Time-dependent entropy of a cooling Bose gas

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Abstract – Exact analytic solutions of a nonlinear boson diffusion equation with suitable initial conditions that account for evaporative cooling of ultracold atoms, plus boundary conditions at the singularity $\mu < 0$ are presented, and used to calculate the time-dependent entropy of a cold quantum gas.

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Introduction. – Thermalization processes are of great interest in physics across the energy scale from pico- to teraelectronvolts. Basic examples for bosonic systems are the fast approach of gluons to local equilibrium in the initial stages of a relativistic heavy-ion collision at the Large Hadron Collider, or the thermal equilibration of cold bosonic atoms such as $^{23}\text{Na}$ or $^{87}\text{Rb}$ in the course of evaporative cooling [1]. Due to the possibility of condensate formation at sufficiently low temperatures, cold quantum gases are particularly intriguing.

Whereas numerical approaches to thermalization can be based on a quantum Boltzmann collision term, it is interesting to have a simple, exactly solvable model to better understand the physics of a given system. For bosons, such a model has been proposed in ref. [2], and adapted to cold quantum gases in ref. [3]. The model is based on a nonlinear boson diffusion equation (NBDE) which has been derived from the Boltzmann collision term. For energy-independent transport coefficients, it is structurally simple, but still complicated to solve exactly due to the nonlinearity in the drift term, which causes the system to reach the Bose-Einstein equilibrium for sufficiently large times.

Analytical solutions for physically meaningful nonlinear partial differential equations are of great interest in several fields of physics, but are rarely available. Notable examples in 1 + 1 dimensions are the Korteweg-de Vries equation [4,5], which is of third order in the spatial variable and has soliton solutions, and Burgers’ equation [6], which has the structure of a one-dimensional Navier-Stokes equation without pressure term. It has been used to describe fluid flow and, in particular, shock waves in a viscous fluid, and it can be solved through Hopf’s transformation [7]. Although few examples of completely integrable NPDEs in 2+1 dimensions are known —such as the Kadomtsev-Petviashvili [8] and the Novikov-Veselov [9] equations, which are analogues of the KdV equation in two spatial dimensions— higher-dimensional problems are usually not integrable.

In case of the nonlinear boson diffusion equation [2,3], no exact solutions in 2 + 1 and higher dimensions are presently known. I discuss it here in 1 + 1 dimensions (energy and time), which is appropriate for a cooling —and eventually, condensing— isotropic three-dimensional thermal cloud of cold atoms. In refs. [2,3] exact solutions were already obtained, but only for initial conditions confined to the energy domain $\epsilon \geq 0$, thus excluding the singularity at $\epsilon = \mu < 0$. Without the singularity, however, the solutions do not reach the Bose-Einstein limit in the infrared for $t \rightarrow \infty$.

In order to attain a thermal equilibrium distribution both in the UV and IR energy regions, one therefore has to include the singularity in the initial conditions. It turns out that the chemical potential in the corresponding solutions is variable, and can attain positive values. To avoid such unphysical behaviour, one must specify boundary conditions at the singularity. This restricts the energy range in the initial conditions to energies larger than the chemical potential. Exact analytic solutions are still possible which have the proper Bose-Einstein equilibrium limit and, moreover, can represent evaporative cooling from an initial temperature $T_i$ to a final temperature $T_f$.
With these exact solutions, I calculate the time-dependent entropy of an equilibrating finite Bose system and compare to the results of a linear relaxation ansatz that enforces equilibration from the initial nonequilibrium distribution to a thermal distribution at a lower temperature $T_1 < T$.

In the next section, the NBDE and its free analytic solutions [3] without and with consideration of the singularity are briefly reviewed. In the following section, the boundary conditions at the singularity are introduced, thus producing realistic physical results. The derivation of the corresponding solutions is discussed in ref. [10] since it exceeds the scope of a letter. Subsequently, evaporative cooling of cold bosonic atoms is modeled through the NBDE solutions. With the exact solutions, the time-dependent entropy of a cooling Bose gas is finally calculated and discussed.

