Volume of the space of qubit channels and some new results about the distribution of the quantum Dobrushin coefficient

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Abstract

The simplest building blocks for quantum computations are the qubit-qubit quantum channels. In this paper we analyse the structure of these channels via their Choi representation. The restriction of a quantum channel to the space of classical states (i.e. probability distributions) is called the underlying classical channel. The structure of quantum channels over a fixed classical channel is studied, the volume of general and unital qubit channels over real and complex state spaces with respect to the Lebesgue measure is computed and explicit formulas are presented for the distribution of the volume of quantum channels over given classical channels. Moreover an algorithm is presented to generate uniformly distributed channels with respect to the Lebesgue measure, which enables further studies. With this algorithm the distribution of trace-distance contraction coefficient (Dobrushin) is investigated numerically by Monte-Carlo simulations, which leads to some conjectures and points out the strange behaviour of the real state space.

Introduction

In quantum information theory, a qubit is the quantum analogue of the classical bit. A qubit can be represented by a $2 \times 2$ self-adjoint positive semidefinite matrix with trace one [10, 12, 13]. The space of qubits with real entries is denoted by $\mathcal{M}_2^\mathbb{R}$ and with complex entries by $\mathcal{M}_2^\mathbb{C}$ respectively. If we do not want to emphasise the underlying field, then we just write $\mathcal{M}_2$. A linear map $Q : \mathcal{M}_2 \to \mathcal{M}_2$ is called a qubit channel if it is a completely positive and trace preserving (CPT) map [12]. A qubit channel is said to be unital if it preserves the identity. Choi has published a tractable representation for completely positive linear maps [2]. To a linear map $Q : \mathbb{K}^{2 \times 2} \to \mathbb{K}^{2 \times 2}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) a block

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matrix
\[
\begin{pmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{pmatrix}
\]
\[Q_{11}, Q_{12}, Q_{21}, Q_{22} \in \mathbb{K}^{2 \times 2}\]  
(1)
is associated, which is called the Choi matrix, such that the action of \( Q \) is given by
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \mapsto aQ_{11} + bQ_{12} + cQ_{21} + dQ_{22}.
\]
Choi’s theorem states that the linear map \( Q : \mathbb{K}^{2 \times 2} \rightarrow \mathbb{K}^{2 \times 2} \) is completely positive if and only if its Choi matrix is positive definite \([2] \). Hereafter, we will use the same symbol for the qubit channel and its Choi matrix. Let
\[
Q = \begin{pmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{pmatrix}
\]
be a qubit channel and we define the underlying classical channel as the restriction of \( Q \) to the space of classical bits (i.e. diagonal matrices). The following Markov chain transition matrix can be associated to the underlying channel of the qubit channel \( Q \)
\[
P = \begin{pmatrix}
\text{Diag}(Q_{11}) \\
\text{Diag}(Q_{22})
\end{pmatrix}
\]
, where \( \text{Diag}(Q_{ii}) \) denotes the diagonal of the submatrix \( Q_{ii} \) in a row vector.

Like many other quantities of interest in quantum information theory the trace distance between states contracts under the action of quantum channels. When \( Q \) is a CPT map, we can define the trace-distance contraction coefficient as
\[
\eta_{\text{Tr}}(Q) = \sup \left\{ \frac{\text{Tr} | Q(\rho) - Q(\sigma) |}{\text{Tr} | \rho - \sigma |} : \rho, \sigma \in \mathcal{M}_2 \right\}
\]
which describes the maximal contraction under \( Q \). This can be regarded as the quantum analogue of the Dobrushin coefficient of ergodicity \([5] \) and has important applications to the problem of mixing time bounds of (quantum) Markov processes, as demonstrated in e.g. \([4, 3, 14, 7] \). To compute the volume of qubit channels and their distributions over classical channels, we use the strategy that was applied by A. Andai to compute the volume of the quantum mechanical state space over \( n \)-dimensional real, complex and quaternionic Hilbert spaces with respect to the canonical Euclidean measure \([1] \).

The paper is organized as follows. In the first section we fix the notations for further computations and we mention some elementary lemmas which will be used in the sequel. In Section 2, the volume of general and unital qubit channels over real and complex state spaces with respect to the canonical Euclidean measure are computed and explicit formulas are given for the distribution of the volume over classical channels. Section 3 deals with the distribution of the trace-distance contraction coefficient. Cumulative distribution function of \( \eta_{\text{Tr}} \) was calculated by Monte-Carlo method on the whole space. Supremum of \( \eta_{\text{Tr}} \) over a fixed classical channel was calculated explicitly. As to the infimum of \( \eta_{\text{Tr}} \) over a fixed classical channel we conjecture that it coincides with the trace-distance contraction coefficient of the considered classical channel. Our conjecture was been confirmed by numerical simulations for unital channels. A kind of anomaly observed in the behaviour of \( \eta_{\text{Tr}} \) over a fixed classical channel in case of real unital channels.
1 Basic lemmas and notations

The following lemmas will be our main tools, we will use them without mentioning, and we also introduce some notations which will be used in the sequel.

The first four lemmas are elementary propositions in linear algebra. For an \(n \times n\) matrix \(A\) we set \(A_i\) to be the left upper \(i \times i\) submatrix of \(A\), where \(i = 1, \ldots, n\).

**Lemma 1.** The \(n \times n\) self-adjoint matrix \(A\) is positive definite if and only if the inequality \(\det(A_i) > 0\) holds for every \(i = 1, \ldots, n\).

**Lemma 2.** The \(n \times n\) self-adjoint matrix \(A\) is positive definite if and only if \(U^*AU\) is positive definite for all unitary matrix \(U\).

