EXPRESSING A GENERAL FORM AS A SUM OF DETERMINANTS

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ABSTRACT. Let $A = (a_{ij})$ be a non-negative integer $k \times k$ matrix. $A$ is a homogeneous matrix if $a_{ij} + a_{kl} = a_{il} + a_{kj}$ for any choice of the four indexes. We ask: If $A$ is a homogeneous matrix and if $F$ is a form in $\mathbb{C}[x_1, \ldots, x_n]$ with deg$(F) = \text{trace}(A)$, what is the least integer, $s(A)$, so that $F = \det M_1 + \cdots + \det M_s(A)$, where the $M_i = (F_{lm})$ are $k \times k$ matrices of forms and deg$F_{lm} = a_{lm}$ for every $1 \leq i \leq s(A)$?

We consider this problem for $n \geq 4$ and we prove that $s(A) \leq kn - 3$ and $s(A) < kn - 3$ in infinitely many cases. However $s(A) = kn - 3$ when the integers in $A$ are large with respect to $k$.

INTRODUCTION

Let $F \in \mathbb{C}[x_1, \ldots, x_n]$ be a general form and $A = (a_{ij})$ a square integer homogeneous matrix with the trace of $A$ (tr$(A)$) equal to the degree of $F$ (deg $F$). In this paper we study representations of $F$ as a sum of determinants of matrices of type $M = (F_{ij})$ where deg $F_{ij} = a_{ij}$.

In case the number of variables is two then any form $F$ in $\mathbb{C}[x_1, x_2]$ of degree $d$ decomposes as a product of linear forms. It follows that if $A$ is any square homogeneous matrix of integers with no negative entries and with tr$(A) = d$, then $F$ is the determinant of a diagonal matrix whose degree matrix is $A$.

In the case of 3 variables, the problem was considered classically by the great American mathematician L.E. Dickson (see [D21]), who proved that a general form of degree $d$ is the determinant of a $d \times d$ matrix of linear forms. The recent paper [CM12], of J. Migliore and the first author, generalizes this fact. Namely, for any given square homogeneous matrix of integers $A$ having trace $d$, there is a simple necessary and sufficient condition on $A$ which tells us when a general form of degree $d$ in three variables is the determinant of a matrix of forms whose degree matrix is $A$.

Thus the case of 4 variables is the first non-trivial case not yet considered. We address the problem in the present paper. We prove our Main Theorem (see below) in the special case involving general forms in 4 variables and non-negative integer matrices $A$, in §2. The proof for greater than 4 variables is in §4.

This problem, of clear algebraic and geometric flavour, turns out to also have an interesting connection with some applications in control theory. Indeed, if the algebraic boundary of a region $\Theta$ in the plane or in space, is described by the determinant of a matrix of linear forms, then the study of systems of matrix inequalities, whose domain is $\Theta$, can be considerably simplified. We refer to the papers [V89] and [HL12], for an account of this theory.
When the number of variables is bigger than 3 one cannot hope to describe a general form of degree \(d\) with just one determinant. For instance, it is a standard fact that a general form \(F\) in four variables, of degree at least 4, cannot be the determinant of a matrix of linear forms. In fact, if we delete one row of the matrix, one sees that the surface \(F = 0\) should contain a curve cut by hypersurfaces of degree \(d - 1\). This is impossible for general surfaces in \(\mathbb{P}^3\) since the celebrated Noether-Lefschetz Theorem prevents a general surface of degree \(d > 3\) from containing curves cut by surfaces of degree \(d - 1\).

As a consequence, we are led to the following, quite natural, question: for a general form \(F\) of degree \(d\), and a given homogeneous square matrix of integers \(A\), with degree (= trace) \(d\), how many matrices of forms, with degree matrix \(A\), are necessary so that \(F\) is the sum of their determinants?

In a previous paper ([CG13]), we showed that a general form in four variables is the sum of two determinants of \(2 \times 2\) matrices with given degree matrices.

When the size of the degree matrix \(A\) grows, one cannot hope to obtain a similar result, with the sum of just two determinants. This is clear from a standard geometrical interpretation of the problem. The interpretation is based on the study of secant varieties using the classical Terracini Lemma. Let us recall a standard construction, already used in [CCG08] and in [CG13].

**Example 0.1.** Inside the projective space \(\mathbb{P}^N\), which parametrizes forms of degree \(d\) in four variables (up to scalar multiplication), the set of points representing forms which are the determinant of a \(k \times k\) matrix, whose degree matrix is fixed, is dense in a projective subvariety \(V\). Our question can be rephrased by asking: what is the minimal \(s\) such that a general point of \(\mathbb{P}^N\) is spanned by \(s\) points of \(V\). In classical Algebraic Geometry, (the closure of) the set of points spanned by \(s\) points of \(V\), is called the \(s\)-th secant variety \(S^s(V)\) of \(V\). Thus, we look for the minimal \(s\) such that \(S^s(V) = \mathbb{P}^N\).

At a general point \(F = \det(G) \in V\), the tangent space to \(V\) at \(F\) corresponds to forms of degree \(d\) in the ideal \(J\), generated by the submaximal minors of \(G\). If the matrix \(A\) is \(k \times k\), with all entries equal to \(a\) (so that \(d = ak\)), then \(J\) is generated by \(k^2\) forms of degree \(ak - 1\).

By the celebrated Terracini Lemma, the tangent space at a general point \(F\) of the \(s\)-th secant variety is spanned by \(s\) secant spaces at the points \(G_i \in V, i = 1, \ldots, s\), such that \(F = \sum G_i\).

Thus, we want to know the minimal \(s\) such that, for general matrices \(G_1, \ldots, G_s \in V\) with degree matrix \(A\), the ideal \(I\), generated by all their submaximal minors, coincides with the polynomial ring \(R = \mathbb{C}[x, y, z, t]\), in degree \(d\).

Just computing the dimensions as vector spaces, we see that

\[
\dim I_d \leq k^2s \dim R_a = k^2sa^3/6 + o(a^3)
\]

while the dimension of \(R_d\) is \(a^3k^3/6 + o(a^3)\).

So, it is immediate to see that, at least when \(a\) grows, if \(I_d = R_d\) then \(s\) must be asymptotically equal to \(k\).

We show that the bound of the previous rough estimate, is always attained. Namely, we prove (see Theorem 2.2 below):

**Theorem. (Main)** Let \(A\) be a homogeneous \(k \times k\) matrix of non-negative integers, with \(tr(A) = d\). Then a general form of degree \(d\) in 4 variables is the sum of \(k\) determinants of matrices in each degree matrix \(A\).
The proof is based on an algebraic analysis of the ideal generated by submaximal determinants. Essentially, we use induction on the degree of \( A \). A fundamental point in the proof is the fact that, by the main result of [CM12], the quotient \( S \) of the polynomial ring \( R \), by the ideal generated by many submaximal minors, satisfies a sort of weak Lefschetz property: multiplication by a general linear form has maximal rank, in degrees close to \( d \).

