CROSSED PRODUCTS BY AUTOMORPHISMS WITH THE TRACIAL QUASI-ROKHLIN PROPERTY

JULIAN BUCK

ABSTRACT. We introduce the tracial quasi-Rokhlin property for an automorphism \( \alpha \) of a unital \( C^* \)-algebra \( A \), which is not assumed to be simple. We show that under suitable hypotheses, the associated crossed product \( C^* \)-algebra \( C^*(\mathbb{Z}, A, \alpha) \) is simple, and there is a bijection between the space of tracial states on \( C^*(\mathbb{Z}, A, \alpha) \) and the \( \alpha \)-invariant tracial states on \( A \). We show that, for a minimal dynamical system \((X, h)\) and a simple, separable, unital \( C^* \)-algebra \( A \), the automorphism \( \beta \) which extends the action of \( h \) on \( C(X) \) has the tracial quasi-Rokhlin property, and hence that \( C^*(\mathbb{Z}, C(X, A), \beta) \) has the structural properties described above.

1. Introduction

The study of \( C^* \)-algebras arising through crossed product constructions has been an area of significant interest in the Elliott classification program for nuclear \( C^* \)-algebras. Two areas where considerable success has been achieved are crossed products associated to minimal dynamical systems and crossed products by automorphisms with various forms of the Rokhlin property. In the first situation, the case of Cantor minimal systems was studied extensively (see for example [8] and [31]), and the techniques of [31] were later considerably generalized first to minimal diffeomorphisms of finite-dimensional compact manifolds (see the long unpublished preprint [24], and also the survey articles [22] and [23]), and later to finite-dimensional compact metric spaces (see [20] and [33] for the best known results). In the second situation, see for example [10], [11], [12], [13], and [19] for results related to the Rokhlin property, and [2], [7], [27], and [30] for results related to various forms of the tracial Rokhlin property.

There is little existing overlap between these two branches of research into crossed products. Most forms of the Rokhlin and tracial Rokhlin properties are formulated for \( C^* \)-algebras containing many projections (such as in the real rank zero case), while the \( C^* \)-algebra \( C(X) \) may have few or no non-trivial projections. We introduce the tracial quasi-Rokhlin property for automorphisms of a unital, separable \( C^* \)-algebra \( A \) which is not assumed to be simple nor contain any non-trivial projections. In fact, the \( C^* \)-algebras in which we will be most interested will be of the form \( C(X, A) \), where \( X \) is an infinite compact metrizable space and \( A \) is a simple, separable, unital, infinite-dimensional \( C^* \)-algebra.

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In Section 2 we define the tracial quasi-Rokhlin property, and show that if \( \alpha \) is an automorphism of \( A \) and \( A \) has no non-trivial \( \alpha \)-invariant ideals, then the crossed product \( C^*(\mathbb{Z}, A, \alpha) \) is simple. Further, an additional technical assumption about \( A \) (which is satisfied for our main algebras of interest) allows us to also show that the restriction mapping \( T(C^*(\mathbb{Z}, A, \alpha)) \to T_\alpha(A) \), between the simplex of tracial states on the crossed product and the simplex of \( \alpha \)-invariant tracial states on \( A \), is a bijection.

In Section 3 we use this condition to show that (with appropriate hypotheses on \( X \) and \( A \)) certain automorphisms \( \beta \) of the algebra \( C(X, A) \), which act minimally on the center \( C(X) \), have the tracial quasi-Rokhlin property. After examining the structure of ideals in \( C(X, A) \) and of its tracial state space, it will follow that the structural theorems of Section 2 apply the associated crossed product \( C^*-\) algebras \( C^*(\mathbb{Z}, C(X, A), \beta) \).

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2. The Tracial Quasi-Rokhlin Property

The following definition is based on Definition 1.1 of [27], and also on the behavior of automorphisms induced by minimal homeomorphisms.

**Definition 2.1.** Let \( A \) be a separable, unital \( C^*- \) algebra, and let \( \alpha \in \text{Aut}(A) \). We say that \( \alpha \) has the tracial quasi-Rokhlin property if for every \( \varepsilon > 0 \), every finite set \( F \subset A \), every \( n \in \mathbb{N} \), and every positive element \( x \in A \) with \( \|x\| = 1 \), there exist \( c_0, \ldots, c_n \in A \) such that:

1. \( 0 \leq c_j \leq 1 \) for \( 0 \leq j \leq n; \)
2. \( c_j c_k = 0 \) for \( 0 \leq j, k \leq n \) and \( j \neq k; \)
3. \( \|\alpha(c_j) - c_{j+1}\| < \varepsilon \) for \( 0 \leq j \leq n - 1; \)
4. \( \|c_j a - ac_j\| < \varepsilon \) for \( 0 \leq j \leq n \) and for all \( a \in F; \)
5. \( c = \sum_{j=0}^{n} c_j \), there exist \( N \in \mathbb{N} \), positive elements \( e_0, \ldots, e_N \in A \), unitaries \( w_0, \ldots, w_N \in A \), and \( d(0), \ldots, d(N) \in \mathbb{Z} \) such that:
   a. \( 1 - c \leq \sum_{j=0}^{N} e_j; \)
   b. \( w_j \alpha^{d(j)}(e_j)w_j^* w_k \alpha^{d(k)}(e_k)w_k^* = 0 \) for \( 0 \leq j, k \leq N \) and \( j \neq k; \)
   c. \( w_j \alpha^{d(j)}(e_j)w_j^* \in \mathbb{Z} \) for \( 0 \leq j \leq N; \)
6. \( \|cx\| > 1 - \varepsilon. \)

The key differences between this definition and Definition 1.1 of [27] are the change from projections to positive contractions, and the statement of condition (5) (as compared to condition (3) in Definition 1.1 of [27]). We also make no assumptions about the simplicity of the algebra \( A \), but it should be noted that this definition is only formulated for cases where the algebra \( A \) is expected to be emph-simply (have no non-trivial \( \alpha \)-invariant ideals), and this will be assumed in the applications that follow.

**Lemma 2.2.** Let \( A \) be a separable, unital \( C^*- \) algebra, let \( \alpha \in \text{Aut}(A) \), and let \( u \) be the canonical unitary of the crossed product \( C^*-\) algebra \( C^*(\mathbb{Z}, A, \alpha) \). Given any \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), let \( c_0, \ldots, c_n \in A \) satisfy:
Lemma 2.3. Let $A$ be a separable, unital $C^*$-algebra, let $\alpha \in \text{Aut}(A)$, and let $a \in C^*(\mathbb{Z}, A, \alpha)$ be positive and non-zero. Then for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $a_j \in A$ for $-N \leq j \leq N$ such that $\|a_0\| = 1$ and

\[
\left\| a - \sum_{j=-N}^{N} a_j u^j \right\| < \varepsilon.
\]

Proof. Let $E : C^*(\mathbb{Z}, A, \alpha) \to A$ be the standard faithful conditional expectation. Set $b = a^{1/2}$, which is positive and non-zero. Then as $E$ is faithful, it follows that $E(a) = E(b^2) = E(b^*b) \neq 0$.

By replacing $a$ with $\|E(a)\|^{-1} a$ if necessary, we may assume that $\|E(a)\| = 1$. Since $C_c(\mathbb{Z}, A, \alpha)$ is dense in $C^*(\mathbb{Z}, A, \alpha)$, there exist $N \in \mathbb{N}$ and $\tilde{b}_j \in A$ for $-N \leq j \leq N$ such that

\[
\left\| (a - E(a)) - \sum_{j=-N}^{N} \tilde{b}_j u^j \right\| < \frac{1}{2}\varepsilon.
\]

Using $E(a - E(a)) = 0$ and $E\left( \sum \tilde{b}_j u^j \right) = E(\tilde{b}_0)$, we estimate

\[
\|\tilde{b}_0\| = \left\| E\left( a - E(a) \right) - E\left( \sum \tilde{b}_j u^j \right) \right\| \leq \left\| (a - E(a)) - \sum \tilde{b}_j u^j \right\| < \frac{1}{2}\varepsilon.
\]

Now set $b_0 = 0$ and $b_j = \tilde{b}_j$ for $1 \leq |j| \leq N$. Then

\[
\left\| (a - E(a)) - \sum_{j=-N}^{N} b_j u^j \right\| \leq \|b_0\| + \left\| (a - E(a)) - \sum_{j=-N}^{N} \tilde{b}_j u^j \right\| < \varepsilon.
\]
By defining \( a_0 = E(a) \) and \( a_j = b_j \) for \( 1 \leq |j| \leq N \), it follows that \( \|a_0\| = 1 \) and
\[
\|a - \sum_{j=-N}^{N} a_j u^j\| = \| (a - E(a)) - \sum_{j=-N}^{N} b_j u^j\| < \varepsilon,
\]
as required.

**Theorem 2.4.** Let \( A \) be a separable, unital \( C^* \)-algebra, let \( \alpha \in \text{Aut}(A) \) have the tracial quasi-Rokhlin property, and suppose that \( A \) is \( \alpha \)-simple. Then \( C^*(\mathbb{Z}, A, \alpha) \) is simple.

**Proof.** Let \( J \subset C^*(\mathbb{Z}, A, \alpha) \) be a non-zero ideal, let \( u \in C^*(\mathbb{Z}, A, \alpha) \) be the canonical unitary in the crossed product, let \( 0 < \varepsilon < \), and let \( a \in J \) be non-negative and positive. By Lemma 2.3 there exist \( n \in \mathbb{N} \) and \( a_k \in A \) for \( -n \leq k \leq n \) such that \( \|a_0\| = 1 \) and
\[
\|a - \sum_{k=-n}^{n} a_k u^k\| < \frac{1}{4} \varepsilon.
\]
Define continuous functions \( f, g : [0, 1] \to [0, 1] \) by
\[
f(t) = \begin{cases} 
0 & t \leq 1 - \frac{\varepsilon}{8} \\
\frac{16}{\varepsilon} (t - 1) + 2 & 1 - \frac{\varepsilon}{8} < t < 1 - \frac{\varepsilon}{16} \\
1 & t \geq 1 - \frac{\varepsilon}{16}
\end{cases}
\]
and
\[
g(t) = \begin{cases} 
0 & t < 1 - \frac{\varepsilon}{16} \\
\frac{16}{\varepsilon} (t - 1) + 1 & t \geq 1 - \frac{\varepsilon}{16}
\end{cases}
\]
Setting \( q = g(a_0^{1/2}) \) and \( r = f(a_0^{1/2}) \), we have the relations \( q, r \geq 0, qr = q, \) and \( \|q\| = \|r\| = 1 \). Now set \( M = \sum_{k \neq 0} \|a_k\| \) and
\[
\varepsilon' = \frac{\varepsilon}{12(M(n+1)^2 + 1)}
\]
and \( F = \{ a_k : -n \leq k \leq n \} \). Apply the tracial quasi-Rokhlin property with \( F, \varepsilon', n, \) and \( q \) to obtain \( c_0, \ldots, c_n \in A \) such that
\[
\begin{align*}
(1) & \quad 0 \leq c_j \leq 1 \text{ for } 0 \leq j \leq n; \\
(2) & \quad c_j c_k = 0 \text{ for } 0 \leq j, k \leq n \text{ and } j \neq k; \\
(3) & \quad \|\alpha(c_j) - c_{j+1}\| < \varepsilon' \text{ for } 0 \leq j \leq n - 1; \\
(4) & \quad \|c_j a_k - a_k c_j\| < \varepsilon' \text{ for } 0 \leq j \leq n \text{ and } -n \leq k \leq n; \\
(5) & \quad \text{with } c = \sum_{j=0}^{n} c_j, \text{ we have } \|qc| > 1 - \varepsilon'.
\end{align*}
\]
Using the mutual orthogonality of the \( c_j \), we have
\[
\begin{align*}
\left\| \sum_{j=0}^{n} c_j a c_j - \sum_{j=0}^{n} \sum_{k=-n}^{n} c_j a_k u^k c_j \right\| & = \left\| \sum_{j=0}^{n} c_j \left( a - \sum_{k=-n}^{n} a_k u^k \right) c_j \right\| \\
& \leq \max_{0 \leq j \leq n} \left\| c_j \left( a - \sum_{k=-n}^{n} a_k u^k \right) c_j \right\| \\
& \leq \left\| a - \sum_{k=-n}^{n} a_k u^k \right\| < \frac{1}{4} \varepsilon.
\end{align*}
\]
Since the $c_j$ approximately commute with the $a_k$, we obtain
\[
\left\| \sum_{j=0}^{n} \sum_{k=-n}^{n} c_j k a_k^k c_j - \sum_{j=0}^{n} \sum_{k=-n}^{n} a_k c_j k a_k^k c_j \right\| = \left\| \sum_{j=0}^{n} \sum_{k=-n}^{n} (c_j k a_k - a_k c_j) k a_k^k c_j \right\| 
\leq \sum_{j=0}^{n} \sum_{k=-n}^{n} \| c_j k a_k - a_k c_j \| 
< 2(n+1)^2 \varepsilon' < \frac{1}{4} \varepsilon.
\]
Next, applying Lemma 2.2 gives
\[
\left\| \sum_{j=0}^{n} \sum_{k=-n}^{n} a_k c_j k a_k^k c_j - \sum_{j=0}^{n} a_k c_j k a_k^k c_j \right\| = \left\| \sum_{j=0}^{n} \sum_{k\neq 0} a_k c_j k a_k^k c_j \right\| 
\leq \sum_{j=0}^{n} \sum_{k\neq 0} \| a_k \| \| c_j k a_k^k c_j \| 
< 3n(n+1)M \varepsilon' < \frac{1}{4} \varepsilon.
\]
Finally, orthogonality of the $c_j$ gives $c^2 = \sum_{j=0}^{n} c_j^2$, and using this we obtain the estimate
\[
\left\| \sum_{j=0}^{n} a_k c_j^2 - ca_0 c \right\| = \left\| \sum_{j=0}^{n} (a_0 c_j - c_j a_0) c_j \right\| \leq \sum_{j=0}^{n} \| a_0 c_j - c_j a_0 \| < (n+1) \varepsilon' < \frac{1}{4} \varepsilon.
\]
Setting $x = \sum_{j=0}^{n} c_j a_k c_j$, it follows that
\[
\| x - ca_0 c \| < \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon = \varepsilon.
\]
We next show that $\| ca_0 c \|$ is sufficiently large. With $f(t)$ as before, we compute
\[
\| a_0^{1/2} r - r \| = \sup_{t \in [0,1]} | tf(t) - f(t) | \leq \frac{1}{8} \varepsilon
\]
Since $rq = q$ and $\| q \| = 1$, it follows that $\| a_0^{1/2} q - q \| < \frac{1}{8} \varepsilon$. This gives
\[
1 - \frac{1}{12} \varepsilon < 1 - \varepsilon' < \|cq c - c_0^{1/2} qc\| \leq \|cq c - c_0^{1/2} qc\| + \| c_0^{1/2} qc \| 
\leq \| q - a_0^{1/2} q \| + \| ca_0^{1/2} c_0 \| 
< \frac{1}{12} \varepsilon + \| ca_0^{1/2} c_0 \|,
\]
and so $\| ca_0^{1/2} \| > 1 - \frac{5}{24} \varepsilon$. Now the assumption $\varepsilon < 1$ gives
\[
\| ca_0 c \| = \|(ca_0^{1/2}) (ca_0^{1/2})^* \| = \| ca_0^{1/2} \| ^2 > (1 - \frac{5}{24} \varepsilon)^2 = (1 - \frac{5}{24} \varepsilon)^2 = \frac{361}{576}.
\]
Now suppose that $J \cap A = 0$. By Theorem 3.1.7 of [26], $A + J$ is a $C^*$-subalgebra of $C^*(\mathbb{Z}, A, \alpha)$, and the assumption that $J \cap A = 0$ implies that the projection map $\pi: A + J \to (A + J)/J$ is isometric when restricted to $A$ (and of course it is norm-reducing in general). Since $ca_0 c \in A$ and $x \in J$, it follows that
\[
\frac{361}{576} < \| ca_0 c \| = \| \pi(ca_0 c) \| = \| \pi(ca_0 c - x) \| \leq \| ca_0 c - x \| < \frac{1}{8},
\]
a contradiction. So there must be a non-zero element in $J \cap A$. Finally, we claim that $J \cap A$ is an $\alpha$-invariant ideal of $A$. To see this, let $b \in J \cap A$. Then $\alpha(b) = ubu^* \in J$ since $J$ is an ideal, and clearly $\alpha(b) \in A$, so $\alpha(b) \in J \cap A$. Thus, $J \cap A$ is a
non-zero $\alpha$-invariant ideal of $A$, which implies that $J \cap A = A$. It follows that
$J = C^*(\Bbb{Z}, A, \alpha)$, and so $C^*(\Bbb{Z}, A, \alpha)$ is simple. \hfill $\square$