**Lemma 3.** Assume that \(A\) is an \(n \times n\) self-adjoint, positive definite matrix with entries \((a_{ij})_{i,j=1,\ldots,n}\) and the vector \(\alpha\) consists of the first \((n-1)\) elements of the last column, that is \(\alpha = (a_{1,n}, \ldots, a_{n-1,n})\). Then for the matrix \(T = \det(A_{n-1})(A_{n-1})^{-1}\) we have

\[
det(A) = a_{nn} \det(A_{n-1}) - \langle \alpha, T\alpha \rangle.
\]

**Proof.** The statement comes from elementary matrix computation, one should expand \(\det(A)\) by minors, with respect to the last row.

**Lemma 4.** Let \(A\) be an \(n \times n\) invertible matrix and for \(1 \leq k \leq n\) define the complementary minor to \((A^{-1})_k\) as the \((n-k)\)-rowed minor obtained from \(A^{-1}\) by deleting all the rows and columns associated with \(A_k\). If \((A^{-1})_{k+1,\ldots,n}\) denotes the complementary minor to \((A^{-1})_k\), then it is true that

\[
det((A^{-1})_{k+1,\ldots,n}) = \frac{\det(A_k)}{\det(A)}.
\]

Note that the previous lemma is the special case of Jacobi’s theorem [6]. We will apply it in the following form.

**Corollary 1.** If \(A\) is an \(n \times n\) invertible matrix, then for the matrix \(T = \det(A)(A^{-1})\) we have

\[
det((T)_{k+1,\ldots,n}) = \det(A_k) \det(A)^{n-1-k}
\]

for every \(1 \leq k \leq n\).

The next two lemmas are about some elementary properties of the gamma function \(\Gamma\) and the beta integral.

**Lemma 5.** Consider the function \(\Gamma\), which can be defined for \(x \in \mathbb{R}^+\) as

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.
\]

This function has the following properties for every natural number \(n \neq 0\) and real argument \(x \in \mathbb{R}^+\).

\[
\Gamma(n) = (n-1)! \quad \Gamma(1 + x) = x\Gamma(x) \quad \Gamma(1/2) = \sqrt{\pi}
\]

\[
\Gamma(n + 1/2) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \quad \Gamma(n/2) = \frac{(n-2)!!}{2^{n/2}} \sqrt{\pi}
\]
Lemma 6. For parameters \( a, b \in \mathbb{R}^+ \) and \( t \in \mathbb{R}^+ \) the integral equalities

\[
\int_0^t x^a(t-x)^b \, dx = t^{1+a+b} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}
\]

\[
G_{a,b} := \int_0^1 x^a(1-x^2)^b \, dx = \frac{1}{2} \frac{\Gamma(b+1)\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a+2}{2}+b+\frac{3}{2}\right)}
\]

hold.

Proof. These are consequences of the formula below for the beta integral

\[
\int_0^1 x^p(1-x)^q \, dx = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}.
\]

Lemma 7. The surface \( F_{n-1} \) of a unit sphere in an \( n \) dimensional space is

\[
F_{n-1} = \frac{n\pi^\frac{n}{2}}{\Gamma\left(\frac{n}{2}+1\right)}.
\]

Proof. It follows from the well-known formula for the volume of the sphere in \( n \) dimension with radius \( r \)

\[
V_n(r) = \frac{r^n\pi^\frac{n}{2}}{\Gamma\left(\frac{n}{2}+1\right)},
\]

since \( F_{n-1} = \left. \frac{dV_n(r)}{dr} \right|_{r=1} \).

When we integrate on a subset of the Euclidean space we always integrate with respect to the usual Lebesgue measure. The Lebesgue measure on \( \mathbb{R}^n \) will be denoted by \( \lambda_n \). The following lemma is the backbone of our investigations.

Lemma 8. Assume that \( T \) is an \( n \times n \) self-adjoint, positive definite matrix, \( l \in \mathbb{R} \) and \( \mu > 0 \). Let \( L \) be an \( m \)-dimensional subspace of the vector space \( \mathbb{K}^n \) and \( x \) is a fixed vector. Let us denote the orthogonal projection onto the orthogonal complement of the subspace \( T(L) \) by \( P_{M^\perp} \). Set

\[
E^R(T,\mu,L,x) := \{ y \in L \mid \langle x+y, T(x+y) \rangle < \mu \}, \quad T_{ij} \in \mathbb{R};
\]

\[
E^C(T,\mu,L,x) := \{ y \in L \mid \langle x+y, T(x+y) \rangle < \mu \}, \quad T_{ij} \in \mathbb{C};
\]

then

\[
\int_{E^R(T,\mu,L,x)} (\mu - \langle x+y, T(x+y) \rangle)^t d\lambda_m(y) = \frac{F_{m-1}G_{m-1,1}^t}{\sqrt{\det(T|_L)}} (\mu - \|z_0\|^2)^{\frac{n}{2}+t}
\]

and

\[
\int_{E^C(T,\mu,L,x)} (\mu - \langle x+y, T(x+y) \rangle)^t d\lambda_{2m}(y) = \frac{F_{2m-1}G_{2m-1,1}^t}{\det(T|_L)} (\mu - \|z_0\|^2)^{m+t},
\]

where \( T|_L \) is the restriction of \( T \) to the subspace \( L \) and \( z_0 := P_{M^\perp} \sqrt{T}x \).
Proof. We prove the statement for the real case only, the other cases can be proved in the same way. The matrix $T$ is supposed to be positive definite thus there exists a unique self-adjoint positive definite matrix $\sqrt{T}$ for which $T = (\sqrt{T})^2$ holds.

