THE MOTIVIC COBORDISM FOR GROUP ACTIONS

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Abstract. Let $k$ be a field of characteristic zero. For a linear algebraic group $G$ over $k$ acting on $k$-schemes, we define the equivariant version of the Voevodsky’s motivic cobordism $MGL$ and show that it is an oriented equivariant cohomology theory on the category of smooth $G$-schemes which satisfies the localization sequence. We give several applications. In particular, we study the motivic cobordism rings for the classifying spaces and the cycle class maps to the singular cohomology of such spaces.

1. Introduction

Let $k$ be a field of characteristic zero. Let $G$ be a linear algebraic group over $k$. In this paper, we use the techniques of $\mathbb{A}^1$-homotopy theory to construct an equivariant version of the motivic cobordism theory discovered for smooth schemes by Voevodsky [40]. We use the notion of Thom and Chern structures on the cohomology theories of motivic spaces to show that the new equivariant motivic cobordism is an oriented equivariant cohomology theory on the category of smooth $G$-schemes. One of the main results about this equivariant motivic cobordism theory is that it satisfies the expected localization sequence.

In order to relate the equivariant motivic cobordism with the equivariant analogue of the geometric cobordism of Levine and Morel [27], studied earlier in [19], we look at the equivariant motivic cobordism from a different approach. This approach allows us to show that the equivariant cobordism theory of [19] is the degree zero part of the equivariant motivic cobordism studied in this paper, if we work with rational coefficients. This is an equivariant analogue of a result of Levine [26]. This also allows us to prove many interesting properties of the equivariant motivic cobordism with rational coefficients. We show that the two approaches give the same answer for the torus action on smooth projective schemes.

We prove the self-intersection formula for the equivariant motivic cobordism. This formula allows us to deduce the localization theorems for the equivariant motivic cobordism for torus action. This is an analogue of the similar localization theorem for the equivariant $K$-theory and generalizes a similar result for the geometric equivariant cobordism in [20].

We prove a decomposition theorem for the equivariant motivic cobordism of smooth projective schemes with torus action. This decomposition is used to give a simple formula for the equivariant and ordinary motivic cobordism of flag varieties.

The representability of motivic cohomology in the stable $\mathbb{A}^1$-homotopy category implies that there is a natural map from the equivariant motivic cobordism to the equivariant higher Chow groups of smooth schemes with group actions which were

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defined by Edidin and Graham [11]. We study the complex realization map from
the equivariant motivic to the complex cobordism ring of smooth varieties over
\( \mathbb{C} \). This realization map is used to study the cycle class map from the equivariant
Chow groups of a smooth projective scheme to its equivariant singular cohomology.
This generalizes a result of Totaro [37].

We show that our equivariant motivic cobordism generalizes to a theory of mo-
tivic cobordism of all quotient stacks and it is a cohomology theory with localiza-
tion sequence on the category of smooth quotient stacks. More applications of the
results presented here and some computations of the motivic cobordism for group
actions will appear in a separate paper.

2. Basic constructions

Let \( k \) be a field of characteristic zero and let \( \text{Sm}_k \) denote the category of smooth
schemes of finite type over \( k \).

A linear algebraic group \( G \) over \( k \) will mean a smooth and affine group scheme
over \( k \). By a closed subgroup \( H \) of an algebraic group \( G \), we shall mean a morphism
\( H \to G \) of algebraic groups over \( k \) which is a closed immersion of \( k \)-schemes. In
particular, a closed subgroup of a linear algebraic group will be of the same type
and hence smooth. Recall from [5, Proposition 1.10] that a linear algebraic group
over \( k \) is a closed subgroup of a general linear group, defined over \( k \).

Let \( \text{Sch}_k^G \) (resp. \( \text{Sm}_k^G \)) denote the category of quasi-projective (resp. smooth) \( k \)-schemes
with \( G \)-action and \( G \)-equivariant maps. An object of \( \text{Sch}_k^G \) will be often be called
a \( G \)-scheme.

Recall that an action of a linear algebraic group \( G \) on a \( k \)-scheme \( X \) is said to be
linear if \( X \) admits a \( G \)-equivariant ample line bundle, a condition which is always
satisfied if \( X \) is normal (cf. [35, Theorem 2.5] for \( G \) connected and [36, 5.7] for \( G \)
general). All \( G \)-actions in this paper will be assumed to be linear. We shall use
the following other notations throughout this text.

(1) \( \text{Nis}_k \): The Grothendieck site of smooth schemes over \( k \) with Nisnevich
topology.

(2) \( \text{Shv}(\text{Nis}_k) \): The category of sheaves of sets on \( \text{Nis}_k \).

(3) \( \Delta^{\text{op}} \text{Shv}(\text{Nis}_k) \): The category of sheaves of simplicial sets on \( \text{Nis}_k \).

(4) \( \mathcal{H}(k) \): The unstable \( A^1 \)-homotopy category of simplicial sheaves on \( \text{Nis}_k \),
as defined in [28].

(5) \( \mathcal{H}_*(k) \): The unstable \( A^1 \)-homotopy category of pointed simplicial sheaves
on \( \text{Nis}_k \), as defined in [28].

(6) \( \mathcal{S}\mathcal{H}(k) \): The stable \( A^1 \)-homotopy category of pointed simplicial sheaves
on \( \text{Nis}_k \) as defined, for example, in [40].

Following the notations of [40], an object of \( \Delta^{\text{op}} \text{Shv}(\text{Nis}_k) \) will be called a motivic
space (or simply a space) and we shall often write this category of motivic spaces
as \( \text{Spc} \). The category of pointed motivic spaces over \( k \) will be denoted by \( \text{Spc}_* \).
For any \( X, Y \in \text{Spc} \), \( S(X,Y) \) denotes the simplicial set of morphisms between
spaces as in [28].

2.1. Admissible gadgets. Let \( G \) be a linear algebraic group over \( k \). All represen-
tations of \( G \) in this text will be assumed to be finite-dimensional. We shall say
that a pair \( (V, U) \) of smooth schemes over \( k \) is a good pair for \( G \) if \( V \) is a \( k \)-rational
representation of $G$ and $U \subseteq V$ is a $G$-invariant open subset on which $G$ acts freely such that the quotient $U/G$ is a smooth quasi-projective scheme. It is known (cf. [28, Remark 1.4]) that a good pair for $G$ always exists.

**Definition 2.1.** A sequence of pairs $\rho = (V_i, U_i)_{i \geq 1}$ of smooth schemes over $k$ is called an admissible gadget for $G$, if there exists a good pair $(V, U)$ for $G$ such that $V_i = V^{\oplus i}$ and $U_i \subseteq V_i$ is $G$-invariant open subset such that the following hold for each $i \geq 1$.

1. $(U_i \oplus V) \cup (V \oplus U_i) \subseteq U_{i+1}$ as $G$-invariant open subsets.
2. $\text{codim}_{U_{i+1}}(U_{i+2} \setminus (U_{i+1} \oplus W_i)) > \text{codim}_{U_{i+1}}(U_{i+1} \setminus (U_i \oplus W_i))$.
3. $\text{codim}_{U_{i+1}}(V_{i+1} \setminus U_{i+1}) > \text{codim}_{V_i}(V_i \setminus U_i)$.
4. The action of $G$ on $U_i$ is free with quotient a quasi-projective scheme.

The above definition is a special case of the more general notion of admissible gadgets in [28, §4.2], where these terms are defined for vector bundles over any given scheme. An example of an admissible gadget for $G$ can be constructed as follows. Choose a faithful $k$-rational representation $W$ of $G$ dimension $n$. Then $G$ acts freely on an open subset $U$ of $V = W^{\oplus n}$. Let $Z = V \setminus U$. We now take $V_i = V^{\oplus i}, U_1 = U$ and $U_{i+1} = (U_i \oplus V) \cup (V \oplus U_i)$ for $i \geq 1$. Setting $Z_i = Z$ and $Z_{i+1} = Z_i \cup (U_i \oplus V)$ for $i \geq 1$, one checks that $V_i \setminus U_i = Z_i$ and $Z_{i+1} = Z_i \oplus U$. In particular, $\text{codim}_{V_i}(V_i \setminus U_i) = i \text{codim}_{V_i}(Z_i)$ and $\text{codim}_{U_{i+1}}(Z_{i+1}) = (i + 1)d - i \text{dim}(Z) - d = i \text{codim}_{V_i}(Z)$, where $d = \text{dim}(V)$. Moreover, $U_i \to U_i/G$ is a principal $G$-bundle.

The definition of equivariant motivic cobordism needs one to consider certain kind of mixed quotient spaces which in general may not be a scheme even if the original space is a scheme. The following well known (cf. [11, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed quotient spaces in this paper are schemes with ample line bundles.

**Lemma 2.2.** Let $H$ be a linear algebraic group acting freely and linearly on a $k$-scheme $U$ such that the quotient $U/H$ exists as a quasi-projective variety. Let $X$ be a $k$-scheme with a linear action of $H$. Then the mixed quotient $X \times U$ for the diagonal action on $X \times U$ exists as a scheme and is quasi-projective. Moreover, this quotient is smooth if both $U$ and $X$ are so. In particular, if $H$ is a closed subgroup of a linear algebraic group $G$ and $X$ is a $k$-scheme with a linear action of $H$, then the quotient $X \times U$ is a quasi-projective scheme.

**Proof.** It is already shown in [11, Proposition 23] using [12, Proposition 7.1] that the quotient $X \times U$ is a scheme. Moreover, as $U/H$ is quasi-projective, [12, Proposition 7.1] in fact shows that $X \times U$ is also quasi-projective. The similar conclusion about $G \times X$ follows from the first case by taking $U = G$ and by observing that $G/H$ is a smooth quasi-projective scheme (cf. [5, Theorem 6.8]). The assertion about the smoothness is clear since $X \times U \to X \times U$ is an $H$-torsor. \qed

In this text, $\text{Sm}_{\text{free}}^G$ will denote the full subcategory of $\text{Sm}_k^G$ whose objects are those schemes $X$ on which $G$ acts freely such that the quotient $X/G$ exists and
is quasi-projective over \( k \). The previous result shows that if \( U \in \text{Sm}_k^G \), then \( X \times U \) is also in \( \text{Sm}_k^G \) for every \( G \)-scheme \( X \).

2.2. The Borel spaces. Let \( X \in \text{Sm}_k^G \). For an admissible gadget \( \rho \), let \( X^i_G(\rho) \) denote the mixed quotient space \( X \times U_i \). If the admissible gadget \( \rho \) is clear from the given context, we shall write \( X^i_G(\rho) \) simply as \( X^i_G \).

We define the motivic Borel space \( X_G(\rho) \) to be the colimit \( \text{colim} \) \( X^i_G(\rho) \), where colimit is taken with respect the inclusions \( U_i \subset U \oplus V \subset U_{i+1} \) in the category of motivic spaces. We can think of \( X_G(\rho) \) as a smooth ind-scheme in \( \text{Spc} \). The finite-dimensional Borel spaces of the type \( X^i_G(\rho) \) were first considered by Totaro \cite{37} in order to define the Chow ring of the classifying spaces of linear algebraic groups. For an admissible gadget \( \rho \), we shall denote the spaces \( \text{colim} U_i \) and \( \text{colim} (U_i/G) \) by \( E_G(\rho) \) and \( B_G(\rho) \) respectively. The definition of the motivic spaces \( X_G \) is based on the following observations.

**Lemma 2.3.** For any \( X \in \text{Sm}_k^G \), the natural map \( X_G(\rho) \overset{\sim}{\to} X \times E_G(\rho) \) is an isomorphism in \( \text{Spc} \).

**Proof.** We first observe that the map \( X \times U_i \to X \times U_{i+1} \) is a closed immersion of smooth schemes and \( \text{colim}_i (X \times U_i) \) is the union of its finite-dimensional subschemes \( (X \times U_i) \)'s. Moreover, \( G \) acts freely on \( \text{colim}_i (X \times U_i) \) such that each \( X \times U_i \) is \( G \)-invariant. Since any \( G \)-equivariant map \( f : \text{colim}_i (X \times U_i) \to Y \) with trivial \( G \)-action on \( Y \) factors through a unique map \( \text{colim}_i (X \times U_i)/G \to Y \), we see that the map \( X_G(\rho) \to \text{colim}(X \times U_i)/G \) is an isomorphism. Thus we only need to show that the natural map \( \text{colim}_i (X \times U_i) \to X \times E_G(\rho) \) is an isomorphism.

To show this, it suffices to prove that these two spaces coincide as representable functors on \( \text{Spc} \). Any object of \( \text{Spc} \) is a colimit of simplicial sheaves of the form \( Y \times \Delta[n] \), where \( Y \) is a smooth scheme. Since \( \text{Hom}_{\text{Spc}}(\text{colim} F, -) = \text{lim} \text{Hom}_{\text{Spc}}(F, -) \), we only need to show that the map

\[
\text{Hom}_{\text{Spc}}(Y \times \Delta[n], \text{colim}_i (X \times U_i)) \to \text{Hom}_{\text{Spc}}(Y \times \Delta[n], X \times E_G(\rho))
\]

is bijective for all \( Y \in \text{Sm}_k \) and all \( n \geq 0 \).

For any \( F \in \Delta^a_{\text{Shv}(\text{Nis}_k)} \), there are isomorphisms

\[
\text{Hom}_{\text{Spc}}(Y \times \Delta[n], F) \cong \mathcal{F}_n(Y) = \text{Hom}_{\text{Shv}(\text{Nis}_k)}(Y, \mathcal{F}_n) = \text{Hom}_{\text{Spc}}(Y, \mathcal{F}_n),
\]

where \( \mathcal{F}_n \) is the \( n \)-th level of the simplicial sheaf \( \mathcal{F} \). Since \( \text{colim}_i (X \times U_i) \) and \( X \times E_G(\rho) \) are constant simplicial sheaves, we are reduced to showing that the map

\[
\text{Hom}_{\text{Spc}}(Y, \text{colim}_i (X \times U_i)) \to \text{Hom}_{\text{Spc}}(Y, X \times E_G(\rho))
\]

is bijective.

On the other hand, it follows from \cite{40} Proposition 2.4 that

\[
\text{Hom}_{\text{Spc}}(Y, \text{colim}_i (X \times U_i)) \cong \text{colim}_i \text{Hom}_{\text{Spc}}(Y, X \times U_i) \\
\cong \text{colim}_i \left[ \text{Hom}_{\text{Spc}}(Y, X) \times \text{Hom}_{\text{Spc}}(Y, U_i) \right] \\
\cong \text{Hom}_{\text{Spc}}(Y, X) \times \left[ \text{colim}_i \text{Hom}_{\text{Spc}}(Y, U_i) \right] \\
\cong \text{Hom}_{\text{Spc}}(Y, X) \times \text{Hom}_{\text{Spc}}(Y, \text{colim}_i U_i) \\
\cong \text{Hom}_{\text{Spc}}(Y, X \times E_G(\rho))
\]
which proves the lemma.

\textbf{Proposition 2.4.} For any two admissible gadgets \( \rho \) and \( \rho' \) for \( G \) and for any \( X \in \mathbf{Sm}^G_k \), there is a canonical isomorphism \( X_G(\rho) \cong X_G(\rho') \) in \( \mathcal{H}(k) \).

\textit{Proof.} This was proven by Morel-Voevodsky [28, Proposition 4.2.6] when \( X = \text{Spec}(k) \) and a similar argument works in the general case as well.

For \( i, j \geq 1 \), we consider the smooth scheme \( \mathcal{V}_{i,j} = (X \times U_i \times V'_j)/G \) and the open subscheme \( \mathcal{U}_{i,j} = (X \times U_i \times V'_j)/G \). For a fixed \( i \geq 1 \), this yields a sequence \((\mathcal{V}_{i,j}, \mathcal{U}_{i,j}, f_{i,j})_{j \geq 1}\), where \( \mathcal{V}_{i,j} \to X^G_{1}(\rho) \) is a vector bundle, \( \mathcal{U}_{i,j} \subseteq \mathcal{V}_{i,j} \) is an open subscheme of this vector bundle and \( f_{i,j} : (\mathcal{V}_{i,j}, \mathcal{U}_{i,j}) \to (\mathcal{V}_{i,j+1}, \mathcal{U}_{i,j+1}) \) is the natural map of pairs of smooth schemes over \( X^G_{1}(\rho) \). Then \((\mathcal{V}_{i,j}, \mathcal{U}_{i,j}, f_{i,j})_{j \geq 1}\) is an admissible gadget over \( X^G_{1}(\rho) \) in the sense of [28, Definition 4.2.1]. Setting \( \mathcal{U}_i = \text{colim}_j \mathcal{U}_{i,j} \) and \( \pi_i = \text{colim}_j \pi_{i,j} \), it follows from [loc. cit., Proposition 4.2.3] that the map \( \mathcal{U}_i \to X^G_{1}(\rho) \) is an \( \mathbf{A}^1 \)-weak equivalence.

Taking the colimit of these maps as \( i \to \infty \) and using [loc. cit., Corollary 1.1.21], we conclude that the map \( \mathcal{U} \to X_G(\rho) \) is an \( \mathbf{A}^1 \)-weak equivalence, where \( \mathcal{U} = \text{colim}_{i,j} \mathcal{U}_{i,j} \). Reversing the roles of \( \rho \) and \( \rho' \), we find that the obvious map \( \mathcal{U} \to X_G(\rho') \) is also an \( \mathbf{A}^1 \)-weak equivalence. This yields the canonical isomorphism \( \pi^i \circ \pi^{-1} : X_G(\rho) \cong X_G(\rho') \) in \( \mathcal{H}(k) \).

\( \square \)

2.2.1. \textit{Admissible gadgets associated to a given }\( G \)-\textit{scheme.} A careful reader may have observed in the proof of Proposition 2.4 that we did not really use the fact that \( G \) acts freely on an open subset \( U_i \) (resp. \( V'_j \)). One only needs to know that for each \( i, j \geq 1 \), the quotients \((X \times U_i)/G \) and \((X \times V'_j)/G \) are smooth schemes and the maps \((X \times U_i)/G \to (X \times U_i)/G \) and \((X \times V'_j)/G \to (X \times V'_j)/G \) are vector bundles with appropriate properties. This observation leads us to the following variant of Proposition 2.4 which will sometimes be useful.

Let \( G \) be a linear algebraic group over \( k \) and let \( X \in \mathbf{Sch}_k^G \). We shall say that a pair \((V, U)\) of smooth schemes over \( k \) is a good pair for the \( G \)-action on \( X \), if \( V \) is a \( k \)-rational representation of \( G \) and \( U \subseteq V \) is a \( G \)-invariant open subset such that \( X \times U \) is an object of \( \mathbf{Sch}_k^G \). We shall say that the sequence of pairs \( \rho = (V_i, U_i)_{i \geq 1} \) of smooth schemes over \( k \) is an \textit{admissible gadget for the }\( G \)-\textit{action on }\( X \), if there exists a good pair \((V, U)\) for the \( G \)-action on \( X \) such that \( V_i = V^{\oplus i} \) and \( U_i \subseteq V_i \) is \( G \)-invariant open subset such that the following hold for each \( i \geq 1 \).

\begin{enumerate}
  \item \((U_i \oplus V) \cup (V \oplus U_i) \subseteq U_{i+1} \) as \( G \)-invariant open subsets.
  \item \( \text{codim}_{U_{i+2}}(U_{i+2} \setminus (U_{i+1} \oplus V)) > \text{codim}_{U_{i+1}}(U_{i+1} \setminus (U_i \oplus V)) \).
  \item \( \text{codim}_{U_{i+1}}(V_{i+1} \setminus U_{i+1}) > \text{codim}_{V_i}(V_i \setminus U_i) \).
  \item \( X \times U_i \in \mathbf{Sch}_k^G \).
\end{enumerate}

Notice that an admissible gadget for \( G \) as in Definition 2.1 is an admissible gadget for the \( G \)-action on every \( G \)-scheme \( X \).
Proposition 2.5. Let $\rho_X$ and $\rho'_X$ be two admissible gadgets for the $G$-action on a smooth scheme $X$. Then there is a canonical isomorphism of motivic spaces

$$\text{colim}_i \left( X \times U_i \right) \cong \text{colim}_j \left( X \times U'_j \right).$$

In view of Proposition 2.4, we shall denote a motivic space $X_G(\rho)$ simply by $X_G$. The motivic space $B_G$ is called the classifying space of the linear algebraic group $G$ following the notations of 28. It follows from 28 Proposition 4.2.3] that the space $E_G(\rho)$ is $A^1$-contractible in $\mathcal{H}(k)$ and Lemma 2.3 implies that $B_G(\rho)$ is the quotient of $E_G(\rho)$ for the free $G$-action. Given $X \in \text{Sm}_k^G$, the motivic Borel space of $X$ will mean the motivic space $X_G \in \text{Spc}$.

Corollary 2.6. Let $H$ be a closed normal subgroup of a linear algebraic group $G$ and let $F = G/H$. Let $f : X \to Y$ be a morphism in $\text{Sm}_k^G$ which is an $H$-torsor for the restricted action. Then there is a canonical isomorphism $X_G \cong Y_F$ in $\mathcal{H}(k)$.

Proof. We first observe from [34 Corollary 12.2.2] that $F$ is also a linear algebraic group over the given ground field $k$. Let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$. The natural morphism $G \to F$ shows that each $V_i$ is a $k$-rational representation of $G$ such that the open subset $U_i$ is $G$-invariant, even though $G$ may not act freely on $U_i$. In particular, $G$ acts on the product $X \times U_i$ via the diagonal action. Since $H$ acts freely on $X$ and $F$ acts freely on $U_i$, it follows that the map $X \times U_i \to X \times U_i$ is a $G$-torsor and hence $\rho = (V_i, U_i)_{i \geq 1}$ is an admissible pair for the $G$-action on $X$.

Since the map $X \times U_i \to Y \times U_i$ is an isomorphism for every $i \geq 1$, we conclude from Proposition 2.5 that $X_G \cong \text{colim}_i \left( X \times U_i \right) \to Y_F(\rho)$ in $\mathcal{H}(k)$. \qed

Corollary 2.7 (Morita isomorphism). Let $H$ be a closed subgroup of a linear algebraic group $G$ and let $X \in \text{Sm}_k^H$. Let $Y$ denote the space $X \times G$ for the action $h \cdot (x, g) = (hx, gh^{-1})$. Then there is a canonical isomorphism $X_H \cong Y_G$ in $\mathcal{H}(k)$.

Proof. Define an action of $H \times G$ on $X \times G$ by

$$\begin{align*}
(h, g) \cdot (x, g') &= (hx, gg'h^{-1})
\end{align*}$$

and an action of $H \times G$ on $X$ by $(h, g) \cdot x = hx$. Then the projection map $X \times G \to X$ is $(H \times G)$-equivariant and a $G$-torsor. Hence there is canonical isomorphism $X_H \cong (X \times G)_{H \times G}$ in $\mathcal{H}(k)$ by Corollary 2.6.

On the other hand, the projection map $X \times G \to X \times G$ is $(H \times G)$-equivariant and an $H$-torsor. Hence there is a canonical isomorphism $(X \times G)_{H \times G} \cong Y_G$ in $\mathcal{H}(k)$ again by Corollary 2.6. Combining these two isomorphisms, we get $X_H \cong Y_G$ in $\mathcal{H}(k)$. \qed

3. The Bar construction

Let $G$ be a linear algebraic group over $k$ and let $E_G^\bullet$ denote the simplicial scheme

$$E_G^\bullet := \left( \cdots \longrightarrow G \times G \times G \longrightarrow G \times G \Rightarrow G \right).$$
with the face maps $d^i_n : G^{n+1} \to G^n \ (0 \leq i \leq n)$ given by the projections
\[ d^i_n(g_0, \cdots , g_n) = (g_0, \cdots , g_{i-1}, \hat{g}_i, g_{i+1}, \cdots , g_n), \]
where $\hat{g}_i$ means that this coordinate is omitted. The degeneracy maps $s^i_n : G^n \to G^{n+1} \ (0 \leq i \leq n)$ are the various diagonals on $G$.

For any $X \in \text{Sch}_k^G$, let $E^*_G \times X$ denote the product of the simplicial scheme $E^*_G$ with the constant simplicial scheme $X$. Thus, $(E^*_G \times X)^n = G^{n+1} \times X$ in which the face and the degeneracy maps are identity on $X$. Notice that as $G$ is smooth over $k$, the face maps of $E^*_G \times X$ are all smooth. Moreover, the degeneracy maps are all regular closed immersions. In particular, they have finite Tor-dimension. We shall use these facts while defining the algebraic $K$-theory of simplicial schemes.

The group $G$ acts on $E^*_G \times X$ by $g \cdot (g_0, \cdots , g_n, g x) = (g_0 g^{-1}, \cdots , g_n g^{-1}, x)$. It is easy to check that all the face and degeneracy maps are $G$-equivariant with respect to this action and hence $E^*_G$ and $E^*_G \times X$ are $G$-simplicial schemes such that the projections maps $X \leftarrow E^*_G \times X \to E^*_G$ are $G$-equivariant.

We also consider the simplicial scheme
\[
(3.2) \quad X^*_G := \left( \cdots \xrightarrow{G \times X \xrightarrow{G} X} \xrightarrow{G} X \right)
\]
where the face maps $p^i_n : G^n \times X \to G^{(n-1)} \times X$ are given by
\[
(3.3) \quad p^i_n \ (g_1, \cdots , g_n, x) = \begin{cases} 
(g_2, \cdots , g_n, g_1 x) & \text{if } i = 0 \\
(g_1, \cdots , g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots , g_n, x) & \text{if } 0 < i < n \\
(g_1, \cdots , g_{n-1}, x) & \text{if } i = n.
\end{cases}
\]
The degeneracy maps $s^i_n : G^n \times X \to G^{n+1} \times X$ are given by $s^i_n(g_1, \cdots , g_n, x) = (g_1, \cdots , g_{i-1}, e, g_i, \cdots , g_n, x)$, where $e \in G$ is the identity element. One observes again that all the face maps of $X^*_G$ are smooth and all the degeneracy maps are regular closed immersions. The simplicial scheme $X^*_G$ will be called the bar construction associated to the $G$-action on $X$. We shall often denote $X^*_G$ by $B^*_G$ when $X = \text{Spec} \ (k)$, in analogy with the known bar construction associated to the classifying spaces in topology.

It is easy to verify that there is a natural morphism of simplicial schemes
\[
(3.4) \quad \pi_X : E^*_G \times X \to X^*_G; \quad \pi_X (g_0, \cdots , g_n, x) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \cdots , g_{n-1} g_n^{-1}, g_n x)
\]
which makes $X^*_G$ the quotient of $E^*_G \times X$ for the free $G$-action. Hence, $\pi_X$ is a principal $G$-bundle of simplicial schemes and there is a natural isomorphism of simplicial schemes
\[
(3.5) \quad \pi_X : E^*_G \times X \xrightarrow{G} X^*_G.
\]

3.1. Geometric models for the bar construction. Recall from [28, § 4] that if $G$ is a sheaf of groups on Nis/$k$, then a left (resp. right) action of $G$ on a motivic space (simplicial sheaf) $X$ is a morphism $\mu : G \times X \to X$ (resp. $\mu : X \times G \to X$) such that the usual diagrams commute. A (left) action is called (categorically) free
if the morphism $G \times X \to X \times X$ of the form $(g, x) \mapsto (gx, x)$ is a monomorphism. For a $G$-action on $X$, the quotient $X/G$ is the motivic space such that

\[
G \times X \xrightarrow{\mu} X \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
X \quad \quad \quad \quad \quad X/G
\]

is a pushout diagram of motivic spaces.

A principal $G$-bundle ($G$-torsor) over a motivic space $X$ is a morphism $Y \to X$ together with a free $G$-action on $Y$ over $X$ such that the map $Y/G \to X$ is an isomorphism.

Let $G$ be a linear algebraic group over $k$. Since a simplicial smooth scheme is also a simplicial sheaf on Nis/$k$, we see that $E_G^\bullet \times X$ and $X_G^\bullet$ are objects of $\Delta^{op}\text{Shv}(\text{Nis}_k)$. More generally, given a sheaf of sets $F$ on Nis/$k$ with a free $G$-action, we can consider a simplicial sheaf of sets

\[(3.7) \quad E_G^\bullet(F) = \left( \cdots \xrightarrow{=} F \times F \times F \xrightarrow{=} F \times F \xrightarrow{=} F \right)
\]

where the face and the degeneracy maps are given exactly like in (3.1). Then $G$ acts freely on $E_G^\bullet(F)$ and we denote the quotient by $B_G^\bullet(F)$.

Let $\pi : E_G^\bullet \to B_G^\bullet$ denote the principal $G$-bundle of (3.4). This is called the universal $G$-torsor over $B_G^\bullet$. Let $B_G^\bullet \xrightarrow{\phi} B_G$ be a trivial cofibration of motivic spaces with $B_G$ fibrant. We shall assume in the rest of this section that

\[(3.8) \quad H^1_{\text{Nis}}(U, G) \xrightarrow{\cong} H^1_{\text{et}}(U, G) \quad \text{for all} \quad U \in \text{Sm}_k.
\]

This condition is equivalent to saying that all étale locally trivial $G$-torsors on smooth schemes over $k$ are also locally trivial in the Nisnevich topology. Such a condition is always satisfied if $G$ is special. Under this condition, it follows from [28, Proposition 4.1.15] (see also [ibid., p. 131]) that there is a universal $G$-torsor $E_G \to B_G$ such that for any sheaf of sets $F$ on Nis/$k$ with a free $G$-action, there is a Cartesian square of motivic spaces

\[
E_G^\bullet(F) \xrightarrow{\phi_F} E_G \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
B_G^\bullet(F) \quad \quad \quad \quad \quad B_G
\]

where $\phi_F$ is well defined up to a simplicial homotopy. The following result is an easy consequence of [28, Proposition 4.1.20].

