**KOSZUL CYCLES AND GOLOD RINGS**

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Abstract. Let $S$ be the power series ring or the polynomial ring over a field $K$ in the variables $x_1, \ldots, x_n$, and let $R = S/I$, where $I$ is proper ideal which we assume to be graded if $S$ is the polynomial ring. We give an explicit description of the cycles of the Koszul complex whose homology classes generate the Koszul homology of $R = S/I$ with respect to $x_1, \ldots, x_n$. The description is given in terms of the data of the free $S$-resolution of $R$. The result is used to determine classes of Golod ideals, among them proper ordinary powers and proper symbolic powers of monomial ideals. Our theory is also applied to stretched local rings.

**Introduction**

Let $S$ be the power series ring or the polynomial ring over a field $K$ in the variables $x_1, \ldots, x_n$, and let $I \subset S$ be a proper ideal of $S$, which we assume to be a graded ideal, if $S$ is the polynomial ring. Consider a finitely generated $R = S/I$-module $M$. The formal power series ring $P^R_S(M)(t) = \sum_{i \geq 0} \dim_K \text{Tor}_i^R(K, M)t^i$ is called the Poincaré series of $M$. Since $M$ is also an $S$-module, we may as well consider the Poincaré series $P^S_R(M)(t)$, similarly defined. Note that $P^S_M(t)$ is a polynomial, since $S$ is regular. By a result of Serre [11], there is a coefficientwise inequality

$$P^R_K(t) \leq \frac{(1 + t)^n}{1 - t(P^R_K(t) - 1)}.$$ 

The ring $R$ (or $I$ itself) is called Golod, if equality holds. In this case the Poincaré series $P^R_K(t)$ is a rational function, which in general for the Poincaré series of the residue field $K$ is not always the case, see [2]. Interestingly, over Golod rings we have rationality not only for $P^R_K(t)$ but also for Poincaré series of all finitely generated $R$-modules.

In Section 1 of this note we give a canonical and explicit description of the cycles of the Koszul complex whose homology classes generate the Koszul homology of $R$ with respect to $x_1, \ldots, x_n$, see Theorem 1.3. The description is given in terms of the data of the free $S$-resolution of $R$, and allows us to identify interesting classes of Golod ideals.

The same strategy, namely to give a nice description of Koszul cycles, has been applied by the first author and Huneke [8] to show, among other results, that in the polynomial case, the powers $I^k$ of $I$ are Golod for all $k \geq 2$, provided the characteristic of $K$ is zero. In that result, the annoying assumption that the characteristic of $K$ should be zero, arises from the fact that if the authors use the result from [7] which says that $\text{char}(K) = 0$, then the desired Koszul cycles can be described in terms of Jacobians derived from the maps in the graded minimal free $S$-resolution of $R$. To avoid this drawback, we choose a different description of the Koszul cycles which can be given for any characteristic of the base field, and only depends on the given order of the variables.

Apart from an explicit description of Koszul cycles, our approach to prove Golodness for certain classes of ideals and rings is based on the Golod criterion [1, Proposition 1.3], due to the second author of this paper. He showed that if $I \subset J \subset S$ are ideals with
\(J^2 \subseteq I\) and such that the natural maps \(\text{Tor}_i^S(K, S/I) \to \text{Tor}_i^S(K, S/J)\) are zero for all \(i \geq 1\), then \(I\) is Golod. As an easy consequence of our description of the Koszul cycles it is shown in Section 2 that \(\text{Tor}_i^S(K, S/I) \to \text{Tor}_i^S(K, S/d(I))\) is the zero map for all \(i \geq 1\), where \(d(I)\) is the ideal generated by the elements \(d^i(f_j)\) for \(i = 1, \ldots, n\) and \(f_1, \ldots, f_m\) a system of generators of \(I\). The operator \(d^i\) is defined by

\[
d^i(f) = (f(0, \ldots, 0, x_i, \ldots, x_n) - f(0, \ldots, 0, x_{i+1}, \ldots, x_n))/x_i \quad \text{for } f \in \mathfrak{n}.
\]

In combination with the above Golod criterion we then obtain that \(I\) is a Golod ideal, if \((d(I))^2 \subseteq I\). If this is the case we say that \(I\) is \(d\)-Golod. The operators \(d^i\) depend on the order of the variables. If \(I\) happens to be \(d\)-Golod after a permutation \(\sigma\) of the variables, then we say that \(I\) is \(d_\sigma\)-Golod, and we call \(I\) strongly \(d\)-Golod, if it is \(d_\sigma\)-Golod for any permutation \(\sigma\).

As one of the main applications of this approach we obtains (that for a monomial ideal \(I\), all proper ordinary powers, saturated powers or symbolic powers of \(I\) are Golod. The same holds true for the integral closures \(\overline{I}\) of the powers \(I^k\) for \(k \geq 2\), see Theorem 3.1 and Proposition 3.2. The same results can be found in [8] for graded ideals, but under the additional assumption that \(\text{char}(K) = 0\). Here we have no assumptions on the the characteristic but we prove these results only for monomial ideal. However, with our methods, a new class of non-monomial Golod ideals in the formal power series ring is detected, see Proposition 3.4.

As a last application of the techniques presented in this paper we have a result of more general nature. In Theorem 3.5 it is shown that if \(R\) is a stretched local ring or a standard graded stretched \(K\)-algebra, then \(R\) is Golod, if one of the following conditions is satisfied: (i) \(R\) is standard graded, (ii) \(R\) is not Artinian, (iii) \(R\) is Artinian and the socle dimension of \(R\) coincides with its embedding dimension.

