A measurable counterpart to the optical concept of an object

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The traditional optical concept for the object does not provide an experimental feasibility to speak for itself, due to the fact that no measuring instrument catches up with the fluctuation of light fields. Using the theory of coherence, we give it a measurable counterpart, and thus to give a satisfactory explanation of what the concept has said. To our knowledge it is the first time to valuate an object in optics in the terms of observable quantities. As a useful example, the applicability to obtain the full knowledge of an object under the partial coherent illuminations is suggested. This work hit the thought which had been advanced by E. Wolf that one measures the correlation function rather than the optical fields themselves. The suggested applicability is a supplementary to the recently found solution to the determination of phases of the diffracted beam.

I. INTRODUCTION

When one talks about an object in optics, he would refer to the effects which it exerts on the illuminating light fields. To valuate these effects one associates the object to a complex amplitude $A(r, \omega)e^{i\theta(r, \omega)}$ with a term like optical complex transmittance (OCT) or optical complex reflectance (OCR). In standard textbooks (e.g. in Ref. [1]) it has traditionally been defined as when the object was incident by a monochromatic electromagnetic wave with a frequency of $\omega$, it would modulate the transmitted/reflected wave front at the position of $P$ specified by a vector $\rho$ in the way of multiplying $A(\rho, \omega)$ times to its amplitude while applying the phase delay of $\theta(\rho, \omega)$ to its argument.

However when one investigates the problem in terms of observable quantities [2], the concept just mentioned is questionable. On one hand, the time-independent sources of monochromatic wave in the optical region of the electromagnetic spectrum are never encountered in real world, and all optical fields, for both their amplitudes and their phases have always been undergoing random fluctuations [3]. Such field fluctuations, comparing to the respond speed of the instrument, are too rapid to be measured, so one can never compare the value of the optical fields just before and just after it have transmitted through/reflected by the objects. On the other hand, what the quantity of being directly measured is the intensity, which is an averaged square modulus of the light fields over a time span approximately as long as the reciprocal of an instrument’s cut-off frequency. During this time span the phase knowledge of the fluctuating optical fields must have been washed out due to the time integral performed by the measuring instrument, so one is never able to obtain the phase knowledge of the fields before and after it have transmitted through/reflected by the objects. For these reasons the present defined concept has not automatically provided an applicable environment to speak for itself.

In this letter, we justify the original optical concept of object by providing a measurable counterpart to it. Meanwhile, as a useful example, we suggests that when the illuminating sources are partially coherent, the acquisition to the knowledge of $A(\rho, \omega)e^{i\theta(\rho, \omega)}$ can be achieved. It is a supplementary to the recently found solution to the determination of phases of the diffracted beam [2].

The paper is organized as follows. In Sec. III we go over the joint intensity and show it is a measurable complex quantity. Then, in Sec. III we theoretically approve that the cross-spectral density function is the joint intensity when an optical fields was being filtered at the frequency of $\omega$. After that, the effects that an object exerts on the cross-spectral density will be presented in Sec. IV. It is the physical significance this paper gives. Finally in Sec. V we give a measurable counterpart to justify the original optical concept of the object by combing the three conclusions drawn in Sec. III – Sec. IV. In addition, an applicability is also suggest in the last section.

II. JOINT INTENSITY IS A MEASURABLE QUANTITY

For simplicity, we only discuss the object in terms of OCT, although the discussions for OCR are straightforward. The notation relating to the following discussion is given by Fig. II. We begin by going over the mutual coherence function [4] defined as

$$\Gamma(\rho_1, \rho_2, \tau) \equiv \langle V^*(\rho_1, t)V(\rho_2, t+\tau) \rangle. \tag{1}$$
In which $V(\mathbf{\rho}, t)$ is known as the complex analytic function $[8]$ (also in Ref. [8] page 92) representing a fluctuating field at a point $P$ specified by a vector $\mathbf{\rho}$ at time $t$, the asterisk is a complex conjugate operator, and the angular brackets denote the ensemble average. Its case for $\tau = 0$,

$$J(\mathbf{\rho}_1, \mathbf{\rho}_2) \equiv \Gamma(\mathbf{\rho}_1, \mathbf{\rho}_2, 0),$$

