MIRROR SYMMETRY FOR CALABI-YAU
COMPLETE INTERSECTIONS IN FANO TORIC VARIETIES

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Abstract. Generalizing the notions of reflexive polytopes and nef-partitions of Batyrev and Borisov, we propose a mirror symmetry construction for Calabi-Yau complete intersections in Fano toric varieties.

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0. Introduction.

Reflexive polytopes \(\Delta\) in \(\mathbb{R}^d\), introduced by Victor Batyrev in [B1], are determined by the property that they have vertices in \(\mathbb{Z}^d\) and have the origin in their interior with the polar dual polytope

\[\Delta^* = \{ y \in \mathbb{R}^d \mid \langle \Delta, y \rangle \geq -1 \}\]

satisfying the same property. The polar duality gives an involution between the sets of reflexive polytopes: \((\Delta^*)^* = \Delta\). There is a one-to-one correspondence between isomorphism classes of reflexive polytopes \(\Delta\) in \(\mathbb{R}^d\) and \(d\)-dimensional Gorenstein Fano toric varieties given by

\[\Delta \mapsto X_\Delta := \text{Proj}(\mathbb{C}[\mathbb{R}_{\geq 0}(\Delta, 1) \cap \mathbb{Z}^{d+1}]),\]

where the grading is induced by the last coordinate in \(\mathbb{Z}^{d+1}\). The dual pair of reflexive polytopes \(\Delta\) and \(\Delta^*\) corresponds to the Batyrev mirror pair of ample Calabi-Yau hypersurfaces \(Y_\Delta \subset X_\Delta\) and \(Y_{\Delta^*} \subset X_{\Delta^*}\) in Gorenstein Fano toric varieties in [B1]. By taking maximal projective crepant partial resolutions \(\tilde{Y}_\Delta \rightarrow Y_\Delta\) and \(\tilde{Y}_{\Delta^*} \rightarrow Y_{\Delta^*}\) induced by toric blow ups, Batyrev obtained a mirror pair of minimal Calabi-Yau hypersurfaces \(\tilde{Y}_\Delta, \tilde{Y}_{\Delta^*}\).

Generalizing the polar duality of reflexive polytopes, Lev Borisov in [Bo] introduced the notion of nef-partition, which is a Minkowski sum decomposition of the

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A reflexive polytope $\Delta = \Delta_1 + \cdots + \Delta_r$ by lattice polytopes such that the origin $0 \in \Delta_i$ for all $1 \leq i \leq r$. A nef-partition has a dual nef-partition defined as the Minkowski sum decomposition of the reflexive polytope $\nabla = \nabla_1 + \cdots + \nabla_r$ in the dual vector space with $\nabla_j$ determined by $\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$ for all $1 \leq i, j \leq r$, where $\delta_{ij}$ is the Kronecker symbol.

A nef-partition of a reflexive polytope $\Delta = \Delta_1 + \cdots + \Delta_r$ with $r < d$ and $\dim \Delta_i > 0$, for all $1 \leq i \leq r$, defines a nef Calabi-Yau complete intersection $Y_{\Delta_1, \ldots, \Delta_r}$ in the Gorenstein Fano toric variety $X_{\Delta}$ given by the equations:

$$\left( \sum_{m \in \Delta_i \cap \mathbb{Z}^d} a_{i,m} \prod_{v_p \in \mathcal{V}(\Delta^*)} x_p^{(m,v_p)} \right) \prod_{v_p \in \mathcal{V}_i} x_p = 0, \quad i = 1, \ldots, r,$$

with generic $a_{i,m} \in \mathbb{C}$, where $x_p$ are the Cox homogeneous coordinates of the toric variety $X_{\Delta}$ corresponding to the vertices $v_p$ of the polytope $\Delta^*$.

The Batyrev-Borisov mirror symmetry construction is a pair of families of generic nef Calabi-Yau complete intersections $Y_{\Delta_1, \ldots, \Delta_r} \subset X_{\Delta}$ and $Y_{\mathcal{V}_1, \ldots, \mathcal{V}_r} \subset X_{\mathcal{V}}$ in Gorenstein Fano toric varieties corresponding to a dual pair of nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\nabla = \nabla_1 + \cdots + \nabla_r$. By taking maximal projective crepant partial resolutions $\hat{Y}_{\Delta_1, \ldots, \Delta_r} \to Y_{\Delta_1, \ldots, \Delta_r}$ and $\hat{Y}_{\mathcal{V}_1, \ldots, \mathcal{V}_r} \to Y_{\mathcal{V}_1, \ldots, \mathcal{V}_r}$, one obtains the Batyrev-Borisov mirror pair of minimal Calabi-Yau complete intersections.

The topological mirror symmetry test for compact $n$-dimensional Calabi-Yau manifolds $V$ and $V^*$ is a symmetry of their Hodge numbers:

$$h^{p,q}(V) = h^{n-p,q}(V^*), \quad 0 \leq p, q \leq n.$$ For singular varieties Hodge numbers must be replaced by the stringy Hodge numbers $h_{st}^{p,q}$ introduced by V. Batyrev in [B2]. The usual Hodge numbers coincide with the stringy Hodge numbers for nonsingular Calabi-Yau varieties, and all crepant partial resolutions $\hat{V}$ of singular Calabi-Yau varieties $V$ have the same stringy Hodge numbers: $h_{st}^{p,q}(\hat{V}) = h_{st}^{p,q}(V)$. In [BB2], Batyrev and Borisov show that the pair of generic Calabi-Yau complete intersections $V = Y_{\Delta_1, \ldots, \Delta_r}$ and $V^* = Y_{\mathcal{V}_1, \ldots, \mathcal{V}_r}$ pass the topological mirror symmetry test:

$$h_{st}^{p,q}(Y_{\Delta_1, \ldots, \Delta_r}) = h_{st}^{d-r-p,q}(Y_{\mathcal{V}_1, \ldots, \mathcal{V}_r}), \quad 0 \leq p, q \leq d - r.$$ Generalizing the notions of reflexive polytopes and nef-partitions for rational polytopes we introduce the notions of $\mathbb{Q}$-reflexive polytopes and $\mathbb{Q}$-nef-partitions. A $\mathbb{Q}$-reflexive polytope $\Delta$ in $\mathbb{R}^d$ is determined by the properties that $0$ is in the interior of $\Delta$ and

$$\text{Conv}(\text{Conv}(\Delta \cap \mathbb{Z}^d)^* \cap \mathbb{Z}^d) = \Delta^*.$$ We show that a $\mathbb{Q}$-reflexive polytope $\Delta$ corresponds to the Fano toric variety $X_{\Delta}$ at worst canonical singularities. A $\mathbb{Q}$-reflexive polytope has a dual $\mathbb{Q}$-reflexive polytope defined by $\Delta^\circ := (\text{Conv}(\Delta \cap \mathbb{Z}^d))^*$ and the property $(\Delta^\circ)^\circ = \Delta$ gives an involution on the set of $\mathbb{Q}$-reflexive polytopes. The dual lattice polytope $\Delta^\circ$ of a $\mathbb{Q}$-reflexive polytope is called an almost reflexive polytope and there is a similar involution on the set of almost reflexive polytopes. All reflexive polytopes are $\mathbb{Q}$-reflexive and almost reflexive.

