A quantitative Gauss-Lucas theorem

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Abstract. A conjecture of T. Richards is proven which yields a quantitative version of the classical Gauss-Lucas theorem: if $K$ is a convex set, then for every $\varepsilon > 0$ there is an $\alpha_\varepsilon < 1$ such that if a polynomial $P_n$ of degree at most $n$ has $k \geq \alpha_\varepsilon n$ zeros in $K$, then $P'_n$ has at least $k - 1$ zeros in the $\varepsilon$-neighborhood of $K$. Estimates are given for the dependence of $\alpha_\varepsilon$ on $\varepsilon$.

1. Introduction and results

The Gauss-Lucas theorem states that if $K$ is a convex subset of the complex plane and all zeros of a polynomial $P_n$ of degree $n$ lie in $K$, then the same is true for $P'_n$, i.e. all critical points belong to $K$. This is no longer true if a single zero of $P_n$ is allowed to lie outside $K$, for then it may happen that all critical points lie outside $K$ (see e.g. the simple example in the beginning of [11]). It was Boris Shapiro who conjectured that in this latter case even though the critical points may lie outside $K$, most of them lie close to $K$, and he formulated the following as a conjecture.

The asymptotic Gauss-Lucas theorem [11]. If $\varepsilon > 0$ and most of the zeros of $P_n$ (i.e. with the exception of $o(n)$ of the zeros) lie in $K$, then most of the zeros of $P'_n$ lie in the $\varepsilon$-neighborhood $K_\varepsilon$ of $K$.

This suggests that perhaps it is also true that if for some $\alpha$ at least $\alpha n$ of the zeros lie in $K$, then at least $(1 + o(1))\alpha n$ (or at least $\beta n$ with some $\beta$ depending on $\alpha$) of the critical points also lie in $K_\varepsilon$ (as has been mentioned, none may lie in $K$). But for $\alpha < 1/2$ this fails dramatically.

Example. If $P_n(z) = z^n - 1$, and $K$ is the square of side-length 2 and with center at the point $1 + \sin((1/2 - \alpha)\pi/2)$, then $K$ contains for large $n$ at least $\alpha n$ of the

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zeros of \( P_n \) (which are the \( n \)-th roots of unity), but all the critical points are at the origin, so \( K_\varepsilon \) with \( \varepsilon = \frac{1}{2} \sin \left( (1/2 - \alpha) \pi / 2 \right) \) does not contain a single critical point.

Still, the asymptotic Gauss-Lucas theorem suggests that this cannot happen when \( \alpha \) is close to 1. In general, if \( k \geq \alpha n \) of the zeros lie in \( K \), how many critical points can be expected in \( K_\varepsilon \)? The following simple example shows that not more than \( k - 1 \).

**Example.** Let \( K \) be the closed unit disk, \( \varepsilon = 1 \) and \( P_n(z) = z^k (2n - z)^{n-k} \). This \( P_n(z) \) has \( k \) zeros in \( K \) and \( k - 1 \) critical points in \( K_\varepsilon \).

It is remarkable that for \( \alpha \) sufficiently close to 1, the set \( K_\varepsilon \) contains this many critical points, as is shown by the following theorem that was conjectured by T. Richards [5], [6].

**Theorem 1.** For any \( \varepsilon > 0 \) there is an \( \alpha_\varepsilon < 1 \) such that if a polynomial \( P_n \) of degree \( n \) has \( k \geq \alpha_\varepsilon n \) zeros in \( K \), then \( P_n' \) has at least \( k - 1 \) zeros in \( K_\varepsilon \).

An immediate consequence of the theorem is the asymptotic Gauss-Lucas theorem stated above (although one should mention that the asymptotic Gauss-Lucas theorem is true not just for convex sets but also so-called polynomially convex sets, see [12, Corollary 1.9]).

The \( \alpha_\varepsilon \) depends on \( \varepsilon \) and \( K \), and in the next theorem we give quantitative bounds for it in terms of \( \varepsilon \).

**Theorem 2.** There is an absolute constant \( C_1 \) such that \( \alpha_\varepsilon = 1 - C_1 \varepsilon^2 / \text{diam}(K)^2 \) suffices in Theorem 1 for all \( \varepsilon \leq \text{diam}(K) \). On the other hand, there is a \( C_2 \) (that depends on \( K \)) such that any \( \alpha_\varepsilon \) necessarily satisfies \( \alpha_\varepsilon \geq 1 - C_2 \varepsilon \).

Here \( \text{diam}(K) \) denotes the diameter of \( K \). Note that the condition \( \varepsilon \leq \text{diam}(K) \) is a natural one in this question.

**Remark.** One could also consider numbers \( \alpha_\varepsilon^* < 1 \) with the property that if a polynomial \( P_n \) of degree \( n \) has at least \( k \geq \alpha_\varepsilon^* n \) zeros in \( K \), then \( P_n' \) has at least \( (1 + o(1)) k \) zeros in \( K_\varepsilon \). Here \( o(1) \) tends to 0 as \( n \to \infty \). Clearly, one can choose \( \alpha_\varepsilon^* = \alpha_\varepsilon \), so \( \alpha_\varepsilon^* = 1 - C_1 \varepsilon^2 / \text{diam}(K)^2 \) suffices for this number by Theorem 2. On the other hand, the proof of Theorem 2 shows that any such \( \alpha_\varepsilon^* \) necessarily satisfies \( \alpha_\varepsilon^* \geq 1 - C_2 \varepsilon \) provided \( K \) has non-empty interior.

**Remark.** A weaker version of Theorem 2 was proved in [6], where it was shown that the conclusion is true if \( P_n \) has at least \( n (1 - c_{\varepsilon,K} / \log n) \) zeros in \( K \). The proof of Theorem 2 proceeds along similar ideas and verifies, in addition, a conjecture formulated in [6] that certain discrete Cauchy-transforms “cannot supercharge certain curves”.


