THE BUCK-PASSING GAME

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ABSTRACT. We consider a model where agents want to transfer the responsibility of doing a job to one of their neighbors in a social network. This can be considered a network variation of the public good model. The goal of the agents is to see the buck coming back to them as rarely as possible. We frame this situation as a game, called the buck-passing game, where players are the vertices of a directed graph and the strategy space of each player is the set of her out-neighbors. The cost that a player incurs is the expected long term frequency of times she gets the buck. We consider two versions of the game. In the deterministic one the players choose one of their out-neighbors. In the stochastic version they choose a probability vector that determines who of their out-neighbors is chosen. We use the finite improvement property to show that the deterministic buck-passing game admits a pure equilibrium. Under some conditions on the strategy space this is true also for the stochastic version. This is proved by showing the existence of an ordinal potential function. These equilibria are prior-free, that is, they do not depend on the initial distribution according to which the first player having the buck is chosen.

KEYWORDS. Markov chain, Markov chain tree theorem, stationary distribution, Nash equilibrium, prior-free equilibrium, ordinal potential game, finite improvement property, price of anarchy.

1. Introduction

1.1. The problem. In several situations individuals living in a closed community tend to unload on their neighbors the responsibility of fixing a problem. The society and therefore each of its member would benefit
in having the problem fixed, but, since fixing it involves a personal cost, agents tend to transfer the responsibility of doing the work to somebody else, when possible. This can be seen as a network version of the classical free riding problem. In the classical model agents want to enjoy a public good and have somebody else pay for it. Here agents want somebody else to fix a problem and this way derive some positive utility without incurring a cost for fixing the problem. Agents can unload the responsibility of fixing the problem only to their neighbors in an existing social network. Obviously they would like to be bothered by this issue as little as possible, so they will do the best they can not to see it coming back to them in the future. This implies that strategically it is not enough to just pass the buck to one of your neighbors. The neighbor must be chosen in a way that the buck will not cycle back fast to the person who is passing it now. The structure of the social network is therefore important for the solution of this problem.

1.2. Our contribution. We model the situation in terms of a finite directed network, where agents are the vertices and directed edges represent the possibility of an agent to transfer the buck (the responsibility of dealing with a problem) to another agent. Whenever an agent receives the buck, she can transfer it to one of her neighbors in the network. Agents make their choice of designated neighbor to whom to transfer the buck once and for all at the beginning. Their goal is to see the buck come back to them as rarely as possible. More formally the cost that each agent incurs is the asymptotic frequency of times she has the buck.

We look at situations where the first agent with the buck is drawn at random by Nature. We define a game and show that if the network on which it is played is strongly connected, then the game has a pure Nash equilibrium that does not depend on the distribution according to which Nature makes its draw, i.e., it is prior-free. In general the game may have multiple Nash equilibria and some of them may not be prior-free. To prove existence of a pure Nash equilibrium, we use the finite improvement property of the game.

We then look at a stochastic version of the game, where agents choose the probability with which the buck is passed to each of their neighbors. This gives rise to a Markov chain. We find conditions under which the transition matrix of the chain is irreducible and, therefore, the chain has a unique stationary distribution. This distribution is exactly the cost vector of the game. To show that the game has a pure Nash equilibrium, we prove the existence of an ordinal potential function and resort to a classical result of Monderer and Shapley [28] that guarantees that
ordinal potential games admit pure Nash equilibria. The equilibria that we obtain are prior-free. We use the Markov chain tree theorem to give an explicit formula for the ordinal potential function and we give a spectral characterization of it.

1.3. Related literature. The issue of taking responsibility for fixing a problem that affects an entire group of people has been considered in different domains such as economics, finance, behavioral science, medicine, political science, law, philosophy, etc.

Hood [18] studies the issue of responsibility in the "risk industry," considering safety and hazard for food, mobile phones, dangerous behavior of people and animals, etc., and looks at risk perception and amplification, and their political implications. Steffel et al. [37] discuss several hypotheses concerning delegation of choice. Gardiner [15] considers the problem of intergenerational fairness, focusing on climate change and nuclear protection, and proposes a so called "global core precautionary principle" to prevent one generation of individuals from shifting the negative effects of their choices to the following generations.

della Paolera et al. [10] study one and a half century of the Argentinian economy and model macroeconomic policies as a game between past, present, and future generations of political rulers and economic agents. León et al. [24] study the ability of individuals to assign responsibility between various levels of government. Sonnenberg [36] uses some simple game-theoretic model to analyze the practice of physicians to refer patients with difficult-to-handle pathologies to some other specialist, even when it is clear that the patient will not reap any benefit from seeing a new doctor. The model considered by Bolle [6] is closer in spirit to the one of our paper. In his model a finite number of players decide one after another whether to pay the cost of an action that is beneficial to the whole society or to pass the buck to the subsequent player. His paper assumes that each player has incomplete information about the social preferences of the other players, and studies the Bayesian equilibrium of the game.