Lemma 2.5. Let $f \in C([0, 1])$.

(1) For any $\varepsilon > 0$, there is a $\delta > 0$ (depending on both $\varepsilon$ and $f$) such that if $A$ is a unital $C^*$-algebra and $a, b \in A$ satisfy $0 \leq a, b \leq 1$, then $\|ab - ba\| < \delta$ implies $\|f(b)a - af(b)\| < \varepsilon$.

(2) For every $\varepsilon > 0$, there is a $\delta > 0$ (depending on both $\varepsilon$ and $f$) such that if $A$ is a unital $C^*$-algebra and $a, b \in A$ satisfy $0 \leq a, b \leq 1$, then $\|a - b\| < \delta$ implies $\|f(a) - f(b)\| < \varepsilon$.

Proof. The proofs the the same as in Lemma 2.5.11 of [15]. \hfill $\square$

Lemma 2.6. Let $A$ be a separable, unital $C^*$-algebra, let $\alpha \in \text{Aut}(A)$, let $T_\alpha(A)$ denote the space of all $\alpha$-invariant tracial states on $A$, and let $\tau \in T_\alpha(A)$. Then the set $I = \{a \in A : \tau(a^*a) = 0\}$ is an $\alpha$-invariant ideal of $A$.

Proof. The map $a \mapsto \tau(a^*a)$ is clearly a bounded linear functional $A \to \Bbb{C}$, so the set $I = \{a \in A : \tau(a^*a) = 0\}$ is closed. In Section 3.4 of [26] it is shown that $I$ is a closed left ideal of $A$ (using Theorem 3.3.7 there). As $\tau(aa^* = \tau(a^*a)$, it is clear that $a \in I$ if and only if $a^* \in I$. Therefore $I$ is a closed left ideal of $A$ that is closed under adjoints. But then for any $b \in A$ and $a \in I$, we have $b^*a \in A$ and $a^*b \in I$. Since $I$ is a left ideal of $A$, we get $b^*a^* \in I$, and since $I$ is closed under adjoints, it follows that $ab = (b^*a^*)^* \in I$. Therefore, $I$ is an ideal of $A$. Finally, given $a \in I$, the $\alpha$-invariance of $\tau$ implies that

$$\tau((a(a)^*(a)) = \tau(a(a^*)\alpha(a)) = \tau(a(a^*a)) = \tau(a^*a) = 0,$$

and this gives $\alpha(a) \in I$. Therefore, $I$ is $\alpha$-invariant. \hfill $\square$

Proposition 2.7. Let $A$ be a separable, unital $C^*$-algebra, let $\alpha \in \text{Aut}(A)$, and assume that $A$ is $\alpha$-simple. Then given any $\tau \in T_\alpha(A)$ and any $y \in A$ with $\text{sp}(y) = [0, 1]$, and with $\mu$ the spectral measure for $\tau$ on $C^*(y, 1)$, there is an open interval $U \subset [0, 1]$ such that $U \neq \emptyset$ and $\mu(U) < \varepsilon$.

Proof. Since $A$ has no non-trivial $\alpha$-invariant ideals, Lemma 2.6 implies that $\tau(a^*a) = 0$ if and only if $a = 0$, and so $\tau$ is faithful. Let $V \subset [0, 1]$ be any non-empty open interval, let $x_0 \in V$, and choose an $f \in C^*(y, 1) \cong C([0, 1])$ such that $f(x_0) = 1$ and $\text{supp}(f) \subset V$. Then

$$\mu(V) \geq \int_0^1 f \, d\mu = \tau(f) > 0.$$ 

Hence all non-empty open intervals in $[0, 1]$ have positive $\mu$-measure. For $n = 2, 3, 4, \ldots$ define open intervals $U_n \subset [0, 1]$ by $U_n = \left(\frac{n-1}{n}, \frac{n}{n+1}\right)$. Then the collection $(U_n)_{n=1}^\infty$ is pairwise disjoint, and $\mu(U_n) > 0$ for all $n \geq 1$ by the previous argument. By pairwise disjointness it follows that

$$\sum_{n=2}^\infty \mu(U_n) = \mu\left(\bigcup_{n=2}^\infty U_n\right) \leq \mu([0, 1]) = 1$$

and so this series converges. Thus for some $N \in \Bbb{N}$ we must have $\sum_{n=N}^\infty \mu(U_n) < \varepsilon$, and so by setting $U = U_N$ we obtain a non-empty open interval $U \subset [0, 1]$ with $\mu(U) < \varepsilon$. \hfill $\square$
In order for the previous lemma to be useful we must know that our \( C^* \)-algebra \( A \) contains a positive element with spectrum equal to \([0, 1]\). We thus introduce the following definition.

**Definition 2.8.** A \( C^* \)-algebra \( A \) is called **scattered** if every state on \( A \) is atomic; that is, given any state \( \omega \) on \( A \), there exist pure states \( (\omega_j)_{j=1}^{\infty} \) and real numbers \( (t_j)_{j=1}^{\infty} \), satisfying \( t_j \geq 0 \) for all \( j \geq 1 \) and \( \sum_{j=1}^{\infty} t_j = 1 \), such that \( \omega = \sum_{j=1}^{\infty} t_j \omega_j \).

By Theorem 2.2 of \[1\], a \( C^* \)-algebra is scattered if and only if the spectrum of every self-adjoint element of \( A \) is countable. The argument in the fourth fact about scattered \( C^* \)-algebras on page 61 of \[1\] shows that if \( A \) is unital and not scattered, then there is a positive element \( y \in A \) with \( \text{sp}(y) = [0, 1] \). For the case in which we have the most interest the algebras involved are not scattered. (See Proposition 3.24 for the justification of this claim.)

**Proposition 2.9.** Let \( A \) be a separable, unital \( C^* \)-algebra that is not scattered, let \( \alpha \in \text{Aut}(A) \) have the tracial quasi-Rokhlin property, and assume that \( A \) is \( \alpha \)-simple. Then for every \( \varepsilon > 0 \), every finite set \( F \subset A \), every \( n \in \mathbb{N} \), and every \( \tau \in T_\alpha(A) \), there exist \( c_0, \ldots, c_n \in A \) such that

1. \( 0 \leq c_j \leq 1 \) for \( 0 \leq j \leq n \);
2. \( c_j c_k = 0 \) for \( 0 \leq j, k \leq n \) and \( j \neq k \);
3. \( \|\alpha(c_j) - c_{j+1}\| < \varepsilon \) for \( 0 \leq j \leq n - 1 \);
4. \( \|ac_j - c_ja\| \leq \varepsilon \) for \( 0 \leq j \leq n \) and for all \( a \in F \);
5. with \( c = \sum_{j=0}^{n} c_j \), we have \( \tau(1 - c) < \varepsilon \).

**Proof.** Let \( \varepsilon > 0 \), \( F \subset A \) finite, \( n \in \mathbb{N} \), and \( \tau \in T_\alpha(A) \) be given. Since \( A \) is not scattered, there is a \( y \in A \) with \( \text{sp}(y) = [0, 1] \). Let \( \mu \) be the spectral measure for \( \tau \) on \( C^*(y, 1) \cong C([0, 1]) \), so that

\[
\tau(f(y)) = \int_0^1 f \, d\mu
\]

for all \( f \in C([0, 1]) \). By Proposition 2.7, there is a non-empty open interval \( I \subset [0, 1] \) such that \( \mu(I) < \varepsilon \). Since \( I \) is an open interval, there exist \( 0 < t_0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < 1 \) such that \( I = (t_0, t_6) \). Define continuous functions \( f, g : [0, 1] \to [0, 1] \) by

\[
f(t) = \begin{cases} 
0 & 0 \leq t < t_1 \\
\frac{t-t_0}{t_2-t_1} & t_1 \leq t < t_2 \\
1 & t_2 \leq t < t_4 \\
\frac{t-t_3}{t_5-t_4} & t_4 \leq t < t_5 \\
0 & t_5 \leq t \leq 1 
\end{cases}
\]

and

\[
g(t) = \begin{cases} 
0 & 0 \leq t < t_2 \\
\frac{t-t_0}{t_3-t_2} & t_2 \leq t < t_3 \\
\frac{t-t_3}{t_4-t_3} & t_3 \leq t < t_4 \\
0 & t_4 \leq t \leq 1 
\end{cases}
\]

Then \( \text{supp}(f), \text{supp}(g) \subset I \), \( fg = g \), and \( f, g \neq 0 \). Set \( x = g(y) \) and \( b = f(y) \). Then \( 0 \leq x \leq b \leq 1 \) and \( xb = bx = x \). Now for any \( a \in xA(x) \) with \( 0 \leq a \leq 1 \), we have

\[
a = b^{1/2} ab^{1/2} \leq b^{1/2} (||a|| \cdot 1)b^{1/2} \leq b,
\]

and so \( \tau(a) \leq \tau(b) \). It follows that for any \( a \in xA(x) \), we have

\[
\tau(a) \leq \tau(b) = \int_0^1 f \, d\mu \leq \mu(I) < \varepsilon.
\]

Now apply the tracial quasi-Rokhlin property with \( \varepsilon, F, n, \) and \( x \), obtaining \( c_0, \ldots, c_n \in A \) such that:
(1) 0 ≤ c_j ≤ 1 for 0 ≤ j ≤ n;
(2) c_jc_k = 0 for 0 ≤ j, k ≤ n and j ≠ k;
(3) \|α(c_j) - c_{j+1}\| < ε for 0 ≤ j ≤ n - 1;
(4) \|ac_j - c_αa\| < ε for 0 ≤ j ≤ n and for all \(a \in F\);
(5) with \(c = \sum_{j=0}^n c_j\), there exists \(N \in \mathbb{N}\), positive elements \(e_0, \ldots, e_N \in A\), unitaries \(w_0, \ldots, w_N \in A\), and \(d(0), \ldots, d(N) \in \mathbb{Z}\) such that:
  (a) 1 - c ≤ \sum_{j=0}^N e_j;
  (b) \(α^{d(j)}(e_j)α^{d(k)}(e_k) = 0\) for 0 ≤ j, k ≤ N;
  (c) \(j ≠ k\), and \(w_jα^{d(j)}(e_j)w_j^* ∈ xAx\) for 0 ≤ j ≤ N.

Since each \(w_jα^{d(j)}(e_j)w_j^* ∈ xAx\), it follows that \(\sum_{j=0}^N w_jα^{d(j)}(e_j)w_j^* ∈ xAx\), and so

\[
τ\left(\sum_{j=0}^N w_jα^{d(j)}(e_j)w_j^*\right) < ε
\]

Then the linearity and α-invariance of τ imply that

\[
τ(1 - c) ≤ \sum_{j=0}^N τ(e_j) = \sum_{j=0}^N τ(α^{d(j)}(e_j)) = \sum_{j=0}^N τ\left(w_jα^{d(j)}(e_j)w_j^*\right) < ε,
\]

which completes the proof.

\[\square\]

**Theorem 2.10.** Let \(A\) be a separable, unital \(C^*\)-algebra that is not scattered, let \(α \in \text{Aut}(A)\) have the tracial quasi-Rokhlin property, and suppose that \(A\) is α-simple. Then the restriction map \(T(C^*(\mathbb{Z}, A, α)) → T_α(A)\) is bijective.

**Proof.** We first verify that every trace on \(T(C^*(\mathbb{Z}, A, α))\) is α-invariant when restricted to \(A\), so that the restriction map indeed has codomain \(T_α(A)\). For any \(τ ∈ T(C^*(\mathbb{Z}, A, α))\) and any \(a ∈ A\), we have

\[
τ(α(a)) = τ(αuau^*) = τ(au^*u) = τ(a),
\]

and so this is in fact the case.

