Choosing on sequences

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Abstract

The standard economic model of choice assumes that a decision maker chooses from sets of alternatives. A new branch of literature has considered the problem of choosing from lists i.e. ordered sets. In this paper, we propose a new framework that considers choice from infinite sequences. Our framework provides a natural way to model decision making in settings where choice relies on a string of recommendations. We introduce three broad classes of choice rules in this framework. Our main result shows that bounded attention is due to the continuity of the choice functions with respect to a natural topology. We introduce some natural choice rules in this framework and provide their axiomatic characterizations. Finally, we introduce the notion of computability of a choice function using Turing machines and show that computable choice rules can be implemented by a finite automaton.

Keywords: Bounded Attention, Infinite Sequences, Computability, Satisficing

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1 Introduction

The standard economic model of choice assumes that decision maker chooses from sets of alternatives. A new branch of literature has considered the problem of choosing from lists i.e. ordered sets (Rubinstein and Salant (2006)). We extend the model to incorporate infinite lists i.e. sequences. Our motivation is twofold: Firstly, we aim to provide a framework to examine situations where the lists could be “large” and have repetitions. Secondly, we aim to establish a notion of “bounded” attention when alternatives appear in some order. One important application of our framework is when a decision maker (DM) seeks recommendations before making a choice. There are a variety of settings where DMs rely on recommendations to make choices. E-commerce websites like Amazon and entertainment websites like Netflix and YouTube present its users with recommendations. There has been sufficient evidence that these recommendations

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influence choices and companies spend considerable resources in designing these recommendations (see Cheung and Masatlioglu (2021)). To the end users, these recommendations often appear in the form of streams of alternatives. Recommender systems provide personalised recommendations in order to influence choices. On the other hand, individuals also rely on recommendations from their relatives, peer groups etc. for making choices such as which movie to watch or which investment option to go for. Often, these recommendations are received sequentially. A natural way to model decision making in these contexts is by using sequences as primitives vis-à-vis sets. Any string of recommendations can be viewed as a sequence of alternatives that the DM observes in order to make a choice. In the standard choice theory, the observable data consists of choices over sets. It is general in its scope and provides fruitful avenues to study many heuristics as the following examples might indicate:

**Example 1.** A DM wants to watch a movie and relies on the recommendations provided by Netflix to select a movie. She has a threshold frequency and the first movie to reach the threshold frequency in the string of recommendations is selected by her.

**Example 2.** A DM wants to decide on a course of treatment for a knee injury. She wishes to avoid surgery and go for a non-invasive treatment. She relies on the opinions provided to her by doctors whom she approaches sequentially. However, if “sufficient” number of surgery recommendations are provided, she wishes to go for it.

**Example 3.** Consider a computer program that takes as inputs a fixed number $k$ of first “entries” for a coding competition. From the received applications, it picks the most efficient code (assuming there exists a unique such code).

**Example 4.** A rating agency has to decide on a country’s performance rating based on its future projections which are in the form of a sequence of performances. It decides on the rating based on the performance with the highest limiting frequency.

**Example 5.** A DM has to choose a partner based on repeated interactions with a set of potential partners. She has a fixed attention span i.e. finite first-$k$ interactions. She attaches a “utility” number to each alternative and has some threshold in her mind. She chooses the first potential partner within the first-$k$ interaction whose utility number exceeds the threshold. Otherwise she chooses the partner with the maximum utility value from the first-$k$ interactions.

We study choice functions over sequences. A “menu” in our setup corresponds to a sequence of alternatives. Despite the set of alternatives being finite, the set of menus i.e. the set of sequences is infinite which distinguishes our model from that on lists. We introduce two broad classes of choice rules which we term stopping rules and tail-regarding rules. As the name indicates, stopping rules corresponds to a stopping time for a given sequence. Intuitively, the idea is that the DM decides on what to choose within a finite amount of time. This stopping time could possibly vary across sequences. Stopping rules encompass a wide range of behavior as the examples above indicate. We show that these can be
characterized using continuity with respect to a natural topology—the product topology—defined on the set of all sequences and the discrete topology on the set of all alternatives. A subclass of stopping rules pins down the idea of bounded attention by requiring a finite stopping time for all sequences. We term these uniform stopping rules. Interestingly, continuity is sufficient to characterize the class of uniform stopping rules. This result holds due to the finiteness of the set of alternatives and we provide a diagonalization argument to establish the result. Within the class of stopping rules, we adapt the idea of satisficing and introduce two variants: ordinal satisficing and cardinal satisficing. In contrast to the result on choice over sets (Rubinstein (2012)), satisficing is behaviorally not equivalent to “rational” choice in our framework. In fact, it turns out that rational behavior is a special case of ordinal satisficing.

Cardinal satisficing captures the idea that DM may inherently have scores attached to the alternatives and more occurrences of an alternative increases its cumulative score. With a threshold score in mind, DM would choose the first alternative whose cumulative score crosses the threshold. In this way, the order of the sequence can significantly affect the final choice. To behaviorally characterize these models, we introduce the notions of sufficiency and minimal sufficiency. The underlying idea is that a DM “makes up” his mind after viewing a certain minimum number of alternatives in a given sequence. Such segments are “sufficient” to implement the choice from the sequence. These notions form the basis of our axioms that characterize the satisficing models in our framework.

In addition to the preference over alternatives, two natural factors in determining choice in our framework are “frequency” and “position”. An alternative may be more attractive if it is located initially in a sequence than another alternative that appears later. On the other hand, the number of occurrences can affect the choice as highlighted in the idea of cardinal satisficing. We introduce another broad class of heuristics that focus on both these factors. We term these as configuration-dependent rules. These rules are characterized by a neutrality axiom which states that the “identity” of alternatives is irrelevant in making choices. Within this broad class we provide a characterization of what we term rational configuration-dependent rules. The idea behind such rules is that the DM has a underlying preference
order over the configurations of alternatives in a sequence. For instance, an alternative that is spaced out evenly may be preferred over another one that is not. Or an alternative may be preferred over another one if it has more occurrences in, say, the first 10 positions. Finally, we introduce the notion of “computability” of a choice function. We call a choice function computable if a DM can deploy a Turing machine—an abstraction of a modern computer—to make choices. We show that any computable function is implementable via a finite automaton—a less powerful model of computation.

1.1 Related Literature

The idea that a DM may observe alternatives in the form of a list i.e. an ordered set was first modeled by Rubinstein and Salant (2006). Following that, a variety of models based on their framework have been introduced in the literature. Satisficing, first introduced by Simon (1955) has been an influential idea in choice theory and many adaptations have been done in the literature. The list setup provides a natural framework to study satisficing behavior. Kovach and Ülkü (2020) introduced one such model. In their model, the DM makes her choice in two stages. In the first stage, she searches through the list till she sees $k$ alternatives. In the second stage, she chooses from the alternatives she has seen. Another adaptation of satisficing was introduced in Manzini et al. (2019). Their model is interpreted as one of approval against choice. Since our framework is a generalization of lists, satisficing heuristics are a natural choice of study.

As discussed above, an important application of our framework is when alternatives come in the form of streams of recommendations. Our paper is not the first one that interprets alternatives as recommendations. Cheung and Masatlioglu (2021) have introduced a model of decision making under recommendation. However, their setup is on sets and hence complements our contribution. The idea that recommendation influences choices has been widely accepted. Our object of interest— infinite sequences—in the context of choices has been previously studied by Caplin and Dean (2011). Our model differs from their model in terms of incorporating sequences in the domain of choice functions. They enrich the observable choice data by incorporating sequences as the output of the choice function and interpret these sequences as provisional choices of the DM with contemplation time.

Computability has been a recurring theme in economic theory. Computational models of behavior as well as issues of computational complexity have been studied in the settings of infinitely repeated games, contracting, mechanism design etc. (see for instance Rubinstein (1986), Abreu and Rubinstein (1988), Jakobsen (2020), Nisan and Ronen (2001)). We study computable aspects of decision making in our framework. Some notable papers that have studied computability in a choice theoretic framework include Salant (2011), Richter and Wong (1999) and Apesteguia and Ballester (2010).

The organization of the paper is as follows. In the next section we introduce the framework and the notions of stopping and tail-regarding rules. In section 3, we introduce a natural topology on the
domain and co-domain of the choice functions and provide an axiomatic characterization of these rules. In section 4, we introduce two variants of satisficing in our framework and provide their characterization. We introduce configuration-dependent rules in section 5 and discuss computational aspects of choice in section 6.

2 The Framework

2.1 Notation and Definitions

Let $X$ be a finite set of alternatives. The object of interest is the set of sequences containing the alternatives of $X$. Let $S$ be the collection of all such sequences i.e. $S = \{ S : S : \mathbb{N} \to X \}$. Denote by $S(i)$, the $i^{th}$ element of the sequence $S$. The DM in our framework is endowed with a choice function which gives a unique choice for any given sequence.

Definition 1. A choice function on sequences is a map $c : S \to X$ such that $c(S) = S(i)$ for some $i \in \mathbb{N}$.

