A Note on Global Suprema of Band-Limited Spherical Random Functions

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Abstract
In this note, we investigate the behaviour of suprema for band-limited spherical random fields. We prove upper and lower bound for the expected values of these suprema, by means of metric entropy arguments and discrete approximations; we then exploit the Borell-TIS inequality to establish almost sure upper and lower bounds for their fluctuations. Band limited functions can be viewed as restrictions on the sphere of random polynomials with increasing degrees, and our results show that fluctuations scale as the square root of the logarithm of these degrees.

• Keywords and Phrases: Spherical Random Fields, Suprema, Metric Entropy, Almost Sure Convergence
• AMS Classification: 60G60; 62M15, 53C65, 42C15

1 Introduction
The analysis of the behaviour of suprema of Gaussian processes is one of the classical topics in probability theory ([1],[3]); in this note, we shall be concerned with suprema of band-limited random fields defined on the unit sphere $S^2$. More precisely, let $T : S^2 \times \Omega \to \mathbb{R}$ be a measurable zero mean, finite variance Gaussian field defined on for some probability space $\{\Omega, \mathcal{F}, P\}$; we assume $T(.)$ is isotropic, e.g. the vectors

$\{T(x_1), ..., T(x_k)\}$ and $\{T(gx_1), ..., T(gx_k)\}$

have the same law, for all $k \in \mathbb{N}$, $x_1, ..., x_k \in S^2$ and $g \in SO(3)$, the group of rotations in $\mathbb{R}^3$. It is then known that the field $\{T(.)\}$ is necessarily mean square

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continuous ([15]) and the following spectral representation holds:

\[ T(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \]

where the spherical harmonics \( \{Y_{\ell m}\} \) form an orthonormal system of eigenfunctions of the spherical Laplacian, \( \Delta_{S^2} Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m} \) (see [19, 14]), while the random coefficients \( \{a_{\ell m}\} \) form a triangular array of complex-valued, zero-mean, uncorrelated Gaussian variables with variance \( \mathbb{E}|a_{\ell m}|^2 = C_\ell \), the angular power spectrum of the field. In the sequel, we shall adopt the following general model for the behaviour of \( \{C_\ell\} \); as \( \ell \to \infty \), there exist \( \alpha > 2 \) and a positive rational function \( G(\ell) \) such that

\[ C_\ell = G(\ell) \ell^{-\alpha}, \quad 0 < c_1 < G(\ell) < c_2 < \infty \tag{1} \]

Spherical random fields have recently drawn a lot of applied interest, especially in an astrophysical environment (see [6, 14]); closed form expressions for the density of their maxima and for excursion probabilities have been given in ([10], [9], [16]). In particular, the latter references exploit the Gaussian Kinematic Fundamental formula by Adler and Taylor (see [1]) to approximate excursion probabilities by means of the expected value of the Euler-Poincaré characteristic for excursion sets. It is then easy to show that

\[ \mathbb{E} \mathcal{L}_\Phi(A_u(T)) = 2 \{1 - \Phi(u)\} + 4\pi \left\{ \sum_{\ell} \frac{2\ell + 1}{4\pi} C_\ell \frac{\ell(\ell + 1)}{2} \right\} \frac{u \phi(u)}{\sqrt{(2\pi)^3}}, \]

where \( \phi, \Phi \) denote density and distribution function of a standard Gaussian variable, while \( A_u(T) := \{x \in S^2 : T(x) \geq u\} \). It is also an easy consequence of results in Ch.14 of [11] that there exist \( \alpha > 1 \) and \( \mu^+ > 0 \) such that, for all \( u > \mu^+ \)

\[ \mathbb{P} \left\{ \sup_{x \in S^2} T(x) > u \right\} = 2 \{(1 - \Phi(u) + u \phi(u) \lambda) \leq \{4\pi \lambda \} \exp\left(-\frac{\alpha u^2}{2}\right), \tag{2} \]

where

\[ \lambda := \sum_{\ell} \frac{2\ell + 1}{4\pi} C_\ell \frac{\ell(\ell + 1)}{2}, \]

denotes the derivative of the covariance function at the origin, see again ([10], [9], [16]).