In our paper we consider prior-free equilibria. Our model is static, that is, even if the cost depends on the asymptotic behavior of a process over time, players choose their strategy at the beginning of the game and play the same action whenever their turn comes, irrespective of the history of the game. A concept of belief-free equilibrium has been considered by various authors in the framework of extensive form games. The term derives from the fact that a player’s belief about his opponent’s history is not needed for computing a best-reply. The concept has been studied for the repeated prisoner dilemma [31, 13], and
then generalized to larger classes of games, see [12, 27, 19, 39, 40] and several others. In [34] belief-free equilibria are studied in the context of repeated congestion games over traffic networks. In [5] belief-free rationalizability is examined. Heller [17] has shown that belief-free equilibria are not robust, in the sense that only trivial belief-free equilibria may satisfy evolutionary stability.

A prior-free approach has been used by several authors in mechanism design. We refer the reader to the chapter [16] for a survey of the topic and to [9, 33, 11], among others, for some more recent contribution.

The tools that we have used to prove existence of pure Nash equilibria are the existence of an ordinal potential function and the finite improvement property. Both concepts and their relations have been studied by Monderer and Shapley [28].

We deal with efficiency of the equilibria in our game. The typical measures of inefficiency are the price of anarchy, i.e., the ratio of the worst equilibrium social cost over the optimum social cost [22, 21, 30], and the price of stability, i.e., the ratio of the best equilibrium social cost over the optimum social cost [3, 35]. In these papers the social cost is defined as the sum of the social costs of all the players. Since our buck-passing game is a constant-sum game, this definition of social cost would produce trivial results. This is why, in the spirit of Rawls [32], we define the social cost as the cost incurred by the player with the highest cost. Other authors have used different social cost functions, for instance [38, 26, 14].

To prove our main theorems we rely on some classical result on Markov chain theory relating the stationary measure of the chain with the abundance of spanning trees of the underlying graph. Such connection between the stationary measure and such combinatorial objects has been known for a long time, with the celebrated Markov chain tree theorem, attributed to Leighton and Rivest [23] (see [2] for a probabilistic proof). Computing the number of spanning trees of an undirected graph in terms of the Laplacian’s spectral properties goes back to Kirchhoff [20] and has been generalized to weighted directed graphs by Brooks et al. [7], see also [8]. Various generalization and variations of these results have been proposed recently, see, e.g., [4]. For other classical results in Markov chain theory we refer to some books such as [29, 1, 25].

2. The deterministic model

We consider a game played on a finite graph where players are the vertices of the graph. Each player chooses, once and for all, one of her
neighbors. At the beginning of the game one player is randomly drawn according to some distribution and is given a buck. This player will pass the buck to her chosen neighbor. This player, in turn, will pass the buck to her chosen neighbor and so on, and so forth. The cost that each player incurs is the asymptotic frequency of times she has the buck in her hands.

More formally, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite directed graph, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of vertices and $\mathcal{E}$ is the set of edges. Call $S_i$ the set of neighbors of vertex $i$. We then define a finite game as follows: each vertex $i \in \mathcal{V}$ is a player. The strategy set of player $i$ is $S_i$ and $S = \times_{i \in \mathcal{V}} S_i$ is the set of strategy profiles. At time 0 one vertex $i \in \mathcal{V}$ is drawn at random according to the measure $\mu$. Once each player has chosen her strategy and Nature has drawn one player according to $\mu$, the drawn player $i$, say, is given the buck, which is then passed to the player chosen in player $i$’s strategy, and so on and so forth. Define the random variable $\Theta_{i,t}(s)$ as follows:

$$\Theta_{i,t}(s) = \begin{cases} 1 & \text{if at time } t \text{ player } i \text{ has the buck,} \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

where, for every possible $s$,

$$\mathbb{P}(\Theta_{i,0}(s) = 1) = \mu(i). \tag{2.2}$$

The cost function $c_i : S \to \mathbb{R}$ of player $i$ is defined as follows:

$$c_i(s) = \mathbb{E} \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \Theta_{i,t}(s) \right], \tag{2.3}$$

where the expectation is according to the initial measure $\mu$. The profile of cost functions is denoted $c = (c_1, \ldots, c_n)$. We call the above game a buck-passing game (BPG) and we denote it $\Gamma(\mathcal{G}, \mu)$. Notice that the costs of all players add up to 1, so the BPG is equivalent to a zero-sum game.