Consider the map $\Phi : L \to \mathbb{R}^n$, $\Phi(y) := \frac{1}{\sqrt{\mu}} \sqrt{T} (x + y)$ and choose an orthonormal basis of the subspace $L$: $e_1, \ldots, e_m$. The corresponding parametrization of Ran($\Phi$) is

$$z(y_1, \ldots, y_m) = \frac{1}{\sqrt{\mu}} \sqrt{T} \left( x + \sum_{i=1}^m y_i e_i \right)$$

and the induced metric on Ran($\Phi$) can be written as

$$g_{ij} = \left\langle \frac{\partial z}{\partial y_i}, \frac{\partial z}{\partial y_j} \right\rangle = \left\langle \frac{1}{\sqrt{\mu}} \sqrt{T} e_i, \frac{1}{\sqrt{\mu}} \sqrt{T} e_j \right\rangle = \frac{\mu}{\mu} \langle e_i, T e_j \rangle$$

hence the inverse Jacobian of this transformation is $\frac{\mu}{\sqrt{\det(T|_L)}}$. We can write

$$\int_{E^2(T, \mu, L, x)} \left( \mu - (x + y, T(x + y)) \right)^t d\lambda_{m}(y) =$$

$$= \frac{\mu^{m+t}}{\sqrt{\det(T|_L)}} \int_{\Phi(E^2(T, \mu, L, x)))} (1 - ||z||^2)^t d\lambda_{m}(z).$$

The set $\Phi(E^2(T, \mu, L, x))$ is the intersection of the affine subspace Ran($\Phi$) and the unit ball of $\mathbb{R}^n$ centered at the origin (Figure 1). Note that, $\Phi(E^2(T, \mu, L, x))$ is non-empty if and only if the distance of Ran($\Phi$) from the origin is less that one: $d^2 := \frac{1}{\mu} ||z_0||^2 = \frac{1}{\mu} ||P_M \cdot \sqrt{T} x||^2 < 1$. Then we compute the integral in spherical coordinates. The integral with respect to the angles gives the surface

\[
\frac{1}{\sqrt{\mu}} z_0 \quad \sqrt{T} \quad 1
\]

Figure 1: The sketch of the region of integration.
of the sphere \( F_{m-1}(1-d^2)^{(m-1)/2} \) and the radial part is
\[
\frac{\mu^{m+l}}{\sqrt{\det(T|_L)}} F_{m-1}(1-d^2)^{m-1+l} \int_0^{\sqrt{1-d^2}} \left( 1 - \frac{r^2}{1-d^2} \right)^l \left( \frac{r}{\sqrt{1-d^2}} \right)^{m-1} dr
\]
We substitute \( u = \frac{r}{\sqrt{1-d^2}} \) and obtain the desired formula.

Remark 1. If \( A \) is an \( n \times n \) positive definite matrix, \( L \subseteq \mathbb{K}^n \) is a subspace and \( M = \sqrt{A^{-1}}(L) \), then \( M^\perp = \sqrt{A}(L^\perp) \) because
\[
M^\perp = (\sqrt{A^{-1}}(L))^\perp = \ker(P_L \sqrt{A^{-1}}) = \sqrt{A}(L^\perp).
\]
Recall that the Pauli matrices \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) together with \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) form an orthogonal basis of the space of \( 2 \times 2 \) self-adjoint matrices.

2 The volume of qubit channels
To determine the volumes of different qubit quantum channels we use the same method which consist of three parts. First we use an unitary transformation to represent channels in a suitable form for further computations. Then we split the parameter space into lower dimensional parts such that the adequate application of the previously mentioned lemmas leads us to the result.

2.1 General qubit channels
A block matrix \( Q \) of the form (1) corresponds to a qubit channel if and only if \( Q_{11}, Q_{22} \in \mathcal{M}_2, Q_{21} = Q_{12}^\ast, \text{Tr} Q_{12} = 0 \) and \( Q \geq 0 \) which means that the space of qubit channels with real and complex entries can be identified with convex subsets of \( \mathbb{R}^7 \) and \( \mathbb{R}^{12} \), respectively. We introduce the following notations for these sets.
\[
Q_R = \{ Q \in \mathbb{R}^{4 \times 4} | Q : \mathcal{M}^R_2 \rightarrow \mathcal{M}^R_2, Q > 0 \}
Q_C = \{ Q \in \mathbb{C}^{4 \times 4} | Q : \mathcal{M}^C_2 \rightarrow \mathcal{M}^C_2, Q > 0 \}
\]
A general element can be parametrized as
\[
Q = \begin{pmatrix}
a & b & c & d \\
b & 1-a & e & -c \\
c & -e & f & g \\
d & g & 1-f \\
\end{pmatrix},
\]
where \( a, f \in [0,1] \) and \( Q > 0 \). Let us choose the unitary matrix
\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
and define the matrix $A$ as

$$A = U^*QU = \begin{pmatrix} a & c & b & d \\ c & f & e & g \\ b & e & 1-a & -c \\ d & g & -c & 1-f \end{pmatrix}$$

which is positive definite if and only if $Q$ is positive definite hence $A$ gives an equivalent parametrization of $\mathcal{Q}_R$ and $\mathcal{Q}_C$.

**Lemma 9.** Let $A$ be an $n \times n$ positive definite matrix, $T = \det(A)A^{-1}$, $L \subseteq \mathbb{K}^n$ a subspace, $x \in L^\perp$ and $M = \sqrt{T}L$. If $\dim(L^\perp) = 1$, then

$$||P_{M^\perp}\sqrt{T}x||^2 = \frac{\det(A)}{\langle x, Ax \rangle}||x||^4.$$ 

**Proof.** According to Remark $[1]$ $M^\perp = \sqrt{A}(L^\perp)$. If $\dim(L^\perp) = 1$, then $\{b_1 = ||\sqrt{A}x||^{-1}\sqrt{A}x\}$ is an orthonormal basis of $M^\perp$ hence $P_{M^\perp} = b_1 \otimes b_1$. We can write

$$||P_{M^\perp}\sqrt{T}x||^2 = \det(A) \left| \langle b_1, \sqrt{A^{-1}}x \rangle \right|^2 = \frac{\det(A)}{\langle x, Ax \rangle}||x||^4$$

which completes the proof. \square

**Theorem 1.** The volume of the space $\mathcal{Q}_R$ with respect to the Lebesgue measure is

$$V(\mathcal{Q}_R) = \frac{4\pi^3}{105},$$

and the distribution of volume over classical channels can be written as

$$V(a,f) = \frac{128}{45}\pi^2 \times \left\{ \begin{array}{ll}
(af)^{3/2} (5(1-a)(1-f) - af) & \text{if } a + f < 1 \\
((1-a)(1-f))^{3/2}(5af - (1-a)(1-f)) & \text{if } a + f \geq 1.
\end{array} \right.$$ 

**Proof.** The volume element corresponding to the parametrization in the real case is $2^4 d\lambda_7$. A matrix of the form $[4]$ with real entries represents a point of $\mathcal{Q}_R$ if and only if $a, f \in [0,1]$ and $\det(A_i) > 0$ for $i = 1, 2, 3, 4$. First we assume that $a$ and $f$ are given.

