AN ALPERN TOWER INDEPENDENT OF A GIVEN PARTITION

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Abstract. Given a measure-preserving transformation $T$ of a probability space $(X, B, \mu)$ and a finite measurable partition $P$ of $X$, we show how to construct an Alpern tower of any height whose base is independent of the partition $P$. That is, given $N \in \mathbb{N}$, there exists a Rohlin tower of height $N$, with base $B$ and error set $E$, so that $B$ is independent of $P$, and $T(E) \subset B$.

2010 Mathematics Subject Classification: 28D05, 37M25, 60A10

Keywords: Rohlin tower, Alpern tower, independent sets, measure-preserving transformation, probability space.

1. Introduction and Statement of Results

It has long been known that, given an ergodic invertible probability measure preserving system, a Rohlin tower may be constructed with base independent of a given partition of the underlying space ([Roh52], [Roh65]). In [Alp79], meanwhile, S. Alpern proved a 'multiple' Rohlin tower theorem (see [EP97] for an easy proof) whose full statement we will not give, but which has the following corollary of interest:

**Theorem 1.1.** Let $N \in \mathbb{N}$ and $\epsilon > 0$ be given. For any ergodic invertible measure-preserving transformation $T$ of a Lebesgue probability space $(X, \mathcal{B}, \mu)$, there exists a Rohlin tower of height $N$ with base $B$ and error set $E$ with $\mu(E) < \epsilon$, so that $T(E) \subset B$.

A Rohlin tower of height $N$ with base $B$ and error set $E$ is characterized by the collection of sets $\{B, TB, \ldots, T^{N-1}B, E\}$ forming a partition of $X$. If in addition $T(E) \subset B$, we shall say Alpern Tower. It is our goal to show that for ergodic transformations on $(X, \mathcal{B}, \mu)$, given a finite measurable partition $P$ of $X$, an Alpern tower may be constructed with base $B$ independent of $P$. Precisely:

**Theorem 1.** Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, and suppose $P$ is a finite measurable partition of $X$. For any ergodic invertible measure-preserving transformation $T$ of $X$, $N \in \mathbb{N}$, there exists a Rohlin tower of height $N$ with base $B$ and error set $E$ such that $T(E) \subset B$ and $B$ is independent of $P$.

We do not specify the size of the error set; but the process of constructing our tower makes it clear that the error set may be made arbitrarily small.

2. Proof of main result

For the remainder of the paper, $(X, \mathcal{B}, \mu)$ will be a fixed Lebesgue probability space and $T : X \to X$ will be an invertible ergodic measure-preserving transformation on
X. All mentioned sets will be measurable and we will adopt a cavalier attitude toward null sets. In particular, “partition” will typically mean “measurable partition modulo null sets”.

**Definition 2.1.** By a tower over $B$ we will mean a set $B \subset X$, called the base, and a countable partition $B = B_1 \cup B_2 \cup \cdots$, together with their images $T^i B_j$, $0 \leq i < j$, such that the family $\{T^i B_j : 0 \leq i < j\}$ consists in pairwise disjoint sets. If this family partitions $X$, we will say that the tower is exhaustive.

If a tower over $B$ is exhaustive and $B = B_N \cup B_{N+1}$, we shall speak of an exhaustive Alpern tower of height $\{N, N+1\}$, as in such a case, $\{B, TB, \ldots, T^{N-1}B, E = T^N B_{N+1}\}$ partitions $X$ with $T(E) \subset B$. So we may re-phrase Theorem 1 as:

**Theorem 1.** Let $(X, B, \mu)$ be a Lebesgue probability space and suppose $\mathbb{P}$ is a finite measurable partition of $X$. For any ergodic invertible measure-preserving transformation $T$ of $X$, $N \in \mathbb{N}$, one may find an exhaustive Alpern tower of height $\{N, N+1\}$ having base independent of $\mathbb{P}$.

We require a lemma (and a corollary).

**Lemma 2.2.** Let $M \in \mathbb{N}$ and let $\mathbb{P} = \{P_1, \ldots, P_t\}$ be a partition of $X$ with $\mu(P_i) > 0$ for each $i$. There exists a set $S$ of positive measure so that if $x \in S$ with first return $n(x) = n$, say, then $|\{x, Tx, \ldots, T^{n-1}x\} \cap P_i| \geq M, 1 \leq i \leq t$.

**Proof.** For almost every $x$ we may find $K(x)$ so that for each $i$ between 1 and $t$ we have $|\{x, Tx, \ldots, T^{K(x)-1}x\} \cap P_i| \geq M$. Since almost all of $X$ is the countable union (over $k \in \mathbb{N}$) of $\{x : K(x) = k\}$, there exists some fixed $K$ so that the set $A = \{x : K(x) \leq K\}$ has positive measure. If $C \subset A$ has very small measure $\mu(C) < 1/K$ then the average first-return time of $x \in C$ to $C$ is $\frac{1}{\mu(C)} > K$, so we can find $S \subset C$ with $\mu(S) > 0$ so that $S, TS, \ldots, T^{K-1}S$ are pairwise disjoint.

