Lifts of Lorentzian r- Paracontact Structure: A Geometrical Dynamics Meaning

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Abstract

The goal of this paper, using lifting theory it is to produce almost paracomplex structures on the tangent bundle of almost Lorentzian r-paracontact manifold endowed with almost Lorentzian r-paracontact structure. Finally, we discuss the effect over dynamics systems of the produced geometrical structures.

Keywords: paracomplex and Lorentzian paracontact structure, paracomplex and almost Lorentzian paracontact manifold, vertical lift, complete lift, horizontal lift, Geometrical dynamics.

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1 Introduction

It is well known that the method of lift has an important role in modern differentiable geometry. Because, differentiable structures and mechanical systems defined on a manifold $M$ or a physical space can be lifted to the same type of structures on its tangent and cotangent bundles which are phase-spaces of velocities and momentums of a given configuration space. It was studied lifting theory of complex and paracomplex structures in [1-4] and the differential geometry of tangent and cotangent bundles in [5-7].

The paper is structured as follows. In section 2, we recall the fundamental structures about paracomplex, paracontact, r-paracontact, Lorentzian paracontact and Lorentzian r-paracontact manifolds given in [8-12]. In section 3, we give vertical, complete and horizontal lifts of paracomplex structures[4]. In section 4, we produce almost paracomplex structures by means of complete and horizontal lifts of almost Lorentzian r-paracontact structure on almost Lorentzian r-paracontact manifold. Also we shall determine the influence on field of dynamics systems of differential geometric elements.

Along this paper, all mappings and manifolds will be understood to be of class differentiable and the sum is taken over repeated indices. In this study we denote by $\mathcal{E}_r^s(M)$ the set of all tensor fields of class $C^\infty$ and of type $(r, s)$ in $M$. We now put $\mathcal{E}(M) = \sum_{r,s = 1}^{\infty} \mathcal{E}_r^s(M)$; which is the set of all tensor fields in $M$. For example $\mathcal{E}^0_0(M), \mathcal{E}^1_0(M), \mathcal{E}^0_1(M)$ and $\mathcal{E}^1_1(M)$ are the set of functions, vector fields, 1-forms and tensor fields of type (1,1) on $M$, respectively. We give by $\mathcal{F}(M)$ to $\mathcal{E}^0_0(M)$, by $\chi(M)$ to $\mathcal{E}^1_0(M)$ and by $\chi^*(M)$ to $\mathcal{E}^0_1(M)$. Similarly, we respectively denote by $\mathcal{E}^*_r(T(M))$ and $\mathcal{E}(T(M))$ the corresponding sets of tensor fields in the tangent bundle $T(M)$. Also $v, c$ and $h$ will denote the vertical, complete and horizontal lifts to $TM$ of geometric structures on $M$, respectively.

2 Basic Structures

2.1 Paracomplex Geometry

A tensor field of type (1,1) $J$ on $M$ such that $J^2 = I$ is an almost product structure $J$ on manifold $M$ of dimension $2m$. Hence, we said to be an almost product manifold the pair $(M, J)$. If the two eigenbundles $T^+M$ and $T^-M$ associated to the eigenvalues $+1$ and $-1$ of $J$, respectively, have the same rank, then an almost product manifold $(M, J)$ is said an almost paracomplex manifold. The dimension of an almost paracomplex manifold is necessarily even. Equivalently, a splitting of the tangent bundle $TM$ of manifold $M$, into the Whitney sum of
two subbundles on $T^\pm M$ of the same fiber dimension is called an *almost paracomplex structure* on $M$. An almost paracomplex structure on manifold $M$ may alternatively be defined as a $G$-structure on $M$ with structural group $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$.

A *paracomplex manifold* is an almost paracomplex manifold $(M, J)$ such that the $G$-structure defined by the tensor field $J$ is integrable. $(x^\alpha, y^\alpha), 1 \leq \alpha \leq m$ be a real coordinate system on a neighborhood $U$ of any point $p$ of $M$, and $\{(\frac{\partial}{\partial x^\alpha})_p, (\frac{\partial}{\partial y^\alpha})_p\}$ and $\{(dx^\alpha)_p, (dy^\alpha)_p\}$ natural bases over $\mathbb{R}$ of the tangent space $T_pM$ and the cotangent space $T^*_pM$ of $M$, respectively. Hence we shall define as:

$$J(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial y^\alpha}, \quad J(\frac{\partial}{\partial y^\alpha}) = \frac{\partial}{\partial x^\alpha}, \quad (dx^\alpha) = -dy^\alpha, \quad (dy^\alpha) = -dx^\alpha.$$  

