laws of classical electrodynamics throughout, but taking the additional force of the vacuum field into account.

1.2 Stochastic electrodynamics

The theory defined with these constraints is already known with the name stochastic (or random) electrodynamics (SED in the following). It has been developed by a small number of people during the last sixty years. Actually SED may be defined in several slightly different forms, for instance as classical electrodynamics modified by the assumption that the vacuum is not empty but there is a random electromagnetic field (or zeropoint field, ZPF) with spectrum eq.(1) filling the whole space. With the precise definition here proposed SED is an approximation to quantum electrodynamics to lowest nontrivial order in the Planck constant, the zeroth order being purely classical.
electrodynamics. A review of the work made until 1995 is the book by L. de la Peña and A. M. Cetto\cite{1} and new results are included in more recent reviews [2], [3]. The application of similar ideas to optics will be reviewed in Chapter 6.

The origin of SED may be traced back to Walter Nernst, who extended to the electromagnetic field the zeropoint fluctuations of oscillators assumed by Planck in his second radiation theory of 1912. Nernst also suggested that the zeropoint fluctuations might explain some empirical facts, like the stability of atoms and the chemical bond. The proposal was soon forgotten due to the success of Bohr’s model of 1913 and the subsequent development of the (old) quantum theory. Many years later the idea has been put forward again several times (e. g. by Braffort et al. in 1954\cite{5} and by Marshall in 1963\cite{6}).

SED studies the motion of charged particles immersed in ZPF, but the back actions of the charged particles on the ZPF are neglected (indeed the effects would be of higher order in Planck constant), so that the random field of free space is used. Assuming that the field is Lorentz invariant (at not too high frequencies) determines the spectrum, that is the energy per unit volume and unit frequency interval\cite{4}, [1]. It is given by

\[
\rho_{ZPF} (\omega) = \frac{1}{2\pi^2 c^3} \hbar \omega^3, \tag{1}
\]

that corresponds to an average energy $\frac{1}{2} \hbar \omega$ per normal mode. Planck constant $\hbar$ enters the theory via fixing the scale of the assumed universal random radiation. Of course the spectrum eq. (1) implies a divergent energy density and any cutoff would break Lorentz invariance. However we may assume that it is valid for low enough frequencies, the behaviour at high frequencies requiring the inclusion of other vacuum fields and general relativity theory. The spectrum eq. (1) is appropriate for systems at zero Kelvin, but SED may be also studied at a finite temperature, where we should add to eq. (1) the thermal Planck spectrum. In addition SED may provide an interpretation of phenomena where the free spectrum is modified by boundary conditions derived from macroscopic bodies, but the average energy $\frac{1}{2} \hbar \omega$ per normal mode still holds true. These phenomena will be revisited in Chapter 6.

The SED study of some simple systems provides an intuitive picture of several phenomena usually considered as purely quantum, like the stability of the classical (Rutherford) atom, Heisenberg uncertainty relations, entanglement, specific heats of solids, behaviour of atoms in cavities, etc. For this
reason I propose that SED may be considered as a clue in the search for a realistic interpretation of quantum theory.

1.3 Scope of stochastic electrodynamics

Not all predictions that have been claimed to follow from SED derive from the theory as defined above. In some cases additional assumptions are introduced in order to agree with the quantum predictions. In this form most of nonrelativistic quantum mechanics might be derived from SED[2]. However with extra assumptions not resting upon deep arguments the physical bases of the theory become unclear and a realistic interpretation problematic.

In this paper we will study SED strictly as defined in Section 1.1. With that definition there are many examples where SED predicts results in contradiction with quantum mechanics and with experiments, as discussed in Section 6 below. In particular SED deals only with charged particles whilst QM laws are valid for both charged and neutral particles. It has been claimed that the restriction may be avoided taking into account that neutral particles may contain charged parts (e. g. the neutron possesses a magnetic moment). I think this is flawed, the application to those neutral particles might be valid in order to explain the stationary equilibrium state, which is effectively defined over an infinite time and it results independent on the total charge, as seen for instance in eqs.(18) and (19) below. However this is not the case for time dependent properties like eq.(38) where the value of the charge is relevant.

With the definition of Section 1.1 SED is an approximation to QED at the lowest nontrivial order in Planck constant. A different approach is to consider that SED is “the closest classical approximation to quantum theory” [3]. This suggests that there are two different theories, namely classical and quantum, but it is assumed that the validity of classical theory may be extended if we include the hypothesis of a radiation field with a Lorentz invariant spectrum in free space. It seems that this approach would increase, rather than solve, the problem of the “infamous boundary” between classical and quantum theories. Indeed in the standard wisdom (i. e. ignoring SED) the boundary is defined (roughly) by the relative value of Planck constant as compared with the typical magnitude of the action variable for the system. That is, classical theories are an approximation of quantum theories when Planck constant may be negleted. An argument for the need of distinguishing between quantum theory and generalizations of SED is the fact that
there are peculiarities of quantum theory, like discrete spectra, that cannot be achieved by generalized SED. However there are numerical calculation providing hints that this problem might be solved, as we will comment on Section 2.5. In any case a generalization of SED (or equivalently a realistic interpretation of quantum theory) needs to agree with quantum theory exclusively in the predictions of results of actual experiments, either performed or at least possible. But it is not required that predictions for ideal (not realistic) experiments or for unobservable facts should agree.

In this book it is supported the view that the whole of quantum theory should admit a realistic (classical-like) interpretation. That interpretation might be obtained via a generalization of SED taking into account not only the effect of the electromagnetic field on the motion of charged particles, but also the back action of the particles on the field and also all other vacuum fields, including metric fluctuations of spacetime. Attempts in that direction have been made elsewhere[7]. In summary SED may be taken as an approximation to quantum electrodynamics in some limited domain. In particular when the equations of motion are linear.

1.4 Plan of the article

In the following a short review of SED is presented and the analogies and differences between SED and nonrelativistic quantum mechanics (QM in the following) for some simple systems are studied. Most of the results have been reviewed in more detail elsewhere[1]. The novelty here is a more careful comparison of SED with QM and the emphasis on those quantum phenomena that might be better understood via the analogy with the picture provided by SED.

In the second and third sections the harmonic oscillator is revisited, with an application to oscillators in several dimensions in section 4. In sections 5 and 6 SED is applied to other linear systems, namely the free particle and the particle in a homogeneous magnetic field. Section 7 is devoted to the application of SED to some nonlinear systems, showing that in this case some disagreements with QM and with the experiments usually appear. Section 8 presents the conclusions. This chapter includes many calculations and, in order that the reader does not lose the essential points, I will write in italics the relevant aspects for the comparison between SED and QM.
2 The harmonic oscillator. Stationary state

2.1 Equation of motion

The harmonic oscillator in one dimension is the most simple system to be treated within SED (the free particle requires a more careful study in order to avoid divergences). It is not strange that it was the first system studied. In this and the following sections we revisit a well known treatment of the oscillator in SED[8][1], but the study of the aspects that may provide a clue for the interpretation of quantum mechanics is original.

If a charged particle moves in one dimension in a potential well and it is also immersed in electromagnetic noise, it may arrive at a dynamical equilibrium between absorption and emission of radiation. In order to study the equilibrium I shall write the differential equation for the one-dimensional motion of the particle in the non-relativistic approximation. The passage to more dimensions is straightforward. We will neglect magnetic effects of the ZPF and the dependence of the field on the position coordinate, which corresponds to the common electric dipole approximation, plausible in a non-relativistic treatment. Thus the differential equation of motion of the particle in a harmonic oscillator potential is

\[ m \dddot{x} = -m \omega_0^2 x + m \tau \dddot{x} + e E(t), \tag{2} \]

where \( m(e) \) is the particle mass (charge) and \( E(t) \) is the \( x \) component of the electric field of the radiation (the zeropoint field, ZPF). The equation of the mechanical (classical) oscillator is modified by the two latter terms. The second term on the right side of eq.(2), is the damping force due to emission of radiation. It should appear also in the classical electrodynamic treatment. Only the third term is specific of SED because it involves Planck constant (it is of order \( O(\hbar^{1/2}) \)). The parameter \( \tau \) given by

\[ \tau = \frac{2e^2}{3mc^2} \Rightarrow \tau \omega_0 = \frac{2 e^2 \hbar \omega_0}{3 \hbar c mc^2} \ll 1. \tag{3} \]

so that the dimensionless quantity \( \tau \omega_0 \) is very small, it being the product of two small numbers namely the fine structure constant, \( \alpha \equiv e^2/\hbar c \sim 1/137 \), and the nonrelativistic ratio \( \hbar \omega_0/mc^2 \simeq v^2/c^2 \ll 1 \). Thus the two latter terms of eq.(2) may be taken as small, which allows some useful approximations. Eq.(2) is a stochastic differential equation of Langevin’s type with
coloured (non-white) noise. It has been named Braffort-Marshall equation by the early workers on SED\[5\],[6]. Solving an equation of this kind usually means finding the evolution of the probability distribution of the relevant quantities as a function of time, starting from given initial conditions. When the time goes to infinity the probability distributions become independent of the initial conditions, giving rise to the stationary or equilibrium distribution.

### 2.2 Average values of the potential and kinetic energies

Several solutions of the eq.(2) have been published\[8], [1]. The most simple is the stationary solution, which may be found by Fourier transform of eq.(2) as follows. Firstly we define the Fourier transform of the stationary process $E(t)$ in a finite time interval by

$$\tilde{E}(\omega,T) \equiv \frac{1}{\sqrt{4\pi T}} \int_{-T}^{T} E(t) \exp(-i\omega t) \, dt.$$  

(4)

Hence it may be shown that $|\tilde{E}(\omega,T)|^2 / 8\pi$ is the mean (in the time interval $(-T,T)$) energy density per unit frequency interval associated to one electric field component. Thus the total energy density per unit frequency interval, $\rho(\omega)$ eq.(1), should be 6 times that quantity (6 because in the ZPF there are 3 components of the electric field and another 3 of the magnetic field all contributing equally on the average). Consequently we define the spectral density, $S_E(\omega)$, of the field $E(t)$ as follows

$$S_E(\omega) \equiv \lim_{T\to\infty} \left|\tilde{E}(\omega,T)\right|^2 = \frac{4\pi}{3} \rho(\omega) = \frac{2}{3\pi c^3} \hbar \omega^3,$$  

(5)

the equality giving the relation between the spectral density and the energy density of the ZPF, eq.(1). For short the spectral density will be named spectrum in the following.

A Fourier transform similar to eq.(4) of all terms of eq.(2) provides a relation between the spectrum of the field component and the spectrum of the coordinate, $x(t)$, namely

$$m(\omega_0^2 - \omega^2 + i\tau \omega^3) \tilde{x}(\omega) = e\tilde{E}(\omega),$$  

(6)
where $\tilde{x}(\omega)$ and $\tilde{E}(\omega)$ are the Fourier transforms of $x(t)$ and $E(t)$ respectively. Hence the spectrum of $x(t)$ is easily got in terms of the spectrum of $E(t)$ that is

$$S_x(\omega) = \lim_{T \to \infty} \left| \tilde{E}(\omega, T) \right|^2 = \frac{3c^3 \tau}{2m \left[ (\omega_0^2 - \omega^2)^2 + \tau^2 \omega^6 \right]} S_E(\omega),$$

whence we obtain, taking eq.(5) into account,

$$S_x(\omega) = \frac{\hbar \tau \omega^3}{\pi m \left[ (\omega_0^2 - \omega^2)^2 + \tau^2 \omega^6 \right]},$$

(8)

From the spectrum it is trivial to get the quadratic means of the relevant variables namely

$$\langle x^2 \rangle = \int_0^\infty S_x(\omega) \, d\omega, \quad \langle v^2 \rangle = \int_0^\infty \omega^2 S_x(\omega) \, d\omega,$$

(9)

where $\langle \rangle$ means time average, and the quantities in eq.(9) are the coordinate and its velocity, respectively. The spectrum of the velocity is $\omega^2$ times the spectrum of the coordinate because the time derivative leads to multiplication of the Fourier transform times $i\omega$. In our treatment of stationary states in SED an ergodic hypothesis is made, that is ensemble averages are assumed equal to time averages for the stationary stochastic processes involved.

