PRIMITIVE PERMUTATION GROUPS AS PRODUCTS OF POINT STABILIZERS

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Abstract. We prove that there exists a universal constant $c$ such that any finite primitive permutation group of degree $n$ with a non-trivial point stabilizer is a product of no more than $c \log n$ point stabilizers.

1. Introduction

Given a finite group $G$ and a subgroup $H$ of $G$ whose normal closure is $G$, one can show, by a straightforward elementary argument, that $G$ is the setwise product of at least $\frac{\log |G|}{\log |H|}$ conjugates of $H$. A far reaching conjecture of Liebeck, Nikolov and Shalev states [8] that in the case that $G$ is a non-abelian simple group, $\frac{\log |G|}{\log |H|}$ is in fact the right order of magnitude for the minimal number of conjugates of $H$ whose product is $G$, namely, there exists a universal constant $c$ such that for any non-abelian simple group $G$ and any non-trivial $H \leq G$, the group $G$ is the product of no more than $c \log |G| \log |H|$ conjugates of $H$. Later on, in [9], this conjecture was extended to allow $H$ to be any subset of $G$ of size at least 2. Some weaker versions of these conjectures are proved in [8, Theorem 2], [9, Theorem 3], and [4, Theorem 1.3].

Here we look for a universal upper bound on the minimal length of a product covering of a finite primitive permutation group by conjugates of a point stabilizer. We will prove the following logarithmic bound:

**Theorem 1.** There exists a universal constant $c$ such that if $G$ is any primitive permutation group of degree $n$ with a non-trivial point stabilizer $H$ then $G$ is a product of at most $c \log n$ conjugates of $H$.

Note that in most relevant cases, $\frac{\log |G|}{\log |H|} < \log |G : H| = \log n$ (see Lemma 2.1). Thus we do not know whether the bound provided by Theorem 1 is the best possible. In fact, on the basis of currently published results we don’t even know if this bound can be improved for any particular O’Nan-Scott family of primitive groups. We believe that these questions deserve further investigation.

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1 All groups discussed are assumed to be finite.

2 Throughout the paper, log stands for logarithm in base 2.
2. Preliminaries

We collect some preparatory results and notation.

Lemma 2.1. Let $G$ be a group and $H \leq G$ such that $|H| \geq 4$ and $|G : H| \geq 4$. Then $\log |G|/\log |H| \leq \log |G : H|$.

Proof. Set $x := \log |G|$ and $y := \log |H|$. Then the desired inequality reads $x/y \leq x - y$, which is equivalent to $x \geq y + 1 + \frac{1}{y-1}$. Since $y \geq 2$ because $|H| \geq 4$, this is clearly satisfied if $x \geq y + 2$, which is equivalent to $|G : H| \geq 4$. \hfill \Box

Lemma 2.2. Let $G$ be an almost simple group with socle $T$. Let $M$ be a maximal subgroup of $G$ and let $M_0 := T \cap M$. Then $|M_0| \geq 6$.

Proof. We can assume that $T \not\leq T$. Since $G$ is almost simple, we have $M_0 \neq 1$ (2 Theorem 1.3.6]) whence $|M_0| \geq 2$. Moreover, $M_0 \leq M$, so by maximality of $M$, the fact that $T$ is simple, and $1 < M_0 < T$, we get that $M = N_G(M_0)$ and $M_0 = M \cap T = N_T(M_0)$. Suppose, by contradiction, that $2 \leq |M_0| \leq 5$. Then $M_0$ is contained in a Sylow $p$-subgroup $P$ of $T$ where $p \in \{2, 3, 5\}$ according to the case. If $M_0 < P$ then $M_0 < N_P(M_0) \leq N_T(M_0) = M_0$ - a contradiction. Thus $M_0$ is a Sylow $p$-subgroup of $T$. But $2 \leq |M_0| \leq 5$ implies that $M_0$ is abelian so $M_0 \leq C_T(M_0) \leq N_T(M_0) = M_0$. Thus $M_0 = Z(N_T(M_0))$, and by Burnside’s $p$-complement theorem (11 10.21), $M_0$ has a normal $p$-complement in $T$ - a contradiction since $T$ is simple. \hfill \Box

The following lemma is an easy corollary to a major result of [5]. Let $x^G$ denote the conjugacy class of $x$ in $G$.

