Bursts of Radiation and Recoil Effects in Electromagnetism and Gravitation

C. Barrabès*
Laboratoire de Mathématiques et Physique Théorique
CNRS/UPRES-A 6083, Université F. Rabelais, 37200 TOURS, France

P.A. Hogan†
Mathematical Physics Department
National University of Ireland Dublin, Belfield, Dublin 4, Ireland

Abstract

The Maxwell field of a charge $e$ which experiences an impulsive acceleration or deceleration is constructed explicitly by subdividing Minkowskian space–time into two halves bounded by a future null–cone and then glueing the halves back together with appropriate matching conditions. The resulting retarded radiation can be viewed as instantaneous electromagnetic bremsstrahlung. If we similarly consider a spherically symmetric, moving gravitating mass, to experience an impulsive deceleration, as viewed by a distant observer, then this is accompanied by the emission of a light–like shell whose total energy measured by this observer is the same as the kinetic energy of the source before it stops. This phenomenon is a recoil effect which may be thought of as a limiting case of a Kinnersley rocket.

PACS numbers: 04.30.Nk, 04.20.Cv, 98.20.Hw.

*E-mail : barrabes@celfi.phys.univ-tours.fr
†E-mail : phogan@ollamh.ucd.ie
1 Introduction

Perhaps the simplest example in electrodynamics of an impulsive electromagnetic wave is produced when a charge $e$ receives an impulsive acceleration or deceleration. For example the charge $e$ might be moving rectilinearly with constant 3–velocity $v$ relative to the laboratory frame and is suddenly halted. This means that in Minkowskian space–time there is a sudden change in the direction of the 4–velocity of the charge. As a result a spherical impulsive electromagnetic wave is emitted. Although this physical situation has been known for some time [1] no analytical description appears to have ever been given allowing calculations (such as those given in section 4 here) with the explicit Maxwell field of the charge. The construction of this Maxwell field is one of the objectives of the present paper. It involves subdividing Minkowskian space–time into two halves, each having as boundary the future null–cone history $\mathcal{N}$ of the impulsive electromagnetic wave, and then glueing the halves back together whilst maintaining via matching conditions the invariance of a certain quantity. This construction is carried out in section 2 and a coordinate system is obtained in which the metric tensor of the re–attached space–times is continuous across the future null–cone $\mathcal{N}$. This enables us to see clearly that (i) the glueing ensures that the space–time remains flat on $\mathcal{N}$ and thus $\mathcal{N}$ cannot, for example, be the history of a gravitational impulsive wave or shock wave and (ii) the electromagnetic field we construct satisfies Maxwell’s vacuum field equations everywhere off the world–line of the charge $e$ and, in particular on the future null–cone $\mathcal{N}$ minus its vertex. We then define a measure of the intensity of the electromagnetic impulse wave and show that its dependence on the original 3–velocity $v$ of the charge is typical of the velocity dependence of the intensity of electromagnetic bremsstrahlung emitted in beta decay [2]. We thus refer to the impulsive electromagnetic wave as instantaneous electromagnetic bremsstrahlung.

Having described instantaneous bremsstrahlung in electromagnetism one is naturally lead to ask similar questions in the case of the gravitational interaction: What happens when a mass receives an impulsive acceleration or deceleration? In particular will an impulsive gravitational wave accompany the abrupt change in motion of the mass? Our geometrical approach to the electromagnetic case is perfectly adapted to this new situation. In section 3 we glue together two Schwarzschild space–times having as common boundary a future null–cone $\mathcal{N}$, using the same matching conditions as in the electromagnetic case. Also one of the Schwarzschild space–times involves a parameter $v$ which we show allows us to interpret its source as moving rectilinearly with 3–velocity $v$, compared to the source of the second Schwarzschild space–time, as measured by an observer at spatial infinity.
Thus a distant observer sees a uniformly moving source come to a sudden halt. As a result a signal travels outward with history $\mathcal{N}$. We use the theory of light–like signals in general relativity developed by Barrabès–Israel [4] to show that this signal is a light–like shell (burst of neutrinos, for example). This emission of a light–like shell is similar to a rocket exhaust which carries sufficient energy to result in zero recoil velocity. We point out, following (3.20) below, the sense in which our model may be thought of as a limiting case of a Kinnersley rocket. Thus there is no gravitational radiation (i.e. it is not a spherical impulsive gravitational wave). Furthermore we demonstrate that the relative kinetic energy of the source, when travelling with 3–velocity $v$ measured by a distant observer, is converted into the energy of the light–like shell when the source suddenly stops. The physical interpretation of our model and the global structure of spacetime will be discussed.

In section 4 below we illustrate some calculations that can be carried out with the electromagnetic field obtained in section 2. These are based on the observations of Penrose [1] that after a charge $e$ receives an impulsive acceleration the resulting spherical impulsive electromagnetic wave (a) approximates a plane impulsive electromagnetic wave in the far zone and (b) if it then experiences a head–on collision with a plane impulsive gravitational wave, the electromagnetic wave will back–scatter. This sort of phenomenon can be expected because pure unidirectional electromagnetic radiation must have a propagation direction in space–time which is shear–free (and geodesic) [3]. However when the shear–free null geodesics intersect the history of a plane impulsive gravitational wave they must acquire shear [4]. Hence the resulting electromagnetic field cannot be a pure unidirectional radiation field but must have another part to it (the back–scattered radiation field). In section 4 the plane wave limit of the electromagnetic wave of section 2 is obtained and its subsequent head–on collision with a plane impulsive gravitational wave is considered. The electromagnetic field in the region between the waves after collision is derived by solving Maxwell’s vacuum field equations with the appropriate boundary conditions. It is explicitly seen that the impulsive wave looses its plane character and the back–scattered radiation appears. In addition the electromagnetic field in this region after the collision becomes singular where light rays crossing the plane gravitational wave are focussed by that wave.

2 Electromagnetic Bremsstrahlung

Let \( \{X^\mu\} \) with $\mu = 1, 2, 3, 4$ be rectangular Cartesian coordinates and time in Minkowskian space–time, in terms of which the line–element reads (taking
the velocity of light $c = 1$)

$$
\text{Let } X^\mu = x^\mu(u) \text{ be the parametric equations of the time–like world–line in Minkowskian space–time of a charge } e \text{ having } u \text{ as proper–time or arc length along it. The 4–velocity and 4–acceleration of the charge have components } v^\mu(u) \text{ and } a^\mu(u) \text{ respectively (with } \eta_{\mu\nu} v^\mu v^\nu \equiv v_\mu v^\mu = -1 \text{ and consequently with } v_\mu a^\mu = 0). \text{ Let } (X^\mu) \text{ be the coordinates of an event off the world–line of the charge and let } (x^\mu(u)) \text{ be the coordinates of the event on the world–line of the charge where the past null–cone with vertex } (X^\mu) \text{ intersects the world–line. The retarded distance } r \text{ of } (X^\mu) \text{ from the world–line is given by}
$$

$$
r = -v_\mu (X^\mu - x^\mu(u)) . \quad (2.2)
$$

Here $X^\mu - x^\mu(u)$ is null and $r \geq 0$ with equality if and only if $(X^\mu)$ coincides with $(x^\mu(u))$. The Liénard–Wiechert field of the charge evaluated at $(X^\mu)$ is given by the Maxwell tensor

$$
F_{\mu\nu}(X) = \frac{N_{\mu\nu}}{r} + \frac{III_{\mu\nu}}{r^2} , \quad (2.3)
$$

with

$$
N_{\mu\nu} = 2 e q_{[\mu} k_{\nu]} , \quad (2.4)
$$

$$
III_{\mu\nu} = 2 e v_{[\mu} k_{\nu]} . \quad (2.5)
$$

Here square brackets denote skew–symmetrisation, $k^\mu = r^{-1} (X^\mu - x^\mu(u))$ so that $k^\mu$ is null and, by (2.2), $v_\mu k^\mu = -1$. Also