(1)

(2)

Let $z^\alpha = x^\alpha + jy^\alpha$, $\overline{z}^\alpha = x^\alpha - jy^\alpha$, $1 \leq \alpha \leq m$, $j^2 = 1$, be a paracomplex local coordinate system on a neighborhood $U$ of any point $p$ of $M$. We define the vector fields and the dual covector fields as:

$$(\frac{\partial}{\partial z^\alpha})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^\alpha})_p - j(\frac{\partial}{\partial y^\alpha})_p\}, \quad (\frac{\partial}{\partial \overline{z}^\alpha})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^\alpha})_p + j(\frac{\partial}{\partial y^\alpha})_p\},$$

(3)

$$(dz^\alpha)_p = (dx^\alpha)_p + j(dy^\alpha)_p, \quad (d\overline{z}^\alpha)_p = (dx^\alpha)_p - j(dy^\alpha)_p,$$

(4)

which represent the bases of the tangent space $T_pM$ and cotangent space $T^*_pM$ of $M$, respectively. Then, using $j^2 = 1$ it is obtained

$$J(\frac{\partial}{\partial z^\alpha}) = -j\frac{\partial}{\partial z^\alpha}, \quad J(\frac{\partial}{\partial \overline{z}^\alpha}) = j\frac{\partial}{\partial \overline{z}^\alpha},$$

(5)

$$J^*(dz^\alpha) = -jd\overline{z}^\alpha, \quad J^*(d\overline{z}^\alpha) = jdz^\alpha.$$  

(6)

For each $p \in M$, let $T_pM$ be set of tangent vectors $Z_p = Z^\alpha(\frac{\partial}{\partial z^\alpha})_p + \overline{Z}^\alpha(\frac{\partial}{\partial \overline{z}^\alpha})_p$, then $TM$ is the union of these vector spaces. Thus, tangent bundle of a paracomplex manifold $M$ is $(TM, \tau_M, M)$, where canonical projection $\tau_M$ is $\tau_M : TM \rightarrow M$ ($\tau_M(Z_p) = p$) and, in addition this map are surjective submersion. Then now, coordinates $\{z^\alpha, \overline{z}^\alpha, z^\alpha, \overline{z}^\alpha\}$, $1 \leq \alpha \leq m$, are taken into considering as local coordinates for $TM$.

### 2.2 Almost Paracontact (r-Pracontact) Geometry

Let $\overline{M}$ be an $n$-dimensional differentiable manifold. If there exist on $\overline{M}$ a $(1,1)$-tensor field $\varphi$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = I - \eta \otimes \xi,$$

(7)
where $I$ is the identity, then $\overline{M}$ is said to be an almost paracontact manifold. In the almost paracontact manifold, the following relations hold good:

$$\varphi(\xi) = 0, \eta \circ \varphi = 0, \text{rank}(\varphi) = n - 1.$$  \hfill (8)

Every almost paracontact manifold has a positive definite Riemannian metric $G$ such that

$$\eta(X) = G(\xi, X), \ G(\varphi X, \varphi Y) = G(X, Y) - \eta(X)\eta(Y), \ X, Y \in \chi(\overline{M})$$  \hfill (9)

where $\chi(\overline{M})$ denotes the set of differentiable vector fields on $\overline{M}$. In this case, we say that $\overline{M}$ has an almost paracontact Riemannian structure $(\varphi, \xi, \eta, G)$ and $\overline{M}$ is said to be an almost paracontact Riemannian manifold.

This structure generalizes as follows. One takes care of a tensor field $F$ of type $(1,1)$ on a manifold $\overline{M}$ of dimension $(2n + r)$. If there exists on $\overline{M}$ the vector fields $(\xi\alpha)$ and the 1-forms $(\eta\alpha)$ such that

$$\eta\alpha(\xi\beta) = \delta\alpha\beta, \ F(\xi\alpha) = 0, \ \eta\alpha \circ F = 0, \ F^2 = I + \sum_{\alpha=1}^{r} \xi\alpha \otimes \eta\alpha,$$  \hfill (10)

then the structure $(F, \xi\alpha, \eta\alpha)$ is also called a framed $F(3, -1)$-structure, where $(\alpha, \beta = 1, 2, ..., r)$ and $\delta\alpha\beta$ denotes Kronecker delta. A manifold $\overline{M}$ endowed with a framed $F(3, -1)$-structure is called an almost $r$-paracontact manifold. In addition, we say that the slit tangent bundle of a Finsler space carries a natural framed $F(3; -1)$-structure and then the framed $F(3; -1)$-structure induces an almost paracontact structure.

