Analytic investigation of the compatibility condition and the initial evolution of a smooth velocity field for the Navier–Stokes equation in a channel configuration

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Abstract

A partial differential equation has usually a regular solution at the initial time if the initial condition is smooth in space, fulfills the governing equations and is compatible with the boundary condition. In the case of the Navier–Stokes equation, the initial velocity field must also be divergence-free. It is common belief that the initial condition is compatible with the boundary condition if the initial condition fulfills the boundary condition but this is not sufficient. Such a field does not necessarily fulfill the full compatibility condition of the Navier–Stokes equation. If the condition is violated, the solution is not regular at the initial time ($t = 0^+$). This issue has been known for a while but not in the full breadth of the engineering fluid dynamics community. In this paper, a practical calculation method is presented for checking the compatibility condition. Furthermore, a smooth initial condition is presented in a periodic channel flow that violates the compatibility condition and has therefore no smooth solution at the initial instant. The calculations were performed in an analytical framework. The results for a channel configuration show that in the absence of wall–normal velocity the condition is always fulfilled and the problem has a regular solution. If the wall–normal velocity component is non-zero, the condition is usually

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not fulfilled but exceptional cases, fulfilling the compatibility condition can be generated with optimization methods. The generation procedure of such a field is useful to provide correct initial conditions for such time-dependent numerical flow simulations, where the very first instances are important from an engineering point of view. The presented methods can provide insight about the applicability of the chosen initial conditions.

Supplementary material for this article is available online

Keywords: Navier–Stokes equation, initial regularity, compatibility condition, analytic solution

1. Introduction

Several mathematical theories about the existence of a smooth Navier–Stokes solution were developed in the last century. Many of the early theorems were summarized in the book of Ladyzhenskai (1969). The Navier–Stokes differential equations have a unique solution in two dimensions for an arbitrary time duration. Unfortunately, this statement has not been proved yet in three dimensions but it is known that the unique solution must exist for a certain time $T^*$ on periodic domains, as well as on the entire space. If the domain is bounded, a wall is present, the initial condition cannot be arbitrary. This problem was briefly discussed in the aforementioned book, and was later investigated thoroughly among others by Temam (1982). He proved that the solution is regular in a three-dimensional space at $t = 0^+$, if and only if the initial condition fulfills the compatibility condition. In this paper, the term ‘regular’ is used to express that the solution and its time-derivative are continuously differentiable. This condition can be relaxed and replaced by the $m = 3$ case of equation (3.11) in Temam (1982). In an early paper, Synge (1936) already gave the necessary conditions for the plane channel flow in two-dimensions. He presented the a so-called kinematic condition for the vorticity field that ensures stationary velocity field at the walls. Furthermore, he derived a necessary condition that a regular solution must fulfill. He called it the dynamical condition. He also presented a vorticity field in the two-dimensional domain which fulfills the kinematic condition but not the dynamical condition. He interpreted his result as: ‘It appears to be that the instant $t = t_0$, at which the arbitrary distribution of velocity is assigned, is a peculiar or singular instant in the history of the motion. There is an instantaneous adjustment in the values of $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ on the walls, and for all values of $t$ greater than $t_0$ the dynamical condition is satisfied.’

However, the compatibility, or according to Synge’s terminology the dynamical condition is not well-known in the engineering fluid dynamics community and therefore will be presented in section 2. Having this in mind, Temam (2006) also attempted a physical explanation for these mathematical results. The practical consequence of the problem appeared in the field of computational fluid dynamics (CFD). If the velocity is prescribed at the boundaries (Dirichlet boundary condition), there are two different ways to calculate the pressure numerically. The usual method uses the wall-normal momentum equation to obtain a Neumann boundary condition for the pressure. However, using discretization schemes like finite difference methods, the calculated pressure can violate the tangential component of the momentum equation, if the initial velocity field is chosen arbitrarily, since there can be a discontinuity at $t = 0^+$. Using the tangential component of the momentum equation to obtain a boundary condition would
lead to a different numerical result for the pressure field (Orszag and Israeli 1974), unless the compatibility condition is fulfilled. The same problem was investigated thoroughly by Gresho and Sani (1987). The main question asked by the authors was, how the boundary condition for the pressure should be handled numerically. If the compatibility condition is not fulfilled, the Dirichlet and the Neumann boundary condition lead to different numerical solutions. They recommend the usage of Neumann boundary condition at the initial time. Furthermore, they propose multiple methods to handle the problems if the condition is not fulfilled, which is generally the case in CFD simulations. In addition, they showed numerical examples. The problem was approached similarly by Johnston and Liu (2004). They solved the incompressible Navier–Stokes equation using the pressure Poisson equation with a Neumann boundary condition. Gallavotti (2002) discussed the problem with the initial condition in chapter 2.1 of his book. He suggests a theoretically possible numerical procedure to handle the issue by extending the computational domain by a thin additional volume. According to his solution, the velocity is not zero at the original boundary but it is reduced by an extra ‘friction’ term in the thin layer over the boundary. Nevertheless, he admits that the numerical solution of the problem using the suggested method is challenging. It must be mentioned that these techniques were developed to obtain numerical results, but according to the theorem of Temam (1982), the regular solution does not exist in these cases at \( t = 0^+ \). This problem can cause errors at the beginning of simulation, which can be disturbing especially if the initial behavior of the flow is critical. Another example is the energy stability analysis of fluid flows, namely the Reynolds–Orr (Orr 1907) equation. In this case, the stability problem is rewritten as a variational problem. The perturbation velocity field is varied to maximize the temporal growth rate of the kinetic energy. The base flow is stable if this maximum is below zero. However, the velocity field usually does not fulfill the compatibility condition and the using it as an initial condition may lead to nonphysical results, at least in the beginning of the simulation.

Although the Navier–Stokes equation rarely has an analytical solution, there are still several examples. They are derived usually with some approximations or assumptions. In addition, some mathematical papers discussed the extraordinary analytical solutions of the Navier–Stokes equation in the weak form. For example, Heywood (1980) showed that multiple solutions exist in the case of flow through a hole. He pointed out the need for a further auxiliary condition for a unique solution. However, the solution of the equation in a strong form must be unique for certain time. The solution is regular in the beginning, if it satisfies the above-mentioned compatibility condition. In this paper, the evolution of a smooth initial velocity field is calculated analytically using the differential form of the equation. Furthermore, a smooth example is given that violates the compatibility condition, therefore a regular solution does not exist at the initial time. For the sake of simplicity, the domain is a rectangular cuboid, a channel configuration (figure 1). In that case, the steady-state solution is the well-known Poiseuille flow. Recently, Józsa (2019) obtained an analytical solution for a controlled channel flow.