**Nonlinear diffusion equation and free solutions.** – The transport equation (NBDE) for the single-particle occupation probability distributions $n \equiv n_0(\epsilon, t)$ has been derived from the bosonic Boltzmann collision term in ref. [2] as

$$ \frac{\partial n}{\partial t} = -\partial_{\epsilon} \left[ vn(1+n) - n^2 \partial D \frac{\partial}{\partial \epsilon} \right] + \partial^2 n [Dn]. \tag{1} $$

The drift term $\psi(\epsilon, t)$ accounts for dissipative effects, the term $D(\epsilon, t)$ for diffusion of particles in the energy space. The many-body physics is contained in these transport coefficients, which depend on energy, time and the second moment of the interaction.

Spatial and momentum isotropy is a prerequisite for the reduction to $1 + 1$ dimensions, corresponding to *sufficient ergodicity* [11]. For the thermal cloud of cold atoms around a Bose-Einstein condensate (BEC), this is expected to be a reasonable assumption, even though the condensate in a trap is spatially anisotropic. Concerning the role of different spatial dimensions in view of BEC formation, this enters my present formulation only through the density of states, which differs according to the number of spatial dimensions, and the confinement. The model calculations in this work are for a 3d system. One-dimensional systems where no BEC should be formed have not yet been investigated.

In the limit of energy-independent transport coefficients the nonlinear boson diffusion equation for the occupation-number distribution $n(\epsilon, t)$ becomes

$$ \frac{\partial n}{\partial t} = -\psi \partial_{\epsilon} \left[ vn(1+n) - n^2 \partial D \frac{\partial}{\partial \epsilon} \right] + D \partial^2 n \frac{\partial n}{\partial \epsilon^2}. \tag{2} $$

The thermal equilibrium distribution is a stationary solution

$$ n_{\text{eq}}(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/T} - 1} \tag{3} $$

with the chemical potential $\mu < 0$ in a finite boson system and $T = -D/v$. In spite of its simple structure, the NBDE with constant transport coefficients thus preserves the essential features of Bose-Einstein statistics which are contained in the bosonic Boltzmann equation. For a given initial condition $n_i(\epsilon)$, it can be solved exactly using the nonlinear transformation outlined in ref. [3]. The resulting solution can be written as

$$ n(\epsilon, t) = -\frac{D}{v} \partial_{\epsilon} \ln Z(\epsilon, t) - \frac{1}{2} = -\frac{D}{v} \frac{\partial Z}{\partial \epsilon} \frac{1}{2}. \tag{4} $$

where the time-dependent partition function $Z(\epsilon, t)$ obeys a linear diffusion equation

$$ \frac{\partial}{\partial t} Z(\epsilon, t) = D \frac{\partial^2}{\partial \epsilon^2} Z(\epsilon, t). \tag{5} $$

If no boundary conditions are specified, the free partition function becomes

$$ Z_{\text{free}}(\epsilon, t) = a(t) \int_{-\infty}^{+\infty} G_{\text{free}}(\epsilon, x, t) F(x) dx. \tag{6} $$

The physically more interesting case with boundary conditions will be treated in the next section. The partition function is only unique up to multiplication with energy-independent prefactors such as $a(t)$, since these drop out when taking the logarithmic derivative in the calculation of the occupation-number distribution. The initial conditions that are contained in the function $F(x)$ cover the full energy region $-\infty < x < \infty$.

