Integration of the Classical Action for the Quartic Oscillator in 1 + 1 Dimensions

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Received July 11, 2013; revised August 11, 2013; accepted August 18, 2013

Abstract

In this paper, we derive an explicit form in terms of end-point data in space-time for the classical action, i.e. integration of the Lagrangian along an extremal, for the nonlinear quartic oscillator evaluated on extremals.

Keywords: Action; Integral

1. Introduction

The action

\[ S_{qo} (t_a, y_a; t_b, y_b) = \int_{t_a}^{t_b} L_{qo} \left( y(t), \frac{dy}{dt} \right) dt \]  

is integrated along an extremal and expressed in terms of the space-time end-point data.

We begin in a well-known way by adding and subtracting the kinetic energy to the Lagrangian. Thus we obtain from (1.1), after changing the variable of integration in the remaining integral, the following equivalent expression.

\[
S_{qo} (t_a, y_a; t_b, y_b) = \int_{y_a}^{y_b} \left( \frac{m}{2} \frac{dy}{dt} \right)^2 - \frac{k_4 y^4}{4} \right) dt \]  

where \( L_{qo} = \frac{m}{2} \left( \frac{dy}{dt} \right)^2 - \frac{k_4 y^4}{4} \) equals the Lagrangian for the quartic oscillator in 1 + 1 dimensions, is integrated along an extremal and expressed in terms of the space-time end-point data \((t_a, y_a; t_b, y_b)\).

We start in a well-known way by adding and subtracting the kinetic energy to the Lagrangian. Thus we obtain from (1.1), after changing the variable of integration in the remaining integral, the following equivalent expression.

\[ S_{qo} (t_a, y_a; t_b, y_b) = \int_{y_a}^{y_b} \left( \frac{m}{2} \frac{dy}{dt} \right)^2 \]  

where \( E \) is the energy on the extremal (See e.g. Goldstein [1]). Equation (1.2) is the form of the action that we will start from and then derive by integrating the first term in (1.2), which we call the momentum integral, thus the desired expression for \( S_{qo} \) is obtained. (Some authors call this momentum integral the action.) For our convenience, we refer to the second term in (1.2) as the energy term. The derived action \( S_{qo} \) depends only on the end-point data in space-time.

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2. Alternative Derivation of the Quartic Oscillator Solution

To begin with, we must establish the sign conventions implied by (1.2) for the quartic oscillator

\[ y^2 \frac{d}{dy} y(t) dy \bigg|_{y=\text{extremal}} = \int m \left( \pm \sqrt{\frac{2E}{m}} \frac{1 - V^{\omega}(y)}{E} \right) dy \bigg|_{y=\text{extremal}}, \]  

(2.1)

where \( V^{\omega}(y) = \frac{k_4 y^4}{E} \).

Taking advantage of the periodicity of any extremal for the quartic oscillator \( q_0 \), we execute a change of variable to the angular variable \( \theta \) by setting

\[ \sin^2(\theta - \theta_0) = \frac{V^{\omega}(y)}{E} = \frac{k_4 y^4}{4E} \],

(2.2)

where

\[ y(\theta(t_0)) = y(\theta_0) = y(t_0) = y_0 = 0 \]

and

\[ \frac{dy}{dt}(\theta(t_0)) = \frac{dy}{dt}(\theta_0) = \frac{dy}{dt}(t_0) = \sqrt{2E/m} \]

and \( E = \) energy on the extremal. We have opted not to change the symbol for a function when it depends on a variable through a nested function in order to avoid unnecessarily heavy notation. Making the signs explicit, (2.1)-(2.2) yield

\[ y = \left( \frac{4E}{k_4} \right)^{1/4} \sin \left( \theta - \theta_0 \right) \left( \sin^2 \left( \theta - \theta_0 \right) \right)^{1/4}. \]  

(2.3)

and

\[ \frac{dy}{dt} = \frac{2E}{m} \cos \left( \theta - \theta_0 \right). \]  

(2.4)

Note, for future use (2.3) implies

\[ \frac{dy}{dt} = \left( \frac{4E}{k_4} \right)^{1/4} \frac{1}{2} \frac{\cos \left( \theta - \theta_0 \right)}{\left( \sin^2 \left( \theta - \theta_0 \right) \right)^{1/2}} d\theta. \]  

(2.5)

Now, we are in position to present an alternative derivation of the solution to Newton’s equations of motion (2.7) below for the quartic oscillator. It involves a parameterization of time in terms of the angular coordinate. As we shall see, this results in the time being given by a quadrature involving a known function of \( \theta \). Now differentiating (2.4) yields

\[ \frac{d^2 y}{dt^2} = \frac{2E}{m} \left( -\sin(\theta - \theta_0) \right) \frac{d\theta}{dt}. \]  

