The Covariance Extension Equation: A Riccati-Type Approach to Analytic Interpolation

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Abstract—Analytic interpolation problems with rationality and derivative constraints are ubiquitous in systems and control. This article provides a new method for such problems, both in the scalar and matrix case, based on a nonstandard Riccati-type equation. The rank of the solution matrix is the same as the degree of the interpolant, thus providing a natural approach to model reduction. A homotopy continuation method is presented and applied to some problems in modeling and robust control. We also address a question on the positive degree of a covariance sequence originally posed by Kalman.

Index Terms—Analytic interpolation, moment problems, model reduction, robust control, spectral estimation, sensitivity shaping.

I. INTRODUCTION

Analytic interpolation problems abound in systems and control, occurring in spectral estimation, robust control, system identification, and signal processing, to mention a few. In the scalar case, the most general problem formulation goes as follows. Given $m+1$ distinct complex numbers $z_0, z_1, \ldots, z_m$ in the open unit disk $D := \{z | |z| < 1\}$, consider the problem to find a real Carathéodory function mapping the unit disk $D$ to the open right half-plane, i.e., a real function $f$ that is analytic in $D$ and satisfies Re$\{f(z)\} > 0$ there, and which in addition satisfies the interpolation conditions

$$f^{(k)}(z_j) / k! = w_{jk}, \quad j = 0, 1, \ldots, m, \quad k = 0, \ldots, n_j - 1$$

where $f^{(k)}$ is the $k$th derivative of $f$, and the interpolation values $\{w_{jk}; j = 0, 1, \ldots, m, k = 0, \ldots, n_j - 1\}$ are complex numbers in the open right half plane $\mathbb{C}^+$ that occur in conjugate pairs. In addition, we impose the complexity constraint that the interpolant $f$ is rational of degree at most

$$n := \sum_{j=0}^{m} n_j - 1.$$ 

To simplify calculations, we normalize the problem by setting $z_0 = 0$ and $f(0) = \frac{1}{2}$, which can be achieved through a simple Möbius transformation. Since $f$ is a real function, $f^{(k)}(z_j) / k! = w_{jk}$ is an interpolation condition whenever $f^{(k)}(z_j) / k! = w_{jk}$ is. For $m = 0$ and $n_0 = n + 1$, this becomes the rational covariance extension problem introduced by Kalman [1] and completely solved in steps in [2]–[6]. This problem, which is equivalent to determining a rational positive real function of the prescribed maximal degree given a partial covariance sequence, is a basic problem in signal processing and speech processing [7] and system identification [8], [9].

With $n_0 = n_1 = \cdots = n_m = 1$, we have the regular Nevanlinna–Pick interpolation problem with degree constraint [10]–[12] occurring in robust control [13], high-resolution spectral estimation [14], [15], simultaneous stabilization [16], and many other problems in systems and control. In fact, the Nevanlinna–Pick interpolation problem to find a Carathéodory function that interpolates the given data was early used in systems and control [17], [18]. The general Nevanlinna–Pick interpolation problem described previously, allowing derivative constraints, was studied in [19], motivated by $H^\infty$ control problems with multiple unstable poles and/or zeros in the plant. Such problems could not be handled by a classical interpolation approach [20, p. 18].

The early work on the rational covariance extension problem [2]–[4] had nonconstructive proofs based on the topological degree theory. A first attempt to provide an algorithm was presented in [5], where a new nonstandard Riccati-type equation called the covariance extension equation (CEE) was introduced. However, this approach was completely superseded by a convex optimization approach [6], [11], and thus, abandoned. However, in [21], it was shown that the regular Nevanlinna–Pick interpolation problem with degree constraint could also be solved by the CEE, and thus, it was shown that CEE is universal in the sense that it can be used to solve more general analytic interpolation problems by only changing certain parameters. This idea was then used in [22] to attach the general problem presented previously. A first attempt to generalize this method to multivariable analytic interpolation problems was then made in [23], and we shall pursue this inquiry in this article.

To provide basic insight into ideas behind the CEE approach, in Section II, we shall review its application to the rational covariance extension problem, and also bring up the issue of the importance to distinguish between the positive and algebraic degree of partial covariance sequences. This is important since

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the CEE approach provides a simple tool for model reduction. In Section III, we deal with the general scalar problem formulated previously and provide a numerical algorithm based on homotopy continuation in the style of [24]. Section IV is devoted to the multivariable generalization, which turns out to be a challenging problem. The results fall somewhat short of what the scalar case promises, and given some results in [25], we suspect that this is due to problems introduced by the nontrivial case of the multivariable case. In Section V, we illustrate our theory with some numerical examples, and finally, Section VI, conclude this article.

II. PRELIMINARIES ON COVARIANCE EXTENSION

To clarify basic concepts and set notation, we first develop and review the basic theory for the special case that $n = 0$, $z_0 = 0$, and

$$w_{0k} = c_k, \quad k = 0, 1, \ldots, n$$

where the Toeplitz matrix

$$T = \begin{bmatrix}
  c_0 & c_1 & \cdots & c_n \\
  c_1 & c_0 & \cdots & c_{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_n & c_{n-1} & \cdots & c_0
\end{bmatrix}$$

is positive definite. A sequence $(c_0, c_1, \ldots, c_n)$ with the property $T > 0$ is called a positive covariance sequence. To normalize the problem, we set $c_0 = \frac{1}{2}$.

A. Rational Covariance Extension Problem

If $f$ is a Carathéodory function, then

$$\phi_+(z) := f(z^{-1})$$

is a positive real function. The problem is then reduced to finding a rational positive real function

$$\phi_+(z) = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \cdots$$

of degree at most $n$ for which only the first $n$ coefficients $c_1, c_2, \ldots, c_n$ are specified. This is the rational covariance extension problem. In fact

$$\phi(z) := \phi_+(z) + \phi_+(z^{-1}) = \sum_{k=-\infty}^{\infty} c_k z^{-k} > 0, \quad z \in \mathbb{T}$$

where $\mathbb{T}$ is the unit circle $\{z = e^{i\theta} | 0 \leq \theta < 2\pi\}$. Hence, $\phi$ is a power spectral density, and therefore, there is a minimum-phase spectral factor $v(z)$ such that

$$v(z) v(z^{-1}) = \phi(z).$$

It is well-known [9] that passing the normalized white noise $\{u(t)\}_{t \in \mathbb{Z}}$ through a shaping filter with a transfer function $v(z)$, i.e.,

white noise $u \rightarrow \{v(z)\}$

until steady state, the output $\{y(t)\}_{t \in \mathbb{Z}}$ is a stationary process with power spectral density $\phi(e^{i\theta})$, $\theta \in [-\pi, \pi]$. Moreover, the coefficient $(c_0, c_1, c_2, \ldots)$ are the covariances lags

$$c_k = \mathbb{E}\{y(t+k)y(t)\}.$$
and the function $g : \mathbb{R}^{n \times n} \to \mathbb{R}^n$ given by

$$g(P) = u + U\sigma + UT\Phi h.$$  

(16)

The CEE, introduced in [5], is the nonstandard Riccati equation

$$P = \Gamma(P - Ph'h')P') + g(P)g(P)'$$  

(17)

where $'$ denotes transposition. The following theorem was proved in [5].

**Theorem 2:** Let $(c_0, c_1, \ldots, c_n)$ be a positive covariance sequence. Then, for each Schur polynomial (12), there is a unique symmetric, positive semidefinite solution $P$ of the CEE satisfying $h'Ph < 1$. Moreover, for each $\sigma$, there is a unique shaping filter (11) for $(c_0, c_1, \ldots, c_n)$ and a corresponding positive real function (10a), where $a(z), b(z)$, and $\rho$ are given in terms of the corresponding $P$ by

$$a = (I - U)(\Gamma Ph + \sigma) - u$$

$$b = (I + U)(\Gamma Ph + \sigma) + u$$

$$\rho = \sqrt{1 - h'Ph}.$$  

(18)

Here, $a := (a_1, a_2, \ldots, a_n)'$ and $b := (b_1, b_2, \ldots, b_n)'$. Finally

$$\deg v = \deg \phi_+ = \text{rank } P.$$  

(19)

Note that $P$ loose rank, i.e., has rank less than $n$, only on a thin (lower dimensional) subset of parameters $u$ [5].

