On Geometric Connections of Embedded and Quotient Geometries in Riemannian Fixed-rank Matrix Optimization

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Abstract

In this paper, we propose a general procedure for establishing the landscape connections of a Riemannian optimization problem under the embedded and quotient geometries. By applying the general procedure to the fixed-rank positive semidefinite (PSD) and general matrix optimization, we establish an exact Riemannian gradient connection under two geometries at every point on the manifold and sandwich inequalities between the spectra of Riemannian Hessians at Riemannian first-order stationary points (FOSPs). These results immediately imply an equivalence on the sets of Riemannian FOSPs, Riemannian second-order stationary points (SOSPs) and strict saddles of fixed-rank matrix optimization under the embedded and the quotient geometries. To the best of our knowledge, this is the first geometric landscape connection between the embedded and the quotient geometries for fixed-rank matrix optimization and it provides a concrete example on how these two geometries are connected in Riemannian optimization. In addition, the effects of the Riemannian metric and quotient structure on the landscape connection are discussed. We also observe an algorithmic connection for fixed-rank matrix optimization under two geometries with some specific Riemannian metrics. A number of novel ideas and technical ingredients including a unified treatment for different Riemannian metrics and new horizontal space representations under quotient geometries are developed to obtain our results. The results in this paper deepen our understanding of geometric connections of Riemannian optimization under different Riemannian geometries and provide a few new theoretical insights to unanswered questions in the literature.

Keywords: Landscape connection, Riemannian optimization, fixed-rank matrix optimization, embedded geometry, quotient geometry

1 Introduction

Riemannian optimization is a powerful method to tackle a general class of optimization problems with geometric constraints so that the solution is constrained to a Riemannian manifold. One key component in Riemannian optimization is the Riemannian geometry. Over the past decades, numerous Riemannian geometries, including different manifold classes, quotient manifold structures and Riemannian metrics, have been proposed in various problems for having either better geometric properties or faster algorithmic convergences. [EAS98, VAV13, MST16, MMBS14, BS10]
In the literature, two popular choices of manifold classes in Riemannian optimization are embedded submanifold and quotient manifold. The embedded geometry often allows computing and interpreting the geometric notions more straightforwardly; while the optimization methods via quotient geometry can be more versatile as quotient geometry provides more choices of quotient structures and Riemannian metrics. The readers are referred to [AMS09, Bon20, MMBS14, Mey11] for surveys on these topics.

The Riemannian optimization under embedded and quotient geometries are not obviously related (even commented to be fundamentally different in [JBAS10]) and are often studied separately in the literature. It is also unclear how to choose between these two geometries in Riemannian optimization. On the other hand, a few empirical studies on the algorithmic comparisons between the embedded and the quotient geometries in matrix completion and graph-based clustering problems showed that for either the gradient or trust-region based algorithms, these two geometries perform more or less the same in terms of the total computational time [MMBS14, MAS12, DH21]. It has been asked by [Van13] on the reason behind. [VV10] hinted that embedded and quotient approaches are probably related as the manifolds under these two geometries are diffeomorphic to each other. However, it remains elusive in the literature how they are exactly connected in specific Riemannian optimization problems from either an algorithmic or a geometric point of view.

In this work, we make the first attempt to answer these questions by proposing a general framework to investigate the first-order and second-order geometric landscape connections of the optimization problem under the embedded and the quotient geometries. The first-order geometric connection can often be easily established and the general procedure for connecting second-order geometries includes three steps: 1. compute the quadratic form of Riemannian Hessians under two geometries; 2. construct a carefully-designed mapping $L$ between the horizontal space under the quotient geometry to the tangent space under the embedded geometry at proper reference points to connect Riemannian Hessians; 3. establish the spectra connection between Riemannian Hessians via bounding the spectrum of $L$.

We then specifically consider the following fixed-rank matrix optimization problems:

**PSD case:**
\[
\min_{X \in S^{p \times p} \geq 0, \text{rank}(X) = r} f(X), \quad 0 < r \leq p,
\]

**general case:**
\[
\min_{X \in \mathbb{R}^{p1 \times p2}, \text{rank}(X) = r} f(X), \quad 0 < r \leq \min\{p_1, p_2\}.
\]

In the positive semidefinite (PSD) case, without loss of generality, we assume $f$ is symmetric in $X$, i.e., $f(X) = f(X^\top)$; otherwise, we can set $\tilde{f}(X) = \frac{1}{2}(f(X) + f(X^\top))$ and have $\tilde{f}(X) = f(X)$ for all $X \succeq 0$ without changing the problem [BKS16]. In both cases we assume $f$ is twice continuously differentiable with respect to $X$ and the Euclidean metric. Both embedded and quotient geometries have been studied for the sets of fixed-rank matrices and many algorithms have been proposed for (1) and (2) on each individual Riemannian geometry. See Section 1.1 for a review of existing results.

By applying the general procedure, we establish the geometric connections of (1) and (2) under the embedded and a variety of quotient geometries (Theorems 3–7, Corollaries 1, 2) informally summarized as follows.

**Theorem 1** (Informal results). Consider optimization problems (1) and (2) on fixed-rank PSD and general matrix manifolds.

- There exists an equivalence relation on the sets of Riemannian first-order stationary points (FOSPs), Riemannian second-order stationary points (SOSPs) and strict saddles of (1) or (2) under embedded and quotient geometries.
- The spectra of Riemannian Hessians of (1) or (2) under two geometries are sandwiched by each other at Riemannian FOSPs.

To the best of our knowledge, this is the first geometric landscape connection between the embedded and the quotient geometries for fixed-rank matrix optimization.

In addition, the effects of Riemannian metric and quotient structure on the landscape connection are discussed. We also observe an algorithmic connection of (1) and (2) under the embedded and the quotient geometries with some specific Riemannian metrics.

In a broad sense, embedded and quotient geometries are the most common two choices in Riemannian optimization. It is known that the manifolds under two geometries are diffeomorphic to each other, however it is unclear how the geometry-dependent key concepts of optimization problems are related. This paper bridges them from a geometric point of view and illustrates explicitly how they are connected in solving fixed-rank matrix optimization problems.

1.1 Related Literature

This work is related to a range of literature on low-rank matrix optimization, Riemannian/nonconvex optimization, and geometric landscape analysis of an optimization problem.

First, choosing a proper Riemannian geometry is undoubtedly a central topic in Riemannian optimization and numerous geometries have been proposed under different considerations. For example, [VAV13] proposed a homogeneous space geometry on the set of fixed-rank PSD matrices such that the complete geodesics can be obtained. [MS16, MAS12] proposed new Riemannian metrics under fixed-rank quotient geometries tailored to objective functions. Different Riemannian manifold structures have also been considered to have better geometric properties [BS10, AAM14]. In this work, we focus on the choice between embedded and quotient geometries and study its effect on the corresponding Riemannian optimization problem.

Second, for fixed-rank matrix optimization (1) and (2), a number of Riemannian optimization methods, including (conjugate) gradient descent, (Gauss-)Newton, trust-region have been developed under either embedded geometry [HLZ20, LHLZ20, SWC12, Van13, VV10, WCCL16] or quotient geometry [AIDLVH09, AAM14, BAL11, EAS98, HH18, MBS11a, MBS11b, MMBS14, NS12]. We refer readers to [AMS09, Bou20, CW18] for the recent algorithmic development in Riemannian matrix optimization. In addition to Riemannian optimization, a number of other methods including convex relaxation [RFP10, CZ13], non-convex factorization [JNS13, MWCC19, SL15, TBS16], projected gradient descent [JMD10], and penalty method [GS10] have been proposed to solve (1) and (2) as well. A comparison of these different approaches can be found in [CC18, CLC19].

Third, a few attempts have been made to analyze the landscape of a Riemannian matrix optimization problem. For example, [MZZ19, AS21] provided landscape analyses for robust subspace recovery and matrix factorization over the Grassmannian manifold. Under the embedded geometry, [UV20] showed the landscape of (2) is benign when $f$ is quadratic and satisfies certain restricted spectral bounds property. Different from this line of works which focus on the landscape of the problem under a single Riemannian geometry when $f$ is well-conditioned, here we study the geometric landscape connections of (1) and (2) under the embedded and the quotient geometries for a general $f$.

Finally, there are a few recent studies on the geometric connections of different approaches for rank constrained optimization. For example, [HLB20] studied the relationship between Euclidean FOSPs/SOSPs under the factorization formulation and fixed points of the projected gradient descent (PGD) in the general low-rank matrix optimization. They showed while the sets of FOSPs under the factorization formulation can be larger, the sets of SOSPs are contained in the set of fixed...
are often used to denote scalars, vectors, matrices, respectively. For any \((SVD)\) and spectral norm as }

\[\|X\|_F = \sqrt{\sum_i \sigma_i^2(X)}\]

and we say a column orthonormal matrix 

\[U\] of 

\[X\]

is denoted as \(\text{dim}(U)\). For any \(i\), lowercase boldface letters (e.g., \(u, v\)), uppercase boldface letters (e.g., \(U, V\)) are often used to denote scalars, vectors, matrices, respectively. For any \(a, b \in \mathbb{R}\), let 

\[a \wedge b := \min\{a, b\}, \quad a \vee b := \max\{a, b\}.\]

For any matrix \(X \in \mathbb{R}^{p_1 \times p_2}\) with singular value decomposition (SVD) \(\sum_{i=1}^{p_1} \sigma_i(X)u_i v_i^T\), where \(\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_{p_1 \wedge p_2}(X)\), denote its Frobenius norm and spectral norm as \(\|X\|_F = \sqrt{\sum_i \sigma_i^2(X)}\) and \(\|X\| = \sigma_1(X)\), respectively. Also, denote 

\[X^{-1}, X^{-\top}\]

and \(X^\dagger\) as the inverse, transpose inverse, and Moore-Penrose inverse of \(X\), respectively. For any \(X \in \mathbb{R}^{p \times p}\), let 

\[\text{Sym}(X) = (X + X^\top)/2, \quad \text{Skew}(X) = (X - X^\top)/2, \quad \text{tr}(X)\]

be the symmetric part, skew-symmetric part, and the trace of \(X\), respectively. For any \(X \in \mathbb{R}^{p \times p}\) having eigendecomposition 

\[U \Sigma U^\top\]

with non-increasing eigenvalues on the diagonal of \(\Sigma\), let \(\lambda_i(X)\) be the \(i\)-th largest eigenvalue of \(X\), \(\lambda_{\text{min}}(X)\) be the left eigenvalue of \(X\), and \(X^{1/2} = U \Sigma^{1/2} U^\top\). We note 

\[X \succeq 0\]

if \(X\) is a symmetric positive semidefinite (PSD) matrix. Throughout the paper, the SVD (or eigendecomposition) of a rank \(r\) matrix \(X\) (or symmetric matrix \(X\)) refers to its economic version and we say a column orthonormal matrix \(U\) spans the top \(r\) left singular space or eigenspace of \(X\) if 

\[U' = UO\]

for some \(O \in \mathbb{O}_r\), where \(U\) is formed by the top \(r\) left singular vectors or eigenvectors of \(X\). For any \(U \in \mathbb{S}(r, p)\), 

\[P_U = UU^\top\]

represents the orthogonal projector onto the column space of \(U\); we also note \(U_{\perp} \in \mathbb{S}(p - r, p)\) as an orthonormal complement of \(U\). We use bracket subscripts to denote sub-matrices. For example, 

\[X_{[i_1, i_2]}\]

is the entry of \(X\) on the \(i_1\)-th row and \(i_2\)-th column. In addition, \(I_r\) is the \(r\)-by-\(r\) identity matrix. Finally, the dimension of a linear space \(\mathcal{V}\) is denoted as \(\text{dim}(\mathcal{V})\). For any two linear spaces \(\mathcal{V}_1, \mathcal{V}_2\), the sum of \(\mathcal{V}_1\) and \(\mathcal{V}_2\) is denoted by
If every vector in $V_1 + V_2$ can be uniquely decomposed into $v_1 + v_2$, where $v_1 \in V_1, v_2 \in V_2$, then we call the sum of $V_1$ and $V_2$ the direct sum, denoted by $V_1 \oplus V_2$. The direct sum satisfies a key property: $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$. For any two Euclidean spaces $V_1$ and $V_2$ endowed with inner product $g(\cdot, \cdot)$, we say $V_1$ is orthogonal to $V_2$ with respect to $g$ and note $V_1 \perp V_2$, if and only if $g(v_1, v_2) = 0$ for any $v_1 \in V_1, v_2 \in V_2$.

Suppose $f : \mathbb{R}^{p_1 \times p_2} \to \mathbb{R}$ is a differentiable scalar function and $\phi : \mathbb{R}^{p_1 \times p_2} \to \mathbb{R}^{q_1 \times q_2}$ is a differentiable matrix-valued function. Let the Euclidean gradient of $f$ at $X$ be $\nabla f(X)$, i.e., $(\nabla f(X))[i, j] = \frac{\partial f(X)}{\partial X[i, j]}$ for $i \in [p_1], j \in [p_2]$. The Euclidean gradient of $\phi$ is a linear operator from $\mathbb{R}^{p_1 \times p_2}$ to $\mathbb{R}^{q_1 \times q_2}$ such that $(\nabla \phi(X)[Z])[i, j] = \sum_{k \in [p_1], l \in [p_2]} \frac{\partial (\phi(X))[i, j]}{\partial X[k, l]} Z[k, l]$ for any $Z \in \mathbb{R}^{p_1 \times p_2}, i \in [q_1], j \in [q_2]$. For a twice continuously differentiable function $f$, let $\nabla^2 f(X)[\cdot]$ be its Euclidean Hessian, which is the gradient of $\nabla f(X)$ and can be viewed as a linear operator from $\mathbb{R}^{p_1 \times p_2}$ to $\mathbb{R}^{p_1 \times p_2}$ satisfying
\[
(\nabla^2 f(X)[Z])[i, j] = \sum_{k \in [p_1], l \in [p_2]} \frac{\partial (\nabla f(X))[i, j]}{\partial X[k, l]} Z[k, l].
\]

Define the bilinear form of the Hessian of $f$ as $\nabla^2 f(X)[Z_1, Z_2] := \langle \nabla^2 f(X)[Z_1], Z_2 \rangle$ for any $Z_1, Z_2 \in \mathbb{R}^{p_1 \times p_2}$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.

### 2 Riemannian Optimization Under Embedded and Quotient Geometries

In this section, we first give a brief introduction to Riemannian optimization and then discuss how to perform Riemannian optimization under embedded and quotient geometries.

Riemannian optimization concerns optimizing a real-valued function $f$ defined on a Riemannian manifold $M$. The readers are referred to [AMS09, Bon20, HLWY20] for more details. The calculations of Riemannian gradients and Riemannian Hessians are key ingredients to perform continuous optimization over the Riemannian manifold. Suppose $X \in M$, $g_X(\cdot, \cdot)$ is the Riemannian metric, and $T_XM$ is the tangent space of $M$ at $X$. Then the Riemannian gradient of a smooth function $f : M \to \mathbb{R}$ at $X$ is defined as the unique tangent vector, $\nabla f(X) \in T_XM$, such that $g_X(\nabla f(X), \xi_X) = Df(X)[\xi_X], \forall \xi_X \in T_XM$, where $Df(X)[\xi_X]$ is the directional derivative of $f$ at point $X$ along the direction $\xi_X$. The Riemannian Hessian of $f$ at $X \in M$ is a linear mapping $\text{Hess} f(X) : T_XM \to T_XM$ defined as
\[
\text{Hess} f(X)[\xi_X] = \nabla_{\xi_X} \nabla f \in T_XM, \quad \forall \xi_X \in T_XM,
\]
where $\nabla$ is the Riemannian connection on $M$, which is a generalization of the directional derivative on a vector field to Riemannian manifolds [AMS09, Section 5.3] (See Appendix A for more details). The bilinear form of Riemannian Hessian is defined as $\text{Hess} f(X)[\xi_X, \theta_X] := g_X(\text{Hess} f(X)[\xi_X], \theta_X)$ for any $\xi_X, \theta_X \in T_XM$. We say $X \in M$ is a Riemannian FOSP of $f$ iff $\text{grad} f(X) = 0$ and a Riemannian SOSP of $f$ iff $\text{grad} f(X) = 0$ and $\text{Hess} f(X) \succ 0$. Moreover, $X \in M$ is a local minimizer of $f$ if there exists a neighborhood $\mathcal{N}$ of $X$ in $M$ such that $f(X) \leq f(X')$ for all $X' \in \mathcal{N}$. Finally, we call a Riemannian FOSP a strict saddle iff the Riemannian Hessian evaluated at this point has a strict negative eigenvalue.

In this work, we mainly focus on two classes of manifolds: embedded submanifold and quotient manifold. The embedded submanifold can be viewed as a generalization of the notion of surface in $\mathbb{R}^d$ and the Riemannian gradients and Hessians under the embedded geometry can often be concretely written out as every geometric object lies in the embedding space. For example, suppose
\( \mathcal{M} \) is a Riemannian embedded submanifold of the Riemannian manifold \( \tilde{\mathcal{M}} \) and the objective function \( f : \mathcal{M} \to \mathbb{R} \) is the restriction of \( \tilde{f} : \tilde{\mathcal{M}} \to \mathbb{R} \) to the embedded submanifold \( \mathcal{M} \). Then we have the following simple expressions for the Riemannian gradient of \( f \) and the Riemannian connection \[ \text{AMS09}, \text{Eq. (3.37) and Proposition 5.3.2} \]: \( \text{grad} f(X) = \text{P}_{TX\mathcal{M}}(\text{grad} \tilde{f}(X)) \) and \( \nabla_X \eta = \text{P}_{TX\mathcal{M}}(\nabla_{\xi_X} \eta) \), where \( \text{P}_{TX\mathcal{M}}(\cdot) \) is the projection operator onto the tangent space \( TX\mathcal{M} \), \( \xi, \eta \) are two vector fields on \( \mathcal{M} \), \( \text{grad} \tilde{f}(X) \) and \( \nabla' \) are the Riemannian gradient of \( \tilde{f} \) and the Riemannian connection on \( \mathcal{M} \), respectively. On the other hand, the geometric objects under quotient manifolds are more abstract. The following Section 2.1 aims to provide more details on how to perform Riemannian optimization on quotient manifolds.

### 2.1 Riemannian Optimization on Quotient Manifolds

Quotient manifolds are often defined via an equivalence relation “\( \sim \)” that satisfies symmetric, reflexive and transitive properties \[ \text{AMS09} \] Section 3.4.1. The equivalence classes are often abstract objects and cannot be directly applied in numerical computations. Riemannian optimization on quotient manifolds works on representatives of these equivalence classes instead. To be specific, suppose \( \tilde{\mathcal{M}} \) is an embedded submanifold equipped with an equivalence relation \( \sim \). The equivalence class (or fiber) of \( \mathcal{M} \) at a given point \( X \) is defined by the set \( [X] = \{ X_1 \in \tilde{\mathcal{M}} : X_1 \sim X \} \). The set \( \mathcal{M} := \tilde{\mathcal{M}}/\sim := \{ [X] : X \in \tilde{\mathcal{M}} \} \) is called a quotient of \( \tilde{\mathcal{M}} \) by \( \sim \). The mapping \( \pi : \tilde{\mathcal{M}} \to \mathcal{M} \) is called the quotient map or canonical projection and the set \( \mathcal{M} \) is called the total space of the quotient \( \tilde{\mathcal{M}}/\sim \). If \( \tilde{\mathcal{M}} \) further admits a smooth manifold structure and \( \pi \) is a smooth submersion, then we call \( \mathcal{M} \) a quotient manifold of \( \tilde{\mathcal{M}} \).

Due to the abstractness, the tangent space \( T_{[X]}\mathcal{M} \) of \( \mathcal{M} \) at \( [X] \) calls for a representation in the tangent space \( T_X\tilde{\mathcal{M}} \) of the total space \( \tilde{\mathcal{M}} \). By the equivalence relation \( \sim \), the representation of elements in \( T_{[X]}\mathcal{M} \) should be restricted to the directions in \( T_X\tilde{\mathcal{M}} \) without inducing displacement along the equivalence class \( [X] \). This can be achieved by decomposing \( T_X\tilde{\mathcal{M}} \) into complementary spaces \( T_X\mathcal{M} = \mathcal{V}_X\mathcal{M} \oplus \mathcal{H}_X\mathcal{M} \). Here, \( \mathcal{V}_X\mathcal{M} \) is called the vertical space that contains tangent vectors of the equivalence class \( [X] \). \( \mathcal{H}_X\mathcal{M} \) is called the horizontal space of \( T_X\tilde{\mathcal{M}} \), which is complementary to \( \mathcal{V}_X\mathcal{M} \) and provides a proper representation of the abstract tangent space \( T_{[X]}\mathcal{M} \) \[ \text{AMS09} \] Section 3.5.8. Once \( \mathcal{M} \) is endowed with \( \mathcal{H}_X\mathcal{M} \), a given tangent vector \( \eta[\mathcal{X}] \in T_{[\mathcal{X}]}\mathcal{M} \) at \( [\mathcal{X}] \) is uniquely represented by a horizontal tangent vector \( \eta_X \in \mathcal{H}_X\mathcal{M} \) that satisfies \( \text{D}\pi(\mathcal{X})[\eta_X] = \eta[\mathcal{X}] \) \[ \text{AMS09} \] Section 3.5.8. The tangent vector \( \eta_X \in \mathcal{H}_X\mathcal{M} \) is also called the horizontal lift of \( \eta[\mathcal{X}] \) at \( \mathcal{X} \).

Next, we introduce the notion of Riemannian quotient manifolds. Suppose the total space \( \tilde{\mathcal{M}} \) is endowed with a Riemannian metric \( g_X \), and for every \( [\mathcal{X}] \in \mathcal{M} \) and every \( \eta[\mathcal{X}], \theta[\mathcal{X}] \in T_{[\mathcal{X}]}\mathcal{M} \), the expression \( g_X(\eta_X, \theta_X) \), i.e., the inner product of the horizontal lifts of \( \eta[\mathcal{X}], \theta[\mathcal{X}] \) at \( \mathcal{X} \), does not depend on the choice of the representative \( \mathcal{X} \). Then the metric \( g_X \) in the total space induces a metric \( g[\mathcal{X}] \) on the quotient space, i.e., \( g[\mathcal{X}](\eta[\mathcal{X}], \theta[\mathcal{X}]) := g_X(\eta_X, \theta_X) \). The quotient manifold \( \mathcal{M} \) endowed with \( g[\mathcal{X}] \) is called a Riemannian quotient manifold of \( \mathcal{M} \) and the quotient mapping \( \pi : \tilde{\mathcal{M}} \to \mathcal{M} \) is called a Riemannian submersion \[ \text{AMS09} \] Section 3.6.2. Optimization on Riemannian quotient manifolds is particularly convenient because computation of representatives of Riemannian gradients and Hessians in the abstract quotient space can be directly performed by means of their analogous in the total space. To be specific, suppose \( \tilde{f} : \tilde{\mathcal{M}} \to \mathbb{R} \) is an objective function in the total space and is invariant along the fiber, i.e., \( f(X_1) = f(X_2) \) whenever \( X_1 \sim X_2 \). Then \( f \) induces a function \( f : \mathcal{M} \to \mathbb{R} \) on the quotient space and the horizontal lift of the Riemannian gradient of \( f \) can be obtained as follows \[ \text{Mey11} \] Section 3.4.2:

\[
\text{grad} f([X]) = P^H_X(\text{grad} \tilde{f}(X)).
\] (4)

Here \( P^H_X(\cdot) \) denotes the projection operator onto the horizontal space at \( \mathcal{X} \) (\( P^H_X(\cdot) \) depends on the
metric \( \tilde{g}_X \) and \( \text{grad} \tilde{f}(X) \) denotes the Riemannian gradient of \( \tilde{f} \) at \( X \) in the total space. The following Lemma 1 shows that if the horizontal space is canonically chosen, i.e., \( \mathcal{H}_X \tilde{M} \) is the orthogonal complement of \( \mathcal{V}_X \tilde{M} \) in \( T_X \tilde{M} \) with respect to \( \tilde{g}_X \), then \( \text{grad} \tilde{f}(X) \) automatically lies in the horizontal space at \( X \).

**Lemma 1.** (\cite[Section 3.6.2]{ams09}) Suppose \( \mathcal{H}_X \tilde{M} := \{ \eta_X \in T_X \tilde{M} : \tilde{g}_X(\eta_X, \theta_X) = 0 \text{ for all } \theta_X \in \mathcal{V}_X \tilde{M} \}. \) Then \( \text{grad} f([X]) = \text{grad} f(X) \).

Finally, the Riemannian connection on the quotient manifold \( M \) can also be uniquely represented by the Riemannian connection in the total space \( \tilde{M} \). Suppose \( \eta, \theta \) are two vector fields on \( M \) and \( \eta_X \) and \( \theta_X \) are the horizontal lifts of \( \eta[X] \) and \( \theta[X] \) in \( \mathcal{H}_X \tilde{M} \). Then the horizontal lift of \( \nabla_{\theta[X]} \eta \) on the quotient manifold is given by \( \nabla_{\theta[X]} \eta = P^M_X(\nabla_{\theta_X} \eta) \), where \( \nabla_{\theta_X} \eta \) is the Riemannian connection on the total space. Combining (3), we have the horizontal lift of the Riemannian Hessian of \( f \) on \( M \) satisfies

\[
\text{Hess} f([X])[\theta[X], \eta_X] = P^M_X(\nabla_{\theta_X} \text{grad} f)
\]

for any \( \theta_X \in T_X M \) and its horizontal lift \( \theta_X \). We also define the bilinear form of the horizontal lift of the Riemannian Hessian as \( \text{Hess} f([X])[\theta[X], \eta_X] := \tilde{g}_X(\text{Hess} f([X])[\theta[X]], \eta_X) \) for any \( \theta_X, \eta_X \in \mathcal{H}_X \tilde{M} \). Then, by recalling the definition of the Riemannian metric \( g[X] \) in the quotient space, we have

\[
\text{Hess} f([X])[\theta[X], \eta_X] = \tilde{g}_X(\text{Hess} f([X])[\theta[X]], \eta_X) = g[X] (\text{Hess} f([X])[\theta[X]], \eta_X) = \text{Hess} f([X])[\theta[X], \eta_X].
\]

So \( \text{Hess} f([X]) \) is completely characterized by \( \text{Hess} f([X]) \) in the lifted horizontal space.

### 3 A General Procedure for Establishing Geometric Connections in Riemannian Optimization

In this section, we present a general procedure for connecting landscape properties of an optimization problem (not necessarily restricted to fixed-rank matrix optimization) under different Riemannian geometries. For convenience of presentation, we focus on the connection between embedded and quotient geometries, while this procedure can be applied in broader settings.

Suppose \( M^e \), endowed with the Riemannian metric \( g_X \), is a Riemannian embedded submanifold of the Riemannian manifold \( \tilde{M} \). Consider the following optimization problem under \( M^e \)

\[
(\text{Opt. under Embedded Geometry}) \quad \min_{X \in M^e} f(X),
\]

where \( f : M^e \rightarrow \mathbb{R} \) is twice continuously differentiable and is the restriction of \( \tilde{f} : \tilde{M} \rightarrow \mathbb{R} \) to the embedded submanifold \( M^e \). Suppose \( \tilde{M}^q \) is another smooth manifold and there exists a submersion \( \ell : Z \in M^q \rightarrow \ell(Z) \in M^e \) such that (6) can be reformulated on \( \tilde{M}^q \) as

\[
\min_{Z \in \tilde{M}^q} \tilde{h}(Z) := f(\ell(Z)).
\]

We will see concrete examples of \( M^e, \tilde{M}^q \) and the mapping \( \ell \) later in the context of fixed-rank matrix optimization. The transformation between (6) and (7) can be regarded as a generalization of the classic technique of changing of variables in the Riemannian optimization setting. However, we shall emphasize that the mapping \( \ell \) is not necessarily a bijection. Hence, it is nontrivial, and
generally impossible, to explicitly characterize the connections between the desired points, e.g., the FOSPs, SOSPs, strict saddles, and (local) minimizers, of \( (9) \) and those of \( (10) \). If we further assume \( \ell \) defines an equivalence relation \( \sim \) on \( \mathcal{M}^q \), i.e., \( \ell(Z_1) = \ell(Z_2) \) whenever \( Z_1 \sim Z_2 \), and \( \mathcal{M}^q := \mathcal{M}^q / \sim \) with metric \( g_{|Z} \) is a Riemannian quotient manifold and diffeomorphic to \( \mathcal{M}^e \), then \( \bar{h}(Z) \) induces a function \( h([Z]) \) on the quotient manifold \( \mathcal{M}^q \) and \( (9) \) can be transformed to an optimization on the quotient manifold \( \mathcal{M}^q \):

\[
\text{(Opt. under Quotient Geometry)} \quad \min_{[Z] \in \mathcal{M}^q} h([Z]). \tag{8}
\]

Compared with \( (10) \), problem \( (8) \) contains an additional quotienting step and this makes it possible to connect the geometric properties of problems \( (9) \) and \( (10) \).

First, since \( \ell \) induces a diffeomorphism \( \bar{\ell} \) between \( \mathcal{M}^q \) and \( \mathcal{M}^e \) [AMRT12 Proposition 3.5.23], a simple fact stated in the following lemma shows there is an equivalence relationship between the sets of local minimizers of \( (9) \) and \( (10) \).