A $\mathbb{Q}$-nef-partition is a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ of a $\mathbb{Q}$-reflexive polytope into polytopes in $\mathbb{R}^d$ such that $0 \in \Delta_i$ for all $1 \leq i \leq r$, and

$$\text{Conv}(\Delta \cap \mathbb{Z}^d) = \text{Conv}(\Delta_1 \cap \mathbb{Z}^d) + \cdots + \text{Conv}(\Delta_r \cap \mathbb{Z}^d).$$
We prove that a \(\mathbb{Q}\)-nef-partition \(\Delta = \Delta_1 + \cdots + \Delta_r\) has a dual \(\mathbb{Q}\)-nef-partition \(\nabla = \nabla_1 + \cdots + \nabla_r\) determined by \(\langle \Delta_i, \Delta_j \rangle \geq -\delta_{ij}\) for all \(1 \leq i, j \leq r\). This dual pair of \(\mathbb{Q}\)-nef-partitions corresponds to a pair of \(\mathbb{Q}\)-nef Calabi-Yau complete intersections \(Y_{\Delta_1, \ldots, \Delta_k} \subset X_\Delta\) and \(Y_{\nabla_1, \ldots, \nabla_k} \subset X_{\nabla}\) in Fano toric varieties. We expect that this pair passes the topological mirror symmetry test as in [BB02].

In [BB01], Batyrev and Borisov introduce the notion of reflexive Gorenstein cones \(\sigma \subset \mathbb{R}^d\), which canonically correspond to Gorenstein Fano toric varieties \(X_\sigma = \text{Proj}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^d])\) such that \(\mathcal{O}_{X_\sigma}(1)\) is an ample invertible sheaf and there is a positive integer \(r\) such that \(\mathcal{O}_{X_\sigma}(r)\) is isomorphic to the anticanonical sheaf of \(X_\sigma\). The zeros \(Y_\sigma\) of generic global sections of \(\mathcal{O}_{X_\sigma}(1)\) are called generalized Calabi-Yau manifolds. The dual cone

\[
\sigma^\vee = \{ y \in \mathbb{R}^d \mid \langle x, y \rangle \geq 0 \ \forall x \in \sigma \}
\]

of a reflexive Gorenstein cone \(\sigma\) is again reflexive, and the dual pair \(\sigma\) and \(\sigma^\vee\) corresponds to the mirror pair of generalized Calabi-Yau manifolds \(Y_\sigma\) and \(Y_{\sigma^\vee}\), which are ample hypersurfaces in the respective Gorenstein Fano toric varieties.

Combining the ideas of [BB01] with the notion of almost reflexive polytopes, we introduce the notion of almost reflexive Gorenstein cones \(\sigma\). Their dual cones \(\sigma^\vee\) are no longer Gorenstein, but there is a canonically defined grading on \(\sigma^\vee \cap \mathbb{Z}^d\). This allows us to associate to an almost reflexive Gorenstein cone \(\sigma \subset \mathbb{R}^d\) the Fano toric variety \(X_\sigma = \text{Proj}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^d])\). The reflexive rank one sheaf \(\mathcal{O}_{X_\sigma}(1)\) corresponds to an ample \(\mathbb{Q}\)-Cartier divisor and there is a positive integer \(r\) such that \(\mathcal{O}_{X_\sigma}(r)\) is isomorphic to the anticanonical sheaf on \(X_\sigma\). In particular, we have a generalized Calabi-Yau manifold \(Y_\sigma\) given by generic global sections of \(\mathcal{O}_{X_\sigma}(1)\). There is an involution on the set of almost reflexive Gorenstein cones \(\sigma \mapsto \sigma^\star\). For a dual pair of almost reflexive Gorenstein cones \(\sigma\) and \(\sigma^\star\) we expect that the correspondence between generalized Calabi-Yau manifolds \(Y_\sigma \leftrightarrow Y_{\sigma^\star}\) corresponds to the mirror involution in \(N = 2\) super conformal field theory.

1. \(\mathbb{Q}\)-reflexive and almost reflexive polytopes.

In this section, we first review the definition of reflexive polytopes due to V. Batyrev in [B1], and then construct a natural generalization of these notions for rational and lattice polytopes.

Let \(M\) be a lattice of rank \(d\) and \(N = \text{Hom}(M, \mathbb{Z})\) be its dual lattice with a natural paring \(\langle \_ , \_ \rangle : M \times N \to \mathbb{Z}\). Denote \(M_\mathbb{R} = M \otimes \mathbb{R}\), \(N_\mathbb{R} = N \otimes \mathbb{R}\).

**Definition 1.1.** A \(d\)-dimensional lattice polytope \(\Delta \subset M_\mathbb{R}\) is called a canonical Fano polytope if \(\text{int}(\Delta) \cap M = \{0\}\).

**Definition 1.2.** [B1] A \(d\)-dimensional lattice polytope \(\Delta \subset M_\mathbb{R}\) is called reflexive (with respect to \(M\)) if \(0 \in \text{int}(\Delta)\) and the dual polytope

\[
\Delta^* = \{ n \in N_\mathbb{R} \mid \langle m, n \rangle \geq -1 \ \forall m \in \Delta \}
\]

in the dual vector space \(N_\mathbb{R}\) is also a lattice polytope. The pair \(\Delta\) and \(\Delta^*\) is called a dual pair reflexive polytopes and it satisfies \(\Delta = (\Delta^*)^\star\).

**Definition 1.3.** A compact toric variety \(X\) is called

- **Fano** if the anticanonical divisor \(-K_X\) is ample and \(\mathbb{Q}\)-Cartier,
- **Gorenstein** if \(K_X\) is Cartier.
Proposition 1.4. There is a bijection between isomorphism classes of canonical Fano polytopes and Fano toric varieties with canonical singularities given by $\Delta \mapsto X_\Delta$. In particular, Gorenstein Fano toric varieties correspond to reflexive polytopes.

Generalizing the notion of a reflexive polytope we introduce:

Definition 1.5. A $d$-dimensional polytope $\Delta$ in $M_\mathbb{R}$ is called $\mathbb{Q}$-reflexive (with respect to $M$) if $0 \in \text{int}(\Delta)$ and
\[
\text{Conv}(\text{Conv}(\Delta \cap M))^* \cap N = \Delta^*.
\]

Remark 1.6. For a $\mathbb{Q}$-reflexive polytope $\Delta$ its dual $\Delta^*$ is a lattice polytope, whence reflexive polytopes are $\mathbb{Q}$-reflexive. It follows from (1) that a $\mathbb{Q}$-reflexive polytope is rational, i.e., its vertices lie in $M_\mathbb{Q}$. These properties together with the next ones suggest the name of $\mathbb{Q}$-reflexive.

