Constant Term of Coleman Power Series and Euler Systems in Function Fields

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Abstract

We calculate the constant term of Coleman power series and use it to prove an analogue of Iwasawa Main Conjecture in function fields of characteristic $p > 0$ using Euler systems. This result is proved by a similar method of classical proof of Iwasawa Main Conjecture, go back to Kolyvagin and Rubin. We construct Euler systems from theta function of Drinfeld modules and establish “finite-singular comparison equation” even when $p$ is not invertible the coefficients. This paper is a revised version of the author’s master’s thesis.

1 Introduction

In 1986 Francisco Thaine discovered the way how to bound the ideal class groups of real abelian extentions of $\mathbb{Q}$. This result was already known as a corollary of the Iwasawa Main Conjecture which was proved by Mazur and Wiles, but Thaine’s proof is easier than theirs. After this, Karl Rubin discovered a new method to prove the Gras conjecture, which was also known as the corollary of the Iwasawa Main Conjecture, by using Euler systems. The way how to use Euler systems to prove the Gras conjecture in classical case $K = \mathbb{Q}(\mu_n)$ in number fields, see [La, appendix].

In this paper we describe a structure of ideal class groups of cyclotomic extensions over a global function fields using an analogue of Rubin’s method over a cyclotomic number field. Let $v$ be a prime of a global function field
$F, \mathcal{O}(M_{A/\nu^n I,F}), n \geq 0$ a cyclotomic extension over $F$ (see section 3 for the definition of the notation) and $M$ a power of prime $p$. We consider Euler systems constructed by Siegel units (Definition 2.3) in $\mathcal{O}(M_{A/\nu^n I,F})$ and their Kolyvagin derivative classes $\{\kappa(\nu_r, \psi \nu_r)\}_r$ (see section 4.1). Then the classes $\{\kappa(\nu_r, \psi \nu_r)\}_r$ satisfy the following equality named “the finite singular comparison equality”.

**Theorem 1.1** (finite-singular comparison equalities)

$$\left[\kappa(\nu_r, \psi \nu_r)\right]_v = \sum_{\lambda|v} \psi_v(\kappa(\nu_r, \psi \nu_r)) \lambda$$

where $[\cdot]_v$ is a projection of a $v$-part of a principal ideal onto $\mathcal{I}_v/M_\mathcal{I}_v$.

This is the key point of proof of Gras conjecture over a global function field.

**Theorem 1.2** (Gras conjecture)

For any character $\chi$ in $\Delta := \text{Gal}(\mathcal{O}(M_{A/\nu^n I,F})/F)$,

$$\text{char} \left(C_\infty(\chi)\right) = \text{char} \left(\mathcal{E}_\infty(\chi)/\mathcal{E}_\infty(\chi)\right)$$

We will see all about it in section 6 and Theorem 6.6.

We remark that if $M$ is prime to $p$ then this result was proved by Hassan Oukhaba and Stéphane Vigué by a similar method. However their method cannot be applied to the case $M$ is a power of $p$. In this paper we develop a method with which we can apply the theory of Euler systems to such a case by using the theory of the Coleman power series. We prove that the constant term of Coleman power series associated with Siegel units is equal to another Siegel unit (see Proposition 5.10) and calculate the constant term of Coleman power series to prove Theorem 1.1.

**Theorem 1.3**

Let $K$ be a local field, $K'$ a finite unramified abelian extension over $K$, $\pi$ an uniformizer, $\mathcal{F}$ the Lubin-Tate module corresponding to $\pi$, $\xi = (\xi_n)_n \in \limleftlim_n \mathcal{F}(n)$ a generator of the Tate module. Put $K'_n := K'[\xi_n]$, $K'_\infty := \bigcup K'_n$, $G := \text{Gal}(K'_\infty/K)$. Let $u = (u_n)_n$ be an element in $\limleftlim_n \mathcal{O}K'_n$, $\text{Col}_u(T)$ the Coleman power series associated to $u$. Then, for any character $\chi : G \to \mathbb{Q}/\mathbb{Z}$
whose order is finite and any finite cyclic extension $K'_\chi = (K'_\infty)^{Ker\chi}$ over $K'$, the following equality holds.

$$(u_{K'_\chi}, \chi)_{K'_\chi}/K = \chi \circ \text{rec}_{K'}(\text{Col}_u(0))$$

where $(\cdot, \cdot)_{L/K}$ is a symbol which is introduced in section 5.2.

In section 2 we will recollect some basic facts on Drinfeld module and Siegel units. The Siegel units were originally introduced by Masahiro Asahi in his master’s thesis. Proposition 2.8 and Proposition 2.10 were proved by him for the first time. In section 3 we will construct our Euler systems using Siegel units and Kolyvagin’s derivative class and give the accurate statements of Theorem 1.1. In section 5 we will calculate the constant term of Coleman power series and give a proof of Theorem 1.1. In the final section we will prove the Gras conjecture in global function fields.

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2 Drinfeld module and Siegel units

2.1 Drinfeld module

Let $p$ be an odd prime number and $q$ a power of $p$. Let $\mathbb{F}_q$ be the finite field of $q$ elements. Let $C$ be a nonsingular projective geometrically irreducible curve over $\mathbb{F}_q$ and $F$ the function field of $C$. We regard the closed points of $C$, as the primes of $F$ and let $F_v$, where $v$ is closed point of $C$ be the completion of $F$ at $v$, $\mathcal{O}_v$ the valuation ring of $F_v$, $\pi_v \in \mathcal{O}_v$ a prime element, $k(v)$ the residue field of $\mathcal{O}_v$. Fix a closed point $\infty$ of $C$. Let $p_\infty$ be the order of $k(\infty)$. Since $C \setminus \{\infty\}$ is affine, we can identify it with Spec $A$ where $A = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$. From now on, unless otherwise stated, all primes of $F$ are assumed to be different from $\infty$. Next we define Drinfeld module and Drinfeld modular variety [Dr].

**Definition 2.1 (Drinfeld module)**

Let $S$ be an $A$-scheme. A Drinfeld module of rank 1 over $S$ is an scheme $E$ in $A$-modules over $S$ which satisfies the following conditions:

1. Zariski locally on $S$, the scheme $E$ is isomorphic to the additive group scheme $\mathbb{G}_a_S$ as a commutative group scheme.
(2) We denote the $A$-action on $E$ by $act : A \to \text{End}_{S_{-gp}}(E)$. For every $a \in A \setminus \{0\}$, the morphism $act(a) : E \to E$ is finite, locally free of constant degree $|a|_{\infty}$ where $|a|_{\infty}$ is absolute value at $\infty$.

(3) The $A$-action on Lie $E$ induced by $act$ coincides with the $A$-action on Lie $E$ which comes from the structure homomorphism $E \to S$.

When we say a Drinfeld module, we assume that its rank is equal to 1.

**Definition 2.2 (Drinfeld Modular Variety)**

Let $N$ be a nonzero torsion $A$-module. Put $U_N := \text{Spec } A \setminus \text{Supp } N$. Let $S$ be a $U_N$-scheme and $E = (E, act)$ a Drinfeld module over $S$. A **level $N$ structure** on $E$ is a monomorphism

$$\text{lev} : N_S \hookrightarrow E$$

of schemes in $A$-modules over $S$, where $N_S$ is a constant group scheme. Let us consider the following contravariant functor from the category of $U_N$-schemes to the category of sets:

$$\begin{align*}
(U_N\text{-schemes}) & \to \quad (\text{Sets}) \\
S & \mapsto (\text{the set of isomorphism classes of } (E, act, \text{lev}))
\end{align*}$$

If $N \neq 0$, there exists a $U_N$-scheme $M_N$ such that the functor is isomorphic to the functor which makes $S$ correspond with $\text{Hom}_{U_N}(S, M_N)$. The $U_N$-scheme isomorphic is unique up to canonical isomorphisms due to Yoneda’s lemma. We call $M_N$ the **Drinfeld Modular Variety** of rank 1 with level $N$-structures.

For two torsion $A$-modules $N, N'$ and an embedding $N \hookrightarrow N'$, let $r_{N,N'}$ denote the following morphism from $M_{N'}$ to $M_N \times_{U_N} U_{N'}$:

$$r_{N,N'} : M_{N'} \to M_N \times_{U_N} U_{N'}$$

$$\begin{align*}
(E, act, \text{lev}) & \mapsto (E, act, \text{lev}|_N)
\end{align*}$$

where $\text{lev}|_N$ is restriction of $\text{lev}$ on $N$. Similarly, for two torsion $A$-modules $N, N''$ and a surjection $N \twoheadrightarrow N''$ we call it $\pi$ for this section only whose kernel
is of finite length, let \( m_{N,N''} \) denote the following morphism from \( M_N \) to \( \mathcal{M}_{N''} \times U_N \):

\[
m_{N,N''} : \quad \mathcal{M}_N \rightarrow \mathcal{M}_{N''} \times U_N
\]

where \( E'' = E/\text{act}(\text{Ker} \, \pi) \) and \( \text{act}'', \text{lev}'' \) are the maps induced by \( \text{act} \) and \( \text{lev} \), respectively.

### 2.2 Theta function

We define the theta function in this section. Let \( S \) be a reduced scheme over \( \text{Spec} \, A \) and \( (E, \text{act}) \) a Drinfeld module on \( S \). We regard \( S \) as a closed subscheme in \( E \) via the zero section \( S \hookrightarrow E \).

**Proposition 2.3**

For any nonzero element \( a \in A \), we denote by \( N_a \) the norm map

\[
\Gamma(E \setminus \text{Ker} \, \text{act}(a), \mathcal{O}_E^\times) \rightarrow \Gamma(E \setminus S, \mathcal{O}_E^\times)
\]

with respect to the finite flat morphism

\[
\text{act}(a) : E \setminus \text{Ker} \, \text{act}(a) \rightarrow E \setminus S.
\]

Then there exist a function \( f \in \Gamma(E \setminus S, \mathcal{O}_E^\times) \), unique up to \( \mu_{p^{\infty}-1}(S) \) which satisfies the following:

1. \( N_a(f) = f \) for any nonzero element \( a \in A \)
2. The order of \( f \) in \( S \), \( \text{ord}_S(f) \) is equal to \( p^{\infty} - 1 \)

**Proof**

See [KY, Lemma 2.1].

From the above, the function \( f^{p^{\infty}-1} \in \Gamma(E \setminus S, \mathcal{O}_E^\times) \) is uniquely defined.

**Definition 2.4 (Theta function)**

For a Drinfeld module \( (E, \text{act}) \), let \( f \) be an element satisfying the condition in Proposition \ref{prop:theta_function}. Let

\[
\vartheta_{E/S} := f^{p^{\infty}-1}.
\]

We call \( \vartheta_{E/S} \) the **theta function** of \( (E, \text{act}) \).
Fact 2.5
Let $S, S'$ be schemes over $\text{Spec } A$, $\text{mor} : S' \to S$ a morphism from $S'$ to $S$, $\text{mor}_E : E \times_{S} S' \to E$ base change with respect to $\text{mor}$. Then we have

$$\text{mor}_E^* \vartheta_{E/S} = \vartheta_{E \times_{S} S'/S'}.$$ 

Fact 2.6
Let $E, E'$ be Drinfeld modules on $S$ and $\text{iso} : E \to E'$ be an isogeny from $E$ to $E'$. Let $N_{\text{iso}}$ denote the norm map corresponding to the isogeny. Then

$$N_{\text{iso}} \vartheta_{E/S} = \vartheta_{E'/S}.$$ 

2.3 Siegel unit

Let $N$ be a nonzero torsion $A$-module as above, $E_N \to M_N$ the universal Drinfeld module, and $\text{lev} : N_{M_N} \hookrightarrow E_N$ the universal level structure. Let us apply the argument in the last paragraph to the case when $S = M_N$. Via the zero section we regard $M_N$ as a closed subscheme of $E_N$. Let us remark that $M_N$ is smooth over $U_N$, in particular $M_N$ is reduced. Therefore Proposition 2.3 assures the existence of a theta function $\vartheta_{E_N/M_N} \in \Gamma(E_N \setminus M_N, \mathcal{O}_{E_N \setminus M_N}^\times)$.

Definition 2.7 (Siegel unit)
For an element $b$ in $N \setminus \{0\}$, let $\text{lev}_b : M_N \to E_N$ denote the restriction of $\text{lev}$ to $M_N$, where we regard $M_N = \{b\} \times M_N$ as a subscheme of $N_{M_N} = \bigcup_{b \in N} M_N$. Let us denote

$$g_{N,b} := \text{lev}_b^* \vartheta_{E_N/M_N} \in \Gamma(M_N, \mathcal{O}_{M_N}^\times)$$

and call it Siegel unit.

Let $N'$ be an $A$-module of finite length, $N$ an $A$-submodule of $N'$. Then both $M_N, M_{N'}$ are not empty. Let $N''$ be an $A$-module of finite length and $\alpha : N \to N''$ a surjection. Then we can prove the follow proposition using Fact 2.5, 2.6.
Proposition 2.8 (distribution equation)
For any \( b \in \mathbb{N}\setminus\{0\}, b'' \in \mathbb{N}''\setminus\{0\} \), the following equalities hold.
\[
\begin{align*}
    r_{N',N}^* g_{N,b} &= g_{N',b} \\
    m_{N,N''}^* g_{N'',b''} &= \prod_{b \in \mathbb{N}, \alpha(b) = b''} g_{N,b}
\end{align*}
\]

Proposition 2.9
If there exist two different maximal ideals of \( A \) which divide \( \text{Ann}_A(b) := \{a \in A | ab = 0\} \), then \( g_{N,b} \) is an unit of the integral closure of \( A \) in \( \Gamma(M_N, \mathcal{O}_{M_N}) \).

Let \( I \neq A \) be an ideal of \( A \). Let \( v \) be a prime of \( F \) not dividing \( I \). Take \( N = A/v^nI \). For any integers \( n \geq 0 \), we can construct the Siegel unit \( g_{A/v^nI,\{1\}} \) as above. For simplicity, we write \( m_n \) for \( m_{A/v^nI,A/v^{n-1}I} \).

Proposition 2.10
The following equality between Siegel units holds
\[
m_{n,*} g_{A/v^nI,\{1\}} = \begin{cases} 
    g_{A/v^{n-1}I,\{1\}} & (\text{if } n \geq 2), \\
    (1 - T_v) g_{A/I,\{1\}} & (\text{if } n = 1).
\end{cases}
\]
where \( T_v \) is a Hecke operator determined by \( v \) and satisfies \( r_{A/I,A/vI}^* = T_v^* m_1^* \).

