A geometric interpretation of integrable motions

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Abstract. Integrability, one of the classic issues in galactic dynamics and in general in celestial mechanics, is here revisited in a Riemannian geometric framework, where newtonian motions are seen as geodesics of suitable “mechanical” manifolds. The existence of constants of motion that entail integrability is associated with the existence of Killing tensor fields on the mechanical manifolds. Such tensor fields correspond to hidden symmetries of non-Noetherian kind. Explicit expressions for Killing tensor fields are given for the $N = 2$ Toda model, and for a modified Hénon-Heiles model, recovering the already known analytic expressions of the second conserved quantity besides energy for each model respectively.

Keywords: Integrability, newtonian dynamics, galactic dynamics

1. Introduction

The problem of integrability in classical mechanics has been very seminal. Motivated by celestial mechanics, it has stimulated a wealth of analytical methods and results. For example, the weaker request of only approximate integrability over finite times, or the existence of integrable regions in the phase space of a globally non-integrable system, have led to the development of classical perturbation theory, with all its important achievements. However, deciding whether a given Hamiltonian system is globally integrable or not still remains a difficult task, for which a general constructive framework is lacking. Besides its theoretical interest, the problem of integrability is still relevant to a number of open problems among which we can mention a long standing one in galactic dynamics: the quest for the third integral of motion besides energy and angular momentum (Binney and Tremaine, 1987). In fact, the apparent absence of dynamical chaos in several models describing the motions of test stars in mean-field galactic gravitational potentials suggests that these models might be integrable.

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The aim of the present paper is to draw attention to the rephrasing of this classical problem of integrability in a Riemannian geometric language. Such a possibility exists because Hamiltonian flows can be identified with geodesic flows on Riemannian manifolds equipped with suitable metrics. The Riemannian geometric framework has hitherto proved very useful to tackle Hamiltonian chaos (Casetti, Pettini and Cohen, 2000), to understand the origin of chaos in non-Anosov flows (Pettini, 1993; Cerruti-Sola and Pettini, 1996; Cerruti-Sola, Franzosi and Pettini, 1997; Pettini and Valdettaro, 1995; Casetti, Clementi and Pettini, 1996) and also to analytically compute Lyapunov exponents (Casetti, Clementi and Pettini, 1996).

The existence of conservation laws, and of conserved quantities along the trajectories of a Hamiltonian system, is related with the existence of symmetries. The link is made by Noether theorem (Arnold, 1978). A symmetry is seen as an invariance under the action of a group of transformations, and, in the case of continuous symmetries, this can be related also to the existence of special vector fields, Killing vector fields on the mechanical manifold, generating the transformations. However, through Noetherian symmetries, and thus Killing vector fields, only a limited set of conservation laws can be accounted for. This is easily understood because only invariants that are linear functions of the momenta can be constructed by means of Killing vectors, while the energy, an invariant for any autonomous Hamiltonian system, is already a quadratic function of the momenta. The possibility of constructing invariants along a geodesic flow, that are of higher order than linear in the momenta, is related with the existence of Killing tensor fields on the mechanical manifolds (Eisenhart, 1964). In the present paper we discuss all these facts and we show how it is possible to explicitly work out the components of the Killing tensors associated with two integrable models: an $N = 2$ Toda model and a modified version of the Hénon-Heiles model, and hence to obtain the analytic expressions of the second integral of motion besides energy.

In general, the components of any Killing tensor field on a mechanical manifold are solutions of a linear, non-homogeneous system of first order partial differential equations. As the number of these equations always exceeds the number of the unknowns, the system is always overdetermined. The existence of Killing tensors thus requires compatibility. However, compatibility is generically very unusual, hence a possible explanation, at least of qualitative kind, of the exceptionality of integrability with respect to non-integrability.

For the sake of clarity and self-containedness, we briefly recall some basic points about the geometrization of newtonian mechanics, about Killing vector fields and Killing tensor fields in Sections II, III, and IV,
respectively. Section V contains the original results mentioned above about the relationship between integrability and the existence of Killing tensor fields on the mechanical manifolds.

2. Geometric formulation of Hamiltonian dynamics

Let us briefly recall a few basic points about the geometrization of Newtonian dynamics in a Riemannian geometric framework. This applies to dynamical systems described by standard Hamiltonians, i.e.

\[ H(p, q) = \frac{1}{2}a^{ij}(q)p_ip_j + V(q), \]

with the shorthands \( p = (p_1, \ldots, p_N) \) and \( q = (q_1, \ldots, q_N) \). Equivalently, we can describe these systems through Lagrangian functions

\[ L(q, \dot{q}) = \frac{1}{2}a_{ij}(q)\dot{q}_i\dot{q}_j - V(q). \]

According to Maupertuis’ principle of stationary action, among all the possible isoenergetic paths \( \gamma(t) \) with fixed end points, the paths that make vanish the first variation of the action functional

\[ \mathcal{A} = \int_{\gamma(t)} p_i dq_i = \int_{\gamma(t)} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \, dt \]

are natural motions.

