GENERATORS FOR HALL ALGEBRAS OF SURFACES

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Abstract. For a smooth surface $S$, Porta–Sala defined a categorical Hall algebra generalizing previous work in K-theory of Zhao and Kapranov–Vasserot. We construct semi-orthogonal decompositions for categorical Hall algebras of points on $S$. We refine these decompositions in K-theory for a topological K-theoretic Hall algebra.

1. Introduction

1.1. Hall algebras of surfaces. Let $S$ be a smooth surface over $\mathbb{C}$. For an algebraic class $\beta \in H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$, let $\mathcal{M}_\beta$ be the (derived) moduli stack of coherent sheaves on $S$ with support class $\beta$. For classes $\beta$ and $\gamma$, there are maps

\[(1) \quad \mathcal{M}_\beta \times \mathcal{M}_\gamma \xrightarrow{q_{\beta, \gamma}} \mathcal{M}_\beta,\gamma \xleftarrow{p_{\beta, \gamma}} \mathcal{M}_\beta + \gamma,\]

where $\mathcal{M}_\beta,\gamma$ is the corresponding stack of extensions. The map $q_{\beta, \gamma}$ is quasi-smooth and the map $p_{\beta, \gamma}$ is proper, so they induce functors

$$m_{\beta, \gamma} := p_{\beta, \gamma}^*q_{\beta, \gamma}^*: D^b(\mathcal{M}_\beta) \otimes D^b(\mathcal{M}_\gamma) \to D^b(\mathcal{M}_{\beta + \gamma}).$$

Porta–Sala [23] showed that the category $\bigoplus_\beta D^b(\mathcal{M}_\beta)$ is monoidal with respect to the functors $m_{\beta, \gamma}$. Taking the Grothendieck group of this category, we obtain the K-theoretic Hall algebra (KHA) of a surface which has been studied by Zhao [33] for sheaves of dimension zero and by Kapranov–Vasserot [15].

Categorical/ K-theoretic Hall algebras for quivers with potential are local version of these categories/ algebras. Particular cases of (equivariant) KHAs of quivers with potentials, namely preprojective KHAs, are expected to be positive parts of quantum affine groups constructed by Okounkov–Smirnov [20].

It is interesting to see whether KHAs of surfaces have properties similar to (positive parts of) quantum groups, for example if they satisfy a PBW theorem, if they are deformations of (the universal enveloping algebra of) a K-theoretic Lie algebra associated to the surface, or if they can be doubled to a Hopf algebra.

1.2. Semi-orthogonal decompositions of the HA. Let $S$ be a smooth surface over $\mathbb{C}$. For $d \in \mathbb{N}$, let $\mathcal{M}_d$ be the moduli stack of dimension zero sheaves of length $d$ on $S$. Let $\text{HA}(S) := \bigoplus_{d \in \mathbb{N}} D^b(\mathcal{M}_d)$. In Subsection 4.1.6 we define categories $\mathcal{M}(d)_w \subset D^b(\mathcal{M}_d)_w$ for $w \in \mathbb{Z}$. Consider the map

\[(2) \quad \pi_d: \mathcal{M}_d \to M_d := \text{Sym}^d(S).\]

Let $V^d_w$ be the set of partitions $A = (d_i, w_i)_{i=1}^k$ of $(d, w)$ with

$$\frac{w_1}{d_1} > \cdots > \frac{w_k}{d_k}.$$ 

For $A \in V^d_w$, let $\mathcal{M}_A := \otimes_{i=1}^k \mathcal{M}_i(d_i, w_i)$.
Theorem 1.1. There is a semi-orthogonal decomposition
\[ D^b(\mathfrak{M})_w = \left\langle \mathfrak{M}_A \right\rangle, \]
where the right hand side is after all partitions \( A \in V^d_w \). The order of categories is as in Subsection 2.2.6. The semi-orthogonal decomposition holds over \( \mathfrak{M} \) in the following sense. Let \( A < B \) be two partitions and let \( F \in \mathfrak{M}_A \) and \( G \in \mathfrak{M}_B \). Then
\[ R\pi_{d*} (R\text{Hom}_{\mathfrak{M}}(F,G)) = 0. \]

The proof of Theorem 1.1 is as follows. The map \( \pi \) is analytically (and formally) locally on \( \text{Sym}^d(S) \) described using the moduli stack of dimension \( d \) sheaves on an open subset \( U \subset \mathbb{A}^d \). The categorical Hall algebra of \( \mathbb{A}^d \) is equivalent to the preprojective Hall algebra of \( J \), the Jordan quiver, and also equivalent to the Hall algebra of the quiver \( \tilde{J} \) with three loops \( x, y, z \) and potential \( \tilde{W} := xyz - xzy \). We glue the decompositions of \( \text{HA} \left( \tilde{J}, \tilde{W} \right) \) from [21] to decompositions of \( \text{HA}(S) \).

It is interesting to see if Theorem 1.1 holds for Hall algebras of semistable sheaves of support \( \beta \in H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) \) on \( S \). By a result of Toda [27], the moduli stacks \( \pi^\beta : \mathcal{M}_{ss} \to \mathcal{M}_{ss} \) admit descriptions using quivers of potential \( (Q, W) \) over \( \mathcal{M}_{ss} \), and we can thus try to use the results in [21]. In loc. cit., the semi-orthogonal decompositions for \( \text{HA}(Q, W) \) depend on certain Weyl-invariant weights \( \delta_A \) for \( A = (d_i, w_i)_{i=1}^k \) a partition of \( (d, w) \). The main difficulty in proving an analogue of Theorem 1.1 for \( \mathcal{M}_{ss} \) is that the local weights \( \delta_A \) cannot be glued to a global \( \delta_A \in \text{Pic} \left( \mathfrak{M}_{ss} \right) \), so it is not clear how to define \( \mathfrak{M}_A \subset D^b(\mathfrak{M})_n \). In the case of \( \tilde{J} \), the Weyl-invariant weights \( \delta \) are multiples to each other and the categories of generators \( \mathfrak{M} \) do not depend on \( \delta \).

1.3. A PBW theorem for a topological KHA. Denote by \( \text{KHA}(S) \) the Grothendieck group of \( \text{HA}(S) \). Let
\[ \text{Sh}(S) := \bigoplus_{n \geq 1} (K_0(S^n)(z_1, \cdots, z_n))^\mathfrak{S}_n \]
be the algebra considered by Neguț [19], Zhao [33]. Zhao in loc. cit. constructed an algebra morphism
\[ \Phi_S : \text{KHA}(S) \to \text{Sh}(S). \]
There are analogous constructions for topological K-theory by applying Blanc’s topological K-theory [3] to the Porta–Sala monoidal category. Zhao’s construction applies to construct an algebra morphism
\[ \Phi_S^{\text{top}} : \text{KHA}^{\text{top}}(S) \to \text{Sh}^{\text{top}}(S). \]
Denote by \( \text{KHA}^{\text{top}}(S) \) the image of \( \Phi \) and by \( \text{KHA}^{\text{top}}(S)_{d,w} \) its graded \( (d, w) \)-part. In Subsection 5.3.2 we define subspaces \( P(d)_w \subset \text{KHA}^{\text{top}}(S)_{d,w} \otimes \mathbb{Q} \). Using a local argument and [21] Proposition 5.5] for \( \text{HA} \left( \tilde{J}, \tilde{W} \right) \), we prove a PBW-type theorem for \( \text{KHA}^{\text{top}} \). Let \( U^d_w \) be the set of partitions \( A = (d_i, w_i)_{i=1}^k \) of \( (d, w) \) such that
\[ \frac{w_1}{d_1} = \cdots = \frac{w_k}{d_k}. \]
Theorem 1.2. Assume $S$ is a smooth surface with $H^1(S, \mathbb{Q}) = 0$. Let $d \in \mathbb{N}$ and $w \in \mathbb{Z}$. There is a decomposition

$$K_0^{\text{top}}(\mathbb{M}(d)_w)_\mathbb{Q} \cong \bigoplus_{U^d_w} \bigotimes_{i=1}^k \text{Sym}^{\ell_i}(P(d_i)_w),$$

where the right hand side is after all partitions in $U^d_w$ with $\ell_i$ summands $(d_i, w_i)$ for $1 \leq i \leq s$ such that $d_1 < \cdots < d_s$.

Theorem 1.1 and Theorem 1.2 imply a PBW-type decomposition for $\text{KHA}^{\text{top}}(S)_\mathbb{Q}$ for $S$ as above, for example it implies that $\text{KHA}^{\text{top}}(S)_\mathbb{Q}$ is generated by the elements $x_{d_1, w_1} \cdots x_{d_k, w_k}$ for $x_{d_i, w_i} \in P(d_i)_w$ with $\frac{w_1}{d_1} \geq \cdots \geq \frac{w_k}{d_k}$. It would be interesting to obtain a version of Theorem 1.2 for the (algebraic) KHA. This would require an analogue of Theorem 5.1 for algebraic K-theory.

1.4. Previous work on Hall algebras of surfaces. The KHA of $A_2$ was studied by Schiffmann–Vasserot [26] and it is the positive part of $U_q^+ \hat{\mathfrak{gl}}_1$. For a curve $C$, the Hall algebra of (sheaves with compact support on) $T^*C$ is the Hall algebras for Higgs bundles on $C$ studied by Sala–Schiffmann [25], Minets [18].

Toda has studied the relation between Hall algebras and Donaldson–Thomas theory of local surfaces $\text{Tot}_\omega(S)$ in [28], [29], [30], [31].

Kapranov–Vasserot [15] proved a PBW theorem for CoHAs of dimension zero sheaves on surfaces using factorization algebras.

1.5. Structure of the paper. In Section 2, we review semi-orthogonal decompositions for categorical Hall algebras of quivers with potential. In Section 3, we compare some of these categories with categorical Hall algebras constructed from quotients of path algebras. A particular case of these quotients is the preprojective algebra of a quiver, but our results are more general and cover all the quotients that appear in local descriptions of the map (3). In particular, we obtain semi-orthogonal decomposition for $D_b(\mathfrak{M}_{\mathbb{Q}})$, where $\mathfrak{M}_{\mathbb{Q}}$ is the formal completion of $\mathfrak{M}$ along $\mathfrak{M}^{\mathbb{Q}}_p$ for any $p \in M_{\mathbb{Q}}$, see [28] Lemma 5.4.1, Section 7.4. As a particular case of semi-orthogonal decompositions of preprojective algebras, we prove Theorem 1.1 for $A_2$. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2.

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1.7. Notations and conventions. All stacks considered are over $\mathbb{C}$. All surfaces considered are smooth.