The result that \(R\) is Golod, if (iii) is satisfied, has been shown in the recent paper [4]. Our proof of this case can be deduced without any big efforts from the case that \(R\) is standard graded. The latter case is accessible to our theory, since after a suitable extension of the base field, the defining ideal of a standard graded stretched \(K\)-algebra \(R\) turns out to be \(d_\sigma\)-Golod for a suitable permutation of the variables.

1. A description of the Koszul cycles

Let \(K\) be a field, and let \(S_r\) stand for power series ring \(K[[x_r, \ldots, x_n]]\) or the polynomial ring \(K[x_r, \ldots, x_n]\) over \(K\). For \(S_1\) we simply write \(S\). Since \(S_r\) is naturally embedded into \(S\) we may view any element in \(S_r\) also as an element in \(S\). We denote by \(n = (x_1, \ldots, x_n)\) the maximal (resp. the graded) maximal ideal of \(S\).

Let \(f \in \mathfrak{n}\). Then

\[
f = \sum_{(r_1, \ldots, r_n) \in \mathbb{N}_0^n} \alpha_{r_1, \ldots, r_n} x_1^{r_1} \cdots x_n^{r_n},
\]

where the coefficients \(\alpha_{r_1, \ldots, r_n}\) belong to \(K\).

For \(f \in \mathfrak{n}\) and \(r = 1, \ldots, n\), we set

\[
d^r(f) = \frac{f(0, \ldots, 0, x_r, \ldots, x_n) - f(0, \ldots, 0, x_{r+1}, \ldots, x_n)}{x_r}.
\]

Then the following rules hold:

(i) \(f = d^1(f)x_1 + \cdots + d^n(f)x_n\), and
(ii) \(d^r(f) \in S_r\) for \(1, \ldots, n\).
Note that the operators $d^r$ are uniquely determined by (i) and (ii).

The following example demonstrates this definition: let $S = K[[x_1, x_2, x_3, x_4]]$ and $f = x_1^2x_3 + x_1x_2^3 + x_2^2x_3^4 + x_3x_4$. Then $d^1(f) = x_1x_3 + x_2^3, d^2(f) = x_2x_3^4, d^3(f) = x_3x_4$ and $d^4(f) = 0$.

Of course the definition of the $d^r(f)$ depend on the order of the variables. Like partial derivatives, the operators $d_i : S \rightarrow S_i$ are $K$-linear maps, and there is a product rule which is however less simple than that for partial derivatives. Indeed one has

**Lemma 1.1.** Let $f, g \in n$ and $r$ be an integer with $1 \leq r \leq n$. Then

(i) $d^r$ is a $K$-linear map and so $d^r(f + g) = d^r(f) + d^r(g)$;

(ii) $d^r(fg) = d^r(f)d^r(g)x_r + \sum_{i > r}(d^r(f)d^r(g) + d^r(g)d^r(f))x_i$.

**Proof.** (i) is obvious. (ii) follows easily from (1) and the equations

$$(fg)(0, \ldots, 0, x_s, \ldots, x_n) = \sum_{i,j=s}^n d^r(f)d^r(g)x_ix_j,$$

for $s = r$ and $s = r + 1$. \qed

Let $\Omega$ be the free $S$-module of rank $n$ with basis $dx_1, \ldots, dx_n$. We denote by $\Omega$ the exterior algebra $\wedge \Omega$ of $\Omega$. Note that $\Omega$ is a graded $S$-algebra with graded components $\Omega_i = \wedge^i \Omega$. In particular, $\Omega_0 = S$ and $\Omega_i$ is a free $S$-module with basis $dx_{r_1} \wedge \cdots \wedge dx_{r_i}$, $1 \leq r_1 < r_2 < \ldots < r_i \leq n$.

Let $\partial_i : \Omega_i \rightarrow \Omega_{i-1}$ be the $S$-linear map given by

$$\partial_i(dx_{r_1} \wedge \cdots \wedge dx_{r_i}) = \sum_{k=1}^i (-1)^{k+1} x_{r_k} dx_{r_1} \wedge \cdots \wedge dx_{r_{k-1}} \wedge dx_{r_{k+1}} \wedge \cdots \wedge dx_i$$

To simplify our notation we shall write $dx_{r_1} \ldots dx_{r_i}$ for $dx_{r_1} \wedge \cdots \wedge dx_{r_i}$.

The complex $\Omega$ is nothing but the Koszul complex with respect to the sequence $x_1, \ldots, x_n$, and so is a minimal free $S$-resolution of the residue field $K$ of $S$.

Now let $I$ be a proper ideal of $S$ and $(F, \delta)$ a minimal free $S$-resolution of $R = S/I$. Then for each $i = 0, \ldots, n$ we have the following isomorphism

$$\psi_i : K \otimes F_i = H_i(K \otimes F) \cong H_i(\Omega \otimes F) \cong H_i(\Omega \otimes S/I).$$

Tracing through this isomorphism one obtains $i$-cycles in $\Omega \otimes S/I$ whose homology classes form a $K$-basis of $H_i(\Omega \otimes S/I)$.