For a solution without boundary conditions as in refs. [2,3], Green’s function $G_{\text{free}}(\epsilon, x, t)$ of eq. (5) is a single Gaussian

$$ G_{\text{free}}(\epsilon, x, t) = \exp \left( -\frac{(\epsilon - x)^2}{4Dt} \right) , \tag{7} $$

but it becomes more complicated once boundary conditions are considered. The function $F(x)$ depends on the initial occupation-number distribution $n_i$,

$$ F(x) = \exp \left[ -\frac{1}{2D} \left( vx + 2v \int_0^x n_i(y) dy \right) \right] . \tag{8} $$

The definite integral over the initial conditions taken at the lower limit drops out in the calculation of $n_i(\epsilon,t)$ when performing the logarithmic derivative. Hence, the integral can be replaced [10] by the indefinite integral $A_i(x)$ over the initial distribution with $\partial_x A_i(x) = n_i(y)$, such that

$$ F(x) = \exp \left[ -\frac{1}{2D} (vx + 2v A_i(x)) \right] . \tag{9} $$

It is now possible to compute the partition function and the overall solution for the occupation-number distribution function eq. (4) analytically, even in the presence of a singularity in the initial conditions. (The singularity had been excluded in the initial conditions, and hence, in the solution given in ref. [3]).

As initial condition that is appropriate for a schematic description of evaporative cooling, one can start from a
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truncated thermal equilibrium distribution that is cut off at a maximum energy \( \epsilon_i \) beyond which high-velocity atoms are removed

\[
n_i(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T} - 1} \theta(1 - \epsilon/\epsilon_i).
\]  

(10)

If the integration in eq. (6) is now carried out across the singularity at \( x = \mu \), the solutions eq. (4) approach a Bose-Einstein equilibrium distribution for \( t \to \infty \). However, the chemical potential of the equilibrium solution moves to a larger value \( \mu' > 0 \)

\[
\mu' = \frac{D_v}{v} \ln[z^{-1} - \exp(-\epsilon_i/T)],
\]

(11)

with the fugacity \( z = \exp(\mu/T) \). Although this solution that includes the singularity is mathematically correct — analytical and numerical results agree with high accuracy — the shift has no reasonable physical interpretation, it is an artefact of the choice of the free Green’s function eq. (7). Clearly, one has to consider the boundary conditions at the singularity in order to obtain physically meaningful solutions not only at short times when the step in the UV region is smeared out, but also in the IR at \( t \to \infty \) when the thermal distribution is approached. These new solutions will be considered in the next section. Their detailed derivations are given in ref. [10].

Exact solution with boundary conditions. — To solve the problem for constant temperature, but with boundary conditions at the singularity, the chemical potential is treated as a fixed parameter. With \( \lim_{\epsilon_i\mu} n(\epsilon, t) = \infty \forall t \), one obtains \( Z(\mu, t) = 0 \), and the energy range is restricted to \( \epsilon \geq \mu \). This requires a new Green’s function that equals zero at \( \epsilon = \mu + vt \). It can be written as

\[
G(\epsilon, x, t) = G_{\text{free}}(\epsilon - \mu, x, t) - G_{\text{free}}(\epsilon - \mu, -x, t),
\]

(12)

and the partition function with this boundary condition becomes

\[
Z(\epsilon, t) = \int_0^\infty G(\epsilon, x, t) F(x + \mu) dx.
\]

(13)

The function \( F \) remains unaltered with respect to eq. (9), save for a shift of its argument by the chemical potential.

Using again eq. (10) as initial distribution, the occupation-number distribution can still be evaluated exactly in case of constant temperature, but also for \( T_i \neq T_f \) as shown in the next section. The result for constant \( T \) [10]

\[
n(\epsilon, t) = \frac{1}{\exp(\frac{\epsilon - 1}{T}) L(\epsilon, t) - 1}
\]

(14)

is formally similar to a Bose-Einstein distribution. The function \( L(\epsilon, t) \) contains the terms that are responsible for the time evolution towards the equilibrium distribution at temperature \( T = -D/v \). It can be written as

\[
L(\epsilon, t) = \frac{\Sigma_1(\epsilon, t)}{\Sigma_2(\epsilon, t)}.
\]