$$
q^\mu = a^\mu + (a^\nu k_\nu) v^\mu , \quad (2.6)
$$

and this is a space–like 4–vector orthogonal to $k^\mu$. The skew–symmetric tensor $N_{\mu\nu}$ in (2.4) is Petrov type N with degenerate principal null direction $k^\mu$ and thus the leading term in the electromagnetic field of the charge (2.3) describes the radiation part of the field. The presence of this term is due entirely to the acceleration $a^\mu$ of the charge. Suppose now that at $u = 0$ (say) on the world–line of the charge, the charge receives an impulsive acceleration (there is sudden change in the 4–velocity of the charge at $u = 0$ leading to a Dirac delta function $\delta(u)$ in the 4–acceleration of the charge) then one would expect the resulting retarded radiation, described by $N_{\mu\nu}$ above, to take the form of an impulsive electromagnetic wave with history the future null–cone $u = 0$ and with profile $\delta(u)$. However this information cannot readily be
extracted from (2.3) and (2.4)–(2.5) because, among other things, \( r \) in (2.2) is now not defined at \( u = 0 \) since there is no unique tangent to the world-line of the charge at \( u = 0 \). To see this clearly let us take the vertex of the future null-cone \( N(u = 0) \) to be the origin of the coordinates \( \{X^\mu\} \) (and so \( x^\mu(0) = 0 \)). Thus if \( P(X^\mu) \) is an event on \( N \) then \( (X^\mu) \) satisfies
\[
\left( X^1 \right)^2 + \left( X^2 \right)^2 + \left( X^3 \right)^2 - \left( X^4 \right)^2 = 0 , \quad X^4 > 0 . \tag{2.7}
\]
This is given in a useful parametric form by
\[
X^1 + iX^2 = \frac{\sqrt{2} \zeta R_0}{1 + \frac{i}{2} \zeta} , \quad X^3 = \left( \frac{1 - \frac{i}{2} \zeta \bar{\zeta}}{1 + \frac{i}{2} \zeta \bar{\zeta}} \right) R_0 , \quad X^4 = R_0 , \tag{2.8}
\]
with \( \zeta \) a complex variable with complex conjugate \( \bar{\zeta} \) and \( R_0 \) a real variable. Putting \( p_0 = 1 + \frac{i}{2} \zeta \bar{\zeta} \) and substituting (2.8) into the Minkowskian line-element (2.1) yields the induced line-element on \( N \)
\[
ds^2 = 2R_0^2 p_0^{-2} d\zeta d\bar{\zeta} . \tag{2.9}
\]

Figure 1: The future null-cone \( N(u = 0) \) with \( QP = r \) and \( Q'P = r_+ \). The region to the future (past) of the null-cone corresponds to \( u > 0 \) (\( u < 0 \)).

In figure 1, \( QP \) is orthogonal to \( v^\mu \) and \( v_\mu v^\mu = -1 \) while \( Q'P \) is orthogonal to \( v'^\mu \) and \( v'_\mu v'^\mu = -1 \). Then the retarded distances \( r = QP \) and \( r_+ = Q'P \) are given by (using (2.2) with \( x^\mu(u) = x^\mu(0) = 0 \))
\[
QP = r = -v^1 X^1 - v^2 X^2 - v^3 X^3 + v^4 X^4 , \tag{2.10}
\]
and
\[ Q'P = r_+ = -v'^1X^1 - v'^2X^2 - v'^3X^3 + v'^4X^4. \] (2.11)

Substituting from (2.8) into these we obtain
\[ r = R_0 p_0^{-1} \quad \text{and} \quad r_+ = R_0 p_0^{-1}, \] (2.12)
with
\[ p = \frac{1}{2} \bar{\zeta} (v^4 + v^3) - \frac{1}{\sqrt{2}} (v^1 - iv^2) \zeta - \frac{1}{\sqrt{2}} (v^1 + iv^2) \bar{\zeta} + v^4 - v^3, \] (2.13)
and \( p_+ \) is the same function as \( p \) but with \( v^\mu \) in \( p \) replaced by \( v'^\mu \). Hence we see from (2.12) that we have the invariant statement
\[ r_+ p_+^{-1} = r p^{-1}, \] (2.14)
and the line–element (2.9) reads
\[ ds^2 = 2r^2 p^{-2} d\zeta d\bar{\zeta}. \] (2.15)

Thus (2.14) is a change of affine parameter along the generator \( \zeta = \text{constant} \) of \( \mathcal{N} \) (OP in figure 1) which (a) leaves the vertex of \( \mathcal{N} \) fixed and (b) leaves the induced metric on \( \mathcal{N} \) invariant.

We now consider the following type of motion of the charge \( e \): In the frame of reference with respect to which the coordinates \( \{X^\mu\} \) are measured (the laboratory frame) the charge moves with uniform 3–velocity \( v \) in the \( X^3 \)–direction when \( u < 0 \) (i.e. to the past of the null–cone \( \mathcal{N} \)). When \( u > 0 \) (to the future of the null–cone \( \mathcal{N} \)) the charge is taken to be at rest in this frame. Thus for \( u < 0 \) the world–line of the charge is the time–like geodesic with unit tangent (4–velocity)
\[ v^\mu = (0, 0, \gamma v, \gamma), \] (2.16)
with \( \gamma = (1 - v^2)^{-\frac{1}{2}} \). For \( u > 0 \) the world–line of the charge is the time–like geodesic with unit tangent
\[ v'^\mu = (0, 0, 0, 1). \] (2.17)

The line–element of Minkowskian space–time in the region \( u < 0 \) to the past of the null–cone \( \mathcal{N}(u = 0) \) and which we denote by \( \mathcal{M}^- \) is, in coordinates \( (\zeta, \bar{\zeta}, r, u) \),
\[ ds_-^2 = 2r^2 p^{-2} d\zeta d\bar{\zeta} - 2 du dr - du^2, \] (2.18)
with
\[ p = \gamma \left\{ \frac{1}{2} (1 + v) \zeta \bar{\zeta} + 1 - v \right\}. \] (2.19)
This latter follows from (2.13) with the specialisation (2.16). The line–element of Minkowskian space–time in the region \( u > 0 \) to the future of the null–cone \( \mathcal{N}(u = 0) \) and which we denote by \( \mathcal{M}^+ \) is, in coordinates \((\zeta_+, \bar{\zeta}_+, r_+, u_+)\),

\[
ds^2_+ = 2r^2_+ p^2_+ d\zeta_+ d\bar{\zeta}_+ - 2 du_+ dr_+ - du^2_+, \tag{2.20}
\]

with

\[
p_+ = \frac{1}{2} \zeta_+ \bar{\zeta}_+ + 1, \tag{2.21}
\]

and \( u_+ = 0 \) corresponding to \( u = 0 \). We then attach the two halves of Minkowskian space–time \( \mathcal{M}^- \) and \( \mathcal{M}^+ \) on \( u = 0(\leftrightarrow u_+ = 0) \) with the matching conditions (cf. (2.14))

\[
\zeta_+ = \zeta, \quad \bar{\zeta}_+ = \bar{\zeta}, \quad r_+ = r p_+ p^{-1}. \tag{2.22}
\]