Calculating the integral of $S_x(\omega)$ is lengthy but it becomes trivial in the limit $\tau \to 0$ where the integrand is highly peaked at $\omega \simeq \omega_0$. If $\tau$ is small the contribution to the integral comes only from values of $\omega$ close to $\omega_0$ and we may put $\omega \to \omega_0$, except in the difference $\omega - \omega_0$, and then to extend the integral to the whole real line. With this substitution the integrand becomes a Dirac’s delta in the limit $\tau \to 0$ and the integral becomes trivial, that is

$$\langle x^2 \rangle = \int_0^\infty S_x(\omega) \, d\omega \simeq \int_{-\infty}^\infty \frac{\hbar \tau \omega_0^3}{\pi m \left[ 4\omega_0^2 (\omega - \omega_0)^2 + \tau^2 \omega_0^6 \right]} \, d\omega$$

$$\simeq \frac{\hbar}{2m \omega_0} \int_{-\infty}^\infty \delta(\omega - \omega_0) \, d\omega = \frac{\hbar}{2m \omega_0},$$

(10)
whence the mean potential energy is
\[ \langle V \rangle = \frac{1}{2} m\omega_0^2 \langle x^2 \rangle = \frac{1}{4} \hbar \omega_0. \]

The contribution of the high frequencies, \( \langle x^2 \rangle_{hf} \), may be approximated by the integral of the spectrum eq. (8) with zero substituted for \( \omega_0 \). However, in order to exclude the low frequency part, calculated in eq. (10), we shall put \( 2\omega_0 \) as lower limit of the integral, that is
\[ \langle x^2 \rangle_{hf} \simeq \int_{2\omega_0}^{\infty} \frac{\hbar \tau \omega^3}{\pi m [\omega^4 + \tau^2 \omega^6]} d\omega \]
\[ \simeq -\frac{\hbar \tau}{\pi m} \log \left( \frac{\tau \omega_0}{\tau \omega_0} \right), \tag{11} \]
which is positive (see eq. (3)). We see that the result depends but slightly on the lower limit of the integral (provided it is of order \( 2\omega_0 \)).

A similar procedure might be used for the quadratic mean velocity, by performing the integral of the velocity spectrum. However that integral is divergent and we shall assume that there is some frequency cut-off, \( \omega_c \). The result of the integral is the sum of two terms. One of them comes from frequencies near \( \omega_0 \) and it is independent of the cut-off in the limit \( \tau \to 0 \) giving
\[ \langle v^2 \rangle = \int_0^{\omega_c} \omega^2 S_x (\omega) d\omega \simeq \frac{\hbar \omega_0}{2m} \Rightarrow \frac{1}{2} m \langle v^2 \rangle = \frac{1}{4} \hbar \omega_0. \tag{12} \]
The other term comes from the high frequency region and it is divergent when the cut-off goes to infinity. It may be approximated as in the case of \( \langle x^2 \rangle \), although here we may put zero as lower limit of the integral, that is
\[ \langle v^2 \rangle_{hf} \simeq \int_0^{\omega_c} \frac{\hbar \tau \omega^5}{\pi m [\omega^4 + \tau^2 \omega^6]} d\omega = \frac{\hbar}{2\pi m \tau} \log \left( 1 + \tau^2 \omega_c^2 \right). \tag{13} \]
However that term is not very relevant because for those frequencies the non-relativistic approximation breaks down (see below the discussion of the velocity dispersion in the free particle case). Adding eqs. (10) and (12) gives the total mean energy to zeroth order in the small quantity \( \tau \omega_0 \), namely
\[ \langle U \rangle = \left\langle \frac{1}{2} m\omega_0^2 x^2 + \frac{1}{2} mv^2 \right\rangle = \frac{1}{2} \hbar \omega_0. \tag{14} \]
An alternative definition of the energy is possible in terms of the canonical momentum, $p$, which avoids problems of divergence. The momentum is defined by

$$ p \equiv mv - \frac{e}{c}A \equiv \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2. \quad (15) $$

Now we take into account that the potential vector, whose $x$ component we label $A$, contains two parts one coming from the ZPF and the other one from the particle self-field, the latter producing the radiation reaction. These two terms give rise to the latter two terms of eq. (2). Taking this relation into account it is straightforward to get the spectrum of the canonical momentum, that is

$$ \frac{d}{dt}p = -m \omega_0^2 x \Rightarrow S_p (\omega) = \frac{m^2 \omega_0^4}{\omega^2} S_x (\omega) \quad (16) $$

Hence we get

$$ \langle p^2 \rangle = m^2 \omega_0^4 \int_0^{\infty} \omega^{-2} S_x (\omega) d\omega = \frac{m \hbar \omega_0}{2} \Rightarrow \frac{\langle p^2 \rangle}{2m} = \frac{1}{4} \hbar \omega_0, \quad (17) $$

in the limit $\tau \to 0$. We see that the energy defined from the velocity is divergent (a cut-off was needed), whilst the one derived from the canonical momentum is finite. Thus the use of the canonical momentum in the definition of the energy seems more convenient. We may expect that in a more correct relativistic treatment the former would be also convergent and not too different from the latter.

### 2.3 Probability distributions of position, momentum and energy

In order to fully define the stationary state of the oscillator immersed in ZPF it is necessary to get the probability distributions, not just the mean values. Before doing that we need to clarify the meaning of the probability distributions involved. Up to now we have considered averages over infinite time intervals, see eq.(5). However we assume that the time dependent quantities are stochastic processes, that is probability distributions of functions
of time. Thus we should write \( x(t, \lambda) \) (as is standard in the mathematical theory of stochastic processes) rather than just \( x(t) \), where \( \lambda \in \Lambda \) and there is a probability distribution on the set \( \Lambda \). For a fixed value of \( t \) this provides a probability distribution of the random variable \( x(t) \). We assume that the probability distribution of each component, \( E(t, \lambda) \), of the ZPF (in free space) is Gaussian with zero mean and also that it is a stationary ergodic process, that is any time average (over an infinite time interval) equals the ensemble average over the probability distribution of \( \Lambda \) at any single time.

Eq.\((\ref{eq:2})\) is linear, whence the Gaussian character of \( E(t, \lambda) \) gives rise to Gaussian distributions (with zero mean) for both positions and velocities. Thus eq.\((\ref{eq:2})\) fixes completely the normalized probability distribution of the positions to be

\[
W(x) \, dx = \sqrt{\frac{m\omega_0}{\pi \hbar}} \exp \left[ -\frac{m\omega_0 x^2}{2\hbar} \right] \, dx.
\]

Similarly eq.\((\ref{eq:17})\) fixes the distribution of momenta, that is

\[
W(p) \, dp = \sqrt{\frac{m}{\pi \hbar \omega_0}} \exp \left[ -\frac{p^2}{2m\hbar \omega_0} \right] \, dp,
\]

which is also normalized. The distribution of velocities is similar to that one, with \( mv \) substituted for \( p \) (modulo ignoring the part due to high frequencies).

In order to get the distribution of energy, \( U \), to lowest order in Planck constant \( \hbar \) we take into account that, as eqs.\((\ref{eq:18})\) and \((\ref{eq:19})\) already contain \( \hbar \), the relation between \( x, v \) and \( U \) should be written to zeroth order in \( \hbar \), that is using the classical relation. Then we get the following exponential distribution of energies, \( U \),

\[
W(U) \, dU = \int W(x) \, dx \int W(p) \, dp \delta \left( U - \frac{1}{2} \frac{m\omega_0 x^2}{\hbar} - \frac{1}{2m} \frac{p^2}{\hbar \omega_0} \right) \, dU, \quad U \geq 0.
\]

where \( \delta () \) is Dirac’s delta. Hence the fluctuation of the energy is

\[
\sqrt{\langle U^2 \rangle - \langle U \rangle^2} = \langle U \rangle = \frac{1}{2} \hbar \omega_0.
\]

The distributions of positions and momenta, eqs.\((\ref{eq:18})\) and \((\ref{eq:19})\) agree with the QM predictions, but this is not the case for the energy because QM predicts a sharp energy, in disagreement with the SED eq.\((\ref{eq:20})\). Below we shall
study this discrepancy, that is very relevant for our realistic interpretation of quantum theory.

Eqs. (18) and (19) show that the Heisenberg uncertainty relations,
\[ \Delta x \Delta p \geq \hbar / 2, \] (22)
aver a natural way in SED. Indeed the probability distributions eqs. (18) and (19) correspond to what in quantum language is called a “minimum uncertainty wavepacket”, that is the quantum state where the Heisenberg inequality, eq. (22), saturates i.e. it becomes an equality.

Calculating the corrections due to the finite value of the parameter \( \tau \) in eqs. (10) to (14) is straightforward although lengthy\[8,1\] and it will not be reproduced here. A relevant point is that the correction is not analytical in \( \tau \) (or in the fine structure constant \( \alpha \)), but the leading term agrees with the radiative corrections of quantum electrodynamics (Lamb shift). An advantage of the SED calculation is that the radiative corrections (to the nonrelativistic treatment) may be got exactly whilst in quantum electrodynamics the required perturbative techniques allow only an expansion in powers of \( \tau \) (or \( \alpha \)), once a ultraviolet cut-off is introduced. In any case the radiative corrections depend on the high frequency region of integrals like eq. (8), where the non-relativistic approximation breaks down. Therefore the calculation of these corrections has a purely academic interest.

2.4 Comparison between the stationary state in SED and the ground state in QM

A conclusion of the study of the stationary state of the oscillator in SED is that it is rather similar to the ground state of the oscillator en QM. Indeed the probability distribution of positions and momenta in the stationary state of SED agree with the predictions of QM for the ground state, in the limit \( \tau \to 0 \), eqs. (18) and (19), whilst the corrections for finite \( \tau \), that depend on the small quantity \( \tau \omega_0 \), correspond to the radiative corrections of quantum electrodynamics. However the probability distribution of the energy does not agree with QM. In the following I study more carefully this discrepancy.

2.4.1 John von Neumann’s theorem against hidden variables

Firstly I should mention that the conflict between the QM prediction and the SED eq. (20) is an example of the general argument used by von Neumann[?].
in his celebrated theorem of 1932 proving that hidden variable theories are incompatible with QM. That theorem prevented research in hidden variables theories until Bell’s rebuttal in 1966[10]. J. von Neumann starts with the assumption that any linear relation between quantum observables should correspond to a similar linear relation between the possible (dispersion free) values in a hypothetical hidden variables theory. In our case the energy $U$ is a linear combination of $v^2$ and $x^2$. Thus as the energy predicted by quantum mechanics, $U = \frac{\hbar \omega_0}{2}$, is sharp, any pair of values of $v^2$ and $x^2$ in the hidden variables theory should fulfil, according to von Neumann’s hypothesis,

$$m(v^2 + \omega_0^2 x^2) = \hbar \omega_0,$$  \hspace{1cm} (23)

which is not compatible with the distributions eqs.(18) and (19) (for instance the possible value $v^2 = 2\hbar \omega_0/m$ is incompatible with eq.(23) because it would imply $x^2 \geq 0$). Bell’s rebutted von Neumann pointing out that the contradiction only arises when two of the quantum observables involved do not commute and in this case the measurement of the three observables should be made in, at least, two different experiments. Thus a contextual hidden variables theory is possible, that is a theory where it is assumed that the value obtained in the measurement depends on both the state of the observed system and the full experimental context.

2.4.2 The apparent contradiction between QM and SED

In our case the apparent contradiction between SED eq.(20) and the QM prediction of a sharp energy dissapears if we take into account how the energy of a state is defined operationally (i. e. how it may be measured.) In SED the stationary state corresponds to a dynamical equilibrium between the oscillator and the ZPF. Checking empirically whether a dynamical equilibrium exists requires a long time, ideally infinite time. If we define the energy of the oscillator in equilibrium as the average over an infinite time, it would be obviously sharp. In fact the probability distribution of the “mean energies over time intervals of size $\Delta t$” has a smaller dispersion as greater is $\Delta t$, and will be dispersion free in the limit $\Delta t \to \infty$. Thus it is natural to assume that the ground state energy as defined by QM actually corresponds to measurements made over infinitely long times. This fits fairly well with the quantum energy-time uncertainty relation

$$\Delta U \Delta t \geq \hbar /2,$$  \hspace{1cm} (24)
which predicts that the measured energy does possess a dispersion $\Delta U$ if the measurement involves a finite time $\Delta t$. Thus no contradiction exists between SED and QM for the energy in the ground state.

It is remarkable that QM and SED lead to the same result via rather different paths. In fact in QM the state vector of the ground state of a system is an eigenstate of the Hamiltonian, which implies a nil dispersion of the state energy, but the uncertainty relation gives rise to some uncertainty for any actual measurement. This leads us to propose that the ground state of a physical system in QM corresponds to a dynamical equilibrium between emission of radiation to the vacuum fields and absorption from them. The instantaneous energy is a badly defined concept. Indeed the SED distribution eq. (20) derives from the (classical) definition of total energy in terms of positions and momenta, but it does not possesses any operational (measurable) meaning.

2.5 Spectrum of the light emitted or absorbed by the SED oscillator

There is another trivial agreement between the SED and QM predictions for the oscillator, namely the spectrum of emitted or absorbed light. In fact the standard quantum method to derive the spectrum of a system starts solving the stationary Schrödinger equation and then calculating the frequencies using the rule

$$\omega_{jk} = \frac{E_j - E_k}{\hbar} = (j - k)\omega_0,$$

where the eigenvalues of the oscillator Hamiltonian, $E_n = n\hbar\omega_0$, have been taken into account. However in the oscillator there is a selection rule that, within the electric dipole approximation, forbids transitions except if $j - k = \pm 1$, whence the spectrum has a single frequency that agrees with the classical one. Actually the spectrum contains also the frequencies $n\omega_0$, that correspond to electric multipole transitions, although these transitions have low probability. The multipoles of the fundamental frequency may be found also in SED calculations if the electric dipole approximation is not made, that is if the following Lorentz force is substituted for the last term of eq. (2)

$$F_x = e \left[ E + \frac{\mathbf{F}}{c} \times \mathbf{B} \right]_x,$$

15
E and B being the electric and magnetic fields of the ZPF. To be consistent the other terms of eq.(2) should be also changed to become relativistic in order to be consistent. Then the differential equation of motion becomes nonlinear and it is far more difficult to solve, but this may be achieved numerically and good agreement with quantum predictions is obtained for the spectrum of emitted or absorbed light\[11\].