Next, we show that the restriction map is injective. Let \(τ ∈ T(C^*(\mathbb{Z}, A, α))\), let \(ε > 0\) be given, let \(a ∈ A\) be non-zero, let \(k ∈ \mathbb{N} \setminus \{0\}\), and let \(u ∈ C^*(\mathbb{Z}, A, α)\) be the canonical unitary. Set \(F = \{a\}\) and choose \(n ∈ \mathbb{N}\) such that \(n > k\) and

\[
\frac{1}{n} < \frac{ε^2}{16k^2(∥a^*a∥ + 1)}.
\]

Apply Lemma 2.5(1) with \(f(x) = \sqrt{x}\) to obtain \(δ_1(ε) > 0\) such that for all \(b, e ∈ A\) with 0 ≤ b, e ≤ 1 and \(∥be - eb∥ < δ_1(ε)\), we have

\[
∥b^{1/2}e - eb^{1/2}∥ < \frac{ε}{8n}.
\]

Similarly, apply Lemma 2.5(2) with the same \(f\) to obtain \(δ_2(ε) > 0\) such that for all \(b, e ∈ A\) with 0 ≤ b, e ≤ 1 and \(∥e - b∥ < δ_2(ε)\), we have

\[
∥e^{1/2} - b^{1/2}∥ < \frac{ε}{8nk(∥a∥ + 1)}.
\]

Define

\[
δ = \min\left\{\frac{1}{2n^3 + n^2 + 1}, δ_1(ε), δ_2(ε), \frac{ε^2}{4(τ(α^*a) + 1)}\right\} > 0
\]

and apply Proposition 2.9 with \(δ, F, n, \) and \(τ\) (identifying \(τ\) with its image in \(T_α(A)\) under the restriction map) to obtain \(c_0, \ldots, c_n ∈ A\) such that:
(1) $0 \leq c_j \leq 1$ for $0 \leq j \leq n$;
(2) $c_j c_k = 0$ for $0 \leq j, k \leq n$ and $j \neq k$;
(3) $\|\alpha(c_j) - c_{j+1}\| < \delta$ for $0 \leq j \leq n - 1$;
(4) $\|c_j a - ac_j\| < \delta$ for $0 \leq j \leq n$;
(5) with $c = \sum_{j=0}^{n} c_j$, we have $\tau(1 - c) < \delta$.

By the choice of $\delta$, and since automorphisms commute with continuous functional calculus, we further obtain

$$\|\alpha(c_j^{1/2}) - c_j^{1/2}\| < \frac{\varepsilon}{8nk(\|a\| + 1)}$$

for $0 \leq j \leq n - k$, and

$$\|c_j^{1/2} a - ac_j^{1/2}\| < \frac{\varepsilon}{8n}$$

for $0 \leq j \leq n$. It is easy to see that $0 \leq c \leq 1$ and hence also $0 \leq 1 - c \leq 1$. Then $(1 - c)^{1/2}$ is a well-defined positive element of $A$ that satisfies $1 - c \leq 1$.

Observing that that continuous functions $f_0, f_1 : [0, 1] \to [0, 1]$ given by $f_0(t) = t^2$ and $f_1(t) = t$ satisfy $f_0 \leq f_1$, continuous functional calculus gives $(1 - c)^2 \leq (1 - c)$. It follows that $\tau((1 - c)^2) \leq \tau(1 - c)$ and so the Cauchy-Schwarz inequality yields

$$|\tau(au^k(1 - c))|^2 \leq \tau((1 - c)^*(1 - c))\tau((au^k)(au^k)^*)$$
$$= \tau((1 - c)^2)\tau((au^k)^*(au^k))$$
$$= \tau((1 - c)^2)\tau(u^{-k}a^*au^k)$$
$$= \tau((1 - c)^2)\tau(a^*a)$$
$$\leq \tau(1 - c)\tau(a^*a)$$
$$< \delta \tau(a^*a).$$

Hence $|\tau(au^k(1 - c))| < \sqrt{\delta \tau(a^*a)} < \frac{1}{2} \varepsilon$.

Next, let $e, b \in A$ be positive and orthogonal. We compute

$$\|b^{1/2}e\|^2 = \|(b^{1/2}e)^*(b^{1/2}e)\| = \|ebe\| = 0,$$

which implies that $b^{1/2}e = 0$. This gives

$$\|e^{1/2}b^{1/2}\|^2 = \|(e^{1/2}b^{1/2})^*(e^{1/2}b^{1/2})\| = \|b^{1/2}eb^{1/2}\| = 0,$$

which implies that $e^{1/2}b^{1/2} = 0$ as well. In particular, for $0 \leq j \leq n - k$, we have $c_j^{1/2} c_{j+k}^{1/2} = 0$, and so $\tau(c_j^{1/2} au^k c_j^{1/2}) = \tau(au^k c_j^{1/2} c_{j+k}^{1/2}) = 0$. For $0 \leq j \leq n - k$, we also have the inequality

$$\|\alpha(c_j^{1/2}) - c_{j+k}^{1/2}\| \leq \sum_{i=0}^{k-1} \|\alpha^{k-i}(c_j^{1/2}) - \alpha^{k-i-1}(c_{j+i+1}^{1/2})\|$$
$$= \sum_{i=0}^{k-1} \|c_j^{1/2} - c_{j+i+1}^{1/2}\| < k\delta.$$
It follows that for $0 \leq j \leq n - k$,
\[
|\tau(au^kc_j)| = |\tau(au^kc_j^{1/2}c_j^{1/2})|
\]
\[
= |\tau(au^kc_j^{1/2}u^kc_j^{1/2})|
\]
\[
\leq |\tau(au^kc_j^{1/2}c_j^{1/2}) - \tau(ac_j^{1/2}u^kc_j^{1/2})| + |\tau(ac_j^{1/2}u^kc_j^{1/2})|
\]
\[
= |\tau(a(\alpha k(c_j^{1/2}) - c_j^{1/2})c_j^{1/2})| + |\tau((ac_j^{1/2} - e_j^{1/2})u^kc_j^{1/2})|
\]
\[
\leq \|\tau\| \|a(\alpha k(c_j^{1/2}) - c_j^{1/2})c_j^{1/2}\| + \|\tau\| \|ac_j^{1/2} - e_j^{1/2}\|u^kc_j^{1/2}\|
\]
\[
\leq \|a\| |\alpha k(c_j^{1/2}) - c_j^{1/2}| + |ac_j^{1/2} - e_j^{1/2}|\|
\]
\[
< \|a\| k \left(\frac{\varepsilon}{8nk(\|a\| + 1)} + \frac{\varepsilon}{8n}\right)
\]
\[
< \frac{\varepsilon}{4n}.
\]

For $0 \leq k \leq n - 1$ the $\alpha$-invariance of $\tau$ implies that
\[
|\tau(c_{j+1} - \tau(c_j)| = |\tau(c_{j+1}) - \tau(\alpha(c_j))| = |\tau(c_{j+1} - \alpha(c_j))| \leq \|c_{j+1} - \alpha(c_j))\| < \delta,
\]
and so we obtain
\[
\left|(n+1)\tau(c_0) - \sum_{j=0}^{n} \tau(c_j)\right| \leq \sum_{j=1}^{n} \left|\tau(c_j) - \tau(c_0)\right| \leq \sum_{j=1}^{n} \sum_{i=0}^{j-1} \left|\tau(c_j - i) - \tau(c_{j-i-1})\right|
\]
\[
< \sum_{j=1}^{n} j\delta \leq n^2 \delta.
\]

Now, since $0 \leq c \leq 1$, we have $\sum_{j=1}^{n} \tau(c_j) \leq 1$. Combining this with the previous result gives
\[
(n+1)\tau(c_0) < n^2 \delta + \sum_{j=0}^{n} \tau(c_j) \leq n^2 \delta + 1,
\]
and this implies that
\[
\tau(c_0) < \frac{n^2 \delta + 1}{n+1} < \frac{1}{\frac{n}{n+1}} < \frac{1}{n}.
\]

Further, since $|\tau(c_j) - \tau(c_0)| < n\delta$ for $1 \leq j \leq n$ (this follows by iterating one of the previous inequalities with the triangle inequality), we conclude that for $0 \leq j \leq n$, we have
\[
\tau(c_j) < n\delta + \tau(c_0) < n\delta + \frac{n^2 \delta + 1}{n+1} < \frac{(2n^2 + n)\delta + 1}{n+1} < \frac{\frac{1}{n} + 1}{n+1} = \frac{1}{n}.
\]

Now $0 \leq c_j \leq 1$ implies that $c_j^2 \leq c_j$ by the same functional calculus argument that was used to show $(1-c)^2 \leq 1-c$, and consequently $0 \leq \tau(c_j^2) \leq \tau(c_j)$. Applying Theorems 3.3.2 and 3.3.7 of [26] gives
\[
|\tau(au^kc_j)|^2 \leq \|\tau\| \|((au^kc_j)^*a^*(au^kc_j)) = \tau((u^kc_j)^*a^*(u^kc_j))
\]
\[
\leq \|a^*a\| \|((u^kc_j)^*(u^kc_j))
\]
\[
= \|a^*a\| \tau(c_j^2) \leq \|a^*a\| \tau(c_j) \leq \frac{\|a^*a\|}{n} \leq \frac{\varepsilon^2}{16k^2}.
\]
which implies $|\tau(a^k c_j)| < \frac{\varepsilon}{4k}$. Finally, we compute
\[
|\tau(a^k)| \leq |\tau(a^k (1-c))| + |\tau(a^k c)|
\]
\[
< \frac{1}{2} \varepsilon + \sum_{j=0}^{n-k} |\tau(a^k c_j)| + \sum_{j=n-k+1}^n |\tau(a^k c_j)|
\]
\[
< \frac{1}{2} \varepsilon + \sum_{j=0}^{n-k} \frac{\varepsilon}{4k} + \sum_{j=n-k+1}^n \frac{\varepsilon}{4k}
\]
\[
\leq \frac{1}{2} \varepsilon + \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon
\]
\[
= \varepsilon.
\]

Since $\varepsilon > 0$ was arbitrary, it follows that $\tau(a^k) = 0$. Now if $k \in \mathbb{Z}$ with $k < 0$, then the previous argument implies that $\tau(a^{* k} - k) = 0$, and therefore
\[
\tau(a^k) = \tau(u^k a) = \tau((a^{* k} u^{-k})^*) = \overline{\tau(a^{* k} u^{-k})} = 0.
\]

Thus for any $\tau \in T(C^*(\mathbb{Z}, A, \alpha))$, any non-zero $a \in A$, and any $k \in \mathbb{Z} \setminus \{0\}$, we have $\tau(a^k) = 0$. Let $E: C^*(\mathbb{Z}, A, \alpha) \to A$ be the standard conditional expectation. Then for any element $\sum_{j=-N}^N a_j u^j \in C_c(\mathbb{Z}, A, \alpha)$, we have
\[
\tau\left( \sum_{j=-N}^N a_j u^j \right) = \tau(a_0) = \tau\left( E\left( \sum_{j=-N}^N a_j u^j \right) \right),
\]
and so $\tau = \tau \circ E$ on a dense subset of $C^*(\mathbb{Z}, A, \alpha)$. This implies that the restriction map $T(C^*(\mathbb{Z}, A, \alpha)) \to T_\alpha(A)$ is injective.

For surjectivity, let $\tau \in T_\alpha(A)$, and let $E$ be the standard conditional expectation introduced above. We claim that $\overline{\tau} = \tau \circ E$ is a tracial state on $C^*(\mathbb{Z}, A, \alpha)$ that satisfies $\overline{\tau}|_A = \tau$. It is clear that $\overline{\tau}$ is a positive linear map since both $\tau$ and $E$ are positive, and we compute $\overline{\tau}(1) = \tau(E(1)) = \tau(1) = 1$. Let $a = a_0 u^m$ and $b = b_0 u^n$ for some $a_0, b_0 \in A$ and $m, n \in \mathbb{Z}$. Then we obtain the formulas
\[
ab = a_0 u^m b_0 u^n = a_0 \alpha^m(b_0) u^{m+n}
\]
and
\[
ba = b_0 u^n a_0 u^m = b_0 \alpha^n(a_0) u^{m+n}
\]
If $m \neq n$, then $E(ab) = 0 = E(ba)$, and consequently $\overline{\tau}(ab) = 0 = \overline{\tau}(ba)$. So assume that $m = -n$, which implies $E(ab) = a_0 \alpha^{-n}(b_0)$ and $E(ba) = b_0 \alpha^n(a_0)$. Using the $\alpha$-invariance of $\tau$ and the trace property, we obtain
\[
\tau(a_0 \alpha^{-n}(b_0)) = \tau(\alpha^{-n}(\alpha^n(a_0)) b_0) = \tau(a_0 \alpha^n(b_0)) = \tau(b_0 \alpha^n(a_0)),
\]
which implies that
\[
\overline{\tau}(ab) = \tau(E(ab)) = \tau(E(ba)) = \overline{\tau}(ba).
\]
Since the dense subset $C_c(\mathbb{Z}, A, \alpha)$ of $C^*(\mathbb{Z}, A, \alpha)$ is linearly spanned by elements of the form $a u^n$ for $a \in A$ and $n \in \mathbb{Z}$, it follows that $\overline{\tau}$ is a tracial state on $C^*(\mathbb{Z}, A, \alpha)$. Since $E(a) = a$ for all $a \in A$, we clearly have $\overline{\tau}|_A = \tau$, which completes the proof that the restriction map $T(C^*(\mathbb{Z}, A, \alpha)) \to T_\alpha(A)$ is surjective, and hence a bijection. \qed
3. Automorphisms of \(C(X,A)\) with the Tracial Quasi-Rokhlin Property

Our next goal is to study the automorphisms for a sort of noncommutative minimal dynamical system, where the commutative \(C^*\)-algebra \(C(X)\) is tensored with a simple, separable, unital, infinite-dimensional \(C^*\)-algebra \(A\). We prove that automorphisms of such algebras which take the action of a minimal homeomorphism when restricted to the central subalgebra \(C(X)\) satisfy the tracial quasi-Rokhlin property (under some additional technical assumptions). After further consideration of the structure of these algebras, it will follow that our results for crossed products by automorphisms with the tracial quasi-Rokhlin property in Section 2 will apply to their associated crossed product \(C^*\)-algebras.