**Lemma 3.1.** Let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$ and let $F = \text{colim}_i U_i$. Then the horizontal morphisms in the Cartesian diagram (3.9) are simplicial weak equivalences.

**Proof.** Assume that $\rho$ is given by a good pair $(V, U)$ and let $x \in U$ be a $k$-rational point. Using the condition (3.8) and [28, Proposition 4.1.20], we have to show that
if \( S \) is smooth henselian local and \( E = S \times G \to S \) is the trivial principal \( G \)-bundle over \( S \), then the morphism \( E \times U_i \to S \) splits for some \( i \gg 0 \) in order to prove that \( \phi_F \) is a simplicial weak equivalence. To find such a splitting, it suffices to find a \( G \)-equivariant morphism \( G \to U \). But it is given by the \( G \)-orbit \( Gx \to U \).

To show that \( \bar{\phi}_F \) is a simplicial weak equivalence, we need to show using \([28, \text{Lemma 3.1.11}]\) that for every smooth henselian local ring \( R \), the map \( E_G(F)(R) \to \mathcal{E}_G(R) \) is a weak equivalence of simplicial sets.

The vertical maps in (3.9) are local fibrations with fiber \( G \) (cf. \([28, \text{Definition 2.1.11, Lemma 4.1.12}]\)). These local fibrations yield for us a commutative diagram of simplicial sets

\[
\begin{align*}
G(R) & \longrightarrow E_G(F)(R) \longrightarrow B_G(F)(R) \\
\downarrow & \phantom{\downarrow} \downarrow & \phantom{\downarrow} \downarrow \\
G(R) & \longrightarrow \mathcal{E}_G(R) \longrightarrow B_G(R)
\end{align*}
\]

where the rows are fibration sequences of simplicial sets. It follows from \([17, \text{Proposition 3.6.1}]\) that the rows remain fibration sequences upon taking the geometric realizations.

We have shown above that the right vertical map is a simplicial weak equivalence and hence a weak equivalence of geometric realizations. The left vertical map is an isomorphism. We conclude from the long exact sequence of homotopy groups of fibrations that the middle vertical map is a weak equivalence of geometric realizations. Hence the map \( E_G(F)(R) \to \mathcal{E}_G(R) \) is a weak equivalence of simplicial sets. \( \square \)

The following result shows that for an action of a linear algebraic group \( G \) on a smooth scheme \( X \), the Borel spaces are the geometric models for the associated bar construction. This is a generalization of the geometric construction of \( B_G^\bullet \) given in \([28, \text{§ 4}]\). As a consequence, we get a more conceptual proof of Proposition 2.4.

**Proposition 3.2.** Let \( \rho = (V_i, U_i)_{i \geq 1} \) be an admissible gadget for a linear algebraic group \( G \) over \( k \). Then for any \( X \in \text{Sm}^G_k \), there is a canonical isomorphism \( X_G(\rho) \cong X^\bullet_G \) in \( H_*(k) \). In particular, \( X_G(\rho) \) does not depend on the choice of the admissible gadget \( \rho \).

**Proof.** It suffices to show using Lemma 2.3 that there is a canonical isomorphism \( E_G(\rho) \overset{G}{\times} X \cong X^\bullet_G \) in \( H_*(k) \), where \( E_G(\rho) = \text{colim}_i U_i \).

Taking \( F = E_G(\rho) = \text{colim}_i U_i \) in (3.9), we have a Cartesian diagram

\[
\begin{align*}
\begin{array}{ccc}
E_G(F) & \overset{\bar{\phi}}{\longrightarrow} & \mathcal{E}_G \\
\downarrow & \phantom{\downarrow} & \phantom{\downarrow} \\
B_G(F) & \overset{\phi}{\longrightarrow} & B_G
\end{array}
\end{align*}
\]

of motivic spaces. We thus have a \( G \)-equivariant map \( E_G(F) \overset{\bar{\phi}}{\times} X \overset{(\bar{\phi} \times \text{id})}{\longrightarrow} \mathcal{E}_G \overset{G}{\times} X \), and hence a map of quotients \( E_G(F) \overset{\phi}{\times} X \overset{\phi}{\longrightarrow} \mathcal{E}_G \overset{G}{\times} X \). We first show that this map is a simplicial weak equivalence.
To do this, we consider the commutative diagram

\[
\begin{array}{ccc}
E^\bullet_G(F)^G \times X \xrightarrow{\phi_X} E^\bullet_G \times X \\
B^\bullet_G(F) \xrightarrow{\phi} B_G \\
\end{array}
\]

where the vertical maps are local fibrations with fiber \(X\). To show that \(\phi_X\) is a simplicial weak equivalence, we need to show that the map \(\left( E^\bullet_G(F)^G \times X \right)(R) \to \left( E_G^\bullet \times X \right)(R) \) is a weak equivalence of simplicial sets for every smooth henselian local ring \(R\).

To show this, we use the commutative diagram

\[
\begin{array}{ccc}
X(R) \longrightarrow \left( E^\bullet_G(F)^G \times X \right)(R) \longrightarrow (B^\bullet_G(F))(R) \\
\downarrow \downarrow \downarrow \\
X(R) \longrightarrow \left( E_G^\bullet \times X \right)(R) \longrightarrow B_G(R) \\
\end{array}
\]

where the rows are fibration sequences of simplicial sets. We have shown in Lemma 3.1 that the right vertical map is a weak equivalence. One now argues as in the proof of Lemma 3.1 to conclude that the map \(\phi_X\) is a simplicial weak equivalence.

We have thus shown that the map \(E^\bullet_G(F)^G \times X \xrightarrow{\phi_X} E_G^\bullet \times X\) is a simplicial weak equivalence. By taking \(F = G\), we similarly see that the map \(X^\bullet_G \to E_G^\bullet \times X\) is a simplicial weak equivalence. Combining the two, we get a simplicial weak equivalence \(X^\bullet_G \xrightarrow{\sim} E^\bullet_G(F) \times X\).

We next think of \(F = E_G(\rho)\) as a constant simplicial sheaf and consider the \(G\)-equivariant map of simplicial sheaves \(u : F \to E^\bullet_G(F)\) given in degree \(n\) by \(u_n(a) = (a, \cdots, a)\). This in turn gives a map \(F^G \times X \xrightarrow{u_X} E^\bullet_G(F)^G \times X\). To finish the proof of the proposition, it suffices now to show that this map is an \(\mathbb{A}^1\)-weak equivalence. By [28, Proposition 2.2.14], it is sufficient to show that each \(u_{X,n} : (F \times X)/G \to (F^{n+1} \times X)/G\) is an \(\mathbb{A}^1\)-weak equivalence. In order to do so, it suffices to show that the projection \((F^{n+1} \times X)/G \xrightarrow{p_{X,n}} (F^n \times X)/G\) is an \(\mathbb{A}^1\)-weak equivalence for each \(n > 0\).

However, it follows from [28, Lemma 4.2.9] that for each \(n, i \geq 1\), the map \((F \times (U_i)^n \times X)/G \to ((U_i)^n \times X)/G\) is an \(\mathbb{A}^1\)-weak equivalence. Taking the colimit as \(i \to \infty\), we see from [3, Proposition 19] that \(p_{X,n}\) is an \(\mathbb{A}^1\)-weak equivalence. This completes the proof of the proposition. □
4. Equivariant motivic cobordism

In this section we define our equivariant motivic cobordism and study its basic properties. We need to work in the motivic stable homotopy category in order to define our equivariant motivic cobordism. We briefly recall this below.

4.1. The motivic stable homotopy category. The motivic stable homotopy category $\mathcal{SH}(k)$ is the homotopy category of motivic $T$-spectra over $k$, where $T$ is the pointed space $(\mathbb{P}^1_k, \infty)$. A $T$-spectrum (or a motivic spectrum) is a sequence of pointed spaces $E = (E_0, E_1, \cdots)$ with the bounding maps $\sigma : T \wedge E_\iota \to E_{\iota+1}$. A morphism of $T$-spectra is a morphism of the sequences of pointed spaces which commute with the bounding maps. The category of motivic spectra is denoted by $\text{Spt}$. We mention the following facts about motivic spectra for the convenience of the reader.

1. The category $\text{Spt}$ has a model structure in which weak equivalence (resp. fibration) is the levelwise $\mathbb{A}^1$-weak equivalence (resp. fibration) and a cofibration is the one which has the left lifting property with respect to all acyclic fibrations. This model structure is proper and cellular.

2. There are isomorphisms of the pointed spaces

$$T \cong \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\} \cong S^1 \wedge \mathbb{G}_m$$

where $S^1$ is the simplicial circle $\Delta^1/\partial \Delta^1$.

3. For any $n \geq 0$, there are functors $F_n : \text{Spc}_\bullet \to \text{Spt}$ and $E_{vn} : \text{Spt} \to \text{Spc}_\bullet$ with $F_n(A)_m = T^{m-n}A$ if $m \geq n$ and zero otherwise, and $E_{vn}(E) = E_n$. These functors form adjoint pairs $(F_n, E_{vn})$. The functor $F_0$ will often be denoted by

$$\Sigma_T^\infty : \text{Spc}_\bullet \to \text{Spt}.$$ 

4. For any $A \in \text{Spc}_\bullet$ and $a \geq b \geq 0$, there is a Nisnevich sheaf $\pi^{a,b}_n(A)$ on $\text{Nis}/k$ which is the sheafification of the presheaf $U \mapsto \text{Hom}_{\mathcal{H}_*(k)}(U_1 \wedge S^{a,b}, A)$. The space $S^{a,b}$ here is the weighted sphere $S^{a,b} \wedge (\mathbb{G}_m)^b$.

5. The stable motivic homotopy category $\mathcal{SH}(k)$ is obtained by localizing the levelwise model structure (as in (1) above) so that the endomorphism $\Sigma_T$ given by $(E_0, E_1, \cdots) \mapsto (T \wedge E_0, T \wedge E_1, \cdots)$ becomes invertible. More precisely, this is obtained as follows.

Given a $T$-spectrum $E = (E_0, E_1, \cdots), U \in \text{Sm}_k$ and a map $f : U_+ \wedge S^{2n+a,n+b} \to E_n$, we get maps

$$U_+ \wedge S^{2n+2+a,n+b+1} = U_+ \wedge S^{2n+a,n+b} \wedge S^{2,1} \to S^{2,1} \wedge E_n = T \wedge E_n \xrightarrow{\sigma} E_{n+1}.$$ 

In particular, there is a sequence of motivic sheaves

$$\cdots \to \pi_{2n+a,n+b}^\mathbb{A}(E_n) \to \pi_{2n+2+a,n+1+b}^\mathbb{A}(E_{n+1}) \to \cdots$$

and this yields the stable homotopy sheaf $\pi_{a,b}^\mathbb{A}(E) := \lim_n \pi_{2n+a,n+b}^\mathbb{A}(E_n)$. The stable homotopy category $\mathcal{SH}(k)$ is the localization of the levelwise model structure on $\text{Spt}$ so that $f : E \to F$ is a stable weak equivalence if the map $f_* : \pi_{a,b}^\mathbb{A}(E) \to \pi_{a,b}^\mathbb{A}(F)$ is an isomorphism for all $a \geq b \geq 0$. The category $\mathcal{SH}(k)$ is a triangulated category symmetric monoidal category in which the shift functor is given by $E \mapsto S^1 \wedge E$ and $L \Sigma_T$ becomes invertible.
with the inverse given by \((E_0, E_1, \cdots) \mapsto (pt, E_0, E_1, \cdots)\). We shall denote \(L \Sigma_T \) still by \(\Sigma_T\) in what follows.

(6) The Quillen pair \((\Sigma^\infty_T, Ev_0) : \text{Spc} \rightleftarrows \text{Spt}\) induces an adjoint pair of derived functors \((\Sigma^\infty_T, Ev_0) : \mathcal{H}_\bullet(k) \rightleftarrows \mathcal{S}\mathcal{H}(k)\).

(7) Let \(\Sigma_s\) and \(\Sigma_t\) denote the endofunctors \(E \mapsto S^1 \wedge E\) and \(E \mapsto \mathbb{G}_m \wedge E\) on \(\mathcal{S}\mathcal{H}(k)\) and let \(\Sigma^{a,b} = \Sigma_{s-b} \circ \Sigma^b_t\) for \(a \geq b \geq 0\). For any \(E \in \mathcal{S}\mathcal{H}(k)\), we define the \(E\)-cohomology theory on \(\mathcal{S}\mathcal{H}(k)\) by

\[
E^{a,b}(F) = \text{Hom}_{\mathcal{S}\mathcal{H}(k)}(F, \Sigma^{a,b}E).
\]

For \(X \in \text{Spc}\), one defines

\[
E^{a,b}(X) = \text{Hom}_{\mathcal{S}\mathcal{H}(k)}(\Sigma^\infty_T X_+, \Sigma^{a,b}E).
\]

4.2. The motivic Thom spectrum. We now recall the construction of the motivic Thom spectrum \(MGL\) defined by Voevodsky [40]. This will play the main role in our definition and further study of the equivariant motivic cobordism.

Recall that for a vector bundle \(p : E \to B\) on a smooth scheme \(B\) with the 0-section \(0_B\), the Thom space of \(E\) is the pointed space \(\text{Th}(E) = E/(E \setminus 0_B) \in \text{Spc}_\bullet\).

There is a canonical isomorphism

\[
\text{Th}(E \oplus \mathcal{O}_B) \cong T \wedge \text{Th}(E).
\]

In particular, for a trivial bundle \(E\) of rank \(n\), one checks easily that \(\text{Th}(E) \cong T^n \wedge B_n\).

It follows from [28 Proposition 4.3.7] that there is a canonical isomorphism \(BGL_n \cong G(n, \infty) = \text{colim}_i G(n, i)\), where \(BGL_n\) is the classifying space (cf. §2.2) of the General linear group of rank \(n\) and \(G(n, i)\) is the Grassmannian scheme of \(n\)-dimensional linear subspaces of \(k^i\). The colimit of the universal rank \(n\) vector bundles on \(G(n, i)\’s defines a unique rank \(n\) vector bundle \(p_n : E_n \to BGL_n\). The rank \(n + 1\) vector bundle \(E_n \oplus \mathcal{O}_{BGL_n} \to BGL_n\) defines a unique map (a closed immersion) \(BGL_n \overset{i_n}{\to} BGL_{n+1}\) such that the diagram

\[
\begin{array}{ccc}
E_n \oplus \mathcal{O}_{BGL_n} & \xrightarrow{\nu_n} & E_{n+1} \\
\downarrow \downarrow & & \downarrow \downarrow \\
BGL_n & \xrightarrow{i_n} & BGL_{n+1}
\end{array}
\]

is Cartesian. This yields a unique map \(\text{Th}(i_n) : T \wedge \text{Th}(U_n) \to \text{Th}(E_{n+1})\). The motivic Thom spectrum \(MGL\) is given by

\[
MGL := (MGL_0, MGL_1, \cdots),
\]

where \(MGL_n = \text{Th}(U_n)\) and the bounding map \(e_n = \text{Th}(i_n) : T \wedge MGL_n \to MGL_{n+1}\). The resulting bi-graded \(MGL\)-cohomology theory on \(\text{Sm}_k\) as defined in (4.1) is called the motivic cobordism theory. This is an oriented ring cohomology theory in the sense of [30].
4.2.1. The ring structure on \( MGL \). Recall from \cite{10} that \( SH(k) \) is a symmetric monoidal category often denoted by \((SH(k), \wedge, S^0)\), where \( S^0 \) is the spectrum \( \Sigma^\infty \) \( (Spec(k)_+) \). We also recall that \( MGL \) is represented by a symmetric spectrum. This spectrum is a commutative ring spectrum in the sense that it is a commutative, associative and unitary monoid in \( (SH(k), \wedge, S^0) \). In particular, the isomorphism \( \Sigma^\infty \) \( ((X \times X')_+) \cong \Sigma^\infty \) \( (X'_+) \wedge \Sigma^\infty \) \( (X'_+) \) implies that there is an external product

\[
MGL^{a,b}(X) \otimes Z MGL^{a',b'}(X') \xrightarrow{ext} MGL^{a+a',b+b'}(X \times X')
\]

for \( X, Y \in \text{Spc} \). For \( X = X' \), this yields the internal product

\[
MGL^{a,b}(X) \otimes Z MGL^{a',b'}(X) \xrightarrow{ext} MGL^{a+a',b+b'}(X \times X) \xrightarrow{\Delta_X} MGL^{a+a',b+b'}(X)
\]

such that \( \alpha \cdot \beta = (-1)^{ab} \beta \cdot \alpha \). The unit element 1 \( \in MGL^{0,0}(pt) \) is the constant map \( \Sigma^\infty(S^0) \to MGL \) mapping \( T^* \) to the base point of \( MGL \). For any \( X \in \text{Spc} \), the unit element 1 \( \in MGL^{0,0} \) is given by the composite map \( \Sigma^\infty(X_+) \to \Sigma^\infty(S^0) \to MGL \) induced by the structure map \( X \to pt \).

4.3. The equivariant motivic cobordism.

**Definition 4.1.** Let \( G \) be a linear algebraic group over \( k \) and let \( X \in \text{Sm}_k \). For \( 0 \leq b \leq a \), we define the equivariant motivic cobordism groups of \( X \) by

\[
MGL^{a,b}_G(X) := MGL^{a,b}(X_G)
\]

where \( X_G \) is a Borel space of the type \( X_G(\rho) \) as in \( \S \) 2.2. We set

\[
MGL^{*,*}_G(X) = \bigoplus_{0 \leq b \leq a} MGL^{a,b}_G(X).
\]

We have seen in Proposition 2.5 that as an object of \( \mathcal{H}(k) \), \( X_G \) does not depend on the choice of an admissible gadget \( \rho \).

**Lemma 4.2.** For \( X, X' \in \text{Sm}_k \), there are external and internal products

\[
MGL^{a,b}_G(X) \otimes Z MGL^{a',b'}_G(X') \xrightarrow{\otimes} MGL^{a+a',b+b'}_G(X \times X');
\]

\[
MGL^{a,b}_G(X) \otimes Z MGL^{a',b'}_G(X) \xrightarrow{\cup} MGL^{a+a',b+b'}_G(X)
\]

such that \( \alpha \cup \beta = (-1)^{ab} \beta \cup \alpha \). In particular, \( MGL^{*,*}_G(X) \) is a bi-graded commutative ring.

**Proof.** Set \( Z = X \times X' \). In view of (4.6), we only need to show that there is a natural morphism \( Z_G \to X_G \times X'_G \) in \( \mathcal{H}(k) \). This will produce the desired map \( MGL^{*,*}_G(X_G \times X'_G) \to MGL^{*,*}_G(Z_G) = MGL^{*,*}_G(Z) \).

Let \( \rho = (V_i, U_i)_{i \geq 1} \) and \( \rho' = (V'_j, U'_j)_{j \geq 1} \) be two admissible gadgets for \( G \). Set \( \mathcal{U}_{k,j} = (Z \times U_i \times U'_j)/G \) and \( \mathcal{U} = colim_{k,j} \mathcal{U}_{k,j} \). We have shown in the proof of Proposition 2.4 that the maps \( \mathcal{U} \to Z_G(\rho) \) and \( \mathcal{U} \to Z_G(\rho') \) are \( \mathbb{A}^1 \)-weak equivalences. In other words, there is a canonical isomorphism \( Z_G \cong \mathcal{U} \) in \( \mathcal{H}(k) \).

On the other hand, there are natural projection maps \( \mathcal{U} \to X_G(\rho) \) and \( \mathcal{U} \to X_G(\rho') \) which yield the desired morphism \( \mathcal{U} \to X_G(\rho) \times X_G(\rho') \) in \( \mathcal{H}(k) \). The internal product on \( MGL^{*,*}_G(X) \) is obtained by composing the external product with the pull-back via the diagonal of \( X \). \( \square \)
5. Basic properties of equivariant motivic cobordism

In this section we derive some basic properties of the equivariant motivic cobordism. The main result is that the equivariant motivic cobordism is an example of an “oriented cohomology theory” on $\text{Sm}_G$. In order to do this, we first study the notion of Thom and Chern structures (cf. [30]) on the motivic cobordism of ind-schemes.

5.1. Ind-schemes as motivic spaces. An ind-scheme $X$ in this text will mean an object of the type $\text{colim}_i X_i$ in $\text{Spc}$, where $\{X_i\}_{i \geq 0}$ is a sequence of cofibrations (monomorphisms)

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots$$

in $\text{Spc}$ such that each $X_i \in \text{Sm}_k$. This definition of ind-schemes is more restrictive than the more general notion where one considers arbitrary filtered colimit of schemes. We shall denote the category of smooth ind-schemes by $\text{ISm}_k$.

A morphism in $\text{ISm}_k$ is a morphism of sequences $f : \{X_i\} \rightarrow \{Y_i\}$ of smooth schemes. We shall say that a morphism $f : Y \rightarrow X$ of ind-schemes has a given property (e.g., étale, smooth, affine, projective) if each map $f_i : Y_i \rightarrow X_i$ of the underlying sequences has that property. The morphism $f$ is called a closed (resp. open) immersion if each $f_i : X_i \rightarrow Y_i$ is a closed (resp. open) immersion of smooth schemes such that $X_i = X_{i+1} \cap Y_i$ for each $i \geq 0$. It is clear that the complement of a closed (resp. open) immersion $Y \hookrightarrow X$ of ind-schemes is the open (resp. closed) ind-subscheme $U = \text{colim}_i (X_i \setminus Y_i)$. For an ind-scheme $X$ and two ind-subschemes $U, V \subset X$, the union $U \cup V$ and intersection $U \cap V$ are defined as the colimits of the individual unions or intersections.

Since a colimit of motivic spaces commutes with finite limits, one has that $\text{colim}_i (X_i \times Y_i) \xrightarrow{\cong} X \times Y$. One also checks that for a morphism of ind-schemes $f : X \rightarrow Y$, there is a canonical isomorphism $\text{colim}_i (Y_i/X_i) \xrightarrow{\cong} Y/X$ in $\text{Spc}$. There are obvious functors $\text{Sm}_k \rightarrow \text{ISm}_k \rightarrow \text{Spc}$ where the first functor $X \mapsto (X \xrightarrow{id} X \xrightarrow{id} \ldots)$ is a full embedding. The following result about the cofibration and weak equivalence between ind-schemes will be used frequently in this text.

**Lemma 5.1.** Let $f : \{X_i\} \rightarrow \{Y_i\}$ be a morphism between two sequences of motivic spaces such that each $f_i : X_i \rightarrow Y_i$ is a cofibration (resp. $\mathbb{A}^1$-weak equivalence). Then $f : \text{colim}_i X_i \rightarrow \text{colim}_i Y_i$ is also a cofibration (resp. $\mathbb{A}^1$-weak equivalence).

**Proof.** It follows from [28 Corollary 1.1.21] that the natural map $\text{hocolim}_i X_i \rightarrow \text{colim}_i X_i$ is a simplicial weak equivalence and hence an $\mathbb{A}^1$-weak equivalence by [15] Proposition 3.3.3. On the other hand, it follows from [15] Theorem 18.5.3 that the map $\text{hocolim}_i X_i \rightarrow \text{hocolim}_i Y_i$ is an $\mathbb{A}^1$-weak equivalence as every motivic space is cofibrant. The assertion about cofibration is obvious. \qed

5.1.1. Vector bundles and projective bundles over ind-schemes. A vector bundle $p : E \rightarrow X$ of rank $n$ over an ind-scheme is a sequence of vector bundles $\{E_i \xrightarrow{\cong} X_i\}$ of rank $n$ such that $E_i = f_i^*(E_{i+1})$ for each $i \geq 0$ and $E = \text{colim}_i E_i$. The maps $i_E : X \hookrightarrow E$ and $j_E : U_E \hookrightarrow E$ will denote the 0-section embedding and its complement respectively. One checks that $U_E = \text{colim}_i U_i$ is an ind-scheme, where $U_i = E_i \setminus X_i$. The pointed space $Th(E) = E/U_E = \text{colim}_i Th(E_i)$ is the
Thom space of the vector bundle $E$ over the ind-scheme $X$. It follows from [28, Example 3.2.2] and Lemma 5.1 that a vector bundle morphism $p : E \to X$ between ind-schemes is an $\mathbb{A}^1$-weak equivalence. A sequence of maps

$$0 \to E' \to E \to E'' \to 0$$

between vector bundles on an ind-scheme $X$ is exact if its restriction to each $X_i$ yields a short exact sequence of associated locally free sheaves.

A vector bundle on an ind-schemes also gives rise to the associated projective bundle $\pi : \mathbb{P}(E) \to X$ in $\text{ISm}_k$ and $\mathbb{P}(E) = \text{colim}_i \mathbb{P}(E_i)$. Moreover, we have the tautological line bundle $\mathcal{O}_{E_i}(-1)$ on $\mathbb{P}(E_i)$ and cofibrations $\mathbb{P}(E_i) \xrightarrow{h_i} \mathbb{P}(E_{i+1})$ such that $\mathcal{O}_{E_i}(-1) = h_i^*(\mathcal{O}_{E_{i+1}}(-1))$. The colimit of these line bundles gives the tautological line bundle $\mathcal{O}_E(-1)$ on $\mathbb{P}(E)$. The following elementary lemma shows that $\text{Th}(E)$ is the colimit of a cofiber sequence of Thom spaces over smooth schemes.

**Lemma 5.2.** Given a Cartesian square

$$
\begin{array}{ccc}
V & \overset{j'}{\longrightarrow} & Y \\
\downarrow i' & & \downarrow i \\
U & \overset{j}{\longrightarrow} & X
\end{array}
$$

of monomorphisms in $\text{Sm}_k$, the map $e : Y/V \to X/U$ is a cofibration in $\text{Spc}$.

**Proof.** Since (5.1) is a Cartesian square of injective maps of smooth schemes, it is easy to check that the map $Z = Y \coprod_U X \to X$ is a monomorphism, i.e., a cofibration in $\text{Spc}$ and the map $Y/V \to Z/U$ is an isomorphism in $\text{Spc}$. So we only need to show that the map $Z/U \to X/U$ is a cofibration. But this follows directly from [15, Lemma 7.2.15].

5.1.2. **Motivic cobordism of ind-schemes.** The motivic cobordism of an ind-scheme (or any motivic space) $X$ is the generalized cohomology

$$MGL^{a,b}(X) = \text{Hom}_{\text{SH}(k)}\left(\Sigma^\infty X_+, \Sigma^{a,b}\text{MGL}\right).$$

Given an ind-scheme $X$ and a closed ind-subscheme $Y$, the motivic cobordism $MGL^b_Y(X)$ is defined to be the motivic cobordism of the motivic space $X/(X \setminus Y)$. It follows from [17, Proposition 6.5.3] that there is a functorial long exact sequence

$$\cdots \to MGL^{-1,b}(X \setminus Y) \xrightarrow{\partial} MGL^{a,b}_Y(X) \to MGL^{a,b}(X) \to MGL^{a,b}(X \setminus Y) \to \cdots.$$ 

Since $MGL$ is a commutative ring spectrum and since $X_+ \wedge pt \cong pt$ for any $X \in \text{Spc}$, we see that given classes $\alpha_1 \in MGL^{a_1,b_1}_Y(X_1)$ and $\alpha_2 \in MGL^{a_2,b_2}_Y(X_2)$, there are maps

$$
\frac{(X_1 \times X_2)}{(X_1 \setminus Y_1) \times X_2} \to MGL \wedge MGL \to MGL
$$
which yields a class $\alpha_1 \times \alpha_2 \in MGL_{X_1 \times X_2}^{a_1+a_2,b_1+b_2}(X_1 \times X_2)$. If $X_1 = X_2 = X$, we can compose with the diagonal map
\[
\frac{X}{X \setminus (Y_1 \cap Y_2)} \to \frac{(X \times X)}{X \times (X \setminus Y_2) \cup (X \setminus Y_1) \times X}
\]
to get a product
\[
MGL_{Y_1}^{a_1,b_1}(X) \times MGL_{Y_2}^{a_2,b_2}(X) \to MGL_{Y_1 \cap Y_2}^{a_1+a_2,b_1+b_2}(X).
\]

5.1.3. Milnor sequence. Suppose there is a sequence of cofibrations of pointed spaces
\[
pt \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots
\]
with colimit $X$ and let $g_i : X_i \to X$ be the natural cofibration. Let $E$ denote a motivic $\Omega$-spectrum which is a commutative ring spectrum. Given a class $\theta$ in $E^{a,b}(X)$, we get a natural class $(\zeta_1) = (g_1^*(\zeta)) \in \prod_i E^{a,b}(X_i)$. This defines an action of $E^{*,*}(X)$ on $\prod_i E^{*,*}(X_i)$ by $(\zeta, (a_i)) \mapsto (\zeta a_i)$. One also checks that there is a natural exact sequence
\[
0 \to \lim_{\leftarrow i} E^{a,b}(X_i) \to \prod_i E^{a,b}(X_i) \xrightarrow{\theta} \prod_i E^{a,b}(X_i) \to \lim_{\rightarrow i} E^{a,b}(X_i) \to 0
\]
where $\theta = (id - (f_i))$ is $E^{*,*}(X)$-linear. In particular, there is a natural action of $E^{*,*}(X)$ on $\lim_{\leftarrow i} E^{*,*}(X_i)$ and $\lim_{\rightarrow i} E^{*,*}(X_i)$. The following result from [17, Proposition 7.3.2] explains the relation between the motivic cobordism of the colimit $X$ and its components.