We recall that an element $z = (z_0, \ldots, z_i) \in (\Omega \otimes F)_i = \bigoplus_{j=0}^i \Omega_j \otimes F_{i-j}$ is a cycle in $(\Omega \otimes F)$ if and only if

$$\delta_j(z_j) = (\prod_{i=0}^{i-1} \partial_{i+1} \otimes \text{id})(z_{i+1}) \quad \text{for all} \quad j.$$  

(2)

Now the isomorphism $\psi_i$ can be describe as follows: let $1 \otimes f \in K \otimes F_i$ and choose a cycle $z = (z_0, \ldots, z_i) \in (\Omega \otimes F)_i$ such that $z_0$ maps to $1 \otimes f$ under canonical epimorphism $\Omega_0 \otimes F_i \rightarrow K \otimes F_i$. Then

$$\psi_i(1 \otimes f) = [\bar{z}_i],$$

where $\bar{z}_i$ denotes the image of $z_i$ in $\Omega_i \otimes S/I$ and $[\bar{z}_i]$ its homology class in $H_i(\Omega \otimes S/I)$.

In order to make this description of $\psi_i$ more explicit one has to choose suitable cycles $z = (z_0, \ldots, z_i) \in (\Omega \otimes F)_i$. There are of course many choices for such cycles.
By applying rule (i), we see that
\[ h_{j_k}(3) \]
In order to prove the assertion it suffices to show that for all \( i \), let
\[ \delta_i(e_{ij}) = \sum_{k=1}^{b_i} \alpha_{kj}^{(i)} e_{i-1k} \]
for all \( i, j \).

The following result is the crucial technical statement of this note.

**Proposition 1.2.** Consider the element \((z_0, \ldots, z_i) \in (\Omega \otimes F)_i \) with \( z_0 = 1 \otimes e_{ij} \) and
\[ z_{i-k} = \sum_{j_k=1}^{b_k} \left( \sum_{1 \leq r_{k+1} < \ldots < r_i \leq n} \sum_{j_{k+1}=1}^{b_{k+1}} \cdots \sum_{j_1=1}^{b_i} d^{r_{k+1}}(\alpha_{j_kj_{k+1}}^{(k+1)}) \cdots d^{r_i}(\alpha_{j_1j_{i-1}}^{(i)}) dx_{r_{k+1}} \cdots dx_{r_i} \right) \otimes e_{k_{jk}} \]
for all \( 0 \leq k < i \).

Then
\[ (\text{id} \otimes \delta_{k+1})(z_{i-k-1}) = (\partial_{i-k} \otimes \text{id})(z_{i-k}) \]
for all \( k \).

**Proof.** Since \( \alpha_{j_1j_{i-1}}^{(i)} = \sum_r d^r(\alpha_{j_1j_{i-1}}^{(i)}) x_r \) it is obvious that \((\text{id} \otimes \delta_{1})(z_0) = (\partial_1 \otimes \text{id})(z_1)\). Thus the assertion is true for \( i-k = 1 \). Now let \( i-k > 1 \). Then
\[ (\text{id} \otimes \delta_{k})(z_{i-k}) = \sum_{j_{k-1}=1}^{b_{k-1}} h_{j_{k-1}} \otimes e_{k-1j_{k-1}}, \]
where
\[ h_{j_{k-1}} = \sum_{1 \leq r_{k+1} < \ldots < r_i \leq n} \sum_{j_{k+1}=1}^{b_{k+1}} \cdots \sum_{j_{k-1}=1}^{b_{k-1}} \alpha_{j_{k-1}j_k}^{(k)} d^{r_{k+1}}(\alpha_{j_kj_{k+1}}^{(k+1)}) \cdots d^{r_i}(\alpha_{j_1j_{i-1}}^{(i)}) dx_{r_{k+1}} \cdots dx_{r_i}. \]
In order to prove the assertion it suffices to show that for all \( j_{k-1} = 1, \ldots, b_{k-1} \) we have
\[ h_{j_{k-1}} = \partial_{i-k+1}(g_{j_{k-1}}), \]
where
\[ g_{j_{k-1}} = \sum_{1 \leq r_{k+1} < \ldots < r_i \leq n} \sum_{j_{k+1}=1}^{b_{k+1}} \cdots \sum_{j_{k-1}=1}^{b_{k-1}} d^{r_{k}}(\alpha_{j_{k-1}j_k}^{(k)}) \cdots d^{r_i}(\alpha_{j_1j_{i-1}}^{(i)}) dx_{r_k} \cdots dx_{r_i}. \]
By applying rule (i), we see that
\[ h_{j_{k-1}} = \sum_{1 \leq r_{k+1} < \ldots < r_i \leq n} a_{r_{k+1} \ldots r_i}, \]
where
\[ a_{r_{k+1} \ldots r_i} = \sum_{r=1}^{n} \sum_{j_k=1}^{b_k} \cdots \sum_{j_{k-1}=1}^{b_{k-1}} d^{r}(\alpha_{j_{k-1}j_k}^{(k)}) d^{r_{k+1}}(\alpha_{j_kj_{k+1}}^{(k+1)}) \cdots d^{r_i}(\alpha_{j_1j_{i-1}}^{(i)}) dx_{r} dx_{r_{k+1}} \cdots dx_{r_i}. \]
Since $\alpha^{(k)}\alpha^{(k+1)} = 0$, Lemma 1.1 implies that

\begin{equation}
\sum_{j_k=1}^{b_k} d^{r+1}(\alpha^{(k)}_{j_k-1j_k})d^{r+1}(\alpha^{(k+1)}_{j_kj_{k+1}})x_{r+1} = \\
\sum_{r+1 < r \leq n} \sum_{j_k=1}^{b_k} d^{r}(\alpha^{(k)}_{j_k-1j_k})d^{r}(\alpha^{(k+1)}_{j_kj_{k+1}})x_r - \\
\sum_{r+1 < r \leq n} \sum_{j_k=1}^{b_k} d^{r+1}(\alpha^{(k)}_{j_k-1j_k})d^{r}(\alpha^{(k+1)}_{j_kj_{k+1}})x_r.
\end{equation}