We notice that our result can also be interpreted as a result for general surfaces with given degree \( d \) in the projective space \( \mathbb{P}^3 \). The Hilbert–Burch Theorem shows that homogeneous \((k-1) \times k\) matrices of forms determine the resolution of ideals of curves which are arithmetically Cohen–Macaulay.

We first extend the idea of trace to matrices of size \((k-1) \times k\) by defining the trace of such a matrix to be the maximal trace of any square \((k-1) \times (k-1)\) submatrix. We can now state our result in terms of surfaces containing curves of given type.

**Corollary 0.2.** Let \( A' \) be a homogeneous \((k-1) \times k\) matrix of non–negative integers. Then a general surface of degree \( d \geq \text{tr}(A') \) in \( \mathbb{P}^3 \) is contained in a linear system generated by \( k \) surfaces, each of which contains an arithmetically Cohen–Macaulay curve whose Hilbert–Burch matrix has degree matrix equal to \( A' \).

As we showed in the previous example, the conclusion of our Main Theorem cannot be improved for certain matrices \( A \) (see also Example 3.5). However, we know that in some specific cases (e.g. when all the entries of \( A \) are 1’s, so that we are looking at determinants of matrices of linear forms) the number of determinants needed to write a general form can be smaller than our bound \( k \). See Remark 2.3 and §3 for a discussion. The problem of finding a sharp bound for the number of determinants needed to express a general form of small degree is still open.

The extension of our Main Theorem to the case of \( n > 4 \) variables is in §4.

**Preliminaries**

We work in the ring \( R = \mathbb{C}[x, y, z, t] \), i.e. the polynomial ring in 4 variables with coefficients in the complex numbers. By *quaternary form*, we mean any homogeneous polynomial in \( R \) and by \( R_n \) we mean the vector space of (quaternary) forms of degree \( n \) in \( R \).

By abuse of notation we will often indicate with the same symbol \( F \), both a form \( F \in \mathbb{C}[x, y, z, w] \) and the surfaces defined by the equation \( F = 0 \).

Fix a degree \( n \). The space \( R_n \) of forms of degree \( n \) has an associated projective space \( \mathbb{P}^N \) with

\[
N := N(n) = \binom{n + 3}{3} - 1.
\]

For any choice of integers \( a_{ij}, 1 \leq i, j \leq k \), consider the numerical \( k \times k \) matrix \( A = (a_{ij}) \).

We will say that a \( k \times k \) matrix \( M = (F_{ij}) \), whose entries are (quaternary) forms, has degree matrix \( A \) if, for all \( i, j \), we have \( \deg(F_{ij}) = a_{ij} \). In this case, we will also write that \( A = \partial M \).

Notice that when, for some \( i, j \), we have \( F_{ij} = 0 \), there are several possible degree matrices for \( M \), since the zero polynomial is considered to have any degree.

Notice that the set of all matrices of forms whose degree matrix is a fixed \( A \), defines a vector space whose dimension is \( \sum \dim(R_{a_{ij}}) \). From the geometrical point
of view, however, we will consider this set as the *product* of projective spaces

\[ V(A) = \mathbb{P}^{r_{11}} \times \cdots \times \mathbb{P}^{r_{kk}} \]

where \( r_{ij} = -1 + \dim(R_{a_{ij}}) \).

We say that the numerical matrix \( A \) is *homogeneous* when, for any choice of the indexes \( i, j, l, m \), we have

\[ a_{ij} + a_{lm} = a_{im} + a_{lj}. \]

All submatrices of a homogeneous matrix are homogeneous.

If a square matrix of forms \( M \) has a homogeneous degree matrix, then the determinant of \( M \) is a homogeneous form. The degree of the determinant is the sum of the numbers on the main diagonal of \( A \), i.e. \( tr(A) \). This number is called the *degree* of the homogeneous square matrix \( A \).

In the *projective* space \( \mathbb{P}^{N} \), which parametrizes all forms of degree \( n \), we have the subset \( U \) of all the forms which are the determinant of a matrix of forms whose degree matrix is a given \( A \). This set is a quasi-projective variety, since it corresponds to the image of the map \( V(A) \to \mathbb{P}^{N} \), which sends every matrix to its determinant (it is undefined when the determinant is the polynomial 0).

We will denote by \( V \) the closure of the image of this map. As explained in the introduction, a general (quaternary) form \( F \) of degree at least 4 cannot be the determinant of a matrix of forms. Thus, \( V \) is not equal to \( \mathbb{P}^{N} \) when the degree of \( A \) is at least 4.

In view of Terracini’s Lemma (mentioned in the Introduction) we need to characterize the tangent space to \( V \) at a general point \( F \).

**Proposition 0.3.** Let \( F \) be a general element in \( V \), \( F = \det M \), where \( M = (F_{ij}) \) is a \( k \times k \) matrix of forms, whose degree matrix is \( A \).

Then, the tangent space to \( V \) at \( F \) coincides with the subspace of \( \mathbb{R}^{n} / \langle F \rangle \), generated by the classes of the forms of degree \( n \) in the ideal \( \langle F, M_{ij} \rangle \), where the \( M_{ij} \) are the submaximal minors of the matrix \( M \).

**Proof.** This is just a direct computation. Namely, over the ring of dual numbers \( \mathbb{C}[\epsilon] \) we want to find when the form \( F + \epsilon G \) is the determinant of a matrix \( (F_{ij} + \epsilon G_{ij}) \), where \( \deg(G_{ij}) = a_{ij} \).

A simple computation shows that this happens exactly when \( G \) sits in the ideal generated by \( F \) and the \( M_{ij} \)'s. \( \square \)

**Remark 0.4.** It follows immediately from the previous propositions, and Terracini’s lemma, that:

- a general form of degree \( n \) is the sum of \( s \) determinants of \( k \times k \) matrices, all having degree matrix \( A \) (i.e. the span of \( s \) general tangent spaces to \( V \) is the whole space \( \mathbb{P}^{N} \)), if and only if
- for a general choice of \( s \) matrices of forms \( M_{1}, \ldots, M_{s} \), of type \( k \times k \), with \( \partial M_{i} = A \) for all \( i \), the ideal generated by all the submaximal minors of all the \( M_{i} \)'s coincides, in degree \( n \), with the whole space \( \mathbb{R}^{n} \).

1. **Some lemmas about \((k - 1) \times k\) matrices of ternary forms**

Let \( A' \) be a non-negative integer homogeneous matrix. By performing permutations of the rows and the columns of \( A' \), we can always assume that the integers in
any row are increasing as we move to the right and that the integers in any column are increasing as we go from bottom to top. A non-negative integer homogeneous matrix whose rows and columns satisfy the condition just described will be called
ordered. Recall that we defined the trace of a (not necessarily square) homogeneous matrix to be the maximum of the traces of its square submatrices.

In this section we collect results about the ideal generated by the maximal minors of some \((k - 1) \times k\) non-negative integer homogeneous matrices of ternary forms, with given degree matrices.