We denote by $S|_k$ the segment with that comprises of the first-$k$ elements of the sequence $S$ and by $S|_k.T$ a “concatenation” of the segment $S|_k$ with some sequence $T \in S$ i.e. $S|_k.T \in S$ with $[S|_k.T](i) = S(i)$ for $i \in \{1, 2, \ldots, k\}$ and $[S|_k.T](k + i) = T(i)$ for $i \in \mathbb{N}$. For a sequence $S$, denote by $S|^{k} \in S$ the subsequence generated from its $k + 1^{th}$ alternative onwards i.e. $[S|^{k}](i) = S(k + i)$ for all $i \in \mathbb{N}$.

2.2 Stopping rules and Tail-regarding rules

Two important classes of choice functions on sequences are what we call stopping rules and tail-regarding rules. As the names suggest, the former are rules that depend on some finite segment of a given sequence to make decisions whereas the latter rules consider the infinite sequence to make decisions. Stopping rules are “reasonable” given the decision maker in question is an individual. For instance, a DM might consider only the first 10 recommendations before deciding which movie to watch or consult only 5 people before deciding which medical treatment to go for. However, the case where the DM is an entity such as a nation or a company that may exist forever, tail-regarding rules seem reasonable as well. In this paper, we examine stopping rules in greater detail. Now, we will define these rules formally.

Definition 2. A stopping rule $c$ is a choice function on sequences such that for every $S \in S$ there exists some $k \in \mathbb{N}$ such that $c(S) = c(S|_k.T)$ for all $T \in S$.

Stopping rules suggest that a DM needs to only look at finitely many terms to make a decision. However, this finite length of observation might vary across sequences. Examples 1-3 are stopping rules.

Definition 3. A tail-regarding rule $c$ is a choice function on sequences such that for every $S \in S$ and any $k \in \mathbb{N}$ $c(S) = c(T_k.S|^{k})$ for all $T \in S$. 

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Tail-regarding rules emphasize the importance of “tails” of a sequence in making a choice. Intuitively, they highlight the fact that a finite initial segment is “irrelevant” for the DM to make a choice. It is important to note that stopping and tail-regarding rules do not exhaust the space of choice functions on sequences as can be seen in Figure 1. Consider a choice rule where the DM picks a particular alternative, say $x$, if it appears in a sequence otherwise selects the first alternative appearing in the sequence. It is easy to see that this rule is neither a tail-regarding nor a stopping rule.

3 Characterization

Now, we will characterize the class of stopping rules. As it turns out, the only condition we require is continuity with respect to the the discrete topology on $X$ and the induced product topology defined on the set of all sequences. The product topology here has a behavioral interpretation. It says that a DM considers two sequences “close” or “similar” if their initial segments are the same. Continuity of the choice function then implies that she cannot display “jumps” in choices for close enough choice problems. This translates into the observation that after a certain finite segment has been observed by the DM, the “tail” of a sequence cannot affect the choice.

Theorem 1. Assume $S$ and $X$ are endowed with the product topology and the discrete topology respectively. Then, a choice function $c : S \rightarrow X$ is a stopping rule if and only if it is continuous.

Proof. First, we show what the product topology on $S$ looks like. We know that $\Pi_S$, the product topology, is the smallest topology with respect to which the projection maps are continuous. Consider any map $M : \{1, \ldots, N\} \rightarrow X$ where $N \in \mathbb{N}$ and define the set $B(M)$ as:

$$B(M) = \{S \in S : \text{ for all } i \in \{1, \ldots, N\}, \ S(i) = M(i)\}$$

Let $\mathcal{B}_S$ be the class of all such sets. Note that a for any $N \in \mathbb{N}$, the number of possible maps $M : \{1, \ldots N\} \rightarrow X$ is $|X|^N$. These sets are what can be interpreted as “open balls” in $S$. Let $\mathcal{T}_S$ be the class of unions of arbitrary subcollections of $\mathcal{B}_S$.

Lemma 1. $\mathcal{T}_S$ is the product topology on $S$

Proof. First, we show that $\mathcal{T}_S$ is indeed a topology over $S$. Notice that $\mathcal{T}_S$ is closed under arbitrary unions by definition. To show that it is closed under finite intersections, let $\bigcap_{i=1}^K B_i$ be a finite intersection such that $B_i \in \mathcal{T}_S$ for all $i \in \{1, \ldots K\}$. Note that each $B_i$ is a union of of some subcollection of $\mathcal{B}_S$ and therefore we can write $B_i = \bigcup_{j \in J_i} B_i^{j_i}$, with $J_i$ being some indexed set, where each $B_i^{j_i}$ corresponds to an “open ball” i.e. is a set of the form $B(M)$ for some $M : \{1, \ldots, N\} \rightarrow X$ and $N \in \mathbb{N}$. Using the
definition of $B(M)$, we know that there exist sets $A_1^i, A_2^i, \ldots$ with $A_j^i \subseteq X$ for all $i \in \mathbb{N}$ such that

$$B_i = \{ S \in \mathcal{S} : \text{for all } j \in \mathbb{N}, S(j) \in A_j^i \}$$

So, we can write $\bigcap_{i=1}^{K} B_i$ as

$$\bigcap_{i=1}^{K} B_i = \{ S \in \mathcal{S} : \text{for all } j \in \mathbb{N}, S(j) \in \bigcap_{i=1}^{K} A_j^i \}$$

Clearly, $\bigcap_{i=1}^{K} B_i = B(M)$ for some $M : \{1, \ldots, N\} \to X$ and $N \in \mathbb{N}$. Therefore, $\mathcal{T}_S$ is closed under finite intersection. Finally, $\mathcal{T}_S$ contains $\mathcal{S}$ and $\emptyset$ as its elements. That $\emptyset \in \mathcal{T}_S$ holds follows from the fact that $\emptyset$ is the union of elements from the empty subcollection of $\mathcal{B}_S$. Further, $\mathcal{S}$ is the union of elements from the full collection $\mathcal{B}_S$. Thus, $\mathcal{T}_S$ is a topology over $\mathcal{S}$.

Now, we argue: $\Pi_S \subseteq \mathcal{T}_S$. For this, fix an arbitrary $i, \pi \in \mathbb{N}$ and $A \subseteq X$. If $A = \emptyset$, then $\pi^{-1}(A) = \emptyset$. As $\emptyset \in \mathcal{T}_S$, $\pi^{-1}(A) \in \mathcal{T}_S$ if $A = \emptyset$. However, if $A \neq \emptyset$, then observe:

$$\pi^{-1}(A) = \bigcup \{ B(M) : M \in X^{\{1, \ldots, i\}} \ ; \ M(i) \in A \}.$$ 

Thus, if $A \neq \emptyset$, then $\pi^{-1}(A) \in \mathcal{T}_S$. That is, $\pi^{-1}(A) \in \mathcal{T}_S$ for every $A \subseteq X$. Hence, $\{ \pi^{-1}(A) : i \in \mathbb{N} ; A \subseteq X \} \subseteq \mathcal{T}_S$ and we have already shown that $\mathcal{T}_S$ is a topology over $\mathcal{S}$. Further, by definition, $\Pi_S$ is the smallest topology that satisfies $\{ \pi^{-1}(A) : i \in \mathbb{N} ; A \subseteq X \} \subseteq \Pi_S$. Therefore, we obtain: $\Pi_S \subseteq \mathcal{T}_S$.

Finally, we argue: $\mathcal{T}_S \subseteq \Pi_S$. For this, fix an arbitrary $I \in \mathbb{N}$ and consider an arbitrary map $M : \{1, \ldots, I\} \to X$. For each $i \in \{1, \ldots, I\}$, let $A_i := \{ M(i) \}$. Then, we have the following:

$$B(M) = \bigcap \{ \pi^{-1}(A_i) : i = 1, \ldots, I \}.$$ 

Since $\Pi_S$ is a topology and $\{ \pi^{-1}(A) : i \in \mathbb{N} ; A \subseteq X \} \subseteq \Pi_S$, it follows that $B(M) \in \Pi_S$. Thus, $\Pi_S$ is a topology over $\mathcal{S}$ such that $\mathcal{B}_S \subseteq \Pi_S$. Moreover, $\mathcal{T}_S$ is the smallest topology over $\mathcal{S}$ such that $\mathcal{B}_S \subseteq \mathcal{T}_S$. Hence, we conclude: $\mathcal{T}_S \subseteq \Pi_S$.

