ERGODICITY OF STOCHASTIC REAL GINZBURG-LANDAU EQUATION DRIVEN BY $\alpha$-STABLE NOISES

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Abstract. We study the ergodicity of stochastic real Ginzburg-Landau equation driven by additive $\alpha$-stable noises, showing that as $\alpha \in (3/2, 2)$, this stochastic system admits a unique invariant measure. After establishing the existence of invariant measures by the same method as in [9], we prove that the system is strong Feller and accessible to zero. These two properties imply the ergodicity by a simple but useful criterion in [16]. To establish the strong Feller property, we need to truncate the nonlinearity and apply a gradient estimate established in [26] (or see [24] for a general version for the finite dimension systems). Because the solution has discontinuous trajectories and the nonlinearity is not Lipschitz, we can not solve a control problem to get irreducibility. Alternatively, we use a replacement, i.e., the fact that the system is accessible to zero. In section 3 we establish a maximal inequality for stochastic $\alpha$-stable convolution, which is crucial for studying the well-posedness, strong Feller property and the accessibility of the mild solution. We hope this inequality will also be useful for studying other SPDEs forced by $\alpha$-stable noises.

Keywords: Stochastic real Ginzburg-Landau equation driven by $\alpha$-stable noises, Galerkin approximation, Strong Feller property, Ergodicity, Accessibility, Stochastic $\alpha$-stable convolution, Maximal inequality.

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1. Introduction

We shall study the ergodicity of stochastic real Ginzburg-Landau equation driven by $\alpha$-stable noises on torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as follows:

\begin{equation}
\begin{aligned}
&dX - \partial_{x}X \, dt - (X - X^3) \, dt = dL_t
\end{aligned}
\end{equation}

where $X : [0, \infty) : \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R}$ and $L_t$ is some cylindrical $\alpha$-stable noises. The more details about Eq. (1.1) will be given in the next section.

For the study of invariant measures and the long-time behaviour of stochastic systems driven by $\alpha$-stable type noises, there seem only several results (cf. [33, 34, 25, 24, 17, 31]). [33, 34] studied the exponential mixing of stochastic spin systems with white $\alpha$-stable noises, while [25, 24] obtained exponential mixing for a family of semi-linear SPDEs with Lipschitz nonlinearity. [17] obtained a nice criterion for the exponential mixing of a family of SDEs forced by Lévy noises, it covers 1D SDEs driven by $\alpha$-stable noises. [31] proved the exponential mixing for a family of 2D SDEs forced by degenerate $\alpha$-stable noises with $0 < \alpha < 2$. [9] obtained the existence of invariant measures for 2D stochastic Navier-Stokes equations forced by $\alpha$-stable noises with $\alpha \in (1, 2)$.

In this paper, we shall study the ergodicity of stochastic real Ginzburg-Landau equation driven by additive $\alpha$-stable noises, showing that as $\alpha \in (3/2, 2)$, the system (1.1) admits a unique invariant measure. After establishing the existence of invariant measures
by the same method as in [9], we prove that the system is strong Feller and accessible to zero. These two properties imply the ergodicity by a simple but useful criterion in [16]. To establish the strong Feller property, we need to truncate the nonlinearity and apply a gradient estimate established in [26] (or see [24] for a general version for the finite dimension systems). Due to the non-Lipschitz nonlinearity and the discontinuous trajectories, unlike the case of SPDEs forced by Wiener noises, we can not solve a control problem to get irreducibility. Alternatively, we use a replacement, i.e., the fact that the system is accessible to zero.

SPDEs with Lévy noises have been intensively studied in recent years ([2, 1, 22, 19, 26, 33, 30, 29, 23, 11, 8, 12]), but most of them assume that the noises are square integrable. This restriction rules out the interesting $\alpha$-stable noises. The loss of the second moment of $\alpha$-stable noises makes many nice analysis tools, such as Burkholder-Davis-Gundy inequality and Da Prato-Kwapień-Zabczyk’s factorization technique ([6]), not available. Consequently, some important estimates such as maximal inequality of stochastic convolution can not be established as in Wiener noises case. Section 3 establishes this inequality using an integration by parts technique other than Da Prato-Kwapień-Zabczyk’s factorization technique, which have their own interest. This maximal inequality of stochastic $\alpha$-stable convolution is crucial for studying the well-posedness of the mild solution, strong Feller property and accessibility property of the processes. We hope that the results in this section will also be useful for studying the other SPDEs forced by $\alpha$-stable type noises ([10]).

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2. Stochastic real Ginzburg-Landau equations driven by additive $\alpha$-stable noises

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be equipped with the usual Riemannian metric, and let $d\xi$ denote the Lebesgue measure on $\mathbb{T}$. Then

$$H := \left\{ x \in L^2(\mathbb{T}, \mathbb{R}) : \int_{\mathbb{T}} x(\xi)d\xi = 0 \right\}$$

is a separable real Hilbert space with inner product

$$\langle x, y \rangle_H := \int_{\mathbb{T}} x(\xi)y(\xi)d\xi, \quad \forall \ x, y \in H.$$ 

For $x \in C^2(\mathbb{T})$, the Laplacian operator $\Delta$ is given by $\Delta x = x''$. Let $(A, \mathcal{D}(A))$ be the closure of $(-\Delta, C^2(\mathbb{T}) \cap H)$ in $H$, which is a positively definite self-adjoint operator on $H$.

Denote $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$, $\{e_k\}_{k \in \mathbb{Z}_*}$ with $e_k = e^{2\pi ik\xi}$ ($k \in \mathbb{Z}_*$) is an orthonormal basis of $H$. For each $x \in H$, it can represented by

$$x = \sum_{k \in \mathbb{Z}_*} x_k e_k; \quad x_k \in \mathbb{C}, x_{-k} = \overline{x_k}.$$  

\[\text{This section includes part of the author’s not published results in [32].}\]
Write
\[ \gamma_k := 4\pi^2|k|^2, \quad k \in \mathbb{Z}_*, \]
it is easy to check \( A\varepsilon_k = \gamma_k\varepsilon_k \) for all \( k \in \mathbb{Z}_* \) and that
\[ \|A^\sigma x\|_H^2 = \sum_{k \in \mathbb{Z}_*} |\gamma_k|^{2\sigma}|x_k|^2, \quad \sigma \geq 0, \]
provided the sum on the right hand side is finite. Denote
\[ V := \mathcal{D}(A^{1/2}), \]
it gives rise to a Hilbert space, which is densely and compactly embedded in \( H \). For each \( x \in V \) with \( x = \sum_{k \in \mathbb{Z}_*} x_k\varepsilon_k \), we have
\[ \|x\|_V^2 = \sum_{k \in \mathbb{Z}_*} \gamma_k|x_k|^2. \]

Let \( z(t) \) be a one-dimensional symmetric \( \alpha \)-stable process with \( 0 < \alpha < 2 \). Its infinitesimal generator \( A \) is given by
\begin{equation}
(2.1) \quad Af(x) := \frac{1}{C_\alpha} \int_{\mathbb{R}} \frac{f(y + x) - f(x)}{|y|^{\alpha+1}} dy, \quad f \in C^2_\alpha(\mathbb{R}),
\end{equation}
where \( C_\alpha = -\int_{\mathbb{R}} (\cos y - 1)\frac{dy}{|y|^{\alpha+1}} \); see [28]. It is well known that \( z(t) \) has the following characteristic function:
\[ \mathbb{E}[e^{i\lambda z(t)}] = e^{-t|\lambda|^{\alpha}}, \quad t \geq 0, \lambda \in \mathbb{R}. \]

We shall study the 1D stochastic real Ginzburg-Landau equation on \( \mathbb{T} \) as the following
\begin{equation}
(2.2) \quad \begin{cases}
 dX_t + [AX_t + N(X_t)]dt = dL_t, \\
 X_0 = x,
\end{cases}
\end{equation}
where
(i) The nonlinear term \( N \) is defined by
\[ N(u) = -(u - u^3), \quad u \in H. \]
(ii) \( L_t = \sum_{k \in \mathbb{Z}_*} \beta_k l_k(t)\varepsilon_k \) is a cylindrical \( \alpha \)-stable processes on \( H \) with \( \{l_k(t)\}_{k \in \mathbb{Z}_*} \) being i.i.d. 1 dimensional symmetric \( \alpha \)-stable process sequence with \( \alpha > 1 \). Moreover, there exist some \( C_1, C_2 > 0 \) so that \( C_1\gamma_k^{-\beta} \leq |\beta_k| \leq C_2\gamma_k^{-\beta} \) with \( \beta > \frac{1}{2} + \frac{1}{2\alpha} \).

**Remark 2.1.** The condition \( \beta > \frac{1}{2} + \frac{1}{2\alpha} \) in (ii) guarantees that the convolution \( Z_t \), defined by (3.1), are in \( V \).

Let \( C > 0 \) be some constant and let \( C_p > 0 \) be some constant depending on some parameter \( p \). We shall often use the following inequalities:
\begin{align*}
(2.3) \quad &\|A^{\sigma_1}x\|_H \leq C_{\sigma_1,\sigma_2}\|A^{\sigma_2}x\|_H, \quad \forall \sigma_1 \leq \sigma_2, \forall x \in H; \\
(2.4) \quad &\|A^\sigma e^{-At}\| \leq C_\sigma e^{-t}, \quad \forall \sigma > 0; \\
(2.5) \quad &\langle x, -N(x) \rangle_H \leq \frac{1}{4}, \quad \forall x \in H; \\
(2.6) \quad &\|N(x)\|_V \leq C(\|x\|_V + \|x\|_V^3), \quad \forall x \in V; \\
(2.7) \quad &\|N(x) - N(y)\|_H \leq C(1 + \|A^{\frac{1}{2}}x\|_H^2 + \|A^{\frac{1}{2}}y\|_H^2)\|x - y\|_H, \quad \forall x, y \in H.
\end{align*}
For all $\sigma \geq \frac{1}{6}$,
\begin{equation}
\|N(x) - N(y)\|_H \leq C(1 + \|A^\sigma x\|_H^2 + \|A^\sigma y\|_H^2)\|A^\sigma (x - y)\|_H, \quad \forall \ x, y \in H;
\end{equation}
\begin{equation}
\|N(x)\|_H \leq C(1 + \|A^\sigma x\|_H^3), \quad \forall \ x \in H.
\end{equation}

We shall show (2.5)-(2.9) in the appendix.

