INJECTIVITY AND UNIVALENCE OF COMPLEX FUNCTIONS VIA MONOTONICITY

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Abstract. In this paper we provide sufficient conditions that ensure the monotonicity, respectively the global injectivity of an operator. Further, some new analytical conditions that assure the injectivity/univalence of a complex function of one complex variable are obtained. We also show that some classical results, such as Alexander-Noshiro-Warschawski and Wolff theorem or Mocanu theorem, are easy consequences of our results.

Keywords: monotone operator; injective operator; complex function; univalent function
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1. Introduction

One of the most celebrated results that provides the univalence of a holomorphic function \( f : D \subseteq \mathbb{C} \rightarrow \mathbb{C} \), \( f = u + iv \), is \( \text{Re} \ f'(z) > 0 \) for all \( z \in D \). However, it can easily be shown (see for instance [1]) that this condition is equivalent to the strict monotonicity of the vector function \( f = (u, v) \), and it is well known that strictly monotone operators are injective. On the other hand the mentioned univalence condition is a particular case of Alexander-Noshiro-Warschawski and Wolff theorem (for \( \gamma = 0 \), see [2, 3, 4, 5]), and the latter cannot be deduced by using the classical strict monotonicity concept of an operator.

Let us mention that several injectivity conditions for operators that are monotone in some sense were obtained recently in [1, 6] and [7]. These results were applied then to obtain some injectivity/univalence results for complex functions of one complex variable. In this paper we deal with operators which are monotone relative to another operator. We obtain some sufficient (analytical) conditions that ensure this monotonicity property. We also show that operators having this monotonicity property are injective under some circumstances. By combining the mentioned results we obtain some analytical conditions that ensure injectivity of an operator. We also extend the well-known injectivity result expressed in terms of positive definiteness of the symmetric part of all Fréchet differentials of operators of class \( C^1 \).

In the last section we apply these results to obtain some unknown injectivity, respectively univalence results for complex functions of one complex variable. As immediate consequences of our main result we obtain Mocanu theorem, respectively Alexander-Noshiro-Warschawski and Wolff theorem concerning on injectivity, respectively univalence of complex functions.

2. Analytical conditions for monotonicity and injectivity

Let \( (H, \langle \cdot, \cdot \rangle) \) be a real Hilbert space identified with its topological dual. Consider the operator \( T : D \subseteq H \rightarrow H \) and let \( A : H \rightarrow H \) be another operator. We say that the operator \( T \) is monotone relative to \( A \) if for all \( x, y \in D \) one has

\[
\langle T(x) - T(y), A(x) - A(y) \rangle \geq 0.
\]

Equality holds only for \( A(x) = A(y) \).

T is called strictly monotone relative to \( A \) if in (1) equality holds only for \( A(x) = A(y) \).

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For $x, y \in H$ let us denote by $(x, y)$ the open line segment with the endpoints $x$, respectively $y$, i.e.

$$(x, y) = \{x + t(y - x) : 0 < t < 1\}.$$ 

Let $D \subseteq H$ be open. For a differentiable operator $T : D \to H$, we denote by $dT_x(\cdot)$ the Fréchet differential of $T$ at $x \in D$. In what follows we provide an analytical condition that ensures the monotonicity of an operator relative to another operator.

**Proposition 2.1.** Let $D \subseteq H$ be an open and convex set, let $T : D \to H$ be an operator of class $C^1$ and let $A : H \to H$ be an operator. Assume that for all $x, y \in D$, with $A(x) \neq A(y)$ and $z \in (x, y)$ one has

$$\langle dT_z(y - x), A(y) - A(x) \rangle > 0.$$ 

Then $T$ is strictly monotone relative to $A$.