**Proposition 5.3** (Milnor exact sequence). There is a natural exact sequence
\[
0 \to \lim_{\leftarrow i} E^{a-1,b}(X_i) \to E^{a,b}(X) \to \lim_{\rightarrow i} E^{a,b}(X_i) \to 0
\]
which is compatible with the action of $E^{*,*}(X)$.

**Proof.** This is proven in [17, Proposition 7.3.2] (see also [32, Lemma A.34] and [6, Corollary 9.3.3]) and we give a sketch.

The cofiber sequence (5.4) yields a sequence of fibrations of simplicial sets
\[
S(X, \Sigma^{a,b} E) \to \cdots \to S(X_i, \Sigma^{a,b} E) \xrightarrow{f_0} S(X_0, \Sigma^{a,b} E) \to S(pt, \Sigma^{a,b} E),
\]
where $S(X, \Sigma^{a,b} E) = \lim_i S(X_i, \Sigma^{a,b} E)$. This yields a Cartesian diagram
\[
\begin{align*}
\begin{array}{ccc}
S(X, \Sigma^{a,b} E) & \xrightarrow{\theta_i} & \prod_i \Hom_{\text{Ssets}}(\Delta^1, S(X_i, \Sigma^{a,b} E)) \\
\downarrow & & \downarrow i^* \\
\prod_i S(X_i, \Sigma^{a,b} E) & \xrightarrow{\theta^*} & \prod_i S(X_i, \Sigma^{a,b} E) \times S(X_i, \Sigma^{a,b} E)
\end{array}
\end{align*}
\]
in which the right vertical map is a fibration which is the restriction of a 1-simplex to its boundary. In particular, we get a fiber sequence
\[(5.7) \quad \prod_i S(S^1 \wedge X_i, \Sigma^{a,b} E) \to S(X, \Sigma^{a,b} E) \to \prod_i S(X_i, \Sigma^{a,b} E).\]

Moreover, the above commutative diagram also shows that the diagram
\[
\begin{array}{ccc}
\prod_i S(S^1 \wedge X_i, \Sigma^{a,b} E) & \to & S(X, \Sigma^{a,b} E) \\
\times & \downarrow & \downarrow \\
S(X, \Sigma^{a',b'} E) & \to & S(X, \Sigma^{a',b'} E) \\
\downarrow & & \downarrow \\
\prod_i S(S^1 \wedge X_i, \Sigma^{a+a',b+b'} E) & \to & S(X, \Sigma^{a+a',b+b'} E) \\
\end{array}
\]

commutes in which the first vertical arrow is the map \((f, g) \mapsto h\) with \(h(a \wedge b) = f(a \wedge b) \wedge g(b) \in \Sigma^{a+a',b+b'} E\). The long exact sequence of homotopy groups corresponding to \((5.7)\) and \([17, \text{Lemma 6.1.2}]\) now complete the proof of the proposition.

The naturality of the Milnor exact sequence with respect to the map of sequences \(\{X_i\} \to \{Y_i\}\) is shown in \([6, \text{Corollary 9.3.3}]\) and follows also from the diagram \((5.6)\).

As a consequence of Proposition 5.3, we get the following form of excision for the motivic cobordism of ind-schemes.

**Corollary 5.4.** Let \(f : U \to X\) be an étale morphism of ind-schemes and let \(Z \subseteq X\) be a closed ind-subscheme such that the map \(f^{-1}(Z) \to Z\) is an isomorphism. Then the map \(f^* : MGL^*_Z(U) \to MGL^*_Z(X)\) is an isomorphism.

**Proof.** Translating the problem in the language of sequences of smooth schemes and applying Lemma 5.2 and Proposition 5.3, we get the following commutative diagram of short exact sequences.
\[(5.8) \quad 0 \to \lim_{i}^{\mathbb{L}} MGL_{Z_i}^{* - 1, *}(X_i) \to MGL^{*}(X) \to \lim_{i}^{\mathbb{L}} MGL^{*, *}(X_i) \to 0.
\]

The vertical maps on the two ends are isomorphisms because \((X, U) \mapsto MGL^{*, *}_{X \setminus U}(X)\) is a cohomology theory on the category of smooth pairs of schemes. We conclude that the middle vertical map is also an isomorphism. \(\square\)

### 5.2. Thom and Chern structures on the cobordism of ind-schemes.

In \([30]\), Panin shows that any ring cohomology theory on the category \(\text{Sm}_k\) can in principle be equipped with three types of structures, namely, the Thom structure, the Chern structure and an orientation and these three structures are equivalent to each other. He further shows in \([31]\) that each of these three structures on a ring cohomology theory on \(\text{Sm}_k\) is equivalent to a trace structure. This trace structure
gives functorial push-forward maps on the cohomology groups under a projective morphism. All these are well known in topology. A ring cohomology theory with these structures is called an oriented cohomology theory. It is known (cf. [30 Example 3.8.7]) that the motivic cobordism on $\text{Sm}_k$ is an oriented cohomology theory. The Thom and the Chern structures for this cohomology are given as follows.

The structure of $T$-spectrum on $MGL$ gives unique maps $\iota_n : T^n \wedge MGL_1 \to MGL_{n+1}$. Since $T \wedge pt = pt$, we get a canonical map

$$(MGL_1, T \wedge MGL_1, T^2 \wedge MGL_1, \cdots) \to (pt, T \wedge MGL_1, T \wedge MGL_2, \cdots)$$

where $MGL_1 \to T \wedge pt = pt$ is the obvious map and $T^n \wedge MGL_1 \to T \wedge MGL_n$ is the map $\iota_n \wedge id_T$. This gives a canonical morphism $\text{th} : Th(\mathcal{O}_{\mathbb{P}^\infty}(-1)) = MGL_1 \to \Sigma^{2,1}MGL$ and equivalently, a canonical element $\text{th} \in MGL^{2,1}(Th(\mathcal{O}_{\mathbb{P}^\infty}(-1)))$.

One defines the Chern class $c_1(\mathcal{O}_{\mathbb{P}^\infty}(-1)) = s^*(\text{th}) \in MGL^{2,1}(\mathbb{P}^\infty)$

where $s : \mathbb{P}^\infty \xrightarrow{i_{\mathbb{P}^\infty}} \mathcal{O}_{\mathbb{P}^\infty}(-1) \to MGL_1$ is the composite map.

Our aim now is to suitably extend the notions of Thom and Chern structures for the motivic cobordism of ind-schemes. This extension will in turn be used to establish some basic properties of the equivariant cobordism.

Let $X = \text{colim}_i X_i$ be an ind-scheme and let $\pi : L \to X$ be a line bundle. We associate the Thom class $\text{th}(L) \in MGL^{2,1}_X(L)$ in the following way. Let $\pi_i : L_i \to X_i$ be the restriction of $L$ to $X_i$ via the cofibration $g_i : X_i \to X$. The existence of the Thom structure on $\text{Sm}_X$ yields an inverse system of Thom classes $\{\text{th}(L_i) \in MGL^{2,1}_X(L_i)\}$. Set $\hat{\text{th}}(L) = \lim_i \text{th}(L_i)$. Since $Th(L)$ is the colimit of a cofiber sequence, we can use Proposition 5.3 to get a short exact sequence

$$0 \to \lim_i^{\leftarrow} MGL^{1,1}_{X_i}(L_i) \to MGL^{2,1}_X(L) \to \lim_i^{\rightarrow} MGL^{2,1}_{X_i}(L_i) \to 0$$

which is compatible with the action of $MGL^{*,*}_X(L)$.

On the other hand, the Thom isomorphism

$$\cup \text{th}(L_i) : MGL^{-1,0}(X_i) \xrightarrow{\cong} MGL^{1,1}_{X_i}(L_i)$$

(cf. [30 Definition 3.1]) implies that each term $MGL^{1,1}_{X_i}(L_i)$ vanishes and hence there is a natural isomorphism

$$(5.9) \quad MGL^{2,1}_X(L) \xrightarrow{\cong} \lim_i^{\rightarrow} MGL^{2,1}_{X_i}(L_i).$$

We conclude that $\hat{\text{th}}(L)$ defines a unique class $\text{th}(L) \in MGL^{2,1}_X(L)$ whose restriction to $MGL^{2,1}_{X_i}(L_i)$ is the Thom class $\text{th}(L_i)$. The element $\text{th}(L)$ is called the Thom class of $L$. The Chern class $c_1(L) \in MGL^{2,1}(X)$ is defined as the class $s^*_L(\text{th}(L))$, where $s_L : X \xrightarrow{\iota_L} L \xrightarrow{\iota_L} Th(L)$ is the composite map. It is easy to see from the above construction of the Thom class (and hence the Chern class) that given a map $f : Y \to X$ of ind-schemes and a line bundle $L$ on $X$, one has
th (f*(L)) = f*(th(L)) and c₁ (f*(L)) = f*(c₁(L)). It follows in turn from this that c₁(1_X) = 0, where 1_X is the trivial line bundle on X.

5.2.1. Projective bundle formula and Chern classes for ind-schemes. Let X = colimᵢ Xᵢ be an ind-scheme and let E be a vector bundle of rank n over X and let π : P(E) → X be the associated projective bundle. Let ξ_E = c₁ (O_E(-1)) be the Chern class of the tautological line bundle. This yields a natural map

\[ \phi_X : MGL^{*,*}(X) \oplus \cdots \oplus MGL^{*,*}(X) \rightarrow MGL^{*,*}(\mathbb{P}(E)) ; \]

\[ \phi_X (\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) = \sum_{j=0}^{n-1} \pi^* (\alpha_j) \cdot \xi^j. \]

It follows from Proposition [5.3] that there is a commutative diagram of short exact sequences

(5.11) \[
0 \rightarrow \lim_{\leftarrow i}^1 (MGL^{*-1,*}(X_i))^\oplus_n \rightarrow (MGL^{*,*}(X))^\oplus_n \rightarrow \lim_{\leftarrow i} (MGL^{*,*}(X_i))^\oplus_n \rightarrow 0 \]

\[ \lim_{\leftarrow i} \phi_i \downarrow \phi_X \downarrow \lim \phi_i \]

\[ 0 \rightarrow \lim_{\leftarrow i}^1 MGL^{*-1,*} (\mathbb{P}(E_i)) \rightarrow MGL^{*,*} (\mathbb{P}(E)) \rightarrow \lim_{\leftarrow i} MGL^{*,*} (\mathbb{P}(E_i)) \rightarrow 0. \]

The left and the right vertical maps in (5.11) are isomorphisms by [30] Theorem 3.9] as each Xᵢ is a smooth scheme over k. We conclude that the middle vertical map is an isomorphism. This in particular yields the projective bundle formula for the vector bundles on ind-schemes. If E is a trivial bundle, then \( \mathbb{P}(E) \) is the pull-back of a map \( \mathbb{P}(E) \rightarrow \mathbb{P}_{k}^{n-1} \) and hence \( \xi_E^n = 0 \), again by [30] Theorem 3.9].

As in the case of schemes, the projective bundle formula gives rise to a theory of Chern classes \{c₀(E), c₁(E), \cdots, cₙ(E)\} of a vector bundle E of rank n on an ind-scheme X such that \( c₀(E) = 1 \) and \( c₁(E) \in MGL^{2i,i}(X) \) is the unique element such that

\[ \xi_E^n = \pi^* (c₁(E)) \cdot \xi_E^{n-1} + \cdots + (-1)^{n-1} \pi^* (cₙ₋₁(E)) \cdot \xi_E + (-1)^n \pi^* (cₙ(E)) = 0 \]

in \( MGL^{2n,n}(\mathbb{P}(E)) \). We set \( c_i(E) = 0 \) for \( i > n \). If E is a vector bundle of rank one, then the map \( \pi : \mathbb{P}(E) \rightarrow X \) is an isomorphism such that \( O_E(-1) \cong E \) and hence the above equation shows that \( c₁(E) \) is same as the one defined before. As another consequence of the projective bundle formula, we get the following extension of [30] Corollary 3.18] to ind-schemes.

**Corollary 5.5.** Let \( E₁ \) and \( E₂ \) be two vector bundles on an ind-scheme X and let \( \mathbb{P}(E₁) \rightarrow \mathbb{P}(E₁ \oplus E₂) \) be the inclusions of the the projective bundles. Then there is a canonical short exact sequence

\[ 0 \rightarrow MGL^{*,*}(\mathbb{P}(E₁)) \rightarrow MGL^{*,*}(\mathbb{P}(E₁ \oplus E₂)) \rightarrow MGL^{*,*}(\mathbb{P}(E₂)) \rightarrow 0. \]

**Proof.** This is an immediate consequence of the projective bundle formula and the homotopy invariance and the proof is exactly like in the case of schemes. We give a sketch.
Let \( U_i \to \mathbb{P}(E_1 \oplus E_2) \) denote the complement of \( \mathbb{P}(E_i) \) for \( i = 1, 2 \). Then there is a natural projection \( p_2 : U_1 \to \mathbb{P}(E_2) \) which is a vector bundle such that there is a factorization \( \mathbb{P}(E_2) \xrightarrow{\eta_2} U_1 \xrightarrow{j_1} \mathbb{P}(E_1 \oplus E_2) \) of \( \iota_2 \) with \( \eta_2 \) being the zero-section. In particular, \( \eta_2 \) is an \( \mathbb{A}^1 \)-weak equivalence. Thus we can replace \( MGL^{*,*}(U_1) \) with \( MGL^{*,*}(\mathbb{P}(E_2)) \) and \( j_1^* \) with \( \iota_2^* \) in the long exact sequence (cf. (5.2))

\[
\cdots \to MGL^{*,*}_{\mathbb{P}(E_1)}(\mathbb{P}(E_1 \oplus E_2)) \to MGL^{*,*}(\mathbb{P}(E_1 \oplus E_2)) \xrightarrow{j_1^*} MGL^{*,*}(U_1) \to \cdots.
\]

Hence one only needs to show that each map \( \iota_i^* \) is surjective. But this follows from the projective bundle formula (5.10) and by noting that the tautological bundle on \( \mathbb{P}(E_1 \oplus E_2) \) restricts to the tautological bundle on each \( \mathbb{P}(E_i) \).

### 5.2.2. Cartan formula for Chern classes.

It follows directly from the above definitions that the Chern classes of vector bundles on ind-schemes satisfy all the standard properties of a Chern class theory except the Cartan formula. To establish this formula, we need to prove some intermediate steps, following the approach of Panin in the case of schemes.

**Lemma 5.6.** Let \( \alpha : E \to F \) be an epimorphism of vector bundles on an ind-scheme \( X \). Then there exists a morphism of ind-schemes \( \pi : Y \to X \) which is a \( \mathbb{A}^1 \)-weak equivalence and such that the epimorphism \( \pi^*(\alpha) \) splits.

**Proof.** This was proven by Panin in the case of schemes. We show that this approach also works for ind-schemes. So let \( \alpha_i : E_i \to F_i \) be the epimorphism of vector bundles on the scheme \( X_i \) where \( X = \text{colim}_i X_i \). In view of Lemma 5.1, all we need to do is to find a sequence of smooth schemes \( \{Y_0 \xrightarrow{\omega_0} Y_1 \xrightarrow{\omega_1} \cdots\} \) and map of sequences \( \pi : \{Y_i\} \to \{X_i\} \) such that the following hold.

1. Each \( \pi_i : Y_i \to X_i \) is an \( \mathbb{A}^1 \)-weak equivalence,
2. There exists a splitting \( \beta_i : \pi_i^*(F_i) \to \pi_i^*(E_i) \) of \( \pi_i^*(\alpha_i) \), and
3. For each \( i \geq 0 \), \( g_i^*(\beta_{i+1}) = \beta_i \).

Given this datum, we take \( Y = \text{colim}_i Y_i \) and \( \beta = \text{colim}_i \beta_i \). Then \( \beta : \pi^*(F) \to \pi^*(E) \) yields a splitting of \( \pi^*(\alpha) \) on the ind-scheme \( Y \).

For each \( i \geq 0 \), consider the vector bundle \( G_i = \text{Hom}_{X_i}(F_i, E_i) \) whose fiber at any point \( x \to X_i \) is the space of linear maps \( (F_i)_x \to (E_i)_x \) of \( k(x) \)-vector spaces. It follows that \( f_i^*(G_{i+1}) = G_i \) where \( f_i : X_i \to X_{i+1} \) is the given cofibration. Setting \( H_i = \text{Hom}_{X_i}(F_i, F_i) \), we have the compatible system of maps \( \alpha_i^* : G_i \to H_i \) given by \( \nu \mapsto \alpha_i \circ \nu \). Let \( Y_i \xrightarrow{\iota_i} G_i \) be the subscheme such that the front and the back faces of the diagram
are Cartesian where \( \theta_i \to H_i \) is the section of the projection \( H_i \to X_i \) corresponding to the identity map of \( F_i \). Notice that all the maps in this diagram are cofibrations. Since \( f_i^*(G_{i+1}) = G_i \), one easily checks that \( f_i^*(Y_{i+1}) = Y_i \). The pairs \((\pi_i, \iota_i)\) uniquely define the maps \( \beta_i : \pi_i^*(F_i) \to \pi_i^*(E_i) \) such that \( \pi_i(\alpha_i) \circ \beta_i = 1_{\pi_i(F_i)} \). Moreover, \( f_i^*(Y_{i+1}) = Y_i \) is equivalent to saying that \( g_i^*(\beta_{i+1}) = \beta_i \). Finally, it is well known that each \( \pi_i \) is an affine bundle and hence an \( \mathbb{A}^1 \)-weak equivalence. \( \square \)

**Lemma 5.7** (Splitting principle). Let \( E \to X \) be a vector bundle of rank \( n \) over an ind-scheme \( X \). Then there exists a morphism of ind-schemes \( \pi : Y \to X \) such that

1. \( \pi^*(E) \) is a direct sum of line bundles and
2. for any morphism of ind-schemes \( f : X' \to X \), the map \( MGL^{*,*}(X') \to MGL^{*,*}(Y \times_X X') \) is split injective.

**Proof.** In view of the projective bundle formula for ind-schemes in §5.2.1, Lemma 5.6 and the known standard techniques in case of schemes, we only need to show that given a vector bundle \( E \) over \( X \), one has a short exact sequence

\[
0 \to \mathcal{O}_E(-1) \to p^*(E) \to E' \to 0
\]

of vector bundles on the projective bundle \( p : \mathbb{P}(E) \to X \). But this is well known for schemes and moreover the maps \( \mathcal{O}_{E_i}(-1) \to p_i^*(E_i) \) are canonical and compatible with the cofibrations \( f_i : X_i \to X_{i+1} \) since \( E_i = f_i^*(E_{i+1}) \). \( \square \)

Using the splitting principle, we can extend the theory Chern classes of vector bundles on ind-schemes as follows.

**Proposition 5.8.** Given an ind-scheme \( X \) and a vector bundle \( E \to X \), there are Chern classes \( c_i(E) \in MGL^{2i,i}(X) \) such that

1. \( c_0(E) = 1, c_i(E) = 0 \) for \( i > \text{rank}(E) \) such that \( c_1(E) \) coincides with the Chern class \( c(E) \) as in §5.2 if \( E \) is a line bundle.
2. \( c_i(E) = c_i(E') \) if \( E \cong E' \) and \( f^*(c_i(E)) = c_i(f^*(E)) \) for a map of ind-schemes \( f : Y \to X \).
3. \( c(E) = c(E') \cdot c(E'') \) if there is a short exact sequence \( 0 \to E' \to E \to E'' \to 0 \) of vector bundles, where \( c(E) = 1 + c_1(E)t + c_2(E)t^2 + \cdots \) is the Chern polynomial.
Proof. We only need to show the Cartan formula \( c(E) = c(E') \cdot c(E'') \) for which we can use the second property and Lemma 5.7 to reduce to the case when \( E = F \oplus L \), where \( L \) is a line bundle and \( F \) is a vector bundle of rank \( n \).

Using the definition of the Chern classes, it suffices to show that

\[
(\xi - c_1(L)) \left( \xi^n - c_1(F)\xi^{n-1} + \cdots + (-1)^n c_n(F) \right) = 0
\]

in \( MGL^{2n+2, n+1} (\mathbb{P}(E)) \), where \( \xi = c_1 (\mathcal{O}_E(-1)) \).

Set \( \alpha(F) = \xi^n - c_1(F)\xi^{n-1} + \cdots + (-1)^n c_n(F) \). Since the tautological line bundle on \( \mathbb{P}(E) \) restricts to the tautological line bundles on \( \mathbb{P}(L) \) and \( \mathbb{P}(F) \), it follows from (5.12) and Corollary 5.5 that \( \xi - c_1(L) \in MGL_{\mathbb{P}(F)}^{2,1} (\mathbb{P}(E)) \) and \( \alpha(F) \in MGL_{\mathbb{P}(F)}^{2n,n} (\mathbb{P}(E)) \). In particular, we conclude from (5.3) that the class \( (\xi - c_1(L)) \alpha(F) \) in \( MGL^{2n+2, n+1} (\mathbb{P}(E)) \) is in the image of the map

\[
MGL_{\mathbb{P}(F)\mathbb{P}(L)}^{2n, n} (\mathbb{P}(E)) \rightarrow MGL_{\mathbb{P}(E)}^{2n+2, n+1} (\mathbb{P}(E)).
\]

The desired assertion (5.14) now follows by observing that \( \mathbb{P}(F) \cap \mathbb{P}(L) = \emptyset \). \( \square \)

5.2.3. Thom classes and Thom isomorphism for ind-schemes. Using the theory of Chern classes, we now define the Thom classes of vector bundles on ind-schemes and prove the Thom isomorphism. More precisely, we prove the following.

Proposition 5.9. Given an ind-scheme \( X \) and a vector bundle \( p : E \rightarrow X \) of rank \( n \), there exists a class \( \text{th}(E) \in MGL_{X}^{2n,n}(E) \) such that

1. \( \text{th}(E) = \text{th}(F) \) if \( E \cong F \);
2. \( f^*(\text{th}(E)) = \text{th}(f^*(E)) \) for a morphism \( f : Y \rightarrow X \) of ind-schemes;
3. the map \( \text{th}_{X}^{E} : MGL_{X}^{a,b}(X) \rightarrow MGL_{X}^{a+2n,b+n}(E) \) given by \( \alpha \mapsto p^*(\alpha) \cdot \text{th}(E) \) (cf. 5.3) is an isomorphism;
4. given the projections \( q_{i} : E_{1} \oplus E_{2} \rightarrow E_{i} \), \( (i = 1,2) \), one has
   \[
   q_{i}^{*} (\text{th}(E_{1})) \cdot q_{2}^{*} (\text{th}(E_{2})) = \text{th}(E_{1} \oplus E_{2});
   \]
5. for a line bundle \( L \) on \( X \), the class \( \text{th}(L) \) coincides with the Thom class defined in § 5.2;
6. for a line bundle \( L \) on \( X \), one has \( \text{th}_{X}^{L}(1) = \text{th}(L) \) and \( s_{L}^{*} \circ \text{th}_{X}^{L}(a) = c_{1}(L) \cdot a \) for every \( a \in MGL_{X}^{*,*}(X) \) where \( s_{L} : X \stackrel{i_{L}}{\rightarrow} L \stackrel{i_{L}}{\rightarrow} Th(L) \) is the composite map.

Proof. Let \( F = E \oplus 1_{X} \) and consider the projective bundle \( \pi : \mathbb{P}(F) \rightarrow X \). As in the case of schemes, there is a short exact sequence \( 0 \rightarrow 1_{\mathbb{P}(F)} \rightarrow \pi^{*}(E) \otimes \mathcal{O}_{F}(1) \rightarrow G \rightarrow 0 \) of vector bundles on \( \mathbb{P}(F) \). It follows from Proposition 5.8 and Corollary 5.5 that \( c_{n} (\pi^{*}(E) \otimes \mathcal{O}_{F}(1)) \in MGL_{X}^{2n,n} (\mathbb{P}(F)) \). On the other hand, we can apply Corollary 5.4 to the inclusions \( X \stackrel{i_{E}}{\hookrightarrow} E \stackrel{e_{E}}{\rightarrow} \mathbb{P}(F) \) to see that the natural map \( e_{E}^{*} : MGL_{X}^{*,*} (\mathbb{P}(F)) \rightarrow MGL_{X}^{*,*}(E) \) is an isomorphism. This gives us a unique element (the Thom class) \( \text{th}(E) = c_{n} (\pi^{*}(E) \otimes \mathcal{O}_{F}(1)) \) in \( MGL_{X}^{2n,n}(E) \). The first and the second properties of these Thom classes follow from their construction and the second point of Proposition 5.8.

To prove the third property, we first notice that the above construction of the Thom class coincides with that in [30] if \( X \) is a smooth scheme. Moreover, since each \( E_{i} \) is the restriction of \( E \) on \( X_{i} \), it follows from the second property of the
Thom classes that \( \text{th}(E_i) = \text{th}(E)|_{E_i} \). We can thus apply Lemma 5.2 and Proposition 5.3 to get the following commutative diagram of short exact sequences.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \lim_{\longleftarrow i} MGL^{a-1,b}(X_i) & \longrightarrow & MGL^{a,b}(X) & \longrightarrow & \lim_{\longleftarrow i} MGL^{a,b}(X_i) & \longrightarrow & 0 \\
& \downarrow \text{th}_{X_i}^{E_i} & \downarrow \text{th}_X^E & & \downarrow \text{th}_{X_i}^E & & \\
0 & \longrightarrow & \lim_{\longleftarrow i} MGL_{X_i}^{a+2n-1,b+n}(E_i) & \longrightarrow & MGL_{X_i}^{a+2n,b+n}(E) & \longrightarrow & \lim_{\longleftarrow i} MGL_{X_i}^{a+2n,b+n}(E_i) & \longrightarrow & 0
\end{array}
\]

The vertical maps on the two ends are isomorphisms by the Thom isomorphism for the motivic cobordism of smooth schemes. It follows that the middle vertical map is an isomorphism too. This proves the property (3). The property (4) follows directly from the Cartan formula in Proposition 5.8 and Corollary 5.5 and the proof works exactly like in the case of schemes (cf. [30, Theorem 3.35]). We now prove the last property to complete the proof of the proposition.

Let \( p : L \to X \) be a line bundle and let us temporarily denote the Thom class as defined in §5.2 by \( \hat{\text{th}}(L) \). We now recall from (5.9) that the natural map \( MGL_{X_i}^{2,1}(L) \to \lim_{\longleftarrow i} MGL_{X_i}^{2,1}(L_i) \) is an isomorphism and by the construction, \( \hat{\text{th}}(L) \in MGL_{X_i}^{2,1}(L) \) is the unique class which restricts to \( \text{th}(L_i) \in MGL_{X_i}^{2,1}(L_i) \) for each \( i \). By the property (2) of the proposition, the class \( \text{th}(L) \) also restricts to \( \text{th}(L_i) \in MGL_{X_i}^{2,1}(L_i) \) for each \( i \). It follows from the isomorphism (5.9) that we must have \( \text{th}(L) = \hat{\text{th}}(L) \).

The fact that \( \text{th}_{X_i}^{L_i}(1) = \text{th}(L) \) now follows because the the horizontal maps in the commutative diagram

\[
\begin{array}{ccccccc}
MGL_{X_i}^{2,1}(L) & \longrightarrow & \lim_{\longleftarrow i} MGL_{X_i}^{2,1}(L_i) \\
\downarrow \text{th}_{X_i}^{L_i} & & \downarrow \text{th}_{X_i}^{L_i} \\
MGL_{X_i}^{0,0}(X) & \longrightarrow & \lim_{\longleftarrow i} MGL_{X_i}^{0,0}(X_i)
\end{array}
\]

are isomorphisms and one knows from [30, Theorem 3.35] that \( \text{th}_{X_i}^{L_i}(1) = \text{th}(L_i) \) for each \( i \geq 0 \).

To prove the last property, one first observes that \( p^* \) is a ring isomorphism and it follows from (5.3) that the map \( MGL_{X_i}^{*,*}(E) \xrightarrow{L_i} MGL^{*,*}(E) \) is \( MGL^{*,*}(E) \)-linear for any vector bundle \( E \) on \( X \). Now the desired assertion follows from property (5) and the definition of the first Chern class of line bundles in §5.2. \( \square \)

6. Gysin map for motivic cobordism of ind-schemes

In this section, we construct the Gysin maps \( \iota_* : MGL^{*,*}(Y) \to MGL^{*,*}(X) \) for a given closed embedding of ind-schemes \( \iota : Y \hookrightarrow X \). For a closed immersion \( Y \hookrightarrow X \) of smooth schemes, let \( N_X(Y) \) denote the normal bundle of \( Y \) in \( X \). Recall
from [31, Definition 2.1] that a commutative square of smooth schemes

\[
\begin{array}{ccc}
Y' & \xrightarrow{g} & Y \\
\downarrow \iota' & & \downarrow \iota \\
X' & \xrightarrow{f} & X
\end{array}
\]

is called transverse if it is Cartesian, the map \(\iota\) is a closed immersion and the map \(N_{X'}(Y') \to g^*(N_X(Y))\) is an isomorphism.

**Definition 6.1.** We shall say that a closed immersion of ind-schemes \(\iota : Y \hookrightarrow X\) is **strict** if the square

\[
\begin{array}{ccc}
Y_i & \xrightarrow{g_i} & Y_{i+1} \\
\downarrow \iota_i & & \downarrow \iota_{i+1} \\
X_i & \xrightarrow{f_i} & X_{i+1}
\end{array}
\]

is transverse for each \(i \geq 0\). An open immersion \(j : Y \hookrightarrow X\) of ind-schemes is strict if the square (6.2) is Cartesian. If (6.1) is commutative diagram of ind-schemes, then we shall say that this square is transverse if it is so at each level \(i \geq 0\) and \(\iota\) is a strict embedding.