For $X$ a quasi-smooth stack, let $D^b(X)$ be the category of bounded complexes of coherent sheaves on $X$ and let $\text{Ind} D^b(X)$ be the Ind-completion of $D^b(X)$ [10], see [25] Section 3.1 for a brief reminder on these topics. For a brief reminder of functoriality of the quasi-smooth stacks used in this paper, see [23] Section 4.2, [28] Section 3.1. We denote by $\text{Hom}$ the morphism spaces in these categories and by $\text{Hom}$ the internal Hom-spaces in these categories. For the stacks considered in
the paper, there is a dualizing complex $\omega_X$ such that the dualizing functor is an equivalence
\[
\mathbb{D}_X(F) := R\text{Hom}_X(F, \omega_X) : D^b(X) \xrightarrow{\sim} D^b(X)^{op},
\]
see \cite[Section 2.2]{12}.

We denote by $\text{MF}^{gr}$ and by $\text{MF}^{gr}_{\text{qcoh}}$ the categories of coherent and quasi-coherent matrix factorizations, respectively, see \cite[Section 2.2]{28} for definitions and functoriality of these categories. These categories also admit dualizing functors $\mathbb{D}$ induced by the canonical sheaf of the ambient variety and they satisfy a Thom-Sebastiani theorem, see \cite[Section 3 and Section 5.1]{24}, \cite[Section 2.5]{9}.

The categories considered are dg and we denote by $\otimes$ the product of dg categories. For stacks $X$ and $Y$ considered in this paper, we have that $D^b(X \times Y) \cong D^b(X) \otimes D^b(Y)$ by \cite{2}.

We use the notation $\mathbb{L}$ for formal completions of $X$ along a specified substack. For a morphism $f$, let $\mathbb{L}_f$ be its cotangent complex. For a stack $X \to \text{Spec} \, \mathbb{C}$, denote by $\mathbb{L}_X$ its cotangent complex.

2. Hall algebras of quivers and dimensional reduction

2.1. Preliminaries.

2.1.1. Let $(Q, W)$ be a symmetric quiver with potential. Let $I$ and $E$ be the sets of vertices and edges, respectively, of $Q$. For $d \in \mathbb{N}^I$, consider the stack
\[
X(d) = R(d)/G(d)
\]
of representations of $Q$ of dimension $d$. Fix maximal torus and Borel subgroups $T(d) \subset B(d) \subset G(d)$. We use the convention that the weights of the Lie algebra of $B(d)$ are negative; it determines a dominant chamber of weights of $G(d)$. Let $M$ be the weight space of $G(d)$, let $M_R := M \otimes \mathbb{Z} \mathbb{R}$, and let $M^+ \subset M$ and $M_R^+ \subset M_R$ be the dominant chambers. When we want to emphasize the dimension vector, we write $M(d)$ etc. Denote by $N$ the coweight lattice of $G(d)$ and by $N_R := N \otimes \mathbb{Z} \mathbb{R}$. Let $(\langle, \rangle)$ be the natural pairing between $N_R$ and $M_R$.

Let $\mathbb{G}_a$ be the Weyl group of $G(d)$. For $\chi \in M(d)^+$, let $\Gamma(\chi)$ be the representation of $G(d)$ of highest weight $\chi$.

Consider the regular function $\text{Tr} \, W : X(d) \to \mathbb{A}^1_\mathbb{C}$. Let $X(d)_0 := X(d) \times_{\mathbb{A}^1_\mathbb{C}} 0$ be its (derived) zero fiber.

Denote by $SG(d) := \ker (\text{det} : G(d) \to \mathbb{C}^*)$.

2.1.2. Assume there is an extra $\mathbb{C}^*$-action on $R(d)$ which commutes with the action of $G(d)$ on $R(d)$ and such that $\text{Tr} \, W$ is of weight 2. Consider the category of graded matrix factorizations $\text{MF}^{gr}(X(d), W)$ for the regular function $\text{Tr} \, W$ \cite[Section 2.2]{28}. It is equivalent to the graded category of singularities $D^b_{\text{sg}}(X(d)_0)$.

2.1.3. The action of $z \cdot \text{Id} \subset G(d)$ on $R(d)$ is trivial. Let $D^b(X(d))_w$ be the category of complexes on which $z \cdot \text{Id}$ acts with weight $w$. We have an orthogonal decomposition $D^b(X(d)) = \bigoplus_{w \in \mathbb{Z}} D^b(X(d))_w$.

More generally, for a stack $X$ with a trivial action of a torus $T$, there is a decomposition
\[
D^b(X) = \bigoplus_{\chi \in \chi(T)} D^b(X)_\chi.
\]
where \( X(T) \) is the character lattice of \( T \). For \( \chi \in X(T) \), denote the projection functor by
\[
\beta^{\chi} : D^b(\mathcal{X}) \to D^b(\mathcal{X})_{\chi}.
\]

There are analogous definitions and decompositions for categories of quasi-coherent sheaves, for \( \text{MF}^{\text{gr}} \), and for \( \text{MF}^{\text{qcoh}} \).

2.1.4. Denote by \( \beta^j_i \) the simple roots of \( G(d) \) for \( i \in I \) and \( 1 \leq j \leq d_i \), where \( d = (d^i)_{i \in I} \in \mathbb{N}^I \), and by \( \rho \) half the sum of positive roots of \( G(d) \). We denote by \( 1_d := z \cdot \text{Id} \) the diagonal cocharacter of \( G(d) \). Consider the real weights
\[
\nu_d := \sum_{j \leq d_i} \beta^j_i,
\]
\[
\tau_d := \frac{\nu_d}{(1_d, \nu_d)}.
\]

2.1.5. For a cocharacter \( \lambda : \mathbb{C}^* \to SG(d) \), consider the maps of fixed and attracting loci
\[
(5) \quad \mathcal{X}(d)^\lambda \xrightarrow{\mathcal{X}(d)^{\lambda_>,0}} p_\lambda \xrightarrow{\mathcal{X}(d)}.\]
The map \( q_{\lambda} \) is an affine bundle map, in particular it is smooth, and the map \( p_\lambda \) is a proper map. We say that two cocharacters \( \lambda \) and \( \lambda' \) are equivalent if \( \lambda \sim \lambda' \) and \( \lambda \) and \( \lambda' \) have the same fixed and attracting stacks as above. For a cocharacter \( \lambda \) of \( SG(d) \), define
\[
(6) \quad n_\lambda = -\langle \lambda, \det \mathcal{F} \rangle \mathcal{X}(d)^{\lambda_>,0} \mathcal{X}(d)^{\lambda_>=0} \cdot \mathcal{X}(d).
\]
Consider the natural inclusion \( \alpha : 0/G(d) \hookrightarrow \mathcal{X}(d) \). We use the notation \( \mathcal{F}|_0 := \alpha^*(\mathcal{F}) \) for the restriction of \( \mathcal{F} \) at the origin. For a cocharacter \( \lambda \), we have an associated ordered partition \( d_1 + \cdots + d_k = d \) such that \( \mathcal{X}(d)^{\lambda} \cong \times_{i=1}^k \mathcal{X}(d_i) \); the order is induced by the choice of \( B(d) \subset G(d) \). Define the length \( \ell(\lambda) := k \).

Define the polytope \( \mathbb{W} \) by
\[
\mathbb{W} := \text{span} [0, \beta] \oplus \mathbb{R} \tau_d \subset M(d)_{\mathbb{R}},
\]
where the span is after all weights \( \beta \) of \( R(d) \). Let \( \delta \in M^{SG(d)}_{\mathbb{R}} \) and define \( N(d; \delta) \) as the subcategory of \( D^b(\mathcal{X}(d)) \) with complexes \( \mathcal{F} \) such that
\[
-\frac{n_\lambda}{2} + \langle \lambda, \delta \rangle \leq \langle \lambda, \mathcal{F}|_0 \rangle \leq \frac{n_\lambda}{2} + \langle \lambda, \delta \rangle,
\]
for all cocharacters \( \lambda \) of \( SG(d) \). Alternatively, it is the subcategory of \( D^b(\mathcal{X}(d)) \) generated by \( O_{\mathcal{X}(d)} \otimes \Gamma(\chi) \) for \( \chi \) a dominant of weight of \( G(d) \) such that
\[
\mathcal{X} + \rho + \delta \in \frac{1}{2} \mathbb{W}.
\]

Let \( M(d; \delta) \) be \( \text{MF}^{\text{gr}}(N(d; \delta), W) \), the category of matrix factorizations with factors in \( N(d; \delta) \). For an ordered partition \( (d_1, \cdots, d_k) \) of \( d \), fix an antidominant cocharacter \( \lambda_{d_1, \cdots, d_k} \) which induces the maps
\[
\times_{i=1}^k \mathcal{X}(d_i) \cong \mathcal{X}^{\lambda} \xrightarrow{q_{\lambda}^*} \mathcal{X}^{\lambda_>,0} \mathcal{X}.
\]
The multiplication is induced by the functor \( p_\lambda q_{\lambda}^* \). We may drop the subscript \( \lambda \) in the functors \( p_\ast \) and \( q^* \) when the cocharacter \( \lambda \) is clear.
2.1.6. Let \( e = (e_i)_{i=1}^l \) and \( d = (d_i)_{i=1}^k \) be two partitions of \( d \in \mathbb{N}^I \). We write \( e \geq d \) if there exist integers

\[
a_0 = 0 < a_1 < \cdots < a_{k-1} \leq a_k = l
\]

such that for any \( 0 \leq j \leq k - 1 \), we have

\[
\sum_{i=a_j+1}^{a_{j+1}} e_i = d_{j+1}.
\]

There is a similarly defined order on pairs \((d, w) \in \mathbb{N}^I \times \mathbb{Z}\).

2.1.7. Let \( (d_i)_{i=1}^k \) be a partition of \( d \). There is an identification

\[
\bigoplus_{i=1}^k M(d_i) \cong M(d),
\]

where the simple roots \( \beta^i_j \) in \( M(d_1) \) correspond to the first \( d_1 \) simple roots \( \beta^i_j \) of \( d \) etc.

2.1.8. Let \( I \) be a set. Assume there is a set \( O \subset I \times I \) such that for any \( i, j \in I \) we have that \((i, j) \in O\), or \((j, i) \in O\), or both \((i, j) \in O\) and \((j, i) \in O\). If \((i, j) \in O\), write \( i > j \).

Let \( T \) be a triangulated category. We will construct semi-orthogonal decompositions

\[
T = \langle A_i \rangle,
\]

by subcategories \( A_i \) with \( i \in I \) such that for any \( i, j \in I \) with \( i > j \) and objects \( A_i \in A_i, A_j \in A_j \), we have that:

\[
\text{RHom}_T(A_j, A_i) = 0.
\]

If there exists a minimal element \( o \) in \( I \), then the inclusion \( A_o \hookrightarrow T \) admits a right adjoint \( R : T \to A_o \).

2.2. Semi-orthogonal decompositions of categorical Hall algebras of quivers with potential. In this Section, we recall the semi-orthogonal decomposition of categorical Hall algebras of quivers with potential from \[22\] Theorem 1.1. We first need to discuss some preliminary notions. We fix a dimension vector \( d \in \mathbb{N}^I \). Denote by \( Q^d \) the subquiver of \( Q \) with vertices \( i \in I \) such that \( d_i > 0 \). We assume that \( Q = Q^d \).