This paper is structured as follows. First, the analytic calculation method of the temporal evolution of the velocity is presented using the vorticity equation. A condition is obtained that can be used to investigate the regularity of the solution at the initial time. The benefit of this method is that the boundary conditions are necessary only for the velocity, since the pressure is eliminated. This is followed by the discussion of the compatibility condition. Then a smooth, analytically calculated example is given that violates the condition and the solution is irregular at the beginning. To the best knowledge of the authors no such example has been given so far in the case of a three-dimensional flow field. Finally, special examples are presented in which case the condition holds and the solution is regular.
2. The calculation method of the analytic solution

2.1. Governing equations

An incompressible Newtonian fluid can be described by the continuity equation
\[ \frac{\partial u_i}{\partial x_i} = 0 \] (1)
and the Navier–Stokes equation in non-dimensional form
\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \] (2)
where \( u_i \) is the non-dimensional velocity, \( p \) is the non-dimensional pressure, \( Re \) is the Reynolds number, \( i = \{ 1, 2, 3 \} \) is a running index of the Einstein summation notation.

\[ Re = \frac{U_0 h}{\nu}, \] (3)
where \( U_0 \) is the velocity scale, \( h \) is the half gap of the channel, \( \nu \) is the kinematic viscosity. The body forces are neglected.

The non-dimensional solution domain is a cuboid: \( x_1 \in [0, L_x], x_2 \in [-1, 1], x_3 \in [0, L_z] \) that represents a channel (see figure 1). The boundary conditions are the following. The velocity field is periodic in the \( x_1 \) and \( x_3 \) directions:
\[ u_i(x_1, x_2, x_3, t) = u_i(x_1 + L_x, x_2, x_3, t), \] (4)
\[ u_i(x_1, x_2, x_3, t) = u_i(x_1, x_2, x_3 + L_z, t). \] (5)

Usually, these directions are called streamwise and spanwise directions, respectively. At \( x_2 = -1 \) and \( x_2 = 1 \), the no-slip wall condition is prescribed, meaning that
\[ u_i(x_1, x_2 = -1, x_3, t) = u_i(x_1, x_2 = 1, x_3, t) = 0. \] (6)
where \( x_2 \) is the wall-normal coordinate. The initial velocity field is given as
\[ u_i(x_1, x_2, x_3, t = 0) = u_{i,0}(x_1, x_2, x_3), \] (7)
which fulfills the continuity equation (1) and the boundary conditions. The initial velocity field example will be presented later.

A possible way to obtain the temporal derivative of the velocity field is using the curl of the Navier–Stokes equation, which known as vortex transport or vorticity equation,
\[ \frac{\partial \omega_j}{\partial t} = -u_j \frac{\partial \omega_i}{\partial x_i} + \omega_i \frac{\partial u_j}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \omega_i}{\partial x_j \partial x_j}, \] (8)
where \( \omega_j \) is the vorticity, the curl of \( u_i \). The benefit of this procedure is the elimination of the pressure. \( \omega_{i,0} \) is the curl of the initial condition \( (u_{i,0}) \) that can be easily calculated for a given initial velocity. The temporal partial derivative of the vorticity at the initial time \( (t = 0) \) can be determined using the vorticity equation (8)
\[ \frac{\partial \omega_i}{\partial t} \big|_{t=0} = -u_{i,0} \frac{\partial \omega_j}{\partial x_j} + \omega_{j,0} \frac{\partial u_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \omega_{i,0}}{\partial x_j \partial x_j}. \] (9)
Furthermore, the well-known vector calculus identity,

$$\nabla \times (\nabla \times \vec{u}) = \nabla (\nabla \cdot \vec{u}) - \Delta \vec{u}$$

(10)

and the symmetry of the derivatives leads to

$$\nabla \times \left( \frac{\partial \vec{\omega}}{\partial t} \right) = -\Delta \left( \frac{\partial \vec{u}}{\partial t} \right),$$

(11)

since the temporal derivative of the velocity must be a solenoid field. $\nabla$ is the Nabla, $\Delta$ is the Laplace operator, $\vec{u}$ is the velocity vector and $\vec{\omega}$ is the vorticity vector. At the initial time, the temporal derivative of vorticity can be obtained from equations (9) and (11) leads to three independent Poisson equations for the temporal derivative of each velocity component:

$$\frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{\partial u_i}{\partial t} \right) |_{t=0} = f_i(x_1, x_2, x_3),$$

(12)

where

$$f_i(x_1, x_2, x_3) = -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \left( -u_l,0 \frac{\partial \omega_k,0}{\partial x_l} + \omega_l,0 \frac{\partial u_k,0}{\partial x_l} + \frac{1}{Re} \frac{\partial^2 \omega_k,0}{\partial x_l \partial x_l} \right)$$

(13)

and $\varepsilon_{ijk}$ is the Levi-Civita symbol.

The velocity boundary conditions of this problem are the same as those of the original problem, equations (4)–(6). Since the Poisson equation is a linear and elliptic differential equation, it has only one solution in the case of the Dirichlet boundary condition. However, it will be shown that for initial values that violate the compatibility condition, the solution of equation (12) is not divergence-free. (It is mentioned that $\vec{f} = [f_1, f_2, f_3]^T$ (13) is undoubtedly divergence-free. However, this does not necessarily mean that $\partial \vec{u}/\partial t$ is also divergence-free. For example, the Laplacian of any linear vector function is a zero vector that is divergence-free, but a linear function is not divergence-free since its divergence is a constant and not necessarily zero.) The unique solution of equation (12) is therefore not acceptable, meaning that the solution of equation (2) must be irregular at $t = 0^+$.  

