DYNAMICAL BEHAVIOUR OF FRACTIONAL-ORDER PREDATOR-PREY SYSTEM OF HOLLING-TYPE

KOLADE M. OWOLABI∗

Faculty of Mathematics and Statistics, Ton Duc Thang University
Ho Chi Minh City, Vietnam

ABSTRACT. In this paper, the local derivative in time is replaced with the Caputo-Fabrizio fractional derivative of order \( \alpha \in (0,1) \). A two-step fractional version of the Adams-Bashforth method is formulated for the approximation of this derivative. To enhance the correct choice of parameters when numerically simulating the full-system, we examine the stability analysis of the main equation. Two important examples are drawn to explore the dynamic richness of the predator-prey model with Holling type. Simulation results at different instances of \( \alpha \) is in agreement with the theoretical findings.

1. Introduction. The study of fractional-order differential equations have been of great interest. Fractional derivative of the Caputo, Caputo-Fabrizio, Riemann-Liouville or the Atangana-Baleanu type of derivatives are important mathematical tools which have been used for developing models in application areas of biology, control, economics and finance, electrical circuits/networks, rheology, nuclear physics, viscoelasticity, chemical physics, fluid flows, signal processing, dynamical phenomena in self-similar and porous structures [5–9,20], just to mention a few.

The Caputo-case model was presented in [24,31] for fractional ordinary differential equations. The Caputo-Fabrizio model type was considered in [7,8,11,12,25,26], the Riemann-Liouville type of fractional reaction-diffusion equations have been considered in [20–24,27–31]. The most recent Atangana-Baleanu fractional derivative that was formulated based on the nonlocal and nonsingular kernels when applied to model various phenomena in science and engineering was discussed in [1–4,10,13–16,32,34–39] and references therein. To our knowledge, application of the Caputo-Fabrizio type to model the dynamic behaviour of predator-prey system with Holling type functional has not been reported. Hence, we are motivated in this work by considering the general time-fractional differential equation

\[
_{0}^{CF}D_{t}^{\alpha}u(t) = f(t, u(t)), \quad u(0) = u_{0}, \quad 0 < \alpha \leq 1
\]

(1)

where \( _{0}^{CF}D_{t}^{\alpha} \) is the Caputo-Fabrizio fractional derivative of order \( \alpha \) defined as [7, 11,12]

\[
_{0}^{CF}D_{t}^{\alpha}u(t) = \frac{M(\alpha)}{1-\alpha} \int_{0}^{t} u'(\tau) \exp \left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau,
\]

(2)

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∗ Corresponding author: kolademathewowolabi@tdtu.edu.vn (K. M. Owolabi).
where $M(\alpha)$ is a normalization function, such that $M(0) = M(1) = 1$. In what follows, a quick tour of some properties of fractional differentiation will be presented.

The Caputo fractional derivative of order $\alpha > 0$ is defined by

$$C_0^\alpha D_t y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\lambda, y(\lambda))(t - \lambda)^{\alpha - 1} d\lambda. \quad (3)$$

The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1]$ for a function $u(t) \in C^1([0, b], \mathbb{R}^n), b > 0$ is given by [20, 28]

$$RL_0^\alpha D_t u(t) = \frac{d^n}{dt^n} T_0^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dt^n} \int_0^t (t - \xi)^{n-\alpha-1} u(\xi) d\xi, \quad (4)$$

for all $t \in [0, b]$ and $n - 1 < \alpha < n$, where $n > 0$ is an integer.

Atangana and Baleanu [9] proposed the following derivatives in the sense of Caputo and Riemann-Liouville. Let $y \in H^1(a, b), a < b, \alpha \in [0, 1]$ then, the definition of the Atangana and Baleanu fractional derivative in Caputo sense is given as [10]

$$ABC_0^\alpha D_t[y(t)] = M(\alpha) \frac{1}{1-\alpha} \int_a^t y'(\tau) E_\alpha \left[-\alpha \frac{(t - \tau)^\alpha}{1-\alpha} \right] d\tau, \quad (5)$$

where $M(\alpha)$ has the same properties as in the case of the Caputo-Fabrizio fractional derivative.