2.2.1. For \( \chi \) a weight in \( M_{\mathbb{R}} \), define its \( r \)-invariant to be the smallest nonnegative real number \( r \) such that \( \chi \in r\mathbb{W} \). There exists a maximal antidominant cocharacter \( \lambda \) such that

\[
\langle \lambda, \chi \rangle = -r(\lambda, N^{\lambda>0}).
\]

The \( r \)-invariant is a measurement of how close a weight is from the polytope \( \frac{1}{2}\mathbb{W} \) used in the definition of the categories \( \mathbb{M}(d; \delta) \).
2.2.2. We define a tree $T$ of partitions which will help us keep track of decompositions of weights in $M(d)_{\mathbb{R}}$ and, in particular, will specify the order in the semi-orthogonal decomposition from Theorem 2.4.

The tree $T = (I, E)$ is defined as follows: the set $I$ of vertices has elements corresponding to partitions $d$ of any dimension vector $d \in \mathbb{N}^I$ and the set $E$ has edges from $d = (d_i)_{i=1}^k$ to $e$ if $e$ is a partition of $d_i \neq d$ for some $1 \leq i \leq k$.

A set $T \subseteq I$ is called a tree of partitions if it is finite and has a unique element $v \in T$ with in-degree zero. Let $\Omega \subseteq T$ be the set of vertices with out-degree zero. Define the Levi group associated to $T$:

$$L(T) := \prod_{\rho \in \Omega} L(\rho).$$

2.2.3. We recall the discussion from [21, Subsection 3.1.2]. Let $\chi \in M^+_{\mathbb{R}}$. Then there exists a tree $T$ of antidominant cocharacters with associated Levi group $L$, see Subsection 2.2.2 such that there exists $\psi \in \frac{1}{2} W$ with

$$\chi = -\sum_{j \in T} r_j N_j + \psi,$$

and if $i, j \in T$ are vertices such that there exists a path from $i$ to $j$, then $r_i > r_j > \frac{1}{2}$. For $d_a$ a summand of a partition of $d$, denote by $M(d_a) \subset M(d)$ in the decomposition from Subsection 2.1.7. The weight $N_j$ is defined as follows. Assume that $j$ is a partition of a dimension vector $d_a \in \mathbb{N}^I$. Let $W_j \subset W$ be the set of weights $\beta$ of $R(d_a)$ with $\langle \lambda_j, \beta \rangle > 0$. Define

$$N_j := \sum_{w \in W_j} \beta.$$

We explain the idea behind the decomposition (7), for more details see [21, Subsection 3.1.2]. If $\chi$ is $r$-invariant, we locate a face of the polytope $r \mathcal{W}$ on which $\chi$ lies. Assume this face is given by the cocharacter $\lambda$ and that $\chi$ lies in the interior of this face. We then write

$$\chi = -rN_{\lambda > 0} + \psi,$$

where $\psi$ is a weight with $r$-invariant $s < r$. Repeating the above procedure for the weight $\psi$ and letting $r := r_1$ and $N_1 := N_{\lambda > 0}$, we obtain the desired decomposition.

2.2.4. We assume that $Q$ is connected and that $Q$ is not $Q^o$. Fix $\delta \in M^o_{\mathbb{R}}$. Let $\chi \in M^+$ and consider the standard form (7):

$$\chi + \rho + \delta = -\sum_{j \in T} r_j N_j + \psi,$$

for $\psi \in \frac{1}{2} \mathcal{W}$. Let $L \cong \times_{i=1}^k G(d_i)$. Write $\chi = \sum_{i=1}^k \chi_i$ with $\chi_i \in M(d_i)^+$ for $1 \leq i \leq k$ and consider the associated partition

$$A = A_\chi := (d_i, w_i)_{i=1}^k,$$

where $w_i = \langle 1_{d_i}, \chi_i \rangle$. Let $S^d_{\psi}(\delta)$ be the set of partitions $A$ for which there exists $\chi \in M^+$ such that $A_\chi = A$. Let $T^d_{\psi}(\delta)$ be the set of partitions $A = (d_i, w_i)_{i=1}^k$ for which there exist $\chi = \sum_{i=1}^k \chi_i \in M^+$ with $\chi_i \in M(d_i)^+$ for $1 \leq i \leq k$, $w_i = \langle 1_{d_i} \rangle$, such that

$$\chi + \rho + \delta = -\frac{1}{2} N_{\lambda > 0} + \psi.$$
for a weight ψ in the interior of $\frac{1}{2}W$. To any such partition $A$ with cocharacter $\lambda$, associate the weight

$$(9) \quad \chi_A := -\sum_{j \in T} r_j N_j - \rho^{\lambda < 0} - \delta.$$  

For $1 \leq i \leq k$, consider weights $\delta_{Ai} \in M(d_i)_{\mathbb{R}}^{\mathbb{E}_{d_i}}$ defined by

$$(10) \quad -\chi_A = \sum_{i=1}^{k} \delta_{Ai} \in M(d)_{\mathbb{R}} \cong \bigoplus_{i=1}^{k} M(d_i)_{\mathbb{R}}.$$  

2.2.5. In the current and next Subsections, we explain the order in the semi-orthogonal decomposition from Theorem 2.1.

Assume first that $Q = Q^0$. Then $\delta$ is a multiple of $\tau_d$. Let $S_{w_i}(\delta)$ be the set of partitions $(1, w_i)_{j=1}^{d_i}$ of $(d, w) \in \mathbb{N} \times \mathbb{Z}$ with $w_1 \geq \cdots \geq w_d$.

Assume that $Q$ is a disconnected quiver and let $\delta \in M(d)_{\mathbb{E}_d}$. If $Q$ is a disjoint union of connected quivers $Q_j$ for $j \in J$, write $d_j$ and $\delta_j$ for the corresponding dimension vector and Weyl invariant weight of $Q_j$ for $j \in J$. Let $C$ be the set of partitions $(w_j)_{j \in J}$ of $w$. Let

$$S_{w_i}(\delta) := \bigcup_C \left( \times_{j \in J} S_{w_j}(\delta_j) \right).$$  

2.2.6. We explain how to compare partitions in $V_{w_i}(\delta)$. Assume first that $Q$ is connected and that $Q$ is not $Q^0$. Consider two partitions $A = (d_i, w_i)_{i=1}^{k}$ and $B = (e_i, v_i)_{i=1}^{l}$ in $S_{w}(\delta)$. Let $\chi_A$ and $\chi_B$ be two weights with associated sets $A$ and $B$:

$$\chi_A + \rho + \delta := -\sum_{j \in T_A} r_{A,j} N_{A,j} + \psi_A,$n
$$\chi_B + \rho + \delta := -\sum_{j \in T_B} r_{B,j} N_{B,j} + \psi_B.$$  

The set $R_{w}(\delta) \subset S_{w}(\delta) \times S_{w}(\delta)$ contains pairs $(A, B)$ for which there exists $c \geq 1$ such that $r_{A,c} \geq r_{B,c}$ and $r_{A,i} = r_{B,i}$ for $i < c$, or for which there exists $c \geq 1$ such that $r_{A,i} = r_{B,i}$ for $i \leq c$, $\lambda_{A,i} = \lambda_{B,i}$ for $i < c$, and $\lambda_{A,c} > \lambda_{B,c}$, or with $A = B$.

Let $O_{w}(\delta) := S_{w}(\delta) \times S_{w}(\delta) \setminus R_{w}(\delta)$.

For $Q^0$, let $R_{w}(\delta) = \{(A, A) \mid A \in S_{w}(\delta)\}$, and let $O_{w}(\delta) := S_{w}(\delta) \times S_{w}(\delta) \setminus R_{w}(\delta)$.

Assume $Q$ is a disconnected quiver. We continue with the notation from the previous Subsection. Consider the set $R_{w}(\delta) \subset S_{w}(\delta) \times S_{w}(\delta)$ for the quiver $Q_j$, let

$$R_{w}(\delta) := \bigcup_C \left( \times_{j \in J} R_{w_j}(\delta_j) \right) \subset S_{w}(\delta) \times S_{w}(\delta),$$  

and let $O_{w}(\delta) := S_{w}(\delta) \times S_{w}(\delta) \setminus R_{w}(\delta)$.

For a partition $d = (d_i)_{i=1}^{k}$ of $d \in \mathbb{N}^k$, denote by $S_{w}(\delta)$ the subset of $S_{w}(\delta)$ with partitions $A = (d_i, w_i)_{i=1}^{k}$ for weights $w_i \in \mathbb{Z}$.

2.2.7. For $A = (d_i, w_i)_{i=1}^{k}$ in $S_{w}(\delta)$, let $N_{A}(\delta) := \otimes_{i=1}^{k} N(d_i, w_i)$ and $M_{A}(\delta) := M_{\mathbb{E}_{d}}(N_A(\delta), W)$, the category of matrix factorizations with factors in $N_A(\delta)$.

Define similarly $M_{A}(\delta)$ for $A$ in $S_{w}(\delta)$. Let $\lambda_A := \lambda_{d_1, \ldots, d_k}$ and $\chi_A := (w_i)_{i=1}^{k}$ be the cocharacter of $SG(d)$ and the weight of $(\mathbb{C}^*)^k$ associated to $A$.  


2.2.8. Let \( a \in \mathbb{Z} \). Fix an ordered partition \( (d_1, \cdots, d_k) \) of \( d \) and let \( \lambda := \lambda_{d_1, \cdots, d_k} \) be an associated cocharacter of this partition. The set of \( A = (d_i, w_i)_{i=1}^k \) in \( S_{w}^d(\delta) \) such that \( (\lambda, g) \geq a \) for \( g \in \mathbb{N} \) is finite. The statement follows by induction on \( \sum_{i=1}^k d_i \) and the fact that \( (\lambda, g) \) is bounded above for any \( A \) in \( S_{w}^d(\delta) \).

2.2.9. We now state [21, Theorem 1.1].

**Theorem 2.1.** Let \( \delta \in M_{\mathbb{Z}}^{\mathbb{C}^d} \). Consider \( A = (d_i, w_i)_{i=1}^k \) in \( S_{w}^d(\delta) \). Let \( \lambda := \lambda_{d_1, \cdots, d_k} \) be the cocharacter associated to \( A \). The functor

\[
p_{A*}q_{A}^*: \mathcal{M}_A(\delta) \rightarrow \text{MF}^{gr}(X(d), W)_w
\]

is fully faithful. There is a semi-orthogonal decomposition

\[
\text{MF}^{gr}(X(d), W)_w = \langle \mathcal{M}_A(\delta) \rangle,
\]

where the right hand side contains all \( A \) in \( S_{w}^d(\delta) \). The order of categories in the semi-orthogonal decomposition respects the order from Subsection 2.1.3, see also Subsection 2.2.8.