Proof
Using the fact that \( m_n^* \) is injective, it is enough to prove that
\[
m_n^* m_{n,*} g_{A/v^nI,\{1\}} = \begin{cases} 
    m_n^* g_{A/v^{n-1}I,\{1\}} & (\text{if } n \geq 2), \\
    m_n^* (1 - T_v) g_{A/I,\{1\}} & (\text{if } n = 1).
\end{cases}
\]
For \( n \geq 1 \), let
\[
\begin{align*}
    \pi_n : A/v^nI &\to A/v^{n-1}I, \\
    \pi_n^\times : (A/v^nI)^\times &\to (A/v^{n-1}I)^\times
\end{align*}
\]
be a natural surjection. There exists a natural isogeny
\[
is \, E_{A/v^nI} \to m_n^* E_{A/v^{n-1}I}.
\]
Recall that $N_{\text{iso}}$ denotes the norm map with respect to iso. By the definition of $\vartheta$ and Proposition 2.8, we obtain

$$N_{\text{iso}} \vartheta E_{A/v^n I}/M_{A/v^n I} = \vartheta m_n^* E_{A/v^n I}/M_{A/v^n I}.$$

(1)

The following diagram is commutative.

![Diagram](image)

Apply $\text{lev}_1^* \text{iso}^*$ to both sides of formula of (1) and calculate, we have

right hand side $= (m_n \text{lev}_1)^* \vartheta m_n^* E_{A/v^n I}/M_{A/v^n I}$

$= m_n^* \text{lev}_1^* \vartheta E_{A/v^n-1 I}/M_{A/v^n-1 I}$

$= m_n^* g_{A/v^n-1 I, \{1\}}$

On the other hand,

left hand side $= \text{lev}_1^* \text{iso}^* N_{\text{iso}} \vartheta E_{A/v^n I}/M_{A/v^n I}$

$= \text{lev}_1^* \text{iso}^* \vartheta_{\text{iso}} \vartheta E_{A/v^n I}/M_{A/v^n I}$

$= \text{lev}_1^* ( \prod_{a \in \text{Ker} \pi_n} t_a^* \vartheta E_{A/v^n I}/M_{A/v^n I} )$

$= \prod_{a \in \text{Ker} \pi_n} ( \text{lev}_1^* t_a^* \vartheta E_{A/v^n I}/M_{A/v^n I} )$

$= \begin{cases} 
\prod_{a \in \text{Ker} \pi_n} ( \text{lev}_a^* \vartheta E_{A/v^n I}/M_{A/v^n I} ) & \text{(if } n \geq 2) \\
\prod_{a \in \text{Ker} \pi_n} ( \text{lev}_a^* \vartheta E_{A/v^n I}/M_{A/v^n I} ) \times ( \text{lev}_b^* \vartheta E_{A/v^n I}/M_{A/v^n I} ) & \text{(if } n = 1) 
\end{cases}$

$= \begin{cases} 
m_n^* m_n^* g_{A/v^n I, \{1\}} & \text{(if } n \geq 2) \\
( \text{lev}_b^* \vartheta E_{A/v^n I}/M_{A/v^n I} ) \times m_n^* m_n^* g_{A/v^n I, \{1\}} & \text{(if } n = 1) 
\end{cases}$
where $t_a$ denotes the action of the element $a$ in $\text{Ker} \pi$, i.e.

$$t_a : E_{A/v^n I} = E_{A/v^n I} \times M_{A/v^n I} \xrightarrow{id \times \text{lev}_a} E_{A/v^n I} \times M_{A/v^n I} \xrightarrow{\text{group law}} E_{A/v^n I}$$

and $b$ is the element of $A/vI$ which is mapped to $(1, 0)$ under the natural isomorphism $A/vI \cong A/I \times A/v$. Remark that we have used the fact that

$$\text{Ker} \pi = \begin{cases} (v^{n-1}A/v^nA) \cong (1 + v^{n-1}A/v^nA) = \text{Ker} \pi^x \quad (\text{if } n \geq 2) \\ A/v \cong (A/v)^x \cup \{b\} = \text{Ker} \pi^x \cup \{b\} \quad (\text{if } n = 1). \end{cases}$$

Then if $n \geq 2$, we have

$$m_n^* g_{A/v^{n-1}I, \{1\}} = m_n^* m_{n,*} g_{A/v^n I, \{1\}}$$

as desired. If $n = 1$, there is an extra term $\text{lev}_b^* \theta_{E_{A/v^n I}/M_{A/v^n I}}$, we will see that it can be written in terms of the Hecke operator $T_v$. We assume that $I$ and $v$ are relatively prime, so $A/vI \cong A/I \oplus A/v$. Via the natural embedding $A/I \hookrightarrow A/vI$ we regard $A/I$ as a direct summand of $A/vI$. Let us consider the morphism

$$r_{A/I, A/vI} : M_{A/vI} \to M_{A/I} \times U_{A/I} \xrightarrow{\text{id} \cup \text{lev}_1} E_{A/vI} \times M_{A/vI}.$$ 

From its definition, we obtain the isomorphism

$$E_{A/vI} \cong E_{A/vI} \times M_{A/vI}.$$ 

Moreover, by using that $b \in A/vI$ is equal to the image of $1 \in A/I$ under the embedding $A/I \hookrightarrow A/vI$, we can prove that the composite of $\text{lev}_b$ and the above isomorphism is equal to,

$$\text{lev}_1 \times \text{id} : M_{A/vI} = M_{A/I} \times M_{A/vI} \hookrightarrow E_{A/I} \times M_{A/vI}.$$ 

Since the Hecke operator satisfies the equality $r_{A/I, A/vI}^* = m_1^* T_v^*$, we have

$$\text{lev}_b^* \theta_{E_{A/vI}/M_{A/vI}} = m_1^* T_v^* \text{lev}_1 \theta_{E_{A/I}/M_{A/I}} = m_1^* T_v^* g_{A/I, \{1\}}.$$
Then put on order,

\[ m_1^* g_{A/I,\{1\}} = m_1^* T_v^* g_{A/I,\{1\}} \times m_1^* m_{1,*} g_{A/vI,\{1\}} \]

\[ = m_1^* (T_v^* g_{A/I,\{1\}} \times m_{1,*} g_{A/vI,\{1\}}) \]

therefore transposition,

\[ m_1^* (1 - T_v^*) g_{A/vI,\{1\}} = m_1^* m_{1,*} g_{A/vI,\{1\}}. \]

In the next section, we write \( M_{N,F} \) for \( M_N \times \text{Spec } F \) for simplicity.

### 3 Euler Systems

In this section we prepare a tower of field extensions analogues to the cyclotomic extension \( \mathbb{Q}(\mu_{p^n}) \) in algebraic number fields. More precisely, we consider \( \mathcal{O}(\mathcal{M}_{A/v^n I,F}) \) and \( \Gamma(\mathcal{M}_{A/v^n I,F}) \) for any \( n \geq 0 \). And then we confirm that Siegel units belong to the tower and satisfy the norm relations of Euler systems.

#### 3.1 Notation

Let \( p, F, \infty, A \) be as in the first paragraph of Section 2. Let \( I \) be an ideal of \( A \) not equal to \( A \). Put \( \hat{A} := \varprojlim_{0 \neq J: \text{ideal of } A} A/J. \) Let \( v \) be a prime of \( F \) not dividing \( I, \mathbb{A}_F \) the ring of adeles of \( F \), and \( \mathbb{A}_F^\times \) the group of ideles of \( F \).

We briefly mention what is \( \mathcal{O}(\mathcal{M}_{A/v^n I,F}) \). First it is proved by Drinfeld that \( \mathcal{M}_{A/v^n I,F} \) is equal to the spectrum of finite abelian extension of \( F \) \cite[8-Theorem1]{Dr}. Hence \( \mathcal{M}_{A/v^n I,F} \) is, as a topological space, consists of a single element, and the structural sheaf is a finite abelian extension of \( F \). Moreover \( \infty \) split completely in \( \mathcal{O}(\mathcal{M}_{A/v^n I,F}) \). We regard the tower of extensions \( \mathcal{O}(\mathcal{M}_{A/v^n I,F}) \) as an analogue of \( \mathbb{Q}(\mu_{p^n}) \) in algebraic field. Next, it is also proved by Drinfeld that the reciprocity map of the global class field theory induces an isomorphism

\[
\text{Gal}(\mathcal{O}(\mathcal{M}_{A/v^n I,F})/F) \cong F^\times F^\times_{\infty} / A_F^\times / \left( \hat{A}^\times \cap (1 + v^n I \hat{A}) \right). \tag{2}
\]

Let \( F_v \) be the completion of \( F \) at \( v \) and \( \mathcal{O}_{F_v} \) the integer ring. Similarly let \( \mathcal{O}(\mathcal{M}_{A/I,F})_w \) denote the completion of \( \mathcal{O}(\mathcal{M}_{A/I,F}) \) at \( w \) and \( \mathcal{O}_w \) the
integer ring. Now let us consider the extension \( \mathcal{O}(\mathcal{M}_{A/I,F}) \) over \( F \). Let \( v \) be a prime of \( F \) not dividing \( I \) which split completely in \( \mathcal{O}(\mathcal{M}_{A/I,F})/F \). Let \( M \in \mathbb{N} \) be a power of \( p_{\infty} \). Let \( \psi_v \) be a continuous surjective homomorphic

\[
\psi_v : \mathcal{O}_{F_v}^{\times} \to (\mathbb{Z}/M\mathbb{Z})
\]

which through \( (\mathcal{O}_{F_v}/p_v^{n_v}) \) for some \( n_v \). Where \( p_v \) is the maximal ideal of the local ring \( \mathcal{O}_{F_v} \).

**Definition 3.1**

Let \( \Psi_M \) denote the set of finite sets of pairs of a prime \( v \) and a homomorphism \( \psi_v \) satisfying the conditions as above. i.e.

\[
\Psi_M := \left\{ (v_1,\psi_{v_1}),..., (v_r,\psi_{v_r}) \right\} \quad \begin{cases} 
\begin{array}{l}
    r \geq 0, \; v_i \; (i=1,...,r) \; \text{is a prime of} \; F \\
    \text{which is prime to} \; I, \\
    \text{split completely in} \; \mathcal{O}(\mathcal{M}_{A/I,F})/F, \\
    \text{and both} \; v_i \; \text{and} \; v_j \; (i \neq j) \; \text{are prime}, \\
    \psi_{v_i} \; \text{is a continuous surjection as above}
\end{array}
\end{cases}
\]

where \( r = 0 \) means \( \emptyset \in \Psi_M \).

For any \( v_i \), fix an integer \( n_i := n_{v_i} \) depending on \( v_i \) as above. Put

\[
(v_r,\psi_{v_r}) := \left\{ (v_1,\psi_{v_1}),..., (v_r,\psi_{v_r}) \right\}
\]

for simplicity. Let “\( (v,\psi_v) \in \Psi_M \)” denote “\( \left\{ (v,\psi_v) \right\} \in \Psi_M \)”.

**3.2 Norm equation**

For an element \( (v_r,\psi_{v_r}) \in \Psi_M \), let us consider the field extension corresponding to it:

\[
\begin{array}{c}
\mathcal{O}(\mathcal{M}_{A/I,F}) \\
\mathcal{O}(\mathcal{M}_{A/v_1^{n_1}I,F}) \\
\vdots \\
\mathcal{O}(\mathcal{M}_{A/v_1^{n_1}...v_r^{n_r}I,F}) \\
F
\end{array}
\]

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Proposition 3.2
\[ \text{Gal} \left( \mathcal{O}(\mathcal{M}_{A/v_1 \cdots v_r, I,F})/\mathcal{O}(\mathcal{M}_{A/I,F}) \right) \cong (A/v_1 \cdots v_r A)^\times \cong \prod_{i=1}^{r} (A/v_i A)^\times \]

Proof
The latter isomorphism is clear since any \( v_i \) is prime to \( v_j (i \neq j) \) so it is followed by Chinese reminder theorem. Let us prove the former. Only in this proof, we put \( v := v_1^{n_1} \cdots v_r^{n_r} \) for simplicity. The above diagram and the Galois group \([2]\) show that the following sequence of abelian groups is exact:

\[
0 \to \text{Gal} \left( \mathcal{O}(\mathcal{M}_{A/vI,F})/\mathcal{O}(\mathcal{M}_{A/I,F}) \right) \to F^\times F^\times \backslash \mathbb{A}^\times_F / (\hat{A}^\times \cap (1 + vI\hat{A})) \\
\to F^\times F^\times \backslash \mathbb{A}^\times_F / (\hat{A}^\times \cap (1 + I\hat{A})) \to 0.
\]

So we will prove

\[
\text{Ker} \left( F^\times F^\times \backslash \mathbb{A}^\times_F / (\hat{A}^\times \cap (1 + vI\hat{A})) \to F^\times F^\times \backslash \mathbb{A}^\times_F / (\hat{A}^\times \cap (1 + I\hat{A})) \right) \\
\cong (A/vA)^\times.
\]

First, by the definition of it, we have

\[
\text{Ker} \left( F^\times F^\times \backslash \mathbb{A}^\times_F / (\hat{A}^\times \cap (1 + I\hat{A})) \to F^\times F^\times \backslash \mathbb{A}^\times_F / (\hat{A}^\times \cap (1 + I\hat{A})) \right) \\
= \hat{A}^\times \cap (1 + I\hat{A})/ \left( (\hat{A}^\times \cap (1 + I\hat{A})) \cap F^\times F^\times \backslash \mathbb{A}^\times_F (\hat{A}^\times \cap (1 + vI\hat{A})) \right) \\
= \hat{A}^\times \cap (1 + I\hat{A})/ \left( F^\times F^\times \cap (\hat{A}^\times \cap (1 + I\hat{A})) \cdot \hat{A}^\times \cap (1 + vI\hat{A}) \right).
\]

Here it is clear that

\[
F^\times F^\times \cap (\hat{A}^\times \cap (1 + I\hat{A})) \cong F^\times F^\times \cap \hat{A}^\times \\
\cong F^\times \cap (F^\times \cdot \hat{A}^\times) \\
= \{ x \in F^\times \mid \text{for any } w \neq \infty, x \in \mathcal{O}_{F_w} \} \\
= \Gamma(C\backslash \infty, \mathcal{O}_C^\times) = A^\times
\]

so recall that \( I \neq A \), we have

\[
\hat{A}^\times \cap (1 + I\hat{A})/ \left( F^\times F^\times \cap (\hat{A}^\times \cap (1 + I\hat{A})) \cdot \hat{A}^\times \cap (1 + vI\hat{A}) \right) \\
\cong \hat{A}^\times \cap (1 + I\hat{A})/ \left( \hat{A}^\times \cap (1 + vI\hat{A}) \right) \\
\cong 1 \cdot (\hat{A}^\times \cap (1 + vI\hat{A})).
\]
Therefore,
\[
\begin{align*}
\text{Ker} \left( F^\times F^\times F_\infty^\times \setminus A_F^\times / (\hat{A}^\times \cap (1 + vI\hat{A})) \right) &\rightarrow (F^\times F^\times F_\infty^\times \setminus A_F^\times / (\hat{A}^\times \cap (1 + I\hat{A})) \\
&\cong (\hat{A}^\times \cap (1 + I\hat{A}))/\hat{A}^\times \cap (1 + vI\hat{A}) \\
&\cong (1 + I\hat{A})/(1 + vI\hat{A}) \\
&\cong (A/vA)^\times.
\end{align*}
\]