Lemma 2.3. Let $T$ be a non-abelian simple group. Then there exist $\alpha, \beta \in T$ such that $T = \alpha^T \beta^T S$, where $S$ is any subset of $T$ of size at least 2. In particular, there exist $\alpha, \beta \in T$ such that $T = \alpha^T \beta^T \gamma^T$ where $\gamma := \beta^{-1} \alpha^{-1}$.

Proof. By [5 Theorem 1.4] there exist $\alpha, \beta \in T$ with $\alpha^T \beta^T \cup \{1\} = T$. If $\alpha^T \beta^T = T$ then we are done. Otherwise, $\alpha^T \beta^T = T - \{1\}$, and since for any $s \in T$, we have $(T - \{1\}) s = T - \{s\}$, we get that for any $s_1 \neq s_2 \in S$ we have $\alpha^T \beta^T s_1 \alpha^T \beta^T s_2 = T$ and $T = \alpha^T \beta^T S$ follows. For proving $T = \alpha^T \beta^T \gamma^T$ (for the same choice of $\alpha, \beta \in T$) we can assume $\alpha^T \beta^T = T - \{1\}$. Hence $\gamma \neq 1$, implying $|\gamma^T| \geq 2$. Now $T = \alpha^T \beta^T \gamma^T$ follows by taking $S = \gamma^T$ in the first claim. \hfill \Box

Notation 1. We denote by $\gamma^H_{\text{cp}}(G)$ the minimal positive integer $m$ such that there exist $m$ conjugates of $H \leq G$ whose product is $G$ ($\gamma^H_{\text{cp}}(G) = \infty$ if $G$ is not a product of conjugates of $H$).

For the proof of Theorem[10] we use the classification of finite primitive permutation groups as given by the O’Nan-Scott theorem, for which we adopt the formulation and notation of [10]. Thus $G$ is assumed to be a primitive permutation group on a set $\Omega$ of size $n = |G : H|$ where $H = G_\alpha$ is the stabilizer of some $\alpha \in \Omega$. The socle of $G$ is denoted $B \cong T^k$ with $k \geq 1$, where $T$ is a simple group. Since $B$ acts transitively on $\Omega$ (being a non-trivial normal subgroup of a primitive group), we have $G = BG_\alpha = BH$. Suppose that $B$ is contained in the product of $t$ conjugates of $H$. Then $G$ is a product of $t$ conjugates of $H$ (see [3 Lemma 7(2)]). Moreover, since $B_\alpha = B \cap H \leq H$, we get that $B$ is certainly contained in the product of $\gamma^H_{\text{cp}}(B)$ conjugates of $H$. These considerations show that $\gamma^G_{\text{cp}}(G) \leq \gamma^B_{\text{cp}}(B)$ while
Let \( G \) be an affine primitive permutation group with a non-trivial point stabilizer \( H \). Then \( G \) is a product of at most \( 1 + c_A \log |G : H| \) conjugates of \( H \), where \( 0 < c_A \leq 3/\log 5 < 1.3 \) is a universal constant.

In order to prove Proposition 3.1, we review some basic properties of affine primitive permutation groups. If \( G \) is an affine primitive permutation group, then it has exactly one minimal normal subgroup \( V \), which is abelian so \( V \cong C_p^l \) for some prime \( p \) and some natural number \( l \). Moreover \( G = VH \) and, viewing \( V \) as the vector space over \( F_p \), then \( H \) acts by conjugation irreducibly as a group of linear transformations on \( V \). When convenient we will use additive notation for \( V \).

