A New Generalized Gamma Function and Its Properties

Inci Ege

Department of Mathematics, University of Aydın Adnan Menderes, Aydın, Türkiye.

Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2022/v37i430443

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdarticle5.com/review-history/87037

Received: 10 March 2022
Accepted: 13 May 2022
Published: 16 May 2022

Original Research Article

Abstract

In this work, we introduce a new generalized Gamma function, which is named as $p$-$v$-Gamma function and provide some properties generalizing those satisfied by the classical Gamma function. We also give some convexity and monotonicity properties. Furthermore, we establish some inequalities related to this new function.

Keywords: Gamma function; polygamma function; $p$-analogue; convexity; monotonicity; inequality.

2010 Mathematics Subject Classification: 33B15; 33E50; 26A51; 26D07.

1 Introduction

The classical Euler’s Gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \lim_{p \to \infty} \frac{p^px}{x(x+1)\ldots(x+p)}.$$ 

This function plays central roles in the theory of special functions and have lots of generalizations. There seems to be so much study of literature. Although it is not possible to list all of these papers, we can give some of them to the readers, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] and the references therein.
Euler, gave an equivalent definition for the function $\Gamma$ (known as $p$-analogue of the Gamma function) as:

$$
\Gamma_p(x) = \frac{p!p^x}{x(x+1)\ldots(x+p)}, \quad x > 0
$$

where $p$ is a positive integer and

$$
\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x),
$$

[13]. It satisfies the identities:

$$
\Gamma_p(x + 1) = \frac{px}{x + p + 1} \Gamma_p(x),
\Gamma_p(1) = \frac{p}{p + 1}.
$$

In [14], the authors define the $p$-analogue of the psi function as the logarithmic derivative of the function $\Gamma_p$, that is:

$$
\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}.
$$

and they gave the series representation of the $\psi_p$ with the relation:

$$
\psi_p(x) = \ln p - \sum_{n=0}^{p} \frac{1}{x + n}.
$$

Also, the authors in [15] introduced a new one-parameter deformation of the classical Gamma function, called $v$-analogue ($v$-deformation or $v$-generalization) of the Gamma function for $x, v > 0$ as:

$$
\Gamma_v(x) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}} e^{-t} dt. \quad (1.1)
$$

Note that when $v = 1$, $\Gamma_v(x) = \Gamma(x)$. They also gave the relation

$$
\Gamma_v(x) = \lim_{n \to \infty} \frac{n! (\frac{x}{v})^n v^{n+2}}{x(x+v)(x+2v)\ldots(x+nv)}. \quad (1.2)
$$

**Definition 1.**

i. A function $f : (a, b) \to \mathbb{R}$ is said to be convex if

$$
f(ax + \beta y) \leq \alpha f(x) + \beta f(y),
$$

(1.3)

ii. A function $f : (a, b) \to \mathbb{R}$ is said to be concave if the inequality (1.3) is reversed,

iii. A function $f : (a, b) \to \mathbb{R}^+$ is said to be logarithmically convex if

$$
\log f(ax + \beta y) \leq \alpha \log f(x) + \beta \log f(y)
$$

for all $x, y \in (a, b)$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. 


iv. A function $f$ is said to be totally monotone if $f$ is continuous on $[0, \infty)$, infinitely differentiable on $(0, \infty)$ and satisfies the condition that:

$$(-1)^n f^{(n)}(x) \geq 0, \quad n = 0, 1, 2, \ldots, x > 0.$$ 

Throughout this work, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of real numbers, positive real numbers and natural numbers respectively.

The main purpose of this paper is to introduce a new generalized Gamma function $\Gamma_{p,v}$, called the $p$-$v$-Gamma function. Our motivation to introduce this new function comes from a natural question if a similar definition of $\Gamma$ can be given for the function $v$. It contributes to giving some generalized properties. We establish recurrent relations for $\Gamma_{p,v}$ in Lemma 1 and Lemma 2. Also, we give the convexity property by Theorem 2. After defining the $p$-$v$-psi and $p$-$v$-polygamma functions we continue to give series representations, monotonicity properties, and some inequalities involving these new functions.

