From D3-branes to Lifshitz spacetimes

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Abstract
We present a simple embedding of a \( z = 2 \) Lifshitz spacetime into type IIB supergravity. This is obtained by considering a stack of D3-branes in type IIB supergravity and deforming the world-volume by a plane wave. The plane wave is sourced by the type IIB axion. The superposition of the plane wave and the D3-branes is 1/4 BPS. The near horizon geometry of this configuration is a five-dimensional \( z = 0 \) Schrödinger spacetime times a 5-sphere. This geometry is also 1/4 BPS. Upon compactification along the direction in which the wave is traveling the five-dimensional \( z = 0 \) Schrödinger spacetime reduces to a four-dimensional \( z = 2 \) Lifshitz spacetime. The compactification is such that the circle is small for weakly coupled type IIB string theory. This reduction breaks the supersymmetries. Further, we propose a general method to construct analytic \( z = 2 \) Lifshitz black brane solutions. The method is based on deforming AdS\(_5\) black strings by an axion wave and reducing to four dimensions. We illustrate this method with an example.

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1. Introduction

The Lifshitz symmetry group is one of the possible symmetry groups that one encounters at quantum critical points in condensed matter systems. Typically such systems are strongly coupled and the idea is more or less that, much like in AdS/CFT, the universal behavior of a strongly coupled field theory near a quantum critical point can be studied holographically by means of a suitably chosen gravitational background such as an asymptotically Lifshitz spacetime. The subject of Lifshitz holography was initiated by [1]. For more details about the basic holographic properties of Lifshitz spacetimes see [2].

Recently, a lot of progress has been made regarding the problem of embedding Lifshitz spacetimes into string theory [3–9]. We present a particularly simple way of embedding
a four-dimensional $z = 2$ Lifshitz spacetime into type IIB string theory. Our construction is inspired by [7] (and also appears in [5]) and works as follows. Take the bosonic part of type IIB supergravity and truncate the 2-forms. Reduce this theory over a 5-sphere of constant radius and subsequently perform a Scherk–Schwarz reduction in which the axion shift symmetry is gauged. The resulting four-dimensional theory admits $z = 2$ Lifshitz solutions. In this construction, the dilaton is a free constant and the axion is a linear function of the compactification circle.

This way of embedding Lifshitz into type IIB supergravity immediately points toward a D3-brane interpretation. We simply take an extremal D3-brane and add to it a solution that is sourced by the axion. This latter solution turns out to be a $1/2$ BPS plane wave. The intersection of the plane wave with the extremal D3-brane is $1/4$ BPS and its near horizon geometry is a five-dimensional $z = 0$ Schrödinger spacetime. Such a spacetime can be reduced to a four-dimensional $z = 2$ Lifshitz spacetime, see for example [4].

Once we have established such a D3-brane/axion wave interpretation of the Lifshitz spacetime we address the question of finding black deformations. We propose a general method to construct analytic $z = 2$ Lifshitz black brane solutions and illustrate it with a simple example.

2. D3-branes and plane waves

Consider type IIB supergravity and truncate away the fermions and the 2-forms. The equations of motion of the resulting theory are (we use the conventions of [10])

$$
R_{AB} = P_A P_B + P_B P_A + \frac{1}{6} F_{C_1...C_4} F^{C_1...C_4}_B, \quad (1)
$$

$$
\mathcal{D}_A P^A = \partial_A P^A + \Gamma^A_{AB} P^B - 2i Q_A P^A = 0, \quad (2)
$$

$$
\ast F_5 = F_5, \quad d F_5 = 0. \quad (3)
$$

By capital Latin indices we denote ten-dimensional indices. We denote by $\mathcal{D}_A$ the covariant derivative that contains besides the Lévi-Civitã connection (and later when we look at Killing spinors the Lorentz connection) also a $U(1)$ connection $Q_A$ which is the pullback of the Kähler connection 1-form of the scalar co-set space $SL(2, \mathbb{R})/SO(2)$. We choose the usual axion–dilaton parametrization of the scalar co-set manifold in which $P_A$ and $Q_A$ are

$$
P_A = \frac{1}{2} \partial_A \phi + \frac{i}{2} \epsilon^\phi \partial_A \chi, \quad (4)
$$

$$
Q_A = \frac{1}{2} \epsilon^\phi \partial_A \chi. \quad (5)
$$

The solution that we will be interested in describes a plane wave traveling along the world-volume of a stack of D3-branes and is given by

$$
\frac{d s^2}{H} = H^{-1/2}(2 dt du + G du^2 + dx^2 + dy^2) + H^{1/2}(dR^2 + R^2 d\Omega^2), \quad (6)
$$

$$
F_{\hat{\mu}_1...\hat{\mu}_5} = l^4 R^{-5/4} \epsilon_{\hat{\mu}_1...\hat{\mu}_5}, \quad F_{\hat{\alpha}_1...\hat{\alpha}_5} = l^4 \omega_{\hat{\alpha}_1...\hat{\alpha}_5}, \quad (7)
$$

$$
\chi = k u, \quad (8)
$$

$$
\phi = \phi_0, \quad (9)
$$
in which the functions $H$ and $G$ are

$$
H = 1 + \frac{l^4}{R^2}, \quad (10)
$$

2
\[ G = \frac{1}{4} e^{2\phi_0} k^2 \left( \frac{l^4}{R^2} - \frac{R^2}{3} \right). \]  