Let $y \in H^1(a, b), a < b, \alpha \in [0, 1]$ then, the definition of the Atangana-Baleanu fractional derivative in Riemann-Liouville sense becomes [10]

$$ABR_0^\alpha D_t[y(t)] = M(\alpha) \frac{d}{dt} \int_a^t y(\tau) E_\alpha \left[-\alpha \frac{(t - \tau)^\alpha}{1-\alpha} \right] d\tau. \quad (6)$$

The rest of this paper is structured as follows. The main model is introduced in Section 2. We proceed with the mathematical analysis of the local derivative in order to ascertain the correct choice of the parameters when numerically simulating the model. Numerical method for the approximation of the Caputo-Fabrizio fractional derivative is presented in Section 3. Simulation experiment for different $\alpha$ values is reported in Section 4. We finally conclude with Section 5.

2. Numerical method for fractional differential equations with the Caputo-Fabrizio fractional derivative. In this section, we introduce the newly formulated numerical scheme that is based on Adams-Bashforth scheme for the approximation of the Caputo-Fabrizio fractional derivative. By following closely the recent idea presented in [10], we consider the general fractional differential equation (1) with the Caputo-Fabrizio fractional derivative (2). By using the fundamental calculus theorem, equation (1) simply transforms into

$$u(t) - u(0) = \frac{1-\alpha}{M(\alpha)} f(t, u(t)) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau, u(\tau)) d\tau \quad (7)$$

in such that

$$u(t_{n+1}) - u(0) = \frac{1-\alpha}{M(\alpha)} f(t_n, u(t_n)) + \frac{\alpha}{M(\alpha)} \int_0^{t_{n+1}} f(t, u(t)) dt \quad (8)$$
where
\[ u(t_n) - u(0) = \frac{1 - \alpha}{M(\alpha)} f(t_{n-1}, u(t_{n-1})) + \frac{\alpha}{M(\alpha)} \int_0^{t_n} f(t, u(t))dt. \] (9)

By subtracting (9) from (8) we have
\[ u(t_{n+1}) - u(t_n) = \frac{1 - \alpha}{M(\alpha)} \{ f(t_n, u_n) - f(t_{n-1}, u_{n-1}) \} + \frac{\alpha}{M(\alpha)} \int_{t_n}^{t_{n+1}} f(t, u(t))dt \]
where
\[ \int_{t_n}^{t_{n+1}} f(t, u(t))dt = \int_{t_n}^{t_{n+1}} \left\{ \frac{f(t_n, u_n)}{h} (t - t_{n-1}) - \frac{f(t_{n-1}, u_{n-1})}{h} (t - t_n) \right\} dt \]
= \frac{3h}{2} f(t_n, u_n) - \frac{h}{2} f(t_{n-1}, u_{n-1}). \] (11)

So that,
\[ u(t_{n+1}) - u(t_n) = \frac{1 - \alpha}{M(\alpha)} \{ f(t_n, u_n) - f(t_{n-1}, u_{n-1}) \} + \frac{3ah}{2M(\alpha)} f(t_n, u_n) \]
- \frac{\alpha h}{2M(\alpha)} f(t_{n-1}, u_{n-1})
which means that
\[ u(t_{n+1}) - u(t_n) = \left( \frac{1 - \alpha}{M(\alpha)} + \frac{3ah}{2M(\alpha)} \right) f(t_n, u_n) + \left( \frac{1 - \alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) f(t_{n-1}, u_{n-1}). \]
Therefore,
\[ u_{n+1} = u_n + \left( \frac{1 - \alpha}{M(\alpha)} + \frac{3ah}{2M(\alpha)} \right) f(t_n, u_n) + \left( \frac{1 - \alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) f(t_{n-1}, u_{n-1}) \]
which is the required two-step Adams-Bashforth scheme for numerical approximation of the Caputo-Fabrizio fractional derivative. It should be noted that if \( \alpha = 1 \), we recover the classical Adams-Bashforth method. The above fractional scheme was studied completely in [10, 25, 26] and the convergence and stability results are summarized in the following theorems.

