TWISTED ARROW CONSTRUCTION FOR SEGAL SPACES

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ABSTRACT. We give an explicit description of the twisted arrow construction for simplicial spaces and demonstrate individually that it preserves the defining properties of a complete Segal space. Moreover, we show that for a Segal space, the natural projection from the twisted arrow Segal space is a left fibration.

1. Introduction

1.1. Twisted arrow categories. Twisted arrow categories have proven extremely useful in the study of a variety of categorical settings, such as computations of (lax) limits and (co)ends [Mac71, Lor21], categorical logic [Law70], configuration spaces [Seg73], algebraic K-theory [Wal85], and the study of exponentiable fibrations [Joh99, BN00].

Category theory has since been generalized to what is now called higher category theory, (∞,1)-category theory or simply ∞-category theory [Ber10]. Similar to the 1-categorical situation, higher categorical analogues of the twisted arrow construction have also found a variety of applications, such as in derived geometry [Lur17, Lur11], algebraic K-theory of higher categories [Bar17, BGN18, BOO+18], computing lax ∞-limits [GHN17], and decomposition spaces [HK22].

As such the twisted arrow ∞-category has already been studied in a variety of settings, such as via quasi-categories [Lur17], 2-Segal spaces [BOO+20] and internal ∞-categories [Mar21].

1.2. Construction of twisted arrow complete Segal spaces. In this work we focus on the twisted arrow construction in the context of complete Segal spaces, which is a prominent model of (∞,1)-categories introduced by Rezk [Rez01] and further studied in a variety of settings [Toë05, JT07]. First, in Theorem 3.8 we give a short and self-contained proof that the twisted arrow construction preserves the three defining characteristics of a complete Segal space (Reedy fibrancy, Segal condition and completeness condition), only assuming basic aspects of the complete Segal space model structure as given in the original paper [Rez01]. The proof for each one of the conditions can already be found separately in the literature (such as Reedy fibrancy in [BOO+20] or the Segal and completeness condition [Mar21]), however using far more advanced and distinct machinery that are not directly compatible with each other.

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Second, we show that the natural projection map from the twisted arrow Segal space is a left fibration (Theorem 4.3), which generalizes several existing results. Indeed, it generalizes an analogous result for the twisted arrow projection of quasi-categories [Lur17, Proposition 5.2.1.3], which implicitly took the completeness condition for granted, which our result demonstrates to be a superfluous assumption. Moreover, combining this result with [Mar21, Proposition 4.2.5] also gives us the same result for internal Segal objects, generalizing twisted internal \(\infty\)-categories defined there. Finally, the twisted arrow construction in particular gives us a parameterized version of slice-Segal spaces (Remark 4.5) and this result hence also generalizes similar results regarding the existence of over-Segal spaces without assuming completeness [Ras17, Theorem 3.44], [RS17, Proposition 8.13].

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2. Twisted Arrow Simplicial Spaces

Before we commence let us recall relevant facts regarding simplicial spaces, following the definitions and notations in [Rez01]. A simplicial space \(W\) is a bisimplicial set, that is a functor \(W : \Delta^{op} \times \Delta^{op} \to \text{Set}\). We denote the category of simplicial spaces by \(s\text{Set}\) and the simplicial space represented by \(([n], [l])\) by \(F(n) \times \Delta[l]\). To simplify notation if \(l = 0\) \((n = 0)\), we use the notation \(F(n)\) \((\Delta[l])\). More generally, for a given simplicial set \(S\) we also denote by \(S\) the simplicial space given as \(S_{nl} = S_l\). This gives us a fully faithful inclusion from simplicial sets to simplicial spaces, which preserves \(\Delta[l]\) and allows us to define a simplicial tensor: For a given simplicial space \(X\) and simplicial set \(S\) we define the simplicial tensor as the Cartesian product \(X \times S\). The tensor also gives us a simplicial enrichment explicitly given by \(\text{Map}_{s\text{Set}}(X, Y)_l = \text{Hom}_{s\text{Set}}(X \times \Delta[l], Y)\). To simplify notation, we denote the simplicial set \(\text{Map}_{s\text{Set}}(F(n), Y)\) by \(Y_n\). Moreover, recall from [JT07, Section 4] the inclusion functor \(p^n_1 : s\text{Set} \to s\text{Set}\) defined as \(p^n_1(S)_{nl} = S_n\), which in particular means \(p^n_1(\Delta[n]) = F(n)\).