## 2 Main Results

We begin this section by presenting a new generalized Gamma function as follows:

**Definition 2.** Let $x, v > 0$ and $p \in \mathbb{N}$. Then the $p$-$v$-Gamma function (also called $p$-$v$-analogue or $p$-$v$-generalization of the Gamma function) is defined as:

$$\Gamma_{p,v}(x) = p! \left( \frac{1}{2} \right)^p v^{p+2} \frac{x(v^p+2)}{x(x+v)(x+2v)\ldots(x+pv)}.$$  \hspace{1cm} (2.1)

Note that $\Gamma_{p,v}(x) \rightarrow \Gamma_v(x)$ as $p \rightarrow \infty$.

**Lemma 1.** Let $x, v > 0$ and $p \in \mathbb{N}$. Then the function $\Gamma_{p,v}$ satisfies the identities:

1. $\Gamma_{p,v}(x+v) = \frac{x^p}{v} \left( \frac{v}{x+pv+v} \right) \Gamma_{p,v}(x)$,  \hspace{1cm} (2.2)
2. $\Gamma_{p,v}(v) = \frac{p}{p+1}$  \hspace{1cm} (2.3)

**Proof.** The result follows immediately by the equation (2.1). \hfill \Box

Also, note that $\Gamma_{p,v}$ satisfies the following commutative diagram:

$$\begin{array}{ccc}
\Gamma_{p,v} & \xrightarrow{p \rightarrow \infty} & \Gamma_v \\
v=1 \downarrow & & \downarrow v=1 \\
\Gamma_p & \xrightarrow{p \rightarrow \infty} & \Gamma
\end{array}$$

Now, we give a recurrent relation for $\Gamma_{p,v}$ which is also a generalization of (2.2).

**Lemma 2.** Let $x, v > 0$ and $p, n \in \mathbb{N}$. Then the function $\Gamma_{p,v}$ satisfies the relation:

$$\Gamma_{p,v}(x+nv) = \left( \frac{p}{v} \right)^n \prod_{i=0}^{n-1} \frac{x+iv}{x+(p+i+1)v} \Gamma_{p,v}(x).$$  \hspace{1cm} (2.4)

3
Proof. By (2.1) we have,
\[
\frac{\Gamma_{p,v}(x + nv)}{\Gamma_{p,v}(x + nv - v)} = \left(\frac{p}{v}\right) \frac{(x + nv - v)(x + nv)(x + nv + v)\ldots(x + nv + (p - 1)v)}{(x + nv)(x + ne + v)(x + nv + 2v)\ldots(x + nv + pv)} \nonumber
\]
\[
= \left(\frac{p}{v}\right) \frac{(x + nv - v)}{(x + nv + pv)}. \nonumber
\]
Then,
\[
\Gamma_{p,v}(x + nv) = \left(\frac{p}{v}\right) \frac{(x + nv - v)}{(x + nv + pv)} \Gamma_{p,v}(x + nv - v). \nonumber
\]
In a similar way, we have
\[
\Gamma_{p,v}(x + nv - v) = \left(\frac{p}{v}\right) \frac{(x + nv - 2v)}{(x + nv + (p - 1)v)} \Gamma_{p,v}(x + nv - 2v). \nonumber
\]
Then we have
\[
\Gamma_{p,v}(x + nv) = \left(\frac{p}{v}\right) \frac{(x + nv - v)}{(x + nv + pv)} \Gamma_{p,v}(x + nv - v) \nonumber
\]
\[
= \left(\frac{p}{v}\right)^2 \frac{(x + nv - v)}{(x + nv + pv)} \frac{(x + nv - 2v)}{(x + (n - 1)v + pv)} \Gamma_{p,v}(x + nv - 2v). \nonumber
\]
Continuing in this way, we obtain
\[
\Gamma_{p,v}(x + nv) = \left(\frac{p}{v}\right)^n \frac{(x + nv - v)(x + nv - 2v)\ldots(x + nv - nv)}{(x + nv + pv)(x + (n - 1)v + pv)\ldots(x + (n - (n - 1)v) + pv)} \Gamma_{p,v}(x) \nonumber
\]
\[
= \left(\frac{p}{v}\right)^n \frac{(x + (n - 1)v)(x + (n - 2)v)\ldots x}{(x + (p + n)v)(x + (p + n - 1)v)\ldots(x + (p + 1)v)} \Gamma_{p,v}(x) \nonumber
\]
\[
= \left(\frac{p}{v}\right)^n \left(\prod_{i=0}^{n-1} \frac{x + iv}{x + (p + i + 1)v}\right) \Gamma_{p,v}(x), \nonumber
\]
and the result follows. \(\square\)

Remark 1. Lemma 2 generalizes Lemma 2.1 of [14].