(11)

The parameters \( l \) and \( k \) relate to the D3-brane charge and the wave momentum, respectively. The 5-sphere is parametrized by the coordinates \( x^a = (\theta_1, \ldots, \theta_4, \phi) \) and \( \epsilon_{\hat{\mu}_1 \ldots \hat{\mu}_5} \) and \( \omega_{a_1 \ldots a_5} \) denote the volume forms on the five-dimensional non-compact spacetime parametrized by \( x^\hat{\mu} = (t, u, x, y, R) \) and the 5-sphere, respectively. The dilaton assumes a constant value \( \phi_0 \) and from now on we will write \( e^{\phi_0} \) as \( g_s \), the type IIB string coupling. When we set \( k \) equal to zero, we recover the well-known extremal D3-brane solution and when we set \( l \) equal to zero we obtain a plane wave spacetime. We note that it is not possible to transform away the string coupling \( g_s \) from the solution by a rescaling of the coordinates \( t \) and \( u \). This can be used to remove \( g_s \) from the metric but then it will reappear in the expression for the axion. The solution (6)–(9) is a special case of the solutions presented in [5] corresponding to constant dilaton and zero NSNS and RR 3-form field strengths.

The near horizon geometry is obtained by considering the solution near \( R = 0 \). Defining \( r = l^2 R^{-1} \) we find

\[ ds^2 = \frac{l^2}{r^2} \left[ \frac{1}{r^2} \left( 2 dt du + \frac{1}{4} g_s^2 k^2 r^2 dt^2 + dx^2 + dy^2 + dr^2 \right) + d\Omega_5^2 \right], \]  

(12)

\[ F_{\hat{\mu}_1 \ldots \hat{\mu}_5} = l^4 \epsilon_{\hat{\mu}_1 \ldots \hat{\mu}_5}, \quad F_{a_1 \ldots a_5} = l^4 \omega_{a_1 \ldots a_5}, \]  

(13)

\[ \chi = ku, \]  

(14)

\[ e^\phi = e^{\phi_0} = g_s. \]  

(15)

The metric describes the direct product of a five-dimensional \( z = 0 \) Schrödinger spacetime and the 5-sphere. This solution of type IIB supergravity has also been studied in [9]. In the near horizon geometry when the parameter \( k \) is nonzero it can be rescaled away by rescaling \( u \) and \( t \).

The symmetries preserved by this solution are translations in \( t, x, y \), rotation of \( x \) and \( y \) together with the following three transformations:

\[ r \to \lambda r, \quad \tilde{x} \to \lambda \tilde{x}, \quad t \to \lambda^2 t, \]  

(16)

\[ \tilde{x} \to \tilde{x} - \tilde{v} u, \quad t \to \tilde{v} \cdot \tilde{x} - \frac{1}{2} \tilde{v}^2 u, \]  

(17)

where \( \tilde{x} = (x, y) \) and \( \tilde{v} \) is a constant vector. The first of these transformations is a dilatation symmetry and the other two are boost transformations with respect to ‘\( u \)-velocities’. The translations in \( u \), that are usually considered part of the \( z = 0 \) Schrödinger algebra, are broken by the axion, though the violation is mild in the sense that the axion transforms into itself up to a symmetry of type IIB supergravity.

In terms of Killing vectors, the symmetries of the five-dimensional \( z = 0 \) Schrödinger spacetime are

\[ H = \frac{\partial}{\partial t}, \]  

(18)

\[ \tilde{P} = \frac{\partial}{\partial \tilde{x}}, \]  

(19)

\[ \tilde{V} = -\tilde{x} \frac{\partial}{\partial t} + u \frac{\partial}{\partial \tilde{x}}, \]  

(20)
This algebra contains the Carroll algebra, made out of $H, \vec{P}, \vec{V}$ and $M$ as a subalgebra. The Carroll algebra is the contraction of the Poincaré algebra in which the speed of light is sent to zero.

For arbitrary values of $z$, the metric of the Schrödinger spacetime takes the form given above with the $du^2$ term replaced by $r^{-2z} \, du^2$. Changing a bit the parameters this metric can be written as

$$ds^2 = \frac{1}{r^2} (2 \, dt \, du + dx^2 + dy^2 + dr^2) \, \pm \, r^{-2z} \, du^2,$$

putting temporarily $l = 1$. Schrödinger spacetimes come in two different forms that differ merely by a sign in front of the $r^{-2z} \, du^2$ term in the metric, but that have the same Lie algebra of Killing vectors. They are nonetheless different spacetimes, i.e. not related by a diffeomorphism. The choice of sign has important consequences for the singular structure of the spacetime. From a combination of a tidal force computation and a study of the geodesic congruence of timelike geodesics with tangent $\text{Pt}$ in which $\delta u$ is a constant of the geodesic motion (\(\delta \equiv P_t \rho^2\)). The vielbeins $e^1_\mu = r^2 g_s^2$ and $u_\mu$ are part of an orthonormal frame that is parallelly propagated along these geodesics. Underlined indices are tangent space indices.

