ISS-based robustness to various neglected damping mechanisms for the 1-D wave PDE

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Abstract
This paper is devoted to the study of the robustness properties of the 1-D wave equation for an elastic vibrating string under four different damping mechanisms that are usually neglected in the study of the wave equation: (i) friction with the surrounding medium of the string (or viscous damping), (ii) thermoelastic phenomena (or thermal damping), (iii) internal friction of the string (or Kelvin-Voigt damping), and (iv) friction at the free end of the string (the so-called passive damper). The passive damper is also the simplest boundary feedback law that guarantees exponential stability for the string. We study robustness with respect to distributed inputs and boundary disturbances in the context of Input-to-State Stability (ISS). By constructing appropriate ISS Lyapunov functionals, we prove the ISS property expressed in various spatial norms.

Keywords Wave equation · Feedback stabilization · Thermoelasticity models · Input-to-state stability · ISS Lyapunov functional · PDEs

1 Introduction

The wave equation is the prototype Partial Differential Equation (PDE) for the description of vibrations in elastic solid and compressible fluid media. Among the numerous physical phenomena that it models are lateral, longitudinal, and torsional oscillations in strings and acoustic oscillations (in gases, liquids and solids; see for instance the

To Eduardo, with gratitude for his deep, elegant ISS and CLF results, in which he had encoded a general vision that enabled their extensions to stochastic, delay, and PDE systems.

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The study of the dynamics of the wave equation with or without boundary control has attracted the interest of many researchers (see for instance [3, 4, 6, 8, 9, 13, 14, 16, 17, 21, 23–26, 28–30, 32, 33, 36, 37]). However, the wave equation is an “idealized” model. In practice there are various damping mechanisms that usually affect the time evolution of the phenomenon, which are neglected in most PDE control designs. For a vibrating string, the following damping mechanisms are usually neglected:

1. friction with the surrounding medium of the string (or viscous damping; see [14]),
2. thermoelastic phenomena (or thermal damping; see [11, 12]),
3. internal friction of the string (or Kelvin-Voigt damping; see [6, 21, 26, 37]).

There is also an additional damping mechanism that may be present when the string has a loose end: a friction mechanism at the loosely attached or damped end. This particular damping mechanism is the so-called “passive damper” and has also been proposed as a boundary feedback controller for the string (see for instance [2, 9, 10, 23, 25, 28, 30]). The passive damper is the simplest boundary feedback law that guarantees exponential stability and it can also guarantee finite-time stability. However, the passive damper has been criticized by many researchers in the literature as a feedback controller that presents severe sensitivity to input delays (see [9, 10] as well as the relevant recent discussion in [1] for the case of systems of first-order hyperbolic PDEs). On the other hand, for a “real” string the passive damper is an additional damping mechanism that is (almost) always present.

The present work is devoted to the study of the robustness properties of the 1-D wave equation under all four of the above damping mechanisms. We study robustness with respect to distributed inputs and boundary disturbances in the context of Input-to-State Stability (ISS). The ISS property is a stability property that was proposed by E. D. Sontag in [32] for finite-dimensional control systems and has been recently extended to the case of infinite-dimensional systems described by PDEs (see [20, 27] and the references therein). The ISS property and the asymptotic gain property for the wave equation under Kelvin-Voigt damping was recently studied in [21, 26]. The study of ISS for systems of 1-D first-order, hyperbolic PDEs is closely related to the study of ISS for 1-D wave equations, since a 1-D wave PDE can be transformed to a system of 1-D first-order, hyperbolic PDEs. The ISS property for systems of first-order hyperbolic PDEs with respect to distributed and boundary inputs was studied in [18, 20]. More specifically, in [18] matrix inequalities are employed in order to guarantee the ISS property in the spatial $L^2$ norm, while in [20] a small-gain analysis is employed in order to guarantee the ISS property in the state sup-norm. Sufficient conditions for the ISS property in the spatial sup-norm and for the integral ISS property in spatial $L^p$ norms with $p \in [1, +\infty)$ are provided in Sect. 5 of [7] for a 1-D wave equation with set-valued boundary damping and boundary disturbances. Moreover, ISS-like estimates for the mechanical energy functional are derived in [19] for a 2-D wave equation with viscous damping and distributed/localized disturbances.

We consider the 1-D wave equation for a vibrating string with one end pinned and the other end free but in the presence of friction. The same model is obtained if one assumes that the applied force on the free end is manipulated with a passive damper (i.e., using the simplest boundary feedback law) with no input delays. In order to study
the robustness properties of the string, we consider different models: (i) a model where only friction with the surrounding medium of the string is taken into account, (ii) a model where thermoelastic phenomena are taken into account as well, and (iii) a model where all damping mechanisms are present. By constructing appropriate ISS Lyapunov functionals for each model, we are in a position to prove the ISS property expressed in different spatial norms. It should be noted that the ISS property for the cases (ii) and (iii) has not been studied in the literature and the obtained results (Theorem 2 and Theorem 3 below) for the cases (ii) and (iii) are completely novel. On the other hand, as mentioned above, the ISS property for the case (i) can be studied using the general results in [18, 20] (although the results in [20] will provide ISS estimates in different spatial norms). However, we were not able to find similar results in the literature to the results provided in the present work (Theorem 1 below) for case (i).

The structure of the paper is as follows. In Sect. 2 we present all different mathematical models for a vibrating string. Section 3 of the present work contains the statements of all main results of the paper as well as a discussion on the obtained results. The proofs of the main results are provided in Sect. 4. Section 5 gives the concluding remarks of the paper. Finally, the Appendix provides a well-posedness analysis for all studied models.

**Notation**

Throughout this paper, we adopt the following notation.

- $\mathbb{R}_+ = [0, +\infty)$ denotes the set of non-negative real numbers.
- Let $S \subseteq \mathbb{R}^n$ be an open set and let $A \subseteq \mathbb{R}^n$ be a set that satisfies $S \subseteq A \subseteq cl(S)$. By $C^0(A; \Omega)$, we denote the class of continuous functions on $A$, which take values in $\Omega \subseteq \mathbb{R}^m$. By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \mathbb{R}^n$, which takes values in $\Omega \subseteq \mathbb{R}^m$ and has continuous derivatives of order $k$. In other words, the functions of class $C^k(A; \Omega)$ are the functions which have continuous derivatives of order $k$ in $S = int(A)$ that can be continued continuously to all points in $\partial S \cap A$. When $\Omega = \mathbb{R}$ then we write $C^0(A)$ or $C^k(A)$. When $I \subseteq \mathbb{R}$ is an interval and $\eta \in C^1(I)$ is a function of a single variable, $\eta'(\rho)$ denotes the derivative with respect to $\rho \in I$.
- Let $I \subseteq \mathbb{R}$ be an interval and let $Y$ be a normed linear space. By $C^0(I; Y)$, we denote the class of continuous functions on $I$, which take values in $Y$. By $C^1(I; Y)$, we denote the class of continuously differentiable functions on $I$, which take values in $Y$.
- Let $I \subseteq \mathbb{R}$ be an interval, let $a < b$ be constants and let $u : I \times [a, b] \to \mathbb{R}$ be given. We use the notation $u[t]$ to denote the profile at certain $t \in I$, i.e., $(u[t])(x) = u(t, x)$ for all $x \in [a, b]$. When $u(t, x)$ is (twice) differentiable with respect to $x \in [a, b]$, we use the notation $u_x(t, x) = u_{xx}(t, x)$ for the (second) derivative of $u$ with respect to $x \in [a, b]$, i.e., $u_x(t, x) = \frac{\partial u}{\partial x}(t, x)$ $(u_{xx}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x))$. When $u[t] \in X$ for all $t \in I$, where $X$ is a normed linear space with norm $\|\cdot\|_X$ and the mapping $I \ni t \to u[t] \in X$ is $C^1$, i.e., the exists a continuous
mapping \( v : I \to X \) with \( \lim_{h \to 0} \left( \left\| \frac{u(t+h) - u(t)}{h} - v(t) \right\|_X \right) = 0 \) for all \( t \in I \), we use the notation \( u_t \) for the mapping \( v : I \to X \). Furthermore, when \( u \in C^1(I; X) \) and the mapping \( I \ni t \to u_t(t) \in X \) is \( C^1 \), i.e., the exists a continuous mapping \( w : I \to X \) with \( \lim_{h \to 0} \left( \left\| \frac{u(t+h) - u(t)}{h} - w(t) \right\|_X \right) = 0 \) for all \( t \in I \), we use the notation \( u_{tt} \) for the mapping \( w : I \to X \). Mixed derivatives are to be understood in this way. For example, when \( u_x \in C^1(I; X) \), i.e., the exists a continuous mapping \( \varphi : I \to X \) with \( \lim_{h \to 0} \left( \left\| \frac{u_{x(t+h)} - u_x(t)}{h} - \varphi(t) \right\|_X \right) = 0 \) for all \( t \in I \), we use the notation \( u_{xt} \) for the mapping \( \varphi : I \to X \).

* Let \( a < b \) be given constants. For \( p \in [1, +\infty) \), \( L^p(a, b) \) is the set of equivalence classes of Lebesgue measurable functions \( u : (a, b) \to \mathbb{R} \) with \( \|u\|_p := \left( \int_a^b |u(x)|^p \, dx \right)^{1/p} < +\infty \). \( L^\infty(a, b) \) is the set of equivalence classes of Lebesgue measurable functions \( u : (a, b) \to \mathbb{R} \) with \( \|u\|_\infty := \text{ess sup}_{x \in (a, b)} (|u(x)|) < +\infty \). For an integer \( k \geq 1 \), \( H^k(a, b) \) denotes the Sobolev space of functions in \( L^2(a, b) \) with all its weak derivatives up to order \( k \geq 1 \) in \( L^2(a, b) \).

### 2 Four different models

The 1-D wave equation

\[
 u_{tt}(t, x) = c^2 u_{xx}(t, x), \quad t \geq 0, \quad x \in (0, 1),
\]  

where \( c > 0 \) is a constant, is a model that is usually employed for the description of the displacement in the \( y \)-direction \( u(t, x) \) at time \( t > 0 \) and position \( x \in (0, 1) \) of a string. The model is accompanied by the boundary conditions

\[
 u(t, 0) = 0, \quad \text{for} \ t \geq 0,
\]

\[
 u_x(t, 1) = U(t), \quad \text{for} \ t \geq 0,
\]

where \( U(t) \) is the control input that corresponds to an external force acting at the right end of the string. The boundary condition (2) means that the left end of the string is pinned down.

A family of boundary feedback laws has been proposed in the literature for the global exponential stabilization of the string (see [23]). The feedback laws are given by the formula

\[
 U(t) = -au_t(t, 1) - \text{the passive damper}
\]

where \( a > 0 \) is a constant (the controller gain). The feedback law (4) (the passive damper) guarantees global exponential stabilization of the string state \( (u, u_t) \) in the norm of \( H^1(0, 1) \times L^2(0, 1) \). Moreover, when \( a = c^{-1} \) the feedback law (4) guarantees finite-time stabilization of the string model (1), (2), (3) (see [2]). It should be noticed that the boundary condition (3), (4) also arises when the free end of the string moves...
under friction ("skin friction", modeling the interaction between the surface of the string and the surrounding fluid).

However, as explained in the Introduction, in practice every string is subject to various damping mechanisms and the evolution of the displacement of the string is not described by the simple equation (1). When friction with the air (or viscous damping) is taken into account then the closed-loop system is given by (2) and

\[ \begin{align*}
  u_{tt}(t, x) &= c^2 u_{xx}(t, x) - \mu u_t(t, x) + f(t, x), \quad t \geq 0, \ x \in (0, 1), \\
  u_x(t, 1) &= -au_t(t, 1) + d(t), \quad \text{for } t \geq 0,
\end{align*} \]

where \( \mu \geq 0 \) is a constant (the air friction coefficient-usually unknown), \( f \) is a distributed disturbance that may be present and \( d \) (a boundary disturbance) can be seen either as the actuator error of the feedback control mechanism or as an unknown external force acting at the end of the string. Model (2), (5), (6) is a model that takes into account viscous damping as well as distributed and boundary disturbances.

A more detailed model is derived when thermoelastic phenomena in the vibrating string are taken into account. In this case the model of the string has an additional state \( \theta \) that corresponds to the deviation of the temperature from the reference temperature. Employing the theory of linear thermoelasticity (see [11, 12]), we obtain a closed-loop system given by (2), (6) and the PDEs

\[ \begin{align*}
  u_{tt}(t, x) &= c^2 u_{xx}(t, x) - \mu u_t(t, x) - b\theta_x(t, x) + f(t, x), \quad t \geq 0, \ x \in (0, 1), \\
  \theta_t(t, x) &= k\theta_{xx}(t, x) - \lambda u_{xt}(t, x), \quad t \geq 0, \ x \in (0, 1),
\end{align*} \]

where, again, \( \mu \geq 0 \) is the air friction coefficient, \( f \) is a distributed disturbance that may be present, \( b, k, \lambda > 0 \) are constants that are related to the reference temperature, the reference density of the string, the elastic moduli of the string, the thermal conductivity, the length, and the coefficient of thermal expansion of the string. Assuming that the ends of the string are kept at the constant reference temperature, we also obtain the temperature (Dirichlet) boundary conditions:

\[ \theta(t, 0) = \theta(t, 1) = 0, \quad \text{for } t \geq 0. \]

Model (2), (6), (7), (8), (9) is a model that takes into account both viscous and thermal damping as well as distributed and boundary disturbances.