**Lemma 2.** If \( X \) is a local minimizer of \( (9) \), then \( \bar{\ell}^{-1}(X) \) is a local minimizer of \( (10) \); if \( [Z] \) is a local minimizer of \( (10) \), then \( X = \ell([Z]) \) is a local minimizer of \( (9) \).

However, for numerical tractable points such as FOSPs and SOSPs, the analysis on the connections is much more involved as the definitions of these points depends deeply on the Riemannian geometry of the underlying manifolds. We show next that with a careful treatment with Riemannian gradients and Hessians of \( (9) \) and \( (10) \), we are able to obtain a much richer geometric connection between the landscapes of these two problems.

### 3.1 Outline of the Procedure

First, we find it is often relatively easy to connect the first-order geometries between \( (9) \) and \( (10) \). For example, by taking derivative on both sides of \( \bar{h}(Z) = f(\ell(Z)) \) along the direction \( \theta_Z \in T_Z \mathcal{M}^q \) and the chain rule, we have

\[
g_Z\left(\nabla \bar{h}(Z)\theta_Z\right) = D\bar{h}(Z)[\theta_Z] = Df(\ell(Z))[D\ell(Z)[\theta_Z]] = g_X\left(\nabla f(\ell(Z)), D\ell(Z)[\theta_Z]\right). \tag{9}
\]

Here \( g_Z \) is the Riemannian metric on \( \mathcal{M}^e \); \( (a) \) is because \( D\ell(Z)[\theta_Z] \in T_{\ell(Z)} \mathcal{M}^e \) [AMS09 Section 3.5.1]. Thus, for reasonable choices of \( \ell \), we hope to find a connection between \( \nabla \bar{h}(Z) \) and \( \nabla f(\ell(Z)) \) based on \( (9) \), and this can further give us the first-order geometric connection.

Next, we present a general three-step procedure to connect the second-order geometries between \( (9) \) and \( (10) \).

- **Step 1: Compute the quadratic forms of Riemannian Hessians.** We first compute the Riemannian Hessians from their definitions. For fixed-rank matrix optimization, we will derive the explicit expressions for the quadratic form of the Riemannian Hessians in Sections \( \S \) and \( \S \) and will see the quadratic forms of the Riemannian Hessians of \( f(X) \) and \( h([Z]) \) always involve the quadratic form of the Riemannian Hessian of \( f \) and the Riemannian gradient of \( f \) or \( \bar{h} \). For concreteness, we assume

\[
\begin{align*}
\text{Hess} f(X)[\xi_X, \xi_X] &= \text{Hess} f(X)[\phi(\xi_X), \phi(\xi_X)] + \Psi_1, & \forall \xi_X \in T_X \mathcal{M}^e, \\
\text{Hess} h([Z])[\theta_Z, \theta_Z] &= \text{Hess} f(\ell(Z))[\varphi(\theta_Z), \varphi(\theta_Z)] + \Psi_2, & \forall \theta_Z \in \mathcal{H}_Z \mathcal{M}^q.
\end{align*} \tag{10}
\]

Here \( \phi : T_X \mathcal{M}^e \to T_X \hat{\mathcal{M}}, \varphi : \mathcal{H}_Z \mathcal{M}^q \to T_{\ell(Z)} \hat{\mathcal{M}} \) and \( \mathcal{H}_Z \mathcal{M}^q \) is the horizontal space of \( T_Z \hat{\mathcal{M}} \); \( \Psi_1 \) and \( \Psi_2 \) incorporate remaining terms of \( \text{Hess} f(X)[\xi_X, \xi_X] \) and \( \text{Hess} h([Z])[\theta_Z, \theta_Z] \) other than the quadratic form of the Riemannian Hessian of \( f \). As we will see in Propositions \( \S \) and \( \S \) we often have \( \phi(\xi_X) = \xi_X \) and \( \varphi(\theta_Z) = D\ell(Z)[\theta_Z] \).
• **Step 2:** Find a proper mapping $\mathcal{L}$ between $\mathcal{H}_{Z\overrightarrow{M}^q}$ and $T_X\mathcal{M}^e$ to connect Riemannian Hessians. To establish the second-order geometric connection, i.e., the connection between $\text{Hess} f(X)$ and $\overline{\text{Hess}} h([Z])$ with $X = \ell(Z)$, a natural idea is to first connect $\text{Hess} f(X)[\phi(\xi_x), \phi(\xi_x)]$ with $\text{Hess} f(\ell(Z))[\varphi(\theta_Z), \varphi(\theta_Z)]$. To do this, we would like to find a mapping $\mathcal{L}$ between $\mathcal{H}_{Z\overrightarrow{M}^q}$ and $T_X\mathcal{M}^e$ such that $\phi(\mathcal{L}(\theta_Z)) = \varphi(\theta_Z)$. Moreover, $\mathcal{L}$ is further constrained to be a bijection so that we can connect the whole spectra of $\text{Hess} f(X)$ with the spectra of $\overline{\text{Hess}} h([Z])$ as we will see in Step 3.

On the other hand, such a mapping $\mathcal{L}$ alone seems not enough for connecting $\text{Hess} f(X)[\xi_x, \xi_x]$ with $\overline{\text{Hess}} h([Z])[\theta_Z, \theta_Z]$ as $\Psi_1$ and $\Psi_2$ in (10) can be complex and distinct from each other (see the forthcoming Propositions 1 and 3). This motivates us to put some assumptions on $X$ or $[Z]$ in order to proceed. In the following, we consider a simple setting to illustrate what assumptions may be needed.

**Example 1.** Consider the optimization problem: $\min_{x \geq 0} f(x)$, where $f$ is a scalar function defined on the nonnegative part of the real line. If we consider the factorization $x = z^2$ and let $h(z) = f(z^2)$, it is easy to see $h(z) = h(-z)$. So we can consider the equivalence classes $[z] = \{z, -z\}$ for $z \in \mathbb{R}$ and take the quotient manifold $\mathcal{M}^q = \mathbb{R}/(+-)\text{, i.e., quotienting out the sign of the real number, as the search space. Suppose the Euclidean inner product is adopted as the Riemannian metric. Then in both cases, the Riemannian Hessians of } f(x) \text{ and } h([z]) \text{ are equal to the corresponding Euclidean Hessians (see also the forthcoming Proposition 1)}:$

$$
\text{Hess} f(x)[\xi_x, \xi_x] = \xi_x^2 f''(x); \quad \overline{\text{Hess}} h([z])[\theta_z, \theta_z] = \theta_z^2 h''(z) = 2\theta_z^2 f'(z^2) + 4z^2 \theta_z^2 f''(z^2),
$$

where $f'$ and $f''$ denote the first and second derivatives of $f$. Given $z_1 \in \mathbb{R}$, $x_1 = z_1^2$ and \(z_1 = 2z\theta_2\) for any $\theta_2 \in \mathbb{R}$, then we have $\overline{\text{Hess}} h([z_1])[\theta_{z_1}, \theta_{z_1}] = \text{Hess} f(x_1)[\mathcal{L}(\theta_{z_1}), \mathcal{L}(\theta_{z_1})]$.

Motivated by Example 1, we find by putting some proper first-order assumptions on $X$ or $[Z]$, we could hope for a nice connection between $\text{Hess} f(X)$ and $\overline{\text{Hess}} h([Z])$. In fact, as we will see in Sections 5 and 6, this intuition applies to all examples we consider in this paper.

• **Step 3:** Establish the spectra connection between Riemannian Hessians via bounding spectrum of $\mathcal{L}$. Suppose one has successfully worked through Steps 1 and 2 and come to the stage that $\overline{\text{Hess}} h([Z])[\theta_Z, \theta_Z] = \text{Hess} f(X)[\mathcal{L}(\theta_Z), \mathcal{L}(\theta_Z)]$ is shown to hold for any $\theta_Z \in \mathcal{H}_{Z\overrightarrow{M}^q}$ at properly chosen $[Z]$ and $X = \ell(Z)$. The following Theorem 2 shows a sandwich inequality between the spectra of $\text{Hess} f(X)$ and $\overline{\text{Hess}} h([Z])$ based on spectrum bounds of $\mathcal{L}$.

**Theorem 2** (Sandwich Inequalities for Spectra of Hessians). Suppose $X \in \mathcal{M}^e$, $Z \in \overrightarrow{M}^q$, $\dim(T_X\mathcal{M}^e) = \dim(\mathcal{H}_{Z\overrightarrow{M}^q}) = p$. Then both $\text{Hess} f(X)$ and $\overline{\text{Hess}} h([Z])$ have $p$ eigenvalues. Moreover, if $\mathcal{L} : \mathcal{H}_{Z\overrightarrow{M}^q} \rightarrow T_X\mathcal{M}^e$ is a bijection satisfying

$$
\overline{\text{Hess}} h([Z])[\theta_Z, \theta_Z] = \text{Hess} f(X)[\mathcal{L}(\theta_Z), \mathcal{L}(\theta_Z)], \quad \forall \theta_Z \in \mathcal{H}_{X\overrightarrow{M}^q}
$$

and

$$
\alpha g_Z(\theta_Z, \theta_Z) \leq g_X(\mathcal{L}(\theta_Z), \mathcal{L}(\theta_Z)) \leq \beta g_Z(\theta_Z, \theta_Z), \quad \forall \theta_Z \in \mathcal{H}_{X\overrightarrow{M}^q}
$$

then

On the other hand, we have $\lambda_k(\text{Hess}(f))$ sandwiched between $\alpha \lambda_k(\text{Hess}(f))$ and $\beta \lambda_k(\text{Hess}(f))$, where $\lambda_k(\text{Hess}(f))$ and $\lambda_k(\text{Hess}(f))$ are the $k$-th largest eigenvalues of $\text{Hess}(f)$ and $\text{Hess}(f)$, respectively.

**Proof of Theorem 2.** First, because $\text{Hess}(f)$ and $\text{Hess}(f)$ are by definition self-adjoint linear maps from $\mathcal{H}_k^q$ and $\mathcal{H}_k^e$ to $\mathcal{H}_k^q$ and $\mathcal{H}_k^e$, respectively, both $\text{Hess}(f)$ and $\text{Hess}(f)$ have $p$ eigenvalues as $\dim(\mathcal{H}_k^q) = \dim(\mathcal{H}_k^q) = p$. Suppose $u_1, \ldots, u_p$ are eigenvectors corresponding to $\lambda_1(\text{Hess}(f)), \ldots, \lambda_p(\text{Hess}(f))$ and $v_1, \ldots, v_p$ are eigenvectors corresponding to $\lambda_1(\text{Hess}(f)), \ldots, \lambda_p(\text{Hess}(f))$. For $k = 1, \ldots, p$, define

$U_k = \text{span}\{u_1, \ldots, u_k\}, \quad U_k^c = \text{span}\{\mathcal{L}^{-1}(u_1), \ldots, \mathcal{L}^{-1}(u_k)\},$

$V_k = \text{span}\{v_1, \ldots, v_k\}, \quad V_k^c = \text{span}\{\mathcal{L}(v_1), \ldots, \mathcal{L}(v_k)\}.$

Let us first consider the case that $\lambda_k(\text{Hess}(f)) \geq 0$. The Max-min theorem for eigenvalues (Lemma 11) yields

$$\lambda_k(\text{Hess}(f)) \geq \min_{u' \in U_k^c, u' \neq 0} \frac{\text{Hess}(f)[u', u']}{\text{Hess}(f)[\mathcal{L}(u'), \mathcal{L}(u')]} \geq \min_{u \in U_k, u \neq 0} \frac{\lambda_k(\text{Hess}(f)) g_X(u, u)}{g_Z(\mathcal{L}^{-1}(u), \mathcal{L}^{-1}(u))} \geq \alpha \lambda_k(\text{Hess}(f)) \geq 0.$$  

On the other hand, we have

$$\lambda_k(\text{Hess}(f)) \geq \min_{v \in V_k, v \neq 0} \frac{\text{Hess}(f)[v, v]}{\text{Hess}(f)[\mathcal{L}(v), \mathcal{L}(v)]} \geq \min_{v \in V_k, v \neq 0} \frac{\lambda_k(\text{Hess}(f)) g_X(v, v)}{g_Z(\mathcal{L}(v), \mathcal{L}(v))} \geq \lambda_k(\text{Hess}(f))/\beta.$$  

So we have proved the result for the case that $\lambda_k(\text{Hess}(f)) \geq 0$. When $\lambda_k(\text{Hess}(f)) < 0$, we have $\lambda_{p+1-k}(-\text{Hess}(f)) = -\lambda_k(\text{Hess}(f)) > 0$. Following the same proof of (14) and (15), we have

$$-\lambda_k(\text{Hess}(f)) = \lambda_{p+1-k}(-\text{Hess}(f)) \geq \alpha \lambda_{p+1-k}(-\text{Hess}(f)) = -\alpha \lambda_k(\text{Hess}(f)) > 0,$$

$$-\lambda_k(\text{Hess}(f)) = \lambda_{p+1-k}(-\text{Hess}(f)) \geq \lambda_{p+1-k}(-\text{Hess}(f))/\beta = -\lambda_k(\text{Hess}(f))/\beta.$$  

This finishes the proof of this theorem. ■

**4 Embedded and Quotient Geometries on Fixed-rank Matrices**

Now we specifically focus on the fixed-rank matrix optimization problems (11) and (12). The set of $p$-by-$p$ rank $r$ PSD matrices $\mathcal{M}_{r^+} := \{X \in \mathbb{S}^{p \times p} \mid \text{rank}(X) = r, X \succeq 0\}$ and the set of $p_1$-by-$p_2$ rank $r$ matrices $\mathcal{M}_r := \{X \in \mathbb{R}^{p_1 \times p_2} \mid \text{rank}(X) = r\}$ are two manifolds of particular interests. In Sections 4.1 and 4.2 we introduce embedded and quotient geometries on $\mathcal{M}_{r^+}$ and $\mathcal{M}_r$, respectively.
4.1 Embedded Geometries for $\mathcal{M}_{r^+}$ and $\mathcal{M}_r$

The following lemma shows that $\mathcal{M}_{r^+}$ and $\mathcal{M}_r$ are smooth embedded submanifolds of $\mathbb{R}^{p \times p}$ and $\mathbb{R}^{p_1 \times p_2}$ and also summarizes commonly used algebraic representations of the corresponding tangent spaces. To emphasizing the embedding natural of $\mathcal{M}_{r^+}$ and $\mathcal{M}_r$, we write them as $\mathcal{M}_{r^+}^e$ and $\mathcal{M}_r^e$, respectively.

Lemma 3. ([HM12 Chapter 5], [VVI0 Proposition 5.2], [Lee13 Example 8.14]) $\mathcal{M}_{r^+}^e$ and $\mathcal{M}_r^e$ are smooth embedded submanifolds of $\mathbb{R}^{p \times p}$ and $\mathbb{R}^{p_1 \times p_2}$ with dimensions $(pr - r(r - 1)/2)$ and $(p_1 + p_2 - r)r$, respectively. The tangent space $T_X\mathcal{M}_{r^+}^e$ at $X \in \mathcal{M}_{r^+}^e$ is

$$ T_X\mathcal{M}_{r^+}^e = \left\{ \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} S & D^T \end{bmatrix} \begin{bmatrix} U & U_\perp \end{bmatrix}^T : S \in \mathbb{S}^{r \times r}, D \in \mathbb{R}^{(p-r) \times r} \right\}, $$

(16)

where $U \in \text{St}(r, p)$ spans the top $r$ eigenspace of $X$.

The tangent space $T_X\mathcal{M}_r^e$ at $X \in \mathcal{M}_r^e$ is

$$ T_X\mathcal{M}_r^e = \left\{ \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} S & D_1^T \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}^T : S \in \mathbb{R}^{r \times r}, D_1 \in \mathbb{R}^{(p_1-r) \times r}, D_2 \in \mathbb{R}^{(p_2-r) \times r} \right\}, $$

(17)

where $U \in \text{St}(r, p_1)$ and $V \in \text{St}(r, p_2)$ span the left and right singular subspaces of $X$, respectively.

In addition, we assume the embedded submanifolds $\mathcal{M}_{r^+}^e$ and $\mathcal{M}_r^e$ are endowed with the natural metric induced by the Euclidean inner product, i.e., $\langle U, V \rangle = \text{trace}(U^T V)$.

4.2 Quotient Geometries for $\mathcal{M}_{r^+}$ and $\mathcal{M}_r$

The versatile choices of fixed-rank matrix factorization yield various Riemannian quotient structures and metrics, which have been explored in the literature on both $\mathcal{M}_{r^+}$ [MBS11b, BS10, JBAS10, MA20] and $\mathcal{M}_r$ [AAM14, MMBS14, MAS12, MBS11a]. Due to the factorization, the total space of fixed-rank matrices under the quotient geometry (i.e., focus of this subsection) can often be written as a product space of some simple smooth manifolds, including the following three examples to be used later:

(1) $\mathbb{R}_+^{p \times r}$: the set of real $p$-by-$r$ full column rank matrices;

(2) $\text{St}(r, p)$: the set of real $p$-by-$r$ matrices with orthonormal columns;

(3) $\mathbb{S}_+(r)$: the set of $r$-by-$r$ real symmetric positive definite matrices.

All these three manifolds are smooth homogeneous spaces and there exists a smooth structure on their product space [AMS09] Section 3.1.6]. In Table 1 we summarize several basic properties of these simple manifolds.

Remark 1. We introduce a weight matrix $W_Y$ while defining $g_Y$ in $\mathbb{R}_+^{p \times r}$ so that various metrics considered in literature are covered. $W_Y$ is required to be in $\mathbb{S}_+(r)$ so that $g_Y(\theta_Y, \theta_Y) \geq 0, \forall \theta_Y \in T_Y\mathbb{R}_+^{p \times r}$ and $g_Y$ is a genuine Riemannian metric [AMS09] Section 3.6]. Common choices of $W_Y$ include $W_Y = I_r$ (flat metric) [JBAS10], $W_Y = (Y^T Y)^{-1}$ (right-invariant metric) [MBS11a], and $W_Y = Y^T Y$ [MAS12].
the invariance mapping is performed via the Lie group factorization and polar factorization) and three quotient geometries for fixed-rank general matrices $T_M$, corresponding vertical and horizontal spaces of $T_M$ Hessians under these quotient geometries (see the forthcoming Remark 5) and show their geometric PSD and general cases, propose several new representations for the horizontal spaces (see the forthcoming Remark 5) for discussing.)

Next, we present two quotient geometries for fixed-rank PSD matrices (based on full-rank factorization and polar factorization) and three quotient geometries for fixed-rank general matrices (based on full-rank factorization, polar factorization, and subspace-projection factorization) in Sections 4.2.1 and 4.2.2, respectively. These quotient geometries have been explored in [6, 10, 17, 18, 19, 20, 21, 22, 23]. Here, we provide a unified way to characterize these quotient geometries, e.g., the introduction of $W_Y$ in the Riemannian metric, in both PSD and general cases, propose several new representations for the horizontal spaces (see the forthcoming Lemmas 7 and 8 and the discussion afterwards), derive explicit formulas for the Riemannian Hessians under these quotient geometries (see the forthcoming Remark 5) and show their geometric connections to the embedded geometry in fixed-rank matrix optimization.

### 4.2.1 Quotient Geometries for $M_{r+}$

Suppose $X \in S^{p \times p}$ is a rank $r$ PSD matrix with economic eigendecomposition $X = U^T \Sigma U$.

1. **Full-rank Factorization $M^{q_1}_{r+}$.** In this factorization, we view $X$ as $X = YY^T$ for $Y \in \mathbb{R}^{p \times r}$. Such a factorization exists, e.g., $Y = U^T \Sigma^{-1/2}$, but is not unique because of the invariance mapping $Y \mapsto YO$ for any $O \in O_r$. To cope with it, we encode the invariance mapping in an abstract search space by defining the equivalence classes $[Y] = \{YO : O \in O_r\}$. Since the invariance mapping is performed via the Lie group $O_r$, we have $M^{q_1}_{r+} := \tilde{M}^{q_1}_{r+}/O_r$, where $\tilde{M}^{q_1}_{r+} = \mathbb{R}^{p \times r}$, is a quotient manifold of $M^{q_1}_{r+}$ [Lee13, Theorem 21.10]. We equip $T_Y \tilde{M}^{q_1}_{r+}$ with the metric $\tilde{g}^+_Y(\eta_Y, \theta_Y) = \text{tr}(W_Y \eta_Y \theta_Y)$ as given in Table 1. In the following Lemma 4, we provide the corresponding vertical and horizontal spaces of $T_Y \tilde{M}^{q_1}_{r+}$ and show under some proper assumptions on $W_Y$, $\tilde{M}^{q_1}_{r+}$ is a Riemannian quotient manifold endowed with the Riemannian metric $\tilde{g}^+_Y$ induced from $\tilde{g}^+_Y$.

**Lemma 4.** (i) Given $U \in \text{St}(r,p)$ that spans the top $r$ eigenspace of $YY^T$ and $P = U^T Y$, the vertical and horizontal spaces of $T_Y \tilde{M}^{q_1}_{r+}$ are given as follows:

$$
\mathcal{V}_Y \tilde{M}^{q_1}_{r+} = \{ \theta_Y : \theta_Y = Y \Omega, \Omega = -\Omega^T \in \mathbb{R}^{r \times r} \} = \{ \theta_Y : \theta_Y = U \Omega P^{-T}, \Omega = -\Omega^T \in \mathbb{R}^{r \times r} \},
$$

$$
\mathcal{H}_Y \tilde{M}^{q_1}_{r+} = \{ \theta_Y : \theta_Y = (US + U \perp D)P^{-T}, SP^{-T} W_Y P^{-1} \in S^{r \times r}, D \in \mathbb{R}^{(p-r) \times (p-r)} \}.
$$

Table 1: Basic Riemannian Geometric Properties for $\mathbb{R}^{p \times r}$, St$(r,p)$ and $\mathbb{S}_+(r)$ [22, 23, 24]. Here for any square matrix $X$, Sym($X$) = $(X + X^T)/2$ and Skew($X$) = $(X - X^T)/2$; $W_Y$ is a weight matrix that specifies the Riemannian metric $g$ (see Remark 1 for discussions).
with \( \dim(V_Y \bar{M}_{r+}^{q_1}) = (r^2 - r)/2 \), \( \dim(H_Y \bar{M}_{r+}^{q_1}) = pr - (r^2 - r)/2 \) and \( V_Y \bar{M}_{r+}^{q_1} \perp H_Y \bar{M}_{r+}^{q_1} \) with respect to \( \bar{g}_Y^{+} \).

(ii) Moreover, \( M_{r+}^{q_1} \) is a Riemannian quotient manifold endowed with the metric \( g_{[Y]}^{+} \) induced from \( \bar{g}_Y^{+} \) if and only if \( W_Y = OW_YO^T \) holds for any \( O \in O_r \).

(2) Polar Factorization \( M_{r+}^{q_2} \). We factorize \( X = U'O(\Sigma'O)(U'O)^T = UBU^T \) with \( O \in O_r, U \in \text{St}(r, p), B \in S_+(r) \) \cite{BS10}. Due to the rotational invariance \( (U, B) \rightarrow (UO, O^T BO) \) for \( O \in O_r \), we define the search space as the following equivalence classes \( \{ (UO, O^T BO) : O \in O_r \} \). This results in the second quotient manifold: \( M_{r+}^{q_2} := \bar{M}_{r+}^{q_2}/O_r \), where \( \bar{M}_{r+}^{q_2} = \text{St}(r, p) \times S_+(r) \). By taking the canonical metrics on \( \text{St}(r, p) \) and \( S_+(r) \) given in Table 1, we endow \( M_{r+}^{q_2} \) with the metric \( \bar{g}_{[U,B]}^{+} \) for \( \bar{g}_{[U,B]}^{+} = \text{tr}(\eta_U^T \theta_U) + \text{tr}(B^{-1} \eta_B B^{-1} \theta_B) \) for \( \eta_U = [\eta_U^T \eta_B^T]^T, \theta_U = [\theta_U^T \theta_B^T]^T \in T_{(U,B)} \bar{M}_{r+}^{q_2} \). \cite{BS10} showed that endowed with the metric \( g_{[U,B]}^{+} \) induced from \( \bar{g}_{[U,B]}^{+} \), \( M_{r+}^{q_2} \) is a Riemannian quotient manifold and has the following vertical and horizontal spaces:

**Lemma 5.** \( \cite{BS10} \) Theorem 1]) \( M_{r+}^{q_2} \) endowed with the metric \( g_{[U,B]}^{+} \) induced from \( \bar{g}_{[U,B]}^{+} \) is a Riemannian quotient manifold with the vertical and horizontal spaces given as:

\[
V_{(U,B)} \bar{M}_{r+}^{q_2} = \{ \theta_{(U,B)} = [\theta_U^T \theta_B^T]^T : \theta_U = U \Omega, \theta_B = B \Omega - \Omega B, \Omega = -\Omega^T \in \mathbb{R}^{r \times r} \},
\]

\[
H_{(U,B)} \bar{M}_{r+}^{q_2} = \{ \theta_{(U,B)} = [\theta_U^T \theta_B^T]^T : \theta_U = U \perp D, \theta_B \in S_+^{r \times r}, D \in \mathbb{R}^{(r-p) \times r} \}.
\]

Here, \( \dim(V_{(U,B)} \bar{M}_{r+}^{q_2}) = (r^2 - r)/2 \), \( \dim(H_{(U,B)} \bar{M}_{r+}^{q_2}) = pr - (r^2 - r)/2 \).

**Remark 2** \( (V_{(U,B)} \bar{M}_{r+}^{q_2} \) and \( H_{(U,B)} \bar{M}_{r+}^{q_2} \) are not orthogonal). In the context of Lemma 2 \( H_{(U,B)} \bar{M}_{r+}^{q_2} \) is complementary but not orthogonal to \( V_{(U,B)} \bar{M}_{r+}^{q_2} \), which means \( H_{(U,B)} \bar{M}_{r+}^{q_2} \) is not a canonical horizontal space for the quotient manifold \( M_{r+}^{q_2} \). In fact, it is easier to find a correspondence between this non-canonical horizontal space \( H_{(U,B)} \bar{M}_{r+}^{q_2} \) and the tangent space \( T_X \bar{M}_{r+}^{q_2} \) under the embedded geometry (see later Proposition 3). Such a correspondence is critical in establishing the landscape connections of embedded and quotient geometries in Riemannian optimization as we have illustrated in Step 2 of the general procedure in Section 3.

In Table 2 we summarize the basic properties of the full-rank factorization and polar factorization based quotient manifolds on fixed-rank PSD matrices.

| Matrix representation | \( \bar{M}_{r+}^{q_1} \) | \( \bar{M}_{r+}^{q_2} \) |
|-----------------------|-------------------------|-------------------------|
| Equivalence classes   | \( \{ YO : O \in O_r \} \) | \( \{ (UO, O^T BO) : O \in O_r \} \) |
| Total space           | \( \mathbb{R}_+^{r \times r} \) | \( \text{St}(r, p) \times S_+(r) \) |
| Tangent space in total space | \( T_Y \mathbb{R}_+^{r \times r} \) | \( T_U \text{St}(r, p) \times T_B S_+(r) \) |
| Metric \( g^{+} \) on total space | \( \text{tr}(W_Y \eta_Y \theta_Y), W_Y \in S_+(r) \) | \( \text{tr}(\eta_U^T \theta_U) + \text{tr}(B^{-1} \eta_B B^{-1} \theta_B) \) |

**Table 2: Basic Properties of Quotient Manifolds \( \bar{M}_{r+}^{q_1}, \bar{M}_{r+}^{q_2} \).**

### 4.2.2 Quotient Geometries for \( M_r \)

In this section, we introduce three quotient structures for \( M_r \) based on three fixed-rank matrix factorizations. For \( X \in M_r \), denote the SVD as \( X = U \Sigma V^T \).
(1) Full-rank Factorization $\mathcal{M}_{\Omega}^q$. In this factorization, we rearrange the SVD of $X$ as $X = (U\Sigma^{1/2})(\Sigma^{1/2}V^T) = LR^T$, where $L \in \mathbb{R}^{n \times p}$, $R \in \mathbb{R}^{p \times p}$. The first quotient geometry for $\mathcal{M}_{\Omega}$ results from the invariance mapping $(L, R) \mapsto (LM, RM^{-T})$ for $M \in \text{GL}(r)$, where $\text{GL}(r) := \{M \in \mathbb{R}^{r \times r} : \text{det}(M) \neq 0\}$ denotes the degree $r$ general linear group. It is thus straightforward to consider the equivalence classes $[L, R] = \{(LM, RM^{-T}) : M \in \text{GL}(r)\}$ as the search space. The set of equivalence classes forms the quotient manifold $\mathcal{M}_{\Omega}^q := \tilde{\mathcal{M}}^q_{\Omega}/\text{GL}(r)$, where $\tilde{\mathcal{M}}^q_{\Omega} := \mathbb{R}^{p_1 \times r} \times \mathbb{R}^{p_2 \times r}$, as $\text{GL}(r)$ is a Lie group [Lee13 Theorem 21.10]. Suppose $W_{(L,R)}, V_{(L,R)}$ are two $r$-by-$r$ positive definite matrices depending on $L$ and $R$, the metric we endow on $T_{(L,R)}\tilde{\mathcal{M}}^q_{\Omega}$ is $g_{(L,R)}^r(\eta_{(L,R)}, \theta_{(L,R)}) = tr(W_{(L,R)}\eta_{(L,R)}^T \theta_{(L,R)}) + tr(V_{(L,R)}\eta_{(L,R)}^T \theta_{(L,R)})$ for $\eta_{(L,R)} = [\eta_L^T \eta_R^T]^T, \theta_{(L,R)} = [\theta_L^T \theta_R^T]^T \in T_{(L,R)}\tilde{\mathcal{M}}^q_{\Omega}$. In the following Lemma 6 we show with some proper assumptions on $W_{(L,R)}$ and $V_{(L,R)}$, $\mathcal{M}_{\Omega}^q$ is a Riemannian quotient manifold.