2. Proof of Theorem 1

For a positive Borel-measure $\mu$ of compact support on the complex plane let

$$C_\mu(z) = \int \frac{1}{t-z} d\mu(t)$$

be its Cauchy-transform. The proof of Theorem 2 is based on the following lemma.

**Lemma 3.** If $\mu$ is a discrete measure of finite support, $\lambda > 0$ and $G$ is a connected component of the level set

$$\Lambda_\lambda(\mu) = \{ z \mid |C_\mu(z)| > \lambda \},$$

then

$$\text{diam}(G) \leq 4 \frac{\|\mu\|}{\lambda},$$

where $\|\mu\|$ denotes the total mass of $\mu$.

Note that the set $\Lambda_\lambda(\mu)$ is open, and so are its connected components.

The lemma proves in a quantitative form the conjecture from [6] mentioned above about “supercharging curves”.

The formulation given in Lemma 3 is sufficient for our purposes, but there is a more general version, see Lemma 4 below.

Consider the special case when $\mu = \mu_N$ is the sum of $N$ unit point masses, so that $\|\mu_N\| = N$. The lemma says that if $A$ is large, then any component of the level set

$$\Lambda_{AN} = \{ z \mid |C_{\mu_N}(z)| > AN \}$$

has diameter $\leq 4/A$, i.e. even the largest diameter tends to 0 (uniformly in $N$ and $\mu_N$) if $A \to \infty$. This should be compared to the fact that the set $\Lambda_{AN}$ does not have to be small in some other metric sense. Indeed, the example given in [1, Theorem 2.2] shows that for every $N$ there is a $\mu_N$ (which is the sum of $N$ unit masses) supported in the unit disk such that the projection of $\Lambda_{(\log N)^{1/2}N}$ onto the real line has linear measure $\geq c$, where $c > 0$ is an absolute constant. Still, in this case the largest diameter of the connected components of $\Lambda_{(\log N)^{1/2}N}$ is at most $\leq 4/(\log N)^{1/2}$ by Lemma 3.

**Proof.** Let $A, B \in G$ be two points in $G$, and let $E$ be a broken line connecting $A$ and $B$ inside $G$. The conformal map $\Phi$ from $\overline{C\setminus E}$ onto the exterior of the unit disk that leaves the point infinity invariant is of the form (around $\infty$)

$$\Phi(z) = \frac{z}{\text{cap}(E)} + c_0 + \frac{c_{-1}}{z} + ..., \tag{2}$$
where cap denotes logarithmic capacity. If $\Omega$ is the unbounded component of $\mathbb{C} \setminus \Lambda_\lambda(\mu)$, then the maximum modulus theorem applied to the function $(1/\Phi(z))/(C_\mu(z)/\lambda)$, which is analytic in $\Omega$, gives that this function is at most 1 in absolute value in $\Omega$, therefore

$$\text{cap}(E) = \lim_{z \to \infty} \frac{|z|}{|\Phi(z)|} \leq \lim_{z \to \infty} \frac{|zC_\mu(z)|}{\lambda} = \frac{\|\mu\|}{\lambda}.$$ 

For a continuum $E$ we have (see Theorem 5.3.2,(a) in [4])

\[\frac{1}{4}\text{diam}(E) \leq \text{cap}(E),\]

so we obtain

\[\text{diam}(E) \leq 4\frac{\|\mu\|}{\lambda}.\]  

Since this is true for any two points $A, B$ of $G$, the lemma follows. □

Let us point out what is behind the preceding lemma. For a positive Borel-measure $\mu$ of compact support on the complex plane let

$$C_\mu^*(z) = \sup_{\epsilon > 0} \left| \int_{|t-z| \geq \epsilon} d\mu(t) \frac{1}{t-z} \right|$$

be the maximal Cauchy-transform. The following extension of Lemma 3 follows from some classical results of X. Tolsa on analytic capacity.

**Lemma 4.** Let $\mu$ be a positive measure of compact support. If $\lambda > 0$ and $G$ is a connected component of the level set

$$\Lambda_\lambda(\mu) = \{ z \mid C_\mu^*(z) > \lambda \},$$

then

\[\text{diam}(G) \leq C\frac{\|\mu\|}{\lambda},\]  

where $\|\mu\|$ denotes the total mass of $\mu$, and $C$ is an absolute constant.

Note that the set $\Lambda_\lambda(\mu)$ is open, hence so are its connected components.

To prove (4) we need the concept of analytic capacity of a set $E$. Actually, there are two notions of analytic capacity in the literature denoted by $\gamma(E)$ and $\gamma_+(E)$, but by the fundamental theorem of X. Tolsa [9, (1.1) and Theorem 1.1] they
are of the same size: \( \gamma(E) \approx \gamma_+(E) \), so in what follows we shall only work with \( \gamma(E) \).

If \( E \) is a compact set, then \( \gamma(E) \) is defined as the supremum

\[
\gamma(E) = \sup_{f} |f'(\infty)|,
\]

where the supremum is taken for all functions \( f \) that are analytic in the unbounded component of \( \mathbb{C} \setminus E \) and \( |f(z)| \leq 1 \) there. Note also that

\[
f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).
\]

The analytic capacity of a Borel-set \( E \) is then defined as the supremum of the analytic capacities of all compact sets lying in \( E \).