**C. Algebraic and Positive Degree**

For the moment, let $\phi_+ (z)$ be any rational function of degree $d$, not necessarily positive real, given by (6). Then, it has a representation (10) with $n$ replaced by $d$. Identifying coefficients of powers of $z$ in $b(z) = 2\phi_+(z)a(z)$ as done in [5], we obtain

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} = 2 \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix} + \begin{bmatrix} 1 \\ 2c_1 \\ \vdots \\ 2c_{d-1} \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$$

(20a)

for nonnegative powers and

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_d \\ c_2 & c_3 & \cdots & c_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_d & c_{d+1} & \cdots & c_{2d-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} c_{d+1} \\ c_{n+2} \\ \vdots \\ c_{2d} \end{bmatrix}$$

(20b)

for negative powers. The coefficient matrix in (20b) is a Hankel matrix that we denote $H_d$. By Kronecker’s theorem [28]

$$d := \deg \phi_+(z) = \text{rank } H_\infty = \text{rank } H_d.$$  

(21)

Therefore, $\phi_+(z)$ can be determined from a finite sequence $(c_0, c_1, \ldots, c_n)$ of covariance lags, where $n := 2d$. We say that $d$ is the algebraic degree of $(c_0, c_1, \ldots, c_n)$.

Therefore, at first blush, given a partial covariance sequence $(c_0, c_1, \ldots, c_n)$, we might assume that (20) solves the rational covariance problem in a minimal-degree form. This idea underlies (at least the early work on) subspace identification [29]–[31], where in general, the biased ergodic estimates

$$c_k = \frac{1}{N-k+1} \sum_{t=0}^{N-k} y_{t+k}y_t$$

(22)

would be used to insure that the corresponding Toeplitz matrix is positive definite. Then, since

$$c_k = h'F^{k-1}g$$

where $\phi_+(z)$ has the realization

$$\phi_+(z) = \frac{1}{2} + h'(zI - F)^{-1}g$$

(23)

$(F, g, h)$ could be determined by minimal factorization of the Hankel matrix $H_d$. However, as pointed out in [8], this is incorrect and may lead to an $\phi_+(z)$ that is not positive real; also see [9, ch. 13]. In [32], simple examples were given where subspace algorithms will fail.

In general, we cannot achieve a solution $\phi_+$ to the rational covariance extension problem of a degree $d$ only half of $n$. By Theorem 2, the best we could do is

$$p := \min \text{rank } P(\sigma)$$

(24)

which we call the positive degree of the covariance sequence $(c_0, c_1, \ldots, c_n)$. Since the algebraic degree can be determined from the rank of the Hankel matrix $H_d$ (also see [28] and [33]), in 1972, Kalman [34] posed the question whether there is a similar matrix-rank criterion for determining the positive degree. However, since then it has been shown [5] that for any $p$ between $[\mathbb{R}]$ and $n$, there is an open set in $\mathbb{R}^n$ of covariance sequences $(c_1, c_2, \ldots, c_n)$ for which $p$ is the positive degree. Hence, it seems that we cannot get a better criterion than (24).

**III. GENERAL SCALAR PROBLEM**

Next, we show that the CEE is universal in the sense that it also solves the general analytic interpolation problem stated in the introduction, by merely adopting the parameters $(u, U)$ to the new interpolation data.

**A. Some Stochastic Realization Theory**

We express the realization (23) of $\phi_+(z)$ in the observable canonical form, where $h$ is defined as in (14),

$$F = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} = J - ah'$$

(25)

where $J$ is the upward shift matrix, and $g$ is an $n$-vector to be determined. Note that this need not be a minimal realization, as there could be cancellations of common zeros of $a(z)$ and $b(z)$.

**Lemma 3:** The vector $g$ in (23) is given by

$$g = \frac{1}{2}(b - a).$$

(26)

**Proof:** From (10a) and (23), we have

$$b(z) = 1 + 2h'(zI - F)^{-1}g$$

to which we apply the matrix inversion lemma to obtain

$$a(z) = 1 - 2h'(2gh' + zI - F)^{-1}g = 1 - 2h'[zI - (J - ah' - 2gh')]^{-1}g.$$
Consequently, since $b(z)$ is the denominator polynomial, we must have $a + 2g = b$, from which (26) follows.

As a preliminary, let us review some facts from the stochastic realizability theory [9, Sec. 6]. In view of (7) and (23), the spectral density $\phi(z)$ may be written as

$$
\phi(z) = [h'(zI - F)^{-1}] M(P) [z^{-1}I - F']^{-1} h 
$$  
(27a)

for any symmetric $n \times n$ matrix $P$, where

$$
M(P) = \begin{bmatrix}
P & FPF' \\
g - FPh & 1 - h'Ph
\end{bmatrix}.
$$  
(27b)

As a straight-forward calculation shows, the left member of (27a) does not depend on $P$, as all terms containing $P$ cancel. However, $M(P)$ does depend on $P$, and by the positive real Lemma (see, e.g., [9, p. 200]), $\phi_+$ is positive real if and only if there is a $P$ such that

$$
M(P) \succeq 0.
$$  
(28)

In this case, $P$ must be positive semidefinite, and there is a minimum-rank factorization

$$
M(P) = \begin{bmatrix}k \\
\rho \end{bmatrix} \begin{bmatrix}k' \\
\rho \end{bmatrix},
$$  
(29)

where $k \in \mathbb{R}^n$ and $\rho \in \mathbb{R}$. Together with (27a), this yields (8) with the spectral factor

$$
v(z) = \rho + h'(zI - F)^{-1}k.
$$  
(30)

There is a unique minimal symmetric solution of (28) in the ordering $\succeq$ of symmetric matrices, and from now on, $P$ will denote precisely this solution. Then, (30) is the minimum-phase spectral factor with all poles and zeros in the open unit disk, i.e., (30) is precisely (11). Moreover, from (29), we also have

$$
P = FPF' + kk' 
$$  
(31a)

$$
g = FPh + pk
$$  
(31b)

$$
\rho^2 = 1 - h'Ph 
$$  
(31c)

from which we have the algebraic Riccati equation

$$
P = FPF' + (g - FPh)(1 - h'Ph)^{-1}(g - FPh)'.
$$  
(32)

Note that $\rho$ must be nonzero, or otherwise $v(z)$ would be identically zero by (11), and thus, the same would hold for the spectral density $\phi(z)$. Therefore, in view of (31c)

$$
h'Ph < 1.
$$  
(33)

Since all eigenvalues of $F$ lie in the open unit disk, the Lyapunov (31a) has a unique solution

$$
P = \sum_{j=0}^{\infty} F^jkk'(F')^j \succeq 0
$$

[9, Proposition B.1.19, B.1.20]. If $(F,k)$ is a reachable pair so that (30) is a minimal realization, then $P > 0$ [9, Proposition B.1.20]. If rank $P = r < n$, there is a transformation $T$ and a positive definite $r \times r$ matrix $P_1$ such that

$$
TPT' = \begin{bmatrix}P_1 & 0 \\
0 & 0
\end{bmatrix}.
$$

Setting

$$
TFT^{-1} = \begin{bmatrix}F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix}, \quad Tk = \begin{bmatrix}k_1 \\
k_2
\end{bmatrix}, \quad (T')^{-1}h = \begin{bmatrix}h_1 \\
h_2
\end{bmatrix}
$$

it follows from (31a) that $F_{21} = 0$ and $k_2 = 0$, and hence, a straightforward calculation yields the minimal realization

$$
v(z) = h_1'(zI - F_{11})^{-1}k_1 + \rho
$$

of degree $r = \text{rank } P$. Moreover

$$
a(z) = \det(zI - F)^{-1} = \det(zI - F_{11})^{-1} \det(zI - F_{22})^{-1}
$$

so $\det(zI - F_{22})^{-1}$ must be the common factor in $\sigma(z)$ and $a(z)$ that is canceled. In view of (13), $b(z)$ has the same common factor, which is canceled in (10a), and hence, degree of $\phi_+(z)$ is also $r$. Thus,

$$
\deg \phi_+(z) = \deg f = \text{rank } P.
$$  
(34)

It is important to note that $P$ looses rank on a thin set where zero cancellation occur. However, by considering the singular values of $P$, we can determine whether $P$ is close to being singular, which can then be used for the approximate model reduction.

**Remark 4:** Note that the algebraic Riccati equation (32) is different from that of Kalman filtering. Indeed, if $\hat{x}(t)$ is the steady-state Kalman filter estimate of a stationary state process $x(t)$, then the algebraic Riccati equation of Kalman filtering solves for the error covariance matrix

$$
\Sigma := E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]'\} = \Pi - P
$$

where $\Pi := E\{x(t)x(t)\}'$, and $P := E\{\hat{x}(t)\hat{x}(t)\}'$ is the matrix $P$ in our present setting [9, Sec. 6.9].

**Lemma 5:** The vectors $g$ and $k$ in (23) and (30) are given by

$$
g = \Gamma Ph + \sigma - a
$$  
(35a)

$$
k = \rho(\sigma - a)
$$  
(35b)

where $P$ is the minimal symmetric solution of (28), or equivalently, (32).