SED may also offer intuitive pictures for cavity quantum electrodynamics, a well established experimental field of research\[1\]. An atom in a cavity get modified its properties, in particular its lifetime. In fact the atom does not decay if the modes having the frequency of the emitted radiation are not possible inside the cavity. In the quantum treatment the intriguing question is how the atom “knows” in advance that it should not decay in these conditions. In SED the explanation is trivial: spontaneous decay is actually stimulated by appropriate modes of the ZPF, and the modes required for the stimulation do not exist inside the cavity. For instance in an early experiment by Haroche et al.?? the excited atoms propagates between two metallic mirrors separated by 1.1 \( \mu \text{m} \) for about 13 natural lifetimes without appreciable decay. The experiment involved a small applied magnetic field in order to demonstrate the anisotropy of spontaneous emission between mirrors. This experiment has been studied within SED via modelling the atom by a harmonic oscillator whence the empirical results have been reproduced quantitatively, but I will not review that work here\[13\], \[14\].

2.6 Lessons for a realistic interpretation of quantum theory

Our study of a particle in a potential well shows that inclusion of the vacuum random electromagnetic field leads to predictions resembling those of quantum electrodynamics.

The quantum ground state of a particle in a potential well corresponds to a stationary state of a particle performing a highly irregular (stochastic) motion driven by vacuum fields.

The spectrum of the field (or the energy per unit volume and unit frequency interval) determines the spectrum for the motion of the particle. It is such that the quadratic mean coordinate position and the quadratic mean momentum agree with quantum predictions and fulfil the Heisenberg uncertainty relations. These are here interpreted as a consequence of the (unavoidable)
random motion of the particles.

No contradiction arises between the exponential distribution of energy in SED and the sharp energy in QM. They are different operational definitions. The former refers to the instantaneous energy (or the mean in a small time interval) but the latter to the mean over an infinite (or very large) time interval. This difference is a good illustration for the flaw in the celebrated von Neumann theorem against hidden variables in quantum mechanics.

Radiative corrections (e. g. Lamb shift) appear naturally in SED with a transparent interpretation, i. e. as a consequence of the interaction between the charged particle and the real vacuum fields.

3 Time-dependent properties of oscillators and free particles

3.1 Evolution of the dynamical variables

In Newtonian mechanics the study of the evolution consists of finding the position as a function of time for given initial conditions, that is initial positions and velocities of the particles involved. Thus the evolution describes a curve in phase space parametrized by time. If there are forces not fully known, which we represent as noise, all we may get is the evolution of the probability distribution in phase space with given initial conditions, i. e. either a point or a probability distribution in phase space. This is the case for the oscillator in SED that we study in the following. In order to calculate the evolution of the oscillator it is convenient to start anew from the equation of motion, eq.(2). We shall work to lowest nontrivial order in the small parameter $\tau$ (see eq.(3)). Thus we may approximate the third order eq.(2) by another one of second order substituting $-m\tau\omega_0^2\dot{x}$ for $m\tau\ddot{x}$ on its right side. That is writing

$$m\ddot{x} = -m\omega_0^2x - m\tau\omega_0^2\dot{x} + eE(t),$$

which agrees with eq.(2) to first order in $\tau$. This second order equation in $x(t)$ is equivalent to two coupled stochastic differential equations of Langevin type, in the variables $x(t)$ and $\dot{x}(t)$. We used eq.(2), rather than (25) because the former was more appropriate for the study of radiative corrections than the latter.

A convenient way to study the motion of the oscillator in SED consists of
introducing new variables, \( a(t) \) and \( b(t) \), as follows

\[
\begin{align*}
    x(t) &= a(t) \cos(\omega_0 t) + b(t) \sin(\omega_0 t) + \xi(t), \\
    \dot{x}(t) &= -a(t)\omega_0 \sin(\omega_0 t) + b(t)\omega_0 \cos(\omega_0 t) + \dot{\xi}(t).
\end{align*}
\] (26)

The rapidly fluctuating quantity \( \xi(t) \) is related to the high frequency part of the spectrum \( S_x(\omega) \) (see comment after eq.(12)) and it will be ignored in the following. The variables \( a \) and \( b \) are constants of the motion in the classical mechanical oscillator and they are slowly varying functions of time, with typical variation time \( 1/(\tau \omega^2_0) \gg \omega_0^{-1} \), see below. An alternative to eq.(26) would be to write the coordinate in terms of the amplitude, \( c(t) \), and the phase, \( \phi(t) \), both slowly varying with time, that is

\[
x(t) = c(t) \cos(\omega_0 t + \phi(t)) + \xi(t),
\]

but the choice eq.(26) is more easy to solve. At the initial time,\( t = 0 \), the parameters \( a(t) \) and \( b(t) \) are easily related to the initial position, \( x_0 \), and momentum, \( p_0 \), that is

\[
a(t_0) = x_0, b(t_0) = \frac{\dot{x}_0}{\omega_0} = \frac{p_0}{m\omega_0}.
\] (27)

Calculating the evolution of the variables \( a(t) \) and \( b(t) \) simplifies if we introduce a complex function \( z(t) \) such that

\[
\begin{align*}
    x(t) &= \text{Re}\left[z(t) \exp(-i\omega_0 t)\right], \\
    \text{Re}z(t) &= a(t), \text{Im}z(t) = b(t).
\end{align*}
\] (28)

The function \( z(t) \) is slowly varying and therefore we may neglect its second (first) derivative in the term of order 0 (order \( \tau \)) in the equation that results from inserting eq.(28) into eq.(25). This gives

\[
-2im\omega_0 \dot{z}(t) = im\tau\omega_0^3 z(t) + eE(t) \exp(i\omega_0 t).
\] (29)

The solution of this equation is trivial and we get

\[
z(t) = \exp\left(-\frac{1}{2}\tau \omega_0^3 t\right) \left[z(0) + \frac{ie}{2m\omega_0} \int_0^t E(t') \exp\left(\frac{1}{2}\tau \omega_0^2 t' + i\omega_0 t'\right) dt'\right].
\]

Hence it is easy to get the ensemble average, or expectation, of \( z(t) \) taking into account that \( E(t) \) is a stochastic process with zero mean. We get

\[
\langle z(t) \rangle = \exp\left(-\frac{1}{2}\tau \omega_0^2 t\right) \langle z(0) \rangle.
\] (30)
Also we may obtain the following quadratic mean
\[
\left\langle \left| z(t) - \exp \left( -\frac{1}{2} \tau \omega_0^3 t \right) z(0) \right|^2 \right\rangle = \frac{e^2}{4m^2\omega_0^2} \exp \left( -\tau \omega_0^3 t \right) F, \quad (31)
\]
where
\[
F \equiv \int_0^t dt' \int_0^t dt'' \langle E(t')E(t'') \rangle \exp \left[ \left( \frac{1}{2} \tau \omega_0^2 + i\omega_0 \right) t' + \left( \frac{1}{2} \tau \omega_0^2 - i\omega_0 \right) t'' \right]. \quad (32)
\]
The selfcorrelation of the process \( E(t) \) is the Fourier transform of the spectral density, that is
\[
\langle E(t')E(t'') \rangle = \frac{1}{2} \int_{-\infty}^{\infty} \frac{2\hbar \omega^3}{3\pi c^3} \exp \left[ i\omega (t'' - t') \right] d\omega, \quad (33)
\]
where eq.(5) has been taken into account. I stress that the process \( E(t) \) is stationary and assumed ergodic, whence time average (over an infinite time) and ensemble average agree. However here we do not assume that \( x(t) \) is stationary but we are investigating its time dependence. If eq.(33) is put in eq.(32) and the integrals in \( t' \) and \( t'' \) performed we get
\[
F = \frac{\hbar}{3\pi c^3} \int_{-\infty}^{\infty} \frac{\omega^3 d\omega}{\tau^2 \omega_0^4 + 4 (\omega - \omega_0)^2} \left| \exp \left( \frac{1}{2} \tau \omega_0^2 + i\omega_0 - i\omega \right) t - 1 \right|^2 \times \left[ \exp(\tau \omega_0^2 t) + 1 - 2 \exp(\frac{1}{2} \tau \omega_0^2 t) \cos(\omega t - \omega_0 t) \right] d\omega.
\]
The integral in \( \omega \) is ultraviolet divergent but the contribution of the high frequencies will be ignored here (see comment after eq.(26)). Thus taking into account that \( \tau \omega_0 << 1 \), the overwhelming contribution to the integral comes from frequencies \( \omega \simeq \omega_0 \) and the integral may be approximated putting \( \omega_0^3 = \omega_0^3 \) whence we obtain
\[
\left\langle \left| z(t) - \exp \left( -\frac{1}{2} \tau \omega_0^3 t \right) z(0) \right|^2 \right\rangle \simeq \frac{\hbar}{2m\omega_0^2} \left[ 1 - \exp \left( -\tau \omega_0^2 t \right) \right]. \quad (34)
\]
Taking eq. (28) into account we may separate the real and the imaginary parts in eqs. (30) and (34). Thus we obtain information about the evolution of the classical constants of the motion \( a = \text{Re} \ z \) and \( b = \text{Im} \ z \), that is

\[
\langle a(t) \rangle = \exp \left( -\frac{1}{2} \tau \omega_0^2 t \right) \langle a(0) \rangle,
\]

\[
\langle [a(t) - \langle a(t) \rangle]^2 \rangle = \left\langle \left( a(t) - \exp \left( -\frac{1}{2} \tau \omega_0^2 t \right) a(0) \right)^2 \right\rangle 
\approx \frac{\hbar}{4m \omega_0^2} \left[ 1 - \exp \left( -\tau \omega_0^2 t \right) \right],
\]

and similar for \( b(t) \) because both variables, \( a(t) \) and \( b(t) \), have similar contributions to eq. (34), which follows from their roles in eqs. (26). Eqs. (35) and the similar ones for \( b(t) \) mean that for any initial distribution of positions and momenta, the oscillator will arrive at a distribution corresponding to the stationary state (corresponding to the quantum ground state).

### 3.2 Diffusion of the probability density in phase space

It is possible to derive differential equations for the probability densities of \( a \) and \( b \). They have the form of Fokker-Planck (or diffusion) equation like

\[
\frac{\partial \rho(a, t)}{\partial t} = \frac{\partial}{\partial a} (A \rho) + \frac{\partial^2}{\partial a^2} (D \rho).
\]

The coefficients of drift, \( A \), and diffusion, \( D \), may be calculated from eqs. (35) as follows

\[
A = \lim_{t \to \infty} \frac{\langle a(t) \rangle}{t} = -\frac{1}{2} \tau \omega_0^2, \quad D = \lim_{t \to \infty} \frac{\langle [a(t) - \langle a(t) \rangle]^2 \rangle}{t} = \frac{\hbar \tau}{4m}.
\]

Whence the Fokker-Planck equation reads

\[
\frac{\partial \rho(a, t)}{\partial t} = \frac{1}{2} \tau \omega_0^2 \frac{\partial}{\partial a} (a \rho) + \frac{\hbar \tau}{4m} \frac{\partial^2 \rho}{\partial a^2},
\]

and a similar one with \( b \) substituted for \( a \). I must point out that the stochastic processes \( a(t) \) and \( b(t) \) are correlated. In fact their crosscorrelation may be easily obtained as follows

\[
\langle a(t)b(t') \rangle = \left\langle \left[ x(t) \cos(\omega_0 t) - \frac{p(t)}{m \omega_0} \sin(\omega_0 t) \right] \left[ x(t') \cos(\omega_0 t') + \frac{p(t')}{m \omega_0} \sin(\omega_0 t') \right] \right\rangle
\]

\[
= -\frac{1}{2} \hbar \sin[\omega_0 (t' - t)] \exp(-\tau \omega_0^2 |t' - t|) = -\langle b(t)a(t') \rangle,
\]
where we have taken into account eqs. (26). Due to this correlation a joint probability density $\rho(a, b, t)$ cannot be got as a product of the individual densities. Consequently deriving the differential equation for $\rho(a, b, t)$ is involved and it will be not made here.

