DIFFERENTIAL CALCULI
ON QUANTUM MINKOWSKI SPACE

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Abstract. The differential calculus on n-dimensional quantum Minkowski space covariant with respect to left action of κ-Poincaré group is constructed and its uniqueness is shown.

I. Introduction
The κ-Poincaré algebra, introduced in [1], provides a Hopf algebra deformation of standard Poincaré algebra which depends on dimensionful parameter κ. Its global counterpart, κ-Poincaré group $P_κ$ has been constructed by Zakrzewski [2]. It is a free *-algebra generated by the hermitean elements $Λ^{µν}, a^α$, subject to the following conditions

\begin{align}
[a^α, a^β] &= i\frac{κ}{κ}(δ^α_0a^β - δ^β_0a^α), \\
[Λ^{µν}, A^{α β}] &= 0, \\
[Λ^{µν}, a^α] &= -i\frac{κ}{κ}((Λ^{µ}_0 - δ^{µ}_0)Λ^{α ν} + (Λ^{α}_0 - δ^{α}_0)g^{µα}), \\
∆(Λ^{µ ν}) &= Λ^{µ α} ⊗ A^α ν, \\
∆(a^α) &= Λ^{µ ν} ⊗ a^ν + a^α ⊗ I, \\
S(Λ^{µ ν}) &= Λ^{ν µ}, \\
S(a^α) &= -Λ^µ_ν a^ν, \\
ε(Λ^{µ ν}) &= δ^{µ}_ν, \\
ε(a^α) &= 0.
\end{align}

It appears ([2], [3]) that one can also define a noncommutative generalization of Minkowski spacetime — the κ-Minkowski space $M_κ$. The κ-Poincaré group acts

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on $\mathcal{M}_\kappa$ covariantly from the left. Once one accepts the idea that the $\kappa$-deformed Poincaré symmetry can have something to do with reality the next step is to find the natural generalizations of standard geometric notions related to Minkowski space. The first step toward this direction was made by Sitarz [4] who showed that one cannot construct four-dimensional differential calculus on $\mathcal{M}_\kappa$ which is covariant with respect to infinitesimal left action of $\mathcal{P}_\kappa$. He sketched also the construction of five-dimensional covariant calculus.

In the present paper we consider the problem of the classification of differential calculi on $\mathcal{M}_\kappa$ which are covariant with respect to the left global action of $\kappa$-Poincaré group on $\mathcal{M}_\kappa$. No restriction is made concerning the dimensionality of spacetime (i.e. the indices $\mu$, $\nu$, etc. in (1) run from 0 to $n - 1$). We show that the lowest dimensional nontrivial left-covariant calculus is $n + 1$-dimensional and is unique. Its construction is given explicitly and the result coincides with the suggestion of Sitarz.

As a main tool we use the beautiful Woronowicz theory of differential calculi ([5]). In Section II we show that the Woronowicz theory can be immediately extended to deal with the problem of covariant differential calculi on quantum spaces. This natural extension provides a nice framework to discuss our problem. In Section III we use the scheme developed in Section II to show that there is a unique $n + 1$-dimensional calculus on $\mathcal{M}_\kappa$ which is left-covariant with respect to the action of $\mathcal{P}_\kappa$. This calculus is explicitly constructed and shown to be lowest-dimensional nontrivial calculus on $\mathcal{M}_\kappa$.

The results obtained here were briefly reported in [6]. Let us also mention that the differential calculi on quantum spacetimes covariant with respect to other deformations of Poincaré group were considered by Podleś ([7]).

II. COVARIANT DIFFERENTIAL CALCULI

Let us first indicate how one can extend the Woronowicz theory of differential calculi ([5]) to the following situation: assume that the quantum group $\mathcal{B}$ acts on quantum space $\mathcal{A}$; one looks for differential calculi on $\mathcal{A}$ on which the covariant action of $\mathcal{B}$ can be defined as ’induced’ by the action of $\mathcal{B}$ on $\mathcal{A}$. All proofs are omitted as being a straightforward extension of those given by Woronowicz.