Definition 1.7. Denote $[\Delta] := \text{Conv}(\Delta \cap M)$ for a polytope $\Delta$ in $M_\mathbb{R}$ (and, similarly in $N_\mathbb{R}$). Also, define $\Delta^\circ := [\Delta]^* = (\text{Conv}(\Delta \cap M))^*$

In this notation, equation (1) is $[[\Delta]^*] = \Delta^*$, or, equivalently, $(\Delta^\circ)^\circ = \Delta$. Hence, we have

Lemma 1.8. If $\Delta \subset M_\mathbb{R}$ is $\mathbb{Q}$-reflexive, then $\Delta^\circ = (\text{Conv}(\Delta \cap M))^* \subset N_\mathbb{R}$ is $\mathbb{Q}$-reflexive and the map $\Delta \mapsto \Delta^\circ$ is an involution on the set of $\mathbb{Q}$-reflexive polytopes.

We will call the pair of rational polytopes $\Delta$ and $\Delta^\circ$ as the dual pair of $\mathbb{Q}$-reflexive polytopes.

Remark 1.9. A $\mathbb{Q}$-reflexive polytope $\Delta$ is completely determined by the convex hull $[\Delta]$ of its lattice points since $\Delta = [[\Delta]^*]^*$.

Definition 1.10. A $d$-dimensional lattice polytope $\Delta$ in $N_\mathbb{R}$ is called almost reflexive (with respect to $N$) if $0 \in \text{int}(\Delta)$ and
\[
\text{Conv}(\text{Conv}(\Delta^* \cap M))^* \cap N = \Delta.
\]

Lemma 1.11. A polytope $\Delta$ in $M_\mathbb{R}$ is $\mathbb{Q}$-reflexive if and only if $\Delta^\circ$ in $N_\mathbb{R}$ is almost reflexive. In particular, reflexive polytopes are almost reflexive.

Definition 1.12. For a polytope $\Delta$ in $N_\mathbb{R}$ define $\Delta^* := [\Delta] = \text{Conv}(\Delta^* \cap M)$

Remark 1.13. In the new notation, equation (2) is $[[\Delta]^*]^* = \Delta$, or, equivalently, $(\Delta^*)^* = \Delta$.

Lemma 1.14. If $\Delta \subset N_\mathbb{R}$ is almost reflexive, then $\Delta^* := \text{Conv}(\Delta^* \cap M)$ is almost reflexive and the map $\Delta \mapsto \Delta^*$ is an involution on the set of almost reflexive polytopes.

We will call the pair of lattice polytopes $\Delta$ and $\Delta^*$ as the dual pair of almost reflexive polytopes.

A $\mathbb{Q}$-reflexive polytope has the following properties.

Lemma 1.15. Every facet of a $\mathbb{Q}$-reflexive polytope contains a lattice point.

Proof. Suppose that $\Delta$ is $\mathbb{Q}$-reflexive. Since $\Delta^*$ is a lattice polytope, every facet of $\Delta$ is determined by $\{m \in M_\mathbb{R} \mid \langle m, v \rangle = -1\}$ for a vertex $v \in \Delta^*$. If this facet does not contain a lattice point then $\text{Conv}(\Delta \cap M)$ is contained in the half-space $\{m \in M_\mathbb{R} \mid \langle m, v \rangle \geq 0\}$. But then $(\text{Conv}(\Delta \cap M))^*$ is unbounded, contradicting that $\text{Conv}((\text{Conv}(\Delta \cap M))^* \cap N) = \Delta^*$ is a polytope. \qed
Lemma 1.16. If $\Delta$ is a $\mathbb{Q}$-reflexive polytope in $M_{\mathbb{R}}$, then
(a) $\text{int}(\Delta) \cap M = \{0\}$,
(b) $\Delta^* = [\Delta^\circ]$,
(c) $\text{int}(\Delta^*) \cap N = \{0\}$.

Proof. The property (a) follows from the fact that $\Delta^*$ is a lattice polytope, while the property (b) is equation (1) in the new notation. Part (c) holds since $\Delta^\circ$ is $\mathbb{Q}$-reflexive in $N_{\mathbb{R}}$ and applying properties (a) and (b) to $\Delta^\circ$ we get $\text{int}(\Delta^*) \cap N = \text{int}(\Delta^\circ) \cap N = \{0\}$. □

By part (c) of the above lemma, we get

Corollary 1.17. If $\Delta$ is a $\mathbb{Q}$-reflexive polytope, then $\Delta^*$ is a canonical Fano polytope.

2. $\mathbb{Q}$-nef-partitions.

In this section, we generalize the construction of nef-partitions of L. Borisov in [Bo] in the context of $\mathbb{Q}$-reflexive polytopes.

Definition 2.1. [Bo] A nef-partition of a reflexive polytope $\Delta$ is a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ by lattice polytopes such that $0 \in \Delta_i$ for all $i$.

Theorem 2.2. [Bo] Let $\Delta = \Delta_1 + \cdots + \Delta_r$ be a nef-partition. If
$$\nabla_j = \{ y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall x \in \Delta_i, \ i = 1, \ldots, r \}$$
for $j = 1, \ldots, r$, where $\delta_{ij}$ is the Kronecker symbol, then $\nabla = \nabla_1 + \cdots + \nabla_r$ is a nef-partition. Moreover,
$$\Delta_i = \{ x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall y \in \nabla_j, \ j = 1, \ldots, r \}$$
for $i = 1, \ldots, r$.

The nef-partitions $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\nabla = \nabla_1 + \cdots + \nabla_r$ are called a dual pair of nef-partitions.

Remark 2.3. The name nef-partition comes from two words: nef and partition. The nef part comes from the property that each summand $\Delta_i$ in the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ defines a nef (numerically effective) divisor
$$D_{\Delta_i} = \sum_{\rho \in \Sigma_{\Delta}(1)} (-\min(\Delta_i, v_\rho)) D_\rho = \sum_{v_\rho \in \nabla_i} D_\rho$$
on the Gorenstein Fano toric variety $X_\Delta$, where $D_\rho$ are the torus invariant divisors in $X_\Delta$ corresponding to the rays $\rho$ of the normal fan $\Sigma_{\Delta}$ of the polytope $\Delta$, and $v_\rho$ are the primitive lattice generators of $\rho$. The partition part corresponds to the fact that the anticanonical divisor has its support $\bigcup_{\rho \in \Sigma_{\Delta}(1)} D_\rho$ partitioned into the union of supports $\bigcup_{v_\rho \in \nabla_i} D_\rho$ of the nef-divisors $D_{\Delta_i}$.

Remark 2.4. It was an original idea of Yu. I. Manin (see [BvS, Sect. 6.2]) to partition the disjoint union $\bigcup_{\rho \in \Sigma_{\Delta}(1)} D_\rho$ of torus invariant divisors into a union of sets which support the nef-divisors $D_{\Delta_i}$. L. Borisov translated this idea into Minkowski sums and found a canonical way of creating dual nef-partitions.