(15)

\[
\Sigma_1(\epsilon, t) = \text{erfc}\left(\frac{2\epsilon - \epsilon_i - \epsilon + tv}{\sqrt{4Dt}}\right) - \exp\left(\frac{\mu - \epsilon_i}{T}\right) \times \text{erfc}\left(\frac{\epsilon - \epsilon_i + tv}{\sqrt{4Dt}}\right)
\]

(16)

and

\[
\Sigma_2(\epsilon, t) = \text{erfc}\left(\frac{\epsilon - \epsilon_i + tv}{\sqrt{4Dt}}\right) - \exp\left(\frac{\mu - \epsilon_i}{T}\right) \times \text{erfc}\left(\frac{\epsilon - 2\mu + \epsilon_i + tv}{\sqrt{4Dt}}\right).
\]

(17)

The complementary error functions are defined as

\[
\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.
\]

(18)

As a consequence of the boundary condition, no unphysical shift in the chemical potential occurs in this special solution of the NBDE. The results for a specific parameter set that is adapted to cold quantum gases [3] are shown in fig. 1. To be able to compare directly with my earlier results in ref. [3] that employed restricted initial conditions, the same parameters are used. The transport coefficients are \( D = 8 \times 10^4 \text{peVs}^{-1} \) and \( v = -1 \times 10^3 \text{peVs}^{-1} \), with an equilibrium temperature \( T = 8 \text{peV} \approx 93 \text{nK} \). These values are motivated by experimental results for temperatures and time scales in ultracold \(^{87}\text{Rb}\). At \( \epsilon_i = 7 \text{peV} \approx 1.7 \text{kHz} \), the initial thermal distribution is truncated, and the chemical potential is chosen as \( \mu = -0.68 \text{peV} \). The temperature \( T \) is kept constant in this particular calculation. As is evident from the time-dependent analytical solutions shown in fig. 1 at \( t = 0.001-40 \text{ms} \), the steep

Fig. 1: Equilibration of a finite Bose system based on the nonlinear evolution according to eq. (2) starting from a truncated Bose-Einstein distribution eq. (10), upper curve with cut-off \( \epsilon_i = 7 \text{peV} \approx 1.7 \text{kHz} \). The transport coefficients are \( D = 8 \times 10^4 \text{peVs}^{-1} \) and \( v = -1 \times 10^3 \text{peVs}^{-1} \). The temperature \( T = -D/v = 8 \text{peV} \approx 93 \text{nK} \) is kept constant in this calculation. The time evolution is shown at \( t = 0.001, 0.01, 0.1, 1, 4 \) and 40 ms (with decreasing dash length).
cutoff in the UV at $\epsilon = \epsilon_i$ is smeared out at short times as in the solution without boundary conditions. At larger times $t \gtrsim 1$ ms, the solutions do not approach a new equilibrium distribution with $\mu' > 0 > \mu$ as in the free case, but at $t \approx 40$ ms return to the original equilibrium with $T = -D/v$, since this is the only temperature that exists in the present formulation.

In the next section, different initial and final temperatures $T_i \neq T_f$ are considered to schematically account for evaporative cooling.

**Analytical solutions for cooling.** — If the temperature $T_i$ in the initial conditions eq. (10) differs from the final equilibrium temperature $T_f = -D/v$ as is the case in evaporative cooling [11–13], the analytic solutions of the NBDE with boundary conditions become more involved, but it is still possible to derive them. It is instructive to first consider the time evolution of the solutions from the initial nonequilibrium distribution $n_i(\epsilon)$ to the final equilibrium distribution $n_{eq}(\epsilon)$ in a linear relaxation ansatz $\partial n_{rel}/\partial t = (n_{eq} - n_{rel})/\tau_{eq}$ with

$$n_{rel}(\epsilon,t) = n_i(\epsilon)e^{-t/\tau_{eq}} + n_{eq}(\epsilon)(1 - e^{-t/\tau_{eq}}), \quad (19)$$

which enforces equilibration towards the thermal distribution $n_{eq}(\epsilon)$ with the bosonic equilibrium time $\tau_{eq}$, for which I have used the value $\tau_{eq} = 4D/(9\epsilon^2) \simeq 3.6$ ms (see footnote ¹). This simplified model can be used to compare with the nonlinear solution for cooling.

