BILINEAR FOURIER MULTIPLIERS AND THE RATE OF DECAY OF THEIR DERIVATIVES

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Abstract. We prove that if $m$ is a bounded function on $\mathbb{R}^{2n}$ satisfying
$$|\{x \in \mathbb{R}^{2n} : |\partial^\alpha m(x)| > \lambda\}| \lesssim \lambda^{-4}, \quad \lambda > 0,$$
for sufficiently many multiindices $\alpha$ then the associated bilinear multiplier operator is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. We also obtain a fractional variant of this result which generalizes the boundedness criterion for multipliers with bounded derivatives by Grafakos, He and the author. In addition, we show that our result is sharp, in the sense that no upper bound for the measure of level sets of partial derivatives of $m$ that is weaker (for small $\lambda$) than the one displayed above suffices to guarantee the aforementioned boundedness.

1. Introduction and overview of the results

Assume that $\sigma$ is a bounded function on $\mathbb{R}^n$. We denote by $S_\sigma$ the linear multiplier operator defined as
$$S_\sigma f(x) = \int_{\mathbb{R}^n} \sigma(\xi) \hat{f}(\xi)e^{2\pi i x \cdot \xi} \, d\xi, \quad x \in \mathbb{R}^n,$$
for any Schwartz function $f$ on $\mathbb{R}^n$. Here, $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} \, dx$ stands for the Fourier transform of the function $f$. One of the key questions about the operator $S_\sigma$ is how it acts on different function spaces. Related to this, it is a well-known consequence of the Plancherel identity that $S_\sigma$ admits a bounded extension from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Conversely, for a general bounded function $\sigma$, the operator $S_\sigma$ does not need to be bounded on $L^p(\mathbb{R}^n)$ if $p \neq 2$.

In connection with various problems involving product-type operations, the bilinear variant of the operator $S_\sigma$ also comes into play. For
a given bounded function \( m \) on \( \mathbb{R}^{2n} \), we define the bilinear multiplier operator \( T_m \) as

\[
T_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta, \quad x \in \mathbb{R}^n,
\]

where \( f \) and \( g \) are Schwartz functions on \( \mathbb{R}^n \). In this paper we focus on the study of the \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \) boundedness of the operator \( T_m \). While this is, in a sense, a bilinear analogue of the \( L^2 \)-boundedness of the linear operator \( S_\sigma \), it is not true that \( T_m \) is bounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \) for every bounded function \( m \). Since finding a reasonable characterization of those functions \( m \) for which the aforementioned boundedness holds seems to be out of reach, various sufficient conditions have been established as a substitute. Let us start with a brief overview of these results.

A criterion by Coifman and Meyer \cite{CoifmanMeyer} says that if the function \( m \) satisfies

\[
|\partial^\alpha m(\xi, \eta)| \leq C_\alpha |(\xi, \eta)|^{-|\alpha|}
\]

for sufficiently many multiindices \( \alpha \) then \( T_m \) is bounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) into \( L^1(\mathbb{R}^n) \) (the same assumption in fact yields the boundedness of \( T_m \) for a much larger family of Lebesgue spaces). We note that condition (1.1) resembles the classical Mikhlin condition \cite{Mikhlin} for linear operators, and the result of \cite{CoifmanMeyer} essentially says that linear Mikhlın multipliers on \( \mathbb{R}^2 \) are bounded bilinear multipliers on \( \mathbb{R}^n \times \mathbb{R}^n \).

Generalizations of the Coifman-Meyer result in the spirit of the classical Hörmander multiplier theorem \cite{Hörmander} are available in the literature as well. They were initiated by Tomita \cite{Tomita} and investigated further by various authors \cite{Far/article, HeHonzik12, HeHonzik14, HeHonzik15, HeHonzik16, HeHonzik18, HeHonzik23, HeHonzik24}. In order to describe this type of results, let us now introduce the notion of the fractional Laplace operator and of fractional Sobolev spaces. For any \( s > 0 \) we let \( (I - \Delta)^{\frac{s}{2}} \) be the operator defined via the Fourier transform as

\[
[(I - \Delta)^{\frac{s}{2}} f]^{\wedge}(\xi) = (1 + 4\pi^2 |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi), \quad \xi \in \mathbb{R}^{2n}.
\]

Given \( 1 < r < \infty \), the fractional Sobolev space \( L^r_s(\mathbb{R}^{2n}) \) is then the space of all functions \( f \) on \( \mathbb{R}^{2n} \) for which

\[
\|f\|_{L^r_s(\mathbb{R}^{2n})} = \|(I - \Delta)^{\frac{s}{2}} f\|_{L^r(\mathbb{R}^{2n})} < \infty.
\]

The bilinear variant of the Hörmander multiplier theorem by Grafakos, He and Honzík \cite{HeHonzik12} Theorem 1] asserts that whenever

\[
\sup_{k \in \mathbb{Z}} \|m(2^k \cdot) \phi\|_{L^r_s(\mathbb{R}^{2n})} < \infty
\]
holds for some $1 < r < \infty$ and $s > \max\{n/2, 2n/r\}$ then $T_m$ admits a bounded extension from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Here, $\phi$ is a smooth function supported in the annulus $\{\xi \in \mathbb{R}^{2n} : 1/2 < |\xi| < 2\}$ and satisfying $\sum_{k \in \mathbb{Z}} \phi(2^k \cdot) = 1$. In fact, it is not difficult to observe that fixing the value $r = 4$ results in an equivalent formulation of this theorem; see Lemma 4.3 below for more details.

A different type of a bilinear multiplier theorem appeared in [11] in connection with the study of rough bilinear singular integrals, and was further improved in [13, Theorem 1.3]. The latter theorem asserts that if $m$ is a bounded function on $\mathbb{R}^{2n}$ with bounded partial derivatives of all orders which, in addition, belongs to the space $L^q(\mathbb{R}^{2n})$ for some $q < 4$ then $T_m$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. It was also pointed out in [13] that the assumption $q < 4$ is essentially sharp, in the sense that the same conclusion does not hold if $q > 4$. We complete this picture by showing that this result fails in the limiting case $q = 4$ as well; the corresponding counterexample is given in Section 6.

The proofs of the multiplier theorems mentioned above [11, 12, 13] rely on the use of techniques of multiresolution analysis. Roughly speaking, the idea of this approach is to express a given multiplier as an infinite linear combination of simpler functions which satisfy certain self-similarity properties; namely, they are obtained from a few generating functions by using translation and dilation. The main goal of this paper is to find a criterion for the $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness of bilinear Fourier multipliers which is (in a certain sense) the best possible on each given scale of this decomposition. To get a better understanding of the kind of questions we have in mind, we can think of the following problem, which is discussed in detail in Section 5.

**Question 1.1.** Assume that $\Psi$ is a smooth function on $\mathbb{R}^{2n}$ supported in a small ball centered at the origin. Let $m$ be a function of the form

\begin{equation}
 m(\xi, \eta) = \sum_{(k,l) \in \mathbb{Z}^{2n}} c_{k,l} \Psi(\xi - k, \eta - l), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n,
\end{equation}

where $\{c_{k,l}\}_{(k,l) \in \mathbb{Z}^{2n}}$ is a sequence of complex numbers. Which conditions on the size of the coefficients $c_{k,l}$ (or, more precisely, on the cardinality of the level sets $\{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n : |c_{k,l}| > \lambda\}$) suffice to guarantee the $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness of $T_m$, and which do not? In particular, is there the weakest sufficient condition of this type?

**Answer 1.2.** Such a condition indeed exists, and it has the form

\begin{equation}
 \text{card}\{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n : |c_{k,l}| > \lambda\} \lesssim \lambda^{-4}, \quad \lambda > 0,
\end{equation}
where \( \text{card } E \) stands for the cardinality of a set \( E \). See Proposition 5.1 for more details.

A crucial tool for proving the above-mentioned result is an elementary lemma about expressing a given function defined on \( \mathbb{Z}^{2n} = \mathbb{Z}^n \times \mathbb{Z}^n \) as a sum of two functions with disjoint supports, the first one having uniformly bounded \( \ell^2 \)-norms over all rows and the second one having uniformly bounded \( \ell^2 \)-norms over all columns. We discuss this problem in Section 2 where we show that such a decomposition is indeed possible if the function satisfies the estimate (1.3), and that no condition on the cardinality of level sets of that function weaker than (1.3) can guarantee the existence of such a decomposition.

Applying the previous result on each scale of the wavelet decomposition of the multiplier \( m \), we derive the following criterion for the \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n) \) boundedness of the operator \( T_m \), expressed in terms of the rate of decay of the derivatives of the multiplier.

**Theorem 1.3.** Let \( m \) be a function on \( \mathbb{R}^{2n} \) satisfying

\[
C_s(m) := \sup_{\lambda > 0} \lambda \left| \{ x \in \mathbb{R}^{2n} : |(I - \Delta)^{\frac{s}{2}} m(x)| > \lambda \} \right|^\frac{1}{4} < \infty
\]

for some \( s > n/2 \). Then the associated operator \( T_m \) admits a bounded extension from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \) and

\[
\| T_m(f, g) \|_{L^1(\mathbb{R}^n)} \leq CC_s(m) \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}.
\]

Let us point out that the non-fractional variant of this result mentioned in the abstract is readily obtained by applying Theorem 1.3 with \( s \) equal to an even integer larger than \( n/2 \).

Customary examples of functions \( m \) for which Theorem 1.3 is of interest are, for instance, functions of the form (1.2), or, more generally, functions with bounded partial derivatives (up to a certain order). In such a situation, only the behavior of the function

\[
\lambda \mapsto \left| \{ x \in \mathbb{R}^{2n} : |(I - \Delta)^{\frac{s}{2}} m(x)| > \lambda \} \right|
\]

for \( \lambda \) near zero is relevant. The next theorem shows that the assumption (1.4) is sharp when we restrict ourselves to small values of \( \lambda \).

**Theorem 1.4.** Assume that \( \mu : (0, 1) \rightarrow [0, \infty) \) is a non-increasing function satisfying

\[
\sup_{\lambda \in (0,1)} \lambda (\mu(\lambda))^{\frac{1}{4}} = \infty.
\]

Given \( s \in 2\mathbb{N} \), there is a smooth function \( m \) on \( \mathbb{R}^{2n} \) with bounded partial derivatives of all orders which fulfills

\[
\left| \{ x \in \mathbb{R}^{2n} : |(I - \Delta)^{\frac{s}{2}} m(x)| > \lambda \} \right| \leq \mu(\lambda), \quad \lambda \in (0, 1).
\]
and for which the associated operator $T_m$ is unbounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

Let us now continue with a few remarks about the statement of Theorem 1.3. It is not difficult to observe that if $m$ is a function on $\mathbb{R}^{2n}$ belonging to the fractional Sobolev space $L^4_s(\mathbb{R}^{2n})$ for some $s > n/2$ then $m$ satisfies (1.4). Thus, Theorem 1.3 yields the following corollary.