The conclusion of our calculation is that the classical constants of the motion, like the parameters $a$ and $b$ or the energy, $U$, perform a slow random motion with typical relaxation time $1/(\tau \omega_0^2)$. In particular the energy is related to these parameters as follows

$$U(t) = \frac{1}{2}m\omega_0^2 x(t)^2 + \frac{1}{2}m\dot{x}(t)^2 = \frac{1}{2}m\omega_0^2 [a(t)^2 + b(t)^2],$$

(37)

where eqs. (26) have been taken into account. The change of the classical constants of the motion of the oscillator in SED is obviously due to the two latter terms of eq. (2). The term involving the vacuum field produces diffusion, characterized by $D$, and the radiation reaction term gives rise to drift, characterized by $A$. The diffusion rate is independent of the particle’s velocity whence $D$ is a constant, but the drift increases with the velocity with the result that $A$ is proportional to $a$. The effect of the diffusion is reduced to some extent by the drift, with the consequence that the probability densities remain localized. In fact, when time increases indefinitely the densities approach the stationary solution studied in section 2. In particular the stationary solution of eq. (36) (with normalized $\rho$, which implies that $\rho$ vanishes for $a \to \pm \infty$) is

$$\rho = \sqrt{\frac{\hbar}{\pi m\omega_0^2}} \exp \left(-\frac{\hbar m\omega_0^2 a^2}{\pi m\omega_0^2}\right),$$

and similar for $b$. The energy tends to the density given by eq. (20).

### 3.3 States of the oscillator in stochastic electrodynamics and in quantum mechanics

Every nonnegative definite function in phase space may be taken as an initial probability density and thus be considered a state of the SED oscillator. We see that the set of states in SED is quite different from the set of states in QM (given by a density operator each). In particular the pure states in SED are those whose initial conditions correspond to points in phase space, whilst the pure states in QM correspond to state vectors (or wave functions). The comparison between SED and QED becomes more clear if we define
the quantum states by means of functions in phase space, which might be achieved via the Wigner function formalism, to be revisited in Chapter 5. Thus in the case of pure quantum states it is obvious that only a small fraction of them correspond to states in SED. Actually, asides from the ground state there are only two interesting pure quantum states that correspond precisely to SED states, namely coherent states and squeezed states. The latter are relevant in case of radiation (squeezed states of light), but not so much for matter oscillators and they will not be studied here.

Coherent states in SED appear as solutions of the oscillator eq.(2) obtained by combining the stationary solution of the equation with the general solution of the homogenous equation, that is

\[ \ddot{x} + \omega_0^2 x + \tau \omega_0^2 \dot{x} = 0 \Rightarrow x \simeq A \cos (\omega_0 t + \phi) \exp (-\tau \omega_0 t), \]

where I have approximated \( \ddot{x} \simeq -\omega_0^2 \dot{x} \) and neglected a small shift, of order \( \tau \), in the frequency \( \omega_0 \). Hence, taking eq.(18) into account, we see that the solution of eq.(2) leads to the following time dependent probability distribution of positions

\[ W(x, t) \simeq \sqrt{\frac{m \omega_0}{\pi \hbar}} \exp \left[ -\frac{m \omega_0}{2 \hbar} \left[ x - A \cos (\omega_0 t + \phi) \exp (-\tau \omega_0^2 t) \right]^2 \right], \quad (38) \]

which contains two integration constants, \( A \) and \( \phi \). It must be stressed that this expression for the probability density derives from eq.(2) and the ZPF spectrum eq.(3) with the approximation of putting \( \tau \to 0 \) except in the exponential decay. It may be seen that when \( \tau = 0 \) the evolution of the position probability density eq.(38) fully agrees with the one of the coherent states of quantum mechanics, whilst the expression for finite \( \tau \) contains the most relevant contribution of the radiative corrections of quantum electrodynamics to these states (a decay towards the stationary state with relaxation time \( (\tau \omega_0^2)^{-1} \))[8].

In summary we see that a few states of the oscillator in SED correspond to pure quantum states in a phase-space representation. But no pure state of SED corresponds to a state of QM. Also most of the pure states of QM do not correspond to states of SED. However for mixed states the agreement is greater, and actually all (mixed) quantum states possessing a positive Wigner function closely correspond to mixed states of SED with the same phase-space distribution.
In spite of these differences the quantum theory of the harmonic oscillator admits a realistic interpretation via SED, provided that the same predictions may be obtained for actual experiments. In particular the quantum states, solutions of Schrödinger equation, might be just mathematical auxiliary functions used in the QM formalism but not required in the SED approach for the prediction of the same results.

3.4 The free particle

For a free particle the differential equation of motion is like the oscillator’s eq. (2) with $\omega_0 = 0$, that is

$$m \ddot{x} = m \tau \ddot{x} + e E(t). \quad (39)$$

This fact may suggest studying the free particle as the limit of an oscillator whose characteristic frequency decreases to zero. However this method is not appropriate because there is a qualitative difference between the two systems. In fact the motion in the oscillator is always bound which is not the case for the free particle. Thus we shall study the motion of the free particle starting from eq. (39). It is a third order equation and therefore has three independent solutions, but one of them is runaway that is the energy increases without limit, which is physically nonsense. The reason is that the radiation reaction term, the former term on the right side of eq. (39), is a linearized approximation not valid for a free particle (in the oscillator the runaway solution is effectively cut-off by the potential and the approximation of eq. (2) is good enough). Thus we shall substitute the following integro-differential equation for eq. (39)

$$\ddot{x} = -\frac{e}{m \tau} \exp \left(\frac{t}{\tau}\right) \int_t^\infty E(t') \exp \left(-\frac{t'}{\tau}\right) dt'.$$

It has the same solutions as eq. (39) except the runaway ones. Hence, it is trivial to get the following equations of evolution for the velocity and the coordinate, respectively, that is

$$v(t) = v_0 - \frac{e}{m \tau} \int_0^t \exp \left(\frac{s}{\tau}\right) ds \int_s^\infty E(u) \exp \left(-\frac{u}{\tau}\right) du,$$

$$x(t) = x_0 + v_0 t - \frac{e}{m \tau} \int_0^t ds \int_0^s \exp \left(\frac{u}{\tau}\right) du \int_u^\infty E(w) \exp \left(-\frac{w}{\tau}\right) dw,$$
where $x_0$ is the initial position and $v_0$ the initial velocity at time $t = 0$. Hence, taking into account that the ensemble average of $E(t)$ is zero, it is trivial to get the mean position and velocity of a particle, that is

$$\langle x(t) \rangle = x_0 + v_0 t. \quad (41)$$

The most interesting quantities are the dispersions of velocity and position with time. The velocity dispersion may be got from the first eq. (40) putting $v_0 = 0$. We obtain, taking eq. (3) into account,

$$\langle v(t)^2 \rangle = \frac{3c^3}{2m\tau} \int_0^t \exp \left( \frac{s}{\tau} \right) ds \int_s^\infty \exp \left( -\frac{u}{\tau} \right) du \left(42\right)$$

$$\times \int_0^t \exp \left( \frac{s'}{\tau} \right) ds' \int_{s'}^\infty \exp \left( -\frac{u'}{\tau} \right) du' \langle E(u) E(u') \rangle.$$ 

The $E(t)$ selfcorrelation is the Fourier transform of the spectrum (see eq. (?)), that is

$$\langle E(u) E(u') \rangle = \int_0^\infty S_x(\omega) \cos[\omega(u-u')] d\omega$$

$$= \frac{1}{2} \int_0^\infty |S_x(\omega)| \exp[i\omega(u-u')] d\omega. \quad \left(43\right)$$

I point out that this relation is correct because $E(t)$ is a stationary process, but it is not possible to get eq. (42) from the spectrum of $v(t)$ because in the free particle case we cannot get the spectrum of $v(t)$ from that of $E(t)$ (as we made in the derivation of eq. (8) for the equilibrium state of oscillator, where both $x(t)$ and $v(t)$ are stationary processes). Inserting eq. (43) in eq. (42) we get, after changing the order of the integrations,

$$\Delta v^2 \equiv \langle v(t)^2 \rangle = \frac{e^2}{2m^2 \tau^2} \int_{-\infty}^\infty |S_x(\omega)| d\omega \left| \int_0^t \exp \left( \frac{s}{\tau} \right) ds \int_s^\infty \exp \left( -\frac{u}{\tau} + i\omega u \right) du \right|^2$$

$$= \frac{\hbar}{2\pi m \tau} \int_{-\infty}^\infty |\omega| d\omega \left| 1 - \exp(i\omega t) \right|^2 = \frac{\hbar \tau}{\pi m} \int_0^\infty \frac{\omega d\omega}{1 + \tau^2 \omega^2} \left| 1 - \cos(\omega t) \right|^2.$$ 

Thus the velocity dispersion gives an ultraviolet divergent integral that may be made convergent by introducing a cut-off frequency $\omega_c$. Thus we get

$$\Delta v^2 = \frac{\hbar \tau}{\pi m} \int_0^{\omega_c} \frac{\omega d\omega}{1 + \tau^2 \omega^2} \left| 1 - \cos(\omega t) \right| \left(44\right)$$

$$\sim \frac{\hbar}{2\pi m \tau} \left[ \log(1 + \omega_c^2 \tau^2) + \tau^2 \right], \text{ for } t >> \tau.$$

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The dispersion $\Delta v$ becomes rapidly independent of $t$, but greater than the velocity of light. In order that $\Delta v < c$ we must have

$$\omega_c < \sqrt{\frac{3\pi mc^2}{\alpha \hbar}} \approx 0.2 \frac{c}{\lambda_C} < < \frac{1}{\tau},$$

$\lambda_C$ being the Compton wavelength. This implies that eq.(44) may be rewritten

$$\Delta v^2 \sim \frac{\hbar \tau \omega_c^2}{2\pi m} + \frac{\hbar^2}{2\pi m \tau^2 t^2}, \text{ for } t \gg \tau. \quad (45)$$

A correct calculation would require a relativistic theory, which will not be attempted here. Nevertheless the result obtained shows that the particle performs a random motion with relativistic speed although the mean velocity remains a constant (see eq.(41)). Also the result suggests that in a relativistic calculation the most relevant wavelengths would be those not too far from the Compton one. The increase of the velocity of a free charged particle by the action of the ZPF has been proposed as possible origen of the observed ultrahigh-energy X rays coming to Earth from outside the Solar System\[15\].

It is interesting to compare the velocity dispersion of the free particle in SED with the particle immersed in Rayleigh-Jeans (classical) radiation. Taking into account that the ZPF and the Rayleigh-Jeans radiation correspond to $\frac{1}{2} \hbar \omega$ and $kT$ per normal mode, respectively, the replacement $\frac{1}{2} \hbar \omega \rightarrow kT$ in eq.(44) leads to

$$\Delta v^2 = \frac{\tau kT}{\pi m} \int_0^{\omega_c} \frac{d\omega}{1 + \tau^2 \omega^2} [1 - \cos(\omega t)]$$

$$\sim \frac{kT}{m} \text{ for } t \gg \tau.$$

We see that the velocity dispersion of the charged free particle does not increase indefinitely but becomes, after a long enough time, a constant corresponding to the kinetic energy $kT/2$ (which is the equipartition of the energy of classical statistical mechanics.)

The dispersion of position on the free particle according to SED may be obtained by a similar method, that is inserting the latter eq.(43) in eq.(40).
We obtain
\[
\Delta x^2 = \frac{\hbar \tau}{2\pi m} \int_{-\infty}^{\infty} \frac{|\omega| d\omega}{1 + \tau^2 \omega^2} \left[ \int_0^t \left| 1 - \exp(i\omega s) \right| ds \right]^2
\]
\[
= \frac{\hbar \tau}{\pi m} \int_0^{\infty} \frac{\omega d\omega}{1 + \tau^2 \omega^2} \left[ t^2 - \frac{2t \sin(\omega t)}{\omega} + \frac{2 - 2\cos(\omega t)}{\omega^2} \right]
\]
\[
\simeq \frac{2\hbar \omega_0^2 \tau^2}{2\pi m} + \frac{2\hbar \tau}{\pi m} \log \left( \frac{t^2}{\tau} \right) - C - 1, \quad t >> \tau,
\] (46)

where C=0.577... is the Euler constant.

The former term, which dominates at long times, is a consequence of the velocity dispersion, as may be easily seen by a comparison with eq.(44). The (canonical) momentum has no dispersion as shown by the former eq.(16) when we put \( \omega_0 = 0 \). This agrees with the quantum prediction that the momentum of a free particle is a constant. For a particle with zero canonical momentum, the typical distance from the original position increases with about one tenth the velocity of light so that a relativistic treatment would give a quite different picture.

The picture that emerges, and gives hints for the interpretation of the free particle in QM, is as follows. The free particle possesses a conserved canonical momentum with an associated inertial motion but, superimposed to this, it has a random motion with a velocity close to that of light. This produces an apparently contradictory behaviour that derives from the spectrum, \( S_E(\omega) \propto \omega^3 \), of the zeropoint field: At short times the motion is governed by the high frequencies where \( S_E(\omega) \) is large thus inducing a rapid erratic motion, at long times it is governed by the low frequencies where \( S_E(\omega) \) is small whence the memory of the initial velocity is lost very slowly. This fact contrast with what happens in Brownian motion and what our intuition may suggest, namely that memory of the initial conditions should be quickly lost. The special behaviour of diffusion in SED is a consequence of the fact that the spectrum \( S_E(\omega) \) is very different from the popular (Brownian) white noise, \( S_{white}(\omega) \simeq \text{constant.} \)

### 3.5 Commutation rules

Stochastic electrodynamics also provides a clue for the interpretation of commutation rules, that is the essential ingredient in the canonical formulation of quantum mechanics. In fact I will define the commutator at two times,
of a stationary stochastic process, \( x(t) \), via the *sinus* Fourier transform of the spectrum. Then I shall show that this stochastic commutator applied to the SED oscillator closely resembles the quantum commutator in the Heisenberg picture of QM.