Thus in particular the line–elements induced on \( \mathcal{N} \) by its embedding in \( \mathcal{M}^- \) and \( \mathcal{M}^+ \), calculated from (2.18) with \( u = 0 \) and from (2.20) with \( u_+ = 0 \) are the same induced line–elements. A more convenient form for (2.18) for our purposes is obtained by introducing the coordinates \( \xi, \phi \) via

\[
\xi = \frac{1 - \frac{1}{2} \zeta \bar{\zeta}}{1 + \frac{1}{2} \zeta \bar{\zeta}}, \quad \phi = \frac{1}{2i} \log \left( \frac{\zeta}{\bar{\zeta}} \right), \tag{2.23}
\]

which results in (2.18) becoming

\[
ds^2_- = k^2 r^2 \left\{ \frac{d\xi^2}{1 - \xi^2} + (1 - \xi^2) d\phi^2 \right\} - 2 du dr - du^2, \tag{2.24}
\]

with \( k^{-1} = \gamma (1 - v \xi) \). Also if \( \xi_+, \phi_+ \) are given by (2.23) with \( \zeta, \bar{\zeta} \) replaced by \( \zeta_+, \bar{\zeta}_+ \) we find that (2.20) becomes

\[
ds^2_+ = r^2_+ \left\{ \frac{d\xi^2_+}{1 - \xi^2_+} + (1 - \xi^2_+) d\phi^2_+ \right\} - 2 du_+ dr_+ - du^2_+. \tag{2.25}
\]

The matching conditions (2.22) now read: on \( u = 0(u_+ = 0) \)

\[
\xi_+ = \xi, \quad \phi_+ = \phi, \quad r_+ = k r. \tag{2.26}
\]

To interpret physically what the geometrical construction above is describing we note the following: The line–elements (2.24) and (2.25) are two versions of the Minkowski line–element which can be transformed one into the other by the coordinate transformation

\[
\xi_+ = \frac{\xi - v}{1 - v \xi}, \quad \phi_+ = \phi, \quad r_+ = r, \quad u_+ = u. \tag{2.27}
\]
Here \( v \) is a real parameter with \( 0 < v < 1 \). The transformation (2.27) is in fact a Lorentz transformation as it can be shown if one rewrites the line elements (2.24) and (2.25) in terms of the rectangular Cartesian coordinates and time \{x, y, z, t\} and \( \{x_+, y_+, z_+, t_+\} \). The relations between these two sets of coordinates and the coordinates \( \{\xi, \phi, r, u\} \) and \( \{\xi_+, \phi_+, r_+, u_+\} \) are given by

\[
x = rk \sqrt{1 - \xi^2} \cos \phi , \\
y = rk \sqrt{1 - \xi^2} \sin \phi , \\
z = \gamma v u + rk \xi , \\
t = \gamma u + rk ,
\] (2.28)

and

\[
x_+ = r_+ \sqrt{1 - \xi_+^2} \cos \phi_+ , \\
y_+ = r_+ \sqrt{1 - \xi_+^2} \sin \phi_+ , \\
z_+ = r_+ \xi_+ , \\
t_+ = u_+ + r_+ .
\] (2.30)

In terms of the coordinates \( \{x, y, z, t\} \) and \( \{x_+, y_+, z_+, t_+\} \) the line elements (2.24) and (2.25) take the usual form (2.1). Furthermore the relationship between \( \{x, y, z, t\} \) and \( \{x_+, y_+, z_+, t_+\} \) corresponding to (2.27) is obtained directly from substituting (2.27) into (2.32)–(2.35) and using (2.28)–(2.31). The result is the Lorentz transformation

\[
x_+ = x , \quad y_+ = y , \quad z_+ = \gamma (z - vt) , \quad t_+ = \gamma (t - v z) ,
\] (2.36)

where, as always, \( \gamma = (1 - v^2)^{-\frac{1}{2}} \).

We now wish to calculate the Maxwell field due to the charge \( e \) performing the motion described above. To this end we first express the line–element of \( \mathcal{M}^- \cup \mathcal{M}^+ \) in continuous coordinates \( \{\Xi, \Phi, R, U\} \). By this we mean coordinates in which the metric tensor components are continuous across \( \mathcal{N} \). In such a coordinate system we find that the line–element can be written

\[
ds^2 = K^2 R^2 \left\{ \frac{d\Xi^2}{1 - \Xi^2} + (1 - \Xi^2) \ d\Phi^2 \right\} - 2 dU \ dR - dU^2 ,
\] (2.37)

with \( K^{-1} = \gamma (1 - v \Xi) \). The coordinate ranges are \(-1 \leq \Xi \leq +1, 0 \leq \Phi < 2\pi, 0 \leq R < +\infty, -\infty < U < +\infty \). Here \( \mathcal{N} \) is given by \( U = 0, \mathcal{M}^- \) by \( U < 0 \) and \( \mathcal{M}^+ \) by \( U > 0 \). It is interesting to note that in these coordinates the
metric tensor components are independent of $U$. Also when $U > 0$, (2.27) is transformed to (2.37) with

$$r_+ = k R \varphi, \quad (2.38)$$
$$u_+ = \gamma U + k (1 - \varphi) R, \quad (2.39)$$
$$\xi_+ = \varphi^{-1} (\chi + \Xi), \quad (2.40)$$
$$\phi_+ = \Phi, \quad (2.41)$$

where

$$\varphi = \left( 1 + 2 \chi \Xi + \chi^2 \right)^{\frac{1}{2}}, \quad \chi = \frac{\gamma v U}{KR}. \quad (2.42)$$

Clearly when $U < 0$, (2.24) is transformed into (2.37) by the identity transformation

$$r = R, \quad u = U, \quad \xi = \Xi, \quad \phi = \Phi. \quad (2.43)$$

We see from (2.38)–(2.42) and (2.43) that on $\mathcal{N}(U = 0 \iff u = 0 \iff u_+ = 0)$

$$r_+ = KR = kr, \quad \xi_+ = \xi, \quad \phi_+ = \phi, \quad (2.44)$$

which agrees with the matching conditions (2.26). In addition it follows from (2.37) that the space–time $\mathcal{M}^- \cup \mathcal{M}^+$ is flat everywhere. In particular the Riemann tensor vanishes on $\mathcal{N}$ for the matching conditions (2.44) and thus $\mathcal{N}$ is not the history of a light–like signal such as an impulsive gravitational wave, a gravitational shock wave or a light–like shell of matter. Hence our geometrical construction has not introduced extra gravitational effects and describes a pure electromagnetic phenomenon. This had to be checked as it is well known that glueing two flat spacetimes on a null hypersurface can lead to a Riemann tensor which does not vanish on this null hypersurface, see for instance [1].

The electromagnetic field due to the charge $e$ in $\mathcal{M}^-$ and in $\mathcal{M}^+$ is the Coulomb field. Thus the electromagnetic 4–potential on $\mathcal{M}^- \cup \mathcal{M}^+$ is given via the 1–form field (in the continuous coordinates $\{\Xi, \Phi, R, U\}$)

$$A = \frac{e}{r_+} (du_+ + dr_+) \vartheta(U) + \frac{e}{R} (dU + dR) (1 - \vartheta(U)), \quad (2.45)$$

with $r_+, u_+$ given in terms of $\Xi, R, U$ by (2.38), (2.29) and where $\vartheta(U)$ is the Heaviside step function (equal to zero when $U < 0$ and equal to unity when $U > 0$). Thus

$$A = A_{\text{Coul}}^+ \vartheta(U) + A_{\text{Coul}} (1 - \vartheta(U)), \quad (2.46)$$

where $A_{\text{Coul}}^+$ is the Coulomb potential 1–form in $\mathcal{M}^+$ due to a charge $e$ with geodesic world–line $r_+ = 0$ and $A_{\text{Coul}}$ is the Coulomb potential 1–form in
$\mathcal{M}^-$ due to a charge $e$ with geodesic world–line $r = 0$. We shall denote the corresponding Coulomb field 2–forms (the exterior derivatives of $A^+_{\text{Coul}}$ and $A_{\text{Coul}}$) by $f^+_{\text{Coul}}$ and $f_{\text{Coul}}$ respectively. Noting from (2.38), (2.29) that

