VECTOR INVARIANTS OF PERMUTATION GROUPS IN CHARACTERISTIC ZERO

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Abstract. We consider a finite permutation group acting naturally on a vector space $V$ over a field $k$. A well known theorem of Göbel asserts that the corresponding ring of invariants $k[V]^G$ is generated by invariants of degree at most $\binom{\dim V}{2}$. In this note we show that if the characteristic of $k$ is zero then the top degree of vector coinvariants $k[V^m]_G$ is also bounded above by $\binom{\dim V}{2}$, which implies the degree bound $\binom{\dim V}{2} + 1$ for the ring of vector invariants $k[V^m]^G$. So Göbel’s bound almost holds for vector invariants in characteristic zero as well.

1. Introduction

Let $G$ be a finite group, $k$ a field and $V$ a finite dimensional vector space over $k$ on which $G$ acts. The action of $G$ on $V$ induces an action on the symmetric algebra $k[V]$ on $V^*$ given by $gf(v) = f(g^{-1}v)$ for $g \in G$, $f \in k[V]$ and $v \in V$. Let $k[V]^G$ denote the ring of invariant polynomials in $k[V]$. This is a finitely generated graded subalgebra of $k[V]$ and a central goal in invariant theory is to determine $k[V]$ by computing generators and relations. We let $\beta(G, V)$ denote the maximal degree of a polynomial in a minimal homogeneous generating set for $k[V]^G$. It is well known by [5, 1, 2] that $\beta(G, V) \leq |G|$ if $|G| \in k^*$. If the characteristic of $k$ divides $|G|$, then the invariant ring is more complicated and there is no bound that applies to all $V$. But it is possible to bound $\beta(G, V)$ using both $|G|$ and dimension of $V$ (see [6]). The Hilbert ideal $I(G, V)$ is the ideal $k[V]^G \cap k[V]$ in $k[V]$ generated by all invariants of positive degree. The algebra of coinvariants $k[V]_G$ is the quotient ring $k[V]/I(G, V)$. Both Hilbert ideal and the algebra coinvariants are subjects of interest as it is possible to extract information about the invariant ring from them. Since $G$ is finite, $k[V]_G$ is finite dimensional as a vector space and the highest degree in which $k[V]_G$ is non-zero is called the top degree of coinvariants. This degree plays an important role in computing the invariant ring and is closely related to $\beta(G, V)$ when $|G| \in k^*$ (see [4]).

In this paper we study the case where $G$ is a permutation group acting naturally on $V$ by permuting a fixed basis of $V$. By a well known theorem of Göbel [3], $\beta(G, V) \leq \binom{n}{2}$, where $n$ is the dimension of $V$. This bound applies in all characteristics and it is known to be sharp as for the alternating group $A_n$, we have $\beta(A_n, V) = \binom{n}{2}$. Now we consider $m$ direct copies $V^m = V \oplus V \oplus \ldots \oplus V$ of $V$ with the action of $G$ extended diagonally. We show that, if $k$ has characteristic zero, the top degree of the coinvariant ring $k[V^m]^G$ is also bounded above by $\binom{n}{2}$. Our method relies on polarizing polynomials in the Hilbert ideal $I(G, V)$ and obtaining enough monomials in $I(G, V^m)$ to bound the top degree of $k[V^m]^G$. This implies that $\beta(G, V^m) \leq \binom{n}{2} + 1$. If polarization of a generating set for $k[V]^G$ gives a generating set for $k[V^m]^G$, then a generating set for $I(G, V^m)$ can be obtained by

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polarizing any generating set for \( I(G, V) \). But in general, one should not expect to get a Gröbner basis for \( I(G, V^m) \) from a Gröbner basis for \( I(G, V) \) by polarization.

2. Polarization and the Hilbert ideal

In this section we prove that to compute the leading monomial of a polarization of a polynomial it is sufficient to polarize the leading monomial of this polynomial. We identify \( \mathbb{k}[V] \) with \( \mathbb{k}[x_1, \ldots, x_n] \) and \( \mathbb{k}[V^m] \) with \( \mathbb{k}[x_i^{(j)} \mid i = 1, \ldots, n, j = 1, \ldots, m] \). We use lexicographic order on \( \mathbb{k}[V^m] \) with

\[
x_1^{(1)} > x_1^{(2)} > \cdots > x_1^{(m)} > x_2^{(1)} > x_2^{(2)} > \cdots > x_2^{(m)} > \cdots > x_n^{(m)}
\]

and the order on \( \mathbb{k}[V] \) is obtained by setting \( m = 1 \). For an ideal \( I \) we denote the lead term ideal of \( I \) with \( L(I) \) and the leading monomial of a polynomial \( f \) is denoted by \( \text{LM}(f) \). We introduce extra variables \( t_1, \ldots, t_m \) and define the algebra homomorphism

\[
\Phi : \mathbb{k}[V] \to \mathbb{k}[V^m][t_1, \ldots, t_m], \quad x_i \mapsto \sum_{j=1}^{m} x_i^{(j)} t_j.
\]