2.2.10. We explain how to construct the adjoint of \( p_{A*}q_{A}^* \) from Theorem 2.1.

The torus \( T = (\mathbb{C}^*)^k \) acts trivially on \( \times_{i=1}^k X(d_i) \), where the ith factor of \( T \) is the group from Subsection 2.1.3. For an ordered tuplet \( A = (d_i, w_i)_{i=1}^k \) with \( k \geq 2 \) in \( S_{w}^d(\delta) \), let \( \lambda = \lambda_{d_1, \cdots, d_k} \) and \( \chi_A := (w_i)_{i=1}^k \) be the cocharacter of \( SG(d) \) and the weight of \( T \) associated to \( A \). Let

\[
\text{MF}^{gr}(X(d), W)_\text{below} \subset \text{MF}^{qcoh}_{\text{gr}}(X(d), W)
\]

be the subcategory of objects \( F \) such that \( \beta_w(F) \) is coherent for all weights \( w \in \mathbb{Z} \) of \( \lambda \) and it is zero for \( w \) small enough. Define similarly \( \text{MF}^{gr}(X(d), W)_\text{above} \). The functor

\[
F := p_{A*}q_{A}^*: \text{MF}^{gr}\left(\times_{i=1}^k X(d_i), W\right)_{\chi_A} \rightarrow \text{MF}^{gr}(X(d), W)_w
\]

is fully faithful and has a right adjoint

\[
(11) \quad R := q_{A*}p_{A}^! : \text{MF}^{gr}(X(d), W)_w \rightarrow \text{MF}^{gr}\left(\times_{i=1}^k X(d_i), W\right)_{\text{above}}.
\]

The proof that the functor is fully faithful is as in [21, Proposition 7.16]. Its image lies in \( \text{MF}_{\text{above}}^{gr} \) by [28, Lemma 2.2.3] for the cocharacter \( \lambda^{-1} \). We check that

\[
L := \mathcal{D}_{\chi A} \mathcal{R} \mathcal{D}_{\chi A} : \text{MF}^{gr}(X(d), W)_w \rightarrow \text{MF}^{gr}\left(\times_{i=1}^k X(d_i), W\right)_{\text{below}}
\]

is a left adjoint to \( F \). First, we have that

\[
(12) \quad \mathcal{D}_{\chi A} F = F \mathcal{D}_{\chi A}.
\]

Indeed, for \( A \) in \( \text{MF}^{gr}\left(\times_{i=1}^k X(d_i), W\right) \), we have that

\[
\mathcal{R} \mathcal{H} \mathcal{O} m_{\chi A}(p_{A*}q_{A}^* A, \omega_{\chi A}) = p_{q}^{*} \mathcal{R} \mathcal{H} \mathcal{O} m_{\chi A}(A, \omega_{\chi A}).
\]

Let \( \omega_{p} = \det L_{p}[^{\dim p}] \), \( \omega_{q} = \det L_{q}[^{\dim q}] \), and let \( C \) be in \( \text{MF}^{gr}\left(\times_{i=1}^k X(d_i), W\right) \). By the formulas in [28, Subsection 2.2] and using that \( Q \) is symmetric,

\[
p_{q}^{*} C = p_{*} q^{*} (C \otimes \omega_{p} \otimes \omega_{q}) = p_{*} q^{*} C.
\]

Next, let \( A \) be as above and let \( B \) be in \( \text{MF}^{gr}(X(d), W) \). The statement (12) to be proved is:

\[
\mathcal{R} \mathcal{H} \mathcal{O} m_{\chi A}(B, F A) = \mathcal{R} \mathcal{H} \mathcal{O} m_{\chi A}(\mathcal{D}_{\chi} \mathcal{R} \mathcal{D}_{\chi} B, A).
\]
This follows from the natural isomorphisms:

\[ R\text{Hom}_{\chi} (\mathbb{D}_X R \mathbb{D}_X B, A) \cong R\text{Hom}_{\chi} (\mathbb{D}_X A, R \mathbb{D}_X B) \cong R\text{Hom}_{\chi} (\mathbb{D}_X F A, \mathbb{D}_X B) \cong R\text{Hom}_{\chi} (B, F A), \]

where the first and fourth isomorphisms are induced by duality, the second by adjunction of \( F \) and \( R \), and the third by \([13]\). The functor \( F \) has thus left adjoints

\[ \Delta_{\lambda} := \beta_{\chi} L : \text{MF}^{\text{gr}}(\mathcal{X}(d), W) \rightarrow \text{MF}^{\text{gr}} \left( \bigotimes_{i=1}^{k} \mathcal{X}(d_i), W \right), \]

The functor \( L \) in \([12]\) has image in \( \text{MF}^{\text{gr}}_{\text{below}} \) and so \( \Delta_{\lambda}(\mathcal{F}) = 0 \) for all but finitely many sets \( \lambda \) in \( S_{w}(\delta) \) by the discussion in Subsection \([2.2.8]\).

2.2.11. By Theorem \([2.1]\), the inclusion \( \mathcal{M}(d; \delta) \subset \text{MF}^{\text{gr}}(\mathcal{X}(d), W) \) has a right adjoint \( R \). Then \( L := \mathbb{D}_X R \mathbb{D}_X \) is a left adjoint to the inclusion.

3. Comparison of Hall algebras via dimensional reduction

3.1. Koszul equivalence.

3.1.1. Consider a quiver \( \tilde{Q} = (I, \tilde{E}) \) where \( \tilde{E} = E \cup C \). Let \( Q = (I, E) \) and \( Q' = (I, C) \). The group \( \mathbb{C}^* \) acts on representations of \( \tilde{Q} \) by scaling the linear maps corresponding to edges in \( C \) with weight 2. Consider a potential \( \tilde{W} \) of \( \tilde{Q} \) on which \( \mathbb{C}^* \) acts with weight 2. The set \( C \) is called a cut for \( (\tilde{Q}, \tilde{W}) \) in the literature.

Denote by \( \widehat{\mathcal{X}}(d) \) the moduli stack of representations of dimension \( d \) for the quiver \( \tilde{Q} \) and by \( \mathcal{X}(d) \) the analogous stack for the quiver \( Q \). We consider the category of graded matrix factorizations \( \text{MF}^{\text{gr}}(\widehat{\mathcal{X}}(d), W) \) with respect to the action of the group \( \mathbb{C}^* \) mentioned above. Denote the representation space of \( Q' \) by \( C(d) \), so \( C(d) \) is the vector space

\[ C(d) := \prod_{c \in C} \text{Hom} \left( \mathbb{C}^{d_{s(c)}}, \mathbb{C}^{d_{t(c)}} \right), \]

where \( s, t : C \rightarrow I \) are the source and target maps. The space \( C(d) \) has a natural action of \( G(d) := \prod_{i \in I} GL(d_i) \) and thus there is a natural \( G(d) \)-equivariant vector bundle \( \mathcal{O}_{R(d)} \otimes C(d) \). We abuse notation and also denote by \( C(d) \) the corresponding vector bundle on the stack \( \mathcal{X}(d) \).

Write

\[ \tilde{W} = \sum_{c \in C} cW_c, \]

where \( W_c \) is a path of \( Q \). Define the algebra \( P = \mathbb{C}[\tilde{Q}]/I \), where \( I \) is the two-sided ideal generated by \( W_c \) for \( c \in C \). The potential \( \tilde{W} \) induces a section of the dual vector bundle \( C(d)^{\vee} \), and thus a map \( \partial : C(d) \rightarrow \mathcal{O}_{\mathcal{X}(d)} \). The moduli stack of representations of \( P \) of dimension \( d \) is the Koszul stack

\[ \mathfrak{P}(d) := \text{Spec} \left( \mathcal{O}_{R(d)} \left[ C(d)[1]; \partial \right] \right) / G(d). \]

There is an equivalence of categories \([14], [28] \text{ Theorem 2.3.3}] \), called the Koszul equivalence or dimensional reduction:

\[ \Phi : D^b(\mathfrak{P}(d)) \cong \text{MF}^{\text{gr}}(\widehat{\mathcal{X}}(d), \tilde{W}). \]
3.1.2. Assume we are in the setting on Subsection 3.1 and that \( \tilde{Q} \) is symmetric. Let \( d \in \mathbb{N}^I \). Recall that the representation space of \( Q' \) is \( C(d) \). We also denote by \( C(d) \) the natural vector bundle on \( \mathcal{X}(d) \). The dual \( C(d)^\vee \) is naturally isomorphic to \( \overline{C}(d) \), the representation space of the opposite quiver \( Q' \). There is a vector bundle \( C(d)^{\lambda \leq 0} \cong (\overline{C}(d)^{\lambda \geq 0})^\vee \) on \( \mathcal{X}(d)^{\lambda \geq 0} \). For a cocharacter \( \lambda \) of \( SG(d) \), define

\[
\Psi(d)^{\lambda \geq 0} := \text{Spec} \left( \mathcal{O}_{R(d)^{\lambda \geq 0}} \left[ C(d)^{\lambda \leq 0}[1]; \partial \right] \right) / G(d).
\]

For dimension vectors \( d \) and \( e \), let \( \lambda := \lambda_{d,e} \) be the cocharacter corresponding to the partition \( (d, e) \) of \( d + e \). Define \( \Psi(d, e) := \Psi(d + e)^{\lambda \geq 0} \). There are quasi-smooth maps \( q_{d,e} \) and proper maps \( p_{d,e} \), see [32, Section 2.2] for the case of preprojective algebras:

\[
(17) \quad \Psi(d) \times \Psi(e) \xleftarrow{q_{d,e}} \Psi(d, e) \xrightarrow{p_{d,e}} \Psi(d + e).
\]

Recall the Koszul equivalence (16):

\[
\Phi : D^b(\Psi(d)) \cong \text{MF}_{gr}(\overline{\mathcal{X}(d)}, \tilde{W}).
\]

We drop the subscripts for the maps \( p \) and \( q \) when they are clear. Let \( \lambda \) be a cocharacter of \( SG(d) \). Define the line bundle on \( \overline{\mathcal{X}(d)}^\lambda \):

\[
(18) \quad \omega_\lambda := \text{det} \left( C(d)^{\lambda \leq 0} / C(d)^{\lambda} \right)
\]

**Proposition 3.1.** Let \( d \) be a dimension vector and \( \lambda \) a cocharacter of \( SG(d) \). Define the functor

\[
\tilde{m} := p_\ast q_\ast (- \otimes \omega_\lambda) : \text{MF}_{gr}(\overline{\mathcal{X}(d)}^\lambda, \tilde{W}) \to \text{MF}_{gr}(\overline{\mathcal{X}(d)}, \tilde{W}).
\]

The following diagram commutes:

\[
\begin{array}{ccc}
D^b(\Psi(d)^{\lambda}) & \xrightarrow{p_\ast q_\ast} & D^b(\Psi(d)) \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
\text{MF}_{gr}(\overline{\mathcal{X}(d)}^\lambda, \tilde{W}) & \xrightarrow{\tilde{m}} & \text{MF}_{gr}(\overline{\mathcal{X}(d)}, \tilde{W}).
\end{array}
\]