Lemma 6. (i) Suppose $U \in \text{St}(r, p_1)$ and $V \in \text{St}(r, p_2)$ span the top $r$ left and right singular subspaces of $LR^T$, respectively and $P_1 = U^T L, P_2 = V^T R$. Then the vertical and horizontal spaces of $T_{(L,R)}\mathcal{M}_{\Omega}^q$ are given as follows:

\[ V_{(L,R)} \tilde{\mathcal{M}}^q_{\Omega} = \left\{ (\theta_{(L,R)}, \eta_{(L,R)}) : \theta_L = USP_2^T, \theta_R = -VS^T P_1^T, S \in \mathbb{R}^{r \times r} \right\}, \]

\[ H_{(L,R)} \tilde{\mathcal{M}}^q_{\Omega} = \left\{ \theta_{(L,R)} : \theta_L = (USP_2^T P_1^T + U \cap \text{Di} (P_2^-)), D_1 \in \mathbb{R}^{(p_1 - r) \times r}, \theta_R = (VS^T P_1^T V \cap \text{Di} (P_2^-)), D_2 \in \mathbb{R}^{(p_2 - r) \times r}, S \in \mathbb{R}^{r \times r} \right\}, \]

with $\dim(V_{(L,R)} \tilde{\mathcal{M}}^q_{\Omega}) = r^2, \dim(H_{(L,R)} \tilde{\mathcal{M}}^q_{\Omega}) = (p_1 + p_2 - r)r$ and $V_{(L,R)} \tilde{\mathcal{M}}^q_{\Omega} \perp H_{(L,R)} \tilde{\mathcal{M}}^q_{\Omega}$ with respect to $g_{(L,R)}^r$.

(ii) Moreover, $\mathcal{M}_{\Omega}^q$ is a Riemannian quotient manifold endowed with metric $g_{(L,R)}^r$ induced from $\tilde{g}_{(L,R)}^r$ if and only if $W_{(L,R)} = MW_{LM, RM^{-T}}M^T$ and $V_{(L,R)} = M^{-T} V_{LM, RM^{-T}} M^{-1}$ hold for any $M \in \text{GL}(r)$.

Remark 3. Similarly to $W_Y$ for the PSD case, we introduce $W_{(L,R)}$ and $V_{(L,R)}$ in $\tilde{g}_{(L,R)}^r$ to accommodate various metric choices considered in literature. Common choices include: $W_{(L,R)} = (LM)^{-1}, V_{(L,R)} = (R^T R)^{-1}$ [MBS11a, MBS11b] and $W_{(L,R)} = R^T R, V_{(L,R)} = L^T L$ [MAS12]. Distinct from the quotient geometry on $\mathcal{M}_{\Omega}^q$, the flat metric, i.e., $W_{(L,R)} = I_r$ and $V_{(L,R)} = I_r$, is no longer proper as it does not yield a valid Riemannian metric in the quotient space.

(2) Polar Factorization $\mathcal{M}^q_{\Omega}$. We consider another factorization: $X = U'O'(O^T \Sigma'O')^{1/2} = UBV^T$, where $O \in \mathbb{O}_r, U \in \text{St}(r, p_1), B \in \mathbb{S}_+(r), V \in \text{St}(r, p_2)$. The rotational invariance mapping here is $(U, B, V) \mapsto (UO, O^T BO, VO)$ for $O \in \mathbb{O}_r$. This gives us the equivalence classes $[U, B, V] = \{(UO, O^T BO, VO) \in \mathbb{O}_r \}$ and the second quotient manifold $\mathcal{M}^q_{\Omega} = \tilde{\mathcal{M}}^q_{\Omega}/\mathbb{O}_r$, where $\tilde{\mathcal{M}}^q_{\Omega} = \text{St}(r, p_1) \times \mathbb{S}_+(r) \times \text{St}(r, p_2)$. By picking natural metrics for $\text{St}(r, p_1)$ and $\mathbb{S}_+(r)$ (and $\text{St}(r, p_2)$ in Table 4) we endow $T_{(U, B, V)} \tilde{\mathcal{M}}^q_{\Omega}$ with the product metric $g_{(U, B, V)}(\eta_{(U, B, V)}, \theta_{(U, B, V)}) = tr(\eta_U^T \theta_U) + tr(\eta_B^T \theta_B) + tr(\eta_V^T \theta_V)$ for $\eta_{(U, B, V)} = [\eta_U^T \eta_B^T \eta_V^T]^T, \theta_{(U, B, V)} = [\theta_U^T \theta_B^T \theta_V^T]^T \in T_{(U, B, V)} \tilde{\mathcal{M}}^q_{\Omega}$. In the following Lemma 7 we show $\mathcal{M}^q_{\Omega}$ is a Riemannian quotient manifold and provide expressions for its vertical and horizontal spaces.

Lemma 7. $\mathcal{M}^q_{\Omega}$ endowed with the metric $g_{(U, B, V)}$ induced from $g_{(U, B, V)}$ is a Riemannian quotient manifold with the following vertical and horizontal spaces:

\[ V_{(U, B, V)} \tilde{\mathcal{M}}^q_{\Omega} = \left\{ (\theta_{(U, B, V)}, \eta_{(U, B, V)}) : \theta_U = U \Omega, \theta_B = B \Omega - B \Omega, \theta_V = V \Omega, \Omega = -\Omega^T \in \mathbb{R}^{r \times r} \right\}, \]

\[ H_{(U, B, V)} \tilde{\mathcal{M}}^q_{\Omega} = \left\{ (\theta_{(U, B, V)}, \eta_{(U, B, V)}) : \theta_U = U \cap \text{Di} (D_1), \theta_B = \eta_B \in \mathbb{S}^{r \times r}, \theta_V = V \cap \text{Di} (D_2), D_1 \in \mathbb{R}^{(p_1 - r) \times r}, D_2 \in \mathbb{R}^{(p_2 - r) \times r} \right\}, \]

(18)
with \( \text{dim}(\nu_{U, B, V} \bar{M}^{q_2}) = (r^2 - r)/2, \text{dim}(\mathcal{H}_{U, B, V} \bar{M}^{q_2}) = (p_1 + p_2 - r)r. \)

(3) **Subspace-projection Factorization** \( M^{q_3} \). The third quotient geometry is based on the factorization \( X = U' \Sigma' V'^T = UY^T \), where \( U \in \text{St}(r, p_1) \) and \( Y \in \mathbb{R}^{p_2 \times r} \). This factorization is called the subspace-projection factorization \([\text{MMBS14}]\) as \( U \) represents the column space of \( X \) and \( Y \) is the left projection coefficient matrix of \( X \) on \( U \). Here the rotational invariance mapping is \((U, Y) \mapsto (UO, YO)\) for \( O \in \mathbb{O}_r \) and the equivalence classes are \([U, Y] = \{(UO, YO) : O \in \mathbb{O}_r\}\). This results in the third quotient manifold we are interested in: \( \bar{M}^{q_3} := \bar{M}_r^{q_3}/\mathbb{O}_r \), where \( \bar{M}^{q_3} = \text{St}(r, p_1) \times \mathbb{R}^{p_2 \times r} \). By taking the canonical metrics on \( \text{St}(r, p_1) \) and \( \mathbb{R}^{p_2 \times r} \), we endow \( \bar{M}^{q_3} \) with the metric \( \bar{g}_r(U, Y)(\eta(U, Y), \theta(U, Y)) = \text{tr}(\eta_U^T \theta_U) + \text{tr}(\bar{W}_Y \eta_Y^T \theta_Y) \) for \( \eta(U, Y) = [\eta_U^T \quad \eta_Y^T]^T, \theta(U, Y) = [\theta_U^T \quad \theta_Y^T]^T \in T(U, Y) \bar{M}^{q_3} \). In the following Lemma 8 we provide the vertical and horizontal spaces of \( T(U, Y) \bar{M}^{q_3} \) and show with some proper assumptions on \( \bar{W}_Y \), \( \bar{M}^{q_3} \) is a Riemannian quotient manifold.

**Lemma 8.** (i) The vertical and horizontal spaces of \( T(U, Y) \bar{M}^{q_3} \) are

\[
\nu_{U, Y} \bar{M}^{q_3} = \{ \theta(U, Y) = [\theta_U^T \quad \theta_Y^T]^T : \theta_U = U \Omega, \theta_Y = Y \Omega, \Omega = -\Omega^T \in \mathbb{R}^{r \times r} \}, \\
\mathcal{H}_{U, Y} \bar{M}^{q_3} = \{ \theta(U, Y) = [\theta_U^T \quad \theta_Y^T]^T : \theta_U = U_D, \theta_Y = Y_D, D \in \mathbb{R}^{(p_1 - r) \times r} \}. 
\]

Here, \( \text{dim}(\nu_{U, Y} \bar{M}^{q_3}) = (r^2 - r)/2 \), \( \text{dim}(\mathcal{H}_{U, Y} \bar{M}^{q_3}) = (p_1 + p_2 - r)r \).

(ii) Moreover, \( \bar{M}^{q_3} \) is a Riemannian quotient manifold endowed with metric \( g_{[U, Y]} \) induced from \( \bar{g}_r(U, Y) \) if and only if \( \bar{W}_Y = OWYO^T \) holds for any \( O \in \mathbb{O}_r \).

**Remark 4.** In Lemmas 7 and 8, we introduce new horizontal spaces that are distinct from the canonical horizontal spaces in the literature in \( M^{q_2} \) and \( M^{q_3} \) \([\text{MMBS13, MMBS14, MBS11a, AAMI14}]\). These new horizontal spaces admit closed-form expressions, which makes developing a correspondence between the embedded manifold tangent space \( T_X M^{q_3}_r \) and the non-canonical horizontal spaces \( \mathcal{H}_{U, B, V} \bar{M}^{q_2}, \mathcal{H}_{U, Y} \bar{M}^{q_3} \) easier and facilitates the later landscape analysis (see later in Propositions 6 and 7).

In Table 3, we summarize the basic properties of the full-rank factorization, polar factorization, and subspace-projection factorization based quotient manifolds of the general fixed-rank matrices.

## 5 Geometric Connections of Embedded and Quotient Geometries in Fixed-rank PSD Matrix Optimization

In this section, we apply the general procedure in Section 3 to connect the landscape properties of optimization \( \mathbb{D} \) under the embedded and the quotient geometries. First, under the two quotient geometries introduced in Section 4.2.1 the optimization problem \( \mathbb{D} \) can be reformulated as follows,

\[ \min_{Y \in \mathbb{R}^{p_2 \times r}} \bar{h}_{r+}(Y) := f(YY^T), \quad (20a) \]

\[ \min_{U \in \text{St}(r, p_1), B \in \mathbb{B}_+(r)} \bar{h}_{r+}(U, B) := f(UBU^T). \quad (20b) \]

Since \( \bar{h}_{r+}(Y) \) and \( \bar{h}_{r+}(U, B) \) are invariant along the fibers of \( \bar{M}^{q_1}_{r+} \) and \( \bar{M}^{q_2}_{r+} \), they induce functions \( h_{r+}([Y]) \) and \( h_{r+}([U, B]) \) on quotient manifolds \( M^{q_1}_{r+} \) and \( M^{q_2}_{r+} \), respectively. Next, we provide the
Expression for Riemannian gradients and Hessians of $\mathcal{M}_r$ under both geometries (Step 1), construct the bijective maps between $T_X \mathcal{M}_r$ and $\mathcal{H}_Y \tilde{\mathcal{M}}_r$, $\mathcal{H}_{(U,B)} \tilde{\mathcal{M}}_r$ (Step 2), and give their spectrum bounds (Step 3).

**Proposition 1 (Riemannian Gradients and Hessians of $\mathcal{M}_r$).** The Riemannian gradients and Hessians of $\mathcal{M}_r$ under the embedded and the quotient geometries introduced in Section 4 are:

- **On $\mathcal{M}_r$:** Suppose $X \in \mathcal{M}_r$, $U$ spans the top $r$ eigenspace of $X$, $\xi_X = [U \ U_\perp] \begin{bmatrix} S & D^T \\ 0 & 0 \end{bmatrix} [U \ U_\perp]^T \in T_X \mathcal{M}_r$. Then
  \[
  \begin{align*}
  \text{grad} f(X) &= P_U \nabla f(X) P_U + P_{U_\perp} \nabla f(X) P_{U_\perp} + P_U \nabla f(X) P_{U_\perp}, \\
  \text{Hess} f(X)[\xi_X, \xi_X] &= \nabla^2 f(X)[\xi_X, \xi_X] + 2\langle \nabla f(X), U \Sigma^{-1} D^T U_\perp \rangle,
  \end{align*}
  \]
  where $\Sigma = U^T X U$.

- **On $\mathcal{M}_r$:** Suppose $Y \in \mathbb{R}_+^{p \times r}$ and $\theta_Y \in \mathcal{H}_Y \tilde{\mathcal{M}}_r$. Then
  \[
  \begin{align*}
  \text{grad} g_Y([Y]) &= \nabla f(Y Y^T) Y W_Y^{-1}, \\
  \text{Hess} g_Y([Y])[\theta_Y, \theta_Y] &= \nabla^2 f(Y Y^T)[\theta_Y \theta_Y^T + \theta_Y Y \theta_Y^T + \theta_Y Y^T + \theta_Y Y^T] + 2\langle \nabla f(Y Y^T), \theta_Y \theta_Y \rangle \\
  &\quad + 2\langle \nabla f(Y Y^T) Y D W_Y^{-1} \theta_Y, \theta_Y Y \rangle + \langle D W_Y \left( \text{grad} g_Y([Y]) \right) \theta_Y^T \theta_Y \rangle / 2,
  \end{align*}
  \]

- **On $\mathcal{M}_r$:** Suppose $U \in \text{St}(r, p)$, $B \in \mathbb{S}^+(r)$ and $\theta_{(U,B)} = [\theta_U^T \theta_B^T]^T \in \mathcal{H}_{(U,B)} \tilde{\mathcal{M}}_r$. Then
  \[
  \begin{align*}
  \text{grad} h_B([U, B]) &= \begin{bmatrix} \text{grad}_U h_B([U, B]) \\ \text{grad}_B h_B([U, B]) \end{bmatrix} = \begin{bmatrix} 2P_{U_\perp} \nabla f(UBU^T) UB \\ BU^T \nabla f(UBU^T) UB \end{bmatrix}, \\
  \text{Hess} h_B([U, B])[\theta_{(U,B)}, \theta_{(U,B)}] &= \nabla^2 f(UBU^T)[UB \theta_U^T + U \theta_B U^T + \theta_U BU^T, UB \theta_U^T + U \theta_B U^T + \theta_U BU^T] + 2\langle \nabla f(UBU^T), \theta_U B \theta_U^T \rangle \\
  &\quad + \langle \nabla f(UBU^T) U, 4 \theta_U \theta_U + UB \theta_B^{-1} \theta_B - 2 \theta_U U \theta_U B - 2 UB \theta_U^T \theta_U \rangle.
  \end{align*}
  \]
Remark 5 (Quadratic Form of Riemannian Hessians). In Proposition 1 we only give the quadratic expressions of the Hessians as we use them exclusively throughout the paper. It is easy to obtain general bilinear expressions by noting that \( \text{Hess}_f(X)[\xi , \theta X] = \left( \text{Hess}_f(X)[\xi + \theta X , \xi + \theta X] - \text{Hess}_f(X)[\xi - \theta X , \xi - \theta X] \right)/4 \).

We also note the Riemannian Hessian expressions under the fixed-rank quotient geometries have been explicitly or implicitly developed in [JBAS10, MMS11, Mey11, MMBS14] for some specific problems. Most of these works only provide the linear form of the Riemannian Hessians, which often do not admit a closed-form expression due to the horizontal projection involved in (5). Here we provide explicit formulas for the quadratic form of the Riemannian Hessians and the closed-form expressions are critical in establishing the landscape connections of embedded and quotient geometries in fixed-rank matrix optimization.

Proposition 2 (Bijection Between \( \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \) and \( T_X M_{r+}^e \)). Suppose \( Y \in \mathbb{R}_s^{d \times r} \), \( X = YY^\top \), the eigenspace of \( X \) is \( U \), and \( P = U^\top Y \). For any \( \theta_Y \in \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \) and \( \xi_X \in [\mathcal{U} \ \mathcal{U}_\perp] \left[ \begin{array}{ll} S & D^\top \\ D & 0 \end{array} \right] [\mathcal{U} \ \mathcal{U}_\perp]^\top \in T_X M_{r+}^e ,
\]

where \( S \) in \( \theta_Y \) is uniquely determined by the linear equation system \( S^\top + S^\top = S \) and \( S^\top P - W_Y P - S^\top \). Then we can find a linear bijective mapping \( \mathcal{L}_Y^+ \) between \( \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \) and \( T_X M_{r+}^e \),

\[
\mathcal{L}_Y^+ : \theta_Y \in \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \longrightarrow \xi_X^\theta_Y \in T_X M_{r+}^e \quad \text{and} \quad (\mathcal{L}_Y^+)^{-1} : \xi_X \in T_X M_{r+}^e \longrightarrow \theta_Y \in \mathcal{H}_Y \mathcal{M}^{q_1}_{r+},
\]

such that \( \mathcal{L}_Y^+(\theta_Y) = Y^\top \theta_Y^T + \theta_Y Y^\top \) holds for any \( \theta_Y \in \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \).

Finally, \( \mathcal{L}_Y^+ \) satisfies the following spectrum bound:

\[
2\sigma_1(\text{PW}_Y^{-1} P^\top) \| g_{Y^\top}^{\mathcal{L}_Y^+}(\theta_Y , \theta_Y) \|_F \leq 4\sigma_1(\text{PW}_Y^{-1} P^\top) \| \xi_X^\theta_Y \|_F \quad \forall \theta_Y \in \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} .
\]

Proof of Proposition 2. The proof is divided into two steps: in Step 1, we show \( \xi_X^\theta_Y \) and \( \theta_Y \) are well defined for any \( \theta_Y \) and \( \xi_X \); in Step 2, we show \( \mathcal{L}_Y^+ \) is a bijection and prove its spectrum bounds.

Step 1. First, it is clear for any \( \theta_Y \in \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \), \( \xi_X^\theta_Y \) is well defined. To show \( \theta_Y^\top \) is well defined for any \( \xi_X \in T_X M_{r+}^e \), we need to show the linear system \( \tilde{S}^\top + \tilde{S}^\top = S \), \( SP - W_Y P - S^\top \). \( \mathcal{H}_Y \mathcal{M}^{q_1}_{r+} \) has a unique solution with respect to \( \tilde{S} \). Simple calculations assert that

\[
\{S : S^\top = S, SP - W_Y P - S^\top \}
\]

Observing facts that

\[
P^{-1} W_Y P^{-1} \tilde{S} + \tilde{S}SP - W_Y P - S^\top = S
\]

is a Sylvester equation with respect to \( \tilde{S} \), and \( P^{-1} W_Y P^{-1} \) and \( P^{-1} W_Y P^{-1} \) have disjoint spectra, we know from [Bha13 Theorem VII.2.1] that \( \tilde{S} \) has a unique solution, denoted \( S' \).

Since \( S \) is symmetric and \( P^{-1} W_Y P^{-1} \) is a PSD matrix, we have

\[
P^{-1} W_Y P^{-1} S' + S'P^{-1} W_Y P - P^{-1} W_Y P^{-1} S = 0
\]

\[
S'^\top P^{-1} W_Y P^{-1} + P^{-1} W_Y P^{-1} S'^\top - SP^{-1} W_Y P^{-1} = 0.
\]
By summing two equations in (28), we get $P^{-T}WYP^{-1}(S' + S'^T - S) + (S' + S'^T - S)P^{-T}WYP^{-1} = 0$, i.e., $S' + S'^T - S$ is a solution to the new Sylvester equation $P^{-T}WYP^{-1}S_1 + S_1P^{-T}WYP^{-1} = 0$ with respect to $S_1$. Now, we know again by [Bha13] Theorem VII.2.1 that $0$ is the unique solution to the system $P^{-T}WYP^{-1}S_1 + S_1P^{-T}WYP^{-1} = 0$. Thus, $S' + S'^T = S$, and from (28), it further holds $S'P^{-T}WYP^{-1} = P^{-T}WYP^{-1}S'^T$. This, together with (28) and the uniqueness of $S'$, asserts that $S'$ is the unique solution to the linear system $SP^{-T}WYP^{-1} = P^{-T}WYP^{-1}S'^T$, $S + S'^T = S$. This finishes the proof of this part.

**Step 2.** Note that both $\mathcal{H}_Y \bar{M}^q_{r+}$ and $T_X M_{r+}^e$ are of dimension $(p - (r^2 - r))/2$. Let $L^r_{Y+} : \xi_X \rightarrow Y^e \in \mathcal{H}_Y \bar{M}^q_{r+}$. Moreover, for any $\xi_X = [U \ U_1] S \ [D \ 0] \ U \ U_1 \in T_X M_{r+}^e$, we have

$$L^r_{Y+}(L^r_{Y+}(\xi_X)) = L^r_{Y+}(\theta^r Y_X) = [U \ U_1][P\theta^r Y_X U + U^T\theta^r Y X P^T \ U_1^T \theta^r Y X P^T \ 0][U \ U_1]^T = \xi_X.$$  

(29)

Therefore, $L^r_{Y+}$ is a bijection and $L^r_{Y+} = (L^r_{Y+})^{-1}$. At the same time, it is easy to check $L^r_{Y+}$ satisfies $L^r_{Y+}(\theta_Y) = Y\theta_Y + \theta_Y Y^T$ by observing $Y = UP$.

Next, we provide the spectrum bounds for $L^r_{Y+}$. For any $\theta_Y = (US + U_1D)P^{-T} \in \mathcal{H}_Y \bar{M}^q_{r+}$, we have

$$g^r Y + (\theta_Y, \theta_Y) = \text{tr}(WYP^{-1/2}W^{1/2} = \|\theta_Y W^{1/2}\|_F^2 \leq \|P^{-T}W^{1/2}(\|S\|_F^2 + \|D\|_F^2) = \|P^{-T}W^{-1}P^{-1}\|_F^2(\|S\|_F^2 + \|D\|_F^2) = (\|S\|_F^2 + \|D\|_F^2)/\sigma_1(PW^{-1}P^T)),$$

and

$$\langle S^T, S \rangle = \langle PW^{-1}P^TSP^{-T}WYP^{-1}, S \rangle = \langle (PW^{-1}P^T)^{1/2}SPW^{-1}P^T)^{-1/2}, (PW^{-1}P^T)^{1/2}SPW^{-1}P^T)^{-1/2} \rangle \geq 0,$$

here (a) is because $SP^{-T}WYP^{-1} = P^{-T}WYP^{-1}S^T$ by the construction of $H_Y \bar{M}^q_{r+}$. Thus

$$\|L^r_{Y+}(\theta_Y)\|_F^2 = \|\theta^r Y_X\|_F^2 \|P\theta^r Y U + U^T\theta_Y P^T\|_F^2 + 2\|U_1^T \theta_Y P^T\|_F^2$$

$$\|\theta^r Y_X\|_F^2 \|P\theta^r Y U + U^T\theta_Y P^T\|_F^2 + 2\|U_1^T \theta_Y P^T\|_F^2$$

$$\geq 2(\|S\|_F^2 + \|D\|_F^2) \geq 2\sigma_1(PW^{-1}P^T)g^r Y + (\theta_Y, \theta_Y),$$

and

$$\|L^r_{Y+}(\theta_Y)\|_F^2 = \|\theta^r Y_X\|_F^2 \leq 4\|U^T\theta_Y P^T\|_F^2 + 2\|U_1^T \theta_Y P^T\|_F^2$$

$$\leq 4\|U^T\theta_Y P^T\|_F^2 + 2\|\theta_Y P^T\|_F^2$$

$$\leq 4\|\theta_Y P^T\|_F^2 = 4\|\theta_Y W^{1/2}W^{-1/2}P^T\|_F^2$$

$$\leq 4\sigma_1(PW^{-1}P^T)g^r Y + (\theta_Y, \theta_Y).$$

This finishes the proof of this proposition.
Proposition 3 (Bijection Between $\mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$ and $T_X\mathcal{M}_{r+}^{e}$). Suppose $\mathbf{U} \in \text{St}(r, p)$, $\mathbf{B} \in \mathbb{S}_+(r)$ and $X = \mathbf{U}\mathbf{U}^T$. For any $\theta(\mathbf{U}, \mathbf{B}) = [\theta_U^T \ \theta_B^T]^T \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$ and $\xi_X = [U \ U_\bot] \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} [U \ U_\bot]^T \in T_X\mathcal{M}_{r+}^{e}$, define

$$\xi_{\mathbf{X}}^{\theta(\mathbf{U}, \mathbf{B})} := [U \ U_\bot] \begin{bmatrix} \theta_B \\ U_\bot^T \theta_U \mathbf{B} \\ 0 \end{bmatrix} [U \ U_\bot]^T \in T_X\mathcal{M}_{r+}^{e},$$

$$\theta_{\mathbf{X}}^{\xi_X(\mathbf{U}, \mathbf{B})} := [(U_\bot \mathbf{D} \mathbf{B}^{-1})^T \ S]^T \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}.$$  

Then we can find a linear bijective mapping $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}$ between $\mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$ and $T_X\mathcal{M}_{r+}^{e}$,

$$\mathcal{L}^+_{\mathbf{U}, \mathbf{B}} : \theta(\mathbf{U}, \mathbf{B}) \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta} \rightarrow \xi_{\mathbf{X}}^{\theta(\mathbf{U}, \mathbf{B})} \in T_X\mathcal{M}_{r+}^{e},$$

$$\left(\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}\right)^{-1} : \xi_X \in T_X\mathcal{M}_{r+}^{e} \rightarrow \theta_{\mathbf{X}}^{\xi_X(\mathbf{U}, \mathbf{B})} \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}.$$  

such that $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B})) = \mathbf{U}\mathbf{B}\theta_U^T + \mathbf{U}\theta_B \mathbf{U}^T + \theta_U \mathbf{B} \mathbf{U}^T$ holds for any $\theta(\mathbf{U}, \mathbf{B}) \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$.

Finally, $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}$ satisfies the following spectrum bound: $\forall \theta(\mathbf{U}, \mathbf{B}) \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$,

$$\sigma_1^2(X) \bar{g}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}), \theta(\mathbf{U}, \mathbf{B})) \leq \|\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}))\|_F^2 \leq 2\sigma_1^2(X) \bar{g}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}), \theta(\mathbf{U}, \mathbf{B})).$$  

Proof of Proposition 3. First, it is easy to see $\xi_{\mathbf{X}}^{\theta(\mathbf{U}, \mathbf{B})}$ and $\theta_{\mathbf{X}}^{\xi_X(\mathbf{U}, \mathbf{B})}$ are well defined given any $\theta(\mathbf{U}, \mathbf{B}) \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$, and $\xi_X \in T_X\mathcal{M}_{r+}^{e}$.

Next, we show $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}$ is a bijection. Notice $\mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$ is of dimension $(pr - (r^2 - r)/2)$, which is the same with $T_X\mathcal{M}_{r+}^{e}$. Suppose $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}} : \xi_X \in T_X\mathcal{M}_{r+}^{e} \rightarrow \theta_{\mathbf{X}}^{\xi_X(\mathbf{U}, \mathbf{B})} \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$. For any $\xi_X = [U \ U_\bot] \begin{bmatrix} S & D^T \\ D & 0 \end{bmatrix} [U \ U_\bot]^T \in T_X\mathcal{M}_{r+}^{e}$, we have

$$\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\xi_X)) = \mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\theta_{\mathbf{X}}^{\xi_X(\mathbf{U}, \mathbf{B})}) = [U \ U_\bot] \begin{bmatrix} \theta_B \\ U_\bot^T \theta_U \mathbf{B} \\ 0 \end{bmatrix} [U \ U_\bot]^T \in T_X\mathcal{M}_{r+}^{e}. \tag{34}$$

Since $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}$ and $\mathcal{L}^{+\prime}_{\mathbf{U}, \mathbf{B}}$ are linear maps, (34) implies $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}$ is a bijection and $\mathcal{L}^{+\prime}_{\mathbf{U}, \mathbf{B}} = (\mathcal{L}^+_{\mathbf{U}, \mathbf{B}})^{-1}$. At the same time, it is easy to check $\mathcal{L}^{+\prime}_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B})) = \mathbf{U}\mathbf{B}\theta_U^T + \mathbf{U}\theta_B \mathbf{U}^T + \theta_U \mathbf{B} \mathbf{U}^T$ holds for any $\theta(\mathbf{U}, \mathbf{B}) \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$.