If \(\iota : Y \hookrightarrow X\) is a strict closed immersion, then the normal bundles \(\{N_{X_i}(Y_i)\}\) define a vector bundle \(N_X(Y)\) on the ind-scheme \(Y\) (cf. §5.1.1) of rank \(d\) where \(d = \text{codim}_{X_i}(Y_i)\) is called the codimension of \(Y\) in \(X\). This vector bundle will be called the normal bundle of \(Y\) in \(X\). It is easy to check that if (6.1) is a transverse square of ind-schemes, then \(\iota' : Y' \hookrightarrow X'\) is a strict closed embedding and the map \(N_{X'}(Y') \to g^*(N_X(Y))\) is an isomorphism. Note also that if (6.1) is a Cartesian square of ind-schemes such that \(\iota\) is a strict embedding and \(f\) is an open immersion (not necessarily strict), then it is transverse. If (6.1) is a transverse square of ind-schemes such that \(\iota\) and \(f\) are strict closed embeddings, then we shall say that \(X'\) and \(Y\) intersect transversely in \(X\). In such a case, the sequence \(\{Y_i \cap X'_i\}\) defines a strict closed ind-subscheme \(Y \cap X'\) of \(X\).

It is also easy to see that if \(\iota : Y \hookrightarrow X\) is a strict closed embedding of ind-schemes with normal bundle \(N\), then \(Y \times \{0\} \to X \times \mathbb{A}^1\) is also a strict closed embedding with the normal bundle \(1_Y \oplus N\). Let \(M'\) denote the blow-up of \(X \times \mathbb{A}^1\) along \(Y \times \{0\}\) and let \(M = M' \setminus Bl_{Y \times \{0\}}(X \times \{0\}) = M \setminus Bl_Y(X)\). Then \(M'\) is an ind-scheme \(\{M'_0 \xrightarrow{h_0} M'_1 \xrightarrow{h_1} \cdots\}\) with the open ind-subscheme \(M = \{M_0 \xrightarrow{h_0} M_1 \xrightarrow{h_1} \cdots\}\) (cf. [13, Corollary II.7.15]). We obtain the following **deformation to normal cone** diagram of ind-schemes.
where the two top squares and the bottom left square are transverse. This induces the maps of motivic spaces

\[ (6.4) \quad \text{Th}(N) \xrightarrow{j_0} M/(M \setminus (Y \times \mathbb{A}^1)) \xrightarrow{j_1} X/(X \setminus Y). \]

It follows from [28, Theorem 3.2.23] and Lemma 5.1 that the maps \( j_0 \) and \( j_1 \) are \( \mathbb{A}^1 \)-weak equivalences. In particular, we get a functorial \( \mathbb{A}^1 \)-weak equivalence

\[ (6.5) \quad t_{X,Y} = (j_0)^{-1} \circ j_1 : X/(X \setminus Y) \to \text{Th}(N). \]

Let \( G : V = M \setminus (Y \times \mathbb{A}^1) \hookrightarrow M \) and \( G' : V' = M' \setminus (Y \times \mathbb{A}^1) \hookrightarrow M' \) denote the open inclusions.

**Lemma 6.2.** The map

\[ (j_0^*, G'^*): MGL^* (M') \to MGL^* (\mathbb{P}(1 \oplus N)) \oplus MGL^* (V') \]

is injective.

**Proof.** We first consider the commutative diagram

\[ (6.6) \quad MGL^*_{Y \times \mathbb{A}^1}(M') \xrightarrow{j_0^*} MGL^*_{Y} (\mathbb{P}(1 \oplus N)) \]

The two vertical maps are isomorphisms by Corollary 5.4 and we have seen above that the bottom horizontal map is an isomorphism. It follows that \( j_0^* \) is an isomorphism.

The lemma now follows by using the commutative diagram

\[ (6.7) \quad MGL^*_{Y \times \mathbb{A}^1}(M') \xrightarrow{j_0^*} MGL^*_{Y} (\mathbb{P}(1 \oplus N)) \]

Corollary 5.5 and (5.2). \qed
**Definition 6.3.** Given a strict closed embedding \( \iota : Y \hookrightarrow X \) of ind-schemes of codimension \( d \), we define the Gysin map \( \iota_* : MGL^{a,b}(Y) \to MGL^{a+2d,b+d}(X) \) as the composite

\[
MGL^{a,b}(Y) \xrightarrow{\text{th}^N_Y} MGL^{a+2d,b+d}(N) \xrightarrow{t^\iota_{X,Y}} MGL^{a+2d,b+d}(X) \xrightarrow{v^\iota_{X,Y}} MGL^{a+2d,b+d}(X)
\]

where \( X \xrightarrow{v^\iota_{X,Y}} X/(X \setminus Y) \) is the quotient map.

Since all the maps in this sequence are \( MGL^*,* \)-linear, we see that the Gysin map is \( MGL^*,* \)-linear. In particular, we have the projection formula

\[
(6.9) \quad \iota_* (\iota^*(a)) = a \cdot \iota_*(1).
\]

**Proposition 6.4.** The Gysin maps \( \iota_* : MGL^*,*(Y) \to MGL^*,*(X) \) satisfy the following functoriality properties.

1. **Base Change:** If \( (6.1) \) is a transverse square of ind-schemes, then the diagram

\[
\begin{array}{ccc}
MGL^*,*(Y) & \xrightarrow{\iota_*} & MGL^*,*(Y') \\
\downarrow & & \downarrow \\
MGL^*,*(X) & \xrightarrow{f^*} & MGL^*,*(X')
\end{array}
\]

commutes.

2. **Identity:** \( \text{id}_* = \text{id} \).

3. **Gysin exact sequence:** For a strict closed embedding \( \iota : Y \hookrightarrow X \) of codimension \( d \), with the complement \( j : U \hookrightarrow X \), the sequence

\[
\cdots \to MGL^{a-1,b}(U) \xrightarrow{\partial} MGL^{a-2d,b-d}(Y) \xrightarrow{\iota_*} MGL^{a,b}(X) \xrightarrow{j^*} MGL^{a,b}(U) \xrightarrow{\partial} \cdots
\]

is exact.

4. **Section of a projective bundle:** If \( E \) is a rank \( n \) vector bundle on \( Y \) and \( s : Y \to \mathbb{P}(1 \oplus E) \) is the zero-section of the projective bundle \( p : \mathbb{P}(1 \oplus E) \to Y \), then \( s_* = (-) \cdot (\text{th}(E)) \circ p^* \).

5. **Smooth divisor:** If \( \iota : D \hookrightarrow X \) is a strict embedding of a smooth divisor, then \( \iota_* (1) = c_1(L(D)) \).

6. **Functoriality:** For strict closed embeddings \( Z \hookrightarrow Y \hookrightarrow X \) of ind-schemes, one has \( \iota_* \circ \iota'_* = (\iota \circ \iota')_* \).

**Proof.** Since the isomorphism \( t^\iota_{X,Y} \) in (6.5) and the map \( v^\iota_{X,Y} \) in (6.8) are functorial, the commutativity of (6.10) is a direct consequence of the functoriality of the Thom classes from Proposition 5.9 and the transversality condition. The property (2) follows directly from the definition and the property (3) is a direct consequence of the above definition, the Thom isomorphism and (5.2). The property (4) follows immediately from the definition of the Gysin map and that of the Thom class in Proposition 5.9 using Corollary 5.4.

To prove property (5), we consider the diagram (6.3) and set \( F' = j \circ F \) and let \( s : D \to \mathbb{P}(1 \oplus N) \) denote the closed embedding. The transversality of the squares and property (1) yield a commutative diagram.
(6.11) \[ MGL^{*,*}(D) \xrightarrow{k^*_1} MGL^{*,*}(D \times A^1) \xrightarrow{k^*_1} MGL^{*,*}(D) \]

Since \( k^*_1 \) is an isomorphism, it follows from the functoriality of the Chern classes (cf. Proposition 5.8) that it is enough to show that
\[ F'_s(1) = c_1(L'), \] where \( L' = L(D \times A^1) \).

Using property (4) and the definition of the Thom class, we get
\[ j_0^*(c_1(L')) = c_1(\mathcal{O}_{1+\mathbb{N}}(1) \otimes p^*(\mathcal{N})) = s^*(1) = s^* \circ k^*_0(1) = j_0^* \circ F'_s(1). \]

Since the elements \( c_1(L') \) and \( F'_s(1) \) vanish in \( MGL^{*,*}(V') \), we conclude from Lemma 6.2 that \( F'_s(1) = c_1(L') \) which proves (6.12) and hence property (5).

We prove property (6) in several steps by imitating the proof for the case of smooth schemes. In the first step, we show that if \( X \) is an ind-scheme and \( \{D_j\}_{1 \leq j \leq n} \) is a collection of strict smooth divisors which intersect transversely in \( X \) with \( Y = \bigcap_{j=1}^n D_j \), then
\[ \iota_*(1) = c_n \left( \bigoplus_{j=1}^n L_j \right) \]
where \( \iota : Y \hookrightarrow X \) is the strict closed embedding and \( L_j = L(D_j) \).

To prove this, we set \( N = \bigoplus_{j=1}^n \iota^*(L_j) \) and consider again the deformation diagram
\[ (6.14) \]

Let \( p : \mathbb{P}(1 + N) \to Y \) be the projection map. It is easy to check that the proper transform \( M'_j \) of \( D_j \times A^1 \) in \( M' \) are all strict smooth divisors which intersect transversely with \( Y \times A^1 = \bigcap_{j=1}^n M'_j \). Moreover, each \( M'_j \) intersects \( \mathbb{P}(1 + N) \) transversely such that the intersection \( P_j = \mathbb{P}(1 + N_j) \) is a smooth divisor in \( \mathbb{P}(1 + N) \), where \( N_j \) is the direct sum of all line bundles on \( Y \) except \( \iota^*(L_j) \). The line bundle \( L(P_j) \) is isomorphic to the line bundle \( p^* \circ \iota^*(L_j) \otimes \mathcal{O}_{1+\mathbb{N}}(1) \). It follows from Proposition 5.8 that \( c_n (p^*(N) \otimes \mathcal{O}_{1+\mathbb{N}}(1)) \) is the product of the Chern classes \( c_1 (p^* \circ \iota^*(L_j) \otimes \mathcal{O}_{1+\mathbb{N}}(1)) \).

The line bundle \( L'_j = L(M'_j) \) restricts to \( L(P_j) \) over \( \mathbb{P}(1 + N) \) and hence is isomorphic to the line bundle \( p^* \circ \iota^*(L_j) \otimes \mathcal{O}_{1+\mathbb{N}}(1) \). We also have \( j_0^*(L'_j) \cong L_j \). Therefore, we have the relation \( j_0^*(c_1(L'_j)) = c_1 (p^* \circ \iota^*(L_j) \otimes \mathcal{O}_{1+\mathbb{N}}(1)) \) in
\[ MGL^{*,*}(\mathbb{P}(1 \oplus N)) \text{ and the relation } j^!_* \left( c_1(L_j) \right) = c_1(L_j) \text{ in } MGL^{*,*}(X). \]

It follows from property (4) that

\begin{equation}
(6.15) \quad s_*(1) = \bigcup_{j=1}^n c_1(p^* \circ \iota^*(L_j) \otimes O_{1\oplus N}(1)).
\end{equation}

Next we show that

\begin{equation}
(6.16) \quad F'_*(1) = \bigcup_{j=1}^n c_1(L'_j).
\end{equation}

To prove this, we first observe that \( F'_*(1) \) vanishes in \( MGL^{*,*}(V') \) as follows from (5.2). On the other hand, \( c_1(L'_j) \) vanishes in \( MGL^{*,*}(M' \setminus M'_j) \) and hence comes from \( MGL_{M'_j}^{*,*}(M') \) by (5.2) again. It follows from (5.3) that \( \bigcup_{j=1}^n c_1(L'_j) \) comes from \( MGL_{Y \times \mathbb{A}^1}^{*,*}(M') \) and hence vanishes in \( MGL^{*,*}(V') \). We now compute

\[ j_0^*(\bigcup_{j=1}^n c_1(L'_j)) = \bigcup_{j=1}^n (j_0^*(L'_j)) = \bigcup_{j=1}^n c_1(p^* \circ \iota^*(L_j) \otimes O_{1\oplus N}(1)) \]

where the equalities \( \dagger \) and \( \dagger\dagger \) follow from (6.15) and property (1) respectively. The relation (6.16) now follows from Lemma 6.2.

Finally, we have

\[ c_n \left( \bigoplus_{j=1}^n L_j \right) = \bigcup_{j=1}^n c_1(L_j) = j_1^* \left( \bigcup_{j=1}^n c_1(L'_j) \right) = j_1^* \circ F'_*(1) = \iota_* \circ k_1^*(1) = \iota_*(1) \]

where the equality \( \dagger \) follows from property (1). This completes the proof of (6.13).

The second step is to show the following. Let \( Y \) be an ind-scheme and let \( 0 \to N \to M \to F \to 0 \) be a short exact sequence of vector bundles on \( Y \) such that \( \text{rank}(F) = d \). Let \( \iota : \mathbb{P}(1 \oplus N) \to \mathbb{P}(1 \oplus M) \) be the inclusion map and let \( p : \mathbb{P}(1 \oplus M) \to Y \) be the projection. Then one has

\begin{equation}
(6.17) \quad \iota_*(1) = c_d(p^* F \otimes O_{1\oplus M}(1)).
\end{equation}

Using the splitting principle (Lemma 5.7) and property (1), we can assume that \( F = \bigoplus_{j=1}^d L_j \) is a direct sum of line bundles, in order to prove (6.17). Let \( M_j \) be the preimage of the direct sum of all summands of \( F \) except \( L_j \) and set \( D_j = \mathbb{P}(1 \oplus M_j) \). Then \( \{ D_j \} \) is a collection of strict smooth divisors on \( \mathbb{P}(1 \oplus M) \) which intersect transversely with the intersection \( \mathbb{P}(1 \oplus N) \). Moreover, the line bundle \( L(D_j) \) is isomorphic to the line bundle \( p^*(L_j) \otimes O_{1\oplus M}(1) \). It follows from (6.13)
that
\[
\begin{align*}
\ell_\ast(1) &= \bigcup_{j=1}^n c_1(L(D_j)) \\
&= \bigcup_{j=1}^n c_1(p^\ast(L_j) \otimes \mathcal{O}_{1 \oplus M}(1)) \\
&= cd \left( \left( \bigoplus_{j=1}^n p^\ast(L_j) \right) \otimes \mathcal{O}_{1 \oplus M}(1) \right) \\
&= cd \left( p^\ast(F) \otimes \mathcal{O}_{1 \oplus M}(1) \right)
\end{align*}
\]
which proves (6.17).

In the third step, we prove the composition property in the special case of the strict inclusions \( Y \xrightarrow{s'} \mathbb{P}(1 \oplus N) \xrightarrow{\iota} \mathbb{P}(1 \oplus M) \) in the situation of the second step.

Let \( \iota \circ s' \) denote the section of the projection map \( p \) using the identification \( Y = \mathbb{P}(1) \). We then show that
\[
(6.18) \quad s_\ast = \ell_\ast \circ s'_\ast.
\]

To prove this, we note that both sides are \( MGL^\ast \ast(Y) \)-linear maps and hence it suffices to show that \( s_\ast(1) = \iota_\ast \circ s'_\ast(1) \). Let \( n = \text{rank}(N) \). We then have in \( MGL^\ast \ast(\mathbb{P}(1 \oplus M)) \):
\[
\begin{align*}
\ell_\ast \circ s'_\ast(1) &= \iota_\ast \left[ c_n \left( p^\ast(N) \otimes \mathcal{O}_{1 \oplus N}(1) \right) \right] \\
&= \iota_\ast \left[ \iota^\ast \left( c_n \left( p^\ast(N) \otimes \mathcal{O}_{1 \oplus M}(1) \right) \right) \right] \\
&= \ell_\ast(1) \cup c_n \left( p^\ast(N) \otimes \mathcal{O}_{1 \oplus M}(1) \right) \\
&= \left[ c_d \left( p^\ast(F) \otimes \mathcal{O}_{1 \oplus M}(1) \right) \right] \cup \left[ c_n \left( p^\ast(N) \otimes \mathcal{O}_{1 \oplus M}(1) \right) \right] \\
&= c_{d+n} \left( p^\ast(M) \otimes \mathcal{O}_{1 \oplus M}(1) \right) \\
&= s_\ast(1)
\end{align*}
\]
which proves (6.18).

In the final step, we complete the proof of the composition property of the Gysin map. So let \( Z \xrightarrow{s} Y \xrightarrow{\iota} X \) be given strict closed embeddings of ind-schemes. We consider the deformation diagram
\[
(6.19) \quad \begin{array}{ccc}
Z & \xrightarrow{k_0} & Z \times \mathbb{A}^1 \xrightarrow{k_1} Z \\
\underset{s'}{\downarrow} & & \underset{\iota'}{\downarrow} \\
\mathbb{P}(1 \oplus N_Y(Z)) & \xrightarrow{j_0'} & M'_Y \\ & \underset{s_Y}{\downarrow} & \underset{\iota}{\downarrow} \\
\mathbb{P}(1 \oplus N_X(Z)) & \xrightarrow{j_0} & M'_X \\
\end{array}
\]
where \( s = s_Y \circ s' \), \( F^X = J \circ F^Y \) and \( M'_X \) is the proper transform of \( Y \times \mathbb{A}^1 \) in \( M'_X \). Moreover, all the squares are transverse. Using this transversality of the left squares, we get the relations
\[
\begin{align*}
s'_Y \circ s' \circ k_0^\ast &= s_Y \circ (j_0')_\ast \circ F^Y = (j_0^X)_\ast \circ J_\ast \circ F^Y \\
&= s_\ast \circ k_0^\ast = (j_0^X)_\ast \circ (J \circ F^Y)_\ast.
\end{align*}
\]
We have shown in (6.18) that \( s_Y \circ s'_* = s_* \). Thus we get

\[
(j_0^X)^* \circ [(J \circ F^Y)_* - J_* \circ F'_Y] = 0.
\]

Since \((J \circ F^Y)_*\) and \(J_* \circ F'_Y\) both vanish in \(MGL_{*t}(V'_{K})\) where \( V'_{K} = M'_{K} \setminus (Z \times A^1) \), it follows from Lemma 6.2 that \((J \circ F^Y)_* = J_* \circ F'_Y\).

Using the transversality of the right squares in the above diagram, we get the relations

\[
(t \circ t')_* \circ k_1^* = (j_1^X)^* \circ (J \circ F^Y)_* = (j_1^X)^* \circ J_* \circ F'_Y = \iota_* \circ (j_1^Y)^* \circ F'_Y = \iota_* \circ \iota'_* \circ k_1^*.
\]

Since \(k_1\) is an \(A^1\)-weak equivalence, we conclude that \((t \circ t')_* = \iota_* \circ \iota'_*\). This completes the proof of the composition property and hence the proof of the proposition. \qed

7. Basic properties of equivariant cobordism

Let \(G\) be a linear algebraic group over \(k\). The following result describes the basic properties of the equivariant motivic cobordism. Recall that a bigraded ring \(R = \bigoplus_{i,j \geq 0} R_{i,j}\) is called commutative if for \(a \in R_{i,j}, b \in R_{i',j'}\), one has \(ab = (-1)^{ii'}ba\).

Let \(\textbf{R}^*\) denote the category of commutative bigraded rings. We need the following elementary result in order to establish some basic properties of the equivariant cobordism.

Lemma 7.1. Let \(U \in \text{Sm}^G_{free/k}\) and let \(i : V_1 \hookrightarrow V_2\) be a closed (or open) immersion in \(\text{Sm}^G_{free/k}\). Consider a transverse square

\[
\begin{array}{ccc}
Y' & \xrightarrow{g} & Y \\
\downarrow{i'} & & \downarrow{i} \\
X' & \xrightarrow{f} & X
\end{array}
\]

in \(\text{Sm}^G_k\) where \(i\) is a closed embedding. Then the squares

\[
\begin{array}{ccc}
Y \times V_1 & \xrightarrow{i_Y} & Y \times V_2 \\
\downarrow{i_1} & & \downarrow{i_2} \\
X \times V_1 & \xrightarrow{i_X} & X \times V_2
\end{array} \quad \begin{array}{ccc}
Y' \times U & \xrightarrow{\overline{\iota}} & Y \times U \\
\downarrow{\overline{\iota'}} & & \downarrow{\overline{\iota}} \\
X' \times U & \xrightarrow{\iota'} & X \times U
\end{array}
\]

are transverse in \(\text{Sm}^G_k\).

Proof. This is an elementary exercise. We give only give a sketch and leave the details for the readers. One first checks that if (7.1) is any Cartesian square in
\( \text{Sm}^G_{\text{free}/k} \), then the squares

\[
\begin{array}{ccc}
Y'/G & \xrightarrow{g} & Y/G \\
\downarrow{\tau} & \downarrow{\pi} & \downarrow{\pi'} \\
X'/G & \xrightarrow{f} & X/G
\end{array}
\]

are also Cartesian. In particular, the map \( T_{X'/G} \rightarrow f^* T_{X/G} \) of relative tangent bundles is an isomorphism. From this, it follows immediately that

\[
\pi'^* N_{X/G} \rightarrow N_{X'/G}(X'/G)
\]

if \( f \) is a closed immersion.

To prove the lemma, it suffices to show that if (7.1) is a transverse square in \( \text{Sm}^G_{\text{free}/k} \), then the associated square of quotients in (7.3) is also transverse. To do this, we now only need to show the appropriate isomorphism of the normal bundles. We consider the commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{g} & Y \\
\downarrow{\pi'} & \downarrow{\pi} & \downarrow{\pi} \\
Y'/G & \xrightarrow{\overline{f}} & Y/G \\
\downarrow{\pi} & \downarrow{\pi} & \downarrow{\pi} \\
X'/G & \xrightarrow{f} & X \\
\downarrow{\pi} & \downarrow{\pi} & \downarrow{\pi} \\
X'/G & \xrightarrow{\overline{f}} & X/G.
\end{array}
\]

Let \( N \) (resp. \( N' \)) denote the normal bundle of \( Y/G \) (resp. \( Y'/G \)) in \( X/G \) (resp. \( X'/G \)). Since \( \pi' \) is a smooth covering, it suffices to show that the map

\[
\pi'^* N' \rightarrow \pi'^* (\overline{g}^* N)
\]

is an isomorphism. But this easily follows from the fact that the back face of the above cube is transverse and we have shown above that the two side faces are also transverse.

\[ \square \]

**Theorem 7.2.** The equivariant motivic cobordism \( MGL^*_G(-) \) is an oriented cohomology theory on \( \text{Sm}^G_k \) in the following sense.

1. **Contravariance:** \( X \mapsto MGL^*_G(X) \) is a functor \( (\text{Sm}^G_k)^{\text{op}} \rightarrow R^* \).
2. **Homotopy Invariance:** For a \( G \)-equivariant vector bundle \( p : E \rightarrow X \), the map \( p^* : MGL^*_G(X) \rightarrow MGL^*_G(E) \) is an isomorphism.
3. **Chern classes:** For a \( G \)-equivariant vector bundle \( E \) on \( X \), there are equivariant Chern classes \( c_i^G(E) \in MGL^G_{2i}(X) \) such that \( c_i^G(E) = 1, c_i^G(E) = 0 \) for \( i > \text{rank}(E) \), \( f^* (c_i^G(E)) = c_i^G(f^*(E)) \) for a morphism \( f : Y \rightarrow X \) in \( \text{Sm}^G_k \) and \( c_i^G(E) = c_i^G(E') \cdot c_i^G(E'') \) for an exact sequence of equivariant vector bundles \( 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \) on \( X \).
(4) Projective bundle formula: For an equivariant vector bundle $E$ of rank $n$ on $X$, the map

$$\Phi_X : MGL_G^*(X) \oplus \cdots \oplus MGL_G^*(X) \to MGL_G^* (\mathbb{P}(E)) ;$$

$$\Phi (a_0, \cdots , a_{n-1}) = \sum_{i=0}^{n-1} \pi^*(a_i) : \xi^i$$

is an isomorphism, where $\pi : \mathbb{P}(E) \to X$ is the projection map and $\xi = c^G_1 (\mathcal{O}_E(-1)).$

(5) Gysin map: For a closed embedding $\iota : Y \hookrightarrow X$ in $\text{Sm}_k^G$ of codimension $d$, there is a Gysin map $\iota_* : MGL_G^{a,b} (Y) \to MGL_G^{a+2d,b+d} (X)$ such that given a transverse square

$$\begin{array}{ccc}
  Y' & \overset{g}{\to} & Y \\
  \iota' \downarrow & & \downarrow \iota \\
  X' & \overset{f}{\to} & X
\end{array}$$

in $\text{Sm}_k^G$ where $\iota$ is a closed embedding, the diagram

$$\begin{array}{ccc}
  MGL_G^{a,b} (Y) & \overset{\iota^*}{\to} & MGL_G^{a,b} (Y') \\
  \iota_* \downarrow & & \downarrow \iota'_* \\
  MGL_G^{a,b} (X) & \overset{f^*}{\to} & MGL_G^{a,b} (X')
\end{array}$$

commutes. Moreover, given $G$-equivariant closed embeddings $Z \overset{\iota'}{\hookrightarrow} Y \hookrightarrow X$, one has $(\iota \circ \iota')_* = \iota_* \circ \iota'_*.$

(6) Gysin exact sequence: For a closed embedding $\iota : Y \hookrightarrow X$ of codimension $d$ in $\text{Sm}_k^G$ with the complement $j : U \hookrightarrow X$, the sequence

$$\cdots \to MGL_G^{a-1,b} (U) \overset{\partial}{\to} MGL_G^{a-2d,b-d} (Y) \overset{\iota_*}{\to} MGL_G^{a,b} (X) \overset{\partial}{\to} MGL_G^{a,b} (U) \overset{\partial}{\to} \cdots$$

is exact.

(7) Change of groups: If $H \subseteq G$ is a closed subgroup and $X \in \text{Sm}_k^G$, then there is a natural restriction map $\rho^G_H, X : MGL_G^{*,*} (X) \to MGL_H^{*,*} (X).$ In particular, there is a natural forgetful map

$$\rho^G_X : MGL_G^{*,*} (X) \to MGL^{*,*} (X).$$

(8) Morita Isomorphism: If $H \subseteq G$ is a closed subgroup and $X \in \text{Sm}_k^H$, then there is a canonical isomorphism $MGL_G^{*,*} (X \times G) \cong MGL_H^{*,*} (X).$

(9) Free action: If $X \in \text{Sm}_G^{free/k}$, then the map $MGL_G^{*,*} (X/G) \to MGL_G^{*,*} (X)$ is an isomorphism.

Proof. Any map $f : Y \to X$ in $\text{Sm}_k^G$ gives rise to the corresponding map of ind-schemes $f_G : Y_G \to X_G$ which in turn induces the map $f^* = f_G^* : MGL_G^{*,*} (X) = MGL^* (X_G) \to MGL^* (Y_G) = MGL_G^{*,*} (Y).$ If $p : E \to X$ is a $G$-equivariant vector bundle of rank $n$, then we have seen in the proof of Lemma 7.1 that the
map $p : E_G \to X_G$ is a vector bundle of rank $n$ over the ind-scheme $X_G$. In particular, it is an $\mathbb{A}^1$-weak equivalence. This proves property (2).

If $E$ is a $G$-equivariant vector bundle on $X$, then equivariant Chern classes $c_i^G(E)$ are defined by

$$
(7.9) \quad c_i^G(E) := c_i(E_G) \in MGL^{2i,i}(X_G) = MGL_G^{2i,i}(X).
$$

The fact that this defines a Chern class theory for equivariant bundles in $MGL_G^{*,*}(-)$ follows immediately from Proposition 5.8.

Let $E$ be a $G$-equivariant vector bundle of rank $n$ on $X$ and let $p : \mathbb{P}(E) \to X$ be the associated equivariant projective bundle. Let $p_G : \mathbb{P}(E_G) \to X_G$ denote the projective bundle associated to the vector bundle $E_G$ on $X_G$. The desired projective bundle formula then follows from (5.10) and the canonical isomorphism of ind-schemes $\mathbb{P}(E_G) \simeq \mathbb{P}(E)_G$.

If $i : Y \hookrightarrow X$ is an equivariant closed embedding, then it follows from Lemma 7.1 that $i_G : Y_G \hookrightarrow X_G$ is a strict closed embedding of ind-schemes. The equivariant Gysin map $i_* : MGL_G^{a,b}(Y) \to MGL_G^{a+2d,b+d}(X)$ is defined as in Proposition 6.4. Given a transverse square (7.6) in $\text{Sm}_G^*$, it follows from Lemma 7.1 that corresponding square of Borel spaces is also transverse. The commutativity of the square (7.7) and the composition property now follow from Proposition 6.4.

To prove the Gysin exact sequence, we first note that if $i : Y \hookrightarrow X$ is an equivariant closed embedding with complement $j : U \hookrightarrow X$, then Lemma 7.1 implies that $i_G : Y_G \hookrightarrow X_G$ is a closed embedding with complement $U_G$. The exact sequence (6) now follows from Proposition 6.4.