A more complicated model can be obtained if Kelvin-Voigt (internal) damping is taken into account. In this case we replace the PDE (7) by the PDE

\[ u_{tt}(t, x) = c^2 u_{xx}(t, x) + \sigma u_{xxt}(t, x) - \mu u_t(t, x) - b\theta_x(t, x) + f(t, x) \]

for \( t \geq 0, \ x \in (0, 1) \), where \( \sigma > 0 \) is the coefficient of Kelvin-Voigt damping (a constant). However, in this case, the closed-loop system is not only affected by the actuator error (or external force) \( d \) but is also affected by the derivative \( \dot{d} \). In order to simplify things, we ignore in this case the control actuator errors (or external force) and consider the model (8), (10) with boundary conditions (2), (9) and
Model (2), (8), (9), (10), (11) is a model that takes into account viscous, Kelvin-Voigt and thermal damping as well as distributed disturbances. It should be noticed that the PDEs (8), (10) arise also in the study of a different phenomenon: sound propagation. Indeed, the so-called acoustic approximation for a compressible fluid (linearization around the steady state of constant density and temperature and zero fluid velocity) moving in a single spatial direction gives the PDEs (equations of viscous thermoacoustics-see [22]):

\[
\begin{align*}
\rho_t + \gamma v_x &= 0 \quad (12) \\
v_t + \frac{c^2}{\gamma} \rho_x + b \theta_x &= \sigma v_{xx} \quad (13) \\
\theta_t + \lambda v_x &= k \theta_{xx} \quad (14)
\end{align*}
\]

where \(c, \gamma, b, k, \lambda, \sigma > 0\) are constants (depending on the physical properties of the fluid at the reference density and reference temperature), \(v\) is the fluid velocity and \(\rho, \theta\) are the deviations of density, temperature, respectively, from their reference values. Differentiating (12) with respect to \(x\) and (13) with respect to \(t\) we get the PDE:

\[
v_{tt} = c^2 v_{xx} + \sigma v_{xxt} - b \theta_{xt} \quad (15)
\]

Setting \(v = u_t\), the PDEs (14), (15) give us (8) and the PDE \((u_{tt} - c^2 u_{xx} - \sigma u_{xxt} + b \theta_x)_t = 0\), which gives (10) with \(f(t, x) = f(x)\) and \(\mu = 0\) (since there is no surrounding medium like air, there is no friction with the surrounding medium).

Table 1 shows the four different models for a vibrating string under the feedback law (4).

Model (2), (8), (9), (10), (11) is a model that takes into account viscous, Kelvin-Voigt and thermal damping as well as distributed disturbances. It should be noticed that the PDEs (8), (10) arise also in the study of a different phenomenon: sound propagation. Indeed, the so-called acoustic approximation for a compressible fluid (linearization around the steady state of constant density and temperature and zero fluid velocity) moving in a single spatial direction gives the PDEs (equations of viscous thermoacoustics-see [22]):

\[
\begin{align*}
ux(t, 1) &= -au_t(t, 1), \text{ for } t \geq 0. \quad (11)
\end{align*}
\]
1. Does the string exhibit exponential stability in the absence of disturbances and in some appropriate state norm for Model (B), Model (C) and Model (D)?
2. Does the string satisfy an ISS property with respect to the disturbances and in some appropriate state norm for Model (B), Model (C) and Model (D)?

The next section of the paper is devoted to the answers of these questions.

### 3 Main results

The answers to the questions that were posed in the previous section are positive. In other words, for all models the string exhibits exponential stability in the absence of disturbances and for all models the string satisfies an ISS property with respect to the disturbances. The following theorems give precise information for each model and are the main results of the present work.

**Theorem 1** (String with Viscous Damping): For every $a, c > 0$, $\mu \geq 0$ there exist constants $\omega, G, \gamma_1, \gamma_2 > 0$ such that every solution

$$u \in C^1\left(\mathbb{R}_+; H^1(0, 1)\right) \cap C^0\left(\mathbb{R}_+; H^2(0, 1)\right)$$

of (2), (5), (6) corresponding to inputs $f \in C^0\left(\mathbb{R}_+; L^2(0, 1)\right)$, $d \in C^0\left(\mathbb{R}_+\right)$ with

$$u_t \in C^1\left(\mathbb{R}_+; L^2(0, 1)\right) \cap C^0\left(\mathbb{R}_+; H^1(0, 1)\right) \text{ and } u_t(t, 0) = 0 \text{ for } t \geq 0$$

satisfies the following estimate for all $t \geq 0$:

$$\left(\|u_t[t]\|_2^2 + \|u_x[t]\|_2^2\right)^{1/2} \leq G \exp(-\omega t) \left(\|u_t[0]\|_2^2 + \|u_x[0]\|_2^2\right)^{1/2} + \gamma_1 \sup_{0 \leq s \leq t} \|f[s]\|_2 + \gamma_2 \sup_{0 \leq s \leq t} \|d(s)\| \tag{16}$$

**Theorem 2** (String with Viscous and Thermal Damping): For every $a, c > 0$, $\mu \geq 0$, $b, k, \lambda > 0$ there exist constants $\omega, G, \gamma_1, \gamma_2 > 0$ such that every solution

$$u \in C^1\left(\mathbb{R}_+; H^1(0, 1)\right) \cap C^0\left(\mathbb{R}_+; H^2(0, 1)\right)$$

$$\theta \in C^1\left(\mathbb{R}_+; L^2(0, 1)\right) \cap C^0\left(\mathbb{R}_+; H^2(0, 1)\right)$$

of (2), (6), (7), (8), (9) corresponding to inputs $f \in C^0\left(\mathbb{R}_+; L^2(0, 1)\right)$, $d \in C^0\left(\mathbb{R}_+\right)$ with

$$u_t \in C^1\left(\mathbb{R}_+; L^2(0, 1)\right) \cap C^0\left(\mathbb{R}_+; H^1(0, 1)\right) \text{ and } u_t(t, 0) = 0 \text{ for } t \geq 0$$
satisfies the following estimate for all \( t \geq 0 \):

\[
\left( \| u_t[t] \|_2^2 + \| u_x[t] \|_2^2 + \| \theta[t] \|_2^2 \right)^{1/2} 
\leq G \exp(-\mu t) \left( \| u_t[0] \|_2^2 + \| u_x[0] \|_2^2 + \| \theta[0] \|_2^2 \right)^{1/2} 
+ \gamma_1 \sup_{0 \leq s \leq t} (\| f[s] \|_2) + \gamma_2 \sup_{0 \leq s \leq t} (|d(s)|)
\]

\( (17) \)

**Theorem 3** (String with Viscous, Thermal, and Kelvin-Voigt Damping): For every \( a, c > 0, \mu \geq 0, \sigma, b, k, \lambda > 0 \) there exist constants \( \omega, G, \gamma > 0 \) such that every solution

\[
\begin{align*}
    u \in C^1 \left( \mathbb{R}_+; L^2(0, 1) \right) \cap C^0 \left( \mathbb{R}_+; H^2(0, 1) \right) \\
    \theta \in C^1 \left( \mathbb{R}_+; L^2(0, 1) \right) \cap C^0 \left( \mathbb{R}_+; H^2(0, 1) \right)
\end{align*}
\]

of (2), (8), (9), (10), (11) corresponding to input \( f \in C^0 \left( \mathbb{R}_+; L^2(0, 1) \right) \) with

\[
\begin{align*}
    u_t \in C^1 \left( \mathbb{R}_+; L^2(0, 1) \right) \cap C^0 \left( \mathbb{R}_+; H^2(0, 1) \right), \\
    u_{xx} \in C^1 \left( \mathbb{R}_+; L^2(0, 1) \right), u_t(\cdot, 1) \in C^1(\mathbb{R}_+) \\
    u_t(t, 0) = 0, u_{xx}(t, 1) = -a \frac{d}{dt}(u_t(t, 1)) \text{ for } t \geq 0
\end{align*}
\]

satisfies the following estimate for all \( t \geq 0 \):

\[
\left( \| u_t[t] \|_2^2 + \| u_x[t] \|_2^2 + \| u_{xx}[t] \|_2^2 + |u_t(t, 1)|^2 + \| \theta[t] \|_2^2 \right)^{1/2} 
\leq G \exp(-\mu t) \left( \| u_t[0] \|_2^2 + \| u_x[0] \|_2^2 + \| u_{xx}[0] \|_2^2 + |u_t(0, 1)|^2 + \| \theta[0] \|_2^2 \right)^{1/2} 
+ \gamma \sup_{0 \leq s \leq t} (\| f[s] \|_2)
\]

\( (18) \)

**Remark** (i) The reader should notice that ISS estimates (16), (17), (18) for each of the models (model (B), model (C) and model (D); recall Table 1) actually guarantee the so-called “exponential ISS” property (see [27]), i.e., the effect of the initial condition is expressed by an exponential \( KL \) function and the gain functions of the disturbances are linear. The ISS estimates (16), (17), (18) show global exponential stability in the absence of disturbances.

(ii) The ISS estimates (16), (17), (18) for each of the models (model (B), model (C) and model (D); recall Table 1) are expressed in different state spatial norms. More specifically, ISS estimate (16) for model (B) involves the norm of the space \( H^1(0, 1) \times L^2(0, 1) \), ISS estimate (17) for model (C) involves the norm of the space \( H^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \), while ISS estimate (18) for model (D) involves a norm of the...
space $H^2(0, 1) \times C^0([0, 1]) \times L^2(0, 1)$ where the norm of the state component $u_t \in C^0([0, 1])$ is not the standard sup-norm but the norm $\|u_t\| = (\|u_t\|^2 + |u_t(1)|^2)^{1/2}$.

(iii) Each of the above theorems (Theorem 1, Theorem 2 and Theorem 3) assumes different regularity properties for the solution (going from less regular solutions to more regular solutions). This is expected, since the models (model (B), model (C) and model (D); recall Table 1) are quite different models involving very different differential operators. A detailed well-posedness analysis for all studied models is provided in the Appendix. However, it should be noted that the linearity of all studied models and the provided ISS estimates, guarantee uniqueness of solutions for the corresponding initial-boundary value problems. In other words, ISS estimates also guarantee uniqueness of solutions for a linear model.

(iv) Each of the above theorems (Theorem 1, Theorem 2 and Theorem 3) provide qualitative results for the solutions of the corresponding closed-loop systems. In other words, the above theorems do not provide formulae that show how large are the constants that are involved in the corresponding ISS estimates. However, it should be noticed that the proofs of the above theorems are constructive and provide estimates for the constants.

(v) The proofs of the above theorems follow a similar methodology: the construction of an ISS Lyapunov functional. However, for each of the closed-loop systems (model (B), model (C) and model (D); recall Table 1) the ISS Lyapunov functionals are different.

(vii) As pointed out above, when $a = c^{-1}$ the feedback law (4) guarantees finite-time stabilization of the string model (1), (2), (3) (see [2]). However, this is not the case when the closed-loop system is described by one of the more complicated models (model (B), model (C) and model (D); recall Table 1): finite-time stability is a feature that is not preserved under perturbations of the original string model.

(viii) It should be noted that the ISS property for model (C) and model (D) has not been studied in the literature and the obtained results (Theorem 2 and Theorem 3) for model (C) and model (D) are completely novel. On the other hand, as mentioned above, the ISS property for model (B) can be studied using the general results in [18, 20] after transforming model (B) to a system of first-order hyperbolic PDEs. The results in [20] will provide sufficient conditions for the ISS property in the norm $\|u_t(t)\|_\infty + \|u_x(t)\|_\infty$, while the results in [18] will provide matrix inequalities that guarantee ISS estimates in the norm $\left(\|u_{t}(t)\|^2 + \|u_{x}(t)\|^2\right)^{1/2}$. However, we were not able to find a similar result to Theorem 1 in the literature.

4 Proofs of main results

The proofs of the main results use a similar notation (for example, there is a constant $M > 0$, there are functionals $E$, $V$, etc.). However, the reader should not be tempted, by an overlapping notation for different quantities, to compare the different quantities in the proofs of Theorem 1, Theorem 2 and Theorem 3. To understand this, we point out that while the functional $E$ in the proof of Theorem 1 is the mechanical energy of the string, the functionals $E$ in the proof of Theorem 2 and Theorem 3 below are not
the mechanical energy of the string and are not functionals that may be considered as some kind of energy of the string (e.g., the sum of mechanical energy and the thermal energy of the string).

We start by providing the proof of Theorem 1.

**Proof of Theorem 1** Consider an arbitrary solution \( u \in C^1(\mathbb{R}_+; H^1(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1)) \) of (2), (5), (6) corresponding to (arbitrary) inputs \( f \in C^0(\mathbb{R}_+; L^2(0, 1)) \), \( d \in C^0(\mathbb{R}_+) \) with \( u_t \in C^1(\mathbb{R}_+; L^2(0, 1)) \cap C^0(\mathbb{R}_+; H^1(0, 1)) \) and \( u_t(t, 0) = 0 \) for all \( t \geq 0 \). Let \( r > 0 \) be a constant and define the functionals \( E, \Phi : H^1(0, 1) \times L^2(0, 1) \to \mathbb{R}_+ \) by means of the following formulae:

\[
E(u, w) := \frac{1}{2} \int_0^1 w^2(x)dx + \frac{c^2}{2} \int_0^1 u_x^2(x)dx, \\
\Phi(u, w) := \frac{1}{2} \int_0^1 \exp(rx) (w(x) + cu_x(x))^2 dx \\
+ \frac{1}{2} \int_0^1 \exp(-rx) (w(x) - cu_x(x))^2 dx,
\]

for all \( u \in H^1(0, 1), w \in L^2(0, 1) \) \( (19) \)

It should be noticed at this point that \( E(u[t], u_t[t]) \) is the mechanical energy of the string at time \( t \geq 0 \), while \( \Phi(u[t], u_t[t]) \) is the value of a Lyapunov functional for the simple model (A). When disturbances are absent and when \( \mu = 0 \) (no viscous damping), the function \( \Phi(u[t], u_t[t]) \) can be used for the derivation of an exponential stability estimate for the string.