Let \(A' = (a'_{ij})\) be such a \((k - 1) \times k\) non-negative ordered integer homogeneous matrix. Notice that the trace of \(A'\) is equal to

\[
tr(A') = a_{12} + a_{23} + \cdots + a_{k-1,k}.
\]

**Remark 1.1.** Let \(A' = (a_{ij})\) be as above. We will denote by \(T(A')\) the number

\[
T(A') = tr(A') + a_{11} = a_{11} + a_{23} + \cdots + a_{k-1,k}.
\]

The number \(T(A')\) has the following property: for a general matrix \(G\) of ternary forms, with \(\partial G = A'\), the Hilbert–Burch theorem implies that the maximal minors of \(G\) generate, in the ring \(R' := \mathbb{C}[x, y, z]\), the homogeneous ideal \(I_{k-1}(G)\) of a set of points \(Z \subset \mathbb{P}^2\) (see the paper [CGO88], to which we refer for facts about the Hilbert–Burch matrices of ternary forms).

The Betti numbers of a minimal free resolution of \(I_{k-1}(G)\) are fixed by the degree matrix \(A'\). The number \(T(A')\) is exactly the maximal degree of a syzygy appearing in the resolution of \(I_{k-1}(G)\).

It is well known that the Hilbert function of \(R'/I_{k-1}(G)\) is equal to the number of points in \(Z\) for all degrees \(n \geq T(A') - 2\).

Moreover, a general linear form \(L\) in \(R'\) represents a line in \(\mathbb{P}^2\) which meets no point of \(Z\). Thus the multiplication by \(L\) is an isomorphism

\[
(R'/I_{k-1}(G))_{d-1} \rightarrow (R'/I_{k-1}(G))_d
\]

whenever the degree \(d\) is at least \(T(A') - 1\). It follows that for any ideal \(J \supset I_{k-1}(G)\), multiplication by a general linear form \(L\) gives a surjective map \((R/J)_{d-1} \rightarrow (R/J)_d\) for \(d \geq T(A') - 1\).

**Example 1.2.** Let

\[
A' = \begin{pmatrix}
5 & 6 & 8 & 9 \\
5 & 6 & 8 & 9 \\
2 & 3 & 5 & 6
\end{pmatrix}
\]

Since \(tr(A') = 20\) we have \(T(A') = 25\).

For a general matrix \(G\) of ternary forms, with \(\partial G = A'\) and \(L\) a general linear form we thus have that the map

\[
(R'/I_{k-1}(G))_{d-1} \rightarrow (R'/I_{k-1}(G))_d
\]

induced by \(L\), is an isomorphism as soon as \(d \geq 24\).

We are now ready for our main Lemma. In order to present its proof in a reasonable fashion we need some other pieces of notation. These extend the notation introduced in the previous section.

**Notation 1.3.** Let \(B_1, B_2\) be ordered non-negative integer homogeneous matrices of size \(k - 1 \times k\). We will say that

\[\text{condition } M_k(B_1', B_2^{k-j})\text{ holds}\]
if for a general choice of $k$ matrices of ternary forms $G_1, \ldots, G_k$ with $\partial G_i = B_i$ for all $i \leq j$ and $\partial G_i = B_2$ for all $i > j$, the ideal generated by all the maximal minors $I_{k-1}(G_1) + \cdots + I_{k-1}(G_k)$ coincides with the ring $R' := \mathbb{C}[x, y, z]$ in degree $s$.

When $B_1 = B_2$, we will write that condition $M_s(B^k)$ holds.

**Lemma 1.4.** With the previous notation, condition $M_{T(A')}((A')^k)$ holds. I.e. for a general choice of matrices of ternary forms $G_1, \ldots, G_k$ with $\partial G_i = A'$ for all $i$, then the ideal generated by all the maximal minors of the $G_i$ i.e. $I_{k-1}(G_1) + \cdots + I_{k-1}(G_k)$ coincides with $R'$ in degree $T(A')$.

**Proof.** It’s enough to exhibit $k$ matrices with the desired property which, by semicontinuity, implies the result for a generic choice.

First assume that all the rows of $A'$ are equal. In this case, the Lemma follows from the main result of [CM12]. To see this, we add a new row to $A'$, equal to all the other rows of $A'$. We get a square $k \times k$ ordered homogeneous matrix of non-negative integers, which we denote by $A$. Notice that $tr(A) = T(A')$.

By [CM12], Theorem 3.6, we know that a general ternary form of degree equal to the trace of $A$, is the determinant of a matrix of forms $G$, with $\partial G = A$. This implies by 0.3 and 0.4) that the ideal generated by all the $(k-1) \times (k-1)$ minors of $G$ coincides with $R'$ in degree $T(A')$. If we take $G_i = A = A'$, the matrix $G$ with the $i$-th row canceled, we thus have an instance of $A$ matrices coming from $G$ by erasing one row at a time. Thus the Lemma is true for the $k$ matrices coming from $G$.

Now, let us consider the general case. Since $A'$ is homogeneous and ordered, the $i$-th row of $A'$ is obtained from the last row of $A'$ by adding a fixed non-negative integer to every entry. We define the **diameter** $d(A')$ of $A'$ to be the (constant) difference between the entries in the first row of $A'$ and the entries in the last row.

We do induction on $d(A')$, and notice that the case $d(A') = 0$ is exactly the case where all the rows of $A'$ are equal.

Assume that $d(A') > 0$ and the Lemma is true for all matrices with diameter smaller than $d(A')$. Let $m \geq 1$ be the number of rows of $A'$ which are equal to the first row. Then by subtracting 1 from the entries in the first $m$ rows of $A'$ we get a new matrix $A''$ which is still an ordered non-negative integer homogeneous matrix, with diameter $d(A'') = d(A') - 1$. Then, by induction, the Lemma holds for $A''$, i.e. $M_{T(A''')}((A'')^k)$ holds. Notice that $T(A'') = T(A') - (m + 1)$.

For $j = 0, \ldots, m$ call $A_j$ the matrix obtained by adding 1 to the entries in the first $j$ rows of $A''$. Each $A_j$ is again an ordered non-negative integer homogeneous matrix. Moreover $A_0 = A'$ and $A_m = A'$. For $T(A_j) = T(A'') + j + 1$ for $j > 0$. We will prove by induction that condition $M_s((A_{j-1})^j, (A_j)^k-j)$ holds for $j = 1, \ldots, m$ and $s = T(A'') + j = T(A_j) - 1$.

For $j = 1$, we prove that $M_{T(A''')}((A'')^k)$ implies $M_{T(A''')}((A'')^k)$ holds for all $i$, and call $S$ the quotient $S = R'/I_{k-1}(G_1)$, then we have that the image of the ideal $I_{k-1}(G_2) + \cdots + I_{k-1}(G_k)$ fills $S$ in degree $T(A'')$. Moreover, by Remark 1.1 we know that the multiplication by a general linear form gives a surjection $S_{T(A'')} \rightarrow S_{T(A'')}$. For $i = 2, \ldots, k$ call $G^L_i$ the matrix obtained by multiplying the entries of the first row of $G_i$ by a general linear form $L$. The $(k-1) \times (k-1)$ minors of each $G^L_i$ are the $(k-1) \times (k-1)$ minors of $G_i$ multiplied by $L$. Thus $I_{k-1}(G^L_2) + \cdots + I_{k-1}(G^L_k)$ is equal to $L(I_{k-1}(G_2) + \cdots + I_{k-1}(G_k))$.
and therefore its image fills $S_{T(A')}+1$. It follows that $R'$ is equal to $I_{k-1}(G_1) + I_{k-1}(G_2^j) + \cdots + I_{k-1}(G_k^j)$ in degree $T(A')+1$. Thus we have a particular set of matrices $G_1, G_2, \ldots, G_k$ with $\partial G_1 = A'' = A_0$ and $\partial G_i^j = A_1$, such that all their maximal minors generate an ideal which coincides with $R'$ in degree $T(A')+1$. By semicontinuity, we see that $M_{T(A')+1}(A_0, A_1^{k-1})$ holds.