Note that, when taking the limit of both sides of the equation (2.4) as \(p \to \infty\), we obtain that
\[
\lim_{p \to \infty} \Gamma_{p,v}(x + nv) = \lim_{p \to \infty} \left(\frac{p}{v}\right)^n \frac{\prod_{i=0}^{n-1} (x + iv)}{\prod_{i=0}^{n-1} (x + (p + i)v)} \Gamma_{p,v}(x) \nonumber
\]
Hence we get
\[
\Gamma_v(x + nv) = 1_{v^{2n}} \prod_{i=0}^{n-1} (x + iv) \Gamma_v(x) \nonumber
\]
or equivalently
\[
\Gamma_v(x) = v^{2n} \frac{\Gamma_v(x + nv)}{x(x + v)\ldots(x + (n - 1)v)}. \nonumber
\]
Theorem 1. Let $x, v > 0$, $r > 1$ and $p \in \mathbb{N}$. Then, the inequality
\[
\Gamma_{p,v}(rx) < \left(\frac{p}{v}\right)^{\frac{x-v}{r}} \Gamma_{p,v}(x)
\] (2.5)
is valid.

Proof. Using the definition of $\Gamma_{p,v}$ we get
\[
\frac{\Gamma_{p,v}(rx)}{\Gamma_{p,v}(x)} = \frac{p!}{\left(\frac{p}{v}\right)^{\frac{x-v}{r}}} v^{p+2} x(x+v)\cdots(x+pv) < \left(\frac{p}{v}\right)^{\frac{x-v}{r}}
\]
and the result follows. \(\square\)

Theorem 2. Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the function $\Gamma_{p,v}$ is convex.

Proof. We have to prove that:
\[
\Gamma_{p,v}(x + y) \geq \left(\Gamma_{p,v}(x)\right)^{\alpha} \left(\Gamma_{p,v}(y)\right)^{\beta}
\] (2.6)
for all $\alpha, \beta > 0$, $\alpha + \beta = 1$ and $x, y > 0$. Using the concavity of the logarithm function we have
\[
x^\alpha y^\beta \leq \alpha x + \beta y.
\] (2.7)
By this, we obtain
\[
\left(\frac{x}{k}\right)^{\alpha} \left(\frac{x}{k}\right)^{\beta} \leq \alpha \left(\frac{x}{k}\right) + \beta \left(\frac{y}{k}\right) = \frac{\alpha x + \beta y}{k}
\]
for all $k \in \mathbb{N}$. Then we have
\[
\left(\frac{x}{k}\right)^{\alpha} \left(\frac{x}{k}\right)^{\beta} \cdots \left(\frac{x}{k}\right)^{\beta} \leq \left(\frac{x}{k} + \frac{\alpha x + \beta y}{2}\right) \left(\frac{x}{k} + \frac{\alpha x + \beta y}{2}\right) \cdots \left(\frac{x}{k} + \frac{\alpha x + \beta y}{2}\right)
\]
By using the equation (2.1) and inequality (2.7) we can write
\[
\Gamma_{p,v}(\alpha x + \beta y) = \left(\frac{p}{v}\right)^{\frac{\alpha x + \beta y}{p+2}} x^{\alpha} y^{\beta} (x+y)^{p+2} v^{p+2}
\leq \left(\frac{p}{v}\right)^{\frac{\alpha x + \beta y}{p+2}} x^{\alpha} y^{\beta} (x+y)^{p+2} v^{p+2}
\]
On the other hand, we have
\[
\left[\Gamma_{p,v}(x)\right]^{\alpha} \left[\Gamma_{p,v}(y)\right]^{\beta} = \left(\frac{p}{v}\right)^{\frac{\alpha x + \beta y}{p+2}} x^{\alpha} y^{\beta} (x+y)^{p+2} v^{p+2}
\]
\[ \left( \frac{p}{v} \right)^{\frac{x + y}{x + v}} v^{p + 2} \frac{1}{x^\alpha (x + v)^\alpha \left( \frac{x}{p} + v \right)^\alpha \cdots y^\beta (y + v)^\beta \left( \frac{y}{p} + v \right)^\beta} \]. \tag{2.9}