The near horizon solution \((12)\) to \((15)\) solves the equations of motion of the following five-dimensional action,

$$S = \frac{1}{16\pi G^N_N} \int d^5 x \sqrt{-g} \left( \hat{R} + \frac{12}{l^2} - \frac{1}{2} \partial_\nu \phi \partial^\nu \phi - \frac{1}{2} e^{2 \hat{\phi}} \partial_\nu \hat{x} \partial^\nu \hat{x} \right),$$

where hatted fields refer to five-dimensional fields. The 5-form has been dualized to a cosmological constant term. This action, as is well known, follows by performing a Freund–Rubin compactification of type IIB supergravity over the 5-sphere.

If we compactify $u \sim u + 2\pi L$, so that $u$ parametrizes a spacelike circle, and reduce to four dimensions, we obtain a $z = 2$ Lifshitz spacetime. To make this more explicit we write the $z = 0$ Schrödinger metric in the form of a Kaluza–Klein reduction Ansatz

$$ds^2 = l^2 \left[ -\frac{4}{k^2 g_s^2} \frac{dr^2}{r^4} + \frac{1}{r^2} (dx^2 + dy^2 + dr^2) + \frac{k^2 g_s^2}{4} \left( du + \frac{4}{k^2 g_s^2} \frac{dt}{r^2} \right)^2 \right].$$

Asymptotically, Lifshitz solutions can be studied as solutions of a four-dimensional action that follows from the above five-dimensional action \((25)\) by Scherk–Schwarz reduction.

---

3 The sign choice in the metric does not strongly affect the calculation of the tidal forces. For example, if we take a congruence of timelike geodesics with tangent $u^\mu$ and with $\dot{x} = y = 0$, then $R_{\mu\nu\rho\tau} u^\mu u^\rho u^\tau = -1 \pm P_t^2 (z - 1) r^{4-2z}$, in which $P_t$ is a constant of the geodesic motion ($u = P_t r^2$). The vielbeins $e^{1}_\mu = r^2 g^2_s$ and $u_\mu$ are part of an orthonormal frame that is parallelly propagated along these geodesics. Underlined indices are tangent space indices. We thus observe that the tidal forces diverge as $r$ goes to infinity. Therefore, the question is can timelike geodesics get to $r = \infty$ in finite proper time or not and it is the answer to this question that strongly depends on the sign choice in the metric.
in which the axion shift symmetry has been gauged (see also [7]). We make the following
reduction Ansatz:

\[ ds^2 = e^{-\phi} ds^2 + e^{2\phi} (du + A)^2, \]  
(27)

\[ \hat{\chi} = \chi + ku, \]  
(28)

\[ \hat{\phi} = \phi, \]  
(29)

where the unhatted fields\(^4\) do not depend on \(u\). The resulting four-dimensional action is

\[ S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (D\chi)^2 - \frac{3}{2} (\partial \Phi)^2 - \frac{1}{4} e^{3\Phi} F^2 - V \right), \]  
(30)

where

\[ D\chi = d\chi - kA, \]  
(31)

\[ F = dA, \]  
(32)

\[ V = \frac{k^2}{2} e^{-3\Phi + 2\phi} - \frac{12}{l^2} e^{-\phi}. \]  
(33)

The equations of motion coming from this action admit the following Lifshitz invariant
solution:

\[ ds^2 = l^3 \frac{k g_s}{2} \left[ -\frac{4}{k^2 g_s^2 r^4} \, dt^2 + \frac{1}{r^2} \, dr^2 + \frac{1}{r^2} (dx^2 + dy^2) \right], \]  
(34)

\[ A = \frac{4}{k^2 g_s^2} \frac{1}{r^2} \, dt, \]  
(35)

\[ e^\phi = l^3 \frac{k g_s}{2}, \]  
(36)

\[ e^\Phi = g_s, \]  
(37)

\[ \chi = \chi_0, \]  
(38)

where \(\chi_0\) is an arbitrary constant. The type IIB coupling constant \(g_s\) is a free parameter. For
\(k = 0\) this metric makes no sense since for that case the reduction cannot be done along \(u\), it
being a lightlike circle for \(k = 0\).

To get this four-dimensional model we first reduced over the 5-sphere and then along a
circle. We could have also done this the other way around. The Scherk–Schwarz reduction
of the IIB axion shift symmetry along a circle gives rise to a massive deformation of nine-
dimensional maximal supergravity [13, 14] and four-dimensional Lifshitz times the 5-sphere
is a solution of that theory.