Since \( r > 0 \), the functional \( \Phi : H^1(0, 1) \times L^2(0, 1) \to \mathbb{R}_+ \) satisfies the following estimate:

\[
\exp(-r) \left( c^2 \|u_x\|_2^2 + \|w\|_2^2 \right) \leq \Phi(u, w) \leq \exp(r) \left( c^2 \|u_x\|_2^2 + \|w\|_2^2 \right), \\
for all \( u \in H^1(0, 1), w \in L^2(0, 1) \) \( (20) \)

Since \( u \in C^1(\mathbb{R}_+; H^1(0, 1)) \), \( u_t \in C^1(\mathbb{R}_+; L^2(0, 1)) \), we get from definitions (19), (20) for all \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t]) = \int_0^1 u_t(t, x)u_{tt}(t, x)dx + c^2 \int_0^1 u_x(t, x)u_{xt}(t, x)dx \quad (22) \\
\frac{d}{dt} \Phi(u[t], u_t[t]) = \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))(u_{tt}(t, x) + cu_{xt}(t, x)) dx \\
+ \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))(u_{tt}(t, x) - cu_{xt}(t, x)) dx \quad (23)
\]
Integrating by parts the integral \( \int_0^1 u_x(t, x) u_{xt}(t, x) \, dx \) and using (22), (5), (6), (2) and the fact that \( u_t(t, 0) = 0 \) for all \( t \geq 0 \), we obtain for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t]) = -\mu \int_0^1 u_t^2(t, x) \, dx \\
+ \int_0^1 u_t(t, x) f(t, x) \, dx - ac^2 u_t^2(t, 1) + c^2 d(t) u_t(t, 1) \tag{24}
\]

Using (23) and (5) we get for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) = c \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x))^2 \right) \, dx \\
- c \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x))^2 \right) \, dx \\
+ \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) f(t, x) \, dx \\
+ \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) f(t, x) \, dx \\
- \mu \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) u_t(t, x) \, dx \\
- \mu \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) u_t(t, x) \, dx \tag{25}
\]

Using the inequalities \( d(t) u_t(t, 1) \leq \frac{a}{2} u_t^2(t, 1) + \frac{1}{2a} |d(t)|^2 \) (which holds since \( a > 0 \)),

\[
(u_t(t, x) \pm cu_x(t, x)) f(t, x) \leq \frac{cr}{4(1 + \mu)} (u_t(t, x) \pm cu_x(t, x))^2 + \frac{1 + \mu}{cr} |f(t, x)|^2,
\]

\[
(u_t(t, x) \pm cu_x(t, x)) u_t(t, x) \geq - \frac{cr}{4(1 + \mu)} (u_t(t, x) \pm cu_x(t, x))^2 - \frac{1 + \mu}{cr} u_t^2(t, x)
\]

and the fact that \( \mu \geq 0 \), we obtain from (24), (25) for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t]) \leq -\mu \int_0^1 u_t^2(t, x) \, dx \\
+ \int_0^1 u_t(t, x) f(t, x) \, dx - \frac{ac^2}{2} u_t^2(t, 1) + \frac{c^2}{2a} |d(t)|^2 \tag{26}
\]
\[
\frac{d}{dt} \Phi(u[t], u_t[t]) \leq c \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x))^2 \right) dx \\
- c \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x))^2 \right) dx \\
+ \frac{cr}{4} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
+ \frac{cr}{4} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx \\
+ \frac{2\mu (1 + \mu)}{cr} \int_0^1 \cosh(rx) u_t^2(t, x) dx + \frac{2(1 + \mu)}{cr} \int_0^1 \cosh(rx) |f(t, x)|^2 dx
\]

Integrating by parts the integrals \(\int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx\), \(\int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx\) and using (27), (6), (2) and the fact that \(u_t(t, 0) = 0\) for all \(t \geq 0\), we obtain for \(t \geq 0\):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) \\
\leq \frac{c}{2} \left( \exp(r) ((1 - ac)u_t(t, 1) + cd(t))^2 - \exp(-r) ((1 + ac)u_t(t, 1) - cd(t))^2 \right) \\
- \frac{cr}{4} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
- \frac{cr}{4} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx \\
+ \frac{2\mu (1 + \mu)}{cr} \int_0^1 \cosh(rx) u_t^2(t, x) dx + \frac{2(1 + \mu)}{cr} \int_0^1 \cosh(rx) |f(t, x)|^2 dx
\]

The fact that \(2u_t(t, x) = (u_t(t, x) + cu_x(t, x)) + (u_t(t, x) - cu_x(t, x))\) implies that

\[
4u_t^2(t, x) \leq (1 + \zeta) (u_t(t, x) + cu_x(t, x))^2 + \left( 1 + \zeta^{-1} \right) (u_t(t, x) - cu_x(t, x))^2,
\]

for all \(\zeta > 0\). Setting \(\zeta = \exp(2rx)\) we get

\[
2u_t^2(t, x) \\
\leq \cosh(rx) \left( \exp(rx) (u_t(t, x) + cu_x(t, x))^2 + \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 \right)
\]

Consequently, we get:

\[
\int_0^1 u_t^2(t, x) dx \leq \frac{\cosh(r)}{2} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
+ \frac{\cosh(r)}{2} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx
\]
Define

\[ M := \max \left( \frac{2(1 + \mu)}{cr} \cosh(r), \frac{2}{ac} \left( \exp(r)(1 - ac)^2 - \exp(-r)(1 + ac)^2 \right) \right) \]  

(30)

Using the inequality \( u_t(t, x) f(t, x) \leq \frac{cr}{4M} u_t^2(t, x) + \frac{M \cosh(r)}{cr} |f(t, x)|^2 \) and (26), (29) we obtain for \( t \geq 0 \):

\[
\begin{align*}
\frac{d}{dt} E(u[t], u_t[t]) &\leq -\mu \int_0^1 u_t^2(t, x) dx + \frac{cr}{8M} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
&\quad + \frac{cr}{8M} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx \\
&\quad - \frac{ac^2}{2} u_t^2(t, 1) + \frac{c^2}{2a} |d(t)|^2 + \frac{M \cosh(r)}{cr} \|f[t]\|_2^2 
\end{align*}
\]

(31)

Define the functional \( V : H^1(0, 1) \times L^2(0, 1) \to \mathbb{R}_+ \) by means of the formula

\[ V(u, w) := \Phi(u, w) + ME(u, w), \]  

(32)

with \( M \) given in (30). Using (19), (20), (21), (28), (31) and definition (32) we conclude that the functional \( V : H^1(0, 1) \times L^2(0, 1) \to \mathbb{R}_+ \) satisfies the following estimates:

\[
\begin{align*}
&\left( \frac{M}{2} + \exp(-r) \right) \left( c^2 \|u_x\|_2^2 + \|w\|_2^2 \right) \leq V(u, w) \\
&\leq \left( \frac{M}{2} + \exp(r) \right) \left( c^2 \|u_x\|_2^2 + \|w\|_2^2 \right), \\
&\text{for all } u \in H^1(0, 1), \ w \in L^2(0, 1) \\
\frac{d}{dt} V(u[t], u_t[t]) &\leq -\frac{c}{2} \left( acM + \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2 \right) u_t^2(t, 1) \\
&\quad + 2c^2 \left( \cosh(r) - ac \sinh(r) \right) u_t(t, 1)d(t) \\
&\quad + c^2 \left( \frac{M}{2a} + \sinh(r)c \right) |d(t)|^2 - \frac{cr}{4} \Phi(u[t], u_t[t]) \\
&\quad - \mu \left( M - \frac{2(1 + \mu)}{cr} \cosh(r) \right) \int_0^1 u_t^2(t, x) dx + \left( 2(1 + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \|f[t]\|_2^2 \\
&\quad \text{for all } t \geq 0 
\end{align*}
\]

(33)

Using the facts that \( \mu \geq 0, M \geq \frac{2(1 + \mu)}{cr} \cosh(r), M \geq \frac{2}{ac} \left( \exp(r)(1 - ac)^2 - \exp(-r)(1 + ac)^2 \right) \) (recall (4.12)) and the inequality

\[
(c\cosh(r) - ac \sinh(r)) u_t(t, 1)d(t) \\
\leq \frac{aM}{8} u_t^2(t, 1) + \frac{2(c\cosh(r) - ac \sinh(r))^2}{aM} |d(t)|^2
\]
we conclude from (34) that the following estimate holds for all \( t \geq 0 \):

\[
\frac{d}{dt} V(u[t], u_t[t]) \leq -\frac{cr}{4} \Phi(u[t], u_t[t]) + \left( 2 (1 + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \| f[t] \|_2^2 \\
+ c^2 \left( \frac{M}{2a} + \frac{4 (\cosh(r) - ac \sinh(r))^2}{aM} + c \sinh(r) \right) |d(t)|^2
\]  

(35)

Definitions (19), (32) and estimate (33) imply that

\[
-\Phi(u, w) = -V(u, w) + M E(u, w) \\
\leq -V(u, w) + \frac{M}{M + 2 \exp(-r)} V(u, w) = -\frac{2 \exp(-r)}{M + 2 \exp(-r)} V(u, w)
\]

for all \( u \in H^1(0, 1) \), \( w \in L^2(0, 1) \). The above inequality implies the following differential inequality for all \( t \geq 0 \):

\[
\frac{d}{dt} V(u[t], u_t[t]) \leq -\frac{cr \exp(-r)}{2(M + 2 \exp(-r))} V(u[t], u_t[t]) \\
+ \left( 2 (1 + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \| f[t] \|_2^2 \\
+ c^2 \left( \frac{M}{2a} + \frac{4 (\cosh(r) - ac \sinh(r))^2}{aM} + c \sinh(r) \right) |d(t)|^2
\]  

(36)

Multiplying both sides of (36) by \( \exp(2\omega t) \) where \( \omega := \frac{cr \exp(-r)}{4(M + 2 \exp(-r))} \), we get for all \( t \geq 0 \):

\[
\frac{d}{dt} \left( \exp(2\omega t) V(u[t], u_t[t]) \right) \leq 2\omega K_1 \exp(2\omega t) \| f[t] \|_2^2 + 2\omega K_2 \exp(2\omega t) |d(t)|^2
\]  

(37)

where \( K_1 := \left( 2 (1 + \mu) + M^2 \right) \frac{\cosh(r)}{2\omega cr} \) and \( K_2 := \frac{c^2}{2\omega} \left( \frac{M}{2a} + \frac{4(\cosh(r) - ac \sinh(r))^2}{aM} + c \sinh(r) \right) \). Differential inequality (37) directly implies the following estimate for all \( t \geq 0 \):

\[
V(u[t], u_t[t]) \leq \exp(-2\omega t) V(u[0], u_0[0]) \\
+ 2\omega K_1 \int_0^t \exp(-2\omega (t-s)) \| f[s] \|_2^2 \, ds + 2\omega K_2 \int_0^t \exp(-2\omega (t-s)) |d(s)|^2 \, ds
\]

which gives

\[ \square \] Springer
Proof of Theorem 2

We next continue with the proof of Theorem 2.

\[ V(u[t], u_t[t]) \leq \exp(-2\omega t) V(u[0], u_t[0]) + K_1 \sup_{0 \leq s \leq t} \left( \| f[s] \|_2^2 \right) + K_2 \sup_{0 \leq s \leq t} \left( \| d(s) \|_2^2 \right) \]  

(38)

Exploiting estimates (38) and (33) we get for all \( t \geq 0 \):

\[
c^2 \| u_x[t] \|_2^2 + \| u_t[t] \|_2^2 \leq \exp(-2\omega t) \frac{M + 2 \exp(r)}{M + 2 \exp(-r)} \left( c^2 \| u_x[0] \|_2^2 + \| u_t[0] \|_2^2 \right) \\
+ \frac{2K_1}{M + 2 \exp(-r)} \sup_{0 \leq s \leq t} \left( \| f[s] \|_2^2 \right) + \frac{2K_2}{M + 2 \exp(-r)} \sup_{0 \leq s \leq t} \left( \| d(s) \|_2^2 \right)
\]

(39)

Estimate (16) with \( G := \sqrt{\frac{M + 2 \exp(r)}{M + 2 \exp(-r)}} \sqrt{\max(1, c^2)} \min(1, c^2) \), \( \gamma_1 := \sqrt{\frac{2K_1}{(M + 2 \exp(-r)) \min(1, c^2)}} \) and \( \gamma_2 := \gamma_1 \sqrt{\frac{K_2}{K_1}} \) is a direct consequence of estimate (39). The proof is complete. \( \square \)

We next continue with the proof of Theorem 2.

**Proof of Theorem 2**

Consider an arbitrary solution \( u \in C^1(\mathbb{R}^+; H^1(0, 1)) \cap C^0(\mathbb{R}^+; H^2(0, 1)) \), \( \theta \in C^1(\mathbb{R}^+; L^2(0, 1)) \cap C^0(\mathbb{R}^+; H^2(0, 1)) \) of (2), (6), (7), (8), (9) corresponding to (arbitrary) inputs \( f \in C^0(\mathbb{R}^+; L^2(0, 1)) \), \( d \in C^0(\mathbb{R}^+) \) with \( u_t \in C^1(\mathbb{R}^+; L^2(0, 1)) \cap C^0(\mathbb{R}^+; H^1(0, 1)) \) and \( u_t(t, 0) = 0 \) for all \( t \geq 0 \). Let \( r > 0 \) be a constant and define the functionals \( E : H^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R}_+ \), \( \Phi : H^1(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R}_+ \) by means of the formulae (20) and

\[
E(u, w, \theta) := \frac{1}{2} \int_0^1 w^2(x)dx + \frac{c^2}{2} \int_0^1 u_x^2(x)dx + \frac{b}{2\lambda} \int_0^1 \theta^2(x)dx,
\]

for all \( u \in H^1(0, 1), \ w, \theta \in L^2(0, 1) \)  

(40)

where \( c, \lambda, b > 0 \) are the constants appearing in (7) and (8). It should be noticed at this point that \( E(u[t], u_t[t], \theta[t]) \) is an appropriate linear combination at time \( t \geq 0 \) of the mechanical energy \( \left( \frac{1}{2} \int_0^1 u_t^2(t, x)dx + \frac{c^2}{2} \int_0^1 u_x^2(t, x)dx \right) \) of the string and the squared \( L^2 \) norm of the deviation of temperature from its reference value (the term \( \int_0^1 \theta^2(t, x)dx \) is not the thermal energy of the string but is the value of a Lyapunov functional used frequently for the heat equation).

