Checking the Quality of Approximation of $p$-values in Statistical Tests for Random Number Generators by Using a Three-Level Test

Hiroshi Haramoto*a, Makoto Matsumotob

aFaculty of Education, Ehime University, Ehime 790-8577, Japan
bGraduate School of Sciences, Hiroshima University, Hiroshima 739-8526, Japan

Abstract

Statistical tests of pseudorandom number generators (PRNGs) are applicable to any type of random number generators and are indispensable for evaluation. While several practical packages for statistical tests of randomness exist, they may suffer from a lack of reliability: for some tests, the amount of approximation error can be deemed significant. Reducing this error by finding a better approximation is necessary, but it generally requires an enormous amount of effort. In this paper, we introduce an experimental method for revealing defects in statistical tests by using a three-level test proposed by Okutomi and Nakamura. In particular, we investigate the NIST test suite and the test batteries in TestU01, which are widely used statistical packages. Furthermore, we show the efficiency of several modifications for some tests.

Keywords: Statistical testing, Pseudorandom number generations, Three-level test

1. Introduction

Statistical testing of pseudorandom number generators (PRNGs) is indispensable for their evaluation and many such test suites exist. Widely used examples are TestU01 by L’Ecuyer and Simard [11], and the test suite of the National Institute of Standards and Technology (NIST) [1].

*Corresponding Author

Email addresses: haramoto@ehime-u.ac.jp (Hiroshi Haramoto), m-mat@math.sci.hiroshima-u.ac.jp (Makoto Matsumoto)
Those suites are easy to apply to PRNGs, and further tests are still being designed. However, implementers and users always face an important problem in determining whether each test yields correct \( p \)-values. Common problems include making tests based on incorrect mathematical analyses, parameter selection through experiments, poor implementations damaging testing credibility, etc. Moreover, some statistical tests yield erroneous results because they use approximation formulas for \( p \)-values with non-negligible error. Therefore, checking accuracy of the approximation formula is important.

The aim of this paper is to develop a method for checking the quality of the approximation for the \( p \)-values of statistical tests by using a three-level test. This method has the merit of being easily conducted experimentally. Furthermore, our criterion only makes use of the uniformity of \( p \)-values, meaning that a wide range of tests can be subjected to the three-level method. Additionally, the result of this test is a \( p \)-value, so it is easy to understand as a figure of merit in statistical tests.

The rest of this paper is organized as follows. In Section 2, we briefly review statistical testing for PRNGs. In section 3, we consider a three-level test for checking the quality of the approximation for the \( p \)-values of statistical tests proposed by Okutomi and Nakamura [16]. In section 4, we present several results for the NIST test suite, SmallCrush and Crush in TestU01. We also present some modifications to those suites. These results support the usefulness of the three-level test.

2. Statistical testing for PRNGs and approximation error

This section gives a brief explanation on statistical testing for PRNGs, especially one-level and two-level tests. You can find further descriptions and explanations in [6, 7, 8, 9, 10, 12]. Our aim is to use these methods to evaluate the precision of the approximations used in statistical tests.

Let \( I \) denote the two element set \( \{0, 1\} \) or the interval \([0, 1)\). Let \( X_1, X_2, \ldots \) be random variables distributed over \( I \), with each \( X_k \) representing the \( k \)-th output of the tested PRNG. A statistical test (called a one-level test) looks for empirical evidence against the null hypothesis

\[
\mathcal{H}_0 : X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} U(I)
\]

with a test statistic

\[
f : I^n \to \mathbb{R}.
\]
Let \( X = (X_1, \ldots, X_n) \). In a statistical test, we assume that the distribution of \( f(X) \) under \( \mathcal{H}_0 \) is well-approximated by a known (cumulative) distribution \( F \). Thus, for our purpose to test the exactness of the approximation under \( \mathcal{H}_0 \), we make the following hypothesis

\[
\mathcal{H}': f(X) \sim F.
\]

Let \( a = (a_1, \ldots, a_n) \in I^n \) be an output sequence of the PRNG. If the \( p \)-value

\[
F(f(a)) = \Pr(f(a) \leq f(X))
\]

is too close to 0 or too close to 1, then either \( \mathcal{H}_0 \) or \( \mathcal{H}' \) is rejected. In usual tests for PRNG, \( \mathcal{H}' \) is assumed and hence the randomness of PRNG (\( \mathcal{H}_0 \)) is rejected. In this manuscript, \( \mathcal{H}_0 \) is assumed and hence the precision of the approximation (\( \mathcal{H}' \)) is rejected.