Now to show $c$ is a stopping rule if and only if it is continuous, first, assume that $c : \mathcal{S} \to X$ is continuous. Fix an arbitrary $S_* \in \mathcal{S}$ and let $y_{S_*} := c(S_*)$. Now, we know that $\{ y_{S_*} \}$ is open in the discrete topology over $X$. By continuity of the map $c$, the following set:

$$c^{-1}(\{ y_{S_*} \}) := \{ S \in \mathcal{S} : c(S) = y_{S_*} \}$$

satisfies $c^{-1}(\{ y_{S_*} \}) \in \Pi_S$. By the lemma above and the definition of $\mathcal{T}_S$, there exists $M : \{1, \ldots, k\} \to X$.
such that $S_e \in B(M) \subseteq c^{-1}([y_S,])$. Now, $S_e \in B(M)$ implies: $M = S_e |_k$ and $B(M) = \{S_e |_k \cdot T : T \in S\}$. Since $B(M) \subseteq c^{-1}([y_S,])$, it follows: $c(S_e |_k \cdot T) = y_S$ for all $T \in S$. Since $y_{S_e} = c(S_e)$ and $S_e$ was arbitrary, we have established: if the map $c : S \rightarrow Y$ is continuous, then $c$ is a stopping rule.

Now, assume that $c : S \rightarrow X$ is a stopping rule. Since $X$ has the topology $2^X$, we must argue that $c^{-1}(A) := \{S \in S : c(S) \in A\} \in \Pi_S$ for any $A \subseteq X$. Since $c^{-1}$ preserves arbitrary unions and $\Pi_S$ is closed under arbitrary unions, it is enough to argue that $c^{-1}(\{y\}) \in \Pi_S$ for any $y \in X$. So, fix an arbitrary $y_* \in X$. If $c^{-1}(\{y_*\}) = \emptyset$, then we have nothing more to argue as $\emptyset \in \Pi_S$. Hence, assume that $c^{-1}(\{y_*\}) \neq \emptyset$. Consider an arbitrary $S_e \in c^{-1}(\{y_*\})$. Since $c$ is a stopping rule, there exists $k(S_e) \in \mathbb{N}$ such that: $c(S) = y_*$ for every $S \in B(S_e |_{k(S_e)})$. This is because $B(S_e |_{k(S_e)}) = \{S_e |_{k(S_e)} \cdot T : T \in S\}$. Thus, we have:

$$\bigcup \{B(S_e |_{k(S)}) : S \in c^{-1}(\{y_*\})\} = c^{-1}(\{y_*\}).$$

Hence, $c^{-1}(\{y_*\}) \in \mathcal{T}_S$ by definition of $\mathcal{T}_S$. By lemma, it follows that $c^{-1}(\{y_*\}) \in \Pi_S$. Since $y_* \in Y$ was arbitrary, we have: $c^{-1}(A) \in \Pi_S$ for any $A \in 2^X$. Thus, if the $c : S \rightarrow X$ is a stopping rule, then it is continuous.

\*\*\* 3.1 Bounded Attention  

Within the class of stopping rules lie what we call \textit{uniform-stopping rules} or \textit{fixed-k rules}. As the name suggests, there exists a fixed $k \in \mathbb{N}$ that is “relevant” to make a choice from a sequence. Note that stopping rules give us a $k$ for every sequence and fixed $k$ rules give us a uniform bound , $k$, across all sequences.

**Definition 4.** A stopping rule is a uniform stopping rule if there exists a $k \in \mathbb{N}$ such that for all $S \in S$ and for all $T \in S$,

$$c(S) = c(S |_k.T)$$

Uniform stopping rules indicate the idea that the DM may have a “bounded” attention span or finite processing capability. There is a \textit{fixed} stage beyond which she does not go irrespective of the choice problem i.e. the sequence. While these rules are clearly a sub-class of stopping rules, it turns out that they are are infact equivalent to stopping rules.

**Theorem 2.** A choice function over sequences is a stopping rule if and only if it is a uniform stopping rule

**Proof.** A uniform stopping rule is by definition a stopping rule. So, we prove the converse. Let $c : S \rightarrow X$ be a stopping rule. Suppose, for the sake of contradiction, $c$ does not have a uniform stopping time. For each input $S \in S$, define $k_d(S) \in \mathbb{N}$ as follows:

$$k_d(S) := \inf\{k \in \mathbb{N} : (\forall T \in S)[c(S |_k.T) = c(S)]\}.$$

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$k_d(S)$ refers to the corresponding stopping time for the sequence $S$. The proof is organized in steps which are as follows.

**Step 1:** We iteratively define a sequence of pairs $\{(k_j, A_j)\}_{j \in \mathbb{N}}$, where $k_j \in \mathbb{N}$ and $A_j \subseteq S$, as follows:

1. Let $k_1 := \inf\{k_d(S) : S \in S\}$ and $A_1 := \{S \in S : k_d(S) = k_1\}$.
2. For any $j \in \mathbb{N} \setminus \{1\}$, assuming $(k_l, A_l)$ have already been defined for every $l \in \{1, \ldots, j-1\}$, let:
   
   $$k_j := \inf\{k_d(S) : S \in S \setminus \bigcup_{l=1}^{j-1} A_l\},$$
   
   and
   
   $$A_j := \{S \in S \setminus \bigcup_{l=1}^{j-1} A_l : k_d(S) = k_j\}.$$ 

The sets $A_j$ refer to the set of all the sequences for which the stopping time is $k_j$. From our supposition that $c$ is stopping rule and $c$ does not have a uniform stopping time, the following properties are immediate:

(a) For each $j \in \mathbb{N}$, $k_j \in \mathbb{N}$ and $A_j \neq \emptyset$.

(b) $k_1 < k_2 < \ldots < k_j < \ldots$ and so on.

(c) $\{A_j : j \in \mathbb{N}\}$ is a partition of $S$.

These properties shall be referred to in the rest of the argument.

**Step 2:** Pick an arbitrary $S_j \in A_j$ for every $j \in \mathbb{N}$. This generates a sequence of sequences $(S_1, S_2, \ldots)$ such that the stopping time for each $S_j$ is $k_j$. Now, we construct a subsequence of this sequence $(\tilde{S}_1, \tilde{S}_2, \ldots)$ that “converges” to some $S^* \in S$. We do this inductively i.e. we show that for any $N \in \mathbb{N}$, there exists infinitely many sequences $(S_{N_1}, S_{N_2}, \ldots)$ in $(S_1, S_2, \ldots)$ that “agree” on the first $N$ terms. For the base case, note that since $X$ is finite, there must exist at least one alternative in $X$ that appears in the first position in infinitely many terms of the sequence $(S_1, S_2, \ldots)$ i.e. in infinitely many sequences. There may be multiple such terms. We pick any one arbitrarily. Consider all such sequences which have the same first term. This forms a subsequence of $(S_1, S_2, \ldots)$. We can write this subsequence as $(S_{11}, S_{12}, \ldots)$ where $S_{1i} = S_{m_i}$ for some $m_i \geq i$. Note that all sequences $S_{11}, \ldots$ have the same first term. Let $\tilde{S}_1 = S_{11}$. For the inductive step, suppose we have found out a subsequence $(S_{k1}, S_{k2}, \ldots)$ of $(S_{(k-1)1}, S_{(k-1)2}, \ldots)$ (and consequently of $(S_1, S_2, \ldots)$) such that all sequences (i.e. terms) in the subsequence “agree” on the first $k$ terms. Let $\tilde{S}_k = S_{k1}$. Now, to show that it must be true for $k + 1$ i.e. we can find a subsequence of $(S_{k1}, S_{k2}, \ldots)$ such that all sequences agree on the first $k + 1$ terms, note that since $X$ is finite, there must exist at least one alternative in $X$ that occurs at the $(k+1)^{th}$ position in infinitely many sequences i.e. infinitely many terms of $(S_{k1}, S_{k2}, \ldots)$. Let that subsequence be $(S_{(k+1)1}, S_{(k+1)2}, \ldots)$ and
\( S_{k+1} = S_{(k+1)1} \). Therefore, we have shown that there exists a subsequence \((\tilde{S}_1, \tilde{S}_2, \ldots)\) of \((S_1, S_2, \ldots)\) such that for any \( k \in \mathbb{N} \), we have

\[
\tilde{S}_k |_k = \tilde{S}_j |_k \quad \forall \, j > k
\]

Therefore \((\tilde{S}_1, \tilde{S}_2, \ldots)\) is a convergent subsequence that converges to some \( S^* \in \mathcal{S} \).

**Step 3:** Since \( c \) is a stopping rule, there must exist \( k^* \in \mathbb{N} \) such that

\[
c(S^*) = c(S^*|_{k^*}.T) \quad \forall \, T \in \mathcal{S}
\] (1)

Note that we can write \((\tilde{S}_1, \tilde{S}_2, \ldots)\) as \((S_{11}, S_{21}, \ldots)\) such that \( S_{ni} \) are elements of the initial sequence \((S_1, S_2, \ldots)\) for all \( i \in \mathbb{N} \). Consider \( n_i > k^* \). Note that by construction, \( S_{ni} \in \mathcal{A}_{ni} \) i.e. for any \( k < n_i \), there exists some \( T' \in \mathcal{S} \) such that \( c(S_{ni}) \neq c(S_{ni}|_{k}.T') \) Let \( k = k^* \). We know that \( S_{ni}|_{k^*} = S^*|_{k^*} \) which implies that \( S_{ni} = S^*|_{k^*}.T^m \) for some \( T^m \in \mathcal{S} \). By (1), we know that \( c(S^*) = c(S_{ni}) \). However, since \( c(S_{ni}) \neq c(S_{ni}|_{k^*}.T') \) for some \( T' \in \mathcal{S} \), we get

\[
c(S_{ni}|_{k^*}.T') = c(S^*|_{k^*}.T') \neq c(S^*)
\]

which contradicts (1). So, there does not exist \( k \in \mathbb{N} \) such that (1) holds i.e. there is no finite stopping time for \( S^* \), a contradiction to \( c \) being a stopping rule. Therefore our supposition is wrong and \( c \) must be a uniform stopping rule.