### 2.2. The compatibility condition

Temam (1982) proved (and later discussed the physical implications (Temam 2006)) that the solution is regular in the initial time, if and only if it fulfills the compatibility condition. In the case of the linear heat equation (Temam 2006), the compatibility condition expresses that the initial value of the problem does not contradict the boundary condition. In contrast, the
compatibility condition in the case of the Navier–Stokes equation is more complicated. The condition is that the expression (Temam 1982)

$$\frac{1}{Re} P(\Delta u_{i,0}) - P \left( u_{i,0} \frac{\partial u_{i,0}}{\partial x_j} \right) + g_i(0) = 0 \quad (14)$$

must be fulfilled at the boundaries; otherwise, the solution is irregular at \( t = 0 \). (The sign of the second term is corrected here since it was erroneously positive in the cited paper.) \( P \) is the Leray projector that gives the divergence-free part of a vector field. The projector can be expressed with the Helmholtz decomposition. \( \Delta \) is the Laplace operator, \( g_i \) is an arbitrary volumetric source term in the momentum equation, and is 0 in this study. Temam (2006) proposed a calculation method to check the compatibility condition. First, the following equation is solved:

$$\Delta p_0 = - \frac{\partial u_{j,0}}{\partial x_i} \frac{\partial u_{i,0}}{\partial x_j}, \quad (15)$$

which is the divergence of the momentum equation. It is also known as the Poisson equation for the pressure. (In the cited paper, the negative sign is omitted, which leads to the same condition, but in this form, \( p \) is the widely used physical pressure.) The Neumann boundary condition for the pressure is

$$\frac{\partial p_0}{\partial x_i} n_i = \left( \frac{1}{Re} \Delta u_{i,0} \right) n_i \quad (16)$$

in the case of a no-slip wall, which can be obtained from the momentum equation (2) at the stationary walls (6) in the wall-normal direction. \( n_i \) is the wall-normal vector. In the presented channel configuration, it is non-zero only for \( i = 2 \). Temam (1982) states as a consequence of the compatibility condition, after solving the problem (15), using (16) that the temporal derivative of the velocity field is smooth at \( t = 0 \), if the tangential components of \( \frac{\partial p_0}{\partial x_i} \) are equal to the tangential components of \( \frac{1}{Re} \Delta u_{i,0} \):

$$\frac{\partial p_0}{\partial x_i} t_i = \left( \frac{1}{Re} \Delta u_{i,0} \right) t_i, \quad (17)$$

where \( t_i \) is an arbitrary vector perpendicular to \( n_i \). The condition is actually the evaluation of the momentum equation at the wall in a tangential direction. The physical meaning of equation (17) is that the pressure field prevents the tangential acceleration (the tangential movement) of the wall. The problem is that this additional condition cannot be prescribed for the Poisson equation (15), since then it would become overdetermined. This condition is a constraint for the initial velocity field. If it does not hold, there is no regular solution at the initial time which fulfills the stationary wall condition and the solenoid property at the same time. Irregular solutions exist but such velocity fields can exist only as mathematical abstractions. It can smoothly evolve physically neither from a previous state nor to a next state. However, all this reasoning is valid only for a fully incompressible flow, which is an idealization. A real fluid is always compressible and the mathematical requirement that the pressure maintains the incompressibility is relaxed.

2.3. Smooth examples that violate the compatibility condition

Next, the temporal evolution of a given velocity field will be calculated analytically using equation (12). The solution is defined by a finite Fourier series, meaning that it is smooth in space. The only analytical solution will be shown to be wrong, meaning that a regular solution
Figure 2. The slice ($z = 0$) of the example velocity field. The parameters are $\alpha = 1$, $\beta = 1$, $Re = 80$ (a) $u_{2,0}$ (19); (b) $u_{3,0}$ equation (20).

does not exist. It will be proven by evaluating the compatibility condition which does not hold. The calculation is presented for the following initial velocity field

$u_{1,0} = 0$  \hspace{1cm} (18)

$u_{2,0} = \cos(\alpha x + \beta z) \left( y^2 - 1 \right)^2$  \hspace{1cm} (19)

$u_{3,0} = -\frac{4y \sin(\alpha x + \beta z) \left( y^2 - 1 \right)}{\beta}$  \hspace{1cm} (20)

where $x = x_1$, $y = x_2$, $z = x_3$, $\alpha = 2\pi/L_x$, $\beta = 2\pi/L_z$. This is one of the simplest initial velocity fields, which is divergence-free and fulfills the boundary conditions. (Of course, $u_{1,0}$ is not necessarily 0. It was set to 0 for the sake of simplicity, since otherwise the analytical terms would become far too long for publication but this assumption does not restrict the message of the paper.) The velocity field is plotted in figure 2 at the plane $z = 0$. The analytical calculation of the temporal derivative of the velocity, its divergence, and the compatibility condition are presented in the appendix. The calculation is straightforward, but the expressions are long. They could become even more lengthy in the case of a more complex initial field. Therefore, Matlab 2019b Symbolic toolbox was used to reduce the solution time and the risk of miscalculation. The code is available as additional material to the paper.

The same calculation was repeated using numerical functions mainly based on the Chebychev collocation method, similarly to the work of Falsaperla et al (2019) and Nagy (2022). In this case, the parameters must have numerical values. The calculations were carried out multiple times for different parameters and the analytical and the numerical computation results were practically identical, meaning that no error was made in the calculation.

This initial velocity field does not fulfill the compatibility condition, and the corresponding problem has no regular solution at $t = 0$. The divergence of the time derivative of the velocity is shown in figure 3 for certain parameters, and its value is clearly non-zero. This means that the
calculated temporal derivative of the velocity field is discontinuous at the initial time, leading again to the conclusion that the solution must be irregular at the initial time. The calculated terms and their derivation are shifted to the appendix because they are long expressions. Using the presented method, the calculated temporal derivative of the velocity field would lead to a solution that fulfills the boundary condition, but violates the divergence-free condition. The two conditions cannot be fulfilled simultaneously that is the consequence of the given initial velocity field violating the compatibility condition.

2.4. Similar examples that fulfill the compatibility condition

Some further analysis was done to investigate, which similar initial fields fulfill the condition and lead therefore to a regular solution. The velocity field components have the form

\[ u_{i,0} = a_{i}(y) \cos(\alpha x + \beta z) + b_{i}(y) \sin(\alpha x + \beta z) \]

(21)

in all cases, where \( a(y) \) and \( b(y) \) are polynomial functions. In this form, the periodic boundary conditions (4) and (5) are automatically fulfilled. The polynomials \( a(y) \) and \( b(y) \) must fulfill the wall boundary condition (6) and the continuity equation (1). A possible choice for \( a(y) \) and \( b(y) \) is the form \( c(y) = \tilde{c}(y)(y^2 - 1) \) for \( u_{1,0} \) and \( c(y) = \tilde{c}(y)(y^2 - 1)^2 \) for \( u_{2,0} \) to fulfill the boundary condition where \( c(y) \) means \( a(y) \) or \( b(y) \) and \( \tilde{c}(y) \) is an arbitrary polynomial. The last velocity component \( u_{3,0} \) cannot be arbitrary, it can be determined simply using the continuity equation (1).