In an analogous way, for $1 < j \leq m$ one proves that $M_{T(A')+j-1}(A_{j-2}^{k-1}, A_{j-1}^{j-1})$ implies $M_{T(A')+j}(A_{j-1}^{j-1}, A_j^{k-j})$. Namely, take $k$ general matrices $G_1, \ldots, G_k$ with $\partial G_i = A_{j-2}$ for $i = 1, \ldots, j-1$ and $\partial G_i = A_{j-1}$ for $i \geq j$. Call $S'$ the quotient $S' = R'/I_{k-1}(G_j)$. Now $M_{T(A')+j-1}(A_{j-2}^{k-1}, A_{j-1}^{j-1})$ implies that the image of the ideal $I_{k-1}(G_1) + \cdots + I_{k-1}(G_{j-1}) + I_{k-1}(G_{j+1}) + \cdots + I_{k-1}(G_k)$ fills $S'$ in degree $T(A')+j-1$. Moreover, since $T(A_{j-1}) = T(A') + j$, by Remark 1.1 we know that the multiplication by a general linear form gives a surjection $S_{T(A')+j-1} \to S_{T(A')+j}$.

For $i = 1, \ldots, j-1$ call $G_i^j$ the matrix obtained by multiplying the entries of the $(j-1)$-th row of $G_i$ by a general linear form $L$. For $i = j+1, \ldots, k$ call $G_i^j$ the matrix obtained by multiplying the entries of the $j$-th row of $G_i$ by the same general linear form $L$. The $(k-1) \times (k-1)$ minors of each $G_i^j$ are the $(k-1) \times (k-1)$ minors of $G_i$ multiplied by $L$. Thus $I_{k-1}(G_1^j) + \cdots + I_{k-1}(G_{j-1}^j) + I_{k-1}(G_{j+1}^j) + \cdots + I_{k-1}(G_k^j)$ is equal to $L(I_{k-1}(G_1) + \cdots + I_{k-1}(G_{j-1}) + I_{k-1}(G_{j+1}) + \cdots + I_{k-1}(G_k))$ and therefore its image fills $S_{T(A')}^j$. It follows that $R'$ is equal to $I_{k-1}(G_1^j) + \cdots + I_{k-1}(G_{j-1}) + I_{k-1}(G_{j+1}^j) + \cdots + I_{k-1}(G_k^j)$ in degree $T(A')+j$. Thus we have a particular set of matrices $G_1^j, \ldots, G_{j-1}^j, G_j, G_{j+1}^j, \ldots, G_k^j$ such that all their maximal minors generate an ideal which coincides with $R'$ in degree $T(A')+j$. Notice that $\partial G_i^j = A_{j-1}$ when $i = 1, \ldots, j-1$, $\partial G_j = A_{j-1}$ and $\partial G_i = A_j$ for $i > j$. Thus, by semicontinuity, we see that $M_{T(A')+j}(A_{j-1}^{j-1}, A_j^{k-j})$ holds.

After $m$ steps, we get that $M_{T(A')}((A')^k)$ implies $M_{T(A')}+(m)(A_{m-1}^{m-1}, A_{m}^{k-m})$. It remains to show that $M_{T(A')}+(m)(A_{m-1}^{m-1}, A_{m}^{k-m})$ implies $M_{T(A')}((A')^k)$.

Take general matrices of forms $G_1, \ldots, G_k$ with $\partial G_i = A_{m-1}$ for $i = 1, \ldots, m$ and $\partial G_i = A_{m} = A'$ for $i \geq m+1$ (recall that $k > m$). Call $S''$ the quotient $R'$ by the ideal $I_{k-1}(G_{m+1}) + \cdots + I_{k-1}(G_k)$. Since $S''$ is a quotient of $R'/I_{k-1}(G_k)$ and $\partial G_i = A_{m} = A'$, by Remark 1.1 we know that the multiplication by a general linear form gives a surjection $S_{T(A')}^m \to S_{T(A')+m+1}$. Now $M_{T(A')+m}(A_{m-1}^{m-1}, A_{m}^{k-m})$ implies that the image of the ideal $I_{k-1}(G_1) + \cdots + I_{k-1}(G_{m-1})$ fills $S''$ in degree $T(A')+m$. For $i = 1, \ldots, m$ call $G_i^j$ the matrix obtained by multiplying the entries of the $m$-th row of $G_i$ by a general linear form $L$. The $(k-1) \times (k-1)$ minors of each $G_i^j$ are the $(k-1) \times (k-1)$ minors of $G_i$ multiplied by $L$. Thus $I_{k-1}(G_1^j) + \cdots + I_{k-1}(G_{m-1}^j) + I_{k-1}(G_m^j) + \cdots + I_{k-1}(G_k^j)$ is equal to $L(I_{k-1}(G_1) + \cdots + I_{k-1}(G_{m-1}) + I_{k-1}(G_m) + \cdots + I_{k-1}(G_k))$ and therefore its image fills $S''$ in degree $T(A')+m+1$. It follows that $R_{T(A')} = I_{k-1}(G_1^j) + \cdots + I_{k-1}(G_{m-1}^j) + I_{k-1}(G_m^j) + \cdots + I_{k-1}(G_k^j)$.

Thus we have a particular set of matrices $G_1^j, \ldots, G_{m-1}^j, G_m, G_{m+1}^j, \ldots, G_k^j$ such that all their maximal minors generate an ideal which coincides with $R'$ in degree $T(A')$. Notice that now $\partial G_i^j = A'$ when $i = 1, \ldots, m$, and also $\partial G_i = A'$ for $i > m$. Thus, by semicontinuity, we see that $M_{T(A')}((A')^k)$ holds.

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**Example 1.5.** We give an explicit description of the previous argument, for a particular $3 \times 4$ matrix.
Assume we want to know that the maximal minors of $k = 4$ general matrices, with degree matrix
\[
A' = \begin{pmatrix} 5 & 6 & 8 & 9 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}
\]
generate the ring $R'$ in degree $T(A') = 25$. i.e. we want to show that $M_{25}((A')^4)$ holds. $A'$ is an ordered homogeneous matrix of non-negative integers, with diameter 3. The first two rows of $A'$ are equal so, in the notation of Lemma 14, $m = 2$. Thus in order to decrease the diameter, we need to subtract 1 from the first two rows. We obtain, in this way, the matrix
\[
A'' = \begin{pmatrix} 4 & 5 & 7 & 8 \\ 4 & 5 & 7 & 8 \\ 2 & 3 & 5 & 6 \end{pmatrix}
\]
which is still ordered and whose diameter is 2. Since $T(A'') = 22$, we may assume by induction that $M_{22}((A'')^4)$ holds.