**Corollary 1.5.** Let $m$ be a function on $\mathbb{R}^{2n}$ belonging to the fractional Sobolev space $L^4_s(\mathbb{R}^{2n})$ for some $s > n/2$. Then the associated operator $T_m$ admits a bounded extension from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and

$$\|T_m(f,g)\|_{L^1(\mathbb{R}^n)} \leq C\|m\|_{L^4_s(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$}

We remark that a variant of Corollary 1.5 with $L^4_s(\mathbb{R}^{2n})$ replaced by $L^q_s(\mathbb{R}^{2n})$ with $q < 4$ and $s > \frac{2n}{q}$ appeared in [12, Remark 2]; the limiting case $q = 4$ was however not included there. Let us also point out a substantial difference between Corollary 1.5 and [13, Theorem 1.3]: while [13, Theorem 1.3] fails for the critical exponent $q = 4$, Corollary 1.5 continues to hold in this borderline case.

It is now natural to seek a comparison of Theorem 1.3 and, in particular, Corollary 1.5 with the previously mentioned bilinear multiplier theorems. We focus on this problem in Section 4 of the present paper. Our main conclusion is that Corollary 1.5 in fact generalizes [13, Theorem 1.3]. The crucial tool for proving this result is an interpolation inequality for intermediate derivatives whose first variant was obtained independently by Gagliardo [10] and Nirenberg [25]. It asserts, in particular, that if $f$ is a function in $L^{p_1}(\mathbb{R}^{2n})$ whose partial derivatives of an integer order $k$ belong to $L^{p_2}(\mathbb{R}^{2n})$ and $l$ is a positive integer less than $k$, then all partial derivatives of $f$ of order $l$ belong to $L^p(\mathbb{R}^{2n})$, where $p$ is given by

$$\frac{k}{p} = \frac{k - l}{p_1} + \frac{l}{p_2}.$$}

This inequality was subsequently studied and extended to more general contexts by various authors, see, e.g., [2, 3, 4, 5, 7, 8]. Our proof relies on a fractional version of the Gagliardo-Nirenberg inequality due to Brezis and Mironescu [2].

Let us finally focus on the comparison of Corollary 1.5 with the bilinear variant of the Hörmander multiplier theorem [12, Theorem 1]. We show that, even though these two multiplier theorems involve the same class of function spaces, they are in fact not comparable. In particular, it is of interest to notice that “near infinity”, the derivatives of any multiplier satisfying the assumptions of [12, Theorem 1] need to be integrable when raised to a certain power less than 4; see Proposition 4.4.
for more details. For this reason, the bilinear variant of the Hörmander multiplier theorem does not imply the $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness of the multiplier of the form $c$ if $c$ belongs to $\ell^4(\mathbb{Z}^{2n})$ but not to any smaller Lebesgue space. The same boundedness can, however, be proved by applying Corollary 1.5. A detailed discussion of this example can be found in Section 5.

Notation. Let us now fix notation that will be used throughout the paper. Given $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we denote by $|x|$ and $|x|_\infty$ the Euclidean and maximum norm of $x$, respectively. Also, given a real number $y$, $\lfloor y \rfloor$ stands for the integer part of $y$, that is, the largest integer which does not exceed $y$.

For $p \in [1, \infty)$, we denote by $L^p(\mathbb{R}^n)$ the space of all Lebesgue measurable functions on $\mathbb{R}^n$ whose absolute value is integrable when raised to the power $p$, while $L^\infty(\mathbb{R}^n)$ is the space of all essentially bounded functions on $\mathbb{R}^n$. Variants of these spaces when the Lebesgue measure on $\mathbb{R}^n$ is replaced by the counting measure on some countable set $\mathcal{C}$ are denoted by $\ell^p(\mathcal{C})$. The Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{S}'(\mathbb{R}^n)$ stands for its dual, the space of tempered distributions. By $\langle m, f \rangle$ we mean the action of a temperate distribution $m$ on a function $f \in \mathcal{S}(\mathbb{R}^n)$. The Fourier transform of a temperate distribution $m$ is denoted by $\hat{m}$, and the inverse Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is denoted either by $\check{f}$, or by $\mathcal{F}^{-1}(f)$.

By writing “$\lesssim$” we mean that the expression on the left-hand side of “$\lesssim$” is dominated by the expression on the right-hand side up to multiplicative constants depending only on unessential quantities. The relation “$\approx$” between two expressions means that they are bounded by each other up to multiplicative constants depending only on unessential quantities.

2. AN ELEMENTARY LEMMA

In this section we prove an elementary combinatorial lemma which will serve as a crucial tool for proving our main result, Theorem 1.3.

Let us start by introducing some necessary terminology. For a totally $\sigma$-finite measure space $(R, \nu)$, we define the Lorentz space $L^{4,\infty}(R, \nu)$ as the collection of all $\nu$-measurable functions $f$ on $R$ satisfying

$$\|f\|_{L^{4,\infty}(R, \nu)} = \sup_{\lambda > 0} \lambda \nu(\{x \in R : |f(x)| > \lambda\})^{\frac{1}{4}} < \infty.$$ 

Alternatively, the quantity $\|f\|_{L^{4,\infty}(R, \nu)}$ can be expressed as

$$\|f\|_{L^{4,\infty}(R, \nu)} = \sup_{t > 0} \lambda \nu(\{x \in R : |f(x)| > \lambda\})^{\frac{1}{4}} = \sup_{t > 0} t^{\frac{1}{2}} f^*_\nu(t),$$
where
\[ f^*_\nu(t) = \inf \{ \rho \geq 0 : \nu(\{ x \in R : |f(x)| > \rho \}) < t \}, \quad t > 0, \]
stands for the non-increasing rearrangement of \( f \) with respect to the measure \( \nu \).

In the special case when \((R, \nu)\) is the Euclidean space \( \mathbb{R}^{2n} \) equipped with the \( 2n \)-dimensional Lebesgue measure \( \lambda_{2n} \), we write \( L^{4,\infty}(\mathbb{R}^{2n}) \) instead of \( L^{4,\infty}(\mathbb{R}^{2n}, \lambda_{2n}) \) for simplicity. In addition, if \((R, \nu)\) is a countable set \( C \) equipped with the counting measure then the corresponding Lorentz space is denoted by \( \ell^{4,\infty}(C) \). We shall also skip the subscript \( \nu \) in the notation for the non-increasing rearrangement of a function if no confusion can arise about the choice of the underlying measure.

Lemma 2.1. Let \( C \) be a countable set, and let \( f \) be a function on \( C \times C = C^2 \) such that \( f \in \ell^{4,\infty}(C^2) \). Then we can write \( C^2 \) as a union of two disjoint sets \( S_1 \) and \( S_2 \) satisfying

\[ \left( \sum_{k : (k,l) \in S_1} |f(k,l)|^2 \right)^{\frac{1}{2}} \leq C \| f \|_{\ell^{4,\infty}(C^2)} \quad \text{for every } k \in C \]
and

\[ \left( \sum_{k : (k,l) \in S_2} |f(k,l)|^2 \right)^{\frac{1}{2}} \leq C \| f \|_{\ell^{4,\infty}(C^2)} \quad \text{for every } l \in C, \]
where \( C > 0 \) is an absolute constant.

Proof. We can assume, without loss of generality, that \( \| f \|_{\ell^{4,\infty}(C^2)} = 1 \). Then

\[ \text{card}\{(k,l) \in C^2 : |f(k,l)| > \lambda\} \leq \lambda^{-4}, \quad \lambda > 0. \]

In particular, if \( \lambda > 1 \) then \( \text{card}\{(k,l) \in C^2 : |f(k,l)| > \lambda\} < 1 \), which implies that the corresponding level sets are empty, and thus \( \| f \|_{\ell^{\infty}(C^2)} \leq 1 \).

We first construct two auxiliary subsets \( \tilde{S}_1 \) and \( \tilde{S}_2 \) of \( C^2 \) (not necessarily disjoint). Let us fix \( k \in C \). If \( \sum_{l \in C} |f(k,l)|^2 \leq 2 \) then we set \((k,l) \in \tilde{S}_1\) for every \( l \in C \). Conversely, if \( \sum_{l \in C} |f(k,l)|^2 > 2 \) then we rearrange those numbers \(|f(k,l)|\) which are positive in decreasing order:

\[ |f(k,l_1)| \geq |f(k,l_2)| \geq |f(k,l_3)| \geq \ldots, \]
where \( l_1, l_2, \ldots \) is a suitable permutation of (a subset of) \( C \). Note that such a rearrangement can be constructed since all level sets \( \{(k,l) \in C^2 : |f(k,l)| > \lambda\} \) with \( \lambda > 0 \) are finite. Then we find \( N_k \in \mathbb{N}, \)
\[ N_k \geq 2 \text{ satisfying } \sum_{i=1}^{N_k-1} |f(k, l_i)|^2 < 2 \text{ and } \sum_{i=1}^{N_k} |f(k, l_i)|^2 \geq 2 \text{ (notice that } N_k \geq 2 \text{ because } \|f\|_{L^\infty(C^2)} \leq 1). \] Then we set \((k, l_i) \in \tilde{S}_1\) for \(i = 1, \ldots, N_k,\) and also \((k, l) \in \tilde{S}_1\) if \(|f(k, l)| = 0.\) We observe that

\[
(2.3) \quad \sum_{l: (k,l) \in \tilde{S}_1} |f(k, l)|^2 = |f(k, l_{N_k})|^2 + \sum_{i=1}^{N_k-1} |f(k, l_i)|^2 \leq 1 + 2 = 3.
\]

Let us now fix \(l \in C.\) If \(\sum_{k \in C} |f(k, l)|^2 \leq 2\) then we set \((k, l) \in \tilde{S}_2\) for every \(k \in C.\) Conversely, if \(\sum_{k \in C} |f(k, l)|^2 > 2\) then we rearrange those numbers \(|f(k, l)|\) which are positive in decreasing order:

\[ |f(k_1, l)| \geq |f(k_2, l)| \geq |f(k_3, l)| \geq \ldots,\]

where \(k_1, k_2, \ldots\) is a suitable permutation of (a subset of) \(C.\) Again, this can be done since all level sets \(\{(k, l) \in C^2 : |f(k, l)| > \lambda\}\) with \(\lambda > 0\) are finite. Then we find \(N_i \in \mathbb{N}, N_i \geq 2\) satisfying \(\sum_{i=1}^{N_i-1} |f(k, l)|^2 < 2\) and \(\sum_{i=1}^{N_i} |f(k, l)|^2 \geq 2\) and we set \((k_i, l) \in \tilde{S}_2\) for \(i = 1, \ldots, N_i,\) and also \((k, l) \in \tilde{S}_2\) if \(|f(k, l)| = 0.\) As previously, we observe that

\[
\sum_{k: (k, l) \in \tilde{S}_2} |f(k, l)|^2 = |f(k_N, l)|^2 + \sum_{i=1}^{N_i-1} |f(k_i, l)|^2 \leq 1 + 2 = 3.
\]

For \(k \in C\) we denote

\[ R_k = \max_{l \in C: (k, l) \notin \tilde{S}_1} |f(k, l)|.\]