The supergravity approximation, i.e. small curvature and weak string coupling, requires

\[ \frac{l}{l_s} \gg 1, \quad g_s \ll 1. \]  
(39)

\(^4\) To avoid heavy notation we use the same symbols to denote ten- and four-dimensional fields. We hope this does
not lead to any confusion.
The radius of the circle over which we compactify from five to four dimensions is given by (in units of string length)
\[ \frac{1}{l_s} \int_0^{2\pi L} du \sqrt{g_{uu}} = \frac{l}{l_s} \pi k L g_s. \] (40)

Using that in type IIB string theory, the axion shift symmetry is broken to a symmetry under integer shifts (which here follows from single valuedness of the axion wavefunction along \( u \)) we conclude that \( 2\pi L k \in \mathbb{Z} \). Hence, the circle is of small radius, i.e. of the order of the string length if \( l g_s / l_s \) is of order unity. Assuming weak curvature implies that then the type IIB string coupling is weak.

We end this section with a few remarks about the solution (6)–(9) and Kaluza–Klein reduction of Schrödinger spacetimes. In the expression for \( G \), equation (11), we have set two integration constants equal to zero. The most general solution for \( G \) is
\[ G = \frac{1}{4} e^{\phi_0} k^2 \left( \frac{l^4}{R^2} - \frac{R^2}{3} \right) + c_1 + \frac{c_2}{R^4}, \] (41)
where the constant \( c_1 \) can be put to zero by the coordinate transformation \( t \rightarrow t - c_1 u / 2 \). The constant \( c_2 \) has been put to zero by hand in (11). For \( k = 0 \) and \( c_2 \neq 0 \) the solution for \( G \) can be obtained from the solution generating technique of Garfinkle and Vachaspati [15].

It is interesting to note that for \( k = 0 \) and \( c_2 \neq 0 \) we have in the near horizon limit, i.e. for \( H = l^4 R^{-4} \), what might be called a five-dimensional \( z = -1 \) Schrödinger spacetime. For \( c_2 > 0 \) this can be ‘Kaluza–Klein reduced’ to a four-dimensional \( z = 3 \) Lifshitz spacetime [16]. In [16], the five-dimensional \( z = -1 \) Schrödinger spacetime is obtained via a scaling limit of the near horizon limit of a boosted black D3-brane.

In general, if we take the Schrödinger metric (24) with the plus sign and with \( u \) periodically identified, then the ‘Kaluza–Klein reduction’ along the spacelike circle\(^5\) parametrized by \( u \) gives rise to a Lifshitz spacetime with dynamical exponent \( 2 - z \). We write Kaluza–Klein reduction in quotation marks because for \( z \neq 0 \) the four-dimensional metric in Einstein frame is only conformal to a Lifshitz spacetime with dynamical exponent \( 2 - z \) and further because for \( z < 0 \) the physical size of the compactification circle goes to zero near the boundary (\( r = 0 \) or \( R = \infty \)) while for \( z > 0 \) the physical size goes to zero for \( r = \infty \) (or \( R = 0 \)). Hence, for \( z \neq 0 \) one would expect that the periodic identification of \( u \) will violate the supergravity approximation as light string winding modes cannot be ignored. The case \( z = 0 \) is special because the four-dimensional spacetime in Einstein frame is a \( z = 2 \) Lifshitz spacetime and further the physical size of the compactification circle is independent of the radial coordinate. This agrees with the fact that for \( z = 0 \), the Killing vectors \( D \) (dilatations) and \( N \) (translations in \( u \)) commute. It has been pointed out in [12] that for \( z < 1 \), the periodic identification of the \( u \) coordinate introduces a null identification in the AdS\(_5\) boundary metric and thus corresponds to some discrete light cone quantization (DLCQ) of the boundary field theory. This is because the \( du^2 \) term in the metric for \( z < 1 \) is subleading to the AdS part.

3. Supersymmetry

The type IIB Killing spinor equations are
\[ \delta \lambda = i P_A \gamma^A \epsilon_C = 0, \] (42)

\( ^5 \) By saying that \( u \) parametrizes a spacelike direction and thus that the compactification circle is spacelike we mean that the tangent of a curve moving only in the \( u \)-direction is spacelike when measured with respect to the bulk metric. We call \( u \) spacelike even when \( u \) parametrizes a null direction with respect to the boundary metric.

6
\[
\delta \psi_A = D_A \epsilon + \frac{i}{480} \gamma^{C_1 \cdots C_5} \gamma_A F_{C_1 \cdots C_5} \epsilon = 0, \tag{43}
\]
in which
\[
D_A \epsilon = \left( \partial_A + \frac{i}{4} \omega_A^{AB} \gamma_{AB} - \frac{i}{2} Q_A \right) \epsilon. \tag{44}
\]
Underlined indices are tangent space indices. The spinors \( \lambda, \psi_\mu \) and \( \epsilon \) are complex combinations of the two type IIB Majorana–Weyl spinors, e.g. \( \epsilon = \epsilon_1 + i \epsilon_2 \) with \( \epsilon_1 \) and \( \epsilon_2 \) Majorana–Weyl. By \( \epsilon_C \) we denote the charge conjugate spinor \( \epsilon_C = \epsilon_1 - i \epsilon_2 \) and similarly for \( \lambda \) and \( \psi_\mu \).