Let $E$ be a Banach space and let $T > 0$ be arbitrary. Denote by $B_b(E)$ the space of bounded measurable functions: $f : E \to \mathbb{R}$. Denote by $D([0, T]; E)$ the space of the functions $f : [0, T] \to E$ which has left limit and is right continuous. Denote by $C([0, T]; E)$ the space of the functions $f : [0, T] \to E$ which is continuous. If $f \in D([0, T]; E)$, it is said to be Càdlàg in $E$.

The main results of this paper are the following three theorems.

**Theorem 2.2.** The following statements hold:

1. For every $x \in H$ and $\omega \in \Omega$ a.s., Eq. (2.2) admits a unique mild solution $X(\omega) \in D([0, \infty); H) \cap D((0, \infty); V)$. Moreover, $X(\omega)$ has the following form:
   \[ X_t(\omega) = e^{-At}x + \int_0^t e^{-A(t-s)}N(X_s(\omega))ds + \int_0^t e^{-A(t-s)}dL_s(\omega), \quad \forall \ t > 0. \]

2. $X$ is a Markov process.

3. For every $x \in V$ and $\omega \in \Omega$ a.s., we have $X(\omega) \in D([0, \infty); V)$. Moreover, for every $T > 0$,
   \[ \sup_{0 \leq t \leq T} \|X_t(\omega)\|_V \leq C, \]
   where $C$ is some constant depending on $T, \alpha, \beta$ and $\omega$.

**Theorem 2.3.** $X$ admits at least one invariant measure. The invariant measures are supported on $V$.

**Theorem 2.4.** $X$ admits a unique invariant measure if $\alpha \in (3/2, 2)$ and $\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}$.

3. **Stochastic convolution of $\alpha$-stable noises ($Z_t$) $\ell \geq 0$**

Consider the following Ornstein-Uhlenbeck process:
\begin{equation}
Z_t = \int_0^t e^{-\gamma(t-s)}dL_s = \sum_{k \in \mathbb{Z}_+} z_k(t)e_k
\end{equation}
where
\[ z_k(t) = \int_0^t e^{-\gamma_0(t-s)}\beta_k dL_k(s), \quad \gamma_k = 4\pi^2|k|^2. \]

We shall prove the following two lemmas in this section: the first one is a maximal inequality of $(Z_t)_{\ell \geq 0}$, while the other claims that $(Z_t)_{0 \leq t \leq T}$ stays, with positive probability, in arbitrary small ball with zero center for all $T > 0$. These two lemmas will play a crucial role in proving strong Feller and accessibility for the solution of Eq. (2.2). [5] established a nice maximal inequality for the stochastic convolution of a family of Lévy noises, but these noises do not include $\alpha$-stable noises.
Lemma 3.1. Let $T > 0$ be arbitrary. For all $0 \leq \theta < \beta - \frac{1}{2\alpha}$ and all $0 < p < \alpha$, we have

\begin{equation}
E \sup_{0 \leq t \leq T} \|A^\theta Z_t\|_H^p \leq C T^{p/\alpha},
\end{equation}

where $C$ depends on $\alpha, \theta, \beta, p$.

Lemma 3.2. Let $\tilde{\theta} \in [0, \beta - \frac{1}{2\alpha})$ be arbitrary. For all $T > 0$ and $\varepsilon > 0$, we have

$$P\left( \sup_{0 \leq t \leq T} \|A^{\tilde{\theta}} Z_t\|_H \leq \varepsilon \right) > 0.$$

Proof of Lemma 3.1. We only need to show the inequality for the case $p \in (1, \alpha)$ since the case of $0 < p \leq 1$ is an immediate corollary by Hölder’s inequality.

Step 1. We claim that for all $\theta \in [0, \beta - \frac{1}{2\alpha})$, $\|A^\theta L_t\|_H$ is a right continuous submartingale such that

\begin{equation}
E\|A^\theta L_t\|_H^p \leq C_{\alpha, p} \left( \sum_{k \in \mathbb{Z}_*} |\beta_k|^{\alpha \gamma_k^\theta} \right)^{p/\alpha} t^{p/\alpha}, \quad t > 0.
\end{equation}

where $p \in (1, \alpha)$.

We first follow the argument in the proof of [26, Theorem 4.4] to show (3.3). Take a Rademacher sequence $\{r_k\}_{k \in \mathbb{Z}_*}$ in a new probability space $(\Omega', \mathcal{F}', P')$, i.e. $\{r_k\}_{k \in \mathbb{Z}_*}$ are i.i.d. with $P\{r_k = 1\} = P\{r_k = -1\} = \frac{1}{2}$. By the following Khintchine inequality: for any $p > 0$, there exists some $C(p) > 0$ such that for arbitrary real sequence $\{h_k\}_{k \in \mathbb{Z}_*}$,

\begin{align*}
\left( \sum_{k \in \mathbb{Z}_*} h_k^2 \right)^{1/2} \leq C(p) \left( E' \left[ \sum_{k \in \mathbb{Z}_*} r_k h_k \right]^p \right)^{1/p}.
\end{align*}

By this inequality, we get

\begin{equation}
E\|A^\theta L_t\|_H^p = E \left( \sum_{k \in \mathbb{Z}_*} \gamma_k^{2\theta} |\beta_k|^2 |l_k(t)|^2 \right)^{p/2} \leq C E' \left( \sum_{k \in \mathbb{Z}_*} r_k \gamma_k^\theta |\beta_k| |l_k(t)| \right)^p
\end{equation}

where $C = C^p(p)$. For any $\lambda \in \mathbb{R}$, by the fact $|r_k| = 1$ and an approximation argument similar as for getting (4.12) of [26], one has

$$E \exp \left\{ i \lambda \sum_{k \in \mathbb{Z}_*} r_k \gamma_k^\theta |\beta_k| |l_k(t)| \right\} = \exp \left\{ -|\lambda|^{\alpha} \sum_{k \in \mathbb{Z}_*} |\beta_k|^{\alpha \gamma_k^\theta} |l_k(t)| \right\}.$$

Now we use (3.2) in [26]: if $X$ is a symmetric random variable satisfying

$$E[e^{i\lambda X}] = e^{-\sigma^\alpha |\lambda|^\alpha}$$

for some $\alpha \in (0, 2)$ and any $\lambda \in \mathbb{R}$, then for all $p \in (0, \alpha)$,

$$E|X|^p = C(\alpha, p) \sigma^p.$$

Since $\sum_{k \in \mathbb{Z}_*} |\beta_k|^{\alpha \gamma_k^\theta} < \infty$, (3.3) holds.
Let us now show that \( \|A^\theta Z_t\|_H \) is a right continuous submartingale. Denote \( L^\alpha_t = \sum_{|k| \leq n} \beta_k l_k(t) e_k \) for all \( n \in \mathbb{N} \) and \( t > 0 \), it is clear that \( A^\theta L^\alpha_t \) is an \( L^p \) martingale with \( p \in (1, \alpha) \). Therefore, \( \|A^\theta L^\alpha_t\|_H \) is a submartingale, i.e.,

\[
\mathbb{E}[\|A^\theta L^\alpha_t\|_H | F_s] \geq \|A^\theta L^\alpha_s\|_H, \quad t > s.
\]

Thanks to the inequality (3.3) with some \( p > 1 \), let \( n \to \infty \), we get

\[
\mathbb{E}[\|A^\theta L_t\|_H | F_s] \geq \|A^\theta L_s\|_H,
\]

i.e., \( \|A^\theta Z_t\|_H \) is a submartingale. Observe that \( A^\theta L_t = \sum_{k \in \mathbb{Z}^+} \gamma^\theta_k \beta_k l_k(t) e_k \). Since

\[
\sum_{k \in \mathbb{Z}^+} \gamma^\theta_k \beta_k |^\alpha < \infty,
\]

it follows from [18, Theorem 2.2] that \( A^\theta L_t \) has a Càdlàg version. Hence, \( \|A^\theta L_t\|_H \) has a right continuous version.

**Step 2.** It follows from Ito’s product formula ([3, Theorem 4.4.13]) that for all \( k \in \mathbb{Z}^+ \),

\[
l_k(t) = \int_0^t \gamma_k e^{-\gamma_k (t-s)} l_k(s) ds + \int_0^t e^{-\gamma_k (t-s)} dl_k(s) + \int_0^t \gamma_k e^{-\gamma_k (t-s)} \Delta l_k(s) ds
\]

where \( \Delta l_k(s) = l_k(s) - l_k(s-) \). Since \( l_k(t) \) is an \( \alpha \)-stable process, \( \Delta l_k(s) = 0 \) for \( s \in [0, t] \) a.s. and thus

\[
\int_0^t \gamma_k e^{-\gamma_k (t-s)} \Delta l_k(s) ds = 0.
\]

Therefore,

\[
(3.5) \quad z_k(t) = \beta_k l_k(t) - \int_0^t \gamma_k e^{-\gamma_k (t-s)} \beta_k l_k(s) ds
\]

Hence, we get

\[
(3.6) \quad Z_t = L_t - Y_t,
\]

where

\[
(3.7) \quad Y_t = \int_0^t A e^{-A(t-s)} L_s ds.
\]

By Step 1 and Doob’s martingale inequality, we have

\[
(3.8) \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|A^\theta L_t\|_H^p \right] \leq \left( \frac{p}{p - 1} \right)^p \mathbb{E} \left[ \|A^\theta L_T\|_H^p \right] \leq CT^{\frac{p}{2}} \| e \in [0, \beta - \frac{1}{2\alpha}], \theta \in [0, \beta - \frac{1}{2\alpha}], \alpha, \beta \text{ and } p \text{. Choose some } 0 < \varepsilon < \min \{ \beta - \frac{1}{2\alpha}, \theta, 1 \}, \text{ we have}
\]

\[
\sup_{0 \leq t \leq T} \|A^\theta Y_t\|_H \leq \sup_{0 \leq t \leq T} \int_0^t \|A^{1-\varepsilon} e^{-A(t-s)} \| \|A^{\theta+\varepsilon} L_s\|_H ds
\]

\[
\leq \sup_{0 \leq t \leq T} \|A^{\theta+\varepsilon} L_t\|_H \sup_{0 \leq t \leq T} \int_0^t \|A^{1-\varepsilon} e^{-A(t-s)} \| ds.
\]