If the \( p \)-value is very small (e.g., less than \( 10^{-10} \)), then it is clear that either \( \mathcal{H}_0 \) or \( \mathcal{H}' \) is rejected. However, it is difficult to judge if the \( p \)-value is suspicious but is not very small (such as \( 10^{-4} \), for example). In order to avoid such difficulties, a two-level test is often used, see \([6, 7]\). A two-level test can be considered as a composite function

\[
I^n \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R},
\]

where \( f \) is the test statistic of the one-level test and \( f^N \) is defined by

\[
f^N(a_1, \ldots, a_N) := (f(a_1), \ldots, f(a_N)) \quad (a_1, \ldots, a_N \in I^n).
\]

At the second level, the function \( g \) corresponds to a Goodness-Of-Fit (GOF) test that compares the empirical distribution of the \( N \) \( p \)-values

\[
F(f(a_1)), \ldots, F(f(a_N))
\]

from the observations \( f(a_1), \ldots, f(a_N) \) with its theoretical distribution; the sample size at the second level is \( N \). If the \( p \)-value at the second level is small, either \( \mathcal{H}_0 \) or \( \mathcal{H}' \) is rejected.

Two-level tests permit one to apply the test with a larger total sample size to increase its power. Hence, if the generator fails the test in this particular way, then the \( p \)-value at the second level tends to become extremely small value as the sample size \( N \) is increased.

However, the \( p \)-value also tends to be very small if the approximations of the \( p \)-values at the first level is not good enough (i.e. if \( \mathcal{H}' \) fails). In this case, computational errors accumulate at each level and two-level tests detect the Lack-Of-Fit of that approximation, leading to rejection even if the generator is good \([6, 10, 13, 17]\).
3. Checking the quality of the approximation of the \( p \)-value by using a three-level test

Although it is easy to extend the level of a statistical test from two to three (or higher) using a technique such as

\[
I^{nNn' \times N} \xrightarrow{f} \mathbb{R}^{NN'} \xrightarrow{g} \mathbb{R}^{NN'} \xrightarrow{h} \mathbb{R},
\]

this type of test is often useless, because the approximation error of the second level may destroy the result: the resulting \( p \)-values tend to be too close to 0.

By contrast, Okutomi and Nakamura proposed a three-level test that can be considered as reliable as a two-level one [16]. The novelty of their method is that it uses an error-free function at the second level. This allows us to increase the sample size by \( N' \) times, and consequently to increase the power while avoiding an accumulation of computational errors. Okutomi and Nakamura originally intended to develop a new statistical test for PRNGs, but their method is useful to check the quality of the approximation of statistical tests.

Let \( f \) be an \( n \)-variable statistic corresponding to the one-level test. Suppose we want to check the quality of the approximation of the distribution of \( f(X) \), namely \( \mathcal{H}' \). Let \( (a_1, \ldots, a_{NN'}) \in I^{nNn'} \) \( (a_i \in I^n) \) be a sequence of \( NN' \) vectors in \( I^n \). At the first level, we compute \( f(a_1), \ldots, f(a_{NN'}) \).

Here we make an assumption: the approximating distribution \( F \) in the hypothesis \( \mathcal{H}' \) is assumed to be continuous. Under \( \mathcal{H}_0 \) and \( \mathcal{H}' \), the probability distribution of \( F(f(X)) \) is uniform in [0, 1]. This is proved by \( \Pr(F(f(X)) \leq p) = \Pr(f(X) \leq F^{-1}(p)) = F(F^{-1}(p)) = p \), where \( F^{-1} \) is the generalized inverse distribution function \( F^{-1}(p) = \inf\{x \in \mathbb{R} \mid F(x) \geq p\} \) and the equalities follow from the continuity of \( F \). Note that in the case of \( I = \{0, 1\} \), \( f(X) \) cannot have a continuous distribution, thus \( \mathcal{H}' \) must have some error. Therefore, we should distinguish the right and left \( p \)-values [11, 12]. In this paper, the assumption \( \mathcal{H}' \) means that \( F \) is an approximation good enough so that the statistical tests behave well. Thus, \( \mathcal{H}' \) includes the assumption that each probability mass is small enough to be negligible by itself.

We fix an arbitrary significance level \( \alpha \in (0, 1) \). The function \( g \), which corresponds to the second level, is the function that counts the number \( T_i \) of \( p \)-values greater than or equal to \( \alpha \) in

\[
F(f(a_{1+(i-1)_N})), \ldots, F(f(a_{iN})).
\]
for $i = 1, \ldots, N'$. Under the hypotheses $H_0$, $H'$ and the continuity of $F$, the distribution of the above $p$-values should be independently uniformly distributed over the interval $[0, 1]$, as shown above. Therefore, $T_i$ should have the binomial distribution $B(N, 1 - \alpha)$.

Finally, at the third level, we compare the empirical distributions of $T_1, \ldots, T_{N'}$ and $B(N, 1 - \alpha)$ via a GOF test. If the resulting $p$-value at the third level is extremely small, it strongly suggests that either $H_0$ or $H'$ fails. In our purpose, we use good PRNGs so that $H_0$ is assumed, and consider that $H'$ is rejected, or equivalently the approximation of $f(X)$ by $F$ is not good enough.