The kinetic energy \( W \) is a homogeneous function of degree two, hence

\[ 2W = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}, \]

and Maupertuis’ principle reads

\[ \delta \mathcal{A} = \delta \int_{\gamma(t)} 2W \, dt = 0. \]

The configuration space \( M \) of a system with \( N \) degrees of freedom is an \( N \)-dimensional differentiable manifold and the lagrangian coordinates \( (q_1, \ldots, q_N) \) can be used as local coordinates on \( M \). The manifold \( M \) is naturally given a proper Riemannian structure. In fact, by introducing the matrix

\[ g_{ij} = 2[E - V(q)]a_{ij} \]

Eq.(2) becomes

\[ \delta \int_{\gamma(t)} 2W \, dt = \delta \int_{\gamma(t)} \left( g_{ij} \dot{q}_i \dot{q}_j \right)^{1/2} \, dt = \delta \int_{\gamma(s)} ds = 0, \]

so that the Newtonian motions fulfill the geodesic condition on the manifold \( M \), provided we define \( ds \) as its arclength. The metric tensor \( g_{ij} \) of \( M \) is defined through its components by Eq.(3). This is known as Jacobi (or kinetic energy) metric. Denoting by \( \nabla \) the canonical Levi-Civita connection on \((M, g_{ij})\), the geodesic equation

\[ \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \]
becomes, in the local coordinates \((q^1, \ldots, q^N)\),

\[
\frac{d^2 q^i}{ds^2} + \Gamma^i_{jk} \frac{dq^j}{ds} \frac{dq^k}{ds} = 0,
\]

where the Christoffel coefficients \(\Gamma^i_{jk}\) are the components of \(\nabla\) defined by

\[
\Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk} \right)
\]

(7)

\[
= -\frac{1}{2W} \left[ \delta^i_k \partial_j V + \delta^i_j \partial_k V - \partial_l V a^l a_{jk} \right] + \frac{1}{2} a^{l[i} \left[ \partial_l a_{jk} + \partial_k a_{lj} - \partial_j a_{lk} \right],
\]

where \(\partial_i = \partial/\partial q^i\). Without loss of generality consider \(g_{ij} = 2[E - V(q)] \delta_{ij}\), so that

\[
\Gamma^i_{jk} = -\frac{1}{2W} \left[ \delta^i_k \partial_j V + \delta^i_j \partial_k V - \delta_{jk} \partial^i V \right].
\]

(8)

and from Eq. (6) we get

\[
\frac{d^2 q^i}{ds^2} + \frac{1}{2(E - V)} \left[ 2 \frac{\partial(E - V)}{\partial q_j} \frac{dq^j}{ds} \frac{dq^i}{ds} - g^{ij} \frac{\partial(E - V)}{\partial q_j} g^{km} \frac{dq^k}{ds} \frac{dq^m}{ds} \right] = 0,
\]

(9)

and, using \(ds^2 = 2(E - V)^2 dt^2\), these equations finally yield

\[
\frac{d^2 q^i}{dt^2} = -\frac{\partial V}{\partial q_i}, \quad i = 1, \ldots, N.
\]

(10)

which are Newton equations.

3. Killing vector fields

On a Riemannian manifold, for any pair of vectors \(V\) and \(W\), the following relation holds

\[
\frac{d}{ds} \langle V, W \rangle = \langle \nabla V, W \rangle + \langle V, \nabla W \rangle
\]

(11)

where \(\langle V, W \rangle = g_{ij} V^i W^j\) and \(\nabla/\partial s\) is the covariant derivative along a curve \(\gamma(s)\). If the curve \(\gamma(s)\) is a geodesic, for a generic vector \(X\) we have

\[
\frac{d}{ds} \langle X, \dot{\gamma} \rangle = \langle \nabla \dot{\gamma}, X \rangle = \langle \nabla X, \dot{\gamma} \rangle \equiv \langle \nabla_\dot{\gamma} X, \dot{\gamma} \rangle
\]

(12)

where \((\nabla_\dot{\gamma} X)^i = \frac{d}{ds} \frac{\partial X^i}{\partial x^l} + \Gamma^i_{jk} \frac{dx^j}{ds} X^k\), so that in components it reads

\[
\frac{d}{ds} (X_i v^i) = v^i \nabla_i (X_j v^j)
\]

(13)
where \( v^i = dx^i/ds \); with \( X_j v^i \nabla v^j = X_j \nabla \dot{v}^j = 0 \) – because geodesics are autoparallel – this can be obviously rewritten as

\[
\frac{d}{ds}(X_i v^i) = \frac{1}{2} v^j v^i (\nabla_i X_j + \nabla_j X_i)
\]

(14)
telling that the vanishing of the l.h.s., i.e. the conservation of \( X_i v^i \) along a geodesic, is guaranteed by the vanishing of the r.h.s., i.e.

\[
\nabla (iX_j) \equiv \nabla_i X_j + \nabla_j X_i = 0, \quad i, j = 1, \ldots, \text{dim}M_E.
\]

(15)
If such a field exists on a manifold, it is called a Killing vector field (KVF). Equation (15) is equivalent to \( \mathcal{L}_X g = 0 \), where \( \mathcal{L} \) is the Lie derivative. On the mechanical manifolds \( (M_E, g_J) \), being the unit vector \( dq_k ds \) – tangent to a geodesic – proportional to the canonical momentum \( p_k = \frac{\partial L}{\partial \dot{q}^k} = q^k \), \( a_{ij} = \delta_{ij} \), the existence of a KVF \( X \) implies that the quantity, linear in the momenta,

\[
J(q, p) = X_k(q) \frac{dq^k}{ds} = \frac{1}{\sqrt{2(E - V(q))}} X_k(q) \frac{dq^k}{dt} = \frac{1}{\sqrt{2W(q)}} \sum_{k=1}^{N} X_k(q)p_k
\]

(16)
is a constant of motion along the geodesic flow. Thus, for an \( N \) degrees of freedom Hamiltonian system, a physical conservation law, involving a conserved quantity linear in the canonical momenta, can always be related with a symmetry on the manifold \( (M_E, g_J) \) due to the action of a KVF on the manifold. These are conservation laws of Noetherian kind. The equation (15) is equivalent to the vanishing of the Poisson brackets

\[
\{H, J\} = \sum_{i=1}^{N} \left( \frac{\partial H}{\partial q^i} \frac{\partial J}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial J}{\partial q^i} \right) = 0,
\]

(17)
the standard definition of a constant of motion \( J(q, p) \). In fact, a linear function of the momenta

\[
J(q, p) = \sum_i C_i(q)p_i
\]