After analyzing multiple initial velocity fields in the proposed form, the following conclusions can be drawn. In the case of \( u_{2,0} = 0 \), the solution of equation (12) is divergence-free in every case meaning that the solution is regular. In the case of \( u_{2,0} \neq 0 \), many attempts were made to obtain an analytical velocity field that fulfills the compatibility condition. The procedure failed since increasing the order of the polynomials leads to a very complicated analytical problem that cannot be solved by hand or on a personal computer. On the other hand, the equations can be easily solved numerically. In the following, an initial velocity field in the proposed polynomial and periodic form is searched which has a regular solution at \( t = 0 \) and has wall-normal velocity components.

First, the asymptotic eigenmodes of the corresponding Orr–Sommerfeld problem are used as initial conditions. They are the solution of the linearised Navier–Stokes equation where the base flow is the well-known Poiseuille flow. The eigenfunctions are obtained with the method of Juniper et al (2013), where the domain is discretized with \( N = 40 \) Chebyshev polynomials. These eigenfunctions have the proposed form (21) and have a wall-normal component. In this case, a spatially non-oscillating base flow, the Poiseuille flow is part of the initial velocity field, beside the wavelike velocity field in the proposed form (21) that is usually called a perturbation.
The smooth temporal evolution of the velocity field is known, if the amplitude of the perturbation is low and the linearity assumption is reasonable. This means that the solution must be regular at \( t = 0 \). It has been verified that these eigenvectors fulfill the compatibility condition and the divergence of the temporal derivative of the velocity field is zero. The question arises whether the compatibility condition holds for the non-linear Navier–Stokes equations too. After increasing the perturbation amplitude to reasonable finite values, the condition is numerically still fulfilled. These initial conditions therefore provide a regular solution for the Navier–Stokes problem.

Another approach was followed to find an initial velocity field that has wall-normal component and fulfills the compatibility condition. The proposed product of polynomial and trigonometric functions (21) was used, the order of polynomial \( \tilde{c}(y) \) was set to \( N_p \)

\[
\tilde{c}(y) = \sum_{i=0}^{N_p} \tilde{k}_i y^i. 
\]  

The steps of finding an initial velocity field fulfilling the compatibility conditions were the following.

(a) Initializing the \( \tilde{k}_i \) coefficients by random values in equation (22), for obtaining the \( a(y) \) and \( b(y) \) polynomials of the first and second velocity components. (The \( j \)th polynomial coefficients of the \( i \)th Fourier coefficient of the \( i \)th velocity component are denoted by \( k_{a,y,j,l} \) and \( k_{b,y,j,l} \). In the subscript \( a \) or \( b \) refers to the Fourier component, \( y \) refers to the velocity component.)

(b) Calculating the Fourier coefficients of the third velocity component by using the continuity equation (1).

(c) Evaluating the right hand side of the Poisson equation (12), \( f_j \) that has the form (A4).

(d) Solving the Poisson equation (12) for \( \partial a_i / \partial t \) in the form (A20), as described in appendix A.

(e) Calculating the divergence of \( \partial a_i / \partial t \) in a similar form to equation (A20).

(f) Evaluating the norm of the Fourier coefficients of the divergence of \( \partial a_i / \partial t \), divided by the norm of the Fourier coefficients of the velocity components,

\[
r = \frac{||a_{d,j}|| + ||b_{d,j}||}{||a_{w,j}|| + ||b_{w,j}||}. 
\]  

(g) Minimizing \( r \) by varying the \( k_i \) coefficients for each velocity component using an optimization algorithm and repeating all steps from step (b). If minimization fails, go to step (a).

Steps (b)–(f) were carried out using the same Chebyshev collocation method. The residual norm was evaluated simply by taking the norm of Fourier coefficients at the grid points. Step (g) was solved by the non-linear optimization \( \text{fsolve} \) function built in MATLAB requiring the residual to be smaller than \( 10^{-8} \). Multiple solutions were obtained; an example is given at \( N_p = 4, \alpha = 1, \beta = 1, Re = 80 \): 

\[
a_{a_1} = (0.8055y^4 + 0.5767y^3 + 0.1829y^2 + 0.2399y + 0.8865)(y^2 - 1) \quad (24) 
\]

\[
b_{a_1} = (0.02868y^4 + 0.4899y^3 + 0.1679y^2 + 0.9787y + 0.7127)(y^2 - 1) \quad (25) 
\]

\[
a_{a_2} = (0.02541y^4 - 0.1133y^3 - 0.04897y^2 + 0.2675y + 0.01717)(y^2 - 1)^2 \quad (26) 
\]

\[
b_{a_2} = (0.2172y^4 + 0.01326y^3 - 0.4185y^2 - 0.0313y + 0.1468)(y^2 - 1)^2 \quad (27) 
\]
These functions (24)–(27) can be used to check numerical codes that cannot handle the problem of irregularity of the solution at the initial time. In this way, other velocity fields can be obtained that have a wall-normal component and still fulfill the compatibility condition. Although these calculations were carried out using numerical techniques, there exist probably analytical initial velocity fields too, where \( u_{2,0} \neq 0 \), and the corresponding Navier–Stokes equation has a regular solution at \( t = 0 \).

3. Conclusion

Solving the Navier–Stokes equation is a challenging problem. In the last century, mathematicians proved that a smooth solution exists for a finite time. However, in the case of physically relevant bounded domains, the solution is regular at the initial time if and only if the compatibility condition holds. In this paper, an analytic example is given in a channel flow, when the initial velocity field is smooth and divergence-free. At the same time, it does not fulfill the compatibility condition, and the corresponding Navier–Stokes equation has no regular solution at \( t = 0^+ \). According to the authors’ best knowledge, this is the first analytic example in the case of a three-dimensional flow-field where the irregularity of the solution was demonstrated analytically. Further single wave functions were investigated and all of them violated the conditions. Therefore, attempts have been made to construct an initial velocity field which does not contradict the condition. In the absence of a wall-normal velocity component, the condition is always fulfilled in the presented configuration. Some further numerical calculations suggest that there are velocity fields with a non-zero wall-normal component that also fulfill the condition. An example is given in the paper. However, we have not been able to construct such a field analytically.