Take \( \psi_{v_i} \) apart as follows:
\[
O \times F_{v_i} \rightarrow (O F_{v_i}/p_{v_i}^n)^\times \rightarrow (Z/MZ)
\]
and put \( \overline{\psi_{v_i}} : (O F_{v_i}/p_{v_i}^n)^\times \rightarrow Z/MZ. \) the latter map. Let \( E(v_r, \psi_{v_r}) \) be a subfield of \( O(M_{A/v_1^n \cdots v_r^n, I, F}) \) over \( O(M_{A/I, F}) \) whose Galois group Gal \((O(M_{A/v_1^n \cdots v_r^n, I, F})/E(v_r, \psi_{v_r}))\) is equal to \( \text{Ker} (\overline{\psi_{v_1}} \times \cdots \times \overline{\psi_{v_r}}). \) For the sake of convenience if \( r = 0, \) we define \( E(v_0, \psi_{v_0}) := O(M_{A/I, F}). \) It is clear that
\[
\text{Gal} (E(v_r, \psi_{v_r})/O(M_{A/I, F})) \cong \prod_{i=1}^{r} (Z/MZ).
\]
Put
\[
G_i := \text{Gal} (E(v_i, \psi_{v_i}/E(v_{i-1}, \psi_{v_{i-1}}))
\]
then there exists a natural isomorphism \( \prod_{i=1}^{r} G_i \cong \text{Gal} (E(v_r, \psi_{v_r}/O(M_{A/I, F})). \)

Let \( g_{A/v_1^n \cdots v_r^n, I, \{1\}} \in O(M_{A/v_1^n \cdots v_r^n, I, F}) \) be a Siegel unit as section 2.3. Put
\[
g(v_r, \psi_{v_r}) := \text{Norm}_{O(M_{A/v_1^n \cdots v_r^n, I, F})/E(v_r, \psi_{v_r})} g_{A/v_1^n \cdots v_r^n, I, \{1\}}.
\]
For any \( G_i \) let us choose an element \( \sigma_i \in G_i \) whose image under the isomorphism \( G_i \cong Z/MZ \) is equal to 1. We define operators \( N_i \) and \( D_i \) as
\[
N_i := \sum_{\tau \in G_i, \ \tau \neq G_i-1} \tau \in Z[G_i]
\]
\[
D_i := \sum_{j=1}^{\#G_i-1} j\sigma^j_i \in Z[G_i]
\]
where we write an action for multiplicative group \( O(M_{A/v_1^n \cdots v_r^n, I, F})^\times \) additively. We will often use this additive notation.
Proposition 3.3
The following equality holds for any $i$.

$$(\sigma_i - 1)D_i = M - N_i$$

Proof
Calculating both sides with taking care of $\sharp G_i = M$, the proposition is clear.

Put $N := \prod_{i=1}^r N_i, D := \prod_{i=1}^r D_i$. Then the Siegel unit $g_{(v_r, \psi_{v_r})}$ holds the following conditions.

**Theorem 3.4 (Norm relations)**
For any $(v_r, \psi_{v_r})$ and $(v_i, \psi_{v_i}) \in (v_r, \psi_{v_r})$,

**ES1** $g_{(v_r, \psi_{v_r})} \in (E(v_r, \psi_{v_r}))^\times$.

**ES2** $g_{(v_r, \psi_{v_r})}$ is a unit.

**ES3** $N_i g_{(v_r, \psi_{v_r})} = (1 - \text{Frob}_i)g_{(v_{r\setminus i}, \psi_{v_{r\setminus i}})}$.

Where Frobi is the Frobenius of $v_i$ in $G_r$, $(v_{r\setminus i}, \psi_{v_{r\setminus i}})$ is an element that $(v_i, \psi_{v_i})$ is removed from $(v_r, \psi_{v_r})$.

Proof
Drinfeld [Dr, §8, Thm.1, Cor.] proved that $\mathcal{M}_{A/v^nI,F}$ is a spectrum of finite abelian extension over function field $F$. We have the following dictionary:

$\mathcal{M}_{A/I,F} \leftrightarrow$ Spec of finite abelian extension over function field $F$

$\mathcal{M}_{A/I} \leftrightarrow$ normalization of $U_{A/I}$ in $\mathcal{M}_{A/I,F}$

$m_{A/v^nI,A/v_i^{n-1}I,*} \leftrightarrow$ norm operator

$T_{v_i}^* \leftrightarrow \text{Frob}_{v_i}$

for a prime $v_i$. This leads (ES2) using Proposition 2.9. ES1 is clear from the definition. Remark that Proposition 2.10 can be written as

$N_i g_{A/v_i^{n-1}I,\{1\}} = \begin{cases} g_{A/v_i^{n-1}I,\{1\}} & \text{if } n \geq 2, \\ (1 - \text{Frob}_i)g_{A/I,\{1\}} & \text{if } n = 1. \end{cases}$
So take $I$ as $(v_{1}^{n_{1}} \cdots v_{r}^{n_{r}}/v_{i}^{n_{i}})I$, we have

$$N_{i}g_{A/v_{i}^{n_{i}}(v_{1}^{n_{1}} \cdots v_{r}^{n_{r}}/v_{i}^{n_{i}})I}/I, \{1\} = (1 - \text{Frob}_{i})g_{A/I,\{1\}}$$

Therefore (ES3) is hold.

4 Finite-singular comparison equalities

In this section, we construct Kolyvagin’s derivative class starting from our Euler system and formulate the “Finite-singular comparison relations” (Theorem 4.3 and 4.4). Our assumption is that $M$ is power of $p$. Note that if $M$ is prime to $p$, similar relations are proved by Oukhaba and S.Viguié in [OS].

4.1 Derivative class

For Theorem 3.4 the following lemma is hold.

Lemma 4.1

Let $(v_{r}, \psi_{v_{r}}) \in \Psi_{M}$. Then

$$Dg_{(v_{r}, \psi_{v_{r}})} \in (E(v_{r}, \psi_{v_{r}})^{\times}/(E(v_{r}, \psi_{v_{r}})^{\times})^{M})^{G}$$

where $G = \prod_{i=1}^{r} G_{i}$.

Proof

We prove this by induction on the cardinality of the finite set $r$ included in $(v_{r}, \psi_{v_{r}})$. Let $\sigma_{i}$ be a fixed generator of $G_{i}$ (1 ≤ $i$ ≤ $r$). When $r = 1$, i.e. $(v_{r}, \psi_{v_{r}}) = (v_{1}, \psi_{v_{1}})$, by the Proposition 3.3 and [ES3] in 3.4

$$(1 - \sigma_{1})Dg_{(v_{1}, \psi_{v_{1}})} = (M - N_{1})g_{(v_{1}, \psi_{v_{1}})}$$

$$\equiv -N_{1}g_{(v_{1}, \psi_{v_{1}})} \mod (E(v_{1}, \psi_{v_{1}})^{\times})^{M}$$

$$= -(1 - \text{Frob}_{v_{1}})g_{A/I,\{1\}} \mod (E(v_{1}, \psi_{v_{1}})^{\times})^{M}$$

Here $\text{Frob}_{v_{1}}$ acts trivially in $\mathcal{O}({\mathcal{M}_{A/I,F}})$ and $g_{A/I,\{1\}} \in \mathcal{O}({\mathcal{M}_{A/I,F}})$, so the last term is 0.
Let us assume for all \( r - 1 \), the claim is hold. In other words, for all \((v_{r-1}, \psi_{v_{r-1}}) \in \Psi_M\) and for all \( \sigma \in \prod_{i=1}^{r-1} G_i \), the following holds:

\[
(1 - \sigma) \prod_{i=1}^{r-1} D_{i\overline{g}(v_{r-1}, \psi_{v_{r-1}})} \equiv 0 \mod (E(v_{r-1}, \psi_{v_{r-1}})^x)^M
\]

Now we will prove that

\[
(1 - \sigma) \prod_{i=1}^{r} D_{i\overline{g}(v_r, \psi_{v_r})} \equiv 0 \mod (E(v_1, \psi_{v_1})^x)^M
\]

for \((v_r, \psi_{v_r}) \in \Psi_M\) and \( \sigma \in G = \prod_{i=1}^{r} G_i \). Since \( G = \prod_{i=1}^{r} G_i \) is generated by \( \sigma_i \in G_i (i = 1, \ldots, r) \), we can assume that \( \sigma = \sigma_i \). Furthermore, by the symmetry, we may assume without loss of generality that \( \sigma = \sigma_r \).

\[
(1 - \sigma_r) \prod_{i=1}^{r} D_{i\overline{g}(v_r, \psi_{v_r})} = (M - N_r) \prod_{i=1}^{r-1} D_{i\overline{g}(v_r, \psi_{v_r})}
\]

\[
\equiv -N_r \prod_{i=1}^{r-1} D_{i\overline{g}(v_r, \psi_{v_r})} \mod (E(v_r, \psi_{v_r}))^M
\]

\[
= - (1 - \text{Frob}_{v_r}) \prod_{i=1}^{r-1} D_{i\overline{g}(v_{r-1}, \psi_{v_{r-1}})}
\]

\[
\mod (E(v_r, \psi_{v_r}))^M
\]

Due to our assumption, the last term equal to 0.

**Lemma 4.2**

For any \((v_r, \psi_{v_r}) \in \Psi_M\), there exists an elements

\[
\kappa(v_r, \psi_{v_r}) \in (\mathcal{O}(\mathcal{M}_{A/I,F})^x / \mathcal{O}(\mathcal{M}_{A/I,F})^x)^M
\]

such that

\[
\kappa(v_r, \psi_{v_r}) \equiv D_{g(v_r, \psi_{v_r})} \mod (E(v_r, \psi_{v_r})^x)^M.
\]
Proof
Now we assume that $M$ is power of $p$, there is no $M$-th root of unity in $E(v_r, \psi_{v_r})$. By taking the Galois cohomology of the following short exact sequence:

\[
0 \to E(v_r, \psi_{v_r})^\times \xrightarrow{\text{power of } M} E(v_r, \psi_{v_r})^\times \\
\quad \xrightarrow{\text{projection}} E(v_r, \psi_{v_r})^\times / (E(v_r, \psi_{v_r})^\times)^M \to 0.
\]

We have a following exact sequence.

\[
0 \to \mathcal{O}(\mathcal{M}_{A/I,F})^\times \xrightarrow{\text{power of } M} \mathcal{O}(\mathcal{M}_{A/I,F})^\times \\
\quad \to (E(v_r, \psi_{v_r})^\times / (E(v_r, \psi_{v_r})^\times)^M)^G \to 0.
\]

The exactness of the last term follows from Hilbert 90. Hence the inclusion of $\mathcal{O}(\mathcal{M}_{A/I,F})^\times$ into $E(v_r, \psi_{v_r})^\times$ leads a natural isomorphism

\[
\mathcal{O}(\mathcal{M}_{A/I,F})^\times / (\mathcal{O}(\mathcal{M}_{A/I,F})^\times)^M \cong (E(v_r, \psi_{v_r})^\times / (E(v_r, \psi_{v_r})^\times)^M)^G.
\]

Lemma 4.1 says $D_{g(v_r, \psi_{v_r})} \in (E(v_r, \psi_{v_r})^\times / (E(v_r, \psi_{v_r})^\times)^M)^G$, Put $\kappa(v_r, \psi_{v_r})$ be an element corresponding to $D_{g(v_r, \psi_{v_r})}$ by the isomorphism above.

We call the family $\{\kappa(v_r, \psi_{v_r})\}_r$ of the above element $\kappa(v_r, \psi_{v_r})$ Kolyvagin’s derivative class.

4.2 Finite-singular comparison equalities

Let $\mathcal{I} := \bigoplus_{\lambda} \mathbb{Z}\lambda$ be the group of fractional ideals of $\mathcal{O}(\mathcal{M}_{A/I,F})$, where $\lambda$ ranges over the primes of $\mathcal{O}(\mathcal{M}_{A/I,F})$. For any prime $v$ of $F$, let us identify $v$ as a prime ideal of $A$, and define $\mathcal{I}_v := \bigoplus_{\lambda | v} \mathbb{Z}\lambda$. It is clear that $\mathcal{I} = \bigoplus_v \mathcal{I}_v$.

Let $y$ be an element of $\mathcal{O}(\mathcal{M}_{A/I,F})^\times$. Let $(y) \in \mathcal{I}$ denotes the principal ideal generated by $y$, $(y)_v \in \mathcal{I}_v$ the $v$-part of $(y)$, $[y] \in \mathcal{I} / M \mathcal{I}$ the projection of $(y)$, and $[y]_v \in \mathcal{I}_v / M \mathcal{I}_v$ the $v$-part of $[y]$. Then the following condition holds.
Theorem 4.3 (finite-singular comparison equalities)

Let \((v_r, \psi_{v_r}) \in \Psi_M, (v, \psi_v) \in \Psi_M\). Then

For any \(i = 1, \ldots, r\), \(v \neq v_i\)
\[\Rightarrow [\kappa(v_r, \psi_{v_r})]_v = 0\]

For some \(i = 1, \ldots, r\) such that \(v = v_i\)
\[\Rightarrow [\kappa(v_r, \psi_{v_r})]_v = \sum_{\lambda | v} \psi_v(t_{\lambda}(\kappa(v_r, \psi_{v_r}))) \lambda\]

where \(t_{\lambda}\) denotes the isomorphism \(O_{\lambda} \xrightarrow{\sim} O_{F_v}\).

The statement is proved at the section 5. This is the main result of this paper.

4.3 The existence of prime

This section, we prepare the proof of Iwasawa main conjecture in function fields. The above, we assume that there exists a pair \((v, \psi_v) \in \Psi_M\). We now prove that there exists a “good” pair in our case. Let \(C\) be a \(p\)-part of ideal class group of \(\mathcal{O}(\mathcal{M}_{A/I,F})\).

Theorem 4.4

Given \(c \in C\), a finite free \(\mathbb{Z}/M\mathbb{Z}\)-submodule \(W\) of \(\mathcal{O}(\mathcal{M}_{A/I,F})^\times / (\mathcal{O}(\mathcal{M}_{A/I,F})^\times)^M\), and surjection
\[\varphi : W \rightarrow (\mathbb{Z}/M\mathbb{Z})[\text{Gal}(\mathcal{O}(\mathcal{M}_{A/I,F})/F)]\]

compatible with the actions of the Galois group. Then there exists a prime \(w\) of \(\mathcal{O}(\mathcal{M}_{A/I,F})\) such that

1. \(w \in \mathfrak{c}\)
2. there exists a prime \(v\) of \(F\) below \(w\) and a map \(\psi_v : \mathcal{O}_v^\times \rightarrow \mathbb{Z}/M\mathbb{Z}\) such that
   (i) \(v\) is split completely in \(\mathcal{O}(\mathcal{M}_{A/I,F})/F\)
   (ii) for any \(y \in W\), \([y]_v = 0\)
   (iii) \(\psi_v(w) = \varphi(w)\).
Proof
Let $H$ be the maximal unramified abelian $p$-extension of $\mathcal{O}(\mathcal{M}_{A/I,F})$, and regard $C$ as $\text{Gal}(H/\mathcal{O}(\mathcal{M}_{A/I,F}))$ by class field theory.

\[ C \xrightarrow{\psi} \text{Gal}(H/\mathcal{O}(\mathcal{M}_{A/I,F})) \]

(class of $\lambda$) $\mapsto$ Frob$_{\lambda}$

where $\lambda$ is a prime of $\mathcal{O}(\mathcal{M}_{A/I,F})$. Let us consider a prime $w$ in $\mathcal{C}$. It is clear that there exists a prime $v$ below $w$ such that it holds (2)-(i). Let us consider these prime $v$. Since $W$ is a finite set so there exists only finite many prime which divides $(y)(y \in W)$, therefore we can take $v$ different from these and not divides $(y)$. Now we construct $\psi_v$ which holds (iii) from $\varphi$.