If $H \subseteq G$ is a closed subgroup and $X \in \text{Sm}_k^G$, then $p : X_H \to X_G$ is a morphism of ind-schemes with fibers $G/H$. This induces the restriction map $r^G_{H,X} : MGL_H^{*,*}(X) \to MGL_H^{*,*}(X)$. Taking $H = \{e\}$ and using the isomorphism $X_{\{e\}} \cong X$, we get the forgetful map $r^G_{X,X}$. If $X \in \text{Sm}_k^H$, then the isomorphism $MGL_H^{*,*}(X \times_G H) \cong MGL_H^{*,*}(X)$ follows from Corollary 2.7. The last property about the free action follows from [28, Lemma 4.2.9].

### 7.1. Self-intersection formula

We now prove the self-intersection formula for the equivariant motivic cobordism. Let $i : Y \hookrightarrow X$ be a closed embedding of codimension $d \geq 0$ in $\text{Sm}_k^G$ and let $N_X(Y)$ denote the equivariant normal bundle of $Y$ in $X$.

**Proposition 7.3.** For any $a \in MGL^{*,*}_G(Y)$, one has $i^* \circ i_*(a) = c^G_d(N_X(Y)) \cdot a$.

**Proof.** It follows from Lemma 7.1 that $i_G : Y_G \hookrightarrow X_G$ is a strict closed embedding of ind-schemes with normal bundle $(N_X(Y))_G$. Using the definitions of the equivariant cobordism and the equivariant Chern classes, it suffices to show that if $i : Y \hookrightarrow X$ is a strict closed embedding of ind-schemes of codimension $d \geq 0$ with normal bundle $N$, then $i^* \circ i_*(a) = c^G_d(N) \cdot a$ for all $a \in MGL^{*,*}(Y)$. So we prove this statement.

To prove this, we consider the diagram (6.3) and make the following claim.

**Claim.** Given any $a \in MGL^{*,*}(Y)$, there exists $b \in MGL^{*,*}(M)$ such that $i_*(a) = j_1^*(b)$ and $(i_N)_*(a) = j_0^*(b)$. 

Proof of the Claim: Set \( b = F^* \circ p^*(a) \in MGL^{*,*}(M) \). We then have
\[
\iota_* (a) = \iota_* \circ (p \circ k_1)^* (a) = \iota_* \circ k_1^* \circ p^*(a) = j_1^* \circ F^* \circ p^*(a) = j_1^* (b)
\]
where the equality \( = \dagger \) follows from the transversality of the top right square in (6.3) and Proposition 6.4. On the other hand, we have
\[
(i_N)_* (a) = (i_N)_* \circ (p \circ k_0)^* (a) = (i_N)_* \circ k_0^* \circ p^*(a) = j_0^* \circ F^* \circ p^*(a) = j_0^* (b)
\]
where the equality \( = \ddagger \) follows from the transversality of the top left square in (6.3) and Proposition 6.4. This proves the claim.

We now prove the self-intersection formula for ind-schemes. There is nothing to prove if \( d = 0 \) and so we assume that \( d \geq 1 \). We first consider the case when \( p_N: N \to Y \) is a vector bundle of rank \( d \) and \( \iota: Y \hookrightarrow N \) is the zero section embedding of ind-schemes. Let \( p: \mathbb{P}(1 \oplus N) \to Y \) be the projectivization of \( N \), giving us the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\iota} & Y \\
\downarrow s & & \downarrow s \\
N & \xrightarrow{j} & \mathbb{P}(1 \oplus N)
\end{array}
\]
which is clearly transverse. In particular, we have \( \iota_* = j^* \circ s_* \) using property (1) of Proposition 6.4. Combining this the property (4) of Proposition 6.4 we get
\[
\iota^* \circ \iota_* (a) = \iota^* [j^* \circ s_*(a)] = \iota^* [j^* (c_d (p^*(N) \otimes O_{1\oplus N}(1)) \cdot p^*(a))] = \iota^* [c_d (j^* (p^*(N) \otimes O_{1\oplus N}(1))) \cdot j^* \circ p^*(a)] = \iota^* (c_d (p_N^*(N))) \cdot \iota^* \circ p_N^*(a) = c_d (N) \cdot a.
\]

In the general case, we fix \( a \in MGL^{*,*}(Y) \) and let \( b \in MGL^{*,*}(M) \) be as in Claim 7.1. We then have
\[
\iota^* \circ \iota_* (a) = \iota^* \circ j_1^* (b) = k_1^* \circ F^* (b) = k_0^* \circ F^* (b) = i_N^* \circ j_0^* (b) = i_N^* \circ (i_N)_* (a) = c_d (N) \cdot a.
\]
This completes the proof of the Self-intersection formula.

8. Equivariant cobordism for torus actions

In this section, we study certain special features of the equivariant motivic cobordism when the underlying group is a torus. Our main result is to show that the equivariant motivic cobordism of smooth projective schemes with a torus action has a simple description. We shall see later in this paper that the equivariant cobordism for the action of a connected reductive group can be described in terms
of the equivariant cobordism for the action of maximal tori of the group. As applications of these results, we shall compute the equivariant and ordinary motivic cobordism of various classes of smooth schemes with group actions.

8.1. Borel-Moore homology associated to motivic cobordism. Let $\text{SP}$ denote the category of pairs $(M, X)$ with $M \in \text{Sm}_k$ and $X \subseteq M$ a closed subset (possibly singular); a morphism $f : (M, X) \to (N, Y)$ is a morphism $f : M \to N$ such that $f^{-1}(Y) \subseteq X$. An object of the form $(M, X)$ is called a smooth pair. Recall from [30] that an oriented cohomology theory $A$ on $\text{SP}$ is a contravariant functor $(M, X) \mapsto A_X(M)$ with values in abelian groups together with a functor $\partial : A(M \setminus X) \to A_X(M)$ which satisfies Gysin exact sequence, excision, homotopy invariance and Chern classes for vector bundles.

Let $\text{SP}'$ denote the category of smooth pairs $(M, X)$ where a morphism $f : (M, X) \to (N, Y)$ is a projective morphism $f : M \to N$ such that $f(X) \subseteq Y$. Let $A$ be an oriented bi-graded ring cohomology theory on $\text{SP}$, an integration with supports on $A$ is an assignment of a bi-graded push-forward map $f_* : A_X(M) \to A_Y(N)$ which satisfies the usual functoriality properties, compatibility with pull-back and Chern class operators (cf. [25, Definition 1.8]). Levine shows that every oriented ring cohomology theory on $\text{SP}$ has a unique integration with supports.

Using the existence of integration with supports, Levine [25] has further shown that any given oriented bi-graded ring cohomology theory $A$ on $\text{SP}$ uniquely extends to an oriented bi-graded Borel-Moore homology theory $H$ on $\text{Sch}_k$ such that the pair $(H, A)$ is an oriented duality theory on $\text{Sch}_k$ in the sense of [24, Definition 3.1]. In particular, the Borel-Moore homology theory $H$ has projective push-forward, pull-back under open immersion and smooth projection, Gysin exact sequence, weak homotopy invariance, exterior product, Chern class operators and the Poincaré duality $H_{a,b}(X) \cong A^{2d-a,d-b}(X)$ if $X$ is smooth of dimension $d$.

For $X \in \text{Sch}_k$, $H(X)$ is defined by choosing a closed embedding $X \subseteq M$ with $M \in \text{Sm}_k$ (which is possible since $X$ is quasi-projective) and setting $H(X) := A_X(M)$. The main point of [25] is to show that this is well defined and has all the properties mentioned above. The projective push-forward is constructed by showing that given smooth pairs $(M, X), (N, Y)$ and a map $f : M \to N$ such that $f|_X : X \to Y$ is projective, the orientation and the excision property of $A$ yield a well-defined map $f_* : A_X(M) \to A_Y(N)$.

Apart from the above, the homology theory $H$ also satisfies the following commutativity property.

Lemma 8.1. Let

$$
\begin{array}{ccc}
U' & \xrightarrow{j'} & X' \\
\downarrow g & & \downarrow f \\
U & \xrightarrow{j} & X
\end{array}
$$

be a Cartesian square in $\text{Sch}_k$ such that $f$ is projective and $j$ is an open immersion. One has then, $j^* \circ f_* = g_* \circ j'^* : H(X') \to H(U)$.

Proof. We can write $f = p \circ i$ where $i : X' \hookrightarrow \mathbb{P}^n \times X$ is a closed embedding and $p : \mathbb{P}^n \times X \to X$ is the projection map. Since $f_* = p_* \circ i_*$, one easily checks that
the assertion of the lemma holds if it holds when $f$ is a closed immersion and a projection $\mathbb{P}^n \times X \to X$. Assume first that $f : X' \to X$ is a closed immersion.

We can embed the above Cartesian diagram into a commutative diagram

\[ (8.1) \]

\[
\begin{array}{ccc}
U' & \xrightarrow{j'} & X' \\
\downarrow{g} & & \downarrow{f} \\
U & \xrightarrow{j} & X \\
\downarrow{i} & & \downarrow{id} \\
M & \xrightarrow{id} & N \\
\end{array}
\]

where $N$ is smooth, $i : X \hookrightarrow N$ is a closed immersion, $J$ is an open immersion and the front square is Cartesian. Since the top and the bottom squares are also Cartesian, the same holds for the back square too. Moreover, the bottom square is clearly transverse whose vertices are smooth. The desired equality $j'^* \circ f_* = g_* \circ j'^*$ now follows immediately from [25, Definitions 2.7-A(4) and 3.1-A(1), A(2)].

If $f : \mathbb{P}^n \times X \to X$ is the projection map, then we only have to replace $N \xrightarrow{id} N$ and $M \xrightarrow{id} M$ in the bottom square of the diagram (8.1) by $\mathbb{P}^n \times N \xrightarrow{p} N$ and $\mathbb{P}^n \times M \xrightarrow{p} M$ respectively. This makes all squares Cartesian and the bottom square transverse with all vertices smooth. One concludes the proof as before using [25, Definitions 2.7-A(4) and 3.1-A(1), A(2)].

The Borel-Moore homology theory associated to the motivic cobordism $MGL$ is denoted by $MGL'$. It is defined by setting $MGL'(X) = MGL_X(M) := MGL(M/U)$, where $(M, X)$ is a smooth pair with $U = M \setminus X$. Our study of the equivariant cobordism of smooth projective schemes with torus action is based on the following result about $MGL'$.

**Proposition 8.2.** Let $X$ be a $k$-scheme with a filtration by closed subschemes

\[ (8.2) \]

\[
\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X
\]

such that there are maps $\phi_m : W_m = (X_m \setminus X_{m-1}) \to Z_m$ for $0 \leq m \leq n$ which are all vector bundles. Assume moreover that each $Z_m$ is smooth and projective. Then there is a canonical isomorphism

\[
\bigoplus_{m=0}^{n} MGL'_{*,*}(Z_m) \cong MGL'_{*,*}(X).
\]

**Proof.** We prove it by induction on $n$. For $n = 0$, the map $X = X_0 \xrightarrow{\phi_0} Z_0$ is a vector bundle over a smooth scheme and hence the proposition follows from the homotopy invariance of the motivic cobordism. We now assume by induction that $1 \leq m \leq n$ and

\[ (8.3) \]

\[
\bigoplus_{j=0}^{m-1} MGL'_{*,*}(Z_j) \cong MGL'_{*,*}(X_{m-1}).
\]
The Gysin exact sequence for the inclusions $i_{m-1} : X_{m-1} \hookrightarrow X_m$ and $j_m : W_m = X_m \setminus X_{m-1}$ of the closed and open subschemes yields a long exact sequence

\[(8.4) \quad \cdots \rightarrow MGL'_{s,*}(X_{m-1}) \xrightarrow{i_{(m-1)*}} MGL'_{s,*}(X_m) \xrightarrow{j^*_m} MGL'_{s,*}(W_m) \xrightarrow{\partial} \cdots \]

Using (8.3), it suffices now to construct a canonical splitting of the pull-back and the maps $X \xrightarrow{j_m} \cdots \rightarrow (8.4)$

This proves the commutativity of (8.6) and hence the proposition. □

Let $V_m \subset W_m \times Z_m$ be the graph of the projection $W_m \xrightarrow{\phi_m} Z_m$ and let $\overline{V}_m$ denote the closure of $V_m$ in $X_m \times Z_m$. Let $Y_m \rightarrow \overline{V}_m$ be a resolution of singularities. Since $V_m$ is smooth, we see that $V_m \xrightarrow{j_m} Y_m$ as an open subset. We consider the composite maps

\[(8.5) \quad p_m : V_m \hookrightarrow W_m \times Z_m \rightarrow W_m, \quad q_m : V_m \hookrightarrow W_m \times Z_m \rightarrow Z_m \quad \text{and} \quad \overline{p}_m : Y_m \rightarrow X_m \times Z_m \rightarrow X_m, \quad \overline{q}_m : Y_m \rightarrow X_m \times Z_m \rightarrow Z_m.

Note that $\overline{p}_m$ is a projective morphism since $Z_m$ is projective. The map $q_m$ is smooth and $p_m$ is an isomorphism. We consider the diagram

\[(8.6) \quad \begin{array}{ccc}
MGL'_{s,*}(Z_m) & \xrightarrow{\overline{q}_m} & MGL'_{s,*}(Y_m) \\
\phi_m^* & \cong & \overline{p}_m^* \\
MGL'_{s,*}(W_m) & \xleftarrow{j^*_m} & MGL'_{s,*}(X_m).
\end{array}
\]

Note that the maps $\overline{p}_m^*$ and $j^*_m$ exist by the above mentioned properties of $MGL'$ and the maps $\overline{q}_m$ and $\phi_m^*$ exist by the standard functoriality of $MGL$ as $Z_m, W_m$ and $Y_m$ are all smooth.

The map $\phi_m^*$ is an isomorphism by the homotopy invariance of the $MGL$-theory. It suffices to show that this diagram commutes. For, the map $s_m := \overline{p}_m^* \circ \overline{q}_m^* \circ \phi_m^{-1}$ will then give the desired splitting of the map $j_m^*$. We now consider the commutative diagram

\[
\begin{array}{ccc}
X_m & \xrightarrow{j_m} & W_m \\
\overline{p}_m \downarrow & & \downarrow p_m \\
Y_m & \xrightarrow{j_m} & W_m \\
\overline{q}_m \downarrow & & \downarrow q_m \\
Z_m & \xrightarrow{(id, \phi_m)} & W_m,
\end{array}
\]

Since the top left square is Cartesian with $Y_m$ smooth and $j_m$ an open immersion, it follows from Lemma 8.1 that $j_m^* \circ \overline{p}_m^* = p_m^* \circ j_m^*$. Now, using the fact that $(id, \phi_m)$ is an isomorphism, we get

\[
j_m^* \circ \overline{p}_m^* \circ \overline{q}_m^* = p_m^* \circ j_m^* \circ \overline{q}_m^* = p_m^* \circ (id, \phi_m)_* \circ (id, \phi_m)^* \circ q_m^* = id_* \circ \phi_m^* = \phi_m^*.
\]

This proves the commutativity of (8.6) and hence the proposition. □
8.2. Equivariant cobordism of filtrable schemes. Recall that a linear algebraic group $T$ over $k$ is said to be a split torus if it is isomorphic to $(\mathbb{G}_m)^r$ as a group scheme over $k$ where $n \geq 1$ is a positive integer, called the rank of the torus. We shall assume all tori to be split in this section.

We recall from [7, Section 3] that a $k$-scheme $X$ with an action of a torus $T$ is called filtrable if the fixed point locus $X^T$ is smooth and projective, and there is an ordering $X^T = \prod_{m=0}^{n} Z_m$ of the connected components of the fixed point locus, a filtration of $X$ by $T$-invariant closed subschemes

\[(8.7)\]
\[\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X\]
and maps $\phi_m : W_m = (X_m \setminus X_{m-1}) \to Z_m$ for $0 \leq m \leq n$ which are all $T$-equivariant vector bundles. It is important to note that the closed subschemes $X_m$'s may not be smooth even if $X$ is so. The following result was proven by Białynicki-Birula [3] when $k$ is algebraically closed and by Hesselink [14] in general.

**Theorem 8.3** (Białynicki-Birula, Hesselink). Let $X$ be a smooth projective scheme with an action of $T$. Then $X$ is filtrable.

8.2.1. Canonical admissible gadgets. Let $T$ be a split torus of rank $r$. For a character $\chi$ of $T$, let $L_\chi$ denote the one-dimensional representation of $T$ where $T$ acts via $\chi$. Given a basis $\{\chi_1, \ldots, \chi_r\}$ of the character group $\widehat{T}$ of $T$ and given $i \geq 1$, we set $V_i = \prod_{j=1}^{r} L_{\chi_j}^i$ and $U_i = \prod_{j=1}^{r} (L_{\chi_j}^i \setminus \{0\})$. Then $T$ acts on $V_i$ by $(t_1, \ldots, t_r)(x_1, \ldots, x_r) = (\chi_1(t_1)(x_1), \ldots, \chi_r(t_r)(x_r))$. It is then easy to see that $\rho = (V_i, U_i)_{i \geq 1}$ is an admissible gadget for $T$ such that $\rho/w \to (\mathbb{P}^{i-1})^r$. Moreover, the line bundle $L_{\chi_j} \times (L_{\chi_j}^i \setminus \{0\}) \to \mathbb{P}^{i-1}$ is the line bundle $\mathcal{O}(\pm 1)$ for each $1 \leq j \leq r$. An admissible gadget for $T$ of this form will be called a canonical admissible gadget in this text.

Let $X \in \text{Sm}_k^T$ be a filtrable scheme with the filtration given by (8.7). Let $\rho = (V_i, U_i)_{i \geq 1}$ be a canonical admissible gadget for $T$ and set

\[X^i = X \times U_i, \quad X_m^i = X_m \times U_i, \quad W_m^i = W_m \times U_i \quad \text{and} \quad Z_m^i = Z_m \times U_i.\]

Given the $T$-equivariant filtration of $X$ as in (8.7), it is easy to see that for each $i \geq 1$, there is an associated system of filtrations

\[(8.8)\]
\[\emptyset = X_{i-1}^i \subseteq X_0^i \subseteq \cdots \subseteq X_n^i = X^i\]
and maps $\phi_m : W_m^i = X_m^i \setminus X_{m-1}^i \to Z_m^i$ for $0 \leq m \leq n$ which are all vector bundles. Moreover, as $T$ acts trivially on each $Z_m$, we have that $Z_m^i \cong Z_m \times (U_i/T) \cong Z_m \times (\mathbb{P}^{i-1})^r$. Since $Z_m$ is smooth and projective, this in turn implies that each $Z_m^i$ is smooth and projective. We conclude that the filtration (8.8) of $X^i$ satisfies all the conditions of Proposition 8.2. In particular, there are split exact sequences

\[(8.9)\]
\[0 \to MGL'_{s,s} (X^i_{m-1}) \to MGL'_{s,s} (X^i_m) \to MGL'_{s,s} (W_m^i) \to 0\]
for all $0 \leq m \leq n$ and $i \geq 1$. 
Lemma 8.4. Let $X \in \mathbf{Sm}_k^T$ be smooth and projective and let $a \geq b \geq 0$. Then for any admissible gadget $\rho = (V_i, U_i)_{i \geq 1}$ for $T$, the inverse system $\{ \text{MGL}^a_b(X \times U_i) \}$ satisfies the Mittag-Leffler condition. In particular, the map

$$\text{MGL}^a_b(X) \to \varprojlim_i \text{MGL}^a_b(X \times U_i)$$

is an isomorphism.

Proof. Since $\text{MGL}^a_b(X) = \text{MGL}^a_b(\text{X}_G(\rho))$, the second assertion follows from the Mittag-Leffler condition using Proposition 5.3. To prove the Mittag-Leffler condition, we first assume that $\rho$ is a canonical admissible pair and prove the stronger assertion that the restriction map $\text{MGL}^a_b(X^{i+1}) \to \text{MGL}^a_b(X^i)$ is surjective for all $i \geq 1$, where $X^i = X \times U_i$.

Since the map $\text{MGL}^a_b(X \times (U_i \oplus W_i)) \to \text{MGL}^a_b(X^i)$ is an isomorphism by the homotopy invariance, we only have to show that the map $\text{MGL}^a_b(X^{i+1}) \to \text{MGL}^a_b(X^i)$ induced by the open immersion is an isomorphism.

It follows from Theorem 8.3 that $X$ is filtrable. Consider a $T$-equivariant filtration of $X$ as in (8.7). Set $\overline{X}_i = X \times (U_i \oplus W_i)$. We show by induction on $m \geq 0$ that the pull-back map $\text{MGL}^a_b(X^{i+1}_m) \to \text{MGL}^a_b(X^i_m)$ induced by the open immersion, is surjective for all $m \geq 0$.

For any $0 \leq m \leq n$, there is a commutative diagram

(8.10) $\text{MGL}^a_b(Z^{i+1}_m) \longrightarrow \text{MGL}^a_b(Z_m) \longrightarrow \text{MGL}^a_b(Z^i_m)$

of the motivic cobordism of smooth schemes where all the vertical arrows are isomorphisms by the homotopy invariance. The left horizontal arrows in both rows are isomorphisms again by the homotopy invariance.

Next, we observe that $T$ acts trivially on each $Z_m$ and hence $Z^i_m \cong Z_m \times (U_i/T) \cong Z_m \times (\overline{\mathbb{P}^{i-1}})^T$. Hence the projective bundle formula for the motivic cobordism (cf. §5.2.1) implies that the map $\text{MGL}^a_b(Z^{i+1}_m) \to \text{MGL}^a_b(Z^i_m)$ is surjective. In particular, all the arrows on the top row in (8.10) are surjective.

We conclude that the map $\text{MGL}^a_b(W^{i+1}_m) \to \text{MGL}^a_b(W^i_m)$ is surjective for all $0 \leq m \leq n$ and all $i \geq 1$. Taking $m = 0$, we see in particular that the map $\text{MGL}^a_b(X_0^{i+1}) \to \text{MGL}^a_b(X^i_0)$ is surjective for all $i \geq 1$. 


Assume now that $m \geq 1$ and that this surjectivity assertion holds for all $j \leq m - 1$. We consider the diagram

\begin{equation}
0 \to MGL_{*,*}'(X_{m-1}^{i+1}) \to MGL_{*,*}'(X_m^{i+1}) \to MGL_{*,*}'(W_m^{i+1}) \to 0
\end{equation}

The left square is commutative by Lemma 8.1 and the right square is commutative by the functoriality of open pull-back (cf. [25, p. 34]). The left vertical arrow is surjective by induction and we have shown above that the right vertical arrow is surjective. The top sequence is exact by (8.9). Suppose we know that the bottom sequence is also exact. It will then follow that the middle vertical arrow is surjective. Thus we are only left with showing that the bottom sequence of (8.12) is exact.

Using the Gysin exact sequence, it is enough to show that the open pull-back $MGL_{*,*}'(X_m^{i}) \to MGL_{*,*}'(W_m^{i})$ is surjective. This is equivalent to showing that the map $MGL_{*,*}'(X_m^{i}) \to MGL_{*,*}'(W_m^{i})$ is surjective. For this, we consider the diagram

\begin{equation}
0 \to MGL_{X_{m-1}^{i}}^{*,*} (X^{i}) \to MGL_{X_{m}^{i}}^{*,*} (X^{i}) \to MGL_{X_{m}^{i}}^{*,*} (W_{m}^{i}) \to 0
\end{equation}

where the top sequence is exact by (8.9). The vertical arrows are the pull-back maps induced by the vector bundle $p : X^{i} \to X_{m}^{i}$ on the smooth scheme $X^{i}$ (cf. [25, Definition 2.7-A(6)]). Hence the right vertical arrow is an isomorphism by the homotopy invariance of the motivic cobordism since $W_{m}^{i}$ is smooth. It follows that the bottom horizontal arrow is surjective. This completes the proof of the first assertion for a canonical admissible pair.

Let us now assume that $\rho = (V_i, U_i)$ is any admissible pair for $T$ and let $\rho' = (V'_i, U'_i)$ be a canonical pair. Set $X^{ii} = X^T \times U'_i$ and $Y^{i,j} = X^T \times (U_i \oplus U'_j)$.

Fix $i_0 \geq 1$. We have shown in the proof of Lemma 9.2 that there exists $s_0 \gg 0$ such that the map $\alpha_{i_0,j}^* : MGL^{a,b}(X^{i_0}) \to MGL^{a,b}(Y^{i_0,j})$ is an isomorphism for all $j \geq s_0$. By reversing the role of the admissible pairs, let $i_1 \gg i_0$ be such that the map $\beta_{i,s_0}^* : MGL^{a,b}(X^{i_0}) \to MGL^{a,b}(Y^{i,s_0})$ is an isomorphism for all $i \geq i_1$. 
Let us now fix an element \( a \in \text{Image} \left( MGL_{a,b}^i(X^i) \to MGL_{a,b}^j(X^j) \right) \). For any \( i \geq i_1 \), we get a commutative diagram

\[
\begin{array}{ccc}
MGL_{a,b}^i(X^i) & \xrightarrow{=} & MGL_{a,b}^i(Y^{i,s_1})
\\ & \downarrow & \downarrow
\\ MGL_{a,b}^{i_i} & \xrightarrow{=} & MGL_{a,b}^{i_1,s_0}
\\ & \downarrow & \downarrow
\\ MGL_{a,b}^{i_0} & \xrightarrow{=} & MGL_{a,b}^{i,s_0}
\end{array}
\]

in which \( s_1 \gg s_0 \) is chosen so that the top left horizontal map is an isomorphism. We have shown above that the extreme right vertical arrow is surjective. An easy diagram chase shows that \( a \in \text{Image} \left( MGL_{a,b}^i(X^i) \to MGL_{a,b}^j(X^j) \right) \). This completes the proof of the lemma. \( \Box \)

The following consequence of Lemma 8.4 improves Corollary 10.3 in the special cases of torus action.

**Corollary 8.5.** For \( X \in \text{Sm}_k^T \) be smooth and projective. Then for \( q \geq 0 \), there is a natural isomorphism

\[
(8.13) \quad \Omega_T^q(X) \xrightarrow{\cong} MGL_T^{2q,q}(X).
\]

**Proof.** Let \( \rho = (V_i, U_i)_{i \geq 1} \) be a canonical admissible gadget for \( T \). It follows from [19, Theorem 6.1] that there is a natural isomorphism \( \lim_i \Omega^q \left( X \times U_i \right) \xrightarrow{\cong} \Omega_T^q(X) \).

On the other hand, the map \( \Omega^q \left( X \times U_i \right) \to MGL_T^{2q,q} \left( X \times U_i \right) \) is an isomorphism for each \( i \geq 1 \) by [26, Theorem 3.1]. The corollary now follows by applying Lemma 8.4. \( \Box \)

Let \( X \in \text{Sm}_k^T \) be smooth and projective with a filtration as in (8.7). Set \( S_m = X \setminus X_{m-1} \) for \( 0 \leq m \leq n \). Let \( V_m \subset W_m \times Z_m \) be the scheme defined in (8.5) and let \( Y_m \to V_m \) be the canonical \( T \)-equivariant resolution of singularities. One easily checks that all the maps in (8.5) then become \( T \)-equivariant.

Let \( \rho = (V_i, U_i)_{i \geq 1} \) be a canonical admissible gadget for \( T \). This yields for every \( 0 \leq m \leq n \) and \( i \geq 1 \), the maps \( j_m^i : (S_m^i, W_m^i) \to (X_i^i, X_m^i) \) in \( \text{SP} \) and the maps \( \overline{f}_m^i : (Y_m^i, Y_m^i) \to (X_i^i, X_m^i) \) in \( \text{SP}' \). Note also that for every \( i \geq 1 \), there is a closed immersion \( \gamma_m^i : X_i^i \hookrightarrow X_i^{i+1} \), which is natural with respect to maps in \( \text{Sch}_k^T \).

**Lemma 8.6.** Let \( X \in \text{Sm}_k^T \) be smooth and projective and fix \( 0 \leq m \leq n \) and \( i \geq 1 \). Consider the notations of (8.5) and (8.6) where now all the maps are
Then the diagram

\[(8.14) \quad MGL^* (Z_{m+1}^i) \xymatrix{ \ar[rr]^-{\varphi^*_{m}} & & MGL^* (Y_{m+1}^i) \ar[ll]_*+<0pt,0pt>^{m} } \]

commutes.

**Proof.** We have shown in the proof of Proposition 8.2 (cf. (8.6)) that the front and the back squares commute. The left and the top squares commute by the functoriality of pull-backs in $SP$. Notice that the map $MGL^* (W_{m+1}^i) \to MGL^* (Y_{m+1}^i)$ induced by the the inclusion $(W_{m+1}^i, W_{m+1}^i) \to (S_{m+1}^i, W_{m+1}^i)$ is an isomorphism. Thus the bottom square commutes again by the functoriality of pull-backs in $SP$. We only need to explain why does the right square commute.

To do this, we consider another diagram

\[(8.15) \quad Y_{m+1}^i \xymatrix{ \ar[rr]^-{P_m^*} & & X_{m+1}^i \ar[ll]_*+<0pt,0pt>^{m} } \]

induced by the maps $(X^i, X_{m}^i) \to (X_{m+1}^i, X_{m+1}^i)$, $(Y^i, Y_{m}^i) \to (Y_{m+1}^i, Y_{m+1}^i)$ in $SP$ and the maps $(Y_{m+1}^i, Y_{m+1}^i) \to (X_{m+1}^i, X_{m+1}^i)$, $(Y_{m+1}^i, Y_{m+1}^i) \to (X_{m+1}^i, X_{m+1}^i)$ in $SP'$.