We now proceed to review relevant aspects of the twisted arrow construction for simplicial sets, as covered in [Lur17, Section 5.2.1]. Let \(Q : \Delta \to \Delta\) be the functor defined as \(Q([n]) = [n]^{op} \star [n]\) and define the functor \(\text{Tw}_{s\text{Set}} : s\text{Set} \to s\text{Set}\) as \(\text{Tw}_{s\text{Set}}(S) = S \circ Q\). This in particular means that \(\text{Tw}_{s\text{Set}}(S)_n = S_{2n+1}\). Moreover, the two natural inclusions \([n]^{op} \hookrightarrow [n]^{op} \star [n]\), \([n] \hookrightarrow [n]^{op} \star [n]\) induce a projection map \(\text{Tw}_{s\text{Set}}(S) \to S^{op} \times S\), which we call the twisted arrow projection. See [Lur17, Construction 5.2.1.1] for more details regarding this definition.

We want to define \(\text{Tw} : s\text{Set} \to s\text{Set}\) such that it satisfies the following two conditions:

(1) It should lift the original twisted arrow construction, meaning for a given simplicial set \(W\), \(\text{Tw}(p^n_1W) = p^n_1\text{Tw}_{s\text{Set}}(W)\).

(2) The functor should be simplicially enriched and tensored.

We claim that these two conditions determine the functor uniquely. First of all, following [Kel05, Theorem 4.51], a simplicially enriched functor \(\text{Tw} : s\text{Set} \to s\text{Set}\) that preserves colimits and tensors is uniquely determined by its restriction \(\text{Tw} \circ F(\bullet) : \Delta \to s\text{Set}\). Moreover, by the first assumption we need to have \(\text{Tw}(F(\bullet)) = p^n_1\text{Tw}_{s\text{Set}}(\Delta(\bullet))\) as functors from \(\Delta\) to \(s\text{Set}\). Combining these two facts we have the following definition.
Definition 2.1. Let $\text{Tw} : sS \to sS$ be the unique simplicially enriched and tensored functor that lifts the following functor

$$
\begin{array}{ccc}
\Delta & \rightarrow & s\text{Set} \\
\downarrow F(\bullet) & & \downarrow p_1^* \\
S & \rightarrow & sS \\
\end{array}
$$

Unwinding the definitions and the fact that $\text{Tw}_{sS}(\Delta[n]) = \Delta[2n+1]$, it follows that $\text{Tw}(F(n) \times \Delta[l]) = F(2n+1) \times \Delta[l]$. Moreover, we have isomorphisms of simplicial sets $\text{Tw}(W)_n \cong W_{2n+1}$. We also have the following alternative characterizations of $\text{Tw}$, the proof of which is a straightforward computation.

Lemma 2.2. $\text{Tw}$ is equal to the functor $\text{Fun}(\Delta^{\text{op}}, s\text{Set}) : sS \rightarrow sS$.

Hence, $\text{Tw}$ can also be obtained by applying $\text{Tw}_{sS}$ level-wise. This in particular means that similar to the case for simplicial sets, for every simplicial space $W$ we have the twisted arrow projection $\text{Tw}W \rightarrow W^{\text{op}} \times W$.

Remark 2.3. If we relax one of the conditions above, we get other possible candidates that are not as suitable. For example, the functor $W_{n,l} \mapsto W_{2n+1,2l+1}$ does lift the twisted arrow construction for simplicial sets, however, it is not simplicial.

3. Preservation Properties of the Twisted Arrow Construction

We now show the twisted arrow construction preserves the defining properties of a complete Segal space.

3.1. Reedy Fibrancy. We commence by showing that $\text{Tw}$ preserves Reedy fibrancy. Recall from [Ree74], but also [Rez01, 2.4] that a simplicial space $W$ is Reedy fibrant if for all $n \geq 0$, the map

$$
\text{Map}(F(n), W) \rightarrow \text{Map}(\partial F(n), W)
$$

is a Kan fibration.