First we show that $W$ is embedded to $\mathcal{O}_{w}^\times/((\mathcal{O}_{w}^\times)^M$ by an injection. For any $y \in W$, $y'$ denotes the image under the lift up to $\mathcal{O}(\mathcal{M}_{A/I,F})^\times$ in a completion of $\mathcal{O}(\mathcal{M}_{A/I,F},w)$. For any $y \in W\setminus W^p$, if there exists an element $y'$ which is not in $(\mathcal{O}_{w}^\times)^p$, $W$ is embedded by an injection in $\mathcal{O}_{w}^\times/(\mathcal{O}_{w}^\times)^M$ due to the claim(ii). But the condition needs

\[ y^{p^d} - y \not\equiv 0 \mod v^2. \]

Here we use the fact that $y^{p^d} - y$ can be divided by $v$ at least 1 time where $p^d$ is the order of residue field of $F_v$. This requires that the morphism $C \to \mathbb{P}_{F_v}^1$ induced by $y'$ is unramified at $v$. This claim, the assumption that $W$ is free over $\mathbb{Z}/M\mathbb{Z}$ and $y' \in W\setminus W^p$, it is show that $y' \not\in F_v^p$. Therefore $W$ is embedded in $\mathcal{O}_{w}^\times/(\mathcal{O}_{w}^\times)^M$ under an injection with the finite exception of $v$. Let $w,v$ be as above (and replace if we need). Fix an isomorphism $\mathcal{O}_w \cong \mathcal{O}_{F_v}$. There exists a homomorphism $\varphi'$ such that the following diagram holds:

\[
\begin{array}{ccc}
W & \xrightarrow{\text{injection}} & \mathcal{O}_{w}^\times/(\mathcal{O}_{w}^\times)^M \cong \mathcal{O}_{F_v}^\times/(\mathcal{O}_{F_v}^\times)^M \\
\downarrow \varphi & & \exists \varphi' \\
\mathbb{Z}/M\mathbb{Z}[\text{Gal}(\mathcal{O}(\mathcal{M}_{A/I,F})/F)] & \xrightarrow{\circ} & \\
\end{array}
\]

Since $(\mathbb{Z}/M\mathbb{Z})[\text{Gal}(\mathcal{O}(\mathcal{M}_{A/I,F})/F)]$ is injective as $\mathbb{Z}/M\mathbb{Z}$-module. We regard in natural $\varphi'$ as $\mathcal{O}_{F_v}^\times \to \mathbb{Z}/M\mathbb{Z}[\text{Gal}(\mathcal{O}(\mathcal{M}_{A/I,F})/F)]$ with $\varphi'((\mathcal{O}_{F_v}^\times)^M) = \ldots$
0. Let it denotes also $\varphi'$. Let $\varrho : \mathbb{Z}/M\mathbb{Z}[\text{Gal}(\mathcal{O}(M_{A/I,F})/F)] \to \mathbb{Z}/M\mathbb{Z}$ be a surjective $\mathbb{Z}/M\mathbb{Z}$-homomorphism such that

$$
\begin{align*}
\mathbb{Z}/M\mathbb{Z}[\text{Gal}(\mathcal{O}(M_{A/I,F})/F)] & \to \mathbb{Z}/M\mathbb{Z} \\
\text{unit of } \text{Gal}(\mathcal{O}(M_{A/I,F})/F) & \mapsto 1 \\
\text{not unit of } \text{Gal}(\mathcal{O}(M_{A/I,F})/F) & \mapsto 0
\end{align*}
$$

Put $\psi_v := \varrho \circ \varphi'$. This is the homomorphism what we need.

5 Calculate the constant term of Coleman power series

In this section we will prove Theorem 4.3. The proof requires a relation of the theta function with the constant term of Coleman power series. In section 5.2 we will introduce a new symbol $(\cdot, \cdot)$.

5.1 Coleman power series

First we prepare the Coleman power series. This section conform to [Co].

Let $K$ be a local field, $\mathcal{O}_K$ an integer ring of $K$. Fix $\pi$ a uniformizer. Let $\mathcal{F}$ denotes the Lubin-Tate module corresponding to $\pi$, $\mathcal{F}(n + 1)$ the set of the root of $\pi^{n+1}$ in $\mathcal{F}$. Let us write $H$ as a maximal unramified abelian extension of $K$, Frob the arithmetic Frobenius of Gal $(H/K)$. Let us consider the tower of extensions:

$$
H_n := H(\mathcal{F}(n + 1)) \ (n = 0, 1, 2, \ldots).
$$

Let $\xi = (\xi_n)_n$ be a family of generator of $\varprojlim_n \mathcal{F}(n)$ as a $\mathcal{O}_K$-module. Then for any $m \leq n \leq 0$,

$$
[\pi^{n-m}]_{\mathcal{F}}(\xi_m) = \xi_n.
$$

Let $\mathcal{O}_H$ be an integer ring of $H$, $\mathcal{O}_H[[T]]$ a ring of formal power series whose coefficients are in $\mathcal{O}_H$. The group Gal $(H/K)$ act it by the action to the coefficients. Similarly let $\mathcal{O}_H((-T))$ be a ring of Laurent series whose coefficients are in $\mathcal{O}_H$ and which have at most finite poles on which the group Gal $(H/K)$ by the action to the coefficients. The followings was proved by Coleman in [Co].
**Theorem 5.1**
Put $\alpha = (\alpha_n)_n \in \varprojlim H_n^\times$ where the inverse limit is taken with respect to norm maps. Then there exists an unique element $f_\alpha(T) \in O_H((T))^\times$ such that

$$(\text{Frob}^{-n}f_\alpha)(\xi_n) = \alpha_n$$

for any $n \geq 0$. Moreover, the map which sends $f_\alpha(T) \in O_H((T))^\times$ to $((\text{Frob}^{-n}f_\alpha)(\xi_n))_n \in \prod H_n^\times$ induces an isomorphism

$$(O_H((T))^\times)^{N=1} \xrightarrow{\sim} \varprojlim H_n^\times$$

of groups where $N$ is called Coleman’s norm operator and $(O_H((T))^\times)^{N=1}$ a subgroup of $(O_H((T))^\times)$ that Coleman’s norm operator acts as the identity.

**Corollary 5.2**
Put $\alpha = (\alpha_n)_n \in \varprojlim O_H^\times$. Then there exists an element $f_\alpha(T)$ in $O_H[[T]]^\times$ uniquely such that

$$(\text{Frob}^{-n}f_\alpha)(\xi_n) = \alpha_n$$

In other words, the map correspond $f_\alpha(T) \in O_H[[T]]^\times$ to $((\text{Frob}^{-n}f_\alpha)(\xi_n))_n \in \prod O_H^\times$ induces a group isomorphic

$$(O_H[[T]]^\times)^{N=1} \xrightarrow{\sim} \varprojlim O_H^\times$$

similarly.

See [Co] for the proof. We call the power series $f_\alpha$ induced by the image of $\alpha$ under the above isomorphism a **Coleman power series**, and denote it by $\text{Col}_\alpha(T)$.

**5.2 The symbol $(\cdot, \cdot)_{L/K}$**
In this section we introduce the symbol $(\cdot, \cdot)_{L/K}$. Let $K$ be a local field, $L$ a finite abelian extension over $K$, $d$ the dimension of extension, $G$ the Galois group $\text{Gal}(L/K)$, $\text{val}_K$ a valuation of $L$ which is normalized at $K$, i.e. $\text{val}_K(K^\times) = \mathbb{Z}$ and $N_{L/K}$ the norm map from $L$ to $K$. Put $U_1 = \text{Ker} N_{L/K}$,
Let us write \( \hat{G} \) as the group of character of \( G \). We will define the symbol \( (\cdot, \cdot)_{L/K} \) for any \( u \in U_1 \) and \( \chi \in \hat{G} \).

Let \( K_\chi \) be an intermediate field of \( L/K \) which corresponds to \( \ker \chi \). It means that \( K_\chi \) satisfies \( \text{Gal}(L/K_\chi) = \ker \chi \). It is clear that \( K_\chi \) is a cyclic extension over \( K \). Write \( d_\chi \) as the degree of extension, \( G_\chi \) the Galois group of it, and \( u_{K_\chi} := N_{L/K_\chi}(u) \). Let us choose \( \sigma \in G \) which satisfies \( \chi(\sigma) = 1/m \) where \( m \) is the order of \( \chi \). we can show that \( N_{K_\chi/K}(u_{K_\chi}) = 1 \) since \( u \in U_1 \).

Therefore using Hilbert 90, there exists \( b_\chi \in K_\chi^\times \) such that

\[
u_{K_\chi} = \frac{\sigma(b_\chi)}{b_\chi}.
\]

It is unique up to \( K^\times \).

**Definition 5.3**

We define the map \( (\cdot, \cdot)_{L/K} : U_1 \times \hat{G} \to \mathbb{Q}/\mathbb{Z} \) as

\[
(u, \chi)_{L/K} = \text{val}_K(b_\chi) \mod \mathbb{Z}
\]

for any \( u \in U_1, \chi \in \hat{G} \).

This is well-defined since \( \text{val}_K \) is normalized. And it is easy to show that

\[
(uu', \chi)_{L/K} = (u, \chi)_{L/K} + (u', \chi)_{L/K}.
\]

**Proposition 5.4**

If \( L/K \) is cyclic, for any \( u \in U_1, \chi, \chi' \in \hat{G} \), the following equality is hold:

\[
(u, \chi \chi')_{L/K} = (u, \chi)_{L/K} + (u, \chi')_{L/K}.
\]

**Proof**

\( L/K \) is cyclic, \( \hat{G} \) is generated by a single element \( \chi_0 \). So we have to show that

\[
(u, \chi_0)_{L/K} = a(u, \chi_0)_{L/K} \quad \text{for any integer } a \geq 0.
\]

Put \( d' := d_{\chi_0} / \gcd(a, d_{\chi_0}) \).

By definition \( d' \) is equal to the order of \( \chi_0^a \). Take \( \sigma \) which satisfies \( \chi(\sigma) = 1/d' \). Then there exists an integer \( c \geq 1 \) such that

\[
a \cdot c \equiv \gcd(a, d) \mod d \mathbb{Z}.
\]

Hence we have \( ac/d \equiv 1/d' \mod \mathbb{Z} \) and \( \chi^a(\sigma^c) = 1/d' \). Put \( a' = a / \gcd(a, d) \), and take \( b_{\chi_0} \in K_{\chi_0}^\times \) which holds \( u_{K_{\chi_0}} = \sigma(b_{\chi_0})/b_{\chi_0} \) as above. Define \( b_{\chi_0, a} \).
as
\[ b_{\chi_0,a} := N_{K_{\chi_0}/K_{\chi_0}} \left( \prod_{i=0}^{a'-1} \sigma^{ci}(b_{\chi_0}) \right). \]

It is easy to calculate that \( u_{K_{\chi_0}} = \sigma^c(b_{\chi_0,a})/b_{\chi_0,a} \), therefore
\[
(u,\chi_0^a)_{L/K} \equiv \text{val}_K(b_{\chi_0,a}) \equiv \frac{d}{da'} a' \text{val}_K(b_{\chi_0,a}) \\
\equiv \text{aval}_K(b_{\chi_0,a}) \equiv a(u,\chi_0)_{L/K} \mod \mathbb{Z}.
\]

**Proposition 5.5**
Assume that \( L \) has no nontrivial \( d_\chi \)-th root of unity and for \( u \in U_1 \), there
exists an element \( a \in \mathcal{O}_K^\times, b \in \mathcal{O}_L^\times \) such that \( u = ab^{d_\chi} \). Then for any \( \chi \in \hat{G} \),
the following equality holds:
\[
(u,\chi)_{L/K} = 0.
\]

**Proof**
It is enough to show in case \( L = K_\chi \). Let \( \sigma \) be a generator of \( \hat{G}_\chi \). If there
exists \( u' \in \mathcal{O}_K^\times \) such that \( u = \sigma(u')/u' \), then we have \( (u,\chi)_{L/K} = 0 \) via the
definition. Therefore we have to prove the existence of \( u' \). Put
\[
u' := \prod_{i=0}^{d_\chi-1} \sigma^i(b^i)
\]
and then we have

\[
(\sigma(u')/u')^{d_\chi} = \left( \prod_{i=0}^{d_\chi-1} \sigma_i(b^d_i) \right) / \left( \prod_{i=0}^{d_\chi-1} \sigma_i(b^d_i) \right)
= \left( \prod_{i=0}^{d_\chi-1} \sigma_i(a^i b^d_i) \right) / \left( \prod_{i=0}^{d_\chi-1} \sigma_i(a^i b^d_i) \right)
= \left( \prod_{i=0}^{d_\chi-1} \sigma_i(a) \right) / \left( \prod_{i=0}^{d_\chi-1} \sigma_i(a) \right)
= \left( \prod_{i=1}^{d_\chi} \sigma_i(a^i b^d_i) \right) / \left( \prod_{i=0}^{d_\chi-1} \sigma_i(a^i b^d_i) \right)
= \left( \prod_{i=0}^{d_\chi} \sigma_i(a) \right) / \left( \prod_{i=0}^{d_\chi-1} \sigma_i(a) \right)
= (\sigma^{d_\chi}(u^d_\chi)) / \left( \prod_{i=0}^{d_\chi-1} \sigma_i(u) \right) = u^{d_\chi}/1
\]

we assume that \( L \) has no nontrivial \( d_\chi \)-th root, so we can erase \( d_\chi \)-th power.

**Lemma 5.6**

Assume that \( L \) be a finite totally ramified abelian extension over \( K \). Let \( N_{L/K} \) denote the norm map. For any \( u \in \mathcal{O}_K^\times \), there exist a finite unramified extension \( K' \) and an unit \( u' \in \mathcal{O}_L^\times \) where \( L' = K'L \) such that

\[
u = N_{L'/K'}(u')
\]

is hold.