We note that \(R_k\) is understood to be 0 if the set \(\{l \in C: (k, l) \notin \tilde{S}_1\}\) is empty. We now rearrange those numbers \(R_k\) that are positive in decreasing order:

\[ R_{\tilde{k}_1} \geq R_{\tilde{k}_2} \geq \ldots,\]

where \(\tilde{k}_1, \tilde{k}_2, \ldots\) is a permutation of (a subset of) \(C,\) indexed by the elements of a set \(I_k \subseteq \mathbb{N}\) (we either have \(I_k = \emptyset,\) or \(I_k = \{1, 2, \ldots, N\}\) for some \(N \in \mathbb{N},\) or \(I_k = \mathbb{N}\)). Such a rearrangement is possible since all level sets \(\{(k, l) \in C^2 : |f(k, l)| > \lambda\}\) with \(\lambda > 0\) are finite. We claim that \(R_{\tilde{k}_i} \lesssim 1/\sqrt{i}\) for \(i \in I_k,\) up to an absolute multiplicative constant. To verify this, we fix \(i \in I_k\) and consider the set

\[ A = \{(k_j, l) : j \in \{1, \ldots, i\}, (k_j, l) \in \tilde{S}_1, f(k_j, l) \neq 0\}.\]

Then \(|f(k, l)| \geq R_{\tilde{k}_i}\) for every \((k, l) \in A.\) Using the definition of the set \(\tilde{S}_1,\) we obtain

\[
\sum_{(k,l) \in A} |f(k, l)|^2 = \sum_{j=1}^{i} \sum_{l: (k_j, l) \in \tilde{S}_1} |f(k_j, l)|^2 \geq 2i.
\]
Also, recalling that
\[ \sup_{t > 0} t^\frac{1}{4} f^*(t) = \|f\|_{\ell^4(\mathbb{R}^2)} = 1, \]
where \( f^* \) denotes the non-increasing rearrangement of \( f \) with respect to the counting measure on \( \mathbb{C}^2 \), we get
\[
\sum_{(k, l) \in A} |f(k, l)|^2 \leq \sum_{j=1}^{\text{card} A} (f^*(j))^2 \leq \sum_{j=1}^{\text{card} A} \frac{1}{\sqrt{j}} \approx \sqrt{\text{card} A}.
\]
So, \( \text{card} A \geq c i^2 \), where \( c > 0 \) is an absolute constant, which yields
\[
R_{\bar{k}_i} \leq f^*(\text{card} A) \leq f^*(ci^2) \lesssim \frac{1}{\sqrt{i}},
\]
as desired.

Similarly, for \( l \in \mathcal{C} \) we denote
\[
C_l = \max_{k: (k, l) \notin \tilde{S}_2} |f(k, l)|
\]
and rearrange those numbers \( C_l \) that are positive in decreasing order:
\[
C_{\tilde{l}_1} \geq C_{\tilde{l}_2} \geq \ldots,
\]
where \( \tilde{l}_1, \tilde{l}_2, \ldots \) is a permutation of (a subset of) \( \mathcal{C} \), indexed by the elements of a set \( J_l \subseteq \mathbb{N} \). An argument as above shows that \( C_{\tilde{l}_i} \lesssim \frac{1}{\sqrt{i}} \) for \( i \in J_l \).

We are now ready to define the sets \( S_1 \) and \( S_2 \). We notice that if \( (k, l) \notin \tilde{S}_1 \cup \tilde{S}_2 \) then \( R_k > 0 \) and \( C_l > 0 \), and therefore \( k = \bar{k}_i \) for some \( i \in \mathbb{N} \) and \( l = \bar{l}_j \) for some \( j \in \mathbb{N} \). Thus, we can set
\[
S_1 = \tilde{S}_1 \cup \{(k, l) \notin \tilde{S}_1 \cup \tilde{S}_2 : (k, l) = (\bar{k}_i, \bar{l}_j) \text{ for } i \geq j\}
\]
and
\[
S_2 = \mathcal{C}^2 \setminus S_1 = (\tilde{S}_2 \setminus \tilde{S}_1) \cup \{(k, l) \notin \tilde{S}_1 \cup \tilde{S}_2 : (k, l) = (\bar{k}_i, \bar{l}_j) \text{ for } i < j\}.
\]

It remains to verify inequalities (2.1) and (2.2). To show (2.1), we fix \( k \in \mathcal{C} \) such that \( (k, l) \in S_1 \setminus \tilde{S}_1 \) for some \( l \in \mathcal{C} \). Then \( k = \bar{k}_i \) for some \( i \in \mathbb{N} \). Now, if \( (k, l) = (\bar{k}_i, \bar{l}_j) \in S_1 \setminus \tilde{S}_1 \) then \( j \leq i \), which means that \( \text{card}\{l \in \mathcal{C} : (k, l) \in S_1 \setminus \tilde{S}_1\} \leq i \). We also have
\[
|f(k, l)| \leq R_{\bar{k}_i} \lesssim \frac{1}{\sqrt{i}} \text{ if } (k, l) \in S_1 \setminus \tilde{S}_1.
\]
Thus,
\[
(2.4) \sum_{l: (k, l) \in S_1 \setminus \tilde{S}_1} |f(k, l)|^2 \lesssim i \cdot \left( \frac{1}{\sqrt{i}} \right)^2 = 1.
\]
A combination of (2.3) and (2.4) yields (2.1). Inequality (2.2) can be proved analogously. □

**Example 2.2.** In this example we show that the assumption \( f \in \ell^{4,\infty}(C^2) \) of Lemma 2.1 is sharp. For simplicity, we work in the setting where \( C = \mathbb{Z} \) but an analogous argument can be applied to cover the general situation as well.

Assume that \( f \) is a function on \( \mathbb{Z}^2 \) satisfying the monotonicity assumption
\[
(2.5) \quad |f(k_1, l_1)| \leq |f(k_2, l_2)| \quad \text{if} \quad \max\{|k_1|, |l_1|\} > \max\{|k_2|, |l_2|\}.
\]

We show that if there are two disjoint subsets \( S_1 \) and \( S_2 \) of \( \mathbb{Z}^2 \) whose union is the entire \( \mathbb{Z}^2 \) and if there is a constant \( C \) for which
\[
\sum_{l: (k,l) \in S_1} |f(k,l)|^2 \leq C \quad \text{for every} \quad k \in \mathbb{Z}
\]
and
\[
\sum_{k: (k,l) \in S_2} |f(k,l)|^2 \leq C \quad \text{for every} \quad l \in \mathbb{Z},
\]
then \( f \in \ell^{4,\infty}(\mathbb{Z}^2) \).

Let us fix \( M \in \mathbb{Z}_{0}^+ \). Then we have
\[
\sum_{k=-M}^{M} \sum_{l=-M}^{M} |f(k,l)|^2 \leq \sum_{k=-M}^{M} \sum_{l: (k,l) \in S_1} |f(k,l)|^2 + \sum_{l=-M}^{M} \sum_{k: (k,l) \in S_2} |f(k,l)|^2 \leq 2C(2M + 1).
\]

Due to the monotonicity assumption (2.5), \( f^*((2M + 1)^2) \leq |f(k,l)| \) whenever \( \max\{|k|, |l|\} \leq M \). This, combined with the previous inequality, yields
\[
(2M + 1)^2 (f^*((2M + 1)^2))^2 \leq 2C(2M + 1),
\]
or
\[
\sup_{M \in \mathbb{Z}_{0}^+} (2M + 1)^{\frac{1}{2}} f^*((2M + 1)^2) < \infty.
\]

Thanks to the monotonicity of the function \( f^* \), this implies
\[
\sup_{t \in [1,\infty)} t^{\frac{1}{2}} f^*(t) < \infty.
\]

Since \( f \) is bounded, we also trivially have
\[
\sup_{t \in (0,1)} t^{\frac{1}{2}} f^*(t) < \infty,
\]
which yields that \( f \in \ell^{4,\infty}(\mathbb{Z}^2) \), as desired.
Finally, let us mention that if a function $f \in \ell^4_{\infty}(\mathbb{Z}^2)$ satisfies (2.5) then an example of a decomposition of $\mathbb{Z}^2$ having the properties as in Lemma 2.1 is

$$S_1 = \{(k, l) \in \mathbb{Z}^2 : |k| \geq |l|\} \quad \text{and} \quad S_2 = \{(k, l) \in \mathbb{Z}^2 : |k| < |l|\}.$$ 

3. Proof of Theorem 1.3

In this section we apply Lemma 2.1 to prove the main result of this paper, Theorem 1.3. To this end, we first introduce the notion of the fractional Lorentz-Sobolev space $L^4_{s, \infty}(\mathbb{R}^{2n})$, where $s > 0$. This space is defined as the collection of all functions $f$ on $\mathbb{R}^{2n}$ for which

$$\|f\|_{L^4_{s, \infty}(\mathbb{R}^{2n})} = \|(I - \Delta)^{\frac{s}{2}}f\|_{L^4_{s, \infty}(\mathbb{R}^{2n})} < \infty.$$ 

It is worth noticing that a function $m$ on $\mathbb{R}^{2n}$ satisfies the assumption (1.4) if and only if $m \in L^4_{s, \infty}(\mathbb{R}^{2n})$, and the constant $C_s(m)$ in (1.4) is equal to $\|m\|_{L^4_{s, \infty}(\mathbb{R}^{2n})}$.

We prove Theorem 1.3 as a consequence of an even sharper result involving a space of Besov type. Let us now introduce this function space as well. We let $\varphi_0$ be a smooth function on $\mathbb{R}^{2n}$ satisfying $\varphi_0(\xi) = 1$ if $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ if $|\xi| \geq \frac{1}{2}$. Further, if $k \in \mathbb{N}$ then we set $\varphi_k(\xi) = \varphi_0(2^{-k} \xi) - \varphi_0(2^{1-k} \xi)$. We denote by $\{\Lambda_k\}_{k=0}^{\infty}$ the inhomogeneous Littlewood-Paley decomposition, defined via the Fourier transform as $\hat{\Lambda}_k f = \hat{\varphi}_k \hat{f}$ for $k \in \mathbb{Z}_0^+$. We then let $B(\mathbb{R}^{2n})$ be the space of all functions $f$ on $\mathbb{R}^{2n}$ satisfying

$$\|f\|_{B(\mathbb{R}^{2n})} = \sum_{k=0}^{\infty} 2^{nk} \|\Lambda_k f\|_{L^4_{s, \infty}(\mathbb{R}^{2n})} < \infty.$$ 

The space $B(\mathbb{R}^{2n})$ is the Besov space with smoothness index $n/2$ and summability index 1 built upon the Lorentz space $L^{4,\infty}(\mathbb{R}^{2n})$. The family of Besov spaces built upon Lorentz spaces and its relation to Lorentz-Sobolev spaces was studied in the recent paper [26] where it was shown, in particular, that $L^{4,\infty}(\mathbb{R}^{2n}) \hookrightarrow B(\mathbb{R}^{2n})$ whenever $s > n/2$.

**Theorem 3.1.** Let $m$ be a function on $\mathbb{R}^{2n}$ belonging to the Besov space $B(\mathbb{R}^{2n})$. Then the associated operator $T_m$ admits a bounded extension from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and

$$\|T_m(f, g)\|_{L^1(\mathbb{R}^n)} \leq C \|m\|_{B(\mathbb{R}^{2n})} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

for some dimensional constant $C$.