Consider, for example, a continuum (connected compact set) \( E \) that has at least two points. The conformal map from the unbounded component \( \Omega \) of \( \mathbb{C} \setminus E \) onto the exterior of the unit disk is of the form (2). Therefore, setting \( f(z) = 1/\Phi(z) \) as a test function in the definition of \( \gamma(E) \) we obtain

\[
\gamma(E) \geq \text{cap}(E).
\]

There is also a converse inequality, namely if \( f \) is as in the definition of \( \gamma(E) \), then

\[
((f(z) - f(\infty))/2)\Phi(z) \text{ is of modulus } \leq 1 \text{ in } \Omega \text{ by the maximal principle, and hence}
\]

\[
|f'(\infty)| \leq 2 \lim_{z \to \infty} z/\Phi(z) = 2\text{cap}(E),
\]

giving \( \gamma(E) \leq 2\text{cap}(E) \). Since for a continuum \( E \) we have (see Theorems 5.3.2,(a) and 5.3.4 in [4])

\[
\frac{1}{4}\text{diam}(E) \leq \text{cap}(E) \leq \frac{1}{2}\text{diam}(E),
\]

we obtain as before

\[
\text{diam}(E) \leq 4\gamma(E).
\]

The reverse inequality \( \gamma(E) \leq \text{diam}(E) \) also follows from the just given discussion, and in view of \( \gamma(E) \approx \gamma_+(E) \) this yields \( \gamma_+(E) \approx \text{diam}(E) \), which is attributed in [9] to P. Jones.

The relevance of all these to Lemma 4 is that by [9, Theorem 1] and [10, Proposition 2.1]

\[
\gamma(\Lambda_\lambda^\ast(\mu)) \leq D \frac{\|\mu\|}{\lambda}
\]

with some absolute constant \( D \). Hence, if \( G \) is a component of \( \Lambda_\lambda^\ast(\mu) \), any two points and \( E \) is a broken line connecting \( A \) and \( B \) in \( G \) as in the proof of Lemma 3, then applying (5) and (6) we obtain Lemma 4.
Proof of Theorem 1. The proof easily follows from Lemma 3 and from Rouché’s theorem (cf. [5], [6]). Since we need a quantitative estimate in Theorem 2, we give some details.

We may assume $\varepsilon < \text{diam}(K)/100$.

Let $P_n(z) = \prod_{j=1}^{n} (z - z_j)$, and assume that $k \geq n/2$ of the zeros, say $z_1, \ldots, z_k$, lie in $K$. For simpler pole and zero counting we assume that the $z_j$’s are different — the general case follows from here by taking limits. We set

$$
\mu_1 = \sum_{j=1}^{k} \delta_{z_j}, \quad \mu_2 = \sum_{j=k+1}^{n} \delta_{z_j}, \quad \mu = \mu_1 + \mu_2,
$$

where $\delta_{z}$ denotes the Dirac mass at $z$.

The relevance of the Cauchy transform to our theorem is that

$$
-C_{\mu}(z) = \sum_{j=1}^{n} \frac{1}{z - z_j} = \frac{P'_n(z)}{P_n(z)}.
$$

In particular, the poles of $C_{\mu}$ are the zeros of $P_n$, and a zeros of $C_{\mu}$ are the zeros of $P'_n$.

Instead of $\varepsilon$ we shall prove the result for $3\varepsilon$. Let $\partial K_\varepsilon$ be the boundary of the set $K_\varepsilon$. First we need that for $z \in K_{3\varepsilon} \setminus K_\varepsilon$, $\varepsilon \leq \text{diam}(K)$, the inequality

$$
|C_{\mu_1}(z)| \geq c_1 n \varepsilon,
$$

holds, where $c_1$ depends only on the diameter of $K$. Indeed, let $z \in K_{3\varepsilon} \setminus K_\varepsilon$, and let $w$ be the closest point to $z$ from $K$. Let $\ell$ be the line that passes through $w$ and is perpendicular to the segment $zw$. Since the open disk about $z$ and of radius $|w - z|$ cannot contain a point of $K$, it follows that $K$ must lie on different side of $\ell$ than $z$. Without loss of generality we may assume that $\ell$ is the imaginary axis, $z$ belongs to the negative half of the real axis, and $K$ lies in the half-plane $\Re z \geq 0$. Then for all $z_j \in K$ we have $\Re(z_j - z) \geq \varepsilon$, and hence

$$
\Re \frac{1}{z_j - z} = \frac{\Re(z_j - z)}{|z_j - z|^2} \geq \frac{\varepsilon}{(3\varepsilon + \text{diam}(K))^2} \geq \frac{\varepsilon}{4\text{diam}(K)^2}, \quad 1 \leq j \leq k,
$$

and (7) follows with $c_1 = 1/8\text{diam}(K)^2$ since $k \geq n/2$.

Now assume that

$$
n - k \leq \frac{\varepsilon^2 c_1}{4 \cdot 5} n,
$$

(8)
which, for $\varepsilon \leq \text{diam}(K)$, also implies the $k \geq n/2$ assumption used above. By Lemma 3 any connected component $G$ of the set

$$
\Lambda = \Lambda_{c_1 n \varepsilon / 2}(\mu_2) = \left\{ z \left| \left| C_{\mu_2}(z) \right| > \frac{1}{2} c_1 n \varepsilon \right. \right\}
$$

satisfies

$$
\text{diam}(G) \leq 4 \frac{n-k}{c_1 n \varepsilon / 2} < \varepsilon / 2.
$$

Thus, if such a component intersects $\partial K_{2\varepsilon}$, then it lies inside the set $K_{3\varepsilon} \setminus K_{\varepsilon}$.