If $A_3$ is fixed, then by Lemma $[8]$ and Lemma $[9]$ we have

$$V(A_3) = \int_{E^3(T_3,(1-f)\det(A_3),L_3,x_3)} 2^4 d\lambda_2$$

$$= 2^4 F_1 G_{1,0} \left( (1-f) - \frac{c^2}{1-a} \right) \frac{\det(A_3)}{\sqrt{\det(T_3|L_3)}}$$

$$= 2^4 F_1 G_{1,0} \frac{((1-a)(1-f) - c^2)}{(1-a)^{3/2}} \det(A_3)^{1/2},$$

where $L_3 = \text{Span}\{(1,0,0)^T, (0,1,0)^T\}$ and $x_3 = (0,0,-c)^T$. 

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If \( A_2 \) is fixed, then

\[
V(A_2) = \int_{E^2(T_2, (1-a) \det(A_2), \mathbb{R}^2, 0)} V(A_3) \, d\lambda_2
\]

\[
= 2^4 F_1 G_{1,0} \frac{((1-a)(1-f) - c^2)^+}{(1-a)^{3/2}}
\]

\[
\times \int_{E^3(T_2, (1-a) \det(A_2), \mathbb{R}^3, 0)} ((1-a) \det(A_2) - \langle y, T_2 y \rangle)^{1/2} \, d\lambda_2(y)
\]

\[
= 2^4 F_1^2 G_{1,0} G_{1,1/2}^2 \frac{((1-a)(1-f) - c^2)^+ \det(A_2)}{(1-a)(1-f) - c^2}.
\]

Observe that \( af - c^2 > 0 \) implies \( (1-a)(1-f) - c^2 > 0 \) whenever \( a + f \leq 1 \) and \( (1-a)(1-f) - c^2 > 0 \) implies \( af - c^2 > 0 \) if \( a + f \geq 1 \) holds. Since

\[
\frac{2^6}{15} F_1^2 G_{1,0} G_{1,1/2} = \frac{128}{45} \pi^2
\]

the volume element corresponding to a fixed \( a \) and \( f \) can be expressed as

\[
V(a, f) = \frac{128}{45} \pi^2 \times \begin{cases} 
(a f)^{3/2} (5(1-a)(1-f) - af) & \text{if } a + f < 1 \\
((1-a)(1-f))^{3/2}(5af - (1-a)(1-f)) & \text{if } a + f \geq 1
\end{cases}
\]

(see Figure 2) thus for the volume of \( Q_R \) we have

\[
V(\mathbb{Q}_R) = \int [0,1]^2 V(a,f) \, da \, df = \frac{4\pi^3}{105} \approx 1.18119
\]

which completes the proof. \( \square \)

**Theorem 2.** The volume of the space \( Q_C \) with respect to the Lebesgue measure is

\[
V(\mathbb{Q}_C) = \frac{2\pi^5}{4725}.
\]
and the distribution of volume over classical channels can be written as

\[ V(a, f) = \frac{16}{45} \pi^5 \times \begin{cases} a^3 f^3 [10(1-a)(1-f) - af]^2 + \\ 15a f (1-a)(1-f) - 9a^2 f^2 \end{cases} \begin{cases} \text{if } a + f < 1 \\ (1-a)^3 (1-f)^3 [10((1-a)(1-f) - af)^2 + \\ 15a f (1-a)(1-f) - 9(1-a)^2 (1-f)^2] \end{cases} \begin{cases} \text{if } a + f \geq 1. \end{cases} \]

**Proof.** The volume element corresponding to the parametrization (2) in the complex case is \( 2^7 \, \text{d} \lambda_4 \). Similar to the real case, a matrix of the form (4) with complex entries represents a point of \( \mathcal{Q}_C \) if and only if \( a, f \in [0,1] \) and \( \det(A_i) > 0 \) for \( i = 1, 2, 3, 4 \). First we assume that \( a \) and \( f \) are given.

If \( A_3 \) is fixed, then by Lemma 8 and Lemma 9 we have

\[ V(A_3) = \int_{E^C(T_3, (1-f) \det(A_3), L_3, x_3)} 2^7 \, \text{d} \lambda_4 = \frac{2^7 F_3 G_{3,0} (1-f) - |c|^2}{\det(T_3) L_3} \det(A_3)^2 = \frac{2^7 F_3 G_{3,0} ((1-a)(1-f) - |c|^2)^2}{(1-a)^3} \det(A_3), \]

where \( L_3 = \text{Span}\{(1,0,0)^T, (0,1,0)^T\} \) and \( x_3 = (0,0,-a)^T \).