$$\frac{1}{r^+} (du_+ + dr_+) = \varphi^{-1} \left( K \gamma v d\Xi + \frac{\gamma}{K R} dU + \frac{1}{R} dR \right) ,$$

we find that the candidate for Maxwell 2–form on $\mathcal{M}^– \cup \mathcal{M}^+$ is

$$F = f^+_{\text{Coul}} \vartheta (U) + f_{\text{Coul}} (1 - \vartheta (U)) + e K \gamma v \delta (U) dU \wedge d\Xi ,$$

where $\delta (U)$ is the Dirac delta function singular on $\mathcal{N} (U = 0)$. The expressions for $f^+_{\text{Coul}}$ and $f_{\text{Coul}}$ are given below in (??) and (??). We remark that in terms of the basis 1–forms on $\mathcal{M}^– \cup \mathcal{M}^+$,

$$\vartheta^1 = \frac{K R d\Xi}{\sqrt{1 - \Xi^2}} , \quad \vartheta^2 = K R \sqrt{1 - \Xi^2} d\Phi ,$$

$$\vartheta^3 = dU , \quad \vartheta^4 = dR + \frac{1}{2} dU ,$$

we can write (2.48) as

$$F = f^+_{\text{Coul}} \vartheta (U) + f_{\text{Coul}} (1 - \vartheta (U)) - e \gamma v \sqrt{\frac{1 - \Xi^2}{R}} \delta (U) \vartheta^1 \wedge \vartheta^3 .$$

We note that the $\delta$–function part is Petrov Type N with degenerate principal null direction given via the 1–form $\vartheta^3$. Thus the principal null direction is that of the vector field $\partial / \partial R$.

To prove that (2.51) is a vacuum Maxwell field on $\mathcal{M}^– \cup \mathcal{M}^+$ excluding $R = 0$, but including the future null–cone $U = 0$, we first return to (2.48) and calculate its Hodge dual

$$*F = *f^+_{\text{Coul}} \vartheta (U) + *f_{\text{Coul}} (1 - \vartheta (U)) + e K \gamma v \delta (U) * (dU \wedge d\Xi) .$$

Now since $d^* f^+_{\text{Coul}} = 0 = d^* f_{\text{Coul}}$, with $d$ standing for exterior differentiation, we have

$$d^* F = \delta (U) dU \wedge ( * f^+_{\text{Coul}} - * f_{\text{Coul}} ) + e \gamma v d [K \delta (U) * (dU \wedge d\Xi)] .$$

Using

$$* (dU \wedge d\Xi) = - (1 - \Xi^2) dU \wedge d\Phi ,$$

$$* (dU \wedge dR) = - K^2 R^2 d\Xi \wedge d\Phi ,$$

$$* (dR \wedge d\Xi) = \left( 1 - \Xi^2 \right) dR \wedge d\Phi ,$$

we find that the candidate for Maxwell 2–form on $\mathcal{M}^– \cup \mathcal{M}^+$ is

$$F = f^+_{\text{Coul}} \vartheta (U) + f_{\text{Coul}} (1 - \vartheta (U)) - e \gamma v \delta (U) dU \wedge d\Xi .$$
and the explicit expressions

\[
f_{\text{Coul}}^+ = e \frac{\gamma}{KR^2 \varphi^3} \left\{ 1 - v \Xi + \chi (\Xi - v) \right\} \, dU \wedge dR + e \frac{\chi}{R \varphi^3} dR \wedge d\Xi \\
+ \frac{e}{R \varphi^3} \left\{ \chi + \gamma^2 v (1 - v \Xi) \right\} \, dU \wedge d\Xi ,
\]

(2.57)

and

\[
f_{\text{Coul}} = e \frac{R}{R^2} dU \wedge dR ,
\]

(2.58)

we find that

\[
\delta (U) \, dU \wedge \left( ^* f_{\text{Coul}}^+ - ^* f_{\text{Coul}} \right) = e (K^2 - 1) \delta (U) \, dU \wedge d\Xi \wedge d\Phi ,
\]

(2.59)

and

\[
d \left[ K \delta (U) \, ^* (dU \wedge d\Xi) \right] = \delta (U) \, \frac{d}{d\Xi} \left\{ (1 - \Xi^2) \, K \right\} \, dU \wedge d\Xi \wedge d\Phi .
\]

(2.60)

Hence (2.53) reads

\[
d^* F = e \delta (U) \left\{ K^2 - 1 + \gamma v \frac{d}{d\Xi} \left\{ (1 - \Xi^2) \, K \right\} \right\} \, dU \wedge d\Xi \wedge d\Phi .
\]

(2.61)

The right side of this equation vanishes since $K^{-1} = \gamma (1 - v \Xi)$. Therefore (2.51) is a vacuum Maxwell field for all $U$ and for $R > 0$. The final (type $N$) term in (2.51) thus represents a spherical impulsive electromagnetic wave having the future null–cone $\mathcal{N} (U = 0)$ as history in Minkowskian space–time.

The tetrad defined via the 1–forms (2.39), (2.40) on $\mathcal{M}^- \cup \mathcal{M}^+$ is a half–null tetrad. The tetrad defined via the 1–forms \{\vartheta^1, \vartheta^2, \omega^3, \omega^4\} with

\[
\omega^3 = dR , \quad \omega^4 = dU + dR ,
\]

(2.62)

is an orthonormal tetrad. The final term in the electromagnetic field (2.51) when written on this orthonormal tetrad reads

\[
\hat{F} = -e \frac{\gamma v \sqrt{1 - \Xi^2}}{R} \delta (U) \left( \vartheta^1 \wedge \omega^4 - \vartheta^1 \wedge \omega^3 \right) .
\]

(2.63)

Thus in the laboratory frame $\hat{F}$ corresponds to an electric 3–vector and a magnetic 3–vector given respectively by

\[
\mathbf{E} = \mathbf{E}_0 \delta (U) , \quad \mathbf{H} = \mathbf{H}_0 \delta (U) ,
\]

(2.64)

with

\[
\mathbf{E}_0 = (\mathcal{E}, 0, 0) , \quad \mathbf{H}_0 = (0, \mathcal{E}, 0) ,
\]

(2.65)
and
\[ E = -\frac{e \gamma v}{R} \sqrt{1 - \Xi^2} \]  
(2.66)

A measure of the intensity of this electromagnetic wave is given by
\[ I = \frac{1}{8\pi} \left( |E_0|^2 + |H_0|^2 \right) = \frac{e^2 \gamma^2 (1 - \Xi^2)}{4\pi R^2} \]  
(2.67)

A measure of the total intensity, \( I_{\text{total}} \), of this wave is got by integrating \( (2.67) \) over the spherical wave–front \( U = 0, R = \text{constant} \). The area element, obtained from \( (2.37) \), is
\[ dA = K^2 R^2 d\Xi d\Phi, \]  
(2.68)

with \( K^{-1} = \gamma (1 - v \Xi), -1 \leq \Xi \leq +1 \) and \( 0 \leq \Phi < 2\pi \). One readily verifies that the total area of the wave–front is \( 4\pi R^2 \) and that
\[ I_{\text{total}} = \frac{e^2}{2} \left\{ \frac{1}{v} \log \left( \frac{1 + v}{1 - v} \right) - 2 \right\}, \]  
(2.69)

for \( 0 < v < 1 \). This is typical of the 3–velocity dependence of the total intensity of electromagnetic bremsstrahlung (see, for example, Jackson’s discussion of beta decay). For the charge \( e \) above the deceleration from 3–velocity \( v \) to zero 3–velocity in the laboratory frame is instantaneous (at \( U = 0 \)) and since \( (2.69) \) has the characteristic 3–velocity dependence of electromagnetic bremsstrahlung we shall refer to the radiation described by \( (2.63) \) as \textit{instantaneous electromagnetic bremsstrahlung}. It is an important example of an impulsive electromagnetic wave because (a) its origin is known (it is due to the sudden change in the 4–velocity of the charge) and (b) it is free from unphysical directional singularities.