Remark 2.1. Despite the fact that the description of the game involves an infinite discrete time horizon, the game is actually a static game, where strategies are chosen at the beginning of the game, once and for all. The cost in Eq. (2.3) is well defined, since the graph is finite and strategies do not vary over time, therefore, from some time on, the buck moves around in a periodic fashion, hence the limit in square brackets exists, although it is random, since it depends on the initial draw of Nature. Obviously the cost is not random.
Any strategy profile \( s \) induces a spanning subgraph \( \mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s) \subset \mathcal{G} \), where
\[
\mathcal{E}_s = \{(i, j): s_i = j\}.
\] (2.4)

It is not difficult to see that the spanning subgraph \( \mathcal{G}_s \) has \( n \) edges. Any subgraph of \( \mathcal{G} \) such that every vertex has an out-degree equal to 1 is induced by some strategy profile.

We assume that players choose their strategy without knowing Nature’s draw. We will focus our analysis on equilibria that do not depend on the initial distribution.

**Definition 2.1.** A strategy profile \( s^* \) is a Nash equilibrium (NE) if for all players \( i \in \mathcal{V} \) and for all strategies \( s_i \in S_i \) we have
\[
c_i(s^*) \leq c_i(s_i, s^*_i),
\] (2.5)
where \( s^*_i \) is the subprofile of strategies of all players different from \( i \).

A prior-free Nash equilibrium (PFNE) is a strategy profile that is a Nash equilibrium for all initial distributions \( \mu \).

Throughout the paper, unless explicitly stated, when we refer to Nash equilibrium, we mean pure Nash equilibrium.

**Definition 2.2.**
1. A directed graph \( \mathcal{G} \) is called strongly connected if for every \( i, j \in \mathcal{V} \), there exists a path from \( i \) to \( j \).
2. A simple cycle is a strongly connected subgraph where each vertex has in-degree and out-degree equal to 1.
3. A directed graph is called unicyclic if it contains exactly one simple cycle, and each vertex \( i \) admits exactly one outgoing edge.

**Theorem 2.3.** Consider a BPG \( \Gamma(\mathcal{G}, \mu) \). If the graph \( \mathcal{G} \) is strongly connected, then the game admits a PFNE, which induces a unicyclic spanning subgraph.

**Proof.** Since the graph \( \mathcal{G} \) is strongly connected, it admits a cycle. Consider a simple cycle \( \gamma \) in \( \mathcal{G} \) having the longest possible length. Again, by strong connectedness, there exists a unicyclic graph \( \kappa \supset \gamma \). Consider the strategy profile \( s^* \) that induces \( \kappa \). We now prove that \( s^* \) is a Nash equilibrium. Every player \( i \in \kappa \setminus \gamma \) has no incentive to deviate, since her cost under \( s^* \) is zero. No player in \( \gamma \) can find a deviation that guarantees a lower cost, since any \( s_i \neq s^*_i \) would produce a new simple cycle containing \( i \) and, by the assumption on \( \gamma \), the length of this new cycle would be smaller or equal to the length of \( \gamma \). It is not difficult to see that \( s^* \) remains an equilibrium, no matter what the initial state of the system is, and therefore it is prior-free. \( \square \)
Remark 2.2. A PFNE of our incomplete information game remains a NE in the complete information game where players are informed about the initial draw of Nature. Vice versa is not true, as the following example shows. Consider Fig. 1. On the left we have the graph $\mathcal{G}$ on which the game is played. Assume that Nature draws the red vertex in the central figure. Notice that the strategy profile described by the solid line in this figure is a Nash equilibrium in the game where every player is informed about Nature’s draw. This is not a PFNE. Indeed, take $\Gamma(\mathcal{G}, \mu)$ where $\mu$ puts positive mass on one of the green vertices. Then, the last of them—in the clockwise order—prefers to change her strategy. Such a deviation gives rise to the PFNE presented—in solid—in the right picture. Notice also that this PFNE is quite unfair: even if the game is symmetric, in such equilibrium only two player pay the whole cost.