In this paper, following [16], we use the parameters $\alpha = 0.01$, $N = 10^3$, and $N' = 10^3$, as well as the following categorization:

- $C_0 = \{0, 1, \ldots, 981\}$,
- $C_i = \{981 + i\}$ ($i = 1, 2, \ldots, 15$),
- $C_{16} = \{997, 998, 999, 1000\}$.

Let $Y_i := \#\{T_j \mid j = 1, \ldots, N', T_j \in C_i\}$ for $i = 0, \ldots, 16$. We compute the $\chi^2$-value

$$h(T_1, \ldots, T_{N'}) := \sum_{i=0}^{16} \frac{(Y_i - N'p_i)^2}{N'p_i},$$

where $p_i = \sum_{j \in C_i} \binom{N}{j}/2^N$. The distribution of this statistic under $H_0$ and $H'$ is approximated by the $\chi^2$-distribution with 16 degrees of freedom.

It might seem to be more natural to use a GOF test on the entire distribution of the $N'$ $p$-values under the uniform distribution hypothesis, by using a test such as a Kolmogorov-Smirnov test. However, if the distribution of the test statistic at the second level is only approximated and the approximation error is significant, a three-level test will detect this approximation error, and tends to give $p$-values nearly zero.

On the other hand, the presented method counts only the number of $p$-values at the second level, which has no approximation error introduced at this level (under the hypotheses $H_0$ and $H'$). This is a reason why the proposed test is better.

4. Experimental results

This section shows the experimental results of the three-level test for the NIST test suite, SmallCrush, and Crush in TestU01. In order to mimic truly
random number sequences at the first level, we adopt Mersenne Twister (MT) [15] and a PRNG from the SHA1 algorithm. Note that MT fails certain tests (e.g. the linear complexity test, the binary matrix rank test with large matrix size) even when the \( p \)-value is computed correctly with no significant error. Thus we need both generators in the following experiments.

4.1. Results of the NIST test suite

The NIST test suite consists of 15 statistical tests for randomness of bit sequences, and its result is a list of 188 \( p \)-values. Since it was published in 2001, many modifications and corrections have been studied. However, the latest version 2.1.2 [1], released in 2014, does not incorporate several modifications.

We will show that the three-level method can reveal known defects of some statistical tests and show the effectiveness of several proposals to increase the reliability of those tests. In addition, through the three-level methods, we deduce new constraints for the Random Excursions test and the Random Excursions Variant test.

First, we consider all tests other than the Random Excursions test and the Random Excursions Variant test. In this experiment, the sample size at the first level \( n \) is fixed to \( 10^6 \), as recommended by NIST. Recall that throughout the experiments, the number of iterations \( N \) in the second level and \( N' \) in the third level are both 1000 with categorizations described in the previous section.

Table 1 shows the results of the three-level test for the original NIST test suite and the modified tests explained later. The Cumulative Sums test and the Serial test have two statistics respectively, with two \( p \)-values written in Table 1. The Non-Overlapping Template Matching test reports 148 \( p \)-values, thus the passing rates are filled in the table.

From our experiments, the \( p \)-values from the Discrete Fourier Transform (DFT) test, the Overlapping Template Matching test, and the Maurer’s Universal Statistical test in the original test suite are much too small. Additionally, the \( p \)-values of the Longest Runs of Ones in a Block test are relatively small. This result indicates that those tests have some flaws. After we applied appropriate modifications, the three-level test reported reasonable \( p \)-values for the four tests as described in the two columns at the right in Table 1.

Note that the current implementation of the NIST test suite uses one-level or two-level tests, differently from the above experiments, and the approxi-
Table 1: Results of the three-level test for the NIST test suite with $n = 10^6$

| Test Name                                | p-value (Original) | p-value (Modified) |
|-------------------------------------------|--------------------|--------------------|
|                                           | MT                 | SHA1               | MT            | SHA1            |
| Frequency                                 | 0.85               | 0.59               | -             | -               |
| Frequency test within a Block             | 0.017              | 0.68               | -             | -               |
| Cumulative Sums Test                      | 0.13, 0.64         | 0.37, 0.43         | -             | -               |
| Runs                                      | 0.56               | 0.47               | -             | -               |
| Longest Run of Ones in a Block           | 3.9E−5             | 1.3E−8             | 0.44          | 0.0011          |
| Binary Matrix Rank                        | 0.30               | 0.13               | -             | -               |
| Discrete Fourier Transform                | 4.1E−119           | 7.2E−116           | 0.19          | 0.026           |
| Non-Overlapping Template Matching         | 148/148            | 148/148            | -             | -               |
| Overlapping Template Matching             | 7.5E−80            | 5.6E−73            | 0.70          | 0.88            |
| Maurer’s Universal Statistical Approx.    | 8.7E−76            | 4.1E−66            | 0.99          | 0.77            |
| Linear Complexity                         | 0.023              | 0.0030             | -             | -               |