The approach to instantaneous electromagnetic bremsstrahlung given in this section leads naturally to the gravitational analogue described in section 3. For a treatment of instantaneous electromagnetic bremsstrahlung that does not involve the “cut and paste” procedure, see Appendix A.

### 3 Recoil Effect in Gravitation

We consider now the analogous situation for a spherically symmetric mass \( m \) in general relativity to that of the charge \( e \) discussed in the previous section. We shall first describe our geometrical model and then physically interpret it. Hence in place of the Minkowskian line–element of \( \mathcal{M}^- \) given by \( (2.24) \) we take the Schwarzschild line–element for \( \mathcal{M}^- \) with a source of mass \( m \):
\[ ds_+^2 = k^2 r^2 \left\{ \frac{d\xi^2}{1 - \xi^2} + (1 - \xi^2) d\phi^2 \right\} - 2 du dr - \left( 1 - \frac{2m}{r} \right) du^2, \]  
(3.1)
with $k^{-1} = \gamma (1 - v \xi)$. Also in place of (2.25) we take $\mathcal{M}^+$ to be the Schwarzschild space–time with source of mass $m_+$:

$$
\begin{align*}
    ds_+^2 &= r_+^2 \left\{ \frac{d\xi_+^2}{1 - \xi_+^2} + (1 - \xi_+^2) d\phi_+^2 \right\} - 2 du_+ dr_+ - \left( 1 - \frac{2m_+}{r_+} \right) du_+^2. \\
    & \quad \text{(3.2)}
\end{align*}
$$

For greater generality, and to enable a comparison with known results, we have assumed that the rest–mass of the Schwarzschild source has changed from $m$ in $\mathcal{M}^-$ to $m_+$ in $\mathcal{M}^+$.

In (3.1) $u = \text{constant}$ are null hypersurfaces generated by the integral curves of the vector field $\partial/\partial r$ and these null hypersurfaces are asymptotically (for large $r$) future null–cones. In (3.2) $u_+ = \text{constant}$ are also null hypersurfaces which asymptotically (for large $r_+$) are future null–cones. In (3.1) we shall take $u \leq 0$ and $\mathcal{M}^-$ as the region of space–time to the past of the null hypersurface $N(u = 0)$. In (3.2) we take $u_+ \geq 0$, with $u_+ = 0 \iff u = 0$, and we take $\mathcal{M}^+$ to be the region of space–time to the future of $N$. By analogy with the electromagnetic case discussed in section 2 we match $\mathcal{M}^-$ and $\mathcal{M}^+$ on $N$ with the matching conditions (2.26) which, by (3.1) and (3.2), ensure that the line–elements induced on $N$ by its embedding in $\mathcal{M}^-$ and $\mathcal{M}^+$ agree.

A physical interpretation of the above geometrical construction can be done along the same lines as in section 2 for the electromagnetic case. In particular one can also perform the same transformation (2.27) and make the same change of coordinates (2.28)-(2.31) and (2.32)-(2.35). Then the line element (3.1) reads

$$
    ds_{\pm}^2 = dx^2 + dy^2 + dz^2 - dt^2 + O \left( \frac{m}{\mathcal{R}} \right), \\
    & \quad \text{(3.3)}
$$

with

$$
\mathcal{R} = \left\{ x^2 + y^2 + \gamma^2(z - vt)^2 \right\}^{\frac{1}{2}}, \\
& \quad \text{(3.4)}
$$

and for the line element (3.2) one gets

$$
    ds_+^2 = dx_+^2 + dy_+^2 + dz_+^2 - dt_+^2 + O \left( \frac{m_+}{\mathcal{R}_+} \right), \\
    & \quad \text{(3.5)}
$$

with

$$
\mathcal{R}_+ = \left\{ x_+^2 + y_+^2 + z_+^2 \right\}^{\frac{1}{2}}. \\
& \quad \text{(3.6)}
$$

The relationship (2.36) between the coordinates $x, y, z, t$ and $x_+, y_+, z_+, t_+$ immediately show that $\mathcal{R}_+ = \mathcal{R}$. If the source of the gravitational field modelled by the space–time with line–element (3.2) is at rest relative to
a distant observer using the rectangular Cartesian coordinates and time \( \{x_+, y_+, z_+, t_+\} \) then the source of the gravitational field modelled by the space–time with line–element (3.1) may be considered moving with 3–velocity \( v \) in the \( z_+ \)–direction relative to this distant observer. As a result the physical situation described in the opening paragraph of this section is the analogue for a mass to that for a charge \( e \) described in section 2. To a distant observer using the plus coordinates the mass \( m \) is initially moving rectilinearly with uniform 3–velocity and is suddenly halted at \( u = 0 (u_+ = 0) \) and experiences a change in its rest–mass. We then ask what type of signal exists on the null hypersurface \( \mathcal{N} \)? The theory of light–like signals in general relativity developed by Barrabès–Israel (BI) [4] is tailor–made to answer this question.

With the space–time \( \mathcal{M}^- \) with line–element (3.1) attached to the space–time \( \mathcal{M}^+ \) with line–element (3.2) on the null hypersurface \( \mathcal{N}(u = 0 \Leftrightarrow u_+ = 0) \) with the matching conditions (2.26) the BI theory enables us to calculate, if it exists, the coefficient of \( \delta(u) \) in the Einstein tensor of \( \mathcal{M}^- \cup \mathcal{M}^+ \). This coefficient, if non–zero, is simply related to the surface stress–energy tensor of a light–like shell with history \( \mathcal{N} \). The theory also enables us to calculate the coefficient of \( \delta(u) \) in the Weyl tensor of \( \mathcal{M}^- \cup \mathcal{M}^+ \) if it exists. This allows us to determine whether or not the light–like signal with history \( \mathcal{N} \) includes an impulsive gravitational wave [11]. For the details of the BI technique the reader must consult [4] and further developments are to be found in [6]. We will merely guide the reader through the present application of the theory.

The local coordinate system in \( \mathcal{M}^- \) with line–element (3.1) is denoted \( \{x^\mu\} = \{\xi, \phi, r, u\} \) while the local coordinate system in \( \mathcal{M}^+ \) with line–element (3.2) is denoted \( \{x^\mu_+\} = \{\xi_+, \phi_+, r_+, u_+\} \). The equation of \( \mathcal{N} \) is \( u = 0 \Leftrightarrow u_+ = 0 \) and thus we take as normal to \( \mathcal{N} \) the null vector field with components \( n_\mu \) given via the 1–form \( n_\mu dx^\mu_\pm = -du \). Since we want the physical properties of \( \mathcal{N} \) observed by the observer using the plus coordinates we take as intrinsic coordinates on \( \mathcal{N} \), \( \{\xi^a\} = \{\xi_+, \phi_+, r_+\} \) with \( a = 1, 2, 3 \). A set of three linearly independent tangent vector fields to \( \mathcal{N} \) is \( \{e_{(1)} = \partial/\partial \xi_+, e_{(2)} = \partial/\partial \phi_+, e_{(3)} = \partial/\partial r_+\} \). The components of these vectors on the plus side of \( \mathcal{N} \) are \( e^\mu_{(a)}|_+ = \delta^\mu_a \). The components of these vectors on the minus side of \( \mathcal{N} \) are

\[
e^\mu_{(a)}|_- = \frac{\partial x^-_\mu}{\partial \xi^a}, \tag{3.7}\]

with the relation between \( \{x^\mu\} \) and \( \{\xi^a\} \) given by the matching conditions (2.26). Hence we find that

\[
e^\mu_{(1)}|_- = (1, 0, -r_+ \gamma v, 0), \tag{3.8}\]
\[
e^\mu_{(2)}|_- = (0, 1, 0, 0), \tag{3.9}\]
\[ e^{\mu}_{(3)}|_- = (0, 0, \gamma (1 - v \xi_+), 0) . \] (3.10)

We need a transversal on \( \mathcal{N} \) consisting of a vector field on \( \mathcal{N} \) which points out of \( \mathcal{N} \). A convenient such (covariant) vector expressed in the coordinates \( \{x^\mu \} \) is \( +N_\mu = (0, 0, 1, \frac{1}{2} - \frac{m+k}{r_+}) \). Thus since \( n^\mu = \delta^\mu_3 \) we have \( +N_\mu n^\mu = 1 \).

