Quantisation of 2+1 gravity for genus 2

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In [1,2] we established and discussed the algebra of observables for 2+1 gravity at both the classical and quantum level, and gave a systematic discussion of the reduction of the expected number of independent observables to $6g - 6 (g > 1)$. In this paper the algebra of observables for the case $g = 2$ is reduced to a very simple form. A Hilbert space of state vectors is defined and its representations are discussed using a deformation of the Euler-Gamma function. The deformation parameter $\theta$ depends on the cosmological and Planck’s constants.

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1. Introduction.

In a previous article [1] we presented the abstract quantum algebra for 2+1 gravity with cosmological constant $\Lambda$:

\begin{align*}
(a_{mk}, a_{jl}) &= (a_{mj}, a_{kl}) = 0 \quad (1.1) \\
(a_{jk}, a_{km}) &= \left( \frac{1}{K} - 1 \right) (a_{jm} - a_{jk} a_{km}) \quad (1.2) \\
(a_{jk}, a_{kl}) &= \left( 1 - \frac{1}{K} \right) (a_{jl} - a_{kl} a_{jk}) \quad (1.3) \\
(a_{jk}, a_{lm}) &= \left( K - \frac{1}{K} \right) (a_{ji} a_{km} - a_{kl} a_{jm}) \quad (1.4)
\end{align*}
where $K = \frac{4\alpha - i\hbar}{4\alpha + i\hbar} = e^{i\theta}$, $\Lambda = -\frac{1}{3\alpha^2}$ is the cosmological constant and $\hbar$ is Planck’s constant. In (1.1-4) $m, j, l, k$ are 4 anticlockwise points of Fig.1. $m, j, l, k = 1 \cdots n$, and the time independent quantum operators $a_{lk}$ correspond to the classical $\frac{n(n-1)}{2}$ gauge invariant trace elements

$$\alpha_{ij} = \alpha_{ji} = \frac{1}{2} \text{Tr} \left( S(t_it_{i+1} \cdots t_{j-1}) \right), \ S \in SL(2, R) \quad (1.5)$$

For $n = 2g + 2$ the map $S : \pi_1(\Sigma) \rightarrow SL(2, R)$ is defined by the integrated anti-De Sitter connection in the initial data Riemann surface $\Sigma$ of genus $g$, and refers to one of the two spinor components, say the upper component, of the spinor group $SL(2, R) \otimes SL(2, R)$ of the gauge group $SO(2, 2)$ of 2+1 gravity with negative cosmological constant [2]. The algebra (1.1-4) is invariant under the quantum action of the mapping class group on traces [1], the lower component yields an independent algebra of traces $b_{ij}$ identical to (1.1-4) but with $K \rightarrow 1/K$. Moreover $(a_{ij}, b_{kl}) = 0 \ \forall \ i, j, k, l$. Here we discuss only the upper component. The homotopy group $\pi_1(\Sigma)$ of the surface is defined by generators $t_i, \ i = 1 \cdots 2g + 2$ and presentation:

$$t_1 t_2 \cdots t_{2g+2} = 1, \ t_1 t_3 \cdots t_{2g+1} = 1, \ t_2 t_4 \cdots t_{2g+2} = 1 \quad (1.6)$$

The first relator in (1.6) implies that $\Sigma$ is closed. The operators in (1.1-4) are ordered with the convention that $t(a_{ij})$ is increasing from left to right where $t(a_{ij}) = \frac{(i-1)(2n-2-i)}{2} + j - 1$.

The case of $g = 1$, the torus, has been studied extensively, both in this approach [2], and others [3,4]. In this approach the algebra (1.1-4) is isomorphic to the quantum algebra of $SU(2)_q$ when $\Lambda \neq 0$ [2]. For $\Lambda = 0$ it has been shown [5] that the metric approach to determining the complex modulus of the torus [3] is classically equivalent to the classical limit of (1.1-4) for $n = 4$. There are similar, recent results for $\Lambda \neq 0$ [6].

