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Injective Modules

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Abstract. In this article, we introduce the concept of $(\rho, m)$-N-injectivity (where $\rho$ is a preradical, $m$ is a positive integer and $N$ is an $R$-module) as a generalization of both $\rho$-injectivity and $\rho$-N-injectivity. This concept unifies several definitions on generalizations of $N$-injectivity, such as nearly $P$-N-injective modules and special $P$-N-injective modules. Many characterizations and properties of $(\rho, m)$-N-injectivity are given. The results of this work unify and extend many results in the literature.

1. Introduction

Throughout this work, $R$ stands a commutative ring with identity element 1 and a module means a unitary left $R$-modules. The class of all $R$-module will be denoted by $R$-Mod and the symbol $\rho$ means a preradical on $R$-Mod (A preradical $\rho$ is defined to be a subfunctor of the identity functor of $R$-Mod). For an $R$-module $M$, the notations $L(M), J(M), E(M)$ and $S = \text{End}_R(M)$ will respectively stand for the prime radical of $M$, the Jacobson radical of $M$, the injective envelope of $M$ and the endomorphism ring of $M$. The notation $\text{Hom}_R(N, M)$ denoted to the set of all $R$-homomorphism from $R$-module $N$ into $R$-module $M$. An $R$-module $M$ is said to be $\rho$-N-injective, if for any $R$-monomorphism $\alpha: A \rightarrow N$ and any $R$-homomorphism $\beta: A \rightarrow M$, there exists an $R$-homomorphism $\gamma: N \rightarrow M$ with $\beta(a) = (\gamma \circ \alpha)(a) \in \rho(M)$, for all $a \in A$. An $R$-module $M$ is said to be $\rho$-injective, if $M$ is $\rho$-N-injective, for all $R$-modules $N$ [9]. A module $M$ is said to be nearly $P$-N-injective if for any $R$-homomorphism $\alpha: A \rightarrow M$ (where $A$ is any cyclic submodule of $N$), there is an $R$-homomorphism $\beta: N \rightarrow M$ such that $\beta(a) - \alpha(a) \in J(M)$, for each $a \in A$. An $R$-module $N$ is called nearly PQ-injective, if $N$ is nearly $P$-N-injective [8]. An $R$-module $M$ is said to be pointwise nearly-injective, if $M$ is nearly $P$-N-injective, for every $R$-module $N$ [2]. Also, in [1] M. S. Abbas and Sh. N. Abd-Alridha introduced the concept of special $P$-N-injective modules. A module $M$ is said to be

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special \(P\)-\(N\)-injective if for any \(R\)-homomorphism \(f : A \to M\), where \(A\) is a cyclic submodule of \(N\), there is an \(R\)-homomorphism \(g : N \to M\) such that \(g(a) f(a) \in L(M)\), for each \(a \in A\). An \(R\)-module \(N\) called special \(PQ\)-injective, if \(N\) is special \(P\)-\(N\)-injective \([1]\). For a submodule \(N\) of an \(R\)-module \(M\) and \(a \in M\), \([N ; a] = \{r \in R \mid ra \in N\}\). For an \(R\)-module \(M\) and \(a \in M\). A submodule \(N\) of an \(R\)-module \(M\) is called essential and denoted by \(N \leq e\) \(M\) if every non zero submodule of \(M\) has nonzero intersection with \(N\).

2. \((\rho, m)\)-\(N\)-Injective Modules

Definition 2.1. Let \(\rho\) be any preradical, let \(m \in \mathbb{Z}^+\) and let \(M, N\) be two \(R\)-modules. An \(R\)-module \(M\) is said to be \((\rho, m)\)-\(N\)-injective, if for any \(m\)-generated submodule \(A\) of \(N\) and any \(R\)-homomorphism \(f : A \to M\), there exists an \(R\)-homomorphism \(g : N \to M\) with \(g(a) f(a) \in \rho(M)\), for any \(a \in A\). If \(M\) is \((\rho, m)\)-\(N\)-injective, for all \(R\)-module \(N\), then \(M\) is called \((\rho, m)\)-injective. If \(M\) is \((\rho, m)\)-\(M\)-injective, then \(M\) is called \((\rho, m)\)-quasi-injective.

Examples and Remarks 2.2.

(1) Nearly \(PQ\)-injectivity is a special case of \((\rho, m)\)-quasi-injectivity, if we take \(\rho = J\) and \(m = 1\).
(2) Special \(PQ\)-injectivity is a special case of \((\rho, m)\)-quasi-injectivity, if we take \(\rho = L\) and \(m = 1\).
(3) Every \(\rho\)-quasi-injective (and hence \(\rho\)-injective) \(R\)-modules are \((\rho, m)\)-quasi-injective, for any preradical \(\rho\) and \(m \in \mathbb{Z}^+\).
(4) The concept of \((\rho, m)\)-quasi-injective modules is a proper generalization of \(\rho\)-quasi-injectivity, for some preradical \(\rho\). For example, let \(R\) be the ring of all continuous functions from the set of rational numbers \(\mathbb{Q}\) to \(\mathbb{Z}_2\). By \([8, \text{p.2}]\) we have that \(R\) is nearly \(PQ\)-injective \(R\)-module, but it is not nearly quasi-injective \(R\)-module and hence \(R\) is a \((\rho, m)\)-quasi-injective \(R\)-module, but it is not \(\rho\)-quasi-injective \(R\)-module, where \(\rho = J\) and \(m = 1\).
(5) \((\rho, m)\)-\(N\)-injectivity is an algebraic property.