is being called joint intensity between $P_1$ and $P_2$. When $\mathbf{\rho}_1 = \mathbf{\rho}_2 = \mathbf{\rho}$, the expression

$$I(\mathbf{\rho}) \equiv J(\mathbf{\rho}, \mathbf{\rho}),$$

defines a quantity which was directly measurable, formally named as the intensity at a position of $P$. After a lengthy but straightforward calculation, the diffraction patterns $I(\mathbf{\rho})$ in in screen $\Sigma_2$, can be carried out as $[7]$:

$$I(\mathbf{\rho}) = \left[ I^{(1)}(\mathbf{\rho}) + I^{(2)}(\mathbf{\rho}) \right]$$

$$\times \left\{ 1 + \frac{2\sqrt{I^{(1)}(\mathbf{\rho})}I^{(2)}(\mathbf{\rho})}{I^{(1)}(\mathbf{\rho}) + I^{(2)}(\mathbf{\rho})} \right\}$$

$$\times |J(\mathbf{\rho}_1, \mathbf{\rho}_2)| \cos |\kappa + \delta(\mathbf{\rho})| \}$$

In which

$$\kappa = \arg J(\mathbf{\rho}_1, \mathbf{\rho}_2)$$

is the argument of $J(\mathbf{\rho}_1, \mathbf{\rho}_2)$, $\delta(\mathbf{\rho}) = \frac{2\pi}{\bar{\lambda}} |(\mathbf{\rho} - \mathbf{\rho}_2) - (\mathbf{\rho} - \mathbf{\rho}_1)|$ is the phase difference at $\mathbf{P}$ resulted from the difference of optical distances form $P$ to $P_2$ and $P$ to $P_1$. The latter can be simplified as

$$\delta(\mathbf{\rho}) = \frac{2\pi \bar{\lambda}}{\lambda d} \mathbf{\rho}$$

under the paraxial approximation, and where $\bar{\lambda}$ is the central wavelength for the light filed under the narrow band assumption. In Eq. $[3]$ $I^{(k)}(\mathbf{\rho})(k = 1, 2)$, are intensity distributions on $\Sigma_2$ when only $P_k(k = 1, 2)$ was opened. On the scale of $\frac{\lambda d}{D}$, $I^{(k)}(\mathbf{\rho})(k = 1, 2)$ varied so slowly comparing to cosine that they can be seen as constance in the terms of

$$I^{(k)}(\mathbf{\rho}) = I^{(k)}(k = 1, 2).$$

On substituting Eqs. $[4]$ $[7]$ From Eq. $[4]$ one sees the observed pattern in in screen $\Sigma_2$ has a sinusoidal profile in the form of

$$I(\mathbf{\rho}) = \left[ I^{(1)} + I^{(2)} \right]$$

$$\times \left\{ 1 + \frac{2\sqrt{I^{(1)}I^{(2)}}}{I^{(1)} + I^{(2)}} \right\}$$

$$\times |J(\mathbf{\rho}_1, \mathbf{\rho}_2)| \cos \left[ \kappa + \frac{2\pi D}{\lambda d} \mathbf{\rho} \right].$$

(8)

Eq. $[8]$ shows that the contrast $\gamma$ of the observed pattern is:

$$\gamma = \frac{2\sqrt{I^{(1)}I^{(2)}}}{I^{(1)} + I^{(2)}} |J(\mathbf{\rho}_1, \mathbf{\rho}_2)|,$$

as defined by

$$\gamma \equiv \frac{I_{\max}(\mathbf{\rho}) - I_{\min}(\mathbf{\rho})}{I_{\max}(\mathbf{\rho}) + I_{\min}(\mathbf{\rho})}.$$ (10)