We construct the transversal on the minus side of \( \mathcal{N} \) with covariant components \( -N_\mu \). To ensure that this is the same vector on the minus side of \( \mathcal{N} \) as \( +N_\mu \) when viewed on the plus side we require

\[ +N_\mu e^{\mu}_{(a)}|_+ = -N_\mu e^{\mu}_{(a)}|_-, \quad +N_\mu + N^\mu = -N_\mu - N^\mu . \] (3.11)

The latter scalar product is zero as we have chosen to use a null transversal.

We find that

\[ -N_\mu = \left( \frac{r_- v}{1 - v \xi_+}, 0, \frac{1}{\gamma (1 - v \xi_+)}, D \right) , \] (3.12)

with

\[ D = \frac{v^2(1 - \xi^2_+)}{2(1 - v \xi_+)} + \frac{1}{2\gamma (1 - v \xi_+)} - \frac{m}{\gamma^2(1 - v \xi_+)^2 r_+} \] (3.13)

Next the transverse extrinsic curvature on the plus and minus sides of \( \mathcal{N} \) is given by

\[ \pm K_{ab} = \pm N_\mu \left( \frac{\partial e^{\mu}_{(a)}|_\pm}{\partial \xi^b} + \pm \Gamma^\mu_{\alpha\beta} e^{\alpha}_{(a)}|_\pm e^{\beta}_{(b)}|_\pm \right) , \] (3.14)

where \( \pm \Gamma^\mu_{\alpha\beta} \) are the components of the Riemannian connection associated with the metric tensor of \( M^+ \) or \( M^- \) evaluated on \( \mathcal{N} \). The key quantity we need is the jump in the transverse extrinsic curvature across \( \mathcal{N} \) given by

\[ \sigma_{ab} = 2 \left( +K_{ab} - -K_{ab} \right) . \] (3.15)

This jump is independent of the choice of transversal on \( \mathcal{N} \). We find that in the present application \( \sigma_{ab} = 0 \) except for

\[ \sigma_{11} = \frac{2}{1 - \xi^2_+} \left( m k^3 - m_+ \right) , \quad \sigma_{22} = 2 \left( 1 - \xi^2_+ \right) \left( m k^3 - m_+ \right) , \] (3.16)

with \( k^{-1} = \gamma (1 - v \xi) \). Now \( \sigma_{ab} \) is extended to a 4–tensor field on \( \mathcal{N} \) with components \( \sigma_{\mu\nu} \) by padding–out with zeros (the only requirement on \( \sigma_{\mu\nu} \) is \( \sigma_{\mu\nu} e^{\mu}_{(a)}|_\pm e^{\nu}_{(b)}|_\pm = \sigma_{ab} \)). With our choice of future–pointing normal to \( \mathcal{N} \) and past–pointing transversal, the surface stress–energy tensor components are \( -S_{\mu\nu} \) with \( S_{\mu\nu} \) given by

\[ 16\pi S_{\mu\nu} = 2 \sigma(\mu n_\nu) - \sigma n_\mu n_\nu - \sigma^\mu g_{\mu\nu} , \] (3.17)
with
\[ \sigma_\mu = \sigma_{\mu\nu} n^\nu , \quad \sigma^\dagger = \sigma_\mu n^\mu , \quad \sigma = g^{\mu\nu} \gamma_{\mu\nu} . \] (3.18)
In the present case \( \sigma_\mu = 0 \) and thus \( \sigma^\dagger = 0 \) and the surface stress–energy tensor takes the form
\[ - S_{\mu\nu} = \rho n_\mu n_\nu . \] (3.19)
Hence the energy density of the light–like shell measured by the distant observer using the plus coordinates is
\[ \rho = \frac{\sigma}{16\pi} = \frac{1}{4\pi r^2_+} \left( m k^3 - m_+ \right) . \] (3.20)
Thus the null–cone \( \mathcal{N} \) is the history of a light–like shell with surface stress–energy given by (3.19). We note that \( m k^3 \) is the “mass aspect” in the terminology of Bondi et al.\[7\], on the minus side of \( \mathcal{N} \). A calculation of the singular \( \delta \)–part of the Weyl tensor for \( M^- \cup M^+ \) reveals that it vanishes. Hence there is no possibility of the light–like signal with history \( \mathcal{N} \) containing an impulsive gravitational wave. We note that \( \rho \) is a monotonically increasing function of \( \xi_+ \). Thus on the interval \( -1 \leq \xi_+ \leq +1 \), \( \rho \) is maximum at \( \xi_+ = +1 \) (in the direction of the motion) and \( \rho \) is minimum at \( \xi_+ = -1 \). This is as one would expect. A burst of null matter predominantly in the direction of motion is required to halt the mass. In this sense the model we have constructed here could be thought of as a limiting case of a Kinnersley rocket \[8\] \[9\].

By integrating (3.20) over the shell with area element \( dA_+ = r^2_+ d\xi_+ d\phi_+ \) and with \( -1 \leq \xi_+ \leq +1, 0 \leq \phi_+ < 2\pi \) we obtain the total energy \( E_+ \) of the shell measured by the distant observer who sees the mass \( m \), moving rectilinearly with 3–velocity \( v \) in the direction \( \xi_+ = +1 \), suddenly halted. Thus
\[ E_+ = \frac{1}{4\pi} \int_0^{2\pi} d\phi_+ \int_{-1}^{+1} \left( m k^3 - m_+ \right) d\xi_+ . \] (3.21)
This results in
\[ E_+ = m \gamma - m_+ . \] (3.22)
So the energy of the light–like shell is the difference in the relative masses before and after the emission of the light–like shell. If one wants to exhibit the conservation of energy one can also interpret (3.22) by saying that, in the reference frame \((x_+, y_+, z_+, t_+)\) where the mass \( m_+ \) is at rest, the energy \( m \gamma \) of the ingoing mass is transferred into the rest energy \( m_+ \) plus the energy \( E_+ \) of the emitted null shell.

When \( v = 0 \) (\( \gamma = 1 \)) the energy of the shell is the difference in the rest–masses (naturally taking \( m_+ < m \)) and this is a well–known result \[4\] which
may also be thought of as a limiting case of the Vaidya solution. If \( v \neq 0 \) and \( m = m_+ \) then

\[
E_+ = m(\gamma - 1).
\]

(3.23)

In this case all of the kinetic energy of the mass \( m \) before stopping is converted into the relativistic shell.

The simplest way of interpreting our solution is that it represents the field outside a moving spherical body (a star for instance) which, at a certain moment of retarded time, suddenly stops as a consequence of some internal process such as a laser-like nuclear reaction. The emission of a sharp burst of null matter (photons or neutrinos) predominantly in the forward direction provides the recoil momentum which is necessary to put the body at rest. As we are only interested in the outside field we have not made any assumption about the structure of the body except that it is spherically symmetric and is initially moving with constant velocity. If we were to describe the global structure of spacetime two possibilities would have to be considered. The first one would correspond to a point–like body, and the maximal analytic extension of spacetime would then show that the light–like shell emerges from the white–hole region and propagates radially to future null infinity. This situation, which appears to be the closest analogue of the charged particle described in section 2, is of limited physical significance. The second possibility would be that the body has some finite radius larger than its gravitational radius. This is more realistic as it avoids the white-hole region and in particular the shell starting from the white-hole singularity. Here only the outside region has to be considered and the light–like shell is directly emitted from the surface of the body. The idea of an extended body remaining rotationally symmetric when suddenly decelerated to rest is, of course, a strong idealization. More realistically, one would expect the body to undergo some deformations and, thereby, to emit gravitational radiation. Such deformation effects, treated with the help of approximation methods, could be the subject of further studies, using the idealized situation considered in this paper as the starting point. The time reverse of both of the above possibilities involves impulsive acceleration arising from a radially imploding shell of null matter (for the first possibility the white–hole is now replaced by a black–hole).