If \( T \leq 1 \), using (2.4) we get

\[
\sup_{0 \leq t \leq T} \int_0^t \|A^{1-\varepsilon} e^{-A(t-s)} \| ds \leq C \sup_{0 \leq t \leq T} \int_0^t (t - s)^{-1+\varepsilon} ds \leq C/\varepsilon.
\]
If $T > 1$, using (2.4) again we get
\[
\sup_{0 \leq t \leq T} \int_t^1 \| A^{1-\varepsilon} e^{-A(t-s)} \| ds \leq \sup_{0 \leq t \leq 1} \int_t^1 \| A^{1-\varepsilon} e^{-A(t-s)} \| ds + \sup_{1 \leq t \leq T} \int_t^1 \| A^{1-\varepsilon} e^{-A(t-s)} \| ds
\]
\[
\leq C/\varepsilon + \sup_{1 \leq t \leq T} \int_{t-1}^t \| A^{1-\varepsilon} e^{-A} \| e^{-A(t-1-s)} \| ds + \sup_{1 \leq t \leq T} \int_{t-1}^1 \| A^{1-\varepsilon} e^{-A} \| e^{-A(t-1-s)} \| ds
\]
\[
\leq 2C/\varepsilon + \| A^{1-\varepsilon} e^{-A} \| \sup_{1 \leq t \leq T} \int_{t-1}^1 e^{-2\pi(t-1-s)} ds,
\]
where the last inequality is by the spectral property of $A$. Collecting above three inequalities, we have
\[
(3.10) \quad \sup_{0 \leq t \leq T} \| A^\theta Y_t \|_H \leq C_\varepsilon \sup_{0 \leq t \leq T} \| A^{\theta + \varepsilon} L_t \|_H.
\]
This, together with (3.8), implies
\[
(3.11) \quad \mathbb{E} \sup_{0 \leq t \leq T} \| A^\theta Y_t \|_H^p \leq CT^{p/\alpha},
\]
where $C$ depends on $\alpha, \beta, \theta, \varepsilon$ and $p$.

Combining the above inequality with (3.8) and (3.6), we immediately get the desired inequality. \hfill \square

**Proof of Lemma 3.2.** Take $0 < \tilde{\theta} < \theta < \beta - \frac{1}{2\alpha}$, since $\{z_k(t)\}_{k \in \mathbb{Z}_*}$ are independent sequence, we have
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \| A^{\tilde{\theta}} Z_t \|_H \leq \varepsilon \right) = \mathbb{P} \left( \sup_{0 \leq t \leq T} \sum_{k \in \mathbb{Z}_*} \gamma_k^{2\tilde{\theta}} |z_k(t)|^2 \leq \varepsilon^2 \right) \geq I_1 I_2
\]
where
\[
I_1 := \mathbb{P} \left( \sup_{0 \leq t \leq T} \sum_{|k| > N} \gamma_k^{2\tilde{\theta}} |z_k(t)|^2 \leq \varepsilon^2/2 \right),
\]
\[
I_2 := \mathbb{P} \left( \sup_{0 \leq t \leq T} \sum_{|k| \leq N} \gamma_k^{2\tilde{\theta}} |z_k(t)|^2 \leq \varepsilon^2/2 \right),
\]
with $N \in \mathbb{N}$ being some fixed large number. By the spectral property of $A$, we have
\[
I_1 \geq \mathbb{P} \left( \sup_{0 \leq t \leq T} \sum_{|k| > N} \gamma_k^{2\tilde{\theta}} |z_k(t)|^2 \leq \varepsilon^2/2 \right)
\]
\[
\geq \mathbb{P} \left( \sup_{0 \leq t \leq T} \| A^{\tilde{\theta}} Z(t) \|_H^2 \leq \gamma_N^{2(\theta - \tilde{\theta})} \varepsilon^2/2 \right)
\]
\[
= 1 - \mathbb{P} \left( \sup_{0 \leq t \leq T} \| A^{\tilde{\theta}} Z(t) \|_H^2 > \gamma_N^{2(\theta - \tilde{\theta})} \varepsilon^2/2 \right)
\]
\[
= 1 - \mathbb{P} \left( \sup_{0 \leq t \leq T} \| A^{\tilde{\theta}} Z(t) \|_H^p > \gamma_N^{p(\theta - \tilde{\theta})} \varepsilon^p/2^{p/2} \right)
\]
\[
\]
This, together with Lemma 3.1 and Chebyshev inequality, implies that as $\gamma_N$ is sufficient large

$$I_1 \geq 1 - C\gamma_N^{-(\theta-\hat{\theta})}p\varepsilon^{-p} > 0,$$

where $p \in (1, \alpha)$ and $C$ depends on $p, \alpha, \beta, T$.

To finish the proof, it suffices to show that

(3.12) \hspace{1cm} I_2 > 0.

Define $A_k := \{ \sup_{0 \leq t \leq T} |z_k(t)| \leq \varepsilon/(\sqrt{2N}\gamma_k^{\hat{\theta}}) \}$, it is easy to have

(3.13) \hspace{1cm} I_2 \geq \mathbb{P} \left( \bigcap_{|k| \leq N} A_k \right) = \prod_{|k| \leq N} \mathbb{P}(A_k).

Recalling (3.5), we have

$$z_k(t) = \beta_k l_k(t) - \int_0^t \gamma_k e^{-\gamma_k(t-s)} \beta_k l_k(s)ds.$$\hspace{1cm} \text{Furthermore, it follows from a straightforward calculation that}

$$\sup_{0 \leq t \leq T} \left| \int_0^t \gamma_k e^{-\gamma_k(t-s)} \beta_k l_k(s)ds \right| \leq |\beta_k| \sup_{0 \leq t \leq T} |l_k(t)| \quad k \geq 1.$$\hspace{1cm} \text{Therefore,}

$$\mathbb{P}(A_k) \geq \mathbb{P} \left( \sup_{0 \leq t \leq T} |l_k(t)| \leq \frac{\varepsilon}{2|\beta_k|\sqrt{2N}\gamma_k^{\hat{\theta}}} \right).$$

By [3] Proposition 3, Chapter VIII], there exist some $c, C > 0$ only depending on $\alpha$ so that

$$\mathbb{P}( \sup_{0 \leq t \leq T} |l(t)| \leq 1) \geq Ce^{-cT}.$$\hspace{1cm} \text{This, together with the scaling property of stable process, implies}

(3.14) \hspace{1cm} \mathbb{P}(A_k) > 0 \quad |k| \leq N,$$

which, combining with (3.13), immediately implies (3.12). \hfill \square

By Lemma 3.1 above and Theorem 2.2 in [18], we have the following lemma which is important for showing the solution of Eq. (2.2) has Càdlàg trajectories.

**Lemma 3.3.** $Z$ defined by (3.1) has a version in $D([0, \infty); V)$.

**Proof.** Observe that $A_1^+ L_t = \sum_{k \in \mathbb{Z}} \gamma_k^{\frac{1}{p}} \beta_k b_k(t) e_k$. By Lemma 3.1 with $\theta = \frac{1}{2}$, we have

$$\sum_{k \in \mathbb{Z}} \gamma_k^{\frac{1}{p}} |\beta_k|^\alpha < \infty.$$\hspace{1cm} \text{This, together with [18, Theorem 2.2], implies that $Z_t$ has a version which has left limit and is right continuous in $V$.} \hfill \square
4. Proof of Theorem 2.2

For all $\omega \in \Omega$, define $Y_t(\omega) := X_t(\omega) - Z_t(\omega)$, then

\begin{equation}
\partial_t Y_t(\omega) + AY_t(\omega) + N(Y_t + Z_t)(\omega) = 0, \quad Y_0(\omega) = x.
\end{equation}

For each $T > 0$, define

\begin{equation}
K_T(\omega) := \sup_{0 \leq t \leq T} \|Z_t(\omega)\|_V, \quad \omega \in \Omega.
\end{equation}

From Lemma 3.2, for every $k \in \mathbb{N}$, there exists some set $N_k \subset \Omega$ such that $\mathbb{P}(N_k) = 0$ and

$$K_k(\omega) < \infty, \quad \omega \notin N_k.$$  

Define $N = \cup_{k \geq 1} N_k$, it is easy to see $\mathbb{P}(N) = 0$ and that for all $T > 0$

\begin{equation}
K_T(\omega) < \infty, \quad \omega \notin N.
\end{equation}

**Lemma 4.1.** The following statements hold:

(i) For every $x \in H$ and $\omega \notin N$, there exists some $0 < T(\omega) \leq 1$, depending on $\|x\|_H$ and $K_1(\omega)$, such that Eq. (4.1) admits a unique solution $Y(\omega) \in C([0, T]; H)$ satisfying for all $\sigma \in \left[\frac{1}{2}, \frac{1}{2}\right]$

$$\|A^\sigma Y_t(\omega)\|_H \leq C(t^{-\sigma} + 1), \quad t \in (0, T(\omega)],$$

where $C$ is some constant depending on $\|x\|_H, \sigma$ and $K_1(\omega)$.

(ii) Let $\sigma \in \left[\frac{1}{2}, \frac{1}{2}\right]$. For every $x \in D(A^\sigma)$ and $\omega \notin N$, there exists some $0 < \hat{T}(\omega) \leq 1$, depending on $\|x\|_H, \sigma, K_1(\omega)$, such that Eq. (4.1) admits a unique solution on $C([0, \hat{T}(\omega)]; D(A^\sigma))$ such that

$$\sup_{0 \leq t \leq \hat{T}(\omega)} \|A^\sigma Y_t(\omega)\|_H \leq 1 + \|A^\sigma x\|_H.$$  

In particular, when $\sigma = 1/2$,

$$\sup_{0 \leq t \leq \hat{T}(\omega)} \|Y_t(\omega)\|_V \leq 1 + \|x\|_V.$$  

**Proof.** We shall omit the variable $\omega$ for the notational simplicity in the proof, since no confusions will arise.