Next, we provide the spectrum bounds for $\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}$. For any $\theta(\mathbf{U}, \mathbf{B}) = [(U_\bot D)^T \ \theta_B]^T \in \mathcal{H}(\mathbf{U}, \mathbf{B})\overline{\mathcal{M}}_{r+}^{\theta}$, we have

$$\|\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}))\|_F^2 \leq \|\theta_B\|_F^2 + 2\|\mathbf{B}\theta_U U_\bot\|_F^2 = \|\theta_B\|_F^2 + 2\|\mathbf{B}\theta_U\|_F^2 \leq 2\sigma_1^2(\mathbf{B}) \left(\|\mathbf{B}^{-1/2}\theta_B \mathbf{B}^{-1/2}\|_F^2 + \|\mathbf{D}\|_F^2\right)$$

$$= 2\sigma_1^2(X) \bar{g}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}), \theta(\mathbf{U}, \mathbf{B})), \tag{35}$$

and

$$\|\mathcal{L}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}))\|_F^2 \geq \|\theta_B\|_F^2 + 2\|\mathbf{B}\theta_U\|_F^2 \geq \sigma_1^2(\mathbf{B}) \left(\|\mathbf{B}^{-1/2}\theta_B \mathbf{B}^{-1/2}\|_F^2 + \|\mathbf{D}\|_F^2\right)$$

$$\geq \sigma_1^2(X) \bar{g}^+_{\mathbf{U}, \mathbf{B}}(\theta(\mathbf{U}, \mathbf{B}), \theta(\mathbf{U}, \mathbf{B})).$$

Next, we present our first main result on the geometric landscape connections of Riemannian optimization (1) under the embedded and the quotient geometries.
Theorem 3 (Geometric Landscape Connections of \(1\) on \(\mathcal{M}_{r+}^e\) and \(\mathcal{M}_{r+}^{q_1}\)). Suppose the conditions in Proposition 2 hold and the \(W_Y\) in \(\tilde{g}_{r+}^+\) satisfies \(W_Y = OW_YO^\top\) for any \(O \in \mathcal{O}_{r}\). Then

\[
\text{grad} f(X) = \left(\text{grad} h_{r+}([Y])W_Y Y^\top + \left(\text{grad} h_{r+}([Y])W_Y Y^\top\right)^\top (I_p - YY^\top)\right)/2,
\]

(35)

\[
\text{grad} h_{r+}([Y]) = 2\text{grad} f(X)YW_Y^{-1}.
\]

Furthermore, if \([Y]\) is a Riemannian FOSP of \(h_{r+}([Y'])\) defined via \(20a\), we have:

\[
\text{Hess} h_{r+}([Y])[\theta_Y, \theta_Y] = \text{Hess} f(X)[\mathcal{L}_{Y}^+ (\theta_Y), \mathcal{L}_{Y}^+ (\theta_Y)], \quad \forall \theta_Y \in \mathcal{H}_Y \mathcal{M}_{r+}^{q_1}.
\]

(36)

Finally, \(\text{Hess} h_{r+}([Y])\) and \(\text{Hess} f(X)\) have \((pr - (r^2 - r)/2)\) eigenvalues; for \(i = 1, \ldots, pr - (r^2 - r)/2,\)

\[
\lambda_i(\text{Hess} h_{r+}([Y])) \text{ is sandwiched between } 2\sigma_r (PW_Y^{-1}P^\top) \lambda_i(\text{Hess} f(X)) \text{ and } 4\sigma_1 (PW_Y^{-1}P^\top) \lambda_i(\text{Hess} f(X)).
\]

Proof of Theorem 3: First, notice \(Y\) lies in the column space spanned by \(U\) and \(YY^\top = P_U\). So \(35\) is by direct calculation from the gradient expressions in Proposition 1.

Next, we prove \(35\). Since \([Y]\) is a Riemannian FOSP of \(h_{r+}([Y'])\), we have

\[
\text{grad} h_{r+}([Y]) = 0, \quad \nabla f(YY^\top) = 0,
\]

(37)

and have \(\text{grad} f(X) = 0\) by \(35\). So \(\nabla f(X) = P_{U\perp} \nabla f(X)P_{U\perp}\). Recall \(P = U^\top Y, X = YY^\top\) and let \(\Sigma = U^\top XU\). Given any \(\theta_Y \in \mathcal{H}_Y \mathcal{M}_{r+}^{q_1}\), we have

\[
\langle \nabla f(X), P_{U\perp}, \theta_Y P^\top \Sigma^{-1} P \theta_Y P_{U\perp} \rangle = \langle \nabla f(X), \theta_Y \theta_Y^\top \rangle,
\]

(38)

where the equality is because \(P\) is nonsingular, \(PP^\top = \Sigma\) and \(\nabla f(X) = P_{U\perp} \nabla f(X)P_{U\perp}\).

Then by Proposition 1

\[
\text{Hess} h_{r+}([Y])[\theta_Y, \theta_Y]
\]

\[
= \nabla^2 f(YY^\top)[\theta_Y \theta_Y^\top + \theta_Y Y^\top + \theta_Y Y^\top] + 2\langle \nabla f(YY^\top), \theta_Y \theta_Y^\top \rangle
\]

\[
+ 2\langle \nabla f(YY^\top) Y Y W_Y^{-1} [\theta_Y], \theta_Y W_Y \rangle + \langle D W_Y \text{grad} h_{r+}([Y]), \theta_Y \theta_Y \rangle/2
\]

\[
= \nabla^2 f(X)[\theta_Y \theta_Y^\top + \theta_Y Y^\top + \theta_Y Y^\top] + 2\langle \nabla f(YY^\top), \theta_Y \theta_Y \rangle
\]

Proposition 2

\[
\nabla^2 f(X)[\mathcal{L}_{Y}^+ (\theta_Y), \mathcal{L}_{Y}^+ (\theta_Y)] + 2\langle \nabla f(X), P_{U\perp} \theta_Y P^\top \Sigma^{-1} P \theta_Y P_{U\perp} \rangle
\]

\[
= \text{Hess} f(X)[\mathcal{L}_{Y}^+ (\theta_Y), \mathcal{L}_{Y}^+ (\theta_Y)],
\]

where the last equality follows from the expression of \(\text{Hess} f(X)\) in \(21\) and the definition of \(\mathcal{L}_{Y}^+\).

Then, by \(25\), \(36\) and Theorem 2 we have \(\text{Hess} h_{r+}([Y])\) and \(\text{Hess} f(X)\) have \((pr - (r^2 - r)/2)\) eigenvalues and \(\lambda_i(\text{Hess} h_{r+}([Y]))\) is sandwiched between \(2\sigma_r (PW_Y^{-1}P^\top) \lambda_i(\text{Hess} f(X))\) and \(4\sigma_1 (PW_Y^{-1}P^\top) \lambda_i(\text{Hess} f(X))\) for \(i = 1, \ldots, pr - (r^2 - r)/2\).

Theorem 4 (Geometric Landscape Connections of \(1\) on \(\mathcal{M}_{r+}^e\) and \(\mathcal{M}_{r+}^{q_1}\)). Suppose \(U \in \text{St}(r, p), B \in \mathcal{S}_{+}(r)\) and \(X = UBU^\top\). Then

\[
\text{grad} f(X) = \frac{\text{grad}_U h_{r+}([U, B])B^{-1}U^\top}{2} + \frac{\text{grad}_U h_{r+}([U, B])B^{-1}U^\top}{2}
\]

\[
+ UB^{-1} \text{grad}_B h_{r+}([U, B])B^{-1}U^\top,
\]

(39)

\[
\text{grad} h_{r+}([U, B]) = \left[2P_U \text{grad} f(X)UB \right],
\]

20
Furthermore, if \([U, B]\) is a Riemannian FOSP of \(h_{r+}([U', B'])\) defined via \(20\), we have:

\[
\text{Hess}_{h_{r+}([U, B])}[[\theta(U, B), \theta(U, B)]] = \text{Hess}_f(X)[\mathcal{L}^r_{U, B}(\theta(U, B)), \mathcal{L}^r_{U, B}(\theta(U, B))], \quad \forall \theta(U, B) \in \mathcal{H}(U, B, \hat{M}^q_{r+}).
\]

Finally, \(\text{Hess}_{h_{r+}([U, B])}\) has \((pr - (r^2 - r)/2)\) eigenvalues and for \(i = 1, \ldots, pr - (r^2 - r)/2\),

\[
\lambda_i(\text{Hess}_{h_{r+}([U, B])}) \text{ is sandwiched between } \sigma^2_r(X)\lambda_i(\text{Hess}_f(X)) \text{ and } 2\sigma^2_r(X)\lambda_i(\text{Hess}_f(X)).
\]

Proof of Theorem 4. The proof of this theorem is similar to the proof of Theorem 3 and is postponed to Appendix B.3. ■

Theorems 3 and 4 immediately show the following equivalence of Riemannian FOSPs, SOSPs and strict saddles of optimization \(1\) under the embedded and the quotient geometries.

**Corollary 1** (Equivalence on Riemannian FOSPs, SOSPs and strict saddles of \(1\) Under Embedded and Quotient Geometries). Suppose \(W_Y = OWYO^\top\) holds for any \(O \in \Theta_r\). Then we have

(a) given \(Y \in \mathbb{R}_+^{p \times T}, U \in \text{St}(r, p)\) and \(B \in \mathbb{S}_+(r)\), if \([Y] \,(U, B)\) is a Riemannian FOSP or SOSP or strict saddle of \(h_{r+}([U', B'])\), then \(X = YY^\top\) (\(X = UBU^\top\)) is a Riemannian FOSP or SOSP or strict saddle of \(1\) under the embedded geometry;

(b) if \(X\) is a Riemannian FOSP or SOSP or strict saddle of \(1\) under the embedded geometry, then there is a unique \([Y] \,(U, B)\) such that \(YY^\top = X\) (\(UBU^\top = X\)) and it is a Riemannian FOSP or SOSP or strict saddle of \(h_{r+}([U', B'])\).

Proof of Corollary 1. Here we prove the Riemannian FOSP, SOS and strict saddle equivalence on \(M_{r+}^e\) and \(M_{r+}^q\), similar proof applies to the equivalence of \(1\) on \(M_{r+}^e\) and \(M_{r+}^q\). First, by the connection of Riemannian gradients in \(35\), the connection of Riemannian FOSPs under two geometries clearly holds.

Suppose \([Y]\) is a Riemannian SOSP of \(h_{r+}([Y'])\) and let \(X = YY^\top\). Given any \(\xi \in T_X M_{r+}^e\), we have \(\text{Hess}_f(X)[\xi_X, \xi_X] \geq 0\) if \(Y \in M_{r+}^e\). The inequality is by the SOSP assumption on \([Y]\). Combining the fact \(X\) is a Riemannian SOSP under this embedded geometry, this shows \(X = YY^\top\) is a Riemannian SOSP under the embedded geometry.

Next, let us show the other direction: suppose \(X\) is a Riemannian SOSP under the embedded geometry, there is a unique \([Y]\) such that \(YY^\top = X\) and it is a Riemannian SOSP of \(h_{r+}([Y'])\). To see this, first the uniqueness of \([Y]\) is guaranteed by the fact \(f : Y \in M_{r+}^q \to YY^\top \in M_{r+}^e\) induces a diffeomorphism between \(M_{r+}^e\) and \(M_{r+}^q\) \([MA20, \text{ Proposition A.7}]\). In addition, we have shown \([Y]\) is a Riemannian FOSP of \(h_{r+}([Y'])\). Then by \(30\), we have for any \(\theta \in \mathcal{H}(Y, M_{r+}^q), \text{Hess}_{h_{r+}([Y])}[\theta_Y, \theta_Y] = \text{Hess}_f(X)[\mathcal{L}_{Y}^e(\theta_Y), \mathcal{L}_{Y}^e(\theta_Y)] \geq 0\).

Finally the equivalence on strict saddles also follows easily from the sandwich inequality from Theorem 3 and the definition of strict saddle. ■

**Remark 6** (Spectrum Connection of Riemannian Hessians). The sandwich inequalities in Theorems 3 and 4 provide a finer connection on the spectrum of the Riemannian Hessians under the embedded and the quotient geometries at Riemannian FOSPs. These results are useful in transferring a few common geometric landscape properties from one geometry formulation to another. One such example is the so-called strict saddle property \([GHIJY13, LPP+19]\), which states that the function has a strict negative curvature at all stationary points but local minima. With this strict saddle property, various Riemannian gradient descent and trust region methods are guaranteed to escape all strict saddles and converge to a SOSP \([BAC19, CB19, LPP+19, SQW18, SFF19]\).
Remark 7 (Effects of Riemannian Metric and Quotient Structure on Landscape Connection). We can see from Corollary 1 that the choices of the quotient structure and the $\mathbf{W}_Y$ in the Riemannian metric $g_Y^{r^+}$ does not affect the landscape connection of FOSPs and SOSPs under two geometries. Similar phenomenon will also occur in the general fixed-rank matrix optimization. On the other hand, the quotient structure and the $\mathbf{W}_Y$ do affect the gaps of the sandwich inequalities in Theorems 3 and 4. For example, in Theorem 3 the gap coefficients are $2\sigma_r^2(Y)$ and $4\sigma_r^2(Y)$ when $\mathbf{W}_Y = \mathbf{I}_r$, however, they become universal constants 1 and 2 if we choose $\mathbf{W}_Y = 2\mathbf{Y}^\top \mathbf{Y}$. As we will discuss in Remark 7 that when the gap coefficients are some universal constants, there is a surprising algorithmic connection of adopting embedded and quotient geometries in Riemannian fixed-rank matrix optimization.

Remark 8 (Implications on Connections of Different Geometries for Riemannian Optimization). Generally speaking, embedded and quotient geometries are the most common two choices in Riemannian optimization. Compared to quotient geometry, embedded geometry allows computing and interpreting many geometric objects straightforwardly. Theorems 3 and 4 establish a strong geometric landscape connection between two geometries in fixed-rank PSD matrix optimization and this provides an example under which two different geometries are indeed connected in treating the same constraint in Riemannian optimization. Finally, we note although we focus on the geometric connections of (1) under the embedded and the quotient geometries, it is also relative easy to obtain the geometric connections under different quotient geometries based on our results.

6 Geometric Connections of Embedded and Quotient Geometries in Fixed-rank General Matrix Optimization

In this section, we present the geometric landscape connections of optimization problem (2) under the embedded and the quotient geometries. First, the problem (2) can be reformulated under each of the three quotient geometries in Section 4 as follows,

\begin{align}
\text{on } \tilde{\mathcal{M}}_{rq}^1 & : \min_{\mathbf{L} \in \mathbb{R}^{p_1 \times r}_+, \mathbf{R} \in \mathbb{R}^{p_2 \times r}_+} \bar{h}_r(\mathbf{L}, \mathbf{R}) := f(\mathbf{L}\mathbf{R}^\top), \\
\text{on } \tilde{\mathcal{M}}_{rq}^2 & : \min_{\mathbf{U} \in \text{St}(r, p_1), \mathbf{B} \in \mathbb{S}_+^r(r), \mathbf{V} \in \text{St}(r, p_2)} \bar{h}_r(\mathbf{U}, \mathbf{B}, \mathbf{V}) := f(\mathbf{U}\mathbf{B}\mathbf{V}^\top), \\
\text{on } \tilde{\mathcal{M}}_{rq}^3 & : \min_{\mathbf{U} \in \text{St}(r, p_1), \mathbf{Y} \in \mathbb{R}^{p_2 \times r}} \bar{h}_r(\mathbf{U}, \mathbf{Y}) := f(\mathbf{U}\mathbf{Y}^\top).
\end{align}

Since $\bar{h}_r(\mathbf{L}, \mathbf{R})$, $\bar{h}_r(\mathbf{U}, \mathbf{B}, \mathbf{V})$ and $\bar{h}_r(\mathbf{U}, \mathbf{Y})$ are invariant along the fibers of $\tilde{\mathcal{M}}_{rq}^1$, $\tilde{\mathcal{M}}_{rq}^2$ and $\tilde{\mathcal{M}}_{rq}^3$, they induce functions $h_r([\mathbf{L}, \mathbf{R}])$, $h_r([\mathbf{U}, \mathbf{B}, \mathbf{V}])$ and $h_r([\mathbf{U}, \mathbf{Y}])$ on quotient manifolds $\mathcal{M}_{rq}^1$, $\mathcal{M}_{rq}^2$ and $\mathcal{M}_{rq}^3$, respectively. In the following Proposition 4 we provide Riemannian gradients and Hessians of (2) under embedded and quotient geometries.

Proposition 4 (Riemannian Gradients and Hessians of (2)). The Riemannian gradients and Hessians of (2) under the embedded and the quotient geometries introduced in Section 4 are:

- On $\mathcal{M}_{rq}^r$: Suppose $\mathbf{X} \in \mathcal{M}_{rq}^r$, $\mathbf{U}, \mathbf{V}$ span the top $r$ left and right singular subspaces of $\mathbf{X}$, respectively and $\xi_\mathbf{X} = [\mathbf{U} \quad \mathbf{U}_\perp] \begin{bmatrix} \mathbf{S} & \mathbf{D}_2^\top \\ \mathbf{D}_1 & 0 \end{bmatrix} [\mathbf{V} \quad \mathbf{V}_\perp]^\top \in T_\mathbf{X} \mathcal{M}_{rq}^r$. Then
  \begin{equation}
  \begin{aligned}
  \text{grad} f(\mathbf{X}) &= P_{\mathbf{U}} \nabla f(\mathbf{X}) \mathbf{P}_\mathbf{V} + P_{\mathbf{U}_\perp} \nabla f(\mathbf{X}) \mathbf{P}_\mathbf{V} + P_{\mathbf{U}} \nabla f(\mathbf{X}) \mathbf{P}_{\mathbf{V}_\perp}, \\
  \text{Hess} f(\mathbf{X})[\xi_\mathbf{X}, \xi_\mathbf{X}] &= \nabla^2 f(\mathbf{X})[\xi_\mathbf{X}, \xi_\mathbf{X}] + 2\langle \nabla f(\mathbf{X}), \mathbf{U}_\perp \mathbf{D}_1 \Sigma^{-1} \mathbf{D}_2 \mathbf{V}_\perp \rangle,
  \end{aligned}
\end{equation}

where $\Sigma = \mathbf{U}^\top \mathbf{X} \mathbf{V}$.
• On $\mathcal{M}_p^{q_1}$: Suppose $L \in \mathbb{R}^{p_1 \times r}$, $R \in \mathbb{R}^{p_2 \times r}$ and $\theta_{(L,R)} = [\theta_L^T \theta_R^T]^T \in \mathcal{H}_{(L,R),R}$. Then
\[
\text{grad } h_r([L, R]) = \begin{bmatrix} \text{grad}_L h_r([L, R]) \\ \text{grad}_R h_r([L, R]) \end{bmatrix} = \begin{bmatrix} \nabla f(LR^T)RW_{LR}^{-1} \\ (\nabla f(LR^T))^T LV_{LR}^{-1} \end{bmatrix},
\]
\[
\text{Hess } h_r([L, R])[\theta_{(L,R)}, \theta_{(L,R)}] = \nabla^2 f(LR^T)[L\theta_R^T + \theta_L R^T, L\theta_R^T + \theta_L R^T] + 2(\nabla f(LR^T), \theta_L \theta_R^T)
+ \nabla f(LR^T)\mathbb{D}W_{LR}^{-1}[\theta_{(L,R)}], \theta_L W_{LR} + (\nabla f(LR^T))^T LDV_{LR}^{-1}[\theta_{(L,R)}], \theta_R V_{LR}
+ \langle DW_{LR}[\text{grad } h_r([L, R])], \theta_L \theta_R^T \rangle/2.
\] (43)

• On $\mathcal{M}_p^{q_2}$: Suppose $U \in \text{St}(r, p_1)$, $B \in \mathbb{S}_+(r)$, $V \in \text{St}(r, p_2)$ and $\theta_{(U,B,V)} = [\theta_U^T \theta_B^T \theta_V^T]^T \in \mathcal{H}_{(U,B,V)}$. Then
\[
\text{grad } h_r([U, B, V]) = \begin{bmatrix} \text{grad}_U h_r([U, B, V]) \\ \text{grad}_B h_r([U, B, V]) \\ \text{grad}_V h_r([U, B, V]) \end{bmatrix} = \begin{bmatrix} P_{U_\perp} \nabla f(UBV^T)VB + U (\text{Skew}(\Delta)B + B\text{Skew}(\Delta))/2 \\ \\ BSym(\Delta)B \\ \\ P_{V_\perp} \nabla f(UBV^T)^T UB - V (\text{Skew}(\Delta)B + B\text{Skew}(\Delta))/2 \end{bmatrix},
\]
\[
\text{Hess } h_r([U, B, V])[\theta_{(U,B,V)}, \theta_{(U,B,V)}] = \nabla^2 f(UBV^T)[\theta_U BV^T + UB^T \theta_U, \theta_U BV^T + UB^T \theta_U] + 2(\nabla f(UBV^T), \theta_U B \theta_V^T)
+ \langle \Delta, \text{Sym}(U^T \theta_U U^T \theta_U)B + BSym(V^T \theta_V U^T \theta_U) - 2\theta_U \theta_U B \rangle /2
+ \langle \Delta, BSym(V^T \theta_V V^T \theta_U) + \text{Sym}(U^T \theta_U V^T \theta_V)B - 2B \theta_V \theta_V + 2\theta_B B^{-1} \theta_B \theta_V \rangle /2
+ \langle \Delta', 2\theta_B - U^T \theta_U B - \theta_U^T UB/2 - V^T \theta_V B/2 \rangle + \langle \Delta'', 2\theta_B - B \theta_U^T V - BV^T \theta_V/2 - B \theta_U^T U/2 \rangle,
\] (44)
where $\Delta = U^T \nabla f(UBV^T)V$, $\Delta' = \theta_U^T \nabla f(UBV^T)V$ and $\Delta'' = U^T \nabla f(UBV^T)\theta_V$.

• On $\mathcal{M}_p^{q_3}$: Suppose $U \in \text{St}(r, p_1)$, $Y \in \mathbb{R}^{p_2 \times r}$ and $\theta_{(U,Y)} = [\theta_U^T \theta_Y]^T \in \mathcal{H}_{(U,Y),R}$. Then
\[
\text{grad } h_r([U, Y]) = \begin{bmatrix} \text{grad}_U h_r([U, Y]) \\ \text{grad}_Y h_r([U, Y]) \end{bmatrix} = \begin{bmatrix} P_{U_\perp} \nabla f(UY^T)Y \\ (\nabla f(UY^T))^T UW_{Y^{-1}} \end{bmatrix},
\]
\[
\text{Hess } h_r([U, Y])[\theta_{(U,Y)}, \theta_{(U,Y)}] = \nabla^2 f(UY^T)[U\theta_Y + U \theta_Y^T, U \theta_Y + \theta_Y U^T] + 2(\nabla f(UY^T), \theta_U \theta_Y^T) - \langle U^T \nabla f(UY^T)Y, \theta_U^T \theta_Y \rangle
+ \langle (\nabla f(UY^T))^T UD W_{Y^{-1}}[\theta_Y], \theta_Y W_Y \rangle + \langle DW_{Y}[\text{grad } h_r([U, Y])], \theta_Y \theta_Y \rangle/2.
\] (45)

Next, we construct bijective maps between $T_X \mathcal{M}_r$ and $\mathcal{H}_{(L,R),R} \mathcal{M}_r^{q_1}$, $\mathcal{H}_{(U,B,V)} \mathcal{M}_r^{q_2}$, $\mathcal{H}_{(U,Y),R} \mathcal{M}_r^{q_3}$, and give their spectrum bounds.

**Proposition 5** (Bijection Between $\mathcal{H}_{(L,R),R} \mathcal{M}_r^{q_1}$ and $T_X \mathcal{M}_r$). Suppose $L \in \mathbb{R}^{p_1 \times r}$, $R \in \mathbb{R}^{p_2 \times r}$, $X = LR^T$ with its top $r$ left and right singular subspaces spanned by $U$ and $V$, respectively and $P_1 = \ldots$
\[ U^\top L, P_2 = V^\top R. \] For any \( \theta_{(L,R)} = [\theta_L^T \theta_R^T]^T \in \mathcal{H}(L,R)\overline{M}_q \) and \( \xi_X = [U U_L] \begin{bmatrix} S & D_2^T \\ D_1 & 0 \end{bmatrix} [V V_L]^T \in T_X M^e_r, \]

\[
\xi_{(L,R)}^\theta := [U U_L] \begin{bmatrix} P_1 \theta^T_R V & U^\top \theta_L P_2^\top \\ U^\top \theta_L P_2^\top & 0 \end{bmatrix} [V V_L]^T \in T_X M^e_r,
\]

where \( S' \) in \( \theta_{(L,R)}^\xi \) is uniquely determined by the Sylvester equation \( P_1 V L R^1 P_2 V L R^2 = S. \) Then we can find a linear bijective mapping \( L^r_{(L,R)} \) between \( \mathcal{H}(L,R)\overline{M}_q \) and \( T_X M^e_r, \)

\[
L^r_{(L,R)} : \theta_{(L,R)} \in \mathcal{H}(L,R)\overline{M}_q \rightarrow \xi_{(L,R)}^\theta \in T_X M^e_r \quad \text{and} \quad (L^r_{(L,R)})^{-1} : \xi_X \in T_X M^e_r \rightarrow \theta_{(L,R)}^\xi \in \mathcal{H}(L,R)\overline{M}_q,
\]

such that \( L^r_{(L,R)}(\theta_{(L,R)}) = L \theta_R^T + \theta_L R^T \) holds for any \( \theta_{(L,R)} \in \mathcal{H}(L,R)\overline{M}_q \).