If \( A_2 \) is fixed, then

\[ V(A_2) = \int_{E^C(T_2, (1-a) \det(A_2), C^2, 0)} V(A_3) \, \text{d} \lambda_4 = \frac{2^7 F_3 G_{3,0} (1-a)(1-f) - |c|^2}{(1-a)^3} \times \int_{E^C(T_2, (1-a) \det(A_2), C^2, 0)} (1-a) \det(A_2) - \langle y, T_2 y \rangle \, \text{d} \lambda_4 (y) = \frac{2^7 F_3^2 G_{3,0} G_{3,1} ((1-a)(1-f) - |c|^2)^2}{\det(A_2)^2}. \]

The volume corresponding to a fixed \( a \) and \( f \) can be expressed as

\[ V(a, f) = \frac{2^7}{60} F_1 F_2^2 G_{3,0} G_{3,1} \times \begin{cases} a^4 f^3 [10((1-a)(1-f) - af)^2 + \\ 15a f (1-a)(1-f) - 9a^2 f^2] \end{cases} \begin{cases} \text{if } a + f < 1 \\ (1-a)^3 (1-f)^3 [10((1-a)(1-f) - af)^2 + \\ 15a f (1-a)(1-f) - 9(1-a)^2 (1-f)^2] \end{cases} \begin{cases} \text{if } a + f \geq 1. \end{cases} \]

(see Figure 3) thus for the volume of \( \mathcal{Q}_C \) we have

\[ V(\mathcal{Q}_C) = \int_{[0,1]^2} V(a, f) \, \text{d} a \, \text{d} f = \frac{2 \pi^5}{4725} \approx 0.129532. \]
2.2 Unital qubit channels

Identity preserving requires that $Q_{11} = Q_{22} = I$ in the Choi representation which means that the space of unital qubit channels with real and complex entries can be identified with convex subsets of $\mathbb{R}^5$ and $\mathbb{R}^9$, respectively. We introduce the following notations for these sets.

$$Q^1_{\mathbb{R}} = \{ Q \in \mathbb{R}^{4 \times 4} | Q : M_2^{\mathbb{R}} \rightarrow M_2^{\mathbb{R}}, Q > 0, Q(I) = I \}$$
$$Q^1_{\mathbb{C}} = \{ Q \in \mathbb{C}^{4 \times 4} | Q : M_2^{\mathbb{C}} \rightarrow M_2^{\mathbb{C}}, Q > 0, Q(I) = I \}$$

A general element can be parametrized as

$$Q = \begin{pmatrix}
    a & b & c & d \\
    \bar{b} & 1 - a & c & -c \\
    \bar{c} & \bar{c} & 1 - a & -b \\
    \bar{d} & -\bar{c} & -\bar{b} & a
\end{pmatrix}, \quad (5)$$

where $a \in [0, 1]$ and $Q > 0$. Let us choose the unitary matrix

$$U = \begin{pmatrix}
    0 & 0 & 1 & 0 \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}$$

and define the matrix $A$ as

$$A = U^*QU = \begin{pmatrix}
    1 - a & e & b & -c \\
    \bar{e} & 1 - a & c & -b \\
    \bar{b} & \bar{c} & a & d \\
    -\bar{c} & -\bar{b} & \bar{d} & a
\end{pmatrix} \quad (6)$$

which is positive definite if and only if $Q$ is positive definite.
Lemma 10. Let us denote the left upper $k \times k$ submatrix of $A$ by $A_k$. If $L_3 = \text{Span}\{(0,0,1)^T\}$ and $M = \sqrt{A_3^{-1}(L_3)}$, then $\sqrt{A_3^{-1}P_{M^\perp}} A_3^{-1} = \begin{pmatrix} A_2^{-1} & 0 \\ 0^T & 0 \end{pmatrix}$.

Proof. According to Remark 1, $M^\perp = \sqrt{A(L^\perp)}$. If $u_1$ and $u_2$ are vectors in $L_3^\perp$ for which $\langle u_1, A_3 u_j \rangle = \delta_{ij}$ holds, then $\{ \sqrt{A_3} u_1, \sqrt{A_3} u_2 \}$ is an orthonormal basis of $M^\perp$ hence

$$P_{M^\perp} = \sqrt{A_3} u_1 \otimes \sqrt{A_3} u_1 + \sqrt{A_3} u_2 \otimes \sqrt{A_3} u_2 = \sqrt{A_3} (u_1 \otimes u_1 + u_2 \otimes u_2) \sqrt{A_3}$$

which implies that $\sqrt{A_3^{-1}P_{M^\perp}} A_3^{-1} = u_1 \otimes u_1 + u_2 \otimes u_2$. Let us define the matrix $B = \begin{pmatrix} A_2 & 0 \\ 0^T & 1 \end{pmatrix}$. It is easy to see that $\langle x, By \rangle = \langle x, A_3 y \rangle$ holds for each $x, y \in L_3^\perp$. We can choose $u_i = \sqrt{B^{-1}} e_i$, $i = 1, 2$, where $(e_i)_j = \delta_{ij}$, $i, j = 1, 2$ is the standard basis of $L_3^\perp$. We can write $u_1 \otimes u_1 + u_2 \otimes u_2 = \sqrt{B^{-1}} (e_1 \otimes e_1 + e_2 \otimes e_2) \sqrt{B^{-1}} = \begin{pmatrix} A_2^{-1} & 0 \\ 0^T & 0 \end{pmatrix}$ which completes the proof.

Theorem 3. The volume of the space $Q_2^1$ with respect to the Lebesgue measure is

$$V(Q_2^1) = \frac{4\pi^2}{15},$$

and the distribution of volume over classical channels can be written as

$$V(a) = 8\pi^2 a^2 (1 - a)^2.$$

Proof. The volume element corresponding to the parametrization (5) in the real case is $2^4 \, d\lambda_5$. A matrix of the form (5) with real entries represents a point of $Q_2^3$ if and only if $a \in [0,1]$ and $\det(A_i) > 0$ for $i = 1, 2, 3, 4$. First we assume that $a$ is given.

If $A_3$ is fixed, then by Lemma 9 and Lemma 10 we have

$$V(A_3) = \int_{B^5(T_3, a \det(A_3), L_3, x_3)} 2^4 \, d\lambda_1 = \frac{2^4 F_0}{\sqrt{\det(A_2)}} \left( a - \left< x_3, \begin{pmatrix} A_2^{-1} & 0 \\ 0^T & 0 \end{pmatrix} x_3 \right> \right)^{1/2} \det(A_3)^{1/2},$$

where $L_3 = \text{Span}\{(0,0,1)^T\}$ and $x_3 = (-c, -b, 0)^T$.