Let \(M\) be an \(R\)-module and let \(m \in \mathbb{Z}^+\). Let \(\bar{x} = (x_1, x_2, \ldots, x_m) \in M^m\). The annihilator of \(\bar{x}\) in \(R^m\) is denoted by \(\text{ann}_{R^m}(\bar{x})\) and defined as follows: \(\text{ann}_{R^m}(\bar{x}) = \{r = (r_1, r_2, \ldots, r_m) \in R^m \mid \sum_{i=1}^m r_i x_i = 0\}\).

The following theorem gives many characterizations of \((\rho, m)\)-\(N\)-injective modules.

Theorem 2.3. Let \(M\) and \(N\) be two \(R\)-modules, \(m \in \mathbb{Z}^+\) and \(S = \text{End}_R(M)\). Then the following statements are equivalent:
(1) \(M\) is \((\rho, m)\)-\(N\)-injective;
(2) If \(\bar{x} = (x_1, x_2, \ldots, x_m) \in M^m\) and \(\bar{y} = (y_1, y_2, \ldots, y_m) \in N^m\) such that \(\text{ann}_{R^m}(\bar{y}) \subseteq \text{ann}_{R^m}(\bar{x})\), then there is an \(R\)-homomorphism \(g : N \to M\) with \(g(y_i) - x_i \in \rho(M)\), for all \(i = 1, 2, \ldots, m\).
(3) For each \(\bar{x} = (x_1, x_2, \ldots, x_m) \in M^m\), \(\bar{y} = (y_1, y_2, \ldots, y_m) \in N^m\) such that \(\text{ann}_{R^m}(\bar{y}) \subseteq \text{ann}_{R^m}(\bar{x})\) and for each \(h \in S\), there exists an \(R\)-homomorphism \(\alpha \in \text{Hom}_R(N, M)\) such that \(\alpha(y_i) - \alpha(x_i) \in \rho(M)\), for all \(i = 1, 2, \ldots, m\).
For each \( R\)-homomorphism \( f: A \to M \) where \( A \) is any submodule of \( N \) and for each set \( \{a_1, a_2, ..., a_m\} \subseteq A \), there exists an \( R\)-homomorphism \( g: N \to M \) such that \( g(a_i) - f(a_i) \in \rho(M) \), for all \( i = 1,2, ..., m \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( \vec{x} = (x_1, x_2, ..., x_m) \in M^m \) and \( \vec{y} = (y_1, y_2, ..., y_m) \in N^m \) such that \( \text{ann}_R^m(\vec{y}) \subseteq \text{ann}_R^m(\vec{x}) \). Define \( f: (y_1, y_2, ..., y_m) \to M \) by \( f(\sum_{i=1}^m r_i y_i) = \sum_{i=1}^m r_i x_i \), for all \( r_i \in R \) and \( i = 1,2, ..., m \). Clearly, \( f \) is a well-defined \( R\)-homomorphism. By \((\rho, m)\)-injectivity of \( M \), there exists an \( R\)-homomorphism \( g: N \to M \) such that \( g(z) - f(z) \in \rho(M) \), for all \( z \in (y_1, y_2, ..., y_m) \). Thus \( g(y_i) - f(y_i) \in \rho(M) \) and hence \( g(y_i) - x_i \in \rho(M) \), for all \( i = 1,2, ..., m \).

(2) \( \Rightarrow \) (3). Let \( \vec{x} = (x_1, x_2, ..., x_m) \in M^m \) and \( \vec{y} = (y_1, y_2, ..., y_m) \in N^m \) such that \( \text{ann}_R^m(\vec{y}) \subseteq \text{ann}_R^m(\vec{x}) \). By hypothesis, \( g(y_i) - x_i \in \rho(M) \), for all \( i = 1,2, ..., m \), for some an \( R\)-homomorphism \( g: N \to M \). Put \( (y_i) - x_i = t_i \), where \( t_i \in \rho(M) \), for all \( i = 1,2, ..., m \). Let \( h \in S \); thus \( h(x_i) = h(g(y_i)) - t_i = (h \circ g)(y_i) - t_i \). Put \( \alpha = h \circ g \). Since \( \alpha \in \text{Hom}_R(N, M) \) and \( h(t_i) \in \rho(M) \) it follows that there is an \( R\)-homomorphism \( \alpha \in \text{Hom}_R(N, M) \) such that for all \( i = 1,2, ..., m \), we have \( h(x_i) = \alpha(\gamma_i) \in \rho(M) \).