**Proof.** Let $x, y \in D$ such that $A(x) \neq A(y)$. We show that $\langle T(y) - T(x), A(y) - A(x) \rangle > 0$.

Consider the real function $\phi : [0, 1] \to \mathbb{R}$, $\phi(t) = \langle T(x + t(y - x)), A(y) - A(x) \rangle$. Then $\phi$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$, hence according to mean value theorem, there exists $c \in (0, 1)$ such that $\phi'(c) = \phi(1) - \phi(0)$. Equivalently, we can state

$$\langle dT_{x+c(y-x)}(y-x), A(y) - A(x) \rangle = \langle T(y) - T(x), A(y) - A(x) \rangle.$$ 

On the other hand $c \in (0, 1)$ implies $x + c(y - x) \in (x, y)$ and by the hypothesis of theorem we have

$$\langle dT_{x+c(y-x)}(y-x), A(y) - A(x) \rangle > 0,$$

which shows that

$$\langle T(y) - T(x), A(y) - A(x) \rangle > 0.$$ 

\[\square\]

**Remark 2.1.** Note that the condition $\langle dT_z(y - x), A(y) - A(x) \rangle < 0$ for all $x, y \in D$, $x \neq y$ and $z \in (x, y)$ ensures that $T$ is strictly monotone relative to $-A$.

It can be analogously proved that the condition $\langle dT_z(y - x), A(y) - A(x) \rangle \geq 0$ for all $x, y \in D$ and $z \in (x, y)$, ensures that $T$ is monotone relative to $A$.

For a linear operator $A : H \to H$ we denote by $\ker A$ the set of zeroes of $A$, that is

$$\ker A = \{x \in H : A(x) = 0\}.$$ 

**Corollary 2.1.** Let $D \subseteq H$ be an open and convex set, let $T : D \to H$ be an operator of class $C^1$ and let $A : H \to H$ be a linear operator. Assume that for all $x \in D$ and $y \in H \setminus \ker A$ one has

$$\langle dT_x(y), A(y) \rangle > 0.$$ 

Then $T$ is strictly monotone relative to $A$.

**Proof.** Indeed, let $u, v \in D$, with $A(u) \neq A(v)$. Take $y = v - u$ and $x = w \in (u, v)$. Since $A(u) \neq A(v)$ we have $A(y) \neq 0$, hence $y \in D \setminus \ker A$.

Consequently, the condition $\langle dT_x(y), A(y) \rangle > 0$ for all $y \in H \setminus \ker A$ becomes

$$\langle dT_w(v - u), A(v) - A(u) \rangle > 0, \forall u, v \in D, \text{ with } A(u) \neq A(v).$$

The conclusion follows from Proposition 2.1. \[\square\]

**Remark 2.2.** If we assume that for all $x \in D$ and $y \in H \setminus \ker A$ one has $\langle dT_x(y), A(y) \rangle < 0$, we obtain that $T$ is strictly monotone relative to $-A$.

Next we provide some conditions that ensure the injectivity of an operator which is monotone relative to an operator $A$. 

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2
Proposition 2.2. An operator \( T : D \subseteq H \rightarrow H \) which is strictly monotone relative to \( A : H \rightarrow H \), is injective on \( D \setminus \{ x \in D : \exists y \in D, x \neq y, A(x) = A(y) \} \). If \( A \) is injective on \( D \), that is, for all \( x, y \in D, x \neq y \) one has \( A(x) \neq A(y) \), then \( T \) is also injective on its whole domain.

Proof. Indeed, for \( u, v \in D \setminus \{ x \in D : \exists y \in D, x \neq y, A(x) = A(y) \} \), \( u \neq v \) one has \( A(u) \neq A(v) \), hence
\[
\langle T(u) - T(v), A(u) - A(v) \rangle > 0.
\]
The latter relation shows that \( T(u) \neq T(v) \).

If \( A \) is injective on \( D \), then \( \{ x \in D : \exists y \in D, x \neq y, A(x) = A(y) \} = \emptyset \), hence \( T \) is injective on \( D \).

Combining the results obtained so far we obtain the following.