**Proof**

Since \( N_{L/K}(\mathcal{O}_L^\times) \) is a subgroup of finite index of \( \mathcal{O}_K^\times \), there exists an integer \( n \geq 1 \) such that \( u^n \in N_{L/K}(\mathcal{O}_L^\times) \). Let \( K' \) be an unramified extension of degree \( n \) over \( K \). Then the following homomorphism

\[
\mathcal{O}_K^\times/N_{L'/K'}(\mathcal{O}_L^\times) \to \mathcal{O}_K^\times/N_{L/K}(\mathcal{O}_L^\times)
\]
induced by the norm map $N_{K'/K}$ is an isomorphism. Since we regard $u$ as an element in $\mathcal{O}_{K'}$, we have

$$N_{K'/K} (u) = u^n \in N_{L/K} (\mathcal{O}_L^\times).$$

Therefore we have $u \in N_{L'/K'} (\mathcal{O}_{L'}^\times)$. It implies that there exists an unit $u' \in \mathcal{O}_L^\times$ such that $u = N_{L'/K'} (u')$ holds.

**Lemma 5.7**

Assume that $u \in \mathcal{O}_L^\times$ satisfies $N_{L/K} (u) = 1$ and $(u, \chi)_{L/K} = 0$ for any character $\chi : \text{Gal} (L/K) \to \mathbb{Q}/\mathbb{Z}$. Then there exist a finite unramified extension $K'$, an integer $r \geq 0$, units $\beta_1, \ldots, \beta_r \in \mathcal{O}_L^\times$ and elements $\sigma_1, \ldots, \sigma_r \in \text{Gal} (L'/K')$ such that the following equality is satisfied:

$$u = \prod_{i=1}^{r} \frac{\sigma_i (\beta_i)}{\beta_i}.$$  

**Proof**

We prove it by induction on the number of generators of $\text{Gal} (L/K)$. When $\text{Gal} (L/K) = 1$, the claim is clear. When $\text{Gal} (L/K) \neq 1$, there exists an intermediate extension $M$ of $L/K$ such that the number of generators $\text{Gal} (M/K)$ is less than that of $\text{Gal} (L/K)$ and $L/M$ is a cyclic extension. Then there exists an intermediate extension $N$ of $L/K$ such that the composition

$$\text{Gal} (L/M) \hookrightarrow \text{Gal} (L/K) \twoheadrightarrow \text{Gal} (N/K)$$

is a bijection. Let us apply the inductive hypothesis to $N_{L/M} (u) \in \mathcal{O}_M^\times$. There exist a finite unramified extension $K''$, an integer $r' \geq 0$, units $\beta'_1, \ldots, \beta'_r \in \mathcal{O}_M^\times$ and elements $\sigma'_1, \ldots, \sigma'_{r'} \in \text{Gal} (M''/K'')$ such that

$$N_{L/M} (u) = \prod_{i=1}^{r'} \frac{\sigma'_i (\beta'_i)}{\beta'_i}$$

where $M'' = K'' M$. Let us apply Lemma 5.6 to $\beta'_1, \ldots, \beta'_{r'}$. Then there exist a finite unramified extension $K'$ over $K''$ and units $\beta_1, \ldots, \beta_r \in \mathcal{O}_L^\times$ such that

$$\beta'_i = N_{L'/M'} (\beta_i)$$
for any $i = 1, \ldots, r'$ where $L' = K'L$ and $M' = K'M$. Let $\sigma_1, \ldots, \sigma_{r'} \in \text{Gal}(L'/K')$ denote the lifting of $\sigma_1', \ldots, \sigma_{r'}' \in \text{Gal}(M''/K'') \cong \text{Gal}(M'/K')$ and put
\[
u' := u \cdot \left( \prod_{i=1}^{r'} \frac{\sigma_i(\beta_i)}{\beta_i} \right)^{-1}.
\]
Then we have $N_{L'/M'}(\nu') = 1$. By the assumption $\text{Gal}(L'/M') \cong \text{Gal}(L/M)$ is cyclic. Let us choose a generator $\tau \in \text{Gal}(L'/M')$ then there exists an element $b \in M'^\times$ such that $\nu' = \tau(b)/b$ by Hilbert 90. Moreover we have $(\nu', \chi)_{L'/K'} = 0$ for any character $\chi : \text{Gal}(L'/K') \to \mathbb{Q}/\mathbb{Z}$. Put $m = [L : M]$ and $N' = K'^N$. Let us choose $\chi$ so that $\chi(\tau) = 1/m$. Then there exists an element $a \in M'^\times$ such that $ab \in O_{L'}^\times$. Put $r = r' + 1$ and $\beta_r = ab$. Since $a \in M'^\times$ we have $\nu' = \tau(\beta_r)/\beta_r$. Therefore, put $\sigma_r = \tau$ and we have
\[
u = \prod_{i=1}^{r} \frac{\sigma_i(\beta_i)}{\beta_i}.
\]

5.3 Coleman power series and $(\cdot, \cdot)_{L/K}$

In this section we give a statement in Theorem 5.8 which relates Coleman power series and $(\cdot, \cdot)_{L/K}$. Let $K$ be a local field, and take an uniformizer $\pi$. Let $F$ be a Lubin-Tate module of $\pi$ over $O_K$, and fix a generator $\xi = (\xi_n)_{n \in \lim_{n} F(n)}$ of Tate module. Let $K'$ be a finite unramified extension over $K$. Put $K'_n := K'[\xi_n], K'_\infty := \cup_n K'_n, G := \text{Gal}(K'_\infty/K)$. Let $\text{rec}_K$ be the reciprocity map given by the class field theory
\[	ext{rec}_K : O_K^\times \overset{\sim}{\to} G.
\]

By Corollary 5.2 there exists the Coleman power series $\text{Col}_u(T) \in O_{K'}[[T]]$ such that $\text{Col}_u(\xi_n) = u_n$ for $u = (u_n)_n \in \lim_{n} O_{K'_n}^\times$. Let $L$ be a finite extension over $K'$ contained in $K'_\infty$. Put $u_L := N_{K'_n/L}(u_n)$ where $n$ is large enough. Then the following equality holds.

**Theorem 5.8**

Assume that $u_{K'} = 1$. Then, for any character $\chi : G \to \mathbb{Q}/\mathbb{Z}$ whose order is finite and any finite cyclic extension $K'_\chi = (K'_\infty)^{\text{ker}\chi}$ over $K'$, the following equality holds.
\[(u_{K'_\chi}, \chi)_{K'_\chi/K'} = \chi \circ \text{rec}_{K'}(\text{Col}_u(0)).\]
Proof
Let \( b_u \in \mathcal{O}_K^\times \) denote an elements which satisfies \((u_{K'}^\prime, \chi)_{K'/K'} = \chi \circ \text{rec}_{K'}(b_u)\).
We have to prove \( b_u = \text{Col}_u(0) \). Put \( u' = (\text{rec}_{K'}(\xi_n)/\xi_n)_n \in \lim_{\rightarrow n} \mathcal{O}_{K_n}^\times \) and \( u'' := (u''_n)_n = (u_n/u'_n)_n \). Then we have
\[
\text{Col}_{u'}(T) = \frac{\text{Col}_{\text{rec}_{K'}(b_u)(\xi_n)_n}(T)}{\text{Col}(\xi_n)_n(T)} = \frac{b_u T + (\dim \geq 2)}{T}.
\]
It implies \( \text{Col}_{u'}(0) = b_u \). Therefore we have
\[
\text{Col}_{u''}(0) = \text{Col}_u(0)/\text{Col}_{u'}(0) = \text{Col}_u(0)/b_u.
\]
So we have to prove \( \text{Col}_{u''}(0) = 1 \). Let us fix an integer \( n \geq 0 \). Let \( \mathcal{F}[\pi^n] \) denotes the finite flat group scheme of \( \pi^n\)-torsion points of \( \mathcal{F} \) over \( \mathcal{O}_K \) and \( R_n \) the coordinate ring of \( \mathcal{F}[\pi^n] \). Then \( R_n \) is local and finite flat \( \mathcal{O}_K \) algebra whose rank is equal to the \( n \)-th power of order of the residue field of \( K \). We regard \( \xi_n \) as a rational point of \( \mathcal{O}_{K_n} \) at \( \mathcal{F}(n) \) and then we have
\[
\text{Spec} \mathcal{O}_{K_n} \to \mathcal{F}(n).
\]
This is an open immersion and the image is equal to \( \mathcal{F}(n)\setminus \mathcal{F}(n - 1) \). So there exists a surjection
\[
r_n : R_n \twoheadrightarrow \mathcal{O}_{K_n}.
\]
Since \( R_n \) is local, \( r_n \) induces \( R_n^\times \to \mathcal{O}_{K_n}^\times \). We also denote it by \( r_n \). Let \( r'_n \) denotes the surjection \((R_n \otimes \mathcal{O}_{K'})^\times \to \mathcal{O}_{K_n}^\times \) induced by \( r_n \). Let \( \tilde{u}_n \in (R_n \otimes \mathcal{O}_{K'})^\times \) denotes a lifting of \( u''_n \) which satisfies \( r'_n(\tilde{u}_n) = u''_n \). Our claim is that the following equality holds independently of a choice of \( \tilde{u}_n \):
\[
N_{R_n \otimes \mathcal{O}_{K'}/\mathcal{O}_{K'}}(\tilde{u}_n) \equiv \text{Col}_{u''}(0) \mod \pi^{n+1} \mathcal{O}_{K'}.
\]
If it is hold for any \( n \), we have \( N_{R_n \otimes \mathcal{O}_{K'}/\mathcal{O}_{K'}}(\tilde{u}_n) = \text{Col}_{u''}(0) \). Applying Lemma 5.7 to \( u'' \), there exist an integer \( r \), elements \( \beta_1, \ldots, \beta_r \in \mathcal{O}_{K_n}^\times \) and \( \sigma_1, \ldots, \sigma_r \in \text{Gal}(K_n/K) \) such that \( u''_n = \prod_{i=1}^{r} \sigma_i(\beta_i)/\beta_i \). Let \( \tilde{\beta}_i \in (R_n \otimes \mathcal{O}_{K'})^\times \) denotes a lifting of \( \beta_i \) which satisfies \( r'_n(\tilde{\beta}_i) = \beta_i \). Then we have
\[
\tilde{u''}_n = \prod_{i=1}^{r} \sigma_i(\tilde{\beta}_i)/\tilde{\beta}_i.
\]
On the other hand it is clear that $N_{R_n \otimes O_{K'}} \circ \sigma_i = N_{R_n \otimes O_{K'}}$ for all $\sigma_i \in \text{Gal}(K_n/K)$. Therefore we have

$$N_{R_n \otimes O_{K'}} \left( \frac{\sigma_i(\tilde{\beta}_i)}{\beta_i} \right) = 1$$

for all $0 \leq i \leq r$. It implies $N_{R_n \otimes O_{K'}}(\tilde{u}_n) = 1$. At last we prove the claim (3). To prove the independence of a choice of $\tilde{u}_n$, we now show that

$$x \in \text{Ker} r_n \Rightarrow N_{R_n \otimes O_{K'}/O_{K'}}(x) \equiv 1 \mod \pi^{n+1}O_K.$$ 

Let $x \in R_n \otimes O_{K'}/O_K$ be an element which satisfies $r_n(x) = 1$ for any $n \geq 0$, $t_n$ a surjection from $R_n \otimes O_{K'}$ to $R_{n-1} \otimes O_{K'}$ induced by $\mathcal{F}[\pi^n] \hookrightarrow \mathcal{F}[\pi^{n+1}]$ where $R_{-1} := O_K$. Since $R_n \cong O_K[[T]]/(\left[\pi^{n+1}(T)\right])$, we have

$$(R_n \otimes O_{K'})^\times \cong (O_{K'}[[T]]/(\left[\pi^{n+1}(T)\right]))^\times.$$

$x$ satisfies $r_n(x) = 1$ so there exist $g(T)$ such that

$$x - 1 \equiv h(T)g(T) \mod [\pi^{n+1}](T)$$

where $h(T) := [\pi^{n+1}](T)/[\pi^n](T)$. It is clear that there exist a polynomial $h_1(T)$ such that $h(T) \equiv \pi h_1(T) \mod [\pi^n](T)$ and then $t_n(h(T)g(T)) \equiv 0 \mod \pi R_{n-1} \otimes O_{K'}$. Hence we have the following:

$$t_n(x) \equiv 1 \mod \pi R_{n-1} \otimes O_{K'}.$$

Next it is also clear that $r_{n-1}(N_{R_n \otimes O_{K'}/R_{n-1} \otimes O_{K'}}(x)) = N_{K'/K_{n-1}}(r_n(x)) = 1$. Moreover if $x$ satisfies $r_n(x) = 1$ and $t_n(x) \equiv 1 \mod \pi R_{n-1} \otimes O_{K'}$ then

$$t_{n-1}(N_{R_n \otimes O_{K'}/R_{n-1} \otimes O_{K'}}(x)) \equiv 1 \mod \pi^2 R_{n-2} \otimes O_{K'}.$$ 

By repeating the following discussion, we have

$$t_0(N_{R_n \otimes O_{K'}/R_0 \otimes O_{K'}}(x)) \equiv 1 \mod \pi^{n+1}R_0 \otimes O_{K'}.$$

We remark that $N_{R_n \otimes O_{K'}/R_{n-1} \otimes O_{K'}}(y) = (t_0(y))(N_{K_0/O_K}(r_n(y)))$ for any $y \in R_0 \otimes O_{K'}$. Hence we have

$$N_{R_n \otimes O_{K'}/O_{K'}}(x) \equiv 1 \mod \pi^{n+1}O_K.$$
Let us choose $f_n$ the image of $\text{Col}_{u^n}(T)$ under the natural projection $O_{K'}[[T]]^\times \to (O_{K'}[[T]]/([\pi^{n+1}](T)))^\times$. Therefore we have

$$N_{R_n \otimes O_{K'}/O_{K'}}(\tilde{u}_n) \equiv N_{R_n \otimes O_{K'}/O_{K'}}(f_n) \mod \pi^{n+1}O_{K'}$$

and then we have to prove $N_{R_n \otimes O_{K'}/O_{K'}}(f_n) = \text{Col}_{u^n}(0)$ in $K'$. For $0 \leq i \leq n$, $\xi_i$ defines

$$\text{Spec} O_{K_i} \hookrightarrow \mathcal{F}(i) \hookrightarrow \mathcal{F}(n).$$

Moreover $\mathcal{F}(n+1)$ has the unit point $\text{Spec} O_K \hookrightarrow \mathcal{F}(n)$ so we have following maps:

$$R_n \to O_{K_i} \text{ (for all } 0 \leq i \leq n)$$
$$R_n \to O_K$$

Here it is well known that the map

$$R_n \otimes K' \to K' \times \prod_{i=0}^{n} K'_i$$

induced by the map $R_n \to O_K \times \prod_{i=0}^{n} O_{K_i}$ is isomorphism. Then the image of $f_n$ under the map $R_n \otimes K' \to K'_i$ is equal to $\text{Col}_{u^n}(\xi_i)$ and under the map $R_n \otimes K' \to K'$ is equal to $\text{Col}_{u^n}(0)$. Therefore we have

$$N_{R_n \otimes O_{K'}/O_{K'}}(f_n) = N_{R_n \otimes K'/K'}(f_n)$$

$$= N_{K'/K'} \text{Col}_{u^n}(0) \times \prod_{i=0}^{n} N_{K'_i/K'} \text{Col}_{u^n}(\xi_i)$$

$$= \text{Col}_{u^n}(0) \times \prod_{i=0}^{n} N_{K'_i/K'}(u^n_i)$$

$$= \text{Col}_{u^n}(0) \times \prod_{i=0}^{n} u_{K'}$$

$$= \text{Col}_{u^n}(0)$$

since $u_{K'} = 1$ and this is end of the proof.
Let \( u \in \lim_{\leftarrow n} \mathcal{O}_{K_n}^\times \) be an element of Tate module which satisfies \( u_K = 1 \). Assume that there exists a positive integer \( d \) and \( a \in \mathcal{O}_{K'}^\times \) such that \( u_{K'} = a^d \). Let \( \chi : G \to \mathbb{Q}/\mathbb{Z} \) be a character of finite order \( d \). Assume that \( K'_{\chi} \) has no nontrivial \( d \)-th root. Then \( N_{K'/K}(a) = 1 \), by using Hilbert 90, there exists an element \( b \in \mathcal{O}_{K'}^\times \) such that \( a = \text{Frob}(b)/b \) where \( \text{Frob} \in \text{Gal}(K'/K) \) is an arithmetic Frobenius. Then Theorem 5.8 shows the following. This proposition is used to prove Theorem 4.3 mainly.