Information error in p-values provided by those tests are not large. For example, Table 2 and Table 3 show p-values provided by the NIST test suite.
Table 2: \( p \)-values of one-level tests and a two-level test for MT

| Test Name                                | 1st    | 2nd    | 3rd    | 4th    | 5th    | second level \( (N = 10^3) \) |
|------------------------------------------|--------|--------|--------|--------|--------|-------------------------------|
| Longest Run of Ones in a Block          | 0.15   | 0.39   | 0.64   | 0.029  | 0.47   | 0.88                          |
| Discrete Fourier Transform               | 0.48   | 0.44   | 0.31   | 0.89   | 0.66   | 0.41                          |
| Overlapping Template Matching           | 0.58   | 0.69   | 0.18   | 0.47   | 0.99   | 0.15                          |
| Maurer’s Universal Statistical          | 0.78   | 0.96   | 0.083  | 0.40   | 0.38   | 0.99                          |

Table 3: \( p \)-values of one-level tests and a two-level test for SHA1

| Test Name                                | 1st    | 2nd    | 3rd    | 4th    | 5th    | second level \( (N = 10^3) \) |
|------------------------------------------|--------|--------|--------|--------|--------|-------------------------------|
| Longest Run of Ones in a Block          | 0.65   | 0.50   | 0.69   | 0.44   | 0.052  | 0.64                          |
| Discrete Fourier Transform               | 0.73   | 0.038  | 0.13   | 0.77   | 0.34   | 0.034                         |
| Overlapping Template Matching           | 0.21   | 0.75   | 0.91   | 0.087  | 0.76   | 0.14                          |
| Maurer’s Universal Statistical          | 0.32   | 0.33   | 0.63   | 0.89   | 0.090  | 0.083                         |

Let us explain the modifications. We begin by considering the DFT test. Let \( X_k \) be the \( k \)-th bit of the tested sequence. The DFT test computes the discrete Fourier coefficients

\[
F_i = \sum_{k=0}^{n-1} (2X_k - 1) \exp(-2\pi \sqrt{-1}ki/n), \quad i = 0, 1, \ldots, n/2 - 1.
\]

The \( p \)-value of the DFT test is approximated by

\[
\Pr((o_h - 0.95n/2)/\sqrt{0.05 \cdot 0.95n/d < Z}), \quad Z \sim N(0,1)
\]

for a realization \( o_h \) of the number \( O_h \) of \( |F_j| \)'s that are smaller than some constant \( h \). The latest version of the NIST test suite uses the parameter \( d = 4 \) proposed by Kim et al. \[5\]. Subsequently, Pareschi et al. \[17\] proposed \( d = 3.8 \) for \( n \approx 10^6 \), which we use here as modification.

The Overlapping Template Matching test uses a \( \chi^2 \) GOF test that compares the empirical distribution of occurrences of a certain bit template with the theoretical one. NIST once used the probabilities derived by an approximation formula, and now it adopts more accurate values derived by \[4\]. However, the C-code overlappingTemplateMatching.c changes the new values
to the former wrong ones. We thus remove this instruction from the original code (lines 40–44), which is the modification.

The Maurer’s Universal Statistical test detects whether the sequence can be significantly compressed without loss of information. The original test adopts an asymptotic measure. We use the modification by Coron [2], a variant test statistic which enables better detection of defects in the tested sequence.

The Longest Runs of Ones in a Block test also uses a $\chi^2$ GOF test. The NIST test suite uses approximation values to four decimal places instead of the theoretical probabilities. We modify these values by more accurate ones to fifteen decimal places.

Unlike the other tests, the Random Excursions test and the Random Excursions Variant test do not always yield $p$-values. We review the algorithms of those tests and explain why this happens.

Both tests are based on considering successive sums of the bits as a one-dimensional random walk. Let $X_1, \ldots, X_n$ be random variables distributed over \{0, 1\}. The Random Excursions and Random Excursions Variant tests compute the partial sums

$$S_i := \sum_{k=1}^{i} (2X_k - 1), \quad i = 1, \ldots, n,$$

called the $i$-th state of the random walk. For an integer $x$, we say that $S_i$ takes the value $x$ if $S_i = x$. Consider the sequence $(0, S_1, \ldots, S_n, 0)$, and let $J$ be the number of 0’s minus one in this sequence. We call a subsequence of $(0, S_1, \ldots, S_n, 0)$ a cycle if it has length no less than two, it starts with 0, ends with 0, and contains no 0 between the first 0 and the last 0. Hence $J$ is the total number of cycles in $(0, S_1, \ldots, S_n, 0)$. Let $x$ be an integer among $x = \pm 1, \pm 2, \pm 3, \pm 4$. For each $x$ among the eight, the Random Excursions test uses the test statistic consisting of six integers $\nu_k(x)$ ($k = 0, 1, 2, 3, 4, 5$). For $k < 5$, $\nu_k(x)$ is the number of cycles in which the frequency of the value $x$ in the states is exactly $k$. For $k = 5$, $\nu_5(x)$ is the number of cycles in which the frequency of the value $x$ is 5 or more. Thus, $\sum_{k=0}^{5} \nu_k(x) = J$ holds. The corresponding $\chi^2$ statistic is