Then by the equations (2.8) and (2.9) we get

\[ \Gamma_{p,v}(ax + \beta y) \leq [\Gamma_{p,v}(x)]^\alpha [\Gamma_{p,v}(y)]^\beta, \]

and since every logarithmically convex function is also convex \cite{16}, the result follows.

\[ \square \]

**Corollary 1.** Let \( x, v > 0 \) and \( p \in \mathbb{N} \). Then, the inequality

\[ \Gamma_{p,v}(x + y) \leq \left( \frac{p}{v} \right)^{\frac{x + y}{v}} \sqrt{\Gamma_{p,v}(x)\Gamma_{p,v}(y)} \] \tag{2.10}

is valid.

**Proof.** By letting \( \alpha = \beta = \frac{1}{2} \), writing \( 2x \) instead of \( x \) and \( 2y \) instead of \( y \) in the equation (2.6) we have

\[ \Gamma_{p,v}(x + y) \leq \sqrt{\Gamma_{p,v}(2x)\Gamma_{p,v}(2y)}. \]

Now, using the inequality (2.5) for \( r = 2 \) we get

\[ \Gamma_{p,v}(x + y) \leq \sqrt{\left( \frac{p}{v} \right)^{\frac{x + y}{v}} \Gamma_{p,v}(x)\Gamma_{p,v}(y)}, \]

and the result follows. \[ \square \]

We define the \( p,v \)-psi (\( p,v \)-digamma) function as the logarithmic derivative of the function \( \Gamma_{p,v} \); that is

\[ \psi_{p,v}(x) = \frac{d}{dx} \ln \Gamma_{p,v}(x) = \frac{\Gamma_{p,v}'(x)}{\Gamma_{p,v}(x)}. \]

By using the equation (2.4) we get the following recurrent relation for \( \psi_{p,v} \).

**Corollary 2.** Let \( x, v > 0 \) and \( p, n \in \mathbb{N} \). Then the function \( \psi_{p,v} \) satisfies the relation:

\[ \psi_{p,v}(x + nv) - \psi_{p,v}(x) = \sum_{i=0}^{n-1} \frac{p^i + v}{(x + iv)(x + (p + i + 1)v)}. \tag{2.11} \]

**Proposition 1.** Let \( x, v > 0 \) and \( p \in \mathbb{N} \). Then, the function \( \psi_{p,v} \) has the series representation:

\[ \psi_{p,v}(x) = \frac{1}{v} \ln \left( \frac{p}{v} \right) - \sum_{n=0}^{p-1} \frac{1}{x + nv}. \tag{2.12} \]

**Proof.** By using the equation (2.1) we can write

\[ \Gamma_{p,v}(x) = \left( \frac{p}{x + v} \right)^{\frac{x + y}{2}} v^{p + 2} x^\alpha (x + v)^\alpha \left( \frac{x}{p} + v \right)^\alpha \cdots y^\beta (y + v)^\beta \left( \frac{y}{p} + v \right)^\beta. \tag{2.13} \]
Then,
\[
\ln \Gamma_{p,v}(x) = \frac{x}{v} \ln \left( \frac{p}{v} \right) + (p + 2) \ln v - \left( \ln x + \ln(x + v) + \ln \left( \frac{x}{2} + v \right) \ldots \ln \left( \frac{x}{p} + v \right) \right).
\]

Now, by differentiating both sides of the last equation with respect to \( x \) we get
\[
\psi_{p,v}(x) = \frac{1}{v} \ln \left( \frac{p}{v} \right) - \left( \frac{1}{x} + \frac{1}{x + v} + \frac{1}{2} \frac{1}{x + v} + \ldots + \frac{1}{p} \frac{1}{x + v} \right)
\]
\[
= \frac{1}{v} \ln \left( \frac{p}{v} \right) - \sum_{n=0}^{p} \frac{1}{x + nv},
\]
and the result follows.