**Proof:** In the same way as in the proof of Lemma 3, the matrix inversion lemma yields

$$
\frac{a(z)}{\sigma(z)} = 1 - h'(\rho^{-1}kh' + zI - F)^{-1}\rho^{-1}k
$$

$$
= 1 - h'[zI - J + (a + \rho^{-1}k)h']^{-1}\rho^{-1}k
$$

and consequently

$$
J - (a + \rho^{-1}k)h' = J - \sigma h' = \Gamma
$$

since the denominator is $\sigma(z)$. Hence, (35b) follows. Moreover, $\Gamma = F - \rho^{-1}kh'$, which, together with (35b), yields

$$
(1 - h'Ph)(\sigma - a) = g - JPh + ah'Ph
$$

from which (35a) follows.

In view of (35)

$$
P = (\Gamma + \rho^{-1}kh')P(\Gamma + \rho^{-1}kh')' + kk'
$$

from which it follows that

$$
P - \Gamma PT' = \rho^{-2}kk' + \rho^{-1}\Gamma Phk' + \rho^{-1}k'PT'
$$

$$
= (\Gamma Ph + \rho^{-1}k)(\Gamma Ph + \rho^{-1}k)' - \Gamma Phh'PT'
$$
which in turn yields
\[ P = \Gamma(P - Phh'P)\Gamma' + gg' \] (36)
by (35). By introducing interpolation data, we shall derive the appropriate CEE from (36).

**B. CEE for General Interpolation Data**

We return to the general interpolation condition (1), where now
\[ f(z) := \phi_+(z^{-1}) = \frac{1}{2}b_1(z) \] (37)
\[ a_+(z) := z^na(z^{-1}) \]
being the reversed polynomial, and in view of (6)
\[ f(z) = \frac{1}{2} + c_1z + c_2z^2 + c_3z^3 + \cdots . \] (38)
Given the interpolation values, we form the \((n + 1) \times (n + 1)\) matrix
\[ W := \begin{bmatrix} W_0 & \cdots & \vdots & \cdots & W_m \end{bmatrix} \] (39a)
where, for \(j = 0, 1, \ldots, m\)
\[ W_j := \begin{bmatrix} w_{j0} & w_{j1} & w_{j2} & \cdots & w_{j1} & w_{j0} \end{bmatrix} \] (39b)
In the same format, we also define
\[ Z := \begin{bmatrix} Z_0 & \cdots & \cdots & \cdots & \cdots & \cdots & Z_m \end{bmatrix}, \quad Z_j := \begin{bmatrix} z_j & 1 & \cdots & \cdots & \cdots & \cdots & z_j \end{bmatrix} \] (40)
and the \((n + 1)\)-dimensional column vector
\[ e := [e_1, e_2, \ldots, e_1, e_2]^T \] (41)
where \(e_j^n := [1, 0, \ldots, 0] \in \mathbb{R}^n\) for each \(j = 0, 1, \ldots, m\).
Clearly \(Z\) is a stability matrix with all eigenvalues in \(D\), and therefore, the Lyapunov equation
\[ X = ZXZ^* + ee^* \] (42)
where \(Z^*\) is the Hermitian conjugate of \(Z\), having a unique solution \(X\) [9, Proposition B.1.19]. The following result can, for example, be found in [11], [19], and [35].

**Proposition 6:** There exists a (strict) Carathéodory function \(f\) satisfying (1) if and only if
\[ \Sigma = WX + XW^* \] (43)
is positive definite.
The matrix \(\Sigma\) is called the **generalized Pick matrix**.

In view of (38)
\[ f(Z) = \frac{1}{2}I + c_1Z + c_2Z^2 + c_3Z^3 + \cdots = W \] (44)
[36], which together with \(b_1(Z) = 2f(Z)\alpha(Z)\) yields
\[ b_1(Z)e = 2W\alpha(Z)e. \]
Therefore
\[ V \begin{bmatrix} 1 \n a \end{bmatrix} = 2WV \begin{bmatrix} 1 \\ a \end{bmatrix} \]
where the matrix
\[ V := [e, Ze, Z^2e, \ldots, Z^ne] \] (45)
is nonsingular by reachability. Therefore, by Lemma 3
\[ \begin{bmatrix} 0 \\ g \end{bmatrix} = (V^{-1}W - \frac{1}{2}I) \begin{bmatrix} 1 \\ a \end{bmatrix} = V^{-1}(W - \frac{1}{2}I)V \begin{bmatrix} 1 \\ a + g \end{bmatrix} \] (46)
or equivalently
\[ (W + \frac{1}{2}I)V \begin{bmatrix} 0 \\ g \end{bmatrix} = (W - \frac{1}{2}I)V \begin{bmatrix} 1 \\ a + g \end{bmatrix} \] (47)
Since \((W + \frac{1}{2}I)\) is nonsingular, it follows from Lemma 5 that
\[ \begin{bmatrix} 0 \\ g \end{bmatrix} = V^{-1}TV \begin{bmatrix} 1 \\ \Gamma Ph + \sigma \end{bmatrix} \] (48)
where
\[ T := (W + \frac{1}{2}I)^{-1}(W - \frac{1}{2}I). \] (49)
Now defining the \(n\)-vector \(u\) and the \(n \times n\)-matrix \(U\) via
\[ \begin{bmatrix} u \\ U \end{bmatrix} := \begin{bmatrix} 0 & I_n \end{bmatrix} V^{-1}TV \] (50)
where \(I_n\) denotes the \(n \times n\) identity matrix to distinguish it from the \((n + 1) \times (n + 1)\) identity matrix \(I\), (48) yields
\[ g = u + U\sigma + UT Ph \] (51)
which inserted into (36) yields precisely the CEE
\[ P = \Gamma(P - Phh'P)\Gamma' \]
\[ + (u + U\sigma + UT Ph)(u + U\sigma + UT Ph)' \] (52)
but now with \((u, U)\) exchanged for (50). Furthermore, by (26), (31b), (31c), and (35)
\[ a = (I - U)(\Gamma Ph + \sigma) - u \]
\[ b = (I + U)(\Gamma Ph + \sigma) + u \]
\[ \rho = \sqrt{1 - h'/h} \] (53)
in analogy with (18).

**C. Main Theorems**

Let the first column in (39b) be denoted \((w_{j0}, w_{j1}')\)' and form the \(n\)-vector
\[ w = (w_{0}, w_{10}, w_{1}, w_{20}, w_{2}, \ldots, w_{m0}, w_{m}') \] (54)
where \(w_{00} = \frac{1}{2}\) has been removed since it is a constant and not a variable, and let \(W_a\) be the space of all \(w\) such that \(\Sigma\) in (43) is positive definite. Moreover, let \(S_a\) be the space of the Schur polynomial of the form (12).

The proof of the following proposition will be deferred to the appendix.

**Proposition 7:** There is map \(u = \omega(w)\) sending \(w\) to \(u\), which is a diffeomorphism. Moreover, there is a linear map \(L\) such that \(U = Lw\).
The CEE (52) can be written as
\[ P = \Gamma P \Gamma' + R(p) \]  
(55)
where \( R(p) \) is a function of the first column \( p = Ph \) in the matrix variable \( P \). Hence, once \( p \) has been determined, \( P \) can be solved from the Lyapunov equation (55), since \( \Gamma \) is a stability matrix. Consequently, the CEE contains \( n \) independent variables, the same number as the real dimension of \( w \).

Note that (52) can be reformulated as
\[ P = (\Gamma + \sigma h')P(\Gamma + \sigma h')' - (\Gamma Ph + \sigma)(\Gamma Ph + \sigma)' + gg' + \rho^2 \sigma \sigma' \]
where \( g \) is given by (51). Since \( g = \frac{1}{2}(b - a) \) and \( \Gamma Ph + \sigma = \frac{1}{2}(a + b) \), this can be rewritten as
\[ P - JPJ' = -\frac{1}{2}(ab' + ba') + \rho^2 \sigma \sigma' \]  
(56)
where \( a \) and \( b \) are given by (53).

Let \( \mathcal{P}_n \) be the \( 2n \)-dimensional space of pairs \( (a, b) \in S_n \times S_n \) such that \( f = \frac{b}{a} \) is a Carathéodory function. Moreover, for each \( \sigma \in S_n \), let \( \mathcal{P}_n(\sigma) \) be the submanifold of \( \mathcal{P}_n \) for which (13) holds. It was shown in [37] that \( \{ \mathcal{P}_n(\sigma) | \sigma \in S_n \} \) is a foliation of \( \mathcal{P}_n \), i.e., a family of smooth nonintersecting submanifolds, called leaves, which together cover \( \mathcal{P}_n \).

**Theorem 8:** Let \( \sigma \in S_n \). Then, for each \( w \in \mathcal{W}_+ \), there is a unique \( (a, b) \in \mathcal{P}_n(\sigma) \) such that (37) satisfies the interpolation conditions (1) and the positivity condition (13). In fact, the map sending \((a, b) \in \mathcal{P}_n(\sigma) \) to \( w \in \mathcal{W}_+ \) is a diffeomorphism.