### 2.3 Lorentzian Almost Paracontact (r-Paracontact) Geometry

Let $\overline{M}$ be an $n$-dimensional differentiable manifold equipped with a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a $(1,1)$-tensor field, $\xi$ a vector field and $\eta$ is a 1-form on $\overline{M}$ satisfying

$$\eta(\xi) = -1, \ \varphi^2 = I + \eta \otimes \xi,$$  \hfill (11)

where $I$ is the identity. In manifold $\overline{M}$, the following relations hold good:

$$\varphi(\xi) = 0, \ \eta \circ \varphi = 0, \ \text{rank}(\varphi) = n - 1.$$  \hfill (12)

If $\overline{M}$ has a Lorentzian metric $G$ such that

$$G(\varphi X, \varphi Y) = G(X, Y) + \eta(X)\eta(Y), \ X, Y \in \chi(\overline{M})$$  \hfill (13)
we say that $\overline{M}$ has a Lorentzian almost paracontact structure $(\varphi, \xi, \eta, G)$ and $\overline{M}$ is said to be a Lorentzian almost paracontact manifold.

This structure generalizes as follows. One takes care of a tensor field $F$ of type $(1,1)$ on a manifold $\overline{M}$ of dimension $(2n + r)$. If there exists on $\overline{M}$ the vector fields $(\xi_\alpha)$ and the 1-forms $(\eta_\alpha)$ such that

$$\eta_\alpha(\xi_\beta) = -\delta_\beta^\alpha, \quad F(\xi_\alpha) = 0, \quad \eta_\alpha \circ F = 0, \quad F^2 = I - \sum_{i=1}^{r} \xi_\alpha \otimes \eta_\alpha, \quad (14)$$

then the structure $(F, \xi_\alpha, \eta_\alpha)$ is also called a Lorentzian almost $r$-paracontact structure, where $(\alpha, \beta = 1, 2, ..., r)$ and $\delta_\beta^\alpha$ denotes Kronecker delta. A manifold $\overline{M}$ endowed with $(F, \xi_\alpha, \eta_\alpha)$-structure is called a Lorentzian almost $r$-paracontact manifold.

### 3 Lifting theory of Paracomplex Structures

#### 3.1 Vertical Lifts

The vertical lift of paracomplex function $f \in \mathcal{F}(M)$ to $TM$ is the function $f^v$ in $\mathcal{F}(TM)$ given by

$$f^v = f \circ \tau_M, \quad (15)$$

where $\tau_M : TM \rightarrow M$ canonical projection. We have $rang(f^v) = rang(f)$, since

$$f^v(Z_p) = f(\tau_M(Z_p)) = f(p), \quad \forall Z_p \in TM. \quad (16)$$

The vertical lift of a vector field $Z \in \chi(M)$ to $TM$ is the vector field $Z^v \in \chi(TM)$ given by

$$Z^v(f^c) = (Zf)^v, \quad \forall f \in \mathcal{F}(M). \quad (17)$$

If $Z = Z^\alpha \frac{\partial}{\partial z^\alpha} + \overline{Z}^\alpha \frac{\partial}{\partial \overline{z}^\alpha}$ we have

$$Z^v = (Z^\alpha)^v \frac{\partial}{\partial z^\alpha} + (\overline{Z}^\alpha)^v \frac{\partial}{\partial \overline{z}^\alpha}, \quad 1 \leq \alpha \leq m. \quad (18)$$

The vertical lift of a 1-form $\omega \in \chi^*(M)$ to $TM$ is the 1-form $\omega^v \in \chi^*(TM)$ given by

$$\omega^v(Z^c) = (\omega Z)^v, \quad \forall Z \in \chi(M). \quad (19)$$

The vertical lift of the paracomplex 1-form $\omega$ given by

$$\omega = \omega_\alpha dz^\alpha + \overline{\omega}_\alpha d\overline{z}^\alpha \quad (20)$$
\[ \omega^v = (\omega_\alpha)^v dz^\alpha + (\overline{\omega}_\alpha)^v d\overline{z}^\alpha, 1 \leq \alpha \leq m. \] (21)