The introduction of the stochastic commutator is suggested by the Fourier transform of the spectrum, \( S_x(\omega) \). It consists of two terms, that is

\[
\int_0^\infty S_x(\omega) \exp \left[ i \omega (t' - t) \right] d\omega = \langle x(t) x(t') \rangle + \frac{1}{2} [x(t), x(t')],
\]

\[
\langle x(t) x(t') \rangle = \int_0^\infty S_x(\omega) \cos [\omega (t' - t)] d\omega,
\]

\[
[x(t), x(t')] = 2i \int_0^\infty S_x(\omega) \sin [\omega (t' - t)] d\omega,
\]

where the spectrum is defined to be zero for negative frequencies, that is \( S_x(\omega) = 0 \) if \( \omega < 0 \). The real part, \( \langle x(t) x(t') \rangle \), is the selfcorrelation function of the stochastic process so that it is plausible that the imaginary part, \( [x(t), x(t')] \), is also relevant and we define it as the commutator. The factor 2 is chosen in order to be similar to the quantum commutator. I point out that the cosinus Fourier transform of the spectrum is the selfcorrelation only for stationary processes that are ergodic, as is proved by the Wiener-Khinchine theorem.

The relation between spectrum and stochastic commutator is also suggested by the fact that in QM there is a similar relation between the spectrum and the two-times commutator of the coordinate operator in the Heisenberg picture. That relation is fulfilled for the ground state of a particle in any potential well. For the proof here I consider a one-dimensional (quantum) problem defining the spectrum, \( S_x(\omega) \), as follows

\[
S_x(\omega) \equiv \sum_n |\langle \psi_0 | \hat{x} (0) | \psi_n \rangle|^2 \delta (\omega - \omega_0 n),
\]

where \( \hat{x} \) is the quantum position operator of the particle. The coefficients of the Dirac’s deltas are proportional to the transition probabilities in QM from the ground state to all possible excited states. (Although I stress that in QED the deltas are approximations of highly peaked functions with a finite width when radiative corrections are taken into account). The analogy with the latter eq. (47) is shown as follows. From the Heisenberg equation of motion

\[
\hat{x} (t) = \exp(i \hat{H} t/\hbar) \hat{x} (0) \exp(-i \hat{H} t/\hbar).
\]
we may obtain the expectation value of the commutator in the ground state,
\[
\langle [\hat{x}(0) , \hat{x}(t)] \rangle = \langle \psi_0 | \hat{x}(0)\hat{x}(t) | \psi_0 \rangle - \langle \psi_0 | \hat{x}(t)\hat{x}(0) | \psi_0 \rangle .
\]
After introducing the resolution of the identity between \( \hat{x}(0) \) and \( \hat{x}(t) \) and between \( \hat{x}(t) \) and \( \hat{x}(0) \) in terms of eigenvectors of the Hamiltonian \( \hat{H} \), this gives
\[
\langle [\hat{x}(0), \hat{x}(t)] \rangle = 2i \sum_n |\langle \psi_0 | \hat{x}(0) | \psi_n \rangle|^2 \sin (\omega_0 n t) = 2i \int S_x (\omega) \sin (\omega t) d\omega ,
\]
where in the latter equality we have taken eq.(48) into account. This equality, similar to the stochastic latter eq.(47), played an important role in the origin of quantum mechanics. In fact the derivative with respect to \( t \) leads to
\[
2im \sum_n \omega_0 n |\langle \psi_0 | \hat{x}(0) | \psi_n \rangle|^2 \cos (\omega_0 n t) = [\hat{x}(0), \hat{p}(t)],
\]
that in the limit \( t \to 0 \) becomes an example of the well known Thomas-Reiche-Kuhn sum rule. The rule is usually applied to atoms where a sum over the three coordinates of the \( Z \) electrons is performed, so that it reads
\[
2m \sum_n \omega_0 n \left| \sum_{j=1}^Z r_j \middle| \psi_0 \right|^2 = -i \sum_{k=1}^{3Z} [\hat{x}_k(0), \hat{p}_k(0)] = 3Zh .
\]
For a stationary process both the selfcorrelation and the commutator depend only on the time difference \((t - t')\). In this case the latter eq.(47) may be easily inverted via a time integral. In fact we get
\[
\int_{-\infty}^{\infty} \sin [\nu (t - t')][x(t), x(t')] dt \\
= \int_{-\infty}^{\infty} \sin [\nu (t - t')][dt2i \int_0^{\infty} S_x (\omega) \sin (\omega (t' - t)) d\omega \\
= -2i \int_0^{\infty} S_x (\omega) d\omega \int_{-\infty}^{\infty} \sin [\nu (t - t') \sin [\omega (t - t')] dt = -i\pi S_x (\nu) ,
\]
where in the latter equality we take into account that \( S_x (\nu) = 0 \) for \( \nu < 0 \).
All stationary properties of the SED oscillator studied in Section 2 may be equally well obtained either from the spectrum eq. (58), from the self-correlation or from the commutator, the latter being

\[
[x(0), x(t)] = 2i\hbar \int_0^{\infty} \frac{\tau \omega^3 \sin \omega t \, d\omega}{\pi m (\omega_0^2 - \omega^2)^2 + \tau^2 \omega^6} = i\hbar \int_{-\infty}^{\infty} \frac{\tau \omega^3 \exp[i\omega t] \, d\omega}{\pi m (\omega_0^2 - \omega^2)^2 + \tau^2 \omega^6}. \tag{49}
\]

This equality shows the advantage of the commutator with respect to the self-correlation in the stochastic process associated to the SED oscillator. In fact the latter integral may be performed analytically via the method of residues, whilst getting the self-correlation requires approximations. Thus I propose that the reason for the use of commutators in QM is the fact that the basic stochastic processes involved have spectra that are odd with respect to the change \( \omega \rightarrow -\omega \).

Performing the integral eq. (49) is now straightforward. For \( t > 0 \) we shall take into account the three simple poles in the upper half plane of the complex variable \( \omega \), that is

\[
\omega = \pm \omega_0 + \frac{1}{2} i \tau \omega_0^2 + O(\tau^2 \omega_0^3), \quad \omega = \pm i \left( \frac{1}{\tau} + \tau \omega_0^2 \right) + O(\tau^2 \omega_0^3).
\]

For \( t < 0 \) we shall use the poles in the lower half plane. The contribution of the poles in the imaginary axis should be neglected because it contains an exponential of the form \( \exp(-|t|/\tau) \) that is zero except for extremely small values of time. (Actually the term derives from the high-frequency part of the spectrum and should be cut-off, as discussed in Section 1.4.) Thus the result may be written, to order \( O(\tau \omega_0^3) \),

\[
[x(0), x(t)] = \frac{i\hbar}{m\omega_0} \left\{ \sin(\omega_0 t) + \tau \omega_0 \frac{t}{|t|} \cos(\omega_0 t) \right\} \exp \left(-\frac{1}{2} \tau \omega_0^2 |t| \right), \tag{50}
\]

Similarly, taking eqs. (47) and (16) into account we may obtain the commutator of the canonical momentum, that is

\[
[p(0), p(t)] = i\hbar m\omega_0 \sin(\omega_0 t) \exp \left(-\frac{1}{2} \tau \omega_0^2 |t| \right). \tag{51}
\]
In the limit $\tau \to 0$ the commutator eq.(50) agrees with the one derived from elementary quantum mechanics for the corresponding (time dependent) operators in the Heisenberg picture.

The commutator of the particle coordinate in the SED oscillator may be derived from the commutator of the electric field of the zeropoint radiation taking eqs.(5) and (47) into account. Actually the relation between the commutator of the vacuum field and the commutator of a charged particle immersed in the field may be obtained in the context of quantum mechanics without passing through the spectrum. Indeed this was made long ago by Schiller for quantum commutators[1],[4].

Our definition of commutator may be generalized to two different stationary stochastic processes as follows:

**Definition 1** Given two stationary stochastic process, $x(t)$ and $y(t)$, I define the (stochastic) commutator of these processes, $[x(t), y(t')]$, as $2i$ times the Hilbert transform of the crosscorrelation, $\langle x(t)y(t') \rangle$.

The Hilbert transform, $g(u)$, of a function $f(t), t \in (-\infty, \infty)$ is defined by

$$g(u) = \frac{1}{\pi} P \int_{-\infty}^{\infty} f(t) \frac{1}{u-t} dt, f(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} g(u) \frac{1}{u-t} du,$$

where $P$ means principal part and the second equality corresponds to the inverse transform. However the inverse transform does not always recovers the original. For instance the Hilbert transform of a constant is zero and the inverse of zero is also zero. The relevant property for us is that the Hilbert transforms changes $\sin(\omega t)$ into $\cos(\omega u)$ and $\cos(\omega t)$ into $-\sin(\omega u)$, provided that $\omega \neq 0$.

After that it is possible to define the derivative of a commutator with respect to time, that is

$$\frac{d}{dt} [x(t), x(t')] = \lim_{t'' \to t'} \frac{[x(t), x(t'')] - [x(t), x(t')]}{t'' - t'}$$

$$= \lim_{t'' \to t'} \frac{[x(t), x(t'')] - x(t')}{t'' - t'} = \left[ x(t), \frac{dx(t')}{dt'} \right],$$

where the linearity of the commutator has been used. Hence taking eq.(50) into account we get to zeroth order in $\tau$

$$[x(0), p(t)] = i\hbar \cos(\omega_0 t) \Rightarrow [x(0), p(0)] = i\hbar,$$
the latter being the fundamental commutation rule of quantum mechanics.

The stochastic commutator provides a hint for a realistic interpretation of the quantum commutation rules as a disguised form of establishing the properties of some peculiar stochastic processes. The main peculiarity is the fact that the spectra of the processes are usually odd with respect to a change \( \omega \rightarrow -\omega \) of the frequency.

4 Coupled oscillators in SED

The generalization of the harmonic oscillator in SED to many dimensions is straightforward using the appropriate extension of eq.(2). In the following I will study two simple examples of coupled oscillators. Firstly a system of two one-dimensional oscillators at a long distance as an example of van der Waals force. The system is interesting because it shows that a phenomenon similar to quantum entanglement appears also in SED. The second example is an array of coupled three-dimensional oscillators at a finite temperature, that reproduces Debye theory of the specific heat of solids.

4.1 A model for quantum entanglement

Entanglement is a quantum property of systems with several degrees of freedom, which appears when the total state vector cannot be written as a product of vectors associated to one degree of freedom each. In formal terms a typical entangled state fulfils

\[
|\psi(1,2)\rangle = \sum_{m,n} c_{mn} \, |\psi_m(1)\rangle \, |\psi_n(2)\rangle,
\]

(52)

where 1 and 2 correspond to two different degrees of freedom, usually belonging to different subsystems. The essential condition is that the state eq.(52) cannot be written as a product, that is the sum cannot be reduced to just one term via a change of basis in the Hilbert space. Entanglement appears as a specifically quantum form of correlation, which is claimed to be dramatically different from the correlations of classical physics. The latter may be usually written in the form

\[
\rho(1,2) = \sum_{m,n} w_{mn} \rho_m(1) \rho_n(2),
\]

(53)
where the quantities $\rho \geq 0$ are probability densities and the coefficients play the role of weights fulfilling $w_{mn} \geq 0$, in sharp contrast with eq. (52) where $|\psi\rangle$ are vectors in a Hilbert space and $c_{mn}$ are complex numbers.

In the last decades entanglement has been the subject of intense study, and a resource for many applications, specially in the domain of quantum information. In this case the relevant entanglement usually involves spin or polarization. Entanglement is quite common in nonrelativistic quantum mechanics of many-particle systems, e.g. for electrons in atoms or molecules. However it is most relevant when the state-vectors $|\psi_m(1)\rangle$ and $|\psi_n(2)\rangle$ of eq. (52) belong to different systems placed far from each other. A study of entanglement and its relation with “local realism” will be made in Chapter 4 and examples of photon entanglement will be provided in Chapter 6. Here I will illustrate, with a simple example, that entanglement might be understood as a correlation induced by quantum vacuum fluctuations acting in two different places.

4.2 London-van der Waals forces

I shall study the London theory of the van der Waals forces in a simple model of two one-dimensional oscillating electric dipoles. Each dipole consists of a particle at rest and another particle (which we will name electron) with mass $m$ and charge $e$. In the model it is assumed that every electron moves in a harmonic oscillator potential and there is an additional interaction between the electrons. Thus the Hamiltonian is

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_0^2x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_0^2x_2^2 - Kx_1x_2,$$

(54)

where $x_1(x_2)$ is the position of the electron of the first (second) dipole with respect to the equilibrium position. The positive parameter $K < m\omega_0^2$ depends on the distance between the dipoles, but the dependence is irrelevant for our purposes. (For a more complete study of this problem within SED see Refs. [1], [2]). We shall work both the QM and the SED calculations.