**Corollary 2.3.** Let $M \in \mathbb{N}$ and $\mathbb{P} = \{P_1, \ldots, P_t\}$ be a partition of $X$ with $\mu(P_i) > 0$ for each $i$. There is a tower having base $S = S_1M \cup S_2M+1 \cup \cdots$ where for each $x \in S_i$, $|\{x, Tx, \ldots, T^{r-1}x\} \cap P_i| \geq M$ for all $1 \leq i \leq t$.

**Proof.** Let $S, K$ be as in Lemma 2.2 and choose any $k \geq K$.

We turn now to the proof of Theorem 1. Fix a partition $\mathbb{P} = \{P_1, \ldots, P_t\}$, an arbitrary natural number $N$, and $\epsilon > 0$. Set $m_i = \mu(P_i)$, and assume (without loss of generality) that $0 < m_1 \leq m_2 \leq \cdots \leq m_t$. Select and fix $M > \frac{2N+1}{m_1}$. Let $S$ be as in Corollary 2.3 for this $M$; hence $S = S_1M \cup S_2M+1 \cup \cdots$ (Some $S_i$ may be empty, of course.) For each non-empty $S_R$, partition $S_R$ by $\mathbb{P}$-name of length $R$. (Recall that $x, y$ in $S_R$ have the same $\mathbb{P}$-name of length $R$ if $T^ix$ and $T^iy$ lie in the same cell of $\mathbb{P}$ for $0 \leq i < R$.) Let $C$ be the base of one of the resulting columns; hence, every $x \in C$ has the same $\mathbb{P}$-name of length $R$ (for some $R \geq tM$), and the length $R$ orbit of each $x \in C$ meets each $P_i$ at least $M$ times.

Partition $C$ into pieces $C^{(1)}, C^{(2)}, \ldots, C^{(t)}$ whose measures will be determined later. Then partition each $C^{(i)}$ into $N$ equal measure pieces, $C^{(i)} = C^{(i)}_1 \cup C^{(i)}_2 \cup \cdots \cup C^{(i)}_N$.

Now we fix $(R, C)$ and focus our attention on the height $R$ column over a single $C^{(i)}$ and its height $R$ subcolumns over $C^{(i)}_j$, $1 \leq j \leq N$. We refer to the sets $T^rC^{(i)}_j$, $0 \leq r < R$, as levels and to the sets $T^rC^{(i)}_j$ as rungs. We are going to build a portion
of $B$ by carefully selecting some rungs from the subcolumns under consideration. As we move through the various subcolumns, we need to have gaps of length $N$ or $N+1$ between selections. Now to specifics. We want to have our $C^{(i)}$-selections form a “staircase” of height $N$ starting at level $N^2 - N$. That is, at height $(N-1)N$, the rung over $C_1^{(i)}$ is the only one selected; at height $N(N-1) + 1$, the rung over $C_2^{(i)}$ is the only one selected; etc., so that at height $N^2 - 1$, the rung over $C_N^{(i)}$ is the only one selected.

This is easy to accomplish. First, we select each base rung $C_j^{(i)}$, $j = 1, 2, \ldots, N$ (i.e., the rungs in the zeroth level). Over $C_1^{(i)}$, we then select $N - 1$ additional rungs with gaps of length $N$; that is, we select the rungs at heights $N$, $2N$, $\ldots$, $(N-1)N$. Over $C_2^{(i)}$ we select $N - 2$ rungs with gap $N$, then a rung with gap $N+1$. We continue in this fashion, choosing one less gap of length $N$ and one more of length $N+1$ in each subsequent subcolumn. In the last subcolumn (that over $C_N^{(i)}$) we are thus choosing rungs with gaps of length $N+1$ a total of $N-1$ times. See the left side of Figure 1 for the case $N = 4$.

Now we perform a similar procedure moving down from the top, so as to obtain a staircase starting at height $R - (N^2 - 1)$. Note that there are either $N$ or $N-1$ unselected rungs at the top of each subcolumn. See the right side of Figure 1.

**Figure 1. Bottom, Top of Tower for $N = 4$**
following rung. As we want to match stride with the staircase already selected at the top, the total number of levels skipped in the middle section will be constrained to a certain residue class modulo \( N \), and as we want the selected rungs to form a portion of an Alpern tower of height \( \{ N, N + 1 \} \), we cannot skip any two levels with fewer than \( N \) levels between them.

Some terminology: an appearance of \( P_j \) in \( C^{(i)} \) is just a level of \( C^{(i)} \) that is contained in \( P_j \). A selection of \( P_j \) is just a selected rung in a subcolumn of \( C^{(i)} \) that is contained in \( P_j \). The net skips of \( P_j \) in the tower over \( C^{(i)} \) is defined as

\[
S_j(C^{(i)}) = (\# \text{ of appearances of } P_j) - (\# \text{ of selections of } P_j).
\]

For example, looking at Figure 1, one sees that 4 zeroth level rungs are selected. So if the zeroth level belongs to \( P_j \), the zeroth level contribution to \( S_j(C^{(i)}) \) is \(-3\) (one appearance and 4 selections).