Since \( r > 0 \), the functional \( \Phi : H^1(0, 1) \times L^2(0, 1) \rightarrow \mathbb{R}_+ \) satisfies estimate (21). Moreover, since \( u \in C^1(\mathbb{R}^+; H^1(0, 1)) \), \( u_t \in C^1(\mathbb{R}^+; L^2(0, 1)) \), \( \theta \in C^1(\mathbb{R}^+; L^2(0, 1)) \) we get from definitions (20), (40) for all \( t \geq 0 \) formula (23) as well as the following formula:
\[
\frac{d}{dt} E(u[t], u_t[t], \theta[t]) = \int_0^1 u_t(t, x) u_{tt}(t, x) dx + c^2 \int_0^1 u_x(t, x) u_{xt}(t, x) dx + \frac{b}{\lambda} \int_0^1 \theta(t, x) \theta_t(t, x) dx
\]

Integrating by parts the integral \( \int_0^1 u_x(t, x) u_{xt}(t, x) dx \) and using (41), (6), (7), (8), (2) and the fact that \( u_t(t, 0) = 0 \) for all \( t \geq 0 \), we obtain for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t], \theta[t]) = -\mu \int_0^1 u_t^2(t, x) dx + \int_0^1 u_t(t, x) f(t, x) dx - a c^2 u_t^2(t, 1) + c^2 d(t) u_t(t, 1)
\]

Using the inequality \( d(t) u_t(t, 1) \leq \frac{a}{2} u_t^2(t, 1) + \frac{1}{2a} |d(t)|^2 \) (which holds since \( a > 0 \)), (9) and integrating by parts the integral \( \int_0^1 \theta(t, x) \theta_{xx}(t, x) dx \), we obtain from (42) for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t], \theta[t]) \leq -\mu \int_0^1 u_t^2(t, x) dx + \int_0^1 u_t(t, x) f(t, x) dx - \frac{a c^2}{2} u_t^2(t, 1) + \frac{c^2}{2a} |d(t)|^2 - \frac{kb}{\lambda} \int_0^1 \theta_{x}^2(t, x) dx
\]

Using (23) and (7) we get for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) = c \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x))^2 \right)_x dx \\
- c \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x))^2 \right)_x dx \\
+ \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) (f(t, x) - b \theta_x(t, x) - \mu u_t(t, x)) dx \\
+ \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) (f(t, x) - b \theta_x(t, x) - \mu u_t(t, x)) dx
\]

 Integrating by parts the integrals \( \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x))^2 \right)_x dx \), \( \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x))^2 \right)_x dx \) and using (44), (6), (2) and the fact that \( u_t(t, 0) = 0 \) for all \( t \geq 0 \), we obtain for \( t \geq 0 \):
Using the inequalities

\[
(u_t(t, x) \pm c u_x(t, x)) f(t, x) \leq \frac{cr}{4(1+b+\mu)} (u_t(t, x) \pm c u_x(t, x))^2 + \frac{1+b+\mu}{cr} |f(t, x)|^2
\]

and the facts that \( \mu \geq 0, b > 0 \), we obtain from (45) the following estimate for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) \leq -\frac{cr}{4} \int_0^1 \exp(rx) (u_t(t, x) + c u_x(t, x))^2 \, dx
\]

\[
-\frac{cr}{4} \int_0^1 \exp(-rx) (u_t(t, x) - c u_x(t, x))^2 \, dx
\]

\[
-\frac{c}{2} \left( \exp(-r)(1+ac)^2 - \exp(r)(1-ac)^2 \right) u_t^2(t, 1)
\]

\[
+ \sinh(r)c^3 |d(t)|^2 + 2e^2 (\cosh(r) - ac \sinh(r)) u_t(t, 1)d(t)
\]

\[
+ \frac{2(1+b+\mu)}{cr} \int_0^1 \cosh(rx) |f(t, x)|^2 \, dx + \frac{2b(1+b+\mu)}{cr} \int_0^1 \cosh(rx) \theta_x^2(t, x) \, dx
\]

\[
+ \frac{2\mu(1+b+\mu)}{cr} \int_0^1 \cosh(rx) u_t^2(t, x) \, dx
\]

(46)

Define

\[
M := \max \left( \max \left( 1, \frac{2\lambda}{k} \right), \frac{2(1+b+\mu)}{cr} \cosh(r), \frac{2}{ac} \left( \exp(r)(1-ac)^2 - \exp(-r)(1+ac)^2 \right) \right)
\]

(47)

\[
V(u, w, \theta) := \Phi(u, w) + M E(u, w, \theta), \quad \text{for all } u \in H^1(0, 1), \ w, \theta \in L^2(0, 1)
\]

(48)
Using (43), (46) and definition (48), we get for $t \geq 0$:

\[
\frac{d}{dt} V(u[t], u_t[t], \theta[t]) \leq -\frac{cr}{4} \int_0^1 \exp(rx) (u(t, x) + cu_x(t, x))^2 \, dx \\
- \frac{cr}{4} \int_0^1 \exp(-rx) (u(t, x) - cu_x(t, x))^2 \, dx + \frac{2(1 + b + \mu)}{cr} \cosh(r) \| f[t] \|^2_2 \\
- \frac{c}{2} (acM + \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2) u_t^2(t, 1) \\
+ c^2 \left( c \sinh(r) + \frac{M}{2a} \right) |d(t)|^2 + 2c^2 (\cosh(r) - ac \sinh(r)) u_t(t, 1)d(t) \\
- \mu \left( M - \frac{2(1 + b + \mu)}{cr} \cosh(r) \right) \int_0^1 u_t^2(t, x) \, dx + M \int_0^1 u_t(t, x) f(t, x) \, dx \\
- b \left( \frac{kM}{\lambda} - \frac{2(1 + b + \mu)}{cr} \cosh(r) \right) \int_0^1 \theta_x^2(t, x) \, dx
\]  

Using the inequality $\int_0^1 u_t(t, x) f(t, x) \, dx \leq \frac{cr}{4M \cosh(r)} \int_0^1 u_t^2(t, x) \, dx + \frac{M \cosh(r)}{cr}$, inequality (29), the fact that $M \geq \frac{2(1 + b + \mu)}{cr} \cosh(r)$ (recall (4.29)), the fact that $\frac{kM}{\lambda} \geq \frac{2(1 + b + \mu)}{cr} \cosh(r)$ (recall (4.29)) and the facts that $\mu \geq 0$, $b > 0$, we conclude from (49) the following estimate for $t \geq 0$:

\[
\frac{d}{dt} V(u[t], u_t[t], \theta[t]) \leq -\frac{cr}{8} \int_0^1 \exp(rx) (u(t, x) + cu_x(t, x))^2 \, dx \\
- \frac{cr}{8} \int_0^1 \exp(-rx) (u(t, x) - cu_x(t, x))^2 \, dx \\
- \frac{c}{2} \left( acM + \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2 \right) u_t^2(t, 1) \\
+ c^2 \left( c \sinh(r) + \frac{M}{2a} \right) |d(t)|^2 + 2c^2 (\cosh(r) - ac \sinh(r)) u_t(t, 1)d(t) \\
+ \left( 2(1 + b + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \| f[t] \|^2_2 - b \frac{kM}{\lambda} \int_0^1 \theta_x^2(t, x) \, dx
\]

Wirtinger’s inequality and (9) imply that $\int_0^1 \theta_x^2(t, x) \, dx \geq \pi^2 \int_0^1 \theta^2(t, x) \, dx$. The previous inequality and the inequality

\[
(\cosh(r) - ac \sinh(r)) u_t(t, 1)d(t) \\
\leq \frac{aM}{8} u_t^2(t, 1) + \frac{2 (\cosh(r) - ac \sinh(r))^2}{aM} |d(t)|^2
\]

in conjunction with (50) give the following estimate for $t \geq 0$:
\[
\frac{d}{dt} V(u[t], u_t[t], \theta[t]) \\
\leq - \frac{cr}{8} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
- \frac{cr}{8} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx \\
- \frac{c}{2} \left( \frac{acM}{2} + \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2 \right) u_t^2(t, 1) \\
+ c^2 \left( \frac{4 \cosh(r) - ac \sinh(r)}{aM} + \frac{M}{2a} \right) |d(t)|^2 \\
+ \left( 2(1 + b + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \|f[t]\|^2 - \frac{bkM}{2\lambda} \pi^2 \int_0^1 |\theta(t, x)| dx \\
\] (51)

The fact that \( \frac{acM}{2} + \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2 \geq 0 \) (recall (4.29)) and definition (20) combined with estimate (51) give us the following estimate for \( t \geq 0 \):

\[
\frac{d}{dt} V(u[t], u_t[t], \theta[t]) \leq - \frac{cr}{4} \Phi(u[t], u_t[t]) - \frac{bkM}{2\lambda} \pi^2 \|\theta[t]\|^2 \\
+ c^2 \left( \frac{4 \cosh(r) - ac \sinh(r)}{aM} + \frac{M}{2a} \right) |d(t)|^2 \\
+ \left( 2(1 + b + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \|f[t]\|^2 \\
\] (52)

Definition (48), definition (40) and estimate (21) imply the following inequality:

\[
V(u[t], u_t[t], \theta[t]) \leq \left( 1 + \frac{M}{2} \exp(r) \right) \Phi(u[t], u_t[t]) + \frac{bM}{2\lambda} \|\theta[t]\|^2 \\
\] (53)

Combining (52) and (53), we obtain for \( t \geq 0 \)

\[
\frac{d}{dt} V(u[t], u_t[t], \theta[t]) \leq -2 \omega V(u[t], u_t[t], \theta[t]) \\
+ c^2 \left( \frac{4 \cosh(r) - ac \sinh(r)}{aM} + \frac{M}{2a} \right) |d(t)|^2 \\
+ \left( 2(1 + b + \mu) + M^2 \right) \frac{\cosh(r)}{cr} \|f[t]\|^2 \\
\] (54)

where \( \omega := \frac{1}{2} \min \left( \frac{cr}{2(1 + M \exp(r))}, k \pi^2 \right) \). Multiplying both sides of (54) by \( \exp(2\omega t) \) we obtain for \( t \geq 0 \):
\[
\frac{d}{dt} \left( \exp(2\omega t)V(u[t], u_t[t], \theta[t]) \right) \leq 2\omega K_1 \exp(2\omega t) \| f[t]\|^2_2 + 2\omega K_2 \exp(2\omega t) |d(t)|^2
\]

(55)

where \( K_1 := \left( 2 (1 + b + \mu) + M^2 \right) \frac{\cosh(r)}{2\omega \omega} \) and \( K_2 := \frac{c^2}{2\omega} \left( \frac{M}{2a} + \frac{4(\cosh(r) - ac \sinh(r))^2}{aM} \right) + c \sinh(r) \). Differential inequality (55) directly implies the following estimate for all \( t \geq 0 \):

\[
V(u[t], u_t[t], \theta[t]) \leq \exp(-2\omega t) V(u[0], u_t[0], \theta[0]) + 2\omega K_1 \int_0^t \exp(-2\omega(t - s)) \| f[s]\|^2_2 \, ds + 2\omega K_2 \int_0^t \exp(-2\omega(t - s)) |d(s)|^2 \, ds
\]

which gives

\[
V(u[t], u_t[t], \theta[t]) \leq \exp(-2\omega t) V(u[0], u_t[0], \theta[0]) + K_1 \sup_{0 \leq s \leq t} \left( \| f[s]\|^2_2 \right) + K_2 \sup_{0 \leq s \leq t} \left( |d(s)|^2 \right)
\]

(56)

Definition (40), definition (48) and estimate (21) in conjunction with (56) allow us to conclude that the following estimate holds for all \( t \geq 0 \):

\[
\min \left( \min \left( 1, c^2 \right) \left( \frac{M}{2} + \exp(-r) \right), \frac{bM}{2\lambda} \right) \left( \| u_t[t]\|^2_2 + \| u_x[t]\|^2_2 + \| \theta[t]\|^2_2 \right) 
\leq \exp(-2\omega t) \max \left( \max \left( 1, c^2 \right) \left( \frac{M}{2} + \exp(r) \right), \frac{bM}{2\lambda} \right) 
\left( \| u_t[0]\|^2_2 + \| u_x[0]\|^2_2 + \| \theta[0]\|^2_2 \right) 
\]

\[+ K_1 \sup_{0 \leq s \leq t} \left( \| f[s]\|^2_2 \right) + K_2 \sup_{0 \leq s \leq t} \left( |d(s)|^2 \right)
\]

(57)

Estimate (17) with appropriate constants \( G, \gamma_1, \gamma_2 > 0 \), is a direct consequence of estimate (57). The proof is complete. \( \square \)

We end this section with the proof of Theorem 3.

**Proof of Theorem 3** Consider an arbitrary solution \( u \in C^1(\mathbb{R}_+; L^2(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1)), \theta \in C^1(\mathbb{R}_+; L^2(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1)) \) corresponding to (arbitrary) input \( f \in C^0(\mathbb{R}_+; L^2(0, 1)) \) with \( u_t \in C^1(\mathbb{R}_+; L^2(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1)), u_{xx} \in C^1(\mathbb{R}_+; L^2(0, 1)), u_x(t, 1) \in C^1(\mathbb{R}_+) \) and \( u_t(t, 0) = 0, u_{xx}(t, 1) = -a \frac{d}{dt}(u_t(t, 1)) \) for all \( t \geq 0 \). By virtue of (11) and the facts that \( u_{xx} \in C^1(\mathbb{R}_+; L^2(0, 1)) \), \( u_t(t, 1) \in C^1(\mathbb{R}_+) \) and \( u_x(t, x) = u_x(t, 1) - \int_x^1 u_{ss}(t, s) \, ds \) for all \( x \in [0, 1] \), we conclude that \( u_x \in C^1(\mathbb{R}_+; L^2(0, 1)) \). Let \( r > 0 \) be a constant and define the functionals \( E : H^1((0, 1) \times C^0([0, 1])) \times L^2(0, 1) \rightarrow \mathbb{R}_+ \), \( \square \) Springer
\[ \Phi : H^1(0, 1) \times C^0([0, 1]) \to \mathbb{R}_+, \quad W : H^2(0, 1) \times L^2(0, 1) \to \mathbb{R}_+ \] by means of the formulae

\[ E(u, w, \theta) := \frac{1}{2} \int_0^1 w^2(x)dx + \frac{c^2}{2} \int_0^1 u_x^2(x)dx + \frac{b}{2\lambda} \int_0^1 \theta^2(x)dx + \frac{a\sigma}{2} w^2(1), \]

for all \( u \in H^2(0, 1), \ w \in C^0([0, 1]), \ \theta \in L^2(0, 1) \) (58)

\[ W(u, w) := \frac{1}{2} \int_0^1 (w(x) - \sigma u_{xx}(x))^2dx, \]

for all \( u \in H^2(0, 1), \ w \in C^0([0, 1]) \) (59)

\[ \Phi(u, w) := \frac{1}{2} \int_0^1 \exp(rx) (w(x) + cu_x(x))^2dx \\
+ \frac{1}{2} \int_0^1 \exp(-rx) (w(x) - cu_x(x))^2dx + \frac{B}{2} w^2(1) \]

for all \( u \in H^2(0, 1), \ w \in C^0([0, 1]) \) (60)

where \( B := a\sigma (\exp(r)(1 - ac) + \exp(-r)(1 + ac)) \) and \( c, \lambda, b, a, \sigma > 0 \) are the constants appearing in (8), (10) and (11). It should be noticed at this point that \( E(u[t], u_t[t], \theta[t]) \) is an appropriate linear combination at time \( t \geq 0 \) of the potential energy of the string (the term \( \frac{c^2}{2} \int_0^1 u_x^2(t, x)dx \)), a measure of the kinetic energy of the string (the term \( \frac{1}{2} \int_0^1 u_t^2(t, x)dx + \frac{a\sigma}{2} u_t^2(1, t) \)) and the squared \( L^2 \) norm of the deviation of temperature from its reference value (the term \( \int_0^1 \theta^2(t, x)dx \) is not the thermal energy of the string but is the value of a Lyapunov functional used frequently for the heat equation). The functional \( W(u, w) \) is a functional that allows the derivation of bounds for the \( L^2 \) norm of the second spatial derivative of \( u \), i.e., \( \|u_{xx}\|_2 \). The functional \( W(u, w) \) was not used in the proofs of Theorem 1 and Theorem 2 because the ISS estimates (16), (17) do not provide bounds for the \( L^2 \) norm of the second spatial derivative of \( u \). Finally, the functional \( \Phi(u, w) \) is not the same functional used in the proofs of Theorem 1 and Theorem 2 (due to the additional term \( Bw^2(1)/2 \)) but when \( \sigma \to 0^+ \) it becomes equal to the Lyapunov functional for the simple model (A) that was employed in the proofs of Theorem 1 and Theorem 2.