(3) \( \Rightarrow \) (4). Let \( f: A \to M \) be a \( R\)-homomorphism, where \( A \) is any submodule of \( N \), and let \( \{a_1, a_2, ..., a_m\} \subseteq A \). Let \( \vec{y} = (a_1, a_2, ..., a_m) \in N^m \). Put \( x_i = f(a_i) \), for all \( i = 1,2, ..., m \). Now we will prove that \( \text{ann}_R^m(\vec{y}) \subseteq \text{ann}_R^m(\vec{x}) \), where \( \vec{x} = (x_1, x_2, ..., x_m) \in M^m \). Let \( \vec{r} = (r_1, r_2, ..., r_m) \in \text{ann}_R^m(\vec{y}) \), thus \( \sum_{i=1}^m r_i a_i = 0 \). Since \( f(\sum_{i=1}^m r_i a_i) = f(0) = 0 \) and \( f(\sum_{i=1}^m r_i a_i) = \sum_{i=1}^m r_i f(a_i) = \sum_{i=1}^m r_i x_i \), \( \sum_{i=1}^m r_i x_i = 0 \) and hence \( \vec{r} \in \text{ann}_R^m(\vec{x}) \). Thus \( \text{ann}_R^m(\vec{y}) \subseteq \text{ann}_R^m(\vec{x}) \). Let \( I_M: M \to M \) be the identity an homomorphism. Since \( I_M \in S \) it follows from (3) that there exists an \( R\)-homomorphism \( g \in \text{Hom}_R(N, M) \) such that \( I_M(x_i) - g(a_i) \in \rho(M) \), for all \( i = 1,2, ..., m \). Thus \( g(a_i) - x_i \in \rho(M) \) and hence for all \( i = 1,2, ..., m \), we have \( g(a_i) - f(a_i) \in \rho(M) \).

(4) \( \Rightarrow \) (1). Let \( f: A \to M \) be a left \( R\)-homomorphism, where \( A \) is a \( m\)-generated submodule of \( N \). Let \( A = (a_1, a_2, ..., a_m) \). Since \( \{a_1, a_2, ..., a_m\} \subseteq A \), there is an \( R\)-homomorphism \( g: N \to M \) with \( g(a_i) - f(a_i) \in \rho(M) \), for all \( i = 1,2, ..., m \). For each \( x = \sum_{i=1}^m r_i a_i \in A \), where \( r_i \in R \), then \( g(x) - f(x) = r_i (\sum_{i=1}^m g(a_i) - \sum_{i=1}^m f(a_i)) \in \rho(M) \), for all \( i = 1,2, ..., m \). Therefore, \( M \) is a \((\rho, m)\)-injective \( R\)-module.

**Corollary 2.4.** [8, Theorem 1.3, p.3] Let \( M \) and \( N \) be two \( R\)-modules and \( S = \text{End}_R(M) \), then the following conditions are equivalent:

(1) \( M \) is a nearly \( p\)-\( N\)-injective \( R\)-module.

(2) If \( a \in M \) and \( b \in N \) with \( \text{ann}_R(b) \subseteq \text{ann}_R(a) \), then there exists an \( R\)-homomorphism \( g: N \to M \) such that \( g(b) - a \in J(M) \).

(3) If \( a \in M \) and \( b \in N \) with \( \text{ann}_R(b) \subseteq \text{ann}_R(a) \), then for any \( h \in S \), there is an \( R\)-homomorphism \( \alpha \in \text{Hom}_R(N, M) \) with \( h(b) - \alpha(a) \in J(M) \).

(4) For each \( R\)-homomorphism \( f: A \to M \) (where \( A \) is any submodule of \( N \) ) and each \( a \in A \), there exists an \( R\)-homomorphism \( g: N \to M \) such that \( g(a) - f(a) \in J(M) \).

**Proof.** By taking \( m = 1 \) and \( \rho = J \) (the Jacobson radical functor) and applying Theorem 2.3.

**Corollary 2.5.** [1, Theorem 10, p.99] Let \( M \) and \( N \) be two \( R\)-modules and \( S = \text{End}_R(M) \). Then the following statements are equivalent:

(1) \( M \) is a special \( P\)-\( N\)-injective \( R\)-module.

(2) If \( a \in M \) and \( b \in N \) with \( \text{ann}_R(b) \subseteq \text{ann}_R(a) \), then there exists an \( R\)-homomorphism \( g: N \to M \) such that \( g(b) - a \in J(M) \).

(3) If \( a \in M \) and \( b \in N \) with \( \text{ann}_R(b) \subseteq \text{ann}_R(a) \), then for any \( h \in S \), there is an \( R\)-homomorphism \( \alpha \in \text{Hom}_R(N, M) \) with \( h(b) - \alpha(a) \in J(M) \).