The results of the paper are direct consequences of Temam’s (1982) mathematical proof about the regularity of the solution. Yet, the physical consequence is surprising. There exist many mathematically smooth velocity fields that fulfill the boundary conditions and the continuity equation, but violate the governing equation in a non-trivial way. In these cases, the Navier–Stokes equation has no regular solution at the initial time and this may have serious consequences for the initial part of time-dependent simulations. However, as Synge (1936) pointed out, there is an instantaneous adjustment in the spatial derivatives of the velocity field and the condition is satisfied as the solution evolves.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Appendix A. The calculation of the temporal derivative in an analytic example

First, the curl of the initial velocity (18)–(20) is calculated:

\[
\omega_{1,0} = \frac{\sin (\alpha x + \beta z) \left( \beta^2 y^4 - 2 \beta^2 y^2 + \beta^2 - 12 y^2 + 4 \right)}{\beta} \quad \text{(A1)}
\]

\[
\omega_{2,0} = \frac{4 \alpha y \cos (\alpha x + \beta z) \left( y^2 - 1 \right)}{\beta} \quad \text{(A2)}
\]

\[
\omega_{3,0} = -\alpha \sin (\alpha x + \beta z) \left( y^2 - 1 \right)^2 \quad \text{(A3)}
\]

The temporal derivative of the vorticity (\(\frac{\partial \omega_i}{\partial t}\)|_{t=0}) is obtained from equation (9). These terms are not shown here. Then the curl of \(-\partial \omega_i / \partial t\)|_{t=0} is calculated according to equation (13). It is convenient to express the term in a Fourier series in the following form:

\[
f_i(y) = \sum_{j=1}^{2} \left( A_{ij}(y) \cos \left( j(\alpha x + \beta z) \right) + B_{ij}(y) \sin \left( j(\alpha x + \beta z) \right) \right),
\]

where \(A_{ij}(y), B_{ij}(y)\) are polynomials. (It is noted that in a general case the terms \(\cos \left( j(\alpha x - \beta z) \right)\) and \(\sin \left( j(\alpha x - \beta z) \right)\) from the Fourier series cannot be neglected but in the case of the presented initial velocity field their coefficients would be zero. This statement is valid for each calculation here.)

The coefficients are in the presented case:

\[
A_{1,0} = 0 \quad \text{(A5)}
\]

\[
A_{1,1} = 0 \quad \text{(A6)}
\]

\[
B_{1,1} = 0 \quad \text{(A7)}
\]

\[
A_{1,2} = 0 \quad \text{(A8)}
\]

\[
B_{1,2} = -8 \alpha \left( y^2 - 1 \right)^2 \left( y^2 + 1 \right) \quad \text{(A9)}
\]

\[
A_{2,0} = 0 \quad \text{(A10)}
\]

\[
A_{2,1} = -\frac{1}{\text{Re}} \left( \left( -\alpha^4 - 2 \alpha^2 \beta^2 - \beta^4 \right) y^4 
+ \left( 2 \alpha^4 + 4 \alpha^2 \beta^2 + 24 \alpha^2 + 2 \beta^4 + 24 \beta^2 \right) y^2 
- \alpha^4 - 2 \alpha^2 \beta^2 - 8 \alpha^2 - \beta^4 - 8 \beta^2 - 24 \right) \quad \text{(A11)}
\]

\[
B_{2,1} = 0 \quad \text{(A12)}
\]

\[
A_{2,2} = 8 y \left( 3 y^2 + 1 \right) \left( y - 1 \right) \left( y + 1 \right) \quad \text{(A13)}
\]

\[
B_{2,2} = 0 \quad \text{(A14)}
\]

\[
A_{3,0} = 0 \quad \text{(A15)}
\]

\[
A_{3,1} = 0 \quad \text{(A16)}
\]

\[
B_{3,1} = \frac{4 y \left( \alpha^2 + \beta^2 \right) \left( -\alpha^2 \beta^2 + \alpha^2 y^2 + \beta^2 y^2 + 12 \right)}{\text{Re} \beta} \quad \text{(A17)}
\]
\[ A_{3,2} = 0 \]  \hspace{1cm} (A18)

\[ B_{3,2} = \frac{4 \left( 2\alpha^2 y^6 - 2\alpha^2 y^4 - 2\alpha^2 y^2 + 2\alpha^2 - 15 y^4 + 6 y^2 + 1 \right)}{\beta} \]  \hspace{1cm} (A19)

The temporal derivative of the velocity must have a similar form:

\[ \frac{\partial u_k}{\partial t} \bigg|_{t=0} = a_{k,0}(y) + \sum_{j=1}^{2} \left\{ a_{k,j}(y) \cos \left( j(\alpha x + \beta z) \right) + b_{k,j}(y) \sin \left( j(\alpha x + \beta z) \right) \right\}. \]  \hspace{1cm} (A20)

Its Laplacian is

\[ \frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{\partial u_k}{\partial t} \bigg|_{t=0} \right) = \frac{d^2 a_{k,0}(y)}{dy^2} + \sum_{j=1}^{2} \left( -j^2 (\alpha^2 + \beta^2) a_{k,j}(y) + \frac{d^2 a_{k,j}(y)}{dy^2} \right) \times \cos \left( j(\alpha x + \beta z) \right) \\
+ \sum_{j=1}^{2} \left( -j^2 (\alpha^2 + \beta^2) b_{k,j}(y) + \frac{d^2 b_{k,j}(y)}{dy^2} \right) \times \sin \left( j(\alpha x + \beta z) \right). \]  \hspace{1cm} (A21)

The solution of the Poisson equation (12) for the temporal derivative of velocity is equivalent to solving \((A4) = (A21)\). The five terms (constant, sine, cosine, harmonics) in the three equations can be treated separately. The original Poisson equations become five independent 1D boundary value problems for each component of the vector field in the series expansion. For example:

\[ \frac{d^2 a_{1,0}(y)}{dy^2} = A_{1,0}(y) \]  \hspace{1cm} (A22)

or

\[ (-\alpha^2 - \beta^2) a_{1,1}(y) + \frac{d^2 a_{1,1}(y)}{dy^2} = A_{1,1}(y) \]  \hspace{1cm} (A23)