Since \( u \in C^1(\mathbb{R}_+; L^2(0, 1)), \ \theta \in C^1(\mathbb{R}_+; L^2(0, 1)), \ u_t \in C^1(\mathbb{R}_+; L^2(0, 1)), \ u_x \in C^1(\mathbb{R}_+; L^2(0, 1)), \ u_{xx} \in C^1(\mathbb{R}_+; L^2(0, 1)), \ u_t(\cdot, 1) \in C^1(\mathbb{R}_+) \), we get from definitions (58), (59), (60) for all \( t \geq 0 \) the following formulae:

\[ \frac{d}{dt} E(u[t], u_t[t], \theta[t]) = \int_0^1 u_t(t, x)u_{tt}(t, x)dx + c^2 \int_0^1 u_x(t, x)u_{xt}(t, x)dx \\
+ \frac{b}{\lambda} \int_0^1 \theta(t, x)\theta_t(t, x)dx + a\sigma u_t(t, 1) \frac{d}{dt} (u_t(t, 1)) \]

(61)

\[ \frac{d}{dt} W(u[t], u_t[t]) = \int_0^1 (u_t(t, x) - \sigma u_{xx}(t, x))(u_{tt}(t, x) - \sigma u_{xxt}(t, x))dx \]

(62)
\[
\frac{d}{dt} \Phi(u[t], u[t]) = \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x)) \right)^2 dx \\
+ \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x)) \right)^2 dx + Bu(t, 1) \frac{d}{dt} (u_t(t, 1))
\]

Using (63) and (10) we get for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t], \theta[t]) = -\mu \int_0^1 u_t^2(t, x) dx + \int_0^1 u_t(t, x) f(t, x) dx \\
+ \sigma \int_0^1 u_t(t, x) u_{xx}(t, x) dx \\
- ac^2 u_t^2(t, 1) - b \int_0^1 (u_t(t, x) \theta(t, x)) dx \\
+ \frac{kb}{\lambda} \int_0^1 \theta(t, x) \theta_{xx}(t, x) dx + a \sigma u_t(t, 1) \frac{d}{dt} (u_t(t, 1))
\]

Integrating by parts the integral \( \int_0^1 u_t(t, x) u_{xx}(t, x) dx \) and using (61), (8), (10), (11), (2) and the fact that \( u_t(t, 0) = 0 \) for all \( t \geq 0 \), we obtain for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t], \theta[t]) = -\mu \int_0^1 u_t^2(t, x) dx + \int_0^1 u_t(t, x) f(t, x) dx \\
- \sigma \int_0^1 u_t^2(t, x) dx - ac^2 u_t^2(t, 1) - \frac{kb}{\lambda} \int_0^1 \theta^2(t, x) dx
\]

Using (63) and (10) we get for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) = c \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x)) \right)^2 dx \\
- c \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x)) \right)^2 dx + Bu(t, 1) \frac{d}{dt} (u_t(t, 1))
\]

Integrating by parts the integrals \( \int_0^1 u_t(t, x) u_{xx}(t, x) dx \), \( \int_0^1 \theta(t, x) \theta_{xx}(t, x) dx \) and using (64), (9), (2) and the facts that \( u_t(t, 0) = 0 \), \( u_{xx}(t, 1) = -a \frac{d}{dt} (u_t(t, 1)) \) for all \( t \geq 0 \), we obtain for \( t \geq 0 \):

\[
\frac{d}{dt} E(u[t], u_t[t], \theta[t]) = -\mu \int_0^1 u_t^2(t, x) dx + \int_0^1 u_t(t, x) f(t, x) dx \\
- \sigma \int_0^1 u_t^2(t, x) dx - ac^2 u_t^2(t, 1) - \frac{kb}{\lambda} \int_0^1 \theta^2(t, x) dx
\]
Integrating by parts the integrals \( \int_0^1 \exp(rx) \left( \frac{1}{2} (u_t(t, x) + cu_x(t, x))^2 \right)_x dx \), \( \int_0^1 \exp(-rx) \left( \frac{1}{2} (u_t(t, x) - cu_x(t, x))^2 \right)_x dx \) and using (66), (11), (2) and the fact that \( u_t(t, 0) = 0 \) for all \( t \geq 0 \), we obtain for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) = -\frac{cr}{2} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
- \frac{cr}{2} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx \\
+ \frac{c}{2} \left( \exp(r)(1-ac)^2 - \exp(-r)(1+ac)^2 \right) u_t^2(t, 1) + B u_t(t, 1) \frac{d}{dt} (u_t(t, 1)) \\
+ \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) (f(t, x) + \sigma u_{xt}(t, x)) dx \\
- \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) (b \theta_x(t, x) + \mu u_t(t, x)) dx \\
+ \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) (f(t, x) + \sigma u_{xt}(t, x)) dx \\
- \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) (b \theta_x(t, x) + \mu u_t(t, x)) dx \tag{67}
\]

Integrating by parts the integrals \( \int_0^1 \exp(\pm rx) (u_t(t, x) \pm cu_x(t, x)) u_{xxt}(t, x) dx \) and using (67), (11), (2) and the facts that \( u_t(t, 0) = 0, u_{xt}(t, 1) = -a \frac{d}{dx} (u_t(t, 1)) \) for all \( t \geq 0 \), \( B = a \sigma \left( \exp(r)(1-ac) + \exp(-r)(1+ac) \right) \), we obtain for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) = -\frac{cr}{2} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 dx \\
- \frac{cr}{2} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 dx \\
- \frac{c}{2} \left( \exp(-r)(1+ac)^2 - \exp(r)(1-ac)^2 \right) u_t^2(t, 1) \\
- 2\sigma \int_0^1 \cosh(rx) u_{xxt}^2(t, x) dx + 2c\sigma \int_0^1 \sinh(rx) u_{xx}(t, x)u_{x}(t, x) dx \\
+ \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) (f(t, x) - \sigma ru_{xt}(t, x)) dx \\
- \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x)) (b \theta_x(t, x) + \mu u_t(t, x)) dx \\
+ \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) (f(t, x) + \sigma ru_{xt}(t, x)) dx \\
- \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x)) (b \theta_x(t, x) + \mu u_t(t, x)) dx \tag{68}
\]
Using the following inequalities

\[
(u_t(t, x) \pm c u_x(t, x)) f(t, x) \\
\leq \frac{cr}{4(1 + \mu + b + \sigma r)} (u_t(t, x) \pm c u_x(t, x))^2 + \frac{1 + \mu + b + \sigma r}{cr} |f(t, x)|^2 \\
(u_t(t, x) \pm c u_x(t, x)) \theta_x(t, x) \\
\geq -\frac{cr}{4(1 + \mu + b + \sigma r)} (u_t(t, x) \pm c u_x(t, x))^2 - \frac{1 + \mu + b + \sigma r}{cr} \theta_x^2(t, x) \\
(u_t(t, x) \pm c u_x(t, x)) u_t(t, x) \\
\geq -\frac{cr}{4(1 + \mu + b + \sigma r)} (u_t(t, x) \pm c u_x(t, x))^2 - \frac{1 + \mu + b + \sigma r}{cr} u_t^2(t, x) \\
|u_t(t, x) \pm c u_x(t, x)| |u_{xt}(t, x)| \\
\leq \frac{cr}{4(1 + \mu + b + \sigma r)} (u_t(t, x) \pm c u_x(t, x))^2 + \frac{1 + \mu + b + \sigma r}{cr} u_{xt}^2(t, x)
\]

we obtain from (68) the following estimate for \( t \geq 0 \):

\[
\frac{d}{dt} \Phi(u[t], u_t[t]) \leq -\frac{cr}{4} \int_0^1 \exp(rx) (u_t(t, x) + c u_x(t, x))^2 \, dx \\
-\frac{cr}{4} \int_0^1 \exp(-rx) (u_t(t, x) - c u_x(t, x))^2 \, dx \\
-\frac{c}{2} \left( \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2 \right) u_t^2(t, 1) \\
-2cr \int_0^1 \sinh(rx) u_{xx}(t, x) u_{xt}(t, x) \, dx \\
+ \frac{2(1 + \mu + b + \sigma r)}{cr} \int_0^1 \cosh(rx) \left( |f(t, x)|^2 + b \theta_x^2(t, x) + \mu u_t^2(t, x) + \sigma r u_{xt}^2(t, x) \right) \, dx
\]

Equation (62) in conjunction with (10) implies for all \( t \geq 0 \):

\[
\frac{d}{dt} W(u[t], u_t[t]) \\
= \int_0^1 (u_t(t, x) - \sigma u_{xx}(t, x)) \left( \epsilon^2 u_{xx}(t, x) - \mu u_t(t, x) - b \theta_x(t, x) + f(t, x) \right) \, dx \\
= -\sigma \epsilon^2 \int_0^1 u_{xx}^2(t, x) \, dx + (\epsilon^2 + \mu \sigma) \int_0^1 u_t(t, x) u_{xx}(t, x) \, dx - \mu \int_0^1 u_t^2(t, x) \, dx \\
- b \int_0^1 u_t(t, x) \theta_x(t, x) \, dx + \int_0^1 u_t(t, x) f(t, x) \, dx \\
+ \sigma b \int_0^1 u_{xx}(t, x) \theta_x(t, x) \, dx - \sigma \int_0^1 u_{xx}(t, x) f(t, x) \, dx
\]
Using the inequalities

\[ u_t(t, x)u_{xx}(t, x) \leq \frac{\sigma c^2}{2(c^2 + (1 + \mu + b)\sigma)} u_{xx}^2(t, x) + \frac{c^2 + (1 + \mu + b)\sigma}{2c^2} u_t^2(t, x) \]

\[ u_{xx}(t, x)\theta_x(t, x) \leq \frac{\sigma c^2}{2(c^2 + (1 + \mu + b)\sigma)} u_{xx}^2(t, x) + \frac{c^2 + (1 + \mu + b)\sigma}{2c^2} \theta_x^2(t, x) \]

\[ u_{xx}(t, x) f(t, x) \geq -\frac{\sigma c^2}{2(c^2 + (1 + \mu + b)\sigma)} u_{xx}^2(t, x) - \frac{c^2 + (1 + \mu + b)\sigma}{2c^2} |f(t, x)|^2 \]

\[ u_t(t, x)\theta_x(t, x) \geq -\frac{\sigma c^2}{2(c^2 + (1 + \mu + b)\sigma)} u_t^2(t, x) - \frac{c^2 + (1 + \mu + b)\sigma}{2c^2} \theta_x^2(t, x) \]

\[ u_t(t, x) f(t, x) \leq \frac{\sigma c^2}{2(c^2 + (1 + \mu + b)\sigma)} u_t^2(t, x) + \frac{c^2 + (1 + \mu + b)\sigma}{2c^2} |f(t, x)|^2 \]

we obtain from (70) the following estimate for \( t \geq 0 \):

\[
\frac{d}{dt} W(u[t], u_t[t]) \leq -\frac{\sigma c^2}{2} \int_0^1 u_{xx}^2(t, x) dx + Q \int_0^1 u_t^2(t, x) dx + \frac{\mu(1 + \mu + b)\sigma}{2c^2} \int_0^1 u_t^2(t, x) dx \\
+ \frac{(\sigma + 1)b}{2c^2} \left( c^2 + (1 + \mu + b)\sigma \right) \int_0^1 \theta_x^2(t, x) dx \\
+ \frac{\sigma + 1}{2c^2} \left( c^2 + (1 + \mu + b)\sigma \right) \int_0^1 |f(t, x)|^2 dx
\]

(71)

where \( Q := \frac{c^2 + (1 + b)\sigma}{2c^2} + \frac{\sigma c^2(1 + b)}{2(c^2 + (1 + \mu + b)\sigma)} \). Define:

\[
R := \frac{c}{8Q \cosh(r)}
\]

(72)

\[
V(u, w, \theta) := \Phi(u, w) + R W(u, w) + M E(u, w, \theta),
\]

for all \( u \in H^2(0, 1), w \in C^0([0, 1]) , \theta \in L^2(0, 1) \)

(73)

where \( M > 0 \) is a sufficiently large constant that satisfies

\[
M \geq \frac{2(1 + \mu + b + \sigma r)}{c} \cosh(r) + \frac{4 \sinh^2(r)}{R}
\]

\[
M \geq \frac{(1 + \mu + b)\sigma R}{2c^2} + \frac{2(1 + \mu + b + \sigma r)}{cr} \cosh(r)
\]

\[
M \geq \frac{2k}{k} \left( \frac{2(1 + \mu + b + \sigma r)}{cr} \cosh(r) + \frac{(\sigma + 1)}{2\sigma c^2} \left( c^2 + (1 + \mu + b)\sigma \right) R \right)
\]

\[
M \geq \frac{1}{ac} \left( \exp(r)(1 - ac)^2 - \exp(-r)(1 + ac)^2 \right)
\]

\[
M > R - 2 \exp(-r)
\]

(74)
Definitions (58), (59), (60) and (73) guarantee that the following inequalities hold for all $u \in H^2(0, 1)$, $w \in C^0([0, 1])$, $\theta \in L^2(0, 1)$:

$$V(u, w, \theta) \leq C_2 \left( \|w\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + w^2(1) + \|u_{xx}\|_2^2 \right)$$

$$V(u, w, \theta) \geq C_1 \left( \|w\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + w^2(1) + \|u_{xx}\|_2^2 \right)$$

(75)

where

$$C_1 := \min \left( \frac{M - R}{2} + \exp(-r), c^2 \left( \frac{M}{2} + \exp(-r) \right), \frac{bM - a\sigma M + B}{2}, \frac{R \sigma^2}{4} \right)$$

$$C_2 := \max \left( \frac{M}{2} + \exp(r) + R, c^2 \left( \frac{M}{2} + \exp(r) \right), \frac{bM - a\sigma M + B}{2}, \frac{R \sigma^2}{4} \right)$$

(76)

Notice that due to (74), the constants $C_1, C_2$ defined by (76) are positive, i.e., $0 < C_1 \leq C_2$.