(4) For each \( R\)-homomorphism \( f: A \to M \) (where \( A \) is any submodule of \( N \)) and each \( a \in A \), there exists an \( R\)-homomorphism \( g: N \to M \) such that \( g(a) - f(a) \in J(M) \).
(2) If \( a \in M \) and \( b \in N \) with \( \text{ann}_R(b) \subseteq \text{ann}_R(a) \), then there exists an \( R \)-homomorphism \( g: N \to M \) such that \( g(b) = a \in L(M) \).

(3) For each \( a \in M, b \in N \) with \( \text{ann}_R(b) \subseteq \text{ann}_R(a) \) and for each \( h \in S \), there exists an \( R \)-homomorphism \( \alpha \in \text{Hom}_R(N, M) \) such that \( h(b) = \alpha(a) \in L(M) \).

(4) For each \( R \)-homomorphism \( f: A \to M \) (where \( A \) is any submodule of \( N \)) and each \( a \in A \), there exists an \( R \)-homomorphism \( g: N \to M \) such that \( g(a) = f(a) \in L(M) \).

**Proof.** By taking \( m = 1 \) and \( \rho = L \) (the prime radical functor) and applying Theorem 2.3.

The following corollary gives many characterizations of \((\rho, m)\)-quasi-injective modules, and its proof immediately from Theorem 2.3.

**Corollary 2.6.** The following statements are equivalent for an \( R \)-module \( M \), where \( m \in \mathbb{Z}^+ \) and \( S = \text{End}_R(M) \).

(1) \( M \) is \((\rho, m)\)-quasi-injective.

(2) If \( \bar{x} = (x_1, x_2, \ldots, x_m), \bar{y} = (y_1, y_2, \ldots, y_m) \in M^m \) such that \( \text{ann}_R(m)(\bar{y}) \subseteq \text{ann}_R(m)(\bar{x}) \), then there is an \( R \)-homomorphism \( g: M \to M \) such that \( g(y_i) = x_i \in \rho(M) \), for all \( i = 1, 2, \ldots, m \).

(3) If \( \bar{x} = (x_1, x_2, \ldots, x_m), \bar{y} = (y_1, y_2, \ldots, y_m) \in M^m \) such that \( \text{ann}_R(m)(\bar{y}) \subseteq \text{ann}_R(m)(\bar{x}) \) and for each \( h \in S \), then there is an \( R \)-homomorphism \( \alpha \in S \) with \( h(x_i) = \alpha(y_i) \in \rho(M) \), for all \( i = 1, 2, \ldots, m \).

(4) For each \( R \)-homomorphism \( f: A \to M \), where \( A \) is any submodule of \( M \) and for each set \( \{a_1, a_2, \ldots, a_m\} \subseteq A \), there is an \( R \)-homomorphism \( g: M \to M \) with \( g(a_i) = f(a_i) \in \rho(M) \), for all \( i = 1, 2, \ldots, m \).

(5) For each \( m \)-generated ideal \( I \) of \( R \) and each \( \text{ann}_R(R) \)-homomorphism \( f: I \to M \), there exists \( \alpha \in M \) such that \( f(r) = -ra \in \rho(M) \), for all \( r \in I \).

**Proof.** Clearly from Theorem 2.3, we can prove the equivalence of (1), (2), (3), and (4).

(1) \( \Rightarrow \) (5). Let \( I \) be a \( m \)-generated ideal of \( R \) and \( f: I \to M \) be any \( R \)-homomorphism. \((\rho, m)\)-injectivity of \( M \) implies existence of an \( R \)-homomorphism \( g: R \to M \) such that \( f(\bar{r}) - g(\bar{r}) \in \rho(M) \), for all \( r \in I \) and hence \( f(r) - rg(1) \in \rho(M) \). Put \( a = g(1) \), thus there exists \( \alpha \in M \) with \( f(r) = -ra \in \rho(M) \), for all \( r \in I \).

(5) \( \Rightarrow \) (1). Let \( f: I \to M \) be any \( R \)-homomorphism, where \( I \) is any \( m \)-generated ideal of \( R \). By (2), there is \( \alpha \in M \) such that \( f(r) - ra \in \rho(M) \), for all \( r \in I \). Define \( g: R \to M \) by \( g(x) = xa \), for all \( x \in R \). It
is clear that $g$ is an $R$-homomorphism. For all $r \in I$, we have that $f(r) - g(r) = f(r) - ra \in \rho(M)$. Thus $M$ is a $(\rho, m)$-$R$-injective $R$-module.

In the following proposition, we give a new characterization of $(\rho, m)$-$R$-injective modules.

**Proposition 2.8.** Let $M$ be an $R$-module, $m \in \mathbb{Z}^+$ and $S = \text{End}_R(M)$. Then $M$ is $(\rho, m)$-$R$-injective if and only if $\text{ann}_M^m(\text{ann}_R^m(\bar{y})) \subseteq \bar{y}M + (\rho(M))^m$, for any $\bar{y} \in R^m$.