The following proposition plays a crucial role in the proofs of Theorems 1.3 and 3.1.
Proposition 3.2. Let $k \in \mathbb{N}$ and let $m \in L^{4,\infty}(\mathbb{R}^{2n})$ be a function whose Fourier transform is supported in the ball $\{x \in \mathbb{R}^{2n} : |x| < 2^k\}$. Then there is a dimensional constant $C$ such that

$$\|T_m(f, g)\|_{L^1(\mathbb{R}^n)} \leq C2^{nk}2^{nk} \|m\|_{L^{4,\infty}(\mathbb{R}^{2n})} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$ 

We will need several auxiliary results to prove Proposition 3.2. We start with the following straightforward lemma.

Lemma 3.3. Let $\Phi$ be a nonnegative function on $\mathbb{R}^n$ satisfying

$$\sup_{x \in \mathbb{R}^n} \Phi(x)(1 + |x|)^\gamma < \infty$$

for some $\gamma > n$. Then

$$\sup_{x \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} \Phi(x - k) < \infty.$$ 

Proof. Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we set

$$\lfloor x \rfloor = ([x_1], [x_2], \ldots, [x_n]) \in \mathbb{Z}^n.$$ 

Then $|x - \lfloor x \rfloor| < \sqrt{n} \leq n$, which yields that for any $k \in \mathbb{Z}^n$,

$$1 + |x - k| \geq 1 + ||x - \lfloor x \rfloor| - |\lfloor x \rfloor - k||$$

$$\geq \begin{cases} 
1 + \frac{|x| - |\lfloor x \rfloor|}{2} & \text{if } ||\lfloor x \rfloor - k|| \geq 2n \\
1 & \text{if } ||\lfloor x \rfloor - k|| < 2n \\
\end{cases}$$

$$\geq \frac{1}{2n + 1}(1 + ||\lfloor x \rfloor - k||).$$

Thus,

$$\sup_{x \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} \Phi(x - k) \lesssim \sup_{x \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} (1 + |x - k|)^{-\gamma}$$

$$\lesssim \sup_{x \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} (1 + |k - \lfloor x \rfloor|)^{-\gamma}$$

$$= \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{-\gamma} < \infty.$$ 

A key tool for proving Proposition 3.2 is a representation of functions in terms of product-type wavelets. In particular, we will make use of the following fact, due to Meyer [20] [21]:

There exist real-valued functions $\Psi_F, \Psi_M \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\Psi_F}$ is compactly supported, $\widehat{\Psi_M}$ is compactly supported away from the origin.
and, if we denote

\[
\Psi_\beta(x) = \prod_{r=1}^{2n} \Psi_F(x_r - \beta_r), \quad \beta = (\beta_1, \ldots, \beta_{2n}) \in \mathbb{Z}^{2n}, \\
x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n},
\]

and

\[
\Psi_{G_\beta}^G(x) = \prod_{r=1}^{2n} \Psi_{G_r}(x_r - \beta_r), \quad \beta \in \mathbb{Z}^{2n}, \quad x \in \mathbb{R}^{2n},
\]

where \(G = (G_1, \ldots, G_{2n}) \in \{F, M\}^{2n} \setminus \{(F, \ldots, F)\}\), then the family of functions

\[
\Psi_{j,G_\beta} = \begin{cases} 
\Psi_\beta(x), & j = 0, \quad G = (F, \ldots, F) \\
2^{(j-1)n} \Psi_{\beta}^G(2^{j-1}x), & j \in \mathbb{N}, \quad G \in \{F, M\}^{2n} \setminus \{(F, \ldots, F)\}
\end{cases}
\]

forms an orthonormal basis in \(L^2(\mathbb{R}^{2n})\).

In addition, the same family of functions is also an unconditional basis in any Lebesgue space \(L^p(\mathbb{R}^{2n})\) with \(1 < p < \infty\). Thus, setting

\[
J = \{(j, G) : \quad j = 0 \text{ and } G = (F, \ldots, F), \\
or j \in \mathbb{N} \text{ and } G \in \{F, M\}^{2n} \setminus \{(F, \ldots, F)\}\},
\]

any function \(f \in L^p(\mathbb{R}^{2n})\) can be expressed in the form

\[
f = \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} \langle f, \Psi_{j,G_\beta} \rangle \Psi_{j,G_\beta},
\]

unconditional convergence being in \(S'(\mathbb{R}^{2n})\). Also, we have the equivalence

\[
\|f\|_{L^p(\mathbb{R}^{2n})} \approx \left( \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} |\langle f, \Psi_{j,G_\beta} \rangle| \chi_{j\beta} |^{2j/n} \right)^{1/2},
\]

where \(\chi_{j\beta}\) denotes the characteristic function of the cube centered at \(2^{-j}\beta\) with side-length \(2^{1-j}\). For the proof of this statement, see, e.g., [29, Theorem 3.12].

The final ingredient needed for the proof of Proposition 3.2 is the following lemma, which can be obtained from (3.1) using real interpolation.

**Lemma 3.4.** Suppose that \(m \in L^{4,\infty}(\mathbb{R}^{2n})\). Let \((j,G) \in J\) and let \(a^{j,G}_\beta = \{a^{j,G}_{\beta}\}_{\beta \in \mathbb{Z}^{2n}}\) be the sequence defined by \(a^{j,G}_{\beta} = \langle m, \Psi_{\beta}^G \rangle\) for \(\beta \in \mathbb{Z}^{2n}\). Then

\[
\|a^{j,G}\|_{L^{4,\infty}(\mathbb{R}^{2n})} \lesssim 2^{-\frac{m}{p}} \|m\|_{L^{4,\infty}(\mathbb{R}^{2n})}.
\]
Proof. Let \( S \) be the sublinear operator defined as
\[
Sf = \left( \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} \left| \langle f, \Psi_{\beta}^{j,G} \rangle 2^{jn} \widetilde{\chi}_{j\beta} \right|^2 \right)^{\frac{1}{2}},
\]
where \( \widetilde{\chi}_{j\beta} \) denotes the characteristic function of the cube centered at \( 2^{-j}\beta \) with side-length \( 2^{-j} \). Since \( \widetilde{\chi}_{j\beta} \leq \chi_{j\beta} \), it follows from (3.1) that the operator \( S \) is bounded on \( L^p(\mathbb{R}^{2n}) \) for every \( p \in (1, \infty) \). The Marcinkiewicz interpolation theorem (see, e.g., [1, Chapter 4, Theorem 4.13]) then yields that \( S \) is bounded on \( L^{4,\infty}(\mathbb{R}^{2n}) \), that is,
\[
\left\| \left( \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} \left| \langle m, \Psi_{\beta}^{j,G} \rangle 2^{jn} \widetilde{\chi}_{j\beta} \right|^2 \right)^{\frac{1}{2}} \right\|_{L^{4,\infty}(\mathbb{R}^{2n})} \lesssim \| m \|_{L^{4,\infty}(\mathbb{R}^{2n})}.
\]
Therefore, fixing a pair \( (j, G) \) and using that the supports of the functions \( \widetilde{\chi}_{j\beta} \) do not overlap in \( \beta \) (except perhaps for the boundary of the corresponding cubes), we obtain
\[
\left\| \sum_{\beta \in \mathbb{Z}^{2n}} a_{\beta}^{j,G} \widetilde{\chi}_{j\beta} \right\|_{L^{4,\infty}(\mathbb{R}^{2n})} \lesssim \| m \|_{L^{4,\infty}(\mathbb{R}^{2n})}.
\]
Now, for any \( \lambda > 0 \),
\[
\left\{ x \in \mathbb{R}^{2n} : \sum_{\beta \in \mathbb{Z}^{2n}} |a_{\beta}^{j,G}| \widetilde{\chi}_{j\beta}(x) > \lambda \right\} = 2^{-2jn} \text{card}\{ \beta \in \mathbb{Z}^{2n} : |a_{\beta}^{j,G}| > \lambda \},
\]
which implies that
\[
\left\| \sum_{\beta \in \mathbb{Z}^{2n}} a_{\beta}^{j,G} \widetilde{\chi}_{j\beta} \right\|_{L^{4,\infty}(\mathbb{R}^{2n})} = \sup_{\lambda > 0} \lambda^{\frac{1}{4}} \left\{ x \in \mathbb{R}^{2n} : \sum_{\beta \in \mathbb{Z}^{2n}} |a_{\beta}^{j,G}| \widetilde{\chi}_{j\beta}(x) > \lambda \right\}^{\frac{1}{4}} = 2^{-jn} \sup_{\lambda > 0} \lambda^{\frac{1}{4}} \left( \text{card}\{ \beta \in \mathbb{Z}^{2n} : |a_{\beta}^{j,G}| > \lambda \} \right)^{\frac{1}{4}} = 2^{-jn} \| a^{j,G} \|_{\ell^{4,\infty}(\mathbb{Z}^{2n})}.
\]
Combining this equality with (3.2) yields the conclusion. \( \square \)

Proof of Proposition 3.2. Let \( (j, G) \in J \) and let \( a_{\beta}^{j,G} \) be the sequence defined in Lemma 3.4. We observe that
\[
a_{\beta}^{j,G} = \langle m, \Psi_{\beta}^{j,G} \rangle = \langle \hat{m}, (\Psi_{\beta}^{j,G})^* \rangle,
\]
where \( \hat{m} \) is the Fourier transform of \( m \).
and that for every \( j \in \mathbb{N} \), the inverse Fourier transform of \( \Psi^{j,G}_{\beta} \) is supported in the annulus \( \{ x \in \mathbb{R}^{2n} : K_1 2^j < |x| < K_2 2^j \} \), where \( K_1 \) and \( K_2 \) are suitable dimensional constants. Using the support properties of \( \hat{m} \), we thus deduce that \( a^{j,G}_{\beta} = 0 \) whenever \( j \in \mathbb{N} \) satisfies \( K_1 2^j \geq 2^k \).