(i). We shall apply Banach fixed point theorem. Let $0 < T \leq 1$ and $B > 0$ be some constants to be determined later. Take $\sigma = \frac{1}{6}$ and define

$$S = \{u \in C([0, T]; H) : u_0 = x, \sup_{0 \leq t \leq T} t^\sigma \|A^\sigma u_t\|_H \leq B\}.$$  

Given any $u, v \in S$, define $d(u, v) = \sup_{0 \leq t \leq T} t^\sigma \|A^\sigma (u_t - v_t)\|_H$, then $(S, d)$ is a closed metric space. Define a map $F : S \rightarrow C([0, T]; H)$ as the following: for any $u \in S$,

$$\mathcal{F}u_t = e^{-At}x + \int_0^t e^{-A(t-s)}N(u_s + Z_s)ds, \quad t \in [0, T],$$

we aim to show as $T$ is sufficient small and $B$ is sufficiently large,

(a) $\mathcal{F}u \in S$ for $u \in S$,
(b) $d(\mathcal{F}u, \mathcal{F}v) \leq \frac{1}{2} d(u, v)$ for $u, v \in S$. 

It is obvious \((F_0 u) = x\). By \((2.4)\), \((2.8)\), \((2.9)\) and Young’s inequality, we have

\[
\|A^\sigma (F u)_t\|_H \leq C t^{-\sigma} \|x\|_H + \int_0^t \|A^\sigma e^{-A(t-s)}\| \|N(u_s + Z_s)\|_H ds \\
\leq C t^{-\sigma} \|x\|_H + C \int_0^t (t-s)^{-\sigma}(1 + \|A^\sigma u_s + A^\sigma Z_s\|_H^3)ds \\
\leq C t^{-\sigma} \|x\|_H + C \int_0^t (t-s)^{-\sigma}(1 + K_1^3 + \|A^\sigma u_s\|_H^3)ds.
\]

Hence,

\[
t^\sigma \|A^\sigma (F u)_t\|_H \leq C \|x\|_H + C t^\sigma \int_0^t (t-s)^{-\sigma}(1 + K_1^3 + s^{-3\sigma}B^3)ds.
\]

As \(T > 0\) is sufficiently small and \(B\) is sufficiently large, \((a)\) immediately follows from the above inequality.

Given any \(u, v \in S\), it follows from \((2.4)\) and \((2.8)\)

\[
t^\sigma \|A^\sigma (F u)_t - A^\sigma (F v)_t\|_H \\
\leq C T^\sigma \int_0^t (t-s)^{-\sigma} \|N(u_s + Z_s) - N(v_s + Z_s)\|_H ds \\
\leq C T^\sigma \int_0^t (t-s)^{-\sigma}(1 + K_1^3 + \|A^\sigma u_s\|_H^3 + \|A^\sigma v_s\|_H^3)\|A^\sigma u_s - A^\sigma v_s\|_H ds \\
\leq C(1 + K_1^3)T^\sigma \int_0^t (t-s)^{-\sigma} s^{-\sigma} [s^{-\sigma} \|A^\sigma u_s - A^\sigma v_s\|_H] ds \\
+ C T^\sigma \int_0^t (t-s)^{-\sigma} s^{-3\sigma} B^3 [s^{-\sigma} \|A^\sigma u_s - A^\sigma v_s\|_H] ds
\]

where the last inequality is by the fact \(u, v \in S\). This inequality implies

\[
\sup_{0 \leq t \leq T} t^\sigma \|A^\sigma (F u)_t - A^\sigma (F v)_t\|_H \\
\leq C[(1 + K_1^3)T^{1-\sigma} + B^2 T^\sigma] \sup_{0 \leq t \leq T} t^\sigma \|A^\sigma u_t - A^\sigma v_t\|_H.
\]

Choosing \(T\) small enough, we immediately get \((b)\) from the above inequality. Combining \((a)\) and \((b)\), Eq. \((1.1)\) has a unique solution in \(S\) by Banach fixed point theorem.

Let \(Y. \in S\) be the solution obtained by the above Banach fixed point theorem, for every \(\sigma \in [\frac{1}{6}, \frac{1}{2}]\) and \(t \in (0, T]\), by \((2.4)\) and \((2.9)\) we have

\[
\|A^\sigma Y_t\|_H \leq C t^{-\sigma} \|x\|_H + C \int_0^t (t-s)^{-\sigma} \|N(Y_s + Z_s)\|_H ds \\
\leq C t^{-\sigma} \|x\|_H + C \int_0^t (t-s)^{-\sigma}(\|A^\sigma Y_s\|_H^3 + K_1^3)ds \\
\leq C t^{-\sigma} \|x\|_H + C \int_0^t (t-s)^{-\sigma}s^{-\frac{3}{2}}(B^3 + K_1^3)ds.
\]

This inequality clearly imply the desired inequality.
Let us show the uniqueness. Let \( u, v \in C([0, T]; H) \) be two solutions satisfying the inequality, it follows from (2.7) that for all \( t \in [0, T] \),
\[
\| u_t - v_t \|_H \leq \int_0^t \| N(u_s + Z_s) - N(v_s + Z_s) \|_H ds
\]
\[
\leq \int_0^t (1 + K_1^2 + \| A^4 u_s \|_H^2 + \| A^4 v_s \|_H^2) \| u_s - v_s \|_H ds
\]
\[
\leq \int_0^t (1 + K_1^2 + C s^{-1/2}) \| u_s - v_s \|_H ds,
\]
the above inequality implies \( \| u_t - v_t \|_H = 0 \) for all \( t \in [0, T] \).

(ii). Let \( 0 < \hat{T} \leq 1 \) be some constant to be determined later. For every \( \sigma \in [\frac{1}{6}, \frac{1}{2}] \), define
\[
S = \{ u \in C([0, \hat{T}]; D(A^\sigma)) : u_0 = x, \ \sup_{0 \leq t \leq \hat{T}} \| A^\sigma u_t \|_H \leq 1 + \| A^\sigma x \|_H \}.
\]
Given any \( u, v \in S \), define \( d(u, v) = \sup_{0 \leq t \leq \hat{T}} \| A^\sigma (u_t - v_t) \|_H \), then \( (S, d) \) is a closed metric space.

Define a map \( F : S \to C([0, \hat{T}]; D(A^\sigma)) \) as the following: for any \( u \in S \),
\[
(Fu)_t = e^{-At} x + \int_0^t e^{-A(t-s)} N(u_s + Z_s) ds, \quad 0 \leq t \leq \hat{T}.
\]
By (2.4) and (2.9), we have
\[
\| A^\sigma (Fu)_t \|_H \leq \| A^\sigma x \|_H + C \int_0^t (t-s)^{-\sigma} (1 + \| A^\sigma u_s \|_H^3 + K_1^3) ds
\]
\[
\leq \| A^\sigma x \|_H + C \int_0^t (t-s)^{-\sigma} (1 + \| A^\sigma x \|_H^3 + K_1^3) ds
\]
Choosing \( \hat{T} > 0 \) so small that \( \| A^\sigma (Fu)_t \|_H \leq 1 + \| A^\sigma x \|_H \) for all \( 0 \leq t \leq \hat{T} \), we get \( F : S \to S \) from the previous inequality.

For all \( u, v \in S \), as \( \hat{T} > 0 \) is sufficiently small, by a similar calculation we have
\[
\sup_{0 \leq t \leq \hat{T}} \| A^\sigma (Fu)_t - A^\sigma (Fv)_t \|_H \leq \frac{1}{2} \sup_{0 \leq t \leq \hat{T}} \| A^\sigma u_t - A^\sigma v_t \|_H,
\]
i.e., \( d(Fu, Fv) \leq \frac{1}{2} d(u, v) \). By Banach fixed point theorem, we complete the proof of (ii).

Lemma 4.2. The following statements hold:

(a) For every \( x \in H \) and \( \omega \notin N \), Eq. (1.1) admits a unique global solution on \( Y_\omega(\omega) \in C([0, \infty); H) \cap C((0, \infty); V) \).

(b) If \( x \in V \), then \( Y_\omega(\omega) \in C([0, \infty); V) \).

Proof. For national simplicity, we shall omit the variable \( \omega \) in the proof. Thanks to (1.2), (1.3) and Lemma 4.1 to get a global solution, it suffices to show the following a’priori estimate:
\[
\| Y_\omega \|^2_H \leq e^{-2(2\pi-3)t} \| x \|^2_H + \int_0^t e^{-(2\pi-3)(t-s)} (\| Z_\omega \|^2_H + C \| Z_s \|_V^4) ds.
\]
To this end, let us first show the following auxiliary inequality:

\begin{equation}
\langle -N(u + v), u \rangle_H \leq \frac{3}{2} \|u\|^2_H + \frac{1}{2} \|v\|^2_H + C\|v\|_{V}^4.
\end{equation}

In fact, it follows from the following Young’s inequalities: for $a, b \geq 0$,

\[ ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \quad ab \leq \frac{a^4}{4} + \frac{3b^4}{4}, \]

that

\[ \langle -N(u + v), u \rangle_H = \int_T |u(\xi)|^2 d\xi + \int_T u(\xi)v(\xi) d\xi - \int_T |u(\xi)|^4 d\xi \]
\[ \quad - 3 \int_T u^3(\xi)v(\xi) d\xi - 3 \int_T u^2(\xi)v^3(\xi) d\xi - \int_T u(\xi)v^3(\xi) d\xi \]
\[ \leq \|u\|^2_H + \frac{1}{2}\|u\|^2_H + \frac{1}{2}\|v\|^2_H - \|u^2\|^2_H + \|u^2\|^2_H + C\|v^2\|^2_H \]
\[ = \frac{3}{2}\|u\|^2_H + \frac{1}{2}\|v\|^2_H + C\|v^2\|^2_H. \]

This, together with Sobolev embedding $\|u^2\|^2_H \leq C\|v\|_{V}^4$, immediately implies (1.5).