**Proof:** The Carathéodory function \( f \) can be written as
\[ f(z) = \int_{\pi}^{\pi} e^{i\theta + z} e^{-i\theta - z} \Re\{\varphi(e^{i\theta})\} \frac{d\theta}{2\pi} \]
where \((e^{i\theta} + z)(e^{i\theta} - z)^{-1}\) is a Herglotz kernel. Hence, the interpolation problem can be formulated as the generalized moment problem to find the Carathéodory function (37) satisfying the moment conditions
\[ \int_{-\pi}^{\pi} \alpha_{jk}(e^{i\theta}) \Re\{\varphi(e^{i\theta})\} \frac{d\theta}{2\pi} = w_{jk} \]  
(57)
where
\[ \alpha_{00}(z) = \frac{z + z_j}{z - z_j} \quad j = 0, 1, \ldots, m \]
\[ \alpha_{jk}(z) = \frac{2z}{(z - z_j)(z + z_j)} \quad j = 1, \ldots, n_j - 1, \quad k = 1, \ldots, n_{j-1} \]
(see, e.g., [12]). Then, by [38, Th. 3.4], there is a diffeomorphic map sending \( aa^* \) to \( w \). However, there is a smooth bijection between \( aa^* \) and \( a \); see, e.g., [4, Sec. III]. Given \( a \) and \( b \), \( a \) is uniquely determined via the linear relation (13). Note that \( \rho^2 \) is just the appropriate normalizing scalar factor once \( (a, \sigma) \) has been chosen.

**Theorem 9:** For each \((\sigma, w) \in S_n \times \mathcal{W}_+ \), the CEE (52) has a unique positive semidefinite solution \( P \) with the property \( h'Ph < 1 \), and (53) is the corresponding unique solution of the analytic interpolation problem to find a rational Carathéodory function (37) of degree at most \( n \) satisfying the interpolation conditions (1). Moreover
\[ \deg f = \text{rank } P. \]  
(58)

**Proof:** For each \((\sigma, w) \in S_n \times \mathcal{W}_+ \), by Theorem 8, there is a unique \((a, b) \in \mathcal{P}_n(\sigma) \), which means there is a unique rational positive real function \( \phi_+(z) \) given by (10a). By the construction in Section III-A, the algebraic Riccati equation (32) has a unique minimal solution \( P \geq 0 \) satisfying (33). By transforming (32) to (36) and inserting (51), there is a unique positive semidefinite solution to (52) satisfying (33). Relation (58) follows from (34).

Finally, we observe as in [24] that \( P \) can be eliminated from (56) by multiplying by \( z^{-j-i} \) and summing over all \( i, j = 1, 2, \ldots, n \), leading to an equation in merely the independent vector variable \( p \). In fact, we recover (13), which in a matrix form can be written as
\[ S(a) \begin{bmatrix} \frac{1}{b} \\ s \end{bmatrix} = 2(1 - h'p) \begin{bmatrix} s \end{bmatrix} \]  
(59)
where
\[ S(a) = \begin{bmatrix} 1 & \cdots & a_{n-1} & a_n \\ a_1 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ a_n \end{bmatrix} + 1 \begin{bmatrix} 1 & \cdots & a_{n-1} \\ 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ a_n \end{bmatrix} \]
and
\[ s = \begin{bmatrix} 1 + \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2 \\ \sigma_1 + \sigma_1 \sigma_2 + \cdots + \sigma_n - \sigma_1 \sigma_n \\ \vdots \\ \sigma_{n-1} + \sigma_1 \sigma_n \end{bmatrix} \]
(60)
in \( n \) variables \( p_1, p_2, \ldots, p_n \).

**D. Back to Rational Covariance Extension**

Next, we show how the results presented in Section II-B follow from Theorem 9. With \( m = 0, z_0 = 0, \) and \( w \) given by (3), we have
\[ W = \begin{bmatrix} \frac{1}{2} & 0 \\ c_1 & \frac{1}{2} \\ \vdots & \vdots \\ c_n \end{bmatrix} \]
and
\[ Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]
Since, therefore, \( V = I \) and
\[ D = \begin{bmatrix} 1 & c \\ c & C \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ c & I_n - C^{-1} \end{bmatrix} \]
where
\[
C = \begin{bmatrix}
1 & 1 \\
c_1 & c_2 \\
\vdots & \vdots \\
c_{n-1} & c_{n-2} \\
c_n & 1
\end{bmatrix}, \quad c = \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{n-1} \\
c_n
\end{bmatrix}
\]

(50) yields
\[
u = C^{-1} c, \quad U = I_n - C^{-1}.
\]

To see that (61) is equivalent to (15), we identify negative powers of \(z\) in
\[
(1 + c_1 z^{-1} + \cdots + c_n z^{-n})(1 - u_1 z^{-1} - u_2 z^{-2} - \cdots) = 1
\]
to obtain
\[
c_k = u_k + \sum_{j=1}^{k-1} c_{k-j} u_j, \quad k = 1, 2, \ldots, n
\]
which is equivalent to
\[
Cu = c, \quad C(I_n - U) = I_n.
\]

**E. Algorithm for Solving the CEE**

We shall use a homotopy continuation method to solve the CEE, i.e., determine the unique positive semidefinite \(P\) with the property \(h'Ph < 1\) that satisfies (52) (see Theorem 9). For \(u = 0\), the CEE takes the form
\[
P = \Gamma(P - Phh'P)\Gamma' = \Gamma(P - Phh'P)\Gamma'
\]
which has the unique solution \(P = 0\). We would like to make a continuous deformation of \(u\) to go between the solutions of (52) and (63). To this end, we choose
\[
u(\lambda) = \lambda u, \quad \lambda \in [0, 1].
\]

Then, \(U(\lambda) = \lambda U\) (see Proposition 7). Define \(w(\lambda) := \omega^{-1}(\lambda u)\) in terms of the diffeomorphism in Proposition 7 for all \(\lambda \in [0, 1]\). It follows from (49) that \(W = (I - T)^{-1} - \frac{1}{2} I\), and therefore, the corresponding deformation is
\[
W(\lambda) = (I - \lambda T)^{-1} - \frac{1}{2} I.
\]

We want to show that \(W(\lambda)\) remains in \(W_p\) along the trajectory, i.e., that \(W(\lambda)\) satisfies \(\Sigma > 0\) in (43) for all \(\lambda \in [0, 1]\). To this end, a straightforward calculation yields
\[
\Sigma(\lambda) := W(\lambda)X + XW(\lambda)^* = (I - \lambda T)^{-1}(X - \lambda^2TXT^*)(I - \lambda T^*)^{-1}.
\]

However, \(X - \lambda^2TXT^* \geq 0 \quad X - \lambda T^* > 0 \quad \Sigma(\lambda) > 0\) as claimed.

To solve the reduced CEE in terms of \(p = (p_1, p_2, \ldots, p_n)\), we use the homotopy
\[
H(p, \lambda) := \begin{bmatrix}
I_n & 0
\end{bmatrix} S(a(p, \lambda)) \begin{bmatrix}
1 \\
b(p, \lambda)
\end{bmatrix} - 2(1 - h'p) s = 0
\]
where
\[
a(p, \lambda) = (I - \lambda U)(\Gamma p + \sigma) - \lambda u \quad (66a)
\]
\[
b(p, \lambda) = (I + \lambda U)(\Gamma p + \sigma) + \lambda u \quad (66b)
\]

which also has a unique solution \(p(\lambda)\) for all \(\lambda \in [0, 1]\).

By the implicit function theorem, we have the differential equation
\[
\frac{dp}{d\lambda} = \left[\frac{\partial H(p, \lambda)}{\partial p}\right]^{-1} \frac{\partial H(p, \lambda)}{\partial \lambda}, \quad p(0) = 0
\]
where
\[
\frac{\partial H(p, \lambda)}{\partial \lambda} = \begin{bmatrix}
I_n & 0 \\
0 & (S(a(p, \lambda)) - S(b(p, \lambda)))
\end{bmatrix}
\]
\[
\frac{\partial H(p, \lambda)}{\partial p} = \begin{bmatrix}
0 \\
[S(a(p, \lambda)) + S(b(p, \lambda))]
\end{bmatrix}
\]
and
\[
g(p, \lambda) = u(\lambda) + U(\lambda) \sigma + U(\lambda) \Gamma p.
\]

The differential equation (67) has a unique solution \(p(\lambda)\) on the interval \(\lambda \in [0, 1]\), so by solving the Lyapunov equation
\[
P - \Gamma P\Gamma' = -\Gamma p(1) p(1)\Gamma'
\]
we obtain the unique solution of (52) [9, Proposition B.1.19]. To solve the differential equation (67), we use predictor–corrector steps [40].