The \textit{vertical lift} of a paracomplex tensor field of type \((1,1)\) \(F \in \mathfrak{S}_1^1(M)\) to tangent bundle \(TM\) of a paracomplex manifold \(M\) is the tensor field \(F^v \in \mathfrak{S}_1^1(TM)\) given by

\[ F^v(Z^c) = (FZ)^v, \quad \forall Z \in \chi(M). \] (22)

Clearly, we have

\[ F^v = (F_\alpha^\beta)^v \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + (F_\alpha^\beta)^v \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha, 1 \leq \alpha, \beta \leq m, \] (23)

where \(F = F_\alpha^\beta \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + F_\alpha^\beta \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha\).

The vertical lifts of paracomplex tensor fields have the general properties

\[ i) \quad (f.g)^v = f^v.g^v, (f+g)^v = f^v + g^v, \]
\[ ii) \quad (X+Y)^v = X^v+Y^v, (fX)^v = f^vX^v, X^v(f^v) = 0, [X^v,Y^v] = 0, \]
\[ iii) \quad \chi(U) = Sp \{ \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^\alpha} \}, \chi(TU) = Sp \{ \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^\alpha}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \overline{x}^\alpha} \}, \]
\[ iv) \quad (\omega + \theta)^v = \omega^v + \theta^v, (f\omega)^v = f^v\omega^v, (f\omega)^v = f^v\omega^v, \omega^v(Z^v) = 0, \]
\[ v) \quad \chi^v(U) = Sp \{ dz^\alpha, d\overline{z}^\alpha \}, \chi^v(TU) = Sp \{ dz^\alpha, d\overline{z}^\alpha, dz^\alpha, d\overline{z}^\alpha \}, \]
\[ vi) \quad F^v(Z^v) = 0, \]

for all \(f \in \mathcal{F}(M), X, Y \in \chi(M), \omega, \theta \in \chi^v(M), F \in \mathfrak{S}_1^1(M)\). Where \(1 \leq \alpha \leq m, [.,.]\) is Lie bracket, \(\overline{d}\) denotes the differential operator on \(TM\).

### 3.2 Complete Lifts

The \textit{complete lift} of paracomplex function \(f \in \mathcal{F}(M)\) to \(TM\) is the function \(f^c \in \mathcal{F}(TM)\) given by

\[ f^c = z^\alpha (\frac{\partial f}{\partial z^\alpha})^v + \overline{z}^\alpha (\frac{\partial f}{\partial \overline{z}^\alpha})^v, \] (24)

where we denote by \((z^\alpha, z^\alpha, \overline{z}^\alpha, \overline{z}^\alpha)\) local coordinates of a chart-domain \(TU \subset TM\). Furthermore, for \(Z_p \in TM\) we have

\[ f^c(Z_p) = z^\alpha(Z_p)(\frac{\partial f}{\partial z^\alpha})^v(p) + \overline{z}^\alpha(Z_p)(\frac{\partial f}{\partial \overline{z}^\alpha})^v(p). \] (25)

The \textit{complete lift} of a vector field \(Z \in \chi(M)\) to \(TM\) is the vector field \(Z^c \in \chi(TM)\) given by

\[ Z^c(f^c) = (Zf)^c, \quad \forall f \in \mathcal{F}(M). \] (26)
Obviously, we obtain
\[ Z^c = (Z^\alpha)^v \frac{\partial}{\partial z^\alpha} + \{Z^\beta\}_c \frac{\partial}{\partial z^\alpha} + \{Z^\alpha\}^c \frac{\partial}{\partial z^\alpha} + \{\overline{Z}^\alpha\}^c \frac{\partial}{\partial z^\alpha}, 1 \leq \alpha \leq m, \]  
(27)

where \( Z = Z^\alpha \frac{\partial}{\partial z^\alpha} + \{\overline{Z}^\alpha\} \frac{\partial}{\partial \overline{z^\alpha}}. \)

The complete lift of a 1-form \( \omega \in \chi^*(M) \) to \( TM \) is the 1-form \( \omega^c \in \chi^*(TM) \) given by
\[ \omega^c(Z^c) = (\omega Z)^c, \quad \forall Z \in \chi(M). \]  
(28)