Theorem 2.1. Let \( D \subseteq H \) be an open and convex set, let \( T : D \rightarrow H \) be an operator of class \( C^1 \) and let \( A : H \rightarrow H \) be an operator injective on \( D \). Assume that one of the following conditions hold.

(a) For all \( x, y \in D \) with \( A(x) \neq A(y) \) and \( z \in (x, y) \) one has
\[
\langle dT_z(y - x), A(y) - A(x) \rangle > 0.
\]
(b) \( A \) is linear and for all \( x \in D \) and \( y \in H \setminus \ker A \) one has
\[
\langle dT_x(y), A(y) \rangle > 0.
\]

Then \( T \) is injective.

Proof. The conclusion follows from Proposition 2.1 and Proposition 2.2 respectively from Corollary 2.1 and Proposition 2.2.

Remark 2.3. Since \( A \) is injective on \( D \) if and only if \( -A \) is injective on \( D \), according to Remark 2.1 respectively Remark 2.2 the conditions

(a) For all \( x, y \in D \), \( A(x) \neq A(y) \) and \( z \in (x, y) \) one has
\[
\langle dT_z(y - x), A(y) - A(x) \rangle < 0,
\]
respectively

(b) \( A \) is linear and for all \( x \in D \) and \( y \in H \setminus \ker A \) one has
\[
\langle dT_x(y), A(y) \rangle < 0,
\]
also assure the injectivity of \( T \).

Consider now \( H = \mathbb{R}^n \) endowed with the usual euclidian scalar product, let \( T : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( T = (t_1, t_2, \ldots, t_n) \) be an operator of class \( C^1 \) and let \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear operator. For \( x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \in D \) we denote by \( J_T(x^0) \) the Jacobian matrix of \( T \) in \( x^0 \), i.e.
\[
J_T(x^0) = \begin{pmatrix}
\frac{\partial t_1}{\partial x_1}(x^0) & \frac{\partial t_1}{\partial x_2}(x^0) & \cdots & \frac{\partial t_1}{\partial x_n}(x^0) \\
\frac{\partial t_2}{\partial x_1}(x^0) & \frac{\partial t_2}{\partial x_2}(x^0) & \cdots & \frac{\partial t_2}{\partial x_n}(x^0) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial t_n}{\partial x_1}(x^0) & \frac{\partial t_n}{\partial x_2}(x^0) & \cdots & \frac{\partial t_n}{\partial x_n}(x^0)
\end{pmatrix}.
\]

Note that the linear operator \( A \) can be identified with a real square matrix \( (a_{ij})_{1 \leq i, j \leq n} \). Let us denote by \( A^T \) the transpose of \( A \). For a given square matrix \( B \) of order \( n \) we denote the submatrix obtained by deleting the last \( n - m \) rows and the last \( n - m \) columns by \( (B)_{1 \leq i, j \leq m} \).
**Theorem 2.2.** Let $D \subseteq \mathbb{R}^n$ be an open and convex set, let $T : D \rightarrow \mathbb{R}^n$ be an operator of class $C^1$ and let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator with $\det A \neq 0$. Assume that for all $x \in D$ one of the following conditions hold.

(a) $\det(A^\top J_T(x) + J_T^\top(x)A)_{1 \leq i,j \leq m} > 0$, $\forall m \in \{1, 2, \ldots, n\}$.

(b) $(-1)^m \det(A^\top J_T(x) + J_T^\top(x)A)_{1 \leq i,j \leq m} > 0$, $\forall m \in \{1, 2, \ldots, n\}$.

Then $T$ is injective.