(18)
if conserved, can be associated with the vector of components

\[
X_k = [E - V(q)]C_k(q).
\]

(19)
The explicit expression of the system of equations (15) is obtained by writing in components the covariant derivatives associated with the connection coefficients (8) and it finally reads

\[
[E - V(q)] \left[ \frac{\partial C_i(q)}{\partial q^j} + \frac{\partial C_j(q)}{\partial q^i} \right] - \delta_{ij} \sum_{k=1}^{N} \frac{\partial V}{\partial q^k} C_k(q) = 0,
\]

(20)
or equivalently

\[
\frac{1}{2} \sum_{k=1}^{N} p_k^2 \left[ \frac{\partial C_i(q)}{\partial q^j} + \frac{\partial C_j(q)}{\partial q^i} \right] - \delta_{ij} \sum_{k=1}^{N} \frac{\partial V}{\partial q^k} C_k(q) = 0, \quad (21)
\]

which, according to the principle of polynomial identity, yields the following conditions for the coefficients \( C_i(q) \)

\[
\frac{\partial C_i(q)}{\partial q^j} + \frac{\partial C_j(q)}{\partial q^i} = 0 \quad i \neq j, \quad i, j = 1, \ldots, N
\]

\[
\frac{\partial C_i(q)}{\partial q^i} = 0 \quad i = 1, \ldots, N \quad (22)
\]

\[
\sum_{k=1}^{N} \frac{\partial V}{\partial q^k} C_k(q) = 0 .
\]

One can easily check that the same conditions stem from Eq.(17). As an elementary example, we can give the explicit expression of the components of the Killing vector field associated with the conservation of the total momentum \( P(q, p) = \sum_{k=1}^{N} p_k \).

In this case the coefficients are \( C_i(q) = 1 \), so that the momentum conservation can be geometrically related with the action of the vector field of components

\[
X_i = E - V(q), \quad i = 1, \ldots, N \quad (23)
\]

on the mechanical manifold. At least this class of invariants has a geometric counterpart in a symmetry of \( (M_E, g_J) \).

However, in order to achieve a fully geometric rephrasing of integrability, we need something similar for any constant of motion. If a one-to-one correspondence is to exist between conserved physical quantities along a Hamiltonian flow and suitable symmetries of the mechanical manifolds \( (M_E, g_J) \), then integrability will be equivalent to the existence of a number of symmetries at least equal to the number of degrees of freedom \( (= \dim M_E) \).

If a Lie group \( G \) acts on the phase space manifold through completely canonical transformations, and there exists an associated momentum mapping\(^1\), then every Hamiltonian having \( G \) as a symmetry group, with respect to its action, admits the momentum mapping as constant of motion (Abraham and Marsden, 1987). These symmetries are usually\(^1\) 

\(^1\) This happens whenever this action corresponds to the lifting to the phase space of the action of a Lie group on the configuration space.
referred to as hidden symmetries because, even though their existence is ensured by integrability, they are not easily recognizable\(^2\).

4. Killing tensor fields

Let us now extend what has been presented in the previous section about KVF's, trying to generalize the form of the conserved quantity along a geodesic flow from \( J = X^i v^i \) to \( J = K_{j_1 j_2 \ldots j_r} v^{j_1} v^{j_2} \ldots v^{j_r} \), with \( K_{j_1 j_2 \ldots j_r} \) a tensor of rank \( r \). Thus, we look for the conditions that entail

\[
\frac{d}{ds}(K_{j_1 j_2 \ldots j_r} v^{j_1} v^{j_2} \ldots v^{j_r}) = v^j \nabla_j (K_{j_1 j_2 \ldots j_r} v^{j_1} v^{j_2} \ldots v^{j_r}) = 0 . \quad (24)
\]

In order to work out from this equation a condition for the existence of a suitable tensor \( K_{j_1 j_2 \ldots j_r} \), which is called a Killing tensor field (KTF), let us first consider the 2\( r \)-rank tensor \( K_{j_1 j_2 \ldots j_r} v^{i_1} v^{i_2} \ldots v^{i_r} \) and its covariant derivative along a geodesic, i.e.

\[
v^j \nabla_j (K_{j_1 j_2 \ldots j_r} v^{i_1} v^{i_2} \ldots v^{i_r}) =
\]

\[
= v^j \left( \frac{\partial K_{j_1 j_2 \ldots j_r}}{\partial x^j} - K_{l_1 l_2 \ldots l_r} \Gamma_{j_1 j_2 \ldots j_r}^{l_1 l_2 \ldots l_r} - \ldots - K_{j_1 j_2 \ldots j_r} \Gamma_{j_1 j_2 \ldots j_r}^{l_1} \right) v^{i_1} \ldots v^{i_r} +
\]

\[
+ K_{j_1 j_2 \ldots j_r} \left( v^j \frac{\partial v^{i_1}}{\partial x^j} + \Gamma_{j_1 j_2 \ldots j_r}^{i_1 j_1} v^{i_1} \right) v^{i_2} \ldots v^{i_r} + \ldots
\]

\[
+ K_{j_1 j_2 \ldots j_r} v^{i_1} \ldots v^{i_{r-1}} \left( v^j \frac{\partial v^{i_r}}{\partial x^j} + \Gamma_{j_1 j_2 \ldots j_r}^{i_r j_1} v^{i_1} \right)
\]

\[
= v^{i_1} v^{i_2} \ldots v^{i_r} v^j \nabla_j K_{j_1 j_2 \ldots j_r} . \quad (25)
\]

where we have again used \( v^j \nabla_j v^{i_k} = 0 \) along a geodesic, and a standard covariant differentiation formula (Doubrovine et al., 1979). Now, by contraction on the indices \( i_k \) and \( j_k \) the 2\( r \)-rank tensor of the r.h.s. of Eq.(25) provides a new expression for the r.h.s. of Eq.(24) which reads

\[
\frac{d}{ds}(K_{j_1 j_2 \ldots j_r} v^{j_1} v^{j_2} \ldots v^{j_r}) = v^{j_1} v^{j_2} \ldots v^{j_r} v^j \nabla_j K_{j_1 j_2 \ldots j_r} , \quad (26)
\]

where \( \nabla_j K_{j_1 j_2 \ldots j_r} = \nabla_j K_{j_1 j_2 \ldots j_r} + \nabla_j K_{j_2 j_3 \ldots j_r} + \ldots + \nabla_j K_{j_r j_1 j_2 \ldots j_{r-1}} \), as it can be easily understood by rearranging the indices of the summations in the contraction of the 2\( r \)-rank tensor in the last part of Eq. (25);