![Figure 1](image)

**Figure 1.** Left: the graph. Middle: A NE that is not prior-free. Right: a PFNE

2.1. **The finite improvement property.** Following the definition of Monderer and Shapley [28], we say that a game satisfies the finite improvement property if the following holds

**Definition 2.4.** Given a game $(\mathcal{V}, S, c)$, a path is a sequence of strategy profiles $(s^0, s^1, \ldots)$ such that $s^{k+1}$ differs from $s^k$ only in one coordinate. A path $(s^k)_k$ is an improvement path with respect to $\Gamma$ if

$$c_i(s^{k+1}) - c_i(s^k) < 0,$$

where $i$ is the coordinate where $s^k$ and $s^{k+1}$ differ. We call the difference in costs in Eq. (2.6) the improvement of player $i$.

The game $(\mathcal{V}, S, c)$ has the finite improvement property (FIP) if every improvement path is finite.

We now show that the FIP holds whenever the graph is strongly connected.
Theorem 2.5. If \( \mathcal{G} \) is strongly connected, then, for any prior distribution \( \mu \), the BPG \( \Gamma(\mathcal{G}, \mu) \) has the FIP. Moreover any improvement paths involves less than \( 3n/2 \) steps.

We call \( U \subset S \) the subset of strategy profiles that induce a spanning connected subgraph, which is actually a unicyclic subgraph. This subset of profiles is important for the following reasons:

- If \( s \in U \), there cannot exist an improvement of any player \( i \) that produces a strategy profile \( s' \notin U \). In other words, if an improvement path reaches \( U \), then is remains there forever.
- Every NE in \( U \) is prior-free. Indeed, consider the deterministic Markov chain in which the buck follows the edges of \( \mathcal{G}_s \) for some \( s \in U \). Such a chain has a unique ergodic component, so, the costs associated to \( s \) do not depend on \( \mu \).

Proof of Theorem 2.5. The idea underlying the proof is that

1. An improvement path starting in \( S \setminus U \) after finitely many steps reaches either \( U \) or a pure Nash equilibrium in \( S \setminus U \).
2. An improvement path starting in \( U \) remains there forever and does not pass twice by the same strategy profile.

This, together with Theorem 2.3 and the finiteness of \( S \), implies that after finitely many steps the improvement path reaches a NE, and then it stops.

Let \( s' \) be obtained from \( s \) via an improvement for player \( i \). By definition of improvement path, this implies that \( i \) lies in one of the cycles of \( \mathcal{G}_s \). The improvement \( s \to s' \) can be of one of the following two types.

(1) In the graph \( \mathcal{G}_s \), the vertex \( s'_i \) is in the same component as the vertex \( i \). Since \( s \to s' \) is part of an improvement path, \( \mathcal{G}_{s'} \) and \( \mathcal{G}_s \) have the same cycles, except for the cycle that contains \( i \), which is longer in \( \mathcal{G}_{s'} \) than in \( \mathcal{G}_s \). This type of improvement is the only one that can occur once the path reaches \( U \).

(2) In the graph \( \mathcal{G}_s \), the vertex \( s'_i \) is not in the same component as the vertex \( i \). This means that, to achieve the improvement, player \( i \) breaks the cycle where she was, so, \( \mathcal{G}_{s'} \) has one less cycle than \( \mathcal{G}_s \). As a consequence, every vertex \( j \) that, in \( \mathcal{G}_s \), was in the same connected component as \( i \), has now zero cost under \( s' \).

By the finiteness of \( \mathcal{V} \), improvements of the first type can occur at most \( n \) times, and improvements of the second type can occur at most \( n/2 \) times. \( \square \)
2.2. **Fairness of equilibria.** Usually efficiency of equilibria of cost games is considered in terms of the social cost, i.e., the sum of the costs incurred by each player. The typical measures of inefficiency are the *price of anarchy* (PoA) (the ratio of the social cost of the worst equilibrium over the optimum social cost) or the *price of stability* (PoS) (the ratio of the social cost of the best equilibrium over the optimum social cost). As we mentioned at the beginning, the game has constant sum, therefore it does not make sense to measure efficiency of equilibria in the above sense. It is possible to use a different social cost to measure inefficiency. There are examples of this in the literature [38, 26, 14]. In our case, we consider a Rawlsian social cost, that is to say the cost of the player who pays the most, see [32].

**Definition 2.6.** We define the *social cost* of a strategy profile as

\[
SC(s) = \max_{i \in V} c_i(s),
\]

the *price of anarchy* as

\[
\text{PoA} = \frac{\max_{s \in \text{NE}} SC(s)}{\min_{s \in S} SC(s)},
\]

and the *price of stability* as

\[
\text{PoA} = \frac{\min_{s \in \text{NE}} SC(s)}{\min_{s \in S} SC(s)},
\]

where NE is the class of Nash equilibria.