For $g > 1$ there are very few results apart from those of Moncrief [3] who studies the second order, metric formalism and achieves very general results. In this article the case $g = 2$ of the algebra (1.1-4) is studied in detail. In [7] we determined for $n \leq 6$, i.e. $g \leq 2$ a set of $p$ linearly independent central elements $A_{nm}$, $m = 1 \cdots p$ where $n = 2p$ or $n = 2p + 1$, and analysed the trace identities which follow from the presentation (1.5) of the homotopy group $\pi_1(\Sigma)$ and a set of rank identities. These identities together generate a two-sided ideal. For generic $g$ there are precisely $6g - 6$ independent elements which satisfy the algebra (1.1-4). The reduction from
\[
\frac{n(n-1)}{2} = (g + 1)(2g + 1) \text{ to } 6g - 6 \text{ results from the use of the above mentioned identities} [7] \text{ but is highly non unique. For } g = 2 \text{ the reduction from the original 15 elements } a_{ij} \text{ to 6 independent elements has been the subject of a long study. Here this reduction is implemented explicitly in terms of a set of 6 independent operators which satisfy a particularly simple algebra. There are many such possibilities but a convincing set is described as follows:}

We group the vertices of the hexagon into 3 sectors, see Fig. 2, the vertices labeled \(2b\) and \(2b - 1\) belonging to the sector \(b, b = 1 \cdots 3\). Accordingly we define the sector function \(s[2b] = s[2b - 1] = b\). A convenient choice for the 6 independent elements is given by 3 commuting angles \(\varphi_{-b} = -\varphi_b, \ b = \pm 1 \cdots - 3\) defined by:

\[
a_{2b-1,2b} = \frac{\cos \varphi_b}{\cos \frac{\theta}{2}} \quad b = 1 \cdots 3
\]

and commuting operators \(M_{ab}\) with the properties:

\[
M_{ab} = M_{ba} \quad a, b = \pm 1 \cdots \pm 3
\]

\[
M_{a, -a} = 1, \ M_{a, -b} M_{b, c} = M_{ac}
\]

(1.8)

The \(M_{ab}\) act as raising and lowering operators on the \(\varphi_a:\)

\[
M_{\pm a, b} \varphi_a = (\varphi_a \mp \theta) M_{\pm a, b}
\]

(1.9)

It can be checked that the 12 remaining \(a_{ik}\) are represented by:

\[
a_{kj} = \frac{1}{K + 1} K^{\frac{k+j}{2}} \sum_{n,m=\pm 1} \exp(-i(n(\tilde{k} + 1)\varphi_a + m\varphi_b)) \times
\]

\[
\times \frac{\sin \left(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}\right) \sin \left(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2}\right)}{\sin(n\varphi_a) \sin(m\varphi_b)} M_{na, mb}
\]

(1.10)

where we set \(\tilde{k} = k \mod 2 + \frac{1}{2}\) and \(a = s(k), \ b = s(j), \ a, b, c \) in cyclical order. Under these conditions the \(a_{ik}\) satisfy the trace and rank identities. These identities can all be derived from:

\[
a_{12}a_{34} + K^{-2}a_{23}a_{14} - K^{-1}a_{13}a_{24} - a_{56} = 0
\]

(1.11)

by repeated commutation with the elements of the algebra (1.1-4). For example two useful identities are:
\[ K^3 a_{12}a_{46} + Ka_{24}a_{16} - K(1 + K^3)a_{34}a_{45} - K^2a_{14}a_{26} + (1 - K + K^2)a_{35} = 0 \] (1.12)

\[ (1 + K^3)((1 + K)a_{34}a_{56}a_{45} - Ka_{34}a_{46} - a_{56}a_{35}) + K^2a_{14}a_{25} - K^3a_{12}a_{45} - Ka_{24}a_{15} + K(1 + K^2 - K)a_{36} = 0 \]

and their images under cyclical permutations of the indices 1 \cdots 6. These identities are certainly not all independent. By heavy use of computer algebra we were able to show that (1.11-12) and their images follow from (1.7-10).