$$\chi^2 := \sum_{k=0}^{5} \frac{(\nu_k(x) - J \pi_k(x))^2}{J \pi_k(x)},$$

9
where $\pi_k(x)$ is the probability that the state $S_i$ visits the value $x$ exactly $k$ times in a cycle, under $H_0$. For the test statistic to have approximately a chi-square distribution, the expectation $J\pi_k(x)$ for each $k$ should not be too small, say $J\pi_k(x) \geq 5$. The NIST test suite discards the sample if $J < 500$ because the minimum value of $\pi_k(x)$’s is $\pi_4(4) \approx 0.0105$. Thus, each test yields eight $p$-values (one for each $x$) when $J \geq 500$, and yields no result when $J < 500$.

The Random Excursions Variant test computes the number $\xi(x)$ of times that $x$ occurs across all $J$ cycles for $x = \pm 1, \pm 2, \ldots, \pm 9$. The limiting distribution of $\xi(x)$ is known to be normal with mean $J$ and variance $J(4|x| - 2)$ for each $x$: thus, the test suite uses the statistic

$$Z := (\xi(x) - J) / (\sqrt{J(4|x| - 2)}).$$

The constraint is also $J \geq 500$.

In the Random Excursions test and the Random Excursions Variant test, $J$ is the sample size in computing $p$-values. Hence, the approximations of the statistics of the tests by a chi-square distribution and a normal distribution are getting better when the number $J$ is increased.

Since these tests discard some parts of the output of PRNG, the formalism of the three-level test does not apply as it is. However, for the both tests, the first level procedure yields a sequence of $p$-values which are uniform i.i.d in $[0, 1]$ under the hypotheses $H_0$ and $H'$. We iterate the first-level tests until we obtain $N (= 1000)$ sample $p$-values. Then, the rest of the three-level test works in the same manner.

We show the results of the three-level test for the Random Excursions test in Table 4, and for the Random Excursions Variant test in Table 5. Note that we use the sample size at the first level $n = 10^7$ to decrease the number of tests in which the test procedure is discontinued.

From our experiments, the Random Excursions test for $x = 4$ shows some flaw up to $J = 1500$. For the safety, we recommend a stronger constraint $J \geq 2000$ than $J \geq 500$ which NIST specified, with a larger sample size $n = 10^7$. For the Random Excursions Variant test, from the too small $p$-values for $x = \pm 9$, we recommend a constraint $J \geq 1000$.

4.2. Results for SmallCrush and Crush in TestU01

We examine the quality of the approximation of the $p$-values of SmallCrush and Crush batteries in TestU01. SmallCrush battery consists of 10 statistical tests (16 statistics). Of those tests, the smarsa_BirthdaySpacings
Table 4: \(p\)-values of the three-level test of the Random Excursions test

| \(x\) | \(J \geq 500\) MT | \(J \geq 1000\) MT | \(J \geq 1500\) MT | \(J \geq 2000\) MT | \(J \geq 500\) SHA1 | \(J \geq 1000\) SHA1 | \(J \geq 1500\) SHA1 | \(J \geq 2000\) SHA1 |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(-4\) | 1.0E–10        | 2.7E–03        | 2.6E–02        | 1.1E–03        | 9.1E–20        | 2.3E–02        | 2.4E–01        | 4.1E–01        |
| \(-3\) | 3.2E–06        | 3.4E–04        | 1.2E–01        | 8.4E–02        | 8.1E–07        | 1.9E–01        | 5.8E–01        | 4.7E–01        |
| \(-2\) | 4.5E–01        | 3.8E–01        | 3.1E–01        | 3.2E–01        | 3.1E–01        | 1.9E–01        | 2.1E–04        | 1.1E–01        |
| \(-1\) | 6.4E–02        | 8.6E–01        | 8.6E–01        | 4.1E–01        | 8.6E–01        | 2.9E–01        | 1.6E–01        | 2.8E–01        |
| 1     | 9.0E–02        | 5.7E–01        | 2.9E–01        | 2.5E–01        | 8.9E–01        | 5.7E–01        | 1.9E–01        | 4.7E–01        |
| 2     | 3.8E–02        | 5.8E–02        | 4.0E–02        | 8.2E–02        | 3.8E–02        | 4.0E–02        | 1.3E–01        | 1.7E–01        |
| 3     | 3.5E–06        | 2.5E–08        | 8.5E–04        | 1.5E–03        | 6.4E–08        | 2.5E–08        | 6.8E–05        | 5.5E–03        |
| 4     | 6.7E–16        | 4.7E–18        | 5.7E–03        | 9.7E–02        | 6.7E–16        | 4.7E–18        | 6.0E–10        | 5.3E–02        |