Time-dependent results for the relaxation-time approximation (RTA) are shown in fig. 2. Here, the initial distribution corresponds to a temperature of $T_i \approx 232$ nK, which is above the critical temperature for $^{87}$Rb.

¹This result refers to a $\delta$-function initial distribution, see ref. [2].

that accounts schematically for evaporative cooling is again placed at $\epsilon_i \approx 1.7$ kHz. The equilibrium solution at a temperature $T_i = -D/v \approx 93$ K—below $T_{crit}$ for reasonable density values—is approached by the relaxation-time solutions that are shown at $t = 1.3$ and 7 ms.

The new thermal distribution at the lower temperature is reached within $t \approx 14$ ms. Due to the linear approximation of a nonlinear system, the RTA-solutions do not account for the correct physical behaviour of the system at the cut: The distribution functions remain discontinuous.

In contrast, the analytical solutions of the nonlinear NBDE produce a more realistic account of the thermalization. The partition function with boundary conditions at the singularity $\epsilon = \mu$ has been derived in ref. [10] for $T_i \neq T_f$ as

$$Z(\epsilon,t) = \sqrt{4Dt} \exp \left( -\frac{\mu}{2T_i} \right) \sum_{k=0}^{\infty} \left( \frac{T_i}{k} \right) (-1)^k \times \left[ e^{\alpha_k^2Dt} A_k(\epsilon,t) - e^{\alpha_k(\mu-\epsilon)} A_k(\epsilon,t) \right] + \exp \left( \frac{(\mu - \epsilon)k}{T_i} \right) \exp \left( \frac{Dt}{4T_i^2} \right) \times \left[ \exp \left( \frac{\epsilon - \mu}{2T_i} \right) A_d(\epsilon,t) - \exp \left( \frac{\mu - \epsilon}{2T_i} \right) A_d(\epsilon,t) \right], \quad (20)$$

with $\alpha_k = 1/T_i - k/T_i$, and the auxiliary functions

$$A_k(\epsilon,t) = \text{erf} \left( \frac{\epsilon - \mu + 2Dt\alpha_k}{\sqrt{4Dt}} \right) - \text{erf} \left( \frac{\epsilon - \epsilon_i + 2Dt\alpha_k}{\sqrt{4Dt}} \right), \quad (21)$$

Fig. 2: Schematic representation of evaporative cooling in a bosonic system from temperature $T_i = 20$ peV $\approx 232$ nK (upper solid curve, truncated at 7 peV) to an equilibrium distribution with temperature $T_i = 8$ peV $\approx 93$ nK (lower solid curve). Discontinuous time-dependent single-particle occupation-number distribution functions at $t = 1.3$ and 7 ms (decreasing dash lengths) are shown using exact solutions of the NBDE (2).
\[ \Lambda_2(\epsilon, t) = \text{erf} \left( \frac{\mu - \epsilon + 2Dt\alpha_k}{\sqrt{4Dt}} \right) \]
\[ - \text{erf} \left( \frac{2\mu - \epsilon - \epsilon_i + 2Dt\alpha_k}{\sqrt{4Dt}} \right), \] (22)
\[ \Lambda_3(\epsilon, t) = \text{erfc} \left( \frac{\epsilon - \epsilon_i + tv}{\sqrt{4Dt}} \right), \] (23)
\[ \Lambda_4(\epsilon, t) = \text{erfc} \left( \frac{\epsilon - 2\mu + \epsilon_i + tv}{\sqrt{4Dt}} \right). \] (24)

The derivative \( \partial \mathcal{Z} / \partial \epsilon \) can also be obtained analytically, such that the time-dependent occupation-number distribution function for evaporative cooling can be calculated from eq. (4). For \( T_i = \tilde{T}_i \), the result of eq. (14) is recovered.