The periodic boundary conditions are automatically fulfilled because of the form of the expressions. The no-slip boundary condition (6) can be written as \( a_{k,j}(y = 1) = a_{k,j}(y = -1) = 0 \) for \( j = \{0, 1, 2\} \) and similarly, \( b_{k,j}(y = 1) = b_{k,j}(y = -1) = 0 \) for \( j = \{1, 2\} \). The 15 ordinary differential equations were solved using Matlab symbolic and \( a_{k,j}, b_{k,j} \) polynomials are obtained. The solutions of the Poisson equations are:

\[ a_{1,0} = 0 \]  \hspace{1cm} (A24)

\[ a_{1,1} = 0 \]  \hspace{1cm} (A25)

\[ b_{1,1} = 0 \]  \hspace{1cm} (A26)

\[ a_{1,2} = 0 \]  \hspace{1cm} (A27)
\[ b_{1,2} = \frac{2\alpha y^6}{\alpha^2 + \beta^2} - \frac{\alpha (-4\alpha^6 - 12\alpha^4\beta^2 + 2\alpha^4 - 12\alpha^2\beta^4 + 4\alpha^2\beta^2 + 6\alpha^2 - 4\beta^4 + 2\beta^2 + 6\beta^2 - 45)}{2(\alpha^2 + \beta^2)^3} \]
\[ - \frac{\alpha y^4 (2\alpha^2 + 2\beta^2 - 15)}{(\alpha^2 + \beta^2)^3} - \frac{\alpha y^2 (2\alpha^4 + 4\alpha^2\beta^2 + 6\alpha^2 + 2\beta^4 + 6\beta^2 - 45)}{(\alpha^2 + \beta^2)^3} \]
\[ - \frac{\alpha e^{-2\sqrt{\alpha^2 + \beta^2}} (16\alpha^4 + 32\alpha^2\beta^2 + 84\alpha^2 + 16\beta^4 + 84\beta^2 + 45)}{2 (e^{-2\sqrt{\alpha^2 + \beta^2}} + e^{2\sqrt{\alpha^2 + \beta^2}}) (\alpha^2 + \beta^2)^3} \]
\[ - \frac{\alpha e^{2\sqrt{\alpha^2 + \beta^2}} (16\alpha^4 + 32\alpha^2\beta^2 + 84\alpha^2 + 16\beta^4 + 84\beta^2 + 45)}{2 (e^{-2\sqrt{\alpha^2 + \beta^2}} + e^{2\sqrt{\alpha^2 + \beta^2}}) (\alpha^2 + \beta^2)^3} \]
\[ (A28) \]
\[ a_{2,0} = 0 \] (A29)
\[ a_{2,1} = \frac{2y^2 (\alpha^2 + \beta^2 + 6)}{Re} - \frac{8e^{(\gamma+1)\sqrt{\alpha^2 + \beta^2}}}{Re} \frac{y^4 (\alpha^2 + \beta^2)}{e^{2\sqrt{\alpha^2 + \beta^2} + 1}} \]
\[ - \frac{\alpha^2 + \beta^2 + 4}{Re} - \frac{8e^{-y\sqrt{\alpha^2 + \beta^2}}}{Re} \frac{y^4 (\alpha^2 + \beta^2)}{e^{-\sqrt{\alpha^2 + \beta^2} + e^{\sqrt{\alpha^2 + \beta^2}}}} \] (A30)
\[ b_{2,1} = 0 \] (A31)
\[ a_{2,2} = \frac{2y^3 (2\alpha^2 + 2\beta^2 - 15)}{(\alpha^2 + \beta^2)^3} - \frac{6y^5}{\alpha^2 + \beta^2} \]
\[ + \frac{y (2\alpha^4 + 4\alpha^2\beta^2 + 6\alpha^2 + 2\beta^4 + 6\beta^2 - 45)}{(\alpha^2 + \beta^2)^3} \]
\[ + \frac{3e^{-2y\sqrt{\alpha^2 + \beta^2}} (8\alpha^2 + 8\beta^2 + 15)}{(e^{-2\sqrt{\alpha^2 + \beta^2}} - e^{2\sqrt{\alpha^2 + \beta^2}}) (\alpha^2 + \beta^2)^3} \]
\[ - \frac{3e^{2y\sqrt{\alpha^2 + \beta^2}} (8\alpha^2 + 8\beta^2 + 15)}{(e^{-2\sqrt{\alpha^2 + \beta^2}} - e^{2\sqrt{\alpha^2 + \beta^2}}) (\alpha^2 + \beta^2)^3} \] (A32)
\[ b_{2,2} = 0 \] (A33)
\[ a_{3,0} = 0 \] (A34)
\[ a_{3,1} = \frac{2y^2 (\alpha^2 + \beta^2 + 6)}{Re} - \frac{8e^{(\gamma+1)\sqrt{\alpha^2 + \beta^2}}}{Re} \frac{y^4 (\alpha^2 + \beta^2)}{e^{2\sqrt{\alpha^2 + \beta^2} + 1}} \]
\[ - \frac{\alpha^2 + \beta^2 + 4}{Re} - \frac{8e^{-y\sqrt{\alpha^2 + \beta^2}}}{Re} \frac{y^4 (\alpha^2 + \beta^2)}{e^{-\sqrt{\alpha^2 + \beta^2} + e^{\sqrt{\alpha^2 + \beta^2}}}} \] (A35)
\[ b_{3,1} = 0 \] (A36)
\[ a_{3,2} = \frac{2y^3 (2\alpha^2 + 2\beta^2 - 15)}{(\alpha^2 + \beta^2)^3} - \frac{6y^2}{(\alpha^2 + \beta^2)^2} + \frac{y (2\alpha^3 + 4\alpha^2\beta + 6\alpha^2 + 2\beta^2 + 6\beta^2 - 45)}{(\alpha^2 + \beta^2)^3} \]

\[ + \frac{3e^{-2y\sqrt{\alpha^2 + \beta^2}} (8\alpha^2 + 8\beta^2 + 15)}{(\alpha^2 + \beta^2)^3} - \frac{3e^{2y\sqrt{\alpha^2 + \beta^2}} (8\alpha^2 + 8\beta^2 + 15)}{(\alpha^2 + \beta^2)^3} \]

\[ = 3e^{-2y\sqrt{\alpha^2 + \beta^2}} (8\alpha^2 + 8\beta^2 + 15) \]

\[ \left( e^{-2\sqrt{\alpha^2 + \beta^2}} - e^{2\sqrt{\alpha^2 + \beta^2}} \right) (\alpha^2 + \beta^2)^3 \]

\[ b_{3,2} = 0. \]  

(A37)

The divergence of \( \frac{\partial u_a}{\partial r} \) (equation (A20)) is taken, its Fourier coefficients are:

\[ da_0 = 0 \]  

(A39)

\[ da_1 = \frac{8e^{-y\sqrt{\alpha^2 + \beta^2}} \left( (e^{2y+1}) \sqrt{\alpha^2 + \beta^2} - e^{\sqrt{\alpha^2 + \beta^2}} \right) C_1}{Re (e^{4\sqrt{\alpha^2 + \beta^2}} - 1)} \]  

(A40)

\[ C_1 = 3e^{2\sqrt{\alpha^2 + \beta^2}} + \sqrt{\alpha^2 + \beta^2} - e^{2\sqrt{\alpha^2 + \beta^2}} \sqrt{\alpha^2 + \beta^2} + 3 \]  