**Proof.** Define the stacks

\[
\mathcal{W} = \left( R(d)^{\lambda \geq 0} \oplus C(d)^{\lambda} \right) / G^{\lambda \geq 0}
\]

\[
\mathcal{Y} = \left( R(d)^{\lambda \geq 0} \oplus C(d)^{\lambda \leq 0} \right) / G^{\lambda \geq 0}
\]

\[
\mathcal{Z} = \left( R(d)^{\lambda \geq 0} \oplus C(d) \right) / G^{\lambda \geq 0}.
\]

There are natural maps, see for example the settings of [28, Lemma 2.4.4, Lemma 2.4.7]:

\[
q_1 : \mathcal{W} \to \overline{\mathcal{X}(d)}^\lambda
\]

\[
q_2 : \mathcal{W} \to \mathcal{Y}
\]

\[
p_1 : \mathcal{Z} \to \mathcal{Y}
\]

\[
p_2 : \mathcal{Z} \to \overline{\mathcal{X}(d)}.
\]
The map $q$ is quasi-smooth, so the following diagram commutes by [28, Lemma 2.4.7]:

$$
\begin{array}{ccc}
D^b(\mathfrak{P}(d)^\lambda) & \xrightarrow{\Phi} & MF^\text{gr}\left(\mathcal{X}(d)^\lambda, \mathcal{W}\right) \\
\downarrow q^* & & \downarrow q_1^* \\
D^b(\mathfrak{P}(d)^{\lambda \geq 0}) & \xrightarrow{\Phi} & MF^\text{gr}\left(\mathcal{Y}, \mathcal{W}\right)
\end{array}
$$

The map $p$ is proper, so the following diagram commutes by [28, Lemma 2.4.4]:

$$
\begin{array}{ccc}
D^b(\mathfrak{P}(d)^{\lambda > 0}) & \xrightarrow{\Phi} & MF^\text{gr}\left(\mathcal{Y}, \mathcal{W}\right) \\
\downarrow p^* & & \downarrow p_1^* \\
D^b(\mathfrak{P}(d)) & \xrightarrow{\Phi} & MF^\text{gr}\left(\mathcal{Z}, \mathcal{W}\right)
\end{array}
$$

There are natural maps $s, t$ in the following cartesian diagram:

$$
\begin{array}{ccc}
\mathcal{X}(d)^{\lambda > 0} & \xrightarrow{t} & \mathcal{Z} \\
\downarrow s & & \downarrow p_1 \\
\mathcal{W} & \xrightarrow{q_2} & \mathcal{Y}.
\end{array}
$$

By proper base change, $p_1^* q_2^* = t_2 s^*$. Further, $s^* q_1^* \omega_\lambda = s^* \det L_{q_2} = \det L_t$ and so $t_1(F) = t_1(F \otimes s^* q_1^* \omega_\lambda)$ by [28, Subsection 2.2.2]. Let $E$ be a complex in $D^b(\mathfrak{P}(d)^\lambda)$.

We compute

$$
p_1^* q_1^* (E) = p_2^* p_1^* q_2^* q_1^* (E) = p_2^* t_1 s^* q_1^* (E) = \tilde{p}_1^* (\tilde{q}_1^* (E \otimes \omega_\lambda)).
$$

\[\square\]

3.2. Semi-orthogonal decompositions for preprojective-like HAs.

3.2.1. Let $\lambda$ be a cocharacter of $SG(d)$ and recall the definition of $\omega_\lambda$ from (18).

We claim that:

$$
2\langle \lambda, \omega_\lambda \rangle = \langle \lambda, L_{\mathcal{X}(d)}^{\lambda \leq 0} |_{\mathcal{X}(d)} \rangle - \langle \lambda, L_{\mathfrak{P}(d)}^{\lambda \leq 0} |_{\mathfrak{P}(d)} \rangle.
$$

This follows from the following equalities in $K_0(BG(d))$:

$$
\begin{align*}
[L_{\mathcal{X}(d)}] &= [R(d)] + [C(d)] - [g(d)] \\
[L_{\mathfrak{P}(d)}] &= [R(d)] - [C(d)] - [g(d)] \\
[\omega_\lambda] &= [C(d)^{\lambda \leq 0}].
\end{align*}
$$
3.2.2. Consider $A = (d_i, w_i)_{i=1}^k$ an ordered partition of $(d, w)$ and with associated cocharacter $\lambda = \lambda_{d_1, \ldots, d_k}$. Let $A'$ be the ordered partition $(d_i, v_i)_{i=1}^k$, where $v_i$ are the corresponding weights of $(\mathbb{Z}_{d_i}^{\oplus} \otimes \omega_\lambda$.

Let $\delta \in M_S^d$. The set of all such $A'$ for $A \in S_w^d(\delta)$ is denoted by $V_w^d(\delta)$. The set $V_w^d(\delta)$ has an order induced from the order on $S_w^d(\delta)$. The set of all such $A'$ for $A \in T_w^d(\delta)$ is denoted by $U_w^d(\delta)$.

3.2.3. The category $\mathcal{M}(d; \delta)$ of $D^b(\mathfrak{P}(d))$ is generated by complexes $\mathcal{F}$ such that for any cocharacter $\lambda$ of $SG(d)$, we have that

$$-\frac{n_\lambda}{2} + \langle \lambda, \delta \rangle \leq \langle \lambda, \iota_* \mathcal{F}|_0 \rangle \leq \frac{n_\lambda}{2} + \langle \lambda, \delta \rangle,$$

where $\iota: \mathfrak{P}(d) \to \mathcal{X}(d)$ and $n_\lambda$ is the weight for the quiver $\tilde{Q}$, see [28].

This category corresponds to the category $\mathcal{M}(d; \delta)$ of $\text{MF}_{gr}(\tilde{X}(d), W)$ by [28, Subsection 5.3, Lemma 5.3.8].

3.2.4. For $A = (d_i, w_i)_{i=1}^k$ in $V_w^d(\delta)$ or $U_w^d(\delta)$, let $\mathcal{M}_A(\delta) := \otimes_{i=1}^k \mathcal{M}(d_i, v_i)$ be a subcategory of $D^b(\times_{i=1}^k \mathfrak{P}(d_i))_{\chi}$, where $\chi := (w_i)_{i=1}^k$ is the weight corresponding to $A$.

Using the Thom-Sebastiani theorem, the Koszul equivalence induces an equivalence:

$$\Phi: \mathcal{M}_A(\delta) \cong \mathcal{M}_{A^o}(\delta) \subset \text{MF}_{gr}(\times_{i=1}^k \tilde{X}(d_i), W)_{\chi},$$

where $A^o \in S_w^d(\delta)$ or $T_w^d(\delta)$ with $(A^o)' = A$.

3.2.5. By Theorem 3.1, Proposition 3.1, and the discussion in Subsection 3.2.2, we have that:

**Corollary 3.2.** There is a semi-orthogonal decomposition

$$D^b(\mathfrak{P}(d))_w = \langle \mathcal{M}_A(\delta) \rangle,$$

where the right hand side contains all sets $A$ in $V_w^d(\delta)$. The order of the summands of the semi-orthogonal decomposition is as in Subsection 3.2.2, see also Subsection 3.2.3.

3.2.6. We continue with the notation from the previous Subsection. Consider an ordered tuple $A = (d_i, w_i)_{i=1}^k$ with $k \geq 2$ in $V_w^d(\delta)$. Let $\lambda = \lambda_{d_1, \ldots, d_k}$ and $\chi = (w_i)_{i=1}^k$ be the cocharacter of $SG(d)$ and the character of $T$ associated to $A$. We explain the analogous results of Subsection 2.2.10 in this setting. By the Koszul equivalence, the categories $D^b(\mathfrak{P}(d))$ have a dual functor $\mathcal{D}$. The fully faithful functor

$$p_{\lambda, q^*}: D^b(\times_{i=1}^k \mathfrak{P}(d_i))_{\chi} \to D^b(\mathfrak{P}(d))_w$$

has a right adjoint

$$R := q_s p^!: D^b(\mathfrak{P}(d))_w \to \text{Ind} D^b(\times_{i=1}^k \mathfrak{P}(d_i))_{\text{above}},$$

where the category $\text{Ind} D^b_{\text{above}}$ is defined as in Subsection 2.2.10. Indeed, using Proposition 3.1 and taking the adjoint of the two functors $m'$ and $p_* q^*$, we have that

$$\Phi q_s p^!(\mathcal{E}) = (\tilde{q}_s \tilde{p}^! \Phi(\mathcal{E})) \otimes \omega_{\chi}^{-1}$$

for $\mathcal{E}$ in $D^b(\mathfrak{P}(d))$. Thus the functor $q_s p^!$ has image in $\text{Ind} D^b_{\text{above}}$ by the discussion for $\text{MF}_{gr}$ in Subsection 2.2.10.
Further, by the argument in Subsection 2.2.10 the functor \(L = \mathbb{D}q\lambda Rq\) is a left adjoint of \(p_{\lambda,s}q^\lambda\) and has image in \(\text{Ind}D^b\) below. Thus, for any weight \(\chi\),

\[
\beta_\chi L : D^b(\mathfrak{P}(d))_w \to D^b\left(\times_{i=1}^k \mathfrak{P}(d_i)\right)_\chi.
\]

The inclusion of \(M_A(\delta)\) in \(D^b(\mathfrak{P}(d))_w\) has a left adjoint by the argument in Subsection 2.2.11. Thus the fully faithful functor

\[
p_{\lambda,s}q^\lambda : M_A(\delta) \to D^b(\mathfrak{P}(d))_w
\]

has left adjoints

\[
\Delta_A : D^b(\mathfrak{P}(d))_w \to M_A(\delta).
\]

3.2.7. Let \(\mathcal{F}\) be in \(D^b(\mathfrak{P}(d))_w\). By the discussion in Subsection 2.2.8 and the fact that \(L\) has image in \(\text{Ind}D^b\) below, \(\Delta_A(\mathcal{F}) = 0\) for all but finitely many \(A\) in \(V^d_w(\delta)\).

3.2.8. Let \(J\) be the Jordan quiver, let \(\tilde{J}\) be the quiver with three loops \(x, y, z\), and let \(\tilde{W} := xyz - zyx\). The stacks \(\mathfrak{P}(d)\) from Subsection 3.1.1 recover the moduli stacks of points on \(\mathbb{A}^2_C\) which we denote by \(\mathfrak{c}_d\) for \(d \in \mathbb{N}\). Denote its coarse space by \(C_d\).