If \( \omega = \omega_\alpha dz^\alpha + \overline{\omega}_\alpha d\overline{z}^\alpha \) we calculate
\[ \omega^c = (\omega_\alpha)^c dz^\alpha + \overline{(\omega_\alpha)^c} d\overline{z}^\alpha + (\omega_\alpha)^v dz^\alpha + \overline{(\omega_\alpha)^v} d\overline{z}^\alpha, 1 \leq \alpha \leq m. \]  
(29)

The complete lift of a paracomplex tensor field of type \((1,1)\) \( F \in \mathbb{Z}^1(M) \) to \( TM \) is the tensor field \( F^c \in \mathbb{Z}^1(TM) \) given by
\[ F^c(Z^c) = (FZ)^c, \quad \forall Z \in \chi(M). \]  
(30)

The complete lift of the paracomplex tensor field of type \((1,1)\) \( F \) is
\[ F^c = (F^\alpha_\beta)^v \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + (F^\alpha_\beta)^v \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha + (F^\alpha_\beta)^v \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + (F^\alpha_\beta)^\gamma \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha, \]  
(31)

where \( F = F^\alpha_\beta \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + \overline{F}^\alpha_\beta \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha. \)

The complete lift of \( J \) given by \( J = j \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha - j \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha \) and being a paracomplex tensor field of type \((1,1)\) is
\[ J^c = j \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + j \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha - j \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha - j \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha. \]  
(32)

Because of \((J^c)^2 = I, J^c \) is an almost paracomplex structure for tangent bundle \( TM. \)

The complete lifts of paracomplex tensor fields obey the general properties

i) \((f + g)^c = f^c + g^c, (f \cdot g)^c = f^c \cdot g^c, (X + Y)^c = X^c + Y^c, (f \cdot X)^c = f^c \cdot X^c, (X \cdot f)^c = f^c \cdot X^c,\)

ii) \(X^c(f^v) = X^v(f^c) = (X^v)^c, X^c(f^c) = (X^c(f^c))^c,\)

iii) \(\chi(U) = Sp \{ \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^\alpha} \}, \chi(TU) = Sp \{ \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^\alpha}, \frac{\partial}{\partial \overline{z}^\alpha} \},\)

iv) \((\omega + \theta)^c = \omega^c + \theta^c, \omega^c(Z^c) = (\omega Z)^c,\)

v) \(\omega^c(Z^v) = \omega^v(Z^c) = (\omega Z)^v, \omega^c(Z^c) = (\omega Z)^c,\)

vi) \(F^v(Z^c) = F^c(Z^v) = (F(Z))^v, F^c(Z^c) = (F(Z))^c,\)
for all \( f, g \in \mathcal{F}(M), X, Y, Z \in \chi(M), \omega, \theta \in \chi^*(M), F \in \mathfrak{S}_1^1(M) \). Where \( 1 \leq \alpha \leq m \), \( \partial \) denotes the differential operator on \( TM \) and \([,] \) is Lie bracket.

### 3.3 Horizontal Lifts

The *horizontal lift* of \( f \in \mathcal{F}(M) \) to \( TM \) is the function \( f^h \in \mathcal{F}(TM) \) given by

\[
f^h = f^c - \gamma(\nabla f), \quad (\gamma(\nabla f) = \nabla \gamma),
\]

where \( \nabla \) is an affine linear connection on \( M \) with local components \( \Gamma^\beta_\alpha \), \( \nabla f \) is gradient of \( f \) and \( \gamma \) is an operator given by

\[
\gamma : \mathfrak{S}_s^r(M) \to \mathfrak{S}_{s-1}^r(TM).
\]

Thus, it is \( f^h = 0 \) since

\[
\nabla \gamma f = z^\alpha \left( \partial \frac{\partial f}{\partial z^\alpha} \right)^v + z^\alpha \left( \partial \frac{\partial f}{\partial z^\alpha} \right)^v.
\]

The *horizontal lift* of a vector field \( Z \in \chi(M) \) to \( TM \) is the vector field \( Z^h \in \chi(TM) \) given by

\[
Z^h f^v = (Zf)^v.
\]

Obviously, we have

\[
Z^h = Z^\alpha D_\alpha + \overline{Z}^\beta \overline{D}_\beta, \quad 1 \leq \alpha, \beta \leq m
\]

such that \( D_\alpha = \frac{\partial}{\partial z^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial z^\beta} \) and \( \overline{D}_\alpha = \frac{\partial}{\partial \overline{z}^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial \overline{z}^\beta} \), \( 1 \leq \alpha, \beta \leq m \), where \( Z = Z^\alpha \frac{\partial}{\partial z^\alpha} + \overline{Z}^\beta \frac{\partial}{\partial \overline{z}^\beta} \).