Choose now an oriented Jordan curve (i.e. a homeomorphic image of the unit circle) $\Gamma$ in $K_{3\varepsilon} \setminus K_\varepsilon$ that avoids the set $\Lambda$ and that circles $K$ once in the counterclockwise direction. The existence of $\Gamma$ follows from the fact that each component of $\Lambda$ has diameter $< \varepsilon / 2$. We shall give a rigorous proof for the existence, but first let us finish the proof of Theorem 1. Thus, on $\Gamma$ we have $|C_{\mu_2}(z)| \leq c_1 n \varepsilon / 2$, which is smaller than the absolute value $|C_{\mu_1}(z)| \geq c_1 n \varepsilon$ established above. Thus, by Rouché’s theorem, the difference

$$
\Delta = (\text{number of zeros inside } \Gamma - \text{number of poles inside } \Gamma)
$$

is the same for $C_{\mu_1}(z)$ and for $C_{\mu_1}(z) + C_{\mu_2}(z) = C_{\mu}(z)$. By the Gauss-Lucas theorem this difference is $-1$ for $C_{\mu_1}(z)$ (all poles and zeros of $(\prod_{j=1}^{k} (z-z_j))' / (\prod_{j=1}^{k} (z-z_j))$ lie in $K$), hence this difference is again $-1$ for $C_{\mu}(z)$. By the assumption of the theorem the number of poles of $C_{\mu}$ inside $\Gamma$ is at least $k$, therefore $C_{\mu}$, and hence also $P'_n(z)$, has at least $k-1$ zeros inside $\Gamma$. Since $\Gamma$ lies inside $K_{3\varepsilon}$, it follows that $P'_n$ has at least $k-1$ zeros inside $K_{3\varepsilon}$, and that completes the proof of the theorem.

The existence of $\Gamma$ is intuitively clear, but for completeness we give a rigorous proof. To do that, define the polynomial convex hull $Pc(S)$ for a compact $S \subset \mathbb{C}$ as the complement $\mathbb{C} \setminus \Omega$ of the unbounded component $\Omega$ of the complement $\mathbb{C} \setminus S$ of $S$. This is nothing else than the union of $S$ with the bounded components of $\mathbb{C} \setminus S$. The boundary of the polynomial convex hull is called the outer boundary of $S$ and is denoted by $\partial_{\text{out}} S$. Clearly, $\partial_{\text{out}} S = \partial_{\Omega}$.

We may assume without loss of generality that $n \varepsilon / 2$ is not a critical value of $C_{\mu_2}$ i.e. $C'_{\mu_2}(z) \neq 0$ on the set

$$
\left\{ z \left| \left| C_{\mu_2}(z) \right| = \frac{c_1}{2} n \varepsilon \right. \right\}
$$

(if this is not the case, just decrease $\varepsilon$ by a tiny amount — note that $C_{\mu_2}$ has only finitely many critical values). But then every component $G$ of $\Lambda$ is bounded by a finite number of disjoint analytic Jordan curves, and so the outer boundary $\partial_{\text{out}} G$ of $G$ is also an analytic Jordan curve.
Figure 1. The sets $K_{\epsilon}, K_{2\epsilon}$ and $K_{3\epsilon}$ and their boundaries (in particular, $L=\partial K_{2\epsilon}$), some components of the level set $\Lambda_{c_1 \varepsilon n/2}$ (shaded regions) and a possible path for $\Gamma$ (the thick path).

For simpler notation we set $L=\partial K_{2\epsilon}$ with its counterclockwise orientation. We define $\Gamma$ so that

- $\Gamma$ consists of parts of $L$ and parts of the outer boundaries of some components of $\Lambda$,
- $\Gamma$ circles $K$ once in the counterclockwise direction (i.e. the index of any point $z\in K$ with respect to $\Gamma$ is 1),
- $\Gamma$ does not have a point common with the interior of $\Lambda$, and
- $\Gamma$ lies in $K_{3\epsilon}\setminus K_{\epsilon}$.

See Figure 1.

Let $G_1, \ldots, G_n$ be those connected components of $\Lambda$ which intersect $L$ (if there are no such components, then $L=\partial K_{2\epsilon}$ oriented counterclockwise is suitable for $\Gamma$). Note that for two such $G_j$ the polynomially convex hulls $\text{Pc}(\overline{G_j})$ of their closures are either disjoint or one of them is part of the other one. Discard those $G_j$ for which $\text{Pc}(\overline{G_j})$ is part of some other $\text{Pc}(\overline{G_k})$, and we may assume that $G_j$, $1\leq j \leq m$, are those components that remain. Then $\text{Pc}(\overline{G_j})$, $1<j\leq m$, are disjoint, they have diameter $<\varepsilon/2$ (see (9)), and $L\cap \Lambda$ is part of $\cup_{j=1}^m \text{Pc}(\overline{G_j})$. As has been said, $\partial_{\text{out}} \overline{G_j}$ are analytic Jordan curves.

For each $j=0,1,\ldots, m$ we shall construct an oriented Jordan curve $\Gamma_j$ with the properties: either $\Gamma_j=L$, or $\Gamma_j$ has the following structure. There are subarcs $A_tB_t|L|$, $1\leq t\leq r=r_j$, of $L$ in the counterclockwise orientation of $L$, so that their numbering reflects counterclockwise orientation, i.e. $A_1B_1A_2B_2\ldots A_rB_rA_{r+1}B_{r+1}\ldots$
A quantitative Gauss-Lucas theorem

follow each other in the counterclockwise orientation on \( L \), where the indices are considered mod \( r \) (i.e. \( A_{r+1} = A_1 \)). Each arc \( \overline{A_tB_t} \) lies outside (the interior of) \( \bigcup_{s=1}^j \text{Pc}(G_s) \), and the curve \( \Gamma_j \) consists of these arcs as well as for each \( t \) of a subarc of some \( \partial_{\text{out}} G_{k_t} \) that connects \( B_t \) and \( A_{t+1} \), i.e. each arc \( \overline{B_tA_{t+1}} \) of \( \Gamma_j \) is a subarc of the outer boundary of some \( \overline{G_{k_t}} \), where the \( k_t \)'s are different for different \( t \)'s. The orientation of the arcs \( \overline{A_tB_t} \) on \( \Gamma_j \) coincides with their (counterclockwise) orientation on \( L \), while each \( \overline{B_tA_{t+1}} \) is oriented from \( B_t \) to \( A_{t+1} \). In other words, the structure of \( \Gamma_j \) is as follows: an arc of \( L \) is followed by an arc of some \( \partial_{\text{out}} G_k \), followed by another arc of \( L \) followed by an arc of some other \( \partial_{\text{out}} G_k' \) etc., and the arcs of \( L \) follow each other in the same order on \( \Gamma_j \) (in the orientation of the latter) as on \( L \) (in its counterclockwise orientation).