We will need the auxiliary matrix
\[
A_1 = \begin{pmatrix} 5 & 6 & 8 & 9 \\ 4 & 5 & 7 & 8 \\ 2 & 3 & 5 & 6 \end{pmatrix}
\]
for which $T(A_1) = 24$. Following the proof of the Lemma, we show that $M_{22}((A'')^4)$ implies $M_{23}((A'')^3, A_1^4)$ which in turn implies $M_{24}(A_1^3, (A')^2)$, which finally implies $M_{25}((A')^4)$.

Indeed, take 4 general matrices $G_1, G_2, G_3, G_4$ whose degree matrix is $A''$. By $M_{22}((A'')^4)$, we know that $I_3(G_1) + \cdots + I_3(G_4)$ fills $R'$ in degree 22. Moreover the multiplication by a general linear form gives an isomorphism $(R'/I_3(G_1))_{22} \rightarrow (R'/I_3(G_1))_{23}$. Thus if $G_1^i, i = 2, 3, 4$ is the matrix obtained from $G_i$ by multiplying the first row by a general linear form $L$, we see that $\partial G_1^i = A_1$ and $I_3(G_1) + I_3(G_2^i) + I_3(G_3^i) + I_3(G_4^i)$ coincides with $R'$ in degree 23. Thus, by semicontinuity, $M_{23}((A')^3, A_1^4)$ holds. Now take new general matrices $H_1, H_2, H_3, H_4$ with $\partial H_1 = A''$ and $\partial H_2 = \partial H_3 = \partial H_4 = A_1$, so that, by $M_{23}((A'')^3, A_1^4)$, the ideal $I_3(H_1) + \cdots + I_3(H_4)$ coincides with $R'$ in degree 23. Since $\partial H_2 = A_1$ and $T(A_1) = 24$, the multiplication by a general linear form determines an isomorphism $(R/I_3(H_2))_{23} \rightarrow (R/I_3(H_2))_{24}$. Take a general linear form $X$. Call $H_2^X$ the matrix obtained by multiplying the first row of $H_1$ by $X$, and call $H_2^X$ (resp. $H_3^X$) the matrix obtained by multiplying the second row of $H_3$ (resp. $H_4$) by $X$. Since $I_3(H_1^X) + I_3(H_2^X) + I_3(H_3^X) = X(I_3(H_1) + I_3(H_3) + I_3(H_4))$, then $I_3(H_1^X) + I_3(H_2) + I_3(H_3^X) + I_3(H_4^X)$ coincides with $R'$ in degree 24. Notice that $\partial H_1^X = \partial H_2 = A_1$ while $\partial H_3^X = \partial H_4^X = A'$. Thus, by semicontinuity, $M_{24}(A_1^3, (A')^2)$ holds. Finally, take new general matrices $K_1, K_2, K_3, K_4$ with $\partial K_1 = \partial K_2 = A_1$ and $\partial K_3 = \partial K_4 = A'$, so that, by $M_{24}(A_1^3, (A')^2)$, the ideal $I_3(K_1) + \cdots + I_3(K_4)$ coincides with $R'$ in degree 24. Since $\partial K_3 = A'$ and $T(A') = 24$, the multiplication by a general linear form determines an isomorphism $(R'/I_3(K_3))_{24} \rightarrow (R'/I_3(K_3))_{25}$ and consequently a surjection $S_{24} \rightarrow S_{25}$, where $S = R'/I_3(K_3) + I_3(K_4))$. Take a general linear form $Y$. Call $K_1^Y$ (resp. $K_2^Y$) the matrix obtained by multiplying the second row of $K_1$ (resp. $K_2$) by $Y$. Since $I_3(K_1^Y) + I_3(K_2^Y) = Y(I_3(K_1) + I_3(K_2))$, then $I_3(K_1^Y) + I_3(K_2^Y) + I_3(K_3) + I_3(K_4)$ coincides with $R'$ in degree 25. Notice that $\partial K_1^Y = \partial K_2^Y = \partial K_3 = \partial K_4 = A'$. Thus, by semicontinuity, $M_{25}((A')^4)$ holds.
2. THE PROOF OF THE MAIN THEOREM

Let \( R = \mathbb{C}[x,y,z,t] \). The technical Lemma in the previous section gives us a Lefschetz-type property for certain quotients of \( R \).

**Proposition 2.1.** Let \( A' = (a_{ij}) \) be a homogeneous \((k-1) \times k\) matrix of non-negative integers. Let \( G_1, \ldots, G_k \) be a general choice of matrices of quaternary forms such that \( \partial G_i = A' \) for all \( i \).

If \( J \) is the ideal generated by all the maximal minors of the \( G_i \)'s, then the multiplication map \((R/J)_{n-1} \to (R/J)_n\) by a general linear form is surjective when \( n \geq T(A') \).

**Proof.** The proof follows immediately from Lemma 1.4. Indeed, since condition \( M_{T(A')}((A')^k) \) holds, the residues of the matrices \( G_i \)'s in \( R' = R/(x) = \mathbb{C}[y,z,t] \) have maximal minors which generate \( R' \) in all degrees \( n \geq T(A') \). Thus, modulo \( J \), every element in \((R/J)_n\) is divisible by \( x \). Hence, multiplication by \( x \) surjects onto \((R/J)_n\). \( \square \)

We have all the ingredients to prove the main result, which we recall here:

**Theorem 2.2.** Let \( A \) be a homogeneous \( k \times k \) matrix of non-negative integers, with degree \( d \). Then a general form of degree \( d \) in \( 4 \) variables is the sum of \( k \) determinants of matrices of forms, with degree matrix \( A \).

**Proof.** Let \( A = (a_{ij}) \) be a square, homogeneous \( k \times k \) matrix of non-negative integers.

We will prove the Theorem by induction on the degree (= trace) \( d \) of \( A \).

If \( d = 0 \), then all the entries of \( A \) are 0 and the Theorem is obvious.

Assume, by induction, that the Theorem holds for all matrices with trace < \( d \) and assume that \( A \) has trace \( d \). We also assume that \( A \) is ordered. Then, since \( a_{k1} \geq 0 \), \( a_{1k} = a_{11} + a_{kk} \) and \( a_{kk} = \max\{a_{ij}\} \), we see that, after reflecting \( A \) along its anti-diagonal, if necessary, we may also assume \( a_{11} > 0 \). This is so because the determinant of a matrix with degree matrix \( A \) is equal to the determinant of any matrix with degree matrix obtained by reflecting across the anti-diagonal of \( A \).