Since \( m \in L^{4,\infty}(\mathbb{R}^{2n}) \), we can write \( m = m_1 + m_2 \), where \( m_1 \in L^3(\mathbb{R}^{2n}) \) and \( m_2 \in L^5(\mathbb{R}^{2n}) \). As the family \( \{ \Psi^{j,G}_{\beta} \} \) is an unconditional basis in both \( L^3(\mathbb{R}^{2n}) \) and \( L^5(\mathbb{R}^{2n}) \), we deduce that \( m \) can be represented as

\[
m = \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} a^{j,G}_{\beta} \Psi^{j,G}_{\beta},
\]

unconditional convergence being in \( \mathcal{S}'(\mathbb{R}^{2n}) \). Consequently, whenever \( f \) and \( g \) are Schwartz functions on \( \mathbb{R}^n \), we have the pointwise identity

\[
T_m(f, g)(x) = \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} a^{j,G}_{\beta} T_{\Psi^{j,G}_{\beta}}(f, g)(x)
= \sum_{(j,G) \in J} \sum_{\beta \in \mathbb{Z}^{2n}} a^{j,G}_{\beta} T_{\Psi^{j,G}_{\beta}}(f, g)(x), \quad x \in \mathbb{R}^n.
\]

Let \( U \) be any subset of \( \{(j, G) \in J : j = 0 \text{ or } K_1 2^j \leq 2^k \} \) and let \( V \) be any finite subset of \( \mathbb{Z}^{2n} \). By the Fatou lemma,

\[
\| T_m(f, g) \|_{L^1(\mathbb{R}^n)} \leq \sup_{U, V} \left\| \sum_{(j,G) \in U} \sum_{\beta \in V} a^{j,G}_{\beta} F^{-1}(\Psi^{j,G}_{\beta}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta)) \right\|_{L^1(\mathbb{R}^n)}
\leq \sup_{U, V} \sum_{(j,G) \in U} \left\| \sum_{\beta \in V} a^{j,G}_{\beta} F^{-1}(\Psi^{j,G}_{\beta}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta)) \right\|_{L^1(\mathbb{R}^n)}.
\]

We fix \( (j, G) \in U \). We shall use the notation \( \beta = (k, l) \), \( (k, l) \in \mathbb{Z}^n \times \mathbb{Z}^n = \mathbb{Z}^{2n} \),

\[
a^{j,G}_{\beta} = a_{k,l} \quad \text{and} \quad \Psi^{j,G}_{\beta}(\xi, \eta) = \omega_{1,k}(\xi) \omega_{2,l}(\eta),
\]

where the multiplicative constants in the definition of the functions \( \omega_{1,k} \) and \( \omega_{2,l} \) are chosen in such a way that \( \| \omega_{1,k} \|_{L^\infty(\mathbb{R}^n)} \approx \| \omega_{1,l} \|_{L^\infty(\mathbb{R}^n)} \approx 2^{3r} \).

According to Lemma \[2.1\] there are two pairwise disjoint sets \( S_1, S_2 \) in \( \mathbb{Z}^{2n} \) such that \( S_1 \cup S_2 = \mathbb{Z}^{2n} \),

\[
\left( \sum_{(k, l) \in S_1} |a_{k,l}|^2 \right)^{\frac{1}{2}} \leq C \| a \|_{L^1(\mathbb{Z}^{2n})} \quad \text{for every } k \in \mathbb{Z}^n.
\]
and
\[
\left( \sum_{k: (k,l) \in S_2} |a_{k,l}|^2 \right)^{\frac{1}{2}} \leq C\|a\|_{\ell^4(\mathbb{Z}^n)} \quad \text{for every } l \in \mathbb{Z}^n.
\]

Let us now use Lemma 3.3 to derive two preliminary estimates that will be needed later on. Assume that \( j \in \mathbb{N} \) and \( f \in S(\mathbb{R}^n) \). Then the function \( \Phi(\xi) = \prod_{r=1}^n \Psi^{2, r}_{G_r}(\xi_r) \) belongs to the Schwartz space \( S(\mathbb{R}^n) \), and therefore satisfies the assumption of Lemma 3.3. Consequently,

(3.3) \[
\sum_{k \in \mathbb{Z}^n} \|\omega_{1,k} \hat{f}\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{k \in \mathbb{Z}^n} |\omega_{1,k}(\xi)|^2 \, d\xi
\]
\[
\approx 2^{jn} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{k \in \mathbb{Z}^n} \Phi(2^j \xi - k) \, d\xi
\]
\[
\approx 2^{jn} \|f\|_{L^2(\mathbb{R}^n)}^2.
\]

To derive the second estimate, we apply Lemma 3.3 with the functions

\[
\Phi_1(\xi) = (1 + |\xi|)^{-2n} \quad \text{and} \quad \Phi_2(\xi) = (1 + |\xi|)^{2n} \prod_{r=1}^n \Psi^{2, r}_{G_r}(\xi_r), \quad \xi \in \mathbb{R}^n.
\]

We notice that

\[
|\omega_{1,k}(\xi)|^2 = 2^{(j-1)n} \prod_{r=1}^n \Psi^{2, r}_{G_r}(2^{j-1} \xi_r - k_r)
\]
\[
= 2^{(j-1)n} \Phi_1(2^{j-1} \xi - k) \Phi_2(2^{j-1} \xi - k).
\]

Therefore,

(3.4) \[
\sum_{l \in \mathbb{Z}^n} \left\| \sum_{k: (k,l) \in S_1 \cap V} a_{k,l} \omega_{1,k} \hat{f} \right\|_{L^2(\mathbb{R}^n)}^2
\]
\[
\approx 2^{jn} \sum_{l \in \mathbb{Z}^n} \left\| \sum_{k: (k,l) \in S_1 \cap V} a_{k,l} [\Phi_1(2^{j-1} \xi - k)]^{\frac{1}{2}} [\Phi_2(2^{j-1} \xi - k)]^{\frac{1}{2}} \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)}^2
\]
\[
\leq 2^{jn} \sum_{l \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \left( \sum_{k: (k,l) \in S_1 \cap V} |a_{k,l}|^2 \Phi_1(2^{j-1} \xi - k) \right) \left( \sum_{k \in \mathbb{Z}^n} \Phi_2(2^{j-1} \xi - k) \right) \, d\xi
\]
\[
\leq 2^{jn} \sum_{k \in \mathbb{Z}^n \cap V} \sum_{l: (k,l) \in S_1 \cap V} |a_{k,l}|^2 \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \Phi_1(2^{j-1} \xi - k) \, d\xi
\]
\[
\leq 2^{jn} \left\| a \right\|_{\ell^4(\mathbb{Z}^n)}^2 \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{k \in \mathbb{Z}^n} \Phi_1(2^{j-1} \xi - k) \, d\xi
\]
\[
\approx 2^{jn} \left\| a \right\|_{\ell^4(\mathbb{Z}^n)}^2 \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{k \in \mathbb{Z}^n} \Phi_1(2^{j-1} \xi - k) \, d\xi
\]
Applying the inequalities we have just derived, we conclude that

\[
\| \sum_{\beta \in V} a^{jG}_\beta \mathcal{F}^{-1}(\Psi^{jG}_\beta(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta)) \|_{L^1(\mathbb{R}^n)}
\]

\[
\leq \| \sum_{(k,l) \in S_1 \cap V} a_{k,l} \mathcal{F}^{-1}(\omega_{1,k} \hat{f}) \mathcal{F}^{-1}(\omega_{2,l} \hat{g}) \|_{L^1(\mathbb{R}^n)}
\]

\[
+ \| \sum_{(k,l) \in S_2 \cap V} a_{k,l} \mathcal{F}^{-1}(\omega_{1,k} \hat{f}) \mathcal{F}^{-1}(\omega_{2,l} \hat{g}) \|_{L^1(\mathbb{R}^n)}
\]

\[
\leq \sum_{l \in \mathbb{Z}^n} \| \sum_{k: (k,l) \in S_1 \cap V} a_{k,l} \mathcal{F}^{-1}(\omega_{1,k} \hat{f}) \mathcal{F}^{-1}(\omega_{2,l} \hat{g}) \|_{L^1(\mathbb{R}^n)}
\]

\[
+ \sum_{k \in \mathbb{Z}^n} \| \sum_{l: (k,l) \in S_2 \cap V} a_{k,l} \mathcal{F}^{-1}(\omega_{1,k} \hat{f}) \mathcal{F}^{-1}(\omega_{2,l} \hat{g}) \|_{L^1(\mathbb{R}^n)}
\]

\[
\lesssim \left( \sum_{l \in \mathbb{Z}^n} \| \omega_{2,l} \hat{g} \|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^n} \| \omega_{1,k} \hat{f} \|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}
\]

\[
+ \left( \sum_{k \in \mathbb{Z}^n} \| \omega_{1,k} \hat{f} \|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \left( \sum_{l \in \mathbb{Z}^n} \| \omega_{2,l} \hat{g} \|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim 2^{in} \| a \|_{\ell_4, \infty(\mathbb{Z}^n)} \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}.
\]

Using this and Lemma 3.4 we obtain

\[
\| T_m(f, g) \|_{L^1(\mathbb{R}^n)} \leq \sup_U \left( \sum_{(j,G) \in U} 2^{jn} \| a^{jG} \|_{\ell_4, \infty(\mathbb{Z}^n)} \right) \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}
\]

\[
\lesssim \left( \sum_{(j,G) \in J} 2^{jn} \right) \| m \|_{L^{4, \infty}(\mathbb{R}^n)} \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}
\]

\[
\approx 2^{kn} \| m \|_{L^{4, \infty}(\mathbb{R}^n)} \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}.
\]
Proof of Theorem 3.1. We observe that the Littlewood-Paley decomposition \( \{ \Lambda_k m \}_{k=0}^{\infty} \) of \( m \) has the property that \( \hat{\Lambda_k m} \) is supported in the ball \( \{ x \in \mathbb{R}^n : |x| < 2^k + 2 \} \) for each \( k \in \mathbb{Z}_0^+ \). Proposition 3.2 then yields
\[
\| T_m(f, g) \|_{L^1(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty} \| T_{\Lambda_k m}(f, g) \|_{L^1(\mathbb{R}^n)} \\
\leq \sum_{k=0}^{\infty} 2^{nk} \| \Lambda_k m \|_{L^4,\infty(\mathbb{R}^n)} \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)} \\
= \| m \|_{B(\mathbb{R}^n)} \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}.
\]

Proof of Theorem 1.3. This is a consequence of Theorem 3.1 and of the embedding \( L^4,\infty(\mathbb{R}^2,n) \hookrightarrow B(\mathbb{R}^2,n) \) which holds whenever \( s > \frac{n}{2} \) according to [26, Theorem 1.2]. □

4. Comparison of Corollary 1.5 with other bilinear multiplier theorems

We start this section by showing that each multiplier satisfying the assumptions of [13, Theorem 1.3] falls under the scope of Corollary 1.5 as well. More precisely, we show that whenever \( m \) is a function belonging to \( L^q(\mathbb{R}^2,n) \) for some \( 1 < q < 4 \) whose partial derivatives up to the order \( \lfloor \frac{2n}{4-q} \rfloor + 1 \) are bounded, then \( m \in L^4,\infty(\mathbb{R}^2,n) \) for some \( s > \frac{n}{2} \). The proof of this statement is based on a fractional variant of the classical Gagliardo-Nirenberg interpolation inequality due to Brezis and Mironescu [2, Theorem 1]. We start with a few preliminaries on fractional Sobolev spaces.