By using this identity, we obtain for $a_{r_{k+1},\ldots,r_i}$ the expression

\begin{equation}
a_{r_{k+1},\ldots,r_i} = \\
\sum_{1 \leq r < r_{k+1}} \sum_{j_{i-1}=1}^{b_{i-1}} \cdots \sum_{j_1=1}^{b_1} d^{r}(\alpha^{(k)}_{j_{i-1}j_i})d^{r}(\alpha^{(k+1)}_{j_ij_{i+1}})\cdots d^{r_i}(\alpha^{(k)}_{j_{i-1}j_i})x_r dx_{r_{k+1}} \cdots dx_{r_i}
\end{equation}

Next we use the analogue to formula (4) corresponding to the fact that $\alpha^{(k+1)}\alpha^{(k+2)} = 0$. Then the sum in the bottom row of the previous expression for $a_{r_{k+1},\ldots,r_i}$ can be rewritten as

\begin{equation}
\sum_{r_{k+2} < r \leq n} \sum_{j_k=1}^{b_k} \cdots \sum_{j_1=1}^{b_1} d^{r+2}(\alpha^{(k)}_{j_k-1j_k})d^{r}(\alpha^{(k+1)}_{j_{i-1}j_i})\cdots d^{r_i}(\alpha^{(k+2)}_{j_{i-1}j_i})x_r dx_{r_{k+1}} \cdots dx_{r_i}
\end{equation}

Proceeding this way, we obtain that

\begin{equation}
h_{j_{k-1}} = \sum_{1 \leq r_{k+1} < \ldots < r_i \leq n} \sum_{l=k}^{i} (-1)^{l-k} c_{r_{k+1},\ldots,r_i},
\end{equation}

where

\begin{equation}
c_{r_{k+1},\ldots,r_i} = \\
\sum_{r_{i+1} < r \leq n} \sum_{j_k=1}^{b_k} \cdots \sum_{j_1=1}^{b_1} d^{r+1}(\alpha^{(k)}_{j_{i+1}j_{i+1}})\cdots d^{r_i}(\alpha^{(l)}_{j_{i-1}j_{i}})d^{r}(\alpha^{(l)}_{j_{i-1}j_{i}}).\end{equation}

Here, by definition, $r_k = 1$ and $r_{i+1} = n$.\]
On the other hand,  
\[
\partial_{i-k+1}(g_{j_{k-1}}) = \partial_{i-k+1}\left( \sum_{1 \leq r'_k < \ldots < r'_i \leq n} b_{r'_k} \ldots \sum_{j_{i-1}=1} b_{r'_i} d_x^{r'_i}(\alpha^{(i)}_{j_{i-1}j_i}) dx_{r'_i} \ldots dx_{r'_i} \right)
\]

\[
= \sum_{1 \leq r'_k < \ldots < r'_i \leq n} \sum_{j_{i-1}=1} b_{r'_k} \ldots \sum_{j_{i-1}=1} b_{r'_i} (-1)^{s+1} d_x^{r'_i}(\alpha^{(i)}_{j_{i-1}j_i}) \ldots\]

\[
\ldots d_x^{r'_i}(\alpha^{(i)}_{j_{i-1}j_i}) dx_{r'_i} \ldots dx_{r'_i} dx_{r'_{i-s+2}} \ldots dx_{r'_i}
\]

Comparing this with (5) the desired equality (3) follows. \(\Box\)

As a consequence of Proposition 1.2 we obtain

**Theorem 1.3.** For \(i = 1, \ldots, n\) and \(j = 1, \ldots, b_i\) let  
\[
z_{ij} = \sum_{1 \leq r_1 < \ldots < r_i \leq n} \sum_{j_{i-1}=1}^b \ldots \sum_{j_{i-1}=1}^b d_x^{r_1}(\alpha^{(1)}_{j_0j_1}) \ldots d_x^{r_i}(\alpha^{(i)}_{j_{i-1}j_i}) dx_{r_1} \ldots dx_{r_i},
\]

and denote its image in \(\Omega \otimes S/I\) by \(\bar{z}_{ij}\). Then for all \(i\) and \(j\), the element \(\bar{z}_{ij}\) are cycles of \(\Omega \otimes S/I\), and the homology classes \([\bar{z}_{ij}]\) with \(j = 1, \ldots, b_i\) form a \(K\)-basis of \(H_i(\Omega \otimes S/I)\).

**Proof.** The elements \(1 \otimes e_{ij}\) with \(1 \leq j \leq b_i\) form a \(K\)-basis of \(K \otimes F_i\). Proposition 1.2 together with (2) implies that \(\psi_i(1 \otimes e_{ij}) = \pm [\bar{z}_{ij}]\), and this yields the assertion. \(\Box\)

2. A Golod criterion

Let \(S\) be as before and \(R = S/I\) with \(I \subset n\). As before we assume that \(I\) is a graded ideal, if \(S\) is the polynomial ring.