The above two results highlight the fact that a “bounded” attention span of the DM is an implication of the continuity of her choice function. An important observation is that our results go through even if we do not restrict the choice to be an element of the sequence i.e. \( c(S) \neq S(i) \) for any \( i \in \mathbb{N} \). This bears the interpretation similar to the model of Cheung and Masatlioglu (2021) that the choice can be outside of the recommendations provided to the DM. In fact, our result is more general and allows for choice to land in any arbitrary finite set \( Y \) that is possibly distinct from the set of alternatives \( X \).\(^1\)

### 4 Satisficing Stopping Rules

Satisficing, first introduced by Herbert Simon (see Simon (1955)), has been a hugely influential model of decision making and has been studied widely in the literature (see Kovach and Ülkü (2020), Aguiar et al. (2016), Tyson (2015) and Papi (2012), among others). Satisficing behavior incorporates search of the DM until a “good” enough alternative is observed. While some existing models endogenize the search

\(^1\) In this case, we refer to a function \( d : X^N \to Y \) as a decision rule. While elements of \( X^N \) have a “temporal” interpretation in the sense of alternatives appearing sequentially in discrete time, our results hold even for functions of the form \( d : X^Z \to Y \) where \( Z \) is a countable set. For instance, \( Z \) can be a countably infinite tree which gives the decision rule a “spatial” interpretation.
order of the DM (see Aguiar et al. (2016)), others treat it as observable in the form of a list and vary the threshold (see Kovach and Ülkü (2020)). In this section, we propose two models similar in spirit and provide their behavioral characterization.

4.1 Preliminaries

There is a large literature on limited attention and a variety of modeling approaches have been used (see Masatlioglu et al. (2012), Caplin and Dean (2015) and Manzini and Mariotti (2014)). As mentioned in the above section, our notion of stopping rules also indicate a limited attention span of the DM. From the informational aspect of decision making, we argue that there exists a point by which the DM makes up her mind regarding what to choose. We capture this idea using the concepts of sufficiency and minimal sufficiency of finite segments in a sequence. Let \( S_k \) be the set of all segments of length \( k \). Then we define a sufficient segment as follows

**Definition 5.** A segment \( M \in S_m \) is sufficient if \( c(M.T) = c(M.T') \) for all \( T, T' \in S \)

The intuitive content of the above definition is as follows. As the DM faces a sequence \( S \in S \), there comes a point \( k \in \mathbb{N} \) when the segment \( S|_k \) has enough information for the decision maker to have made up his mind about the choice i.e. \( S|_k \) is informationally “sufficient” to enforce a decision. However, the acquired information will not be sufficient until a certain point in time. This motivates the notion of minimal sufficiency. Formally,

**Definition 6.** A segment \( M \in S_m \) is minimal sufficient if it is sufficient and for any \( k < m \), \( M|_k \) is not sufficient

Minimal sufficiency captures the idea of “critical” length of a segment to enforce a decision. By critical, we mean that if the segment is smaller than that length, it can no longer guarantee the same choice irrespective of the tail. Note that the definition of stopping rules indicates that every sequence must have a corresponding minimal sufficient segment that “implements” the choice. Let us denote the class of sufficient and minimal sufficient segments as \( S \) and \( MS \) respectively. If \( M = S|_k \) for some \( k \in \mathbb{N} \) and \( M \in S \cup MS \), then we will abuse notation and denote the choice of \( S \) by \( c(M) \) i.e. \( c(S) = c(M) \). Also denote by \( M(X) \) the set of alternatives that appear in the segment in \( M \).

To illustrate the idea of sufficiency and minimal sufficiency, let us consider example 1 of the introduction. Suppose the DM has a threshold of 3 and consider the sequence \( S = (a \ b \ c \ a \ b \ c.\ldots) \) i.e. it consist of “cycles” of alternatives \( a, b \) and \( c \). Here, minimal sufficient segment is of length 7 i.e. where \( a \) is the first alternative to appear 3 times. Any segment of length less than 7 is not minimal sufficient and any segment of length more than 7 is sufficient.
4.2 Cardinal Satisficing

We generalize the idea of example 1 and equip the DM with two objects. The first one is a weight function \(w : X \to \mathbb{R}_+\) that assigns a a number to every alternative. The weights can be thought of as some scores the DM assigns to the alternatives that are indicative of the relative importance of alternatives. For instance, a DM may give a higher score to “action” movies over the ones belonging to the genre “drama”. The second object that the DM is endowed with is a threshold number \(v \in \mathbb{R}_+\). The threshold can be thought of as the satisficing component or the cutoff that the DM uses to make decisions. In what we term as the Cardinal Satisficing Rule (CSR), the DM parses through a sequence and selects the first alternative whose “cumulative” weight crosses the threshold. Since in this heuristic, the “intensity” of the weights can affect the choice, it is termed as “cardinal”. For any given sequence \(S \in \mathcal{S}\) and a position \(N \in \mathbb{N}\), we define the cumulative weight of an alternative \(x\) as

\[
W^N_S(x) = |\{i \in \{1, \ldots, N\} : S(i) = x\}|w(x)
\]

Now, we can define SSR formally as follows

**Definition 7.** A stopping rule \(c\) is a Cardinal Satisficing Rule if there exists \(v \in \mathbb{R}_+\) and \(w : X \to \mathbb{R}_+\) such that for any \(S \in \mathcal{S}\),

\[
c(S) = \{x : W^N_S(x) \geq v > W^N_S(y)\}
\]

for all \(y \neq x\) and some \(N \in \mathbb{N}\).

Before we state our axioms, we will introduce two concepts: favorable deletion and favorable shift. For any sequence \(S \in \mathcal{S}\) and \(k \in \mathbb{N}\), let \(\hat{S}^k \in \mathcal{S}\) be the sequence which is defined as follows:

\[
\hat{S}^k(i) = \begin{cases} 
S(k + 1) & \text{if } i = k; \\
S(k) & \text{if } i = k + 1; \\
S(i) & \text{otherwise.}
\end{cases}
\]

That is, the sequence \(\hat{S}^k\) is obtained from \(S\) by interchanging its \(k^{th}\) and \(k + 1^{th}\) elements. We call \(\hat{S}^k\) a favorable shift of \(S\) with respect to an alternative \(x\) if \(S(k + 1) = x\).

For any \(S \in \mathcal{S}\) and \(k \in \mathbb{N}\) define \(\tilde{S}^k\) as

\[
\tilde{S}^k(i) = \begin{cases} 
S(i) & \text{if } i < k; \\
S(i - 1) & \text{if } i > k
\end{cases}
\]

The sequence \(S^k\) is obtained from \(S\) by dropping the alternative located at the \(k^{th}\) position. We call \(\tilde{S}^k\) as a favorable deletion \(S\) with respect to an alternative \(x\) if \(S(k) \neq x\). The notions of favorable
shift and favorable deletion with respect to an alternative capture the idea of bringing it “closer” to the DM. In other words, lowering the position in which an alternative appears in a sequence is considered as “favorable” for it.

Let the class of all favorable shifts of $S$ with respect to $x$ be denoted by $\mathcal{FS}(S, x)$. Further, the class of all favorable deletions of $S$ with respect to $x$ be denoted by $\mathcal{FD}(S, x)$. For any $S \in \mathcal{S}$ and $x \in X$, a favorable transformation of $S$ with respect to $x$ is any favorable shift or favorable deletion. The class of all favorable transformations of $S$ with respect to $x$ shall be denoted by $\mathcal{F}(S, x)$. Therefore, $\mathcal{F}(S, x) = \mathcal{FS}(S, x) \cup \mathcal{FD}(S, x)$ by definition.

4.2.1 Axioms

Our characterization of CSR relies upon two axioms. The first axiom is an adaptation of the idea of monotonicity to the setting of sequences.

**Axiom 1** (Monotonicity). Let $S \in \mathcal{S}$ and $c(S) = x$. Then $c(S') = x$ for all $S' \in \mathcal{F}(S, x)$.

Intuitively, the axiom requires the DM to make the same choice if the chosen alternative is brought “closer” to him in the sequence i.e. if a new sequence is more favorable for an alternative that was previously chosen, then it should continue being chosen in a new sequence. The second axiom relies on the notions of sufficient and minimal sufficient segments.

**Axiom 2** (Informational Dominance). Let $M \in \mathcal{MS}$ and $N \in \mathcal{S}$ such that $c(M) = x$, $c(N) = z$ and $x \not\in N(X)$. Then $c([M|_{k:N}].T) \neq x$ for any $k < m$ and all $T \in \mathcal{S}$.