When working on compact domains as the sphere, it is often of great interest to focus on sequences of band-limited random fields; for instance, a very powerful tool for data analysis is provided by fields which can be viewed as a sequence of wavelet transforms (at increasing frequencies) of a given isotropic spherical field \( T \). More precisely, take \( b(\cdot) \) to be a \( C^\infty \) function, compactly supported in \([\frac{1}{2}, 2]\); having in mind the wavelets interpretation, it would be natural to impose the partition of unity property \( \sum_{\ell} b^2(\frac{\ell}{2^j}) = 1 \), but this condition however plays no
role in our results to follow. Let us now focus on the sequence of band-limited spherical random fields

\[ \beta_j(x) := \sum_{\ell=2^{j-1}}^{2^{j+1}} b(\ell/2^j) a_{\ell m} Y_{\ell m}(x), \]

which have a clear interpretation as wavelet components of the original field, and as such lend themselves to a number of statistical applications, see for instance [4], [8], [17], [18]. Band-limited spherical fields have also been widely studied in other context of mathematical physics, although in such cases \( b(\cdot) \) is not necessarily assumed to be smooth, see for instance [23] and the references therein.

In the sequel, it will be convenient to normalize the variance of \( \{\beta_j(x)\} \) to unity, and thus focus on

\[ \tilde{\beta}_j(x) := \frac{\beta_j(x)}{\sqrt{\sum_{\ell} b(\ell/2^j)^2 2\ell+1 C_{\ell}}}. \]

The sequence of fields \( \{\tilde{\beta}_j(x)\} \) has covariance functions

\[ \rho_j(x, y) = \sum_{\ell} b(\ell/2^j)^2 2\ell+1 C_{\ell} P_\ell(\langle x, y \rangle) \frac{1}{\sum_{\ell} b(\ell/2^j)^2 2\ell+1 C_{\ell}}, \]

and second spectral moments

\[ \lambda_j := \sum_{\ell} b(\ell/2^j)^2 2\ell+1 C_{\ell} P_\ell(1) = \frac{\sum_{\ell} b(\ell/2^j)^2 2\ell+1 C_{\ell} \ell(\ell+1)}{\sum_{\ell} b(\ell/2^j)^2 2\ell+1 C_{\ell}}, \]

see [16]. For fixed \( j \), as in (2) it follows from results in [1] that there exist \( \alpha > 1 \) and \( \mu^+ > 0 \) such that, for all \( u > \mu^+ \)

\[ \mathbb{P} \left\{ \sup_{x \in S^2} \tilde{\beta}_j(x) > u \right\} \leq 2 \left\{ (1 - \Phi(u) + u \phi(u) \lambda_j) \right\} \leq \{ 4\pi \lambda_j \exp(-\alpha u^2/2). \quad (3) \]

However, here for \( j \to \infty \) we also have \( \lambda_j \to \infty \), whence the previous result clearly becomes meaningless. Intuitively, sample paths become rougher and rougher as \( j \) grows, hence any fixed threshold is crossed with probability tending to one. In [16], uniform bounds for band-limited fields have indeed been established, covering even non-Gaussian circumstances; however these bounds require a further averaging in the space domain for the fields considered, and this averaging ensures the uniform boundedness of \( \lambda_j \); in these circumstances, the multiplicative constant on the right-hand side of (3) can be simply incorporated into the exponential choosing a different constant \( 1 < \alpha' < \alpha \).

There is, however, a question that naturally arises for the cases where \( \lambda_j \) diverges - e.g., whether it is possible to provide bounds on global suprema,
allowing the thresholds to grow with frequency. This is a natural question for a number of statistical applications, for instance when considering thresholding estimates or multiple testing. Loosely speaking, the issue we shall be concerned with is then related to the existence of a growing sequence $\tau_j$ and positive constants $c_1, c_2, c'_1, c'_2$ such that
\[
c_1 \leq \mathbb{E} \left\{ \frac{\sup_{x \in S^2} \tilde{\beta}_j(x)}{\tau_j} \right\} \leq c_2, \quad \text{and} \quad \mathbb{P} \left\{ c'_1 \leq \frac{\sup_{x \in S^2} \tilde{\beta}_j(x)}{\tau_j} \leq c'_2 \right\} = 1.
\]
In fact, we shall be able to be more precise with our lower bounds. To make this statement more precise, it will be convenient to write $\ell_j := 2^j$; we shall then establish the following