For any \( f \in \mathbb{k}[V] \), we write

\[
\Phi(f) = \sum_{(k_1, \ldots, k_m) \in \mathbb{N}^m} f_{k_1, \ldots, k_m} t_1^{k_1} \cdots t_m^{k_m},
\]

with polynomials \( f_{k_1, \ldots, k_m} \in \mathbb{k}[V^m] \). This process is known as polarization and for an \( m \)-tuple \( \underline{k} = (k_1, \ldots, k_m) \) let \( \text{Pol}_{\underline{k}}(f) \) denote the coefficient \( f_{k_1, \ldots, k_m} \). We set

\[
\text{Pol}(f) = \{ \text{Pol}_{\underline{k}}(f) \mid \underline{k} \in \mathbb{N}^m, f_{\underline{k}} \neq 0 \}.
\]

The importance of polarization for invariant theory comes from the fact that if \( f \in \mathbb{k}[V]^G \) implies \( \text{Pol}(f) \subseteq \mathbb{k}[V^m]^G \). In addition, Kohls-Sezer \cite{4} observed that for every polynomial \( f \in I(G, V) \) in the Hilbert ideal we have \( \text{Pol}(f) \subseteq I(G, V^m) \).

**Lemma 1.** Assume that \( \text{char}(\mathbb{k}) = 0 \). Let \( M, M' \) be monomials in \( \mathbb{k}[V] \) with \( M' < M \). Then for any \( \underline{k} \in \mathbb{N}^m \) we have

\[
\text{LM}(\text{Pol}_{\underline{k}}(M')) < \text{LM}(\text{Pol}_{\underline{k}}(M))
\]

with respect to the lexicographic order fixed above.

**Proof.** Fix \( \underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m \) and let \( M = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{k}[V] \). We have

\[
\Phi(M) = (x_1^{(1)} t_1 + \cdots + x_1^{(m)} t_m)^{a_1} \cdots (x_n^{(1)} t_1 + \cdots + x_n^{(m)} t_m)^{a_n}.
\]

Let \( h = \text{LM}(\text{Pol}_{\underline{k}}(M)) \) and write \( h = \prod_{1 \leq i \leq n, 1 \leq j \leq m} (x_i^{(j)})^{b_{i,j}} \). Note that \( h \) contains with multiplicities \( k_1 \) variables from the first summand of variables \( \{x_1^{(1)}, \ldots, x_n^{(1)}\} \). Since \( x_1^{(1)} \) is the highest ranked variable among them we have \( b_{1,1} = \min\{k_1, a_1\} \).

More generally, \( h \) contains with multiplicities \( k_j \) variables from the set \( \{x_1^{(j)}, \ldots, x_n^{(j)}\} \). Inductively, multiplicity of the \( x_i^{(j)} \) is \( b_{i,j} \) for \( 1 \leq l < i \). Moreover, out of \( a_l \) factors \( (x_i^{(l)} t_1 + \cdots + x_i^{(m)} t_m)^{a_l} \), \( b_{i,l} \) of them contribute \( a_l \) to \( h \) for \( 1 \leq l < j \). Since \( x_i^{(j)} \) is the highest rank monomial in \( \{x_1^{(j)}, \ldots, x_n^{(j)}\} \), we get a recursive relation

\[
b_{i,j} = \min\{k_j - \sum_{l=1}^{i-1} b_{l,j}, \ a_i - \sum_{l=1}^{j-1} b_{l,j}\}.
\]
Note that the coefficient of \( h \) in \( \text{Pol}_k(M) \) is

\[
\prod_{1 \leq i \leq n} \frac{a_i!}{b_{i,1}! \cdots b_{i,m}!}
\]

which is non-zero because \( \text{char}(k) = 0 \). We may take \( M' = x_1^{a_1} \cdots x_k^{a_{k-1}} x_k^{a_k} \cdots x_n^{a_n} \) with \( a_k < a_1 \). Set \( h' = \text{LM}(\text{Pol}_k(M')) \) and write \( h = \prod_{1 \leq i \leq n, 1 \leq j \leq m} (x_i^{(j)})^{b_{i,j}} \).

As in the case for \( b_{i,j} \), \( b_{i,j}' \) depends only on \( k_l \) for \( 1 \leq l \leq j \) and \( a_l \) for \( 1 \leq l \leq i \).