**IV. MULTIVARIABLE ANALYTIC INTERPOLATION**

Next, we consider the multivariable version of the problem stated in Section I. More precisely, let \(F\) be an \(\ell \times \ell\) matrix-valued real rational function, analytic in the unit disk \(\mathbb{D}\), which satisfies the interpolation condition
\[
\frac{1}{k!} F^{(k)}(z_j) = W_{jk}, \quad j = 0, 1, \ldots, m
\]
\[
k = 0, \ldots, n_j - 1
\]
and the positivity condition
\[
F(e^{i\theta}) + F(e^{-i\theta}) > 0, \quad -\pi \leq \theta \leq \pi.
\]

We restrict the complexity of the rational function \(F(z)\) by requiring that its McMillan degree be at most \(\ell n\), where
\[
n = \sum_{j=0}^m n_j - 1.
\]

Without loss of generality, we may assume that \(z_0 = 0\) and \(W_0 = \frac{1}{2} I\). Then, \(F(z)\) has a realization
\[
F(z) = \frac{1}{2} I + z H(I - z F)^{-1} G
\]
where \(H \in \mathbb{R}^{\ell \times \ell}, F \in \mathbb{R}^{\ell \times \ell}, G \in \mathbb{R}^{\ell \times \ell}, (H, F)\) is an observable pair, and the matrix \(F\) has all its eigenvalues in \(\mathbb{D}\).

In analogy with the construction in Section III-B, we form the \((\ell(n + 1) \times \ell(n + 1))\) matrix
\[
W := \begin{bmatrix}
W_0 & \cdots \\
& \ddots \\
& & W_m
\end{bmatrix}
\]
with
\[
W_j = \begin{bmatrix}
W_{j0} & W_{j1} & \cdots & W_{jn_j}
\end{bmatrix} \in \mathbb{C}^{\ell_n \times \ell_n}
\] (75)

for each \( j = 0, 1, \ldots, m \). Let \( X \) be the unique solution of the Lyapunov equation (42). The inverse problem to determine the interpolant \( F \) has a solution if and only if the Pick-type condition
\[
W(X \otimes I_e) + (X \otimes I_e)W^* > 0
\] (76)
is satisfied, where \( \otimes \) denotes the Kronecker product.

### A. Multivariable Stochastic Realization Theory

Following the pattern in Section IV-A, we define
\[
\Phi_+(z) := F(z^{-1}) = \frac{1}{z} I + H(z I - F)^{-1} G
\] (77)
which is (strictly) positive real. Moreover, define the minimum-phase spectral factor \( V(z) \) satisfying
\[
V(z) V(z^{-1})' = \Phi(z) := \Phi_+(z) + \Phi_+(z^{-1})'
\] (78)
which then has a realization of the form
\[
V(z) = H(z I - F)^{-1} K + R
\] (79)

[9, ch. 6]. Now, by the usual coordinate transformation \((H, F, G) \rightarrow (H T^{-1}, T F^{-1}, T G)\), we can choose \((H, F)\) in the observer canonical form
\[
H = \text{diag}(h_{t_1}, h_{t_2}, \ldots, h_{t_{\ell}}) \in \mathbb{R}^{\ell \times n \ell}
\]
with \(h_{t_i} := (1, 0, \ldots, 0) \in \mathbb{R}^{\nu} , \) and
\[
F = J - AH \in \mathbb{R}^{n \ell \times n \ell}
\] (80)
where \(J := \text{diag}(J_{t_1}, J_{t_2}, \ldots, J_{t_{\ell}})\) with \( J_{t_i} \) the \( \nu \times \nu \) shift matrix
\[
J_{t_i} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\]
and \( A \in \mathbb{R}^{n \ell \times \ell} \). The numbers \( t_1, t_2, \ldots, t_{\ell} \) are the observability indices of \( \Phi_+(z) \), and
\[
t_1 + t_2 + \cdots + t_{\ell} = n \ell.
\] (81)
Moreover, define
\[
\Pi(z) := \text{diag}(\pi_{t_1}(z), \pi_{t_2}(z), \ldots, \pi_{t_{\ell}}(z))
\] (82)
where \( \pi_{t_i}(z) = (z^{t_i - 1}, \ldots, z, 1) \)
\[
D(z) := \text{diag}(z^{t_1}, z^{t_2}, \ldots, z^{t_{\ell}}).
\] (83)
and
\[
A(z) = D(z) + \Pi(z) A.
\] (84)

**Lemma 10:** The rational matrix functions (77) and (79) have the matrix fraction representations
\[
\Phi_+(z) = \frac{1}{2} A(z)^{-1} B(z)
\] (85a)
where
\[
B(z) = D(z) + \Pi(z) B \quad \text{with} \quad B = A + 2 G
\] (85b)
and
\[
V(z) = A(z)^{-1} \Sigma(z) R
\] (86a)
where
\[
\Sigma(z) = D(z) + \Pi(z) \Sigma \quad \text{with} \quad \Sigma = A + K R^{-1}.
\] (86b)

**Proof:** Since \( \Pi(z)(z I - J) = D(z) H \)
\[
\Pi(z)(z I - F) = \Pi(z)(z I - J) + \Pi(z) A H = A(z) H
\]
and hence,
\[
H(z I - F)^{-1} = A(z)^{-1} \Pi(z).
\] (87)
Then, (85) and (86) follow from (77) and (79), respectively. It follows from the stochastic realization theory [9, Ch. 6] that
\[
K = (G - F P H')(R')^{-1} (G - F P H')', \quad \text{(88a)}
\]
\[
R R' = I - H P H'
\] (88b)
where, by (86b)
\[
\Gamma = J - \Sigma H
\] (89)
and consequently
\[
G = \Gamma P H' + \Sigma - A.
\] (90)

We are now in a position to derive the multivariable version of (36), namely
\[
P = \Gamma (P - P H' H P) \Gamma' + G G'.
\] (92)
In fact, noting that \( F = \Gamma + K R^{-1} H \) and \( G - \Gamma P H' = K R^{-1} \), we see that (89) can be written as
\[
P = (\Gamma + K R^{-1} H) P (\Gamma + K R^{-1} H)' + K K' \quad \text{(92)}
\]
where we have also used (88b). Then, inserting \( K R^{-1} = G - \Gamma P H' \), we obtain (92).

### B. Multivariable CEE

Next, we introduce the interpolation condition (70).

**Lemma 11:** The interpolation condition (70) can be written as
\[
F(Z \otimes I_e) = W
\] (93)
where the matrices \( W \) and \( Z \) are given by (74) and (40), respectively.
Proof: Since $F(z)$ is analytic in $\mathbb{D}$, it has a representation

$$F(z) = \sum_{k=0}^{\infty} C_k z^k$$

where $C_0 = \frac{1}{2} I_\ell$. A straightforward calculation yields

$$F(Z_j \otimes I_\ell) = \sum_{k=0}^{\infty} (Z_j)^k \otimes C_k = W_j$$

where $W_j$ is given by (75). Then, (93) follows from (40) and (74).

Analogously to the situation in Section III-B, (85) provides us with the representation

$$F(z) = \frac{1}{2} A_\star(z)^{-1} B_\star(z)$$

in terms of the reversed matrix polynomials

$$A_\star(z) = D(z)A(z)z^{-1} = I_\ell + D(z)I(z^{-1})A$$

$$B_\star(z) = D(z)B(z)z^{-1} = I_\ell + D(z)I(z^{-1})B$$

where $D(z)$ is given by (83). Then, the interpolation condition (93) takes the form

$$2A_\star(Z \otimes I_\ell)W = B_\star(Z \otimes I_\ell).$$

(95)

In view of (84) and (85b), we have the polynomial representations

$$A_\star(z) = I_\ell + A_1 z + A_2 z^2 + \cdots + A_\ell z^\ell$$

$$B_\star(z) = I_\ell + B_1 z + B_2 z^2 + \cdots + B_\ell z^\ell$$

where $\ell$ is the largest observability index. Introducing $Q := A + G$, it follows from (85b) that $A = Q - G$ and $B = Q + G$, so the interpolation condition (95) can be written as

$$G_\star(Z \otimes I_\ell) = Q_\star(Z \otimes I_\ell)T$$

(96)

where

$$G_\star(z) = G_1 z + G_2 z^2 + \cdots + G_\ell z^\ell$$

$$Q_\star(z) = I_\ell + Q_1 z + Q_2 z^2 + \cdots + Q_\ell z^\ell$$

and

$$T := (W - \frac{1}{2} I)(W + \frac{1}{2} I)^{-1} = \begin{bmatrix} T_0 & \vdots & \vdots & \vdots & T_m \end{bmatrix}$$

(100)

where

$$T_j = \begin{bmatrix} T_{j0} & T_{j1} & T_{j2} & \cdots & T_{jn_j-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T_{j0} & T_{j1} & \cdots & \cdots & T_{j0} \end{bmatrix}$$

(101)

for $j = 0, 1, \ldots, m$. Now, (97) yields

$$Z \otimes G_1 + Z^2 \otimes G_2 + \cdots + Z^\ell \otimes G_{\ell}$$

$$= (I_{\ell(n+1)} + Z \otimes Q_1 + Z^2 \otimes Q_2 + \cdots + Z^\ell \otimes Q_\ell)T.$$  