**Proof.** ($\Rightarrow$) Let $\bar{y} = (y_1, y_2, ..., y_m) \in R^m$ and let $\bar{x} = (x_1, x_2, ..., x_m) \in \text{ann}_M^m(\text{ann}_R^m(\bar{y}))$, thus $\sum_{i=1}^m r_i x_i = 0$, for all $\bar{r} = (r_1, r_2, ..., r_m) \in \text{ann}_R^m(\bar{y})$. Let $\bar{k} = (k_1, k_2, ..., k_m) \in \text{ann}_R^m(\bar{y})$, thus $\sum_{i=1}^m k_i x_i = 0$ and hence $\bar{k} \in \text{ann}_R^m(\bar{x})$ and this implies that $\text{ann}_R^m(\bar{y}) \subseteq \text{ann}_R^m(\bar{x})$. Since $M$ is $(\rho, m)$-$R$-injective $R$-module it follows from Proposition 2.7 that there exists an $R$-homomorphism $g: R \to M$ such that $g(y_i) - x_i \in \rho(M)$, for all $i = 1, ..., m$ and hence $y_ig(1) - x_i \in \rho(M)$. For all $i = 1, ..., m$. Put $y_i g(1) - x_i = t_i$, where $t_i \in \rho(M)$, thus $x_i = y_i g(1) - t_i$. Since $g(1) \in \rho(M)$, it follows that $\bar{x} = \bar{y} g(1) - \bar{t} \in \bar{y} M + (\rho(M))^m$, where $\bar{t} = (t_1, t_2, ..., t_m)$. Therefore, $\text{ann}_M^m(\text{ann}_R^m(\bar{y})) \subseteq \bar{y} M + (\rho(M))^m$, for any $\bar{y} \in R^m$.

($\Leftarrow$) Suppose that $\text{ann}_M^m(\text{ann}_R^m(\bar{y})) \subseteq \bar{y} M + (\rho(M))^m$, for all $\bar{y} \in R^m$. Let $I$ be any $m$-generated ideal of $R$, say $I = (x_1, x_2, ..., x_m)$ and let $f: I \to M$ by any $R$-homomorphism. Put $\bar{x} = (x_1, x_2, ..., x_m)$, thus $\bar{x} \in R^m$. Let $\bar{r} \in \text{ann}_R^m(\bar{x})$, thus $\sum_{i=1}^m r_i x_i = 0$. Since $\sum_{i=1}^m r_i f(x_i) = f(\sum_{i=1}^m r_i x_i) = f(0) = 0$ implies $(f(x_1), f(x_2), ..., f(x_m)) \in \text{ann}_M^m(\text{ann}_R^m(\bar{x}))$. By hypothesis, $\text{ann}_M^m(\text{ann}_R^m(\bar{x})) \subseteq \bar{x} M + (\rho(M))^m$ and hence $(f(x_1), f(x_2), ..., f(x_m)) \in \bar{x} M + (\rho(M))^m$. Thus $(f(x_1), f(x_2), ..., f(x_m)) = \bar{x} a + \bar{t}$, for some $a \in M$ and $\bar{t} \in (\rho(M))^m$ and hence $f(x_i) - x_i a \in \rho(M)$, for all $i = 1, ..., m$. Let $b \in I$, thus $b = \sum_{i=1}^m s_i x_i$, for some $s_i \in R$ ($i = 1, ..., m$). Thus $f(b) = f(\sum_{i=1}^m s_i x_i) = \sum_{i=1}^m s_i f(x_i) a = \sum_{i=1}^m s_i (f(x_i) - x_i a)$. Since $f(x_i) - x_i a \in \rho(M)$, for all $i = 1, ..., m$, $\sum_{i=1}^m s_i (f(x_i) - x_i a) \in \rho(M)$ and hence there is $a \in M$ with $f(r) - ra \in \rho(M)$, for all $r \in I$. By Proposition 2.7, $M$ is a $(\rho, m)$-$R$-injective $R$-module.

**Corollary 2.9.** [7, Proposition 2.2.7, p.65] An $R$-module $M$ is nearly P-injective if and only if $\text{ann}_M^m(\text{ann}_R^m(x)) \subseteq x M + J(M)$, for all $x \in R$.

**Proof.** By taking $m = 1$, $\rho = J$ and applying Theorem 2.8.

**Corollary 2.10.** An $R$-module $M$ is special P-R-injective if and only if $\text{ann}_M(\text{ann}_R(x)) \subseteq x M + L(M)$, for all $x \in R$.