It follows from (1.5) and Poincare inequality $\|x\|_H \leq \frac{1}{2\pi}\|x\|_V$ that

\[ \|Y_i\|^2_H = \|x\|^2_H - 2 \int_0^t \|Y_s\|^2_{V} ds - 2 \int_0^t \langle N(Y_s + Z_s), Y_s \rangle_H ds \]
\[ \leq \|x\|^2_H - (2\pi - 3) \int_0^t \|Y_s\|^2_{H} ds + \int_0^t \|Z_s\|^2_{H} ds + C \int_0^t \|Z_s\|^4_{V} ds. \]

This implies (1.4) immediately.

It follows from Lemma 4.1 that Eq. (1.1) admits a unique local solution $Y \in C([0, T]; H) \cap C((0, T]; V)$ for some $T > 0$. Thanks to (ii) of Lemma 4.1 and (1.4), we can extend this solution $Y \in C([0, T]; H) \cap C((0, T]; V)$ to be $Y \in C([0, \infty); H) \cap C((0, \infty); V)$.

We now prove the new $Y \in C([0, \infty); H) \cap C((0, \infty); V)$ is unique. Suppose there are two solutions $Y^1, Y^2 \in C([0, \infty); H) \cap C((0, \infty); V)$. Thanks to the uniqueness of $Y$ on $[0, T]$, we have $Y^1_t = Y^2_t$. For any $T_0 > T$, it follows from the continuity that

\[ \sup_{T \leq t \leq T_0} \|Y^1_t\| \leq \hat{C}, \quad \sup_{T \leq t \leq T_0} \|Y^2_t\| \leq \hat{C}, \]

where $\hat{C} > 0$ depends on $T_0, Y^1, Y^2$ and $\omega$. Hence, for all $t \in [T, T_0]$, by (2.4), (2.3) and (2.7),

\[ \|Y^1_t - Y^2_t\| \leq \int_T^t (t - s)^{-1/2} \|N(Y^1_s + Z_s) - N(Y^2_s + Z_s)\|_H ds \]
\[ \leq C \int_T^t (t - s)^{-1/2}(1 + K_t^2 + ||A^Y_{Y^1_s}||_{H}^2 + ||A^Y_{Y^2_s}||_{H}^2)\|Y^1_s - Y^2_s\|_H ds \]
\[ \leq \hat{K} \int_T^t (t - s)^{-1/2}\|Y^1_s - Y^2_s\|_V ds, \]

where $\hat{K} := (1 + K_t^2 + 2\hat{C}^2)$. This immediately implies $Y^1_t = Y^2_t$ for all $t \in [T, T_0]$. Since $T_0$ is arbitrary, we get the uniqueness of the solution $Y \in C([0, \infty); H) \cap C((0, \infty); V)$.

If $x \in V$, it follows from (a) that Eq. (1.1) admits a unique solution $Y \in C([0, \infty); H) \cap C((0, \infty); V)$. By (ii) of Lemma 4.1, Eq. (1.1) admits a unique solution $Y \in C([0, T]; V)$.
for some $\hat{T} > 0$. Since $C([0, \hat{T}]; V)$ is a subset of $C([0, \hat{T}]; H) \cap C((0, \hat{T}]; V)$, $Y_t = \hat{Y}_t$ for all $t \in [0, \hat{T}]$. Hence, $Y \in C([0, \infty); V)$.

**Proof of Theorem 2.2.** Let us study Eqs. (2.2) and (4.1) for a negligible set defined in (4.3). Since $Z. \in D(0, \infty); V)$, it is of course $Z. \in D(0, \infty); H)$. By (a) of Lemma 4.2, $X. = Y. + Z.$ is the unique solution to Eq. (2.2) in $D(0, \infty); H) \cap D((0, \infty); V)$. The Markov property follows from the uniqueness immediately. (3) immediately follows from (4.3) and (b) of Lemma 4.2.

5. **Proof of Theorem 2.3**

To show the existence of invariant measures, we follow the method in [9]. To this end, let us consider the Galerkin approximation of Eq. (2.2).

Recall that $\{e_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $H$, define

$$H_m := \text{span}\{e_k; |k| \leq m\}$$

equipped with the norm adopted from $H$. It is clear that $H_m$ is a finite dimensional Hilbert space. Given any $m > 0$, let $\pi_m : H \rightarrow H_m$ be the projection from $H$ to $H_m$.

It is well known that for all fixed $m \in \mathbb{N}$, $H_m$ and $V_m$ are equivalent since we have

$$C_1\|x\|_H \leq \|x\|_V \leq C_2 \|x\|_H \quad \forall x \in H_m,$$

where $C_1, C_2$ are both only depends on $m$.

The Galerkin approximation of (2.2) is as the following:

$$(5.1) \quad dX^m_t = [AX^m_t + N^m(X^m_t)]dt + dL^m_t, \quad X^m_0 = x^m,$$

where $X^m_t = \pi_m X_t$, $N^m(X^m_t) = \pi_m [N(X^m_t)]$, $L^m_t = \sum |k| \leq m \beta_k l_k(t) e_k$. Eq. (5.1) is a dynamics evolving in $H_m$.

**Theorem 5.1.** The following statements hold:

(i) For every $x \in W$ with $W = H, V$ and $\omega \in \Omega$ a.s., Eq. (5.1) has a unique mild solution $X^{m,x^m}(\omega) \in D((0, \infty), W_m)$ such that

$$\sup_{0 \leq t \leq T} \|X^{m,x^m}_t(\omega)\|_W \leq C, \quad T > 0,$$

where $C$ depends on $\|x\|_W$, $T$ and $K_T(\omega)$.

(ii) For every $x \in W$ with $W = H, V$ and $\omega \in \Omega$ a.s., we have

$$\lim_{m \rightarrow \infty} \|X^{m,x^m}_t(\omega) - X^x_t(\omega)\|_W = 0, \quad t \geq 0.$$

**Proof.** We omit the variable $\omega$ for the notational simplicity in the proof. By the same method as proving Theorem 2.2, we can show (i). It remains to show (ii). Since the two cases $W = H$ and $W = V$ can be shown by the same method, we only prove the case $W = V$.

As $t = 0$, (ii) is obvious. For $t > 0$, by (3) of Theorem 2.2 and (i) we have

$$(5.2) \quad \sup_{0 \leq s \leq t} \|X_s\|_V \leq \hat{C}, \quad \sup_{0 \leq s \leq t} \|X^m_s\|_V \leq \hat{C},$$

where $\hat{C} > 0$ depends on $\|x\|_V$, $t$ and $K_t$. Observe

$$(5.3) \quad X^m_t - X_t = I_1(t) + I_2(t) + I_3(t) + I_4(t),$$

where $I_1, I_2, I_3, I_4$ are defined as

$$I_1(t) = \int_0^t (AX^m_s + N^m(X^m_s)) ds,$$

$$I_2(t) = \int_0^t dL^m_s,$$

$$I_3(t) = \sum_{|k| \leq m} \beta_k \int_0^t l_k(s) e_k ds,$$

$$I_4(t) = \sum_{|k| > m} \beta_k \int_0^t l_k(s) e_k ds.$$
where $I_1(t) := e^{-At}(x^m - x)$, $I_2(t) := Z_t - Z_t^m$,

$$I_3(t) := \int_0^t e^{-A(t-s)}(I - \pi_m)N(X_s)ds,$$

$$I_4(t) := \int_0^t e^{-A(t-s)}[N^m(X^m_s) - N^m(X_s)]ds.$$ 

It is clear that as $m \to \infty$,

$$\|I_1(t)\|_V \to 0, \quad \|I_2(t)\|_V \to 0.$$ 

By (2.9), we have

$$(5.4) \quad \|(I - \pi_m)N(X_s)\|_H \to 0.$$ 

This, together with (2.4) and dominated convergence theorem, implies that as $m \to \infty$,

$$\|I_3(t)\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}}\|(I - \pi_m)N(X_s)\|_Hds \to 0.$$ 

It remains to estimate $I_4(t)$. By (2.4), (2.9) and (5.2),

$$\|I_4(t)\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}}\|N^m(X_s) - N^m(X^m_s)\|_Hds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}}\|N(X_s) - N(X^m_s)\|_Hds$$

$$\leq C\tilde{K} \int_0^t (t-s)^{-\frac{1}{2}}\|X_s - X^m_s\|_Vds$$

$$\leq C\tilde{K} \left( \int_0^t (t-s)^{-\frac{1}{2}}ds \right)^{\frac{1}{q}} \left( \int_0^t \|X_s - X^m_s\|_V^qds \right)^{\frac{1}{q}},$$

where $\tilde{K} = \sup_{0 \leq s \leq t,m} (2 + \|X_s\|_V^p + \|X^m_s\|_V^p) \leq 2 + 2\tilde{C}^2$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p < 2$.