**Theorem 2.1.** Let \( u(t) \) be a solution of \( 0^\alpha D_0^\beta u(t) = f(t, u(t)) \) where \( f \) is a continuous function bounded for the Caputo-Fabrizio fractional derivative, we have
\[ u_{n+1} = u_n + \left( \frac{1 - \alpha}{M(\alpha)} + \frac{3ah}{2M(\alpha)} \right) f(t_n, u_n) + \left( \frac{1 - \alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) f(t_{n-1}, u_{n-1}) + R_n^\alpha \]
where \( \|R_n^\alpha\| \leq M. \)

**Theorem 2.2.** Let \( u(t) \) be a solution of \( 0^\alpha D_0^\beta u(t) = f(t, u(t)) \), for every \( n \in N \)
\[ \|u_{n+1} - u_n\|_\infty < \frac{1 - \alpha}{M(\alpha)} \|f(t_n, u_n) - f(t_{n-1}, u_{n-1})\|_\infty + \frac{\alpha h^{n+1}(n + 1)!}{4M(\alpha)} \]
such that if \( \|f(t_n, u_n) - f(t_{n-1}, u_{n-1})\|_\infty \to 0 \) as \( n \to \infty \), then \( \|u_{n+1} - u_n\|_\infty \to 0 \) as \( n \to \infty \).
3. Analysis of the main equation with local derivative. The main model considered in this paper is the dynamics of a predator-prey system with Holling type function response given as:

\[ \frac{du_1}{dt} = \varphi u_1(t) \left( 1 - \frac{u_1(t)}{\kappa} \right) - \frac{u_1(t)u_2(t)}{\phi + \psi u_1(t) + u_1^2(t)} \]
\[ \frac{du_2}{dt} = u_2(t) \left( \frac{\epsilon u_1(t)}{\phi + \psi u_1(t) + u_1^2(t)} - \sigma \right), \]

where \( u_1 \) and \( u_2 \) being a function of time that represent the species population densities for respective prey and predator. The carrying capacity of the prey is denoted by \( \kappa \), while the death rate of the predator is given by \( \sigma \), the growth rate of prey and maximum predation rate are denoted by \( \varphi > 0 \) and \( \epsilon > 0 \), respectively. The parameter \( \phi > \alpha \) stands for half-saturation constant, \( \psi \) is given in such a way that system (13) does not vanish for positive \( u_1 \) and \( b > -2\sqrt{\phi} \), we shall study this dynamic in strictly first quadrant where all the parameters are biologically feasible.

In this section, we demonstrate the existence and stability of positive equilibria of system (13). From above equation, it is easy to see that if there is a positive equilibrium, then we obtain

\[ \frac{\epsilon u_1}{\phi + \psi u_1 + u_1^2} \sigma = 0. \]

For convenience, we let \( f_1 = (\epsilon - \psi \sigma)^2 - 4\phi \sigma^2 \), \( f_2 = \epsilon - \psi \sigma \), \( f_3 = \frac{\epsilon - \psi \sigma}{2\sigma} \), and \( f_4 = \frac{\epsilon - \psi \sigma}{\sigma} \). After some algebraic manipulations, we can see that the set \( S = \{(\epsilon, \psi, \sigma, \phi, \kappa) : 0 < f_3 < \kappa, f_1 = 0 \} \) is a saddle-node bifurcation surface. Obviously, system (13) has four equilibrium points \( E_0(0, 0), E_1(\kappa, 0), E_2(\bar{u}_1, \bar{u}_2) \) and \( E_3(\bar{u}_1^*, \bar{u}_2^*) \), where