Now, we introduce a generalization of nef-partition in the context of $\mathbb{Q}$-reflexive polytopes.
Definition 2.5. A $\mathbb{Q}$-nef-partition of a $\mathbb{Q}$-reflexive polytope $\Delta$ is a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ into polytopes in $M_{\mathbb{R}}$ such that $0 \in \Delta_i$ for all $i$, and $\text{Conv}(\Delta \cap M) = \text{Conv}(\Delta_1 \cap M) + \cdots + \text{Conv}(\Delta_r \cap M)$.

A $\mathbb{Q}$-nef-partition has the following property.

Lemma 2.6. Let $\Delta = \Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition, and let $F$ be a facet of $\Delta$ and $F = F_1 + \cdots + F_r$ be the induced decomposition by faces $F_i$ of $\Delta_i$, for $i = 1, \ldots, r$. Then $\text{Conv}(F \cap M) = \text{Conv}(F_1 \cap M) + \cdots + \text{Conv}(F_r \cap M)$.

Proof. Let $F$ be a facet of $\Delta$ with the induced decomposition $F = F_1 + \cdots + F_r$. Then the inclusion $\text{Conv}(F \cap M) \subseteq \text{Conv}(F_1 \cap M) + \cdots + \text{Conv}(F_r \cap M)$ is clear. To show the other inclusion, notice that $\text{Conv}(\Delta \cap M) = \text{Conv}(\Delta_1 \cap M) + \cdots + \text{Conv}(\Delta_r \cap M)$. Let $v$ be the vertex of $\Delta^*$ such that $\langle F, v \rangle = \min \langle \Delta, v \rangle = -1$. We have $\min \langle \Delta_i, v \rangle \leq \min \langle \Delta, v \rangle$ for all $i$ and

$$\min \langle \Delta, v \rangle = \sum_{i=1}^r \min \langle \Delta_i, v \rangle \leq \sum_{i=1}^r \min \langle \Delta_i, v \rangle = \min \langle \Delta, v \rangle. \quad (3)$$

Hence, $\min \langle \Delta_i, v \rangle = \min \langle \Delta, v \rangle$ for all $i$, since $\min \langle \Delta, v \rangle = \min \langle \Delta, v \rangle = -1$ by Lemma 1.15. Since the faces $F_i$ and $G_i$ are determined by the minimal value of $v$ on $\Delta_i$ and $|\Delta_i|$, respectively, we conclude that $G_i \subseteq F_i$, whence $\text{Conv}(F_1 \cap M) + \cdots + \text{Conv}(F_r \cap M)$.

Definition 2.7. For a $\mathbb{Q}$-nef-partition $\Delta_1 + \cdots + \Delta_r$ in $M_{\mathbb{R}}$ define the polytopes

$$\nabla_j = \{ y \in N_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall x \in \text{Conv}(\Delta_i \cap M), i = 1, \ldots, r \} \quad (4)$$

for $j = 1, \ldots, r$.

Proposition 2.8. Let $\Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition in $M_{\mathbb{R}}$, then

$$(\Delta_1 + \cdots + \Delta_r)^* = \text{Conv}(\nabla_1 \cap N, \ldots, \nabla_r \cap N),$$

where $\nabla_1, \ldots, \nabla_r$ are defined by (4).

Proof. Let $v$ be a vertex of $\Delta^*$, where $\Delta = \Delta_1 + \cdots + \Delta_r$ is a $\mathbb{Q}$-nef-partition. Then by (3) and Lemma 1.15, we have $\sum_{i=1}^r \min \langle \Delta_i, v \rangle = -1$, whence the integer $\min \langle \Delta_i, v \rangle = -1$ for some $j$ and $\min \langle \Delta_i, v \rangle = 0$ for $i \neq j$ since $\min \langle \Delta, v \rangle \leq 0$ by $0 \in \Delta_i$ for all $i$. Hence, every vertex $v$ of $\Delta^*$ is contained in some $\nabla_j \cap N$, and $\Delta^* \subseteq \text{Conv}(\nabla_1 \cap N, \ldots, \nabla_r \cap N)$.

To show the opposite inclusion, let $y \in \nabla_j \cap N$ for some $j$. Then $\min \langle \Delta, y \rangle = \sum_{i=1}^r \min \langle \Delta_i, y \rangle \geq \sum_{i=1}^r \delta_{ij} = -1$, whence $y \in [\Delta]^* = \Delta^*$. Since $y$ is a lattice point, we get $y \in [\Delta] = \Delta^*$ by part (b) of Lemma 1.16.

Proposition 2.9. Let $\Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition in $M_{\mathbb{R}}$, then

$$(\nabla_1 + \cdots + \nabla_r)^* = \text{Conv}(\Delta_1 \cap M, \ldots, \Delta_r \cap M),$$

where $\nabla_1, \ldots, \nabla_r$ are defined by (4).

Proof. One inclusion $\text{Conv}(\langle \Delta_1, \ldots, \Delta_r \rangle) \subseteq (\nabla_1 + \cdots + \nabla_r)^*$ holds since $\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$ for all $i, j$ by (4).

The opposite inclusion holds because $[\Delta_i]$ are lattice polytopes with $0 \in [\Delta_i]$ and $0$ is the only interior lattice point in $[\Delta_1] + \cdots + [\Delta_r]$ by Definition 2.5 and part (a) of Lemma 1.16.
**Definition 2.10.** Let $P_1, \ldots, P_r$ be polytopes in $M$. Consider the lattice $\bar{M} = M \oplus \mathbb{Z}^r$, where $\{e_1, \ldots, e_r\}$ is the standard basis of $\mathbb{Z}^r$. The cone

$$C_{P_1, \ldots, P_r} := \mathbb{R}_{\geq 0} \cdot \text{Conv}(P_1 + e_1, \ldots, P_r + e_r)$$

is called the Cayley cone associated to the $r$-tuple of polytopes $P_1, \ldots, P_r$.

**Lemma 2.11.** [M4, Lem. 1.6] Let $P_1, \ldots, P_r$ be polytopes in $M$ such that $P = P_1 + \cdots + P_r$ is $d$-dimensional and $0 \in \text{int}(P)$ Then the dual of the Cayley cone associated to $P_1, \ldots, P_r$ is

$$C^\vee_{P_1, \ldots, P_r} = \mathbb{R}_{\geq 0} \cdot \text{Conv}\left(\left\{x - \sum_{i=1}^r \min(\Delta_i, x)e_i^* \mid x \in \mathcal{V}(P^*)\right\}, \{e_1^*, \ldots, e_r^*\}\right),$$

where $\mathcal{V}(P^*)$ is the set of vertices of $P^*$ and $\{e_1^*, \ldots, e_r^*\}$ is the basis of $\mathbb{Z}^r \subset N \oplus \mathbb{Z}^r$ dual to $\{e_1, \ldots, e_r\}$.

Using this lemma from Propositions 2.8 and 2.9 we get

**Proposition 2.12.** Let $\Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition in $M$, then

$$C_{\Delta_1, \ldots, \Delta_r} = \mathcal{C}_2[\Delta_1, \ldots, \Delta_r], \quad C^\vee_{\Delta_1, \ldots, \Delta_r} = \mathcal{C}^\vee[\Delta_1, \ldots, \Delta_r],$$

where $\nabla_1, \ldots, \nabla_r$ are defined by (4).