4.3 Quantum theory of the model

An exact quantum calculation is not difficult. We take $x_j, p_j$ and $H$ as operators in the Hilbert space of the full system, fulfilling the standard com-
mutation relations

\[ [\hat{x}_j, \hat{x}_l] = [\hat{p}_j, \hat{p}_l] = 0, [\hat{x}_j, \hat{p}_l] = i\hbar \delta_{jl}. \]  

(55)

Now we introduce the new operators

\[ \hat{x}_+ (t) = \frac{1}{\sqrt{2}} [\hat{x}_1 (t) + \hat{x}_2 (t)], \hat{x}_- (t) = \frac{1}{\sqrt{2}} [\hat{x}_1 (t) - \hat{x}_2 (t)], \]

\[ \hat{p}_+ (t) = \frac{1}{\sqrt{2}} [\hat{p}_1 (t) + \hat{p}_2 (t)], \hat{p}_- (t) = \frac{1}{\sqrt{2}} [\hat{p}_1 (t) - \hat{p}_2 (t)]. \]  

(56)

It is easy to derive the commutation relations of the new operators, that are similar to eqs.(55) with the subindices +,− substituted for 1,2. The Hamiltonian eq.(54) in terms of the new operators is

\[ \hat{H} = \frac{\hat{p}_+^2}{2m} + \frac{1}{2} (m\omega_0^2 + K) \hat{x}_+^2 + \frac{\hat{p}_-^2}{2m} + \frac{1}{2} (m\omega_0^2 - K) \hat{x}_-^2. \]

This is equivalent to two uncoupled harmonic oscillators with the same mass, \( m \), and frequencies

\[ \omega_+ = \sqrt{\omega_0^2 + K/m}, \omega_- = \sqrt{\omega_0^2 - K/m}, (K < m\omega_0^2) \]

respectively. Thus the wavefunction of the two-electron system is

\[ \psi = \psi (x_+) \psi (x_-) = \sqrt{\frac{m}{\pi \hbar \omega_+ \omega_-}} \exp \left[ -\frac{m}{2\hbar} (\omega_+ x_+^2 + \omega_- x_-^2) \right] \]

\[ = \sqrt{\frac{m}{\pi \hbar \omega_+ \omega_-}} \exp \left\{ -\frac{m}{4\hbar} \left[ (\omega_+ + \omega_-) (x_1^2 + x_2^2) + 2(\omega_+ - \omega_-) (x_1 x_2) \right] \right\}, \]

and the interaction energy of the system is

\[ \Delta E = \frac{\hbar}{2} \left( \sqrt{\omega_0^2 - K/m} + \sqrt{\omega_0^2 - K/m - 2\omega_0} \right) \]

\[ = -\frac{\hbar K^2}{4m^2 \omega_0^3} + O(K^4), \]

that to lowest nontrivial order in the coupling constant \( K \) gives

\[ \psi = \sqrt{\frac{m\omega_0}{\pi \hbar}} \left( 1 + \frac{2K x_1 x_2}{m\omega_0} \right) \exp \left[ -\frac{m\omega_0}{2\hbar} (x_1^2 + x_2^2) \right], \]  

(57)
which may be written in terms of the wavefunctions of the ground state, \( \psi_0(x) \), and the first excited state, \( \psi_1(x) \), of the simple oscillator as follows

\[
\psi = \psi_0(x_1) \psi_0(x_2) + \frac{K}{m\omega_0^2} \psi_1(x_1) \psi_1(x_2).
\]

(The function is not normalized because the normalization was lost when we truncated at first order the expansion in powers of \( K \)). In quantum language this wavefunction \( \psi \) may be interpreted saying that the two-system state is a sum of two amplitudes, one of them corresponds to both oscillators being in the ground state and the other one to both being in the first excited state. It is true that eq.(57) is not an irreducible sum of products like eq.(52). However, it cannot be factorized in terms of wavefunctions of individual electrons and therefore it is not a classical correlation that might be represented as eq.(53). Therefore it may be considered an entangled state involving two distant systems.

Although quantum mechanics usually does not offer intuitive pictures of the phenomena, in this case it is difficult to refrain from interpreting the entanglement in this example as a correlation of the (random) motions of the electrons. Indeed the modulus squared of the wavefunction eq.(57) gives the probability density for the positions of the electrons, which is larger when the electrons are far from each other so that their mutual repulsion energy is smaller. Then the correlation (entanglement) lowers the energy giving rise to an attractive force between the oscillators. Of course this explanation departs from the Copenhagen interpretation (see Chapter 3), that should not speak about the probability that one electron is in the region \( x_1 > 0 \) and the other one is in the region \( x_2 > 0 \). Instead it compels us to say something like “if we perform a measurement of the simultaneous positions of the electrons the probability that we get one of them in the region \( x_1 > 0 \) and the other one is in the region \( x_2 > 0 \) is given by the modulus squared of eq.(57).” (Simultaneous measurements are possible because the observables commute.) In any case the origin of the correlation is not clear in quantum mechanics.

4.4 The model in stochastic electrodynamics

In sharp contrast with QM the interpretation offered by SED is transparent: the random motion of the electrons is induced by the ZPF, and the correlation is produced by the interaction. The SED calculation is as follows. The
differential equations of motion may be obtained from eq. (54). I shall write them including the forces due to the random ZPF and the radiation reaction, see eq. (2), that is

\[ m \ddot{x}_1 = -m \omega_0^2 x_1 - K x_2 + \frac{2e^2}{3c^3} \ddot{x}_1 + eE_1 (t), \]
\[ m \ddot{x}_2 = -m \omega_0^2 x_2 - K x_1 + \frac{2e^2}{3c^3} \ddot{x}_2 + eE_2 (t). \]  

(59)

The approximation of neglecting the \( x \) dependence of the field, \( E(x,t) \), is not good if the dipoles are at a long distance (on the other hand the Hamiltonian eq. (54) is not valid for short distances). However we may neglect the \( x \) dependence within each dipole, that is we will approximate \( E(x_1, t) \approx E(a, t) \), \( E(x_2, t) \approx E(b, t) \), where \( a \) and \( b \) are the positions of the first and second dipole, respectively. Also we will simplify the notation writing \( E_1 (t) \) for \( E(a, t) \) and \( E_2 (t) \) for \( E(b, t) \). Furthermore, as we assume that the distance between dipoles is large, we shall take the stochastic processes \( E_1 (t) \) and \( E_2 (t) \) as uncorrelated.

The coupled eqs. (59) may be decoupled via writing new equations which are the sum and the difference of the former, and introducing the new position variables

\[ x_+ (t) = \frac{1}{\sqrt{2}} [x_1 (t) + x_2 (t)], x_- (t) = \frac{1}{\sqrt{2}} [x_1 (t) - x_2 (t)], \]  

(60)

and similarly definitions for \( E_+ (t) \) and \( E_- (t) \). We get

\[ m \ddot{x}_+ = -(m \omega_0^2 - K)x_+ + \frac{2e^2}{3c^3} \ddot{x}_+ + eE_+ (t), \]
\[ m \ddot{x}_- = -(m \omega_0^2 + K)x_- + \frac{2e^2}{3c^3} \ddot{x}_- + eE_- (t), \]  

(61)

where the stochastic processes \( E_+ (t) \) and \( E_- (t) \) are statistically independent as a consequence of \( E_1 (t) \) and \( E_2 (t) \) being uncorrelated. With the method used to solve eqs. (10) and (12) we get

\[ \langle x^2_\pm \rangle = \frac{\hbar}{2m \sqrt{\omega_0^2 + K/m}}, \langle v^2_\pm \rangle = \frac{\hbar \sqrt{\omega_0^2 + K/m}}{2m}. \]  

(62)

The Hamiltonian eq. (54) may be written in terms of \( x_+ (t), x_- (t) \) leading to

\[ H = \frac{p_+^2}{2m} + \frac{1}{2} m \omega_0^2 x_+^2 + \frac{p_-^2}{2m} + \frac{1}{2} m \omega_0^2 x_-^2 - \frac{1}{2} K \left( x_+^2 - x_-^2 \right). \]
Hence, defining $p_\pm = mv_\pm$, it is easy to get the total energy, $\langle H \rangle$, taking eqs.\([62]\) into account. The result is in agreement with the quantum eq.\([??]\). The joint probability distribution of positions is Gaussian and factorizes because eqs.\([61]\) are decoupled. That is

$$\rho (x_+, x_-) \, dx_+ \, dx_- = \rho_+ (x_+) \, \rho_- (x_-) \, dx_+ \, dx_-.$$ 

The densities $\rho_\pm$ should be normalized whence we get

$$\rho_\pm (x) = \sqrt{\frac{2m}{\pi \hbar}} (\omega_0^2 \mp \frac{K}{m})^{-1/4} \exp \left[ -\frac{m}{2\hbar} \sqrt{\omega_0^2 \mp \frac{K}{m} x^2} \right].$$

Hence it is easy to get the joint probability in terms of the variables $x_1$ and $x_2$ taking eqs.\([60]\) into account. The result is in agreement with the quantum prediction, eq.\([57]\).

In the equation of motion\([59]\) I have assumed that the ZPF components, $E_1 (t)$ and $E_2 (t)$, acting upon the two particles are uncorrelated. This is a good approximation if the particles are at a distance which is large in comparison with wave length, $\lambda \approx \frac{c}{\omega_0}$, corresponding to the typical frequencies involved. However if the distance is of that order or smaller, the ZPF components will be correlated, which would cause a much stronger correlation between the particle’s motions. We might speculate that correlations induced by the ZPF are related to quantum statistics, that is behaviour of particles as either bosons or fermions. But this possibility will not be further discussed in this book.

The conclusion of our study of the two coupled oscillators in SED is the suggestion that quantum entanglement is a correlation between the quantum fluctuations of different systems, mediated by the vacuum fields, these fields not being apparent in the quantum formalism.

### 4.5 Specific heats of solids

An application of SED at a finite temperature is the calculation of the specific heat of solids, which we summarize in the following\([16]\). We shall consider a solid as a set of positive ions immersed in an electron gas. As is well known the electrons contribute but slightly to the specific heat at not too high temperatures. In SED we shall study the motion of the ions under the action of three forces. The first one derives from the interaction with the neighbour ions and the electron gas, that may be modelled by an oscillator.
potential which increases when the distance between neighbour ions departs from the equilibrium configuration. The second is the random background radiation with Planck spectrum (including the ZPF) and the third one is the radiation reaction. This gives rise to a discrete set of coupled third order differential equations that may be decoupled by the introduction of normal mode coordinates. After that, every equation is similar to eq.(2) and may be solved in analogous form. The net result is that the mean (potential plus kinetic) energy in equilibrium becomes

$$E(\omega) = \frac{1}{2} \hbar \omega \coth \left( \frac{\hbar \omega}{2kT} \right),$$

(63)

where $\omega$ is the frequency of the mode. With an appropriate distribution, $\rho(\omega)$, of modes this leads to the quantum result derived by Debye[7], the specific heat being the derivative of the total energy with respect to the temperature.

There are other interesting results of SED at a finite temperature, in particular about magnetic properties. They may be seen in the books of de la Peña et al.[1], [2] and references therein.

The SED calculation of the specific heat of solids provides another argument for the continuity (as opposed to discreteness) of the energies of quantum oscillators. If it is hard to accept that electromagnetic radiation consists of particles (photons) in a realistic interpretation of quantum physics, it is still harder to assume that quantized oscillations of the ions in a solid (phonons) are particles. It is more plausible to assume that the energies of the normal modes of the set of ions have a continuous, although random, distribution of energies such that the average for a mode is given by eq.(63). Also it is plausible that the mean energy of a vibration mode of every ion is the same as the mean energy of the radiation mode having the same frequency, that is the result here obtained.

5 The particle in a homogeneous magnetic field

Another linear problem that has been extensively studied within SED is the motion of a charged particle in a homogeneous magnetic field[1]. The most relevant result is the prediction of diamagnetic properties of a free charge
(without magnetic moment), which departs from classical physics and agrees with QM. Here I shall revisit the SED calculation of the free charged particle in a homogeneous field of magnitude $B$.

### 5.1 Classical theory

The classical motion may be got from Newton's law with the Lorentz force, that is

$$m \ddot{r} = -(e/c) \hat{r} \times \mathbf{B}. \quad (64)$$

If we choose the Z axis in the direction of the $\mathbf{B}$ the motion in that direction is uniform and in the perpendicular plane it is given by

$$x = R \cos [2\omega_0 (t - t_0)] + x_0,$$
$$y = R \sin [2\omega_0 (t - t_0)] + y_0, \omega_0 \equiv \frac{eB}{2mc}, \quad (65)$$

with four integration constants, namely $\{R, t_0, x_0, y_0\}$. The motion is circular with radius $R$ and constant (Larmor) angular frequency $\omega_0$. The total energy $E$ may be identified with the Hamiltonian, that is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \omega_0 (xp_y - yp_x) + \frac{1}{2} m\omega_0^2 (x^2 + y^2). \quad (66)$$

Taking Hamilton equations into account we get

$$E = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) = 2mR^2\omega_0^2. \quad (67)$$

Actually in a classical electrodynamical calculation we should include the radiation reaction (similar to the second term of the right side in the oscillator eq.(2)). This term would give rise to a loss of energy by radiation whence the system will eventually arrive at the state of minimal energy, that is zero (when $R = 0$). This shows that no diamagnetic effects can be expected to occur in classical physics.