**Theorem 1** There exist positive constants $\gamma_1, \gamma_2 \geq 1$, such that
\[
1 \leq \liminf_j \frac{\mathbb{E} \left\{ \sup_{x \in S^2} \tilde{\beta}_j(x) \right\}}{\sqrt{4 \log \ell_j}} \leq \limsup_j \frac{\mathbb{E} \left\{ \sup_{x \in S^2} \tilde{\beta}_j(x) \right\}}{\sqrt{4 \log \ell_j}} \leq \gamma_1,
\]
and
\[
\mathbb{P} \left( 1 \leq \liminf_j \frac{\sup_{x \in S^2} \tilde{\beta}_j(x)}{\sqrt{4 \log \ell_j}} \leq \limsup_j \frac{\sup_{x \in S^2} \tilde{\beta}_j(x)}{\sqrt{4 \log \ell_j}} \leq \gamma_2 \right) = 1.
\]

The corresponding upper bounds are proved in Section 2, while the proofs for the lower bounds are collected in Section 3.

The random functions $\{\tilde{\beta}_j(.)\}$ can be viewed as restrictions to the sphere of linear combinations of polynomials with increasing degree $p_j = 2^j \ell_j \quad (14)$. Our results can then be summarized by simply stating that as $j \to \infty$, the supremum of $\{\tilde{\beta}_j(.)\}$ grows as twice the square root of the logarithm of $p_j$.

## 2 Metric Entropy and Upper Bounds

The result we shall give in this Section is the following.

**Proposition 2** There exist a positive constant $c$ such that, for all $j \in \mathbb{N}$
\[
\mathbb{E} \left\{ \sup_{x \in S^2} \tilde{\beta}_j(x) \right\} \leq c \sqrt{4 \log \ell_j}.
\]