Using (65), (69), (71) and definition (73), we get for $t \geq 0$:

$$\frac{d}{dt} V(u[t], u_x(t), \theta(t)) \leq -\frac{cr}{4} \int_0^1 \exp(rx) (u_t(t, x) + cu_x(t, x))^2 \, dx$$

$$-\frac{cr}{4} \int_0^1 \exp(-rx) (u_t(t, x) - cu_x(t, x))^2 \, dx$$

$$-\frac{c}{2} (2acM + \exp(-r)(1 + ac)^2 - \exp(r)(1 - ac)^2) u_t^2(t, 1)$$

$$- \left( \frac{c}{2} \left( \frac{M - R}{2} + \exp(-r) \right), \frac{bM - a\sigma M + B}{2}, \frac{R \sigma^2}{4} \right) \int_0^1 u_{xx}^2(t, x) \, dx - \frac{\sigma^2 c^2 R}{2} \int_0^1 u_{xx}^2(t, x) \, dx$$

$$+ M \int_0^1 u_t(t, x) f(t, x) \, dx + QR \int_0^1 u_t^2(t, x) \, dx$$

$$- 2\sigma \int_0^1 \sinh(rx) u_{xx}(t, x) u_{xt}(t, x) \, dx$$

$$+ \left( \frac{c^2 + (1 + \mu + b)\sigma}{2\sigma c^2} \frac{R}{r} + \frac{2(1 + \mu + b + \sigma r)}{cr} \frac{\cosh(r)}{\cosh(r)} \right) \|f[t]\|_2^2$$

$$- \mu \left( \frac{M - (1 + \mu + b)\sigma R}{2c^2} - \frac{2(1 + \mu + b + \sigma r)}{cr} \frac{\cosh(r)}{\cosh(r)} \right) \int_0^1 u_t^2(t, x) \, dx$$

$$- b \left( \frac{km}{\lambda} - \frac{2(1 + \mu + b + \sigma r)}{cr} \frac{\cosh(r)}{\cosh(r)} - \frac{(\sigma + 1)}{2\sigma c^2} \frac{R}{r} + (1 + \mu + b)\sigma \right) \int_0^1 \theta_x^2(t, x) \, dx$$

(77)

Using inequalities (74) and the inequalities

$$Mu(t, x) f(t, x) \leq QR u_t^2(t, x) + \frac{M^2}{4QR} |f(t, x)|^2$$

$$u_{xx}(t, x) u_{xt}(t, x) \geq -\frac{c R}{8 \sinh(r)} u_{xx}^2(t, x) - \frac{2 \sinh(r)}{c R} u_{xt}^2(t, x)$$
we obtain from (77) for all $t \geq 0$:

$$
\frac{d}{dt} V(u[t], u_i[t], \theta[t]) \leq -\frac{cr}{8} \int_0^1 \exp(r x) (u_i(t,x) + cu_x(t,x))^2 \, dx
- \frac{cr}{8} \int_0^1 \exp(-r x) (u_i(t,x) - cu_x(t,x))^2 \, dx - \frac{ac^2 M}{2} u_i^2(t,1)
- \sigma \left( M - \frac{2(1 + \mu + b + \sigma r)}{c} \cosh(r) - \frac{4 \sinh^2(r)}{R} \right) \int_0^1 u_{xx}^2(t,x) \, dx
- \frac{\sigma c^2 R}{4} \int_0^1 u_{xx}^2(t,x) \, dx + 2QR \int_0^1 u_i^2(t,x) \, dx
- \frac{bkM}{2\lambda} \int_0^1 \theta_x^2(t,x) \, dx + K \|f[t]\|_2^2
$$

(78)

where $K := \frac{\sigma^2}{2ac^2} \left( \frac{c^2}{2} + (1 + \mu + b)\sigma \right) R + \frac{2(1 + \mu + b + \sigma r)}{8} \cosh(r) + \frac{M^2}{4QR}$. Combining (29) with (78) and using definition (72) and inequalities (74), we obtain the following estimate for $t \geq 0$:

$$
\frac{d}{dt} V(u[t], u_i[t], \theta[t]) \leq -\frac{cr}{8} \int_0^1 \exp(r x) (u_i(t,x) + cu_x(t,x))^2 \, dx
- \frac{cr}{8} \int_0^1 \exp(-r x) (u_i(t,x) - cu_x(t,x))^2 \, dx - \frac{ac^2 M}{2} u_i^2(t,1)
- \frac{\sigma c^2 R}{4} \|u_{xx}[t]\|_2^2 - \frac{bkM}{2\lambda} \|\theta_x[t]\|_2^2 + K \|f[t]\|_2^2
$$

(79)

Wirtinger’s inequality and (9) imply that $\int_0^1 \theta_x^2(t,x) \, dx \geq \pi^2 \int_0^1 \theta^2(t,x) \, dx$. The previous inequality and estimate (79) give the following estimate for $t \geq 0$:

$$
\frac{d}{dt} V(u[t], u_i[t], \theta[t]) \leq -\frac{cr}{8} \exp(-r) \|u_i[t]\|_2^2 - \frac{c^2 r}{8} \exp(-r) \|u_x[t]\|_2^2
- \frac{ac^2 M}{2} u_i^2(t,1) - \frac{\sigma c^2 R}{4} \|u_{xx}[t]\|_2^2 - \frac{bkM}{2\lambda} \pi^2 \|\theta[t]\|_2^2 + K \|f[t]\|_2^2
$$

(80)

Consequently, we obtain from (80) for $t \geq 0$:

$$
\frac{d}{dt} V(u[t], u_i[t], \theta[t])
\leq -\varphi \left( \|u_i[t]\|_2^2 + \|u_x[t]\|_2^2 + u_i^2(t,1) + \|u_{xx}[t]\|_2^2 + \|\theta[t]\|_2^2 \right) + K \|f[t]\|_2^2
$$

(81)

where $\varphi := \min \left( \frac{cr}{8} \exp(-r), \frac{c^2 r}{8} \exp(-r), \frac{ac^2 M}{2}, \frac{\sigma c^2 R}{4}, \frac{bkM}{2\lambda} \pi^2 \right)$. Using (75) and (81) we get for $t \geq 0$:

$$
\frac{d}{dt} V(u[t], u_i[t], \theta[t]) \leq -2\omega V(u[t], u_i[t], \theta[t]) + K \|f[t]\|_2^2
$$

(82)
where \( \omega := \frac{\varphi}{2C_2} \). Multiplying both sides of (82) by \( \exp(2\omega t) \) we get for \( t \geq 0 \):

\[
\frac{d}{dt} \left( \exp(2\omega t) V(u[t], u_t[t], \theta[t]) \right) \leq K \exp(2\omega t) \| f[t] \|_2^2
\]

Differential inequality (83) directly implies the following estimate for all \( t \geq 0 \):

\[
V(u[t], u_t[t], \theta[t]) \leq \exp(-2\omega t) V(u[0], u_t[0], \theta[0]) + K \int_0^t \exp(-2\omega(t-s)) \| f[s] \|_2^2 \, ds
\]

which gives

\[
V(u[t], u_t[t], \theta[t]) \leq \exp(-2\omega t) V(u[0], u_t[0], \theta[0]) + \frac{K}{2\omega} \sup_{0 \leq s \leq t} \left( \| f[s] \|_2^2 \right)
\]

Estimate (18) with appropriate constants \( G, \gamma > 0 \), is a direct consequence of estimate (84) and inequalities (75). The proof is complete. 

**Remark** It should be noticed that when \( \sigma \to 0^+ \), the constants \( K := \frac{\sigma+1}{2\sigma c^2} \left( c^2 + (1 + \mu + b) \sigma \right) R + \frac{2(1+\mu+b+\sigma r)}{c r} \cosh(r) + \frac{M^2}{QR} \omega := \frac{\varphi}{2C_2} \) and \( C_1 \) defined by (76) satisfy \( C_1 \to 0^+, K \to +\infty \) and \( \omega \to 0^+ \). Therefore, it becomes clear from (84) and (75) that the gain \( \gamma \) in the ISS estimate (18) of the distributed disturbance input \( f \) becomes unbounded as \( \sigma \to 0^+ \). This is expected because when \( \sigma \to 0^+ \), model (D) “tends” to model (C) for which the ISS estimate (17) does not allow the derivation of bounds for \( \| u_{xx}[t] \|_2 \) and \( |u_t(t, 1)| \). On the other hand, the asymptotic analysis when \( \sigma \to +\infty \) of the gain \( \gamma \) in the ISS estimate (18) of the input \( f \) is not easy and is a topic for future research.

**5 Concluding remarks**

The study of the robustness properties of the 1-D wave equation for an elastic vibrating string was performed under four different damping mechanisms that are usually neglected in the study of the wave equation: (i) friction with the surrounding medium of the string (or viscous damping), (ii) thermoelastic phenomena (or thermal damping), (iii) internal friction of the string (or Kelvin-Voigt damping), and (iv) friction at the free end of the string (the so-called passive damper).

The study is by no means complete. Future work may consider different boundary conditions for the temperature at the ends of the string or may also consider the heat exchange between the body of the string and the surrounding fluid. Moreover, from a control perspective, the “ultimate goal” would be to use the ISS analysis for the design of robust stabilizers (robust even in the presence of small input delays). We cannot be
sure that such a goal is feasible (although some stabilization results are given in [28]) but it is certainly a topic that must be studied in the future.

Declarations

Conflict of interest The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers’ bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Appendix A Well-posedness of 1-D wave models

We next present the well-posedness analysis for all studied models. However, it should be noted that such an analysis provides sufficient and not necessary conditions for the existence/uniqueness of a solution with specific regularity properties. On the other hand, Theorem 1, Theorem 2 and Theorem 3 assume less regularity properties for the external inputs. For example, the well-posedness analysis given below for Model (D) requires that \( f \in C^1(\mathbb{R}_+; L^2(0, 1)) \) is a sufficient condition for the existence/uniqueness of a solution for Model (D), while Theorem 3 simply requires that \( f \in C^0(\mathbb{R}_+; L^2(0, 1)) \). This is important because one cannot exclude the possibility that a solution for Model (D) with the regularity properties that are specified in Theorem 3 exists for less regular inputs (see the discussion on pages 105-110 in [31]).

The results that follow use a similar notation (for example, the state is \( U \), the state space is \( X \), the linear unbounded operator is \( A \), etc.). However, the reader should not be tempted, by an overlapping notation for different quantities, to compare the different quantities in the analysis of each one of the models.

(1) Well-posedness analysis of Model (B).

Model (B) consists of equations (2), (5), (6). Applying the transformation

\[
\begin{align*}
    w(t, x) &= u_t(t, x) \\
    v(t, x) &= u(t, x) - d(t)x, \quad \text{for } t \geq 0, x \in [0, 1]
\end{align*}
\]

(A1)

and assuming that \( d \in C^1(\mathbb{R}_+) \), we obtain the following first-order in time model

\[
U_t + AU = F
\]

(A2)

where

\[
\begin{align*}
    U &= (v, w) \\
    F(t, x) &= (-\dot{d}(t)x, f(t, x)), \quad \text{for } t \geq 0, x \in [0, 1]
\end{align*}
\]

(A3)

\[
A : D(A) \to X
\]

is the linear unbounded operator defined by
\[ AU = \left( -w, -c^2 v'' + \mu w \right), \text{ for } U = (v, w) \in D(A) \]  
\[ (A5) \]

with \( X \) being the Hilbert space

\[ X = \left\{ v \in H^1(0, 1) : v(0) = 0 \right\} \times L^2(0, 1) \]  
\[ (A6) \]

with scalar product

\[
(U, \bar{U}) = c^2 \int_0^1 v'(x)\bar{v}'(x)dx + \int_0^1 v(x)\bar{v}(x)dx + \int_0^1 w(x)\bar{w}(x)dx,
\]

\[ \text{for } U = (v, w) \in X, \quad \bar{U} = (\bar{v}, \bar{w}) \in X \]  
\[ (A7) \]

and \( D(A) \subset X \) being the following linear subspace of \( X \)

\[ D(A) = \left\{ (v, w) \in H^2(0, 1) \times H^1(0, 1) : v(0) = w(0) = v'(1) + aw(1) = 0 \right\} \]  
\[ (A8) \]

Let \( I : X \rightarrow X \) be the identity operator. We next show that \( A + I \) is a maximal monotone operator (see [5]). Indeed, we have for all \( U = (v, w) \in D(A) \)

\[
((A + I)U, U) = -c^2 \int_0^1 w'(x)v'(x)dx - \int_0^1 w(x)v(x)dx - c^2 \int_0^1 v''(x)w(x)dx
\]

\[
+ (\mu + 1) \|w\|_2^2 + c^2 \|v'\|_2^2 + \|v\|_2^2
\]

\[
= -c^2 w(1)v'(1) - \int_0^1 w(x)v(x)dx + (\mu + 1) \|w\|_2^2 + c^2 \|v'\|_2^2 + \|v\|_2^2
\]

\[
\geq ac^2 w^2(1) - \|w\|_2 \|v\|_2 + (\mu + 1) \|w\|_2^2 + c^2 \|v'\|_2^2 + \|v\|_2^2
\]

\[
\geq ac^2 w^2(1) + \left( \mu + \frac{1}{2} \right) \|w\|_2^2 + c^2 \|v'\|_2^2 + \frac{1}{2} \|v\|_2^2 \geq 0
\]

In the above we have used (A5), (A7), (A8), integration by parts, the Cauchy-Schwarz inequality and the fact that \( \|w\|_2 \|v\|_2 \leq \frac{1}{2} \|w\|_2^2 + \frac{1}{2} \|v\|_2^2 \). Moreover, for every \( \bar{F} = (\bar{F}_1, \bar{F}_2) \in X \) the equation \( (A + 2I)U = \bar{F} \in X \) has a unique solution \( U = (v, w) \in D(A) \). Indeed, the equation \( (A + 2I)U = \bar{F} \in X \) gives \( w = 2v - \bar{F}_1 \) and \( v(x) = \tilde{v}(x) + \frac{a\bar{F}_1(1)}{2a+1}x \) for \( x \in [0, 1] \), where \( \tilde{v} \in H^2(0, 1) \) is the unique solution of the boundary value problem

\[
-c^2 \tilde{v}''(x) + 2(\mu + 2)\tilde{v}(x) = (\mu + 2)\bar{F}_1(x) + \bar{F}_2(x) - 2(\mu + 2)\frac{a\bar{F}_1(1)}{2a+1}x,
\]

\[ \text{for } x \in (0, 1) \text{ a.e., with } \tilde{v}(0) = 0 \text{ and } \tilde{v}'(1) = -2a\tilde{v}(1) \]  
\[ (A9) \]

The fact that the boundary-value problem (A9) has a unique solution follows from the methodology described on pages 221-229 in [5] and the Lax-Milgram Theorem (Corollary 5.8 on page 140 in [5]).
Thus $A + I$ is a maximal monotone operator and consequently (using the Hille-Yosida Theorem) $A$ is the generator of a continuous semigroup of contractions $S(t) : X \to X$. It follows from Theorem 7.10 on page 198 in [5] that for every $F \in C^1(\mathbb{R}_+; X)$ and for every $U_0 \in D(A)$ there exists a unique solution $U \in C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(A))$ of the initial-value problem (A.2) with initial condition $U(0) = U_0$.