We will use the following Golod criterion given in [1] by the second author.

**Theorem 2.1.** Let \(I \subset J \subset S\) be ideals with \(J^2 \subset I\) and such that the natural maps  
\[
Tor_i^S(K, S/I) \rightarrow Tor_i^S(K, S/J)
\]

are zero for all \(i \geq 1\). Then \(I\) is Golod.

Let \(I = (f_1, \ldots, f_m)\). We define \(d(I)\) to be the ideal generated by the elements \(d^i(f_j)\) with \(i = 1, \ldots, n\) and \(j = 1, \ldots, m\). Note that \(I \subset d(I)\) and that \(d(I)\) does not depend on the particular choice of the generators, but of course on the order of the variables. If \(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\) is a relabeling the variables given by the permutation \(\sigma\), then for \(i = 1, \ldots, n\) we set  
\[
d^i_\sigma(f) = \sigma(d^i(f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}))),
\]

and let \(d_\sigma(I)\) be ideal generated by the elements \(d^i_\sigma(f_j)\).

The important conclusion that arises from our description of the Koszul cycles is the following
Corollary 2.2. Let $I$ be a proper ideal of $S$ and $\sigma$ be any permutation of the integers $1, \ldots, n$. Then the natural map

$$\text{Tor}^S_i(K, S/I) \to \text{Tor}^S_i(K, S/d_\sigma(I))$$

induced by the surjection $S/I \to S/d_\sigma(I)$ is zero for all $i \geq 1$.

Proof. We first notice that for any $S$-module $M$ the Koszul homology $H_i(\Omega; M)$ is functorially isomorphic to $\text{Tor}^S_i(K, M)$. Thus it suffice to show that the map $H_i(\Omega; S/I) \to H_i(\Omega; S/d_\sigma(I))$ is the zero map for all $i \geq 1$. It is enough to do this for the basis $[z_{ij}]$ elements of $H_i(\Omega; S/I)$ as given in Theorem 1.3. These cycles have coefficients in $d_\sigma(I)$ and hence their image, already in $\Omega \otimes S/d_\sigma(I)$, is zero.  

Now combining Theorem 2.1 with Corollary 2.2 we obtain

Theorem 2.3. Let $I \subset S$ be a proper ideal such that $d_\sigma(I)^2 \subset I$ for some permutation $\sigma$. Then $I$ is a Golod ideal.

We say that $I$ is $d_\sigma$-Golod, if $I$ satisfies the condition of the theorem, and simply say that $I$ $d$-Golod, if $I$ is $d_\sigma$-Golod for $\sigma = \text{id}$. Finally we say that $I$ is strongly $d$-Golod, if $I$ is $d_\sigma$-Golod for all permutations $\sigma$ of the set $[n] = \{1, 2, \ldots, n\}$.

For monomial ideals $d(I)$ can be easily computed. If $u$ is a monomial and $r$ is the smallest integer such that $x_r$ divides $u$, then $d^r(u) = u/x_r$ and $d^i(u) = 0$ for all $i \neq r$. Thus, for example, if $I = (x_1x_2, x_2^2)$, then $d(I) = (x_2)$, and $d_\sigma(I) = (x_1, x_2)$ for $\sigma$ the permutation $\sigma$ with $\sigma(1) = 2$ and $\sigma(2) = 1$.

Obviously, one has the following implications:

$$I \text{ is strongly } d\text{-Golod} \Rightarrow I \text{ is } d_\sigma\text{-Golod} \Rightarrow I \text{ is Golod}.$$

In the above example, $I$ is $d$-Golod, but not $d_\sigma$-Golod. In particular, $d$-Golod does not imply strongly $d$-Golod. Also, a Golod ideal is in general not $d_\sigma$-Golod for any $\sigma$. For example, the ideal $I = (x_1x_2)$ is Golod, but not $d_\sigma$-Golod.

On the other hand, if $\text{char}(K) = 0$ and $I$ is a monomial ideal, then $I$ is strongly Golod in the sense of [8] if and only if it is strongly $d$-Golod. This follows from the remarks at the beginning of Section 3 in [8], where it is observed that a monomial ideal $I$ is strongly Golod if and only if for all monomial generators $u, v \in I$ and all integers $i$ and $j$ with $x_i | u$ and $x_j | v$ it follows that $uv/x_i x_j \in I$. Indeed, it is obvious that strongly Golod implies strongly $d$-Golod. Conversely, let $u, v \in I$ be monomials with $x_i | u$ and $x_j | v$. If $x_j | u$ or $x_i | v$, then clearly $uv/x_i x_j \in I$. Suppose now that $x_j \nmid u$ and $x_i \nmid v$. Then $i \neq j$, and we may assume that $i < j$. Choose any permutation $\sigma$ of $[n]$ such that $\sigma(1) = i$ and $\sigma(2) = j$. Then $d_\sigma(u) = u/x_i$ and $d_\sigma(v) = v/x_j$, and since $I$ is $d_\sigma$-Golod we get that $uv/x_i x_j \in I$.

3. Applications

Let $I$ and $J$ be ideals of $S$. The ideal $\bigcup_{j \geq 1} I : J^j$ is called the saturation of $I$ with respect to $J$. For $J = \mathfrak{n}$, this saturation is denoted by $I$. The $k$th symbolic power of $I$, denoted by $I^{(k)}$, is the saturation of $I^k$ with respect to the ideal which is the intersection of all associated, non-minimal prime ideals of $I^k$.