Lemma 3.2. Let \( G \) be an affine primitive permutation group with point stabilizer \( H \) and minimal normal subgroup \( V \cong C_p^l \). Let \( h \in H \) and \( v \in V \). Set \( w := v^{h^{-1}} - v \) and \( k := \lceil \log p \rceil \). Then \( \langle w \rangle \) is contained in a product of \( k + 1 \) conjugates of \( H \).

Proof. We can assume \( w \neq 0_V = 1_G \) for which the claim clearly holds. Then \( w \) is of order \( p \), and any element of \( \langle w \rangle \) is of the additive form \( sw \) where the integer \( s \) satisfies \( 0 \leq s \leq p - 1 \). Since \( k := \lceil \log p \rceil \), the base 2 representation of \( s \) takes the form \( s = \sum_{j=0}^{k-1} b_j 2^j \) (\( b_j \in \{0, 1\} \) for all \( 0 \leq j \leq k - 1 \)). Now note that \( w = v^{h^{-1}} - v = v^{-1} h v h^{-1} \in H^v H \). Similarly, for any \( c \in F_p \) we have \( cw = (cv)^{h^{-1}} - cv \in H^{cv} H \). Thus, identifying the powers \( 2^j \) with elements of \( F_p \), we see that \( sw \in (H^v H) (H^{2v} H) (H^{2^2 v} H) \cdots (H^{2^{k-1} v} H) \), for any \( 0 \leq s \leq p - 1 \), where we pick \( 0_V \) from the \( j \)-th factor \( (H^{2^j v} H) \) in the product if \( b_j = 0 \) and \( 2^j \) otherwise.

Finally, for the choice \( u_0 = 2k^{-1} v, u_{k-2} = u_{k-3} = 2k^{-2} v \) and in general \( u_{k-j} = u_{k-j+1} + 2k^{-j+1} v \) for all \( 2 \leq j \leq k \). where \( u_0 = 0_V \), we get that \( \Pi_H \) is equal to a product of \( k + 1 \) conjugates of \( H \).

Lemma 3.3. For each prime number \( p \) define \( f(p) := \lceil \log p \rceil / \log p \). Then \( f(p) \) has a global maximum at \( p = 5 \). Consequently

\[
\lceil \log p \rceil \leq (3/\log 5) \log p, \text{ for every prime } p.
\]
Proof. First check that $1 + 1/\log 11 < 1.29 < 3/\log 5$. Then, using this, we get:

$$f(p) \leq (\log p + 1)/\log p = 1 + 1/\log p < 3/\log 5 = f(5), \forall p \geq 11,$$

and for $p = 2, 3, 7$ we verify explicitly that $f(p) < f(5)$. Hence $f(p)$ has a global maximum $f(5) = 3/\log 5$ at $p = 5$. Finally, $\lceil \log p \rceil = f(p) \log p \leq f(5) \log p$. $\square$

Proof of Proposition 3.1 Using the notation introduced after the statement of the proposition, $\log |G : H| = \log |V| = \log p^l = l \log p$. Using Inequality 3.1, we obtain:

$$1 + l[\log p] \leq 1 + (3/\log 5)l \log p = 1 + (3/\log 5) \log |G : H|.$$

Thus, it is enough to show that $G$ is a product of at most $1 + l[\log p]$ conjugates of $H$.

Fix a non-zero vector $v \in V$. If $v$ is central in $G$ then $V = \langle v \rangle$ by minimality of $V$. It follows that $H$ is a non-trivial normal subgroup of $HV = G$ since $V$ is central - a contradiction to $H$ being core-free. Therefore $v$ is not central, and there is some $h \in H$ with $v^{h^{-1}} \neq v$. Set $w := v^{h^{-1}} - v$.