\(^2\) An interesting account of these hidden symmetries can be found in (Olshanetsky and Perelomov, 1981) where it is surmised that integrable motions of \( N \) degrees of freedom systems are the “shadows” of free motions in symmetric spaces (for example euclidean spaces \( \mathbb{R}^n \), hyperspheres \( S^n \), hyperbolic spaces \( \mathbb{H}^n \)) of sufficiently large dimension \( n > N \).
(a direct check for the case $N = r = 2$ is immediate). The vanishing of Eq. (26), entailing the conservation of $K_{j_1j_2...j_r} v^{j_1} v^{j_2} \ldots v^{j_r}$ along a geodesic flow, is therefore guaranteed by the existence of a tensor field fulfilling the conditions

$$\nabla_{(j} K_{j_1j_2...j_r)} = 0,$$

(27)

these equations generalize Eq. (15) and give the definition of a KTF on a Riemannian manifold. These $N^r + 1$ equations in $(N + r - 1)!/r!(N - 1)!$ unknown independent components\(^3\) of the Killing tensor constitute an overdetermined system of equations. Thus, a-priori, we can expect that the existence of KTFs has to be rather exceptional.

If a KTF exists on a Riemannian manifold, then the scalar

$$K_{j_1j_2...j_r} \frac{dq^{j_1}}{ds} \frac{dq^{j_2}}{ds} \ldots \frac{dq^{j_r}}{ds}$$

(28)

is a constant of motion for the geodesic flow on the same manifold.

Let us consider, as a generalization of the special case of rank one given by Eq. (18), the following constant of motion

$$J(q, p) = \sum_{\{i_1, i_2, \ldots, i_N\}} C_{i_1i_2...i_N} p_{i_1}^{i_1} p_{i_2}^{i_2} \ldots p_{i_N}^{i_N},$$

(29)

which, with the constraint $i_1 + i_2 + \ldots + i_N = r$, is a homogeneous polynomial of degree $r$. The index $i_j$ denotes the power with which the momentum $p_j$ contributes. If $r < N$ then necessarily some indices $i_j$ must vanish. By repeating the procedure developed in the case $r = 1$, and by identifying

$$J(q, p) \equiv K_{j_1j_2...j_r} \frac{dq^{j_1}}{ds} \frac{dq^{j_2}}{ds} \ldots \frac{dq^{j_r}}{ds}$$

(30)

we get the relationship between the components of the Killing tensor of rank $r$ and the coefficients $C_{i_1i_2...i_N}$ of the invariant $J(q, p)$, that is

$$K_{1,2,...,N}^{i_1, i_2, \ldots, i_N} = 2^{r/2} [E - V(q)]^r C_{i_1i_2...i_N}.$$  

(31)

With the only difference of a more tedious combinatorics, also in this case it turns out that the equations (27) are equivalent to the vanishing of the Poisson brackets of $J(q, p)$, that is

$$\{H, J\} = 0 \iff \nabla_{(j} K_{j_1j_2...j_r)} = 0.$$  

(32)

\(^3\) This number of independent components, i.e. the binomial coefficient $\binom{N+r-1}{r}$, is due to the totally symmetric character of Killing tensors.
Thus, the existence of Killing tensor fields, obeying Eq. (27), on a mechanical manifold \((M, g_J)\) provide the rephrasing of integrability of Newtonian equations of motion or, equivalently, of standard Hamiltonian systems, within the Riemannian geometric framework.

At first sight, it might appear too restrictive that prime integrals of motion have to be homogeneous functions of the components of \(p\). However, as we shall discuss in the next Section, the integrals of motion of the known integrable systems can be actually cast in this form. This is in particular the case of total energy, a quantity conserved by any autonomous Hamiltonian system.

5. Explicit KTFs of known integrable systems

The first natural question to address concerns the existence of a KT field, on any mechanical manifold \((M, g_J)\), to be associated with total energy conservation. Such a KT field actually exists and coincides with the metric tensor \(g_J\), in fact it satisfies\(^4\) by definition Eq. (27).

One of the simplest case of integrable system is represented by a decoupled system described by a generic Hamiltonian

\[
H = \sum_{i=1}^{N} \left[ \frac{p_i^2}{2} + V_i(q_i) \right] = \sum_{i=1}^{N} H_i(q_i, p_i)
\]

for which all the energies \(E_i\) of the subsystems \(H_i, \, i = 1, \ldots, N\), are conserved. On the associated mechanical manifold, \(N\) KT fields of rank 2 exist, they are given by

\[
K_{jk}^{(i)} = \delta_{jk} \{ V_i(q_i)[E - V(q)] + \delta^j_i [E - V(q)]^2 \}.
\]

In fact, these tensor fields fulfil Eq. (27) which explicitly reads

\[
\nabla_k K_{lm}^{(i)} + \nabla_l K_{mk}^{(i)} + \nabla_m K_{kl}^{(i)} = 0
\]

\[ \quad k, l, m = 1, \ldots, N. \]