So far, we have worked with fractional Sobolev spaces defined via the Fourier transform. Let us now introduce another variant of these spaces. For any \( s > 0 \) and \( 1 \leq p \leq \infty \), we set \( k = \lfloor s \rfloor \) and define the functional
\[
|f|_{W^{s,p}(\mathbb{R}^2,n)} = \begin{cases} 
\| D_k f \|_{L^p(\mathbb{R}^2,n)} & \text{if } s \in \mathbb{N} \\
\int_{\mathbb{R}^2,n} \int_{\mathbb{R}^2,n} \frac{|D_k^x f(x) - D_k^y f(y)|^p}{|x-y|^{2n+(s-k)p}} \, dx \, dy & \text{if } s \notin \mathbb{N} \text{ and } p < \infty \\
\sup_{x \neq y} \frac{|D_k^x f(x) - D_k^y f(y)|}{|x-y|^{s-k}} & \text{if } s \notin \mathbb{N} \text{ and } p = \infty.
\end{cases}
\]

The fractional Sobolev space \( W^{s,p}(\mathbb{R}^2,n) \) is then defined as the collection of all \( k \)-times weakly differentiable functions \( f \) on \( \mathbb{R}^2,n \) satisfying
\[
\| f \|_{W^{s,p}(\mathbb{R}^2,n)} = \| f \|_{L^p(\mathbb{R}^2,n)} + |f|_{W^{s,p}(\mathbb{R}^2,n)} < \infty.
\]
In general, the space $W^{s,p}(\mathbb{R}^{2n})$ does not coincide with $L^p_s(\mathbb{R}^{2n})$ but we have the chain of embeddings

$$W^{s_2,p}(\mathbb{R}^{2n}) \hookrightarrow L^p_s(\mathbb{R}^{2n}) \hookrightarrow W^{s_1,p}(\mathbb{R}^{2n}),$$

where $0 < s_1 < s < s_2$ and $1 < p < \infty$. The first embedding in (4.1) implies that the statement of Corollary 1.5 continues to hold when the space $L^4_s(\mathbb{R}^{2n})$ is replaced by $W^{s,4}(\mathbb{R}^{2n})$.

**Corollary 4.1.** Let $m$ be a function on $\mathbb{R}^{2n}$ belonging to $W^{s,4}(\mathbb{R}^{2n})$ for some $s > \frac{n}{2}$. Then the associated operator $T_m$ admits a bounded extension from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and

$$\|T_m(f,g)\|_{L^1(\mathbb{R}^n)} \leq C\|m\|_{W^{s,4}(\mathbb{R}^{2n})}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$ 

It follows from [2, Theorem 1] that

$$\|m\|_{W^{s,4}(\mathbb{R}^{2n})} \lesssim \|m\|_{L^q(\mathbb{R}^{2n})}^{\frac{1}{2}}\|m\|_{W^{s,\infty}(\mathbb{R}^{2n})}^{1-\frac{1}{2q}}$$

holds whenever $s > 0$, $1 < q < \infty$ and $\tilde{s} = \frac{4s}{4-q}$. A combination of this inequality and of Corollary 4.1 yields the following fractional variant of [13, Theorem 1.3].

**Corollary 4.2.** Let $1 < q < 4$ and $s > 2n\frac{1-q}{4}$. Let $m$ be a function on $\mathbb{R}^{2n}$ belonging to $L^q(\mathbb{R}^{2n}) \cap W^{s,\infty}(\mathbb{R}^{2n})$. Then the associated operator $T_m$ admits a bounded extension from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and

$$\|T_m(f,g)\|_{L^1(\mathbb{R}^n)} \leq C\|m\|_{L^q(\mathbb{R}^{2n})}^{\frac{1}{2}}\|m\|_{W^{s,\infty}(\mathbb{R}^{2n})}^{1-\frac{1}{2q}}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}.$$ 

Let us now focus on the comparison of Corollary 1.5 with the bilinear variant of the Hörmander multiplier theorem [12, Theorem 1]. As mentioned in Section 1, we first observe that it suffices to formulate the latter theorem with the integrability index $r = 4$.

**Lemma 4.3.** Let $\phi$ be a smooth function on $\mathbb{R}^{2n}$ supported in the annulus $\{\xi \in \mathbb{R}^{2n}: 1/2 < |\xi| < 2\}$. Assume that $1 < r < \infty$ and $s > \max\{n/2, 2n/r\}$. Then there is $\tilde{s} > \frac{n}{2}$ such that

$$\sup_{k \in \mathbb{Z}}\|m(2^k \cdot)\phi\|_{L^1(\mathbb{R}^{2n})} \lesssim \sup_{k \in \mathbb{Z}}\|m(2^k \cdot)\phi\|_{L^1(\mathbb{R}^{2n})}.$$ 

**Proof.** We first assume that $1 < r < 4$. Then the embedding $L^4_s(\mathbb{R}^{2n}) \hookrightarrow L^r_s(\mathbb{R}^{2n})$ holds with $\tilde{s} = s - 2n/r + n/2 > n/2$, see, e.g., [28, Section 2.7.1]. This implies (4.2). On the other hand, if $r > 4$ then we let $\Phi$ be a smooth function supported in the annulus $\{\xi \in \mathbb{R}^{2n}: 1/4 < |\xi| < 4\}$ such that $\Phi(\xi) = 1$ whenever $1/2 < |\xi| < 2$. The Kato-Ponce inequality [17, Theorem 1] then yields that for each $k \in \mathbb{Z}$,

$$\|m(2^k \cdot)\phi\|_{L^1(\mathbb{R}^{2n})} = \|(I - \Delta)^{\frac{s}{2\tilde{s}}}m(2^k \cdot)\phi\|_{L^1(\mathbb{R}^{2n})}.$$
\[ = \|(I - \Delta)^{s/2} [m(2^k \cdot) \phi] \|_{L^4(\mathbb{R}^{2n})} \]
\[ \lesssim \|\Phi\|_{L^{r_0}(\mathbb{R}^{2n})} \|(I - \Delta)^{s/2} m(2^k \cdot) \phi \|_{L^r(\mathbb{R}^{2n})} \]
\[ + \|(I - \Delta)^{s/2} \Phi \|_{L^{r_0}(\mathbb{R}^{2n})} \|m(2^k \cdot) \phi\|_{L^r(\mathbb{R}^{2n})} \]
\[ \lesssim \|(I - \Delta)^{s/2} m(2^k \cdot) \phi\|_{L^r(\mathbb{R}^{2n})} \]
\[ = \|m(2^k \cdot) \phi\|_{L^r(\mathbb{R}^{2n})}, \]

where \( r_0 > 4 \) is a real number satisfying \( 1/r + 1/r_0 = 1/4 \). Since \( s > n/2 \) in this case, inequality (4.2) follows. □

Let us now show that Corollary 1.5 and the bilinear variant of the Hörmander multiplier theorem [12, Theorem 1] are not comparable. Clearly, any nontrivial constant function satisfies the assumptions of [12, Theorem 1] but not of Corollary 1.5. On the other hand, [12, Theorem 1] imposes stronger assumptions on the decay of derivatives of the multiplier near infinity than Corollary 1.5. A precise formulation of this fact is the content of the following proposition. An example which further illustrates it is provided in the next section.

We recall that, for any \( s > 0 \), the operator \((I - \Delta)^{s/2}\) is defined via the Fourier transform as
\[ \hat{(-\Delta)^{s/2} f}(\xi) = (4\pi^2|\xi|^2)^{s/2} \hat{f}(\xi), \quad \xi \in \mathbb{R}^{2n}. \]

**Proposition 4.4.** Assume that \( m \) is a function on \( \mathbb{R}^{2n} \) supported away from the origin and satisfying
\[ \sup_{k \in \mathbb{Z}} \|(I - \Delta)^{s/2} m(2^k \cdot) \phi\|_{L^4(\mathbb{R}^{2n})} < \infty \]
for some \( s > n/2 \), where \( \phi \) is a smooth function supported in the annulus \( \{\xi \in \mathbb{R}^{2n} : 1/2 < |\xi| < 2\} \) and satisfying \( \sum_{k \in \mathbb{Z}} \phi(2^k \cdot) = 1 \). Then
\[ \|(I - \Delta)^{s/2} m\|_{L^q(\mathbb{R}^{2n})} < \infty \]
for every \( q \in (\max\{\frac{2n}{s}, 1\}, 4) \).

**Proof.** Since \( m \) is supported away from the origin, there is \( l \in \mathbb{Z} \) such that \( m = 0 \) on the support of \( \phi(2^{-k} \cdot) \) when \( k < l \). Therefore,
\[ m = \sum_{k=-\infty}^{\infty} \phi(2^{-k} \cdot) m = \sum_{k=l}^{\infty} \phi(2^{-k} \cdot) m. \]

Let \( q \in (\max\{\frac{2n}{s}, 1\}, 4) \) and let \( \Phi \) be a smooth function which is equal to one on the annulus \( \{\xi \in \mathbb{R}^{2n} : 1/2 < |\xi| < 2\} \) and supported
in \( \{ \xi \in \mathbb{R}^{2n} : 1/4 < |\xi| < 4 \} \). Using \([44] \) and the Kato-Ponce inequality \([17] \), we obtain

\[
\|(-\Delta)^{\frac{s}{2}} m\|_{L^q(\mathbb{R}^{2n})} \leq \sum_{k=1}^{\infty} \|(-\Delta)^{\frac{s}{2}} [\Phi(2^{-k} \cdot) m]\|_{L^q(\mathbb{R}^{2n})}
\]

\[
= \sum_{k=1}^{\infty} \|(-\Delta)^{\frac{s}{2}} [\Phi(2^{-k} \cdot) \phi(2^{-k} \cdot) m]\|_{L^q(\mathbb{R}^{2n})}
\]

\[
\lesssim \sum_{k=1}^{\infty} \|(-\Delta)^{\frac{s}{2}} [\phi(2^{-k} \cdot) m]\|_{L^4(\mathbb{R}^{2n})} \|\Phi(2^{-k} \cdot)\|_{L^4(\mathbb{R}^{2n})}
\]

\[
+ \sum_{k=1}^{\infty} \|\phi(2^{-k} \cdot) m\|_{L^{\frac{4q}{q-4}}(\mathbb{R}^{2n})} \|(-\Delta)^{\frac{s}{2}} [\Phi(2^{-k} \cdot)]\|_{L^4(\mathbb{R}^{2n})}
\]

\[
\lesssim \sum_{k=1}^{\infty} 2^{-k(s-\frac{2n}{q})} \|(-\Delta)^{\frac{s}{2}} [m(2^k \cdot) \phi]\|_{L^4(\mathbb{R}^{2n})} \|\Phi\|_{L^{\frac{4q}{q-4}}(\mathbb{R}^{2n})}
\]

\[
+ \sum_{k=1}^{\infty} 2^{-k(s-\frac{2n}{q})} \|m(2^k \cdot) \phi\|_{L^{\frac{4q}{q-4}}(\mathbb{R}^{2n})} \|(-\Delta)^{\frac{s}{2}} \Phi\|_{L^4(\mathbb{R}^{2n})}
\]

\[
\lesssim \sup_{k \in \mathbb{Z}} \|(-\Delta)^{\frac{s}{2}} [m(2^k \cdot) \phi]\|_{L^4(\mathbb{R}^{2n})} + \sup_{k \in \mathbb{Z}} \|m(2^k \cdot) \phi\|_{L^\infty(\mathbb{R}^{2n})}
\]

\[
\lesssim \sup_{k \in \mathbb{Z}} \|(I - \Delta)^{\frac{s}{2}} [m(2^k \cdot) \phi]\|_{L^4(\mathbb{R}^{2n})},
\]

where the last inequality follows from the fact that the Sobolev space \( L^4_s(\mathbb{R}^{2n}) \) is continuously embedded into \( L^\infty(\mathbb{R}^{2n}) \). \( \square \)

5. Proof of Theorem \([44] \)

In this section we focus on proving the sharpness of Theorem \([33] \) as stated in Theorem \([44] \). The following proposition is an important step towards achieving this goal.