Before we proceed to the proof, we make the following observation regarding Reedy fibrancy of $\text{Tw}W$. Let us define $\partial_{\text{Tw}}F(2n+1)$ as the following coequalizer diagram

$$
\begin{array}{ccc}
\bigoplus_{0 \leq i < j \leq n} F(2n-3) & \rightarrow & \bigoplus_{0 \leq i \leq n} F(2n-1) \\
& & \rightarrow \partial_{\text{Tw}}F(2n+1) \\
\end{array}
$$

Using the fact that $\text{Tw}$ preserves colimits and $\partial F(n)$ is precisely given via an analogous coequalizer diagram [GJ09, Proposition 2.3], it follows that $\text{Map}(\partial F(n), \text{Tw}W) \cong \text{Map}(\partial_{\text{Tw}}F(2n+1), W)$, which means in order to establish the Reedy fibrancy of $\text{Tw}(W)$ we only need to show the map

$$
\text{Map}(F(2n+1), W) \rightarrow \text{Map}(\partial_{\text{Tw}}F(2n+1), W)
$$

is a Kan fibration.

Proposition 3.1. Let $W$ be a Reedy fibrant simplicial space. Then $\text{Tw}W$ is also Reedy fibrant.
Proof. Following the explanation above, we only need to establish that $\text{Map}(\partial T_w F(2n + 1), W) \to \text{Map}(\partial T_w F(2n + 1), W)$ is a Kan fibration. However, $W$ is Reedy fibrant and Reedy model structure is simplicial and so we only need to show that $\partial T_w F(2n + 1) \to F(2n + 1)$ is a cofibration, meaning we want to show that for all $k \geq 0$, the map $i : \partial T_w F(2n + 1)_k \to F(2n + 1)_k$ is injective.

We can explicitly describe $\partial T_w F(2n + 1)_k$ as a quotient of $\bigcup_{0 \leq i \leq n} F(2n - 1)_k$. Take two $k$-cells $\alpha, \beta$ in two copies of $F(2n - 1)$. If $\alpha, \beta$ are in the same copy then $i$ is injective by construction and so there is nothing to check. If $\alpha$ and $\beta$ live in two different copies, then by definition we have the following pullback square

$$
\begin{array}{c}
F(2n - 3)_k \\
\downarrow \\
\alpha \in F(2n - 1)_k \\
\downarrow \\
F(2n + 1)_k
\end{array}
\xrightarrow{f} 
\begin{array}{c}
F(2n - 1)_k \ni \beta \\
\downarrow \\
F(2n + 1)_k
\end{array}
$$

Now, the fact that $\alpha$ and $\beta$ map to the same element in $F(2n + 1)_k$ and the universal property of pullbacks implies that there exists a $\gamma \in F(2n - 3)_k$ such that $g(\gamma) = \alpha$ and $f(\gamma) = \beta$ and so $\alpha \sim \beta$, meaning $\alpha = \beta$ in $\partial T_w F(2n + 1)_k$. This gives us the desired injection and so we are done. □

3.2. Segal Condition. Next we want to show that the twisted arrow Segal space is a Segal space. Recall that a Reedy fibrant simplicial space $W$ is a Segal space if the maps $W_n \to W_1 \times \cdots \times W_0 \cdots \times W_0$ are Kan equivalences for all $n \geq 2$.

**Proposition 3.2.** If $W$ is a Segal space, then $T_w W$ is also a Segal space.

**Proof.** By Proposition 3.1 $T_w W$ is a Reedy fibrant simplicial space. Hence, we just need to show that $T_w(W)_n \to T_w(W)_{n-1} \times T_w(W)_{n-1}$ is a Kan equivalence for $n \geq 2$. By induction it suffices to show that $T_w(W)_n \to T_w(W)_{n-1} \times T_w(W)_{n-1}$ is an equivalence, which unwinds to showing that $W_{2n+1} \to W_{2n-1} \times W_{2n-1}$ is an equivalence. Now, the fact that the Kan model structure is right proper and the following commutative diagram

$$
\begin{array}{ccc}
W_{2n-1} & \to & W_1 \\
\downarrow & & \downarrow \\
W_{2n-1} & \to & W_1 \\
\end{array}
\quad \begin{array}{cc}
\xrightarrow{\text{Tw}(d_0)} & \xleftarrow{\pi_1} \\
W_3 \times W_1 \times W_1 & \cong W_{2n-1} \times W_1 \times W_1
\end{array}
$$

implies that we have the following Kan equivalence

$$
W_{2n-1} \times W_3 \to W_{2n-1} \times W_1 \times W_1 \times W_1 \cong W_{2n-1} \times W_1 \times W_1.
$$

This means in the following commutative triangle
the right hand map is a weak equivalence. Moreover, the left hand diagonal map is an equivalence by the Segal condition and so the desired result follows by 2-out-of-3. \(\square\)

Remark 3.3. The result also follows as a special case of [BOO+20, Theorem 2.9] as every Segal space is in particular a 2-Segal space.