We claim that there are $l$ elements $h_1, \ldots, h_l \in H$ such that $B := \{w^{h_1}, \ldots, w^{h_l}\}$ is a vector space basis of $V = C_p^l$. Note that $w \neq 0_V$, this claim is immediate for $l = 1$, and hence we assume $l \geq 2$. Suppose by contradiction that $1 \leq m < l$ is the maximal integer such that there exist $h_1, h_2, \ldots, h_m \in H$ for which $B = \{w^{h_1}, \ldots, w^{h_m}\}$ is linearly independent. It follows that for any $h \in H$, $w^h \in \text{Span}(B)$. Thus $\text{Span}(B) = \text{Span}(\{w^h| h \in H\})$. This shows that $\text{Span}(B)$ is a proper non-trivial $H$-invariant subspace of $V$, contradicting the fact that $H$ acts irreducibly on $V$. Thus there exists a basis of $V$ of the form $B := \{w^{h_1}, \ldots, w^{h_l}\}$.

For each $v \in V$ there exist $s_1, \ldots, s_l \in \mathbb{F}_p$ for which $v = \sum_{i=1}^l s_i w^{h_i}$. Applying Lemma 3.2 to each $w^{h_i}$ separately, we get that each $v \in V$ belongs to $\prod_i \Pi_i$, where each $\Pi_i$ is a product of $\lceil \log p \rceil + 1$ conjugates of $H$. But, as in the proof of Lemma 3.2 this shows that $V \subseteq \prod_{i=1}^{\lceil \log p \rceil} \Pi_i$ for any choice of $u_1, \ldots, u_{\lceil \log p \rceil} - 1 \in V$, and one can choose these elements so that the product $\prod_{i=1}^{\lceil \log p \rceil} \Pi_i$ is a product of at most $l[\log p] + 1$ conjugates of $H$. $\square$

From Proposition 3.1 it follows that if $G$ is an affine primitive permutation group then $\gamma^H_{cp}(G) \leq c_1 \log n$ where the constant $c_1$ satisfies $0 < c_1 < 2.3$.

4. Type II. G is an Almost Simple Primitive Permutation Group

In this case we have $k = 1$ and $B = T$. Note that $T$ is a non-abelian simple group acting transitively on $\Omega$. Furthermore, $T$ does not act regularly on $\Omega$ by [10].

First suppose that $|G| < n^9$. By [11] Theorem 3, since $T_\alpha$ is a subset of $T$ of size at least 2 (because $T$ does not act regularly), there exists a constant $c_1$ such that $T$ is a product of less than $c_1 \log |T|$ conjugates of $T_\alpha$. Now $|T| \leq |G| < n^9$ implies that $\gamma^T_{cp}(T) < 9 \cdot c_1 \log n$.

Assume that $|G| \geq n^9$. By [12] one of the following holds:

(1) $T = A_m$, where $m \geq 5$ and either
   (a) $\Omega$ is the set of all subsets of size $k$ of $\{1, \ldots, m\}$, $n = \binom{m}{k}$ or
   (b) $\Omega$ is the set of all partitions of $\{1, \ldots, m\}$ into $a$ subsets of size $b$ where $ab = m$, $a > 1$, $b > 1$; $n = m!/((b!)^a!)$.
(2) $T$ is a classical simple group acting on an orbit of subspaces of the natural module, or (in the case $T = \text{PSL}(d,q)$) on pairs of subspaces of complementary dimensions.