These \( \Gamma_j \) will have the properties:

1) \( \Gamma_j \) consists of parts of \( L \) and parts of the outer boundaries of \( \overline{G_1}, ..., \overline{G_j} \),

2) \( \Gamma_j \) circles \( K \) once in the counterclockwise orientation,

3) \( \Gamma_j \) does not have a point common with the interior of \( \text{Pc}(G_1), ..., \text{Pc}(G_j) \), and

4) \( \Gamma_j \) lies in \( K_{3\varepsilon} \setminus K_{\varepsilon} \).

Then clearly, \( \Gamma = \Gamma_m \) will satisfy all the requirements.

To do this, first choose points \( X_1, ..., X_M \in L \setminus \bigcup_{j=1}^m \text{Pc}(G_j) \) in the counterclockwise direction on \( L \) such that the length \( \ell(X_sX_{s+1}) \) of the oriented arc \( X_sX_{s+1} \) of \( L \) from \( X_s \) to \( X_{s+1} \) satisfies \( 2\varepsilon \leq \ell(X_sX_{s+1}) < 6\varepsilon \) for all \( s = 1, ..., M \), where we take the indices mod \( M \) (i.e. \( X_{M+1} = X_1 \)). Indeed, let \( X_1 \in L \setminus \bigcup_j \text{Pc}(G_j) \) be arbitrary, and then consider the points \( P, Q \in L \) such that \( \ell(X_1XP) = 2\varepsilon \) and \( \ell(X_1XQ) = 2\varepsilon \), and \( X_1, P, Q \) follow each other in this order on \( L \). Since the diameter of a convex arc is at least as large as \( 1/\pi \)-times its length, \(^{(1)}\) (see [8] or [2, Sec. 44, (5)]) it follows that \( X_1XQ \) has diameter \( > \varepsilon/2 \). Therefore, this arc cannot lie entirely in a \( \text{Pc}(G_j) \) because these latter have diameter smaller than \( \varepsilon/2 \). But then \( X_1XQ \) is impossible, for then \( X_1XQ \) would be the union of more than one of its non-empty disjoint closed subsets (namely of those \( X_1XQ \cap \text{Pc}(G_j) \) that are not empty) which is impossible since \( X_1XQ \) is connected. As a consequence, there is an \( X_2 \in L \) \( P \) \( \bigcup_j \text{Pc}(G_j) \) giving the choice of \( X_2 \). Now do the same

\(^{(1)}\) This is usually stated for closed curves, but the arc-case then follows by simply connecting the two endpoints of the arc by a segment.
construction starting from $X_2$ to get $X_3$, then from $X_3$ to get $X_4$, etc. until we get to an $X_M$ for which $\ell(X_M X_1)_L < 6\varepsilon$ from $X_1$.

After this we turn to the construction of the Jordan curves $\Gamma_j$ for all $j$. Let $\Gamma_0 = L$ oriented counterclockwise, and suppose that for some $0 \leq j < m$ the $\Gamma_j$ has already been constructed. If $\Gamma_j \cap \text{Pc}(G_{j+1}) = \emptyset$ (which is equivalent to the fact that $\Gamma_j \cap \text{Pc}(G_{j+1})$ can contain only boundary points of $\text{Pc}(G_{j+1})$), then set $\Gamma_{j+1} = \Gamma_j$. If this is not the case, then let $A_0$ be a point in $\Gamma_j \cap G_{j+1}$. Note that since different $\text{Pc}(G_k)$ are disjoint, every point of $\Gamma_j \cap \text{Pc}(G_{j+1})$ lies in one of the arcs $A_i B_i \mid L$. So this $A_0$ lies in one of the arcs $X_s X_{s+1}_L$ of $L$. Then, by the construction of the points $X_k$ and by $\text{diam}(G_j + 1) < \varepsilon/2$, the intersection $\Gamma_j \cap \text{Pc}(G_{j+1})$ is part of the arc $X_{s-1} X_{s+2} \mid L$ of $L$. Now let $A$ and $B$ be the first and last points in the orientation of $\Gamma_j$ that lie in $\text{Pc}(G_{j+1})$ (i.e., $A, B \in \Gamma_j \cap \text{Pc}(G_{j+1})$), $AA_0B$ follow each other on $\Gamma_j$ in this order, and the arc $BA \mid \Gamma_j$ of $\Gamma_j$ from $B$ to $A$ does not intersect $\text{Pc}(G_{j+1})$ except for its endpoints $B, A$). Then $A \neq B$ (since $A_0 \in G_{j+1}$ and $G_{j+1}$ is open), $A$ and $B$ lie on the outer boundary $\partial_{\text{out}} G_{j+1}$ of $G_{j+1}$, and since this outer boundary is a Jordan curve, there is a Jordan arc $J$ on that boundary that connects $A$ and $B$ (actually, there are two such arcs, it does not matter which one we choose). Orient $J$ so that $J$ is an arc from $A$ to $B$, see Figure 2. The points $A$ and $B$ also lie on $\Gamma_j$, and they divide $\Gamma_j$ into two Jordan arcs $J_1$ and $J_2$, say $J_1$ is the arc from $A$ to $B$ (in the orientation inherited from $\Gamma_j$). Replace now the arc $J_1 = AB \mid \Gamma_j$ on $\Gamma_j$ from $A$ to $B$ by $J$ to get the Jordan-curve $\Gamma_{j+1}$ (note that $J$ does not intersect the other arc $J_2 = BA \mid \Gamma_j$ of $\Gamma_j$ because of the definition of the points $A$ and $B$, so $J \cup J_2$ is, indeed, a Jordan-curve). It is clear that this $\Gamma_{j+1}$ has the structure described above. Properties 1), 3) and 4) are obvious for $\Gamma_{j+1}$ from the induction hypothesis and from the fact that $J$ lies in the $\varepsilon/2$-neighborhood of the arc $X_{s-1} X_{s+2} \mid L$ of $L$ (recall that the points $A$ and $B$ belong to $L$).