This axiom states that if a minimal sufficient segment $M$ “implements” an alternative $x$ and another sufficient segment $N$ that does not contain $x$ implements some other alternative $z$, then concatenating any truncation of $M$ with $N$ prevents $x$ from being chosen. In other words, it asserts that a sufficient segment not containing an alternative can “dominate” a non-minimal sufficient segment in an informational sense. To illustrate, consider the example discussed above. We showed that for a threshold of 3, the minimal sufficient segment for the sequence $S = (a \ b \ c \ a \ b \ c \ a \ b \ c \ldots)$ is $M = (a \ b \ c \ a \ b \ c \ a)$. Consider another sequence $S' = (b \ b \ c \ b \ c \ldots)$. It is easy to see that the segment $N = (b \ b \ c \ b \ b)$ is sufficient. Informational dominance says that for any sequence which contains any truncation of $M$ concatenated with the segment $N$ as its initial segment, the choice cannot be equal to $a$. Now, we are ready to state our result.

**Theorem 3.** A stopping rule $c$ is a Cardinal Satisficing Rule if and only if it satisfies Monotonicity and Informational Dominance

**Proof.** (Necessity): Given a a choice function is a Cardinal Satisficing Rule, we know that there exists a $v \in \mathbb{R}^+$ and $w : X \rightarrow \mathbb{R}^+$ such that for any $S \in \mathcal{S}$, we have $c(S) = \{x : W_S^N(x) \geq v > W_S^N(y)\}$ for some
$N \in \mathbb{N}$ and for all $y \neq x$. To show it satisfies Monotonicity consider any $S$ and its favourable deletion with respect to $c(S)$, say $S'$. Let $N_1 \in \mathbb{N}$ be the position of $S$ where $W^N_S(c(S)) \geq v > W^N_S(y)$ for all $y \neq c(S)$. Note that $S'$ is generated by “deleting” a term of $S$ that is not equal to $c(S)$, for $N_2 = N_1 - 1$ we have $W^{N_2}_S(c(S)) \geq v > W^N_S(y)$ for all $y \neq c(S)$ and therefore $c(S') = c(S)$. By similar argument, we can see that $c(S) = c(S')$ where $S'$ is a favorable shift of $S$ with respect to $c(S)$. To show that $c$ satisfies Informational Dominance, consider a minimal sufficient segment $M$ of length $m$ such that $c(M) = x$ and a sufficient segment $N$ of length $n$ such that $c(N) = y$ and $x \notin N(X)$. Assume for contradiction that $c([M|k]T) = x$ for some $k < m$. Then there exists $N_1 \in N$ such that $W^{N_1}_T(x) \geq v > W^N_S(y)$. Note that since $M|k$ is not minimal sufficient and $x \notin N(X)$, we must have $N_1 > k+n$. But, since the segment $N$ is sufficient, we must have $W^{N_2}_S(y) \geq v$ for some $N_2 < N_1$, a contradiction.

**Sufficiency**: Let $c$ be a stopping rule that satisfies Monotonicity and Informational Dominance. First, we construct the “revealed” critical frequency of each alternative. Fix $x \in X$. Note that $x$ is chosen from the constant sequence $S^x = (x, x, \ldots)$ i.e. $c(S^x) = x$. Now, by the definition of a stopping rule, there exists $k \in \mathbb{N}$ such that $c(S^x) = c([S^x|k]T)$ for all $T \in S$. Let $n_x = \inf \{k : c(S^x) = c(S^x|k)\}$. Since $\mathbb{N}$ is well ordered, we know that $n_x \in \mathbb{N}$.

Consider any non-constant sequence $S$ such that $c(S) = x$ (we do not need to prove anything for the case of constant sequences). Denote by $\#x(S) = |\{j \in \{1, \ldots, i\} : [S]_i(j) = x\}|$, i.e. the number of appearances of $x$ in a segment $S|_i$ of $S$. Now, denote by $i(S, a) = \{i \in \mathbb{N} : \#a(S|_i) = n_a\}$ i.e. the position at which an alternative $a$ reaches $n_a$ appearances in $S$. We show that $i(S, x) < i(S, y)$ for all $y \neq x$. Assume for contradiction that $i(S, y) < i(S, x)$ for some $y \neq x$. Let $S'$ be a sequence generated from $S$ by deleting all the first terms in the first $i(S, x)$ positions that are not equal to $x$ or $y$ i.e. finitely many favorable deletions with respect to $x$ and $y$. By Monotonicity, $c(S') = x$. Note that first $i(S', x)$ terms contain $n \geq n_y$ number of $y$’s and $n_x$ terms of $x$’s ($n + n_x = i(S', x)$). Now, consider finitely favourable shifts of $S'$ with respect to $x$ to generate $S''$ such that its first $n_x$ terms are all $x$ followed by $n$ terms that are $y$. Again, by Monotonicity, we have $c(S'') = x$. 

![Figure 2: Informational Dominance](image-url)
Denote this segment of y’s as N and the segment of x’s as Mx where M is the segment of \((n_x - 1)\) x’s. So, we can write \(S'' = [Mx.N].T\) where \(T \in \mathcal{S}\) and \(T(j) = S(i(S') + j)\) for all \(j \in \mathbb{N}\). By the definition of \(n_x\) we know that there exists some \(T \in \mathcal{S}\) such that \(c(M,T) \neq x\). Also, by the definition of \(n_y\), we know that \(c(N.T) = y\) for all \(T \in \mathcal{S}\). In other words, \(Mx\) is a minimal sufficient segment and \(N\) is a sufficient segments. Using Informational Dominance, we know that \(c(MNxT) \neq x\) for all \(T \in \mathcal{S}\). It must be that \(c(MNxT) = y\) for all \(T \in \mathcal{S}\). Suppose not i.e. \(c(MNxT) = z\) for some \(z \neq x, y\) and \(T \in \mathcal{S}\). Then, by Monotonicity, it must be that \(c(NxT) = z\), a contradiction since \(N\) contains \(n_y\) first y’s. Therefore \(c(MNxT) = y\) for all \(T \in \mathcal{S}\). Now, notice that we can generate the earlier sequence \(S'\) by successively moving y’s to the left i.e. a finitely many favourable shifts with respect to y and by Monotonicity, we must have \(c(S') = y\), a contradiction. Therefore \(i(S, x) < i(S, y)\).

Let \(v = 1\) and \(w(x) = \frac{1}{n_x}\) for all \(x \in X\). Consider a choice function such that \(c^*(S) = \{x : W^N_S(x) \geq v > W^N_y(y)\}\) for all \(S \in \mathcal{S}\). We will show that \(c^*\) and \(c\) coincide. Consider any arbitrary \(S \in \mathcal{S}\) and let \(c(S) = z\). We know that \(i(S, z) < i(S, y)\) for all \(y \neq z\). Let \(i(S, z) = N\). By construction, we know that \(W^N_S(z) \geq v > W^N_y(y)\) for all \(y \neq z\) and therefore \(c^*(S) = z\). Since \(S\) was chosen arbitrarily, we have shown that \(c^* = c\).

### 4.3 Ordinal satisficing

In this section we introduce another variation of satisficing which we term ordinal satisficing. We use the term “ordinal” because the DM in this model endowed with a strict preference ordering \(\succ\) over the set of alternatives, \(X\). She has a threshold alternative, say \(a^* \in X\) that reflects the satisficing component. For any sequence she considers a fixed number of alternatives- her bounded attention span- and picks the first alternative that is ranked above \(a^*\). If no such alternative exists in her attention span, she chooses the \(\succ\)-maximal element among the alternatives considered. To illustrate, consider an example where \(X = \{a, b, c\}\) with \(a \succ b \succ c\), \(a^* = b\) and \(k = 2\). For the sequence \(S = (b, c, a, \ldots)\), the choice is \(b\) whereas the choice from the sequence \(S' = (a, b, c, \ldots)\) is \(a\).

**Definition 8.** A choice function \(c\) is a **Ordinal Satisficing Rule (OSR)** if there exists \((\succ, a^*, k)\) with \(a^* \in X\), \(\succ\) a linear order over \(X\) and \(k \in \mathbb{N}\) such that \(c(S) = x\) if

(i) If \(x \succ a^*\) and \(x\) is the first such alternative in the first \(k\) positions; or

(ii) \(x\) is the \(\succ\)-maximal element in the first \(k\) positions

Before we state the axioms, we need to define the concept of a **decisive** element.

**Definition 9.** A set \(D \subseteq X\) is the set of decisive alternatives of \(X\) if \(D = \{x \in X : (\forall M \in \mathcal{MS})[x \in X(M) \implies c(M) = x]\}\). Let \(D' = X \setminus D\)
The idea behind a “decisive” alternative is that whenever it is present in a minimal sufficient segment, it is chosen. Intuitively, it dominates attention of the DM and enforces its choice. For any given choice function, the set of decisive alternatives may be empty. However, we will show that in the heuristic we are about to present, it is non-empty. Further, in the special case where the DM is an attention-constrained preference maximizer, this set will be a singleton.