If \( \omega = \omega_\alpha dz^\alpha + \overline{\omega}_\alpha d\overline{z}^\alpha \) we obtain

\[
\omega^h = \omega_\alpha \eta^\alpha + \overline{\omega}_\alpha \overline{\eta}^\alpha, \quad 1 \leq \alpha, \beta \leq m
\]

such that \( \eta^\alpha = \overline{d} \overline{z}^\alpha + \Gamma^\alpha_\beta \overline{d}z^\alpha \), \( \overline{\eta}^\alpha = d\overline{z}^\alpha + \overline{\Gamma}^\alpha_\beta dz^\alpha \).

The *horizontal lift* of a paracomplex tensor field of type \((1,1)\) \( F \in \mathfrak{S}_1^1(M) \) to \( TM \) is the tensor field \( F^h \in \mathfrak{S}_1^1(TM) \) given by

\[
F^h(Z^h) = (FZ)^h, \quad F^h(Z^v) = (F(Z))^v.
\]

The horizontal lift of the paracomplex tensor field of type \((1,1)\) \( F \) is

\[
F^h = F^\beta_\alpha \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + (\Gamma^\alpha_\beta F^\beta_\alpha - \Gamma^\beta_\alpha F^\beta_\alpha) \frac{\partial}{\partial z^\beta} \otimes dz^\alpha + F^\beta_\alpha \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha + F^\beta_\alpha \frac{\partial}{\partial \overline{z}^\alpha} \otimes d\overline{z}^\alpha,
\]
where \(1 \leq \alpha, \beta \leq m\), \(F = \sum_{\alpha} F^{\beta}_{\alpha} \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + \sum_{\alpha} \bar{F}^{\beta}_{\alpha} \frac{\partial}{\partial \bar{z}^\alpha} \otimes d\bar{z}^\alpha\)

With respect to the adapted frame, we have

\[
F^h = F^{\beta}_{\alpha} D_{\beta} \otimes \theta^\alpha + F^{\beta}_{\alpha} V_{\beta} \otimes \eta^\alpha + \sum_{\alpha} \bar{F}^{\beta}_{\alpha} \bar{D}_{\beta} \otimes \bar{\theta}^\alpha + \sum_{\alpha} \bar{F}^{\beta}_{\alpha} \bar{V}_{\beta} \otimes \bar{\eta}^\alpha.
\]  

(41)

The properties of horizontal lifts of paracomplex tensor fields are

i) \((f + g)^h = 0, (fg)^h = 0\),

ii) \((Z + W)^h = Z^h + W^h, Z^h(f^v) = (Zf)^v\),

\((\frac{\partial}{\partial z^\alpha})^h = D_{\alpha}, (\frac{\partial}{\partial \bar{z}^\alpha})^h = \bar{D}_{\alpha}\),

iii) \((\omega + \theta)^h = \omega^h + \theta^h, \omega^h(Z^h) = 0, \omega^h(Z^v) = (\omega Z)^v\),

\((dz^\alpha)^h = \eta^\alpha, (d\bar{z}^\alpha)^h = \bar{\eta}^\alpha\),

iv) \(F^h(Z^h) = (FZ)^h, F^h(Z^v) = (F(Z))^v\),

for all \(f, g \in \mathcal{F}(M), Z, W \in \chi(M), \omega, \theta \in \chi^*(M), F \in \mathfrak{z}\mathfrak{z}(M)\).

Where \(1 \leq \alpha \leq m\), \(\partial\) denotes the differential operator on \(TM\) and \([,]\) is Lie bracket. The set of local vector fields \(\{D_{\alpha}, \bar{D}_{\alpha}, V_{\alpha} = \frac{\partial}{\partial z^\alpha}, \bar{V}_{\alpha} = \frac{\partial}{\partial \bar{z}^\alpha}\}\) is called adapted frame to \(\nabla\). The dual coframe \(\{\theta^\alpha = dz^\alpha, \bar{\theta}^\alpha = d\bar{z}^\alpha, \eta^\alpha, \bar{\eta}^\alpha\}\) is called adapted coframe to \(\nabla\).