The relations (1.8-9) follows from the single sector factorisation for all \(a, b = \pm 1 \cdots \pm 3:\)

\[ M_{ab} = M_a M_b = M_{ba} = M_b M_a \] (1.13)

\[ M_{-a} = M_a^{-1} \] (1.14)

\[ M_{\pm a} \varphi_a = (\varphi_a \mp \theta) M_{\pm a} \] (1.15)

It is clear that (1.13-15) can be formally satisfied by setting \(M_a = \exp(-\theta \frac{\partial}{\partial \varphi_a})\) and therefore \(M_{ab} = \exp(-\theta(\frac{\partial}{\partial \varphi_a} + \frac{\partial}{\partial \varphi_b}))\), in turn (1.10) becomes:

\[ a_{kj} = \frac{1}{2 \cos \left(\frac{\theta}{2}\right)} \sum_{n,m=\pm 1} \sin \left(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}\right) \sin \left(\frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2}\right) \times \sin \varphi_a \sin m\varphi_b \]

\[ \times \exp \left( -i(n(k+1)\varphi_a + m(j\varphi_b + \theta(np_a + mp_b)) \right) \] (1.16)

where we have used the Baker-Hausdorff formula [8]:

\[ \exp A \exp B = \exp \left( A + B + \frac{AB - BA}{2} \right) = \exp(AB - BA) \exp B \exp A \] (1.17)

valid when \(AB - BA\) is a \(C\)-number and \(M_a = \exp(-i\theta p_a)\). Note that, from (1.7) and (1.16), all of the 15 original \(a_{ij}\) are expressed in terms of the 3 angles \(\varphi_a\) and their conjugate momenta \(p_a\).
The treatment of (1.16) can be further simplified by noting that
\[ a_{kj} = U_{kj}^{-1} A_{ab} U_{kj} \]
where:
\[ U_{kj} = \exp \left( \frac{i(k + 1)\varphi_a^2 + \tilde{j}\varphi_b^2}{2\theta} \right) \]  
(1.18)

\[ A_{ab} = \frac{1}{2 \cos \left( \frac{\theta}{2} \right)} \sum_{n,m=\pm 1} \sin \left( \frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_a}{2} \right) \sin \left( \frac{\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_a}{2} \right) \times \]
\[ \times \exp \left( -i\theta(np_a + mp_b) \right) \]

\[ A_{ab} \] is an operator which is a function of the sectors \( a, b \) only and is independent of the position of \( k, j \) within \( a, b \).

The discussion of the representations of (1.13-15) is considerably simplified by the introduction of the deformed Euler Gamma-function \( \Gamma(z, \theta) \) (see Appendix for the definition and a list of properties) which extends to the complex domain the symbol:

\[ \left[ n! \right] = \prod_{p=1}^{n} \frac{\sin \frac{p\theta}{2}}{\sin \frac{\theta}{2}} \]  
(1.19)

In particular \( \Gamma(n + 1, \theta) = \left[ n! \right] \) and \( \Gamma(z, \theta) \) is a meromorphic analytic function of \( z \) with poles at \( z = -s - \frac{2\pi r}{\theta}, \quad s, r \geq 0 \text{ and integer and zeroes at } z = s + \frac{2\pi r}{\theta}, \quad s, r \geq 1 \text{ and integer.} \)

2. Representations.

The \( a_{ij} \) expressed by (1.7) are by definition all hermitian operators. We denote by \( \phi_a \) the generic eigenvalue of the operator \( \varphi_a \) and set \( \phi = \{ \phi_1, \phi_2, \phi_3 \}, \quad z_a = \cos \phi_a, \quad z = \{ z_1, z_2, z_3 \} \) where the \( z_a \) are real and restricted to a domain \( D^3 \subset \mathbb{R}^3 \).

Let \( T \) with \( T^2 = 1 \) be the antilinear conjugacy operator \( \Psi(z) \xrightarrow{T} \Psi^*(z) \). A measure \( \sigma(z)d^3z \) with \( \sigma(z) \geq 0 \) and real turns \( H \) into a Hilbert space \( H \) with norm:

\[ |\Psi|^2 = \int_{D^3} |\Psi(z)|^2 \sigma(z)d^3z \]  
(2.1)