Multiplying both sides from the right by $(e \otimes I_\ell)$ and observing that, in view of the rule

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

(102)

which holds for arbitrary matrices of appropriate dimensions

$$(Z^k \otimes G_k)(e \otimes I_\ell) = (Z^k e) \otimes G_k = (Z^k e \otimes I_\ell) G_k$$

we have

$$V \begin{bmatrix} G_1 \\ \vdots \\ G_t \end{bmatrix} = \hat{T} + (Z \otimes Q_1 + Z^2 \otimes Q_2 + \cdots + Z^\ell \otimes Q_\ell)\hat{T}$$

(103)

where $V$ is the $\ell(n+1) \times \ell$ matrix

$$V := [Ze \otimes I_\ell \cdots (Z^\ell e) \otimes I_\ell]$$

(104)

and $\hat{T}$ is the $(n+1) \times \ell$ matrix

$$\hat{T} := T(e \otimes I_\ell).$$

(105)

Here

$$\hat{T} = \begin{bmatrix} \hat{T}_0 \\ \vdots \\ \hat{T}_m \end{bmatrix}, \quad \text{where} \quad \hat{T}_j = \begin{bmatrix} T_{j0} \\ T_{j1} \\ \vdots \\ T_{jn_j-1} \end{bmatrix}.$$

(106)

Next let $N_1, N_2, \ldots, N_t$ be the $\ell \times \ell n$ matrices defined by

$$D(z)I(z^{-1}) = N_1 z + N_2 z^2 + \cdots + N_t z^t.$$  

(107)

Then, $A_j = N_j A, B_j = N_j B, G_j = N_j G$, and $Q_j = N_j Q$ for $j = 1, 2, \ldots, t$, and therefore, (103) takes the form

$$VNG = \hat{T} + (Z \otimes N_1 Q + \cdots + Z^\ell \otimes N_t Q)\hat{T}$$

(108)

where

$$N = \begin{bmatrix} N_1 \\ \vdots \\ N_t \end{bmatrix} \in \mathbb{R}^{\ell \times \ell n}, \quad N_k = \begin{bmatrix} e_{t_1}^k \\ e_{t_2}^k \\ \vdots \\ e_{t_\ell}^k \end{bmatrix}.$$  

(109)

Here, $e_{t_k}^k$ is a $1 \times \ell$ row vector with the $k$th element being 1 and the others 0 whenever $k \leq j$, and a zero row vector of dimension $1 \times j$ when $k > j$. Now, $ VN $ is an $\ell (n+1) \times \ell n$ matrix in which the top $\ell$ rows are zero, since $z_0 = 0$, i.e., it takes the form

$$VN = \begin{bmatrix} 0_{\ell \times \ell n} \\ L \end{bmatrix}.$$

(110)

To derive the multivariable CEE, we would like to solve (108) for $G$ and insert it in (92). This would be possible if the square matrix $L$ is nonsingular, in which case $VN$ would have a pseudoinverse $(VN)^\dagger$.

Lemma 12: The $\ell n \times \ell n$ matrix $L$ defined by (110) is nonsingular if and only if all observability indices are the same, i.e., $t_1 = t_2 = \cdots = t_\ell = n$.

Proof: Ordering the observability indices as

$$t_1 \geq t_2 \geq \cdots \geq t_\ell$$
and setting \( t := t_1 \), we have \( t \geq n \) by (81). Since \((Z, e)\) is a reachable pair
\[
\text{rank } [Ze \ Z^2e \ \cdots \ Z^ne] = n. \tag{111}
\]
First assume that \( t = n \). Then, since \( \text{rank}(A \otimes B) = \text{rank}(A)\text{rank}(B) \)
\[
V = [Ze \ Z^2e \ \cdots \ Z^ne] \otimes I_\ell \in \mathbb{C}^{\ell n \times \ell n}
\]
has rank \( \ell n \), and so does \( N \in \mathbb{R}^{\ell n \times n^t} \). Therefore, Sylvester's inequality
\[
\text{rank } V + \text{rank } N - \ell n \leq \text{rank } VN \leq \min \{\text{rank } V, \text{rank } N\}
\]
(see, e.g., [9, p.741]) implies that \( VN \) has rank \( \ell n \), and hence, \( L \) is nonsingular.

Next assume that \( t > n \). Then, the first \( t \) columns of \( N \) can be written \( I_\ell \otimes (e_1^t)' \), so the first \( t \) columns of the \( VN \) form the matrix
\[
\left(\begin{array}{c}
[Ze \ Z^2e \ \cdots \ Z^ne] \\
\end{array}\right) (I_\ell \otimes (e_1^t)')
\]
which in view of (111) has rank \( n < t \). Hence, the columns of \( VN \) are linearly dependent, and thus, \( L \) is singular.

Consequently, assuming that all observability indices are the same, we can solve (108) for \( G \) to obtain
\[
G = (VN)^\dagger \hat{T} + (VN)^\dagger (Z \otimes N_1Q + \cdots + Z^t \otimes N_tQ)\hat{T}, \tag{112}
\]
Since \( Q = A + G, (91) \) yields
\[
G = u + U(\Gamma PH' + \Sigma), \tag{113}
\]
where \( u := (VN)^\dagger \hat{T} \) and \( U : \mathbb{R}^{\ell n \times \ell t} \to \mathbb{R}^{\ell n \times \ell t} \) is the linear operator
\[
Q \mapsto (VN)^\dagger (Z \otimes N_1Q + \cdots + Z^t \otimes N_tQ)\hat{T}.
\]
Then, inserting (113) into (92), we obtain the multivariable CEE
\[
P = \Gamma(P - PH'HP')\Gamma' + (u + U\Sigma + UTPH')(u + U\Sigma + UTPH')', \tag{114}
\]

C. Main Results in the Multivariable Case

We redefine \( S_n \) for the multivariable case to be the class of \( \ell \times \ell \) matrix polynomials (84) such that \( \det A(z) \) has all its zeros in the open unit disk \( \mathbb{D} \). Moreover, let \( \mathcal{W}_+ \) be the values in (70) that satisfy the generalized Pick condition (76). In the present matrix case, the relation (13) becomes
\[
A(z)B(z)^{-1} + B(z)A(z)^{-1} = 2\Sigma(z)RR'\Sigma(z)^{-1}'. \tag{115}
\]
Let \( \mathcal{P}_n \) be the space of pairs \((A, B) \in S_n \times S_n \) such that \( A(z)^{-1}B(z) \) is positive real. Then, the problem at hand is to find, for each \( \Sigma \in S_n \), a pair \((A, B) \in \mathcal{P}_n \) such that (115) and (70) hold.

Clearly \( S_n \) consists of subclasses with different Jordan structures \( J \) defined via (80). In each such subclass, \( D(z) \) and \( \Pi(z) \) in (84), as well as \( N_1, N_2, \ldots, N_t \), are the same.

From this calculation, we have the following theorem. For the details of the proof of the last statement (118), we refer to [5].

Theorem 13: Given \((\Sigma, W) \in S_n \times \mathcal{W}_+ \), where \( \Sigma(z) \) has all its observability indices equal. Then, there is a positive semidefinite solution \( P \) to the CEE (114) such that \( HPH' < I \). To any such \( P \), there corresponds a unique analytic interpolant (94), where the matrices \( A \) and \( B \) are given by
\[
A = (I - U)(\Gamma PH' + \Sigma) - u, \tag{116}
B = (I + U)(\Gamma PH' + \Sigma) + u.
\]
The matrix polynomials \( A(z) \) and \( B(z) \) have the same Jordan structure as \( \Sigma(z) \), and they satisfy (115) with
\[
R = (I - HPH')^{1/2}. \tag{117}
\]
Finally
\[
\text{deg } F = \text{rank } P. \tag{118}
\]

This result is considerably weaker than the scalar version
Theorem 9. Theorem 13 does not guarantee that a solution to (114) is unique. In fact, if there were two solutions to (114), there would be two interpolants, a unique one for each solution \( P \). Moreover, the condition on the observability indices restricts the classes of Jordan structures that are feasible.

Theorem 14: Given \((\Sigma, W) \in S_n \times \mathcal{W}_+ \), where \( \Sigma(z) = \sigma(z)I \) with \( \sigma(z) \) a scalar Schur polynomial. Then, there is a unique positive semidefinite solution \( P \) to the CEE (114) such that \( HPH' < I \) and a corresponding unique analytic interpolant (94), where \( A(z) \) and \( B(z) \) have the same Jordan structure as \( \Sigma(z) \), and the matrices \( A \) and \( B \) are obtained as in Theorem 13. Finally, \( \text{deg } F(z) = \text{rank } P \).

The observability indices of \( \Sigma(z) \) in Theorem 14 are all the same. Moreover, for this case, existence and uniqueness of the underlying multivariable analytic interpolation problem have already been established [41], [44]. Then, the proof of Theorems 8 and 9 can be modified for the resulting setting mutatis mutandis.