Since $n = |G : G_\alpha| = |T : T_\alpha|$, and since $G$ is almost simple, we have by Lemma 2.7 (i) that $|G : T| \leq |\text{Out}(T)| < n$. This gives $|T| > 2^n = 8^{\frac{\log|T|}{\log|T_\alpha|}}$, implying $\frac{\log|T|}{\log|T_\alpha|} < \frac{8}{2}$. If $T_\alpha$ is maximal in $T$, we can conclude from \cite{S} Theorem 2] that there exist a universal constant $c_2$ and a universal function $f : \mathbb{N} \to \mathbb{N}$, such that for all $T$ satisfying $|T| > f(2)$ it holds that $\gamma_{cp}^\alpha(T) \leq c_2 \frac{\log|T|}{\log|T_\alpha|}$. Now we have claim that this conclusion is in fact valid even if $T_\alpha$ is not maximal in $T$. More precisely, we claim that \cite{S} Theorem 2 is valid for all subgroups belonging to the families listed in \cite{S} Lemma 3.1 in the case $T = A_n$, and in \cite{S} Lemma 4.3 in the case that $T$ is a classical group. Note that these families include the $(T, T_\alpha)$ of \cite{S} listed above. Our claim is based on a close examination of the use of the maximality assumption in the proof of \cite{S} Theorem 2. We find that the maximality assumption is used only in two places. First, in appealing to \cite{S} Theorem 1 in order to discard cases of simple groups of Lie type of small Lie rank. Here we replace \cite{S} Theorem 1] by \cite{N} Theorem 1.3, which applies to any subset of $T$ of size at least 2. The second use of the maximality assumption is to identify the possible isomorphism types for maximal subgroups of the remaining simple groups, according to the O'Nan-Scott classification in the alternating case and the Aschbacher classification in the classical case. These are precisely the families listed in \cite{S} Lemma 3.1 and in \cite{S} Lemma 4.3. The rest of the proof of Theorem 2 of \cite{S} carries through even when the subgroups in question are not actually maximal.

Finally, by Lemma 2.1 and Lemma 2.2 we get that $\gamma_{cp}^H(G) \leq c_{II} \log n$ for some universal constant $c_{II} > 0$, for all primitive almost simple $G$.

5. Type III(a). $G$ is a primitive permutation group of diagonal type.

Here $B_\alpha$ is the diagonal subgroup of $B$ ($\Delta$ in the notation of Proposition 5.1 below) and $n = |G : G_\alpha| = |T|^k - 1$, where $k \geq 2$.

Proposition 5.1. Let $T$ be a non-abelian simple group, $k$ a positive integer, $B := T^k$. Set $\Delta := \{(t, t, \ldots, t) : t \in T\} \leq B$. Then $k \leq \gamma_{cp}^\Delta(B) \leq 3k - 2$.

Proof. Suppose $B$ is a product of $m$ conjugates of $\Delta$. Then $\Delta \cong T$ implies $|T|^k = |B| \leq |\Delta|^m = |T|^m$. This proves $k \leq \gamma_{cp}^\Delta(B)$. For proving $\gamma_{cp}^\Delta(B) \leq 3k - 2$, choose $\alpha, \beta, \gamma \in T$ as in Lemma 2.3. Set $a := \alpha^{-1}$ and $b := \gamma$. Then $T = \alpha T \beta^T \gamma^T = (a^{-1})^T (ab^{-1})^T b^T$.

Let $i \in \{1, \ldots, k\}$. Let $\tau_i : T \to T^k$ be the map that sends $t \in T$ to the element of $T^k$ that has $t$ in the $i$-th component and 1 elsewhere. We denote $T_i := \tau_i(T)$. Consider $D_i := \Delta \Delta^{\tau_i(a)} \Delta^{\tau_i(b)} \Delta$. We prove that $D_i \supseteq T_i$. An element of $D_i$ has the form

$$(xyzw, xyzw, \ldots, xyzw, xy^a z^bw, xyzw, \ldots, xyzw)$$

for some $w \in T$. This gives $|D_i| \geq 2^{k-1} |T|$.
where \( x, y, z, w \in T \) are arbitrary. In order to prove that \( D_i \supseteq T_i \), choose arbitrary \( x, y, z \in T \) and \( w = (xyz)^{-1} \). Then for the \( i \)-th component we have
\[
xy^a z^b w = xy^a z^b (xyz)^{-1} = xa^{-1}yab^{-1}bz^{-1}y^{-1}x^{-1}
\]
\[
= (xa^{-1}x^{-1})((xy)(ab^{-1})(y^{-1}x^{-1}))(b(z^{-1}y^{-1}x^{-1}))
\]
\[
\in (a^{-1})^T(ab^{-1})^Tb^T = T.
\]
Since \( \{ (x, xy, xyz) \mid x, y, z \in T \} = T^3 \), we can deduce \( D_i \supseteq T_i \).