**Proposition 5.1.** Assume that \( \Psi \) is a nontrivial smooth function supported in the set \( \{ x \in \mathbb{R}^{2n} : |x|_\infty < \frac{1}{10} \} \). Given a sequence \( c = \{ c_{k,l} \}_{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n} \) of complex numbers, consider the function \( m \) defined by

\[
m(\xi, \eta) = \sum_{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n} c_{k,l} \Psi(\xi - k, \eta - l), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

If \( c \in \ell^{1,\infty}(\mathbb{Z}^{2n}) \) then the operator \( T_m \) admits a bounded extension from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \), and there is a constant \( C \) depending on \( n \) and \( \Psi \) such that

\[
\|T_m(f, g)\|_{L^1(\mathbb{R}^n)} \leq C\|c\|_{\ell^{1,\infty}(\mathbb{Z}^{2n})} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.
\]
Conversely, assume that 

\[ d_{k_1,l_1} \leq d_{k_2,l_2} \quad \text{if} \quad |(k_1,l_1)|_\infty > |(k_2,l_2)|_\infty \]

and does not belong to \( \ell^{4,\infty}(\mathbb{Z}^{2n}) \). Then there exists a sequence \( c = \{c_{k,l}\}_{(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^n} \) of real numbers such that \( |c_{k,l}| = d_{k,l} \) for all \((k,l)\) \in \(\mathbb{Z}^n \times \mathbb{Z}^n\), for which the associated operator \(T_m\) is unbounded from \(L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) to \(L^1(\mathbb{R}^n)\).

Before we get to the proof of Proposition 5.1, we point out that the sufficiency part of it cannot be obtained by making use of [13, Theorem 1.3], nor by using the bilinear variant of the Hörmander multiplier theorem [12, Theorem 1]. Indeed, if a function of the form (5.1) satisfies the assumptions of either of these theorems then the corresponding sequence \(c\) necessarily belongs to \(\ell^q(\mathbb{Z}^{2n})\) for some \(q < 4\). This is straightforward to verify in the former case while in the latter case, the conclusion follows by applying Proposition 4.4. Clearly, the requirement that \(c \in \ell^q(\mathbb{Z}^{2n})\) (and, the more so, stronger than requiring \(c \in \ell^{4,\infty}(\mathbb{Z}^{2n})\)).

**Proof of Proposition 5.1.** We first verify that if \(c \in \ell^{4,\infty}(\mathbb{Z}^{2n})\) then the function \(m\) given by (5.1) belongs to the Lorentz-Sobolev space \(L^{4,\infty}_s(\mathbb{R}^{2n})\) for some \(s > \frac{n}{2}\). To this end, we fix an even integer \(s > \frac{n}{2}\) and observe that

\[
\|m\|_{L^{4,\infty}_s(\mathbb{R}^{2n})} \leq \sup_{|\alpha| \leq s} \|\partial^\alpha m\|_{L^{4,\infty}(\mathbb{R}^{2n})} \lesssim \|c\|_{\ell^{4,\infty}(\mathbb{Z}^{2n})}.
\]

Applying Theorem 1.3 we obtain the first part of the proposition.

Let us now focus on the second part of the proposition. The assumption \(d \notin \ell^{4,\infty}(\mathbb{Z}^{2n})\) tells us that

\[
\sup_{t>0} t^{\frac{4}{n}} d^*(t) = \infty.
\]

Since \(d\) is a bounded sequence, we have

\[
\sup_{t \in (0,1^{2n})} t^{\frac{4}{n}} d^*(t) < \infty,
\]

and thus

\[
\sup_{t \in [1^{2n}, \infty)} t^{\frac{4}{n}} d^*(t) = \infty.
\]

Furthermore, thanks to the monotonicity of the function \(d^*\), the last equality implies

\[
\sup_{N \in \mathbb{N}} N^{\frac{4}{n}} d^*((4N)^{2n}) = \infty.
\]
Therefore, we can find an increasing sequence \( \{b_K\}_{K \in \mathbb{N}} \) of positive integers satisfying \( b_{K+1} > 2b_K \) for every \( K \in \mathbb{N} \) and

\[
\lim_{K \to \infty} b_K^2 d^\ast((4b_K)^{2n}) = \infty.
\]

We set \( \rho_K = (4b_K)^{2n} \), \( K \in \mathbb{N} \). Then

\[
\lim_{K \to \infty} \rho_K^\frac{1}{2} d^\ast(\rho_K) = \infty.
\]

We denote by \( \{a_l(t)\}_{l \in \mathbb{Z}^n} \) the sequence of Rademacher functions indexed by the elements of the countable set \( \mathbb{Z}^n \). For any given \( t \in [0, 1] \) we define the function

\[
m_t(\xi, \eta) = \sum_{(j,k) \in \mathbb{Z}^{2n}} a_{j+k}(t) d_{j,k} \Psi(\xi - j, \eta - k), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

Further, let \( \varphi \) be a Schwartz function on \( \mathbb{R}^n \) whose Fourier transform is supported in the set \( \{\xi \in \mathbb{R}^n : |\xi|_\infty < \frac{1}{5}\} \) and which satisfies \( \hat{\varphi}(\xi) = 1 \) if \( |\xi|_\infty \leq \frac{1}{10} \). Given any \( K \in \mathbb{N} \), we denote \( I_K = \{b_K, b_K + 1, \ldots, 2b_K - 1\} \) and define \( f_K = g_K \) to be the functions on \( \mathbb{R}^n \) whose Fourier transform satisfies

\[
\hat{f}_K(\xi) = \sum_{j \in I_K^n} \hat{\varphi}(\xi - j), \quad \xi \in \mathbb{R}^n.
\]

Then

\[
m_t(\xi, \eta) \hat{f}_K(\xi) \hat{g}_K(\eta) = \sum_{j \in I_K^n} \sum_{k \in I_K^n} a_{j+k}(t) d_{j,k} \Psi(\xi - j, \eta - k).
\]

This yields

\[
T_{m_t}(f_K, g_K)(x) = (\mathcal{F}^{-1} \Psi)(x, x) \sum_{j \in I_K^n} \sum_{k \in I_K^n} a_{j+k}(t) d_{j,k} e^{2\pi i x \cdot (j+k)}
\]

\[
= (\mathcal{F}^{-1} \Psi)(x, x) \sum_{l \in I_K^n + I_K^n} a_l(t) e^{2\pi i x \cdot l} \sum_{j \in I_K^n : l - j \in I_K^n} d_{j,l-j}
\]

for \( x \in \mathbb{R}^n \). By Fubini’s theorem and Khintchine’s inequality,

\[
\int_0^1 \|T_{m_t}(f_K, g_K)\|_{L^1(\mathbb{R}^n)} dt = \int_{\mathbb{R}^n} \int_0^1 |T_{m_t}(f, g)(x)| dt dx
\]

\[
\approx \int_{\mathbb{R}^n} |(\mathcal{F}^{-1} \Psi)(x, x)| dx \left( \sum_{l \in I_K^n + I_K^n} \left( \sum_{j \in I_K^n : l - j \in I_K^n} d_{j,l-j} \right)^2 \right)^{\frac{1}{2}} \geq d^\ast((4b_K)^{2n}) \left( \sum_{l \in I_K^n + I_K^n} \text{card}\{j \in I_K^n : l - j \in I_K^n\}^2 \right)^{\frac{1}{2}} \geq d^\ast(\rho_K)(b_K)^{2n} \geq \rho_K^\frac{3}{2} d^\ast(\rho_K).
\]
Here, we have used that $d_{j,k} \geq d^*((4b_K)^{2n})$ if $(j,k) \in I_n^K \times I_n^K$, a fact which follows from the monotonicity assumption (5.2). Estimate (5.4) now yields that there is a sequence $\{t_K\}_{K \in \mathbb{N}}$ of numbers in $[0,1]$ such that

$$\|T_{m_{tK}}(f_K,g_K)\|_{L^1(\mathbb{R}^n)} \gtrsim \rho_{K}^{\frac{3}{4}} d^*(\rho_{K}).$$

Let us define the sequence $c$ as

$$c_{j,k} = \begin{cases} a_{j+k}(t_K)d_{j,k} & \text{if } (j,k) \in I_n^K \times I_n^K \text{ for some } K \in \mathbb{N} \\ d_{j,k} & \text{if } (j,k) \in \mathbb{Z}^{2n} \setminus \bigcup_{K \in \mathbb{N}} (I_n^K \times I_n^K). \end{cases}$$

We notice that this definition is correct since the sets $I_K^n$ are pairwise disjoint.

Let $m$ be the function given by (5.1). Using the support properties of $f_K$ and $g_K$, it is not difficult to observe that

$$T_m(f_K,g_K) = T_{m_{tK}}(f_K,g_K)$$

for every $K \in \mathbb{N}$. Consequently,

$$\|T_{m}(f_K,g_K)\|_{L^1(\mathbb{R}^n)} \gtrsim \rho_{K}^{\frac{3}{4}} d^*(\rho_{K}).$$

Assume, for the sake of contradiction, that $T_m$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Then

$$\rho_{K}^{\frac{3}{4}} d^*(\rho_{K}) \lesssim \|f_K\|_{L^2(\mathbb{R}^n)} \|g_K\|_{L^2(\mathbb{R}^n)} \lesssim \rho_{K}^{\frac{1}{4}}.$$

This yields

$$\sup_{K \in \mathbb{N}} \rho_{K}^{\frac{1}{4}} d^*(\rho_{K}) < \infty,$$

which contradicts (5.3). The proof is complete. □

Having Proposition 5.1 at our disposal, we can now prove Theorem 1.4.