**Proof.** Let $x \in D$. Then we have $\langle dT_x(y), A(y) \rangle = \langle A^\top dT_x(y), y \rangle = y^\top A^\top J_T(x)y$. This shows that the positive definiteness, respectively negative definiteness of $A^\top J_T(x)$ is equivalent to

$$\langle dT_x(y), A(y) \rangle > 0, \forall y \in \mathbb{R}^n \setminus \{0\},$$

respectively

$$\langle dT_x(y), A(y) \rangle < 0, \forall y \in \mathbb{R}^n \setminus \{0\}.$$

On the other hand, $y^\top A^\top J_T(x)y = ((A^\top J_T(x))^\top)^\top y = (J_T^\top(x)Ay)^\top y = y^\top ((J_T^\top(x)Ay)^\top)^\top = y^\top J_T^\top(x)Ay$, hence

$$y^\top A^\top J_T(x)y = \frac{1}{2}y^\top (A^\top J_T(x) + J_T^\top(x)A)y.$$

Observe that

$$\det(A^\top J_T(x) + J_T^\top(x)A)_{1 \leq i,j \leq m} > 0, \forall m \in \{1, 2, \ldots, n\}.$$

is actually Sylvester’s criterion for positive definiteness of the symmetric matrix

$$A^\top J_T(x) + J_T^\top(x)A,$$

meanwhile the condition $(-1)^m \det(A^\top J_T(x) + J_T^\top(x)A)_{1 \leq i,j \leq m} > 0$, $\forall m \in \{1, 2, \ldots, n\}$ is actually Sylvester’s criterion for negative definiteness of the symmetric matrix

$$A^\top J_T(x) + J_T^\top(x)A.$$

But the latter relations are equivalent to the positive definiteness, respectively negative definiteness of $A^\top J_T(x)$. Hence, we have

$$\langle dT_x(y), A(y) \rangle > 0, \forall y \in \mathbb{R}^n \setminus \{0\},$$

respectively

$$\langle dT_x(y), A(y) \rangle < 0, \forall y \in \mathbb{R}^n \setminus \{0\}.$$ 

Since $\det A \neq 0$ we obtain that $A$ is injective. The conclusion follows from Theorem 2.1, respectively Remark 2.3. \qed

3. **Injective complex functions**

Let us denote by $\mathbb{C}$ the set of complex numbers, that is

$$\mathbb{C} = \{ z = x + iy : x, y \in \mathbb{R}, i^2 = -1 \}.$$

For $z = x + iy \in \mathbb{C}$ we denote by $\text{Re} z, \text{Im} z, \overline{z}$, respectively $|z|$ the real part, imaginary part, conjugate and absolute value respectively, that is $\text{Re} z = x, \text{Im} z = y, \overline{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$. Obviously $z\overline{z} = |z|^2$. Note that the real linear space $\mathbb{C}$ becomes a real Hilbert space with the inner product

$$\langle z, w \rangle = \text{Re} z\overline{w}.$$

This real Hilbert space may be identified with the real Hilbert space $\mathbb{R}^2$ endowed with the euclidean scalar product, therefore we can identify $z \in \mathbb{C}$ by $(\text{Re} z, \text{Im} z) \in \mathbb{R}^2$. 

4
Let $D \subseteq \mathbb{C}$ be open. For a complex function of one complex variable $f : D \to \mathbb{C}$, $f(z) = u(x, y) + iv(x, y)$, $\forall z = x + iy \in D$ of class $C^1(D)$, we denote by $J_f(z_0)$ the Jacobian matrix of $f$ in $z_0 = x_0 + iy_0$, i.e.,

$$J_f(z_0) = \begin{pmatrix} u'_x(x_0, y_0) & u'_y(x_0, y_0) \\ v'_x(x_0, y_0) & v'_y(x_0, y_0) \end{pmatrix}.$$ 

If we consider $f$ as the vector function $(u, v)$ then its differential in $z_0 = x_0 + iy_0$ can be defined as

$$df_{(x_0, y_0)}(p, q) = J_f(z_0) \cdot \begin{pmatrix} p \\ q \end{pmatrix},$$

hence for $w = p + iq$ the differential of $f$ in $z_0$ becomes

$$df_{z_0}(w) = (u'_x(x_0, y_0)p + u'_y(x_0, y_0)q) + i(v'_x(x_0, y_0)p + v'_y(x_0, y_0)q).$$

The partial derivatives of $f$ are defined as:

$$\frac{\partial f}{\partial x}(z_0) = u'_x(x_0, y_0) + iv'_x(x_0, y_0),$$

respectively

$$\frac{\partial f}{\partial y}(z_0) = u'_y(x_0, y_0) + iv'_y(x_0, y_0).$$

Let us introduce the following notations:

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0) \right),$$

respectively

$$\frac{\partial f}{\partial \overline{z}}(z_0) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(z_0) + i \frac{\partial f}{\partial y}(z_0) \right).$$

The main result of this section is the following general injectivity result.