**Definition 10.** $\mathcal{MS}_{D'} = \{ M \in \mathcal{MS} : X(M) \subseteq D' \}$ and $\mathcal{MS}_D = \mathcal{MS} \setminus \mathcal{MS}_{D'}$

The above definition partitions the set of minimal sufficient segments into the ones that contain at least one decisive elements and the ones that do not contain any. Now, we are ready state our axioms.

### 4.3.1 Axioms

**Axiom 3 (Replacement).** Let $M \in \mathcal{MS}_{D'}$. Consider any $M' = (M \setminus \{x\}) \cup \{y\}$ such that $y \in D'$. Then $M' \in \mathcal{S}$

The above axiom is related to the informational implications of replacing a non-decisive alternative with another non-decisive alternative. It says that such a replacement does not affect the informational content of a segment. In other words, the new segment generated by replacing one alternative retains its “sufficiency”

**Axiom 4 (Sequential-α).** Consider any $M, M' \in \mathcal{MS}_{D'}$ such that $M'(X) \subseteq M(X) \cup c(M') \in M(X)$. Then $c(M') = c(M)$

This axiom is the related to the classic condition-α (Sen (1969)) that characterizes rational choice functions on sets. Here, it is restricted to the sequences that contain only non-decisive alternatives. Intuitively, it says that if a non-decisive alternative is revealed “superior” to another non-decisive alternative, then the reverse cannot hold true for those alternatives.

**Axiom 5 (Sequential-No Binary Cycles (NBC)).** Consider any $x, y, z \in X$ and $M, M', M'' \in \mathcal{MS}_{D'}$ such that $X(M) = \{x, y\}, X(M') = \{y, z\}$ and $X(M'') = \{x, z\}$. If $c(M) = x$ and $c(M') = y$, then $c(M'') \neq z$

This is a mild condition related to the no binary cycles condition of Manzini and Mariotti (2007). It says that the minimal sufficient segments that contain pairs of alternatives cannot display cycles in choices. Now, we are ready to state our result.

**Theorem 4.** A stopping rule $c$ is an OSR if and only if it satisfies Replacement, Sequential-α and Sequential-NBC.

**Proof.** Necessity is easy to establish. So we prove the sufficiency. Suppose $c$ satisfies Replacement Invariance, Sequential-α and Sequential NBC. We will use the following two lemmas in establishing that $c$ is an OSR.
Lemma 2. \(|M| = |M'|\) for any \(M,M' \in \mathbb{MS}_D\).

Proof. Suppose not. W.L.O.G let \(|M| > |M'|\). We know by continuity that \(M\) and \(M'\) are finite. Therefore, we know that the restriction of \(M\) to \(|M'|\) i.e. \(M|_{|M'|}\) is also finite. So, we can reach from \(M'\) to \(M|_{|M'|}\) in finite number of “steps” of replacement i.e. there exists a chain of segments \(M_1, \ldots, M_n\) with \(M_1 = M'\) and \(M_n = M|_{|M'|}\) such that \(|\{i : M_j(i) \neq M_{j+1}(i)\}| = 1\) for all \(j \in \{1, \ldots, n - 1\}\). In other words \(M_j\) and \(M_{j+1}\) differ only in one position for all segments in the chain. By Replacement, we know all the segments in the chain are sufficient and therefore \(M|_{|M'|} \in \mathcal{S}\). Since \(|M|_{|M'|} < |M|\) and \(|M|_{|M'|}(i) = M(i)\) for all \(i\), this is a contradiction to \(M \in \mathbb{MS}\).

We have established that all minimal sufficient segments that do not contain any decisive alternatives are of the same length. Let that length be \(i^D\).

Lemma 3. Consider any \(M \in \mathbb{MS}_D\) and let \(i^D = \inf\{i \in \mathbb{N} : M(i) \in D\}\). Then \(|M| \leq i^D'\) and \(|M| = i^D\).

Proof. Suppose not i.e. there exists a \(M \in \mathbb{MS}_D\) such that \(|M| > i^D\). By continuity, we know that \(M\) is finite. Consider any \(M' \in \mathbb{MS}_D\). By lemma 2, we know that \(|M'| = i^D'\). As in lemma 2, consider a chain of segments \(M_1, \ldots, M_n\) such that \(M_1 = M'\) and \(M_n = M|_{|M'|}\) such that every successive element in the chain differs by an alternative in exactly one position. By Replacement, we know that \(M_n\) is a sufficient segment. Since \(|M_n| < M\) and \(M_n(i) = M(i)\) for all \(i\), this is a contradiction to \(M \in \mathbb{MS}\). Therefore, \(|M| < i^D\). Now, we will show that \(|M| = i^D\). Assume for contradiction that \(|M| > i^D\) (note that the argument for the case \(|M| < i^D\) is trivial by the definition of \(\mathbb{MS}_D\)’). W.L.O.G let \(M(i^D) = x\). By definition of \(D\), we know that \(c(M) = x\). Since \(M \in \mathbb{MS}\), there exists a sequence \(T\) such that \(c([M]_{|M|-1}.T) \neq c(M) = x\). Let \(\tilde{M}\) be the minimal sufficient segment of the sequence \([M]_{|M|-1}.T\). By definition of \(D\), \(\tilde{M} < i^D\). Since \(|\tilde{M}| < |M|\) and \(\tilde{M}(i) = M(i)\) for all \(i\), this is a contradiction to \(M \in \mathbb{MS}_D\). Therefore \(|M| = i^D\).

Remark. The above lemmas show that all the minimal sufficient segments that do not contain any decisive alternatives have the same length. The DM considers all alternatives in his attention span and then decides. Whereas any occurrence of a decisive alternative in a minimal sufficient segment makes the DM stop and pick that alternative.

We know that continuity implies \(c\) is a uniform stopping rule. Now, we consider the following cases:

(i) \(k = 1\). This implies that size of any minimal sufficient segment is 1. Therefore \(D = X\). Consider any linear order \(\succ \in X \times X\) and \(a^* = \min(X, \succ)\). We can show that \((k, \succ, a^*)\) explain the choice data.

(ii) \(k \geq 2\). Consider any \(x, y \in D'\). Define \(\succ\) as follows: \(x \succ y\) if there exists \(M \in \mathbb{MS}_D\) such that \(\{x, y\} \in X(M)\) and \(c(M) = x\). We first show that \(\succ\) is a linear order over \(D'\). Assume
for contradiction that $\succ$ is not asymmetric. Then there exists $M, M' \in \mathbb{MS}_{D'}$ such that $x, y \in X(M) \cap X(M')$, $c(M) = x$ and $c(M') = y$. Consider $M''$ such that $X(M'') = \{x, y\}$ (such $M''$ exists due to continuity and the fact that $k \geq 2$). By Sequential-$\alpha$ $c(M'') = x$ and $c(M'') = y$, a contradiction. Therefore $\succ$ is asymmetric. Now, consider any $x, y \in D'$. By Continuity there exists $M \in \mathbb{MS}_{D'}$ such that $X(M) = \{x, y\}$. By definition of $\succ$ we have either $x \succ y$ or $y \succ x$. Therefore $\succ$ is weakly connected. To show $\succ$ is transitive, we consider two cases:

(a) $k = 2$. This implies $|M| = 2$ for all $M \in \mathbb{MS}_{D'}$. Consider any $x, y, z \in D'$ and suppose $x \succ y$ and $y \succ z$. Therefore we know that there exists $M, M' \in \mathbb{MS}_{D'}$ such that $X(M) = \{x, y\}$, $c(M) = x$ and $X(M') = \{y, z\}$, $c(M')$. Consider $M'' \in \mathbb{MS}_{D'}$ such that $X(M'') = \{x, y\}$. By Sequential-$\alpha$ c, we know that $c(M'') \neq z$, implying $c(M'') = x$.

(b) $k > 2$. Consider any $x, y, z \in D'$. Let $x \succ y$ and $y \succ z$. Consider $M \in \mathbb{MS}_{D'}$ such that $X(M) = \{x, y, z\}$. By sequential-$\alpha$, we know that $c(M) \neq z$ and $c(M) \neq y$. Therefore $c(M) = x$ implying $x \succ z$.

We have shown that $\succ$ is a linear order over $D'$. Now, let $a^* = \max(D', \succ)$ and consider any linear order $\succ \in X \times X$ such that $\succ \subset \succ$ and $x \succ y$ for all $x \in D$ and $y \in D'$.