Observe that $\left< x_3, \begin{pmatrix} A_2^{-1} & 0 \\ 0^T & 0 \end{pmatrix} x_3 \right> = \langle y, \sigma_1 A_2^{-1} \sigma_1 y \rangle = \langle y, A_2^{-1} y \rangle$, where $y = (b, c)^T$ because $\sigma_1 A_2^{-1} \sigma_1 = A_2^{-1}$ and $A_2^{-1}$ is a matrix with real entries.
If $A_2$ is fixed, then
\[ V(A_2) = \int_{E^2(T_2, a \det(A_2), \mathbb{R}^2, 0)} V(A_3) \, d\lambda_2 \]
\[ = \frac{2^4 F_0}{\sqrt{\det(A_2)}} \int_{E^2(T_2, a \det(A_2), \mathbb{R}^2, 0)} (a - \langle y, A_2^{-1} y \rangle)^{1/2} \det(A_3)^{1/2} \, d\lambda_2(y) \]
\[ = \frac{2^4 F_0}{\det(A_2)} \int_{E^2(T_2, a \det(A_2), \mathbb{R}^2, 0)} a \det(A_2) - \langle y, T_2 y \rangle \, d\lambda_2(y) \]
\[ = 2^4 F_0 F_1 G_{1,1} a^2 (\det(A_2))^{1/2}. \]

The volume corresponding to a fixed $a \in [0, 1]$ can be written as
\[ V(a) = 2^4 F_0 F_1 G_{1,1} a^2 \int_{-1}^{1-a} \sqrt{(1-a)^2 - c^2} \, d\lambda_1(c) \]
\[ = 2^4 F_0 F_1 G_{1,1} a^2 (1-a)^2 = 8\pi^2 a^2 (1-a)^2 \]
(see Figure 4) thus the volume of $Q^1_3$ is
\[ V(Q^1_3) = 8\pi^2 \int_0^1 a^2 (1-a)^2 \, da = \frac{4\pi^2}{15} \approx 2.63189 \]
which completes the proof.

**Theorem 4.** The volume of the space $Q^1_3$ with respect to the Lebesgue measure is
\[ V(Q^1_3) = \frac{2\pi^4}{315}, \]
and the distribution of volume over classical channels can be written as
\[ V(a) = 2^2 \pi^4 a^4 (1-a)^4. \]

**Proof.** The volume element corresponding to the parametrization in the complex case is $2^7 \, d\lambda_3$. Similar to the real case, a matrix of the form \[ \begin{pmatrix} a & x_3 \\ 0 & 0 \end{pmatrix} \]
with complex entries represents a point of $Q^2_3$ if and only if $a \in [0, 1]$ and $\det(A_i) > 0$ for $i = 1, 2, 3, 4$. First we assume that $a$ is given.

If $A_3$ is fixed, then by Lemma 8 and Lemma 10 we have
\[ V(A_3) = \int_{E^2(T_3, a \det(A_3), L_3, x_3)} 2^7 \, d\lambda_2 = \]
\[ = \frac{2^6 F_1}{\det(A_2)} \left( a - \begin{pmatrix} x_3, \begin{pmatrix} A_3^{-1} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \right)_+ \det(A_3), \]
where $L_3 = \text{Span}\{(0, 0, 1)^T\}$ and $x_3 = (-c, -b, 0)^T$. 

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Similar to the real case \( \langle x_3, \begin{pmatrix} A_2^{-1} & 0 \\ 0 & 0 \end{pmatrix} x_3 \rangle = \langle y, \sigma_1 A_2^{-1} \sigma_1 y \rangle \), where \( y = (b, c)^T \), but \( \langle y, \sigma_1 A_2^{-1} \sigma_1 y \rangle \neq \langle y, A_2^{-1} y \rangle \) because \( A_2^{-1} \neq A_2^{\top} \) in the complex case.

If \( A_2 \) is fixed, then

\[
V(A_2) = \int_{E^2(\mathbb{C})} V(A_3) \, d\lambda_4 = 2^6 F_1 \int_{E^2(\mathbb{C})} (a - \langle y, \sigma_1 A_2^{-1} \sigma_1 y \rangle)_+ (a - \langle y, A_2^{-1} y \rangle) \, d\lambda_4(y).
\]

Let us substitute \( y = \sqrt{a} \sqrt{A_2} z \) and obtain

\[
V(A_2) = 2^6 F_1 a^4 \det(A_2) \int_{\{z : ||z|| < 1\}} (1 - \langle z, Bz \rangle)_+ (1 - ||z||^2) \, d\lambda_4(z),
\]

where \( B = \sqrt{A_2} \sigma_1 A_2^{-1} \sigma_1 \sqrt{A_2} \) is a self-adjoint matrix that is unitary equivalent to a diagonal matrix and \( \det(B) = 1 \). As a unitary coordinate transformation does not change the value of the previous integral hence

\[
V(A_2) = 2^6 F_1 a^4 \det(A_2)
\times \int_{\{z : ||z|| < 1\}} \left( 1 - \lambda |z_1|^2 - \frac{1}{\lambda} |z_2|^2 \right)_+ (1 - |z_1|^2 - |z_2|^2) \, d\lambda_4(z),
\]

where \( \lambda \) denotes the largest eigenvalue of \( B \). Then we compute the integral above in the Descartes product of two polar coordinate systems. The integral with respect to the angles gives \( F^2_1 \) and the radial part can be written as

\[
V(A_2) = 2^6 F_1 a^4 \det(A_2) \int_{R^2} \left( 1 - \lambda r_1^2 - \frac{1}{\lambda} r_2^2 \right)_+ (1 - r_1^2 - r_2^2)_+ r_1 r_2 \, dr_1 \, dr_2
\]

\[
= \frac{2^5 \pi^3}{3} a^4 \det(A_2) \frac{3\lambda - 1}{\lambda(1 + \lambda)}. \]