It is easy to see that the top, bottom, left and right squares are Cartesian. Moreover, it follows from Lemma 7.1 that the bottom square is transverse with all vertices smooth. Since $P_m$ is projective and $\gamma$ is a closed immersion, it follows from the standard properties of an oriented cohomology theory having integration with supports (cf. [25] Definition 2.7-A(4))] that the left square of (8.14) commutes. \(\square\)
The motivic cobordism for group actions

Let \( X \in \text{Sm}_k^T \) be smooth and projective with the \( T \)-equivariant filtration \([8.7]\). This equivariant filtration on \( X \) induces a commutative diagram in \( \text{SP} \)

\[
\begin{array}{c}
(X^i, X^i_{m-1}) \xrightarrow{\gamma^i_{m-1}} (X^i, X^i_m) \xrightarrow{\gamma^i_m} (S^i_m, W^i_m) \\
(X^{i+1}, X^{i+1}_{m-1}) \xrightarrow{\gamma^i_{m-1}} (X^{i+1}, X^{i+1}_m) \xrightarrow{\gamma^i_{W_m}} (S^{i+1}_m, W^{i+1}_m).
\end{array}
\]

If we consider the associated diagram of motivic cobordism with supports and use the identification \( MGL'(X^i_m) = MGL_{X^i_m}(X^i) \), we obtain a commutative diagram

\[
\begin{array}{c}
0 \longrightarrow MGL^*_{X^i_{m-1}} (X^{i+1}) \xrightarrow{(\gamma^i_{m-1})^*} MGL^*_{X^i_m} (X^{i+1}) \xrightarrow{(\gamma^i_m)^*} MGL^* (W^{i+1}_m) \longrightarrow 0 \\
0 \longrightarrow MGL^*_{X^i_{m-1}} (X^i) \xrightarrow{(\gamma^i_{m-1})^*} MGL^*_{X^i_m} (X^i) \xrightarrow{(\gamma^i_m)^*} MGL^* (W^i_m) \longrightarrow 0.
\end{array}
\]

Notice that \((\gamma^i_{m-1})^*\) is same as \((\gamma^i_{m-1})_\ast\) under the identification \( MGL'(X^i_m) = MGL_{X^i_m}(X^i) \) (cf. [25] Definition 1.8-(5)). The two rows are exact and we have shown in the proof of Proposition \([8.2]\) (cf. \([8.4]\)) that the map \((\gamma^i_{m})^*\) is split by \( s^i_m := (\overline{p}_m)^* \circ (\overline{q}_m)^* \circ ((\phi^i_m)^*)^{-1} \) for each \( i \geq 1 \) (cf. diagram \([8.14]\)). In particular, the two rows form split short exact sequences. We now show that

\[
(\gamma^i_m)^* \circ (\gamma^i_{W_m})^* = (\gamma^i_m)^* \circ s^i_{m+1}.
\]

To see this, it is equivalent to showing that

\[
(\gamma^i_m)^* \circ (\overline{p}_m)^* \circ (\overline{q}_m)^* = (\overline{p}_m)^* \circ (\overline{q}_m)^* \circ ((\phi^i_m)^*)^{-1} \circ (\gamma^i_{W_m})^* \circ (\phi^i_{m+1})^*.
\]

On the other hand, it follows from Lemma \([8.6]\) that

\[
((\phi^i_m)^*)^{-1} \circ (\gamma^i_{W_m})^* \circ (\phi^i_{m+1})^* = (\gamma^i_{m})^* \circ (\phi^i_m)^* \circ (\gamma^i_{Z_m})^*.
\]

Applying Lemma \([8.6]\) again, we get

\[
(\gamma^i_m)^* \circ (\gamma^i_{Z_m})^* = (\gamma^i_m)^* \circ (\gamma^i_{W_m})^* \circ (\gamma^i_{Z_m})^*.
\]

This shows \([8.19]\) and hence \([8.18]\).

**Lemma 8.7.** Let \( X \in \text{Sm}_k^T \) be smooth and projective with the \( T \)-equivariant filtration \([8.7]\). Then for every \( 0 \leq m \leq n \), there is a canonical split exact sequence

\[
0 \to \lim_{i \to m} MGL^*_{X^i_{m-1}} (X^i) \xrightarrow{\gamma^i_{m-1}} \lim_{i \to m} MGL^*_{X^i_m} (X^i) \xrightarrow{j^i_m} \lim_{i \to m} MGL^* (W^i_m) \to 0.
\]
Proof. It follows from (8.17) and the left exactness of the inverse limit that there is a sequence as above which is exact except possibly at the right end. But (8.18) shows that \( j_m^* \circ s_m^* \) is identity on \( \lim_{\leftarrow i} MGL^* (W^i_m) \). This proves the lemma. \( \square \)

**Lemma 8.8.** Let \( T \) be a split torus of rank \( n \) acting trivially on a smooth scheme \( X \) of dimension \( d \) and let \( \{ \chi_1, \cdots, \chi_n \} \) be a chosen basis of \( \hat{T} \). Then the assignment \( t_j \mapsto e_1^T (L_{X_j}) \) induces an isomorphism of graded rings

\[
MGL^* (X) \left[ [t_1, \cdots, t_n] \right] \overset{\cong}{\to} MGL^*_T (X)
\]

where \( MGL^* (X) \left[ [t_1, \cdots, t_n] \right] \) is the graded power series ring over \( MGL^* (X) \).

**Proof.** Let \( \rho = (V_i, U_i)_{i \geq 1} \) be a canonical admissible gadget for \( T \). Since \( T \) acts trivially on \( X \), we have \( X \times U_i \cong \mathbb{P} \times (U_i / T) \cong \mathbb{P} \times (\mathbb{P}^{i-1})^n \). The projective bundle formula for the motivic cobordism implies that the map \( MGL^* \left( X \times U_{i+1} \right) \to MGL^* \left( X \times U_i \right) \) is surjective.

We have seen in §8.2.1 that for each \( 1 \leq j \leq n \), \( L_{X_j} \) defines the line bundle \( \mathcal{O}(\pm 1) \) on each factor of the product \( (\mathbb{P}^{i-1})^n \). Let \( \zeta_j \) denote the first Chern class of this line bundle on \( \mathbb{P}^{i-1} \). Applying Proposition 5.3 and the non-equivariant projective bundle formula once again, we see that

\[
MGL^{a,b}_T (X) = \prod_{p_i, \cdots, p_n \geq 0} MGL^{a-2(\sum_{i=1}^n p_i), b-(\sum_{i=1}^n p_i)} (X) \zeta_1^{p_1} \cdots \zeta_n^{p_n}.
\]

Taking sum over \( a \geq b \geq 0 \), we see that the map \( MGL^* (X) \left[ [t_1, \cdots, t_n] \right] \overset{\cong}{\to} MGL^*_T (X) \) is an isomorphism of graded rings. \( \square \)

We now prove our main result on the description of the equivariant cobordism of smooth and projective schemes with torus action.

**Theorem 8.9.** Let \( T \) be a split torus of rank \( n \) acting on a smooth and projective scheme \( X \) and let \( i : X^T = \coprod_{m=0}^n Z_m \hookrightarrow X \) be the inclusion of the fixed point locus. Then there is a canonical isomorphism

\[
\bigoplus_{m=0}^n MGL_*^* (Z_m) \overset{\cong}{\to} MGL_*^* (X).
\]

of bi-graded \( S(T) \)-modules. In particular, there is a canonical isomorphism of bi-graded \( S(T) \)-modules.

\[
MGL_*^* (X) \overset{\cong}{\to} MGL_*^* (X) \left[ [t_1, \cdots, t_n] \right]
\]

**Proof.** Let \( \rho = (V_i, U_i)_{i \geq 1} \) be a canonical admissible gadget for \( T \). By inducting on \( 0 \leq m \leq n \) and using the homotopy invariance, it follows from Lemma 8.7 that there is a canonical isomorphism

\[
\bigoplus_{m=0}^n \lim_{\leftarrow i} MGL^* (Z^i_m) \overset{\cong}{\to} \lim_{\leftarrow i} MGL^* (X^i).
\]
Applying Lemma 8.4, we get
\[ \bigoplus_{m=0}^{n} MGL_{T}^{\ast}(Z_m) \xrightarrow{\cong} MGL_{T}^{\ast}(X). \]
We have already shown in Proposition 8.2 that there is a canonical isomorphism of \(\mathbb{L}\)-modules
\[ \bigoplus_{m=0}^{n} MGL^{\ast,\ast}(Z_m) \xrightarrow{\cong} MGL^{\ast,\ast}(X). \]
The last assertion follows from these two isomorphisms and Lemma 8.8 since \(T\) acts trivially on \(X^T\).

9. **Equivariant motivic cobordism : Another approach**

In order to study the topological \(K\)-theory of the classifying spaces, Atiyah and Hirzebruch \([2]\) had defined three different notions of the topological \(K\)-theory of the classifying space \(BG\) of a Lie group \(G\). One of these notions is the theory \(K^{*}(BG)\), which is defined as the generalized cohomology of \(BG\) given by the topological \(K\)-theory spectrum. This is analogous to our equivariant motivic cobordism \(MGL_{G}^{*,*}(k) = MGL^{*,*}(BG)\), discussed above. The other one is the \(K\)-theory \(k^{*}(BG)\) which is defined in terms of the projective limit of the usual \(K\)-theory of the finite skeleta of the \(CW\)-complex \(BG\). The relations between these two notions and their applications have been the subject of study in several works of Atiyah and his coauthors.

Motivated by the above topological construction, we consider a similar approach in the algebraic context in this section. An outcome of this approach is that one is able to invent another notion of equivariant motivic cobordism based the \(k^{*}(BG)\)-theory of \([2]\). We denote this version of equivariant motivic cobordism by \(mgl_{G}^{*,*}(-)\). We shall prove certain results which compare the two notions \(MGL_{G}^{*,*}(-)\) and \(mgl_{G}^{*,*}(-)\). In particular, we shall show that these two coincide if we consider cohomology theory with rational coefficients. This allows us to compare the equivariant cobordism rings defined in this paper with the one studied earlier in \([19]\).

9.1. **\(mgl_{G}^{*,*}\)-theory.** Let \(G\) be a linear algebraic group over \(k\). The following two results form the basis for our definition of \(mgl_{G}^{*,*}(X)\) for \(X \in \text{Sm}_{k}^{G}\).

**Lemma 9.1.** Let \(V \in \text{Sm}_{k}\) and let \(i : W \hookrightarrow X\) be a closed subset with complement \(j : U \hookrightarrow X\). Given any \(a \geq b \geq 0\), there exists a integer \(N \gg 0\) such that the open pull-back \(MGL^{a,b}(V) \xrightarrow{j_{*}} MGL^{a,b}(U)\) is an isomorphism if \(\text{codim}_{V}(W) > N\).

**Proof.** Let \(X \hookrightarrow \text{CH}^{d}(X, j)\) denote the higher Chow groups of Bloch \([3]\) for \(X \in \text{Sch}_{k}\). There is an isomorphism \(\text{CH}^{d}(X, j) \cong \text{CH}_{d-i}(X, j)\) if \(X\) is smooth of dimension \(d\). To prove the lemma, we first claim that given any \(m \geq 0\), the map \(\text{CH}^{d}(V, j) \xrightarrow{j}\rightarrow \text{CH}^{d}(U, j)\) is an isomorphism for all \(i \leq m\) and for all \(j \geq 0\) if \(\text{codim}_{V}(W) > m\).

To see this, we set \(d = \text{codim}_{V}(W)\) and consider the localization exact sequence
\[ \cdots \rightarrow \text{CH}_{\text{dim}(X)-i}(W, j) \xrightarrow{i_{*}} \text{CH}^{i}(V, j) \xrightarrow{j}\rightarrow \text{CH}^{i}(U, j) \rightarrow \text{CH}_{\text{dim}(X)-i}(W, j-1) \rightarrow \cdots. \]
Suppose that \( d > m \). Then for any \( i \leq m \), we have \( i < d \). This in turn implies that \( \dim(X) - i + j > \dim(W) + j \) for all \( j \geq 0 \) and hence \( \text{CH}_{\dim(X) - i}(W, j) = 0 \) for all \( j \geq 0 \). The above exact sequence shows that the map \( j^* \) is an isomorphism for all \( j \geq 0 \) if \( d > m \). This proves the claim.

Having proved the claim, we can now use the following motivic Atiyah-Hirzebruch spectral sequence of Hopkins and Morel [26]:

\[
\begin{align*}
E_2^{p,a-p}(V) &= \text{CH}^{b-a+p}(V, 2b - a) \otimes \mathbb{L}^{a-p} \implies \text{MGL}^{a,b}(V) \\
E_2^{p,a-p}(U) &= \text{CH}^{b-a+p}(U, 2b - a) \otimes \mathbb{L}^{a-p} \implies \text{MGL}^{a,b}(U).
\end{align*}
\]

The differentials for this spectral sequence are given by \( d_r^{a,p} : E_r^{p,a} \to E_r^{p+r,a-r+1} \).

There is a finite filtration

\[
\text{MGL}^{a,b}(V) = F^0\text{MGL}^{a,b}(V) \supseteq \cdots \supseteq F^n\text{MGL}^{a,b}(V) \supseteq F^{n+1}\text{MGL}^{a,b}(V) = 0
\]

such that \( F^p\text{MGL}^{a,b}(V)/F^{p+1}\text{MGL}^{a,b}(V) = E_{np}^{a,p} \) for some \( n_p \gg 0 \). The same holds for \( \text{MGL}^{a,b}(U) \). We see from this that there are only finitely many higher Chow groups which completely determine \( \text{MGL}^{a,b}(V) \) and \( \text{MGL}^{a,b}(U) \) for given integers \( a \geq b \geq 0 \). Hence by the above claim, we can choose some \( N \gg 0 \) such that all these finitely many higher Chow groups of \( V \) and \( U \) are isomorphic if \( \text{codim}_W(W) > N \). In particular, we get \( j^* : E_{np}^{a,p}(V) \xrightarrow{\sim} E_{np}^{a,p}(U) \) for all \( 0 \leq p \leq n \). By a descending induction on the filtration, we get \( j^* : \text{MGL}^{a,b}(V) \xrightarrow{\sim} \text{MGL}^{a,b}(U) \).

\[\Box\]

**Lemma 9.2.** Let \( \rho = (V_i, U_i) \) and \( \rho' = (V'_i, U'_i) \) be two admissible gadgets for \( G \) and let \( a \geq b \geq 0 \). Then for any \( X \in \text{Sm}^G_k \), there is a canonical isomorphism

\[
\lim_i \text{MGL}^{a,b} \left( X \times U_i \right) \cong \lim_i \text{MGL}^{a,b} \left( X \times U'_i \right).
\]

**Proof.** Fix \( a \geq b \geq 0 \). For any \( i, j \geq 1 \), we have the canonical maps

\[
\alpha_{ij} : X \times (U_i \oplus U'_j) \to X \times (U_i \oplus V'_j) \xrightarrow{p_{ij}} X \times U_i
\]

where the first map is the open immersion and the second map is a vector bundle. In particular, the map \( p_{ij}^* \) is an isomorphism on the motivic cobordism. It follows from the property (iv) of an admissible gadget (cf. Definition 2.11 and Lemma 9.1) that given \( i \geq 1 \), the map

\[
(9.1) \quad \alpha_{ij}^* : \text{MGL}^{a,b} \left( X \times U_i \right) \to \text{MGL}^{a,b} \left( X \times (U_i \oplus U'_j) \right)
\]

is an isomorphism for all \( j \gg 0 \). Taking the limit, we see that the map

\[
\alpha^* : \lim_i \text{MGL}^{a,b} \left( X \times U_i \right) \to \lim_i \lim_j \text{MGL}^{a,b} \left( X \times (U_i \oplus U'_j) \right)
\]
is an isomorphism. By reversing the roles of the admissible gadgets, we see that the map
\[ \beta^*: \lim_{i \leftarrow} MGL^{a,b} \left( X^G \times U'_i \right) \to \lim_{i \leftarrow} \lim_{j \leftarrow} MGL^{a,b} \left( X^G \times (U_i \oplus U'_j) \right) \]
is also an isomorphism. The map \( \beta^* \circ \alpha^* \) is the desired isomorphism. This proves the lemma. □

To prove the second part, suppose that the inverse system \( \{ MGL^{a,b} \left( X^G \times U'_i \right) \} \) satisfies the Mittag-Leffler condition.

Fix \( i_0 \geq 1 \) and let \( s_0 \gg 0 \) be such that the map \( \alpha^*_{i_0,j} \) is an isomorphism for all \( j \geq s_0 \). Let \( i_0 \gg i_0 \) be such that the map
\[ \beta^*_{i,s_0} : MGL^{a,b} \left( X'_{i,s_0} \right) \to MGL^{a,b} \left( Y_{i,s_0} \right) \]
is an isomorphism for all \( i \geq i_1 \).

Let \( s_1 \geq s_0 \) be such that for all \( j \geq s_1 \), we have
\[ \text{Image} \left( MGL^{a,b} \left( X'_i \right) \to MGL^{a,b} \left( X'_{s_0} \right) \right) = \text{Image} \left( MGL^{a,b} \left( X'_{s_1} \right) \to MGL^{a,b} \left( X'_{s_0} \right) \right). \]

Let us now fix an integer \( i \geq i_1 \) and an element \( a \in \text{Image} \left( MGL^{a,b} \left( X_{i_0} \right) \to MGL^{a,b} \left( X_{i_0} \right) \right). \)

We get a commutative diagram

\[ \text{Definition 9.3.} \text{ Let } G \text{ be a linear algebraic group over } k \text{ and let } X \in \text{Sm}_k^G. \text{ We let } mgl^{a,b}_G(X) \text{ be the group} \]
\[ mgl^{a,b}_G(X) = \lim_{i \leftarrow} MGL^{a,b} \left( X^G \times U_i \right) \]

where \( \rho = (V_i, U_i) \) is an admissible gadget for \( G \).

It follows from Lemma 9.2 that \( mgl^{a,b}_G(X) \) is well defined. It also follows from the definition of \( MGL^{a,b}_G(X) \) that there is a natural map
\[ \phi_X : MGL^{a,b}_G(X) \to mgl^{a,b}_G(X) \]
and this map is surjective by Proposition 5.3. We set \( mgl^{a,b}_G(X) = \bigoplus_{a \geq 0} mgl^{a,b}_G(X). \)

9.2. Geometric equivariant cobordism. Motivated by the work of Quillen [33] on complex cobordism, Levine and Morel [27] gave a geometric construction of the algebraic cobordism and showed that this is a universal oriented Borel-Moore homology theory in \( \text{Sch}_k \). Based on the work of Levine and Morel, a theory of equivariant algebraic cobordism was constructed in [19]. This theory of equivariant cobordism was subsequently used in [20], [21], [23] and [18] to compute the ordinary algebraic cobordism of many classes of smooth schemes. We recall the definition of this equivariant cobordism.

Let \( G \) be a linear algebraic group and let \( X \in \text{Sm}_k^G \). For any integer \( i \geq 1 \), let \( V_i \) be a representation of \( G \) and let \( U_i \) be a \( G \)-invariant open subset of \( V_i \) such that the codimension of \( V_i \setminus U_i \) is at least \( i \) and \( G \) acts freely on \( U_i \) such that the quotient \( U_i/G \) is a quasi-projective scheme. Let \( \Omega^G_i(X) \) denote the quotient
\[\Omega^q(X \times U_i)/F^i \Omega^q(X \times U_i),\] where \(\Omega^q(X)\) is the algebraic cobordism of Levine-Morel and \(F^i \Omega^q(X)\) is the subgroup of \(\Omega^q(X)\) generated by cobordism cycles which are supported on the closed subschemes of \(X\) of codimension at least \(i\). It is known that \(\Omega^q_G(X)\) is independent of the choice of the pair \((V, U_i)\) and there is a natural surjection \(\Omega^q_G(X) \twoheadrightarrow \Omega^q_G(X)\) if \(i' \geq i\). The geometric equivariant cobordism group \(\Omega^q_G(X)\) is defined by

\[\Omega^q_G(X) := \lim_{\leftarrow i} \Omega^q_G(X)_i.\]

It was shown in [19] that this geometric version of the equivariant cobordism has all the properties of an oriented cohomology theory on \(\text{Sm}_k\) except that it does not in general have the localization sequence. We also remark that the equivariant geometric cobordism \(\Omega^*_G(-)\) in [19] is defined for all schemes in \(\text{Sch}_k\) and is an example of a Borel-Moore homology theory.

9.3. Basic properties of \(mgl^*_G(-)\). Recall from [31] that the motivic cobordism theory \(MGL\) has push-forward maps for projective morphisms between smooth schemes. We need the following property of this map in order to define the push-forward map on the \(mgl^*_G(-)\).

Lemma 9.4. Consider the Cartesian square

\[\begin{array}{ccc}
Y' & \xrightarrow{f'} & Y \\
\downarrow{g'} & & \downarrow{g} \\
X' & \xrightarrow{f} & X
\end{array}\]

in \(\text{Sm}_k\) such that \(f\) is projective. Suppose that either

1. \(g\) is a closed immersion and \((9.5)\) is transverse, or,
2. \(g\) is smooth.

One has then \(g^* \circ f_* = f'_* \circ g'^*: MGL^*(X') \rightarrow MGL^*(Y)\).

Proof. We can write \(f = p \circ i\), where \(X' \hookrightarrow \mathbb{P}^n \times X\) is a closed immersion and \(p: \mathbb{P}^n \times X \rightarrow X\) is the projection. This yields a commutative diagram

\[\begin{array}{ccc}
Y' & \xrightarrow{i'} & \mathbb{P}^n \times Y & \xrightarrow{p'} & Y \\
\downarrow{g'} & & \downarrow{h} & & \downarrow{g} \\
X' & \xrightarrow{i} & \mathbb{P}^n \times X & \xrightarrow{p} & X
\end{array}\]

where both squares are Cartesian.

First suppose that \(g\) is a closed immersion and \((9.5)\) is transverse. Since the right square is transverse and so is the big outer square, it follows that the left square is also transverse. In particular, we have \(h^* \circ i_* = i'_* \circ g'^*\) by [31] Definition 2.2-(2)]. On the other hand, we have \(g^* \circ p_* = \iota'_* \circ h^*\) by [31] Definition 2.2-(3)]. Combining these two, we get

\[g^* \circ f_* = g^* \circ p_* \circ i_* = \iota'_* \circ h^* \circ i_* = p'_* \circ \iota'_* \circ g'^* = f'_* \circ g'^*\]
where the first and the last equalities follow from the functoriality property of the push-forward (cf. [31 Definition 2.2-(1)]).

Now suppose that \( g \) is smooth. For the above proof to go through, only thing we need to know is that the left square in the above diagram is still transverse. But this is an elementary exercise using the fact that \( g, h, g' \) are all smooth and \( T(g') = i^\ast(T(h)) \), where \( T(f) \) denotes the relative tangent bundle of a smooth map \( f \). This proves the lemma. \( \square \)

The following result describes the basic properties of \( \text{mgl}^{*}(-) \).

**Theorem 9.5.** The equivariant cobordism theory \( \text{mgl}^{*}(-) \) on \( \text{Sm}^G_k \) satisfies the following properties.

1. **Functoriality:** The assignment \( X \mapsto \text{mgl}^{*}(-)(X) \) is a contravariant functor on \( \text{Sm}^G_k \).
2. **Push-forward:** Given a projective map \( f : X' \to X \in \text{Sm}^G_k \), there is a push-forward map \( f_* : \text{mgl}^{a,b}_G(X') \to \text{mgl}^{a+2d+b}_G(X) \) where \( d = \dim(X) - \dim(X') \). If \( X'' \xrightarrow{f} X' \xrightarrow{f} X \) are projective, then \( (f \circ f')_* = f_* \circ f'_* \). If the square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

is transverse in \( V_G \) where \( f \) is a closed immersion, one has \( g^* \circ f_* = f'_* \circ g'^* : \text{mgl}^{*}_G(Y') \to \text{mgl}^{*}_G(Y) \).
3. **Homotopy Invariance:** If \( f : E \to X \) is a \( G \)-equivariant vector bundle, then \( f^* : \text{mgl}^{*}_G(X) \xrightarrow{\sim} \text{mgl}^{*}_G(E) \).
4. **Chern classes:** For any \( G \)-equivariant vector bundle \( E \to X \) of rank \( r \), there are equivariant Chern class operators \( c^G_m(E) : \text{mgl}^{*}_G(X) \to \text{mgl}^{*}_G(X) \) for \( 0 \leq m \leq r \) with \( c^G_0(E) = 1 \). These Chern classes have same functoriality properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.
5. **Exterior Product:** There is a natural product map

\[
\text{mgl}^{a,b}_G(X) \otimes \mathbb{Z} \text{mgl}^{a',b'}_G(X') \to \text{mgl}^{a+a',b+b'}_G(X \times X').
\]

In particular, \( \text{mgl}^{*}(-) \) is a bi-graded ring for every \( X \in \text{Sm}^G_k \).
6. **Projection formula:** For a projective map \( f : X' \to X \in \text{Sm}^G_k \), one has for \( x \in \text{mgl}^{*}_G(X) \) and \( x' \in \text{mgl}^{*}_G(X') \), the formula : \( f_*(x' \cdot f^*(x)) = f_*(x') \cdot x \).
7. **Projective bundle formula:** For an equivariant vector bundle \( E \) of rank \( n \) on \( X \), the map

\[
\Phi_X : \text{mgl}^{*}_G(X) \oplus \cdots \oplus \text{mgl}^{*}_G(X) \to \text{mgl}^{*}_G(\mathbb{P}(E));
\]

\[
\Phi(a_0, \cdots, a_{n-1}) = \sum_{i=0}^{n-1} \pi^i(a_i) \cdot \xi^i
\]
is an isomorphism, where $\pi : \mathbb{P}(E) \to X$ is the projection map and $\xi = c_1^G(\mathcal{O}_E(-1))$.

(8) Change of groups: If $H \subseteq G$ is a closed subgroup and $X \in \text{Sm}^G_k$, then there is a natural restriction map $r^G_{H,X} : mgl_{G,H}^*(-)(X) \to mgl_{H}^*(-)(X)$. In particular, there is a natural forgetful map

$$r^G_X : mgl^*_G(X) \to MGL^*G(X).$$

(9) Morita Isomorphism: If $H \subseteq G$ is a closed subgroup and $X \in \text{Sm}^H_k$, then there is a canonical isomorphism $mgl^*H(X \times G) \cong mgl^*_H(X)$.

(10) Free action: If $X \in \text{Sm}^G_{free/k}$, then there is a natural isomorphism

$$MGL^*G(X/G) \cong mgl^*G(X).$$

(11) Comparison with geometric cobordism: For any $X \in \text{Sm}^G_k$ and $p \geq 0$, there is a natural isomorphism $\Omega^p_{G}(X) \cong mgl^{2p,p}_G(X)$.

Proof. The contravariant functoriality of $mgl^*G(-)$ follows from the similar property of motivic cobordism. If $f : X' \to X$ is a projective morphism, and if $\rho = (V_i, U_i)$ is an admissible pair for $G$, then it follows from [19, Lemma 5.1] that $f_i : X' \times U_i \to X \times U_i$ is projective for all $i \geq 1$. Moreover, we have seen in the proof of Lemma 7.1 that the diagram

$$
\begin{array}{ccc}
X' \times U_i & \xrightarrow{f_{i+1}^*} & X' \times U_{i+1} \\
\downarrow f_i & & \downarrow f_{i+1} \\
X \times U_i & \xrightarrow{\iota_{X,i}} & X \times U_{i+1}
\end{array}
$$

is transverse, where the horizontal maps are closed immersions. It follows from Lemma 7.1 that $f_{i+1}^* \circ (f_{i+1})_* = (f_i)_* \circ f_{i+1}^*$. Taking the inverse limit of $(f_i)_*$, we get the push-forward map $f_* : mgl^a_G(X') \to mgl^{a+2d, b+d}_G(X)$. The other part of property (2) follows from the similar property of the motivic cobordism on $\text{Sm}_k$ by [3] Definition 2.2-(2) and Lemma 9.4.

The proof of properties (3) through (6) is same as the proof of [19, Theorem 5.2] and the proof of the projective bundle formula follows directly from the similar result in the non-equivariant case. The proof of Property (8) is straightforward and the proof of property (9) follows like [19, Proposition 5.4].

To prove property (10), we observe in the case of free action of $G$ on $X$ that there are maps $X \times U_i \to X \times V_i \to X/G$ in $\text{Sm}_k$. It follows from the homotopy invariance of the motivic cobordism, Lemma 9.1 and the fourth property of an admissible gadget that the induced map $MGL^*G(X/G) \to MGL^*G\left(X \times U_i\right)$ is an isomorphism for all $i \gg 0$. In particular, the map

$$MGL^*G(X/G) \to \lim_{i \to \infty} MGL^*G\left(X \times U_i\right)$$
is an isomorphism. The property (11) follows directly from from [26, Theorem 3.1]. □

Remark 9.6. We have seen in Theorem 7.2 that the equivariant motivic cobordism theory $\text{MGL}^*_{G}(-)$ has localization sequences. However, the proof of this localization sequence fails in the case of $\text{mgl}^*_{G}(-)$. In fact, it was shown by Buhštaber and Miščenko [8] that if $k^*(X)$ is the projective limit of the topological $K$-theory of the finite skeleta of a CW-complex $X$, then $k^*(-)$ does not satisfy the localization sequence. Using the results of Buhštaber-Miščenko and Landweber [24], one can find such an example also for the complex cobordism. Because of this, one does not expect localization sequence to be true for $\text{mgl}^*_{G}(-)$. This is one serious drawback of this theory. We shall show however that the localization sequence does hold for $\text{MGL}^*_{G}(-)$ with the rational coefficients.