Collecting the estimates for $I_1(t), \ldots, I_4(t)$, we get

$$\lim_{m \to \infty} \sup_m \|X_t - X^m_t\|_V \leq C\tilde{K} t^{\frac{1}{q} - \frac{1}{2}} \lim_{m \to \infty} \left( \int_0^t \|X_s - X^m_s\|_V^qds \right)^{\frac{1}{q}}$$

$$\leq C\tilde{K} t^{\frac{1}{q} - \frac{1}{2}} \left( \int_0^t \lim_{m \to \infty} \sup_m \|X_s - X^m_s\|_V^qds \right)^{\frac{1}{q}},$$

where the last inequality is thanks to the fact $\sup_{0 \leq s \leq t} \|X_s - X^m_s\|_V \leq 2\tilde{C}$ and Fatou’s theorem. The above inequality implies

$$\lim_{m \to \infty} \sup_m \|X_t - X^m_t\|_V = 0.$$ 

Before proving Theorem 2.3, let us have a fast review about purely jump Lévy processes as following. Let $\{(l_j(t))_{t \geq 0}, j \in \mathbb{Z}_+\}$ be a sequence of independent one dimensional purely jump Lévy processes with the same characteristic function, i.e.,

$$Ee^{i\xi_1 l_j(t)} = e^{-t\psi_j(t)}, \quad \forall t \geq 0, j \in \mathbb{Z}_+,$$
where $\psi(\xi)$ is a complex valued function called Lévy symbol given by

$$\psi(\xi) = \int_{\mathbb{R}\setminus\{0\}} (e^{\xi y} - 1 - i\xi y 1_{|y| \leq 1}) \nu(dy),$$

where $\nu$ is the Lévy measure and satisfies that

$$\int_{\mathbb{R}\setminus\{0\}} 1 \wedge |y|^2 \nu(dy) < +\infty.$$

For $t > 0$ and $\Gamma \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, the Poisson random measure associated with $l_j(t)$ is defined by

$$N^{(j)}(t, \Gamma) := \sum_{s \in (0, t]} 1_\Gamma(l_j(s) - l_j(s-)).$$

The compensated Poisson random measure is given by

$$\tilde{N}^{(j)}(t, \Gamma) = N^{(j)}(t, \Gamma) - t\nu(\Gamma).$$

By Lévy-Itô’s decomposition (cf. [3, p.108, Theorem 2.4.16]), one has

$$l_j(t) = \int_{|x| \leq 1} x \tilde{N}^{(j)}(t, dx) + \int_{|x| > 1} x N^{(j)}(t, dx).$$

**Proof of Theorem 2.3.** We follow the argument in [9, (3.6)]. Write

$$f(u) := (||u||_{H}^2 + 1)^{1/2}, \quad u \in H_m,$$

it follows from Itô formula ([3] or [9]) that

$$f(X^m_t) =: f(x^m) - I^m_1(t) + I^m_2(t) + I^m_3(t) + I^m_4(t),$$

where

$$I^m_1(t) := \int_0^t \frac{\|X^m_s\|_V^2}{(\|X^m_s\|_H^2 + 1)^{1/2}} ds + \int_0^t \frac{\langle X^m_s, N(X^m) \rangle_H}{(\|X^m_s\|_H^2 + 1)^{1/2}} ds,$$

$$I^m_2(t) := \sum_{|j| \leq m} \int_0^t \int_{|y| \leq 1} [f(X^m_s + y \beta_j e_j) - f(X^m_s)] \tilde{N}^{(j)}(ds, dy),$$

$$I^m_3(t) := \sum_{|j| \leq m} \int_0^t \int_{|y| \leq 1} \left[ f(X^m_s + y \beta_j e_j) - f(X^m_s) - \frac{\langle X^m_s, y \beta_j e_j \rangle_0}{(\|X^m_s\|_H^2 + 1)^{1/2}} \right] \nu(dy) ds,$$

$$I^m_4(t) := \sum_{|j| \leq m} \int_0^t \int_{|y| > 1} [f_n(X^m_s + y \beta_j e_j) - f_n(X^m_s)] N^{(j)}(ds, dy).$$

It follows from (2.5) that

$$I^m_1(t) \geq \int_0^t \frac{\|X^m_s\|_V^2}{(\|X^m_s\|_H^2 + 1)^{1/2}} ds - \frac{t}{4}.$$

We apply the same argument as in [9] to $I^m_2$, ..., $I^m_4$ and get

$$\mathbb{E}[\sup_{0 \leq t \leq T} |I^m_2(t)|] \leq C T^{1/2},$$

$$\mathbb{E}[\sup_{0 \leq t \leq T} |I^m_3(t)|] \leq C T,$$

$$\mathbb{E}[\sup_{0 \leq t \leq T} |I^m_4(t)|] \leq C T,$$

where $C$ is some constant depending on $\alpha, \beta$ and $T > 0$ is arbitrary.
Collecting the estimates about $I_1^n, \ldots, I_4^n$, we immediately get
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} (\|X_t^n\|_H^2 + 1)^{1/2} \right] + \mathbb{E} \int_0^T \frac{\|X_s^n\|_V^2}{(\|X_s^n\|_H^2 + 1)^{1/2}} ds \\
\leq (\|x\|_H^2 + 1)^{1/2} + CT + CT^{1/2}.
\]
It follows from Theorem 5.1 that for all $t > 0$,
\[
\lim_{m \to \infty} \|X_t^n\|_H = \|X_t\|_H \quad a.s., \\
\lim_{m \to \infty} \|X_t^n\|_V = \|X_t\|_V \quad a.s.
\]
By Fatou’s Lemma, we have
\[
\mathbb{E}\left[ \sup_{t \in [0, T]} (\|X_t\|_H^2 + 1)^{1/2} \right] + \mathbb{E} \int_0^T \frac{\|X_s\|_V^2}{(\|X_s\|_H^2 + 1)^{1/2}} ds \leq (\|x\|_H^2 + 1)^{1/2} + CT + CT^{1/2}.
\]
This easily implies
\[
\mathbb{E} \left( \int_0^T \|X_s\|_V ds \right) \leq \mathbb{E} \left( \int_0^T \frac{\|X_s\|_V (\|X_s\|_H + 1)}{(\|X_s\|_H^2 + 1)^{1/2}} ds \right) \\
\leq C \mathbb{E} \left( \int_0^T \frac{\|X_s\|_V^2 + 1}{(\|X_s\|_H^2 + 1)^{1/2}} ds \right) \\
\leq C(1 + \|x\|_H + T).
\]
This, together with the classical Bogoliubov-Krylov’s argument, implies the existence of invariant measures and that the support of invariant measures is $V$. \qed

6. STRONG FELLER PROPERTY

For all $f \in B_b(H)$, define
\[
P_tf(x) = \mathbb{E}[f(X_t^x)],
\]
for all $t \geq 0$ and $x \in H$. By Theorem 2.2, $(P_t)_{t \geq 0}$ is a Markov semigroup on $B_b(H)$. The main result of this section is

**Theorem 6.1.** $(P_t)_{t \geq 0}$, as a semigroup on $B_b(H)$, is strong Feller.

To prove this theorem, we need to use the following theorem which will be proven later.

**Theorem 6.2.** $(P_t)_{t \geq 0}$, as a semigroup on $B_b(V)$, is strong Feller.

**Proof of Theorem 6.1.** Let $T_0 > 0$ be arbitrary, it suffices to show that for all $t \in (0, T_0]$ and $x \in H$
\[
\lim_{\|y-x\|_H \to 0} P_tf(y) = P_tf(x).
\]
Define $\Omega_N := \{\sup_{0 \leq t \leq T_0} \|Z(t)\|_V \leq N\}$, it follows from Lemma 3.1 and Chebyshev’s inequality that
\[
(6.1) \quad \mathbb{P}(\Omega_N^c) \leq c/N,
\]
where $c$ is some constants depending on $\alpha$ and $T_0$. 16
Let $x, y \in H$ be arbitrary and let $C > 0$ be some constant depending on $\|x\|_H, \|y\|_H$ and $N$, whose exact values may vary from line to line.

For $\omega \in \Omega_N$, denote by $Y^x(\omega)$ and $Y^y(\omega)$ the solutions to Eq. (4.1) with initial data $x$ and $y$ respectively. For the notational simplicity, we shall omit the variable $\omega$ in functions if no confusions arise.

By (i) of Lemma 4.1, there exists some constant $0 < t_0 \leq 1$, depending on $\|x\|_H, \|y\|_H$ and $N$, such that for all $0 < t \leq t_0$

\[\|A^\frac{1}{2}Y^x_t\|_H \leq Ct^{-1/6}, \quad \|A^\frac{1}{2}Y^y_t\|_H \leq Ct^{-1/6}.\]

Observe that

\[X^x_t - X^y_t = I_1 + I_2,\]

where

\[I_1(t) := e^{-At}x - e^{-At}y, \quad I_2(t) := \int_0^t e^{-A(t-s)}[N(X^x_s) - N(X^y_s)]ds.\]

It follows from (2.4) that

\[\|I_1(t)\|_V \leq \tilde{c}t^{-\frac{1}{2}}\|x - y\|_H,\]

where $\tilde{c}$ is some constant. Using (2.4), (2.8) and (2.3), we get

\[
\|I_2(t)\|_V \leq \int_0^t \|A^{1/2}e^{-A(t-s)}\|_H \|N(X^x_s) - N(X^y_s)\|_H ds
\]

\[
\leq C \int_0^t (t-s)^{-\frac{1}{2}}(1 + \|A^\frac{1}{2}X^x_s\|_H^2 + \|A^\frac{1}{2}X^y_s\|_H^2)\|X^x_s - X^y_s\|_V ds.
\]

By (6.2),

\[\|A^\frac{1}{2}X^x_s\|_H \leq \|A^\frac{1}{2}Y^x_s\|_H + \|A^\frac{1}{2}Z_s\|_H \leq C(s^{-1/6} + 1)\]

Similarly, $\|A^\frac{1}{2}X^y_s\|_H \leq C(s^{-1/6} + 1)$. Since $0 < s \leq t \leq t_0 \leq 1$, we further get

\[1 + \|A^\frac{1}{2}X^x_s\|_H^2 + \|A^\frac{1}{2}X^y_s\|_H^2 \leq Cs^{-\frac{1}{2}}.\]

Hence,

\[\|I_2(t)\|_V \leq C \int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{4}}\|X^x_s - X^y_s\|_V ds.
\]

For any $r \in (0, t_0]$, define

\[\Phi_r = \sup_{0 \leq t \leq r} t^\frac{1}{4}\|X^x_t - X^y_t\|_V,\]

by (i) of Lemma 4.1

\[\Phi_r \leq \sup_{0 \leq t \leq r} t^\frac{1}{4}(\|Y^x_t\|_V + \|Y^y_t\|_V) + 2r^\frac{1}{4}N < \infty.\]

It follows from (6.3) and the bounds of $I_1, I_2$ that

\[\Phi_r \leq \tilde{c}\|x - y\|_H + C \sup_{0 \leq t \leq r} [t^\frac{1}{4}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{3}{2}}ds] \Phi_r \]

\[\leq \tilde{c}\|x - y\|_H + Cr^\frac{1}{4}\Phi_r.\]