We will show that the D3-brane/plane wave intersection as well as its near horizon geometry are 1/4 BPS solutions. The integrability condition for the gravitino equation is
\[
[D_A, D_B] \epsilon = \left( \frac{1}{4} R_{AB}^{\underline{\alpha} \underline{\beta}} \gamma_{\underline{\alpha} \underline{\beta}} + \frac{1}{2} (P_A P_B^\ast - P_B P_A^\ast) \right) \epsilon = (\nabla_A S_B - \nabla_B S_A - (S_A S_B - S_B S_A)) \epsilon, \tag{45}
\]
where \( R_{AB}^{\underline{\alpha} \underline{\beta}} \) is the curvature 2-form. By \( S_A \) we denote
\[
S_A = - \frac{i}{480} \gamma^{C_1 \cdots C_5} \gamma_A F_{C_1 \cdots C_5}. \tag{46}
\]
The integrability condition for the dilatino equation can be obtained by acting with \( D_A \) on the right-hand side of \( \delta \lambda \) and using the gravitino equation to rewrite \( D_A \epsilon \). Using that \( \lambda \) has \( U(1) \) weight 3/2 and \( P \) has \( U(1) \) weight 2, we find (we actually used the charge conjugate dilatino equation)
\[
\gamma^B (D_A P_B^\ast - P_B^\ast S_A) \epsilon = 0. \tag{47}
\]
Substituting the complete solution (6)–(9) into these two integrability conditions leads to the following two supersymmetry projectors:
\[
P_1 \epsilon = P_2 \epsilon = 0, \tag{48}
\]
where the projectors \( P_1 \) and \( P_2 \) are given by
\[
P_1 = \frac{1}{2} (1 - \gamma_{12}), \tag{49}
\]
\[
P_2 = \frac{1}{2} (1 - \gamma_{34}). \tag{50}
\]
With two projectors the number of preserved supersymmetries is \( \text{Tr} - \text{Tr} P_1 - \text{Tr} P_2 + \text{Tr}(P_1 P_2) \). From this we find that the solution is 1/4 BPS. The \( P_1 \) projector is equivalent to demanding \( \gamma_1^2 \epsilon = 0 \) (the 1/2 BPS axion wave projector) and \( P_1 \) and \( P_2 \) together imply the D3-brane projector \( \frac{1}{2} (1 + i \gamma_{12}) \). The projectors \( P_1 \) and \( P_2 \) are in agreement with the supersymmetry analysis of [5].

The doubling of the supersymmetries that occurs in the near horizon geometry of the extremal D3-brane does not occur in the case where the extremal D3-brane has a plane wave superposed on it. This can be explicitly verified by substituting the near horizon solution into the above two integrability conditions. Doing so leads again to the projectors \( P_1 \) and \( P_2 \). Even more explicitly one can solve the Killing spinor equations and confirm that the only nonzero solutions are
\[
\epsilon = e^{i \frac{1}{2} \epsilon A \dot{A}} r^{-1} \eta, \tag{51}
\]
We use the following metric on the tangent space \( \eta_\mu = 0, \eta_\underline{\alpha} = 1 \) and \( \eta_\underline{\alpha} = 1 \) with the rest of the metric diagonal, i.e. \( \eta_\underline{\alpha} = 1 \) etc. Further we use \( \epsilon_A^\underline{\alpha} = \epsilon_A^1 \cdots \epsilon_A^{10} \) where \( \epsilon_A^1 \cdots \epsilon_A^{10} \) is the tangent space Levi-Civita symbol satisfying \( \epsilon_A^1 \epsilon_A^2 \cdots \epsilon_A^{10} = +1 \). The chirality matrix is \( \gamma_{11} = \gamma^{11}_{\underline{\alpha} \underline{\beta}} \) and \( \epsilon \) satisfies \( \gamma_{11} \epsilon = -\epsilon \).
where \( \eta \) satisfies \( P_1 \eta = P_2 \eta = 0 \) and which depends on the 5-sphere coordinates. For explicit expressions for Killing spinors on spheres see for example [17].

When the \( u \) coordinate is compact we need to impose boundary conditions on the fields. For the axion the natural choice is to demand that it transforms into itself up to a symmetry of the theory. In order for the axion to remain single valued we must take integer shifts and divide type IIB out by these discrete transformations. The Killing spinor has a dependence on the \( u \) coordinate and so we can likewise demand that when going once around the compact circle \( \epsilon \) transforms into itself (or possibly minus itself since it is a spinor) up to a symmetry of the theory. The spinors of type IIB supergravity do not transform under the axion shift symmetry and so all we can ask for is that \( \epsilon \) transforms into itself up to a sign. This means that we need to impose

\[
g_s k L = 2n \quad \text{(anti-periodic boundary conditions),} \\
g_s k L = 4n \quad \text{(periodic boundary conditions),}
\]

where \( n \in \mathbb{Z} \). At the same time in order for the axion to transform by an integer amount when going once around the \( u \) circle we need \( 2\pi L k = m \in \mathbb{Z} \). It follows that supersymmetry is preserved if

\[
g_s = 4\pi \frac{n}{m} \quad \text{(anti-periodic boundary conditions),} \\
g_s = 8\pi \frac{n}{m} \quad \text{(periodic boundary conditions).}
\]

This means that for generic values of the string coupling supersymmetry will be broken once \( u \) is compactified.