Applying a projection technique from [BN] to the Cayley cones in the last proposition we get the following one.

**Proposition 2.13.** Let $\Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition in $M$, then

$$(\text{Conv}(\Delta_1, \ldots, \Delta_r))^* = \text{Conv}(\nabla_1 \cap N) + \cdots + \text{Conv}(\nabla_r \cap N),$$

$$(\text{Conv}(\nabla_1, \ldots, \nabla_r))^* = \text{Conv}(\Delta_1 \cap M) + \cdots + \text{Conv}(\Delta_r \cap M),$$

where $\nabla_1, \ldots, \nabla_r$ are defined by (4).

**Proposition 2.14.** Let $\Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition in $M$, and let $\nabla_j$ be defined by (4). Then $\text{Conv}(\nabla_1, \ldots, \nabla_r)$ is a $\mathbb{Q}$-reflexive polytope and

$$(\Delta_1 + \cdots + \Delta_r)^0 = \text{Conv}(\nabla_1, \ldots, \nabla_r).$$

**Proof.** By Definitions 1.7, 2.5, and Proposition 2.13 we have

$$(\Delta_1 + \cdots + \Delta_r)^0 = [\Delta_1 + \cdots + \Delta_r]^* = ([\Delta_1] + \cdots + [\Delta_r])^* = \text{Conv}(\nabla_1, \ldots, \nabla_r).$$

**Proposition 2.15.** Let $\Delta_1 + \cdots + \Delta_r$ be a $\mathbb{Q}$-nef-partition in $M$, and let $\nabla_1, \ldots, \nabla_r$ be defined by (4). Then $\nabla_1 + \cdots + \nabla_r$ and $\text{Conv}(\Delta_1, \ldots, \Delta_r)$ are $\mathbb{Q}$-reflexive polytopes and

$$(\nabla_1 + \cdots + \nabla_r)^0 = \text{Conv}(\Delta_1, \ldots, \Delta_r).$$

**Proof.** First, we claim

$$\text{Conv}([\Delta_1], \ldots, [\Delta_r]) = [\text{Conv}(\Delta_1, \ldots, \Delta_r)].$$

Indeed, take $x \in \text{Conv}(\Delta_1, \ldots, \Delta_r) \cap M$, then $x \in (\sum_{i=1}^r [\nabla_i])^*$ by Proposition 2.13, and we have

$$-1 \leq \min\left(x, \sum_{i=1}^r [\nabla_i]\right) = \sum_{i=1}^r \min(x, [\nabla_i]) \leq 0,$$
Corollary 2.20. If 

\[ \text{Corollary 2.19.} \]

\[ \square \]

Next, we show that \( \nabla_1 + \cdots + \nabla_r \) is \( \mathbb{Q} \)-reflexive. Using Propositions 2.9, 2.13, we have

\[
\text{Conv}([\Delta_1], \ldots, [\Delta_r]) = (\nabla_1 + \cdots + \nabla_r)^* \subseteq [\nabla_1 + \cdots + \nabla_r]^* \subseteq \langle [\nabla_1] + \cdots + [\nabla_r] \rangle = \text{Conv}(\Delta_1, \ldots, \Delta_r).
\]

Applying (5) to this, we get

\[
[\nabla_1 + \cdots + \nabla_r]^* = \text{Conv}([\Delta_1], \ldots, [\Delta_r]) = (\nabla_1 + \cdots + \nabla_r)^*,
\]

showing that the polytope \( \nabla_1 + \cdots + \nabla_r \) is \( \mathbb{Q} \)-reflexive. Then, by Definition 1.7 and the properties of \( \mathbb{Q} \)-reflexive polytopes, the dual \( \mathbb{Q} \)-reflexive polytope is \( (\nabla_1 + \cdots + \nabla_r)^\circ = \text{Conv}(\Delta_1, \ldots, \Delta_r). \)

Finally, we establish the existence of the dual \( \mathbb{Q} \)-nef-partition:

**Theorem 2.16.** Let \( \Delta_1 + \cdots + \Delta_r \) be a \( \mathbb{Q} \)-nef-partition, then \( \nabla_1 + \cdots + \nabla_r \) is a \( \mathbb{Q} \)-nef-partition, where \( \nabla_1, \ldots, \nabla_r \) are defined by (4). Moreover,

\[ \Delta_i = \{ x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq -\delta_{ij} \forall y \in \text{Conv}(\nabla_j \cap N), j = 1, \ldots, r \}. \]

**Proof.** Proposition 2.15 gives \( \mathbb{Q} \)-reflexivity of \( \nabla_1 + \cdots + \nabla_r \). To show that \( \nabla_1 + \cdots + \nabla_r \) is a \( \mathbb{Q} \)-nef-partition notice

\[
(\text{Conv}(\Delta_1, \ldots, \Delta_r))^* = [\nabla_1] + \cdots + [\nabla_r] \subseteq [\nabla_1 + \cdots + \nabla_r]
\]

by Proposition 2.13. Applying part (b) of Lemma 1.16, we see that \( [\nabla_1 + \cdots + \nabla_r] = (\text{Conv}(\Delta_1, \ldots, \Delta_r))^* \) since \( \text{Conv}(\Delta_1, \ldots, \Delta_r) = (\nabla_1 + \cdots + \nabla_r)^\circ \) by Proposition 2.15. From the above inclusions we get the required equality \( [\nabla_1 + \cdots + \nabla_r] = [\nabla_1] + \cdots + [\nabla_r] \) in the definition of a \( \mathbb{Q} \)-nef-partition. The last part of this theorem follows by Proposition 2.8 since 0 is the only interior lattice point in \( [\nabla_1] + \cdots + [\nabla_r] \).

The Minkowski sums \( \Delta_1 + \cdots + \Delta_r \) and \( \nabla_1 + \cdots + \nabla_r \) in Theorem 2.16 will be called a dual pair of \( \mathbb{Q} \)-nef-partitions.

**Definition 2.17.** A \( \mathbb{Q} \)-nef-partition \( \Delta_1 + \cdots + \Delta_r \) in \( M_{\mathbb{R}} \) is called proper if \( \Delta_i \neq 0 \) for all \( 1 \leq i \leq r \).

**Corollary 2.18.** Let \( \Delta_1 + \cdots + \Delta_r \subset M_{\mathbb{R}} \) and \( \nabla_1 + \cdots + \nabla_r \subset N_{\mathbb{R}} \) be a dual pair of \( \mathbb{Q} \)-nef-partitions. Then, for \( i = 1, \ldots, r \), one has \( \Delta_i = 0 \) if and only if \( \nabla_i = 0 \).

**Proof.** If \( \Delta_i = 0 \), then \( \nabla_i = 0 \) by Definition 2.5, since \( [\Delta_1] + \cdots + [\Delta_r] \) spans \( M_{\mathbb{R}} \). The opposite implication follows from Theorem 2.16.