Choose $r$ so small that $Cr^\frac{1}{4} \leq \frac{1}{2}$, we get

\[\Phi_r \leq 2\tilde{c}\|x - y\|_H.\]
this immediately implies
\[(6.4) \quad \|X^x_t - X^y_t\|_V \leq 2c t^{-\frac{1}{6}} \|x - y\|_H, \quad 0 < t \leq r.\]

By the Markov property, for all $0 < t \leq T_0$, we have
\[
|P_t f(x) - P_t f(y)| = |E[P_{t-s} f(X^x_s) - P_{t-s} f(X^y_s)]| \leq J_1 + J_2
\]
where $s = \frac{t}{2} \wedge r$ and
\[
J_1 := |E \{ [P_{t-s} f(X^x_s) - P_{t-s} f(X^y_s)]\Omega_N \} |,
J_2 := |E \{ [P_{t-s} f(X^x_s) - P_{t-s} f(X^y_s)]\Omega_N \} |.
\]

By $(6.1)$,
\[
J_1 \leq 2c \|f\|_\infty/N.
\]
It follows from $(6.4)$, Theorem 6.2 and dominated convergence theorem that
\[
J_2 \to 0, \quad \|x - y\|_H \to 0.
\]
Combining the estimates of $J_1$ and $J_2$, we immediately conclude the proof. \(\square\)

Let us now discuss the method of proving Theorem 6.2. To show the strong Feller property of the semigroup $(P_t)_{t \geq 0}$ on some function space $B_b(W)$, the noise $(L_t)_{t \geq 0}$ under the norm $\|\cdot\|_W$ need to be sufficiently strong to get a gradient estimate for the OU semigroup corresponding to $(Z_t)_{t \geq 0}$. If $W = H$, $(L_t)_{t \geq 0}$ is not strong enough. Therefore, we choose $W = V$ to make the norm of $L_t$ larger.

Because the nonlinearity $N$ is not bounded, we need to use a well known truncation technique, i.e., considering the equation with truncated nonlinearity as follows:
\[(6.5) \quad dX^\rho_t + [AX^\rho_t + N^\rho(X^\rho_t)]dt = dL_t, \quad X^\rho_0 = x \in V.
\]
where $\rho > 0$, $N^\rho(x) = N(x) \chi(\frac{\|x\|_V}{\rho})$ for all $x \in V$ and $\chi : \mathbb{R} \to [0, 1]$ is a smooth function such that
\[
\chi(z) = 1 \quad \text{for } |z| \leq 1, \quad \chi(z) = 0 \quad \text{for } |z| \geq 2.
\]
By (2.6), for all $x \in V$,
\[(6.6) \quad \|N^\rho(x)\|_V \leq C(\|x\|^3_V + \|x\|_V) \chi(\frac{\|x\|_V}{\rho}) \leq C(\rho^3 + \rho).
\]
One can easily check that $N^\rho$ is a Lipschitz function from $V$ to $V$. Hence, Eq. (6.5) admits a unique solution $X^\rho \in D([0, \infty); V)$. For every $f \in B_b(V)$, define
\[
P_t^\rho f(x) = E[f(X^\rho_{t,x})], \quad t \geq 0, \quad x \in V,
\]
$(P_t^\rho)_{t \geq 0}$ is a Markov semigroup.

To establish the gradient estimate of $(P_t^\rho)_{t \geq 0}$, let us first define the derivative of $f \in C^1_b(V)$: given an $h \in V$,
\[
D_h f(x) := \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon}.
\]
By Riesz representation theorem, for every $x \in V$, there exists some $Df(x) \in V$ such that
\[
D_h f(x) = \langle Df(x), h \rangle_V, \quad h \in V.
\]
We define
\[(6.7)\]
\[\|Df\|_\infty = \sup_{x \in V} \|Df(x)\|_V.\]

**Proposition 6.3.** Let $f \in B_b(V)$. For all $\alpha \in (3/2, 2)$, there exists some $\theta \in [1/\alpha, 1)$ such that
\[(6.8)\]
\[\|DP_t^\theta f\|_\infty \leq C t^{-\theta} \|f\|_\infty, \quad t > 0,
\]
where $C > 0$ depends on $\rho, \alpha$ and $\theta$.

**Proof.** Observe that $L_t = \sum_{k \in \mathbb{Z}_*} \tilde{\beta}_k l_k(t) e_k$ is represented in the space $V$ by
\[L_t = \sum_{k \in \mathbb{Z}_*} \tilde{\beta}_k l_k(t) \tilde{e}_k,
\]
where $\tilde{\beta}_k = \gamma_k^{1/2} \beta_k$, $\tilde{e}_k = \gamma_k^{-1/2} e_k$ for $k \in \mathbb{Z}_*$. \{\tilde{e}_k\}_{k \in \mathbb{Z}_*}$ is an orthonormal basis of $V$.

Recall the condition in (ii) of Eq. (2.2): there exists some $C_1, C_2 > 0$ such that
\[(6.9)\]
\[C_1 \gamma_k^{-\beta} \leq |\beta_k| \leq C_2 \gamma_k^{-\beta},
\]
it is easy to check that as $\beta < 3/2 - 1/\alpha$, we have
\[(6.10)\]
\[|\tilde{\beta}_k| \geq C \gamma_k^{-(\theta - \frac{1}{\alpha})},
\]
where $\theta \in [1/\alpha, 1)$ and $C > 0$ is some constant. Note that (6.10) is Hypothesis (5.2) of [26], so we get (5.19) of [26], i.e., there exists some constant $C > 0$ depending on $\alpha$ such that
\[\|D R_t f\|_\infty \leq C t^{-\theta} \|f\|_\infty, \quad f \in B_b(V).
\]
where $(R_t)_{t \geq 0}$ is the semigroup corresponding to the OU process $(Z_t)_{t \geq 0}$.

To make (6.9) and (6.10) be both satisfied, we need
\[(6.11)\]
\[\frac{1}{2} + \frac{1}{2\alpha} < \beta < \frac{3}{2} - \frac{1}{\alpha}.
\]
To make the condition (6.11) be satisfied, we need
\[\alpha \in (3/2, 2).
\]
Recall that $N^\rho$ is a bounded Lipschitz function (see (6.6)), by Lemma 5.9 of [26], we have
\[P_t^\rho f(x) = R_t f(x) + \int_0^t R_{t-s}[\langle N^\rho, DP_s^\rho f \rangle_V](x)ds
\]
and the desired inequality. \qed

Define
\[(6.12)\]
\[\tau_x := \inf\{t > 0; \|X_t^x\|_V \geq \rho\},
\]
by (3) of Theorem 2.2, $\tau_x$ is a stopping time. For all $t < \tau_x$, Eqs. (2.2) and (6.5) have the same solutions. Thanks to the following two points: one is the semigroup $(P_t^\rho)_{t \geq 0}$ has a gradient estimate, the other is the stopping time can be estimated, we can prove the strong Feller property of $(P_t)_{t \geq 0}$.
Proof of Theorem 6.2. Without loss of generality, we assume $\|f\|_\infty = 1$. Let $T_0 > 0$ be arbitrary, it suffices to show that for all $t \in (0, T_0]$ and $x \in V$

$$\lim_{\|y-x\|_V \to 0} P_t f(y) = P_t f(x). \tag{6.13}$$

Recall
$$K_{T_0}(\omega) := \sup_{0 \leq t \leq T_0} \|Z_t(\omega)\|_V, \quad \omega \in \Omega,$$
by Lemma 3.1 and Markov inequality we have
$$\mathbb{P}(K_{T_0} > \rho/2) \leq \frac{C}{\rho}, \tag{6.14}$$
where $C$ is some constant depending on $\alpha$ and $T_0$.

Choose $\rho$ so large that $\|x\|_V \leq \sqrt{\rho}$ and define
$$A := \{K_{T_0} \leq \rho/2\},$$
By (ii) of Lemma 3.1 there exists some $0 < t_0 \leq T_0$ depending on $\rho$ such that for all $\omega \in A,$
$$\sup_{0 \leq t \leq t_0} \|Y^x_t(\omega)\|_V \leq 1 + \|x\|_V. \tag{6.15}$$
Observe that
$$\mathbb{P}(\sup_{0 \leq t \leq t_0} \|X^x_t\|_V \geq \rho) \leq \mathbb{P}(\sup_{0 \leq t \leq t_0} \|Y^x_t\|_V + \sup_{0 \leq t \leq T_0} \|Z_t\|_V \geq \rho)
\leq \mathbb{P}(K_{T_0} > \rho/2) + \mathbb{P}(\sup_{0 \leq t \leq t_0} \|Y^x_t\|_V > \rho/2, A) \tag{6.16}$$
By (6.15), for all $\omega \in A,$ we have
$$\sup_{0 \leq t \leq t_0} \|Y^x_t(\omega)\|_V \leq 1 + \|x\|_V \leq 1 + \sqrt{\rho} < \rho/2.$$ 
So,
$$\mathbb{P}(\sup_{0 \leq t \leq t_0} \|Y^x_t\|_V > \rho/2, A) = 0. \tag{6.17}$$
Hence,
$$\mathbb{P}(\sup_{0 \leq t \leq t_0} \|X^x_t\|_V \geq \rho) \leq \mathbb{P}(K_{T_0} > \rho/2) \leq C/\rho, \tag{6.18}$$
where the last inequality is by (6.14). It follows from the above inequality that for all $t \in [0, t_0]$
$$\mathbb{P}_x(\tau_x \leq t) = \mathbb{P}(\sup_{0 \leq s \leq t} \|X^x_s\|_V \geq \rho) \leq C/\rho. \tag{6.19}$$
Since Eq. (2.2) and Eq. (6.5) both have a unique mild solution, for all $t \in [0, \tau_x)$, we have
$$X^{p,x}_t = X^x_t \quad a.s.. \tag{6.20}$$