Let \( B \) be the matrix obtained from \( A \) by subtracting 1 from the first row. Then \( B = (b_{ij}) \) is again a homogeneous matrix of non-negative integers, whose trace is \( d - 1 \). Thus the Theorem holds for \( B \). Hence, by Proposition 1.3 and Remark 1.4 for a general choice of \( k \) matrices of quaternary forms \( M_1, \ldots, M_k \), with \( \partial M_i = B \), the ideal generated by all the \((k-1) \times (k-1)\) minors of the matrices \( M_i \)'s coincides with \( R \) in degree \( d - 1 \). Now, if we forget the first rows of the matrices \( M_i \), we get \( k \) matrices \( G_1, \ldots, G_k \), of size \((k-1) \times k\), whose degree matrix \( B' \) equals \( B \) with the first row canceled (which is equal to \( A \) with the first row canceled). As \( A \) is ordered, \( T(B') \) is at most \( d \). Thus, by Proposition 2.1 all the \((k-1) \times (k-1)\) minors of \( G_1, \ldots, G_k \) generate an ideal \( J \) such that multiplication by a general linear form \( L \) determines a surjective map \((R/J)_{d-1} \to (R/J)_d \). Call \( M'_i \) the matrix obtained from \( M_i \) by multiplying the first rows by \( L \). It follows that the ideal generated by the \((k-1) \times (k-1)\) minors of the matrices \( M'_1, \ldots, M'_k \)'s coincides with \( R \) in degree \( d \). By semicontinuity, this last property holds for a general choice of \( k \) matrices \( H_1, \ldots, H_k \), with \( \partial H_i = A \) for all \( i \).

By Proposition 0.3 and Remark 0.4, the Theorem follows. \( \square \)
Remark 2.3. It is very reasonable to ask when the bound given in Theorem 2.2 is sharp.

As we observed in the Introduction (immediately after formula (11)), the bound is sharp when all the entries of the matrix $A$ are equal to a number $a$, which is sufficiently large with respect to $k$. Standard arithmetic shows that, indeed, the bound is sharp whenever $\min\{a_{ij}\} \gg k$. In all these cases, a general form of degree $d = \text{tr}(A)$ cannot be written as a sum of determinants of fewer than $k$ matrices of forms, with degree matrix $A$. Here the word "general" means that forms requiring less than $k$ summands are contained in a (non-trivial) Zariski closed subset of the space of all forms of degree $d$.

On the other hand, we will provide, in the next section, examples of degree matrices $A$ (with some small entry) and degrees $d$ such that fewer than $k$ summands are sufficient for general forms of degree $d$.

The problem of finding the complete range in which our theorem is sharp seems, at least technically, rather laborious.

3. IMPROVEMENTS AND OPEN QUESTIONS

In this section we show how the main theorem can sometimes be improved. We also give some open questions on the subject.

Assume that we are dealing with $3 \times 3$ matrices of forms in 4 variables. Then Theorem 2.2 above states that for any homogeneous matrix $A$ of degree $d$, with non-negative entries, a general form of degree $d$ is the sum of three determinants of matrices of forms, whose degree matrix is $A$.

We want to refine this statement and show that when the minimal entry of $A$ is 1, then we can write a general form of degree $d$ as the sum of determinants of two matrices whose degree matrix is $A$.

We will get the proof by using the Lefschetz property of Artinian complete intersection rings.

**Theorem 3.1.** Let $A$ be a $3 \times 3$ homogeneous matrix of non-negative integers, of degree $d$, whose minimal entry is 1. Then a general form of degree $d$ in 4 variables is the sum of the determinants of two matrices of forms, whose degree matrix is $A$.

**Proof.** Assume that $A = (a_{ij})$ is ordered. We have $a_{31} = 1$. We will prove the statement by induction on the biggest entry $a_{13}$ of $A$.

Assume $a_{13} = 1$. Then all the entries of $A$ are 1 and the degree of $A$ is 3. It is classical that a general cubic form is the determinant of a single $3 \times 3$ matrix of linear forms, since the corresponding surface contains a twisted cubic curve (see e.g., [1]). Thus the statement trivially holds in this case.

Assume now $a_{13} > 1$, so that $a_{11} + a_{33} = a_{31} + a_{13} > 2$. After taking the reflection along the anti-diagonal, we may assume that $a_{11} > 1$, so that all the entries in the first row are bigger than 1. Assume that the statement is true for all matrices with trace smaller than the trace $d$ of $A$.

Let $B$ be the matrix obtained by erasing the first row of $A$. By the Hilbert-Burch Theorem, the 3 minors of a general matrix of forms, with degree matrix $B$, vanish along an arithmetically Cohen-Macaulay curve $C$, which is contained in a complete intersection of surfaces of degrees $u = a_{22} + 1 = a_{21} + a_{32}$ and $t = 1 + a_{32} = a_{21} + a_{33}$. Thus, if $F_1, F_2$ are two general matrices of forms, with degree matrix equal to $B$, then the ideal $J$ generated by the $2 \times 2$ minors of $F_1, F_2$
is a quotient of a complete intersection artinian ideal generated by forms of degrees $u, u, t, t$. Since $(u + u + t + t - 4)/2 = a_{22} + a_{32}$ which is at most equal to $a_{22} + a_{33}$, it follows that for $n \geq a_{21} + a_{22} + a_{23}$ the multiplication map by a general linear form $(R/J)_n \to (R/J)_{n-1}$ surjects.

Now, let $A'$ be the $3 \times 3$ matrix obtained from $A$ by decreasing the first row by 1 and reordering (if necessary). Then $A'$ satisfies the inductive hypothesis, for its degree is smaller than $d$. Thus the statement holds for $A'$. In particular, by Remark 0.4 if we take two general matrices $G_1, G_2$ of forms, with degree matrix $A'$, then their $2 \times 2$ minors generate the ring $R$ in degree $d - 1$. Moreover, if $J$ is the ideal generated by the $2 \times 2$ minors obtained after deleting the first row in both $G_1, G_2$, then the multiplication by a general linear form $(R/J)_{d-1} \to (R/J)_d$ surjects. Thus, by multiplying the first row of both $G_1, G_2$ by a general linear form, we get two matrices whose degree matrix is $A$ and whose $2 \times 2$ minors generate $R$ in degree $d$. The statement follows from Remark 0.4. □

**Remark 3.2.** For a specific $k \times k$ matrix $A$ containing small positive integers one can check directly, with the aid of the Computer Algebra package [DGPS11], if the $k - 1 \times k - 1$ minors of $k_0 < k$ general matrices of forms, whose degree matrix is $A$, are sufficient to generate the polynomial ring $R$ in degree equal to the trace of $A$.

With this procedure, one can prove for instance that a general form of degree 6 in 4 variables is the sum of two determinants of $3 \times 3$ matrices of forms, all of whose entries have degree 2.

We see then that for some specific homogeneous matrices of non-negative integers $A = (a_{ij})$, the minimal number $s(A)$ of determinants of $k \times k$ matrices of forms with fixed degree matrix $A$, which are necessary to write a general form in 4 variables of degree $= tr(A)$, can be smaller than $k$. We show how one can produce a sharp conjecture for $s(A)$, at last when all the entries of $A$ are positive, by making more precise the construction already outlined in Example 0.1.