As a first application we prove a result analogue to [8, Theorem 2.3], whose proof follows very much the line of arguments given there. The new and important fact is that no assumptions on the characteristic of the base field are made.
Theorem 3.1. Let $I, J \subset S$ be ideals. Assume that $\sigma$ is a permutation of the integers $[n]$. Then the following holds:

(a) If $I$ and $J$ are $d_{\sigma}$-Golod, then $I \cap J$ and $IJ$ are $d_{\sigma}$-Golod.
(b) If $I$ and $J$ are $d_{\sigma}$-Golod and $d_{\sigma}(I)d_{\sigma}(J) \subset I + J$, then $I + J$ is $d_{\sigma}$-Golod.
(c) If $I$ is a strongly $d$-Golod monomial ideal and $J$ is an arbitrary monomial ideal such that $I : J = I : J^2$, then $I : J$ is strongly $d$-Golod.
(d) If $I$ is a monomial ideal, then $I^k$, $I^{(k)}$ and $\bar{I}$ are strongly $d$-Golod for all $k \geq 2$.
(e) If $I \subset J$ are monomial ideals and $I$ is $d_{\sigma}$-Golod, then $IJ$ is $d_{\sigma}$-Golod.

Proof. For the proofs of (a) and (b) we may assume that $\sigma = \text{id}$. The proof for a general permutation $\sigma$ is the same.

(a) By assumption, $d(I)^2 \subset I$ and $d(J)^2 \subset J$. Hence, since $d(I \cap J) \subset d(I) \cap d(J)$, it follows that $d(I \cap J)^2 \subset d(I)^2 \cap d(J)^2 \subset I \cap J$, and this shows that $I \cap J$ is $d$-Golod.

Now let $f \in I$ and $g \in J$. Then Lemma 1.1(ii) implies that $d^*(fg) \in d(I)d(J)$. Therefore, $(d(I)d(J)) \subset (d(I)d(J))d(I) \cap d(I)d(J) \subset I \cap J$, and hence, $(d(I)d(J))^2 \subset d(I)^2d(J)^2 \subset I \cap J$, as desired.

(b) By assumption, $d(I)^2 \subset I$, $d(J)^2 \subset J$ and $d(I)d(J) \subset I + J$, it follows that $d(I + J)^2 = (d(I) + d(J))^2 = d(I)^2 + d(I)d(J) + d(J)^2 \subset I + J$, which shows that $I + J$ is $d$-Golod.

(c) Let $w_1, w_2 \in I : J$ be two monomials and $i$ and $j$ be integers with $x_i | w_1$ and $x_j | w_2$. We must show that $w_1w_2/x_ix_j \in I : J$.

Assume that $u, v \in J$ are arbitrary. Therefore $w_1u \in I$ and $w_2v \in I$. Since $I$ is strongly $d$-Golod, it follows that $(w_1w_2/x_ix_j)uv = (w_1u/x_i)(w_2v/x_j) \in I$. Hence, $w_1w_2/x_ix_j \in I : J^2 = I : J$.

(d) We first show that $I^k$ is strongly $d$-Golod for all $k \geq 2$. For this we have to show that $I^k$ is $d_{\sigma}$-Golod for all $k \geq 2$ and all permutations $\sigma$ of $[n]$. We prove this for $\sigma = \text{id}$. The proof for general $\sigma$ is the same. So now let $w = w_1 \cdots w_k$ be a monomial generator of $I^k$ with $w_j \in I$ for $j = 1, \ldots, k$, and let $r$ be the smallest integer which divides $w$. We may assume that $x_r$ divides $w_1$. Then $d^*(w) = 0$ for $i \neq r$ and $d^*(w) \in I^{k-1}$. It follows that $d(I^k) \subset I^{k-1}$, and hence $d(I^k)^2 \subset I^{2(k-1)} \subset I^k$.

The remaining statements of (d) now result from the following more general fact: Let $I$ be a strongly $d$-Golod ideal and $J$ an arbitrary monomial ideal. Then the saturation of $I$ with respect to $J$ is strongly $d$-Golod.

For the proof of this we observe that, due to the fact that $S$ is Noetherian, there exists an integer $t_0$ such that $\bigcup_{I \geq 0} I : J^t = I : J^{t_0}$ for $s \geq t_0$. Thus if $L = J^{t_0}$. Then $\bigcup_{I \geq 0} I : J^t = I : L = I : L^2$, and the claim follows from (c).

(e) Let $u_1, u_2 \in I$ and $v_1, v_2 \in J$ be monomials. Assume that $i$ is the smallest integer such that $x_{\sigma(i)} | u_1v_1$ and $j$ is the smallest integer such that $x_{\sigma(j)} | u_2v_2$. We need to show that $u_1u_2/v_{\sigma(i)}x_{\sigma(j)} \in IJ$.