Consider reducing the type IIB theory from ten to nine dimensions. Performing a Scherk–Schwarz reduction of the axion shift symmetry requires the supersymmetry parameter \( \epsilon \) to be independent of the circle coordinate \( u \) [14]. This is because the \( u \) dependence of the IIB fields is dictated by their transformation properties under the symmetries of type IIB supergravity and under the condition that the lower dimensional equations of motion do not depend on \( u \). If we reduce the type IIB Killing spinor equations, we obtain the Killing spinor equations of maximal nine-dimensional supergravity. Therefore, four-dimensional Lifshitz times the 5-sphere is a non-supersymmetric solution of nine-dimensional maximal supergravity (mass deformed by \( k \)) for if it were a supersymmetric solution it would have to uplift to ten dimensions to a supersymmetric solution with a \( u \)-independent Killing spinor and this is not possible.

The analysis of sections 2 and 3 can be generalized as follows. In type IIB supergravity one can define besides the RR axion, two other axionic scalars [18]: there is one axionic scalar for each Killing vector of the scalar co-set manifold. Therefore, one can use one of the other two axions and perform a similar analysis to what was done in sections 2 and 3. In the notation of [18], the scalar co-set manifold is parametrized in terms of the scalars \( T \) and \( \chi' \), where \( \chi' \) is an axion. The solutions then take the form \( T = \text{cst} \) and \( \chi' \) proportional to \( u \). These are again 1/4 BPS intersections of a D3-brane with a wave whose near horizon geometry is a five-dimensional \( z = 0 \) Schrödinger spacetime.

4. Black deformations

In this section, we will use the D3-brane/axion wave picture to find four-dimensional analytic \( z = 2 \) Lifshitz black brane solutions. The basic observation is the following. If before we turn on the axion, we have a metric of the form

\[
ds_{(10)}^2 = H^{-1/2} \left( 2A_1 \, du \, dt + A_2 \, du^2 + dx^2 + dy^2 \right) + H^{1/2} \left( \frac{dR^2}{F} + R^2 \, d\Omega_5^2 \right),
\]
with \( H = l^4 R^{-4} \), i.e. in the near horizon limit of the D3-brane, then the effect of adding the axion wave is to deform the function \( A_2 \) while leaving the functions \( A_1 \) and \( F \) unmodified. Metrics of this form have an everywhere null Killing vector which is \( \partial_t \). We will take the function \( A_2 \) to be proportional to \( k \).

For \( k = 0 \), in the near horizon limit of the D3-brane, the five-dimensional metric described by \( t, u, x, y, R \) will be an asymptotically AdS (AAdS₅) black string solution with the world-volume of the string described by \( t \) and \( u \). If we define the coordinates \( T \) and \( z \) by \( \sqrt{2}u = T + z \) and \( \sqrt{2}t = -T + z \), then the string stretches in the \( z \) direction and is infinitely long. The transverse space, parametrized by \( R, x, y \), has translational and rotational symmetries in the \( x \) and \( y \) directions. The effect of turning on the axion is to deform the asymptotics of the spacetime from an AAdS₅ metric to an asymptotically \( z = 0 \) Schrödinger metric. It also breaks the string’s world-volume boost symmetry\(^7\). If we then compactify the direction parametrized by \( u \), and reduce along it, we obtain a four-dimensional black hole solution whose transverse space has translational and rotational symmetries in the \( x \) and \( y \) directions that is asymptotic to a \( z = 2 \) Lifshitz spacetime. In other words, a \( z = 2 \) Lifshitz black brane solution.

One may wonder if it is possible to deform the usual black hole deformation of the D3-brane by the axion wave, i.e. to start with the metric

\[
\begin{align*}
\begin{bmatrix} \text{d}s^2 \end{bmatrix}_{(10)} &= H^{-1/2}(-F \, dt^2 + dz^2 + dx^2 + dy^2) + H^{1/2} \left( \frac{dR^2}{F} + R^2 \, d\Omega_5^2 \right),
\end{align*}
\]

where \( F = 1 - M R^{-4} \) (and \( H = l^4 R^{-4} \) working again only in the near horizon limit) and to deform this by an axion that is proportional to \( T + z \). The difficulty one faces is that the spacetime now has only asymptotic null Killing vectors while the generator of the horizon is only null at the horizon and timelike everywhere outside. Deforming this spacetime such that at infinity we have a wave traveling in the (null) \( u \) direction, with \( \sqrt{2}u = T + z \), and such that we preserve translational symmetries in the \( T, z, x, y \) directions as well as rotational symmetries in the \( (x, y) \) plane is expected to be a more difficult task than the black string approach discussed above\(^8\). If one were able to solve this problem, then upon reduction along \( u \) one would obtain \( z = 2 \) Lifshitz black brane solutions. This gives some physical understanding as to why it is difficult to construct generic Lifshitz black brane solutions.