Going back to the original state variables (and using (A1)), we conclude that for every $d \in C^2(\mathbb{R}_+)$, $f \in C^1(\mathbb{R}_+; L^2(0, 1))$ and for every $u_0 \in H^2(0, 1)$, $w_0 \in H^1(0, 1)$ with $u_0(0) = w_0(0) = 0$, $u_0'(1) = -aw_0(1) + d(0)$ there exists a unique solution $u \in C^1(\mathbb{R}_+; H^1(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1))$ with $u_t \in C^1(\mathbb{R}_+; L^2(0, 1)) \cap C^0(\mathbb{R}_+; H^1(0, 1))$ of the initial-boundary value problem (2), (5), (6) with $u[0] = u_0$, $u_t[0] = w_0$ that additionally satisfies $u_t(t, 0) = 0$ for all $t \geq 0$.

(2) Well-posedness analysis of Model (C).
Model (C) consists of equations (2), (6), (7), (8), (9). Applying the transformation (A1) and assuming that $d \in C^1(\mathbb{R}_+)$, we obtain the first-order in time model (A2), where

$$U = (v, w, \theta)$$
$$F(t, x) = (-\dot{d}(t)x, f(t, x), 0), \text{ for } t \geq 0, x \in [0, 1]$$

and $A : D(A) \to X$ is the linear unbounded operator defined by

$$AU = \left(-w, -c^2v'' + \mu w + b\theta', -k\theta'' + \lambda w'\right), \text{ for } U = (v, w, \theta) \in D(A)$$

with $X$ being the Hilbert space

$$X = \left\{ v \in H^1(0, 1) : v(0) = 0 \right\} \times L^2(0, 1) \times L^2(0, 1)$$

with scalar product

$$(U, \bar{U}) = c^2 \int_0^1 v(x)\bar{v}(x)dx + \int_0^1 u(x)\bar{u}(x)dx$$
$$+ \int_0^1 w(x)\bar{w}(x)dx + \frac{b}{\lambda} \int_0^1 \theta(x)\bar{\theta}(x)dx,$$

for $U = (v, w, \theta) \in X$, $\bar{U} = (\bar{v}, \bar{w}, \bar{\theta}) \in X$

and $D(A) \subset X$ being the following linear subspace of $X$

$$D(A) = \left\{(v, w, \theta) \in H^2(0, 1) \times H^1(0, 1) \times H^2(0, 1) : v(0) = w(0) = \theta(0) = 0, v'(1) + aw(1) = \theta(1) = 0 \right\}$$
Let $I : X \to X$ be the identity operator. We next show that $A + I$ is a maximal monotone operator (see [5]). Indeed, we have for all $U = (v, w, \theta) \in D(A)$

$$((A + I)U, U) = -c^2 \int_0^1 w'(x)v'(x)dx - \int_0^1 w(x)v(x)dx - c^2 \int_0^1 v''(x)w(x)dx$$

$$+ (\mu + 1) \|w\|_2^2 + b \int_0^1 \theta'(x)w(x)dx - \frac{bk}{\lambda} \int_0^1 (x)\theta''(x)dx$$

$$+ b \int_0^1 \theta(x)w'(x)dx + c^2 \|v''\|_2^2 + \|v\|_2^2 + \frac{b}{\lambda} \|\theta\|_2^2$$

$$= -c^2 w(1)v'(1) - \int_0^1 w(x)v(x)dx + (\mu + 1) \|w\|_2^2$$

$$+ \frac{bk}{\lambda} \|\theta'\|_2^2 + c^2 \|v''\|_2^2 + \|v\|_2^2 + \frac{b}{\lambda} \|\theta\|_2^2$$

$$\geq a c^2 w^2(1) - \|w\|_2^2 \|v\|_2^2 + (\mu + 1) \|w\|_2^2 + c^2 \|v''\|_2^2 + \|v\|_2^2 + \frac{bk}{\lambda} \|\theta'\|_2^2 + \frac{b}{\lambda} \|\theta\|_2^2$$

$$\geq a c^2 w^2(1) + \left(\mu + \frac{1}{2}\right) \|w\|_2^2 + c^2 \|v''\|_2^2 + \frac{1}{2} \|v\|_2^2 + \frac{bk}{\lambda} \|\theta'\|_2^2 + \frac{b}{\lambda} \|\theta\|_2^2 \geq 0$$

In the above we have used (A12), (A14), (A15), three integrations by parts, the Cauchy-Schwarz inequality and the fact that $123$.

Moreover, for every $\tilde{F} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3) \in X$ the equation $(A + 2I)U = \tilde{F} \in X$ has a unique solution $U = (v, w, \theta) \in D(A)$. Indeed, the equation $(A + 2I)U = \tilde{F} \in X$ gives $w = 2v - \tilde{F}_1$ and $v(x) = \tilde{v}(x) + \frac{a \tilde{F}_1(1)}{2a+1} x$ for $x \in [0, 1]$, where $(\tilde{v}, \theta) \in H^2(0, 1) \times H^2(0, 1)$ is the unique solution of the boundary-value problem

$$-c^2 \tilde{v}''(x) + 2(\mu + 2) \tilde{v}(x) + b \theta'(x) = \varphi_1(x)$$

$$-k \theta''(x) + 2\lambda \tilde{v}'(x) + 2\theta(x) = \varphi_2(x)$$

for $x \in (0, 1)$ a.e. with $\tilde{v}(0) = \theta(0) = 0$

and $\tilde{v}'(1) + 2a \tilde{v}(1) = \theta(1) = 0$  \hspace{1cm} (A16)

with $\varphi_1(x) = \tilde{F}_2(x) + (\mu + 2) \tilde{F}_1(x) - 2(\mu + 2) \frac{a \tilde{F}_1(1)}{2a+1} x$ and $\varphi_2(x) = \tilde{F}_3(x) + \lambda \tilde{F}_1'(x) - \frac{2a \tilde{F}_1(1)}{2a+1}$. Let $Y$ be the Hilbert space

$$Y = \left\{ (\tilde{v}, \theta) \in H^1(0, 1) \times H^1(0, 1) : \tilde{v}(0) = \theta(0) = \theta(1) = 0 \right\}$$

with the usual scalar product

$$((\tilde{v}, \theta), (r, p)) = \int_0^1 \tilde{v}'(x)r'(x)dx + \int_0^1 \tilde{v}(x)r(x)dx$$

$$+ \int_0^1 \theta'(x)p'(x)dx + \int_0^1 \theta(x)p(x)dx,$$

for $(\tilde{v}, \theta) \in Y$, $(r, p) \in Y$  \hspace{1cm} (A18)
The fact that the boundary-value problem (A16) has a unique solution $(\tilde{v}, \theta) \in H^2(0, 1) \times H^2(0, 1)$ for every $(\varphi_1, \varphi_2) \in L^2(0, 1) \times L^2(0, 1)$ is a direct consequence of the methodology described on pages 221-229 in [5] and the Lax-Milgram Theorem (Corollary 5.8 on page 140 in [5]) applied to the continuous, coercive, bilinear form $\alpha$ on $Y$ defined by the formula

\[
\alpha ((\tilde{v}, \theta), (r, p)) = 2ac^2\tilde{v}(1)r(1) + c^2 \int_0^1 \tilde{v}'(x)r'(x)dx + 2(\mu + 2) \int_0^1 \tilde{v}(x)r(x)dx + b \int_0^1 \theta'(x)r(x)dx + \frac{bk}{2k} \int_0^1 \theta'(x)p'(x)dx + b \int_0^1 \tilde{v}'(x)p(x)dx + b \int_0^1 \theta(x)p(x)dx
\]

for all $(\tilde{v}, \theta) \in Y$, $(r, p) \in Y$.

Thus $A + I$ is a maximal monotone operator and consequently (using the Hille-Yosida Theorem) $A$ is the generator of a continuous semigroup of contractions $S(t) : X \to X$.

It follows from Theorem 7.10 on page 198 in [5] that for every $F \in C^1(\mathbb{R}_+; X)$ and for every $U_0 \in D(A)$ there exists a unique solution $U \in C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(A))$ of the initial-value problem (A2) with initial condition $U(0) = U_0$.

Going back to the original state variables (and using (A1)), we conclude that for every $d \in C^2(\mathbb{R}_+), f \in C^1(\mathbb{R}_+; L^2(0, 1))$ and for every $u_0 \in H^2(0, 1), w_0 \in H^1(0, 1), \theta_0 \in H^2(0, 1)$ with $u_0(0) = w_0(0) = \theta(0) = \theta(1) = 0, u_0'(0) = -aw_0(1) + d(0)$ there exists a unique solution $u \in C^1(\mathbb{R}_+; H^1(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1)), \theta \in C^1(\mathbb{R}_+; L^2(0, 1)) \cap C^0(\mathbb{R}_+; H^2(0, 1))$ of the initial-boundary value problem (2), (6), (7), (8), (9) with $u[0] = u_0, u_t[0] = w_0, \theta[0] = \theta_0$ that additionally satisfies $u_t(t, 0) = 0$ for all $t \geq 0$.

(3) Well-posedness analysis of Model (D).

Model (C) consists of equations (2), (8), (9), (10), (11). Applying the transformation

\[
w(t, x) = u_t(t, x) - \sigma u_{xx}(t, x) + gu(t, x) \\
v(t, x) = u_t(t, x) \\
y(t) = u_t(t, 1)
\]

for $t \geq 0$, $x \in [0, 1]

(A19)

we obtain the first-order in time model (A2), where

\[
U = (u, v, \theta, w, y)
\]

(A21)

\[
F(t, x) = (0, f(t, x), 0, f(t, x), 0), \text{ for } t \geq 0, x \in [0, 1]
\]

(A22)

and $A : D(A) \to X$ is the linear unbounded operator defined by

\[
\alpha ((\tilde{v}, \theta), (r, p)) = 2ac^2\tilde{v}(1)r(1) + c^2 \int_0^1 \tilde{v}'(x)r'(x)dx + 2(\mu + 2) \int_0^1 \tilde{v}(x)r(x)dx + b \int_0^1 \theta'(x)r(x)dx + \frac{bk}{2k} \int_0^1 \theta'(x)p'(x)dx + b \int_0^1 \tilde{v}'(x)p(x)dx + b \int_0^1 \theta(x)p(x)dx
\]

for all $(\tilde{v}, \theta) \in Y$, $(r, p) \in Y$.

Thus $A + I$ is a maximal monotone operator and consequently (using the Hille-Yosida Theorem) $A$ is the generator of a continuous semigroup of contractions $S(t) : X \to X$.

It follows from Theorem 7.10 on page 198 in [5] that for every $F \in C^1(\mathbb{R}_+; X)$ and for every $U_0 \in D(A)$ there exists a unique solution $U \in C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(A))$ of the initial-value problem (A2) with initial condition $U(0) = U_0$.

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\]

for $t \geq 0$, $x \in [0, 1]

(A19)

we obtain the first-order in time model (A2), where

\[
U = (u, v, \theta, w, y)
\]

(A21)

\[
F(t, x) = (0, f(t, x), 0, f(t, x), 0), \text{ for } t \geq 0, x \in [0, 1]
\]

(A22)

and $A : D(A) \to X$ is the linear unbounded operator defined by
\[ AU = \left( -\sigma u'' - h, -\sigma v'' + gv + \kappa h + b\theta', -k\theta'' + \lambda v', \kappa h + b\theta', a^{-1} v'(1) \right) \]

for \( U = (u, v, \theta, w, y) \in D(A) \), where \( h = w - gu \) (A23)

with \( X \) being the Hilbert space

\[ X = \left( L^2(0, 1) \right)^4 \times \mathbb{R} \] (A24)

with scalar product

\[
(U, \tilde{U}) = \int_0^1 u(x)\tilde{u}(x)dx + \int_0^1 v(x)\tilde{v}(x)dx \\
+ \frac{b}{\lambda} \int_0^1 \theta(x)\tilde{\theta}(x)dx + \int_0^1 w(x)\tilde{w}(x)dx + a\sigma y\tilde{y},
\]

for \( U = (u, v, \theta, w, y) \in X \), \( \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{w}, \tilde{y}) \in X \) (A25)

and \( D(A) \subseteq X \) being the following linear subspace of \( X \)

\[
D(A) = \left\{ (u, v, \theta, w, y) \in Z : \begin{align*}
u(0) = \theta(0) = v(0) &= 0 \\
u(1) - y = u'(1) + a\sigma y &= \theta(1) = 0
\end{align*} \right\}
\]

where \( Z = (H^2(0, 1))^3 \times L^2(0, 1) \times \mathbb{R} \) (A26)

Let \( I : X \to X \) be the identity operator. We next show that \( A + \beta I \) is a maximal monotone operator (see [5]) for a sufficiently large constant \( \beta > 0 \). Indeed, we have for all \( U = (u, v, \theta, w, y) \in D(A) \):

\[
((A + \beta I)U, U) = \int_0^1 u(x) \left( -\sigma u''(x) + gu(x) - w(x) \right) dx \\
+ \int_0^1 v(x) \left( -\sigma v''(x) + gv(x) + \kappa(w(x) - gu(x)) + b\theta'(x) \right) dx \\
+ \frac{b}{\lambda} \int_0^1 \theta(x) \left( -k\theta''(x) + \lambda v'(x) \right) dx + \beta \|u\|_2^2 + \beta \|v\|_2^2 \\
+ \beta \frac{b}{\lambda} \|\theta\|_2^2 + \beta \|w\|_2^2 + a\sigma \beta y^2 \\
+ \int_0^1 w(x) \left( \kappa(w(x) - gu(x)) + b\theta'(x) \right) dx + \sigma y v'(1) \\
= a\sigma y u(1) + \sigma \|u'\|_2^2 - (1 + \kappa g) \int_0^1 u(x)w(x)dx + b \int_0^1 w(x)\theta'(x)dx \\
+ \sigma \|v'\|_2^2 + \sigma^{-1}e^2 \int_0^1 v(x)w(x)dx - \kappa g \int_0^1 v(x)u(x)dx + \frac{bk}{\lambda} \|\theta'\|_2^2 \\
+ (\beta + g) \|u\|_2^2 + (\beta + g) \|v\|_2^2 + \beta \frac{b}{\lambda} \|\theta\|_2^2 + \left( \beta + \sigma^{-1}e^2 \right) \|w\|_2^2 + a\sigma \beta y^2
\]
In the above we have used (A23), (A25), (A26) and integration by parts. Using the fact that \( y u(1) \geq -\beta y^2 - \frac{u^2(1)}{4\beta} \) and the Cauchy-Schwarz inequality we obtain:

\[
((A + \beta I)U, U) \geq -\frac{a\sigma}{4\beta} u^2(1) + \sigma \|u'\|_2^2 \quad - \|1 + \kappa g\| u_2 \|w_2\| - \|b \| w_2 \|\theta'\|_2 \quad - \|1 + \kappa g\| u_2 \|w_2\| - \|b \| w_2 \|\theta'\|_2 \\
+ \sigma \|u'\|_2^2 + \kappa \|v\|_2 \|w_2\| + \kappa |g| \|u\|_2 \|v\|_2 + \frac{bk}{\lambda} \|\theta'\|_2^2 \\
+ (\beta - g) \|u\|_2^2 + (\beta + g) \|v\|_2^2 + \frac{\beta b}{\lambda} \|\theta\|_2^2 + (\beta + \kappa) \|w\|_2^2
\]