3.3. The Homotopy Category of Twisted Arrow Segal Spaces. Before we proceed to the completeness condition, we will separately analyze the homotopy category of a twisted arrow Segal space, in the sense of [Rez01, Section 5]. Concretely, for a given Segal space \(W\), we want to construct an equivalence \(F_W : \text{HoTw}(W) \to \text{Twho}(W)\).

We will start with the construction of \(F_W\). Unwinding definitions, the set of objects of \(\text{HoTw}(W)\) is given by \(W_{10}\), whereas the set of objects of \(\text{Twho}(W)\) is given by \(W_{10}/\sim\), where \(\sim\) is the homotopy relation. Hence, \(F_W\) on objects maps a morphism to its homotopy class.

Similarly, a morphism in \(\text{HoTw}(W)\) is given by a homotopy class \([\sigma]\), where \(\sigma \in W_{30}\), whereas a morphism in \(\text{Twho}(W)\) is given by a pair of homotopy classes of morphisms \(([g],[h])\), where \(g,h \in W_{10}\). So, we define \(F_W\) on morphisms as \(F_W([\sigma]) = ([d_2d_2\sigma],[d_0d_0\sigma])\).

We now verify that \(F_W\) indeed preserves identities and composition. For a given object \(f : x \to y\) in \(\text{HoTw}(W)\) we have

\[ F([s_1s_1f]) = ([d_2d_2s_1s_1f],[d_0d_0s_1s_1f]) := ([id_x],[id_y]). \]

and for two given composable morphisms \([\sigma],[\sigma']\) in \(\text{HoTw}(W)\) and choice of composition \(\sigma''\) of \(\sigma,\sigma'\) we have

\[ F([\sigma']) \circ F([\sigma]) = ([d_2d_2\sigma'],[d_0d_0\sigma']) \circ ([d_2d_2\sigma],[d_0d_0\sigma]) = ([d_2d_2\sigma'',[d_0d_0\sigma'']) = F([\sigma'']). \]

Having defined \(F_W\), we now prove it is an equivalence. Notice, by construction, \(F_W\) is surjective on objects, hence we will now proceed to proving that it is also fully faithful.

Lemma 3.4. The functor \(F_W : \text{HoTw}(W) \to \text{Twho}(W)\) is fully faithful.

Proof. Throughout this proof fix two objects \(f,g\) in \(\text{HoTw}(W)\). First we show that \(F_W\) is full. Let \(([k],[h])\) be a morphism from \(F_W(f)\) to \(F_W(g)\) and fix representatives \(k,h \in W_{10}\). By the Segal condition, the triple \((k,f,h)\) in \(W_1 \times W_1 \times W_1\) gives us an element \(\sigma \in W_{30}\), such that \(d_0d_0\sigma = k\) and \(d_2d_2\sigma = h\), meaning that \(F_W([\sigma]) = ([k],[h])\), proving that \(F_W\) is full.

We now show \(F_W\) is faithful. Let \([\sigma],[\sigma']\) be two morphisms from \(f\) to \(g\), such that \(F_W([\sigma]) = F_W([\sigma'])\). We need to show that \([\sigma] = [\sigma']\), which by definition means constructing a path in the space \(* \times W_3\) from \(\sigma\) to \(\sigma'\).
Before we can proceed we need to better understand the data of such a path. By definition a path is a map \( \Delta[1] \to \ast \times \prod_{W_1 \times W_1} W_3 \), which unwinds to a map
\[
H = F(3) \times \Delta[1] \prod_{F(1) \times \Delta[1]} (F(1) \prod_{F(1) \times \Delta[1]} F(1)) \to W.
\]
As the Reedy model structure is left proper, the projection map \( F(3) \times \Delta[1] \to H \) is a Reedy equivalence. Moreover, by definition the inclusion
\[
H' = (G(3) \times \Delta[1]) \prod_{G(3) \times G(3)} (F(3) \prod_{F(3) \times \Delta[1]} F(3)) \to F(3) \times \Delta[1]
\]
is an equivalence in the Segal space model structure and so the composition \( H' \to H \) is one as well.