Notation 3.1. Throughout, we let \(X\) be an infinite compact metrizable space, and let \(h: X \to X\) be a minimal homeomorphism. The corresponding minimal dynamical system \((X,h)\) will sometimes be denoted simply by \(X\), with the homeomorphism \(h\) understood. We denote by \(M_h(X)\) the space of \(h\)-invariant Borel probability measures on \(X\). Whenever necessary, it will be assumed that \(X\) is a metric space with metric \(d\). In this case, for \(x \in X\) and \(\varepsilon > 0\), we will denote the \(\varepsilon\)-ball centered at \(x\) by \(B(x,\varepsilon) = \{y \in X: d(x,y) < \varepsilon\}\).

We denote the boundary of a set \(A \subset X\) by \(\partial A\). In particular, if \(U \subset X\) is open then \(\partial U = \overline{U} \setminus U\), and if \(C \subset X\) is closed then \(\partial C = C \setminus \text{int}(C)\). We take \(A\) to be a simple, unital, separable, infinite-dimensional nuclear \(C^*\)-algebra. Form the algebra \(C(X,A)\), which we frequently identify with \(C(X) \otimes A\) in the canonical way. For \(f \in C(X)\) and \(a \in A\), we denote by \(f \otimes a\) the element of \(C(X,A)\) given by \((f \otimes a)(x) = f(x)a\) for all \(x \in X\), noting that these elements span \(C(X,A)\). We identify \(C(X)\) with the central subalgebra of \(C(X,A)\) given by \(\{f \otimes 1: f \in C(X)\}\).

We will eventually want it to be the case that \(C(X,A)\) has cancellation of projections and order on projections determined by traces. This will in fact occur for many reasonable choices of \(A\). The proof of the following proposition uses heavy machinery, and it is likely that its conclusion applies to a more general class of algebras \(A\).

Proposition 3.2. Let \((X,h)\) and \(A\) be as in Notation 3.1. Assume in addition that \(A\) has tracial rank zero and satisfies the Universal Coefficient Theorem. Then \(C(X,A)\) has cancellation of projections, and order on projections over \(C(X,A)\) is determined by traces.

Proof. Since \(A\) has tracial rank zero and satisfies the Universal Coefficient Theorem, Lin’s classification theory (see [18]) implies that \(A\) is a simple infinite-dimensional AH-algebra with no dimension growth. Write \(A \cong \lim_{\longrightarrow} A_n\), where the \(A_n\) are homogeneous algebras and the direct system has no dimension growth, and observe that
\[C(X,A) \cong C(X) \otimes A \cong C(X) \otimes \left(\lim_{\longrightarrow} A_n\right) \cong \lim_{\longrightarrow} C(X) \otimes A_n \cong \lim_{\longrightarrow} C(X,A_n).\]
Hence \(C(X,A)\) itself is a simple, infinite-dimensional inductive limit of homogeneous algebras with no dimension growth. Now Corollary 1.9 of [20] implies that...
the associated direct system has strict slow dimension growth. By Theorem 3.7 of [25], it follows that \( C(X, A) \) has cancellation and order on projections over \( C(X, A) \) is determined by traces.

Unfortunately, the assumption that \( A \) has real rank zero (in addition to the other standard hypotheses) is not sufficient to guarantee that \( C(X, A) \) has cancellation. We are thankful to Efren Ruiz for pointing out the following result.

**Proposition 3.3.** Let \((X, h)\) and \(A\) be as in Notation 3.1 and assume that \(A\) is purely infinite. Then \(C(X, A)\) does not have cancellation of projections.

**Proof.** Let \( p \in A \) be a non-zero projection. Since \(A\) is purely infinite, there is a unital embedding \( \iota : \mathcal{O}_2 \to (1 \otimes p)C(X, A)(1 \otimes p) \subset C(X, A) \).

Then \( 0 + \iota(1) \sim \iota(1) + \iota(1) \), but 0 is not Murray-von Neumann equivalent to \( \iota(1) \). \(\square\)

**Remark 3.4.** It is possible that \(A\) being infinite-dimensional, stably finite, and having real rank zero is sufficient to imply that \(C(X, A)\) has cancellation of projections and order on projections determined by traces. On the other hand, real rank zero is certainly not necessary. If \(X\) is connected and \(Z\) is the Jiang-Su algebra, then \(C(X, Z)\) has no nontrivial projections and so both properties hold.

**Proposition 3.5.** Let \((X, h)\) and \(A\) be as in Notation 3.1. For \(x \in X\), denote by \(\mu_x \in M(X)\) (where \(M(X)\) is the space of all Borel probability measures on \(X\)) the point-mass measure concentrated at \(x\). Then \(T(C(X, A))\) is the weak*-closed convex hull of the set

\[ \{\mu_x \otimes \tau : x \in X, \tau \in T(A)\} . \]

**Proof.** It suffices to prove that for any \(\varepsilon > 0\), any finite set \(\mathcal{F} \subset C(X, A)\), and any \(\tau \in T(C(X, A))\), there exist \(x_1, \ldots, x_n \in X\), \(\tau_1, \ldots, \tau_n \in T(A)\), and \(\lambda_1, \ldots, \lambda_n \in [0, 1]\) with \(\sum_{j=1}^{n} \lambda_j = 1\) such that

\[ \left| \tau(b) - \sum_{j=1}^{n} \lambda_j \mu_{x_j} \otimes \tau_j(b) \right| < \varepsilon \]

for all \(b \in \mathcal{F}\). Let \(\varepsilon > 0\), a finite set \(\mathcal{F} \subset C(X, A)\), and \(\tau \in T(C(X, A))\) be given. Choose \(\delta > 0\) such that whenever \(E \subset X\) with \(\text{diam}(E) < \delta\), we have \(\|b(x) - b(y)\| < \varepsilon\) for any \(b \in \mathcal{F}\) and all \(x, y \in E\) (this can be done since \(\mathcal{F}\) is finite). Choose an open cover \(\{E_j\}_{j=1}^{n}\) for \(X\) such that \(\text{diam}(E_j) < \delta\) for \(1 \leq j \leq n\). Choose a partition of unity \(\{g_j\}_{j=1}^{n}\) subordinate to this cover. For \(1 \leq j \leq E_j\), choose \(x_j \in E_j\). By the choice of the sets \(E_j\) we then have

\[ \left\| b - \sum_{j=1}^{n} g_j \otimes b(x_j) \right\| < \varepsilon \]

for each \(b \in \mathcal{F}\). For \(1 \leq j \leq n\) define \(\sigma_j(a) = \tau(g_j \otimes a)\). Then each \(\sigma_j\) is clearly a positive linear functional on \(A\) that satisfies the trace property, and so there is a
\[ \lambda_j \geq 0 \text{ and } \tau_j \in T(A) \text{ such that } \sigma_j = \lambda_j \tau_j. \text{ Now for any } b \in F, \text{ we have} \]
\[
\left| \tau(b) - \sum_{j=1}^{n} \lambda_j \mu x_j \otimes \tau_j(b) \right| = \left| \tau(b) - \sum_{j=1}^{n} \lambda_j \tau_j(b(x_j)) \right| \\
= \left| \tau(b) - \sum_{j=1}^{n} \tau(g_j \otimes b(x_j)) \right| \leq \left\| b - \sum_{j=1}^{n} g_j \otimes b(x_j) \right\| < \varepsilon, \\
\]
which completes the proof.

\[ \square \]

**Lemma 3.6.** Let \((X, h)\) and \(A\) be as in Notation 3.1. Let \(\alpha : X \rightarrow Aut(A)\) (where \(\alpha(x)\) will be denoted \(\alpha_x\)) be a map which is continuous in the strong operator topology. (In other words, for each \(a \in A\) the mapping \(x \rightarrow \alpha_x(a)\) is norm-continuous.) Then the map \(\alpha^{-1} : X \rightarrow Aut(A)\) given by \(\alpha^{-1}(x) = \alpha_x^{-1}\) is continuous in the strong operator topology.

**Proof.** This is a straightforward calculation. \(\square\)

**Proposition 3.7.** Let \((X, h)\) and \(A\) be as in Notation 3.1. Let \(\alpha : X \rightarrow Aut(A)\) be a map which is continuous in the strong operator topology. Define a map \(\beta : C(X, A) \rightarrow C(X, A)\) by \(\beta(f)(x) = \alpha_x(f \circ h^{-1}(x))\) for each \(x \in X\). Then \(\beta \in Aut(C(X, A))\).

**Proof.** We first verify that \(\beta(f)\) is continuous for \(f \in C(X, A)\). Let \(\varepsilon > 0\) be given, let \(f \in C(X, A)\), and let \(x \in X\). Since \(f \circ h^{-1}(x) \in A\) and \(\alpha\) is continuous in the strong operator topology, there exists \(\delta_1 > 0\) such that \(d(x, y) < \delta_1\) implies \(|\alpha_x(f \circ h^{-1}(x)) - \alpha_y(f \circ h^{-1}(x))| < \varepsilon/2\). Since \(f\) is continuous, there exists \(\delta_2 > 0\) such that \(d(x, y) < \varepsilon/2\) implies \(\|f(x) - f(y)\| < \varepsilon/2\). Also, since \(h\) is a homeomorphism, there is a \(\delta_3 > 0\) such that \(d(x, y) < \delta_3\) implies \(h^{-1}(x), h^{-1}(y)\) is \(\delta_3\). Now let \(\delta = \min\{\delta_1, \delta_2, \delta_3\}\). Then for all \(y \in X\) with \(d(x, y) < \delta\), we have
\[
\|\beta(f)(x) - \beta(f)(y)\| = \|\alpha_x(f \circ h^{-1}(x)) - \alpha_y(f \circ h^{-1}(y))\| \\
< \frac{\varepsilon}{2} + \|\alpha_x\| \|f \circ h^{-1}(x) - f \circ h^{-1}(y)\| \\
< \varepsilon.
\]
Thus \(\beta(f)\) is continuous at \(x\). Since this holds for any \(x \in X\), it follows that \(\beta(f) \in C(X, A)\).

Since the operations on \(C(X, A)\) are given pointwise, each \(\alpha_x\) is an automorphism on \(A\) for \(x \in X\), and the map \(f \mapsto f \circ h^{-1}\) is an automorphism of \(C(X)\), it follows easily that for all \(f, g \in C(X, A)\), we have \(\beta(f + g) = \beta(f) + \beta(g)\), \(\beta(fg) = \beta(f)\beta(g)\), and \(\beta(f^*) = \beta(f)^*\). This implies that \(\beta\) is a \(*\)-homomorphism.

Next suppose that \(f \in ker(\beta)\). Then \(\beta(f)(x) = 0\) for all \(x \in X\), and so \(\alpha_x(f \circ h^{-1}(x)) = 0\) for all \(x \in X\). Since each \(\alpha_x\) is an automorphism of \(A\), this implies that \(f \circ h^{-1}(x) = 0\) for each \(x \in X\), and hence \(f \circ h^{-1} = 0\). As \(h\) is a homeomorphism, it follows that \(f = 0\). Now let \(f \in C(X, A)\). Define \(g : X \rightarrow A\) by \(g(x) = \alpha_x^{-1}(f \circ h(x))\). That \(g\) is continuous follows from the same argument that shows \(\beta\) is continuous, using Lemma 3.6. Now for each \(x \in X\), \(\beta(g)(x) = \alpha_x^{-1}((f \circ h) \circ h^{-1}(x))) = f(x)\), and so \(\beta(g) = f\). It follows that \(\beta\) is bijective, and hence \(\beta \in Aut(C(X, A))\). \(\square\)

**Proposition 3.8.** Let \((X, h)\) and \(A\) be as in Notation 3.1. Let \(\alpha : X \rightarrow Aut(A)\) be continuous in the strong operator topology. For \(k \in \mathbb{Z} \setminus \{0\}\), we define \(\alpha^{(k)} : X \rightarrow Aut(A)\) by \(\alpha^{(k)}(x) = \alpha_{x \circ \alpha_{h^{-1}(x)} \circ \cdots \circ \alpha_{h^{-1}(k-1)}}(x)\) if \(k \geq 1\) and \(\alpha^{(k)}(x) = \alpha_{h(x)} \circ \cdots \circ \alpha_{h^2(x)}\) if \(k \leq -1\).
α_{h^i(x)} if \( k < 0 \), henceforth denoting \( \alpha^{(k)}(x) \) by \( \alpha^{(k)}_x \). Then \( \alpha^{(k)} \) is continuous in the strong operator topology. Moreover, the map \( \alpha^{(k)}_x : X \to \text{Aut}(A) \), defined by \( \alpha^{-1}_x = \alpha^{-1}_{h^{-(k-1)}(x)} \circ \cdots \circ \alpha^{-1}_{h^{-1}(x)} \circ \alpha^{-1}_x \) for \( k \geq 1 \) and \( \alpha^{-1}_x = \alpha^{-1}_{h^{k+1}(x)} \circ \cdots \circ \alpha^{-1}_x \) for \( k < 0 \), is continuous in the strong operator topology and satisfies \( \alpha^{-1}_x = (\alpha^{(k)}_x)^{-1} \) for all \( x \in X \).