It follows that \( B = T_1 \cdots T_k = \Delta T_2 \cdots T_k \subseteq \Delta D_2 \cdots D_k = D_2 \cdots D_k \). Therefore
\[
B = D_2 \cdots D_k = (\Delta \Delta \tau_2(a) \Delta \tau_2(b) \Delta) \cdots (\Delta \Delta \tau_k(a) \Delta \tau_k(b) \Delta)
\]
\[
= (\Delta \Delta \tau_2(a) \Delta \tau_2(b)) \cdots (\Delta \Delta \tau_k(a) \Delta \tau_k(b)) \Delta,
\]
and \( B \) is a product of \( 3(k - 1) + 1 = 3k - 2 \) conjugates of \( \Delta \). \( \square \)

By Proposition 5.1, we have \( \gamma_{\mathcal{G}_P}(B) \leq 3k - 2 \). On the other hand
\[
\log |G : G_\alpha| = (k - 1) \log |T| \geq (k - 1) \log 60 > 5(k - 1).
\]
Comparing the numbers we see that \( B \) is the product of less than \( \log |G : G_\alpha| \) conjugates of \( B_\alpha \), and so we have \( \gamma_{\mathcal{G}_P}(G) \leq c_{III(a)} \log n \) with \( 0 < c_{III(a)} \leq 1 \).

6. **Type III(b).** \( G \) is a primitive permutation group of product action type.

Let \( R \) be a primitive permutation group of type II or III(a) on a set \( \Gamma \). For \( \ell > 1 \), let \( W = R \wr S_\ell \), and take \( W \) to act on \( \Omega = \Gamma^\ell \) in its natural product action. Then for \( \gamma \in \Gamma \) and \( \alpha = (\gamma, \ldots, \gamma) \in \Omega \) we have \( W_\alpha = R_\gamma \wr S_\ell \), and \( n = |\Gamma|^\ell \). If \( K \) is the socle of \( R \) then the socle \( B \) of \( W \) is \( K^\ell \), and \( B_\alpha = (K_\gamma)^\ell \neq 1 \). If \( G \) is primitive of type III(b), then \( G \) satisfies \( B \leq G \leq W \) and acts transitively on the \( \ell \)-factors of \( B = K^\ell \). In particular, \( \text{soc}(G) = \text{soc}(W) = K^\ell \). By the discussion of cases II and III(a) we know that \( K \) is the product of at most \( \{ c_{III, c_{III(a)}} \} \cdot \log |K : K_\gamma| \) conjugates of \( K_\gamma \). Since \( B = K^\ell \) and \( B_\alpha = (K_\gamma)^\ell \), we get that \( B \) is the product of at most \( \{ c_{III, c_{III(a)}} \} \cdot \log |K : K_\gamma| \) conjugates of \( B_\alpha \). Now \( |G : G_\alpha| = |\Gamma|^\ell \), and, since \( K \) acts transitively on \( \Gamma \), \( |\Gamma| = |K : K_\gamma| \). Hence
\[
\log n = \log |G : G_\alpha| = \log |\Gamma|^\ell = \ell \log |K : K_\gamma|,
\]
and we have proved that \( \gamma_{\mathcal{G}_P}(G) \leq c_{III(b)} \log n \) with \( 0 < c_{III(b)} \leq \max \{ c_{III, c_{III(a)}} \} \).

7. **Type III(c).** \( G \) is a primitive permutation group of twisted wreath product type.