Eq. $[10]$ also provides an experimental way to obtain the contrast of the diffracted patterns. This fact combined with Eq. $[4]$ tells one that the modules $|J(\mathbf{\rho}_1, \mathbf{\rho}_2)|$ is obtainable because $\gamma, I^{(k)}(\mathbf{\rho}_k), (k = 1, 2)$ in Eq. $[3]$ can all be determined by the experiment. Besides, from Eq. $[8]$ one can see that the argument of $J(\mathbf{\rho}_1, \mathbf{\rho}_2)$, namely $\kappa$, can also be determined by observing the pattern’s maxima position $P_m$ in a way of

$$\kappa = -\frac{2\pi D}{\lambda d} \mathbf{\rho}_m,$$ (11)

where $\mathbf{\rho}_m$ stands for the position of the pattern’s maxima position $P_m$. In a word, from Eqs. $[9]$ $[11]$ one can say that the joint intensity (Eq[2]) is a measurable quantity since all terms in the right side of equation,

$$J(\mathbf{\rho}_1, \mathbf{\rho}_2) = \gamma \frac{I^{(1)} + I^{(2)}}{2\sqrt{I^{(1)}I^{(2)}}} I(\mathbf{\rho}_1) I(\mathbf{\rho}_2) e^{j\kappa},$$ (12)

are obtainable. This is the first conclusion for latter reference. Although a special case when $I^{(1)} = I^{(2)}$ and $I(\mathbf{\rho}_1) = I(\mathbf{\rho}_2)$ had discussed to the measurability for the mutual coherence function in Ref. $[8]$, a more general case in form of Eq. $[12]$ is still needed for the later discussions.

III. CROSS-SPECTRAL DENSITY FUNCTION AND ITS RELATION TO THE JOINT INTENSITY

Suppose the field is statistically stationary, at least in the wide sense, the Fourier transform of Eq. $[11]$

$$W(\mathbf{\rho}_1, \mathbf{\rho}_2, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Gamma(\mathbf{\rho}_1, \mathbf{\rho}_2, \tau) e^{j\omega \tau} d\tau$$ (13)
is known as the cross-spectral density function \[6, 8\]. It can be expressed \[14, 16\] in the coherent-mood representation \[17\] (also see Ref. \[8\], page 214, and Ref. \[18\]):

\[
W(\mathbf{p}_1, \mathbf{p}_2, \omega) = \sum_n \lambda_n(\omega) \phi_n^*(\mathbf{p}_1, \omega) \phi_n(\mathbf{p}_2, \omega); \tag{14}
\]

in which \(\lambda_n(\omega) > 0\) may be shown to be the eigenvalues and \(\phi(\mathbf{p}, \omega)\) the eigenfunctions of the homogeneous Fredholm integral equation:

\[
\int_D W(\mathbf{p}_1, \mathbf{p}_2, \omega) \phi_n(\mathbf{p}_1, \omega) d^3 \mathbf{p}_1 = \lambda_n(\omega) \phi_n(\mathbf{p}_2, \omega), \tag{15}
\]

Where \(D\) is the domain containing all pairs of \(P_1\) and \(P_2\). Its eigenfunctions may be taken to form an orthonormal set as

\[
\int_D \phi_n^*(\mathbf{p}, \omega) \phi_n(\mathbf{p}, \omega) d^3 \mathbf{p} = \delta_{mn}, \tag{16}
\]

\(\delta_{mn}\) being the Kronecker symbol (\(\delta_{mn} = 1\) when \(m = n\), \(\delta_{mn} = 0\) when \(m \neq n\)).

Let us consider a situation for a beam which has passed through a narrow filter with central frequency of \(\omega\), then the fluctuating fields at the point of \(P\) may be seen as a random process among an ensemble of \(\{U(\mathbf{p}, \omega)\}\) \[8\], in which \(U(\mathbf{p}, \omega)\) is a frequency dependent sample function. Let \(U(\mathbf{p}, \omega)\) to be a monochromatic realization \[8\] of the ensemble, and the fields at the point of \(P_1\) and \(P_2\) can be constructed as

\[
U(\mathbf{p}_1, \omega) = \sum_n a_n(\omega) \phi_n(\mathbf{p}_1, \omega), \tag{17}
\]

and

\[
U(\mathbf{p}_2, \omega) = \sum_n a_n(\omega) \phi_n(\mathbf{p}_2, \omega); \tag{18}
\]

due to the orthonormal properties of \(\phi_{m,n}(\mathbf{p}, \omega)\). The \(a_{m,n}(\omega)\) are random coefficients satisfied

\[
\langle a_n^*(\omega) a_n(\omega) \rangle = \alpha_n \delta_{mn}. \tag{19}
\]