The matrix $A'$ obtained from $A$ by erasing the first row is the degree Hilbert-Burch matrix of arithmetically normal curves in $\mathbb{P}^3$, which fill a dense open subset of an irreducible component $\text{Hilb}(A')$ of the Hilbert scheme. The dimension of $\text{Hilb}(A')$ can be computed from the entries of $A'$. See [E75], Theorem 2.

Now one can construct the incidence variety:

$$Z = \{(F, C) : C \in \text{Hilb}(A') \text{ and } F \text{ a surface of degree } d \text{ containing } C\}.$$  

The fiber of the projection $Z \to \text{Hilb}(A')$ over $C$ is equal to $\mathbb{P}(H^0(I_C(d)))$. These are projective spaces of the same dimension independent of $C$. One can easily compute this dimension from the resolution induced by $A'$

$$0 \to \oplus^{k-1}O(-b_j) \to \oplus^kO(-a_i) \to I_C \to 0.$$  

In particular, $Z$ is irreducible and one can compute the dimension of $Z$ as a function of $d$ and the entries of $A'$.

Call $V(A)$ the closure of the image of the projection of $Z$ to the space $\mathbb{P}^N_A$ which parametrizes surfaces of degree $d$ in $\mathbb{P}^3$. Recall that $N_d = \binom{d+3}{d} - 1$. $V(A)$ is exactly the closure of the locus of surfaces of degree $d = \deg(A)$, containing a curve $C \in \text{Hilb}(A')$, i.e. it is the locus of those surfaces whose equation is the determinant of a single matrix of forms $G$, with $\partial G = A$. 


The closure of the set of forms which are the sum of $s$ determinants of matrices $G_1, \ldots, G_s$ with $\partial G_i = A$ for all $i$, corresponds to the $s$-th secant variety of $V(A)$.

The expected value for the dimension of the $s$-th secant variety of $V(A)$ is equal to the minimum between $s \dim(V(A)) + s - 1$ and the dimension of the whole space $N_d$. In particular, as soon as $s \dim(V(A)) + s - 1$ is bigger than or equal to $N_d$, i.e. as soon as

$$s \geq \frac{(d+3)}{3} \dim(V(A)) + 1$$

then one expects the $s$-th secant variety of $V(A)$ to fill $\mathbb{P}^{N_d}$. This means that a general form of degree $d$ should be the sum of $s$ determinants of matrices $G_1, \ldots, G_s$ with $\partial G_i = A$ for all $i$.

When the dimension of the $s$-th secant variety of $V(A)$ is different from the expected value, then $V(A)$ is said to be $s$-defective. Thus one should consider the following problem:

**Problem.** Are there homogeneous matrices $A$ of non-negative integers such that the corresponding variety $V(A)$ is defective? Can one classify them?

If one believes that $V(A)$ is not defective, then there is a conjecture for the minimal integer $s(A)$ such that a general form of degree $d = \deg(A)$ can be written as the sum of $s(A)$ determinants of matrices with degree matrix $A$.

Of course, the conjectured bound depends on the dimension of $V(A)$. Clearly, $V(A)$ is irreducible and one can compute its dimension once one knows the dimension of a general fiber of the projection $Z \to V(A)$. For a general choice of $C \in \text{Hilb}(A')$ we know that $C$ is a smooth curve (see [SS]). Since $d$ is bigger than the degree of a maximal generator of $I_C$ by Bertini we know that a general $F \in V(A)$ is a smooth surface in $\mathbb{P}^3$, hence is regular. It follows that the fiber of $Z \to V(A)$ over a general point $F$ is given by the union of a finite number of linear systems on $F$, each of them composed of arithmetically Cohen-Macaulay curves with the same Betti numbers as $C$. It follows that for $(F,C) \in Z$ general, the dimension of the fiber equals the dimension of the space of sections of the normal bundle $N_{C|F}$ of $C$ in $F$, which is equal to the dimension of the linear system $L_C$ on $F$ that contains the divisor $C$.

This last dimension can be obtained as follows: take a general surface $F'$ of minimal degree $a$ passing through $C$. The residue $C'$ of $C$ in the intersection $F \cap F'$ is also an arithmetically Cohen-Macaulay curve. Every curve which is in the linear system of $C$ is directly linked to $C'$ by a complete intersection of type $d,a$ with $a < d$. From this we see that the dimension of the linear system $L_C$ equals the dimension of the space of surfaces of degree $a$ passing through $C'$. This last number can be computed, since one can compute a minimal resolution for the ideal sheaf of $C'$, via the mapping cone procedure (see the description on page 4 of [M98]).

Summing up, we obtain a conjectured number for $s(A)$.

**Example 3.3.** Let us compute the conjectured value for $s(A)$ when $A$ is a $k \times k$ matrix of linear forms. We will assume that $V(A)$ is not defective. Notice that $\deg(A) = k$ in this case.

Let $C$ be an arithmetically Cohen-Macaulay curve as above. The minimal resolution of $I_C$ looks like

$$0 \to O^{k-1}(-k) \to \oplus^k O(-k+1) \to I_C \to 0$$
from which one gets $\dim(\text{Hilb}(A')) = 2k^2 - 2k$, which is equal to $4 \deg(C)$. Then one easily computes that the dimension of a general fiber of $Z \to V(A)$ is equal to $h^0(I_C(k)) - 1 = 3k$, so that $\dim(Z) = 2k^2 + k$.

We now compute, for $(F, C)$ general in $Z$, the dimension of the linear system $L_C$ on $F$, which contains $C$. Consider the residue $C'$ of $C$ in the intersection $F \cap F'$, where $F'$ is a surface of minimal degree $k - 1$ passing through $C$. $C'$ is an arithmetically Cohen-Macaulay curve whose resolution, computed via the mapping cone, is equal to the resolution of $C$. Thus, the space of surfaces of degree $k - 1$ through $C'$ has dimension $k - 1$.

We obtain $\dim(V(A)) = 2k^2 + 1$.

The computation in the previous example yields the following

**Conjecture.** A general form of degree $k$ in 4 variables is the sum of

$$s = \left[ \frac{k}{12} + \frac{1}{2} + \frac{10k}{12k^2 + 12} \right]$$

determinants of $k \times k$ matrices of linear forms.

We checked this Conjecture, using a computer aided procedure with the package [DGPS11], for some initial values of $k$.

**Remark 3.4.** Notice that in the case of matrices of linear forms, the conjectured minimal number of determinants needed for writing a general form of degree $k$, is always smaller than $k$.

**Example 3.5.** Let us perform the previous computation for a $3 \times 3$ degree matrix $A$ with all entries equal to $a$. Assuming that $V(A)$ is not defective and following the standard procedure we outlined above, we see that the dimension of $V(A)$ is

$$\dim(V(A)) = \frac{9a^3 + 54a^2 + 99a - 48}{6}$$

while the space of forms of degree $3a$ in $\mathbb{P}^3$ has dimension

$$\theta(3a) = \frac{27a^3 + 54a^2 + 33a}{6}.$$

In particular, notice that $\theta(3a) = \dim(V(A))$ when $a = 1$, consistent with the fact that $V(A)$ is the space of all cubic surfaces. Indeed a general cubic surface contains a twisted cubic curve and consequently a general cubic form is the determinant of a $3 \times 3$ matrix of linear forms.