Finally, we have the following spectrum bounds for \( L^r_{(L,R)} \):

\[
\gamma \cdot \Delta_{(L,R)}(\theta_{(L,R)}(\theta_L, \theta_R)) \leq \| L^r_{(L,R)}(\theta_{(L,R)}(\theta_L, \theta_R)) \|_F^2 \leq 2 \Gamma \cdot \Delta_{(L,R)}(\theta_{(L,R)}(\theta_L, \theta_R)), \quad \forall \theta_{(L,R)} \in \mathcal{H}(L,R)\overline{M}_q,
\]

where \( \Delta := \sigma_r(P_2 W_L R P_2^T) \wedge \sigma_r(P_1 V_L R P_1^T) \) and \( \Gamma := \sigma_1(P_2 W_L R P_2^T) \vee \sigma_1(P_1 V_L R P_1^T). \)

**Proof of Proposition 5** First, the uniqueness of \( S' \) in \( \theta_{(L,R)}^\xi \) is guaranteed by the fact \( P_1 V L R^1 P_2 \) and \( -P_2 W_L R P_2^T \) have disjoint spectra and [Bla13] Theorem VII.2.1. Thus, \( \theta_{(L,R)}^\xi \) and \( \xi_{(L,R)} \) are well defined given any \( \theta_{(L,R)} \) and \( \xi_X. \)

Next, we show \( L^r_{(L,R)} \) is a bijection. Notice both \( \mathcal{H}(L,R)\overline{M}_q \) and \( T_X M^e_r \) are of dimension \( (p_1 + p_2 - r)r \). Suppose \( L^r_{(L,R)} : \xi_X \in T_X M^e_r \rightarrow \theta_{(L,R)}^\xi \in \mathcal{H}(L,R)\overline{M}_q. \) For any \( \xi_X = [U U_L] \begin{bmatrix} S & D_2^T \\ D_1 & 0 \end{bmatrix} [V V_L]^T \in T_X M^e_r, \) we have

\[
L^r_{(L,R)}(L^r_{(L,R)}(\xi_X)) = L^r_{(L,R)}(\theta_{(L,R)}^\xi) = [U U_L] \begin{bmatrix} P_1 \theta_{(L,R)}^\xi V & U^\top \theta_L P_2^\top \\ U^\top \theta_L P_2^\top & 0 \end{bmatrix} [V V_L]^T = \xi_X.
\]

Since \( L^r_{(L,R)} \) and \( L^r_{(L,R)} \) are linear maps, [ES] implies \( L^r_{(L,R)} \) is a bijection and \( (L^r_{(L,R)})^{-1} \). At the same time, it is easy to check \( L^r_{(L,R)} \) satisfies \( L^r_{(L,R)}(\theta_{(L,R)}(\theta_L, \theta_R)) = L \theta_R^T + \theta_L R^T \) by observing \( L = UP_1, R = VP_2. \)

Next, we provide the spectrum bounds for \( L^r_{(L,R)} \). For any \( \theta_{(L,R)} = [\theta_L^T \theta_R^T]^T \in \mathcal{H}(L,R)\overline{M}_q, \)

\[
\Delta(\theta_{(L,R)}(\theta_L, \theta_R)) = \text{tr}(W L R \theta_L^T \theta_L) + \text{tr}(V R \theta_R^T \theta_R) = \| \theta_L W_L \|^2_F + \| \theta_R V_L \|^2_F \leq (\| S P_2 W_L R P_2^T \|^2_F + \| D_1 \|^2_F) \sigma_1^2(P_2^T W_L R P_2^T) + (\| SV^T P_1 V_L R P_1^T \|^2_F + \| D_2 \|^2_F) \sigma_1^2(P_1^T V_L R P_1^T) \leq \frac{1}{\sigma_r(P_2 W_L R P_2^T) \wedge \sigma_r(P_1 V_L R P_1^T)}(\| S P_2 W_L R P_2^T \|^2_F + \| D_1 \|^2_F + \| SV^T P_1 V_L R P_1^T \|^2_F + \| D_2 \|^2_F), \]

(49)
and

\[ \langle P_1 V_{L,R}^{-1} P_{L}^T S, SP_2 W_{L,R}^{-1} P_{L}^T \rangle = \langle (P_1 V_{L,R}^{-1} P_{L}^T)^{1/2} S (P_2 W_{L,R}^{-1} P_{L}^T)^{1/2}, (P_1 V_{L,R}^{-1} P_{L}^T)^{1/2} S (P_2 W_{L,R}^{-1} P_{L}^T)^{1/2} \rangle \geq 0. \]  

(50)

Thus

\[ \| \mathcal{L}_Y^r(\theta_{(L,R)}) \|^2_F = \| \xi_X^{\theta_{(L,R)}} \|^2_F = \| P_1 \theta_R^T V + U^T \theta_L P_{L}^T \|^2_F + \| P_1 \theta_R V \|^2_F + \| U^T \theta_L P_{L}^T \|^2_F = \| P_1 V_{L,R}^{-1} P_{L}^T S + SP_2 W_{L,R}^{-1} P_{L}^T \|^2_F + \| D_1 \|^2_F + \| D_2 \|^2_F \]

(50)

\[ \geq \| SP_2 W_{L,R}^{-1} P_{L}^T \|^2_F + \| S^T P_1 V_{L,R}^{-1} P_{L}^T \|^2_F + \| D_1 \|^2_F + \| D_2 \|^2_F \]

\[ \geq (\sigma_r(P_2 W_{L,R}^{-1} P_{L}^T) \wedge \sigma_r(P_1 V_{L,R}^{-1} P_{L}^T)) \mathcal{g}^r_{(L,R)}(\theta_{(L,R)}, \theta_{(L,R)}) , \]

and

\[ \| \mathcal{L}_Y^r(\theta_{(L,R)}) \|^2_F = \| \xi_X^{\theta_{(L,R)}} \|^2_F = \| P_1 \theta_R^T V + U^T \theta_L P_{L}^T \|^2_F + \| P_1 \theta_R V \|^2_F + \| U^T \theta_L P_{L}^T \|^2_F \leq 2(\| P_1 \theta_R^T V \|^2_F + \| U^T \theta_L P_{L}^T \|^2_F) \]

(50)

\[ \leq 2(\sigma^2_r(W_{L,R}^{-1/2} P_{L}^T) \wedge \sigma^2_r(V_{L,R}^{-1/2} P_{L}^T)) \mathcal{g}^r_{(L,R)}(\theta_{(L,R)}, \theta_{(L,R)}) \]

(50)

\[ = 2(\sigma_1(P_2 W_{L,R}^{-1} P_{L}^T) \wedge \sigma_1(P_1 V_{L,R}^{-1} P_{L}^T)) \mathcal{g}^r_{(L,R)}(\theta_{(L,R)}, \theta_{(L,R)}) \]

This finishes the proof of this proposition. ■

**Proposition 6** (Bijection Between \( \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2} \) and \( T_X \mathcal{M}^e_r \)). Suppose \( U \in St(r, p_1), B \in S_+(r), \) \( V \in St(p_2, r) \) and \( X = UBV^T \). For any \( \theta_{(U,B,V)} = [\theta^T_U \ \theta^T_B \ \theta^T_V]^T \in \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2} \) and \( \xi_X \in [U \quad U_\perp \begin{bmatrix} S & D_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}]^T \in T_X \mathcal{M}^e_r \), define

\[ \xi_X^{\theta_{(U,B,V)}} := [U \quad U_\perp \begin{bmatrix} U^T \theta_U B + \theta_U B \theta^T_B V & B \theta^T_B V \_ \_ \_ \\ U^T \theta_U B & 0 \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}]^T \in T_X \mathcal{M}^e_r, \]

(51)

\[ \theta^{\xi_X}_{(U,B,V)} := [(U \_ \_ \_ D_1 B^{-1} + U \Omega')^T \ S' \ (V \_ \_ \_ D_2 B^{-1} - V \Omega')^T]^T \in \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2}, \]

where \( S', \Omega' \) are uniquely determined by the linear equation system: \( \Omega' B + S' - BU'^T = S, \ \Omega' = -\Omega'^T, S' = S'^T \). Then we can find a linear bijective mapping \( \mathcal{L}_{U,B,V}^r \) between \( \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2} \) and \( T_X \mathcal{M}^e_r \),

\[ \mathcal{L}_{U,B,V}^r : \theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2} \rightarrow \xi_X^{\theta_{(U,B,V)}} \in T_X \mathcal{M}^e_r, \]

\[ (\mathcal{L}_{U,B,V}^r)^{-1} : \xi_X \in T_X \mathcal{M}^e_r \rightarrow \theta^{\xi_X}_{(U,B,V)} \in \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2}, \]

such that \( \mathcal{L}_{U,B,V}^r(\theta_{(U,B,V)}) = U B V^T + U \theta_B V^T + UB \theta^T_B B \) holds for any \( \theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2} \).

Finally, we have the following spectrum bounds for \( \mathcal{L}_{U,B,V}^r \): for all \( \theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)} \bar{\mathcal{M}}_{\rho^2} \),

\[ \sigma^2_r(X) g^r_{(U,B,V)}(\theta_{(U,B,V)}, \theta_{(U,B,V)}) \leq \| \mathcal{L}_{U,B,V}^r(\theta_{(U,B,V)}) \|^2_F \leq 2\sigma^2_r(X) g^r_{(U,B,V)}(\theta_{(U,B,V)}, \theta_{(U,B,V)}), \]

(52)
Proof of Proposition 6. The proof is divided into two steps: in Step 1, we show $\xi_{\mathbf{X}}^{\theta_{(U,B,V)}}$ and $\theta_{(U,B,V)}^{\ell_{\mathcal{X}}}$ are well defined for any $\theta_{(U,B,V)}$ and $\xi_{\mathbf{X}}$; in Step 2, we show $\ell_{U,B,V}^{r}$ is a bijection and prove its spectrum bounds.

Step 1. First, it is clear for any $\theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)}\overrightarrow{M}_{r}$, $\xi_{\mathbf{X}}^{\theta_{(U,B,V)}}$ is well defined. To show $\theta_{(U,B,V)}^{\ell_{\mathcal{X}}}$ is well defined given any $\xi_{\mathbf{X}} \in T_{X}\mathcal{M}_{r}$, we need to show the equation system: (i) $\Omega_{1}B + S_{1} - B\Omega_{1}^{T} = S$, (ii) $\Omega_{1} = -\Omega_{1}^{T}$, (iii) $S_{1} = S_{1}^{T}$ with respect to $S_{1}, \Omega_{1}$ has a unique solution. By (i) we have $S_{1} = S - \Omega_{1}B + B\Omega_{1}^{T}$. By plugging it into (iii), we have $B\Omega_{1}^{T} - \Omega_{1}B = (S^{T} - S)/2$. Combining it with (ii) states $\Omega_{1} + \Omega_{1}B = (S - S^{T})/2$. So we conclude

\[(S_{1}, \Omega_{1}) : \Omega_{1}B + S_{1} - B\Omega_{1}^{T} = S, \Omega_{1} = -\Omega_{1}^{T}, S_{1} = S_{1}^{T}\]

\[\subseteq \{(S_{1}, \Omega_{1}) : \Omega_{1} + \Omega_{1}B = (S - S^{T})/2, S_{1} = S - \Omega_{1}B + B\Omega_{1}^{T}\} \]

Note that $\Omega_{1} + \Omega_{1}B = (S - S^{T})/2$ is a Sylvester equation with respect to $\Omega_{1}$. Since $B$ and $-B$ have distinct spectra, the system has a unique solution [Bha13, Theorem VII.2.1] and we denote it by $\Omega'$. Let $S' = S - \Omega'B + B\Omega'^{T}$. If we can show $\Omega' = -\Omega'^{T}$ and $S' = S^{T}$, then we can conclude $(S', \Omega')$ is the unique solution of the linear equation system $\Omega_{1}B + S_{1} - B\Omega_{1}^{T} = S, \Omega_{1} = -\Omega_{1}^{T}, S_{1} = S_{1}^{T}$.

Let us first show $\Omega' = -\Omega'^{T}$. We know $\Omega'$ satisfies $B\Omega' + \Omega' B = (S - S^{T})/2$ and $\Omega'^{T}B + B\Omega'^{T} = (S^{T} - S)/2$. By summing these two equations we have $B(\Omega' + \Omega'^{T}) + (\Omega' + \Omega'^{T})B = 0$. This is a new Sylvester equation $B\Omega + \Omega B = 0$ with respect to $\Omega$ and we know again by [Bha13, Theorem VII.2.1] that $0$ is the unique solution to this system. So we have $\Omega' + \Omega'^{T} = 0$, i.e., $\Omega' = -\Omega'^{T}$. Then

\[S' = S - \Omega'B + B\Omega'^{T} = S + \Omega'^{T}B + B\Omega'^{T} = S + (S^{T} - S)/2 = (S^{T} + S)/2 = S^{T} + B\Omega' + \Omega'B = S^{T} - B\Omega'^{T} + \Omega'B = S'^{T}.\]

So we have shown $\xi_{\mathbf{X}}^{\theta_{(U,B,V)}}$ is well defined for any $\theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)}\overrightarrow{M}_{r}$.

Step 2. Notice $\mathcal{H}_{(U,B,V)}\overrightarrow{M}_{r}$ is of dimension $(p_{1} + p_{2} - r)r$ and it is the same with dim$(T_{X}\mathcal{M}_{r})$.

Suppose $\ell_{U,B,V}^{r} : \xi_{\mathbf{X}} \in T_{X}\mathcal{M}_{r} \rightarrow \ell_{\mathcal{X}}^{\theta_{(U,B,V)}} \in \mathcal{H}_{(U,B,V)}\overrightarrow{M}_{r}$. For any $\xi_{\mathbf{X}} = [U \quad U_{1}] \begin{bmatrix} S & D_{1}^{T} \\ D_{1} & 0 \end{bmatrix} \begin{bmatrix} V & V_{1} \end{bmatrix}^{T} \in T_{X}\mathcal{M}_{r}$, we have

\[\ell_{U,B,V}^{r}(\ell_{U,B,V}^{r}(\xi_{\mathbf{X}})) = \ell_{U,B,V}^{r}(\ell_{\mathcal{X}}^{\theta_{(U,B,V)}}) = [U \quad U_{1}] \begin{bmatrix} U^{T} \ell_{U}^{\ell_{\mathcal{X}}} B + \ell_{B}^{\ell_{\mathcal{X}}} V & \ell_{B}^{\ell_{\mathcal{X}}} V \end{bmatrix} \begin{bmatrix} U^{T} \ell_{U}^{\ell_{\mathcal{X}}} B \\ U_{1} \end{bmatrix} \begin{bmatrix} V & V_{1} \end{bmatrix}^{T} \]

\[= \xi_{\mathbf{X}}. \]

(53)

Since $\ell_{U,B,V}^{r}$ and $\ell_{U,B,V}^{r}$ are linear maps, (53) implies $\ell_{U,B,V}^{r}$ is a bijection and $\ell_{U,B,V}^{r} = (\ell_{U,B,V}^{r})^{-1}$. At the same time, it is easy to check $\ell_{U,B,V}^{r}(\theta_{(U,B,V)}) = \theta_{U}B^{T} + U\theta_{B}V^{T} + UB\theta_{V}^{T}$ holds for any $\theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)}\overrightarrow{M}_{r}$.

Next, we provide the spectrum bounds for $\ell_{U,B,V}^{r}$. For any $\theta_{(U,B,V)} = [\phi_{U}^{T} \quad \phi_{B}^{T} \quad \phi_{V}^{T}]^{T} \in \mathcal{H}_{(U,B,V)}\overrightarrow{M}_{r}$ with $\theta_{U} = U_{1}D_{1} + U\Omega, \theta_{V} = V_{1}D_{2} - V\Omega, \Omega = -\Omega^{T}, \theta_{B} \in \mathbb{S}^{r \times r}$. We have

\[\ell_{U,B,V}^{r}(\theta_{(U,B,V)}, \theta_{(U,B,V)}) = ||\phi_{U}||_{F}^{2} + ||\phi_{V}||_{F}^{2} + \text{tr}(B^{-1}\theta_{B}B^{-1}\theta_{B})\]

\[= ||\phi_{U}||_{F}^{2} + ||\phi_{V}||_{F}^{2} + ||B^{1/2}\theta_{B}B^{-1/2}||_{F}^{2}\]

\[= ||D_{1}||_{F}^{2} + ||D_{2}||_{F}^{2} + 2||\Omega||_{F}^{2} + ||B^{-1/2}\theta_{B}B^{-1/2}||_{F}^{2}, \]

(54)
and
\[
\langle \Omega B - B\Omega^T, \theta_B \rangle = \langle B\Omega^T - \Omega B, \theta_B \rangle \quad : \quad \Omega B + \theta_B - B\Omega^T = 0
\]
Thus,
\[
\|L_{U,B,B}(\theta_{(U,B,V)})\|_F^2 = \|\xi_X^{(U,B,V)}\|_F^2 = \|U^T\theta_B + \Omega B\|^2_F + \|\Omega B - \theta_B - B\Omega^T\|^2_F + \|B\theta_V V_\perp\|^2_F + \|U^T\theta_B\|^2_F
\]
\[
= \|\Omega B - B\Omega^T\|^2_F + \|\theta_B\|^2_F + \|BD_2\|^2_F + \|D_1B\|^2_F
\]
\[
\Omega := \Omega^22\|B\|^2_F + 2\|B\Omega^2\| + \|B\|_F^2 + \|BD_2\|^2_F + \|D_1B\|^2_F
\]
(56)
\[
(a) \geq \sigma^2_r(X)\bar{g}_{r(U,B,V)}((\theta_{(U,B,V)}), (\theta_{(U,B,V)}),
\]
where in (a) we use the fact \(\sigma_r(B) = \sigma_r(X)\), and
\[
\|L_{U,B,B}(\theta_{(U,B,V)})\|_F^2 = \|\xi_X^{(U,B,V)}\|_F^2 = \|\theta_B + \Omega B\|^2_F + \|\Omega B - \theta_B - B\Omega^T\|^2_F + \|\theta_B\|^2_F + \|BD_2\|^2_F + \|D_1B\|^2_F
\]
\[
\leq 2\sigma^2_r(X)\bar{g}_{r(U,B,V)}((\theta_{(U,B,V)}), (\theta_{(U,B,V)}).\]
This finishes the proof of this proposition. 

**Proposition 7** (Bijection Between \(H_{(U,Y),\overline{M}_{p_1}^{T}}\) and \(T_XM_{p_1}^{e}\)). Suppose \(U \in \text{St}(r, p_1)\), \(Y \in \mathbb{R}^{p_2 \times r}\) and \(X = UY^T\) with top \(r\) right singular subspace spanned by \(V\). For any \(\theta_{(U,Y)} = [\theta_U^T, \theta_Y^T]^T \in H_{(U,Y),\overline{M}_{p_1}^{T}}\) and \(\xi_X = [U \quad U_\perp]\begin{bmatrix} S & D_1^2 & 0 \\ \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}^T \in T_XM_{p_1}^{e}\), define
\[
\xi_X^{(U,Y)} := [U \quad U_\perp] \begin{bmatrix} \theta_U^TV & \theta_Y^TV_\perp \\ U_\perp \theta_U Y^TV & 0 \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}^T \in T_XM_{p_1}^{e},
\]
\[
\theta^{(U,Y)}_X := [(U_\perp D_1(Y^TV)^{-1})^T (VS^T + V_\perp D_2)^T]^T \in H_{(U,Y),\overline{M}_{p_1}^{T}}.
\]
Then we can find a linear bijective mapping \(L_{U,Y}: H_{(U,Y),\overline{M}_{p_1}^{T}} \rightarrow T_XM_{p_1}^{e}\) \((L_{U,Y})^{-1}: T_XM_{p_1}^{e} \rightarrow H_{(U,Y),\overline{M}_{p_1}^{T}}\) such that \(L_{U,Y}(\theta_{(U,Y)}) = U\theta_U^T + \theta_Y^TY^T\) holds for any \(\theta_{(U,Y)} \in H_{(U,Y),\overline{M}_{p_1}^{T}}\).

Finally, we have the following spectrum bounds for \(L_{U,Y}^r: \) for all \(\theta_{(U,Y)} \in H_{(U,Y),\overline{M}_{p_1}^{T}}\),
\[
(\sigma_r(Y)^2 - \frac{1}{\sigma_1(W_Y)})\bar{g}_{r(U,Y)}((\theta_{(U,Y)}), (\theta_{(U,Y)})) \leq \|L_{U,Y}(\theta_{(U,Y)})\|_F^2 \leq (\sigma_1(Y) \frac{1}{\sigma_r(W_Y)})\bar{g}_{r(U,Y)}((\theta_{(U,Y)}), (\theta_{(U,Y)})),
\]
(58)

**Proof of Proposition 7** First, it is easy to see \(\xi_X^{(U,Y)}\) and \(\theta^{(U,Y)}_X\) are well defined given any \(\theta_{(U,Y)} \in H_{(U,Y),\overline{M}_{p_1}^{T}}\) and \(\xi_X \in T_XM_{p_1}^{e}\).

Next, we show \(L_{U,Y}^r\) is a bijection. Notice \(H_{(U,Y),\overline{M}_{p_1}^{T}}\) is of dimension \((p_1 + p_2 - r)r\), which is the same with \(T_XM_{p_1}^{e}\). Suppose \(L_{U,Y}: \xi_X \in T_XM_{p_1}^{e} \rightarrow \theta^{(U,Y)}_X \in H_{(U,Y),\overline{M}_{p_1}^{T}}\). For any \(\xi_X = [U \quad U_\perp]\begin{bmatrix} S & D_1^2 & 0 \\ \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}^T \in T_XM_{p_1}^{e}\), we have
\[
L_{U,Y}(L_{U,Y}^r(\xi_X)) = L_{U,Y}(\theta^{(U,Y)}_X) = [U \quad U_\perp] \begin{bmatrix} \theta^{(U,Y)}_X^TV & \theta^{(U,Y)}_X^TV_\perp \\ U_\perp \theta^{(U,Y)}_X Y^TV & 0 \end{bmatrix} \begin{bmatrix} V & V_\perp \end{bmatrix}^T = \xi_X.
\]
Proof of Theorem 5. Furthermore, if \( r \) and \( Y \) lies in the column space of \( V \).

Next, we provide the spectrum bounds for \( L_{U,Y} \) under the embedded and quotient geometries.

Next, we prove (61). Since \( \bar{g}(u,y) = [(U_D)^\top \theta_y]^\top \in H_{(U,Y)}\mathcal{M}_r^{\theta_3} \), we have
\[
\|L_{U,Y}(\theta(u,y))\|^2_F = \|\theta_YV\|^2_F + \|U_{\theta}^\top Y\bar{V}\|^2_F
\]
\[
= \|\theta_Y\|^2_F + \|DYV\|^2_F \geq \|D\|^2_F\sigma_1^2(Y) + \|\theta_YW_{1,2}^2\|^2_F/\sigma_1(W_Y)
\]
\[
\geq (\sigma_2^2(Y) \cdot \frac{1}{\sigma_1(W_Y)})\bar{g}(u,y)\theta_u(u,y), \theta(u,y))
\]
where (a) is because \( Y \) lies in the column space of \( V \), and
\[
\|L_{U,Y}(\theta(u,y))\|^2_F = \|\theta_Y\|^2_F + \|DYV\|^2_F \leq (\sigma_2^2(Y) \cdot \frac{1}{\sigma_1(W_Y)})\bar{g}(u,y)\theta_u(u,y), \theta(u,y)). \]

Now, we are ready to present our main results on the geometric landscape connection of Riemannian fixed-rank matrix optimization \( 2 \) under the embedded and quotient geometries.

Theorem 5 (Geometric Landscape Connections of \( 2 \) on \( M_r^{\theta_3} \) and \( M_r^{\theta_4} \)). Suppose the conditions in Proposition 2 hold and the \( W_{L,R} \) and \( V_{L,R} \) in \( \bar{g}_{(L,R)}(u,y) \) satisfy \( W_{L,R} = MW_{LM,LM}^{-1}M^\top \), \( V_{L,R} = M^{-1}V_{LM,LM}^{-1}M^{-1} \) for any \( M \in GL_r \). Then
\[
\text{grad} f(X) = \frac{\text{grad}_L h_r([L,R])W_{L,R}L^\top + \text{grad}_R h_r([L,R])V_{L,R}L^\top (I_{p_2} - RR^\top)}{\text{grad}_r h_r([L,R])}
\]
\[
= \frac{\text{grad} f(X)RW_{L,R}L^\top}{\text{grad} f(X)V^\top L}\left[\begin{array}{c}
\text{grad} f(X)W_{L,R}L^\top \\
\text{grad} f(X)\top V^\top L\end{array}\right].
\]

Furthermore, if \( [L,R] \) is a Riemannian FOSP of \( h_r([L',R']) \) defined via \( 41a \), we have:
\[
\text{Hess} h_r([L,R])[\theta(L,R), \theta(L,R)] = \text{Hess} f(X)[L_{L,R}(\theta(L,R)), L_{R,R}(\theta(L,R))], \quad \forall \theta(L,R) \in H_{(L,R)}\mathcal{M}_r^{\theta_4}.
\]

Finally, \( \text{Hess} h_r([L,R]) \) and \( \text{Hess} f(X) \) have \((p_1+p_2-r)\) eigenvalues and for \( i = 1, \ldots, (p_1+p_2-r) \), we have \( \lambda_i(\text{Hess} h_r([L,R])) \) is sandwiched between \( \gamma \lambda_i(\text{Hess} f(X)) \) and \( \Gamma \lambda_i(\text{Hess} f(X)) \), where \( \gamma \) and \( \Gamma \) are given in \( 41k \).

Proof of Theorem 5. First, recall \( U \) and \( V \) span the top \( r \) left and right singular subspaces of \( LR^\top \), respectively, and we have \( LL^\top = P_U, RR^\top = P_V \). So (62) is by direct calculation from the gradient expressions in Proposition 3.

Next, we prove (62). Since \( [L,R] \) is a Riemannian FOSP of \( h_r([L',R']) \), we have
\[
\text{grad} h_r([L,R]) = 0, \quad \text{grad} f(LR^\top) = 0 \quad \text{and} \quad \langle \text{grad} f(LR^\top) \rangle^\top L = 0
\]
and have \( \text{grad} f(X) = 0 \) by (62). So \( \nabla f(X) = P_U \nabla f(X)P_{V^\perp} \). Recall \( P_1 = U^\top L, P_2 = V^\top R, \quad X = LR^\top \) and let \( \Sigma = U^\top XV \). Given any \( \theta(L,R) = [\theta_L \theta_R]^\top \in H_{(L,R)}\mathcal{M}_r^{\theta_4} \), we have
\[
\langle \nabla f(X), P_U \theta_L P_2 \Sigma^{-1} P_1 \theta_R P_{V^\perp} \rangle = \langle \nabla f(X), \theta_L \theta_R^\top \rangle,
\]
where the equality is because \( P_1, P_2 \) are nonsingular, \( P_1 P_2 \Sigma = \Sigma \) and \( \nabla f(X) = P_U \nabla f(X)P_{V^\perp} \).

28
Then by Proposition 4

\[
\text{Hess } h_r([L, R])\theta_{(L,R), \theta_{(L,R)}} = \nabla^2 f(LR^\top)[L\theta_R^\top + \theta_L R^\top, L\theta_R^\top + \theta_L R^\top] + 2\langle \nabla f(LR^\top), \theta_L \theta_R^\top \rangle \\
+ \langle \nabla f(LR^\top)RDW_{L\theta_R}^{-1}[\theta_{(L,R)}], \theta_L W_{L\theta_R} \rangle + \langle (\nabla f(LR^\top))^\top \text{LDV}_{L\theta_R}^{-1}[\theta_{(L,R)}], \theta_R V_{L\theta_R} \rangle \\
+ \langle \text{DW}_{L\theta_R}[\text{grad } h_r([L, R])], \theta_L \theta_R / 2 \rangle + \langle \text{DV}_{L\theta_R}[\text{grad } h_r([L, R])], \theta_R^2 \theta_R / 2 \rangle \\
= \nabla^2 f(X)[\mathcal{L}_{L,R}(\theta_{(L,R)}), \mathcal{L}_{L,R}(\theta_{(L,R)})] + 2\langle \nabla f(X), \mathcal{P}_{U\perp \theta_L P_2 \Sigma^{-1} P_1 \theta_R^\top P_{V\perp} \rangle
\]

where the last equality follows from the expression of \( \text{Hess } f(X) \) in (62) and the definition of \( \mathcal{L}_{L,R} \).

Then, by (47), (61) and Theorem 2, we have

\[
\text{Hess } h_r([L, R]) \quad \text{and} \quad \text{Hess } f(X) \quad \text{have disjoint spectra and } [Bha13, \text{Theorem VII.2.1}]. \quad \text{Since }
\]

Then, by Proposition 4:

\[
\text{Hess } h_r([L, R]) \quad \text{and} \quad \text{Hess } f(X) \quad \text{are sandwiched between } \gamma \lambda_i(\text{Hess } f(X)) \quad \text{and} \quad 2\gamma \lambda_i(\text{Hess } f(X)) \quad \text{for i = 1, ..., (p_1 + p_2 - r)r, where } \gamma \quad \text{and} \quad \Gamma \quad \text{are given in (47).}
\]

**Theorem 6** (Geometric Landscape Connections of \( \mathcal{M}_c^r \) on \( \mathcal{M}_f^r \)). Suppose \( U \in \text{St}(r, p_1) \), \( B \in \mathcal{S}_+(r) \), \( V \in \text{St}(p_2, r) \) and \( X = UBV^\top \). Then

\[
\nabla f(X) = \mathcal{P}_{U\perp \text{grad } U} h_r([U, Y])B^{-1}V^\top + U \Delta_1 V^\top + \langle P_{V\perp \text{grad } V} h_r([U, B, V])B^{-1}U^\top \rangle
\]

where \( \Delta_1 \) is uniquely determined by the equation system:

\[
B \text{Skew}^\top(\Delta_1) + \text{Skew}^\top(\Delta_1)B = 2\text{grad } U h_r([U, Y])^\top U,
\]

and

\[
\nabla h_r([U, B, V]) = \begin{bmatrix}
P_{U\perp \text{grad } U} f(X)VB + U(\text{Skew}(\Delta_2)B + B \text{Skew}(\Delta_2))^\top / 2 \\
\text{BSym}(\Delta_2)
\end{bmatrix}
\]

where \( \Delta_2 = U^\top \text{grad } f(X)V \).

Furthermore, if \([U, B, V] \) is a Riemannian FOSP of \( h_r([U', B', V']) \) defined via (41b), we have:

for any \( \theta_{(U,B,V)} \in \mathcal{H}_{(U,B,V)} \),

\[
\text{Hess } h_r([U, B, V]) \quad \text{has disjoint spectra and for i = 1, ..., (p_1 + p_2 - r)r, we have} \quad \lambda_i(\text{Hess } f(X)) \quad \text{is sandwiched between} \quad 2\sigma_i^2(X) \lambda_i(\text{Hess } f(X)).
\]

**Proof of Theorem 6** First, recall \( \Delta = U^\top \nabla f(UBV^\top)V \) and notice \( B \text{Skew}^\top(\Delta_1) + \text{Skew}^\top(\Delta_1)B = 2\text{grad } U h_r([U, Y])^\top U \). \( B \text{Skew}^\top(\Delta_1) + \text{Skew}^\top(\Delta_1)B \) is a Sylvester equation with respect \( \text{Skew}(\Delta_1)^\top \), which has a unique solution as \( B, -B \) have disjoint spectra and [Bha13, Theorem VII.2.1]. Since \( \Delta_1 = \text{Skew}(\Delta_1) + \text{Sym}(\Delta_1) \), \( \Delta_1 \) is uniquely determined by the equation system \( B \text{Skew}^\top(\Delta_1) + \text{Skew}^\top(\Delta_1)B = 2\text{grad } U h_r([U, Y])^\top U, \text{Sym}(\Delta_1) = B^{-1} \text{grad } B h_r([U, Y])B^{-1} \). Finally, because \( \Delta \) is a solution to this equation system, we have \( \Delta_1 = \Delta = U^\top \nabla f(UBV^\top)V \). The rest proofs of (64) and (65) are by direct calculation from the gradient expressions in Proposition 4.
Next, we prove \((66)\). Since \([U, B, V]\) is a Riemannian FOSP of \(h_r([U', B', V'])\), we have

\[
\nabla h_r([U, B, V]) = 0, \quad \nabla f(X) \equiv 0, \quad \nabla f(UBV^T)V = 0 \quad \text{and} \quad U^T \nabla f(UBV^T) = 0.
\]

So \(\nabla f(X) = P_{U, V} \nabla f(X) P_{U, V}\). Given any \(\theta_{(U, B, V)} = [\theta^T_U \theta^T_B \theta^T_V] \in \mathcal{H}_{(U, B, V), \mathcal{M}^p_r}\), we have

\[
\langle \nabla f(X), P_{U, V} \theta_U B B^{-1} B \theta_V P_{U, V} \rangle = \langle \nabla f(X), \theta_U B \theta_V \rangle,
\]

where the equality is because \(\nabla f(X) = P_{U, V} \nabla f(X) P_{U, V}\).