Now considering the joint intensity:

\[
J^{(\omega)}(\mathbf{p}_1, \mathbf{p}_2) \equiv \langle U^*(\mathbf{p}_1, \omega) U(\mathbf{p}_2, \omega) \rangle \tag{20}
\]

(superscript notes the beam has passed through a narrow filter with central frequency of \(\omega\)). By referring to Eqs. \[1\] and \[2\] and Eqs. \[17\] and \[18\] one has

\[
J^{(\omega)}(\mathbf{p}_1, \mathbf{p}_2) = \left[ \sum_m a_m^*(\omega) \phi_m(\mathbf{p}_1, \omega) \right] \times \left[ \sum_n a_n(\omega) \phi_n(\mathbf{p}_2, \omega) \right]
\]

\[
= \sum_m \sum_n \langle a_m^*(\omega) a_n(\omega) \rangle \phi_m(\mathbf{p}_1, \omega) \phi_n(\mathbf{p}_2, \omega).
\]

On substituting Eq. \[19\] from Eq. \[21\] and by letting

\[
\alpha_n = \sqrt{\lambda_n}, \tag{22}
\]

one can derive joint intensity of Eq. \[21\] out to be

\[
J^{(\omega)}(\mathbf{p}_1, \mathbf{p}_2) = \sum_n \lambda_n \phi_n(\mathbf{p}_1, \omega) \phi_n(\mathbf{p}_2, \omega). \tag{23}
\]

Comparing Eq. \[23\] with Eq. \[13\] one may get

\[
J^{(\omega)}(\mathbf{p}_1, \mathbf{p}_2) = W(\mathbf{p}_1, \mathbf{p}_2, \omega). \tag{24}
\]

To this end we are able to give a physical meaning to Eq. \[13\]. That is the cross-spectral density function \(W(\mathbf{p}_1, \mathbf{p}_2, \omega)\) is the joint intensity (Eq. \[3\] when an optical fields was being filtered at the central frequency of \(\omega\). This is the second conclusion of this paper \[12\]. Similar to Eq. \[3\] the equal position case of Eq. \[24\]

\[
I(\mathbf{p}, \omega) = J^{(\omega)}(\mathbf{p}, \mathbf{p}) \tag{25}
\]

defines the intensity at point \(P\) after the light was thus filtered.

IV. THE MODULATION TO THE CROSS-SPECTRAL DENSITY FUNCTION BY AN OBJECT

In free space, the mutual coherence function (Eq. \[1\]) satisfies the wave equations \[19\] (also known as Wolf equation \[20\]),

\[
\nabla_j^2 \Gamma - \frac{1}{c^2} \frac{\partial^2 \Gamma}{\partial \tau^2} = 0, \quad j = 1, 2; \tag{26}
\]

where \(\nabla_j^2\) is the Laplacian operator with respect to \(\mathbf{p}_j\). By substitute the inverse form of Eq. \[18\]

\[
\Gamma(\mathbf{p}_1, \mathbf{p}_2, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(\mathbf{p}_1, \mathbf{p}_2, \omega) e^{-j\omega \tau} d\omega, \tag{27}
\]

into Eq. \[26\] one obtains

\[
\nabla_j^2 W(\mathbf{p}_1, \mathbf{p}_2, \omega) + k^2 W(\mathbf{p}_1, \mathbf{p}_2, \omega) = 0. \tag{28}
\]

On substituting Eq. \[14\] from Eq. \[28\] with respect to \(j = 2\), multiplying the resulted equation by \(\phi_m(\mathbf{p}_2, \omega)\), integrating its both sides with respect to the variable \(\mathbf{p}_2\) over the domain \(D\), and using the orthonormality relation Eq. \[16\] one obtains

\[
\nabla_j^2 \phi_n(\mathbf{p}_1, \omega) + k^2 \phi_n(\mathbf{p}_1, \omega) = 0. \tag{29}
\]