By elementary matrix computation, we get

\[
\lambda = 1 + \frac{2\Im(e)^2}{\det(A_2)} + \sqrt{\left( 1 + \frac{2\Im(e)^2}{\det(A_2)} \right)^2 - 1}
\]

thus

\[
V(A_2) = \frac{2^5 \pi^3}{3} a^4 \det(A_2) \left( 1 + \frac{2\Im(e)^2}{\det(A_2)} \left( \frac{\Im(e)^2}{\det(A_2) + \Im(e)^2 - 1} \right) \right).
\]
The volume corresponding to a fixed $a \in [0,1]$ can be written as
\[
V(a) = \int_{|e|^2 \leq (1-a)^2} V(A_2) \, d\lambda_2(e)
\]
\[
= \frac{2^5 \pi^3}{3} a^4 (1 - a)^4 
\times \int_0^1 \int_0^{2\pi} \left( 1 + \frac{2r^2 \sin^2 \phi}{1 - r^2} \left( \sqrt{\frac{r^2 \sin^2 \phi}{1 - r^2 \cos^2 \phi} - 1} \right) \right) (1 - r^2) r \, d\phi \, dr
\]
\[
= 2^2 \pi^4 a^4 (1 - a)^4
\]
(see Figure 4) thus the volume of $Q^1_{\mathbb{C}}$ is
\[
V(Q^1_{\mathbb{C}}) = 2^2 \pi^4 \int_0^1 a^4 (1 - a)^4 \, da = \frac{2\pi^4}{315} \approx 0.61847
\]
which completes the proof.

Figure 4: Graph of $V(a)$ for $Q^1_\mathbb{R}$ (solid) and $Q^1_{\mathbb{C}}$ (dashed).

One might think about the generalization of the presented results, although in a more general setting several complications occur. For example, in the case of unital qubit channels one should integrate over the Birkhoff polytope, which would cause difficulties since even the volume of the polytope is still unknown.

3 The trace-norm contraction coefficient

The way of integration presented in the previous sections suggests an efficient method for generating uniformly distributed points in the space of qubit channels. This method makes the numerical study of different channel related quantities possible. As an example, the distribution of $\eta^{\text{Tr}}$ is investigated numerically by Monte-Carlo simulations over different kind of quantum channels.
3.1 Monte-Carlo simulations

Simulations were implemented in MATLAB 2014a and random vectors within a sphere were generated, as described by Knuth [8].

Algorithm 1. The next scheme describes for the case of \( \mathcal{Q}_R \) how the algorithm works, where \( x \sim \mathcal{U}(B) \) denotes that \( x \) is uniformly distributed on the set \( B \). The other cases (\( \mathcal{Q}_C \), \( \mathcal{Q}_R^k \) and \( \mathcal{Q}_C^k \)) can be treated in a similar way.

1. **Step 1:** Generate \( a, f \sim \mathcal{U}([0,1]) \) independently.
2. **Step 2:** Generate \( x_1 \sim \mathcal{U}(-\sqrt{a}J, \sqrt{a}J) \) and set \( A_2 = \begin{pmatrix} a & x_1 \\ x_1 & f \end{pmatrix} \).
3. **Step 3:** Generate \( y_2 \sim \mathcal{U}(\{r \in \mathbb{R}^2 : ||r|| \leq \sqrt{3} \}) \) and set \( A_3 = \begin{pmatrix} x_2^T & 1-a \\ x_2 & 1 \end{pmatrix} \), where \( x_2 = \sqrt{2}y_2 \).
4. **Step 4:** Compute the projection \( P \) onto the subspace \( \text{Span}(\{\sqrt{3}e_3\}) \) and set \( z = -x_1(1-f)^{-1/2}P\sqrt{A_3^{-1}}e_3 \).
5. **Step 5:** If \( ||z|| > 1 \), then goto Step 2.
6. **Step 6:** Generate \( y_3 \sim \mathcal{U}(\{r \in \mathbb{R}^2 : ||r|| \leq \sqrt{1-||z||^2} \}) \) and set \( A = \begin{pmatrix} x_3^T & 1-f \\ x_3 & 1 \end{pmatrix} \), where \( x_3 = \sqrt{1-f}\sqrt{A_3}(e_1, e_2) + z \).
7. **Step 7:** Apply the transform \( Q = UAU^\ast \), where \( U \) is given by \( [3] \).

The first step is omitted when our goal is to generate a random qubit channel over the classical channel parametrized by \( a \) and \( f \). Step 5 is needed just because up to this point it was not guaranteed that \( c^2 \leq (1-a)(1-f) \).

Any \( \rho \in \mathcal{M}_2 \) can be represented in the Pauli bases as \( \rho = \frac{1}{3}(I + x \cdot \sigma) \) by a unique \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \) with \( ||x|| \leq 1 \), where \( x \cdot \sigma = \sum_{j=1}^{3} x_j \sigma_j \). A qubit channel \( Q : \mathcal{M}_2 \rightarrow \mathcal{M}_2 \) is represented in the Pauli bases as

\[
Q \left( \frac{1}{2} (I + x \cdot \sigma) \right) = \frac{1}{2} (I + (v + Tx) \cdot \sigma),
\]

where \( v \in \mathbb{R}^3 \) and \( T \) is a \( 3 \times 3 \) real matrix. This representation is suitable for calculating the trace-norm contraction coefficient because \( \eta^{\text{Tr}}(Q) \) can be expressed as \( \eta^{\text{Tr}}(Q) = ||T||_\infty \), where \( ||.||_\infty \) denotes the Schatten-\( \infty \) norm [10]. It means that the trace distance contraction coefficient of a qubit channel \( Q \) given by [2] is the largest singular value of the following matrix.