**Proof.** By taking $m = 1$, $\rho = L$ and applying Theorem 2.8.

**Proposition 2.11.** Let $m \in \mathbb{Z}^+$. For an $R$-module $M$, the following statements are equivalent.

1. $M$ is $(\rho, m)$-injective.
2. $M$ is $(\rho, m)$-E(M)-injective.
3. For each $R$-monomorphism $\alpha: M \to E(M)$ and for each $A = \{a_1, a_2, ..., a_m\} \subseteq M$, there exists an $R$-homomorphism $\beta: E(M) \to M$ such that $(\beta \alpha)(a_i) - a_i \in \rho(M)$, for all $i = 1, 2, ..., m$.

**Proof.** (1) $\Rightarrow$ (2). This is clear.
Let \( \alpha: M \to E(M) \) be any \( R \)-monomorphism and let \( A = \{a_1, a_2, \ldots, a_m\} \subseteq M \). Define \( \beta: \alpha(M) \to M \) by \( \beta(\alpha(x)) = x \), for all \( x \in M \), thus \( \beta \) is a well-defined \( R \)-homomorphism. Let \( L = \langle \alpha(a_1), \alpha(a_2), \ldots, \alpha(a_m) \rangle \).

Clearly, \( L \) is an \( m \)-generated submodule of \( E(M) \). Define \( \lambda: L \to M \) by \( \lambda(a) = \beta(\alpha(a)) \), for all \( a \in L \).

Proposition 2.14. Let \( M \) and \( N \) be two \( R \)-modules. If \( M \) is \( (\rho, m) \)-\( N \)-injective, then \( M \) is \( (\rho, m) \)-\( A \)-injective for each submodule \( A \) of \( N \).

Proof. Let \( A \) be any submodule of \( N \), \( B \) be any \( m \)-generated submodule of \( A \) and \( f: B \to M \) be any \( R \)-homomorphism. Let \( i_B \) be the inclusion \( R \)-homomorphism from \( B \) into \( A \) and \( i_A \) be the inclusion \( R \)-homomorphism from \( A \) into \( N \). Since \( B \) is \( m \)-generated \( R \)-submodule of \( N \) and \( M \) is \( (\rho, m) \)-\( N \)-injective, there is an \( R \)-homomorphism \( g: N \to M \) such that \( (g \circ i_A) (i_B(b)) = f(b) \in \rho(M) \), for all \( b \in B \).

Directly from Proposition 2.14, we have:

Corollary 2.15. Let \( N \) be any submodule of an \( R \)-module \( M \). If \( N \) is \( (\rho, m) \)-\( M \)-injective, then \( N \) is \( (\rho, m) \)-quasi-injective.
**Proposition 2.16.** Any direct summand of \((\rho, m)\)-\(N\)-injective \(R\)-module is \((\rho, m)\)-\(N\)-injective.

**Proof.** Let \(M\) be any \((\rho, m)\)-\(N\)-injective \(R\)-module and \(A\) be any direct summand submodule of \(M\). Thus there exists submodule \(A_1\) of \(M\) such that \(M = A \oplus A_1\). Let \(B\) be any \(m\)-generated submodule of \(N\) and \(f: B \rightarrow A\) be any \(R\)-homomorphism. Define \(g: B \rightarrow M = A \oplus A_1\) by \(g(b) = (f(b), 0)\), for all \(b \in B\). It is clear that \(g\) is an \(R\)-homomorphism and since \(M\) is a \((\rho, m)\)-\(N\)-injective \(R\)-module, there exists an \(R\)-homomorphism \(h: N \rightarrow M\) such that \(h(b) - g(b) \in \rho(M)\) for all \(b \in B\). Let \(\pi_A\) be the natural projection \(R\)-homomorphism of \(M = A \oplus A_1\) into \(A\). Put \(h_1 = \pi_A \circ h: N \rightarrow A\). Thus \(h_1\) is an \(R\)-homomorphism and for any \(b \in B\) we get that \(h_1(b) - f(b) = (\pi_A \circ h)(b) - \pi_A ((f(b), 0)) = \pi_A (h(b)) - \pi_A (g(xb)) = \pi_A (h(b) - g(b)) \in \rho(A)\). Therefore, \(A\) is a \((\rho, m)\)-\(N\)-injective \(R\)-module.

The following corollary is immediate from Proposition 2.16.

**Corollary 2.17.** Any direct summand of \((\rho, m)\)-quasi-injective \(R\)-module is also \((\rho, m)\)-quasi-injective.

In the last part of this section, we will study the direct sum of \((\rho, m)\)-quasi-injective modules.

The following example shows that there is \(m \in \mathbb{Z}^+\) and a preradical \(\rho\) such that the direct sum of two \((\rho, m)\)-quasi-injective modules need not be \((\rho, m)\)-quasi-injective module.

**Example 2.18.** Let \(M = Q \oplus \mathbb{Z}_p\) as \(\mathbb{Z}\)-module. By [8, Example 2.2, p.5], \(M\) is not \((J, 1)\)-quasi-injective \(\mathbb{Z}\)-module, but \(Q\) and \(\mathbb{Z}_p\) are \((J, 1)\)-quasi-injective \(\mathbb{Z}\)-module. Therefore, \(M\) is not \((\rho, m)\)-quasi-injective \(\mathbb{Z}\)-module, but \(Q\) and \(\mathbb{Z}_p\) are \((\rho, m)\)-quasi-injective \(\mathbb{Z}\)-module, where \(m = 1\) and \(\rho = J\).