Let \( P \) be a transitive permutation group of degree \( k \), acting on \( \{ 1, \ldots, k \} \), and let \( Q \leq P \) be the stabilizer of 1. Let \( \varphi : Q \rightarrow \text{Aut}(T) \) be a homomorphism such that \( \varphi(Q) \) contains all the inner automorphisms of \( T \). Let
\[
B_0 = \{ f : P \rightarrow T : f(pq) = f(p)\varphi(q) \quad \forall p, q \in Q \}.
\]
Then \( B_0 \) is a group with pointwise multiplication. Let \( L = \{ l_1, \ldots, l_k \} \subseteq P \) be an arbitrary fixed left transversal of \( Q \) in \( P \). By definition of \( B_0 \), a function \( f \in B_0 \) is determined by its values on \( L \). On the other hand, the values of \( f \) on \( L \) can be
arbitrary, and therefore we get $B_0 \cong T^k$. More specifically, for $\ell \in L$ and $t \in T$ define $f_{\ell,t} : P \to T$ by:

$$f_{\ell,t}(x) := \begin{cases} t^{\varphi(t^{-1}x)} & \text{if } x \in \ell Q, \quad \forall x \in P, \\ 1 & \text{if } x \not\in \ell Q \end{cases}$$

We claim that $f_{\ell,t} \in B_0$. Indeed, let $p \in P$, $q \in Q$, and consider $f_{\ell,t}(pq)$. If $pq \not\in \ell Q$ then $p \not\in \ell Q$ and $f(pq) = f(p) = 1$ so $f(pq) = f(p)^{\varphi(q)}$ holds. If $pq \in \ell Q$ then $p \in \ell Q$ and there exists $q_0 \in Q$ such that $p = t^\ell q_0$. Hence

$$f_{\ell,t}(pq) = f_{\ell,t}\left(t^\ell q_0 q\right) = t^{\varphi(t^{-1}t^\ell q_0 q)} = t^{\varphi(q_0)} = f(p)^{\varphi(q)} = f_{\ell,t}(p)^{\varphi(q)}.$$ 

Furthermore, if $\ell = l_i$ then $f_{\ell,t}$ corresponds to the element of $T^k$ that has $t$ in the $i$-th component and 1 elsewhere. To see this we just have to check that $f_{l_i,t_i}(l_j)$ satisfies $f_{l_i,t_i}(l_j) = t$ if $i = j$ and $f_{l_i,t_i}(l_j) = 1$ if $i \neq j$ and this is immediate from the definition. Thus we can construct an explicit isomorphism $B_0 \to T^k$ which maps $\{f_{l_i,t_i} | t_i \in T\}$ onto $T_i$, where $T_i$ is the $i$-th direct factor of $T^k$, $1 \leq i \leq k$. From now on we identify $\{f_{l_i,t_i} | t_i \in T\}$ with $T_i$. Furthermore, $P$ acts on $B_0$ in the following way: if $f \in B_0$ and $p \in P$ define $f^p(x) := f(px)$ for all $x \in P$. The semidirect product $G := B_0 \rtimes P$ with respect to this action is called the twisted wreath product $(P \times T)$. Then $G$ acts transitively by right multiplication on the set $\Omega$ of size $n = |B_0|$ of all right cosets of $P$. This action is not always primitive. If it is, $G$ belongs to class III($\ell$). In this case $B = B_0 \cong T^k = T_1 \times \cdots \times T_k$ is a normal subgroup (the unique minimal one) in $G$ and acts regularly on $\Omega$, and we take $G_\alpha = P$. We have $n = |T|^k$.