The time-dependent analytic distribution functions that solve the NBDE exactly with boundary conditions at the singularity are displayed in fig. 3 for the same initial conditions as in fig. 2. They agree precisely with numerical solutions of the basic equation, and predict the time dependence of the entropy —without the cut— is about four times larger than the one of the final distribution, whereas the initial nonequilibrium distribution —with the cut at \( \epsilon = \epsilon_i \)— carries only half the entropy of the initial thermal distribution at the lower temperature \( \tilde{T}_i \). The detailed time dependence of \( S(t) \) is then subject of a corresponding model calculation.

If the final temperature \( T_f \) is below the critical value for condensate formation \( T_{\text{crit}} \) such that particles occupy the condensed state, the total entropy still equals the entropy of the atoms in the thermal cloud. As emphasized in ref. [14], this is the case even though the entropy of the particles in the ground state is nonzero, because the latter is cancelled by the so-called correlation entropy due to the fixed number of particles distributed among the quantum states.

It is therefore sufficient for a calculation of the total time-dependent entropy \( S(t) \) to consider only the thermal cloud. For the corresponding numerical calculation, the analytical solutions based on eqs. (4) and (20) for constant chemical potential \( \mu \) and boundary conditions at the singularity \( \epsilon = \mu \) are used. The entropy in a bosonic system for an average number of particles \( n(\epsilon, t) \) per single-particle state can be written as [15]
\[ S(t) = \int_0^\infty g(\epsilon)[\ln(1 + n(\epsilon, t))] \, d\epsilon, \] (25)
where the density of states for a three-dimensional isotropic Bose gas without external potential obeys the power law
\[ g(\epsilon) = g_0 \epsilon^{\delta}, \] (26)
with \( (\hbar = c = k_B = 1) \)
\[ g_0 = (2m)^{3/2}V/(4\pi^2), \] (27)
as obtained from the substitution of a summation over the quantum numbers of the associated states with an energy integration [1].

The spatial dimensionality and the external confinement thus enter the present formulation only through the density of states, enabling future considerations of their effect.

\[ \text{Time-dependent entropy}. – \text{A condition that physically reasonable solutions of the nonlinear boson diffusion equation should fulfil is the increase of the entropy with time towards the equilibrium value that is determined by the final Bose-Einstein distribution. This is indeed the case for the solutions of the NBDE at constant temperature \( T \) shown in fig. 1: The entropy \( S_i \) of the initial nonequilibrium distribution rises monotonically in the course of the time evolution to the final equilibrium entropy \( S_{\text{eq}} \) by about a factor of three.} \]
The first term in the entropy eq. (25) is usually referred to as wave entropy, it yields the largest contribution when the single-particle state is occupied by many particles, as in the IR. The second term is the particle entropy, which is more relevant in case of low occupation \( n(\epsilon, t) < 1 \), as in the UV.

Results for the time-dependent entropy are shown in fig. 4, with the same parameters as in fig. 3 for the analytical solutions of the NBDE. Here, the equilibrium value of the entropy at the initial temperature \( T_i \approx 232 \) nK is \( S_{eq)/g_0(T_i) \approx 256.35 \) peV\(^3/2\). With a sharp cutoff at \( \epsilon_i = 7 \) peV \( \approx 1.7 \) kHz to account for evaporative cooling, the entropy of the initial (cooled) nonequilibrium distribution is reduced to \( S_i/g_0(T_i) \approx 30.96 \) peV\(^3/2\).

Thermalization during the time evolution then occurs through the analytical solutions of the NBDE, and the result for the rising entropy \( S(t)/g_0 \) is shown in the solid curve in fig. 4. The new equilibrium value of the entropy at the final temperature \( T_f \approx 93 \) nK following evaporative cooling and thermalization is \( S_{eq)/g_0(T_f) \approx 61.66 \) peV\(^3/2\), dotted horizontal line.