The weight function \( \sigma(z) \) can be determined from the hermiticity of the \( a_{ij} \) (1.7) as follows.
Let \( p_a = -i \frac{\partial}{\partial \phi_a}, \ a = 1, 2, 3 \) satisfying the CCR:

\[
(\varphi_a, \varphi_b) = 0, \ (\varphi_a, p_b) = i \delta_{ab}, \ (p_a, p_b) = 0, \ a, b, = 1, 2, 3 \tag{2.2}
\]

it follows by conjugation that:

\[
(\varphi_a^{\dagger}, \varphi_b^{\dagger}) = 0, \ (\varphi_a^{\dagger}, p_b^{\dagger}) = i \delta_{ab}, \ (p_a^{\dagger}, p_b^{\dagger}) = 0, \ a, b, = 1, 2, 3 \tag{2.3}
\]

but also that

\[
\varphi_a^{\dagger} = T \varphi_a T, \ a_{2a,2a-1} = T a_{2a,2a-1} T, \ a = 1, 2, 3 \tag{2.4}
\]

The hermiticity relation between \( O, O^{\dagger} \) namely \( \langle \Psi, O \Phi \rangle^* = \langle O^{\dagger} \Psi, \Phi \rangle \) implies

\[
(-i \frac{\partial}{\partial z_a})^{\dagger} = -i \sigma^{-1} \frac{\partial}{\partial z_a} \sigma. \text{ But } \frac{\partial}{\partial z_a} = -\frac{1}{\sin \phi_a} \frac{\partial}{\partial \phi_a} \text{ whereby:}
\]

\[
p_a^{\dagger} = \left( -i \frac{\partial}{\partial \phi_a} \right)^{\dagger} = \left( i \sin \phi_a \frac{\partial}{\partial z_a} \right)^{\dagger} = i \sigma^{-1} \frac{\partial}{\partial z_a} \sigma \sin \phi_a^{\dagger} = \tag{2.5}
\]

\[
= i \sigma^{-1} \frac{\partial}{\partial z_a} \sigma T \sin \phi_a T = -i T \sigma^{-1} \frac{\partial}{\partial z_a} \sigma \sin \phi_a T = -T \rho^{-1} p_a \rho T
\]

where \( \rho(\phi) = C \sin \phi_1 \sin \phi_2 \sin \phi_3 \sigma(z), \ C \) being a normalization constant, the operator \( \rho = \rho(\varphi, \varphi_2, \varphi_3) = \rho(\varphi) \) is now to be determined by extending (2.4) to all \( i, k \) as \( a_{ik} = a_{ik}^{\dagger} = T a_{ik} T \).

From (1.16-18) we obtain by conjugation:

\[
a_{kj} = \frac{1}{2 \cos \left( \frac{\theta}{2} \right)} U_{k_j}^{\dagger} \sum_{n,m=\pm 1} \exp \left( i \theta (n p_a^{\dagger} + m p_b^{\dagger}) \right) \times 
\]

\[
\sin \left( \frac{\theta}{4} + \frac{n \varphi_a^{\dagger} + m \varphi_b^{\dagger} + \varphi_c^{\dagger}}{2} \right) \sin \left( \frac{\theta}{4} + \frac{n \varphi_a^{\dagger} + m \varphi_b^{\dagger} - \varphi_c^{\dagger}}{2} \right) 
\]

\[
\times \frac{\sin(n \varphi_a^{\dagger} \sin m \varphi_b^{\dagger})}{U_{k_j}^{\dagger}^{-1}} 
\] \tag{2.6}

We apply now [1.17] and reorder the operators in (2.6) by bringing the exponential factor to the right thus finding:

\[
a_{kj} = \frac{1}{2 \cos \left( \frac{\theta}{2} \right)} U_{k_j}^{\dagger} \sum_{n,m=\pm 1} \sin \left( \frac{5 \theta}{4} + \frac{n \varphi_a^{\dagger} + m \varphi_b^{\dagger} + \varphi_c^{\dagger}}{2} \right) \sin \left( \frac{5 \theta}{4} + \frac{n \varphi_a^{\dagger} + m \varphi_b^{\dagger} - \varphi_c^{\dagger}}{2} \right) 
\]

\[
\times \exp \left( i \theta (n p_a^{\dagger} + m p_b^{\dagger}) \right) U_{k_j}^{\dagger}^{-1} 
\]
and

\[ a_{kj} = \frac{1}{2 \cos^2(\frac{\theta}{2})} T U_{kj}^{-1} \sum_{n,m=\pm 1} \sin(\frac{5\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}) \sin(\frac{5\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2}) \times \]
\[ \times \exp \left( i\theta (np_a + mp_b) \right) U_{kj} T \]