**Corollary 2.19.** If \( \Delta_1 + \cdots + \Delta_r \subset M_{\mathbb{R}} \) is a proper \( \mathbb{Q} \)-nef-partition, then its dual \( \mathbb{Q} \)-nef-partition \( \nabla_1 + \cdots + \nabla_r \) is proper.

**Corollary 2.20.** If \( \Delta_1 + \cdots + \Delta_r \subset M_{\mathbb{R}} \) is a proper \( \mathbb{Q} \)-nef-partition, then \( \Delta_i \cap M \neq \{0\} \) for all \( 1 \leq i \leq r \).
3. Almost reflexive Gorenstein cones.

In this section, we generalize the notion of reflexive Gorenstein cones.

Definition 3.1. [BB01] Let $M$ and $\bar{N}$ be a pair of dual lattices of rank $\bar{d}$. A $\bar{d}$-dimensional polyhedral cone $\sigma$ with a vertex at $0 \in M$ is called Gorenstein, if it is generated by finitely many lattice points contained in the affine hyperplane $\{ x \in M \mid \langle x, h_\sigma \rangle = 1 \}$ for $h_\sigma \in \bar{N}$. The unique lattice point $h_\sigma$ is called the height (or degree) vector of the Gorenstein cone $\sigma$. A Gorenstein cone $\sigma$ is called reflexive if both $\sigma$ and its dual $\sigma^\vee = \{ y \in \bar{N}_\mathbb{R} \mid \langle x, y \rangle \geq 0 \forall x \in \sigma \}$ are Gorenstein cones. In this case, they both have uniquely determined $h_\sigma \in \bar{N}$ and $h_{\sigma^\vee} \in \bar{M}$, which take value 1 at the primitive lattice generators of the respective cones. The positive integer $r = \langle h_{\sigma^\vee}, h_\sigma \rangle$ is called the index of the reflexive Gorenstein cones $\sigma$ and $\sigma^\vee$.

As in [BN], denote $\sigma(i) := \{ x \in \sigma \mid \langle x, h_\sigma \rangle = i \}$, for $i \in \mathbb{N}$. The basic relationship between reflexive polytopes and reflexive Gorenstein cones is provided by the following:

Proposition 3.2. [BB01, Pr. 2.11] Let $\sigma$ be a Gorenstein cone. Then $\sigma$ is a reflexive Gorenstein cone of index $r$ if and only if the polytope $\sigma(r) - h_{\sigma^\vee}$ is a reflexive polytope with respect to the lattice $\bar{M} \cap h_{\sigma^\vee}^+ = \{ x \in M \mid \langle x, h_\sigma \rangle = 0 \}$.

Generalizing the notion of reflexive Gorenstein cones we introduce:

Definition 3.3. A Gorenstein cone $\sigma$ in $\bar{M}_\mathbb{R}$ is called almost reflexive, if there is $r \in \mathbb{N}$ such that $\sigma(r)$ has a unique lattice point $h$ in its relative interior and $\sigma(r) - h$ is an almost reflexive polytope with respect to the lattice $\bar{M} \cap h_{\sigma^\vee}^+$. We will denote $h$ by $h_{\sigma^\vee}$. The positive integer $r$ will be called the index of the almost reflexive Gorenstein cone $\sigma$.

Lemma 3.4. Reflexive Gorenstein cones are almost reflexive.

Proof. This follows from Proposition 3.2 and Lemma 1.11. $\square$

Definition 3.5. For an almost reflexive Gorenstein cone $\sigma$ in $\bar{M}_\mathbb{R}$ define $\sigma^\vee(i) = \{ y \in \sigma^\vee \mid \langle h_{\sigma^\vee}, y \rangle = i \}$, for $i \in \mathbb{N}$.

Denote $[\sigma^\vee] := \mathbb{R}_{\geq 0}[\sigma^\vee(i)] = \mathbb{R}_{\geq 0}\text{Conv}(\sigma^\vee(i) \cap \bar{M})$

Lemma 3.6. Let $\Delta$ be a polytope in $M_\mathbb{R}$ with $0 \in \text{int}(\Delta)$, and let $\sigma_\Delta = \mathbb{R}_{\geq 0}(\Delta, 1) \subset \bar{M}_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}$. Then $\sigma^\vee = \sigma_{\Delta^\vee} = \mathbb{R}_{\geq 0}(\Delta^\vee, 1) \subset \bar{N}_\mathbb{R} = N_\mathbb{R} \oplus \mathbb{R}$.

Corollary 3.7. A Gorenstein cone $\sigma$ in $\bar{M}_\mathbb{R}$ is almost reflexive of index 1 if and only if the polytope $\sigma^\vee(i) - h_\sigma$ is $\mathbb{Q}$-reflexive with respect to the lattice $\bar{N} \cap h_{\sigma^\vee}^+$. $\square$

Proof. Combine Lemmas 1.11 and 3.6.

Corollary 3.8. If $\sigma$ in $\bar{M}_\mathbb{R}$ is an almost reflexive Gorenstein cone of index 1, then $[\sigma^\vee]$ is an almost reflexive Gorenstein cone of index 1.

Proposition 3.9. If $\sigma$ in $\bar{M}_\mathbb{R}$ is an almost reflexive Gorenstein cone of index $r$, then $[\sigma^\vee]$ is an almost reflexive Gorenstein cone of index $r$. $\square$
Proof. Use the techniques in the proof of [BBo1, Pr. 2.11]. \qed

Almost reflexive Gorenstein cones have the following property.

Lemma 3.10. Let \( \sigma \subset \hat{M}_\mathbb{R} \) be an almost reflexive Gorenstein cone. Then \([\sigma^\vee] = \sigma\).

Definition 3.11. For an almost reflexive Gorenstein cone \( \sigma \), denote \( \sigma^\bullet := [\sigma^\vee] \).

Corollary 3.12. The map \( \sigma \mapsto \sigma^\bullet \) is an involution on the set of almost reflexive Gorenstein cones: \((\sigma^\bullet)^\bullet = \sigma\).

Cayley cones corresponding to a dual pair of \( \mathbb{Q} \)-nef-partitions are related to almost reflexive Gorenstein cones as follows:

Proposition 3.13. Let \( \Delta_1 + \cdots + \Delta_n \subset M_\mathbb{R} \) and \( \nabla_1 + \cdots + \nabla_r \subset N_\mathbb{R} \) be a dual pair of \( \mathbb{Q} \)-nef-partitions. Then the Cayley cones \( C_{[\Delta_1],\ldots,[\Delta_n]} \) and \( C_{[\nabla_1],\ldots,[\nabla_r]} \) is a dual pair of almost reflexive Gorenstein cones:

\[
C_{[\Delta_1],\ldots,[\Delta_n]}^\bullet = [C_{[\Delta_1],\ldots,[\Delta_n]}] = C_{[\nabla_1],\ldots,[\nabla_r]}.
\]

Proof. This follows directly from Proposition 2.12 since the height vectors of the Cayley cones \( C_{[\Delta_1],\ldots,[\Delta_n]} \) and \( C_{[\nabla_1],\ldots,[\nabla_r]} \) are \( e_1^* + \cdots + e_n^* \) and \( e_1 + \cdots + e_r \), respectively. \qed

4. Basic toric geometry.

This section will review some basics of toric geometry.