The following proposition is given in [15]:

**Proposition 2.** Let \( x, v > 0 \). Then the \( v \)-digamma function \( \psi_v \) has the series representation:
\[
\psi_v(x) = -\ln v + \gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{nv} - \frac{1}{x + nv} \right).
\]
(2.14)

Note that, \( \gamma \) is the Euler-Mascheroni constant in the proposition 2:
\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right).
\]

**Theorem 3.** Let \( x, v > 0 \) and \( p \in \mathbb{N} \). Then, the function \( \psi_{p,v} \) satisfies the limit property:
\[
\lim_{p \to \infty} \psi_{p,v}(x) = \psi_v(x).
\]

**Proof.** By the equation (2.12) and the Proposition 2 we have
\[
\lim_{p \to \infty} \psi_{p,v}(x) = \lim_{p \to \infty} \left[ \frac{1}{v} \ln \left( \frac{p}{v} \right) - \frac{1}{x} - \sum_{n=1}^{p} \frac{1}{x + nv} \right]
\]
\[
= \lim_{p \to \infty} \left[ \frac{1}{v} \left( \ln p - \ln v - \sum_{n=1}^{p} \frac{1}{n} \right) - \frac{1}{x} - \sum_{n=1}^{p} \frac{1}{x + nv} \right]
\]
\[
= \frac{1}{v} \lim_{p \to \infty} \left[ \ln p - \sum_{n=1}^{p} \frac{1}{n} \right] - \frac{1}{x} - \lim_{p \to \infty} \left( \sum_{n=1}^{p} \frac{1}{x + nv} - \sum_{n=1}^{p} \frac{1}{nv} \right)
\]
\[
= -\frac{1}{v} \gamma - \frac{\ln v}{v} - \frac{1}{x} + \lim_{p \to \infty} \sum_{n=1}^{p} \left( \frac{1}{nv} - \frac{1}{x + nv} \right)
\]
\[
= \psi_v(x),
\]
and the result follows. \( \square \)

By using the equation (2.12) we get the following.
Corollary 3. Let \( x, v > 0 \) and \( p \in \mathbb{N} \). Then, the following identities hold:

i. \( \psi_{p,v}(v) = \frac{1}{v} \left[ \ln \left( \frac{p}{v} \right) - H_{p+1} \right] \)

where \( H_p \) is the \( p \)-th harmonic number, i.e. the sum of the reciprocals of the first \( p \) natural numbers:

\[
H_p = 1 + \frac{1}{2} + \ldots + \frac{1}{p} = \sum_{n=1}^{p} \frac{1}{n}.
\]

ii. The function \( \psi_{p,v} \) is increasing on \((0, \infty)\).

iii. The function \( \psi_{p,v} \) is totally monotone on \((0, \infty)\).

Note that, for an alternative proof of convexity of the function \( \Gamma_{p,v} \), we can use monotonicity. Since \( \psi_{p,v} \) is increasing on \((0, \infty)\), we have \( \psi'_{p,v}(x) > 0 \) for \( x > 0 \). Then \( (\ln \Gamma_{p,v}(x))'' > 0 \), i.e. \( \Gamma_{p,v} \) is logarithmically convex. It follows that \( \Gamma_{p,v} \) is convex. Also, since every totally monotone function \( f \) is logarithmically convex \([17, 18]\), we get that the function \( \psi_{p,v} \) is logarithmically convex and so it is convex.

Now, we define the \( p,v \)-polygamma function of order \( m \) as the \((m+1)\)-th derivative of the logarithm of the function \( \Gamma_{p,v} \) as

\[
\psi_{p,v}^{(m)}(x) = \frac{d^m}{dx^m} \ln \Gamma_{p,v}(x).
\]

Thus

\[
\psi_{p,v}^{(0)}(x) = \psi_{p,v}(x).
\]

Proposition 3. Let \( x, v > 0 \) and \( p, m \in \mathbb{N} \). Then the function \( \psi_{p,v}^{(m)} \) has the series representation:

\[
\psi_{p,v}^{(m)}(x) = \sum_{n=0}^{p} \frac{(-1)^{m+1} m!}{(x + nv)^{m+1}}.
\] \( (2.15) \)

Proof. By differentiating \( m \) times of the equation (2.12) with respect to \( x \) we get the result. \( \square \)

Note that

\[
\lim_{n \to \infty} \psi_{p,v}^{(m)}(x) = \psi_{v}^{(m)}(x),
\]

where \( \psi_{v}^{(m)} \) is the \( v \)-polygamma function of order \( m \) and has the series representation,

\[
\psi_{v}^{(m)}(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(x + nv)^{m+1}}.
\]

given in \([15]\). By using the equation (2.15) we get the following.