Let $\mathcal{A}$ be an algebra with unity (quantum space). The starting point in Woronowicz construction is the universal bimodule $\mathcal{A}^2 \subset \mathcal{A} \otimes \mathcal{A}$ defined by

$$\mathcal{A}^2 = \{ \sum_k a_k \otimes b_k \in \mathcal{A} \otimes \mathcal{A} | \sum_k a_k b_k = 0 \},$$

$$c(\sum_k a_k \otimes b_k) = \sum_k c a_k \otimes b_k,$$

$$\sum_k a_k \otimes b_k c = \sum_k a_k \otimes b_k c.$$ (2)

The universal differential $D : \mathcal{A} \rightarrow \mathcal{A}^2$ is given by $da = I \otimes a - a \otimes I$.

It can be easily shown that any other calculus is obtained from the universal one by dividing by an appropriately chosen sub-bimodule $\mathcal{N} \subset \mathcal{A}^2$.

Let now $\rho_L$ be a left action of a quantum group $\mathcal{B}$ on $\mathcal{A}$, i.e. a homomorphism $\rho_L : \mathcal{A} \rightarrow \mathcal{B} \otimes \mathcal{A}$ obeying

$$(\text{id} \otimes \rho_L) \circ \rho_L = (\Delta \otimes \text{id}) \circ \rho_L,$$

$$(\varepsilon \otimes \text{id}) \circ \rho_L = \text{id}.$$ (3)
Let us define the action \( \tilde{\rho}_L \) of \( \mathcal{B} \) on \( \mathcal{A}^2 \) as follows

\[
q = \sum_i x_i \otimes y_i \in \mathcal{A} \otimes \mathcal{A},
\]

\[
\rho_L(x_i) = \sum_k a_{ik}^k \otimes x_i^k \in \mathcal{B} \otimes \mathcal{A}, \tag{4a}
\]

\[
\rho_L(y_i) = \sum_l b_{il}^l \otimes y_i^l \in \mathcal{B} \otimes \mathcal{A},
\]

then

\[
\tilde{\rho}_L(q) = \sum_{i,k,l} a_{ik}^k b_{il}^l \otimes x_i^k \otimes y_i^l. \tag{4b}
\]

Obviously, \( \tilde{\rho}_L : \mathcal{A} \otimes \mathcal{A} \to \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \), however, it is straightforward to show that \( \tilde{\rho}_L : \mathcal{A}^2 \to \mathcal{B} \otimes \mathcal{A}^2 \). Following the same lines as in [5], one easily proves the following properties of \( \tilde{\rho}_L \)

(i) \( \tilde{\rho}_L \) for \( x \in \mathcal{A}, \ y \in \mathcal{A}^2 \)

\[
\tilde{\rho}_L(xy) = \rho_L(x)\tilde{\rho}_L(y),
\]

\[
\tilde{\rho}_L(yx) = \tilde{\rho}_L(y)\rho_L(x)
\]

(ii) \( \tilde{\rho}_L \circ D = (id \otimes D) \circ \rho_L \)

(iii) \( (id \otimes \tilde{\rho}_L) \circ \tilde{\rho}_L = (\Delta \otimes id) \circ \tilde{\rho}_L, \)

\[
(\varepsilon \otimes id) \circ \tilde{\rho}_L = id.
\]

Property (iii) means that \( \tilde{\rho}_L \) is the left action of \( \mathcal{B} \) on \( \mathcal{A}^2 \) while (i), (ii) can be summarized by saying that \( \tilde{\rho}_L \) is the lift of \( \rho_L \) to \( \mathcal{A}^2 \) (\( \tilde{\rho}_L \) is the left action of \( \mathcal{B} \) on universal differential calculus on \( \mathcal{A} \) induced from the left action \( \rho_L \)). Moreover, let us note that the following formula holds

\[
\tilde{\rho}_L(\sum_i x_i D y_i) = \sum_i \rho_L(x_i)(id \otimes D)\rho_L(y_i) \tag{6}
\]

which is a counterpart of (1.15) of [5].