(2.6)

Or from Newton’s equation of motion for the quartic oscillator

\[ m \frac{d^2 y}{dt^2} = -k_4 y^3, \]  

(2.7)

we obtain

\[ \frac{-k_4 y^3}{m} = \frac{2E}{m} \sin \left( \theta - \theta_0 \right) \frac{d\theta}{dt}. \]  

(2.8)

Thus, it follows from (2.3) that we obtain the equation that yields \( t \) involving \( \theta \)

\[ dt = \left( k_4 E \right)^{-1/4} m^{1/2} 2^{-1} \left( \sin^2 \left( \theta - \theta_0 \right) \right)^{1/4} d\theta. \]  

(2.9a)

Or, it’s integrated form which yields \( t \) (in quadrature) involving a known function of \( \theta \)

\[ t(\theta) - t_0 = \frac{\theta}{\phi} \left( k_4 E \right)^{-1/4} m^{1/2} 2 \left( \sin^2 \left( \theta - \theta_0 \right) \right)^{1/4} d\theta'. \]  

(2.9b)

The inverse of (2.9a) is given by

\[ d\theta = \left( \frac{2k_4}{m} \right)^{1/2} \left( y^2 \right)^{1/2} dt, \]  

(2.10a)

and it’s integrated form is given by

\[ \theta(t) - \theta_0 = \frac{\theta}{\phi} \left( \frac{2k_4}{m} \right)^{1/2} \left( y^2(t) \right)^{1/2} dt', \]  

(2.10b)

where the integration is along an extremal.

The equivalence to the linearization map given in [2] is specified by setting \( \theta - \theta_0 = \omega(t - t_0) \), where \( k_2 = m\omega^2 \) is spring constant of the harmonic oscillator \( h o \) and \( t \) equals the time of the \( ho \) corresponding to \( t \) of the \( qo \).

Then (2.9b) and (2.10b) are equivalent to one half of the linearization map in [2]. The other half of the linearization map is given by

\[ \left( \frac{y^2(t)}{4E/k_4} \right)^{1/2} y(t) = \sin(\theta - \theta_0) = \frac{y_h(t)}{(2E/k_4)^{1/2}}. \]  

(2.11)

Equation (2.2) plus equation (2.4) imply

\[ \frac{4E}{k_4} \left[ y_o^4 + y_o^4 - 2 \left( y_o^2 \right)^{3/2} y_o \left( y_o^2 \right)^{3/2} y_o \cos(\theta_o - \theta_o) \right] \]  

(2.12)

where \( \theta_o - \theta_o \) is given by (2.10b).

Finally, in this paragraph, given the end-point data
how does one determine all other quantities. 

One is given \( (y_a, t_a) \) and \( (y_b, t_b) \) on a \( qo \) extremal. The linearization map yields \( x_a \) and \( x_b \) on the corresponding \( ho \) extremal as well as \( E_{ho} = E_{qo} = E \). This implies from (2.12) the \( ho \) time differences \( \hat{t}_a - \hat{t}_b \) and \( \hat{t}_b - \hat{t}_a \), where \( \hat{t} \) refers to \( ho \) times, are known. Now we can set \( t_a = \hat{t}_a \).

From [4], as a result of mapping extremals for the \( ho \) \( 1 \rightarrow 1 \) onto extremals to the \( qo \), we have from [4],

\[
\sin(\theta(t) - \theta_0) = (4E/k_a)^{1/2} \left[ (y_a^2)^{1/2} \frac{y_b \sin(\theta(t) - \theta_0)}{\sin(\theta_0 - \theta_0)} + (y_b^2)^{1/2} \frac{y_a \sin(\theta_0 - \theta(t))}{\sin(\theta_0 - \theta_0)} \right],
\]

(2.13)

and

\[
\cos(\theta(t) - \theta_0) = (4E/k_b)^{1/2} \left[ (y_b^2)^{1/2} \frac{y_a \cos(\theta(t) - \theta_0)}{\sin(\theta_0 - \theta_0)} - (y_a^2)^{1/2} \frac{y_b \cos(\theta_0 - \theta(t))}{\sin(\theta_0 - \theta_0)} \right],
\]

(2.14)

Now (2.13) and (2.14) imply e.g.

\[
\tan(\theta_h - \theta_0) = \frac{(y_b^2)^{1/2} \frac{y_b \sin(\theta_h - \theta_0)}{\sin(\theta_0 - \theta_0)}}{(y_a^2)^{1/2} \frac{y_a \cos(\theta_h - \theta_0)}{\sin(\theta_0 - \theta_0)} - (y_a^2)^{1/2} \frac{y_a \cos(\theta_0 - \theta_h)}{\sin(\theta_0 - \theta_0)}},
\]

where \( \hat{t}_a - \hat{t}_b = (\hat{t}_b - \hat{t}_a) \) and \( \omega \hat{t} = \theta(t) \) yields \( \theta_0 \).