### 5.2 Quantum theory

The QM treatment starts from a quantum Hamiltonian operator which may be got from eq.(66) by promoting the classical coordinates and momenta to
operators in a Hilbert space (for a detailed study see[17]). The Z component
of the angular momentum operator and the Hamiltonian commute and we
may search for simultaneous eigenvectors having eigenvalues
\[ L_z \equiv xp_y - yp_x \rightarrow m_l \hbar, m_l = 0, \pm 1, \pm 2, \ldots, \]
\[ H \rightarrow E_r = (2r + 1) \hbar |\omega_0|, r = 0, 1, 2, \ldots \]
(68)

We see that the quantum ground state, given by \( r = 0 \) and \( E_r = \hbar |\omega_0| \),
has an infinite degeneracy because this energy is shared by states with all
possible values of \( m_l \). For this reason it is common to add to the Hamiltonian
a two-dimensional oscillator potential with characteristic frequency \( \omega_1 (> 0) \).
Then the energy eigenvalues have an additional term \((2n + 1) \hbar \omega_1\)
with \( n = 2r - m_l \geq 0\), which breaks the degeneracy, the ground state now corresponding
to \( r = n = m_l = 0 \). From eq.(68) we may get the most relevant parameter,
that is the magnetic moment. In the ground state it is
\[ M = -\nabla_B E = -\hbar \nabla_B |\omega_0| = -M_B \frac{B}{B}, M_B = \frac{\hbar |e|}{2mc} \]
(69)
where \( M_B \) is the Bohr magneton. This (or the ground state energy, second
eq(68)) is the result that we may expect to reproduce in SED.

5.3 SED treatment

In SED we should add the action of the ZPF (plus the radiation reaction)
to the force derived from the homogeneous magnetic field, see eq.(64). We
will study only the motion in the XY plane. If \( u \equiv \dot{x} \) and \( v \equiv \dot{y} \) are
the components of the velocity vector the equations of motion are
\[ m \ddot{u} = \frac{e}{c} vB + m\tau \ddot{u} + eE_u, \]
\[ m \ddot{v} = -\frac{e}{c} uB + m\tau \ddot{v} + eE_v, \]
(70)
where the first term is the component of the Lorentz force, the second is
the radiation reaction and the third one the action of the ZPF (in the long
wavelength approximation, see eq.(2)). The components of the electric ZPF,
\( E_u(t) \) and \( E_v(t) \), are assumed statistically independent stochastic processes.

The small value of \( \tau \ll 1/\omega_0 \) allows an approximation similar to the one
made in the free particle case, Section 3.4. We may substitute \( evB/(cm\tau) \) for
\( \ddot{u} \) and similar for \( \ddot{v} \), thus obtaining two first order equations from eqs.(70).
Then the solution is straightforward and we get, with steps similar to those involved in the solution of eq. (2) lead to
\[ \langle u^2 \rangle = \int_0^\infty \frac{\hbar \tau \omega^3 (4\omega_0^2 + \omega^2)}{\pi m (4\omega_0^2 - \omega^2)^2 + 4\tau^2 \omega^6} d\omega \approx \frac{\hbar |\omega_0|}{m}, \] (71)
and the same result for \( \langle v^2 \rangle \). Actually the integral in eq. (71) is ultraviolet divergent so that a high frequency cutoff, \( \omega_c \), should be included. It may be seen that for small \( \tau \), i.e. \( \tau \omega_0 \ll 1 \), the main contributions to the integral eq. (71) come either from frequencies \( \omega \) close to \( 2 \omega_0 \) or for high frequencies \( \omega \gg 2 \omega_0 \) (a similar case happens in the oscillator, see eqs. (10) and (12)). The former contribution, given by eq. (71), is independent of both the cut-off frequency and the precise value of \( \tau \). The latter, high frequencies, contribution may be obtained neglecting \( \omega_0 \) in comparison with \( \omega \), and putting \( 4\omega_0 \) as lower limit of the integral in order to exclude the frequency region around \( 2\omega_0 \) calculated in eq. (71). Thus we get
\[ \langle u^2 \rangle_{hf} \approx \int_{4\omega_0}^{\infty} \frac{\hbar \tau \omega^5 d\omega}{\pi m (\omega^4 + 4\tau^2 \omega^6)} = \frac{\hbar}{8\pi m \tau} \log (1 + 4\tau^2 \omega_c^2) \approx \frac{\hbar \tau}{2\pi m} \omega_c^2, \]
where we have assumed \( 2\tau \omega_c \ll 1 \). A comparison with eq. (45) shows that this contribution is the same for a free particle. Indeed it is independent of the magnetic field (which does appear in eq. (71)).

The mean energy in SED is obtained inserting eq. (71) in the expression of the energy (see eq. (67)) giving
\[ \langle E \rangle = \frac{1}{2} m \langle u^2 + v^2 \rangle = \hbar |\omega_0|, \] (72)
in agreement with the quantum result. Hence there is also agreement for the magnetic moment, eq. (69).

Another interesting result from SED is the mean value of the angular momentum that is
\[ \langle L_z \rangle = \langle xp_y - yp_x \rangle = m \langle xv - yu \rangle = -\frac{e}{|e|} \hbar, \] (73)
independently of the magnitude of the magnetic field and the mass of the particle. I omit the proof that is straightforward. Thus the angular momentum
is parallel to the magnetic field if the charge is negative and antiparallel if it is positive. We saw that the magnetic moment is always antiparallel to the magnetic field. The results eqs. (72) and (73) correspond to the limit $\tau \to 0$. In both cases there are corrections for finite $\tau$ which would require a relativistic treatment. If an appropriate cutoff is introduced, say $\omega_c = mc^2/\hbar$, the high frequencies contribution is small.

There is however a disagreement between QM and SED for the angular momentum in the stationary state. SED predicts a finite value given by eq. (73) but in QM there are many possible angular momenta in the ground state as shown in eq. (68). On the other hand if we include an additional oscillator potential with characteristic frequency $\omega_1$, then the quantum prediction for the ground state angular momentum is zero that also disagrees with the SED result. Thus in QM there are two features whose realistic interpretation is difficult. Firstly eq. (69) that strongly suggests that the angular momentum in the ground state is the same of the SED prediction, eq. (73), rather than the degeneracy eq. (68). Secondly that an additional oscillator potential no matter how small breaks the degeneracy, but leading to zero angular momentum, rather than the most intuitive value eq. (73). These facts show that a realistic interpretation of the angular momentum in quantum mechanics is difficult. A possible solution is proposed in Section 6.4.

In summary the SED treatment of the particle in a homogeneous magnetic field reproduces the most relevant results of QM and provides a realistic interpretation for the QM prediction of a diamagnetic behaviour of the charged particle in the presence of an homogeneous magnetic field. However there is disagreement for the angular momentum.

6 SED application to nonlinear systems

Several nonlinear systems have been studied in stochastic electrodynamics that provide some results in semiquantitative agreement with quantum mechanics, but badly fail in other cases. Actually SED reproduces quantum results, and agrees with experiments, in a limited domain, namely for systems of charged particles that may be treated linearly and within a nonrelativistic approximation. In sharp contrast the treatment of nonlinear systems gives results that frequently disagree with quantum predictions. The explanation of this fact is that SED, as defined in the introduction section, is an approximation to QED to lowest order in Planck constant $\hbar$, but quantum mechanics
gives predictions for nonlinear systems that involve $\hbar$ to higher order. Therefore to be valid for all physical systems, SED should be generalized, likely including all vacuum fields and taking into account the back action of the particles on the fields. I propose that this would lead to quantum theory, but the steps needed are not known. There is here a paradox, namely we might foresee that the final theory should be rather cumbersome due to the large number of fields involved and the nonlinearity of equations, but it is the case that quantum theory has a relatively simple formalism. This is the magic of quantum theory and one of the reasons for the difficulty of getting a realistic interpretation of it.

In the following I comment on some calculations for nonlinear systems. The best method for the SED study in these cases is to get the evolution of the classical mechanical ‘constants of the motion’, one of them being the total energy. These parameters are no longer constant due to the interaction with the ZPF and the radiation reaction, but may be slowly varying. The method was used in Section 3 for the oscillator and it will be illustrated in the following for a nonlinear system, the rigid planar rotor. After that I will comment on the hydrogen atom in SED and the problem of equilibrium between radiation and matter.

### 6.1 The planar rigid rotator

The planar rigid rotor is the most simple nonlinear system studied in SED [18]. A model of the rotor is a particle of mass $m$ and charge $e$ constrained to move in the $XY$ plane always at a distance $R$ of a fixed point. Thus the problem has a single degree of freedom and the SED equation of motion may be written in terms of the polar angle $\phi$ as follows

$$mR \ddot{\phi} = -m\tau R \dot{\phi} + m\tau \ddot{\phi} + eE,$$

$$E = -\cos \phi E_x(t) + \sin \phi E_y(t).$$

The former two terms of the right side give the tangential component of the radiation reaction force, $m\tau R$, and the tangential component of the force due to the ZPF, respectively. In terms of the angular velocity, $\omega = \dot{\phi}$, the equation becomes

$$\dot{\omega} = -\tau \omega^3 + \tau \dot{\omega} + \frac{e}{mR} E. \quad (74a)$$
Eq. (74a) may be solved perturbatively in two steps. In the first step we solve the classical equation of motion \( \dot{\omega} = 0 \), which trivially gives \( \omega = \omega_0 = \text{constant} \). That constant becomes slowly varying when we take into account the radiation reaction and the action of the ZPF. In order to get that variation eq. (74a) may be solved substituting \( \omega_0 \) for \( \omega \) in the perturbation, that is in all terms of the right side. The solution with initial condition \( \omega (0) = \omega_0 \) becomes

\[
\omega(t) = \omega_0 + \int_0^t dt' \left[ -\tau \omega_0^3 + \frac{e}{mR} (-\cos (\omega_0 t') E_x (t) + \sin (\omega_0 t') E_y (t')) \right]
\]

The former term within the integral sign represents drift and the latter term diffusion. The diffusion constant may be calculated via the limit

\[
D = \lim_{t \to \infty} \frac{1}{t} \left\langle \left[ \int_0^t dt' \frac{e}{mR} (-\cos (\omega_0 t') E_x (t') + \sin (\omega_0 t') E_y (t')) \right]^2 \right\rangle
= \left( \frac{e}{mR} \right)^2 \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \int_0^t dt'' \cos [\omega_0 (t' - t'')] \left\langle E_x (t') E_x (t'') \right\rangle,
\]

where I have taken into account that \( \left\langle E_x (t') E_x (t'') \right\rangle = \left\langle E_y (t') E_y (t'') \right\rangle \) and that \( \left\langle E_x (t') E_y (t'') \right\rangle = 0 \). The field correlation may be easily got from the spectrum, eq. (5) as follows

\[
\left\langle E_x (t') E_x (t'') \right\rangle = \frac{2\hbar}{3\pi c^3} \int_0^\infty u^3 \cos [u (t' - t'')] du.
\]

When this is inserted in eq. (75) the variables \( t' \) and \( t'' \) may be changed to \( w \equiv (t' + t'') / 2 \) and \( t' - t'' \equiv s \). With good approximation the integration may be performed from 0 to \( t \) for the \( w \) integral and for the whole real line for the variable \( s \). Then the limit \( t \to \infty \) in eq. (75) is trivial and we get, taking the definition of \( \tau \), eq. (3), into account,

\[
D = \frac{\hbar \tau}{\pi mR^2} \int_0^\infty u^3 du \int_{-\infty}^\infty \cos (us) \cos (\omega_0 s) ds
= \frac{\hbar \tau}{\pi mR^2} \int_0^\infty u^3 du \pi [\delta (\omega_0 + u) + \delta (\omega_0 - u)] = \frac{\hbar \tau}{mR^2} \omega_0^3.
\]

From the diffusion constant and the damping it is possible to obtain the following Fokker-Planck equation for the probability density, \( \rho (\omega_0) \), of frequencies of the rotor

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \omega_0} \left( \tau \omega_0^3 \rho \right) + \frac{1}{2} \frac{\partial^2}{\partial \omega_0^2} \left( \frac{\hbar \tau}{mR^2} \omega_0^3 \rho \right),
\]

43
whose (regular) stationary solution is
\[ \rho = \frac{2mR^2}{\hbar} \exp\left(-\frac{2mR^2\omega_0}{\hbar}\right). \]

A similar method may be used for the three-dimensional rotor[18] and the result is
\[ \rho = \left(\frac{2mR^2}{\hbar}\right)^2 \omega_0 \exp\left(-\frac{2mR^2\omega_0}{\hbar}\right). \]

(76)

6.2 Comparison between SED and QM

The predictions of SED for the rigid rotor disagree with those of QM at least in four respects, that will be illustrated in the following for the particular case of the three-dimensional rigid rotor:

1. The distribution of positions or momenta in the minimal energy state.
   In quantum mechanics the eigenstates of the angular momentum squared and the Hamiltonian of the rotor are, respectively,
   \[ L^2 = \hbar^2 l(l + 1), \quad E_l = \frac{\hbar^2}{2l} l(l + 1), \quad l = 0, 1, 2... \]
   (77)
   so that the ground state corresponds to \( L^2 = E = 0 \). In contrast the stationary solution in SED is given by eq.(76) where there is a spherical distribution of angular momenta given by
   \[ W(L) L dL = \frac{4}{\hbar^2} \exp\left(-\frac{2L}{\hbar}\right) L dL. \]
   (78)

2. The set of states. As in the oscillator studied in section 3.3 the set of possible states is quite different in QM and SED.

