1 Introduction

The inflationary scenario allows to solve the horizon and flatness problems and predicts the generation of density (scalar) perturbations and of gravitational waves (tensor perturbations). Due to the Sachs-Wolfe effect those perturbations can be observed in the cosmic microwave background (CMB). The COBE satellite has measured these anisotropies for the first time. Forthcoming high precision observations, especially the MAP and PLANCK satellites, will determine the temperature correlations with a precision of a few percent. Therefore predictions from inflationary models should be made on the few percent level as well.

Almost all (analytical) predictions for perturbation spectra from inflation rely on the slow roll approximation. So far, no systematic, quantitative analysis on the error of the slow roll approximation has been performed, neither for the power spectra, nor for the temperature two-point correlations. We consider this work as a first step in such an error analysis. We compare the results from slow roll inflation, i.e. $a(t) \sim \exp(Ht)$, $H \sim \text{const}$, with the exact solutions from power law inflation, i.e. $a(t) \sim t^p$, $p = \text{const}$.

In a previous work, in response to contrary claims, we showed that the contribution of tensor perturbations with respect to scalar perturbations to the CMB anisotropies is small for the equation of state $\rho \approx -p$ during inflation.
However, inflation occurs already if $\rho + 3p < 0$, which does not necessarily lead to slow roll inflation.

Power law inflation provides exact solutions for the time evolution of cosmological perturbations and inflation can occur although the slow roll conditions are violated. It is therefore interesting to investigate the difference of the exact predictions with the slow roll predictions.

Here, we concentrate on the so-called consistency check, which relates the scalar and tensor CMB quadrupole.

2 Observables

The observable quantities are the temperature two-point correlation functions, respectively their momenta $C^S_{l},$ $C^T_{l}$. We define them through:

$$
\langle \left( \frac{\delta T}{T} \right)^S_{(\varepsilon_1)} \left( \frac{\delta T}{T} \right)^S_{(\varepsilon_2)} \rangle = \frac{1}{4\pi} \sum_l (2l + 1) C^S_{l} P_l (\cos \delta),
$$

where $\varepsilon_a$ denotes the beam direction. The temperature fluctuations $\delta T/T(\varepsilon)$ are related to the primordial cosmological perturbations by the Sachs-Wolfe effect. We evaluated the scalar fluctuations for purely adiabatic perturbations (i.e. no entropy perturbations) and for large scale modes, such that the sudden decoupling approximation applies. We also assumed that the perfect fluid approximation holds.

If the perturbations are of quantum-mechanical origin then the power spectra of the Bardeen potential and of the tensor fluctuations are respectively given by $P_{\Phi}(k) = A_S k^{n_S - 4}$ and $P_{h}(k) = A_T k^{n_T - 3}$. They are related to the $C_l$'s by

$$
C^S_{l} = \frac{4\pi}{9} \int_{0}^{\infty} \frac{dk}{k} [j_l(k\eta_0)]^2 A_S k^{n_S - 1},
$$

$$
C^T_{l} = \frac{9\pi}{4} (l - 1)(l + 1)(l + 2) \int_{0}^{\infty} \frac{dk}{k} I^2_l(k\eta_0) A_T k^{n_T},
$$

where,

$$
I_l(k\eta_0) = \int_{k\eta_0}^{k\eta_0} \frac{j_2(x) j_l(k\eta_0 - x)}{x(k\eta_0 - x)^2} dx,
$$

$j_l$ being a spherical Bessel function and $\eta_0$ the time of reception of CMB photons.
3 Slow roll inflation

Slow roll inflation (at leading order) is controlled by three parameters:

\[ \epsilon \equiv 3\dot{\varphi}^2/(2\dot{\varphi}^2/2 + V)^{-1} = -\dot{H}/H^2, \]  
(5)

\[ \delta \equiv -\ddot{\varphi}/(H\dot{\varphi}) = -\dot{\epsilon}/(2H\epsilon) + \epsilon, \]  
(6)

\[ \xi \equiv (\dot{\epsilon} - \dot{\delta})/H. \]  
(7)

The universe is inflating as soon as \( \epsilon < 1 \). The slow roll approximation holds true for \( \epsilon \ll 1, \delta \ll 1, \) and \( \xi = \mathcal{O}(\epsilon^2, \delta^2, \epsilon\delta) \). There are important examples that do not satisfy all three conditions, e.g. inflation with a Coleman-Weinberg potential.