Corollary 4. Let \( x, v > 0 \) and \( p, m \in \mathbb{N} \). Then, the following identities hold:

i. The function \( \psi_{p,v}^{(m)} \) is positive and decreasing on \((0, \infty)\) if \( m \) is odd.

ii. The function \( \psi_{p,v}^{(m)} \) is negative and increasing on \((0, \infty)\) if \( m \) is even.

Finally, as an application to the definition of \( \psi_{p,v} \), we give the following theorems.

Theorem 4. The following inequalities are valid for \( x, v > 0 \) and \( p \in \mathbb{N} \):

\[
\frac{1}{v} \ln \left( \frac{px}{v(x + pv + v)} \right) - \frac{pv + v}{x(x + pv + v)} < \psi_{p,v}(x) < \frac{1}{v} \ln \left( \frac{px}{v(x + pv + v)} \right).
\]
Proof. If we apply the mean value theorem to the function 

\[ f(x) = \ln \Gamma_{p,v}(x) \]

on \((x, x + v)\), then there is a point \(x_0 \in (x, x + v)\) such that

\[ \ln \Gamma_{p,v}(x + v) - \ln \Gamma_{p,v}(x) = v \psi_{p,v}(x_0). \]

By the monotonicity of the function \( \psi_{p,v} \) on \((0, \infty)\) we get

\[ \psi_{p,v}(x) < \psi_{p,v}(x_0) < \psi_{p,v}(x + v) \tag{2.16} \]

and also by (2.2) we get \( \psi_{p,v}(x_0) = \frac{1}{v} \ln \frac{pv}{v(x + pv + v)} \). Now using the equation (2.11) for \( n = 1 \) in the inequalities (2.16) we get

\[ \psi_{p,v}(x) < \frac{1}{v} \ln \left( \frac{pv}{v(x + pv + v)} \right) < \frac{pv + v}{x(x + pv + v)} + \psi_{p,v}(x), \]

and the result follows.

\[ \square \]

Theorem 5. Let \( x, v > 0 \) and \( p \in \mathbb{N} \). Then, the function

\[ x \to x \psi_{p,v}(x) \]

is convex.

Proof. We have

\[ (x \psi_{p,v}(x))'' = 2 \psi_{p,v}'(x) + x \psi_{p,v}''(x). \]

Then by using the equation (2.15) we have

\[ (x \psi_{p,v}(x))'' = 2 \sum_{n=0}^{p} \frac{1}{(x + nv)^2} - x \sum_{n=0}^{p} \frac{2}{(x + nv)^3} = \sum_{n=0}^{p} \frac{2nv}{(x + nv)^3} \geq 0, \]

and the proof is completed.

\[ \square \]

Theorem 6. Let \( x \in [0, 1], v > 0, p, m \in \mathbb{N} \) and \( a, b \) \((a \geq b)\) be positive real numbers. Also let \( c, d \) be positive real numbers such that \( 0 < cb^{m+1} \leq da^{m+1} \). Then the function

\[ x \to \left( \ln \frac{[\Gamma_{p,v}(a + bx)]^c}{[\Gamma_{p,v}(b + ax)]^d} \right)^{(m)} \]

i. decreasing if \( m \) is odd,

ii. increasing if \( m \) is even.

Proof. Let \( g(x) = \frac{[\Gamma_{p,v}(a + bx)]^c}{[\Gamma_{p,v}(b + ax)]^d} \) and \( h(x) = \ln g(x) \). Then,

\[ h^{(m+1)}(x) = \ln g(x)^{(m)} = c \ln(\ln(\Gamma_{p,v}(a + bx)))^{(m+1)} - d \ln(\ln(\Gamma_{p,v}(b + ax)))^{(m+1)} \]

\[ = cb^{m+1} \psi_{p,v}(a + bx) - da^{m+1} \psi_{p,v}(b + ax). \]