The categories \(M_A(\delta)\) do not depend on the weight \(\delta \in M^S_{R^d}\), and we will assume that \(\delta = 0\) and drop it from the notation. We want to describe in more detail the sets \(V^d_w\) and \(U^d_w\). Let \(\chi\) be a dominant weight in \(M(d)\) and write

\[
\chi + \rho = -\sum_{j \in T} r_j N_j + \psi
\]
as in (1). Let \(\lambda\) be the corresponding antidominant cocharacter with associated partition \((d_1, \cdots, d_k)\). Consider the weight

\[
\theta_\chi := -\sum_{j \in T} r_j N_j - \rho^\lambda < 0 + \omega_\lambda.
\]

Define \(g_j\) analogously to \(N_j\) for the adjoint representation of \(GL(d)\) for \(j \in T\). Then

\[
\theta_A := -\sum_{j \in T} \left(3r_j - \frac{3}{2}\right) g_j + c\tau_d = \sum_{i=1}^k w_i \tau_d_i
\]
is a dominant weight and so \(\frac{w_1}{d_1} > \cdots > \frac{w_k}{d_k}\). Conversely, consider a partition \((d_i, w_i)_{i=1}^k\) with \(\frac{w_1}{d_1} > \cdots > \frac{w_k}{d_k}\) and let \(\psi_A := \sum_{i=1}^k w_i \tau_d_i\). Then \(\psi_A\) is a dominant weight and \(\psi_A\) is a linear combination of \(g_j\) for a tree \(T\):

\[
\psi_A = \sum_{i=1}^s w_i \tau_d_i = -\sum_{j \in T} \left(3r_j - \frac{3}{2}\right) g_j + c\tau_d.
\]

Then \(r_i > r_j\) for \(i, j \in T\) such that there exists a path from \(i\) to \(j\), otherwise \(\psi_A\) is not dominant.

Thus the set \(V^d_w\) contains partitions \((d_i, w_i)_{i=1}^k\) with \(\frac{w_1}{d_1} > \cdots > \frac{w_k}{d_k}\). By the above discussion and Corollary 3.2 we see that Theorem 1.1 holds for \(\mathbb{A}^2_C\).

Corollary 3.3. There is a semi-orthogonal decomposition

\[
D^b(\mathfrak{c}_d)_w = \langle M_A \rangle,
\]
where the right hand side contains all partitions $A \in V^d_w$. The semi-orthogonal decomposition holds over $C_d$ in the following sense. Let $A < B$ be two partitions and let $\mathcal{F} \in \mathcal{M}_A$ and $\mathcal{G} \in \mathcal{M}_B$. Then

$$R\pi_{d*} (R\text{Hom}_{\mathbb{C}}(\mathcal{F}, \mathcal{G})) = 0.$$  

Note that the statement (24) holds from the semi-orthogonality property because $C_d$ is affine. Similarly, the set $U^d_w$ contains partitions $A = (d_i, w_i)_{i=1}^{k}$ such that $\psi_A$ from (22) is a multiple of $\tau_d$, which implies that

$$\frac{w_1}{d_1} = \cdots = \frac{w_k}{d_k}. \tag{25}$$

Conversely, all the partitions $A$ satisfying (26) are in $U^d_w$.

4. Hall algebras of surfaces

4.1. Moduli of sheaves on surfaces via quivers.

4.1.1. Let $S$ be a smooth surface and let $d \in \mathbb{N}$. Consider the coarse moduli space morphism

$$\pi_d : \mathcal{M}_d \to M_d := \text{Sym}^d(S).$$

Choose $k$ distinct points $p_1, \ldots, p_k$ with multiplicities $d_1, \ldots, d_k$ and let $p \in \text{Sym}^d(S)$ be the corresponding point. Denote by $\mathcal{M}_{d,p}$ the formal completion of $\mathcal{M}_d$ along $\pi_d^{-1}(p)$. Note that

$$\mathcal{M}_{d,p} \cong \mathcal{M}_d \times_{M_d} \mathcal{M}_{d,p}, \tag{26}$$

where $\mathcal{M}_{d,p}$ is the completion of $M_d$ at $p$, see [13, Section 2.1].

Denote by $J'$ the doubled quiver of $J$. Consider the quiver $Q = \bigsqcup_{i=1}^{k} J'$ with vertex set $I = \{1, \ldots, k\}$ and consider the dimension vector $d = (d_i)_{i=1}^{k} \in \mathbb{N}^k$. The quiver $Q$ is the Ext quiver of the polystable sheaf $\bigoplus_{i=1}^{k} \mathcal{O}_{p_i}^{\oplus d_i}$. Consider the (Koszul) stack

$$\mathcal{P}(d) := \times_{i=1}^{k} \mathcal{C}(d_i), \tag{27}$$

where the completions for the spaces on the right hand side are at zero. Let $\mathcal{P}^{cl}(d)$ be the coarse space of $\mathcal{P}(d)$. The corresponding coarse space map is $\pi_{cl} : \times_{i=1}^{k} \pi_{d_i}$. By [28, Lemma 5.4.1, Section 7.4], we have canonical isomorphisms $\nu$ which extend to isomorphisms over analytic neighborhoods of $p \in M_d$ and $0 \in \mathcal{P}(d)$:

$$\begin{array}{cccc}
\mathcal{M}_d & \overset{\sim}{\longrightarrow} & \mathcal{M}_{d,p} & \overset{\sim}{\rightarrow} & \mathcal{P}(d) \\
\pi_d & \downarrow & \pi_d & \downarrow & \pi_{cl} \\
M_d & \overset{\sim}{\leftarrow} & \mathcal{M}_{d,p} & \overset{\sim}{\rightarrow} & \mathcal{P}^{cl}(d).
\end{array} \tag{28}
$$

4.1.2. For $d = (d_1, \ldots, d_k)$ a partition of $d$, let $M_d \subset M_d$ be the locus of points $p \in M_d$ corresponding to (not necessarily distinct) points $p_1, \ldots, p_k$ with multiplicities $d_1, \ldots, d_k$. By a dévissage argument, the category $D^b(\mathcal{M}_d)$ is generated by locally free sheaves on $\pi_{cl}^{-1}(M_d)$ for all partitions $d$. For $U \subset M_d$ an open analytic subset, denote by $D^b(U) \subset D^b(\mathcal{M}_d)$ the category of coherent (analytic) sheaves generated by restrictions of coherent sheaves on $D^b(\mathcal{M}_d)$. 

For $U$ a small contractible open subset of $M_d$, the category $D^b_o(\pi^{-1}_d(U))$ is generated by sheaves $\mathcal{O}_{\pi^{-1}_d(U \cap M_d)} \otimes \Gamma(\chi)$ for $\chi$ a weight of $GL(d)$ and $\underline{d}$ a partition of $d$.

4.1.3. Assume $S = \mathbb{A}^2_C$ and let $p \in M_d$ as above. We obtain a commutative diagram:

\[
P(d) := \times_{i=1}^k C(d_i) \longrightarrow C(d)
\]

\[
P(d) := \times_{i=1}^k C(d_i) \longrightarrow C(d).
\]

The thick subcategory of $D^b(\mathfrak{P}(d))$ generated by restrictions of coherent sheaves on $\mathfrak{C}(d)$ is $D^b(\mathfrak{P}(d))$.

For $p \in M_d$, define $D^b_o(\mathfrak{M}_{d,p})$ as the thick subcategory of $D^b(\mathfrak{M}_{d,p})$ generated by restrictions of sheaves $\mathcal{O}_{\pi^{-1}_d(M_d)} \otimes \Gamma(\chi)$ for $\chi$ a dominant weight of $GL(d)$ and $\underline{d}$ a partition of $d$. The category $D^b_o(\mathfrak{M}_{d,p})$ depends only on the Ext quiver of $p$. Indeed, this follows from the isomorphisms in (28) and (29) for analytic open neighborhoods of the point 0 in $P(d)$, the point $p$ in $C(d)$, and $p$ in $M_d$.

The semi-orthogonal decomposition (23) induces a semi-orthogonal decomposition:

\[
D^b(\mathfrak{M}(d)) = \langle \mathfrak{M}_{A,p} \rangle,
\]

where we denote by $\mathfrak{M}_{A,p}$ the subcategory of $D^b(\mathfrak{P}(d))$ generated by restrictions of sheaves in $\mathfrak{M}_A$. Further, the discussion in Subsection 4.2.3 applies to the categories from (30) and provides adjoint functors $\Delta_A$ to the multiplication functors for $A \in V^d$.

4.1.4. The Hall products for the surface $S$ and the preprojective algebra of the quiver $Q$ are compatible. Let $b, c \in \mathbb{N}$, let $a = b + c$, and let $p \in M_b$, $q \in M_c$, and $r := p \oplus q \in M_a$. Let $Q$ be the Ext-quiver (for $S$) of $a$ and let $d$ and $e$ be the dimension vector of $Q$ corresponding to $b$ and $c$. Denote by $\mathfrak{P}$ the derived stack defined in (27) for $a$. Then there is a commutative diagram

\[
\begin{array}{c}
\mathfrak{M}_b \times \mathfrak{M}_c \leftarrow \mathfrak{M}_{b,p} \times \mathfrak{M}_{c,q} \overset{\sim}{\longrightarrow} \mathfrak{P}(d) \times \mathfrak{P}(e) \\
\mathfrak{M}_b \times \mathfrak{M}_c \leftarrow \mathfrak{M}_{b,c} \overset{\sim}{\longrightarrow} \mathfrak{P}(d) \leftarrow \mathfrak{P}(d+e) \\
\mathfrak{M}_a \leftarrow \mathfrak{M}_{a,r} \overset{\sim}{\longrightarrow} \mathfrak{P}(d+e).
\end{array}
\]

4.1.5. For any coherent sheaf $\mathcal{F}$ on $S$, there is an inclusion $\mathbb{C}^* \hookrightarrow \text{Aut}(\mathcal{F})$ given by scaling. Thus the inertia stack of $\mathfrak{M}_d$ contains a natural copy of $\mathbb{C}^*$. Denote by $D^b(\mathfrak{M}_d)_w$ the category of complexes on $\mathfrak{M}_d$ on which $\mathbb{C}^*$ acts with weight $w$. 


4.1.6. For \( p \in M_d \), consider the stack \( \hat{M}_d \) defined in [27]. Recall the natural isomorphism

\[
\nu : \hat{M}_{d,p} \cong \hat{M}_d.
\]

There is thus an equivalence of categories:

\[
\nu^{-1} : D^b \left( \hat{M}_d \right) \cong D^b \left( \hat{M}_{d,p} \right).
\]

We have that \( \hat{M}(d)_p \cong \nu^{-1} \left( \hat{M}(d) \right) \). Consider the restriction functor:

\[
\Psi_p : D^b \left( \hat{M}_d \right) \rightarrow D^b \left( \hat{M}_{d,p} \right).
\]

Define \( M(d) \) as the subcategory of \( D^b \left( \hat{M}_d \right) \) generated by complexes \( F \) such that for any point \( p \) in \( M_d \), we have that

\[
\Psi_p(F) \in \hat{M}(d)_p \subset D^b \left( \hat{M}_{d,p} \right).
\]

For \( A = (d_i, w_i)_{i=1}^k \in V_d^w \) or \( U_d^w \), define \( M_A := \otimes_{i=1}^k \hat{M}(d_i)_{w_i} \).