**Proposition 2.7.** For any BPG defined on a strongly connected graph we have

1. PoS = 1.
2. PoA ≤ n/2.

*Proof.* To prove the result about the PoS, consider that, by the construction described in the proof of Theorem 2.3, there exists a NE supported on a unicycle that includes the longest possible cycle in $G$. This is also the best that a social planner can achieve.

For the PoA, given the strong connectedness of $G$, at least two players must have a positive cost in equilibrium. If they are on the same cycle, they pay the same cost, so nobody can pay more than 1/2. When the graph contains a Hamiltonian cycle, under the social optimum, every player pays 1/n. This gives the bound on the PoA.

2.3. **Some examples.** Here we provide some instances that show that the bound for the PoA in Proposition 2.7 is tight. Moreover we show one instance where PoA = PoS = 1. In this section we consider directed graphs where $(i, j) \in E$ iff $(j, i) \in E$.
Figure 2. The complete graph, the cycle and the ring-star graph with 6, 10, and 7 vertices, respectively. For ease of representation in the first picture we used bidirectional arrows instead of drawing two arrows for each couple of vertices. These graphs admit a Hamiltonian cycle.

The complete graph. Call $K_n$ the complete graph of size $n$. The following proposition is straightforward.

**Proposition 2.8.** Consider the BPG $\Gamma(\mathcal{G}, \mu)$ with $\mathcal{G} = K_n$. We have that

1. If $s \in S$ is a NE, then $\mathcal{G}_s$ is a Hamiltonian cycle. Hence $s$ is prior-free.
2. PoA = 1

The cycle. Call $C_n$ the undirected cycle with $n$ vertices. In this case the strategy of each player can be only her left or right neighbor. This configuration can give rise to very unfair equilibria, as the following propositions shows. Its proof is straightforward and it is illustrated in Fig. 1.

**Proposition 2.9.** Consider the BPG $\Gamma(\mathcal{G}, \mu)$, with $\mathcal{G} = C_n$. Then

1. All $s \in S$ induce either a Hamiltonian cycle or the disjoint union of unicycles whose cycle has length 2.
2. PoA = $n/2$.

The ring-star graph. Call $R_n$ the graph on the right of Fig. 2. In this graph $(n - 1)$ are connected to their left and right neighbors and to a central player, who is therefore connected to all the other players.

**Proposition 2.10.** Consider the BPG $\Gamma(\mathcal{G}, \mu)$, with $\mathcal{G} = R_n$. Then

1. If $s^\star$ is a PFNE, then $s^\star \in U$.
2. PoA = $n/3$.

The proof of Proposition 2.10 is in the Appendix.
3. The stochastic model

In this section we present a generalization of the model studied in Section 2. Given a graph $G$, for each $i \in V$, call $\Sigma_i$ the set of probability vectors $\pi_i$ over the set $V$ such that $\pi_{ij} = 0$ if there is no edge from $i$ to $j$. Throughout the paper player $i$'s strategy space $S_i$ is assumed to be a non-empty closed subset of $\Sigma_i$.

Some interesting properties arise when we impose some constraints to the strategies of all players. In particular we assume that $G$ is strongly connected and that for each player $i$ the strategy $\pi_i$ is such that $\pi_{ij} > 0$ if there is an edge from $i$ to $j$, i.e., $(i,j) \in E$.

**Lemma 3.1.** Let $\eta > 0$. Assume that $G$ is strongly connected and that for each player $i \in V$ the vector $\pi_i$ satisfies

$$
\pi_{ij} \begin{cases} 
\geq \eta & \text{if } (i,j) \in E, \\
0 & \text{if } (i,j) \notin E.
\end{cases}
$$

Then the transition matrix $\Pi = [\pi_{ij}]$ is irreducible, so the associated Markov chain has a unique stationary distribution.

**Proof.** Since the graph is strongly connected, Eq. (3.1) implies that it is possible to go from any $i$ to any $j$ in finite time with positive probability. This is just the definition of irreducible transition matrix, see [25, Section 1.3]. 

For every $i \in V$, the cost function of player $i$ is the one defined in Eq. (2.3), except that now the expectation is taken with respect to the initial measure $\mu$ and to the transition matrix $\Pi$. We call this game $\Gamma(G, S, \mu)$ stochastic buck-passing game (SBPG).