Everything else follows from the development in Part 3.

3. Integration of \( \int_{y_a}^{y_b} \frac{m \, dy_{qo}}{dt} \, dy_{qo} \) \( \text{extremal} \)

The problem of integrating (1.2) is the problem of integrating (2.1). Therefore, using (2.2), (2.4), and (2.5), we obtain

\[
\int_{y_a}^{y_b} m \, \frac{d \theta}{dt} \, dy_{qo} \bigg|_{\text{extremal}} = \int_{y_a}^{y_b} \frac{2E}{m} \sqrt{\left( 1 - \frac{k_a y^4}{4E} \right)} \, dy_{qo} \bigg|_{\text{extremal}}
\]

\[
= \frac{2}{\omega} \int_{\theta_0}^{\theta_h} \cos(\theta - \theta_0) \left( \frac{4E}{k_a} \right)^{1/4} \left( \frac{1}{2} \right) \left( \sin^2(\theta' - \theta_0) \right)^{1/4} \cos(\theta' - \theta_0) \, d\theta'
\]

(3.1)

Effecting the integration by parts, where \( \frac{d}{d\theta} \left( \frac{d\theta}{d\theta'} \right) g = \frac{df}{d\theta} + \frac{dg}{d\theta}, f = (\sin^2(\theta - \theta_0))^{3/4} \) and \( g = \frac{2 \cos(\theta - \theta_0)}{3 \sin(\theta - \theta_0)} \) yields

\[
\int_{y_a}^{y_b} m \, \frac{dy_{qo}}{dt} \, dy_{qo} \bigg|_{\text{extremal}} = m \frac{2E}{m} \sqrt{\frac{4E}{k_a}} \frac{1}{2} \frac{1}{3} \frac{1}{2} \left( \frac{2}{m^{1/2}} \right) \left( \frac{k_a E^{1/4}}{1} \right) \left( \frac{m^{1/2}}{2} \right) \left( \frac{1}{k_a E^{1/4}} \right)
\]

\[
= \frac{2}{\omega} \left( \sin^2(\theta - \theta_0) \right)^{1/4} \frac{d\theta}{\sin(\theta - \theta_0)} + \left( \sin^2(\theta - \theta_0) \right)^{3/4} \frac{\cos(\theta - \theta_0)}{\sin(\theta - \theta_0)} \bigg|_{\theta_0}^{\theta_h}
\]

(3.2)

Finally, from (2.9b), we have

\[
\int_{y_a}^{y_b} m \, \frac{dy_{qo}}{dt} \, dy_{qo} \bigg|_{\text{extremal}} = \frac{4E}{3} (t_h - t_a) + \frac{2m^{1/2} (E)^{3/4}}{3k_a^{1/4}} \left( \sin^2(\theta - \theta_0) \right)^{3/4} \frac{\cos(\theta - \theta_0)}{\sin(\theta - \theta_0)} \bigg|_{\theta_0}^{\theta_h}
\]

(3.3)

where \( \theta - \theta_0 \) is given by (2.10b).
4. Determination of a $S_{qo}$

The developments in Part 2 and Part 3 lead directly to the following determination of $S_{qo}$.

It follows from (3.3) that (1.2) is given by

$$S_{qo}(t_a, y_t, t_b, y_b) = \int_{y_a}^{y_b} m \frac{d}{dt} \left[ -E(t_b - t_a) \right]_{\text{extremal}}$$

$$= 4E \left( \frac{1}{3} \right) \left( \frac{m k_4}{2} \right)^{1/2} \left( y^2(t) \right)^{1/2} \cos \left( \theta - \theta_0 \right) \sin \left( \theta - \theta_0 \right) - E(t_b - t_a)$$

$$= \frac{1}{3} \left( \frac{m k_4}{2} \right)^{1/2} \left( y^2(t) \right)^{3/2} \cos \left( \theta - \theta_0 \right) \sin \left( \theta - \theta_0 \right)$$

Therefore, using (2.10b), we obtain

$$S_{qo}(t_a, y_t, t_b, y_b)$$

$$= \frac{1}{3} \left( \frac{m k_4}{2} \right)^{1/2} \left( y^2(t) \right)^{3/2} \cos \left( \theta - \theta_0 \right) \sin \left( \theta - \theta_0 \right)$$

This is expressed in the endpoint variables as required. This implies

$$\frac{\partial S_{qo}}{\partial y_b} = p_{qo_b} = \left( \frac{m k_4}{2} \right)^{1/2} \left( y^2(t) \right)^{1/2}$$

$$= \left( 2mE \right)^{1/2} \cos \left( \frac{2k_4}{m} \right) \left( y^2(t) \right)^{1/2}$$

$$\frac{\partial S_{qo}}{\partial t_b} = \left( \frac{m k_4}{2} \right)^{1/2} \left( y^2(t) \right)^{1/2}$$