**Theorem 3.1.** Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \to \mathbb{C}$ be a function of class $C^1$. Assume that there exist $w_1, w_2 \in \mathbb{C}$ such that $\text{Re} \, w_1 \text{Im} \, w_2 \neq \text{Re} \, w_2 \text{Im} \, w_1$ and for all $z \in D$ the following condition holds:

$$\text{Re} \left( \frac{\partial f}{\partial z}(z) w_1 + \frac{\partial f}{\partial \overline{z}}(z) \overline{w_1} \right) + \text{Im} \left( \frac{\partial f}{\partial z}(z) w_2 + \frac{\partial f}{\partial \overline{z}}(z) \overline{w_2} \right) > 0 \quad (2)$$

$$\left| \frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \overline{z}}(z)w_2 + iw_1 \right|.$$ 

Then $f$ is injective.

**Proof.** One can assume that $w_1 = a + ib$, $w_2 = c + id$, $a, b, c, d \in \mathbb{R}$. It can easily be deduced, that (2) is equivalent to

$$\sqrt{\left( (u'_x c + u'_y d) + (v'_x a + v'_y b) \right)^2 + \left( (u'_x c + v'_y d) - (u'_x a + u'_y b) \right)^2}.$$ 

By taking the square of both sides we obtain

$$4(u'_x a + u'_y b)(v'_x c + v'_y d) > \left( (u'_x c + u'_y d) + (v'_x a + v'_y b) \right)^2,$$

or equivalently

$$4 \text{Re} \, df_z(w_1) \cdot \text{Im} \, df_z(w_2) > (\text{Re} \, df_z(w_2) + \text{Im} \, df_z(w_1))^2, \forall z \in D.$$ 

The latter relation can be written as

$$\det \left( \begin{pmatrix} 2 \text{Re} \, df_z(w_1) \\ \text{Re} \, df_z(w_2) + \text{Im} \, df_z(w_1) \end{pmatrix} \begin{pmatrix} \text{Re} \, df_z(w_2) + \text{Im} \, df_z(w_1) \end{pmatrix} \right) > 0, \forall z \in D. \quad (3)$$
Let us denote by $L$ the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then (3) becomes

$$\det(L^T J_f(z) + J_f(z) L) > 0, \forall z \in D.$$ 

We show next that $\text{Re} df_z(w_1) > 0$ for all $z \in D$, or $\text{Re} df_z(w_1) < 0$ for all $z \in D$. Observe that (3) assures that $\text{Re} df_z(w_1) \neq 0$ for all $z \in D$. Assume for instance that $\text{Re} df_z(w_1) > 0$ and $\text{Re} df_z(w_1) < 0$ for some $z_1, z_2 \in D$. Then, the intermediate value theorem, applied to the function $g : D \rightarrow \mathbb{R}$, $g(z) = \text{Re} df_z(w_1)$, provides the existence of $z_3 \in D$ such that $\text{Re} df_z(x_1) = 0$, contradiction.

In conclusion one of the following conditions is fulfilled.

(i) $\text{Re} df_z(w_1) > 0$ and $\det(L^T J_f(z) + J_f(z) L) > 0$ for all $z \in D$, or

(ii) $\text{Re} df_z(w_1) < 0$ and $\det(L^T J_f(z) + J_f(z) L) > 0$ for all $z \in D$.