9.4. Comparison of $\text{MGL}^*_{G}(-)$ and $\text{mgl}^*_{G}(-)$ theories. We have seen before that there is a natural transformation of contravariant functors $\text{MGL}^*_{G}(-) \to \text{mgl}^*_{G}(-)$ on $\text{Sm}^G_k$. When it comes to computing the equivariant cobordism $\text{MGL}^*_{G}(X)$, it is often desirable to have these two functors isomorphic for $X$. Although we can not expect this to be the case in general, we have the following version of Lemma 8.4 in the case of torus action. We shall show later in this text that this isomorphism always holds with rational coefficients.

Corollary 9.7. Let $T$ be a split torus and let $X \in \text{Sm}^T_k$ be smooth and projective. Then for any $a \geq b \geq 0$, the map

$$\phi_X : \text{MGL}^{a,b}_T(X) \to \text{mgl}^{a,b}_T(X)$$

is an isomorphism.

10. EQUIVARIANT COBORDISM WITH RATIONAL COEFFICIENTS

In this section, we study the equivariant motivic cobordism with rational coefficients and show in this case that the equivariant cobordism of a smooth scheme with a group action can be computed in terms of the limit of the ordinary motivic cobordism groups of smooth schemes. This allows us to show in particular that if $G$ is a connected reductive group, then the equivariant cobordism of any $G$-scheme can be written in terms of the Weyl group invariants of the equivariant cobordism of the given scheme for the action of a maximal torus. An equivariant analogue of the Levine-Morel algebraic cobordism was studied in [19]. We also give the precise relation between the two versions of equivariant cobordism in this section.

Recall from [1, Remark III.6.5] that given an abelian group $R$, there is a Moore space $M_R \in \text{HoSets}_\bullet$, where $\text{HoSets}_\bullet$ is the unstable homotopy category of pointed simplicial sets. We can consider $M_R$ as an object of $\mathcal{H}_*(k)$ via the obvious functor $\text{HoSets}_\bullet \to \mathcal{H}_*(k)$. For any spectrum $E \in \mathcal{SH}(k)$, there is a Moore spectrum $E_R = E \wedge M_R$.

Definition 10.1. Let $G$ be a linear algebraic group over $k$. For any $X \in \text{Sm}^G_k$, the equivariant motivic cobordism of $X$ with coefficients in the group $R$ is given by

$$\text{MGL}^{a,b}_G(X; R) = \text{MGL}^{a,b}(X_G; R) := \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty X_+, \Sigma^{a,b}\text{MGL}_R)$$
where \( X_G \) is a Borel space of the type \( X_G(\rho) \) as in \( \S 2.2 \). We also consider \( mgl^{\ast,*}_G(-; R) \):
\[
mgl^{a,b}_G(X; R) = \lim_{i \to \infty} MGL^{a,b}(X \times U_i; R).
\]

We set
\[
MGL^{\ast,*}_G(X; R) = \bigoplus_{0 \leq b \leq a} MGL^{a,b}_G(X; R) \text{ and } mgl^{\ast,*}_G(X; R) = \bigoplus_{0 \leq b \leq a} mgl^{a,b}_G(X; R).
\]

Note that since every \( X \in \text{Sm}_k \) is compact as a motivic space, it follows that there is a short exact sequence
\[
0 \to MGL^{a,b}(X) \otimes_{\mathbb{Z}} R \to MGL^{a,b}(X; R) \to \text{Tor}(MGL^{a+1,b}(X), R) \to 0.
\]

In particular, for any \( R \subseteq \mathbb{Q} \), the natural map \( MGL^{a,b}(X) \otimes_{\mathbb{Z}} R \to MGL^{a,b}(X; R) \) is an isomorphism. But this is no longer true for the equivariant motivic cobordism because of the fact that the spaces \( X_G \) are not in general compact. The following result gives a simple description of the rational equivariant motivic cobordism.

**Theorem 10.2.** Let \( G \) be a linear algebraic group and let \( X \in \text{Sm}_k^G \). Then for all \( a \geq b \geq 0 \), the natural map
\[
MGL^{a,b}_G(X; \mathbb{Q}) \to mgl^{a,b}_G(X; \mathbb{Q})
\]
is an isomorphism.

**Proof.** Let \( \rho = (V_i, U_i) \) be an admissible gadget for \( G \). In view of Proposition \( \S 5.3 \) it is enough to show that for any \( 0 \leq b \leq a \), the inverse system \( \{MGL^{a,b}(X_G(\rho, i); \mathbb{Q})\}_{i \geq 1} \) satisfies the Mittag-Leffler condition.

Let us denote the smooth scheme \( X_G(\rho, i) \) in short by \( X_i \) and let \( d_i \) denote the dimension of \( X_i \). Recall that \( X_G = \text{colim}_i X_i \). It follows from \( \S 29 \) Corollary 10.6] that for any \( i \geq 1 \), there is a natural isomorphism
\[
MGL^{a,b}(X_i; \mathbb{Q}) \cong \bigoplus_{b-j} H_2^{i, 2b-a}(X_i; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{L}_Q^{b-j}
\]
where \( H^{\ast,*}(X) \) is the motivic cohomology of \( X \in \text{Spc} \) given by the motivic Eilenberg-MacLane spectrum and \( \mathbb{L}_Q \) is the rationalization of the Lazard ring \( \mathbb{L} \).

We fix an integer \( i \geq 1 \). It follows from the localization sequence for the motivic cohomology and the third property of an admissible gadget (cf. Definition \( \S 2.1 \) that for any integer \( j \in [b, d_i + 2b - a] \), there exists \( m(i, j) \gg i \) such that for all \( l \geq m(i, j) \), the restriction map
\[
H^{d_i + 2b - a, j}(X_{l+1}; \mathbb{Q}) \to H^{d_i + 2b - a, j}(X_X \times (U_i \oplus W_i); \mathbb{Q})
\]
is an isomorphism. On the other hand, the homotopy invariance property of the motivic cohomology shows that the map
\[
H^{d_i + 2b - a, j}(X_G \times (U_i \oplus W_i); \mathbb{Q}) \to H^{d_i + 2b - a, j}(X_i; \mathbb{Q})
\]
is also an isomorphism. Setting \( m(i) = \max_{j=b} m(i, j) \), we see that for all \( l \geq m(i) \), the restriction map
\[
\bigoplus_{j=b} H_2^{2j+a-2b,j}(X_i; \mathbb{Q}) \to \bigoplus_{j=b} H_2^{2j+a-2b,j}(X_{m(i)}; \mathbb{Q})
\]
is an isomorphism. Since \( i \mapsto d_i \) is an strictly increasing function, it follows from \( \text{(10.3)} \) that the image of the restriction map \( \text{MGL}^{a,b}(X_i; \mathbb{Q}) \rightarrow \text{MGL}^{a,b}(X_l; \mathbb{Q}) \) does not depend on the choice of \( l \geq m(i) \). In other words, the inverse system \( \{ \text{MGL}^{a,b}(X_i; \mathbb{Q}) \}_{i \geq 1} \) satisfies the Mittag-Leffler condition. □

We have seen in Remark \( \text{[10]} \) that the \( mgl^*_G(-) \)-theory is not expected to satisfy the localization sequence with the integral coefficients. The following result rectifies this problem if one is working with the rational coefficients.

**Corollary 10.3.** Let \( \iota : Y \hookrightarrow X \) be a closed immersion in \( \text{Sm}_k^G \) of codimension \( d \). There is then a long exact localization sequence

\[
\cdots \rightarrow \text{mgl}_G^{a-2d,b-d}(Y; \mathbb{Q}) \xrightarrow{\iota^*} \text{mgl}_G^{a,b}(X; \mathbb{Q}) \rightarrow \text{mgl}_G^{a,b}(X \setminus Y; \mathbb{Q}) \xrightarrow{\partial} \text{mgl}_G^{a-2d+1,b-d}(Y; \mathbb{Q}) \rightarrow \cdots.
\]

In particular, there is an exact sequence

\[
\Omega_G^{q-d}(Y; \mathbb{Q}) \rightarrow \Omega_G^q(X; \mathbb{Q}) \rightarrow \Omega_G^q(X \setminus Y; \mathbb{Q}) \rightarrow 0.
\]

for all \( q \geq 0 \).

**Proof.** The first long exact sequence follows from Theorems \( [7,2] \) (6) and \( [10,2] \). The second exact sequence follows from the first, together with Theorem \( [9,5] \) (11) and \( [19, \text{Proposition 5.3}] \). □

11. REDUCTION OF ARBITRARY GROUPS TO TORI

The theme of this section is to study the question of how to reduce the problem of computing the equivariant motivic cobordism for the action of a linear algebraic group \( G \) to the case when the underlying group is a torus. We prove various results in this direction and compute the motivic cobordism of the classifying spaces of some reductive groups. We begin with the following result.

**Proposition 11.1.** Let \( G \) be a connected reductive group over \( k \). Let \( B \) be a Borel subgroup of \( G \) containing a maximal torus \( T \) over \( k \). Then for any \( X \in \text{Sm}_k^G \), the restriction map

\[
\text{MGL}_B^{*,*}(X) \xrightarrow{r_B^X} \text{MGL}_T^{*,*}(X)
\]

is an isomorphism.

**Proof.** By the Morita Isomorphism of Theorem \( [7,2] \) we only need to show that

\[
\text{MGL}_B^{*,*} \left( \frac{T}{B \times X} \right) \cong \text{MGL}_B^{*,*}(X).
\]

By \( [10, \text{XXII, 5.9.5}] \), there exists a characteristic filtration \( B^n = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\} \) of the unipotent radical \( B^n \) of \( B \) such that \( U_{i-1}/U_i \) is a vector group, each \( U_i \) is normal in \( B \) and \( TU_i = T \times U_i \). Moreover, this filtration also implies that for each \( i \), the natural map \( B/TU_i \rightarrow B/TU_{i-1} \) is a torsor under the vector bundle \( U_{i-1}/U_i \times B/TU_{i-1} \) on \( B/TU_{i-1} \). Hence, the homotopy invariance (cf. Theorem \( [7,2] \)) gives an isomorphism

\[
\text{MGL}_B^{*,*}(B/TU_{i-1} \times X) \cong \text{MGL}_B^{*,*}(B/TU_i \times X).
\]
Proposition 11.2. Let \( B \) be a possibly non-reductive group over \( k \). Let \( G = H \ltimes G^u \) be the Levi decomposition of \( G \) (which exists since \( k \) is of characteristic zero). Then the restriction map

\[
MGL_B^\ast(X) \to MGL_B^\ast(B/T \times X).
\]

The canonical isomorphism of \( B \)-varieties \( B \times X \cong (B/T) \times X \) now proves (11.2) and hence (11.1).

\[\square\]

**Proposition 11.2.** Let \( G \) be a possibly non-reductive group over \( k \). Let \( G = H \ltimes G^u \) be the Levi decomposition of \( G \) (which exists since \( k \) is of characteristic zero). Then the restriction map

\[
(11.3) \quad MGL_G^\ast(X) \to MGL_H^\ast(X)
\]

is an isomorphism.

**Proof.** Since the ground field is of characteristic zero, the unipotent radical \( G^u \) of \( G \) is split over \( k \). Now the proof is exactly same as the proof of Proposition 11.1 where we just have to replace \( B \) and \( T \) by \( G \) and \( H \) respectively. \[\square\]

11.0.1. **Action of Weyl group.** Let \( G \) be a linear algebraic group and let \( H \subseteq G \) be a closed normal subgroup with quotient \( W \). If \( \rho = (V_i, U_i)_{i \geq 1} \) is an admissible gadget for \( G \), then it is also an admissible gadget for \( H \). If \( X \in \text{Sm}_k^G \), then \( W \) acts on the ind-scheme \( X_G(\rho) = \text{colim}_i (X \times H U_i) \) such that each closed subscheme \( X \times H U_i \) is \( W \)-invariant. In particular, \( W \) acts on \( MGL^\ast(X_G(\rho)) = MGL^\ast(X) \).

One example where such a situation occurs is when \( H \) is a maximal torus of a reductive group and \( G \) is its normalizer. The quotient \( W \) is then the Weyl group of \( G \). In that case, \( MGL_G^\ast(X) \) becomes a \( \mathbb{Z}[W] \)-module.

11.1. **Rational results.** Let \( G \) be a connected reductive group and let \( T \) be a split maximal torus of \( G \). Let \( N \) be the normalizer of \( T \) in \( G \) with the associated Weyl group \( W = N/T \). We first consider the case of equivariant cobordism with rational coefficients and give the most complete result in this case.

**Theorem 11.3.** Let \( G \) be a connected reductive group and let \( T \) be a split maximal torus of \( G \) with the Weyl group \( W \). Then for any \( X \in \text{Sm}_k^G \), the restriction map \( r^G_{T, X} : MGL_G^\ast(X) \to MGL_T^\ast(X) \) induces an isomorphism

\[
MGL_G^\ast(X; \mathbb{Q}) \cong (MGL_T^\ast(X; \mathbb{Q}))^W.
\]

**Proof.** In view of Theorem 10.2, we can replace \( MGL_G^\ast(X; \mathbb{Q}) \) with \( mgl_G^\ast(X; \mathbb{Q}) \). Using the definition of \( mgl_G^\ast(X; \mathbb{Q}) \) and the fact that the inverse limit commutes with taking \( W \)-invariants, it suffices to show that for an admissible gadget \( \rho = (V_i, U_i)_{i \geq 1} \) for \( G \), the map \( MGL^\ast(X \times U_i; \mathbb{Q}) \to (MGL^\ast(X \times U_i; \mathbb{Q}))^W \) is an isomorphism for all \( i \geq 1 \). But this follows at once from (10.3) and Corollary 8.7. \[\square\]
11.2. Motivic cobordism of classifying spaces. Let $G$ be a connected reductive group over $k$ with a split maximal torus $T$ and the Weyl group $W$. In such a case, one says that $G$ is split reductive. Let $B$ denote a Borel subgroup of $G$ containing the maximal torus $T$. Let $d$ denote the dimension of the flag variety $G/B$. In this case, every character $\chi$ of $T$ yields a line bundle $L_\chi$ on the classifying space $BT$ which restricts to a line bundle $L_\chi := G \times^B L_\chi$ on the flag variety $G/B$ via the maps

(11.4) \[ G/B \overset{\iota}{\hookrightarrow} BT \overset{\pi}{\longrightarrow} BG \]

in $\operatorname{ISm}_k$. Recall that the torsion index of $G$ is defined as the smallest positive integer $t_G$ such that $t_G$ times the class of a point in $H^2(G/B, \mathbb{Z})$ belongs to the subring of $H^*(G/B, \mathbb{Z})$ generated by the first Chern classes of line bundles $L_\chi$ (e.g., $t_G = 1$ for $G = GL_n$, see [38] for computations of $t_G$ for other groups). If $G$ is simply connected then this subring is generated by $H^2(G/B, \mathbb{Z})$. We shall denote the ring $\mathbb{Z}[t_G^{-1}]$ by $R$.

Let $X \in \operatorname{Sm}^G_k$ and let $p_X : X \to \text{Spec}(k)$ denote the structure map. Let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$. Set $X_G^i = X \times U_i$ and $X_G = X_G(\rho)$. This gives rise to the commutative diagram in

(11.5) \[ G/B \overset{\iota}{\hookrightarrow} X_B \overset{\pi_X}{\longrightarrow} X_G \]

\[ G/B \overset{\iota}{\hookrightarrow} BB \overset{\pi}{\longrightarrow} BG \]

in $\operatorname{ISm}_k$, where $\iota$ and $\iota_X$ are strict closed embeddings. A similar commutative diagram exists for each $X_G^i$. This in turn yields a commutative diagram

(11.6) \[ G/B \overset{\iota}{\hookrightarrow} X_B^i \overset{\pi_X^i}{\longrightarrow} X_G^i \]

\[ G/B \overset{\iota}{\hookrightarrow} U_i/B \overset{\pi_i}{\longrightarrow} U_i/G \]

\[ G/B \overset{\iota}{\hookrightarrow} BB \overset{\pi}{\longrightarrow} BG. \]

Lemma 11.4. Let $f : X \to Y$ be a morphism in $\operatorname{Sm}^G_{\text{free}/k}$. Then the diagram of quotients

\[ X/B \overset{\iota}{\hookrightarrow} X/G \]

\[ Y/B \overset{\iota}{\hookrightarrow} Y/G \]

is Cartesian such that the horizontal maps are smooth and projective. If $f$ is a closed immersion, then this diagram is transverse.
Proof. The top horizontal map is an étale locally trivial smooth fibration with fiber $G/B$. Hence this map is proper by the descent property of properness. Since this map is also quasi-projective, it must be projective. The same holds for the bottom horizontal map. Proving the other properties is an elementary exercise and can be shown using the commutative diagram
\[
\begin{array}{ccc}
X & \longrightarrow & X/B \\
\downarrow f & & \downarrow \\
Y & \longrightarrow & Y/B
\end{array}
\]
\[
\begin{array}{ccc}
X/B & \longrightarrow & X/G \\
\downarrow & & \downarrow \\
Y/B & \longrightarrow & Y/G.
\end{array}
\]
One easily checks that the left and the big outer squares are Cartesian and transverse if $f$ is a closed immersion. Since all the horizontal maps are smooth and surjective, the right square must have the similar property. □

Proposition 11.5. For any $X \in \text{Sm}^G_k$, there is a push-forward map
\[
(\pi_X)_*: \text{mgl}^{a,b}_T(X) \to \text{mgl}^{a-2d,b-d}_G(X)
\]
which is contravariant for smooth maps and covariant for projective maps in $\text{Sm}^G_k$. This map satisfies the projection formula $(\pi_X)_*(x \cdot r^G_{T,X}(y)) = (\pi_X)_*(x) \cdot y$ for $x \in \text{mgl}^{a,b}_T(X)$ and $y \in \text{mgl}^{a,b}_G(X)$.

Proof. Let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$. For any $j \geq i \geq 1$, Lemma 11.4 yields a transverse square
\[
\begin{array}{ccc}
X^i_B & \longrightarrow & X^i_G \\
\downarrow s^i_{B,X} & & \downarrow s^i_{G,X} \\
X^j_B & \longrightarrow & X^j_G
\end{array}
\]
where the vertical maps are closed immersions and the horizontal maps are smooth and projective. It follows from Lemma 11.4 that there is a projective system of push-forward maps $\{(\pi_X)_*: \text{mgl}^{a,b}_T(X^i_B) \to \text{mgl}^{a-2d,b-d}_G(X^i_G)\}$. Taking the limit, we get the desired map $(\pi_X)_*: \text{mgl}^{a,b}_T(X) \to \text{mgl}^{a-2d,b-d}_G(X)$ and we can replace $T$ by $B$ using Proposition 11.1. The covariant functoriality is obvious and the contravariant functoriality follows directly from Lemmas 11.4 and 9.4. The projection formula for $(\pi_X)_*$ and $r^G_{T,X}$ follows from the projection formula for the maps $X^i_B \xrightarrow{\pi_X} X^i_G$ and observing that $r^G_{T,X}$ is the inverse limit of the pull-back maps $(\pi_X)^*$. □

Lemma 11.6. Let $X \in \text{Sm}^G_k$ be projective and let $\rho = (V_i, U_i)_{i \geq 1}$ be an admissible gadget for $G$. Then for any $a \geq b \geq 0$, the projective system $\{\text{MGL}^{a,b}_G(X^i_G; R)\}_{i \geq 1}$ satisfies the Mittag-Leffler condition.

Proof. It was shown in [18, Proposition 4.8, Lemma 4.2] that there is an element $a_0 \in \text{mgl}^{2d,d}(BB; R)$ such that $i^*(a_0)$ is the class of a rational point in $\text{MGL}^{2d,d}(G/B; R) = \Omega^d(G/B; R)$. Moreover, it was also shown in [18, Proposition 4.8] that $\alpha_0 = \pi_*(a_0) \in \text{mgl}^{a,b}_G(BG; R)$ is an invertible element, where $\pi_* := (\pi_k)_*$ is the push-forward map of Proposition 11.5.
Setting \( a = p^b_{B,X}(a_0) \), we see that there is a class \( a \in \text{mgl}_{B}^{d,d}(X; R) \) such that \( t'_X(a) \) is the class of a rational point in \( MGL_{B}^{d,d}(G/B; R) = \Omega^d(G/B; R) \). It follows moreover from Proposition \([11.5]\) that \( \alpha = (\pi_X)_*(a) \in \text{mgl}_{G}^{*}(X; R) \) is an invertible element.

For any \( j \geq i \geq 1 \), we have a diagram

\[
\begin{align*}
\text{MGL}^{*}(X^j_G; R) & \xrightarrow{(s_{G,X}^j)^*} \text{MGL}^{*}(X^j_B; R) \xrightarrow{(s_{B,X}^j)^*} \text{MGL}^{*}(X^j_G; R) \\
\Downarrow (s_{G,X}^j)^* & \quad \Downarrow (s_{B,X}^j)^* \\
\text{MGL}^{*}(X^j_G; R) & \xrightarrow{(\pi_X)^*} \text{MGL}^{*}(X^j_B; R) \xrightarrow{(\pi_X)^*} \text{MGL}^{*}(X^j_G; R).
\end{align*}
\]

The left square clearly commutes and the right square commutes by Lemma \([9.4]\). Notice that the map \( \text{MGL}^{*}(X^j_B; R) \rightarrow \text{MGL}^{*}(X^j_G; R) \) is an isomorphism as shown in Proposition \([11.1]\). Setting \( s_{G,X}^j : X^j_G \hookrightarrow X_G \) and taking limit over \( j \geq i \), we see that \( (s_{G,X}^j)^* \circ (\pi_X)_* = (\pi_X^i)_* \circ (s_{B,X}^j)^* \) for every \( i \geq 1 \). Letting \( a_i = (s_{B,X}^j)^*(a) \) and \( \alpha_i = (s_{G,X}^j)^*(\alpha) \), we conclude that \( (\pi_X^i)_*(a_i) = \alpha_i \) for every \( i \geq 1 \). Moreover, \( \alpha_i \in \text{MGL}^{*}(X^j_G; R) \) is invertible.

To verify the required Mittag-Leffler condition, fix any \( i \geq 1 \). It follows from Lemma \([8.3]\) that there exists \( i' \gg i \) such that

\[
\text{Image} \left( (s_{B,X}^{i'})^* \right) = \text{Image} \left( (s_{B,X}^{i})^* \right) \quad \text{for all} \quad j \geq i'.
\]

Given any \( x \in \text{MGL}^{*}(X^j_G; R) \), we get for any \( j \geq i' \):

\[
(s_{G,X}^{i'})^*(\alpha_{i'} \cdot x) = (s_{G,X}^{i})^* \circ (\pi_X^{i'})_* (a_{i'} \cdot (\pi_X^{i'})^*(x)) = (\pi_X^i)_* \circ (s_{B,X}^{i'})^* (a_{i'} \cdot (\pi_X^{i'})^*(x)) = (\pi_X^i)_* \circ (s_{B,X}^{i})^*(y) = (s_{G,X}^{i})^* \circ (\pi_X^i)_*(y),
\]

where the first equality is from the projection formula and the second equality follows from \([11.10]\) for some \( y \in \text{MGL}^{*}(X^j_B; R) \). Since \( \alpha_{i'} \in \text{MGL}^{*}(X^{i'}_G; R) \) is invertible, we see that \( \text{Image} \left( (s_{B,X}^{i'})^* \right) \subseteq \text{Image} \left( (s_{B,X}^{i})^* \right) \). Since the other inclusion is obvious, this verifies the desired Mittag-Leffler condition. \( \square \)

As an immediate consequence of Lemma \([11.6]\) and Proposition \([5.3]\) we get the following generalization of Corollary \([0.7]\).

**Corollary 11.7.** Let \( G \) be a connected and split reductive group over \( k \) and let \( X \in \text{Sm}^G_k \) be projective. Then for any \( a \geq b \geq 0 \), the map

\[ \phi_X : \text{MGL}_{G}^{a,b}(X; R) \rightarrow \text{mgl}_{G}^{a,b}(X; R) \]

is an isomorphism.

**Lemma 11.8.** Let \( G \) be a connected and split reductive group over \( k \) and let \( X \in \text{Sm}^G_k \) be projective. Let \( a \in \text{mgl}_{B}^{d,d}(X; R) \) be the element constructed in the proof of Lemma \([11.0]\) and let \( \theta_X : \text{mgl}_{G}^{a,b}(X; R) \rightarrow \text{mgl}_{G}^{a,b}(X; R) \) be the map \( x \mapsto (\pi_X)_*(a \cdot x) \). Then the maps

\[
\left( \text{mgl}_{G}^{a,b}(X; \mathbb{Q}) \right)^W \xrightarrow{\theta_X} \text{mgl}_{G}^{a,b}(X; \mathbb{Q}) \xrightarrow{r_{F,X}^G} \left( \text{mgl}_{G}^{a,b}(X; \mathbb{Q}) \right)^W
\]
are isomorphisms.

Proof. We shall follow the notations that we used in the proof of Lemma \[11.6\]. Let \( \rho = (V_i, U_i)_{i \geq 1} \) be an admissible gadget for \( G \). For any \( i \geq 1 \), let \( a_i = (s_{i,B,X})^*(a) \) and let \( \theta^i_X : MGL^{a,b}(X_B^i) \to MGL^{a,b}(X_G^i) \) be the map \( x \mapsto (\pi^i_X)_*(a_i \cdot x) \). It suffices to show that the maps

\[
(11.11) \quad (MGL^{a,b}(X_B^i; \mathbb{Q}))^W \xrightarrow{\theta^i_X} MGL^{a,b}(X_G^i; \mathbb{Q}) \xrightarrow{(\pi^i_X)^*} (MGL^{a,b}(X_B^i; \mathbb{Q}))^W
\]

are isomorphisms for every \( i \geq 1 \).

Let \( \tilde{a}_i \in H^{2d,d}(X_B^i; \mathbb{Q}) \) be the image of the element \( a_i \in MGL^{2d,d}(X_B^i; \mathbb{Q}) \) under the natural transformation of functors \( MGL^{*,*}(-) \to H^{*,*}(-) \). By \[10.3\], it suffices to show that the map

\[
(11.12) \quad (H^{a,b}(X_B^i; \mathbb{Q}))^W \xrightarrow{\theta^i_X} H^{a,b}(X_G^i; \mathbb{Q}); \quad x \mapsto (\pi^i_X)_*(\tilde{a}_i \cdot x)
\]

is an isomorphism.

It follows from \[22\] Lemma 6.4] that there is a natural isomorphism

\[
\phi^i_X : H^*(G/B, \mathbb{Q}) \otimes \mathbb{Q} H^{*,*}(X_G^i; \mathbb{Q}) \xrightarrow{\approx} H^{*,*}(X_B^i, \mathbb{Q}),
\]

where \( H^*(X) = \bigoplus_{i \geq 0} H^{2i,i}(X) \). Let \( \iota_i : G/B \to X_B^i \) be the fiber of the map \( X_B^i \xrightarrow{\pi^i_X} X_G^i \). It is well known that the map \( (H^*(G/B; \mathbb{Q}))^W \to \mathbb{Q} \), given by \( x \mapsto (p_{G/B})_*(\iota_i^*(a_i) \cdot x) \) is an isomorphism. It follows from the construction of the map \( \phi^i_X \) (cf. \[22\] Lemma 6.4]) that the map in \[11.12\] is an isomorphism. \( \square \)

Proposition 11.9. Let \( G \) be a connected reductive group with a split maximal torus \( T \) and the associated Weyl group \( W \). Then for any \( X \in \text{Sm}_k^G \) which is projective, the map

\[
r^G_{T,X} : MGL^{*,*}_G(X; R) \to (MGL^{*,*}_T(X; R))^W
\]

is injective.

Proof. In view of Corollary \[11.7\], we can replace \( MGL^{*,*}_G(X; R) \) and \( MGL^{*,*}_T(X; R) \) by \( mgl^{*,*}_G(X; R) \) and \( mgl^{*,*}_T(X; R) \) respectively. Let \( B \) be a Borel subgroup of \( G \) containing the maximal torus \( T \). Using the definition of \( mgl^{*,*}_G(X; R) \) and the fact that the inverse limit commutes with taking \( W \)-invariants, it suffices to show that for an admissible gadget \( \rho = (V_i, U_i)_{i \geq 1} \) for \( G \), the map

\[
(\pi^i_X)^* : MGL^{*,*}(X_G^i; R) \to (MGL^{*,*}(X_B^i; R))^W
\]

is injective for all \( i \geq 1 \).

To show this, let \( x \in MGL^{*,*}(X_G^i; R) \). Let \( a_i \in MGL^{*,*}(X_B^i; R) \) and \( \alpha_i \in MGL^{*,*}(X_G^i; R) \) be as in the proof of Lemma \[11.6\]. Using the projection formula, we see that \( (\pi^i_X)_*(a_i \cdot (\pi^i_X)^*(x)) = \alpha_i \cdot x \). Since \( \alpha_i \in MGL^{*,*}(X_G^i; R) \) is invertible, we see that \( (\pi^i_X)^* \) is injective. \( \square \)

For any \( X \in \text{Sm}_k^G \), let \( MGL^*(X) = \bigoplus_{i \geq 0} MGL^{2i,i}(X) \) and for any \( X \in \text{Sm}_k^G \), let \( MGL^*_G(X) = \bigoplus_{i \geq 0} MGL^{2i,i}_G(X) \). We can prove a stronger form of Proposition \[11.9\] in the following case.
Theorem 11.10. Let $G$ be a connected reductive group with a split maximal torus $T$ and the associated Weyl group $W$. Let $X \in \text{Sm}^G(k)$ be projective such that $T$ acts on $X$ with only finitely many fixed points. Then the map
\[ r^G_T, X : MGL^*_{G}(X; R) \rightarrow (MGL^*_T(X; R))^W \]
is an isomorphism.