**Corollary 5.9**

\[
(\frac{u_{K'}^d/a}{\chi})_{K'/K'} = \chi \circ \text{rec}(\text{Col}_u(0)/b^d)
\]

**Proof**

Now \( N_{K'/K}(a) = 1 \), so there exists an element \( \tilde{u} \in \lim_{\leftarrow n} \mathcal{O}_{K_n}^\times \) such that \( \tilde{u}_{K'} = a \). Via the Coleman power series \( \text{Col}_{\tilde{u}} \) associated with \( \tilde{u} \), it is easy to show that \( \text{Frob}(\text{Col}_{\tilde{u}}(0))/\text{Col}_{\tilde{u}}(0) = \tilde{u}_{K'} = a \), there exists \( c \in \mathcal{O}_{K}^\times \) such that \( \text{Col}_{\tilde{u}}(0) = bc \). Since \( N_{K'/K}(a/\tilde{u}_{K'}^d) = (a/\tilde{u}_{K'}^d)^d = 1 \), we have

\[
(\frac{a/\tilde{u}_{K'}^d}{\chi})_{K'/K'} = 0
\]

by Proposition 5.5. Using Theorem 5.8, we have

\[
(\frac{u_{K'}^d/a}{\chi})_{K'/K'} = \left(\frac{(\frac{u_{K'}^d/\tilde{u}_{K'}^d}{\chi})_{K'/K'}}{\chi} \right)_{K'/K'}
\]

\[
= (\frac{u_{K'}^d/\tilde{u}_{K'}^d}{\chi})_{K'/K'} - (\frac{a/\tilde{u}_{K'}^d}{\chi})_{K'/K'}
\]

\[
= \chi \circ \text{rec}(\text{Col}_u(0)/\text{Col}_{\tilde{u}}(0)^d)
\]

\[
= \chi \circ \text{rec}(\text{Col}_u(0)/b^d c^d).
\]

Here the order of \( \chi \) is \( d \), the last term is equal to \( \chi \circ \text{rec}(\text{Col}_u(0)/b^d) \).

**5.4 Proof of Theorem 4.3**

Let us go back to our setting. We prove Theorem 4.3. Let \( v_i \) \((1 \leq i \leq r)\) be a prime of \( F \) which does not divide the ideal \( I, \infty \) and split completely in \( \mathcal{O}(\mathcal{M}_{A/I,F})/F \). Fix a prime \( w_i \) of \( \mathcal{O}(\mathcal{M}_{A/I,F}) \) above \( v_i \). Let us denote \( w_i^j \) \((1 \leq j \leq r)\) the prime of \( \mathcal{O}(\mathcal{M}_{A/\prod w_i^j 1,F}) \) above \( v_i \). Let \( \mathcal{O}_{w_i^j} \) be the integral ring of the completion of \( F \) at \( v_i \), \( \mathcal{O}_{w_i^j} \) the integral ring of the completion...
of $\mathcal{O}(\mathcal{M}_{A/I,F})$ at $w_i^j$. Remark that $\mathcal{O}_{w_i^j} \cong \mathcal{O}_{v_i}$ since $v_i$ is split completely in $\mathcal{O}(\mathcal{M}_{A/I,F})/F$. We consider a Drinfeld module $E_{A/I} \cong \text{Spec } R[T]$ whose zero section $\mathcal{M}_{A/I} \to E_{A/I}$ satisfies $T \mapsto 0$. Make $E_{A/I}$ base change from $\mathcal{M}_{A/I}$ to $\mathcal{O}_{w_i^j}$, and put

$$E_{\mathcal{O}_{w_i^j}} := E_{A/I} \times_{\mathcal{M}_{A/I}} \text{Spec } \mathcal{O}_{w_i^j}.$$  

Remark that $E_{\mathcal{O}_{w_i^j}} \cong \text{Spec } \mathcal{O}_{w_i^j}[T]$. The following maps is induced by group law of $E_{\mathcal{O}_{w_i^j}} \times E_{\mathcal{O}_{w_i^j}} \to E_{\mathcal{O}_{w_i^j}}$

$$\mathcal{O}_{w_i^j}[T] \mathcal{O}_{w_i^j}[T] \otimes_{\mathcal{O}_{w_i^j}[T]} \mathcal{O}_{w_i^j}[T] \otimes_{\mathcal{O}_{w_i^j}[T]} \mathcal{O}_{w_i^j}[T] \to \mathcal{O}_{w_i^j}[T, S].$$

We regard $F(T, S)$ as a formal power series by $\mathcal{O}_{w_i^j}[T, S] \subseteq \mathcal{O}_{w_i^j}[[T, S]]$ and $F$ a formal group law over $\mathcal{O}_{w_i^j}$ here it satisfies the qualification of formal group law since zero section satisfies $T \mapsto 0$. By the definition of Drinfeld module, for an uniformizer $\pi$ of $\mathcal{O}_{v_i}$, it is clear that

$$[\pi]_F(T) \equiv T^{q_{w_i^j}} \mod \pi$$

where $q_{w_i^j}$ is the order of residue field of $\mathcal{O}_{w_i^j}$. So $F$ is the Lubin-Tate module of $\pi$. Let $F(n)$ be the set of $\pi^{n+1}$-th root of unity, $\xi := (\xi_n)_n$ a generator of Tate module $\lim_{\leftarrow n} F(n)$. Put $\mathcal{O}_{w_i^j}(n) := \mathcal{O}_{w_i^j}[F(n)]$. Then we have

$$\mathcal{M}_{A/v^n I} \cong \text{Spec } \mathcal{O}_{w_i^j}(n).$$

Repost in section 2

Let $m_{n_j}$ denotes the map repeated above for $n_j$ times

$$m_{n_j} : \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} \to \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j-1} I}.$$
Here \( v_j \) is split completely in \( \mathcal{O}(\mathcal{M}_{A/I,F})/F \), in particular \( v_j \) is a principle ideal. So there exists an element \( \pi v_j \in A \) such that
\[
v = \pi v_j A.
\]

Let \((\pi v) : E/v_1^{n_1} \cdots v_j^{n_j} I \to E/v_1^{n_1} \cdots v_j^{n_j} I\) be the map given by the multiplication by \( \pi v_j \). Then we have
\[
m_{n_j}^* E_{A/I} \cong E_{A/v_1^{n_1} \cdots v_j^{n_j} I}/(\text{lev}(\pi v_j^n)) \cong E_{A/v_1^{n_1} \cdots v_j^{n_j} I},
\]
where the last isomorphism is induced by \( \pi v_j^n \). Let \( \phi_{n_j} \) be the composition
\[
(A/v_1^{n_1} \cdots v_j^{n_j-1} I \times \pi v_j A/A) \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} \cong (A/v_1^{n_1} \cdots v_j^{n_j-1} I \times A/v_1^{n_1} \cdots v_j^{n_j} A) \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I}
\]
\[
\text{lev} \ E_{A/v_1^{n_1} \cdots v_j^{n_j} I} \cong m_{n_j}^* E_{A/v_1^{n_1} \cdots v_{j-1}^{n_j-1} I}
\]
and \( \phi_{n_j} \), the composition
\[
(A/v_1^{n_1} \cdots v_{j-1}^{n_j-1} I) \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} \xrightarrow{\text{lev}} E_{A/v_1^{n_1} \cdots v_j^{n_j} I} \cong m_{n_j}^* E_{A/v_1^{n_1} \cdots v_{j-1}^{n_j-1} I}.
\]

Let us embed \( \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} \) into
\[
(A/v_1^{n_1} \cdots v_{j-1}^{n_j-1} I \times \pi v^A A/A) \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} = \bigoplus_{b \in (A/v_1^{n_1} \cdots v_{j-1}^{n_j-1} I \times \pi v^A A/A)} \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I}.
\]

We now calculate what a Siegel unit will be.
\[
g_{A/v_1^{n_1} \cdots v_j^{n_j} I, \{1\}} = \text{lev}^* \partial E_{A/v_1^{n_1} \cdots v_j^{n_j} I}/\mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I}
\]
\[
= \phi_{n_j}((\pi v_j^{-n_j}, \pi v_j^{-n_j})*m_{n_j}^* \partial E_{A/v_1^{n_1} \cdots v_j^{n_j} I}/\mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I}
\]
\[
= \phi_{n_j}((\pi v_j^{-n_j}, 0) + (0, \pi v_j^{-n_j}))*m_{n_j}^* \partial E_{A/v_1^{n_1} \cdots v_j^{n_j} I}/\mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I}
\]
\[
= \phi_{n_j}((0, \pi v_j^{-n_j})*m_{n_j}^* \partial E_{A/v_1^{n_1} \cdots v_j^{n_j} I}/\mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I}
\]
where \( t \phi_{n_j}((\pi v_j^{-n_j}) \), appearing the last term is translation by \( t \phi_{n_j}((\pi v_j^{-n_j}) \),
\[
t \phi_{n_j}((\pi v_j^{-n_j}) : E_{A/v_1^{n_1} \cdots v_j^{n_j} I} \xrightarrow{\text{id} \times \text{lev}^*} E_{A/v_1^{n_1} \cdots v_j^{n_j} I} \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} \times \mathcal{M}_{A/v_1^{n_1} \cdots v_j^{n_j} I} \xrightarrow{\text{group law}} E_{A/v_1^{n_1} \cdots v_j^{n_j} I}.
\]

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And take care that
\[ \phi'_{n_j}(\pi_{v_j}^{-n_j}) = \phi'_{n_j}(\text{Frob}_{v_j}^{-n_j}(1)) = \text{Frob}_{v_j}^{-n_j}\phi'_{n_j}(1). \]

On the other hands what will \( \phi_{n_j}(0, \pi_{v_j}^{-n_j})^*m_{n_j}^* \) be. It is easy to show that
\[ m_{n_j}\phi_{n_j}(0, \pi_{v_j}^{-n_j}) : \mathcal{M}_A/\mathcal{v}_1^{n_1}...\mathcal{v}_j^{n_j}I \to m_{n_j}^*E_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_{j-1}^{n_{j-1}}} \to E_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_{j-1}^{n_{j-1}}}I \]
\[ \text{Spec } \mathcal{O}_{u_j}(n) \quad \text{Spec } \mathcal{O}_{u_j}[T] \]
so we have
\[ \phi_{n_j}(0, \pi_{v_j}^{-n_j})^*m_{n_j}^* : \mathcal{O}_{u_j}[T] \to \mathcal{O}_{u_j}(n) \]
\[ T \quad \mapsto \quad \xi_{n_j} \]

This asserts that it is the map which substitute the value of point for \( T \).
Corollary 5.2 asserts that there exists the Coleman power series. By summarizing, we have
\[ \phi_{n_j}(0, \pi_{v_j}^{-n_j})^*m_{n_j}^* t^{\vartheta_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_j^{n_j}}/\mathcal{M}_A/\mathcal{v}_1^{n_1}...\mathcal{v}_j^{n_j}I} \cdot \mathcal{I}(1) = \text{Col}_u(\xi_{n_j}) \]
where \( u = (g_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_j^{n_j}I})_{n_j} \in \lim_{\leftarrow n_j} \mathcal{O}_{u_j}. \)
Especially when \( n_j = 0, \)
\[ g_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_{j-1}^{n_{j-1}}}I.(1) = \text{Col}_u(0). \]

Thus we obtain the following proposition:

**Proposition 5.10**

\[ t^{\vartheta_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_j^{n_j}}/\mathcal{M}_A/\mathcal{v}_1^{n_1}...\mathcal{v}_j^{n_j}I} = \text{Col}_u(T) \]

Especially the constant term of the Coleman power series is equal to the Siegel unit \( g_{A/\mathcal{v}_1^{n_1}...\mathcal{v}_{j-1}^{n_{j-1}}}I.(1) = g(v_{j-1}, \psi_{v_{j-1}}). \)

At the end of this section, we prove Theorem 4.3.