Moreover there exist another positive constant $C$ such that
\[
\mathbb{P} \left( \limsup_j \frac{\sup_{x \in S^2} \tilde{\beta}_j(x)}{\sqrt{4 \log \ell_j}} \leq C + \frac{1}{\sqrt{2}} \right) = 1.
\]
Proof. Define the canonical (Dudley) metric on $S^2$ as follows:
\[
d_j(x, y) = \sqrt{\mathbb{E} \left( \beta_j(x) - \beta_j(y) \right)^2} = \sqrt{2 - 2\rho_j(x, y)},
\]
see [1]. Note that since $\beta_j(x)$ is isotropic, all the distances can be measured from one fixed point (say the north pole). Therefore,
\[
d_j^2(x, y) = 2(1 - \rho_j((\cos \theta)))
\]
where $\theta := \arccos \langle x, y \rangle$ is the usual geodesic distance on the sphere, and
\[
1 - \rho_j(\cos \theta) = \frac{\sum_{\ell=2j-1}^{2j+1} b^2 \left( \frac{\ell}{2} \right) \frac{(2\ell+1)}{4\pi} C_\ell \left( 1 - P_\ell(\cos \theta) \right)}{\sum_{\ell=2j-1}^{2j+1} b^2 \left( \frac{\ell}{2} \right) \frac{(2\ell+1)}{4\pi} C_\ell}
\]
Now fix $\theta < 1/(K\ell)$ and use Hilb’s asymptotics ([21]) to obtain
\[
P_\ell(\cos \theta) = \sqrt{\frac{\theta}{\sin \theta}} J_0(\frac{\ell+1}{2}\theta) + \delta(\theta), \quad \delta(\theta) = O(\theta^2),
\]
where
\[
J_0(z) := \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2},
\]
is the standard Bessel function of zeroth order; note also that
\[
\lim_{K \to \infty} \sup_{\theta \leq (K\ell)^{-1}} \left| \frac{1 - J_0((\ell + 1/2)\theta)}{\ell^2\theta^2} - \frac{1}{4} \right| = \lim_{K \to \infty} \sup_{x \leq K^{-1}} \left| \frac{1 - J_0(x)}{x^2} - \frac{1}{4} \right| = 0,
\]
which means that for all $\delta > 0$, there exist $K_\delta$ small enough so that
\[
\frac{1}{4} - \delta \ell^2\theta^2 \leq 1 - J_0((\ell + 1/2)\theta) \leq \frac{1}{4} - \delta \ell^2\theta^2, \quad \text{for all } \theta < \frac{K_\delta}{\ell}.
\]
Combining these bounds with Hilb’s asymptotics, we get for $\theta < \frac{K_\delta}{\ell}$
\[
\frac{1}{4} - \delta \ell^2\theta^2 + O(\theta^2) \leq 1 - P_\ell(\cos \theta) \leq (\frac{1}{4} + \delta)\ell^2\theta^2 + O(\theta^2).
\]
It follows that
\[
1 - \rho_j(\cos \theta) = \frac{\sum_{\ell=2j-1}^{2j+1} b^2 \left( \frac{\ell}{2} \right) \frac{(2\ell+1)}{4\pi} C_\ell \left( 1 - P_\ell(\cos \theta) \right)}{\sum_{\ell=2j-1}^{2j+1} b^2 \left( \frac{\ell}{2} \right) \frac{(2\ell+1)}{4\pi} C_\ell}
\]
\[
\leq \frac{1}{4} \theta^2(2j^2 + (1/2)^2) + O(\theta^2)
\]
and likewise
\[
1 - \rho_j(\cos \theta) \geq \frac{1}{4} \theta^2(2j^2 + 1/2)^2 + O(\theta^2),
\]
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thus implying that, for some constants $c_1, c_2 > 0$
\[c_1\theta^2 2^j \leq 1 - \rho_j(\cos \theta) \leq c_2\theta^2 2^j.\]
We hence get
\[c'_1\theta^2 \leq \frac{d_j^2(0, (\theta, \phi))}{\ell_j^2} \leq c'_2\theta^2,
\]
and more generally for $\xi_1, \xi_2 \in S^2$
\[c'_1 d_S^2(\xi_1, \xi_2) \leq \frac{d_j^2(\xi_1, \xi_2)}{\ell_j^2} \leq c'_2 d_S^2(\xi_1, \xi_2),
\]
where $d_S(\xi_1, \xi_2) := \arccos(\langle \xi_1, \xi_2 \rangle)$ is the standard spherical distance. Now for $\varepsilon < C$ and $\theta < \frac{C}{\ell_j}$, define the sequence of $d_j$-balls $B_{d_j}(\xi_{jk}, \varepsilon) = \{u \in S^2 : d_j(\xi_{jk}, u) \leq \varepsilon\}$, which can be rewritten as
\[B_{d_j}(\xi_{jk}, \varepsilon) = \{\xi \in S^2 : d_j(\xi_{jk}, \xi) = \ell_j d_S(\xi_{jk}, \xi) \leq \varepsilon\}.
\]
Hence $B_{d_j}(\xi_{jk}, \varepsilon)$ is a spherical cap of radius $\sim \frac{\varepsilon}{\ell_j}$, with Euclidean volume
\[B_d(\xi_{jk}, \varepsilon) \sim \frac{\varepsilon^2}{\ell_j^2}.
\]
It follows that the number of $d_j$-balls needed to cover the sphere is asymptotic to $N_j(\varepsilon) \sim \frac{\ell_j^2}{\varepsilon^2}$. Consequently, by Theorem 1.3.3. of [1], for any $\delta \in (0, \pi]$ there exists a universal constant $K^*$ such that
\[
E \left(\sup_x \beta_j(x)\right) \leq K^* \int_0^\delta \sqrt{\log N_j(\varepsilon)} \, d\varepsilon
\leq K^* \left\{ \int_0^{C/\ell_j} \sqrt{\log N_j(\varepsilon)} \, d\varepsilon + \int_{C/\ell_j}^\delta \sqrt{\log N_j(\varepsilon)} \, d\varepsilon \right\}.
\]
Clearly for $\varepsilon > C/\ell_j$ one has $N_j(\varepsilon) \leq c\ell_j^4$, whence
\[\int_{C/\ell_j}^\delta \sqrt{\log N_j(\varepsilon)} \, d\varepsilon \leq c' \sqrt{4 \log \ell_j}.
\]
On the other hand
\[\int_0^\delta \sqrt{\log N_j(\varepsilon)} \, d\varepsilon = \int_0^\delta \sqrt{2 \log \left(\frac{\ell_j}{\varepsilon}\right)} \, d\varepsilon = \ell_j \int_{\sqrt{2 \log \frac{L_j}{\ell_j}}}^\infty v^2 \exp \left(-\frac{v^2}{2}\right) \, dv,
\]
with the change of variables $\tilde{x}_j = \exp (-v^2/2)$, whence