As for property 2), note first of all that $J_1$ consists of subarcs of $L$ and of some subarcs $J_k$ of some $\partial_{\text{out}} G_k$'s. Each of the latter ones connect some two points $C_k, D_k$ of $L$ that lie in between $A$ and $B$ in the counterclockwise orientation on $L$. If $\Delta_{\varepsilon}(C_k)$ is the disk of radius $\varepsilon$ about $C_k$, then $J_k \subset \partial_{\text{out}} G_k \subset \Delta_{\varepsilon}(C_k)$ (recall that $G_k$ has diameter $< \varepsilon/2$ and $C_k \in G_k$), so the arcs $J_k$ and $C_kD_k \mid L$ can be continuously deformed into each other within $\Delta_{\varepsilon}(C_k)$. Since $\Delta_{\varepsilon}(C_k)$ is also part of the $\varepsilon$-neighborhood of $AB \mid L$, we obtain that $J_1$ can be continuously deformed into $AB \mid L$, within the $\varepsilon$-neighborhood of $AB \mid L$. Clearly the same is true for $J$ and $AB \mid L$ (for the same reason), hence $\Gamma_j$ and $\Gamma_{j+1}$ can be continuously deformed into
Figure 2. The points $A$ and $B$ and the arcs $J$ and $J_1$ in the definition of $\Gamma_{j+1}$. In the figure we assume that $1 \leq k \leq j$, and then $J_1$ consists of the arc $AB \big|_L$ of $L$, one of the subarcs of the boundary of $G_k$ that connects $B_t$ with $A_{t+1}$, and from the arc $A_{t+1}B \big|_L$. When defining $\Gamma_{j+1}$ from $\Gamma_j$, these three arcs are replaced by the single arc $J$ connecting $A$ and $B$ on the boundary of $G_{j+1}$ (as has been said, there are two choices for $J$, in the figure we chose the longer one). Note also if we had $k > j$ in the figure, then $J_1$ would be simply the arc of $L$ from $A$ to $B$.

each other in $K_{3\varepsilon} \setminus K_\varepsilon$. Since $\Gamma_j$ circles $K$ once in the counterclockwise direction by the induction hypothesis, the same is true of $\Gamma_{j+1}$, proving 2). \qed

3. Proof of Theorem 2

The first part follows from the just given proof for Theorem 1. Indeed, we have seen that if $\varepsilon \leq \text{diam}(K)/100$ and (see (8))

$$n - k \leq \frac{1}{4 \cdot 5 \cdot 8} \frac{\varepsilon^2}{\text{diam}(K)^2},$$

then $k - 1$ critical points are guaranteed in $K_{3\varepsilon}$, so

$$C_1 = \frac{1}{9 \cdot 4 \cdot 5 \cdot 8}.$$
suffices in Theorem 2 for such $3\varepsilon$. Now to cover the range $3 \text{diam}(K)/100 \leq 3\varepsilon \leq \text{diam}(K)$, just divide this $C_1$ by $100^2/3^2$.

We shall prove the second part first for a square of side-length 2.

We shall use some basic notions and results from logarithmic potential theory (see for example the books [3], [4] and [7]), among others the notion of equilibrium measure and of balayage (for the latter see [7, Sec. II.4] or [3, Ch. IV]). In particular, we shall use that if $R_N(z)$ is a polynomial of degree $N$ with leading coefficient 1, $\mu$ is the normalized counting measure on its zeros, then the equilibrium measure of a level set $L = \{ z | |R_N(z)| = \tau \}$ is the balayage $\hat{\nu}$ of $\nu$ out of the bounded components of $\mathbb{C} \setminus L$ (i.e. “onto” $L$). Indeed, on $L$ the logarithmic potential

$$U^\mu(z) = \int \log \frac{1}{|z-t|} d\hat{\mu}(t)$$

coincides with

$$U^\mu(z) = \int \log \frac{1}{|z-t|} d\mu(t) = \frac{1}{N} \log \frac{1}{|R_N(z)|} = \frac{1}{N} \log \frac{1}{\tau}$$

(the logarithmic potential does now change on $L$ when forming balayage out of the components of $\mathbb{C} \setminus L$), i.e. it is constant on $L$, and that characterizes equilibrium measures among unit measures on $L$.

For an integer $s \geq 2$ set

$$R(z) = z^s(1-z).$$

This has $s-1$ critical points at 0 and one critical point at $s/(s+1)$. Let

$$\rho_0 = \left( \frac{s}{s+1} \right)^s \frac{1}{s+1}$$

be the value of $R$ at the critical point $s/(s+1)$. Then the level set

$$L_{\rho_0} := \{ z | |R(z)| = \rho_0 \}$$

passes through the point $s/(s+1)$ and consists of two loops, say $\ell_0$ around 0 and $\ell_1$ around 1, that meet at the point $s/(s+1)$ (see Figure 3). If we set

$$\mu_0 = \frac{s}{s+1} \delta_0 + \frac{1}{s+1} \delta_1$$

(the normalized zero counting measure of the zeros of $R$), then, as has been mentioned, the equilibrium measure $\omega_{L_{\rho_0}}$ of $L_{\rho_0}$ is the balayage of $\mu_0$ out of the two bounded domains encircled by $L_{\rho_0}$. During this balayage process $(s/(s+1))\delta_0$ is moved entirely to $\ell_0$, and $(1/(s+1))\delta_1$ is moved entirely to $\ell_1$, hence