Notice that, when the set $S$ is such that transition matrix $\Pi$ is always irreducible, the vector cost is nothing else than the unique stationary measure of the Markov chain and the measure $\mu$ does not play any role.
role in its computation. A by-product of this is that every pure NE is also prior-free. A remarkable property of the SBPG is that, under the assumption that the chain is irreducible for every strategy profile, a pure NE exists. This is stated more rigorously in the following theorem.

**Theorem 3.2.** Consider a SBPG $\Gamma(\mathcal{G}, S, \mu)$ such that $\mathcal{G}$ is strongly connected and for all $i \in \mathcal{V}$ the set $S_i$ is a set of vectors that satisfy Eq. (3.1). Then the game admits a pure PFNE.

Monderer and Shapley [28] proved that an ordinal potential game admits a pure Nash equilibrium. So, Theorem 3.2 will be proved by showing that the game is an ordinal potential game.

**Remark 3.1.** Even when $S_i = \Sigma_i$ for all $i \in \mathcal{V}$, the SBPG is not the mixed extension of the buck-passing game (MBPG) studied in Section 2, as the following example shows. Let $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{G}$ be the complete graph, and $\mu$ be any probability measure on $\mathcal{V}$. Consider the following strategy profile (see Fig. 4):

$$
\pi_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right), \quad \pi_2 = (1, 0, 0), \quad \pi_3 = (0, 1, 0).
$$

(3.2)

![Figure 4](image)

**Figure 4.** Left: A complete graph with 3 vertices. Right: The strategy in Eq. (3.2).

In the SBPG this strategy profile gives rise to an irreducible transition matrix, whose stationary distribution is

$$
\begin{pmatrix}
\frac{2}{5} & \frac{2}{5} & \frac{1}{5}
\end{pmatrix},
$$

(3.3)

which is equal to the cost vector induced by $\pi$. 

In the MBPG the mixed strategy profile of Eq. (3.2) produces the following cost for player 1:

$$c_1(\pi) = \frac{1}{2} \mathbb{E} \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \Theta_{1,t}(2, 1, 2) \right] + \frac{1}{2} \mathbb{E} \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \Theta_{1,t}(3, 1, 2) \right]$$

$$= \frac{11}{22} + \frac{11}{23} = \frac{5}{12} \neq \frac{2}{5}.$$  

(3.4)

3.1. **Ordinal potential games.** We first recall the definition of potential and ordinal potential games. Then we apply these concepts to BPGs.

**Definition 3.3.** 1. A game \((\mathcal{V}, S, C)\) is a **potential game** if there exists a potential function \(\Psi : S \to \mathbb{R}\) such that for every player \(i\) and every pair of strategy profiles \(s = (s_1, \ldots, s_i, \ldots, s_n)\) and \(s' = (s_1, \ldots, s'_i, \ldots, s_n)\), we have

$$c_i(s) - c_i(s') = \Psi(s) - \Psi(s').$$  

(3.5)

2. A game \((\mathcal{V}, S, C)\) is an **ordinal potential game** if there exists a potential function \(\Psi : S \to \mathbb{R}\) such that for every player \(i\) and every pair of strategy profiles \(s = (s_1, \ldots, s_i, \ldots, s_n)\) and \(s' = (s_1, \ldots, s'_i, \ldots, s_n)\), we have

$$c_i(s) - c_i(s') < 0 \iff \Psi(s) - \Psi(s') < 0.$$  

(3.6)

**Definition 3.4.** Given a directed graph \(\mathcal{G}\), a spanning tree \(\tau\) rooted at \(i\) is an acyclic subgraph of \(\mathcal{G}\) such that all the vertices except \(i\) have out-degree 1. We call \(\mathcal{T}(i)\) the class of spanning trees rooted at \(i\).

**Definition 3.5.** Given a strategy \(\pi\) and a subset \(C \subset \mathcal{C}\), we define

$$w_\pi(C) = \prod_{(i,j) \in C} \pi_{ij}.$$  

(3.7)

**Theorem 3.6.** A SBPG \(\Gamma(\mathcal{G}, S, \mu)\) that satisfies Eq. (3.1) admits the following ordinal potential function

$$\Psi(\pi) := - \sum_{j \in \mathcal{V}} \sum_{\tau \in \mathcal{T}(j)} w_\pi(\tau).$$  

(3.8)

If, in addition, \(S\) is finite, then the SBPG has the FIP, i.e., a NE can be found in finite time by following any path of myopic unilateral deviations.