We will now first discuss the black string deformation of the D3-brane and then deform it further by the axion wave. Keeping the matter fields undeformed, i.e. \( \chi = k u \), \( F_{\alpha_1...\alpha_5} = l^4 \omega_{\alpha_1...\alpha_5} \) with the remaining components of \( F_5 \) determined by self-duality and with \( \phi \) some function of \( R \), the most general metric that is of the form (56) that is a solution to the type IIB equations of motion (and thus also to the equations of motion of (25)), assuming first that \( k = 0 \), is given by

\[
\begin{align*}
A_1 &= F^{1/2}, \\
A_2 &= c_1 A_1 + c_2 A_1 \log A_1, \\
F &= 1 - M R^{-4},
\end{align*}
\]

with the dilaton given by

\[
e^\phi = g_s F^{\pm 1/2}.
\]

\(^7\) The setup here is reminiscent of that of [19, 20] which discusses waves traveling along extremal asymptotically flat black string solutions.

\(^8\) For more details about the problem of combining black holes and plane waves we refer to [21, 22].
The metric has a curvature singularity at $R = M^{1/4}$. This however does not imply that the four-dimensional solution obtained after reduction must also have a curvature singularity at the same value of $R = M^{1/4}$. The constant $c_1$ can always be set to zero by the coordinate transformation $t \rightarrow t - c_1 u/2$. We will take $c_2 = 0$ so that $A_1$ will be proportional to $k$. The two possible signs in the solution for the dilaton are a manifestation of the S-duality invariance (for $\chi = 0$) of the theory.

When $k \neq 0$, the solution remains the same except that now $A_2$ is

$$A_2 = -\frac{g_s^2 k^2 l^4}{4 R^2} + \frac{g_s^2 k^2 l^4}{2 M^{1/2}} A_1 \arcsin \frac{M^{1/2}}{R^2},$$

(62)

when we take the minus sign in expression (61) for the dilaton and

$$A_2 = \frac{g_s^2 k^2 l^4}{4 M^{1/2}} A_1 \left[ (1 - \log 2) \arcsin \frac{M^{1/2}}{R^2} - \left( \arcsin \frac{M^{1/2}}{R^2} \right) \log A_1 + \frac{1}{2} \text{Cl}_2 \left(2 \arcsin A_1\right) \right],$$

(63)

when we take the plus sign in (61). In expression (63) the function $\text{Cl}_2(x)$ is the Clausen function that is defined by

$$\text{Cl}_2(x) = -\int_0^x \log \left(2 \sin \frac{t}{2}\right) \, dt.$$

(64)

Since the axion is nonzero whenever $k \neq 0$ the two possible sign choices for the dilaton are no longer related by S-duality, so that we obtain different solutions for each. These two functions $A_2$ have the property that for $M = 0$ they reduce to $g_s^2 k^2 l^4 R^{-2}/4$ which is the near horizon limit of (11) while they vanish for $k = 0$. The expression for $A_2$ in (63) has been written such that each term is positive for $M^{1/4} < R < \infty$. To take the $M = 0$ limit of (63) it is useful to use the following two results for Clausen functions [23]:

$$\text{Cl}_2(\pi - x) = \text{Cl}_2(x) - \frac{1}{2} \text{Cl}_2(2x),$$

(65)

$$\text{Cl}_2(x) = x - x \log x + O(x^2),$$

(66)

with $x = 2 \arcsin \left(\sqrt{1 - A_1^2}\right) = \pi - 2 \arcsin A_1$.

We next compactify the $u$ direction and reduce to four dimensions. To do so we write the metric (56) in the form of a Kaluza–Klein reduction Ansatz as

$$dx^2_{(10)} = e^{-\Phi} \, ds^2 + e^{2\Phi} \, (du + A)^2 + r^2 \, d\Omega_2^2,$$

(67)

where

$$ds^2 = f^2 \frac{1}{r} A_2^{1/2} \left[ -\frac{1}{r^2 A_2} \, F \, dr^2 + \frac{1}{r^2} (dx^2 + dy^2) + \frac{dr^2}{r^2 F} \right],$$

(68)

$$A = \frac{F^{1/2}}{A_2} \, dr,$$

(69)

$$e^{2\Phi} = f^2 \frac{1}{r^2 A_2},$$

(70)

with $r = f^2 R^{-1}$. The asymptotic form is given by (34)–(36). The asymptotic $z = 2$ Lifshitz solution (68)–(70) and (61) (as well as with the four-dimensional axion constant) thus obtained solves the equations of motion of the four-dimensional action (30). The four-dimensional result is particularly sensitive to the zeros of $A_2$. In fact, these points form curvature singularities. The compactification circle becomes lightlike for points, where $A_2 = 0$. Ideally, we would like this to happen inside the region bounded by the horizon. Furthermore, after the reduction
cannot be negative. The solution (62) has one zero in between the horizon and infinity and therefore possesses a naked singularity. The second solution (63) has one zero that coincides with the horizon, so that the curvature singularity coincides with the horizon.