\[
\int_0^\delta \sqrt{\log N_j(\varepsilon)} \, d\varepsilon = \ell_j \left( -1 \right) \exp \left( -\frac{v^2}{2} \right) \left| \frac{\ell_j}{2\log \frac{\delta}{\ell_j}} \right| + \int_{\frac{2\log \frac{\delta}{\ell_j}}{2}}^\infty \exp \left( -\frac{v^2}{2} \right) \, dv
\]

\[
\leq \ell_j \left( \frac{\delta}{\ell_j} \sqrt{2 \log \frac{\ell_j}{\delta}} \right) + \left( 2 \log \left( \frac{\ell_j}{\delta} \right) \right)^{-1/2} \frac{\delta}{\ell_j}
\]

\[
= \delta \left( \sqrt{2 \log \left( \frac{\ell_j}{\delta} \right)} + \left( 2 \log \left( \frac{\ell_j}{\delta} \right) \right)^{-1/2} \right) \leq \sqrt{4 \log \ell_j}.
\]

Taking the same $C$ as in the entropy upper bound, and using the Borell-TIS inequality (cf. [1]) we have

\[
\mathbb{P} \left( \sup_x \tilde{\beta}_j(x) > (C + \varepsilon) \sqrt{4 \log \ell_j} \right) \leq \mathbb{P} \left( \|\tilde{\beta}_j\| > E \|\tilde{\beta}_j\| + \varepsilon \sqrt{4 \log \ell_j} \right)
\]

\[
\leq \exp \left( -\frac{4\varepsilon^2 \log \ell_j}{2} \right)
\]

\[
= \exp \left( -\log \frac{\ell_j^2}{2\varepsilon^2} \right)
\]

\[
= \frac{1}{\ell_j^2} \to 0, \quad \forall \varepsilon > 0
\]

Now for (5), taking $\varepsilon > \frac{1}{\sqrt{2}}$ in the above expression, we obtain summable probabilities, and then by a simple application of the Borel-Cantelli Lemma we have that

\[
\mathbb{P} \left( \lim \sup_j \frac{\|\tilde{\beta}_j\|}{\sqrt{4 \log \ell_j}} \geq C + \frac{1}{\sqrt{2}} \right) \leq \lim_{j \to \infty} \sum_{j'=j}^\infty \mathbb{P} \left( \frac{\|\tilde{\beta}_{j'}\|}{\sqrt{4 \log \ell_{j'}}} \geq C + \frac{1}{\sqrt{2}} \right) = 0.
\]

\[\blacksquare\]

### 3 Discretization and Lower Bounds

As explained in the Introduction, this Section is devoted to the proofs for the lower bounds that follow.

**Proposition 3** We have

\[
\lim_{j} \inf \frac{\mathbb{E} \left\{ \sup_{x \in S^2} \tilde{\beta}_j(x) \right\}}{\sqrt{4 \log \ell_j}} \geq 1.
\]
and
\[
P \left( \liminf_j \frac{\sup_{x \in S^2} \tilde{\beta}_j(x)}{\sqrt{4 \log \ell_j}} \geq 1 \right) = 1. \tag{7}
\]