(A41)

\[ db_1 = 0 \]  

(A42)

\[ da_2 = \frac{e^{-2(y-1)\sqrt{\alpha^2 + \beta^2}} \left( e^{2y\sqrt{\alpha^2 + \beta^2}} + 1 \right)}{(e^{4\sqrt{\alpha^2 + \beta^2}} - 1) (\alpha^2 + \beta^2)^3/2} C_2 \]  

(A43)

\[ C_2 = \left( 16e^{4\sqrt{\alpha^2 + \beta^2}} - 16 \right) (\alpha^2 + \beta^2)^3 - \left( \left( 90e^{4\sqrt{\alpha^2 + \beta^2}} + 90 \right) (\alpha^2 + \beta^2)^3/2 \right) \]

\[ + \alpha^2 \left( 45e^{4\sqrt{\alpha^2 + \beta^2}} - 45 \right) + \alpha^4 \left( 84e^{4\sqrt{\alpha^2 + \beta^2}} - 84 \right) \]

\[ + \beta^2 \left( 45e^{4\sqrt{\alpha^2 + \beta^2}} - 45 \right) + \beta^4 \left( 84e^{4\sqrt{\alpha^2 + \beta^2}} - 84 \right) \]

\[ - \alpha^2 \left( 48e^{4\sqrt{\alpha^2 + \beta^2}} + 48 \right) (\alpha^2 + \beta^2)^3/2 \]

\[ - \beta^2 \left( 48e^{4\sqrt{\alpha^2 + \beta^2}} + 48 \right) (\alpha^2 + \beta^2)^3/2 + \alpha^2 \beta^2 \left( 168e^{4\sqrt{\alpha^2 + \beta^2}} - 168 \right) \]  

(A44)

\[ db_2 = 0. \]  

(A45)

For the parameters \( \alpha = 1, \beta = 1, Re = 80 \) the divergence was plotted as the function of \( x, y \) at \( z = 0 \) in figure 3.
Appendix B. The analytical calculation of the compatibility condition

The compatibility condition is obtained similarly. The pressure Poisson equation (15) is solved using the Neumann boundary condition for the pressure (16). The compatibility condition (17) is evaluated at the walls \((y = \pm 1)\) only in the two main tangential directions: the streamwise and spanwise directions.

The first step of this calculation was the evaluation of the right hand side which has a similar form to equation (A4), but it is a scalar function in this case.

\[
-\frac{\partial u_{j,0}}{\partial x_i} \frac{\partial u_{i,0}}{\partial x_j} = \sum_{j=1}^{2} \left\{ c_{aj}(y) \cos \left( j(\alpha x + \beta z) \right) + c_{bj}(y) \sin \left( j(\alpha x + \beta z) \right) \right\},
\]

where \(c_{aj}(y)\) and \(c_{bj}(y)\) are the Fourier coefficients. The coefficients in the presented case are:

\[
\begin{align*}
c_{a0} &= -4 \left( y^2 - 1 \right)^2 \left( 7y^2 - 1 \right) \\
c_{a1} &= 0 \\
c_{b1} &= 0 \\
c_{a2} &= -4 \left( y^2 - 1 \right)^2 \left( y^2 + 1 \right) \\
c_{b2} &= 0.
\end{align*}
\]

Before solving the Poisson equation for the pressure, the boundary condition must be evaluated at the top \((y = 1)\) and bottom wall \((y = -1)\), since the boundary conditions are non-homogeneous in contrary to the calculation of the temporal derivative of the velocity. Furthermore, the Fourier coefficients of the boundary conditions are single constant values and not polynomial functions. The values at the top:

\[
\begin{align*}
b_{ta0} &= 0; \quad b_{ta1} = \frac{8}{\text{Re}}; \quad b_{tb1} = 0; \quad b_{ta2} = 0; \quad b_{tb2} = 0,
\end{align*}
\]

and at the bottom:

\[
\begin{align*}
b_{ba0} &= 0; \quad b_{ba1} = \frac{8}{\text{Re}}; \quad b_{bb1} = 0; \quad b_{ba2} = 0; \quad b_{bb2} = 0.
\end{align*}
\]

It can be seen that the Fourier coefficients of the pressure gradient at the top and bottom wall are the same in the presented calculation because of the symmetry property of the initial velocity field but in general they can be different. The next step is solving the Poisson equation for the pressure. The pressure is also assumed in a finite Fourier series

\[
p = p_{a0}(y) + \sum_{j=1}^{2} \left\{ p_{aj}(y) \cos \left( j(\alpha x + \beta z) \right) + p_{bj}(y) \sin \left( j(\alpha x + \beta z) \right) \right\}.
\]

For each polynomial coefficient a second order boundary value problem should be solved together with the two Neumann boundary conditions at the both ends. This leads to 5 separated problems. (It is mentioned that the solution of some differential equations is zero since both the right hand side and the boundary conditions are homogeneous in the presented case.) The first differential equation is
\[ \frac{d^2 p a_0(y)}{dy^2} = c a_0(y) \]  

(B10)

and the boundary conditions are

\[ \frac{d p a_0(y)}{dy} \bigg|_{y=1} = b t a_0; \quad \frac{d p a_0(y)}{dy} \bigg|_{y=-1} = b b a_0. \]  

(B11)

The next differential equation is

\[ (-\alpha^2 - \beta^2) p a_1(y) + \frac{d^2 p a_1(y)}{dy^2} = c a_1(y) \]  

(B12)

and the boundary conditions are

\[ \frac{d p a_1(y)}{dy} \bigg|_{y=1} = b t a_1; \quad \frac{d p a_1(y)}{dy} \bigg|_{y=-1} = b b a_1. \]  

(B13)

After solving the five equations, the Fourier coefficients of the pressure are in the case of the presented initial velocity field are:

\[ p a_0 = \frac{-y^8}{2} + 2y^6 - 3y^4 + 2y^2 + C_1 \]  

(B14)

\[ p a_1 = 8 e^{-(y-1)\sqrt{\alpha^2+\beta^2}} \left( e^{2y \sqrt{\alpha^2+\beta^2}} - 1 \right) \frac{\text{Re} \left( e^{2\sqrt{\alpha^2+\beta^2}} + 1 \right)}{\sqrt{\alpha^2 + \beta^2}} \]  

(B15)

\[ p b_1 = 0 \]  