(2.7)

We define the maps:

\[ \varphi_a, \varphi_b, \varphi_c \xrightarrow{\Delta(na, mb)} \varphi_a + n\theta, \varphi_b + m\theta, \varphi_c \]  

(2.8)

where as before \( n, m \) take all values \( \pm 1 \) and \( a, b, c \) are any permutation of \( 1, 2, 3 \). From \( a_{ik} = a^\dagger_{ik} = T a_{ik} T \) and by comparing (2.7) with (1.16) we find the recursion relation in the eigenvalues \( \varphi, z \):

\[ \Delta(na, mb) \sigma(z_1, z_2, z_3) = \sigma(z_1, z_2, z_3) \times \]
\[ \times \sin(\frac{\theta}{4} - \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}) \sin(\frac{\theta}{4} - \frac{n\varphi_a + m\varphi_b - \varphi_c}{2}) \]
\[ \times \sin(\frac{5\theta}{4} + \frac{n\varphi_a + m\varphi_b + \varphi_c}{2}) \sin(\frac{5\theta}{4} + \frac{n\varphi_a + m\varphi_b - \varphi_c}{2}) \]

(2.9)

A solution of (2.9) is then provided by:

\[ \sigma(z_1, z_2, z_3) = P(\phi) \prod_{m_1m_2m_3=\pm 1} \Gamma \left( -\frac{1}{4} + \frac{q\pi}{\theta} + \frac{m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3}{2\theta}, 2\theta \right) \]  

(2.10)

where \( q \) is arbitrary and integer and \( P(\phi) \) is invariant under (2.8), otherwise arbitrary, in (2.10) the product is carried on all independent sign choices of \( m_1, m_2, m_3 \).

By using (A.7) we see that:

\[ E(\phi, \theta, q + 1) = S(\phi) E(\phi, \theta, q) \]

where:

\[ S(\phi) = 2^q (2 \sin \theta)^\frac{q\pi}{\theta} \prod_{m_1, m_2, m_3=\pm 1} \sin \pi \left( -\frac{1}{4} + \frac{q\pi}{\theta} + \frac{m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3}{2\theta} \right) \]
\[ E(\phi, \theta, q) = \prod_{m_1, m_2, m_3=\pm 1} \Gamma \left( -\frac{1}{4} + \frac{q\pi}{\theta} + \frac{m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3}{2\theta}, 2\theta \right) \]  

(2.11)
Since \( S(\phi) \) is invariant under (2.8) it can be absorbed into \( P(\phi) \) hence the appearance of \( q \) does not signal any new arbitrariness. It is however convenient in our discussion to have a solution which depends explicitly on \( q \).

The function \( \rho(\phi) \) is periodic of period \( 2\pi \) and odd in \( \phi_1, \phi_2, \phi_3 \) if we have (see (2.10) and (A.7)): 

\[
\frac{\rho(\phi_1 + 2\pi, \phi_2, \phi_3)}{\rho(\phi_1, \phi_2, \phi_3)} = \frac{P(\phi_1 + 2\pi, \phi_2, \phi_3)}{P(\phi_1, \phi_2, \phi_3)} \times \prod_{m_2m_3 = \pm 1} \sin \pi \left( \frac{-1}{4} + q\frac{\pi}{\theta} + \frac{\phi_1 + m_2\phi_2 + m_3\phi_3}{2\theta} \right) = 1
\]

This be achieved by setting

\[
\theta = \frac{2q - 1}{2t + 1} 2\pi , \ t \ \text{integer} \quad (2.12)
\]

and \( P(\varphi) = 1 \).

We list here the basic properties of \( E(\phi, \theta, q) \):

1). \( E(\phi, \theta, q) \) is even in each of the \( \phi_1, \phi_2, \phi_3 \).

2). \( E(\phi, \theta, q) \) is periodic of period \( 2\pi \) in each of the \( \phi_1, \phi_2, \phi_3 \).

3). \( E(\phi, \theta, q) \) is real but not necessarily positive for \( \phi_1, \phi_2, \phi_3 \) all real. It follows by analytic continuation that \( E(\phi^\dagger, \theta, q) = E(\phi, \theta, q)^\dagger \).