Let \( \delta = 2(N - 1)(N - 2) \) and choose \( \gamma \) with

\[
\frac{\delta}{m_1} + N > \gamma \geq \frac{\delta}{m_1} \quad \text{and} \quad (t - 1)\delta + \gamma \equiv R \pmod{N}.
\]

Over \( C^{(i)} \), we skip a quantity of “middle” levels belonging to each \( P_j \) (for \( j \neq i \)) sufficient to ensure that \( S_j(C^{(i)}) = \delta \) for \( j \neq i \) and \( S_i(C^{(i)}) = \gamma \). (Note that \( P_j \) cannot have been skipped more than \( \delta \) times in the outer rungs.) This is not delicate; one can just enact the selection greedily. That is to say, travel up the tower, beginning at level \( N^2 \), skipping rungs that belong to cells requiring additional skips whenever there’s been no too-recent skip. Since each \( P_j \) appears at least \( M > \frac{3N^3}{m_1} \) times, and we need only \( \gamma + (t - 1)\delta \leq \frac{2N^2}{m_1} \) net skips, we’ll find all the skips we need.

We have not specified the relative masses of the bases of the columns \( C^{(i)} \). Set

\[
b_j = \frac{\mu(P_j)(\gamma + (t - 1)\delta) - \delta}{\gamma - \delta}
\]

and put \( \mu(C^{(i)}) = b_i\mu(C) \), \( 1 \leq i \leq t \). Our choice of \( \gamma \) ensures that \( b_i \geq 0 \) for each \( i \), and one easily checks that \( \sum b_i = 1 \), so this is coherent.

Let \( B_C \) be the union of the rungs selected from the columns over \( C \) (this includes each of the rungs selected from each of the \( N \) subcolumns over \( C^{(i)} \), \( 1 \leq i \leq t \)) and put \( B = \bigcup_C B_C \). (Here \( C \) runs over the bases of the columns corresponding to every \( \mathcal{P} \)-name of length \( R \) for every \( R \geq tM \).) It is clear that \( B \) forms the base of an Alpern tower of height \( \{ N, N + 1 \} \). It remains to show that \( B \) is independent of \( \mathcal{P} \), which we will do by constructing a set \( A \), disjoint from \( B \), such that both \( A \) and \( A \cup B \) can be shown to be independent of \( \mathcal{P} \).

Here is how \( A \) is constructed. Consider again the tower over \( C^{(i)} \). This tower had \( R \) levels and \( RN \) rungs, some of which were selected for the base \( B \). We now choose \( \gamma + (t - 1)\delta \) additional rungs for the set \( A \). For each \( j \neq i \), \( \delta \) of these rungs should be contained in \( P_j \), with the remaining \( \gamma \) contained in \( P_i \). (We don’t worry about gaps and whatnot; just choose any such collection of rungs disjoint from the family of \( B \) selections.) Denote the union of the these additional rungs (in all of the columns over \( C^{(i)} \), \( 1 \leq i \leq t \)) by \( A_C \). Finally, put \( A = \bigcup_C A_C \).
That $A \cup B$ is independent of $\mathbb{P}$ is a consequence of the fact that for each $C^{(i)}$, the number appearances of $P_j$ in the column over $C^{(i)}$ is precisely the number of $B$-selections from $P_j$ plus the number of $A$-selections from $P_j$. Accordingly, the relative masses of the cells of $\mathbb{P}$ restricted to $A \cup B$ are equal to the relative frequencies of the appearances of the cells of $\mathbb{P}$ in the column over $C^{(i)}$. Therefore, since the proportion of the column that is selected for $A \cup B$ is independent of $C^{(i)}$ (in fact is always equal to $\frac{1}{N}$), and since the columns over the various $C^{(i)}$ exhaust $X$, $A \cup B$ is independent of $\mathbb{P}$ (in fact $\mu(P_j \cap (A \cup B)) = \frac{1}{N}\mu(P_j)$, $1 \leq j \leq t$).

That $A$ is independent of $\mathbb{P}$, meanwhile, is a consequence of equation (2.1). Fixing $C$ and recalling that $b_i = \frac{\mu(C^{(i)})}{\mu(C)}$, that there were $\delta P_j$-rungs in the column over $C^{(i)}$ selected for $A$, $i \neq j$, and that there were $\gamma P_j$-rungs in the column over $C^{(i)}$ selected for $A$, the relative mass of $P_i$ among the $A$-selections in the tower over $C$ is

$$r_i = \frac{b_i \gamma + (1 - b_i) \delta}{\gamma + (t - 1) \delta}.$$ 

But, solving for $\mu(P_i)$ in equation (2.1), one gets that

$$\mu(P_i) = \frac{b_i \gamma + (1 - b_i) \delta}{\gamma + (t - 1) \delta}$$

as well. So the intersection of $A$ with the column over $C$ is independent of $\mathbb{P}$. That this is true for every $C$ gives independence of $A$ from $\mathbb{P}$ simpliciter. ■

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