**Proof.** We start showing that, for all \( \delta > 0 \),
\[
\lim_j P \left( \frac{\sup_{x \in S^2} \tilde{\beta}_j}{\sqrt{4 \log \ell_j}} > 1 - \delta \right) = 1. \tag{8}
\]
Note first that \( \sup \tilde{\beta}_j \geq \sup_k \tilde{\beta}_{j,k} \), where \( \{ \tilde{\beta}_{j,k} \} \) is any discrete sample taken from \( \tilde{\beta}_j \). Now, let us choose a grid of points such that the distance between them is at least \( 2^{-j(1-\delta)} \), for some \( \delta > 0 \) - e.g., a \( 2^{-j(1-\delta)} \)-net, see [5]. Note that the vectors \( \beta_j \) and \( \tilde{\beta}_j \) both have cardinality of order \( 2^{2j(1-\delta)} \). By using the correlation inequality given in Lemma 10.8 of [14], we have
\[
E \tilde{\beta}_{j,k} \tilde{\beta}_{j,k'} \leq C \frac{1}{(1 + 2^{5\delta})^M},
\]
where \( M \in \mathbb{N} \) can be chosen arbitrarily large. The idea of the proof is to approximate these subsampled coefficients by means of a triangular array of Gaussian i.i.d. random variables, say \( \hat{\beta}_{j,k} \). More precisely, let \( \Sigma_j \) be the covariance matrix of the Gaussian vector \( \tilde{\beta}_j \); then define \( \hat{\beta}_j = \Sigma_j^{-1/2} \tilde{\beta}_j \), which is clearly a vector of i.i.d. Gaussian variables, and let \( \lambda_{j,\max} \) and \( \lambda_{j,\min} \) be the largest and the smallest eigenvalues of the matrix \( \Sigma_j \). Then
\[
\lambda_{j,\max}, \lambda_{j,\min} = 1 + O(\varepsilon_j),
\]
for a deterministic sequence \( \{ \varepsilon_j \} \) which goes to zero faster than any polynomial (nearly exponentially). Indeed
\[
\lambda_{j,\max} = \sup_x x' \Sigma_j x = \sup_x x' (\Sigma_j - I + I) x = \sup_x x' (\Sigma_j - I) x + 1
\]
\[
\leq \ell_j^{4(1-\delta)} \frac{C_M}{1 + 2^{5\delta}M} + 1,
\]
where the bound follows crudely from the cardinality of the off-diagonal terms in the matrix. Similarly,
\[
\lambda_{j,\min} = \inf_x x' \Sigma_j x = \inf_x x' (\Sigma_j - I + I) x
\]
\[
= \inf_x x' (\Sigma_j - I) x + 1 \geq 1 - \sup |x' (\Sigma_j - I) x |
\]
\[
\geq 1 - \ell_j^4 \frac{C_M}{1 + \ell_j^M}.
\]

As a consequence, writing $\| \cdot \|_2$ for the Euclidean inner product in the appropriate dimension we have

$$E \left( \sup |\tilde{\beta}_{j,k} - \hat{\beta}_{j,k}| \right) \leq \sqrt{E \| \tilde{\beta}_j . - \hat{\beta}_j . \|_2^2}$$

$$= \sqrt{E \| (I - \Sigma^{-1/2})\tilde{\beta}_j . \|_2^2} \leq |1 - \lambda_{\text{max}}| \sqrt{E \| \tilde{\beta}_j . \|_2^2}$$

$$\leq \ell_j^{4(1-\delta)} \frac{C_M}{1 + \ell_j^{3M}} \cdot \ell_j^2 = \ell_j^{2+4(1-\delta)} \frac{C_M}{1 + \ell_j^{3M}} = O(\ell_j^{6-\delta M}).$$

We can now exploit a classical result by Berman (7) to conclude that

$$P \left( \sup_{x \in S^2} \tilde{\beta}_{j,k} \sqrt{4 \log \ell_j} < 1 - \varepsilon \right) \rightarrow 0,$$

as $j \to \infty$, for all $\varepsilon > 0$. Thus (8) is established; (6) follows immediately, given that $\delta$ is arbitrary. To establish (7), we use again the Borel-Cantelli Lemma, so that we need to prove that, for all $\varepsilon > 0$

$$\sum_j P \left( \sup_{x \in S^2} \tilde{\beta}_{j,k} \sqrt{4 \log \ell_j} < 1 - \varepsilon \right) < \infty.$$