Since \( u \in H^2(0, 1) \) with \( u(0) = 0 \) we get \( u^2(1) = 2 \int_0^1 u(x)u'(x)dx \leq 2 \|u\|_2 \|u'\|_2 \). Using this inequality and the inequalities \( \|u\|_2 \|w\|_2 \leq \|u\|_2 \|w\|_2 \leq \|u\|_2^2 + \|w\|_2^2 \), \( \|v\|_2 \|w\|_2 \leq \|v\|_2^2 + \|w\|_2^2 \), \( \|\theta\|_2 \|w\|_2 \leq \frac{k}{\lambda} \|\theta\|_2^2 + \frac{\lambda}{4k} \|w\|_2^2 \), we get:

\[
((A + \beta I)U, U) \geq -\frac{a\sigma}{2\beta} \|u\|_2 \|u'\|_2 + \sigma \|u'\|_2^2 + \|v\|_2^2 \\
+ (\beta - |g| - 1 - 2\kappa |g|) \|u\|_2^2 + (\beta - |g| - \kappa - \kappa |g|) \|v\|_2^2 \\
+ (\beta - \frac{b\lambda}{4k} - 1 - \kappa |g|) \|w\|_2^2
\]

Finally, using the inequality \( \frac{a\sigma}{2\beta} \|u\|_2 \|u'\|_2 \leq \sigma \|u'\|_2^2 + \frac{a^2\sigma}{16\beta^2} \|u\|_2^2 \) we get:

\[
((A + \beta I)U, U) \geq \left( \beta - |g| - 1 - 2\kappa |g| - \frac{a^2\sigma}{16\beta^2} \right) \|u\|_2^2 \\
+ (\beta - |g| - \kappa - \kappa |g|) \|v\|_2^2 + \left( \beta - \frac{b\lambda}{4k} - 1 - \kappa |g| \right) \|w\|_2^2
\]

The above inequality shows that for \( \beta \geq 1 + \kappa + (2\kappa + 1) |g| + \frac{a^2\sigma}{16\beta^2} + \frac{b\lambda}{4k} \) we have \( ((A + \beta I)U, U) \geq 0 \) for all \( U = (u, v, \theta, w, y) \in D(A) \).

We next show that for sufficiently large \( \beta > 0 \), the range of the linear operator \( (A + (\beta + 1)I) \) is \( X \). We show that for every \( \tilde{F} = \{ \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4, \tilde{F}_5 \} \in X \) the equation \( (A+(\beta+1)I)U = \tilde{F} \in X \) has a unique solution \( U = (u, v, \theta, w, y) \in D(A) \) provided that \( \beta > 0 \) is sufficiently large. The equation \( (A+(\beta+1)I)U = \tilde{F} \in X \) gives

\[
y = \frac{a\tilde{F}_5}{1+a(\beta+1)} - \tilde{v}'(1) \quad u(x) = \tilde{u}(x) - a \left( \tilde{v}(1) + \frac{a\tilde{F}_5}{1+a(\beta+1)} \right) x, w(x) = \frac{\tilde{F}_5(x) - b\tilde{v}'(x)}{\beta+1+k} \\
+ \frac{\kappa g}{\beta+1+k} \left( \tilde{u}(x) - a \left( \tilde{v}(1) + \frac{a\tilde{F}_5}{1+a(\beta+1)} \right) x \right) \text{ and } v(x) = \tilde{v}(x) + \frac{a\tilde{F}_5}{1+a(\beta+1)} \text{ for } x \in [0, 1],
\]

where \( (\tilde{u}, \tilde{v}, \theta) \in (H^2(0, 1))^3 \) is a solution of the boundary-value problem

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\[-\sigma \ddot{u}'(x) + (\beta + 1) \left( 1 + \frac{g}{\beta + 1 + \kappa} \right) \ddot{u}(x) \]
\[-a(\beta + 1) \left( 1 + \frac{g}{\beta + 1 + \kappa} \right) \dot{v}(1) x + \frac{b}{\beta + 1 + \kappa} \theta'(x) = \varphi_1(x) \]
\[-\sigma \ddot{v}'(x) + (\beta + 1 + g) \dot{v}(x) + \frac{b(\beta + \kappa)}{\beta + 1 + \kappa} \theta'(x) \]
\[-\frac{\kappa g(\beta + \kappa)}{\beta + 1 + \kappa} \ddot{u}(x) + \frac{a \kappa g(\beta + \kappa)}{\beta + 1 + \kappa} \ddot{v}(1) x = \varphi_2(x) \]
\[-k \theta''(x) + \lambda \dot{v}'(x) + (\beta + 1) \theta(x) = \varphi_3(x) \]

for \( x \in (0, 1) \) a.e. with \( \ddot{u}(0) = \dot{v}(0) = \theta(0) = 0 \)
and \( \ddot{u}'(1) = \dot{v}'(1) + a(\beta + 1) \dot{v}(1) = \theta(1) = 0 \) \hspace{1cm} (A27)

with \( \varphi_3(x) = \tilde{F}_3(x) - \frac{a \lambda \tilde{F}_5}{1 + a(\beta + 1)}, \varphi_1(x) = \tilde{F}_1(x) + \frac{1}{\beta + 1 + \kappa} \tilde{F}_4(x) + \frac{a^2(\beta + 1) \tilde{F}_5}{1 + a(\beta + 1)} \)
\((1 + \frac{g}{\beta + 1 + \kappa}) x \) and \( \varphi_2(x) = \tilde{F}_2(x) - \frac{1}{\beta + 1 + \kappa} \tilde{F}_4(x) - \left( \beta + 1 + g + \frac{a \kappa g(\beta + \kappa)}{\beta + 1 + \kappa} \right) \frac{a \tilde{F}_5 x}{1 + a(\beta + 1)}. \)

Let \( Y \) be the Hilbert space
\[
Y = \left\{ (\ddot{u}, \dot{v}, \theta) \in \left( H^1(0, 1) \right)^3 : \ddot{u}(0) = \dot{v}(0) = \theta(0) = \theta(1) = 0 \right\} \hspace{1cm} (A28)
\]

with the usual scalar product
\[
((\ddot{u}, \dot{v}, \theta), (q', r', p)) = \int_0^1 \ddot{u}(x) q'(x) dx + \int_0^1 \ddot{u}(x) q(x) dx \\
+ \int_0^1 \dot{v}'(x) r'(x) dx + \int_0^1 \dot{v}(x) r(x) dx + \int_0^1 \theta'(x) p'(x) dx + \int_0^1 \theta(x) p(x) dx \\
\text{for } (\ddot{u}, \dot{v}, \theta) \in Y, \ (q', r', p) \in Y \hspace{1cm} (A29)
\]

We show that the boundary-value problem (A27) has a unique solution \((\ddot{u}, \dot{v}, \theta) \in \left( H^2(0, 1) \right)^3 \) for every \((\varphi_1, \varphi_2, \varphi_3) \in \left( L^2(0, 1) \right)^3 \) provided that \( \beta > 0 \) is sufficiently large. Let \( R > 0 \) be an arbitrary constant and consider the continuous, bilinear form \( \alpha \) on \( Y \) defined by the formula
\[ \alpha ((\tilde{u}, \tilde{v}, \theta), (q, r, p)) = \frac{\sigma \sqrt{15\sigma}}{a^2 \sqrt{\beta + 1}} \int_0^1 \tilde{u}'(x)q'(x)dx \]
\[ + \frac{\sqrt{15\sigma} (\beta + 1)}{a^2} \left(1 + \frac{g}{\beta + 1 + \kappa}\right) \int_0^1 \tilde{u}(x)q(x)dx \]
\[ - \frac{\sqrt{15\sigma} (\beta + 1)}{a} \left(1 + \frac{g}{\beta + 1 + \kappa}\right) \tilde{v}(1) \int_0^1 xq(x)dx \]
\[ + \frac{b \sqrt{15\sigma}}{a^2 (\beta + 1 + \kappa) \sqrt{\beta + 1}} \int_0^1 \theta'(x)q(x)dx \]
\[ + \frac{b (\beta + 1)}{\lambda (\beta + 1 + \kappa)} \int_0^1 \vartheta(x)p(x)dx \]

for all \((\tilde{u}, \tilde{v}, \theta) \in Y, (q, r, p) \in Y\). (A30)

We notice that continuity of the bilinear form \(\alpha\) on \(Y\) defined by (A30) is established by means of the inequality \(|\tilde{v}(1)| \leq \sqrt{2} \|\tilde{v}\|_2 \|\tilde{v}'\|_2 \leq \|\tilde{v}\|_2 + \|\tilde{v}'\|_2\), which holds since \(\tilde{v} \in H^1(0, 1)\) with \(\tilde{v}(0) = 0\) (recall (A28)). Completing the squares and using the Cauchy-Schwarz inequality and the fact that \(|\tilde{v}(1)| \leq \sqrt{2} \|\tilde{v}\|_2 \|\tilde{v}'\|_2\), we are in a position to establish the following inequality for all \((\tilde{u}, \tilde{v}, \theta) \in Y\):

\[ \alpha ((\tilde{u}, \tilde{v}, \theta), (\tilde{u}, \tilde{v}, \theta)) \geq \frac{\sigma \sqrt{15\sigma}}{a^2 \sqrt{\beta + 1}} \|\tilde{u}'\|_2^2 + \frac{\sigma a(\beta + 1)}{4} \tilde{v}'^2(1) + \frac{\sigma}{2} \|\tilde{v}'\|_2^2 \]
\[ + \frac{\|\tilde{u}\|_2^2}{2a^2 k \sqrt{\beta + 1}} (\beta + 1) a^2 k \sqrt{15\sigma} \]
\[ - \frac{\|\tilde{u}\|_2^2}{2a^2 k \sqrt{\beta + 1}} \left(\frac{2\sqrt{15\sigma} + a^2 \kappa^2}{\sqrt{\beta + 1}} a^2 k |g| + k \kappa a^4 |g| \sqrt{\beta + 1} + 15\sigma b \lambda + 10 \kappa a |g|^2\right) \]
\[ + \frac{1}{6\sigma} (\sigma (\beta + 1) - (6\sigma + 3\sigma \kappa + a^2 |g|) |g|) \|\tilde{v}\|_2^2 \]
\[ + \frac{bk (\beta + \kappa)}{2\lambda (\beta + 1 + \kappa)} \|\theta '\|_2^2 + \frac{b (\beta + 1) (\beta + \kappa)}{\lambda (\beta + 1 + \kappa)} \|\theta \|_2^2 \] (A31)

It follows from (A31) that the bilinear form \(\alpha\) on \(Y\) defined by (A30) is coercive when

\[ \beta + 1 > \frac{2a^2 \sqrt{\beta + 1}}{\sqrt{15\sigma}} a^2 k |g| + 15\sigma b \lambda + 10ak |g|^2 \]
\[ \beta + 1 > \frac{2\sqrt{15\sigma} + a^2 \kappa^2}{a^2 k \sqrt{15\sigma}} a^2 k |g| + 15\sigma b \lambda + 10ak |g|^2 \]

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The above inequalities are satisfied when $\beta > 0$ is sufficiently large (notice that left hand sides of the above inequalities tend to $+\infty$ as $\beta \to +\infty$). Following the methodology described on pages 221-229 in [5] and using the Lax-Milgram Theorem (Corollary 5.8 on page 140 in [5]) applied to the continuous, coercive, bilinear form $\alpha$ on $Y$ defined by (A30) we conclude that the boundary-value problem (A27) has a unique solution $(\tilde{u}, \tilde{v}, \theta) \in (H^2(0, 1))^3$ for every $(\varphi_1, \varphi_2, \varphi_3) \in (L^2(0, 1))^3$ provided that $\beta > 0$ is sufficiently large.

Thus $A + \beta I$ is a maximal monotone operator when $\beta > 0$ is sufficiently large and consequently (using the Hille-Yosida Theorem) $A$ is the generator of a continuous semigroup of contractions $S(t) : X \to X$. It follows from Theorem 7.10 on page 198 in [5] that for every $F \in C^1([0, T] \times X)$ and for every $U_0 \in D(A)$ there exists a unique solution $U \in C^1([0, T] \times X) \cap C^0([0, T] \times D(A))$ of the initial-value problem (A2) with initial condition $U(0) = U_0$.

Going back to the original state variables (and using (A19)), we conclude that for every $f \in C^1([0, T] \times L^2(0, 1))$ and for every $u_0 \in H^2(0, 1), v_0 \in H^2(0, 1)$, $\theta_0 \in H^2(0, 1)$ with $u_0(0) = v_0(0) = \theta(0) = \theta(1) = 0$, $u_0'(1) = -a v_0(1)$ there exists a unique solution $u \in C^1([0, T] \times L^2(0, 1)) \cap C^0([0, T] \times H^2(0, 1))$, $\theta \in C^1([0, T] \times L^2(0, 1)) \cap C^0([0, T] \times H^2(0, 1))$ with $u_t \in C^1([0, T] \times L^2(0, 1)) \cap C^0([0, T] \times H^2(0, 1))$, $u_{xx} \in C^1([0, T] \times L^2(0, 1))$, $u_t(t, 1) = -a \frac{d}{dt} u_t(t, 1)$ for all $t \geq 0$.

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