**Proof.** First, assume that \( k \geq 1 \). We proceed by induction on \( k \). When \( k = 1 \) the map \( \alpha^{(1)}_x : X \to \text{Aut}(A) \) is simply \( \alpha_x \), which is continuous in the strong operator topology by assumption. Suppose that \( \alpha^{(k)}_x \) is continuous in the strong operator topology for some \( k \geq 1 \). Let \( \varepsilon > 0 \) be given, let \( a \in A \), and let \( x \in X \). Then there is a \( \delta_1 > 0 \) such that \( d(x,y) < \delta_1 \) implies \( \| \alpha_x^{(k)}(a) - \alpha_y^{(k)}(a) \| < \frac{1}{2} \varepsilon \). Further, with \( b = \alpha_x^{(k)}(a) \), the strong operator continuity of \( \alpha = \alpha^{(1)}_x \) gives a \( \delta_2 > 0 \) such that \( d(x,y) < \delta_2 \) implies \( \| \alpha_x(b) - \alpha_y(b) \| < \frac{1}{2} \varepsilon \). Let \( \delta = \min \{ \delta_1, \delta_2 \} \). Then \( d(x,y) < \delta \) implies that

\[
\left\| \alpha_x^{(k+1)}(a) - \alpha_y^{(k+1)}(a) \right\| \leq \left\| \alpha_x^{(k+1)}(a) - \alpha_y \circ \alpha_x^{(k)}(a) \right\| + \left\| \alpha_y \circ \alpha_x^{(k)}(a) - \alpha_y^{(k+1)}(a) \right\|
\]

\[
= \left\| \alpha_x(a_x^{(k)}(a)) - \alpha_y(a_x^{(k)}(a)) \right\| + \left\| \alpha_y(a_x^{(k)}(a)) - \alpha_y^{(k)}(a) \right\|
\]

\[
\leq \left\| \alpha_x(b) - \alpha_y(b) \right\| + \left\| \alpha_x^{(k)}(a) - \alpha_y^{(k)}(a) \right\|
\]

\[
< \frac{1}{2} \varepsilon + \frac{1}{2} = \varepsilon.
\]

It follows that \( \alpha^{(k+1)}_x \) is continuous at \( x \) in the strong operator topology. Since this holds for all \( x \in X \), \( \alpha^{(k+1)}_x \) is continuous in the strong operator topology. By induction, \( \alpha^{(k)}_x \) is continuous in the strong operator topology for all \( k \geq 1 \). To obtain continuity for all \( k \in \mathbb{Z} \setminus \{ 0 \} \), note that \( g = h^{-1} \) is also a homeomorphism, and for any \( k \geq 1 \) we have

\[
\alpha^{-1}_x = \alpha_{h^k} \circ \cdots \circ \alpha_{h^1} = \alpha_{h^{-1}}(x) \circ \cdots \circ \alpha_{h^{-k}}(x).
\]

Applying the above argument to the map \( \gamma^{(k)} : X \to \text{Aut}(A) \) given by \( \gamma^{(k)}(x) = \alpha_x \circ \alpha_{g^{-1}(x)} \circ \cdots \circ \alpha_{g^{-k}(x)} \) shows that \( \gamma^{(k)}_x = \alpha_x \circ \alpha_{g^{-1}}(x) \) is continuous at \( x \) in the strong operator topology for \( k \geq 1 \). Since \( \alpha_{g^{-1}} \) is also continuous at \( x \) in the strong operator topology, so is \( \alpha_x^{(k)} = \alpha_{g^{-1}} \circ \gamma^{(k)}_x \) Thus \( \alpha^{(k)}_x \) is continuous in the strong operator topology for all \( k \in \mathbb{Z} \).

Finally, \( \alpha^{-1}_x \) is continuous in the strong operator topology by Lemma 3.6, and so an argument analogous to the one above, with \( \alpha^{-1} \) in place of \( \alpha \), shows that \( \alpha^{-1}_x \) is continuous in the strong operator topology for all \( k \in \mathbb{Z} \). Further, it is easy to see that for any \( x \in X \), \( \alpha_x^{(k)} \circ \alpha_x^{(k)} = \text{id}_A = \alpha_x^{(k)} \circ \alpha_x^{(k)} \). \( \square \)

**Corollary 3.9.** Let \((X, h)\) and \( A \) be as in Notation 3.1 and let \( \beta \in \text{Aut}(C(X, A)) \) be the automorphism of Proposition 3.7. For \( n \in \mathbb{Z} \setminus \{ 0 \} \), the automorphism \( \beta^n \in \text{Aut}(C(X, A)) \) is given explicitly by \( \beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x)) \) for all \( x \in X \).

**Proof.** We consider first the case where \( n \geq 1 \), and proceed by induction on \( n \). Observe that for all \( x \in X \), we have

\[
\beta^1(f)(x) = \beta(f)(x) = \alpha_x(f \circ h^{-1}(x)) = \alpha_x^{(1)}(f \circ h^{-1}(x))
\]
and so the base case holds. Next, suppose that \( \beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x)) \) for some \( n \geq 1 \). Then for all \( x \in X \), we compute

\[
\begin{align*}
\beta^{n+1}(f)(x) &= \beta^n(\beta(f))(x) \\
&= \alpha_x^{(n)}((\beta(f)) \circ h^{-n}(x)) \\
&= \alpha_x^{(n)}(\beta(f)(h^{-n}(x))) \\
&= \alpha_x^{(n)}(\alpha_{h^{-n}(x)}(f \circ h^{-1}(h^{-n}(x)))) \\
&= \alpha_x^{(n)} \circ \alpha_{h^{-n}(x)}(f \circ h^{-1-n}(x)) \\
&= \alpha_x^{(n+1)}(f \circ h^{-n+1}(x)).
\end{align*}
\]

It follows that the result holds for all \( n \geq 1 \). To extend this result to all \( n \in \mathbb{Z} \setminus \{0\} \), we first observe that \( \psi \in \text{Aut}(C(X, A)) \), given by \( \psi(f)(x) = \alpha_{h(x)}^{-1}(f \circ h(x)) \), satisfies \( \psi \circ \beta(f)(x) = f(x) = \beta \circ \psi(f)(x) \) for all \( f \in C(X, A) \) and \( x \in X \), and hence \( \psi \circ \beta = \text{id}_{C(X, A)} = \beta \circ \psi \). This gives \( \psi = \beta^{-1} \). Further, an induction argument entirely analogous to the one above shows that for \( k \geq 1 \), \( \psi^k(f)(x) = \alpha_x^{(-k)}(f \circ h^k(x)) \) for all \( f \in C(X, A) \) and \( x \in X \). But \( \psi = \beta^{-1} \) implies that \( \beta^{-k}(f)(x) = \alpha_x^{(k)}(f \circ h^k(x)) \) for \( k \geq 1 \). Letting \( n = -k \), it follows that \( \beta^n(f)(x) = \alpha_x^{(n)}(f \circ h^{-n}(x)) \) for \( n < 0 \).}

\[ \square \]

**Lemma 3.10.** Let \((X, h)\) and \(A\) be as in Notation 3.1 and let \( \alpha : X \rightarrow \text{Aut}(A) \) be continuous in the strong operator topology. Assume in addition that \( C(X, A) \) has order on projections determined by traces. Let \( p_0 \in A \) be a non-zero projection, let \( k \in \mathbb{Z} \), and let \( \alpha^{(k)} \) be as in Proposition 3.8. Then for any projection \( p \in A \) with the property that

\[
\inf_{\tau \in T(A)} \tau(p_0) - \sup_{\tau \in T(A)} \tau(p) > 0,
\]

the function \( q_{p,k} : X \rightarrow A \) given by \( q_{p,k}(x) = \alpha_x^{(k)}(p) \) is a projection in \( C(X, A) \) that satisfies \( q_{p,k} \precsim 1 \otimes p_0 \).

**Proof.** It is clear that \( q_{p,k} \) is continuous, that \( q_{p,k}^* = q_{p,k} \), and that \( q_{p,k}^2 = q_{p,k} \). Therefore, \( q_{p,k} \) is a projection in \( C(X, A) \). By Proposition 3.8, in order to show that \( \lambda(q_{p,k}) < \lambda(1 \otimes p_0) \) for all \( \lambda \in T(C(X, A)) \), it suffices to show that \( \lambda(q_{p,k}) < \lambda(1 \otimes p_0) \) for all \( \lambda \) of the form \( \lambda = \mu \otimes \tau \), where \( \mu \in M(X) \) and \( \tau \in T(A) \). We first observe that for any \( x \in X \), \( \alpha_x^{(k)} \in \text{Aut}(A) \) implies that \( \tau \circ \alpha_x^{(k)} \in T(A) \). Then for any \( \mu \in M(X) \), any \( \tau \in T(A) \), and \( \lambda = \mu \otimes \tau \), we have

\[
\begin{align*}
\lambda(q_{p,k}) &= \int_X \tau(q_{p,k}(x)) \ d\mu = \int_X \tau(\alpha_x^{(k)}(p)) \ d\mu \\
&\leq \sup_{\tau \in T(A)} \int_X \sigma(p) \ d\mu \\
&< \inf_{\tau \in T(A)} \int_X \sigma(p_0) \ d\mu \\
&\leq \int_X \tau(p_0) \ d\mu = \lambda(1 \otimes p_0).
\end{align*}
\]

As mentioned above, this is sufficient to imply that \( \lambda(q_{p,k}) < \lambda(1 \otimes p_0) \) for all \( \lambda \in C(X, A) \). Since order on projections over \( C(X, A) \) is determined by traces, we conclude that \( q_{p} \precsim 1 \otimes p_0 \). \( \square \)
Lemma 3.11. Let \((X,h)\) and \(A\) be as in Notation 3.1 and assume in addition that \(C(X,A)\) has cancellation of projections. Let \(p,q \in C(X,A)\) be projections with \(p \preceq q\). Then there is a unitary \(w \in C(X,A)\) such that \(wpw^* \leq q\).

Proof. Since \(C(X,A)\) has cancellation, there exists a projection \(e \in C(X,A)\) such that \(e \leq q\) and partial isometries \(s,t \in C(X,A)\) such that \(s^*s = p, ss^* = e, t^*t = 1 - p\), and \(tt^* = 1 - e\). Define \(w = s + t\). It is straightforward to check that \(s^*t = st^* = ts^* = t^*s = 0\), from which it follows that \(w^*w = (s^* + t^*)(s + t) = s^*s + t^*t = p + (1 - p) = 1\) and \(ww^* = (s + t)(s^* + t^*) = ss^* + tt^* = e + (1 - e) = 1\), so \(w\) is unitary. Moreover,

\[
wpw^* = (s + t)p(s^* + t^*) = sps^* + tpt^* + spt^* + tps^* = ss^*ss^* + t(1 - t^*)t^* + ss^*st^* + t(1 - t^*)t^* = e^2 + tt^* - tt^*t^* = e + (1 - e) - (1 - e)^2 = e,
\]

as required. \(\Box\)

Definition 3.12. Let \((X,h)\) and \(A\) be as in Notation 3.1. For an open set \(V \subset X\) and a projection \(p \in A\), the hereditary subalgebra of \(C(X,A)\) determined by \(V\) and \(p\), denoted by \(\text{Her}(V,p)\), is defined to be the hereditary subalgebra of \(C(X,A)\) generated by all functions \(f \in C(X,A)\) such that \(\text{supp}(f) \subset V\) and \(f \leq 1 \otimes p\).

We wish to show that given a hereditary subalgebra determined by some non-zero projection \(p\) and non-empty open set \(V\), any sufficiently small positive central element \(f \otimes 1\) of \(C(X,A)\) can be decomposed into positive elements which, upon translating by \(\beta\) and conjugating by unitaries, are mutually orthogonal elements of the hereditary subalgebra. To do this requires us to be able to carry out a similar decomposition at the level of the space \(X\). Definition 3.14, which first appeared in [5], gives a property which allows such a decomposition. In order to state it, and for some results which come later, we need definitions describing certain smallness properties for closed sets.

Definition 3.13. Let \((X,h)\) be as in Notation 3.1 and let \(F \subset X\) be closed.

1. \(F\) is universally null if \(\mu(F) = 0\) for all \(\mu \in M_h(X)\).
2. \(F\) is topologically \(h\)-small if there is some \(m \in \mathbb{Z}_+\) such that whenever \(d(0), d(1), \ldots, d(m)\) are \(m + 1\) distinct elements of \(\mathbb{Z}\), then \(h^{d(0)}(F) \cap h^{d(1)}(F) \cap \cdots \cap h^{d(m)}(F) = \emptyset\).

In Corollary 2.11 of [5], it is shown that a topologically \(h\)-small set is universally null. This fact will be used repeatedly in the proof of Theorem 3.19.

Definition 3.14. Let \((X,h)\) be as in Notation 3.1. We say \((X,h)\) has the dynamic comparison property if whenever \(U \subset X\) is open and \(C \subset X\) is closed with \(\partial C, \partial U\) universally null and \(\mu(C) < \mu(U)\) for every \(\mu \in M_h(X)\), then there are \(M \in \mathbb{N}\), continuous functions \(f_j : X \to [0,1]\) for \(0 \leq j \leq M\), and \(d(0), \ldots, d(M) \in \mathbb{Z}\) such that \(\sum_{j=0}^{M} f_j = 1\) on \(C\), and such that the sets \(\text{supp}(f_j \circ h^{-d(j)})\) are pairwise disjoint subsets of \(U\) for \(0 \leq j \leq M\).
In [5], it is shown this property holds for a large class of minimal dynamical systems \((X, h)\) that includes all finite-dimensional examples. It will also be used in the proof of our main result.

**Proposition 3.15.** Let \((X, h)\) and \(A\) be as in Notation 3.1. Let \(\beta \in \text{Aut}(C(X, A))\) be the automorphism of Proposition 3.7. Assume that \((X, h)\) has the dynamic comparison property and that \(A\) is a non-elementary \(C^*\)-algebra with real rank zero and order on projections determined by traces. Then for every non-zero projection \(p_0 \in A\) and every non-empty open set \(V \subset X\), there exist \(M \in \mathbb{N}\) and \(\varepsilon > 0\) such that whenever \(g_0 \in C(X)\) is positive and satisfies \(\mu(\text{supp}(g_0)) < \varepsilon\) for all \(\mu \in M_h(X)\), then there exist for \(0 \leq k \leq M\) positive elements \(a_k \in C(X, A)\), unitaries \(w_k \in C(X, A)\), and \(r(k) \in \mathbb{Z}\) such that:

1. \(\sum_{k=0}^{M} a_k \geq g_0 \otimes 1;\)
2. the elements \(\beta^r(k)(a_k)\) are mutually orthogonal, and \(\text{supp}(\beta^r(k)(a_k)) \subset V\) for each \(k;\)
3. with \(b_k = w_k \beta^r(k)(a_k) w_k^*\), the \(b_k\) are mutually orthogonal positive elements in \(\text{Her}(V, p_0)\).