$$\omega_{L_{\rho_0}}(\ell_0) = \frac{s}{s+1}, \quad \omega_{L_{\rho_0}}(\ell_1) = \frac{1}{s+1}.$$
Consider now the square with center at the origin and of side-length 2, and shift it horizontally so that its right-hand side passes through the point $1 - 2/3(s + 1)$. This will be our set $K$. If we also set $\varepsilon = 1/3(s + 1)$, then the “right-hand side” $K_\varepsilon$ of $K_\varepsilon$ passes through the point $1 - 1/3(s + 1)$ (see Figure 4). The point is that the loop $\ell_0$ lies inside $K$, but $K$ also contains some part of $\ell_1$, hence

$$\omega_{L_{\rho_0}}(K) = (1 + 3\tau) \frac{s}{s + 1}$$

with some $\tau > 0$. But then for some $\rho^* > \rho_0$ we shall have

$$\omega_{L_{\rho^*}}(K) \geq (1 + 2\tau) \frac{s}{s + 1}$$

as well (note that, as $\rho^* \searrow \rho_0$, $\omega_{L_{\rho^*}}$ converges in the weak* topology to $\omega_{L_{\rho_0}}$). We fix this $\rho^*$. For it the level set $L_{\rho^*}$ is an analytic Jordan curve (this is the case for all the level sets $L_{\rho}$ with $\rho > \rho_0$).

In what follows we need the following lemma for the integrals of $R^n$ with large $n$.

**Lemma 5.** If $z \in L_{\rho}$ with some $\rho > \rho_0$, then

$$\int_{s/(s+1)}^z R^n(u)du = (1 + o(1)) \frac{R^{n+1}(z)}{(n+1)R'(z)},$$

where $o(1)$ tends uniformly to 0 in $z \in L_{\rho}$ (with any fixed $\rho > \rho_0$) as $n \to \infty$.

Taking this lemma for granted for the time being, we continue the proof, and set

$$S_n(z) = (n(s+1)+1) \int_{s/(s+1)}^z R^n(u)du - (\rho^*)^n;$$
which is a polynomial of degree \( n(s+1)+1 \) with leading coefficient 1. We claim that if \( \rho_0 < \rho_1 < \rho^* < \rho_2 \), then for large \( n \) all the zeros of \( S_n \) lie in the strip in between the level sets \( L_{\rho_1} \) and \( L_{\rho_2} \). Indeed, in view of (11) for \( z \in L_{\rho_1} \) we have

\[
S_n(z) = O \left( \rho_1^{n+1} - (\rho^*)^n \right),
\]

so \( S_n \) has no zero inside \( L_{\rho_1} \) by Rouché’s theorem (if \( n \) is sufficiently large). On the other hand, if \( z \in L_{\rho_2} \), then again (11) gives that

\[
(n(s+1)+1) \left| \int_{s/(s+1)}^z R^n(u)du \right| = (1+o(1))(s+1) \left| \frac{R^{n+1}(z)}{R'(z)} \right| > c\rho_2^{n+1}
\]

with some \( c>0 \) (that is uniform in \( z \in L_{\rho_2} \)), hence, by Rouché’s theorem, for both \( S_n(z) \) and \( S_n(z)+(\rho^*)^n \) the number of zeros inside \( L_{\rho_2} \) is the same as the number

\[
D = (\text{number of zeros inside } L_{\rho_2} - \text{number of poles inside } L_{\rho_2})
\]

for the function

\[
(n(s+1)+1) \frac{R^{n+1}(z)}{(n+1)R'(z)},
\]

which is clearly \((n+1)(s+1)-s = n(s+1)+1\). Thus, all zeros of \( S_n \) lie inside \( L_{\rho_2} \), and the claim follows.

Let

\[
S_n(z) = \prod_{j=1}^{n(s+1)+1} (z-w_{j,n}),
\]
and consider the zero counting measure
\[ \nu_n = \frac{1}{n(s+1)+1} \sum_{j=1}^{n(s+1)+1} \delta_{w_{j,n}}. \]

We claim that these converge in the weak* topology to the equilibrium measure \( \omega_{L_{\rho^*}} \), and to do that it is enough to show that if \( \nu \) is a weak* limit of \( \{\nu_n\} \), say \( \nu_n \to \nu \) as \( n \to \infty, n \in \mathbb{N} \), then \( \nu = \omega_{L_{\rho^*}} \). We have just shown that \( \nu \) is supported on \( L_{\rho^*} \). Furthermore, if \( z \) lies outside \( L_{\rho^*} \), then from (11) and from what we have just shown about the location of the zeros of \( S_n \), it follows that
\[ \int \log \frac{1}{|z-t|} d\nu_n(t) = \int \log \frac{1}{|z-t|} d\nu_n(t) \]
\[ = \lim_{n \to \infty, n \in \mathbb{N}} \frac{1}{n(s+1)+1} \log \frac{1}{|S_n(z)|} = \frac{1}{(s+1)} \log \frac{1}{|R(z)|}. \]

However, the right-hand side is the same as
\[ \int \log \frac{1}{|z-t|} d\mu_0(t) = \int \log \frac{1}{|z-t|} d\omega_{L_{\rho^*}}(t), \]
where, in the last step, we used that the equilibrium measure of the level set \( L_{\rho^*} \) is the balayage of the measure \( \mu_0 \) from the inner domain of \( L_{\rho^*} \), hence its logarithmic potential outside \( L_{\rho^*} \) coincides with the logarithmic potential of \( \mu_0 \). Thus, the logarithmic potentials of the measures \( \nu \) and \( \omega_{L_{\rho^*}} \), both of which are supported on \( L_{\rho^*} \), coincide outside \( L_{\rho^*} \), and the equality \( \nu = \omega_{L_{\rho^*}} \) follows from Carleson’s unicity theorem [7, Theorem II.4.13].