Let \( X_\Sigma \) be a \( d \)-dimensional toric variety associated with a finite rational polyhedral fan \( \Sigma \) in \( N_\mathbb{R} \). Denote by \( \Sigma(1) \) the finite set of the 1-dimensional cones \( \rho \) in \( \Sigma \), which correspond to the torus invariant divisors \( D_\rho \) in \( X_\Sigma \). By \([C] \), every toric variety can be described as a categorical quotient of a Zariski open subset of an affine space by a subgroup of a torus. Consider the polynomial ring \( S(\Sigma) := \mathbb{C}[x_\rho : \rho \in \Sigma(1)] \), called the homogeneous coordinate ring of the toric variety \( X_\Sigma \), and the corresponding affine space \( \mathbb{C}^{\Sigma(1)} = \text{Spec}(\mathbb{C}[x_\rho : \rho \in \Sigma(1)]) \). The ideal \( B = (\prod_{\rho \in \Sigma} x_\rho : \sigma \in \Sigma) \) in \( S \) is called the irrelevant ideal. This ideal determines a Zariski closed set \( V(B) \) in \( \mathbb{C}^{\Sigma(1)} \), which is invariant under the diagonal group action of the subgroup

\[
G = \left\{ (\mu_\rho) \in (\mathbb{C}^\times)^{\Sigma(1)} \mid \prod_{\rho \in \Sigma(1)} \mu_\rho^{u,v_\rho} = 1 \forall u \in M \right\}
\]

of the torus \((\mathbb{C}^\times)^{\Sigma(1)}\) on the affine space \( \mathbb{C}^{\Sigma(1)} \), where \( v_\rho \) denotes the primitive lattice generator of the 1-dimensional cone \( \rho \). The toric variety \( X_\Sigma \) is isomorphic to the categorical quotient \((\mathbb{C}^{\Sigma(1)} \setminus V(B))/G\), which is induced by a toric morphism \( \pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma \), constant on \( G(\Sigma) \)-orbits (see [CLS, Thm. 5.1.10]).

The coordinate ring \( S(\Sigma) \) is graded by the the Chow group

\[
A_{d-1}(X_\Sigma) \simeq \text{Hom}(G, \mathbb{C}^\times),
\]

and \( \deg(\prod_{\rho \in \Sigma(1)} x_\rho^{b_\rho}) = [\sum_{\rho \in \Sigma(1)} b_\rho D_\rho] \in A_{d-1}(X_\Sigma) \). For a torus invariant Weil divisor \( D = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho \), there is a one-to-one correspondence between the monomials of \( \mathbb{C}[x_\rho : \rho \in \Sigma(1)] \) in the degree \([\sum_{\rho \in \Sigma(1)} b_\rho D_\rho] \in A_{d-1}(X_\Sigma) \) and the lattice points inside the polytope

\[
\Delta_D = \{ m \in M_\mathbb{R} \mid \langle m, v_\rho \rangle \geq -b_\rho \forall \rho \in \Sigma(1) \}
\]
by associating to \( m \in \Delta_D \) the monomial \( \prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho}+(m,v_{\rho})} \). If we denote the homogeneous degree of \( S(\Sigma) \) corresponding to \( \beta = [D] \in A_{d-1}(X) \) by \( S(\Sigma)_{\beta} \), then by [C, Prop. 1.1], we also have a natural isomorphism
\[
H^0(X, \mathcal{O}_{X}(D)) \simeq S(\Sigma)_{\beta}.
\]
In particular, every hypersurface in \( X_{\Sigma} \) of degree \( \beta = \sum_{\rho \in \Sigma(1)} b_{\rho} D_{\rho} \) corresponds to a polynomial
\[
\sum_{m \in \Delta_D \cap \mathbb{N}} a_m \prod_{\rho \in \Sigma(1)} x_{\rho}^{b_{\rho}+(m,v_{\rho})}
\]
with the coefficients \( a_m \in \mathbb{C} \). By [CLS, Prop. 5.2.8], all closed subvarieties of \( X_{\Sigma} \) correspond to homogeneous ideals \( I \subseteq S(\Sigma) \), and [M3, Thm. 1.2] shows that a closed subvariety in a toric variety \( X_{\Sigma} \) can be viewed as a categorical quotient as well. A complete intersection in the toric variety \( X_{\Sigma} \) (in homogeneous coordinates) is a closed subvariety \( V(I) \subset X_{\Sigma} \) corresponding to a radical homogeneous ideal \( I \subseteq S(\Sigma) \) generated by a regular sequence of homogeneous polynomials \( f_1, \ldots, f_k \in S(\Sigma) \) such that \( k = \dim X_{\Sigma} - \dim V(I) \) (see [M3, Sect. 1]).

Every rational polytope \( \Delta \) in \( M_{\mathbb{R}} \) determines the Weil \( \mathbb{Q} \)-divisor
\[
D_{\Delta} = \sum_{\rho \in \Sigma(1)} (- \min(\Delta, v_{\rho})) D_{\rho} \in W\text{Div}(X_{\Sigma}) \otimes_{\mathbb{Z}} \mathbb{Q}
\]
on \( X_{\Sigma} \).

**Definition 4.1.** Let \( X \) be a complete variety. A \( \mathbb{Q} \)-Cartier divisor \( D \in \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \) on \( X \) is called nef (numerically effective) if \( D \cdot C \geq 0 \) for all irreducible curves \( C \subset X \). We will call such divisors \( \mathbb{Q} \)-nef.

**Lemma 4.2.** Let \( X_{\Sigma} \) be a compact toric variety. Then the divisor \( D_{\Delta} \) is \( \mathbb{Q} \)-nef if and only if its support function \( \psi_{\Delta} = - \min(\Delta, \underline{\cdot}) \) is convex piecewise linear with respect to the fan \( \Sigma \).

**Proof.** By [F, p. 68], we know that \( \mathcal{O}_{X_{\Sigma}}(nD_{\Delta}) \) is generated by global sections for some sufficiently large \( n \in \mathbb{N} \) if and only if \( \psi_{\Delta} \) is convex piecewise linear on \( \Sigma \). On the other hand, we showed in [M1, Thm. 1.6] that, for a compact toric variety \( X_{\Sigma} \), the invertible sheaf \( \mathcal{O}_{X_{\Sigma}}(D) \) is generated by global sections if and only if \( D \) is nef. \( \square \)

**Lemma 4.3.** [M3, Lem 2.1] Let \( X_{\Sigma} \) be a compact toric variety associated to a fan \( \Sigma \) in \( N_{\mathbb{R}} \). Suppose \( \Delta_1 \) and \( \Delta_2 \) are rational polytopes in \( M_{\mathbb{R}} \) then \( D_{\Delta_1 + \Delta_2} \) is a \( \mathbb{Q} \)-nef divisor on \( X_{\Sigma} \) iff \( D_{\Delta_1} \) and \( D_{\Delta_2} \) are \( \mathbb{Q} \)-nef on \( X_{\Sigma} \).