**Proof.** Set \(\delta = \inf_{\tau \in T(A)} \tau(p_0) > 0\), and choose \(N \in \mathbb{N}\) such that \(N > 1\) and \(1/N < \delta/2\). Then by Theorem 1.1 of [34] there exist \(2^N + 1\) mutually orthogonal projections \(q_0, \ldots, q_{2^N}\) such that \(q_0 \lesssim q_1 \sim \cdots \sim q_{2^N}\) and \(\sum_{j=0}^{2^N} q_j = 1\). We immediately obtain \(\tau(q_1) = \cdots = \tau(q_{2^N})\) for all \(\tau \in T(A)\). Then for \(1 \leq j \leq 2^N\) and each \(\tau \in T(A)\), we have

\[
1 = \tau(1) = \sum_{i=0}^{2^N} \tau(q_i) \geq \sum_{i=1}^{2^N} \tau(q_i) = 2^N \tau(q_j),
\]

and so \(\tau(q_j) \leq 1/2^N\). This gives \(\tau(q_j) < 1/N < \delta/2\) for \(1 \leq j \leq 2^N\), and hence that

\[
\inf_{\tau \in T(A)} \tau(p_0) - \sup_{\tau \in T(A)} (q_j) > \delta - \delta/2 = \delta/2 > 0
\]

for all \(\tau \in T(A)\). In particular, we clearly have \(\tau(q_j) < \tau(p_0)\) for \(1 \leq j \leq 2^N\) and for all \(\tau \in T(A)\), and since the order on projections in \(A\) is determined by traces, we conclude that \(q_j \lesssim p_0\) for \(1 \leq j \leq 2^N\). Since \(q_0 \lesssim q_1\), we actually obtain \(q_j \lesssim p_0\) for \(0 \leq j \leq 2^N\).

Set \(J = 2^N\), and let \(\sigma = \inf_{\mu \in M_h(X)} \mu(V) > 0\). Choose \(J\) distinct points \(x_0, \ldots, x_J \in V\) and for each \(j\) consider the nested sequence of neighborhoods \((B(x_j, 1/k))_{k=1}^{\infty}\). Choose \(K_{j+1} \in \mathbb{N}\) so large that the sets \(B(x_j, 1/K_{j+1})\) are pairwise disjoint subsets of \(V\) for \(0 \leq j \leq J\).

For each \(0 \leq j \leq J\), apply the same argument as in the proof of Lemma 1.4 of [5] to find a \(K_j \in \mathbb{N}\) so large that \(\mu(B(x_j, 1/K_j)) < \sigma/(J + 1)\) for every \(\mu \in M_h(X)\). Let \(K = \max\{K_0, \ldots, K_{j+1}\}\), and for \(0 \leq j \leq 2^N\) set \(V_j = B(x_j, 1/K)\). Then for \(0 \leq j \leq J\), we have \(\mu(V_j) < \sigma/(J + 1)\) for every \(\mu \in M_h(X)\), and the sets \(V_j\) are pairwise disjoint subsets of \(V\). Using Proposition 3.9 of [5], for \(0 \leq j \leq J\) choose open sets \(W_j\) such that \(x_j \in W_j \subset \overline{W}_j \subset V_j\) with \(\partial W_j\) topologically \(h\)-small. Now set

\[
\varepsilon = \min_{0 \leq j \leq M} \inf_{\mu \in M_h(X)} \mu(W_j) > 0.
\]

Choose an open set \(E \subset X\) such that \(\mu(E) < \varepsilon\) for all \(\mu \in M_h(X)\), and let \(g_0 \in C(X)\) be positive such that \(C_0 = \text{supp}(g_0) \subset E\). Apply Proposition 3.9 of [5]
to obtain a closed set $C$ with $C_0 \subseteq C \subseteq E$ and $\partial C$ topologically $h$-small. Then for $0 \leq j \leq J$, we have $\mu(C) < \varepsilon < \mu(W_j)$ for all $\mu \in M_b(X)$, with the sets $\partial C$ and $\partial W_j$ universally null. By assumption, $(X, h)$ has the dynamic comparison property. Thus for each $0 \leq j \leq J$ there exist $M_j \in \mathbb{N}$, continuous functions $f_{j,i}: X \to [0, 1]$ for $0 \leq i \leq M_j$, and $r_j(i) \in \mathbb{Z}$ for $0 \leq i \leq M_j$, such that $\sum_{i=0}^{M_j} f_{j,i} = 1$ on $C$ (and hence also on $C_0 = \text{supp}(g_0)$) and such that the sets $\text{supp}(f_{j,i} \circ h^{-r_j(i)})$ are pairwise disjoint subsets of $W_j \subseteq V_j$ for $0 \leq i \leq M_j$.

For $0 \leq j \leq J$ and $0 \leq i \leq M_j$, define $q_{j,i}: X \to A$ by $q_{j,i}(x) = \alpha_x^{(r_j(i))}(q_j)$. From the inequality computed earlier, we have

$$\inf_{\tau \in T(A)} \tau(p_0) - \sup_{\tau \in T(A)} \tau(q_{j,i}) \geq \delta/2 > 0.$$ 

Then by Lemma 3.10, each $q_{j,i}$ is an element of $C(X, A)$ and $q_{j,i} \preceq 1 \oplus p_0$. Hence by Lemma 3.11, there exist unitaries $w_{j,i} \in C(X, A)$ for $0 \leq j \leq J$, $0 \leq i \leq M_j$ such that $w_{j,i}q_{j,i}w_{j,i}^* \preceq 1 \oplus p_0$. Now for $0 \leq j \leq J$ and $0 \leq i \leq M_j$ set $a_{j,i} = f_{j,i} \otimes q_j$ and $b_{j,i} = w_{j,i}^*b_{j,i}(a_{j,i})w_{j,i}^*$.

Let $x \in X$. If $x \notin C$, then $(g_0 \otimes 1)(x) = 0 \leq \sum_{j=0}^{J} \sum_{i=0}^{M_j} a_{j,i}(x)$. If $x \in C$, then we compute

$$\sum_{j=0}^{J} \sum_{i=0}^{M_j} a_{j,i}(x) = \sum_{j=0}^{J} \sum_{i=0}^{M_j} f_{j,i}(x)q_j = \sum_{j=0}^{J} q_j \left( \sum_{i=0}^{M_j} f_{j,i}(x) \right) = \sum_{j=0}^{J} q_j = 1.$$ 

It follows that $g_0 \otimes 1 \leq \sum_{j=0}^{J} \sum_{i=0}^{M_j} a_{j,i}$. Next, for any $x \in X$, we have

$$\beta^{r_j(i)}(a_{j,i})(x) = \alpha_x^{(r_j(i))}(f_{j,i} \circ h^{-r_j(i)}(x)q_j) = (f_{j,i} \circ h^{-r_j(i)}(x)) \alpha_x^{(r_j(i))}(q_j) = (f_{j,i} \circ h^{-r_j(i)}(x))q_{j,i}(x).$$

This gives $\text{supp}(\beta^{r_j(i)}(a_{j,i})) \subseteq \text{supp}(f_{j,i} \circ h^{-r_j(i)}) \subseteq V$ which implies the sets $\text{supp}(\beta^{r_j(i)}(a_{j,i}))$ are pairwise disjoint, and hence that the elements $\beta^{r_j(i)}(a_{j,i})$ are mutually orthogonal. Since $\text{supp}(b_{j,i}) \subseteq \text{supp}(\beta^{r_j(i)}(a_{j,i}))$, it follows immediately that the $b_{j,i}$ are also mutually orthogonal. Moreover, as $0 \leq f_{j,i} \leq 1$ and $w_{j,i}q_{j,i}w_{j,i}^* \preceq p_0$, it follows that $0 \leq b_{j,i} \leq 1 \oplus p_0$. Therefore, the $b_{j,i}$ are mutually orthogonal positive elements in $\text{Her}(V, p_0)$. Now simply order the $a_{j,i}, w_{j,i}, d_j(i)$, and $b_{j,i}$ as $a_k, w_k, d(k)$, and $b_k$ for $0 \leq k \leq M$, where $M + 1 = \sum_{j=0}^{J} M_j$. \hfill $\square$

**Lemma 3.16.** Let $E \subseteq C$ be open, let $f: E \to C$ be continuous, let $A$ be a unital $C^*$-algebra, and set $Q = \{ b \in A : b$ is normal with $\text{sp}(b) \subseteq E \}$. Then $\varphi: Q \to A$ given by $\varphi(b) = f(b)$ is norm-continuous.

*Proof.* This is easily adapted from Lemma 2.5.11 of [15]. \hfill $\square$

**Proposition 3.17.** Let $(X, h)$ and $A$ be as in Notation 3.1. Let $g \in C(X, A)$ be a non-zero positive element with $\|g\| = 1$. Then there is an open set $V \subseteq \text{supp}(g)$, a non-zero projection $p_0 \in A$, and a unitary $w \in C(X, A)$ such that $wfw^* \in gC(X, A)g$ for all $f \in \text{Her}(V, p_0)$.

*Proof.* Let $\varepsilon > 0$ be given, and assume that $\varepsilon < 1$. Since $\|g\| = 1$ and $X$ is compact, there exists $x_0 \in \text{supp}(g)$ such that $\|g(x_0)\| = 1$. Let $a = g(x_0)$ (note that $a \geq 0$
since $g$ is positive) and define continuous functions $k_1, k_2 : [0, 1] \to [0, 1]$ by
\[
k_1(t) = \begin{cases} \frac{32}{16t} & 0 \leq t \leq 1 - \frac{\varepsilon}{32} \\ 1 & 1 - \frac{\varepsilon}{32} < t \leq 1 \end{cases}
\]
and
\[
k_2(t) = \begin{cases} 0 & 0 \leq t \leq 1 - \frac{\varepsilon}{32} \\ \frac{64}{\varepsilon} (t - 1) + 1 & 1 - \frac{\varepsilon}{64} < t \leq 1 \end{cases}
\]
Setting $a_1 = k_1(a)$ and $a_2 = k_2(a)$, we observe that $a_2a_1 = a_2$ and
\[
\|a - a_1\| = \sup_{t \in [0, \|a\|]} |t - k_1(t)| < \frac{1}{16}\varepsilon.
\]
This gives $\|a_2a - a_2\| = \|a_2a - a_2a_1\| \leq \|a - a_1\| < \frac{1}{16}\varepsilon$. Since $A$ has real rank zero, there is a non-zero projection $q \in \overline{a_2Aa_2}$. Then $a_2a_1 = a_2$ implies that $qa_1 = q$. We thus obtain $\|qa - q\| = \|qa - qa_1\| \leq \|a - a_1\| < \frac{1}{16}\varepsilon$, and similarly $\|aq - q\| < \frac{1}{16}\varepsilon$. Now choose a neighborhood $U$ of $x_0$ such that $\|g(x) - g(x_0)\| < \frac{1}{4}\varepsilon$ for all $x \in U$. Using the compactness of $X$, choose an open set $W \subset U$ with $\overline{W} \subset U$, and set $K = \overline{W}$. Then for all $x \in K$,
\[
\|gg(x) - q\| \leq \|gg(x) - qg(x_0)\| + \|qg(x_0) - q\|
\leq \|g(x) - g(x_0)\| + \|qa - q\|
< \frac{1}{8}\varepsilon + \frac{1}{8}\varepsilon
= \frac{1}{4}\varepsilon.
\]
So for all $x \in K$, we have
\[
\|g(x)gg(x) - q\| \leq \|g(x)gg(x) - g(x)q\| + \|g(x)q - q\|
\leq \|g(x)\| \|gg(x) - q\| + \|g(x)q - q\|
< \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon
= \frac{1}{2}\varepsilon.
\]
Set $E = (-\infty, 1/2) \cup (1/2, \infty)$, and $Q = \{b \in A : \|b\| = \chi(1/2, \infty)\}$. Apply Lemma 2.10 to obtain a continuous function $\varphi : Q \to A$ such that $\varphi(b) = \chi(1/2, \infty)(b)$ for all $b \in Q$. Next observe that for all $x \in K$, $\|g(x)gg(x) - q\| < \frac{1}{2}\varepsilon < \frac{1}{2}$ implies that $g(x)gg(x) \in Q$. Thus we may define a function $\psi : K \to Q$ by $\psi(x) = g(x)gg(x)$. Further, for $x, y \in K$ we have
\[
\|\psi(x) - \psi(y)\| = \|g(x)gg(x) - g(y)gg(y)\|
\leq \|g(x)gg(x) - q\| + \|q - g(y)gg(y)\|
< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon
= \varepsilon,
\]
which implies that $\psi$ is continuous on $K$. Now setting $p^{(0)} = \varphi \psi$ gives a continuous function $p^{(0)} : K \to A$ with $p^{(0)}(x) = \chi(1/2, \infty)(g(x)gg(x)) \in g(x)Ag(x)$ for all $x \in K$. Extend $p^{(0)}$ to a continuous function $p : X \to A$ such that $\text{supp}(p) \subset \text{supp}(g)$. Choose $\delta > 0$ so small that $\delta < 1$ and $d(x, x_0) < \delta$ implies $p(x)$ is a projection. Set $V_0 = B(p(x_0), \delta)$ and $V = p^{-1}(V_0)$. Then $x_0 \in V \subset \overline{V}$, and $\|p(x) - p(x_0)\| \leq \frac{1}{2} < 1$ for all $x \in \overline{V}$ by the continuity of $p$. Let $p_0 = p(x_0)$ and $F = \overline{V}$.
Set $p_F = p|_F$ and let $e : F \to A$ be the constant function $e(x) = p_0$. Then $p_F$ and $e$ are projections in $C(F, A)$, and satisfy $\|p_F(x) - e(x)\| = \|p(x) - p_0\| \leq \delta$ for all $x \in F$. This implies that $\|p_F - e\| < 1$, and so by Lemma 2.5.1 of [15], there is a unitary $u \in C(F, A)$ such that $up_Fu^* = e$ and $\|1 - u\| \leq \sqrt{\delta} \|p_F - e\|$. This norm estimate further implies that $\|1 - u\| < \sqrt{\delta}$, and so $u \in U_0(C(F, A))$. (Recall that for a unital $C^*$-algebra $B$, $U_0(B)$ denotes the connected component of $U(B)$ containing $1_B$.) Since the restriction map $U_0(C(X, A)) \to U_0(C(F, A))$ is surjective, there is a $w \in U_0(C(X, A))$ such that $w|_F = u$. If $f \in \text{Her}(V, p_0)$, then supp$(f) \subset F$ and $f \leq 1 \otimes p_0$. Then for any $x \in \text{supp}(f)$, we have $w(x)f(x)w(x)^* = w(x)p_0u_x^* = u(x)p_0u_x = p(x)$. Thus for every $f \in \text{Her}(V, p_0)$, supp$(f) \subset F \subset \text{supp}(g)$ and $f(x) \in g(x)Ag(x)$ for all $x \in X$. 