Note that (i), respectively (ii) are equivalent to

(a) $\det(L^T J_f(z) + J_f(z) L)_{1 \leq i,j \leq m} > 0, \forall m \in \{1, 2\}$, respectively

(b) $(-1)^m \det(L^T J_f(z) + J_f(z) L)_{1 \leq i,j \leq m} > 0, \forall m \in \{1, 2\}$.

Since $\text{Re} w_1 \text{Im} w_2 \neq \text{Re} w_2 \text{Im} w_1$, we obtain that $L$ is invertible, hence injective. According to Theorem 2.2, $f$ is injective.

\[ \square \]

Remark 3.1. One can easily deduce that for $z \in D$ and $w \in \mathbb{C}$ we have

$$df_z(w) = \frac{\partial f}{\partial z}(z) w + \frac{\partial f}{\partial \overline{z}}(z) \overline{w},$$

hence the condition (2) in the hypothesis of Theorem 3.1 can be replaced by

$$\text{Re} df_z(w_1) + \text{Im} df_z(w_2) > |df_z(w_2) - idf_z(w_1)|, \forall z \in D.$$ 

The next Corollary can be viewed as an extension of Mocanu’s injectivity result.

**Corollary 3.1.** Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \rightarrow \mathbb{C}$ be a function of class $C^1$. Assume that there exist $\gamma \in \mathbb{R}$ such that, for all $z \in D$ it holds:

$$\text{Re} \left( \frac{\partial f}{\partial z}(z)e^{i\gamma} \right) > \left| \frac{\partial f}{\partial \overline{z}}(z) \right|.$$ 

Then $f$ is injective.

**Proof.** Take $w_1 = e^{i\gamma}$ and $w_2 = ie^{i\gamma}$. Then an easy computation shows that

$$2 \text{Re} \left( \frac{\partial f}{\partial z}(z)e^{i\gamma} \right) = \text{Re} \left( \frac{\partial f}{\partial z}(z)w_1 + \frac{\partial f}{\partial \overline{z}}(z)\overline{w_1} \right) + \text{Im} \left( \frac{\partial f}{\partial z}(z)w_2 + \frac{\partial f}{\partial \overline{z}}(z)\overline{w_2} \right).$$

On the other hand

$$2 \left| \frac{\partial f}{\partial \overline{z}}(z) \right| = \left| \frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \overline{z}}(z)\overline{w_2 + iw_1} \right|.$$ 

Hence,

$$\text{Re} \left( \frac{\partial f}{\partial z}(z)e^{i\gamma} \right) > \left| \frac{\partial f}{\partial \overline{z}}(z) \right|$$

is equivalent to

$$\text{Re} \left( \frac{\partial f}{\partial z}(z)w_1 + \frac{\partial f}{\partial \overline{z}}(z)\overline{w_1} \right) + \text{Im} \left( \frac{\partial f}{\partial z}(z)w_2 + \frac{\partial f}{\partial \overline{z}}(z)\overline{w_2} \right) >$$

$$\left| \frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial \overline{z}}(z)\overline{w_2 + iw_1} \right|.$$ 

Since $\text{Re} w_1 \text{Im} w_2 - \text{Re} w_2 \text{Im} w_1 = \cos^2 \gamma + \sin^2 \gamma = 1$, obviously $\text{Re} w_1 \text{Im} w_2 \neq \text{Re} w_2 \text{Im} w_1$.

The conclusion follows from Theorem 3.1. \[ \square \]
Remark 3.2. Note that for $\gamma = 0$ in Corollary 3.1, we obtain Mocanu’s injectivity theorem, see [8].

Corollary 3.2. Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \to \mathbb{C}$ be a function of class $C^1$. Assume that there exist $\gamma \in \mathbb{R}$ such that, for all $z \in D$ it holds:

$$\text{Re} \left( \frac{\partial f}{\partial z}(z) e^{i\gamma} \right) > \left| \frac{\partial f}{\partial z}(z) \right|.$$ 

Then $f$ is injective.