The type IIB string coupling goes to zero at the horizon in the case where $A_2$ is given by (63). Clearly, the supergravity approximation breaks down near the curvature singularity, i.e. before we get to the horizon due to the large curvature. Therefore, in the region near the horizon $\alpha'$ corrections to type IIB supergravity will be important.

Even though there is definitely room for improvement, especially to find examples where the curvature singularity is strictly behind the event horizon\(^9\) we believe that these ideas provide a complementary view on the more numerical approaches taken in [24–27]. From a four-dimensional point of view the problem to find Lifshitz black brane solutions has to do with the fact that it is rather difficult to decouple the differential equations. In our model (30), the four-dimensional equations can be uplifted to five dimensions and then using the Ansatz

$$d\tilde{s}^2 = \frac{R^2}{l^2} (2A_1 \, du \, dt + A_2 \, du^2 + dx^2 + dy^2) + \frac{l^2}{R^2} \frac{dR^2}{F}$$

for the five-dimensional metric allows one to decouple these differential equations. It would be interesting to study this further from the four-dimensional point of view in a wider class of models and to see if similar decouplings can be obtained.

5. Discussion

The D3-brane/axion wave interpretation of the $z = 2$ Lifshitz spacetime may be of interest for a number of open problems. For example, it has been noted in [1] that Lifshitz spacetimes suffer from divergent tidal forces. This issue has been further elaborated on in [28]. These divergent tidal forces persist to exist in the five-dimensional $z = 0$ Schrödinger uplift of the four-dimensional $z = 2$ Lifshitz spacetime. It would be interesting to understand better, if only qualitatively, what the right attitude toward this singularity is. We may now have a simple starting point to address this issue. For example, it would be interesting to study the effect of introducing the axion wave from the point of view of the world-volume theory of the D3-branes. This will be some deformation of $\mathcal{N} = 4$ SYM which may well have an infrared singularity corresponding to the singularity of the $z = 0$ Schrödinger spacetime.

Another potential application of this way of embedding Lifshitz into string theory concerns the problem of holographic renormalization. The five-dimensional $z = 0$ Schrödinger spacetime is asymptotically AdS$_5$ [12]. Therefore, we can apply the standard holographic renormalization techniques for AAdS$_5$ spacetimes to construct the counterterm action for the five-dimensional AdS-gravity-axion–dilaton system. Part of this analysis, where it concerns the conformal anomaly, has been done in [29]. Once the complete counterterm action is known for the five-dimensional AdS-gravity-axion–dilaton system we may be able to obtain the $z = 2$ Lifshitz counterterms by dimensional reduction. It would be interesting to compare such an analysis with existing proposals for the holographic renormalization of asymptotically Lifshitz spacetimes, see e.g. [30, 31].

As discussed at the end of section 2 (based on [12]), the Scherk–Schwarz reduction of the bulk $z = 0$ Schrödinger spacetime to the $z = 2$ Lifshitz spacetime is expected to correspond on the boundary theory to some DLCQ of the field theory dual of the $z = 0$

\(^9\) In the last paragraph of section 3, we mentioned that the solution (6)–(9) can be generalized by considering other coordinate systems on the scalar co-set manifold. Here, we just mention that these solutions can be black deformed in much the same way as was done above. None of the solutions obtained in this way have the curvature singularity behind the event horizon.
Schrödinger spacetime. It would be interesting to study the Scherk–Schwarz reduction of a five-dimensional \( z = 0 \) Schrödinger spacetime in more detail in particular in relation to the bulk-boundary correspondence.

Regarding the construction of analytic Lifshitz black brane solutions, we would like to mention that the approach taken here is not expected to be limited to the particular theory that we work with, but to apply more generally. The general recipe that this approach suggests is to take some five-dimensional supergravity theory that possesses AdS solutions and that contains an axion and then to look for AAdS\(_5\) black string solutions with a null Killing vector on its world-volume and to deform them by an axion wave. The axion profile must be that one which in the case of pure AdS\(_5\) deforms it to a \( z = 0 \) Schrödinger spacetime. Reducing this to four dimensions along the direction in which the wave is traveling should then provide us with \( z = 2 \) Lifshitz black brane solutions of various theories. It would be interesting to see if this method can be used to construct examples where the curvature singularity is strictly inside the region bounded by the horizon. This would be highly advantageous as black brane solutions in a holographic context are often the background on which some field theory is defined. Having analytic control of such a background is without doubt desirable. Finally, it would be interesting to see if this method can be used to learn more about the attractor mechanism with Lifshitz scaling (see [32] and references therein) where it concerns asymptotically Lifshitz spacetimes.

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