\[ \bar{u}_1 = \frac{\epsilon - \psi \sigma - \sqrt{(\epsilon - \psi \sigma)^2 - 4\phi \sigma^2}}{2\sigma}, \quad \bar{u}_2 = \frac{\epsilon - \psi \sigma + \sqrt{(\epsilon - \psi \sigma)^2 - 4\phi \sigma^2}}{2\sigma}, \]
\[ \bar{u}_1^* = \varphi \left( \phi + \psi \bar{u}_1 + \bar{u}_1^2 \right) \left( 1 - \frac{\bar{u}_1}{\kappa} \right), \quad \bar{u}_2^* = \varphi \left( \phi + \psi \bar{u}_2 + \bar{u}_2^2 \right) \left( 1 - \frac{\bar{u}_2}{\kappa} \right). \]

We also let \( f_0 = \frac{\epsilon - \sqrt{(\epsilon - \psi \sigma)^2 - 4\phi \sigma^2}}{2\sigma} \bar{u}_1 \). From the Jacobian or community matrix at point \((u_1, u_2)\) we obtain the characteristic equation

\[ \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0, \]

where \( a_{11} = \varphi - 2\frac{\psi u_1}{\kappa} - u_2 \frac{\phi - u_1^2}{(\phi + \psi u_1 + u_1^2)^2} \), \( a_{12} = \frac{\psi u_1}{(\phi + \psi u_1 + u_1^2)^2} \), \( a_{21} = u_2 \frac{\phi - u_1^2}{(\phi + \psi u_1 + u_1^2)^2} \)

and \( a_{22} = \frac{\epsilon u_1}{\phi + \psi u_1 + u_1^2} - \sigma \). From qualitative analysis, we obtain the following result.

**Theorem 3.1.** If \( \psi^2 < 3\phi \) and \( \bar{u}_1 < \kappa \), \( \min \left\{ u_1^*, f_0, 1/2(\sqrt{12\phi - 3\psi^2} - \psi) \right\} \), then the dynamical predator-prey system (13) has three equilibrium points, \( E_0(0, 0) \) (saddles), \( E_1(\kappa, 0) \) (saddle nodes), and globally asymptotically stable point \( E_2(\bar{u}_1, \bar{u}_2) \) respectively.

**Proof.** To prove that system (13) is globally asymptotically stable at \( E_2(\bar{u}_1, \bar{u}_2) \), we require to show that no periodic orbits exist in \( \mathbb{R}^+ = \{(u_1, u_2) | u_1 \geq 0, u_2 \geq 0 \} \).
With substitution $dt = (\phi + \psi_1 + \psi^2_1) d\xi$ in (13), we obtain
\[
\frac{du_1}{d\xi} = \varphi u_1 \left(1 - \frac{u_1}{\kappa}\right) (\phi + \psi + u^2_1) - u_1 u_2,
\]
\[
\frac{du_2}{d\xi} = \epsilon u_1 u_2 - \sigma u_2 (\phi + \psi + u^2_1).
\]
(15)

By following [17], we apply Dulac function $D(u_1, u_2) = \frac{u_1 - 1}{u_1} + \frac{u_2 - 1}{u_2}$ for dynamic system (15), then
\[
\text{div}_{(\sim)} = \frac{\partial}{\partial u_1} [A(u_1, u_2)D(u_1, u_2)] + \frac{\partial}{\partial u_2} [B(u_1, u_2)D(u_1, u_2)]
\]
\[
= \frac{\varphi}{\kappa u_2} (-3u^2_1 - 2u_1 (\psi - \kappa) + \kappa \psi - \phi),
\]
where
\[
A(u_1, u_2) = \varphi u_1 \left(1 - \frac{u_1}{\kappa}\right) (\phi + \psi + u^2_1) - u_1 u_2,
\]
and
\[
B(u_1, u_2) = \epsilon u_1 u_2 - \sigma u_2 (\phi + \psi + u^2_1).
\]