Now, assume that \( \mathcal{N} \subset \mathcal{A}^2 \) is a subbimodule such that

\[
\tilde{\rho}_L(\mathcal{N}) \subset \mathcal{B} \otimes \mathcal{N}. \tag{7}
\]

Then the differential calculus \( (\Gamma, d) \) determined by \( \mathcal{N} \) has the following property

\[
\sum_i x_i d y_i = 0 \Rightarrow \sum_i \rho_L(x_i)(id \otimes d)\rho_L(y_i) = 0. \tag{8}
\]

Therefore

\[
\tilde{\rho}_L(\sum_i x_i d y_i) = \sum_i \rho_L(x_i)(id \otimes d)\rho_L(y_i) \tag{9}
\]
is well defined linear mapping from $\Gamma$ into $\mathcal{B} \otimes \Gamma$. Formulae (i$_L$)–(iii$_L$) and (6) hold upon replacing $A^2$ by $\Gamma$ and $D$ by $d$.

We shall say that $(\Gamma, d)$ is left-covariant with respect to the action of $\mathcal{B}$. All the above results can be extended mutatis mutandis to right actions. Let $\rho_R : A \to A \otimes \mathcal{B}$ be right action of $\mathcal{B}$ on $A$

\[(\rho_R \otimes \text{id}) \circ \rho_R = (\text{id} \otimes \Delta) \circ \rho_R, \quad \text{(10)}\]

For

\[q = \sum_i x_i \otimes y_i \in A \otimes A,\]

\[\rho_R(x_i) = \sum_k x_i^k \otimes a_i^k \in A \otimes \mathcal{B}, \quad \text{(11a)}\]

\[\rho_R(y_i) = \sum_l y_i^l \otimes b_i^l \in A \otimes \mathcal{B}, \quad \text{(11b)}\]

we put

\[\tilde{\rho}_R(q) = \sum_{i,k,l} x_i^k \otimes y_i^l \otimes a_i^k b_i^l. \quad \text{(12)}\]

Again $\tilde{\rho}_R : A^2 \to A^2 \otimes \mathcal{B}$ obeys

(i$_R$) for $x \in A, y \in A^2$

\[\tilde{\rho}_R(xy) = \rho_R(x)\tilde{\rho}_R(y), \quad \text{id} \circ \tilde{\rho}_R = \tilde{\rho}_R \circ \rho_R \quad \text{(13)}\]

as well as

\[\tilde{\rho}_R(\sum_i x_i D y_i) = \sum_i \rho_R(x_i) (D \otimes \text{id}) \rho_R(y_i). \quad \text{(14)}\]

Now assume $\mathcal{N} \in A^2$ to be a sub-bimodule such that $\tilde{\rho}_R(\mathcal{N}) \subset \mathcal{N} \otimes \mathcal{B}$. Then, for the calculus $(\Gamma, d)$ determined by $\mathcal{N}$, (i$_R$)–(iii$_R$) and (13) hold with appropriate replacements $A^2 \to \Gamma, D \to d$.

Finally, let the pair $(\rho_L, \rho_R)$ of actions of $\mathcal{B}$ on $A$ be given. We assume that $\rho_L, \rho_R$ commute

\[(\text{id} \otimes \rho_R) \circ \rho_L = (\rho_L \otimes \text{id}) \circ \rho_R. \quad \text{(15)}\]

We say that $(\Gamma, d)$ is bicovariant with respect to the action of $\mathcal{B}$ on $A$ if it is left- and right-covariant. Then, it has all properties of left- and right-covariant calculi together with the following one (cf. (1.20) of [5])

\[(\text{id} \otimes \tilde{\rho}_R) \circ \tilde{\rho}_L = (\tilde{\rho}_L \otimes \text{id}) \circ \tilde{\rho}_R. \quad \text{(16)}\]
Let us now discuss the problem of infinitesimal action of $B$ on $A$. Let $\chi$ be any element of the Hopf algebra dual to $B$. We put

$$
\chi_{\rho_L} = (\chi \otimes \text{id}) \circ \rho_L,
$$

$$
\tilde{\chi}_{\rho_L} = (\chi \otimes \text{id}) \circ \tilde{\rho}_L.
$$

(16)

The first definition, introduced by Woronowicz ([8]), coincides with the one used by Majid and Ruegg ([3]). The second one is equivalent to the proposal of Sitarz ([4])

$$
\tilde{\rho}_L(\chi (\otimes \text{id}) \circ \chi_{\rho_L}(x)(\text{id} \otimes d)\chi_{\rho_L}(y))
$$

where $\Delta \chi = \chi_{(1)} \otimes \chi_{(2)}$. Analogous definitions can be given for $\rho_R$ and $\tilde{\rho}_R$.