Set, for each $1 \leq i \leq k$, $Q_i := l_iQl_i^{-1}$. We prove that $Q_i$ leaves $T_i$ invariant with respect to the action of $P$ on $B_0$. For this we have to show that if $p \in Q_i$, namely, $p = l_iql_i^{-1}$ for some $q \in Q$, then, for all $x \in P$, $x \in l_iQ$ if and only if $px \in l_iQ$. But $px \in l_iQ$ means $l_iql_i^{-1}x \in l_iQ$, and this is true if and only if $ql_i^{-1}x \in Q$, which, since $q \in Q$, is equivalent to $T_i^{-1}x \in Q$, which is equivalent to $x \in l_iQ$.

Thus the action of $P$ on $B_0$ induces an action of $Q_i$ on $T_i$ for each $1 \leq i \leq k$. Note that for $p = l_iql_i^{-1} \in Q_i$ and $x = l_iq_0 \in l_iQ$, we get

$$f_{l_i,t_i}^p(x) = f_{l_i,t_i}(px) = t^{\varphi(l_i^{-1}px)} = t^{\varphi(l_i^{-1}ql_i^{-1}q_0)} = t^{\varphi(q_0)} = f_{l_i,t_i}(q_0) = f_{l_i,\varphi(a),T_i}(x).$$

Since $f_{l_i,t_i}^p(x) = f_{l_i,\varphi(a),T_i}(x)$ clearly holds also for any $x \not\in l_iQ$ we get $f_{l_i,t_i}^p = f_{l_i,\varphi(a),T_i}$ and so $Q_i$ acts on $T_i$ as $\varphi(Q)$. By Lemma 2.3 there exist $t_{i,1}, t_{i,2}, t_{i,3} \in T_i$ for each $1 \leq i \leq k$ such that $T_i = t_{i,1}t_{i,2}t_{i,3}$. For each $j \in \{1,2,3\}$ set $O_{i,j} := \{p^{-1}t_{i,j}p | p \in Q_i\}$. In $G := B_0 \rtimes P$, the action of $P$ on $B_0$ is the restriction of the conjugation action of $G$ on itself, and hence $O_{i,j}$ is the orbit of $t_{i,j}$ under the action of $Q_i$ on $T_i$. Since, by assumption, $\text{Inn}(T) \leq \varphi(Q)$, $O_{i,j}$ is a normal subset of $T_i$, and in particular contains $t_{i,j}^2$, the conjugacy class of $t_{i,j}$ in $T_i$. Set $X_{i,j} := t_{i,j}^{-1}O_{i,j}$, $1 \leq j \leq 3$. Since the $O_{i,j}$’s are normal sets,

$$T_i = t_{i,1}^{-1}t_{i,2}^{-1}t_{i,3}^{-1}t_{i,1}t_{i,2}t_{i,3} = t_{i,1}^{-1}t_{i,2}^{-1}t_{i,3}^{-1}O_{i,1}O_{i,2}O_{i,3} = X_{i,1}X_{i,2}X_{i,3}. $$
Moreover $X_{i,j} = \{t_{i,j}^{-1}p^{-1}t_{i,j}p : p \in Q_i\} \subseteq P^{t_{i,j}}P$ for $j = 1, 2, \text{ and } X_{i,3} = t_{i,3}^{-1}O_{i,3} = O_{i,3}t_{i,3}^{-1} = \{p^{-1}t_{i,3}p^{-1}t_{i,3}^{-1} : p \in Q_i\} \subseteq PP^{-1}$. It follows that

$$T_i = X_{i,1}X_{i,2}X_{i,3} \subseteq PP^{-1}P.$$ 

Thus $B = T_1 \cdots T_k$ is contained in a product of at most $5k$ conjugates of $P$, and since $n = |T|^k$ we have $k = \log n / \log |T|$. Hence

$$\gamma^P_{cp}(G) \leq 5k = 5 \log n / \log |T| \leq \frac{5}{\log 60} \log n < \log n.$$

We have proved that if $G$ is a primitive group of twisted wreath product type then $\gamma^H_{cp}(G) \leq c_{III}(c) \log n$ where $0 < c_{III}(c) < 1$.

This completes the proof of Theorem 1.

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