Clearly

$$P \left( \sup_{x \in S^2} \tilde{\beta}_{j,k} \sqrt{4 \log \ell_j} < 1 - \varepsilon \right) \leq P \left( \sup_{x \in S^2} \tilde{\beta}_{j,k} \sqrt{4 \log \ell_j} < 1 - \varepsilon \right),$$

whence it suffices to prove that

$$\sum_j P \left( \sup_{x \in S^2} \tilde{\beta}_{j,k} \sqrt{4 \log \ell_j} < 1 - \varepsilon \right) < \infty.$$

Now

$$P \left( \sup_{x \in S^2} \tilde{\beta}_{j,k} \sqrt{4 \log \ell_j} < 1 - \varepsilon \right) = P \left( \sup_k \left( \tilde{\beta}_{j,k} - \hat{\beta}_{j,k} + \hat{\beta}_{j,k} \right) \sqrt{4 \log \ell_j} < 1 - \varepsilon \right)$$
≤ \mathbb{P}\left(\frac{\sup_k \hat{\beta}_{j,k} - \sup_k (\hat{\beta}_{j,k} - \bar{\beta}_{j,k})}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon\right)

≤ \mathbb{P}\left(\frac{\sup_k \hat{\beta}_{j,k} - \sup_k |\beta_{j,k} - \bar{\beta}_{j,k}|}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon\right)

= \mathbb{P}\left(\frac{\sup_k \hat{\beta}_{j,k}}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon + \frac{\sup_k |\beta_{j,k} - \bar{\beta}_{j,k}|}{\sqrt{4 \log \ell_j}}\right)

≤ \mathbb{P}\left(\frac{\sup_k \hat{\beta}_{j,k}}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\frac{\sup_k |\beta_{j,k} - \bar{\beta}_{j,k}|}{\sqrt{4 \log \ell_j}} > \varepsilon \frac{\varepsilon}{2}\right)

≤ \mathbb{P}\left(\frac{\sup_k \hat{\beta}_{j,k}}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon \frac{\varepsilon}{2}\right) + O(\varepsilon^{6 - \varepsilon M}).

The second term above is clearly summable, for all fixed \( \varepsilon > 0 \), by simply taking \( M \) large enough. To check summability of the first term we write

\[ \mathbb{P}\left(\frac{\sup_k \hat{\beta}_{j,k}}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon \frac{\varepsilon}{2}\right) = \prod_k \mathbb{P}\left(\frac{\hat{\beta}_{j,k}}{\sqrt{4 \log \ell_j}} < 1 - \varepsilon \frac{\varepsilon}{2}\right) \]

\[ = \left(\mathbb{P}\left(\hat{\beta}_{j,1} < \left(1 - \varepsilon \frac{\varepsilon}{2}\right)\sqrt{4 \log \ell_j}\right)\right)^{\ell_j^2} \]

\[ = \left(1 - \mathbb{P}\left(\hat{\beta}_{j,1} > \left(1 - \varepsilon \frac{\varepsilon}{2}\right)\sqrt{4 \log \ell_j}\right)\right)^{\ell_j^2} \]

\[ \leq \left(1 - \frac{1}{\left(1 - \varepsilon \frac{\varepsilon}{2}\right)^2 \sqrt{4 \log \ell_j}}\right)^{\ell_j^2} \left(1 - \frac{1}{\left(1 - \varepsilon \frac{\varepsilon}{2}\right)^2 \sqrt{4 \log \ell_j}}\right)^{\ell_j^2} \]

\[ \leq \left(1 - \frac{1}{2\left(1 - \varepsilon \frac{\varepsilon}{2}\right)^2 \sqrt{4 \log \ell_j}}\right)^{\ell_j^2} \]

where we have used Mill’s inequality for standard Gaussian variables, \( \mathbb{P}\{Z > z\} \geq \frac{1}{\sqrt{2\pi}z^2} \phi(z) \). Since \( (1 - \varepsilon \frac{\varepsilon}{2})^2 < 1 \), this term decays exponentially, and it is hence summable. The proof of (7) is hence concluded.

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