(B16)

\[ p a_2 = \frac{y^6}{\alpha^2 + \beta^2} + \frac{3e^{\sqrt{\alpha^2+\beta^2}} (8\alpha^2 + 8\beta^2 + 15)}{2 \left( e^{2\sqrt{\alpha^2+\beta^2}} - e^{2\sqrt{\alpha^2+\beta^2}} \right) (\alpha^2 + \beta^2)^{7/2}} \]

\[ -4\alpha^2 - 12\alpha^2 \beta^2 + 2\alpha^4 - 12\alpha^2 \beta^4 + 4\alpha^2 \beta^2 + 6\alpha^2 - 4\beta^4 + 2\beta^4 + 6\beta^2 - 45 \]

\[ -\frac{y^4 (2\alpha^2 + 2\beta^2 - 15)}{2(\alpha^2 + \beta^2)^2} = \frac{y^2 (2\alpha^4 + 4\alpha^2 \beta^2 + 6\alpha^2 + 2\beta^4 + 6\beta^2 - 45)}{2(\alpha^2 + \beta^2)^4} \]

\[ + \frac{3e^{-2\sqrt{\alpha^2+\beta^2}} (8\alpha^2 + 8\beta^2 + 15)}{2 \left( e^{-2\sqrt{\alpha^2+\beta^2}} - e^{-2\sqrt{\alpha^2+\beta^2}} \right) (\alpha^2 + \beta^2)^{7/2}} \]  

(B17)

\[ p b_2 = 0. \]  

(B18)

Finally, the compatibility condition (17) is evaluated: the difference between the two sides of equation (17) is calculated. Due to the symmetry property of the example, the Fourier coefficients at the top and bottom walls are almost the same; only the signs are different in some components. Here, only the values at the top are presented. The Fourier coefficients of the difference at the top \((y = 1)\) evaluating the compatibility condition in the streamwise \((x)\) direction:

\[ CCx a_0 = 0 \]  

(B19)

\[ CCx a_1 = 0 \]  

(B20)

\[ CCx b_1 = \frac{8\alpha \left( e^{2\sqrt{\alpha^2+\beta^2}} - 1 \right)}{\text{Re} \left( e^{2\sqrt{\alpha^2+\beta^2}} + 1 \right) \sqrt{\alpha^2 + \beta^2}} \]  

(B21)

\[ CCx a_2 = 0 \]  

(B22)
\[ CC_{xb_2} = -2 \alpha C_3 \quad (B23) \]

\[ C_3 = \frac{2 \alpha^2 + 2 \beta^2 - 15}{2(\alpha^2 + \beta^2)^2} - \frac{1}{\alpha^2 + \beta^2} + \frac{2 \alpha^4 + 4 \alpha^2 \beta^2 + 6 \alpha^2 + 2 \beta^4 + 6 \beta^2 - 45}{2(\alpha^2 + \beta^2)^3} \]
\[ + \frac{-4 \alpha^6 - 12 \alpha^4 \beta^2 + 2 \alpha^4 - 12 \alpha^2 \beta^4 + 4 \alpha^2 \beta^2 + 6 \alpha^2 - 4 \beta^6 + 2 \beta^4 + 6 \beta^2 - 45}{4(\alpha^2 + \beta^2)^4} \]
\[ + \frac{24 \alpha^2 + 24 \beta^2 + 45}{2\left(e^{2\sqrt{\alpha^2+\beta^2}} - 1\right)(\alpha^2 + \beta^2)^{3/2}} + \frac{3 e^{4\sqrt{\alpha^2+\beta^2}} \left(8 \alpha^2 + 8 \beta^2 + 15\right)}{2\left(e^{2\sqrt{\alpha^2+\beta^2}} + 1\right)(\alpha^2 + \beta^2)^{3/2}} \]
\[ (B24) \]

\[ (B25) \]

The difference coefficients at the top \((y = 1)\) evaluating the compatibility condition in the spanwise \((z)\) direction:

\[ CC_{za_0} = 0 \quad (B27) \]

\[ CC_{za_1} = 0 \quad (B28) \]

\[ CC_{zb_1} = \frac{8 \beta \left(e^{2\sqrt{\alpha^2+\beta^2}} - 1\right)}{\text{Re} \left(e^{2\sqrt{\alpha^2+\beta^2}} + 1\right) \sqrt{\alpha^2 + \beta^2}} - \frac{24}{\text{Re} \beta} \quad (B29) \]

\[ CC_{za_2} = 0 \quad (B30) \]

\[ CC_{zb_2} = -2 \beta C_3. \quad (B31) \]

The coefficients are zero if the compatibility condition is fulfilled. However, it is not the case here.

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**References**

Falsaperla P, Giacobbe A and Mulone G 2019 Nonlinear stability results for plane Couette and Poiseuille flows Phys. Rev. E 100 013113

Gallavotti G 2002 *Foundations of Fluid Dynamics* (Berlin: Springer)

Gresho P M and Sani R L 1987 On pressure boundary conditions for the incompressible Navier–Stokes equations Int. J. Numer. Methods Fluids 7 1111–45

Heywood J G 1980 Auxiliary flux and pressure conditions for Navier–Stokes problems Approximation Methods for Navier–Stokes Problems ed R Rautmann (Berlin: Springer) pp 223–34

Johnston H and Liu J-G 2004 Accurate, stable and efficient Navier–Stokes solvers based on explicit treatment of the pressure term J. Comput. Phys. 199 221–59

Józsa T I 2019 Analytical solutions of incompressible laminar channel and pipe flows driven by in-plane wall oscillations Phys. Fluids 31 083605
Juniper M P, Hanifi A and Theofilis V 2013 Modal stability theory: lecture notes from the FLOW-NORDITA summer school on advanced instability methods for complex flows, Stockholm, Sweden, 2013 Appl. Mech. Rev. 66 024804
Ladyzhenskiaia O A 1969 The Mathematical Theory of Viscous Incompressible Flow 2nd edn (New York: Gordon and Breach)
Nagy P T 2022 Enstrophy change of the Reynolds–Orr solution in channel flow Phys. Rev. E 105 035108
Orr W M 1907 The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. Part II: a viscous liquid Proc. R. Ir. Acad. A 27 69–138 (available at: www.jstor.org/stable/20490591)
Orszag S A and Israeli M 1974 Numerical simulation of viscous incompressible flows Annu. Rev. Fluid Mech. 6 281–318
Synge J L 1936 Conditions satisfied by the vorticity and the stream-function in a viscous liquid moving in two dimensions between fixed parallel planes Proc. London Math. Soc. s2-40 23–36
Temam R 1982 Behaviour at time $t = 0$ of the solutions of semi-linear evolution equations J. Differ. Equ. 43 73–92
Temam R 2006 Suitable initial conditions J. Comput. Phys. 218 443–50