From the above discussion it follows then that in order to check whether a calculus on $A$ is consistent with the action of $B$ on $A$ it is sufficient to check the property $\tilde{\rho}_L(N) \subset B \otimes N$ ($\tilde{\rho}_R(N) \subset N \otimes B$).

This simplifies considerably if $A$ itself is a quantum group and $N$ defines (say) left-covariant calculus on it. Then $N = r^{-1}(A \otimes R)$ where $R$ is a right ideal in $\ker \varepsilon$; any element of $A \otimes R$ can be written as

$$
t = \sum_i a_i \otimes x_i b_i
$$

(18)

where $a_i, b_i \in A$ and $x_i$ are generators of $R$. From the very definition of the operation $r^{-1}$ the following formula follows immediately ([5])

$$
r^{-1}(t) = \sum_{i,l} a_i S(b'_{i}) r^{-1}(I \otimes x_i) b''_{i}
$$

(19a)

where

$$
\Delta (b_i) = \sum_l b'_{il} \otimes b''_{il}.
$$

(19b)

The properties (5) applied to the universal calculus imply

$$
\tilde{\rho}_L(r^{-1}(t)) = \sum_{i,l} \rho_L(a_i S(b'_{i})) \tilde{\rho}_L(r^{-1}(I \otimes x_i)) \rho_L(b''_{i}).
$$

(20)

Therefore it is sufficient to check that

$$
\tilde{\rho}_L(r^{-1}(I \otimes x_i)) \subset B \otimes N
$$

(21)

for all generators $x_i$ of $R$.

Let us now pass to the external algebra. Given $\tilde{\rho}_L : \Gamma \to B \otimes \Gamma$ we define $\tilde{\rho}_L^{\otimes 2} : \Gamma^{\otimes 2} \to B \otimes \Gamma^{\otimes 2}$ by

$$
\tilde{\rho}_L^{\otimes 2}(\omega_1 \otimes \omega_2) = \sum_{k,l} a_{1k} a_{2l} \otimes \omega_{1k} \otimes \omega_{2l}
$$

(22)
extended by linearity; here $\omega_i \in \Gamma$ and

$$\tilde{\rho}_L(\omega_i) = \sum_k a_{ik} \otimes \omega_{ik}, \quad i = 1, 2. \quad (23)$$

Let us assume that $A$ is a quantum group, $(\Gamma, d)$ — a bicovariant calculus on it and let $\sigma$ be the module homomorphism defined in Proposition 3.1 of [5]. Then $\Gamma^{\wedge 2}$ is defined as

$$\Gamma^{\wedge 2} = \Gamma \otimes \Gamma / \ker(I - \sigma) \quad (24)$$

and, in order to have a consistent action of $B$ on $\Gamma^{\wedge 2}$ we must only check that

$$(\text{id} \otimes \sigma) \circ \tilde{\rho}_L^{\otimes 2} = \tilde{\rho}_L^{\otimes 2} \circ \sigma. \quad (25)$$

Due to the property

$$\tilde{\rho}_L^{\otimes 2}(xy) = \rho_L(x)\tilde{\rho}_L^{\otimes 2}(y), \quad x \in A, \ y \in \Gamma \otimes \Gamma \quad (26)$$

it is sufficient to verify (25) for the basic elements only.

### III. Bicovariant calculi on $\kappa$-Minkowski space

The $n$-dimensional $\kappa$-Minkowski space $M_\kappa$ is an $*$-algebra with unity generated by $n$ hermitian elements $x^\mu$ subject to the following conditions ([2], [3])

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu). \quad (27)$$

$M_\kappa$ can be equipped with the structure of the quantum group by putting

$$\Delta x^\mu = I \otimes x^\mu + x^\mu \otimes I,$$
$$S(x^\mu) = -x^\mu,$$
$$\varepsilon(x^\mu) = 0. \quad (28)$$

The left action of $n$-dimensional $\kappa$-Poincaré group $P_\kappa$ on $M_\kappa$ can be defined as follows

$$\rho_L(I) = I \otimes I,$$
$$\rho_L(x^\mu) = \Lambda^{\mu^\nu} x^\nu + a^\mu \otimes I \quad (29)$$

extended by linearity and multiplicativity.