When $a = 2, \ldots, 8$, the quotient $\theta(3a)/(\dim(V(A)) + 1)$ sits between 1 and 2. It could happen that in these cases the general form of degree $3a$ is the sum of two determinants of $3 \times 3$ matrices of forms of degree $a$. Remark [5.2] shows that this does indeed happen for $a = 2$. Using the same procedure, we checked that in all the cases $a = 3, \ldots, 8$, a general form of degree $3a$ is the sum of two determinants of $3 \times 3$ matrices of forms of degree $3a$.

For $a > 8$, the quotient $\theta(3a)/(\dim(V(A)) + 1)$ sits between 2 and 3. In these cases, at least 3 determinants are needed for obtaining a general form of degree $3a$. However, our Main Theorem shows that 3 determinants are always sufficient.

Thus, the example shows that our Main Theorem is sharp.

A similar computation, for the case of a $k \times k$ degree matrix $A$ with all entries equal to $a$ and with $a \gg k$, shows that the minimal number $s(A)$ of determinants
required to obtain a general form of degree $ka$ cannot be smaller than $k$ and hence, by our Main Theorem, must be equal to $k$.

**Remark 3.6.** One might ask: What happens when some entry of the degree matrix $A = (a_{ij})$ is negative?

If $G$ is a matrix of forms with $\partial G = A$ and $a_{ij} < 0$ for some $i, j$, then necessarily the corresponding entry $g_{ij}$ of $G$ is the 0 polynomial.

Assume that $A$ is ordered and $a_{ii} < 0$ for some $i$. Then $a_{ij} = a_{ji} = 0$ for all $j \leq i$. It follows that any such matrix of forms $G$ with degree matrix $A$ has a block of zeroes which touches the main diagonal. Consequently, $\det(G) = 0$. In particular, no non-zero forms can be the sum of any number of determinants of matrices $G$ with $\partial G = A$.

When the $a_{ii}$’s are all non-negative but still there exists some entry $a_{ij} < 0$ (so that $j < i$ when $A$ is ordered), the question about the minimal number of determinants of matrices of forms $G$ with $\partial G = A$, which are necessary to express a general form of degree $\deg(A)$, is still open.

4. Extension to a larger number of variables.

When the number of variables increases we can find similar results on the number of determinants that one needs in order to express a general form. Unfortunately the required number of determinants grows exponentially.

Indeed, as we noted in the introduction, for a fixed $k \times k$ homogeneous matrix $A$ of non-negative integers, the question amounts to asking for the minimal $s$ such that for a general choice of matrices $G_1, \ldots, G_s$ with degree matrix $A$, the ideal $I$ generated by all the $(k-1) \times (k-1)$ (submaximal) minors of the $G_i$’s coincides with the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$ in degree $d = \deg(A)$.

Assume that all the entries of $A$ are equal to $a$, so that $d = ka$. Since $A$ has $k^2$ submaximal minors, then

$$\dim I_d \leq k^2 s \dim R_a = \frac{k^2 sa^{n-1}}{(n-1)!} + o(a^{n-1})$$

while the dimension of $R_d$ is $a^{n-1}k^{n-1}/(n-1)! + o(a^{n-1})$.

So, it is immediate to see that, at least when $a$ grows, in order to have $I_d = R_d$ then $s$ must be asymptotically equal to $k^{n-3}$.

With a procedure which is similar to the proof of the Main Theorem (but with a much heavier notation!), and using induction on the number of variables, we can prove:

**Theorem 4.1.** Let $A$ be a homogeneous $k \times k$ matrix of non-negative integers, with degree $d$. Then a general form of degree $d$ in $n \geq 3$ variables is the sum of $k^{n-3}$ determinants of matrices of forms, with degree matrix $A$.

**Proof.** We make induction on the number $n$ of variables. The case $n = 3$ is the main result in [CM12], while the case $n = 4$ is Theorem 2.2 above.

Assume the Theorem is true for forms in $n-1$ variables. We show how the argument of Lemma 1.4, Proposition 2.1 and Theorem 2.2 can be modified, to provide a proof of the statement for forms in $n$ variables.

Fix the matrix $A$ and assume it is ordered. Call $d$ the degree of $A$. Forgetting the first row of $A$, we obtain a $(k-1) \times k$ non-negative integer matrix $A'$, with $T(A') \leq d$. 

The first step consists in proving that given a general set of $k^{n-3}$ matrices of forms $G_1, \ldots, G_{k^{n-3}}$ in the ring $R' = \mathbb{C}[x_1, \ldots, x_{n-1}]$ with $n - 1$ variables, with $\partial G_i = A'$ for all $i$, the ideal generated by all the minors of the $G_i$’s coincides with $R'$ in degree $T(A')$. This is true by the inductive assumption, when all the rows of $A$ are equal. Indeed, a special instance of the $G_i$’s can be obtained by taking $k^{n-4}$ matrices $H_1, \ldots, H_{k^{n-4}}$ of forms in $R'$, with $\partial H_i = A$ for all $i$, and taking the $G_i$’s equal to the matrices obtained by erasing one line from the $H_j$’s, in all possible ways. When the rows of $A$ are different, we make induction on the diameter of $A$, exactly as in the proof of Lemma 0.3 with the unique difference that one passes from condition $MT(A''_{i+j-1}) = (A'^{k^{n-4}(j-1)}_{i+1}, A''_{i+k})$ to condition $\partial H_i = A$ for all $i$. Moreover, by forgetting the first rows of the matrices $M_i$, we get $k$ matrices $G_1, \ldots, G_k$, of size $(k-1) \times k$, whose degree matrix $B'$ satisfies $T(B') \leq d$. Thus, by the surjectivity proved above, all the $(k-1) \times (k-1)$ minors of $G_1, \ldots, G_k$ generate an ideal $J$ such that the multiplication by a general linear form $L$ determines a surjective map $(R/J)_{d-1} \rightarrow (R/J)_d$. Then, call $M'_i$ the matrix obtained from $M_i$ by multiplying the first row by $L$. It follows that the ideal generated by the $(k-1) \times (k-1)$ minors of the matrices $M'_1, \ldots, M'_{k^{n-3}}$ coincides with $R$ in degree $d$. By semicontinuity, this last property holds for a general choice of $k^{n-3}$ matrices $H$, with $\partial H_i = A$ for all $i$.

The Theorem follows by Proposition 0.3 and Remark 0.4.

□

Remark 4.2. As we observed in the statement of the previous Theorem, $k^{n-3}$ is almost always a sharp bound.

With an argument analogous to the discussion in Example 3.5, one can show that the bound of Theorem 4.4 is sharp when the $k \times k$ degree matrix has all entries equal to a positive integer $a$ and $a \gg k$. In these cases, forms that can be written as the sum of fewer than $k^{n-3}$ determinants are contained in a (non-trivial) Zariski closed subset of the space of all forms of degree $d$.

For example, when $A$ is a $4 \times 4$ matrix with entries equal to $a \gg 4$, working in five variables, it follows that one needs 16 determinants in order to express a general form of degree $4a$.

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