Now in view of (10) and of the convergence \( \nu_n \to \omega_{L_{\rho^*}} \) in the weak* topology, for all large \( n \) the polynomial \( S_n(z) \) of degree \( n(s+1)+1 \) has at least
\[ (1+\tau) \frac{s}{s+1}(n(s+1)+1) \geq (1+\tau)sn \]
zeros in \( K \), but \( S'_n(z) = (n(s+1)+1)R^n(z) \) has only \( sn \) zeros (the ones at the origin) in \( K_\varepsilon \). This shows that the number \( \alpha_\varepsilon = \alpha_{1/3(s+1)} \) for \( K \) must be greater than
\[ \frac{(1+\tau)sn}{n(s+1)+1} > \frac{s}{s+1} - \frac{3}{3(s+1)} = 1 - 3\varepsilon. \]
Finally, for \( \varepsilon \) not of the form \( 1/3(s+1) \) select the largest \( s \) for which \( \varepsilon < 1/3(s+1) \).

This proves the second part of Theorem 2 for a square of side-length 2.

If \( K \) is a convex set with non-empty interior, then the argument is the same. Clearly, it is sufficient to prove the claim for a homothetic copy of \( K \). Now take a disk inside \( K \) and translate it so that we obtain a disk \( D \) which still lies in \( K \), but
contains a boundary point $M$. By scaling, rotating and translating we may achieve that $M$ is the point $1 - 2/3(s+1)$, the tangent line to $K$ at $M$ is vertical and $D$ is sufficiently large and lies to the left of that tangent line (which is necessarily the tangent line to $D$, as well). Now we are in the position that we can use the just given proof (which was for squares) using the same function $R$ as before (in this situation $\ell_0$ lies again in $K$).

Finally, if $K$ has empty interior, then it is a segment, say $K = [-1, 1]$. For an $s \geq 1$ set

$$S_n(z) = (z^2 - 1)^{sn}(z - i)^n,$$

which has $k = 2sn$ zeros in $[-1, 1]$, and

$$S'_n(z) = n(z^2 - 1)^{sn-1}(z - i)^{n-1} \left( s(z - i)2z + (z^2 - 1) \right)$$

has $2sn - 2$ zeros in $[-1, 1]$, $n - 1$ zeros at $i$ and two other zeros lying outside $[-1, 1]$, the closest of which to $[-1, 1]$ is

$$\frac{i}{s + \sqrt{s^2 - 2s - 1}},$$

which is of distance $1/(s + \sqrt{s^2 - 2s - 1}) > 1/2s$ from $[-1, 1]$. Thus, if $\varepsilon = 1/2s$, then $S_n$ has at most $k - 2$ critical points in $K_\varepsilon$, therefore $\alpha_1/2s$ must be bigger than $2sn/(2sn + n) = 2s/(2s + 1)$, which proves the second part of Theorem 2 for the segment $K = [-1, 1]$.

We still need to prove Lemma 5.

**Proof of Lemma 5.** Let $\rho > \rho_0$ be fixed, $z \in L_\rho$, and select $\rho_0 < \rho_1 < \rho$ close to $\rho$. The mapping $\xi \rightarrow R(\xi)$ maps $L_\rho$ into the circle $C_\rho = \{ w \mid |w| = \rho \}$, and $L_{\rho_1}$ into the circle $C_{\rho_1}$ in a $(s+1)$–to–1 fashion. We may assume that $R(z) = \rho$ (if this is not the case then just multiply $R$ by a suitable number $\theta$ of modulus 1 to achieve that and then divide the integral by $\theta^n$). The inverse image of the segment $[\rho_1, \rho]$ under this mapping consists of $(s+1)$ Jordan arcs, one of which, say $J$, has $z$ as one of its endpoints. Let $z_1$ be the other endpoint of $J$. Then $z_1 \in L_{\rho_1}$. If the path of the integration lies in the inner domain of $L_{\rho_1}$, then it is immediate that

$$\int_{s/(s+1)}^{z_1} R^n(u) du = O(\rho_1^n).$$

We also have

$$\frac{1}{R'(z)} \int_{z_1}^{z} R^n(u)R'(u) du = \frac{R^{n+1}(z)}{(n+1)R'(z)} - \frac{R^{n+1}(z_1)}{(n+1)R'(z_1)} = \frac{R^{n+1}(z)}{(n+1)R'(z)} - O(\rho_1^n).$$
Thus, it is left to show that
\begin{equation}
\int_{J} R^n(u) \left(1 - \frac{R'(u)}{R'(z)}\right) du = O\left(\frac{\rho^2}{n^2}\right),
\end{equation}
because the right-hand side is
\begin{equation}
o\left(\frac{R^{n+1}(z)}{(n+1)R'(z)}\right).
\end{equation}

If we make the substitution $t=R(u)$ in the integral on the left of (12), the integral becomes
\begin{equation}
\int_{\rho}^{\rho_1} t^n \left(1 - \frac{R'(R^{-1}(t))}{R'(R^{-1}(\rho))}\right) \frac{1}{R'(R^{-1}(t))} dt
\end{equation}
with some local branch of $R^{-1}$, which, in view of
\begin{equation}
\left|\frac{1}{R'(R^{-1}(t))} - \frac{1}{R'(R^{-1}(\rho))}\right| \leq C|t-\rho|,
\end{equation}
is in absolute value at most
\begin{equation}
C \int_{\rho_1}^{\rho} t^n(\rho-t)dt \leq C \int_{0}^{\rho} t^n(\rho-t)dt = C \frac{\rho^{n+2}}{(n+1)(n+2)}
\end{equation}
(apply integration by parts). \(\square\)

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