First we show that there exists at least one decisive alternative

**Lemma 4.** $D$ is non-empty

**Proof.** Assume for contradiction that $D$ is empty i.e. $X = D'$ By lemma 1, all minimal sufficient segments are of the same length i.e. $M = M'$ for all $M \in \mathbb{MS}$. Since $\succ$ is a linear order over $X$, we have a unique maximal element. W.L.O.G let it be $x$. By Sequential-$\alpha$, we know that $c(M) = x$ for all $M \in \mathbb{MS}$ such that $c \in X(M)$. By definition of $D$, we must have $x \in D$, a contradiction. \qed

Now, we will show that $(k, \succ, a^*)$ explains the choice data. Let $c(S) = x$. There are two possible cases: (i) The segment of $S_k$ does not contain any alternative from $D$. In this case, by construction, $x$ is the $\succ$-maximal element of $X(S_k)$. (ii) The initial segment $S_k$ contains at least one alternative from $D$. By lemma 2, $x \in D$ and is the first alternative from $D$ to feature in $S_k$ \qed

It is interesting to note that rational behavior within the limited attention span is a special case of our satisficing model i.e. when $D$ is a singleton. This is in contrast with satisficing over sets where satisficing is equivalent to preference maximization (See Rubinstein (2012)). Note that in the case where $|D| > 1$, the identified preference order is not unique i.e. any ordering between the alternatives of $D$ can explain the choice data.
5 Configuration-dependent rules

In this section, we define a broad class of rules that we call configuration dependent rules. The underlying idea is that the decision is made using the “configuration” of the alternatives i.e. the pattern of their occurrence in a sequence. Such choice rules are described by what we call a *bitstream* processor. A bitstream is any sequence \( b \in \{0,1\}^\mathbb{N} \) i.e. a sequence of 0’s and 1’s. Let \( b(i) \) denote the \( i^{th} \) component of the bitstream \( b \). We call a collection of bitstreams \( B \) is feasible if it satisfies the following condition

\[
|\{b \in B | b(i) = 1\}| = 1 \quad \forall i \in \mathbb{N}
\]

In other words for any arbitrary position \( i \in \mathbb{N} \), there is exactly one bitstream that contains 1 at its \( i^{th} \) position. Denote by \( \mathcal{B} \) the set of all bitstreams and \( \mathcal{F} \) the set of all feasible collections of bitstreams.

**Definition 11.** A *bitstream processor* is a map \( f: \mathcal{B} \rightarrow \mathcal{B} \) such that \( f(B) \in B \) for all \( B \in \mathcal{B} \).

A bitstream processor selects a bitstream from a feasible collection of bitstreams. One can think of configuration dependent rules “encrypting” any given sequence into a feasible collection sequences of 0’s and 1’s and feeding it into a bitstream processor which then selects one bitstream out of the ones fed into it. This selected bitstream is then “decrypted” into an alternative which is the final choice.

Let \( x(S) = \{a \in \{0,1\}^\mathbb{N} : a(i) = 1 \text{ if } S(i) = x, \ 0 \text{ otherwise}\} \). This corresponds to the “configuration” of the alternative \( x \) in sequence \( S \). Note that any sequence \( S \) corresponds to a feasible collection of bitstreams. Consider any bijection \( \sigma: \mathcal{X} \rightarrow \mathcal{X} \) and \( S, S' \in \mathcal{S} \).

\[
[S'(i) = \sigma(S(i)) \ \forall i \in \mathbb{N}] \implies [c(S') = \sigma(c(S))]
\]

This axiom states that if a sequence is “transformed” into a new sequence by relabelling the alter-
natives, then the choice from the new sequence must respect this transformation. In other words, the choice function is “neutral” with respect to the identity of the alternatives.

**Theorem 5.** A choice function is a configuration dependent rule if and only if it satisfies Neutrality.

*Proof.* Suppose \( c \) is a configuration dependent rule. Then there exists an \( f \) such that \( c(S) = x \iff f(B(S)) = x(S) \). Consider any \( S, S' \) such that \( c(S) = x \) and for all \( i \in \mathbb{N}, S'(i) = \sigma(S(i)) \). We know that \( B(S) = B(S') \) and \( \sigma(x)(S') = x(S) \). Therefore \( f(B(S')) = \sigma(x)(S') \) which implies \( c(S') = \sigma(x) = \sigma(c(S)) \).

To show the other direction, consider a choice rule \( c \) that satisfies Neutrality. Define the relation \( \sim_\sigma \) as follows: \( S \sim_\sigma S' \) if and only if there exists a bijection \( \sigma : X \to X \) such that \( S'(i) = \sigma(S(i)) \forall i \in \mathbb{N} \). Note that \( \sim_\sigma \) is an equivalence relation and hence partitions \( S \). Now, consider any arbitrary \( S \in S \) such that \( c(S) = y \) for some \( y \in X \). Define \( f \) as \( f(B(S)) = y(S) \). Consider any \( S' \) such that \( S \sim_\sigma S' \), for some bijection \( \sigma : X \to X \). By Neutrality, we know that \( c(S') = \sigma(c(S)) \). Also, since \( B(S) = B(S') \) and \( y(S) = \sigma(y)(S') \), we have \( f(B(S')) = \sigma(y)(S') \), by construction. Hence we have defined an \( f \) such that \( c(S) = x \) if and only if \( f(B(S)) = x(S) \). Therefore \( c \) is a configuration dependent rule. \( \square \)

### 5.1 Rational Configuration Dependent Rules

Configuration dependent rules subsume many possible behaviors. One can think of rules that utilise the information on positioning of alternatives to make choices. Alternatively, one may be interested in rules that focus on frequency of alternatives appearing in the sequence. Rules that utilise any combination of the frequency and positioning of alternatives can also be analysed under the umbrella of configuration dependent rules. In this section, we provide and characterize a natural class of rules that we term “rational” configuration dependent rules.

The idea behind rational configuration dependent rules is that the DM chooses rational i.e. she has a preference over the patterns or configurations of alternatives. This is captured by endowing her with a linear order\(^2\) over the set \( \{0, 1\}^\mathbb{N} \) which she uses to make choices. The heuristic can be formally defined as follows.

**Definition 13.** A choice function \( c \) is a rational configuration dependent rule if there exists a linear order \( \triangleright \) over \( \{0, 1\}^\mathbb{N} \) such that for all \( S \in S \)

\[
c(S) = \{x : x(S) \triangleright y(S) \text{ for all } y \neq x, y \in X(S)\}
\]

As mentioned above, configuration dependent rules can be used to describe behavior where the information about location of alternatives can be used to make choices. Rational configuration-dependent rules...
rules, in particular, are useful to this effect. For instance, consider a DM that always picks the first alternative from a sequence. To describe this behavior, let us denote by $O$ the set of all $\{0, 1\}^N$ sequences. Let $O_1$ and $O_2$ form a partition of $O$ where $O_1 = \{a \in O : a(1) = 1\}$ i.e. the set of all 0, 1 sequences that have 1 at its first position and $O_2 = O \setminus O_1$. Such behavior can be explained as a rational configuration-dependent rule by a linear order $\succ$ over $O$ with $x \succ y$ for any $x \in O_1$ and $y \in O_2$.

Rational configuration-dependent rules are characterized using a condition that resembles the well-known Strong Axiom of Revealed Preference (SARP). To state our axiom, we need the notion of an equivalence relation between two sequences with respect to an alternative.

**Definition 14.** For any $x \in X$, let $\sim_x \in S \times S$ such that $S \sim_x S'$ if and only if $S(i) = x \implies S'(i) = x$ for all $i \in \mathbb{N}$.

The above defined binary relation says that two sequences are related via the relation $\sim_x$ if the configuration of the alternative $x$ is the same for both. Now, we are ready to state our axiom.

**Axiom 7** (Acylicity). For any $x_1, x_2 \ldots x_n$ such that $x_i \in X$ and $S_1, S_2 \ldots S_n$ such that $S_j \sim_{x_{j+1}} S_{j+1}$ for all $j \in \{1, \ldots n - 1\}$, $S_n \sim_{x_1}$

\[ c(S_1) = x_1 \ldots c(S_{n-1}) = x_{n-1} \implies c(S_n) \neq x_n \]

This axiom is closely related to SARP and it says that if an alternative $x_1$’s configuration is directly or indirectly “revealed” preferred to another alternative $x_n$’s configuration, then the converse cannot hold. Now, we state our result.

**Theorem 6.** A choice function $c$ is a rational configuration-dependent rule if and only if it satisfies Neutrality and Acyclicity.

**Proof.** Define the following “revealed” relation over configurations $\succ^c$ as follows: For any $a, b \in \{0, 1\}^N$, $a \succ^c b$ if there exists a sequence $S$ with $x(S) = a$ and $y(S) = b$ for some $x, y, X$. First, we show that $\succ^c$ is asymmetric. Suppose not, then there exist $a, b \in \{0, 1\}^N$ such that $a \succ^c b$ and $b \succ^c a$, i.e. there exist $S, S' \in S$ and $x, y, w, z \in X$ with $x(S) = w(S') = a$, $c(S) = x$ and $y(S) = z(S') = b$, $c(S') = z$. There are four possible cases:

(i) $x = w$ and $y = z$. Note that $S \sim_y S'$ and $S' \sim_x S$. By Acyclicity, $c(S') \neq y$.

(ii) $x \neq w$ and $y = z$. Let $\sigma : X \to X$ be such that $\sigma(x) = w$ and $\sigma(a) = a$ for all $a \neq x$. Let $S''$ be such that $S''(i) = \sigma(S(i))$ for all $i \in \mathbb{N}$. By Neutrality, $c(S'') = w$. Note that $S'' \sim_y S'$ and $S' \sim_w S''$. By Acyclicity, $c(S') \neq y = z$.