We want to find a left-covariant (with respect to action of $P_\kappa$) calculi on $M_\kappa$. As the first step let us note that $M_\kappa$ is a subgroup of $P_\kappa$. Indeed, $\Pi : P_\kappa \to M_\kappa$ given by

$$\Pi(a^\mu) = x^\mu, \quad \Pi(\Lambda^{\mu^\nu}) = \delta^\mu_0 I \quad (30)$$

is an epimorphism obeying

$$\Delta_M \circ \Pi = (\Pi \otimes \Pi) \circ \Delta_P. \quad (31a)$$

Moreover, it is immediate to check that

$$(\Pi \otimes \text{id}) \circ \rho_L = \Delta_M. \quad (31b)$$
Let \( \tilde{\rho}_L \) be the extension of \( \rho_L \) to \( \mathcal{M}_\kappa^2 \). Equations (8), (31) and the results contained in [5] imply that any calculus on \( \mathcal{M}_\kappa \) left-covariant with respect to action of \( \mathcal{P}_\kappa \) is also left-covariant with respect to action of \( \mathcal{M}_\kappa \) on itself. Therefore the relevant sub-bimodule \( \mathcal{N} \) is of the form \( r^{-1}(\mathcal{M}_\kappa \otimes \mathcal{R}) \) where \( \mathcal{R} \) is a right ideal in \( \ker \varepsilon_M \).

Let \( \mathcal{R} \) be any ideal in \( \ker \varepsilon_M \). Any \( a \in \mathcal{R} \) can be written as

\[
a = \sum_{\mu_0, \mu_k} c_{\mu}(x^0)^{\mu_0} \prod_{k=1}^{n-1} (x^k)^{\mu_k}.
\] (32)

Let us call \( |\mu| = \mu_0 + \sum_{k=1}^{n-1} \mu_k \); obviously, \( c_\mu = 0 \) for \( |\mu| = 0 \); further, let

\[
\mu(a) = \max_{c_\mu \neq 0} |\mu|,
\]

\[
\varepsilon(\mathcal{R}) = \min_{a \in \mathcal{R}} \mu(a).
\] (33)

Obviously, \( \varepsilon(\mathcal{R}) \geq 1 \); let us first assume that \( \varepsilon(\mathcal{R}) = 1 \). This means that \( c_0 x^0 + c_k x^k \in \mathcal{R} \) for some (not all zero) constants \( c_0, c_k \). But

\[
\tilde{\rho}_L(r^{-1}(I \otimes c_\mu x^\mu)) = c_\mu A_{\mu \nu} \otimes r^{-1}(I \otimes x^\nu).
\] (34)

Therefore \( x^\mu \in \mathcal{R} \) for all \( \mu \), i.e. \( \mathcal{R} = \ker \varepsilon_M \) and the corresponding calculus is trivial.

As the next step let us take \( \varepsilon(\mathcal{R}) = 2 \). It is straightforward to check that

\[
\tilde{\rho}_L(r^{-1}(I \otimes x^{\mu \nu})) = A_{\mu \alpha} A_{\nu \beta} \otimes r^{-1}(I \otimes x^{\mu \nu})
\] (35)

where

\[
x^{\mu \nu} \equiv x^\mu x^\nu + \frac{i}{\kappa} (g^{\mu \nu} x^0 - g^{0 \mu} x^\nu).
\] (36)

Due to the fact that \( \Lambda \)’s commute among themselves we can write a standard representation theory of Lorentz group. First of all, we note that \( r^{-1}(I \otimes x^{\mu \nu}) \) transform as a second order symmetric \( (x^{\mu \nu} = x^{\mu \nu}) \) due to (27), (36) tensor. It carries \( D^{(1,1)} \otimes D^{(0,0)} \) representation of Lorentz group. Let us first take all \( x^{\mu \nu} \) as generators of \( \mathcal{R} \). Then \( (x^0)^2 = x^{00} \in \mathcal{R} \) and \( x^1 x^0 = x^{10} \in \mathcal{R} \), i.e. \( [(x^0)^2, x^1] \in \mathcal{R} \); therefore \( x^1 x^0 + x^0 x^1 \in \mathcal{R} \) and \( x^0 x^1 \in \mathcal{R} \). However, \( x^{0 i} = x^{0 i} - \frac{i}{\kappa} x^i \in \mathcal{R} \) which implies \( x^i \in \mathcal{R} \). Then Poincaré invariance implies \( x^0 \in \mathcal{R} \) and our calculus is trivial.