\[
T = \begin{pmatrix}
\Re(d + e) & \Im(d + e) & \Re(b - g) \\
-\Im(d + e) & \Re(d - e) & -\Im(b - g) \\
2\Re(e) & 2\Im(e) & a - f
\end{pmatrix}
\] (7)

3.2 Distribution of \( \eta^{\text{Tr}} \) on the whole space

Empirical cumulative distribution functions (CDF) of \( \eta^{\text{Tr}} \) on the space of qubit channels are presented in Figure [3] for \( \mathcal{Q}_R \), \( \mathcal{Q}_C \), \( \mathcal{Q}_R^k \) and \( \mathcal{Q}_C^k \). In each case,
$10^4$ random qubit channels were generated independently and confidence band corresponding to the confidence level 99.995\% ($\alpha = 5 \times 10^{-5}$) was calculated by Greenwood’s formula \[9\].

\[\begin{align*}
\begin{array}{c}
\text{(a) Empirical CDF of } \eta^{\text{Tr}} \text{ on } Q_1^R.
\end{array}
\end{align*}\]

\[\begin{align*}
\begin{array}{c}
\text{(b) Empirical CDF of } \eta^{\text{Tr}} \text{ on } Q_C.
\end{array}
\end{align*}\]

Figure 5: Empirical CDF of $\eta^{\text{Tr}}$ and confidence band ($n = 10^4$, $\alpha = 5 \times 10^{-5}$).

### 3.3 Distribution $\eta^{\text{Tr}}$ over classical channels

Three natural questions arise about the distribution of trace-distance contraction coefficient over a fixed classical channel:

i. What is the supremum of $\eta^{\text{Tr}}$ over a fixed classical channel?

ii. What is the infimum of $\eta^{\text{Tr}}$ over a fixed classical channel?

iii. What is the typical value (the mode) of $\eta^{\text{Tr}}$ over a fixed classical channel?

The set of qubit channels over the classical channel $\begin{pmatrix} 1-a & a \\ 1-f & f \end{pmatrix}$ with respect to the parametrization \[9\] is denoted by $Q_R(a, f)$, $Q_C(a, f)$, $Q_R^1(a)$ and $Q_C^1(a)$. The next Theorem answers the first question.
**Theorem 5.** Let \( a, f \in [0,1] \) be arbitrary real numbers. For all \( x \in (|a - f|, \sqrt{(1-a)f + a(1-f)}) \) there exists a qubit channel \( Q \in \mathcal{Q}_R(a, f) \subset \mathcal{Q}_C(a, f) \) for which \( \eta^{\text{Tr}}(Q) = x \).

**Proof.** Let \( a, f \in [0,1] \) and \( x \in (|a - f|, \sqrt{(1-a)f + a(1-f)}) \) be arbitrary. Consider the following qubit channel

\[
Q = \begin{pmatrix}
a & 0 & 0 & d \\
0 & 1-a & e & 0 \\
0 & e & f & 0 \\
d & 0 & 0 & 1-f
\end{pmatrix},
\]

where \( d, e \in \mathbb{R} \). In order to guarantee the positivity of the matrix above, the following constraints must be held.

\[
e^2 \leq (1-a)f \\
d^2 \leq a(1-f)
\]

According to \( \left[7\right] \), \( \eta^{\text{Tr}}(Q) = \max(|d+e|, |d-e|, |a-f|) \), where \( |d \pm e| \leq |d| + |e| \leq \sqrt{(1-a)f} + \sqrt{a(1-f)} \) which completes the proof.

**Corollary 2.** For unital channels \( f = 1-a \) hence the supremum of \( \eta^{\text{Tr}}(Q) \) on the set \( \mathcal{Q}_R^1 \subset \mathcal{Q}_C^1 \) is equal to \( \sqrt{(1-a)^2 + a^2} = 1 \) which means that the theoretical upper bound of \( \eta \) can be reached over any classical channel.

**Conjecture 1.** We conjecture that \( \inf\{\eta(Q) : Q \in \mathcal{Q}_C(a, f)\} = |a - f| \) which is equal to the trace-distance contraction coefficient of the underlying classical channel.

It seems that there is no chance to give explicit formula for the mode of \( \eta^{\text{Tr}} \) over a fixed classical channel. Instead of this, Monte-Carlo simulations were done for the case of unital channels. The interval \([0,1] \) was divided into 100 equidistant parts. Infimum, expectation and mode was estimated from a sample of size \( n = 1000 \) in each point. Infimum of \( \eta^{\text{Tr}} \) in \( a \in [0,1] \) was estimated by the following formula.

\[
\inf\{\eta^{\text{Tr}}(Q) : Q \in \mathcal{Q}_C^1(a)\} \approx \min(|2a - 1|, \text{smallest element in the sample})
\]

Smoothed density histogram was applied to estimate the mode. Confidence band corresponding to the confidence level \( 99.995\% \) \( (\alpha = 5 \times 10^{-5}) \) was calculated for the expected value. The estimated infimum of \( \eta^{\text{Tr}} \) is displayed by dotted line in Figure 6. We can see that the estimated infimum coincides with the trace-distance contraction coefficient of the underlying classical channel which confirms Conjecture 1 for unital channels. The mode shows irregular behaviour in case of real unital channels (Figure 6a). Small deviations of mode from infimum can be observed near \( a \approx 0.1 \) and \( a \approx 0.9 \) and the distribution of \( \eta^{\text{Tr}} \) changes dramatically near \( a \approx 0.33 \) and \( a \approx 0.67 \). We can see in Figure 6b that qubit channels over the complex field are condensed near the extremal \( \eta^{\text{Tr}} = 1 \) isosurface.
Figure 6: Minimal value (dotted) of $\eta_{Tr}$, mode of $\eta_{Tr}$ (thick), expectation of $\eta_{Tr}$ (dashed) and confidence band (solid) corresponding to the expectation ($n = 1000$, $\alpha = 5 \times 10^{-5}$).

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