Let $y \in V$ be such that $\|x - y\|_V \leq 1$ and choose $\rho > 0$ be sufficiently large so that $\|x\|_V, \|y\|_V \leq \sqrt{\rho}$. Let $t \in (0, t_0]$ observe
$$|P_t f(x) - P_t f(y)| = |\mathbb{E}[f(X^x_t)] - \mathbb{E}[f(X^y_t)]| = I_1 + I_2 + I_3, \tag{6.21}$$
where
$$I_1 := |\mathbb{E}[f(X^x_t)1_{\tau_x > t}] - \mathbb{E}[f(X^y_t)1_{\tau_y > t}]|,$$
$$I_2 := |\mathbb{E}[f(X^x_t)1_{\tau_x \leq t}]|, \quad I_3 := |\mathbb{E}[f(X^y_t)1_{\tau_y \leq t}]|. \tag{20}$$
It follows from (6.19) that
\[ I_2 \leq \frac{C}{\rho}, \quad I_3 \leq \frac{C}{\rho} \]

It remains to estimate \( I_1 \). It follows from (6.20), Proposition 6.3 and (6.19) that
\[
I_1 = |E[f(X_{t_\tau}^x)1_{\tau > t} - E[f(X_{t_\tau}^y)1_{\tau > t}]] |
\leq |E[f(X_{t_\tau}^x) - E[f(X_{t_\tau}^y)]]| + |E[f(X_{t_\tau}^x)1_{\tau \leq t}]| + |E[f(X_{t_\tau}^y)1_{\tau \leq t}]|
\leq \tilde{C}t^{-\theta}\|x - y\|_V + 2C/\rho
\]

where \( \tilde{C} \) depends on \( \alpha, \rho, \) and \( \theta \). For all \( \epsilon > 0 \), choosing
\[
\rho \geq \max\left\{ \frac{12C}{\epsilon}, 2\|x\|^2 + 2 \right\}, \quad \delta < \frac{\epsilon t^\theta}{2C},
\]
as \( \|x - y\|_V \leq \delta \), we have
\[
|P_t f(x) - P_t f(y)| < \epsilon, \quad t \in (0, t_0).
\]

As \( t_0 < t \leq T_0 \), it follows from Markov property and the strong Feller property above that
\[
P_t f(y) - P_t f(x) = P_{t_0}[P_{t-t_0} f](y) - P_{t_0}[P_{t-t_0} f](x) \to 0
\]
as \( \|y - x\|_V \to 0 \).

### 7. Proof of Theorem 2.4

Here we can not use the classical Doob’s Theorem to get the ergodicity because we are not able to prove the irreducibility. Alternatively, we shall use a simple but useful criterion in [16]. Let us first introduce the conception of accessibility.

**Definition 7.1 (Accessibility).** Let \((X_t)_{t \geq 0}\) be a stochastic process valued on a metric space \(E\) and let \((P_t(x, \cdot))_{x \in E}\) be the transition probability family. \((X_t)_{t \geq 0}\) is said to be accessible to \(x_0 \in E\) if the resolvent \( R_\lambda \) satisfies
\[
R_\lambda(x, U) := \int_0^\infty e^{-\lambda t}P_t(x, U)dt > 0
\]
for all \( x \in E \) and all neighborhoods \( U \) of \( x_0 \), where \( \lambda > 0 \) is arbitrary.

The simple but useful criterion we shall use is the following theorem.

**Theorem 7.2** (Corollary 7.8, [16]). If \((X_t)_{t \geq 0}\) is strong Feller at an accessible point \( x \in E \), then it can have at most one invariant measure.

**Proof of Theorem 2.4**. For all \( \epsilon > 0 \) and \( t > 0 \), define \( \Omega_{\epsilon, t} = \{ \sup_{0 \leq s \leq t} \| Z_s \|_V \leq \epsilon \} \), it follows from Lemma 5.2 that
\[
P(\Omega_{\epsilon, t}) > 0.
\]

Recall (4.4), for all \( \omega \in \Omega_{\epsilon, t} \) we get
\[
\| Y_t(\omega) \|_H^2 \leq e^{-(2\pi^{-3})t}\| x \|_H^2 + \int_0^t e^{-(2\pi^{-3})(t-s)}(\| Z_s(\omega) \|_H^2 + C\| Z_s(\omega) \|_V^4)ds
\]
\[
\leq e^{-(2\pi^{-3})t}\| x \|_H^2 + C(\epsilon^2 + \epsilon^4).
\]

For all \( r > 0 \), denote
\[
B_H(r) := \{ x \in H; \| x \|_H < r \}.
\]
For all \( R > 0 \), it follows from the previous inequality that for all \( \delta > 0 \), we can choose \( T := T_{R,\delta} \) sufficiently large and \( \varepsilon := \varepsilon_{R,\delta} \) sufficiently small so that, as \( t \geq T \), for all \( x \in B_H(R) \) and \( \omega \in \Omega_{\varepsilon,t} \),
\[
\left\|X_t^x(\omega)\right\|_H \leq \left\|Y_t^x(\omega)\right\|_H + \left\|Z_t(\omega)\right\|_H \\
\leq e^{-(\pi - \frac{3}{2})R} + C(\varepsilon^4 + \varepsilon^2 + \varepsilon) < \delta.
\]
Since \( \mathbb{P}(\Omega_{\varepsilon,t}) > 0 \), we have for all \( x \in B_H(R) \)
\[(7.1)\quad P(t; x, B_H(\delta)) > 0, \quad t \geq T.\]
This clearly implies for all \( x \in B_H(R) \) and \( \lambda > 0 \),
\[
\mathcal{R}_\lambda(x, B_H(\delta)) > 0.
\]
Since \( R > 0 \) is arbitrary, the above inequality is true for all \( x \in H \) and thus \( (X_t)_{t \geq 0} \) is accessible to 0.

Of course, we can apply Theorem 7.2 to get the ergodicity immediately. However, following the spirit in [16], we can give a clear and short proof as follows.

If \( \mu \) is an invariant measure, it follows from Theorem 2.3 that \( \mu \) is supported on \( V \). Therefore, there exists some (large) \( R > 0 \) such that
\[(7.2)\quad \mu(B_H(R)) > 0.\]
The inequalities (7.1) and (7.2) immediately imply
\[(7.3)\quad \mu(B_H(\delta)) = \int_H P(T; x, B_H(\delta))\mu(dx) > 0, \quad \forall \delta > 0.\]
Assume that Eq. (2.2) admits two invariant measures \( \mu_1 \) and \( \mu_2 \). It is well known ([16]) that there are two sets \( A_1 \) and \( A_2 \) such that \( \mu_1(A_1) = 1 \), \( \mu_2(A_2) = 1 \) and \( A_1 \cap A_2 = \emptyset \). Observe that
\[(7.4)\quad \int_H P(t; x, A_1)\mu_1(dx) = \mu_1(A_1) = 1,\]
\[(7.5)\quad \int_H P(t; x, A_1)\mu_2(dx) = \mu_2(A_1) = 0.\]
It follows from (7.2) that \( \mu_1(B_H(\delta)) > 0 \) for all \( \delta > 0 \). By the strong Feller property (Theorem 6.1) and (7.4), we have
\[P(t; 0, A_1) = 1.\]
On the other hand, by strong Feller property and \( \mu_2(B_H(\delta)) > 0 \) again,
\[
\int_H P(t; x, A_1)\mu_2(dx) \geq \int_{B_H(\delta)} P(t; x, A_1)\mu_2(dx) > 0.
\]
This is contradictory to (7.5). So Eq. (2.2) admits a unique invariant measure. \( \Box \)
8. Appendix: Some estimates about the nonlinearity $N$.

Let us show (2.5)-(2.9). It follows from Young’s inequality that
\[
\langle x, -N(x)\rangle_H = \langle x, x - x^3\rangle_H = \int_T |x(\xi)|^2 d\xi - \int_T |x(\xi)|^4 d\xi \leq \int_T \frac{1}{4} d\xi \leq \frac{1}{4}.
\]

By Sobolev embedding theorem and (2.3), we have
\[
\|N(x)\|_V^2 \leq C \int_T |\partial_\xi x(\xi)|^2 d\xi + C \int_T |x(\xi)|^4 |\partial_\xi x(\xi)|^2 d\xi \\
\leq C\|x\|_V^2 + C\|x\|_H^4 \|x\|_V^2 \\
\leq C\|x\|_V^2 + C\|A^{\frac{1}{2}} x\|_H^2 \|x\|_V^2 \\
\leq C\|x\|_V^2 + C\|x\|_V^6.
\]

For (2.8), it follows from Sobolev embedding theorem and Young’s inequality that
\[
\|N(x) - N(y)\|_H = \|x - y\|_H + \|(x - y)(x^2 + y^2 + xy)\|_H \\
\leq \|x - y\|_H + C(\|x\|_L^2 + \|y\|_L^2) \|x - y\|_L^6 \\
\leq C\|A^{\frac{1}{2}}(x - y)\|_H + C(\|A^{\frac{1}{2}} x\|_H^2 + \|A^{\frac{1}{2}} y\|_H^2) \|A^{\frac{1}{2}}(x - y)\|_H \\
\leq C(1 + \|A^{\frac{1}{2}} x\|_H^2 + \|A^{\frac{1}{2}} y\|_H^2) \|A^{\frac{1}{2}}(x - y)\|_H \\
\leq C(1 + \|A^\sigma x\|_H^2 + \|A^\sigma y\|_H^2) \|A^\sigma(x - y)\|_H,
\]

where the last inequality is by (2.3). Let $y = 0$ and apply Young’s inequality, we immediately get (2.9) from (2.8).

It remains to show (2.7). By Sobolev embedding theorem again, we have
\[
\|N(x) - N(y)\|_H = \|x - y\|_H + \|(x - y)(x^2 + y^2 + xy)\|_H \\
\leq \|x - y\|_H + C(\|x\|_L^2 + \|y\|_L^2) \|x - y\|_H \\
\leq C(1 + \|A^{\frac{1}{2}} x\|_H^2 + \|A^{\frac{1}{2}} y\|_H^2) \|x - y\|_H.
\]

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