The conserved quantities \(J_{(i)}(q, p)\) are then obtained by saturation of the tensors \(K^{(i)}\) with the velocities \(dq/ds\)

\[
J_{(i)}(q, p) = \sum_{j,k=1}^{N} K_{jk}^{(i)} \frac{dq^j}{ds} \frac{dq^k}{ds} = V_i(q_i) \frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} + \frac{p_i^2}{2} = E_i.
\]

\(^4\) A property of the canonical Levi-Civita connection, on which the covariant derivative is based, is just the vanishing of \(\nabla g\).
This equation suggests that to require that the constants of motion have to be homogeneous polynomials of the momenta is not so restrictive as it might appear, in fact, through the following constant quantity

\[
\frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} = 1
\]

homogeneous of second degree in the momenta, any even degree polynomial of the momenta can be made homogeneous. The possibility of inferring the existence of a conservation law from the existence of a KTF on \((M, g_J)\) is thus extended to the constants of motion given by a sum of homogeneous polynomials whose degrees differ by an even integer

\[
J(p, q) = P^{(r)}(p) + P^{(r-2)}(p) + \ldots + P^{(r-2n)}(p) \in C^\infty(q)[p_1, \ldots, p_N]
\]

\[
homdeg P^s = s \quad s = r, r - 2, \ldots, r - 2\left[\frac{r}{2}\right]
\]

so that it can be recast in the homogeneous form

\[
J(p, q) = P^{(r)}(p) + P^{(r-2)}(p) \frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} + \ldots +
\]

\[
+ P^{(r-2n)}(p) \left[ \frac{1}{E - V(q)} \sum_{k=1}^{N} \frac{p_k^2}{2} \right]^n
\]

5.1. Nontrivial integrable models

Nontrivial examples of nonlinear integrable Hamiltonian systems are provided by the following Hamiltonians

\[
H = \sum_{i=1}^{N} \left\{ \frac{p_i^2}{2} + \frac{a}{b} \left[ e^{-b(q_{i+1} - q_i)} - 1 \right] \right\}
\]

known as the Toda model (Toda, 1970), which is integrable for any given pair of the constants \(a\) and \(b\), and

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2} \left( \sum_{i=1}^{N} q_i^2 \right)^2 - \sum_{i=1}^{N} \lambda_i q_i^2
\]

which is completely integrable for any \(\lambda_1, \ldots, \lambda_N\) (Choodnovsky and Choodnovsky, 1978). Recursive formulae are available for all the constants of motion of the Toda model at any \(N\) (Hénon, 1974), and
also for the second Hamiltonian the exact form of first integrals is known (Choodnovsky and Choodnovsky, 1978). In both cases, the first integrals are polynomials of given parity of the momenta so that, on the basis of what we have said above, each invariant \( J^{(i)} \), \( i = 1, \ldots, N \) can be derived from a KTF on \((M, g_{iJ})\). Thus, integrability of these systems admits a Riemannian-geometric interpretation.

5.2. The special case of \( N=2 \) Toda model

Let us consider the special case of a two-degrees of freedom Toda model described by the integrable Hamiltonian

\[
H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{24} \left[ e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y} \right] - \frac{1}{8}.
\]  

From what is already reported in the literature (Hénon, 1974), we know that a third order polynomial of the momenta has to be found eventually, therefore we look for a rank-3 KT fulfilling

\[
\nabla_i K_{jkl} + \nabla_j K_{ikl} + \nabla_k K_{ijl} + \nabla_l K_{ijk} = 0 \quad i, j, k, l = 1, 2 \tag{43}
\]

where, associating the label 1 to \( x \) and the label 2 to \( y \), \( \{(i, j, k, l)\} = \{(1, 1, 1, 1); (1, 1, 1, 2); (1, 1, 2, 2); (1, 2, 2, 2); (2, 2, 2, 2)\} \). The computation of the Christoffel coefficients according to Eq.(8) yields

\[
\begin{align*}
\Gamma^1_{11} &= \frac{-\partial_x V}{2[E - V(x, y)]}, & \Gamma^1_{22} &= \frac{\partial_y V}{2[E - V(x, y)]}, & \Gamma^1_{11} &= \frac{-\partial_y V}{2[E - V(x, y)]}, \\
\Gamma^2_{22} &= \frac{-\partial_y V}{2[E - V(x, y)]}, & \Gamma^2_{12} &= \frac{-\partial_x V}{2[E - V(x, y)]}, & \Gamma^2_{12} &= \frac{-\partial_x V}{2[E - V(x, y)]}.
\end{align*} 
\]  

From Eq.(43) we get the system

\[
\begin{align*}
\nabla_1 K_{111} &= 0 \\
\nabla_1 K_{122} + \nabla_2 K_{112} &= 0 \\
\nabla_2 K_{111} + 3\nabla_1 K_{211} &= 0 \\
\nabla_1 K_{222} + 3\nabla_2 K_{122} &= 0 \\
\nabla_2 K_{222} &= 0
\end{align*} \tag{45}
\]

whence

\[
\partial_x K_{111} - 3\Gamma^1_{11} K_{111} - 3\Gamma^2_{11} K_{211} = 0
\]