If $x_{\sigma(i)} | u_1$ and $x_{\sigma(j)} | u_2$, then $(u_1u_2/x_{\sigma(i)}x_{\sigma(j)})v_1v_2 \in IJ$ since $I$ is $d_{\sigma}$-Golod. If $x_{\sigma(i)} | v_1$ and $x_{\sigma(j)} | v_2$, then $u_1u_2(v_1v_2/x_{\sigma(i)}x_{\sigma(j)}) \in IJ$ since $I \subset J$. If $x_{\sigma(i)} | u_1$ and $x_{\sigma(j)} | v_2$, then $u_1/x_{\sigma(i)}(u_1u_2v_2/x_{\sigma(j)}) \in IJ$ since $I$ is $d_{\sigma}$-Golod. The case $x_{\sigma(i)} | v_1$ and $x_{\sigma(j)} | u_2$ is similar.

Let $I$ be a monomial ideal. We denote by $\bar{I}$ the integral closure of $I$. The following result and its proof are completely analogous to that of [8, Proposition 3.1], where a similar result is shown for strongly Golod monomial ideals.

Proposition 3.2. Let $I$ be a monomial ideal which is $d_{\sigma}$-Golod. Then $\bar{I}$ is $d_{\sigma}$-Golod. In particular, $I^k$ is strongly $d$-Golod for all $k \geq 2$. 

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A monomial ideal $I$ of polynomial ring $S = K[x_1, \ldots, x_n]$ is called stable if for all monomial $u \in I$ one has $x_iu/x_{m(u)}$ for all $i \leq m(u)$, where $m(u)$ is the largest integer $j$ such that $x_j$ divides $u$.

We use Theorem 3.1 to reprove and generalize a result of Aramova and [3] who showed that stable monomial ideals are Golod.

**Corollary 3.3.** Let $I$ be a stable monomial ideal. Then $IJ$ is a Golod ideal for any $J$ with $I \subset J$. In particular, $I$ is Golod.

**Proof.** Let $\sigma$ be the permutation reversing the order of the variables. Then $d_\sigma(u) = u/x_{m(u)}$. Hence for all monomials $u, v \in I$ one has $d_\sigma(u)d_\sigma(v) \in I$, since $I$ is stable. It follows that $I$ is $d_\sigma$-Golod. Therefore the desired result follows from Theorem 3.1(e). \qed

In the following proposition we present new family of $d$-Golod ideals which are not necessarily monomial ideals.

**Proposition 3.4.** Let $J_i \subset K[[x_1, \ldots, x_n]]$ be an ideal for $i = 1, \ldots, n$, and $J \subset S = K[x_1, \ldots, x_n]$ be the ideal generated by $\sum_{i=1}^n x_i J_i$. Then $J^k$ is $d$-Golod for all $k \geq 2$.

**Proof.** Note that for any $k \geq 2$ we have

$$J^k = \sum_{a_1 + \ldots + a_n = k} x_1^{a_1} \cdots x_n^{a_n} J_1^{a_1} \cdots J_n^{a_n}$$

where the $a_i$ are non-negative integers. By convention, $x_i^{a_i} J_i^{a_i} = S$ if $a_i = 0$. Thus we see that $J^k$ is generated by elements of the form

$$x_1^{a_1} \cdots x_n^{a_n} f_1 \cdots f_n$$

with integers $a_i \geq 0$ such that $a_1 + \ldots + a_n = k$ and with $f_i \in J_i^{a_i}$ for $i = 1, \ldots, n$.

One has

$$d_i(f) = \begin{cases} 0, & \text{if there exists } j < i \text{ with } a_j > 0 \text{ or } a_i = 0, \\ x_i^{a_i-1} \cdots x_n^{a_n} f_j \cdots f_n, & \text{otherwise.} \end{cases}$$

Hence, we see that $d_i(f)d_j(g) \in J^k$ for all $f, g \in J^k$ and all $i, j$ with $1 \leq i, j \leq n$, and this shows that $J^k$ is $d$-Golod for all $k \geq 2$. \qed

The last application we have in mind is of more general nature and deals with stretched local rings. Let $(R, m, K)$ be a Noetherian local ring with maximal ideal $m$ and residue field $K$ or a standard graded $K$-algebra with graded maximal ideal $m$. The ring $R$ is said to be stretched if $m^2$ is a principal ideal.

We set $n = \dim_K m/m^2$ and $\tau = \dim_K \text{Soc}(R)$, where $\text{Soc}(R) = (0 :_R m)$ is the socle of $R$. Moreover, if $R$ is Artinian, we let $s$ be the largest integer such that $m^s \neq 0$. Note the $s + 1$ is the Loewy length of $R$.

Stretched local rings have been introduced by Sally [10]. She showed that the Poincaré series of $K$ is a rational function. Indeed, she showed that

$$P^R_K(t) = \begin{cases} 1/(1 - nt), & \text{if } \tau = n \\ 1/(1 - nt + t^2), & \text{if } \tau \neq n \end{cases}$$

Very recently in [4] it was shown that all finitely generated modules over a stretched Artinian local ring $R$ have a rational Poincaré series with a common denominator by studying the algebra structure of the Koszul homology of $R$. Among other results they
proved in [4, Theorem 5.4] that $R$ is Golod, if $\tau = h$. By using our methods we give an alternative proof of the result and show

**Theorem 3.5.** Let $(R, m, K)$ be a stretched local ring or a stretched standard graded $K$-algebra. Then $R$ is Golod if one of the following conditions is satisfied:

(i) $R$ is standard graded, (ii) $R$ is not Artinian, or (iii) $R$ is Artinian and $\tau = n$.

The following lemma will be needed for the proof of the theorem.

**Lemma 3.6.** Let $R$ be a stretched standard graded $K$-algebra with $n \geq 2$ and $s \geq 3$ in Artinian case. Then the following holds:

(i) $\tau = n$, if $R$ is Artinian.