It is not surprising that the recoil effect described above does not produce an impulsive gravitational wave since the lowest multipole of an isolated body that contributes to gravitational radiation is the quadrupole and the change of motion that we have considered here only brings a dipole moment. Although the light–like shell above is (geometrically) spherically symmetric its energy density (3.20) is not, and is concentrated in the forward direction of motion. We have pointed out following (3.20) that in effect it is the axial
symmetry of the energy density (3.20) which explains why the burst of null matter can result in zero recoil velocity of the moving source. Impulsive gravitational waves which are spherical, in the sense that they have a future null–cone history, do exist but they contain line singularities \[1\] and thus are not really spherically symmetric. If a system with multipole moments experiences an explosion, such as a supernova, which suddenly changes those moments then the sudden change in the quadrupole moment is the dominant contribution to an asymptotically spherical impulsive gravitational wave \[1\].

4 Asymptotic Collision with Back–Scatter

We illustrate here some interesting calculations that can be carried out using the instantaneous electromagnetic bremsstrahlung described in section 2 by the final term in (2.51). As Penrose \[1\] says:“this retarded radiation approximates a purely impulsive electromagnetic wave in the far zone, but after an encounter with a gravitational plane (impulsive) wave, the electromagnetic wave back–scatters and so ceases to satisfy Huygen’s principle”. Since we have in our possession in (2.51) an analytic expression for the electromagnetic wave, we wish to use it to give an analytic description of the remainder of the situation envisaged by Penrose and in particular to exhibit the back–scattered electromagnetic radiation. We therefore first must take the “plane wave limit” of (2.51). To do this we make use of a device invented by Robinson and Trautman \[12\] and developed in the context of Liénard–Wiechert electromagnetic fields by Hogan and Ellis \[13\]. We want to find expression for a limit of (2.51) which captures the idea that we are viewing (2.51) at a large distance from the charge \(e\) and over not too large regions of space. This can be achieved by first making the coordinate transformation \[12\]

\[
U = \lambda \bar{U} , \quad R = \gamma \lambda^{-2} + \lambda^{-1} \bar{V} , \tag{4.1}
\]

\[
\Xi = \lambda^2 \bar{X} , \quad \Phi = \lambda^2 \bar{Y} , \tag{4.2}
\]

where \(\lambda\) is a real parameter. If this is applied to the line–element (2.37) of \(\mathcal{M}^- \cup \mathcal{M}^+\) and then the limit \(\lambda \to 0\) taken, the result is the line–element

\[
ds^2 = d\bar{X}^2 + d\bar{Y}^2 - 2d\bar{U}d\bar{V} . \tag{4.3}
\]

If in addition (4.1),(4.2) is applied to the Maxwell field (2.51) in its form (2.48) and at the same time the charge \(e\) is rescaled according to \[13\]

\[
e = \lambda^{-2} \bar{e} . \tag{4.4}
\]
then for small $\lambda$ we find, using (2.59) and (2.60) that

$$f_{\text{Coul}}^+ = O\left(\lambda^2\right), \quad f_{\text{Coul}}^- = O\left(\lambda^2\right), \quad (4.5)$$

while

$$ek\gamma v\delta(U)\,dU \wedge d\Xi = \bar{e}v\delta\left(\bar{U}\right)\,d\bar{U} \wedge d\bar{X} + O\left(\lambda^2\right). \quad (4.6)$$

Thus in the limit $\lambda \to 0$ the electromagnetic field (2.48) becomes

$$F = \bar{e}v\delta\left(\bar{U}\right)\,d\bar{U} \wedge d\bar{X}. \quad (4.7)$$

This is a plane impulsive electromagnetic wave with history the null hyperplane $\bar{U} = 0$ in Minkowskian space–time with line–element (4.3) and with propagation direction $\partial/\partial\bar{V}$ in this space–time. We note that evidence of the origin of the wave (4.7) in the presence of the velocity parameter $v$ has survived the plane wave limit.

We now consider the head–on collision of this plane wave with a plane impulsive gravitational wave. The space–time picture of the process is given in figure 2.

The half space–time $\bar{V} < 0$ has line–element (4.3) while the half space–time $\bar{V} > 0$ has line–element

$$ds^2 = (1 - p\bar{V})^2d\bar{X}^2 + (1 + p\bar{V})^2d\bar{Y}^2 - 2d\bar{U}d\bar{V}, \quad (4.8)$$

where $p$ is constant. These are both regions of Minkowskian space–time. The Riemann tensor of the space–time in figure 2 has one non–vanishing Newman–Penrose component

$$\Psi_4 = p\delta\left(\bar{V}\right), \quad (4.9)$$

and is therefore type N in the Petrov classification with $\partial/\partial\bar{U}$ as degenerate principal null direction. Thus if $p \neq 0$ then $\bar{V} = 0$ in figure 2 is the history of a plane impulsive gravitational wave. Also in figure 2, $\bar{U} = 0, \bar{V} < 0$ is the history of the plane impulsive electromagnetic wave (4.7) which collides head–on with the gravitational wave. We thus have no Maxwell field within regions I, II and III and we solve Maxwell’s vacuum field equations in region IV assuming no Maxwell field on the boundary $\bar{V} = 0, \bar{U} > 0$ and an impulsive electromagnetic wave on the boundary $\bar{U} = 0, \bar{V} > 0$. In terms of the basis $\mathbf{1}$–forms in the region $\bar{V} > 0$,

$$\tau^1 = (1 - p\bar{V})\,d\bar{X}, \quad \tau^2 = (1 + p\bar{V})\,d\bar{Y}, \quad \tau^3 = d\bar{U}, \quad \tau^4 = d\bar{V}, \quad (4.10)$$
Figure 2: The 2–plane $\bar{X}, \bar{Y} = \text{constants}$ in the space–time having the line–element
\[ ds^2 = (1 - p \bar{V} \partial(\bar{V}))^2 d\bar{X}^2 + (1 + p \bar{V} \partial(\bar{V}))^2 d\bar{Y}^2 - 2 d\bar{U} d\bar{V}. \]
$\bar{U} = 0, \bar{V} < 0$ is the history of the incoming plane electromagnetic impulsive wave (4.7) while $\bar{V} = 0$ is the history of the plane impulsive gravitational wave with which it collides head–on. The dotted line is $\bar{V} = |p|^{-1}$.