\footnote{This is derived from an \( N = 3 \) Hamiltonian (40) by means of two canonical transformations of variables removing translational invariance, see for example (Boccaletti and Pucacco, 1996); the third order expansion of this new Hamiltonian yields the Hénon-Heiles model of Eq.(49) with \( C = D = 1 \).}
\[
\begin{align*}
\partial_x K_{122} + \partial_y K_{211} - \Gamma_{11} K_{122} - \Gamma_{12} K_{222} - 4\Gamma_{12} K_{112} - \\
4\Gamma_{12} K_{212} - \Gamma_{22} K_{111} - \Gamma_{22} K_{211} &= 0 \\
\partial_y K_{111} + 3\partial_x K_{211} - 6\Gamma_{12} K_{111} - 6\Gamma_{12} K_{112} - \\
6\Gamma_{11} K_{212} - 6\Gamma_{22} K_{112} &= 0 \\
\partial_x K_{222} + 3\partial_y K_{122} - 6\Gamma_{12} K_{222} - 6\Gamma_{22} K_{112} - \\
6\Gamma_{11} K_{222} - 6\Gamma_{22} K_{212} &= 0 \\
\partial_y K_{222} - 3\Gamma_{22} K_{122} - 3\Gamma_{22} K_{222} &= 0
\end{align*}
\]

with the Christoffel coefficients given by Eq.(44), where one has to replace \(V(x, y)\) with the potential part of the Hamiltonian (42) and \(\partial_x V, \partial_y V\) with its derivatives. The general method of solving a linear, non-homogeneous system of first-order partial differential equations in more than one dependent variables is sketched in Appendix. However, finding the explicit solution to the system of equations (46) is much facilitated because we already know a-priori that this system is compatible and thus admits a solution, and we also have strong hints about the solution itself because the general form of the integrals of the Toda model is known (Hénon, 1974). The KTF, besides the metric tensor, for the model (42) is eventually found to have the components

\[
K_{111} = 2(E - V)^2[3\partial_y V + 4(E - V)]
= 8(E - V)^3 + \frac{1}{2}(E - V)^2[e^{2y - 3x} + e^{2y + 2\sqrt{3}x} - 2e^{-4y}]
\]

\[
K_{122} = 2(E - V)^2[\partial_y V - 4(E - V)]
= -24(E - V)^3 + \frac{1}{2}(E - V)^2[e^{2y - 3x} + e^{2y + 2\sqrt{3}x} - 2e^{-4y}]
\]

\[
K_{112} = -2(E - V)^2\partial_x V = \frac{\sqrt{3}}{6}(E - V)^2(e^{2y + 2\sqrt{3}x} - e^{2y - 2\sqrt{3}x})
\]

\[
K_{222} = -6(E - V)^2\partial_x V = \frac{\sqrt{3}}{2}(E - V)^2(e^{2y + 2\sqrt{3}x} - e^{2y - 2\sqrt{3}x}),
\]

as can be easily checked by substituting them into Eqs.(46). Hence, the second constant of motion, besides energy, is given by

\[
J(x, y, p_x, p_y) = K_{ijk} \frac{dq^i}{ds} \frac{dq^j}{ds} \frac{dq^k}{ds} = K_{ijk} \frac{dq^i}{dt} \frac{dq^j}{dt} \frac{dq^k}{dt} \frac{1}{2\sqrt{2}[E - V(x, y)]^3}
\]

\[
= \frac{1}{2\sqrt{2}[E - V(x, y)]^3}(K_{111} p_x^3 + 3K_{122} p_x p_y^2 + 3K_{112} p_x^2 p_y + K_{222} p_y^3)
= 8p_x(p_x^2 - 3p_y^2) + (p_x + \sqrt{3}p_y)e^{2y - 3x} - 2p_x e^{-4y} + (p_x - \sqrt{3}p_y)e^{2y + 2\sqrt{3}x}
\]

(48)
which coincides with the expression already reported in the literature (Lieberman and Lichtenberg, 1992) for the Hamiltonian (42).

5.3. THE GENERALIZED HENON-HEILES MODEL

Let us now consider the two-degrees of freedom system described by the Hamiltonian

\[
H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} (x^2 + y^2) + D x^2 y - \frac{1}{3} C y^3 .
\] (49)

This model, originally derived to describe the motion of a test star in an axisymmetric galactic mean gravitational field, provided one of the first numerical evidences of the chaotic transition in nonlinear Hamiltonian systems (Hénon and Heiles, 1964). Hénon and Heiles considered the case \( C = D = 1 \). The existence of a chaotic layer in the phase space of this model means lack of global integrability. However, by means of the Painlevé method, it has been shown (Chang, Tabor and Weiss, 1982) that for special choices of the parameters \( C \) and \( D \) this system is globally integrable.

Let us now tackle integrability of this model from the viewpoint of the existence of KT fields on the manifold \((M, g_J)\).

We first begin with the equations for a Killing vector field. By means of Eqs.(22) we look for possible coefficients \( C_1(x, y) \), \( C_2(x, y) \), thus obtaining

\[
C_1 = C_1(y), \quad C_2 = C_2(x)
\]

\[
\frac{dC_1(y)}{dy} + \frac{dC_2(x)}{dx} = 0 \tag{50}
\]

\[x(1 + 2Dy)C_1(y) + (y + Dx^2 - Cy^2)C_2(x) = 0 \tag{52}
\]

From the second equation of Eqs.(50) it follows that

\[
\frac{dC_1(y)}{dy} = -\frac{dC_2(x)}{dx} = \text{cost}. \tag{51}
\]

whence, denoting with \( \alpha \) the constant, the possible expressions for \( C_1(y) \) and \( C_2(x) \) are only of the form \( C_1(y) = -\alpha y + \beta \), \( C_2(x) = \alpha x + \gamma \), that, after substitution into the last equation of Eqs. (50), imply

\[(x + 2Dxy)(-\alpha y + \beta) + (y + Dx^2 - Cy^2)(\alpha x + \gamma) = 0, \tag{52}\]

which has only a non-trivial solution for \( C = D = 0 \). On the other hand, for these values of the parameters the potential simplifies to \( V(x, y) = \frac{1}{2} x^2 + \frac{1}{2} y^2 \) whence the existence of the Killing vector field \( X \) of components \( X_1 = y \) and \( X_2 = -x \) which is due to the invariance under rotations in the \( xy \) plane.
Let us now consider the case of a rank-2 KTF. Equations (43) become
\[ \nabla_i K_{jk} + \nabla_j K_{ik} + \nabla_k K_{ij} = 0, \quad i, j, k = 1, 2 \] (53)
where, associating again the label 1 to \( x \) and the label 2 to \( y \), \( \{(i, j, k)\} = \{(1, 1, 1); (1, 1, 2); (1, 2, 2); (2, 2, 2)\} \). The Christoffel coefficients are still given by Eq. (44), where we have to use the potential part of Hamiltonian (49) so that \( \partial_x V(x, y) = x + 2Dxy \) and \( \partial_y V(x, y) = y + Dx^2 - Cy^2 \).