(iii) $x = w$ and $y \neq z$. Let $\sigma : X \to X$ be such that $\sigma(z) = y$ and $\sigma(a) = a$ for all $a \neq x$. Let $S''$ be such that $S''(i) = \sigma(S'(i))$ for all $i \in \mathbb{N}$. By Neutrality, $c(S'') = y$. Note that $S'' \sim_y S$ and $S \sim_x S''$. By Acyclicity, $c(S) \neq x$.
(iv) $x \neq w$ and $y \neq z$. Let $\sigma : X \to X$ be such that $\sigma(z) = y$, $\sigma(w) = x$ and $\sigma(a) = a$ for all $a \neq z, w$.

Let $S''$ be such that $S''(i) = \sigma(S'(i))$ for all $i \in \mathbb{N}$. By Neutrality, $c(S'') = y$. Note that $S \sim_y S''$ and $S'' \sim_x S$. By Acyclicity, $c(S) \neq x$.

Similarly, by Acyclicity and Neurality, we can show that the “revealed” relation $\preceq^c$ is also acyclic.

Now, define an indirect “revealed” relation $a \triangleright^i b$ as follows: For any $a, b \in \{0, 1\}^\mathbb{N}$, $a \triangleright^i b$ if there exists a chain of alternatives $a_1, \ldots, a_n \in \{0, 1\}^\mathbb{N}$ with $a = a_1$ and $a_n = b$ such that $a \triangleright^c a_1 \triangleright^c \ldots a_n \triangleright^c b$. Note that by construction $\triangleright^i$ is asymmetric and transitive. By Zorn's lemma, there exists a linear order $\triangleright$ such that $\triangleright^i \subseteq \triangleright$. Now, define $\tilde{c}(S) = \{x : x(S) \triangleright y(S) \text{ for all } y \neq x\}$ for all $S \in \mathcal{S}$. Consider any $S$ and W.L.O.G. let $c(S) = x$. Let $x(S) = a$ and now it is easy to see that $c(S) = \tilde{c}$.

6 Computation

In the context of individual decision making, computational models have been used to study boundedly rational behavior (see for instance Salant (2011)). Such models often deploy a finite state machine or a finite automaton to describe choice behavior or strategies. The computational complexity corresponds in this setup corresponds to the state complexity i.e. the number of states used to implement a choice function or strategy. The automaton is formally defined as follows.

Definition 15. A finite automaton is a tuple $A \equiv (Q, \Sigma, \delta)$ where $Q$ is a finite set of states, $\Sigma$ is a finite set of symbols called the input alphabet and $\delta : Q \times X \to Q$ is a transition function.

An automaton starts in an initial state $q_0 \in Q$ and reads elements of $\Sigma$ one at a time (in some order). For every input element and the current state, the transition function determines the next state of the automaton. Within the set of states are the terminal or absorbing states, denoted by $F$. Once the automaton enters one of these states, it remains in that state irrespective of the subsequent inputs. Salant (2011) describes implementation of a choice function on lists using an automaton with an output function. It reads the elements of a list in order and stops either at end of the list or at some intermediate step, depending on the choice function it is implementing. Once it reaches its terminal state i.e. it stops, the output function determines the choice from that list. We can denote such an automaton by $A_O$ where $O : F \to \Sigma$. We can define implementability of a choice function on sequences by an automaton in our framework in fairly straightforward manner.

Definition 16. A choice function $c : \mathcal{S} \to X$ is automaton-implementable if there exists an automaton $A_O$ such that for any input $S \in \mathcal{S}$, the output generated by it is $c(S)$.

While an automaton highlights the computational aspects of choice, it does not entirely capture the notion of “computability”. A more general model of computation—and perhaps the most general known
till date—is the Turing machine. A Turing machine is defined formally as follows.

**Definition 17.** A turing machine is a tuple $TM = (Q, \Sigma, \delta)$, where $Q$ is a finite set of states, $\Sigma$ is a finite set of symbols called the alphabet and $\delta : Q \times \Sigma^2 \rightarrow Q \times \Sigma \times \{L, S, R\}$ is a transition function.

Our formulation of a Turing machine contains two *tapes* which are infinite one directional line of “cells”. We denote the two tapes as input and output tapes. Each tape is equipped with a tape head. The tape head of the input tape reads the symbols on the tape one cell at a time whereas the tape head of the output tape can potentially write symbols to the tape one cell at a time. Both the tapes contain $\texttt{\textless}$ in its first position, the start symbol that *initializes* the machine. It contains a “register” that holds a single element of $Q$ at a time with the intial state being $q_0$. The transition function maps the current state and the symbol on the current entries of the tapes to a new state and instructions for the heads. The input head can move to the next entry of the tape, or move to the previous entry of the tape, or stay in case a terminal state is reached. The machine stops when a terminal state is reached and the output of the machine, denoted by $TM(I)$ where $I$ is the input sequence, is entry under head on the output tape. Let $\mathcal{I}$ denote a set of possible inputs and $TM_{\mathcal{I}}$ be any Turing machine that stops in finite time for all $I \in \mathcal{I}$ i.e. for every $I \in \mathcal{I}$, $T(I)$ is computed in finite number of steps.

The idea of a decision maker as a Turing machine has been discussed in Camara (2021). He defines a choice correspondence to be *tractable* if it can be computed in “reasonable” time by a Turing machine. In a similar spirit, we define a choice function over sequences to be computable if it can be implemented by a Turing Machine in finite time (finite number of steps). It is defined formally as follows.

**Definition 18.** A choice function $c : S \rightarrow X$ is computable if there exists a Turing machine $TM_S$ such that $c(S) = TM_S(S)$ for all $S \in S$.

It is clear that a choice function that is implementable via an automaton is implementable via a Turing machine. However, below we show that choice functions that are computable are in fact implementable by a finite automaton, a result that may be of independent interest to theoretical computer scientists.

**Theorem 7.** A choice function on sequences is computable if and only if it is an automaton-implementable rule.

**Proof.** The if part is trivial so we prove the only if part. We use the following lemma to prove our assertion.

**Lemma 5.** An automaton that processes input strings of length $k$ from an alphabet $\Sigma$ has a state complexity of $O(|\Sigma|^k)$

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3 For more on the computational differences between a finite automaton and a Turing machine, one may refer to Hopcroft and Ullman (1979). The Turing machine we introduce here is one of the many variants. However, most of them are formally equivalent (see Arora and Barak (2009)).
Figure 3: Cardinal satisficing with $X = \{x, y\}$, $v = 2$ and $w(x) = w(y) = 1$

Proof. Consider an arbitrary input string from whose elements are from $\Sigma$. Let $q_0$ be the initial state. The maximum possible states the automaton can transition to after reading the first element is $|\Sigma|$. For every possible first element of the string, the automaton can transition to $|\Sigma|$ states. Therefore, after reading the second element the possible states are $|\Sigma|^2$. Proceeding inductively, we can observe that the maximum possible states by the end of the $k$ length string, we can have $|\Sigma|^k$ states. Hence, the maximum possible states is $\sum_{i=1}^{k} |\Sigma|^i + 1$

Consider a computable choice function $c : S \rightarrow X$. Since it can be implemented by a Turing machine $T_S$, it is a stopping rule. By Theorem 2, it must be a uniform-stopping rule. Therefore there exists a $k \in \mathbb{N}$ such that the choice function decides on the output by viewing strings of length $k$. By the lemma above, this choice function is automaton-implementable by an automaton of state complexity $|X|^k$.

We end this section by presenting a simple automaton to represent a cardinal satisficing model presented in section 4. In this example, $X = \{x, y\}$, the threshold $v = 2$ and the weights of both the alternatives are 1 each. The initial state is denoted by $q_0$ whereas $2_x$ and $2_y$ denote the terminal states. Depending on whether the first element of a sequence is $x$ or $y$, the automaton enters state $1_x$ or $1_y$ respectively. If the second element of the sequence is the same as the first one, the automaton transitions to state $2_x$ or $2_y$ respectively and terminates. If it differs from the first element, it transitions to the state $1_x1_y$. Subsequently, depending on whether the third element is $x$ or $y$, it transitions to state $2_x$ or $2_y$ respectively and terminates.
7 Final Remarks

This paper introduces a new framework of choice that considers decision making from infinite sequences. Our framework provides a natural setting to study decision-making situations where the DM faces alternatives sequentially such as recommendation streams. We have shown that bounded attention of a DM follows from the continuity of his choice function over sequences with respect to a natural topology. While we have introduced some natural heuristics in this framework, one can think of more such heuristics. Future work involves a characterization of tail-regarding rules and stochastic choice rules in this framework. We introduced the notion of computability of a choice function using Turing machines and showed that any computable choice function can be implemented by a finite automaton-a less powerful model of computation than Turing machines. State complexity of some natural heuristics is a topic for future research.

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