we find that the Maxwell 2–form $F^+$ and its dual $^*F^+$ in the region $\bar{V} > 0$ is given by
\[
F^+ - i^*F^+ = -\bar{e} v (1 - p^2 \bar{V}^2)^{-1/2} \delta \left( \bar{U} \right) \left( \tau^1 + i \tau^2 \right) \wedge \tau^3 \\
+ \bar{e} p v (1 - p^2 \bar{V}^2)^{-3/2} \delta \left( \bar{U} \right) \left( \tau^1 - i \tau^2 \right) \wedge \tau^4. \quad (4.11)
\]
Thus the Maxwell 2–form in $\bar{V} > 0$ is
\[
F^+ = \bar{e} v (1 - p \bar{V})^{1/2} (1 + p \bar{V})^{-1/2} \delta \left( \bar{U} \right) d\bar{U} \wedge d\bar{X} \\
- \bar{e} p v (1 - p \bar{V})^{-1/2} (1 + p \bar{V})^{-3/2} \delta \left( \bar{U} \right) d\bar{V} \wedge d\bar{X}. \quad (4.12)
\]
If the gravitational wave is removed by putting $p = 0$ then (4.12) becomes the plane electromagnetic wave (4.7). The first term in (4.12) is type N in the Petrov classification with degenerate principal null direction $\partial/\partial \bar{V}$ and is an impulsive electromagnetic wave with history $\bar{U} = 0, \bar{V} > 0$ and with propagation direction $\partial/\partial \bar{V}$. The geodesic integral curves of $\partial/\partial \bar{V}$ generate
$\bar{U} = 0, \bar{V} > 0$ and have expansion $p^2\bar{V}/(1 - p^2\bar{V}^2)$ and shear $p/(1 - p^2\bar{V}^2)$. Thus $\bar{U} = 0, \bar{V} > 0$ is not a hyperplane and so this impulsive wave is not a plane wave. The second term in (4.12), which is non-zero for $\bar{U} > 0$, is type N in the Petrov classification with degenerate principal null direction $\partial/\partial \bar{U}$. Thus this term describes electromagnetic back-scattered radiation in region IV ($\bar{U} > 0, \bar{V} > 0$) of figure 2 with propagation direction $\partial/\partial \bar{U}$.

The integral curves of $\partial/\partial \bar{U}$ are twist-free, expansion-free, shear-free null geodesics generating the null hyperplanes $\bar{V} = \text{constant}$. Hence the histories of the wave fronts of this back-scattered electromagnetic radiation are null hyperplanes. Finally we note that the Maxwell field described by (4.11) in region $\bar{V} > 0$ becomes singular when $\bar{V} = |p|^{-1}$ where the null geodesics tangent to $\partial/\partial \bar{V}$ are focussed after passing through the gravitational wave (see figure 2).

Section 2 above and this section taken together give analytical expression to a fascinating space–time diagram (figure 4) in [1]. We note that the Maxwell field in region $\bar{U} > 0, \bar{V} > 0$ of figure 2, which is given by (4.11), not only contains the back-scattered radiation but is only valid for $0 \leq \bar{V} < |p|^{-1}$.

Acknowledgment

We thank Professor W. Israel for helpful discussions. This collaboration has been funded by the Ministère des Affaires Étrangères, D.C.R.I. 220/SUR/R.

References

[1] R. Penrose, General Relativity, Papers in Honour of J.L. Synge (Clarendon Press, Oxford, 1972), p.101.

[2] J. D. Jackson, Classical Electrodynamics (John Wiley, New York, 1967), p. 526; see also L. Landau and E. Lifchitz, Théorie du Champ (ed. MIR, Moscow, 1966), p 233.

[3] I. Robinson, J. Math. Phys. 2, 290 (1961).

[4] C. Barrabès and W. Israel, Phys. Rev. D43, 1129 (1991).

[5] J. L. Synge, Relativity:the special theory (North–Holland, Amsterdam, 1965), p.386.

[6] C. Barrabès and P. A. Hogan, Phys. Rev. D58, 044013 (1998).
A Alternative Approach to Electromagnetic Bremsstrahlung

The rectangular Cartesian coordinates \( \{ X^\mu \} \) used in section 2 are continuous across the future null–cone \( \mathcal{N} \) which is the history of the spherical impulsive electromagnetic wave constructed there. It may therefore be of some interest to see how the wave is constructed in terms of \( \{ X^\mu \} \). With \( \mathcal{N} \) given by (2.7) the distances \( r \) and \( r_+ \) in (2.10) and (2.11) for the 4–velocities (2.16) and (2.17) are given by

\[
r = \gamma \left( X^4 - v X^3 \right) , \quad r_+ = X^4 .
\tag{A.1}
\]

The jump in the Coulomb potential across \( \mathcal{N} \) is

\[
[A^\mu_{\text{Coul}}] = e \left( \frac{v^\mu}{r_+} - \frac{v^\mu}{r} \right) .
\tag{A.2}
\]

Written out explicitly this reads

\[
\begin{align*}
[A^1_{\text{Coul}}] &= [A^2_{\text{Coul}}] = 0 , \\
[A^3_{\text{Coul}}] &= -\frac{e v}{X^4 - v X^3} , \\
[A^4_{\text{Coul}}] &= -\frac{e v X^3}{X^4 (X^4 - v X^3)} .
\end{align*}
\tag{A.3}
\]
Now the 4–potential (2.46) in coordinates \(\{X^\mu\}\) is

\[
A^\mu = A_{\text{Coul}}^\mu \theta(u) + A_{\text{Coul}}^{\mu} (1 - \theta(u)) ,
\]

(A.6)

\[
e \frac{v^\mu}{r^+} \theta(u) + e \frac{v^\mu}{r} (1 - \theta(u)) ,
\]

(A.7)

with

\[
u = \frac{1}{2} \eta_{\mu\nu} X^\mu X^\nu .
\]

(A.8)

By (2.7) \(u = 0\) (with \(X^4 > 0\)) is the equation of \(\mathcal{N}\). The electromagnetic tensor corresponding to the 4–potential (A.6) takes the form

\[
F^{\mu\nu} = F_{\text{Coul}}^{\mu\nu} \theta(u) + F_{\text{Coul}}^{\mu\nu} (1 - \theta(u)) + \tilde{F}^{\mu\nu} \delta(u) ,
\]

(A.9)

with

\[
\tilde{F}^{\mu\nu} = [A_{\text{Coul}}^{\mu}] u^{\nu} - [A_{\text{Coul}}^{\nu}] u^{\mu} .
\]

(A.10)

To make the connection between the electromagnetic wave described in (A.9) and that in section 2 we proceed as follows: The delta function part of (A.9) is given equivalently by the 2–form

\[
\tilde{F} = \frac{1}{2} \tilde{F}_{\mu\nu} dX^\mu \wedge dX^\nu = [A_{\text{Coul}}^{\mu}] dX^\mu \wedge du .
\]

(A.11)

By (A.4) and (A.5) we can write

\[
[A_{\text{Coul}}^{\mu}] dX^\mu = - \frac{e v}{1 - v \xi} d\xi = - e k \gamma v d\xi ,
\]

(A.12)

with

\[
\xi = \frac{X^3}{X^4} ,
\]

(A.13)

and \(k^{-1} = \gamma (1 - v \xi)\). Now (A.12) in (A.11) shows that (A.11), multiplied by \(\delta(u)\), is the spherical impulsive electromagnetic wave constructed in section 2 above. Furthermore on \(\mathcal{N}\) we have

\[
X^3 = \frac{\xi \rho}{\sqrt{1 - \xi^2}} , \quad X^4 = \frac{\rho}{\sqrt{1 - \xi^2}} ,
\]

(A.14)

with \(\rho = \sqrt{(X^1)^2 + (X^2)^2}\). Putting \(X^1 = \rho \cos \phi\), \(X^2 = \rho \sin \phi\) and substituting this and (A.14) into (2.1) yields the induced line–element on \(\mathcal{N}\),

\[
d\xi^2 = \frac{\rho^2}{(1 - \xi^2)} \left\{ \frac{d\xi^2}{1 - \xi^2} + (1 - \xi^2) d\phi^2 \right\} .
\]

(A.15)
Thus from (A.1) and (A.13) we recover our matching condition $r_+ = k r$ and, in addition,

$$\rho = r_+ \sqrt{1 - \xi^2}, \quad (A.16)$$

and so in (A.13) we have

$$\frac{\rho^2}{(1 - \xi^2)} = r_+^2 = k^2 r^2. \quad (A.17)$$

It is by no means obvious that (A.9) satisfies Maxwell’s equations. The verification that it does is given in section 2 above.