To improve the situation we can only, due to condition (21), subtract \( D^{(0,0)} \) or \( D^{(1,1)} \). Obviously, subtracting \( D^{(1,1)} \) gives larger calculus, so we will substract \( D^{(0,0)} \). It is not difficult to check then that the following lemma holds.

**Lemma.** Let \( \mathcal{R} \subset \ker \varepsilon_M \) be right ideal generated by the elements

\[
x^{\mu \nu} + \frac{i}{\kappa} (g^{\mu \nu} x^0 - g^{0 \mu} x^\nu) - \frac{1}{n} g^{\mu \nu} \left( x^2 + \frac{i}{\kappa} (n-1) x^0 \right).
\] (37)

Then

(a) \( \mathcal{R} \) defines a left-\( \mathcal{P}_\kappa \)-covariant calculus on \( \mathcal{M}_\kappa \),
(b) \( a \in \mathcal{R} \) implies \( S(a)^* \in \mathcal{R} \),
(c) \( \ker \varepsilon_M / \mathcal{R} \) is spanned by \( x^\mu \) and by

\[
\varphi \equiv x^2 + \frac{i}{\kappa} (n-1) x^0.
\] (38)
Now, we can construct the relevant calculus. The left-invariant forms are
\begin{align}
\tau^\mu &= \pi r^{-1} (I \otimes x^\mu) = dx^\mu, \\
\tau &= \pi r^{-1} (I \otimes \varphi) = d\varphi - 2x_\mu dx^\mu
\end{align}
(39)

and they appear to be also right invariant. The commutation rules are easily derived according to the standard procedure of [5]
\begin{align}
[\tau^\mu, x^\nu] &= \frac{i}{\kappa} g^{0\mu} \tau^\nu - \frac{i}{\kappa} g^{0\nu} \tau^\mu + \frac{1}{n} g^{0\mu} \tau, \\
[\tau, x^\mu] &= -\frac{n}{\kappa^2} \tau^\mu
\end{align}
(40)

while the hermicity properties read
\begin{align}
(\tau^\mu)^* &= \tau^\mu, \\
\tau^* &= -\tau.
\end{align}
(41)

The left action of \(P_\kappa\) on \(M_\kappa\) is easily calculated to be
\begin{align}
\tilde{\rho}_L(\tau^\mu) &= \Lambda^\mu_\nu \otimes \tau^\nu, \\
\tilde{\rho}_L(\tau) &= I \otimes \tau.
\end{align}
(42)

In order to construct the external algebra we first verify property (25) for the bimodule homomorphism \(\sigma\): 
\(\sigma(\tau^\mu \otimes \tau^\nu) = \tau^\nu \otimes \tau^\mu\), \(\sigma(\tau \otimes \tau^\mu) = \tau^\mu \otimes \tau\), \(\sigma(\tau^\mu \otimes \tau) = \tau \otimes \tau^\mu\). The external algebra implied by \(\sigma\) takes the standard form
\begin{align}
\tau^\mu \wedge \tau^\nu &= -\tau^\nu \wedge \tau^\mu, \\
\tau \wedge \tau^\mu &= -\tau^\mu \wedge \tau.
\end{align}
(43)

Moreover,
\begin{align}
d\tau^\mu &= 0, \\
d\tau &= -2d\tau^\mu \wedge d\tau_\mu.
\end{align}
(44)

From the discussion carried out above it follows that the \(n + 1\)-dimensional calculus described by (39)–(43) is the lowest dimensional nontrivial calculus on \(M_\kappa\) covariant with respect to the left action of \(P_\kappa\). This is due to the fact that all differential calculi with \(\mu(R) \geq 3\) have higher dimensions.

Finally, let us compare our calculus with that proposed by Sitarz ([4]). Our equations (40) agree with equations (60) of [4] under the identification: \(x^\mu \rightarrow ix^\mu\), \(\tau \rightarrow \frac{1}{\kappa} \varphi\). In the twodimensional case there is also in agreement provided the replacement \(\tau \rightarrow \frac{2}{\kappa} \varphi\) is made; also the multiplication rules for one-forms (equations (58) of [4]) coincide in this case.

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