The following lemma will be the main ingredient in the proof of Theorem 3.6.
Lemma 3.7 (Markov chain tree theorem). Let $\Pi$ be an irreducible transition matrix. Then its unique stationary distribution $\rho_\Pi$ can be computed as

$$\rho_\Pi(i) = \frac{\sum_{\tau \in \mathcal{T}(i)} w_\pi(\tau)}{\sum_{i \in V} \sum_{\tau \in \mathcal{T}(i)} w_\pi(\tau)}.$$  \hspace{1cm} (3.9)

A proof of the Markov chain tree theorem can be found, for instance, in [2] or [1, Theorem 9.10 and Corollary 9.11].

Proof of Theorem 3.6. Consider the fraction on the right hand side of Eq. (3.9). Its numerator does not depend on $\pi_i$. Consider now two irreducible transition matrices $\Pi, \Pi'$ where $\pi_j = \pi'_j$ for all $j \neq i$. We have

$$\rho_\Pi(i) > \rho_{\Pi'}(i) \iff \sum_{i \in V} \sum_{\tau \in \mathcal{T}(i)} w_\pi(\tau) < \sum_{i \in V} \sum_{\tau \in \mathcal{T}(i)} w_{\pi'}(\tau)$$  \hspace{1cm} (3.10)

Since $c_i(\pi_1, \ldots, \pi_i, \ldots, \pi_n) = \rho_\Pi(i)$, we have the result. \hfill \Box

Proof of Theorem 3.2. The existence of a NE follows from the fact that the SBPGs are ordinal potential games, [28].

To prove that there exists a PFNE consider that, when the transition matrix $\Pi$ is irreducible, the cost is the same for every possible initial distribution $\mu$. \hfill \Box

3.2. A spectral characterizations of the ordinal potential function. We now provide a representation of the potential of BPGs that uses the spectrum of the Markov chain induced by the strategy profile. We assume that, for every strategy profile $\pi$, the associated transition matrix $\Pi$ is irreducible. This holds, for instance, when Eq. (3.1) is satisfied. In Section 3.1 we showed that the ordinal potential evaluated at $\pi \in S$ measures the abundance of spanning trees in the weighted graph induced by the Markov chain $\Pi$. Therefore, an alternative way to compute this abundance immediately gives an alternative representation of the ordinal potential function. The tool that we use is the celebrated matrix-tree theorem, which generalizes the original work of Kirchhoff [20], see, e.g, [7, 8]. Call

$$L_\pi = I - \Pi$$  \hspace{1cm} (3.11)

the Laplacian associated with the strategy profile $\pi$.

Theorem 3.8. If the transition matrix $\Pi$ is irreducible, then

$$\Psi(\pi) = -\prod_{i=2}^{n} \lambda^{(i)}_\pi,$$  \hspace{1cm} (3.12)
where \( \{\lambda_\pi^{(1)}, \ldots, \lambda_\pi^{(n)}\} \) is the spectrum of \( L_\pi \) and \( \lambda_\pi^{(1)} = 0 \). Moreover, \( \lambda_\pi^{(1)} = 0 \) and all the other elements of the spectrum are positive.

The proof of Theorem 3.8 is in the Appendix.

4. Conclusions and open problems

We have introduced a class of games that represent the desire of agents to unload their responsibilities to other agents. We have given two different variations, one deterministic and one stochastic of the game. In the deterministic version each player passes the buck to one neighbor of her choice. In the stochastic version each player chooses a distribution that determines who, among her neighbors, receives the buck. In both cases we showed existence of a pure Nash equilibrium when the graph on which the game is played is strongly connected.

4.1. Open problems. In the game that we have considered the players choose their strategy at the beginning of the game and play the same action whenever they receive the buck. We plan to consider a sequential version of the game where every time a player has the buck, she can decide what to do, based on the history of the game.

In the model that we considered the network is known to all players and their equilibrium actions are heavily based on this knowledge. We want to consider a variation of the model where players have only a local knowledge of the network, that is, they just know who their neighbors are. Players will then use some adaptive strategies, whose convergence properties we want to study.

Another issue that is worth further investigation is more of combinatorial flavor. The presence and the abundance of Hamiltonian cycles in a random graph, or even the length of the maximal cycle, is a classical topic in combinatorics and theoretical computer science. In the same vein one may ask questions about the number of NE in a random graph, or about the presence of a NE with some particular feature.

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Proofs of Section 2.