At the leading order we find

\[ A_S k^{n_S - 1} = \frac{9}{25\pi\epsilon} \frac{H^2}{m_{Pl}^2} \left| k = aH \right|, \quad A_T k^{n_T} = \frac{16}{\pi} \frac{H^2}{m_{Pl}^2} \left| k = aH \right|. \]  
(8)

With \( n_T \approx -2\epsilon \) the so-called consistency check follows:

\[ C_T^2/C_S^2 \approx -6.93n_T. \]  
(9)

The integrations in (3) have been performed numerically at \( n_T = 0 \) and expressions in Eq. (2) have been evaluated for \( n_S = 1 \).

For \( \epsilon \to 0 \) (no roll), \( A_S \) diverges, which means that linear perturbation theory does not apply for very small values of \( \epsilon \). Thus, nowhere in the parameter space \( (\epsilon, \delta, \xi) \) the slow roll approximation is exact. Including higher orders in \( \epsilon, \delta, \xi \) and/or introducing new parameters does not change this conclusion.

4 Power law inflation

This special model is equivalent to scalar field inflation with the potential \( V(\varphi) = V_0 \exp(\pm 4\sqrt{m_{Pl}\varphi}/\sqrt{p}) \). For power law inflation \( \epsilon = \delta = 1/p, \xi = 0 \). Thus, \( p > 1 \) is sufficient for inflation. In the limit \( p \gg 1 \) power law inflation and slow roll inflation with \( \epsilon = \delta \) agree at the leading order in \( \epsilon \).

The exact evolution of the cosmological fluctuations for power law inflation \([a(\eta) = l_0|\eta|^{p+1}; p = (\beta + 1)/(\beta + 2)]\) gives rise to

\[ A_S = \frac{l_{Pl}^3}{l_0^3} \frac{9}{25\pi\epsilon} f(\beta), \quad n_S = 2\beta + 5 \]  
(10)

with \( f(\beta) = 4\pi/[2^{\beta+2} \cos(\beta\pi)\Gamma(\beta + 3/2)]^2 \) and

\[ A_T = \frac{l_{Pl}^3}{l_0^3} \frac{16}{\pi} f(\beta), \quad n_T = 2\beta + 4. \]  
(11)
Now, the ratio of tensor to scalar contributions to the temperature fluctuations is given by:
\[ \frac{C_T}{C_S} \approx -6.93F(n_T)n_T/(1 - n_T/2), \]  
where \( F(n_T) \) denotes a numerical integration, which is normalized such that \( F(0) = 1 \). We have used the relation \( \epsilon = -n_T/(2 - n_T) \) and have put all other dependence on \( n_T \) and on \( n_S = n_T + 1 \) into the function \( F(n_T) \).

5 Discussion and conclusion

The exact result and the slow roll result differ by a factor \( F(n_T)/(1 - n_T/2) \). For inflation \( p > 1 \), which translates into \( 0 > n_T = -2/(p - 1) > -\infty \) we find that the slow roll result might differ considerably, e.g. for \( p = 2 \) the error is a factor \( F(-2)/2 \approx 0.34 \). Only for \( p > 100 \), i.e. for \( 1 > n_S > 0.99 \), the error of the slow roll approximation is less than 1%.

For small numbers \( l \) the cosmic variance introduces an uncertainty of \( \Delta C_l = \sqrt{2/2l + 1}C_l \), for the quadrupole \( \Delta C_2 = 0.63C_2 \). Thus, even at the largest scales the error from the slow roll approximation might be as big as the cosmic variance. For intermediate values \( l < 30 \) the error from the slow roll approximation is even more important.

Our main conclusion is that the consistency check, Eq. (9), is not valid for an arbitrary inflationary model. When the slow roll approximation does not apply, as for power law inflation, we expect significant modifications.

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