3. The spectrum. In QM the spectrum consists of the set of frequencies
   \[ \omega_{ij} = (\hbar/2I) [j(j + 1) - l(l + 1)] \rightarrow (\hbar/I) (l + 1), \]
   (79)
   the latter frequencies corresponding to the transitions allowed in the atomic dipole approximation. In sharp contrast SED predicts a continuous spectrum, although the most intense absorption from the stationary state eq.(77) corresponds to the maximum absorption, that may be shown to be \( \omega = \hbar/I \)
[11], in agreement with the QM result for the transition from the ground to
the first excited state, see eq. (79). However QM predicts a sharp frequency whilst the SED prediction corresponds to a wide band. In experiments the frequency is not sharp, but it is less wide than the SED prediction. The disagreement between the QM prediction and experiments is usually explained because the rigid rotator is not a realistic model of a molecule. For instance molecules are not completely rigid. The disagreement with experiments is greater in SED and it cannot be explained as easily as in QM.

4. The specific heat. There is also a discrepancy as shown in the comparison between quantum and SED treatments [18]. This point will not be discussed here.

6.3 A difficulty with the angular momentum

The disagreement between the quantum prediction, eq. (77), and the SED prediction, eq. (78), for the rigid rotor is actually general and it puts a problem for any realistic model of the rotation in quantum physics. For instance if we want to get a picture of a rotating molecule. The quantum ground state of the rigid rotor possesses zero angular momentum and spherical symmetry, but these two properties are contradictory for any realistic interpretation. For the sake of clarity let us consider a more realistic example, for instance the molecule of carbon oxide, CO, which may be modelled by a three-dimensional rigid rotor. It consists of an oxygen atom and a carbon atom at a distance which is very well known empirically. The ground state of this molecule possesses zero angular momentum and therefore (according to the quantum formalism) spherical symmetry. Discarding explanations which are bizarre for any realistic interpretation, like saying that “the form of the molecule emerges during the act of measurement”, the meaning of spherical symmetry is unclear. The unique meaning compatible with a physical picture is that the molecule is rotating randomly in such a way that the probability distribution of the orientations of the axis in space possesses spherical symmetry. However this is in conflict with the quantum prediction that the total angular momentum is zero, dispersion-free. Therefore the mean square angular momentum is also zero. The situation is quite common, it involves many molecules, atoms or nuclei. It seems that either the standard quantum prediction is wrong (e. g. the ground state is not physically achievable) or a realistic physical model is not possible.

A possible solution to the dilemma is that the quantum formalism actually provides the total angular momentum of the molecule plus the vacuum fields
that interact with it. If the ground state correspond to an equilibrium of the system (e. g. the molecule) with the vacuum fields it is plausible to assume that there is a continuous exchange of angular momentum between the system and the vacuum fields so that the total angular momentum (a conserved quantity) remains always zero. This is the case in the SED treatment of the planar rigid rotor of the previous section. In fact, eq. (74a) may be interpreted as the equation for the balance of angular momentum. In fact the equation may be rewritten as the Z component of the angular momentum vector, that is

\[ \frac{d}{dt}(I\omega) = -\tau I\omega^3 + \tau \dot{I}\omega + e (\mathbf{R} \times \mathbf{E})_z, \]

where the change of the rotor angular momentum equals the radiated momentum (the first two terms) minus the momentum absorbed from the ZPF.

In summary there is no real contradiction between the fact that the SED predicts a distribution of angular momenta of the rotor alone and our interpretation of the QM prediction that the angular momentum of rotor plus field is strictly zero. I do not believe that SED is the correct reinterpretation of QM, but I think that it illustrates adequately the possible solution to the problem.

A similar solution may be given to the strange, if not paradoxical, quantum prediction that a charged particle in a homogeneous magnetic field has zero component of the angular momentum in the direction of the field, but the energy is precisely the product of the field times the Bohr magneton. The SED results are more intuitive, namely the energy of the equilibrium state, eq. (72), agrees with the quantum ground energy, but there is a component of the angular momentum in the direction of the field, see eq. (73).

### 6.4 The hydrogen atom

The hydrogen atom is the most relevant nonlinear system within elementary quantum mechanics, therefore a crucial test for the validity of SED. Once the stationary state of the harmonic oscillator had been solved with success, several authors devoted a big effort during the 1960’s to study the hydrogen atom in SED. Several approximation methods were proposed for calculating the stationary state of the atom (modelled as two particles with opposite charge, one of them at rest). The most successful method devised for the study of a charged particle in a potential well rests upon the assumption that the classical constants of the motion change slowly. That is the motion is close
to the classical one, the action of the ZPF and the radiation reaction giving rise to a slow diffusion in the space of classical orbits. As every classical orbit is determined by the initial position and velocity, \( \{ r_0, v_0 \} \), the final result of the calculation is a probability distribution in the phase space of positions and velocities, \( \{ r, v \} \), that is the same as the distribution of initial positions and velocities, \( \{ r_0, v_0 \} \), if the state is stationary. This is similar to what happens in the planar rigid rotator.

In the case of the hydrogen atom the result of the calculation did not provide a stationary solution. In fact the prediction was that the atom is not stable but ionizes spontaneously due to the orbits passing close to the nucleus\[19\].

That work has been criticized because such orbits cannot be treated with a non-relativistic approximation, and a relativistic treatment could produce an important change in the results. Actually the prediction of spontaneous ionization made by SED analytical calculation is not a too strong argument against the SED prediction. In fact the result depends crucially on the electron orbits passing close to the nucleus, that would require a relativistic treatment. Also quantum theory predicts that the free atom is unstable against ionization at any finite temperature, no matter how small. This trivially follows from the fact that the quantum partition function is divergent, that is
\[
Z = \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} (2l + 1) \exp \left( -\frac{E_0}{n^2} \right) \to \infty.
\]

Therefore it is not too relevant if an approximation method used in SED has an effect (spontaneous ionization) similar to the effect of a thermal radiation in QM.

Furthermore numerical solutions of the hydrogen atom in SED have been made\[20\] since 2003 that explain the stability of the atom. They led to stationary distribution fairly close to the quantum prediction for the position distribution in the ground state. However more powerful calculations made in 2015\[21\] predict a ionization of the atom. Numerical calculations have the advantage that do not require approximations in the differential equations, like the neglect of the dependence on position of the ZPF (the electric dipole approximation). However the numerical methods have uncertainties that may explain the discrepancy as commented above for the early analytical treatment. See also \[22\].
6.5 Thermal equilibrium between radiation and matter. SED derivation of Planck Law.

Several authors have claimed that Planck’s law may be derived from classical postulates, usually within the framework of SED [2], [23]. A derivation of the thermal radiation should follow from the study of the thermal equilibrium between radiation and matter. In the framework of standard quantum theory it leads to Planck’s law, but here we are considering the question whether it may be obtained from classical electrodynamics. The difficulty is related to the fact that the equilibrium radiation-matter should involve nonlinear systems. In particular the study of equilibrium requires a balance between absorption of energy from the radiation at a frequency and emission at a different frequency. Only in these conditions it is possible to study the distribution of energy amongst the different frequencies that is the essential purpose of a radiation law. If we deal only with linear (harmonic) oscillators both the absorption and emission of radiation take place at the same frequency.

The problem of thermal equilibrium was extensively studied in the first decades of the 20th century and the conclusion was uncontroversial in my opinion: If one assumes classical dynamics then thermal equilibrium is achieved when the particles have the Maxwell-Boltzmann distribution and the radiation the Rayleigh-Jeans spectrum [24]. Thus there is a contradiction between the derivation reported by van Vleck and the derivations claiming that the classical equilibrium spectrum is given by Planck’s law. It was suggested that early derivations [24] involved Newtonian dynamics and that a study with relativistic dynamics might led to Planck’s law. However it has been shown that thermal equilibrium of relativistic particles also leads to the Rayleigh-Jeans law [25]. A different question is whether Planck law may be derived for systems of charged particles immersed in the ZPF field plus additional radiation and we assume thermal equilibrium of that radiation with the particles. In these conditions Planck spectrum is obtained [23], [26].

A related result is the classical derivation of the Davies-Unruh effect initially derived from quantum electrodynamics [27], [28]. It is interpreted in quantum theory as the production of photons with Planck distribution of frequencies when a detector moves in the vacuum with accelerated motion. The result may be got in SED with the interpretation that the spectrum of the ZPF appears as thermal when seen from an accelerated reference frame [29], [4].
7 SED as a clue for a realistic interpretation of quantum mechanics

We have seen that calculations of several linear systems within SED provide a remarkable agreement with the predictions of QM. On the other hand the realistic interpretation of SED is rather obvious. Thus the question arises, offers SED the realistic interpretation of QM which we are searching for?. Unfortunately the answer is in the negative, the difficulties of SED for the interpretation of phenomena associated to nonlinear systems seem unsourmountable.

I propose that geting a realistic interpretation of QM would be possible accepting the general ideas of SED but rejecting many of the particular assumptions. The general ideas to be retained are the following: 1) Nuclei, atoms or molecules (but maybe not elementary particles like electrons) are bodies with well defined size and form following definite, but highly irregular, trajectories. (If the bodies are composite, like atoms, they may suffer deformations). 2) The motion is strongly influenced by the fluctuations of the vacuum fields.

A summary of the clues provided by SED for a realistic interpretation of (non-relativistic) quantum mechanics follows.

The attempt to interpret the quantum mechanics of particles alone is misleading if quantum fields, in particular vacuum fields, are not included.

The quantum ground state of a particle in a potential well corresponds to a stationary state of the particle performing a highly irregular (stochastic) motion driven by vacuum fields. There is a dynamical equilibrium between absorption from and emission of radiation to the vacuum fields. The interaction gives rise to probability distributions of coordinates and momenta of the particle that agree with quantum predictions for linear systems, but for nonlinear ones there is disagreement. Radiative corrections (e. g. Lamb shift) have a transparent interpretation.

The study of coupled oscillators at zero Kelvin provides an intuitive picture of entanglement as a correlation between quantum fluctuations mediated by the vacuum fields. At a finite temperature it gives a simple realistic interpretation of the Debye theory of specific heats of solids.

The motion of particles is highly irregular due to the interaction with the vacuum fields. In particular the free particle possesses a conserved canonical momentum with an associated inertial motion but, superimposed to this, it
has a random motion with high velocity that cannot be studied adequately in
the nonrelativistic approximation. This derives from the spectrum, \( S_E(\omega) \propto \omega^3 \), of the vacuum radiation fields: At short times the motion is governed
by the high frequencies where \( S_E(\omega) \) is large thus inducing a rapid erratic
motion, at long time it is governed by the low frequencies where \( S_E(\omega) \) is
small, whence the memory of the initial velocity is lost slowly. This behaviour
is very different from Brownian motion.

The spectrum of the radiation absorbed or emitted by a particle in a potential
well badly fails to reproduce the quantum spectrum, except in the trivial
case of the harmonic oscillator whose spectrum consists of a single frequency.
This fact suggests that the back action of the particles on the vacuum radi-
ation field and/or the inclusion of many fields would be essential for the
prediction of spectra, e. g. of atoms.

The stochastic commutator provides a hint for a realistic interpretation
of the quantum commutation rules as a disguised form of stabilizing the
properties of some peculiar stochastic processes. The main peculiarity is the
fact that the spectra of the processes is usually odd with respect to a change
\( \omega \rightarrow -\omega \) of the frequency. Thus I propose that the reason for the success of
formulating QM with noncommuting mathematical objects is the fact that the
basic stochastic processes involved have spectra that are odd with respect to
the change \( \omega \rightarrow -\omega \).

The SED calculation of the specific heat of solids provides an argument
for the continuity of the energies of ions. Quantized oscillations of the ions
in a solid (phonons) are not particles, the energies of the normal modes of
the set of ions having a continuous distribution of energies. The mean energy
of a vibration mode of the ions is the same as the mean energy of a radiation
mode with the same frequency.

The prediction that the angular momentum is zero, dispersion-free, in
some cases is possibly the most paradoxical prediction of QM. In fact a
molecule, or the electron in the ground state of a hydrogen atom, are pre-
dicted to be in a state with spherical symetry. A realistic interpretation is
possible only if we assume that there is a random rotation having zero an-
gular momentum on the average, but the mean squared angular momentum
being different from zero. Our study suggests a realistic interpretation assum-
ing that the dispersion-free momentum refers to the addition of two highly
correlated random angular momenta, namely those of the material system
plus the vacuum fields, these modified by the presence of matter.
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