**Proposition 2.19.** Let \(m \in \mathbb{Z}^+\). If \(M\) and \(N\) are two \((\rho, m)\)-injective \(R\)-modules, then \(M \oplus N\) is a \((\rho, m)\)-injective \(R\)-module.

**Proof.** Let \(B\) be any \(R\)-module and let \(f: A \rightarrow M \oplus N\) be any \(R\)-homomorphism, where \(A\) is any submodule of \(B\). Let \(\{a_1, a_2, \ldots, a_m\} \subseteq A\) and let \(\pi_M\) be the canonical projection. By \((\rho, m)\)-\(B\)-injectivity of \(M\) and Theorem 2.3, there exists an \(R\)-homomorphism \(g_1: B \rightarrow M\) such that \(g_1 (a_i) - (\pi_M f)(a_i) \in \rho(M)\), for all \(i = 1, \ldots, m\). By the same way, there exists an \(R\)-homomorphism \(g_2: B \rightarrow N\) such that \(g_2 (a_i) - (\pi_N f)(a_i) \in \rho(N)\), for all \(i = 1, \ldots, m\), where \(\pi_N: M \oplus N \rightarrow N\) is the canonical projection. Define \(h: B \rightarrow M \oplus N\) by \(h(b) = (g_1 (b), g_2 (b))\), for all \(b \in B\). It is clear that \(h\) is an \(R\)-homomorphism. Thus for each \(i = 1, \ldots, m\), we have that \(h(a_i) - f (a_i) = (g_1 (a_i), g_2 (a_i)) - f (a_i) = (g_1 (a_i), g_2 (a_i)) - \left(\pi_M (f (a_i)), \pi_N (f (a_i))\right) = (g_1 (a_i) - \pi_M (f (a_i)), g_2 (a_i) - \pi_N (f (a_i))) \in \rho(M) \oplus \rho(N) = \rho(M \oplus N)\) (by [3, Proposition 2, p.76]). Therefore, \(M \oplus N\) is a \((\rho, m)\)-injective, by Theorem 2.3.

**Theorem 2.20.** Let \(m \in \mathbb{Z}^+\). Then an \(R\)-module \(M\) is \((\rho, m)\)-injective if and only if \(M \oplus E(M)\) is a \((\rho, m)\)-quasi-injective \(R\)-module.
Proof. ($\Rightarrow$) Let $M$ be a $(\rho, m)$-injective $R$-module. By Proposition 2.19, $M \oplus E(M)$ is $(\rho, m)$-injective and hence $M \oplus E(M)$ is a $(\rho, m)$-quasi-injective $R$-module.

($\Leftarrow$) Suppose that $M \oplus E(M)$ is a $(\rho, m)$-quasi-injective $R$-module. Thus $M \oplus E(M)$ is $(\rho, m)$-$(M \oplus E(M))$-injective. By Proposition 2.14, $M \oplus E(M)$ is $(\rho, m)$- $(M \oplus E(M))$-injective. By Proposition 2.16, $M$ is $(\rho, m)$-$E(M)$-injective and hence Proposition 2.11 implies that $M$ is $(\rho, m)$-$N$-injective, for all $R$-module $N$.

Corollary 2.21. Let $m \in \mathbb{Z}^+$ and let $M$ be any $m$-generated $R$-module. Then $M$ is $(\rho, m)$-injective if and only if $M \oplus E(M)$ is $(\rho, m)$-quasi-injective $R$-module.

Proof. By Theorem 2.20 and Corollary 2.12.

Theorem 2.22. The following statements are equivalent.

(1) Direct sum of any two $(\rho, m)$-quasi-injective $R$-modules is $(\rho, m)$-quasi-injective.

(2) Every $(\rho, m)$-quasi-injective $R$-module is $(\rho, m)$-injective.

Proof. (1) $\Rightarrow$ (2) Let $M$ be any $(\rho, m)$-quasi-injective $R$-module and let $E(M)$ be the injective envelope of $M$. By hypothesis, $M \oplus E(M)$ is a $(\rho, m)$-quasi-injective $R$-module and hence Theorem 2.20 implies that $M$ is a $(\rho, m)$-injective $R$-module.

(2) $\Rightarrow$ (1) Let $M_1$ and $M_2$ be any two $(\rho, m)$-quasi-injective $R$-modules. Hence the hypothesis implies that $M_1$ and $M_2$ are $(\rho, m)$-injective $R$-modules. Thus $M_1 \oplus M_2$ is a $(\rho, m)$-injective $R$-module (by Proposition 2.19). Hence $M_1 \oplus M_2$ is $(\rho, m)$-quasi-injective $R$-module.