Proof of Proposition 2.10. We start by proving the existence of a NE $\mathbf{s}^*$ that induces a cycle of size 3. Let the vertices in the cycle have labels $\{1, \ldots, n-1\}$ and let the center of the star be the vertex having label $n$. Define $\mathbf{s}^* = (s_1^n, \ldots, s_n^n)$ as follows:

$$s_1^n = 2, \quad s_2^n = n, \quad s_3^n = 2, \quad s_n^n = 1, \quad s_i^n = n, \quad \text{for all } i \notin \{1, 2, 3, n\}. \quad (5.1)$$

This is the situation represented in Fig. 3. Only players 1, 2 and $n$ have a positive cost and may be potentially interested in changing their strategies. Actually player 1 is indifferent between $s_1^n$ and the possible alternative strategy $n - 1$, and she prefers $s_1^n$ to strategy $n$. Player 2 prefers $s_2^n$ to the alternative strategies 1 or 3. Finally, player $n$ is indifferent between $s_n^n$ and 3 and prefers $s_n^n$ to any of the alternatives $j \in \{2, 4, 5, \ldots, n-1\}$.

To prove that the strategy profile $\mathbf{s}^*$ achieves $\text{PoA} = n/3$ we need to prove the following claim.

Claim 5.1. A NE $\mathbf{s}'$ that induces a cycle of length 2 involving the players $i$ and $j$, can exist only if the initial distribution $\mu$ is such that $c_i(\mathbf{s}') = c_j(\mathbf{s}') = 0$.

Proof. There are only two ways to form a cycle of length 2:

1. The vertices of the 2-cycle are two consecutive vertices of the ring. In this case, at least one of the two vertices, say $i$, is such that $s_i'^n \neq i$. Therefore, if $\mu$ puts some mass on $i$, then $i$ prefers the strategy $n$ to $s_i'^n$.
2. The vertices of the 2-cycle are some $i \in \{1, \ldots, n-1\}$ and $n$. In this case, if the strategy of a neighbor $j \neq n$ of $i$ is $s_j = i$, then player $n$ prefers to deviate to $j$. If the strategy $s_j = n$, then $i$ prefers to deviate to $j$. 

Notice that one of the above scenarios must occur.

This implies that the numerator in Eq. (2.8) cannot exceed \(1/3\). Moreover, it also implies that the NE \(s'\) cannot be prior-free. Notice that—thanks to the geometry of \(R_n\)— if a strategy profile induces two or more cycles, then one of them has length 2. Therefore, the claim above immediately implies the first point of Proposition 2.10.

**Proofs of Section 3.** To prove Theorem 3.8 we will use the following theorem, see [7, 8].

**Theorem 5.2 (Matrix-tree theorem).** Let \(i \in \mathcal{V}\) be the state of an irreducible Markov chain having Laplacian \(L\). Then

\[
\sum_{\tau \in \mathcal{F}(i)} w(\tau) = \det L_{(ii)},
\]

Moreover, we need a definition and a corollary of Theorem 5.2.

**Definition 5.3.** We call adjugate of \(L\) the matrix \(\text{adj}(L)\) such that

\[
\text{adj}(L)_{ij} = (-1)^{i+j} \det L_{(ij)}.
\]

**Corollary 5.4.** The potential \(\Psi\) can be represented as

\[
\Psi(\pi) = -\text{tr} \text{adj}(L)_{ij}
\]

**Proof of Theorem 3.8.** Consider the spectral decomposition of \(L_\pi\)

\[
L_\pi = UJU^{-1}
\]

for some Jordan matrix \(J\) and some orthogonal matrix \(U\). We have

\[
\text{tr} \text{adj}(L)_{ij} = \text{tr}(\text{adj}(U) \text{adj}(J) \text{adj}(U^{-1}))
\]

\[
= \text{tr}(\det(U)U^{-1} \text{adj}(J) \det(U^{-1})U)
\]

\[
= \text{tr}(U^{-1} \text{adj}(J)U)
\]

\[
= \text{tr} \text{adj}(J)
\]

\[
= \prod_{i=2}^{n} \lambda_{(i)}^{(j)},
\]

where the first equality is due to the fact that adjugate and product commute; the second equality stems from the fact that the adjugate of a full rank matrix is the inverse of that matrix times its determinant; the third equality derives from \(\det(M^{-1}) = [\det(M)]^{-1}\), when \(M\) is full rank; the fourth is just the invariance of trace with respect to change of basis; for the last one consider that \(L_\pi\) is the Laplacian of an irreducible chain, so the only nonzero element on the diagonal of \(\text{adj}(J)\) is the cofactor \((1|1)\).