4.2. Proof of Theorem 1.1. We use induction on \( d \). The stacks \( \hat{M}_d \) have dualizing functors \( \mathbb{D} \) [12 Subsection 2.2.1]. If \( d = 1 \), then \( D^b \left( \mathfrak{M}_1 \right)_w \cong D^b(S) \cong \hat{M}(1)_w \) for any \( w \in \mathbb{Z} \).

Let \( F \) be a complex in \( D^b \left( \mathfrak{M}_d \right) \), let \( A = (d_i, w_i)_{i=1}^k \in V_d^w \) with \( k \geq 2 \), and let \( \lambda \) be its associated cocharacter. We have that

\[
\mathbb{D}q_{\lambda}^* p_{\lambda}^! \mathbb{D}F \in \text{Ind} D^b \left( \times_{i=1}^k \hat{M}_{d_i} \right)_\lambda.
\]

Indeed, from the local statement [20] and [28], the statement is true over a small neighborhood \( U \) of \( p \in M_d \), by embedding \( S \) in a projective surface, we see that \( M_d \) can be covered by a finite number of such open sets \( U \). Thus, from Subsection 3.2.7 we have that \( \Delta_A(F) = 0 \) for all but finitely \( A \) in \( V_d^w \).

Consider \( A = (d_i, w_i)_{i=1}^k \) and \( B = (e_i, v_i)_{i=1}^s \) in \( V_d^w \) such that \( (d, w) < A < B \).

Let \( \lambda \) and \( \mu \) be the associated cocharacters for \( A \) and \( B \) and let \( \chi = (w_i)_{i=1}^k \) be the weight corresponding to \( A \). Using the induction hypothesis and the argument in Subsection 2.2.11 the inclusion of \( M_A \) in \( D^b \left( \times_{i=1}^k \hat{M}_{d_i} \right)_\chi \) has a left adjoint

\[
\Phi_A : D^b \left( \times_{i=1}^k \hat{M}_{d_i} \right)_\chi \rightarrow M_A.
\]

The functor

\[
p_{\lambda}q_{\lambda}^* : M_A \rightarrow D^b \left( \mathfrak{M}_d \right)_w
\]

is fully faithful and has a left adjoint

\[
(31) \quad \Delta_A := \Phi_A \beta_{\lambda} \mathbb{D}q_{\lambda}^* p_{\lambda}^! \mathbb{D} : D^b \left( \mathfrak{M}_d \right)_w \rightarrow M_A.
\]

Both statements follow from the analogous statements for a formal completion \( \hat{M}_{d,p} \) for \( p \in M_d \), see the discussion in the Subsections 3.2.6 and 4.1.3.

Next, let \( F_i \in \hat{M}(d_i)_{w_i} \) for \( 1 \leq i \leq k \). We claim that:

\[
\Delta_{BP_{\lambda_i}q_{\lambda_i}^*} \left( \mathbb{D}F_{i=1}^k \right) = 0.
\]

Once again, this follows from the analogous statement for a formal completion \( \hat{M}_{d,p} \) for \( p \in M_d \), see Subsections 3.2.6 and 4.1.3.
Let $\mathcal{W}$ be the left complement of the categories $\mathcal{M}_A$ for all sets $A = (d_i, w_i)_{i=1}^k$ in $V_d^w$ with $k \geq 2$. There is a semi-orthogonal decomposition
\begin{equation}
D^b(\mathcal{M}_d)_w = \left\langle \mathcal{M}_A, \mathcal{W} \right\rangle,
\end{equation}
with summands corresponding to all $A$ in $V_d^w$ with $k \geq 2$. For $p \in M_d$, define $\hat{\mathcal{W}}_p$ as the subcategory of $D^b(\hat{\mathcal{M}}_{d,p})_w$ generated by restrictions of sheaves in $\mathcal{W}$. The restriction of the semi-orthogonal decomposition (32) implies an analogous semi-orthogonal decomposition
\begin{equation}
D^b(\hat{\mathcal{M}}_{d,p})_w = \left\langle \hat{\mathcal{M}}_A, \hat{\mathcal{W}}_p \right\rangle
\end{equation}
for $p \in M_d$. Indeed, consider partitions $(d, w) < A < B$ and let $\mathcal{F} \in \mathcal{M}_A$ and $\mathcal{G} \in \mathcal{M}_B$. Then
\begin{equation}
R\pi_{d*} (R\mathcal{H}om_{\mathcal{M}}(\mathcal{F}, \mathcal{G})) = 0
\end{equation}
from the semi-orthogonal decomposition (32) and Corollary 3.3 applied for all points $p \in M_d$. Next, the category $\mathcal{W}$ contains sheaves $\mathcal{F}$ such that $\Delta_A(\mathcal{F}) = 0$ for all partitions $A = (d_i, w_i)_{i=1}^k \in V_d^w$ with $k \geq 2$. The restriction of the functor $\Delta_A$ to $\hat{\mathcal{M}}_{d,p}$ is the analogously defined functor
\begin{equation}
\Delta_A : \hat{\mathcal{M}}_{d,p} \to \hat{\mathcal{M}}_{A,p}.
\end{equation}
This construction further shows that $\hat{\mathcal{W}}_p = (\hat{\mathcal{M}}(d)_p)_w$ and thus that $\mathcal{W} = \mathcal{M}(d)_w$, and it also shows that
\begin{equation}
R\pi_{d*} (R\mathcal{H}om_{\mathcal{M}}(\mathcal{F}, \mathcal{G})) = 0
\end{equation}
for $\mathcal{F} \in \mathcal{M}(d)_w$ and $\mathcal{G} \in \mathcal{M}_A$.

5. PBW THEOREM FOR SURFACES

The proof of Theorem 1.2 follows closely the proof of [21, Proposition 5.5]. We will explain how to modify the proof in loc. cit. to obtain the proof of Theorem 1.2. All the K-theoretic spaces in this Section are over the rational numbers $\mathbb{Q}$.

5.1. The Shuffle algebra.

5.1.1. Let $\text{Sh}(S)$ be the shuffle algebra of the surface $S$ considered by Neguţ [19], Zhao [33, Section 5]. Its underlying $\mathbb{N}$-graded vector space is
\begin{equation}
\text{Sh}(S) := \bigoplus_{n \geq 1} (K_0(S^n)(z_1, \ldots, z_n))^{S_n}.
\end{equation}
Fix $n, m \geq 1$. For $1 \leq i, j \leq n + m$, let $\Delta_{ij} \subset S^{n+m}$ be the inclusion of the locus with equal points on the $i$th and $j$th copies of $S$. Define
\begin{equation}
\zeta_{ij}(z) := 1 + \frac{z \cdot [\mathcal{O}_{\Delta_{ij}}]}{(1 - z)(1 - z[\omega_S])} \in K_0(S^{n+m})(z),
\end{equation}
see [19] Equation 3.6, Proposition 5.24. Let $f$ and $g$ be elements of $\text{Sh}(S)$ of degrees $n$ and $m$, respectively. The product on the shuffle algebra is defined by
\begin{equation}
(f \ast g)(z_1, \ldots, z_{n+m}) := \text{Sym} \left( f(z_1, \ldots, z_n)g(z_{n+1}, \ldots, z_{n+m}) \prod_{1 \leq i \leq n} \prod_{n+1 \leq j \leq n+m} \zeta_{ij} \left( \frac{z_i}{z_j} \right) \right),
\end{equation}
where the right hand side is the symmetrization after all cosets in $\mathcal{S}_{n+m}/\mathcal{S}_n \times \mathcal{S}_m$. There is an algebra morphism

$$\Phi_S : \text{KHA}(S) \to \text{Sh}(S)$$

defined in [33 Section 5.2]. Let $S = A^2_\mathbb{C}$ and consider the map

$$t_d : \mathcal{C}(d) \to \mathfrak{gl}(d)^2/\text{GL}(d).$$

In [21 Subsection 5.1.2], we constructed algebra morphisms by setting the potential to zero, so, for example, a morphism KHA $(\tilde{J}, \tilde{W}) \to$ KHA $(\tilde{J}, 0)$. Using dimensional reduction, the morphism is the same as the pushforward (up to an equivariant factor, see Proposition 5.1.1):

$$\iota_s : \bigoplus_{d \in \mathbb{N}} G_0 (\mathcal{C}(d)) \to \bigoplus_{d \in \mathbb{N}} K_0 (\mathfrak{gl}(d)^2/\text{GL}(d)).$$

Write $\mathcal{C}(d) = C(d)/\text{GL}(d)$. Let $k_d : D(d) \to C(d)$ be the locus of diagonal matrices and let $\mathcal{D}(d) = D(d)/T(d)$. Consider the map

$$K_0 (\mathfrak{gl}(d)^2/\text{GL}(d)) \cong K_0 (\mathfrak{gl}(d)^2/\text{T}(d)) \cong K_0 (\mathfrak{gl}(d)^2/T(d)) \cong K_0 (\mathfrak{gl}(d)^2/\text{T}(d)) \cong K_0 (\mathfrak{gl}(d)^2/\text{GL}(d)) \cong K_0 (\mathfrak{gl}(d)^2/\text{GL}(d)) \cong K_0 (\mathfrak{gl}(d)^2/\text{GL}(d)) \cong K_0 (\mathfrak{gl}(d)^2/\text{GL}(d)).$$

Write $k^* := \bigoplus_{d \in \mathbb{N}} k^*_d$. Then $\Phi_{\mathbb{A}^2_\mathbb{C}} = k^* \iota_s$.

5.1.2. There is a K-theoretic Hall algebra defined using topological K-theory by applying the Blanc topological K-theory functor $K^{\text{top}} X$ to the Porta–Sala monoidal category [23]. For a quotient stack $X = X/G$ with $X$ a possibly singular variety and $G$ a reductive group, $K^{\text{top}} (X)$ is the Atiyah-Segal equivariant K-theory of $X$, while $G^{\text{top}} (X)$ is the Borel-Moore equivariant K-homology (also called Spanier-Whitehead) of $X$, see [11 Theorem 3.9, Remark 0.1].

We denote by $\text{KHA}^{\text{top}}(S) := \bigoplus_{d \in \mathbb{N}} G_0^{\text{top}} (\mathfrak{M}_d)$.

5.1.3. There is a topological Chern character

$$\text{ch}^{\text{top}} : \text{G}_0^{\text{top}} (\mathfrak{M}_d) \to \prod_{i \in \mathbb{Z}} H^{2i}_{BM} (\mathfrak{M}_d, \mathbb{Q})$$

factoring the usual Chern character $\text{ch} : G_0 (\mathfrak{M}_d) \to \prod_{i \in \mathbb{Z}} H^{2i}_{BM} (\mathfrak{M}_d, \mathbb{Q})$. The map $\text{ch}^{\text{top}}$ is injective, see the explanations about the map (37).

Davison [6 Theorem A, Subsection 4.1.1], [5 Theorems B and C] proved a BBDG Decomposition Theorem for the complex $R\pi_{d!}\omega_{\mathfrak{M}_d}$ which implies a PBW-type Theorem for the CoHA of points of $S$, see also [14 Theorem 7.1.6]. The CoHA is generated by the cohomology of BPS sheaves, which are certain constructible sheaves on $M_d$ for every $d \geq 1$. The BPS summand in dimension $d$ of $R\pi_{d!}\omega_{\mathfrak{M}_d}$ is IC$_S$, where $S \to M_d = \text{Sym}^d(S)$ is the small diagonal, see also the proof of Proposition 5.2.