Since the multiplicities of the variables \( x_1, \ldots, x_{k-1} \) in \( M \) and \( M' \) are the same, we get that \( b_{i,j} = b_{i,j}' \) for \( i < k \). On the other hand since \( \sum_{1 \leq l \leq m} b_{k,l} = a_k > a_{k'} = \sum_{1 \leq l \leq m} b_{k',l} \), the equality \( b_{k,l} = b_{k,l}' \) fails for some \( 1 \leq l \leq m \). Let \( j \) denote the smallest index such that \( b_{k,j} \neq b_{k,j}' \). Then we have

\[
b_{k,j}' = \min\{k_j - \sum_{l=1}^{k-1} b_{l,j}', a_k' - \sum_{l=1}^{j-1} b_{k,l}'\} \\
= \min\{k_j - \sum_{l=1}^{k-1} b_{l,j}, a_k - \sum_{l=1}^{j-1} b_{k,l}\} \\
\leq \min\{k_j - \sum_{l=1}^{k-1} b_{l,j}, a_k - \sum_{l=1}^{j-1} b_{k,l}\} \\
= b_{k,j}
\]

So we get that \( h' < h \).

\[\square\]

Remark 2. The coefficient of the highest ranked monomial \( h \) which is given by Equation (2) is non-zero over all fields \( k \) with \( a_i! \in k^* \) for all \( i \). So the assertion of the previous lemma is true for all pairs of monomials \( M', M \) with \( M = x_1^{a_1} \cdots x_n^{a_n} \) satisfying \( a_i < \text{char}(k) \) for all \( i \).

Remark 3. Consider the lexicographic monomial order with a slightly different ordering of the variables

\[
x_1^{(1)} > x_2^{(1)} > \ldots > x_1^{(1)} > x_2^{(2)} > \ldots > x_n^{(2)} > \ldots > x_n^{(m)}.
\]

For a fixed \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), the set of variables in \( \{x_1^{(j)}, \ldots, x_i^{(j)}\} \) and in \( \{x_1^{(1)}, \ldots, x_1^{(m)}\} \) that is smaller than \( x_i^{(j)} \) remains unchanged. So the recursive description of the \( \text{LM}(\text{Pol}_k(M)) \) in Equation (2) and consequently the assertion of the lemma carry over to this ordering as well.

Since \( \text{Pol}(f) \subseteq I(G, V^m) \) for all \( f \in I(G, V) \) by [3], Lemma 12], the previous lemma immediately implies the following.

**Proposition 4.** Assume that \( \text{char}(k) = 0 \). Let \( f \in I(G, V) \). Then we have

\[
\text{LM}(\text{Pol}_k(\text{LM}(f))) \in L(I(G, V^m))
\]

for all \( k \in \mathbb{N}^m \).

We now prove our main result.

**Theorem 5.** Let \( G \leq S_n \) be a permutation group acting naturally on \( V = k^n \). Assume that \( \text{char}(k) = 0 \) or \( \text{char}(k) > n \). Then we have

\[
\beta(G, V^m) \leq \binom{n}{2} + 1.
\]
Therefore, the top degree of the coinvariants well. This implies that 
$I$ contains the set of monomials we write $Φ(x_i) = (x_i^{(1)} t_1 + \cdots + x_i^{(m)} t_m)^i$. So we get
\[
\text{Pol}_k(x_i) = \frac{i!}{k_1! \cdots k_m!} \prod_{j=1}^m \left( \left( x_i^{(j)} \right)^{k_j} \right).
\]
Note that since $i \leq n$, the coefficient is non-zero. It follows that $L(I(G, V^m))$ contains the set of monomials
\[
\left\{ \prod_{j=1}^m \left( x_i^{(j)} \right)^{k_j} \mid 1 \leq i \leq n, \sum_{j=1}^m k_j = i \right\}.
\]
Therefore, the top degree of the coinvariants $k[V^m]_G$ bounded above by \( \binom{n}{2} \) as well. This implies that $I(G, V^m)$ is generated by polynomials of degree at most \( \binom{n}{2} + 1 \). Now we apply a standard argument to get a bound for $β(G, V^m)$ as follows. Let $f_1, \ldots, f_s$ be generators for $I(G, V^m)$ of degree at most \( \binom{n}{2} + 1 \). We may assume these generators lie in $k[V^m]_G$. Let $f \in k[V^m]_G$ with degree $> \binom{n}{2} + 1$. Write $f = \sum_{i=1}^s q_i f_i$ with $q_i \in [V^m]_G$. Let $Tr : k[V^m] \to k[V^m]_G$ denote the transfer map defined by $Tr(h) = \sum_{\sigma \in G} \sigma(h)$ for $h \in k[V^m]$. Then we have $Tr(f) = [G]f = \sum_{i=1}^s Tr(q_i) f_i$. Therefore $f$ is in the algebra generated by invariants of strictly smaller degree. So we get $β(G, V^m) \leq \binom{n}{2} + 1$ as desired.

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