Now we can see that time-independent function \(\phi_n(\mathbf{p}_1, \omega)\) is a monochromatic electromagnetic wave which had been traditionally used to define the concept of complex objects because it obeys Helmholtz equation (Eq. \[24\]). Now, attaching an object at \(P_1\) and applying the traditional concept, Eq. \[17\] changes into

\[
U'(\mathbf{p}_1, \omega) = \sum_m a_m(\omega) [A(\mathbf{p}_1, \omega) e^{j\theta(\mathbf{p}_1, \omega)} . \phi_m(\mathbf{p}_1, \omega)].
\]

\[
(30)
\]
On substituting Eq. 30 from Eq. 20 for $U(p_1,\omega)$ and by going over the deriving procedure from Eq. 17 to Eq. 24 one can get the cross-spectral density function for this situation as

$$W'(p_1, p_2, \omega) = \langle U^{*}(p_1) U_2(p_2) \rangle = A(p_1, \omega) e^{j\theta(p_1)} W(p_1, p_2, \omega).$$

(31)

Eq. 31 states that When placing a complex object immediate after point $P_1$, it changes the cross-spectral density function of $W(p_1, p_2, \omega)$ into $W'(p_1, p_2, \omega)$. This is the third conclusion and it is also the physical significance of this paper. Here we note again that Eq. 30 is the bridge between the gap of the traditional concept and a measurable one.

V. A MEASURABLE COUNTERPART TO THE OPTICAL CONCEPT OF THE OBJECT

Combining the three conclusions drawn in Sec I - IV we can give a counterpart to the optical concept of an object from the perspective of measurable quantities: One can associate an object to a complex amplitude $A(p, \omega) e^{j\theta(p,\omega)}$, it means when an object is placed at the position of $P_1$ in the Youngs interference experiment setup shown by Fig. 1, it modules the measurable cross-spectral density function $W(p_1, p_2, \omega)$ in the way of multiplying $A(p_1, \omega)$ times to its amplitude while applying the phase delay of $\theta(p_1, \omega)$ to its argument.

Meanwhile, through the above discussion, the applicability to obtain the complex transmittance is suggested by the experimental setup of Fig. 1. Filtering the incident partial coherent light by a narrowband filter with central frequency of $\omega$, then according to the first two conclusion of this paper, the cross-spectral density function of $W(p_1, p_2, \omega)$ and $W'(p_1, p_2, \omega)$, which equals to the joint intensity of Eq. 2 before and after the complex object was attached to the point $P_1$, can be measured according to Eq. 12 in a way of:

$$W(p_1, p_2, \omega) = \left[ I^{(1)}(\omega) + I^{(2)}(\omega) \right]$$

$$\times \frac{\sqrt{I^{(1)}(\omega) I^{(2)}(\omega)}}{2 \sqrt{I^{(1)}(\omega) I^{(2)}(\omega)}} e^{j\kappa(\omega)},$$

(32)

and

$$W'(p_1, p_2, \omega) = \left[ I^{(1)}(\omega) + I^{(2)}(\omega) \right]$$

$$\times \frac{\sqrt{I^{(1)}(\omega) I^{(2)}(\omega)}}{2 \sqrt{I^{(1)}(\omega) I^{(2)}(\omega)}} e^{j\kappa(\omega)}.$$ 

(33)

In which the superscripts have the same meaning which we have already introduced when describing Eq. 9 and the prime "$'$" denotes the values after the object being placed. By such information, one can obtain $A(p_1, \omega) e^{j\theta(p_1, \omega)}$ from

$$A(p_1, \omega) = \left| \frac{W'(p_1, p_2, \omega)}{W(p_1, p_2, \omega)} \right|,$$

(34)

and

$$\theta(p_1, \omega) = \kappa'(\omega) - \kappa(\omega).$$

(35)

In summary, we provide a measurable counterpart to the traditional imaginarily defined optical concept for the object. Meanwhile the applicability to retrieval the complex knowledge of the object is suggested. This research hit the thought advanced by Wolf that one measures the correlation function rather than the optical fields themselves. 2 21 2 21. The suggested applicability can be seen as a supplementary to the recently found solution to the determination of phases of the diffracted beam 3.

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$U(r, \omega)$ is not a Fourier component of the fluctuating field but is the space-dependent part of the statistical ensemble $\{V(r, t) = U(r, \omega) e^{-j\omega t}\}$ of monochromatic realizations, all of frequency $\omega$. That is why $U(r, \omega)$ can be bounded. To appreciate such realizations, one can refer to Ref. [10]. Also, in Ref. [11], authors referred to $U(r, \omega)$ as the associated field.

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