In order to prove our main result, we require a well-known tool in applications of topological dynamics to $C^*$-algebras: the Rohlin tower construction.

**Theorem 3.18.** Let $(X, h)$ be as in Notation 3.1. Let $Y \subset X$ be a closed set with $\text{int}(Y) \neq \emptyset$. For $y \in Y$, define $r(y) = \min \{ m \geq 1 : h^m(y) \in Y \}$. Then $\sup_{y \in Y} r(y) < \infty$, so there are finitely many distinct values $n(0) < n(1) < \cdots < n(l)$ in the range of $r$. For $0 \leq k < l$, set

$$Y_k = \{ y \in Y : r(y) = n(k) \} \quad \text{and} \quad Y_k^o = \text{int}(\{ y \in Y : r(y) = n(k) \}).$$

Then:

1. the sets $h^j(Y_k^o)$ are pairwise disjoint for $0 \leq k \leq l$ and $0 \leq j \leq n(k) - 1$;
2. $\bigcup_{k=0}^{l} Y_k = Y$;
3. $\bigcup_{k=0}^{l} \bigcup_{j=0}^{n(k)-1} h^j(Y_k) = X$.

**Proof.** Proofs of some or all of the statements in this theorem can be found in [22], [23], and [24] (as well as other places). □

We are now in position to prove that our automorphisms $\beta$ satisfy the tracial quasi-Rokhlin property.

**Theorem 3.19.** Let $(X, h)$ and $A$ be as in Notation 3.1. Suppose in addition that $(X, h)$ has the dynamic comparison property, that $A$ is non-elementary with real rank zero and order on projections determined by traces such that $C(X, A)$ has cancellation of projections and order on projections determined by traces, and that $\beta \in \text{Aut}(C(X, A))$ is the automorphism of Proposition 3.7. Then $\beta$ has the tracial quasi-Rokhlin property.

**Proof.** Let $\varepsilon > 0$, let $F \subset C(X, A)$ be finite, let $n \in \mathbb{N}$, and let $g \in C(X, A)$ be positive with $\|g\| = 1$. By Proposition 3.17 there is non-zero projection $p_0 \in A$, an open set $V \subset \text{supp}(g)$, and a unitary $u \in C(X, A)$ such that $ufu^* \in gc(X, A)g$ for all $f \in \text{Her}(V, p_0)$. By Proposition 3.15 there is an $M \in \mathbb{N}$ and a $\delta > 0$ such that for any positive element $g_0 \in C(X)$ with $\mu(\text{supp}(g_0)) < \delta$ for all $\mu \in M_h(X)$, there exist for $0 \leq k \leq M$ positive elements $a_k \in C(X, A)$, unitaries $w_k \in C(X, A)$, and $r(k) \in \mathbb{Z}$ such that $\sum_{k=0}^{M} a_k \geq g_0 \otimes 1$, the elements $\beta^{r(k)}(a_k)$ are mutually orthogonal, and such that with $b_k = w_k \beta^{r(k)}(a_k)w_k^*$, the $b_k$ are mutually orthogonal elements of $\text{Her}(V, p_0)$. By the continuity of $g$ and the compactness of $X$, there exist $x_0 \in X$ with $\|g(x_0)\| = 1$ and an open neighborhood $G$ of $x_0$ such that $\|g(x)\| > 1 - \frac{1}{2}\varepsilon$ for all $x \in G$. Choose open neighborhoods $G_0, G_1, G_2$ of $x_0$ such that $G_2 \subset \overline{G_2} \subset G_1 \subset \overline{G_1} \subset G_0 \subset G$, $\mu(G_0) < \delta$ for all $\mu \in M_h(X)$, $\partial G_2$ is
topologically $h$-small, and $\|g(x)\| > 1 - \varepsilon$ for all $x \in G_2$. To see this can be done, observe that such a neighborhood $G_0$ exists by applying Corollary 1.5(4) of [3] to the closed set $\{x_0\}$, while $G_1$ and $G_2$ exist using local compactness, continuity, and Proposition 3.9 of [5]. Choose continuous functions $g_0, g_1 : X \to [0, 1]$ such that $g_1 = 1$ on $\overline{G_2}$, supp$(g_1) \subset G_1$, $g_0 = 1$ on $\overline{G_1}$, and supp$(g_0) \subset G_0$. Apply Proposition 3.14 with $g_0$ to obtain the $a_k$, $w_k$, and $r(k)$ described above. Set $\sigma = \min \{\frac{1}{8}\inf_{u \in M_h(X)} \mu(G_2), \varepsilon\} > 0$ and choose $K \in \mathbb{N}$ so large that $\frac{1}{8} \sigma < \frac{1}{8} \varepsilon$. Apply Lemma 4.4 of [3] with $N = nK$ to obtain a closed set $Y \subset X$ such that int$(Y) \neq \emptyset$, $\partial Y$ is topologically $h$-small, and the sets $Y, h(Y), \ldots, h^{nK}(Y)$ are pairwise disjoint.

Adopt the notation of Theorem 3.18 and let $M = (l + 1) \sum_{k=0}^{l} n(k)$. Then:

1. the sets $h^j(Y_k^o)$ are pairwise disjoint for $0 \leq k \leq l$ and $0 \leq j \leq n(k) - 1$;
2. $\bigcup_{k=0}^{l} Y_k = Y$;
3. $\bigcup_{k=0}^{l} h^{n(k)-1} h^j(Y_k) = X$;
4. $\partial h^j(Y_k)$ is topologically $h$-small for $0 \leq k \leq l$ and $0 \leq j \leq n(k) - 1$;
5. for $0 \leq k \leq l$, there exists an open set $U_k \subset Y_k^o$ such that $U_k \subset Y_k^o$, $\partial U_k$ is topologically $h$-small, and $\mu(Y_k^o \setminus \overline{U_k}) < \frac{\sigma}{8M}$ for all $\mu \in M_h(X)$;
6. for $0 \leq k \leq l$, there exists an open set $W_k \subset U_k$ such that $\overline{W_k} \subset U_k$, $\partial W_k$ is topologically $h$-small, and $\mu(U_k \setminus \overline{W_k}) < \frac{\sigma}{8M}$ for all $\mu \in M_h(X)$.

Properties (1)-(3) follow immediately from Theorem 3.18 and property (4) is given by Lemma 4.5 of [5]. For (5), we observe that $\partial Y_k^o = \partial Y_k$ has $\mu(\partial Y_k^o) = 0$ for all $\mu \in M_h(X)$, and then, for each $k$, apply Corollary 1.5(3) of [4] to $Y_k^o$, obtaining open sets $U_k^{(0)}$ such that $U_k^{(0)} \subset Y_k^o \subset U_k^o$ and $\mu(\overline{U_k^{(0)}}) < \frac{\sigma}{8M}$ for all $\mu \in M_h(X)$. Now, apply Proposition 3.9 of [5] to obtain, for each $k$, an open set $U_k$ with $\overline{U_k^{(0)}} \subset U_k \subset Y_k^o$ and such that $\partial U_k$ is topologically $h$-small. (Notice that $U_k$ plays the role of the open set $V$ there, and we ignore the other conclusions.) Then $\overline{U_k^{(0)}} \subset U_k$ clearly implies that $\mu(Y_k^o \setminus \overline{U_k}) \leq \mu(\overline{U_k^{(0)}}) < \frac{\sigma}{8M}$ for all $\mu \in M_h(X)$. The argument to obtain (6) is entirely analogous to that for (5), with $U_k, W_k^{(0)}$, and $W_k$ in place of $Y_k^o, U_k^{(0)}$, and $U_k$ respectively.

Now for $0 \leq k \leq l$ set $s(k) = \max \{m \geq 1 : mn \leq n(k) - 1\}$. Note that $s(k) \geq K$ by the choice of $Y$. For $0 \leq k \leq l$ and $0 \leq j \leq s(k)$, choose continuous functions $c_{k,j} : X \to [0, 1]$ such that $c_{k,j} = 1$ on $h^j(\overline{W_k})$, and supp$(c_{k,j}) \subset h^j(U_k)$. Next set $c_{k,j} = (k,j)$ for $0 \leq k \leq l$ and $0 \leq j \leq s(k)$. Finally, define $c_0, \ldots, c_n \in C(X, A)$ by setting

$$c_0 = \sum_{k=0}^{l} \sum_{i=0}^{s(k)} c_{k,i}$$

and $c_{j+1} = \beta(c_j)$ for $0 \leq j \leq n - 1$. It follows immediately from these definitions that:

1. $0 \leq c_{j} \leq 1$ for $0 \leq j \leq n$;
2. $c_{j} c_{k} = 0$ for $0 \leq j, k \leq n$ and $j \neq k$;
3. $\|\beta(c_j) - c_{j+1}\| = 0$ for $0 \leq j \leq n - 1$;
4. $\|c_{j} f - f c_{j}\| = 0$ for $0 \leq j \leq n$ and for all $f \in F$. 

Now set $c = \sum_{j=0}^{n} c_j$ and $C = \text{supp}(1 - c)$. Also set

$$F = X \setminus \bigcup_{k=0}^{l} \bigcup_{j=0}^{s(k)n} h^{j}(W_k).$$

Then we immediately have $C \subset F$, and we observe that

$$\partial F = \partial \bigcup_{k=0}^{l} \bigcup_{j=0}^{s(k)n} h^{j}(W_k) \subset \bigcup_{k=0}^{l} \bigcup_{j=0}^{s(k)n} \partial h^{j}(W_k).$$

Since each set $h^{j}(W_k)$ is topologically $h$-small, it follows that $F$ is topologically $h$-small by Lemma 2.3(3) of [5]. Also, $\partial Y_k$ topologically $h$-small for $0 \leq k \leq l$ implies that $\mu(\partial Y_k) = 0$ for all $\mu \in M_h(X)$, and so $\mu(Y_k) = \mu(Y_k^0)$. Since the $Y_k^0$ are pairwise disjoint, we obtain for each $\mu \in M_h(X)$ the inequality

$$\mu(Y) = \mu\left( \bigcup_{k=0}^{l} Y_k \right) \geq \mu\left( \bigcup_{k=0}^{l} Y_k^0 \right) = \sum_{k=0}^{l} \mu(Y_k^0) = \sum_{k=0}^{l} \mu(Y_k).$$

Further, the $h$-invariance of $\mu$ and the pairwise disjointness of the sets $h^{j}(Y)$ for $0 \leq j \leq nK$ imply that

$$1 \geq \sum_{j=0}^{nK} \mu(h^{j}(Y)) = \sum_{j=0}^{nK} \mu(Y) = nK \mu(Y)$$

for every $\mu \in M_h(X)$, and so we have $\mu(Y) < 1/(nK)$. Observing that $\mu(\partial U_k) = mu(\partial W_k) = 0$ for all $\mu \in M_h(X)$, it follows that, for any $\mu \in M_h(X)$,

$$\mu(F) \leq \mu\left( X \setminus \bigcup_{k=0}^{l} \bigcup_{j=0}^{s(k)n} h^{j}(W_k) \right)$$

$$\leq \sum_{k=0}^{l} \sum_{j=s(k)n+1}^{n(k)-1} \mu(h^{j}(Y_k)) + \sum_{k=0}^{l} \sum_{j=0}^{s(k)n} (\mu(h^{j}(U_k \setminus W_k)) + \mu(h^{j}(Y_k \setminus U_k))))$$

$$= \sum_{k=0}^{l} \sum_{j=s(k)n+1}^{n(k)-1} \mu(Y_k) + \sum_{k=0}^{l} \sum_{j=0}^{s(k)n} (\mu(U_k \setminus W_k) + \mu(Y_k \setminus U_k))$$

$$\leq (n + 1) \mu(Y) + M \left( \frac{\sigma}{8M} + \frac{\sigma}{8M} \right)$$

$$< \frac{n + 1}{nK} + \frac{1}{\sigma}$$

$$< \frac{2}{K} + \frac{1}{\sigma}$$

$$< \frac{1}{\sigma}.$$ 

Thus for all $\mu \in M_h(X)$ we have

$$\mu(F) < \sigma < \inf_{\mu \in M_h(X)} \mu(G) \leq \mu(G_2),$$

with both $\partial F$ and $\partial G_2$ topologically $h$-small, and hence universally null. Then by the dynamic comparison property there exist $N \in \mathbb{N}$, continuous functions $f_j^{(0)} : X \to [0, 1]$ for $0 \leq j \leq N$, and $d(0), \ldots, d(N) \in \mathbb{Z}$ such that $\sum_{j=0}^{N} f_j^{(0)} = 1$ on $F$, and such that the sets $\text{supp}(f_j^{(0)} \circ h^{-d(j)})$ are pairwise disjoint subsets of $G_2$. 

for $0 \leq j \leq N$. Define continuous functions $f_j : X \to A$ by $f_j = f_j^{(0)} \otimes 1$. Then $1 - c \leq \sum_{j=0}^{N} f_j$ since $C \subset F$, and for $0 \leq j \leq N$, the elements $\beta^{d(j)}(f_j)$ are mutually orthogonal positive elements in $(g_1 \otimes 1)C(X,A)(g_1 \otimes 1)$. For $0 \leq j \leq N$ and $0 \leq k \leq M$, define $e_{j,k} = f_j\beta^{-d(j)}(a_k)$. Since the $\beta^{d(j)}(f_j)$ are mutually orthogonal elements of $(g_1 \otimes 1)C(X,A)(g_1 \otimes 1)$, it follows that $\sum_{j=0}^{N} \beta^{d(j)}(f_j \otimes 1) \leq g_0 \otimes 1$. Moreover, since $\beta^{d(j)+r(k)}(e_{j,k}) = \beta^{r(k)+d(j)}(f_j)\beta^{r(k)}(a_k)$ and the $f_j$ are central, the elements $\beta^{d(j)+r(k)}(e_{j,k})$ are mutually orthogonal. Now let $u_{j,k} = uw_k$ for $0 \leq j \leq N$, $0 \leq k \leq M$. Then $u_{j,k}\beta^{d(j)+r(k)}(e_{j,k})u_{j,k}^* = \beta^{d(j)+r(k)}(f_j)uw_k\beta^{r(k)}(a_k)uw_k^* = \beta^{d(j)+r(k)}(f_j)ub_k u^*$. Since $\beta^{d(j)+r(k)}(f_j) \in C(X)$ and $ub_k u^* \in gC(X,A)g$, it follows that $u_{j,k}e_{j,k}u_{j,k}^* \in gC(X,A)g$. Finally, we compute

$$
\sum_{j=0}^{N} \sum_{k=0}^{M} e_{j,k} = \sum_{j=0}^{N} \sum_{k=0}^{M} f_j\beta^{-d(j)}(a_k) = \sum_{j=0}^{N} f_j\beta^{-d(j)} \left( \sum_{k=0}^{M} a_k \right) \\
\geq \sum_{j=0}^{N} f_j\beta^{-d(j)}(g_0 \otimes 1) \\
= \sum_{j=0}^{N} \beta^{-d(j)}(\beta^{d(j)}(f_j)(g_0 \otimes 1)) \\
= \sum_{j=0}^{N} \beta^{-d(j)}(\beta^{d(j)}(f_j)) \\
= \sum_{j=0}^{N} f_j \geq 1 - c.
$$

Now re-order the elements $e_{j,k}, u_{j,k}$, and $d(j)+r(k)$ as $e_i, u_i$, and $t(i)$ for $0 \leq i \leq I$, where $I = (M + 1)(N + 1)$. It follows that $1 - c \leq \sum_{i=0}^{I} e_i, \beta^{t(i)}(e_i) \beta^{d(j)}(e_j) = 0$ for $0 \leq i, j \leq I$ and $i \neq j$, and $u_i u_i^* \in gC(X,A)g$ for $0 \leq i \leq I$. Finally, as $\inf_{u \in M_h(X)} \mu[G_2 - \mu(C)] \geq \inf_{u \in M_h(X)} \mu[G_2 - \mu(F)] > 0$, there is an $x \in G_2$ such that $x \notin C$. Then $(1 - c)(x) = 0$, and so $c(x) = 1$. It follows that $\|c(x)g(x)c(x)\| = \|g(x)\| > 1 - \varepsilon$, which implies that $\|cg\| > 1 - \varepsilon$. Thus, $\beta$ has the tracial quasi-Rokhlin property.