Recently there have been several results [25], [43–47] on the question of existence and uniqueness of the multivariate analytic interpolation problem, mostly for the covariance extension problem \((m = n_0 = n + 1) \), but there are so far only partial results and for special structures of the prior (in our case \( \Sigma(z) \)). Especially the question of uniqueness has proven elusive. Perhaps, as suggested in [25], this is due to the Jordan structure, and this could be the reason for the condition on the observability indices required in Theorem 13. In any case, as long as our algorithm delivers a solution to the CEE, we will have a solution to the analytic interpolation problem, unique or not. An advantage of our method is that (118) can be used for model reduction, as will be illustrated in Section V.

D. Algorithm for the Multivariable CEE

As in the scalar case, we shall use a homotopy continuation method. We assume from now on that \( t := t_1 = t_2 = \ldots, t_\ell = n \). When \( u = 0, \hat{T} = 0 \), and hence, \( U = 0 \). Then, the modified Riccati equation (114) becomes
\[
P = \Gamma(P - PH'HP')\Gamma' + (u + U\Sigma + UTPH')(u + U\Sigma + UTPH')', \tag{119}
\]
which has the solution \( P = 0 \). We would like to make a continuous deformation of \( u \) to go from this trivial solution to the solution of (114), so we choose \( u(\lambda) = \lambda u \) with \( \lambda \in [0, 1] \). The corresponding deformation of \( U \) is \( U(\lambda) \), and \( T \) is deformed to \( \lambda T \). Since (100) implies that \( W = (I - T)^{-1} - \frac{1}{2}I \), the value matrix (74) will then vary as \( W(\lambda) = (I - \lambda T)^{-1} - \frac{1}{2}I \). Then, the proof that \( W(\lambda) \in \mathcal{W}_+ \) is mutatis mutandis is the same as in Section III-E. Hence, \( W(\lambda) \) satisfies (76) along the whole trajectory.
Analogously with the scalar case, we reduce the problem to solving for the $n \ell \times \ell$ matrix

$$p = PH'.$$

To this end, we note that the matrix version of (115) is

$$S(A)M(B) + S(B)M(A) = 2S(\Sigma)(I_{n+1} \otimes RR')M(\Sigma)$$

where

$$S(A) = \begin{bmatrix} I & A_1 & \cdots & A_n \\ I & \cdots & A_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ I & \cdots & A_1 & I \end{bmatrix}, \quad M(A) = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}.$$

From (88b), (91), and (116), we have

$$A + B = 2(\Gamma PH' + \Sigma) = 2(JPH' + \Sigma RR').$$

Since $e_n^*J_n = 0$, and hence, $N_nJ = 0$, this yields the relation

$$A_n + B_n = 2\Sigma_n RR'$$

between $A_n = N_nA$, $B_n = N_nB$, and $\Sigma_n = N_n\Sigma$. This is the same as the last block row in (120), which can, therefore, be deleted, leaving us with

$$[I_{n\ell} \ 0_{n\ell \times \ell}] (S(A)M(B) + S(B)M(A)) = 2[I_{n\ell} \ 0_{n\ell \times \ell}] S(\Sigma)(I_{n+1} \otimes RR')M(\Sigma).$$

Consequently, we use the homotopy

$$H(p, \lambda) := [I_{n\ell} \ 0_{n\ell \times \ell}] (S(A)M(B) + S(B)M(A) - 2S(\Sigma)(I_{n+1} \otimes (I - Hp))M(\Sigma)) = 0$$

where

$$A = A(p, \lambda) := \Gamma_p + \Sigma - \lambda u - \lambda U(\Gamma_p + \Sigma)$$

$$B = B(p, \lambda) := \Gamma_p + \Sigma + \lambda u + \lambda U(\Gamma_p + \Sigma)$$

depend on $(p, \lambda)$, thus reducing the problem to solving the differential equation

$$\frac{d}{d\lambda} \vec{\text{vec}}(p(\lambda)) = \left[ \frac{\partial \text{vec}(H(p, \lambda))}{\partial \text{vec}(p)} \right]^{-1} \frac{\partial \text{vec}(H(p, \lambda))}{\partial \lambda} \quad (123)$$

with $\vec{\text{vec}}(p(0)) = 0$

[40], which has the solution $\hat{p}(\lambda)$ for $0 \leq \lambda \leq 1$. The solution of (114) is then obtained by finding the unique solution of the Lyapunov equation

$$P - \Gamma P \Gamma' = -\Gamma p(1)p(1)'\Gamma'$$

$$(u + U(\Gamma p(1) + \Sigma))(u + U(\Gamma p(1) + \Sigma))'.$$
Continuity of the trajectory shows the feasibility of the homotopy continuation method.

Moreover, we obtain a solution of \( P \) of the CEE with the singular values

\[
0.7435, 0.1328, 0.0794, 0.0630, 0.0023, 0.0003, 6 \times 10^{-6}
\]

the last three of which are close to zero. Consequently, \( P \) has approximately rank 4. Therefore, in view of (58) and the fact that \( \deg v = \deg f \), we can reduce the degree of \( v(z) \) to 4 to obtain the reduced system \( \hat{v}(z) \). Fig. 4 shows the given spectral factor \( v(z) \) together with the degree 7 solution and the approximate degree 4 approximation \( \hat{v}(z) \).

More precisely,

\[
\hat{v}(z) = \hat{\rho} \hat{\sigma}(z) / \hat{\alpha}(z) \quad \text{with} \quad \hat{\rho} = 0.5247 \quad \text{and} \quad \hat{\sigma}(z) = z^4 + 1.5973z^3 + 1.7783z^2 + 1.4073z + 0.7157 \\
\hat{\alpha}(z) = z^4 + 0.9341z^3 + 1.112z^2 + 0.7007z + 0.3939
\]

where the last three spectral zeros of \( \sigma(z) \) have been removed to obtain \( \hat{\sigma}(z) \). Likewise, computing the degree 5 and 6 approximations show that the corresponding solutions \( P \) also have rank approximately 4.

### B. Robust Control With Sensitivity Shaping

Given a plant

\[
P(z) = \frac{\left(z - 1.1e^{\frac{20}{9}} \pi i\right) \left(z - 1.1e^{-\frac{20}{9}} \pi i\right)}{z(z - 1.1)(z^2 + 1.21)}
\]

and the feedback configuration in Fig. 5, we need to find a controller \( C \) such that the system is internally stable and satisfies the following specifications:

\[
\begin{align*}
|S(e^{j\theta})| &\leq 1 \text{dB}, \quad \theta \in [0, 0.3]\text{(rad/s)} \\
|S(e^{j\theta})| &\leq 0.5 \text{dB}, \quad \theta \in [2.5, \pi]\text{(rad/s)}
\end{align*}
\]

(126)

where

\[
\|S\|_\infty < 5 \approx 13.98 \text{dB}
\]

is the sensitivity function. From the robust control literature [13], we know that a necessary and sufficient condition for the internal stability is that we have no unstable pole-zero cancellation between \( P \) and \( C \) in the sensitivity function and that the sensitivity function is stable.

The plant \( P(z) \) has three real unstable poles at \( \pm 1.1i \) and 1.1 and three unstable zeros at \( \approx 1\text{dB} \) and \( 0.5\text{dB} \), with multiplicities two, one, and one, respectively. Since the system should be internally stable, the sensitivity function must satisfy the interpolation conditions

\[
S(\pm 1.1i) = 0, \quad S(1.1) = 0
\]

\[
S(\infty) = 1, \quad S'(\infty) = 0, \quad S\left(1.1e^{\frac{20}{9}} \pi i\right) = 1.
\]

(127)

Since \( \|S\|_\infty < 5 \), the function \( g(z) := S(z)/5 \) maps the exterior of the disk into the unit disk, so

\[
f(z) := \frac{1 + g(z^{-1})}{1 - g(z^{-1})} = \frac{5 + S(z^{-1})}{5 - S(z^{-1})}
\]

maps the disk into the right half plane, and hence, \( f \) is a Carathéodory function. To find such a function \( f \) satisfying the given specifications (126) and interpolation constraints (127) is an analytic interpolation problem of the type stated in Section I. Since there are seven interpolation conditions, we can construct an interpolant of degree six by choosing six spectral zeros.

Note that the zeros of \( f(z) + f(z^{-1}) \) are the zeros of

\[
\Gamma(z) := 25 - S(z)S(z^{-1}).
\]

Next, we will show how to achieve the given specifications by choosing suitable zeros of \( \Gamma(z) \). Suppose \( \Gamma(z) \) has one spectral zero \( \lambda \) near \( z = e^{i\theta} \), then \( |S(e^{j\theta})| \approx 5 \) by the continuity of \( \Gamma(z) \) at \( z = e^{i\theta} \). So, by choosing a spectral zero near \( z = e^{i\theta} \), we can elevate the frequency response of \( S \) at \( \theta \) to about 5. More details can be found in [39].