Since \( x \in [0, 1] \) and \( a \geq b \) we have \( a + bx \geq b + ax \). Now using the Corollary 4, we get the followings:
If \( m \) is odd then \( 0 < \psi_{p,v}^{(m)}(a + bx) \leq \psi_{p,v}^{(m)}(b + ax) \). Then since \( 0 < cb^{m+1} \leq da^{m+1} \) we can write
\[
ck^{m+1} \psi_{p,v}^{(m)}(a + bx) \leq dk^{m+1} \psi_{p,v}^{(m)}(a + bx) \leq dk^{m+1} \psi_{p,v}^{(m)}(b + ax).
\]
So, \( h^{(m+1)}(x) \leq 0 \). It means that the function \( h^{(m)} \) is decreasing on \([0, 1] \) if \( m \) is odd.
Similarly, if \( m \) is even we have
\[
ck^{m+1} \psi_{p,v}^{(m)}(a + bx) \geq dk^{m+1} \psi_{p,v}^{(m)}(a + bx) \geq dk^{m+1} \psi_{p,v}^{(m)}(b + ax),
\]
this implies that the function \( h^{(m)} \) is increasing on \([0, 1] \) if \( m \) is even, and the result follows.

3 Conclusions

In the first section, we have given the necessary definitions for our main results. In the main section, we have introduced a new generalized Gamma function \( \Gamma_{p,v} \), called the \( p-v \)-Gamma function. We have proved a recurrent relation convexity property and related results for \( \Gamma_{p,v} \). In addition, we have defined \( \psi_{p,v} \) and \( \psi_{p,v}^{(m)} \) functions, called the \( p-v \)-psi(digamma) and \( p-v \)-polygamma functions respectively. Also, we have given some series representations, monotonicity properties, and inequalities involving these new functions.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Díaz R, Pariguan E. On hypergeometric functions and Pochhammer \( k \)-symbol. Divulg. Mat. 2007;15(2):179–192.
[2] Duran U, Acikgoz M. A Study on novel extensions for the \( p \)-adic Gamma and \( p \)-adic beta Functions. Mathematical and Computational Applications. 2019;24(2):53.
[3] Ege, İ. On Defining the \( (p, q, k) \)-generalized Gamma Function. Note di Matematica. 2019;39(1):107-116.
[4] Ege, İ. Some Inequalities for the Extension of \( k \)-Gamma Function. Asian Research Journal of Mathematics. 2019;14(4):1-8.
[5] Gehlot KS, Nisar KS. Extension of two parameter Gamma, beta functions and its properties. Applications and Applied Mathematics: An International Journal (AAM). 2020;15(3):4.
[6] Gehlot KS, Ram C. Pochhammer symbol of ultra Gamma function and its applications to hypergeometric functions; 2018.
[7] Kadak U, Mohiuddine SA. Generalized statistically almost convergence based on the difference operator which includes the \( (p, q) \)-Gamma function and related approximation theorems. Results in Mathematics. 2018;73(1):1-31.
[8] Kim T, Kim DS. Degenerate Laplace transform and degenerate gamma function. Russian Journal of Mathematical Physics. 2017;24(2):241–248.
[9] Laforgia A, Pierpaolo N. On some inequalities for the gamma function. Advances in Dynamical Systems and Applications. 2013;8(2):261–267.
[10] Nantomah K, Prempeh E, Twum SB. On a \( (p,k) \)-analogue of the Gamma function and some associated Inequalities, Moroccan Journal of Pure and Applied Analysis. 2016;2(2):79-90.
[11] Salem A. Monotonic functions related to the q-Gamma function. Monatshefte fr Mathematik. 2016;179(2):281–292.

[12] Yin L, Huang L, Lin X. Complete monotonicity of some functions involving $k$-digamma function with application. Journal of Mathematical Inequalities. 2012;15(1):229–238.

[13] Apostol TM. Introduction to analytic number theory. Springer-Verlag; 1976.

[14] Krasniqi V, Shabani AS. Convexity properties and inequalities for a generalized Gamma function. Applied Mathematics E-Notes. 2010;10:27-35.

[15] Djabang E, Nantomah K, Iddrisu M. On a $v$-analogue of the Gamma Function and some associated inequalities. J. Math. Comput. Sci. 2021;11(1):74–86.

[16] Niculescu C, Persson LE. Convex functions and their applications, New York: Springer. 2006:23.

[17] Fink AM. Kolmogorov-Landau inequalities for monotone functions. Journal of Mathematical Analysis and Applications. 1982;90(1):251–258.

[18] Merkle M. Completely monotone functions: A digest. Analytic Number Theory Approximation Theory and Special Functions Springer, New York, NY. 2014;347-364.

© 2022 Ege: This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar).
https://www.sdiarticle5.com/review-history/87037