We now use this equivalence to construct a path from \( \sigma \) to \( \sigma' \). To simplify notation we will denote \( d_0 \sigma = k, d_2 \sigma = h, d_0 \sigma' = k', d_2 \sigma' = h' \). The assumption implies that \([k] = [k']\) and \([h] = [h']\), meaning there are paths \( \gamma_k, \gamma_h : \Delta[1] \to W_1 \) going from \( k \) to \( k' \) and \( h \) to \( h' \).

Now, the data of \((\gamma_k, f, \gamma_h, \sigma, \sigma')\), where \( f \) denotes the constant path, precisely assemble into a morphism \( H' \to W \), which lifts to a morphism \( H \to W \), as \( W \) is a Segal space. This is precisely the desired path from \( \sigma \) to \( \sigma' \) and we are done. \( \square \)

**Proposition 3.5.** If \( W \) is a Segal space, then the functor \( F_W : \text{Tw} \text{Ho}(W) \to \text{Ho} \text{Tw}(W) \) is an equivalence.

**Proof.** The functor \( F_W \) is surjective by construction and fully faithful by Lemma 3.4, hence we are done. \( \square \)

3.4. Completeness Condition. In this subsection we show that \( \text{Tw} \) also preserves the completeness condition. Recall, from [Rez01], that a Segal space \( W \) is complete if the morphism \( W_0 \to W_{\text{hoequiv}} \) is a Kan equivalence. Here \( W_{\text{hoequiv}} \) is the collection of path-components in \( W_1 \) consisting of equivalences. Further details can be found in [Rez01].

In order to proceed we need a better understanding of \( \text{Tw}(W)_{\text{hoequiv}} \) relying on our understanding of the homotopy category from the previous subsection.

**Lemma 3.6.** If \( W \) is a Segal space, then a morphism \( \sigma \)
\[
\begin{array}{ccc}
\bullet & \xleftarrow{k} & \bullet \\
\downarrow{f} & & \downarrow{g} \\
\bullet & \xrightarrow{h} & \bullet
\end{array}
\]
in the twisted arrow Segal space \( \text{Tw}W \) is a homotopy equivalence if \( k \) and \( h \) are homotopy equivalences in \( W \).

**Proof.** We need to show that the class \([\sigma]\) in \( \text{Ho} \text{Tw}(W) \) is an isomorphism if and only if \([k],[h]\) are isomorphisms in \( \text{Ho}(W) \). By Proposition 3.5, \([\sigma]\) is an isomorphism if and only if its image in \( \text{Tw} \text{Ho}(W) \) is an isomorphism. By definition of the twisted arrow category, this is equivalent to \([k]\) and \([h]\) being isomorphisms, and hence we are done. \( \square \)

**Proposition 3.7.** Let \( W \) be a Segal space. Then we have the following pullback square
Proof. We denote the pullback by $Pb$ and show that $Pb = \text{Tw}(W)_{\text{hoequiv}}$. Notice that both $Pb \rightarrow \text{Tw}(W)_1$ and $\text{Tw}(W)_{\text{hoequiv}} \rightarrow \text{Tw}(W)_1$ are inclusions of path-components, as they are stable under pullback. Hence, it suffices to show that $\text{Tw}(W)_{\text{hoequiv}}$ and $Pb$ include the same path components of $\text{Tw}(W)_1$.

If $f \in (Pb)_0$, then by Lemma 3.6, $f \in (\text{Tw}(W)_{\text{hoequiv}})_0$. Conversely, if $f \in (\text{Tw}(W)_{\text{hoequiv}})_0$, then by universal property of pullback, $f \in (Pb)_0$.

Thus, $(Pb)_0 = (\text{Tw}(W)_{\text{hoequiv}})_0$, which further implies, $Pb = \text{Tw}(W)_{\text{hoequiv}}$.  

We are finally ready to prove the main result of this section.