Proof. In view of Proposition 11.9 we only need to prove the surjectivity assertion. We can identify $MGL^*_G(X; R)$ and $mgl^*_G(X; R)$ using Corollary 11.7.

It follows from Theorem 9.5 and [22, Proposition 6.7] that for any $r \in \mathbb{L}_R[[t_1, \ldots, t_n]]$, using Lemma 8.8 and Theorem 8.9, we see that $MGL^*_G(X; R) \cong \mathbb{L}_R[[t_1, \ldots, t_n]]$. Using Lemma 8.8 and Theorem 8.9, we see that $MGL^*_G(X; R)$ is injective.

We now consider the commutative diagram
\[
\begin{array}{ccc}
(MGL^*_T(X; R))^W & \xrightarrow{(\pi_X)_*} & MGL^*_G(X; R) \\
\downarrow f & & \downarrow f \\
(MGL^*_T(X; \mathbb{Q}))^W & \xrightarrow{(\pi_X)_*} & MGL^*_G(X; \mathbb{Q}) \\
\end{array}
\]

(11.13)

Let $a \in MGL^*_T(X; R)$ and $\alpha = (\pi_X)_*(a) \in MGL^*_G(X; R)$ be as in the proof of Lemma 11.6. Recall that $\alpha \in MGL^*_G(X; R)$ is invertible. For any $x \in MGL^*_G(X; R)$ and $y = r^G_T, X(x)$, it follows from Proposition 11.5 that
\[
\begin{align*}
r^G_T, X \circ (\pi_X)_*(a \cdot y) &= r^G_T, X \circ (\pi_X)_*(a \cdot r^G_T, X(x)) \\
&= r^G_T, X(a \cdot x) \\
&= r^G_T, X(a) \cdot y.
\end{align*}
\]

In particular, it follows from Lemma 11.8 that for any $y \in (MGL^*_T(X; R))^W$, one has
\[
r^G_T, X(\alpha) \cdot f(y) = r^G_T, X \circ (\pi_X)_*(a \cdot f(y)) = f \left( r^G_T, X \circ (\pi_X)_*(a \cdot y) \right).
\]

Equivalently, we get $f(y) = f \left( r^G_T, X(\alpha^{-1}) \cdot (r^G_T, X \circ (\pi_X)_*(a \cdot y)) \right)$. Since we have shown above that $f$ is injective, we get
\[
y = r^G_T, X(\alpha^{-1}) \cdot (r^G_T, X \circ (\pi_X)_*(a \cdot y)) = r^G_T, X(\alpha^{-1} \cdot ((\pi_X)_*(a \cdot y))).
\]

This proves the required surjectivity. \hfill \Box

As an immediate consequence of Theorem 11.10 we obtain the following generalization of Totaro’s theorem [38, Theorem 1.3] to the case of motivic cobordism of the classifying spaces of reductive groups.

Corollary 11.11. Let $G$ be a connected reductive group with a split maximal torus $T$ and let $B$ be a Borel subgroup containing $T$. Then
\begin{enumerate}
\item $MGL^*(BG; R) \cong (MGL^*(BT; R))^W$ and
\item $MGL^*(BT; R) \cong (MGL^*_T(G/B; R))^W$.
\end{enumerate}

Proof. The first part follows straightaway from Theorem 11.10 by taking $X = \text{Spec}(k)$. The second part follows by applying Theorem 11.10 to $X = G/B$ and then using the identification $MGL^*_G(G/B) \cong MGL^*_T(k) = MGL^*(BT)$ by Theorem 7.2 and Proposition 11.1. \hfill \Box
As another application of the above results, we get the following computation of the $T$-equivariant motivic cobordism of the flag variety $G/B$.

**Corollary 11.12.** Let $G$ be a connected and split reductive group over $k$ with a split maximal torus $T$. Let $B$ be a Borel subgroup of $G$ containing $T$. Then the natural map

$$
\Psi_{G/B}: S(T; R) \otimes_{S(G; R)} S(T; R) \rightarrow MGL^*_T(G/B; R)
$$

$$
\Psi(a \otimes b) = a \cdot r^G_{T,G/B}(b);
$$

is an isomorphism of $S(T; R)$-algebras.

**Proof.** The above map is defined using the identification $MGL^*_G(G/B) = MGL^*_G(k) = S(T)$. Now, the desired isomorphism follows from [18, Theorem 4.6], Theorem 9.5 (11) and Corollary 11.7. □

12. **Realizations of equivariant motivic cobordism**

We shall assume in this section that the ground field $k$ is a subfield of the field of complex numbers and we fix an embedding $\sigma: k \hookrightarrow \mathbb{C}$. Recall that there is a functor $\text{HoSets}_\bullet \rightarrow \mathcal{H}_\bullet(k)$, where $\text{HoSets}_\bullet$ denote the unstable homotopy category of simplicial sets which is equivalent to the homotopy category of pointed topological spaces $\text{Top}_\bullet$ via the geometric realization functor. Let $\text{SH}$ denote the stable homotopy category the pointed topological spaces.

12.0.1. **Topological realization functor.** There is a topological realization functor $\text{Sm}_k \rightarrow \text{Top}$ which takes a scheme $X$ over $k$ to the space $X^{an} = X(\mathbb{C})$ of the complex valued points on $X$ via the embedding $\sigma$. Then $X^{an}$ is a complex manifold.

Every motivic space $Y \in \text{Spc}$ can be written as $\left( \colim_{X \Delta^n \rightarrow Y} X \times \Delta^n \right) \sim Y$. This gives a topological realization functor

$$
\mathbf{R}_\mathbb{C}: \text{Spc}_\bullet \rightarrow \text{Top}_\bullet;
$$

$$
\mathbf{R}_\mathbb{C}(Y) = \colim_{X \Delta^n \rightarrow Y} (X^{an} \times |\Delta^n|).
$$

It is clear from this that for any ind-scheme $X$, $\mathbf{R}_\mathbb{C}(X)$ is complex manifold, which may be infinite-dimensional. We shall write $\mathbf{R}_\mathbb{C}(X)$ often as $X^{an}$ if $X$ is an ind-scheme.

Let $\text{Sp}(\text{Top}, \mathbb{CP}^1)$ denote the category of $\mathbb{CP}^1$-spectra in the category $\text{Top}_\bullet$. Let $(\text{SH}, \mathbb{CP}^1)$ denote the stable homotopy category of pointed $\mathbb{CP}^1$-spectra under the stable equivalence. It is known, as can be found in [32, §A.7], that the above topological realization functor descends to an exact functor $\mathbf{R}_\mathbb{C}: \text{SH}(k) \rightarrow (\text{SH}, \mathbb{CP}^1)$. There is a suspension functor $\Sigma_{\text{top}}: (\text{SH}, \mathbb{CP}^1) \rightarrow (\text{SH}, S^1)$ from the category of $\mathbb{CP}^1$-spectra to the category of $S^1$-spectra which is given by

$$
(\Sigma_{\text{top}}M_\bullet)_{2n} = M_n \quad \text{and} \quad (\Sigma_{\text{top}}M_\bullet)_{2n+1} = S^1 \wedge M_n = \Sigma M_n.
$$

This functor induces an equivalence of the stable homotopy categories. This stable homotopy category is denoted by $\text{SH}$. In other words, there is an exact functor $\mathbf{R}_\mathbb{C}: \text{SH}(k) \rightarrow \text{SH}$.

**Notation:** Let $X \in \text{Sm}_k$ and let $E \in \text{SH}$. For the rest of this section, we shall denote the generalized cohomology $E^*(X^{an})$ in short by $E^*(X)$. 


12.0.2. **Complex cobordism.** Recall that the complex cobordism $MU$ is obtained by applying the suspension functor $\Sigma_{top}$ to the $\mathbb{CP}^1$-spectrum $(MU_0, MU_1, \ldots)$, where $MU_n = Th(E_n)$, where $E_n$ is the universal rank $n$-bundle on the classifying space $BU_n$. The rank $(n+1)$-bundle $O_{BU_n} \oplus E_n$ defines a unique map $BU_n \xrightarrow{\sim} BU_{n+1}$ such that one has a commutative diagram like (1.4). This gives the bounding maps $\mathbb{CP}^1 \wedge Th(E_n) \to Th(E_{n+1})$ of the spectrum $MU$. The weak equivalence of topological spaces $E_n/(E_n \setminus \{0\}) \to Th(E_n)$ and the homotopy equivalence $BU_n \cong BGL_n$ shows that $R_\mathbb{C}(MGL) = MU$, where $MGL$ is the motivic Thom spectrum (cf. (1.5)). In fact, this is an isomorphism of ring spectra. In particular, for any ind-scheme $X$, there is a natural map

$$
\text{Hom}_{\mathcal{SH}(k)} \left( \Sigma^\infty X_+, \Sigma^{a,b} MGL \right) \to \text{Hom}_{\mathcal{SH}} \left( \Sigma^\infty_{\mathbb{CP}^1} X^\an_+, R_\mathbb{C} \Sigma^{a,b} MU \right)
$$

for any $a \geq b \geq 0$. Using the canonical isomorphism $R_\mathbb{C}(\mathbb{G}_m) \cong S^1$, we see that there is a natural homomorphism

$$
t_X : MGL^{a,b}(X) \to MU^a(X^\an).
$$

Recall that if $G$ is a complex Lie group acting on a finite-dimensional $CW$-complex $X$, the equivariant complex cobordism of $X$ is defined as

$$
MU^*_G(X) := MU^* \left( X^G \times EG \right)
$$

where $EG \to BG$ is the universal principal $G$-bundle over the classifying space $BG$. It is known that $MU^*_G(X)$ does not depend on the choice of the universal principal bundle $EG \to BG$.

Let $G$ be a linear algebraic group over $k$ and let $X \in \text{Sm}_k^G$. Then $G^\an$ is a complex Lie group and it follows from Lemma 2.3 and (12.2) that there is a natural homomorphism

$$
t^G_X : MGL^{a,b}_G(X) \to MU^a_G(X).
$$

In particular, we get a natural ring homomorphism $t^G_X : MGL^*_G(X) \to MU^*_G(X)$ which takes an element $x \in MGL^{2a,a}_G(X)$ to an element $t^G_X(x) \in MU^{2a}_G(X)$.

12.1. **Equivariant motivic cohomology and the Cycle class maps.** For any abelian group $A$, let $H_A$ denote the motivic Eilenberg-MacLane $T$-spectrum in $\mathcal{SH}(k)$ as defined by Voevodsky [10]. For a linear algebraic group $G$ over $k$ and $X \in \text{Sm}_k^G$, one defines the *equivariant motivic cohomology* of $X$ by

$$
H^{a,b}_G(X; A) := \text{Hom}_{\mathcal{SH}(k)} \left( \Sigma^\infty X_+^G, \Sigma^{a,b} H_A \right)
$$

where $\rho = (V_i, U_i)$ is an admissible gadget for $G$. One also defines the analogue of $mgl^*_G(X)$ as

$$
h^{a,b}_G(X; A) := \lim_{i \to 1} H^{a,b}_G \left( X^G \times U_i; A \right).
$$

It is shown in [22] that $h^{r,*}_G(\cdot)$ is well-defined and it is an example of an oriented cohomology theory on $\text{Sm}_k^G$. Moreover, it follows from the proof of Theorem 10.2 that the map $H^{a,b}_G(X; A) \to h^{a,b}_G(X; A)$ is in fact an isomorphism. In particular, the analogue of Theorem 9.5 holds verbatim for the equivariant motivic cohomology $H^{r,*}_G(\cdot)$.
It was shown by Voevodsky in [39, Proposition 3.8] that $R_C(H_A)$ is isomorphic to the topological Eilenberg-MacLane spectrum in $SH$. In particular, one obtains a commutative diagram

$$
\begin{array}{ccc}
MGL^{a,b}(X; A) & \xrightarrow{c^G_X} & MU_G^a(X; A) \\
\downarrow & & \downarrow \\
H_G^{a,b}(X; A) & \xrightarrow{c^G_X} & H_G^a(X; A)
\end{array}
$$

of the equivariant cohomology theories on $\text{Sm}_k^G$. The horizontal maps are called the cycle class maps.

12.1.1. Totaro’s refined cycle class map. In order to study the cycle class maps and the natural maps between the equivariant versions of cobordism and the ordinary cohomology, we need to recall the following notation for the tensor product while dealing with projective systems of modules over a commutative ring. Let $A$ be a commutative ring and let $\{L_i\}$ and $\{M_i\}$ be two projective systems of $A$-modules. Following [37], one defines the topological tensor product of $L$ and $M$ by

$$
L \hat{\otimes}_A M := \lim_{\leftarrow i} (L_i \otimes_A M_i).
$$

Given a linear algebraic group $G$, $X \in \text{Sm}_k^G$ and an $L$-module $A$, the proof of Lemma [9.2] shows that the tensor product

$$
mgl^{\ast, \ast}_G(X) \hat{\otimes}_L A = \lim_{\leftarrow i} \left( MGL^{\ast, \ast}(X \times G U_i) \otimes_L A \right)
$$

is independent of the choice of an admissible gadget $\rho = (V_i, U_i)$ for $G$. In case of the natural map $MGL^\ast_G(X) \to mgl^{\ast, \ast}_G(X)$ being an isomorphism, we shall also use the notation $MGL^\ast_G(X) \hat{\otimes}_L A$ for $mgl^{\ast, \ast}_G(X) \hat{\otimes}_L A$.

We mentioned earlier that the present definition of the Chow groups of the classifying space of a linear algebraic group $G$ over $k$ was invented by Totaro [37]. He also showed that the cycle class map $c^G_X : CH^\ast(BG) \to H^\ast(BG)$ in fact factors through a refined cycles class map $\tilde{c}^G_X : CH^\ast(BG) \to MU^\ast(BG) \hat{\otimes}_L \mathbb{Z}$. It is known that $MU^\ast(X) \hat{\otimes}_L \mathbb{Z}$ naturally maps to $H^\ast(X)$ and is a more refined topological invariant of a topological space $X$ than its singular cohomology $H^\ast(X)$.

Recall that that $R$ denotes the ring $\mathbb{Z}[t_G^{-1}]$, where $t_G$ is the torsion-index of $G$. As a consequence of Corollary [14.7] Theorem 9.5 and [19] Proposition 7.2, we obtain the following generalization of [37] Theorem 2.1.

**Theorem 12.1.** For a linear algebraic group $G$ over $k$ and $X \in \text{Sm}_k^G$ projective, there is a natural refined cycle class map

$$
\tilde{c}^G_X : CH^\ast_G(X; R) \to MU^\ast_G(X; R) \hat{\otimes}_{LR} R.
$$

such that its composite with the natural map $MU^\ast_G(X; R) \hat{\otimes}_{LR} R \to H^\ast_G(X; R)$ is the usual cycle class map $c^G_X$. 

12.2. Comparison of equivariant motivic and complex cobordism. As a consequence of the results of Quillen [33], Levine-Morel [27] and Levine [26], one knows that the topological realization map $MGL^*(k) \to MU^*\langle pt \rangle$ is an isomorphism. Based on this isomorphism, we now prove some comparison results for the equivariant motivic and complex cobordisms of schemes with group actions. As a consequence, we verify some conjectures of Totaro about the cycle class maps. The following result is the topological analogue of Theorem 11.10. This result for the singular cohomology was earlier proven by Holm and Sjamaar [16] Proposition 2.1.

**Theorem 12.2.** Let $G$ be a connected reductive group over $k$ with a split maximal torus $T$ and the associated Weyl group $W$. Let $X \in \text{Sm}^G_k$ be projective such that $T$ acts on $X$ with only finitely many fixed points. Then the map

$$r_{T,X}^{G,\text{top}} : MU^*_G(X;R) \to (MU^*_T(X;R))^W$$

is an isomorphism.

**Proof.** This is essentially proven by an easy topological translation of the proof of Theorem 11.10. We give a sketch of the main steps. The topological version of (11.5) gives rise to the commutative diagram

$$
\begin{align*}
MU^*(B_G) & \xrightarrow{\pi_X^*} MU^*(B_T) \xrightarrow{\pi_Y^*} MU^*(G/B) \\
\varphi_{G,X}^* & \downarrow \quad \downarrow \varphi_{T,X}^* \\
MU^*_G(X) & \xrightarrow{\iota_X^*} MU^*_T(X) \xrightarrow{\iota_X^*} MU^*(G/B).
\end{align*}
$$

It was shown in [18] Lemma 4.2 that there are elements $\{\rho_{w,X} : w \in W\}$ in $MU^*_T(X)$ such that $\{\iota_X^*(\rho_{w,X}) : w \in W\}$ forms an $L$-basis of $MU^*(G/B)$. Moreover, we can choose $\rho_{w_0,X} = 1$, where $w_0 \in W$ is the longest length element. It follows from [18] Lemma 4.3] that the map

$$\Psi_X^{\top} : MU^*(BT) \otimes_{MU^*(BG)} MU^*_G(X) \to MU^*_T(X);$$

$$\Psi_X^{\top}(x \otimes y) = \varphi_{T,X}^* (x) \cdot \pi_X^*(y)$$

is $W$-equivariant and an isomorphism of $MU^*(BT)$-modules. In particular, we get

$$\Psi_X^{\top}(1 \otimes y) = \Psi_X^{\top} (\iota_X^*(\rho_{w_0,X}) \otimes y) = \pi_X^*(y).$$

Hence, to show that $r_{T,X}^{G,\text{top}}$ is injective, it suffices to show that the map $MU^*_G(X) \xrightarrow{1 \otimes id} (MU^*(G/B) \otimes MU^*_G(X))^W$ is injective. But to do this, we only have to observe from the projection formula for the map $p_{G/B} : G/B \to \text{pt}$ that $p_{G/B}^* (\rho \cdot p_{G/B}^*(x)) = p_{G/B}^* (\rho) \cdot x = x$, where $\rho \in MU^*(G/B)$ is the class of a point. This gives a right inverse of the map $p_{G/B}^*$ and hence a right inverse of $1 \otimes id = p_{G/B}^* \otimes id$.

To prove the surjectivity of the map $r_{T,X}^{G,\text{top}}$, we first note from our assumption and the topological analogue of Theorem 8.9 that $MU^*_T(X) \cong (\mathbb{L}[[t_1, \ldots, t_n]])^r$, where $n = \text{rank}(T)$ and $r$ is the number of $T$-fixed points on $X$. In particular, the map $MU^*_T(X;R) \to MU^*_T(X;\mathbb{Q})$ is injective.
Now, the surjectivity argument of Theorem [11.10] goes through verbatim, where one has to replace \(a \in MGL^*_T(X; R)\) with \(t^*_X(a) \in MU^*_T(X; R)\) and \(\alpha \in MGL^*_G(X; R)\) with \(t^*_X(\alpha) \in MU^*_G(X; R)\) and then use the fact that the surjectivity result holds over the rationals by [19, Theorem 8.8]. \(\square\)

12.2.1. Totaro’s conjectures. It was conjectured (cf. [37, Introduction]) that the refined cycle class \(\text{CH}^*(BG) \to MU^*(BG)\) should be an isomorphism. Totaro modified this conjecture to an expectation that this map should be an isomorphism after localization at certain prime \(p\). We shall show below that the refined cycle class map is in fact an isomorphism after inverting the torsion index of the group \(G\). We first have the following stronger result.

**Theorem 12.3.** Let \(G\) be a connected reductive group over \(k\) with a split maximal torus \(T\). Let \(X \in \text{Sm}_k^G\) be projective such that \(T\) acts on \(X\) with only finitely many fixed points. Then the maps

\[
t^*_X : MGL^*_G(X; R) \to MU^*_G(X; R)
\]

and

\[
c^*_X : \text{CH}^*_G(X; R) \to H^*_G(X; R)
\]

are isomorphisms. In particular, \(MU^*_G(X; R)\) and \(H^*_G(X; R)\) have no element in odd degrees.

**Proof.** It follows from Theorems [8.3, 9.5, Corollary 9.7, 18, Thorem 3.7] that the map \(MGL^*_T(X) \xrightarrow{t^*_X} MU^*_T(X)\) is an isomorphism.

A much simpler argument shows that the map \(\text{CH}^*_T(X) \xrightarrow{c^*_X} H^*_T(X)\) also is an isomorphism. To see this quickly, take a canonical admissible gadget \((V_i, U_i)\) for \(T\) and observe that \(X_i = X \times^T U_i\) is then a smooth cellular scheme and hence the map \(\text{CH}^*(X_i) \to H^*(X_i)\) is an isomorphism. It follows that the map \(c^*_X\) is an isomorphism.

The isomorphism of \(t^*_X\) follows immediately from the isomorphism of \(t^*_T\), combined with Theorems [11.10, 12.1]. The isomorphism of \(c^*_X\) follows from the isomorphism of \(c^*_T\), combined with [22, Corollary 5.9, 18, Lemma 3.6] and [16, Proposition 2.1]. \(\square\)

**Theorem 12.4.** Let \(G\) be a connected reductive group over \(k\) with a split maximal torus \(T\). Let \(X \in \text{Sm}_k^G\) be projective such that \(T\) acts on \(X\) with only finitely many fixed points. Then the maps

\[
(12.10) \quad \text{CH}^*_G(X; R) \xleftrightarrow{c^*_X} MU^*_G(X; R) \widehat{\otimes}_{\mathbb{Z}_p} R \to H^*_G(X; R)
\]

are isomorphisms.

**Proof.** It follows from Theorem [12.3] that the composite map \(\text{CH}^*_G(X; R) \xrightarrow{c^*_X} H^*_G(X; R)\) is an isomorphism. Thus, we only need to show that the refined cycle class map \(\overline{c^*_X}\) (cf. Theorem [12.1]) is surjective.

Let \(\rho = (V_i, U_i)\) be an admissible gadget for \(G\). Let us denote the mixed space \(X \xrightarrow{G} \overline{\text{Sm}}_k\) in short by \(X_i\). It follows from our assumption, Theorem [8.3]
and [18, Lemma 3.6] that $H^*_T(X)$ is torsion-free. It follows from [16, Proposition 2.1] that $H^*_G(X; R)$ is torsion-free. It follows subsequently using the Atiyah-Hirzebruch spectral sequence (cf. [21, Corollary 2], [37, Lemma 2.2]) that the map $MU^*_G(X; R) \rightarrow \varprojlim_i MU^*_i(X; R)$ is an isomorphism and moreover, for each positive integer $i$, there is $j \geq i$ such that

\[(12.11) \quad \text{Image (} MU^*_G(X; R) \rightarrow MU^*_i(X; R) \text{)} = \text{Image (} MU^*_G(X; R) \rightarrow MU^*_j(X; R) \text{)} .\]

We conclude from Corollary 11.7 and Theorem 12.3 that there is a commutative diagram of $L_R$-modules

\[(12.12) \quad MGL^*_G(X; R) \xrightarrow{\tilde{c}_X^G} MU^*_G(X; R) \]

\[\xrightarrow{\varprojlim_i} \quad \lim_i MGL^*(X; R) \xrightarrow{\varprojlim_i} \lim_i MU^*(X; R) \]

in which all arrows are isomorphisms.

We now show the surjectivity of $\tilde{c}_X^G$. Using the isomorphisms in (12.12) and [19, Proposition 7.2], we need to show that the map

\[(12.13) \quad \lim_i MGL^*(X; R) \xrightarrow{\varprojlim_i} \lim_i MU^*(X; R) \]

is surjective. Since the bottom horizontal arrow in (12.12) is an isomorphism, it suffices to show that the map

\[(12.14) \quad \lim_i MU^*(X; R) \xrightarrow{\varprojlim_i} \lim_i MU^*(X; R) \]

is surjective. To show this, it suffices to show that $\varprojlim_i \mathbb{L}^{<0}_R MU^*(X; R) = 0$. Using the surjectivity $\varprojlim_i \mathbb{L}^{<0}_R MU^*(X; R) \rightarrow \varprojlim_i \mathbb{L}^{<0}_R MU^*(X; R)$, it is enough to show that $\varprojlim_i \mathbb{L}^{<0}_R MU^*(X; R) = 0$.

To prove this last assertion, it suffices to show that the projective system $\{\mathbb{L}^{<0}_R \otimes_{L_R} MU^*(X; R)\}$ satisfies the Mittag-Leffler condition. On the other hand, it follows from (12.11) that the projective system $\{MU^*(X; R)\}$ satisfies the Mittag-Leffler condition. From this, it follows immediately that the same holds for $\{\mathbb{L}^{<0}_R \otimes_{L_R} MU^*(X; R)\}$. We have thus proven the surjectivity of $\tilde{c}_X^G$. This completes the proof of the theorem. \[\square\]

**Corollary 12.5.** Let $G$ be a connected split reductive group over $k$. Then the maps $MGL^*(BG; R) \rightarrow MU^*(BG; R)$ and $CH^*(BG; R) \rightarrow MU^*(BG; R) \otimes_{L_R} R$ are isomorphisms.

### 13. Motivic cobordism of quotient stack

In this section, we show how one can use equivariant motivic cobordism to define the motivic cobordism for quotient stacks. The motivic cohomology of such stacks
was earlier defined by Edidin-Graham [11]. Our definition is based on the following result.

Proposition 13.1. Let $G$ and $H$ be two linear algebraic groups acting on two smooth schemes $X$ and $Y$ respectively such that $[X/G] \cong [Y/H]$ as stacks. There is then a canonical isomorphism $X_G \cong Y_H$ of motivic spaces in $\mathcal{H}(k)$.

Proof. Let $\mathcal{X}$ denote the stack $[X/G] \cong [Y/H]$. Let $\rho = (V_i, U_i)_{i \geq 1}$ and $\rho' = (V'_i, U'_i)_{i \geq 1}$ be admissible gadgets for $G$ and $H$ respectively. We set $\overline{X}_G(\rho) = [(X \times V_i)/G]$ and $\overline{Y}_H(\rho') = [(Y \times V'_i)/H]$.

This yields representable morphisms of stacks

$$
(13.1) \quad X^i_G(\rho) \hookrightarrow \overline{X}_G(\rho) \xrightarrow{f_i} [X/G] \cong \mathcal{X} \quad \text{and} \quad Y^j_H(\rho') \hookrightarrow \overline{Y}_H(\rho') \xrightarrow{g_i} [Y/H] \cong \mathcal{X}.
$$

For each $i, j \geq 1$, we consider the fiber product diagrams of stacks

$$
\begin{array}{ccc}
U_{i,j} & \xrightarrow{p_{i,j}} & X^i_G(\rho) \\
\downarrow g_{i,j} & & \downarrow f_i \\
Y^j_H(\rho') & \xrightarrow{g_j} & \mathcal{X}
\end{array}
\quad \begin{array}{ccc}
V_{i,j} & \xrightarrow{v_{i,j}} & X^i_G(\rho) \\
\downarrow \rho_{i,j} & & \downarrow f_i \\
Y^j_H(\rho') & \xrightarrow{g_j} & \mathcal{X}
\end{array}
$$

Since each $g_j$ in the diagram on the right is a vector bundle map of stacks with fiber $V'_j$, we see that each $\rho_{i,j}$ is a vector bundle with fiber $V'_j$. In particular, each $V_{i,j}$ is a smooth scheme and $U_{i,j} \subseteq V_{i,j}$ is an open subscheme.

For a fixed $i \geq 1$, we get a sequence $(V_{i,j}, U_{i,j}, f_{i,j})_{j \geq 1}$ of pairs of smooth schemes where $V_{i,j} \to X^i_G(\rho)$ is a vector bundle, $U_{i,j} \subseteq V_{i,j}$ is an open subscheme and $f_{i,j} : (V_{i,j}, U_{i,j}) \to (V_{i,j+1}, U_{i,j+1})$ is the natural map of pairs of smooth schemes over $X^i_G(\rho)$. It follows moreover from the property of $\rho'$ being an admissible gadget for $H$ that $(V_{i,j}, U_{i,j}, f_{i,j})_{j \geq 1}$ is an admissible gadget over $X^i_G(\rho)$ in the sense of [28, Definition 4.2.1].

Setting $U_i = \text{colim}_j U_{i,j}$ and $p_i = \text{colim}_j p_{i,j}$, we conclude from [28, Proposition 4.2.3] that $U_i \xrightarrow{p_i} X^i_G(\rho)$ is an $\mathbb{A}^1$-weak equivalence. Taking the colimit of these maps as $i \to \infty$, we conclude that $U \xrightarrow{p} X_G(\rho)$ is an $\mathbb{A}^1$-weak equivalence, where $U = \text{colim}_i U_{i,j}$. By reversing the roles of $\rho$ and $\rho'$, we see that the map $U \xrightarrow{q} Y_H(\rho')$ is also an $\mathbb{A}^1$-weak equivalence. This concludes the proof. \hfill $\square$

Definition 13.2. Let $\mathcal{X}$ be a smooth stack of finite type over $k$ which is isomorphic to a stack of the form $[X/G]$ where $G$ is a linear algebraic group acting on a smooth scheme $X$ over $k$. We define the motivic cobordism of $\mathcal{X}$ as

$$
MGL^{a,b}(\mathcal{X}) := MGL^{a,b}_G(X).
$$

It follows from Proposition [13.1] that $MGL^{a,b}(\mathcal{X})$ is well defined. We let $MGL^{*,*}(\mathcal{X})$ to be the sum $\bigoplus_{a,b} MGL^{a,b}(X)$. It follows from Theorem [7.2] that the association $\mathcal{X} \mapsto MGL^{*,*}(\mathcal{X})$ is a contravariant functor from the category of smooth quotient stacks into the theory of bigraded commutative rings. Furthermore, this functor satisfies homotopy invariance, localization, theory of Chern classes and projective bundle formula.
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