Corollary 2.23. The following statements are equivalent.

(1) Direct sum of any two special $PQ$-injective $R$-modules is special $PQ$-injective.

(2) Every special $PQ$-injective $R$-module is special $P$-$N$-injective, for every $R$-module $N$.

Proof. By taking $m = 1$ and $\rho = L$ (the prime radical functor) and applying Theorem 2.22.

Corollary 2.24. [8, Proposition 2.5, p.7] Direct sum of any two nearly $PQ$-injective $R$-modules is nearly $PQ$-injective if and only if every nearly $PQ$-injective $R$-module is pointwise nearly injective.

Proof. By taking $m = 1$ and $\rho = J$ (the Jacobson radical functor) and applying Theorem 2.22.

We say that a preradical $\rho$ on $R$-$Mod$ is said to be a J-preradical if $\rho(M) \subseteq J(M)$, for all $M \in R$-$Mod$.

In the following theorem, we characterize rings over which every semisimple $R$-module is $\rho$-injective (where $\rho$ is a J-preradical). Also, this theorem gives a characterization of semi-simple Artinian rings in terms of $\rho$-injective modules.

Theorem 2.25. Let $\rho$ be a J-preradical. For a ring $R$, the following assertions are equivalent.

(1) $R$ is a semi-simple Artinian ring.

(2) Any $R$-module is $\rho$-injective.

(3) Any cyclic $R$-module is $\rho$-injective.

(4) Any semi-simple $R$-module is $\rho$-injective.

Proof. (1) $\Rightarrow$ (2). Since over semi-simple Artinian ring $R$ we have that every $R$-module is injective [5], every $R$-module is $\rho$-injective.
(2) \implies (3) and (2) \implies (4) are obvious.

(3) \implies (1) Let \( M \) be any simple \( R \)-module. By hypothesis, \( M \) is a \( \rho \)-injective \( R \)-module. Since \( \text{J}(M) = 0 \) [6, p.218] and \( \rho \) is a \( \text{J} \)-preradical, implies \( \rho(M) \subseteq \text{J}(M) = 0 \) and hence \( M \) is an injective \( R \)-module. Thus, we have that every simple \( R \)-module is injective and hence [6, Exercise18, p.272] implies that \( R \) is a regular ring. Since the Jacobson radical of every cyclic module over a regular ring is zero by [6, p.272] and since \( \rho \) is a \( \text{J} \)-preradical, we have for any cyclic \( R \)-module \( N, \rho(N) = 0 \), and so every cyclic \( R \)-module is injective. Therefore, \( R \) is a semi-simple Artinian ring [10].

In terms of \((\rho, m)\)-quasi-injective \( R \)-modules, a new characterization of semi-simple Artinian ring is given in the following proposition, which is a generalization of Faith’s and Utumi’s result [4].

**Theorem 2.26.** Let \( \rho \) be a \( \text{J} \)-preradical. Then the following statements are equivalent for a ring \( R \).

1. \( R \) is a semi-simple Artinian ring.
2. Every \( R \)-module is \((\rho, 1)\)-quasi-injective.
3. Every cyclic \( R \)-module is \((\rho, 1)\)-quasi-injective and direct sum of any two \((\rho, 1)\)-quasi-injective \( R \)-modules is \((\rho, 1)\)-quasi-injective.

**Proof.** (1) \implies (2) and (2) \implies (3) are obvious.

(3) \implies (1) Let \( M \) be any cyclic \( R \)-module. By (3), \( M \) is \((\rho, 1)\)-quasi-injective \( R \)-module and \( M \oplus \text{E}(M) \) is \((\rho, 1)\)-quasi-injective. Hence Corollary 2.21 implies that \( M \) is \( \rho \)-injective. Therefore, \( R \) is a semi-simple Artinian ring, by Theorem 2.25.

**Corollary 2.27.[8, Proposition 2.10, p.10]** The following statements are equivalent for a ring \( R \).

1. \( R \) is a semi-simple Artinian ring.
2. Every \( R \)-module is nearly \( \text{PQ} \)-injective.
3. Every cyclic \( R \)-module is nearly \( \text{PQ} \)-injective and direct sum of any two nearly \( \text{PQ} \)-injective \( R \)-modules is nearly \( \text{PQ} \)-injective.

**Proof.** By taking \( \rho = \text{J} \) (the Jacobson radical functor) and applying Theorem 2.26.

**Corollary 2.28.** The following statements are equivalent for a ring \( R \).

1. \( R \) is a semi-simple Artinian ring.
2. All \( R \)-modules are special \( \text{PQ} \)-injective.
3. All cyclic \( R \)-modules are special \( \text{PQ} \)-injective and for any two special \( \text{PQ} \)-injective \( R \)-modules, the direct sum of them is special \( \text{PQ} \)-injective.

**Proof.** By taking \( \rho = \text{L} \) (the prime radical functor) and applying Theorem 2.26.

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