5.1.4. Let $(U_j)_{j \in J}$ be a cover of $M_d$ with open subsets. Using the results from Subsection 5.1.3 we obtain that the restriction maps induce an injection

$$G_0^{\text{top}} (\mathfrak{M}_d) \to \bigoplus_{j \in J} G_0^{\text{top}} (\pi_d^{-1}(U_j)).$$
For a surface $S$, there is also a topological $K$-theoretic shuffle algebra $\text{SH}^{\text{top}}(S)$ and an algebra morphism $\Phi_S^{\text{top}} : \text{KHA}^{\text{top}}(S) \to \text{SH}^{\text{top}}(S)$ defined as above.

Let $U \subset S$ be a small open analytic subset of a point $p$ such that $\text{Sym}^d(U) \subset M_d$ is an open neighborhood of $(p, \cdots, p) \in M_d$ as in (28). One way to obtain $\Phi_S^{\text{top}}$ is to glue the analogous of the composition of the local maps (33) and (34):

$$k_{d,i}^{\top} : \text{G}^{\top}_i \left( \pi_i^{-1} \left( \text{Sym}^d(U) \right) \right) \rightarrow \left( \text{K}^{\top}_i \left( U^d \right) \left( z_1, \cdots, z_d \right) \right)^{\otimes_d}$$

for $i \in \mathbb{Z}$. To obtain a cover of $M_d$, we also need to use the corresponding maps for the open neighborhoods $\chi^i_{\top} \text{Sym}^d(U_i) \subset M_d$ associated to small neighborhoods $p_i \in U_i \subset S$. We denote the image of $\Phi_S^{\text{top}}$ by $\text{KHA}^{\text{top}}(S)$.

5.2. A coproduct-type map. Let $C > A$ be in $U_w^d$. The construction (31) also provides a functor

$$\Delta_{AB} : M_A \rightarrow M_C.$$

Consider pairs $(b, t)$, $(c, s)$, $(e, v)$, $(f, u)$, and $(d, w)$ in $\mathbb{N} \times \mathbb{Z}$ such that

$$(e, v) + (f, u) = (b, t) + (c, s) = (d, w).$$

We denote by $A$ the two term partition $(e, v)$, $(f, u)$, and by $B$ the two term partition $(b, t)$, $(c, s)$. Assume that $A$ and $B$ are in $U_w^d$. Let $S$ be the set of partitions $C$ of $U_w^d$ with terms $(a_i, \alpha_i)$ for $1 \leq i \leq 4$, some of them possibly zero, such that

$$(a_1, \alpha_1) + (a_2, \alpha_2) = (e, v),$$

$$(a_3, \alpha_3) + (a_4, \alpha_4) = (f, u),$$

$$(a_1, \alpha_1) + (a_3, \alpha_3) = (b, t),$$

$$(a_2, \alpha_2) + (a_4, \alpha_4) = (c, s),$$

Define $m \boxtimes m := m \cdot (1 \boxtimes \text{sw} \boxtimes 1)$.

**Theorem 5.1.** The following diagram commutes:

$$
\begin{array}{ccc}
K_0^{\top}(M_A) & \xrightarrow{m} & K_0^{\top}(M(d)_w) \\
\downarrow_{\Delta_{AC}} & & \downarrow_{\Delta_B} \\
\bigoplus_{C \in S} K_0^{\top}(M_C) & \xrightarrow{\tilde{m} \boxtimes m} & K_0^{\top}(M_B).
\end{array}
$$

**Proof.** The construction of the maps $(\Phi_S)_{d} = k_{d,i}^{\top}$ is local on $M_d$. For an analytic open subset $O \subset M_d$, let $M_A(O)$ be the subcategory of $D^b_{\text{top}}(\pi_1^{-1}(O))$, see Subsection 4.1.2 generated by restrictions of sheaves on $M_A \subset D^b(M_d)$. By the discussion in Subsection 5.1.4 it suffices to show the analogous statement for $U \subset S$ an open analytic subset as in Subsection 5.1.4:

$$
\begin{array}{ccc}
K_0^{\top}(M_A(O)) & \xrightarrow{m} & K_0^{\top}(M(d; O)_w) \\
\downarrow_{\Delta_{AC}} & & \downarrow_{\Delta_B} \\
\bigoplus_{C \in S} K_0^{\top}(M_C(O)) & \xrightarrow{\tilde{m} \boxtimes m} & K_0^{\top}(M_B(O)),
\end{array}
$$

where $O := \text{Sym}^d(U) \subset M_d$. We may assume that $U \subset \mathbb{A}^2_C$. The argument follows as in the global case $\mathbb{A}^2_C$ using explicit shuffle formulas for the maps involved in the quiver with zero potential $(\tilde{J}, 0)$, see [21 Theorem 5.3], [22 Theorem 5.2]. Note
that the formula for $\tilde{\boxtimes} m$ is more complicated in loc. cit., but it simplifies to the above description for the quiver $\tilde{J}$ and by Proposition 3.1.

5.3. Primitive generators of the KHA.

5.3.1. We first discuss two preliminary results.

**Proposition 5.2.** Let $S$ be a surface with $H^1(S, \mathbb{Q}) = 0$. Then $H_{BM}^{\text{odd}}(M_d, \mathbb{Q}) = 0$.

**Proof.** Recall the map

$$\pi_d : M_d \rightarrow M_d.$$ 

We use the Decomposition Theorem of [5, Theorem C, Subsection 1.2.3] for the sheaf $R\pi_d^*(\omega_{M_d})$. Then the summands have even shifts, which follows by the direct computation for the stack of commuting matrices, or alternatively from the computation of the BPS sheaves of $(\tilde{J}, \tilde{W})$ [7, Theorem 5.1]. The summands are of the form $IC_{M_d}(L)$ for certain local systems $L$, and these summands appear also in the Decomposition Theorem for the complex $r_{d*}^*(\omega_{M_d})$, see for example [4, Proposition 1.5], also see [17, Proposition 3.5, Theorem 4.6] for a stronger statement in the local case. The only full dimensional support summand is $IC_{M_d}$. By induction on $d \in \mathbb{N}$ and after taking the dual, it suffices to check that $IH_{c}^{\text{odd}}(M_d, \mathbb{Q}) = 0$. The variety $M_n$ has finite quotient singularities, so it suffices to show that $IH_{c}^{\text{odd}}(M_d, \mathbb{Q}) \cong \bigoplus H_{c}^{\text{odd}}(S, \mathbb{Q})$. This is true because

$$H_{c}^{\text{odd}}(M_d, \mathbb{Q}) \cong \bigoplus \left( H_{c}^{\text{odd}}(S, \mathbb{Q}) \right)^{\otimes d}.$$ 

Let $X$ and $Y$ be possibly singular varieties. Assume $X$ has an action of a reductive group $G$ and let $\mathfrak{X} = X/G$. Recall the Atiyah-Hirzebruch isomorphism

$$G_i^{\text{top}}(Y)_{\mathbb{Q}} \cong \bigoplus_{j \in \mathbb{Z}} H_{i+2j}^{BM}(Y, \mathbb{Q}).$$

Using Totaro’s approximations for the stack $\mathfrak{X}$, there is an inclusion map

(37) $$G_i^{\text{top}}(\mathfrak{X})_{\mathbb{Q}} \hookrightarrow \prod_{j \in \mathbb{Z}} H_{i+2j}^{BM}(\mathfrak{X}, \mathbb{Q}).$$

Using Proposition 5.2 and the Künneth Theorem for $G_0^{\text{top}}$, we obtain:

**Corollary 5.3.** Let $S$ be a surface with $H^1(S, \mathbb{Q}) = 0$ and let $d, e \in \mathbb{N}$. Then $G_0^{\text{top}}(M_d \times M_e)_{\mathbb{Q}} \cong G_0^{\text{top}}(M_d)_{\mathbb{Q}} \otimes G_0^{\text{top}}(M_e)_{\mathbb{Q}}$.

5.3.2. Using induction, Theorem 1.1 and Corollary 5.3 we see that

(38) $$\bigotimes_{i=1}^k K_0^{\text{top}}(M(d_i)_{w_i})_{\mathbb{Q}} \cong K_0^{\text{top}}(M_{A})_{\mathbb{Q}}$$

for $A = (d_i, w_i)_{i=1}^k$ in $V^d_w$ or $U^d_w$.

We define inductively on $(d, w)$ a (split) subspace

$$\ell_{d,w} : P(d)_{w} \hookrightarrow K_0^{\text{top}}(M(d)_{w})_{\mathbb{Q}}$$
with a surjection
\[ \pi_{d,w} : K_0^{\text{top}}(\mathcal{M}(d)_w)_{\mathbb{Q}} \to P(d)_w \]
such that \( \pi_{d,w} \circ d_w = \text{id} \). Let \( P(1)_w := K_0^{\text{top}}(\mathcal{M}(1)_w)_{\mathbb{Q}} \). Let \( A = (d_i, w_i)_{i=1}^k \in U_w^d \) with \( k \geq 2 \). Let \( \pi_A \) be the natural projection
\[ \pi_A := \otimes_{i=1}^k \pi_{d_i, w_i} : K_0^{\text{top}}(\mathcal{M}_A)_{\mathbb{Q}} \to \otimes_{i=1}^k P(d_i)_w. \]

Let \( A = (d_i, w_i)_{i=1}^k \in U_w^d \) with \( A > (d, w) \), or alternatively with \( k \geq 2 \). Let \( P_A := \otimes_{i=1}^k P(d_i)_w \), and let \( \pi_A \) be the natural projection:
\[ \pi_A := \otimes_{i=1}^k \pi_{d_i, w_i} : K_0^{\text{top}}(\mathcal{M}_A)_{\mathbb{Q}} \to P_A. \]

For \( \sigma \in \mathfrak{S}_k \), denote by \( \sigma(A) \) the partition \( (d_{\sigma(i)}, w_{\sigma(i)})_{i=1}^k \). Let \( K_A \) be the kernel of the map
\[ \left( \bigoplus_{\sigma \in \mathfrak{S}_k} \pi_{\sigma(A)} \right) \left( \bigoplus_{\sigma \in \mathfrak{S}_k} \sigma \right) \Delta_A : K_0^{\text{top}}(\mathcal{M}(d)_w)_{\mathbb{Q}} \to \bigoplus_{\sigma \in \mathfrak{S}_k} P_{\sigma(A)}. \]

Define
\[ P(d)_w := \bigcap_{A > (d, w)} K_A. \]

Given the construction of these spaces of primitive generators \( P(d)_w \), Theorem 5.1 follows formally from Theorem 5.1 exactly as in [21, Proposition 5.5], [22, Theorem 5.13].

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