# 4 Lifts of Almost Lorentzian r-Paracontact Structures

In this section, we produce almost paracomplex structures on tangent bundle \(T(\bar{M})\) of almost Lorentzian \(r\)-paracontact manifold \(\bar{M}\) having the structure \((F, \xi^\alpha, \eta^\alpha)\).

## 4.1 Complete Lifts

**Theorem 4.1.** Let \(\bar{M}\) be a differentiable manifold endowed with almost \(r\)-paracontact structure \((F, \xi^\alpha, \eta^\alpha)\), then

\[
\bar{J} = F^c + \sum_{\alpha=1}^r \xi^\alpha \otimes \eta^{\alpha v} - \xi^c \otimes \eta^{\alpha c}
\]

is almost paracomplex structure on \(T(\bar{M})\).

**Proof:** From (10) and the vertical and complete lifts of paracomplex tensor fields we have

\[
(F^2)^c = (I + \sum_{\alpha=1}^r \xi^\alpha \otimes \eta^\alpha)^c,
\]

(42)

\[
(F^c)^2 = I + \sum_{\alpha=1}^r \xi^\alpha \otimes \eta^{\alpha c} + \xi^c \otimes \eta^{\alpha c},
\]

(43)
and

\[ \mathbf{F}^c(\xi^c_\alpha) = 0, \mathbf{F}^c(\xi^c_\alpha) = 0, \]
\[ \eta^{av} \circ \mathbf{F}^c = 0, \eta^{ac} \circ \mathbf{F}^v = 0, \eta^{ac} \circ \mathbf{F}^c = 0, \]
\[ \eta^{av}(\xi^c_\beta) = 0, \eta^{av}(\xi^c_\beta) = \delta^a_\beta, \eta^{ac}(\xi^c_\beta) = \delta^a_\beta, \eta^{ac}(\xi^c_\beta) = 0. \] (44) (45) (46)

Consider an element \( \tilde{J} \) of \( \mathbb{J}_1(T \overline{M}) \) given by

\[ \tilde{J} = \mathbf{F}^c + \sum_{\alpha=1}^r (\xi^v_\alpha \otimes \eta^{av} - \xi^c_\alpha \otimes \eta^{ac}). \] (47)

Using (44), (45) and (46) we find equation

\[ (\tilde{J})^2 = (\mathbf{F}^c + \sum_{\alpha=1}^r (\xi^v_\alpha \otimes \eta^{av} - \xi^c_\alpha \otimes \eta^{ac}))^2 \] (48)

By means of (44), (45) and (46), we obtain

\[ (\tilde{J})^2 = I. \] (49)

So, \( \tilde{J} \) is an almost paracomplex structure in \( T(\overline{M}) \). Hence the proof is completed.

**Theorem 4.2.** Let \( \overline{M} \) be a differentiable manifold endowed with almost Lorentzian \( r \)-paracontact structure \((\mathbf{F}, \xi_\alpha, \eta^\alpha)\), then an almost paracomplex structure on \( T(\overline{M}) \) is calculated by

\[ \hat{J} = \mathbf{F}^c - \sum_{\alpha=1}^r \xi^v_\alpha \otimes \eta^{av} - \xi^c_\alpha \otimes \eta^{ac}. \]

**Proof:** By means of the equation (44) and the vertical and complete lifts of paracomplex tensor fields we have

\[ (\mathbf{F}^2)^c = (I - \sum_{\alpha=1}^r \xi_\alpha \otimes \eta_\alpha)^c. \] (50)
\[(F^c)^2 = I - \sum_{\alpha=1}^{r} \xi^v_\alpha \otimes \eta^{ac}_\alpha + \xi^c_\alpha \otimes \eta^{\alpha v}, \quad (51)\]

and

\[F^c(\xi^v_\alpha) = 0, \quad F^c(\xi^c_\alpha) = 0, \quad (52)\]

\[\eta^{\alpha v} \circ F^c = 0, \quad \eta^{ac} \circ F^v = 0, \quad \eta^{ac} \circ F^c = 0, \quad (53)\]

\[\eta^{\alpha v}(\xi^v_\beta) = 0, \quad \eta^{\alpha v}(\xi^c_\beta) = -\delta^\alpha_\beta, \quad \eta^{ac}(\xi^v_\beta) = -\delta^\alpha_\beta, \quad \eta^{ac}(\xi^c_\beta) = 0. \quad (54)\]