Then by Proposition 4 and recall \(\Delta' = \theta_U \nabla f(UBV^T)\) and \(\Delta'' = U^T \nabla f(UBV^T)\theta_V\), we have

\[
\text{Hess } h_r([U, B, V]) |_{\theta_{(U, B, V)}} = \nabla^2 f(UBV^T)[\theta_U B V^T + U \theta_B V^T + U \theta_B V^T + U \theta_B V^T + U \theta_B V^T + 2 \langle \nabla f(UBV^T), \theta_U B \theta_V \rangle] = \nabla^2 f([U, B, V]) [\mathcal{L}_{U, B, V}(\theta_{(U, B, V)}) + 2 \langle \nabla f(X), P_{U, V} \theta_U B B^{-1} B \theta_V P_{U, V} \rangle}
\]

where the last equality follows from the expression of Hess \(f(X)\) in \(42\) and the definition of \(\mathcal{L}_{U, B, V}\).

Then, by \(52\), \(65\) and Theorem 2 we have \(\text{Hess } h_r([U, B, V])\) has \((p_1 + p_2 - r)r\) eigenvalues and \(\lambda_i(\text{Hess } h_r([U, B, V]))\) is sandwiched between \(\sigma^2(X) \lambda_i(\text{Hess } f(X))\) and \(2 \sigma^2(X) \lambda_i(\text{Hess } f(X))\) for \(i = 1, \ldots, (p_1 + p_2 - r)r\).

**Theorem 7** (Geometric Landscape Connections of \(2\) on \(\mathcal{M}^p_r\) and \(\mathcal{M}^q\)). Suppose the conditions in Proposition 7 hold and the \(W_Y\) in \(\widetilde{g}(U, Y)\) satisfies \(W_Y = O W_{Y0} O^T\) for any \(O \in \mathcal{O}_r\). Then

\[
\text{grad } f(X) = \text{grad } h_r([U, Y]) Y^T + \left(\text{grad } h_r([U, Y]) W_Y U^T\right)^T
\]

Furthermore, if \([U, Y]\) is a Riemannian FOSP of \(h_r([U', Y'])\) defined via \(44\), we have:

\[
\text{Hess } h_r([U, Y]) [\theta_{(U, Y)}, \theta_{(U, Y)}] = \text{Hess } f(X) [\mathcal{L}_{U, Y}(\theta_{(U, Y)}), \mathcal{L}_{U, Y}(\theta_{(U, Y)})], \quad \forall \theta_{(U, Y)} \in \mathcal{H}_{(U, Y), \mathcal{M}^p_r}.
\]

Finally, \(\text{Hess } h_r([U, Y])\) has \((p_1 + p_2 - r)r\) eigenvalues and for \(i = 1, \ldots, (p_1 + p_2 - r)r\), we have \(\lambda_i(\text{Hess } h_r([U, Y]))\) is sandwiched between \((\sigma^2(Y) \lambda_i(\text{Hess } f(X))\) and \((\sigma^2(Y) \lambda_i(\text{Hess } f(X))\).

**Proof of Theorem 7.** The proof is similar to the proof of Theorem 5 and is postponed to Appendix B.4. ■

By Theorems 5, 6 and 7, we have the following Corollary 2 on the equivalence of Riemannian FOSPs, SOSP’s and strict saddles of optimization 2 under the embedded and the quotient geometries. The proof is given in Appendix B.4.
Corollary 2 (Equivalence on Riemannian FOSPs, SOSP and strict saddles of (2) Under Embedded and Quotient Geometries). Suppose $W_{L,R} = M W_{LM, RM}^{-1} M^T$, $V_{L,R} = M^{-1} V_{LM, RM}^{-1} M$ hold for any $M \in \text{GL}(r)$ and $W_Y = OW_{YO} O^T$ holds for any $O \in \text{O}_r$. Then we have

(a) given $L \in \mathbb{R}_{p_1}^{p_1 \times r}$, $R \in \mathbb{R}_{p_2}^{p_2 \times r}$, $U \in \text{St}(r,p_1)$, $B \in \mathbb{S}_+(r)$, $V \in \text{St}(r,p_2)$ and $Y \in \mathbb{R}_{p_2}^{p_2 \times r}$, if $[L,R]$ $([U,B,V]$ or $[U,Y])$ is a Riemannian FOSP or SOSP or strict saddle of $h_r([L',R'])$ $(h_r([U',B',V'])$ or $h_r([U',Y']))$, then $X = LR^T (X = UBV^T$ or $X = UY^T)$ is a Riemannian FOSP or SOSP or strict saddle of (2) under the embedded geometry;

(b) if $X$ is a Riemannian FOSP or SOSP or strict saddle of (2) under the embedded geometry, then there is a unique $[L,R]$ $([U,B,V]$ or $[U,Y])$ such that $LR^T = X$ $(UBV^T = X$ or $UY^T = X)$ and it is a Riemannian FOSP or SOSP or strict saddle of $h_r([L',R'])$ $(h_r([U',B',V'])$ or $h_r([U',Y']))$.

Remark 9 (Algorithmic Connection of Embedded and Quotient Geometries). Contrast to the geometric connection of embedded and quotient geometries in fixed-rank matrix optimization, the algorithmic connection between two geometries is more subtle. First, the algorithms under the quotient geometry are performed in the horizontal space and they depend on the quotient structure and the Riemannian metric we pick. Thus, it is hard to expect for a universal algorithmic connection under two geometries. On the other hand, we find that by taking some specific metrics under the quotient geometry, there are indeed some algorithmic connections. This is particularly true when the metrics are chosen in the way such that the sandwich gap coefficients in the geometric connection are some universal constants (see Remark 7). For example, if we pick $W_Y = 2Y^T Y$ in $\tilde{g}_Y^+$ and $W_{L,R} = R^T R$, $V_{L,R} = L^T L$ in $\tilde{g}_{(L,R)}$, then we have the following gradient flows of (1) under $M_{r+}^+$, $M_{r+}$ and (2) under $M_r^+$, $M_{r+}$:

- (PSD case)
  
  under $M_{r+}^+$:
  \[
  \frac{dX}{dt} = -\text{grad} f(X) = -P_U \nabla f(X) - \nabla f(X) P_U + P_U \nabla f(X) P_U, 
  \]

  under $M_{r+}$:
  \[
  \frac{dX}{dt} = -\text{grad} h_{r+}([Y]) Y^T - Y \left( \text{grad} h_{r+}([Y]) \right)^T = -P_U \nabla f(X) - \nabla f(X) P_U;
  \]

- (general case)
  
  under $M_{r+}^+$:
  \[
  \frac{dX}{dt} = -\text{grad} f(X) = -P_U \nabla f(X) - \nabla f(X) P_V + P_U \nabla f(X) P_V, 
  \]

  under $M_{r+}$:
  \[
  \frac{dX}{dt} = -\text{grad} h_r([L,R]) R^T - L \left( \text{grad} h_r([L,R]) \right)^T = -P_U \nabla f(X) - \nabla f(X) P_V.
  \]

We can see that the gradient flows of (1) and (2) under these embedded and the quotient geometries only differ one term, which has magnitude smaller than the other terms. Some empirical evidence which shows the remarkably similar algorithmic performance under these embedded and quotient geometries was provided in [12], and here our geometric connection results provide more theoretical insights for this empirical observation.

Remark 10 (Geometric Connection of Non-convex Factorization and Quotient Manifold Formulations for (1) and (2)). As we have discussed in the Introduction, another popular approach for handling the rank constraint in (1) or (2) is via factorizing $X$ into $YY^T$ or $LR^T$ and then treating the new problem as unconstrained optimization in the Euclidean space. In the recent work [12], they showed for both (1) and (2) the geometric landscapes under the factorization and embedded submanifold formulations are almost equivalent. By combining their results and the results in this paper, we also have a geometric landscape connection of (1) and (2) under the factorization and quotient manifold formulations.
7 Conclusion and Discussions

In this paper, we propose a general procedure for establishing geometric connections of a Riemannian optimization problem under different geometries. By applying it to problems (1) and (2) under the embedded and quotient geometries, we establish an exact Riemannian gradient connection under two geometries at every point on the manifold and sandwich inequalities between the spectra of the Riemannian Hessians at Riemannian FOSPs. These results immediately imply an equivalence on the sets of Riemannian FOSPs, SOSPs and strict saddles of (1) and (2) under embedded and quotient geometries.

There are many interesting extensions to the results in this paper to be explored in the future. First, as we have mentioned in the Example 1 in Section 3.1, our results on the connection of Riemannian Hessians under the embedded and the quotient geometries are established at FOSPs. It is interesting to explore whether it is possible to connect the landscapes under two geometries at non-stationary points. Second, although we have a unified treatment on various Riemannian metrics in the quotient geometry, different quotient structures are still treated case-by-case in the theoretical analysis. It is an interesting future work to unify that part as well. Third, other than the geometries covered in this paper, there are many other embedded and quotient geometries for fixed-rank matrices, such as the ones in [AIDLVH09, GP07, VAV13] for $\mathcal{M}_{r,+}$, it will be interesting to study the geometric landscape connection of (1) and (2) under these geometries. Fourth, another common manifold that has both embedded and quotient representations is the Stiefel manifold [EAS98]. We believe our general procedure in Section 3 can also be used to establish the geometric connection in that setting as well. Finally, our ultimate goal is to better understand the connections and comparisons of different Riemannian geometries and give some guidelines on how to choose them given a Riemannian optimization problem. Some progress and discussions on how to choose Riemannian geometries in quotient geometries can be found in [MS16, VAV13]. While there is still not too much study on how to choose different quotient structures and manifold classes. It is an important future work to develop a general theory to connect any two different Riemannian geometries on a given Riemannian optimization problem from either an algorithmic or a geometric point of view.

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A Additional Preliminaries on Riemannian Connection

In this section, we provide a more detailed discussion on Riemannian connection as it is critical in deriving the Riemannian Hessians. First, a vector field $\xi$ on a manifold $\mathcal{M}$ is a mapping that assigns each point $X \in \mathcal{M}$ a tangent vector $\xi_X \in T_X \mathcal{M}$. The Riemannian gradient is a typical example of vector field. The general definition of the Riemannian connection follows from the fundamental theorem of differential geometry [Pet06, Chapter 2.1] and can be characterized by the so-called Koszul formula given as follows [AMS09, Theorem 5.3.1]:

$$2g_X(\nabla_\xi \eta, \theta) = Dg_X(\eta, \theta)[\xi] + Dg_X(\xi, \theta)[\eta] - Dg_X(\eta, \xi)[\theta] + g_X(\theta, [\xi, \eta]) + g_X(\eta, [\theta, \xi]) - g_X(\xi, [\eta, \theta]),$$

(71)
where $\xi, \eta, \theta$ are vector fields on $\mathcal{M}$ and $[\xi, \eta]$ denotes the Lie bracket between vector fields $\xi$ and $\eta$. If $\mathcal{M}$ is an open subset of a vector space, the Lie bracket of vector fields is given by

$$[\xi, \eta] = D\eta[\xi] - D\xi[\eta], \quad (72)$$

where $D\eta[\xi]$ is the classical direction derivative of a vector field in a vector space and its evaluation at $X$ is denoted by $(D\eta[\xi])_X$ and given as follows:

$$(D\eta[\xi])_X = \lim_{t \to 0} \left( \eta_{X + t \xi} - \eta_X \right) / t \quad \text{[Mey11, Section 3.5].}$$

In the following Proposition 8, we present the Koszul formulas associated with each simple manifold in Table 1. These formulas will be used in deriving Riemannian Hessians under the quotient geometries given in Sections 4.2.1 and 4.2.2.

**Proposition 8** (Koszul Formulas Associated with $\mathbb{R}^{p \times r}$, $\text{St}(r, p)$ and $\text{S}_+(r)$). Consider manifolds $\mathbb{R}^{p \times r}$, $\text{St}(r, p)$ and $\text{S}_+(r)$ endowed with metrics $g$ given in Table 1. For vector fields $\xi, \eta, \theta$ on $\mathbb{R}^{p \times r}$ or $\text{St}(r, p)$ or $\text{S}_+(r)$, we have the following Koszul formulas:

$$\mathbb{R}^{p \times r} : 2g_Y(\nabla_\xi \eta, \theta) = 2g_Y(D\eta[\xi], \theta) + \text{tr}(DW_Y[\xi] \eta^T \theta) - \text{tr}(DW_Y[\eta] \xi^T \theta),$$

$$\text{St}(r, p) : 2g_U(\nabla_\xi \eta, \theta) = 2g_U(D\eta[\xi], \theta),$$

$$\text{S}_+(r) : 2g_B(\nabla_\xi \eta, \theta) = 2g_B(D\eta[\xi], \theta) - \text{tr}(B^{-1} \xi B^{-1} \eta B^{-1} \theta) - \text{tr}(B^{-1} \eta B^{-1} \xi B^{-1} \theta). \quad (73)$$

Here, the operation on vector fields is performed elementwise on the associated tangent vectors. For example, $\eta^T \theta$ is mapping that assign each point $Y \in \mathbb{R}^{p \times r}$ to the matrix $\eta^T Y\theta$, i.e., $(\eta^T \theta)_Y = \eta^T Y\theta$.

**Proof of Proposition 8** The Koszul formulas associated with $\text{St}(r, p)$ and $\text{S}_+(r)$ can be found in [Mey11, Section 3.5.1] and [Mey11, Appendix B], respectively. Next, we derive the Koszul formula associated with the $\mathbb{R}^{p \times r}$ by using the general formula (71). First, for any vector fields $\eta, \theta, \xi$ on $\mathbb{R}^{p \times r}$, we have

$$(Dg_Y(\eta, \theta)[\xi])_{Y'} = \lim_{t \to 0} \frac{\text{tr}(W_{Y + t \xi, Y', t \xi, Y'} - \text{tr}(W_{Y Y', t \xi, Y'}))}{t}$$

$$= \lim_{t \to 0} \frac{\text{tr}(W_{Y + t \xi, Y', t \xi, Y'} - \text{tr}(W_{Y Y', t \xi, Y'}))}{t}$$

$$+ g_Y(D\eta[\xi], \theta_{Y'}) + g_Y(D\eta[\xi], \theta_{Y'}) + g_Y(\eta_{Y'}, D\eta[\xi]) \quad \text{[71].}$$

This proves

$$Dg_Y(\eta, \theta)[\xi] = \text{tr}(DW_Y[\xi]\eta^T \theta) + g_Y(D\eta[\xi], \theta) + g_Y(\eta, D\theta[\xi]). \quad (75)$$

Then by (71) and the fact $\mathbb{R}^{p \times r}$ is an open subset of $\mathbb{R}^{p \times r}$, we have

$$2g_Y(\nabla_\xi \eta, \theta) = Dg_Y(\eta, \theta)[\xi] + Dg_Y(\xi, \theta)[\eta] - Dg_Y(\eta, \xi)[\theta] + g_Y(\theta, [\xi, \eta]) + g_Y(\eta, \theta, [\xi]) - g_Y(\xi, \theta, [\eta]). \quad \text{[74].}$$

$$2g_Y(\nabla_\xi \eta, \theta) = \text{tr}(DW_Y[\xi]\eta^T \theta) + g_Y(D\eta[\xi], \theta) + g_Y(D\eta[\xi], D\eta[\xi]) + g_Y(D\eta[\xi], \theta) + g_Y(\theta, D\eta[\xi]) - g_Y(\xi, D\eta[\xi]) \quad \text{[76].}$$

Then by (71) and the fact $\mathbb{R}^{p \times r}$ is an open subset of $\mathbb{R}^{p \times r}$, we have

$$2g_Y(\nabla_\xi \eta, \theta) = 2g_Y(D\eta[\xi], \theta) + 2g_Y(DW_Y[\xi]\eta^T \theta) - \text{tr}(DW_Y[\eta]\eta^T \theta) - \text{tr}(DW_Y[\xi]\eta^T \theta). \quad \text{[76].}$$
B Additional Proofs

B.1 Additional Proofs in Section 3

Proof of Lemma 2: Since \( \bar{l} \) is a diffeomorphism between \( \mathcal{M}^0 \) and \( \mathcal{M}^\ell \), \( \bar{l} \) and \( \bar{l}^{-1} \) are both open maps [Lee13, Proposition 2.15]. If \( X \) is a minimizer of \( \Phi \) in a neighborhood of \( X \) say \( \mathcal{N}_1 \), \( \bar{l}^{-1} \) maps \( \mathcal{N}_1 \) to an open neighborhood of \( 
abla^{-1}(X) \) say \( \mathcal{N}_2 \), and \( \bar{l}^{-1}(X) \) is a minimizer of \( \Phi \) for all \( \{Z\} \in \mathcal{N}_2 \). This implies \( \bar{l}^{-1}(X) \) is a local minimizer of \( \Phi \). This argument also applies to another direction. 

B.2 Additional Proofs in Section 4

Proof of Lemma 4: It has been shown in [JBA10, Section 4] that the vertical space of \( \mathcal{M}^{q_1}_r \) is \( \{\theta_Y: \theta_Y = Y\mathbf{\Omega}, \mathbf{\Omega} = -\mathbf{\Omega}^T\} \) with dimension \((r^2 - r)/2\). By observing \( Y = UP \), it is easy to check

\[
\{\theta_Y: \theta_Y = YU^T, \mathbf{\Omega}^T = -\mathbf{\Omega}^T\} = \{\theta_Y: \theta_Y = YP^{-1}\mathbf{\Omega}^T, \mathbf{\Omega}^T = -\mathbf{\Omega}^T\} = \{\theta_Y: \theta_Y = Y\mathbf{\Omega}, \mathbf{\Omega} = -\mathbf{\Omega}^T\} = V_Y\mathcal{M}^{q_1}_r.
\]

In addition, notice \( \dim(\mathcal{H}_Y\mathcal{M}^{q_1}_r) = pr-(r^2-r)/2 \) and \((r^2-r)/2 + (pr-(r^2-r)/2) = pr\), then \( \mathcal{H}_Y\mathcal{M}^{q_1}_r \) is the horizontal space if we can show it is orthogonal to \( V_Y\mathcal{M}^{q_1}_r \) with respect to \( \bar{g}_Y^+ \). For \( \eta_Y = U\mathbf{\Omega}P^{-1} \in V_Y\mathcal{M}^{q_1}_r \) and \( \theta_Y = (US + U_D)P^{-1} \in \mathcal{H}_Y\mathcal{M}^{q_1}_r \),

\[
\bar{g}_Y^+(\eta_Y, \theta_Y) = \langle \Omega, SP^{-1}WP^{-1} \rangle = -\langle \Omega, SP^{-1}WP^{-1} \rangle = -\bar{g}_Y^+(\eta_Y, \theta_Y),
\]

where (a) is because \( \Omega = -\mathbf{\Omega}^T \) and (b) is because \( SP^{-1}WP^{-1} \) is symmetric by the construction of \( \mathcal{H}_Y\mathcal{M}^{q_1}_r \). (77) implies \( \bar{g}_Y^+(\eta_Y, \theta_Y) = 0 \). This finishes the proof for the first part.

From Section 2.3, we know to show \( \mathcal{M}^{q_1}_r \) is a Riemannian quotient manifold endowed with the Riemannian metric \( g_Y^+ \) induced from \( \bar{g}_Y^+ \); it is enough to show \( \bar{g}_Y^+(\eta_Y, \theta_Y) = \bar{g}_{YO}^+(\eta_{YO}, \theta_{YO}) \) for any \( O \in \mathcal{O}_r \), where \( \eta_{YO}, \theta_{YO} \) are the horizontal lifts of \( \eta_{[Y]} \) and \( \theta_{[Y]} \) at \( Y \) and \( YO \), respectively. By Lemma (3i), we have the the horizontal lifts of \( \eta_{[Y]} \) at \( Y \) and \( YO \) are related as \( \eta_{YO} = \eta_YO \). So

\[
\bar{g}_{YO}^+(\eta_{YO}, \theta_{YO}) = \text{tr}(WYO\eta_{YO}\theta_{YO}) = \text{tr}(WYO\theta_{YO}^T\eta_{YO}),
\]

and \( \bar{g}_{YO}^+(\eta_{YO}, \theta_{YO}) = \bar{g}_Y^+(\eta_Y, \theta_Y) \) holds for any \( \eta_Y, \theta_Y \) if and only if \( W = OWYO\mathbf{\Omega}^T \) holds for any \( O \in \mathcal{O}_r \). This shows \( \mathcal{M}^{q_1}_r \) is a Riemannian quotient manifold. 

Proof of Lemma 6: The vertical space of \( \mathcal{M}^{q_1}_r \) was given in [AAM14, Eq. (7)] and it has dimension \( r^2 \). Moreover, we can check \( \dim(\mathcal{H}(L,R)\mathcal{M}^{q_1}_r) = (p_1 + p_2 - r) \). If we can show \( \mathcal{V}_{(L,R)}\mathcal{M}^{q_1}_r \perp \mathcal{H}(L,R)\mathcal{M}^{q_1}_r \) with respect to \( \bar{g}_{(L,R)}^+ \), then it implies \( \mathcal{H}(L,R)\mathcal{M}^{q_1}_r \) is a valid horizontal space choice.

Suppose \( \theta_{(L,R)} = \begin{bmatrix} USP_2^{-1} - VS^T P_1^{-1} \end{bmatrix} \in \mathcal{V}_{(L,R)}\mathcal{M}^{q_1}_r \) and \( \eta_{(L,R)} = \begin{bmatrix} (USP_2^{-1}W_{-1}L^{-1}P_2^{-1} + U_D)P_1^{-1} \,(VS^T P_1^{-1}L^{-1}P_1^{-1} + V_D)P_1^{-1} \end{bmatrix} \in \mathcal{H}(L,R)\mathcal{M}^{q_1}_r \), then by some simple calculations we have \( \bar{g}_{(L,R)}^+(\eta_{(L,R)}, \theta_{(L,R)}) = \text{tr}(S^TS) - \text{tr}(S^TS) = 0 \).

To show \( \mathcal{M}^{q_1}_r \) is a Riemannian quotient manifold endowed with the Riemannian metric \( g_{(L,R)}^+ \), it is enough to show \( \bar{g}_{(L,R)}^+(\eta_{(L,R)}, \theta_{(L,R)}) = \bar{g}_{(LM,RM^{-1})}^+(\eta_{(LM,RM^{-1})}, \theta_{(LM,RM^{-1})}) \) for any \( M \in \text{GL}(r) \), where \( \eta_{(L,R)} \), \( \theta_{(L,R)} \) and \( \eta_{(LM,RM^{-1})} \), \( \theta_{(LM,RM^{-1})} \) are the horizontal lifts of \( \eta_{(L,R)} \) and \( \theta_{(L,R)} \) at \( (L,R) \) and \( (LM, RM^{-1}) \), respectively. By Lemma (3ii), we have the horizontal lifts of
\[ \theta_{[LR]} (\theta_{[LM, RM^{-\top}}) \text{ are related as } \theta_{[LM, RM^{-\top}} = [(\theta_{LM})^T (\theta_{RM^{-\top}})^T]^T \text{ given } \theta_{[LR]} = [(\theta_L^T \theta_R^T)^T]^T. \]

So
\[ \bar{g}_{[LM, RM^{-\top}}(\eta_{[LM, RM^{-\top}}), \theta_{[LM, RM^{-\top}}) \]
\[ = \text{tr} \left( W_{LM, RM^{-\top}}(\eta_{[LM, RM^{-\top}})\theta_{LM}) + \text{tr} (V_{LM, RM^{-\top}}(\eta_{RM^{-\top}})\theta_{RM^{-\top}}), \right) \]

and it is equal to \( \bar{g}_{[LR]}(\eta_{[LR]}, \theta_{[LR]}) \) for any \( \eta_{[LR]}, \theta_{[LR]} \) if and only if \( W_{LR} \) and \( V_{LR} \) satisfy the assumption given in the lemma. This shows \( M_{\theta}^q \) is a Riemannian quotient manifold endowed with the Riemannian metric \( g_{[LR]} \). This finishes the proof. \( \blacksquare \)

**Proof of Lemma 7.** The vertical space of \( M_{\theta}^q \) was given in [MMB13] Eq. (10)]. Distinct from [MMB13], here the horizontal space in [iS] is chosen in a non-canonical way, i.e., it is not orthogonal to \( V_{(U, B, V)} \). Moreover, for \( \eta_{(U, B, V)} \in V_{(U, B, V)}, \theta_{(U, B, V)} \in H_{(U, B, V)}, \) we have
\[ \eta_{(U, B, V)} + \theta_{(U, B, V)} \in T_{(U, B, V)} M^2. \]

Finally, given \( \xi_{(U, B, V)} = \left[ (\Omega^T + U_\perp) \left( V_{\perp} (\Omega^T + U_\perp) \right)^T \right] \), we can find \( \eta_{(U, B, V)} = \left[ (\Omega_1^T + \Omega_1^T) \left( V_{\perp} (\Omega^T + U_\perp) \right)^T \right] \in V_{(U, B, V)} M_{\theta}^q, \theta_{(U, B, V)} = \left[ (\Omega_1^T + \Omega_1^T) \left( V_{\perp} (\Omega^T + U_\perp) \right)^T \right] \in H_{(U, B, V)}, M_{\theta}^q \) such that \( \eta_{(U, B, V)} + \theta_{(U, B, V)} = \xi_{(U, B, V)}, \) where \( \Omega_1, \Omega_2, \theta, \eta \) are uniquely determined by the equation system \( \Omega_1 + \Omega_2 = \Omega, \theta = S - (\Omega_1 - \Omega_1^T), D_1 = D_1^T, D_2 = D_2^T \). This proves \( T_{(U, B, V)} M_{\theta}^q = V_{(U, B, V)} M_{\theta}^q \cup H_{(U, B, V)} M_{\theta}^q \).

The claim \( M_{\theta}^q \) endowed with metric \( g_{[U, B, V]} \) induced from \( \bar{g}_{[LR]} \) is a Riemannian quotient manifold has appeared in [MMB13] Section 3] without a proof, here we provide the proof for completeness. The result follows if we can show
\[ \bar{g}_{[U, B, V]}(\eta_{(U, B, V)}, \theta_{(U, B, V)}) = \bar{g}_{[U_0, O^T BO, VO]}(\eta_{U_0, O^T BO, VO}, \theta_{U_0, O^T BO, VO}) \]
holds for any \( U \in O_\perp, \) where \( \eta_{(U, B, V)}, \theta_{(U, B, V)} \) and \( \theta_{U_0, O^T BO, VO}, \theta_{U_0, O^T BO, VO} \) are the horizontal lifts of \( \eta_{(U, B, V)} \) and \( \theta_{(U, B, V)} \) at \( (U, B, V) \) and \( (U_0, O^T BO, VO) \), respectively. By Lemma [iv], we have the horizontal lifts of \( \theta_{(U, B, V)} \) at \( (U, B, V) \) and \( (U_0, O^T BO, VO) \) are related as \( \theta_{U_0, O^T BO, VO} = \left[ (\theta_U O)^T (O^T \theta_B O)^T (\theta_V O)^T \right] \text{ given } \theta_{(U, B, V)} = \left[ \theta_U^T \theta_B^T \theta_V^T \right]^T. \)

So
\[ \bar{g}_{[U_0, O^T BO, VO]}(\eta_{U_0, O^T BO, VO}, \theta_{U_0, O^T BO, VO}) \]
\[ = \text{tr}(\eta_U^T \theta_U O) + \text{tr}((O^T BO)^{-1} O^T \eta_B O^{-1} O^T \theta_B O) + \text{tr}(O^T \eta_V \theta_V O) \]
\[ = \text{tr}(\eta_U^T \theta_U + \text{tr}(B^{-1} \eta_B B^{-1} \theta_B) + \text{tr}(\eta_V^T \theta_V) = \bar{g}_{[U, B, V]}(\eta_{(U, B, V)}, \theta_{(U, B, V)}). \]

This shows \( M_{\theta}^q \) endowed with the metric \( g_{[U, B, V]} \) is a Riemannian quotient manifold. This finishes the proof of this lemma. \( \blacksquare \)

**Proof of Lemma 8.** The vertical space of \( M_{\theta}^q \) was given in [AAM14] Eq. (33)]. Distinct from [AAM14], here the horizontal space in [iS] is chosen in a non-canonical way, so we need to verify \( H_{(U, Y)} M_{\theta}^q \) in [iS] is a valid horizontal space choice. To show this, we need to show \( T_{(U, Y)} M_{\theta}^q = V_{(U, Y)} M_{\theta}^q \cup H_{(U, Y)} M_{\theta}^q \), where \( T_{(U, Y)} M_{\theta}^q = T_{U ST(r, p_1) \times \mathbb{R}^{P_{2x}}}. \)

First, it is easy to check \( \dim(V_{(U, Y)} M_{\theta}^q) = (r^2 - r)/2 \), \( \dim(H_{(U, Y)} M_{\theta}^q) = (p_1 + p_2 - r)r \) and \( \dim(T_{(U, Y)} M_{\theta}^q) = (p_1 + p_2 - r)r + (r^2 - r)/2 \). So \( \dim(V_{(U, Y)} M_{\theta}^q) + \dim(H_{(U, Y)} M_{\theta}^q) = \dim(T_{(U, Y)} M_{\theta}^q) \). Moreover, for \( \eta_{(U, Y)} \in V_{(U, Y)} M_{\theta}^q, \theta_{(U, Y)} \in H_{(U, Y)} M_{\theta}^q \), we have \( \eta_{(U, Y)} + \theta_{(U, Y)} = \left[ \theta_U \theta_B \theta_V \right]^T. \)
Finally, given $\xi_{(U,Y)} = \left[\left(\Omega U + U_1 D_1\right)^\top \xi_{(U,Y)}^\top\right]^\top$, we can find $\eta_{(U,Y)} = \left[\left(\Omega U + U_1 D_1\right)^\top \xi_{(U,Y)}^\top\right]^\top \in \mathcal{V}_{(U,Y)}$, $\theta_{(U,Y)} = \left[\left(\Omega U + U_1 D_1\right)^\top \xi_{(U,Y)}^\top\right]^\top \in \mathcal{H}_{(U,Y)}$, such that $\eta_{(U,Y)} + \theta_{(U,Y)} = \xi_{(U,Y)}$, where $\Omega U_1, D_1, \theta_y$ are uniquely determined as by the equation system $\Omega U_1 = \Omega, D_1 = D, \theta_y = \xi_y - \Omega \Omega_1$. This proves $T_{(U,Y)} \mathcal{M}_{q}^{q_3} = \mathcal{V}_{(U,Y)} \mathcal{M}_{q}^{q_3} \oplus \mathcal{H}_{(U,Y)}$.