It is clear to see that $4(\psi - \kappa)^2 + 12(\kappa \psi - \phi) < 0$ for $0 < \kappa < \frac{\sqrt{12\phi - 3\psi^2 - \psi}}{2}$ and $\psi^2 < 3\phi$. Then, by Dulac’s theorem $\text{div}_{(\sim)} < 0$, and the local stability of equilibrium point $E_2(\tilde{u}_1, \tilde{u}_2)$. Hence the point $E_2(\tilde{u}_1, \tilde{u}_2)$ is globally asymptotically stable.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Dynamic behaviour of fractional system (16) with $\alpha = 0.50$. Other parameters are as fixed in (17).}
\end{figure}

4. **Numerical experiments.** In this section, we explore the dynamic richness of the fractional predator-prey model with Holling type-IV functional response. The local time-derivative in system (13) is replaced with the Caputo-Fabrizio fractional derivative, to have
\[
\mathcal{C}^\alpha_0 D^\alpha_t u_1(t) = \varphi u_1(t) \left(1 - \frac{u_1(t)}{\kappa}\right) - \frac{u_1(t) u_2(t)}{\phi + \psi u_1 + u^2_1(t)}
\]
\[ C^F \frac{0}{D} \alpha^t u_2(t) = u(t) \left( \frac{\epsilon u_1(t)}{\phi + \psi u_1(t) + u_1^2(t)} - \sigma \right) \], \quad (16)

where \( d_1, d_2 \) are the diffusivity constants, \( C^F \frac{0}{D} \alpha^t u(t) \) remains as earlier defined. We simulate with parameters
\[
\phi = 0.1; \psi = 0.001; \varphi = 5; \kappa = 0.7; \epsilon = 1; \sigma = 1 \quad (17)
\]

for different values of \( \alpha \) to obtain the results in Figures 1-4. A strange attractor of the species is reported in Figure 4 for \( \alpha = 0.48 \).
We extend our simulation experiment by considering the time-fractional reaction-diffusion version with the Caputo-Fabrizio derivative, given as

\begin{align*}
\mathcal{C}_0^F D^\alpha_t u_1(x, t) & = \frac{\partial^2 u_1(x, t)}{\partial x^2} \varphi u_1(x, t) \left( 1 - \frac{u_1(x, t)}{\kappa} \right) - \frac{u_1(x, t) u_2(x, t)}{\phi + \psi u_1(x, t) + u_1^2(x, t)} \\
\mathcal{C}_0^F D^\alpha_t u_2(x, t) & = u(x, t) \left( \frac{\epsilon u_1(x, t)}{\phi + \psi u_1(x, t) + u_1^2(x, t)} - \sigma \right). \tag{18}
\end{align*}
In this case, the solution is sought as a function of position $x$ and time $t$. The second-order partial derivative is approximated with the second-order central finite difference scheme, see [19]. In the simulation, we compute with the zero-flux boundary condition and initial function given in [33]. The diffusivity coefficients are set
to be $d_1 = 0.007$ and $d_2 = 0.1$. As displayed in Figures 5 to 9, it is obvious that both species undergo a spatiotemporal oscillation in phase. It should be mentioned that as $\alpha$ approaches 1, there exists a stable distribution. The biological implication is that, whenever the trajectories is approaching the boundary (washout)
equilibrium point $E_1(\kappa, 0)$ as $t \to +\infty$, which implies that the prey population with initial condition will obviously get to the stable density $\kappa$, while that of the predator population will tend to extinction as shown in Figures 1-3.

5. **Conclusion.** In this paper, a fractional version of the Adams-Bashforth scheme is applied to numerically approximate the Caputo-Fabrizio derivative which was used to study the dynamic complexities of a predator-prey system with Holling-type functional response. Mathematical analysis of the local derivative system is examined to guarantee the good choice of the parameters. Our findings on stability show that the system is globally asymptotically stable. Simulation experiment results obtained for different instances of fractional order $\alpha$ confirm the theoretical findings.

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*E-mail address: koladematthewowolabi@tdtu.edu.vn*