Take an element \(\hat{J}\) of \(\mathfrak{S}_1(T\overline{M})\) defined by

\[\hat{J} = F^c - \sum_{\alpha=1}^{r} (\xi^v_\alpha \otimes \eta^{\alpha v} - \xi^c_\alpha \otimes \eta^{ac}) \quad (55)\]

Similarly proof of Theorem 4.1, using (43), (44), (45) and (46)and (47) we have the equation

\[(\hat{J})^2 = I. \quad (56)\]

Thus, for \(\hat{J}\) is an almost paracomplex structure in \(T(\overline{M})\), the proof is completed.

### 4.2 Horizontal Lifts

**Theorem 4.3.** Let \((F, \xi_\alpha, \eta^\alpha)\) be an almost r-paracontact structure in \(\overline{M}\) with an affine connection \(\nabla\). Then an almost paracomplex in \(T(\overline{M})\) is given by

\[\tilde{J}^* = F^h + \sum_{\alpha=1}^{r} (\xi^v_\alpha \otimes \eta^{\alpha v} - \xi^c_\alpha \otimes \eta^{ac}). \]

**Proof:** Taking into consideration the equation given by (10) and the horizontal lifts of para-complex tensor fields, we have

\[(F^2)^h = (I + \sum_{\alpha=1}^{r} \xi_\alpha \otimes \eta^\alpha)^h, \quad (57)\]

\[(F^h)^2 = I + \sum_{\alpha=1}^{r} \xi^h_\alpha \otimes \eta^{\alpha v} + \xi^v_\alpha \otimes \eta^{\alpha h}, \quad (58)\]

and

\[F^h(\xi^h_\alpha) = 0, \quad F^c(\xi^c_\alpha) = 0, \quad (59)\]

\[\eta^{\alpha h} \circ F^h = 0, \quad \eta^{\alpha v} \circ F^h = 0, \quad (60)\]

\[\eta^{\alpha h}(\xi^v_\beta) = 0, \quad \eta^{\alpha h}(\xi^c_\beta) = \delta^\alpha_\beta, \quad \eta^{\alpha v}(\xi^h_\beta) = \delta^\alpha_\beta. \quad (61)\]
Given an element $\tilde{J}$ of $\Im^1_{\mathcal{M}}(T\mathcal{M})$ defined by

$$\tilde{J}^* = \mathbf{F}^h + \sum_{\alpha=1}^{r} (\xi^v_{\alpha} \otimes \eta^{\alpha v} - \xi^h_{\alpha} \otimes \eta^{\alpha h}).$$

Taking care of the above equations, it is clear that

$$(\tilde{J}^*)^2 = I.$$  \hspace{1cm} (63)

Consequently $\tilde{J}^*$ is an almost paracomplex structure in $T(\mathcal{M})$. Thus the proof is finished.

**Theorem 4.4.** Let $(\mathbf{F}, \xi_{\alpha}, \eta^\alpha)$ be an almost Lorentzian $r$-paracontact structure in $\mathcal{M}$ with an affine connection $\nabla$. Then structure

$$\hat{J}^* = \mathbf{F}^h - \sum_{\alpha=1}^{r} (\xi^v_{\alpha} \otimes \eta^{\alpha v} - \xi^h_{\alpha} \otimes \eta^{\alpha h}).$$

is an almost paracomplex in $T(\mathcal{M})$.

**Proof:** It is similar to the proofs of the above theorems.

\section{5 Corollary}

Taking into consideration the above theorems, we conclude that when we consider an almost Lorentzian $r$-paracontact structure on the base manifold, the structure defined on the tangent bundle $T(\mathcal{M})$ is an almost paracomplex. Therefore, by means of [13], it is possible to obtain paracomplex Hamiltonian formalisms in classical mechanics and field theory on $T(\mathcal{M})$, is the tangent manifold of a differentiable manifold $\mathcal{M}$ endowed with almost Lorentzian $r$-paracontact structure $(\mathbf{F}, \xi_{\alpha}, \eta^\alpha)$.

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