**Theorem 3.8.** If $W$ is a complete Segal space, then $\text{Tw}W$ is a complete Segal space.

Proof. By Proposition 3.2 we only need to check the completeness condition. By Proposition 3.7 and the pasting property of pullbacks, we have the following diagram, where the left hand square is a pullback

\[
\begin{array}{ccc}
\text{Tw}(W)_0 & \longrightarrow & \text{Tw}(W)_{\text{hoequiv}} \\
\downarrow & & \downarrow \\
W^0_0 \times W_0 & \cong & W^0_{\text{hoequiv}} \times W_{\text{hoequiv}} \\
\end{array}
\]

Since $W$ is a complete Segal space, $W^0_0 \times W_0 \cong W^0_{\text{hoequiv}} \times W_{\text{hoequiv}}$ is an equivalence of spaces, which implies that $\text{Tw}(W)_0 \cong \text{Tw}(W)_{\text{hoequiv}}$ is an equivalence of spaces. 

4. **The Twisted Arrow Construction as a Left Fibration**

In this final section we demonstrate that if $W$ is a Segal space, then the twisted arrow projection $\text{Tw}W \rightarrow W^{op} \times W$ is a left fibration. First we need some technical lemmas.

**Lemma 4.1.** If $W$ is a Segal space, then $\text{Tw}W \rightarrow W^{op} \times W$ is a Reedy fibration.

Proof. We need to show that the map

\[
\text{Map}(F(k), \text{Tw}W) \rightarrow (W^k_0 \times W_k)_{\text{Map}(\partial F(k), W^{op} \times W)} \times \text{Map}(\partial F(k), \text{Tw}W)
\]
is a Kan fibration.

By construction the left hand side is given by Map(F(2k+1), W). Moreover, by the observation in the beginning of Subsection 3.1, we have Map(∂F(k), TwW) ≃ Map(∂TwF(2k+1), W) and so the right hand side simplifies to

\[
\text{Map}(\partial Tw F(2k+1) \coprod_{\partial F(k)} F(k) \coprod_{\partial F(k)} F(k), W).
\]

As W is Reedy fibrant, it hence suffices to show that

\[
\partial Tw F(2k+1) \coprod_{\partial F(k)} F(k) \coprod_{\partial F(k)} F(k) \to F(2k+1)
\]

is a cofibration or, in other words, a level-wise injection of sets. However, this directly follows from the same argument given in Proposition 3.1 and so we are done.

Lemma 4.2. If W is a Segal space, then the following diagram is a homotopy pullback square,

\[
\begin{array}{ccc}
Tw(W)_1 & \longrightarrow & Tw(W)_0 \\
\downarrow & & \downarrow \\
W_{1}^{\text{op}} \times W_1 & \longrightarrow & W_{0}^{\text{op}} \times W_0
\end{array}
\]

Proof. By definition Tw(W)_0 = W_1, and so the strict pullback is given by Tw(W)_1 ≃ W_1 × W_1 × W_1. The desired equivalence Tw(W)_1 = W_{3} ≃ W_1 × W_1 × W_1 now follows from the Segal condition.

Theorem 4.3. If W is a Segal space, then TwW \to W_{\text{op}} \times W is a left fibration.

Proof. By Lemma 4.1, the map is a Reedy fibration and so it suffices to check the appropriate homotopy pullbacks. By [Ras17, Lemma 3.29], as W is a Segal space, it is sufficient to check the homotopy pullback condition for n = 1, which we have done in Lemma 4.2.

Remark 4.4. Following this result, [Ras17, Lemma 3.38] gives us an alternative proof that the twisted arrow construction preserves complete Segal spaces. In fact this is the approach used in [Mar21, Proposition 4.2.5], via other methods, to obtain a similar result.

Remark 4.5. The previous result implies that for a given object x in a Segal space W, the fiber of TwW \to W_{\text{op}} \times W via the inclusion \{x\} \times W \hookrightarrow W_{\text{op}} \times W is a left fibration Tw(W)_x \to W, such that (Tw(W)_x)_0 = W_1 × \Delta[0].

This means Tw(W)_x \to W simply recovers the under-Segal space left fibration W_{x/} \to W as described in [Ras17, Definition 3.41] and so Theorem 4.3 is a direct generalization of [Ras17, Theorem 3.44].

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