(ii) $\tau = n - 1$, if $R$ is not Artinian.

**Proof.** Since $R$ is standard graded, $R_i R_i = R_{i+1} \neq 0$ for $2 \leq i < s$ if $R$ is Artinian, and $R_i R_i = R_{i+1} \neq 0$ for all $i \geq 2$ if $R$ is not Artinian. Since dim$_K R_i = 1$ for all $i \geq 2$ with $R_i \neq 0$, it follows that $R_s \subsetneq \text{Soc}(R) \subsetneq R_1 \oplus R_s$ if $R$ is Artinian, and $\text{Soc}(R) \subsetneq R_1$ if $R$ is not Artinian. Thus in order to prove (i) and (ii) we must show that dim Soc($R$) $\cap$ $R_1 = h - 1$.

After an extension of the base field we may assume that $K$ is algebraically closed. Indeed, a base field extension does not change the Hilbert function, so it does not change the socle dimension of the $R$.

We proceed by induction on $n$. We first assume that $n = 2$. Since dim$_K R_1 >$ dim$_K R_2$ and since $K$ is algebraically closed, [9, Lemma 2.8] implies that there exists a non-zero linear form $x_1$ such that $x_1^2 = 0$. Assume that $x_1 R_1 \neq 0$. Then $R_2 = x_1 R_1$, and $R_3 = R_1 R_2 = x_1 R_1^2 = x_1^2 R_1 = 0$, contradicting the assumption that $s \geq 3$. Thus $x_1 R_1 = 0$.

Then it follows that $(R_1)^2 = R_1^2$. Therefore the standard graded $K$-algebra $R' = K[R_1]$ is a stretched $K$-algebra of embedding dimension $n - 1$. By induction hypothesis, dim$_K \text{Soc}(R') \cap R_1' = n - 2$. Let $x_2, \ldots, x_{n-1}$ span the $K$-vector space $\text{Soc}(R') \cap R_1'$.

These elements $x_i$ are also socle elements of $R$ and together with $x_1$, they span a vector space of dimension $n - 1$, as desired.

**Remark 3.7.** Suppose that $R = S/I$ is a stretched standard graded $K$-algebra, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring with $K$ is an algebraically closed field, and where $I \subset n^2$. The proof of Lemma 3.6 shows that after a suitable linear change of coordinates one has that

(i) $I = (x_1, \ldots, x_{n-1})^2 + x_n (x_1, \ldots, x_{n-1}) + (x_n^{s+1})$, if $R$ is Artinian;

(ii) $I = (x_1, \ldots, x_{n-1})^2 + x_n (x_1, \ldots, x_{n-1})$, if $R$ is not Artinian.

**Proof of Theorem 3.5.** Let us first assume that $R$ is a standard graded $K$-algebra. After a suitable base field extension, which does not affect the Golod property, we may assume that $I$ is generated as described in Remark 3.7. We order the variables as follows: $x_n, x_1, \ldots, x_{n-1}$, and let $\sigma$ be the corresponding permutation of $[n]$. Then $d_\sigma(I) = (x_1, \ldots, x_{n-1}, x_n^i)$ in the Artinian case, and $d_\sigma(I) = (x_1, \ldots, x_{n-1})$ in the non-Artinian case. Clearly $d_\sigma(I)^2 \subset I$ in both cases. Thus in any case $R$ is a Golod ring. This proves case (i).

In order to prove Golodness of $R$ in the case (ii), we consider the associated graded ring $G(R)$ of $R$, which, as can be seen from its Hilbert function, is a standard graded stretched $K$-algebra. We claim that $G(R)$ is a Koszul algebra. Koszulness of a standard graded $K$-algebra is characterized by the property that $P_R^R(t) = 1/H_R(-t)$, where $H_R(t)$ denotes
the Hilbert series for $R$. Therefore, it is enough to prove the claim for a suitable base extension, because a base extension does not change the Poincaré series of a $K$-algebra, nor does it change its Hilbert series, as already noticed before. Hence after this base field extension we may assume that $I$ is generated by quadratic monomials as described in Remark 3.7. By a result of Fröberg [5], this implies that $G(R)$ is Koszul. Now by another result of Fröberg [6] it follows that $P^R_K(t) = P^{G(R)}_K(t)$. By case (i), $G(R)$ is Golod, and hence

$$P^R_K(t) = P^{G(R)}_K(t) = \frac{(1 + t)^n}{1 - t(P^S_{G(R)}(t) - 1)} \geq \frac{P^S_R(t)}{1 - t(P^S_R(t) - 1)}.$$  

The coefficientwise inequality in these formulas follows from the well-known fact that there is the coefficientwise inequality $P^S_R(t) \leq P^S_{G(R)}(t)$. Since the opposite inequality $P^S_K(t)/(1 - t(P^S_R(t) - 1) \geq P^R_K(t)$ always holds, we have equality and $R$ is Golod.

Finally suppose that (iii) is satisfied. Then $G(R)$ is a stretched Artinian $K$-algebra, and hence by Lemma 3.6 we have $\tau = n$ for $G(R)$. By our assumption, $\tau = n$ also for $R$. Thus (6) implies that $P^R_k(t) = P^{G(R)}_k(t)$. As in (ii) it follows from this equation that $R$ is Golod, since $G(R)$ is Golod.  

□

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