After using (2.11) this checks with $m$ times (2.4) for $p_{qo_b}$ and $\frac{\partial}{\partial t_a}$ obviously checks. The $a$-differentiations parallel the $b$-differentiations and yield

$$\frac{\partial S_{qo}}{\partial y_a} = -p_{qo_a} = -\left( \frac{m k_4}{2} \right)^{1/2} \left( y^2(t) \right)^{1/2}$$

$$= -\left( 2mE \right)^{1/2} \cos \left( \frac{2k_4}{m} \right) \left( y^2(t) \right)^{1/2}$$
5. Equivalent Actions

Here, we present two examples of equivalent actions as variations on this result. By equivalent we mean they are equal in value on extremals and they both produce the same Hamilton equations.

First Variation:

\[ \delta S_{\text{var}} = -\frac{1}{3} \left( \frac{mk_4}{2} \right)^{1/2} \left[ \left( y_\theta^2 \right)^{3/2} \sin \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \left( \frac{2k_4}{m} \right) \left( y^2(t') \right)^{1/2} \, dt' \right] \left( y_\theta^2 \right)^{1/2} - \frac{1}{3} E = + E. \]  

(4.4)

This variation follows from the identities

\[ \sin (\theta - \theta_0) = \sin (\theta \pm \theta_{\text{max}} - \theta_0) = \cos (\theta - \theta_{\text{max}}), \]

\[ \cos (\theta - \theta_0) = \cos (\theta \pm \theta_{\text{max}} - \theta_0) = - \sin (\theta - \theta_{\text{max}}), \]

which implies that (4.2) transforms to the expression

\[ S_{\text{var}}(t_a, y_a, t_b, y_b) = \frac{1}{2} m \left. \frac{dy}{dt} \right|_{\text{extremal}} - E(t_b - t_a) \big|_{\text{extremal}} \]

\[ = \frac{1}{3} \left( \frac{mk_4}{2} \right)^{1/2} \left[ \left( y_\theta^2 \right)^{3/2} \sin \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \left( \frac{2k_4}{m} \right) \left( y^2(t') \right)^{1/2} \, dt' \right] \left( y_\theta^2 \right)^{1/2} + \frac{1}{3} E (t_b - t_a). \]

(5.2)

Second Variation:

\[ S_{\text{var}}(t_a, y_a, t_b, y_b) = \frac{1}{2} m \left. \frac{dy}{dt} \right|_{\text{extremal}} - E(t_b - t_a) \big|_{\text{extremal}} \]

\[ = \frac{1}{3} \left( \frac{mk_4}{2} \right)^{1/2} \left[ \left( y_\theta^2 \right)^{3/2} \cos \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \left( \frac{2k_4}{m} \right) \left( y^2(t') \right)^{1/2} \, dt' \right] \left( y_\theta^2 \right)^{1/2} + \frac{1}{3} E (t_b - t_a). \]

(5.3)

Comment: The signs and the limits of integration have to be carefully watched in these calculations.

The identity

\[ -\left( y_\theta^2 \right)^{3/2} \cos (\theta_{\text{max}} - \theta_0) = -3y_\theta y_{\text{max}} \left( y_{\text{max}}^2 \right)^{1/2}, \]

\[ +2 \left( y_{\text{max}}^2 \right)^{3/4} \cos (\theta_{\text{max}} - \theta_0), \]

follows from \( \left( y_\theta^2 \right)^{1/2} y_\theta = \left( y_{\text{max}}^2 \right)^{1/2} y_{\text{max}} \cos (\theta_{\text{max}} - \theta_0). \)

Similarly for the \( a \) endpoint, thus we obtain the result reported in [4].

The results given in [4] were obtained before the integration result reported here in Part IV was obtained.

6. Conclusions

One can parallel the development in Parts 3 and 4 for an hierarchy with potential energies

\[ V_{2n}(y) = \frac{1}{2n} k_{2n} y^{2n} \]

(6.1)

Starting with setting

\[ \sin^2 (\theta - \theta_0) = \frac{V_{2n}(y)}{E} = \frac{k_{2n} y^{2n}}{2nE}, \]

(6.2)

one can parallel Part 3.

Then integration by parts in these cases is effected by

\[ \frac{d}{d\theta} fg = \frac{df}{d\theta} g + f \frac{dg}{d\theta}, \]

and

\[ g = \frac{n \cos (\theta - \theta_0)}{(n+1) \sin (\theta - \theta_0)}. \]

This then parallels the development in Part 4.
The linearization map for these cases is given in [2].

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