4). \( E(\phi, \theta, q) \) is real and positive if at least one of the \( \phi_1, \phi_2, \phi_3 \) is imaginary and the others real. This follows from the possibility of arranging (2.11) in pairs of conjugate factors.

In this case we may choose \( \sigma(z) = E(\phi, \theta, q) \). The discussion of the positivity of the function \( \sigma(z) \) for arbitrary \( z \) is rather involved. A particular solution is provided by restricting all \( z_a \) to the hyperbolic domain \( z_a > 1 \), i.e. all \( \phi_a \) pure imaginary. In this case all \( a_{kj} \) from (1.7) and (1.16) are represented by unbounded hermitian operators. This, and the inclusion of the other \( SL(2, R) \) component, will be discussed elsewhere [9].

**Appendix.**

Here we give the definition and a comprehensive list of properties of the deformed Euler Gamma function:
\[ \Gamma(z, \theta) = \left( \frac{\theta}{2 \sin \frac{\theta}{2}} \right)^{z-1} \Gamma(z) \prod_{n=1}^{\infty} \left( \frac{\theta}{2\pi n} \right)^{2z-1} \frac{\Gamma \left( z + \frac{2\pi n}{\theta} \right)}{\Gamma \left( 1 - z + \frac{2\pi n}{\theta} \right)} \]  \hspace{1cm} (A.1)

\[ \lim_{\theta \to 0} \Gamma(z, \theta) = \Gamma(z) \] \hspace{1cm} (A.2)

\[ \Gamma(1, \theta) = 1 \] \hspace{1cm} (A.3)

\[ \Gamma(z + 1, \theta) = \Gamma(z, \theta) \frac{\sin \frac{\theta z}{2}}{\sin \frac{\theta}{2}}, \Gamma(n + 1, \theta) = \left[ n \right]! \text{ n integer } > 0 \] \hspace{1cm} (A.4)

\[ \Gamma \left( z + \frac{2\pi}{\theta}, \theta \right) = 2 \sin(\pi z) \left( 2 \sin \frac{\theta}{2} \right)^{-\frac{2\pi}{\theta}} \Gamma(z, \theta) \] \hspace{1cm} (A.5)

\[ \Gamma(z, \theta) \Gamma(1 - z, \theta) = \frac{2\pi \sin \frac{\theta}{2}}{\theta \sin(\pi z)} \] \hspace{1cm} (A.6)

\[ \Gamma(z, \theta) \Gamma \left( \frac{2\pi}{\theta} - z, \theta \right) = \frac{\pi}{\theta \sin \frac{\theta z}{2}} \left( 2 \sin \frac{\theta}{2} \right)^{2 - \frac{2\pi}{\theta}} \] \hspace{1cm} (A.7)

\[ \Gamma(z, \theta) \Gamma \left( 1 + \frac{2\pi}{\theta} - z, \theta \right) = \frac{2\pi}{\theta} \left( 2 \sin \frac{\theta}{2} \right)^{1 - \frac{2\pi}{\theta}} \] \hspace{1cm} (A.8)

Setting \( \theta' = \frac{4\pi^2}{\theta} \) we have the duality property:

\[ \Gamma(z, \theta) = \Gamma \left( \frac{\theta z}{2\pi}, \theta' \right) \left( 2 \sin \frac{\theta'}{2} \right)^{-\frac{\theta}{\theta'}} - 1 \left( 2 \sin \frac{\theta}{2} \right)^{1 - z} \frac{\theta'}{2\pi} \] \hspace{1cm} (A.9)

(A.9) is meaningless in the limit \( \theta \to 0 \) and therefore the standard Euler Gamma function \( \Gamma(z) \) has no dual symmetry. From (A.9) it follows that the function \( \Gamma(z, a, b) = a\Gamma \left( \frac{bz}{2\pi a}, \frac{2\pi a}{b} \right) \left( 2 \sin \frac{\pi a}{b} \right)^{bz-1} \) is symmetrical i.e \( \Gamma(z, a, b) = \Gamma(z, b, a) \). Duality exchanges (A.4) with (A.5) and (A.6) with (A.7).

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Fig. 1

Fig. 2.
This figure "fig1-1.png" is available in "png" format from:

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