The dashed curve is the wave entropy. It is most relevant for large occupation numbers, which are present at all times in the IR region \( \epsilon < \epsilon_i \), and therefore, this contribution shows a rather weak time dependence. The dotted curve is the particle entropy, which is initially smaller than the wave entropy, because the occupation in the UV beyond the cut is negligible at small times. In the course of thermalization, however, it quickly exceeds the wave entropy at \( t \approx 0.7 \) ms for the parameter set used in this work, and rises subsequently. Hence, the entropy at large times is mostly determined by the contribution of the thermal tail.

Whereas the analytical solutions of the NBDE reach the thermal equilibrium values in the IR rather fast within about 1.4 ms as shown in fig. 3, it takes more time to build up the thermal tail in the UV beyond the cut: The rise of the particle entropy in the nonlinear model occurs fairly slowly, such that the total entropy reaches the equilibrium value only within about 60 ms.

These results for the time-dependent entropy as obtained from the analytic solutions of the NBDE are compared in fig. 4 with the corresponding relaxation ansatz, dot-dashed curve. Although thermalization in the IR occurs more slowly when using this ansatz as compared to the nonlinear model — see figs. 2 and 3 — the exponential Boltzmann-like slope in the UV appears faster. The rapid rise of the entropy at relatively short times in the relaxation-time approximation is a consequence of this enforced fast buildup of a thermal tail in such a linear approach. As expected, the system’s entropy approaches the same equilibrium value, but at shorter times than the nonlinear solution.

**Conclusion.** — New exact solutions of the nonlinear boson diffusion equation have been explored which take account of the singularity in the initial conditions at \( \epsilon = \mu < 0 \), and the necessary boundary conditions at the singularity. Differently from earlier results that were calculated with the free Green’s function, these solutions converge towards the Bose-Einstein equilibrium. Hence, they properly account for thermalization not only in the UV, but also in the IR region. The analytic solutions are in excellent agreement with numerical approaches.

The analytic results of the NBDE are tailored to describe equilibration processes that occur in quantum gases in the course of evaporative cooling, and subsequent re-thermalization. They are applied schematically to bosonic atoms like \(^{87}\)Rb and their evaporative cooling, which is a precondition for condensate formation. For cooling from an initial temperature \( T_i \) to a final temperature \( T_f \), the average single-particle occupation-number distributions are calculated as a function of time, and compared to a relaxation ansatz.

Using both the linear and the nonlinear model of thermalization, the time-dependent entropy is calculated, and the contributions of wave and particle entropy are discussed. Whereas cooling drastically reduces the entropy, the subsequent re-thermalization causes a gradual increase of the entropy towards the equilibrium value, which coincides with the Bose-Einstein result and is significantly below the initial value before cooling.

Further refinements of the model such as variable transport coefficients and time-dependent chemical potentials

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Fig. 4: Time evolution of the entropy \( S(t)/g_0 \) in an equilibrating Bose system in the course of evaporative cooling from \( T_i = 232 \) nK to \( T_f = 93 \) nK as calculated from the analytical solution of the NBDE (2) with constant chemical potential \( \mu \), solid curve. \( S(t = 0) \) is the total entropy following evaporative cooling. The dashed curve is the wave entropy, the dotted curve is the particle entropy, and the dot-dashed curve is the result from the linear relaxation ansatz. The dotted horizontal line indicates the equilibrium value at \( T_f = 93 \) nK. The entropy of the analytical solution reaches the thermal value at \( t \approx 60 \) ms.
are conceivable, but may not allow for analytic solutions. Extensions of the NBDE itself to higher dimensions in order to account for anisotropic systems should also be investigated. Direct comparisons of the results to data from cold-atom experiments would be most welcome.

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