In order to apply our structure theorems from Section 2 to $C^*(Z,C(X,A),\beta)$, we require information about the ideals of $C(X,A)$.

**Lemma 3.20.** Let $(X, h)$ and $A$ be as in Notation 3.1. If $F \subset X$ is closed, then $I_F = \{ f \in C(X,A) : f|_F = 0 \}$ is an ideal in $C(X,A)$. Moreover, given any ideal $I \subset C(X,A)$, $I = I_F$ for some closed set $F \subset X$.

**Proof.** For $F \subset X$ closed, it is clear that $I_F$ is an ideal in $C(X,A)$. Now let $I \subset C(X,A)$ be an ideal. Define $F \subset X$ by $F = \{ x \in X : f(x) = 0 \text{ for all } f \in I \}$, which is certainly a closed subset of $X$. Set $I_F = \{ f \in C(X,A) : f|_F = 0 \}$, which we have already shown is an ideal of $C(X,A)$. From the definition of $F$ it is clear that $I \subset I_F$. To prove the converse, let $x_0 \in X \setminus F$. We claim that $\{ g(x_0) : g \in I \}$ is dense in $A$. To see this, let $\delta > 0$ be given, and let $a \in A$. Since $x_0 \notin F$, there is
a function $g_0 \in I$ such that $g_0(x_0) \neq 0$. Then the ideal $\overline{Aq_0(x_0)A}$ is non-zero and so equals $A$ by the simplicity of $A$. It follows that there exist $b_1, \ldots, b_n, c_1, \ldots, c_n \in A$ such that $\|a - \sum_{j=1}^{n} b_j g_0(x_0) c_j\| < \delta$. Define a function $g \in C(X, A)$ by $g = \sum_{j=1}^{n} (1 \otimes b_j) g_0 (1 \otimes c_j)$. Then $f \in I$ as $g_0 \in I$ and $1 \otimes b_j, 1 \otimes c_j \in C(X, A)$, and $\|g_{x_0} - a\| < \delta$. Now let $\varepsilon > 0$ be given and let $q \in I_F$. For each $x \in X$, choose $f_x \in I$ such that $\|f_x(x) - q(x)\| < \frac{1}{4} \varepsilon$. This can be done by taking $f_x = 0$ whenever $x \in F$, and for $x \notin F$, $f_x$ can be obtained from the previous claim. Next for each $x \in X$ choose an open neighborhood $U_x$ of $x$ such that $\|f_x(x) - f_x(y)\| < \frac{3}{4} \varepsilon$ and $\|q(x) - q(y)\| < \frac{1}{4} \varepsilon$ for all $y \in U_x$. We obtain an open cover $\{U_x : x \in X\}$ of $X$, which has a finite subcover $\{U_{x_1}, \ldots, U_{x_N}\}$. Let $f_1, \ldots, f_n$ be the functions corresponding to the points $x_1, \ldots, x_n$. Choose a partition of unity $\varphi_1, \ldots, \varphi_N$ subordinate to this cover, let $g_j = \varphi_j f_j$ for $1 \leq j \leq N$, and set $g = \sum_{j=1}^{N} g_j$. Then $g \in I$, and for $1 \leq j \leq N$ and every $x \in X$ we have

$$
\|q(x) - f_j(x)\| \leq \|q(x) - q(x_j)\| + \|q(x_j) - f_j(x_j)\| + \|f_j(x_j) - f_j(x)\| < \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon + \frac{1}{4} \varepsilon = \frac{3}{4} \varepsilon.
$$

For $x \in X$, let $J(x) = \{j : x \in U_j\}$. Then for every $x \in X$, we have

$$
\|q(x) - g(x)\| = \left\|q(x) - \sum_{j=1}^{N} \varphi_j(x) f_j(x)\right\| = \left\|\sum_{J(x)} \varphi_j(x) q(x) - \sum_{J(x)} \varphi_j(x) f_j(x)\right\| = \left\|\sum_{J(x)} \varphi_j(x) (q(x) - f_j(x))\right\| \leq \sum_{J(x)} \varphi_j(x) \|q(x) - f_j(x)\| \leq \left(\sum_{J(x)} \varphi_j(x)\right) \max_{J(x)} \{\|q(x) - f_j(x)\|\} < \frac{3}{4} \varepsilon.
$$

It follows that $\|q - f\| < \varepsilon$, and hence $q \in I$ as $I$ is closed. Therefore $I_F \subset I$, which completes the proof. \hfill \Box

**Proposition 3.21.** Let $(X, h)$ and $A$ be as in Notation [3.3.1] Then the $C^*$-algebra $C(X, A)$ has no non-trivial $\beta$-invariant ideals.

**Proof.** Let $I \subset C(X, A)$ be a non-trivial ideal. By Lemma [3.29] there is a closed set $F \subset X$ such that $I = \{f \in C(X, A) : f(x) = 0 \text{ for all } x \in F\}$. Then $F \neq \varnothing$ and $F \neq X$ as $I$ is non-trivial. Suppose that $I$ is $\beta$-invariant. Then $\beta(I) \subset I$, and so for any $f \in I$, we have $\beta(f) \in I$. Then for any $x \in F$, $f(x) = 0$ and $\beta(f)(x) = 0$. But $0 = \beta(f)(x) - \alpha_x(f \circ h^{-1}(x))$ implies that $f \circ h^{-1}(x) = 0$ since $\alpha_x \in \text{Aut}(A)$. Thus $f(x) = 0$ for all $x \in F \cap h^{-1}(F)$. The $\beta$-invariance of $I$ further implies that $\beta^n(f) \in I$ for all $n \in N$, and recalling that $\beta^n(f)(x) = \alpha_x^n(f \circ h^{-n}(x))$ (this is Corollary [3.3.9] and that $\alpha^n \in \text{Aut}(A)$, it follows that for any $f \in I$, we have
Definition 3.23. A topological space is simple. □

Corollary 3.22. Let \((X,h), A, \) and \(\beta\) be as in Theorem 3.19 Then the crossed product \(C^\ast\)-algebra \(C^\ast(Z,C(X,A),\beta)\) is simple.

Proof. By Proposition 3.21, \(C(X,A)\) has no non-trivial \(\beta\)-invariant ideals. Since \(\beta\) has the tracial quasi-Rokhlin property, Theorem 2.4 implies that \(C^\ast(Z,C(X,A),\beta)\) is simple. □

Definition 3.23. A topological space \(X\) is topologically scattered if every closed subset \(Y\) of \(X\) contains a point \(y\) that is relatively isolated in \(Y\).

It is a standard result (see [28]) that a compact Hausdorff space \(X\) is topologically scattered if and only if every Radon measure on \(X\) is atomic; that is, if and only if for any Radon measure \(\nu\) on \(X\), there exist point-mass measures \((\nu_j)_{j=1}^\infty\) and real numbers \((t_j)_{j=1}^\infty\), satisfying \(t_j \geq 0\) for all \(j \geq 1\) and \(\sum_{j=1}^\infty t_j = 1\), such that

\[
\nu = \sum_{j=1}^\infty t_j \nu_j.
\]

Definition 2.8 can be thought of as a noncommutative version of this one, with an atomic state playing the role of a “noncommutative atomic Radon measure”.

Proposition 3.24. Given any infinite compact metrizable space \(X\) that has a minimal homeomorphism \(h: X \to X\) and any simple, separable, unital \(C^\ast\)-algebra \(A\), the \(C^\ast\)-algebra \(C(X,A)\) is not scattered.

Proof. First note that as \(X\) has a minimal homeomorphism, it cannot be topologically scattered. Indeed if we take \(Y = X\), then for \(X\) to be topologically scattered it must contain at least one isolated point \(y\), which is impossible since the \(h\)-orbit of \(y\) is dense in \(X\). Therefore \(X\) has a non-atomic radon measure \(\nu\). Define a state \(\varphi_\nu\) on \(C(X)\) by

\[
\psi_\nu(f) = \int_X f d\nu.
\]

We claim that \(\psi_\nu\) is a non-atomic state. If it were atomic, we could write \(\psi_\nu = \sum_{i=1}^\infty \delta_\varphi_i\) for some sequence of pure states \((\varphi_i)_{i=1}^\infty\) and some sequence of nonnegative real numbers \((\delta_i)_{i=1}^\infty\) such that \(\sum_{i=1}^\infty \delta_i = 1\). By the Riesz Representation Theorem, we would obtain \(\nu = \sum_{i=1}^\infty \nu_i\) for some sequence of point-mass measures \(\nu_i\), a contradiction. Now let \(\omega\) be any non-zero state on \(A\), and suppose the state \(\psi_\nu \otimes \omega\) is atomic. By Theorem IV.4.14 of [22], we may write \(\psi_\nu \otimes \omega = \sum_{i=1}^\infty t_i \varphi_i \otimes \omega_i\) for some sequences of pure states \((\varphi_i)_{i=1}^\infty\) on \(C(X)\) and \((\omega_i)_{i=1}^\infty\) on \(A\), and for some sequence of nonnegative real numbers \((t_i)_{i=1}^\infty\) such that \(\sum_{i=1}^\infty t_i = 1\). Then for any \(f \in C(X)\), we have

\[
(\psi_\nu \otimes \omega)(f \otimes 1) = \sum_{i=1}^\infty t_i \varphi_i(f)
\]

which implies that \(\psi_\nu = \sum_{i=1}^\infty t_i \varphi_i\), a contradiction to \(\psi_\nu\) being non-atomic. □

Corollary 3.25. Let \((X,h), A, \) and \(\beta\) be as in Theorem 3.19 Then the restriction map \(T(C^\ast(Z,C(X,A),\beta)) \to T_\beta(C(X,A))\) is a bijection.
Proof. By Proposition 3.24, \( C(X, A) \) is not a scattered \( C^* \)-algebra, and by Proposition 3.21, \( C(X, A) \) has no \( \beta \)-invariant ideals. Since \( \beta \) has the tracial quasi-Rokhlin property, the given bijection follows from Theorem 2.10. □

In the case where the homeomorphism \( h \) is uniquely ergodic (there is a unique \( h \)-invariant Borel probability measure on \( X \), and hence a unique \( h \)-invariant tracial state on \( C(X) \)) and where \( A \) has a unique tracial state, we obtain the following nice corollary.

**Corollary 3.26.** Let \((X, h), A, \) and \( \beta \) be as in Theorem 3.19 and assume that \( h \) is uniquely ergodic and that \( A \) has a unique tracial state. Then \( C^*(Z, C(X, A), \beta) \) has a unique tracial state.

We summarize the results of this section for crossed product \( C^* \)-algebras by automorphisms with the tracial quasi-Rokhlin property.

**Theorem 3.27.** Let \( X \) be an infinite compact metrizable space, let \( h: X \to X \) be a minimal homeomorphism, and let \( A \) be a simple, separable, unital, non-elementary \( C^* \)-algebra with real rank zero, such that \( C(X, A) \) has cancellation of projections and order on projections determined by traces. Let \( \beta \in \text{Aut}(C(X, A)) \) be defined as in Proposition 3.7. Suppose that \((X, h)\) has the dynamic comparison property.

Then the crossed product \( C^* \)-algebra \( C^*(Z, C(X, A), \beta) \) is simple, and there is a bijection \( T(C^*(Z, C(X, A), \beta)) \to T_\beta(C(X, A)). \)

If \( A = \mathbb{C} \), then \( C^*(Z, C(X, A), \beta) \) is just \( C^*(Z, X, h) \), whose structure has been extensively studied in [24], [20], and [33] (among other places), as discussed in the Introduction. In particular, in the case where \( X \) has finite covering dimension and projections separate traces, then \( C^*(Z, X, h) \) has tracial rank zero. Even if one omits the assumption that projections separate traces, one of the main theorems of [33] shows that \( C^*(Z, X, h) \) is stable under tensoring with the Jiang-Su algebra. Whether these results can be extended to the situation we have studied in this paper is a question of considerable interest. Hua (see [9]) has shown that in the case where \( X \) is the Cantor set and \( A \) has tracial rank zero, the crossed product \( C^*(Z, C(X, A), \beta) \) has tracial rank zero under some additional technical assumptions. However, the case for more general spaces \( X \) remains open. Aspects related to this will be considered in subsequent papers [4] and [6].

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