test and one of the \texttt{sknuth\_Collision} tests are based on a Poisson distribution, meaning that their distributions of \(p\)-values are not uniform. We thus assess the quality of the approximation of the \(p\)-values of the remaining 14 test statistics. Table 6 indicates that all 14 tests have the approximations of \(p\)-values which are sufficiently accurate.

Crush battery consists of 96 tests and reports 144 \(p\)-values. We check the quality of the approximation of the 76 tests (90 statistics), whose statistics have continuous distributions, ignoring those whose statistics are discrete, namely the \texttt{smarsa\_CollisionOver} test (No.3–10), the \texttt{smarsa\_BirthdaySpacings} test (No.11–17), the \texttt{snpair\_ClosePairs} test (No.18–20), the \texttt{snpair\_ClosePairsBitMatch} test (No.21–22), and one of the test statistics of the \texttt{sknuth\_CollisionPermut} test (No.39–40), where the numbers correspond to the enumeration of the tests in the user’s guidebook [12].

To reduce the computation time, we check Crush using the following procedure. We apply the three-level test with Mersenne Twister to each test. If the \(p\)-value is smaller than \(10^{-10}\), we check the test with a PRNG from SHA1. The test (i.e. hypothesis \(H'\)) will be rejected if both \(p\)-values are smaller than \(10^{-10}\).

Table 7 shows the tests rejected by both Mersenne Twister and SHA1. Because the \texttt{sspectral\_Fourier3} test has three statistics, three corresponding \(p\)-values are listed in the right-most column in Table 7, each of which is smaller than \(10^{-300}\). The \texttt{sstring\_Run} test has two statistics, thus we listed two \(p\)-values in the table. Similarly to the case of the NIST test suite, the
Table 5: \( p \)-values of the Random Excursions Variant test

| \( x \) | \( J \geq 500 \) | \( J \geq 1000 \) |
|-----|-----|-----|
|     | MT  | SHA1 | MT  | SHA1 |
| -9  | 1.5E-07 | 9.3E-10 | 2.5E-01 | 5.3E-01 |
| -8  | 2.8E-07 | 3.6E-05 | 3.3E-01 | 8.1E-01 |
| -7  | 1.3E-08 | 2.2E-05 | 3.9E-01 | 2.4E-01 |
| -6  | 8.9E-03 | 5.9E-03 | 6.0E-01 | 6.9E-01 |
| -5  | 5.1E-02 | 1.6E-02 | 9.2E-01 | 7.9E-01 |
| -4  | 7.4E-04 | 1.7E-01 | 4.5E-01 | 6.6E-01 |
| -3  | 8.6E-03 | 4.7E-03 | 5.2E-01 | 8.7E-01 |
| -2  | 2.5E-02 | 8.3E-01 | 3.1E-01 | 1.2E-02 |
| -1  | 4.9E-01 | 7.1E-04 | 7.2E-01 | 9.3E-01 |
| 1   | 3.4E-01 | 8.8E-01 | 2.0E-01 | 7.0E-01 |
| 2   | 3.4E-02 | 1.6E-02 | 6.1E-02 | 1.3E-01 |
| 3   | 6.0E-01 | 1.6E-03 | 7.0E-01 | 8.5E-01 |
| 4   | 8.5E-02 | 1.4E-02 | 2.1E-01 | 2.5E-01 |
| 5   | 1.5E-01 | 1.5E-03 | 5.5E-01 | 5.6E-01 |
| 6   | 1.8E-03 | 5.5E-05 | 1.5E-01 | 1.5E-01 |
| 7   | 2.8E-03 | 1.0E-06 | 6.7E-01 | 5.3E-01 |
| 8   | 2.7E-04 | 1.0E-05 | 8.7E-02 | 9.0E-01 |
| 9   | 1.2E-07 | 2.6E-09 | 5.5E-01 | 4.7E-01 |

approximation error in the \( p \)-value by TestU01 is not that large even if we find \( \varepsilon \) values in these three-level tests.