When we choose the spectral zeros at \( 0.98e^{\pm \frac{20}{9}} \pi i, 0.97e^{\pm \frac{1}{2}} \pi i, 0 \), and \( -0.1, 0 \), we obtain the sensitivity function and the controller as

\[
S(z) = \frac{z^6 - 0.0414z^5 + 1.1873z^4 - 0.8951z^3}{z^6 - 0.0414z^5 + 1.5522z^4 - 0.0209z^3 + 0.5729z^2 + 0.0192z - 0.0219}
\]

Fig. 5. Feedback configuration.
The trajectory of poles as $\hat{\lambda}$ varies from 0 to 1.

respectively. The frequency response of $S$ is illustrated in Fig. 6, from which we can see that the specifications are indeed fulfilled.

**C. Model Reduction in the Multivariable Case**

Consider a system with a $2 \times 2$ transfer function

$$V(z) = A(z)^{-1} \Sigma(z) R$$

(128)

of dimension ten and with observability indices $t_1 = t_2 = 5$, where

$$R = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with

$$A_{11} = z^5 - 0.11z^4 - 0.08z^3 + 0.05z^2 - 0.05z - 0.13$$

$$A_{12} = -0.02z^4 - 0.15z^3 + 0.1z^2 - 0.09z - 0.09$$

$$A_{21} = 0.11z^4 + 0.09z^3 - 0.03z^2 - 0.1z + 0.12$$

$$A_{22} = z^5 + 0.07z^4 + 0.19z^3 - 0.03z^2 - 0.13z + 0.05$$

and

$$\Sigma(z) = (z - 0.1)(z - 0.9)(z - 0.37)(z + 0.4)(z + 0.95)I_2.$$ 

Fig. 7 shows the location of poles and zeros ("2" means there are two zeros at the same position). Clearly there is no pole zero cancellation. Let $F$ be the matrix-valued Carathéodory function $F(z) := \Phi_+(z^{-1})$, where $\Phi_+$ is the positive real function satisfying (78).

$$C(z) = \frac{0.3648z^3 + 0.08142z^2 + 0.434z}{z^3 + 1.059z^2 + 1.142z + 0.411}$$

Next, passing normalized (vector-valued) white noise through the filter

$$\text{white noise } u \longrightarrow \hat{V}(z) \longrightarrow y$$

with transfer function $V(z)$, we generate a vector-valued stationary process $y$ with an observed record $y_0, y_1, y_2, \ldots, y_N$, and from this output data, we estimate the $2 \times 2$ matrix valued covariance sequence

$$\hat{C}_k = \frac{1}{N - k + 1} \sum_{t=k}^{N} y_t y_{t-k}.$$ (129)

We want to determine a matrix-valued Carathéodory function $\hat{F}$ satisfying the interpolation conditions

$$\frac{1}{k!} \hat{F}^{(k)}(0) = \hat{C}_k, \quad k = 0, 1, \ldots, 5.$$ (130)

This is a matrix-valued covariance extension problem, which takes the form (70) with $\ell = 2, m = 0$, and $n_0 = 0$. Using the homotopy method of Section IV-D, the poles move as $\hat{\lambda}$ varies from 0 to 1 as shown in Fig. 8.

The modified Riccati equation has a solution $P$ with eigenvalues

$$5.8 \times 10^{-6}, 2.03 \times 10^{-4}, 1.8 \times 10^{-3}, 3.9 \times 10^{-3}, 6 \times 10^{-3}$$

0.03, 0.365, 0.4879, 0.7895, 0.8967.

The first six eigenvalues are very small, so we can reduce the degree of this system from 10 to 4 by choosing the first three covariance lags $\hat{C}_0, \hat{C}_1,$ and $\hat{C}_2$ and removing six zeros of $\Sigma(z)$. We choose two double zeros at 0.9 and -0.95. The reduced system

$$\hat{V}(z) = H \hat{A}(z)^{-1} \hat{\Sigma}(z) \hat{R}$$
has observability indices $t_1 = t_2 = 2$, and

$$H = \begin{bmatrix} \frac{535}{378} & -\frac{363}{3758} \\ -\frac{363}{3758} & \frac{891}{523} \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} 1623/1138 & 1177/1625 \\ 1997/3448 & 7279/6408 \end{bmatrix}$$

$$\hat{A}(z) = \begin{bmatrix} z^2 - 0.01968z + 0.09216 & -0.1574z - 0.08796 \\ 0.03346z + 0.1083 & z^2 + 0.1314z + 0.4366 \end{bmatrix}$$

$$\hat{\Sigma}(z) = (z - 0.9)(z + 0.95)I_2.$$  

The singular values of the true system (128) together with those of the estimated systems of degree 10 and 4 are shown in Fig. 9.

Here, the estimated degree 10 system estimates the true system (128) perfectly, as the black curves of the given spectral factor are completely covered by the red estimate curves. However, the estimated system of degree 4 approximates the true system well.

The singular values of the true system (128) together with those of the estimated systems of degree 10 and 4 are shown in Fig. 9.

VI. CONCLUDING REMARKS

We have shown that the modified Riccati equation introduced in [5] for solving the covariance extension problem can be used for very general analytic interpolation problems (with both rationality and derivative constraints) by merely changing certain parameters computed from data. A robust and efficient numerical algorithm based on homotopy continuation has been provided. There are still some open questions in the multivariable case. The most general formulation of the multivariable analytic interpolation with rationality constraints has been marred by difficulties to establish existence, and in particular, uniqueness in the various parameterizations [2], [25], [41]–[47], and we have encountered similar difficulties here. Our approach attacks these problems from a different angle and might put new light on these challenges. Therefore, future research efforts will be directed toward settling these intriguing open questions in the context of the modified Riccati equation (114).

APPENDIX A

PROOF OF PROPOSITION 7

From (45), (49), and (50), we have

$$u = \begin{bmatrix} 0 \\ I_n \end{bmatrix} V^{-1} T e \quad (131)$$

where

$$T = \text{diag}(D_0, \ldots, D_m) = (W + \frac{1}{2} I)^{-1}(W - \frac{1}{2} I) \quad (132a)$$

with

$$D_j = \begin{bmatrix} d_{j0} \\ d_{j1} & d_{j1} \\ \vdots & \ddots & \ddots \\ d_{j1} & \cdots & d_{j1} & d_{j0} \end{bmatrix} \quad (132b)$$

Consequently

$$u = Md \quad (133)$$

where $d$ is the $n$-vector

$$d = \begin{bmatrix} d_0' \\ d_{10}' \\ d_1' \\ \vdots \\ d_{m0}' \\ d_m' \end{bmatrix}'$$

and $M$ is the nonsingular $n \times n$ matrix obtained by deleting the first row and the first column in $V^{-1}$. We want to establish a diffeomorphism $d = \varphi(w)$ from the $n$-vector (54), i.e.,

$$w = (w_0', w_{10}', w_1', w_{20}', \ldots, w_{m0}', w_m')'$$

to $d$. To this end, we compute $D_j$ to obtain

$$D_j := \left( W_j + \frac{1}{2} I \right)^{-1} \left( W_j - \frac{1}{2} I \right)$$

$$= \begin{bmatrix} (w_{j0} + \frac{1}{2})^{-1}(w_{j0} - \frac{1}{2}) & 0 \\ C_j^{-1}w_j(w_{j0} + \frac{1}{2})^{-1} & C_j^{-1}(C_j - I) \end{bmatrix}.$$
Therefore
\[
\begin{bmatrix}
d_j \circ_0 \\
d_j \\
\end{bmatrix} = \begin{bmatrix}
(w_j \circ_0 + \frac{1}{2})(w_j \circ_0 - \frac{1}{2}) \\
\frac{1}{2} - 1
\end{bmatrix}
\]
from which, we have \( w_j = C_j(w_j \circ_0 + \frac{1}{2})d_j \), \( w_j \circ_0 = \frac{1}{2}(1 + d_j)(1 + d_j \circ_0)^{-1} \) and
\[
S_j := C_j^{-1}(C_j - I) = \begin{bmatrix}
d_j \circ_0 \\
\vdots \\
d_{j_{n-2}} \\
\vdots \\
d_j \circ_0
\end{bmatrix}.
\]
Hence, we have the smooth maps
\[
w_j = (I - S_j)^{-1}(1 - d_j)(1 - d_j \circ_0)^{-1} d_j
\]
defining a diffeomorphism \( d = \varphi(w) \) from \( w \) to \( d \). Thus, since the matrix \( M \) in (133) is nonsingular, \( u = M \varphi(w) \) is the sought diffeomorphism \( \omega \).

Finally, it follows from (132) that there is a linear map \( N \) such that \( D = N(d) = N(M^{-1}u) \), and hence, there is a linear map \( L \) such that \( U = Lu \), as claimed.

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