Let \( (\mathbf{v}_r, \psi_{\mathbf{v}_r}) = \{(v_1, \psi_{v_1}), ... , (v_r, \psi_{v_r})\} \) be an element in \( \Psi_M \) and take an element \( (v, \psi_v) \in \Psi_M. \) If \( v \neq v_i \) for any \( i = 1, ... , r, \) it is clear that
\[ [\kappa(v_r, \psi_{\mathbf{v}_r})]_v = 0 \]
since \( v_i \) is unramified in \( \mathcal{O}(\mathcal{M}_{A/v_{1}^{r} \cdots v_{r}^{r}, I,F})/\mathcal{O}(\mathcal{M}_{A/I,F}) \). We will show the other case. We may assume without loss of generality that \( v = v_r \). Let \( w_r \) denote a prime of \( \mathcal{O}(\mathcal{M}_{A/I,F}) \) above \( v_r \), \( \mathcal{O}(\mathcal{M}_{A/I,F})_{w_r} \) the completion of \( \mathcal{O}(\mathcal{M}_{A/I,F}) \) at \( w_r \), \( w_r^{(i)} \) the prime of \( E(v_i, \psi_{v_i}) \) above \( w_r \) and \( E_{w_r^{(i)}} \) the completion of \( E(v_i, \psi_{v_i}) \) at \( w_r^{(i)} \) for any \( i = 1, \ldots, r \). Put

\[
\begin{align*}
    u_{w_r^{(i)}} &= \prod_{i=1}^{r-1} D_{i} g(v_r, \psi_{v_r}) \\
u_{w_r} &= N_{E_{w_r^{(r-1)}}/E_{w_r^{(r-1)}}}(u_{w_r^{(r)}}).
\end{align*}
\]

In the same way of proof of Lemma \ref{lemma} we have \( u_{w_r^{(r)}} = 1 \). Moreover we have

\[
\begin{align*}
u_{w_r^{(r-1)}} &= N_{w_r^{(r-1)}}^{/w_r^{(r-1)}}(u_{w_r^{(r-1)}}) \\
&= N_{r} \prod_{i=1}^{r-1} D_{i} g(v_r, \psi_{v_r}) \\
&= (1 - \text{Frob}_{v_r}) \prod_{i=1}^{r-1} D_{i} g(v_{r-1}, \psi_{v_{r-1}}) \\
&\equiv 0 \mod (E_{w_r^{(r-1)}})^{M}.
\end{align*}
\]

So there exists an element \( a \in E_{w_r^{(r-1)}} \) such that \( a^M = u_{w_r^{(r-1)}} \). Let \( \chi \) be a character \( G \to \mathbb{Q}/\mathbb{Z} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\chi \circ \text{rec} & \mathbb{Q}/\mathbb{Z} & \\
\mathcal{O}(\mathcal{M}_{A/I,F})^{\times} & \mathbb{Q}/\mathbb{Z} & \\
\psi_{v_r} : \mathcal{O}_{v_r}^{\times} & \to \mathbb{Z}/M\mathbb{Z} & \\
\end{array}
\]

Applying Corollary \ref{corollary} to \( u = (u_{w_r^{(r-1)}}) \), we have

\[
(u_{w_r^{(r-1)}}/a, \chi)_{w_r^{(r-1)}} = \chi \circ \text{rec}(\text{Col}_u(0)/b^M)
\]

where \( b \) is the element in \( u_{w_r^{(r-1)}}^{\chi} \) such that \( a = \text{Frob}_{v_r}(b)/b \). By definition, the element \( \kappa(v_r, \psi_{v_r}) \) is of the form

\[
\kappa(v_r, \psi_{v_r}) = \prod_{i=1}^{r} D_{i} g(v_r, \psi_{v_r}) \cdot c
\]

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where \( c \) is an element in \((E^\times_{w(r-1)})^M\) which holds

\[
(1 - \sigma_r)c = \left(\prod_{i=1}^{r-1} D_{i}\mathcal{S}(\nu_r,\psi_{\nu})\right) = \frac{u_{E_{w(r)}}}{\alpha}.
\]

Remark that \( \prod_{i=1}^{r} D_{i}\mathcal{S}(\nu_r,\psi_{\nu}) \) is an unit, it is clear that

\[
[k(\nu_r,\psi_{\nu})]_{\nu_r} = [c]_{\nu_r} = \oplus \text{val}_{\nu_r}(c)\lambda.
\]

And we have \((u_{E_{w(r)}}/\alpha, \chi_{E_{w(r)}}/\alpha, \nu_r, \psi_{\nu_r})\) since the definition. The above implies that \((u_{E_{w(r)}}/\alpha, \chi_{E_{w(r)}}/\alpha, \nu_r, \psi_{\nu_r})\) is equal to the coefficient of the \( w_{r} \)-part of \([k(\nu_r,\psi_{\nu})]_{\nu_r} \). On the other hand, using Proposition 5.10 we have

\[
\text{Col}_u(0)/\mathcal{M}^M = k(\nu_r,\psi_{\nu_r})
\]

and then via the definition of \( \chi \), we have

\[
\chi \circ \text{rec}(\text{Col}_u(0)/\mathcal{M}^M) = \psi_{\nu_r}(k(\nu_r,\psi_{\nu_r})).
\]

This completes the proof of Theorem 4.3.

6 Proof of Iwasawa Main Conjecture

In this section we describe the \( p \)-part of the ideal class group in function field (Theorem 6.6) using Theorem 4.3 and 4.4. The theorem is called Gras conjecture. We will prove this by imitating the proof of Gras conjecture in algebraic number fields.

6.1 Preparation of Iwasawa theory and Iwasawa main conjecture

Let us introduce some notations that we will use in Iwasawa theory. Let \( F, p, p, A \) be as above. Let \( I \) be an ideal of \( A \). Put \( K := O(M_A/I,F) \) for simplicity. Let \( A_K \) be the integral closure of \( A \) in \( K \). Fix a prime \( q \) of \( A \)
not dividing $I$. (until the previous section we denoted by $q$ the order of the field of $F$. Here we have changed the notation.) Put $I_n := q^{n+1}I, K_n := \mathcal{O}(\mathcal{M}_{A/q^{n+1}I, F}) \ (n \geq 0)$ for simplicity. Let $K_\infty$ denotes $\bigcup_{n \geq 0} K_n$ and $K^{(p)}_\infty$ a subfield of $K_\infty$ whose the Galois group $\Gamma := \text{Gal} (K^{(p)}_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p$. Put $\Delta := \text{Gal} (K_0/F)$. Assume that the order of $\Delta$ is prime to $p$. $(K_n)_{q'}$ denotes a completion of $K_n$ at the prime $K_n$ of $q'$ above $q$. Define these symbols as follows:

$C_n$ : The $p$-part of the ideal class group of $K_n$.

$E_n$ : The unit group of $K_n$.

$\mathcal{E}_n \subset E_n$ : The group generated by Siegel units of $K_n$.

$U_n$ : The unit group of $(K_n)_{q'}$.

$\overline{E}_n$ : The closure of $E_n \cap U_n$ in $U_n$.

$\overline{\mathcal{E}}_n$ : The closure of $\mathcal{E}_n \cap U_n$ in $U_n$.

And put

$C_\infty := \varprojlim C_n$

$E_\infty := \varprojlim E_n$

$\mathcal{E}_\infty := \varprojlim \mathcal{E}_n$

$U_\infty := \varprojlim U_n$

$\overline{E}_\infty := \varprojlim \overline{E}_n$

$\overline{\mathcal{E}}_\infty := \varprojlim \overline{\mathcal{E}}_n$

where the inverse limit is taken with respect to the norm map. Especially $C_\infty$ is equal to the $p$-part of ideal class group of $K^{(p)}_\infty$. For a character $\chi$ in $\Delta$, put

$e(\chi) := \frac{1}{#\Delta} \sum_{\delta \in \Delta} \chi(\delta)^{-1} \delta.$

Define Iwasawa algebra $\Lambda$ as

$\Lambda := \mathbb{Z}_p[\text{Im } \chi][[\Gamma]].$
Let $e(\chi)C_n, e(\chi)E_n, e(\chi)U_n, e(\chi)\overline{E}_n$ denote $C_n(\chi), E_n(\chi), U_n(\chi), \overline{E}_n(\chi)$ for short. Similarly when $n = \infty$. Then the following fact is known [OS].

**Fact 6.1**

$C_\infty(\chi), E_\infty(\chi)/\overline{E}_\infty(\chi)$ are finite generated $\Lambda$-modules.

Now we impose the following assumption on $C_\infty(\chi), E_\infty(\chi)$ and $\overline{E}_\infty(\chi)$.

**Hypothesis 6.2**

$C_\infty(\chi), E_\infty(\chi)/\overline{E}_\infty(\chi)$ is a torsion $\Lambda$-module. This means that $\Gamma$ satisfies the following conditions:

Let $m_\Lambda$ be the maximal ideal of $\Lambda$. There exist elements $f_C, f_E \in m_\Lambda$ such that

$$\sharp(C_\infty(\chi)/f_C(C_\infty(\chi)) < \infty \quad \text{and} \quad \sharp((E_\infty(\chi)/\overline{E}_\infty(\chi))/f_E(E_\infty(\chi)/\overline{E}_\infty(\chi))) < \infty$$

From now on we assume hypothesis 6.2. Now we recall the notion of pseudo-isomorphism of finite generated torsion $\Lambda$-modules and that of characteristic ideals.

**Definition 6.3 (pseudo-isomorphic)**

Let $N, N'$ be finite generated torsion $\Lambda$-modules. We say that $N$ and $N'$ are pseudo-isomorphic when there exists a homomorphism $N \to N'$ of $\Lambda$-modules whose kernel and cokernel are finite. If $N$ and $N'$ are pseudo-isomorphic then we write $N \sim N'$.

This is well-known fact about finite generated torsion $\Lambda$-modules.

**Fact 6.4**

For any finite generated torsion $\Lambda$-module $N$, there are finitely many $f_i (i = 1, \ldots, r) \in m_\Lambda$ unique up to rearrangement such that

$$N \sim \bigoplus_{i=1}^{r} \Lambda/f_i\Lambda.$$

**Definition 6.5 (characteristic ideal)**

For the above $f_i$, put $f := \prod_i f_i$. We call the ideal $f\Lambda$ of $\Lambda$ characteristic ideal of $N$, and write char $(N)$.
In the setting, our main result is as follows.

**Theorem 6.6 (Iwasama main conjecture)**
For any character $\chi$ in $\Delta$,

$$\text{char} \left( C_\infty(\chi) \right) = \text{char} \left( E_\infty(\chi) / (E_\infty(\chi)^\chi) \right)$$

The goal of this section is to prove the theorem. We will prove it in section 6.3

### 6.2 Iwasawa theory

First we write $a_n \approx b_n$ when $a_n/b_n$ is bounded from below and above independently of $n$, where $a_n$ and $b_n$ are sequences of positive integers.

**Proposition 6.7**
For any $n \geq 0$, $\sharp(C_n) \approx \left[ E_n : E_n \right]$.

**Proof**
We refer the reader to [La, Appendix-Lemma 6.6]. The equality between the index number of unit groups and ideal class number in global function fields is proved by L. Yin [Yi].

Next for any $n \geq 0$, put $\Gamma_n := \text{Gal}(K_\infty^{(\rho)} / K_n)$. Let $J_n$ be an ideal of $\Lambda$ generated by $\{ \gamma - 1 \mid \gamma \in \Gamma_n \}$, and $\gamma$ a generator of $\Gamma$. By the definition, we have $J_n = (\gamma^{p^n} - 1)\Lambda$. Put $\Lambda_n := \Lambda / J_n\Lambda$. Then we have $\Lambda_n \cong \mathbb{Z}_p[\text{Im} \chi][\text{Gal}(K_n / K_0)]$. For $\Lambda$-module $N$, let $N_{\Lambda_n}$ denotes a tensor product $N \otimes_\Lambda \Lambda_n$ and regard it as a $\Lambda_n$ module. Let us consider the pseudo-isomorphisms described above:

$$(C_\infty(\chi)) \sim \bigoplus_{i=1}^k \Lambda / f_i \Lambda,$$

$$(E_\infty(\chi) / (E_\infty(\chi)^\chi)) \sim \bigoplus_{i=1}^l \Lambda / h_i \Lambda,$$

and for each $f_i, h_i$ corresponding to the above, we define

$$f_\chi := \prod_{i=1}^k f_i,$$

$$h_\chi := \prod_{i=1}^l h_i.$$
Clearly \( \text{char}(C_\infty(\chi)) = f_\chi \Lambda, \text{char} \left( E_\infty(\chi)/(E_\infty(\chi)) \right) = h_\chi \Lambda. \)

**Lemma 6.8**
Assume that \( a_1, a_2 \in \Lambda, a_1|a_2, \sharp((\Lambda/a_1\Lambda)_{\Lambda_n}) \approx \sharp((\Lambda/a_2\Lambda)_{\Lambda_n}). \) Then
\[
a_1\Lambda = a_2\Lambda.
\]

**Proof**
We refer the reader to [La] Appendix-Corollary 7.3.

**Lemma 6.9**
Let \( \chi \) be a non trivial character of \( \Delta. \) Then there exists an ideal \( \mathfrak{A} \) of \( \Lambda \) of finite index such that

1. for any \( n \geq 0, \) there exist at least two coprime elements \( \eta, \eta' \in \mathfrak{A} \) such that \( \Lambda_n/\eta\Lambda_n, \Lambda_n/\eta'\Lambda_n \) are always finite.
2. for any \( \eta \in \mathfrak{A} \) and \( n \geq 0, \) there exist \( \Lambda \) homomorphism \( \theta_{n,\eta} : E_n(\chi) \to \Lambda_n \) such that
   \[
   \theta_{n,\eta}(E_n(\chi)) = \eta h_\chi \Lambda_n.
   \]
3. for any \( n \geq 0, \) there exists an element \( c_1, \ldots, c_k \in C_n(\chi) \) such that
   \[
   \mathfrak{A} \text{Ann}(c_i) \subseteq f_i\Lambda_n.
   \]

**Proof**
We refer the reader to [La] Appendix-Theorem 6.3, Corollary 6.4, Corollary 6.5.

**6.3 Proof of Theorem 6.6**
In this section we fix a positive integer \( n. \) Let \( \lambda' \) be a prime of \( F \) which split completely in \( K_n/F. \) Recall that for \( y \in K_n^\times, \) \( (y)_{\lambda'} \in I_{\lambda'} \) denote the \( \lambda' \)-part of \( (y) \in I, \) \( [y]_{\lambda'} \in I_{\lambda'}/M I_{\lambda'} \) the image under the projection where \( M \) is a power of \( p. \) Let \( \lambda \) be a prime of \( K_n \) above \( \lambda'. \) Then \( I_{\lambda'}(\chi) := e(\chi)I_{\lambda'} \) is free \( \Lambda_n \) module of degree one over \( \Lambda_n \cong \mathbb{Z}_p[[\text{Im } \chi]][\text{Gal}(K_n/K_0)], \) generated by \( \lambda(\chi) := e(\chi)\lambda. \) we define the map \( \nu_{\lambda,\chi} : K_n^\times \to \Lambda_n \cong I_{\lambda'}(\chi) \) satisfies
\[
\nu_{\lambda,\chi}(y)\lambda(\chi) = e(\chi)(y)_{\lambda'}
\]
and the map $\overline{\nu}_{\lambda,\chi}: K_n^\times/(K_n^\times)^M \to \Lambda_n/\Lambda_n$ satisfies
\[
\overline{\nu}_{\lambda,\chi}(y)\lambda(\chi) = e(\chi)[y]_{\lambda'}.\]

From the section 5.2, for the set $\Psi$, $(v, \psi_\nu) \in \Psi_M$ and field extension $K_n/F$, there exists a Kolyvagin’s derivative class $\kappa(v, \psi_\nu) \in K_n^\times/(K_n^\times)^M$. Be careful that the ideal $I$ in the section 4 is replaced with $q^{n+1}I$ here, similarly $O(M_{A/I,F})$ replaced with $K_n$. We prepare two lemmas.

**Lemma 6.10**

Let $M$ be a sufficiently large power of $p$ and $(v, \psi_\nu) \in \Psi_M$. Take a prime $\lambda'$ of $F$ which divides $v$ and a prime $\lambda$ of $K_n$ above $\lambda'$. Let $B_n$ be a subgroup of $C_n$ generated by $v/\lambda'$. Let $c$ denotes the class of ideal $e(\chi)\kappa(v, \psi_\nu)$ of $K_n^\times/(K_n^\times)^M$ generated by $e(\chi)\kappa(v, \psi_\nu)$. Assume that $\eta, a \in \Lambda_n$ satisfy the following conditions:

1. $\Lambda_n/a\Lambda_n$ is finite.
2. $\text{Ann}(c) \subseteq \Lambda_n$ satisfies $\eta \text{Ann}(c) \subseteq a\Lambda_n$ in $C_n(\chi)/B_n(\chi)$.