Among these rejected tests, we find that two of them can be modified to pass the three-level test. These are the \texttt{svaria} \_\texttt{SampleCorr} test and the \texttt{sstring} \_\texttt{Run} test. The improvements are shown in Table 8. The \texttt{svaria} \_\texttt{SampleCorr} test computes a correlation between \( X_1, \ldots, X_n \) which are random variables distributed over \([0, 1]\). TestU01 assumes that the statistic

\[
\frac{1}{n-k} \sum_{j=1}^{n-k} (X_j X_{j+k} - 1/4),
\]

has the normal distribution with mean 0 and variance \( 1/12(n-k) \). Fishman
Table 6: $p$-values of the three-level test of the SmallCrush

distribution corr

| test name                  | distribution | MT   | SHA1 |
|----------------------------|--------------|------|------|
| sknuth_Collision           | normal       | 0.057| 0.76 |
| sknuth_Gap                 | $\chi^2$     | 0.059| 0.37 |
| sknuth_SimplePoker         | $\chi^2$     | 0.47 | 0.75 |
| sknuth_CouponCollector     | $\chi^2$     | 0.62 | 0.94 |
| sknuth_MaxOft              | normal       | 0.0047| 0.47 |
| sknuth_MaxOft              | $\chi^2$     | 0.017| 0.62 |
| svaria_WeightDistrib       | $\chi^2$     | 0.50 | 0.049|
| smarsa_MatrixRank          | $\chi^2$     | 0.29 | 0.90 |
| sstring_HammingIndep       | normal       | 0.019| 0.095|
| swalk_RandomWalk1 (H)      | $\chi^2$     | 0.0020| 0.043|
| swalk_RandomWalk1 (M)      | $\chi^2$     | 0.011| 0.48 |
| swalk_RandomWalk1 (J)      | $\chi^2$     | 0.26 | 0.90 |
| swalk_RandomWalk1 (R)      | $\chi^2$     | 0.83 | 0.12 |
| swalk_RandomWalk1 (C)      | $\chi^2$     | 0.23 | 0.40 |

[3] shows that the statistic

$$\frac{1}{n-k} \sum_{j=1}^{n-k} (X_j - 1/2)(X_{j+k} - 1/2)$$

converges to normal with mean 0 and variance $1/144(n-k)$. We modified

the original statistic $\frac{1}{n-k} \sum_{j=1}^{n-k} (X_jX_{j+k} - 1/4)$ to $\frac{1}{n-k} \sum_{j=1}^{n-k} (X_j - 1/2)(X_{j+k} - 1/2)$.

The sstring_Run test is a variant of the run test applicable to a bit sequence, which yields two $p$-values: the test statistics are based on a normal distribution and a $\chi^2$ distribution.

Let $Y$ be the total number of bits needed to obtain $2n$ runs. Under $H_0$, we have $Y = \sum_{i=1}^{2n} X_i + 2n$ where $X_i$ are independent geometric random variables with parameter $1/2$. TestU01 adopts the statistic $(Y - 4n)/\sqrt{8n}$ and assumes that it can be approximated by the standard normal distribution. However, the expectation of $X_i$ is 1 and the variance of $X_i$ is 2, so

$$E[Y] = \sum_{i=1}^{2n} E[X_i] + 2n = 4n, \quad V[Y] = \sum_{i=1}^{2n} V[X_i] = 4n.$$
Table 7: Rejected tests in Crush and their \( p \)-values (\( \varepsilon \) : the \( p \)-value < \( 10^{-300} \))

| test name          | parameters | MT      | SHA1       |
|--------------------|------------|---------|------------|
| svaria_SampleCorr  | \( n = 5 \times 10^8, k = 1 \) | 1.8E−222 | 5.5E−237   |
| smarsa_Savir2      | \( n = 2 \times 10^7, m = 2^{20}, t = 30 \) | 2.7E−49  | 9.9E−32    |
| scomp_LempelZiv    | \( n = 2^{25} \) | \( \varepsilon \) | \( \varepsilon \) |
| sspectral_Fourier3 | \( n = 2^{14} \times 50000 \) | \( \varepsilon, \varepsilon, \varepsilon \) | \( \varepsilon, \varepsilon, \varepsilon \) |
| sstring_Run       | \( n = 10^9 \) | \( \varepsilon, \varepsilon \) | \( \varepsilon, \varepsilon \) |

Thus the appropriate statistic is \( (Y - 4n)/\sqrt{4n} \), this is the modification.

The other test statistic is

\[
\sum_{i=1}^{k} \frac{(X_{0,i} - np_i)^2}{np_i(1-p_i)} + \sum_{i=1}^{k} \frac{(X_{1,i} - np_i)^2}{np_i(1-p_i)},
\]

where \( X_{0,i} \) and \( X_{1,i} \) are the number of runs of 0’s and 1’s of length \( i \) for \( i = 1, \ldots, k \), where \( k \) is some positive integer, and \( p_i = 2^{-i} \). TestU01 assumes that the statistic has approximately the \( \chi^2 \) distribution with \( 2(n-1) \) degrees of freedom for a \( \chi^2 \) GOF test. However, the factor \( 1 - p_i \) in the denominator seems to be unnecessary, so our modification removes them.