To show $\mathcal{M}_{q}^{q_3}$ is a Riemannian quotient manifold, we only need to show $\bar{g}^r_{(U,Y)}(\eta_{(U,Y)}, \theta_{(U,Y)}) = \bar{g}^r_{(U,Y)}(\eta_{(U,Y)}, \theta_{(U,Y)})$ for any $O \in \mathbb{R}$, where $\eta_{(U,Y)}, \theta_{(U,Y)}$ are the horizontal lifts of $\eta_{(U,Y)}$ and $\theta_{(U,Y)}$ at $(U, Y)$ and $(U_1, Y_1)$, respectively. By Lemma 9, we have the horizontal lift of $\eta_{(U,Y)}$ at $(U, Y)$ and $(U_1, Y_1)$ are related as $\eta_{(U,Y)} = \left[\left(\eta_{U} O_{1}^\top \eta_{Y} O_{1}^\top\right)^\top\right]$ given $\eta_{(U,Y)} = \left[\left(\eta_{U} \eta_{Y}^\top\right)^\top\right]$. So
\[
\bar{g}^r_{(U,Y)}(\eta_{(U,Y)}, \theta_{(U,Y)}) = \text{tr}(O^\top \eta_{U}^\top \eta_{Y} O_{1}^\top) + \text{tr}(W Y O O^\top \eta_{Y}^\top \eta_{Y} O_{1}^\top),
\]
and it is equal to $\bar{g}^r_{(U,Y)}(\eta_{(U,Y)}, \theta_{(U,Y)})$ if and only if $W Y = O W Y O^\top$ holds for any $O \in \mathbb{R}$. This shows $\mathcal{M}_{q}^{q_3}$ is a Riemannian quotient manifold endowed with metric $\bar{g}^r_{(U,Y)}$ induced from $\bar{g}^r_{(U,Y)}$. This finishes the proof of this lemma. 

B.3 Additional Proofs in Section 5

Proof of Proposition 1. The proof is divided into four steps: in Step 1, we derive the expressions for Riemannian gradients; in Step 2, we derive the Riemannian Hessian of $\bar{g}^r_{(U,Y)}$ under the embedded geometry; in Step 3, we derive the Riemannian Hessian for $h_{r+}(\bar{Y})$; in Step 4, we derive the Riemannian Hessian for $h_{r+}(\bar{U}, B)$.

Step 1. First, the Riemannian gradient expression of $\bar{g}^r_{(U,Y)}$ under the embedded geometry can be found in [LLZ21 Proposition 1]. Next, we compute the Riemannian gradients under the quotient geometry. By definition
\[
\langle \text{grad} \bar{h}_{r+}(Y) W Y, \eta_Y \rangle = \bar{g}^r_Y(\text{grad} \bar{h}_{r+}(Y), \eta_Y) = D \bar{h}_{r+}(Y)[\eta_Y] = \langle \nabla \bar{h}_{r+}(Y), \eta_Y \rangle
\]
\[
= 2\langle \nabla f(YY^\top)Y, \eta_Y \rangle, \quad \forall \eta_Y \in T_Y \mathcal{M}_{q_{1}}, \tag{78}
\]
where $\nabla \bar{h}_{r+}(Y)$ denotes the Euclidean gradient of $\bar{h}_{r+}$ at $Y$, and
\[
\langle \text{grad}_U \bar{h}_{r+}(U, B), \eta_U \rangle + \langle B^{-1} \text{grad}_B \bar{h}_{r+}(U, B) B^{-1}, \eta_B \rangle
\]
\[
= \bar{g}^r_{\bar{U}, \bar{B}}(\text{grad} \bar{h}_{r+}(U, B), \eta_{(U,B)})
\]
\[
= D \bar{h}_{r+}(U, B)[\eta_{(U,B)}] = \langle \nabla U \bar{h}_{r+}(U, B), \eta_U \rangle + \langle \nabla_B \bar{h}_{r+}(U, B), \eta_B \rangle
\]
\[
= 2\langle \nabla f(UBU^\top)UB, \eta_U \rangle + \langle U^\top \nabla f(UBU^\top)U, \eta_B \rangle, \quad \forall \eta_{(U,B)} \in T_{(U,B)} \mathcal{M}_{q_{1}}. \tag{79}
\]
From (78) and (79), we have the following expressions for $\text{grad} \bar{h}_{r+}(Y)$ and $\text{grad} \bar{h}_{r+}(U, B)$, respectively:
\[
\text{grad} \bar{h}_{r+}(Y) = \nabla \bar{h}_{r+}(Y) W_Y^{-1} = 2\nabla f(YY^\top)YW_Y^{-1},
\]
\[
\text{grad} \bar{h}_{r+}(U, B) = \begin{bmatrix} \text{grad}_U \bar{h}_{r+}(U, B) \\ \text{grad}_B \bar{h}_{r+}(U, B) \end{bmatrix} = \begin{bmatrix} 2P_{U_{\text{srt}(r,p)}}(\nabla f(UBU^\top)UB) \\ BU^\top \nabla f(UBU^\top)UB \end{bmatrix}. \tag{80}
\]
Since $\mathcal{V}_{Y} \mathcal{M}_{q_{1}} \perp \mathcal{H}_{Y} \mathcal{M}_{q_{1}}$, by Lemma 1 we have $\text{grad} h_{r+}([Y]) = \bar{g}^r_{(U,Y)}$. At the same time
\[
\text{grad} h_{r+}([U, B]) = P^H_{(U,B)}(\text{grad} \bar{h}_{r+}(U, B)) = \begin{bmatrix} 2P_{U_{\text{srt}(r,p)}}(\nabla f(UBU^\top)UB) \\ BU^\top \nabla f(UBU^\top)UB \end{bmatrix},
\]
36
here $P^H_{(U,B)}(\cdot)$ is the projection operator onto $\mathcal{H}(U,B)\bar{M}^q_{r+}$ and it satisfies $P^H_{(U,B)}(\eta_{(U,B)}) = [(P_{U\perp}\eta_U)^\top \eta_B^\top]^\top$
for any $\eta_{(U,B)} \in T_{(U,B)}\bar{M}^q_{r+}$.

**Step 2.** Suppose $X$ has eigendecomposition $U'\Sigma'U'^\top$ with $U = U'O$ and $O \in O_r$. Then for

$$\xi_X = [U \ U_{\perp}] \begin{bmatrix} S & D^\top \end{bmatrix} [U \ U_{\perp}]^\top = [U' \ U_{\perp}] \begin{bmatrix} OS^\top & OD^\top \end{bmatrix} [U' \ U_{\perp}]^\top,$$

by [LLZ21] Proposition 2] we have in the PSD case

$$\text{Hess} f(X)[\xi_X, \xi_X] = \nabla^2 f(X)[\xi_X, \xi_X] + 2\langle \nabla f(X), U_{\perp}D^\top \Sigma'^{-1}OD^\top U_{\perp}^\top \rangle$$

where $\Sigma = O^\top \Sigma' O = U^\top XU$.

**Step 3.** We first compute

$$\text{Dgrad} h_{r+}([Y])[\theta_Y] = \text{Dgrad} \tilde{h}_{r+}(Y)[\theta_Y] \equiv \lim_{t \to 0} \left( \nabla \tilde{h}_{r+}(Y + t\theta_Y)^{-1}W^{-1}_{Y+\theta_Y} - \nabla \tilde{h}_{r+}(Y)W^{-1}_Y \right)/t$$

$$= \nabla \tilde{h}_{r+}(Y)DW^{-1}_Y[\theta_Y] + \nabla^2 \tilde{h}_{r+}(Y)[\theta_Y]W^{-1}_Y.$$

By the definition for the bilinear form of the Riemannian Hessian, we have

$$\text{Hess} h_{r+}([Y])[\theta_Y, \theta_Y] = \tilde{g}^{r+}_Y \left( \text{Hess} h_{r+}([Y])[\theta_Y], \theta_Y \right)$$

$$= \tilde{g}^{r+}_Y \left( \text{Dgrad} h_{r+}([Y]), \theta_Y \right)$$

$$= \tilde{g}^{r+}_Y \left( \text{Dgrad} h_{r+}([Y])[\theta_Y], \theta_Y \right) + \text{tr} \left( DW_Y \left[ \text{grad} h_{r+}([Y]) \right] \theta_Y^\top \theta_Y \right)/2$$

$$= \tilde{g}^{r+}_Y \left( \text{Dgrad} h_{r+}([Y])[\theta_Y], \theta_Y \right) + 2\text{tr} \left( W_Y \left( \nabla f(YY^\top)YDW^{-1}_Y[\theta_Y] \right)^\top \theta_Y \right)$$

$$+ \text{tr} \left( D_{W_Y} \left[ \text{grad} h_{r+}([Y]) \right] \theta_Y^\top \theta_Y \right)/2.$$

Finally, we have $\nabla^2 \tilde{h}_{r+}(Y)[\theta_Y, \theta_Y] = \nabla^2 f(YY^\top)[\theta_Y, \theta_Y, \theta_Y, \theta_Y, \theta_Y] + 2\langle \nabla f(YY^\top), \theta_Y \theta_Y \rangle$ from [LLZ21] Proposition 2] and plugging it into (82), we get the expression for the Riemannian Hessian of $h_{r+}([Y])$.

**Step 4.** Notice $\bar{M}^q_{r+}$ is a product space of two smooth manifolds, so the Riemannian connection on $\bar{M}^q_{r+}$ is also a product of the Riemannian connection on each individual manifold. For convenience, we denote $\tilde{g}^{r+}_U(\eta_U, \theta_U) = \text{tr}(\eta_U^\top \theta_U)$ as the $U$ part inner product of $\tilde{g}^{r+}_{(U,B)}$ and $\tilde{g}^{r+}_B(\eta_B, \theta_B) = \text{tr}(B^{-1}\eta_B B^{-1}\theta_B)$ as the $B$ part inner product of $\tilde{g}^{r+}_{(U,B)}$. We have

$$\text{Hess} h_{r+}([U,B])[\theta_{(U,B)}, \theta_{(U,B)}]$$

$$= \tilde{g}^{r+}_{(U,B)} \left( \text{Hess} h_{r+}([U,B])[\theta_{(U,B)}], \theta_{(U,B)} \right)$$

$$= \tilde{g}^{r+}_{(U,B)} \left( \text{Dgrad} h_{r+}([U,B])[\theta_{(U,B)}], \theta_{(U,B)} \right)$$

$$= \tilde{g}^{r+}_{(U,B)} \left( \text{Dgrad} h_{r+}([U,B])[\theta_{(U,B)}], \theta_{(U,B)} \right) + \tilde{g}^{r+}_B \left( \text{Dgrad} h_{r+}([U,B])[\theta_{(U,B)}], \theta_B \right) - \text{tr} \left( B^{-1}\text{Sym} \left( \theta_B B^{-1}\text{grad} h_{r+}([U,B]) \right) \right) B^{-1}\theta_B.$$
Next, we compute \( \text{Dgrad}_U h_{r+}([U, B])[\theta_{(U, B)}] \) and \( \text{Dgrad}_B h_{r+}([U, B])[\theta_{(U, B)}] \) separately in (33).

\[
\text{Dgrad}_U h_{r+}([U, B])[\theta_{(U, B)}] = 2 \lim_{t \to 0} \left( (I_p - P_{U + t\theta_U}) \nabla f \left( (U + t\theta_U)(B + t\theta_B)(U + t\theta_U)^\top \right) \right) (U + t\theta_U)(B + t\theta_B) - P_{U+} \nabla f(UBU^\top UB)/t
= - 4\text{Sym}(U\theta_U)\nabla f(UBU^\top UB) + 2P_{U+} \nabla f(UBU^\top UB)\theta_B + \nabla f(UBU^\top UB)\theta_B
+ 2P_{U+} \nabla^2 f(UBU^\top UB)[UB\theta_U + UB\theta_U + \theta_BUB]UB,
\]

and

\[
\text{Dgrad}_B h_{r+}([U, B])[\theta_{(U, B)}] = \lim_{t \to 0} \left( (B + t\theta_B)(U + t\theta_U)^\top \nabla f \left( (U + t\theta_U)(B + t\theta_B)(U + t\theta_U)^\top \right) \right) (U + t\theta_U)(B + t\theta_B)
- BU^\top \nabla f(UBU^\top UB) / t
= (\theta_B U^\top + B\theta_U)\nabla f(UBU^\top UB) + BU^\top \nabla f(UBU^\top UB)(\theta_B U + \theta_U U^\top)
+ BU^\top \nabla^2 f(UBU^\top UB)[UB\theta_U + UB\theta_U + \theta_BUB]UB.
\]

By observing \( f(X) \) is symmetric in \( X, \theta_{(U, B)} \in \mathcal{H}_{(U, B)} \mathcal{X}^{(2)}_{r+} \) and plugging (34) and (35) into (33), we have

\[
\text{Hess} h_{r+}([U, B])[\theta_{(U, B)}, \theta_{(U, B)}] = \nabla^2 f(UBU^\top UB)[UB\theta_U + UB\theta_U + \theta_BUB] + 2\langle \nabla f(UBU^\top UB), \theta_B UB \rangle
- 4\langle \text{Sym}(U\theta_U)\nabla f(UBU^\top UB), \theta_U \rangle + 2\langle \nabla f(UBU^\top UB)\theta_B, \theta_U \rangle
+ \nabla^2 f(UBU^\top UB)[UB\theta_U + UB\theta_U + \theta_BUB] + \text{tr}(B^{-1}(\theta_B U^\top + B\theta_U)\nabla f(UBU^\top UB)\theta_B)
+ \text{tr}(UBU^\top UB)\theta_B + \theta_B U U^\top \theta_B - \text{tr}(B^{-1}\text{Sym}(\theta_B B^{-1}\text{grad}_B \text{h}_{r+}([U, B])) B^{-1} \theta_B)
= \nabla^2 f(UBU^\top UB)[UB\theta_U + UB\theta_U + \theta_BUB] + \theta_B UB\theta_U] + 2\langle \nabla f(UBU^\top UB), \theta_B UB \rangle
+ \langle \nabla f(UBU^\top UB), \theta_B U U^\top \theta_B - \theta_B U U^\top \theta_B - 2UB\theta_U UB - 2UB\theta_U UB \rangle.
\]

**Proof of Theorem 4.** First, (39) is by direct calculation from the gradient expressions in Proposition 1. Next, we prove (40). Since \([U, B] \) is a Riemannian FOSP of \( h_{r+}([U', B']) \), we have

\[
\text{grad} h_{r+}([U, B]) = 0 \quad \text{and} \quad \nabla f(UBU^\top UB) = 0
\]

and have \( \text{grad} f(X) = 0 \) by (39). So \( \nabla f(X) = P_{U+} \nabla f(X) P_{U+} \). Given any \( \theta_{(U, B)} = [\theta_U \theta_B]^\top \in \mathcal{H}_{(U, B)} \mathcal{X}^{(2)}_{r+} \), we have

\[
\langle \nabla f(X), P_{U+} \theta_B BB^{-1} B\theta_U P_{U+} \rangle = \langle \nabla f(X), \theta_B B\theta_U \rangle,
\]

where the equality is because \( \nabla f(X) = P_{U+} \nabla f(X) P_{U+} \).
Then by Proposition 4,
\[
\Hess h_{r+}(\theta) = \nabla^2 f(UBU^\top)[UB\theta_U + \theta_U UB^\top, UB\theta_U + \theta_U UB^\top]
\]
\[
+ 2\langle \nabla f(UBU^\top), \theta_U UB\theta_U \rangle
\]
\[
+ \langle \nabla f(UBU^\top)U, 4\theta_U UB - UB\theta_U + 2\theta_U UB^\top \theta_U \rangle
\]
\[
\Rightarrow \nabla^2 f(UBU^\top)[UB\theta_U + \theta_U UB^\top, UB\theta_U + \theta_U UB^\top] + 2\langle \nabla f(UBU^\top), \theta_U UB\theta_U \rangle
\]
Proposition 4
\[
\nabla^2 f(x)[\mathcal{L}_{UB}^+(\theta(x)), \mathcal{L}_{UB}^+(\theta(x))] + 2\langle \nabla f(x), P_{UB} \theta_U BB^{-1}B\theta_U P_{UB} \rangle
\]
\[
= \Hess f(x)[\mathcal{L}_{UB}^+(\theta(x)), \mathcal{L}_{UB}^+(\theta(x))],
\]
where the last equality follows from the expression of $\Hess f(x)$ in [21] and the definition of $\mathcal{L}_{UB}^+$. Then, by [23], [40] and Theorem 2, we have $\Hess h_{r+}(\theta)$ has $(pr - (r^2 - r))/2$ eigenvalues and $\lambda_i(\Hess h_{r+}(\theta))$ is sandwiched between $\sigma_i^2(\theta)\lambda_i(\Hess f(x))$ and $2\sigma_i^2(\theta)\lambda_i(\Hess f(x))$ for $i = 1, \ldots, pr - (r^2 - r)/2$.

B.4 Additional Proofs in Section 6

Proof of Proposition 4. The proof is divided into five steps: in Step 1, we derive the expressions for Riemannian gradients; in Step 2, we derive the Riemannian Hessian of (2) under the embedded geometry; in Step 3, we derive the Riemannian Hessian for $h_r([L, R])$; in Step 4, we derive the Riemannian Hessian for $h_r([U, B, V])$; in Step 5, we derive the Riemannian Hessian for $h_r([U, Y])$.

Step 1. First, the Riemannian gradient expression of (2) under the embedded geometry can be found in [LLZ11] Proposition 1. Next, we compute $\grad h_r(L, R)$, $\grad h_r(U, B, V)$ and $\grad h_r(U, Y)$ from their definitions:

\[
\langle \grad_L h_r(L, R) W_{L,R}, \eta_L \rangle + \langle \grad_R h_r(L, R) V_{L,R}, \eta_R \rangle
\]
\[
= g_{LR}^r(\grad h_r(L, R), \eta_{(L,R)})
\]
\[
= D_\eta h_r(L, R) = \langle \nabla L h_r(L, R), \eta_L \rangle + \langle \nabla R h_r(L, R), \eta_R \rangle
\]
\[
= \langle \nabla f(LR^T)R, \eta_L \rangle + \langle \nabla f(LR^T) \rangle^T L, \eta_R \rangle, \quad \forall \eta_{(L,R)} \in T_{(L,R)}\mathcal{M}^q_1
\]
where $\nabla L h_r(L, R)$ denotes the Euclidean gradient of $h_r(L, R)$ with respect to $L$,

\[
\langle \grad_U h_r(U, B, V), \eta_U \rangle + \langle B^{-1} \grad_B h_r(U, B, V) B^{-1}, \eta_B \rangle + \langle \grad_V h_r(U, B, V), \eta_V \rangle
\]
\[
= g_{UBV}^r(\grad h_r(U, B, V), \eta_{(U,B,V)}) = D_\eta h_r(U, B, V)[\eta_{(U,B,V)}]
\]
\[
= \langle \nabla U h_r(U, B, V), \eta_U \rangle + \langle \nabla B h_r(U, B, V), \eta_B \rangle + \langle \nabla V h_r(U, B, V), \eta_V \rangle
\]
\[
= \langle \nabla f(UBV^T)VB, \eta_U \rangle + \langle U^T \nabla f(UBV^T)V, \eta_B \rangle + \langle \nabla f(UBV^T) \rangle^T UB, \eta_V \rangle, \quad \forall \eta_{(U,B,V)} \in T_{(U,B,V)}\mathcal{M}^{q_2}
\]
and

\[
\langle \grad_U h_r(U, Y), \eta_U \rangle + \langle \grad_Y h_r(U, Y) W_Y, \eta_Y \rangle
\]
\[
= g_{UY}^r(\grad h_r(U, Y), \eta_{(U,Y)}) = D_\eta h_r(U, Y)[\eta_{(U,Y)}]
\]
\[
= \langle \nabla U h_r(U, Y), \eta_U \rangle + \langle \nabla Y h_r(U, Y), \eta_Y \rangle
\]
\[
= \langle \nabla f(UY^T)Y, \eta_U \rangle + \langle (\nabla f(UY^T))^T U, \eta_Y \rangle, \quad \forall \eta_{(U,Y)} \in T_{(U,Y)}\mathcal{M}^{q_1}_1
\]
From \((88), (89)\) and \((90)\), we have
\[
\text{grad } \tilde{h}_r(L, R) = \left[ \nabla f(LR^\top)RW_{LR}^{-1} \right],
\]
\[
\text{grad } \tilde{h}_r(U, B, V) = \left[ P_{TV(r,p_2)}(\nabla f(UBV^\top)^\top UB) \right] B,
\]
\[
\text{grad } \tilde{h}_r(U, Y) = \left[ P_{TV(r,p_2)}(\nabla f(U^\top Y)^\top)YW_{Y}^{-1} \right].
\]

By Lemma 6, \(\nabla_{(L,R)} \tilde{M}_r^{\Omega_f}\) is orthogonal to \(H_{(L,R)}^{\Omega_f}\) with respect to \(g_r^{(L,R)}\), we have \(\text{grad } h_r([L, R]) = \text{grad } \tilde{h}_r(L, R)\) by Lemma [1].

Next, we compute \(\text{grad } h_r([U, B, V])\). Given \(\eta_{(U,B,V)} = [(U\Omega_1 + U\bot D_1)^\top S (V\Omega_2 + V_\bot D_2)^\top] \in T_{(U,B,V)}\tilde{M}_r^{\Omega_f}\), and suppose \(P_{H}^{\Omega_f}(\eta_{(U,B,V)}) = [(U\Omega + U\bot D_1)^\top \tilde{S} (-V\Omega + V_\bot D_2)^\top] \in H_{(U,B,V)}\tilde{M}_r^{\Omega_f}\). By definition
\[
\tilde{\Omega} = \arg \min_{\Omega'} \tilde{g}_r(U,B,V) \left( \left[ \begin{array}{c} U(\Omega_1 - \Omega') + U\bot (D_1 - D_1') \\ V(\Omega_2 + \Omega') + V\bot (D_2 - D_2') \end{array} \right] \right)
\]
\[
\tilde{\Omega} = \arg \min_{\Omega'} \|\Omega_1 - \Omega'\|_F^2 + \|\Omega_2 + \Omega'\|_F^2 + \sum_{i=1}^2 \|D_i - D_i'\|_F^2 + \|B^{-1/2}(S - S')B^{-1/2}\|_F^2.
\]

So we have \((\tilde{\Omega}, \tilde{S}, \tilde{D}_1, \tilde{D}_2) = ((\Omega_1 - \Omega_2)/2, S, D_1, D_2)\), i.e.,
\[
P_{H}^{\Omega_f}(\eta_{(U,B,V)}) = \left[ P_{U\bot} \eta_U + U(U^\top \eta_U - V^\top \eta_V)/2 \right], \quad \eta_{(U,B,V)} \in T_{(U,B,V)}\tilde{M}_r^{\Omega_f}.
\]

Then recall the definition of the projection operator \(P_{TV(r,p_2)}\) from Table 1 we have
\[
\text{grad } h_r([U, B, V]) = P_{TV(r,p_1)}(\text{grad } \tilde{h}_r(U, B, V)) = \left[ P_{U\bot} \nabla f(UBV^\top)VB + U\Omega \right.
\]
\[
\left. + B\text{Sym}(U^\top \nabla f(UBV^\top)VB) \right),
\]

where
\[
\Omega = \left( \text{Skew } (U^\top \nabla f(UBV^\top)VB) - \text{Skew } (V^\top \nabla f(UBV^\top)UB) \right)/2
\]
\[
= \text{Skew}(U^\top \nabla f(UBV^\top)VB)/2 + B\text{Skew}(U^\top \nabla f(UBV^\top)VB)/2.
\]

Finally,
\[
\text{grad } h_r([U, Y]) = P_{TV(r,p_1)}(\text{grad } \tilde{h}_r(U, Y)) = \left[ P_{U\bot} P_{TV(r,p_1)}(\nabla f(U^\top Y)^\top Y) \right]
\]
\[
\left. \left( \nabla f(U^\top Y)^\top UY_{Y}^{-1} \right) \right).
\]
Step 2. Suppose $X$ has SVD $U'S'V'$ with $U = U'O_1$, $V = V'O_2$ ($O_1, O_2 \in \mathbb{O}_r$). Then for

$$
\xi_X = \left[ U \quad U_\perp \right] \begin{bmatrix} S & D_2 \nabla f(X) \end{bmatrix} \left[ V \quad V_\perp \right]^T = \begin{bmatrix} O_1 & O_1D_2 \end{bmatrix} \left[ O_1 & O_1D_2 \right]^T,
$$

by [LLZ21] Proposition 2 we have in the general case

$$
\text{Hess}f(X)[\xi_X, \xi_X] = \nabla^2 f(X)[\xi_X, \xi_X] + 2\langle \nabla f(X), U_\perp D_1 O_1^T \Sigma^{-1} O_1 D_2 V_\perp \rangle
$$

$$
= \nabla^2 f(X)[\xi_X, \xi_X] + 2\langle \nabla f(X), U_\perp D_1 \Sigma^{-1} D_2 V_\perp \rangle,
$$

where $\Sigma = O_1^T \Sigma' O_2 = U^T X V$.