Proof. Take $w_1 = e^{i\gamma}$ and $w_2 = -ie^{i\gamma}$. Then $\text{Re} w_1 \text{ Im} w_2 - \text{Re} w_2 \text{ Im} w_1 = -\cos^2 \gamma - \sin^2 \gamma = -1$, hence $\text{Re} w_1 \text{ Im} w_2 \neq \text{Re} w_2 \text{ Im} w_1$. The rest of the proof is analogous to the proof of Corollary 3.1. □

Let $D$ be open and connected. Recall that a function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is called holomorphic on $D$ if $f$ is derivable at every point of $D$. Note, that in the case when $f$ is holomorphic on $D$, one has

$$\frac{\partial f}{\partial z}(z) = 0,$$

for all $z \in D$, and

$$df_z(w) = \frac{\partial f}{\partial z}(z) w = f'(z) w,$$ 

for all $z \in D$ and $w \in \mathbb{C}$.

A holomorphic function which is also injective is called univalent. The next result is an extension of the univalence result of Alexander-Noshiro-Warschawski and Wolff.

Corollary 3.3. Let $D \subseteq \mathbb{C}$ be open and convex and let $f : D \to \mathbb{C}$ be a holomorphic function. Assume that there exist $w_1, w_2 \in \mathbb{C}$ such that $\text{Re} w_1 \text{ Im} w_2 \neq \text{Re} w_2 \text{ Im} w_1$ and for all $z \in D$ the following condition holds:

$$\text{Re} f'(z) w_1 + \text{Im} f'(z) w_2 > |f'(z)||w_2 - iw_1|.$$

Then $f$ is univalent.

Proof. According to Remark 3.1 the condition

$$\text{Re} \left( \frac{\partial f}{\partial z}(z) w_1 + \frac{\partial f}{\partial z}(z) \overline{w_1} \right) + \text{Im} \left( \frac{\partial f}{\partial z}(z) w_2 + \frac{\partial f}{\partial z}(z) \overline{w_2} \right) > \left| \frac{\partial f}{\partial z}(z)(w_2 - iw_1) + \frac{\partial f}{\partial z}(z)w_2 \overline{w_1} \right|,$$

is equivalent to

$$\text{Re} df_z(w_1) + \text{Im} df_z(w_2) > |df_z(w_2) - idf_z(w_1)|.$$ 

On the other hand the latter relation is exactly

$$\text{Re} f'(z) w_1 + \text{Im} f'(z) w_2 > |f'(z)||w_2 - iw_1|.$$ 

The conclusion follows from Theorem 3.1. □

From the previous result one can easily obtain Alexander-Noshiro-Warschawski and Wolff univalence theorem (see [9] and [11, 15]).

Corollary 3.4. Let $D \in \mathbb{C}$ be open and convex and let $f : D \to \mathbb{C}$ be a holomorphic function. Assume that there exist $\gamma \in \mathbb{R}$ such that, for all $z \in D$ it holds:

$$\text{Re} f'(z) e^{i\gamma} > 0.$$

Then $f$ is univalent.
Proof. Indeed, let \( w_1 = e^{i\gamma}, \ w_2 = ie^{i\gamma} \). Then
\[
\text{Re} f'(z)w_1 + \text{Im} f'(z)w_2 = 2 \text{Re} f'(z)e^{i\gamma}.
\]
Obviously \( |f'(z)||w_2 - iw_1| = 0 \), hence \( \text{Re} f'(z)e^{i\gamma} > 0 \) is equivalent to
\[
\text{Re} f'(z)w_1 + \text{Im} f'(z)w_2 > |f'(z)||w_2 - iw_1|.
\]
Note that \( \text{Re} w_1 \text{Im} w_2 - \text{Re} w_2 \text{Im} w_1 = \cos^2 \gamma + \sin^2 \gamma = 1 \), hence \( \text{Re} w_1 \text{Im} w_2 \neq \text{Re} w_2 \text{Im} w_1 \). The conclusion follows from Corollary 3.3. □

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