Table 8 shows the \( p \)-values for those test statistics. The results indicate that the above modifications are satisfactory in improving the reliability of the tests.

Table 8: \( p \)-values of the original tests and their modifications (\( \varepsilon \) : the \( p \)-value < \( 10^{-300} \))

| test name          | \( p \)-value (Original) | \( p \)-value(Improved) |
|--------------------|--------------------------|-------------------------|
| svaria_SampleCorr  | MT          | SHA1       | MT      | SHA1 |
|                    | 1.8E−222    | 5.5E−237   | 0.498  | 0.825 |
| sstring_Run (normal)| \( \varepsilon \) | \( \varepsilon \) | 0.657  | 0.302 |
| sstring_Run (chi-squared) | \( \varepsilon \) | \( \varepsilon \) | 0.715  | 0.0479 |

We discuss on the rest three tests, for which we are not able to give satisfactory modifications. The \texttt{smarsa_Savir2} test is a modified version of the Savir test proposed by Marsaglia \[14\]. Let \( U_1, U_2, \ldots, U_t \) be independent uniform random variables over \((0, 1)\). For a given \( m \), the random integers \( I_1, \ldots, I_m \). \( U_{I_1}, U_{I_2}, \ldots, U_{I_m} \) are
\(I_2, \ldots, I_t\) are defined by \(I_1 = \lceil mU_1 \rceil\), \(I_2 = \lceil I_1 U_2 \rceil\), \(\ldots, I_t = \lceil I_{t-1} U_t \rceil\). It thus generates \(n\) values of \(I_t\) and compares their empirical distribution with the theoretical one via a \(\chi^2\)-test.

TestU01 recommends the values of \(m\) and \(t\) that satisfy \(m \approx 2^t\) and Crush adopts \(m = 2^{20}\) and \(t = 30\). Table 9 shows the \(p\)-values obtained with \(n = 2 \times 10^7\), \(m = 2^{20}\) and various values of \(t\).

The \(p\)-values are slightly suspicious but not too small. Therefore, it is necessary to investigate the tests mathematically, but we are not able to manage this at present. Tentatively, we propose to take \(t = 9\) for a compromise between the reliability of the test and choosing a larger value of \(t\).

Table 9: \(p\)-values of the smarsa_Savir2 test for various \(t\)'s

| \(t\) | MT     | SHA1   | \(t\) | MT     | SHA1   |
|------|--------|--------|------|--------|--------|
| 5    | 2.7E−07| 6.9E−06| 12   | 1.9E−06| 1.6E−12|
| 6    | 4.2E−12| 3.4E−05| 13   | 1.6E−07| 2.3E−08|
| 7    | 2.3E−10| 3.5E−08| 14   | 3.8E−06| 4.9E−16|
| 8    | 5.2E−06| 1.5E−09| 15   | 4.7E−16| 1.7E−17|
| 9    | 1.1E−06| 1.6E−05| 20   | 2.2E−13| 2.0E−18|
| 10   | 3.2E−04| 2.1E−13| 25   | 1.3E−28| 1.2E−17|
| 11   | 5.6E−14| 1.6E−15| 30   | 2.7E−49| 9.9E−32|

The scomp_LempelZiv test measures the compressibility of the bit sequence using the Lempel-Ziv compression algorithm. TestU01 uses approximations of the mean and variance obtained by simulation. The spectral_Fourier3 test is a kind of DFT tests proposed by Erdmann. However, the authors of TestU01 claim that those tests tend not to be very sensitive. Indeed, the resulting \(p\)-values of those tests are smaller than \(10^{-300}\), so more mathematical justifications for those tests are needed.

5. Concluding remarks

We introduced a three-level test to check the quality of the approximation for the \(p\)-values in statistical tests for PRNGs. We find that some statistical tests use approximation with some flaw. We list some of such tests from NIST and TestU01. This does not mean that these tests are erroneous,
but the reliability of the tests is increased if the approximation is improved. We give three satisfactory modifications to three tests in Crush, and propose new parameters for several tests from this viewpoint. In this study, we need to assume that the approximated statistics are continuous, because our three-level test is based on the uniformity of \( p \)-values in \([0, 1]\) at the first level. This condition is not essential: if the distribution of \( p \)-values can be computed exactly, we can conduct the three-level test with an appropriate GOF test at the third level. For example, the exact probability formula of the \texttt{smarsa_BirthdaySpacings} test is presented in [6]. It indicates the possibility of calculating the exact distribution of its \( p \)-values. In future work, we hope to assess the reliability of all of the remaining tests in Crush battery.

According to the original proposal presented in [16], we employ a \( \chi^2 \) GOF test at the third level. However, a Kolmogorov-Smirnov (KS) test seems to be more appropriate and more powerful. An accurate approximation of the KS distribution is now available [18], so we should experiment with this method to obtain more decisive conclusions.

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