The KTF equations are then
\begin{align*}
\nabla_1 K_{11} &= 0 \\
2\nabla_1 K_{12} + \nabla_2 K_{11} &= 0 \\
\nabla_1 K_{22} + 2\nabla_2 K_{12} &= 0 \\
\nabla_2 K_{22} &= 0
\end{align*}
whence
\begin{align*}
\partial_x K_{11} - 2\Gamma_{11}^1 K_{11} - 2\Gamma_{11}^2 K_{21} &= 0 \\
2\partial_x K_{12} + \partial_y K_{11} - 4\Gamma_{12}^1 K_{11} - (4\Gamma_{12}^1 + 2\Gamma_{11}^1)K_{12} - 2\Gamma_{11}^2 K_{22} &= 0 \\
\partial_x K_{22} + 2\partial_y K_{12} - 2\Gamma_{22}^1 K_{11} - (4\Gamma_{12}^2 + 2\Gamma_{22}^2)K_{12} - 4\Gamma_{12}^2 K_{22} &= 0 \\
\partial_y K_{22} - 2\Gamma_{22}^1 K_{12} - 2\Gamma_{22}^2 K_{22} &= 0.
\end{align*}
(55)

Since the Hamiltonian (49) is not integrable for a generic choice of the parameters \( C \) and \( D \), we can reasonably expect that the generic property of the above overdetermined system of equations is incompatibility, i.e. only the trivial solution \( K_{ij} = 0 \) exists for the overwhelming majority of the pairs \( (C, D) \). However, the existence of special choices of \( C \) and \( D \) for which the Hamiltonian is integrable suggests that this overdetermined system can be compatible in special cases. For example, when \( D = 0 \) the variables \( x \) and \( y \) in (49) are decoupled and thus two KT fields of rank 2 exist according to Eq.(34).

A non trivial solution for the system (55) must exist at least for the pair \( (C = -6, D = 1) \). In fact, in this case the modified Hénon-Heiles model is known to be integrable (Chang, Tabor and Weiss, 1982). An explicit solution for the system (55) is eventually found to be given by
\begin{align*}
K_{11} &= (3 - 4y)(E - V(x, y))^2 + x^2(x^2 + 4y^2 + 4y + 3)(E - V(x, y)) \\
K_{12} &= 2x(E - V(x, y)) \\
K_{22} &= \frac{1}{2}(x^2 + 4y^2 + 4y + 3)(E - V(x, y)).
\end{align*}
(56)
The associated constant of motion is therefore

\[ J(x, y, p_x, p_y) = \frac{1}{(E - V(x, y))^2}(K_{11}p_x^2 + 2K_{12}p_xp_y + K_{22}p_y^2) = \]

\[ = x^4 + 4x^2y^2 - p_x^2y + 4p_xp_yx + 4x^2y + 3p_x^2 + 3x^2. \quad (57) \]

This expression is identical to that reported in (Chang, Tabor and Weiss, 1982), worked out for the same values of \( C \) and \( D \) with a completely different method based on the Painlevé property.

6. Concluding remarks

Let us now summarize the meaning of the results presented above and point out the open problems.

- Besides qualitative and quantitative descriptions of chaos, within the framework of Riemannian geometrization of newtonian mechanics also integrability has its own place. The idea of associating KTFs with integrability is not new, though this has been essentially developed in the context of classical General Relativity, see for example (Baleanu, 1998; Gibbons et al., 1993; Sommers, 1973) and references quoted therein. Recently, also an extension to classical newtonian mechanics has been considered in (Rosquist and Pucacco, 1995), where integrability conditions for quadratic invariants were obtained, and where the authors concluded saying “It is of considerable interest to develop these techniques and use them to look for fixed energy invariants of physically interesting models such as the Hénon-Heiles potential and others”: this is just what has been done in our present work.

- The reduction of the problem of integrability of a given Hamiltonian system to the existence of suitable KTFs on \((M_E, g)\) offers several reasons of interest, in particular we have seen that the system of equations in the unknown components of a KTF of a preassigned rank is overdetermined, thus – at a qualitative level – integrability seems a rather exceptional property, and the larger \( N \) the “more exceptional” it seems to be, because of the fastly growing mismatch between the number of unknowns and the number of equations. In principle, the existence of compatibility conditions for systems of linear, first-order partial differential equations could allow to decide about integrability prior to any explicit attempt at solving the equations for the components of a KTF. Even better,
there are geometric constraints to the existence of KTFs, early results in this sense are reported in (Yano and Bochner, 1953), so that it seems possible, at least in some cases, to devise purely geometric criteria of non-integrability. For example, hyperbolicity of compact manifolds excludes (Yano and Bochner, 1953) the existence of KTFs, and this is consistent with the property of geodesic flows on compact hyperbolic manifolds of being strongly chaotic (Anosov flows).

− In the present paper, before working out the second invariant besides energy for two integrable models, we already knew that a KTF had to exist and of which rank (because of the degree of the polynomial invariant), thus we knew that the system of equations to be solved was compatible. Whereas, in general we lack a criterion to restrict the search for KTFs to a small interval of ranks, and this constitutes a practical difficulty. Nevertheless, since the involution of two invariants translates into the vanishing of special brackets – the Schouten brackets (Sommers, 1973) – between the corresponding Killing tensors, a shortcut to prove integrability, for a large class of systems fulfilling the conditions of the Poincaré-Fermi theorem (Poincaré, 1892; Fermi, 1923), might be to find only one KTF of vanishing Schouten brackets with the metric tensor. In fact, for quasi-integrable systems with \( N \geq 3 \), the Poincaré-Fermi theorem states that generically only energy is conserved, thus if another constant of motion is known to exist (apart from Noetherian ones, like angular momentum) then the system must be integrable and in fact there must be \( N \) constants of motion.