Then there exists a homomorphism $\varphi: W \to \Lambda_n/\Lambda_n \cong \mathbb{Z}/M\mathbb{Z}[\text{Gal}(K/F)]$ which keep the action of galois group such that.

$$a \varphi(e(\chi)\kappa(v, \psi_\nu)) = \eta \overline{\nu}_{\lambda,\chi}(\kappa(v, \psi_\nu)).$$

**Proof**

Let $\beta$ be one of lifting of $e(\chi)\kappa(v, \psi_\nu)$ to $K_n^\times$. Via the definition of $\nu_{\lambda,\chi}$, we have

$$e(\chi)(\beta) = e(\chi)(\beta)_{\lambda'} + \sum_{\lambda'' \neq \lambda'} e(\chi)(\beta)_{\lambda''} = \nu_{\lambda,\chi}(\beta)\lambda(\chi) + \sum_{\lambda'' \neq \lambda'} e(\chi)(\beta)_{\lambda''}$$

Here Theorem 4.3 implies that when $\lambda''$ does not divide $v$, $(\beta)_{\lambda''} \in M\mathcal{F}_{\lambda''}$.

Since we assume that $M$ is a sufficiently large power of $p$, $M$ annihilates $C_n(\chi)$. Therefore $\nu_{\lambda,\chi}(\beta)\lambda(\chi) \in \text{Ann}(c)$ since $\nu_{\lambda,\chi}(\beta)\lambda(\chi)$ is zero in $C_n(\chi)/B_n(\chi)$ by the definition of $B_n(\chi)$. The second assumption implies

$$\nu_{\lambda,\chi}(\beta)\lambda(\chi) \in a\Lambda_n.$$
The first assumption implies that \( \nu_{\lambda,\chi}(\beta)\lambda(\chi)/a \) is well-defined. Let us write it by \( \sigma \). Recall that \( W \) is generated by \( e(\chi)\kappa(\nu,\psi) \) over \( \Lambda_n \cong \mathbb{Z}_p[\text{Gal}(K_n/K_0)] \). Define a homomorphism \( \varphi : W \to \Lambda_n/M\Lambda_n \) by

\[
\varphi\left(\rho(e(\chi)\kappa(\nu,\psi))\right) = \rho\sigma
\]

for any \( \rho \in \Lambda_n \). It is clear that this keeps the galois action. Therefore we check that it is well-defined and is independent of the way of taking \( \beta \). Assume that \( \rho e(\chi)\kappa(\nu,\psi) = 0 \) i.e. there exists an element \( x \) in \( \Lambda_n \) such that \( \rho\beta = xM \). Especially \( \rho|e(\chi)\kappa(\nu,\psi)[\lambda] = 0 \). We have to prove \( \rho \sigma \in M\Lambda_n \). We assume that \( M \) is large enough, so we can assume \( (M/\sharp C(\chi))(\mathcal{I}_{\lambda'}(\chi)/M\mathcal{I}_{\lambda'}(\chi)) \) is included in \( \Lambda_n[e(\chi)\kappa(\nu,\psi)] \). Therefore we have \( \rho \in (\sharp C_n(\chi))\Lambda_n \). Thus

\[
e(\chi)(x) = \sum_{\lambda''} e(\chi)(x)_{\lambda''}
\]

\[
= M^{-1}e(\chi)(\rho\beta)_{\lambda'} + \sum_{\lambda'' \text{ divides } (\nu/(\lambda'))} e(\chi)(x)_{\lambda''}
\]

\[
+ \sum_{\lambda'' \text{ does not divide } (\nu)} e(\chi)(\rho\beta)_{\lambda''}
\]

\[
\equiv M^{-1}e(\chi)(\rho\beta)_{\lambda'} \mod \bigoplus_{\lambda'' \text{ divide } (\nu/(\lambda'))} \mathcal{I}_{\lambda''}(\chi), \sharp C_n(\chi)\mathcal{I}_{\lambda''}(\chi).
\]

Here \( \sharp C_n(\chi) \) vanish \( C_n(\chi) \), hence \( M^{-1}e(\chi)(\rho\beta)_{\lambda'} \) is equal to zero in \( C_n(\chi)/B_n(\chi) \). Therefore \( M^{-1}\nu_{\lambda,\chi}(\rho\beta)e = 0 \) and we have

\[
\rho\sigma a = \eta\nu_{\lambda,\chi}(\rho\beta) \in Ma\Lambda_n.
\]

Dividing both sides by \( a \), we have \( \rho\sigma \in M\Lambda_n \).

**Lemma 6.11**

For any character \( \chi \) of \( \Delta \), \( \text{char}(C_{\infty}(\chi)) \) divides \( \text{char}\left(E_{\infty}(\chi)/(E_{\infty}(\chi))\right) \) i.e.

\[
f_{\chi}|h_{\chi}
\]

**Proof**

If \( \chi = 1 \), it is clear that \( C_n(\chi) = (1/2^\Delta) \sum_{\delta \in \Delta} \chi(\delta)^{-1}\delta = C_n \). And \( \gamma \) denotes the topological generator of \( \Gamma \), it is easy to show that \( C_n/C_n^{\gamma^{-1}} = C_0 \). Since
the definition and the fact that $C_0 = 1$, we have $C_n = C_n^{\gamma - 1}$. Via the usual change $\gamma - 1 \mapsto X$, $Z_p[[\Gamma]] \cong Z_p[[X]]$ in Iwasawa theory, $C_n$ is regarded as a $Z_p[[X]]$-module. The above imply that

$$C_n = (X)C_n$$

where $(X)$ is the maximal ideal of local ring, especially it is Jacobson’s radical. By the Nakayama’s lemma, we have $C_n = 0$. Therefore $C_\infty = 0$ and $f_\chi$ is a unit.

Assume that $\chi \neq 1$. Remark that $E_n(\chi)$ is generated by $g_{A/q^{n+1}I,1}(\chi) = e(\chi)g_{A/q^nI,1}$. Let us consider an ideal of $\Lambda$ which have finite index in $C_n(\chi)$, satisfying the conditions in Lemma 6.9. Moreover take another element $c_{k+1}$ in $C_n(\chi)$. (For example, $c_{k+1} = 0$).

There exists an elements $\eta \in C$ such that $\Lambda_n/\eta \Lambda_n$ is finite, and for there exists a homomorphism $\theta_{n,\eta}$ over $\Lambda$ such that $\theta_{n,\eta}(E_n(\chi)) = \eta h_\chi \Lambda_n$. Here without loss of generality, we assume that $\theta_{n,\eta}$ satisfies

$$\theta_{n,\eta}(g_{A/q^{n+1}I,1}(\chi)) = \eta h_\chi.$$

Let $M \in \mathbb{Z}$ be a power of $p$ large enough. From now we use Theorem 4.4 to construct $\lambda_i$ satisfying the following conditions each $i$ with $1 \leq i \leq k+1$,

1. $\lambda_i \in \mathfrak{c}_i$,
2. the pairing $(\lambda'_i, \psi_{\lambda'_i}) \in \Psi_M$ of prime $\lambda_i$ of $F$ below $\lambda'_i$ and continuous surjective homomorphism $\psi_{\lambda'_i}$ which $\lambda'_i$ split completely in $K_n/F$,
3. If $i = 1$ then $\eta h_\chi\kappa(\lambda'_1, \psi_{\lambda'_1}) = \eta h_\chi$
4. If $2 \leq i \leq k+1$ then $f_{i-1}\eta h_\chi\kappa(\lambda'_i, \psi_{\lambda'_i}) = \eta h_\chi\kappa(\lambda'_{i-1}, \psi_{\lambda'_{i-1}})$

where $(\lambda'_i, \psi_{\lambda'_i}) = \{(\lambda'_1, \psi_{\lambda'_1}), \ldots, (\lambda'_i, \psi_{\lambda'_i})\}$.

First we will show to choose $\lambda_i$ by induction about $i$. When $i = 1$, let $\iota : E_n(\chi)/E_n(\chi)^M \rightarrow E_n(\chi)/E_n(\chi)^M$ be a natural embedding. Applying Theorem 4.4 to the case when $\mathfrak{c} = \mathfrak{c}_1$, $W = (E/E^M)(\chi)$ and $\varphi$ as the composition of

$$W \xrightarrow{\iota} E(\chi)/E(\chi)^M \xrightarrow{\theta_{n,\eta}} \Lambda_n/\Lambda_n \xrightarrow{\cong} \mathbb{Z}/\mathbb{Z}[\text{Gal}(K/F)].$$
Then there exist a prime $\lambda_1 \in \mathfrak{c}$ and a pairing $(\lambda'_1, \psi_{\lambda'_1})$ where $\lambda'_1$ is below to $\lambda_1$. It is clear from the choice of $\lambda_1$ that (1) and (2) are satisfied. We will prove that (3) also holds. Theorem 4.3 and 4.4 imply

$$\overline{\nu_{\lambda_1, \chi}(\kappa(\lambda'_1, \psi_{\lambda'_1}))}(\chi) = e(\chi)[\kappa(\lambda'_1, \psi_{\lambda'_1})]_{\lambda_1}$$

$$= e(\chi)\psi_{\lambda'_1}(\kappa_0)$$

$$= \varphi(\kappa_0)\lambda_1(\chi)$$

$$= \theta_{n, \eta}(g_0)\lambda_1(\chi)$$

$$= \eta \nu_{\lambda_1, \chi}(\lambda_1) = \eta h_\chi(\lambda_1).$$

So $\overline{\nu_{\lambda_1, \chi}(\kappa(\lambda'_1, \psi_{\lambda'_1}))} = \eta h_\chi$.

Continue this induction process, assume that we can choose $\lambda_1, \ldots, \lambda_{i-1}$.

Put $\left( v_{i-1}' \psi_{v_{i-1}'} \right) = \{(\lambda'_1, \psi_{\lambda'_1}), \ldots, (\lambda'_{i-1}, \psi_{\lambda'_{i-1}})\}$. Let $W_i$ be a sub $\Lambda_n$ module in $K_n^m/(K_n^m)^M$ generated by $e(\chi)\kappa(v_{i-1}', \psi_{v_{i-1}'})$. It follow from Lemma 6.10 that there exists a map $\varphi_i : W_i \to \Lambda_n/M\Lambda_n$ such that

$$f_{i-1}\varphi_i(e(\chi)\kappa(v_{i-1}', \psi_{v_{i-1}'}) = \eta \nu_{\lambda_{i-1}, \chi}(\kappa(v_{i-1}', \psi_{v_{i-1}'})).$$

Applying Theorem 4.4 with $c = c_i, W = W_i, \varphi = e(\chi)\varphi_i$, there exists a prime $\lambda_i \in \mathfrak{c}$ and a pairing $(\lambda'_i, \psi_{\lambda'_i})$ where $\lambda'_i$ is below to $\lambda_i$. We will show that (4) holds. Similarly when $i = 1$, we can calculate

$$f_{i-1}\overline{\nu_{\lambda_i, \chi}(\kappa(v_{i}', \psi_{v_i}')) = f_{i-1}e(\chi)[\kappa(v_{i}', \psi_{v_i}')]_{\lambda_i}$$

$$= f_{i-1}\psi_{\lambda'_i}(e(\chi)\kappa(v_{i-1}', \psi_{v_{i-1}'})$$

$$= f_{i-1}\varphi_i(e(\chi)\kappa(v_{i-1}', \psi_{v_{i-1}'})\lambda_i(\chi)$$

$$= \eta \nu_{\lambda_{i-1}, \chi}(\kappa(v_{i-1}', \psi_{v_{i-1}'})\lambda_i(\chi).$$

Therefore we have $f_{i-1}\overline{\nu_{\lambda_i, \chi}(\kappa(v_{i}', \psi_{v_i}')) = \eta \nu_{\lambda_{i-1}, \chi}(\kappa(v_{i-1}', \psi_{v_{i-1}'})}$. From the above we construct $\lambda_i$ inductively.
What we have to prove is that \( f' \) divides \( h' \). It is easy to calculate that

\[
\eta^{k+1}h' = \eta^k \nu_{\lambda_1, \chi}(\kappa(v_1', \psi v_1'))
\]

\[
= \eta^{k-1}f_1 \nu_{\lambda_2, \chi}(\kappa(v_2', \psi v_2'))
\]

\[
= \ldots
\]

\[
= \left( \prod_{j=1}^k f_j \right) \nu_{\lambda_{k+1}, \chi}(\kappa(v_{k+1}', \psi v_{k+1}'))
\]

so \( f' = \prod_{j=1}^k f_j \) divides \( \eta^{k+1}h' \). If \( f' \) does not divide \( h' \), \( f' \) divides \( \eta^{k+1} \). But Lemma 6.9 implies that there exists \( \eta' \in \mathcal{C} \) prime to \( \eta \). Do the same discussion for \( \eta' \), \( f' \) have to divide \( \eta'^{k+1} \). It is contradiction. As a consequence, \( f' \) divides \( h' \).

**Proof (of Theorem 6.6)**

It is easy to show the theorem using Lemma 6.8, 6.11, and Proposition 6.7.

Put

\[
f := \prod_{\chi} f_{\chi} = \prod_{\chi} \text{char}(C_{\infty}(\chi))
\]

\[
h := \prod_{\chi} h_{\chi} = \prod_{\chi} \left( \text{char}(E_{\infty}(\chi)/(E_{\infty}(\chi))) \right).
\]

For any \( m \geq 0 \), we have

\[
\sharp(\Lambda_m/f \Lambda_m) \approx \prod_{\chi} \sharp(\Lambda_{m}/f_{\chi} \Lambda_{m}) \approx \prod_{\chi} \sharp(C_{m}(\chi)) = \sharp(C_m)
\]

\[
\sharp(\Lambda_m/h \Lambda_m) \approx \prod_{\chi} \sharp(\Lambda_{m}/h_{\chi} \Lambda_{m}) \approx \prod_{\chi} \left[ \text{char}(E_{m}(\chi)/(E_{m}(\chi))) = \frac{[E_m : E_n]}{[E_{m}(\chi) : E_{m}(\chi)]} \right]
\]

Proposition 6.7 implies that \( \sharp(\Lambda_{m}/f \Lambda_{m}) \approx \sharp(\Lambda_{m}/h \Lambda_{m}) \). Lemma 6.11 implies that \( f|g \). Since \( f, g \in \Lambda \), we have \( f \Lambda = g \Lambda \) by using Lemma 6.8. Using Lemma 6.11 once more, we have \( f_{\chi} \Lambda = g_{\chi} \Lambda \) for any \( \chi \). Therefore we have

\[
\text{char} \left( C_{\infty}(\chi) \right) = \text{char} \left( E_{\infty}(\chi)/(E_{\infty}(\chi)) \right)
\]

This is what we need to prove.
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