Step 3. For convenience, we denote $\bar{g}_L(\eta_L, \theta_L) = \text{tr}(W_{L,R}^T \eta_L^T \theta_L)$ as the $L$ part inner product of $\bar{g}_L(\eta_L, \theta_L)$ and $\bar{g}_R(\eta_R, \theta_R) = \text{tr}(V_{L,R}^T \eta_R^T \theta_R)$ as the $R$ part inner product of $\bar{g}_L(\eta_L, \theta_L)$. First following similar arguments as in (74) and (76), for three vector fields $\eta, \theta, \xi$ on $M^q$, we have

$$
D\bar{g}_L(\eta_L, \theta_R)[\xi] = \text{tr}(DW_{L,R}[\xi]\eta_L^T \theta_L) + \text{tr}(DV_{L,R}[\xi]\eta_R^T \theta_R) + \bar{g}_L(\eta_L \theta_L) + \bar{g}_R(\eta, D_\theta[\xi])
$$

and

$$
2\bar{g}_L(\nabla \eta, \theta) = 2\bar{g}_L(\partial_\eta)[\xi] \theta_L + \text{tr}(DW_{L,R}[\xi]\eta_L^T \theta_L) + \text{tr}(DV_{L,R}[\xi]\eta_R^T \theta_R) + \text{tr}(DW_{L,R}[\eta] \xi_L^T \theta_L)
$$

+ \text{tr}(DV_{L,R}[\eta] \xi_R^T \theta_R) - \text{tr}(DV_{L,R}[\eta] \xi_L^T \theta_L) - \text{tr}(DV_{L,R}[\theta] \eta_R^T \xi_R).
$$

Then

$$
\text{Hess} h_r([L, R])[\theta(L,R), \theta(L,R)]
$$

$$
= \bar{g}_L(\nabla h_r([L, R])[\theta(L,R), \theta(L,R)])
$$

$$
= \bar{g}_L([L, R]) \left( \text{D} h_r([L, R])[\theta(L,R), \theta(L,R)] \right)
$$

$$
= \bar{g}_L \left( D h_r([L, R])[\theta(L,R), \theta(L,R)] \right) + \bar{g}_R \left( D \text{grad} \ h_r([L, R])[\theta(L,R), \theta(L,R)] \right) + \text{tr}(DW_{L,R}[\text{grad} \ h_r([L, R])[\theta(L,R), \theta(L,R)]]) / 2 + \text{tr}(DV_{L,R}[\text{grad} \ h_r([L, R])[\theta(L,R), \theta(L,R)]) / 2.
$$

Next, we compute $D h_r([L, R])[\theta(L,R)]$.

$$
D h_r([L, R])[\theta(L,R)]
$$

$$
= \lim_{t \to 0} \left( \nabla f(L + t \theta_L)(R + t \theta_R) \right) - \nabla f(LR^T)RW_{L,R}^{-1}
$$

$$
= \nabla f(LR^T) \theta_L W_{L,R}^{-1} + \nabla f(LR^T) R \text{D} \theta_L W_{L,R}^{-1}[\theta(L,R)] + \nabla f(LR^T) [\theta_R^T + \theta_L R^T] RW_{L,R}^{-1}.
$$

Similarly we have

$$
D h_r([L, R])[\theta(L,R)]
$$

$$
= \left( \nabla f(LR^T) \right)^T \theta_L V_{L,R}^{-1} + \left( \nabla f(LR^T) \right)^T LD V_{L,R}^{-1}[\theta(L,R)] + \left( \nabla^2 f(LR^T) [\theta_R^T + \theta_L R^T] \right)^T LW_{L,R}^{-1}.
$$

41
Plugging (95) and (96) into (94), we have
\[
\begin{aligned}
\text{Hess}_r([L,R])&[\theta_L, \theta_L] \\
&= \nabla^2 f(LR^\top) [\theta_L^2 L + \theta_L \theta_L^\top + 2\nabla f(LR^\top), \theta_L \theta_L^\top] \\
&+ \langle \nabla f(LR^\top) R DW_{L,L}[\theta_L], \theta_L W_{L,L} \rangle + \langle (\nabla f(LR^\top))^\top LD V_{L,L}[\theta_L], \theta_R V_{L,L} \rangle \\
&+ \text{tr}(DW_{L,L}[\text{grad}_r([L,R])]\theta_L \theta_L^\top)/2 + \text{tr}(DV_{L,L}[\text{grad}_r([L,R])]\theta_R \theta_R)/2.
\end{aligned}
\]

**Step 4.** For convenience, we denote \(g^r_{U,B,V}(\eta_U, \theta_U) = \text{tr}(\eta_U \theta_U)\) as the \(U\) part inner product of \(g^r_{U,B,V}\), \(g^r_{B}(\eta_B, \theta_B) = \text{tr}(B^{-1}\eta_B B^{-1} \theta_B)\) as the \(B\) part inner product of \(g^r_{U,B,V}\) and \(g^r_{V}(\eta_V, \theta_V) = \text{tr}(\eta_V \theta_V)\) as the \(V\) part inner product of \(g^r_{U,B,V}\). Then
\[
\begin{aligned}
\text{Hess}_r([U,B,V])&[\theta_{U,B,V}] \\
&= g^r_{U,B,V}
\left(\text{Hess}_r([U,B,V])[\theta_{U,B,V}], \theta_{U,B,V}\right) \\
&= g^r_{U,B,V}
\left(\text{Dgrad}_U h_r([U,B,V])[\theta_{U,B,V}], \theta_{U,B,V}\right) \\
&= g^r_{U,B,V}
\left(\text{Dgrad}_B h_r([U,B,V])[\theta_{U,B,V}], \theta_{U,B,V}\right) \\
&= g^r_{V}
\left(\text{Dgrad}_V h_r([U,B,V])[\theta_{U,B,V}], \theta_{U,B,V}\right) \\
&= \text{tr}(B^{-1}\text{Sym}(\theta_B B^{-1}\text{grad}_B h_r([U,B,V]) B^{-1} \theta_B) + \text{tr}(\theta_B B^{-1}\text{grad}_B h_r([U,B,V]) B^{-1} \theta_B) \\
&= \text{tr}(\theta_B B^{-1}\text{grad}_B h_r([U,B,V]) B^{-1} \theta_B) + \text{tr}(\theta_B B^{-1}\text{grad}_B h_r([U,B,V]) B^{-1} \theta_B).
\end{aligned}
\]

(97)

Recall the horizontal projection operator \(P^H_{U,B,V}\) given in (62), and let \(P^H\), \(P^H\) and \(P^H\) be the restriction of \(P^H_{U,B,V}\) to the \(U\), \(B\) and \(V\) components, respectively. Moreover, define \(U_t = U + t\theta_U, V_t = V + t\theta_V, B_t = B + t\theta_B\). By (91), we have
\[
\begin{aligned}
\Delta_U := \lim_{t\to 0} \left(\text{grad}_U h_r(U_t, B_t, V_t) - \text{grad}_U h_r(U, B, V)\right)/t \\
&= P_{U,St}(r,p_1) \left(\nabla f(UBV^\top) \theta_B + \nabla f(UBV^\top) \theta_B + \nabla^2 f(UBV^\top) \theta_B + \nabla f(UBV^\top) \theta_B + \nabla^2 f(UBV^\top) \theta_B\right) \\
&\quad - 2\text{Sym}(\theta_B) \nabla f(UBV^\top) \theta_B + \theta_B \text{Skew}(U_t \nabla f(UBV^\top) \theta_B) + \text{Skew}(U_t \nabla f(UBV^\top) \theta_B).
\end{aligned}
\]
\[
\begin{aligned}
\Delta_V := \lim_{t\to 0} \left(\text{grad}_V h_r(U_t, B_t, V_t) - \text{grad}_V h_r(U, B, V)\right)/t \\
&= P_{V,St}(r,p_2) \left((\nabla f(UBV^\top))^\top \theta_B + (\nabla f(UBV^\top))^\top \theta_B\right) \\
&\quad + (\nabla^2 f(UBV^\top) \theta_B + \nabla f(UBV^\top) \theta_B) \theta_B + (\nabla^2 f(UBV^\top) \theta_B + \nabla f(UBV^\top) \theta_B) \theta_B \\
&\quad - 2\text{Sym}(\theta_B) (\nabla f(UBV^\top))^\top \theta_B \\
&\quad + \theta_B \text{Skew} (V_t \nabla f(UBV^\top) \theta_B) + \theta_B \text{Skew} (V_t \nabla f(UBV^\top) \theta_B).
\end{aligned}
\]

(98)

Then
\[
\begin{aligned}
\text{Dgrad}_U h_r([U,B,V])[\theta_{U,B,V}] \\
&= \lim_{t\to 0} \left(P^H_{U,\theta_U} \left(\text{grad}_U h_r(U_t, B_t, V_t)\right) - P^H_{U,\theta_U} \left(\text{grad}_U h_r(U, B, V)\right)\right)/t \\
&= P^H_{U,\theta_U} \left(\lim_{t\to 0} \left(\text{grad}_U h_r(U_t, B_t, V_t) - \text{grad}_U h_r(U, B, V)\right)/t\right) \\
&= - (\theta_U V^\top + U \theta_U^\top) \text{grad}_V h_r(U, B, V)/2 \\
&\Rightarrow P^H_{U}(\Delta_U) - \text{Sym}(\theta_U^\top) \text{grad}_U h_r(U, B, V) - (\theta_U V^\top + U \theta_U^\top) \text{grad}_V h_r(U, B, V)/2.
\end{aligned}
\]

(99)
\[ \text{and similarly} \]
\[
\text{D}_{\nabla \theta_{U,B,V}} \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V) = \lim_{t \to 0} \left( \frac{P^H_{\nabla\theta_{U,B,V}}(\nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V)) - P^H_{\nabla\theta_{U,B,V}}(\nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V))}{t} \right)
\]
\[= P^H_{\nabla\theta_{U,B,V}} \left( \lim_{t \to 0} \left( \nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) - \nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \right)/t \right) - \text{Sym}(\nabla \theta_{U,B,V}) \nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) / 2
\]
\[= P^H_{\nabla\theta_{U,B,V}}(\nabla \theta_{U,B,V}) \nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) / 2. \quad (100)\]

In addition
\[
\text{D}_{\partial \nabla_{\theta_{U,B,V}}} \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V) = \lim_{t \to 0} \left( \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) - \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \right)/t
\]
\[= \partial \nabla_{\theta_{U,B,V}} \left( \text{Sym}(\nabla \theta_{U,B,V}) + \nabla \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \right) + \text{Sym}(\nabla \theta_{U,B,V}) \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V) \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) / 2 \quad (101)\]
\[= \partial \nabla_{\theta_{U,B,V}}(\nabla \theta_{U,B,V}) \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) / 2. \]

Let \( \Delta = \nabla \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V), \Delta' = \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V), \) and \( \Delta'' = \nabla \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V). \) By (99), (100), and (101) and observing the fact \( \theta_{U,B,V} \in \mathcal{H}(U,B,V) \mathcal{M}_{1,2}^2, \) we have
\[
\text{By} (\nabla \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V)) \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \]
\[= \text{Sym}(\nabla \theta_{U,B,V}) + \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \]
\[= \partial \nabla_{\theta_{U,B,V}}(\nabla \theta_{U,B,V}) \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) / 2 \quad (102)\]
and similarly
\[
\text{By} (\nabla \partial \nabla_{\theta_{U,B,V}} \mathcal{H}(U,B,V)) \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \]
\[= \text{Sym}(\nabla \theta_{U,B,V}) + \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) \]
\[= \partial \nabla_{\theta_{U,B,V}}(\nabla \theta_{U,B,V}) \partial \nabla_{\theta_{U,B,V}} h_r(U,B,V) / 2 \quad (103)\]
By plugging (102), (103) and (104) into (97), we finally have

\[
\begin{align*}
\text{Hess } h_r([\mathbf{U}, \mathbf{V}]) &\big|_{\theta(U,B,V)} - \text{tr} \left( \mathbf{B}^{-1} \text{Sym} \left( \mathbf{B}^{-1} \text{grad}_B h_r([\mathbf{U}, \mathbf{V}]) \mathbf{B}^{-1} \right) \right) \\
&= \nabla^2 f(\mathbf{UBV}^T)[\theta_U, \mathbf{B} \mathbf{V}^T + \mathbf{U} \theta_B \mathbf{V}^T + \mathbf{UB} \theta_V, \mathbf{U} \theta_B \mathbf{V}^T] + 2\langle \Delta, \mathbf{B} \theta_B^{-1} \theta_B \rangle \\
&\quad + \langle \Delta', \theta_B \rangle + \langle \Delta'', \theta_B \rangle - \langle \Delta, \mathbf{B} \theta_B^{-1} \theta_B \rangle \\
&= \nabla^2 f(\mathbf{UBV}^T)[\theta_U, \mathbf{B} \mathbf{V}^T + \mathbf{U} \theta_B \mathbf{V}^T + \mathbf{UB} \theta_V, \mathbf{U} \theta_B \mathbf{V}^T] + \langle \Delta, \mathbf{B} \theta_B^{-1} \theta_B \rangle + \langle \Delta', \theta_B \rangle + \langle \Delta'', \theta_B \rangle. \\
\end{align*}
\]

(104)

By plugging (102), (103) and (104) into (97), we finally have

\[
\begin{align*}
\text{Hess } h_r([\mathbf{U}, \mathbf{V}]) &\big|_{\theta(U,B,V)} - \text{tr} \left( \mathbf{B}^{-1} \text{Sym} \left( \mathbf{B}^{-1} \text{grad}_B h_r([\mathbf{U}, \mathbf{V}]) \mathbf{B}^{-1} \right) \right) \\
&= \nabla^2 f(\mathbf{UBV}^T)[\theta_U, \mathbf{B} \mathbf{V}^T + \mathbf{U} \theta_B \mathbf{V}^T + \mathbf{UB} \theta_V, \mathbf{U} \theta_B \mathbf{V}^T] + 2\langle \Delta, \mathbf{B} \theta_B^{-1} \theta_B \rangle \\
&\quad + \langle \Delta', \theta_B \rangle + \langle \Delta'', \theta_B \rangle - \langle \Delta, \mathbf{B} \theta_B^{-1} \theta_B \rangle \\
&= \nabla^2 f(\mathbf{UBV}^T)[\theta_U, \mathbf{B} \mathbf{V}^T + \mathbf{U} \theta_B \mathbf{V}^T + \mathbf{UB} \theta_V, \mathbf{U} \theta_B \mathbf{V}^T] + \langle \Delta, \mathbf{B} \theta_B^{-1} \theta_B \rangle + \langle \Delta', \theta_B \rangle + \langle \Delta'', \theta_B \rangle.
\end{align*}
\]

(105)

Step 5. For convenience, we denote \( \bar{g}_U(\eta_U, \theta_U) = \text{tr}(\eta_U^T \theta_U) \) as the \( \mathbf{U} \) part inner product of \( \bar{g}_{U,Y}(\eta_U, \theta_Y) = \text{tr}(\mathbf{W}_Y \eta_Y^T \theta_Y) \) as the \( \mathbf{Y} \) part inner product of \( \bar{g}_{U,Y} \). Then

\[
\begin{align*}
\text{Hess } h_r([\mathbf{U}, \mathbf{Y}]) &\big|_{\theta(U,Y)} - \text{tr} \left( \mathbf{B}^{-1} \text{Sym} \left( \mathbf{B}^{-1} \text{grad}_B h_r([\mathbf{U}, \mathbf{Y}]) \mathbf{B}^{-1} \right) \right) \\
&= \bar{g}_{U}(\nabla_{\theta(U,Y)} \text{grad}_B h_r([\mathbf{U}, \mathbf{Y}]) - \nabla_{\theta(U,Y)} \text{grad}_B h_r([\mathbf{U}, \mathbf{Y}]), \theta_Y) \\
&\quad + \text{tr} \left( \mathbf{D} \mathbf{W}_Y \left( \text{grad}_B h_r([\mathbf{U}, \mathbf{Y}]) \right) \theta_Y^T \theta_Y \right) / 2.
\end{align*}
\]

(106)

Next, we compute \( \text{D} \text{grad}_U h_r([\mathbf{U}, \mathbf{Y}]) \big|_{\theta(U,Y)} \) and \( \text{D} \text{grad}_Y h_r([\mathbf{U}, \mathbf{Y}]) \big|_{\theta(U,Y)} \) separately. We have

\[
\begin{align*}
\text{D} \text{grad}_U h_r([\mathbf{U}, \mathbf{Y}]) &\big|_{\theta(U,Y)} \\
&= \lim_{t \to 0} \left( \mathbf{I}_{\mathbf{U}} - \mathbf{P}_U + \mathbf{t} \theta_U \right) \nabla f \left( \left( \mathbf{U} + \mathbf{t} \theta_U \right) (\mathbf{Y} + \mathbf{t} \theta_Y)^T \right) (\mathbf{Y} + \mathbf{t} \theta_Y) - \mathbf{P}_U \nabla f(\mathbf{U} \mathbf{Y}^T) \mathbf{Y} / t \\
&= \mathbf{P}_U \nabla f(\mathbf{U} \mathbf{Y}^T) \theta_Y + \nabla^2 f(\mathbf{U} \mathbf{Y}^T)[\theta_U, \mathbf{U} \theta_Y + \mathbf{t} \mathbf{U} \theta_Y^T] \mathbf{Y} - \mathbf{U} \mathbf{U}^T \nabla f(\mathbf{U} \mathbf{Y}^T) \mathbf{Y} - \mathbf{U} \mathbf{U}^T \nabla f(\mathbf{U} \mathbf{Y}^T) \mathbf{Y}, \\
\end{align*}
\]

(107)

and

\[
\begin{align*}
\text{D} \text{grad}_Y h_r([\mathbf{U}, \mathbf{Y}]) &\big|_{\theta(U,Y)} \\
&= \lim_{t \to 0} \left( \nabla f \left( \left( \mathbf{U} + \mathbf{t} \theta_U \right) (\mathbf{Y} + \mathbf{t} \theta_Y)^T \right) \right)^T (\mathbf{U} + \mathbf{t} \theta_U) \mathbf{W}_{\theta(U,Y)}^{-1} - (\nabla f(\mathbf{U} \mathbf{Y}^T))^T \mathbf{U} \mathbf{W}_{\theta(Y)}^{-1} ) / t \\
&= (\nabla f(\mathbf{U} \mathbf{Y}^T))^T \theta_U \mathbf{W}_{\theta(Y)}^{-1} + (\nabla f(\mathbf{U} \mathbf{Y}^T))^T \mathbf{U} \mathbf{W}_{\theta(Y)}^{-1} [\theta_Y] + (\nabla^2 f(\mathbf{U} \mathbf{Y}^T)[\theta_U, \mathbf{U} \theta_Y + \mathbf{t} \mathbf{U} \theta_Y^T] \mathbf{Y})^T \mathbf{U} \mathbf{W}_{\theta(Y)}^{-1},
\end{align*}
\]

(107)
By plugging \([106]\) and \([107]\) into \([105]\) and observing the fact \(\theta_{(U,Y)} \in H_{(U,Y)}\overline{M}^{\theta_3}_{r}\), we have

\[
\text{Hess } h_r([U, Y])[\theta(U, Y), \theta(U, Y)] \\
= \nabla^2 f(UY^T)[U\theta_Y^T + \theta_U Y^T] + 2\langle \nabla f(UY^T), \theta_U \theta_Y^T \rangle - \langle U^T \nabla f(UY^T) Y, \theta_U^T \theta_U \rangle \\
+ \langle (\nabla f(UY^T))^T UD W_{Y^{-1}}[\theta_Y], \theta_Y W_{Y} + \langle D W_{Y} \overline{\text{grad}} h_r([U, Y]), \theta_Y \theta_Y \rangle/2.
\]

This finishes the proof of this proposition. 

**Proof of Theorem 7**  
First, recall \(X = UY^T\) and \(V\) spans the right singular subspace of \(X\), so \(Y\) lies in the column space of \(V\) and \(YY^T = P_Y\). Thus, \([09]\) is by direct calculation from the gradient expressions in Proposition 4.

Next, we prove \([10]\). Since \([U, Y]\) is a Riemannian FOSP of \(h_r([U', Y'])\), we have

\[
\overline{\text{grad}} h_r([U, Y]) = 0 \quad \text{and} \quad (\nabla f(UY^T))^T U = 0 \tag{108}
\]

and have \(\nabla f(X) = 0\) by \([59]\). So \(\nabla f(X) = P_{U^T} \nabla f(X) P_{V^T}\). Let \(\Sigma = U^T X V\). Given any \(\theta(U, Y) = [\theta_U^T, \theta_Y^T]^T \in H_{(U,Y)}\overline{M}^{\theta_3}_{r}\), we have

\[
\langle \nabla f(X), P_{U^T} \theta_U Y^T V^{-1} \theta_Y^T P_{V^T} \rangle = \langle \nabla f(X), \theta_U \theta_Y^T \rangle, \tag{109}
\]

where the equality is because \(Y^T V = U^T X V = \Sigma\) and \(\nabla f(X) = P_{U^T} \nabla f(X) P_{V^T}\).

Then by Proposition 4

\[
\text{Hess } h_r([U, Y])[\theta(U, Y), \theta(U, Y)] \\
= \nabla^2 f(UY^T)[U\theta_Y^T + \theta_U Y^T] + 2\langle \nabla f(UY^T), \theta_U \theta_Y^T \rangle - \langle U^T \nabla f(UY^T) Y, \theta_U^T \theta_U \rangle \\
+ \langle (\nabla f(UY^T))^T UD W_{Y^{-1}}[\theta_Y], \theta_Y W_{Y} + \langle D W_{Y} \overline{\text{grad}} h_r([U, Y]), \theta_Y \theta_Y \rangle/2.
\]

\[
\text{Hess } f(X)[L_{U,Y}(\theta(U,Y)), L_{U,Y}(\theta(U,Y))] + 2\langle \nabla f(X), P_{U^T} \theta_Y Y^T V^{-1} \theta_Y^T P_{V^T} \rangle
\]

where the last equality follows from the expression of \(\text{Hess } f(X)\) in \([12]\) and the definition of \(L_{U,Y}\).

Then, by \([52]\), \([10]\) and Theorem 2 we have \(\text{Hess } h_r([U, Y])\) has \((p_1 + p_2 - r)r\) eigenvalues and \(\lambda_i(\text{Hess } h_r([U, Y]))\) is sandwiched between \((\sigma_r^2(Y)^i \wedge \frac{1}{\sigma_1(W_{Y})}) \lambda_i(\text{Hess } f(X))\) and \((\sigma_r^2(Y)^i \wedge \frac{1}{\sigma_1(W_{Y})}) \lambda_i(\text{Hess } f(X))\) for \(i = 1, \ldots, (p_1 + p_2 - r)r\). 

**Proof of Corollary 2**  
Here we prove the Riemannian FOSP, SOSP and strict saddle equivalence of \([2]\) on \(M^c_r\) and \(M^{\theta_3}_r\), similar proof applies to the equivalence on \(M^c_r\) and \(M^{\theta_2}_r\) (or \(M^{\theta_3}_r\)). First, by the connection of Riemannian gradients in \([60]\), the connection of Riemannian FOSPs under two geometries clearly holds.

Suppose \([L, R]\) is a Riemannian SOSP of \(h_{r+}([L', R'])\) and let \(X = LR^T\). Given any \(\xi_X \in T_X\overline{M}^{\theta_3}_r\), we have \(\text{Hess } f(X)[\xi_X, \xi_X] \geq 0\), where the inequality is by the SOSP assumption on \([L, R]\). Combining the fact \(X\) is a Riemannian FOSP, this shows \(X = LR^T\) is a Riemannian SOSP under the embedded geometry.

Next, let us show the other direction: suppose \(X\) is a Riemannian SOSP under the embedded geometry, then there is a unique \([L, R]\) such that \(LR^T = X\) and it is a Riemannian SOSP of \(h_{r}([L', R'])\). To see this, first the uniqueness of \([L, R]\) is guaranteed by the fact \(f : (L, R) \in \overline{M}^{\theta_3}_r \to LR^T \in \overline{M}^{\theta_3}_r\) induces a diffeomorphism between \(M^{\theta_3}_r\) and \(M^{\theta_3}_r\). In addition, we have shown
\( [L, R] \) is a Riemannian FOSP of \( h_r([L', R']) \). Then by \[61\], we have for any \( \theta_{(L, R)} \in \mathcal{H}_{(L, R)}\mathcal{M}^{q_1}_{r+} \), \( \text{Hess} h_r([L, R]) [\theta_{(L, R)}, \theta_{(L, R)}] = \text{Hess} f(X) [\mathcal{L}_{L, R}^\theta(\theta_{(L, R)}), \mathcal{L}_{L, R}^\theta(\theta_{(L, R)})] \geq 0 \).

Finally, the strict saddle equivalence follows from the sandwich inequality in Theorem \[5\] and the definition of strict saddle. □

C Additional Lemmas

The following lemma provides the connection of horizontal lifts of tangent vectors at different reference points for the quotient geometries we consider.

**Lemma 9.** (i) Suppose \( \theta_{[Y]} \) is a tangent vector to the quotient manifold \( \mathcal{M}^{q_1}_{r+} \) at \( [Y] \). If \( W = OW_{YO}O^\top \) holds for any \( O \in \mathbb{O}_r \), then the horizontal lifts of \( \theta_{[Y]} \) at \( Y \) and \( YO \) for \( O \in \mathbb{O}_r \) are related as \( \theta_{YO} = \theta_YO \).

(ii) Suppose \( \theta_{[U, B]} \) is a tangent vector to the quotient manifold \( \mathcal{M}^{q_2}_{r+} \) at \( [U, B] \). The horizontal lifts of \( \theta_{[U, B]} \) at \( (U, B) \) and \( (UO, O^\top BO) \) are related as \( \theta_{(UO, O^\top BO)} = [(\theta_UO)^\top (O^\top \theta_BO)^\top]^\top \) given \( \theta_{(U, B)} = [\theta_U^\top \theta_B^\top]^\top \).

(iii) Suppose \( \theta_{[L, R]} \) is a tangent vector to the quotient manifold \( \mathcal{M}^{q_3}_{r+} \) at \( [L, R] \). If \( W_{LR} = MW_{LM, RM^{-1}M^{-1}} \) and \( V_{LR} = M^{-1}V_{LM, RM^{-1}M^{-1}} \) hold for any \( M \in \text{GL}(r) \), then the horizontal lifts of \( \theta_{[L, R]} \) at \( (L, R) \) and \( (LM, RM^{-1}) \) are related as \( \theta_{(LM, RM^{-1})} = [(\theta_LM)^\top (\theta_RM^{-1})^\top]^\top \) given \( \theta_{(L, R)} = [\theta_L^\top \theta_R^\top]^\top \).

(iv) Suppose \( \theta_{[U, B, V]} \) is a tangent vector to the quotient manifold \( \mathcal{M}^{q_4}_{r+} \) at \( [U, B, V] \). The horizontal lifts of \( \theta_{[U, B, V]} \) at \( (U, B, V) \) and \( (UO, O^\top BO, VO) \) are related as \( \theta_{(UO, O^\top BO, VO)} = [(\theta_UO)^\top (O^\top \theta_BO)^\top (O^\top \theta_VO)^\top]^\top \) given \( \theta_{(U, B, V)} = [\theta_U^\top \theta_B^\top \theta_V^\top]^\top \).

(v) Suppose \( \theta_{[U, Y]} \) is a tangent vector to the quotient manifold \( \mathcal{M}^{q_5}_{r+} \) at \( [U, Y] \). The horizontal lifts of \( \theta_{[U, Y]} \) at \( (U, Y) \) and \( (UO, YO) \) are related as \( \theta_{(UO, YO)} = [(\theta_UO)^\top (\theta_YO)^\top]^\top \) given \( \theta_{(U, Y)} = [\theta_U^\top \theta_Y^\top]^\top \).

**Proof of Lemma 9** Since the proofs for the claims (ii)(iii)(iv)(v) are similar to the proof of the first claim, for simplicity we only present the proof for the first one. Let \( l : \mathcal{M}^{q_1}_{r+} \to \mathbb{R} \) be an arbitrary smooth function and define \( \bar{l} := l \circ \pi : \mathcal{M}^{q_1}_{r+} \to \mathbb{R} \), where \( \pi \) is the quotient mapping on the manifold \( \mathcal{M}^{q_1}_{r+} \). Let \( z : Y \to YO \), so we have

\[
\bar{l}(Y) = l(z(Y)), \quad \forall Y \in \mathbb{R}^{p \times r}.
\]

By taking differential with respect to \( Y \) along direction \( \theta_Y \in \mathcal{H}_Y\mathcal{M}^{q_1}_{r+} \) on both sides of \[110\], we have

\[
D\bar{l}(Y)[\theta_Y] = Dl(z(Y))[Dz(Y)[\theta_Y]].
\]

Moreover, by chain rule and the definition of the horizontal lift of a tangent vector, we have

\[
D\bar{l}(Y)[\theta_Y] = Dl(\pi(Y))[D\pi(Y)[\theta_Y]] = Dl(\pi(Y)))[\theta_Y].
\]

In addition,

\[
Dz(Y)[\theta_Y] = \lim_{t \to 0}(z(Y + t\theta_Y) - z(Y))/t = \theta_YO.
\]
Then,
\[
\text{D}(\pi(YO))\theta_Y = \text{D}(\pi(Y))\theta_Y
\]

Here (a) is by chain rule and the definition of \( z \). Since the above equation holds for any \( l \), this implies \( D\pi(YO)\theta_Y = \theta_Y \). Finally, by Lemma 2(i), we have \( H_{YO}M_{r+}^{\eta} = \{ \theta_Y : \theta_Y = (US + U_D)P_T, D \in \mathbb{R}^{(p-r)\times r}, SP_T W_Y P_T^{-1} \in \mathbb{S}^{r \times r} \} \), where \( U \in \text{St}(r, p) \) spans the top \( r \) eigenspace of \( YY^T \), and \( P = U^TY \). Then, it holds that \( H_{YO}M_{r+}^{\eta} = \{ \theta'_{YO} : \theta'_{YO} = (US + U_D)P_T, D \in \mathbb{R}^{(p-r)\times r}, SP_T W_Y P_T^{-1} \in \mathbb{S}^{r \times r} \} \). Thus, we have \( \theta_{YO} \in H_{YO}M_{r+}^{\eta} \) given \( \theta_Y \in H_{YO}M_{r+}^{\eta} \) and conclude \( \theta_{YO} \) is the unique horizontal lift of \( \theta_Y \) at \( YO \).

**Lemma 10.** Suppose \( f : M \to \mathbb{R} \) is a smooth function defined on the Riemannian quotient manifold \( M = \hat{M}/\sim \) and \( M \) is endowed with the Riemannian metric \( \bar{g}_X \), which induces a Riemannian metric \( g[X] \) on \( M \). Then \( \text{Hess}(f([X]))\theta_X, \eta_X] = \bar{g}_X \left( \text{grad} f, \eta_X \right) \) for any \( \theta_X, \eta_X \in T_XM \).

**Proof of Lemma 10.** Suppose the vertical and horizontal spaces on \( M \) are \( V_X \hat{M} \) and \( H_X \hat{M} \) with \( V_X M \oplus H_X M = T_XM \). Let us define \( V_X' \hat{M} \) to be the subspace in \( T_XM \) that is orthogonal to \( H_X M \) with respect to \( \bar{g}_X \). In general, \( V_X' \hat{M} \) is not equal to \( V_X \hat{M} \) unless we pick the horizontal space in the canonical way.

Since \( \text{grad} f \) by definition belongs to \( T_XM \), we have for \( \theta_X, \eta_X \in H_X \hat{M} \),
\[
\text{Hess}(f([X]))\theta_X, \eta_X] = \bar{g}_X \left( \text{Hess}(f([X]))\theta_X, \eta_X \right) = \bar{g}_X \left( P_X^H \left( \text{grad} f \right), \eta_X \right)
\]

where (a) is because \( \eta_X \in H_X \hat{M} \) and \( P_X^H \) and \( P_X^{V'} \) denote the projection operators onto \( H_X \hat{M} \) and \( V_X' \hat{M} \), respectively.

**Lemma 11.** (Max-min Theorem for Eigenvalues [Bha13, Corollary III.1.2]) Suppose \( A \) is a Hermitian operator on the Hilbert space \( \mathcal{H} \) with dimension \( p \) and inner product \( g(\cdot, \cdot) \) and \( A \) has eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \). If \( \mathcal{C}_k \) denotes the set of \( k \)-dimensional subspaces of \( \mathcal{H} \), then
\[
\lambda_k = \max_{C \in \mathcal{C}_k} \min_{u \in C, u \neq 0} g(u, Au)/g(u, u).
\]

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