Every rational polytope \( \Delta \) in \( M_{\mathbb{R}} \) corresponds to a projective toric variety \( X_{\Delta} := X_{\Sigma_\Delta} \), whose fan \( \Sigma_\Delta \) (called the normal fan of \( \Delta \)) is the collection of cones
\[
\sigma_F = \{ y \in N_{\mathbb{R}} \mid \langle x, y \rangle \leq \min(\Delta, y) \forall x \in F \}.
\]
The support function \( \psi_{\Delta} = - \min(\Delta, \underline{\cdot}) \) is strictly convex piecewise linear with respect to the fan \( \Sigma_\Delta \). In this case, the divisor \( D_{\Delta} \) is ample, and, in particular, \( \mathbb{Q} \)-nef. From Lemma 4.3, we get

**Corollary 4.4.** Let \( X_{\Delta} \) be a Fano toric variety, and suppose \( \Delta = \Delta_1 + \cdots + \Delta_r \) is a Minkowski sum decomposition by rational polytopes. Then the divisors \( D_{\Delta_i} \) are \( \mathbb{Q} \)-nef on \( X_{\Delta} \) for all \( 1 \leq i \leq r \).
There is an alternative way to describe projective toric varieties using the Proj
functor, which is simple but less useful in the context of complete intersections.
Consider the cone
\[ K = \{(t\Delta, t) \mid t \in \mathbb{R}_{\geq 0}\} \subset M_\mathbb{R} \oplus \mathbb{R}. \]
The projective toric variety \( X_\Delta \) can be represented as \( \text{Proj}(\mathbb{C}[K \cap (M \oplus \mathbb{Z})]) \).
Moreover, if \( \beta \in A_{d-1}(X_\Delta) \) is the class of the ample divisor \( D_\Delta = \sum_{\rho \in \Sigma_\Delta(1)} b_\rho D_\rho \),
then there is a natural isomorphism of graded rings
\[ \mathbb{C}[K \cap (M \oplus \mathbb{Z})] \cong \bigoplus_{i=0}^{\infty} S(\Sigma_\Delta)_{i\beta}, \]
送 \( \chi^{(m,i)} \in \mathbb{C}[K \cap (M \oplus \mathbb{Z})]_{\beta} \) to \( \prod_{\rho \in \Sigma_\Delta(1)} x_\rho^{ib_\rho + (m,v_\rho)} \). In particular, a hypersurface given by a polynomial in homogeneous coordinates
\[ \sum_{m \in \Delta \cap M} a_m \prod_{\rho \in \Sigma_\Delta(1)} x_\rho^{ib_\rho + (m,v_\rho)} = 0 \]
corresponds to \( \sum_{m \in \Delta \cap M} a_m \chi^{(m,i)} = 0 \).

5. Mirror Symmetry Construction.

In this section, we propose a generalization of the Batyrev-Borisov Mirror Symmetry constructions.

A proper \( \mathbb{Q} \)-nef-partition \( \Delta = \Delta_1 + \cdots + \Delta_r \) with \( r < d \) defines a \( \mathbb{Q} \)-nef Calabi-Yau complete intersection \( Y_{\Delta_1, \ldots, \Delta_r} \) in the Fano toric variety \( X_\Delta \) given by the equations:
\[ \left( \sum_{m \in \Delta_i \cap M} a_{i,m} \prod_{\rho \in \Sigma_\Delta(1)} x_\rho^{(m,v_\rho)} \right) \prod_{\rho \in \Sigma_\Delta(1)} x_\rho = 0, \quad i = 1, \ldots, r, \]
where \( x_\rho \) are the homogeneous coordinates of the toric variety \( X_\Delta \) corresponding to the vertices \( v_\rho \) of the polytope \( \Delta^* \).

Following the Batyrev-Borisov Mirror Symmetry construction, we naturally expect that Calabi-Yau complete intersections corresponding to a dual pair of \( \mathbb{Q} \)-nef-partitions pass the topological mirror symmetry test:

**Conjecture 5.1.** Let \( Y_{\Delta_1, \ldots, \Delta_r} \subset X_\Delta \) and \( Y_{\nabla_1, \ldots, \nabla_r} \subset X_\nabla \) be a pair of generic Calabi-Yau complete intersections in \( d \)-dimensional Fano toric varieties corresponding to a dual pair of \( \mathbb{Q} \)-nef-partitions \( \Delta = \Delta_1 + \cdots + \Delta_r \) and \( \nabla = \nabla_1 + \cdots + \nabla_r \). Then
\[ h_{st}^{p,q}(Y_{\Delta_1, \ldots, \Delta_r}) = h_{st}^{d-r-p,q}(Y_{\nabla_1, \ldots, \nabla_r}), \quad 0 \leq p, q \leq d - r. \]

Assuming that this conjecture holds, by taking maximal projective crepant partial resolutions we obtain the mirror pair of minimal Calabi-Yau complete intersections \( \tilde{Y}_{\Delta_1, \ldots, \Delta_k} \), \( \tilde{Y}_{\nabla_1, \ldots, \nabla_k} \).

For an almost reflexive Gorenstein cone \( \sigma \) in \( \tilde{M}_\mathbb{R} \), we have the Fano toric variety \( X_\sigma = \text{Proj}(\mathbb{C}[\sigma^* \cap \tilde{N}]) \), whose fan consists of cones generated by the faces of the almost reflexive polytope \( \sigma (\gamma) - h_{\sigma^*} \) in \( \tilde{M}_\mathbb{R} \cap h_{\sigma}^\perp \). A generalized Calabi-Yau manifold is defined as the ample \( \mathbb{Q} \)-Cartier hypersurface \( Y_\sigma \subset X_\sigma \) given by the equation
\[ \sum_{n \in \sigma^* \cap \tilde{N}} a_n \chi^n = 0 \]
with generic $a_n \in \mathbb{C}$, where $\chi^n$ are the elements in the graded semigroup ring $\mathbb{C}[\sigma^\vee \cap \bar{N}]$ corresponding to the lattice points $n \in \sigma^\vee \cap \bar{N}$.

**Conjecture 5.2.** The involution $\sigma \mapsto \sigma^*$ on the set of almost reflexive Gorenstein cones corresponds to the mirror involution of $N = 2$ super conformal field theories associated to the generalized Calabi-Yau manifolds $Y_\sigma$ and $Y_{\sigma^*}$.

In the case, when a $\mathbb{Q}$-nef Calabi-Yau complete intersection $Y_{\Delta_1, \ldots, \Delta_r}$ does not have the property that $0 \in \Delta_i$ for all $1 \leq i \leq r$ (i.e., the Minkowski sum $\Delta_1 + \cdots + \Delta_r$ is not a $\mathbb{Q}$-nef-partition), one can still associate to it the mirror in the form of the generalized Calabi-Yau manifold corresponding to the dual of the Cayley cone $C_{\Delta_1, \ldots, \Delta_r}$.

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