**Proof of Theorem 1.4.** Assume that $\Psi$ is a function as in Proposition 5.1 which satisfies, in addition, the pointwise estimate $|(I-\Delta)^{\frac{s}{2}}\Psi| \leq 1$. Also, let $m$ be any function of the form (5.1). Since $s \in 2\mathbb{N}$, we have that all functions of the form $(I-\Delta)^{\frac{s}{2}}[\Psi(\xi - k, \eta - l)]$ for some $(k,l) \in \mathbb{Z}^{2n}$ are compactly supported in a set of measure less than 1 and their supports are pairwise disjoint in $k$ and $l$. Therefore,

$$\{x \in \mathbb{R}^{2n} : |(I-\Delta)^{\frac{s}{2}}m(x)| > \lambda\}$$

$$\leq \text{card}\{(k,l) \in \mathbb{Z}^{2n} : |c_{k,l}| > \lambda\}, \quad \lambda \in (0,1).$$

Let $\tau(\lambda)$ be the right-continuous function on $(0,1)$ that is equal to $[\mu(\lambda)]$ for a.e. $\lambda \in (0,1)$. Since the function $\tau$ is right-continuous, non-increasing and has values in $\mathbb{Z}_0^+$, there is a sequence $d = \{d_{k,l}\}_{(k,l) \in \mathbb{Z}^{2n}}$ satisfying

$$\{x \in \mathbb{R}^{2n} : |\tau(\lambda)| > \lambda\} \leq \text{card}\{d_{k,l} \in \mathbb{Z}_0^+ : |d_{k,l}| > \lambda\}, \quad \lambda \in (0,1).$$

Let $\tau(\lambda)$ be the right-continuous function on $(0,1)$ that is equal to $[\mu(\lambda)]$ for a.e. $\lambda \in (0,1)$. Since the function $\tau$ is right-continuous, non-increasing and has values in $\mathbb{Z}_0^+$, there is a sequence $d = \{d_{k,l}\}_{(k,l) \in \mathbb{Z}^{2n}}$.
of non-negative numbers satisfying the monotonicity assumption \( \text{(5.2)} \) for which
\[
\text{card}\{(k, l) \in \mathbb{Z}^{2n} : d_{k,l} > \lambda\} = \tau(\lambda), \quad \lambda \in (0, 1).
\]

Since \( \sup_{\lambda \in (0, 1)} \lambda(\mu(\lambda))^{\frac{1}{4}} = \infty \) and \( \mu \) is non-increasing, we necessarily have \( \lim_{\lambda \to 0^+} \mu(\lambda) = \infty \). Therefore, \( \tau(\lambda) \) is comparable to \( \mu(\lambda) \) for all but countably many \( \lambda \in (0, \lambda_0) \), with \( \lambda_0 > 0 \) sufficiently small. Consequently,
\[
\sup_{\lambda \in (0, \lambda_0)} \lambda(\tau(\lambda))^{\frac{1}{4}} \gtrsim \sup_{\lambda \in (0, \lambda_0)} \lambda(\mu(\lambda))^{\frac{1}{4}} = \infty,
\]

since
\[
\sup_{\lambda \in [\lambda_0, 1)} \lambda(\mu(\lambda))^{\frac{1}{4}} \leq (\mu(\lambda_0))^{\frac{1}{4}} < \infty.
\]

This implies that \( d \) does not belong to \( \ell^{4, \infty}(\mathbb{Z}^{2n}) \), and so, owing to Proposition \( [5.1] \), there is a sequence \( \{c_{k,l}\}_{(k,l) \in \mathbb{Z}^{2n}} \) of real numbers such that \( |c_{k,l}| = d_{k,l} \) for all \( (k, l) \in \mathbb{Z}^{2n} \) and for which the associated operator \( T_m \), with \( m \) defined by \( \text{(5.1)} \), is unbounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \). In addition, using \( \text{(5.5)} \) and \( \text{(5.6)} \), we deduce that
\[
|\{x \in \mathbb{R}^{2n} : |(I - \Delta)^{\frac{1}{2}} m(x)| > \lambda\}| \leq \tau(\lambda) \leq \mu(\lambda), \quad \lambda \in (0, 1),
\]
as desired. \( \square \)

6. A COUNTEREXAMPLE

We finish the paper by showing that, unlike Corollary \( \text{[1.5]} \) \( \text{[13, Theorem 1.3]} \), does not hold in the borderline case \( q = 4 \).

**Theorem 6.1.** There is a smooth bounded function \( m \) on \( \mathbb{R}^{2n} \) belonging to the space \( L^4(\mathbb{R}^{2n}) \) which has bounded partial derivatives of all orders and for which the associated operator \( T_m \) is unbounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \).

For every \( N \in 2\mathbb{N} \) we consider the “interval”
\[
I_N = \{b_N, b_N + 1, \ldots, b_N + 2^{N+1} - 1\},
\]
where \( \{b_N\}_{N \in 2\mathbb{N}} \) is an increasing sequence of integers to be specified later. We also set
\[
S_N = I_N^2 \subseteq \mathbb{Z}^{2n}.
\]

**Lemma 6.2.** We have
\[
\sum_{l \in S_N + S_N} (\text{card}\{j \in S_N : l - j \in S_N\})^2 \gtrsim 2^{3n(N^2 + \frac{N}{2})}.
\]
Proof. We set
\[ J_N = I_N + I_N = \{2b_N, 2b_N + 1, \ldots, 2b_N + 2^{N^2+N/2+1} - 2\} \]
and observe that \( S_N + S_N = J_N \). Consider the set \( K_N \) defined as
\[ K_N = \{2b_N + 2^{N^2+N/2-1}, 2b_N + 2^{N^2+N/2-1} + 1, \ldots, 2b_N + 2^{N^2+N/2} - 1\}. \]
Then \( K_N \subseteq J_N \). Now, if \( l = (l_1, \ldots, l_n) \in K_N \) and \( j = (j_1, \ldots, j_n) \), then \( j \) satisfies \( j \in S_N \) and \( l - j \in S_N \) if and only if \( b_N \leq j_i \leq l_i - b_N \) for every \( i = 1, \ldots, n \). Therefore,
\[ \text{card}\{j \in S_N : l - j \in S_N\} \geq \prod_{i=1}^n (l_i - 2b_N + 1) \gtrsim 2^n(N^2+N/2). \]
Altogether,
\[ \sum_{l \in S_N + S_N} (\text{card}\{j \in S_N : l - j \in S_N\})^2 \geq \sum_{l \in K_N} (\text{card}\{j \in S_N : l - j \in S_N\})^2 \gtrsim \sum_{l \in K_N} 2^{2n(N^2+N/2)} \approx 2^{3n(N^2+N/2)}. \]
\[ \square \]

Proof of Theorem 6.1. Let \( \psi \) be a smooth function on \( \mathbb{R}^n \) supported in the set \( \{\xi \in \mathbb{R}^n : |\xi| < 1/10\} \) such that \( \psi(\xi) = 1 \) if \( |\xi| \leq 1/20 \). Let \( \{a_l(t)\}_{l \in \mathbb{Z}^n} \) be the sequence of Rademacher functions indexed by the elements of the countable set \( \mathbb{Z}^n \). For a given \( t \in [0, 1] \) we define
\[ m_t(\xi, \eta) = \sum_{N \in 2\mathbb{N}} 2^{-nN/2} \sum_{j \in S_N} \sum_{k \in S_N} a_{j+k}(t) \psi(2^N \xi - j) \psi(2^N \eta - k) \]
\[ = \sum_{N \in 2\mathbb{N}} F_{t,N}(\xi, \eta), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n, \]
where the sequence \( \{b_N\}_{N \in 2\mathbb{N}} \) appearing in the definition of the sets \( I_N \) and \( S_N \) is chosen in such a way that the supports of the functions \( F_{t,N} \) are pairwise disjoint in \( N \). Also, let \( \varphi \) be a Schwartz function on \( \mathbb{R}^n \) whose Fourier transform is supported in the set \( \{\xi \in \mathbb{R}^n : |\xi| < 1/20\} \). For any \( N \in 2\mathbb{N} \) we consider the functions \( f_N = g_N \) whose Fourier transform satisfies
\[ \hat{f}_N(\xi) = 2^{nN/2} \sum_{j \in S_N} \hat{\varphi}(2^N \xi - j). \]
Then, by Plancherel’s theorem,

\[(6.1) \quad \| f_N \|_{L^2(\mathbb{R}^n)}^2 = \| g_N \|_{L^2(\mathbb{R}^n)}^2 \approx 2^{n_N} - nN^2 2^{-nN} \text{ card } S_N = 1.\]

We have

\[m_t(\xi, \eta) \hat{f}_N(\xi) \hat{g}_N(\eta) = 2^{n_N} - \frac{3nN^2}{2} \sum_{j \in S_N} \sum_{k \in S_N} a_{j+k}(t) \hat{\varphi}(2^N \xi - j) \hat{\varphi}(2^N \eta - k),\]

and so

\[T_{m_t}(f_N, g_N)(x) = 2^{n_N} - \frac{3nN^2}{2} \sum_{l \in S_N+S_N} a_l(t) e^{2\pi i x \cdot \frac{l}{2N}} \text{ card}\{j \in S_N : l - j \in S_N\}.\]

By Fubini’s theorem and Khintchine’s inequality,

\[
\int_0^1 \| T_{m_t}(f_N, g_N) \|_{L^1(\mathbb{R}^n)} dt = \int_{\mathbb{R}^n} \int_0^1 |T_{m_t}(f_N, g_N)(x)| dt \, dx
\]

\[= 2^{-\frac{nN}{2}} \frac{3nN^2}{2} \int_{\mathbb{R}^n} \left( \varphi \left( \frac{x}{2N} \right) \right)^2 \left( \sum_{l \in S_N+S_N} \text{ (card}\{j \in S_N : l - j \in S_N\}\right)^2 \, dx
\]

\[\approx 2^{-\frac{nN}{2}} \frac{3nN^2}{2} \left( \sum_{l \in S_N+S_N} \text{ (card}\{j \in S_N : l - j \in S_N\}\right)^2 \right)^{\frac{1}{2}}
\]

\[\geq 2^{2nN},\]

where the last inequality follows from Lemma 6.2. Therefore, there is \(t_N \in [0,1]\) such that

\[\| T_{m_N}(f_N, g_N) \|_{L^1(\mathbb{R}^n)} \geq 2^{nN}.\]

Let \(m\) be the function defined as

\[m = \sum_{N \in 2\mathbb{N}} F_{t_N,N}.\]

Then \(T_m(f_N, g_N) = T_{m_N}(f_N, g_N)\) for every \(N \in 2\mathbb{N}\), and so

\[(6.2) \quad \| T_m(f_N, g_N) \|_{L^1(\mathbb{R}^n)} \geq 2^{nN}.\]

A combination of (6.1) and (6.2) thus yields that \(T_m\) is not bounded from \(L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) to \(L^1(\mathbb{R}^n)\).

Since \(\psi\) is a smooth function and the supports of \(\psi(2^N \xi - j)\) are pairwise disjoint in \((N,j)\), we deduce that \(m\) is a smooth function. To verify that \(m\) has bounded partial derivatives of all orders, let us fix a
multiindex \( \alpha \). Since the functions \( F_{t,N,N} \) have pairwise disjoint supports in \( N \), it is enough to verify that the functions \( \partial^\alpha F_{t,N,N} \) are pointwise bounded by a constant independent of \( N \). This is indeed true, as

\[
N|\alpha| = \sqrt{nN} \cdot \frac{|\alpha|}{\sqrt{n}} \leq \frac{nN^2}{2} + \frac{|\alpha|^2}{2n},
\]

which implies

\[
|\partial^\alpha F_{t,N,N}(\xi, \eta)| \leq C_{\alpha,\psi} 2^{-\frac{nN^2}{4} + N|\alpha|} \leq C_{\alpha,\psi} 2^{\frac{|\alpha|^2}{2n}}, \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

Finally, let us show that \( m \) belongs to \( L^4(\mathbb{R}^{2n}) \). We have

\[
\|F_{t,N,N}\|_{L^4(\mathbb{R}^{2n})}^4 \lesssim 2^{-2nN^2} 2^{-2nN} (\text{card} \, S_N)^2 \approx 2^{-nN},
\]

and so

\[
\|m\|_{L^4(\mathbb{R}^{2n})} \leq \sum_{N \in 2^N} \|F_{t,N,N}\|_{L^4(\mathbb{R}^{2n})} \lesssim \sum_{N \in 2^N} 2^{-\frac{nN}{4}} < \infty,
\]

as desired. \( \square \)

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