− At variance with Killing vectors, which are associated with Noetherian symmetries and conservation laws, Killing tensors no longer have a simple geometrical interpretation (Gibbons et al., 1993; Rosquist, 1989), therefore the associated symmetries are non-Noetherian and hidden.

The present paper contributes the subject of a Riemannian geometric approach to integrability with constructive examples that non-trivial constants of motion besides energy can be derived from KTFs for two degrees of freedom integrable Hamiltonians of physical interest. This approach to integrability deserves further attention and investigation. In fact, among the other reasons of interest, by considering, for example, the standard Hénon-Heiles model \( (C = D = 1) \), we could wonder whether the regular regions of phase space correspond to a local fulfilment of the compatibility conditions of the system (55), this
would lead to a better understanding of the relationship between geometry and stability of Newtonian mechanics. Moreover, we could imagine that, by suitably defining weak and strong violations of these compatibility conditions, we could better understand the geometric origin of weak and strong chaos in Hamiltonian dynamics (Pettini and Landolfi, 1990; Pettini and Cerruti-Sola, 1991) and, perhaps, this might even suggest a starting point to develop a “geometric perturbation theory” complementary to the more standard canonical perturbation theory.

Finally, the celebrated problem of the third integral in galactic dynamics could find here a new constructive, and hopefully useful, approach.

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Appendix

Let us now briefly sketch a classical method (Forsyth, 1959) of solving systems of linear, first-order, partial differential equations in several dependent variables, denoted by $z_1, \ldots, z_m$, and two independent variables, denoted by $x$ and $y$. Writing $X_i = \partial z_i / \partial x$ and $Y_i = \partial z_i / \partial y$, equations (46) and (55) are in the form

$$X_i = B_i + \sum_{s=1}^{m} A_{is} Y_s$$  \hspace{1cm} (58)

with an obvious meaning of the coefficients $A_{is}$ and $B_i; i = 1, \ldots, m$.

The first step consists of “diagonalizing” the above system, writing equivalent first order equations, or systems of equations, in only one dependent variable. Thus equations (58) are multiplied by $\lambda_1, \ldots, \lambda_m$, a set of multipliers, and summed to give

$$\sum_{i=1}^{m} \lambda_i B_i - \sum_{i=1}^{m} \lambda_i X_i + \sum_{i,s=1}^{m} \lambda_i A_{is} Y_s = 0 \, .$$  \hspace{1cm} (59)

Since the derivatives must fulfill the obvious relations

$$dz_i - X_i dx - Y_i dy = 0 \, ,$$  \hspace{1cm} (60)
by combining Eqs.(59) and (60) the following system of ordinary equations is formed

\[ \sum_{i=1}^{m} \lambda_{i} dz_{i} = dx = -\frac{\lambda_{s} dy}{\sum_{i=1}^{m} \lambda_{i} A_{is}} , \]  

(61)

so that, by solving the critical equation \( \det(A + \mu I) = 0 \), where \( I \) is the identity matrix, the numbers \( \lambda_{1}, \ldots, \lambda_{m} \) can be eliminated among these equations obtaining a set of ratios \( \lambda_{1} : \lambda_{2} : \ldots : \lambda_{m} \), then the quantities \( \alpha_{i} \) defined by

\[ \alpha_{i} = \frac{\lambda_{i}}{\sum_{i=1}^{m} \lambda_{i} B_{i}} \]  

(63)

are uniquely determined. The above system of ordinary equations is finally rewritten as

\[ \alpha_{1} dz_{1} + \ldots + \alpha_{m} dz_{m} = dx \]

\[ dy = \mu dx \]  

(64)

i.e. two linear characteristic equations for each root \( \mu \) of the critical equation. Now, if \( u(x, y, z_{1}, \ldots, z_{m}) = \text{const} \) is an integral of the equations (64), then it fulfils also the system of \( m \) equations

\[ \frac{\partial u}{\partial z_{1}} + \alpha_{1} \frac{\partial u}{\partial x} + \alpha_{1} \mu \frac{\partial u}{\partial y} = 0 \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ \frac{\partial u}{\partial z_{m}} + \alpha_{m} \frac{\partial u}{\partial x} + \alpha_{m} \mu \frac{\partial u}{\partial y} = 0 \]  

(65)

since \( i = 1, \ldots, m \), there are \( m \) equations in \( m + 2 \) variables; an integral of this system, involving any of the dependent variables, is an integral of the original system of equations, and if each of the roots of the critical equation leads to an integral, then the ensemble of these integrals provides an integral equivalent of the original system. With the substitutions

\[ z_{1} = w_{1} \]

\[ z_{i} = c_{i} + (w_{1} - c_{1})w_{1} \, , \quad i = 2, \ldots, m \]  

(66)

the following single equation is constructed

\[ \frac{\partial u}{\partial w_{1}} + U_{1} \frac{\partial u}{\partial x} + U_{2} \frac{\partial u}{\partial y} = 0 \]  

(67)
where $U_1 = \alpha_1 + \sum_{i=2}^{m} \alpha_i w_i$ and $U_2 = \alpha_1 \mu + \sum_{i=2}^{m} \mu \alpha_i w_i$. Finally, the integration of the equation (67) proceeds by integrating the characteristic equations

$$dw_1 = \frac{dx}{U_1} = \frac{dy}{U_2}$$

(68)

taking $w_2, \ldots, w_m$ as non-varying quantities, and then proceeding in a standard way (Courant and Hilbert, 1962).
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